Chance-Constrained Optimal Covariance Steering with Iterative Risk Allocation

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Abstract—This paper extends the optimal covariance steering problem for linear stochastic systems subject to chance constraints to account for optimal risk allocation. Previous works have assumed a uniform risk allocation to cast the optimal control problem as a semi-definite program (SDP), which can be solved efficiently using standard SDP solvers. We adopt the Iterative Risk Allocation (IRA) formalism from [1], which uses a two-stage approach to solve the optimal risk allocation problem for covariance steering. The upper-stage of IRA optimizes the risk, which is proved to be a convex problem, while the lower-stage optimizes the controller with the new constraints. This is done iteratively so as to find the optimal risk allocation that achieves the lowest total cost. The proposed framework results in solutions that tend to maximize the terminal covariance, while still satisfying the chance constraints, thus leading to less conservative solutions than previous methodologies. We also introduce two novel convex relaxation methods to approximate quadratic chance constraints as second-order cone constraints. We finally demonstrate the approach to a spacecraft rendezvous problem and compare the results.

I. INTRODUCTION

In this paper we address the problem of finite-horizon stochastic optimal control of a discrete linear time-varying (LTV) system with time-independent white-noise Gaussian diffusion. The control task is to steer the state from an initial Gaussian distribution to a final Gaussian distribution with known statistics. In addition to the boundary conditions, we consider chance constraints that restrict the probability of violating the state constraints to be less than a certain threshold. Hard state constraints are difficult to impose in stochastic systems because the noise can be unbounded, so chance constraints are used to deal with this problem by imposing a small, but finite, probability of violating the constraints. In the literature, there are two kinds of chance constraints; individual and joint [2]. Individual chance constraints limit the probability of violating each constraint, while joint chance constraints limit the probability of violating any constraint over the whole time horizon. In this paper, we consider the case of joint chance constraints, because they are a more natural choice for most applications.

The control of stochastic systems can be best formulated as a problem of controlling the distribution of trajectories over time. Moreover, Gaussian distributions are completely characterized by their first and second moments, so the control problem can be thought of as one of steering the mean and the covariance to their terminal values. The problem of covariance control has a history dating back to the ’80s, with the works of Hotz and Skelton [3], [4]. Much of the early work focused on the infinite horizon problem, where the state covariances asymptotically approach their terminal values. Only recently has the finite-horizon problem drawn attention, with much of the early work focusing on the covariance steering (CS) problem, namely, with the problem of steering an initial distribution to a final distribution at a specific final time step subject to LTV dynamics. The problem could be thought of as a linear-quadratic Gaussian (LQG) problem with a condition on the terminal covariance [5]. Moreover, it has been shown that the finite-horizon controller can be constructed as a state-feedback controller and the problem can be formulated as a convex program [5], [6], or as the solution of a pair Lyapunov differential equations coupled through their boundary conditions [7], [8]. Alternatively, for certain special cases one can solve the CS problem directly by solving an LQ stochastic problem with a particular choice of cost weights [9]. Other approaches [10], [11] use an affine disturbance feedback controller having two components, one that steers the mean state and the other that steers the covariance.

In general, the theory of steering marginal distributions has a long history stemming from the problem of Schrödinger bridges and optimal mass transport [8], [12]–[14]. Recent work has focused on incorporating physical constraints on the system, such as state chance constraints [15] and obstacles in path-planning environments [11], input hard constraints [16], incomplete state information [17], and extensions in the context of stochastic model predictive control [18] and nonlinear systems [19]–[21].

In this work, we extend the Covariance Steering Chance Constraint (CSCC) problem, to account for optimal risk allocation. By risk allocation we mean allocating the probability of violating each individual chance constraint at each time step. For example, if there are M chance constraints and N time steps, there would be NM total allocations for the whole problem. Previous works [9], [11], [15], [16], [18] have assumed a constant risk allocation, so that the resulting problem can be turned into a semi-definite program (SDP). Here, however, we adopt a two-stage algorithm that optimizes the risk distribution over all time steps, and subsequently optimizes the controller by solving a SDP. Other works have tried to optimize the risk using techniques such as ellipsoidal relaxation [22] and particle control [23]. However, ellipsoidal
relaxation techniques are overly conservative and lead to highly suboptimal solutions. Particle control methods are computationally too demanding, since the number of decision variables grows with the number of samples. The two-stage risk allocation scheme proposed in this paper is computed iteratively until the cost is within a given tolerance of the minimum, from which we get the optimal risk allocation for the problem, as well as the optimal controller.

Previous works on chance constrained optimization and CS use polyhedral chance constraints, since they can be represented as intersections of linear inequalities \[ E \]. This formulation results in some favorable properties that help with the optimization. However, in many applications the constraints are in the form of a conical region (e.g., line-of-sight (LOS) constraints). Approximating such cone constraints with intersecting planes would make the problem rather large for high accuracy approximations. In this work, we also present a way to approximate such cone chance constraints (as special cases of general quadratic constraints) in terms of two-sided polyhedral constraints. We then apply this formulation to the case of LOS chance constraints, and compare with a polyhedral approximation. Additionally, we present a geometric relaxation of the cone chance constraints, which is less conservative than the two-sided approximation. To illustrate the proposed risk allocation algorithm we use as an example a spacecraft rendezvous problem between two spacecraft, in which the approaching spacecraft has to remain within a predetermined LOS region during the whole maneuver. Both polyhedral and cone LOS constraints are investigated and compared.

The paper is structured as follows: In Section II we define the general stochastic optimal control problem for steering a distribution from an initial Gaussian to a terminal Gaussian with joint state chance constraints. In Section III we review the two-stage risk allocation formalism, and formulate the SDP for the optimal controller as well as the proposed iterative risk allocation algorithm. In Section IV we present two different convex relaxations of quadratic chance constraints, one in terms of two-sided linear constraint relaxation, and the other based on a geometrical construction. Finally, in Section V we implement the theory to the spacecraft rendezvous and docking problem with both polyhedral and cone chance constraints.

II. PROBLEM STATEMENT

We consider the following discrete-time stochastic time-varying system subject to noise

\[ x_{k+1} = A_k x_k + B_k u_k + D_k w_k, \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m \), with time steps \( k = 0, \ldots, N-1 \), where \( N \) representing the finite horizon. The uncertainty \( w \in \mathbb{R}^r \) is a zero-mean white Gaussian noise with unit covariance, i.e., \( \mathbb{E}[w_k] = 0 \) and \( \mathbb{E}[w_k w_k^\top] = I \delta_{k_1,k_2} \). Additionally, we assume that \( \mathbb{E}[x_k w_k^\top] = 0 \), for \( 0 \leq k_1 \leq k_2 \leq N \). The initial state \( x_0 \) is a random vector drawn from the normal distribution

\[ x_0 \sim \mathcal{N} (\mu_0, \Sigma_0), \]

where \( \mu_0 \in \mathbb{R}^n \) is the initial state mean and \( \Sigma_0 \in \mathbb{R}^{n \times n} > 0 \) is the initial state covariance. The objective is to steer the trajectories of \([1]\) from the initial distribution \([2]\) to the terminal distribution

\[ x_f \sim \mathcal{N} (\mu_f, \Sigma_f), \]

where \( \mu_f \in \mathbb{R}^n \) and \( \Sigma_f > 0 \) are the state mean and covariance at time \( N \), respectively. The cost function to be minimized is

\[ J(u_0, \ldots, u_{N-1}) := \mathbb{E}\left[ \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k \right], \]

where \( Q_k \geq 0 \) and \( R_k > 0 \) for all \( k = 0, \ldots, N-1 \). Additionally, and over the whole horizon, we impose the following joint chance constraint that limits the probability of state violation to be less than a pre-specified threshold, i.e.,

\[ P \left( \bigwedge_{k=0}^{N} x_k \notin \mathcal{X} \right) \leq \Delta, \]

where \( P(\cdot) \) denotes the probability of an event, \( \mathcal{X} \subseteq \mathbb{R}^n \) is the state constraint set, and \( \Delta \in (0, 0.5] \).

Remark 1: We assume that the system \([1]\) is controllable, that is, for any \( x_0, x_f \in \mathbb{R}^n \), and no noise \( (w_k \equiv 0, k = 0, \ldots, N-1) \), there exists a sequence of control inputs \( \{u_k\}_{k=0}^{N-1} \) that steer the system from \( x_0 \) to \( x_f \).

First, we provide an alternative description of the system \([1]\) in order to solve the problem at hand. Using \([9]\), \([11]\), \([15]\), \([16]\), \([18]\), we can reformulate \([1]\) as

\[ X = Ax_0 + BU + DW, \]

where \( X := [x_0^\top, \ldots, x_N^\top]^\top \in \mathbb{R}^{(N+1)n}, U := [u_0^\top, \ldots, u_{N-1}^\top]^\top \in \mathbb{R}^{Nm}, \) and \( W := [w_0^\top, \ldots, w_{N-1}^\top]^\top \in \mathbb{R}^{Nr} \) are the state, input, and disturbance sequences, respectively. The matrices \( A, B, \) and \( D \) are defined accordingly \([9]\). Using this notation, we can write the cost function compactly as

\[ J(U) = \mathbb{E}(X^\top Q X + U^\top R U), \]

where \( Q \) and \( R \) are defined accordingly. Note that since \( Q_k \geq 0 \) and \( R_k > 0 \) for all \( k = 0, \ldots, N-1 \), it follows that \( Q \geq 0 \) and \( R > 0 \). The initial and terminal conditions \([2]\) and \([3]\) can be written as

\[ \mu_0 = E_0 E[X], \quad \Sigma_0 = E_0 \Sigma_X E_0, \]

and

\[ \mu_f = E_N E[X], \quad \Sigma_f = E_N \Sigma_X E_n, \]

where \( \Sigma_X := E[X^2] - E[X]^2 \), and \( E_k := [0_{n,kn}, I_n, 0_{n,(N-k)n}] \) picks out the \( k \)th component of a vector. Consequently, the state chance constraints \([5]\) can be written as

\[ P \left( \bigwedge_{k=1}^{N} E_k X \notin \mathcal{X} \right) \leq \Delta. \]

In summary, we wish to solve the following stochastic optimal control problem.
Problem 1: Given the system (5), find the optimal control sequence $U^* := U^*_{N-1}$ that minimizes the objective function (7), subject to the initial state (8), terminal state (9), and the state chance constraints (10).

III. CHANCE CONSTRAINED COVARIANCE STEERING WITH RISK ALLOCATION

A. Lower-Stage Covariance Steering

Borrowing from the work in [11], we adopt the control policy

$$u_k = v_k + K_k y_k,$$  \hspace{1cm} (11)

where $v_k \in \mathbb{R}^m$, $K_k \in \mathbb{R}^{m \times n}$, and $y_k \in \mathbb{R}^n$ is given by

$$y_{k+1} = A_k y_k + D_k u_k,$$ \hspace{1cm} (12a)

$$y_0 = x_0 - \mu_0.$$ \hspace{1cm} (12b)

Remark 2: The proposed control scheme [11]-[12] leads to a convex programming formulation of Problem 1 as follows.

Using (11)-(12), we can write the control sequence as

$$U = V + K(Ay_0 + DW),$$ \hspace{1cm} (13)

where $V := [v_1^T, \ldots, v_N^T] \in \mathbb{R}^{m \times N}$ and $K \in \mathbb{R}^{m \times (N+1)n}$ a matrix containing the gains $K_k$. It follows that the dynamics can be decoupled into a mean and error state as follows

$$\bar{X} = \mathbb{E}[X] = A\mu_0 + BV,$$ \hspace{1cm} (14)

$$\bar{X} = X - \bar{X} = (I + BK)(A\mu_0 + DW).$$ \hspace{1cm} (15)

Additionally, the cost function takes the form

$$J(V, K) = (A\mu_0 + BV)^T \bar{Q}(A\mu_0 + BV) + V^T \bar{R}V + \text{tr} \left[ (I + BK)^T \bar{Q} (I + BK) + K^T \bar{R}K \right] \Sigma_Y,$$ \hspace{1cm} (16)

where $\Sigma_Y := \Sigma \Sigma_0 A^T + DD^T$. The terminal constraints can be reformulated as

$$\mu_f = E_N(A\mu_0 + BV),$$ \hspace{1cm} (17a)

$$\Sigma_f = E_N(I + BK)\Sigma_Y (I + BK)^T E_N^T.$$ \hspace{1cm} (17b)

Qualitatively speaking, $V$ steers the mean of the system to $\mu_f$, while $K$ steers the covariance to $\Sigma_f$. In order to make the problem convex, we relax the terminal covariance constraint (17b) to $\Sigma_f \geq E_N(I + BK)\Sigma_Y (I + BK)^T E_N^T$, which can be written as the linear matrix inequality (LMI)

$$\begin{bmatrix} \Sigma_f & E_N(I + BK)\Sigma_Y^{1/2} \\ \Sigma_Y^{1/2} (I + BK)^T E_N^T & I \end{bmatrix} \geq 0.$$ \hspace{1cm} (18)

B. Polyhedral Chance Constraints

When dealing with the risk allocation problem, it is customary to assume that the state constraint set $X$ is a convex polytope $X^p$, so that

$$X^p := \bigcap_{j=1}^M \{x : \alpha_j^T x \leq \beta_j\},$$ \hspace{1cm} (19)

where $\alpha_j \in \mathbb{R}^n$ and $\beta_j \in \mathbb{R}$. Under this assumption, the probability of violating the state constraints (10) can be written as

$$P \left( \bigcap_{k=1}^N \bigcap_{j=1}^M \alpha_j^T E_k X > \beta_j \right) \leq \Delta.$$ \hspace{1cm} (20)

Equation (20) represents the objective that the joint probability of violating any of the $M$ state constraints over the horizon $N$ is less than or equal to $\Delta$. Using Boole’s Inequality [25], [26], one can decompose a joint chance constraint into individual chance constraints as follows

$$P \left( \alpha_j^T E_k X \leq \beta_j \right) \geq 1 - \delta_k^j,$$ \hspace{1cm} (21)

with

$$\sum_{k=1}^N \sum_{j=1}^M \delta_k^j \leq \Delta,$$ \hspace{1cm} (22)

where each $\delta_k^j$ represents the probability of violating the $j$th constraint at time step $k$. Notice that the probability in (21) is of a random variable with mean $\alpha_j^T E_k X$ and covariance $\alpha_j^T E_k \Sigma X E_k^T \alpha_j$. Thus, (21) can be equivalently written as

$$\Phi \left( \frac{\beta_j - \alpha_j^T E_k X}{\sqrt{\alpha_j^T E_k \Sigma X E_k^T \alpha_j}} \right) \geq 1 - \delta_k^j,$$ \hspace{1cm} (23)

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution. Simplifying (23) and noting that $\Sigma_X = (I + BK)\Sigma_Y (I + BK)^T$ yields

$$\alpha_j^T E_k (A\mu_0 + BV) + ||\Sigma_Y^{1/2} (I + BK)^T E_N^T \alpha_j|| \Phi^{-1}(1 - \delta_k^j) \leq \beta_j.$$ \hspace{1cm} (24)

Remark 3: Since $\Sigma_X > 0$, it follows that $\Sigma_Y > 0$, and $\Sigma_Y^{1/2}$ in (24) can be computed from its Cholesky decomposition.

The expression in (24) gives $NM$ inequality constraints for the optimization problem. In summary, Problem 1 is converted into a convex programming problem.

Problem 2: Given the system (14) and (15), find the optimal control sequences $V^*$ and $K^*$ that minimize the cost function (16) subject to the terminal state constraints (17a) and (18), and the individual chance constraints (24).

Remark 4: Note that it is not possible to decouple the mean and covariance controllers in the presence of chance constraints, because of (24).

C. Risk Allocation Optimization

Since $\delta_k^j$ are decision variables in (24), the constraints are bilinear, which makes it difficult to solve this problem. As mentioned previously, in order to transform Problem 2 to a more tractable form, the allocation of the risk levels $\delta_k^j$ may be assumed to be fixed to some pre-specified values, usually uniformly. In this case, $\delta_k^j$ are no longer decision variables and the problem can be efficiently solved as an SDP. However, a better approach is to allocate $\delta_k^j$ concurrently when solving the
optimization Problem 2, so as to minimize the total cost. This gives rise to a natural two-stage optimization framework [1].

According to the approach in [1], the upper stage optimization finds the optimal risk allocation \( \delta := [\delta_1, \delta_2, \ldots, \delta_M] \in \mathbb{R}^{NM} \), and the lower stage solves the CS problem for the optimal controller \( U^* = U_{N-1}^* \) given the risk allocation \( \delta \) from the upper-stage.

Let the value of the objective function after the lower-stage optimization for a given risk allocation \( \delta \) be \( J^*(\delta) \), that is,

\[
J^*(\delta) = \min_{V, K} J(V, K),
\]

where \( J(V, K) \) is given in [16]. The upper-stage optimization problem can then be formulated as follows.

**Problem 3 (Risk Allocation):**

\[
\min_{\delta} \quad J^*(\delta),
\]

\[
\text{such that } \quad \sum_{k=1}^{N} \sum_{j=1}^{M} \delta_{k,j} \leq \Delta,
\]

\[
\delta_{k,j} > 0,
\]

As shown in [1], Problem 3 is a convex optimization problem, given that the objective function \( J(V, K) \) is convex, and \( \Delta \in (0, 0.5) \).

**D. Iterative Risk Allocation Motivation**

Even though we have formulated the solution of Problem 2 as a two-stage optimization problem, it is not clear yet how to solve Problem 3 efficiently in order to determine the optimal risk allocation. To gain insight into the solution, we first state a theorem about the monotonicity of \( J^*(\delta) \).

**Theorem 1.** The optimal cost from solving Problem 2 is a monotonically decreasing function in \( \delta_{k,j} \), that is,

\[
\frac{\partial J^*}{\partial \delta_{k,j}} \leq 0, \quad k = 1, \ldots, N, \quad j = 1, \ldots, M.
\]

**Proof.** Let \( \delta, \delta' \) be two risk allocations, and let \( \mathcal{R}(\delta), \mathcal{R}(\delta') \) denote the feasible regions, defined by the inequality constraints [24]. If \( \delta_{k,j} \leq \delta'_{k,j} \) for all \( j \) and \( k \), then \( \mathcal{R}(\delta) \subseteq \mathcal{R}(\delta') \). To see this, let us rewrite (24) as follows

\[
\alpha_j^T E_k \bar{X} \leq \beta_j - \left| \sum_{i=1}^{\nu} (I + BK)^T \alpha_j \right| \Phi^{-1}(1 - \delta_{k,\nu}).
\]

Next, and since \( \Delta \in (0, 0.5) \), it follows that \( \delta_{k,j} \in (0, 0.5) \). Also note that in the domain \( z \in [0.5, 1] \) the cumulative distribution function \( \Phi(z) \) forms a convex region \( \mathcal{M} \), as shown in Figure [11]. Additionally, \( \Phi^{-1}(z) \) is a monotonically increasing function, and hence

\[
\frac{\partial \Phi^{-1}(1 - \delta_{k,j})}{\partial \delta_{k,j}} < 0.
\]

Thus, if \( \delta'_{k,j} \geq \delta_{k,j} \), the right hand side of (30) will be larger for \( \delta' \) than it is for \( \delta \). This implies that the inequality constraints are tighter for \( \delta \) than for \( \delta' \), which proves that \( \mathcal{R}(\delta) \subseteq \mathcal{R}(\delta') \). This fact finally implies that \( J^*(\delta) \geq J^*(\delta') \).

**Remark 5:** The chance constraints can be written in yet another form that will prove useful below. Starting from (24), notice that we can write the chance constraints as

\[
\delta_{k,j} \geq 1 - \Phi \left( \| \Sigma Y^{1/2} (I + BK^*)^T E_k \alpha_j \| / \| \Sigma Y^{1/2} (I + BK^*)^T \| \right) =: \tilde{\delta}_{k,j}. \quad (32)
\]

The quantity \( \tilde{\delta}_{k,j} \) represents the true risk experienced by the optimal trajectories, i.e., when using \( (V^*, K^*) \). Clearly, the risk we have selected does not need to be equal to the actual risk once the optimization is completed. When these values are equal we will say that the constraint is active, and is inactive otherwise. Good solutions correspond to cases when the true risk is within a small margin of the allocated risk. Many values of \( \delta_{k,j} \) smaller than their true counterparts would imply an overly conservative solution.

**E. Iterative Risk Allocation Algorithm**

We can exploit Theorem 1 in the context of CS to create an iterative risk allocation algorithm that simultaneously finds the optimal risk allocation \( \delta^* \) and the optimal control pair \( (V^*, K^*) \). To this end, suppose we start with some feasible risk allocation \( \delta_{k,j}^{\nu} \) for all \( k, j \), where \( i \) denotes the iteration number. Using this risk allocation, we then solve Problem 2 to get the optimal controller \( (V_{N-1}^*, K_{N-1}^*) \), which corresponds to the optimal mean trajectory \( \bar{X}_{i,j}^{\nu} \) at iteration \( i \). Next, we construct a new risk allocation \( \delta_{k,j}^{\nu} \) as follows: for all \( k, j \) such that \( \tilde{\delta}_{k,j}^{\nu} \) is active, we keep the corresponding allocation the same, i.e., \( \delta_{k,j}^{\nu} = \delta_{k,j}^{\nu} \). However, for all \( k, j \) such that \( \tilde{\delta}_{k,j}^{\nu} \) is inactive we let \( \delta_{k,j}^{\nu} < \delta_{k,j}^{\nu} \) which corresponds to tightening the constraints. Since this new risk allocation is smaller, it follows from (31) that \( \Phi^{-1}(1 - \delta_{k,j}^{\nu}) > \Phi^{-1}(1 - \delta_{k,j}^{\nu}) \). Furthermore, this implies that

\[
\alpha_j^T E_k \bar{X}_{i,j}^{\nu} \leq \beta_j - \left| \sum_{i=1}^{\nu} (I + BK)^T \alpha_j \right| \Phi^{-1}(1 - \delta_{k,j}^{\nu}) < \beta_j - \left| \sum_{i=1}^{\nu} (I + BK)^T \alpha_j \right| \Phi^{-1}(1 - \delta_{k,j}^{\nu}).
\]

Fig. 1: Convex region \( \mathcal{M} \) of the inverse cumulative distribution function, of the standard normal distribution.
The constraint $[33]$ ensures that the optimal solution for $\delta_{(i)}$ is feasible for $\delta_{(i)}^*$. Furthermore, since $\delta_{k(1)}^l < \delta_{k(1)}^r$, it follows that $R(\delta^*) \subseteq R(\delta)$, so the optimal solution for $\delta_{(i)}^*$ is also the optimal solution for $\delta_{(i)}$, as well, hence $J^*(\delta) = J^*(\delta)$.

Next, we construct a new risk allocation $\delta_{k(i+1)}^l$ from $\delta_{k(i)}^l$ as follows. For all $k, j$ such that $\delta_{k(i)}^j$ is active, let $\delta_{k(i+1)}^j > \delta_{k(i)}^j$, which corresponds to relaxing the constraints. Following the same logic, Theorem 1 implies that $J^*(\delta_{(i)}^j) \geq J^*(\delta_{(i+1)}^j)$. Thus, we have laid out an iterative scheme for a sequence of risk allocations $\{\delta_{(0)}, \delta_{(1)}, \ldots, \delta_{(i)}\}$ that continually lowers the optimal cost.

This leads to Algorithm 1 that solves the optimal risk allocation for the CS problem subject to chance constraints. Note that the algorithm is initialized with a constant risk allocation. To tighten the inactive constraints in Line 9, the corresponding risk is scaled by a parameter $0 < \rho < 1$ that weighs the current risk with the true risk from that solution. Additionally, to loosen the active constraints in Line 13, the corresponding risk is increased proportionally to the residual risk remaining.

**Algorithm 1: Iterative Risk Allocation CS**

1. **Input:** $\delta_k^l \leftarrow \Delta/(NM), \epsilon, \rho$
2. **Output:** $\delta^*, J^*, V^*, K^*$
3. **while** $|J^* - J_{prev}^*| > \epsilon$ **do**
4. $J_{prev}^* \leftarrow J^*$
5. Solve Problem 2 with current $\delta$ to obtain $\delta$
6. $\hat{N} \leftarrow$ number of indices where constraint is active
7. **if** $\hat{N} = 0$ or $\hat{N} = MN$ **then**
8. **break**
9. **foreach** $(k, j)$ such that $j$th constraint at $k$th time step is **inactive** **do**
10. $\delta_k^l \leftarrow \rho(\delta_k^l - 1 - \rho)\delta_k^l$
11. **end**
12. $\delta_{res} \leftarrow \Delta - \sum_{k=1}^{\hat{N}} \sum_{j=1}^{M} \delta_k^j$
13. **foreach** $(k, j)$ such that $j$th constraint at $k$th time step is **active** **do**
14. $\delta_k^l \leftarrow \delta_k^l + \delta_{res}/\hat{N}$
15. **end**
16. **end**

**IV. CONE CHANCE CONSTRAINTS**

In many engineering applications polytopic constraints such as $[19]$ are not realistic. Most often, the constraints have the form of a convex cone, namely, the feasible region is characterized by

$$\mathcal{X}^c := \{x \in \mathbb{R}^n : \|Ax + b\|_2 \leq c^T x + d\}. \quad (34)$$

Cone constraints such as $[34]$ are more realistic, as they better describe the feasible space. As with the case of a polyhedral feasible state space $\mathcal{X}^p$, we want the state to be inside $\mathcal{X}^c$ throughout the whole time horizon. However, since the dynamics are stochastic and similar to $[5]$, this assumption is relaxed to the condition that the probability that the state is not inside this set is less than or equal to $\Delta$. In the context of convex cone state constraints, this condition becomes

$$\mathbb{P}(|Ax_k + b|_2 \leq c^T x_k + d) \geq 1 - \delta_k, \quad k = 1, \ldots, N. \quad (35a)$$

$$\sum_{k=1}^{N} \delta_k \leq \Delta. \quad (35b)$$

**Remark 6:** Although the set $\mathcal{X}^c$ is convex, the chance constraint $\mathbb{P}(x \in \mathcal{X}^c) \geq 1 - \delta$ may not be convex. Specifically, for large $\delta$, it is possible that the chance constraint $[35]$ is non-convex $[27]$.

Since there is no guarantee that $[35]$ will be a convex constraint, we need to make a convex approximation so that $[35]$ holds for all $\Delta \in (0, 0.5]$.

**A. Two-Sided Approximation of Cone Constraints**

Recent work on two-sided affine chance constraints $[27]$ has shown how to relax a general class of quadratic constraints of the form

$$\mathbb{P}((a^T \xi + \hat{b})^2 + (\hat{c}^T \xi + \hat{d})^2 \leq \kappa) \geq 1 - \epsilon, \quad (36)$$

where $\xi$ is a Gaussian random variable, and $\epsilon \in (0, 0.5]$. In $[27]$ the authors proved that $[36]$ can be conservatively approximated by the following convex constraints

$$\mathbb{P}((a^T \xi + \hat{b}) \leq f_1) \geq 1 - \beta \epsilon, \quad (37a)$$

$$\mathbb{P}((\hat{c}^T \xi + \hat{d}) \leq f_2) \geq 1 - (1 - \beta) \epsilon, \quad (37b)$$

$$f_1^2 + f_2^2 \leq \kappa, \quad (37c)$$

where $\beta \in (0, 1)$ represents a constant that balances the trade-off between violating any of the two constraints $[37a]-[37b]$. In order to cast the cone chance constraint $[35a]$ in the form $[36]$, we first replace the constraint in $[35a]$ with the chance constraint

$$\mathbb{P}(|Ax_k + b|_2 \leq c^T x_k + d) \geq 1 - \delta_k, \quad k = 1, \ldots, N. \quad (38)$$

**Remark 7:** The chance constraint $[38]$ is a relaxation of the original chance constraint $[35a]$. The proof of this result is given in Appendix A.

In order to write $[38]$ in the form $[36]$, square both sides of the inequality in $[38]$ and rearrange terms to obtain

$$\mathbb{P}(x_k^T A^T Ax_k + 2b^T Ax_k \leq (c^T x_k + d)^2 - b^T b) \geq 1 - \delta_k. \quad (39)$$

Letting now

$$A^T A = \hat{a} \hat{a}^T + c c^T, \quad (40a)$$

$$A^T b = \hat{b} \hat{c} + \hat{d} c, \quad (40b)$$

$$(c^T x_k + d)^2 - b^T b = \kappa - \hat{b}^2 - \hat{d}^2, \quad (40c)$$
and identifying \( \xi = x_k \) yields
\[
\mathbb{P}(\zeta^T(\bar{a}^T + \bar{c}^T)\xi + 2(\bar{b}a^T + \bar{d}c^T)\xi \leq \kappa - \bar{b}^2 - \bar{d}^2) \geq 1 - \epsilon, \tag{41}
\]
or, after rearranging terms and completing the squares,
\[
\mathbb{P}(\langle \bar{a}^T\xi + \bar{b} \rangle^2 + \langle \bar{c}^T\xi + \bar{d} \rangle^2 \leq \kappa) \geq 1 - \epsilon, \tag{42}
\]
which yields the desired result.

Remark 8: It should be noted that the set of equations \( \mathbf{40} \) does not always have a solution. Specifically, \( \mathbf{30a} \) implies that \( A^TA \) is the sum of two rank-one matrices, which is a restrictive condition. However, it turns out that this condition holds for our problem.

In the case when the cone is centered at the origin, we have that \( b = d = 0 \) and a simple solution for equations \( \mathbf{40} \) yields
\[
A^T A = \bar{a} \bar{a}^T + \bar{c} \bar{c}^T, \quad \kappa = (\bar{c}^T \bar{x}_k)^2, \quad \bar{b} = \bar{d} = 0. \tag{43}
\]

In this sense, \( \bar{a} \) and \( \bar{c} \) denote the unit vectors that parametrize the orientation of the cone. In the context of CS, \( \mathbf{37a}-\mathbf{37c} \) then result in the following four affine chance constraints
\[
\begin{align*}
\mathbb{P}(\bar{a}^T E_k \bar{X} + \bar{b} \leq f_1) & \geq 1 - \beta \delta_k, \quad \tag{44a} \\
\mathbb{P}(\bar{a}^T E_k \bar{X} + \bar{b} \leq -f_1) & \leq (1 - \beta) \delta_k, \quad \tag{44b} \\
\mathbb{P}(\bar{c}^T E_k \bar{X} + \bar{d} \leq f_2) & \geq 1 - (1 - \beta) \delta_k, \quad \tag{44c} \\
\mathbb{P}(\bar{c}^T E_k \bar{X} + \bar{d} \leq -f_2) & \leq (1 - \beta) \delta_k. \quad \tag{44d}
\end{align*}
\]

These constraints are now in the standard affine form, and similar to \( \mathbf{24} \), they can be converted to
\[
\begin{align*}
\bar{a}^T E_k \bar{X} + \bar{b} + \Phi^{-1}(1 - \beta \delta_k)\|\Sigma^{1/2}(I + BK)^T E_k \bar{a}\|_2 & \leq f_1, \quad \tag{45a} \\
\bar{a}^T E_k \bar{X} + \bar{b} + \Phi^{-1}(\beta \delta_k)\|\Sigma^{1/2}(I + BK)^T E_k \bar{a}\|_2 & \geq f_1 \geq 0, \quad \tag{45b} \\
\bar{c}^T E_k \bar{X} + \bar{d} - \Phi^{-1}(1 - (1 - \beta) \delta_k)\|\Sigma^{1/2}(I + BK)^T E_k \bar{c}\|_2 & \leq f_2, \quad \tag{45c} \\
\bar{c}^T E_k \bar{X} + \bar{d} + \Phi^{-1}(1 - (1 - \beta) \delta_k)\|\Sigma^{1/2}(I + BK)^T E_k \bar{c}\|_2 & \geq f_2 \geq 0. \quad \tag{45d}
\end{align*}
\]

As a result, the approximation of the quadratic chance constraints has resulted in four cone constraints at each time step, or \( 4N \) total cone constraints for the whole problem. Since these constraints are now convex, the resulting problem is convex and can be solved using standard SDP solvers similarly to the polyhedral chance constraint case.

B. Geometric Approximation

We limit the following discussion to the three-dimensional case, which often occurs when enforcing position constraints. However, the results can be generalized to \( n \)-dimensional convex cones. For simplicity, let \( b = 0 \) in \( \mathbf{34} \), which corresponds to a cone centered at the origin. From a geometric point of view, one can think of the conical state space \( \mathbf{35a} \) as imposing, at each time step \( k \), that the projection \( \xi_k := AE_k X \in \mathbb{R}^2 \) lies inside the disk \( r_k = c^T E_k X + d \) with probability greater than \( 1 - \delta_k \). However, since \( E_k X \) is a stochastic process, it follows that the radius of the disk is uncertain, therefore, and similar to Section \( \mathbf{V-A} \), we relax the chance constraint such that the Gaussian vector \( \xi \) lie within the mean radius of the disk \( r_k = c^T E_k X + d \).

Using this approximation, the chance constraints \( \mathbf{35a} \) become
\[
\mathbb{P}(\|\xi_k\|_2 \leq r_k) \geq 1 - \delta_k. \tag{46}
\]

Note that the random variable \( \xi_k = AE_k X \) is Gaussian such that \( \xi_k \sim N(\xi_k, \Sigma_{\xi_k}) \) with mean \( \bar{E}_k := AE_k \bar{X} \) and covariance \( \Sigma_{\xi_k} := AE_k \Sigma_X E_k^T A^T \). So far, we have turned the convex cone chance constraint \( \mathbf{35a} \) into the chance constraint \( \mathbf{46} \) that requires the probability of a Gaussian random vector being inside a circle of given radius to be greater than \( 1 - \delta_k \). This problem can be analytically solved as follows.

**Proposition 1.** Let \( \zeta \sim N(0, \Sigma_{\zeta}) \) be a two-dimensional random vector. Then, for any \( a > 0 \),
\[
\mathbb{P}(\zeta^T \Sigma_{\zeta}^{-1} \zeta \leq a^2) = 1 - e^{-\frac{1}{2}a^2}. \tag{47}
\]

**Proof.** The probability density function (PDF) of \( \zeta \) is given by
\[
N(0, \Sigma_{\zeta}) = \frac{1}{2\pi \sqrt{|\det \Sigma_{\zeta}|}} e^{-\frac{1}{2} \zeta^T \Sigma_{\zeta}^{-1} \zeta}. \tag{48}
\]

Then, the probability in \( \mathbf{47} \) is given explicitly by
\[
\mathbb{P}(\zeta^T \Sigma_{\zeta}^{-1} \zeta \leq a^2) = \frac{1}{2\pi |\det \Sigma_{\zeta}|^{1/2}} \int_{\Omega_{\zeta}} e^{-\frac{1}{2} \zeta^T \Sigma_{\zeta}^{-1} \zeta} d\zeta, \tag{49}
\]

where \( \Omega_{\zeta} := \{\zeta : \zeta^T \Sigma_{\zeta}^{-1} \zeta \leq a^2\} \). Changing coordinates such that \( \nu := \Sigma_{\zeta}^{1/2} \zeta = (\rho \cos \phi, \rho \sin \phi) \) so that \( d\nu = |\det \Sigma_{\zeta}|^{-1/2} d\nu \), note that the sets \( \{\zeta^T \Sigma_{\zeta}^{-1} \zeta \leq a^2\} \) and \( \{\|\nu\| \leq a\} \) are equivalent. Thus, the integral in \( \mathbf{49} \) becomes
\[
\mathbb{P}(\zeta^T \Sigma_{\zeta}^{-1} \zeta \leq a^2) = \mathbb{P}(\|\nu\| \leq a) = \frac{1}{2\pi} \int_{\Omega_{\nu}} e^{-\frac{1}{2} \nu^T \nu} d\nu, \tag{50}
\]

where \( \Omega_{\nu} := \{\nu : \|\nu\| \leq a\} \). The last integral is straightforward to evaluate in two dimensions,
\[
\mathbb{P}(\|\nu\| \leq a) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^a e^{-\frac{1}{2} \rho^2} \rho d\rho d\phi = 1 - e^{-\frac{1}{2}a^2}, \tag{51}
\]
which yields the desired result.

**Lemma 1.** Let \( \zeta \sim N(0, \Sigma_{\zeta}) \) be a two-dimensional random vector, let \( \sigma_{\zeta}^2 = \lambda_{max}(\Sigma_{\zeta}) \), and let \( r > 0 \). Then
\[
\mathbb{P}(\|\zeta\|_2 \leq r) \geq 1 - e^{-r^2/2\sigma_{\zeta}^2}. \tag{52}
\]

**Proof.** Since the covariance matrix is positive definite, we can diagonalize it as \( \Sigma_{\zeta} = PDP^T \) where \( D \) is a diagonal matrix containing the eigenvalues \( \lambda_i \) of \( \Sigma_{\zeta} \) and \( P \) is an orthogonal matrix. Since \( \sigma_{\zeta}^2 = \lambda_{max} \lambda_i \), it follows that
\[
D^{-1} = \frac{1}{\sigma_{\zeta}^2} \text{diag}(\sigma_{\zeta}^2 / \lambda_i) \geq \frac{1}{\sigma_{\zeta}^2} I. \tag{53}
\]

From the previous expression, it follows that
\[
\zeta^T \Sigma_{\zeta}^{-1} \zeta = \zeta^T P D^{-1} P^T \zeta \geq \frac{1}{\sigma_{\zeta}^2} \zeta^T P P^T \zeta = \frac{1}{\sigma_{\zeta}^2} \|\zeta\|_2^2. \tag{54}
\]

\[
\zeta^T \Sigma_{\zeta}^{-1} \zeta \geq 1 - \delta_k. \tag{46}
\]
In summary, the convex cone chance constraints (35a) become

\[ \mathbb{P}(\|\zeta\|^2/\sigma^2 \leq \zeta^T \Sigma^{-1} \zeta, \text{ and using (47) it follows that} \]

\[ \mathbb{P}(\|\zeta\|^2/\sigma^2 \leq \mathbb{P}(\zeta^T \Sigma^{-1} \zeta \leq a^2) = 1 - e^{-\frac{1}{2}a^2}. \] (55)

Setting \( r^2 = \sigma^2a^2 \) achieves the desired result. Geometrically, the level sets \( \{ \zeta^T \Sigma^{-1} \zeta = a^2 \} \) define the contours of ellipses having probability \( 1 - e^{-a^2/2} \) and the level sets \( \{ \|\zeta\|^2 = r^2 \} \) are the smallest circles that contain these ellipses. \( \square \)

**Proposition 2.** Let \( \xi \sim \mathcal{N}(\xi, \Sigma) \) be a two-dimensional random vector, let \( \sigma^2 = \lambda_{\max}(\Sigma) \), and let \( r > 0 \). Then

\[ \|\bar{\xi}\|_2 + \sigma^2 \sqrt{2 \log \frac{1}{\delta}} \leq r \Rightarrow \mathbb{P}(\|\xi\|_2 \leq r) \geq 1 - \delta. \] (56)

**Proof.** First, note that for \( \|\bar{\xi}\|_2 \leq r \), the following implications hold

\[ \|\bar{\xi}\|_2 + \sigma^2 \sqrt{2 \log \frac{1}{\delta}} \leq r \Rightarrow \sigma^2 \sqrt{2 \log \frac{1}{\delta}} \leq r - \|\bar{\xi}\|_2 \] (57a)

\[ \Rightarrow 2\sigma^2 \log \frac{1}{\delta} \leq (r - \|\bar{\xi}\|_2)^2 \] (57b)

\[ \Rightarrow 2\sigma^2 \log \delta \geq -(r - \|\bar{\xi}\|_2)^2 \] (57c)

\[ \Rightarrow \log \delta \geq -\frac{(r - \|\bar{\xi}\|_2)^2}{2\sigma^2} \] (57d)

\[ \Rightarrow \delta \geq \exp\left(-\frac{(r - \|\bar{\xi}\|_2)^2}{2\sigma^2}\right) \] (57e)

\[ \Rightarrow 1 - \delta \leq 1 - \exp\left(-\frac{(r - \|\bar{\xi}\|_2)^2}{2\sigma^2}\right). \] (57f)

Since \( \{ \xi : \|\xi\|_2 \leq r \} \subseteq \{ \xi : \|\bar{\xi}\|_2 \leq r \} \), where \( \xi := \bar{\xi} - \xi \), it follows that

\[ \mathbb{P}(\|\xi\|_2 \leq r) \geq \mathbb{P}(\|\bar{\xi}\|_2 + \|\xi\|_2 \leq r) = \mathbb{P}(\|\bar{\xi}\|_2 \leq r - \|\xi\|_2). \] (58)

Since \( \bar{\xi} \) is a zero-mean Gaussian vector, applying Lemma 1 gives

\[ \mathbb{P}(\|\bar{\xi}\|_2 \leq r - \|\bar{\xi}\|_2) \geq 1 - \exp\left(-\frac{(r - \|\bar{\xi}\|_2)^2}{2\sigma^2}\right). \] (59)

Finally, by (57) and (58), we obtain the desired result. \( \square \)

Using Proposition 2 we can now satisfy (46) by enforcing

\[ \sigma_\xi \sqrt{2 \log \frac{1}{\delta_k}} \leq \bar{r}_k - \|\bar{\xi}_k\|_2 =: \bar{R}_k. \] (60)

Note that \( \sigma^2 = \lambda_{\max}(\Sigma) = \lambda_{\max}(AE_k \Sigma E_k^T A^T) \). Therefore, using \( \Sigma = (I + BK)(I + BK)^T \), we get

\[ \sigma^2_k = \|\Sigma_k^{1/2}(I + BK)^T E_k A^T\|_2^2. \] (61)

In summary, the convex cone chance constraints (35a) become

\[ \sqrt{2 \log \frac{1}{\delta_k}} \|\Sigma_k^{1/2}(I + BK)^T E_k A^T\|_2 \leq \bar{R}_k, \quad k = 1, \ldots, N. \] (62)

### V. Spacecraft Rendezvous Example

#### A. IRA-CS with Polytopic Chance Constraints

In this section, we implement the previous theory of CS with optimal risk allocation to the problem of spacecraft proximity operations in orbit. We consider the problem where one of the spacecraft, called the Deputy, approaches and docks with the second spacecraft, called the Chief, such that in the process, the Deputy remains within the line-of-sight (LOS) of the Chief, defined initially to be the polytopic region shown in Figure 2.

![Figure 2: Feasible state space region for spacecraft rendezvous problem.](image)

Assuming that the Chief is in a circular orbit, the relative dynamics of the motion between the two spacecraft are given by the Clohessy-Wiltshire-Hill Equations [28].

\[ \begin{align*}
\dot{x} &= 3\omega^2 x + 2\omega y + F_x/m_c, \\
\dot{y} &= -2\omega x + F_y/m_c, \\
\dot{z} &= -\omega^2 z + F_z/m_c,
\end{align*} \] (63)

where \( m_c \) is the mass of the Chief, \( \omega = \sqrt{\mu/R_0^3} \) is the orbital frequency, and \( F := [F_x, F_y, F_z]^T \) represents the thrust input components to the spacecraft. These equations of motion are written in a relative coordinate system, where the Chief is located at the origin, and \( x, y, z \) represent the position of the Deputy with respect to the Chief. Note that the \( z \) dynamics are decoupled from the \( x - y \) dynamics; furthermore, the \( z \) dynamics are globally asymptotically stable, so in theory we only need to control the planar dynamics. In Figure 2 the blue area represents the planar region. To write the system in state space form, let \( x := [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T \in \mathbb{R}^6 \) to obtain the LTI
system $\dot{x} = Ax + Bu$, where
\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
3\omega^2 & 2\omega & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2\omega & 0 & 0 \\
0 & 0 & -\omega^2 & 0 & 0 & 0
\end{bmatrix}, \quad B = [0_3, I_3]^T,
\] 
(64)
and $u := m_c^{-1}[F_x, F_y, F_z]^T \in \mathbb{R}^3$. To discretize the system, we divide the time interval into $N = 15$ steps, with a time interval $\Delta t = 0.5$ sec. Assuming a zero-order hold (ZOH) on the control yields the discrete system
\[
x_{k+1} = A_dx_k + B_du_k + Gw_k,
\] 
(65)
where $A_d = e^{A\Delta t}$, $B_d = B\Delta t + AB\Delta t^2/2$ and we choose the associated noise characteristics $G = \text{diag}(10^{-4}, 10^{-4}, 5 \times 10^{-8}, 5 \times 10^{-8})$ \cite{29}. We assume that the initial state mean and covariance are $\mu_0 = [0.75, -1, 0.75, 0]_\mathbb{T}$ km and $\Sigma_0 = 10^{-2}\text{diag}(0.1, 0.1, 0.1, 0.01, 0.01, 0.01, 0.01)$, respectively. We wish to steer the distribution from the above initial state to the final mean $\mu_f = 0$ with final covariance $\Sigma_f = \frac{1}{4}\Sigma_0$, while minimizing the cost function \cite{4} with weight matrices $Q = \text{diag}(10, 10, 10, 1, 1, 1)$ and $R = 10^3I_3$. We impose the joint probability of failure over the whole horizon to be $\Delta = 0.03$, which implies that the probability of violating any state constraint over the whole horizon is less than 3%. The control inputs are bounded as $\|u_k\|_\infty \leq 0.08$ km/s$^2$. Note that these bounds are hard constraints as opposed to (soft) chance constraints. To implement this input hard constraint within the CS framework, the algorithm in \cite{16} was used. The details are given in the Appendix. Lastly, in the iterative risk allocation algorithm, we use a scaling parameter $\rho(i) = (0.7)(0.98)^i$ in Line 10 of the algorithm, where $i$ represents the current iteration. The SDP in Problem 2 was implemented in MATLAB using YALMIP \cite{30} along with MOSEK \cite{31} to solve the relevant optimization problems.

Figures 3 and 4 show the optimal trajectories with optimal risk allocation, and Figure 5 shows the two dimensional planar motion. Figure 4 compares the terminal trajectories of CS with a uniform risk allocation with the proposed method. The two solutions look similar and both satisfy the terminal constraints on the mean and the covariance. However, due to the relaxation $\Sigma_N \leq \Sigma_f$, the uniform risk allocation leads to more conservative solutions, as shown in Figure 6. The volume of the final covariance ellipsoid, $V_N \propto \log \det \Sigma_N$ is considerably smaller for the uniform allocation solution compared to the optimal allocation solution (see Table 1). In fact, we see that a consequence of optimal risk allocation is that it maximizes the final covariance given all the constraints, while still being bounded by $\Sigma_f$.

Figures 7 and 8 show the state trajectories and the optimal controls for the polyhedral chance constraints. The control is almost linear but saturates at the first and the last few time steps. Figure 9 shows the a priori allocation of risk, as well as the true risk $\delta$ once the optimization is completed, where $\delta_r$ corresponds to the risk allocated for the right boundary
Fig. 6: Comparison of terminal covariance steering using a uniform and the optimal risk allocation.

Fig. 7: Trajectories of controlled system and their associated standard deviations using iterative risk allocation.

Fig. 8: Optimal control inputs using iterative risk allocation.

Fig. 9: Comparison of allocated risk and true risk using: (a) uniform risk allocation, (b) iterative risk allocation.

and $\delta_u$ for the risk allocated for the top boundary. Notice that in Figure 9a the true risk exposure is much lower than the allocated risk, which confirms the conclusion that the solutions for the uniform allocation case are overly conservative. In fact, the true risk is nearly zero except at the initial and terminal times. Comparing this to Figure 9b we see a close correspondence between the allocated risk and the true risk exposure over the whole horizon for the optimal risk allocation case. It should be noted that although the true risk is still slightly less than the allocated risk, the error between the two is much smaller when compared to that of the uniform risk allocation strategy.

\[
\bar{\Delta} := 1 - P \left[ \bigwedge_{k=1}^{N} \bigwedge_{j=1}^{M} a_j^T E_k X^* \leq \beta_j \right].
\]  

It is clear that the uniform risk allocation does not even come close to the desired design of $\Delta = 0.03$, while the IRA gives

The iterative risk allocation algorithm is robust in the sense that the algorithm will assign risk proportionately to how close the solution trajectories are to the boundaries of the state space. Since solutions are close to the right and top boundaries of the allowable LOS region for most of the horizon, the optimal allocation weighs these respective risks greater than those of the left and bottom boundaries. Thus, IRA assigns an extremely small risk to the right boundary during these time steps and only assigns a larger risk when the trajectories reach their terminal values. Table I shows the true joint probability of failure, defined as
a true probability of failure very close to the desired one.

Finally, we looked at the optimal cost function over each IRA iteration, as in Figure 10. The convergence criterion set in this example is $\epsilon = 10^{-5}$, or when all of the constraints are inactive, which can be proved in [1] to be a sufficient condition for optimality for Problem 3. We see that indeed (29) holds, and the optimization resulted even in a slight decrease of the objective function, converging within 16 iterations. Thus, the iterative risk algorithm optimizes the risk allocation at each time step without increasing the cost.

### B. IRA-CS with Cone Chance Constraints

For the convex cone chance constraint case, we also implemented the method outlined in Section IV, namely the $4N$ constraints in (45). For this example, the following representation of a cone was used

$$
\mathcal{X}^c = \{(x, y, z) : x^2 + z^2 \leq (\lambda y)^2\},
$$

where $\lambda = 1.2$, which corresponds to a 50° cone half-angle. This requirement translates to the individual chance constraints

$$
P(x_k^2 + z_k^2 \leq (\lambda y_k)^2) \geq 1 - \delta_k, \quad k = 1, \ldots, N.
$$

As discussed in Section IV in order to put this in the form of (36), the probabilistic constraint in (68) is relaxed to $\kappa = \lambda \bar{y}_k$, so that at each time step, the state is forced to stay inside a disk with radius $\lambda \bar{y}_k$. Comparing (68) with (34), we see that $A = \text{diag}(1, 0, 1, 0, 0, 0), b = d = 0$ and $c = [0, \lambda, 0, 0]^T$. From (67) and (36) we see that $\hat{a} = [1, 0_1 \times 5]^T$ and $\hat{c} = [0, 0, 1, 0_1 \times 3]^T$. The parameters $\beta = 0.5$ for constant weights and $f_k^1 = f_k^2 = f_k = \lambda \bar{y}_k / \sqrt{2}$ were used in the numerical simulations. Figures 11 and 12 show the optimal trajectories in the three-dimensional space and in the projection on the $x$-$y$ plane, respectively.

It should be noted that for the two-sided approximation of the cone constraint, and since we approximated the quadratic constraints as four linear constraints, the IRA algorithm needs to be adjusted as follows. In Line 5 of Algorithm 1, a constraint is active at time step $k$ if any of the four affine constraints in (45) is active. Similarly, when tightening the constraints in Line 10, the maximum true risk $\tilde{\delta}_k := \max_j \delta_j^k$ is used. This is not needed for the geometric approximation because it approximates each cone chance constraint as a single convex constraint for each $k$, so the standard IRA algorithm is applicable.

### VI. CONCLUSION

In this paper, we have incorporated an iterative risk allocation (IRA) strategy to optimize the probability of violating the state constraints at every time step within the covariance
steering problem of a linear stochastic system subject to chance constraints. For the covariance steering problem, we showed that employing IRA not only leads to less conservative solutions that are more practical, but also tends to maximize the final covariance. Additionally, the use of IRA in the context of CS with chance constraints results in optimal solutions that have a true risk much closer to the intended design requirements, compared to the use of a uniform risk allocation. We also implemented quadratic chance constraints in the form of convex cones, which are more accurate and natural for many engineering applications. Using a two-sided affine approximation, the quadratic chance constraints can be made convex, and a slightly modified IRA algorithm was used to optimize the risk. Lastly, we also used a geometric approximation of the cone chance constraints, which is valid when the state space is three-dimensional, as is often the case when constraining the position of the vehicle, and which is less conservative than the two-sided affine approximation. Both relaxations result in convex programs, where the two-stage IRA algorithm is applicable.

REFERENCES

[1] M. Ono and B. C. Williams, “Iterative risk allocation: A new approach to robust model predictive control with a joint chance constraint,” in 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec 8–11, 2008, pp. 3427–3432.

[2] P. Li, M. Wendt, and G. Wozny, “A probabilistically constrained model predictive controller,” Automatica, vol. 38, pp. 1171–1176, 2002.

[3] A. F. Hotz and R. E. Skelton, “A covariance control theory,” in 24th IEEE Conference on Decision and Control, Fort Lauderdale, FL, Dec 11–13, 1985, pp. 552–557.

[4] A. Hotz and R. E. Skelton, “Covariance control theory,” International Journal of Control, vol. 46, no. 1, pp. 13–32, 1987.

[5] E. Bakolas, “Optimal covariance control for discrete-time stochastic linear systems subject to constraints,” in 55th IEEE Conference on Decision and Control, Las Vegas, NV, Dec 14–16, 2016, pp. 1153–1158.

[6] ——, “Finite-horizon covariance control for discrete-time stochastic linear systems subject to input constraints,” Automatica, vol. 91, pp. 61–69, 2018.

[7] A. Halder and E. D. B. Wendel, “Finite horizon linear quadratic gaussian density regulator with wasserstein terminal cost,” in American Control Conference, Boston, MA, July 6–8, 2016, pp. 7249–7254.

[8] Y. Chen, T. T. Georgiou, and M. Pavon, “Optimal steering of a linear stochastic system to a final probability distribution – Part I,” IEEE Transactions on Automatic Control, vol. 61, no. 5, pp. 1158–1169, 2016.

[9] M. Goldshtein and P. Tsiotras, “Finite-horizon covariance control of linear time-varying systems,” in 56th IEEE Conference on Decision and Control, Melbourne, Australia, Dec 12–15 2017, pp. 3606–3611.

[10] E. Bakolas, “Finite-horizon separation-based covariance control for discrete-time stochastic linear systems,” in 57th IEEE Conference on Decision and Control, Miami Beach, FL, Dec 17–19, 2018, pp. 3299–3304.

[11] K. Okamoto and P. Tsiotras, “Optimal stochastic vehicle path planning using covariance steering,” IEEE Robotics and Automation Letters, vol. 4, no. 3, pp. 2276–2281, 2019.

[12] Y. Chen, T. T. Georgiou, and M. Pavon, “Optimal steering of a linear stochastic system to a final probability distribution – Part II,” IEEE Transactions on Automatic Control, vol. 61, no. 5, pp. 1170–1180, 2016.

[13] ——, “Optimal steering of a linear stochastic system to a final probability distribution – Part III,” IEEE Transactions on Automatic Control, vol. 63, no. 9, pp. 3112–3118, 2018.

[14] Y. Chen, T. T. Georgiou, and A. Tannenbaum, “Vector–valued optimal mass transport,” 2017, arXiv:1611.09946.

[15] K. Okamoto, M. Goldshtein, and P. Tsiotras, “Optimal covariance control for stochastic systems under chance constraints,” IEEE Control System Letters, vol. 2, pp. 266–271, 2018.

[16] K. Okamoto and P. Tsiotras, “Input hard constrained optimal covariance steering,” in 58th IEEE Conference on Decision and Control, Nice, France, Dec 11–13, 2019, pp. 3497–3502.

[17] E. Bakolas, “Covariance control for discrete-time stochastic linear systems with incomplete state information,” in American Control Conference, Seattle, WA, 2017, pp. 432–437.

[18] K. Okamoto and P. Tsiotras, “Stochastic model predictive control for constrained linear systems using optimal covariance steering,” 2019, arXiv:1905.12396.

[19] J. Ridderehov, K. Okamoto, and P. Tsiotras. “Nonlinear uncertainty control with iterative covariance steering,” in 58th IEEE Conference on Decision and Control, Nice, France, Dec 11–13 2019, pp. 3484–3490.

[20] E. Bakolas and A. Tsolovikos, “Greedy finite-horizon covariance steering for discrete-time stochastic nonlinear systems based on the unscented transform,” 2020, arXiv:2003.03679.

[21] Z. Yi, Z. Cao, E. Theodorou, and Y. Chen, “Nonlinear covariance control via differential dynamic programming,” 2019, arXiv:1911.09283.

[22] D. H. van Hessem, “Stochastic inequality constrained closed loop model predictive control with application to chemical process operation,” Ph.D. dissertation, Delft University of Technology, 2004.

[23] L. Blackmore, “A probabilistic particle control approach to optimal, robust predictive control,” in AIAA Guidance, Navigation, Control Conference, Keystone, CO, Aug 21–24, 2006, pp. 1–15.

[24] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge University Press, 2004.

[25] A. Príkopa, “Boole-Bonferroni inequalities and linear programming,” Operations Research, vol. 36, no. 1, pp. 145–162, 1988.

[26] L. Blackmore, H. X. Li, and B. C. Williams, “A probabilistic approach to optimal robust path planning with obstacles,” in American Control Conference, Minneapolis, MN, June 14–16, 2006, pp. 1–7.

[27] M. Lubin, D. Bienstock, and J. Vielma, “Two-sided linear chance constraints and extensions,” 2016, arXiv:1507.01995.

[28] W. Weisel, Spaceflight Dynamics. McGraw-Hill Book Co, 1989.

[29] A. Vinod and M. Oishi, “Affine controller synthesis for stochastic reachability via difference of convex programming,” in 58th IEEE Conference on Decision and Control, Nice, France, 2019, pp. 7275–7280.

[30] J. Lofberg, “YALMIP: A toolbox for modeling and optimization in matlab,” in IEEE International Symposium on Computer Aided Control Systems Design, Taipei, Taiwan, 2004, pp. 284–289.

[31] MOSEK ApS, The MOSEK Optimization Toolbox for MATLAB Manual. Version 8.1, 2017, [Online]. Available: http://docs.mosek.com.

[32] M. J. Wainwright, High-Dimensional Statistics: A Non-Asymptotic Viewpoint. Cambridge University Press, 2019.

APPENDIX

A. CONE CHANCE CONSTRAINT RELAXATION

Theorem 2. The quadratic chance constraint

$$\mathbb{P}(\|Ax + b\|_2 \leq c^T \mu + d) \geq 1 - \delta,$$  \hspace{1cm} (A.1)

where $x \sim N(\mu, \Sigma)$ is a relaxation of the cone chance constraint

$$\mathbb{P}(\|Ax + b\|_2 \leq c^T x + d) \geq 1 - \delta.$$ \hspace{1cm} (A.2)

Proof. Since $x \sim N(\mu, \Sigma)$ it follows that $\xi := \|Ax + b\|_2$ follows a non-central $\chi^2$ distribution with probability density function $f_\chi(x)(\xi)$.[32]. Let $\eta := c^T x + d$ and notice that $\eta \sim N(c^T \mu + d, c^T \Sigma c)$. The chance constraint (A.2) then takes the form $\mathbb{P}(\xi \leq \eta) \geq 1 - \delta$. The probability that one random variable is less than or equal another random variable is given by

$$\mathbb{P}(\xi \leq \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\eta} f_{\xi,\eta}(x, y) \, dx \, dy,$$  \hspace{1cm} (A.3)

where $f_{\xi,\eta}(x, y)$ is the joint probability distribution function of the random variables $\xi$ and $\eta$. Next, write $\eta = c^T \mu + d +$...
z \sqrt{c^T \Sigma c}$, where $z \sim \mathcal{N}(0,1)$ with probability density $f_z(y)$ and let $\tilde{y} = c^T \mu + d$. The inner integral in (A.3) then becomes

$$
\int_{-\infty}^y f_{\xi, \eta}(x, y) \, dx = \int_{-\infty}^{\eta + y \sqrt{c^T \Sigma c}} f_{\xi, z}(x, y) \, dx \\
= \int_{-\infty}^{\eta} f_{\xi, z}(x, y) \, dx + \int_{\eta}^{\eta + y \sqrt{c^T \Sigma c}} f_{\xi, z}(x, y) \, dx.
$$

It follows that

$$
P(\xi \leq \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\eta} f_{\xi, z}(x, y) \, dx \, dy \\
+ \int_{-\infty}^{\infty} \int_{\eta}^{\infty} f_{\xi, z}(x, y) \, dx \, dy \\
\geq \int_{-\infty}^{\infty} \int_{-\infty}^{\eta} f_{\xi, z}(x, y) \, dx \, dy.
$$

(A.4)

Noticing that

$$
\int_{-\infty}^{\infty} f_{\xi, z}(x, y) \, dy = f_\xi(x),
$$

the last expression in (A.4) implies that

$$
P(\xi \leq \eta) \geq \int_{-\infty}^{\eta} f_\xi(x) \, dx = P(\xi \leq \tilde{\eta}),
$$

which achieves the desired result. In other words, if the relaxed chance constraint $P(\xi \leq \eta) \geq 1 - \delta$ is satisfied, then the original chance constraint $P(\xi \leq \eta) \geq 1 - \delta$ is satisfied as well.

### B. Input Hard Constrained Covariance Controller

In the following, and similar to polytopic chance constraints, we assume that the hard input constraints on the control are affine, i.e., they are of the form

$$
\alpha_{u,s}^T F_k U \leq \beta_{u,s}, \quad s = 1, \ldots, N_c.
$$

(B.1)

**Theorem 3** ([16]). The control law

$$
u_k = v_k + K_k z_k,
$$

(B.2)

where $z_k$ is governed by the dynamics

$$
z_{k+1} = A z_k + \phi(w_k),
\quad z_0 = \phi(y_0), \quad y_0 = x_0 - \mu_0,
$$

(B.3)

where $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is an element-wise symmetric saturation function with pre-specified saturation value of the $i$th entry of the input $y_i^{\text{max}} > 0$ as

$$
\phi_i(y) = \max(-y_i^{\text{max}}, \min(y_i, y_i^{\text{max}})).
$$

(B.5)

converts Problem 1 to the following convex programming problem that constrains the control to a maximum saturation value

$$
\min_{V, K, \Omega} J(V, K, \Omega) = tr \left( \hat{Q} [I \quad BK] \Sigma_{XX} \left[ I \quad K^T \Phi^T \right] \right) \\
+ tr(\bar{R} K \Sigma_{UU} K^T) + (A \mu_0 + BV)^T \bar{Q}(A \mu_0 + BV) + V^T \bar{R} V
$$

subject to

$$
P(E_k X \notin \mathcal{X}) \leq \delta, \quad k = 1, \ldots, N
$$

(B.6)

$$
\sum_{k=1}^N \delta_k \leq \Delta,
$$

(B.7)

$$
HF_k V + \Omega^T \sigma \leq h,
$$

(B.8)

$$
HF_k K [A \quad D] = \Omega^T S,
$$

(B.9)

$$
\Omega \geq 0,
$$

(B.10)

$$
\mu_f = E_N (A \mu_0 + BV),
$$

(B.11)

$$
\Sigma_f \geq E_N [I \quad BK] \Sigma_{XX} \left[ I \quad K^T \Phi^T \right] E_N^T,
$$

(B.12)

where $\Omega \in \mathbb{R}^{2(N+1)n \times N_c}$ is a decision (slack) variable,

$$
\Sigma_{XX} = \begin{bmatrix} A & A \mathbb{E} [y_0 \phi(y_0)^T] & \mathbb{E} [\phi(y_0) \phi(y_0)^T] \end{bmatrix} \begin{bmatrix} A^T \\ \mathbb{D}^T \end{bmatrix},
$$

(B.13)

$$
\Sigma_{UU} = \mathbb{E} [\phi(y_0) \phi(y_0)^T] \mathbb{A}^T + \mathbb{D} \mathbb{E} [\phi(W) \phi(W)^T] \mathbb{D}^T.
$$

(B.14)

Further,

$$
H = [\alpha_{u,1}, \ldots, \alpha_{u,N_c}]^T \in \mathbb{R}^{N_c \times m},
$$

(B.15)

$$
h = [\beta_{u,1}, \ldots, \beta_{u,N_c}]^T \in \mathbb{R}^{N_c},
$$

(B.16)

In addition, $S \in \mathbb{R}^{2(N+1)n \times (N+1)n}$ and $\sigma \in \mathbb{R}^{2(N+1)n}$ are constant, given by

$$
S_{2i-1} = e_{2i-1}^T, \quad S_{2i} = -e_{2i}^T, \\
\sigma_{2i-1} = y_i^{\text{max}}, \quad \sigma_{2i} = y_i^{\text{max}},
$$

(B.17)

where $S$ denotes the $i$th row of $S$, and $e_i \in \mathbb{R}^{2(N+1)n}$ is a unit vector with $i$th element 1. Lastly, the chance constraints [B.6] take the form

$$
\alpha_j^T E_k (A \mu_0 + BV) + \|\Sigma_{XX}^{1/2} [I \quad BK]^T E_k^T \alpha_j\| \leq \beta_j,
$$

(B.19)

when $X = X^p$ ([79]).
when $\mathcal{X} = \mathcal{X}^{c}$ [34] with the substitution [40], and

$$\sqrt{2 \log \frac{1}{\delta_k} \| \Sigma_{XX}^{1/2} \begin{bmatrix} I & BK \end{bmatrix}^T E_k^T A^T \|} \leq \bar{R}_k, \quad (B.21)$$

for $\mathcal{X} = \mathcal{X}^{c}$ with the geometric approximation.

**Proof.** It can be shown that

$$\mathbb{E}[\tilde{X} \tilde{X}^\top] = \begin{bmatrix} I & BK \end{bmatrix} \Sigma_{XX} \begin{bmatrix} I \\ K^T B^T \end{bmatrix}, \quad (B.22)$$

from which it follows that the standard deviation of the state vector $x_k = E_k X$ is

$$\sqrt{E_k \mathbb{E}[\tilde{X} \tilde{X}^\top] E_k^\top} = \| \Sigma_{XX}^{1/2} \begin{bmatrix} I & BK \end{bmatrix}^T E_k^\top \| \quad (B.23)$$

The rest of the proof then follows from the results in [16]. \qed