SOLVABILITY OF INCLUSIONS OF HAMMERSTEIN TYPE

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Abstract. A fairly general continuation theorem of Leray-Schauder type for the class of so-called admissible multimaps is set forth. This result is then used to establish a universal rule for solving operator inclusions of Hammerstein type in Lebesgue-Bochner spaces. Examples illustrating the legitimacy of this approach include the initial value problem for perturbation of \( m \)-accretive multivalued differential equations, the anti-periodic problem for semilinear differential inclusions, abstract integral inclusions of Fredholm and Volterra type and the two-point boundary value problem for nonlinear evolution inclusions.

1. Introduction

This paper aims to formulate quite natural and easily verifiable hypotheses, ensuring solvability of the following inclusion of Hammerstein type

\[ u \in (K \circ N_F)(u), \]

in the space \( L^p(I, E) \) of Bochner \( p \)-integrable functions. In inclusion (1), \( N_F \) is the Nemytskiǐ operator associated to a multifunction \( F : I \times E \to E \), while \( K \) is an external set-valued operator of a certain type (defined later).

Consideration of such operator inclusion accompany, of course, attempts to grasp the integro-differential multivalued problems from a unifying topological point of view. These attempts have been made repeatedly (see for instance [5, 6, 7]). Our efforts follow in the footsteps of authors of [7] and tend to generalize (Theorem 5) regarding the existence of solutions to inclusion (1) poses an application example of a fixed point approach. Its proof is based on the principle of Leray-Schauder type ([7, Th.3.2]), which was extended (Theorem 4) to the case of strongly admissible multimaps (in the sense of Górniewicz, [11, Def.40.1]). Just as in [7] the superposition \( K \circ N_F \) may not be a condensing map and our assumptions about \( K \) and \( F \) are formulated so that the Mönch type compactness condition could have been satisfied.

In order to apply Eilenberg-Montgomery type fixed point argument directly to the superposition \( K \circ N_F \) we need to know that this map is pseudo-acyclic. However, the Nemytskiǐ operator \( N_F : L^p \to L^q \) is by no means acyclic. Therefore, the authors of [7] rely on the assumption (SG) that operator \( H := K \circ N_F \) has acyclic values. This is very uncomfortable hypothesis from practical point of view. In general, if \( K \) is nonlinear, then the composite map \( H \) may not have convex values. Unfortunately, even if \( K \) and \( F \) has convex values the map \( H \) may still have values with “awful” geometry, since the class of acyclic mappings is not closed with respect to the composition law. It turns out that it is enough to take into account a relatively weak assumption regarding convexity or decomposability of fibers of the operator \( K \) in addition to the acyclicity of its values, to ensure the fulfillment of condition (SG).

The applicability of our abstract existence result is richly illustrated by numerous examples of differential and integral inclusions, which may be interpreted as a fixed point problem given by (1). These examples include cases where the operator \( K \) is a univalent mild solution operator of the \( m \)-accretive

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quasi-autonomous problem or the mild solution operator of the semilinear inhomogeneous two-point boundary value problem. There were also presented examples in which the map \( K \) is simply linear. Such as those, in which it has the form of Volterra or Hammerstein integral operator. And finally, there is also the case considered, when the map \( K \) constitutes a multivalued strongly upper semicontinuous maximal monotone operator.

2. Preliminaries

Let \((E, \| \cdot \|)\) be a Banach space, \( E^* \) its normed dual and \( \sigma(E, E^*) \) its weak topology. If \( X \) is a subset of a Banach space \( E \), by \((X, \omega)\) we denote the topological space \( X \) furnished with the relative weak topology of \( E \) to \( X \).

The normed space of bounded linear endomorphisms of \( E \) is denoted by \( \mathcal{L}(E) \). Given \( T \in \mathcal{L}(E) \), \( \| T \|_{\mathcal{L}} \) is the norm of \( T \). For any \( \varepsilon > 0 \) and \( A \subset E \), \( B_E(A, \varepsilon) \) (respectively \( D_E(A, \varepsilon) \)) stands for an open (closed) \( \varepsilon \)-neighbourhood of \( A \). The (weak) closure and the closed convex envelope of \( A \) will be denoted by \( \overline{A} \) and \( \omega \overline{A} \), respectively. If \( x \in E \) we put \( \text{dist}(x, A) := \inf \| x - y \| : y \in A \). Besides, for two nonempty closed bounded subsets \( A, B \) of \( E \) the symbol \( h(A, B) \) stands for the Hausdorff distance from \( A \) to \( B \), i.e. \( h(A, B) := \max \{ \sup \{ \text{dist}(x, B) : x \in A \}, \sup \{ \text{dist}(y, A) : y \in B \} \} \).

We denote by \((C(I, E), \| \cdot \|)\) the Banach space of all continuous maps \( I \to E \) equipped with the maximum norm. Let \( 1 \leq p < \infty \). By \((L^p([a, b], E), \| \cdot \|_p)\) we mean the Banach space of all Bochner \( p \)-integrable maps \( f : [a, b] \to E \), i.e. \( f \in L^p([a, b], E) \) iff map \( f \) is strongly measurable and

\[
\| \| f \| \|_p = \left( \int_a^b |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty
\]

if \( p < \infty \) and respectively

\[
\| \| f \| \|_\infty = \text{ess sup}_{t \in [a, b]} \| f(t) \| < \infty
\]

provided \( p = \infty \). Recall that strong measurability is equivalent to the usual measurability in case \( E \) is separable. A subset \( D \subset L^p([a, b], E) \) is called decomposable if for every \( u, w \in D \) and every Lebesgue measurable \( A \subset [a, b] \) we have \( u \cdot 1_A + w \cdot 1_{A^c} \in D \).

Given metric space \( X \), a set-valued map \( F : X \to E \) assigns to any \( x \in X \) a nonempty subset \( F(x) \subset E \). \( F \) is (weakly) upper semicontinuous, if the inverse image \( F^{-1}(A) = \{ x \in X : F(x) \subset A \} \) is open in \( X \) whenever \( A \) is (weakly) open in \( E \). A map \( F : X \to E \) is lower semicontinuous, if the inverse image \( F^{-1}(A) \) is closed in \( X \) for any closed \( A \subset E \). We say that \( F : X \to E \) is upper hemicontinuous if for each \( p \in E^* \), the function \( \sigma(p, F(\cdot)) : X \to \mathbb{R} \cup \{ +\infty \} \) is upper semicontinuous (as an extended real function), where \( \sigma(p, F(x)) = \sup_{y \in F(x)} \langle p, y \rangle \). We have the following characterization (see [2], Prop.2(b)): a map \( F : X \to E \) with convex values is weakly upper semicontinuous and has weakly compact values iff given a sequence \( (x_n, y_n) \) in the graph \( \text{Gr}(F) \) with \( x_n \xrightarrow{w} x \) there is a subsequence \( y_{n_k} \xrightarrow{E} y \in F(x) \) (\( \xrightarrow{\omega} \) denotes the weak convergence). A multifunction \( \tilde{F} : X \to E \) is compact if its range \( F(X) \) is relatively compact in \( E \). It is quasicompact if its restriction to any compact subset \( A \subset X \) is compact. The set of all fixed points of the map \( F : E \to E \) is denoted by \( \text{Fix}(F) \).

Let \( H^q(\cdot) \) denote the Alexander-Spanier cohomology functor with coefficients in the field of rational numbers \( \mathbb{Q} \) (see [18]). We say that a topological space \( X \) is acyclic if the reduced cohomology \( \tilde{H}^q(X) \) is 0 for any \( q \geq 0 \).

An upper semicontinuous map \( F : E \to E \) is called acyclic if it has compact acyclic values. A set-valued map \( F : E \to E \) is strongly admissible (in the sense of Górniewicz, [11], Def.40.1) if there is a Hausdorff topological space \( \Gamma \) and two continuous functions \( p : \Gamma \to E, q : \Gamma \to E \) from which \( p \) is a Vietoris map such that \( F(x) = q(p^{-1}(x)) \) for every \( x \in E \). Clearly, every acyclic map is strongly admissible. Moreover, the composition of strongly admissible maps is strongly admissible ([11], Th.40.6)).
A real function $\gamma$ defined on the family $\mathcal{B}(E)$ of bounded subsets of $E$ is called a measure of non-compactness (MNC) if $\gamma(\Omega) = \gamma(\overline{\Omega})$ for any bounded subset $\Omega$ of $E$. The following example of MNC is of particular importance: given $E_0 \subset E$ and $\Omega \in \mathcal{B}(E_0)$,

$$\beta_{E_0}(\Omega) := \inf \left\{ \varepsilon > 0 : \text{there are finitely many points } x_1, \ldots, x_n \in E_0 \text{ with } \Omega \subset \bigcup_{i=1}^{n} B_{E}(x_i, \varepsilon) \right\}$$

is the Hausdorff MNC relative to the subspace $E_0$. Recall that this measure is regular, i.e. $\beta_{E_0}(\Omega) = 0$ iff $\Omega$ is relatively compact in $E_0$; monotone, i.e. if $\Omega_1 \subset \Omega_2$ then $\beta_{E_0}(\Omega_1) \leq \beta_{E_0}(\Omega_2)$ and invariant with respect to union with compact sets, i.e. $\beta_{E_0}(A \cup \Omega) = \beta_{E_0}(\Omega)$ for any relatively compact $A \subset E_0$.

We recall the following results on account of their practical importance. The first is a weak compactness criterion in $L^p(\Omega, E)$, which originates from [20].

**Theorem 1** ([20] Cor.9]. Let $(\Omega, \Sigma, \mu)$ be a finite measure space with $\mu$ being a nonatomic measure on $\Sigma$. Let $A$ be a uniformly $p$-integrable subset of $L^p(\Omega, E)$ with $p \in [1, \infty)$. Assume that for a.a. $\omega \in \Omega$, the set $\{f(\omega) : f \in A\}$ is relatively weakly compact in $E$. Then $A$ is relatively weakly compact.

**Remark 1.** The genuine formulation of this result assumes the boundedness of the set $A$. However, the fact that $\mu$ is nonatomic means that uniform integrability of $A$ entails its boundedness.

The next property is commonly known as the Convergence Theorem for upper hemicontinuous maps with convex values.

**Theorem 2.** Let $F : X \to E$ be an upper hemicontinuous map from a metric space $X$ to the closed convex subsets of a Banach space $E$. If $I$ is a finite interval of $\mathbb{R}$ and sequences $(x_n : I \to X)_{n \geq 1}$ and $(y_n : I \to E)_{n \geq 1}$ satisfy the following conditions

1. $x_n(t) \xrightarrow{\text{a.e. on } I} x(t)$ as $n \to \infty$.
2. $y_n \xrightarrow{L(I,E)} y$.
3. $y_n(t) \in \overline{\text{co}} B_E(F(B_X(x_n(t), \varepsilon_n)), \varepsilon_n)$ for a.a. $t \in I$, where $\varepsilon_n \to 0^+$ as $n \to \infty$, then $y(t) \in F(x(t))$ a.e. on $I$.

The third result is an immediate consequence of the Lefschetz-type fixed point theorem [11] Th.41.7].

**Theorem 3** ([11] Cor.41.12]). Every acyclic absolute neighbourhood retract has the fixed point property within the class of admissible compact maps.

3. **Fixed point approach to inclusions of Hammerstein type**

The subsequent result constitutes a generalization of the continuation principle [7] Th.3.2] to the case of strongly admissible multimap.

**Theorem 4.** Let $X$ be a closed convex subset of a Banach space $E$ and $U$ a relatively open subset of $X$ whose closure is a retract of $X$. Assume that $H : \overline{U} \to X$ is a strongly admissible multimap and for some $x_0 \in U$ the following two conditions are satisfied:

1. $M \subset \overline{U}$, $M \subset \text{co } (\{x_0\} \cup H(M))$ and $\overline{M} = \overline{C}$ with $C \subset M$ countable
2. $x \notin (1 - \lambda)x_0 + \lambda H(x)$ on $\overline{U} \setminus U$ for all $\lambda \in (0, 1)$.

Then $H$ has a fixed point in $\overline{U}$. 

Proof. We will proceed with accordance to the scheme contained in [16]. Define
\[ \Sigma := \{ x \in \overline{U} : x \in (1 - \lambda)x_0 + \lambda H(x) \text{ for some } \lambda \in [0, 1] \}. \]
Operator \( H \) as a strongly admissible map is compact valued upper semicontinuous. Thus \( \Sigma \) is closed. By (3), \( \Sigma \subset U \). Let \( r : X \to \overline{U} \) be a retraction and \( \theta : X \to [0, 1] \) be an Urysohn function such that \( \theta|_{\overline{X}\setminus U} \equiv 0 \) and \( \theta|_U \equiv 1 \). By \( \psi : [0, 1] \times X \to X \) we denote a map given by \( \psi(\lambda, x) := (1 - \lambda)x_0 + \lambda x \). Now, we are in position to define an auxiliary operator \( \tilde{H} : X \to X \) in the following way, \( \tilde{H}(x) := \psi \circ (\theta \times (H \circ r)) \).
Assume for a moment that \( X, Y \) and \( Z \) are three arbitrary Hausdorff topological spaces. Let \( F : X \to Y \) and \( G : X \to Z \) be two strongly admissible maps. There exist selected pairs \((p_F : \Gamma_F \to X, q_F : \Gamma_F \to Y)\) and \((p_G : \Gamma_G \to X, q_G : \Gamma_G \to Z)\) of \( F \) and \( G \), respectively. Denote by \( \tilde{p} : \Gamma_F \bowtie \Gamma_G \to X \) the fibre product of the diagram
\[ \Gamma_F \xrightarrow{p_F} X \xleftarrow{p_G} \Gamma_G. \]
It is easy to see that this is also a Vietoris map. In particular, the fiber \( \tilde{p}^{-1}(x) = p_F^{-1}(x) \times p_G^{-1}(x) \) is an acyclic set in view of the Künneth Formula for Čech cohomology functor (cf. [11, Th.5.5]). Define \( \tilde{q} : \Gamma_F \bowtie \Gamma_G \to Y \times Z \) to be \( \tilde{q}(u, w) := (q_F(u), q_G(w)) \). Then
\[ \tilde{q} \left( \tilde{p}^{-1}(x) \right) = q_F(p_F^{-1}(x)) \times q_G(p_G^{-1}(x)) = (F \times G)(x). \]
In other words \((\tilde{p}, \tilde{q})\) constitutes a selected pair of the product map \( F \times G \). Accordingly, the map \( \theta \times (H \circ r) \) is strongly admissible. Ultimately, operator \( \tilde{H} : X \to X \) is strongly admissible.
It is easy to verify the following condition
\[ M \subset X, \quad M = \text{co}\left( \{x_0\} \cup \tilde{H}(M) \right) \quad \text{and} \quad \overline{M} = \overline{C} \text{ with } C \subset M \text{ countable} \] (cf. the proof of [16 Th.3.2.]). Following the proof of [16 Th.3.1.], one can indicate a convex relatively compact subset \( \overline{M} \subset X \) such that \( M = \text{co}\left( \{x_0\} \cup \tilde{H}(M) \right) \). Since \( \tilde{H} \) is an upper semicontinuous map and \( \tilde{H}(\overline{M}) \subset \overline{M} \), we infer that \( \tilde{H}(\overline{M}) \subset \overline{M} \). The compact strongly admissible set-valued map \( \tilde{H} : \overline{M} \to \overline{M} \) must have a fixed point \( x \in X \). That is a straightforward consequence of Theorem 3. If \( x \in X \setminus U \), then \( \tilde{H}(x) = \{x_0\} \). But \( x_0 \in U \) - contradiction. Thus, \( x \in U \) and \( x \in (1 - \theta(x))x_0 + \theta(x)H(x) \). Whence \( x \in \Sigma \) and eventually \( x \in H(x) \).

Remark 2. The following properties of \( H : \overline{U} \to X \) imply the Leray-Schauder boundary condition (3) with \( x_0 \in U \):

(i) if \( \lambda(x - x_0) \in H(x) - x_0 \) for \( x \in \partial U \), then \( \lambda \leq 1 \) (Yamamura’s condition),
(ii) \( U \) is convex and \( H(\partial U) \subset \overline{C} \) (Rothe’s condition),
(iii) \( |y - x|^2 \geq |y - x_0|^2 - |x - x_0|^2 \) for each \( x \in \partial U \) and \( y \in H(x) \) (Krasnoselskii-Altman’s condition),
(iv) \( |y - x_0, x - x_0| \leq |x - x_0|^2 \) for each \( x \in \partial U \) and \( y \in H(x) \) if \( E \) is a Hilbert space (Browder’s condition).

Assume that \( p \in [1, \infty] \) and \( q \in [1, \infty) \). Fix a compact segment \( I := [0, T] \) for some end time \( T > 0 \). Let \( F : I \times E \to E \) be a set-valued map. Throughout the paper we will use the following hypotheses on the mapping \( F \):

(\( F_1 \)) for every \((t, x) \in I \times E \) the set \( F(t, x) \) is nonempty and convex,
(\( F_2 \)) the map \( F(\cdot, x) \) has a strongly measurable selection for every \( x \in E \),
(\( F_3 \)) the graph \( \text{Gr}(F(t, \cdot)) \) is sequentially closed in \((E, \|\cdot\|) \times (E, w)\) for a.a. \( t \in I \),
(\( F_4 \)) \( F \) satisfies a sublinear growth condition, i.e. there is \( b \in L^p(I, \mathbb{R}) \) and \( c > 0 \) such that for all \( x \in E \) and for a.a. \( t \in I \),
\[ \|F(t, x)\|^* := \sup \{|y|_E : y \in F(t, x)\} \leq b(t) + c|x|^\frac{p}{p-1}, \]
when \( p \in [1, \infty) \). If \( p = \infty \), then for every \( R > 0 \) there exists \( b_R \in L^q(I, \mathbb{R}) \) such that
\[
\|F(t, x)\|_+ \leq b_R(t) \text{ a.e. on } I, \text{ for all } x \in E \text{ with } |x| \leq R.
\]

\((F_5)\) for every closed separable linear subspace \( E_0 \) of \( E \) the map \( F \big|_{x \in E_0} (t, \cdot) \cap E_0 \) is quasicompact for a.a. \( t \in I \).

Recall that the Nemytskii operator \( N_F : L^q(I, E) \to L^q(I, E) \), corresponding to \( F \), is a multivalued map defined by
\[
N_F(u) := \{w \in L^q(I, E) : w(t) \in F(t, u(t)) \text{ for a.a. } t \in I\}.
\]

Consider also a multivalued external operator \( K : L^q(I, E) \to L^q(I, E) \). Our hypothesis on the multifunction \( K \) is the following:

\((K_1)\) for every compact \( C \subset E \), the map \( K : (L^q(I, C), w) \to (L^q(I, E), \| \cdot \|_p) \) is acyclic,

\((K_2)\) the map \( K : L^q(I, E) \to L^q(I, E) \) is \( L \)-Lipschitz with closed values,

\((K_3)\) for every uniformly \( q \)-integrable possessing relatively weakly compact vertical slices a.e. on \( I \) subset \( C \subset L^q(I, E) \), the map \( K : (C, w) \to (L^q(I, E), \| \cdot \|_p) \) is acyclic.

**Remark 3.**

(i) For every relatively weakly compact \( C \subset L^q(I, E) \), \( K : (C, w) \to (L^q(I, E), \| \cdot \|_p) \) is compact valued upper semicontinuous iff given a sequence \((x_n, y_n)\) in the graph \( \text{Gr}(K) \) with \( x_n \xrightarrow{n \to \infty} x \), there is a subsequence \( y_{k_n} \xrightarrow{n \to \infty} y \in K(x) \). (notice that the space \( C, w \) is sequential as a subset of the angelic space \((L^q(I, E), w)\))

(ii) \((K_3) \Rightarrow (K_1)\).

Before we will be able to set forth a result concerning the existence of solutions to inclusion \([1]\), we have to prove a few auxiliary facts.

**Lemma 1.** Let \( p \in [1, \infty] \). If the multimap \( F : I \times E \to E \) satisfies conditions \((F_1)\)-\((F_5)\), then the Nemytskii operator \( N_F : L^q(I, E) \to L^q(I, E) \) is a weakly upper semicontinuous multivalued map with nonempty convex weakly compact values.

**Proof.** For any \( u \in L^q(I, E) \) one can always define a sequence \((u_n)_{n=1}^\infty \) of simple functions, which converges to \( u \) almost everywhere and for which \(|u_n(t)| \leq 2|u(t)|\) for every \( t \in I \) (cf. the proof of \([9\) Th.III.2.22]). Consequently, vertical slices \([u_n(t)]_{n=1}^\infty \) are relatively compact in \( E \) for a.a. \( t \in I \).

Accordingly to the assumption \((F_2)\) we can indicate a strongly measurable map \( w_n : I \to E \) such that \( w_n(t) \in F(t, u_n(t)) \) for a.a. \( t \in I \). Thanks to condition \((F_4)\) we know that the sequence \((w_n)_{n=1}^\infty \) is \( q \)-integrably bounded. Let \( E_0 \) be a closed separable linear subspace of \( E \) such that \([u_n(t)]_{n=1}^\infty \cup [w_n(t)]_{n=1}^\infty \subset E_0 \) a.e. on \( I \). By \((F_3)\), the vertical slices \([w_n(t)]_{n=1}^\infty \) are relatively compact a.e. on \( I \). In view of Theorem \([1]\) the sequence \((w_n)_{n=1}^\infty \) is relatively weakly compact in \( L^q(I, E) \). Hence we may assume, passing to a subsequence if necessary, that \( w_n \xrightarrow{n \to \infty} w \).

Observe that for a.a. \( t \in I \), the multimap \( F(t, \cdot) \) is compact valued upper semicontinuous. Indeed, consider sequences \((x_n)^n_{n=1} \) and \((y_n)^n_{n=1} \) satisfying \( x_n \xrightarrow{n \to \infty} x \) and \( y_n \in F(t, x_n) \). Put \( E_0 := \text{Span} \left( \{x_n\}_{n=1}^\infty \cup \{y_n\}_{n=1}^\infty \right) \). Then \( y_n \xrightarrow{n \to \infty} y \). Assumption \((F_3)\) implies \( y \in F(t, x) \).

Applying Theorem \([2]\) one gets \( w(t) \in F(t, u(t)) \) for a.a. \( t \in I \). In this way we have shown that the Nemytskii operator \( N_F \) has nonempty values.

Applying similar reasoning one may prove that given a sequence \((u_n, w_n)_{n=1}^\infty \) in the graph \( \text{Gr}(N_F) \) with \( u_n \xrightarrow{n \to \infty} u \), there is a subsequence \( w_{k_n} \xrightarrow{n \to \infty} w \in N_F(u) \). Indeed, since the family \([|u_n(t)|]^n_{n=1} \) is
Corollary 1. Let $E$ be a reflexive Banach space. If the set-valued map $F: I \times E \rightarrow E$ fulfills conditions (F1)-(F3), then the thesis of Lemma [1] holds.

Proof. By (F3) the map $F(t, \cdot)$ is locally bounded a.e. on $I$. Consider a sequence $(x_n, y_n)_{n \geq 1}$ in the graph $\text{Gr}(F(t, \cdot))$ with $x_n \rightarrow x$ in the norm of $E$. Since $E$ is reflexive, there must be a subsequence $y_{k_n}$ converging weakly to $y$. Bearing in mind (F3), i.e. that $\text{Gr}(F(t, \cdot))$ is strongly-weakly closed, we obtain $y \in F(t, x)$. Therefore, $F(t, \cdot)$ is weakly upper semicontinuous and possesses weakly compact values for a.a. $t \in I$.

Retaining the notation of the previous proof it may be observed that $\{w_n(t)\}_{n=1}^{\infty}$ forms a subset of a weakly compact set $F(t, [w_n(t)]_{n=1}^{\infty})$ a.a. $t \in I$. Now, in manner fully analogous to the mentioned proof, we can use Theorem [1] and Theorem [2] to justify the thesis. \[\Box\]

Lemma 2. Assume $(K_1)$ and $(K_2)$ are satisfied. Let $M$ be a countable subset of $L^q(I, E)$ such that $M(t)$ is relatively compact in $E$ for a.a. $t \in I$. Then the image $K(M)$ is relatively compact in $L^q(I, E)$ and $K$ is upper semicontinuous from $M$ furnished with the relative weak topology of $L^q(I, E)$ to $L^p(I, E)$ with its norm topology.

Proof. Let $M := \{w_n\}_{n=1}^{\infty}$. In view of Pettis measurability theorem there exists a closed separable subspace $E_0$ of $E$ with $w_n(t) \in E_0$ a.e. on $I$ for every $n \geq 1$. For each $k \geq 1$ there is a $k$-dimensional linear subspace $E_k \subset E_0$ such that $E_k \subset E_{k+1}$ and $E_0 = \bigcup_{k=1}^{\infty} E_k$. Let $\rho > 0$ be an arbitrarily chosen scalar. Obviously, there exists a subset $\Theta_1 \subset I$ such that $|w_n(t)| \leq \rho$ for every $t \in I \setminus \Theta_1$ and $n \geq 1$. Consequently,

$$\forall n \geq 1 \forall k \geq 1 \forall t \in I \setminus \Theta_1 \quad \text{dist}(w_n(t), E_k) = \text{dist}(w_n(t), D_E(0, 2\rho) \cap E_k).$$

Take $\varepsilon > 0$. Using the formula for the Hausdorff MNC in a separable Banach space ([15, Prop.2.] one sees that $\beta_{E_0}(\{w_n(t)\}_{n=1}^{\infty}) = \lim_{k \rightarrow \infty} \sup_{n \geq 1} \text{dist}(w_n(t), E_k)$.

Thus, in view of Egorov’s Theorem,

$$\exists \Theta_2 \subset I \setminus \Theta_1 \exists k_0 \in \mathbb{N} \forall k \geq k_0 \forall t \in I \setminus (\Theta_1 \cup \Theta_2) \sup_{n \geq 1} \text{dist}(w_n(t), D_E(0, 2\rho) \cap E_k) < \beta_{E_0}(\{w_n(t)\}_{n=1}^{\infty}) + \frac{\varepsilon}{3}.$$

Referring once more to Egorov’s Theorem we can indicate a measurable $\Theta_3 \subset I$ and a simple function $\tilde{w}_n: I \rightarrow E$ such that

$$|w_n(t) - \tilde{w}_n(t)| < \frac{\varepsilon}{3} \quad \text{and} \quad \text{dist}(\tilde{w}_n(t), D_E(0, 2\rho) \cap E_k) < \frac{2}{3} \varepsilon$$

for all $t \in I \setminus (\Theta_1 \cup \Theta_2 \cup \Theta_3)$, $k \geq k_0$ and $n \geq 1$. The latter property comes down eventually to the following:

$$(4) \quad \forall \varepsilon > 0 \exists k_0 \in \mathbb{N} \forall k \geq k_0 \forall n \geq 1 \exists w_{n,k} \in L^q(I, D_E(0, 2\rho) \cap E_k) \text{ such that } \|w_n - w_{n,k}\|_q < \varepsilon.$$

Fix $\varepsilon > 0$ and $k \geq k_0$. The range $\{w_{n,k}\}_{n=1}^{\infty}$ is relatively weakly compact in $L^q(I, E)$ in view of Theorem [1] Condition $(K_1)$ implies the relative compactness of $K(\{w_{n,k}\}_{n=1}^{\infty})$ in $L^p(I, E)$. Thus, making use of (4) and $(K_2)$ we arrive at

$$\beta_{L^p}(K(\{w_{n,k}\}_{n=1}^{\infty})) = \beta_{L^p}(K(\{w_n\}_{n=1}^{\infty})) - \beta_{L^p}(K(\{w_{n,k}\}_{n=1}^{\infty})) \leq \sup_{n \geq 1} h(K(w_n), K(w_{n,k})) \leq L \|w_n - w_{n,k}\|_q \leq \varepsilon L.$$

Since $\varepsilon$ was arbitrary, the image $K(\{w_{n,k}\}_{n=1}^{\infty})$ must be relatively compact.
Assume that \((w_n, v_n) \in \text{Gr}(K)\) with \(w_n \xrightarrow{M} w\). As we have shown above the set \(K(\{w_n\}_{n=1}^{\infty})\) is relatively compact. Thus, there exists a subsequence (again denoted by) \((v_n)_{n=1}^{\infty}\) such that \(v_n \xrightarrow{L^q(E)} v\). Our aim is to show that \(v \in K(w)\). Take \(\varepsilon > 0\). As previously, we can indicate a sequence \((w_{n_k})_{n_k=1}^{\infty}\) and a compact subset \(C_\varepsilon \subset E\) such that \(\{w_{n_k}\}_{n_k=1}^{\infty} \subset L^q(I, C_\varepsilon)\) and \(\|v_{n_k} - w_{n_k}\|_q \leq \frac{\varepsilon}{24}\). View of the weak compactness criterion (Theorem [1]), we may assume that \(w_{n_k} \xrightarrow{L^q(E)} w'\), passing once again to a subsequence if necessary. Clearly, \(\|v - w'\|_q \leq \frac{\varepsilon}{24}\) due to the weak lower semicontinuity of the norm. Choose \(y_n' \in K(w_n')\) in such a way that \(\|v_{n_k} - y_{n_k}'\|_p = \text{dist}(v_{n_k}, K(w_n'))\). Assumption (K1) guarantees that \(y_{n_k}' \xrightarrow{L^p(I)} y' \in K(w')\), up to a subsequence. Of course, there is \(N \in \mathbb{N}\) such that \(\|v_N - v\|_p \leq \frac{\varepsilon}{4}\) and \(\|\gamma_{y_{n_k}}' - \gamma_y\|_p \leq \frac{\varepsilon}{4}\). Now, it is possible to estimate
\[
\text{dist}(v, K(w)) \leq \|v - v_N\|_p + \|v_N - y_{n_k}'\|_p + \|y_{n_k}' - y'\|_p + \text{dist}(y', K(w)) \leq \frac{\varepsilon}{4} + \text{dist}(v_N, K(w_n')) + \frac{\varepsilon}{4} + h(K(w'), K(w)) \\
\leq \frac{\varepsilon}{2} + L\|w_N - w_n\|_q + \|w - w'\|_q \leq \varepsilon.
\]
Since \(\varepsilon\) was arbitrary, it follows that \(v \in K(w)\).

\[\square\]

**Lemma 3.** Let \(X\) be a compact topological space and \(Y\) be a paracompact topological space. Assume that \(F: X \to Y\) is an upper semicontinuous surjective multimap with compact acyclic values and acyclic fibers. Then there is an isomorphism \(H^*(X) \approx H^*(Y)\).

**Proof.** Since \(X\) is compact, the product \(X \times Y\) is a paracompact space. The space \(Y\) is regular and the map \(F\) is usc so the graph \(Gr(F)\) is a closed subset of \(X \times Y\). Thus it is also a paracompact space. The projection \(\pi_1: Gr(F) \to X\) of \(Gr(F)\) onto the domain \(X\) is continuous and surjective. It is easy to see that \(\pi_1\) is a closed map, since \(F\) is compact valued and usc. Moreover, the fibers \(\pi_1^{-1}(\{x\}) = \{x\} \times F(x)\) are compact acyclic. Hence \(\pi_1\) is perfect and consequently a proper map. Analogously, the projection \(\pi_2: Gr(F) \to Y\) is surjective continuous and the preimage \(\pi_2^{-1}(\{y\}) = F^{-1}(\{y\}) \times \{y\}\) is compact acyclic. The map \(\pi_2\) is also closed, since the domain \(X\) is compact. In view of Vietoris-Begle mapping theorem ([13] Th.6.9.15) it follows that \((\pi_1)^{-1} \circ \pi_2^{-1}: H^*(Y) \to H^*(X)\) is an isomorphism. \[\square\]

Recall that for the sake of convenience we had introduced the letter \(H\) to denote the superposition \(K \circ N_F\).

**Lemma 4.** Let (F1)-(F3) be satisfied. Assume that either \(E\) is reflexive and (K1) holds or (K1)-(K2) and (F3) are met. In both cases, the operator \(H: L^p(I, E) \to L^q(I, E)\) is a compact valued upper semicontinuous map.

**Proof.** Assume that \(u_n \xrightarrow{L^p(I, E)} u\). Obviously, there exists a subsequence \((u_{n_k})_{n_k=1}^{\infty}\) such that \(u_{n_k}(t) \xrightarrow{E} u(t)\) for a.a. \(t \in I\). Let \(w_n \in N_F(u_n)\) and \(v_n \in K(w_n)\) for \(n \geq 1\). Observe that the sequence \((w_n)_{n=1}^{\infty}\) is bounded uniformly \(q\)-integrable (or simply bounded for \(p = \infty\)).

If \(E\) is reflexive, then the map \(F(t, \cdot)\) is weakly upper semicontinuous and possesses weakly compact values a.e. on \(I\). Thus, the sets \(\{w_n\}_{n=1}^{\infty}\) are relatively weakly compact for a.a. \(t \in I\). The sequence \((w_n)_{n=1}^{\infty}\) is relatively compact in view of Theorem [1] We may assume, passing again to a subsequence if necessary, that \(w_{n_k} \xrightarrow{L^q(E)} w\). Condition (K3) implies (cf. Remark [3]) that \(v_n \xrightarrow{L^q(E)} v \in K(w)\), up to a subsequence. It is enough to apply Corollary [11] to show that \(w \in N_F(u)\). Eventually, \(v \in H(u)\), i.e. the set-valued map \(H: L^q(I, E) \to L^q(I, E)\) is an upper semicontinuous operator with compact values.

If assumption (F3) is met, then the multimap \(F(t, \cdot)\) is compact valued and upper semicontinuous a.e. on \(I\). In this case the sets \(\{w_{n_k}(t)\}_{n_k=1}^{\infty}\) are relatively compact in \(E\) for a.a. \(t \in I\). Passing to a subsequence if
necessary, we obtain \( w_{k_n} \xrightarrow{L^q(I,E)} w \). By virtue of Lemma\(^2\) there exists a subsequence (again denoted by) 
\((v_{k_n})_{n=1}^{\infty}\) such that \( v_{k_n} \xrightarrow{n \to \infty} v \in K(w)\). Lemma\(^1\) implies that \( w \in N_F(u)\). This means that \( v \in H(u)\). \( \square \)

**Lemma 5.** Let \( U \subseteq L^q(I,E) \) and \( x_0 \in U \). Assume that \( F: I \times E \to E \) satisfies \((F_1)-(F_4)\). Suppose further that operator \( H: \overline{U} \to L^q(I,E) \) with uniformly \( p \)-integrable range (or bounded range if \( p = \infty \)) meets the following condition:

\[(5) \quad M \subseteq \overline{U} \text{ countable and } M \subset \overline{co}\left(\{x_0\} \cup H(M)\right) \implies M(t) \text{ is relatively compact for a.a. } t \in I.\]

Assume also that either \( E \) is reflexive and \((K_3)\) holds or \((K_1)-(K_2)\) and \((F_5)\) are met. In both cases, condition \((2)\) is fulfilled.

**Proof.** Suppose \( M \subseteq \overline{U} \), \( M \subseteq \overline{co}\left(\{x_0\} \cup H(M)\right) \) and \( \overline{M} = \overline{C} \) for some countable subset \( C \subseteq M \). Since \( C \subseteq \overline{co}\left(\{x_0\} \cup H(M)\right) \), there exists a countable set \( \{v_n\}_{n=1}^{\infty} \subseteq H(M) \) such that \( C \subseteq \overline{co}\left(\{x_0\} \cup \{v_n\}_{n=1}^{\infty}\right) \).

Clearly,
\[
C \subseteq \overline{co}\left(\{x_0\} \cup H(M)\right) \subseteq \overline{co}\left(\{x_0\} \cup H(\overline{M})\right) = \overline{co}\left(\{x_0\} \cup H(\overline{C})\right).
\]

Due to assumption \((5)\), the vertical slices \( M(t) \) are relatively compact for a.a. \( t \in I \). Obviously, we can indicate \( w_n \in L^q(I,E) \) such that \( w_n \in K(x_0) \) and \( w_n \in N_F(M) \).

Suppose \( E \) is reflexive. Taking into account that \( \{w_n(t)\}_{n=1}^{\infty} \subseteq F(t, \overline{M}(t)) \) a.e. on \( I \) and that \( F(t, \cdot) \) is weakly upper semicontinuous we see that \( \{w_n(t)\}_{n=1}^{\infty} \) is relatively weakly compact for a.a. \( t \in I \). Since \( M \) forms a subset of uniformly \( p \)-integrable convex hull \( co(\{x_0\} \cup H(M)) \), the sequence \( \{w_n\}_{n=1}^{\infty} \) must be uniformly \( q \)-integrable. It follows from condition \((K_4)\) that the image \( K\left(\{w_n\}_{n=1}^{\infty}\right) \) is compact in \( L^p(I,E) \). Hence the set \( \{v_n\}_{n=1}^{\infty} \) is relatively compact. The latter entails the relative compactness of \( C \) and eventually the compactness of the closure \( \overline{M} \).

If conditions \((K_1)-(K_2)\) and \((F_5)\) are met, then the map \( F(t, \cdot) \) is upper semicontinuous, vertical slices \( \{w_n(t)\}_{n=1}^{\infty} \) are relatively compact a.e. on \( I \) and the image \( K\left(\{w_n\}_{n=1}^{\infty}\right) \) is compact in the space \( L^p(I,E) \) in view of Lemma\(^2\). Therefore, \( M \) must be a relatively compact subset of \( L^p(I,E) \). \( \square \)

**Remark 4.** Clearly, the operator \( H: \overline{U} \to L^p(I,E) \), which is condensing relative to some monotone nonsingular and regular MNC \( \gamma \) defined on the space \( L^p(I,E) \), satisfies condition \((2)\).

The eponymous solvability of operator inclusions of Hammerstein type expresses itself in the following fixed point principle, formulated in the context of the Bochner space \( L^p(I,E) \).

**Theorem 5.** Let \( X \) be a closed convex subset of the space \( L^p(I,E) \). Assume that either

(i) the space \( E \) is reflexive, the operator \( K: L^q(I,E) \to X \) possesses convex or decomposable fibers and satisfies assumption \((K_3)\), the multimap \( F: I \times E \to E \) meets conditions \((F_1)-(F_4)\)

or

(ii) the operator \( K: L^q(I,E) \to X \) possesses compact acyclic values and convex or decomposable fibers and satisfies assumptions \((K_1)-(K_2)\), the set-valued map \( F: I \times E \to E \) meets conditions \((F_1)-(F_3)\).

Suppose further that there exists a radius \( R > 0 \) such that

\[(6) \quad L\left(\|b\|_q + c\left(\|K(0)\|_p^*\right)^\frac{1}{p}\right) \leq R\]

if \( p < \infty \) and respectively

\[(7) \quad L\|b\|_q + \|K(0)\|_p^* \leq R\]

provided \( p = \infty \). If the operator \( H: D_{L^p}(K(0), R) \cap X \to X \) with uniformly \( p \)-integrable range satisfies condition \((5)\) with \( x_0 \in K(0) \), then there exists at least one solution \( x \in D_{L^p}(K(0), R) \cap X \) of the initial inclusion \((1)\).
Prove. Fix \( u \in L^p(I, E) \). The subset \( N_F(u) \) furnished with the relative weak topology of \( L^q(I, E) \) is compact (cf. Lemma 1 or Corollary 1). Moreover, \((N_F(u), w)\) is in fact an acyclic space, given that \( N_F(u) \) is always contractible in the weak topology \( \sigma(L^q(I, E), L^p(I, E^*)) \) (regardless of whether the values of \( F \) are convex or not, because values of the Nemyskii operator are still decomposable). Under assumption \((K_3)\) the multimap \( K: (N_F(u), w) \rightarrow (H(u), \| \cdot \|_p) \) may be regarded as an acyclic operator between compact topological space \((N_F(u), w)\) and a paracompact space \((H(u), \| \cdot \|_p)\). The same can be said if we assume that \( K \) is Lipschitz with compact acyclic values. Observe that the intersection \( K^{-1}(\{v\}) \cap N_F(u) \) is convex in case \( K \) has convex fibers or decomposable if we assume that the fibers of \( K \) are decomposable. Therefore, the fibers of the multimap under consideration are acyclic. In view of Lemma 3 we are allowed to conclude that the reduced Alexander-Spanier cohomologies \( \tilde{H}^*((H(u), \| \cdot \|_p)) \) are trivial, i.e. the image \( H(u) \) is acyclic as a subspace of \( L^p(I, E) \).

From Lemma 3 follows that the operator \( H: D_{L^p}(K(0), R) \cap X \rightarrow X \) is compact valued upper semi-continuous. Considering what we have established so far, it is apparent that \( H \) is an acyclic operator. Lemma 5 guarantees that condition (3) is also satisfied.

Let \( p < \infty \) and \( R > 0 \) be matched according to (6). Take \( u \in D_{L^p}(K(0), R) \). Since \( K(0) \) is compact (both in the case (i) and in the case (ii)), there is \( z_u \in K(0) \) such that \( \| u - z_u \|_p = \text{dist}(u, K(0)) \). Observe that

\[
\| N_F(u) \|_q^\varepsilon \leq \| b \|_q + c \| u \|_p^\varepsilon \leq \| b \|_q + c \left( \| u - z_u \|_p + \| z_u \|_p \right)^\varepsilon \leq \| b \|_q + c \left( R + \| K(0) \|_p \right)^\varepsilon .
\]

Whence

\[
\sup_{v \in H(u)} \text{dist}(v, K(0)) \leq \sup_{w \in N_F(u)} h(K(w), K(0)) \leq \sup_{w \in N_F(u)} L \| w \|_q = L \| N_F(u) \|_q^\varepsilon \leq \| b \|_q + c \left( R + \| K(0) \|_p \right)^\varepsilon .
\]

Eventually,

\[
H(D_{L^p}(K(0), R) \cap X) \subset D_{L^p}(K(0), R) \cap X,
\]

by (6). The latter entails (3). Indeed, fix any \( x_0 \in K(0), \lambda \in (0, 1) \) and \( x \in D_{L^p}(K(0), R) \). Then

\[
\sup_{v \in H(x)} \text{dist}((1 - \lambda) x_0 + \lambda v, K(0)) \leq \sup_{v \in H(x)} \text{dist}(v, K(0)) \leq \lambda R < R.
\]

Thus, \( x \notin (1 - \lambda) x_0 + \lambda H(x) \) provided \( x \in \partial D_{L^p}(K(0), R) \). In analogous manner one can show that Leray-Schauder boundary condition (3) is satisfied under assumption (7).

In view of Theorem 4 we infer that the multifunction \( H \) has a fixed point in \( D_{L^p}(K(0), R) \cap X \). \( \square \)

Remark 5. As it comes to formulation of sufficient conditions for acyclicity of the values of the superposition \( K \circ N_F \) (cf. 7 Remark 4.2), it should be emphasized that condition: for all \( w_0, w_1, w_2 \in L^q(I, E) \), the equality \( K(w_1 1_{[0,1]} + w_0 1_{[1, T]}) = K(w_2 1_{[0,1]} + w_0 1_{[1, T]}) \)

for every \( \lambda \in I \), is much stronger than assumption regarding the decomposability of the fibers of operator \( K \). Similarly, the condition that operator \( K \) is affine is visibly stronger than the fact that \( K \) has convex fibers.

4. Examples

We conclude this paper with examples, which illustrate the wide range of applications of the unified topological approach, developed in the previous section, to integro-differential inclusions.

Example 1. Given an \( m \)-accretive operator \( A: D(A) \subset E \rightarrow E \) in a Banach space \( E \) and a multivalued perturbation \( F: I \times \overline{D}(A) \rightarrow E \) we consider the initial value problem:

\[
\dot{u}(t) \in -Au(t) + F(t, u(t)) \quad \text{on} \; I,
\]

\[
u(0) = u_0.
\]
If \( A \) is \( m \)-accretive and \( U(t)x \) is an integral solution of (8) with \( F \equiv 0 \) and \( u(0) = x \), then the family \( \{U(t)\}_{t \geq 0} \) of nonexpansive mappings \( U(t) : D(A) \to D(A) \) is called the semigroup generated by \(-A\).

**Theorem 6.** Let \( E^* \) be strictly convex and \( A : D(A) \subset E \to E \) be \( m \)-accretive such that \(-A \) generates an equicontinuous semigroup. Assume that \( F : I \times \overline{D(A)} \to E \) satisfies (F_1)-(F_3) together with
\[
\text{(F_4)} \quad \text{there is } \mu \in L^1(I, \mathbb{R}) \text{ such that } ||F(t, x)||^+ \leq \mu(t)(1 + |x|) \text{ for all } x \in E \text{ and for a.a. } t \in I
\]
and
\[
\text{(F_5)} \quad \text{there is a function } \eta \in L^1(I, \mathbb{R}) \text{ such that for all bounded subsets } \Omega \subset E \text{ and for a.a. } t \in I \text{ the inequality holds}
\]
\[
\beta(F(t, \Omega)) \leq \eta(t) \beta(\Omega).
\]

Then the Cauchy problem (8) has an integral solution for every \( u_0 \in \overline{D(A)} \).

**Proof.** As it is well known, we can associate with any \( w \in L^1(I, E) \) a unique integral solution \( S(w) \in C(I, \overline{D(A)}) \) of the quasi-autonomous problem
\[
\begin{align*}
\dot{u}(t) & \in -Au(t) + w(t) \quad \text{on } I, \\
u(0) & = u_0.
\end{align*}
\]

The mapping \( S : L^1(I, E) \to C(I, E) \) satisfies
\[
|S(w_1)(t) - S(w_2)(t)| \leq |S(w_1)(s) - S(w_2)(s)| + \int_s^t |w_1(\tau) - w_2(\tau)|d\tau
\]
for all \( 0 \leq s \leq t \leq T \), which means in particular that \( S \) meets condition (K_2).

Take \( w_1, w_2 \in S^{-1}(\{u\}) \) and fix \( \lambda \in (0, 1) \). For every \( (x, y) \in \text{Gr}(A) \) and \( 0 \leq s \leq t \leq T \) the following inequality holds
\[
|u(t) - x|^2 - |u(s) - x|^2 \leq 2 \int_s^t \langle w_i(\tau) - y, u(\tau) - x \rangle_+ d\tau,
\]
where \( i = 1 \) and \( i = 2 \) for the cases \( S(w_1) \) and \( S(w_2) \), respectively. Since \( E^* \) is strictly convex, the semi-inner products are indistinguishable, i.e. \( \langle x, y \rangle_+ = \langle x, y \rangle_- \). In view of the latter we are allowed to write down the following estimation:
\[
\begin{align*}
|u(t) - x|^2 - |u(s) - x|^2 & \leq \lambda 2 \int_s^t \langle w_1(\tau) - y, u(\tau) - x \rangle_+ d\tau + (1 - \lambda) 2 \int_s^t \langle w_2(\tau) - y, u(\tau) - x \rangle_+ d\tau \\
& \quad + \lambda \int_s^t \langle w_1(\tau) - y, u(\tau) - x \rangle_+ d\tau + (1 - \lambda) \int_s^t \langle w_2(\tau) - y, u(\tau) - x \rangle_+ d\tau \\
& = 2 \int_s^t \langle (\lambda w_1 + (1 - \lambda) w_2)(\tau) - y, u(\tau) - x \rangle_+ d\tau.
\end{align*}
\]

This means that \( u \) constitutes a solution to the quasi-autonomous problem
\[
\begin{align*}
\dot{u}(t) & \in -Au(t) + (\lambda w_1 + (1 - \lambda) w_2)(t) \quad \text{on } I, \\
u(0) & = u_0.
\end{align*}
\]

In other words \( u = S(\lambda w_1 + (1 - \lambda) w_2) \), i.e. the fiber \( S^{-1}(\{u\}) \) is convex.

Consider a compact subset \( C \subset E \) and a sequence \( \{w_n\}_{n=1}^{\infty} \subset L^1(I, C) \) such that \( w_n \xrightarrow{n \to \infty} L^1(I, E) \cdot w \). Since \(-A\) generates an equicontinuous semigroup and the family \( \{w_n\}_{n=1}^{\infty} \) is uniformly integrable, the image \( S(\{w_n\}_{n=1}^{\infty}) \subset C(I, E) \) is equicontinuous. This is the thesis of [12 Th.2.3]. In view of Pettis measurability theorem \( E_0 := \overline{\text{span}} \bigcup_{n=1}^{\infty} \text{Gr}\(w_n(I)\) \) is a separable subspace of \( E \). The arguments contained in the proof of part (b) of [2] Lem.4] justify the following estimate
\[
\beta\left(S(w_n)(t)\right)_{n=1}^{\infty} \leq \int_0^t \beta_{E_0}\left(|w_n(s)|_{n=1}^{\infty}\right) ds
\]
for $t \in I$. It should be stressed here that the veracity of this formula is completely independent of the geometrical properties of the dual space $E^*$, such as the uniform convexity assumed by the author of [2]. As a consequence, we get that $\beta \left( \{ S(w_n(t)) \}_{n=1}^{\infty} \right) = 0$ for all $t \in I$. In view of the Arzela theorem the sequence $(S(w_n))_{n=1}^{\infty}$ must be uniformly convergent to some $v$. The extra condition regarding the geometry of the dual space $E^*$ makes it possible to demonstrate that $S(w) = v$ (cf. [19]). Therefore, operator $S$ meets condition (K1).

Let $R_0 := \sup_{t \in I} |U(t)x_0|$. To indicate a priori bounds on the solutions of (8) consider $w \in N_F(S(w))$. It is easy to see that
\[ |S(w(t))| \leq |S(0(t))| + |S(w(t)) - S(0(t))| \leq |U(t)x_0| + \int_0^t |w(s)| \, ds \leq R_0 + \|\mu\|_1 + \int_0^t |\mu(s)| |S(w(s))| \, ds \]
for $t \in I$. Hence,
\[ |S(w)| \leq (R_0 + \|\mu\|_1)e^{\|\mu\|_1} =: M \]
for any $w \in \text{Fix}(N_F \circ S)$, by the Gronwall inequality. Now, the standard trick allows us to assume that $\|F(t,x)\| \leq \mu(t)(1 + M) = \delta(t)$ a.e. on $I$ with $\delta \in L^1(I, \mathcal{R}_+)$. Otherwise we may always replace the right-hand side $F$ by $F(\cdot, r(\cdot))$ with $r: E \to D_F(0, M) \cap \overline{co}D(A)$ being a retraction.

Let $X := L^\infty(I, \overline{co}D(A))$. Clearly, $X$ is closed and convex in $L^\infty(I, E)$. Define $H: X \to X$ by the superposition $H := S \circ N_F$, where $N_F: X \to L^1(I, E)$ is the Nemytskii operator corresponding to the righ-hand side $F$. According to the above observations on the growth of $F$, if $w \in L^1(I, E)$ then
\[ \|S(w) - S(0)\| \leq (r_0 + \|\mu\|_1)e^{\|\mu\|_1} := R. \]
This means, in particular, that $H(\partial B_{L^1}(S(0), R) \cap X) \subset D_{L^1}(S(0), R) \cap X$. Put $U := B_{L^1}(S(0), R) \cap X$. It follows that operator $H: \overline{U} \to X$ satisfies boundary condition (5) with $x_0 := S(0) \in U$.

We claim that operator $H$ meets condition (5). Let $M \subset \overline{U}$ be a denumerable subset of $\text{co}([S(0)] \cup H(\overline{M}))$. Then there is a subset $(S(w_n))_{n=1}^{\infty} \subset H(\overline{M})$ such that $M \subset \text{co} \left( \{ S(0) \} \cup \{ (S(w_n))_{n=1}^{\infty} \} \right)$ with $w_n \in N_F(u_n)$ and $u_n \in \overline{M}$. Since the family $\{ (S(w_n))_{n=1}^{\infty} \}$ is equicontinuous (cf. [12], Th.2.3)), the mapping $I \ni t \mapsto \beta((S(w_n))_{n=1}^{\infty}) \in \mathcal{R}_+$ must be continuous. Let $E_0$ be a closed and separable subspace of $E$ such that $\{ w_n(t) \}_{n=1}^{\infty} \subset E_0$ for a.a. $t \in I$. Under assumption (F0) the following estimate is easily verifiable:
\[ \beta_{E_0}(\{ w_n(t) \}_{n=1}^{\infty}) \leq 2 \beta(\{ (S(w_n))_{n=1}^{\infty} \}) \leq 2 \beta(F(t, [u_n(t)]_{n=1}^{\infty})) \leq 2 \eta(t) \beta(\{ u_n(t) \}_{n=1}^{\infty}) \leq 2 \eta(t) \beta((S(w_n(t))_{n=1}^{\infty}) \]
for a.a. $t \in I$. As we have noticed previously, the proof of [2, Lem.4], constitutes a justification for the estimation
\[ \beta \left( \{ S(w_n(t)) \}_{n=1}^{\infty} \right) \leq \int_0^t \beta_{E_0} \left( \{ w_n(s) \}_{n=1}^{\infty} \right) \, ds. \]
Hence,
\[ \beta \left( \{ S(w_n(t)) \}_{n=1}^{\infty} \right) \leq 2 \int_0^t \eta(s) \beta((S(w_n(s)))_{n=1}^{\infty}) \, ds \]
for $t \in I$. Consequently, $\sup_{t \in I} \beta((S(w_n(t)))_{n=1}^{\infty}) = 0$ and vertical slices $M(t)$ are relatively compact in $E$ for every $t \in I$.

Combining the theses of Lemma[4] and [5] with the argument taken from the proof of Theorem[5], we infer that operator $H: \overline{U} \to X$ meets all the requirements imposed by Theorem[4]. Therefore, $H$ possesses a fixed point $u \in \overline{U} \cap C(I, E)$. Clearly, $u$ is an integral solution to (8). \qed

**Remark 6.** Note that our result corresponds exactly to the content of [2, Th.2], except for the assumption about the geometry of the dual space $E^*$. As we have seen it was enough to assume that $E^*$ is strictly convex. The counter-example given in [2] (cf. [2, Ex.1]) shows that this geometric condition on $E^*$ cannot be removed.
Example 2. In this example we consider the antiperiodic problem

\[ \dot{x}(t) = A x(t) + F(t, x(t)) \quad \text{on } I, \]
\[ x(0) = -x(T), \]

under the following assumption

(A) \( A : D(A) \subset E \to E \) is the infinitesimal generator of a \( C_0 \)-semigroup \( \{U(t)\}_{t \geq 0} \) of bounded linear operators on \( E \).

It is well known that there are constants \( M > 0 \) and \( \omega \in \mathbb{R} \) such that \( \|U(t)\| \leq Me^{\omega t} \) for any \( t \geq 0 \). Renorming the Banach space \( E \) in an appropriate way one can achieve that \( M = 1 \). Moreover, \( A \) is a densely defined closed linear operator.

Theorem 7. Assume (A). Let \( F : I \times E \to E \) be a multimap satisfying (F1)-(F3) together with (F4) and (F6). Then the antiperiodic problem \( (11) \) possesses a mild solution in each of the following cases:

(i) \( \omega < -T^{-1} \max\{\|\eta\|, \|\mu\|\} \) if \( E \) is separable,
(ii) \( \omega < -T^{-1} \max\{2\|\eta\|, \|\mu\|\} \) if \( E \) is a weakly compactly generated space,
(iii) \( \omega < -T^{-1} \max\{4\|\eta\|, \|\mu\|\} \) if \( E \) is an arbitrary Banach space.

Proof. Define \( K : L^1(I, E) \to C(I, E) \) to be the map which assigns to each \( f \in L^1(I, E) \) the unique mild solution of the following problem

\[ \dot{x}(t) = A x(t) + f(t), \quad \text{a.e. on } I, \]
\[ x(0) = -x(T). \]

\( K \) is a well defined single-valued mapping. Indeed, suppose \( x, y \) are solutions of \( (12) \). Then

\[ |x(t) - y(t)| = \left| -U(t)x(T) + \int_0^t U(t-s)f(s)\,ds + U(t)y(T) - \int_0^T U(t-s)f(s)\,ds \right| \leq \|U(t)\| \|x(T) - y(T)\| \]

and consequently

\[ |x(T) - y(T)| \leq \|U(T)\| |x(T) - y(T)|. \]

Since \( \|U(T)\| < 1 \), the latter means that \( x(T) = y(T) \). Eventually, \( x(t) = y(t) \) for each \( t \in I \).

Put \( R_0 := \sup_{t \in I} \|U(t)\| \). Let \( f, g \in L^1(I, E) \). Then

\[ |K(f)(t) - K(g)(t)| \leq \left| -U(t)K(f)(T) + U(t)K(g)(T) + \int_0^T U(t-s)f(s)\,ds - \int_0^T U(t-s)g(s)\,ds \right| \leq \|U(t)\| |K(f)(T) - K(g)(T)| + \int_0^T \|U(t-s)\| |f(s) - g(s)|\,ds. \]

On the other hand

\[ |K(f)(T) - K(g)(T)| \leq \|U(T)\| |K(f)(T) - K(g)(T)| + \int_0^T \|U(T-s)\| |f(s) - g(s)|\,ds \]

whence

\[ |K(f)(T) - K(g)(T)| \leq R_0 \frac{R_0}{1 - \|U(T)\|} \|f - g\|_1. \]

Ultimately,

\[ \|K(f) - K(g)\| \leq \frac{R_0}{1 - \|U(T)\|} \|f - g\|_1 + R_0 \|f - g\|_1 = \left( \frac{R_0^2}{1 - \|U(T)\|} + R_0 \right) \|f - g\|_1, \]

i.e. operator \( K \) is Lipschitz.
From the assumption \(|\|U(T)\|_{L^2} < 1\) follows immediately that \(-1 \not\in \sigma_p(U(T))\). Taking into account that \(K(0)(T) = -U(T)K(0)(T)\), it is clear that \(K(0)(T) = 0\). Consequently, \(K(0) = 0\). Now, if we denote
\[
L := \frac{R_0^2}{1 - \|U(T)\|_{L^2}} + R_0,
\]
then one easily sees that
\[
(13) \quad K(D_L((0,R))) \subset D_C(0,LR).
\]
We will show that operator \(K\) satisfies also condition \((K_1)\). To see this consider a compact subset \(C \subset E\) and a sequence \((f_n)_{n=1}^\infty \subset L^1(I,C)\) such that \(f_n \overset{L(I,E)}{\rightarrow} f\). Observe that
\[
\beta \left( \left( U(t-s)f_n(s) \right)_{n=1}^\infty \right) \leq \|U(t-s)\|_{L^2} \beta \left( \left( f_n(s) \right)_{n=1}^\infty \right) \leq R_0 \beta(C) \text{ a.e. on } I.
\]
Applying this inequality in the context of Cor.3.1 one gets
\[
\beta \left( \left( K(f_n)(t) \right)_{n=1}^\infty \right) = \beta \left( \left( -U(t)K(f_n)(T) + \int_0^t U(t-s)f_n(s) \, ds \right)_{n=1}^\infty \right) \leq \beta \left( \left( -U(t)K(f_n)(T) \right)_{n=1}^\infty \right) + 2 \int_0^\infty R_0 \beta(C) \, ds
\]
\[
\leq \|U(t)\|_{L^2} \beta \left( \left( K(f_n)(T) \right)_{n=1}^\infty \right),
\]
which means that \(\beta \left( \left( K(f_n)(T) \right)_{n=1}^\infty \right) = 0\). The latter implies \(\beta \left( \left( K(f_n)(t) \right)_{n=1}^\infty \right) = 0\) for each \(t \in I\). To show that the sequence \((K(f_n))_{n=1}^\infty\) is relatively compact in \(C(I,E)\) we need to prove its equicontinuity. Let \(t_0 \in I\) and \(\varepsilon > 0\) be fixed. There is a subset \(J \subset I\) of full measure such that \((f_n(s))_{n=1}^\infty \subset C\) for every \(s \in J\). Therefore, the closure \([f_n(J)]_{n=1}^\infty\) is compact. As we have shown above, the set \([K(f_n)(T)]_{n=1}^\infty\) is also compact. Hence, the families \((U(t-s)f_n(s))_{n=1}^\infty\) and \((U(t-s)K(f_n)(T))_{n=1}^\infty\) are equicontinuous. In other words there is \(0 < \delta < \frac{\varepsilon}{\|C\|_{L^2}}\) such that, for any \(n \geq 1\) and \(s \in J\),

\[
|U(t)K(f_n)(T) - U(t_0)K(f_n)(T)| < \frac{\varepsilon}{3} \quad \text{and} \quad |U(t-s)f_n(s) - U(t_0-s)f_n(s)| < \frac{\varepsilon}{3T}
\]

if \(|t - t_0| < \delta\). These properties yield that, for any \(n \geq 1\),

\[
|K(f_n)(t) - K(f_n)(t_0)| = \left| -U(t)K(f_n)(T) + U(t_0)K(f_n)(T) + \int_0^t U(t-s)f_n(s) \, ds - \int_0^{t_0} U(t_0-s)f_n(s) \, ds \right|
\]

\[
\leq \|U(t)K(f_n)(T) - U(t_0)K(f_n)(T)\| + \int_0^{t_0} |U(t-s)f_n(s) - U(t_0-s)f_n(s)| \, ds
\]

\[
+ \int_{t_0}^t \|U(t-s)\|_{L^2} |f_n(s)| \, ds < \frac{\varepsilon}{3} + \frac{\varepsilon}{3T} + R_0 \|C\|_1 |t - t_0| < \varepsilon.
\]

Thus, the family \((K(f_n))_{n=1}^\infty\) is equicontinuous. In view of the Arzelà theorem, we see that (passing to a subsequence if necessary) \(K(f_n) \overset{C(I,E)}{\rightarrow} y\). Notice that \(U(\cdot)K(f_n)(T) \overset{C(I,E)}{\rightarrow} U(\cdot)y\). Since the linear part of \(K(f_n)\) tends weakly to \(\int_I U(\cdot-s)f(s) \, ds\), one sees that

\[
y(t) \overset{E}{\leftarrow} K(f_n)(t) \overset{E}{\rightarrow} y(t) + \int_0^t U(t-s)f(s) \, ds, \quad \text{for } t \in I.
\]

Hence, \(y = K(f)\), i.e. \(K : L^1(I,C) \rightarrow C(I,E)\) is strongly continuous.
We claim that $K$ has convex fibers. Take $f, g \in K^{-1}(\{x\})$ and fix $\lambda \in (0, 1)$. In particular, $K(f) = K(g)$. We have a representation

\[
K(f)(t) = \lambda K(f)(T) + (1 - \lambda)K(f)(T) = -\lambda U(T)K(f)(T) + \int_0^T U(T - s)\hat{\lambda}f(s)\,ds - (1 - \lambda)U(T)K(f)(T)
\]

\[
+ \int_0^T U(T - s)(1 - \lambda)g(s)\,ds = -U(T)K(f)(T) + \int_0^T U(T - s)\hat{\lambda}f(s) + (1 - \lambda)g(s)\,ds.
\]

Hence

\[
|K(\hat{\lambda}f + (1 - \lambda)g)(T) - K(f)(T)| \leq \|U(T)\|_\varphi |K(\hat{\lambda}f + (1 - \lambda)g)(T) - K(f)(T)|
\]

and ultimately $K(\hat{\lambda}f + (1 - \lambda)g)(T) = K(f)(T) = K(g)(T)$. Therefore, for each $t \in I$

\[
K(\hat{\lambda}f + (1 - \lambda)g)(t) = -U(t)K(\hat{\lambda}f + (1 - \lambda)g)(T) + \int_0^T U(t - s)\hat{\lambda}f(s) + (1 - \lambda)g(s)\,ds
\]

\[
= \lambda \left( -U(t)K(f)(T) + \int_0^T U(t - s)f(s)\,ds \right) + (1 - \lambda) \left( -U(t)K(g)(T) + \int_0^T U(t - s)g(s)\,ds \right)
\]

\[
= \lambda K(f)(t) + (1 - \lambda)K(g)(t) = \lambda x(t) + (1 - \lambda)x(t) = x(t),
\]

i.e. $\hat{\lambda}f + (1 - \lambda)g \in K^{-1}(\{x\})$.

We will see that there are a priori bounds on the solutions to (11). Take $u \in \text{Fix}(K \circ N_F)$, where $N_F : L^\infty(I, E) \to L^1(I, E)$ is the Nemyskii operator, corresponding to $F$. Then

\[
|u(t)| \leq e^{\omega t}|u(T)| + \int_0^t e^{\omega(t-s)}\mu(s)(1 + |u(s)|)\,ds
\]

for $t \in I$. If the right side of the above inequality we treat as a function $\rho$ of the variable $t$, then:

\[
\rho'(t) = \omega e^{\omega t}|u(T)| + \mu(t)(1 + |u(t)|) \leq \omega \rho(t) + \mu(t)(1 + \rho(t))
\]

for a.a. $t \in I$. Whence

\[
|u(t)| \leq \rho(t) \leq \exp\left(\int_0^t \omega + \mu(s)\,ds\right)\left|u(T)\right| + \int_0^t \mu(s)\exp\left(-\int_0^s \omega + \mu(\tau)\,d\tau\right)\,ds
\]

\[
\leq \exp\left(\int_0^t \omega + \mu(s)\,ds\right)\left|u(T)\right| + \int_0^t \mu(s)\,ds
\]

for every $t \in I$. In particular,

\[
|u(T)| \leq e^{\omega T + \|\mu\|_1}|u(T)| + e^{\omega T + \|\mu\|_1}\|\mu\|_1.
\]

Since $\omega T + \|\mu\|_1 < 0$, one gets the estimation

\[
|u(T)| \leq \frac{e^{\omega T + \|\mu\|_1}}{1 - e^{\omega T + \|\mu\|_1}} =: R_1.
\]

Inserting the latter into (14) we obtain

\[
|u| \leq e^{\omega T + \|\mu\|_1}(R_1 + \|\mu\|_1) =: R_2
\]

for every solution $u$ of (11).

Thus without any loss of generality we may assume that $\|F(t, x)||^\ast \leq \mu(t)(1 + R_2) =: \delta(t)$ for all $x \in E$ and for a.a. $t \in I$ with $\delta \in L^1(I, \mathbb{R})$. Otherwise one can replace $F$ by the multimap $\tilde{F}$ such that $\tilde{F}(t, x) := F(t, r(x))$ with $r$ being the radial retraction onto the disc $D_\delta(0, R_2)$. Now, let us choose a radius $R$ in such a way that $R \geq L\|\delta\|_1$. Then

\[
K(N_F(D_{L^\infty(0, R)})) \subset K(D_{L^\infty(0, \|\delta\|_1)})) \subset D_C(0, L\|\delta\|_1) \subset D_{L^\infty(0, R)}),
\]

by (13). In other words the operator $K \circ N_F : D_{L^\infty(0, R)} \to L^\infty(I, E)$ satisfies condition (3).
To complete the proof it is sufficient to show that the multimap \( H = K \circ N_F \) satisfies condition (5). Let \( M \subset D_t(I, (0, R)) \) be a denumerable subset of \( \text{co}(I) \cup H(M) \). Then there is a subset \( \{v_n\}_{n=1}^{\infty} \subset H(M) \) such that \( M \subset \text{co}(I) \cup \{v_n\}_{n=1}^{\infty} \). Assume that \( v_n = K(w_n) \) and \( w_n \in N_F(u_n) \) with \( u_n \in M \). In view of the Pettis measurability theorem there exists a closed linear separable subspace \( E_0 \) of \( E \) such that \( \{v_n\}_{n=1}^{\infty} \subset E_0 \) for a.e. \( t \in I \). Since the function \( I \ni t \mapsto \beta_{E_0} (v_n(s))_{n=1}^{\infty} \in \mathbb{R} \) is measurable (straightforward consequence of [14, Prop.2.]), it must constitute an element of \( L^\infty(I, \mathbb{R}) \). Due to assumption \((F_\eta)\), the following inequality is satisfied:

\[
\beta\left( (U(t-s)w_n(s))_{n=1}^{\infty} \right) \leq \|U(t-s)\| \beta\left( (w_n(s))_{n=1}^{\infty} \right) \leq \|U(t-s)\| \eta(s) \beta\left( (u_n(s))_{n=1}^{\infty} \right) \leq e^{\eta(t)} \eta(s) \beta\left( (v_n(s))_{n=1}^{\infty} \right).
\]

Applying the latter in the context of [15, Cor.3.1] one obtains, for every \( t \in I \)

\[
\beta\left( (v_n(t))_{n=1}^{\infty} \right) = \beta\left( \left\{ -U(t)K(w_n)(T) + \int_0^T U(t-s)w_n(s) \, ds \right\}_{n=1}^{\infty} \right) \leq \beta\left( \left\{ -U(T)K(w_n)(T) \right\}_{n=1}^{\infty} \right) + 2 \int_0^T e^{\eta(t)} \eta(s) \beta_{E_0} \left( (v_n(s))_{n=1}^{\infty} \right) \, ds.
\]

Let \( \rho : I \rightarrow \mathbb{R} \) be the right-hand side of the above estimation. Making use of its definition we arrive at the following differential inequality:

\[
\rho'(t) = \omega e^{\eta(t)} \beta\left( (v_n(T))_{n=1}^{\infty} \right) + 2 \eta(t) \beta_{E_0} \left( (v_n(t))_{n=1}^{\infty} \right) \leq \omega \rho(t) + 4 \eta(t) \beta\left( (v_n(t))_{n=1}^{\infty} \right) \leq (\omega + 4 \eta(t)) \rho(t)
\]

a.e. on \( I \). By the Gronwall inequality,

\[
\beta\left( (v_n(t))_{n=1}^{\infty} \right) \leq \rho(t) \leq \beta\left( (v_n(T))_{n=1}^{\infty} \right) \exp\left( \int_0^T \omega + 4 \eta(s) \, ds \right).
\]

In particular,

\[
\beta\left( (v_n(T))_{n=1}^{\infty} \right) \leq \beta\left( (v_n(T))_{n=1}^{\infty} \right) \exp(T \| \rho \|_1),
\]

which amounts to \( \beta\left( (v_n(T))_{n=1}^{\infty} \right) \equiv 0 \). The latter stems from the assumption (ii). Now, it is clear that \( \beta\left( (v_n(T))_{n=1}^{\infty} \right) \equiv 0 \) for every \( t \in I \). Therefore, \( M(t) \) is relatively compact for \( t \in I \) and condition (5) follows.

If \( E \) is a weakly compactly generated Banach space, then there is a separable subspace \( E_0 \subset E_1 \) and \( \beta((v_n(t))_{n=1}^{\infty}) = \beta_{E_0}((v_n(t))_{n=1}^{\infty}) \) (see [14, Prop.2.]). Therefore the additional factor 2 in the estimate (16) can be avoided. In the case that \( E \) is separable, there is no need to pass to the relative MNC \( \beta_{E_0} \) and inequality (5) holds without factor 2.

By virtue of theorem 5 the operator \( H \) possesses a fixed point \( x \in D_{L^r}(0, R) \). This fixed point constitutes the solution of the antiperiodic problem (11).

**Remark 7.** The preceding result may be treated as a refinement of [11, Th.8.], taking into account the difference in the type of boundary condition, which is rather cosmetic in nature. Specifically, we did not assume neither equicontinuity of the semigroup \( \{U(t)\}_{t=0}^{\infty} \) nor that \( F \) must be integrably bounded.

**Example 3.** For the third example we consider the following so-called Hammerstein integral inclusion:

\[
x(t) \in h(t) + \int_0^T k(t,s)F(s, x(s)) \, ds \quad \text{a.e. on } I
\]

with \( h \in L^1(I, E) \). We shall assume the following hypotheses about the kernel mapping \( k : I^2 \rightarrow \mathcal{L}(E) \):

\( (k_1) \) the function \( k : I \times I \rightarrow \mathcal{L}(E) \) is strongly measurable in a product measure space,

\( (k_2) \) for every \( t \in I \), \( k(t, \cdot) \in L^r(I, \mathcal{L}(E)) \) with \( r \in (1, \infty] \) being the conjugate exponent of \( q \), i.e. \( q^{-1} + r^{-1} = 1 \).
(k₃) the function \( I \ni t \mapsto k(t, \cdot) \in L^r(I, \mathcal{L}(E)) \) belongs to \( L^p(I, L^r(I, \mathcal{L}(E))) \).

**Theorem 8.** Let \( 1 \leq q \leq p < \infty \). Assume that \( k : I^2 \to \mathcal{L}(E) \) satisfies (k₁)-(k₃), while \( F : I \times E \to E \) fulfills (F₁)-(F₄) together with

\[(F'_6) \text{ for every closed separable subspace } E_0 \text{ of } E \text{ there exists a function } \eta_{E_0} \in L^{r(\cdot)}(I, \Bbb{R}) \text{ such that for all bounded subsets } \Omega \subset E_0 \text{ and for a.a. } t \in I \text{ the inequality holds}
\]

\[\beta_{E_0}(F(t, \Omega) \cap E_0) \leq \eta_{E_0}(t) \beta_{E_0}(\Omega).
\]

If there is an \( R > 0 \) such that

\[(18) \quad |||k(t, \cdot)||_r \leq ||k(t, \cdot)||_r + c \left(R + ||h||_p\right)^q \leq R
\]

and

\[(19) \quad \beta\left(|||k(t, \cdot)||_r \eta_{E_0}||_{r(\cdot)}\right) < 1,
\]

then the integral inclusion \([17]\) has at least one \( p \)-integrable solution.

**Proof.** Define the external operator \( K : L^q(I, E) \to L^p(I, E) \) in the following way

\[(20) \quad K(w)(t) := h(t) + \int_0^T k(t, s)w(s) \, ds, \quad t \in I.
\]

Since, \( K \) is affine, it has convex fibers. It is clear that

\[||K(w_1) - K(w_2)||_p \leq \|k(t, \cdot)||_r \|w_1 - w_2\|_q
\]

for any \( w_1, w_2 \in L^q(I, E) \). Thus, \( K \) meets condition (K₂).

To see that (K₁) is also satisfied, consider a compact subset \( C \subset E \) and a sequence \((w_n)_{n=1}^\infty \subset L^q(I, C)\) such that \( w_n \rightharpoonup w \). Observe that

\[\sup_{n \geq 1} |K(w_n)(t)| \leq |h(t)| + \|k(t, \cdot)||_r \sup_{n \geq 1} \|w_n\|_q \text{ a.e. on } I,
\]

which means that the family \( \{K(w_n)\}_{n=1}^\infty \) is \( p \)-integrably bounded. Moreover,

\[\sup_{n \geq 1} |k(t, s)w_n(s)| \leq \|k(t, s)||_{L^p} \|C\|_{L^p} \text{ a.e. on } I
\]

and

\[\beta\left((k(t, s)w_n(s))_{n=1}^\infty\right) \leq \|k(t, s)||_{L^p} \beta\left((w_n(s))_{n=1}^\infty\right) = 0 \text{ a.e. on } I.
\]

Hence, by [13] Cor.3.1, we have

\[\beta\left((K(w_n)(t))_{n=1}^\infty\right) = \beta\left((h(t) + \int_0^T k(t, s)w_n(s) \, ds)_{n=1}^\infty\right) \leq \beta\left(2 \int_0^T \left|k(t, \cdot)w_n(s)\right| \, ds\right)_{n=1}^\infty \leq 2 \int_0^T 0 \, dt = 0
\]

for a.a. \( t \in I \). Employing [13] Cor.3.1 again one obtains:

\[\beta\left(\left\{\int_s^t K(w_n)(\tau) \, d\tau\right\}_{n=1}^\infty\right) \leq 2 \int_s^t 0 \, d\tau = 0
\]

for every \( 0 < s < t < T \). In other words the sets \( \left\{\int_s^t K(w_n)(\tau) \, d\tau\right\}_{n=1}^\infty \) are relatively compact in \( E \). On the other hand, the following estimate holds

\[\int_0^{T-h} |K(w_n)(t + h) - K(w_n)(t)|^p \, dt \leq \int_0^{T-h} \left(\int_0^T \|k(t + h, s) - k(t, s)||_{L^p} \|w_n(s)\|_q \, ds\right)^p \, dt
\]

\[\leq \sup_{n \geq 1} \|w_n\|_q \int_0^{T-h} \|k(t + h, \cdot) - k(t, \cdot)||_p^p \, dt.
\]
Bearing in mind that the singleton set \( \{ t \mapsto k(t, \cdot) \} \) is compact in \( L^p(I, L^r(I, \mathcal{L}(E))) \), the following convergence is self-evident

\[
\sup_{n \geq 1} \left\| w_n \right\|_{\ell^p} \int_0^T \left\| k(t + h, \cdot) - k(t, \cdot) \right\|_{\ell^p} \, dt \xrightarrow{h \to 0^+} 0.
\]

Therefore, the set \( \{ K(w_n) \}_{n=1}^\infty \) is \( p \)-equiintegrable. In view of [10, Th.2.3.6], \( K(w_n) \xrightarrow{n \to \infty} Y \), up to a subsequence. At the same time

\[
y(t) = E \xrightarrow{n \to \infty} K(w_n)(t) = E \xrightarrow{n \to \infty} h(t) + \int_0^T k(t, s)w(s) \, ds, \quad \text{for } t \in I.
\]

Eventually, \( K(w_n) \xrightarrow{n \to \infty} K(w) \).

Let \( R > 0 \) be matched according to (13) and \( H : D_{12}(h, R) \to L^p(I, E) \) be defined as usual as \( H := K \circ N_F \). Take \( u \in D_{12}(h, R) \) and \( w \in N_F(u) \). Then

\[
|K(w)(t)| \leq |h(t)| + \int_0^T \left| k(t, \cdot) \right| |s| |w(s)| \, ds \leq |h(t)| + \int_0^T \left| k(t, s) \right| \left( |b(s)| + c \left| u(s) \right|^{\frac{q}{p}} \right) \, ds
\]

\[
\leq |h(t)| + \int_0^T \left| k(t, \cdot) \right| \left( |b|_{\ell^q} + c \left( R + |h| \right)^{\frac{q}{p}} \right) \leq |h(t)| + \int_0^T \left| k(t, \cdot) \right| \left( |b|_{\ell^q} + c \left( R + |h| \right)^{\frac{q}{p}} \right),
\]

i.e. the range of the operator \( H \) is uniformly \( p \)-integrable. Note that the assumption (13) is nothing but condition (6) formulated in the context of the Hammerstein inclusion (17).

Let \( M \subset D_{12}(h, R) \) be a countable subset of \( \text{co}(h \cup H(M)) \). Then there is a subset \( \{ n_{-1} \} \in H(M) \) such that \( M \subset \text{co} \left( \{ h \} \cup \{ n_{-1} \} \right) \). Assume that \( v_n = K(w_n) \) and \( w_n \in N_F(u_n) \) with \( u_n \in M \). In view of the Pettis measurability theorem there exists a closed linear separable subspace \( E_0 \) of \( E \) such that

\[
\{ h(t) \} \cup \{ v_n(t) \}_{n=1}^{\infty} \cup \{ w_n(t) \}_{n=1}^{\infty} \cup \left( \int_0^T k(t, s)w_n(s) \, ds \right)_{n=1}^{\infty} \subset \overline{M(t)} \subset E_0 \quad \text{for a.a. } t \in I.
\]

Let \( \mu \in L^p(I, \mathbb{R}) \) be such that \( \mu(t) := |h(t)| + \left| k(t, \cdot) \right| \left( |b|_{\ell^q} + c \left( R + |h| \right)^{\frac{q}{p}} \right). \) From (21) it follows that \( |v_n(t)| \leq \mu(t) \) a.e. on \( I \). Since \( \overline{M(t)} \subset \overline{\text{co} \left( \{ h(t) \} \cup \{ v_n(t) \}_{n=1}^{\infty} \right)} \), we infer that \( |u_n(t)| \leq \mu(t) \) a.e. on \( I \) for every \( n \geq 1 \). Eventually,

\[
\sup_{n \geq 1} \left| v_n(t) \right| \leq \sup_{n \geq 1} \left| F(t, u_n(t)) \right|^{\frac{1}{q}} \leq b(t) + c \sup_{n \geq 1} \left| u_n(t) \right|^{\frac{q}{p}} \leq b(t) + c \mu(t)^{\frac{q}{p}} \quad \text{a.e. on } I,
\]

i.e. the family \( \{ v_n \}_{n=1}^{\infty} \) is \( q \)-integrably bounded. Observe that operator \( K \) meets assumptions of [7, Lem.4.3]. Particularly, condition (S1) is satisfied for the kernel \( k : I \to \mathbb{R} \) such that \( \bar{k}(t, s) := \left| k(t, s) \right|_{\ell^p} \).

Therefore,

\[
\beta_{E_0} \left( \{ v_n(t) - h(t) \}_{n=1}^{\infty} \right) = \beta_{E_0} \left( \{ \int_0^T k(t, s)w_n(s) \, ds \}_{n=1}^{\infty} \right) \leq \int_0^T \left| k(t, s) \right|_{\ell^p} \beta_{E_0} \left( \{ w_n(s) \}_{n=1}^{\infty} \right) \, ds.
\]

Under assumption (F'_e) the following estimate holds:

\[
\beta_{E_0} \left( \{ w_n(s) \}_{n=1}^{\infty} \right) \leq \beta_{E_0} \left( F \left( s, \{ u_n(s) \}_{n=1}^{\infty} \right) \cap E_0 \right) \leq \eta_{E_0}(s) \beta_{E_0} \left( \{ u_n(s) \}_{n=1}^{\infty} \right) \leq \eta_{E_0}(s) \beta_{E_0} \left( M(s) \right) \leq \eta_{E_0}(s) \beta_{E_0} \left( \{ v_n(s) - h(s) \}_{n=1}^{\infty} \right).
\]

Using the latter in the context of (22), one gets

\[
\left\| \beta_{E_0} \left( \{ v_n(\cdot) - h(\cdot) \}_{n=1}^{\infty} \right) \right\|_p \leq \left\| \left| k(t, \cdot) \right|_{\ell^p} \eta_{E_0} \right\|_{\ell^p} \left\| \beta_{E_0} \left( \{ v_n(\cdot) - h(\cdot) \}_{n=1}^{\infty} \right) \right\|_p.
\]
Now, assumption (19) entails $\beta_{E_n} \left( (v_n(t) - h(t))_{n=1}^{\infty} \right) = 0$ a.e. on $I$. This means that vertical slices $\{v_n(t)\}_{n=1}^{\infty}$ are relatively compact for a.a. $t \in I$. Consequently, $M(t)$ is relatively compact a.e. on $I$ and the operator $H: D_{L^p}(h, R) \to L^p(I, E)$ satisfies condition 5.

In view of Theorem 5 there exists a solution of the Hammerstein integral inclusion (17), contained in $D_{L^p}(h, R)$.

Remark 8. The case $p = \infty$ is a little bit tricky as far as it comes to proving the relative compactness of $(K(w_n))_{n=1}^{\infty}$. The easiest way to avoid such speculations is to impose the following assumption

\[ (k'_1) \quad \text{the function } I \ni t \mapsto k(t, \cdot) \in L^1(I, \mathcal{L}(E)) \text{ is continuous} \]

and to make use of the classical Arzelà criterion.

Remark 9. The above proven result is essentially [7, Cor.4.5] formulated in the context of the integral inclusion (17). The issue of the existence of continuous solutions to Hammerstein integral inclusion is well established in the literature of the subject. One such result was proved by the author in [17].

Example 4. The following Volterra integral inclusion

\[ x(t) = h(t) + \int_0^t k(t, s)F(s, x(s)) \, ds, \quad \text{a.e. on } I \]

is a special case of the problem (17) with $k: I^2 \to \mathcal{L}(E)$ such that $k(t, s) = 0$ for $t < s$. The subsequent existence result concerning inclusion (23) stems from the application of Theorem 5.

Theorem 9. Let $h \in L^p(I, E)$. Suppose that all assumptions of Theorem 8 are satisfied with the exception of (19). Then the integral inclusion (23) possesses a $p$-integrable solution.

Proof. In the Volterra case, the mapping $K: L^p(I, E) \to L^p(I, E)$ should by defined as follows:

\[ K(w)(t) := h(t) + \int_0^t k(t, s)w(s) \, ds, \quad t \in I. \]

In order to demonstrate the thesis it is sufficient to give reason for condition 5. The proof of this property goes exactly the same as previously until one reaches the estimate (22). Here, we have

\[
\int_0^t \beta_{E_n} \left( (v_n(s) - h(s))_{n=1}^{\infty} \right)^p \, ds \\
\leq \int_0^t \left( \int_0^\infty \left| k(s, \tau) \right| \beta_{E_n} \left( (v_n(\tau))_{n=1}^{\infty} \right) \, d\tau \right)^p \, ds \\
\leq \int_0^t \left( \int_0^\infty \left| k(s, \tau) \right| \beta_{E_n} \left( (v_n(\tau))_{n=1}^{\infty} \right) \, d\tau \right)^p \, ds \\
\leq \int_0^t \beta_{E_n} \left( (v_n(s) - h(s))_{n=1}^{\infty} \right)^p \, ds
\]

for every $t \in I$. Hence, $\beta_{E_n} \left( (v_n(t) - h(t))_{n=1}^{\infty} \right) = 0$ a.e. on $I$ by the Gronwall inequality. This shows that condition 5 is fulfilled in the Volterra case as well.

Remark 10. Of course, modifying the assumptions about the integral kernel $k$ accordingly, it is not difficult to demonstrate the existence of continuous solutions to Volterra inclusion (cf. [17, Th.5.]).

Corollary 2. Let $1 \leq p = q < \infty$ and $h \in L^p(I, E)$. Suppose that all assumptions of Theorem 8 are satisfied with the exception of (18) and (19). Then the integral inclusion (23) possesses a $p$-integrable solution.
Solvability of Inclusions of Hammerstein Type

**Proof.** Exclusion of assumption (13) means for us necessity of showing that \( H : D_L (h, R) \rightarrow L^p (I, E) \) satisfies boundary condition (3) for some radius \( R > 0 \). To this aim, put

\[
R := 2^{1 - \frac{1}{p}} \left( \frac{\|k(t, \cdot)\|_{p'}}{\|b\|_{p} + c|\|h\|_{p}} \right)^{\frac{1}{1 - \frac{1}{p}}} e^{\frac{1}{2 - \frac{1}{q}} c |\|k(t, \cdot)\|_{p'}}.
\]

Assume that \( u \in \partial D_L (h, R) \). If \( \lambda (u - h) \in H (u) - h \), then

\[
\lambda |u(t) - h(t)| \leq \|k(t, \cdot)\|_{p'} (\|b\|_{p} + c|\|h\|_{p}) \left( \int_{0}^{t} (|u(s) - h(s)| + |h(s)|)^p \, ds \right)^{\frac{1}{p}}
\]

for a.a. \( t \in I \). Whence

\[
\int_{0}^{t} |u(s) - h(s)|^p \, ds \leq \lambda^{-p} 2^{p-1} \|k(t, \cdot)\|_{p'} (\|b\|_{p} + c|\|h\|_{p})^p \int_{0}^{t} |k(s, \cdot)|^p \int_{0}^{s} |u(\tau) - h(\tau)|^p \, d\tau \, d\tau
\]

for every \( t \in I \). The Gronwall inequality implies

\[
\int_{0}^{t} |u(s) - h(s)|^p \, ds \leq \lambda^{-p} 2^{p-1} \|k(t, \cdot)\|_{p'} (\|b\|_{p} + c|\|h\|_{p})^p \exp \left( \lambda^{-p} 2^{p-1} c \int_{0}^{t} |k(s, \cdot)|^p \, ds \right)
\]

for \( t \in I \). Eventually, the following estimation holds

\[
(24) \quad \|u - h\|_{p} \leq \frac{1}{\lambda} 2^{1 - \frac{1}{p}} \|k(t, \cdot)\|_{p'} (\|b\|_{p} + c|\|h\|_{p}) e^{\frac{1}{2 - \frac{1}{q}} c |\|k(t, \cdot)\|_{p'}}.
\]

If \( \lambda > 1 \) would be the case, then (24) leads to

\[
\|u - h\|_{p} < 2^{1 - \frac{1}{p}} \|k(t, \cdot)\|_{p'} (\|b\|_{p} + c|\|h\|_{p}) e^{\frac{1}{2 - \frac{1}{q}} c |\|k(t, \cdot)\|_{p'}}.
\]

which is in contradiction with the definition of radius \( R \). Therefore \( \lambda \leq 1 \) and the Yamamuro's condition is satisfied (Remark 2). \( \square \)

**Example 5.** Let \((\mathbb{H}, \cdot|\cdot|_\mathbb{H})\) be a separable Hilbert space and \((E, \cdot|\cdot|_E)\) be a subspace of \(\mathbb{H}\) carrying the structure of a separable reflexive Banach space, which embeds into \(\mathbb{H}\) continuously and densely. Identifying \(\mathbb{H}\) with its dual we obtain \(E \hookrightarrow \mathbb{H} \hookrightarrow E^\ast\), with all embeddings being continuous and dense. Such a triple of spaces is usually called evolution triple (21, p.416). Let us note that since \(E \hookrightarrow \mathbb{H} \hookrightarrow E^\ast\) continuously, there exist constants \(L, M > 0\) such that \(\cdot|\cdot|_\mathbb{H} \leq L \cdot |\cdot|_E\) and \(\cdot|\cdot|_{E^\ast} \leq M |\cdot|_\mathbb{H}\). Without any loss of generality and to simplify our calculations we may take \(L = M = 1\).

In this example we will investigate an abstract evolution inclusion of the following form:

\[
\dot{x}(t) + A(t, x(t)) + F(t, x(t)) \ni h(t), \quad \text{a.e. on } I := [0, 1],
\]

where \(A : I \times E \rightarrow E^\ast\), \(F : I \times \mathbb{H} \rightarrow \mathbb{H}\) and \(h \in L^q (I, E^\ast)\).

Let \(2 < p < \infty\) and \(1 < q < 2\) be Hölder conjugates. We assume an agreement regarding the following notations: \(\mathcal{E} := L^q (I, E^\ast)\), \(\mathcal{E}^* := L^p (I, E)\), \(\mathcal{H} := L^q (I, \mathbb{H})\), \(\mathcal{H}^* := L^p (I, \mathbb{H})\), \(\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\mathbb{H}}\times E^\ast\) (the duality brackets for the pair \((E^\ast, E)\)).

The symbol \(W^1_p\) stands for the Bochner-Sobolev space

\[
W^1_p (0, 1; E, \mathbb{H}) := \{ x \in \mathcal{E} : \dot{x} \in \mathcal{E}^* \},
\]

where the derivative \(\dot{x}\) is understood in the sense of vectorial distributions. Denote by \(N_A : \mathcal{E} \rightarrow \mathcal{E}^*\) the Nemytski operator corresponding to the multimap \(A\). By a solution of (25) we mean a function \(x \in W^1_p\) such that \(\dot{x}(t) + g(t) + f(t, x(t)) \in h(t)\) a.e. on \(I\), \(x(0) = x(1)\) with \(g \in N_A(x)\) and \(f \in N_F(x)\).

Our hypothesis on the operator \(A : I \times E \rightarrow E^\ast\) is as follows:

\[\begin{align*}
(A_1) & \text{ for every } (t, x) \in I \times E \text{ the set } A(t, x) \text{ is nonempty closed and convex,} \\
(A_2) & \text{ the map } t \mapsto A(t, x) \text{ has a measurable selection for every } x \in E, \\
(A_3) & \text{ for a.a. } t \in I, \text{ the operator } A(t, \cdot) : E \rightarrow E^\ast \text{ is hemicontinuous (i.e. } \lambda \mapsto A(t, x + \lambda y) \text{ is usc from } [0, 1] \text{ into } (E^\ast, w) \text{ for all } x, y \in E,)\end{align*}\]
(A4) the map \( x \mapsto A(t, x) \) is monotone,

(A5) there exists a nonnegative function \( a \in L^0(I, \mathbb{R}) \) and a constant \( \hat{c} > 0 \) such that for all \( x \in E \) and for a.a. \( t \in I, \)

\[
\left\| A(t, x) \right\|_{E^*} \leq a(t) + \hat{c} |x|_{E}^2,
\]

(A6) there exists a constant \( d > 0 \) such that for all \( x \in E \) and a.e. on \( I, \)

\[
d|x|_{E}^p \leq \langle A(t, x), x \rangle^\prime := \inf\{ \langle y, x \rangle : y \in A(t, x) \}.
\]

**Theorem 10.** Let \((E, \mathbb{H}, E^*)\) be an evolution triple such that \( E \hookrightarrow \mathbb{H} \) compactly. Assume that conditions (A1)-(A6) and (F1)-(F4) are satisfied. Suppose further that the following inequality holds

\[
c < d.
\]

Then problem (25) has at least one solution. Moreover, these solutions form a compact subset of the space \( (\mathcal{H}, \| \cdot \|_{\mathcal{H}}) \).

**Proof.** Let \( A_h : I \times E \rightarrow E^* \) be defined by \( A_h(t, x) := A(t, x) - h(t) \). Evidently, the multimap \( A_h \) meets conditions (A1)-(A6), with the proviso that

\[
\left\| A_h(t, x) \right\|_{E^*} \leq \left( a(t) + \|h\|_{E^*} \right) + \hat{c} |x|_{E}^2 \quad \text{and} \quad \langle A_h(t, x), x \rangle^\prime \geq d|x|_{E}^p - \|h(t)|_{E^*}|x|_{E}.
\]

Observe that conditions (F1)-(F4) are fulfilled by the multimap \( A_h \). Indeed, assumptions (A1), (A3), (A4) entail the strong-weak sequential closedness of the graph of \( A_h \) while weak upper semicontinuity of the map \( x \mapsto A_h(t, x) \) follows from assumptions (A3) and (A4) (see [10 Prop.3.2.18]). Therefore, the Nemytskii operator \( N_{A_h} : E \rightarrow E^* \) is a convex weakly compact valued weakly upper semicontinuous map (cf. Corollary 1). As such it is maximal monotone (cf. [10 Prop.3.2.19]).

Denote by the symbol \( L : D(L) \rightarrow E^* \) a continuous linear differential operator \( L := \frac{d}{dt} \), with the domain

\[
D(L) := \{ x \in W^1_\mu : x(0) = x(1) \}
\]

being a closed linear subspace of \( W^1_\mu \). In view of [21] Prop.32.10.] this operator is maximal monotone as well. Therefore, the sum \( L + N_{A_h} : D(L) \rightarrow E^* \) must be maximal monotone (by [10 Th.3.2.41]). Since

\[
\frac{\langle \langle Lx, x \rangle \rangle + \langle \langle N_{A_h}(x), x \rangle \rangle^\prime}{\|x\|_{E}} \geq \frac{d\|x\|_{E}^p - \|h\|_{E^*}\|x\|_{E}}{\|x\|_{E}} = \frac{d\|x\|_{E}^p - \|h\|_{E^*}}{\|x\|_{E}} \rightarrow +\infty,
\]

the map \( L + N_{A_h} \) is coercive and the equality \((L + N_{A_h})(D(L)) = E^* \) follows (cf. [10 Cor.3.2.31]). It is an immediate consequence of definition that the set-valued inverse operator \((L + N_{A_h})^{-1} : E^* \rightarrow E \) is also maximal monotone, i.e.

\[
\langle \langle x_1 - y_1, x_2 - y_2 \rangle \rangle \geq 0 \quad \forall (x_1, x_2) \in \text{Gr}((L + N_{A_h})^{-1}) \implies (y_1, y_2) \in \text{Gr}((L + N_{A_h})^{-1}) \subset E^* \times D(L).
\]

It is easy to realize that (25) is in fact an inclusion of Hammerstein type. Since

\[
Lx \in -N_{A_h}(x) + N_{(-F)}(x) \iff x \in (L + N_{A_h})^{-1} \circ N_{(-F)}(x),
\]

it is fully understandable that the external operator \( K : \mathcal{H} \rightarrow \mathcal{H} \) should be defined as \( K := (L + N_{A_h})^{-1} \).

If we set \( H : \mathcal{H} \rightarrow \mathcal{H} \) to be \( H := K \circ N_{(-F)}, \) then the value of \( H \) at point \( u \) is a solution set of the periodic problem

\[
\ddot{x}(t) + A(t, x(t)) + F(t, u(t)) \ni h(t), \quad \text{a.e. on } I
\]

\[
x(0) = x(1).
\]

Notice that \( K^{-1}(\{x\}) \cap N_{(-F)}(u) = (L + N_{A_h})(x) \cap N_{(-F)}(u) \) for any \( x \in H(u) \). This means that operator \( K \) possesses convex fibers as a map from \( N_{(-F)}(u) \) onto the image \( H(u) \), which is exactly what we need to be able to apply Theorem 5.
Let $w_n \rightarrow w_0$ in $\mathcal{H}^*$ and $x_n \in K(w_n)$ for $n \geq 1$. This means that $x_n \in D(L)$ and $Lx_n + z_n = w_n$ for some $z_n \in N_{A_k}(x_n)$. Since $\{w_n\}_{n=1}^\infty$ is bounded in $\mathcal{H}^*$, we see that

$$\|Lx_n\|_{\mathcal{H}^*} \leq \|z_n\|_{\mathcal{H}^*} + \|w_n\|_{\mathcal{H}} \leq \|N_{A_k}(x_n)\|_{\mathcal{H}^*} + \|w_n\|_{\mathcal{H}} \leq \|a\|_q + \|b\|_{\mathcal{H}} + \varepsilon \|x_n\|_{\mathcal{H}}^{p-1} + \sup_{n \geq 1} \|w_n\|_{\mathcal{H}}$$

and

$$d\|x_n\|_{\mathcal{H}}^p - \|h\|_{\mathcal{H}} \cdot \|x_n\|_{\mathcal{H}} \leq \langle \langle z_n, x_n \rangle \rangle + \langle \langle Lx_n, x_n \rangle \rangle = \langle \langle w_n, x_n \rangle \rangle \leq \|w_n\|_{\mathcal{H}} \cdot \|x_n\|_{\mathcal{H}} \leq \sup_{n \geq 1} \|w_n\|_{\mathcal{H}} \cdot \|x_n\|_{\mathcal{H}}$$

i.e.

$$\|x_n\|_{\mathcal{H}}^{p-1} \leq d^{-1} \left( \sup_{n \geq 1} \|w_n\|_{\mathcal{H}} + \|h\|_{\mathcal{H}} \right).$$

Thus, the sequence $(x_n)^n_{n=1}$ is bounded in $W^1_p$ and we may assume, passing to a subsequence if necessary, that $x_n \overset{W^1_p}{\rightarrow} x_0$. Moreover, there is a subsequence (again denoted by) $(z_n)^n_{n=1}$ such that $z_n \overset{\mathcal{H}}{\rightarrow} z_0$. Since $W^1_p$ embeds into $\mathcal{H}$ compactly (see [10, Th.2.3.30]), we infer that $(x_n)_{n=1}^\infty$ tends strongly, up to a subsequence, to $x_0$ in the norm of $\mathcal{H}$. Observe that $\langle \langle \dot{x}_n, x_n - x_0 \rangle \rangle = \langle \langle \dot{x}_0, x_n - x_0 \rangle \rangle$ and so

$$\limsup_{n \rightarrow \infty} \langle \langle z_n, x_n - x_0 \rangle \rangle = \limsup_{n \rightarrow \infty} \langle \langle -\dot{x}_n, x_n - x_0 \rangle \rangle + \limsup_{n \rightarrow \infty} \langle \langle w_n, x_n - x_0 \rangle \rangle$$

$$= \lim_{n \rightarrow \infty} \langle \langle -\dot{x}_n, x_n - x_0 \rangle \rangle + \limsup_{n \rightarrow \infty} \langle \langle w_n, x_n - x_0 \rangle \rangle \subseteq \mathcal{G} \cap \mathcal{H}$$

$$\leq 0 + \limsup_{n \rightarrow \infty} \|w_n\|_{\mathcal{H}} \cdot \|x_n - x_0\|_{\mathcal{H}}$$

$$\leq \sup_{n \geq 1} \|w_n\|_{\mathcal{H}} \cdot \lim_{n \rightarrow \infty} \|x_n - x_0\|_{\mathcal{H}} = 0.$$  

Hence,

$$\limsup_{n \rightarrow \infty} \langle \langle z_n, x_n \rangle \rangle \leq \langle \langle z_0, x_0 \rangle \rangle.$$  

Recalling the decisive monotonicity trick one easily sees that $z_0 \in N_{A_k}(x_0)$. Indeed, by (28) we have

$$\langle \langle z_0 - z, x_0 - x \rangle \rangle = \langle \langle z_0, x_0 \rangle \rangle - \langle \langle z, x_0 \rangle \rangle - \langle \langle z_0 - z, x_0 \rangle \rangle$$

$$\geq \limsup_{n \rightarrow \infty} \langle \langle (z_n, x_n) - (z_n, x_n) - (z_n - z, x) \rangle \rangle = \limsup_{n \rightarrow \infty} \langle \langle z_n - z, x_n - x \rangle \rangle \geq 0$$

for every $(x, z) \in \text{Gr}(N_{A_k})$. Since $N_{A_k}$ is maximal monotone, we get $z_0 \in N_{A_k}(x_0)$. The weak convergence $x_n \overset{W^1_p}{\rightarrow} x_0$ entails $x_n \overset{C(L, H)}{\rightarrow} x_0$. Thus, $x_n(t) \overset{H}{\rightarrow} x_0(t)$ for every $t \in I$. This means that the boundary condition $x(0) = x_0(1)$ is satisfied. Since

$$Lx_0 + z_0 \overset{\mathcal{H}}{\rightarrow} Lx_n + z_n = w_n \overset{\mathcal{H}}{\rightarrow} w_0,$$

it follows that $x_0 \in K(w_0)$. This justifies the claim that the external operator $K : \mathcal{H}^* \rightarrow \mathcal{H}$ is a strongly upper semicontinuous map with compact values. Further, maximal monotonicity of $K$ implies also convexity of its values (cf. [10, Prop.3.2.7]). In other words, operator $K$ satisfies assumption (K1).

It is easy to see that operator $H$ maps bounded sets into relatively compact ones. Indeed, let $x \in H(u)$ and $w \in N_{(-f)}(u)$ be such that $x \in K(w)$. Then

$$\|Lx\|_{\mathcal{H}} \leq \| - N_{A_k}(x) + w \|_{\mathcal{H}}^p \leq \|N_{A_k}(x)\|_{\mathcal{H}}^p + \|w\|_{\mathcal{H}} \leq \|N_{A_k}(x)\|_{\mathcal{H}}^p + \|w\|_{\mathcal{H}}$$

$$\leq \|a\|_q + \|b\|_{\mathcal{H}} + \varepsilon \|x\|_{\mathcal{H}}^{p-1} + \|b\|_q + c\|a\|_q^{p-1}.$$

and

$$d\|x\|_{\mathcal{H}}^p - \|h\|_{\mathcal{H}} \cdot \|x\|_{\mathcal{H}} \leq \langle \langle N_{A_k}(x), x \rangle \rangle - \langle \langle Lx, x \rangle \rangle \leq \|w\|_{\mathcal{H}} \cdot \|x\|_{\mathcal{H}} - d \leq \left( \|b\|_q + c\|a\|_q^{p-1} \right) \|x\|_{\mathcal{H}}.$$
If \( u \in D_{2\ell}(0, R) \), then
\[
\|x\|_{E}^{p-1} \leq d^{-1} \left( \|b\|_{q} + cR^{p-1} + \|h\|_{E} \right).
\]
Taking into account (20), we arrive at the estimate
\[
\|Lx\|_{E} \leq \|a\|_{q} + \|h\|_{E} + \|b\|_{q} + (1 + d^{-1}c)R^{p-1} + d^{-1}c \left( \|b\|_{q} + \|h\|_{E} \right).
\]
Therefore, the image \( H(D_{2\ell}(0, R)) \) is bounded in \( W_{p}^{1} \). Since the embedding \( W_{p}^{1} \hookrightarrow \mathcal{K} \) is compact, the set \( H(D_{2\ell}(0, R)) \) must have a compact closure in the space \( \mathcal{K} \). This means in particular that operator \( H: D_{2\ell}(0, R) \to \mathcal{K} \) satisfies condition (2).

In order to complete the proof we will choose a radius \( R > 0 \) in such a way that \( u \notin \lambda H(u) \) on \( \partial D_{2\ell}(0, R) \) for all \( \lambda \in (0, 1) \). Let
\[
R := \left( \frac{\|b\|_{q} + \|h\|_{E}}{d - c} \right)^{\frac{1}{p}}.
\]
This definition is correct, since we have assumed (26). Suppose that \( u \in \partial D_{2\ell}(0, R) \) and \( \lambda u \in H(u) \), i.e. \( L(\lambda u) + w = -N_{A}(\lambda u) \) for some \( w \in N_{F}(u) \). One easily sees that
\[
d\|\lambda u\|_{E}^{p} - \|\lambda u\|_{E} \leq \langle (N_{A}(\lambda u), \lambda u) \rangle - \langle (-L(\lambda u) - w, \lambda u) \rangle \leq \|N_{F}(u)\|_{E}^{\prime} \|\lambda u\|_{E}
\]
(32)
\[
\leq \left( \|b\|_{q} + c\|u\|_{E}^{p-1} \right) \|\lambda u\|_{E}
\]
and so
\[
d\|u\|_{E}^{p-1} \leq d\|u\|_{E}^{p-1} \leq \lambda^{1-p} \left( \|b\|_{q} + c\|u\|_{E}^{p-1} + \|h\|_{E} \right).
\]
The latter implies that \( \lambda < 1 \). Otherwise, the following inequality would have to be satisfied
\[
R < \left( \frac{\|b\|_{q} + \|h\|_{E}}{d - c} \right)^{\frac{1}{p}},
\]
which contradicts the definition of radius \( R \). Therefore, operator \( H: D_{2\ell}(0, R) \to \mathcal{K} \) satisfies the boundary condition (3).

By virtue of Theorem 5, the solution set of the periodic problem (25) possesses at least one fixed point. Of course, this fixed point constitutes a solution of the periodic problem (25).

The preceding estimation ensures also that \( \text{Fix}(H) \subset D_{2\ell}(0, R) \). Since \( H \) is completely continuous, the fixed point set \( \text{Fix}(H) \) is closed and the image of this set \( H(\text{Fix}(H)) \) possesses a compact closure. Given that \( \text{Fix}(H) \subset H(\text{Fix}(H)) \), the solution set of the periodic problem (25) must be compact as a subset of the space \( \mathcal{K} \).

\[\Box\]

**Corollary 3.** Under assumptions of Theorem 10, the solution set of the periodic problem (25) is nonempty and compact in the norm topology of the space \( C(I, \mathbb{H}) \).

**Proof.** Suppose that \( u_{n} \in H(u_{n}) \). Since the fixed point set \( \text{Fix}(H) \) is bounded in \( W_{p}^{1} \) (as we have actually proved that previously), we may assume, passing to a subsequence if necessary, that \( u_{n} \overset{w_{p}}{\rightharpoonup} u \) and \( (u_{n})_{n \geq 1} \) be a subsequence strongly convergent to \( u \) in \( \mathcal{K} \). We know already that the limit point \( u \) belongs to \( \text{Fix}(H) \). Thus, it is sufficient to show that \( \sup_{t \in I} |u_{n}(t) - u(t)|_{\mathbb{H}} \to 0 \) as \( n \to \infty \).

Let \( w \in N_{F}(u) \) be such that \( \dot{u}(t) + w(t) - h(t) \in -A(t, u(t)) \) a.e. on \( I \). Clearly, there are \( w_{k_{n}} \in N_{F}(u_{k_{n}}) \) fulfilling \( w_{k_{n}} \overset{\mathcal{K}}{\rightharpoonup} w \) and \( \dot{u}_{k_{n}}(t) + w_{k_{n}}(t) - h(t) \in -A(t, u_{k_{n}}(t)) \) for a.a \( t \in I \). Whence
\[
\langle \dot{u}_{k_{n}}(t) + w_{k_{n}}(t) - \dot{u}(t) - w(t), u_{k_{n}}(t) - u(t) \rangle \leq 0,
\]
by (A4). Accordingly,
\[
\langle \dot{u}_{k_{n}}(t) - \dot{u}(t), u_{k_{n}}(t) - u(t) \rangle \leq \langle w(t) - w_{k_{n}}(t), u_{k_{n}}(t) - u(t) \rangle
\]
for a.a. \( t \in I \) and for every \( n \geq 1 \).
Observe that \( u_k(t) \xrightarrow{n \to \infty} u(t) \) a.e. on \( I \), at least for a subsequence. Fix \( t_0 \in I \) such that \( u_k(t_0) \xrightarrow{n \to \infty} u(t_0) \).

Engaging the integration by parts formula in \( W^1_p \) we get
\[
\frac{1}{2} \left( |u_k(1) - u(1)|_{H^1}^2 - |u_k(t_0) - u(t_0)|_{H^1}^2 \right) = \int_{t_0}^1 \langle \dot{u}_k(s) - \dot{u}(s), u_k(s) - u(s) \rangle ds \\
\leq \int_{t_0}^1 (w(s) - w_k(s), u_k(s) - u(s)) ds = \int_{t_0}^1 (w(s) - w_k(s), u_k(s) - u(s))_{H^1} ds \\
\leq \int_0^1 ||w(s) - w_k(s)||_{H^1} ||u_k(s) - u(s)||_{H^1} ds \leq \left( \sup_{n \geq 1} ||w_{k_n} - w||_{\mathcal{E}^*} \right) ||u_k - u||_{\mathcal{E}^*}.
\]

Thus
\[
||u_k(0) - u(0)||_{H^1}^2 = ||u_k(1) - u(1)||_{H^1}^2 \xrightarrow{n \to \infty} 0.
\]

Now, it is clear that
\[
0 \leq \lim_{n \to \infty} \sup_{t \in I} ||u_k(t) - u(t)||_{H^1}^2 \leq 2 \left( \sup_{n \geq 1} ||w_{k_n} - w||_{\mathcal{E}^*} \right) \lim_{n \to \infty} ||u_k - u||_{\mathcal{E}^*} + \lim_{n \to \infty} ||u_k(0) - u(0)||_{H^1} = 0
\]
and so \( (u_k)_{n \geq 1} \) is norm convergent in \( C(I, \mathbb{H}) \).

In many actual parabolic problems the presence of nonmonotone terms depending on lower-order derivatives forces us to think over the situation where the right-hand side \( F \) is defined only on \( I \times E \) and not on \( I \times \mathbb{H} \). Application of fixed point approach in this context in conjunction with lightweight competence leads to over-restrictive assumptions, as illustrated by the following:

**Theorem 11.** Let \( (E, \mathbb{H}, E^*) \) be an evolution triple such that \( E \hookrightarrow \mathbb{H} \) compactly. Assume that the multifunction \( F : I \times E \rightharpoonup H \) is such that

(i) for every \( (t, x) \in I \times E \) the set \( F(t, x) \) is nonempty and convex,

(ii) the map \( F(\cdot, x) \) has a strongly measurable selection for every \( x \in E \),

(iii) the graph \( \text{Gr}(F(t, \cdot)) \) is sequentially closed in \( (E, w) \times (\mathbb{H}, w) \) for a.a. \( t \in I \),

(iv) there is a function \( b \in L^R(I, \mathbb{R}) \) and \( c > 0 \) such that for all \( x \in E \) and for a.a. \( t \in I \),

\[
||F(t, x)||_{L^p} \leq b(t) + c|x|^\frac{p}{2}.
\]

If hypotheses \((A_1)-(A_6)\) hold and \( h \in E^* \) is such that
\[
\begin{align*}
   c + ||b||_{L^p} + ||h||_{E^*} &\leq d, \\
   c + \check{c} + ||a||_{L^p} + ||b||_{L^p} + ||h||_{E^*} &\leq 1,
\end{align*}
\]
then problem (25) has a solution.

**Proof.** Let us use symbols \( K : \mathcal{H}^* \to W^1_p \) and \( N_{(-F)} : W^1_p \to \mathcal{H}^* \) to denote the operators we have defined previously in the proof of Theorem 10. To tackle the problem of finding solutions of evolution inclusion (25) we shall indicate a fixed point of the superposition \( H := K \circ N_{(-F)} \).

Assume that \( u_n \xrightarrow{w^*_{n \to \infty}} \ u \) and \( x_n \in H(u_n) \) for \( n \geq 1 \). Let \( w_n \in \mathcal{H}^* \) be such that \( w_n \in N_{(-F)}(u_n) \) and \( x_n \in K(w_n) \). Put \( J := \{ t \in I : \sup_{n \geq 1} ||u_n(t)||_E = +\infty \} \). Since \( E \) is reflexive and the sequence \( (u_n)_{n \geq 1} \) is relatively weakly compact in the Bochner space \( L^1(I, E) \), it must be uniformly integrable in view of Dunford-Pettis theorem (10 Th.2.3.24). If \( t \in J \), then for every \( \lambda > 0 \) there is \( n_0 \in \mathbb{N} \) such that \( ||u_n(t)||_E \geq \lambda \). Fix \( \lambda > 0 \). The following estimation can be easily verified
\[
\lambda \ell(J) = \int_J \lambda \cdot dt \leq \int_J \lambda \cdot dt \leq \int_J ||u_n(t)||_E \cdot dt \leq \sup_{n \geq 1} \int_J ||u_n(t)||_E \cdot dt.
\]
Now, if $\ell(J) > 0$, then

$$
\lim_{\lambda \to +\infty} \sup_{n \geq 1} \int |u_n(t)| dt = \lim_{\lambda \to +\infty} \lambda \cdot \ell(J) = +\infty,
$$

which contradicts the uniform integrability of the sequence $(u_n)_{n=1}^\infty$. Therefore,

$$
\ell \left( \left\{ t \in I : \sup_{n \geq 1} |u_n(t)| < +\infty \right\} \right) = \ell(I).
$$

Since the embedding $W^1_p \hookrightarrow \mathcal{K}$ is compact, we may assume that $u_n \xrightarrow{\lambda, n \to \infty} u$. Thus, there exists a subset $I_0$ of full measure in $I$ such that $u_{k_n}(t) \xrightarrow{H, n \to \infty} u(t)$ and at the same time $\sup_{n \geq 1} |u_{k_n}(t)| < +\infty$ for $t \in I_0$.

Fix $t_0 \in I_0$. For every bounded subsequence $(u_{k_n}(t_0))_{n=1}^\infty$ there is a weakly convergent in $E$ subsequence $(u_{m_{k_n}}(t_0))_{n=1}^\infty$. Clearly, $u_{m_{k_n}}(t_0) \xrightarrow{E, n \to \infty} u(t_0)$ and eventually $u_{k_n}(t_0) \xrightarrow{E, n \to \infty} u(t_0)$. Therefore, $u_k(t) \xrightarrow{E, n \to \infty} u(t)$ a.e. on $I$.

Using reflexivity of the space $\mathbb{H}$ and the fact that $F$ has sublinear growth and $x \mapsto F(t, x)$ is sequentially closed in $(E, w) \times (\mathbb{H}, w)$ one can easily show that given a sequence $(x_n, y_n)$ in the graph $\text{Gr}(F(t, \cdot))$ with $x_n \xrightarrow{H, n \to \infty} x$, there is a subsequence $y_{k_n} \xrightarrow{\mathbb{H}, n \to \infty} y \in F(t, x)$. This means, in particular, that $F(t, \cdot)$ is weakly sequentially upper hemi-continuous multimap for a.a. $t \in I$.

Observe that $\|w_k\|_{\mathcal{K}} \leq \|b\|_q + c \sup_{n \geq 1} \|u_{k_n}\|_{E^*}^{r-1}$ for every $n \geq 1$. Of course, there is a subsequence (again denoted by) $(w_{k_n})_{n=1}^\infty$ such that $-w_{k_n} \xrightarrow{\lambda, n \to \infty} -w$. Hence, by the Convergence Theorem (cf. [4] Lem.1.), $w(t) \in -F(t, u(t))$ a.e. on $I$.

As we have already shown in the proof of Theorem [10], there must be a subsequence $(x_{k_n})_{n=1}^\infty$ of the sequence $(x_n)_{n=1}^\infty$ such that $x_{k_n} \xrightarrow{W^1_p, n \to \infty} x \in K(w)$. Since $w \in N_{(\mathcal{F})}(u)$, it follows that $x \in H(u)$. The conducted reasoning proves in essence that for every relatively weakly compact $C \subset W^1_p$, the multimap $H : (C, w) \rightarrow (W^1_p, w)$ is compact valued upper semicontinuous.

Fix $u \in W^1_p$. The subset $N_{(\mathcal{F})}(u)$ furnished with the relative weak topology of $\mathcal{K}$ is compact. Moreover, $(N_{(\mathcal{F})}(u), w)$ is an acyclic space. The multimap $K : (N_{(\mathcal{F})}(u), w) \rightarrow (H(u), w)$ may be regarded as an acyclic operator between compact topological space $N_{(\mathcal{F})}(u)$ and a paracompact space $H(u)$ endowed with the relative weak topology of $W^1_p$. The fibers of this map are formed by intersections $(L \cap N_{(\mathcal{F})}(u)) \cap N_{(\mathcal{F})}(u)$. Hence, they are convex. Lemma [3] implies that the reduced Alexander-Spanier cohomologies $\tilde{H}^*((H(u), w))$ are isomorphic to $\tilde{H}^*((N_{(\mathcal{F})}(u), w))$. In other words the set-valued map $H : (W^1_p, w) \rightarrow (W^1_p, w)$ possesses acyclic values.

Assumption [33] entails

$$
\frac{\|b\|_q + \|h\|_{E^*}}{d - c} \leq 1 \quad \text{and} \quad \frac{\|a\|_q + \|b\|_q + \|h\|_{E^*}}{1 - (c + \hat{c})} \leq 1.
$$

Choose a radius $R > 0$ in such a way that

$$
R \in \max \left\{ \left( \frac{\|b\|_q + \|h\|_{E^*}}{d - c} \right) ^{\hat{c}}, \frac{\|a\|_q + \|b\|_q + \|h\|_{E^*}}{1 - (c + \hat{c})} \right\}, 1.
$$

Then

$$
\ell^{-1} \left( \|b\|_q + c R^{p-1} + \|h\|_{E^*} \right) \leq R^{p-1}
$$

and

$$
\|a\|_q + \|b\|_q + \|h\|_{E^*} + (c + \hat{c}) R^{p-1} \leq R.
$$
Take an \( x \in H\left(D_{W^1_p}(0, R)\right) \). Then \( \|x\|_{C^0} \leq R^{-1} \), by (31) and (34). The latter combined with (30) and (25) yields \( \|Lx\|_{C^1} \leq R \). Therefore the length of vector \( x \), measured in the equivalent norm \( \max \|L(\cdot)\|_{C^1} \) of the space \( W^1_p \), is not greater then \( R \). In other words, \( H\left(D_{W^1_p}(0, R)\right) \subseteq D_{W^1_p}(0, R) \).

As we have seen above, the operator \( H: (D_{W^1_p}(0, R), w) \rightarrow (D_{W^1_p}(0, R), w) \) is a strongly admissible multimap. Since \( H\left(D_{W^1_p}(0, R)\right) \) is norm bounded in \( W^1_p \), it is also a compact map. In view of the Dugundji Extension Theorem (8 Th.4.1), the ball \( D_{W^1_p}(0, R) \) constitutes an absolute retract as a convex subset of a locally convex linear space \( (W^1_p, w) \). From Theorem 3, it follows directly that the multimap \( H: (D_{W^1_p}(0, R), w) \rightarrow (D_{W^1_p}(0, R), w) \) must have at least one fixed point.

**Remark 11.** Generic approach to evolution inclusions governed by operators monotone in the sense of Minty-Browder essentially comes down to the observation that the sum of densely defined monotone integral inclusion (17).

**Example 6.** The last two examples are dedicated to boundary value problems in which the nonlinear part \( F \) possesses not necessarily convex values. The first of these concerns solvability of the Hammerstein integral inclusion (17).

**Theorem 12.** Let \( E \) be a separable Banach space and \( 1 \leq q < p < \infty \). Assume that \( k: \tilde{I}^p \rightarrow L^q(E) \) satisfies (k1)-(k3), while \( F: I \times E \rightarrow E \) satisfies the following conditions:

(i) for every \( (t, x) \in I \times E \) the set \( F(t, x) \) is nonempty and closed,

(ii) the map \( F(t, \cdot) \) is lower semicontinuous for each fixed \( t \in I \),

(iii) the map \( F(\cdot, x) \) is measurable with respect to the product of Lebesgue and Borel \( \sigma \)-fields defined on \( I \) and \( E \), respectively.

(iv) there is a function \( b \in L^q(I, \mathbb{R}) \) and \( c > 0 \) such that for all \( x \in E \) and for a.a. \( t \in I \),

\[
\|F(t, x)\|_p := \text{sup}\{|y_E : y \in F(t, x)\} \leq b(t) + c|x|^q,
\]

(v) there exists a function \( \eta \in L^{p/q}(I, \mathbb{R}) \) such that for all bounded subsets \( \Omega \subset E \) and for a.a. \( t \in I \) the inequality holds

\[
\beta(F(t, \Omega)) \leq \eta(t)\beta(\Omega).
\]

If there is an \( R > 0 \) such that (13) holds together with

\[
|||k(t, \cdot)||, |||\eta|||_{p/q} \rightarrow 1,
\]

then the integral inclusion (17) has at least one \( p \)-integrable solution.

**Proof.** Denote by \( S(I, E) \) the space of all Lebesgue measurable functions mapping I to E, equipped with the topology of convergence in measure. Under these assumptions (cf. [15] p.731), the Nemytskii operator \( N_F : S(I, E) \rightarrow S(I, E) \) is lower semicontinuous. Consider a sequence \( (u_n)_{n \geq 1} \) such that \( u_n \xrightarrow{n \to \infty} u \) in measure. If \( w \) is an arbitrary element of \( N_F(u) \), then there exists a sequence \( w_n \in N_F(u_n) \) such that \( w_n \xrightarrow{n \to \infty} w \) in \( S(I, E) \). From every subsequence of \( (w_n)_{n \geq 1} \) we can extract some subsequence \( (w_{k_n})_{n \geq 1} \) satisfying \( w_{k_n}(t) \xrightarrow{n \to \infty} w(t) \) a.e. on I. W.l.o.g we may assume that \( u_{k_n}(t) \xrightarrow{n \to \infty} u(t) \) for a.a. \( t \in I \). Assumption (iv) means that \( |w_{k_n}(t)| \leq b(t) + c|u_{k_n}(t)|^q \) a.e. on I. Since \( \{u_{k_n}(\cdot)|^{l}_{n=1} \) is uniformly integrable, the latter implies that the family \( \{|w_{k_n}(\cdot)|^{l}_{n=1} \) is uniformly integrable. Secondly, under passage to the limit one sees that \( |w(t)| \leq b(t) + c|u(t)|^q \) a.e. \( t \in I \), i.e. \( w \in L^q(I, E) \). Hence, the uniform integrability of \( \{|w_{k_n}(\cdot) - w(\cdot)|^q\}_{n=1}^{l} \) follows. In view of Vitali convergence theorem \( \lim_{n \to \infty} \int_I |w_{k_n}(t) - w(t)|^q dt = 0 \). Consequently, \( w_n \xrightarrow{n \to \infty} w \). Therefore, the multimap
Let Theorem 13.

Example 7. The last example illustrates the application of Rothe-type fixed point argument, as a conclusion stemming from Theorem 4., in order to show the existence of solutions of the periodic problem (25) with non-convex perturbation term $F$.

**Theorem 13.** Let $(E, H, E^*)$ be an evolution triple such that $E \hookrightarrow H$ compactly. Assume that the operator $A \colon I \times E \rightharpoonup E^*$ satisfies $(A_1)$-$(A_5)$, while the map $F \colon I \times H \rightharpoonup H$ fulfills conditions (i)-(iv) of Theorem 4. Suppose further that the inequality (26) holds. Then problem (25) has at least one solution.

**Proof.** In accordance with what we have shown in the proof of Theorem 4, there must exist a continuous selection $f \colon \mathcal{H} \rightharpoonup \mathcal{H}$ of the Nemytskii operator $N_F$. The proof of Theorem 10 leaves no doubt that the external operator $K \colon \mathcal{H} \rightharpoonup \mathcal{H}$, given by $K := (L + N_A)^{-1}$, is an upper semicontinuous multimap with compact convex values. Therefore, the map $H \colon \mathcal{H} \rightharpoonup \mathcal{H}$ such that $H := K \circ f$ is a strongly admissible one. Proof of Theorem 10 also illustrates the hypothesis that operator $H \colon D_{\mathcal{H}}(0, R) \rightharpoonup \mathcal{H}$ satisfies the boundary condition (3), if you adopt that the radius $R$ is not less than $(\|h\|_H + \|h\|_E)/(d-c)^{\frac{1}{2}}$. And finally, the proof of this theorem sets forth reasons for the compactness of the operator $H \colon D_{\mathcal{H}}(0, R) \rightharpoonup \mathcal{H}$. Hence, we may again evoke the fixed point result expressed in Theorem 4 to show the existence of a fixed point of $H$, i.e. a solution of the problem (25).

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