Non-abelian simple groups which occur as the type of a Hopf–Galois structure on a solvable extension

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Abstract
We determine the finite non-abelian simple groups which occur as the type of a Hopf–Galois structure on a solvable extension. In the language of skew braces, our result gives a complete list of finite non-abelian simple groups which occur as the additive group of a skew brace with solvable multiplicative group.

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1 | INTRODUCTION

Given a group Γ, let Perm(Γ) denote the group of all permutations on Γ. Recall that a subgroup Δ of Perm(Γ) is said to be regular if its action on Γ is both transitive and free, or equivalently, if the map

\[ \xi_\Delta : \Delta \longrightarrow \Gamma; \quad \xi_\Delta(\delta) = \delta(1_\Gamma) \]
is bijective. Let us also write

\[
\begin{align*}
\lambda : \Gamma &\rightarrow \text{Perm}(\Gamma); \quad \lambda(\gamma) = (x \mapsto \gamma x), \\
\rho : \Gamma &\rightarrow \text{Perm}(\Gamma); \quad \rho(\gamma) = (x \mapsto \gamma x^{-1}),
\end{align*}
\]

respectively, for the left and right regular representations of \( \Gamma \). Clearly, both \( \lambda(\Gamma) \) and \( \rho(\Gamma) \) are regular subgroups. The holomorph of \( \Gamma \) is defined to be

\[
\text{Hol}(\Gamma) = \rho(\Gamma) \rtimes \text{Aut}(\Gamma) = \lambda(\Gamma) \rtimes \text{Aut}(\Gamma),
\]

which is equal to the normalizer of both \( \lambda(\Gamma) \) and \( \rho(\Gamma) \) in \( \text{Perm}(\Gamma) \).

Regular subgroups lying inside the holomorph are closely related to Hopf–Galois structures (on Galois extensions) and skew braces. Let us first briefly review the connections and some important applications.

Let \( L/K \) be a finite Galois extension with Galois group \( G \). By [9], there is a bijection between Hopf–Galois structures on \( L/K \) and regular subgroups of \( \text{Perm}(G) \) normalized by \( \lambda(G) \). Explicitly, for such a regular subgroup \( \mathcal{N} \), we associate to it the Hopf–Galois structure

\[
\mathcal{H} = \left\{ \sum_{\eta \in \mathcal{N}} \epsilon_n \eta \in L[\mathcal{N}] \left| \sum_{\eta \in \mathcal{N}} \epsilon_n \eta = \sum_{\eta \in \mathcal{N}} \sigma(\epsilon_n) \lambda(\sigma) \eta \lambda(\sigma)^{-1} \text{ for all } \sigma \in G \right. \right\},
\]

whose action on \( L \) is defined by

\[
\left( \sum_{\eta \in \mathcal{N}} \epsilon_n \eta \right) \cdot x = \sum_{\eta \in \mathcal{N}} \epsilon_n (\eta^{-1}(1_G))(x) \text{ for all } x \in L,
\]

and we refer to (the isomorphism class of) \( \mathcal{N} \) as the type of \( \mathcal{H} \). For example, when \( \mathcal{N} \) is taken to be \( \rho(G) \), since it commutes with \( \lambda(G) \) element-wise, we recover the so-called classical Hopf–Galois structure \( K[G] \) of type \( G \). By [2], given any group \( N \) of the same order as \( G \), there is in turn a (not necessarily one-to-one) correspondence between Hopf–Galois structures on \( L/K \) of type \( N \) and regular subgroups of \( \text{Hol}(N) \) which are isomorphic to \( G \).

Hopf–Galois structures are useful in the study of Galois modules. Let \( L/K \) be a finite Galois extension of number fields or \( p \)-adic fields. In the classical setting, one views the ring of integers or valuation ring \( \mathcal{O}_L \) of \( L \) as a module over the associated order of \( L/K \), defined by

\[
\mathfrak{A}_{K[G]} = \{ \alpha \in K[G] \mid \alpha \mathcal{O}_L \subseteq \mathcal{O}_L \},
\]

where \( G \) is the Galois group of \( L/K \). But one can similarly regard \( \mathcal{O}_L \) as a module over the associated order of \( \mathcal{H} \), defined by

\[
\mathfrak{A}_\mathcal{H} = \{ \alpha \in \mathcal{H} \mid \alpha \mathcal{O}_L \subseteq \mathcal{O}_L \},
\]

for any Hopf–Galois structure \( \mathcal{H} \) on \( L/K \). In [3], Byott exhibited extensions \( L/K \) of \( p \)-adic fields for which \( \mathcal{O}_L \) is not free over \( \mathfrak{A}_{K[G]} \), but is free over \( \mathfrak{A}_\mathcal{H} \) for some nonclassical Hopf–Galois structure.
This suggests that we can gain a better understanding of the Galois module structure of $\mathcal{O}_L$ by considering all Hopf–Galois structures on $L/K$ other than the classical one.

We refer the reader to [7, 8] for more details on Hopf–Galois structures.

Let $B$ be a set equipped with two group operations $+$ and $\circ$. Here, we do not require $+$ to be commutative despite the notation. We shall say that $B$ or the triplet $(B, +, \circ)$ is a (left) skew brace if we have the brace relation

$$x \circ (y + z) = x \circ y - x + x \circ z \text{ for all } x, y, z \in B.$$

The groups $(B, +)$ and $(B, \circ)$, respectively, are called the additive and multiplicative groups of $B$. Skew brace originated from [10]. It was shown that given any group $N = (N, +)$, there is a bijection between group operations $\circ$ for which $(N, +, \circ)$ is a skew brace and regular subgroups of $\text{Hol}(N)$. Explicitly, any such regular subgroup $\mathcal{G}$ yields a bijection

$$\xi_G : \mathcal{G} \to N; \quad \xi_G(\sigma) = \sigma(1_N),$$

and we may define $\circ$ via transport by setting

$$x \circ y = \xi_G(\xi^{-1}_G(x) \cdot \xi^{-1}_G(y)) \text{ for all } x, y \in N.$$

Clearly, $(N, \circ)$ is isomorphic to $\mathcal{G}$ and one can check that the brace relation is satisfied. We remark that isomorphism class of skew brace corresponds to conjugacy class of regular subgroup by the action of $\text{Aut}(N)$.

Skew brace was defined in [10] (also see [13]) as a tool to study the Yang–Baxter equation. A (set-theoretical) solution of the Yang–Baxter equation is a set $X$ equipped with a bijective map

$$r : X \times X \to X \times X; \quad r(x, y) = (\sigma_x(y), \tau_y(x))$$

which satisfies the equation

$$(r \times \text{id})(\text{id} \times r)(r \times \text{id}) = (\text{id} \times r)(r \times \text{id})(\text{id} \times r).$$

It is called nondegenerate if the maps $\sigma_x$ and $\tau_x$ are bijective for all $x \in X$. As shown in [10, Theorem 3.1], from any skew brace $B = (B, +, \circ)$, one can construct a nondegenerate solution

$$r : B \times B \to B \times B; \quad r(x, y) = (\gamma_x(y), \gamma^{-1}_x(y)(-x \circ y) + x + (x \circ y)),$$

where we define $\gamma_x(y) = -x + x \circ y$ for any $x, y \in B$. Some sort of converse of this is also true; see [10, Theorem 3.9]. This is the motivation behind the study of skew braces.

We now return to the discussion regarding regular subgroups of the holomorph. Given two finite groups $G$ and $N$ of the same order, let us say that the pair $(G, N)$ is realizable if there exists a regular subgroup of $\text{Hol}(N)$ that is isomorphic to $G$. By the above, realizability of $(G, N)$ is equivalent to the existence of the following:

(a) a Hopf–Galois structure of type $N$ on any Galois $G$-extension;
(b) a skew brace with additive group isomorphic to $N$ and multiplicative group isomorphic to $G$. 
It is natural to ask how $G$ and $N$ must be related, in terms of their group-theoretic properties say, in order for $(G, N)$ to be realizable. For example:

1. If $G$ is cyclic, then $N$ must be supersolvable [18]. (Also see [21].)
2. If $G$ is abelian, then $N$ must be metabelian [5]. (Also see [14, 18].)
3. If $G$ is nilpotent, then $N$ must be solvable [18].

We also have the following conjecture due to Byott.

**Conjecture 1.1.** Let $G$ and $N$ be finite groups of the same order for which the pair $(G, N)$ is realizable. If $G$ is insolvable, then $N$ is also insolvable.

Conjecture 1.1 is known to be true in some special cases; see [4, 18–20] for examples. Let us also remark that Byott has made some significant progress regarding this conjecture in the preprint arXiv:2205.13464.

However, the converse of Conjecture 1.1 is false. In other words, there are realizable pairs $(G, N)$ for which $N$ is insolvable, while $G$ is solvable (see [5, Corollary 1.1] for examples). The purpose of this paper is to investigate the possibilities of $N$ in such a realizable pair $(G, N)$. Focusing on almost simple groups $N$, we shall prove the following.

**Theorem 1.2.** Let $N$ be a finite almost simple group. If there is a solvable group $G$ such that $(G, N)$ is realizable, then the socle of $N$ is isomorphic to one of the following:

(a) $\text{PSL}_3(3)$, $\text{PSL}_3(4)$, $\text{PSL}_3(8)$, $\text{PSU}_3(8)$, $\text{PSU}_4(2)$, $\text{M}_{11}$;
(b) $\text{PSL}_2(q)$ with $q \neq 2, 3$ a prime power.

We remark that our proof of Theorem 1.2 uses a result that relies on the classification of finite simple groups (CFSG).

Restricting to nonabelian simple groups $N$, we are able to prove the converse and obtain a complete classification, as follows.

**Theorem 1.3.** Let $N$ be a finite nonabelian simple group. Then, there is a solvable group $G$ such that $(G, N)$ is realizable if and only if $N$ is isomorphic to one of the following:

(a) $\text{PSL}_3(3)$, $\text{PSL}_3(4)$, $\text{PSL}_3(8)$, $\text{PSU}_3(8)$, $\text{PSU}_4(2)$, $\text{M}_{11}$;
(b) $\text{PSL}_2(q)$ with $q \neq 2, 3$ a prime power.

We have hence obtained a complete classification of the finite nonabelian simple groups $N$ which can occur as

(a) the type of a Hopf–Galois structure on a solvable extension;
(b) the additive group of a skew brace with solvable multiplicative group.

We remark that when $G$ is fixed to be a finite nonabelian simple group, it is known by [4] that $(G, N)$ is realizable if and only if $N$ is isomorphic to $G$. In fact, other than the obvious examples $\rho(G)$ and $\lambda(G)$, there are no other regular subgroups of $\text{Hol}(G)$. In the language of Hopf–Galois structures, this means that

(a) the only Hopf–Galois structures on a Galois $G$-extension are the classical and canonical nonclassical ones (in the sense of [15]).
In the language of skew braces, this means that
(b) the only group operations $\circ$ on $G = (G, +)$ for which $(G, +, \circ)$ is a skew brace are the **trivial** and **almost trivial** ones, given by $x \circ y = x + y$ and $x \circ y = y + x$, respectively.

The same is true when $G$ is fixed to be a finite quasi-simple group [20].

Note that when $N$ is a finite simple abelian group, namely, a cyclic group of prime order, trivially $(G, N)$ is realizable if and only if $G$ is isomorphic to $N$. Similarly when $G$ is fixed to be a finite simple abelian group.

Regarding the proofs of Theorems 1.2 and 1.3, it turns out that factorizations of groups play an important role; see Proposition 2.1. We remark that the proof of the aforementioned facts (1)–(3) given in [18, Theorem 1.3] also exploits the theory of factorizations of groups, though in a way different from Proposition 2.1.

Recall that a group $\Gamma$ is said to be **factorized** by the subgroups $A$ and $B$ if $\Gamma = AB$ holds, and any such factorization is said to be **exact** if $A \cap B = 1$ in addition.

## 2 PRELIMINARIES

Let $G$ and $N$ be two finite groups of the same order. We write $\operatorname{Inn}(N)$ for the inner automorphism group of $N$ and let

$$\text{conj} : N \rightarrow \operatorname{Inn}(N); \quad \text{conj}(\eta) = (x \mapsto \eta x \eta^{-1})$$

denote the natural homomorphism. We have the following proposition.

**Proposition 2.1.** The regular subgroups of $\operatorname{Hol}(N)$ isomorphic to $G$ are exactly the subsets

$$\{\rho(g(\sigma)) \cdot f(\sigma) : \sigma \in G\},$$

where $f : G \rightarrow \operatorname{Aut}(N)$ is any homomorphism and $g : G \rightarrow N$ is any bijection such that $g(\sigma \tau) = g(\sigma) \cdot f(\sigma)(g(\tau))$ for all $\sigma, \tau \in G$. Moreover, in this case, the map $h : G \rightarrow \operatorname{Aut}(N)$ defined by

$$h(\sigma) = \text{conj}(g(\sigma))f(\sigma) \quad (2.1)$$

is a homomorphism, the product $f(G)h(G)$ is a subgroup of $\operatorname{Aut}(N)$ containing $\operatorname{Inn}(N)$, and we have the equality $f(G)\operatorname{Inn}(N) = h(G)\operatorname{Inn}(N)$.

Let us note that $f$ and $h$, respectively, correspond to the projections onto $\operatorname{Aut}(N)$ along $\rho(N)$ and $\lambda(N)$, as given by the semidirect product decompositions of $\operatorname{Hol}(N)$ in (1.1).

**Proof.** See [16, Proposition 2.1] for the first claim, and that $h$ is a homomorphism was verified in [17, Proposition 3.4]. By (2.1) and the bijectivity of $g$, it is clear that $f(G)h(G)$ contains $\operatorname{Inn}(N)$ and $f(G)\operatorname{Inn}(N) = h(G)\operatorname{Inn}(N)$.

To show that $f(G)h(G)$ is a subgroup of $\operatorname{Aut}(N)$, it suffices to check that

$$f(G)h(G) = h(G)f(G).$$
Let \( \sigma, \tau \in G \) be arbitrary. Since \( g \) is bijective, we may write

\[
\begin{align*}
  h(\sigma\tau)^{-1}g(\sigma)^{-1} &= g(\mu) \\
  f(\sigma\tau)^{-1}g(\sigma) &= g(\nu)^{-1}
\end{align*}
\]

for some \( \mu, \nu \in G \). Then, using the relation \( (2.1) \), we obtain

\[
\begin{align*}
  f(\sigma)h(\tau) &= \text{conj}(g(\sigma)^{-1})h(\sigma\tau) \\
                 &= h(\sigma\tau)\text{conj}(h(\sigma)^{-1}(g(\sigma)^{-1})) \\
                 &= h(\sigma\tau)\text{conj}(g(\mu)) \\
                 &= h(\sigma\tau\mu)f(\mu^{-1}),
\end{align*}
\]

and similarly,

\[
\begin{align*}
  h(\sigma)f(\tau) &= \text{conj}(g(\sigma))f(\sigma\tau) \\
                 &= f(\sigma\tau)\text{conj}(f(\sigma\tau)^{-1}(g(\sigma))) \\
                 &= f(\sigma\tau)\text{conj}(g(\nu)^{-1}) \\
                 &= f(\sigma\tau\nu)h(\nu^{-1}).
\end{align*}
\]

Thus, the equality \( f(G)h(G) = h(G)f(G) \) indeed holds. \( \square \)

The assertions concerning \( f(G) \) and \( h(G) \) in Proposition 2.1 are why realizability of \( (G, N) \) is related to factorizations of groups. In particular, as an immediate consequence of Proposition 2.1, we have the following.

**Corollary 2.2.** If \( (G, N) \) is realizable, then there is a subgroup \( P \) of \( \text{Aut}(N) \) containing \( \text{Inn}(N) \) such that \( P = AB \) is factorized by some subgroups \( A \) and \( B \) which are quotients of \( G \) and satisfy \( AI\text{Inn}(N) = BI\text{Inn}(N) \).

Corollary 2.2 is enough to prove Theorem 1.2. But to prove the backward implication of Theorem 1.3, we need some sort of converse to Corollary 2.2. To that end, we shall use the following definition from [6].

**Definition 2.3.** For any group \( \Gamma \), a pair \( \varphi, \psi : G \longrightarrow \Gamma \) of homomorphisms is said to be fixed point free if \( \varphi(\sigma) = \psi(\sigma) \) holds only when \( \sigma = 1_G \).

As shown in [6, Section 2], given any fixed-point free pair \( \varphi, \psi : G \longrightarrow N \) of homomorphisms, the subset

\[
\{\lambda(\varphi(\sigma))\rho(\psi(\sigma)) : \sigma \in G\}
\]

of \( \text{Hol}(N) \) is a regular subgroup isomorphic to \( G \), which implies that \( (G, N) \) is realizable. As noted in [5, Remark 7.2], one can use this to show that:
**Proposition 2.4.** If $N = AB$ is exactly factorized by subgroups $A$ and $B$, then the pair $(A \times B, N)$ is realizable.

**Proof.** Note that $A \times B$ does have the same order as $N$. The claim follows from the above discussion because $\varphi, \psi : A \times B \rightarrow N$ given by $\varphi(a, b) = a$ and $\psi(a, b) = b$ is clearly a fixed-point free pair of homomorphisms.

In proving the backward implication of Theorem 1.3, if $N = AB$ has an exact factorization by solvable subgroups $A$ and $B$, then we can just apply Proposition 2.4. But $\text{PSL}_3(4)$ and $\text{PSU}_4(2)$, for example, do not admit such a factorization and we would need another way to deal with these groups.

For simplicity, let us focus on groups $N$ having trivial center. In this case, the map $\text{conj} : N \rightarrow \text{Inn}(N)$ is an isomorphism, and one sees that the pair $f, h : G \rightarrow \text{Aut}(N)$ of homomorphisms in Proposition 2.1 is fixed point free because the map $g$ there is bijective and sends $1_G$ to $1_N$. Moreover, one can recover $g$ from the pair $f, h$ using (2.1). We may then rephrase Proposition 2.1 as follows, where (2.3) is a generalization of (2.2); see Remark 2.6.

**Proposition 2.5.** Assume that $N$ has trivial center. The regular subgroups of $\text{Hol}(N)$ isomorphic to $G$ are exactly the subsets

$$\{ \rho(\text{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) : \sigma \in G \},$$

(2.3)

where $f, h : G \rightarrow \text{Aut}(N)$ is a fixed-point free pair of homomorphisms such that $f(\sigma) \equiv h(\sigma) \pmod{\text{Inn}(N)}$ holds for all $\sigma \in G$.

**Proof.** It is clear from Proposition 2.1 and (2.1) that every regular subgroup of $\text{Hol}(N)$ isomorphic to $G$ is of the stated shape. The converse is also true by the calculation in [17, Proposition 3.4].

**Remark 2.6.** In the case that $N$ has trivial center, taking $f, h$ to have image lying inside $\text{Inn}(N)$, we may recover the construction (2.2) from (2.3). This is because then we can write

$$f(\sigma) = \text{conj}(\varphi(\sigma)) \text{ and } h(\sigma) = \text{conj}(\psi(\sigma))$$

for homomorphisms $\varphi, \psi : G \rightarrow N$. For any $\sigma \in G$, we see that

$$\rho(\text{conj}^{-1}(h(\sigma)f(\sigma)^{-1})) \cdot f(\sigma) = \rho(\psi(\sigma)\varphi(\sigma)^{-1}) \cdot \text{conj}(\varphi(\sigma))$$

$$= \rho(\psi(\sigma)\varphi(\sigma)^{-1}) \cdot \rho(\varphi(\sigma)\lambda(\varphi(\sigma)))$$

$$= \lambda(\varphi(\sigma))\rho(\psi(\sigma)).$$

This shows that (2.3) and (2.2) yield the same subgroup.

We may now generalize Proposition 2.4 as follows; see Remark 2.8.
**Proposition 2.7.** Assume that $N$ has trivial center and let $P$ be a subgroup of $\text{Aut}(N)$ containing $\text{Inn}(N)$. If $P = AB$ is exactly factorized by subgroups $A$ and $B$ such that $A\text{Inn}(N) = B\text{Inn}(N)$ and $A$ splits over $A \cap \text{Inn}(N)$, then the pair $(A \cap \text{Inn}(N) \rtimes_{\alpha} B, N)$ is realizable for a suitable choice of $\alpha$.

**Proof.** Write $A_0 = A \cap \text{Inn}(N)$ and $B_0 = B \cap \text{Inn}(N)$. Under the hypothesis, there is a subgroup $C$ of $A$ such that $A = A_0 \rtimes C$, and $B/B_0 \cong C$ via

$$\theta : \frac{B}{B_0} \xrightarrow{\cong} \frac{B\text{Inn}(N)}{\text{Inn}(N)} \xrightarrow{\cong} \frac{A\text{Inn}(N)}{\text{Inn}(N)} \xrightarrow{\cong} \frac{A}{A_0} \xrightarrow{\cong} C,$$

where all of the appearing isomorphisms are the natural ones. Since $C$ acts on $A_0$ via conjugation, this induces an action of $B$ on $A_0$. Explicitly, define

$$\alpha : B \longrightarrow \text{Aut}(A_0); \; \alpha(b) = (x \mapsto \theta(bB_0)x\theta(bB_0)^{-1}).$$

Notice that $AB = A\text{Inn}(N)$, where $\supseteq$ holds trivially since $P = AB$ contains $\text{Inn}(N)$, while $\subseteq$ holds because $A\text{Inn}(N) = B\text{Inn}(N)$. We then deduce that

$$|\text{Inn}(N)| = \frac{|AB|}{[A\text{Inn}(N) : \text{Inn}(N)]} = \frac{|A| |B| |A \cap B|}{[A : A_0]} = |A_0||B|.$$

where $A \cap B = 1$ because the factorization $P = AB$ is assumed to be exact. It follows that $A_0 \rtimes_{\alpha} B$ indeed has the same order as $\text{Inn}(N) \cong N$.

Now, let us define $f, h : A_0 \rtimes_{\alpha} B \longrightarrow \text{Aut}(N)$ by setting

$$f(a, b) = a\theta(bB_0) \text{ and } h(a, b) = b \text{ for } a \in A_0, b \in B.$$

Clearly $h$ is a homomorphism, and $f$ is also a homomorphism because

$$f((a_1, b_1)(a_2, b_2)) = f(a_1\alpha(b_1)(a_2), b_1b_2)$$

$$= a_1\theta(b_1B_0)a_2\theta(b_1B_0)^{-1} \cdot \theta(b_1b_2B_0)$$

$$= a_1\theta(b_1B_0) \cdot a_2\theta(b_2B_0)$$

$$= f(a_1, b_1)f(a_2, b_2)$$

for all $a_1, a_2 \in A_0$ and $b_1, b_2 \in B$. Note that the images of $f$ and $h$ are equal to $A$ and $B$, respectively. Since $A \cap B = 1$, we have $f(G) \cap h(G) = 1$.

Let $a \in A_0$ and $b \in B$ be arbitrary. We see that

$$f(a, b) = h(a, b) \Rightarrow a\theta(bB_0) = 1, \; b = 1 \Rightarrow a = 1, \; b = 1,$$

which means that $(f, h)$ is fixed point free. Moreover, since $AB = A\text{Inn}(N)$ as noted above and $A = A_0 \rtimes C$ with $A_0 \subseteq \text{Inn}(N)$, we may write

$$b = c \cdot \text{conj}(\eta) \text{ for some } c \in C \text{ and } \eta \in N.$$
By the definition of $\vartheta$, we have $\vartheta(bB_0) = c$, so then

$$f(a, b) \equiv ac \equiv c \equiv b \equiv h(a, b) \pmod{\text{Inn}(N)}$$

because $a \in \text{Inn}(N)$. The claim now follows from Proposition 2.5. \qed

Remark 2.8. In the case that $N$ has trivial center, taking $P = \text{Inn}(N) \simeq N$, we may recover Proposition 2.4 from Proposition 2.7. Indeed, for any exact factorization $\text{Inn}(N) = AB$, it is trivial that $A\text{Inn}(N) = B\text{Inn}(N)$ and that $A = A \cap \text{Inn}(N)$ splits over itself. Thus, in the proof of Proposition 2.7, we have $C = 1$ and so $\alpha$ is simply the trivial homomorphism. This implies that $A \rtimes_{\alpha} B = A \times B$, and the $f, h$ constructed there are the natural projections onto $A, B$, respectively, just like the $\varphi, \psi$ in the proof of Proposition 2.4.

We remark that part of our proof of Theorem 1.3 involves computations in MAGMA [1], and in our codes, we only compute with representatives $P$ of the conjugacy classes of subgroups in $\text{Aut}(N)$. For each $P$, similarly, we only compute with representatives $A, B$ of the conjugacy classes of subgroups in $P$. By the two propositions below, as far as the conditions in Corollary 2.2 and Proposition 2.7 are concerned, no generality is lost in doing so.

For any $\Gamma \subseteq \text{Aut}(N)$ and $\pi \in \text{Aut}(N)$, let us write $\Gamma_\pi = \pi \Gamma \pi^{-1}$.

**Proposition 2.9.** Let $P$ be a subgroup of $\text{Aut}(N)$ and let $A, B$ be subgroups of $P$. For any $\pi \in \text{Aut}(N)$, the following hold.

(a) If $P$ contains $\text{Inn}(N)$, then $P_\pi$ also contains $\text{Inn}(N)$.
(b) If $P = AB$, then $P_\pi = A_\pi B_\pi$.
(c) If $A \cap B = 1$, then $A_\pi \cap B_\pi = 1$.
(d) If $A\text{Inn}(N) = B\text{Inn}(N)$, then $A_\pi \text{Inn}(N) = B_\pi \text{Inn}(N)$.
(e) If $A$ splits over $A \cap \text{Inn}(N)$, then $A_\pi$ splits over $A_\pi \cap \text{Inn}(N)$.

*Proof.* Parts (a), (d), and (e) hold simply because $\text{Inn}(N)$ is a normal subgroup of $\text{Aut}(N)$, whereas parts (b) and (c) are obvious. \qed

**Proposition 2.10.** Let $P = AB$ be a subgroup of $\text{Aut}(N)$ containing $\text{Inn}(N)$ that is factorized by subgroups $A, B$. For any $\pi_1, \pi_2 \in P$, the following hold.

(a) We also have the factorization $P = A_{\pi_1} B_{\pi_2}$.
(b) If $A \cap B = 1$, then $A_{\pi_1} \cap B_{\pi_2} = 1$.
(c) If $A\text{Inn}(N) = B\text{Inn}(N)$, then $A_{\pi_1} \text{Inn}(N) = B_{\pi_2} \text{Inn}(N)$.

*Proof.* Since $P = AB = BA$, we may write

$$\pi_1 = b_1 a_1, \quad \pi_2 = a_2 b_2, \quad b_1^{-1} a_2 = ab,$$

where $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$. It follows that

$$A_{\pi_1} B_{\pi_2} = A_{b_1} B_{a_2} = b_1 A a b a_2^{-1} = b_1 A b a_2^{-1} = b_1 B a a_2^{-1} = B A = P,$$
which proves (a). This, in turn, implies that
\[ |A \cap B| = \frac{|A||B|}{|AB|} = \frac{|A_{\pi_1}||B_{\pi_2}|}{|A_{\pi_1}B_{\pi_2}|} = \frac{|A_{\pi_1} \cap B_{\pi_2}|}{|A_{\pi_1}||B_{\pi_2}|}, \]
which yields (b). Finally, suppose that \( A\text{Inn}(N) = B\text{Inn}(N) \). Since \( \text{Inn}(N) \) is a normal subgroup of \( \text{Aut}(N) \), we have
\[
A_{\pi_1}\text{Inn}(N) = (A\text{Inn}(N))_{b_1} = (B\text{Inn}(N))_{b_1} = B\text{Inn}(N) = A\text{Inn}(N)
\]
\[
B_{\pi_2}\text{Inn}(N) = (B\text{Inn}(N))_{a_2} = (A\text{Inn}(N))_{a_2} = A\text{Inn}(N),
\]
which shows (c). This completes the proof.

\[\square\]

### 3 PROOF OF THEOREM 1.2

Let \( N \) be a finite almost simple group and assume that there is a solvable group \( G \) for which \( (G, N) \) is realizable. By Corollary 2.2, there is a subgroup \( P \) of \( \text{Aut}(N) \) containing \( \text{Inn}(N) \) such that \( P = AB \) for some subgroups \( A, B \) that are quotients of \( G \). Here, there are two important points.

- Since \( P \) lies between \( \text{Inn}(N) \) and \( \text{Aut}(N) \), it is also almost simple and its socle is isomorphic to that of \( N \). Indeed, suppose that
  \[ S \leq N \leq \text{Aut}(S), \]
  where \( S \) is a nonabelian simple group identified with \( \text{Inn}(S) \). Then as is well known (or see [17, Lemma 4.3]), the natural homomorphism
  \[
  \text{Aut}(N) \longrightarrow \text{Aut}(S); \quad \varphi \mapsto \varphi|_S
  \]
  induced by restriction is injective and it maps \( \text{Inn}(N) \) to \( N \). It follows that \( P \) is also embedded between \( S \) and \( \text{Aut}(S) \), whence \( P \) is an almost simple group with socle isomorphic to \( S \).
- Since \( G \) is solvable, its quotients \( A \) and \( B \) are also solvable.

Almost simple groups that are factorizable as the product of two solvable subgroups have been studied in [12, Proposition 4.1] (its proof is based on a result of [11] and some computations in MAGMA [1]; it requires CFSG). In particular, it tells us that the socle of \( P \), which is isomorphic to the socle of \( N \), must be one of the groups stated in the theorem.

### 4 PROOF OF THEOREM 1.3

In view of Theorem 1.2, it suffices to prove the backward implication. We shall split the finite nonabelian simple groups \( N \) in question into four families, as follows. The citations in the parentheses indicate the propositions to be used to deal with the family.

1. \( N = \text{PSL}_3(3), \text{M}_{11} \) (Proposition 2.4).
(2) $N = \text{PSL}_3(4), \text{PSL}_3(8), \text{PSU}_4(2)$ (Proposition 2.7).

(3) $N = \text{PSU}_3(8)$ (Proposition 2.5).

(4) $N = \text{PSL}_3(q)$ with $q \neq 2, 3$ a prime power (Proposition 2.4 for an even $q$, and Proposition 2.7 for an odd $q$).

**Proof of (1).** By running CODE 1 in the Appendix, we find that $N = AB$ is exactly factorized by some solvable subgroups $A$ and $B$. It then follows from Proposition 2.4 that the pair $(G, N)$ is realizable for the group $G = A \times B$, which is clearly solvable. □

**Proof of (2).** By running CODE 2 in the Appendix, we find that $\text{Aut}(N)$ has a subgroup $P$ containing $\text{Inn}(N)$ such that $P = AB$ is exactly factorized by some solvable subgroups $A$ and $B$ for which

- $A \text{Inn}(N) = B \text{Inn}(N)$;
- $A$ splits over $A \cap \text{Inn}(N)$.

Proposition 2.7 then implies that the pair $(G, N)$ is realizable for a suitable semidirect product $G = (A \cap \text{Inn}(N)) \rtimes B$, which is clearly solvable. □

**Proof of (3).** By Proposition 2.5, we only need to find a solvable group $G$ of the same order as $N$ and also a fixed-point free pair $f, h : G \rightarrow \text{Aut}(N)$ of homomorphisms satisfying

$$f(\sigma) \equiv h(\sigma) \pmod{\text{Inn}(N)}$$

for all $\sigma \in G$. We shall do so using CODE 3 and CODE 4 in the Appendix.

Let us explain what we computed in these codes.

(i) CODE 3: Finding candidates for $f(G)$ and $h(G)$.

We find that up to conjugacy in $\text{Aut}(N)$, there is only one subgroup $P$ of $\text{Aut}(N)$ containing $\text{Inn}(N)$, which admits a factorization $P = AB$ by solvable subgroups $A, B$ such that $|A|, |B|$ divide $|N|$ and the equality $A \text{Inn}(N) = B \text{Inn}(N)$ holds. Without loss of generality, let us take $A$ to have smaller order than $B$. We note that then

$$|A| = 513, |B| = 96768, A \cap B = 1, [P : \text{Inn}(N)] = 9,$$

as computed in the code. The homomorphisms $f, h : G \rightarrow \text{Aut}(N)$ to be constructed will have images $f(G) = A$ and $h(G) = B$.

(ii) CODE 4 PART I: Finding a candidate for $G$.

Note that the subgroup $\rho(N) \rtimes A$ of $\text{Hol}(N)$ is naturally isomorphic to the outer semidirect product $\text{Inn}(N) \rtimes A$, where $A$ acts on $\text{Inn}(N)$ via conjugation in $\text{Aut}(N)$. We construct this semidirect product and find that up to conjugacy, it has only one solvable subgroup $G$ of the same order as $N$.

(iii) CODE 4 PART II: Finding candidates for $\ker(f)$ and $\ker(h)$.

Note that $|N| = 5515776$, and that

$$\frac{|G|}{|A|} = \frac{5515776}{513} = 10752, \quad \frac{|G|}{|B|} = \frac{5515776}{96768} = 57.$$
We find that \( G \) has a unique normal subgroup \( K_f \) of order 10752, and similarly a unique normal subgroup \( K_h \) of order 57. We also have
\[
K_f \cap K_h = 1, \ G/K_f \cong A, \ G/K_h \cong B,
\]
as verified in the code. The homomorphisms \( f, h : G \rightarrow \text{Aut}(N) \) to be constructed will have kernels \( \ker(f) = K_f \) and \( \ker(h) = K_h \).

(iv) **Code 4 Part III:** Finding a desired fixed-point free pair \((f, h)\).

Let \( q_f : G \rightarrow G/K_f \) and \( q_h : G \rightarrow G/K_h \) denote the natural quotient maps. Also, we construct isomorphisms
\[
\varphi_f : G/K_f \rightarrow A \text{ and } \varphi_h : G/K_h \rightarrow B.
\]
For any \( \pi_A \in \text{Aut}(A) \) and \( \pi_B \in \text{Aut}(B) \), we have the homomorphisms
\[
f, h : G \rightarrow \text{Aut}(N); \begin{align*}
f &= \pi_A \circ \varphi_f \circ q_f, \\
h &= \pi_B \circ \varphi_h \circ q_h.
\end{align*}
\]
From their definitions, it is clear that
\[
f(G) = A, \ \ker(f) = K_f, \ h(G) = B, \ \ker(h) = K_h.
\]

Since \( A \cap B = 1 \) and \( K_f \cap K_h = 1 \), we see that \((f, h)\) is fixed point free. We also need (4.1) to hold for all \( \sigma \in G \), or equivalently for any set of generators \( \sigma \in G \). Using the generators of \( G, \text{Aut}(A), \text{Aut}(B) \) given by Magma, we find that (4.1) holds for all generators \( \sigma \) of \( G \) for suitable choices of \( \pi_A \in \text{Aut}(A) \) and \( \pi_B \in \text{Aut}(B) \).

We now conclude from Proposition 2.5 that the pair \((G, N)\) is realizable, where \( G \) is solvable by construction. ∎

**Proof of (4).** We regard \( N \) as a normal subgroup of \( \text{PGL}_2(q) \) via the natural embedding, and similarly \( \text{PGL}_2(q) \) as a subgroup \( \text{Aut}(N) \) by letting it act on \( N \) via conjugation. In other words, we view
\[
N \leq \text{PGL}_2(q) \leq \text{Aut}(N),
\]
where \( N \) is identified with \( \text{Inn}(N) \). Note that
\[
|\text{PSL}_2(q)| = \frac{q(q-1)(q+1)}{\gcd(2, q-1)}, \ |\text{PGL}_2(q)| = q(q-1)(q+1).
\]
As explained in [12, Section 3.1], we may factor \( \text{PGL}_2(q) \) as the product of a Singer cycle and the stabilizer of a one-dimensional subspace. Below, let us describe this factorization explicitly in terms of matrices.
Let $\beta$ be a generator of $\mathbb{F}_{q^2}^\times$ and let $X^2 + cX + d$ be its minimal polynomial over $\mathbb{F}_q$. For any $x_1, y_1, x_2, y_2 \in \mathbb{F}_q$, from the relation $\beta^2 = -d - c\beta$, we have

$$(x_1 + y_1\beta)(x_2 + y_2\beta) = (x_1x_2 + y_1y_2\beta^2) + (x_1y_2 + x_2y_1)\beta$$

$$(x_1x_2 - dy_1y_2) + (x_1y_2 + x_2y_1 - cy_1y_2)\beta.$$ 

By associating $x + y\beta$ to the matrix $\begin{bmatrix} x & -dy \\ y & x - cy \end{bmatrix}$, we see that multiplication in $\mathbb{F}_{q^2}^\times$ corresponds to matrix multiplication, so we obtain a cyclic subgroup

$$\tilde{A} = \left\{ \begin{bmatrix} x & -dy \\ y & x - cy \end{bmatrix} : x, y \in \mathbb{F}_q, (x, y) \neq (0, 0) \right\}$$

of $\text{GL}_2(q)$ of order $q^2 - 1$ generated by $\begin{bmatrix} 0 & -d \\ 1 & -c \end{bmatrix}$. We also have the subgroup

$$\tilde{B} = \left\{ \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} : u, w \in \mathbb{F}_q^\times, v \in \mathbb{F}_q \right\}$$

of $\text{GL}_2(q)$ of order $q(q - 1)^2$; this is the stabilizer of the subspace generated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and is naturally isomorphic to $\mathbb{F}_q \rtimes (\mathbb{F}_q^\times \times \mathbb{F}_q^\times)$. Clearly, $\tilde{A} \cap \tilde{B}$ equals the center $Z$ of $\text{GL}_2(q)$. We then see that

$$\text{PGL}_2(q) = AB \text{ with } A \cap B = 1, \text{ where } A = \tilde{A}/Z \text{ and } B = \tilde{B}/Z$$

are clearly solvable subgroups of order $q + 1$ and $q(q - 1)$, respectively.

First, suppose that $q \equiv 0 \pmod{4}$. We then have $\text{PGL}_2(q) = \text{PSL}_2(q)$, and so, by Proposition 2.4, the pair $(G, N)$ is realizable for $G = A \times B$, which is clearly solvable.

Next, suppose that $q \not\equiv 0 \pmod{4}$. Observe that $d$, which is the norm of $\beta$ over $\mathbb{F}_q$, generates $\mathbb{F}_q^\times$ because $\beta$ is a generator of $\mathbb{F}_{q^2}^\times$ and the norm map for finite fields is surjective. For any $u \in \mathbb{F}_q^\times$, because $\det\begin{bmatrix} 0 & -d \\ 1 & -c \end{bmatrix} = d$ we then see that there exists $\tilde{a} \in \tilde{A}$ for which $\det(\tilde{a}) = u$, and there clearly exists $\tilde{b} \in \tilde{B}$ for which $\det(\tilde{b}) = u$. It follows that $\tilde{A} \cdot \text{SL}_2(q) = \tilde{B} \cdot \text{SL}_2(q)$, and modding out by the center, this implies

$$A \cdot \text{PSL}_2(q) = B \cdot \text{PSL}_2(q).$$

We also deduce that the homomorphisms

$$A \longrightarrow \mathbb{F}_p^\times/\mathbb{F}_p^\times^2, \quad B \longrightarrow \mathbb{F}_p^\times/\mathbb{F}_p^\times^2$$

induced by the determinant map are surjective. Thus, their kernels

$$A_0 = A \cap \text{PSL}_2(q) \text{ and } B_0 = B \cap \text{PSL}_2(q)$$
have index 2 in $A$ and $B$, respectively. In particular, we have

$$|A_0| = \frac{q + 1}{2}, \quad |B_0| = \frac{q(q - 1)}{2}.$$ 

By the Schur–Zassenhaus theorem, we know that $A$ splits over $A_0$ when $|A_0|$ is odd, and that $B$ splits over $B_0$ when $|B_0|$ is odd. From Proposition 2.7, it then follows that $(G, N)$ is realizable for the solvable group

$$G = \begin{cases} 
A_0 \rtimes_\alpha B & \text{when } q \equiv 1 \text{ (mod } 4), \\
B_0 \rtimes_\alpha A & \text{when } q \equiv 3 \text{ (mod } 4),
\end{cases}$$

where $\alpha$ is a suitable choice of homomorphism.

This completes the proof of the theorem.

**APPENDIX: MAGMA codes**

**Code 1:**

```magma
N := ;
//input PSL(3,3) and M11 for N
SolSub := [S | subgroup:S in SolvableSubgroups(N)];
exists{<A,B>:A,B in SolSub|#(A meet B) eq 1 and #N eq #A*#B};
//output: true
```

**Code 2:**

```magma
N := ;
//input PSL(3,4), PSL(3,8), and PSU(4,2) for N
Aut := PermutationGroup(AutomorphismGroup(N));
Inn := Socle(Aut);
PP := [P | subgroup:P in Subgroups(Aut)|Inn subset P'subgroup];
L := [];
//A list to contain the P satisfying the desired conditions.
for P in PP do
    SolSub := [S | subgroup:S in SolvableSubgroups(P)];
    if exists{<A,B>:A,B in SolSub|
        #(A meet B) eq 1 and #P eq #A*#B and
        sub<P|A,Inn> eq sub<P|B,Inn> and
        HasComplement(A,A meet Inn)}
    then Append( L,P);
    end if;
end for;
not IsEmpty(L);
//output: true
```

**Code 3:**

```magma
N := PSU(3,8);
Aut := PermutationGroup(AutomorphismGroup(N));
```
Inn:=Socle(Aut);
PP:=[P’subgroup:P in Subgroups(Aut)|Inn subset P’subgroup];
for p in [1..#PP] do
P:=PP[p];
SolSub:=SolvableSubgroups(P:OrderDividing:=#N);
for a in [1..#SolSub] do
A:=SolSub[a]’subgroup;
for b in [a..#SolSub] do
B:=SolSub[b]’subgroup;
if #A*#B eq #P*#(A meet B) and sub <P|A,Inn> eq sub <P|B,Inn> then
<p,a,b,#A,#B,#(A meet B),#P/#N>;
end if;
end for;
end for;
end for;

//output: <8, 127, 217, 513, 96768, 1, 9>

CODE 4:
//Setting up the P, A, and B that we found in Code 3.
N:=PSU(3,8);
Aut:=PermutationGroup(AutomorphismGroup(N));
Inn:=Socle(Aut);
PP:=[P’subgroup:P in Subgroups(Aut)|Inn subset P’subgroup];
P:=PP[8];
SolSub:=SolvableSubgroups(P:OrderDividing:=#N);
A:=SolSub[127]’subgroup;
B:=SolSub[217]’subgroup;

"Part I";
AutInn:=AutomorphismGroup(Inn);
xi:=hom<A->AutInn|a:->hom<Inn->Inn|x:->a'(-1)*x*a>;
InnxA:=SemidirectProduct(Inn,A,xi);
GG:=SolvableSubgroups(InnxA:OrderEqual:=#N);
#GG;
//output: 1
G:=GG[1]’subgroup;

"Part II";
NorSubf:=NormalSubgroups(G:OrderEqual:=10752);
NorSubh:=NormalSubgroups(G:OrderEqual:=57);
<#NorSubf,#NorSubh>;
//output: <1, 1>
Kf:=NorSubf[1]’subgroup;
Kh:=NorSubh[1]’subgroup;
<#(Kf meet Kh),IsIsomorphic(G/Kf,A),IsIsomorphic(G/Kh,B)>;
//output: <1, true, true>
"Part III";
Qf, qf := quo<G|Kf>;
Qh, qh := quo<G|Kh>;
isof, phi: = IsIsomorphic(Qf,A);
isoh, phi: = IsIsomorphic(Qh,B);
GenG := Generators(G);
GenAutA := Generators(AutomorphismGroup(A));
GenAutB := Generators(AutomorphismGroup(B));
Out, q := quo<Aut|Inn>;
exists{<piA,piB>: piA in GenAutA, piB in GenAutB|
forall{g: g in GenG|(qf*phi*f*piA*q)(g) eq (qh*phi*h*piB*q)(g)}
};
//output: true

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