Noncommutative Topological Half-flat Gravity*

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Abstract

We formulate a noncommutative description of topological half-flat gravity in four dimensions. BRST symmetry of this topological gravity is deformed through a twisting of the usual BRST quantization of noncommutative gauge theories. Finally it is argued that resulting moduli space of instantons is characterized by the solutions of a noncommutative version of the Plebański's heavenly equation.

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1. INTRODUCTION

In recent years, renewed efforts have been performed in the formulation of a noncommutative theory of gravitation, motivated in part by the understanding of the short distance behavior of the spacetime [1]. Some proposals based on the recent developments are given in [2, 3, 4, 5, 6]. In particular, in [4, 5] a Seiberg-Witten map for the tetrad and the Lorentz connection is given, where these fields were taken as components of a SO(4,1) connection in the first work, and of a U(2,2) connection in the second one. In these works a MacDowell-Mansouri (MM) type action was considered, invariant under the subgroup U(1,1)×U(1,1), and the excess of degrees of freedom, additional to the ones of the commutative theory, is handled by means of constraints. In [6], from the chosen constraints, a consistent noncommutative SO(3,1) extension was proposed. In Ref. [7] it has been shown that noncommutative gauge theories, based on the Seiberg-Witten map, for any commutative theory invariant under a gauge group $G$, can be constructed. The resulting noncommutative theory can be regarded as an effective theory, invariant under the noncommutative enveloping algebra transformations of $G$, which are induced by the commutative transformations of $G$.

Following these developments, starting from a SL(2,$\mathbb{C}$) self-dual connection, we have given a formulation for quadratic noncommutative topological gravitation, which contains the SO(3,1) topological invariants, namely the signature and Euler characteristic [8]. In fact, the noncommutative signature can be straightforwardly obtained, but the Euler invariant cannot, as it involves the same difficulty as the MM action, which contains a contraction with the Levi-Civita tensor, instead of the SO(3,1) trace. However, both invariants can be combined into an expression given by the signature with a SO(3,1) self-dual connection, which amounts to the SL(2,$\mathbb{C}$) signature.

Self-dual gravity (for a review, see [9]), from which the hamiltonian Ashtekar’s formulation [10] can be obtained, has very useful properties which have allowed the exploration of quantum gravity in the framework of loop quantum gravity. In Ref. [11] we have considered self-dual gravity in the Plebański formulation [12], in order to make a proposal for a noncommutative theory of gravity, which is fully invariant under the noncommutative gauge transformations. Thus, Plebański formulation is written as a SL(2,$\mathbb{C}$) topological BF formulation (helicity formalism [13]), given by the trace of the two-form $B$, times the field strength [14]. The contact with Einstein gravitation is done through the implementation
of constraints on the $B$-field, which are solved by the square of the tetrad one-form. This theory can be restated in terms of self-dual SO(3,1) fields, the spin connection and the $B$-field. After the identification of the $B$ two-form with the tetrad one-form squared, a variation of this action with respect to the spin connection gives the vanishing of the torsion. The resulting action contains Einstein gravitation plus an imaginary term, which is identically zero due to the Bianchi identities. The noncommutative version is obtained at the level of the SL(2,$\mathbb{C}$) theory, by the application of the Moyal product.

On the other hand, topological gravity in four dimensions has been a very useful toy model, which encodes some of the properties of quantum gravity. Among these theories, topological half-flat gravity represents a cohomology theory of the moduli space $\mathcal{M}$ of the so called half-flat metrics, which mean metrics with the self-dual part of Riemann tensor vanishing. In this theory, the variables for the gravitational field consist of a self-dual two-form $\cal{S}$ and the self-dual spin connection $\omega^+$. The solutions of the equations of motion (called in [12] as $\cal{S}_H$-structure) for these variables determine the classical moduli space of self-dual metrics $\mathcal{M}$. This structure was eventually used in [15], to propose a topological gravity theory describing the intersection theory on the moduli space $\mathcal{M}$. In the case of simply connected spacetimes, the self-dual spin connection can be gauged away and fixed to zero. The remaining equations ($\cal{S}_{H}^\prime$-structure) determine then a moduli space $\mathcal{M}_H$ generated by all solutions of the heavenly equation [16].

In the present paper we propose the noncommutative deformation of the topological half-flat gravity proposed by Kunitomo in Ref. [15]. In section 2 we give a brief overview of topological gravity. Section 3 is devoted to review some aspects of Plebański action and $\cal{S}_H$-structures in self-dual gravity [12, 13]. In section 4 we formulate the noncommutative deformation of the topological half-flat gravity. Finally, in section 5 we give our final comments.

2. TOPOLOGICAL GRAVITY IN FOUR DIMENSIONS

Originally the cohomological field theories were proposed as an interpretation of Donaldson theory, describing the topology of four-dimensional manifolds, in terms of a suitable quantum field theory [17]. Topological gravity in four dimensions was a further proposal by Witten in [18] to construct a gravitational analog for Donaldson theory whose basic variables
were the tetrad and the spin connection. Then some other extra fields were introduced in order to have a theory with a fermionic BRST-like symmetry. The action is then written as a BRST commutator and consequently it is a BRST invariant theory by construction. In Refs. \[19, 20\], Witten’s Lagrangian for topological gravity was obtained from the suitable chosen Lagrangian \( \int_X d^4x \sqrt{|g|} C_{abcd} \tilde{C}^{abcd} \), with \( C_{abcd} \) the Weyl tensor and \( \tilde{C}^{abcd} \) its Hodge dual, plus the gauge-fixing and ghost Lagrangians. Witten’s Lagrangian is then rederived through a genuine BRST procedure. The BRST gauge-fixing procedure implies the introduction of new fields involving the gauge-fixing of diffeomorphism, Weyl and Lorentz symmetries.

The moduli problem of topological gravity considered in \[18\], was the moduli space of tetrads and spin connections satisfying the torsion free condition and the self-duality condition of the Weyl tensor \( \tilde{C}^{abcd} = +C_{abcd} \). Further developments of this proposal were considered in \[21, 22\]. Of particular interest, among other moduli problems, is the topological gravity theory based in the moduli of gravitational instantons \[23\], where the moduli space is now based on the self-duality of the Riemann tensor instead of the Weyl tensor. In these papers, the starting classical action which is BRST gauge-fixed is a linear combination of the Euler and the signature topological invariants of the spacetime manifold \( X \),

\[
S = \int_X d^4x \sqrt{|g|} \left( A \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} + B \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} \right),
\]

where \( A \) and \( B \) are arbitrary constants and \( R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} - i[\omega_\mu, \omega_\nu]^{ab} \).

In all these mentioned moduli problems involving topological gravity in four dimensions in the Palatini formalism whose basic set of variables include the tetrad (or metric) as well as the spin connection. As mentioned in the introduction, it is very difficult to construct a sensible noncommutative extension of gravity theories involving these field variables, and which at the same time preserves all their symmetries. Fortunately, there is a formulation of topological gravity theory based not on the tetrad and in the spin connection, but in this case, the tetrad is changed by the self-dual two-form \( B \) mentioned in the introduction. This theory has been proposed in Ref. \[15\], where it was called topological half-flat gravity. In the present paper we will construct the noncommutative extension of the BRST symmetry of this formulation, following the procedure of Ref. \[24\].
3. SELF-DUAL VARIABLES

One of the main features of the tetrad formalism of the theory of gravitation, is that it introduces local Lorentz SO(3,1) transformations. In this case, the generalized Hilbert-Palatini formulation is written as \( \int e^\mu_a e^\nu_b R_{\mu\nu}^{ab}(\omega) d^4x \), where \( e^\mu_a \) is the inverse tetrad, and \( R_{\mu\nu}^{ab}(\omega) \) is the so(3,1) valued field strength. The decomposition of the Lorentz group as \( \text{SO}(3,1) = \text{SL}(2,\mathbb{C}) \otimes \text{SL}(2,\mathbb{C}) \), and the geometrical structure of four dimensional space-time, makes it possible to formulate gravitation as a complex theory, as in [12, 13]. These formulations take advantage of the properties of the fundamental or spinorial representation of \( \text{SL}(2,\mathbb{C}) \), which allows a simple separation of the action on the fields of both factors of \( \text{SO}(3,1) \), as shown in great detail in [12, 13]. All the Lorentz Lie algebra valued quantities, in particular the connection and the field strength, decompose into the self-dual and anti-self-dual parts, in the same way as the Lie algebra \( \text{so}(3,1) = \text{sl}(2,\mathbb{C}) \oplus \text{sl}(2,\mathbb{C}) \). However, Lorentz vectors, like the tetrad, transform under mixed transformations of both factors and so this formulation cannot be written as a chiral \( \text{SL}(2,\mathbb{C}) \) theory. Various proposals in this direction have been made (for a review, see [9]). In an early formulation, this problem has been solved by Plebański [12], where by means of a constrained Lie algebra valued two-form \( S \), the theory can be formulated as a chiral \( \text{SL}(2,\mathbb{C}) \) invariant BF-theory, \( \text{Tr} \int S \wedge R(\omega) \). In this formulation \( S \) has two \( \text{SL}(2,\mathbb{C}) \) spinorial indices, and it is symmetric on them \( S^{AB} = S^{BA} \), as any such \( \text{sl}(2,\mathbb{C}) \) valued quantity. The constraints are given by \( S^{AB} \wedge S^{CD} = \frac{1}{3} \delta^C_{(A} \delta^D_{B)} S^{EF} \wedge S_{EF} \) and, as shown in [12], their solution implies the existence of a tetrad one-form, which squared gives the two-form \( S \). In the language of \( \text{SO}(3,1) \), this two-form is a second rank antisymmetric self-dual two-form, \( S^{+ab} = \Pi^{+ab}_{cd} S^{cd} \), where \( \Pi^{+ab}_{cd} = \frac{1}{4} \left( \delta^a_c \delta^b_d - i \varepsilon^{ab}_{cd} \right) \). In this case, the constraints can be recast into the equivalent form \( S^{+ab} \wedge S^{+cd} = -\frac{1}{3} \Pi^{+abcd} S^{+ef} \wedge S^{+ef} \), with solution \( S^{ab} = 2 e^a \wedge e^b \).

For the purpose of the noncommutative formulation, we will consider self-dual gravity in a somewhat different way as in the papers [12, 13]. In this section we will fix our notations and conventions.

Let us take the self-dual \( \text{SO}(3,1) \) BF action, defined on a \((3 + 1)\)-dimensional pseudo-riemannian manifold \((X, g_{\mu\nu})\),

\[
I = i \text{Tr} \int_X S^+ \wedge R^+ = i \int_X \varepsilon^{\mu\nu\rho\sigma} S^+_{\mu\nu} R^+_{\rho\sigma \omega}(\omega) d^4x, \tag{2}
\]
where $R_{\rho \sigma ab}^{c d} = \Pi_{\mu \nu}^{c d} R_{\rho \sigma \mu \nu}$ is the self-dual SO(3,1) field strength tensor. This action can be rewritten as

$$I = \frac{1}{2} \int_X \varepsilon^{\mu \nu \rho \sigma} \left( i S_{\mu \nu \rho \sigma} + \frac{1}{2} \varepsilon_{a b c d} S_{a b \rho \sigma} \right) d^4 x. \quad (3)$$

If now we take the solution of the constraints on $S$, which we now write as

$$S_{\mu \nu \rho \sigma} = e^a_{\mu} e^b_{\nu} - e^a_{\nu} e^b_{\mu}, \quad (4)$$

then

$$I = \int_X (\det e R + i \varepsilon^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma}) d^4 x. \quad (5)$$

The real and imaginary parts of this action must be varied independently because the fields are real. The first part represents Einstein action in the Palatini formalism, from which, after variation of the Lorentz connection, a vanishing torsion $T_{\mu \nu}^a = 0$ turns out. As a consequence, the second term vanishes due to Bianchi identities.

The action (2) can be written as

$$I = i \int_X \varepsilon^{\mu \nu \rho \sigma} \left[ S_{\mu \nu}^{+} R_{\rho \sigma 0 i} (\omega^+) + S_{\mu \nu}^{+} R_{\rho \sigma 0 j} (\omega^+) \right] d^4 x, \quad (6)$$

where $R_{\mu \nu}^{0 i} (\omega^+) = \partial_{\mu} \omega_{\nu}^0 - \partial_{\nu} \omega_{\mu}^0 + \omega_{\mu}^{a c} \omega_{\nu}^b - \omega_{\nu}^{a c} \omega_{\mu}^b$. From the decomposition SO(3,1)=SL(2, C)×SL(2, C), it turns out that $\omega_{\mu}^i = \omega_{\mu}^{0 i}$ is a SL(2, C) connection. Further, if we take into account self-duality, $\varepsilon_{c d}^{a b} \omega_{\mu}^{a c + d} = 2i \omega_{\mu}^{a b}$, we get $\omega_{\mu}^{+ ij} = -i \varepsilon_{k j}^{i k} \omega_{\mu}^k$. Therefore,

$$R_{\mu \nu}^{0 i} (\omega^+) = \partial_{\mu} \omega_{\nu}^i - \partial_{\nu} \omega_{\mu}^i + 2 i \varepsilon_{j k}^{i} \omega_{\mu}^j \omega_{\nu}^k = R_{\mu \nu}^i (\omega), \quad (7)$$

$$R_{\mu \nu}^{ij} (\omega^+) = \partial_{\mu} \omega_{\nu}^{+ ij} - \partial_{\nu} \omega_{\mu}^{+ ij} + 2 (\omega_{\mu}^{i j} \omega_{\nu}^{+ j} - \omega_{\nu}^{i j} \omega_{\mu}^{+ j}) = -i \varepsilon_{k j}^{i k} R_{\mu \nu}^j (\omega), \quad (8)$$

where $R_{\mu \nu}^i$ is the SL(2, C) field strength.

Similarly, we define $S_{\mu \nu}^i = S_{\mu \nu}^{+ 0 i}$, which transforms in the SL(2, C) adjoint representation. From it we get, $S_{\mu \nu}^{+ ij} = -i \varepsilon_{k j}^{i k} S_{\mu \nu}^k$. Thus, the action (6) can be written as a SL(2, C) BF-action

$$I = i \int_X \varepsilon^{\mu \nu \rho \sigma} \left[ S_{\mu \nu}^{+ 0 i} R_{\rho \sigma 0 i} (\omega^+) + S_{\mu \nu}^{+ ij} R_{\rho \sigma 0 j} (\omega^+) \right] d^4 x$$

$$= -4i \int_X \varepsilon^{\mu \nu \rho \sigma} S_{\mu \nu}^i R_{\rho \sigma i} (\omega) d^4 x. \quad (9)$$

Therefore, if we choose the algebra $s\ell(2, \mathbb{C})$ to satisfy $[T_i, T_j] = -2 \varepsilon_{i j}^k T_k$ and $Tr(T_i T_j) = -2 \delta_{i j}$, we have that (2) can be rewritten as the self-dual action [12],

$$I = 2i Tr \int_X S \wedge R = 2i \int_X S_i \wedge R^i, \quad (10)$$
which is invariant under the SL(2, \( \mathbb{C} \)) transformations

\[
\delta_{\lambda} \omega_{\mu} = \partial_{\mu} \lambda + i[\lambda, \omega_{\mu}] \quad \text{and} \quad \delta_{\lambda} S_{\mu\nu} = i[\lambda, S_{\mu\nu}].
\]

If the variation of this action with respect to the SL(2, \( \mathbb{C} \)) connection \( \omega \) is set to zero, we get the equations

\[
\varepsilon^{\mu\nu\rho\sigma} D_\nu S_{\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu S_{\rho\sigma} + 2i \varepsilon^{ijk} \omega_j S_{\rho\sigma}^k \right) = 0.
\]

(11)

Taking into account separately both real and imaginary parts, we get, in terms of the SO(3,1) connection,

\[
\varepsilon^{\mu\nu\rho\sigma} D_\nu S_{\rho\sigma}^{ab} = \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu S_{\rho\sigma}^{ab} + \omega_\nu^{ac} S_{\rho\sigma}^{bc} - \omega_\nu^{bc} S_{\rho\sigma}^{ac} \right) = 0,
\]

(12)

which after the identification (4), can be written as

\[
\varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu e^a_\rho e^b_\sigma - \partial_\nu e^b_\rho e^a_\sigma + \omega_\nu^{ac} e^b_\rho e^c_\sigma - \omega_\nu^{bc} e^a_\rho e^c_\sigma \right) = \varepsilon^{\mu\nu\rho\sigma} \left( T^a_{\nu \rho} e^b_\sigma - T^b_{\nu \rho} e^a_\sigma \right) = 0.
\]

(13)

From which the vanishing torsion condition once more turns out.

The underlying self-dual structure encoded in (10), including constraints, can be written in terms of the Lagrangian \( L(S_H) \) given by,

\[
L(S_H) := \int X S_i \wedge R^i + \frac{1}{2} C_{ij} \left( \frac{1}{2} S^i \wedge S^j - v \delta^{ij} \right).
\]

(14)

Similarly for the anti-self-dual structure, a Lagrangian \( L(S_{H'}) \) can be defined \[12, 13\]. Here \( C_{ij} \) is the self-dual Weyl tensor, which is symmetric and traceless. Further, \( v \) is the volume form given by \( v = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \) and \( \delta_{ij} = \text{diag}(1, 1, 1) \).

Equations of motion and constraints can be written in the following way:

\[
\delta \omega^i : \quad D S^i := d S^i + \varepsilon^{ijk} \omega^j \wedge S^k = 0,
\]

(15)

\[
\delta S^i : \quad R^i := d \omega^i + \frac{1}{2} \varepsilon^{ijk} \omega^j \wedge \omega^k = 0,
\]

(16)

\[
\delta C_{ij} : \quad \frac{1}{2} S^i \wedge S^j = \delta^{ij} \cdot v.
\]

(17)

The Eqs. (15), (16) and (17) define a self-dual Einsteinian substructure \( S_H \) of the complete structure \( V \) composed by the combinations of self-dual and anti-self-dual substructures, \textit{i.e.} \( V := S_H \cup S_{H'} \), which are discussed in detail in Refs. \[12, 13\].
4. NONCOMMUTATIVE TOPOLOGICAL HALF-FLAT GRAVITY

Topological half-flat gravity is based on the moduli space \( \mathcal{M} \) given by the solutions to the S\( \mathcal{H} \) structure. In this section we will consider a noncommutative deformation of it. Thus the basic field variables are the noncommutative self-dual spin connection \( \hat{\omega}^i \) and the noncommutative self-dual two-form \( \hat{S}^i \), defined on the noncommutative spacetime \( \hat{A}, (X) \equiv X_\theta \).

The Lagrangian must be invariant under a noncommutative BRST-like transformation, \( \delta Q \hat{O} = -i\{Q; \hat{O}\} \), where \( \hat{O} \) is a functional of the noncommutative fields, \( Q \) is the BRST charge, and \( \{\cdot; \cdot\} \) is the BRST noncommutative commutator.

We will consider the fundamental BRST multiplet \((\hat{\omega}^I, \hat{S}^I, \hat{\psi}^I, \hat{\Psi}^I; \hat{\gamma}^I, \hat{\gamma}^\mu)\), which consists of the bosonic and ghost fields. As usual in noncommutative theories, all fields are promoted to be valued on the enveloping algebra of the adjoint representation \( \text{ad} \) of \( \mathfrak{s}\ell(2, \mathbb{C}) \), that is on \( \mathcal{U}(\mathfrak{s}\ell(2, \mathbb{C}), \text{ad}) \). Indices \( I, J, \) etc. run over all possible generators of \( \mathcal{U}(\mathfrak{s}\ell(2, \mathbb{C}), \text{ad}) \). Here \( \hat{\omega}^I \) is the spin connection one-form, \( \hat{\psi}^I \) is a ghost one-form field associated by the BRST symmetry to \( \hat{\omega}^I \), \( \hat{\Psi}^I \) is a ghost two-form field, and \( \hat{\gamma}^I \) and \( \hat{\gamma}^\mu \) are the ghost fields useful to fix the Lorentz and diffeomorphism symmetries respectively.

The equations of the noncommutative self-dual \( \hat{S}\mathcal{H} \)-structure are given by:

\[
\delta \hat{\omega}^I : \quad D\hat{S}^I := d\hat{S}^I + \frac{1}{2} h^{IJK} \hat{\omega}^J \wedge \hat{S}^K = 0, \quad (18)
\]

\[
\delta \hat{S}^I : \quad \hat{R}^I := d\hat{\omega}^I + h^{IJK} \hat{\omega}^J \wedge \hat{\omega}^K = 0. \quad (19)
\]

Solutions of these equations constitute the moduli space \( \hat{\mathcal{M}} \) of noncommutative `gravitational instantons`.

In local coordinates of \( X_\theta \), under the assumption \( h^{IJK} t^I = t^J \cdot t^K \), the curvature two form is given by

\[
\hat{R}^I_{\mu\nu} = \partial_\mu \hat{\omega}^I_\nu - \partial_\nu \hat{\omega}^I_\mu - i[\hat{\omega}^I_\mu, \hat{\omega}^I_\nu]^I = \partial_\mu \hat{\omega}^I_\nu - \partial_\nu \hat{\omega}^I_\mu - ih^{IJK} \hat{\omega}^J_\mu \wedge \hat{\omega}^K_\nu + ih^{JKI} \hat{\omega}^K_\nu \wedge \hat{\omega}^J_\mu
\]

\[
= \partial_\mu \hat{\omega}^I_\nu - \partial_\nu \hat{\omega}^I_\mu + \frac{1}{2} f^{IJK} \{\hat{\omega}^I_\mu, \hat{\omega}^K_\nu\} - \frac{i}{2} d^{IJK} [\hat{\omega}^I_\mu, \hat{\omega}^K_\nu], \quad (20)
\]

where \( h^{IJK} = \frac{i}{2} f^{IJK} + \frac{1}{2} d^{IJK} \), with \( d^{IJK} = d(i^{IJK}) \). Here \([A; B] \equiv A \ast B - B \ast A\), \([A \ast B] \equiv A \ast B + B \ast A\), and \( \ast \) is given by the external product of the Moyal product and the matrix.
multiplication. Also \( [t^I, t^J] = i f^{IJK} t^K \), \( \{ t^I, t^J \} = d^{IJK} t^K \) and the covariant derivative is given by \( D_\mu \Theta(x) = \partial_\mu \Theta + i [\Theta, \omega_\mu] \), for any \( \Theta \).

Before the full constraints are implemented, actually we do not have still the full BRST symmetry. Thus this preliminary BRST symmetry is simply the \( S \)-symmetry from Ref. [15]. Thus the infinitesimal transformations for bosonic and ghost fields, now denoted by \( \delta_S \hat{\mathcal{O}} \), are:

\[
\delta_S \hat{\omega}^I = \hat{\psi}^I, \quad \delta_S \hat{\mathcal{S}}^I = \hat{\Psi}^I, \quad (21)
\]

\[
\delta_S \hat{\psi}^I_\mu = D_\mu \hat{\gamma}^I + \hat{\gamma}^\lambda * \partial_\nu \hat{\omega}^I_\nu + \partial_\mu \hat{\gamma}^\lambda * \hat{\omega}^I_\lambda \equiv \hat{\delta}_G^{(\gamma)} \hat{\omega}^I_\mu \quad (22)
\]

\[
\delta_S \hat{\Psi}^I = D_\mu \hat{\gamma}^I + \hat{\gamma}^\lambda * \partial_\lambda \hat{\mathcal{S}}^I + \partial_\mu \hat{\gamma}^\lambda * \hat{\mathcal{S}}^I_\mu \equiv \hat{\delta}_G^{(\gamma)} \hat{\mathcal{S}}^I, \quad (23)
\]

\[
\delta_S \hat{\gamma}^I = 0, \quad \delta_S \hat{\gamma}_\mu = 0, \quad (24)
\]

where \( D_\mu \hat{\gamma}^I = D^a_\mu \hat{\gamma}^I + h^{IJK} [\hat{\gamma}^J, \hat{\omega}^K_\mu] \) and \( \hat{\delta}_G^{(\gamma)} \) denotes the Lorentz and diffeomorphism infinitesimal transformation, with transformation parameter given by \( \gamma \).

The anti-ghost fields are: \( \hat{\chi}_I \) (two-form), \( \hat{\chi}_I \) (one-form) and \( \hat{\chi}_{IJ} \) (zero-form), while the auxiliary fields \( \hat{\mu}_I \) (two-form), \( \hat{\pi}_I \) (one-form) and \( \hat{\pi}_{IJ} \) (zero-form) transform as follows:

\[
\delta_S \hat{\chi}_I = \hat{\mu}_I, \quad \delta_S \hat{\mu}_I = \hat{\delta}_G^{(\gamma)} \hat{\chi}_I, \quad (25)
\]

\[
\delta_S \hat{\pi}_I = \hat{\pi}_I, \quad \delta_S \hat{\pi}_I = \hat{\delta}_G^{(\gamma)} \hat{\chi}_I, \quad (26)
\]

\[
\delta_S \hat{\chi}_{IJ} = \hat{\pi}_{IJ}, \quad \delta_S \hat{\pi}_{IJ} = \hat{\delta}_G^{(\gamma)} \hat{\chi}_{IJ}. \quad (27)
\]

Then we propose the \( \delta_Q \)-invariant Lagrangian

\[
L = \delta_Q \int \left( \hat{\chi}_I \hat{\chi}^I + \hat{\mathcal{S}}^I + D \hat{\mathcal{S}}^I + \frac{1}{2} \hat{\chi}_{IJ} \hat{\mathcal{S}}^I \hat{\mathcal{S}}^J \right)
\]

\[
= \int \left( \hat{\mu}_I \hat{\mu}^I + \hat{\pi}_I \hat{\pi}^I + D \hat{\mathcal{S}}^I + \frac{1}{2} \hat{\pi}_{IJ} \hat{\mathcal{S}}^I \hat{\mathcal{S}}^J - \hat{\chi}_I \hat{\mu}^I \right)
\]

\[
- \hat{\chi}_I \hat{\pi}^I \left( D \hat{\mathcal{S}}^I - h_{JK} \hat{\mathcal{S}}^J \hat{\mathcal{S}}^K \right) - \hat{\chi}_{IJ} \hat{\mathcal{S}}^I \hat{\mathcal{S}}^J \right), \quad (28)
\]
where we have used the BRST transformation previously defined. Here $\hat{\wedge}$ is a noncommutative wedge product given, for instance, by: $\hat{\Psi} \hat{\wedge} \hat{\Phi} = \hat{\int}_{\mu_1 \ldots \mu_p} (x) \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$. for any $\hat{\Psi} \in \Lambda^p (T^* X_\theta)$ and $\hat{\Phi} \in \Lambda^q (T^* X_\theta)$.

The noncommutative gravitational instantons described by the $\hat{\mathcal{S}}_H$ structure are not independent but, they are related via the Bianchi identities $D \hat{\mathcal{R}}^I = 0$ and $D^2 \hat{\mathcal{S}}^I = h^{IJK} \hat{\mathcal{R}}^J \wedge \hat{\mathcal{S}}^K$. This gives rise to the incorporation of a gauge symmetry, which mixed with the above preliminary BRST symmetry, leads to the full BRST symmetry determined by the BRST charge $Q$, which, with help of Ref. [24], can be easily shown that satisfies $Q^2 = 0$, i.e. $\delta_Q \hat{\mathcal{O}} = 0$ for any $\hat{\mathcal{O}}$. This gauge symmetry is fixed through the incorporation of the ghost multiplet $(\hat{\gamma}, \hat{\xi}_IJ, \hat{\lambda}_I, \hat{\eta}_I)$.

For simply connected spacetimes $X_\theta$, the noncommutative spin connection $\hat{\omega}_\mu^I$ can be gauged away and therefore can be removed from the action [28]. Other fields can also be integrated out from the action and we have,

$$L' = \int \left( \hat{\pi}_I \wedge d\hat{S}^I + \frac{1}{2} \hat{\pi}_{IJ} \hat{\wedge} \hat{S}^I - \hat{\chi}_I \hat{\wedge} \hat{\Psi}^I - \hat{\chi}_{IJ} \hat{\wedge} \hat{\Psi}^I \right),$$

with the BRST transformations for the involved fields $(\hat{\mathcal{S}}^I, \hat{\Psi}^I, \hat{\gamma}^\mu, \hat{\omega}^\mu, \hat{\chi}_I, \hat{\xi}_IJ, \hat{\lambda}_I, \hat{\eta}_I)$ given by:

$$\delta_Q \hat{\mathcal{S}}^I = \hat{\Psi}^I + \hat{\delta}^{D}_\gamma \hat{\mathcal{S}}^I, \quad \delta_Q \hat{\Psi}^I = \hat{\delta}^{D}_\gamma \hat{\Psi}^I + \hat{\delta}^{D}_\chi \hat{\Psi}^I,$$

$$\delta_Q \hat{\gamma}^\mu = \hat{\gamma}^\mu + \hat{\gamma}^\lambda \hat{\wedge} \hat{\omega}^\mu, \quad \delta_Q \hat{\omega}^\mu = -\hat{\gamma}^\mu + \hat{\gamma}^\lambda \hat{\wedge} \hat{\omega}^\mu,$$

$$\delta_Q \hat{\chi}_I = \hat{\pi}_I + \hat{\delta}^{D}_\gamma \hat{\chi}_I, \quad \delta_Q \hat{\xi}_IJ = \hat{\pi}_{IJ} + \hat{\delta}^{D}_\gamma \hat{\xi}_{IJ}, \quad \delta_Q \hat{\lambda}_I = \hat{\delta}^{D}_\gamma \hat{\lambda}_I + \hat{\delta}^{D}_\chi \hat{\lambda}_I,$$

$$\delta_Q \hat{\eta}_I = -\hat{\eta}_I + \hat{\delta}^{D}_D \hat{\eta}_I,$$

The lagrangian $L'$ has associated the partition function,

$$Z = \int (D\Phi) \exp \left( - L' \right),$$

where $(D\Phi)$ abbreviates the measure of all noncommutative fields of the theory.
The BRST invariance of $\delta_Q L' = 0$, implies that the partition function $Z$ and the correlation functions of BRST-invariant observables $\hat{O}$, i.e. functionals of the noncommutative fields contained in $L'$,

$$Z(\hat{O}) = \langle \hat{O} \rangle = \int (D\Phi) \exp (-L') \cdot \hat{O}, \quad (36)$$

are topological invariants and constitute a noncommutative deformation of the gravitational Donaldson invariants $[18]$. These invariants are independent on the metric, but they do depend on the differentiable structure of the noncommutative differentiable manifold $X_\theta$. From inspection of the BRST transformation (24), the obvious invariants and their descendents are those functionals constructed from the BRST-invariant fields $\hat{\gamma}^i$ and $\hat{\gamma}^\mu$, such that $\delta_Q \hat{\gamma}^i = 0$ and $\delta_Q \hat{\gamma}^\mu = 0$.

Moreover, for similar reasons, $Z$ and $Z(\hat{O})$ are independent of a rescaling of the volume form, $v \rightarrow tv$. Thus, for simplicity, we can evaluate the path integral for large values of $t$. e.g. $t \rightarrow \infty$. In this limit the path integral is dominated by the classical minima, and it is now concentrated in the noncommutative instanton configuration, determined by the following system of equations $\hat{S}_H$:

$$\delta \hat{\omega}' : \quad d\hat{S}' = 0, \quad (37)$$

$$\delta \hat{S}' : \quad \hat{\gamma}' = 0. \quad (38)$$

These equations are the noncommutative $\hat{S}_H$-structure in the “strong heaven” gauge $[16]$, which determines the moduli space of the heavenly configurations $\hat{\mathcal{M}}_H$. After some computations, following closely Ref. $[16]$, we can carry over the above equations into the equivalent form of a noncommutative deformation of the first and second heavenly equations. After the analysis, the resulting first heavenly equation is,

$$\partial^2_{pq} \Omega \ast \partial^2_{qs} \Omega - \partial^2_{qr} \Omega \ast \partial^2_{ps} \Omega = 1, \quad (39)$$

which is a nonlinear PDE for the holomorphic function $\Omega = \Omega(p,q,r,s)$ in $X_\theta$, in a local chart $\{x^\mu\} = \{p,q,r,s\}$. An equivalent description of the moduli space $\hat{\mathcal{M}}_H$ can be given in the local chart $\{x^\mu\} = \{x,y,p,q\}$ with $x := \partial_p \Omega$ and $y := \partial_q \Omega$. In this case we get a noncommutative version of the second heavenly equation given by

$$\partial^2_{xx} \Theta \ast \partial^2_{yy} \Theta - (\partial^2_{xy} \Theta) \ast (\partial^2_{xy} \Theta) + \partial^2_{xp} \Theta + \partial^2_{yq} \Theta = 0, \quad (40)$$

for a holomorphic function $\Theta = \Theta(x,y,p,q)$. 

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5. CONCLUSIONS

In the present paper, starting from previous results [8] concerning the noncommutative deformation of the classical topological invariants given by the Euler number and signature of a four smooth manifold $X$, we deform the topological half-flat gravity proposed in Ref. [15]. We first discuss the convenience of having a $SL(2,C)$ chiral gravity theory, in order to find a noncommutative Lagrangian for gravity preserving all original symmetries of the theory. An interesting feature is that this procedure can be implemented to find a noncommutative deformation of this topological gravity.

Topological half-flat gravity is defined as the intersection theory in the moduli space $\mathcal{M}$. This moduli problem becomes important as far as the other topological gravity theories are cohomological theories of the moduli space of metrics, with an action of the form (11), which involves the Levi-Civita tensor $\varepsilon_{abcd}$, and which seems to be very difficult to implement in the formulation of a noncommutative theory of gravitation, without spoiling the full original symmetry.

Topological gravity possesses a fermionic BRST symmetry, which we have noncommutatively deformed following the procedure of Ref. [24]. This results in a noncommutative topological gravity, invariant under the full set of symmetries including the BRST symmetry. For simply connected spaces $X_\theta$, we can gauge away the self-dual spin connection and set $\hat{\omega}_\mu^I$ to zero. We have shown, by using the stationary phase approximation in evaluating the partition function, that the main contribution of the Feynman path integral comes from the moduli space $\hat{\mathcal{M}}_H$, characterized as the space of solutions of the noncommutative formulation of the heavenly equations (39) or (40). Thus in noncommutative topological half-flat gravity, the gravitational analogs of Donaldson invariants can be computed as the intersection form in $\hat{\mathcal{M}}_H$ determined by the heavenly equation. A detailed analysis of this result is left for future work.

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