Sharp Spectral Asymptotics for 2-dimensional Schrödinger operator with a strong but degenerating magnetic field. II

Victor Ivrii *

March 29, 2022

Abstract

I consider the same operator as in part I [Ivr10] assuming however that \( \mu \geq C h^{-1} \) and \( V \) is replaced by \((2l + 1) \mu h F + W\) with \( l \in \mathbb{Z}^+ \). Under some non-degeneracy conditions I recover remainder estimates up to \( O(\mu^{-\frac{1}{2}} h^{-1} + 1) \) but now case \( \mu \geq C h^{-\nu} \) is no more forbidden and the principal part is of magnitude \( \mu h^{-1} \).

6 Modified V. I. \( \mu \leq \epsilon h^{-\nu} \)

6.1 Introduction

This paper is a continuation of [Ivr10] which is considered as Part I. I consider spectral asymptotics of the magnetic Schrödinger operator

\[
(6.1) \quad A = \frac{1}{2} \left( \sum_{j,k} P_j g^{jk}(x) P_k - V \right), \quad P_j = D_j - \mu V_j, \quad V = (2l + 1) \mu h F + W
\]

where \( g^{jk}, V_j, W \) are smooth real-valued functions of \( x \in \mathbb{R}^2, l \in \mathbb{Z}^+ \) (i.e. \( l = 0, 1, \ldots \)) and \( (g^{jk}) \) is positive-definite matrix, \( 0 < h \ll 1 \) is a Planck parameter and \( \mu \gg 1 \) is a coupling parameter. I assume that \( A \) is a self-adjoint operator and all the conditions are satisfied in the ball \( B(0, 1), F = F_{12} g^{-\frac{1}{2}}, F_{12} = \partial_{x_1} V_2 - \partial_{x_2} V_1, g = \text{det}(g^{jk})^{-1}. \)

*Work was partially supported by NSERC grant OGP0138277.
Further, exactly as in \[ Ivr10 \], I assume that
\begin{equation}
F \asymp |x_1|^{\nu-1}, \quad \nu \in \mathbb{Z}^+, \nu \geq 2
\end{equation}
and thus with no loss of the generality I can assume that
\begin{equation}
V_1 = 0, \quad V_2 \asymp x_1^\nu.
\end{equation}
Furthermore, I assume that either
\begin{equation}
\pm W \geq \epsilon_0, \quad \text{as } l \geq 0
\end{equation}
and as \( l = 0 \) only sign “+” is interesting or
\begin{equation}
|\partial_{x_2} W/f| \geq \epsilon_0, \quad f \text{ def } Fx_1^{1-\nu}.
\end{equation}

Also as in \[ Ivr10 \], I am interested in the asymptotics of \( \int e(x, x, 0)\psi(x) \, dx \) where \( e(x, y, \tau) \) is the Schwartz kernel of the spectral projector \( E(\tau) \) of \( A \) and \( \psi \in C_0^\infty(B(0, \frac{1}{2})) \) is a cut-off function and I expect the main part of it to be \( \int \mathcal{E}^{MW}(x, 0)\psi(x) \, dx \) where \( \mathcal{E}^{MW} \) is defined by (0.8)\(^1\) which is of magnitude \( \mu h^{-1} \). I am assuming without mention that \( \psi \) is supported in the small but fixed vicinity of \( \{x_1 = 0\} \).

In the sharp contrast to the analysis of Part I the case \( \mu \geq Ch^{-\nu} \) is not “forbidden” anymore as well as zone \( Z'' = \{ |x_1| \geq \gamma_1 \text{ def } C(\mu h)^{-1/(\nu-1)} \} \). On the contrary, as \( \mu h \geq C \) this zone becomes the main contributor to the principal part of asymptotics which now is of magnitude \( \mu h^{-1} \) instead of \( h^{-2}(\mu h)^{-1/(\nu-1)} \) as it was in \[ Ivr10 \]. Actually I will time to time slightly change the definition of \( \gamma_1 \), replacing it by \( \gamma_1 = \epsilon(\mu h)^{-1/(\nu-1)} \) and back and changing respectively definition of zones.

Section 6 is devoted to the case of \( \mu \leq \epsilon h^{-\nu} \). Analysis in zone \( Z' \text{ def } \{ |x_1| \leq 2\gamma_1 \} \) remains basically the same and the main attention is paid here to the formally forbidden zone \( Z'' \). The main results here are theorems 6.10, 6.11 and 6.17.

As \( \mu \geq \epsilon h^{-\nu} \) this separation to zones is no more reasonable and will be modified. In section 7 I analyze the case of \( \epsilon h^{-\nu} \leq \mu \leq Ch^{-\nu} \). The main results here are theorems 7.3 and 7.4.

Further, in section 8 analyze the case of \( \mu \geq Ch^{-\nu} \). The main results here are theorems 8.9, 8.10, 8.11 and 8.12.

Finally, appendix A is devoted to asymptotics of some one-dimensional Schrödinger operators associated with (6.1).

\(^1\) References by default are to \[ Ivr10 \].
6.2 Simple Rescaling

As in [Ivr10] the simple rescaling arguments help us to get the easy but not sharp results.

6.2.1 In this and the next subsubsection I assume that \( \mu \leq Ch^{-\nu} \). Rescaling arguments in the zone \( Z' \) work exactly in the same manner as in [Ivr10] leading to the asymptotics of \( \int e(x, x, 0) \psi'(x) \, dx \) with the principal part \( \int E_{MW}(x, 0) \psi'(x) \, dx \) and the remainder estimate \( O(h^{-1}) \) where \( \psi'(x) \) and \( \psi''(x) \) are cut-off functions supported in zones \( Z' \) and \( Z'' \) (defined as above) respectively; one can take \( \psi'(x) = \psi(x) \psi_0(x_1/\gamma_1) \), \( \psi'' = \psi - \psi' \) where \( \psi_0 \in C_0^\infty \) is supported in \((-1, 1)\) and equals 1 in \([-\frac{1}{2}, \frac{1}{2}]\).

However the contribution of the previously forbidden zone \( Z'' \) to the remainder estimate is

\[
O\left( \int_{\{\gamma_1 \leq \gamma \leq 1\}} \gamma^{-2} \, d\gamma \right) = O(\gamma_1^{-1})
\]

which is \( O(h^{-1}) \) due to assumption \( \mu \leq Ch^{-\nu} \) and the contribution of \( Z'' \) to the principal part is

\[
(6.6) \quad \int E_{MW}(x) \psi''(x) \, dx = \frac{1}{4\pi} \mu h^{-1} l_+ \int \psi'' |F| \sqrt{g} \, dx, \quad l_+ \overset{\text{def}}{=} l + \frac{1}{2} (-1 \pm 1)
\]

under condition \((6.4)_\pm\).

Under condition \((6.5)\) the above arguments remain true for the contribution of the subzone \( Z'' \cap \{|W| \geq C\gamma\} \); for the contribution of the zone \( Z'' \cap \{|W| \leq C\gamma\} \) one needs to take in account correction term \( 2 \sum_m \kappa_m \mu h_{\text{eff}}^{1+2m} \) for the case \( \mu h_{\text{eff}} \geq 1 \), \( h_{\text{eff}} \leq 1 \) where in the rescaling and division arguments \( \mu_{\text{eff}} = \mu \gamma_{\text{eff}}^{1/2}, h_{\text{eff}} = h \gamma_{\text{eff}}^{-1/2} \) and the number of balls is \( O(1) \) for each \( \gamma \). Then the total contribution of this correction terms is \( O(\mu h) \) as \( \nu \geq 3 \) and \( O(\mu h \log h) \) as \( \nu = 2 \).

6.2.2 Replacing \( \psi \) by \( x_1 \psi \) in the above arguments one gains factor \( \gamma \) in each integrand; then the total contribution of the zone \( Z' \) to the remainder estimate becomes

\[
O\left( \int \mu^{-1} h^{-1} \gamma^{1-\nu} \times \gamma \times \gamma^{-2} \, d\gamma \right) = O(\mu^{-1/\nu} h^{-1})
\]

which is exactly what I want. On the other hand, the contribution of zone \( Z'' \) to the remainder estimate becomes \( O(\gamma^{-1} \, d\gamma) = O(|\log h|) \) which is what we want as \( \mu \leq C(h \log h)^{-\nu} \) only. To fix it under condition \((6.4)_\pm\) one can notice that zone \( Z'' \) is the spectral gap and therefore the contribution of the individual ball to the remainder estimate is \( O(\gamma h_{\text{eff}}^{s}) \) with \( h_{\text{eff}} = h / \gamma \) rather than \( O(1) \) and therefore the total contribution of zone \( Z'' \) to the remainder estimate is \( O(1) \).

\(^2\)See section 6 of [Ivr1].
As before, under condition (6.5) these arguments are applicable in the subzone $Z'' \cap \{|W| \geq C\gamma\}$ with $h_{\text{eff}} = h/(\gamma |W|^{1/2})$ as long as $h_{\text{eff}} \leq 1$. This leads to $O(1)$ estimate of the contribution of the subzone $Z'' \cap \{|W| \geq C\gamma, |W|^{1/2} \gamma \geq h\}$ to the remainder. One can see easily that the integral of $\gamma^{-1}$ taken over subzones $Z'' \cap \{|W| \geq C\gamma, |W|^{1/2} \gamma \leq h\}$ and $Z'' \cap \{|W| \leq C\gamma\}$ is $O(1)$ as well. Thus rescaling arguments provide remainder estimate $O(\mu^{-1/\nu} + 1)$ if $\psi$ contains an extra factor $x_1$ and under condition (6.5) correction terms are taken into account.

Therefore

(6.7) As $\mu \leq Ch^{-\nu}$ in what follows one can assume without any loss of the generality that $\psi(x) = \psi_1(x_1)\psi_2(x_2)$.

6.2.3 As $\mu \geq Ch^{\nu}$ arguments of subsubsection 6.2.1 work as $\{|x_1| \geq Ch\}$ providing $O(h^{-1})$ contribution of this zone to the remainder estimate while the contribution of zone $\{|x_1| \leq Ch\}$ will be $O(\mu h^{\nu-1})$. The main part of the asymptotics will be the same as above. Moreover, arguments of subsubsection 6.2.1 work as $\{|x_1| \geq Ch\}$ providing $O(1)$ contribution of this zone to the remainder estimate as $\psi$ is replaced by $x_1 \psi$ while the contribution of zone $\{|x_1| \leq Ch\}$ will be $O(\mu h^{\nu})$.

In the next section I will improve these latter results.

6.3 Estimates. I

In section 2 and subsections 4.1–4.4 of [Ivr10] various properties of operator $A$ were proven in the outer and inner zones $Z_{\text{out}} = \{\bar{\gamma} \leq |x_1| \leq 2\bar{\gamma}_1\}$ and $Z_{\text{inn}} = \{|x_1| \leq 2\bar{\gamma}_1\}$ with $\bar{\gamma} \overset{\text{def}}{=} C\mu^{-1/\nu}$ as long as $\bar{\gamma} \leq \bar{\gamma}_1$ i.e. $\mu \leq \epsilon h^{-\nu}$. These properties were proven first in section 2 under assumption

(6.8) $C \leq \mu \leq \epsilon(h|\log h|)^{-\nu}$

using standard microlocal analysis with logarithmic uncertainty principle and then in subsections 4.1–4.4 under assumption

(6.9) $\epsilon(h|\log h|)^{-\nu} \leq \mu \leq \epsilon h^{-\nu}$

applying microlocal analysis for $h$-pseudo-differential operators with respect to $x_2$ with operator-valued symbols – operators in the auxiliary space $\mathbb{H} = L^2(\mathbb{R}_{x_1})$; I remind that in the case (6.9) localization was done with respect to $\xi_2$ rather $x_1$. 
Therefore in both cases (6.8), (6.9) in the redefined outer zone

\[ Z_{\text{out}} = \{ \gamma \leq |x_1| \leq \gamma' = \epsilon \gamma_1 \} \]  

(with the small constant \( \epsilon \)) all these arguments remain true leading us eventually to the following statements:

**Proposition 6.1.** Let conditions (6.2) and (6.4) be fulfilled. Let \( \psi = \psi(x_2) \) be supported in \( B(0, \frac{1}{2}) \) and let \( \varphi = \varphi(\xi_2) \) be supported in the strip

\[ Y_{\gamma} = \{ \mu \gamma' \leq |\xi_2| \leq 2\mu \gamma' \} \]

with \( C_1 \gamma \leq \gamma \leq \epsilon_1 \gamma_1 \). Then

(i) As \( \mu \leq \epsilon h^{-\nu} \) estimates

\[ |F_{t \to h^{-1}} \mathcal{X}(t) \Gamma(Qe)| \leq Ch^s \]

and

\[ R' = |\Gamma(Qe) - h^{-1} \int_{-\infty}^{0} \left( F_{t \to h^{-1}} \mathcal{X}(t) \Gamma(Qe) \right) d\tau| \leq C \mu^{-1} \gamma_1^{1-\nu} h^{-1} \]

hold with \( Qe = \varphi(hD_2)(e\psi), e = e(x,y,\tau), |\tau| \leq \epsilon, T \in [T_0,T_1], T_0 = Ch|\log h|, T_1 = \epsilon \mu^{-1} \gamma_1^{1-\nu}; \)

(ii) Moreover, under condition (6.8) statement (i) holds with \( Q = \psi_1 \psi, \psi_1 = \psi_1(x_1) \) supported in \( Z_{\gamma} = \{ \gamma \leq |x_1| \leq 2\gamma \} \).

**Corollary 6.2.** Let conditions (6.2) and (6.4) be fulfilled. Let \( \psi = \psi(x_2) \) be supported in \( B(0, \frac{1}{2}) \) and \( \varphi = \varphi(\xi_2) \) be supported in the outer zone defined in the terms of \( \xi_2 \)

\[ Y_{\text{out}} = \{ C_0 \leq |\xi_2| \leq \epsilon (\mu h^{-\nu})^{-1/(\nu-1)} \} \]

Then

(i) As \( \mu \leq \epsilon h^{-\nu} \) estimate

\[ R' \leq C \mu^{-1/\nu} h^{-1} \]

holds.

(ii) Moreover, under condition (6.8) statement (i) holds with \( Q = \psi_1 \psi, \psi_1 = \psi_1(x_1) \) supported in \( Z_{\text{out}} \).
On the other hand, under condition (6.4) the whole zone \( Z' = Z_{\text{inn}} \cup Z_{\text{out}} \) will be forbidden leading us to the following statement not having analogues in [Ivr10]:

**Proposition 6.3.** Let conditions (6.2) and (6.4) be fulfilled. Let \( \psi = \psi(x) \), \( \psi_1 = \psi_1(x_1) \) be supported in \( B(0, \frac{1}{2}) \) and \( Z' \) respectively and let \( \varphi = \varphi(\xi_2) \) be supported in the zone

\[
Y' = \{ |\xi_2| \leq \epsilon (\mu h^\nu)^{-1/(\nu-1)} \}.
\]

Then

(i) \( |Q_e| \leq C h^s \) with \( Q_e = \varphi(hD_2)(e\psi) \), \( e = e(x,y,\tau), |\tau| \leq \epsilon \) as \( \mu \leq \epsilon h^{-\nu} \);

(ii) Moreover, under condition (6.8) statement (i) holds with \( Q = \psi_1 \psi \), \( \psi_1 = \psi_1(x_1) \) supported in \( Z' \).

Therefore as \( \mu \leq \epsilon h^{-\nu} \) and condition (6.4) is fulfilled one needs to discuss the contribution of the inner zone \( Z_{\text{inn}} = \{ |x_1| \leq \tilde{\gamma}_1 \} \) or equivalently \( Y_{\text{inn}} = \{ |\xi_2| \leq C_0 \} \) to the remainder estimate. Furthermore one needs to consider the contribution of the previously forbidden zone \( Z'' = \{ |x_1| \geq \bar{\gamma}_1 \} \) or equivalently \( Y'' = \{ |\xi_2| \geq \epsilon (\mu h^\nu)^{-1/(\nu-1)} \} \) to the remainder estimate.

The inner zone is analyzed exactly as in section 2 and subsections 4.1–4.4 of [Ivr10] leading us eventually to

**Proposition 6.4.** Let conditions (6.2) and (6.4) be fulfilled. Let \( \psi = \psi(x_2) \) and \( \psi_1 = \psi_1(x_1) \) be supported in \( B(0, \frac{1}{2}) \) and \( Z_{\text{inn}} \) respectively and let \( \varphi = \varphi(\xi_2) \) be supported in \( Y_{\text{inn}} = \{ |\xi_2| \leq C_0 \} \). Then all the results of section 2 and subsections 4.1–4.4 of [Ivr10] remain true; in particular

(i) As \( \mu \leq Ch^{s-\nu} \)

\[
R'' \overset{\text{def}}{=} |\Gamma(Q_e) - h^{-1} \sum_j \int_{-\infty}^{0} (F_{t \rightarrow h^{-1} \tau} X_{T_j}(t) \Gamma(Q_j e)) \, d\tau| \leq C \mu^{-1/\nu} h^{-1}
\]

with \( Q_e = \varphi(hD_2)(e\psi) \), \( e = e(x,y,\tau) \), \( Q = \sum_j Q_j \) and \( |\tau| \leq \epsilon \) where partition \( Q_j \) and \( Ch |\log h| \leq T_j \) are defined following formula (3.28) in [Ivr10];

(ii) Moreover, under nondegeneracy condition

\[
(6.18)_m \sum_{1 \leq k \leq m} |\partial_{x_2}^k \left( \frac{W}{f} \right)| \geq \epsilon_0.
\]

3) These two definitions are essentially equivalent under condition (6.8) but in the case (6.9) one needs always use definition in the frames of \( \xi_2 \).
\( \mathcal{R}' \) does not exceed \( C\mu^{-1/\nu}h^{-1} \) as \( \mu \leq \epsilon h^{-\nu} \);

(iii) On the other hand, in the general case \( \mathcal{R}' \) does not exceed \( C\mu^{-1/\nu}h^{-1} + Ch^{-\delta} \) as \( \mu \leq \epsilon h^{-\nu} \);

(iv) Furthermore, under condition (6.8) all statements (i)--(iii) hold with \( Q = \psi_1 \psi \).

**Remark 6.5.** In frames of proposition 6.4 estimate (6.12) holds for \( Q = Q_m \) and \( T \in [T_m, T'_m] \) with \( T'_m \) defined by (2.98) (it was denoted by \( T_1 \) then).

### 6.4 Estimates. II

To investigate zone \( \mathcal{Z}' \) I will apply the theory of operators with operator-valued symbols. However, as \( \mu \leq \epsilon (h|\log h|)^{-\nu} \) one can apply a usual microlocal analysis with logarithmic uncertainty principle.

So, let us consider \( A \) as \( h \)-pseudo-differential operator \( \mathcal{A}(x_2, hD_2) \) with operator-valued symbol \( \mathcal{A}(x_2, \xi_2) \). However, before doing this one can assume without any loss of the generality that \( g^{11} = 1, g^{12} = 0 \) and therefore

\[
(6.19) \quad \mathcal{A}(x_2, \xi_2) = \frac{1}{2} \left( h^2 D_1^2 + \sigma^2(x) \left( \xi_2 - \mu V_2(x) \right)^2 - (2l + 1) \mu h F - W(x) \right).
\]

with \( \phi(x) = 1 \) as \( x_1 = 0 \); then \( f = \sigma \phi \).

Further, for given \( x_2 \) by change of variable \( x_1 \) one can transform \( \mathcal{A} \) unitarily to the similar operator with \( \phi = 1 \) and with

\[
\sigma = 1 \quad \text{as } x_1 = 0;
\]

but this new operator is multiplied from the left and the right by \( \alpha(x) \). So operator \( \mathcal{A}(x_2, \xi_2) \) is unitary equivalent to

\[
(6.21) \quad \mathcal{A}'(x_2, \xi_2) = \frac{1}{2} \alpha(x) \left( h^2 D_1^2 + \sigma^2(x) \left( \xi_2 - \mu \frac{1}{\nu} x_1^\nu \right)^2 - (2l + 1) \mu h \sigma(x) x_1^{\nu-1} - W_0(x) \right) \alpha(x).
\]

Note that \( W_0 = W/f \) as \( x_1 = 0 \) and thus conditions (6.4), (6.5) and (6.18) are reformulated in terms of \( W_0 \) obviously.

Proposition A.3(ii) of Appendix A implies that under condition (6.4) \( \mathcal{Y}' \setminus \mathcal{Y}_0' = \{ \epsilon(\mu h^\nu)^{-1/(\nu-1)} \leq |\xi_2| \leq 2C(\mu h^\nu)^{-1/(\nu-1)} \} \) is microhyperbolic with respect to \( \xi_2 \) and thus one can extend \( \mathcal{Y}' \) to zone \( \mathcal{Y} \) resulting in the following statement:
Proposition 6.6. Let conditions (6.2) and (6.4)± be fulfilled. Then estimate \( R' \leq C \) holds as \( R' \) is defined by (6.13) with \( Qe = \varphi(h D_2)(e \psi) \), \( \varphi \) supported in the zone \( Y'' \setminus Y''_0 \), \( T \in [T_0, T_1] \), \( T_0 = Ch \log h \), \( T_1 = \epsilon (\mu h)^{-1/(\nu - 1)} \), \( \mu \leq e h^{-\nu} \).

Furthermore, proposition A.3(i) implies that under condition (6.4)± zone \( Y'''_0 = \{ |\xi_2| \geq C(\mu h)^{-1/(\nu - 1)} \} \) is forbidden on energy levels \( |\tau| \leq \epsilon \) as long as \( \mu \leq e h^{-\nu} \) is forbidden; namely

\[
|F_{t-h^{-1}\tau_x}(t)(Qu)(x, y, t)| \leq C Th^s \quad \forall \tau : |\tau| \leq \epsilon
\]
as \( Q \psi = \varphi(h D_2)(w \psi) \) with \( \varphi \) supported in the zone \( Y'''_0 \) and therefore its contribution to the remainder \( R' \) defined by (6.13) is negligible as well:

Proposition 6.7. Let conditions (6.2) and (6.4)± be fulfilled. Then estimate \( R' \leq Ch^s \) holds as \( R' \) is defined by (6.13) with \( Qe = \varphi(h D_2)(e \psi) \), \( \varphi \) supported in the zone \( Y'''_0 \), \( T \geq T_0 = Ch \log h \), \( \mu \leq e h^{-\nu} \).

The analysis of all zones under condition (6.5) will be done in subsection 6.7.

6.5 Calculations. I

In this subsection I will change partition: instead of \( Z' \) and \( Z'' \) I will consider \( \tilde{Y}' \) and \( Y'''_0 \) obtained if I redefine \( \tilde{\gamma}_1 = C(\mu h)^{-1/(\nu - 1)} \); respectively change definitions and notations of zones \( Y_{out}, Z_{out}, Z', Z''. \)

After estimates were derived in two previous subsections under assumption \( Ch^{-1} \leq \mu \leq e h^{-\nu} \) and condition (6.4)± calculations in zone \( Y' \) are done exactly as in section 3 and subsection 4.4 of [Ivr10].

On the other hand, calculations in zone \( Y'''_0 \) as \( \mu \leq e h^{-\nu} \) are rather obvious under assumptions \( Ch^{-1} \leq \mu \leq e h^{-\nu} \) and (6.4)±. Therefore I arrive to the intermediate estimate

\[
(6.23) \quad |\int \left( (\varphi(h D_2)e)(x, x, 0) - (2\pi h)^{-1} \int e(x_1, x_2, \xi_2, 0)\varphi(\xi_2) d\xi_2 \right) \psi_2(x_2) dx| \leq R
\]
where \( R \) is an estimate already derived in the corresponding conditions (also see below) and \( \varphi \in C^\infty_0 (-\epsilon', \epsilon') \) with sufficiently small constant \( \epsilon' \).

Then the same estimate holds with \( \psi(x_2) \) replaced by \( \psi(x) \) such that \( \psi(x) = \psi_2(x_2) \) as \( |x_1| \leq C_1 \epsilon' \) because this transition leads to a negligible error. I take \( \psi \) also satisfying \( \psi(x) = 0 \) as \( |x_1| \geq 2C_1 \epsilon' \). Then in the latter estimate I can replace \( \varphi \) by 1. Really, then the error would be

\[
(6.24) \quad |\int \left( (1 - \varphi(h D_2))e)(x, x, 0) - (2\pi h)^{-1} \int e(x_1, x_2, \xi_2, 0)(1 - \varphi(\xi_2)) d\xi_2 \right) \psi(x) dx| \]
and replacing $\psi$ by $\psi'$ equal to $\psi$ as $|x_1| \geq 2C_2^{-1} \epsilon'$ and equal to 0 as $|x_1| \leq C_2^{-1} \epsilon'$ leads to a negligible error. However, to expression (6.24) modified this way one can apply the theory of operators with non-degenerating magnetic field and then to estimate expression (6.24) by $C$.

Thus I derived (6.23) with $\varphi$ replaced by 1 and $\psi_2(x_2)$ replaced by some “special” function $\psi(x)$. Then due to rescaling arguments like in subsubsection 6.2.2 the same estimate holds for a general function $\psi(x)$ supported in $\{|x_1| \leq 2C_1 \epsilon'\}$. Thus I arrive to

**Proposition 6.8.** Let conditions (6.2) and (6.4)$_+$ be fulfilled. Then

(i) As either $\mu \leq \epsilon h^{-\nu}$ or condition (6.18)$_m$ is fulfilled and $\mu \leq \epsilon h^{-\nu}$ the following estimate holds

$$R_I \overset{\text{def}}{=} \left| \int (e(x,x,0) - (2\pi h)^{-1} \int e(x_1,x_1;x_2,0) d\xi_2) \psi(x) dx \right| \leq C \mu^{-\frac{1}{2}} h^{-1}$$

where here and below $e(x_1,y_1;x_2,\xi_2,\tau)$ is the Schwartz kernel of the spectral projector of operator $A(x_2,\xi_2)$ defined by (6.19) and $\delta > 0$ is an arbitrarily small exponent;

(ii) In the general case with $\mu \leq \epsilon h^{-\nu}$ estimate

$$R_I \leq C \mu^{-\frac{1}{2}} h^{-1} + Ch^{-\delta}$$

holds.

I remind that in both statements of proposition 6.8 the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$ (as $\mu \geq h^{-1}$).

On the other hand, under condition (6.4)$_-$ zone $Y'$ becomes forbidden and thus I arrive to

**Proposition 6.9.** Let conditions (6.2) and (6.4)$_-$ be fulfilled and $l \geq 1$. Then for $Ch^{-1} \leq \mu \leq \epsilon h^{-\nu}$ estimate $R_I \leq C$ holds while the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$.

### 6.6 Calculations. II

Transition to the auxiliary operator $A_0$ without increasing error estimates could be done easily in zone $Y_{\text{out}}$ exactly as it was done in the proof of propositions 3.3 and 3.4 while arguments of 3.8 etc work in zone $Y_{\text{inn}}$.

On the other hand, this transition in zone $Y_{0}''$ is obvious under condition (6.4)$_{\pm}$, and I arrive to two theorems below as $\mu \leq h^{-\nu} |\log h|^{-K}$ and function $\psi$ is “special” in the sense
of the previous subsection. Then the same arguments as there extend theorem to general $\psi$.

Furthermore, under condition (6.4)$_\pm$ the case $h^{-\nu}|\log h|^{-K} \leq \mu \leq \epsilon h^{-\nu}$ is analyzed exactly as in section 4 of Part I leading to the extension of these theorems to $\mu \leq \epsilon h^{-\nu}$:

**Theorem 6.10.** Let conditions (6.2) and (6.4)$_+$ be fulfilled. Then

(i) As either $\mu \leq h^{-\nu}$ or condition (6.18)$_m$ is fulfilled and $\mu \leq \epsilon h^{-\nu}$

$$R^* \overset{\text{def}}{=} |\int (e(x, x, 0) - \tilde{E}^{\text{MW}}(x, 0)) \psi(x) dx - \int \mathcal{E}_{\text{corr}}^{\text{MW}}(x_2, 0) \psi(0, x_2) dx_2|$$

does not exceed $C\mu^{-1/\nu}h^{-1}$ where

$$\mathcal{E}_{\text{corr}}^{\text{MW}}(x, \tau) \overset{\text{def}}{=} (2\pi h)^{-1} \int e_0(x_1, x_1; x_2, \xi_2, \tau, h) dx_1 d\xi_2 - \int \tilde{E}_0^{\text{MW}}(x, \tau) dx_1,$$

$\mathcal{E}^{\text{MW}}$ is Magnetic Weyl approximation$^4$ and here and below $e_0(x_1, y_1; x_2, \xi_2, \tau)$ is the Schwartz kernel of the spectral projector of operator $A_0(x_2, \xi_2)$ defined by (6.19) and with $\alpha, \phi, \sigma, W$ restricted to $\{x_1 = 0\}$ and $\tilde{E}_0^{\text{MW}}$ is Magnetic Weyl approximation for this operator.

(ii) In the general case with $\mu \leq \epsilon h^{-\nu}$ estimate $R^* \leq C\mu^{-1/\nu} + Ch^{-\delta}$ holds.

**Theorem 6.11.** Let conditions (6.2) and (6.4)$_-$ be fulfilled and $l \geq 1$. Then as $Ch^{-1} \leq \mu \leq \epsilon h^{-\nu}$ estimate $R^* \leq C$ holds while the principal part of asymptotics has magnitude $\asymp \mu h^{-1}$.

**Remark 6.12.** Obviously the same approximate expressions (3.52), (3.52)$^*$, (3.52)$^{**}$ hold for the part of $\mathcal{E}_{\text{corr}}^{\text{MW}}$ "associated" with $\mathcal{Y}_{\text{inn}}$;

6.7 Estimates under condition (6.5)

I start from the remainder estimate in zone $\mathcal{Y}'$ which is trivial:

**Proposition 6.13.** Let conditions (6.2), (6.20) and (6.5) be fulfilled. Then

(i) Estimate (6.13) holds with $Qe = \varphi(hD^2)(e\psi)$, $\varphi$ supported in the strip $\mathcal{Y}_\gamma$ with the same restrictions to $\gamma$ and the same $T_0, T_1$ as in proposition 6.1(i);

(ii) Furthermore, the same estimate holds as $\varphi$ is supported in zone $\mathcal{Y}_{\text{inn}}$ and $\gamma = \tilde{\gamma}_0 = \mu^{-1/\nu}$;

(iii) Therefore $R'$ defined by (6.13) does not exceed $C\mu^{-1/\nu}h^{-1}$ as $\varphi$ is supported in zone $\mathcal{Y}'$ and $T = T_0$.

$^4$ See e.g. (0.8).
6 MODIFIED V. I. \( \mu \leq eh^{-\nu} \)

Now let us analyze zone \( Y_0'' \) under condition (6.5):

**Proposition 6.14.** Let conditions (6.2), (6.20) and (6.5) be fulfilled. Then estimate \( R' \leq C \) holds as \( R' \) is defined by (6.13) with \( Qe = \varphi(hD_2)(e\psi) \), \( \varphi \) supported in the zone \( Y_0'' \).

**Proof.** (i) Let us note first that estimate (6.31)

\[
|F_{t \mapsto h^{-1}\tau}(\bar{x}_T(t) - \bar{x}_T(t))(Qu)(x, y, t)| \leq Ch^3 \quad \forall \tau : |\tau| \leq \epsilon
\]

holds with \( T_1 = \epsilon \mu^{-1}\gamma^{-\nu}, \bar{T} = Ch|\log h| \) as \( Qu = \varphi(hD_2)(u\psi) \), \( \varphi \) supported in the strip \( Y_{(\gamma)} = \{ \mu\gamma^\nu \leq |\xi_2| \leq 2\gamma^\nu \} \) with \( \gamma \geq C\gamma_1 \).

Really, us consider a partial trace \( \Gamma'(Qu) \) (with respect to \( x_1 \)). Due to proposition A.3 the propagation speed with with respect to \( x_2 \) does not exceed \( C|\xi_2|^{-1} \simeq C(\mu\gamma^\nu)^{-1} \) and the propagation speed with respect to \( \xi_2 \) does not exceed \( C^5 \); moreover, under condition (6.5) this propagation speed with respect to \( \xi_2 \) is greater than \( \epsilon \).

On the other hand, an obvious estimate

\[
|F_{t \mapsto h^{-1}\tau}\bar{x}_T_0(t)\Gamma(Qu)(t)| \leq C\mu\gamma^\nu h^{-1} \times T_0 = C\mu\gamma^\nu|\log h|
\]

holds where the first factor is \( \mu_{\text{eff}}h_{\text{eff}}^{-1}\gamma^{-1} \); furthermore, due to (6.29) this estimate holds for the left-hand expression with \( T_0 \) replaced by \( T_1 \).

Therefore the contribution of the strip \( Y_{(\gamma)} \) to the remainder estimate does not exceed

\[
C\mu\gamma^\nu|\log h| \times T_0^{-1} = C|\log h|
\]

and therefore the total contribution of \( Y_0'' \) to the remainder estimate does not exceed \( C|\log h| \int \gamma^{-1} d\gamma \simeq C|\log h|^2 \).

This estimate is as good as I need for \( \mu \leq Ch^{-\nu}|\log h|^{-2\nu} \). However for \( Ch^{-\nu}|\log h|^{-2\nu} \leq \mu \leq eh^{-\nu} \) I would like to improve it getting rid of two logarithmic factors.

(ii) Getting rid off one of them is easy: rescaling \( t \mapsto t/T, (x_j - y_j) \mapsto (x_j - y_j)/T, \mu \mapsto \mu T, h \mapsto h/T \) estimates for Schrödinger operator with strong non-degenerate magnetic field [Ivr1], section 6 (with arbitrary parameters \( \mu \) and \( h \) such that \( \mu h \geq C \) I arrive to two following inequalities

\[
|F_{t \mapsto h^{-1}\tau}\bar{x}_T(t)\Gamma(Qu)| \leq C\mu \left( \frac{h}{T} \right)^s
\]

\[
|F_{t \mapsto h^{-1}\tau}\bar{x}_T(t)\Gamma(Qu)| \leq C\mu
\]

\footnote{Under some assumptions this would be equivalent to the estimate of the the average propagation speed with respect to \( x_1 \) of \( Qu \) by \( C\gamma(\mu\gamma^\nu)^{-1} \); further one can estimate average propagation speed with respect to \( x_2 \) of \( Qu \) by \( C(\mu\gamma^\nu)^{-1} \) as well.}
as $h \leq T \leq 1$, $|\tau| \leq \epsilon$ under condition $|W| + |\nabla W| \geq \epsilon_0$. Then using our standard scaling 
$x_1 \mapsto x_1/\gamma$, $x_2 \mapsto (x_2 - y_2)/\gamma$, $\mu \mapsto \mu_{\text{eff}} = \mu \gamma^\nu$, $h \mapsto h_{\text{eff}} = h/\gamma$ and $T \mapsto T/\gamma$ I arrive to estimate (6.30) without logarithmic factor

\[(6.30)^* \quad |F_{t \rightarrow h^{-1} \tau} \chi_T(t) \Gamma(Qu)(t)| \leq C \mu \gamma^\nu\]

as $|\tau| \leq \epsilon$, $T/\gamma \leq \epsilon \mu \gamma^\nu$ $\iff$ $T \leq T'_1 = \epsilon \mu \gamma^\nu + 1$. Further, and to (6.29) this estimate holds as $h \leq T \leq T_1 = \epsilon \mu \gamma^\nu$ provided $T'_1 \geq Ch$ i.e. $\gamma \geq \bar{\gamma}_1$.

Then the contribution of the strip $\mathcal{Y}_\gamma$ to the remainder $\mathcal{R}'$ is $C$ and therefore the total estimate is $C |\log h|$.

(iii) To get rid off the second logarithmic factor I need to further increase $T_1$ in the previous arguments and for this purpose I need for each $\gamma$ to make $x_2$-partition of $\mathcal{Y}_\gamma$ of the size

\[(6.34) \quad \ell = \epsilon |V(0, x_2)| + \bar{\ell}, \quad \bar{\ell} \geq C \gamma.\]

Consider first elements $\mathcal{U}_{\gamma, \ell}$ with $\ell \geq C \bar{\ell}$. For every such element on levels $\tau$ with $|\tau| \leq \epsilon \ell$ after rescaling

\[(6.35) \quad x_2 \mapsto x_2 \ell^{-1}, \quad h \mapsto h' = h \ell^{-\frac{1}{2}}, \quad t \mapsto t \ell^{-1}, \quad \mu \mapsto \mu' = \mu \ell^{\frac{1}{2}}\]

I am in the elliptic situation.

Therefore contribution of each such element to the remainder estimate does not exceed $C \mu'(h')^\ast$ and therefore the total contribution of such elements is negligible as $\bar{\ell} = h^\delta$.

So I need to consider only elements $\mathcal{U}'_{\gamma} = \mathcal{U}_{\gamma, \ell}$ with $\ell \times \bar{\ell} = h^\delta$. For such elements after rescaling (6.35) I can apply estimate (6.30)$^\ast$; then scaling back I get the same estimate (6.30)$^\ast$ again but with $Q = \psi'(x_2) \varphi(h D_2)$ supported in $\mathcal{U}'_{\gamma}$, $|\tau| \leq \epsilon \ell$ and $Ch |\log h| \ell^{-1} \leq T \leq T_1 = \epsilon \mu \gamma^\nu + 1$). Furthermore, applying (6.29) I can increase $T_1$ to $\epsilon \mu \gamma^\nu$.

So far I gained nothing: the estimate I proved alone would bring me the same final remainder estimate $C |\log h|$ as before but now I can further increase $T_1$ and thus reduce the remainder estimate.

Namely, let us consider propagation in the time direction in which $|\xi_2|$ increases. If only propagation with respect to $\xi_2$ was considered, until time $\epsilon_3 \mu$ it would be confined to zone

\[
\{ \epsilon_0 \leq |\xi_2| (\mu \gamma^\nu + |t|)^{-1} \leq C \} \subset \left\{ \frac{1}{2} \mu \gamma^\nu \leq |\xi_2| \leq \epsilon_1 \mu \right\}
\]

and thus to $\{|x_1| \leq \epsilon_3\}$.

\[\text{(6)} \quad \text{It is consistent with the fact that support of } \psi' \text{ is of the length } \ell \text{ but now } \bar{T} = Ch |\log h| \ell.\]
However let us note that the propagation speed with respect to $x_2$ does not exceed $C\ell/|\xi_2|$ as $\ell \geq C|V|+\ell$. Therefore one can prove easily that propagation, which started in the zone $\{|x_2| \leq \frac{1}{2}, |V| \leq h^\delta\}$ as I have assumed, until time $T_1^* = \mu \gamma^\nu h^{-\delta_1}$ is confined to a bit larger zone $\{|x_2| \leq \frac{2}{3}, |V| \leq h^{\delta/2}\}$ of the same type.

Therefore estimate (6.30)* holds with $Ch^{1-2\delta} \leq T \leq T_1^*$. Then due to the Tauberian approach contribution of each partition element $U'$ to the remainder estimate does not exceed $C\mu \gamma^\nu T_1^{*-1} = Ch^{\delta_1}$ and the contribution of the whole strip $\mathcal{Y}$ does not exceed $Ch^{\delta_1}$ as well and of the whole zone $\mathcal{Y}_0''$ does not exceed $Ch^{\delta_2}$.

Clearly, at some moment I increased slightly $T_0$ but after summation over partition was done I can (using negligibility of the trace on $[Ch|\log h|, h^{1-\delta}]$ time interval on energy levels $|\tau| \leq \epsilon$) return to original $T$.

\section*{6.8 Calculations under condition (6.5)}

Calculations in zone $\mathcal{Y}'$ are exactly as in [Ivr10]. However one should be more careful with calculations in zone $\mathcal{Y}_0''$.

Let me remind that according to subsection 6.2 [Ivr1] in the nondegenerate case with $\mu h \geq C$ the operator in question is reduced to one-dimensional $\mu^{-1} h$-pdo $B(x_2, \mu^{-1} hD_2, h^2)$ \footnote{Where $x_2$ is not our original $x_2$.} with the “main symbol” $B(x_2, \xi_2, 0) = W \circ \Psi$ and therefore the contribution of the partition element to the final answer will be given as in subsection 6.6 by magnetic Weyl expression $\int E^{MW}(x, 0)\psi(x) \, dx$ plus correction terms $\mu h^{1+2m} \int \kappa_{l,m}(x)\psi(x) \, dx$, $m = 0, 1, \ldots$

After rescaling $\mu \mapsto \mu \gamma^\nu$, $h \mapsto h/\gamma$, $dx \mapsto \gamma^{-2} dx$ these terms are transformed into

\begin{equation}
\mu h^{1+2m} \int \kappa_{l,m}(x, \gamma)\psi(x)\gamma^{-2m-3} \, dx
\end{equation}

integrated over zone $\{\gamma_1 \leq \gamma \leq \epsilon\}$.

One can see easily that if there was an extra factor $\gamma$ one would be able to rewrite this expression (6.36) modulo $O(1)$ into the similar expression with integration over $\{\gamma \leq \epsilon\}$ as $2m + 2 < \nu$ \footnote{thus resulting in exactly expression $\kappa_{l,m}\mu h^{1+2m}$ as in non-degenerate case.} or to simply skip it as $2m + 2 > \nu$ or to get a term which is $O(\mu h^\nu|\log h|)$ as $2m + 2 = \nu$. To gain this extra factor one needs to consider the difference of expressions $\int e(x, x, 0)\psi(x) \, dx$ for two operators with $g^{jk}(x), f(x), V(x)$ coinciding as $x_1 = 0$. As this second operator it is natural to pick up the simplest one i.e.

\begin{equation}
A_0 = \frac{1}{2} \left( h^2 D_1^2 + (hD_2 - \mu x_1^\nu/\nu)^2 - (2l + 1) \mu h x_1^{\nu-1} - W(x_2) \right).
\end{equation}

Therefore I arrive to
Proposition 6.15. Under condition \((6.5)\) estimate

\[
(6.38) \quad \left| \int \left( e(x, x, 0) - e_0(x, x, 0) - \mathcal{E}^{\text{MW}}(x, 0) + \mathcal{E}_0^{\text{MW}}(x, 0) \right) \psi(x) \, dx - \sum \kappa_{l,m} \mu h^{1+2m} \right| \leq C \mu^{1/\nu} h^{-1}
\]

holds as \(\mu \leq h^{-\nu} \log h^{-K}\) where \(e_0\) and \(\mathcal{E}_0^{\text{MW}}\) are defined for operator \(A_0\).

(6.39) Now in what follows I can consider operator \(A_0\) instead of \(A\).

Then I can apply the standard method of successive approximations with unperturbed operator \(A(y_2, hD_2)\) and plug the results of successive approximations into expression

\[
(6.40) \quad h^{-1} \int_{-\epsilon}^{0} \left( F_{t \rightarrow h^{-1} \tau} \chi_T(t) \Gamma(Qu) \right) \, d\tau
\]

which calculates exactly contribution of the “problematic” eigenvalue \(\lambda_l\) of the corresponding one-dimensional Schrödinger operator; I remind that \(T = \bar{T} = Ch|\log h|\).

Thus while the main part of asymptotics is estimated by \(C \mu h^{-2} \gamma^\nu T = C \mu h^{-1} \gamma^\nu |\log h|\), each next term seemingly acquires factor

\[
(6.41) \quad Ch^{-1} (\mu h \gamma^{\nu-1})^{1/2} T^2 \asymp Ch (\mu h \gamma^{\nu-1})^{1/2} |\log h|^2;
\]

since the propagation speed with respect to \(x_2\) is estimated by \(C_0 (\mu h \gamma^{\nu-1})^{1/2}\) such factor could be larger than 1.

In fact however, \(C_0 (\mu h \gamma^{\nu-1})^{1/2}\) is the estimate for the instant propagation speed only. Using instead the mentioned reduction to a one-dimensional \(\mu^{-1} h\)-pdo one can find that the propagation speed with respect to \(x_2\) is estimated by \(C_0 \mu^{-1}\) if magnetic field is non-degenerate and then in the canonical coordinates for time \(T = \bar{T}\) the shift of \((x_2', \xi_2')\) will be estimated by \(C_0 (\mu^{-1} h |\log h|)^{1/2}\) which is the smallest distance allowed by the logarithmic uncertainty principle\(^9\) and this would persist if one returns back to the original \((x_2, \mu^{-1} \xi_2)\); so one would be able to estimate \((x_2 - y_2)\) on the time interval in question by \(C_0 (\mu^{-1} h |\log h|)^{1/2}\).

\(^9\) Since \(\mu^{-1} h\)-Fourier Integral Operators are involved later one needs the same distance in each \((x, \xi)\) direction.
In the degenerate case described here one must replace $\mu$, $h$ by $\mu \gamma^\nu$, $h/\gamma$ respectively and then multiply by $\gamma$ thus producing final estimate for $|x_2 - y_2|

(6.42) \quad \varrho \overset{\text{def}}{=} C\left(\mu^{-1} h \gamma^{1-\nu} |\log h| \right)^{1/2} \asymp C h \gamma^{1/2(\nu-1)} \gamma^{-\nu} h^{\nu-1} \gamma \log h \frac{1}{2}

and therefore each next term acquires factor $\varrho |\log h|$. Then $m$-th term of the final answer is estimated by

(6.43) \quad C \mu h^{-1} \varrho^{m-1} |\log h|^K \asymp C \mu h^{m-2} \gamma^{\nu - \frac{1}{2}(\nu-1)(m-1)} \gamma \frac{1}{2} h^{\nu} \gamma h \gamma |\log h|^K.

After integration over $|\gamma^{-1} d\gamma$ with $\gamma_1 \leq \gamma \leq \epsilon$ expression (6.43) results in $C \mu h^{m-2} \gamma \nu h |\log h|^K$ as $\nu - \frac{1}{2}(\nu-1)(m-1) \leq 0$ or in $C (\mu^{-1} h)^{(m-3)/2} |\log h|^K$ otherwise. One can check easily that in either case the answer is $O(|\log h|^K)$ as $m \geq 3$ and only terms with $m = 1, 2$ should be considered more carefully under condition (6.8).

On the other hand, the main term appears as (6.40) with $u$ replaced by $\bar{u}$ and modulo negligible one can rewrite it with any $T \geq \bar{T}$, in particular with $T = \infty$ which leads to

(6.44) \quad (2\pi h)^{-1} \int e_0(x_1, x_1, 0; x_2, \xi_2) \psi(x_1) \varphi(\xi_2) \, dx_2 \, d\xi_2

where I remind that $e_0(x_1, y_1, 0; x_2, \xi_2)$ is the Schwartz kernel of the spectral projector of one-dimensional Schrödinger operator $A_0(x_2, \xi_2)$.

Let us consider terms with $m = 2$ i.e. expression (6.40) with $u$ replaced by $\bar{u}_1$; similarly to analysis of (i) one can estimate contribution of $O((x_2 - y_2)^2)$ terms in the perturbation $A(x_2, hD_2) - A(y_2, hD_2)$ by $C |\log h|^K$. Therefore one should consider only $A(x_2, hD_2) - A(y_2, hD_2) = (x_2 - y_2) B_1(y_2)$ in which case $\bar{u}_1$ is defined by (3.23) without the last term since $B_1$ commutes with $(x_2 - y_2)$:

(6.45) \quad u \mapsto \bar{u}_1 = -ih \sum_{\xi = \pm} \varsigma \bar{G}^{x} B_1 \bar{G}^{x} [\bar{A}, x_2 - y_2] \bar{G}^{x} \delta(t) \delta(x_2 - y_2) \delta(x_1 - y_1).

One needs to multiply this by $h^{-1} \psi$, integrate with respect to $\tau$ and apply $\Gamma$ to it. Obviously since for odd $\nu$ operators $\bar{G}^{x}$ and $[\bar{A}, x_2 - y_2]$ are even and odd respectively as $x_1 \mapsto -x_1$, $\xi_2 \mapsto -\xi_2$ the answer would be 0 if $\psi$ is even with respect to $x_1$.

To cover the case of even $\nu$ and general $\psi$ let us note that $B_1$ commutes with $\bar{G}^{x}$ considered as operators in the auxiliary space $L^2(\mathbb{R}_{x_1}^1)$. Then if $\bar{G}^{x}$ commuted with $\psi$, taking trace and integrating with respect to $\tau$ would result in

$$\text{const} \cdot \partial_{\xi_2} B_1 \sum_{\varsigma = \pm} \varsigma \, \text{Tr}(\bar{G}^{x} \psi)$$
which after integration over $\xi_2$ results in 0.

However $\bar{G}^\varsigma$ does not commute with $\psi$, so instead of 0 one gets

$$\text{const} \cdot B_1 \sum_{\varsigma = \pm} \text{Tr} \varsigma \left( \bar{G}^\varsigma (\partial_{\xi_2} \bar{G}^\varsigma) \left( \bar{G}^\varsigma [\bar{A}, \psi] \right) \right)$$

and to this expression one can apply the same type of transformations and calculations as in the proof of proposition 6.15 resulting in the expression $\sum_m \kappa_{l,m} \mu h^{1+2m}$ where coefficients $\kappa_{l,m}$ are changed as needed.

Therefore combining with the results for zone $\mathcal{Y}''$ I arrive to

**Proposition 6.16.** For a model operator

(6.46) \( | \int \left( e_0(x,x,0) - (2\pi h)^{-1} \int e_0(x_1,x_1,0;x_2,\xi_2) d\xi_2 \right) \psi(x) dx - \sum \kappa_{l,m} \mu h^{1+2m} | \leq C \mu^{-1/\nu} h^{-1} \)

as $\mu \leq C h^{-\nu} | \log h |^{-K}$.

Further, combining this with proposition 6.14 I get as $\mu \leq h^{-\nu} | \log h |^{-K}$ estimate (6.47):

**Theorem 6.17.** Under condition (6.5) estimate

(6.47) \( | \int \left( e(x,x,0) - (2\pi h)^{-1} \int e_0(x_1,x_1,0;x_2,\xi_2) d\xi_2 

- E^{\text{MW}}(x,0) + E^{\text{MW}}_0(x,0) \right) \psi(x) dx - \sum \kappa_{l,m} \mu h^{1+2m} | \leq C \mu^{-1/\nu} h^{-1} \)

holds as $\mu \leq e h^{-\nu}$.

**Proof.** To finish the proof of this theorem one needs to cover the case $h^{-\nu} \log h |^{-K} \leq \mu \leq e h^{-\nu}$, getting rid of the term $| \log h |^K$ in the error estimates.

The first problematic error comes from the correction terms in proposition 6.15, namely from the terms of the type $\mu h^{1+2m} \int \chi_{l,m}(x_2) \gamma^{\nu-2m-3+k} dx$ with $k \geq 1$, $\nu - 2m - 3 + k = -1$ and this error term is $O(1)$ unless $k = 1$, $\nu = 2m + 1$ in which case it is $\kappa'_l \mu h^{\nu} | \log h |$. This is possible only for odd $\nu$ in which case operator $\mathcal{A}_0$ is even with respect to $x_1 \mapsto -x_1$, $\xi_2 \mapsto -\xi_2$ but perturbation contains exactly one factor $x_1$ and therefore it is odd and after integration with respect to $x_1$, $\xi_2$ this correction term results in 0 if $\psi$ is even with respect to $x_1$.
Further, one needs to consider terms corresponding to \( m = 3 \) in the successive approximations leading to proposition 6.16 and there one can replace \( A_0(x_2, \xi_2) - A_0(y_2, \xi_2) \) by \( B_1(x_2 - y_2) \), and also terms corresponding to \( m = 2 \) in the same successive approximations and there one can replace \( A_0(x_2, \xi_2) - A_0(y_2, \xi_2) \) by \( B_2(x_2 - y_2)^2 \).

To calculate the contribution of such terms one can apply the same approach as in the proof of proposition 6.15 and the contribution of \( \gamma \)-admissible partition element with respect to \( x_1 \) will be

\[
\sum_m \mu h^{1+2m} \int \kappa_{l,m,k}(x_2) \psi(x) \gamma^{\nu-2m-3+k} \, dx
\]

with \( k \geq 0 \); however since this expression should be \( O(|\log h|^K) \) all the terms but those with \( \nu \leq 2m + 1 \), \( k \geq 1 \) should vanish; further, the total contribution of all remaining terms save those with \( \nu = 2m + 1 \) and \( k = 1 \) is \( O(1) \), which leaves us with no “bad” terms for even \( \nu \) and with one “bad” term \( \kappa' \mu h^\nu \log h \) for odd \( \nu \), \( m = (\nu - 1)/2 \). However, parity considerations with respect to \( x_1 \) show that this term should vanish as well. \( \Box \)

**Remark 6.18.** (i) All the coefficients \( \kappa_{l,*} \) and \( \kappa_{l,*} \) vanish for \( l = 0 \).
(ii) Obviously the same approximate expressions \((3.52), (3.52)^*, (3.52)^*\) as in [Ivr10] hold for part \( \mathcal{E}_{\text{corr}} \) “associated” with \( \mathcal{Y}_{\text{nn}} \);

### 7 Modified V. II. \( \epsilon_0 h^{-\nu} \leq \mu \leq C_0 h^{-\nu} \)

Now I will consider the intermediate case

\[
(7.1) \quad \epsilon_0 h^{-\nu} \leq \mu \leq C_0 h^{-\nu}
\]

with arbitrarily small constant \( \epsilon_0 \) and arbitrarily large constant \( C_0 \); this case which described the largest possible values in [Ivr10] now is no more than transition to the next section.

#### 7.1 Estimates

Let us denote by \( \lambda_n(\xi_2) \) eigenvalues of operator

\[
(7.2) \quad a^0 = \frac{1}{2} \left( D_1^2 + (\xi_2 - x_1^\nu/\nu)^2 - (2l + 1)x_1^{\nu-1} \right);
\]

then \( \Lambda_n(x_2, \xi_2) = \lambda_n(\xi_2) - \frac{1}{2} W(x_2) \) are eigenvalues of \( a = a^0 - W(x_2) \).
My main nondegeneracy assumption will be

\[(7.3) \quad |\Lambda_n| + (|\xi_2| + 1)|\partial_{\xi_2}\Lambda_n| + |\partial_{xx}\Lambda_n| \geq \epsilon_0 \quad \forall n, \xi_2, \]

may be coupled with (6.4)\pm. This condition (7.3) follows from (6.5); further, it follows from (6.4)\pm for $|\xi_2| \geq C$. On the other hand, since $\lambda_n \to 0$ and $\xi_2 \partial_{\xi_2}\lambda_n \to 0$ as $|\xi_2| \to \infty$, condition (7.3) implies that $|W| + |\partial_{xx}W| \geq \epsilon_0$ and therefore locally one of conditions (6.4)\pm, (6.5) must be fulfilled.

Obviously, under conditions (7.1), (7.3) for each $\xi_2$ number of eigenvalues of one-dimensional operator

\[(7.4) \quad A_0 = \frac{1}{2} \left( h^2 D_1^2 + (\xi_2 - \mu x_1^\nu / \nu)^2 - (2l + 1)x_1^{\nu - 1} - W \right) \]

below level $c_0$ does not exceed $C$.

Further, note that condition (7.3) for eigenvalues of $A_0$ is equivalent to the same condition for eigenvalues of $a$. Then I easily arrive to

**Proposition 7.1.** Under conditions (7.1), (7.3) contribution to the remainder estimate of the zone $\{|\xi_2| \leq C\}$ is $O(1)$.

Furthermore, analysis in the zone $\mathcal{Y}_0''$ under condition (7.1) does not differ from the analysis as $\mu \leq \epsilon h^{-\nu}$. Namely

\[(7.5) \quad \text{Under conditions (7.1) and (6.4)\pm operator } A_0 \text{ and thus operator } A \text{ is elliptic in the zone } \mathcal{Y}_0'' \text{ and the contribution of } \mathcal{Y}_0'' \text{ to the remainder estimate is negligible.} \]

\[(7.6) \quad \text{Similarly, under conditions (7.1) and (6.5) operator } A \text{ is microhyperbolic in the zone } \mathcal{Y}_0'' \text{ and the contribution of } \mathcal{Y}_0'' \text{ to the remainder estimate is } O(1). \]

Therefore

**Proposition 7.2.** Let conditions (7.1), (7.3) and one of conditions (6.4)\pm, (6.5) be fulfilled. Then the remainder estimate is $O(1)$ where the principal part is defined by (6.40).
7.2 Calculations

Calculations in this case also do not differ from those in section 6 leading to the following statements

**Theorem 7.3.** Let conditions (7.1), (7.3) and (6.4) ± be fulfilled. Then $R_I$ defined by (6.25) and $R^*$ defined by (6.27) do not exceed $C$.

**Theorem 7.4.** Let conditions (7.1) and (6.5) be fulfilled. Then left-hand expressions of (6.38), (6.46) and (6.47) do not exceed $C$.

8 Modified V. III. $\mu \geq C_0 h^{-\nu}$

Now I consider the previously forbidden case

$$\mu \geq C_0 h^{-\nu}$$

with sufficiently large constant $C_0$. In this case all zones should be redefined. Also the difference between $l = 0$ and $l \geq 1$ becomes crucial.

8.1 Estimates. I

As $|\xi| \approx \mu^{-\nu}, \gamma \geq C_1(\mu^{-1}h)^{(\nu+1)}$ let us consider first eigenvalues $\Lambda_n(x_2, \xi_2)$ of operator $A(x_2, \xi_2)$. Then proposition A.3 implies instantly that

$$\Lambda_n(x_2, \xi_2) \approx (n-l) \mu h^{\gamma-1}$$

and signs of the left and right-hand expressions coincide and

$$\Lambda_l(x_2, \xi_2) = \omega_l h^2 \gamma^{-2} - \frac{1}{2} W(x_2) + O\left(h^2 \gamma^{-1} + h^2 (\mu^{-1}h)^{2\gamma-4-2\nu}\right), \quad \omega_l > 0 \text{ as } l \geq 1.$$

Therefore

$$\mu h^\nu \text{def} \{C_0(\mu h^\nu)^{1/(\nu+1)} \leq |\xi| \leq \epsilon h^\nu\} \text{ is elliptic and its contribution to the remainder estimate is } O(h^s).$$

On the other hand,
(8.5) Under condition (6.4)± zone $Y = \{ |\xi_2| \geq C \mu h^\nu \}$ is elliptic as well and its contribution to the remainder estimate is $O(h^s)$ as well for $l \geq 0$.

Therefore as $l \geq 1$ and condition (6.4)± is fulfilled, one needs to analyze only two remaining zones $X_1 = \{ \epsilon \rho_1 \leq |\xi_2| \leq C \rho_1 \}$, $\rho_1 = \mu h^\nu$ and $X_0 = \{ |\xi_2| \leq C_0 \rho_0 \}$, $\rho_0 = (\mu h^\nu)^{1/(\nu+1)}$.

In the zone $X_1$ propagation speed with respect to $x_2$ is in average $\approx \rho_1^{-1}$ (with $\rho = \rho_1$) due to proposition A.3 again and the propagation speed with respect to $\xi_2$ is in average $O(1)$ and therefore one can take

\[ T_0 = C h |\log h|, \quad T_1 = \epsilon_1 \rho_1 \]

and for $T \in [T_0, T_1]$ propagation on the energy levels $\tau \in [-\epsilon, \epsilon]$ which started in $B(0, \frac{1}{\epsilon})$ does not leave $B(0, 1)$ but the shift with respect to $x_2$ is $\approx \rho_1^{-1} T$ and it satisfies logarithmic uncertainty principle and thus the spectral trace is negligible.

**Remark 8.1.** One should be more careful as $\mu \geq h^{-M}$ with arbitrarily large $M$ and use $\log \mu$ instead of $|\log h|$.

Therefore

\[ |F_{t \rightarrow h^{-1}\tau,\bar{\chi}_T}(t)(Qu)| \]

does not exceed $C h^{-1} \rho T_0 = C \rho |\log h|$ where $Q$ is a partition element corresponding to $X_1$, $|\tau| \leq \epsilon$. Therefore due to Tauberian arguments the contribution of this zone to the remainder is $O(h^{-1} T_0 / T_1) = O(|\log h|)$. One can get rid off this superficial logarithmic factor both in the estimate of (8.7) and in the remainder estimate; standard details I leave to the reader. So,

**Proposition 8.2.** Let $l \geq 1$ and conditions (6.4)± and (8.1) be fulfilled. Then as $Q$ is supported in the zone $X_1$ expression (8.7) does not exceed $C \rho_1$ and the contribution of $X_1$ to the remainder estimate is $O(1)$.

Therefore I am left with the zone $X_0 = \{ |\xi_2| \leq C_0 (\mu h^\nu)^{1/(\nu+1)} \}$. Let us fix $x_2$. I don’t know if eigenvalue $\lambda_n(\xi_2)$ of $a^0(\xi_2)$ vanishes in $X_0$ (may be even with some of its derivatives)\(^{10}\) but I know that if it happens then $n \leq c_1$; moreover due to the analyticity of $\lambda_n(\xi_2)$ it can happen only in no more then $C_1$ points and due to proposition A.3 and the analyticity of $\lambda_n(\xi_2)$

\[ \lambda_n(\eta) \sim \alpha(\eta - \bar{\eta})^r \]

\(^{10}\) It clearly happens for even $\nu$ and $n < l$. 

for some $\alpha \neq 0$ and $r = 1, 2, \ldots$ near each such point $\bar{\eta}$, $\alpha$ and $r$ depend on $\bar{\eta} = \bar{\eta}_{n,k}$ $k = 1, \ldots, K$ (depending on $\nu, l$ as well). Further, two eigenvalues do not vanish simultaneously.

But then condition (6.4)± will provide non-degeneracy. Really, in our assumptions an ellipticity is broken only in the strips of the type

$$Y = \{ |\xi_2 - \bar{\eta}\rho_0| \approx C\Delta \}, \quad \Delta = \rho_0^{1-2/r}, \tag{8.9}$$

and the average propagation speed with respect to $x_2$ is of magnitude $\rho_0^{-1} |\xi_2 - \bar{\eta}|^{r-1} \approx \rho_0^{(2-r)/r}$ there and therefore one can take

$$T_1 = \epsilon \rho_0^{-(2-r)/r}, \quad T_0 = Ch|\log h|\rho_0^{-(2-r)/r}\Delta^{-1} \approx h|\log h|, \quad \Delta = \rho_0^{1-2/r}. \tag{8.10}$$

Therefore for $Q$ supported in the strip $Y$ expression (8.7) does not exceed $Ch^{-1} \Delta \times T_0 = C|\log h|\rho_0^{-1} \Delta^{-1} \approx h|\log h|$. Furthermore, using standard methods one can easily get rid off the superficial logarithmic factor both in the estimate of (8.7) and the remainder estimate:

**Proposition 8.3.** Let $l \geq 1$ and conditions (6.4)± and (8.1) be fulfilled. Then as $Q$ is supported in the strip $Y$ described by (8.9), expression (8.7) does not exceed $C\rho_0^{-(2-r)/r}$ and the contribution of $Y$ to the remainder estimate is $O(1)$.

Therefore I arrive to

**Proposition 8.4.** Let $l \geq 1$ and conditions (6.4)± and (8.1) be fulfilled. Then the remainder estimate is $O(1)$ while the principal part is given by (6.40) for different strips with any $T \in [T_0, T_1]$ defined by (8.10) for strip $Y$ under conditions (8.8) – (8.9) and by (8.6) for strip $X_1$.

I would like to note that

**Proposition 8.5.** Let $l \geq 1$ and conditions (6.4)− and (8.1) be fulfilled. Then

(i) Zone $X_1$ is elliptic and its contribution to the remainder estimate is $O(h^s)$;

(ii) Furthermore if also condition

$$\lambda_n(\eta) \neq 0 \quad \forall n, \eta \tag{8.11}$$

is fulfilled \(^{11}\) then the remainder estimate is $O(h^s)$.

---

\(^{11}\) However I cannot check condition (8.11).
8.2 Estimates. II

Let us consider the special case \( l = 0 \); I remind that then only eigenvalue \( \lambda_0(\eta) \) should be considered and that condition (6.4)_- leads then to the asymptotics with the principal part 0 and remainder estimate \( O(h^n) \) and therefore is excluded from the further consideration.

Further, as \( \nu \) is odd \( \lambda_0 = 0 \) identically, condition (6.4)_+ provides ellipticity everywhere. Thus I arrive to

**Proposition 8.6.** Let \( l = 0, \nu \) be odd and conditions (6.4)_+ and (8.1) be fulfilled. Then the remainder estimate is \( O(h^n) \) while the principal part is given by (6.40).

On the other hand, as \( l = 0, \nu \) is even and condition (6.4)_+ holds due to proposition A.7 ellipticity is violated only in the strip

\[(8.12) \quad \mathcal{Y} = \{\varepsilon_1 \Delta \leq |\xi_2 - \eta \rho_0| \leq C\Delta\}, \quad \eta \asymp |\log \rho_0|^{\nu/(\nu+1)}, \quad \Delta = \rho_0 |\log \rho_0|^{-1/(\nu+1)}\]

where as before \( \rho_0 = (\mu h^n)^{1/(\nu+1)} \). In this strip propagation speed with respect to \( x_2 \) is \( \asymp \Delta^{-1} \) and again

\[(8.13) \quad T_0 = Ch|\log h|, \quad T_1 = \varepsilon \Delta\]

and expression (8.7) does not exceed \( Ch^{-1} \Delta T_0 = C\Delta |\log h| \) and the remainder estimate is \( O(|\log h|) \). Further, by the standard arguments one can get rid off the superficial logarithmic factors. Thus

**Proposition 8.7.** Let \( l = 0, \nu \) be even and conditions (6.4)_+ and (8.1) be fulfilled. Then the remainder estimate is \( O(1) \) while the principal part is given by (6.40) with \( T_0, T_1 \) defined by (8.13).

8.3 Estimates. III

Now I want to derive estimates under condition (6.4)_± replaced by (6.5). Without condition (6.4)_± some zones cease to be elliptic and should be reexamined:

\[(8.14) \quad \text{As } l \geq 1 \text{ these zones are } \{|\xi_2| \geq C\mu h^n\} \text{ and also}\]

\[(8.15) \quad \text{As } l \geq 1 \text{ these zones are } \text{“inner parts” of the strips described by (8.9), namely, } \mathcal{Y} = \{|\xi_2 - \bar{\eta} \rho_0| \leq \varepsilon_1 \Delta\} .\]

\[(8.16) \quad \text{As } l = 0, \nu \text{ even this zone is } \{|\xi_2| \geq C\rho_0 |\log \rho_0|^{\nu/(\nu+1)}\};\]
\[(8.17) \text{As } l = 0, \nu \text{ odd this zone is } \{ |\xi_2| \leq \epsilon \mu \}. \]

Since condition (6.5) provides \[T_0 = Ch|\log h| \] anyway contribution of (8.9)-type strips to the remainder estimate will be \(O(1)\) again. The standard partition-rescaling arguments in all other zones bring contribution of all other zones to \(O(\log \mu)\); however additional arguments of the proof of proposition 6.14 allow us to reduce it to \(O(1)\). Therefore

**Proposition 8.8.** Let conditions (8.1) and (6.5) be fulfilled. Then the remainder estimate is \(O(1)\) while the principal part of the asymptotics is given by (6.40) for different zones with any \(T \in [T_0, T_1]\), \(T_0 = Ch|\log h|\) and \(T_1\) defined as in propositions 8.2–8.7.

### 8.4 Calculations. I

In this subsection I give the principal parts of asymptotics already derived under condition (6.4) in more explicit form.

First of all, consider method of successive approximations fixing \(x_2 = y_2\). Then while contribution of the strip of the width \(\Delta\) in \(\xi_2\) to the principal part is of magnitude \(\Delta h^{-1}\), each next term of successive approximations acquires factor \(|\partial_\xi_2 \Lambda_n| T \times T/h\) with \(T = T_0\) where \(\Lambda_n\) is an eigenvalue of \(\mathcal{A}\). Further one needs to consider only strips where ellipticity fails and then \(\Delta \approx |\partial_\xi_2 \Lambda_n|^{-1}\).

So, the first, the second and the the third terms do not exceed

\[(8.18)_{1-3} \quad Ch^{-1}|\partial_\xi_2 \Lambda_n|^{-1}, \quad C|\log h|^2, \quad Ch|\partial_\xi_2 \Lambda_n| : |\log h|^4\]

respectively.

Actually the second term in the successive approximations is \(O(1)\). Really, considering the second term which corresponds to the linear part \((x_2 - y_2)\partial_{y_2} \mathcal{A}(y_2, hD_2)\) of the perturbation one can rewrite it as the result of direct calculations in the form including \(\partial_{x_2} \partial_\xi_2 \Lambda_n = 0\); on the other hand considering the second term corresponding to the rest \((x_2 - y)^2 \mathcal{B}(x_2, y_2, hD_2)\) of the perturbation one can estimate it easily by \(O(h^\delta)\).

Now I can rewrite the principal part of the asymptotics as

\[(8.19) \quad (2\pi h)^{-1} \int e(x_1, x_1, 0; x_2, \xi_2) \psi(x) d\xi_2 dx\]

with error not exceeding already achieved remainder estimate which is either \(O(1)\) or \(O(h^\infty)\) (where remainder estimate \(O(h^\infty)\) corresponds to the elliptic case and no successive approximations are needed at all).

Let us consider the contribution of the strips where ellipticity is broken to the error; I remind it does not exceed the minimum of all three expressions in \((8.18)_{1-3}\). Then \((8.18)_3\)
is obviously \( O(1) \) in all cases with the singular exception of the strip (8.9) with \( r = 1 \), \( \rho h \geq |\log h|^{-K} \). However in this case (8.18) is \( O(1) \) unless \( |\log h|^{-K} \leq \rho h \leq |\log h|^K \) and one can still handle this case getting rid off the superficial logarithmic factors in (8.18) by the standard arguments. Thus I arrive to

**Theorem 8.9.** *Let conditions (6.4) and (8.6) be fulfilled. Then*

(i) *Asymptotics with the principal part given by (8.19) holds with the remainder estimate \( O(1) \);*

(ii) *Furthermore, as \( l = 0 \), \( \nu \) is odd this asymptotics holds with the remainder estimate \( O(h^{\infty}) \).*

Furthermore, fixing \( W \) at \( x_1 = 0 \) and \( \alpha = 1 \) and thus replacing \( A \) by \( A^0 \) to the pilot model operator, I can apply the method of successive approximation again; then each next term gets an extra factor \( C_\gamma T_0 h^{-1}|\log h| \) with \( \gamma = (\mu^{-1}|\xi_2|)^{1/\nu} \) and only strips where ellipticity breaks should be counted. Also one can see easily that

(8.20) The error does not exceed the second term \( Ch^{-2}T_0 \Delta \gamma^{12} \). Furthermore, for odd \( \nu \) and perturbation, which is odd with respect to \( x_1 \), the second term is 0 and therefore the error does not exceed the sum of the second term with a perturbation \( O(x_1^3) \) and the third term with a perturbation \( O(x_1) \) i.e. \( Ch^{-3}T_0^2 \Delta \gamma^{2}^{12} \).

Thus, I just list the different cases:

(8.21) *As \( l \geq 1 \) and condition (6.4) is fulfilled the main contribution to the error is provided by the zone \( \mathcal{X}_1 \) with \( \xi_2 \asymp \mu h^{\nu} \) and \( \gamma \asymp h \) and of the width \( \Delta \asymp \mu h^{\nu} \); so the error is \( O(\mu h^{\nu}) \). The contributions of (8.9)-type strips are much smaller;*

(8.22) *As \( l \geq 1 \) and condition (6.4) is fulfilled the main contribution to the error is provided by (8.9)-type strips with the largest possible \( r \); then \( \xi_2 = O((\mu h^{\nu})^{1/(\nu+1)}) \), \( \gamma \asymp (\mu^{-1}h)^{1/(\nu+1)} \) and \( \Delta \asymp (\mu h^{\nu})^{(r-2)/r(\nu+1)} \); so the error is \( O((\mu h^{\nu})^{-\delta}) \) with \( \delta = 2/r(\nu + 1) \) anyway;*

(8.23) *As \( l = 0 \), \( \nu \) is even and condition (6.4) is fulfilled the main contribution to the error is provided by \( \mathcal{X}_1 \) with \( \xi_2 \asymp (\mu h^{\nu})^{1/(\nu+1)} |\log(\mu h^{\nu})|^{\nu/(\nu+1)} \), \( \gamma \asymp (\mu^{-1}h)^{1/(\nu+1)} |\log(\mu h^{\nu})|^{1/(\nu+1)} \) and of the width \( \Delta \asymp (\mu h^{\nu})^{1/(\nu+1)} |\log(\mu h^{\nu})|^{-1/(\nu+1)} \); so the error is \( O(1) \) anyway;*

\[^{12}\text{I skip superficial logarithmic factors one can easily get rid off by the standard arguments.}\]
(8.24) As \( l = 0, \nu \) is odd and condition \((6.4)_{+}\) is fulfilled the error is just \( O(h^{\infty}) \).

Thus I arrive to asymptotics with the principal part

\[
(2\pi h)^{-1} \int e_0(x_1, x_1, 0; x_2, \xi_2) \psi(x) d\xi_2 dx
\]

and remainder estimates described in Theorem 8.10 below:

**Theorem 8.10.** Let condition \((8.1)\) be fulfilled. Then

(i) As \( l \geq 1 \) and condition \((6.4)_{+}\) is fulfilled asymptotics with the principal part given by \((8.25)\) holds with the remainder estimate \( O(\mu h^\nu) \);

(ii) As either \( l \geq 1 \) and condition \((6.4)_{-}\) is fulfilled or \( l = 0, \nu \) is even and condition \((6.4)_{+}\) is fulfilled asymptotics with the principal part given by \((8.25)\) holds with the remainder estimate \( O(1) \);

(iii) Furthermore, as \( l = 0, \nu \) is odd and condition \((6.4)_{+}\) is fulfilled the same asymptotics holds with the remainder estimate \( O(h^{\infty}) \).

**8.5 Calculations. II**

In this subsection I give in more explicit form the principal parts of asymptotics already derived under condition \((6.5)\). Basically I need to reconsider only the external formerly elliptic zones described by \((8.14)\)–\((8.17)\). The analysis in the first of them is not different from the analysis under condition \((6.4)_{\pm}\); analysis in the second one repeats the proof of theorem 6.17; analysis in two latter is rather obvious. Thus I arrive to two following theorems:

**Theorem 8.11.** Let conditions \((6.5)\) and \((8.6)\) be fulfilled. Then asymptotics with the principal part \((8.19)\) holds with the remainder estimate \( O(1) \).

**Theorem 8.12.** Let conditions \((6.5)\) and \((8.6)\) be fulfilled. Then

(i) As \( l \geq 1 \) estimate

\[
(8.26) \quad R^{**} \overset{\text{def}}{=} | \int \left( e(x, x, 0) - (2\pi h)^{-1} \int e_0(x_1, x_1, 0; x_2, \xi_2) d\xi_2 
- \mathcal{E}^{MW}(x, 0) + \mathcal{E}_{0}^{MW}(x, 0) \right) \psi(x) dx - \sum_{k,l,m} \kappa_{l,m} h^{1+2m} | \leq C \mu h^\nu
\]

holds;
(ii) As $l = 0$ estimate

\[
\left| \int (e(x,x,0) - (2\pi h)^{-1} \int e_0(x_1,x_1,0;x_2,\xi_2) d\xi_2 - \mathcal{E}^{\text{MW}}(x,0) + \mathcal{E}^{\text{MW}}_0(x,0) ) \psi(x) dx \right| \leq C
\]

holds.

A Appendix: Eigenvalues of 1D operators

A.1 General observations

In this Appendix $\lambda_n(\eta)$ $(n = 0, 1, \ldots)$ denote eigenvalues of one-dimensional pilot-model Schrödinger operators with $\mu = h = 1$

(A.1) \[ a^0(\eta) = D^2 + (\eta - x^\nu/\nu)^2 - (2l + 1)x^{\nu-1} \]

or more general operator

(A.2) \[ a(\eta) = (1 + \alpha_1 x + \beta_1^2 x^2)D^2 + (1 + \alpha_2 x + \beta_2^2 x^2)(\eta - x^\nu/\nu)^2 - (2l + 1)(1 + \alpha_3 x)x^{\nu-1} \]

with $\nu = 2, 3, \ldots$ and $\beta_j > \alpha_j^2/2$.

One can prove easily the following statement:

**Proposition A.1.** Let $l \in \mathbb{R}$. Then

(i) As $|\eta| \leq C_0$ the spacing between two consecutive eigenvalues $\lambda_n$ and $\lambda_{n+1}$ with $n \leq c_0$ is $\asymp 1$;

(ii) For operator (A.1) with odd $\nu$ $\lambda_n(-\eta) = \lambda_n(\eta)$;

(iii) For even $\nu$ and $\eta \leq 0$ $\lambda_n(\eta) \geq (1 - \epsilon)\eta^2 - C_1 \quad \forall n = 0, 1, \ldots$.

However, the case of even $\nu$ and $\eta \to -\infty$ is rather exceptional:

**Proposition A.2.** As $\eta \geq C_0$ (and thus also as $\eta \leq -C_0$ and $\nu$ is odd)

(i) The spacing between eigenvalues with $n \leq c_0$ is $\asymp (1 + |\eta|)^{(\nu-1)/\nu}$;

(ii) As $n < l$ ($l < n \leq c_0$) $\lambda_n(\eta)$ is less than (greater than respectively) $\epsilon(n - l)(1 + |\eta|)^{(\nu-1)/\nu}$.

**Proof.** Proof follows from the proof of proposition A.3 below.

---

13) Thus leaving the special case $n = l \in \mathbb{Z}^+$ for the further analysis.
A.2 Asymptotic behavior of $\lambda_l(\eta)$ as $\eta \to \infty$ as $l \geq 1$

In this subsection I prove Proposition A.3.

**Proposition A.3.** (i) For operator (A.1) with $l \geq 1$ as $\eta \to +\infty$ (and thus also as $\eta \to -\infty$ and $\nu$ is odd)

(A.3) \[ \lambda_l(\eta) = \kappa \eta^{-2/\nu} + O(\eta^{-(\nu+3)/\nu}) \]

with $\kappa > 0$;

(ii) For operator (A.2) with $l \geq 1$ as $\eta \to +\infty$ (and thus as $\eta \to -\infty$ and $\nu$ is odd)

(A.4) \[ \partial_{\alpha_j} \lambda_l(\eta) |_{\alpha=0} = \kappa_j \eta + O(\eta^{-1/\nu}) \]

with $\kappa_1 = \kappa_2 = -\kappa_3/2$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$ and furthermore

(A.5) \[ \sum_{1 \leq j \leq 3} \partial_{\alpha_j} \lambda_l(\eta) |_{\alpha=0} = \kappa_4 \eta^{1/\nu} \lambda_l + O(\eta^{-2/\nu}). \]

**Proof.** (i) Let us plug $\eta = \gamma^\nu/\nu$ with $\gamma \gg 1$ where in the case even $\nu$ this is the only scenario and in the case of odd $\nu$ analysis of scenario $\xi_2 = -\gamma^\nu/\nu$ is done by the symmetry. Then after shift $x \mapsto x + \gamma$ operator $a^0(\eta)$ is transformed into operator

\[
D^2 + x^2 \left( \gamma^{\nu-1} + \frac{1}{2}(\nu-1)x\gamma^{\nu-2} + \frac{1}{6}(\nu-1)(\nu-2)x^2\gamma^{\nu-3} + \ldots \right)^2 - (2l + 1) \left( \gamma^{\nu-1} + (\nu-1)x\gamma^{\nu-2} + \frac{1}{2}(\nu-1)(\nu-2)x^2\gamma^{\nu-3} + \ldots \right)
\]

and after rescaling $x \mapsto x\gamma^{(1-\nu)/2}$ this operator is transformed into $\gamma^{\nu-1}b_{\epsilon}$ where

\[
b_{\epsilon} = D^2 + x^2 \left( 1 + \frac{1}{2}(\nu-1)x\epsilon + \frac{1}{6}(\nu-1)(\nu-2)x^2\epsilon^2 + \ldots \right)^2 - (2l + 1) \left( 1 + (\nu-1)x\epsilon + \frac{1}{2}(\nu-1)(\nu-2)x^2\epsilon^2 + \ldots \right)
\]

with $\epsilon = \gamma^{-(\nu+1)/2}$. Then

\[
b_{\epsilon} = D^2 + x^2 - (2l + 1) + \epsilon (\nu-1) \left( x^3 - (2l + 1)x \right) + \underbrace{\epsilon^2 (\nu-1) \left( \left( \frac{7}{12} \nu - \frac{11}{12} \right)x^4 - \frac{1}{2}(2l + 1)(\nu-2)x^2 \right) + O(\epsilon^3)}_{h_2}
\]

with $h_0$, $h_1$, and $h_2$.
and let us denote by $\Lambda_\varepsilon$ and $U_\varepsilon$ its eigenvalue close to 0 and the corresponding eigenfunction. Then

$$(A.6) \quad \Lambda_\varepsilon = \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \ldots \quad \text{and} \quad U_\varepsilon = u_0 + u_1 \varepsilon + u_2 \varepsilon^2 \ldots$$

where obviously $u_0 = v_l$ is a Hermite function, $\omega_1 = \omega_3 = \cdots = 0$ and

$$(A.7) \quad h_0 u_1 + h_1 u_0 = 0 \quad h_0 u_2 + h_1 u + h_2 u_0 = \omega_2 u_0.$$ 

Then

$$(A.8) \quad \omega_2 = \langle h_1 u + h_2 u_0, u_0 \rangle = -\langle u, h_0 u \rangle + \langle h_2 u_0, u_0 \rangle.$$ 

It is known that $(x - iD)v_k = (2k + 2)^{1/2}v_{k+1}$, $(x + iD)v_k = (2k)^{1/2}v_{k-1}$ and therefore

$$xv_l = \frac{1}{2} \left((2l + 2)^{1/2}v_{l+1} + (2l)^{1/2}v_{l-1}\right),$$

$$x^2v_l = \frac{1}{4} \left((2l + 2)^{1/2}(2l + 4)^{1/2}v_{l+2} + (4l + 1) + (2l + 2)^{1/2}(2l - 2)^{1/2}v_{l-2}\right),$$

$$(x^2 - 2l - 1)v_l = \frac{1}{4} \left((2l + 2)^{1/2}(2l + 4)^{1/2}v_{l+2} - 2(2l + 1)v_l + (2l + 2)^{1/2}(2l - 2)^{1/2}v_{l-2}\right),$$

$$x(x^2 - 2l - 1)v_l = \frac{1}{8} \left((2l + 2)^{1/2}(2l + 4)^{1/2}(2l + 6)^{1/2}v_{l+3} - (2l + 2)^{1/2}(2l - 2)^{1/2}v_{l+1} - (2l + 2)^{1/2}(2l - 2)^{1/2}(2l - 4)^{1/2}v_{l-3}\right),$$

which imply

$$\langle h_0 u, u \rangle = \frac{1}{64}(\nu - 1)^2 \left(\frac{1}{6}(2l+2)(2l+4)(2l+6) + \frac{1}{2}(2l+2)(2l-2)^2 - \frac{1}{2}(2l)(2l+4)^2 - \frac{1}{6}(2l)(2l-2)(2l-4)\right) = \frac{1}{16}(\nu - 1)^2 \left(-2l^2 - 2l + 3\right).$$

On the other hand

$$\langle h_2 u_0, u_0 \rangle = (\nu - 1) \left(\left(\frac{7}{12}\nu - \frac{11}{12}\right)\|x^2 u_0\|^2 - \frac{1}{2}(\nu - 2)(2l + 1)\|x u_0\|^2\right) = (\nu - 1) \left(\frac{7}{12}\nu - \frac{11}{12}\right) \cdot \frac{1}{16} \left((2l + 2)(2l + 4) + (4l + 2)^2 + (2l)(2l - 2)\right) - \frac{1}{4}(\nu - 1)(\nu - 2) \cdot (2l + 1)^2 = (\nu - 1)(7\nu - 11) \cdot \frac{1}{16} (2l^2 + 2l + 1) - \frac{1}{4}(\nu - 1)(\nu - 2)(2l + 1)^2.$$
and
\[ \omega_2 = \frac{1}{16}(\nu - 1) \left( (7\nu - 11)(2l^2 + 2l + 1) - 4(\nu - 2)(4l^2 + 4l + 1) + (\nu - 1)(-2l^2 - 2l + 3) \right) = \frac{1}{2}(\nu - 1)l(l + 1), \]

Therefore \( \Lambda_\varepsilon = \omega_2 \varepsilon^2 + O(\varepsilon^4) \) as \( \varepsilon \to 0 \) (because \( \omega_3 = 0 \) as well) which implies statement (i) with \( \kappa = \omega_2 \nu^{-2/\nu} \).

(ii) After obvious transformations
\[ \partial_\alpha \lambda_i(\eta) \big|_{\alpha=\beta=0} = \gamma^{\nu-1}\langle k_j U_\varepsilon, U_\varepsilon \rangle \]
with
\[
\begin{align*}
k_1 &= (\gamma + \varepsilon x)D^2, \\
k_2 &= x^2 \left( 1 + \frac{1}{2}(\nu - 1)\varepsilon x + \frac{1}{6}(\nu - 1)(\nu - 2)x^2 \varepsilon^2 + \ldots \right)^2 \\
k_3 &= -(2l + 1) \left( 1 + (\nu - 1)\varepsilon x + \frac{1}{2}(\nu - 1)(\nu - 2)x^2 \varepsilon^2 + \ldots \right)
\end{align*}
\]
and therefore
\[ \langle k_j U_\varepsilon, U_\varepsilon \rangle = \gamma\langle k'_j u_0, u_0 \rangle + O(\varepsilon^2 \gamma) \]
with \( k'_1 = D^2, k'_2 = x^2, k'_3 = -(2l + 1) \) which implies (A.4).

Known equalities \( \langle x^2 v_l, v_l \rangle = \langle D^2 v_l, v_l \rangle = (2l + 1)/2 \) imply that \( \kappa_1 = \kappa_2 = -\kappa_3/2 \).

Further, \( \sum_{1 \leq j \leq 3} \langle k_j U_\varepsilon, U_\varepsilon \rangle = \gamma \lambda_l + O(\varepsilon^2) \) which implies (A.5).

A.3 More general operators

Now I consider operator
\[
(A.9) \quad A(y, \eta) \overset{\text{def}}{=} \beta \left( \alpha h^2 D^2 \alpha + \alpha^{-2}(\eta - \mu x^\nu/\nu)^2 - (2l + 1)\mu h x^{\nu-1} \right) \beta
\]
with
\[
(A.10) \quad \alpha = \alpha(x, y), \quad \beta = \beta(x, y), \quad \alpha(0, y) = 1, \quad c_0^{-1} \leq \beta \leq c_0.
\]

Let \( \lambda_n \) be eigenvalues of \( A \). Changing \( x \mapsto \gamma(\mu^{-1}h)^{1/(\nu+1)}x \) and \( \eta \mapsto (\mu h^\nu)^{1/(\nu+1)} \) respectively I arrive to operator (A.9) again with \( \mu = h = 1 \) and \( \alpha, \beta \) replaced by \( \alpha((\mu^{-1}h)^{1/(\nu+1)}x, y), \beta((\mu^{-1}h)^{1/(\nu+1)}x, y) \) and with a factor \( (\mu h^\nu)^{2/(\nu+1)} \).
Proposition A.4. Let conditions (A.9), (A.10) be fulfilled. Then
(i) $\lambda_n(\eta) \geq C_0(\mu h)^{2/(\nu+1)}$ as $n \geq C$;
(ii) As $|\eta| \leq C_0(\mu h)^{2/(\nu+1)}$ the spacing between consecutive eigenvalues with $n \leq c_0$ is
\[ \approx (\mu h)^{2/(\nu+1)} \]
and
\[ |\partial_y^p \partial_q^q \lambda_n(y,\eta)| \leq C_{pq}(\mu^{-1}h)^{p/(\nu+1)}(\mu h)^{(2-q)/(\nu+1)}; \]
(iii) For even $\nu$ and $\eta \leq 0$
\[ \lambda_n(y,\eta) \geq (1 - \epsilon)\eta^2 - C_1, \quad n = 0, 1, \ldots \]

Proposition A.5. As $\eta \geq C_0(\mu h)^{1/(\nu+1)}$ (and thus also as $\eta \leq -C_0(\mu h)^{1/(\nu+1)}$ and $\nu$ is odd)
(i) The spacing between eigenvalues with $n \leq c_0$ is
\[ \approx |\eta|^{(\nu-1)/\nu}(\mu h)^{1/\nu}; \]
(ii) As $n < l$ $(l < n \leq c_0)$ $\lambda_n(y,\eta)$ is less than (greater than respectively) $\epsilon(n-l)((\mu h)^{2/(\nu+1)} + |\eta|^{(\nu-1)/\nu}(\mu h)^{1/\nu})$ and these eigenvalues satisfy
\[ |\partial_y^p \partial_q^q \lambda_n(y,\eta)| \leq C_{pq}(\mu^{-1}h)^{p/(\nu+1)}|\eta|^{-q}|\lambda_n(y,\eta)|; \]
(iii) As $\eta \geq C_0(\mu h)$ (and thus as $\eta \leq -C_0(\mu h)$ and $\nu$ is odd) $|\lambda_l(y,\eta)| \leq \epsilon_0$.

An extra analysis is needed for our purposes as $n = l$ and
\[ \mu h^{\nu} \geq C_1 \]
with arbitrarily large $C_1$.

Proposition A.6. Let condition (A.13) be fulfilled and $l \geq 1$. Then as $\eta \geq C_0(\mu h)^{1/(\nu+1)}$
\[ \lambda_l(y,\eta) \approx (\mu h^{\nu}/\eta)^{2/\nu} \quad \text{and} \quad \eta \partial_\eta \lambda_l(y,\eta) \approx (\mu h^{\nu}/\eta)^{2/\nu}. \]

A.4 Case of $\lambda_l$ as $l = 0$

Here cases of odd and even $\nu$ differ drastically. Note first that
\[ a^0(\eta) = (iD + \xi_2 - x^{\nu}/\nu)(-iD + \xi_2 - x^{\nu}/\nu) \]
and as $\nu$ is odd operator $a^0(\eta)$ has the bottom eigenvalue $\lambda_0(\eta)$ with eigenfunction defined from $(-\partial + \xi_2 - x^{\nu}/\nu)v = 0$ i.e. $v = \exp(\xi_2 x - x^{\nu+1}/\nu(\nu + 1))$ and therefore $\lambda_0(\eta)$ is identically 0.
Similarly, as $\beta = 1$ operator $A$ defined by (A.9) is equal modulo $O(h^2)$ to operator

(A.16) $B(y, \eta) \overset{\text{def}}{=} h^2 \alpha^2 D + \alpha^{-2}(\eta - \mu x^\nu / \nu)^2 - \mu hx^{\nu-1} = (ihD\alpha + \alpha^{-1}(\eta - \mu x^\nu / \nu))(-\alpha ihD + \alpha^{-1}(\eta - \mu x^\nu / \nu))$

and I arrive to the statement (i) of

**Proposition A.7.** (i) For odd $\nu$ the bottom eigenvalue of $B(y, \eta)$ is 0;
(ii) For even $\nu$ the bottom eigenvalue of $B(y, \eta)$ is $(\mu h^{\nu})(\mu h^{\nu})^{-1/\nu+1})$ where

(A.17) $C^{-1} \exp(-C\eta^{(\nu+1)/\nu}) \leq \Lambda(y, \eta) \leq C \exp(-c\eta^{(\nu+1)/\nu})$,

(A.18) $\eta^{1/\nu} \leq -\partial_\eta(\log \Lambda(y, \eta)) \leq C\eta^{1/\nu}$.

**Proof.** I need to consider the case of even $\nu$ only. The same representation (A.15) shows that $\lambda_0(y, \eta) > 0$. However, since this eigenfunction is fast decaying outside of the potential well, one can do the same shift and rescaling as before and using arguments of [HeMa] to prove that $\Lambda_0(y, \eta) \sim k \exp(-k_2\eta^{\nu+1/\nu})$. Also one can prove easily that $\partial_\eta \Lambda_0(y, \eta) \sim -k_3\eta^{1/\nu} k \exp(-k_2\eta^{(\nu+1)/\nu})$ as $\eta \geq C$ with $k_3 = kk_2(1 + \nu)/\nu$. Estimates (A.17), (A.18) follow from this.

**References**

[HeMa] B. HELFFER, A. MARTINEZ. *Phase transition in the semiclassical regime*, Rev. Math. Phys. 12 (2000), no. 11, 1429–1450.

[Ivr1] V. IVRII. *Microlocal Analysis and Precise Spectral Asymptotics*, Springer-Verlag, SMM, 1998, xv+731.

[Ivr10] V. IVRII. *Sharp Spectral asymptotics for two-dimensional Schrödinger operator with a strong degenerating magnetic field.*, (to appear).

March 29, 2022

Department of Mathematics, University of Toronto, 100, St.George Str., Toronto, Ontario M5S 3G3 Canada ivrii@math.toronto.edu Fax: (416)978-4107