UNIFORM IN TIME WEAK PROPAGATION OF CHAOS ON THE TORUS

François Delarue∗ and Alvin Tse†

ABSTRACT. We address the long time behaviour of weakly interacting diffusive particle systems on the $d$-dimensional torus. Our main result is to show that, under certain regularity conditions, the weak error between the empirical distribution of the particle system and the theoretical law of the limiting process (governed by a McKean-Vlasov stochastic differential equation) is of the order $O(1/N)$, uniform in time on $[0, \infty)$, where $N$ is the number of particles in the interacting diffusion. This comprises general interaction terms with a small enough mean-field dependence together with interactions terms driven by an $H$-stable potential. Our approach relies on a systematic analysis of the long-time behaviour of the derivatives of the semigroup generated by the McKean-Vlasov SDE, which may be explicitly computed through the linearised Fokker-Planck equation. Ergodic estimates for the latter hence play a key role in our analysis. We believe that this strategy is flexible enough to cover a wider broad of situations. To wit, we succeed in adapting it to the super-critical Kuramoto model, for which the corresponding McKean-Vlasov equation has several invariant measures.

1. Introduction

In this paper, we are concerned with the large size and the large time behaviour of a weakly interacting particle system with toroidal data. Denoting by $N$ the number of particles, the system has the following generic form

\[
\begin{aligned}
Y_{t}^{i,N} &= \eta^{i} + \int_{0}^{t} b(Y_{s}^{i,N}, \mu_{s}^{N}) \, ds + W_{t}^{i}, \quad i \in \{1, \cdots, N\}, \quad t \geq 0, \\
\mu_{s}^{N} &:= \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_{s}^{i,N}},
\end{aligned}
\]

where $b$ is an $\mathbb{R}^d$-valued function defined on $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$, $\mathbb{T}^d$ denoting the $d$-dimensional torus and $\mathcal{P}(\mathbb{T}^d)$ the space of probability measures on $\mathbb{T}^d$, which we equip (unless specified differently) with the $W_1$-Wasserstein distance

\[
W_{1}(\mu, \nu) = \inf_{\pi} \left\{ \int_{\mathbb{T}^d \times \mathbb{T}^d} d\pi(x, y) \, d\pi(x, y) \right\},
\]

the infimum being over all the probability measures $\pi$ on the product space $\mathbb{T}^d \times \mathbb{T}^d$ that have $\mu$ and $\nu$ as respective marginal measures. In (1.1), $W^{i}$, $i = 1, \cdots, N$, are independent $d$-dimensional Brownian motions and $\eta^{i}$, $i = 1, \cdots, N$, are $N$ $\mathbb{R}^d$-valued random variables, with the two tuples $(\eta^{1}, \cdots, \eta^{N})$ and $(W^{1}, \cdots, W^{N})$ being independent. Most of the time, the random variables $\eta^{i}$, $i = 1, \cdots, N$, are also assumed to be independent and identically distributed (I.I.D.) with a common law $\mu_{init}$, but this might not be the case in some of our results (in those cases we emphasise it very clearly). In physical applications, this type of processes arises when we consider interacting particle systems with periodic boundary conditions. (See, for example, [68], for a discussion on the Boltzmann equation under binary collision supplemented with

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periodic boundary conditions, together with \cite{26,46,59} for synchronisation models of noisy oscillators or liquid crystals.)

1.1. **State of the art.** In (1.1), every individual particle $Y^{i,N}$ evolves according to some diffusive motion depending on the behaviour of others. It is well-known that as the population size $N$ grows to infinity, (1.1), when subjected to I.I.D. initial conditions $\eta^1, \cdots, \eta^N$, behaves like the following McKean-Vlasov SDE (an SDE in which the coefficients depend on the evolving law):

$$
\begin{align*}
X_t &= \eta + \int_0^t b(X_s, \mathcal{L}(X_s)) \, ds + W_t, \quad t \geq 0, \\
\mathcal{L}(\eta) &= \text{Law}(\eta) = \mu_{\text{init}},
\end{align*}
$$

where $(\eta, W)$ is a copy of $(\eta^1, W^1)$. Existence and uniqueness of a solution to both (1.1) and (1.3) are usually proven under a Lipschitz condition on $b$ (w.r.t. the product topology on $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$) (see for instance \cite{62}). However, a weaker (although useless in our setting) condition is known: It suffices for $b$ to be globally bounded and merely Lipschitz continuous in the measure argument with respect to the total variation distance (see \cite{44,47,55} and the references therein). In any case, the flow of marginal laws $(m(t; \mu_{\text{init}}) := \mathcal{L}(X_t))_{t \geq 0}$ is known to satisfy (at least in a distributional sense) the nonlinear Fokker-Planck equation:

$$
\partial_t m(t; \mu) = \frac{1}{2} \Delta m(t; \mu) - \text{div} \left[ m(t; \mu) b(\cdot, m(t; \mu)) \right], \quad t \geq 0,
$$

with $m(0; \mu) = \mu$ as initial condition.

**Propagation of chaos.** A particular characteristic of the limiting behaviour is that any finite subset of particles becomes asymptotically independent of each other. This phenomenon is known as propagation of chaos. More precisely, on any finite time interval $[0, T]$ and, for any fixed $k \in \mathbb{N}$,

$$(Y^{1,N}, \ldots, Y^{k,N}) \Rightarrow (X^1, \ldots, X^k), \quad \text{as } N \to \infty,$$

where $\{X^i\}_{i \in \mathbb{N}}$ are i.i.d. copies of (1.3) and $\Rightarrow$ denotes weak convergence of random variables taking values in the space $\mathcal{C}([0, T]; (\mathbb{T}^d)^k)$. The main reference in this direction is \cite{62}, where propagation of chaos is proved by means of a coupling argument now known as ‘Sznitman’s coupling’. The proof works under the same aforementioned Lipschitz property of $b$ (we refer to \cite{47} for a similar analysis but under weaker assumptions) and yields a quantitative convergence estimate which we describe in the next paragraph. Another result from \cite{62} asserts that propagation of chaos is equivalent to weak convergence of the measure-valued random variables $(\mu_{\text{init}}^N)_{0 \leq t \leq T}$ to the limiting laws $(\mathcal{L}(X_t))_{0 \leq t \leq T}$. This paves the way for another approach based on compactness, the first step of which is to prove tightness of $(\pi_t^N := \mathcal{L}(\mu_{\text{init}}^N))_{0 \leq t \leq T}$ and then to identify the limit by showing that $(\pi_t^N)_{0 \leq t \leq T}$ converges weakly to $(\delta_{\mathcal{L}(X_t)})_{0 \leq t \leq T}$. We refer the reader to \cite{40,52,62} for classical results in this direction. For sure, this strategy of proof does not reveal quantitative bounds regarding the rate of convergence of (1.1) to (1.3).

**Strong versus weak errors in finite time.** The analysis of quantitative estimates of propagation of chaos, at least over a finite time horizon, may be divided into two categories: strong propagation of chaos and weak propagation of chaos. In the short review of the two that we provide below, we do not make any distinction between the Euclidean and toroidal settings. This is only when we come to the long time behaviour of both (1.1) and (1.3) that compactness of the torus makes a substantial difference.

Strong propagation of chaos mainly concerns with errors in $W_p$ norms (the latter being obtained by a mere generalisation of $W_1$ in (1.2), changing the $L^1$ norm with respect to $\pi$ into an $L^p$ norm) between (1.1)
and (1.3). In the case where \( b \) depends on the measure component linearly, i.e., is of the form \( b(x, \mu) := \int_{\mathbb{R}^d} B(x, y) \mu(dy) \), with \( B \) being Lipschitz continuous in both variables, it follows from a simple calculation ([62]) that \( \sup_{t \in [0, T]} W_2(\mathcal{L}(Y_t^{1, N}), \mathcal{L}(X_t)) = O(N^{-1/2}) \). The case with a general measure dependence in \( b \) is exemplified in [18, Ch. 1], and more generally, in [38]. Technically, the proof works in the same way, but the rate of strong propagation of chaos then deteriorates with the dimension \( d \), since it is then needed to estimate the Wasserstein distance between the empirical law of I.I.D. samples and the limiting measure. This follows from results such as ([31] or [35]) in which the dimension explicitly shows up. In fact, the rate \( N^{-1/2} \) may be retrieved in this more general setting by requiring a strong form of smoothness of the drift \( b \) with respect to the measure argument (see [30, Lemma 5.10] and [63]). Certainly, the analysis of the strong error is not limited to models with smooth interactions; we refer for instance to the recent work [43] in which quantitative estimates are established for models with singular interactions. Also, it is worth saying that the strong error may be also quantified through a relevant form of central limit theorem (see [52, 61, 64], or in the perspective of large deviation principle, see [13, 28, 61]).

On the other hand, weak propagation of chaos addresses the statistical behaviour of the empirical distribution of the particle system. Typically, the objective is to provide a rate for quantities of the form

\[
\left| \mathbb{E}[\Phi(\mu_t^N)] - \Phi(m(t ; \mu_{\text{init}})) \right|,
\]

where \( \Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \) is a test function chosen within a suitable class. This new direction of research has been introduced in independent works [5, 45, 53, 54]. These works presented novel weak estimates of propagation of chaos for various forms of test functions \( \Phi \): \( \Phi \) is linear in measure in [5], i.e. \( \Phi(\mu) := \int_{\mathbb{T}^d} F(x) \mu(dx) \) for some function \( F : \mathbb{T}^d \to \mathbb{R} \); \( \Phi \) is a polynomial function in [53, 54], i.e. a product of linear functions; \( \Phi \) is a quite general nonlinear function in [45]. Under appropriate smoothness conditions on the test function and on the coefficients of (1.1), this gives a rate of convergence of \( O(1/N) \), plus the error due to the approximation of the functional of the initial law (as for the latter, see [54, Lem. 4.6] for a dimension-dependent estimate and [25, Th. 2.11] for an \( O(1/N) \) bound). The key idea of the analysis (highlighted in [45, Th. 9.2.1] and [54, Th. 6.1]) is to work with a semigroup that acts on the space of functions of measures (expounded in the next section). A similar idea has been used in [16] in order to study the convergence problem for mean field games, up to the difference that the equation for the semigroup then becomes a nonlinear equation. Also, another recent work [25] provides an extension of [5, 45, 54] in the form of a weak error expansion.

**Long time analysis.** Propagation of chaos is said to be uniform whenever the quantitative estimates (of propagation of chaos) are uniform in time. This problem is hence more challenging as it involves two large parameters, \( N \) and \( t \). It is thus related to the long time behaviour of the McKean-Vlasov and non-linear Fokker-Planck equations (1.3) and (1.4) themselves.

Actually, the ergodic analysis of McKean-Vlasov and corresponding nonlinear Fokker-Planck equations has been an intense topic of research on its own for more than twenty years. In the earlier papers [3, 4, 6, 10, 23, 22], convergence towards a (unique) invariant measure has been mostly studied for drifts of the form \( b(x, \mu) = -\nabla V(x) - \int \nabla W(x - y) \mu(dy) \), with confinement and interaction potentials \( V \) and \( W \) satisfying suitable convexity properties, \( W \) being symmetric. In this framework, a key conceptual feature is that the Fokker-Planck equation (1.4) can be regarded as a gradient flow on the space of probability measures. However, as demonstrated in [42], uniqueness of the invariant measure may be easily lost under a small modification of the shape of the potentials, which obviously raises challenging questions about the long-time behaviour of the Fokker-Planck equation. In fact, regardless of the precise form of the drift \( b \), a possible strategy to force
uniqueness and in turn to get convergence towards the hence unique invariant measure is to assume that the mean field interaction is small enough (see for instance [9, 14, 34]). When convergence towards a unique invariant measure is no longer true, a case-by-case stability analysis of all the existing stationary solutions may be carried out, depending on the shape of the dynamics. We refer for instance to [7, 41] for results on the super-critical (toroidal) Kuramoto model, which we revisit in Section 4. More examples may be found in [21, 29, 67].

Uniform rate of convergence. Uniform in time propagation of chaos is even more challenging. To wit, it may fail even in cases where there exists a globally attractive invariant measure to the Fokker-Planck equation (1.4). In the Euclidean setting, a well-known example by now may be found in [50]. Therein, the center of mass of the N particles in (1.1) is shown to behave (for a suitable choice of b) like a Brownian motion of intensity of 1/√N. Of course, the latter becomes macroscopic in size for time t larger than N, which precludes any relevant comparison with the invariant measure of the Fokker-Planck equation. A similar phenomenon may occur on the torus when the invariant measure is not the Lebesgue measure; once again, a prototype example is the super-critical Kuramoto model (see for instance [8]).

That said, several positive results have already been proven under relevant conditions. For instance, [33] shows that sup_{t≥0} W_1(\mathcal{L}(Y_t^{1,N}), \mathcal{L}(X_t)) = O(N^{-1/2}) in the Euclidean setting, when b takes the aforementioned special form b(x, \mu) = -∇V(x) - ∫ W(x-y)\mu(dy), for relevant conditions on V and W that force W to be small enough (as claimed by the authors, the method could also be extended to more general drifts, provided that the mean field interaction remains small enough). A similar result is also available in [60]. Lastly, in the recent contribution [2], the weak error (1.5) is shown to be O(N^{-1} + N^{-1/d} \exp(-λt)), for λ > 0, when the McKean-Vlasov dependence is small enough with respect to the confinement properties of the drift. The result is stated for test functions Φ that are linear in the measure argument.

1.2. Our contribution. Our contribution here is to provide, in the periodic setting, a uniform O(N^{-1}) bound for the weak error (1.5) when Φ is a general (smooth enough) test function and b belongs to one of the following three classes:

1. b is a general function on T^d × \mathcal{P}(T^d) with a small enough dependence with respect to the measure argument μ;
2. b has the form b(x, \mu) = -∫_{T^d} ∇W(x-y)\mu(dy), for W being an H-stable (periodic symmetric) potential, see [58], i.e., all the Fourier coefficients of W are non-negative;
3. d = 1 and b has the form b(x, \mu) = -2\pi\kappa ∫_{T^d} \sin(2\pi(x-y))\mu(dy), for some possibly large value κ > 0.

Description of the three cases. Obviously, the condition imposed in (1) is reminiscent of those required in [33] and [2] on the smallness of the mean field dependence. The three papers indeed share a common idea: the threshold for the intensity of the mean field dependence is determined in terms of the ergodic properties of the (linear) Fokker-Planck equation obtained by ‘removing’ the mean field term in (1.4). For sure, there is no clear definition of the latter, as it may depend on the shape of b. For instance, when b is ‘naturally’ split in the form b(x, \mu) = b_0(x) + b_1(x, \mu) (which is, for instance, the case when b(x, \mu) = -∇V(x) - ∫ ∇W(x-y)\mu(dy)), the mean field term that should be removed is not the whole ‘b’, but instead just ‘b_1’. This plays a key role when working on R^d, since the Fokker-Planck equation that would be obtained by removing the whole ‘b’ would obviously have no ergodic properties. Also, this makes an obvious difference with the periodic setting.

\footnote{If the invariant measure is the Lebesgue measure, it is invariant by rotation. Accordingly, propagation of chaos may be uniform even when the center of mass is rotating fast.}
On the torus, we can indeed benefit from the mixing properties of the Laplace operator (which hold true because of the compactness of the torus). In other words, the drift $b$ must satisfy additional confinement properties when working on the Euclidean space (see, for instance [2, Assumption $(H)$]).

Condition (2) is proper to the periodic setting. The aforementioned counter-example provided in [50] clearly demonstrates that the same result cannot hold in the Euclidean case. When the setting is periodic and $W$ is $H$-stable, the McKean-Vlasov SDE has the Lebesgue measure as unique invariant measure and the latter is globally attractive. As explained in footnote 1, there is then no contradiction in having uniform propagation of chaos.

Despite being more restrictive, example (3) is actually the most challenging. The key fact is that $\kappa > 0$ may be large, in which case (3) is known as the synchronised (or super-critical) Kuramoto model. Reformulated with the notations of example (2), this corresponds to $W(x) = -2\pi \kappa \cos(2\pi x)$. Obviously, it is not $H$-stable. Consistently, the McKean-Vlasov equation has several invariant measures. These are the Lebesgue measure, which is then unstable, and a collection of non-trivial measures obtained by rotating a common density profile

Consistently, the McKean-Vlasov equation has several invariant measures. These are the Lebesgue measure, which is then unstable, and a collection of non-trivial measures obtained by rotating a common density profile on the torus, see [7, 41]. As mentioned before, uniform propagation of chaos then fails, see [8]. In fact, the best result that has been proven so far is due to [27]. It says that, with high probability, the empirical distribution of (1.1) stays close to the collection of non-trivial invariant measures up to times that are subexponential in $N$. We complement this result here by proving that the weak error (1.5) is of order $O(N^{-1})$, uniformly in time, for functionals $\Phi$ that are rotation invariant (see Definition 4.2) and for initial distributions $\mu_{\text{init}}$ that are at a positive distance from the Lebesgue measure.

About the test function. In all our results, the function $\Phi$ is allowed to be any arbitrary sufficiently smooth ‘nonlinear’ functional on $P(\mathbb{T}^d)$ (with the additional constraint that it is rotation invariant in example (3)). The need for smoothness should not come as a surprise, since the weak error provides an estimate for the difference $\mathcal{L}(\mu_t^n) - \delta_{\mu(t)}$ when acting, by duality, on a suitable space of real-valued functions defined on $P(\mathbb{T}^d)$. Also, all the aforementioned results for the analysis of the weak error have similar requirements. To make it clear, we assume here that $\Phi$ has two ‘linear functional’ (or ‘flat’) derivatives with respect to the measure argument (see the next section for the details) that are Hölder continuous in the spatial variables (the derivatives of $\Phi$ at a measure $\mu$ are functions on the torus; Hölder continuity is thus interpreted in terms of the standard distance on the torus). When specialising to a linear function $\Phi(\mu) = \int_{\mathbb{T}^d} F(x)\mu(\mathrm{d}x)$, this says that $F$ has to be merely Hölder continuous. In contrast, $F$ is required to be twice differentiable (with bounded derivatives) in [2]. Our strategy of allowing such weaker conditions combines two main ingredients. First, we exploit in a systematic manner the smoothing effect of the Laplace operator in (1.4) (using Schauder’s estimates); Second, we introduce a mollification method for $\Phi$ that permits to work with a smoother $\Phi$, provided that the resulting bounds for the weak error only depend on the regularity properties of the un mollified $\Phi$. To the best of our knowledge, such a regularisation argument for functions defined on the space of probability measures is new and is interesting on its own.

The advantage of working with a general function $\Phi$ (as opposed to linear or polynomial functions) is two-fold. Firstly, we may choose, as a typical instance of nonlinear function:

$$
\Phi(\mu) = \|\mu - \mu_0\|_{L^{(d/2+\varepsilon),2}}^2, \quad \mu \in P(\mathbb{T}^d),
$$

for some $\varepsilon > 0$, where $\mu_0$ is a fixed ‘target’ probability measure on $\mathbb{T}^d$ and $\|\cdot\|_{L^{(d/2+\varepsilon),2}}$ is the norm on the dual of the standard Sobolev space $H^{d/2+\varepsilon}(\mathbb{T}^d)$ (see Proposition 2.4). We find it very useful, especially when $\mu_0$ is chosen as $m(t;\mu)$ in (1.4) or as $m(\infty;\mu) := \lim_{t \to \infty} m(t;\mu)$. In particular, this permits to retrieve a
bound for the strong error, but in another topology than \( W_1 \) (the comparison being not easy, except when \( d = 1 \), in which case \( \| \cdot \|_{-(1/2+\epsilon),2} \) is finer than \( W_1 \)). Secondly, this allows us to implement some localisation arguments in the form of cut-off functions on \( \mathcal{P}(\mathbb{T}^d) \). This is especially useful in the analysis of the super-critical Kuramoto model, in which the stable invariant measures are not globally attractive and the state space has to be divided in basins of attraction (see, for instance, the proof of Proposition 4.14).

**Strategy of proof.** Our approach is heavily based on the so-called master equation satisfied by the semigroup generated by the McKean-Vlasov equation (1.3). Equivalently, the latter is the semigroup \((P_t)_{t \geq 0} \) generated by the (deterministic) Fokker-Planck equation (1.4), whose action on a (bounded and measurable) test function \( \Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \) reads:

\[
P_t \Phi : \mathcal{P}(\mathbb{T}^d) \ni \mu \mapsto \Phi(m(t; \mu)), \quad t \geq 0,
\]

where \((m(t; \mu))_{t \geq 0} \) solves (1.4).

A key fact is that \((P_t \Phi)_{t \geq 0} \) is a classical solution of the aforementioned master equation whenever \( \Phi \) is smooth enough. In PDE theory, the trajectories \((m(t; \mu))_{t \geq 0} \) should be regarded as the characteristics of the master equation. In a probabilistic language, the master equation could be called a forward Kolmogorov equation even though the underlying dynamics (as given by the Fokker-Planck equation) are completely deterministic. A systematic analysis of the smoothness of \((P_t \Phi)_{t \geq 0} \) is provided in [12, 37, 66] (see also [16] for a similar study in a nonlinear setting). In any case, the formulation of the master equation requires an appropriate form of differential calculus on the space of probability measures. As we already alluded, we choose here to work with the extrinsic derivative, which is also known as the ‘linear functional’ or ‘flat’ derivative, as opposed to the intrinsic Wasserstein derivative (see [1, 19]). Our choice is mostly dictated by our methodology, as it seems to be the most natural one in our framework. Anyway, the two derivatives are strongly connected (see for instance [19, Chapter 5]) and, accordingly, the master equation would be the same if we had to work with the intrinsic derivative (for instance [12] makes use of the Wasserstein derivative). The reader may find the form of the master equation in (2.7) below, paying attention to the fact that the writing therein is backwards in time.

The analysis of the weak error is then carried out in two steps:

1. The first one is to ‘test’ \((P_t \Phi)_{t \geq 0} \) onto the empirical distribution of the \( N \)-particle system; the resulting bound is shown to be \( O(1/N) \), with the leading constant in the symbol \( O(\cdot) \) depending on the bounds for the derivatives of order 1 and 2 of \((P_t \Phi)_{t \geq 0} \).

2. The second step is to provide uniform-in-time bounds for the derivatives of \((P_t)_{t \geq 0} \); this is the main challenge in any of the three cases addressed in the article.

**Linearised Fokker-Planck equation.** The derivatives of \((P_t \Phi)_{t \geq 0} \) may be explicitly computed by linearising the Fokker-Planck equation (1.4). This is a well-known fact in PDE theory: The derivatives of the solution of a transport equation can be expressed in terms of the derivatives of the corresponding characteristics with respect to the initial point. In our setting, the derivatives of the characteristics are indeed obtained by linearising the Fokker-Planck equation with respect to the measure argument. See for instance the results exposed in Subsection 3.1, which are mostly borrowed from [66].

Thus, not only does the long-time behaviour of the Fokker-Planck equation matter in our analysis, but also the long-time asymptotics of the linearised Fokker-Planck equation are important. Actually, this is consistent with the analysis of an equilibrium, since the linearised operator (driving the linearised equation) may provide some insight about the possible local stability of the corresponding invariant measure. However, local stability
is not enough for our purpose. Indeed, the empirical distribution of (1.1) may leave any neighbourhood of an equilibrium, even if the latter is stable. This follows from the stochastic nature of the empirical distribution: Even if the noise to which the empirical distribution is subjected gets smaller and smaller as \( N \) increases, it may induce some deviation on the long run. We thus need more in our analysis. In this respect, the three aforementioned cases addressed in the paper may be classified as follows:

1. When the mean field dependence is small enough, we can directly prove that, for any initial distribution \( \mu \) of the Fokker-Planck equation (1.4), the linearised equation (at \( (m(t; \mu))_{t \geq 0} \)) ‘sends’ to 0 any initial condition \( q \) that is orthogonal to the constant function 1. Moreover, the rate of convergence is exponential, uniformly with respect to \( \mu \). This suffices for our purpose and this implies, as a corollary, that there is a unique invariant measure, which is globally attractive.

2. When \( b \) derives from an \( H \)-stable potential, the Lebesgue measure \( \text{Leb}_T \) is already known to be invariant and globally attractive, with an exponential rate. It is then sufficient to address the linearised equation at \( \text{Leb}_T \). The latter can be proven to ‘send’ to 0 any initial condition \( q \) that is orthogonal to 1, with an exponential rate.

3. In the super-critical Kuramoto model, there are two main issues. The first one is that \( \text{Leb}_T \), which is invariant, is unstable. All the other invariant measures, obtained by rotation of a common profile, form a collection that attracts exponentially fast any solution of (1.4) that starts at a given positive distance of \( \text{Leb}_T \). A key point in our analysis is thus to prove that the empirical distribution of (1.1) cannot spend long journeys in the neighborhood of \( \text{Leb}_T \). Roughly speaking, this permits to proceed as if the collection of the non-trivial invariant measures were globally attractive. In a similar manner to case (2) right above, the main point is then to address the long-time behaviour of the linearised Fokker-Planck equation at any of those non-trivial invariant measures. Here is then the second issue: in contrast with what happens in (1) and (2), the linearised operator has a non-constant zero, which is the spatial derivative \( p' \) of the density \( p \) of the invariant measure in hand. Accordingly, the best we can prove is that the linearised equation at \( p' \) ‘sends’ to the span of \( p' \) any initial condition \( q \) that is orthogonal to 1. This is the point where the rotation invariance of the test function \( \Phi \) comes in: In that case, only the part that is orthogonal to \( p' \) (in the solution of the linearised equation) matters in the formula for the derivatives of \( (P_t \Phi)_{t \geq 0} \).

Further prospects. For sure, our strategy shares some similarities with several of the aforementioned works. For instance, the linearised Fokker-Planck equation has already been used in the derivation of the finite horizon estimates in [16, 54]. Beyond the obvious fact that our work addresses the infinite time horizon \([0, \infty)\), we feel that it also brings a new perspective on the existing literature. In particular, our choice to make a systematic use of the master equation permits to track, in a transparent manner, the impact of the linearised equation onto the estimates of uniform propagation of chaos. This is possible, thanks to the explicit representation formula that we use for the derivatives of the semigroup. Our choice to have three different types of examples treated in the same paper precisely aims at demonstrating that the method is flexible enough to cover several types of situations, including some with several invariant measures. Moreover, we are confident that a similar approach could be adapted to other cases, including cases with a non-constant diffusion coefficient. Even more, although we restrict ourselves here to the periodic case as we already find it very rich, we feel that our method could be applied to the Euclidean setting, provided that a suitable form of confining drift is added to the dynamics. In this regard, it is worth mentioning that the approach used in [2] to handle the Euclidean case is of a somewhat different essence (in addition to the fact that the
class of test functions is not the same). Therein, the linearisation procedure is mostly achieved at the level of the McKean-Vlasov equation itself. In other words, the characteristics of the related master equation are no longer seen as the solutions of (1.4), but as the trajectories of (1.3). This may not lead to the same quantitative analysis and this leaves some space to our method in the Euclidean case.

1.3. Organisation and notations.

Organisation of the paper. We start the paper by reviewing the theory of calculus in Wasserstein spaces and the master equation in Section 2. We also provide the key semi-group expansion that serves as bounding the weak error in a systematic way (see Lemma 2.2). Subsequently, in Section 3, we address the ergodic properties of the linearised Fokker-Planck equations and then establish the property of uniform weak propagation of chaos for McKean-Vlasov equations with a small enough mean-field interaction and with a drift that derives from an $H$-stable potential (see Theorems 3.8, 3.12 and 3.17). Finally, in Section 4, we explore a special case related to the Kuramoto model in which the associated Fokker-Planck equation does not have a unique invariant measure (see Theorem 4.3).

Useful Notations. The scalar product between two vectors $a, b \in \mathbb{R}^d$ is denoted by $a \cdot b$. For each $i \in \mathbb{R}^d$, $e_i$ denotes the vector with 1 in the $i$th component and 0 elsewhere. For any vector $x \in \mathbb{R}^d$, $x_i$ denotes the $i$th component of $x$. For $a, b \in \mathbb{R}$, $a \lor b$ denotes $\max\{a, b\}$ and $a \land b$ denotes $\min\{a, b\}$. The set $\mathbb{N}$ is the set of integers, including $\{0\}$. For any real $s$, we call $[s]$ the floor part of $s$. For two probability measures $\mu$ and $\nu$ on $\mathbb{T}^d$, we call $\text{dist}_{TV}(\mu, \nu) = \sup_{\|f\|_{\infty} \leq 1} \int_{\mathbb{T}^d} f(x)d(\mu - \nu)$ the total variation distance between $\mu$ and $\nu$, where $\|f\|_{\infty}$ is the essential sup norm of $f : \mathbb{T}^d \to \mathbb{R}$. For $z \in \mathbb{C}$, $\overline{z}$ is the complex conjugate of $z$. Moreover, $i$ is the complex number such that $i^2 = -1$.

2. Main method of proof in this paper

In this section, we introduce the framework of calculus on Wasserstein spaces and the master equation, which constitute the main ingredients needed in our approach.

2.1. Master equation. As explained in the introduction, our framework of analysis depends on the so-called master equation for the semigroup $(P_t\Phi)_{t \geq 0}$ defined in (1.6). In turn, this requires a notion of differentiation w.r.t. measures in $\mathcal{P}(\mathbb{T}^d)$, called linear functional derivatives, which appears in various works in the literature, such as [16, 19, 25, 30, 57]. A function $\mathcal{V} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is said to be continuously differentiable if there exists a continuous function $\frac{\delta \mathcal{V}}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d),$

$$\mathcal{V}(m') - \mathcal{V}(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}((1-s)m + sm', y) (m' - m)(dy)ds. \quad (2.1)$$

The function $\frac{\delta \mathcal{V}}{\delta m}$ is said to be the linear functional derivative of $\mathcal{V} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. It is uniquely defined up to an additive constant, which is fixed by the convention

$$\int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(m, y)m(dy) = 0. \quad (2.2)$$

Throughout, the space $\mathcal{P}(\mathbb{T}^d)$ is equipped with the $\mathcal{W}_1$ distance, where we recall that convergence for $\mathcal{W}_1$ is equivalent to weak convergence on $\mathcal{P}(\mathbb{T}^d)$. Notice in particular that (2.1) may be reformulated as

$$\mathcal{V}(m') - \mathcal{V}(m) = \int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(m, y) (m' - m)(dy) + o(\mathcal{W}_1(m, m')), \quad (2.3)$$
with the symbol $o(\cdot)$ being uniform with respect to $m, m' \in \mathcal{P}(\mathbb{T}^d)$.

By induction, we then introduce higher-order derivatives: for any integer $p \geq 2$ and any $m, m' \in \mathcal{P}(\mathbb{T}^d)$ and $y \in (\mathbb{T}^d)^{p-1}$,

$$
\frac{\delta^{p-1} \mathcal{V}}{\delta m^{p-1}}(m', y) - \frac{\delta^{p-1} \mathcal{V}}{\delta m^{p-1}}(m, y) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m^p}(1-s)m + sm', y, y') (m' - m)(dy') \, ds,
$$

provided that the $(p-1)$-th order derivative is well defined. These derivatives are required to be continuous on the corresponding products spaces (for the product topology). In order to ensure uniqueness, they are required to satisfy

$$
\int_{\mathbb{T}^d \times \mathcal{P}^p} \frac{\delta \mathcal{V}}{\delta m^p}(m, y_1, \ldots, y_p) \, m(dy_p) = 0, \quad y_1, \ldots, y_{p-1} \in \mathbb{T}^{d-1}.
$$

As discussed earlier, there is a related notion called Wasserstein derivative ([11]), which has been intensively used for McKean-Vlasov and nonlinear Fokker-Planck equations possibly in the form of the so-called $L$-derivative (see [12, 15, 19]). In short, the Wasserstein derivative is the gradient field of the linear functional derivative, which is usually written in the form $\partial_x \mathcal{V}(m, y) = \partial_y [\mathcal{V}/\delta m](m, y)$, where $\partial_x \mathcal{V}$ denotes the Wasserstein derivative. A more detailed review is however out of the scope of this paper, as we shall work with linear functional derivatives exclusively. [19, Propositions 5.48 and 5.51] serve as a dictionary to enable us to pass from one to the other.

In the rest of the article, the vector field $b$ is assumed to be Lipschitz continuous on $\mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$. As a result, existence and uniqueness hold for (1.3) and the marginal law of the solution depends on the initial condition only through the statistical distribution of the latter. Then, we may call $(m(t; \mu))_{t \geq 0}$ the evolving law of the process $X$ in (1.3) when starting at law $\mu$; it solves (1.4) in a distributional sense.

For a bounded measurable function $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, we let (see (1.6))

$$
\mathcal{U}(t, \mu) := \mathcal{P}(\Phi(\mu)) = \Phi(m(t; \mu)), \quad t \geq 0, \quad \mu \in \mathcal{P}(\mathbb{T}^d).
$$

It is proven in Theorem 7.2 of [12] that $\mathcal{U}$ satisfies the master equation given by

$$
\begin{cases}
\partial_t \mathcal{U}(t, \mu) = \int_{\mathbb{T}^d} \left[ \sum_{i=1}^d \partial_{x_i} \frac{\delta \mathcal{U}}{\delta m}(t, \mu)(x) b_i(x, \mu) + \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i} \frac{\delta \mathcal{U}}{\delta m}(t, \mu)(x) \right] \mu(dx), \quad t \geq 0, \\
\mathcal{U}(0, \mu) = \Phi(\mu),
\end{cases}
$$

provided that $b$ and $\Phi$ satisfy the following two conditions:

(Diff-$b$): for every $i, j \in \{1, \ldots, d\}$, the derivatives

$$
\partial_{x_i} b(x, \mu), \quad \partial_{x_i} \partial_{x_j} b(x, \mu), \quad \partial_{(y_i)} \frac{\delta b}{\delta m}(x, \mu, y_1), \quad \partial_{(y_i)} \partial_{(y_j)} \frac{\delta b}{\delta m}(x, \mu, y_1)
$$

exist and are globally Lipschitz continuous w.r.t. the Euclidean and $W_1$ norms (which implies that they are globally bounded by an obvious compactness argument).

(Diff-$\Phi$): for every $i, j \in \{1, \ldots, d\}$, the derivatives

$$
\partial_{(y_i)} \frac{\delta \Phi}{\delta m}(\mu, y_1), \quad \partial_{(y_2)} \partial_{(y_1)} \frac{\delta \Phi}{\delta m^2}(\mu, y_1, y_2), \quad \partial_{(y_1)} \partial_{(y_1)} \frac{\delta \Phi}{\delta m}(\mu, y_1)
$$
exist and are globally Lipschitz continuous w.r.t.

The application of [12, Theorem 7.2] requires a modicum of care. Firstly, the equation therein is set in the Euclidean setting, which requires, when translated in the periodic setting, to check that the various derivatives are indeed periodic functions in the extra argument \( y \). We refer for instance to the appendix in [16] and to [51] (the latter being more detailed, but restricted to the \( 1d \) case). In this respect, it is worth noticing that in [12, Theorem 7.2], the measure argument \( \mu \) in (2.7) is taken in the space \( P_2(\mathbb{R}^d) \) of probability measures on \( \mathbb{R}^d \) with a finite second moment. In our setting, such a restriction on the integrability properties of the measure no longer exists since the torus is compact. Secondly, the various hypotheses in (2.7) are directly stated in terms of the \( L \)-derivative, but the aforementioned two propositions [19, Propositions 5.48 and 5.51] can be used to reformulate the set of assumptions in terms of the linear functional derivative, hence the form of (Diff-b) and (Diff-\( \Phi \)).

Very importantly, the result of [12, Theorem 7.2] says that:

**Proposition 2.1.** For every \( i, j \in \{1, \ldots, d\} \), the derivatives

\[
\partial\mu(t, \mu), \quad \frac{\partial U}{\partial \mu}(t, \mu, y), \quad \partial_{(y_1)}\partial_{(y_2)} \frac{\partial^2 U}{\partial \mu \partial y_1}(t, \mu, y_2), \quad \partial_{(y_1)}\partial_{(y_2)} \frac{\partial^2 U}{\partial \mu \partial y_1}(t, \mu, y_1)
\]

exist and are globally Lipschitz continuous w.r.t. \((\mu, y_1, y_2)\) for the Euclidean and \( W_1 \) norms, uniformly in time \( t \) in compact subsets, and are continuous in time (and hence jointly continuous w.r.t. all the parameters).

In fact, [12, Theorem 7.2] does not directly give the announced result, since the derivatives therein are proven to be Lipschitz continuous in the measure argument for the \( W_2 \)-distance. The adaptation to the \( W_1 \)-distance may be found in [20, Theorem 5.10], noticing that the result therein is in fact more general since the dynamics are forward-backward. In particular, the restriction that \( T \) has to be small enough (in the latter statement) can be easily removed in our simpler forward setting. Also, our function \( U(t, \mu) \) corresponds in the notation of [20] to \( \int U(t, x, \mu) \mu(dx) \).

### 2.2. Expansion along the particle system

The starting point of our analysis is to make use of the identity \( \Phi(\mu) = U(0, \mu) \), as given by the initial condition of (2.6). Recalling the notation \( \mu^N_t \) from (1.1), this gives the decomposition

\[
\Phi(\mu^N_t) - \Phi(\mathcal{L}(X_t)) = U(0, \mu^N_t) - U(t, \mu_{\text{init}}) = (U(t, \mu^N_0) - U(t, \mu_{\text{init}})) + (U(0, \mu^N_t) - U(t, \mu^N_0)).
\]

To treat the second term on the second line, we define, for every \( t > 0 \), the finite dimensional projection \( U_t : [0, t] \times (T^{d})^N \to \mathbb{R} \) by

\[
U_t(s, x_1, \ldots, x_N) := U\left(t, \frac{1}{N} \sum_{i=1}^N \delta_{x_i}\right).
\]

Then

\[
U(0, \mu^N_t) - U(t, \mu^N_0) = U_t(t, Y^1, \ldots, Y^N) - U_t(0, Y^1, \ldots, Y^N).
\]

We can now apply Itô’s formula to this equality. By combining our Proposition 2.1 with [24, Proposition 3.1], we can conclude that \( U_t \) is differentiable in the time component and twice-differentiable in the space components. Moreover, the same result ([24, Proposition 3.1]) expresses the first and second order partial derivatives of \( U_t \) in terms of the derivatives of \( U \). Very much in the spirit of [19, (5.131)], this allows us to
use (2.7) to obtain a cancellation of all the terms apart from one term which gives us the rate of convergence of 1/N:

$$
\mathbb{E}[U(0, \mu^N_1) - U(t, \mu^N_0)] = \frac{1}{N} \sum _{i,j=1}^{d} \int_0^t \mathbb{E} \left[ \int _{\mathbb{T}^d} \left( \partial _{(y_{2j})} \partial _{(y_{2i})} \frac{\delta ^2U}{\delta m^2}(t - s, \mu^N_s)(z, z) \right) \mu^N_s \text{d}z \right] \text{d}s.
$$

(2.9)

More importantly, this formula holds regardless of the assumptions on the initial data $Y^{1,N}_0, \ldots, Y^{N,N}_0$. In particular, they are not required to be I.I.D. This is an important fact that allows to understand the dynamics of the semi-group generated by the particle system, since the marginal empirical distribution $(\mu^N_t)_{t \geq 0}$ is a Markov process itself (see for instance Subsection 4.4).

In fact, the I.I.D. assumption becomes useful in order to estimate the first term in the expansion (2.8). Indeed, by [25, Theorem 2.11] (which we can apply in our context thanks to Proposition 2.1), we then have

$$
\mathbb{E}[U(t, \mu^N_0) - U(t, \mu_{\text{init}})] = \frac{1}{N} \int_0^1 \int_0^1 \mathbb{E} \left[ \int _{\mathbb{T}^d} \left( \partial _{(y_{2j})} \partial _{(y_{2i})} \frac{\delta ^2U}{\delta m^2}(t, m^N_{s,s}) \right) \phi(t, m^N_{s,s}) \right] \text{d}s \text{d}s,
$$

(2.10)

where $\eta_1$ is as in (1.1), $\tilde{\eta}$ is independent of $(\eta_1, \ldots, \eta_N)$ with law $\mu_{\text{init}}$ and

$$
m^N_{s,s} := \frac{s s_1}{N} (\delta_{\eta} - \delta_{\eta}) + \mu_{\text{init}} + s (\mu^N_0 - \mu_{\text{init}}), \quad s, s_1 \in [0,1].
$$

By combining the above equation with (2.8) and (2.9), we deduce the following lemma:

**Lemma 2.2.** Under (Diff-\(b\)) and (Diff-\(\Phi\)), for any integer $N \geq 1$ such that $(Y^{1,N}, \ldots, Y^{N,N})$ are I.I.D. with common law $\mu_{\text{init}} := \mathcal{L}(X_0)$ and for any $t \geq 0$,

$$
\mathbb{E}[\Phi(\mu^N_t)] - \Phi(\mathcal{L}(X_t)) = \frac{1}{N} \int_0^1 \int_0^1 \mathbb{E} \left[ \frac{\delta ^2U}{\delta m^2}(t, m^N_{s,s}) \phi(t, m^N_{s,s}) \right] \text{d}s \text{d}s
$$

$$
+ \frac{1}{N} \sum _{i,j=1}^{d} \int_0^t \mathbb{E} \left[ \int _{\mathbb{T}^d} \left( \partial _{(y_{2j})} \partial _{(y_{2i})} \frac{\delta ^2U}{\delta m^2}(t - s, \mu^N_s)(z, z) \right) \mu^N_s \text{d}z \right] \text{d}s.
$$

(2.11)

Equation (2.11) is in fact a key in our analysis. Recall now that we want to prove that the left-hand side is bounded by $1/N$, uniformly in $t \geq 0$. In order to achieve this goal, we must be able:

1. to bound $\frac{\delta ^2U}{\delta m^2}(t, \cdot)(\cdot, \cdot)$, uniformly in time;
2. to bound $\partial _{(y_{2j})} \partial _{(y_{2i})} \frac{\delta ^2U}{\delta m^2}(t, \cdot)(\cdot, \cdot)$, with an integrable decay in time.

Observe that, implicitly, those bounds are required to be uniform in space. Obviously, this is a very strong constraint. Anyhow, we succeed in relaxing it in the analysis of the super-critical Kuramoto model provided in Section 4. This asks us to prove that the empirical measure stays for sufficiently long time in the region where the above bounds are true and then to manage possible excursions of the empirical measure outside this region.

The next step is thus to get a tractable representation of the second-order derivatives in the right-hand side of (2.11). This is precisely the point where the linearisation of the Fokker-Planck equation (1.4) comes in. Indeed, it is shown in [66] that, under certain regularity conditions on $b$ and $\Phi$ (see Theorem 3.2), $\frac{\delta ^2U}{\delta m^2}$ can be represented as

$$
\frac{\delta ^2U}{\delta m^2}(t, \mu)(z_1, z_2) = \frac{\delta ^2\Phi}{\delta m^2}(m(t; \mu))(m^{(1)}(t; \mu, \delta_{z_1}), m^{(1)}(t; \mu, \delta_{z_2})) + \frac{\delta \Phi}{\delta m}(m(t; \mu))(m^{(2)}(t; \mu, \delta_{z_1}, \delta_{z_2})),
$$
for \( m^{(1)} \) and \( m^{(2)} \) solutions to linearised equations that are defined next by (3.4) and (3.5) respectively. To apply (2.11), one would need to differentiate this formula w.r.t. \( z_1 \) and \( z_2 \) and this is the main goal of Section 2.1, which proves, in Theorem 3.5, that

\[
(\partial_{z_2})_j(\partial_{z_1})_i \delta^2U_{m2}(t,\mu)(z_1,z_2) = \delta^2m(t;\mu)\left(d^{(1)}_i(t;\mu,z_1),d^{(1)}_j(t;\mu,z_2)\right) + \delta\Phi(m(t;\mu))\left(d^{(2)}_i(t;\mu,z_1,z_2)\right),
\]

for some linearised Kolmogorov equations \( d^{(1)} \) and \( d^{(2)} \) defined by (3.6) and (3.11) respectively. The core of the analysis is thus to provide ergodic Sobolev estimates on \( m^{(1)},m^{(2)},d^{(1)} \) and \( d^{(2)} \).

2.3. Main assumptions.

2.3.1. Functional spaces. We shall use two types of functional spaces in our analysis: \( W^{s,\infty}(\mathbb{T}^d) \) spaces (and their duals), and \( W^{s,2}(\mathbb{T}^d) \) spaces (and their duals), for \( s > 0 \).

We start with \( W^{s,\infty}(\mathbb{T}^d) \) spaces. Following [11, 32], we define the following notations.

1. For any integer \( n \geq 0 \), we call \( W^{n,\infty}(\mathbb{T}^d) \) the space of functions \( f \) that are \((n-1)\)-times differentiable and whose \((n-1)\text{th}-\text{derivative}\) is Lipschitz continuous. The derivatives up to order \( n-1 \) are denoted by \((\nabla f)^{(k)}_{k=1,\ldots,n-1}\) with each \( \nabla f \) taking values in \((\mathbb{R}^d)^k\). The function \( \nabla^{n-1}f \) itself has a generalised derivative \( \nabla^n f \in L^\infty(\mathbb{T}^d; (\mathbb{R}^d)^n) \). The \( W^{n,\infty}(\mathbb{T}^d) \)-norm is written \( ||f||_{n,\infty} := \sum_{k=0}^n ||\nabla^k f||_{\infty} \).

2. For any integer \( n \geq 0 \) and any real \( \alpha \in (0,1) \), we call \( W^{n+\alpha,\infty}(\mathbb{T}^d) \) the space of functions that are \( n \)-times differentiable such that their \( n \text{th}-\text{derivatives} \) are \( \alpha \)-Hölder continuous. The \( W^{n+\alpha,\infty}(\mathbb{T}^d) \)-norm is written as \( ||f||_{n+\alpha,\infty} := \sum_{k=0}^n ||\nabla^k f||_{\alpha,\infty} \), where \( ||\cdot||_{\alpha,\infty} \) is the standard Hölder norm

\[
||f||_{\alpha,\infty} = \sup_{x \in \mathbb{T}^d} |f(x)| + \sup_{x,y \in \mathbb{T}^d;x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.
\]

3. For any integer \( n \geq 0 \) and any real \( \alpha \in (0,1) \), we call \( (W^{n+\alpha,\infty}(\mathbb{T}^d))' \) the dual space of \( W^{n+\alpha,\infty}(\mathbb{T}^d) \).

We merely write \( ||\cdot||_\infty \) for \( ||\cdot||_0,\infty \). In the text, we make use of the following interpolation inequality:

\[
||\phi||_{\alpha+\eta,\infty} \leq ||\phi||_{\alpha,\infty}^{\eta/\gamma} ||\phi||_{\alpha+\gamma,\infty}^{\eta/\gamma}, \quad (2.12)
\]

which holds for any \( a \geq 0, \eta, \gamma \in [0,1] \) with \( \eta \leq \gamma \). In the above inequality, \( \phi \in W^{a+\gamma,\infty}(\mathbb{T}^d) \) (see [11]).

In order to introduce \( W^{s,2}(\mathbb{T}^d) \), we feel more convenient to use Fourier analysis. For a function \( f \in L^2(\mathbb{T}^d) \), we denote its Fourier coefficients by

\[
f^n := \int_{\mathbb{T}^d} f(x)e^{-2\pi in.x}dx, \quad n \in \mathbb{Z}^d.
\]

For \( s > 0 \), we then call \( W^{s,2}(\mathbb{T}^d) \) the space of functions \( f \in L^2(\mathbb{T}^d) \) such that \( ||f||_{s,2}^2 := \sum_{n \in \mathbb{Z}^d}(1+n^2)^s|f^n|^2 < \infty \). The \( W^{s,2}(\mathbb{T}^d) \)-norm is \( ||\cdot||_{s,2} \). The dual space is identified with \( W^{-s,2}(\mathbb{T}^d) \), which is defined in similar manner, by extending the notation \( \langle q^n \rangle_{n \in \mathbb{Z}^d} \) for the Fourier coefficients of a Schwartz distribution \( \tau \) (acting on smooth functions of \( \mathbb{T}^d \)). Then, \( W^{-s,2}(\mathbb{T}^d) \) is the space of distributions \( q \) such that \( ||q||_{-s,2}^2 := \sum_{n \in \mathbb{Z}^d}(1+n^2)^s|q^n|^2 < \infty \). The \( W^{-s,2}(\mathbb{T}^d) \)-norm is \( ||\cdot||_{-s,2} \). For brevity, we write \( ||\cdot||_{2} \) for \( ||\cdot||_{0,2} \).

For any vector field \( f = (f^1, \cdots, f^d) \), we write \( ||f|| = \max_{i=1, \cdots, d} ||f^i|| \) for any norm on the space in which the \( f^i \)'s are taken. Most of the time, the duality product between a function \( f \) and a distribution \( q \) is merely denoted by \( \langle f, q \rangle \), with the spaces to which \( f \) and \( q \) belong being implicitly understood (it is
if, for any \( (\text{Reg}^b)(n,k) \) and \( (\text{Lip}^b)(n,k) \) hold for the same values of \( n, \infty \) and \( n-1, \infty \). For \( z \in \mathbb{T}^d \), we define \( D_z \) as the Dirac distribution at point \( z \) and \( D_z' \) for the opposite of its derivative. Therefore, \( D_z' \) is acting on continuously differentiable functions \( f \) in the following way: \( \langle D_z', f \rangle = f'(z) \). For a time-dependent function \( f \) and a time-dependent distribution \( q \), we often write \( f(t, x) \) for the former and \( q(t) \) for the latter.

Finally, for any \( n \in \mathbb{N} \setminus \{0\} \) and any \( n \)-times differentiable function \( \Phi \) on \( \mathcal{P}(\mathbb{T}^d) \), we write \( \frac{\delta^n \Phi}{\delta m^n}(\mu)(q_1, \ldots, q_n) \), for distributions \( q_1, \ldots, q_n \) on \( \mathbb{T}^d \), to denote

\[
\frac{\delta^n \Phi}{\delta m^n}(\mu)(q_1, \ldots, q_n) := \left\langle \frac{\delta^n \Phi}{\delta m^n}(\mu, \ldots), q_1 \otimes q_2 \otimes \cdots \otimes q_n \right\rangle,
\]

if the duality product makes sense, where we recall that the function in the left-hand side of the duality product is defined on \( (\mathbb{T}^d)^n \). Note that \( q_1 \otimes q_2 \otimes \cdots \otimes q_n \) is the tensor product of \( q_1, \ldots, q_n \) (see [65, Definition 40.3], which includes a form of Fubini’s identity for tensor products of distributions).

2.3.2. Main assumptions. Throughout the article, we use the following set of assumptions, with \( n \) and \( k \) denoting two integers:

- **(Reg-\( b \)-(\( n, k \)))**: We say that \( b \) satisfies \( (\text{Reg}^b)(n,k) \) if, for any \( i \in \{1, \ldots, d\} \), the function \( b_i \) is \( k \) times differentiable with respect to the measure argument \( m \), and for any \( m \in \mathcal{P}(\mathbb{T}^d) \) and \( \ell \in \{0, \ldots, k\} \), the function

\[
(\mathbb{T}^d)^{\ell+1} \ni (x, y_1, \ldots, y_\ell) \mapsto \frac{\delta^\ell b_i}{\delta n^\ell}(x, m, y_1, \ldots, y_\ell)
\]

has continuous crossed derivatives \( \frac{\partial^{n_0} \partial^{n_1}_{y_1} \cdots \partial^{n_\ell}_{y_\ell}}{\partial^{n} m}(x, m, y_1, \ldots, y_\ell) \) for any \( n_0, n_1, \ldots, n_\ell \in \{0, \ldots, n\} \), with all these crossed derivatives being bounded w.r.t. \((x, y_1, \ldots, y_\ell)\), uniformly in \( m \).

- **(Lip-\( b \)-(\( n, k \)))**: We say that \( b \) satisfies \( (\text{Lip}^b)(n,k) \) if it satisfies \( (\text{Reg}^b)(n,k) \) and, for any \( i \in \{1, \ldots, d\} \) and \( \ell \in \{1, \ldots, k\} \), for any \( n_0, n_1, \ldots, n_\ell \in \{0, \ldots, n\} \), the derivatives \( \frac{\partial^{n_0} \partial^{n_1}_{y_1} \cdots \partial^{n_\ell}_{y_\ell}}{\partial^{n} m} \) are Lipschitz continuous in \( m \) with respect to \( \mathcal{W}_1 \).

Below, we do not require that \( (\text{Reg}^b)(n,k) \) and \( (\text{Lip}^b)(n,k) \) hold for the same values of \( n \) and \( k \). Typically, we require \( (\text{Reg}^b)(4,2) \) and \( (\text{Lip}^b)(3,2) \), noticing that the latter subsumes \( (\text{Diff}^b) \). Moreover, we often use \( (\text{Reg}^b)(n,k) \) and \( (\text{Lip}^b)(n,k) \) in the following manner. We indeed observe that \( (\text{Reg}^b)(n,k) \) implies that, for each \( i \in \{1, \ldots, d\} \) and \( \ell \in \{1, \ldots, k\} \),

\[
\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \sup_{\|q_1\|_{(n,\infty)^d} \leq \ldots \|q_\ell\|_{(n,\infty)}} \left\| \frac{\delta^\ell b_i}{\delta n^\ell}(\cdot, m)(q_1, \ldots, q_\ell) \right\|_{(n,\infty)} < +\infty.
\]

Similarly, \( (\text{Lip}^b)(n,k) \) implies that, for each \( i \in \{1, \ldots, d\} \) and \( \ell \in \{1, \ldots, k-1\} \),

\[
\sup_{\|q_1\|_{(n,\infty)^d} \leq \ldots \|q_\ell\|_{(n,\infty)}} \left( \mathcal{W}_1(\mu_1,\mu_2)^{-1} \right) \left\| \frac{\delta^\ell b_i}{\delta n^\ell}(\cdot, \mu_1)(q_1, \ldots, q_\ell) - \frac{\delta^\ell b_i}{\delta n^\ell}(\cdot, \mu_2)(q_1, \ldots, q_\ell) \right\|_{(n,\infty)}
\]

is finite.
We proceed similarly with the test function $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$. For an integer $k$ and a real $s \geq 0$, with $n := \lfloor s \rfloor$ and $\alpha := s - n$, we let

(Reg-$\Phi$-$(s, k)$): We say that $\Phi$ satisfies (Reg-$\Phi$-$(s, k)$) if $\phi$ is $k$ times differentiable with respect to the measure argument $m$, and for any $m \in \mathcal{P}(\mathbb{T}^d)$ and $\ell \in \{0, \ldots, k\}$, the function

$(\mathbb{T}^d)_{\ell} \ni (y_1, \ldots, y_{\ell}) \mapsto \frac{\partial^\ell \Phi}{\partial m^\ell}(m, y_1, \ldots, y_{\ell})$

has continuous crossed derivatives $\frac{\partial^{n_1}_{y_1} \cdots \partial^{n_{\ell}}_{y_{\ell}} \delta \Phi}{\partial m^{n_1}}(m, y_1, \ldots, y_{\ell})$ for any $n_1, \ldots, n_{\ell}$ in $\{0, \ldots, n\}$, with all these crossed derivatives being bounded, continuous and $\alpha$-Hölder continuous if $\alpha > 0$, w.r.t. $(y_1, \ldots, y_{\ell})$, uniformly in $m$.

In the sequel, we will use several values of $(n, k)$ in (Reg-$\Phi$-$(n, k)$). In the final statements, we just require (Reg-$\Phi$-$(\alpha, 2)$), for some $\alpha \in (0, 1)$ (being possibly arbitrary small). In the proofs, we use (Reg-$\Phi$-$(4, 3)$), but we eventually recover the result under the sole (Reg-$\Phi$-$(\alpha, 2)$) by a (non-trivial) mollification argument, which is proven in the appendix. The analogue of (2.14), but for $\phi$ satisfying (Reg-$\Phi$-$(4, 3)$), is straightforward. The analogue of (2.14), but for $\phi$ satisfying (Reg-$\Phi$-$(\alpha, 2)$), requires some care. We indeed observe that, under the latter assumption, the mapping $y_1 \mapsto [(\partial^2 \Phi/\partial m^2)(m, y_1, \cdot) : y_2 \mapsto (\partial^2 \Phi/\partial m^2)(m, y_1, \cdot) \in W^{\alpha/2, \infty}(\mathbb{T}^d)]$ is $\alpha/2$-Hölder continuous, from which we deduce that (Reg-$\Phi$-$(\alpha, 2)$) implies

$$\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \sup_{\|q_1\|_{(\alpha/2, \infty)'}, \|q_2\|_{(\alpha/2, \infty)'} \leq 1} \left| \frac{\partial^\ell \Phi}{\partial m^\ell}(., m)(q_1, q_2) \right| < +\infty.$$  (2.16)

In words, the Hölder exponent used in the duality is $\alpha/2$ and not $\alpha$.

**Remark 2.3.** It is an interesting task to compare our set of assumptions with the set of assumptions used in [66], from which we borrow some results below. It is clear from (2.14) that (Reg-$b$-$(n, k)$) implies (Int-$b$-$(n, k)$) in [66]. By (2.15), we also have (Lip-$b$-$(n, k)$) as defined therein. We may compare in a similar fashion our assumption on $\Phi$ with the set of assumptions used in [66, Theorem 4.5]: in the end, the assumptions in the latter statement are satisfied under our conditions (Reg-$b$-$(4, 3)$), (Lip-$b$-$(4, 2)$) and (Reg-$\Phi$-$(4, 3)$).

**2.4. Examples.** We now give some examples of $b$ and $\Phi$ that satisfy the above assumptions (as in [66]).

2.4.1. **Linear interaction.** Let $n \in \mathbb{N}$. Suppose that for each $i \in \{1, \ldots, d\}$, $F_i : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ is $n$-times continuously differentiable and that $G : \mathbb{T}^d \to \mathbb{R}$ is $n$-times continuously differentiable. We then define functions $b_i : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ and $\Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ by

$$b_i(x, \mu) := \int_{\mathbb{T}^d} F_i(x, y) \mu(dy), \quad \Phi(\mu) := \int_{\mathbb{T}^d} G(y) \mu(dy).$$

It can be shown easily by the definition of linear functional derivatives (along with the condition of normalisation) that, for any integer $k \geq 1$,

$$\frac{\partial^k b_i}{\partial m^k}(x, \mu, y_1, \ldots, y_k) = (-1)^k \left( \int_{\mathbb{T}^d} F_i(x, y) \mu(dy) - F_i(x, y_k) \right),$$

$$\frac{\partial^k \Phi}{\partial m^k}(\mu, y_1, \ldots, y_k) = (-1)^k \left( \int_{\mathbb{T}^d} G(y) \mu(dy) - G(y_k) \right).$$
It is easily seen ([66]) that $b$ satisfies \((\text{Reg-b}(n,k))\) and \((\text{Lip-b}(n,k))\), whereas $\Phi$ satisfies \((\text{Reg-\Phi}(n,k))\). Note that $k$ is arbitrary in $\mathbb{N}$ since the dependence on measure is linear for functions $b$ and $\sigma$.

2.4.2. A completely non-linear example. The following ‘completely non-linear’ example will be very useful.

**Proposition 2.4.** For given $\alpha \in (0,1)$ and $\nu_0 \in \mathcal{P}(\mathbb{T}^d)$, the function $\Phi$ below satisfies \((\text{Reg-\Phi}(\alpha/2,2))\):

$$
\Phi(\mu) = \|\mu - \nu_0\|^2_{(d+\alpha)/2,2}, \quad \mu \in \mathcal{P}(\mathbb{T}^d).
$$

One could increase the absolute value of the Sobolev index (towards a wider space of distributions) to increase the smoothness of $\Phi$. Since this is nonetheless useless for our purpose, we find it better to restrict ourselves to this statement, which we use in the following analysis.

**Proof.** Throughout the proof, we let $s := (d + \alpha)/2$. Then, it is obvious to see that (using the standard notation $\bar{z}$ for the conjugate of $z$)

$$
\Phi(\mu) = \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|^2)^s} (\mu^n \bar{\nu}^n + \nu_0^n \bar{\nu}_0^n - \mu^n \nu_0^n - \nu_0^n \bar{\mu}^n).
$$

(2.17)

Throughout the proof, we use the fact that

$$
\sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|^2)^s} < \infty, \quad \sum_{n \in \mathbb{Z}^d} \frac{|n|^{\alpha/2}}{(1 + |n|^2)^s} < \infty.
$$

(2.18)

Writing the product $\mu^n \bar{\nu}^n$ in the form $\int_{\mathbb{T}^d} e^{-i2\pi n \cdot (\theta - \theta')} \mu(d\theta)\mu(d\theta')$ (and similarly for the other products) and using (2.18), we get that

$$
\frac{\delta \Phi}{\delta m}(\mu)(x) = \Phi^{(1)}(\mu, x) - \int_{\mathbb{T}^d} \Phi^{(1)}(\mu, y) \mu(dy),
$$

$$
\Phi^{(1)}(\mu, x) = \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|^2)^s} \int_{\mathbb{T}^d} \left(e^{-i2\pi n \cdot (\theta - x)} + e^{-i2\pi n \cdot (x - \theta)}\right) (\mu - \nu_0)(d\theta), \quad x \in \mathbb{T}^d.
$$

We then compute the derivative $\delta^2 \Phi/\delta m^2$ in a similar manner. We have

$$
\frac{\delta^2 \Phi}{\delta m^2}(\mu)(x, x') = \Phi^{(2)}(\mu)(x, x') - \int_{\mathbb{T}^d} \Phi^{(2)}(x, y) \mu(dy) - \left(\Phi^{(1)}(\mu, x') - \int_{\mathbb{T}^d} \Phi^{(1)}(\mu, y) \mu(dy)\right).
$$

$$
\Phi^{(2)}(x, x') = \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|^2)^s} \left(e^{-i2\pi n \cdot (x - x')} + e^{-i2\pi n \cdot (x' - x)}\right), \quad x, x' \in \mathbb{T}^d.
$$

Then, we can find a constant $C$, depending on $\alpha$ and allowed to vary from term to term, such that for any $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $x, x' \in \mathbb{T}^d$,

$$
|\Phi^{(1)}(\mu, x) - \Phi^{(1)}(\mu, x')| \leq C|x - x'|^{\alpha/2} \sum_{n \in \mathbb{Z}^d} \frac{|n|^{\alpha/2}}{(1 + |n|^2)^s} \leq C|x - x'|^{\alpha/2}.
$$

Proceeding in a similar way with $\Phi^{(2)}$, the conclusion easily follows. \qed
2.5. About constants. Unless otherwise specified, $C$ is a generic constant that only depends on $n$, $k$, $T$, $b$ and $\Phi$, whose value varies from line to line. The dimension $d$ is assumed to be constant, so any dependence on it will not be indicated. If the dependence of $C$ needs to be explicit when necessary, we enclose $C$ with round brackets containing all the parameters on which $C$ depends.

3. Uniform weak propagation of chaos for McKean-Vlasov equations

This section is dedicated to the analysis of the general $d$-dimensional case. Most of our analysis is based upon a linearisation argument, which prompts us to introduce the following linearised operator:

$$L_m q = \frac{1}{2} \Delta q - \text{div}(b(\cdot, m)q) - \text{div}\left(m \frac{\partial b}{\partial m}(\cdot, m)(q)\right),$$

(3.1)

for a probability measure $m \in \mathcal{P}(\mathbb{T}^d)$ and a distribution $q$ on $\mathbb{T}^d$. In our analysis, we make use of the dual operator, which has the following nonlocal form:

$$L_m^* \phi = \frac{1}{2} \Delta \phi + b(\cdot, m) \cdot \nabla \phi + \int_{\mathbb{T}^d} \frac{\partial b}{\partial m}(y, m)(x) \cdot \nabla \phi(y) \, m(dy),$$

for a smooth test function $\phi$ on $\mathbb{T}^d$.

3.1. Second order mixed spatial derivatives of the second order linear functional derivative of $\mathcal{U}$. We first invoke a local estimate for forward Kolmogorov equations proven in [66, Theorem 2.3] (see also [16, Subsection 3.3]).

**Theorem 3.1.** For $n \geq 1$, assume (Reg-\textbf{b}(n, 1)) is in force. For $T > 0$, $q_0 \in (W^{n, \infty}(\mathbb{T}^d))^\prime$ and $r \in L^\infty([0, T], (W^{n, \infty}(\mathbb{T}^d))^\prime)$, the Cauchy problem defined by

$$\begin{cases}
\partial_t q(t) - L_{m(t, \mu)} q(t) - r(t) = 0, \\
q(0) = q_0,
\end{cases}$$

(3.2)

interpreted in the weak sense as

$$\langle \phi(t, \cdot), q(t) \rangle - \langle \phi(0, \cdot), q(0) \rangle = \int_0^t \langle \partial_s \phi(s, \cdot), q(s) \rangle \, ds + \int_0^t \langle L_{m(t, \mu)}^* \phi(s, \cdot), q(s) \rangle \, ds + \int_0^t \langle \phi(s, \cdot), r(s) \rangle \, ds,$$

for $t \in [0, T]$ and for each $\phi \in C^\infty([0, T] \times \mathbb{T}^d)$, has a unique solution in $L^\infty([0, T], (W^{n, \infty}(\mathbb{T}^d))^\prime)$ such that

$$\sup_{t \in [0, T]} \|q(t)\|_{(n, \infty)^\prime} \leq C \left(\|q_0\|_{(n, \infty)^\prime} + \sup_{t \in [0, T]} \|r(t)\|_{(n, \infty)^\prime}\right),$$

for some constant $C > 0$, independent of the inputs $q_0$ and $r$ (but depending on $T$ and on $b$ through the bounds in (Reg-\textbf{b}(n, 1))).

We now recall Theorem 4.5 from [66], which gives a representation of the second order linear functional derivative of $\mathcal{U}$ in terms of solutions of forward Kolmogorov equations. See Remark 2.3 for the connection between our conditions and the assumptions in [66, Theorem 4.5].
Theorem 3.2. Assume (Reg-b-(4, 2)), (Lip-b-(3, 2)) and (Reg-Φ-(4, 3)). Then, $U$ is twice differentiable with respect to $m$ and the first and second-order derivatives $\frac{\delta U}{\delta m}$ and $\frac{\delta^2 U}{\delta m^2}$ are given by
\[
\frac{\delta U}{\delta m}(t, \mu)(z) = \frac{\delta \Phi}{\delta m}(m(t; \mu))(m(t; \mu, \delta_z)),
\]
\[
\frac{\delta^2 U}{\delta m^2}(t, \mu)(z_1, z_2) = \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu))(m^{(1)}(t; \mu, \delta_{z_1}), m^{(1)}(t; \mu, \delta_{z_2})) + \frac{\delta \Phi}{\delta m}(m(t; \mu))(m^{(2)}(t; \mu, \delta_{z_1}, \delta_{z_2})),
\]
where, for any $\mu, \nu \in \mathcal{P}(\mathbb{T}^d)$, $m^{(1)}(\cdot; \mu, \nu) \in \cap_{T>0}L^\infty([0, T], (W^{2, \infty}(\mathbb{T}^d))^\prime)$ satisfies the Cauchy problem
\begin{align}
\begin{cases}
\partial_t m^{(1)}(t; \mu, \nu) - L_{m(t; \mu)} m^{(1)}(t; \mu, \nu) = 0, & t \geq 0,
m^{(1)}(0; \mu, \nu) = \nu - \mu,
\end{cases}
\end{align}
and, for any $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{T}^d)$, $m^{(2)}(\cdot; \mu, \nu_1, \nu_2) \in \cap_{T>0}L^\infty([0, T], (W^{3, \infty}(\mathbb{T}^d))^\prime)$ satisfies the Cauchy problem
\begin{align}
\begin{cases}
\partial_t m^{(2)}(t; \mu, \nu_1, \nu_2) - L_{m(t; \mu)} m^{(2)}(t; \mu, \nu_1, \nu_2) \\
+ \text{div}\left(m^{(1)}(t; \mu, \nu_1) \frac{\delta m}{\delta m}(\cdot, m(t; \mu))(m^{(1)}(t; \mu, \nu_2))\right) \\
+ \text{div}\left(m^{(1)}(t; \mu, \nu_2) \frac{\delta m}{\delta m}(\cdot, m(t; \mu))(m^{(1)}(t; \mu, \nu_1))\right) \\
+ \text{div}\left(m(t; \mu) \frac{\delta^2 m}{\delta m^2}(\cdot, m(t; \mu))(m^{(1)}(t; \mu, \nu_1), m^{(1)}(t; \mu, \nu_2))\right) = 0, & t \geq 0,
m^{(2)}(0; \mu, \nu_1, \nu_2) = \nu_2 - \nu - \mu.
\end{cases}
\end{align}
We differentiate each of the two terms on the right hand side of (3.3) respectively, each with respect to $z_1$ and $z_2$.

Theorem 3.3. Assume (Reg-b-(4, 2)), (Lip-b-(3, 2)) and (Reg-Φ-(4, 3)). Then, for any $i, j \in \{1, \ldots, d\}$, $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $z_1, z_2 \in \mathbb{T}^d$,
\[
(\partial_{z_1})_j(\partial_{z_2})_i \left\{ \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu))(m^{(1)}(t; \mu, \delta_{z_1}), m^{(1)}(t; \mu, \delta_{z_2})) \right\} = \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu))(d^{(1)}_i(t; \mu, z_1), d^{(1)}_j(t; \mu, z_2)),
\]
it being understood that the argument in brackets in the left-hand side is indeed differentiable with respect to $z_1$ and $z_2$ and where $d^{(1)}_i(\cdot; \mu, z) \in \cap_{T>0}L^\infty([0, T], (W^{2, \infty}(\mathbb{T}^d))^\prime)$ satisfies the Cauchy problem
\begin{align}
\begin{cases}
\partial_t d^{(1)}_i(t; \mu, z) - L_{m(t; \mu)} d^{(1)}_i(t; \mu, z) = 0, & t \geq 0,
d^{(1)}_i(0; \mu, z) = (D'_i)_i(z),
\end{cases}
\end{align}
and $(D'_i)_i$ is defined by
\begin{align}
\langle \xi, (D'_i)_i \rangle = \partial_{z_i} \xi(z).
\end{align}

Proof. Existence of solution to (3.6) in $\cap_{T>0}L^\infty([0, T], (W^{2, \infty}(\mathbb{T}^d))^\prime)$ is guaranteed by Theorem 3.1. For given $T > 0$ and $i \in \{1, \cdots, d\}$, we define, for $t \in [0, T], \mu \in \mathcal{P}(\mathbb{T}^d)$, $z \in \mathbb{T}^d$ and $h \in \mathbb{R} \setminus \{0\}$,
\begin{align}
\rho^{(1)}_i(t; \mu, z, h) := \frac{1}{h} \left( m^{(1)}(t; \mu, \delta_z + h \epsilon_i) - m^{(1)}(t; \mu, \delta_z) \right) - d^{(1)}_i(t; \mu, z), & t \in [0, T].
\end{align}
Applying (3.4) to $\nu = \delta_z$ and $\nu = \delta_{z+he}$ respectively, followed by a comparison with (3.6), we have
\[
\begin{cases}
\partial_t \rho_i^{(1)}(t; \mu, z, h) - L_{m(t; \mu)} \rho_i^{(1)}(t; \mu, z, h) = 0, & t \in [0, T], \\
\rho_i^{(1)}(0; \mu, z, h) = h^{-1} (\delta_{z+he_i} - \delta_z) - (D'_z).
\end{cases}
\]

Note from Taylor-Lagrange inequality that
\[
\|\rho_i^{(1)}(0; \mu, z, h)\|_{-2, \infty} = \sup_{\|\xi\|_{2, \infty} \leq 1} \left\langle \xi, \rho_i^{(1)}(0; \mu, z, h) \right\rangle = \sup_{\|\xi\|_{2, \infty} \leq 1} \left[ \frac{\xi(z + he_i) - \xi(z)}{h} - \partial_z \xi(z) \right] \leq \frac{|h|}{2}.
\]
By Theorem 3.1, we get
\[
\lim_{h \to 0} \sup_{t \in [0, T]} \|\rho_i^{(1)}(t; \mu, z, h)\|_{(2, \infty)'} = 0. \quad (3.9)
\]
Therefore, using the smoothness of $\delta^2 \Phi / \delta m^2$,
\[
(\partial_{z_1}) \{ \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu)) \left( m^{(1)}(t; \mu, \delta_{z_1}), m^{(1)}(t; \mu, \delta_{z_2}) \right) \} = \lim_{h \to 0} \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu)) \left( \frac{m^{(1)}(t; \mu, \delta_{z_1+he}) - m^{(1)}(t; \mu, \delta_{z_1})}{h}, m^{(1)}(t; \mu, \delta_{z_2}) \right)
\]
\[
= \frac{\delta^2 \Phi}{\delta m^2}(m(t; \mu)) \left( d^{(1)}_i(t; \mu, z_1), m^{(1)}(t; \mu, \delta_{z_2}) \right). \quad (3.10)
\]
The result follows by repeating the same procedure on $z_2$. \hfill \Box

**Theorem 3.4.** Assume (Reg-b-(4, 2)), (Lip-b-(3, 2)) and (Reg-\(\Phi-(4, 3)\)). Then for any $i, j \in \{1, \ldots, d\}$, $\mu \in \mathcal{P}(\mathbb{T}^d)$ and $z_1, z_2 \in \mathbb{T}^d$,
\[
(\partial_{z_2}) (\partial_{z_1}) \{ \frac{\delta \Phi}{\delta m}(m(t; \mu)) \left( m^{(2)}(t; \mu, \delta_{z_1}, \delta_{z_2}) \right) \} = \frac{\delta \Phi}{\delta m}(m(t; \mu)) \left( d^{(2)}_{i,j}(t; \mu, z_1, z_2) \right),
\]
it being understood that the argument in brackets in the left-hand side is indeed differentiable with respect to $z_1$ and $z_2$ and where $d^{(2)}_{i,j}(\cdot; \mu, z_1, z_2) \in \cap_{T>0} L^\infty ([0, T], (W^{3, \infty}(\mathbb{T}^d))')$ satisfies the Cauchy problem
\[
\begin{align*}
\partial_t d^{(2)}_{i,j}(t; \mu, z_1, z_2) - L_{m(t; \mu)} d^{(2)}_{i,j}(t; \mu, z_1, z_2) + & \text{div} \left( d^{(1)}_j(t; \mu, z_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(d^{(1)}_i(t; \mu, z_1)) \right) \\
+ & \text{div} \left( m(t; \mu) \frac{\partial^2 b}{\partial m^2}(\cdot, m(t; \mu))(d^{(1)}_i(t; \mu, z_1), d^{(1)}_j(t; \mu, z_2)) \right) = 0, & t \geq 0,
\end{align*}
\]
\[
d^{(2)}_{i,j}(t; \mu, z_1, z_2) = 0. \quad (3.11)
\]
**Proof.** As before, existence of solution to (3.11) in $\cap_{T>0} L^\infty ([0, T], (W^{3, \infty}(\mathbb{T}^d))')$ is guaranteed by Theorem 3.1. For given $T > 0$ and $i \in \{1, \ldots, d\}$, we define, for $t \in [0, T]$, $\mu, \nu_2 \in \mathcal{P}(\mathbb{T}^d)$, $z \in \mathbb{T}^d$ and $h \in \mathbb{R} \setminus \{0\}$,
\[
\Pi_i(t; \mu, z, \nu_2, h) := \frac{1}{h} \left( m^{(2)}(t; \mu, \delta_{z+he_i}, \nu_2) - m^{(2)}(t; \mu, \delta_z, \nu_2) \right) - \Theta_i(t; \mu, z, \nu_2), \quad t \in [0, T],
\]
where \( \Theta_i(\cdot; \mu, z, \nu_2) \in L^\infty([0, T], (W^{3, \infty}(\mathbb{T}^d))') \) satisfies the Cauchy problem (for which the existence of solution follows again from Theorem 3.1)

\[
\begin{align*}
\partial_t \Theta_i(t; \mu, z, \nu_2) &= -L_{m(\mu)} \Theta_i(t; \mu, z, \nu_2) + \text{div} \left( m^{(1)}(t; \mu, \nu_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(d_i^{(1)}(t; \mu, z)) \right) \\
&\quad + \text{div} \left( d_i^{(1)}(t; \mu, z) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(m^{(1)}(t; \mu, \nu_2)) \right) \\
&\quad + \text{div} \left( m(t; \mu) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h), m^{(1)}(t; \mu, \nu_2)) \right) = 0, \quad t \in [0, T], \\
\Theta_i(0; \mu, z, \nu_2) &= 0.
\end{align*}
\]

Therefore, with the same notation as in (3.8), \( \Pi_i(\cdot; \mu, z, \nu_2, h) \in L^\infty([0, T], (W^{3, \infty}(\mathbb{T}^d))') \) satisfies

\[
\begin{align*}
\partial_t \Pi_i(t; \mu, z, \nu_2, h) &= -L_{m(\mu)} \Pi_i(t; \mu, z, \nu_2, h) + \text{div} \left( m^{(1)}(t; \mu, \nu_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h)) \right) \\
&\quad + \text{div} \left( \rho_i^{(1)}(t; \mu, z, h) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(m^{(1)}(t; \mu, \nu_2)) \right) \\
&\quad + \text{div} \left( m(t; \mu) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h), m^{(1)}(t; \mu, \nu_2)) \right) = 0, \quad t \in [0, T], \\
\Pi_i(0; \mu, z, \nu_2, h) &= 0.
\end{align*}
\]

By estimate (3.9), along with condition (Reg-b-(4, 2)) (which trivially implies (Reg-b-(3, 2))),

\[
\begin{align*}
\lim_{h \to 0} \sup_{t \in [0, T]} \left\| \text{div} \left[ m^{(1)}(t; \mu, \nu_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h)) \right] \right\|_{(3, \infty)'} &= \lim_{h \to 0} \sup_{t \in [0, T]} \sup_{\|\xi\|_{3, \infty} \leq 1} \left\langle \xi, \text{div} \left[ m^{(1)}(t; \mu, \nu_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h)) \right] \right\rangle \\
&= \lim_{h \to 0} \sup_{t \in [0, T]} \sup_{\|\xi\|_{3, \infty} \leq 1} \left\langle \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h)) \cdot \nabla \xi, m^{(1)}(t; \mu, \nu_2) \right\rangle = 0.
\end{align*}
\]

Similarly, one can show that

\[
\lim_{h \to 0} \sup_{t \in [0, T]} \left\| \text{div} \left[ m(t; \mu) \frac{\delta^2 b}{\delta m^2}(\cdot, m(t; \mu))(\rho_i^{(1)}(t; \mu, z, h), m^{(1)}(t; \mu, \nu_2)) \right] \right\|_{(3, \infty)'} = 0.
\]

We then conclude by (3.12), (3.13) and Theorem 3.1 that

\[
\lim_{h \to 0} \sup_{t \in [0, T]} \| \Pi_i(t; \mu, z, \nu_2, h) \|_{(3, \infty)'} = 0.
\]

Consequently, by repeating the same argument as (3.10), it follows that

\[
(\partial_{z_1}) \frac{\delta \Phi}{\delta m}(m(t; \mu)) \left( m^{(2)}(t; \mu, \delta_{z_1}, \delta_{z_2}) \right) = \frac{\delta \Phi}{\delta m}(m(t; \mu)) \left( \Theta_i(t; \mu, z_1, \delta_{z_2}) \right).
\]

By repeating the same analysis on the variable \( z_2 \), the proof is complete.

We conclude this subsection by stating our main representation formula.
Theorem 3.5. Assume (Reg-$\Phi$-(4,2)), (Lip-$\Phi$-(3,2)) and (Reg-$\Phi$-(4,3)). Then for every $i,j \in \{1, \ldots, d\}$, $(\partial_{z_2}^2)_{(\partial_{z_1})_j} \frac{\partial^2 U}{\partial m^2}(t,\mu)_{(z_1,z_2)}$ exists and for every $\mu \in P_{2}(\mathbb{R}^d)$, $(\partial_{z_2}^2)_{(\partial_{z_1})_j} \frac{\partial^2 U}{\partial m^2}(t,\mu)_{(z_1,z_2)}$ is uniformly bounded in $t, z_1, z_2$ and Lipschitz continuous in $z_1$ and $z_2$, uniformly in time in segments. Moreover, it can be represented by

\[
(\partial_{z_2}^2)_{(\partial_{z_1})_j} \frac{\partial^2 U}{\partial m^2}(t,\mu)_{(z_1,z_2)} = \frac{\partial^2 \Phi}{\partial m^2}(m(t;\mu)) \left(d^{(1)}_{1,j}(t;\mu,z_1),d^{(1)}_{2,j}(t;\mu,z_2)\right) + \frac{\partial \Phi}{\partial m}(m(t;\mu)) \left(d^{(2)}_{1,j}(t;\mu,z_1,z_2)\right),
\]

(3.14)

where $d^{(1)}_{i,j}(\cdot;\mu,z)$ is defined in (3.6) and $d^{(2)}_{i,j}(\cdot;\mu,z_1,z_2)$ is defined in (3.11).

Proof. Existence of the derivative in the left-hand side of (3.14) is a consequence of Proposition 2.1. Formula (3.14) follows from assumptions (Reg-$\Phi$-(4,2)), (Lip-$\Phi$-(3,2)) and (Reg-$\Phi$-(4,3)), by combining Theorem 3.3 and Theorem 3.4.

3.2. From ergodic estimates on the tangent processes to uniform propagation of chaos. As suggested in the previous subsection, the ergodic properties of the operators $(L_m(t;\mu))_{\mu \in P(\mathbb{T}^d)}$ play a key role in the long-term behaviour of the various tangent processes appearing in Theorems 3.2 and 3.5. Our goal here is to clarify those ergodic properties. For $b$ satisfying (Reg-$\Phi$-(4,2)) and (Lip-$\Phi$-(3,2)) as before, we define the following two assumptions.

(Erg-$(K,\gamma,\alpha)$): For $K \geq 0$, $\gamma \geq 0$, $\alpha \in [0,2]$ and $\mu \in P(\mathbb{T}^d)$, we say that $L_m(t;\mu)$ in (3.1) satisfies (Erg-$(K,\gamma,\alpha)$) if there exist two constants $C, \lambda \geq 0$, only depending on $b, K, \gamma$ and $\alpha$, with $\lambda$ being required to be strictly positive if $\gamma > 0$, such that, for any $k \in \{0,1,2\}$, with $k \geq \alpha$, $q_0 \in (W^{k,\infty}(\mathbb{T}^d))^\prime$, with $\langle q_0, 1 \rangle = 0$ and $r \in L^\infty([0,\infty), (W^{k,\infty}(\mathbb{T}^d))^\prime)$ satisfying

\[
\begin{align*}
\left\{ (r(t),1) = 0 \\
\|r(t)\|_{-k,\infty} \leq Ke^{-\gamma t},
\end{align*}
\]

the unique solution in $L^\infty([0,T], (W^{k,\infty}(\mathbb{T}^d))^\prime)$ of the Cauchy problem

\[
\begin{align*}
\partial_t q(t) - L_m(t;\mu)q(t) - r(t) &= 0, & t \geq 0, \\
q(0) &= q_0,
\end{align*}
\]

(3.16)

satisfies, for any $\mu \in P(\mathbb{T}^d)$,

\[
\|q(t)\|_{-k,\infty} \leq C \frac{1}{1 + \lambda t^{\alpha/2}} e^{-\lambda t} \max \left\{ 1, \|q_0\|_{-k,\infty} \right\},
\]

(3.17)

(Erg): We say that $b$ satisfies (Erg) if, for any choice of $K \geq 0$, $\gamma > 0$ and $\alpha \in [0,2)$, $L_m(t;\mu)$ satisfies (Erg-$(K,\gamma,\alpha)$) for any $\mu \in P(\mathbb{T}^d)$, with the constants $C$ and $\lambda$ in (Erg-$(K,\gamma,\alpha)$) being uniform with respect to $\mu \in P(\mathbb{T}^d)$.

Proposition 3.6. Assume $b$ satisfies (Reg-$\Phi$-(4,2)), (Lip-$\Phi$-(3,2)) and (Erg). Let $d^{(1)}_{i}(\cdot;\mu,z)$ be defined by (3.6), $m^{(1)}(\cdot;\mu,\nu)$ be defined by (3.4). Then, for any $\alpha \in [0,1]$, $z \in \mathbb{T}^d$ and $\mu, \nu \in P(\mathbb{T}^d)$, $d^{(1)}_{i}(\cdot;\mu,z)$ and
\( m^{(1)}(\cdot; \mu, \nu) \) both satisfy
\[
\sup_{\mu, \nu \in P(T^d)} \sup_{t \geq 0} \| m^{(1)}(t; \mu, \nu) \|_{(0, \infty)'} \leq C^{(1)} e^{-\lambda^{(1)} t},
\]
\[
\sup_{z \in T^d} \sup_{\mu \in P(T^d)} \| d_i^{(1)}(t; \mu, z) \|_{(1-\alpha, \infty)'} \leq \frac{C^{(1)}}{1 + t^{\alpha/2}} e^{-\lambda^{(1)} t}, \quad t > 0,
\]
for some constants \( C^{(1)}, \lambda^{(1)} > 0 \), only depending on \( b, \alpha \) and the values of \( C \) and \( \lambda \) in \((\text{Erg}-(0,1,0))\) and \((\text{Erg}-(0,1,\alpha))\).

**Proof.** This result is immediate from \((\text{Erg})\) applied to \((3.4)\) (with \((k, \alpha)\) therein given by \((0, 0)\)) and \((3.6)\) (with \((k, \alpha)\) therein given by \((1, \alpha)\)). \(\square\)

**Proposition 3.7.** Assume \( b \) satisfies \((\text{Reg}-(4, 2))\), \((\text{Lip}-(3, 2))\) and \((\text{Erg})\). Let \( d_i^{(1)}(\cdot; \mu, z) \) be defined by \((3.6)\), \( m^{(1)}(\cdot; \mu, \nu) \) be defined by \((3.4)\). Then, for any \( \alpha \in [0, 2] \), \( z_1, z_2 \in T^d \) and \( \mu, \nu_1, \nu_2 \in P(T^d) \), \( d_{i,j}^{(2)}(\cdot; \mu, z_1, z_2) \) and \( m^{(2)}(\cdot; \mu, \nu_1, \nu_2) \) both satisfy
\[
\sup_{\mu, \nu_1, \nu_2 \in P(T^d)} \sup_{t \geq 0} \| m^{(2)}(t; \mu, \nu_1, \nu_2) \|_{(0, \infty)'} \leq C^{(2)} e^{-\lambda^{(2)} t},
\]
\[
\sup_{z_1, z_2 \in T^d} \sup_{\mu, \nu_1, \nu_2 \in P(T^d)} \| d_{i,j}^{(2)}(t; \mu, z_1, z_2) \|_{(2-\alpha, \infty)'} \leq \frac{C^{(2)}}{1 + t^{\alpha/2}} e^{-\lambda^{(2)} t}, \quad t > 0,
\]
where \( C^{(2)}, \lambda^{(2)} > 0 \) depend on \( b, \alpha \) and the values of \( C \) and \( \lambda \) in \((\text{Erg}-(K, \gamma, 0))\) and \((\text{Erg}-(K, \gamma, \alpha))\), when \( K \) and \( \gamma \) therein are some constants that depend explicitly on \( b, \alpha, C^{(1)} \) and \( \lambda^{(1)} \).

**Proof.** We shall establish the bounds for \( d_{i,j}^{(2)}(\cdot; \mu, z_1, z_2) \) by using \((\text{Erg})\) with \( k = 2 \) therein. The bound for \( m^{(2)}(\cdot; \mu, \nu_1, \nu_2) \) can be proven in a similar way. We notice from Proposition 3.6 (with \( \alpha = 0 \)) that
\[
\sup_{z_1, z_2 \in T^d} \sup_{\mu \in P(T^d)} \| - \text{div} \left[ d_i^{(1)}(t; \mu, z_2) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(d_j^{(1)}(t; \mu, z_1)) \right] \|_{(2, \infty)'} \leq C_2 e^{-2\lambda t},
\]
\(\text{(3.18)}\)
for a possibly new value of the constant \( C_2 \) appearing in the statement of Proposition 3.6. Similarly,
\[
\sup_{z_1, z_2 \in T^d} \sup_{\mu \in P(T^d)} \left\| - \text{div} \left[ m(t; \mu) \frac{\delta^2 b}{\delta m^2}(\cdot, m(t; \mu))(d_i^{(1)}(t; \mu, z_1), d_j^{(1)}(t; \mu, z_2)) \right. \right. \quad \text{div} \left[ d_i^{(1)}(t; \mu, z_1) \frac{\delta b}{\delta m}(\cdot, m(t; \mu))(d_j^{(1)}(t; \mu, z_2)) \right] \right\|_{(2, \infty)'} \leq C_2 e^{-2\lambda t}, \quad t > 0.
\]
\(\text{(3.19)}\)
Finally, note that
\[
\left\langle \mathbb{1}, d_{i,j}^{(2)}(0; \mu, z_1, z_2) \right\rangle = 0.
\]
\(\text{(3.20)}\)
Therefore, by \((3.18), (3.19), (3.20)\) and \((\text{Erg})\) (note that the latter clearly satisfies \( r(t), \mathbb{1} = 0 \) with an obvious choice for \( r \) therein), the result follows thanks to the shape of \((3.11)\). \(\square\)
We now return to the original problem of the weak error estimate between particle system \((1.1)\) and its mean-field limiting equation \((1.3)\). We are now in a position to prove a meta-form for our result (concrete examples being given next).

Theorem 3.8. Assume that \(b\) satisfies \((\text{Reg-}b-(4,2))\), \((\text{Lip-}b-(3,2))\) and \((\text{Erg})\) and that \(\Phi\) satisfies \((\text{Reg-} \Phi-(\alpha,2))\) for some \(\alpha \in (0,1]\). Then, there exists a constant \(C > 0\) such that, for any \(\mu_{\text{init}} \in \mathcal{P}(\mathbb{T}^d)\) and any \(N \geq 1\),

\[
\sup_{t \geq 0} \left| \mathbb{E}[\Phi(\mu_i^n)] - \Phi(\mathcal{L}(X_i)) \right| \leq \frac{C}{N},
\]

(3.21)

Proof. First Step. We first assume that \(\Phi\) satisfies \((\text{Reg-} \Phi-(4,3))\). The proof relies on Lemma 2.2. We address separately each of the two terms in the right-hand side of (2.11). Firstly, by Proposition 3.7, it is clear that

\[
\sup_{t \geq 0} \left| \int_0^t \int_0^1 \mathbb{E} \left[ \frac{\delta^2 \mathcal{U}}{\delta m^2}(t,m_{i,j}^N) \right] \right| dsdt \leq C,
\]

for a constant \(C\) that is independent of \(N\) and \(\mu\). Importantly, \(C\) only depends on \(\Phi\) through the \(L^\infty\) and Hölder bounds in \((\text{Reg-} \Phi-(\alpha,2))\). Next, we bound the second term in the right-hand side of (2.11). By (3.14),

\[
(\partial_{z_2})_j(\partial_{z_1})_i \frac{\delta^2 \mathcal{U}}{\delta m^2}(t,\mu)(z_1,z_2) = \frac{\delta^2 \Phi}{\delta m^2}(m(t;\mu)) \left( d_{i,j}^{(1)}(t;\mu, z_1), d_{j,i}^{(1)}(t;\mu, z_2) \right) + \frac{\delta \Phi}{\delta m}(m(t;\mu)) \left( d_{i,j}^{(2)}(t;\mu, z_1), d_{j,i}^{(2)}(t;\mu, z_2) \right).
\]

It follows from Propositions 3.6 (with \(1 − \alpha/2\) instead of \(\alpha\)) and 3.7 (with \(2 − \alpha/2\) instead of \(\alpha\)) that

\[
\sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} \sup_{z_1 \in \mathbb{T}^d} \left\| d_{i,j}^{(1)}(t;\mu, z_1) \right\|_{(\alpha/2, \infty)^1} \leq \frac{C}{1 \wedge t^{(1-\alpha/2)/2}} e^{-\lambda t},
\]

and

\[
\sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} \sup_{z_1,z_2 \in \mathbb{T}^d} \left\| d_{i,j}^{(2)}(t;\mu, z_1, z_2) \right\|_{(\alpha/2, \infty)^1} \leq \frac{C}{1 \wedge t^{(2-\alpha/2)/2}} e^{-\lambda t},
\]

for some constants \(C, \lambda > 0\) that are independent of \(N\) and \(\mu\). Importantly, using the fact the distributions right above are evaluated in \((W^{\alpha/2, \infty}(\mathbb{T}^d))^\prime\), we have that \(C\) only depends on \(\Phi\) through the \(L^\infty\) and Hölder bounds in \((\text{Reg-} \Phi-(\alpha,2))\), see (2.16).

This completes the proof when \(\Phi\) satisfies \((\text{Reg-} \Phi-(4,3))\), since we then have

\[
\sup_{t \geq 0} \left| \int_0^t \int_{\mathbb{T}^d} \left( \partial_{y_2} \partial_{y_1} \frac{\delta^2 \mathcal{U}}{\delta m^2}(t,\mu_s^N)(z, z) \right) \mu_s^N(dz) \right| ds \leq \sup_{t \geq 0} \left| \int_0^t \int_{\mathbb{T}^d} \left( \partial_{y_2} \partial_{y_1} \frac{\delta^2 \mathcal{U}}{\delta m^2}(t,\mu_s^N)(z, z) \right) \mu_s^N(dz) \right| ds < \infty.
\]

Second Step. We now prove the result under the weaker assumption \((\text{Reg-} \Phi-(\alpha,2))\). The proof consists in a mollification argument. By Theorem 5.1 in the appendix (to the best of our knowledge, the result is new), we can find a sequence \((\Phi_n)_{n \geq 1}\) that converges uniformly to \(\Phi\), such that each \(\Phi_n\) satisfies \((\text{Reg-} \Phi-(4,3))\) (for constants that depend on \(n\)) and all the functions \(\Phi_n\) satisfy \((\text{Reg-} \Phi-(\alpha,2))\) with \(L^\infty\) and Hölder bounds therein independent of \(n\). Applying (3.21) to each \(\Phi_n\) and then letting \(n\) tend to \(\infty\), we complete the proof.
Part of our effort in the rest of the section is to provide explicit conditions under which \((\text{Erg})\) is satisfied. Meanwhile, we find it useful to notice that \((\text{Erg})\) implies the existence and uniqueness of a stationary measure to the underlying McKean-Vlasov equation:

**Proposition 3.9.** Under the assumption of Theorem 3.8, the McKean-Vlasov equation (1.3) has a unique invariant measure \(\nu_\infty\) and it is exponentially stable, in the sense that there exist two constants \(C, \lambda > 0\) such that, for any \(\mu \in \mathcal{P}(\mathbb{T}^d)\),

\[
\text{dist}_{TV}(m(t; \mu), \nu_\infty) \leq Ce^{-\lambda t}, \quad t \geq 0.
\]

**Proof.** For a smooth test function \(f\) on the torus, we choose \(\Phi(m) = \langle f, m \rangle\) in the statement of (3.2). We then observe that, for any two measures \(\mu_1, \mu_2 \in \mathcal{P}(\mathbb{T}^d)\),

\[
|\langle f, m(t; \mu_1) - m(t; \mu_2) \rangle| = \left| \int_0^1 \langle f, m^{(1)}(t; \lambda \mu_1 + (1 - \lambda)\mu_2) \rangle d\lambda \right|.
\]

By Proposition 3.6, we deduce that

\[
|\langle f, m(t; \mu_1) - m(t; \mu_2) \rangle| \leq C(1)\|f\|_{\infty}e^{-\lambda(1)t}.
\]  (3.23)

By choosing \(\mu_1\) and \(\mu_2\) as two candidates for being an invariant measure, this shows that an invariant measure (if it exists) must be unique.

Existence follows by choosing \(\mu_1 = \mu\) and then \(\mu_2 = m(s; \mu)\) in (3.23). By the flow property of McKean-Vlasov dynamics, we have \(m(t; m(s; \mu)) = m(t + s; \mu)\), from which we deduce that

\[
|\langle f, m(t + s; \mu) - m(t; \mu) \rangle| \leq C(1)\|f\|_{\infty}e^{-\lambda(1)t}.
\]

By completeness of \(\mathcal{P}(\mathbb{T}^d)\) (equipped with the total variation distance), we deduce that \((m(t; \mu))_{t \geq 0}\) has a limit. We call it \(\nu_\infty\). Writing the above in the form

\[
|\langle f, m(s; m(t; \mu)) - m(t; \mu) \rangle| \leq C_1\|f\|_{\infty}e^{-\lambda t},
\]

and letting \(t\) tend to \(\infty\), it is easy to deduce that \(m(s; \nu_\infty) = \nu_\infty\). Finally, by choosing \(\mu_1 = \nu_\infty\) in (3.23), we get that the invariant measure is indeed exponentially stable. \(\square\)

By combining the last two statements with Proposition 2.4, we get the following result for the strong error.

**Corollary 3.10.** Under the assumption of Theorem 3.8, we call the unique invariant measure of (1.3) \(\nu_\infty\). Then, there exist two constants \(C, \lambda > 0\) such that, for any \(N \geq 1\),

\[
\mathbb{E} \left[ \|\mu_t^N - \nu_\infty\|_{-d+\alpha/2,2}^2 \right] \leq C \left( \frac{1}{N} + e^{-\lambda t} \right), \quad t \geq 0.
\]

### 3.3 Ergodic Sobolev estimates.

In view of (3.14), the main aim of this subsection is to introduce the main tools for a uniform control over time for \(d_i^{(1)}(\cdot; \mu, z)\) and \(d_i^{(2)}(\cdot; \mu, z_1, z_2)\), as well as \(m^{(1)}(\cdot; \mu, \delta_{z_2})\) and \(m^{(2)}(\cdot; \mu, \delta_{z_1}, \delta_{z_2})\). We hence provide estimates on backward PDEs in the spaces \(W^{s,\infty}(\mathbb{T}^d)\), for \(s \in [1,4]\), which we use next in some duality arguments.
**Theorem 3.11.** Let $t > 0$, $\xi \in W^{1,\infty}(\mathbb{T}^d)$ and $V$ be a vector field from $[0, t] \times \mathbb{T}^d$ into $\mathbb{R}^d$ that is Hölder continuous in time, uniformly in space, and Lipschitz in space, uniformly in time (i.e., $\sup_{s \in [0, t]} \|V(s, \cdot)\|_{1, \infty} < \infty$). Then the Cauchy problem

$$
\begin{cases}
\partial_s w(s, \cdot) + \frac{1}{2} \Delta_x w(s, \cdot) + V(s, \cdot) \cdot \nabla_x w(s, \cdot) = 0, & s \in [0, t], \\
w(t, \cdot) = \xi,
\end{cases}
$$

(3.24)

admits a unique solution $(w(s, \cdot))_{0 \leq s \leq t}$ that is continuous on $[0, t] \times \mathbb{T}^d$ and classical on $[0, t] \times \mathbb{T}^d$. Moreover, there are constants $C, \lambda > 0$ (only depending on $\sup_{s \in [0, t]} \|V(s, \cdot)\|_{1, \infty}$ and hence independent of $t$) such that

(i) \quad \|w(s, \cdot) - \int_{\mathbb{T}^d} w(s, y) \, dy\|_\infty \leq Ce^{-\lambda(t-s)}, \quad \forall s \in [0, t],

(3.25)

(ii) for any $\alpha, \beta \in [0, 1]$ (also allowing $C$ and $\lambda$ to depend on $\alpha, \beta$),

\[ \|\nabla^\beta w(s, \cdot)\|_{\beta, \infty} \leq \frac{C}{1 \wedge (t-s)^{(\alpha+\beta)/2}} \|\xi\|_{1-\alpha, \infty} e^{-\lambda(t-s)}, \quad \forall s \in [0, t]. \]

(3.26)

Now, suppose that, for $k \in \{2, 3\}$, $\xi \in W^{k, \infty}(\mathbb{T}^d)$ and $\sup_{s \in [0, t]} \|V(s, \cdot)\|_{k, \infty} < \infty$, the derivative $\nabla^\ell V$, for $\ell \in \{1, k-1\}$ is Hölder continuous in time, uniformly in space. Then, for any $\alpha, \beta \in [0, 1]$, there exist constants $C, \lambda > 0$, depending only on $\alpha, \beta$ and $\sup_{s \in [0, t]} \|V(s, \cdot)\|_{k, \infty}$, such that

(iii) \quad \|\nabla^k_x w(s, \cdot)\|_{\beta, \infty} \leq \frac{C}{1 \wedge (t-s)^{(\alpha+\beta)/2}} \|\xi\|_{k-\alpha, \infty} e^{-\lambda(t-s)}, \quad \forall s \in [0, t].

(3.27)

**Proof.** First Step. The well-posedness of (3.24) in the classical sense is a standard fact under the time-space Hölder regularity of $V$ (see for instance [36, Thm. 5, Chap. 3]). Estimate (i) is a direct consequence of [17, Lem. 7.4], since the latter is precisely established in $L^\infty$.

Estimate (ii) is more subtle. We denote by $(p(s, s', x, x'))_{0 \leq s < s', x, x' \in \mathbb{T}^d}$ the transition density of the operator driving the equation (3.24). We know from [39, Thm 3.5] (also see [36, Thm. 11, Chap. 1] and [36, (4.19), Chap. 9], but with a less precise discussion on the dependence of the constant $C$ below upon the input of the equation) that there exists a bounded density $g$ on the torus such that, for any $0 \leq s < s'$ with $s' - s \leq 1$,

$$
|\nabla^j_x p(s, s', x, x')| \leq C(s' - s)^{-(d+j)/2} g \left( \frac{x' - x}{C(s' - s)^{1/2}} \right),
$$

(3.28)

for $j \in \{1, 2\}$ and for a constant $C \geq 1$ only depending on $d$ and on the time-space Hölder norm of $V$ on $[0, t]$. We observe that, for $s \leq t$ and $x \in \mathbb{T}^d$,

$$
w(s, x) = \int_{\mathbb{T}^d} p(s, x, (s + 1) \wedge t, y) w((s + 1) \wedge t, y) \, dy,
$$

which follows directly from the definition of the transition density. The above may be rewritten in the form:

$$
w(s, x) = \int_{\mathbb{T}^d} p(s, x, (s + 1) \wedge t, y) \left( w((s + 1) \wedge t, y) - c \right) \, dy + c, \quad \text{for } c = \int_{\mathbb{T}^d} w((s + 1) \wedge t, z) \, dz.
$$

(3.29)
Taking \( j \) derivatives in \( x \), we deduce from \((i)\) that, for \( s \leq t - 1 \),
\[
|\nabla^j_x w(s, x)| \leq \int_{T^d} |\nabla^j_y p(s, x, s + 1, y)| \left| w(s + 1, y) - \int_{T^d} w(s + 1, z)dz \right| dy
\]
\leq C\|\xi\|_\infty e^{-\lambda(t-s)} \int_{T^d} |\nabla^j_y p(s, x, s + 1, y)| dy.
\tag{3.30}

We now apply \((3.28)\) and obtain that
\[
|\nabla^j_x w(s, x)| \leq C\|\xi\|_\infty e^{-\lambda(t-s)},
\tag{3.31}
\]
for \( j \in \{1, 2\} \) and \( s + 1 \leq t \) (apply \((3.28)\) with \( s' = s + 1 \)). This implies \((3.26)\) when \( t - s \geq 1 \) and \((\alpha, \beta) \in \{(1, 0), (1, 1)\} \). Obviously, this implies \((3.26)\) when \( t - s \geq 1 \) for any other value of \((\alpha, \beta) \in [0, 1]^2\).

Now, for any \( \alpha \in [0, 1] \) and for \( t - s \leq 1 \), we take \( j \) derivatives in \((3.29)\), with \( c \) therein being equal to \( \xi(x_0) \) for a fixed \( x_0 \). Taking \( x_0 = x \) in the resulting identity and using \((3.28)\), we get
\[
\|\nabla^j_x w(s, \cdot)\|_\infty \leq \frac{C}{1 \wedge (t-s)(\alpha+j-1)/2} \|\xi\|_{1-\alpha, \infty},
\tag{3.32}
\]
for \( j \in \{1, 2\} \). By the interpolation estimate \((2.12)\) with \( a = 1, \eta = \beta \) and \( \gamma = 1 \), we get \((3.26)\) for \( t - s \leq 1 \) and for any \((\alpha, \beta) \in [0, 1]^2\).

**Second Step.** It now remains to study \((3.27)\) under the additional assumption that \( \xi \in W^{k, \infty}(T^d) \) and that \( V \) has time-space Hölder continuous derivatives up to order \( k - 1 \), for \( k \in \{2, 3\} \). By \[36, \text{Thm. 10, Chap. 3}\], \( w \) has space derivatives up to order \( k + 1 \).

We start with the case \( k = 2 \). Taking the derivative with respect to \( x \) in the equation satisfied by \( w \), we get that \( \partial_x w \) solves the equation
\[
\partial_t \partial_x w + \frac{1}{2} \Delta_x \partial_x w + V \cdot \nabla_x \partial_x w + \partial_x V \cdot \nabla_x w = 0,
\tag{3.33}
\]
with \( \partial_x \xi \) as value at \( t \). Accordingly, we have the following representation:
\[
\partial_x w(s, x) = \int_{T^d} \partial_s \xi(y)p(s, x, (s + 1) \wedge t, y) dy + \int_s^{(s+1)/t} \int_{T^d} (\partial_x V \cdot \nabla_x w)(r, y)p(s, x, r, y) d\nu dr dy.
\]
\[
:= T_1(s, x) + T_2(s, x).
\]

Obviously, \( T_1 \) leads to the same bound as in the first step, up to the fact that it is for \( \partial_x w \) and that \( \|\xi\|_{1-\alpha, \infty} \) has to be replaced by \( \|\xi\|_{2, \infty} \). If there were not the second term \( T_2 \), we would directly obtain \((3.27)\).

Therefore, the only difficulty is \( T_2 \). In order to handle it, we recall that \( \partial_x V \) is bounded and that \( \nabla_x w \) is already known to decay exponentially fast with time (see \((3.26)\) with \( \alpha = \beta = 0 \)). Subsequently, by taking the derivative of \( T_2 \) with respect to \( x \), then applying \((3.28)\) with \( j = 1 \) and noticing that the resulting singularity is integrable, we deduce that \( \|\nabla_x T_2(s, \cdot)\|_\infty \leq C\|\xi\|_{1, \infty} \exp(-\lambda(t-s)) \), for \( C \) only depending on the Hölder norms of \( V \) and \( \nabla_x V \). As for \( T_2 \), by subtracting \((\partial_x V \cdot \nabla_x w)(r, x_0)p(s, r, x, y) \) in the integrand for a fixed \( x_0 \), applying \((3.28)\) with \( j = 2 \) and then taking \( x_0 = x \), we get
\[
\|\nabla^2_x T_2(s, \cdot)\|_\infty \leq \int_s^{(s+1)/t} C(r-s)^{-1} \left((r-s)^{\gamma/2} + \frac{(r-s)^{1/2}}{1 \wedge (t-r)^{1/2}}\right)\|\xi\|_{1, \infty} e^{-\lambda(t-r)} dr.
\]
where $\gamma$ is used here to denote the Hölder exponent of $\nabla_x V$ and the second term inside the parentheses comes from (3.26) with $\alpha = 0$ and $\beta = 1$. Hence, we obtain that
\[
\|\nabla^2_T T_2(s, \cdot)\|_\infty \leq C\|\xi\|_{1,\infty}e^{-\lambda(t-s)}.
\]
Combining with the estimates for $T_1$, we get (3.27) for $k = 2$. The case $k = 3$ is handled in the same way.

\[\square\]

3.4. Small McKean-Vlasov interaction.

**Theorem 3.12.** Let $b$ satisfy (Reg-$b$-$\langle 4, 2 \rangle$) and (Lip-$b$-$\langle 3, 2 \rangle$). Then, there exists $\epsilon_0 > 0$ (only depending on any bound for $\sup_{m \in P(\mathbb{T}^d)} \|b(\cdot, m)\|_{1,\infty}$) such that $b$ satisfies (Erg) if
\[
\sup_{m \in P(\mathbb{T}^d)} \left\| \frac{\delta b_i}{\delta m}(\cdot) \right\|_{2,\infty} < \epsilon_0.
\]
Accordingly, the conclusions of Theorem 3.8 and Corollary 3.10 hold in that case.

We explain in Remark 3.13 below how to improve the definition of threshold (3.34).

**Proof. First Step.** For $K, \gamma, \alpha$ as in (Erg-$\langle K, \gamma, \alpha \rangle$), for $k \in \{0, 1, 2\}$, with $k \geq \alpha$, for $r$ as in (3.15) and for an initial condition $q_0 \in (W^{k, \infty}(\mathbb{T}^d))^r$, with $\langle q_0, \mathbb{1} \rangle = 0$, we consider the solution $q$ to (3.16) within the space $\cap_{t \geq 0} L^\infty([0, T], (W^{k, \infty}(\mathbb{T}^d))^r)$.

**Second Step.** We check that $q$ belongs to $L^\infty([0, \infty), (W^{k, \infty}(\mathbb{T}^d))^r)$. We adopt a duality approach similar to the proof of Lemma 7.6 in [17]. For a smooth function $\xi$ on $\mathbb{T}^d$, we consider the following Cauchy problem
\[
\begin{align*}
\partial_t w + \frac{1}{2} \Delta_x w + b(x, m(s; \mu)) \cdot \nabla_x w &= 0, \quad (s, x) \in [0, t] \times \mathbb{T}^d, \\
w(t, x) &= \xi(x).
\end{align*}
\]
We check that the above problem fits the assumption of Theorem 3.11. Since $b$ is bounded, the path $(m(s; \mu))_{0 \leq s \leq t}$ is $1/2$-Hölder continuous in $s$ with respect to $\mathcal{W}_1$. By (Lip-$b$-$\langle 3, 2 \rangle$), the transport coefficient in (3.35) is 3-times differentiable with time-space continuous derivatives. Theorem 3.11 hence guarantees the existence of a solution to (3.35) with derivatives up to order 5 in space (since $\xi$ itself is smooth). Therefore, we can expand the duality product $\langle w(s, \cdot), q(s) \rangle$ (recalling (3.15) for the regularity of $r$). By (3.1) and (3.2),
\[
\begin{align*}
\langle \xi, q(t) \rangle &= \left\langle w(0, \cdot) - \int_{\mathbb{T}^d} w(0, y)dy, q(0) \rightangle + \int_0^t \left\langle w(s, \cdot) - \int_{\mathbb{T}^d} w(s, y)dy, r(s) \right\rangle ds \\
&\hspace{1cm} + \int_0^t \int_{\mathbb{T}^d} \left\langle \frac{\delta b}{\delta m}(x, m(s, \mu)) (\cdot) \cdot \nabla_x w(s, x), q(s) \right\rangle m(s, \mu)(dx) ds \\
&=: T_1 + T_2 + T_3.
\end{align*}
\]
By estimates (3.25) (which suffices if $k = 0$ and $\alpha = 0$), (3.26) (the latter with $(\alpha, \beta)$ therein given by: $(\alpha, 0)$ if $k = 1$ and $\alpha \in [0, 1]$; $(\alpha - 1, 1)$ if $k = 2$ and $\alpha \in [1, 2]$) and (3.27) (with $(\alpha, \beta)$ therein given by $(\alpha, 0)$ if $k = 2$ and $\alpha \in [0, 1]$),
\[
|T_1| \leq C_k(\alpha, b) \frac{1}{1 + t^{\alpha/2}} \|\xi\|_{k-\alpha, \infty} \|q_0\|_{(k, \infty)^r},
\]
(3.37)
where $C_k(\alpha, b)$ only depends on $\alpha$ and $\sup_{m \in P(\mathbb{T}^d)} \|b(\cdot, m)\|_{k, \infty}$. Letting $\epsilon := \sup_{m \in P(\mathbb{T}^d)} \|\delta b_i / \delta m(\cdot, m)(\cdot)\|_{k, \infty}$, we have

$$|T_3| \leq \int_0^t \sup_{x \in \mathbb{T}^d} \left| \frac{\delta b}{\delta m}(x, m(s, \mu)) \cdot \nabla_x w(s, x, q(s)) \right| ds \leq \sum_{i=1}^d \int_0^t \|\partial_{x_i} w(s, \cdot)\|_{\infty} \|q(s)\|_{(k, \infty)'} ds.$$  

Subsequently, by (3.26) (with $(\alpha, \beta)$ therein given by: $(1, 0)$ if $k = 0$ and $\alpha = 0$; $(\alpha, 0)$ if $k \in \{1, 2\} \text{ and } \alpha \in [0, 1)$; $(\alpha - 1, 0)$ if $k = 2 \text{ and } \alpha \in [1, 2)$), we have (with $C_1(\alpha, b)$ depending on $\alpha$ and $\sup_{m \in P(\mathbb{T}^d)} \|b(\cdot, m)\|_{1, \infty}$):

$$|T_3| \leq C_1(\alpha, b) \|\xi\|_{k-\alpha, \infty} \epsilon \left( \sup_{0 \leq s \leq t} \left\{ \|q(s)\|_{(k, \infty)'} e^{-\lambda(t-s)/2} \right\} \right) \int_0^t \frac{e^{-\lambda(t-s)/2}}{1 \wedge (t-s)^{1/2}} ds \quad (3.38)$$

Similarly, by (3.25) (which suffices if $k = 0 \text{ and } \alpha = 0$), (3.26) (with $(\alpha, \beta)$ therein given by: $(\alpha, 0)$ if $k = 1 \text{ and } \alpha \in [0, 1)$; with $(\alpha - 1, 1)$ if $k = 2 \text{ and } \alpha \in [1, 2)$) and (3.27) (with $(\alpha, \beta)$ therein given by: $(\alpha, 0)$ if $k = 2 \text{ and } \alpha \in [0, 1)$), along with the fact that $r \in L^\infty([0, \infty), (W^{k, \infty}(\mathbb{T}^d))')$, we establish the bound

$$|T_2| \leq C_k(\alpha, b) \|\xi\|_{k-\alpha, \infty} \left( \sup_{s \geq 0} \left\{ \|r(s)\|_{(k, \infty)'} e^{-\lambda(t-s)/2} \right\} \right) \int_0^t \frac{e^{-\lambda(t-s)/2}}{1 \wedge (t-s)^{\alpha/2}} ds \quad (3.39)$$

By combining (3.36), (3.37), (3.38) and (3.39), we have

$$\|q(t)\|_{(k-\alpha, \infty)'} \leq C_k(\alpha, b) \frac{\|q_0\|_{(k, \infty)'} 1 \wedge t^{\alpha/2}}{1 - C_1(\alpha, b) \epsilon} + C_1(\alpha, b) \epsilon \left( \sup_{0 \leq s \leq t} \|q(s)\|_{(k, \infty)'} \right) + C_k(\alpha, b) \left( \sup_{s \geq 0} \|r(s)\|_{(k, \infty)'} \right). \quad (3.40)$$

Choosing $\alpha = 0$ and setting $\epsilon_0 = C_1(b)^{-1}$, with $C_k(b) = C_k(0, b)$, we deduce that, for any $\epsilon \in (0, \epsilon_0)$,

$$\sup_{s \geq 0} \|q(s)\|_{(k, \infty)'} \leq \frac{1}{1 - C_1(\alpha, b) \epsilon} \left[ C_k(b) \|q_0\|_{(k, \infty)'} + C_k(b) \left( \sup_{s \geq 0} \|r(s)\|_{(k, \infty)'} \right) \right]. \quad (3.41)$$

This shows that $q$ in $L^\infty([0, \infty), (W^{k, \infty}(\mathbb{T}^d))')$. Plugging the above bound in (3.40), we get the small time asymptotics in (3.17). This is also enough to get (3.17) with $\lambda = 0$ therein (which suffices to conclude the proof when $\gamma = 0$ in (3.15)).

In order to get the exponential decay when $\gamma > 0$ in (3.15), it suffices to apply (3.36) between times $1$ and $t$, for $t \geq 1$. Using the decay of $w$ given by Theorem 3.11 with the same rate $\lambda$ as therein, we get the following variant of (3.40):

$$\|q(t)\|_{(k-\alpha, \infty)'} \leq C_k(\alpha, b) \|q(1)\|_{(k, \infty)'} e^{-\lambda(t-1)} + C_1(\alpha, b) \epsilon \left( \sup_{1 \leq s \leq t} \left\{ \|q(s)\|_{(k, \infty)'} e^{-\lambda(t-s)/2} \right\} \right) + C_k(\alpha, b) \left( \sup_{1 \leq s \leq t} \left\{ \|r(s)\|_{(k, \infty)'} e^{-\lambda(t-s)/2} \right\} \right) \quad (3.42)$$

By inserting (3.40) and (3.41), we get that the first term in the right-hand side is bounded by $C \exp(-\lambda t)$, for a constant $C$ as in (Erg-$(K, \gamma, \alpha)$). As for the last term in the right-hand side, we know from the decay of
that is bound by $C \exp(-ct)$, for $C$ as in the statement and for a new constant $c$ only depending on $\gamma$ and $\lambda$. Without any loss of generality, we may assume that $c \leq \lambda/2$. We deduce at the end that

$$e^{ct} \| q(t) \|_{(k-\alpha, \infty)'} \leq C \max \left\{ 1, \| q_0 \|_{(k, \infty)'} \right\} + C_1(\alpha, b) \varepsilon \left( \sup_{1 \leq s \leq t} \left\{ \| q(s) \|_{(k, \infty)'} e^{c_s} \right\} \right).$$

Choosing first $\alpha = 0$, we obtain a bound for the last term in the right-hand side if $C_1(0, b)\varepsilon < 1$. In turn, this gives a bound for the left-hand side regardless of the value of $\alpha$ in the statement. The conclusion follows. \[ \square \]

**Remark 3.13.** A careful inspection of the proof would permit to improve the condition (3.34). In short, what really matters is that

$$C \sum_{i=1}^{d} \sup_{m \in \mathcal{P}(\mathbb{T}^d)} \left\| \delta b / \delta m (c, m) (c) \right\|_{1, \infty} < 1,$$

where $C$ and $\lambda$ are any constants that satisfy (3.26) whenever $\alpha = \beta = 0$ and the velocity field is chosen as

$$V(s, x) = b(x, m(s)),$$ (4.43)

for any flow $(m(s))_{s \geq 0}$ of probability measures solving the Fokker-Planck equation (1.4) for some initial condition $m(0) \in \mathcal{P}(\mathbb{T}^d)$. In fact, our proof does not directly show that the condition (4.43) is the right one. The main point is to notice that, in our proof, we used the $W^{k, \infty}(\mathbb{T}^d)$-norm of $\delta b / \delta m$ in the derivation of (3.38). When $k \in \{0, 1\}$, this is consistent with (4.43), but this is more demanding when $k = 2$ (whence the 2-Sobolev norm in (3.34)). This is precisely where some work is needed. The argument is divided into three steps:

1. For small $t$, the integral on the first line of (3.38) can be made small; accordingly, we may require $t$ instead of $\varepsilon$ to be small in (4.40) and hence get a bound for $\| q(t) \|_{(2, \infty)'}$ for $t$ less than some $t_0 > 0$ (even if $\varepsilon$ is large).
2. Once we have a bound for $\| q(t) \|_{(2, \infty)'}$ for $t \leq t_0$, (4.40) (with $\alpha = 1$) gives a bound for $\| q(t_0) \|_{(1, \infty)'}$ (regardless of the value of $\varepsilon$).
3. The last step is to rewrite (3.42), with $k = 1$ and $\alpha = 0$ and with time 1 replaced by time $t_0$. Importantly, when $k = 1$ and $\alpha = 0$, (3.38) holds with the simpler integral $\int_0^t \exp(-\lambda(t-s)/2)ds$, which is less than $2/\lambda$. This permits to replace $C_1(\alpha, b)$ by $C_1(\alpha, b)/\lambda$ in (3.42). Moreover, by (3.39) with $k = 2$ and $\alpha = 1$, we can replace $\| r(s) \|_{(1, \infty)'}$ in the right-hand side of (3.42) by $\| r(s) \|_{(2, \infty)'}$. This gives an exponential decay for $\| q(t) \|_{(1, \infty)'}$ when $k = 2$, when (4.43) holds true. The rest of the proof is unchanged and we get an exponential decay for $\| q(t) \|_{(1, \infty, \infty)'}$ when $\alpha \in (0, 1)$.

Now, it is interesting to see what (3.36) means when $\alpha = \beta = 0$ therein and for the same choice of velocity field as in (4.44). It may be reformulated as

$$\| \nabla x w(s, \cdot) \|_{0, \infty} \leq C \| \xi \|_{1, \infty} e^{-\lambda(t-s)}, \quad \forall s \in [0, t],$$ (4.45)

for any time $t \geq 0$ and any boundary condition $\xi$, it being understood that the constants $C$ and $\lambda$ are implicitly independent of the choice of the initial condition in (1.4). This leads us to the following interpretation of the constraint (4.43): the McKean-Vlasov interaction should not be strong enough with respect to the rate underpinning the exponential mixing properties of the free (i.e., with frozen McKean-Vlasov interaction) system. In this regard, it is worth observing that the hence chosen pair $(C, \lambda)$ in (4.45) is not required to satisfy the other assertions in the statement of Theorem 3.11. The mixing is hence studied within a precise
functional space. The latter observation permits in fact to draw a clear parallel with the framework addressed in the paper [2]. In (1.9) of that paper, the value of \( \lambda \) is given by \( \lambda_1 \) in the so-called assumption (H), which (reformulated in terms of \( V \) in (3.44)) would impose that

\[
(\nabla_x V)_{\text{sym}} \leq -\lambda_1 I_d,
\]

(3.46)

where, on the left-hand side, the index \( \text{sym} \) denotes the symmetric part of the matrix and, on the right-hand side, \( I_d \) denotes the \( d \)-dimensional identity matrix. Of course, \( \lambda_1 \) is (strictly) positive. Surely, assumption (3.46) is meaningful in the Euclidean setting (which is the case addressed in [2]), but does not make any sense on the torus. Still, it is worth noticing that, in the Euclidean case, condition (3.46) forces (3.45). The proof relies on a systematic analysis of the derivatives, with respect to the space variable \( x \), of the flow generated by the SDE

\[
dX_s^x = V(s, X_s^x)ds + dB_s, \quad s \in [0, t]; \quad X_0 = x.
\]

We feel that even a sketchy presentation of it would be useful for the reader. In short, it suffices to write

\[
d\nabla_x X_t^x = \partial_s V(X_t^x) \nabla_x X_t^x dt, \quad t \geq 0; \quad \nabla_x X_0^x = I_d,
\]

which yields (under (3.46))

\[
|\nabla_x X_t^x| \leq e^{-\lambda_1 t}, \quad t \geq 0.
\]

Back to the statement of Theorem 3.11, we then notice that

\[
|\nabla_x w(t, x)| = \| \mathbb{E} \left[ \nabla_x \xi(X_t^x) \nabla X_t^x \right] \| \leq e^{-\lambda_1 t}, \quad t \geq 0,
\]

(3.47)

which permits to recover (3.45).

Once again, the above computation is irrelevant in our periodic setting. However, this clearly demonstrates that the setting addressed in [2] fits within our own approach. Even more, the condition \( \lambda_1 > \|B[2]\|_{\text{spectral}} \) postulated in [2, (H)] (where \( B[2] \) is the gradient with respect to the second variable, \( \| \cdot \|_{\text{spectral}} \) is the spectral norm and \( B : \mathbb{T}^d \times \mathbb{T}^d \rightarrow \mathbb{R}^d \) is a new notation for the drift in [2] that avoids any confusion with our own \( b \)) is consistent with (3.43) by noticing that, for the same choice of mean-field term,

\[
b(x, m) = \int B(x, y)m(dy)
\]

as in [2], the linear functional derivative \( [\delta b/\delta m](m, x, y) \) is \( B(x, y) - b(x, m) \) (see Subsection 2.4). Accordingly, the condition \( \lambda_1 > \|B[2]\|_{\text{spectral}} \) becomes \( \lambda > \sup_{(x, y) \in \mathbb{T}^d} \sup_{m \in \mathcal{P}(\mathbb{T}^d)} \| \partial_y(\delta b/\delta m) \|_{\text{spectral}} \) in our notation.

Back to (3.43), (3.47) says that the constant \( C \) therein can be chosen as 1. Then, we see that the two constraints are really close, except for the fact that our choice of norms here is a bit stronger (which mostly comes from the fact that our approach is written out for more general forms of drifts \( b \)).

3.5. Mean-field term driven by an \( H \)-stable potential. The main purpose of this new subsection is to provide a class of examples for which the conclusion of Theorem 3.8 remains true even though the mean field interaction is not small with respect to the mixing properties of the free motion. We assume that

\[
b(x, m) = -\kappa \int_{\mathbb{T}^d} \nabla W(x - y)m(dy), \quad x \in \mathbb{T}^d, \quad m \in \mathcal{P}(\mathbb{T}^d),
\]

(3.48)
for a positive constant $\kappa$ and a sufficiently smooth potential $W : \mathbb{T}^d \to \mathbb{R}$ that is coordinate-wise even, i.e.

$$W(x_1, \cdots, -x_i, \cdots, x_d) = W(x_1, \cdots, x_i, \cdots, x_d), \quad (x_1, \cdots, x_d) \in \mathbb{T}^d.$$ 

This example has received a lot of attention in the literature. Below, we borrow several results from [21].

The very first point with coefficients $b$ given in terms of the interaction kernel $W$ as in (3.48) is to notice that, necessarily, the uniform distribution $\text{Leb}_{\mathbb{T}^d}$ (the Lebesgue measure on the torus) is an invariant measure. This follows from the simple fact that $b(x, \text{Leb}_{\mathbb{T}^d}) = 0$. Accordingly, the linearised operator $L_{\text{Leb}_{\mathbb{T}^d}}$ at $\text{Leb}_{\mathbb{T}^d}$ has the simpler form (see [21, Subsection 3.2]):

$$L_{\text{Leb}_{\mathbb{T}^d}}(\cdot) = \frac{1}{2} \Delta(\cdot) + \kappa \Delta(W \ast \cdot). \quad (3.49)$$

Since $W$ is coordinate-wise even, $L_{\text{Leb}_{\mathbb{T}^d}}$ is symmetric.

In order to check (Erg), we follow the setting illustrated in [21]. Recalling the notation $(\hat{W}^n)_{n \in \mathbb{Z}^d}$ for the Fourier coefficients of $W$, we introduce the following notion of $H$-stability (see [21, Definition 2.1]).

**Definition 3.14.** The potential $W$ is called $H$-stable if, for any $n \in \mathbb{Z}^d$, $\hat{W}^n \geq 0$. The latter is equivalent to the condition that $\forall m \in \mathcal{P}(\mathbb{T}^d)$, $\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} W(x - y)m(dx)m(dy) \geq 0$.

We then have the following two propositions, which are mostly taken from [21].

**Proposition 3.15.** Assume that $W$ is $H$-stable. Then $\text{Leb}_{\mathbb{T}^d}$ is the unique invariant measure and it is exponentially stable, in the sense of Proposition 3.9.

**Proof.** This directly follows from Proposition [21, Proposition 3.1], except for the fact that the initial condition of the Fokker-Planck equation therein is assumed to be smooth enough. Obviously, this is not an issue, since the solution of the Fokker-Planck equation becomes regular in positive time. The formulation used in [21, Proposition 3.1] is a way, therein, to track the influence of the initial condition onto the rate of convergence.

The second result is taken from [21, Subsection 3.2].

**Proposition 3.16.** Assume that $W$ is $H$-stable. Then the non-trivial eigenfunctions of the operator $L_{\nu_{\infty}}$ form an orthonormal basis of the space $\{f \in L^2(\mathbb{T}^d) : \langle f, 1 \rangle = 0\}$ and all the corresponding eigenvalues $(\lambda_k \in \mathbb{Z}^d \setminus \{0\})_{k \in \mathbb{Z}^d \setminus \{0\}}$ are strictly negative, with a non-zero spectral gap, i.e.

$$-\lambda = \sup_{k \in \mathbb{Z}^d \setminus \{0\}} \lambda_k < 0.$$

We show below that the combination of Propositions 3.15 and 3.15 suffices to get (Erg), which is thus another path to uniform propagation of chaos. The case $\kappa < 0$ is much more delicate. When its absolute value is small, uniform propagation of chaos remains true: this is a consequence of Theorem 3.12. When the absolute value is large, uniqueness of the invariant measure may be lost. In fact, this is the purpose of the next section to address a specific instance of this phenomenon.

The following is our main statement.

---

2Here, the description that $W$ is smooth means $W$ has derivatives of any order. In fact, the proof shows that it would be enough for $W$ to be 6-times continuously differentiable.
Theorem 3.17. Let $b$ be given by (3.48) for a smooth $H$-stable potential $W$. Then $b$ satisfies (Erg) and the conclusions of Theorem 3.8 and Corollary 3.10 hold.

Proof. We provide a sketch of the proof only, since part of the material is in common with the proof of Theorem 3.12.

First Step. The first point is to observe that, for any $\tau > 0$, the analysis performed in the proof of Theorem 3.12 remains true on the time interval $[0, \tau]$, whether (3.34) holds or not. Indeed, it suffices to first work with small $\tau$, in which case the integral on the first line of (3.38) becomes small and plays the role of $\epsilon$ (see Remark 3.13). Then, by iterating the argument a finite number of time, we recover the result for any $\tau > 0$. Back to (Erg-(K, $\gamma$, $\alpha$)), this says that (3.17) holds (under (3.15) and (3.16)) for $t \in (0, \tau]$, with the constants being independent of the choice of $\mu$.

Second Step. We adapt Theorem 3.11 to our setting in order to prove (3.17) for $t \geq \tau$, for a fixed $\tau > 0$. To do so, we recall from the existence of a spectral gap that, for any smooth $u : \mathbb{T}^d \to \mathbb{R}$ with $\langle u, 1 \rangle = 0$,

$$-(L_{\text{Leb}}u, u) \geq \lambda \langle u, u \rangle.$$  

Now, for a smooth function $\xi$ on $\mathbb{T}^d$ and for $t > 0$, we consider $(w(s))_{0 \leq s \leq t}$, which denotes the solution to the equation

$$
\begin{cases}
\partial_s w(s, \cdot) + L_{\text{Leb}}w(s, \cdot) = 0, & s \in [0, t], \\
 w(0) = \xi.
\end{cases}
$$

Then $\langle w, 1 \rangle$ is constant and

$$
\|w(s) - \langle w, 1 \rangle\|_2 \leq \|\xi\|_2 e^{-\lambda(t-s)}, \quad s \in [0, t].
$$

Recalling the shape of $L_{\text{Leb}}$ in (3.49) and that $\int_{\mathbb{T}^d} \Delta W(x - y)dy = 0$, we deduce that the zero-order term in (3.49) satisfies, for any $k \geq 0$,

$$
\sup_{x \in \mathbb{T}^d} |\nabla_x^k \Delta (W \ast w)(s, x)| \leq C(k) \|\xi\|_\infty e^{-\lambda(t-s)},
$$

and, in turn, by the maximum principle,

$$
\sup_{(s, x) \in [0, t] \times \mathbb{T}^d} |w(s, x)| \leq C\|\xi\|_\infty.
$$

We now split $w$ into $w := w_1 + w_2$, with

$$
\begin{cases}
\partial_s w_1(s, \cdot) + \frac{1}{2} \Delta w_1(s, \cdot) = 0, & s \in [0, t], \\
w_1(0) = \xi,
\end{cases}
$$

and

$$
\begin{cases}
\partial_s w_2(s, \cdot) + \frac{1}{2} \Delta w_2(s, \cdot) + \frac{1}{2} (\Delta W \ast w)(s, \cdot) = 0, & s \in [0, t], \\
w_2(0) = 0.
\end{cases}
$$

Obviously, $w_1$ satisfies all the conclusions of Theorem 3.11. In particular, $w_1$ satisfies (3.25) and thus $w_2$ also satisfies (3.51). This makes it possible to treat $w_2$. It suffices to represent $w_2$ in terms of the underlying (Gaussian) transition kernel on an interval of length $\tau$, by following the arguments of (3.29). The new point here comes from the presence of a source term, but the latter satisfies (3.52).

We deduce that $w$ satisfies the conclusion of Theorem 3.11.
Third Step. It now remains to adapt the proof of Theorem 3.12. We start from the same equation (3.16), with the same solution \( q \), when driven by any \( L_{m(t;\mu)} \). Instead of considering \( w \) as the solution of (3.35), we choose \( w \) as the solution of (3.50). This leads to a new expansion in (3.36). Whilst \( T_1 \) and \( T_2 \) are the same, \( T_3 \) takes the following new form (the reader may replace \( b \) by its expression in (3.48), but this is not needed).

\[
T_3 = \int_0^t \left\langle \left( b(\cdot, m(s; \mu)) - b(\cdot, \text{Leb}_{T^d}) \right) \cdot \nabla_x w(s, \cdot), q(s) \right\rangle \, ds \\
+ \int_0^t \int_{T^d} \left( \frac{\delta b}{\delta m}(x, m(s; \mu))(\cdot) \cdot \nabla_x w(s, x), q(s) \right) \left( m(s; \mu) - \text{Leb}_{T^d} \right) (dx) \, ds \\
+ \int_0^t \int_{T^d} \left( \frac{\delta b}{\delta m}(x, m(s; \mu))(\cdot) - \frac{\delta b}{\delta m}(x, \text{Leb}_{T^d})(\cdot) \right) \cdot \nabla_x w(s, x), q(s) \right\rangle \text{Leb}_{T^d}(dx) \, ds.
\]  

(3.53)

On the top line, \( b(\cdot, \text{Leb}_{T^d}) \) is zero, but we find it useful to write it down in order to emphasize the remainder form of \( T_3 \). The key point is then to use Proposition 3.15: in all three differences right above, it provides an additional term of the form \( \exp(-\lambda s) \). Following the derivation of (3.38), we obtain

\[
|T_3| \leq C|\xi|_{k-\alpha, \infty} e^{-\lambda t/2} \sup_{0 \leq s \leq t} \|q(s)\|_{(k, \infty)} \int_0^t e^{-\lambda(t-s)/2} \, ds.
\]  

(3.54)

For \( t \) large enough, we can regard \( e^{-\lambda t/2} \) as playing the role of \( \epsilon \) in (3.40). We can hence choose \( \tau \) to be large enough such that \( e^{-\lambda \tau \epsilon/2} \leq \epsilon \), for some value of \( \epsilon \) that is eventually chosen as in the proof of Theorem 3.12. By the first step, we already have a bound for \( \|q(s)\|_{(k, \infty)} \), if \( s \leq \tau \). We then get (3.41). The rest of the proof is even easier, since we already have the exponential decay in the estimate of \( T_3 \). \( \square \)

4. Model without a unique invariant measure

The purpose of this section is to show that the method developed in the previous sections can be adapted to the more challenging case when the mean-field interaction is no longer small and, accordingly, the Fokker-Planck equation may not have a unique invariant measure. As explained in the introduction, we do not provide any general results here, but merely address a specific model, known as the Kuramoto model, which has received a lot of attention in statistical physics and in neurosciences. It is in fact a particular case of example (3.48), but for \( \kappa < 0 \). Kuramoto’s model is indeed defined in dimension \( d = 1 \), by choosing (in (1.1))

\[
b(y, \mu) = -2\pi \kappa \int_T \sin(2\pi(y - x)) \mu(dx),
\]  

(4.1)

where \( \kappa \) is a coupling constant, which is usually chosen to be positive in the literature (the negative case generally has fewer interest). Here, the normalisation constant \( \pi \) in front of \( \kappa \) comes from the fact that the standard form of the Kuramoto model has a slightly different form:

\[
d\tilde{Y}_t^i = \tilde{\eta}_i - \frac{\kappa}{N} \sum_{j=1}^N \sin(\tilde{Y}_t^i - \tilde{Y}_t^j) \, dt + dW^i_t,
\]

for a new collection of initial conditions \( (\tilde{\eta}_i)_{1 \leq i \leq N} \) and a new collection of Brownian motions \( ((W^i_t)_{t \geq 0})_{1 \leq i \leq N} \). Obviously, the connection between the above equation and (1.1) is obtained by observing that, for a convenient choice of the initial condition, \( ((2\pi)^{-1}Y_{4\pi^2 t})_{t \geq 0} \) solves (1.1) with \( b \) as in (4.1). Having the right normalisation in (4.1) for the constant \( \kappa \) is important, since Kuramoto’s model is precisely known to exhibit
a phase transition when $\kappa = 1$ (see for instance [7]). When $\kappa \in (0, 1]$, the Fokker-Planck equation (1.4) has a unique invariant probability measure, which is given by the uniform measure on the torus. When $\kappa > 1$, it has an infinite number of invariant measures, namely the trivial one (i.e. the invariant measure on the torus) and a collection of non-trivial ones, all of them being obtained by rotation of a common non-constant profile $\psi$ (i.e., $p_\infty$ is a non-constant density on $T$ and $p_{\infty, \psi} := p_\infty(\cdot - \psi)$ is an invariant measure for any $\psi \in T$).

In the rest of the section, we focus on the regime $\kappa > 1$. In that case, propagation of chaos cannot hold at time of order $t = N$ when $\kappa > 1$ (see [8]). We refer to [50] for a similar phenomenon in the Euclidean setting. In fact, the result of [8] has just been revisited by [27] (in the even more complex case when the interactions are subjected to a non-complete graph). The main idea therein is to show that, even though the particle system may strongly deviate from an invariant profile in time of order $N$, it stays close to the whole collection $\mathcal{I} := \{p_{\infty, \psi}, \psi \in T\}$ for a time period that is nearly exponential in $N$. However, the rate of convergence is not addressed in [27]. Using the techniques developed in the previous section, we show here that, if the initial condition is not the uniform measure, we can retain a weak error of size $1/N$ in a long time period provided that we force the test function $\Phi$ in (2.11) to be invariant by rotation (see Theorem 4.3 below). We stress that the latter requirement on $\Phi$ is fully consistent with the point of view used in [27]. Our improvement is thus twofold: Not only do we get an explicit rate for the weak error, but we also manage to get a bound that holds uniformly in time (not only up until times that are exponential in $N$). Surely, the difference with respect to [27] and the references therein is that, in our work, the error is addressed in a weaker form. Therein, the convergence is indeed understood for the sup norm over the trajectory.

Things become more subtle whenever the system is initialized from the uniform distribution. By propagation of chaos in a finite time, the empirical measure of the particle system then stays close to the uniform distribution over any finite interval, with the related weak error still being of the order $1/N$ for functionals $\Phi$ as in Theorem 3.8. However, this error deteriorates with time. To wit (and this is an important ingredient in our proof), we show in Lemma 4.16 below that the empirical measure leaves, with a large probability, any sufficiently small neighbourhood of the uniform distribution in a time that is at most polynomial in $N$. As a result, another study would be necessary to handle this case specifically. Once again, our main motivation here is mostly to prove that our approach is robust enough to accommodate cases when invariant measures are not unique, whence our choice to focus on the Kuramoto model with $\kappa > 1$.

Throughout the section, we use freely the same general notations as in the previous section. In particular, $m(\cdot; \mu)$ denotes the solution to (1.4) with $\mu$ as initial condition and with $b$ as in (4.1). Here, it takes the form

$$\partial_t m(t; \mu) - \frac{1}{N} \partial_x^2 m(t; \mu) - \partial_x \left( m(t; \mu) \left( J \ast m(t; \mu) \right) \right) = 0,$$

(4.2)

where $\ast$ denotes the standard convolution product and $J(\cdot) = 2\pi \kappa \sin(2\pi \cdot)$.

Thanks to the diffusive structure of (1.4), $m(\cdot; \mu)$ is absolutely continuous in positive time, we therefore let $p(t; \mu) = (d/dx)m(t; \mu): x \mapsto p(t; \mu)$ (which we also write $p(t, x; \mu)$) be the density of $m(t; \mu)$, for $t > 0$. $p(t; \mu)(\cdot)$ is a smooth function of $x$, uniformly in $t \in [t_0, \infty)$ for any $t_0 > 0$. Similarly, we still denote by $\mu^N$ the flow of empirical distributions defined in (1.1). Also, we recall that each $p_{\infty, \psi}$ is (strictly) positive. Lastly, except when it is initialised with another condition (in which case it is explicitly stated), the (common) law of the I.I.D. initial positions $\eta_1, \cdots, \eta_N$ in (1.1) is denoted by $\mu_{\text{init}}$.

---

3Most of the time, we shall identify the densities that belong to $\mathcal{I}$ together with the probability measures that are driven by those densities.
4.1. Main result. As mentioned above, Kuramoto's model exhibits two different kinds of equilibrium point. While the invariant measures in $\mathcal{I}$ are stable, the uniform measure is unstable. Below, we thus focus on initial conditions that are sufficiently far away from the uniform measure, namely we let, for any $\eta \in (0, 1)$,

$$\mathcal{Q}_\eta = \{ \mu \in \mathcal{P}(\mathbb{T}) : |\mu^1| \geq \eta \}, \quad \mu^1 := \int_\mathbb{T} \exp(-i2\pi \theta) \mu(d\theta).$$

The following result (whose proof is postponed to the end of the subsection and is in fact mostly taken from [8] and [41]) shows that $\mathcal{Q}_\eta$ is attracted at an exponential rate by $\mathcal{I}$.

**Proposition 4.1.** For any $\eta \in (0, 1)$, and any integer $k \geq 1$, there exist an exponent $\beta > 0$ and a constant $C$, both depending on $K$, $\eta$ and $k$, such that

$$\forall t \geq 1, \quad \sup_{\mu \in \mathcal{Q}_\eta} \inf_{\psi \in \mathbb{T}} \| p(t; \mu) - p_{\infty, \psi} \|_{k, \infty} \leq C \exp(-\beta t).$$

Proposition 4.1 plays a key role in our analysis. Notice that the constraint $t \geq 1$ may be easily changed into $t \geq t_0$ for any $t_0 > 0$, in which case the constant $C$ may depend on $t_0$ as well. This does not make any conceptual difference, since the main interest of the result is about the long time behaviour of $p(\cdot; \mu)$.

In order to state our main result precisely, we need the following additional definition.

**Definition 4.2.** We say that a function $\Phi : \mathcal{P}(\mathbb{T}) \to \mathbb{R}$ is rotation invariant if, for any $\mu \in \mathcal{P}(\mathbb{T})$ and $\psi \in \mathbb{T}$,

$$\Phi(\mu \circ \tau^{-1}_\psi) = \Phi(\mu),$$

where $\mu \circ \tau^{-1}_\psi$ is the image of $\mu$ by the translation $\tau_\psi : \mathbb{T} \ni x \mapsto x + \psi$.

We now have all the ingredients to formulate the main theorem of this section.

**Theorem 4.3.** Assume that $b$ satisfies (Reg-$b$-(4, 2)), (Lip-$b$-(3, 2)) and (Erg) and that $\Phi$ is rotation invariant and satisfies (Reg-$\Phi$-$(\alpha, 2)$) for some $\alpha \in (0, 1]$. Then, for any $\eta \in (0, 1)$, there exists a constant $C > 0$ such that, for any $\mu_{\text{init}} \in \mathcal{Q}_\eta$ and any $N \geq 1$,

$$\sup_{t \geq 0} \left| \mathbb{E}[\Phi(\mu_{\text{init}}^N)] - \Phi(m(t; \mu_{\text{init}})) \right| \leq \frac{C}{N}.$$

As for examples of functions $\Phi$ that satisfy our assumption, the most useful example is certainly the following one (very much in the spirit of Proposition 2.4).

**Proposition 4.4.** Let $p_{\infty, +}$ denote the unique element of $\mathcal{I}$ whose first Fourier coefficient $p_{\infty, +}^1$ is a positive real and let $\mu_{\infty, +} := p_{\infty, +} \cdot \text{Leb}_\mathbb{T}$. For $\varepsilon \in (0, 1]$ and for a smooth non-decreasing cut-off function $\varphi : [0, 1] \to [0, 1]$ that is equal to 0 on $[0, \delta/2]$ and 1 on $[\delta, 1]$, for some $\delta \in (0, 1)$, let $\Phi$ be defined by

$$\Phi(\mu) := \varphi(\mu^1) \left\| \mu \circ \tau^{-1}_{\mu^1} - \mu_{\infty, +} \right\|_{-((1+\varepsilon)/2, 2; 2},^2 + 1 - \varphi(\mu^1),$$

where $\| \cdot \|_{-(1+\varepsilon)/2, 2}$ is defined as in Proposition 2.4 with $d = 1$ therein. Then $\Phi$ satisfies the assumption of Theorem 4.3.

Here, we need to say a few words about the meaning of $p_{\infty, +}$. Recall that the elements of $\mathcal{I}$ are obtained by rotation. Therefore, the collection of their first Fourier coefficients coincides with a circle, whose radius is in fact non-zero (see for instance [8, Subsection 1.2]). Consequently, we may indeed choose $p_{\infty, +} \in \mathcal{I}$ such
that \( p_{\infty,+}^1 > 0 \). Accordingly, we also notice that, in the notation \( \mu \circ \tau_{\mu^1/|\mu^1|}^{-1} \), we identify \( \mu^1/|\mu^1| \) with the unique element \( \psi \in \mathbb{T} \) such that \( \mu^1 = |\mu^1| \exp(i2\pi \psi) \). In particular, the first Fourier coefficient \( (\mu \circ \tau_{\mu^1/|\mu^1|})^1 \) of \( \mu \circ \tau_{\mu^1/|\mu^1|}^{-1} \), which is equal to \( \exp(-i2\pi \psi)\mu^1 = |\mu^1| \), is positive (at least when \( \mu^1 \neq 0 \)), which explains why we compare \( \mu \circ \tau_{\mu^1/|\mu^1|}^{-1} \) with \( \mu_{\infty,+} \). We refer to [49, Lemma 2.8] for another projection onto \( I \).

Moreover, we notice that this is precisely the role of the cut-off function \( \varphi \) in the definition of \( \Phi \) to remove the measures \( \mu \) for which \( \mu^1 = 0 \). In fact, the cut-off function has no real consequence on our result. Actually, what matters is that \( \Phi(\mu) \) is small if and only if \( \mu \) is close enough to \( I \). Precisely, we can find a constant \( c > 0 \) such that

\[
\Phi(\mu) \geq c \inf_{\psi \in I} |\mu - \mu_\psi|^2_{(1+\varepsilon)/2,2}.
\] (4.3)

The proof of the above lower bound is quite easy. It suffices to prove it for \( |\mu^1| \) bounded away from zero. To do so, we may observe that \( |\mu \circ \tau_{\mu^1/|\mu^1|}^{-1} - \mu_0|^2_{-(1+\varepsilon)/2,2} \) is lower-bounded by \( \inf_{\psi \in I} |\mu - \mu_\psi|^2_{-(1+\varepsilon)/2,2} \). The following is a straightforward corollary.

**Corollary 4.5.** For any \( \eta \in (0,1) \) and \( \varepsilon \in (0,1] \), there exist two (positive) constants \( c \) and \( C \) such that, for \( \mu_{\text{init}} \) in \( Q_\eta \),

\[
\forall t \geq 0, \quad \mathbb{E} \left[ \inf_{\psi \in I} |\mu^N - \mu_\psi|^2_{-(1+\varepsilon)/2,2} \right] \leq \frac{C}{N} + C \exp(-ct).
\]

**Proof of Corollary 4.5.** We take for granted the statements of Theorem 4.3 and Propositions 4.1 and 4.4. By (4.3) and with \( \Phi \) as in Proposition 4.4, it suffices to prove that \( \Phi(m(t; \mu)) \) decays exponentially fast for any \( \mu \in Q_\eta \). In order to do so, we observe from Proposition 4.1 that, for some \( \psi \in \mathbb{T} \), \( |p(t; \mu) - p_{\infty,\psi}|_2 \leq C \exp(-ct) \), for \( t \geq 1 \). Hence, it suffices to show that

\[
\left\| p_{\infty,\psi} - p_{\infty,+} \circ \tau_{\psi(t; \mu)} \right\|_2 \leq C \exp(-ct),
\]

or equivalently that

\[
\left\| p_{\infty,+} \circ \tau_{\psi(t; \mu)1} - p_{\infty,+} \circ \tau_{\psi(t; \mu)1} \right\|_2 \leq C \exp(-ct),
\]

at least for \( t \) sufficiently large. Since \( p_{\infty,+}^1 > 0 \), we know that \( |(p(t, \mu))_1| \) is lower-bounded by a positive constant, uniformly over all \( t \) greater than some \( t_0 > 0 \). In turn, we have

\[
\left| \frac{(p(t, \mu))_1}{|p(t, \mu)|} - \frac{(p_{\infty,\psi})_1}{|p_{\infty,\psi}|} \right| \leq C \exp(-ct),
\]

for some possibly new value of \( C \), which gives the expected result. \( \square \)

**Proofs of the auxiliary Propositions 4.1 and 4.4.** We present the proofs of Propositions 4.1 and 4.4 here. The reader may skip them ahead on a first reading.

**Proof of Proposition 4.4.** We use a simplified notation \( \tilde{\mu} \) for \( \mu \circ \tau_{\mu^1/|\mu^1|}^{-1} \). The tricky point is then to study the smoothness of the mapping

\[
\bar{\Phi}(\mu) = \| \tilde{\mu} - \nu_0 \|_{(1+\varepsilon)/2,2}^2.
\]
at least when \( \mu^1 \) stays away from 0, for a fixed \( \nu_0 \). We observe that we have the following:

\[
\tilde{\mu}^n = \int_T e^{-i2\pi nx} d\tilde{\mu}(x) = \left( \frac{|\mu^1|}{\mu^1} \right)^n \int_T e^{-i2\pi nx} d\mu(x) = \left( \frac{|\mu^1|}{\mu^1} \right)^n \mu^n.
\]

Therefore, following (2.17), we get

\[
\Phi(\mu) = \sum_{n \in \mathbb{N}} \frac{1}{(1 + n^2)^{(1+\varepsilon)/2}} \left[ \mu^n \tilde{\mu}^n + \nu_0^n \nu_0^n - \left( \frac{|\mu^1|}{\mu^1} \right)^n \mu^n \nu_0^n - \left( \frac{|\mu^1|}{\mu^1} \right)^n \nu_0^n \mu^n \right],
\]

which we can then rewrite in the form

\[
\Phi(\mu) = \sum_{n \in \mathbb{N}} \frac{1}{(1 + n^2)^{(1+\varepsilon)/2}} \left[ \mu^n \tilde{\mu}^n + \nu_0^n \nu_0^n - (\Psi(\mu)) \mu^n \nu_0^n - (\bar{\Psi}(\mu))^n \nu_0^n \mu^n \right],
\]

with \( \Psi(\mu) = |\mu^1|/\mu^1 \). On the open subset \( \{ \mu^1 \neq 0 \} \) (for the \( \mathcal{W}^1 \) topology), the function \( \Psi \) is infinitely differentiable with respect to \( \mu \). The power \( n \) creates additional factors that are handled in the same way as in the proof of Proposition 2.4. As a result, we get that, for the same values of \( k, \Phi \) satisfies the same properties as in Proposition 2.4, but on any domain where \( \mu^1 \) stays away from 0.

We close this subsection with the following proof.

**Proof of Proposition 4.1.** The proof is achieved in two steps, which are mostly adapted from [7] and [41].

**First step.** The first step is to show that

\[
\lim_{t \to \infty} \sup_{\mu \in \mathcal{Q}_\eta} \inf_{\psi \in \mathcal{T}} \left\| p(t; \mu) - p_{\infty, \psi} \right\|_2 = 0.
\]

In order to do so, we follow the proof of Proposition 1.7 in [7]. We recall indeed that the McKean-Vlasov equation (4.2) may be regarded as a gradient flow, with potential

\[
\mathcal{F}(p) = \frac{1}{2} \int_T p(x) \ln(p(x)) \, dx - \frac{\kappa}{2} \int_T \int_T p(x) \cos(2\pi(x-x'))p(x') \, dx \, dx'.
\]

Recall that the solution of (4.2) has a smooth density in positive time. In particular,

\[
\forall t > 0, \quad \mathcal{F}(p(t; \mu)) < \infty.
\]

Moreover, the gradient flow structure says that, for any \( 0 < t_1 < t_2 \),

\[
\mathcal{F}(p(t_2; \mu)) - \mathcal{F}(p(t_1; \mu)) = -\int_{t_1}^{t_2} \int_T \left( 2\pi \kappa \int_T \sin(2\pi(x-x'))m(r; \mu)(dx') + \frac{1}{2} \frac{p'(r; x; \mu)}{p(r; x, \mu)} \right)^2 m(r; \mu)(dx) \, dr.
\]

Then, we know from Proposition 4.4 in [41] that, after some time \( t_{n, \varepsilon} \) (independent of the choice of \( \mu \in \mathcal{Q}_\eta \)), \( p(t; \mu) \notin B_{L^2(T)}(1, \varepsilon) \) for a given \( \varepsilon > 0 \) (here, \( 1 \) is regarded as the constant function that is equal to 1 on \( T \), and \( B_{L^2(T)}(1, \varepsilon) \) is the \( L^2(T) \)-ball of center \( 1 \) and radius \( \varepsilon \)). Also, after the same time \( t_{n, \varepsilon} \), we know that \( p(t; \mu) \) belongs to a compact subset \( \mathcal{K} \) of \( \mathcal{C}^1(T, (0, +\infty)) \), which may be chosen independently of \( \mu \) (the lower bound on \( p(t; \mu) \) is a mere consequence of Harnack’s inequality, which can be found in, e.g., Proposition 7.37 of [48]).
Accordingly, non-trivial invariant measures appear very often in the sequel. For that reason, we prefer to write \( p_\psi \) instead of \( p_\infty,\psi \).

### 4.2. Linearised operator

Following our agenda, we now address the linearisation of the nonlinear Fokker-Planck equation at a (non-trivial) invariant measure (which hence has the form \( p_\infty,\psi \) for some phase \( \psi \in \mathbb{T} \)). Accordingly, non-trivial invariant measures appear very often in the sequel. For that reason, we prefer to write \( p_\psi \) instead of \( p_\infty,\psi \).
The linearised version (3.1) may be written in the form

$$\partial_t q(t) - L_m(t, \mu) q(t) = 0,$$

with the convenient notation

$$L_m q = \frac{1}{2} \partial_{xx} q + \partial_x \left( q(J * m) + m(J * q) \right),$$

for any two distributions $m$ and $q$ acting on smooth functions on the torus (notice that $L_m q$ always makes sense as a distribution since $J * m$ and $J * q$ themselves should be smooth functions). When $m = p_\psi$ for some element $\psi \in \mathbb{T}$, we merely write $L_\psi$ for $L_m$. Notice in particular that, by choosing $m_\epsilon = p_\psi$ (which is hence independent of $\epsilon$) in (4.2) and then by taking the derivative with respect to $\psi$ (which coincides with the derivative in $x$),

$$L_\psi p'_\psi = 0,$$

(4.5)

where $p'_\psi = (d/dx) p_\psi$. The above identity was already used in [7] and in the subsequent works of the same authors. It plays a key role here in our analysis as well.

We start with the long-run analysis of the linearised operator, which is the most demanding step. In fact, the proof is made easier by all the existing results on the Kuramoto model, but the reader must realise that this preliminary step is the cornerstone of the whole analysis in this subsection.

**Proposition 4.6.** For some smooth initial condition $q_0$ on $\mathbb{T}$, with $\int_\mathbb{T} q_0(x) dx = 0$, let $q$ denote the solution of the equation

$$\partial_t q - L_\psi q = 0, \quad t \geq 0,$$

(4.6)

and, for any $t \geq 0$, let $Q(t, \cdot)$ denote the (unique) periodic primitive of $q(t, \cdot)$ satisfying $\int_\mathbb{T} Q(t, x) dx = 0$.

Then there exist two positive constants $\lambda$ and $C$, only depending on $\kappa$, together with a constant $\tilde{q}_{1/2}$, depending on $q(0, \cdot)$ and $\psi$, such that, for any $t \geq 0$,

$$\int_\mathbb{T} |Q(t, x)|^2 dx \leq C \int_\mathbb{T} |Q(0, x)|^2 dx,$$

(4.7)

$$\int_\mathbb{T} |Q(t, x) - \tilde{q}_{1/2} (p_\psi(x) - 1)|^2 dx \leq Ce^{-\lambda t} \int_\mathbb{T} |Q(0, x)|^2 dx.$$

(4.8)

**Remark 4.7.** In fact, (4.7) follows from (4.8). Indeed, by expanding the square in the right-hand side below and by using the fact that $Q(t, \cdot)$ is centred, notice that

$$\int_\mathbb{T} \frac{Q^2(t, x)}{p_\psi(x)} dx = \int_\mathbb{T} \frac{|Q(t, x) - \tilde{q}_{1/2} p_\psi(x)|^2}{p_\psi(x)} dx + \tilde{q}_{1/2}^2,$$

(4.9)

from which we deduce that, by taking $t = 0$ and by using the fact that $p_\psi$ is lower-bounded, $\tilde{q}_{1/2}^2 \leq C \int_\mathbb{T} |Q(0, x)|^2 dx$.

**Proof.** Surely, the fact that Equation (4.6) has a unique solution is a mere consequence of the analysis performed in the previous section (since the Kuramoto model fits all the regularity conditions that are necessary here). Since $\int_\mathbb{T} q(t, x) dx = 0$ for any $t \geq 0$, it makes sense to define $Q(t, \cdot)$ as in the statement.

Thanks to Remark 4.7, it suffices to focus on the proof of (4.8). The proof mostly relies on the work of [7]. Following the notation introduced in [7, (1.23)], we indeed let

$$\langle u, v \rangle_\psi := \int_\mathbb{T} \frac{\bar{U}(x) \bar{V}(x)}{p_\psi(x)} dx,$$

(4.10)
for any two distributions \( u \) and \( v \) on \( \mathbb{T} \) and any two \( \bar{U} \) and \( \bar{V} \) in \( L^2(\mathbb{T}) \) such that \( \bar{U}' = u \), \( \bar{V}' = v \) and \( \langle \bar{U}, p^{-1}_\psi \rangle = \langle \bar{V}, p^{-1}_\psi \rangle = 0 \). Then we know from [7, (2.14), (2.16), (2.37)] that there exists a constant \( \lambda \), only depending on \( \kappa \), such that

\[
\frac{d}{dt} \left\langle q(t, \cdot), q(t, \cdot) \right\rangle_\psi + \lambda \left\langle q(t, \cdot) - \bar{q}_{1/2}(t) p'_\psi, q(t, \cdot) - \bar{q}_{1/2}(t) p'_\psi \right\rangle_\psi \leq 0,
\]

(4.11)

for any \( t \geq 0 \), with

\[
\bar{q}_{1/2}(t) = \frac{\left\langle q(t, \cdot), p'_\psi \right\rangle_\psi}{\left\langle p'_\psi, p'_\psi \right\rangle_\psi}.
\]

By [7, (2.14)] again, we observe that

\[
\frac{d}{dt} \left\langle q(t, \cdot), p'_\psi \right\rangle_\psi = \left\langle L_\psi q(t, \cdot), p'_\psi \right\rangle_\psi = \left\langle q(t, \cdot), L_\psi p'_\psi \right\rangle_\psi = 0,
\]

with the last equality following from (4.5). Hence, we can write \( \bar{q}_{1/2}(t) \) as \( \bar{q}_{1/2} \). Moreover, applying (4.11) to \( q(t, \cdot) - \bar{q}_{1/2} p'_\psi \), we deduce that

\[
\left\langle q(t, \cdot) - \bar{q}_{1/2} p'_\psi, q(t, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi \leq \left\langle q(0, \cdot) - \bar{q}_{1/2} p'_\psi, q(0, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi e^{-\lambda t}, \quad t \geq 0.
\]

(4.12)

By (4.10), the right-hand side in (4.12) reads

\[
\left\langle q(0, \cdot) - \bar{q}_{1/2} p'_\psi, q(0, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi = \int_T \frac{|Q(0, x) - \bar{q}_{1/2} p_\psi(x) - k(0)|^2}{p_\psi(x)} \, dx,
\]

where \( k(0) \) is a centring constant that forces the mean of \((Q(0, \cdot) - \bar{q}_{1/2} p_\psi - k(0))/p_\psi \) to be zero. In particular, the left-hand side is less than

\[
\left\langle q(0, \cdot) - \bar{q}_{1/2} p'_\psi, q(0, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi \leq \int_T \frac{|Q(0, x) - \bar{q}_{1/2} p_\psi(x)|^2}{p_\psi(x)} \, dx \leq \int_T \frac{|Q(0, x)|^2}{p_\psi(x)} \, dx \leq C \int_T |Q(0, x)|^2 \, dx,
\]

(4.13)

where the penultimate inequality follows from (4.9) and the last one from the fact that \( p_\psi \) is lower-bounded, with the constant depending only on \( \kappa \).

Back to (4.12), we can handle the left-hand side in a similar manner, by writing

\[
\left\langle q(t, \cdot) - \bar{q}_{1/2} p'_\psi, q(t, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi = \int_T \frac{|Q(t, x) - \bar{q}_{1/2} p_\psi(x) - k(t)|^2}{p_\psi(x)} \, dx,
\]

for a new centring constant \( k(t) \). Using now an upper bound for \( p_\psi \) and assuming without any loss of generality that the constant \( C \) right above is large enough (as long as it only depends on \( \kappa \)), we obtain that

\[
\left\langle q(t, \cdot) - \bar{q}_{1/2} p'_\psi, q(t, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi \geq C^{-1} \int_T |Q(t, x) - \bar{q}_{1/2} p_\psi(x) - k(t)|^2 \, dx.
\]

Using the fact that \( Q(t, \cdot) - \bar{q}_{1/2} (p_\psi - 1) \) has zero mean, we deduce that

\[
\left\langle q(t, \cdot) - \bar{q}_{1/2} p'_\psi, q(t, \cdot) - \bar{q}_{1/2} p'_\psi \right\rangle_\psi \geq C^{-1} \int_T |Q(t, x) - \bar{q}_{1/2} (p_\psi(x) - 1)|^2 \, dx.
\]

(4.14)

Combining (4.12), (4.13) and (4.14), we get (4.8).
The following result is the analogue of Theorem 3.11 in Section 3.3.

**Proposition 4.8.** Let $t > 0$, $\psi \in \mathbb{T}$ and $\xi \in W^{1,\infty}(\mathbb{T})$ and let $L_\psi^*$ denote the adjoint if $L_\psi$ is in $L^2(\mathbb{T})$. Then the Cauchy problem
\[
\begin{aligned}
\partial_t w + L_\psi^* w &= 0, \quad s \in [0, t], \\
w(t, x) &= \xi(x),
\end{aligned}
\tag{4.15}
\]
admits a unique classical solution $(w(s, \cdot))_{0 \leq s \leq t}$. Moreover, there exist constants $C, \lambda > 0$ (only depending on $\kappa$ and hence independent of $t$) such that
\[
\begin{aligned}
(i) \quad &\left\| w(s, \cdot) - \int_\mathbb{T} w(s, y)dy \right\|_\infty \leq C \left( \|\xi\|_\infty e^{-\lambda(t-s)} + ||\langle \xi, p_\psi^* \rangle|| \right), \quad \forall s \in [0, t]; \\
(ii) \quad &\left\| \partial_w w(s, \cdot) \right\|_{\beta, \infty} \leq \frac{C}{1 \wedge (t-s)^{(\alpha+\beta)/2}} \left( \|\xi\|_{1-\alpha, \infty} e^{-\lambda(t-s)} + ||\langle \xi, p_\psi^* \rangle|| \right), \quad \forall s \in [0, t].
\end{aligned}
\]
Now suppose that, for $k \in \{2, 3\}$, $\xi \in W^{k,\infty}(\mathbb{T})$. Then, for any $\alpha, \beta \in [0, 1]$, there exist constants $C, \lambda > 0$, depending only $\alpha, \beta$ and $\kappa$, such that
\[
(iii) \quad \left\| \partial_w^k w(s, \cdot) \right\|_{\beta, \infty} \leq \frac{C}{1 \wedge (t-s)^{(\alpha+\beta)/2}} \left( \|\xi\|_{k-\alpha, \infty} e^{-\lambda(t-s)} + ||\langle \xi, p_\psi^* \rangle|| \right), \quad \forall s \in [0, t].
\]

**Proof.** Existence and uniqueness of a classical solution to the Cauchy problem is standard. The rest of the proof is similar to the proofs of Theorems 3.11 and 3.12, but with some differences that we clarify below.

**First Step.** The first step is to observe that all the three bounds hold when $t = s = 1$. The proof follows from the same argument as the one used in the proof of Theorem 3.17. By expanding the operator $L_\psi^*$, we indeed observe that the equation satisfied by $w$ may be rewritten in the form
\[
\begin{aligned}
\partial_t w(s, \cdot) + \frac{1}{2} \partial_{xx}^2 w(s, \cdot) + V \partial_x w(s, \cdot) + \int_\mathbb{T} W(\cdot, y)\bar{w}(s, y)dy &= 0, \quad s \in [0, t],
\end{aligned}
\tag{4.16}
\]
where $\bar{w}(s, x) = w(s, x) - \int_\mathbb{T} w(s, y)dy$ and $V$ and $W$ are smooth functions on $\mathbb{T}$ and $\mathbb{T}^2$ respectively, whose derivatives up to any order are bounded in terms of $\kappa$ only. By a standard application of the maximum principle combined with Gronwall’s lemma, it is easy to show that
\[
\sup_{\max(0,t-1) \leq s \leq t} \|w(s, \cdot)\|_\infty \leq C\|\xi\|_\infty.
\tag{4.17}
\]
Since $W$ is smooth, this provides a bound for the derivatives of any order of the third term in the left-hand side of (4.16). Following the proof of Theorem 3.17, we then split $w$ into $w = w_1 + w_2$, with
\[
\begin{aligned}
\partial_s w_1(s, \cdot) + \frac{1}{2} \partial_{xx}^2 w_1(s, \cdot) + V \partial_x w_1(s, \cdot) &= 0, \quad s \in [0, t]; \quad w_1(t, \cdot) = \xi, \\
\partial_s w_2(s, \cdot) + \frac{1}{2} \partial_{xx}^2 w_2(s, \cdot) + \int_\mathbb{T} W(\cdot, y)\bar{w}(s, y)dy &= 0, \quad s \in [0, t]; \quad w_2(t, \cdot) = 0.
\end{aligned}
\tag{4.18}
\]
By Theorem 3.11, we get all the required bounds on $w_1(s, \cdot)$ and its derivatives, at least for $t - s \leq 1$.\]
As for the equation satisfied by \( w_2 \), we know that the source term therein (with \( \bar{w} \) being frozen) is smooth. The solution \( w_2(s, \cdot) \) thus has bounded (spatial) derivatives of any order, for \( t - s \leq 1 \), with the bounds being independent of \( t \).

**Second Step.** The rest of the proof is hence dedicated to the case when \( t \geq 1 \). We start with the proof of (i), using the same notations \( q(0, \cdot), \bar{q} \) and \( \bar{q}_{1/2} \) as in the statement and the proof of Proposition 4.6. By Proposition 4.6 and Remark 4.7, we deduce that there exist \( \lambda \) and \( C \) as in the statement (but the values of which are allowed to vary from line to line) such that

\[
\langle \xi, q(t, \cdot) \rangle \leq \langle \xi, (q(t, \cdot) - \bar{q}_{1/2} p'_\psi) \rangle + |\bar{q}_{1/2}| \langle \xi, p'_\psi \rangle \\
\leq |\langle \xi', (Q(t, \cdot) - \bar{q}_{1/2}[p_\psi - 1]) \rangle| + C\|Q(0, \cdot)\|_2 \langle \xi, p'_\psi \rangle \\
\leq C\|Q(0, \cdot)\|_2 (\|\xi\|_1 \infty e^{-\lambda t} + \langle \xi, p'_\psi \rangle).
\]

Next, we use the Sobolev bound

\[
\|Q(0, \cdot)\|_2 = \left( \int_T \left| Q(0, x) - \int_T Q(0, y) dy \right|^2 dx \right)^{1/2} \leq \left( \int_T \int_T |Q(0, x) - Q(0, y)|^2 dx dy \right)^{1/2} \leq C\|q(0, \cdot)\|_1,
\]

where we used the equality \( \int_T Q(0, x) dx = 0 \) together with the obvious identity \( Q(x, y) = \int_y^x q(0, z) dz \). By the same duality argument as in the proof of Theorem 3.12, we observe that

\[
\frac{d}{ds} \langle w(s, \cdot), q(s, \cdot) \rangle = 0,
\]

which implies that \( \langle w(0, \cdot), q(0, \cdot) \rangle = \langle \xi, q(t, \cdot) \rangle \). We therefore deduce that

\[
|\langle w(0, \cdot), q(0, \cdot) \rangle| \leq C\|q(0, \cdot)\|_1 (\|\xi\|_1 \infty e^{-\lambda t} + \langle \xi, p'_\psi \rangle).
\]

By choosing \( q(0, x) = q^n(x) - 1 \), where \( (q^n)_{n \geq 1} \) is standard mollifier of the Dirac mass at some point \( x_0 \in T \), and letting \( n \) tend to \( \infty \) and then taking the supremum over \( x_o \), we obtain that

\[
\sup_{x \in T} \left| w(0, x) - \int_T w(0, y) dy \right| \leq C \left( \|\xi\|_1 \infty e^{-\lambda t} + \langle \xi, p'_\psi \rangle \right). \tag{4.20}
\]

Obviously, the above bound does not exactly fit (i). The first point to recover (i) is to replace \( \|\xi\|_1 \infty \) by \( \|\xi\|_\infty \). In order to so, we apply (4.20), but on the interval \( [0, t - 1] \) and with \( \xi = w(t - 1, \cdot) \) itself. For a new value of \( C \) (that depends on the same parameters as before),

\[
\sup_{x \in T} \left| w(0, x) - \int_T w(0, y) dy \right| \leq C \left( \|w(t - 1, \infty)\|_1 \infty e^{-\lambda t} + \langle w(1, \infty), p'_\psi \rangle \right) \leq C \left( \|\xi\|_\infty e^{-\lambda t} + \langle \xi, p'_\psi \rangle \right), \tag{4.21}
\]

where we used in the last line the fact that \( \langle \xi, p'_\psi \rangle = \langle w(1, \cdot), p'_\psi \rangle \), which follows from (4.5) and then from the same duality argument as in (4.19). In the above, we also used the bound \( \|w(t - 1, \infty)\|_1 \infty \leq C\|\xi\|_\infty \), which follows from (ii) (in small time, as already studied in the first step).

**Third Step.** The end of the proof is similar to the end of the second step in the proof of Theorem 3.17. Indeed, by Theorem 3.11, \( w_1 \) satisfies all the required bounds. In particular, \( w_2 = w - w_1 \) also satisfies (4.21). Subsequently, we use the same expansion as in (3.29), by taking benefit of the fact that the source term in the equation for \( w_2 \) is smooth and that all its derivatives can be upper-bounded by (4.21). We then obtain (ii) and (iii).

\( \square \)
The following proposition is one key step in our proof.

**Proposition 4.9.** For fixed \( \kappa > 1 \) and \( \eta \in (0, 1) \), the drift (4.1) satisfies (Erg) up to the change that, in (Erg-\((K, \gamma, \alpha))\), \( \mu \) is taken in \( \mathcal{Q}_\eta \) and that

1. (3.17) holds when \( t \leq 1 \);
2. (3.17) holds when \( t \geq 1 \), but with \( q(t) \) replaced by \( q(t) - q_\infty p'(t; \mu) \), where: \( q_\infty \) is here a real number depending on the input of the Cauchy problem (3.16), but independent of \( t \), and \( q_\infty \) is bounded by \( C \max \{1, \|q_0\|_{(2, \infty)'}\} \), for \( C \) only depending on \( \eta, \kappa, K \) and \( \gamma \).

**Proof.** Throughout the proof, we merely write \( m(t) \) for \( m(t; \mu) \) and \( p(t) \) for \( p(t; \mu) \) (recalling that \( \mu \) is given in the definition of (Erg-\((K, \gamma, \alpha)))\). Moreover, following the statement of Proposition 4.1, we may call \( \psi \in T \) such that \( p_\psi \) is the limit of \( p(t) \) as \( t \) tends to \( \infty \). The proof is then an adaptation of the proof of Theorem 3.17, which is itself an adaptation of the proof of Theorem 3.12.

**First Step.** The first step is completely similar to the first step in the proof of Theorem 3.17. The point is to observe that, for any \( \tau > 0 \), the analysis performed in the proof of Theorem 3.12 remains true on the time interval \( [0, \tau] \), whether (3.34) holds or not. This says that, in (Erg-\((K, \gamma, \alpha)))\), (3.17) holds (under (3.15) and (3.16)) for \( t \in (0, \tau] \), with the constants being independent of the choice of \( \mu \).

**Second Step.** As for the second step, we also follow the second step of the proof of Theorem 3.17. We thus start from the same equation (3.2) (with the same solution \( q \)), but instead of considering \( w \) as the solution of (3.35), we choose \( w \) as the solution of (4.15). As in the proof of Theorem 3.17, this leads to a new expansion in (3.36). In short, \( T_1 \) and \( T_2 \) are the same, but \( T_3 \) takes the new form (very much in the spirit of (3.53)):

\[
T_3 = \int_0^t \left( \left( b(\cdot, m(s)) - b(\cdot, p_\psi) \right) \partial_x w(s, \cdot), q(s) \right) ds
+ \int_0^t \int_{T^d} \left( \delta b \left( x, m(s) \right) \partial_x w(s, x), q(s) \right) \left( p(s, x) - p_\psi(x) \right) dx ds
+ \int_0^t \int_{T^d} \left( \delta b \left( x, m(s) \right) \partial_x w(s, x), q(s) \right) p_\psi(x) dx ds,
\]

where, in the argument of \( b \), we have identified the density \( p_\psi \) with the measure of density \( p_\psi \).

Here comes now the subtlety. Unless we assume \( \langle \xi, F_\psi \rangle = 0 \), the conclusions (i), (ii) and (iii) in Proposition 4.8 just hold without the exponential decay. Although this seems to be a serious drawback to follow the second step of the proof of Theorem 3.17, this can be easily circumvented. Indeed, by the first step, we can easily replace the initial point 0 in \( T_3 \) by any time \( \tau > 0 \) and then assume accordingly that \( t \geq \tau \). By Proposition 4.1, this permits to render the distance between \( p(s) \) (and \( m(s) \)) as small as needed by taking \( \tau \) large enough, uniformly in \( \mu \in \mathcal{Q}_\eta \). Taking \( \tau \geq 1 \) large enough, we get the following variant of (3.40):

\[
\|q(t)\|_{(k-\alpha, \infty)'} \leq C\|q_0\|_{(k, \infty)'} + \frac{1}{2} \left( \sup_{\tau \leq s \leq t} \|q(s)\|_{(k, \infty)'} \right) + C(1 + t) \left( \sup_{0 \leq s \leq t} \|r(s)\|_{(k, \infty)'} \right), \quad t \geq 1,
\]

with the additional \( t \) on the right-hand side coming from the fact that we do not have any integrable decay for \( w \) and its derivatives in Proposition 4.8. As for the constant \( C \), it only depends on \( \kappa, \eta, k \) and \( \alpha \). Choosing first \( \alpha = 0 \), restoring the value of \( \alpha \) and allowing \( C \) to depend on \( \kappa, \eta, \alpha, K \) and \( \gamma \), we deduce that, for \( t \geq 1 \),

\[
\|q(t)\|_{(k-\alpha, \infty)'} \leq C \max \{1 + t, \|q(0)\|_{(k, \infty)'}\}, \quad t > 0.
\]
Third Step. We now assume \( \langle \xi, p'_\psi \rangle = 0 \) in order to benefit from the exponential decay in (i), (ii) and (iii) in Proposition 4.8. As in the proof of Theorem 3.12, we refer back to (3.36), by initialising the dynamics at some positive time. Here, we choose \( t/2 \) as initial time, for \( t \geq 1 \). The equation (3.16), which is regarded on the sole \([t/2, \infty)\), may be split into \( q(s) = q_1(s) + q_2(s) \), for \( s \geq t/2 \), where \( q_1 \) is the solution of (3.16) with \( q_1(t/2) = q(t/2) \) as initial condition and with a zero source term, and \( q_2 \) is the solution of (3.16) with a zero initial condition and with \( r \) as source term. Proceeding as in the second step, \( q_1 \) and \( q_2 \) also satisfy (4.24).

The term \( q_2 \) can be easily addressed by returning to (3.36) (initialised from \( t/2 \)). The term \( T_1 \) therein is zero. Similar to (3.40) (but with \( s \geq t/2 \) in the last supremum) and thanks to the exponential decay of \( r \), the term \( T_2 \) is less than \( C \exp(-\lambda t/4) \| \xi \|_{k-a,\infty} \). Furthermore, by combining (4.24) (for \( q_2 \) and Proposition 4.1, the term \( T_3 \) (as given by (4.22)) is also less than \( C \exp(-\lambda t/4) \| \xi \|_{k-a,\infty} \). This is enough to prove that \( \| q_2(t) \|_{(k-a,\infty)'} \leq C \exp(-\lambda t/4) \max \{1, \| q_0 \|_{(k,\infty)'} \} \).

In order to handle \( q_1 \), we must revisit (3.36). Since the source term in the equation of \( q_2 \) is null, the corresponding \( T_2 \) in (3.36) is 0. Recalling the condition \( \langle \xi, p'_\psi \rangle = 0 \), we get the following variant of (4.24):

\[
\left\| q_1(t) - \frac{\langle q_1(t), p'_\psi \rangle}{\langle p'_\psi, p'_\psi \rangle} \right\|_{(k-a,\infty)'} \leq C e^{-\lambda t/2} \max \{1, \| q(t/2) \|_{(k,\infty)'} \}.
\]

Next, we prove that \( \langle q_1(t), p'_\psi \rangle / \langle p'_\psi, p'_\psi \rangle \) converges exponentially fast to a constant \( q_\infty \). We write \( \langle q_1(t), p'_\psi \rangle = -\langle Q_1(t, \cdot), p'_\psi \rangle \), with \( Q_1 \) being the primitive of \( q_1 \) (in space) with a zero mean. Invoking Proposition 4.6 (with \( q = q_1 \)), we know that \( \langle Q_1(t, \cdot), p'_\psi \rangle \) converges exponentially fast to some constant. The rate of convergence, as given by (4.8), depends on

\[
\| Q_1(t/2 + 1, \cdot) \|_2 \leq C \| q_1(t/2 + 1, \cdot) \|_{\infty}.
\]

We now provide an upper bound for the above right-hand side. Since the source term in the equation for \( q_1 \) is zero, we can apply the second step twice. We first apply it on the interval \([t/2, t/2 + 1]\) with \( k = \alpha = 1 \), which gives \( \| q_1(t/2 + 1, \cdot) \|_{\infty} \leq C \max \{1, \| q_1(t/2 + 1, \cdot) \|_{(1,\infty)'} \} \). We then reapply it on the interval \([t/2, (t+1)/2]\) with \( k = 2 \) and \( \alpha = 1 \), which yields \( \| q_1((t+1)/2, \cdot) \|_{(1,\infty)'} \leq C \max \{1, \| q(t/2, \cdot) \|_{(2,\infty)'} \} \). We end up with \( \| q_1((t+1)/2, \cdot) \|_{(1,\infty)'} \leq C \max \{1 + t, \| q(0, \cdot) \|_{(2,\infty)'} \} \). Inserting the latter in (4.26), we deduce from (4.25) and from the second step again that, for possibly new values of \( C \) and \( \lambda \),

\[
\left\| q_1(t) - q_\infty p'_\psi \right\|_{(k-a,\infty)'} \leq C e^{-\lambda t/2} \max \{1, \| q_0 \|_{(k,\infty)'} \}.
\]

It then remains to recall from Proposition 4.1 that \( p(t) \) converges exponentially fast to \( p_\psi \). We deduce that

\[
\left\| q_1(t) - q_\infty p(t) \right\|_{(k-a,\infty)'} \leq C e^{-\lambda t/2} \max \{1, \| q_0 \|_{(k,\infty)'} \}, \quad t \geq 1.
\]

Using the bound for \( q_2 \), we can easily replace \( q_1 \) by \( q \). By the first step, the constant \( q_\infty \) is bounded by \( C \max \{1, \| q_0 \|_{(2,\infty)'} \} \). This completes the proof.
the following bound holds:
\[ \|m^{(1)}(t; \mu, \nu)\|_{(0, \infty)} + \|m^{(2)}(t; \mu, \nu_1, \nu_2)\|_{(0, \infty)} \leq C, \quad t \geq 0. \] (4.27)

Moreover, we can find two real numbers \( q^{(1)}_\infty(\mu, z) \) and \( q^{(2)}_\infty(\mu, z_1, z_2) \) such that
\[
\begin{align*}
\min(1, t^{\alpha/2}) \|d^{(1)}(t; \mu, z)\|_{(1-\alpha, \infty)} &+ \min(1, t^{\beta/2}) \|d^{(2)}(t; \mu, z_1, z_2)\|_{(2-\beta, \infty)} \leq C, \\
\min(1, t^{\beta/2}) \|d^{(1)}(t; \mu, z)\|_{(1-\alpha, \infty)} &+ \min(1, t^{\beta/2}) \|d^{(2)}(t; \mu, z_1, z_2)\|_{(2-\beta, \infty)} \leq Ce^{-\lambda t},
\end{align*}
\] (4.28)
for \( t > 0 \), where
\[
\begin{align*}
d^{(1)}(t; \mu, z) &= d^{(1)}(t; \mu, z) - q^{(1)}_\infty(\mu, z)p(t; \mu), \\
d^{(2)}(t; \mu, z_1, z_2) &= d^{(2)}(t; \mu, z_1, z_2) - q^{(1)}_\infty(\mu, z_2)\partial_x d^{(1)}(t; \mu, z_1) - q^{(2)}_\infty(\mu, z_2)p(t; \mu).
\end{align*}
\] (4.29)

**Proof. First Step.** We start with the proof of (4.28). Throughout the proof, we fix \( \eta \in (0, 1) \). By Theorem 3.3, we know that, for any \( \mu \in \mathcal{Q}_\eta \) and \( z \in \mathbb{T} \), \( d^{(1)}(t; \mu, z) \) solves the linearised equation
\[ \partial_t d^{(1)}(t; \mu, z) - L_{m(t; \mu)}d^{(1)}(t; \mu, z) = 0, \] (4.30)
with \( d^{(1)}(0; \mu, z) = D'_2 = -\partial_z(\delta_z) \) and \( m(0; \mu) = \mu \). Then, by item (2) in Proposition 4.9, with \( k = 1 \) in (3.17), we get the two bounds for \( d^{(1)}(t; \mu, z) \) and \( \tilde{d}^{(1)}(t; \mu, z) \), with \( q^{(1)}_\infty(\mu, z) \) being given by Proposition 4.9.

As for \( d^{(2)} \), things are more complicated. By Theorem 3.4, we indeed know that, for any \( \mu \in \mathcal{Q}_\eta \) and any \( z_1, z_2 \in \mathbb{T}^d \), \( d^{(2)}(t; \mu, z_1, z_2) \) solves the equation
\[ \partial_t d^{(2)}(t; \mu, z_1, z_2) - L_{m(t; \mu)}d^{(2)}(t; \mu, z_1, z_2)
- \partial_x \left( d^{(1)}(t; \mu, z_1)(J * d^{(1)}(t; \mu, z_2)) + d^{(1)}(t; \mu, z_2)(J * d^{(1)}(t; \mu, z_1)) \right) = 0, \] (4.31)
with \( d^{(2)}(0; \mu, z_1, z_2) = 0 \). Next, if we want to apply item (2) in Proposition 4.9, with \( k = 2 \) in (3.17), we have to choose the following in (3.16):
\[
\begin{align*}
r(t) &= \partial_x \left( d^{(1)}(t; \mu, z_1)(J * d^{(1)}(t; \mu, z_2)) + d^{(1)}(t; \mu, z_2)(J * d^{(1)}(t; \mu, z_1)) \right).
\end{align*}
\]
However, we cannot prove that \( \|r(t)\|_{(2, \infty)} \) decays exponentially fast, since (4.28) just provides an exponential bound for \( \tilde{d}^{(1)}(t; \mu, z_1) \) and \( \tilde{d}^{(1)}(t; \mu, z_2) \), and not for \( d^{(1)}(t; \mu, z_1) \) and \( d^{(1)}(t; \mu, z_2) \). Instead, we focus on
\[
\tilde{d}^{(2)}(t; \mu, z_1, z_2) := d^{(2)}(t; \mu, z_1, z_2) - q^{(1)}_\infty(\mu, z_2)\partial_x d^{(1)}(t; \mu, z_1),
\] (4.32)
for \( q^{(1)}_\infty(\mu, z) \) as in (4.29). We easily see that \( (\partial_x d^{(1)}(t; \mu, z))_{t \geq 0} \) solves (in a weak sense) the equation
\[ \partial_t \partial_x d^{(1)}(t; \mu, z) - L_{m(t; \cdot)} \partial_x d^{(1)}(t; \mu, z) - \partial_x \left( d^{(1)}(t; \mu, z)(J * m^{(1)}(t; \mu)) + m^{(1)}(t; \mu)(J * d^{(1)}(t; \mu, z)) \right) = 0, \] (4.33)
in the space \( \cap_{T > 0} L_\infty([0, T], (W^{2, \infty}(\mathbb{T}^d))') \cap_{T > 1} L_\infty([1/T, T], (W^{2-2\alpha, \infty}(\mathbb{T}^d))') \), with \( -\partial_x^2(\delta_z) \) as initial condition. Then, choosing \( z = z_1 \) in (4.33), multiplying by \( q^{(1)}_\infty(\mu, z_2) \) and then subtracting to (4.31), we obtain
\[
\partial_t d^{(2)}(t; \mu, z_1, z_2) - L_{m(t; \mu)}d^{(2)}(t; \mu, z_1, z_2) - r(t) = 0,
\] with
\[
r(t) = \partial_x \left( d^{(1)}(t; \mu, z_1)(J * \tilde{d}^{(1)}(t; \mu, z_2)) + \tilde{d}^{(1)}(t; \mu, z_2)(J * d^{(1)}(t; \mu, z_1)) \right).
\] (4.34)
We first evaluate the \((W^{2,\infty}(\mathbb{T}))'\) norm of \(r(t)\). Using the fact that the convolution kernel \(J\) is odd, we get, for any \(\xi \in W^{2,\infty}(\mathbb{T})\),

\[
\langle r(t), \xi \rangle = -\left\langle \tilde{d}^{(1)}(t; \mu, z_1)(J \ast \tilde{d}^{(1)}(t; \mu, z_2)) + \tilde{d}^{(1)}(t; \mu, z_2)(J \ast \tilde{d}^{(1)}(t; \mu, z_1)), \xi' \right\rangle \\
= \left\langle \tilde{d}^{(1)}(t; \mu, z_1), J \ast (\tilde{d}^{(1)}(t; \mu, z_2)\xi') \right\rangle - \left\langle \tilde{d}^{(1)}(t; \mu, z_2), (J \ast \tilde{d}^{(1)}(t; \mu, z_1))\xi' \right\rangle.
\]

Invoking (4.28) with \(\alpha = 0\) (using in addition the smoothness of \(J\) and noticing that the bound obviously holds at time \(t = 0\)), we then have

\[
\|r(t)\|_{(2,\infty)'} \leq Ce^{-\lambda t}, \quad t \geq 0,
\]

where the values of \(C\) and \(\lambda\) are allowed to vary as long as they only depend on \(\kappa, \alpha, \beta\) and \(\eta\). By item (2) in Proposition 4.9, with \(k = 2\) in (3.17), we deduce that there exists a constant \(d^{(2)}_{\infty}(\mu, z_1, z_2)\) such that

\[
\min(1, t^{\beta/2})\|\tilde{d}^{(2)}(t; \mu, z_1, z_2)\|_{(2-\beta,\infty)'} \leq Ce^{-\lambda t}, \quad t > 0,
\]

with

\[
\tilde{d}^{(2)}(t; \mu, z_1, z_2) = \tilde{d}^{(2)}(t; \mu, z_1, z_2) - q^{(2)}(t; \mu, z_1, z_2)p'(t; \mu).
\]

This completes the proof of (4.28).

**Second Step.** We now turn to the proof of (4.27), which is quite similar to the first step. By Theorem 3.2, we know that, for any \(\mu \in \mathcal{Q}_\eta\) and \(\nu \in \mathcal{P}(\mathbb{T})\), \((m^{(1)}(t; \mu, \nu))_{t \geq 0}\) solves the same equation (4.30) but with \(\nu - \mu\) as initial condition. Therefore, we have the bound for \(m^{(1)}(t; \mu, \nu)\) in (4.27), using item (2) in Proposition 4.9, with \(k = 0\) in (3.17). Also, we have the same for \(\tilde{m}^{(1)}(t; \mu, \nu)\), with an obvious definition for the latter.

We now turn to \((m^{(2)}(t; \mu, \nu_1, \nu_2))_{t \geq 0}\). For \(t \in [0, 1]\), the bound follows from Proposition 3.7, recalling that there is no need for ergodicity in finite time. In order to address the case \(t > 1\), we write the analogue of (4.33) but for \((\partial_t m^{(1)}(t; \mu, \nu))_{t \geq 0}\) and the analogue of \(r\) in (4.34). We have \(\|r(t)\|_{-1,\infty} \leq C\exp(-\lambda t)\).

By (2) in Proposition 4.9, with \(k = 1\) in (3.17), we recover (4.35) but for \(\tilde{m}^{(2)}(t; \mu, \nu_1, \nu_2)\) (with an obvious definition for it), with \(2 - \beta\) being replaced by 0. We get a bound for \(\|m^{(2)}(t; \mu, \nu_1, \nu_2)\|_{(0,\infty)}, \; t \geq 1\).

In the rest of the subsection, \(\mathcal{U}\) denotes the same functional as in (2.6). We start with the following lemma.

**Lemma 4.11.** Let \(\Phi: \mathcal{P}(\mathbb{T}) \to \mathbb{R}\) be a rotation-invariant function that satisfies (Diff-\(\Phi\)) (see Definition 4.2). Then, for any probability measure \(\mu \in \mathcal{P}(\mathbb{T})\) and any distribution \(q \in (W^{1,\infty}(\mathbb{T}))'\) with \(\langle q, 1 \rangle = 0\):

\[
\frac{\delta \Phi}{\delta m} (\mu)(\mu') = 0, \quad \text{and} \quad \frac{\delta^2 \Phi}{\delta m^2} (\mu)(\mu', q) + \frac{\delta \Phi}{\delta m} (\mu)(q') = 0,
\]

where \(\mu'\) and \(q'\) are the derivatives, in the sense of distributions, of \(\mu\) and \(q\).

**Proof.** We start from the very definition of rotation-invariant function. It says that, for any \(\psi \in \mathbb{T}\),

\[
\frac{d}{d\psi} \Phi(\mu \circ \tau_{\psi}^{-1}) = 0.
\]

The left-hand side writes in the form \(-\langle [\delta \Phi/\delta m](\mu), \mu' \rangle\). We deduce the first identity in the statement.

Assume now that \(\mu\) has a (strictly) positive continuous density \(\mu'dx\) and consider an element \(q \in L^{\infty}(\mathbb{T})\) such that \(\langle q, 1 \rangle = 0\). Then, for \(\varepsilon\) small enough, \(\mu' dx + \varepsilon q\) may be regarded as a density on \(\mathbb{T}\). With a slight abuse of notation, we write \(\mu + \varepsilon q \in \mathcal{P}(\mathbb{T})\). Replacing \(\mu\) by \(\mu + \varepsilon q\) in the first identity in the statement and then
taking the derivative with respect to $\varepsilon$ at $\varepsilon = 0$, we deduce that the second identity in the statement holds at any pair $(\mu, q)$ satisfying the prescribed conditions. Using the density of $L^\infty(T)$ in $(W^1,\infty)'(T)$ together with the fact that $\Phi$ satisfies \textbf{(Diff-$\Phi$)}, the second identity also holds at any pair $(\mu, q)$, with $\mu$ as before and $q \in (W^{1,\infty})'(T)$ such that $\langle q, 1 \rangle = 0$. Approximating (for the 1-Wasserstein topology) any $\mu \in \mathcal{P}(T)$ by probability measures with a positive density and then invoking again the regularity properties of $\Phi$, we finally obtain that the second identity is satisfied at any $\mu \in \mathcal{P}(T)$ and $q \in (W^{1,\infty}(T))'$ with $\langle q, 1 \rangle = 0$. \hfill $\Box$

We deduce the following important proposition.

\textbf{Proposition 4.12.} Assume that $\Phi$ satisfies \textbf{(Reg-$\Phi$-(4,3))}. Then, for any $\alpha \in (0,1)$ and $\eta \in (0,1)$, there exist two positive constants $\lambda$ and $C$, with $\lambda$ only depending on $\kappa$, $\alpha$ and $\eta$, and $C$ only depending on $\kappa$, $\alpha$, $\eta$ and the bounds in \textbf{(Reg-$\Phi$-(\alpha,2))}, such that, for any $\mu \in \mathcal{Q}_\eta$, any $t > 0$ and any $z_1, z_2 \in T$,

$$\left| \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) \right| \leq C, \quad t \geq 0,$$

$$\min(1, t^{1-\alpha/4}) \left| \partial_{z_2} \partial_{z_1} \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) \right| \leq Ce^{-\lambda}, \quad t > 0.$$

\textbf{Proof. First Step.} We start with the proof of (4.37). It is a mere consequence of the representation formula in Theorem 3.2 and the bound (4.27) in the statement of Proposition 4.10.

\textbf{Second Step.} We now turn to the proof of (4.38). We recall the following formula from Theorem 3.5:

$$\partial_{z_2} \partial_{z_1} \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) = \frac{\partial^2 \Phi}{\partial \mu^2}(m(t; \mu)) \left( d^{(1)}(t; \mu, z_1), d^{(1)}(t; \mu, z_2) \right) + \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( d^{(2)}(t; \mu, z_1, z_2) \right),$$

for $\mu \in \mathcal{P}(T)$, $t \geq 0$ and $z_1, z_2 \in T$. Using the same notation as in the statement of Proposition 4.10, we rewrite the above identity in the form

$$\partial_{z_2} \partial_{z_1} \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) = \frac{\partial^2 \Phi}{\partial \mu^2}(m(t; \mu)) \left( d^{(1)}(t; \mu, z_1), d^{(1)}(t; \mu, z_2) \right) + q^{(1)}(\mu, z_2) \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( p'(t; \mu), d^{(1)}(t; \mu, z_2) \right) + \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( d^{(2)}(t; \mu, z_1, z_2) \right).$$

With the same notation as in (4.32) and (4.33), we get

$$\partial_{z_2} \partial_{z_1} \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) = \frac{\partial^2 \Phi}{\partial \mu^2}(m(t; \mu)) \left( d^{(1)}(t; \mu, z_1), d^{(1)}(t; \mu, z_2) \right) - q^{(1)}(\mu, z_2) \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( \partial_{z_2} d^{(1)}(t; \mu, z_2) \right) + \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( d^{(2)}(t; \mu, z_1, z_2) \right).$$

By the first identity in (4.36), we can remove for free $q^{(2)}(\mu, z_1, z_2)p'(t; \mu)$ in the second term on the last line. Using again the notation from the statement of Proposition 4.10, we obtain

$$\partial_{z_2} \partial_{z_1} \frac{\partial^2 U}{\partial \mu^2}(t, \mu)(z_1, z_2) = \frac{\partial^2 \Phi}{\partial \mu^2}(m(t; \mu)) \left( d^{(1)}(t; \mu, z_1), d^{(1)}(t; \mu, z_2) \right) + \frac{\partial \Phi}{\partial m}(m(t; \mu)) \left( d^{(2)}(t; \mu, z_1, z_2) \right).$$

The end of the proof is similar to the derivation of (3.22), by using (4.28) and combining with the shape of \textbf{(Reg-$\Phi$-(\alpha,2))} (see, in particular, (2.16)). \hfill $\Box$
4.4. Semi-group generated by the empirical distribution. We now address the weak error, as in the statement of Theorem 4.3. By the same regularisation argument as in the proof of Theorem 3.8, we can assume that $\Phi$ satisfies (Reg-$\Phi$-(4.3)). Following the statement of Proposition 4.12, we must then prove that the constant $C$ that we obtain in the end in the main inequality of Theorem 4.3 only depends on $\Phi$ through the bounds in (Reg-$\Phi$-(\(\alpha, 2\))), for a fixed value of $\alpha$ as in the statement of Theorem 4.3.

Throughout the proof, we make use of the semi-group generated by the empirical distribution. We hence call $\mathcal{P}_N(\mathbb{T})$ the collection of probability measures $\mu$ that are uniformly distributed on some finite state $\{x_1, \cdots, x_N\}$, with $(x_1, \cdots, x_N) \in \mathbb{T}^N$. In particular, we have that, with probability 1, $\mu^N$ is in $\mathcal{P}_N(\mathbb{T})$.

With this notation, we can define the analogue of the mapping $\mathcal{U}$, but for the particle system (1.1), namely:

$$\mathcal{U}^N(t, \mu) = \mathbb{E}[\Phi(\mu^N) \mid \mu_0^N = \mu], \quad \mu \in \mathcal{P}_N(\mathbb{T}).$$

This definition requires a modicum of care, since we must check that $\mathcal{U}^N(t, \cdot)$ is indeed a function of $\mu \in \mathcal{P}_N(\mathbb{T})$ (in other words, it is a symmetric function of $(x_1, \cdots, x_N)$). This follows from the exchangeability of the particle system (1.1): for any permutation $\sigma$ on $\{1, \cdots, N\}$, $(Y^1_{t,N}, \cdots, Y^N_{t,N})_{t \geq 0}$ starting from $(x_1, \cdots, x_N)$ has the same law as $(Y^1_{t,N}, \cdots, Y^N_{t,N})_{t \geq 0}$ starting from $(x_{\sigma(1)}, \cdots, x_{\sigma(N)})$. Accordingly, the function $\mathcal{U}^N(t, \cdot)$ is a symmetric function of $(x_1, \cdots, x_N)$ and hence may be regarded as a function of $\sum_{i=1}^N \delta_{x_i}/N$. Notice in particular that, by the Markov property of $(Y^1_{t,N}, \cdots, Y^N_{t,N})_{t \geq 0}$, for any $0 \leq s \leq t$, we have

$$\mathbb{E}[\Phi(\mu^N_{t-s})] = \mathcal{U}^N(t-s, \mu^N_{s-N}), \quad (4.39)$$

with the notation in the middle term being rather abusive, but making sense thanks to the symmetry of $\mathcal{U}^N$.

We then proceed step by step. The first step is dictated by the analysis of the strong error in [27] and consists of a preliminary form of the result up until time $\exp(N^{1/2})$, for $\mu$ close enough to $\mathcal{I} = \{p_\psi, \psi \in \mathbb{T}\}$.

**Lemma 4.13.** There exist $\delta > 0$, only depending on $\kappa$, and a constant $C \geq 0$, only depending on $\delta$, $\kappa$ and the bounds for $\Phi$ in (Reg-$\Phi$-(\(\alpha, 2\))), such that, for any $N \geq 1$ and any $\mu \in \mathcal{P}_N(\mathbb{T})$ satisfying $\text{dist}_{\|\cdot\|_{-1,2}}(\mu, \mathcal{I}) \leq \delta$,

$$\forall t \in [0, \exp(N^{1/2})], \quad \|\mathcal{U}^N(t, \mu) - \mathcal{U}(t, \mu)\| \leq \frac{C}{N},$$

where $\text{dist}_{\|\cdot\|_{-1,2}}(\mu, \mathcal{I}) = \inf_{p_\psi \in \mathbb{T}} \|\mu - p_\psi\|_{-1,2}$. Here, we may choose $\delta$ such that $\{\mu : \text{dist}_{\|\cdot\|_{-1,2}}(\mu, \mathcal{I}) \leq \delta\} \subset \mathcal{Q}_{1/2}$.

**Proof.** The proof relies on some auxiliary results obtained in [27].

**First Step.** We apply [27, (98-99)]. It says that there exist a time $T > 0$, a real $\varepsilon > 0$ and a constant $C$, all independent of $N$ and the initial conditions, such that, for any integer $n \geq 1$,

$$\mathbb{P}\left(\sup_{0 \leq t \leq nT} \text{dist}_{\|\cdot\|_{-1,2}}(\mu^N_t, \mathcal{I}) \leq \varepsilon \|\mu_0^N - \text{proj}(\mu_0^N)\|_{-1,2} \leq \frac{\varepsilon}{2}\right) \geq \left[1 - \exp(-CN)\right]^n, \quad (4.40)$$

where $\text{proj} : \mathcal{P}(\mathbb{T}) \to \mathbb{T}$ is the projection mapping defined in [49, Lemma 2.8] and [27, Lemma 3.5] (with the small difference\(^4\) that the function $\mathcal{T}$ is defined here as $\mathbb{R}/\mathbb{Z}$, whilst it is defined as $\mathbb{R}/(2\pi\mathbb{Z})$ in [27, 49]).

\(^4\)There is another technical difference. In [27, 49], the dual space is not $W^{-1,2}(\mathbb{T})$, but $H_0^{-1} = \{f \in W^{1,2}(\mathbb{T}) : (f, 1) = 0\}'$. Of course, any element of $W^{-1,2}(\mathbb{T})$ may be canonically projected onto an element of $H_0^{-1}$ by discarding its constant Fourier mode.
Notice that the conditioning in the left-hand side is implicitly required in [27] (see (84)). More importantly, there is no need to assume that \( \mu^N_0 \) is the \( N \)-sample of a common distribution and the result holds for an arbitrary initial condition \( \mu^N_0 = \mu \in \mathcal{P}_N(\mathbb{T}) \).

On \( \mathcal{I} \), the projection \( \text{proj} \) reduces to the trivial mapping \( \text{proj}(p_\psi) = \psi, \psi \in \mathbb{T} \). Therefore, if some probability measure \( \nu \in \mathcal{P}(\mathbb{T}) \) is close to \( \mathcal{I} \) (for \( \| \cdot \|_{-1,2} \)), then it is close to some \( p_\psi \in \mathcal{I} \) and, by continuity of \( \text{proj} \) with respect to \( W^{-1,2}(\mathbb{T}) \), \( \text{proj}(\nu) \) is close to \( \psi \). In the end, \( \nu \) is close to \( p_{\text{proj}(\nu)} \). This continuity result, combined with a standard compactness argument, may be formulated as follows. For the same \( \varepsilon \) as above, we can find some \( \delta_\varepsilon > 0 \) such that \( \text{dist}_{\| \cdot \|_{-1,2}}(\nu, \mathcal{I}) \leq \delta_\varepsilon \) implies \( \| \nu - p_{\text{proj}(\nu)} \|_{-1,2} \leq \varepsilon/2 \). Hence, taking \( \mu \in \mathcal{P}_N(\mathbb{T}) \) with \( \text{dist}_{\| \cdot \|_{-1,2}}(\mu, \mathcal{I}) \leq \delta_\varepsilon \), we get

\[
\mathbb{P}\left( \sup_{0 \leq t \leq nT} \text{dist}_{\| \cdot \|_{-1,2}}(\mu^N_t, \mathcal{I}) \leq \varepsilon \mid \mu^N_0 = \mu \right) \geq 1 - \exp(-CN)^n.
\]

Following [27, (100)], we deduce that, for \( n = \exp(N^{1/2}) \) and for a new value of \( C \),

\[
\mathbb{P}\left( \sup_{0 \leq t \leq nT} \text{dist}_{\| \cdot \|_{-1,2}}(\mu^N_t, \mathcal{I}) \leq \varepsilon \mid \mu^N_0 = \mu \right) \geq 1 - \frac{C}{N}.
\]

This may be rewritten in the form

\[
\mathbb{P}\left( \tau_N \leq \exp(N^{1/2}) \mid \mu^N_0 = \mu \right) \leq \frac{C}{N}, \tag{4.41}
\]

where \( \tau_N \) is the stopping time

\[
\tau_N = \inf\left\{ t \geq 0 : \text{dist}_{\| \cdot \|_{-1,2}}(\mu^N_t, \mathcal{I}) \geq \varepsilon \right\}.
\]

**Second Step.** Following (2.8), we have the following expansion, which holds for any \( t \geq 0 \),

\[
\mathbb{E}[\Phi(\mu^N_t) \mid \mu^N_0 = \mu] - \mathcal{U}(t, \mu) = \mathbb{E}[\Phi(\mu^N_t) \mid \mu^N_0 = \mu] - \mathbb{E}[\mathcal{U}(t - t \wedge \tau_N, \mu^N_{t \wedge \tau_N}) \mid \mu^N_0 = \mu]
\]

\[
+ \mathbb{E}[\mathcal{U}(t - t \wedge \tau_N, \mu^N_{t \wedge \tau_N}) - \mathcal{U}(t, \mu^N_0) \mid \mu^N_0 = \mu]. \tag{4.42}
\]

If we restrict ourselves to \( t \in [0, \exp(N^{1/2})] \), then (4.41) says that the first difference in the right-hand side is bounded by \( C/N \), by recalling that \( \Phi \) and thus \( \mathcal{U} \) are bounded and allowing \( C \) to depend on \( \| \Phi \|_\infty \).

As for the second term in the right-hand side of (4.42), we may follow (2.9), by expanding the term \( \mathcal{U}(t - s, \mu^N_s) \) up until time \( t \wedge \tau_N \). Using the fact that, up until time \( \tau_N \), the \( \| \cdot \|_{-1,2} \)-distance between \( \mu^N \) and \( \mathcal{I} \) remains less than \( \varepsilon \) and assuming without any loss of generality that \( \varepsilon \) is small enough so that \( \{ \mu : \text{dist}_{\| \cdot \|_{-1,2}}(\mu, \mathcal{I}) \leq \varepsilon \} \subset \mathcal{Q}_{1/2} \), we get from Proposition 4.12 that this term is also bounded by \( C/N \), with \( C \) now depending on \( \kappa \) and the bounds for \( \Phi \) in (Reg-\( \Phi \)-\( (\alpha, 2) \)).

The result is now extended to initial conditions \( \mu \in \mathcal{P}_N(\mathbb{T}) \cap \mathcal{Q}_\eta \), for any \( \eta > 0 \), but on a (slightly) smaller time scale.

**Proposition 4.14.** For any \( \eta \in (0, 1) \), there exists a constant \( C \), only depending on \( \eta, \kappa \) and the bounds for \( \Phi \) in (Reg-\( \Phi \)-\( (\alpha, 2) \)), such that, for any \( N \geq 1 \) and \( \mu \in \mathcal{Q}_\eta \cap \mathcal{P}_N(\mathbb{T}) \),

\[
\forall t \in [0, \exp(N^{1/4})], \quad |\mathcal{U}^N(t, \mu) - \mathcal{U}(t, \mu)| \leq \frac{C}{N}. \tag{4.43}
\]
Proof. First step. Lemma 4.13 says that, for $\delta$ as therein, the result holds up until time $\exp(N^{1/2})$ if $\text{dist}_{1,2}(\mu, I) \leq \delta$. Then, for any $\mu \in \mathcal{Q}_\eta \cap \mathcal{P}_N(\mathbb{T})$, Proposition 4.1 says that there exists a (fixed hence independent of $N$) time $T$, only depending on $\delta$, $\eta$ and $\kappa$, such that

$$\forall t \geq T, \quad \text{dist}_{1,2}(m(t; \mu), I) \leq \frac{\delta}{2}. \quad (4.44)$$

Next, we claim that (the proof is given in the third step below)

$$\mathbb{P}(\|m(T; \mu) - m^N\|_{1,2} > \frac{\delta}{2} \mid \mu^N_0 = \mu) \leq \frac{C}{N}, \quad (4.45)$$

where $C$ only depends on $\eta$, $\kappa$ and the bounds for $\Phi$ in $(\text{Reg-}\Phi-(\alpha, 2))$, from which we deduce that

$$\mathbb{P}(\text{dist}_{1,2}(\mu^N, I) > \delta \mid \mu^N_0 = \mu) \leq \frac{C}{N}. \quad (4.46)$$

By (4.39), we have, for $t \geq T$,

$$\overline{U}^N(t, \mu) = \mathbb{E}[U^N(t - T, \mu^N_T) \mid \mu^N_0 = \mu],$$

when $\mu^N_0 = \mu$. Hence, by (4.43) (for $\text{dist}_{1,2}(\mu, I) \leq \delta$) and (4.46), for $t \in [T, \exp(N^{1/2})/2],$

$$\overline{U}^N(t, \mu) - \mathbb{E}[U(t - T, \mu^N_T)1_{\{\text{dist}_{1,2}(\mu^N, I) \leq \delta\}} \mid \mu^N_0 = \mu] \leq \frac{C}{N}. \quad (4.47)$$

Second Step. We complete the proof of (4.43), recalling that the latter is already known to hold on $[0, T]$ (since there is no need for any ergodic estimates in finite time). In order to proceed, assume for a while that we are given a function $\Xi : \mathcal{P}(\mathbb{T}) \to [0, 1]$ such that

1. $\Xi$ matches 1 on $\mathcal{Q}_{1/2}$ (the latter containing $\{\nu \in \mathcal{P}(\mathbb{T}) : \text{dist}_{1,2}(\nu, I) \leq \delta\}$) and 0 outside $\mathcal{Q}_{3/4};$
2. $\Xi$ is a smooth functional of $\mu \in \mathcal{P}(\mathbb{T}).$

Then, since $\mathcal{Q}_{1/2}$ contains $\{\nu \in \mathcal{P}(\mathbb{T}) : \text{dist}_{1,2}(\nu, I) \leq \delta\}$, we can apply (4.47). Since $\Xi$ is zero outside $\mathcal{Q}_{3/4}$, we know from Proposition 4.12 that $U(t - T, \cdot)\Xi(\cdot)$ satisfies $(\text{Reg-}\Phi-(\alpha, 2))$ with explicit bounds that are uniform with respect to $t \geq T$. We then apply the finite-horizontal version of (4.43) to $\Phi(\cdot) = U(t - T, \cdot)\Xi(\cdot)$. By (4.47), we get, for any $t \geq T$ and $\mu \in \mathcal{Q}_\eta \cap \mathcal{P}_N(\mathbb{T}),$

$$\overline{U}^N(t, \mu) - U(t - T, m(T; \mu))\Xi(m(T; \mu)) \leq \frac{C}{N},$$

for $C$ only depending on $\eta$, $\kappa$ and the bounds for $\Phi$ in $(\text{Reg-}\Phi-(\alpha, 2))$. Noticing that $U(t - T, m(T; \mu)) = U(t, \mu)$ and that $\Xi(m(T; \mu)) = 1$ for our choice of $T$ (see (4.44)), we get the announced result.
The function $\Xi$ is constructed as in the statement of Proposition 4.4. We consider a smooth non-decreasing cut-off function $\varphi : [0, 1] \to [0, 1]$ that is equal to 1 on $[0, 1/2]$ and to 0 on $[3/4, 1]$. We then let $\Xi(\mu) = \varphi(\mu^1)$.

**Third Step.** We now prove (4.45). It is again a consequence of the finite time horizon of (4.43), but with a difference of choosing (temporarily) the functional $\Phi$ in the definition of $U$ as in Proposition 2.4 with $(d + \alpha)/2 = 1$ and $\nu_0 = \mu_T$ therein. The result then follows from Markov’s inequality.

Our last step is to extend the previous result to times greater than $\exp(N^{1/4})$. The key idea is that, in long time, the empirical measure necessarily visits the set $Q_\eta$ quite often, for any $\eta \in (0, 1)$.

**Proposition 4.15.** For any $\eta \in (0, 1)$, there exists a constant $C$, only depending on $\eta, \kappa$ and the bounds for $\Phi$ in (Reg-$\Phi$-$\eta, 2$)), such that, for any $N \geq 1$ and $\mu \in Q_\eta \cap P_N(T)$,

$$\forall t \geq 0, \quad \left| \overline{U}^N(t, \mu) - U(t, \mu) \right| \leq \frac{C}{N}. \tag{4.48}$$

**Proof.** By Proposition 4.14, it suffices to prove the result for $t \geq \exp(N^{1/4})$. For $T_N = \exp(N^{1/4})/2$, we call

$$\tau_N := \inf\left\{ s > 0 : \left| \int_T e^{-i2\pi \theta} \mu_{s+t-T_N}(d\theta) \right| \geq \eta \right\}.$$ 

We prove below (see Lemma 4.16) that $P(\tau_N \geq N^{1/4}) \leq C/N$, for $C$ as in the statement and for $\eta$ small enough. By the strong version of the Markov property (4.39) (noticing that $N^{1/4} \leq \exp(N^{1/4})/2$ for any $N \geq 1$),

$$\overline{U}^N(t, \mu) = \mathbb{E}\left[ \overline{U}^N\left( T_N + T_N \wedge \tau_N, \mu_{T_N + T_N \wedge \tau_N}^N \right) \right] \bigg| \mu_0^N = \mu,$$

and then, assuming without any loss of generality that $\eta$ is small enough,

$$\left| \overline{U}^N(t, \mu) - \mathbb{E}\left[ \overline{U}^N\left( T_N + T_N \wedge \tau_N, \mu_{T_N + T_N \wedge \tau_N}^N \right) \right] \bigg| \mu_0^N = \mu \right| \leq \frac{C}{N}.$$ 

On the event $\{\tau_N < T_N\}$, $\overline{U}_{T_N + T_N \wedge \tau_N}^N \in Q_\eta$. Therefore, by Proposition 4.14, we can replace $\overline{U}^N$ by $U$ inside the expectation and hence get

$$\left| \overline{U}^N(t, \mu) - \mathbb{E}\left[ U\left( T_N + T_N \wedge \tau_N, \mu_{T_N + T_N \wedge \tau_N}^N \right) \right] \bigg| \mu_0^N = \mu \right| \leq \frac{C}{N}. \tag{4.49}$$

By Proposition 4.1, we know, for $s \geq T_N$ and $\nu \in Q_\eta$,

$$U(s, \nu) = \Phi(L(X_s|X_0 \sim \nu)) = \Phi(I) + O\left( \frac{1}{N} \right) = U(t, \mu) + O\left( \frac{1}{N} \right), \tag{4.50}$$

with the Landau symbol $O(\cdot)$ being independent of $\nu$, from which we deduce that we can replace the expectation in (4.49) by $U(t, \mu)$. This completes the proof. In the above, we used the slightly abusive notation $L(X_s|X_0 \sim \nu)$ to denote the law of $X_s$ in (1.3) when the law of $X_0$ is $\nu$.

**4.5. End of the proof of Theorem 4.3.** We now have all the ingredients to complete the proof of Theorem 4.3. For an $N$-sample $(Y_{i,N}^0)_{i=1,\ldots,N}$ with law $\mu_{\text{init}} \in Q_\eta$, for some $\eta > 0$, we let $\mu_0^N = (1/N) \sum_{i=1}^N \delta_{Y_{i,N}^0}$.

**Proof of Theorem 4.3. First Step.** We start with the following (quite standard) computation:

$$\mathbb{E}\left[ \left| \mu_0 - \mu_0^N \right|^2 \right] = \sum_{n \geq 0} \frac{1}{(1 + n^2)} \mathbb{E}
\left[ \int_T e^{-i2\pi \theta} (\mu_0 - \mu_0^N)(d\theta) \right]^2 \leq \frac{1}{N} \sum_{n \geq 0} \frac{1}{(1 + n^2)} \leq \frac{c}{N}, \quad \tag{4.51}$$
is hence completed provided that we prove the following lemma, which we deduce that, for any \( \varrho > 0 \),

\[
P(\|\mu_0 - \mu^N_0\|_{-1,2} \geq \varrho) \leq \frac{e}{\varrho^2 N}.
\]

Now, we can indeed choose \( \varrho \) such that, for any two probability measures \( \nu_1, \nu_2 \in \mathcal{P}(\mathbb{T}) \) with \( \|\nu_1 - \nu_2\|_{-1,2} \leq \varrho \), it holds that \( |\nu_1^1 - \nu_2^1| \leq \eta/2 \) (we recall that \( \nu_1^1 \) and \( \nu_2^1 \) are the 1-Fourier modes of \( \nu_1 \) and \( \nu_2 \)), from which we get that, for a constant \( C \) depending on \( \eta \),

\[
P(\mu_0^N \in \mathcal{Q}_{\eta/2}) \leq \frac{C}{N}.
\]

Therefore, by Proposition 4.15, there exists a constant \( C \), only depending on \( \kappa, \eta \) and the bounds for \( \Phi \) in \((\text{Reg-} \Phi)(\alpha, 2)\), such that

\[
\forall t \geq 0, \quad \left| \mathbb{E}[\mathcal{U}^N(t, \mu_0^N)] - \mathbb{E}[\mathcal{U}(t, \mu_0^N) \mathbf{1}_{\{\mu_0^N \in \mathcal{Q}_{\eta/2}\}}] \right| \leq \frac{C}{N}.
\]

We deduce that it suffices to show that

\[
\left| \mathbb{E}\left[\left(\mathcal{U}(t, \mu_{\text{init}}) - \mathcal{U}(t, \mu_0^N)\right) \mathbf{1}_{\{\mu_0^N \in \mathcal{Q}_{\eta/2}\}}\right]\right| \leq \frac{C}{N}. \tag{4.52}
\]

**Second Step.** In order to prove (4.52), we may argue as in the second step of the proof of Proposition 4.14. Indeed, we can consider a smooth function \( \Xi : \mathcal{P}(\mathbb{T}) \to [0, 1] \) such that \( \Xi \) is 1 on the set \( \mathcal{Q}_{\eta/2} \) and 0 outside the set \( \mathcal{Q}_{\eta/4} \). Then, instead of proving (4.52), it suffices to show that

\[
\left| \mathbb{E}\left[\mathcal{U}(t, \mu_{\text{init}})\Xi(\mu_{\text{init}}) - \mathcal{U}(t, \mu_0^N)\Xi(\mu_0^N)\right]\right| \leq \frac{C}{N}. \tag{4.53}
\]

Thanks to the cut-off function \( \Xi \), the function \( (t, \mu) \mapsto \mathcal{U}(t, \mu)\Xi(\mu) \) satisfies the conclusion of Corollary 4.12, even though \( \mu \notin \mathcal{Q}_\eta \). This suffices to apply (2.10) with \( \mathcal{U}(t, \mu_{\text{init}})\Xi(\mu_{\text{init}}) \) instead of \( \mathcal{U}(t, \mu_{\text{init}}) \) therein. \( \square \)

The proof of Theorem 4.3 is hence completed provided that we prove the following lemma, which we invoked in the proof of Lemma 4.15:

**Lemma 4.16.** There exist a constant \( \eta \in (0, 1) \) and constant \( C \), both independent of \( N \), such that, for any initial distribution \( \mu \in \mathcal{P}(\mathbb{T}) \) and any \( t \geq 0 \), the distribution of the stopping time \( \tau_N := \inf\{s > 0 : |\int_{\mathbb{T}} e^{-i2\pi \theta} \mu_{s+t}^N(d\theta)| \geq \eta\} \) satisfies \( \mathbb{P}(\tau_N \geq N^{1/4}) \leq C/N \).

**Proof.** First Step. We go back to the shape of particle system (1.1) with \( b \) as in (4.1):

\[
dY^j_t \sim N = -\frac{2\pi \kappa}{N} \sum_{k=1}^{N} \sin\left(2\pi (Y_t^j - Y_t^k)\right) dt + dB_t^j, \quad t \geq 0.
\]
Let $E_t^{j,N} = \exp(i2\pi N_t^j)$. Then, recalling the notation $\overline{z}$ for denoting the complex conjugate of a complex number $z \in C$, we obtain that

$$
\frac{dE_t^{j,N}}{N} = -\frac{2\pi^2 \kappa \mu^j_t}{N} E_t^{j,N} \sum_{k=1}^{N} \left( E_t^{j+1,N} \overline{E_t^{k,N}} - E_t^{j,N} \overline{E_t^{k+1,N}} \right) dt - 2\pi^2 \ell^2 E_t^{j,N} dt + i2\pi \ell E_t^{j,N} dW_t^j
$$

where $\mu_t^{\ell,N} = \sum_{j=1}^{N} E_t^{j,N}$ is the $\ell$-Fourier mode of $\mu_t^N$. Taking the mean over $j \in \{1, \cdots, N\}$, we get

$$
d\mu_t^{\ell,N} = -2\pi^2 \kappa \ell \left( \mu_t^{\ell+1,N} - \mu_t^{\ell-1,N} \right) dt - 2\pi^2 \ell^2 \mu_t^{\ell,N} dt + i2\pi \ell \sum_{j=1}^{N} E_t^{j,N} dW_t^j.
$$

Choosing $\ell = 1$ and using the fact that $\kappa > 1$, we deduce that there exists a constant $c > 1$, only depending on $\kappa$, such that

$$
d[|\mu_t^{1,N}|^2] = (c^{-1} - c|\mu_t^{2,N}|)|\mu_t^{1,N}|^2 dt + \frac{c^{-1}}{N} dt + dK_t^1 + dM_t^1,
$$

where $(K_t^1)_{t \geq 0}$ is a non-decreasing absolutely continuous process and $(M_t^1)_{t \geq 0}$ is a martingale satisfying $[d/dt](M_t^1)_t \leq c/N$. Similarly, choosing $\ell = 2$, we get

$$
d[|\mu_t^{2,N}|^2] = (c|\mu_t^{1,N}|^2 - c^{-1}|\mu_t^{2,N}|^2) dt - dK_t^2 + \frac{c}{N} dt + dM_t^2,
$$

where $(K_t^2)_{t \geq 0}$ is a non-decreasing absolutely continuous process and $(M_t^2)_{t \geq 0}$ is a martingale satisfying $[d/dt](M_t^2)_t \leq c/N$. We now let

$$
\Lambda_t^N = |\mu_t^{2,N}|^2 - c^4|\mu_t^{1,N}|^2, \quad t \geq 0.
$$

Using the expansions (4.54) and (4.55), we obtain

$$
d\Lambda_t^N = \left[ (c|\mu_t^{1,N}|^2 - c^{-1}|\mu_t^{2,N}|^2) - c^4(c^{-1} - c|\mu_t^{2,N}|)|\mu_t^{1,N}|^2 \right] dt - \frac{c^3}{N} dt + dK_t + dM_t,
$$

where $(K_t)_{t \geq 0}$ is a non-decreasing absolutely continuous process and $(M_t)_{t \geq 0}$ is a martingale satisfying $[d/dt](M_t)_t \leq C(c)/N$, in which $C(c)$ is a constant that only depends on $c$. We then consider the same stopping time $\tau_N$ as in the statement, but with $\eta = c^{-4}/4$. As long as $t \leq \tau_N$, we have (notice that the term below is nothing but the first term in the expansion of $d\Lambda_t^N$)

$$
(c|\mu_t^{1,N}|^2|\mu_t^{2,N}| - c^{-1}|\mu_t^{2,N}|^2) - c^4(c^{-1} - c|\mu_t^{2,N}|)|\mu_t^{1,N}|^2
\leq -c^3|\mu_t^{1,N}|^2 - c^{-1}|\mu_t^{2,N}|^2 + c^4|\mu_t^{1,N}|^2 + c^5|\mu_t^{2,N}|^2|\mu_t^{1,N}|^2
\leq -c^3|\mu_t^{1,N}|^2 - c^{-1}|\mu_t^{2,N}|^2 + \frac{5}{4}c|\mu_t^{1,N}|^2|\mu_t^{2,N}|
= -\frac{3}{8}c^{-1} \left( |\mu_t^{2,N}|^2 + c^4|\mu_t^{1,N}|^2 \right) - \frac{5}{8} \left( c^{3/2}|\mu_t^{1,N}| - c^{-1/2}|\mu_t^{2,N}| \right)^2 \leq -\frac{3}{8}c^{-1} \Lambda_t^N.
$$

By modifying the definition of $(K_t)_{t \geq 0}$ in (4.57) and by assuming that $c^3 \geq 2c$, we then get

$$
d\Lambda_t^N = -\frac{3}{8}c^{-1} \Lambda_t^N dt - \frac{c}{N} dt + dK_t + dM_t.
Therefore,
\[
A_t^N = \exp\left(-\frac{3}{8}c^{-1}t\right)\left[A_0^N - \frac{c}{N} \int_0^t \exp\left(\frac{3}{8}c^{-1}s\right)ds - \int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dK_s + \int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right]
\]
\[
\leq \exp\left(-\frac{3}{8}c^{-1}t\right)\left[A_0^N + \int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right].
\]
(4.58)

We now observe that, for any integer \(n \geq 0\),
\[
\sup_{n \leq t \leq n+1} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right] \leq \sup_{n \leq t \leq n+1} \left[\int_0^t \exp\left(\frac{3}{8}c^{-1}(s - n)\right) dM_s\right].
\]

Accordingly, by Burkholder-Davis-Gundy inequalities, we deduce that, for any integer \(p \geq 1\), there exists a constant \(C_p(c)\), depending on \(p\) and \(c\), such that
\[
\mathbb{E} \left[\sup_{n \leq t \leq n+1} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right]^p\right] \leq \frac{1}{N^{p/2}} C_p(c) \left(\int_n^{n+1} \exp\left(\frac{3}{4}c^{-1}(s - n)\right) ds\right)^{p/2}.
\]

In turn, by Markov’s inequality, we deduce that, for any \(\varepsilon > 0\),
\[
\mathbb{P}\left(\sup_{n \leq t \leq n+1} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right] \geq \varepsilon\right) \leq \frac{C_p(c)}{\varepsilon p N^{p/2}},
\]
for a new value of the constant \(C_p(c)\), and then
\[
\mathbb{P}\left(\bigcup_{n=0}^{\lfloor N^{1/4} \rfloor} \left\{\sup_{n \leq t \leq n+1} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right] \geq \varepsilon\right\}\right) \leq \frac{C_p(c)}{\varepsilon p N^{p/2-1/4}},
\]
where \(\lfloor N^{1/4} \rfloor\) denotes the floor of \(N^{1/4}\). Choosing \(p\) large enough and using a new value of the constant \(C(c)\), we end up with
\[
\mathbb{P}\left(\bigcup_{n=0}^{\lfloor N^{1/4} \rfloor} \left\{\sup_{n \leq t \leq n+1} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right] \geq \varepsilon\right\}\right) \leq \frac{C(c)}{\varepsilon p N^{1/2}}.
\]

The value of \(p\) that appears in the right-hand side is hence fixed. This prompts us to introduce the event
\[
A_N(\varepsilon) := \left\{\sup_{0 \leq t \leq \lfloor N^{1/4} \rfloor} \left[\exp\left(-\frac{3}{8}c^{-1}t\right)\left|\int_0^t \exp\left(\frac{3}{8}c^{-1}s\right) dM_s\right|\right] \geq \varepsilon\right\}.
\]

On the latter event, we have, by (4.58),
\[
\forall t \in \left[0, N^{1/4} \wedge \tau_N\right], \quad A_t^N \leq \exp\left(-\frac{3}{8}c^{-1}t\right) A_0^N + \varepsilon.
\]

Back to the definition of (4.56), this yields
\[
\forall t \in \left[0, N^{1/4} \wedge \tau_N\right], \quad |\mu_t^2|^2 - b^4|\mu_t^{1,N}|^2 \leq \exp\left(-\frac{3}{8}c^{-1}t\right) A_0^N + \varepsilon,
\]
that is
\[ \forall t \in [0, N^{1/4} \land \tau_N], \quad |\mu^2_{t,N}|^2 \leq \exp\left(\frac{3}{8}c^{-1}t\right)\Lambda_0^N + \varepsilon + c^4|\mu^1_{t,N}|^2 \leq \exp\left(\frac{3}{8}c^{-1}t\right) + \varepsilon + c^4|\mu^1_{t,N}|^2, \]
where we used the obvious inequality \( \Lambda_0^N \leq 1. \)

Second Step. We thus introduce the following times. First, we call \( t_0(\varepsilon) \) the smallest (deterministic) time such that \( \exp(-(3/8)c^{-1}t_0(\varepsilon)) \leq \varepsilon \). Second, we let
\[ \sigma_N(\varepsilon) := \inf\{t \geq t_0 : |\mu^1_{t,N}| \geq \varepsilon c^{-4}\}. \]
Then, assuming that \( \varepsilon \in (0, 1/4) \) and recalling that \( \eta = c^{-4}/4 \) in the definition of \( \tau_N \), we obviously have \( \sigma_N(\varepsilon) \leq \tau_N \lor t_0(\varepsilon) \), which implies
\[ \forall t \in [t_0(\varepsilon), N^{1/4} \land \sigma_N(\varepsilon) \land \tau_N], \quad |\mu^2_{t,N}|^2 \leq 3\varepsilon, \]
at least if the above interval is not empty. Subsequently, plugging the latter into (4.54), we obtain
\[ \forall t \in [t_0(\varepsilon), N^{1/4} \land \sigma_N(\varepsilon) \land \tau_N], \quad d[|\mu^1_{t,N}|^2] = (c^{-1} - \sqrt{3}\varepsilon c^{-1/2})|\mu^1_{t,N}|^2dt + \frac{c^{-1}}{N}dt + d\tilde{K}_t^1 + dM^1_t, \]
for a non-decreasing absolutely continuous process \( (\tilde{K}_t^1)_{t \geq 0} \).

So far, \( \varepsilon \) has been a free parameter. Now, we can choose it such that \( c^{-1} - \sqrt{3}\varepsilon c^{-1/2} = c^{-1}/2 \). For this given value of \( \varepsilon \) (which is now frozen in terms of \( c \)), we get
\[ \forall t \in [t_0(\varepsilon), N^{1/4} \land \sigma_N(\varepsilon) \land \tau_N], \quad |\mu^1_{t,N}|^2 \geq \exp\left(\frac{c^{-1}}{2}t\right)\left[\frac{c^{-1}}{2} \int_{t_0(\varepsilon)}^t \exp\left(-\frac{c^{-1}}{2}s\right)ds + \int_{t_0(\varepsilon)}^t \exp\left(-\frac{c^{-1}}{2}s\right)dM^1_s\right]. \]

In particular, for all \( t \geq 0 \),
\[
\mathbb{E}\left[\mathbb{1}_{\{\tau_N > t_0(\varepsilon)\}} |\mu^1_{t_0(\varepsilon),N} - \mu^1_{t_0(\varepsilon),\sigma_N(\varepsilon)}|^2\right] \\
\geq \frac{c^{-1}}{N} \mathbb{E}\left[\mathbb{1}_{\{\tau_N > t_0(\varepsilon)\}} \int_{t_0(\varepsilon)}^{t_0(\varepsilon) \lor (t \land N^{1/4} \land \sigma_N(\varepsilon))} \exp\left(\frac{c^{-1}}{2}(t-s)\right)ds\right] \\
= \frac{2}{N} \mathbb{E}\left[\mathbb{1}_{\{\tau_N > t_0(\varepsilon)\}} \left(\exp\left(\frac{c^{-1}}{2}(t-t_0(\varepsilon))\right) - \exp\left(\frac{c^{-1}}{2}(t-t_0(\varepsilon) \lor [t \land N^{1/4} \land \sigma_N(\varepsilon)]\right)\right)\right].
\]
Choosing \( t = N^{1/4} \), we obtain that
\[ 1 \geq \frac{2}{N} \exp\left(\frac{c^{-1}}{2}(N^{1/4} \land t_0(\varepsilon))\right) \mathbb{P}\left(\{\sigma_N(\varepsilon) \geq N^{1/4}, \tau_N > t_0(\varepsilon)\}\right), \]
at least if \( N^{1/4} \geq t_0(\varepsilon) \), which yields
\[ \mathbb{P}\left(\{\sigma_N(\varepsilon) \geq N^{1/4}, \tau_N > t_0(\varepsilon)\}\right) \leq \frac{N}{2} \exp\left(-\frac{c^{-1}}{2}(N^{1/4} \land t_0(\varepsilon))\right). \]
Recalling that \( \tau_N \geq \sigma_N(\varepsilon) \) if \( \tau_N > t_0(\varepsilon) \), with \( \eta = c^{-4}/4 \) in the definition of \( \tau_N \), this completes the proof. \( \square \)
5. APPENDIX

We introduce here a new type of regularisation for a function \( \Phi \) defined on the space of probability measures on \( \mathbb{T}^d \). The statement below is tailored to our needs, but we strongly believe that the method of proof, which consists in convoluting \( \Phi \) when expressed in terms of the Fourier coefficients of the measure argument, is of a wider scope.

**Theorem 5.1.** Let \( \Phi : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R} \) satisfy (Reg-\( \Phi(\alpha, 2) \)) for \( \alpha \in [0, 1) \). Then there exists a sequence of smooth functions \( (\Phi_n)_{n \geq 1} \), satisfying (Reg-\( \Phi(\alpha, 4, 3) \)) for each \( n \geq 1 \), and converging to \( \Phi \), uniformly on \( \mathbb{T}^d \), such that the bounds satisfied by \( (\Phi_n)_{n \geq 1} \) in (Reg-\( \Phi(\alpha, 2) \)) are uniform in \( n \geq 1 \).

The proof relies on the following reformulation of Bochner-Herglotz’ theorem on the torus (see the proof of [56, Proposition 5]).

**Lemma 5.2.** Let \( \mu \) be a probability measure on \( \mathbb{T}^d \), with Fourier coefficients \( (\hat{\mu}^n = \int_{\mathbb{T}^d} e^{-i2\pi n \cdot \theta} \mu(d\theta))_{n \in \mathbb{Z}^d} \), where the index \( n \) is written below in the form \( n = (n_1, \ldots, n_d) \). Then, for any integer \( N \geq 1 \), the function

\[
  f_N(\theta; \mu) = \frac{1}{N^d} \sum_{1 \leq n_1, n_2 \leq N, j \in (1, \ldots, d)} \hat{\mu}^{n-n'} \exp(i2\pi(n-n') \cdot \theta), \quad \theta \in \mathbb{T}^d,
\]

is a density on \( \mathbb{T}^d \). Moreover, the Fourier coefficients of \( f_N \) are given by

\[
  \hat{f}_N(\mu) = \begin{cases} 
    \hat{\mu}^n \prod_{j=1}^d (1 - \frac{|n_j|}{N}), & \text{max}_{j=1, \ldots, d} |n_j| < N, \quad n \in \mathbb{Z}^d, \\
    0, & \text{otherwise}
  \end{cases}
\]

Below, we let

\[
  m_N(\mu) := f_N(\cdot; \mu) \cdot \text{Leb}_{\mathbb{T}^d}.
\]

We have the following very useful lemma:

**Lemma 5.3.** With the same notation as in (5.1), \( \lim_{N \to \infty} \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} W_1(m_N(\mu), \mu) = 0. \)

**Proof.** (Lemma 5.3.) We notice from the expression of the coefficients \( (\hat{f}_N(\mu))_{n \in \mathbb{Z}^d} \) that, for any \( \mu \in \mathcal{P}(\mathbb{T}^d) \),

\[
  \left\| m_N(\mu) - \mu \right\|_{(d/2+1, 2)}^2 = \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|^2)^{d/2+1}} |m_N(\mu)^n - \hat{\mu}^n|^2
\]

\[
  \leq \sum_{\max |n_i| \leq N} \frac{1}{(1 + |n|^2)^{d/2+1}} |m_N(\mu)^n - \hat{\mu}^n|^2 + \sum_{\exists i: |n_i| \geq N} \frac{1}{(1 + |n|^2)^{d/2+1}} |m_N(\mu)^n - \hat{\mu}^n|^2
\]

\[
  \leq \sum_{\max |n_i| \leq N} \frac{1}{(1 + |n|^2)^{d/2+1}} \left( \prod_{j=1}^d (1 - \frac{|n_j|}{N}) \right)^2 \sum_{\exists i: |n_i| \geq N} \frac{1}{(1 + |n|^2)^{d/2+1}},
\]

where \( |n|^2 = n_1^2 + \cdots + n_d^2 \), from which we deduce that

\[
  \lim_{N \to \infty} \sup_{\mu \in \mathcal{P}(\mathbb{T}^d)} \left\| m_N(\mu) - \mu \right\|_{(d/2+1, 2)} = 0.
\]
By a standard convolution argument, we note that, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that, for any function $f \in W^{1,\infty}(\mathbb{T}^d)$, with $\|f\|_{1,\infty} \leq 1$, we can find another function $f_\varepsilon \in W^{d/2+1,2}(\mathbb{T}^d)$ satisfying

$$\|f - f_\varepsilon\|_\infty \leq \varepsilon, \quad \|f_\varepsilon\|_{d/2+1,2} \leq C_\varepsilon.$$

Therefore,

$$\sup_{\|f\|_{1,\infty} \leq 1} \langle f, m^N(\mu) - \mu \rangle \leq 2\varepsilon + \sup_{\|f\|_{d/2+1,2} \leq C_\varepsilon} \|m^N(\mu) - \mu\|_{-(d/2+1,2)}.$$

For $N$ that is large enough (depending on $\varepsilon$), the right-hand side is less than $3\varepsilon$. This completes the proof. \hfill $\square$

We now turn to the proof of Theorem 5.1.

**Proof of Theorem 5.1.** First step. For a given $\mathbf{x} = (x^n)_{n:|n_j| \leq N-1} \in \mathbb{R}^{(2N-1)^d}$ with the constraint that $x^0 = 1$ (with $\mathbf{0} = (0, \cdots, 0) \in \mathbb{Z}^d$) and $\varepsilon > 0$, we call

$$f_{N,\varepsilon}(\theta; \mathbf{x}) := \varepsilon + (1 - \varepsilon) \Re \left[ \frac{1}{N^d} \sum_{1 \leq n, n' \leq N} x_{n}^{n-n'} \exp \left( i2\pi(n - n') \cdot \theta \right) \right] = 1 + (1 - \varepsilon) \Re \left[ \frac{1}{N^d} \sum_{1 \leq n, n' \leq N} x_{n}^{n-n'} \exp \left( i2\pi(n - n') \cdot \theta \right) \right].$$

We observe from the statement of Lemma 5.2 that, for any probability measure $\mu$ on $\mathbb{T}^d$,

$$f_{N,\varepsilon}(\theta; (\hat{\mu}^n)_{n:|n_j| \leq N-1}) = \varepsilon + (1 - \varepsilon) f_{N}(\theta; \mu),$$

from which we have the obvious lower bound:

$$\forall \theta \in \mathbb{T}^d, \quad f_{N,\varepsilon}(\theta; (\hat{\mu}^n)_{n:|n_j| \leq N-1}) \geq \varepsilon.$$

We then deduce that there exists a real $\eta_{\varepsilon,N} > 0$ such that, for any choice of $\mu$ and for any tuple $\mathbf{x}$ with the property that $\sup_{n:|n_j| \leq N-1} |x^n - \hat{\mu}^n|_\infty \leq \eta_{\varepsilon,N}$, $f_{N,\varepsilon}(\cdot; \mathbf{x})$ has positive values and is hence a density. Following (5.1), we then let

$$m_{N,\varepsilon}(\mathbf{x}) = f_{N,\varepsilon}(\cdot; \mathbf{x}) \cdot \text{Leb}_{\mathbb{T}^d}.$$

In order to define our mollified version of $\Phi$, we are also given a smooth density $\varrho_{\varepsilon,N}$ on $\mathbb{R}$ with support included in $[-(\varepsilon \wedge \eta_{\varepsilon,N})/2, (\varepsilon \wedge \eta_{\varepsilon,N})/2]$. This allows us to let

$$\Phi_{N,\varepsilon}(\mu) := \int_{\mathbb{R}^{(2N-1)^d}} \Phi \left( m_{N,\varepsilon}(\mathbf{x}) \right) \prod_{n \neq 0} \varrho_{N,\varepsilon}(\hat{\mu} - x^n) \delta_1(dx^0) \prod_{n \neq 0} dx^n,$$

where $n$ in the above two products is an element of $\mathbb{Z}^d$ satisfying $|n_j| \leq N - 1$ for $j \in \{1, \cdots, d\}$. Interestingly,

$$\Phi_{N,\varepsilon}(\mu) = \int_{\mathbb{R}^{(2N-1)^d}} \Phi \left( m_{N,\varepsilon}(\hat{\mu} - x^n) \right) \prod_{n \neq 0} \varrho_{N,\varepsilon}(x^n) \delta_1(dx^0) \prod_{n \neq 0} dx^n$$

$$= \int_{\mathbb{R}^{(2N-1)^d}} \Phi \left( \varepsilon \text{Leb}_{\mathbb{T}^d} + (1 - \varepsilon) (m_N(\mu) - m_{N,0}(\mathbf{x})) \right) \prod_{n \neq 0} \varrho_{N,\varepsilon}(x^n) \delta_1(dx^0) \prod_{n \neq 0} dx^n,$$

(5.3)
where we used the fact that
\[ f_{N,\varepsilon}(\theta; (\hat{\mu}^n - x^n)_{n:|n_j| \leq N-1}) = \varepsilon + (1 - \varepsilon) \left[ f_N(\theta; \mu) - f_N,0(\theta; x) \right]. \]

By Lemma 5.3, we clearly have that \( \Phi_{N,\varepsilon} \) converges to \( \Phi \), as \((N, \varepsilon)\) converges to \((\infty, 0)\).

**Second step.** The fact that \( \Phi_{N,\varepsilon} \) is a smooth functional of \( \mu \) is a direct consequence of (5.2). The Fourier coefficient \( \hat{\mu}^n \) is a mere linear functional of \( \mu \) and the function \( \vartheta_{N,\varepsilon} \) is smooth. Obviously, the function \( \Phi \) in (5.2) plays no main role in the smoothness of \( \Phi_{N,\varepsilon} \) except for the fact that, as it is bounded, it does not yield any integrability problem.

**Third step.** We now check that the bounds satisfied by \((\Phi_n)_{n \geq 1}\) in (Reg-\(\Phi_\cdot(\alpha, 2)\)) are uniform in \(n \geq 1\). Obviously, we have \( \|\Phi_{N,\varepsilon}\|_\infty \leq \|\Phi\|_\infty \). Next, we compute the first order derivatives \( \delta \Phi_{N,\varepsilon}/\delta m \). In order to do so, we notice that, for \( x \in \mathbb{R}^{(2N-1)d} \) and \( \mu, \nu \in \mathcal{P}(\mathbb{T}^d) \),

\[
\Phi\left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\nu) - m_{N,0}(x) \right) \right) - \Phi\left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)
= (1 - \varepsilon) \int_0^1 \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( \lambda m_N(\nu) + (1 - \lambda) m_N(\mu) - m_{N,0}(x) \right) \right)(y)(m_N(\nu - \mu))(dy),
\]

with the obvious notation (although \( \nu - \mu \) is not a probability measure) \( m_N(\nu - \mu) = f_N(\cdot; \nu - \mu) \cdot \text{Leb}_{T^d} \) and \( f_N(\theta; \nu - \mu) \) as in the statement of Lemma 5.2.

As a result, using the continuity of \( \delta \Phi/\delta m \), we deduce that

\[
\Phi\left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\nu) - m_{N,0}(x) \right) \right) - \Phi\left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)
= (1 - \varepsilon) \int_0^1 \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) m_N(\mu) - m_{N,0}(x) \right)(y)(m_N(\nu - \mu))(dy) + o(\text{dist}_{TV}(\mu, \nu)),
\]

which suffices to identify \( \delta \Phi_{N,\varepsilon}/\delta m \) through the linear functional that appears in the above right-hand side (see, for instance, [19, Remark 5.47]). By Lemma 5.2 and Parseval’s identity,

\[
\int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)(y)(m_N(\nu - \mu))(dy)
= \sum_{n \in \mathbb{Z}^d} \left[ \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)(y)e^{i2\pi n \cdot y} dy \right] m_N(\nu - \mu)^n
= \sum_{n:|n_j| < N} \left[ \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)(y)e^{i2\pi n \cdot y} dy \right] \left( \nu^n - \mu^n \right) \prod_{j=1}^d \left( 1 - \frac{|n_j|}{N} \right)
= \sum_{n:|n_j| < N} \int_{T^d} \left[ \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)(y)e^{-i2\pi n \cdot (\theta - y)} dy \right] \prod_{j=1}^d \left( 1 - \frac{|n_j|}{N} \right) d(\nu - \mu)(\theta),
\]

which shows that

\[
\frac{\delta \Phi_{N,\varepsilon}(\mu)}{\delta m}(\theta) = \sum_{n:|n_j| < N} \prod_{j=1}^d \left( 1 - \frac{|n_j|}{N} \right) \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(x) \right) \right)(\theta - y)e^{-i2\pi n \cdot y} dy.
\]
By reverting the computations, we get

$$\frac{\delta \Phi_{N,\varepsilon}(\mu)}{\delta m}(y) = \int_{T^d} \frac{\delta \Phi}{\delta m} \left( \varepsilon \text{Leb}_{T^d} + (1 - \varepsilon) \left( m_N(\mu) - m_{N,0}(\mathbf{x}) \right) \right)(y) \left( m_N(\delta \theta) \right) \, \text{d}y \quad (5.4)$$

The $\alpha$-Hölder norm of the right-hand side with respect to $\theta$ can be easily estimated in terms of the $\alpha$-Hölder norm of $\delta \Phi/\delta m$, which fits our objective. Differentiating once again with respect to $\mu$, we may prove a similar result for the $\alpha$-Hölder regularity of $[\delta^2 \Phi_{N,\varepsilon}/\delta m^2](\mu, \theta')$ with respect to $\theta, \theta'$. 

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Université Côte d’Azur, CNRS, Laboratoire J.A.Dieudonné, Parc Valrose, France-06108 NICE Cedex 2.
Email address: delarue@unice.fr

Université Paris-Est, Cermics (ENPC), INRIA, F-77455 MARNE-LA-VALLÉE, FRANCE.
Email address: alvin.tse@enpc.fr