Relational Lattice Axioms

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Relational lattice is a formal mathematical model for Relational algebra. It reduces the set of six classic relational algebra operators to
two: natural join and inner union. We continue to investigate Relational lattice properties with emphasis onto axiomatic definition.
New results include additional axioms, equational definition for set difference (more generally anti-join), and case study
demonstrating application of the relational lattice theory for query transformations.

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1. INTRODUCTION

Classic Relational Algebra employs six basic operations: projection, restriction, join, union, difference, and
renaming. Relational Lattice [1,2] represents them in terms of two mathematically attractive binary
operations: natural join and inner union. By abuse of set notation we define these operations as follows:

- natural join
  \[ A(x,y) \land B(y,z) \triangleq \{(x,y,z) | (x,y)\in A \land (y,z)\in B\} \]

- inner union
  \[ A(x,y) \lor B(y,z) \triangleq \{(y) | \exists x (x,y)\in A \lor \exists z (y,z)\in B\} \]

It has been demonstrated that these operations satisfy lattice axioms, hence the notation change in favor of
standard mathematical symbols for lattice meet and join. Please note that certain terminological
incompatibility still remains, and we call relational operations (natural) join and (inner) union versus
correspondingly lattice meet, and join. The fact that join term is used in both worlds to denote the exactly
the opposite operation is unfortunate artifact of Relational Model legacy, which we still would like to keep.

We'll employ Prover9 [3] as our primary theorem prover tool. Consequently, we'll use symbols \(^{(caret)}\) for join, and \(v\) (letter \(v\)) for union. Also, in order to avoid Mace4 interpretation of constants, we have
chosen to prefix relations 00, 01, 10 and 11 with letter \(R\).
2. EXAMPLE

Relational lattices can be generated in fairly straightforward way. We start with several basic relations and generate a closure by calculating all the possible joins and unions. For example, suppose we have five relations \( A=\{(x=1)\} \), \( B=\{(x=1),(x=2)\} \), \( C=\{(y=a)\} \), \( D=\{(y=a),(y=b)\} \), and \( R00 \), where the last relation is an empty relation with empty header. Then by joining \( A \) and \( D \) we'll generate new relation \( \{(x=1,y=a),(x=1,y=b)\} \) which we add to the lattice. Continuing this process we'll produce the lattice on fig.1.

Figure 1. The lattice of relations generated by five relations \( A=\{(x=1)\} \), \( B=\{(x=1),(x=2)\} \), \( C=\{(y=a)\} \), \( D=\{(y=a),(y=b)\} \), and \( R00 \). The lattice bottom element is \( R01 \) -- the relation with empty header and nonempty content\(^1\).

3. FUNDAMENTAL AXIOMS

In axiomatic description relation variables don't assumed to have any structure, other than the one specified by lattice operations. The brackets are used strictly for algebraic purpose of grouping operations. We would never be forced to specify columns explicitly (like in the introduction section in case of the relations \( A(x,y) \) and \( B(y,z) \)); consequently, comma symbol would never make an appearance. Therefore, the case for variable is no longer important, and we can switch to lower case letters \( x, y, z \), etc for relation variables.

\(^1\)Relations \( R00 \) and \( R01 \) are known in database literature under colorful names of Dee and Dum
We partition Relational axioms into several groups. In this section we list standard lattice axioms (SLA)

\[ x \land y = y \land x. \]

\[ (x \land y) \land z = x \land (y \land z). \]

\[ x \land (x \lor y) = x. \]

\[ x \lor y = y \lor x. \]

\[ (x \lor y) \lor z = x \lor (y \lor z). \]

\[ x \lor (x \land y) = x. \]

together with *Fundamental Decomposition Identity* (FDA)

\[ x = (x \land R00) \lor (x \land R11). \]

where \( R11 \) is the universal relation. In the lattice example from the previous section (Fig.1) for all practical purposes \( R11 \) can be identified with the relation \{\((x=1,y=a),(x=1,y=b),(x=2,y=a),(x=2,y=b)\)\}.

Informally, FDA asserts that any relation is a union of relation’s content and header.  The dual of FDA

\[ x = (x \lor R00) \land (x \lor R11). \]

is invalid axiom. In the lattice example from the previous section let

\[ x \equiv \{(x=1,y=a),(x=1,y=b),(x=2,y=a)\}\]

then \( x \lor R00 \) evaluates to \( R01 \), and \( x \lor R11 \) to \( R11 \). Consequently, \( R01 \land R11 = R11 \neq x \).

If both sides of the dual of FDA are joined with \( R00 \), then we obtain a weaker axiom which (for the lack of better name) we call FDA\(^{-1}\)

\[ R00 \land (x \lor R11) = x \land R00. \]

Unlike previously considered dual of FDA, FDA\(^{-1}\) is a legitimate axiom.

It is easy to see that FDA is independent from SLA. Choose \( R00 = R11 \) to be an arbitrary element in a lattice. Then, \( x \land R00 = x \land R11 \) therefore, \( (x \land R00) \lor (x \land R11) \) evaluates to \( x \land R00 \) which would not reduce to \( x \) if \( R00 \neq x \). To witness that FDA\(^{-1}\) is independent from the SLA+FDA take the standard 2-element boolean algebra \( \{1,0\} \), \( 0 \land 1 = 1 \), \( 0 \lor 1 = 0 \), and assume \( R00 = R11 = 0 \). Then, the right side of the FDA\(^{-1}\) reduces to 0 by absorption. Choose \( x = 1 \) on the right side.

Speaking universal algebra language, so far we have introduced two binary operations – join and union, and two constants -- \( R00 \) and \( R11 \) (which can be viewed as 0-ary operations). What about two more constants, which were introduced in [2]: \( R10 \) (lattice top) and \( R01 \) (lattice bottom)? They were defined to satisfy the equations

\[ Dots \text{ are essential part of Prover9 syntax} \]
x ^ R01 = x.
x v R10 = x.
Let's substitute x = R01 into FDA. It immediately follows that
R00 v R11 = R01.
Likewise, one can prove
R00 ^ R11 = R10.
These two theorems exhaust all the interesting propositions about R10 and R01. For that reason we have chosen not to mention R10 and R01 (and their defining axioms) at all.

FDA⁻¹ has two more cousins which can be derived from SLA+FDA+FDA⁻¹:
R11 v (x ^ R00) = x v R11.
R00 v (x ^ R11) = x v R00.
In appendix A we list assertions and goal, so that user could be able to reproduce Prover9 generated proof.

We complete the section with one more theorem
R00 ^ x = R00 ^ y <-> R11 v x = R11 v y.
see appendix B. Informally, given a condition onto relation headers, it can be rewritten in terms of domains and vice versa. This theorem is widely leveraged in the next section.

4. DISTRIBUTIVITY AXIOMS

It is easy to spot the “pentagon” N₁ sublattices on Fig.1. Therefore, Relational lattice in general is not distributive. SLA+FDA+FDA⁻¹ axiom system, however, possesses some rudimentary distributivity properties such as
(R00 ^ x) v (R00 ^ y) = R00 ^ (x v y).
-- see appendix C. By associativity and absorption we also have
(R00 ^ x) ^ (R00 ^ y) = R00 ^ (x ^ y).
These assertions establish that the mapping x → R00 ^ x is a lattice homomorphism (LH^). The dual mapping x → R00 v x is not a homomorphism.

Likewise, the mapping x → R11 v x is a homomorphism (LHv). The dual mapping x → R11 ^ x is not a homomorphism, either.

Relational lattice honors conditional distributivity laws that are stronger than aforementioned LH^ and LHv theorems. In previous work ([2]) we discovered two distributivity criteria and proved them within set theoretic framework. The first criteria for distributivity of join over union translates into the following implication
\[(R00 \land (x \lor y) = R00 \land (x \lor z)) \rightarrow (x \land (y \lor z) = (x \land y) \lor (x \land z)).\]

This axiom, which we'll refer to as SDC, is independent from SLA+FDA+FDA\(^{-1}\); please refer to appendix D for counterexample. The dual criteria for distributivity of union over join

\[
(R00 \land (x \lor y) = R00 \land (x \lor z)) \land (R00 \land (x \lor z) = R00 \land (y \lor z)) \rightarrow (x \lor (y \land z) = (x \lor y) \land (x \lor z)).
\]

is a theorem in the SLA+FDA+FDA\(^{-1}\)+SDC system; please refer to appendix E.

These (and the assertions that follows) can be rewritten in terms of condition over domains, rather than relation headers based upon the theorem that we derived at the end of section 3.

Unfortunately, SLA+FDA+FDA\(^{-1}\)+SDC system is still too weak. Distributivity Constraint on relation Headers (DCH)

\[
R00 \land (x \land (y \lor z)) = R00 \land ((x \land y) \lor (x \land z)).
\]

is one more axiom (Appendix F). The dual assertion

\[
R00 \land (x \lor (y \land z)) = R00 \land ((x \lor y) \land (x \lor z)).
\]

is a theorem (Appendix G).

Closely related to DCH is the law of Distributivity of Empty Relations

\[
(x \land R00) \land ((y \land R00) \lor (z \land R00)) = ((x \land R00) \land (y \land R00)) \lor ((x \land R00) \land (z \land R00)).
\]

\[
(x \land R00) \lor ((y \land R00) \land (z \land R00)) = ((x \land R00) \lor (y \land R00)) \land ((x \land R00) \lor (z \land R00)).
\]

see appendix H.

5. DISTRIBUTIVITY IN DATE&DARWEN ALGEBRA

Date & Darwen introduced the OR operation [4] as follows

\[
A(x,y) \lor B(y,z) \overset{\text{def}}{=} \{(x,y,z) \mid (x,y)\in A \land (z \in Z) \} \cup \{(x,y,z) \mid x\in X \land (y,z)\in B \}
\]

It is represented in relational lattice terms as

\[
x + y \overset{\text{def}}{=} (x \land (y \lor R11)) \lor (y \land (x \lor R11)).
\]

where we found it convenient to switch the notation to the plus symbol in order to continue leveraging Prover9 system. Associativity of the OR operation and distributivity of join over the OR

\[
x + (y + z) = (x + y) + z.
\]

\[
x \land (y + z) = (x \land y) + (x \land z).
\]

are theorems in the SLA+FDA+FDA\(^{-1}\)+SDC+DCH system (Appendixes I and J). Unfortunately, we were unable to find the proof, nor counterexample for distributivity of the OR operation over join

\[
x + (y \land z) = (x + y) \land (x + z).
\]
6. ANTIJOIN

Among six basic Relational algebra operations only five are directly representable via join and union. Set difference is the exception. Yet, set difference is a special case of anti-join, which we can define equationally in relational lattice theory. Let's lower abstraction level a little, and introduce relational variables $E$ and $D$ which we informally associate with the familiar tables $\text{Emp}(\text{ename}, \text{deptno})$ and $\text{Dept}(\text{deptno})$ (Fig.2).

![Diagram](image1.png)

**Figure 2.** The record ($\text{Ename}=\text{SMITH, Deptno}=10$) doesn’t match any record in the department table: it belongs to the set $E \setminus D$ (where we extended set difference notation for anti-join).

Once again, we are trying to define anti-join/set difference equationally, following pretty much the same idea how arithmetic difference is defined (that is as a solution of the equation $x + a = b$ in classic algebra). Therefore, let's introduce variable $\text{EmD}^3$ for antijoin $E \setminus D$. It satisfies the following system of equations:

$$(E \land D) \lor \text{EmD} = E.$$

$$(E \land D) \land \text{EmD} = (E \land D) \land \text{R00}.$$

These equations informally assert that the relation $E$ is a disjoint union of $\text{EmD}$ and join of $E$ and $D$.

Anti-join is well defined thanks to the following *uniqueness* theorem. Given “alternative” anti-join variable $\text{EmD1}$

$$(E \land D) \lor \text{EmD1} = E.$$

$$(E \land D) \land \text{EmD1} = (E \land D) \land \text{R00}.$$

one can formally prove the equality $\text{EmD1} = \text{EmD}$.

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³We apologize to mathematically inclined readers for introducing programming flavored variable names
7. CASE STUDY: REDUNDANT JOIN ELIMINATION

Eliminating redundant joins [5] is one of the “obvious” query transformations which we are going to test our method against. Assume foreign key constraint between the E and D tables from the previous section, that is the empty EmD relation. Formally,

\[ \text{EmD} \lor \text{R00} = \text{R00}. \]

Consider a query where we join E and D first, and then project the result to a subset of columns of E. We define the “subset of columns of E” as a formal relation E0 that satisfies\(^4\)

\[ (E \land E0) \land \text{R00} = E \land \text{R00}. \]

Now we can formally express our query as

\[ E0 \lor (E \land D). \]

Our goal is to prove

\[ E0 \lor (E \land D) = E0 \lor E. \]

Once again, we plug all the above equations into Prover9 theorem prover, and effortlessly derive the desired result (Appendix L).

8. ACKNOWLEDGMENTS

Jan Hidders spotted an error in the distributivity of union over join criteria, and is credited with the idea of splitting the + - associativity theorem in half.

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\(^4\)Note that usually we associate relation headers with empty relations, so that we may want to add one more condition

\[ E0 \land \text{R00} = E0. \]

This constraint, however, is non essential in our case study, so we'll drop it from our consideration.
APPENDIX A. FDA\(^1\) COUSIN IDENTITIES

R11 \(\lor (x \land R00) = x \lor R11\). [goal].
\(x \land y = y \land x\). [assumption].
\(x \land (x \lor y) = x\). [assumption].
\(x \lor y = y \lor x\). [assumption].
\(x = (x \land R00) \lor (x \land R11)\). [assumption].
R00 \(\land (x \lor R11) = x \land R00\). [assumption].
R00 \(\lor (x \land R11) = x \lor R00\). [goal].
\(x \land y = y \land x\). [assumption].
\(x \lor y = y \lor x\). [assumption].
\(x \land (x \lor y) = x\). [assumption].
\(x \lor (x \land y) = x\). [assumption].
\(x = (x \land R00) \lor (x \land R11)\). [assumption].

APPENDIX B. EQUIVALENCE OF CONDITIONS OVER HEADERS AND DOMAINS

R11 \(\lor x = R11 \lor y \iff R00 \land x = R00 \land y\). [goal].
\(x \land y = y \land x\). [assumption].
\(x \land (x \lor y) = x\). [assumption].
\(x \lor y = y \lor x\). [assumption].
\(x = (x \land R00) \lor (x \land R11)\). [assumption].
R00 \(\land (x \lor R11) = x \land R00\). [assumption].

APPENDIX C. DISTRIBUTIVITY FOR R00 ELEMENT

\((R00 \land x) \lor R00 \land y = R00 \land (x \lor y)\). [goal].
\(x \land y = y \land x\). [assumption].
\((x \land y) \land z = x \land (y \land z)\). [assumption].
\(x \land (x \lor y) = x\). [assumption].
\(x \lor y = y \lor x\). [assumption].
\((x \lor y) \lor z = x \lor (y \lor z)\). [assumption].
\(x \lor (x \land y) = x\). [assumption].
\(x = (x \land R00) \lor (x \land R11)\). [assumption].
R00 \(\land (x \lor R11) = x \land R00\). [assumption].

APPENDIX D. MODEL FOR SDC

![5-element “diamond” lattice M3 as a model satisfying SLA+FDA+FDA\(^1\) but invalidating SDC axiom.]

APPENDIX E. PROOF FOR DISTRIBUTIVITY OF UNION OVER JOIN

\(R00 \land (x \lor y) = R00 \land (x \lor z) \rightarrow x \land (y \lor z) = (x \land y) \lor (x \land z)\). [assumption].
\(R00 \land (x \lor y) = R00 \land (x \lor z) \& R00 \land (x \lor z) = R00 \land (y \lor z) \rightarrow x \lor (y \lor z) = (x \lor y) \lor (x \lor z)\). [goal].
\(x \land y = y \land x\). [assumption].
\((x \land y) \land z = x \land (y \land z)\). [assumption].
\(x \land (x \lor y) = x\). [assumption].
\(x \lor y = y \lor x\). [assumption].
(x v y) v z = x v (y v z).  [assumption].
x v (x ^ y) = x.  [assumption].
x = (x ^ R00) v (x ^ R11).  [assumption].
R00 ^ (x v R11) = x ^ R00.  [assumption].

APPENDIX F. MODEL FOR DISTRIBUTIVITY CONSTRAINT ON RELATION HEADERS

![Figure 4. Familiar 5-element “pentagon” N₅ as a model satisfying SLA+FDA+FDA⁻¹+SDC but invalidating DCH axiom.]

APPENDIX G. DUAL OF DISTRIBUTIVITY CONSTRAINT ON RELATION HEADERS

R00 ^ (x v y) = R00 ^ (x v z) -> x ^ (y v z) = (x ^ y) v (x ^ z).  [assumption].
R00 ^ (x v (y ^ z)) = R00 ^ ((x v y) ^ (x v z)).  [goal).
x ^ y = y ^ x.  [assumption].
(x ^ y) ^ z = x ^ (y ^ z).  [assumption].
x ^ (x v y) = x.  [assumption].
x v y = y v x.  [assumption].
(x v y) v z = x v (y v z).  [assumption].
x v (x ^ y) = x.  [assumption].
R00 ^ (x ^ (y v z)) = R00 ^ ((x ^ y) v (x ^ z)).  [assumption].

APPENDIX H. DISTRIBUTIVITY OF UNION OVER JOIN FOR EMPTY RELATIONS

R00 ^ (x v y) = R00 ^ (x v z) -> x ^ (y v z) = (x ^ y) v (x ^ z).  [assumption].
(x ^ R00) ^ ((y ^ R00) v (z ^ R00)) = ((x ^ R00) ^ (y ^ R00)) v ((x ^ R00) ^ (z ^ R00)).  [goal].
x ^ y = y ^ x.  [assumption].
(x ^ y) ^ z = x ^ (y ^ z).  [assumption].
x ^ (x v y) = x.  [assumption].
R00 ^ (x ^ (y v z)) = R00 ^ ((x ^ y) v (x ^ z)).  [assumption].

APPENDIX I. ASSOCIATIVITY OF D&D ‘OR’ OPERATOR

This is one of the proofs where Prover9 is pushed to its limits. We splitted the problem in half and proved

(x + y) + z = (((x ^ ((y ^ z) v R11)) v (y ^ ((x ^ z) v R11)))) v (z ^ ((x ^ y) v R11)).

We also represented SDC and DCH in the domain form

(R11 v (x v y) = R11 v (x v z)) -> (x ^ (y v z) = (x ^ y) v (x ^ z)).
R11 v (x ^ (y v z)) = R11 v ((x ^ y) v (x ^ z)).

(R11 v (x v y) = R11 v (x v z) -> x ^ (y v z) = (x ^ y) v (x ^ z).  [assumption].
(x + y) + z = (((x ^ ((y ^ z) v R11)) v (y ^ ((x ^ z) v R11)))) v (z ^ ((x ^ y) v R11)).  [goal].
x ^ y = y ^ x.  [assumption].
\[(x \land y) \land z = x \land (y \land z).\]  \[\text{assumption}].
\[x \land (x \lor y) = x.\]  \[\text{assumption}].
\[x \lor y = y \lor x.\]  \[\text{assumption}].
\[(x \lor y) \lor z = x \lor (y \lor z).\]  \[\text{assumption}].
\[x \lor (x \land y) = x.\]  \[\text{assumption}].
\[x = (x \land R0) \lor (x \land R1).\]  \[\text{assumption}].
\[R0 \lor (x \land R1) = x \lor R0.\]  \[\text{assumption}].
\[R1 \lor (x \land (y \lor z)) = R1 \lor ((x \land y) \lor (x \land z)).\]  \[\text{assumption}].
\[x \lor y = (x \lor R1) \lor (y \lor (x \lor R1)).\]  \[\text{assumption}].

**APPENDIX J. DISTRIBUTIVITY OF D&D ‘\textcolor{red}{\textsc{OR}}’ OPERATOR**

\[R1 \lor (x \lor y) = R1 \lor (x \lor z) \rightarrow x \land (y \lor z) = (x \land y) \lor (x \land z).\]  \[\text{assumption}].
\[x \land (y + z) = (x \land y) + (x \land z).\]  \[\text{goal}].
\[x \land y = y \land x.\]  \[\text{assumption}].
\[(x \land y) \land z = x \land (y \land z).\]  \[\text{assumption}].
\[x \land (x \lor y) = x.\]  \[\text{assumption}].
\[x \lor y = y \lor x.\]  \[\text{assumption}].
\[(x \lor y) \lor z = x \lor (y \lor z).\]  \[\text{assumption}].
\[x \lor (x \land y) = x.\]  \[\text{assumption}].
\[x = (x \land R0) \lor (x \land R1).\]  \[\text{assumption}].
\[R0 \lor (x \land R1) = x \lor R0.\]  \[\text{assumption}].
\[R0 \lor (x \land (y \lor z)) = R0 \lor ((x \land y) \lor (x \land z)).\]  \[\text{assumption}].
\[R0 \lor (y \land (x \lor z)) = R0 \lor ((x \lor y) \land (x \lor z)).\]  \[\text{assumption}].
\[(E \land D) \lor \text{EmD} = E.\]  \[\text{assumption}].
\[(E \land D) \land \text{EmD} = (E \land D) \land R0.\]  \[\text{assumption}].
\[(E \land D) \lor \text{EmD} = E.\]  \[\text{assumption}].
\[(E \land D) \land \text{EmD} = (E \land D) \land R0.\]  \[\text{assumption}].

**APPENDIX K. ANTI-JOIN UNIQUENESS**

\[R0 \land (x \lor y) = R0 \land (x \lor z) \rightarrow x \land (y \land z) = (x \land y) \land (x \land z).\]  \[\text{assumption}].
\[x \land y = y \land x.\]  \[\text{assumption}].
\[(x \land y) \land z = x \land (y \land z).\]  \[\text{assumption}].
\[x \land (x \lor y) = x.\]  \[\text{assumption}].
\[x \lor y = y \lor x.\]  \[\text{assumption}].
\[(x \lor y) \lor z = x \lor (y \lor z).\]  \[\text{assumption}].
\[x \lor (x \land y) = x.\]  \[\text{assumption}].
\[x = (x \land R0) \lor (x \land R1).\]  \[\text{assumption}].
\[R0 \land (x \land R1) = x \land R0.\]  \[\text{assumption}].
\[R0 \land (x \land (y \lor z)) = R0 \land ((x \land y) \land (x \land z)).\]  \[\text{assumption}].
\[R0 \land (y \land (x \lor z)) = R0 \land ((x \lor y) \land (x \lor z)).\]  \[\text{assumption}].
\[(E \land D) \lor \text{EmD} = E.\]  \[\text{assumption}].
\[(E \land D) \land \text{EmD} = (E \land D) \land R0.\]  \[\text{assumption}].
\[(E \land D) \lor \text{EmD} = E.\]  \[\text{assumption}].
\[(E \land D) \land \text{EmD} = (E \land D) \land R0.\]  \[\text{assumption}].

**APPENDIX L. REDUNDANT JOIN ELIMINATION**

\[R0 \land (x \lor y) = R0 \land (x \lor z) \rightarrow x \land (y \lor z) = (x \land y) \land (x \land z).\]  \[\text{assumption}].
\[x \lor y = y \lor x.\]  \[\text{assumption}].
\[(x \lor y) \lor z = x \lor (y \lor z).\]  \[\text{assumption}].
\[x \lor (x \land y) = x.\]  \[\text{assumption}].
\[x \land y = y \land x.\]  \[\text{assumption}].
\[(x \land y) \land z = x \land (y \land z).\]  \[\text{assumption}].
\[x \land (x \lor y) = x.\]  \[\text{assumption}].
\[x \lor y = y \lor x.\]  \[\text{assumption}].
\[(x \lor y) \lor z = x \lor (y \lor z).\]  \[\text{assumption}].
\[x \lor (x \land y) = x.\]  \[\text{assumption}].
\[x = (x \land R0) \lor (x \land R1).\]  \[\text{assumption}].
\[R0 \lor (x \land R1) = x \lor R0.\]  \[\text{assumption}].
\[R0 \lor (x \land (y \lor z)) = R0 \lor ((x \land y) \lor (x \land z)).\]  \[\text{assumption}].
\[R0 \lor (y \land (x \lor z)) = R0 \lor ((x \lor y) \land (x \lor z)).\]  \[\text{assumption}].
\[(E \land D) \lor \text{EmD} = E.\]  \[\text{assumption}].
\[(E \land D) \land \text{EmD} = (E \land D) \land R0.\]  \[\text{assumption}].
\[(E \land E0) \lor \text{R00} = E \lor \text{R00}.\]  \[\text{assumption}].
\[\text{EmD} \lor \text{R00} = \text{R00}.\]  \[\text{assumption}].