Acausal quantum theory for non-Archimedean scalar fields

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Abstract: We construct a family of quantum scalar fields over a $p$–adic spacetime which satisfy $p$–adic analogues of the Gårding–Wightman axioms. Most of the axioms can be formulated the same way in both, the Archimedean and non-Archimedean frameworks; however, the axioms depending on the ordering of the background field must be reformulated, reflecting the acausality of $p$–adic spacetime. The $p$–adic scalar fields satisfy certain $p$–adic Klein-Gordon pseudo-differential equations. The second quantization of the solutions of these Klein-Gordon equations corresponds exactly to the scalar fields introduced here.

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1. Introduction

Ever since the advent of Quantum Mechanics, the question of its compatibility with Special Relativity was raised. The occurrence of non-locality in the quantum world and its implications regarding the relativistic causal structure was the central theme in the well-known works by Einstein, Podolsky and Rosen, and Bell. These issues are still debated today, but there is a increasing amount of research pointing towards the fact that quantum mechanics is incompatible at a fundamental level not only with the causal structure furnished by Special Relativity (through light cones), but with any other possible causal ordering. In [6], it is concluded that the description of non-localities requires fine-tuning of the system’s parameters, thus violating a basic principle of any causal model. In [40], quantum correlations incompatible with a definite causal order are constructed (although they prove that a causal order emerges in the classical limit), and the experimental existence of these correlations is reported in [47]. See also [45] for the incompatibility of Quantum Mechanics with some non-local causal models. Applications of the absence of a predefined causal structure to quantum computations are given in [7].

Motivated by these considerations, one could wonder whether it is possible to construct a quantum field theory (QFT) on a spacetime devoid of any a priori causal structure. The notions of spacelike and timelike intervals which, from an operational point of view, characterize the causal structure, are intimately tied to the existence of a total order on the field number \( \mathbb{R} \) compatible with the algebraic field operations, so a possibility is to start from a non-ordered number field. Leaving aside the case of finite fields, the most obvious choice is to consider the non-Archimedean field of \( p \)-adic numbers \( \mathbb{Q}_p \). The corresponding spacetime would be \( \mathbb{Q}_p^4 \). In this way, \( (p-\text{adic}) \) time no longer acts as an ordering parameter. While this is completely consistent with the requirement of covariance, it raises some questions about its meaning in Quantum Mechanics; for some theoretical points of view about the possibility of quantum processes without a time parameter see [64,46].

The spacetime \( \mathbb{Q}_p^4 \) is acausal in the broad sense of lacking a causal structure, but also in the particular, technical, sense that for any pair of points on it, the causal character. There are other possible orderings (chronological, horismos) that will be not considered here, although they are related, see [53].
there exists no causal curve connecting them (which, in particular, also implies that it is achronal). The question of the intrinsic (a)cascuality of spacetime has been studied sometime ago \[31\], and is a topic of obligated discussion when dealing with the possibility of ‘travels in time’ \[34,35\]. Acausal (portions of) spacetimes appears often in relation with wormholes in General Relativity \[38\].

There have been problems in constructing the $S$ matrix for interacting massive scalar fields in this setting \[14\], but it should be stressed that these are due to the interaction along closed timelike curves, which do not exist at all in the framework of a globally acausal spacetime such as the one presented here, where the very notion of ‘timelike’ does not make sense.

A problem present in any acausal theory is the characterization of microcausality or local commutativity, that is, the vanishing of the commutator of field operator-valued distributions when the test functions have support in spacelike separated regions. It is not clear \textit{a priori} that a theory without a causal structure will allow for vanishing commutators even restricting the domain of the involved operators, but we will show below that a similar property holds when the test functions are supported in the $p$–adic unit ball. Thus, there is no room for phenomena arising in the non-Archimedean case, such as the connection of spacelike regions by large timelike loops. It is also reasonable to expect that the consideration of $p$–adics numbers could also cure the divergences in 1–loop effective Lagrangians that appear in the real Euclidean case \[5\], although no attempt is made here to pursue this direction of research.

Another, different, kind of motivation for studying quantum field theory in the $p$–adic setting comes from the conjecture of Vladimirov and Volovich stating that spacetime has a non-Archimedean nature at the Planck scale, \[61\], see also \[55\]. The existence of the Planck scale implies that below it the very notion of measurement as well as the idea of ‘infinitesimal length’ become meaningless, and this fact translates into the mathematical statement that the Archimedean axiom is no longer valid. Before Volovich, some authors explored the possibility of constructing theories of the spacetime using background fields different from $\mathbb{R}$ and $\mathbb{C}$; for instance, in \[12\] Everett and Ulam study the Lorentz group over $\mathbb{Q}_p$ in the hope that ‘spaces of this sort might be useful in some future models of nuclear or subnuclear theories’, see also \[55\], \[56\] Chapter 6\] and references therein. Volovich’s conjecture propelled a wide variety of investigations in cosmology, quantum mechanics, string theory, QTF, etc., and the influence of this conjecture is still relevant nowadays, see e.g. \[1, 4–11, 10, 9, 15–19, 28–37, 57–61, 65, 67\]. In a completely different framework, that of the physics of complex systems, the paradigm asserting that the space of states of several complex systems has an ultrametric structure has also originated a large amount of research, see \[42, 27\] and references therein. These two ideas are the main motivations driving the development of $p$–adic mathematical physics. In particular, during the last thirty years $p$–adic QFT has been studied intensively, a topic whose importance has been highlighted by Varadarajan in \[55\].

In this article we present a second-quantization, based on Segal’s formalism, for $p$–adic free scalar fields whose evolution is described by a certain class of Klein-Gordon type pseudo-differential operators. In order to guarantee that the resulting theory has some physical content, we show that the corresponding quantum non-Archimedean scalar fields satisfy $p$–adic versions of Gårding–Wightman’s axioms. Most of them can be formulated in a way valid in both the
Archimedean and non-Archimedean cases, but some of them must be appropriately re-formulated in the \( p \)-adic setting by introducing new mathematical ideas and re-interpreting some classical constructions that are not directly available in the \( p \)-adic context. For instance, the absence of an ordering in the background number field implies some profound modifications in the usual interpretation of notions such as the timelike or spacelike character of \( p \)-adic spacetime events, and the introduction of new mathematical objects such as the \( p \)-adic restricted Lorentz group, that we will discuss below. As another example, our \( p \)-adic spectral condition does not provide a definition of energy and momentum operators, because this would require a theory of semigroups, with \( p \)-adic time, for operators acting on complex-valued functions, and such a theory does not exist at the moment. However, the outcomes of our analysis are consistent with the requirement that the mathematical description of physical reality must not depend on the background number field, see [62]. This property is due to the particular nature of the Klein-Gordon field, notice that the same is not true for the Schrödinger equation, as the number \( i \) does not have an analog in an arbitrary field.

Thus, the main conclusion is that there seems to be no obstruction to the existence of a mathematically rigorous quantum field theory (QFT) for free fields in the \( p \)-adic framework, based on an acausal spacetime. It must be remarked that we deal with free fields, omitting interactions. The reason for this is that, due to Haag’s theorem, interactions require a more technical treatment, but having a consistent theory for the free case is the first step towards a complete \( p \)-adic QFT.

We have remarked some features derived from the fact that the spacetime is \( p \)-adic. Let us now make some comment about those originated in the configuration space of the fields. A key fact is that we work with complex-valued fields. This allow us to use the tools from classical functional analysis, in particular Segal quantization. On the other hand, it is also possible to work with \( p \)-adic valued fields. In this setting, Khrennikov developed a theory of Gaussian integration of non-Archimedean-valued functions on infinite-dimensional non-Archimedean spaces and a calculus of pseudo-differential operators which is suitable for the second-quantization representation in non-Archimedean quantum field theory, see [23]-[25] and references therein. Mathematically speaking, this is a completely different setting from ours: for instance, \( p \)-adic Hilbert spaces are radically different to their complex counterparts.

The construction of a quantum field theory over a \( p \)-adic spacetime raises the question about the physical meaning of the prime \( p \). Once a choice for \( p \) is made, we can construct \( \mathbb{Q}_p^4 \) (endowed with the maximum norm) and then give it a geometric structure through a quadratic form \( q \). The geometry of the resulting spacetime, the quadratic space \((\mathbb{Q}_p^4, q)\), depends crucially on both, \( p \) and \( q \). We choose the simplest case in which the quadratic form is the unique elliptic form of dimension four and a prime number \( p \equiv 1 \mod 4 \). The first choice is motivated by the need for ellipticity when doing the explicit computation of the fundamental solutions (and the corresponding propagators) of the Klein-Gordon equation. Notice that the naive choice \( q(k) = k_0^2 - (k_1^2 + k_2^2 + k_3^2) \) is excluded because it is not elliptic. It is possible to develop a theory based on this form, but at the cost of facing greater technical difficulties. However, as we will see, our choice for \( q \) retains all the essential features of a relativistic
theory, so it is justifiable from a physical point of view. Regarding the choice of
$p$, the quantum fields introduced here will strongly depend on the geometry of
the hypersurface $V = \{ k \in \mathbb{Q}_p^4; q(k) = 1 \}$, and if we pick $p \equiv 1 \text{ mod } 4$, then we
can guarantee that $\sqrt[\alpha]{\omega(k)} \neq 0$ for any $k \in U_q$, where $U_q \subset \mathbb{Q}_p^3$ is a certain open
and compact subset (depending on $q$) that will be defined later on. Notice that,
due to these choices, we are actually defining a family of quantizations, a fact
that could be viewed as an advantage over the rigidity of the classical case.

Thus, given a prime number $p \equiv 1 \text{ mod } 4$ and a $p$-adic elliptic quadratic form
$q$ of dimension 4, we will denote by $O(q)$ the orthogonal group of $q$. As stated,
the $p$-adic Minkowski spacetime is, by definition, the quadratic space $(\mathbb{Q}_p^4, q)$,
so the Lorentz group of spacetime is $O(q)$. In this article, ‘time’ is a
$p$-adic variable, so the notions of past and future are not clearly defined. However, the
$p$-adic implicit function theorem allows us to determine $k_0$, from
$q(k_0, k) = 1$, as $k_0 = \pm \sqrt[\alpha]{\omega(k)}$, where $\sqrt[\alpha]{\omega(k)}$ is a
$p$-adic analytic function defined in $U_q$, and in this way we can define the mass shells:

$$V^\pm = \left\{ (k_0, k) \in \mathbb{Q}_p \times \mathbb{Q}_p^3; k_0 = \pm \sqrt[\alpha]{\omega(k)} , k \in U_q \right\} .$$

In the $p$-adic setting the usual geometric notion of cone does not make sense,
because it depends on the fact that the real numbers form an ordered field. Therefore,
the notion of closed forward light cone is replace by the notion of ‘closed
forward semigroup’, which is the topological closure of the additive semigroup
generated by $V^+$. This notion allow us to construct a spectral measure attached
to a strongly continuous unitary representation of the $p$-adic Poincaré group
as in the classical case, see Theorem$^2$.

We will denote by $\mathcal{F}$ the Fourier transform operator associated to the quadra-
tic form $q$. The $p$-adic Klein-Gordon operator attached to $q$ with unit mass is
defined as

$$\Box_{q,\alpha} \varphi = \mathcal{F}^{-1} \left( |q - 1|_p^{\alpha} \mathcal{F} \varphi \right)$$

where $\varphi$ is a test function and $\alpha$ is a fixed positive number.

In conventional QFT there have been some studies devoted to the optimal
choice of the space of test functions. In [22], Jaffe discussed this topic (see also
[52] and [33]); his conclusion was that, rather than an optimal choice, there
exists a set of conditions that must be satisfied by the candidate space, and any
class of test functions with these properties should be considered as valid. The
main condition is that the space of test functions must be a nuclear countable
Hilbert one. In this article, we use the following Gel’fand triple: $\mathcal{H}_\infty(\mathbb{K}) \subset
L^2_{\mathbb{K}} \subset \mathcal{H}_\infty^\ast(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$. This triple was introduced in [55]. The space
$\mathcal{H}_\infty(\mathbb{K})$ is a nuclear countable Hilbert space, which is invariant under the action
of a large class of pseudo-differential operators. This space can be considered
the ‘true’ non-Archimedean analogue of the classical Schwartz space, as we will
repeatedly justify in what follows. In fact, our results could be summarized by
saying that the Gårding–Wightman axioms make sense in the $p$-adic context if
we replace the Schwartz space of the classical framework by $\mathcal{H}_\infty(\mathbb{C})$.

The $p$-adic Klein-Gordon equation

$$\Box_{q,\alpha} u (t, x) = 0$$ (1)
admits solutions of plane wave type, more precisely, the functions
\[ \exp 2\pi i \left\{ tE^\pm - sx_1l_1 - px_2l_2 + spx_3l_3 \right\}_p, \]
where \( \{ \cdot \}_p \) denotes the \( p \)-adic fractional part, \( l = (l_1, l_2, l_3) \in \mathbb{Q}_p^3 \) is a fixed vector, and \( E^\pm = \pm \sqrt{\omega(k)} \) (here \( \sqrt{\omega(k)} \) is the \( p \)-adic dispersion) are weak solutions of (1), see Theorem 3. The general solution of (1), up to multiplication by a non-zero complex constant, is
\[ \int_{U_q} \left( \chi_p \left( -\sqrt{\omega(k)}t + k \cdot x \right) a(k) + \chi_p \left( \sqrt{\omega(k)}t - k \cdot x \right) a^*(-k) \right) \frac{d^3k}{|\sqrt{\omega(k)}|_p}. \]
where \( \chi_p(\cdot) = \exp \left( 2\pi i \{ \cdot \}_p \right) \) is the standard additive character of \( \mathbb{Q}_p \), \( U_q \subset \mathbb{Q}_p^3 \) is an open and compact subset, \( k \cdot x \) denotes a suitable bilinear form, and \( a(k), a^*(-k) \) are test functions, see Theorem 3. The solutions (2) can be quantized using the techniques described below, and the corresponding Klein-Gordon fields satisfy the corresponding Wightman axioms, see Theorem 2.

The \( p \)-adic Klein-Gordon equations in the form used in this article were introduced by the third author, see [67, Chapter 6] and references therein, where also the problem of the second quantization of their solutions was posed [67, Chapter 7]. The resulting field theory has a strong number-theoretic flavor. For instance, the calculation of the Green functions is related to the meromorphic continuation of Igusa’s local zeta functions, see Theorem 4 and the references [21], [27, Chapter 10], [67, Chapter 5].

Finally, let us remark that there are a lot of open questions related to \( p \)-adic quantum fields and their underlying mathematical techniques that remain to be studied within the present framework. Among them, probably the most important one is the reconstruction theorem, which depends on an appropriate definition of Wightman distributions, and, of course, the inclusion of non-trivial interactions, that will be discussed elsewhere. The corresponding theory for non-elliptic quadratic forms \( q \), though much more difficult, is also of interest.

2. Preliminaries

Along this article \( p \) will denote a prime number different from 2. Due to physical considerations we will formulate all our results in dimension 4, however, many of our results are still valid in arbitrary dimension.

2.1. The field of \( p \)-adic numbers. In this section we summarize the essential aspects and basic results on \( p \)-adic analysis that we will use through the article. For a detailed exposition of \( p \)-adic analysis the reader may consult [21, 53, 60].

The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( |\cdot|_p \), which in turn is defined as
\[ |x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-\gamma} & \text{if } x = p^\gamma a/b, \end{cases} \]
where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := ord(x)$, with $ord(0) := +\infty$, is called the $p$–adic order of $x$. Any $p$–adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where $x_j \in \{0, \ldots, p-1\}$ and $x_0 \neq 0$. Any non-zero $p$–adic number $x$ can be written uniquely as $x = p^{ord(x)}ac(x)$, with $|ac(x)|_p = 1$, $ac(x)$ is called the angular component of $x$.

By using expansion (3), we define the fractional part of $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\ p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0. \end{cases}$$

As a topological space, $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [2,60]. The balls and spheres are compact subsets.

We extend the $p$–adic norm to $\mathbb{Q}_p^4$ by taking

$$||x||_p := \max_{0 \leq i \leq 3} |x_i|_p, \quad \text{for } x = (x_0, x_1, x_2, x_3) \in \mathbb{Q}_p^4.$$

We define $ord(x) = \min_{0 \leq i \leq 3} \{ord(x_i)\}$, then $||x||_p = p^{-ord(x)}$. The metric space $(\mathbb{Q}_p^4, ||\cdot||_p)$ is a complete ultrametric space. Thus $(\mathbb{Q}_p^4, ||\cdot||_p)$ is a locally compact topological space.

For $l \in \mathbb{Z}$, denote by $B_l^4(a) = \{x \in \mathbb{Q}_p^4 : ||x - a||_p \leq p^l\}$ the ball of radius $p^l$ with center at $a = (a_0, a_1, a_2, a_3) \in \mathbb{Q}_p^4$, and take $B_l^4(0) := B_l^4$. Note that $B_l^4(a) = B_l(a_0) \times \cdots \times B_l(a_3)$, where $B_l(a_i) := \{x \in \mathbb{Q}_p : ||x - a_i||_p \leq p^l\}$ is the one-dimensional ball of radius $p^l$ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^4$ equals the product of four copies of $B_0 := \mathbb{Z}_p$, the ring of $p$–adic integers. For $l \in \mathbb{Z}$, denote by $S_l^4(a) = \{x \in \mathbb{Q}_p^4 : ||x - a||_p = p^l\}$ the sphere of radius $p^l$ with center at $a \in \mathbb{Q}_p^4$, and take $S_l^4(0) := S_1^4$.

**Remark 1.** The natural map $\mathbb{Z}_p \to \mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with $p$ elements, is called the reduction modulo $p$, denoted by $\bar{\cdot}$. We will identify $\mathbb{F}_p = \{\bar{0}, \bar{1}, \ldots, \bar{p-1}\}$, where the addition and multiplication are defined modulo $p$. We will distinguish between $\{0, 1, \ldots, p-1\} \subset \mathbb{Z}_p$ and $\mathbb{F}_p$. Later on, we will also use the symbol $\bar{\cdot}$ to mean conjugation of complex numbers, but it will clear from the context which case it is being used.

**Note 1.** Let us collect here some conventions.

(i) We denote by $\Omega(||x||_p)$ the characteristic function of $B_0^4$. For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of $A$.

(ii) From now on, we denote by $d^4x$ the Haar measure of the locally compact group $(\mathbb{Q}_p^4, +)$ normalized so that the volume of $\mathbb{Z}_p^4$ is one.

(iii) We will use the notation $x = (x_0, x_1, x_2, x_3) = (x, \bar{x}) \in \mathbb{Q}_p \times \mathbb{Q}_p^3$ from now up to Section 5.5.
2.2. Some function spaces.

2.2.1. The Bruhat-Schwartz space. We take \( \mathbb{K} \) to mean \( \mathbb{R} \) or \( \mathbb{C} \). A \( \mathbb{K} \)-valued function \( \varphi \) defined on \( \mathbb{Q}_p^4 \) is called locally constant, if for any \( x \in \mathbb{Q}_p^4 \) there exists an integer \( l(x) \in \mathbb{Z} \) such that

\[
\varphi(x + x') = \varphi(x) \text{ for } x' \in B^4_{l(x)}. \tag{4}
\]

A function \( \varphi : \mathbb{Q}_p^4 \rightarrow \mathbb{K} \) is called a Bruhat-Schwartz function (or a test function), if it is locally constant with compact support. The \( \mathbb{K} \)-vector space of Bruhat-Schwartz functions is denoted by \( \mathcal{D}_\mathbb{K}(\mathbb{Q}_p^4) := \mathcal{D}_\mathbb{K} \). Let \( \mathcal{D}'_\mathbb{K}(\mathbb{Q}_p^4) := \mathcal{D}'_\mathbb{K} \) denote the space of all continuous functionals (distributions) on \( \mathcal{D}_\mathbb{K} \). The space \( \mathcal{D}'_\mathbb{K} \) coincides with the algebraic dual of \( \mathcal{D}_\mathbb{K} \), i.e. any linear functional on \( \mathcal{D}_\mathbb{K} \) is continuous. For an in-depth discussion the reader may consult [2], [53], [60].

Remark 2. Most of the time we will work in dimension four, with spaces like \( \mathcal{D}_\mathbb{K}(\mathbb{Q}_p^4) \) and \( \mathcal{D}'_\mathbb{K}(\mathbb{Q}_p^4) \), in these cases we will use the abbreviated notation \( \mathcal{D}_\mathbb{K} \), \( \mathcal{D}'_\mathbb{K} \). In a few occasions we will work in dimensions different from 4, then we will use the notation \( \mathcal{D}_\mathbb{K}(\mathbb{Q}_p^n) \), \( \mathcal{D}'_\mathbb{K}(\mathbb{Q}_p^n) \). A similar rule will be used for other function spaces.

2.2.2. The spaces \( L^r \). Given \( r \in [1, +\infty) \), we denote by \( L^r(\mathbb{Q}_p^4, d^4x) := L^r_\mathbb{K} \), the \( \mathbb{K} \)-vector space of all the \( \mathbb{K} \)-valued functions \( g \) satisfying \( \int_{\mathbb{Q}_p^4} |g(x)|^r \, d^4x < \infty \).

2.3. Fourier transform. Set \( \chi_p(y) = \exp(2\pi i \{y\}_p) \) for \( y \in \mathbb{Q}_p \). The map \( \chi_p(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuous map from \( \mathbb{Q}_p \) into the unit circle satisfying \( \chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1) \), \( y_0, y_1 \in \mathbb{Q}_p \).

We set

\[
\mathfrak{B}(x, y) = x_0y_0 - sx_1y_1 - px_2y_2 + spx_3y_3,
\]

where \( s \in \mathbb{Z} \) is a quadratic non-residue module \( p \), i.e. the congruence \( x^2 \equiv s \mod p \) does not have solution. Then \( \mathfrak{B}(x, y) \) is a symmetric non-degenerate \( \mathbb{Q}_p \)-bilinear form on \( \mathbb{Q}_p^4 \times \mathbb{Q}_p^4 \), and

\[
q(x) := \mathfrak{B}(x, x) = x_0^2 - sx_1^2 - px_2^2 + spx_3^2, \quad x \in \mathbb{Q}_p^4
\]

is a non-degenerate quadratic form on \( \mathbb{Q}_p^4 \). In addition, \( q(x) \) is the unique (up to linear equivalence) elliptic quadratic form in dimension four, here elliptic means that \( q(x) = 0 \Leftrightarrow x = 0 \) (notice that this is not equivalent to the non-degeneracy of \( \mathfrak{B} \), as the equation \( q(x) = 0 \) could have its own solutions, not coming from vectors orthogonal to all the vectors in \( \mathbb{Q}_p^4 \)).

We identify the \( \mathbb{Q}_p \)-vector space \( \mathbb{Q}_p^4 \) with its algebraic dual \( (\mathbb{Q}_p^4)^* \) by means of \( \mathfrak{B}(\cdot, \cdot) \). We now identify the dual group (i.e. the Pontryagin dual) of \( (\mathbb{Q}_p^4, +) \) with \( (\mathbb{Q}_p^4)^* \) by taking \( x^* (x) = \chi_p(\mathfrak{B}(x, x^*)) \). The Fourier transform is defined by

\[
(Fg)(k) = \int_{\mathbb{Q}_p^4} g(x) \chi_p(\mathfrak{B}(x, k)) \, d\mu(x), \quad \text{for } g \in L^1_\mathbb{K},
\]
where $d\mu(x)$ is a Haar measure on $\mathbb{Q}_p^4$. Let $\mathcal{L}(\mathbb{Q}_p^4)$ be the space of complex-valued continuous functions $g$ in $L^1_{\mathbb{C}}$ whose Fourier transform $\mathcal{F}g$ is integrable. The measure $d\mu(x)$ can be normalized uniquely in such manner that

$$(\mathcal{F}(\mathcal{F}g))(x) = g(-x)$$

for every $g$ belonging to $\mathcal{L}(\mathbb{Q}_p^4)$.

We say that $d\mu(x)$ is a self-dual measure relative to $\chi_p(\mathcal{B}(\cdot,\cdot))$. Notice that $d\mu(x) = C(q)d^4x$ where $C(q)$ is a positive constant and $d^4x$ is the normalized Haar measure on $\mathbb{Q}_p^4$. For further details about the material presented in this section the reader may consult [63].

We will also use the notation $\mathcal{F}_x \to \xi \hat{g}$ and $\hat{g}$ for the Fourier transform of $g$.

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'_{\mathbb{C}}$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}\varphi)$$

for all $\varphi \in \mathcal{D}_{\mathbb{C}}$.

The Fourier transform $T \to \mathcal{F}[T]$ is a linear isomorphism from $\mathcal{D}'_{\mathbb{C}}$ onto itself. Furthermore, $T(\xi) = \mathcal{F}[\mathcal{F}[T](\xi)]$.

Note 2. Along this article we will use the notation $q(x) = x_0^2 - q_0(x)$, where $q_0(x) = sx_1^2 + px_2^2 - spx_3^2$ is an elliptic quadratic form. The bilinear form corresponding to $q_0$ will be denoted $\mathcal{B}_0(\cdot,\cdot)$. Then $\mathcal{B}(x, y) = x_0y_0 - \mathcal{B}_0(x, y)$.

2.4. The $p$–adic Minkowski space. Take $q(x)$ as before, and define

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -s & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & sp \end{bmatrix}.$$  

Then $q(x) = x^\top Gx$, where $\top$ denotes the transpose of a matrix, and $x$ is identified with the column vector $[x_0, x_1, x_2, x_3]$. The orthogonal group of $q$ is defined as

$$O(q) = \{A \in GL_4(\mathbb{Q}_p); \mathcal{B}(Ax, Ay) = \mathcal{B}(x, y)\}$$

$$= \{A \in GL_4(\mathbb{Q}_p); A^\top GA = G\}.$$  

Notice that any $A \in O(q)$ satisfies $\det A = \pm 1$. We call the quadratic space $(\mathbb{Q}_p^4, q)$ the $p$–adic Minkowski space, and we define the $p$–adic Lorentz group to be $O(q)$. Later on, we will introduce the $p$–adic restricted Lorentz group and the $p$–adic restricted Poincaré group.

Remark 3. Special relativity in the $p$–adic framework was discussed in [12], however, our definitions of Lorentz group and ‘light cones’ are completely different to the ones used in this article. In [57]-[58], the authors investigated the representations of the $p$–adic Poincaré group, our notion of Lorentz group agrees with the one used in these works.
2.5. The Dirac distribution supported on a hypersurface. Take $f \in \mathbb{Q}_p[x_0, x_1, x_2, x_3]$ to be a non-constant polynomial. The hypersurface attached to $f$ is the set

$$H := H(f) = \{x \in \mathbb{Q}_p^4; f(x) = 0\}.$$ 

We say that $H$ is a non-singular hypersurface, if

$$\nabla f(x) \neq 0 \text{ for any } x \in H. \quad (5)$$

By using the $p$–adic implicit function theorem, see e.g. [21], [48], one shows, like in the case $\mathbb{R}^4$, that $H$ is a $p$–adic manifold embedded in $\mathbb{Q}_p^4$. More exactly, $H$ is a closed submanifold of $\mathbb{Q}_p^4$ (which is a $p$–adic manifold of dimension 4) of codimension 1. For further details about $p$–adic manifolds the reader may consult [21], [48].

The condition (5) implies the existence of a 3-form $\lambda$ (whose restriction to $H$ is unique) satisfying

$$dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = df \wedge \lambda. \quad (6)$$

Usually $\lambda$ is called a Gel’fand-Leray form for $H$. We denote by $d\lambda$ the measure induced by $\lambda$ on $H$. For the details about the construction of $d\lambda$, the reader may consult [21, Chapter 7]. This construction is similar to one done in the real case, [15, Chapter III].

The linear functional

$$\mathcal{D}_K \rightarrow \mathcal{K}$$

$$\varphi \rightarrow (\delta_H, \varphi) = \int_H \varphi(x) d\lambda$$

gives rise to a distribution $\mathcal{D}_K'$, which is called the Dirac distribution $\delta_H$ supported on $H$.

Denote $\mathbb{Q}_p^\times = \mathbb{Q}_p - \{0\}$. For $t \in \mathbb{Q}_p^\times$, we set

$$V_t := V_t(q) = \{x \in \mathbb{Q}_p^4; q(x) = t\}.$$ 

Then $V_t$ is a non-singular hypersurface in $\mathbb{Q}_p^4$. The orthogonal group $O(q)$ acts transitively on $V_t$. On each non-empty orbit $V_t$ there is a non-zero, positive measure which is invariant under $O(q)$ and unique up to multiplication by a positive constant, see [41, Proposition 2-2].

For each $t \in \mathbb{Q}_p^\times$, let $d\mu_t$ be a measure on $V_t$ invariant under $O(q)$. Since $V_t$ is closed in $\mathbb{Q}_p^4$, it is possible to consider $d\mu_t$ as a measure on $\mathbb{Q}_p^4$ supported on $V_t$, and by the using the Caratheodory theorem, we can identify $d\mu_t$ with a positive distribution, i.e. if $\phi$ is a non-negative function, then $(d\mu_t, \phi) \geq 0$. The Rallis-Schiffman result above mentioned can be reformulated as follows: on each non-empty orbit $V_t$ there is a non-zero, positive distribution which is invariant under $O(q)$ and unique up to multiplication by a positive constant.

Now, since $\delta_{V_t}$ is invariant under $O(q)$, see [67, Lemma 156] for a similar calculation, we conclude that $d\mu_t$ agrees (up to a positive constant) with $\delta_{V_t}$. From now on we identify $\delta_{V_t}$ with $d\mu_t$.

Note 3. From now on, we will use $\delta(f)$ to denote the Dirac distribution supported on the non-singular hypersurface attached to the polynomial $f$.
2.6. The spaces $\mathcal{H}_\infty$. The Bruhat-Schwartz space $\mathcal{D}_K$ is not invariant under the action of pseudodifferential operators. In [65], see also [27, Chapter 10], the third author introduced a class of nuclear countably Hilbert spaces which are invariant under the action of a large class of pseudo-differential operators. In this section, we review some basic results about these spaces that we will use in the remaining sections.

Note 4. We set $\mathbb{R}_+ := \{ x \in \mathbb{R} : x \geq 0 \}$, $[\xi]_p := \max(1, \| \xi \|_p)$ and consider $\mathbb{N}$ to be the set of non-negative integers.

We define for $f, g \in \mathcal{D}_K$, with $K = \mathbb{R}, \mathbb{C}$, the following scalar product:

$$\langle f, g \rangle_l := \int_{\mathbb{R}^4_+} |\xi|_p^l \hat{f}(\xi) \overline{\hat{g}(\xi)} d^4 \xi,$$

for $l \in \mathbb{N}$, where the bar denotes the complex conjugate. We also set $\| f \|_l^2 = \langle f, f \rangle_l$. Notice that $\| \cdot \|_l \leq \| \cdot \|_m$ for $l \leq m$. Let denote by $\mathcal{H}_l(Q_p^4, K) = : \mathcal{H}_l(K)$ the completion of $\mathcal{D}_K$ with respect to $\langle \cdot, \cdot \rangle_l$. Then $\mathcal{H}_m(K) \hookrightarrow \mathcal{H}_l(K)$ is a continuous embedding for $l \leq m$. We set

$$\mathcal{H}_\infty(Q_p^4, K) := \mathcal{H}_\infty(K) = \bigcap_{l \in \mathbb{N}} \mathcal{H}_l(K).$$

Notice that $\mathcal{H}_0(K) = L^2_K$ and that $\mathcal{H}_\infty(K) \subset L^2_K$. With the topology induced by the family of seminorms $\| \cdot \|_l$, $\mathcal{H}_\infty(K)$ becomes a locally convex space, which is metrizable. Indeed,

$$d(f, g) := \max_{l \in \mathbb{N}} \left\{ 2^{-l} \left( \frac{\|f - g\|_l}{1 + \|f - g\|_l} \right) \right\}, \text{ for } f, g \in \mathcal{H}_\infty(K),$$

is a metric for the topology of the convex topological space $\mathcal{H}_\infty(K)$. A sequence $\{f_l\}_{l \in \mathbb{N}} \in (\mathcal{H}_\infty(K), d)$ converges to $f \in \mathcal{H}_\infty(K)$, if and only if, $\{f_l\}_{l \in \mathbb{N}}$ converges to $f$ in the norm $\| \cdot \|_l$ for all $l \in \mathbb{N}$. From this observation it follows that the topology of $\mathcal{H}_\infty(K)$ coincides with the projective limit topology $\tau_P$. An open neighborhood base at zero of $\tau_P$ is given by the choice of $\epsilon > 0$ and $l \in \mathbb{N}$, and the sets

$$U_{\epsilon, l} := \{ f \in \mathcal{H}_\infty(K) : \| f \|_l < \epsilon \}.$$

The space $\mathcal{H}_\infty(K)$ endowed with the topology $\tau_P$ is a countably Hilbert space in the sense of Gel’fand and Vilenkin, see e.g. [108, Chapter I, Section 3.1] or [39, Section 1.2]. Furthermore $(\mathcal{H}_\infty(K), \tau_P)$ is metrizable and complete and hence a Fréchet space, cf. [65, Lemma 3.3]. In addition, the completion of the metric space $(\mathcal{D}_K(Q_p^4), d)$ is $(\mathcal{H}_\infty(K), d)$, and this space is a nuclear countably Hilbert space, see [65, Lemma 3.4, Theorem 3.6] or [27, Chapter 10].

For $m \in \mathbb{N}$ and $T \in \mathcal{D}'_K$, we set

$$\| T \|_{m}^{-2} := \int_{\mathbb{R}^4_+} [\xi]_m^{-m} |\hat{T}(\xi)|^2 d^4 \xi.$$

Then $\mathcal{H}_m(K) := \mathcal{H}_{-m}(Q_p^4, K) = \{ T \in \mathcal{D}'_K : \| T \|_{-m}^{-2} < \infty \}$ is a Hilbert space over $K$. Denote by $\mathcal{H}_m'(K)$ the strong dual space of $\mathcal{H}_m(K)$. It is useful to suppress
the correspondence between $\mathcal{H}_m^*(\mathbb{K})$ and $\mathcal{H}_m(\mathbb{K})$ given by the Riesz theorem. Instead we identify $\mathcal{H}_m^*(\mathbb{K})$ and $\mathcal{H}_m(\mathbb{K})$ by associating $T \in \mathcal{H}_m(\mathbb{K})$ with the functional on $\mathcal{H}_m(\mathbb{K})$ given by

$$[T, g] := \int_{\mathbb{Q}_2^4} \overline{T(\xi)}g(\xi) d^4\xi.$$  

(7)

Notice that $|[T, g]| \leq ||T||_{-m}||g||_m$. Now by a well-known result in the theory of countable Hilbert spaces, see [16], $\mathcal{H}_0^*(\mathbb{K}) \subset \mathcal{H}_1^*(\mathbb{K}) \subset \ldots \subset \mathcal{H}_m^*(\mathbb{K}) \subset \ldots$ and

$$\mathcal{H}_\infty^*(\mathbb{K}) = \bigcup_{m \in \mathbb{N}} \mathcal{H}_m(\mathbb{K}) = \{ T \in \mathcal{D}'_{\mathbb{K}} : ||T||_{-l} < \infty, \text{ for some } l \in \mathbb{N} \}$$  

(8)
as vector spaces. Since $\mathcal{H}_\infty(\mathbb{K})$ is a nuclear space, the weak and strong convergence are equivalent in $\mathcal{H}_\infty^*(\mathbb{K})$, see e.g. [16]. We consider $\mathcal{H}_\infty^*(\mathbb{K})$ endowed with the strong topology. On the other hand, let $B : \mathcal{H}_\infty^*(\mathbb{K}) \times \mathcal{H}_\infty(\mathbb{K}) \to \mathbb{K}$ be a bilinear functional. Then $B$ is continuous in each of its arguments if and only if there exist norms $|| \cdot ||^{(a)}_m$ in $\mathcal{H}_m(\mathbb{K})$ and $|| \cdot ||^{(b)}_l$ in $\mathcal{H}_l(\mathbb{K})$ such that $|B(T, g)| \leq M||T||^{(a)}_m||g||^{(b)}_l$ with $M$ a positive constant independent of $T$ and $g$, see e.g. [16]. This implies that (7) is a continuous bilinear form on $\mathcal{H}_\infty^*(\mathbb{K}) \times \mathcal{H}_\infty(\mathbb{K})$, which we will use as a paring between $\mathcal{H}_\infty^*(\mathbb{K})$ and $\mathcal{H}_\infty(\mathbb{K})$.

Remark 4. The spaces $\mathcal{H}_\infty(\mathbb{K}) \subset L^2_{\mathbb{K}} \subset \mathcal{H}_\infty^*(\mathbb{K})$ form a Gel’fand triple (also called a rigged Hilbert space), i.e. $\mathcal{H}_\infty(\mathbb{K})$ is a nuclear space which is densely and continuously embedded in $L^2_{\mathbb{K}}$ and $||g||_{L^2_{\mathbb{K}}}^2 = [g, g]$. This Gel’fand triple was introduced in [65].

The following result will be used later on:

Lemma 1. With the above notation, the following assertions hold:

(i) $\mathcal{H}_l(\mathbb{K}) = \{ f \in L^2_{\mathbb{K}} : ||f||_{l} < \infty \} = \{ T \in \mathcal{D}'_{\mathbb{K}} : ||T||_{l} < \infty \}$;
(ii) $\mathcal{H}_\infty(\mathbb{K}) = \{ f \in L^2_{\mathbb{K}} : ||f||_{l} < \infty, \text{ for any } l \in \mathbb{N} \}$;
(iii) $\mathcal{H}_\infty^*(\mathbb{K}) = \{ T \in \mathcal{D}'_{\mathbb{K}} : ||T||_{l} < \infty, \text{ for any } l \in \mathbb{N} \}$.

For the proof the reader may consult ([64, Lemma 3.2]) or [27] Lemma 10.8.

3. Fundamental Solutions for Pseudo-differential Operators of Klein-Gordon Type

3.1. Some preliminary results. For $\alpha > 0$, $m \in \mathbb{Q}_p^\times$, and $q$ as before, we define the following pseudo-differential operator:

$$\Box_{q, \alpha, m} = \mathcal{F}^{-1} \circ |q - m^2|_p^\alpha \circ \mathcal{F},$$

(9)

where $|q - m^2|_p^\alpha$ denotes the multiplication operator by the function $|q - m^2|_p^\alpha$. We call operators of type 9, $p$-adic Klein-Gordon pseudo-differential operators. These operators were introduced by Zúñiga-Galindo, see [67] Chapter 6] and the references therein.

In this section, we consider operators $\Box_{q, \alpha, m}$ with domain

$$Dom(\Box_{q, \alpha, m}) = \{ T \in \mathcal{D}'_{\mathbb{K}} : |q - m^2|_p^\alpha \mathcal{F}T \in \mathcal{D}'_{\mathbb{K}} \}.$$
Remark 5. Notice that
\[ \Box_{q,\alpha,m}(T(mx)) = |m|^{2\alpha}_p (\Box_{q,\alpha,1}T)(mx) \] for any \( T \in \text{Dom}(\Box_{q,\alpha,m}) \).

Consequently, we may normalize the mass \( m \) to one. From now on we assume that \( m = 1 \), and we use the notation \( \Box_{q,\alpha} \) instead of \( \Box_{q,\alpha,1} \).

Definition 1. We say that \( E_{q,\alpha} \in \mathcal{D}'_C \) is a fundamental solution for
\[ \Box_{q,\alpha} u = \varphi, \] (10)
if \( u = E_{q,\alpha} \ast \varphi \) is a solution of (11) in \( \mathcal{D}'_C \), for any \( \varphi \in \mathcal{D}_C \).

From now on, by an abuse of language, we will say that \( E_{q,\alpha} \) is a fundamental solution of \( \Box_{q,\alpha} \).

Lemma 2. \( E_{q,\alpha} \) is a fundamental solution of \( \Box_{q,\alpha} \) if and only if
\[ |q - 1|^\alpha_p \mathcal{F}(E_{q,\alpha}) = 1 \] (11)
in \( \mathcal{D}'_C \).

Proof. If \( E_{q,\alpha} \) is a fundamental solution of \( \Box_{q,\alpha} \), then
\[ (|q - 1|^\alpha_p \mathcal{F}(E_{q,\alpha}) - 1) \cdot \mathcal{F}\varphi = 0, \]
for any test function in \( \mathcal{D}_C \), which implies (11). Now, if (11) holds, by using the fact that the product of two distributions, if it exists, is commutative and associative (see e.g. [53, p. 127, Theorem 3.19]), we get that
\[ (|q - 1|^\alpha_p \mathcal{F}\varphi) \cdot \mathcal{F}(E_{q,\alpha}) = \mathcal{F}\varphi \]
for any test function \( \varphi \).

3.2. The \( p \)-adic submanifold \( V \). Since \( q(k) = k_0^2 - sk_1^2 - pk_2^2 + spk_3^2 \), where \( s \in \mathbb{Z}^*_p = \mathbb{Z}_p - \{0\} \) a quadratic non-residue mod \( p \), is an elliptic quadratic form (i.e. \( q(k) = 0 \iff k = 0 \)), we have
\[ |q(k)|_p \geq \left( \inf_{x \in S^4_p} |q(x)|_p \right) \|k\|_p^2, \] (12)
see e.g. [67, Lemma 25]. Set
\[ V := \{ k = (k_0, k) \in \mathbb{Q}_p \times \mathbb{Q}^3_p; q(k) = 1 \}. \]

By using (12), and the fact that \( \inf_{x \in S^4_p} |q(x)|_p = p^{-1} \), we get that \( V \subseteq \mathbb{Z}_p^4 \), which implies that \( V \) is a compact submanifold of \( \mathbb{Z}_p^4 \) of codimension 1. Let us emphasize that \( V \) is bounded (in contrast to the classical case). Given \( (\tilde{k}_0, \tilde{k}) \in V \) with \( \tilde{k}_0 \neq 0 \), by applying the \( p \)-adic implicit function theorem, see e.g. [21], there exist
open and compact subsets $U^0_j \subset \mathbb{Z}_p$, $U^1_j \subset \mathbb{Z}_p^3$ such that $(\tilde{k}_0, \tilde{k}) \in U_j = U^0_j \times U^1_j$, and a $p$–adic analytic function $h_j(x) : U^1_j \to U^0_j$ such that

$$V \cap U_j = \{(k_0, k) \in U_j ; k_0 = h_j(k)\}. $$

Notice that $k_0 = -h_j(k)$ is also a ‘local parametrization’ of $V$. By using the compactness of $V$, there exists a finite number of analytic functions $\pm h_j(k) : U^1_j \to \pm U^0_j$, $j = 1, \ldots, N$ such that

$$V = \bigcup_{j=1}^{N} \{(k_0, k) \in U^0_j \times U^1_j ; k_0 = h_j(k)\} \bigcup \bigcup_{j=1}^{N} \{(k_0, k) \in -U^0_j \times U^1_j ; k_0 = -h_j(k)\} \bigcup W,$$

where $W = \{(0, k) : q_0(k) = 1\}$. We set $U_q := \bigcup_{j=1}^{N} U^1_j \subset \mathbb{Z}_p^3$. We now define in $U_q$, two analytic functions as follows:

$$U_q \to \mathbb{Q}_p$$

$$k \to \pm \sqrt{1 + sk^2_1 + pk^2_2 - spk^2_3} =: \pm \sqrt{\omega(k)},$$

where $\pm \sqrt{\omega(k)}|_{U_j} = \pm h_j(k)$.

3.2.1. A notion of positivity. We set $\mathbb{F}_p^x = [\mathbb{F}_p^x]^+ \cup [\mathbb{F}_p^x]_-$, where $[\mathbb{F}_p^x]^+ := \{\mathbf{1}, \ldots, \mathbf{2k-1}\}$ and $[\mathbb{F}_p^x]_- = \{\mathbf{2k+1}, \ldots, p-\mathbf{1}\}$. We define the elements of $[\mathbb{F}_p^x]^+$ as positive and the elements of $[\mathbb{F}_p^x]_-$ as negative. Notice that since $p \neq 2$,

$$[\mathbb{F}_p^x]^+ \to [\mathbb{F}_p^x]_-$$

$$\bar{\mathbf{1}} \to -\bar{\mathbf{1}} \mod p$$

is a bijection. Now, we say that a non-zero $p$–adic number

$$a = p^{-L} (a_0 + a_1 p + \ldots),$$

with $L \in \mathbb{Z}$ and $a_0 \neq 0$,

is positive (denoted as $a > 0$) if $a_0 \in [\mathbb{F}_p^x]^+$, otherwise we say that $a$ is negative (denoted as $a < 0$). This is a well-defined and useful notion of ‘positivity’ in $\mathbb{Q}_p^x$. However, this notion of positivity is not compatible with the field operations, consequently, this notion does not give rise to an order in $\mathbb{Q}_p^x$. We also recall that in the case $p \neq 2$, the equation $x^2 = a$ has two solutions in $\mathbb{Q}_p$ if an only $L$ is even and the congruence $z^2 \equiv \bar{\mathbf{1}} \mod p$ has two solutions, one in $[\mathbb{F}_p^x]^+$ and the other in $[\mathbb{F}_p^x]_-$. We denote them as $\pm \sqrt{\omega_0} \in \mathbb{F}_p^x$. Then

$$x = p^{-\frac{L}{2}} \left(\sqrt{\omega_0} + b_1 p + b_2 p^2 + \ldots\right),$$
where the $b$'s are recursively determined by $\sqrt{a_0}$, i.e. $b_1 = f_1(\sqrt{a_0})$, $b_2 = f_2(\sqrt{a_0}, b_1)$, ..., and

$$-x = -p^{-\frac{x}{2}} (\sqrt{a_0} + b_1 p + b_2 p^2 + \ldots) = p^{-\frac{x}{2}} (p - \sqrt{a_0} + (p - 1 - b_1) p + (p - 1 - b_2) p^2 + \ldots).$$

We now define

$$V^+ = \{(k_0, k) \in V; k_0 > 0 \text{ and } k_0 = \sqrt{\omega(k)}\},$$

$$V^- = \{(k_0, k) \in V; k_0 < 0 \text{ and } k_0 = -\sqrt{\omega(k)}\}.$$

We call $V^+$ the positive mass shell and $V^-$ the negative mass shell. Therefore

$$V = V^+ \bigcup V^- \bigcup W.$$ 

Consequently, $W$ has $d\lambda$-measure zero, so $\int_W \varphi d\lambda \equiv 0$ for any $\varphi \in \mathcal{D}_C$.

3.3. The distributions $\delta_{V^\pm}$

Remark 6. Set $q(k_0, k) := k_0^2 - q_0(k)$, then

$$W = \{(k_0, k) \in \mathbb{Z}_p^3; q(0, k) = 1\} = \{k \in \mathbb{Z}_p^3; -q_0(k) = 1\}.$$

A necessary and sufficient condition to have $W \neq \emptyset$ is that

$$-q_0(k) \equiv 1 \mod p \text{ i.e. } -sk_1^2 \equiv 1 \mod p. \quad (13)$$

The sufficiency of condition (13) follows from the Hensel-Newton lemma, see e.g. [17, Lemma 1]. The existence of solutions for congruence (13) requires the computation of the following Legendre symbol:

$$\left(\frac{-s^{-1}}{p}\right) = \begin{cases} 1 & \text{if congruence (13) has a solution,} \\ -1 & \text{if congruence (13) has no solution.} \end{cases}$$

By using the fact that the Legendre symbol is a multiplicative function and that $\left(\frac{2}{p}\right) = -1$, we get that

$$\left(\frac{-s^{-1}}{p}\right) = \begin{cases} -1 & \text{if } p \equiv 1 \mod 4 \Leftrightarrow W = \emptyset \\ 1 & \text{if } p \equiv 3 \mod 4 \Leftrightarrow W \neq \emptyset. \end{cases}$$

Taking these results into account, we will set $p \equiv 1 \mod 4$ from now on, so $W = \emptyset$. 

Proof. Recall that $V$ is a tubular neighborhood of $f$ where $A$ is defined. Let $\delta_\pm$ denote $\delta (q - 1)$ where implicitly we are choosing for each $\phi \in a$. Indeed, the solution set of the equation $p \equiv 1 \mod q$. Notice that $E_V = 0$. Indeed, the solution set of the equation $E_v = 0$, where $E_V$ is the set of $\phi(k) = 1$ and $b \equiv \phi \bmod p$. This $b$ is not unique. We now define the following tubular neighborhood of $V$: Let $b = (b_0, b_1, b_2, b_3) \in V$, with $b_0 \in \mathbb{Z}_p^3$. Then

$$ (\delta(q(k) - 1), \phi(k)\Omega(p||k - b||_p)) = p^{3} \int_{\mathbb{Z}_p^3} \phi(b_0 + pf(0, u_1, u_2, u_3), b_1 + pu_1, b_2 + pu_2, b_3 + pu_3)du_1du_2du_3, $$

where $f(0, u_1, u_2, u_3)$ is a $p$–adic analytic function on the ball $\mathbb{Z}_p^3$.

Proof. Recall that

$$ (\delta(q(k) - 1), \phi(k)\Omega(p||k - b||_p)) = \int_{V \cap (b + p\mathbb{Z}_p^3)} \phi(k)\frac{dk_1dk_2dk_3}{|k_0|_p}. $$

Remark 7. Take $\tau \in \mathbb{F}_p^4$ satisfying $q(\tau) \equiv 1 \mod p$. Since $\nabla q(\tau) \not\equiv 0 \bmod p$, by the Hensel-Newton lemma, see e.g. [17, Lemma 1], there exists $b \in \mathbb{Z}_p^4$ such that $q(b) = 1$ and $b \equiv \tau \bmod p$. This $b$ is not unique. We now define the following tubular neighborhood of $V$: Let $b = (b_0, b_1, b_2, b_3) \in V$, with $b_0 \in \mathbb{Z}_p^3$. Then

$$ (\delta(q(k) - 1), \phi(k)\Omega(p||k - b||_p)) = p^{3} \int_{\mathbb{Z}_p^3} \phi(b_0 + pf(0, u_1, u_2, u_3), b_1 + pu_1, b_2 + pu_2, b_3 + pu_3)du_1du_2du_3, $$

where $f(0, u_1, u_2, u_3)$ is a $p$–adic analytic function on the ball $\mathbb{Z}_p^3$. 

Lemma 3. Let $b = (b_0, b_1, b_2, b_3) \in V$, with $b_0 \in \mathbb{Z}_p^3$. Then

$$ (\delta(q(k) - 1), \phi(k)\Omega(p||k - b||_p)) = p^{3} \int_{\mathbb{Z}_p^3} \phi(b_0 + pf(0, u_1, u_2, u_3), b_1 + pu_1, b_2 + pu_2, b_3 + pu_3)du_1du_2du_3, $$

where $f(0, u_1, u_2, u_3)$ is a $p$–adic analytic function on the ball $\mathbb{Z}_p^3$. 

Proof. Recall that

$$ (\delta(q(k) - 1), \phi(k)\Omega(p||k - b||_p)) = \int_{V \cap (b + p\mathbb{Z}_p^3)} \phi(k)\frac{dk_1dk_2dk_3}{|k_0|_p}. $$
Now, by changing variables as \( k = b + pz \),

\[
(\delta(q(k) - 1), \phi(k)\mathcal{O}(p||k - b||_p)) = p^{-3} \int_{\{q(b+pz)=1\} \cap \mathbb{Z}_p^4} \phi(b+pz)dz_1dz_2dz_3, \quad (14)
\]

where we are assuming that \( z_0 \) is an analytic function of the variables \( z_1, z_2, z_3 \).

We set

\[
u = F(z), \text{ with } u_0 = \frac{1}{p}(q(b+pz) - 1), u_i = z_i \text{ for } i = 1, 2, 3. \quad (15)
\]

Then \( JacF(z) \equiv 2b_0 + 2pz_0 \equiv \bar{b}_0 \neq 0 \mod p \), by [21] Lemma 7.4.3, \( F \) gives rise to an analytic isomorphism from \( \mathbb{Z}_p^4 \) into itself which preserves the Haar measure, in this coordinate system \( \{q(b+pz) = 1\} \cap \mathbb{Z}_p^4 \) becomes \( \{u_0 = 0\} \times \mathbb{Z}_p^3 \), and (14) takes the form

\[
(\delta(q(k) - 1), \phi(k)\mathcal{O}(p||k - b||_p)) = p^{-3} \int_{\mathbb{Z}_p^4} \phi(b_0 + pu_0, b_1 + pu_1, b_2 + pu_2, b_3 + pu_3)du_0du_1du_2du_3,
\]

where \( f(0, u_1, u_2, u_3) : \mathbb{Z}_p^3 \rightarrow \mathbb{Z}_p \) is a \( p \)-adic analytic function.

**Remark 8.** Let us comment about some related results.

(i) In the case \( b_0 \in p\mathbb{Z}_p, b_1 \in \mathbb{Z}_p^\times \), a calculation similar to the one done in the proof of Lemma 3 shows that

\[
(\delta(q(k) - 1), \phi(k)\mathcal{O}(p||k - b||_p)) = p^{-3} \int_{\mathbb{Z}_p^4} \phi(b_0 + pu_0, b_1 + pg(u_0, u_2, u_3), b_2 + pu_2, b_3 + pu_3)du_0du_1du_2du_3,
\]

where \( g(u_0, u_2, u_3) : \mathbb{Z}_p^3 \rightarrow \mathbb{Z}_p \) is a \( p \)-adic analytic function.

(ii) In the case \( b_0 \in p\mathbb{Z}_p, b_1 \in p\mathbb{Z}_p, b_2 \in \mathbb{Z}_p^\times \), we have

\[
\{q(k) = 1\} \cap [p\mathbb{Z}_p \times p\mathbb{Z}_p \times [b_2 + p\mathbb{Z}_p] \times [b_3 + p\mathbb{Z}_p]] = p(\kappa_0^2 - spk_1^2 - k_2^2 + sk_3^2 = 1) \cap [\mathbb{Z}_p \times \mathbb{Z}_p \times [b_2 + p\mathbb{Z}_p] \times [b_3 + p\mathbb{Z}_p]] = \emptyset.
\]

A similar result is valid in the cases where \( b_0 \in p\mathbb{Z}_p, b_1 \in p\mathbb{Z}_p, b_2 \in p\mathbb{Z}_p, b_3 \in \mathbb{Z}_p^\times \), and where \( b_0 \in p\mathbb{Z}_p, b_1 \in p\mathbb{Z}_p, b_2 \in p\mathbb{Z}_p, b_3 \in p\mathbb{Z}_p \).

### 3.4. Fundamental solutions.

The existence of fundamental solutions for operators \( \Box_{q,a} \) is closely related to the meromorphic continuation of the Igusa local zeta function attached to the polynomial \( q - 1 \), which is the distribution defined as

\[
(\delta_q - 1)^p_{\prime}, \theta = \int_{q_i \setminus V} [q(x) - 1]^p_{\prime}\theta(x)d^4x \quad \text{for } \Re(s) > 0, \theta \in \mathcal{D}_C \quad (17)
\]

Here we use that for \( a > 0 \) and \( s \in \mathbb{C} \), \( a^s = e^{sa} \). Integrals of type (17) admit meromorphic continuations to the whole complex plane as rational functions of \( p^{-s} \), see [21] Theorem 8.2.1.
For further calculations, we rewrite (17) as

\[
|q(x) - 1|_p^s \theta(x) = \int_{Q^4_p \setminus E_V} |q(x) - 1|_p^s \theta(x) d^4x + \int_{E_V \setminus V} |q(x) - 1|_p^s \theta(x) d^4x
\]

=: (I_0(s), \theta) + (I_1(s), \theta).

A fundamental solution \( E_{q, \alpha} \) for operator \( \Box_{q, \alpha} \) is obtained by computing the Laurent expansion of the local zeta function \( |q - 1|_p^s \) at \( s = -\alpha \), see [21, Lemma 5.3.1]. Indeed, if

\[
|q - 1|_p^s = \sum_{j = \lfloor -\alpha \rfloor} |c_j(s + \alpha)|, \quad \text{where } c_j \in \mathcal{D}\, C, \quad \text{with } -j_0 \in \mathbb{Z}, \tag{18}
\]

then \( \hat{E}_{q, \alpha} = c_0 \).

**Note 5.** Given two subsets \( A, B \) in \( \mathbb{Q}_p^4 \), we denote the distance between them as

\[
\text{dist}(A, B) := \inf_{x \in A, y \in B} \|x - y\|_p.
\]

**Lemma 4.** For any \( \theta \in \mathcal{D}\, C \), the function \((I_0(s), \theta)\) is holomorphic in the whole complex plane.

**Proof.** The result follows, by using a well-known result about the analyticity of integrals depending on a complex parameter, see [21 Lemma 5.3.1], from the fact that there exists a positive constant \( \varepsilon = \varepsilon(q) \), such that

\[
|q(x) - 1|_p \geq \varepsilon \text{ for any } x \in \mathbb{Q}_p^4 \setminus E_V. \tag{19}
\]

If (19) is false, there exists a sequence \( \{y_n\}_{n \in \mathbb{N}} \) in \( \mathbb{Q}_p^4 \setminus E_V \) such that \( |q(y_n) - 1|_p \to 0 \) as \( n \to \infty \), which means that

\[
\text{dist}(V, \mathbb{Q}_p^4 \setminus E_V) = 0, \tag{20}
\]

because, since \( V \) is compact, there exists \( x_0 \in V \) such that

\[
\text{dist}(V, \mathbb{Q}_p^4 \setminus E_V) = \inf_{y \in \mathbb{Q}_p^4 \setminus E_V} \|x_0 - y\|_p = \inf_{y \in \mathbb{Q}_p^4 \setminus E_V} \text{dist}(V, y).
\]

The assertion (20) is not true. Indeed, since \( V \) is compact and \( \mathbb{Q}_p^4 \setminus E_V \) is closed (because \( E_V \) is open and closed), we have \( \text{dist}(V, \mathbb{Q}_p^4 \setminus E_V) > 0 \).

**Remark 9.** Notice the following computation:

\[
(I_1(s), \theta) = \int_{E_V \setminus V} |q(x) - 1|_p^s \theta(x) d^4x = \sum_{\alpha, \pi \in \mathbb{Z}_p^4} \int_{\mathbb{Z}_p^4} |q(x) - 1|_p^s \theta(x) d^4x
\]

\[
= p^{-4} \sum_{\alpha, \pi \in \mathbb{Z}_p^4} \int_{\mathbb{Z}_p^4} \left| q(b + pz) - 1 \right|_p^s \theta(b + pz) d^4z
\]

\[
=: p^{-4} \sum_{\alpha, \pi \in \mathbb{Z}_p^4} (I_b(s), \theta). \tag{21}
\]
Lemma 5. With the above notations and setting
\[ I_b(s) = \sum_{j=0}^{\infty} c_j (I_b, \alpha) (s + \alpha)^j, \]
where \( c_j (I_b, \alpha) \in \mathcal{D}_c \),
for \( \bar{b} \in \mathbb{F}_p^4 \), \( q(\bar{b}) \equiv 1 \mod p \), the coefficient \( c_0 \in \mathcal{D}_c \) in expansion (18) is given by
\[ (c_0, \theta) = \int_{\mathbb{F}_p^4} \frac{|q(x) - 1|_{p^{-a}} \theta(x) d^4x}{p^{-a} + p^{-4}} \sum_{q(\bar{b}) \equiv 1 \mod p} (c_0 (I_b, \alpha), \theta). \]

Proof. The formula follows from Lemma 4 and Remark 9.

We now compute the coefficients \( c_0 (I_b, \alpha) \) for some \( b \), the calculation of the missing cases is similar to the one presented here.

Lemma 6. Assume that \( \bar{b}_0 \not\equiv 0 \mod p \). If \( \alpha \neq 1 \), then
\[ (c_0 (I_b, \alpha), \theta) = p^\alpha \int_{\mathbb{Z}_p^4} |u_0|_{p^{-a}} (\Theta_b(u_0) - \Theta_b(0)) du_0 + \frac{p^\alpha (1 - p^{-1})}{1 - p^{-1 + \alpha}} \Theta_b(0), \]
where \( \Theta_b = T_{I_b, \alpha} (\theta) \in \mathcal{D}_c (\mathbb{Q}_p) \), and \( T_{I_b, \alpha} \) is a linear operator from \( \mathcal{D}_c (\mathbb{Q}_p) \) into \( \mathcal{D}_c (\mathbb{Q}_p) \), and
\[ \Theta_b (0) = p^3 (\delta (q(k) - 1), \theta(k) \Omega(p||k - b||_p)). \]

In addition,
\[ (1 \vee c_0 (I_b, \alpha), \theta) = \frac{p^\alpha (1 - p^{-1})}{1 - p^{-1 + \alpha}} \Theta_b(0). \]  \hspace{1cm} (22)

If \( \alpha = 1 \), then
\[ (c_0 (I_b, 1), \theta) = p \int_{\mathbb{Z}_p^4} |u_0|_{p^{-1}} (\Theta_b(u_0) - \Theta_b(0)) du_0 - \frac{p - 1}{2} \Theta_b(0). \]

Moreover,
\[ (1 \vee c_0 (I_b, 1), \theta) = -\frac{p - 1}{2} \Theta_b(0). \]  \hspace{1cm} (23)

Proof. By changing variables as \( u = F(z) \), see (15), we get
\[ (I_b(s), \theta) = \int_{\mathbb{Z}_p^4} |q(b + pz) - 1|_{p^s} \theta(b + pz) d^4z \]
\[ = p^{-s} \int_{\mathbb{Z}_p^4} |u_0|_{p^s} \theta(b_0 + pf(u_0, \ldots, u_3), b_1 + pu_1, b_2 + pu_2, b_3 + pu_3) du_0 du_1 du_2 du_3 \]
where \( f(u_0, \ldots, u_3) \) is a \( p \)-adic analytic function on \( \mathbb{Z}_p^4 \). Set
\[ \Theta_b(u_0) := \int_{\mathbb{Z}_p^4 \setminus D} \theta(b_0 + pf(u_0, \ldots, u_3), b_1 + pu_1, b_2 + pu_2, b_3 + pu_3) du_1 du_2 du_3, \]
where $D = \{ b_0 + pf(u_0, \ldots, u_3) = 0 \}$. Then $\Theta_b(u_0) \in \mathcal{D}_C(\mathbb{Q}_p)$ and $\Theta_b(0) = p^3(\Omega(p\|k-b\|_p)\delta(q(k) - 1), \theta(k))$, see [10]. Notice that for $u_0$ given, the set \( \{ b_0 + pf(u_0, \ldots, u_3) = 0 \} \) has measure zero, and that $b_0 + pf(u_0, \ldots, u_3)$ is locally constant in $u_0$ on $\mathbb{Z}_p^3 \setminus D$, this last fact is verified by using the $p$-adic Taylor expansion, see e.g. [48]. Therefore

\[
(I_b(s), \theta) = p^{-s} \int_{\mathbb{Z}_p^3} \left| w_0 \right|^s \left( \Theta_b(u_0) - \Theta_b(0) \right) du_0 + \frac{p^{-s}(1 - p^{-1})}{1 - p^{-1-s}} \Theta_b(0).
\]

If $\alpha \neq 1$, then $(c_0, (I_b, \alpha), \theta)$ is obtained by replacing $s = -\alpha$ in (24). In the case $\alpha = 1$, the computation of $(c_0, (I_b, 1), \theta)$ is achieved by computing the Laurent expansion of $(I_b(s), \theta)$ around $(s + 1)$, which follows from the formula:

\[
\frac{p^{-s}(1 - p^{-1})}{1 - p^{-1-s}} = \left( \frac{p - 1}{\ln p} \right) \frac{1}{s + 1} + \frac{p - 1}{2} + O(s + 1),
\]

where $O(s + 1)$ denotes a holomorphic function. Finally formulae (22)-(23) follow from the fact that in the coordinate system $(u_0, \ldots, u_3)$, $u_0 = 0$ is a local equation of $V$.

**Remark 10.** Lemma [6] is valid for general $b$, but there are small variations in the formulae for the $c_0 (I_b, \alpha)$s. In the case $b_0 \equiv 0 \pmod{p}$, $b_1 \not\equiv 0 \pmod{p}$, the statement of Lemma [6] and the corresponding proof are similar to ones presented here, see Remark [8]. We outline the calculations for the case $b_0 \equiv 0 \pmod{p}$, $b_1 \equiv 0 \pmod{p}, b_2 \not\equiv 0 \pmod{p}$. In this case, we use the following change of variables:

\[
u = G(z) \quad \text{with} \quad u_0 = z_0, u_1 = z_1, u_2 = \frac{q(pz) - 1}{p^2}, u_3 = z_3.
\]

Then

\[
J\text{adj}_G(z) = \det \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{p^2} \frac{\partial u_0}{\partial z_0} & \frac{1}{p^2} \frac{\partial u_0}{\partial z_1} & \frac{1}{p^2} \frac{\partial u_0}{\partial z_2} & \frac{1}{p^2} \frac{\partial u_0}{\partial z_3} \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{p^2} \frac{\partial u_2}{\partial z_2} = -2 \left( b_2 + pz_2 \right),
\]

and thus $J\text{adj}_G(z) \equiv -2b_2 \equiv b_2 \not\equiv 0 \pmod{p}$, and by Lemma 7.4.3 in [21], $G$ gives rise to an analytic isomorphism from $\mathbb{Z}_p^4$ to itself which preserves the Haar measure. By changing variables in integral $(I_b(s), \theta)$, we get that

\[
(I_b(s), \theta) =
\]

\[
p^{-2s} \int_{\mathbb{Z}_p^3} \left| w_2 \right|^s \theta(b_0 + pz_0, b_1 + pu_1, b_2 + ph(u_0, \ldots, u_3), b_3 + pu_3) du_0 du_1 du_2 du_3.
\]

Now the calculations proceed as in the proof of Lemma [6].

**Remark 11.** Set $\delta_b(x) := p^{4k} \Omega(p^k \|x\|_p)$. We recall the definition of the product of two distributions: given $F, G \in \mathcal{D}_C$, their product is defined as $(F \cdot G, \varphi) = \lim_{k \to \infty}(G, (F \cdot \delta_b)\varphi)$, if the limit exist for all $\varphi \in \mathcal{D}_C$. If the product $F \cdot G$ exists then the product $G \cdot F$ exists and they are equal.
Lemma 7. \(|q - 1|_p^\alpha \delta(q - 1), \psi) = 0\) for any \(\psi \in \mathcal{D}_\mathbb{C}\) and for any \(\alpha > 0\).

Proof. By Remark [11] \((q - 1|_p^\alpha \delta(q - 1), \psi) = \lim_{k \to \infty} (\delta(q - 1), (|q - 1|_p^\alpha * \delta_k)\psi)\).

Now, \((|q - 1|_p^\alpha * \delta_k)(x) = p^{4k} \int_{x + p^k \mathbb{Z}_p^4} |q(y) - 1|_p^\alpha d^4y\).

Since \(V \subseteq \mathbb{Z}_p^4\) has measure zero, we may assume without loss of generality that \(x \notin V\). Now, if \(z \in \mathbb{Z}_p^4\) then \(q(x + p^k z) - 1 = q(x) - 1 + p^k A\), with \(A \in \mathbb{Z}_p^4\) and \(q(x) - 1 \neq 0\), then by taking \(k\) sufficiently large, we have \(|q(x + p^k z) - 1|_p^\alpha = |q(x) - 1|_p^\alpha\), consequently \((|q - 1|_p^\alpha * \delta_k)(x) = |q(x) - 1|_p^\alpha\) for \(k\) sufficiently large.

Finally, \((|q - 1|_p^\alpha \delta(q - 1), \psi) = (\delta(q - 1), |q - 1|_p^\alpha \psi) = 0\) because \(\text{supp}\ \delta(q - 1) = V\).

Remark 12. For any locally constant function \(h\), it holds that \(h|q - 1|_p^\alpha \delta(q - 1) \in \mathcal{D}'_\mathbb{C}\), see e.g. [53, p. 126, Proposition 3.16]. Then \((h|q - 1|_p^\alpha \delta(q - 1), \psi) = (|q - 1|_p^\alpha \delta(q - 1), \psi) = 0\) for any \(\psi \in \mathcal{D}_\mathbb{C}\).

Remark 13. Let us make some comments about orthogonal invariance in this setting.

(i) Let \(\varphi \in \mathcal{D}_\mathbb{C}\) and let \(T \in \mathcal{D}'_\mathbb{C}\). We define the action of \(A \in \mathcal{O}(q)\), by putting \((A \varphi)(x) = \varphi(A^{-1} x)\),

and the action of \(A\) on \(T\), by putting \((AT) \varphi = (T, A^{-1} \varphi)\).

We say that \(T\) is invariant under \(\mathcal{O}(q)\), if \(AT = T\) for any \(A \in \mathcal{O}(q)\).

(ii) \(T\) is invariant under \(\mathcal{O}(q) \Leftrightarrow \hat{T}\) is invariant under \(\mathcal{O}(q)\). We first notice that by using \(B(A^{-1} y, A^{-1} k) = B(y, k)\) for any \(A \in \mathcal{O}(q)\), we have

\[
\left( A^{-1} \varphi \right)(k) = \int_{\mathbb{Q}_p^4} \chi_p(B(x, k))(A^{-1} \varphi)(x) \ d\mu(x)
\]

\[
= \int_{\mathbb{Q}_p^4} \chi_p(B(x, k)) \varphi(Ax) \ d\mu(x) = \int_{\mathbb{Q}_p^4} \chi_p(B(A^{-1} y, A^{-1} (Ak))) \varphi(y) \ d\mu(y)
\]

\[
= \int_{\mathbb{Q}_p^4} \chi_p(B(y, Ak)) \varphi(y) \ d\mu(y) = \hat{\varphi}(Ak),
\]

i.e. \(\left( A^{-1} \varphi \right) = A^{-1} \hat{\varphi}\). Now, assuming that \(AT = T\) for any \(A \in \mathcal{O}(q)\), we have

\[
\left( A T, \varphi \right) = \left( \hat{T}, A^{-1} \varphi \right) = \left( T, A^{-1} \hat{\varphi} \right) = (AT, \hat{\varphi})
\]

\(= (T, \hat{\varphi}) = (\hat{T}, \varphi)\).

Here, it is worth to mention that our definition of Fourier transform using the bilinear form \(B\) plays a crucial role.

(iii) By a result of Rallis-Schifman, the distribution \(\delta(q - 1)\) is the unique (up to multiplication by complex constants) distribution supported on \(V\) invariant under \(\mathcal{O}(q)\), [41].
Theorem 1. There exist fundamental solutions $E_{q,\alpha}$ for operators $\Box_{q,\alpha}$ which are invariant under the action of $O(q)$. Furthermore, the distributions $E_{q,\alpha}$ satisfy the following:

(i) $\mathcal{F}(E_{q,\alpha}) = \mathcal{F}(E_{q,\alpha}^0) + C\delta(q - 1)$, (25)

where $C$ is a non-zero complex constant and $\mathcal{F}(E_{q,\alpha}^0)$, $\delta(q - 1)$ are distributions invariant under $O(q)$.

(ii) $1_V \mathcal{F}(E_{q,\alpha}) = C\delta(q - 1)$. (26)

In particular, the restriction of $\mathcal{F}(E_{q,\alpha})$ to $V$ is unique up to multiplication for a non-zero complex constant.

Proof. The existence of fundamental solutions for operators $\Box_{q,\alpha}$ is guaranteed by Theorem 134 in [67]. If $E_{q,\alpha}^0$ is a fundamental solution for $\Box_{q,\alpha}$, then, by Lemmas 2, 7, $E_{q,\alpha}^0 + C\mathcal{F}^{-1}[\delta(q - 1)]$ is also a fundamental solution for any non-zero complex constant $C$. Therefore, the Fourier transform of any fundamental solution may be written as

$$\mathcal{F}[E_{q,\alpha}] = \mathcal{F}[E_{q,\alpha}^0] + C\delta(q - 1),$$  (27)

for some fundamental solution $E_{q,\alpha}^0$ and some non-zero complex constant $C$.

Remark 14. In fact, if there is another fundamental solution $E'_{q,\alpha}$ of $\Box_{q,\alpha}$, invariant under $O(q)$, satisfying

$$\mathcal{F}[E_{q,\alpha}] = \mathcal{F}[E'_{q,\alpha}] + C\delta(q - 1),$$  (28)

then from (27) and (28) we get that $\mathcal{F}[E'_{q,\alpha} - E_{q,\alpha}^0]$ is a distribution supported on $V$ and invariant under $O(q)$, and consequently $\mathcal{F}[E'_{q,\alpha} - E_{q,\alpha}^0] = C_0\delta(q - 1)$, for some constant $C_0$.

By Lemmas 5, 6 and Remark 10 there exists a fundamental solution $E_{q,\alpha}^0$, such that $\mathcal{F}[E_{q,\alpha}^0]$ is a linear combination of distributions of any of the types

$$\int_{Q^4_p \setminus E_V} |q(x) - 1|^{-\alpha}\Theta(x) d^4x \text{ or } p^\alpha \int_{Z^p} |u_0|^{-\alpha}(\Theta_b(u_0) - \Theta_b(0)) du_0,$$

with $\Theta_b(u_0)$ defined as in Lemma 6. In addition, we have

$$1_V \mathcal{F}[E_{q,\alpha}^0] = 0 \text{ in } \mathcal{D}'(Q^4_p).$$

The rest of assertions announced follows from Remark 13 by the following assertion:

Claim. The distribution $E_{q,\alpha}^0$ is invariant under $O(q)$.

We first note that

$$A|q - 1|^p = |q - 1|^p \text{ for any } A \in O(q), \text{ and } \text{Re}(s) > 0, \quad (29)$$
because \( q(A^{-1}y) = q(y) \) for any \( A \in \mathcal{O}(q) \), and any \( y \in \mathbb{Q}^4_p \). Now, we rewrite \(^9\) as

\[
(q - 1)_p^\alpha A^{-1} \varphi = (q - 1)_p^\alpha \varphi \quad \text{for} \quad A \in \mathcal{O}(q), \varphi \in \mathcal{D}_C, \quad \text{and} \quad \text{Re}(s) > 0,
\]

and use that \( A^{-1} \varphi \in \mathcal{D}_C \) for \( \varphi \in \mathcal{D}_C \), and that the distribution \( |q - 1|^\alpha \) admits a meromorphic continuation to the whole complex plane to conclude that \(^9\) is valid for any \( s \). We now recall that \( \mathcal{F}[E^s_{q,\alpha}] = c_0 \in \mathcal{D}_C \), where

\[
(q - 1)_p^\alpha \varphi = \sum_{j=-j_0}^{\infty} (c_j, \varphi) (s + \alpha)^j = (A[q - 1]^\alpha, A^{-1} \varphi)
\]

\[
= \sum_{j=-j_0}^{\infty} (c_j, A^{-1} \varphi) (s + \alpha)^j,
\]

then \( (c_0, \varphi) = (c_0, A^{-1} \varphi) \), which implies that \( c_0 \) is invariant under \( \mathcal{O}(q) \), and consequently, \( E^0_{q,\alpha} \) is invariant under \( \mathcal{O}(q) \).

4. Klein-Gordon type operators acting on \( \mathcal{H}_\infty \)

**Lemma 8.** Let \( f(k) \in \mathbb{Q}_p[k_0, k_1, k_2, k_3] \) be a non-constant homogeneous polynomial of degree \( c \) and \( \alpha > 0 \). Then there exists a positive constant \( \Lambda = \Lambda(f, \alpha) \) such that

\[
|f(k) - 1|_p^\alpha \leq \Lambda[k]_p^\alpha \quad \text{for} \quad k \in \mathbb{Q}_p^4.
\]

**Proof.** We first note that \( |f(k)| \leq \left[ \max \{|f(k)|_p, 1\} \right]^\alpha \). We now use that \( |f(k)|_p \leq C(f) [k]_p^\alpha \) for \( k \in \mathbb{Q}_p^4 \), to obtain

\[
|f(k) - 1|_p^\alpha \leq \left[ \max \{C(f) [k]_p^\alpha, 1\} \right]^\alpha \leq \left[ \max \{C(f), 1\} \right]^\alpha \left[ \max \{k]_p^\alpha, 1\} \right]^\alpha = \Lambda[k]_p^\alpha.
\]

**Remark 15.** For \( \alpha \in \mathbb{R} \), we set \( \left\lfloor \alpha \right\rfloor := \min \{\gamma \in \mathbb{Z}; \gamma \geq \alpha\} \), the ceiling function.

**Lemma 9.** The mapping

\[
\Box_{q,\alpha} : \mathcal{H}_\infty(\mathbb{K}) \to \mathcal{H}_\infty(\mathbb{K})
\]

is a well-defined continuous linear operator between locally convex spaces.

**Proof.** Take \( \mathbb{K} = \mathbb{C} \). Let us first prove that \( \Box_{q,\alpha} \) is a well-defined linear operator. Let \( h \in \mathcal{H}_{1+\{4\alpha\}}(\mathbb{C}) \), then by the Lemma 8 with \( c = 2 \), we have

\[
\|\Box_{q,\alpha} h\|_2^2 = \int_{\mathbb{Q}_p^4} [\xi]_p^2 |\Box_{q,\alpha} h(k)|^2 d^4k = \int_{\mathbb{Q}_p^4} [\xi]_p^2 |q(k) - 1|_p^{2\alpha} |\widehat{h}(k)|^2 d^4k
\]

\[
\leq C \int_{\mathbb{Q}_p^4} [\xi]_p^{1+4\alpha} |\widehat{h}(k)|^2 d^4k \leq C \int_{\mathbb{Q}_p^4} [\xi]_p^{1+\{4\alpha\}} |\widehat{h}(k)|^2 d^4k = C\|h\|_2^2.
\]
By Lemma \( \Pi (i) \), \( \Box_{\lambda,\alpha} h \in \mathcal{H}_t(\mathbb{C}) \), i.e. \( \Box_{\lambda,\alpha} \) is a well-defined, linear, and continuous operator from \( \mathcal{H}_{t+\{4\alpha\}}(\mathbb{C}) \) into \( \mathcal{H}_t(\mathbb{C}) \) for any \( l \in \mathbb{N} \). In turn, this implies that \( \Box_{\lambda,\alpha} \) is a well-defined linear operator from \( \mathcal{H}_\infty(\mathbb{C}) \) into \( \mathcal{H}_\infty(\mathbb{C}) \). To establish the continuity, we use the fact that \( (\mathcal{H}_\infty(\mathbb{C}), d) \) is a metric space. Take a sequence \( \{ \varphi_n \}_{n \in \mathbb{N}} \subset \mathcal{H}_\infty(\mathbb{C}) \) such that \( \varphi_n \to \varphi \), with \( \varphi \in \mathcal{H}_\infty(\mathbb{C}) \), which is equivalent to say that \( \varphi_n \to \varphi \), for all \( r \in \mathbb{R} \). Take \( l \in \mathbb{N} \) and \( \varphi, \varphi_n \in \mathcal{H}_{t+\{4\alpha\}}(\mathbb{C}) \), then by the continuity of \( \Box_{\lambda,\alpha} : \mathcal{H}_{t+\{4\alpha\}}(\mathbb{C}) \to \mathcal{H}_t(\mathbb{C}) \), we have \( \Box_{\lambda,\alpha} \varphi_n \to \Box_{\lambda,\alpha} \varphi \), and since \( l \) is arbitrary in \( \mathbb{N} \), we conclude that \( \Box_{\lambda,\alpha} \varphi_n \to \Box_{\lambda,\alpha} \varphi \).

We know turn to the case \( K = \mathbb{R} \). Since \( (\Box_{\lambda,\alpha} \varphi)(x) = (\Box_{\lambda,\alpha} \varphi)(x) \) for \( \varphi \in \mathcal{H}_\infty(\mathbb{R}) \), the statement is also valid in \( \mathcal{H}_\infty(\mathbb{R}) \).

Remark 16. The preceding lemma remains valid if we replace \( |q(k) - 1|_p^\alpha \) by \( g(\|k\|_p) |q(k) - 1|_p^\alpha \), where \( g : \mathbb{R}_+ \to \mathbb{C} \) is any continuous function.

Remark 17. We recall that \( V \) is a \( p \)-adic compact submanifold of \( \mathbb{Z}_p^4 \) of codimension one. We denote by \( d\lambda \) the measure corresponding to the distribution \( \delta (q - 1) \) as before. Then \( (V, \mathcal{B}(V), d\lambda) \) is a measure space, where \( \mathcal{B}(V) \) is the Borel \( \sigma \)-algebra generated by the open compact subsets of \( V \), and thus the space \( L^2_\mathcal{B}(V, d\lambda) \) is well-defined.

**Proposition 1.** The mapping

\[
R : \mathcal{H}_t(\mathbb{C}) \to L^2_\mathcal{B}(V^+, d\lambda)
\]

\[
f \to \hat{f} \big|_{V^+}
\]

determines a well-defined operator satisfying

\[
\|R(f)\|_{L^2_\mathcal{B}(V^+, d\lambda)} \leq C \|f\|_t
\]

for any \( l \in \mathbb{N} \). Consequently, \( R \) induces a continuous operator from \( \mathcal{H}_\infty(\mathbb{C}) \) into \( L^2_\mathcal{B}(V^+, d\lambda) \).

**Proof.** Since \( \mathcal{D}_C \) is dense in \( \mathcal{H}_t(\mathbb{C}) \) for any \( l \in \mathbb{N} \), in order to prove \( \Box_{\lambda,\alpha} \) we may assume without loss of generality that \( f \in \mathcal{D}_C \) and that \( \hat{f} \big|_{V^+} \) is not the constant function zero. Notice that

\[
\|R(f)\|_{L^2_\mathcal{B}(V^+, d\lambda)}^2 = \int_{U_{\mathbb{Q}}} \left| \hat{f} \left( \sqrt{\omega(k)} \right) \right|^2 \frac{d^3k}{\sqrt{\omega(k)}},
\]

where \( \left| \sqrt{\omega(k)} \right|_p \neq 0 \), cf. Remark \( \Box \). For \( m \in \mathbb{Q}_p^\times \), we set

\[
V_m = \{ (k_0, k) \in \mathbb{Q}_p^4; q(k_0, k) = m \}.
\]

We recall that \( q(k_0, k) = k_0 - q_0(k) \). Then \( V_m \) is a \( p \)-adic compact submanifold of \( \mathbb{Q}_p^4 \) of codimension one. In the case in which \( V_m \neq \emptyset \), we denote by \( d\lambda(m) \) the measure on \( V_m \) induced by the Gel’fand-Leray form on \( V_m \). Then \( dk_0d^3k = d\lambda(m) \, dm \), where \( dm \) is the normalized Haar measure of \( \mathbb{Q}_p \).
Claim C. For \( \hat{f}(k_0, k) \in D_C \), the \( \mathbb{R} \)-valued function defined by

\[
\int_{Q_p^1} \int_{V_m} |\hat{f}(k_0, k)|^2 \, d\lambda(m) \, dm
\]

is in \( D_{\Omega} (Q_p) \) and

\[
\|f\|^2_0 = \int_{Q_p^1} |\hat{f}(k_0, k)|^2 \, dk_0 \, d^3k = \int_{Q_p^1} \int_{V_m} |\hat{f}(k_0, k)|^2 \, d\lambda(m) \, dm.
\] (32)

This claim is a very particular version of a general theorem on integration over the fibers in the framework of \( p \)-adic manifolds, see [21, Theorem 7.6.1].

Claim D. There exists a positive constant \( C_0 \) such that

\[
\|f\|^2_0 \geq C_0 \int_{V_q} \left| \hat{f} \left( \sqrt{\omega(k)}, k \right) \right|^2 \frac{d^3k}{\sqrt{\omega(k)}_p^p}.
\] (33)

Estimation (30) follows from (31)-(33). The fact that operator \( R \) extends to \( H_\infty(C) \) follows from (30), by using a classical argument based on convergence of sequences due to the fact that the topology of \( H_\infty(C) \) is metrizable.

Proof of Claim D. In order to prove the Claim we proceed as follows. We set \( G_M := 1 + p^M Z_p \), for \( M \geq 1 \). Then \( G_M \) is a multiplicative subgroup of the group of squares of \( Q_p^1 \). This is a compact subgroup so its Haar measure, denoted as \( \text{vol}(G_M) \), is finite. Now, we notice that

\[
\int_{Q_p^1} \int_{V_m} |\hat{f}(k_0, k)|^2 \, d\lambda(m) \, dm \geq \int_{G_M} \int_{V_m} |\hat{f}(k_0, k)|^2 \, d\lambda(m) \, dm
\]

\[
= \int_{G_M} \int_{V_m} |\hat{f}(k_0, k)|^2 \frac{d^3k \, dm}{|m + q_0(k)|^p_2}.
\] (34)

We now use the fact that

Claim E. The mapping

\[
\sqrt{\cdot} : G_M \to G_M
\]

\[
m \to \sqrt{m}
\]

and its inverse are \( p \)-adic analytic functions, for \( M \) sufficiently large.

We change variables in the last integral in (34) as \( y_0 = \frac{k_0}{\sqrt{m}}, y = \frac{k}{\sqrt{m}} \), then \( dk_0 \, d^3k = dy_0 \, d^3y \) and

\[
\int_{G_M} \int_{V_m} |\hat{f}(k_0, k)|^2 \frac{d^3k \, dm}{|m + q_0(k)|^p_2}
\]

\[
= \int_{G_M} \int_{V_m} |\hat{f}(\sqrt{m}y_0, \sqrt{m}y)|^2 \frac{d^3y \, dm}{|1 + q_0(y)|^p_2}.
\]
Finally since $\hat{f}$ is locally constant and $\sqrt{m}$ is a unit for every $m \in G_M$, we have for $M$ sufficiently large that

$$\int_{G_M} \int_{V} \left| \hat{f}(\sqrt{m}y_0, \sqrt{m}y) \right|^2 \frac{d^3ydm}{|1 + q_0(y)|^2_p} \geq \text{vol}(G_M) \int_{V^+} \left| \hat{f}(y_0, y) \right|^2 \frac{d^3y}{|1 + q_0(y)|^2_p}. $$

Proof of Claim E.

We first notice that $(1 + p^M Z_p)^2 = 1 + 2p^M Z_p = 1 + p^M Z_p$ for $M \geq 2$, see Lemma 8.4.1 in [21]. This means that the mapping

$$G_M \to G_M$$

$$x \to x^2$$

is well-defined and surjective. Then for any $m \in G_M$, the equation $x^2 = m$ has a solution $\sqrt{m}$ in $G_M$. Notice that there is another solution $-\sqrt{m} = -1 + (\text{higher order terms})$ which does not belong to $G_M$. Consequently the mapping

$$G_M \to G_M$$

$$m \to \sqrt{m}$$

is well-defined. The fact that the mappings (35)–(36) are $p$–adic analytic follows from the implicit function theorem.

Remark 18. The preceding Proposition remains valid if we replace $R(f) = \hat{f}|_{V^+}$ by $R(f)(k) = g([k]_p) \hat{f}(k)|_{V^+}$, where $g$ is any continuous function $g : \mathbb{R}_+ \to \mathbb{C}$.

Lemma 10. There exist a positive constant $C$ such that

$$\frac{1}{|1 + q_0(k)|^2_p} \leq C \text{ for any } k \in \mathbb{Q}^3_p.$$

Proof. The hypothesis $p \equiv 1 \mod 4$ implies $W = \{ k \in \mathbb{Z}^3_p; 1 + q_0(k) = 0 \} = \emptyset$, see Remark 5

Claim A. $|1 + q_0(k)|^2_p > C_1$ for any $C_1 \in (0, p)$ and for any $||k||_p \geq p$.

We recall that $q_0(k)$ and $q(k_0, k)$ are elliptic quadratic forms and that

$$|q_0(k)|_p = |q(0, k)|_p \geq \left( \inf_{x \in S^2} |q(0, x)|_p \right) \|k\|_p^2 = p^{-1}\|k\|_p^2$$

for any $k \in \mathbb{Q}^3_p$, (37) see (12). Now, $p^{-1}\|k\|_p^2 > 1$ if and only if $\|k\|_p \geq p$, and by applying the ultrametric property of the norm $\| \cdot \|_p$, we get from (37), that for $\|k\|_p \geq p$,

$$|1 + q_0(k)|_p = \max \{1, q_0(k)\} \geq p^{-1}\|k\|_p^2 \geq p > C_1 \text{ for any } C_1 \in (0, p).$$

Claim B. There exist a constant $C_0$ such that $\inf_{k \in \mathbb{Z}^3_p} |1 + q_0(k)|_p \geq C_0 > 0$.

This assertion follows from the fact that $|1 + q_0(k)|_p > 0$ for any $k \in \mathbb{Z}^3_p$. The statement of the lemma is a consequence of Claims A and B.
Lemma 11. The mapping

\[ R : L^2_C \left( \mathbb{Q}_p^4, d^4 x \right) \to L^2_C \left( V^+, d\lambda \right) \]

\[ g \to \hat{g} |_{V^+} \]

satisfies \( \|R(g)\|_{L^2_C(V^+, d\lambda)} \leq C \|g\|_{L^2_C(\mathbb{Q}_p^4, d^4 x)} \). Here \( \hat{g}(k) \) denotes the 3-dimensional Fourier transform defined with respect to the bilinear form

\[-\mathcal{B}_0(x, y) = -sx_1y_1 - px_2y_2 + spx_3y_3.\]

Proof. The results follows from Lemma 10, by using that

\[ |k_0|_p = \left| \sqrt{\omega(k)} \right|_p = |1 + q_0(k)|_p^p \text{ for } k \in U_q. \]

Remark 19. Some observations about the functional spaces involved here:

(i) Let \( X \) be a locally compact totally disconnected space. We denote by \( \mathcal{D}_C(X) \) the \( \mathbb{C} \)-vector space of locally constant functions with compact support. We recall that \( V^+ \subset \mathbb{Q}_p^4 \) is an open and compact subset, then \( \mathbb{Q}_p^4 \setminus V^+ \) is open and closed subset, and thus \( V^+ \) and \( \mathbb{Q}_p^4 \setminus V^+ \) are locally compact totally disconnected spaces. The following exact sequence holds:

\[ 0 \to \mathcal{D}_C(V^+) \to \mathcal{D}_C(\mathbb{Q}_p^4) \to \mathcal{D}_C(\mathbb{Q}_p^4 \setminus V^+) \to 0, \]

see e.g. [21, p. 99].

(ii) It is well-known that the \( \mathbb{C} \)-space of finite-valued simple functions is dense in \( L^2_C(V^+, d\lambda) \). By using the fact that \( d\lambda = \frac{d^4 k}{\sqrt{\omega(k)}} \) is an inner regular measure, one can show that any finite-valued simple function can be approximated in the \( L^2_C(V^+, d\lambda) \)-norm by an element of \( \mathcal{D}_C(V^+) \). i.e. \( \mathcal{D}_C(V^+) \) is dense in \( L^2_C(V^+, d\lambda) \).

(iii) The mapping

\[ L^2_C \left( \mathbb{Q}_p^4, d^4 k \right) \xrightarrow{R} L^2_C \left( V^+, d\lambda \right) \]

\[ f \to \hat{f} |_{V^+} \]

is a well-defined continuous mapping, more precisely,

\[ \left\| \hat{f} |_{V^+} \right\|_{L^2_C(V^+, d\lambda)} \leq C \left\| \hat{f} \right\|_{L^2_C(\mathbb{Q}_p^4, d^4 k)} = C \|f\|_{L^2_C(\mathbb{Q}_p^4, d^4 k)}. \]

Indeed, (39) holds when \( \hat{f} \in \mathcal{D}_C(\mathbb{Q}_p^4) \) and \( \hat{f} |_{V^+} \in \mathcal{D}_C(V^+) \), see Claim D, then (39) follows by the fact that \( \mathcal{D}_C(V^+) \) is dense in \( L^2_C(V^+, d\lambda) \) and that \( \mathcal{D}_C(\mathbb{Q}_p^4) \) is dense in \( L^2_C(\mathbb{Q}_p^4, d^4 k) \).

Remark 20. Regarding the spaces of integrable functions introduced in the preceding Remark, we note the following.
Lemma 12. For \( f \in L^2_C(V^+, d\lambda) \), we define \( T_{V^+}(f) \in \mathcal{D}_C' \) by

\[
\langle T_{V^+}(f), \varphi \rangle = \int_{V^+} f(x) \varphi(x) \, d\lambda(x) \quad \text{for } \varphi \in \mathcal{D}_C.
\]

Then we have the following sequence of continuous mappings:

\[
\mathcal{H}_\infty(C) \xrightarrow{R} L^2_C(V^+, d\lambda) \xrightarrow{T_{V^+}} \mathcal{H}_\infty^+(C),
\]

where the map \( R \) is defined as in Proposition 7.
Proof. The support of $T_{V^+}(f)$ is compact since it is contained in $V$, which is a compact subset of $Q^A_p$. The Fourier transform of $T_{V^+}(f)$ in $D'\mathbb{C}$ is the locally constant function

$$\hat{f}(k) = \int_{V^+} \chi_p(\mathcal{B}(x,k)) f(x) d\lambda(x),$$

for a similar calculation the reader may see, for instance, [2, Theorem 4.9.3]. Now, identifying $f$ with the induced distribution $T_{V^+} f$ on $V^+$, by using the definition of $\mathcal{H}^*_\infty(\mathbb{C})$ (see [51]), the Cauchy-Schwarz inequality, and the fact that $\int_{Q^A_p} [k]^{-l} d^4k < \infty$ for $l \geq 5$, we have

$$\|f\|_{L^2(V^+, d\lambda)}^2 = \int_{Q^A_p} |k|^{-l} |\hat{f}(k)|^2 d^4k = \int_{Q^A_p} [k]^{-l} \left| \int_{V^+} \chi_p(\mathcal{B}(x,k)) f(x) d\lambda(x) \right|^2 d^4k \leq C(l) \int_{V^+} |f(x)|^2 d\lambda(x) = C(l) \|f\|^2_{L^2(V^+, d\lambda)},$$

which implies that $T_{V^+}(f) \in \mathcal{H}^*_\infty(\mathbb{C})$.

5. Free non-Archimedean quantum fields

5.1. The Segal quantization. We start by reviewing some well-known facts about quantization. For an in-depth discussion the reader may consult [51, 44], see also [8, 3, 43, 52] for more physically-oriented approaches. Our presentation follows closely the book of Reed and Simon [44]. In particular, our notation mimics the one used in that book. We set $\mathcal{H} = L^2_\mathbb{C}(V^+, d\lambda)$ and denote by $\langle \cdot, \cdot \rangle$ the inner product of $\mathcal{H}$. We assume that $\langle f, \alpha g \rangle = \alpha \langle f, g \rangle$, for $\alpha \in \mathbb{C}$, and $f, g \in \mathcal{H}$. We define the Fock space over $\mathcal{H}$ as $\mathfrak{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$, where $\mathcal{H}^{(n)} = \otimes_{k=1}^{n} \mathcal{H}$, by definition $\mathcal{H}^{(0)} = \mathbb{C}$. We denote by $S_n : \mathcal{H}^{(n)} \to S\mathcal{H}^{(n)}$, the symmetrization operator, and define $S = \bigoplus_{n=0}^{\infty} S_n$, see [43, Section II.4]. The symmetric Fock space over $\mathcal{H}$ (also called the boson Fock space over $\mathcal{H}$) is defined as $\mathfrak{F}_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$, where $\mathcal{H}^{(n)} = S_n \mathcal{H}^{(n)}$. We call $\mathcal{H}^{(n)}$ the $n$-particle subspace of $\mathfrak{F}_s(\mathcal{H})$. We use the same symbol $\langle \cdot, \cdot \rangle$ to denote the inner product of $\mathfrak{F}_s(\mathcal{H})$.

We now fix a vector $f$ in $\mathcal{H}$. For the vectors of the form $\eta = \psi_1 \otimes \ldots \otimes \psi_n$, we define a map $b^- (f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n-1)}$ by $b^- (f)(\eta) = \langle f, \psi_1 \rangle \psi_2 \otimes \ldots \otimes \psi_n$. Then $b^- (f)$ extends to a bounded map (of norm $\|f\|_\mathcal{H}$) of $\mathcal{H}^{(n)}$ in to $\mathcal{H}^{(n-1)}$. In the case $n = 0$, we define $b^- (f) : \mathcal{H}^{(0)} \to 0$. The adjoint $b^+ (f) : \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}$ of $b^- (f)$ is defined as $b^+ (f)(\psi_1 \otimes \ldots \otimes \psi_n) = f \otimes \psi_1 \otimes \ldots \otimes \psi_n$. The map $f \to b^+ (f)$ is linear, but $f \to b^- (f)$ is anti-linear.

The boson Fock space is invariant under $b^- (f)$ but not under $b^+ (f)$. A vector $\psi = \{ \psi^{(n)} \}_{n \in \mathbb{N}} \in \mathfrak{F}_s(\mathcal{H})$ is called a finite particle vector if $\psi_n = 0$ for all but finitely many $n$. The set of all finite vectors is denoted as $F_0$. We set the vector $\mathcal{T}_0 = (1, 0, 0, \ldots)$ to be the vacuum.

Let $A$ be a self-adjoint operator on $\mathcal{H}$ with domain of essential self-adjointness $D$. Let $D_A = \{ \psi \in F_0; \psi^{(n)} \in \otimes_{k=1}^{n} D \text{ for each } n \}$. We define the operator $\Gamma (A)$ (the second quantization of $A$) on $\mathcal{D}_A \cap \mathcal{H}^{(n)}_s$ as

$$A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + I \otimes I \otimes \cdots \otimes A,$$
where $I$ is the identity operator. The operator $\Gamma(A)$ is essentially self-adjoint on $D_A$. In the case $A = I$, the second quantization $N = \Gamma(A)$ (the number operator) is essentially self-adjoint on $F_0$ and for $\phi \in H^{(n)}$, $N\phi = n\phi$.

The annihilation operator $a^-(f)$ on $\mathcal{F}_s(\mathcal{H})$ with domain $F_0$ is given by

$$a^-(f) = \sqrt{N + 1} b^-(f).$$

For $\psi$, $\eta$ in $F_0$,

$$\langle \sqrt{N + 1} b^-(f)\psi, \eta \rangle = \langle \psi, Sb^+(f)\sqrt{N + 1}\eta \rangle,$$

which implies that

$$(a^-)(f)^* |_{F_0} = Sb^+(f)\sqrt{N + 1},$$

where ‘*’ denotes the adjoint operator. The operator $(a^-)(f)^*$ is called the creation operator. Both $a^-(f)$ and $(a^-)(f)^*$ are closable, the corresponding closures are denoted as $a^-(f)$ and as $(a^-)^*$.

**Definition 2.** For $f \in \mathcal{H}$, the Segal quantum field operator $\Phi_S$ on $F_0$ is defined as

$$\Phi_S(f) = \frac{1}{\sqrt{2}}[a^-(f) + a^-]^*]. \quad (40)$$

The mapping from $\mathcal{H}$ into the self-adjoint operators on $\mathcal{F}_s(\mathcal{H})$ given by $f \mapsto \Phi_S(f)$ is called the Segal quantization over $\mathcal{H}$. Notice that the Segal quantization is a real linear map.

**Remark 21.** By using the fundamental properties of the Segal quantization, see [44, Theorem X.41], we obtain the following facts (among others):

(i) For each $f \in \mathcal{H}$, $\Phi_S(f)$ is essentially self-adjoint on $F_0$.

(ii) The commutation relations: for each $\psi \in F_0$, and $f, g \in \mathcal{H}$,

$$\Phi_S(f)\Phi_S(g)\psi - \Phi_S(g)\Phi_S(f)\psi = \sqrt{-1} \text{Im} (\langle f, g \rangle) \psi,$$

that is, $[\Phi_S(f), \Phi_S(g)] = \sqrt{-1} \text{Im} (\langle f, g \rangle) I$, on $F_0$.

5.1.1. The free Hermitian field of unit mass. We define for each $f \in \mathcal{H}_\infty (\mathbb{R}),$

$$\Phi(f) = \Phi_S(Rf),$$

with $R$ defined as in Lemma [11] and for each $g \in \mathcal{H}_\infty (\mathbb{C}),$

$$\Phi(g) = \Phi(\text{Re} g) + \sqrt{-1} \Phi(\text{Im} g). \quad (42)$$

We call the mapping $g \mapsto \Phi(g)$ the free Hermitian scalar field of unit mass.

**Remark 22.** By extending the mapping $R$ as in Remark [13] the field $f \mapsto \Phi(f)$ remains well-defined. We emphasize that the presence of $R$ (in any of its forms) means that we are working on-shell.
5.1.2. The \( p \)-adic restricted Poincaré group. As we do not have the structure of light cones available, we must choose a substitute for them. Here we will base our treatment on the mass shells \( V^\pm \).

We define the \( p \)-adic restricted Lorentz group as
\[
\mathcal{L}_+^p = \{ \Lambda \in \mathcal{O}(q); \Lambda (V^\pm) = V^\pm \}.
\]
This group is non trivial since transformations of the form
\[
\begin{bmatrix}
1 & 0 \\
0 & \hat{F}
\end{bmatrix} \in \mathcal{O}(q); \hat{F} \in \mathcal{O}(q_0)
\]
belong to \( \mathcal{L}_+^p \). A further justification for choosing \( V^\pm \) as a replacement for the light cones comes from the fact that the distributions \( \delta_\pm (q - 1) \) are invariant under \( \mathcal{L}_+^p \), see \[67, Lemma 163\].

We define the \( p \)-adic restricted Poincaré group \( \mathcal{P}_+^p \) as the set of pairs \( (a, \Lambda) \), where \( a \in \mathbb{Q}_p^4 \) and \( \Lambda \in \mathcal{L}_+^p \), with the group operation
\[
(a, \Lambda_1) (b, \Lambda_2) = (a + \Lambda_1 b, \Lambda_1 \Lambda_2).
\]

The group \( \mathcal{P}_+^p \) acts naturally on \( \mathbb{Q}_p^4 \) by setting \( (a, \Lambda) x = \Lambda x + a \). With the topology inherited from \( \left( \mathbb{Q}_p^4; \| \cdot \|_p \right) \), \( \mathcal{L}_+^p \) and \( \mathcal{P}_+^p \) become locally compact topological groups.

On \( L^2(V^+, d\lambda) \), we define the following projective representation of the restricted Poincaré group:
\[
(U (a, \Lambda) \psi)(k) = \chi_p (\mathcal{B} (a, k)) \psi (A^{-1} k).
\]

5.2. The \( p \)-adic Wightman axioms. We present here a \( p \)-adic counterpart of the classical Wightman axioms, see e.g. \[51,44\], and references therein. We use units where the rationalized Planck's constant and the speed of light are equal to one. We take \( \mathcal{H} = \mathcal{F}_s \left( L^2_p (V^+, d\lambda) \right) \), \( \mathfrak{U} = \Gamma (U (\cdot, \cdot)) \), with \( U (\cdot, \cdot) \) being defined as in \[43\], \( \Phi \) as in \[42\], and \( D = F_0 \). A \( p \)-adic Hermitian scalar quantum field theory is a quadruple \( \{ \mathcal{H}, \mathfrak{U}, \Phi, D \} \) which satisfies the following properties:

Relativistic invariance of states. \( \mathcal{H} \) is a separable Hilbert space and \( \mathfrak{U} (\cdot, \cdot) \) is a strongly continuous unitary representation on \( \mathcal{H} \) of the \( p \)-adic restricted Poincaré group.

Spectral condition. We define the closed forward semigroup \( \overline{S(V^+)} \) as the topological closure of the additive semigroup generated by the vectors of \( V^+ \). Notice that since \( V^+ \subset \mathbb{Z}_p^4 \), \( \overline{S(V^+)} \) is a compact subset of \( \mathbb{Z}_p^4 \). Furthermore, since \( \mathcal{L}_+^p (V^+) = V^+ \), we have \( \mathcal{L}_+^p \left( \overline{S(V^+)} \right) = \overline{S(V^+)} \). The \( p \)-adic counterpart of the spectral condition is the following: there exists a projection-valued measure \( E_{V^+} \) on \( \mathbb{Q}_p^4 \) corresponding to \( \mathfrak{U}(a, I) \) having support in \( \overline{S(V^+)} \).

Remark 23. In the classical case by using a Stone type theorem, see \[33, Theorem VIII.12\], one shows the existence of four commuting operators \( P_0, P_1, P_2, P_3 \), on a suitable Hilbert space so that \( \mathfrak{U}(a, I) = e^{i \sum \alpha_j P_j} \). In the \( p \)-adic case, we...
do not have a complete theory of semigroups, with $p$-adic time, for operators acting on complex-valued functions. For this reason, at the moment, we do not have a definition for the $p$-adic counterparts of the operators $P_0$, $P_1$, $P_2$, $P_3$, and consequently, we do not know their spectra.

**Existence and uniqueness of the vacuum.** There exists a unique vector $τ_0 \in H$ such that $U(a,I)τ_0 = τ_0$ for all $a \in \mathbb{Q}_p^4$, this vector is called the vacuum.

**Invariant domains for fields.** There exists a dense subspace $D \subset H$ and a map from $H_\infty(C)$ to the unbounded operators on $H$ such that:

(i) For each $f \in H_\infty(C)$, we have that $D \subset \text{Dom}(Φ(f))$, $D \subset \text{Dom}(Φ(f)^*)$,

and $Φ(f)^* \upharpoonright D = Φ(\overline{f}) \upharpoonright D$.

(ii) $τ_0 \in D$, and $Φ(f)D \subset D$ for any $f \in H_\infty(C)$.

(iii) For a fixed $ψ \in D$, the map $f \rightarrow Φ(f)ψ$ is linear.

**Regularity of the field.** For any $ψ_1$ and $ψ_2$ in $D$, the map $f \rightarrow \langle ψ_1, Φ(f)ψ_2 \rangle_H$ is an element of $H_\infty^*(C)$. In the Archimedean case this is just a tempered distribution, here it turns out to be an element of $H_\infty^*(C)$, providing yet another argument to consider this space as the correct replacement in the $p$-adic framework of the Schwartz space $S$.

**Poincaré invariance of the field.** For each $(a,Λ) \in P_+^\uparrow$, $U(a,Λ)D \subset D$, and for all $f \in H_\infty(C)$, $ψ \in D$,

$U(a,Λ)Φ(f)U(a,Λ)^{-1}ψ = Φ((a,Λ)f)ψ$,

where $p(a,Λ)f(x) = f(Λ^{-1}(x - a))$.

**Local commutativity.** The $p$-adic local commutativity property states that if $f, g$ are in $D_C(\mathbb{Z}_p^4)$, then

$[Φ(f), Φ(g)]ψ = (Φ(f)Φ(g) - Φ(g)Φ(f))ψ = 0$,

for all $ψ \in D$. In the Archimedean case, the commutator vanishes whenever the test functions $f, g$ are supported on two respective spacelike-separated subsets, that is, $f(x)g(y) = 0$ whenever $x - y$ does not belong to the interior of the light cone. This subset can be characterized as the ‘ball of radius 0’ of Minkowski spacetime in the sense of the theory of indefinite quadratic forms (see, e.g., [20] and references therein). Our result can be seen as the equivalent statement in the $p$-adic case, with the unit ball playing this role.

**Cyclicity of the vacuum.** The set $D_0$ of finite linear combinations of vectors of the form $Φ(f_1)\cdots Φ(f_n)τ_0$ is dense in $H$.

**Theorem 2.** The following hold true:

(i) The quadruple

$\{ζ_s(L_τ^2(V^+, dλ)), Γ(U(\cdot, \cdot)), Φ, F_0\}$

satisfies the $p$-adic Wightman axioms.
(ii) For each \( f \in \mathcal{H}_\infty (\mathbb{C}) \),
\[
\Phi (\square_{q,a} f) = 0.
\]

Proof. In the proof of the first part \( \text{(i)} \), we use the notation
\[
\tilde{\mathfrak{s}}_s = \tilde{\mathfrak{s}}_s (L^2 \left( V^+, d\lambda \right) ) = \otimes_{n=0}^{\infty} \mathcal{H}_s^{(n)}.
\]

Relativistic invariance of states. We first note that \( \tilde{\mathfrak{s}}_s \) is separable because \( L^2 (V^+, d\lambda) \) is separable, see Remark \( (20) \ (i) \). On the other hand, since \( V^+ \) is invariant under \( L^+_1 \), \( U \left( \cdot, \cdot \right) \) is a strongly continuous unitary representation of \( \mathcal{P}_s \) on \( L^2_\mathcal{C} (V^+, d\lambda) \), see \( (23) \). By definition \( \Gamma (U) \) is the unitary operator on \( \tilde{\mathfrak{s}}_s \) given on \( \mathcal{H}_s^{(n)} \) by \( \otimes_{n=1}^{\infty} U \left( \cdot, \cdot \right) \), consequently \( \Gamma (U) : \mathcal{H}_s^{(n)} \to \mathcal{H}_s^{(n)} \) determines a strongly continuous unitary representation of \( \mathcal{P}_s \) on \( \mathcal{H}_s^{(n)} \). Notice that \( \Gamma (U) \) is strongly continuous in \( F_0 \), and since \( F_0 \) is dense in \( \tilde{\mathfrak{s}}_s \) we conclude that \( \Gamma (U) \) is a strongly continuous unitary representation of \( \mathcal{P}_s \) on \( \tilde{\mathfrak{s}}_s \).

Spectral condition. We show that the four parameter group \( \Gamma (U (a, I)) \) has associated a projection-valued measure supported on \( S (V^+) \). The argument needed is exactly the classical one, see \([44], \text{p. 213}\). The notion of closed forward semigroup, which is the \( p \)-adic counterpart of the closed forward light cone, allows us to carry out the calculations as in the classical case. We first notice that \( L^2_\mathcal{C} (V^+, d\lambda) \) is already a spectral representation of \( U (a, I) \) since
\[
\langle \varphi, U (a, I) \varphi \rangle_{L^2_\mathcal{C} (V^+, d\lambda)} = \int_{V^+} \chi_p (\mathfrak{B} (a, k)) | \varphi (k) |^2 d\lambda (k). \tag{44}
\]

Notice that if we define for \( \varphi, \theta \in L^2_\mathcal{C} (V^+, d\lambda) \), the set function
\[
B \to \int_{V^+} \varphi (k) \chi_p (\mathfrak{B} (a, k)) \theta (k) d\lambda (k),
\]

\( B \) being a Borel set in \( V^+ \), and denote the corresponding projection-valued measure as \( d (\varphi, E_k \varphi) \), in the case \( \varphi = \theta \), then \( (44) \) can be rewritten as
\[
\langle \varphi, U (a, I) \varphi \rangle_{L^2_\mathcal{C} (V^+, d\lambda)} = \int_{V^+} \chi_p (\mathfrak{B} (a, k)) d (\varphi, E_k \varphi). \]

Now, since \( \Gamma (U (a, I)) \mid \mathcal{H}_s^{(n)} = \otimes_{n=1}^{\infty} U (a, I) \), if \( \varphi^{(n)} \in \mathcal{H}_s^{(n)} \) with \( n > 0 \), then
\[
\langle \varphi^{(n)}, U (a, I) \varphi \rangle = \int_{V^+} \cdots \int_{V^+} \chi_p \left( \mathfrak{B} \left( a, \sum_{i=1}^{n} k_i \right) \right) | \varphi^{(n)} (k_1, \ldots, k_n) |^2 \prod_{k=1}^{n} d\lambda (k) = \int_{V^+} \chi_p (\mathfrak{B} (a, l)) d \mu_{\varphi^{(n)}} (l),
\]

where
\[
\mu_{\varphi^{(n)}} (A) = \int \cdots \int \sum_{k_i \in A} | \varphi^{(n)} (k_1, \ldots, k_n) |^2 \prod_{k=1}^{n} d\lambda (k),
\]

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A being a Borel set in $S(V^+)$. Since $\lambda$ is supported on $V^+ \subset S(V^+)$ and $S(V^+)$ is an additive semigroup, then $\mu_{\varphi(n)}$ is supported on $S(V^+)$, for any $\varphi(n) \in H_s^{(n)}$. We now take $\Psi = \{\psi^{(n)}\}_n \in \mathfrak{F}_s$ and denote by $\mu_{\Psi}$ the spectral measure so that

$$
\langle \Psi, \Gamma(U(a, I)) \Psi \rangle = \int \chi_p(\mathfrak{B}(a, k)) \, d\mu_{\Psi}(k),
$$

then $\mu_{\Psi} = \sum_{n=0}^{\infty} \mu_{\varphi(n)}$ since $\Gamma(U(a, I)) : H_s^{(n)} \rightarrow H_s^{(n)}$.

Existence and uniqueness of the vacuum. The argument in the $p$-adic case is the same as the Archimedean one, see [44, p. 213].

Invariant domains for fields. By Proposition 1, we have

$$
H_\infty(C) \rightarrow L^2_\infty(V^+, d\lambda) \rightarrow F_0 \rightarrow \mathfrak{F}_s(L^2_\infty(V^+, d\lambda)),
$$

(45)

where all the arrows denote continuous mappings. By using sequence (55), $D_C(V^+) \subset D_C(Q^p_0)$, and since $D_C(Q^p_0) \subset H_\infty(C)$, $\mathcal{F}(D_C) = D_C$, and $D_C(V^+)$ is dense in $L^2_\infty(V^+, d\lambda)$, we conclude that $R(H_\infty(C))$ is dense in $L^2_\infty(V^+, d\lambda)$, and hence $\otimes_{n=0}^{\infty} S_n(\otimes_n R(H_\infty(C)))$ in $\mathfrak{F}_s(L^2_\infty(V^+, d\lambda))$.

If $f$ is real-valued, we use that $\Phi_S(f)$ is essentially self-adjoint on $F_0$, the fact that $\Phi_S(f) : F_0 \rightarrow F_0$, and sequence (45), jointly with the density of $R(H_\infty(C))$ to obtain that $\Phi(f) \rvert_{F_0}$ is essentially self-adjoint, and $\Phi(f) : F_0 \rightarrow F_0$. If $f$ is complex-valued, the results follow from the previous discussion by using the definition of $\Phi(f)$.

Regularity of the field. Suppose that $\psi_1, \psi_2 \in F_0$ and that $f_n \rightarrow f \in H_\infty(C)$ (i.e. $f_n \rightarrow f$ for any $l \in \mathbb{N}$), with $f_n$ real-valued. Then (40) implies that

$$
\hat{f}_n \rvert_{V^+} \rightarrow \hat{f} \rvert_{V^+},
$$

i.e. $R(f_n) \rightarrow R(f)$ in $\mathfrak{F}_s$, see sequence (55). Now by using Segal’s quantization, cf. Theorem X.41-(d) in [23], we have $\Phi(f_n) \psi \rightarrow \Phi(f) \psi$ for all $\psi \in F_0$, therefore

$$
\langle \psi_1, \Phi(f_n) \psi_2 \rangle \rightarrow \langle \psi_1, \Phi(f) \psi_2 \rangle.
$$

By treating the real and imaginary parts of $f$ separately, we obtain that $\langle \psi_1, \Phi(f) \psi_2 \rangle$ is a complex-valued bilinear form in $F_0 \times F_0$, and that

$$
|\langle \psi_1, \Phi(f) \psi_2 \rangle| \leq ||\psi_1|| ||\Phi(f) \psi_2||.
$$

(46)

We now estimate $||\Phi(f) \psi_2||$. By the definition of $\Phi(f)$, it is sufficient to consider that $f$ is real-valued. By taking $\psi_2 = \{\psi^{(n)}_2\}_{n \in \mathbb{N}^+}$, $x_i \in Q^p_0$ for $i \in \{1, \ldots, n\}$, $y \in V^+$, and using that

$$
(\Phi(f) \psi_2)_{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int_{V^+} \frac{\hat{f}(y) \psi^{(n+1)}_2(y, x_1, \ldots, x_n)}{\sqrt{2\pi}} \, d\lambda(y)
$$

$$
+ \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} f(x_i) \psi^{(n-1)}_2(x_1, \ldots, \hat{x}_i, \ldots, x_n),
$$

where $\hat{x}_i$ denotes the $i$-th coordinate of $\hat{x}$ (i.e., $\hat{x}_i = \hat{x}_{(i)}$), and $\hat{x}_{(i)}$ denotes the $i$-th coordinate of $\hat{x}$ (i.e., $\hat{x}_{(i)} = \hat{x}_{(i)}$), we conclude that

$$
(\Phi(f) \psi_2)_{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int_{V^+} \frac{\hat{f}(y) \psi^{(n+1)}_2(y, x_1, \ldots, x_n)}{\sqrt{2\pi}} \, d\lambda(y)
$$

$$
+ \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} f(x_i) \psi^{(n-1)}_2(x_1, \ldots, \hat{x}_i, \ldots, x_n),
$$

(47)

where $\hat{x}_i$ denotes the $i$-th coordinate of $\hat{x}$ (i.e., $\hat{x}_i = \hat{x}_{(i)}$), and $\hat{x}_{(i)}$ denotes the $i$-th coordinate of $\hat{x}$ (i.e., $\hat{x}_{(i)} = \hat{x}_{(i)}$), we conclude that

$$
(\Phi(f) \psi_2)_{(n)}(x_1, \ldots, x_n) = \sqrt{n + 1} \int_{V^+} \frac{\hat{f}(y) \psi^{(n+1)}_2(y, x_1, \ldots, x_n)}{\sqrt{2\pi}} \, d\lambda(y)
$$

$$
+ \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{n} f(x_i) \psi^{(n-1)}_2(x_1, \ldots, \hat{x}_i, \ldots, x_n),
$$

(47)
where \( \tilde{x}_i \) means that \( x_i \) is omitted, we have

\[
\frac{(n + 1)}{2} \int \left| \int_{V^+} f(y) \psi_2^{(n+1)} (y, x_1, \ldots, x_n) \, d\lambda \right|^2 \prod_{j=1}^n d^4 x_j +
\]

\[
\frac{1}{2n} \int \left| \sum_{i=1}^n \tilde{f}(x_i) \psi_2^{(n-1)} (x_1, \ldots, \tilde{x}_i, \ldots, x_n) \right|^2 \prod_{j=1}^n d^4 x_j =: I_0 + I_1.
\]

To estimate \( I_0 \), we use the Cauchy-Schwartz inequality, estimation \((30)\), and Remark \((20)\) (iii) to get:

\[
I_0 \leq \frac{(n + 1)}{2} \left\{ \int_{V^+} \left| \int f(y) \, d\lambda \right|^2 \prod_{j=1}^n d^4 x_j \right\} \times
\]

\[
\left\{ \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \psi_2^{(n+1)} (y, x_1, \ldots, x_n) \right|^2 d\lambda \prod_{j=1}^n d^4 x_j \right\},
\]

\[
\leq C_1(n) \| f \|^2 \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \left| \psi_2^{(n+1)} (y, x_1, \ldots, x_n) \right|^2 d\lambda \prod_{j=1}^n d^4 x_j,
\]

\[
\leq C_1(n) \| f \|^2 \left\| \psi_2^{(n+1)} \right\|^2_{\mathcal{H}_{n+1}^s},
\]

for any \( l \in \mathbb{N} \). For \( I_1 \), we have

\[
I_1 \leq \frac{1}{2n} \left( n \| f \|_0 \left\| \psi_2^{(n-1)} \right\|^2_{\mathcal{H}_{n-1}^s} \right) = n \| f \|_0 \left\| \psi_2^{(n-1)} \right\|^2_{\mathcal{H}_{n-1}^s},
\]

Consequently,

\[
\| \Phi (f) \psi_2 \| \leq \sqrt{2} \| f \|_l \| \psi_2 \| \text{ for any } l \in \mathbb{N},
\]

which implies that

\[
f \to \langle \psi_1, \Phi (f) \psi_2 \rangle \text{ is an element of } \mathcal{H}_\infty^s (\mathbb{C}),
\]

see \((8)\).

**Poincaré invariance of the field.** The proof is identical to that of Theorem X.42 in \[(44)\].

**Cyclicity of the vacuum.** The cyclicity of the vacuum for \( \Phi (\cdot) \) follows from Theorem X. 41 (parts (b) and (d)) in \[(44)\], by using the fact that the mapping

\[
R : \mathcal{D}_C (\mathbb{Q}_p^4) \to L^2_\mathbb{C} (V^+, d\lambda)
\]

\[
f \to \tilde{f} |_{V^+} \quad (47)
\]

has a dense range. Indeed, by using that \( \mathcal{D}_C (V^+) \) is dense in \( L^2_\mathbb{C} (V^+, d\lambda) \), see Remark \[(19)\] and the sequence \[(38)\], we conclude that \( \mathcal{D}_C (\mathbb{Q}_p^4) \) is dense in \( L^2_\mathbb{C} (V^+, d\lambda) \). Finally, \[(47)\] follows from the fact that \( F(\mathcal{D}_C (\mathbb{Q}_p^4)) = \mathcal{D}_C (\mathbb{Q}_p^4) \).
Local commutativity. Segal’s quantization can be performed on the field $\Phi(f)$, $f \in \mathcal{H}_\infty(\mathbb{C})$, see [44, Theorem X.41]. Local commutativity in this context means that
\[
[\Phi(f), \Phi(g)] \psi = \Phi(f) \Phi(g) \psi - \Phi(g) \Phi(f) \psi = 0,
\]
for any $f, g \in \mathcal{H}_\infty(\mathbb{C})$ with support on an appropriate domain, and for all $\psi \in F_0$. Without loss of generality we may suppose that $f$ and $g$ in [48] are real-valued since $\Phi$ is linear. Since the range of $R : \mathcal{D}_\mathbb{C} \to L^2_\mathbb{C}(V^+, d\lambda)$ is dense in $L^2_\mathbb{C}(V^+, d\lambda)$, we may assume that $f, g$ belong to $\mathcal{D}_\mathbb{C}$, cf. [44, Theorem X.41-(d)]. By using the Segal quantization, cf. [44, Theorem X.41-(c)], we have
\[
[\Phi(f), \Phi(g)] \psi = \sqrt{-1} \text{Im} (Rf, Rg)_{L^2_\mathbb{C}(V^+, d\lambda)} \psi
= \frac{1}{2} \left\{ \int_{V^+} \left\{ \bar{\hat{f}}(k)\hat{g}(k) - \hat{f}(k)\bar{\hat{g}}(k) \right\} d\lambda(k) \right\} \psi.
\]
Now, we define
\[
\Delta(x) = \int_{V^+} \{ \chi_p(-B(x,k)) - \chi_p(B(x,k)) \} d\lambda(k),
\]
which is a well-defined function in $\mathbb{Q}_p^4$ because $V^+$ is open and compact. Then
\[
[\Phi(f), \Phi(g)] \psi = \frac{1}{2} \left\{ \int_{\mathbb{Q}_p^2} \int_{\mathbb{Q}_p^2} \Delta(x-y) f(x)g(y) d^2x d^2y \right\} \psi.
\]
Therefore, the study of the local commutativity in the $p$–adic quantum field theory of a scalar field becomes the study of the vanishing of $\Delta(x)$ as a distribution on $\mathcal{D}_\mathbb{C}(\mathbb{Q}_p^4) \times \mathcal{D}_\mathbb{C}(\mathbb{Q}_p^4)$. It is then enough to observe that $\Delta(x) \equiv 0$ if $x \in \mathbb{Z}_p^4$, because $\chi_p|_{\mathbb{Z}_p^4} \equiv 1$.

Finally, to prove the second part [41] notice that, since $\Box_{q,a} : \mathcal{H}_\infty(\mathbb{C}) \to \mathcal{H}_\infty(\mathbb{C})$, see Lemma [41], $\Phi(\Box_{q,a} f)$, $f \in \mathcal{H}_\infty(\mathbb{C})$, is well-defined, and since $\mathcal{H}_\infty(\mathbb{C}) \subset L^2_\mathbb{C}(\mathbb{Q}_p^4, d^4k)$, we have $\mathcal{F}(\Box_{q,a} f) = [q - 1]_p^{\frac{3}{4}} \mathcal{F}(f)$, so $R(\Box_{q,a} f) = 0$, and consequently $\Phi(\Box_{q,a} f) = 0$, for all $f \in \mathcal{H}_\infty(\mathbb{C})$.

5.3. Conjugated fields. We take $\mathcal{H} = L^2_\mathbb{C}(V^+, d\lambda)$ as before. Recall that $(k_0, k) \in V^+$ if and only if $(k_0, -k) \in V^+$. By using this fact, we define
\[
C : \mathcal{H} \to \mathcal{H}
\]
\[
f(k_0, k) \to \overline{f(k_0, -k)}.
\]
Then $C$ induces a conjugation on $\mathcal{H}$, i.e. $C$ gives an antilinear isometry satisfying $C^2 = I$. We set $\mathcal{H}_C := \{ f \in \mathcal{H} : Cf = f \}$.

We recall that $\omega(k) : U_q \to \mathbb{Q}_p$ is a non-vanishing analytic function. We define
\[
\mu(k) = \begin{cases} \sqrt{\omega(k)} & \text{if } k \in U_q, \\ 0 & \text{if } k \in \mathbb{Q}_p^3 \setminus U_q. \end{cases}
\]
Then $\mu (k) \in \mathcal{D}'_{R}(\mathbb{R}^3_p)$.

We now define the canonical fields corresponding to $C$ as follows:

$$\varphi (f) = \frac{1}{\sqrt{\pi}} \left\{ (a^- (Rf))^* + a^- (C Rf) \right\}, \text{ for } f \in \mathcal{H}_\infty (\mathbb{C}),$$

$$\pi (f) = \frac{\sqrt{-1}}{\sqrt{2}} \left\{ (a^- (\mu Rf))^* - a^- (C \mu Rf) \right\}, \text{ for } f \in \mathcal{H}_\infty (\mathbb{C}).$$

We call $f \to \varphi (f)$ the canonical free field over $\mathcal{H}_C$ of mass 1, and $f \to \pi (f)$ the canonical conjugate momentum over $\mathcal{H}_C$ of mass 1. These maps are complex linear and $\varphi (f)$, $\pi (f)$ are self-adjoint if and only if $Rf \in \mathcal{H}_C$.

The distribution $\delta (x_0 - t_0) g (x)$ is defined as the direct product of the distributions $\delta (x_0 - t_0)$ and $g (x)$:

$$\delta (x_0 - t_0) \times g (x) : \mathcal{D}_C (\mathbb{Q}_p^3) \times \mathcal{D}_C (\mathbb{Q}_p^3) \to \mathbb{C} \sum_i \phi_i (x_0) \theta_i (x) = \sum_i \phi_i (t_0) \int_{\mathbb{Q}_p^3} g (x) \theta_i (x) d^3x,$$

see e.g. [60]. If $g \in L^2 (\mathbb{Q}_p^3, d^3x)$, then the Fourier transform of the distribution $\delta (x_0 - t_0) g (x)$ is $\chi_3 (k_0, t_0, \hat{g} (k))$, where $\hat{g} (k) \in L^2 (\mathbb{Q}_p^3, d^3k)$ is the 3-dimensional Fourier transform with respect to the bilinear form $-\mathfrak{K}_0 (x, k)$. By using Lemma [11] we can extend the projection $R$ to the distributions of the form $\delta (x_0 - t_0) g (x)$, $g \in L^2 (\mathbb{Q}_p^3, d^3x)$, and thus we extend the class of functions on which $\varphi (\cdot)$ and $\pi (\cdot)$ are defined to include these distributions.

In the case $t_0 = 0$, with $g$ real-valued, we have

$$\left( C R \delta g \right) (k_0, k) = \overline{R \hat{g} (k_0, -k)} = \overline{R \hat{g} (k_0, -k)} = \hat{g} (-k) = \hat{g} (k) = R \left( \delta g \right).$$

Consequently, $R (\delta g)$ and $\mu R (\delta g)$ are in $\mathcal{H}_C$, and $\varphi (\delta g)$, $\pi (\delta g)$ are self-adjoint if $g \in L^2 (\mathbb{Q}_p^3, d^3x)$ is real. We call the maps $g \to \varphi (\delta g)$ and $g \to \pi (\delta g)$ the time-zero fields.

From now on, we will only use ‘test functions’ of the form $\delta g$ with $g \in L^2 (\mathbb{Q}_p^3, d^3x)$ in $\varphi (\cdot)$ and $\pi (\cdot)$, and write $\varphi (g)$ and $\pi (g)$ instead of $\varphi (\delta g)$ and $\pi (\delta g)$. If $f$ and $g$ are functions from $L^2 (\mathbb{Q}_p^3, d^3x)$, by using Theorem X.43-(c), we have

$$[\varphi (f), \pi (g)] \psi = \sqrt{-1} \int_{V^+} \overline{f(k) \hat{g}(k) \mu(k) d\lambda(k)} \psi, \text{ for all } \psi \in F_0. \quad (51)$$

5.4. Transferring fields from $\mathfrak{F}_s (L^2_v (V^+, d\lambda))$ to $\mathfrak{F}_s (L^2_u (U_q, d^3k))$. We use the notation

$$a^\dagger (f) = (a^- (f))^*, \quad a (f) = (a^- (C f))^*.$$
As we already mentioned, each function \( f(k) = f\left(\sqrt{\omega(k)}, k\right) \in L^2_\mathbb{C}(V^+, d\lambda) \) is a function on \( U_q \). We take

\[
(Jf)(k_0, k) = \frac{f\left(\sqrt{\omega(k)}, k\right)}{\sqrt{\omega(k)_{p}}}
\]
as before. Then \( J \) is a unitary isometry of \( L^2_\mathbb{C}(V^+, d\lambda) \) onto \( L^2_\mathbb{C}(U_q, d^3k) \). The annihilation and creation operators on \( \mathfrak{g}_s (L^2_\mathbb{C}(U_q, d^3k)) \), \( \tilde{a} (\cdot) \) and \( \tilde{a}^\dagger (\cdot) \) are related to \( a (\cdot) \) and \( a^\dagger (\cdot) \) by the formulas:

\[
\tilde{a} (Jf) = \Gamma (J) a (f) \Gamma (J)^{-1}, \\
\tilde{a}^\dagger (Jf) = \Gamma (J) a^\dagger (f) \Gamma (J)^{-1}.
\]

By using the unitary map \( \Gamma (J) \), we carry the quantum fields over \( \mathfrak{g}_s (L^2_\mathbb{C}(U_q, d^3k)) \) as follows:

\[
\tilde{\Phi} (f) = \Gamma (J) \Phi (f) \Gamma (J)^{-1} = \frac{1}{\sqrt{2}} \left\{ \tilde{a} \left( \tilde{C} \frac{Rf}{\sqrt{\omega(k)_{p}}} \right) + \tilde{a}^\dagger \left( \frac{Rf}{\sqrt{\omega(k)_{p}}} \right) \right\}
\]

for \( f \in \mathcal{H}_\infty (\mathbb{R}) \), and

\[
\tilde{\varphi} (f) = \Gamma (J) \varphi (f) \Gamma (J)^{-1} = \frac{1}{\sqrt{2}} \left\{ \tilde{a} \left( \frac{R(f\delta)}{\sqrt{\omega(k)_{p}}} \right) + \tilde{a}^\dagger \left( \frac{R(f\delta)}{\sqrt{\omega(k)_{p}}} \right) \right\}
\]

for \( f \in L^2_\mathbb{C}(\mathbb{Q}^3_p, d^3x) \), where \( \tilde{C} = \Gamma (J) C \Gamma (J)^{-1} \) acts by \( \left( \tilde{C}g \right)(k) = g(-k) \).

We drop the tilde \( \tilde{\cdot} \), and from now on, we work with fields on \( \mathfrak{g}_s (L^2_\mathbb{C}(U_q, d^3k)) \), for \( f, g \) real-valued. Then, formula (51) becomes

\[
[\varphi (f), \pi (f)] = \sqrt{-1} \int_{U_q} f(x)g(x)d^3x,
\]

which is the canonical commutation relation in \( L^2_\mathbb{C}(U_q, d^3x) \).

### 5.5. Some classical calculations

In this section, we discuss in a \( p \)-adic frame the annihilation and creation operators introduced above, to show that they conform to the common usage in the Physics literature. We start by defining

\[
D_0 = \left\{ \psi; \psi \in F_0, \psi^{(n)} \in \mathcal{D}_C(U_q^{(n)}) \text{ for all } n \right\}
\]

and for each \( l \in \mathbb{Q}_p^3 \) (we do not use bold letters for 3-dimensional vectors) an operator \( a (l) \) on \( \mathfrak{g}_s (L^2_\mathbb{C}(U_q, d^3x)) = \oplus_{n=0}^{\infty} \mathcal{H}_s^{(n)} \) with domain \( D_0 \) by

\[
(a (l) \psi)^{(n)} (k_1, \ldots, k_n) = \sqrt{n + 1} \psi^{(n+1)} (l, k_1, \ldots, k_n), \quad n \geq 0.
\]
The formal adjoint of \(a(l)\) is given by

\[
(a(l)\,\!^\dagger\!\psi)^{(n)}(k_1,\ldots,k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(l - k_j) \psi^{(n-1)}(k_1,\ldots,\tilde{k}_j,\ldots,k_n),
\]

for \(n \geq 1\), and by definition \((a(l)\,\!^\dagger\!\psi)^{(n)}(k_1,\ldots,k_n) = 0\) for \(n = 0\). This operator is a well-defined quadratic form on \(D_0 \times D_0\); if \(\psi_2 = \{\psi_2^{(n)}\}_{n \in \mathbb{N}}\), \(\psi_1 = \{\psi_1^{(n)}\}_{n \in \mathbb{N}} \in E_0\), then the quadratic form

\[
\langle \psi_2, a(l)\,\!^\dagger\!\psi_1 \rangle = \sum_{n=1}^{\infty} \left( \psi_2^{(n)}, a(l)\,\!^\dagger\!\psi_1 \right)^{(n)}_{H_1^{(n)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{U_{q^{-1}}} \psi_2^{(n)}(k_1,\ldots,k_{j-1},l,k_{j+1},\ldots,k_n) \times \psi_1^{(n-1)}(k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_n) \prod_{i=1}^{n} d^3k_i
\]

is well-defined. The formulas

\[
a(g) = \int_{U_q} a(k) g(-k) \ d^3k \quad \text{and} \quad a^\dagger(g) = \int_{U_q} a^\dagger(k) g(k) \ d^3k, \quad (52)
\]

hold for all \(g(k) \in \mathcal{D}_c(U_q)\), if the equalities are understood in the sense of quadratic forms, i.e.

\[
\langle \psi_2, a(g) \psi_1 \rangle := \int_{U_q} \langle \psi_2, a(k) \psi_1 \rangle g(-k) \ d^3k
\]

and

\[
\langle \psi_2, a(g) \psi_1 \rangle := \int_{U_q} \langle \psi_2, a^\dagger(k) \psi_1 \rangle g(k) \ d^3k.
\]

On the other hand, since \(a(l) : D_0 \to D_0\), the powers of \(a(l)\) are well-defined on \(D_0\). Then

\[
\langle \psi_1, (a(l)\,\!^\dagger\!)^n \psi_2 \rangle = \langle (a(l))^n \psi_1, \psi_2 \rangle,
\]

for each \(n\), where the equality is to be understood in the sense of quadratic forms, and

\[
\langle \psi_1, \left( \prod_{i=N_1+1}^{N_2} a^\dagger(l_i) \right) \left( \prod_{i=1}^{N_1} a(l_i) \right) \psi_2 \rangle
\]
is a well-defined quadratic form on $D_0 \times D_0$. In addition, if $f_i \in \mathcal{D}_c(U_q)$, then the following expressions are well-defined as quadratic forms: The product

$$
\left( \prod_{i=N_1+1}^{N_2} a^\dagger (f_i) \right) \left( \prod_{i=1}^{N_1} a (f_i) \right) = 
\int_{\mathcal{U}_q^{3N_2}} \left( \prod_{i=N_1+1}^{N_2} a^\dagger (k_i) \right) \left( \prod_{i=1}^{N_1} a (-k_i) \right) \left( \prod_{i=1}^{N_2} f_i (k_i) \right) d^3k_1 \cdots d^3k_{N_2},
$$

the number operator

$$
N = \int_{\mathcal{U}_q} a^\dagger (k) a (k) \, d^3k,
$$

and the free Hamiltonian of unit mass,

$$
H_0 = \int_{\mathcal{U}_q} \mu (k) a^\dagger (k) a (k) \, d^3k.
$$

Finally, by using quadratic forms on $D_0$ we can express the free scalar field and the time zero fields in terms of $a^\dagger (k)$ and $a (k)$ (i.e. by using (52) with $g$ real-valued):

$$
\Phi (t, x) = 
\frac{1}{\sqrt{2}} \int_{\mathcal{U}_q} \left\{ \chi_p \left( \sqrt{\omega (k)} t - \mathfrak{B}_0 (k, x) \right) a^\dagger (k) + \chi_p \left( -\sqrt{\omega (k)} t + \mathfrak{B}_0 (k, x) \right) a (k) \right\} \frac{d^3k}{\sqrt{\omega (k)}}.
$$

$$
\varphi (x) = 
\frac{1}{\sqrt{2}} \int_{\mathcal{U}_q} \left\{ \chi_p (-\mathfrak{B}_0 (k, x)) a^\dagger (k) + \chi_p (\mathfrak{B}_0 (k, x)) a (k) \right\} \frac{d^3k}{\sqrt{\omega (k)}}.
$$

$$
\pi (x) = \sqrt{-1} \frac{1}{\sqrt{2}} \int_{\mathcal{U}_q} \left\{ \chi_p (-\mathfrak{B}_0 (k, x)) a^\dagger (k) - \chi_p (\mathfrak{B}_0 (k, x)) a (k) \right\} \frac{d^3k}{\sqrt{\omega (k)}}.
$$

5.6. A $p$-adic Klein-Gordon equation. In this section, we consider the inhomogeneous $p$–adic Klein-Gordon equation:

$$
\Box_{q,\alpha} u (t, x) = h (t, x), \quad (53)
$$

where $(t, x) \in \mathbb{Q}_p \times \mathbb{Q}^3_p$ and $h (t, x) \in \mathcal{D}_c(\mathbb{Q}_p \times \mathbb{Q}^3_p)$. We use the techniques and results of [67] Chapter 6. By a solution (or weak solution) we understand a distribution from $\mathcal{D}'_{q_0}(\mathbb{Q}_p \times \mathbb{Q}^3_p)$ satisfying (53). We denote by $E_{q_0}^q (t, x)$, the fundamental solution of (53) obtained in Theorem [67].
Theorem 3. The following hold true:

(i) The equation
\[ \Box_{q,\alpha} u(t, x) = 0 \]  

admits plane waves, this means that if \((E^\pm, \kappa) \in V^\pm\), that is, they form a fixed pair of solutions to \(E^\pm = \pm \sqrt{\omega(\kappa)}\), then \(\chi_p \{ -B((t, x), (E^\pm, \kappa)) \} \) is a weak solution of (54).

(ii) The distributions
\[ \int_{U_q} \chi_p \left\{ -B\left( (t, x), \left( \sqrt{\omega(k)}, k \right) \right) \right\} \frac{d^3k}{|\sqrt{\omega(k)}|_p} + \]
\[ \int_{U_q} \chi_p \left\{ B\left( (t, x), \left( -\sqrt{\omega(k)}, k \right) \right) \right\} \frac{d^3k}{|\sqrt{\omega(k)}|_p} \]

are the unique weak solutions of (54) (up to the multiplication by a non-zero complex constant) which are invariant under \(L^\uparrow_+\).

(iii) The distributions
\[ u(t, x; A, B, C) = E^0_q(t, x) \ast h(t, x) + \]
\[ C \int_{U_q} \chi_p \left\{ -\sqrt{\omega(k)} t + \mathfrak{B}_0(k, x) \right\} A(k) + \chi_p \left( \sqrt{\omega(k)} t + \mathfrak{B}_0(k, x) \right) B(k) \]
\[ \times \frac{d^3k}{|\sqrt{\omega(k)}|_p}, \]

where \(C\) is a non-zero complex number, and \(A(k), B(k) \in \mathcal{D}_C(Q_3^3)\), are weak solutions of (54).

Proof.

(i) Since \(F_{k \to \frac{t}{\sqrt{\omega(k)}}}^{-1}(\delta (k_0 - E^\pm, k - \kappa)) = \chi_p \{ -B((E^\pm, \kappa), (t, x)) \}\), the condition \(E^\pm = \pm \sqrt{\omega(\kappa)}\) implies that \(k_0^\pm = \pm \sqrt{\omega(k)}\), so \(\delta (k_0 - E^\pm, k - \kappa)\) is supported on \(V^\pm \subset V\). The result follows from the fact that the weak solutions of (54) are exactly the distributions from \(\mathcal{D}_C(Q_3^3)\) whose Fourier transform is supported on \(V\), see [67, Lemma 169].

(ii) The distributions of the form \(C\delta_V\), for \(C \in \mathbb{C}\), are the unique solutions of (54) which are invariant under \(O(q)\), see [67, Lemma 169] and [11, Proposition 2-2]. By writing \(C\delta_V = C\delta_{V^+} + C\delta_{V^-}\) in \(\mathcal{D}_C(Q_3^3 \times Q_3^3)\) and using the fact that \(\delta_{V^\pm}\) are invariant under \(L^\uparrow_+\) = \{\(A \in O(q): A(V^\pm) = V^\pm\)\}, see [67, Lemma 163], we conclude that \(C\delta_{V^+} + C\delta_{V^-}\) are the unique weak solutions of (54) which are invariant under \(L^\uparrow_+\). The announced formula follows by computing the inverse Fourier transform of \(\delta_{V^\pm}\).

(iii) The result follows from the second part by using Theorem 1.
Remark 24. Notice that \[\sqrt{\omega(k)}_p A(k), \sqrt{\omega(k)}_p B(k)\] are test functions, and also
\[
\int_{U^4} \chi_p \left( \sqrt{\omega(k)}t + 2B_0(k, x) \right) B(k) \frac{d^3k}{\sqrt{\omega(k)}_p} = \int_{U^4} \chi_p \left( \sqrt{\omega(k)}t - 2B_0(k, x) \right) B(-k) \frac{d^3k}{\sqrt{\omega(k)}_p},
\]
so the unique weak solution of \(\Box_{q,\alpha} u = 0\) (with \(C = 1/\sqrt{2}\)) invariant under \(L^p\) corresponds to the free scalar field \(\Phi(t, x)\), with \(a(k) = \sqrt{\omega(k)}_p A(k), a^\dagger(k) = \sqrt{\omega(k)}_p B(k)\). As we have seen, these solutions can be quantized using the machinery of the second quantization in such a way that Wightman axioms are satisfied.

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