On justification of Sobolev’s formula for diffraction by wedge

May 29, 2014

A.I. Komech

Faculty of Mathematics of Vienna University
and the Institute for the Information Transmission Problems of RAS (Moscow)
e-mail: alexander.komech@univie.ac.at

A.E. Merzon and J.E. De la Paz Méndez

Institute of Physics and Mathematics
University of Michoacán of San Nicolas de Hidalgo
Morelia, Michoacán, México
e-mail: anatoli@ifm.umich.mx.

1 Introduction

In this paper we develop our results [5]-[8] on scattering of plane waves by two-dimensional wedge

\[ W := \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 = \rho \cos \theta, \ y_2 = \rho \sin \theta, \ \rho \geq 0, \ 0 \leq \theta \leq \phi \} \quad (1.1) \]

of the magnitude \( \phi \in (0, \pi) \). In these papers the scattering was studied for the harmonic incident waves

\[ u_{\text{in}}(y, t) = e^{-i\omega_0(t-n_0 \cdot y)} f(t - n_0 \cdot y) \quad \text{for} \ t \in \mathbb{R} \ \text{and} \ y \in Q, \quad (1.2) \]

where \( n_0 = (\cos \alpha, \sin \alpha) \) and \( Q := \mathbb{R}^2 \setminus W \) is the angle of the magnitude

\[ \Phi := 2\pi - \phi, \quad \Phi \in (\pi, 2\pi). \quad (1.3) \]

The boundary \( \partial Q = Q_1 \cup Q_2 \cup 0 \), where

\[ Q_1 := \{(y_1, 0) : y_1 > 0\} \quad \text{and} \quad Q_2 := \{(\rho \cos \phi, \rho \sin \phi) : \rho > 0\}. \quad (1.4) \]

---

1 Supported partly by Alexander von Humboldt Research Award, Austrian Science Fund (FWF): P22198-N13, and RFBR.
2Supported by CONACYT and CIC of UMSNH and FWF-project P22198-N13.
Further, the profile function $f$ is a Heaviside-type smooth function:

$$f \in C^\infty(\mathbb{R}), \quad \text{supp } f \subset [0, \infty), \quad \text{and } f(s) = 1 \text{ for } s \geq s_0 \quad (1.5)$$

where $s_0 > 0$. The diffraction is described by the mixed problem

$$\begin{cases}
\Box u(y, t) = 0, \quad y \in Q; \\
Bu(y, t)|_{Q_1 \cup Q_2} = 0, \quad t \in \mathbb{R} \\
u(y, t) = u_{in}(y, t), \quad y \in Q, \quad t < 0.
\end{cases} \quad (1.6)$$

Here $\Box = \partial_t^2 - \Delta$, $B = (B_1, B_2)$, and $Bu|_{Q_1 \cup Q_2} = (B_1 u|_{Q_1}, B_2 u|_{Q_2})$, where $B_{1,2}$ are equal to either the identity operator $I$ or to $\partial/\partial n$ where $n$ is the outward normal to $Q$. The $DD$-problem corresponds to $B_1 = B_2 = I$, the $NN$-problem corresponds to $B_1 = B_2 = \partial/\partial n$, and the $DN$-problem corresponds to $B_2 = I, B_1 = \partial/\partial n$.

The uniqueness, existence, the formula for solution to (1.6) and the Limiting Amplitude Principle were proved in [5]-[8]. Now we generalize these results to the case of nonsmooth and nonperiodic incident wave

$$u_{in}(y, t) = F(t - n_0 \cdot y), \quad y \in \mathbb{R}^2, \quad t \in \mathbb{R}, \quad (1.7)$$

where $F$ is a tempered distribution with support in $\mathbb{R}^2$.

Our main results are the formulas for solutions to the nonstationary problems (1.6)

$$u = u_{in} + F_\delta \ast J, \quad (y, t) \in Q \times \mathbb{R}, \quad (1.8)$$

where $J$ is a suitable distribution corresponding to the type of boundary conditions either $DD$, or $NN$, or $DN$. Here $F_\delta(y, t) := F(t) \delta(y)$, and the convolution is well defined in the sense of distributions (see Theorem 3.4 for the $DD$ case).

As an application, we reproduce the Sobolev formula obtained in [10] for $F(s) = h(s)$ (Heaviside function) in the case of the $DD$-problem. We also obtain similar formulas for the $NN$- and $DN$ problems.

Moreover, we give the explicit formula for solution for $F(s) = \delta(s)$. We study also the case when $F$ is a locally summable function such that

$$F(s) = 0, \quad s < 0, \quad \sup(1 + |s|)^p|F(s)| < \infty, \quad s \in \mathbb{R} \quad (1.9)$$

for some $p \in \mathbb{R}$. We analyze the stabilization of solutions as $t \to \infty$. Namely, we prove that the solution locally tends to a limit as $t \to \infty$ if and only if $F(s) \to C, \ s \to \infty$.

We also generalize the Limiting Amplitude Principle which was proved in [2] for smooth Heaviside-type incident waves: in (1.2) for the $DD$ case, and in [1, 6] and [8] for the $DN$- and $NN$ cases respectively. Namely, we consider the incident waves with $F(s) - a^0 e^{-i\omega_0 s} \to 0$, as $s \to \infty$, and write the corresponding nonstationary solutions in the form

$$u(y, t) = A(y, t) e^{-i\omega_0 t}. \quad (1.10)$$

We prove that $A(y, t) \to A_\infty(y)$ as $t \to \infty$, where $A_\infty(y)$ is a solution to the corresponding stationary Helmholtz equation.
The key role in this asymptotic analysis plays the Sommerfeld-Malujinetz type representation for the diffracted wave

$$u_d(\rho, \theta, t) := \frac{i}{4\Phi} \int \frac{Z(\beta + i\theta) F(t - \rho \cosh \beta) d\beta}{\beta + i\theta}, \quad \theta \in \Theta := [\phi, 2\pi] \setminus \{\theta_1, \theta_2\}$$  \hspace{1cm} (1.10)

in the case of locally summable incident wave. We will use it for an analysis of asymptotic behavior of the diffracted wave near the wave front and for large times. The representation was obtained first in [2]-[8] for the Heaviside-type smooth incident wave (1.2) using the method of complex characteristics [14]-[1]. Here we extend this representation to locally summable incident waves. The representation was used in [17], and [8] to find the convergence rate to the limiting amplitude.

Let us comment on previous works. The scattering by wedge of incident wave (1.7) was considered for the first time by Sobolev [9] - [11] in 1930’, by Keller and Blank [12] in 1951, by Kay [13] in 1953 and Rottbrand [20, 22] in 1998.

The problem obviously reduces to the Heaviside incident wave $F(s) = h(s)$. For this step function Sobolev construct in [9] the solution in the form

$$u(y, t) = g(\zeta(y, t)).$$  \hspace{1cm} (1.11)

Here $\zeta(y, t)$ is an “algebraic” function defined by equation

$$bt - m(\zeta)y_1 - n(\zeta)y_2 - \chi(\zeta) = 0,$$  \hspace{1cm} (1.12)

where $m(\zeta)$, $n(\zeta)$ and $\chi(\zeta)$ are suitable complex analytic functions related by

$$m^2(\zeta) + n^2(\zeta) = 1.$$  \hspace{1cm} (1.13)

Sobolev refers the formula (1.11) as the Sobolev-Smirnov representation and relates it to a dilation invariance of the wave equation.

The problem is solved explicitly using conformal mappings onto unit circle and Schwarz’s reflection principle: antisymmetric reflections in the DD case, and symmetric reflections in the NN case.

The resulting formula coincides with our formula (1.8) as we will prove in this paper.

In the next paper [10] (mainly included in [11]) Sobolev relates this process of the reflections with the wave propagation on the logarithmic Riemann surface which is in spirit of the Sommerfeld ideas cited in [11].

In these papers Sobolev introduced the famous discontinuous “weak solutions” to the wave equations which appear the cornerstone for the Theory of Distributions developed later by L. Schwartz.

Keller and Blank [12] also considered diffraction of the Heaviside incident wave by a wedge developing Busemann’s “Conical Flow Method” which is in the same spirit as the Sobolev approach: the dilation invariance of the wave equation allows to reduce the problem to the Laplace equation on a circle with piecewise constant boundary values. The obtained solution coincides with the Sobolev formula (and with our solution) as we will prove elsewhere.

Kay’s approach [13] relies on separation of variables in the wave equation in suitable variables. Any solution of the wave equation is represented in the form of series in the
Whittaker functions [31, p. 279]. The author proves that the series coincide with the Keller-Blank solution in the case of the Heaviside incident wave (see p. 434 of [13]).

Rottbrand [20, 22] considered the diffraction of the plane wave (1.7) with
\[
F(s) = \int_0^s g(\tau) d\tau
\]
where \(g \in L^1(\mathbb{R})\), \(\text{supp } g \subset (0, \infty)\).

The problem is reduced by a conformal map to the Rawlins’s mixed problem which is solved in [20] using the Wiener-Hopf technique.

The solution for the incident wave with \(F(s) = s^{-1/2}\) has been constructed by Borovikov [23] who used this formula to reproduce the solution of Sobolev.

The formulas obtained in [9]-[11] and [12], [13], and [20] appear quite different. It is instructive to note that in all these works the classes of solutions are not specified, and the uniqueness of solutions is not analyzed.

In our paper we construct the solution in a suitable space of distributions for incident wave (1.7) with any tempered distribution \(F\) with the support in \(\mathbb{R}\). Moreover, we prove that the solution is unique in this class, and is given by convolution (1.8). Let us stress, that we deduce the existence and uniqueness of solutions from our previous results [1, 2].

We obtain the Sobolev formula for the theta-function incident wave. This justification of the diffraction formula was one of our main motivation in writing this paper. The coincidence with the Keller-Blank formula requires more calculations and will be published elsewhere.

Let us outline the plan of our paper. In Section 2 we reduce the problem (1.6) by the Fourier-Laplace transform. In Section 3 we obtain the convolution formula for the solution to (1.6). In Sections 4 and 5 we study asymptotics of the solutions as \(t \to \infty\). In Section 6 we check that our general formula coincides with the Sobolev result for the Heaviside function. In Appendix we calculate some Fourier transforms.

## 2 Formulation of the scattering problem

The front of the incident wave \(u_{in}(y, t)\) at any moment of time \(t \leq 0\) is the straight line \(\{y : t - n_0 \cdot y = 0\}\) in \(\mathbb{R}^2\). For \(n_0 \cdot y > t\), we have \(u_{in}(y, t) = 0\) by (1.9). We impose the following conditions on the vector \(n_0\). First, we suppose that \(\phi - \pi/2 < \alpha < \pi/2\). Then the front of \(u_{in}(y, t)\) lies in \(Q\) for \(t < 0\).

Second, we suppose that the incident wave is reflected by both sides of the wedge. This is equivalent to the condition \(0 < \alpha < \phi\). These two conditions on the vector \(n_0\) are expressed by the inequalities:
\[
\max(0, \phi - \pi/2) < \alpha < \min(\pi/2, \phi).
\]

(2.1)

The extension of our results to another angles \(\phi\) and \(\alpha\) does not pose any new difficulties. In particular, the formulas (1.10)-(8.3) remain valid for all angles \(\Phi\) and \(\alpha\) (see Figure 1).
Let us denote by $u(y, t)$ a solution of problem (1.6) and by
\[ u_s(y, t) := u(y, t) - u_{in}(y, t) \] (2.2)
the scattered wave. Then $u_s$ is a solution to the following mixed problem:
\[
\begin{align*}
\square u_s(y, t) &= 0, \quad y \in Q, \\
Bu_s(y, t)|_{Q_1 \cup Q_2} &= -Bu_{in}(y, t)|_{Q_1 \cup Q_2}, \quad t \in \mathbb{R}, \\
u_s(y, t) &= 0, \quad y \in Q, \quad t < 0.
\end{align*}
\] (2.3)

Let us define the meaning of this mixed problem. For a function $u(t) \in S(\mathbb{R})$ we denote its Fourier transform
\[ \hat{u}(\omega) := \mathcal{F}_{t \to \omega} u(t) := \int_{\mathbb{R}} e^{i\omega t} u(t) \, dt, \quad \omega \in \mathbb{R}. \] (2.4)
This transform is extended by the continuity to tempered distributions $u \in S'(\mathbb{R})$. For the case $\text{supp} u \subset \mathbb{R}^+$ the distribution $\hat{u}(\omega)$ admits an analytic extension to the upper half plane $\mathbb{C}^+ := \{ z \in \mathbb{C} : \text{Im } z > 0 \}$ and
\[ |\hat{f}(\omega)| \leq C(1 + |\omega|)^m |\text{Im } \omega|^{-N}, \quad \omega \in \mathbb{C}^+ \] (2.5)
for some $m, N \geq 0$ by the Paley-Wiener Theorem. We will call this analytic continuation as the Fourier-Laplace transform of $f$. Conversely, if an analytic function $G(\omega)$ in $\mathbb{C}^+$ satisfies (2.5) then there exists its boundary value as $\text{Im } \omega \to 0^+$ in the sense of $S'(\mathbb{R})$, see [24, Thm I.5.2].

Let us introduce functional spaces of solutions. First we define spaces of test functions. For $\varphi(y, t) \in C^\infty(\overline{Q} \times \mathbb{R})$ let us denote
\[ \| \varphi \|_{m,N} = \sup_{(y,t) \in \overline{Q} \times \mathbb{R}, |\alpha| \leq m} (1 + |y| + |t|)^N |\partial_y^\alpha \varphi(y,t)| \] (2.6)
Similarly, for $\varphi(y) \in C^\infty(\overline{Q})$ let us denote
\[ \| \varphi \|_{m,N} = \sup_{y \in \overline{Q}, |\alpha| \leq m} (1 + |y|)^N |\partial_y^\alpha \varphi(y)| \] (2.7)
Definition 2.1. We denote the countably-normed spaces:

(i) \( S(Q \times \mathbb{R}) := \{ \varphi(y,t) \in C^\infty(Q \times \mathbb{R}) : \| \varphi \|_{m,N} < \infty, \ m,N > 0 \} \).

(ii) \( S'(\overline{Q}) := \{ \varphi(y) \in C^\infty(\overline{Q}) \} : \| \varphi \|_{m,N} < \infty, \ m,N > 0 \} \).

Now we define the space \( S'(Q \times \overline{\mathbb{R}^+}) \) of tempered distributions with supports in \( Q \times \overline{\mathbb{R}^+} \):

Definition 2.2. \( S'(Q \times \overline{\mathbb{R}^+}) \) is the space of linear continuous functionals on \( S(Q \times \mathbb{R}) \) with supports in \( Q \times \overline{\mathbb{R}^+} \).

For each \( u \in S'(Q \times \overline{\mathbb{R}^+}) \) there exist \( m, N \geq 0 \) such that

\[
|\langle u(y,t), \varphi(y,t) \rangle| \leq C\| \varphi \|_{m,N}(1 + |\omega|)^m|\text{Im} \omega|^{-N}, \ \varphi \in S(Q)
\]  

(2.8)

This follows from definition of the topology in countable-normed spaces as noted in [25, Ch. I, §4]. We will use the following Paley-Wiener Theorem for distributions which is a straightforward generalization of [24, Thm I.5.2].

Lemma 2.3. (i) Let \( u \in S'(Q \times \overline{\mathbb{R}^+}) \). Then its Fourier transform \( \hat{u}(y,\omega) \) extends to an analytic function on \( \mathbb{C}^+ \) with values in \( S'(Q) \), and there exist \( m, N \geq 0 \) s.t.

\[
|\langle \hat{u}(y,\omega), \varphi(y) \rangle| \leq C\| \varphi \|_{m,N}(1 + |\omega|)^m|\text{Im} \omega|^{-N}, \ \varphi \in S(Q)
\]  

(2.9)

(ii) Conversely, let \( \hat{u}(y,\omega) \) be an analytic function of \( \omega \in \mathbb{C}^+ \) with values in \( S'(Q) \) and the bound \( (2.9) \) holds for some \( m, N \geq 0 \). Then \( \hat{u}(y,\omega) \) is the Fourier-Laplace transform of a distribution \( u \in S'(Q \times \overline{\mathbb{R}^+}) \).

Definition 2.4. We denote by \( HP(\mathbb{C}^+, S'(Q)) \) the space of holomorphic functions in \( \mathbb{C}^+ \) with values in \( S'(Q) \) satisfying the bound \( (2.9) \) for some \( m, N \).

Let us introduce the space of solutions to \( (2.3) \).

Definition 2.5. (see Def 2.1 [2]) (i) \( E_\varepsilon \) is the Banach space of functions \( u(y) \in C(\overline{Q}) \cap C^1(\overline{Q}) \) with finite norm

\[
\|u\|_\varepsilon = \sup_{y \in Q} |u(y)| + \sup_{y \in \hat{Q}} \{ y \} |\nabla u(y)| < \infty
\]  

(2.10)

where \( \{ y \} := \frac{|y|}{1+|y|} \) and \( \hat{Q} := Q \setminus 0 \).

(ii) \( \mathcal{M}_\varepsilon \) is the space of tempered distributions \( u(y,t) \in S'(\overline{Q} \times \overline{\mathbb{R}^+}) \), such that its Fourier-Laplace transform \( \hat{u}(y,\omega) \) is a holomorphic function of \( \omega \in \mathbb{C}^+ \) with the values in \( E_\varepsilon \).

For \( u \in \mathcal{M}_\varepsilon \) the Fourier transform in the system \( (2.3) \) gives

\[
\begin{align*}
( -\Delta - \omega^2 ) \hat{u}_s(y,\omega) &= 0 \quad y \in Q \\
\hat{u}_s(y,\omega) &= -\hat{F}(\omega)e^{i\omega y_1 \cos \alpha}, \quad y \in Q_1 \\
\hat{u}_s(y,\omega) &= -\hat{F}(\omega)e^{-i\omega y_2 \cos(\alpha+\Phi)/\sin \Phi}, \quad y \in Q_2 
\end{align*}
\]  

(2.11)

in the case of DD-problem, and similar equations hold for NN and DN -problems (see Appendix A1).

Let us note that the boundary conditions in \( (2.11) \) are well defined for \( \hat{u}_s(y,\omega) \in E_\varepsilon \) in contrast to the boundary conditions in \( (2.3) \) which are not well defined for tempered distributions \( u_s(y,t) \). This suggests the following definition.

Definition 2.6. We call an \( u_s(y,t) \in \mathcal{M}_\varepsilon \) a solution to \( (2.3) \) if \( \hat{u}_s(y,\omega) \) is the solution to \( (2.11) \).
3 Existence and uniqueness

In this section we prove the uniqueness and existence of solution to the scattering problem (1.6) in class $M_{\varepsilon}$, using methods and results of [2]-[8]. We will assume that

$$F \in S'(\mathbb{R}), \quad \text{supp } F \subset \mathbb{R}^+. \hspace{1cm} (3.1)$$

We will prove the existence and uniqueness of a solution to the problem (2.3) with any fixed boundary operators $B_1$ and $B_2$.

3.1 Uniqueness

**Theorem 3.1.** A solution to problem (2.3) is unique in the class $M_{\varepsilon}$ for any $\varepsilon \in (0,1)$.

**Proof.** Let $u_s(y, t) \in M_{\varepsilon}$ satisfy (2.3). By definition (2.6) it suffices to prove the uniqueness of solution $\hat{u}(y, \omega)$ to problems (2.11) for any $\omega \in \mathbb{C}^+$ in the space $E_{\varepsilon}$. This uniqueness is proved in Sections 7 and 8 of [2] for DD-problem, in [4], [6] for DN-problem and in [8] for NN-problem.

3.2 Existence

Let us recall the functions $S_s(y, \omega)$, $S_d(y, \omega)$, $S_r(y, \omega)$ introduced in [3], [6] and in [8] for DD, DN and NN-problems respectively which are the densities of scattered, diffracted and reflected waves respectively and

$$S_r(\rho, \theta, \omega) := \begin{cases} -e^{i\omega \rho \cos(\theta - \theta_1)} , & \phi < \theta < \theta_1 \\ 0, & \theta_1 < \theta < \theta_2 \\ -e^{i\omega \rho \cos(\theta - \theta_2)} , & \theta_2 < \theta < 2\pi \end{cases} \quad \rho > 0, \quad \omega \in \mathbb{Q}^+. \hspace{1cm} (3.2)$$

$$S_d(\rho, \theta, \omega) := \int_{\mathbb{R}} e^{i\omega \rho \cosh \beta} Z(\beta + i\theta) d\beta, \quad \theta \neq \theta_{1,2}$$

$$S_s(\rho, \theta, \omega) := S_r(\rho, \theta, \omega) + S_d(\rho, \theta, \omega), \quad \theta \neq \theta_{1,2}$$

Here

$$\theta_1 := 2\phi - \alpha, \quad \theta_2 := 2\pi - \alpha, \hspace{1cm} (3.3)$$

are the “critical” directions, and

$$Z(\beta) = -H(-\frac{i\pi}{2} + \beta) + H(-\frac{5i\pi}{2} + \beta), \quad \beta \in \mathbb{Q}; \hspace{1cm} (3.4)$$

where $H$ is the Malyuzhinets type kernels $H$ for DD, NN and DN-problems (see Appendix A2). The formulas for $S_r$ for the DN and NN-problems are given in the Appendix A3.

By [3] Thm 8.1, the function $S_s(y, \omega) \in C_b(\mathbb{Q} \times \mathbb{C}^+)$, and it is analytic in $\omega \in \mathbb{C}^+$. This implies that

$$S_s \in HP(\mathbb{Q}^+, S'(\overline{Q})). \hspace{1cm} (3.5)$$
\[ \mathcal{Z}(\beta) := Z(\beta) + Z(-\beta), \quad l(\lambda) = \begin{cases} \ln(\lambda + \sqrt{\lambda^2 - 1}), & \lambda \geq 1 \\ 0, & \lambda \in (0, 1); \end{cases} \tag{3.6} \]

In Appendix A4 we calculate the inverse Fourier transforms \( F_{\omega \to t} \) of \( S_r \) and \( S_d \) which we denote by \( \mathcal{J}_r(\rho, \theta, t) \), and \( \mathcal{J}_d(\rho, \theta, t) \) respectively:

\[
\begin{align*}
\mathcal{J}_r(\rho, \theta, t) &= \begin{cases} 
\delta(t - \rho \cos(\theta - \theta_1)), & \phi < \theta < \theta_1 \\
0, & \theta_1 < \theta < \theta_2 \\
\delta(t - \rho \cos(\theta - \theta_2)), & \theta_2 < \theta < 2\pi 
\end{cases}, \\
\mathcal{J}_d(\rho, \theta, t) &= \frac{i}{4\Phi} \frac{Z(l(t/\rho) + i\theta)}{\sqrt{t^2 - \rho^2}} h(t - \rho), \quad \rho > 0, \quad t \in \mathbb{R}, \tag{3.7}
\end{align*}
\]

\[
\mathcal{J}_s(\rho, \theta, t) = \mathcal{J}_r(\rho, \theta, t) + \mathcal{J}_d(\rho, \theta, t), \quad \theta \neq \theta_{1,2}
\]

where \( h(\cdot) \) denotes the Heaviside function. Let us note that \( \mathcal{J}_r(\rho, \theta, t) = \mathcal{J}_d(\rho, \theta, t) = 0 \) for \( t < 0 \). For the NN and DN-problems the functions \( \mathcal{J}_r \) and \( \mathcal{J}_d \) are calculated in Appendix A5.

For our application it is crucially important that

\[ \mathcal{J}_d, \mathcal{J}_r, \mathcal{J}_s \in S'(Q \times \mathbb{R}^+). \tag{3.8} \]

This follows immediately from Lemma 2.3 \((3.5)\) and the fact that \( \mathcal{J}_s \) is the inverse Fourier transform of \( S_s \).

Now we prove main theorem on existence of solution to \((2.3)\) and its convolution representation.

By Definition 2.6 the problem \((2.3)\) is equivalent to \((2.11)\). Let us recall basic results of \([2] - [8]\).

**Lemma 3.2.** Let the incident wave \((1.2)\) corresponds to the smooth profile function \((1.5)\). Then

i) The unique solution \( \hat{u}_s(y, \omega) \in E_\varepsilon \) to \((2.11)\) is given by

\[ \hat{u}_s(\rho, \theta, \omega) = \hat{F}(\omega)S_s(\rho, \theta, \omega), \quad \omega \in \mathfrak{C}^+. \tag{3.9} \]

where \( \hat{F}(\omega) = \hat{f}(\omega - \omega_0) \).

ii) The parameter \( \varepsilon \) is given by

\[ \varepsilon = \begin{cases} 1 - \frac{\pi}{2\Phi} & \text{for DD and NN cases}, \\
1 - \frac{\pi}{\Phi} & \text{for DN case}. \tag{3.10} \end{cases} \]

The formula \((3.9)\) is proved in [3, (3.15)], while \((3.10)\) are found in Section 10 of [3] for the DD-problem, in Section 6 of [8] for the NN-problem and in Section 16 of [6] for the DN-problem. This lemma implies, in particular, that

\[ S_s(\cdot, \omega) \in E_\varepsilon, \quad \omega \in \mathfrak{C}^+, \tag{3.11} \]

since for any \( \omega \in \mathfrak{C}^+ \) we can choose a smooth profile function \((1.5)\) such that \( \hat{f}(\omega - \omega_0) \neq 0 \).
Corollary 3.3. The function $S_s(y, \omega)$ is a solution to problem (2.11) with $\hat{F} \equiv 1$.

Our main result is the following theorem.

**Theorem 3.4.** Let $F$ satisfy (3.1). Then

i) There exists a generalized solution $u_s(y, t) \in M_\varepsilon$ to problem (2.3) with $\varepsilon$ given by (3.10).

ii) The solution is given by the convolution

$$ u_s = F_\delta * J_s, \quad (y, t) \in \overline{Q} \times \mathbb{R}, \quad (3.12) $$

where $F_\delta(y, t) := F(t)\delta(y)$, and the convolution is well defined in the sense of distributions.

Proof. i) For any distribution (3.1) it is natural to define the solution to (2.11) again by (3.9). Indeed, $\hat{u}_s(\cdot, \omega) \in E_\varepsilon$ for $\omega \in \mathcal{C}^+$ since $S_s(\cdot, \omega) \in E_\varepsilon$ by (3.11). Moreover, $\hat{u}_s$ is a solution to (2.11) by Corollary 3.3. It remains to prove that

$$ u_s(y, t) := F_{\omega \rightarrow t}^{-1} \hat{u}_s(y, \omega) \in M_\varepsilon. \quad (3.13) $$

It suffices to check that $u_s \in S'(\overline{Q}) \times \mathbb{R}^+$, or equivalently, $\hat{u}_s \in HP(\mathcal{C}^+, S'(\overline{Q}))$. First, we note that $\hat{F} \in HP(\mathcal{C}^+, \mathcal{S}'(\overline{Q}))$ by Paley-Wiener Theorem [24, Thm I.5.2]. Second, $S_s \in HP(\mathcal{C}^+, \mathcal{S}'(\overline{Q}))$ by (3.5). Hence, the product (3.9) also belongs to $HP(\mathcal{C}^+, \mathcal{S}'(\overline{Q}))$ since HP is algebra.

ii) The convolution representation (3.12) follows from (3.9). The convolution is well defined since the intersection of the supports of $F_\delta(y', t')$ and $J_s(y - y', t - t')$ is a bounded set for any fixed $y \in \overline{Q}$ and $t \in \mathbb{R}$. 

3.3 Sommerfeld type representation of diffracted wave

Let us substitute the splitting from the last line of (3.7) into (3.12). Then we obtain the corresponding splitting

$$ u_s = u_r + u_d. \quad (3.14) $$

By (3.9), we obtain for the DD-case

$$ u_r = F_{\omega \rightarrow t}^{-1}[\hat{F} S_r] = F_\delta * J_r, \quad u_d = F_{\omega \rightarrow t}^{-1}[\hat{F} S_d] = F_\delta * J_d. \quad (3.15) $$

Similar formulas hold for NN and DN-cases. The explicit expressions of $u_r$ for all types of boundary conditions are given in Appendix A6.

**Lemma 3.5.** Suppose that

$$ F \in \mathcal{L}_{1_{loc}}^{1}(\mathbb{R}) \quad (3.16) $$

and (1.9) holds. Then the diffracted wave $u_d$ admits the representation (1.10) with a suitable kernel $Z$ for any boundary conditions of the DD, NN or DN-types.
Proof. It suffice to prove that
\[ \hat{u}_d(\rho, \theta, \omega) = \hat{F}(\omega)S_d(\rho, \theta, \omega) = \frac{i}{4\Phi} \int_{\mathbb{R}} e^{i\omega t} \left( \int_{\mathbb{R}} Z(\beta + i\theta) F(t - \rho \cosh \beta) d\beta \right) dt, \quad \omega \in \mathbb{C}^+. \]  

(3.17)

Let us denote
\[ q = \frac{\pi}{2\Phi}. \]  

(3.18)

From (3.4) and (8.2), (8.3) we get the decay
\[ |Z(\beta + i\theta)| \leq C(\theta) e^{-2q|\beta|}, \quad \theta \in \Theta \]  

(3.19)

for the DD- and NN problems, and
\[ |Z(\beta + i\theta)| \leq C(\theta) e^{-q|\beta|}, \quad \theta \in \Theta \]  

(3.20)

for the DN-problem. Hence (1.9) implies by the Fubini Theorem,
\[ \frac{i}{4\Phi} \int_{\mathbb{R}} e^{i\omega t} \left( \int_{\mathbb{R}} Z(\beta + i\theta) F(t - \rho \cosh \beta) d\beta \right) dt \]
\[ = \frac{i}{4\Phi} \int_{\mathbb{R}} Z(\beta + i\theta) \left( \int_{\mathbb{R}} e^{i\omega t} F(t - \rho \cosh \beta) dt \right) d\beta = \hat{F}(\omega)S_d(\rho, \theta, \omega), \quad \omega \in \mathbb{C}^+, \]  

(3.21)

by formula (3.2) for $S_d$.

4 Stabilization of the diffracted wave

Let $l_k$ be the critical rays $l_k := \{(\rho, \theta_k) : \rho > 0\}, k = 1, 2$.

Lemma 4.1. Let (2.1), (3.16) and (1.9) hold. Then for any type of boundary conditions (DD, NN and DN) and $t \in \mathbb{R}$ there exist the limits
\[ u_d(\rho, \theta_k \pm 0, t) := \lim_{\varepsilon \to 0^+} u_d(\rho, \theta_k \pm \varepsilon, t), \quad \rho > 0, \quad k = 1, 2 \]

in the sense of distribution, and the jumps of $u_d$ on the critical rays for the DD-problem are given by
\[ [u_d]_k(\rho, t) := u(\rho, \theta_k + 0, t) - u(\rho, \theta_k - 0, t) = (-1)^{k+1} F(t - \rho), \quad \rho > 0, \quad k = 1, 2. \]  

(4.1)

Proof. We will use the representation (1.10) and we will consider DD case for the concreteness. The cases of NN and DN-problems are analyzed similarly (see Appendix A6). Formulas (3.4) and (8.2) imply the following representation:
\[ Z(\beta + i\theta) = -\coth \left( q\beta + ic_0 \right) \mp \coth \left( q\beta + ic_1 \right) \pm \coth \left( q\beta + ic_2 \right) + \coth \left( q\beta + ic_3 \right) \]  

(4.2)

for DD and NN-cases respectively, where
\[ c_k := q(\theta - p_k); \quad p_0 = \alpha, p_1 = \theta_1, p_2 = \theta_2, p_3 = 2\pi + \alpha. \]  

(4.3)
First let us consider the case when $F$ is a smooth function satisfying (1.9). Then the Sokhotski-Plemelj formulas imply

$$[u_d]_k(\rho, t) = \int_{-1}^{1} F(t - \rho \cosh \beta) \left[ \coth(q\beta + i\theta) - \coth(q\beta - i\theta) \right] d\beta = (-1)^{k+1} F(t - \rho)$$

since $\coth(q\beta + i\theta)$ with $k = 0$ and $k = 3$ are continuous on the critical rays for $\alpha$ satisfying (2.1). For $F$ satisfying (1.9), (4.1) holds in the sense of distributions.

**Theorem 4.2.** Let the incident wave profile $F$ satisfy (1.9), and

$$F(s) \to C, \quad s \to \infty. \quad (4.4)$$

Then

i) The diffracted wave converges in the long time limit:

$$u_d(\rho, \theta, t) \underset{t \to \infty}{\longrightarrow} u_d(\theta, \infty) := \frac{iC}{4\Phi} \int_{\mathbb{R}} Z(\beta + i\theta)d\beta, \quad \rho > 0, \quad \theta \in \Theta, \quad (4.5)$$

and in particular,

$$[u_d]_k(\rho, t) \underset{t \to \infty}{\longrightarrow} C, \quad \text{for} \quad \rho > 0, \quad k = 1, 2. \quad (4.6)$$

ii) Conversely, (4.6) implies (4.4).

**Proof.** We can use (1.10) by Lemma 3.5.

i) Conditions (1.9) and (1.10) imply that

$$u_d(\rho, \theta, t) = \frac{i}{4\Phi} \int_{-l(t/\rho)}^{l(t/\rho)} Z(\beta + i\theta)F(t - \rho \cosh \beta)d\beta, \quad \theta \in \Theta \quad (4.7)$$

where $l(\cdot)$ is defined by (3.6). Then (4.5) follows from (3.19) by the Lebesgue Dominate Convergence Theorem. The convergence (4.6) follows from (4.1) and (4.4).

ii) (4.4) follows from (4.1) and (4.6).

**Corollary 4.3.** For any type of boundary conditions (DD, NN or DN) the function $u_d(\theta, \infty)$ is the piecewise constant function of $\theta \in \Theta$ with the jumps at $\theta = \theta_1$ and $\theta = \theta_2$.

**Proof.** For the DD and NN cases formula (4.2) implies that $Z(\beta)$ is holomorphic on $\mathbb{C} \setminus P$ where $P = \cup_{l=0,1,2,3}\{ip \pm 2ik\Phi : \quad k \in \mathbb{Z}\}$. Moreover, $p_0, p_3 \not\in \Theta$ by (2.1). Hence, $Z(\beta + i\theta)$ may have a pole $\beta \in \mathbb{R}$ only at $\beta = 0$, and it holds only for $\theta = \theta_1$ or $\theta = \theta_2$. Therefore, the corollary follows from the decay (3.19) and the Cauchy Theorem.

For the DN-case the proof is similar relying on the decay (3.20).
5 Limiting Amplitude Principle

Consider the incident wave

$$F^0(t) := a^0 e^{-i\omega_0 t} h(t), \quad t \in \mathbb{R}$$

where $\omega_0 \neq 0$ (the case $\omega_0 = 0$ is covered by Theorem (4.2)). By (1.10) the corresponding diffracted wave is given by

$$u^0_d(\rho, \theta, t) = i e^{-i\omega_0 t} \int_{-l(t/\rho)}^{l(t/\rho)} e^{i\omega_0 \rho \cosh \beta} Z(\beta + i\theta) d\beta$$

where $Z$ is given by (1.2) for the DD- and NN problems (and by (8.3) for the DN-problem), and $l(\cdot)$ is defined by (3.6). The limiting amplitude of this wave is

$$A^0(\rho, \theta) = \frac{i}{4\Phi} \int_{\mathbb{R}} a^0 e^{i\omega_0 \rho \cosh \beta} Z(\beta + i\theta) d\beta. \quad \forall \theta \in \Theta$$

(5.1)

since $l(t/\rho) \to \infty$, as $t \to \infty$ while $Z$ satisfies (3.19) for DD and NN-problems and (3.20) for DN-problem.

Let us define the amplitude $A_d(\rho, \theta, t)$ of the diffracted wave (1.10) by

$$A_d(\rho, \theta, t) := e^{-i\omega_0 t} \frac{i}{4\Phi} \int_{-l(t/\rho)}^{l(t/\rho)} Z(\beta + i\theta) F(t - \rho \cosh \beta) d\beta, \quad \rho > 0, \quad \theta \in \Theta, \quad t > 0.$$ (5.2)

In the following theorem we prove that the amplitude is asymptotically close to the amplitude (5.1) if $F$ is asymptotically close to $F^0$.

Theorem 5.1. (Limiting Amplitude Principle) Suppose that

$$R(t) := F(t) - F^0(t) \to 0, \quad t \to \infty.$$ (5.3)

Then for any $\delta > 0$

$$A_d(\rho, \theta, t) - A_0(\rho, \theta) \to 0, \quad t \to \infty,$$

uniformly in bounded $\rho > 0$ and $\theta - \theta_k \geq \delta$.

Proof. By definitions (5.1) and (5.2),

$$A_d(\rho, \theta, t) - A^0(\rho, \theta) = -\frac{i}{4\Phi} \int_{|\beta| \geq l(t/\rho)} F^0(-\rho \cosh \beta) Z(\beta + i\theta) d\beta + \frac{i}{4\Phi} \int_{-l(t/\rho)}^{l(t/\rho)} e^{i\omega_0 t} Z(\beta + i\theta) R(t - \rho \cosh \beta) d\beta.$$
uniformly in $\rho > 0$ and $\theta \in \Theta$. It remains to prove that

$$R_1(\rho, \theta, t) := \int_{-l(t/\rho)}^{l(t/\rho)} e^{i\omega_0 t} Z(\beta + i\theta) R(t - \rho \cosh \beta) d\beta \to 0, \quad t \to \infty$$

uniformly in bounded $\rho > 0$ and $\theta - \theta_k \geq \delta > 0$. First, (3.19), (3.20) and (5.3) imply that for any $\varepsilon > 0$ there exists $\beta(\varepsilon)$ s.t.

$$\int_{|\beta| \geq \beta(\varepsilon)} |Z(\beta + i\theta) R(t - \rho \cosh \beta)| d\beta < \varepsilon/2 \quad (5.4)$$

uniformly in $\rho > 0$ and $\theta \in \Theta$. Second, (5.3) implies that for $0 < \rho \leq b < \infty$ and $\theta - \theta_k \geq \delta > 0$ there exists $t(\varepsilon, \delta, b)$ such that

$$2\beta(\varepsilon) |Z(\beta + i\theta) R(t - \rho \cosh \beta)| < \varepsilon/2, \quad |\beta| < \beta(\varepsilon), \quad \theta \in \Theta, \quad t > t(\varepsilon, \delta, b). \quad (5.5)$$

Then for $0 < \rho \leq b < \infty$ and $t > t(\varepsilon, \delta, b)$ we have

$$|R_1(\rho, \theta, t)| \leq \int_{|\beta| \leq \beta(\varepsilon)} |Z(\beta + i\theta) R(t - \rho \cosh \beta)| d\beta + \int_{|\beta| \geq \beta(\varepsilon)} |Z(\beta + i\theta) R(t - \rho \cosh \beta)| d\beta < \varepsilon$$

by (5.5) and (5.4).

6 Application to the Sobolev problem

In this section we reproduce the Sobolev formula for dispersion of the $\theta$-function. First, we calculate the diffracted wave for $F(s) = h(s)$. Let us choose below the branch of ln with

$$\text{Im } \ln z \in (-\pi, \pi), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \quad (6.1)$$

**Proposition 6.1.** Let $F(s) = \theta(s)$ and $t > \rho > 0$. Then

i) for the DD and NN-problems respectively

$$u_d(\rho, \theta, t) = \frac{i}{2\pi} \left[ -\ln U_0 \mp \ln U_1 \pm \ln U_2 + \ln U_3 \right], \quad \theta \in \Theta \quad (6.2)$$

where

$$U_k = \frac{b^q e^{ic_k} - b^{-q} e^{-ic_k}}{(b^q e^{-ic_k} - b^{-q} e^{ic_k})}, \quad k = 0, 1, 2, 3; \quad b = \frac{t}{\rho} + \sqrt{\left(\frac{t}{\rho}\right)^2 - 1}, \quad (6.3)$$

and $c_k$ are given by (4.3)

ii) For the DN-problem

$$u(\rho, \theta, t) = \frac{i}{2\pi} \left[ \ln V_0 - \ln V_1 + \ln V_2 - \ln V_3 \right] \quad (6.4)$$

where

$$V_k = \frac{(b^q e^{ic_k} + 1)(b^q e^{-ic_k} - 1)}{(b^q e^{ic_k} - 1)(b^q e^{-ic_k} + 1)}, \quad k = 0, 1, 2, 3. \quad (6.5)$$
Proof. i) Formula (1.10) with \( F(s) = \theta(s) \) gives

\[
\int_{\rho}^{\rho/c} \int_{-l(t/\rho)}^{l(t/\rho)} Z(\beta + i\theta) d\beta = \frac{i}{4\Phi} \sum_{k=0}^{3} s_k \int_{-l(t/\rho)}^{l(t/\rho)} \coth(\beta + ic_k) d\beta = \frac{i}{4\Phi} \sum_{k=0}^{3} s_k \int_{-l(t/\rho)}^{l(t/\rho)} \frac{d\beta}{\sinh(\beta + ic_k)},
\]

since \( Z \) and \( l(t/\rho) \) are defined by (4.2) and (3.6) respectively, \( c_k \in (-\pi, 0) \cup (0, \pi) \), and \( b > 1 \). Now (6.2) is proved.

ii) Similarly to (6.6),

\[
\int_{\rho}^{\rho/c} \int_{-l(t/\rho)}^{l(t/\rho)} Z(\beta + i\theta) d\beta = \frac{i}{4\Phi} \sum_{k=0}^{3} s_k \int_{-l(t/\rho)}^{l(t/\rho)} \frac{d\beta}{\sinh(\beta + ic_k)} = \frac{i}{2\Phi} \sum_{k=0}^{3} s_k \ln \left( \frac{b^c e^{ic_k} + 1}{b^c e^{-ic_k} - 1} \right),
\]

where \( \arg (\cdot) \in (-\pi, \pi) \). Now (6.4) is proved.

Corollary 6.2. Let the incident wave \( F \) be the Heaviside function. Then the diffracted wave \( u_d \) admits the following limits as \( t \to \infty \):

i) for the DD-case,

\[
u_d(\rho, \theta, t) \to \begin{cases} 
0, & \theta \in (\phi, \theta_1) \cup (\theta_2, 2\pi), \\
-1, & \theta \in (\theta_1, \theta_2).
\end{cases}
\]

ii) for the NN-case,

\[
u_d(\rho, \theta, t) \to \begin{cases} 
\frac{2\pi}{\Phi} - 2, & \theta \in (\phi, \theta_1) \cup (\theta_2, 2\pi), \\
\frac{2\pi}{\Phi} - 1, & \theta \in (\theta_1, \theta_2).
\end{cases}
\]

iii) for the DN-case,

\[
u_d(\rho, \theta, t) \to \begin{cases} 
0, & \theta \in (\phi, \theta_1), \\
-1, & \theta \in (\theta_1, \theta_2), \\
-2, & \theta \in (\theta_2, 2\pi).
\end{cases}
\]

Proof. i) First, let us note that

\[
I(\rho, c) := \lim_{t \to +\infty} \int_{-l(t/\rho)}^{l(t/\rho)} \coth(\beta + ic) d\beta = \frac{i}{2\pi} \ln(-e^{2ic})
\]
Hence,

\[ I(\rho, c) = \begin{cases} \frac{-c}{\pi} + \frac{1}{2}, & c \in (0, \pi), \\ \frac{-c}{\pi} - \frac{1}{2}, & c \in (-\pi, 0). \end{cases} \] (6.12)

Now (6.8) for the DD-problem and (6.9) for the NN-problem follow from (6.2), (6.11), (3.3) and (4.3). The limits (6.10) follow similarly.

**Corollary 6.3.** As \( t \to \infty \), the total solution \( u(y, t) \) of problem (1.6)

i) Tends to 0 as \( t \to \infty \) for the DD and the DN-problems and

ii) Tends to \( 2\pi/\Phi \) as \( t \to \infty \) for the NN-problem.

**Proof.** In the case of the DD-problem

\[ u(\rho, \theta, t) \to 0, \ t \to \infty \] (6.13)

since \( u_{in}(\rho, \theta, t) \to 1 \) by (1.7) and \( u_r(\rho, \theta, t) \to -1 \) for \( \theta \in (\phi, \theta_1) \cup (\theta_2, 2\pi) \) by (8.10).

Similarly for DN-problem (6.13) holds since \( u_r \to -1 \) for \( \theta \in (\phi, \theta_1) \) and \( u_r \to 1 \) for \( \theta \in (\theta_2, 2\pi) \) by (8.11).

In the case of the NN-problem \( u(\rho, \theta, t) \to \frac{2\pi}{\Phi} \). In fact, it follows from (6.9) and the fact that \( u_r(\rho, \theta, t) \to 1, \ t \to \infty \) for \( \theta \in (\phi, \theta_1) \cup (\theta_2, 2\pi) \) and \( u_r(\rho, \theta, t) \to 0, \ t \to \infty \) for \( \theta \in (\theta_1, \theta_2) \) by (8.11).

**Remark 6.4.** In the case of the NN-problem the limit does not vanish and depends on \( \Phi \) in contrast to the case of DD and DN-problems.

**Example 6.5.** Let us consider \( F(s) = \delta(s) \). Then \( \hat{F} = 1 \), and

i) (3.13) implies that \( u_d \) equals \( J_d \) which is given by (3.7); ii) \( \hat{u}_s \) equals \( S_s = \hat{J}_s \) which is the solution to (2.11).

### 7 Sobolev’s formula

Now we compare our solution to problem (1.6) with the Sobolev [9] formula (34)] solutions.

**Lemma 7.1.** The solution \( u \) to the problem (1.6) given by (2.2), (3.14) and (8.10) with the pulse \( u_{in}(s) = F(s) = h(s) \) coincides with the Sobolev solution.

**Proof.** We will identify our solution with the Sobolev one (for the DD-problem) in the form of Petrashen’ et al [18] formula (29.3)]: for \( \tau \in [0, 1] \)

\[ U(\varphi, \tau) = \frac{1}{2\pi i} \ln \left[ \frac{1 - p(\tau)e^{-i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 + \pi)}}{1 - p(\tau)e^{i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 + \pi)}} \right] \left[ \frac{1 - p(\tau)e^{i\frac{\pi}{\alpha_1}(\varphi + \varphi_0 - \pi)}}{1 - p(\tau)e^{-i\frac{\pi}{\alpha_1}(\varphi + \varphi_0 - \pi)}} \right] \]

(7.1)
where the branch of $\ln(\cdot)$ is specified by (6.1). In our notations

$$\varphi_0 = \alpha - \pi + \alpha_1, \quad \alpha_1 = \Phi, \quad \varphi = \theta - 2\pi + \alpha_1, \quad \tau = \frac{t}{\rho}, \quad p(\tau) = b^{-2q}. \quad (7.2)$$

The critical directions $\theta_1$ and $\theta_2$ correspond to

$$\varphi_1 = \pi - \varphi_0, \quad \varphi_2 = \pi + 2\alpha_1 - \varphi_0$$

Further, in the new variables (4.3) reads

$$c_0 = \frac{\pi}{2\alpha_1}(\varphi - \varphi_0 + \pi), \quad c_1 = \frac{\pi}{2\alpha_1}(\varphi + \varphi_0 - \pi), \quad c_2 = \frac{\pi}{2\alpha}(\varphi + \varphi_0 + \pi - 2\alpha_1), \quad c_3 = \frac{\pi}{2\alpha}(\varphi - \varphi_0 - \pi) \quad (7.3)$$

Substituting these values into (6.2) we obtain the diffracted wave in the variables of [18]: for $\tau < 1$ and all $\varphi \in (0, \alpha_1)$,

$$u_d(\varphi, \tau) = \frac{1}{2\pi i} \left[ \ln \frac{1 - p(\tau) e^{-i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 + \pi)}}{1 - p(\tau) e^{i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 + \pi)}} - \ln \frac{1 - p(\tau) e^{-i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 - \pi)}}{1 - p(\tau) e^{i\frac{\pi}{\alpha_1}(\varphi - \varphi_0 - \pi)}} \right] \quad (7.4)$$

and $u_d(\varphi, \tau) = 0$ for $\tau > 1$. Finally, (2.2), (3.14), (1.7) and (8.10) imply that in the variables of [18] the total solution $u$ to problem (2.3) for the DD-case is given by

$$u(\varphi, \tau) = \begin{cases} 
1 + u_d(\varphi, \tau), & \varphi \in (\varphi_1, \varphi_2) \\
u_d(\varphi, \tau), & \varphi \in (0, \alpha_1) \setminus [\varphi_1, \varphi_2]. 
\end{cases} \quad \tau \in [0, 1] \quad (7.5)$$

Comparing with (7.1), we obtain that

$$u(\varphi, \tau) = U(\varphi, \tau), \quad \tau \in (0, 1), \quad \varphi \in (0, \alpha_1) \setminus [\varphi_1, \varphi_2], \quad \delta(\varphi, \tau_0) > 0 \quad (7.6)$$

since $\frac{\pi}{\alpha_1} = \frac{\pi}{\Phi} = q$ by (3.18).

For $\varphi \in (\varphi_1, \varphi_2)$ the coincidence follows from continuity of the functions $u$ and $U$.

In fact, (7.1), (7.4), and (7.5) imply that for any $(\varphi_0, \tau_0) \in (0, \alpha_1) \times [0, 1)$ there exist $\delta(\varphi_0, \tau_0) > 0$ and $k(\varphi_0, \tau_0) \in \mathbb{Z}$ s.t.

$$U(\varphi, \tau) - u_d(\varphi, \tau) = k((\varphi_0, \tau_0)), \quad |(\varphi, \tau) - (\varphi_0, \tau_0)| < \delta.$$ 

Hence, $U \equiv u$ by continuity of these functions since $k = 0$ for $\varphi \in (0, \alpha_1) \setminus [\varphi_1, \varphi_2]$ and $\tau \in [0, 1)$ by (7.6).

**Remark 7.2.** The coincidence with the Keller-Blank formula requires more cumbersome calculations and will be established in other paper.
8 Appendix

A1 “Stationary” scattering problem take the forms
\[
\begin{align*}
(\Delta + \omega^2) \hat{u}_s(y, \omega) &= 0, & y \in Q \\
\frac{\partial}{\partial y_2} \hat{u}_s(y, \omega) &= -i\omega \hat{F}(\omega) \sin \alpha e^{i\omega y_1 \cos \alpha}, & y \in Q_1 \\
\frac{\partial}{\partial n_2} \hat{u}_s(y, \omega) &= i\omega \hat{F}(\omega) \sin(\Phi + \alpha)e^{-i\omega y_2 \frac{\cos(\Phi+\alpha)}{\sin \Phi}}, & y \in Q_2 
\end{align*}
\] (8.1)
for the NN-problem and
\[
\begin{align*}
(\Delta - \omega^2) \hat{u}_s(y, \omega) &= 0, & y \in Q \\
\frac{\partial}{\partial y_2} \hat{u}_s(y, \omega) &= -i\omega g(\omega) \sin \alpha e^{i\omega y_1 \cos \alpha}, & y \in Q_1 \quad \omega \in \mathbb{C}^+.
\end{align*}
\]
for the DN-problem.

A2 Malyuzhinetz integral kernels are represented by
\[
H(\beta) = \coth(q(\beta + \frac{i\pi}{2} - i\alpha)) \mp \coth(q(\beta - \frac{3i\pi}{2} + i\alpha))
\] (8.2)
for DD and NN-problems respectively and by
\[
H(\beta) = \frac{1}{\sinh[q(\beta + \frac{i\pi}{2} - i\alpha)]} + \frac{1}{\sinh[q(\beta - \frac{3i\pi}{2} + i\alpha)]}
\] (8.3)
for DN-problem (see also [4]-[8]) where the DN and NN-problems were considered in details). Here \( q = \frac{\pi}{2\delta} \).

A3. The densities of the “stationary” reflected waves \( S_r \) are represented by
\[
S_r(\rho, \theta, \omega) = \begin{cases} 
-e^{i\omega \rho \cos(\theta - \theta_1)}, & \phi \leq \theta \leq \theta_1 \\
0, & \theta_1 < \theta < \theta_2 \\
e^{i\omega \rho \cos(\theta - \theta_2)}, & \theta_2 \leq \theta \leq 2\pi
\end{cases}, \quad \omega \in \mathbb{C}^+.
\] (8.4)
for the DN-problem and by
\[
S_r(\rho, \theta, \omega) = \begin{cases} 
e^{i\omega \rho \cos(\theta - \theta_1)}, & \phi \leq \theta \leq \theta_1 \\
0, & \theta_1 < \theta < \theta_2 \\
e^{i\omega \rho \cos(\theta - \theta_2)}, & \theta_2 \leq \theta \leq 2\pi
\end{cases}, \quad \omega \in \mathbb{C}^+.
\] (8.5)
for the NN-problem

A4. Inverse Fourier transform.

Lemma 8.1. i) The inverse Fourier transforms of the functions \( S_d \) and \( S_r \) given by (3.2) are the functions \( J_d(y, \omega) \) and \( J_r(y, \omega) \), given by (3.7).
Proof. i) We should prove that for $\theta \in \Theta$

$$J_d(\rho, \theta, t) = F_{\omega \to t}^{-1} S_d(\rho, \theta, \omega) = F_{\omega \to t}^{-1} \omega \mapsto t S_d(\rho, \theta, \omega) = F_{\omega \to t}^{-1} \omega \mapsto t \left[ i 4 \Phi \int R e^{i \omega \rho \cosh \beta Z(\beta + i \theta)} d\beta \right]. \quad (8.6)$$

First, we note that

$$i 4 \Phi \int R e^{i \omega \rho \cosh \beta Z(\beta + i \theta)} d\beta = \int_0^\infty e^{i \omega \rho \cosh \beta Z(\beta + i \theta)} d\beta,$$

where $Z$ is defined by (3.6). The integrals converge by (3.19).

Changing the variables $t := \rho \cosh \beta, \ d\beta = \frac{dt}{\sqrt{t^2 - \rho^2}}$ we obtain:

$$i 4 \Phi \int R e^{i \omega \rho \cosh \beta Z(\beta + i \theta)} d\beta = F_{t \to \omega} [J_d(\rho, \theta, t)] \quad (8.7)$$

where $J_d$ is given by (3.7). Hence, (8.6) follows.

The representation (8.5) for $J_r$ follows from (3.7) directly.

The representations for $J_r$ for other cases:

$$J_r(\rho, \theta, t) := \begin{cases} 
\delta(t - \rho \cos(\theta - \theta_1)), & \theta \in (\phi, \theta_1) \\
0, & \theta \in (\theta_1, \theta_2) \\
-\delta(t - \rho \cos(\theta - \theta_2)), & \theta \in (\theta_2, 2\pi) \end{cases} \quad t \geq 0, \ J_r(\rho, \theta, t) = 0, \ t < 0. \quad (8.8)$$

for the DN-problem and

$$J_r(\rho, \theta, t) := \begin{cases} 
-\delta(t - \rho \cos(\theta - \theta_1)), & \theta \in (\phi, \theta_1) \\
0, & \theta \in (\theta_1, \theta_2) \\
\delta(t - \rho \cos(\theta - \theta_2)), & \theta \in (\theta_2, 2\pi) \end{cases} \quad t \geq 0, \ J_r(\rho, \theta, t) = 0, \ t < 0. \quad (8.9)$$

For the DN-problem

A5. Expressions for the reflected waves

$$u_r(y, t) := \begin{cases} 
\mp F(t - n_1 \cdot y), & \phi \leq \theta \leq \theta_1 \\
0, & \theta_1 < \theta < \theta_2, \\
\mp F(t - n_2 \cdot y), & \theta_2 \leq \theta \leq 2\pi, \end{cases} \quad (8.10)$$

for DD and NN-problems respectively and

$$u_r(y, t) := \begin{cases} 
F(t - n_1 \cdot y), & \phi \leq \theta \leq \theta_1 \\
0, & \theta_1 < \theta < \theta_2, \\
-F(t - n_2 \cdot y), & \theta_2 \leq \theta \leq 2\pi, \end{cases} \quad (8.11)$$

for the DN-problem.

A6. Jumps of the diffracted wave in the cases of NN and DN problems.
For the DN-problem the function $Z$ from (4.2) takes the form:

$$Z(\beta + i\theta) = -\frac{1}{\sinh(q\beta + ic_0)} - \frac{1}{\sinh(q\beta + ic_1)} - \frac{1}{\sinh(q\beta + ic_2)} + \frac{1}{\sinh(q\beta + ic_3)}$$

(8.12)

The jumps of the diffracted wave $u_d$ on the critical rays for the NN and DN-problems are given by

$$[u_d]_k(\rho, t) = F(t - \rho), \quad \rho > 0, \quad k = 1, 2$$

(8.13)

(cf. with (4.1)).

References

[1] A.I. Komech, A.E.Merzon, Relation between Cauchy data for the scattering by a wedge, Russian Journal of Mathematical Physics 14 (2007) no. 3, 279-303.

[2] Komech AI , Mauser NJ, Merzon AE. On Sommerfeld representation and uniqueness in scattering by wedges. Mathematical Methods in the Applied Sciences 28 (2005), 147-183.

[3] A.I.Komech, A.E.Merzon. Limiting Amplitude Principle in the Scattering by Wedges. Mathematical Methods in the Applied Sciences 29 (2006), 1147-1185

[4] J.E.de la Paz Mendez, A.Merzon, Scattering of a plane wave by hard-soft wedges. em Recent Progress in Operator Theory and Its Applications. Series: Operator Theory: Advances and Applications, (2012), 220, 207-227.

[5] A. Merzon. Well-posedness of the problem of nonstationary diffr action of Sommer- feld. Proceeding of the International Seminar "Day on Diffraction-2003". University of St.Petersburg (2003) 151-162.

[6] J.E de la Paz Mendez, A.E. Merzon, DN-Scattering of a plane wave by wedges. Mathematical Methods in the Applied Sciences, 34, No. 15, (2011), 1843-1872 (http://onlinelibrary.wiley.com/doi/10.1002/mma.1484/abstract).

[7] Anatoli Merzon, José Eligio De la Paz M. DN-Problema de dispersión de una onda plana sobre una cuña. Principio de Amplitud Límite. Editorial Académica Española (2012-10-03)

[8] A.Esquivel, A.E.Merzon. NN-problem (in preparation)

[9] S.L. Sobolev, Theory of diffraction of plane waves, Proceedings of Seismological Institute, no. 41, Russian Academy of Science, Leningrad, 1934.

[10] S.L. Sobolev, General theory of diffraction of waves on Riemann surfaces, Tr. Fiz.-Mat. Inst. Steklova 9 (1935), 39-105. [Russian] (English translation: S.L. Sobolev, General theory of diffraction of waves on Riemann surfaces, p. 201-262 in: Selected Works of S.L. Sobolev, Vol. I, Springer, New York, 2006.)
[11] S.L. Sobolev, Some questions in the theory of propagations of oscillations, Chap XII, in: *Differential and Integral Equations of Mathematical Physics*, F. Frank and P. Mizes (eds), Leningrad-Moscow (1937) pp 468-617. [Russian]

[12] Keller J, Blank A. Diffraction and reflection of pulses by wedges and corners. *Communications on Pure and Applied Mathematics* 1951; 4(1):75-95.

[13] Kay I. The diffraction of an arbitrary pulse by a wedge. *Communications on Pure and Applied Mathematics* 1953; 6:521-546.

[14] Komech AI. Elliptic boundary value problems on manifolds with piecewise smooth boundary. *Math. USSR Sbornik*. 1973; 21(1): 91-135.

[15] Komech AI, Merzon AE. General boundary value problems in region with corners. *Operator Theory: Adv. Appl.* 1992; 57: 171-183.

[16] Komech A, Merzon A, Zhevandrov P. A method of complex characteristics for elliptic problems in angles and its applications. *American Mathematical Society Translation* 2002; 206(2):125-159.

[17] A. Choque, Yu. Karlovich, A. Merzon and P. Zhevandrov. On the convergence of the amplitude of the diffracted nonstationary wave in scattering by wedges. Russian Journal of Mathematical Physics, 2012, V.19, N 3, pp.373-384.

[18] Petrashen’ CI, Nikolaev VG, Kouzov DP. On the series method in the theory of diffraction of waves by polygonal regions. *Nauchnie ZapiFi LGU*. 1958; 246(5): 5-70 (in Russian).

[19] Felsen LB, Marcuvitz N. *Radiation and Scattering of Waves*. Oxford Univ Pr: Oxford, 1996.

[20] Rottbrand K. Time-dependent plane wave diffraction by a half-plane: explicit solution for Rawlins’ mixed initial boundary value problem. *Z.Angew. Math. Mech.* 1998; 78(5): 321-335.

[21] Meister E, Passow A, Rottbrand K. New results on wave diffraction by canonical obstacles. *Operator Theory: Adv. Appl.* 1999; 110: 235-256.

[22] Rottbrand K. Exact solution for time-dependent diffraction of plane waves by semi-infinite soft/hard wedges and half-planes. Preprint 1984 Technical University Darmstadt. 1998

[23] Borovikov V.A. *Diffraction at Poligons and Polyhedrons*. Nauka, Moscow. (1966)

[24] Komech A.I. Linear partial differential equations with constant coefficients. In Egorov YuE, Komech AI, Shubin MA. *Elements of the Modern Theory of Partial Differential Equations*. Springer: Berlin, 1999: 127-260.

[25] Gel’fand I.M., Shilov G. E. *Generalized functions. Vol. 2. Spaces of fundamental and generalized functions*, Boston, MA, (1968)
[26] Merzon AE. On Ursell’s problem. *Proceedings of the Third International Conference on Mathematical and Numerical Aspects of Wave Propagation*. SIAM – INRIA, edited by Gary Cohen. 1995; 613-623.

[27] Komech A.I., Merzon A.E., Zhevandrov P.N., On completeness of Ursell’s trapping modes. *Russian Journal of Mathematical Physics*. 1996; 4(4): 457-485.

[28] Fedoryuk MV. *Asymptotics: Integrals and Series*. Nauka: Moscow, 1987.

[29] Vladimirov, V. S. (1979), Generalized functions in mathematical physics (in English), Moscow: Mir Publishers, p. 362, ISBN 0-8285-0001-0, MR 0564116, Zbl 0515.46034. A textbook on the theory of generalized functions and their applications to mathematical physics and several complex variables.

[30] Vladimirov, V.S. (1983), Equations of mathematical physics (in English) (2nd ed.), Moscow: Mir Publishers, p. 464, MR 0764399, Zbl 0207.09101 (Zentralblatt review of the first English edition).

V.S.Vladimirov

A. Choque, Yu. Karlovich, A. Merzon and P. Zhevandrov. On the convergence of the amplitude of the diffracted nonstationary wave in scattering by wedges. *Russian Journal of Mathematical Physics*, 2012, V.19, N 3, pp.373-384.

[31] E.L. Whittaker, G.N. Watson, *A Course of Modern Analysis*, Macmillan, New York, 1948.