Abstract

For a compact Riemann surface $M$ of genus $g \geq 2$, we study the functional equations of the Selberg zeta functions attached with the Tate motives $f$. We prove that certain functional equations hold if and only if $f$ has the absolute automorphy.

Key Words: Selberg zeta functions, functional equations, Tate motives

AMS Subject Classifications: 11M06, 11M41, 11F72

Introduction

For a compact Riemann surface $M$ of genus $g \geq 2$ the standard Selberg zeta function $Z_M(s)$ is constructed as

$$Z_M(s) = \prod_{P \in \text{Prim}(M)} \prod_{n=0}^{\infty} \left(1 - N(P)^{-s-n}\right),$$

where Prim$(M)$ denotes the set of primitive closed geodesics and the norm $N(P)$ is defined by

$$N(P) = \exp(\text{length}(P)).$$

It has the functional equation under $s \to 1 - s$:

$$Z_M(1 - s) = Z_M(s) \exp \left(4 - 4g \int_{0}^{s-\frac{1}{2}} \pi t \tan(\pi t) dt \right).$$
This functional equation was proved by Selberg [6,7] and the following symmetric version was found later:
\[
\hat{Z}_M(1-s) = \hat{Z}_M(s),
\]
where
\[
\hat{Z}_M(s) = Z_M(s)\Gamma_M(s)
\]
with
\[
\Gamma_M(s) = (\Gamma_2(s)\Gamma_2(s+1))^{2g-2}.
\]
This double gamma function \(\Gamma_2(s)\) is the normalized one used in [3] and we will recall the proof of the symmetric functional equation for \(Z_M(s)\) in the text.

Now, the simple Euler product
\[
\zeta_M(s) = \prod_{P \in \text{Prim}(M)} (1 - N(P)\cdot P^{-s})^{-1} = \frac{Z_M(s+1)}{Z_M(s)}
\]
was also studied later and it is a more natural analog of the usual Euler product for the Riemann zeta function
\[
\zeta(s) = \prod_{p: \text{primes}} (1 - p^{-s})^{-1}.
\]
Especially the proof of the prime number theorem
\[
\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty)
\]
applied to \(\zeta_M(s)\) gives the prime geodesic theorem
\[
\pi_M(x) \sim \frac{x}{\log x} \quad (x \to \infty),
\]
where
\[
\pi_M(x) = \#\{P \in \text{Prim}(M) \mid N(P) \leq x\}.
\]

The functional equation of \(\zeta_M(s)\) has the following form:
\[
\zeta_M(-s) = \zeta_M(s)^{-1}(2 \sin(\pi s))^{4-4g}.
\]

In this paper we study the functional equations for \(\zeta_{M(f)}(s)\) with Tate motives \(f\). Here we define \(\zeta_{M(f)}(s)\) as
\[
\zeta_{M(f)}(s) = \prod_k \zeta_M(s - k)^{a(k)}
\]
for a Laurent polynomial
\[
f(x) = \sum_{k \in \mathbb{Z}} a(k)x^k \in \mathbb{Z}[x, x^{-1}].
\]
It may be suggestive to consider $x = T$ the Tate twist. Of course $\zeta_{M(1)}(s) = \zeta_M(s)$ in our notation.

We describe results on $\zeta_{M(f)}(s)$ only for “odd” $f$ here in Introduction. See the text concerning the “even” cases.

**Theorem** Let $M$ and $f$ be as above. For each integer $D$ the following conditions are equivalent.

1. $\zeta_{M(f)}(D - s) = \zeta_{M(f)}(s)$.
2. $f(x^{-1}) = -x^{-D} f(x)$.

**Remark** Condition (2) is called the *absolute automorphy* [5]. In the paper [5] the definition of absolute automorphic forms are described in a more general setting for any function $f$ on positive real numbers, and the theory of absolute zeta functions $\zeta_f(s)$ is developed, which are the automorphic $L$-functions constructed from the absolute automorphic forms $f$. It is in the framework of absolute mathematics [2,4].

For example, let $f(x) = (x - 1)^r$ for an odd integer $r \geq 1$. Then we see that

$$f(x^{-1}) = -x^{-r} f(x).$$

Hence Theorem gives the functional equation of $\zeta_{M(f)}(s)$ as

$$\zeta_{M(f)}(r - s) = \zeta_{M(f)}(s).$$

A remarkable point is that we need no “gamma factors” here. In the simplest case $r = 1$ we get the functional equation for

$$\zeta_{M(f)}(s) = \frac{\zeta_M(s - 1)}{\zeta_M(s)}$$

as

$$\zeta_{M(f)}(1 - s) = \zeta_{M(f)}(s).$$

We remark that the study of the functional equations for

$$Z_{M(f)}(s) = \prod_k Z_M(s - k)^{a(k)}$$

is quite similar.

We add a few more comments on $Z_{M(f)}(s)$. Let

$$f(x) = \sum_k a(k)x^k \in \mathbb{Z}[x, x^{-1}]$$

satisfying

$$f(x^{-1}) = Cx^{-D} f(x)$$
with \( C = \pm 1 \). Then
\[
Z_{M(f)}(s) = \prod_k Z_M(s - k)^{a(k)}
\]
has the functional equation
\[
Z_{M(f)}(D + 1 - s) = Z_{M(f)}(s)^C S_{M(f)}(s)^C,
\]
where
\[
S_{M(f)}(s) = \prod_k S_M(s - k)^{a(k)}
\]
with
\[
S_M(s) = \frac{\Gamma_M(s)}{\Gamma_M(1 - s)} = (S_2(s)S_2(s + D))^{2 - 2g}.
\]
Here
\[
S_2(s) = \frac{\Gamma_2(2 - s)}{\Gamma_2(s)}
\]
is the normalized double sine function of [3]. For example \( f(x) = x^{-1} - 1 \) \((C = -1, D = -1)\) gives the functional equation for
\[
Z_{M(f)}(s) = \frac{Z_M(s + 1)}{Z_M(s)} = \zeta_M(s)
\]
as
\[
\zeta_M(-s) = \zeta_M(s)^{-1}(2 \sin(\pi s))^{4 - 4g}
\]
where the result
\[
S_{M(f)}(s) = \frac{S_M(s + 1)}{S_M(s)} = \left( \frac{S_2(s + 2)}{S_2(s)} \right)^{2 - 2g} = (2 \sin(\pi s))^{4 - 4g}
\]
is used. Similarly we obtain the functional equation for
\[
Z_{M(f^2)}(s) = \frac{Z_M(s + 2)Z_M(s)}{Z_M(s + 1)} = \frac{\zeta_M(s + 1)}{\zeta_M(s)} = \zeta_M(f)(s)
\]
as
\[
\zeta_M(f)(-1 - s) = \zeta_M(f)(s)
\]
that is
\[
Z_{M(f^2)}(-1 - s) = Z_{M(f^2)}(s)
\]
with no gamma factors.
1 Selberg zeta functions

We describe the needed functional equations for $Z_M(s)$ and $\zeta_M(s)$ with simple proofs. Let $\Gamma_r(s)$ be the normalized gamma function of order $r$ defined by

$$\Gamma_r(s) = \exp \left( \frac{\partial}{\partial w} \zeta_r(w, s) \bigg|_{w=0} \right)$$

with the Hurwitz zeta function of order $r$

$$\zeta_r(w, s) = \sum_{n_1, \ldots, n_r \geq 0} (n_1 + \cdots + n_r + s)^{-w}.$$  

The normalized sine function $S_r(s)$ of order $r$ is constructed as

$$S_r(s) = \Gamma_r(s)^{-1} \Gamma_r(r-s)^{(-1)^r}.$$  

see [3] for detailed properties with proofs.

**Theorem 1.** (1) Let

$$\hat{Z}_M(s) = Z_M(s) \Gamma_M(s)$$

with

$$\Gamma_M(s) = (\Gamma_2(s) \Gamma_2(s+1))^{2g-2}.$$  

Then

$$\hat{Z}_M(1-s) = \hat{Z}_M(s).$$  

(2)

$$\zeta_M(-s) = \zeta_M(s)^{-1} (2 \sin(\pi s))^{4-4g}.$$  

**Proof.** (1) From the functional equation for $Z_M(s)$ due to Selberg [6,7]

$$Z_M(1-s) = Z_M(s) \exp \left( (4 - 4g) \int_{0}^{\pi/2} \pi t \tan(\pi t) dt \right)$$

we see that it is sufficient to show the identity

$$\exp \left( (4 - 4g) \int_{0}^{\pi/2} \pi t \tan(\pi t) dt \right) = \frac{\Gamma_M(s)}{\Gamma_M(1-s)}.$$  

We first show that

$$\exp \left( (4 - 4g) \int_{0}^{\pi/2} \pi t \tan(\pi t) dt \right) = (S_2(s) S_2(s+1))^{2-2g}. \quad (1.1)$$
Since both sides are equal to 1 at \( s = \frac{1}{2} \) (note that \( S_2(\frac{3}{2}) = \Gamma_2(\frac{3}{2})\Gamma_2(\frac{1}{2})^{-1} = S_2(\frac{1}{2})^{-1} \)), it suffices to show the coincidence of logarithmic derivatives. The left hand side becomes

\[
(4 - 4g)\pi \left( s - \frac{1}{2} \right) \tan \left( \pi \left( s - \frac{1}{2} \right) \right) = (2 - 2g)\pi (1 - 2s) \cot(\pi s).
\]

Concerning the right hand side, the differential equation

\[
S_2''(s) = \pi (1 - s) \cot(\pi s) S_2(s)
\]

proved in [3] gives

\[
(2 - 2g) \left( \frac{S_2'(s)}{S_2(s)} + \frac{S_2(s + 1)}{S_2(s + 1)} \right) = (2 - 2g) (\pi (1 - s) \cot(\pi s) + \pi (-s) \cot(\pi (s + 1))) = (2 - 2g)\pi (1 - 2s) \cot(\pi s).
\]

Thus we obtain (1.1).

Next from (1.1) we get

\[
\exp \left( (4 - 4g) \int_0^{s - \frac{1}{2}} \pi t \tan(\pi t) dt \right) = (S_2(s) S_2(s + 1))^{2 - 2g}
\]

\[
= \left( \frac{\Gamma_2(2 - s)}{\Gamma_2(s)} \cdot \frac{\Gamma_2(1 - s)}{\Gamma_2(s + 1)} \right)^{2 - 2g}
\]

\[
= \frac{(\Gamma_2(s) \Gamma_2(s + 1))^{2g - 2}}{(\Gamma_2(1 - s) \Gamma_2(2 - s))^{2g - 2}}
\]

\[
= \frac{\Gamma_M(s)}{\Gamma_M(1 - s)}.
\]

Hence we have the functional equation

\[
Z_M(1 - s) = Z_M(s) \frac{\Gamma_M(s)}{\Gamma_M(1 - s)}
\]

that is

\[
\hat{Z}_M(1 - s) = \hat{Z}_M(s)
\]

as desired.

(2) Since

\[
\zeta_M(s) = \frac{Z_M(s + 1)}{Z_M(s)}
\]

we have

\[
\zeta_M(-s)\zeta_M(s) = \frac{Z_M(1 - s)}{Z_M(-s)} \cdot \frac{Z_M(s + 1)}{Z_M(s)}
\]

\[
= \frac{Z_M(1 - s)}{Z_M(s - s)} \cdot \frac{Z_M(s + 1)}{Z_M(s + 1)}
\]

\[
= \frac{Z_M(1 - s)}{Z_M(s)} \cdot \frac{Z_M(s + 1)}{Z_M(-s)}
\]

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Hence (1) gives
\[
\zeta_M(-s)\zeta_M(s) = \frac{\Gamma_M(s)}{\Gamma_M(1-s)} \cdot \frac{\Gamma_M(-s)}{\Gamma_M(s+1)}
\]
\[
= (S_2(s)S_2(s+1))^{2-2g}(S_2(s+1)S_2(s+2))^{2g-2}
\]
\[
= \left(\frac{S_2(s+2)}{S_2(s)}\right)^{2g-2}.
\]

Recall the relations proved in [3]:
\[
S_2(s+2) = S_2(s+1)S_1(s+1)^{-1}
\]
\[
= S_2(s+1)(-2\sin(\pi s))^{-1}
\]
and
\[
S_2(s) = S_2(s+1)S_1(s)
\]
\[
= S_2(s+1)(2\sin(\pi s)).
\]
Thus we get the functional equation for \(\zeta_M(s)\):
\[
\zeta_M(-s)\zeta_M(s) = (2\sin(\pi s))^{4-4g}
\]
that is
\[
\zeta_M(-s) = \zeta_M(s)^{-1}(2\sin(\pi s))^{4-4g}.
\]

## 2 Functional equation for \(\zeta_{M(f)}(s)\)

Let
\[
\zeta_{M(f)}(s) = \prod_k \zeta_M(s-k)^{a(k)}
\]
for
\[
f(x) = \sum_k a(k)x^k \in \mathbb{Z}[x, x^{-1}].
\]

We prove the following theorem.

**Theorem 2.** For each integer \(D\) the following conditions are equivalent:

1. \(\zeta_{M(f)}(D-s) = \zeta_{M(f)}(s)\).
2. \(f(x^{-1}) = -x^{-D}f(x)\).
3. \(a(D-k) = -a(k)\) for all \(k\).
Proof. We first show the equivalence (2) $\iff$ (3). Let
\[
f(x) = \sum_k a(k)x^k.
\]
Then
\[
x^D f(x^{-1}) = \sum_k a(k)x^{D-k} = \sum_k a(D-k)x^k,
\]
where we needed the exchange $k \longleftrightarrow D-k$. Hence
\[
x^D f(x^{-1}) = -f(x)
\]
is equivalent to
\[
a(D-k) = -a(k) \quad \text{for all } k.
\]
Next we show the equivalence (1) $\iff$ (2). Since
\[
\zeta_{M(f)}(D-s) = \prod_k \zeta_M((D-s) - k)^{a(k)}
\]
\[
= \prod_k \zeta_M((D-k) - s)^{a(k)}
\]
\[
= \prod_k \zeta_M(k - s)^{a(D-k)},
\]
the functional equation for $\zeta_M(s)$ gives
\[
\zeta_{M(f)}(D-s) = \prod_k (\zeta_M(s-k)^{-1}(2\sin(\pi s))^{4-4g})^{a(D-k)}
\]
\[
= \left( \prod_k \zeta_M(s-k)^{-a(D-k)} \right)^{(2\sin(\pi s))^{(4-4g)f(1)}},
\]
where we used
\[
f(1) = \sum_k a(k) = \sum_k a(D-k).
\]
Hence we have the following expression
\[
\frac{\zeta_{M(f)}(D-s)}{\zeta_{M(f)}(s)} = \left( \prod_k \zeta_M(s-k)^{-a(D-k)-a(k)} \right)^{(2\sin(\pi s))^{(4-4g)f(1)}}. \tag{2.1}
\]
From this expression the equivalence (1) $\iff$ (3) is shown as follows. First the condition (3) (or equivalently (2)) implies $f(1) = 0$ and that $a(D-k) + a(k) = 0$ for all $k$. Hence (2.1) gives
\[
\frac{\zeta_{M(f)}(D-s)}{\zeta_{M(f)}(s)} = 1,
\]
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which is (1).

Now assume (1). Then from (2.1) we have the identity
\[
\prod_k \zeta_M(s-k)^{a(D-k)+a(k)} = (2\sin(\pi s))^{(4-4g)}f(1).
\] (2.2)

Since \(\zeta_M(s)\) is non-zero holomorphic in \(\text{Re}(s) > 1\), the left hand side of (2.2) is non-zero holomorphic for sufficiently large \(\text{Re}(s)\). Hence looking at the left hand side at sufficiently large \(s \in \mathbb{Z}\) we see \(f(1) = 0\). Then (2.2) gives
\[
\prod_k \zeta_M(s-k)^{a(D-k)+a(k)} = 1.
\] (2.3)

We remark that (2.3) is actually written as
\[
\prod_{k \leq K} \zeta_M(s-k)^{a(D-k)+a(k)} = 1
\] (2.4)

for some \(K \in \mathbb{Z}\), since \(f(x) \in \mathbb{Z}[x, x^{-1}]\). Hence we have the identity
\[
\zeta_M(s-K)^{a(D-K)+a(K)} = \prod_{k<K} \zeta_M(s-k)^{-a(D-k)-a(k)}.
\] (2.5)

Look at (2.5) at \(s = K + 1\), then the right hand side is
\[
\prod_{k<K} \zeta_M(1 + (K-k))^{-a(D-k)-a(k)},
\]
which is a finite non-zero value. Hence looking at the left hand side of (2.5) we see that
\[
a(D-K) + a(K) = 0
\]
since \(\zeta_M(s-K)\) has a simple pole at \(s = K + 1\). Thus (2.5) becomes
\[
\prod_{k \leq K-1} \zeta_M(s-k)^{a(D-k)+a(k)} = 1
\] (2.6)

Inductively we see (3). \(\square\)

Theorem 2 treated “odd” \(f\). The next theorem deals with the other case for “even” \(f\).

**Theorem 3.** For each integer \(D\) the following conditions are equivalent:

1. \(\zeta_M(df)(D-s) = \zeta_M(df)(s)^{-1}(2\sin(\pi s))^{(4-4g)}f(1)\).
2. \(f(x^{-1}) = x^{-D}f(x)\).
3. \(a(D-k) = a(k)\) for all \(k\).
Proof. The equivalence (2) \(\iff\) (3) is shown exactly as in the proof of Theorem 2. Now we show (1) \(\iff\) (3). Notice that

\[
\zeta_M(f)(D - s) = \prod_k \zeta_M((D - s) - k)^{a(k)}
\]

\[
= \prod_k \zeta_M((D - k) - s)^{a(k)}
\]

\[
= \prod_k \zeta_M(k - s)^{a(D - k)}
\]

\[
= \prod_k (\zeta_M(s - k)^{-1}(2\sin(\pi s))^{4-4g})^{a(D - k)}
\]

\[
= \left(\prod_k \zeta_M(s - k)^{-a(D - k)}\right)(2\sin(\pi s))^{(4-4g)f(1)},
\]

where we used that

\[
\sum_k a(D - k) = f(1).
\]

Proof of (3) \(\implies\) (1). From (3) we have

\[
\zeta_M(f)(D - s) = \left(\prod_k \zeta_M(s - k)^{-a(k)}\right)(2\sin(\pi s))^{(4-4g)f(1)}
\]

\[
= \zeta_M(f)(s)^{-1}(2\sin(\pi s))^{(4-4g)f(1)},
\]

which is (1).

Proof of (1) \(\implies\) (3). Since

\[
\zeta_M(f)(D - s) = \left(\prod_k \zeta_M(s - k)^{-a(D - k)}\right)(2\sin(\pi s))^{(4-4g)f(1)}
\]

as above, we have

\[
\frac{\zeta_M(f)(D - s)}{\zeta_M(f)(s)^{-1}(2\sin(\pi s))^{(4-4g)f(1)}} = \prod_k \zeta_M(s - k)^{a(k) - a(D - k)}.
\]

Hence from the assumption (1) we get

\[
\prod_k \zeta_M(s - k)^{a(k) - a(D - k)} = 1,
\]

which can be written as

\[
\prod_{k \leq K} \zeta_M(s - k)^{a(k) - a(D - k)} = 1
\]
that is
\[ \zeta_M(s - K)^{a(D - K)} = \prod_{k < K} \zeta_M(s - k)^{a(D - k) - a(k)}. \]

Then we obtain \( a(D - K) = a(K) \) and inductively \( a(D - k) = a(k) \) for all \( k \) exactly as in the proof of Theorem 2.

**Example.** Let \( f(x) = (x - 1)^r \) for an even integer \( r \geq 0 \). Then we see that
\[ f(x^{-1}) = x^{-r} f(x). \]

Hence we obtain the functional equation
\[ \zeta_{M(f)}(r - s) = \zeta_{M(f)}(s)^{-1} \times \begin{cases} (2 \sin(\pi s))^{4g} & (r = 0), \\ 1 & (r \geq 2, \text{ even}). \end{cases} \]

Of course the \( r = 0 \) case gives the functional equation of \( \zeta_M(s) \).

**Remark.** Let \( f(x) = (x - 1)^r \) for an integer \( r \geq 0 \). Then \( \zeta_M(s) \) is written explicitly as
\[ \zeta_{M(f)}(s) = \prod_{k=0}^{r} \zeta_M(s - k)^{(-1)^r - k}(\zeta). \]

In this case another suggestive notation would be
\[ \zeta_{M(f)}(s) = \zeta_{M \otimes \mathbb{G}_m}^r(s) \]

since \((x - 1)^r\) is the counting function of \( \mathbb{G}_m^r \); see [1,2,4].

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