ON THE LEVELT’S THEOREM

LOTFI SAIDANE

Abstract. Let \((E)\) be a homogeneous linear differential equation Fuchsian of order \(n\) over \(\mathbb{P}^1(\mathbb{C})\). The idea of Riemann (1857) was to obtain the properties of solutions of \((E)\) by studying the local system. Thus, he obtained some properties of Gauss hypergeometric functions by studying the associated rank 2 local system over \(\mathbb{P}^1(\mathbb{C}) \setminus \{3\ points\}\). For example, he obtained the Kummer transformations of the hypergeometric functions without any calculation. The success of the Riemann’s methods is due to the fact that the irreducible rank 2 local system over \(\mathbb{P}^1(\mathbb{C}) \setminus \{3\ points\}\) is linearly “rigid” in the sense of Katz [16]. This result constitute one of the best studied example of linear rigid system, it was proved by the Levelt’s theorem [2] Theorem 1.2.3. In this work we propose a partial generalization of the Levelt’s theorem.

1. Introduction

Let \(n \in \mathbb{N}_{\geq 1}\) and \(a_1, ... , a_n \in \mathbb{C}(z)\). We consider the homogeneous linear differential equation of order \(n\) on \(\mathbb{P}^1(\mathbb{C})\):

\[(E) \quad y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1}y' + a_n y = 0.\]

We denote by \(S = \{\varpi_1, ..., \varpi_s\}\) the non empty set, in \(\mathbb{P}^1(\mathbb{C})\), of its singularities, we required that will be all regular. We fix a base point \(z_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S\) and denote by \(G\) the fundamental group \(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, z_0)\). Then \(G\) is a free group generated by the classes of homotopy of loops \(\gamma_i\) starting at \(z_0\) and making a turn in the direct sense of \(\varpi_i\), in a neighborhood not containing \(\varpi_j, j \neq i\), then returning to \(z_0\) such that :

\[\prod_{i \in \{1, ..., s\}} \gamma_i = 1.\]

Because \(z_0\) is a regular point for the equation \((E)\), the Cauchy conditions are satisfied, therefore there are \(n\) solutions (local) of \((E)\) holomorphic near \(z_0\), linearly independent over \(\mathbb{C}\). Let \(V\) be a \(\mathbb{C}\)-vector space spanned by these solutions. The group representation

\[M_{(E)} : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, z_0) \rightarrow GL(V),\]

is called the monodromy representation of \((E)\). For \(i \in \{1, ..., s\}\), we put \(M_i = M_{(E)}(\gamma_i)\). Then, \(M_i \in GL_n(\mathbb{C})\) and

\[\prod_{i \in \{1, ..., s\}} M_i = I_n.\]

The matrices \(M_i\) are called (local) monodromy matrix, they constitute a local (complex) system of order \(n\) on \(\mathbb{P}^1(\mathbb{C}) \setminus S\). The group generated by the matrices \(M_i\) is called the “monodromy group” of \((E)\), related to the basis of local solutions at \(z_0\).

Date: October 2009.
1991 Mathematics Subject Classification. 12H05.
Key words and phrases. Linearly rigid system, hypergeométric operator, monodromy.
**Question:** If we change the basis of local solutions, is the new monodromy group isomorphic to the former?
In other words: The local system defined by the $M_i$ is it "linearly rigid" (in the sens below)?
Let $r \in \mathbb{N}_{\geq 2}$ and $g_1, g_2, \ldots, g_r$ elements of $GL_n(\mathbb{C})$ satisfying
\[ g_1 g_2 \ldots g_r = Id_n. \]
We say that the $r$–tuple \{ $g_1, g_2, \ldots, g_r$ \} is linearly rigid if for any conjugate $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_r$ of $g_1, g_2, \ldots, g_r$ in $GL_n(\mathbb{C})$ satisfying:
\[ \tilde{g}_1 \tilde{g}_2 \ldots \tilde{g}_r = Id_n, \]
there is $u$ in $GL_n(\mathbb{C})$ such that $\tilde{g}_i = u g_i u^{-1}$ for $i = 1, 2, \ldots, r$. For example, the couple $(g, g^{-1})$, $g \in GL_n(\mathbb{C})$ is linearly rigid.
The group $< g_1, \ldots, g_r >$ is said irreducible if and only if it acts irreducibly on $\mathbb{C}^n$. Katz [10] theorem 1.1.2, characterized the Jordan normal forms of irreducible linearly rigid local systems. The Levelt theorem, [2] Theorem 1.2.3 shows that the local system associated to a hypergeometric equation is irreducible linearly rigid.

2. Pseudo-reflection

**Definition 1.** We say that $h \in GL(n, \mathbb{C})$ is a pseudo-reflection if the rank of $(h - Id_n)$ is 1.

**Lemma 1.** Let $n, p \in \mathbb{N}_{\geq 2}$ and $A_1, A_2, \ldots, A_p \in GL_n(\mathbb{C})$ such that, for all $i, j \in \{1, 2, \ldots, p\}$, $i < j$, the operator $A_i A_j^{-1}$ is a pseudo-reflection. Then, up to conjugation, $A_1, A_2, \ldots, A_p$ have the same $(n - 1)$ first rows or columns.

**Proof.** We will furnish a proof for $p = 3$, the general case being similar. Assume that $n \geq 3$.
We set $W_1 = \ker(A_1 - A_2)$ and $W_2 = \ker(A_2 - A_3)$. Because $A_1 A_2^{-1}$ and $A_2 A_3^{-1}$ are pseudo-reflections, we deduce that $W_1$ and $W_2$ are $n - 1$ dimensional subspaces of $\mathbb{C}^n$. If $W_1 = W_2$, we choose a basis of $W_1$ and we complete it by one vector to obtain a basis of the total space. With respect to this basis the matrices $A_1$, $A_2$ and $A_3$ have the same $(n - 1)$ first columns.
Assume that $W_1 \neq W_2$. Because $n \geq 3$, the vector space $W_1 \cap W_2$ has dimension $n - 2$. Let $\{e_1, \ldots, e_{n-2}\}$ a basis of the space $W_1 \cap W_2$.
* If $A_1 A_2^{-1}$ is a reflection, there is $e_n$ such that $(A_1 - A_2)(e_n) = \nu e_n$, with $\nu \neq 0$. We choose $e_{n-1}$ in $W_1$ so that $\{e_1, \ldots, e_{n-1}\}$ is a basis of $W_1$. The system $\xi = \{e_1, \ldots, e_n\}$ is then a basis of $\mathbb{C}^n$. For $i \in \{1, 2, 3\}$, $j \in \{1, 2, \ldots, n\}$, We denote by $A_{i,j}$, the $j - th$ column of the matrix $A_i$, relatively to the basis $\xi$. Because $\text{rank}(A_1 - A_2) = \text{rank}(A_2 - A_3) = 1$ and $W_1 \neq W_2$, there exist $\lambda$ and $\beta$ non-zero in $\mathbb{C}$ such that
\[ A_{1,n-1} - A_{3,n-1} = \lambda(A_{1,n} - A_{3,n}) \]
and
\[ A_{2,n-1} - A_{3,n-1} = \beta(A_{2,n} - A_{3,n}). \]
We have, by hypothesis $A_{1,n-1} = A_{2,n-1}$ and $A_1 e_n = A_2 e_n + \nu e_n$. Substituting these relations in the previous equalities, we obtain, by abuse of writing,
\[ \lambda(A_{2,n} + \nu e_n - A_{3,n}) = \beta(A_{2,n} - A_{3,n}). \]
Therefore, there is at least $\alpha \in \mathbb{C}$, $\alpha \neq 0$ such that
\[ A_{2,n} - A_{3,n} = \alpha e_n, \]
then
\[ A_{1,n-1} - A_{3,n-1} = \lambda \alpha e_n, \]
and consequently, the $(n-1)$ first row of the matrices $A_1, A_2$ and $A_3$ are identical.

* If $A_1 A_2^{-1}$ is idempotent ($1$ is the unique eigenvalue), then the image of $A_1 - A_2$ is contained in its kernel $W_1$. Let $w$ be a generator of $\text{Im} (A_1 - A_2)$, then, there exist a vector $e_n$ in $\mathbb{C}^n$ such that $(A_1 - A_2) e_n = w$. If $w \in W_1 \cap W_2$, we set $e_1 = w$ so that $\{e_1, ..., e_{n-2}\}$ is a basis of the space $W_1 \cap W_2$. We choose $e_{n-1}$ in $W_1$ such that $\{e_1, ..., e_{n-1}\}$ is a basis of $W_1$. Then, the system $W = \{e_1, ..., e_n\}$ is a basis of $\mathbb{C}^n$. If $w \notin W_1 \cap W_2$, we set $e_{n-1} = w$. Thus, there exist a unique index $m \in \{1, n-1\}$, such that $w = e_m$. Under these conditions, the $(n-1)$ first columns of $A_1 - A_2$ are zero and the last column is equal to $e_m$. The $(n-2)$ first columns of $A_1 - A_3$ and $A_2 - A_3$ are zero. Using the fact that $A_1 - A_3$ and $A_2 - A_3$ are of rank $1$, we obtain that all components of their two last columns, except the $m-th$ row, are zero. Therefore, except the $m-th$ row, all rows of $A_1, A_2$ and $A_3$ are identical.

If $n = 2$ and $W_1 \neq W_2$. We repeated the same argument by replacing $W_1 \cap W_2$ by $\{0\}$.

The two following results modify and generalize, in part, the theorem 1.2.1 of [2].

**Theorem 1.** Let $n, p \in \mathbb{N}_{\geq 2}$ and $A_1, A_2, ..., A_p \in M_n(\mathbb{C})$ having the same $(n-1)$ first rows or columns and one eigenvalue in common. Then these matrices stabilize at least a line or a hyperplane of $\mathbb{C}^n$.

**Proof.** We note that a system of matrices stabilizing the same hyperplane if and only if their transpose matrices, as endomorphism of the dual space, stabilize the same line, which mean they have an eigenvector in common. Without loss of generality, we may assume that $(n-1)$ first rows of the matrices $A_i$ are identical. If $\lambda$ denotes the common eigenvalue, then the matrices $A_1 - \lambda I_n$, $A_2 - \lambda I_n$, ..., $A_p - \lambda I_n$ have the same $(n-1)$ first rows and have rank less than equal $n - 1$:
- If the $(n-1)$ first rows of these matrices are linearly independent, then the last row of each of these matrices is a linear combination of the previous. Let $v$ be a nonzero vector orthogonal to the $(n-1)$ first rows. Then $v$ is orthogonal to the last row of each of these matrices. Thus, $v$ is a common eigenvector with eigenvalue $\lambda$, of the matrix $A_1, .., A_p$.
- If the $(n-1)$ first rows of these matrices are linearly dependent, then the $(n-1)$ first columns of the transpose matrices are linearly dependent. Let $c_1, c_2, ..., c_{n-1}$ be the coefficients of a non trivial linear dependence relation. Thereby, $v = (c_1, c_2, ..., c_{n-1}, 0)$ is a common eigenvector of all matrices $A_1^T, ..., A_p^T$ with eigenvalue $\lambda$. Therefore, the matrices $A_1, .., A_p$ stabilize, simultaneously, a hyperplane.

If the matrix $A_1, .., A_p$ have $(n-1)$ common columns, then their transpose have $(n-1)$ common rows, thus the above reasoning leads to the conclusion. \(\square\)

**Theorem 2.** Let $n, p \in \mathbb{N}_{\geq 2}$, $A_1, A_2, ..., A_p \in M_n(\mathbb{C})$ having the same $(n-1)$ first rows or columns and stabilize a same non trivial subspace of $\mathbb{C}^n$. Then, $\cap_{i=1}^p \text{spec} A_i \neq \emptyset$.

**Proof.** We may suppose that, relatively to some basis $B = \{e_1, ..., e_n\}$ of $\mathbb{C}^n$, the matrix $A_1, .., A_p$ have the same $(n-1)$ first columns. Denote by $E$ the subspace
of $\mathbb{C}^n$ generated by $\{e_1, ..., e_{n-1}\}$. Let $W$ be a nontrivial subspace of $\mathbb{C}^n$ stable under the action of $A_i$. Suppose that $W \subset E$ and $\dim_{\mathbb{C}} W = r \in \{1, ..., n-1\}$. Let $\{w_1, ..., w_r\}$ a basis of $W$, we complete so that $B' = \{w_1, ..., w_n\}$ is a basis of $\mathbb{C}^n$. Since $A_i e_j = A_k e_j$, for $i, k \in \{1, ..., p\}$ and $j \in \{1, ..., n-1\}$, we deduce that

$$A_i w_j = A_k w_j, \text{ for } i, k \in \{1, ..., p\} \text{ and } j \in \{1, ..., r\},$$

This proves that, knowing the stability of $W$, and relatively to the basis $B'$ the matrix $A_i$ have the form below

$$A_i = \begin{pmatrix} A_{(r,r)} & 0 \\ o & *(r,n-r) \end{pmatrix},$$

where $A_{(r,r)}$ is an order $r$ matrix common to all $A_i$, $*(r,n-r)$ (resp. $*(n-r,n-r)$) is an element of $M_{(r,n-r)}(\mathbb{C})$ (resp. $M_{(n-r,n-r)}(\mathbb{C})$). Therefore, the polynomial $\det(A_{(r,r)} - \lambda I_r)$ divides all the characteristic polynomials of all $A_i$. The complex roots of $\det(A_{(r,r)} - \lambda I_r)$ are in $\cap_{i=1}^p \text{spec} A_i$.

Assume that $W \not\subset E$, then there is a basis $\{f_{n-p+1}, ..., f_n\}$ of $W$ and a free system $\{g_1, ..., g_{n-p}\}$ of $E$ such that $\{g_1, ..., g_{n-p}, f_{n-p+1}, ..., f_n\}$ is a basis of $\mathbb{C}^n$. With respect to this basis the matrix $A_i$ have the form:

$$A_i = \begin{pmatrix} A_{(n,n-p)} & 0 \\ 0 & *(p,p) \end{pmatrix},$$

where $A_{(n,n-p)}$ is a matrix with $n$ rows and $(n-p)$ columns common to all $A_i$ and $*(p,p)$ is a matrix of order $p$. Thus, we deduce that the $A_i$ have at least one common eigenvalue. □

**Corollary 1** (Beukers Théorème 1.2.1). Let $H$ be a subgroup of $GL_n(\mathbb{C})$ generated by two matrices $A$ and $B$ satisfying $AB^{-1}$ is a pseudo-reflection. Then $H$ is linearly irreducible if and only if, the spectrum of $A$ and $B$ are linealy disjoint.

**Proof.** The lemma for $p = 2$, proves that $A$ and $B$ have $(n-1)$ rows or columns in commun. The theorems furnished the result. □

### 3. Levelt’s Theorem

The following result modifies, slightly, and generalizes in part, the Levelt’s theorem for case $p \geq 2$ (see [2] theorem 1.2.3).

**Theorem 3.** Let $n, p \in \mathbb{N}_{\geq 2}$ and $\alpha_i = \{\alpha_{i,1}, ..., \alpha_{i,n}\} \subset \mathbb{C}^*$, $1 \leq i \leq p$, satisfying $\cap_{i=1}^p \alpha_i = \emptyset$. Then, there exist $A_1, A_2, ..., A_p \in GL_n(\mathbb{C})$ having the same $(n-1)$ first columns (unique up to a same isomorphism) such that for every $i$, $\text{spec} A_i = \alpha_i$.

**Proof.** Existence:
For every $(i, j) \in \{1, ..., p\} \times \{1, ..., n\}$, we define $A_{i,j}$ by

$$\prod_{j=1}^p (X - \alpha_{i,j}) = X^n + \sum_{k=0}^{n-1} A_{i,n-k} X^k,$$

By hypothesis, the $\alpha_{i,j}$ are non zero, than the matrices defined by

$$A_i = \begin{pmatrix} 0 & 0 & 0 & -A_{i,n} \\ 1 & 0 & & \\ 0 & 1 & & -A_{i,1} \end{pmatrix},$$
LEVELT Theorem 5

The characteristic polynomial of $A_i$ is

$$
\det(X I_n - A_i) = X^n + \sum_{k=0}^{n-1} A_{i,n-k} X^k
= \prod_{j=1}^{n}(X - \alpha_{i,j}).
$$

which proves the existence.

Uniqueness:

Let $A_1, A_2, \ldots, A_p \in GL(n, \mathbb{C})$ having the same $(n-1)$ first columns. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{C}^n$, relatively to it we have, for all $i, j \in \{1, \ldots, p\}, k \in \{1, \ldots, n-1\}$:

$$A_i e_k = A_j e_k.$$

We denote by $W$ the vectorial suspace of $\mathbb{C}^n$ generated by $\{e_1, \ldots, e_{n-1}\}$. We have:

$$\dim_{\mathbb{C}}(W \cap A_1 W \cap \ldots \cap A_p^{n-2} W) \geq 1.$$

We suppose that the dimensional of $W \cap A_1 W \cap \ldots \cap A_p^{n-1} W$ is greater than or equal to 2. Therefore, we have

$$\dim_{\mathbb{C}}(W \cap A_1 W \cap \ldots \cap A_p^{n-2} W) = 1,$$

Then, there is a vector $v$ in $W$ such that the system $\{v, A_1 v, \ldots, A_p^{n-2} v\}$ is a basis of $W$, we complete with a vector to a basis of the total space $\mathbb{C}^n$. With respect to the latter matrices $A_j$ have the form

$$A_i = \begin{pmatrix}
0 & 0 & -A_{i,n} \\
1 & 0 & . \\
0 & 1 & -A_{i,1}
\end{pmatrix},$$

where $A_{i,j}$ are determined by the spectrum $\{\alpha_{i,1}, \ldots, \alpha_{i,n}\}$ of $A_i$ as follows:

$$\prod_{j=1}^{n}(X - \alpha_{i,j}) = X^n + \sum_{k=0}^{n-1} A_{i,n-k} X^k,$$

which completed the proof. \hfill \Box

References

[1] Beauville, A., ”Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann”, Séminaire N. Bourbaki, 1992-1993, exp. no 765, 103-119, (1993).
[2] Beukers, F., ”hypergeometric functions in one variable”, manuscrit, 14 Avril 2006.
[3] Bertrand, D., ”Extensions de D-modules et groupes de Galois différentiels”, Springer L.N.1454, 1990, 125-141.
[4] Bertrand, D., ”Unipotent radicals of differential Galois groups”, institut de mathématiques de jussieu, prépublication 239, Février 2000.
[5] Beukers, F.; Heckman, G., ”Monodromy for the hypergeometric function _nF_{n-1}”; Invent.math. 95, 325-354 (1989).
[6] Birkhoff, G., ”The generalized Hilbert problem for linear differential equations and the allied problem for linear difference and _q-difference equations”, Proc, Amer , Acad. 49, 521-568, (1913).
[7] Birkhoff, G., ”Collected Mathematical Papers”, volume 1. Dover Publications, New York, 1968.
[8] Boussel, K., ”Opérateurs hypergéométriques réductibles: décomposition et groupes de Galois différentiels”, Ann. Fac. St. Toulouse, 5, 299-362, 1996.
[9] Deligne, P., ”Equations différentielles à points singuliers réguliers, Lecture Notes in Math. 163, Springer-Verlag, Berlin-Heidelberg-New York (1970)
[10] Dekkers, W., "The matrix of a connection having regular singularities on a vector bundle of rank 2 on $P^1(C)$", Lecture Notes in Math., 712, 33-43, Springer-Verlag, Berlin-Heidelberg-New York, (1979).

[11] Grothendieck, A., "Sur la classification des fibrés holomorphes sur la sphère de Riemann", Amer. J. Math., 79:121-138, 1957.

[12] Hille, E., "Ordinary Differential Equations In The Complex Plane", Pure & Applied Mathematics, A Wiley-Interscience Series Of Texts, Monographs & Tracts.

[13] Ince, E. L., "Ordinary Differential Equations", Dover Publications, New York, 1956.

[14] Katz, N. M., "On the calculation of some differential Galois groups", Invent. Math., 87, pp 13-61, 1987.

[15] Katz, N. M., "Exponential sums and differential equations", Princeton University Press, 1991.

[16] Katz, N. M., "Rigid Local Systems", Annals of Math. Studies 139, Princeton 1996.

[17] Lappo-Danilevskii, I., "Mémoire sur la théorie des systèmes des équations différentielles linéaires", Chelsea, New York (1953).

[18] Röhr, H., "Das Riemann-Hilbertische Problem der Theorie der linearen Differentialgleichungen. Math. Annalen 133, 1-25, (1957)

[19] Van der Put, M., "reduction modulo p of differential equations", indag. Mathem.,N.S.,7 (3),367-387.

[20] Van der Put, M., Singer, S., "Galois theory of linear differential equations", Springer-Verlag Gmbh, janvier 2003

[21] Varadarajan, V. S., "Meromorphic differential equations", Expo. Math. 9 (1991), 97-188.

Lotfi Saidane, Département de Mathématiques, Faculté des Sciences de Tunis, Campus Universitaire, 1060 Tunis, Tunisie.

E-mail address: lotfi.saidane@fst.rnu.tn