Introduction to the Physics of Saturation

Yuri V. Kovchegov

Department of Physics, The Ohio State University, Columbus, OH 43210, USA

Abstract
We present a brief introduction to the physics of parton saturation/Color Glass Condensate (CGC).

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1. Introduction

One of the most interesting features of quantum chromodynamics (QCD) is the property of the asymptotic freedom: the strong coupling constant is small at large momenta/short distances, and it is large at small momenta/large distances \[^1\] \[^2\]. Finding the scale that determines the value of the characteristic running QCD coupling is one of the the central questions for high energy scattering physics, important for the theoretical description of both the hadronic and nuclear scattering processes.

A naive answer to this question would be to say that in high energy scattering the large center-of-mass energy \(s\) determines the scale of strong coupling making it small: \(\alpha_s(s) \ll 1\). While such statement would be true for several \(s\)-channel processes, the dominant contribution to total cross sections in high energy hadronic and nuclear scattering comes from the \(t\)-channel exchanges, for which the scale of the QCD coupling constant is not given by \(s\), but by the typical transverse momentum in the problem. For proton-proton scattering one may estimate the typical transverse momentum to be of the order of the inverse transverse size of the protons, which is roughly the QCD

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confinement scale \( \Lambda_{QCD} \). People performing such estimate would pessimistically conclude that the QCD coupling in high energy hadronic scattering runs as \( \alpha_s(\Lambda_{QCD}^2) \) and is therefore not small, \( \alpha_s(\Lambda_{QCD}^2) \sim 1 \). With the coupling constant of order-one, we would have little chance of describing the total hadronic and nuclear cross sections from first principles, at least certainly not with the help of QCD perturbation theory.

Traditional perturbative QCD (pQCD) approaches are well-aware of the above problem, and try to avoid it by separating hard sub-events where the coupling is small from the full event with large QCD coupling. For instance, in hadronic scattering pQCD may be used to calculate jet production cross section, where the hard partonic scattering is factorized from the non-perturbative distribution and fragmentation functions. The high transverse momentum \( p_T \) of the produced hard parton insures applicability of pQCD: \( \alpha_s(p_T^2) \ll 1 \). Similarly, in deep inelastic scattering (DIS) pQCD can describe structure functions at high photon virtuality \( Q^2 \), since there \( \alpha_s(Q^2) \ll 1 \), but pQCD is expected to fail at low-\( Q^2 \). Since jet production events in hadronic collisions constitute a small percentage of the total cross section, pQCD approach describes only rare events, and is not applicable to the description of the bulk particle production and dynamics.

Saturation physics provides a new way of tackling the problem of total hadronic and nuclear cross sections. It is based on the theoretical observation that small-\( x \) hadronic and nuclear wave functions, and, therefore, the scattering cross sections as well, are described by an internal momentum scale known as the saturation scale and denoted by \( Q_s \). This intrinsic momentum scale grows with the center-of-mass energy \( s \) in the problem, and with the increasing atomic number of a nucleus \( A \) (in the case of a nuclear wave function) approximately as

\[
Q_s^2 \sim A^{1/3} s^{\lambda}
\]

where the best current theoretical estimates of \( \lambda \) give \( \lambda = 0.2 \div 0.3 \). Therefore, for hadronic collisions at high energy and/or for collisions of large ultrarelativistic nuclei, saturation scale becomes large, \( Q_s^2 \gg \Lambda_{QCD}^2 \). Since for total cross sections \( Q_s \) is usually the only momentum scale in the problem, we expect it to give the scale of the running QCD coupling constant, making it small

\[
\alpha_s(Q_s^2) \ll 1
\]
and allowing for first-principles calculations of total hadronic and nuclear cross sections, along with extending our ability to calculate particle production and to describe diffraction in a small-coupling framework.

Below we present a short review of the saturation physics, which is also known as the Color Glass Condensate (CGC) physics. For more extensive descriptions of the subject we refer the readers to the review articles [5, 6, 7].

2. Brief Review of Saturation Physics

Traditional approach to saturation physics consists of two stages, corresponding to two different levels of approximation. The first approximation corresponds to the classical gluon field description of nuclear wave functions and scattering cross sections. It resums all multiple rescatterings in the nucleus, but lacks rapidity-dependence. The latter is included through quantum corrections, which are resummed by the non-linear evolution equations. This constitutes the second level of approximation. We will present both stages below.

2.1. Classical Gluon Fields

Imagine a single large nucleus, which was boosted to some ultrarelativistic velocity, as shown in Fig. 1. We are interested in the dynamics of small-x gluons in the wave function of this relativistic nucleus. The small-x gluons interact with the whole nucleus coherently in the longitudinal direction: therefore, only the transverse plane distribution of nucleons is important for the small-x wave function. As one can see from Fig. 1 after the boost the
nucleons, as “seen” by the small-$x$ gluons, appear to overlap with each other in the transverse plane, leading to high parton density. Large occupation number of color charges (partons) leads to classical gluon field dominating the small-$x$ wave function of the nucleus. This is the essence of the McLerran-Venugopalan (MV) model \cite{8}. According to the MV model, the dominant gluon field is given by the solution of the classical Yang-Mills equations

$$D_\mu F^{\mu\nu} = J^\nu \quad (3)$$

where the classical color current $J^\nu$ is generated by the valence quarks in the nucleons of the nucleus from Fig. 1.

The equations (3) were solved for a single nucleus exactly \cite{9,10}, resulting in the unintegrated gluon distribution $\phi(x, k_T^2)$ (multiplied by the phase space factor of the gluon’s transverse momentum $k_T$) shown in Fig. 2 as a function of $k_T$. (Note that in the MV model $\phi(x, k_T^2)$ is independent of Bjorken-$x$.) Fig. 2 demonstrates the emergence of the saturation scale $Q_s$:

![Figure 2: Unintegrated gluon distribution $\phi(x, k_T^2)$ of a large nucleus due to classical gluon fields (solid line). Dashed curve denotes the lowest-order perturbative result.](image)

as one can see from Fig. 2 the majority of gluons in this classical distribution have transverse momentum $k_T \approx Q_s$. Since in this classical approximation $Q_s^2 \sim A^{1/3}$, for large enough nucleus all of its small-$x$ gluons would have large transverse momenta $k_T \approx Q_s \gg \Lambda_{QCD}$, justifying applicability of perturbative approach to the problem. Note that the gluon distribution slows down its growth with decreasing $k_T$ for $k_T < Q_s$ (from power-law of $k_T$ to a logarithm): the distribution saturates, justifying the name of the saturation scale.
2.2. Nonlinear Evolution

While the classical gluon fields of the MV model exhibit many correct qualitative features of saturation physics, and give predictions about $A$-dependence of observables which may be compared to the data, they do not lead to any rapidity/Bjorken-$x$ dependence of the corresponding observables, which is essential in the data on nuclear and hadronic collisions. To include rapidity dependence one has to calculate quantum corrections to the classical fields described above.

new parton is emitted as energy increases
it could be emitted off anyone of the $N$ partons

Figure 3: Nonlinear small-$x$ evolution of a hadronic or nuclear wave functions. All partons (quarks and gluons) are denoted by straight solid lines for simplicity.

The inclusion of quantum corrections is accomplished by the small-$x$ evolution equations. The first small-$x$ evolution equation was constructed before the birth of the saturation physics. This is the Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution equation \[ 11, 12 \]. This is a linear evolution equation, which is illustrated by the first term on the right hand side of Fig. 3. Consider a wave function of a high-energy nucleus or hadrons: it contains many partons, as shown on the left of Fig. 3. As we make one step of evolution by boosting the nucleus/hadron to higher energy, either one of the partons can split into two partons, leading to an increase in the number of partons proportional to the number of partons $N$ at the previous step,

$$\frac{\partial N(x, k_T^2)}{\partial \ln(1/x)} = \alpha_s K_{BFKL} \otimes N(x, k_T^2), \quad (4)$$

with $K_{BFKL}$ an integral kernel. Clearly the BFKL equation (4) introduces Bjorken-$x$/rapidity dependence in the observables it describes.

The main problem with the BFKL evolution is that it leads to the power-law growth of the total cross sections with energy, $\sigma_{tot} \sim s^{\alpha_p-1}$, with the
BFKL pomeron intercept $\alpha_P - 1 = (4\alpha_s N_c \ln 2)/\pi > 0$. Such power-law cross section increase violates the Froissart bound, which states that the total hadronic cross section can not grow faster than $\ln^2 s$ at very high energies. Moreover, power-law growth of cross sections with with energy violates the black disk limit known from quantum mechanics: high-energy total scattering cross section of a particle on a sphere of radius $R$ is bounded by

$$\sigma_{\text{tot}} \leq 2 \pi R^2.$$  \hfill (5)

(Note the factor of 2 which is due to quantum mechanics, this is not simply a hard sphere from classical mechanics!)

We see that something has to modify Eq. (4) at high energy. The modification is illustrated on the far right of Fig. 3 at very high energies partons may start to recombine with each other on top of the splitting. The recombination of two partons into one is proportional to the number of pairs of partons, which, in turn, scales as $N^2$. We end up with the following non-linear evolution equation:

$$\frac{\partial N(x, k_T^2)}{\partial \ln(1/x)} = \alpha_s K_{\text{BFKL}} \otimes N(x, k_T^2) - \alpha_s [N(x, k_T^2)]^2.$$  \hfill (6)

This is the Balitsky-Kovchegov (BK) evolution equation \cite{13,14}, which is valid for QCD in the limit of large number of colors $N_c$. An equation of this type was originally suggested by Gribov, Levin and Ryskin in \cite{3} and by Mueller and Qiu in \cite{15}, though at the time it was assumed that the quadratic term is only the first non-linear correction with higher order terms possibly appearing as well: in \cite{13,14} the exact form of the equation was found, and it was shown that in the large-$N_c$ limit Eq. (6) does not have any higher-order terms in $N$. Generalization of Eq. (6) beyond the large-$N_c$ limit is accomplished by the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) \cite{16,17} evolution equation, which is a functional differential equation.

The physical impact of the quadratic term on the right of Eq. (6) is clear: it slows down the small-$x$ evolution, leading to parton saturation and to total cross sections adhering to the black disk limit of Eq. (5). The effect of gluon mergers becomes important when the quadratic term in Eq. (6) becomes comparable to the linear term on the right-hand-side. This gives rise to the saturation scale $Q_s$, which now grows with energy (on top of its increase with $A$), as was advertised in Eq. (1) above.
We summarize our knowledge of high energy QCD in Fig. 4 in which different regimes are plotted in the \((Q^2, Y = \ln 1/x)\) plane, by analogy with DIS. For hadronic and nuclear collisions one can think of typical transverse momentum \(p_T\) of the produced particles instead of \(Q^2\). Also rapidity \(Y\) and Bjorken-\(x\) variable are interchangeable. On the left of Fig. 4 we see the region with \(Q^2 \leq \Lambda^2_{QCD}\) in which the coupling is large, \(\alpha_s \sim 1\), and small-coupling approaches do not work. In the pessimistic view of high energy scattering described in the Introduction, this is exactly where the total hadronic and nuclear cross sections would be. In the perturbative region, \(Q^2 \gg \Lambda^2_{QCD}\), we see the standard DGLAP evolution and the linear BFKL evolution. The BFKL equation evolves gluon distribution toward small-\(x\), where parton density becomes large and parton saturation sets in. Transition to saturation is described by the non-linear BK and JIMWLK evolution equations. Most importantly this transition happens at \(Q^2_s \gg \Lambda^2_{QCD}\) where the small-coupling approach is valid.

### 2.3. Some CGC Phenomenology

One of the important predictions of saturation/CGC physics was the so-called geometric scaling of the total DIS cross section. It was argued that
with $Q_s(x)$ being the only scale in the problem, DIS structure functions and cross sections should depend only on one variable – $Q^2/Q_s^2(x)$, instead of being functions of two variables $x$ and $Q^2$. This prediction was supported by detailed calculations based on BK evolution \[18, 19\] and was confirmed by an analysis of the HERA DIS data \[20\].

Another prediction of the non-linear evolution (6) concerns particle production in proton-nucleus ($d+Au$) collisions. It follows from Eq. (6) combined with the formulas for particle production in CGC that the nuclear modification factor $R_{pA}^{pA}$ should decrease as one goes toward more forward rapidity \[21, 22, 23\]. This prediction is illustrated in Fig. 5 and was confirmed by RHIC experiments.

### 3. Recent Progress in Saturation Physics

In recent years saturation physics has become more precision-oriented, with steps taken to improve its quantitative predictions. Running coupling corrections to the BFKL/BK/JIMWLK evolution have been calculated in \[24, 25\]. It led to an interesting result, in which the fixed coupling $\alpha_s$ in Eq. (6) was replaced by a “triumvirate” of the running couplings at different

![Figure 5: Nuclear modification factor as a function of $k_T/Q_s$ as predicted by saturation/CGC physics. Different curves correspond to different rapidities, with lower curves corresponding to higher rapidity.](image-url)
scales \[24, 25]\:

\[
\alpha_\mu \Rightarrow \frac{\alpha_s(\ldots) \alpha_s(\ldots)}{\alpha_s(\ldots)}.
\]  

(7)

Such behavior has never been seen in field theories outside small-\(x\) physics. Among other recent developments, next-to-leading-order BK equation has been found in \[26\], resulting in a rather complicated but useful expression.

There is hope that we have finally managed to significantly advance both the qualitative and quantitative understanding of QCD at high energies.

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