Revisiting arithmetic solutions to the $W = 0$ condition

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Abstract

The gravitino mass is expected not to be much smaller than the Planck scale for a large fraction of vacua in flux compactifications. There is no continuous parameter to tune even by hand, and it seems that the gravitino mass can be small only as a result of accidental cancellation among period integrals weighted by integer-valued flux quanta. DeWolfe et.al. (2005) proposed to pay close attention to vacua where the Hodge decomposition is possible within a number field, so that the precise cancellation takes place as a result of algebra. We focus on a subclass of those vacua—which with complex multiplications—and explore more on the idea in this article. It turns out, in Type IIB compactifications, that those vacua admit non-trivial supersymmetric flux configurations if and only if the reflex field of the Weil intermediate Jacobian is isomorphic to the quadratic imaginary field generated by the axidilaton vacuum expectation value. We also found that flux statistics is highly enriched on such vacua, as F-term conditions become linearly dependent.
1 Introduction

It has been known for more than a decade [1] that it is difficult to achieve both i) small cosmological constant and ii) natural supersymmetric unification under iii) the current understanding of string compactification. The supersymmetric unification scenarios prefer that the Kaluza–Klein scale of the internal geometry is not comparable to the string scale, but slightly lower [2], and then in such a geometric phase, there is no good reason for the vacuum expectation value (vev) of the superpotential $\langle W \rangle$ to be much smaller than $O(1)$ in Planck units. That would immediately imply large gravitino mass, and also large supersymmetry breaking either in the F-term or D-term for the cosmological constant to be small.\(^1\)

To be more specific, consider Type IIB string theory compactified on a Calabi–Yau three-fold $X$ with Ramond–Ramond (RR) and NS–NS three-form fluxes $F^{(3)}$ and $H^{(3)}$. The superpotential and Kähler potential of the effective theory in 3+1-dimensions are given by [4]

$$W = \frac{M_{Pl}^3}{\sqrt{4\pi}} \int_X G \wedge \Omega, \quad K = \frac{M_{Pl}^2}{g_s} \simeq -2 \ln \left( \frac{\text{vol}(X)}{g_s^{3/2} \ell_s^6} \right) - \ln \left( i \int_X \Omega \wedge \overline{\Omega} \right) - \ln(-i(\phi - \phi^*)), \quad (1)$$

where $\phi$ is the dilaton chiral multiplet, $\text{Im}(\langle \phi \rangle) = g_s^{-1}$, $G := F^{(3)} - \phi H^{(3)}$, and the three-form $\Omega(z)$ on $X$ depends on the complex structure moduli chiral multiplets denoted collectively by $z$. Everything in $\int_X G \wedge \Omega$ has been made dimensionless in the convention adopted in (1); once an integral basis $\{e_{i=1,\cdots,b_3(X)}\}$ of $H^3(X; \mathbb{Z})$ is chosen, then the fluxes are parametrized by integers (flux quanta) $\{n_{i=1,\cdots,b_3}\}$ and $\{m_{i=1,\cdots,b_3}\}$, where

$$G = \sum_{i=1}^{b_3} (n^i - \phi m^i) e_i; \quad (2)$$

period integrals

$$\Pi_i(z) = \int_{\gamma_i} \Omega(z) \quad (3)$$

are evaluated over the three-cycles $\gamma_i$ that are Poincaré dual to $e_i$; we therefore have a dimensionless combination $\int_X G \wedge \Omega(z) = \sum_i (n^i - \phi m^i) \Pi_i$. The gravitino mass $m_{3/2}$ is then

\(^1\)When the Kähler potential and superpotential of the 4D effective theory is in the no-scale scenario, this argument for the F-term/D-term supersymmetry breaking is not applied. Also the anomaly mediated contributions to gauginos cancel, when the Kähler potential is in the sequestered form in addition (e.g., [3]), so that the gravitino mass can be much larger than the gaugino mass. When the deviations from those assumptions are small, then the phenomenological requirement on the vev of $\int_X G \wedge \Omega$ is relaxed by that amount.
given by
\[ \frac{m_{3/2}}{M_{\text{Pl}}} = e^{\frac{2\pi M_{\text{Pl}}}{M_{\text{Pl}}}} \frac{W}{M_{\text{Pl}}} = \pi^{1/4} \left( \frac{M_6}{M_{\text{Pl}}} \right)^{3/2} \frac{\int_X \langle G \wedge \Omega \rangle}{\sqrt{\int_X i \langle \Omega \wedge \Omega \rangle}} (g_s)^{1/2}, \] (4)

where \( M_{\text{Pl}} \simeq 2.4 \times 10^{18} \text{ GeV} \) is the reduced Planck scale, and \( M_6 := 1/R_6 := (\langle \text{vol}(X) \rangle)^{-1/6}. \)

When we think of supersymmetric unification scenarios where the Standard Model gauge groups originate from 7-branes wrapped on a 4-cycle with volume \( \langle \text{vol}(D7) \rangle := R_G^4 \), the three observable parameters\(^2\) \( M_{\text{Pl}}, M_{\text{GUT}} \) and \( \alpha_{\text{GUT}} \) are expressed in terms of three microscopic parameters of compactifications, \( (g_s^4, R_6) \) and \( R_G \), which means that the factor \( M_6/M_{\text{Pl}} \) can be expressed purely in terms of observable parameters (e.g., [5]).

\[ \pi^{1/4} \left( \frac{M_6}{M_{\text{Pl}}} \right)^{3/2} \simeq \frac{8.3 \times 10^{-4}}{c^2} \left( \frac{M_{\text{GUT}}}{2 \times 10^{16} \text{ GeV}} \right)^2 \left( \frac{1/24}{\alpha_{\text{GUT}}} \right)^{1/2}; \] (5)

c is a geometry-dependent factor of order unity in the relation \( M_{\text{GUT}} = c/R_G \). Since the value of \( g_s \) cannot be arbitrarily small even in Type IIB compactifications,\(^3\) the vev of \( \int_X G \wedge \Omega \) needs to be small for the gravitino mass to be smaller than, say, \( 10^{-6} \times M_{\text{Pl}} \).

Once a set of flux quanta \( \{n^i, m^i\} \) is specified, then the vev of \( W \propto \int_X G \wedge \Omega \) is determined. Since the flux quanta can change only by an integer, it is not possible to tune the vev of \( \int_X G \wedge \Omega \) continuously to a small value. At best we can hope that the \( b_3 \) contributions \( \sum_{i=1}^{b_3} (n^i - \phi m^i) \Pi_i = \int_X G \wedge \Omega \) almost cancel one another accidentally. When \( b_3 \) complex numbers of order unity are generated randomly, and are summed up, the probability that the absolute value of the sum is \( \epsilon \) or less is \( \mathcal{O}(\epsilon^2) \), which is small,\(^4\) but still non-zero and positive. In a large set of string flux vacua, therefore, there may still be a choice of flux quanta \( \{n^i, m^i\} \) where \( \langle \int_X G \wedge \Omega \rangle \) just happens to be small, and our local universe just happens to be described by such a flux configuration. Certainly such a story cannot be ruled out, but one can hardly say that it is a natural scenario of moduli stabilization behind supersymmetric unification. This is a problem that has been known for more than a decade [1]. While the absence of positive support in the LHC data at 13TeV does not kill low-energy

\(^2\) \( M_{\text{GUT}} \) is the energy scale of gauge coupling unification, and \( \alpha_{\text{GUT}} \) the value of the unified gauge coupling constant.

\(^3\) If the SU(5) symmetry breaking is due to a line bundle in the geometric phase, \( 1 \lesssim (R_G/\ell_s)^4 = g_s/\alpha_{\text{GUT}} \). The lower bound on \( g_s \) is relaxed by \( (2\pi)^{-4} \), if we require \( 1 \lesssim R_G/\sqrt{\alpha'}. \) The other geometric phase condition \( 1 \lesssim (R_6/\ell_s) \), or \( 1 \lesssim R_6/\sqrt{\alpha'}. \) respectively, is slightly weaker. In F-theory, where the up-type quark Yukawa couplings are generated from \( E_6 \) algebra [6], the value of \( g_s \) is not an independent parameter.

\(^4\) For \( m_{3/2} \sim 10^3 \text{ TeV}, (g_s^4/\epsilon) \sim 10^{-9} \) is necessary.
supersymmetry scenarios immediately, this decade-old problem alone is enough to suggest that we are supposed to abandon either ii) natural supersymmetric unification or iii) the current understanding of flux compactifications of string theory. If we are to keep ii), some drastic idea is necessary in modifying the current understanding of flux compactification.

Incidentally, there are some pioneering papers seeking for chances of arithmetics to play some roles in string theory (for examples, [7–15]). DeWolfe et.al. [14] in particular (see also [13]), introduced an idea that arithmetics of complex structure moduli vev plays some role in getting the vev\(^5\) \(\langle W \rangle \propto \int_X \langle G \wedge \Omega \rangle = 0\). The idea is to focus on a subset \(\mathcal{M}_{\text{alg}} \subset \mathcal{M}_{\text{cpx}} \times \mathcal{M}_{\text{dil.}}\) that of the dilaton chiral multiplet. \(\mathcal{M}_{\text{alg}} = \mathcal{M}_{\text{cpx}} \times \mathcal{M}_{\text{dil.}}\) is defined to be the set of points in \(\mathcal{M}_{\text{cpx}} \times \mathcal{M}_{\text{dil.}}\) where all of \(\langle \phi \rangle \in H_{g=1}\), \(\langle \Pi_i \rangle\) and \(\langle D_a \Pi_i \rangle\) for \(a = 1, \cdots, h^{2,1}(X)\) and \(i = 1, \cdots, b_3(X)\) take values in some number field \(K_{\text{tot}}\). For a given set of vevs \(\langle z, \phi \rangle\), the \(\langle W \rangle = 0\) condition can be regarded as an extra linear condition on flux quanta. If \(\langle z, \phi \rangle \notin \mathcal{M}_{\text{alg}}\), only the trivial flux configuration satisfies the \(\langle W \rangle = 0\) condition; for non-trivial fluxes, \(\langle W \rangle \neq 0\) without an accidental fine cancellation as discussed earlier. For an algebraic case \(\langle z, \phi \rangle \in \mathcal{M}_{\text{alg}}\), however, the \(\sum_{i=1}^{b_3}(n^i - \phi m^i)\Pi_i = 0 \in K_{\text{tot}}\) condition can be regarded as a finite number of \(\mathbb{Q}\)-coefficient linear conditions on \(n^i\)'s and \(m^i\)'s; this is because \(K_{\text{tot}}\) is \(d_{K_{\text{tot}}} := [K_{\text{tot}} : \mathbb{Q}]\)-dimensional vector space over \(\mathbb{Q}\). When the space of fluxes satisfying the F-term condition at \(\langle z, \phi \rangle\) forms a lattice \(\mathbb{Z}^\kappa\), the \(\langle W \rangle = 0\) condition is therefore satisfied by fluxes in a sublattice \(\mathbb{Z}^{\kappa-d_{K_{\text{tot}}}}\) [14]. The F-term conditions can also be regarded as \(d_{K_{\text{tot}}} \times (h^{2,1} + 1)\) of \(\mathbb{Q}\)-coefficient linear conditions on the flux quanta in this case, and the lattice \(\mathbb{Z}^\kappa\) has a rank [14]

\[
\kappa = 2b_3 - d_{K_{\text{tot}}} \times (b_3/2).
\]

The rank \(\kappa_0\) of the \(W = 0\) sublattice is then given by

\[
\kappa_0 = \kappa - d_{K_{\text{tot}}} = 2b_3 - d_{K_{\text{tot}}} \times (b_3/2) - d_{K_{\text{tot}}}.
\]

There is no top-down justification for not thinking of string vacua outside \(\mathcal{M}_{\text{alg}}\). While there is higher chance of finding a flux vacuum with \(\langle W \rangle = 0\) for \(\langle z, \phi \rangle \in \mathcal{M}_{\text{alg}}\), it appears

\(^5\)So far as the Kähler moduli are stabilized by non-perturbative effects, this \(\int_X \langle G \wedge \Omega \rangle\) is by far the dominant contribution to \(\langle W \rangle\). Although \(\langle W \rangle\) does not have to be strictly zero phenomenologically, \(\langle W \rangle\) will be precisely zero in the approximation of ignoring volume stabilization (non-perturbative effects) and supersymmetry breaking, if there is any natural solution to small \(\langle W \rangle\).
to be a sheer fine tuning in the current understanding of string compactification to focus on $M_{\text{alg}}$ in the first place. A perspective behind the arithmetic idea above is to rely more on bottom-up clues i) and ii) at the beginning of introduction, and to believe that the current understanding of string compactification overlooks something; future development of string theory will justify focusing on $M_{\text{alg}}$ or even on its smaller subset. Whether such a strategy is fruitful is yet to be seen. For now, the authors of this article intend to elaborate more on the arithmetic idea for the $\langle W \rangle = 0$ problem, and prepare for further developments in the future.

In this article, we explore two questions associated with this idea. The first question is what the subset $M_{\text{alg}}^X$ is like in $M_{\text{cpx}}^X$, and what the number field $K_{\text{tot}}^X$ is like there. The second question is how the estimation (6, 7) of the rank $\kappa$ and $\kappa_0$ of supersymmetric flux quanta is at work. It appears from (6, 7) that there is virtually no chance for a non-trivial supersymmetric flux to exist ($\kappa_0 > 0$) when the number field $K_{\text{tot}}$ is an extension over $\mathbb{Q}$ of degree-4 or higher; in fact, the authors of [14] discovered that there are some cases hinting that truth is stranger than the naive estimation (6, 7). We focus on a subset $M_{\text{CM}}$ of $M_{\text{alg}}$, where plenty of math literature is available, and address the two questions above.

This article is organized as follows. Sections 2–3.1 provide an update of what is being understood about CM points $M_{\text{CM}}$, which is a subset of $M_{\text{alg}}$ that has particularly good properties from the perspective of arithmetic geometry, so that we have a better feeling of what the subset $M_{\text{CM}}$ of $M_{\text{cpx}}$ is like. In sections 3.2 and 3.3, we pick up a family of (possibly) infinitely many CM points in the form of Borcea–Voisin Calabi–Yau threefolds, and study conditions of existence of non-trivial supersymmetric flux configurations. The analysis is also generalized for CM-type Calabi–Yau threefolds that are not Borcea–Voisin type in section 4. The formula for $\kappa$ and $\kappa_0$ for the CM points turn out to be quite different from the estimate (6, 7) for generic points in $M_{\text{alg}}$; conditions for CM points to have $\kappa > 0$ or $\kappa_0 > 0$ are stated in terms of reflex fields, which is also a concept important in arithmetic geometry. Discussions in section 5 include important observations that are not contained in the earlier sections.

In this preprint, the appendices A and B provide a quick overview of mathematical facts in field theory, Hodge theory and complex multiplications, on which the main text is based. We did not try to make this article fully self-contained, however. Such materials as class field

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$K_{\text{tot}}^X$ is the number field generated by the vev of $\Pi_i$’s and $D_a \Pi_i$’s over $\mathbb{Q}$; $K_{\text{tot}} = K_{\text{tot}}^X(\phi)$.

Applying the same logic as in (6, 7) to F-theory flux compactification, we would have $\kappa = [2(h^{3,1} + 1) + h_{H}^{2,2}] - d_{K_{\text{tot}}} \times h^{3,1}$ and $\kappa_0 = \kappa - d_{K_{\text{tot}}}$. Since $h_{H}^{2,2}$ scales as $4h^{3,1} + \text{const.}$ for large $h^{3,1}$ [16–18], cases with $d_{K_{\text{tot}}} = [K_{\text{tot}} : \mathbb{Q}] \leq 6$ do not have to be ruled out in this line of reasoning.
theory, Shimura variety, Mumford–Tate group, field of definition and more, are necessary in appreciating the result of our study in sections 3–5, but not so much in following up the analysis; for this reason, we just provide references for those materials, and decided not to provide explanations or sometimes even definitions of jargons in this article.

**Note added:** The Phys. Rev. D version of this article has adopted a style of presentation that will be more friendly to physicists. Following the suggestion from the referee, we also have inserted a brief discussion on orientifold projection as section 3.3 in the journal version. On the other hand, some of the materials in this preprint version, the entire section 2.2 and part of the appendix B.1, have been dropped from the journal version; they are mostly concerned about the first question mentioned above, which involves too heavy mathematics for Phys. Rev. D.

## 2 Math Background

### 2.1 CM Points and Arithmetics

Let us first focus on the complex structure moduli space $M_{X_{cpx}}$ of a family of Calabi–Yau threefolds; a member of this family—a Calabi–Yau threefold—is denoted by $X$. Whenever we refer to “a Calabi–Yau threefold” in this article, it is implied that a specific complex structure (a point in $M_{X_{cpx}}$) is chosen already. The complex structure of a Calabi–Yau threefold $X$ introduces a Hodge decomposition on the vector space $H^3(X; \mathbb{Q})$ over $\mathbb{Q}$. $H^{3,0}(X; \mathbb{C})$ is the one-dimensional subspace of $H^3(X; \mathbb{C})$ generated by $\Omega$, and $H^{0,3}(X; \mathbb{C})$ by $\overline{\Omega}$; let \{\sum_{a=1,\ldots,h^21}\} ⊂ H^3(X; \mathbb{Q}) ⊗ \mathbb{C}$ be a basis of $H^{2,1}(X; \mathbb{C})$. Overall,

$$\left\{\Omega, \sum_{a=1,\ldots,h^21}, \sum_{a=1,\ldots,h^21}, \overline{\Omega}\right\}$$

forms a $\mathbb{C}$-basis of the vector space $H^3(X; \mathbb{Q}) \otimes \mathbb{C}$. Here, $\dim_{\mathbb{C}}(M_{X_{cpx}}) = h^{2,1}(X)$, and $b_3 = 2(h^{2,1}(X) + 1)$. When we say that all the complex numbers $D_a\Pi_i$ with $a = 1, \ldots, h^{2,1}$ and $i = 1, \ldots, b_3$ take values in a number field $K_{tot}$, we mean by this condition that there exists a $\mathbb{C}$-basis \{\sum_{a=1,\ldots,h^21}\} of $H^{2,1}$ so that $\langle \gamma_i, \sum_a \rangle =: \sum_{ai} \in K_{tot}$.

It is a highly non-trivial condition that a number field $K_{tot}$ exists so that all the complex numbers $\Pi_i$, $\sum_{ai}$, $\overline{\sum_{ai}}$ and $\overline{\Pi}_i$ are contained within $K_{tot}$. In the large complex structure moduli
region, for example, the components of $\Omega$ are in the form

$$ (\Pi_i)^T = (1, t^a, \partial_t \mathcal{F}, 2 \mathcal{F} - t^c \partial_t \mathcal{F})^T, $$

where $t^a$'s ($a = 1, \ldots, h^{2,1}$) are a set of local coordinates on $\mathcal{M}_{cpx}^X$ and $\kappa_{abc}, n_{ab}, c_a, \chi(X^c)/2$ are known to be integers, and $n_\beta \in \mathbb{Q}$. It is hopeless to try to argue whether a complex number given by a series expansion is algebraic or transcendental. When we require that $\Sigma_{ai}$'s are also algebraic, we have little idea how to find out systematically where in $\mathcal{M}_{cpx}^X$ the conditions are satisfied.

The Gepner point\(^8\) in the $h^{2,1} = 101$-dimensional family of quintic Calabi–Yau threefolds is known to be in $\mathcal{M}_{alg}^X$ (e.g., [19, 20]), and so is the Gepner point in the $h^{2,1} = 1$-dimensional family of the mirror quintics. The corresponding number fields $K_{tot}^X$ are cyclotomic fields. Complex structures of other Gepner models are also expected to have this property (cf. [21]).

Given the enormous variety in the topology of Calabi–Yau threefolds, however, collection of one point from $\mathcal{M}_{cpx}^X$ from some choices of topology appears to be a very small subset of all the possible complex structures in Type IIB compactifications. Another class of examples is to construct $X$ through an orbifold of a product of three elliptic curves $E_i = \mathbb{C}/\mathbb{Z} \times \mathbb{Z}$, each one of which has a period $\tau_i$ that is algebraic [14]. How far can we go beyond this list?

Math literatures available at this moment do not seem to say much about $\mathcal{M}_{alg}^X$. There is, however, a subset of $\mathcal{M}_{alg}^X$ that has long attracted interest of mathematicians for its significance in number theory. It is the set of CM points in $\mathcal{M}_{cpx}^X$, which we review shortly, and there is plenty of math literatures available, so that there is a better hope to understand this subset—denoted by $\mathcal{M}_{CM}^X$—systematically. It is also expected in string theory that all the rational conformal field theories with $(c, \tilde{c}) = (9, 9)$ central charges and $\mathcal{N} = (2, 2)$ supersymmetry on world-sheet correspond to the points in $\mathcal{M}_{CM}^X$ [12]. While the bottom-up idea for the $W = 0$ problem does not motivate to take the Kähler moduli vev also to be of CM-type, it would not sound too stupid to expect that better understanding on string theory in the future indicates that we must use rational CFT’s for a consistent string compactifications; this would not only justify to choose the complex structure vevs to be of CM-type, but also predict that the Kähler moduli vevs also are.

\(^8\)Gepner points in moduli spaces of Calabi–Yau compactifications are specific points both in the Kähler and complex structure moduli spaces. Because we only refer to complex structure of Calabi–Yau’s in this article, we use an expression “a Gepner point in $\mathcal{M}_{cpx}^X$” for what is actually be “a point in $\mathcal{M}_{cpx}^X$ on which a Gepner model is projected to.”
An elliptic curve of CM-type is an elliptic curve $E_\tau$ whose complex structure parameter $\tau$ in the upper complex half plane $\mathcal{H}_{g-1}$ is a solution to a non-trivial $\mathbb{Q}$-coefficient quadratic polynomial equation

$$a\tau^2 + b\tau + c = 0, \quad a, b, c \in \mathbb{Z}. \quad (11)$$

The collection of such $\tau$’s in $\mathcal{H}_{g-1}$, denoted by $\mathcal{M}_{\text{CM}}^{\text{ell}}$, is a very small subset of $\mathcal{M}_{\text{alg}}^{\text{ell}} = \mathcal{H}_{g-1} \cap \mathbb{Q}$ because $\tau \in \mathcal{M}_{\text{alg}}^{\text{ell}}$ can be a solution to a $\mathbb{Q}$-coefficient polynomial of any degree, not necessarily quadratic. The subset $\mathcal{M}_{\text{CM}}^{\text{ell}}$ of $\mathcal{M}_{\text{alg}}^{\text{ell}}$ has a very nice property, however; $j(\tau)$ is algebraic, and hence one can choose all the coefficients of the defining equation of $E_\tau$ to be algebraic numbers (e.g., III.1.4, [22]):

$$y^2 + xy = x^3 - \frac{36}{j(\tau) - 1728} x - \frac{1}{j(\tau) - 1728}. \quad (12)$$

This means that $E_\tau$ has a model over a number field $K = \mathbb{Q}(\tau, j(\tau))$. The converse is also known to be true (Thm. IIc, [23]): when $E_\tau$ is defined over a number field, and $\tau \in \mathcal{M}_{\text{alg}}^{\text{ell}}$, then $\tau \in \mathcal{M}_{\text{CM}}^{\text{ell}}$. Elliptic curves with complex multiplication can therefore be characterized as those where both $\tau$ and $j(\tau)$ are algebraic.

This observation is not limited to the case of elliptic curves. The definition of CM type on complex structure has been generalized from elliptic curves to Abelian varieties of higher dimensions, K3 surfaces and Calabi–Yau threefolds (see the appendix B.1 for a brief review, or references therein for more). At least it is now known, when $X$ is either an Abelian variety or a K3 surface in $\mathcal{M}_{\text{alg}}$, that $X$ with a complex structure of CM-type is defined over a number field [24–27], and the converse is also true [28–30].

When a $d$-dimensional variety $X$ is defined over an algebraic number field, the Hasse–Weil $L$-function is defined in association with its $H^d(X)$, and even the zeta function $\zeta(X, s)$ is for $X$. When the Hodge structure on $H^d(X)$ is of CM-type, furthermore, it is expected that the $L$-function factorizes into a product of $L$-functions each one of which is associated with an embedding of the CM endomorphism field of $H^d(X)$ into $\mathbb{C}$. While it is not obvious whether the zeta function $\zeta(X, s)$ and its factorization property have a role to play in string theory, it will not be too surprising even if they are relevant to consistency of flux compactification of string theory.

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9 For some Calabi–Yau threefolds, see [31, 32].

10 See (II.10, [33]) for the case of elliptic curves, (§19, [25]) for Abelian varieties, (§1, [26]) for K3 surfaces, and [34] for higher dimensions. See also [9, 10].

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2.2 André–Oort Conjecture and Coleman–Oort Conjecture

There are infinitely many CM points in the moduli space $M_{\text{cpx}} = \text{SL}(2,\mathbb{Z}) \backslash \mathcal{H}_{g=1}$ of elliptic curves. Any CM field $K$ with $[K : \mathbb{Q}] = 2$ (i.e., a quadratic imaginary field) appears as the endomorphism field at infinitely many CM points. There are also infinitely many CM points for a given CM type $(F, \Phi_F)$ with $[F : \mathbb{Q}] = 2g$ in the moduli space $M_{\text{cpx}}^{4g} = \text{Sp}(2g;\mathbb{Z}) \backslash \mathcal{H}_g$ of Abelian varieties with complex $g$-dimensions. An enormous literature exists on this subject, because of its relevance to the class field theory \cite{35–38}. There are also infinitely many CM-type K3 surfaces, with infinitely many variations for a given CM field $K$ so far as $[K : \mathbb{Q}] \leq 20$; much less seems to be known, though, about the list of CM fields available in the period domain $D(T_0)$ of a given even lattice $T_0$ with signature $(2,2n - 2)$ (cf. the appendix B.1 and references therein).

Those CM points arising in an infinite series are packed into a Shimura variety $\text{Sh}(MT, h)$ associated with its Mumford–Tate group $MT$; a homomorphism $\tilde{h}$ in (67) just needs to be chosen from one of those infinitely many CM points. In the case of Abelian varieties, it is given by (e.g., Rmk. 3.5, \cite{40})

$$MT = N_{\Phi'}(\text{Res}_{K'/\mathbb{Q}}(\mathbb{G}_m)) = N_{\Phi'}((K^r)^\times), \quad (13)$$

whereas $MT = K^\times = \text{Res}_{K/\mathbb{Q}}(\mathbb{G}_m)$ in the case of K3 surfaces. The Mumford–Tate group is implemented as an algebraic subgroup over $\mathbb{Q}$ of $\mathbb{G}\text{Sp}(2g)$ [resp. $\mathbb{G}\text{O}(2,2n - 2)$] in the case of Abelian varieties [resp. K3 surfaces], and such an implementation $(K^r)^\times \to MT \subset \mathbb{G}\text{Sp}(2g)$ [resp. $(K)^\times \cong MT \subset \mathbb{G}\text{O}(2,2n - 2)$] maps the Shimura variety of CM points into the moduli space of Abelian varieties [resp. K3 surfaces] through

$$\text{Sh}(MT, h) \hookrightarrow \text{Sh}(\mathbb{G}\text{Sp}(2g), \mathcal{H}_g) \to \text{Sp}(2g;\mathbb{Z}) \backslash \mathcal{H}_g = M_{\text{cpx}}^{4g}, \quad (14)$$

$$\text{Sh}(MT, \tilde{h}) \hookrightarrow \text{Sh}(\mathbb{G}\text{O}(2,2n - 2), D(T_0)) \to \text{Isom}(T_0) \backslash D(T_0) = M^{K3(T_0)}_{\text{cpx}}, \quad (15)$$

11The authors found \cite{39,40} and \cite{41} useful in learning Shimura variety, connected Shimura variety, and Mumford–Tate group, respectively.

12 For an extension field $E$ over $F$, $\text{Res}_{E/F}(\mathbb{G}_m)$ is an Abelian group identical to $E^\times = E \backslash \{0\}$. With the notation $\text{Res}_{E/F}(\mathbb{G}_m)$, however, one emphasizes that this group is being regarded as an algebraic group defined over $F$.

13The group $\mathbb{G}\text{Sp}(2g)$ [resp. $\mathbb{G}\text{O}(2,2n - 2)$] is the set of linear transformations that preserve the skew symmetric bilinear form on $H^1(A;\mathbb{Q})$ associated with the polarization of $A$ [resp. the intersection form on the lattice $T_0$] up to scalar multiplication. This fudge factor of scalar multiplication is introduced so that the image of $h$ in (67) can be accommodated. Using representations of $\mathbb{C}^\times \cong \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ instead of those of pure complex phase $S^1 \subset \mathbb{C}^\times$, descriptions in the language of algebraic geometry are made possible.
CM points in a connected Shimura variety (like $\mathcal{M}^A_{cpx}$ and $\mathcal{M}^{K3(T_0)}_{cpx}$) show up in the form of a family, a zero-dimensional Shimura variety $\text{Sh}(MT, \tilde{h})$ (as stated above). André–Oort conjecture (cf [43, 44] and references therein) states, furthermore, that when a set of CM points $\mathcal{S}$ is in a connected Shimura variety $\text{Sh}_K(G, X)$ associated with a Shimura datum $(G, X)$, then the Zariski closure of $\mathcal{S}$ is the union of finitely many connected Shimura subvarieties $\text{Sh}_{K_i}(G_i, X_i)$. If the set of CM points $\mathcal{S}$ contains representatives from infinitely many mutually distinct zero-dimensional Shimura varieties, in particular, some of the $\text{Sh}_{K_i}(G_i, X_i)$’s should have a positive dimension.

We should have a little different story for the complex structure moduli space $\mathcal{M}^X_{cpx}$ of a family of Calabi–Yau threefolds, however. First, the space of polarized rational Hodge structure on $H^3(X; \mathbb{Q})$ with the symplectic intersection form is a coset space (e.g., [19, 45])

$$D = \text{Sp}(b_3; \mathbb{R})^+/U(1) \times U(h^{2,1}), \quad (16)$$

where we understand that the Hodge–Riemann bilinear relations have been enforced. This space is not regarded as a connected Shimura variety, but an example of a more general class of objects called Mumford–Tate domain in [46]. It is still expected (e.g., VIII.B, [46]), just like in the André–Oort conjecture for a Shimura variety, that the Zariski closure of a set of CM points in a Mumford–Tate domain is the union of a finitely many Mumford–Tate subdomains, and hence appears as families of infinite many CM points [19].

Secondly, the moduli space of a family of Calabi–Yau threefolds $\mathcal{M}^X_{cpx}$ is an $h^{2,1}$-dimensional subvariety of $D$; note that $\dim_\mathbb{C}(D) = [(h^{2,1})^2 + 5h^{2,1} + 2]/2 > h^{2,1}$. Even when a CM point with a given Mumford–Tate group $MT \subset \mathbb{G}\text{Sp}(b_3)$ is found in $\mathcal{M}^X_{cpx} \subset D$, other CM points in $D$ that share the same Mumford–Tate group are not necessarily in $\mathcal{M}^X_{cpx}$. This means that CM points do not necessarily arise as a family of infinitely many in $\mathcal{M}^X_{cpx}$, although they do in $D$. The André–Oort conjecture and its variation for Mumford–Tate domains still have a thing to say, however. If a set $\mathcal{S}$ of CM points in $\mathcal{M}^X_{cpx} \subset D$ contains representatives from infinitely many mutually distinct zero-dimensional Mumford–Tate domains, then its Zariski closure should be the union of a finitely many Mumford–Tate domains, some of which must have a positive dimension. This implies, in particular, that $\mathcal{M}^X_{cpx}$ needs to contain a Mumford–Tate subdomain of a positive dimension; in other words, the orbit of the

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14 We found that [36, 40, 42] are also useful in learning the relation between the class field theory and Shimura variety.

15 In this article, we do not pay much attention to the choice of a discrete group with which to take the quotient, and avoid technical details that are not essential to the central theme of this article.

16 See footnote 15. We do not repeat this reminder in the rest of this article.
group action in the Mumford–Tate subdomain must be aligned with $\mathcal{M}_{\text{cpx}}^X$. Therefore, group theoretical considerations can be exploited in listing up possible Mumford–Tate subdomains contained in $\mathcal{M}_{\text{cpx}}^X \subset D$.

Now, which closed subvariety $Y$ of a Mumford–Tate domain $D$ does admit a Mumford–Tate subdomain? This question has been studied the best in the case of $D = \mathcal{H}_g$ and $Y = \mathcal{T}_g$, the closure of the Torelli locus $\mathcal{T}_g^\circ$, which is the subspace of $\mathcal{H}_g$ corresponding to the Jacobian of non-singular genus $g$ curves (e.g., see [43,47] for a review). This is a trivial question for $g \leq 3$, but is not for $4 \leq g$, since $\mathcal{T}_g$ is a proper subspace of $\mathcal{H}_g$ for those cases. For the range of $4 \leq g \leq 7$, infinitely many CM points have been found in the Torelli locus $\mathcal{T}_g^\circ \subset \text{Sp}(2g; \mathbb{Z}) \backslash \mathcal{H}_g$ [48–50], but it is expected (Coleman–Oort conjecture), that there will be at most finitely many CM points in $\mathcal{T}_g^\circ \subset \text{Sp}(2g; \mathbb{Z}) \backslash \mathcal{H}_g$ at least for large $g$ (maybe for $g \geq 8$) [51,52]; a proof is not known yet. Note that this expectation does not rule out infinitely many CM points present in $\mathcal{T}_g \backslash \mathcal{T}_g^\circ$. We will come back to this point in section 3.

3 Flux Vacua in a Borcea–Voisin Calabi–Yau Threefold

3.1 Borcea–Voisin Construction

Borcea–Voisin Calabi–Yau threefolds were introduced [19] as families of Calabi–Yau threefolds where the alignment between $\mathcal{M}_{\text{cpx}}^X$ and a Mumford–Tate subdomain of the period domain $D$ is more likely\(^\dagger\) than in general families of Calabi–Yau threefolds.\(^\dagger\) Thus, there is a good chance in such families of finding numerous examples of CM-type Calabi–Yau threefolds, where the arithmetic solutions to the $\langle W \rangle = 0$ condition can be implemented.

Here is a brief summary of what is known about Borcea–Voisin Calabi–Yau threefolds [19,57,58], mostly for the purpose of setting the notation to be used in the following. When $S$ and $E$ are a K3 surface and an elliptic curve, respectively, and there is an automorphism $\sigma_S : S \to S$ and $\sigma_E : E \to E$ so that $\sigma_S^*(\Omega_S) \wedge \sigma_E^*(\Omega_E) = \Omega_S \wedge \Omega_E$, we can think of an orbifold $(S \times E)/(\sigma_S, \sigma_E)$ while leaving unbroken supersymmetry in 3+1-dimensions. Since an automorphism $\sigma_E$ of an elliptic curve can only be of either order 2, 3, 4 or 6, the K3 surface $S$ needs to have a non-symplectic automorphism $\sigma_S$ where order is either 2, 3, 4 or

\(^\dagger\)In fact, all the examples examined in [19] are the $g = 0$ cases (reviewed in the following), where the alignment does take place. See also [53].

\(^\dagger\)Another systematic construction of CM-type varieties (Viehweg–Zuo construction) has been reported [54]; see also [55,56] for additional information. The Fermat quintic Gepner point, for example, is not the only CM point in moduli space $\mathcal{M}_{\text{cpx}}^X$ of the quintic threefolds, but this construction provides a set of CM points whose closure is a 2-dimensional subspace in the $(h^{2,1} = 101)$-dimensional $\mathcal{M}_{\text{cpx}}^X$. See [43,47] for variations.
6. The original construction by Borcea and Voisin [19, 57, 58] was for cases with order 2 automorphisms \((\sigma_S, \sigma_E)\), and we also deal with those cases in this article; generalization of the following analysis must be straightforward.\(^\text{19}\)

Nikulin carried out \([59]\) classification\(^\text{20}\) of lattice “polarizations” of a K3 surface that has an order-2 non-symplectic automorphism \(\sigma_S\); to be more precise,\(^\text{21, 22}\) let \(NS_0 \oplus T_0\) be the pair of mutually orthogonal primitive lattices within \(H^2(K3; \mathbb{Z}) \cong \Pi_{3,19}\) where \(\sigma_S\) acts trivially on \(NS_0\) and by \((-1)\times\) on \(T_0\). There are 75 such lattice “polarizations”; detailed information of those lattice pairs are found in \([59]\). Two integers \((r, a)\) extract important properties of those 75 lattice “polarizations”\(^\text{23}\). \(r\) is the rank of \(NS_0\), which means that \(\text{rank}(T_0) = 22 - r\). The Abelian group \(NS_0^*/NS_0\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^a\) in this classification, and \(a\) the number of generators of \((\mathbb{Z}/2\mathbb{Z})^a\) \((NS_0^*\) is the dual lattice of \(NS_0\).

The fixed locus of \(E\) under the order-2 automorphism \(\sigma_E = [(-1)\times]\) consists of four points. The set of points \(S_{\sigma_S}\) of a K3 surface \(S\) that are fixed under an order-2 non-symplectic automorphism \(\sigma_S\), on the other hand, consists of reducible curves disjoint from one another. Apart from two cases\(^\text{24}\) out of the list of 75 lattice “polarizations”,

\[
S_{\sigma_S} = C_g + L_1 + \cdots + L_k,
\]

where \(C_g\) is a curve with genus \(g\) and \(L_i\) are rational curves. It is known that the integers \(k\)

\(^{19}\) For F-theory applications, an orbifold of \(S_1 \times S_2\) of a pair of K3 surfaces can be used instead. There is more freedom in the choice of automorphisms for the orbifold then. That would be more fruitful for particle physics applications, but in the present study, we restrict ourselves to Type IIB compactifications.

\(^{20}\) Similar classification of automorphisms of a K3 surface that are not order-2 has been studied in \([59–77]\) and references therein.

\(^{21}\) In order to preserve unbroken supersymmetry in 3+1-dimensions, we just need that the image of \(\sigma_S \in \text{Aut}(S)\) in the isometry group \(\text{Isom}(T)\) of the transcendental lattice \(T\) has an order \(\text{rank}(T) = 2, 3, 4\) or 6. If \(\sigma_S\) is an order \(m\) element in \(\text{Aut}(S)\), then \(mT|m\), but there can be a non-trivial kernel—symplectic automorphisms—in the projection \(\text{Aut}(S) \supset \mathbb{Z}/(m\mathbb{Z}) \to \mathbb{Z}/(mT\mathbb{Z}) \subset \text{Isom}(T)\). Examples of a K3 surface with such an automorphism group have also been found in the literatures in the previous footnote.

\(^{22}\) We should also keep in mind that a moduli \(zs\) may be in a Noether–Lefschetz locus (see footnote 44) within the period domain \(D(T_0)\), where the transcendental lattice \(T\) of a K3 surface may be a proper subset of the lattice \(T_0\). For this reason, we maintain quotation marks in “polarization” here. See also footnote 25.

\(^{23}\) Here is a few notable examples of those 75 lattice pairs. The \((r, a) = (1, 1)\) case corresponds to a degree-2 K3 surface, \((r, a) = (2, 0)\) to an elliptic K3 surface, \((r, a) = (2, 2)\) to \(NS_0 = U[2]\), and their mirrors are \((r, a) = (19, 1)\) with \(T_0 = U \oplus \langle +2 \rangle\), \((r, a) = (18, 0)\) with \(T_0 = U^{\oplus 2}\), \((r, a) = (18, 2)\) with \(T_0 = U \oplus U[2]\), respectively. Those without a mirror include \((r, a) = (20, 2)\) where \(T_0 = \langle +2 \rangle \oplus \langle +2 \rangle\) and \((r, a) = (18, 4)\) corresponding to a Kummer surface \(S = \text{Km}(E \times F)\).

\(^{24}\) In one case, \(S_{\sigma_S}\) is empty, while \(S_{\sigma_S}\) consists of two elliptic curves in the other. We do not provide separate analysis of flux vacua for Borcea–Voisin Calabi–Yau’s using one of the two cases of lattice polarization of K3 surface in this article.
and $g$ are related to $r$ and $a$ through [59]

$$k = \frac{1}{2}(r - a), \quad g = \frac{1}{2}(22 - r - a).$$  \hspace{1cm} (18)

The fixed point locus of an order-2 automorphism $(\sigma_S, \sigma_E)$ of $S \times E$ therefore consists of $Z := \bigcup_{b=1}^{4} Z_b := \bigcup_{b=1}^{4} (C_{g(b)} + L_{1(b)} + \cdots + L_{k(b)})$, where each $b$ corresponds to one of the four fixed points under $\sigma_E$ in $E$. The $(\sigma_S, \sigma_E)$-twisted sector fields of the orbifold $(S \times E)/(\sigma_S, \sigma_E)$ are localized along $Z$.

By allowing a K3 surface $S$ to take all possible complex structure in the period domain $D(T_0)$ of the lattice $T_0$, we find a $(20 - r)$-dimensional space of complex structure for $S$. There is one more for $E$, $\tau \in H_{g=1}$, as we just use an automorphism of order 2. By turning on vacuum expectation values in the massless fields in the twisted sector, we would have $4g$ more parameters of complex structure deformation of this Calabi--Yau threefold; the complex structure deformation of the $\mathbb{C}^2/(\mathbb{Z}/(2\mathbb{Z}))$ singularity along $Z$ corresponds to global sections of $(1, 0)$ form on the curve $Z$. This is how we obtain [19, 57, 58]

$$h^{2,1} = 1 + (20 - r) + 4g. \hspace{1cm} (19)$$

Although one could think of a family of Calabi--Yau threefolds over an $h^{2,1}$-dimensional moduli space $\mathcal{M}_{\text{grx}}$, we would then have to face the alignment problem referred to at the end of the previous section. Instead, we focus on vacua that sit within a sub-family over the $(20 - r) + 1$-dimensional moduli space

$$(\text{Isom}(T_0) \setminus D(T_0)) \times (\text{SL}(2; \mathbb{Z}) \setminus H_{g=1}). \hspace{1cm} (20)$$

That is to remain in the orbifold limit.\hspace{1cm} (26) This subspace is an example of connected Shimura varieties associated with the group $GO(2, 20 - r) \times GL(2)$.

Let $X$ be the minimal resolution of the $\mathbb{C}^2/(\mathbb{Z}/(2\mathbb{Z}))$ singularity of the orbifold $(S \times E)/(\sigma_S, -1)$. The cohomology group $H^3(X; \mathbb{Q})$ can be obtained as the $(\sigma_S, -1)$-invariant (even) part of $H^3(\text{Bl}_Z(S \times E); \mathbb{Q})$ (Thm. 7.31, [78]). First,

$$H^3(\text{Bl}_Z(S \times E); \mathbb{Q}) \simeq (H^2(S; \mathbb{Q}) \otimes H^1(E; \mathbb{Q})) \oplus H^1(Z; \mathbb{Z}) \hspace{1cm} (21)$$

\hspace{1cm} 25 With this, $T_0$ can be identified with the transcendental lattice $T_S$ and $NS_0$ with the Néron--Severi lattice $NS_S$ of a K3 surface $S$, if $S$ corresponds to a generic (non-Noether--Lefschetz) point in this period domain. Although we might find motivations to maintain distinction between $NS_0$ and $NS_S$ in particle physics applications of F-theory compactifications, we remain simple minded in this article, and are happy to think just of a most generic point in $D(T_0)$.

\hspace{1cm} 26 We do not say anything about Kähler moduli in this article. So, the twisted sector fields corresponding to the resolution of the $\mathbb{C}^2/(\mathbb{Z}/(2\mathbb{Z}))$ singularity may well have non-zero vacuum expectation value. The Kähler moduli therefore have $h^{1,1} = 1 + r + 4(k + 1)$ degrees of freedom.
as a vector space over \( \mathbb{Q} \), and secondly, its even part under \((\sigma_S, -1)\) is
\[
H^3(X; \mathbb{Q}) \cong (T_0 \otimes H^1(E; \mathbb{Q})) \oplus (\bigoplus_{a=1}^4 H^1(C_{g(a)}; \mathbb{Q})) =: V_0' \oplus (\bigoplus_{a=1}^4 V_a),
\]
where each \( a \) component corresponds to one of the four twisted sectors. In the case of a Borcea–Voisin Calabi–Yau threefold \( X \) with \( g \geq 1 \), with its complex structure remaining in the orbifold limit, the rational Hodge structure on \( H^3(X; \mathbb{Q}) \) has rational Hodge substructures on \( V_0' \) and each one of \( V_a \)'s; if \( g = 0 \), then \( V_a \)'s are empty. We refer to \( V_0' \) as the untwisted sector, and \( V_a \)'s the twisted sector. The rational Hodge substructure on \( V_0' \) is level-3, and those of \( V_a \)'s level-1. The level-1 Hodge structure on \( V_a \) is essentially the same as the weight-1 Hodge structure of a \( g \)-dimensional Abelian variety \( \text{Jac}(C_g) \).

The Mumford–Tate domain (16) contains a connected Shimura variety associated with the image of a homomorphism
\[
\mathbb{G} \text{Sp}(b_3) \to \mathbb{G} O(2, 20 - r) \times \mathbb{G} L(2) \times (\mathbb{G} \text{Sp}(2g))^4,
\]
and the moduli subspace (20) of \( \mathcal{M}^X_{\text{cpx}} \) should be mapped identically to the first two factors of a connected Shimura variety
\[
D(T_0) \times \mathcal{H}_1 \times (\mathcal{H}_g)^4,
\]
while the map \( D(T_0) \to \mathcal{T}_g \subset \mathcal{H}_g \subset (\mathcal{H}_g)^4 \) for \( g \geq 1 \) is non-trivial. The condition that the Calabi–Yau threefold \( X \) is of CM-type (see the appendix B.1) is equivalent to all the rational Hodge substructures on \( V_0' \) and \( V_a \)'s being of CM-type. The rational Hodge substructure on \( V_0' \) is of CM-type if and only if those on \( T_0 \) and \( H^1(E; \mathbb{Q}) \) are of CM-type (Prop. 1.2, [19]). So, on a point
\[
(z_S, \tau) \in \mathcal{M}^X_{\text{cpx}} \times \mathcal{M}^X_{\text{ell}} \subset D(T_0) \times \mathcal{H}_1 \subset \mathcal{M}^X_{\text{cpx}},
\]
the Hodge structure on \( V_0' \) is guaranteed to be of CM-type. There remains a question whether a CM \( z_S \) gives rise to an Abelian variety \( \text{Jac}(C_g) \) with sufficiently many CMs or not. Therefore, the alignment problem between \( \mathcal{M}^X_{\text{cpx}} \) and a Mumford–Tate subdomain of \( D \) in (16) can be regarded as composition of the two following problems in the case of Borcea–Voisin Calabi–Yau threefolds: a) the map \( D(T_0) \to \mathcal{T}_g \subset \mathcal{H}_g \), and b) the CM points within \( \mathcal{T}_g \) (on which the Coleman–Oort conjecture has things to say for large \( g \) cases). This alignment problem has been studied in math literatures, and at least it is known that a non-empty
subset of CM points in $D(T_0)$ are mapped to CM points in $\mathcal{T}_g \subset \mathcal{H}_g$ for not a small number of lattice pairs (Cor. 3.5, [21]).

In this article, we work out which rational Hodge structure admits non-trivial supersymmetric flux configurations in sections 3.2 and 3.3, treating as if $\mathcal{H}_g$ (rational Hodge structure on the twisted sector $V_a$'s) is yet another moduli parameter independent of $D(T_0)$; this physics analysis needs to be combined with the math analysis of a) and b), in principle, but we will just make a brief remark on the result of our physics analysis and b) in this article.

3.2 Fluxes Satisfying the F-term Conditions

There are $1 + h^{2,1} = 1 + (20 - r) + 1 + 4g$ chiral multiplets in the Type IIB compactification in question; one of them is the dilaton $\phi$. The complex structure moduli of $X$, denoted collectively by $z$ can be split into the twisted sector moduli $y_{ar} (a = 1, \ldots, 4, r = 1, \ldots, g)$, $\tau$ for the elliptic curve $E_\tau$ and $(20 - r)$ moduli for the K3 surface $S$ denoted collectively by $z_S$. As we seek for flux vacua that sit within the orbifold limit $D(T_0) \times \mathcal{H}_1 \subset \mathcal{M}^{\times} \mathcal{C}_\text{cpx}$, that is, $\langle y_{ar} \rangle = 0$, the F-term conditions with respect to $y_{ar}$'s are concerned about the Hodge structures on the twisted sector $V_a$'s, and those with respect to $\phi, \tau$ and $z_S$'s about the one on the untwisted sector $V'_0$. We can therefore work on the F-term conditions separately on $V_a$'s, and on $V'_0$.

Let us first work on the F-term conditions on the twisted sector (which is irrelevant in the first place in the Borcea–Voisin Calabi–Yau's with $g = 0$). Since the analysis remains precisely the same, word-by-word, for different isolated twisted sectors labeled by $a = 1, \ldots, 4$, we will now drop the label $a$ for a while. Using a symplectic integral basis $\{f^r=1,\ldots, g, f^r_s=1,\ldots , g\}$ of $H^1(\text{Jac}(C_g); \mathbb{Q}) \cong V_a$, flux quanta on $V_a$ are parametrized by rational numbers $\{n_r, m^r\}$ and $\{n^r, m^r\}$ for the RR and NS–NS sectors, respectively. Let $\tau^{rs}(z_S) \in \mathcal{H}_g$ be the complex structure of the Abelian variety $\text{Jac}(C_g)$ so that the $(1,0)$ (or $(2,1)$) component of $V_a$ is spanned by $\{(f^r + \tau^{rs}f^s) \mid r = 1, \ldots, g\}$. The F-term conditions for $y_{as} (s = 1, \ldots, g)$ are written as

\begin{equation}
(n^r - \phi m^r, -(n_r - \phi m_r))(\gamma^{rs}) = 0 \quad s = 1, \ldots, g.
\end{equation}

We have assumed that the Abelian variety $\text{Jac}(C_g)$ has sufficiently many complex multiplications. $\text{Jac}(C_g)$ is further assumed to be isotypic (B.1.14), for now, because we do not

\[\text{The study in [21] does not impose a condition that those non-empty subsets stay away from Noether–Lefschetz loci of } D(T_0).\]
lose generality by doing so; we can just carry out the same analysis for individual isotypic components of \( H^1(\text{Jac}(C_g); \mathbb{Q}) \). The algebra of Hodge endomorphisms of \( V_a \) is then a CM field—denoted by \( K^C \)—with \( [K^C : \mathbb{Q}] = 2g \). This means that the \( g \) conditions above labeled by \( s \) can be re-organized through a linear transformations in a number field \( (K^C)^{nc} \subset \overline{\mathbb{Q}} \) into \( g \) conditions each one of which is associated with an embedding \( \rho_a \in \overline{\Phi} \) of the CM type \( (K^C, \Phi) \) of the Abelian variety \( \text{Jac}(C_g) \) (see B.1.9). Moreover, the \( g \) column vectors (labeled by \( a = 1, \cdots, g \)) after the re-organization has the property explained in the appendix B.2, so there exists a \( \mathbb{Q} \)-basis \( \{y_r = 1, \cdots, g, y_{r}^{1, \cdots, g}\} \) of \( K^C \) such that the F-term conditions are equivalent to

\[
(n^r - \phi m^r) \rho_a(y_r) - (n_r - \phi m_r) \rho_a(y_r^r) = 0 \quad a = 1, \cdots, g.
\]

These conditions can be further brought into

\[
\rho_a \left( n^r y_r - n_r y_r^r \right) = \phi \rho_a \left( m^r y_r - m_r y_r^r \right), \quad \rho_a \in \overline{\Phi}.
\]

Here, \((n^r y_r - n_r y_r^r)\) and \((m^r y_r - m_r y_r^r)\) are regarded as elements in an abstract finite extension field \( K^C \) over \( \mathbb{Q} \), while the axi-dilaton vev \( \phi \) is a complex number. Now here is a

**Proposition 3.2.1.** Let \( A \) be an Abelian variety of \( \text{dim}_C A = g \) with sufficiently many CMs, and \( (K, \Phi) \) its CM type. When there is an element \( x \in \overline{K} \) such that \( \rho_a(x) = \phi \in \mathbb{C} \setminus \mathbb{R} \) for all \( \rho_a \in \overline{\Phi} \), then both \( \phi \) and \( x \) have a degree-2 minimal polynomial over \( \mathbb{Q} \). In particular, \( \mathbb{Q}(\phi) \) is a quadratic imaginary field.

**Proof.** First, note that \( x \notin K_0 \), where \( K_0 \) is the totally real subfield of \( K \), because \( \rho_a(x) \)'s do not fall into \( \mathbb{R} \). This means that \( K = K_0(x) \). Second, the fact that \( K \) is a degree-2 extension over \( K_0 \) implies that there exist \( P, Q \in K_0 \) so that \( x^2 = Px + Q \). The assumption that \( \rho_a(x) = \phi \in \mathbb{C} \) for all \( \rho_a \in \overline{\Phi} \) implies that

\[
\sum_{\rho_a \in \overline{\Phi}} \rho_a(Px + Q) = \sum_{\rho_a \in \overline{\Phi}} (\rho_a(P)\rho_a(x) + \rho_a(Q)),
\]

where we used the fact that the complex conjugate pair of embeddings \( \rho_a \in \overline{\Phi} \) and \( \bar{\rho}_a \in \Phi \) become the same embedding upon restriction to \( K_0 \), and \( \text{Tr}_{K_0/Q}(a) = \sum_{\rho_a \in \Phi} \rho_a(a) = \sum_{\bar{\rho}_a \in \Phi} \bar{\rho}_a(a) \) for \( a \in K_0 \). As \( \text{Tr}_{F/Q}(a) \) takes a value in \( \mathbb{Q} \) for any \( a \in F \) by definition, this shows that \( \phi \in \mathbb{C} \) has a degree-2 minimal polynomial over \( \mathbb{Q} \).
Moreover, we see that $\rho_a(P) \in \mathbb{R}$’s are independent of $a = 1, \cdots, g$ (and so are $\rho_a(Q) \in \mathbb{R}$’s), because $\phi^2 = \rho_a(x^2) = \rho_a(P)\phi + \rho_a(Q)$ holds in the quadratic imaginary field $\mathbb{Q}(\phi)$ for all $a = 1, \cdots, g$. In fact, $\rho_a(P) = g^{-1}\text{Tr}_{K_0/\mathbb{Q}}(P) \in \mathbb{Q}$, and $\rho_a(Q) = g^{-1}\text{Tr}_{K_0/\mathbb{Q}}(Q) \in \mathbb{Q}$ independent of $a$, and the minimal polynomial of $x$ over $\mathbb{Q}$ is the same as that of $\phi$. 

This proposition is used in the analysis for the twisted sectors by setting $x = (n^r y_r - n_r y^r)/(m^r y_r - m_r y^r) \in K^C$. In doing so, we assume that the NS–NS flux is non-trivial in the twisted sector $V_a$, $a = 1, \cdots, 4$, so $x$ is well-defined. For a non-trivial flux to be consistent with the F-term conditions in the twisted sector, therefore, the endomorphism field $K^C$ needs to satisfy

$$K^C = K_0^C(\frac{3}{x_C}), \quad \mathbb{Q}(x_C) \cong \mathbb{Q}(\phi), \quad [\mathbb{Q}(\phi) : \mathbb{Q}] = 2. \quad (31)$$

Using the notion of the reflex field, the condition above can also be stated as

$$(K^C)^r \cong \mathbb{Q}(\phi), \quad [\mathbb{Q}(\phi) : \mathbb{Q}] = 2. \quad (32)$$

The reflex field of a quadratic imaginary field is the quadratic imaginary field itself, and we should also keep in mind that $(K^C)^r$ is the CM field of the unique primitive CM subtype of the CM type $(K^C, \Phi_C)$ of $\text{Jac}(C_g)$ (see the statement B.1.18). This does not mean anything for $g = 0$ and $g = 1$ cases; for the cases with $g \geq 2$, however, the condition (32) with $(K^C)^r = (K^C)^r$ implies that $\text{Jac}(C_g)$ is not a simple Abelian variety, but isogenous to a product of $g$ elliptic curves each one of which is isogenous to $E_\phi$ (cf Prop. 27, [24, 25]). Such a Jacobian variety as $\text{Jac}(C_g) = E_\phi \times \cdots \times E_\phi = (E_\phi)^g$ is not for a non-singular genus $g$ curve $C_g$, but for a degenerate limit of $C_g$ splitting into $g$ elliptic curves, and hence is not within the Torelli locus $T_g^\circ$. Such a conclusion from physics analysis (conditions for presence of supersymmetric flux configurations) is not in conflict with the Coleman–Oort conjecture, but rather in line with it, even for large $g$ cases.

Let us now work on the F-term conditions in the untwisted sector. For simplicity (and for genericity), we assume that the rational Hodge structure on $V_0' \cong T_0 \otimes H^1(E; \mathbb{Q})$ is simple. The F-term conditions on this untwisted sector, from $\tau$, $z_s$’s and $\phi$, require that the $(1, 2)$ and $(3, 0)$ Hodge components of the flux $G = F^{(3)} - \phi H^{(3)}$ vanish. This is precisely equivalent to the analysis for the twisted sector, when the Abelian variety $\text{Jac}(C_g)$ is replaced by the Weil intermediate Jacobian associated with the weight-3 Hodge structure on $V_0' = T_0 \otimes H^1(E; \mathbb{Q})$. Therefore, for a non-trivial flux configuration to exist on $V_0'$ consistently with the F-term

\footnote{We wish to do so, in order to provide mass terms to the chiral multiplets $y_{ar}$’s.}
conditions (cf footnote 28), the following conditions need to be satisfied:

$$\left( K^S K^E \right)^r \simeq \mathbb{Q}(\phi), \quad \left[ Q(\phi) : \mathbb{Q} \right] = 2. \quad (33)$$

When the twisted sector is non-empty (i.e., \( g \geq 1 \)), the reflex fields of the untwisted and the twisted sectors need to be one and the same quadratic imaginary field \( \mathbb{Q}(\phi) \) associated with the dilaton vev.

Suppose now that the endomorphism fields on a CM point satisfy the conditions stated above. First of all, the number field \( K_{tot} \subset \overline{\mathbb{Q}} \) in [14] in this case becomes

$$K_{tot} = K^X_{tot} = (K^S)^{nc} K^E (K^C)^{nc}; \quad (34)$$

\( \mathbb{Q}(\phi) \) is included in \( (K^S)^{nc} \) and \( (K^C)^{nc} \).

Secondly, we are now ready to find the space of flux quanta consistent with the F-term conditions with this complex structure. We can choose arbitrary NS–NS flux quanta \( \{ m_a, m_{ar} \}_{a=1, \ldots, 4; r=1, \ldots, g} \) in the twisted sector without violating the F-term conditions; once the NS–NS flux quanta is specified, however, the RR flux quanta appropriate for getting the right \( \phi \) vev is uniquely determined. Similarly, we can choose NS–NS flux quanta arbitrarily in the untwisted sector, though the RR flux configuration is now uniquely determined to get the right \( \phi \) vev. Overall,

$$\kappa = 2(22 - r) + 4 \times 2g = b_3(X). \quad (35)$$

### 3.3 Fluxes Satisfying both the F-term and \( \langle W \rangle = 0 \) Conditions

Let us now impose the \( \langle W \rangle = 0 \) condition. Since complex structure moduli and dilaton vev have been fixed by the F-term conditions for a given flux configuration, it is a yes-or-no question to ask whether \( \langle W \rangle = 0 \) or not for a given flux configuration. The space of flux configuration satisfying \( \langle W \rangle = 0 \) as well is a subspace of the \( \kappa \)-dimensional space of fluxes satisfying the F-term conditions for a given \( (z, \phi) \in \mathcal{M}^{\text{cpx}} \times \mathcal{M}_{\text{dil}}. \)

As is known well in the literature, the \( \langle W \rangle \propto \int G \wedge \Omega = 0 \) condition and the F-term condition \( D_\phi W = 0 \) can be reorganized into

$$\int_X F^{(3)} \wedge \Omega = 0 \quad \text{and} \quad \int_X H^{(3)} \wedge \Omega = 0. \quad (36)$$

---

29When two fields \( K_1 \) and \( K_2 \) are subfields of a common field \( L \), the **composite field** \( K_1 K_2 \) is the minimal subfield of \( L \) containing \( K_1 K_2 \). This definition of \( K_1 K_2 \) requires that we know a priori that such a field \( L \) exists. We implicitly assume here that the rational Hodge structure on \( V_0' \) is simple, which guarantees that \( L \) exists. When the rational Hodge structure is not simple, as in section 3.3, we need separate treatment.
The additional constraint \( \langle W \rangle = 0 \) therefore implies that there is a linear relation among the algebraic numbers (e.g., \([13, 15, 79, 80]\))

\[
\sum_{i=1}^{2(22-r)} n_i \Pi_i = 0. \tag{37}
\]

If the rational Hodge structure on \( V'_0 = T_0 \otimes H^1(E; \mathbb{Q}) \) were simple, and the Hodge structure on \( T_0 \) and \( H^1(E; \mathbb{Q}) \) were of CM-type, then the algebra of Hodge endomorphisms on \( V'_0 \) would be a CM field of degree \( 2(22-r) = \dim_{\mathbb{Q}} V'_0 \), and the period integrals \( \Pi_{i=1, \ldots, 2(22-r)} \) would be the image of the elements of a \( \mathbb{Q} \)-basis of the CM field under the embedding associated with the \((3, 0)\) Hodge component. The existence of a non-trivial \( \mathbb{Q} \)-linear relation among the images of the embedding of a basis of the CM field is an outright contradiction. The extra condition \( \langle W \rangle = 0 \) requires that the rational Hodge structure on \( V'_0 = T_0 \otimes H^1(E; \mathbb{Q}) \) is not simple.

In the case of Borcea–Voisin Calabi–Yau threefold \( X \), the rational Hodge structure on \( V'_0 \) can be made not to be simple, when the CM point \((z_S, \tau)\) satisfies an extra condition (49). \( V'_0 \) is split into two vector subspaces over \( \mathbb{Q} \), \( V'_0 \cong V_0 \oplus \bar{V}_0 \) with \( \dim_{\mathbb{Q}}(V_0) = \dim_{\mathbb{Q}}(\bar{V}_0) = (22-r) \), and both \( V_0 \) and \( \bar{V}_0 \) have a rational Hodge substructure of \( V'_0 \). Let us verify this claim in the following.

Let \( \{e'_1, \ldots, e'_{22-r}\} \) and \( \{\hat{\alpha}, \hat{\beta}\} \) be a basis of the vector space \( T_0 \) and \( H^1(E; \mathbb{Q}) \) over \( \mathbb{Q} \), respectively, and

\[
\Omega_S = (e'_1 \ldots e'_{22-r}) \left( \begin{array}{c} \epsilon(y_1) \\ \vdots \\ \epsilon(y_{22-r}) \end{array} \right), \quad \Omega_E = (\hat{\alpha} \ \hat{\beta}) \left( \begin{array}{c} 1 \\ \tau \end{array} \right),
\]

where \( \epsilon : K^S \hookrightarrow \mathbb{C} \) is the embedding associated with the \((2, 0)\) Hodge component of \( T_0 \otimes \mathbb{C} \), and \( \{y_1, \ldots, y_{22-r}\} \) is the basis of \( K^S \) that is introduced in the appendix B.2. The corresponding holomorphic three-form on \( X \) is given by

\[
\Omega = \left( e'_{i=1, \ldots, 22-r} \wedge \hat{\alpha} \ e'_{i=1, \ldots, 22-r} \wedge \hat{\beta} \right) \left( \begin{array}{c} \epsilon(y_i) \\ \tau \cdot \epsilon(y_i) \end{array} \right), \tag{39}
\]

where differential forms \( e'_{i} \wedge \hat{\alpha} \) and \( e'_{i} \wedge \hat{\beta} \) on \( S \times E \) should be pulled back to \( \text{Bl}_Z(S \times E) \), and then be regarded\(^{30}\) as those on \( X \). The condition (37) implies that there is a non-trivial set

\(^{30}\) We only discuss in \( \otimes \mathbb{Q} \) and do not pay attention to integrality in this article. So, this abuse of notations does not lead to any practical problem.
of rational numbers \( \{n_i', n_i''\}_{i=1,...,22-r} \) so that

\[
(n'_1 \ldots n'_{(22-r)} \ n''_1 \ldots n''_{22-r}) \left( \epsilon(y_i) \ \epsilon(y_i) \right) = 0. \tag{40}
\]

This means that there is an element

\[
\xi_S := \frac{n'_iy_i}{n_i''y_i} \in K^S
\]

so that \( \epsilon(\xi_S) = \tau \). As a first lesson, we see that

\[
K^E = \mathbb{Q}(\tau) \subset \epsilon(K^S) \subset (K^S)^{nc}
\]

(which also means that \( K_{tot} = (K^S)^{nc}(K^C)^{nc} \)) \( \tag{42} \)

in order for the extra condition \( \langle W \rangle = 0 \) to be satisfied in a non-trivial \( (n'' \neq 0) \) flux vacuum.

To find out the decomposition of the rational Hodge structure on \( V_0' \), as claimed earlier, it is useful to take a \( \mathbb{Q} \)-basis \( \{\alpha_{i=1,...,(22-r)/2}\} \) of the totally real subfield \( K_0^S \), and use \( \{\alpha_{i=1,...,(22-r)/2}, \xi_S\alpha_{i=1,...,(22-r)/2}\} \) as a \( \mathbb{Q} \)-basis of \( K^S \), instead of \( \{y_{i=1,...,(22-r)}\} \). Changing the basis of \( T_0 \) from \( \{e'_i\} \) to \( \{e''_i\} \) accordingly, the component description of the holomorphic three-form \( \Omega \) of \( X \) in \( \langle 39 \rangle \) can be brought into the form of

\[
\Omega = \left( e''_j \wedge \hat{\alpha} \ e''_{j+(22-r)/2} \wedge \hat{\alpha} \ e''_j \wedge \hat{\beta} \ e''_{j+(22-r)/2} \wedge \hat{\beta} \right) \begin{pmatrix}
\epsilon(\alpha_j) \\
\epsilon(\xi_S\alpha_j) \\
\tau \cdot \epsilon(\alpha_j) \\
\tau \cdot \epsilon(\xi_S\alpha_j)
\end{pmatrix}, \tag{43}
\]

where \( j = 1,...,(22-r)/2 \). The \( \mathbb{Q} \)-basis of \( V_0' = T_0 \otimes H^1(E;\mathbb{Q}) \) employed above is now denoted by \( \{e_{i=1,...,2(22-r)}\} \). Using this basis, eigenvectors of \( V_0' \otimes \mathbb{C} \) diagonalizing the \( K^S \times K^E \) algebra on \( V_0' \) simultaneously are expressed as in

\[
\begin{pmatrix}
\Omega \\
\Sigma \nabla \\
\Sigma' \nabla \\
\Sigma_a \nabla \\
\Sigma_a \\
\Sigma_a \\
\Sigma_a
\end{pmatrix} = \begin{pmatrix}
e_1 & \ldots & e_{2(22-r)}
\end{pmatrix} \begin{pmatrix}
M & M & M & M \\
\tau M & \overline{M} & \tau M & \overline{M} \\
\tau M & \overline{M} & \tau M & \overline{M} \\
\tau^2M & \overline{M}^2 & |\tau|^2M & |\tau|^2M
\end{pmatrix}, \tag{44}
\]

where

\[
M := \begin{pmatrix}
\epsilon(\alpha_1) & \ldots & \epsilon(\alpha_{(22-r)/2}) \\
\sigma_2(\alpha_1) & \ldots & \sigma_2(\alpha_{(22-r)/2}) \\
\vdots & \ddots & \vdots \\
\sigma_{(22-r)/2}(\alpha_1) & \ldots & \sigma_{(22-r)/2}(\alpha_{(22-r)/2})
\end{pmatrix}, \tag{45}
\]
and $\sigma_{a=1,\ldots,(22-r)/2}$ are the embeddings $K_0^S \hookrightarrow \mathbb{R}$. A rationale behind this is as follows. First, let $\sigma_{a=1,\ldots,(22-r)/2}^\pm$ be the $(22 - r)$ embeddings $K^S \hookrightarrow \mathbb{C}$, which satisfy $\sigma_{a}^+|_{K^S_0} = \sigma_a$ and $\sigma_{a}^-(\xi_S) = \tau$, $\sigma^-(-\xi_S) = \bar{\tau}$; in this notation, $\epsilon = \sigma_1^+$ and $\bar{\epsilon} = \sigma_1^-$. Let $c^\pm$ be the 2 embeddings $K^E \hookrightarrow \mathbb{C}$, where $K^E = \mathbb{Q}((\tau \times))$ is generated by a complex multiplication operation $(\tau \times)$, and $c^\pm$ maps $(\tau \times)$ to $\tau$ and $\bar{\tau}$, respectively. The $2(22 - r)$ distinct embeddings of the algebra $K^S \times K^E$ into $\mathbb{C}$ are grouped into four, $(\sigma_a^+ \cdot c^+)$, $(\sigma_a^- \cdot c^-)$, $(\sigma_a^+ \cdot c^-)$ and $(\sigma^- \cdot c^+)$ with $a = 1, \ldots, (22 - r)/2$. The vectors $\Omega$ and $\Sigma_{a'=2,\ldots,(22-r)/2}$ in $V_0' \otimes \mathbb{C}$ span eigenspaces of $K^S \times K^E$ corresponding to the embeddings $\sigma_a^+ \cdot c^+$ ($\Omega$ for $a = 1$, and $a' = a$ otherwise). The three other groups of eigenvectors in $V_0' \otimes \mathbb{C}$ correspond to the three other groups of embeddings in the order of appearance.

The last step is to exploit the fact that the complex structure $\tau$ of the elliptic curve $E$ has been assumed to have complex multiplication. Let us denote the minimal polynomial of $\tau$ as $\tau^2 + p\tau + q$, with $p, q \in \mathbb{Q}$. Because

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
q & p & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\tau & \bar{\tau} & \tau & \bar{\tau} \\
\tau & \bar{\tau} & \tau & \bar{\tau} \\
\tau^2 & |\tau|^2 & \tau & \bar{\tau} \\
|\tau|^2 & |\tau|^2 & \tau & \bar{\tau} \\
\end{pmatrix}
= 
\begin{pmatrix}
\tau - \bar{\tau} & \bar{\tau} - \tau & 0 & 0 \\
p\tau + 2q & p\bar{\tau} + 2q & 0 & 0 \\
0 & 0 & \tau - \bar{\tau} & \bar{\tau} - \tau \\
0 & 0 & p\tau + 2q & p\bar{\tau} + 2q \\
\end{pmatrix},
$$

a new basis of $V_0'$

$$
(\xi_1, \ldots, \xi_{2(22-r)}) = (e_1, \ldots, e_{2(22-r)})
\begin{pmatrix}
P & 1 & 1 & 0 \\
q & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
q & p & 0 & 1 \\
\end{pmatrix}
^{-1}
\otimes \mathbb{Q}^{(22-r),(22-r)}
$$

allows us to split the vector space $V_0'$ over $\mathbb{Q}$ into $V_0 = \text{Span}_{\mathbb{Q}} \{\xi_{1=1,\ldots,(22-r)}\}$ and $\mathcal{V}_0 = \text{Span}_{\mathbb{Q}} \{\xi_{c=(22-r)+1,\ldots,2(22-r)}\}$ so that

$$
V_0 \otimes \mathbb{C} = \text{Span}_{\mathbb{C}} \{\Omega, \Sigma_{a'}, \bar{\Omega}, \bar{\Sigma}_{a'}\}, \quad \mathcal{V}_0 \otimes \mathbb{C} = \text{Span}_{\mathbb{C}} \{\Sigma_{a'}, \bar{\Sigma}_{a'}\}.
$$

$V_0$ is the level-3 component, and $\mathcal{V}_0$ the level-1.

The algebra $K^S \times K^E$ acts on $V_0$ and $\mathcal{V}_0$ separately, because we have already seen that the action of $K^S \times K^E$ can be diagonalized within $V_0 \otimes \mathbb{C}$ and $\mathcal{V}_0 \otimes \mathbb{C}$. Furthermore, the generator $(\tau \times)$ of $K^E$ over $\mathbb{Q}$ acts the same way as $\xi_S$ on $V_0$, and as $(\bar{\tau} - \xi_S)$ on $\mathcal{V}_0$. Therefore, the representation of the algebra $K^S \times K^E$ is not faithful on $V_0$ and $\mathcal{V}_0$; the algebra of Hodge endomorphisms on $V_0$ is $K^S$, and the same is true on $\mathcal{V}_0$. Under the assumption that the complex structure of the K3 surface $S$ is not in a Noether–Lefschetz locus of $D(T_0)$ (i.e., the
rational Hodge structure on $T_0$ is simple, and $K^S$ is a field), the Hodge structure on $V_0$ and $V_0'$ are both simple, and moreover, $[K^S : \mathbb{Q}] = (22 - r) = \dim_{\mathbb{Q}} V_0 = \dim_{\mathbb{Q}} V_0'$. We have now managed to keep the promise.

Now, $(22 - r)/2$ F-term conditions are implemented in $V_0 \otimes \mathbb{C}$ and there are $(22 - r)/2$ more F-term conditions in $V_0 \otimes \mathbb{C}$; the $\langle W \rangle = 0$ condition is now purely on the $V_0 \otimes \mathbb{C}$ component. The analysis of F-term conditions in the level-1 $V_0 \otimes \mathbb{C}$ component is just the same as in section 3.2. For a non-trivial flux to exist, so that the moduli of $S$ and $E$ are stabilized, it is necessary that

$$(K^S)^r \cong \mathbb{Q}(\phi),$$

and the quadratic imaginary field $\mathbb{Q}(\phi)$ should be common to that for the twisted sectors. Moreover,

$$\mathbb{Q}(\tau) \cong \mathbb{Q}(\phi)$$

because $K^S = K^S_0(\xi_S)$ and $\tau = \epsilon(\xi_S)$ should be in $\mathbb{Q}(\phi)$. $(22 - r)$ NS–NS flux quanta can be chosen arbitrarily in this $V_0$ component; the $(22 - r)$ RR flux quanta are determined uniquely in order to reproduce a specified $\phi$ vev.

The same is true in the $V_0 \otimes \mathbb{C}$ component, if just the F-term conditions are imposed. The result on $\kappa$ in (35) therefore does not change, even when the Hodge structure on $V_0'$ is not simple. For a vacuum to satisfy the condition $\langle W \rangle = 0$, however, flux in the $V_0$ component should completely vanish. If there were a non-zero flux configuration in $V_0$, then the relation (37) holds in $V_0$, which contradicts against the assumption that the Hodge structure on $V_0$ (and also on $T_0$) is simple. Therefore, we come to the result

$$\kappa_0 = (22 - r) + 8g$$

for a complex structure $(z_S, \tau, \phi) \in D(T_0) \times \mathcal{H}_1 \times \mathcal{M}_{\text{dil}}$ satisfying the extra conditions (49, 50), and

$$\kappa_0 = 0$$

otherwise.

It is likely that all the moduli $z_S$, $\tau$ and $\phi$ are stabilized by a non-trivial flux in $V_0$, when the conditions (49, 50), even though there is no flux in $V_0$ so that the condition $\langle W \rangle = 0$ is satisfied. To see this, note first that the quadratic order fluctuations of $z_S$ within $D(T_0)$ are
contained also in the $(0, 2)$ Hodge component of $T_0 \otimes \mathbb{C}$, and hence this quadratic fluctuations have overlap with the eigenspace $\mathbb{C}^{\Sigma_{a=1}} \subset V_0 \otimes \mathbb{C}$. A non-trivial flux in the $(2, 1)$ component of $V_0 \otimes \mathbb{C}$ therefore gives rise to mass terms of the $z_S$ moduli. The chiral multiplets of $\tau$ moduli and $\phi$ moduli will have a Dirac mass term $\Delta W \propto - (m_i'' \epsilon(y_i)) \phi \cdot (\delta \tau)$. This argument does not rule out a chance of accidental cancellation when all things considered, but such a cancellation is quite unlikely.\textsuperscript{31} The flux configuration we have in mind here can be supersymmetric only when the moduli $(z_S, \tau, \phi)$ have very specific properties (encoded as very specific structure of the corresponding endomorphism fields); small deviation from such special loci would render the flux configuration non-supersymmetric, which is an indication that there is a scalar potential on the small deviation, the usual Noether–Lefschetz argument for the moduli stabilization by fluxes in Type IIB string and F-theory.

4 Non-Borcea–Voisin Cases

While we cannot have high hope of finding a series of infinite CM points in the complex structure moduli space of a family of Calabi–Yau threefolds other than in the form of Borcea–Voisin construction (André–Oort conjecture), there are CM-type Calabi–Yau threefolds (e.g. some of Gepner models) isolated from other CM points. The analysis of the structure of endomorphism fields and the space of supersymmetric fluxes (35, 51) can be generalized easily for cases that are not Borcea–Voisin type. Let $X$ be such a Calabi–Yau threefold.

First, when a complex structure $z \in \mathcal{M}_X \subset \mathcal{M}_{\text{cpx}}$ is given, a rational Hodge structure is induced on the vector space $H^3(X; \mathbb{Q})$. Suppose that

$$H^3(X; \mathbb{Q}) = V_0 \oplus V_1 \oplus \cdots \oplus V_k$$

is a decomposition into simple rational Hodge substructures; $V_0$ is the level-3 component, and all the others level-1 components. The algebra of Hodge endomorphisms of $V_a$ is then a CM field, denoted by $K^a$, and $[K^a : \mathbb{Q}] = \dim_{\mathbb{Q}} V_a$ for $a = 0, \cdots, k$, as we assume a CM-type Calabi–Yau threefold $X$. The number field $K^X_{\text{tot}} \subset \overline{\mathbb{Q}}$ is then the composite field of all the $(K^a)^{\text{nc}}$’s in $\overline{\mathbb{Q}}$.

When we impose just the F-term conditions, analysis can be carried out (almost) separately for the individual components; among the $(1 + h^{2,1}(X))$ F-term conditions of this

\textsuperscript{31}For a rigorous proof of non-zero masses without cancellation, we have to use a set of $(20 - r)$ coordinates in a local patch of $D(T_0)$ around the vev $z_S \in D(T_0)$, to parametrize $\epsilon(y_i)$’s in (38). We do not do that in this article.
theory, $\dim_{\mathbb{Q}}(V^a)/2$ of them are attributed to the $V_a$ component. For a non-trivial flux to be allowed in the $V_a$ component, we must have

$$(K^a)^r \cong \mathbb{Q}(\phi) \quad (54)$$

and the dilaton vev $\phi$ needs to generate a quadratic imaginary field;

$$[\mathbb{Q}(\phi) : \mathbb{Q}] = 2. \quad (55)$$

The reflex fields of all the CM fields $K^a$ should be one and the same quadratic imaginary field $\mathbb{Q}(\phi)$. This means, as in the previous section, that $\dim_{\mathbb{Q}}(V_a) = 2$ and $K^a = \mathbb{Q}(\phi)$ for all of $a = 1, \ldots, k$. The Weil intermediate Jacobian associated with the level-3 simple component $V_0$ must be isogenous to the product of $\dim_{\mathbb{Q}}(V_0)/2$ copies of an elliptic curve with complex multiplication in $\mathbb{Q}(\phi)$. It thus follows that the Weil intermediate Jacobian associated with $H^3(X; \mathbb{Q})$ needs to be isogenous to $(E_{\phi})^{b_3/2}$. The space of fluxes consistent with the F-term conditions at this complex structure $\langle z, \phi \rangle$ has a dimension

$$\kappa = \sum_{a=0}^{k} \dim_{\mathbb{Q}} V_a = b_3(X). \quad (56)$$

Not all the flux configurations in a $\kappa$-dimensional space over $\mathbb{Q}$ end up with a vacuum with $\langle W \rangle = 0$; the necessary and sufficient condition for $\langle W \rangle = 0$ is that the NS–NS and RR flux vanish in the level-3 simple component $V_0$. Therefore, the flux configurations satisfying both the F-term conditions and $\langle W \rangle = 0$ form a subspace with a dimension

$$\kappa_0 = \sum_{a=1}^{k} \dim_{\mathbb{Q}} V_a. \quad (57)$$

Note that the condition $(K^a)^r \cong \mathbb{Q}(\phi)$ does not have to be satisfied for the level-3 ($a = 0$) component for such a non-trivial supersymmetric flux configuration to exist. We should also remind ourselves, however, that there is no guarantee whether all the moduli in $\mathcal{M}_{\text{cpx}} \times \mathcal{M}_{\text{dil}}$ are given supersymmetric masses in this situation.

Four examples of non-Borcea–Voisin CM-type Calabi–Yau threefolds $M_m$ ($m = 5, 6, 8, 10$) are studied in section 5 of [14]. They are the Gepner points in the complex structure moduli spaces of four families of Calabi–Yau threefolds, all with $h^{2,1} = 1$. The field $K_{\text{tot}}$ in the four

\footnote{All the level-1 simple components $V_a$ with $a \geq 1$ are of 2-dimensions over $\mathbb{Q}$, but the level-3 simple component $V_0$ is not necessarily of 2-dimensions (see also footnote 34).}
examples are all cyclotomic fields, $\mathbb{Q}(\zeta_m)$, with $m = 5, 6, 8, 10$, respectively. The characterization (54, 55) of the number fields of Hodge decomposition and the formula of the space of supersymmetric fluxes (56, 57) reproduce all the results for the four examples obtained in [14] in a systematic way.

Let us first look at the two examples $M_m$ with $m = 5, 10$. The cohomology group $H^3(X; \mathbb{Q})$ as a whole forms a simple rational Hodge structure in the two cases; in the CM type $(K, \Phi)$ of the Weil intermediate Jacobian of $H^3(M_m; \mathbb{Q})$, the endomorphism field $K$ is generated by the minimum phase $+2\pi$ monodromy $\theta_m$ around the corresponding Gepner point of the complex structure moduli space, $K \cong \mathbb{Q}(\theta_m) \cong \mathbb{Q}(\zeta_m)$, and $\Phi$ consists of \{$\phi_1, \phi_3$\} in the $m = 5$ case and of \{$\phi_1, \phi_7$\} in the $m = 10$ case. The reflex field remains $\mathbb{Q}(\zeta_m) \subset \overline{\mathbb{Q}}$ in those two cases (e.g., Ex. 3.2. (c), [40]), which means that the condition (54, 55) cannot be satisfied regardless of the value of $\phi$. Therefore $\kappa = \kappa_0 = 0$.

The example $M_{m=8}$ is a little different from the examples with $m = 5, 10$. Its Weil intermediate Jacobian introduces a weight-1 rational Hodge structure on the 4-dimensional vector space $H^3(M_m; \mathbb{Q})$; in the CM type $(K, \Phi)$, the endomorphism field is generated by the minimum $+2\pi$ monodromy $\theta_m$ around the Gepner point, $K = \mathbb{Q}(\theta_m) \cong \mathbb{Q}(\zeta_m)$, and $\Phi$ consists of \{$\phi_1, \phi_3$\}. This CM type is not primitive, but induced from a CM type on a subfield $\mathbb{Q}(\sqrt{-1}) \subset K$. The reflex field of these CM types is $\mathbb{Q}(\sqrt{-1}) \subset \overline{\mathbb{Q}}$, which is a quadratic imaginary field. Therefore, $\kappa = 4$ if the dilaton vev satisfies $\phi \in \mathbb{Q}(\sqrt{-1})$; $\kappa = 0$ otherwise.

There is no non-trivial flux configuration satisfying $\langle W \rangle = 0$ at this Gepner point (i.e., $\kappa_0 = 0$), however, regardless of the value of $\phi$; this is because the weight-3 rational Hodge structure on $H^3(M_m; \mathbb{Q}) \cong V_0$ is simple.\(^{34}\)

In the example $M_{m=6}$, on the other hand, the weight-3 rational Hodge structure on $H^3(M_m; \mathbb{Q})$ is not simple. $H^3(M_m; \mathbb{Q})$ can be split into two 2-dimensional subspaces $V_0 \oplus V_1$ on which the rational Hodge substructures are level-3 and level-1, respectively. The endomorphism field of those two components are both $K = \mathbb{Q}(\theta_m) \cong \mathbb{Q}(\zeta_m)$ generated by the minimum phase $+2\pi$ monodromy $\theta_m$ around the Gepner point; now $\dim_\mathbb{Q}(V_a) = 2 = [K : \mathbb{Q}]$, and hence $M_{m=6}$ is a CM-type Calabi–Yau. Now, $(K^\ast \mathbb{Q})^{\ast} = K^{\ast \mathbb{Q}}(\zeta_6) \cong \mathbb{Q}(\sqrt{-3})$. So, $\kappa = 4$ and $\kappa_0 = 2$ if $\phi \in \mathbb{Q}(\sqrt{-3})$. If $\phi \notin \mathbb{Q}(\sqrt{-3})$, however, $\kappa = \kappa_0 = 0$.

Our results for the four $M_m$’s, derived in this section, perfectly agree with those in [14].

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\(^{33}\) For a cyclotomic field $K = \mathbb{Q}(\theta_m) \cong \mathbb{Q}(\zeta_m)$, the $\varphi(m)$ embeddings of $K$ to $\mathbb{C}$ are denoted by $\phi_j$, $j \in (\mathbb{Z}/(m\mathbb{Z}))^\ast$; $\phi_j : \theta_m \mapsto (\zeta_m)^j$.

\(^{34}\) The weight-1 rational Hodge structure of the Weil and Griffiths intermediate Jacobians are not simple, but the weight-3 rational Hodge structure on $H^3(M_m; \mathbb{Q})$ is simple in this example.
5 Discussions

There are a few things to remark after carrying out the analysis in the preceding sections.

5.0.1. It turns out that the variety of supersymmetric flux configurations at a CM point in $\mathcal{M}_{CM} \subset \mathcal{M}_{alg} \subset \mathcal{M}_{cpx}^{X} \times \mathcal{M}_{dil.}$ is quite different from what is expected under the general formula (6, 7) for points in $\mathcal{M}_{alg}$. The number field $K_{tot}$ obtained from the CM fields of simple Hodge substructures may have an extension degree $d_{K_{tot}} = [K_{tot} : \mathbb{Q}]$ larger than that of the individual CM fields, first of all. Secondly, the dimension $\kappa$ [resp. $\kappa_{0}$] of the $\mathbb{Q}$-vector space of supersymmetric fluxes can be much larger than the expectation (6, 7), when the CM fields satisfy extra conditions (31, 32, 33) [resp. (31, 32, 33, 49, 50)] in the Borcea–Voisin case, and (54, 55) [resp. (54, 55) except (54) for $a = 0$] in non-Borcea–Voisin cases. When those extra conditions are not satisfied, on the other hand, the space of supersymmetric fluxes just becomes trivial, $\kappa = 0 / \kappa_{0} = 0$.

This discrepancy between the cases in $\mathcal{M}_{alg}$ and $\mathcal{M}_{CM}$ is because the general expectation in (6) and (7) was derived based on the assumption that all the F-term conditions (and also the $\langle W \rangle = 0$ condition) give rise to the conditions on the flux quanta that are mutually linearly independent over $\mathbb{Q}$ [14]. We have seen that those conditions of supersymmetric fluxes are far from mutually independent for any point in $\mathcal{M}_{CM}$, as have also been observed in the example $M_{8}$ of [14]. The key observation is the structure of eigenvectors (78) that supports the Hodge decomposition at a CM point. In a given component of a simple Hodge substructure $V_{a}$, there is essentially just one F-term condition in $K_{a}$, or $[K_{a} : \mathbb{Q}] = d_{A_{a}}$ conditions independent over $\mathbb{Q}$. Therefore, a modified version $\kappa = 2b_{3} - \sum_{a=0}^{k} d_{K_{a}} \times (\dim_{\mathbb{Q}} V_{a})/2$ of (6) for a generic point in $\mathcal{M}_{alg}$ should be further replaced by

$$\kappa = 2b_{3} - \sum_{a=0}^{k} d_{K_{a}} \times 1 = 2b_{3} - \sum_{a=0}^{k} (\dim_{\mathbb{Q}} V_{a}) = b_{3}$$

(58)

for a CM point, which was the essence behind the results (35, 56). The formula $\kappa_{0} = \kappa - d_{K_{tot}}$ in (7) is modified to

$$\kappa_{0} = \kappa - d_{K_{a=0}}$$

(59)

to be the version appropriate for a CM-type Calabi–Yau threefold.\textsuperscript{35}

\textsuperscript{35} It is also worth reminding ourselves that the rational Hodge structure on $H^{3}(X; \mathbb{Q})$ cannot be simple for a non-trivial flux vacuum to exist on a CM-type Calabi–Yau threefold $X$; see (37). This observation itself is not entirely new, however.
5.0.2. For a non-trivial supersymmetric flux configuration to exist on a CM-type Calabi–Yau threefold \( X \), it turns out that the endomorphism fields \( K^a \) of all the simple components of the rational Hodge structure of \( H^3(X; \mathbb{Q}) \) have the reflex field isomorphic to a common quadratic imaginary field \( \mathbb{Q}(\phi) \), which is generated by the dilaton vev \( \phi \). Its immediate consequence is that the Abelian variety associated with the level-1 simple components \( V_a \) (for \( a = 1, \ldots, k \)) are elliptic curves isogenous to \( E_\phi \), and defined over a number field that is an Abelian extension over \( \mathbb{Q}(\phi) \). If we introduce fluxes also in the \( V_0 \) component (by giving up the \( \langle W \rangle = 0 \) condition), then the Weil intermediate Jacobian can also be defined over an Abelian extension of \( \mathbb{Q}(\phi) \).

It is also striking that the conditions for existence of non-trivial supersymmetric flux configurations (54) are stated in terms of reflex fields, because a zero-dimensional Shimura variety, a collection of CM points, is specified for a Mumford–Tate group that is an image of the multiplicative group of the reflex field (13). We can therefore think of classifying CM points, preliminary, by their quadratic imaginary reflex field \( \mathbb{Q}(\phi) \), and then by embeddings of \( \mathbb{Q}(\phi)^\times \to \mathbb{G}_{Sp}(b_3) \). Hard math problems, such as a) and b) in the case of Borcea–Voisin Calabi–Yau threefolds and the alignment of \( \mathcal{M}_X^{\text{cpx}} \) with those CM points in the non-Borcea–Voisin cases, should be imposed on top of this preliminary classifications, however.

5.0.3. Points in \( \mathcal{M}_X^{\text{alg}} \) that are not within its subset \( \mathcal{M}_{\text{CM}}^X \) do not need to be taken out of picture in the original idea in [14]. As reviewed briefly in section 2 in this article, CM points \( \mathcal{M}_{\text{CM}}^X \) forms a much smaller subset of \( \mathcal{M}_X^{\text{alg}} \). While there are so numerous points in \( \mathcal{M}_X^{\text{alg}} \), however, the number of supersymmetric flux vacua should still be estimated by (6, 7) for those that are not in \( \mathcal{M}_{\text{CM}}^X \), and there can often be no supersymmetric flux when \( [K_{\text{tot}} : \mathbb{Q}] \) is moderately large. For CM points where the reflex field is \( \mathbb{Q}(\phi) \) that is quadratic imaginary, however, much greater number of supersymmetric flux configurations are allowed, essentially due to the special structure (78) in the Hodge decomposition at CM points (5.0.1). Due to this enrichment of flux vacua on CM points, it is not obvious which side wins in the game of flux vacua counting, CM points, or non-CM algebraic points.

In the context of flux vacua counting, one will also be interested in the fact that the condition (54) for the level-3 simple component \( V_0 \) does not have to be imposed, when we seek for flux vacua with \( \langle W \rangle = 0 \). This means that a larger subset of CM points than those satisfying (54) admit non-trivial supersymmetric flux configurations in all the level-1 simple components, but this opportunity comes with a chance that some of moduli in \( \mathcal{M}_X^{\text{cpx}} \) remain unstabilized; if we set the flux in the level-3 component \( V_0' \) to be trivial in section 3.2, at least
the moduli $\tau \in \mathcal{M}^{\text{cpx}}$ would have remained massless, for example. We did not encounter a single example in this article (within Borcea–Voisin or $M_m$’s) where i) all the moduli in $\mathcal{M}^{X}_{\text{cpx}} \times \mathcal{M}^{X}_{\text{dil}}$ are stabilized without a flux in the simple level-3 component $V_0$, and ii) the condition (54) is not satisfied for the $a = 0$ component at the same time. It is an open curious math question whether and how many such CM points exist.

5.0.4. A greater problem still is if there is any reason to pay attention only to the vacua within $\mathcal{M}^X_{\text{alg}} \subset \mathcal{M}^{X}_{\text{cpx}}$ and forget others in the first place. As texts in section 2 already clarify the way we think, we just do not have any justification based on the current understanding of string theory. Justification may come from developments on string theory in the future, such as consistency analysis of fluxes in world-sheet language. Or possibly string theory may just have to be formulated only for $(c, c)$ rings that sit on CM points. If that is the case, then one would not have to bother about the statistics game above in the first place.

5.0.5. Obviously the presence of dilaton chiral multiplet plays essential roles in stating the result of flux analysis on CM-type Calabi–Yau threefolds. It is therefore an interesting question how the analysis should be modified, when we think of F-theory compactifications on CM-type Calabi–Yau fourfolds. Also, possible enrichment of flux vacua on loci with discrete symmetries in such an arithmetic regime was also a question of interest in [81, 14]. That will be all the more interesting question, when studied in F-theory compactifications. We therefore leave the study involving F-theory to our future works.

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A Field theory

Anything in the appendix A should be found in standard textbooks on field theory; most of materials in the appendix B are also well-known facts written down explicitly (or used implicitly) in the literatures. Our primary sources are [82, 83] for A.1 and [84, 24, 25, 37, 19]
for A.2–B.2 (see also footnotes 11 and 14). Those materials are placed here in the preprint version for convenience of the readers. They may thus be dropped from a journal version, following suggestions from referees and the editor.

A.1 Basics

In this article, it is always assumed that fields have characteristic zero, and a commutative multiplication law.

A.1.1 Algebraic Extension and Algebraic Number

Definition A.1.1. A nonempty subset $F$ of a field $E$ is called a subfield of $E$ if it is a field with the same operations as in $E$.

If $F$ is a subfield of a field $E$, we call $E$ an extension field of $F$. This extension is denoted by $E/F$.

Definition A.1.2. Let $E/F$ be an extension and $S$ be a subset of $E$. The smallest subfield of $E$ containing both $F$ and $S$ is denoted by $F(S)$. If $S = \{\alpha_1, \ldots, \alpha_n\}$ is a finite set, then the extension $F(S)/F$ is said to be finitely generated and denoted by $F(\alpha_1, \ldots, \alpha_n)$. An extension of the form $F(\alpha)/F$ is said to be simple and $\alpha$ is called a primitive element.

Definition A.1.3. Let $F$ be a field and $E$ be an extension field of $F$. If an element $x \in E$ is a root of some polynomial with all the coefficients in $F$, then $x$ is said to be algebraic over $F$. Otherwise $x$ is said to be transcendental over $F$. An extension $E/F$ is called an algebraic extension if every element in $E$ is algebraic over $F$.

Definition A.1.4. An extension field $E$ of a field $F$ can be viewed as a vector space over $F$. The dimension of the vector space is called the degree of the extension and denoted by $[E:F]$. If $[E:F]$ is finite, then $E/F$ is called a finite extension.

Theorem A.1.5. Let a field $K$ be an extension field of a field $F$. Then the following conditions are equivalent:

1. $K$ is a finite extension field of $F$, i.e., $[K:F]<\infty$.
2. $K$ is a finitely generated algebraic extension field of $F$.

Theorem A.1.6. Let $E/F$ and $K/E$ be extensions. Then

$$[K:F] = [K:E][E:F].$$

(60)
If \( \{\alpha_i \mid i = 1, \ldots, [E:F]\} \) is a basis of the vector space \( E \) over \( F \), and \( \{\beta_j \mid j = 1, \ldots, [K:E]\} \) that of \( K \) regarded as a vector space over \( E \), then the set of products \( \{\alpha_i\beta_j \mid i = 1, \ldots, [E:F], \ j = 1, \ldots, [K:E]\} \) is a basis of the vector space \( K \) over \( F \).

Definition A.1.7. For a field \( F \), \( F[x] \) denotes the ring of polynomials in a single variable \( x \) with all the coefficients in \( F \). For a finite algebraic extension \( E/F \), and for an element \( \alpha \in E \), non-zero polynomials \( p_\alpha(x) \in F[x] \) satisfying \( p_\alpha(\alpha) = 0 \in E \) with the smallest degree possible are called minimal polynomials of \( \alpha \) over \( F \). Such a polynomial always exist (because \( \alpha \) is algebraic over \( F \)), and is unique up to overall multiplication of elements in \( F^\times \). Minimal polynomials are always irreducible in \( F[x] \).

Theorem A.1.8. Let \( K/F \) be an extension and let \( \alpha \in K \) be algebraic over \( F \). Then the subfield \( F(\alpha) \) of \( K \) has a structure

\[
F(\alpha) \cong F[x]/(p_\alpha(x)),
\]

where \( p_\alpha \) is a minimal polynomial of \( \alpha \) over \( F \).

In fact, a finite algebraic extension \( K/F \)—not just a subfield \( K(\alpha) \subset K \)—always has a structure like that, when \( \text{char}(F) = 0 \); this useful property is stated as follows:

Lemma A.1.9. When \( \text{char}(F) = 0 \), any finite extension \( K/F \) is a simple extension; that is, there exists an element \( \theta \in K \) so that \( K = F(\theta) \). Using a minimal polynomial of \( \theta \) over \( F \), therefore, the field \( K \) has a structure \( K \cong F[x]/(p_\theta(x)) \). It is always possible to take \( \{1, \theta, \theta^2, \ldots, \theta^{[K:F]-1}\} \) as a basis, when \( K \) is regarded as a \([K:F]\)-dimensional vector space over \( F \).

Example A.1.10. This theorem states that even a field that is generated by multiple elements can be thought of as a simple extension. For example, \( \mathbb{Q}(i\sqrt{2}, i\sqrt{3}) = \mathbb{Q}(i\sqrt{2} + i\sqrt{3}) \).

All the definitions and theorems on algebraic extension so far are for fields that are defined abstractly by the laws of addition and multiplication among their elements. We may sometimes have a little more specific interest, however, in a field \( K \) that is defined as a subfield of \( \mathbb{C} \). For such a field \( K \), \( \text{char}(K) = 0 \) by definition.
Definition A.1.11. A complex number $\alpha \in \mathbb{C}$ is called an algebraic number, if there is a non-zero polynomial $p_{\alpha}(x) \in \mathbb{Q}[x]$ satisfying $p_{\alpha}(x = \alpha) = 0 \in \mathbb{C}$. It is known that all the algebraic numbers form a subfield of $\mathbb{C}$; this subfield is denoted by $\overline{\mathbb{Q}}$. Any finite extension field $K$ of $\mathbb{Q}$ that is defined as a subfield of $\mathbb{C}$ is called a number field.

Any number field is always a subfield of $\overline{\mathbb{Q}}$. While $\mathbb{Q}/\mathbb{Q}$ is an algebraic extension, it is not a finite extension. Thus, $\overline{\mathbb{Q}}$ itself is not a number field.

A.1.2 Embeddings into $\mathbb{C}$

Here is a summary of results on embeddings of a finite extension field $K$ over $\mathbb{Q}$ into a subfield of $\mathbb{C}$. We begin, however, with the following preparation.

Theorem A.1.12. Let $K$ be an algebraic extension over $\mathbb{Q}$, and $\alpha \in K$. For a minimal polynomial $p_{\alpha}(x)$ of $\alpha$ over $\mathbb{Q}$, there are $\deg(p_{\alpha})$ solutions to $p_{\alpha}(x) = 0$ in $\mathbb{C}$. It is known that all the roots of $p_{\alpha}(x) = 0$ come with multiplicity 1.

This property is valid for any algebraic extension over $F$, in place of $\mathbb{Q}$, as long as $\text{char}(F) = 0$, and is called separability.

Theorem A.1.13. Let $K$ be a finite extension over $\mathbb{Q}$. Then there are $[K : \mathbb{Q}]$ distinct embeddings (isomorphism onto the image) $\rho : K \hookrightarrow \mathbb{C}$ over $\mathbb{Q}$. Since all the elements in $K$ are algebraic, the image of such an embedding is always contained within $\overline{\mathbb{Q}}$; $\rho(K) \subset \overline{\mathbb{Q}} \subset \mathbb{C}$.

This is because $K$ can always be regarded as a simple extension over $\mathbb{Q}$ by a primitive element $\theta \in K$ (Lemma A.1.9); let $p_{\theta}(x)$ be its minimal polynomial over $\mathbb{Q}$, and $\{\xi_{i=1,\ldots,[K:Q]}\} \subset \mathbb{C}$ be the roots of $p_{\theta}(x) = 0$ in $\mathbb{C}$. Then $\rho_{i} : K \hookrightarrow \mathbb{C}$ is given by $\rho_{i} : K \ni \theta \mapsto \rho_{i}(\theta) = \xi_{i} \in \mathbb{C}$ for $i = 1, \ldots, [K : \mathbb{Q}]$. Note that all the $[K : \mathbb{Q}]$ roots $\{\xi_{i}\}$ are distinct from one another (separable), and hence the corresponding embeddings are distinct from one another.

Definition A.1.14. Now let $K/F$ be a finite extension with degree $m = [K : F]$. For any element $x \in K$, then, $A(x) : y \mapsto x \cdot y$ for $y \in K$ is an $F$-linear transformation on the vector space $K$ over $F$. $\text{Tr}_{K/F}(x)$ denotes the trace of the $F$-valued $m \times m$ matrix representation of $A(x)$, and is called the trace of $x \in K$. 
A.1.15. Let \( \{ \omega_i = 1, \ldots, m \} \) be a basis of \( K \) as a vector space over \( F \). Then

\[
x \cdot \omega_i = \omega_j [A(x)]_{ji},
\]

where \( [A(x)]_{ji} \) is the \( F \)-valued \( m \times m \) matrix representation of \( A(x) \). Now, let us take \( F = \mathbb{Q} \). The relation (62) among elements in \( K \) still holds as one among their images under the embeddings of \( K \) into \( \overline{\mathbb{Q}} \subset \mathbb{C} \).

\[
\rho_a(x) \rho_a(\omega_i) = \rho_a(\omega_j) [A(x)]_{ji}.
\]

(63)

Since there are \( m \) distinct embeddings \( \rho_{a=1,\ldots,m} : K \rightarrow \overline{\mathbb{Q}} \subset \mathbb{C} \), \( \rho_a(\omega_i), \rho_a(\omega_j) \) and \( \rho_a(x) \) can be regarded as \( \mathbb{C} \)-valued \( m \times m \) matrices (the matrix \( \rho_a(x) \) is diagonal), and the following relation is obtained:

\[
\text{Tr}_{K/\mathbb{Q}}(x) = \text{tr}_{m \times m} [A(x)] = \sum_{a=1}^{m} \rho_a(x);
\]

(64)

each contribution on the right-hand side is an algebraic number in \( \mathbb{Q} \subset \mathbb{C} \), but their sum should be in \( \mathbb{Q} \), because the left-hand side is, by definition.

A.1.3 Normal Closure

Definition A.1.16. Let \( K \) be a number field, i.e., a subfield of \( \overline{\mathbb{Q}} \subset \mathbb{C} \) that is a finite extension over \( \mathbb{Q} \). Let \( \theta \) be a primitive element (i.e., \( K = \mathbb{Q}(\theta) \)), \( p_\theta(x) \) be its minimal polynomial over \( \mathbb{Q} \), and \( \{ \xi_1 = \theta, \xi_2, \ldots, \xi_{[K:Q]} \} \) be the roots of \( p_\theta(x) = 0 \) in \( \mathbb{C} \). The field \( \mathbb{Q}(\xi_1, \ldots, \xi_{[K:Q]}) \subset \overline{\mathbb{Q}} \) is called the smallest splitting field of \( p_\theta(x) \in \mathbb{Q}[x] \) in \( \overline{\mathbb{Q}} \).

A.1.17. Thinking of a number field \( K \) as an abstract finite extension field over \( \mathbb{Q} \), we see that there must be \( [K : \mathbb{Q}] \) embeddings \( \rho_i : K \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C} \), \( i = 1, \ldots, [K : \mathbb{Q}] \) (Thm A.1.13). The embedding \( \rho_i = 1 : K \hookrightarrow \overline{\mathbb{Q}} \) is a trivial identification, and \( \rho_i(K) = K \subset \overline{\mathbb{Q}} \). For other \( \rho_i \)'s, however, it is not guaranteed that \( \rho_i(K) = K \).

A.1.18. The field \( \mathbb{Q}(\xi_1, \ldots, \xi_{[K:Q]}) \) can be regarded as the minimal subfield of \( \overline{\mathbb{Q}} \) that contains all the images \( \cup_{i=1,\ldots,[K:Q]} \rho_i(K) \) of the \( [K : \mathbb{Q}] \) embeddings from \( K \) to \( \overline{\mathbb{Q}} \). Because of this characterization, the smallest splitting field \( \mathbb{Q}(\xi_1, \ldots, \xi_{[K:Q]}) \subset \overline{\mathbb{Q}} \) of \( p_\theta(x) \) in \( \overline{\mathbb{Q}} \) does not depend on the choice of a primitive element \( \theta \).

Theorem A.1.19. For a subfield \( K^{nc} := \mathbb{Q}(\xi_1, \ldots, \xi_{[K:Q]}) \) of \( \overline{\mathbb{Q}} \) for a number field \( K \), any one of the embeddings \( \rho : K^{nc} \hookrightarrow \overline{\mathbb{Q}} \) over \( \mathbb{Q} \) maps \( K^{nc} \) to \( K^{nc} \subset \overline{\mathbb{Q}} \), not outside of \( K^{nc} \).
(though not necessarily as a trivial map on $K^{nc}$)—(*). This is because such an embedding $\rho$ has to send $\xi_i$’s to $\xi_i$’s, possibly with a permutation among them, and cannot do anything more than that.

A.1.20. A subfield $E$ of $\mathbb{C}$ is said to be a **normal extension of** $\mathbb{Q}$, if it has the property (*) referred to above. The minimum subfield in $\mathbb{C}$ of a number field $K$ that is a normal extension over $\mathbb{Q}$ is called the **normal closure of** $K/\mathbb{Q}$ **in** $\mathbb{C}$, and is denoted by $K^{nc}$, as we have done already above. For a number field $K$, therefore, the smallest splitting field $\mathbb{Q}(\xi_1, \cdots, \xi_{[K: \mathbb{Q}]} \subset \overline{\mathbb{Q}}$ of a primitive element $\theta$ such that $K = \mathbb{Q}(\theta)$ is the normal closure of $K$.

For a finite extension field $E$ over $\mathbb{Q}$ that is defined as an abstract field, one can pick any one of embeddings $\rho : E \hookrightarrow \overline{\mathbb{Q}} \subset \mathbb{C}$. The normal closure of $\rho(E)$ in $\overline{\mathbb{Q}}$ does not depend on which one of $[E : \mathbb{Q}]$ embeddings is used. So, we use a notation $E^{nc}$ for $(\rho(E))^{nc}$ in this article.

The definition of a normal extension $E \subset \overline{\mathbb{Q}}$ over $\mathbb{Q}$ is generalized to extensions $E \subset \overline{\mathbb{Q}}$ over an arbitrary number field $F$ by replacing $\mathbb{Q}$ in A.1.19 and A.1.20 with $F$.

**Definition A.1.21.** An algebraic extension $E/F$ is said to be **Galois**, if it is a separable and normal extension. Note that the separability is always guaranteed, when $F$ has $\text{char}(F) = 0$.

**Example A.1.22.** Let $n \in \mathbb{Z}$ and suppose that $|n|$ is not the square of an integer. $K = \mathbb{Q}[x]/(x^2 - n)$ is totally real [resp. totally imaginary] if $n > 0$ [resp. $n < 0$]. This field $K$
has two embeddings into \( \mathbb{C} \); \( \rho_{\pm} : x \mapsto \pm \sqrt{n} \in \mathbb{C} \). On the other hand, \( K = \mathbb{Q}[x]/(x^3 - 2) \) is neither totally real nor totally imaginary; the three embeddings of \( K \) to \( \mathbb{C} \) send \( x \in K \) to one of the three roots of \( x^3 - 2 = 0 \) in \( \mathbb{C} \).

Now, here is the definition of a CM field.

**Definition A.2.3.** A finite extension field \( K \) over \( \mathbb{Q} \) is said to be a CM field, if (i) it contains a subfield \( K_0 \) that is totally real, (ii) \( K \) is a degree-2 extension of \( K_0 \), and (iii) \( K \) itself is totally imaginary. Therefore, \([K : \mathbb{Q}] = [K : K_0][K_0 : \mathbb{Q}] = 2[K_0 : \mathbb{Q}]\) is always an even integer.

**Proposition A.2.4.** Let \( K \) be a CM field with \([K : \mathbb{Q}] = 2n\). Its \( 2n \) embeddings to \( \mathbb{C} \) can be grouped into \( n \) pairs, \((\rho_i, \bar{\rho}_i)\) for \( i = 1, \cdots, n \), so that \( \bar{\rho}_i(x) = (\rho_i(x))^{cc} \), where the superscript \( cc \) is the complex conjugation operation in \( \mathbb{C} \). To see this, let \( K = \mathbb{Q}(\theta) \) for some primitive element \( \theta \in K \). For a minimal polynomial \( p_\theta(x) \in \mathbb{Q}[x] \) for \( \theta \), all the \( 2n \) roots of \( p_\theta(x) = 0 \) have non-zero imaginary parts, and are grouped into \( n \) pairs, \((\xi, \xi^{cc})\) with \( i = 1, \cdots, n \). The embedding \( \rho_i : \theta \mapsto \xi_i \) forms a pair with \( \bar{\rho}_i : \theta \mapsto \xi_i^{cc} \).

**Example A.2.5.** Because the extension degree of a CM field is always even, the simplest CM field is a quadratic extension over \( \mathbb{Q} \); quartic extensions come next. CM fields \( K \) with \([K : \mathbb{Q}] = 2\) are always in the form of \( K \cong \mathbb{Q}[x]/(x^2 + d) \cong \mathbb{Q}(\sqrt{-d}) \), where \( d \) is a positive integer that is not divisible by the square of an integer. Fields defined by \( K = \mathbb{Q}[x]/(ax^2 + bx + c) \) for \( a, b, c \in \mathbb{Q} \) with \( 4ac - b^2 > 0 \) can always be brought into the form of \( \mathbb{Q}[x]/(x^2 + d) \) by redefining \( x \). Such fields are called quadratic imaginary fields. Two embeddings \( \rho_{\pm} \) send \( x \) to \( \pm \sqrt{d} \in \mathbb{C} \). For quadratic imaginary fields, \( K^{nc} \cong K \).

**Example A.2.6.** A CM field \( K \) with \([K : \mathbb{Q}] = 4\) is always in the form of \( K \cong K_0[x]/(x^2 - p - q\eta) \), \( K_0 = \mathbb{Q}[\eta]/(\eta^2 - d) \) for a positive square free integer \( d \), and \( p, q \in \mathbb{Q} \), satisfying \( p \pm q\sqrt{d} < 0 \). The last condition needs to be imposed both for + and −, because the condition (iii) would not be satisfied if \( p + q\sqrt{d} < 0 \) but \( p - q\sqrt{d} > 0 \) (or vice versa).

CM fields \( K \) with \([K : \mathbb{Q}] = 4\) are not always Galois over \( \mathbb{Q} \). It is Galois (i.e., \( K^{nc} \cong K \)) if and only if \((p^2 - dq^2) = r^2 \) for some \( r \in \mathbb{Q} \), or \((p^2 - dq^2) = ds^2 \) for some \( s \in \mathbb{Q} \). When \( q = 0 \), in particular, \( K \) is Galois, \( K = \mathbb{Q}(\sqrt{-p}, \sqrt{d}) \) and \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) \). See Ex. 8.4 (2) of [24, 25] for more information.

**Example A.2.7.** Any cyclotomic field \( K = \mathbb{Q}(\zeta_m) \) is a CM field. \( K_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1}) \).

**A.2.8.** When \( K \) is a CM field, its normal closure \( K^{nc} \) is also a CM field (Prop. 5.12, [37]).
B Hodge Structure with Complex Multiplication

B.1 CM-type Varieties and Hodge structures

A Kähler manifold is specified by a triplet of data \((M, h, J)\), where \(M\) is a manifold, \(J\) the Kähler form on \(M\) and \(h\) the complex structure. The complex structure \(h\) is encoded by specifying a decomposition

\[
H^k(M; \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_{p+q=k} H^{p,q}
\]

into \(h^{p,q}\)-dimensional vector spaces—\((p, q)\)-Hodge components—over \(\mathbb{C}\).

Hodge structure is a notion that extracts the property of complex structure above in the language of linear algebra. A rational Hodge structure\(^{36}\) \(h\) on a vector space \(V_{\mathbb{Q}}\) over \(\mathbb{Q}\) with weight \(k\) is a decomposition

\[
V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes \mathbb{C} = \bigoplus_{p+q=k, p \geq 0, q \geq 0} V_{p,q}, \quad V_{p,q} = V_{q,p}.
\]

Information equivalent to the decomposition is also provided by specifying a representation

\[
\tilde{h} : \mathbb{C}^\times \cong \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m) \to \text{GL}(V_{\mathbb{R}})
\]

on \(V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R}\); for \((a + ib) \in \mathbb{C}^\times\), \(\tilde{h}(a + ib)|_{V_{p,q}}\) multiplies \((a + ib)^p(a - ib)^q\). Once such a representation is given, then \(V_{p,q}\) can be extracted as the eigenspace of the representation \((p, q)\), and the weight is read out through \(\tilde{h}(a + i0) = a^{p+q} = a^k\).

**Definition B.1.1.** A rational Hodge structure on \(V_{\mathbb{Q}}\) is said to be **simple**, if there is no rational Hodge substructure. A **rational Hodge substructure exists** in a rational Hodge structure on \(V_{\mathbb{Q}}\), when there is a vector subspace \(W_{\mathbb{Q}} \subset V_{\mathbb{Q}}\) over \(\mathbb{Q}\) so that \(W_{p,q} := (W_{\mathbb{Q}} \otimes \mathbb{C}) \cap V_{p,q}\) satisfy \((W_{\mathbb{Q}} \otimes \mathbb{C}) \cong \bigoplus_{p+q=k} W_{p,q}\). An existence of such a \(\mathbb{Q}\)-vector subspace is highly non-trivial; the Néron-Severi lattice of the second cohomology of an algebraic K3 surface is an example of such a subspace, but it does not exist for generic (non-algebraic) complex analytic K3 surfaces.

**Proposition B.1.2.** Suppose that there is a rational Hodge substructure on \(W_{\mathbb{Q}} \subset V_{\mathbb{Q}}\). Let us then take a vector subspace \(U_{\mathbb{Q}}\) so that \(W_{\mathbb{Q}} \oplus U_{\mathbb{Q}} \cong V_{\mathbb{Q}}\). With the definition \(U_{p,q} :=\)

\(^{36}\)In this article, we only need to deal with pure Hodge structures, since we exclusively deal with compact and smooth geometry for compactifications.
(U_Q \otimes \mathbb{C}) \cap V^{p,q},$ the Hodge decomposition on $V_Q \otimes \mathbb{C}$ can be split into those of $W_Q \otimes \mathbb{C}$ and $U_Q \otimes \mathbb{C};$

$$V^{p,q} \cong W^{p,q} \oplus U^{p,q}, \quad (W_Q \otimes \mathbb{C}) \cong \oplus_{p,q=k} W^{p,q}, \quad (U_Q \otimes \mathbb{C}) \cong \oplus_{p+q=k} U^{p,q}. \quad (68)$$

**Definition B.1.3.** When a rational Hodge structure on $V_Q$ of weight $k$ is not simple, its rational Hodge substructure on a vector space $W_Q$ is said to be level-$\ell$, if $\max(|p - q|) = \ell$ on $W_Q \otimes \mathbb{C} \cong \oplus_{p+q=k} W^{p,q}.$

**B.1.4.** A compact Kähler manifold $M$ (with its complex structure specified) introduces a rational Hodge structure of weight $k$ on the cohomology group $H^k(M;\mathbb{Q})$, but, in general, not all the mathematically possible rational Hodge structures on the vector space $H^k(M;\mathbb{Q})$ are realized as complex structures of a family of Kähler manifolds with a given topology. While all the rational Hodge structures are realized by complex structures in the case of elliptic curves and K3 surfaces, that is not the case for Calabi–Yau threefolds. It is still useful to extract the key properties of complex structures of geometries and distil in the form of rational Hodge structure, because all the properties of rational Hodge structure are satisfied by complex structures of geometries.

At a generic point in the complex structure moduli space of a family of Calabi–Yau threefolds (with $b_1(M) = b_5(M) = 0$), we expect that the rational Hodge structure on $H^3(M;\mathbb{C})$ is simple. This does not rule out a possibility that the rational Hodge structure on $H^3(M;\mathbb{Q})$ stops being simple at a sublocus in the moduli space; we encounter such examples in the main text.

**Complex multiplication on an elliptic curve**

**B.1.5.** The condition for the existence of complex multiplication for an elliptic curve $E_\tau$, defined by eq. (11), can be translated into the language of Hodge structure on the cohomology group $H^1(E;\mathbb{Q})$. The algebra of endomorphisms of the simple rational Hodge structure

$$K := \text{End}_{\text{Hdg}}(H^1(E;\mathbb{Q})) := \{ \varphi \in \text{End}_{\mathbb{Q}}(H^1(E;\mathbb{Q})) \mid \varphi(H^{1,0}) \subset H^{1,0}, \varphi(H^{0,1}) \subset H^{0,1} \} \quad (69)$$

is known to be a field for any complex structure of $E_\tau$, and moreover, isomorphic either to $\mathbb{Q}$ or to a quadratic imaginary field (that is, $K$ is a CM field with $[K : \mathbb{Q}] = \dim_{\mathbb{Q}}(H^1(E;\mathbb{Q}))$).

The condition for existence of complex multiplication is known to be equivalent to (**); $K \cong \mathbb{Q}(\sqrt{b^2 - 4ac})$ with $b^2 - 4ac < 0$ then. See [7, 8, 12, 13], or math literatures for more information. We say that $\tau \in \mathcal{M}_{\text{cpx}}^{\text{ell}}$ is a **CM point** when the corresponding elliptic curve $E_\tau$ has complex multiplications (is of CM-type).
B.1.6. The CM points $\mathcal{M}_{CM}^{\text{ell}} \subset \mathcal{M}_{\text{cpx}}^{\text{ell}} = (\text{SL}(2; \mathbb{Z}) \backslash \mathcal{H}_{g=1})$, where $\mathcal{H}_{g=1}$ is the upper complex half plane, are classified by exploiting symmetry group actions on them as follows. While all the solutions to non-trivial quadratic polynomials (11) give rise to CM points, and hence there are infinitely many CM points, action of well-motivated symmetry groups are not transitive on all of them. The CM points are classified by their CM fields $K$, first; in the case of elliptic curves, $K$ must be one of quadratic imaginary fields $\mathbb{Q}(\sqrt{-d})$ with a positive square-free integer $d$. Furthermore, all the CM points in $\text{SL}(2; \mathbb{Z}) \backslash \mathcal{H}_{g=1}$ with a given CM field $K = \mathbb{Q}(\sqrt{-d})$ forms a single zero-dimensional Shimura variety $\text{Sh}((\mathbb{Q}(\sqrt{-d}))^\times, \tilde{h})$, where $\tilde{h}$ is a homomorphism (67) corresponding to a rational Hodge structure on $H^1(E; \mathbb{Q})$ with complex multiplication in $K = \mathbb{Q}(\sqrt{-d})$; conversely, the Shimura variety $\text{Sh}(K^\times, \tilde{h})$ of any quadratic imaginary field $K = \mathbb{Q}(\sqrt{-d})$ has a non-empty share in $\mathcal{M}_{CM}^{\text{ell}}$. Each one of those Shimura varieties consists of infinitely many points, because there are infinitely many quadratic polynomial equations (69) whose discriminant is $(-d)$ times the square of an integer.

The Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-d}))$ acts on the Shimura variety $\text{Sh}(\mathbb{Q}(\sqrt{-d})^\times, \tilde{h})$; the action is not transitive. The group $\text{SL}(2; \mathbb{Q}) \subset \text{GL}(2; \mathbb{Q})$ acts on the CM points $\mathcal{M}_{CM}^{\text{ell}}$ as $\mathbb{Q}$-coefficient rational transformations of $\tau$, but it also acts on individual Shimura varieties, not in between Shimura varieties $\text{Sh}(K^\times, \tilde{h})$ of different quadratic imaginary fields $K$.

Complex multiplication on a simple Abelian variety

Definition B.1.7. An Abelian variety $A$ is said to be simple when it has no proper Abelian subvarieties.

B.1.8. Let $A$ be a simple Abelian variety of dim$_\mathbb{C}A = g$. Its complex structure $\tau^{ij} \in \mathcal{H}_g$ determines a weight-1 rational Hodge structure on $H^1(A; \mathbb{Q})$; here, $\tau^{ij}$ is a $\mathbb{C}$-valued $g \times g$ symmetric matrix whose imaginary part is positive definite, and $\mathcal{H}_g$ the set of all such $\tau^{ij}$’s. The rational Hodge structure on $H^1(A; \mathbb{Q})$ of an Abelian variety is known to be simple if and

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37It does not matter which $\tilde{h}$ we use to specify $\text{Sh}(K^\times, \tilde{h})$ in the case of elliptic curves so long as $\tilde{h}$ has this property.

38Since there is a supersymmetric upper bound in the three-brane charge contribution from three-form fluxes, not all those infinitely many CM points are relevant in practical physics questions [14]. In this article, we just simply ignore this subtlety by tensoring $\mathbb{Q}$.

39We only consider Abelian varieties with a principal polarization in this article.
only if $A$ is simple.\textsuperscript{40} The algebra of endomorphisms of the simple rational Hodge structure,

$$L := \text{End}_{Hdg}(H^1(A; \mathbb{Q})) := \{ \varphi \in \text{End}_\mathbb{Q}(H^1(A; \mathbb{Q})) \mid \varphi(H^{p,q}) \subset H^{p,q}\}$$ (70)

is known to be a division algebra,\textsuperscript{41} and the center of $L$ is denoted by $K$. It is known that $K$ is always a field. Furthermore, there are three cases (e.g., \cite{86}); Type I: $K$ is a totally real field, and $L = K$, Type II/III: $K$ is a totally real field, and $[L : K] = 4$, and Type IV: $K$ is a CM field, $[L : K] = q^2$ for some integer $q$, and $q^2[K : \mathbb{Q}] = 2g$. A simple Abelian variety $A$ is said to have sufficiently many CM\textsuperspace{complex multiplication}s or be of CM-type when the algebra $L$ over $\mathbb{Q}$ falls into the Type IV, and moreover, $[K : \mathbb{Q}] = 2g$. Obviously this definition is a generalization of the condition (***) for elliptic curves (1-dimensional Abelian varieties). For a simple Abelian variety $A$ with sufficiently many complex multiplications, the division algebra $L$ agrees with its center $K$, because $q = 1$. See \cite{24, 25, 86} and references therein, or more modern literatures, for more information.

\textbf{B.1.9.} Now, the $2g$-dimensional vector space $H^1(A; \mathbb{Q})$ over $\mathbb{Q}$ can be regarded as a 1-dimensional vector space over $K$, where the scalar multiplication of $\varphi \in K$ is to apply the endomorphism $\varphi$ on $H^1(A; \mathbb{Q})$. By choosing any non-zero element $e \in H^1(A; \mathbb{Q})$, one finds an isomorphism $K \cong H^1(A; \mathbb{Q})$ as a vector space over $K$. A $\mathbb{Q}$-basis $\{e_{i=1,\ldots,2g}\}$ of $H^1(A; \mathbb{Q})$ can readily be regarded as a $\mathbb{Q}$-basis $\{\omega_{i=1,\ldots,2g}\}$ of $K$ and vice versa. With this identification, $[A(\varphi)]_{ji}$ in (62) is regarded as the $\mathbb{Q}$-valued matrix representation (defining representation) of $\varphi \in K$ on $H^1(A; \mathbb{Q})$. Equation (63) implies that $\rho_a(\varphi) \in \mathbb{C}$ with $a = 1, \ldots, 2g$ are the eigenvalues of the matrix $[A(\varphi)]_{ji}$, and the $2g$ column vectors of the matrix $[\rho_a(\omega_i)]^{-1}$ labeled by $a$ are the corresponding eigenvectors of $\rho_a(\varphi)$. One and the same eigenspace decomposition of $H^1(A; \mathbb{Q}) \otimes \mathbb{C}$ is shared by all the endomorphisms $\varphi = A(\varphi) \in K$.

\textbf{B.1.10.} Among the $2g$ eigenspaces, $g$ eigenspaces should correspond to $H^{0,1} \subset H^1(A; \mathbb{Q}) \otimes \mathbb{C}$ and the remaining $g$ of them to $H^{1,0} \subset H^1(A; \mathbb{Q}) \otimes \mathbb{C}$. The $g$ embeddings corresponding to the former set of eigenspaces are grouped into a set $\Phi := \{\rho_{a=1,\ldots,g}\}$, and the remaining $g$ embeddings into $\overline{\Phi} := \{\overline{\rho}_{a=1,\ldots,g}\}$. The set of information $(K, \Phi)$ of a simple Abelian variety $A$ with sufficiently many complex multiplications is called the CM type of $A$. More generally,

\textsuperscript{40}The authors found 1.11.3, 1.11.4 and 1.12.1 of \cite{85} informative.
\textsuperscript{41}Any non-zero element of $L$ has an inverse element in $L$, but the multiplication law is not necessarily commutative. This is the definition of a division algebra.
\textsuperscript{42}This second case can be split further into two different types, and it is customary in math literatures to think of $L$ in four different types. We are not concerned about such a detailed classification in this article, however.
for a CM field $K$ and a set of its embeddings into $\overline{\mathbb{Q}}$, $\Phi = \{\rho_{a=1, \ldots, [K:Q]/2}\}$, the pair $(K, \Phi)$ is called a **CM type** when $\Phi$ contains no pair of embeddings which are complex conjugate to each other.

For a CM type $(K, \Phi)$, a subfield

$$K^r := \mathbb{Q}\left(\left\{\sum_{\rho \in \Phi} \rho(x) \mid x \in K\right\}\right)$$

(71)

declared the **reflex field** of $(K, \Phi)$. By definition, it is a subfield of $K^{nc}$. In the case of simple Abelian variety $A$ with sufficiently many CMs, $K^{nc}, K$ and $K^r$ are not necessarily isomorphic to one another, when $\dim \mathbb{C}A = g > 1$.

**Complex multiplication on an Abelian variety not necessarily simple**

**Definition B.1.11.** Let $A$ and $B$ be abelian varieties. A homomorphism $A \to B$ is called an **isogeny** if it is surjective, and has finite (zero-dimensional) kernel. The existence of an isogeny $A \to B$ is an equivalence relation between $A$ and $B$. When such an isogeny exists, $A$ and $B$ are said to be **isogenous** to each other.

**B.1.12.** For any Abelian variety $A$, there is an isogeny with a product of simple Abelian varieties of the form $(B_1 \times \cdots \times B_1) \times (B_2 \times \cdots \times B_2) \times \cdots = B_1^{h_1} \times B_2^{h_2} \times \cdots$, where $B_1, B_2, \ldots$ are simple Abelian varieties and are not isogenous to each other. The Abelian variety $A$ is simple if and only if just one simple Abelian variety $B_1$ is found in this product, and moreover, $h_1 = 1$.

The algebra $L_A$ of endomorphisms preserving the Hodge structure of an Abelian variety $A$ depends only on its isogeny class. $L_A$ is a division algebra if and only if $A$ is simple. When $A$ is not simple, the algebra has the form of

$$L_A \cong M_{h_1}(D_1) \times M_{h_2}(D_2) \times \cdots, \quad D_i = \text{End}_{\text{Hdg}}(B_i),$$

(72)

where $M_h(D)$ is the algebra of $D$-valued $h \times h$ matrices. Let $K_i$ be the center of the division algebra $D_i$; the center $K_i$ is a field (as in B.1.8), and it is also the center of $M_{h_i}(D_i)$. It is further known [87, 38] that there always exists a maximal subfield $F_i$ of $M_{h_i}(D_i)$ containing $K_i$, where $[F_i : K_i] = h_i q_i$.

**B.1.13.** An Abelian variety that is not simple is said to have **sufficiently many CMs** or be of **CM-type**, if and only if all the simple Abelian varieties $B_i$ have sufficiently many CMs. In this case, all the fields $F_i$ are CM fields, and $[F_i : \mathbb{Q}] = 2 h_i \dim_{\mathbb{Q}} B_i$. 

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B.1.14. An Abelian variety $A$ is said to be **isotypic**, if it is isogenous to a product $B^h$ of a simple Abelian variety $B$. When an isotypic Abelian variety $A$ is of CM type, the set $(F, \Phi_F)$, where $F$ is its CM field and $\Phi_F$ the half of the embeddings of $F$ to $\mathbb{C}$ corresponding to $H^{1,0}(A)$, is said to be its **CM type**. The reflex field of $(F, \Phi_F)$ is defined as in B.1.10.

B.1.15. A CM type $(K, \Phi)$ is said to be **primitive**, if there is no non-trivial subfield $K'$ of $K$ and its CM type $(K', \Phi')$ from which $\Phi$ is induced. $(\Phi$ is induced by inspecting whether an embedding of $K$ falls into $\Phi'$ or $\overline{\Phi}$ when restricted upon $K'$). For a CM field $F$ associated with an Abelian variety $(B)^h$ made up of a simple Abelian variety $B$ with sufficiently many CMs, for example, a CM type $(F, \Phi_F)$ is induced from the CM type $(K, \Phi)$ of $B$; $(F, \Phi_F)$ and $(K, \Phi)$ play the role of $(K, \Phi)$ and $(K', \Phi')$, respectively, in the definition. This characterization of the primitiveness/non-primitiveness of a CM type in terms Abelian varieties can also be used as an alternative definition. This is possible because for a given CM type $(F, \Phi_F)$, one can always find an Abelian variety $A$ whose CM type is $(F, \Phi_F)$, as we will state in B.1.22. When $(K, \Phi)$ is not primitive, the presence of the CM type $(K', \Phi')$ with $h := [K : K'] > 1$ infers that Abelian varieties for $(K, \Phi)$ are not simple, but isogenous to $B^h$ for some Abelian variety $B$ for $(K', \Phi')$. Conversely, when the Abelian variety $A$ for a CM type $(K, \Phi)$ is not primitive, an Abelian subvariety $B$ of $A$ has a CM type $(K', \Phi')$ from which $(K, \Phi)$ is induced.

B.1.16. When a CM type $(K, \Phi)$ is given, then there is a notion of unique primitive CM subtype $(K', \Phi')$, from which $(K, \Phi)$ is induced [88]. $(K', \Phi')$ is constructed by first thinking of a CM type $(K^{nc}, \Phi^{nc})$, where $\Phi^{nc}$ is induced from $\Phi$, and then determine $(K', \Phi')$ as the unique primitive CM subtype of $(K^{nc}, \Phi^{nc})$, which is also the unique primitive CM subtype of $(K, \Phi)$. $(K, \Phi)$ is primitive, if and only if $K = K'$.

B.1.17. For a CM type $(K, \Phi)$, one can think of a CM type $(K^{nc}, (\Phi^{nc})^{-1})$, and then its unique primitive CM subtype, which is denoted by $(K^r, \Phi^r)$. The CM field characterized in this way is denoted by $K^r$, the same as the reflex field of $(K, \Phi)$, because they are actually the same. The CM type $(K^r, \Phi^r)$ is called the **reflex** of the CM type $(K, \Phi)$.

B.1.18. The reflex of the reflex $(K^r, \Phi^r)$ of a CM type $(K, \Phi)$ is denoted by $(K^{rr}, \Phi^{rr})$; it is known that $K^{rr} \subset K$, and $\Phi$ is induced from $\Phi^{rr}$. Since $(K^{rr}, \Phi^{rr})$ is always primitive, by construction, it follows that $(K^{r}, \Phi^{r}) = (K^{rr}, \Phi^{rr})$.

B.1.19. The reflex $(K^r, \Phi^r)$ of a CM type $(K, \Phi)$ agrees with that of $(K', \Phi')$, where $(K', \Phi')$ is the unique primitive CM subtype of $(K, \Phi)$ (§20.1, [25]).
Let $A$ be an Abelian variety that is isogenous to $B^h$, where $B$ is a simple Abelian variety, and $(F, \Phi_F)$ [resp. $(K, \Phi)$] be the CM type of $A$ [resp. $B$]. The reflex $(K^r, \Phi^r)$ of $(F, \Phi_F)$ therefore depends only on $(K, \Phi)$ of the simple Abelian variety.

**B.1.20.** The reflex CM type $(K^r, \Phi^r)$ is not always the same as $(K, \Phi)$. If $K^{nc}/\mathbb{Q}$ is an Abelian extension, then $K' = K^r$. If $(K, \Phi)$ is primitive, then $K' = K$. So the combination of those two conditions is a sufficient condition for $K = K^r$ (Ex. 8.4. (1), [24, 25]). All the quadratic imaginary fields are covered by this sufficient condition for $(K, \Phi) = (K^r, \Phi^r)$.

**Definition B.1.21.** For a CM type $(K, \Phi)$, the **type norm** is the group homomorphism $N_{\Phi} : K^\times \ni x \mapsto \prod_{\rho \in \Phi}(\rho(x)) \in (K^r)^\times$. The **reflex norm** is the group homomorphism $N_{\Phi^r} : (K^r)^\times \ni y \mapsto \prod_{\rho' \in \Phi^r}(\rho'(y)) \in (K^{rr})^\times \subset K^\times$.

**Classification of CM points in $\mathcal{H}_g$**

**B.1.22.** Here is a statement on the classification of $g$-dimensional Abelian varieties with sufficiently many CMs; this is the generalization of the statement B.1.6 for elliptic curves. First, any CM type $(K, \Phi)$ with $[K : \mathbb{Q}] = 2g$ specified abstractly can be realized as the CM type of an Abelian variety. Such an Abelian variety can be constructed by taking a quotient of $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{2g}$ by an ideal of a ring of algebraic integers within $K$, and by introducing complex structure on this torus by $\Phi$. There can be many Abelian varieties that are not mutually isomorphic for a given CM type $(K, \Phi)$, because there is freedom in choosing the ideal $[88, 38, 24, 25]$. A zero-dimensional subvariety $\text{Sh}(N_{\Phi^r}((K^r)^\times), \tilde{h})$ in the moduli space $\text{Sp}(2g; \mathbb{Z}) \backslash \mathcal{H}_g = \mathcal{M}_{\text{CM}}^{A_g}$ consists of the corresponding CM points of a given CM type $(K, \Phi)$. Any CM type $(K, \Phi)$ with $[K : \mathbb{Q}] = 2g$ has its non-empty share of CM points in the form of $\text{Sh}(N_{\Phi^r}((K^r)^\times), \tilde{h})$ in $\mathcal{M}_{\text{CM}}^{A_g}$. Unlike in the case of elliptic curves, however, it is not guaranteed whether there is just one Shimura variety associated with a given CM field $K^r$ in $\mathcal{M}_{\text{CM}}^{A_g}$, or maybe more than one with the same $K^r$.

Note also that this statement is valid whether the CM type $(K, \Phi)$ is primitive or not; the Mumford–Tate group (13) is not necessarily of $2g$-dimensions.

The Galois group $\text{Gal}(\overline{\mathbb{Q}}/K^r)$ acts on individual Shimura varieties $\text{Sh}(N_{\Phi^r}((K^r)^\times), \tilde{h})$; the action is not transitive. The action of $\text{Sp}(2g; \mathbb{Q})$ also preserves the CM type $(K, \Phi)$. The action of $\text{Sp}(2g; \mathbb{Q})$ illustrates nicely that the CM points arise as a family of infinitely many. See [89–93] for developments on the distribution of CM points in the moduli space.
CM-type K3 surface

Let $T$ be the transcendental lattice of a K3 surface $S$. $T$ is a rank-$(22 - \rho)$ lattice, where $\rho$ is the Picard number of $S$. $T \otimes \mathbb{Q}$ admits a weight-2 simple rational Hodge substructure of $H^2(S; \mathbb{Q})$. The algebra of endomorphisms of the rational Hodge structure

$$K := \text{End}_{\text{Hdg}}(T \otimes \mathbb{Q}) := \{ \varphi \in \text{End}_\mathbb{Q}(T \otimes \mathbb{Q}) \mid \varphi(H^{p,q}) \subset H^{p,q} \}$$

is a field. It is known that $K$ is either totally real or a CM field, and that $\dim \mathbb{Q}(T \otimes \mathbb{Q}) = (22 - \rho)$ is divisible by $[K : \mathbb{Q}]$. A K3 surface $S$ is said to be a CM-type K3 surface if the field $K$ is a CM field with $[K : \mathbb{Q}] = \dim \mathbb{Q}(T \otimes \mathbb{Q})$. Once again, this definition is a generalization of (**) for elliptic curves. Since there is no operation of multiplying a complex number in certain way as an action from $S$ to $S$ in this definition, we no longer use “complex multiplication” but just say “CM.” Due to the definition of a CM field, $(22 - \rho)$ is an even number for a CM-type K3 surface. See (§3, [84]) and [41, 94] for more information.

Precisely the same statement as in B.1.9 holds true for the CM field $K$ of a CM-type K3 surface $S$, after replacing $H^1(A; \mathbb{Q})$ by $T \otimes \mathbb{Q}$ and its dimension $2g$ by $22 - \rho$. A statement for a CM-type K3 surface corresponding to B.1.10 is the following: among the $22 - \rho$ embeddings of $K$ into $\mathbb{C}$, two embeddings correspond to the eigenspaces $H^{2,0}$ and $H^{0,2}$ of all $A(\varphi) = \varphi \in K$; those embeddings are denoted by $\epsilon$ and $\bar{\epsilon}$, respectively; all the remaining $(20 - \rho)$ eigenspaces combined correspond to the $H^{1,1}$ component of $T \otimes \mathbb{C}$.

It is known that any CM field $K$ with $[K : \mathbb{Q}] = 2n$, $2 \leq 2n \leq 20$, can be realized as the field $K$ of a K3 surface $S$ defined by (73) [26, 95]; the transcendental lattice $T$ of $S$ then has a rank $2n$. The proof of this statement in [26] (which is for $2n \leq 16$) constructs, for a CM field $K$ and one of its embeddings $\epsilon$, $\mathbb{Q}$-valued intersection forms on $K$ so that $\epsilon$ is associated with the Hodge $(2,0)$ component; a sublattice $T$ needs to be extracted from the vector space $K$ so that the intersection form becomes even and integral. The way the intersection forms are constructed in [26] suggests strongly that there are infinitely many inequivalent rank-$2n$ lattices $T_0$ and its period domain $D(T_0)$ where a given pair $(K, \epsilon)$ can be realized. Once a CM point with $(K, \epsilon)$ is found in a period domain $D(T_0)$, however, there must be infinitely many CM points with the same $(K, \epsilon)$ within $D(T_0)$ of the same lattice $T_0$ (if $2n > 2$). This is because the group $\text{Isom}(T_0 \otimes \mathbb{Q})$ takes one CM point in $D(T_0)$ to elsewhere within the same $D(T_0)$ without changing the pair $(K, \epsilon)$.

Kuga–Satake construction assigns an isogeny class of $2r' - 2$-dimensional Abelian varieties $[KS(T_S)]$ to a rank-$r'$ transcendental lattice with a rational Hodge structure of K3 type. $S$ is a CM-type K3 surface if and only if the Abelian varieties $[KS(T_S)]$ have sufficiently many CMs [84, 30]. Complex multiplications are still realized geometrically on the Abelian varieties $[KS(T_S)]$, though not on the K3 surface $S$ itself.
One will also be interested in finding a list of pairs \((K, \epsilon)\) of CM points that can be realized in the period domain of an even lattice \(T_0\) with signature \((2, 2n - 2)\). One just has to invert the \(T_0 \Rightarrow (K, \epsilon)\) list referred to above, although it is not a simple task to do so in practice. At least it is clear that not all possible pairs \((K, \epsilon)\) with \([K : \mathbb{Q}] = 2n\) are admitted in a given lattice \(T_0\) with signature \((2, 2n - 2)\). Take the \(2n = 2\) case (i.e., \(\rho = 20\)), as an obvious example. A rank-2 transcendental lattice \(T_0\) admits just one CM point, where \(K\) is the quadratic imaginary field uniquely determined by the transcendental lattice. Slightly more non-trivial examples are the cases of \(T_0 = U[2] \oplus U[2]\) (when \(S = \text{Km}(E \times F)\) Kummer surface associated with mutually non-isogenous elliptic curves \(E\) and \(F\)) and \(T_0 = U \oplus U\); a CM field with \([K : \mathbb{Q}] = 4\) is realized on these two \(D(T_0)\)’s only when the field \(K\) is constructed by setting \(q = 0\) in Example A.2.6. To the knowledge of the authors, it is not guaranteed whether a CM point is found in the non-Noether–Lefschetz locus of \(D(T_0)\) of a given lattice \(T_0\); when the rank of \(T_0\) is odd, for example, obviously there cannot be a CM point in the non-Noether–Lefschetz locus of \(D(T_0)\). Once a CM point of a pair \((K, \epsilon)\) is found in \(D(T_0)\) with \([K : \mathbb{Q}] = \dim_{\mathbb{Q}} T_0\), then this CM point is not in a Noether–Lefschetz locus of \(D(T_0)\), because the fact that \(K\) is a field indicates that the rational Hodge structure on \(T_0\) at the CM point is simple.

**CM-type Calabi–Yau threefold**

When the weight-3 rational Hodge structure on \(H^3(M; \mathbb{Q})\) of a Calabi–Yau threefold \(M\) is simple, the algebra

\[
K := \text{End}_{\text{Hdg}}(H^3(M; \mathbb{Q})) := \{ \varphi \in \text{End}_{\mathbb{Q}}(H^3(M; \mathbb{Q})) \mid \varphi(H^{p,q}) \subset H^{p,q} \}
\]

(74)

is a field. The Calabi–Yau threefold \(M\) (and also the rational Hodge structure) is said to be of **CM-type** if \(K\) is a CM field with \([K : \mathbb{Q}] = \dim_{\mathbb{Q}}(H^3(M; \mathbb{Q}))\), which is once again a generalization of the condition (**\) of elliptic curves. Precisely the same statement as in B.1.9 holds true for the CM field \(K\), by replacing \(H^1(A; \mathbb{Q})\) with \(H^3(M; \mathbb{Q})\) and \(2g\) with \(\dim_{\mathbb{Q}}(H^3(M; \mathbb{Q}))\), respectively. Among the \(\dim_{\mathbb{Q}}(H^3(M; \mathbb{Q}))\) embeddings of \(K\) into \(\mathbb{C}\), there are two special ones, denoted by \(\epsilon\) and \(\bar{\epsilon}\), whose corresponding eigenspaces are the \(H^{3,0}\) and \(H^{0,3}\) Hodge components, respectively. All the other embeddings correspond to the eigenspaces that fit within either \(H^{2,1}\) or \(H^{1,2}\) Hodge components.

\[44\] A Noether–Lefschetz locus of \(D(T_0)\) is a subspace of \(D(T_0)\) where \(T \subsetneq T_0\).
CM-type Rational Hodge Structure

When a vector space $V \otimes \mathbb{Q}$ over $\mathbb{Q}$ has a rational Hodge structure that is simple, then the algebra of endomorphisms of the simple rational Hodge structure

$$L := \text{End}_{\text{Hdg}}(V \otimes \mathbb{Q}) := \{ \varphi \in \text{End}_{\mathbb{Q}}(V \otimes \mathbb{Q}) \mid \varphi(V^{p,q}) \subset V^{p,q} \}$$

(75)
is always a division algebra. When $L$ contains a number field $K$ such that $[K : \mathbb{Q}] = \dim_{\mathbb{Q}} V \otimes \mathbb{Q}$, we say that the simple rational Hodge structure is of CM-type.

When a rational Hodge structure on a vector space $V \otimes \mathbb{Q}$ can be decomposed into multiple rational Hodge substructures that are simple, we say that the rational Hodge structure is of CM-type, when all the simple substructures are of CM-type. When a Calabi–Yau threefold $M$ has a complex structure such that the rational Hodge structure on $H^3(M; \mathbb{Q})$ is not simple, it is said to be of CM-type, if and only if the rational Hodge structure on $H^3(M; \mathbb{Q})$ is of CM-type. We may define Abelian varieties of CM-type that are not necessarily simple by requiring that $H^1(A; \mathbb{Q})$ is of CM-type, and this definition is equivalent to the one in B.1.13.

In this article, we are frequently forced to refer to the field $K$ as the algebra of endomorphisms of a simple Hodge structure of Calabi–Yau type (69, 73, 74), the ceter $K$ of the division algebra $L$ for a simple Abelian variety in (70), the field $K$ contained in the algebra $L$ of a simple rational Hodge structure (75) and the maximal subfield $F$ contained in the central simple algebra $M_h(D)$ for an isotypic Abelian variety. We just use the phrase endomorphism field for all of them in this article (as many literatures also do), since the proper expressions such as “the field of endomorphisms of a simple rational Hodge structure” is too long. This is not too much abuse of language for Abelian varieties in the first place, since the $\mathbb{Q}$-endomorphisms preserving the Hodge structure on $H^1(A; \mathbb{Q})$ can be regarded as some endomorphisms of a group variety $A$ tensored with $\mathbb{Q}$.

B.2 Hodge Components and Embeddings of a CM Field

Suppose that a simple rational Hodge structure is given on a vector space $V \otimes \mathbb{Q}$, and that the endomorphism field $K$ is a CM field satisfying $[K : \mathbb{Q}] = \dim_{\mathbb{Q}} V \otimes \mathbb{Q} =: m$. We have seen in B.1.9 that, for $a = 1, \ldots, m$ labeling distinct embeddings $\rho_a : K \hookrightarrow \mathbb{C}$, we can choose a vector $v_a \in V \otimes \mathbb{C}$ so that it is an eigenvector of $A(\varphi) = \varphi \in K$ with an eigenvalue $\rho_a(\varphi) \in \mathbb{C}$ for any $\varphi \in K$. Moreover, when the vector $v_a$ is expressed as $v_a = \sum_{i=1}^{m} e_i c_i^a$, where $\{e_i = 1, \ldots, m\}$ is a $\mathbb{Q}$-basis of $V \otimes \mathbb{Q}$, and $c_i^a \in \mathbb{C}$, these coefficients $c_i^a$’s can be chosen within $K^{\text{nc}} \subset \mathbb{C}$, because all the $m \times m$ entries of the matrix $[\rho_a(\omega_i)]^{-1}$ take their values in $K^{\text{nc}}$. Although the reasoning
here is applicable\textsuperscript{45} to any finite extension field \( F \) over \( \mathbb{Q} \), yet in the present context, it also means that the Hodge decomposition is possible not just in \( V_{\mathbb{Q}} \otimes \mathbb{C} \), but even within \( V_{\mathbb{Q}} \otimes K^{nc} \).

In fact, there is a much stronger result. Let \( \{ x_{p=1, \ldots, m} \} \) be a \( \mathbb{Q} \)-basis of \( K \). In the notation above,

\[
A_{ji}(x_p)c_{ia} = c_{ja}\rho_a(x_p). \tag{76}
\]

We use those relations for \( j = 1 \). Now, thinking that those relations are

\[
[A_{i1}(x_p)]_p \begin{bmatrix} c_{ia} \\ c_{1a} \end{bmatrix}_i = [\rho_a(x_p)]_p, \tag{77}
\]

where a \( \mathbb{Q} \)-valued matrix \([\cdots]_p\) is multiplied to a \( \mathbb{C} \)-valued vector \([\ ]_p\) on the right-hand side, we see that the \( \mathbb{Q} \)-valued matrix must be invertible; this is because \( x_p \)'s in \( K \) (and hence \( \rho_a(x_p) \)'s in \( \rho_a(K) \)) should be linearly independent over \( \mathbb{Q} \). So, we replace the \( \mathbb{Q} \)-basis of \( K \) \( \{ x_{p=1, \ldots, m} \} \) by the one—denoted by \( \{ y_{i=1, \ldots, m} \} \)—obtained by multiply the inverse of the \( \mathbb{Q} \)-valued matrix \([A_{i1}(x_p)]_p\) on \( x_p \)'s. In this new basis, there is a relation

\[
c_{ia}/c_{1a} = \rho_a(y_i). \tag{78}
\]

The coefficients \( c_{ia}/c_{1a} \) for the eigenvector \( v_a \) all take their values in \( \rho_a(K) \subset K^{nc} \), first of all, and \( c_{ia}/c_{1a} \in \mathbb{C} \) for \( a = 1, \cdots, m \) are the images of the embeddings \( \rho_a \) of a common element \( y_i \in K \), secondly.

When the CM field \( K \) is Galois, \( \rho_a \circ (\rho_b)^{-1} \in \text{Gal}(K/\mathbb{Q}) \) maps the algebraic number \( c_{ib}/c_{1b} \in K \subset \overline{\mathbb{Q}} \) to \( c_{ia}/c_{1a} \in K \subset \overline{\mathbb{Q}} \) for all \( i = 1, \cdots, [K : \mathbb{Q}] \) simultaneously, as in \([14]\). Even when the CM field \( K \) is not a Galois extension over \( \mathbb{Q} \), an isomorphism \( \rho_a \circ (\rho_b)^{-1} : \rho_b(K) \to \rho_a(K) \) extends to an isomorphism from \( \overline{\mathbb{Q}} \) to itself (Thm. 2.19, [83]), which can be restricted to an automorphism of a normal extension \( K^{nc} \) over \( \mathbb{Q} \). Thus, \( \rho_a \circ (\rho_b)^{-1} : \rho_b(K) \to \rho_a(K) \) can be realized by restricting some elements in \( \text{Gal}(K^{nc}/\mathbb{Q}) \). Therefore, the simultaneous map of algebraic numbers \( c_{ib}/c_{1b} \in \rho_b(K) \) to \( c_{ia}/c_{1a} \in \rho_a(K) \) can be regarded as a result of an automorphism in \( \text{Gal}(K^{nc}/\mathbb{Q}) \).

\textsuperscript{45}The corresponding statement is this: when multiplication of elements of \( F \) is seen as action on the vector space \( F \) over \( \mathbb{Q} \), the \( \mathbb{Q} \)-representation of \( F \) over \( F_{\mathbb{Q}} \) can be decomposed into 1-dimensional (and hence irreducible) representations when the representation space \( F_{\mathbb{Q}} \) is tensored with the splitting field \( F^{nc} \).
References

[1] F. Denef, “Les Houches Lectures on Constructing String Vacua,” in String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007, pp. 483–610, 2008. arXiv:0803.1194 [hep-th].

[2] E. Witten, “Symmetry Breaking Patterns in Superstring Models,” Nucl. Phys. B258 (1985) 75.

[3] J. A. Bagger, T. Moroi and E. Poppitz, “Anomaly mediation in supergravity theories,” JHEP 04 (2000) 009, [arXiv:hep-th/9911029].

[4] F. Denef, M. R. Douglas, B. Florea, A. Grassi and S. Kachru, “Fixing all moduli in a simple F-theory compactification,” Adv. Theor. Math. Phys. 9 (2005) 861–929, [arXiv:hep-th/0503124].

[5] R. Tatar, Y. Tsuchiya and T. Watari, “Right-handed Neutrinos in F-theory Compactifications,” Nucl. Phys. B823 (2009) 1–46, [arXiv:0905.2289 [hep-th]].

[6] R. Tatar and T. Watari, “Proton decay, Yukawa couplings and underlying gauge symmetry in string theory,” Nucl. Phys. B747 (2006) 212–265, [arXiv:hep-th/0602238].

[7] G. W. Moore, “Attractors and arithmetic,” arXiv:hep-th/9807056.

[8] G. W. Moore, “Arithmetic and attractors,” arXiv:hep-th/9807087.

[9] P. Candelas, X. de la Ossa and F. Rodriguez-Villegas, “Calabi-Yau manifolds over finite fields. 1.,” arXiv:hep-th/0012233.

[10] P. Candelas, X. de la Ossa and F. Rodriguez Villegas, “Calabi-Yau manifolds over finite fields. 2.,” Fields Inst. Commun. 38 (2013) 121–157, [arXiv:hep-th/0402133].

[11] R. Schimmrigk, “Arithmetic of Calabi-Yau varieties and rational conformal field theory,” J. Geom. Phys. 44 (2003) 555 [arXiv:hep-th/0111226]. R. Schimmrigk and S. Underwood, “The Shimura-Taniyama conjecture and conformal field theory,” J. Geom. Phys. 48 (2003) 169 [arXiv:hep-th/0211284].

[12] S. Gukov and C. Vafa, “Rational conformal field theories and complex multiplication,” Commun. Math. Phys. 246 (2004) 181–210, [arXiv:hep-th/0203213].

[13] G. W. Moore, “Strings and Arithmetic,” in Frontiers in number theory, physics, and geometry 2: On random matrices, zeta functions and dynamical systems. Proceedings,
[14] O. DeWolfe, A. Giryavets, S. Kachru and W. Taylor, “Enumerating flux vacua with enhanced symmetries,” *JHEP* 02 (2005) 037, [arXiv:hep-th/0411061].

[15] O. DeWolfe, “Enhanced symmetries in multiparameter flux vacua,” *JHEP* 10 (2005) 066, [arXiv:hep-th/0506245].

[16] S. Sethi, C. Vafa and E. Witten, “Constraints on low dimensional string compactifications,” *Nucl. Phys.* B480 (1996) 213–224, [arXiv:hep-th/9606122].

[17] A. Klemm, B. Lian, S. S. Roan and S.-T. Yau, “Calabi-Yau fourfolds for M theory and F theory compactifications,” *Nucl. Phys.* B518 (1998) 515–574, [arXiv:hep-th/9701023].

[18] A. P. Braun and T. Watari, “The Vertical, the Horizontal and the Rest: anatomy of the middle cohomology of Calabi-Yau fourfolds and F-theory applications,” *JHEP* 01 (2015) 047, [arXiv:1408.6167 [hep-th]].

[19] C. Borcea, “Calabi-Yau Threefolds and Complex Multiplication,” in *Essays on Mirror Manifolds* (S.-T. Yau, ed.). International Press, 1992.

[20] T. Shioda, “Geometry of Fermat Varieties,” in *Number theory related to Fermat’s last theorem* (N. Koblitz, ed.), vol. 26 of *Progr. in Math.*, pp. 45–56. Birkhäuser Boston, 1982.

[21] Y. Goto, R. Livné and N. Yui, “Automorphy of Calabi-Yau threefolds of Borcea-Voisin type over $\mathbb{Q}$,” *Commun. Number Theory Phys.* 7 (2013) 581–670, [arXiv:1212.3399 [math.NT]].

[22] J. Silverman, *The arithmetic of elliptic curves*, vol. 106 of *GTM*. Springer, 1986.

[23] T. Schneider, “Arithmetische Untersuchungen elliptischer Integrale,” *Mathematische Annalen* 113 (1937) 1–13.

[24] G. Shimura and Y. Taniyama, *Complex multiplication of abelian varieties and its applications to number theory*, vol. 6 of *Publications of the Mathematical Society of Japan*. Math. Soc. Japan, 1961, Large fraction of this book is included as sections 1–16 of [25].

[25] G. Shimura, *Abelian varieties with complex multiplication and modular functions*, vol. 46 of *Princeton Math Series*. Princeton U. Press, 1998.
[26] I. Piatetski-Shapiro and R. Shafarevich, “The arithmetic of K3 surfaces,” in Proc. Steklov Inst. Math., vol. 132, 1973. A copy is also available in [96]. Another record Proc. of the Int’l Conference on Number Theory 132 (1975) p.45. [Russian original is Trudy Mat. Inst. Steklov. 132 (1973) 45.

[27] J. Rizov, “Complex multiplication for K3 surfaces,” arXiv:math/0508018 [math.AG].

[28] P. B. Cohen, “Humbert Surfaces and Transcendence Properties of Automorphic Functions,” Rocky Mountain J. Math. 26 (1996) 987–1001.

[29] H. Shiga and J. Wolfart, “Criteria for complex multiplication and transcendence properties of automorphic functions,” J. Reine Angew. Math. 463 (1995) 1–26.

[30] P. B. Tretkoff, “K3 surfaces with algebraic period ratios have complex multiplication,” Int. J. Number Theory 11 (2015) 1709–1724.

[31] P. B. Tretkoff and M. D. Tretkoff, “A transcendence criterion for CM on some families of Calabi-Yau manifolds,” in From Fourier Analysis and Number Theory to Radon Transforms and Geometry (H. M. Farkas, R. C. Gunning, M. I. Knopp and B. A. Taylor, eds.), pp. 475–490. Springer, 2013.

[32] P. Tretkoff, “Transcendence and CM on Borcea–Voisin towers of Calabi–Yau manifolds,” J. of Number Theory 152 (2015) 118–155, [arXiv:1407.2611 [math.NT]].

[33] J. Silverman, Advanced topics in the arithmetic of elliptic curves, vol. 151 of GTM. Springer, 1994.

[34] N. Yui, “Modularity of Calabi–Yau varieties: 2011 and beyond,” in Arithmetic and Geometry of K3 Surfaces and Calabi–Yau Threefolds, vol. 67, pp. 101–139. Springer, 2013. arXiv:1212.4308 [math.NT].

[35] G. Moreland, “Class field theory for number fields and complex multiplication.” A lecture note for the REU program 2016 at U. Chicago, available at http://math.uchicago.edu/~may/REU2016/.

[36] J. S. Milne, “Class field theory,” 1997, chapter 5 is most relevant to this article. A lecture note available at http://www.jmilne.org/math/CourseNotes/cft.html.

[37] G. Shimura, Introduction to the arithmetic theory of automorphic functions. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971.

[38] M. Bilu, “Complex multiplication of abelian varieties.” available at https://www.math.ens.fr/~bilu/Complexmultiplication.pdf.
[39] J. S. Milne, “Introduction to Shimura varieties,” 2004, a lecture note available at http://www.jmilne.org/math/xnotes/svi.html.

[40] M. Kerr, “Shimura varieties: a Hodge-theoretic perspective,” 2010, a lecture delivered at Trieste Summer School on Hodge Theory. available from the summer school web page http://indico.ictp.it/event/a09153 or from the page of the author http://www.math.wustl.edu/~matkerr/.

[41] Y. G. Zarhin, “Hodge groups of K3 surfaces,” J. Reine Angew. Math 341 (1983) 193–220.

[42] P. L. Clark, “Ray class groups and ray class fields: first classically, then adelically,” 2010, an electronic copy is available in an online search.

[43] B. Moonen and F. Oort, “The Torelli locus and special subvarieties,” in Handbook of Moduli vol.II (G. Farkas and I. Morrison, eds.), vol. 25 of Adv. Lect. Math, pp. 549–594. Int. Press, Comervile, MA, 2013. arXiv:1112.0933 [math.AG].

[44] J. Tsimerman, “A proof of the Andre-Oort conjecture for $A_g$,” arXiv:1506.01466 [math.AG].

[45] M. Kerr, “Algebraic and Arithmetic Properties of Period Maps,” in Calabi-Yau Varieties: Arithmetic, Geometry and Physics (R. Lazza, M. Sch"utt and N. Yui, eds.), pp. 173–208. Springer, 2015.

[46] M. Green, P. A. Griffiths and M. Kerr, Mumford-Tate Groups and Domains: Their Geometry and Arithmetic, vol. 183 of Annals Math Studies. Princeton U. Press, 2012, a closely related material is also found in [97].

[47] F. Oort, “CM Jacobians,” 2012, proceedings for a conference “Galois covers and deformations,” available at http://www.staff.science.uu.nl/~oort0109/.

[48] A. J. de Jong and R. Noot, “Jacobians with complex multiplications,” in Arithmetic Algebraic Geometry (G. van der Geer, F. Oort and J. Steenbring, eds.), vol. 89 of Progr. Math., pp. 171–192. Birkhäuser, 1991.

[49] G. Shimura, “On purely transcendental fields automophic functions of several variable,” Osaka J. Math. 1 (1964) 1–14.

[50] F. Oort, “Special points in Shimura varieties, an Introduction,” 2003, available at http://www.staff.science.uu.nl/~oort0109/.
[51] R. Coleman, “Torsion points on curves,” in *Galois representations and arithmetic algebraic geometry* (Y. Ihara, ed.), vol. 12 of *Adv. Studies Pure Math.*, pp. 235–247. North-Holland, 1987.

[52] F. Oort, “Canonical liftings and dense sets of CM-points,” in *Arithmetic Geometry* (F. Catanese, ed.). Cambridge U. Press, 1997. Symposia Mathematica 37.

[53] A. Garbagnati and B. van Geemen, “Examples of Calabi-Yau threefolds parametrised by Shimura varieties,” *Rend. Sem. Mat. Univ. Pol. Trino* 68 (2010) 65–81, [arXiv:1005.0478 [math.AG]].

[54] E. Viehweg and K. Zuo, “Complex multiplication, Griffiths–Yukawa couplings, and rigidity for families of hypersurfaces,” *Journal of Algebraic Geometry* 14 (2005) 481–528, [arXiv:math/0307398 [math.AG]].

[55] C. Rohde, *Cyclic coverings, Calabi-Yau manifolds and complex multiplication*. No. 1975 in Lecture Notes in Mathematics. Springer Science & Business Media, 2009.

[56] B. Moonen, “Special subvarieties arising from families of cyclic covers of the projective line,” *Documenta Math.* 15 (2010) 793–819.

[57] C. Voisin, “Miroirs et involutions sur les surfaces K3,” *Astérisque* 218 (1993) 273–323.

[58] C. Borcea, “K3 surfaces with involution and mirror pairs of Calabi-Yau manifolds,” in *Mirror symmetry, II* (B. Greene and S.-T. Yau, eds.), AMS/IP Studies in Advanced Mathematics, pp. 717–743. Amer Mathematical Society, 1997.

[59] V. V. Nikulin, “Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogroometric applications,” *J. Math. Sci.* 22 (1983) 1401–1475.

[60] S. Vorontsov, “Automorphisms of even lattices arising in connection with automorphisms of algebraic K3-surfaces,” *Vestnik Moskovskogo Universiteta Seriya 1 Matematika Mekhanika* 2 (1983) 19–21.

[61] S. Mukai, “Finite groups of automorphisms of K3 surfaces and the Mathieu group,” *Inventiones mathematicae* 94 (1988) 183–221.

[62] S. Kondo, “Automorphisms of algebraic K3 susfaces which act trivially on Picard groups,” *J. Math. Soc. Japan* 44 (1992) 75–98.

[63] K. Oguiso, “A remark on the global indices of $\mathbb{Q}$–Calabi–Yau 3-folds,” *Mathematical Proceedings of the Cambridge Philosophical Society* 114 (1993) 427–429.
[64] S. Kondo and S. Mukai, “Niemeier lattices, Mathieu groups, and finite groups of symplectic automorphisms of K3 surfaces,” Duke math. J. 92 (1998) 593–604.

[65] K. Oguiso and D.-Q. Zhang, “K3 surfaces with order five automorphisms,” J. of Math.-Kyoto U. 38 (1998) 419–438.

[66] K. Oguiso and D.-Q. Zhang, “K3 surfaces with order 11 automorphisms,” arXiv:math.AG/9907020.

[67] K. Oguiso and D.-Q. Zhang, “On Vorontsov’s theorem on K3 surfaces with non-symplectic group actions,” Proc. of the American Math. Soc. 128 (2000) 1571–1580.

[68] A. Garbagnati and A. Sarti, “Symplectic automorphisms of prime order on K3 surfaces,” J. of Algebra 318 (2007) 323–350.

[69] M. Artebani and A. Sarti, “Non-symplectic automorphisms of order 3 on K3 surfaces,” Mathematische Annalen 342 (2008) 903–921.

[70] S. Taki, “Non-symplectic automorphisms of 3-power order on K3 surfaces,” Proc. of the Japan Academy, Series A, Mathematical Sciences 86 (2010) 125–130.

[71] M. Schütt, “K3 surfaces with non-symplectic automorphisms of 2-power order,” J. of Algebra 323 (2010) 206–223, [arXiv:0807.3708 [math.AG]].

[72] M. Artebani, A. Sarti and S. Taki, “K3 surfaces with non-symplectic automorphisms of prime order,” Math. Zeit. 268 (2011) 507–533.

[73] S. Taki, “Classification of non-symplectic automorphisms of order 3 on K3 surfaces,” Mathematische Nachrichten 284 (2011) 124–135.

[74] S. Taki, “Classification of non-symplectic automorphisms on K3 surfaces which act trivially on the Néron–Severi lattice,” J. of Algebra 358 (2012) 16–26.

[75] S. Taki, “On Oguiso’s K3 surface,” J. pure and appl. algebra 218 (2014) 391–394.

[76] D. A. Tabbaa, A. Sarti and S. Taki, “Classification of order sixteen non-symplectic automorphisms on K3 surfaces,” arXiv:1409.5803 [math.AG].

[77] M. Artebani and A. Sarti, “Symmetries of order four on K3 surfaces,” J. Math. Soc. Japan 67 (2015) 503–533.

[78] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, vol. 10 of Cours Spécialisés. Société Mathématique de France, 2002.
[79] P. S. Aspinwall and R. Kallosh, “Fixing all moduli for M-theory on K3×K3,” *JHEP* 10 (2005) 001, [arXiv:hep-th/0506014].

[80] A. P. Braun, Y. Kimura and T. Watari, “The Noether-Lefschetz problem and gauge-group-resolved landscapes: F-theory on K3 × K3 as a test case,” *JHEP* 04 (2014) 050, [arXiv:1401.5908 [hep-th]].

[81] F. Denef and M. R. Douglas, “Distributions of flux vacua,” *JHEP* 05 (2004) 072, [arXiv:hep-th/0404116].

[82] S. Roman, *Field theory*, vol. 158 of *GTM*. Springer Science & Business Media, 2005.

[83] G. Fujisaki, *Field and Galois Theory*. Iwanami Publ. Co., 1991, written in Japanese.

[84] D. Huybrechts, *Lectures on K3 surfaces*, vol. 158 of *Cambridge studies Adv. Math.* Cambridge U. Press, 2016.

[85] B. B. Gordon, “A survey of the Hodge conjecture for Abelian varieties,” [arXiv:alg-geom/9709030 [math.AG]], this article is the appendix B of [98].

[86] G. Shimura, “On analytic families of polarized abelian varieties and automorphic functions,” *Annals of Mathematics* 78 (1963) 149–192.

[87] I. Reiner, *Maximal orders*. Oxford U. Press, 2003.

[88] J. S. Milne, “Complex multiplication,” 2006, a lecture note available at [http://www.jmilne.org/math/CourseNotes/cm.html](http://www.jmilne.org/math/CourseNotes/cm.html).

[89] B. Edixhoven and A. Yafaev, “Subvarieties of Shimura varieties,” *Annals of Mathematics* 157 (2003) 621–645, [arXiv:math/0105241 [math.AG]].

[90] L. Clozel and E. Ullmo, “Equidistribution de sous-variétés spéciales,” *Annals of mathematics* 161 (2005) 1571–1588.

[91] S.-W. Zhang, “Equidistribution of CM-points on quaternion Shimura varieties,” *International Mathematics Research Notices* 2005 (2005) 3657–3689.

[92] E. Ullmo, “Manin–Mumford, André–Oort, the equidistribution point of view,” in *Equidistribution in Number Theory, An Introduction* (A. Granville and Z. Rudnick, eds.), pp. 103–138. Springer Netherlands, Dordrecht, 2007.

[93] E. Ullmo and A. Yafaev, “Galois orbits and equidistribution of special subvarieties: towards the André-Oort conjecture,” *Annals of Mathematics* 180 (2014) 823–865, [arXiv:1209.0934 [math.NT]].
[94] B. van Geemen, “Real multiplication on K3 surfaces and Kuga Satake varieties,” *Michigan Math. J.* 58 (2008) 375–399, [arXiv:math/0609839 [math.AG]].

[95] L. Taelman, “K3 surfaces over finite fields with given L-function,” *Algebra & Number Theory* 10 (2016) 1133–1146, [arXiv:1507.08547 [math.AG]].

[96] J. Cogdell, ed., *Selected works of Ilya Piatetski-Shapiro.* American Math Soc., 2000.

[97] M. Green, P. Griffiths and M. Kerr, “Mumford-tate domains,” *Bollettino dell'UMI* 9 (2010) 281–307, available at [http://www.math.wustl.edu/~matkerr/](http://www.math.wustl.edu/~matkerr/).

[98] J. D. Lewis, *A survey of the Hodge conjecture*, vol. 10 of *CRM Monograph Series.* American Math Soc., 1999.