ON CHANNELS OF ENERGY FOR THE RADIAL LINEARISED ENERGY CRITICAL WAVE EQUATION IN THE DEGENERATE CASE

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Abstract. Channels of energy estimates control the energy of an initial data from that which it radiates outside a light cone. For the linearised energy critical wave equation they have been obtained in the radial case in odd dimensions, first in 3 dimensions in [DKM13], then for the general case in [DKM20]. We consider even dimensions, for which such estimates are known to fail [CKS14]. We propose a weaker version of these estimates, around a single ground state as well as around a multisoliton. This allows us in [CDKM22] to prove the soliton resolution conjecture in six dimensions.

1. Introduction and results

We consider the linearised energy critical wave equation around the ground state in $N \geq 3$ dimensions:

\begin{equation}
\begin{cases}
\partial_t^2 u_L - \Delta u_L + V u_L = 0 \\
\tilde{u}_L|_{t=0} = (u_0, u_1),
\end{cases}
\end{equation}

where \((u_0, u_1) \in \mathcal{H} = \dot{H}^1 \times L^2(\mathbb{R}^N)\) and

\[ V = -\frac{N+2}{N-2} W \frac{1}{N-2}, \quad W(x) = \left(1 + \frac{|x|^2}{N(N-2)} \right)^{-\frac{N-2}{2}}. \]

This is the linearised equation for

\begin{equation}
\partial_t^2 u - \Delta u = |u|^{\frac{N}{N-2}} u
\end{equation}

around the stationary solutions $\pm W$. By standard arguments, recalled in Section 4, Equation (1.1) is globally well-posed in $\mathcal{H}$, and moreover the outer radiated energy

\begin{equation}
E_{\text{out}} = E_{\text{out}}^+ + E_{\text{out}}^-, \quad E_{\text{out}}^\pm = \lim_{t \to \pm \infty} \int_{|x|>|t|} |
abla_{t,x} u_L(t,x)|^2 dx,
\end{equation}

where $\nabla_{t,x} f = (\partial_t f, \nabla f)$, is well-defined for both time directions $t \to \pm \infty$. For the free wave equation

\begin{equation}
\begin{cases}
\partial_t^2 u_F - \Delta u_F = 0, \\
\tilde{u}_F|_{t=0} = (u_0, u_1),
\end{cases}
\end{equation}

it controls the total energy of the initial data in odd dimensions [DKM11, DKM12]:

\begin{equation}
\|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} \lesssim \sqrt{E_{\text{out}}} \quad \text{if } N \geq 3 \text{ is odd}
\end{equation}

but only for half the data in even dimensions [CKS14]:

\begin{equation}
\|u_0\|_{\dot{H}^1} \lesssim \sqrt{E_{\text{out}}} \quad \text{if } N \equiv 4 \mod 4 \quad \text{and} \quad \|u_1\|_{L^2} \lesssim \sqrt{E_{\text{out}}} \quad \text{if } N \equiv 6 \mod 4
\end{equation}
while the full estimate \( (1.5) \) is known to fail \cite{CKS14}. Extensions to domains of the form \( \{|x| > R + |t|\} \) for \( R > 0 \), obtained in \cite{KLSS15, DKMM22, LSW21}, will be used in the present paper. Other recent results on the asymptotic behaviour of linear waves can be found in \cite{Del21, CL21, LSWW22}.

For the linearised wave equation \((1.1)\), two natural counter examples to estimates like \((1.5)\) and \((1.6)\) are \( AW \) and \( tAW \) (for \( N \geq 5 \)), as they are non-radiative i.e. \( E_{\text{out}} = 0 \). Here \( AW = x \cdot \nabla W + \frac{N-2}{2} W \) is in the radial kernel of \( - \Delta + V \). For odd dimensions, the strong estimates \((1.5)\) and their extensions to domains \( \{|x| > R + |t|\} \) for \( R > 0 \) allowed the authors of \cite{DKM20} to extend this estimate to Equation \((1.1)\) in the radial case:

\[
\| \Pi_{\hat{H}^1} u_0 \|_{\hat{H}^1} + \| \Pi_{L^2} u_1 \|_{L^2} \lesssim \sqrt{E_{\text{out}}} \quad \text{if } N \geq 3 \text{ is odd.}
\]

Above, we used the projectors
\[
(1.8) \quad \Pi_{\hat{H}^1} = \Pi_{\hat{H}^1}(\text{Span}(AW))^\perp, \quad \Pi_{L^2} = \Pi_{L^2}(\text{Span}(AW))^\perp,
\]
where, for \( H \) a Hilbert space, and \( E \) a closed linear subspace of \( H \), we denote by \( \Pi_{H}(E) \) the orthogonal projection onto \( E \) in \( H \).

In this paper, we consider the even dimensional case. We focus on six and eight dimensions, since we believe their extensions to other even dimensions only to be a technical refinement of the present arguments. Our first result is a weaker version of \((1.7)\) involving the space \( Z_\alpha(\mathbb{R}^N) \) associated to the following norm for \( \alpha \in \mathbb{R} \):

\[
(1.9) \quad \| f \|_{Z_\alpha} = \sup_{R > 0} R^{-\frac{N}{2} - \alpha} \left( \int_{R < |x| < 2R} f^2 \, dx \right)^{\frac{1}{2}}.
\]

Note that, \( L^2(\mathbb{R}^6) \subset Z_{-3}(\mathbb{R}^6) \) with \( \| f \|_{Z_{-3}(\mathbb{R}^6)} \lesssim \| f \|_{L^2(\mathbb{R}^6)} \), so that \( Z_{-3}(\mathbb{R}^6) \) is a logarithmic weakening at 0 and \( \infty \) of \( L^2(\mathbb{R}^6) \).

**Theorem 1.1** (Channels of energy around the ground state). Assume \( N = 6 \). There exists \( C > 0 \) such that any radial solution \( u_L \) of \((1.1)\) satisfies:

\[
(1.10) \quad \| \Pi_{L^2} u_1 \|_{L^2} + \| \nabla \Pi_{\hat{H}^1} u_0 \|_{Z_{-3}} \leq C \sqrt{E_{\text{out}}}.
\]

The weaker estimate \((1.10)\) is sufficient to allow the authors to show in \cite{CDKM22} the soliton resolution and the inelasticity of collisions of solitons for Equation \((1.2)\) in the radial case. We believe the extension of \((1.10)\) to even dimensions \( N \equiv 6 \mod 4 \) to be a technical refinement of the present arguments. An analogue in \( N = 8 \) dimensions is given in Theorem \((6.1)\) in Section 6 which we similarly believe to extend to even dimensions \( N \equiv 4 \mod 4 \).

**Remark 1.2.** An analog of the bound on \( u_0 \) in \((1.10)\) for solutions of the free wave equation \((1.3)\) is also valid. Indeed, let \( \Phi \) be a smooth radial function with compact support included in \( \{ r > 0 \} \) such that \( \int \frac{1}{r} \Phi \, dx \neq 0 \). Then there exists a constant \( C > 0 \) such that for all solution \( u_F \) of \((1.4)\) with initial data \( u_0 \)

\[
(1.11) \quad \int \nabla u_0 \cdot \nabla \Phi \, dx = 0 \implies \| u_0 \|_{Z_{-2}} \lesssim \| \nabla u_0 \|_{Z_{-3}} \leq C \sqrt{E_{\text{out}}},
\]

with a proof that is similar to the proof of \((1.10)\). Note that this does not imply the estimate on the projection: \( \| \Pi_{\hat{H}^1}(\{ \Phi \}) u_0 \|_{Z_{-3}} \lesssim \sqrt{E_{\text{out}}} \), which is false.

**Remark 1.3.** The stronger estimates \((1.7)\) fail in six dimensions. Indeed, defining for \( \alpha \in \mathbb{R} \):

\[
\| f \|_{\hat{Z}_\alpha(\mathbb{R}^6)} = \sup_{R > 0} R^{-3-\alpha} \left( \int_{R < |x| < 2R} f^2 \, dx \right)^{\frac{1}{2}},
\]
we show in Appendix \[A\] that the estimate
\[(1.12) \quad \|\nabla u_0\|_{Z^{-3}} \lesssim \sqrt{E_{\text{out}}},\]
fails for solutions of (1.3). This implies that both the projection and the logarithmic loss in (1.11) are necessary, i.e. that the estimates:
\[\|\nabla \Pi_{H^1}^J(V^\perp)u_0\|_{Z^{-3}} \lesssim \sqrt{E_{\text{out}}}, \quad \|\nabla u_0\|_{Z^{-3}} \lesssim \sqrt{E_{\text{out}}},\]
where \(V\) is any finite dimensional subspace of \(H^1\) are both false, as they would imply \((1.12)\) by rescaling the solution and letting the scaling parameter go to 0 or \(\infty\). For the same reason, the logarithmic loss is necessary in (1.10), i.e. it is not possible to replace \(\|\nabla \Pi_{H^1}^J u_0\|_{Z^{-3}}\) by \(\|\nabla \Pi_{H^1}^J u_0\|_{Z^{-3}}\) (or even the smaller quantity \(\|\Pi_{H^1}^J u_0\|_{Z^{-2}}\)) in this inequality.

Our second result extends Theorem 1.1 to the linearised equation around a multisoliton:
\[(1.13) \quad \begin{cases}
d_t^2 u - \Delta u + V_\lambda u = 0, \\
u|_{t=0} = (u_0, u_1) \in \mathcal{H}.
\end{cases}
\]
Above, \(\lambda \in \Lambda_J = \{ \lambda = (\lambda_1, \ldots, \lambda_J) \in (0, \infty)^J \mid \lambda_J < \lambda_{J-1} < \ldots < \lambda_1 \}\) for some \(J \in \mathbb{N}\) and
\[V_\lambda = \sum_{j=1}^J V_{(\lambda_j)}\]
where we use the following notations for \(H^1\) and \(L^2\) rescalings
\[f_\lambda(x) = \frac{1}{\lambda^\gamma} f \left( \frac{x}{\lambda} \right) \quad \text{and} \quad f[\lambda](x) = \frac{1}{\lambda^\lambda} f \left( \frac{x}{\lambda} \right).\]
By standard arguments, recalled in Section \(5\) Equation \((1.13)\) is globally well-posed and the outer radiated energy \((1.3)\) is well-defined.

We define for all \(\lambda \in \Lambda_J\) the scale separation parameter:
\[\gamma(\lambda) = \max_{1 \leq j \leq J-1} \frac{\lambda_{j+1}}{\lambda_j}.\]
and for \(\alpha \in \mathbb{R}\) the Banach space \(Z_{\alpha, \lambda}\) associated to the norm
\[(1.14) \quad \|f\|_{Z_{\alpha, \lambda}} = \sup_{R > 0} \frac{R^{-3-\alpha}}{\inf_{1 \leq j \leq J} (\log \frac{R}{\lambda_j})} \left( \int_{R < |x| < 2R} f^2 \, dx \right)^{\frac{1}{2}}.\]
Note that for any \(J \in \mathbb{N}\) and \(\lambda \in \Lambda_J\) there holds \(L^2(\mathbb{R}^6) \subset Z_{-3, \lambda}(\mathbb{R}^6)\) with \(\|f\|_{Z_{-3, \lambda}(\mathbb{R}^6)} \lesssim \|f\|_{L^2(\mathbb{R}^6)}\), where the implicit constant is independent of \(J\) and \(\lambda\). Thus, \(Z_{-3, \lambda}\) is a weakening of \(L^2\), with a loss that is logarithmic in the distance to the closest soliton. We define:
\[\Pi_{H^1, \lambda} = \Pi_{H^1} \left( \text{Span}(\{ \Lambda W \}_{\lambda_j} \mid 1 \leq j \leq J) \right)^\perp, \quad \Pi_{L^2, \lambda} = \Pi_{L^2} \left( \text{Span}(\{ \Lambda W \}_{\lambda_j} \mid 1 \leq j \leq J) \right)^\perp.\]

**Theorem 1.4** (Channels of energy around a multisoliton). Assume \(N = 6\). For any \(J \in \mathbb{N}\), there exist \(\gamma^*, C > 0\) such that for any \(\lambda \in \Lambda_J\) with \(\gamma(\lambda) \leq \gamma^*\) if \(u\) is a radial solution of (1.13) then:
\[(1.15) \quad \|\Pi_{L^2, \lambda} u_1\|_{L^2} + \|\Pi_{H^1, \lambda} \nabla u_0\|_{Z^{-3, \lambda}} \leq C \left( \sqrt{E_{\text{out}}} + \gamma(\lambda) \| (u_0, u_1) \|_{\mathcal{H}} \right).\]

\(^{1}\) Using Cauchy-Schwarz and the formula \(f(r) = - \int_0^r \partial_s f(s) \, ds\), one can prove the following variant of Hardy's inequality: \(\|u\|_{Z_{-2}} \lesssim \|\nabla u\|_{Z_{-3}}\) and \(\|u\|_{Z_{-2}} \lesssim \|\nabla u\|_{Z_{-3}}\), for any \(u \in H^1\) radial.
Remark 1.5. While in odd dimensions, an analogue of (1.15) with the second term in the left-hand side of (1.15) replaced by the $\dot{H}^1$ norm is true, see Corollary 3.3 in [DKM19], this cannot hold true in six dimensions, even with the weaker norm $\|\Pi_{\dot{H}^1,\lambda} u_0\|_{L^2_r}$ in the left-hand side of (1.15), see Remark 1.3.

The paper is organised as follows. Section 3 contains some notation and basic estimates. Then Section 4 is devoted to the proof of Theorem 1.1; a generalised estimate for odd in time solutions on domains $\{|x| > R + |t-\tilde{t}|\}$ for $R > 0$ is first proved in Lemma 4.1, and then is used to prove the analogue but weakened generalised estimate for even solutions in Lemmas 4.8 and 4.9. Based on these generalised estimates, the proof of Theorem 1.4 is then given in Section 5.

Section 6 treats the eight dimensions, see Theorem 6.1. A counter-example to the non-weakened estimates is given in Appendix A, and a few technical estimates are given in Appendix B.

2. Acknowledgements

This work was supported by the National Science Foundation [DMS-2153794 to C.K.]; the CY Initiative of Excellence Grant "Investissements d’Avenir" [ANR-16-IDEX-0008 to C.C. and F.M.]; and the France and Chicago Collaborating in the Sciences [FACCTS award #2-91336 to C.C. and F.M.]

3. Notations and estimates for free waves

If $u$ is a function of space and time, we write $\vec{u} = (u, \partial_t u)$.

For $R \geq 0$, $p \in (1, \infty)$ we write

$$\|u\|_{L^p_R} = \int_R^\infty (u(r))^p r^5 dr, \quad \|u\|_{\dot{H}^1_R}^2 = \int_R^\infty (\partial_r u(r))^2 r^5 dr.$$  

We let $\mathcal{H}_R = \dot{H}^1_R \times L^2_R$.

Remark 3.1. Let $R > 0$ and $u$ be a radial function defined for $r > R$. Then the extension $u_R$ of $u$ defined by

$$u_R(r) = u(r), \quad r > R, \quad u_R(r) = 3u(2R - r) - 2u(3R - 2r), \quad 0 < r < R,$$

satisfies, for all $p \geq 1$

$$\|u_R\|_{L^p(\mathbb{R}^6)} \leq C \|u\|_{L^p_R}, \quad \|\partial_r u_R\|_{L^p(\mathbb{R}^6)} \leq C \|\partial_r u\|_{L^p_R}$$

where the constant $C$ is independent of $u$, $p$ and $R$.

For $(t, R) \in \mathbb{R} \times (0, \infty)$, we let

$$\mathcal{C}^*_{t, R} = \left\{ (\vec{t}, \vec{\tau}) \in \mathbb{R} \times (0, \infty) : \vec{\tau} > R + |t - \vec{t}| \right\}$$

be the exterior cone. The exterior energy is

$$\|u\|_{E_{t, R}} = \sup_{\vec{t} \in \mathbb{R}} \left\| \vec{u}(\vec{t}) \right\|_{\dot{H}^1_{R+|t-\vec{t}|} \times L^2_{R+|t-\vec{t}|}}.$$

We introduce the Strichartz norms:

$$\|u\|_{L^p_t L^q_r (r > R + |t-\vec{t}|)} = \left( \int_{\vec{t} \in \mathbb{R}} \left( \int_{r > R + |t-\vec{t}|} |u(\vec{t}, r)|^q r^5 dr \right)^{\frac{2}{q}} d\vec{t} \right)^{\frac{1}{p}}.$$
We recall the Strichartz estimates: if \((u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^6)\) and
\[
u(t) = \cos t\sqrt{-\Delta}u_0 + \sin t\sqrt{-\Delta}u_1 + \int_0^t \frac{\sin(t-t')}{\sqrt{-\Delta}} f(t') dt',
\]
with \(f \in L^1_t L^2_r(\mathbb{R}^6)\), we have
\[
\sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{\dot{H}^1 \times L^2} + \|u\|_{L^2_t L^4_r(\mathbb{C})} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1_t L^2_r}.
\]

We next state the analog of (3.2) for exterior cones

**Lemma 3.2.** Assume that \((u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^6)\), \(u\) is the solution of \(\partial_t^2 u - \Delta u = f\), \(u|_{t=0} = u_0\), \(\partial_t u|_{t=0} = u_1\) with \(f \in L^1_t L^2_r\) (all radial). Then, for any \(R_0 > 0\) (with the implicit constant independent of \((t_0, R_0)\)), we have
\[
\|\tilde{u}(t)\|_{E_{t_0,R_0}} + \|u\|_{L^2_t L^4_r(\mathbb{C}_{t_0,R_0})} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1_t L^2_r(\mathbb{C}_{t_0,R_0})}.
\]

**Proof.** Without loss of generality, we assume \(t_0 = 0\). By (3.2) above, we have
\[
\sup_{t \in \mathbb{R}} \|\tilde{u}(t)\|_{\dot{H}^1 \times L^2} + \|u\|_{L^2_t L^4_r(\mathbb{C}_{t_0,R_0})} \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} + \|f\|_{L^1_t L^2_r}.
\]

Recall that by finite speed of propagation, the solution \(u\) restricted to \(\mathbb{C}_{t_0,R_0}\) depends only on \((u_0, u_1)\) on \(r > R_0\), and \(f\) on \(\mathbb{C}_{0,R_0}\). Clearly \(\tilde{f} = f\mathbb{1}_{\mathbb{C}_{0,R_0}} \in L^1_t L^2_r\). Let now \((\tilde{u}_0, \tilde{u}_1)\) be any extension of \((u_0, u_1)\) restricted to \(r > R_0\), \((\tilde{u}_0, \tilde{u}_1) \in \dot{H}^1 \times L^2\), with \(\|(\tilde{u}_0, \tilde{u}_1)\|_{\dot{H}^1 \times L^2} \lesssim \|(u_0, u_1)\|_{\dot{H}^1_{R_0} \times L^2_{R_0}}\). Let \(\bar{u}\) solve \(\square \bar{u} = \tilde{f}\). By finite speed of propagation, \(u = \bar{u}\) on \(\mathbb{C}_{t_0,R_0}\). But then, by definition of the restriction norms, our left-hand side in the statement is smaller than the corresponding global norms for \(\bar{u}\), which can be bounded by (3.2). Hence, the left-hand bounded by \(C \left\{ \|(u_0, u_1)\|_{\dot{H}^1_{R_0} \times L^2_{R_0}} + \|\tilde{f}\|_{L^1_t L^2_r(\mathbb{C}_{t_0,R_0})} \right\}\) which is the desired result. \(\square\)

**Proposition 3.3** (Channels in 6d, with right-hand side). Let \(u \in \mathbb{R} \times \mathbb{R}^6\), solve
\[
\square u = f, \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1,
\]
f, \(u_0, u_1\) radial. Fix \(R > 0\) and write \(u_1 = \frac{u_1}{r^5/2} + u_1^+,\) where \(\int_{-\infty}^{\infty} u_1^+ \frac{1}{r^5} dr = 0\). Then,
\[
\left\| u_1^+ \right\|_{L^2_R} \lesssim \left[ \lim_{t \to \infty} \|\tilde{u}(t)\|_{\dot{H}^1_{R+|t|} \times L^2_R} + \|f\|_{L^1_{|t|} L^2_r(\mathbb{C})} \right].
\]

**Proof.** Let \(\bar{u}\) be the solution of the homogeneous wave equation with initial data \((u_0, u_1)\), and let \(v\) be the solution of the inhomogeneous equation with right-hand side \(f \mathbb{1}_{\mathbb{C}_{0,R}}\), and \((0,0)\) initial data. Then \(u = \bar{u} + v\) on \(\mathbb{C}_{0,R}\), by finite speed of propagation. By [DKMM22, Proposition 3.8], we have
\[
\left\| u_1^+ \right\|_{L^2_R} = \left\| \tilde{u}_1^+ \right\|_{L^2_R} \leq \frac{20}{3} \lim_{t \to \infty} \|\tilde{u} - \bar{u}\|_{\dot{H}^1_{R+|t|} \times L^2_R} \lesssim \frac{20}{3} \left( \lim_{t \to \infty} \|\tilde{u}(t)\|_{\dot{H}^1_{R+|t|} \times L^2_{R+|t|}} + \sup_{t} \left\| \tilde{v}(t) \right\|_{\dot{H}^1_{R+|t|} \times L^2_{R+|t|}} \right) \lesssim \frac{20}{3} \left( \lim_{t \to \infty} \|\tilde{u}(t)\|_{\dot{H}^1_{R+|t|} \times L^2_{R+|t|}} + C\|f\|_{L^1_{|t|} L^2_R(\mathbb{C})} \right),
\]
by (3.2) and Lemma 3.2. \(\square\)
4. Channels of energy around the ground state

It is easy to check that the solution $u$ of (1.1) is globally well-posed in $H$. Indeed, the local well-posedness can be proved by Strichartz estimates (3.2) and the fact that $W$ is in $L^4$. The global well-posedness follows from the linearity of the equation. Furthermore, using finite speed of propagation, the Strichartz estimates of Lemma 3.2 and that $\lim_{R\to\infty} \|\Pi(R < |t| < |x| - R)W\|_{L^2L^4} = 0$, the outer radiated energy (1.3) is well-defined.

4.1. Channels of energy around the ground state for odd in time solutions. We first establish, for odd in time solutions, a more general estimate result than that of Theorem 1.1 for $\partial_t u$, valid for the exterior of any wave cone $\{r \geq R + |t|\}$. The zeros of $-\Delta + V$ play an important role. We recall that there holds

\begin{equation}
-\Delta \Lambda W + V \Lambda W = 0 \quad \text{and} \quad -\Delta \Gamma + V \Gamma = 0,
\end{equation}

where the second function $\Gamma(r) = -\Delta W(r) \int_1^r s^{-5} (\Lambda W(s))^{-2} ds$ satisfies the Wronskian relation:

\begin{equation}
\forall r > 0, \quad \Gamma(r) \frac{d\Delta W(r)}{dr} - \Delta W(r) \frac{d\Gamma(r)}{dr} = r^{-5}
\end{equation}

and the asymptotics for some $c, c' \neq 0$:

\begin{equation}
\Gamma(r) = cr^{-4} + O(r^{-2}) \quad \text{as} \quad r \to 0, \quad \text{and} \quad \Gamma(r) = c' + O(r^{-2}) \quad \text{as} \quad r \to \infty,
\end{equation}

which propagates for higher order derivatives:

\begin{equation}
\frac{d\Gamma(r)}{dr} = -4cr^{-5} + O(r^{-3}) \quad \text{as} \quad r \to 0, \quad \text{and} \quad \frac{d\Gamma(r)}{dr} = dr^{-3} + O(r^{-5}) \quad \text{as} \quad r \to \infty,
\end{equation}

where $d \neq 0$. For $\chi_0$ a smooth cut-off function such that $\chi_0(r) = 1$ for $r \leq 10$ and $\chi_0(r) = 0$ for $r \geq 12$ we define $\tilde{\Gamma}$ as the truncation of the $\Gamma$ profile:

\begin{equation}
\tilde{\Gamma}(r) = \chi_0(r) \Gamma(r)
\end{equation}

We then define for $R \geq 0$ the projection $\Pi_{L^2_R}^\perp u$ as follows:

\begin{equation}
\Pi_{L^2_R}^\perp = \begin{cases} 
\Pi_{L^2_R} (\text{Span}(\Lambda W)) \perp & \text{for} \ R = 0 \text{ or} \ R \geq 1, \\
\Pi_{L^2_R} (\text{Span}(\Lambda W, \tilde{\Gamma})) \perp & \text{for} \ 0 < R < 1.
\end{cases}
\end{equation}

This section is devoted to the proof of the following result.

Lemma 4.1 (Channels of energy for the odd in time part of solutions around a soliton). There exists $C > 0$ (independent of $R$) such that the following holds true for all $R \geq 0$. Let $u_R$ be a radial solution of (1.1) for $r \geq R + |t|$, with $\tilde{u}_R(0) \in H_R$. Then:

\[ \|\Pi_{L^2}^\perp u_R\|^2_{L^2} \leq C \sum_{\pm} \lim_{t \to \pm \infty} \int_{r \geq R + |t|} |\nabla_{t,x} u_R|^2 dx. \]

The proof combines two quantitative lower-bounds, proved in [DKMM22, CKS14], namely Proposition 3.3 and that the bound (1.6) holds true for all solutions of (1.4), and the following rigidity result:

Lemma 4.2 (Rigidity for odd in time non-radiative linear waves around a soliton). Let $R \geq 0$, and $u_R$ be a solution of (1.1). Assume

\begin{equation}
\sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > R + |t|} |\nabla_{t,x} u_R(t, x)|^2 dx = 0.
\end{equation}

Then there exists $c \in \mathbb{R}$ such that $u_1(r) = c\Delta W(r)$ for all $r \geq R$.
Proof. Up to replacing $u_L$ by $\frac{1}{2}(u_L(t) - u_L(-t))$, we can assume that $u_0 = 0$. We fix a large $R' > 0$ and choose $c \in \mathbb{R}$ such that
\[ \int_{R'}^{\infty} (u_1 - cAW) \frac{1}{r^4} r^5 dr = 0. \]
The solution of (1.1) with initial data $(0, u_1 - cAW)$ is $u_L(t) - ctAW$. By Proposition 3.3,
\[ \|u_1 - cAW\|_{L^2_{R'}} \lesssim \|V(u_L - ctAW)\|_{L^1L^2(r>R'+|t|)}. \]
Furthermore, by Strichartz estimates,
\[ \|u_L - ctAW\|_{L^2L^4(r>R'+|t|)} \lesssim \|V(u_L - ctAW)\|_{L^1L^2(r>R'+|t|)} + \|u_1 - cAW\|_{L^2_{R'}}. \]
Combining, we obtain
\[ \|u_L - ctAW\|_{L^2L^4(r>R'+|t|)} \lesssim \|V(u_L - ctAW)\|_{L^1L^2(r>R'+|t|)} \lesssim \|V\|_{L^2L^4(r>R'+|t|)} \|u_L - ctAW\|_{L^2L^4(r>R'+|t|)}. \]
Since $V = -2W \in L^2L^4(r \geq |t|)$, we have
\[ \lim_{R' \to \infty} \|V\|_{L^2L^4(r>R'+|t|)} = 0, \]
and thus (4.8) implies, choosing $R'$ large enough,
\[ u_L(t, r) = ctAW(r), \quad r > R' + |t|. \]
As a consequence of (4.7) and the propagation of the support for solutions of (1.1) with compactly supported initial data (see [DKM20] Proposition 4.7), we obtain that $u_1(r) = cAW(r)$ for all $r \geq R$.

We can now give the proof of Lemma 4.1.

Proof of Lemma 4.1 We reason by contradiction, and assume that there exist sequences $(R_n)_n$ of radii and $(u_{L,n})_n$ of solutions of (1.1), with initial data $(0, u_{1,n})$ (up to replacing $u_{L,n}$ by $\frac{1}{2}(u_{L,n}(t) - u_{L,n}(-t))$) such that $u_{1,n}(r) = 0$ for $r \leq R_n$, and that
\[ \forall n \in \mathbb{N}, \quad \left\| \Pi_{L_n} u_{1,n} \right\|_{L^2_{R_n}} = 1, \quad \text{and} \quad \lim_{n \to \infty} \left( \lim_{t \to \infty} \int_{|x| > R_n + |t|} |\nabla_{t,x} u_{L,n}(t, x)|^2 dx \right) = 0. \]
We decompose $u_{1,n}$ as follows.
\[ u_{1,n} = c_nAW + v_{1,n}, \quad r > R_n \quad \text{where} \quad \int_{|x| > R_n} v_{1,n}AW = 0. \]
We extend $v_{1,n}$ by $v_{1,n}(r) = 0$ for $0 < r < R_n$. We let $v_{L,n}$ be the solution of (1.1) with initial data $(0, v_{1,n})$, and note that $v_{L,n}(t, r) = u_{L,n}(t, r) - c_nAW$ for $r > R_n + |t|$. Extracting subsequences, we can assume that we are in one of the three following cases:

Case 1: $\forall n, \quad R_n = 0$ or $\forall n, \quad R_n \geq 1$

Case 2: $\forall n, \quad R_n \in (0, 1)$ and $\lim_{n \to \infty} R_n = R_\infty \in (0, 1]$

Case 3: $\forall n, \quad R_n \in (0, 1)$ and $\lim_{n \to \infty} R_n = 0$. 

Case 1. We assume $R_n = 0$ or for all $n, R_n \geq 1$. Extracting subsequences, we can also assume $\lim_{n \to \infty} R_n = R_\infty \in \{0\} \cup [1, \infty]$. We note that in this case, $v_{1,n} = \Pi_{L_n} u_{1,n}$, so that
\[ \|v_{1,n}\|_{L^2_{R_n}} = 1 \text{ by (4.9)}. \]
We first claim
\[ v_{1,n} \xrightarrow{n \to \infty} 0 \text{ in } L^2(\mathbb{R}^6). \]
If $R_\infty = \infty$, this is obvious since $v_{1,n}(r) = 0$ for $r \in (0, R_n)$. If not, we introduce (after extraction) the weak limit $v_1$ of $v_{1,n}$ in $L^2$. We will prove that $v_1 = 0$. By weak convergence and the orthogonality condition in (4.10), we have

$$
(4.12) \int_{|x|>R_\infty} v_1 \Lambda W = 0.
$$

We denote by $v_L$ the solution of (1.1) with initial data $(0, v_1)$. Fix now any $R > R_\infty$, and define $v_{1,n}^R = \mathbb{1}_{\{r \geq R\}} v_{1,n}$ and $v_1^R = \mathbb{1}_{\{r \geq R\}} v_1$, and $v_{L,n}^R$ and $v_L^R$ the solutions to (1.1) with initial data $(0, v_{1,n}^R)$ and $(0, v_1^R)$ respectively. Then, $v_{L,n}^R$ coincides with $u_{L,n}^R - c_n t \Lambda W$ for $r \geq R + |t|$ by finite speed of propagation, so that using (4.9):

$$
(4.13) \lim_{n \to \infty} \lim_{t \to \infty} \int_{|r|>R_\infty} |\nabla v_{L,n}^R|^2 + (\partial_t v_{L,n}^R)^2 = 0.
$$

We have that $v_1^R$ is the weak limit of $(v_{1,n}^R)_n$, so that $v_1^R$ belongs to the closure in $L^2$ of the convex hull of $(v_{1,n}^R)_n \geq N$ for any $N$. Hence, using energy estimates, $v_L^R$ belongs to the closure in $L_t^\infty (H_{r \geq |t|})$ of the convex hull of $(v_{1,n}^R)_n \geq N$ for any $N$. This fact and the estimate (4.13) then imply:

$$
\lim_{t \to \infty} \int_{|r|>R_\infty} |\nabla v_L^R|^2 + (\partial_t v_L^R)^2 = 0.
$$

Thus, Lemma 4.2 implies that $v_1^R(r) = c_R \Lambda W(r)$ for all $r \geq R$ for some $c_R \in \mathbb{R}$. Coming back to the definition of $v_1^R$, we see that $c_R = c$ is independent of $R > R_\infty$ so that $v_1(r) = c \Lambda W(r)$ for $r > R_\infty$, and that $c = 0$ using (4.12). Hence (4.11).

Next, we let $v_{F,n}$ be the solution of the free wave equation $(\partial_t^2 - \Delta)v_{F,n} = 0$ with initial data $(0, v_{1,n})$. By (4.11) and [DKM20, Lemma 3.7], we have

$$
\sup_{t \in \mathbb{R}} \|v_{F,n}(t) - v_{L,n}(t)\|_{H_{|t|}} \to 0.
$$

Thus by (4.9) and (4.10),

$$
(4.14) \lim_{n \to \infty} \lim_{t \to \infty} \int_{|x|>R_\infty+|t|} |\nabla_{t,x} v_{F,n}(t, x)|^2 dx = 0.
$$

On the other hand, by the result (1.6) proved in [CKS14] if $R_n = 0$ or Proposition 3.3 if $R_n > 0$, we have

$$
\lim_{t \to \infty} \int_{|x|>R_n+|t|} |\nabla_{t,x} v_{F,n}(t, x)|^2 dx \geq \frac{\|v_{1,n}\|_{L^2_{R_n}}^2}{r_n^4} \|v_{1,n}\|_{L^2_{R_n} \cap L^1_{R_n}}^{-1} \left\| \frac{1}{r_n^2} \int_{|x|>R_n} \frac{1}{r^2} v_{1,n}(x) dx \right\|^2
$$

Recall that $\|v_{1,n}\|_{L^2_{R_n}}^2 = 1$ for all $n$ by (4.9). In the case where $R_n = 0$ for all $n$, we see that it contradicts (4.14). When $R_n > 0$ for all $n$, since $v_{1,n}$ converges weakly to 0 in $L^2$, and $\lim_{n \to \infty} R_n = 0$, we obtain

$$
\lim_{n \to \infty} \frac{1}{r_n^4} \left\| \frac{1}{r_n^2} \int_{|x|>R_n} \frac{1}{r^2} v_{1,n}(x) dx \right\|_{L^2_{R_n}}^{-1} = 0,
$$

and the contradiction follows in this case also.

**Case 2.** We now assume that $R_n \in (0, 1)$ for all $n$, and that $\lim_{n \to \infty} R_n = R_\infty \in (0, 1]$. In this case, by the equality $u_{1,n} = c_n \Lambda W + v_{1,n}$, we have $\Pi_{L_{R_n}}^\perp u_{1,n} = \Pi_{L_{R_n}}^\perp u_{1,n}$, and thus, by (4.9),
\[ \|v_{1,n}\|_{L_{R_n}^2} \geq 1. \] Letting
\[ \tilde{u}_{1,n} = \frac{1}{\|v_{1,n}\|_{L_{R_n}^2}} u_{1,n}, \quad \tilde{v}_{1,n} = \frac{1}{\|v_{1,n}\|_{L_{R_n}^2}} v_{1,n}, \]
and \( \tilde{u}_{L,n} \) the solution of (1.1) with initial data \((0, \tilde{u}_{1,n})\), we see that \( \tilde{u}_{1,n} = \tilde{c}_n \Lambda W + \tilde{v}_{1,n}, \)
\[ \int_{|x| > R_n} \tilde{v}_{1,n} \Lambda W = 0, \quad \| \tilde{v}_{1,n} \|_{L_{R_n}^2} = 1, \lim_{n \to \infty} R_n = R > 0 \]
and the left-hand side inequality in (4.15) follows. Furthermore, denoting by \( \Xi_L = \tilde{\Gamma}_L - t \tilde{\Gamma} \), we have
\[ (4.16) \]
\[ \left\{ \begin{array}{c} (\partial_t^2 - \Delta + V) \Xi_L = t(-\Delta + V) \tilde{\Gamma} \\ \Xi_L = (0,0), \end{array} \right. \]
and thus, since \((-\Delta + V) \tilde{\Gamma} = 0\) for \( r \leq 10 \) and for \( r \geq 12 \) from (4.5), we see by simple energy estimates that
\[ \lim_{t \to \infty} \int_{|x| > |t|} |\nabla_{t,x} \Xi_{L}(t,x)|^2 dx \]
is finite, which yields the right-hand side inequality in (4.15).

**Step 2.**
Next, we expand \( u_{1,n} \) as follows:
\[ u_{1,n} = d_n \Lambda W + \tilde{d}_n \Gamma + \Pi_{L_{R_n}^2}^+ u_{1,n}, \quad r > R_n \]
and by (4.9) and the lower bound in (4.15), we deduce that \( \tilde{d}_n \) is bounded. Using the definition (1.10) of \( v_{1,n} \), we obtain
\[ (4.17) \]
\[ (c_n - d_n) \Lambda W = \tilde{d}_n \Gamma + \Pi_{L_{R_n}^2}^+ u_{1,n} - v_{1,n}, \]
and thus, taking the scalar product with \( \Lambda W \), \( (c_n - d_n) \int_{|x| > R_n} |\Lambda W|^2 = \tilde{d}_n \int_{R_n} \Lambda W \tilde{\Gamma}. \) Since \( \Lambda W \tilde{\Gamma} \in L^1(\mathbb{R}^6) \), we deduce that \( c_n - d_n \) is also bounded. As a consequence, and since \( \tilde{\Gamma} \in L^2(\mathbb{R}^{7.5}(r^{-2.5} dr) \), we obtain that \( v_{1,n} \) is bounded in \( L^2(\mathbb{R}^{7.5}(r^{-2.5} dr) \). With the same argument
as in Case 1, using Lemma 4.2, we see that \( v_{1,n} \) converges weakly to 0 in \( L^2_R \) for all \( R > 0 \), and thus
\[
v_{1,n} \xrightarrow{n \to \infty} 0, \quad \text{weakly in } L^2 \left( r^{7.5}(r)^{-2.5} dr \right).
\]

**Step 3.**

Next, we decompose \( v_{1,n} \) as follows for \( r > R_n \)
\[
v_{1,n} = \tilde{c}_n \tilde{\Gamma} + w_{1,n}, \quad \int |x| > R_n w_{1,n} \Gamma = 0,
\]
and let \( w_{1,n}(r) = 0 \) for \( r \in (0, R_n) \). We claim
\[(4.18) \quad \lim_{n \to \infty} \tilde{c}_n = 0, \quad w_{1,n} \xrightarrow{n \to \infty} 0 \text{ in } L^2.
\]
Indeed, (4.17) implies \( (\tilde{c}_n - \tilde{d}_n) \tilde{\Gamma} + w_{1,n} = (d_n - c_n) \Lambda W + \Pi_{L_{R_n}^+}^1 u_{1,n} \). Thus
\[
\|(4.19) \quad \|w_{1,n}\|_{L^2_{R_n}}^2 + (\tilde{c}_n - \tilde{d}_n)^2 \|\tilde{\Gamma}\|_{L^2_{R_n}}^2 = \|\Pi_{L_{R_n}^+}^1 u_{1,n}\|_{L^2_{R_n}}^2 + (c_n - d_n)^2 \|\Lambda W\|_{L^2_{R_n}}^2.
\]
Since the right-hand side of (4.19) is bounded, we obtain that its left-hand side is also bounded and thus that \( w_{1,n} \) is bounded in \( L^2 \), and that \( \tilde{c}_n - \tilde{d}_n \) is bounded. Since we have proved above that \( \tilde{d}_n \) is bounded, we obtain that \( \tilde{c}_n \) is bounded. Extracting subsequences, we can assume
\[
\lim_{n \to \infty} \tilde{c}_n = \tilde{c}, \quad w_{1,n} \xrightarrow{n \to \infty} w_1 \text{ in } L^2.
\]
Since \( v_{1,n} \xrightarrow{n \to \infty} 0 \) in \( L^2 (r^{7.5}(r)^{-2.5} dr) \), we obtain \( 0 = \tilde{\Gamma} + w_1 \), and thus, since \( w_1 \in L^2 \) and \( \tilde{\Gamma} \notin L^2 \), \( \tilde{c} = 0 \) and \( w_1 = 0 \). Hence (4.18).

**Step 4.**

Let \( w_{L,n} \) be the solution of (1.1) with initial data \((0, w_{1,n})\). From \( v_{1,n} = \tilde{c}_n \tilde{\Gamma} + w_{1,n} \), (4.15) and (4.18), we obtain
\[
(4.20) \quad \lim_{n \to \infty} \left( \lim_{t \to \infty} \int_{|x| > R_n + |t|} \left| \nabla_{t,x} w_{L,n}(t, x) \right|^2 dx \right) = 0.
\]
Let \( w_{F,n} \) be the solution of the free wave equation with initial data \((0, w_{1,n})\). Since \( w_{1,n} \to 0 \) in \( L^2 \), we have
\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}} \|w_{L,n}(t) - w_{F,n}(t)\|_{H^1} = 0.
\]
By (4.20),
\[
(4.21) \quad \lim_{n \to \infty} \left( \lim_{t \to \infty} \int_{|x| > R_n + |t|} \left| \nabla_{t,x} w_{F,n}(t, x) \right|^2 dx \right) = 0.
\]
Furthermore
\[
(4.22) \quad \left| \frac{1}{r^4} \int_{L_{R_n}^+} \partial_t w_{F,n}(0) \frac{1}{r^4} r^5 dr \right| \to 0.
\]
Indeed,
\[
\left| \int_{r \geq R_n} \partial_t w_{F,n}(0) \frac{1}{r^4} r^5 dr \right| = \left| \int_{r \geq R_n} \partial_t w_{F,n}(0) \left( \frac{1}{r^4} - \frac{1}{c} \tilde{\Gamma} \right) r^5 dr \right| \leq C \left\| \frac{1}{r^4} - \frac{1}{c} \tilde{\Gamma} \right\|_{L^2_{R_n}}.
\]
and hence (4.22) using (4.3). By (4.21), (4.22) and Proposition 3.3, we have
\[
\lim_{n \to \infty} \|w_{1,n}\|_{L^2_{R_n}} = 0.
\]
However,
\[ \|u_{1,n}\|_{L^2_{R_n}}^2 \geq \|\Pi_{L^2_{R_n}}^{-1} w_{1,n}\|_{L^2_{R_n}}^2 = \|\Pi_{L^2_{R_n}}^{-1} u_{1,n}\|_{L^2_{R_n}}^2 = 1, \]
which gives a contradiction. □

4.2. Channels of energy around the ground state for even in time solutions. We prove in this Subsection:

**Lemma 4.3.** There exists \( C > 0 \) such that the following holds true. Let \( u_L \) be a radial solution of \((1.1)\). Then:
\[ \|\nabla \Pi_{H^1}^{-1} u_0\|_{L^2}^2 \leq C \sum_{\pm} \lim_{t \to \pm \infty} \int_{r \geq |t|} |\nabla_{t,x} u_L(t,x)|^2 \, dx. \]

Lemma 4.3 is a consequence of a more general estimate for the set \( \{ r \geq R + |t| \} \) for any \( R > 0 \) stated in Lemma 4.5 below. We introduce an element of the generalized kernel of \(-\Delta + V\).

**Lemma 4.4.** There exists a solution \( \Upsilon \) to \((-\Delta + V) \Upsilon = \Lambda W \) such that, for two constants \( c_0 \neq 0 \) and \( c_\infty \neq 0 \):
\[ \Upsilon(r) = c_0 r^{-4} + O(r^{-2}), \quad \Upsilon(r) = c_\infty r^{-2} + O(r^{-4} \log r) \]
(4.23) \[ \frac{\partial \Upsilon}{\partial r}(r) = -4c_0 r^{-5} + O(r^{-3}), \quad \frac{\partial \Upsilon}{\partial r}(r) = -2c_\infty r^{-3} + O(r^{-5} \log r). \]
(4.24)

**Proof.** Solving \((-\Delta + V) \Upsilon = \Lambda W\), we define \( \Upsilon \) by:
\[ \Upsilon(r) = -\Lambda W(r) \int_0^r \Lambda W(s) \Gamma(s) s^5 \, ds - \Gamma(r) \int_r^\infty \Lambda W^2(s) s^5 \, ds. \]

Using that, by the definition of \( \Lambda W \), for some \( c \neq 0 \), \( d \neq 0 \),
\[ \Lambda W(r) = c + O(r^2), \quad \frac{d \Lambda W}{dr}(r) = O(r) \quad r \to 0, \]
\[ \Lambda W(r) = dr^{-4} + O(r^{-6}), \quad \frac{d \Lambda W}{dr}(r) = -4dr^{-5} + O(r^{-7}) \quad r \to \infty, \]
and the asymptotics behaviours of \( \Gamma \) and \( \frac{\partial \Gamma}{\partial r} \) given by (4.3), (4.4), one obtains the desired asymptotic behaviour (4.23) by direct computations. □

Observe that the function \((t,r) \mapsto \Upsilon(r) - \frac{t^2}{2} \Lambda W(r)\) solves \((1.1)\) but \( \bar{\Upsilon}(t) \) fails critically to belong to the spaces \( \mathcal{H}_R, R > 0 \). This is thus a resonance for Equation \((1.1)\).

We define for \( \alpha \in \mathbb{R} \) and \( R > 0 \):
\[ \|f\|_{Z_{\alpha,R}} = \sup_{\rho \geq R} \frac{\rho^{-3-\alpha}}{(\log \rho^2)^{\alpha}} \|f\|_{L^2(\rho \leq r \leq 2\rho)}, \quad \|f\|_{Z_{\alpha,R}} = \sup_{\rho > R} \rho^{-3-\alpha} \|f\|_{L^2(\rho \leq r \leq 2\rho)} \]
and for any set \( E \subset Z_{\alpha,R} \):
\[ d_{Z_{\alpha,R}}(u, E) = \inf_{v \in E} \|u - v\|_{Z_{\alpha,R}}. \]
Then we claim:

**Lemma 4.5.** There exists \( C > 0 \) such that, for any radial solution \( u_L \) of \((1.1)\) with \( \bar{u}(0) \in \mathcal{H} \), for any \( R > 0 \):
\[ d_{Z_{-3,24R}}(\partial_t u_0, \text{Span}(\partial_r \Lambda W, \partial_r \Upsilon))^2 \leq C \sum_{\pm} \lim_{t \to \pm \infty} \int_{r \geq |t|+R} |\nabla_{t,x} u(t,x)|^2 \, dx. \]

**Remark 4.6.** The constant 24 is arbitrary, and used to ease the proof, but the result could be proved for any constant > 1.
Hence, in the space $L^2(\{1 < r < 2\})$, both $\partial_t u_0$ and $\partial_r \tilde{u}_0$ are uniformly bounded, and so is $c_n \partial_t \Lambda W + c'_n \partial_r \mathcal{Y}$. As $\partial_t \Lambda W$ and $\partial_r \mathcal{Y}$ are non collinear in $L^2(\{1 < r < 2\})$, we get that $c_n$ and $c'_n$ are bounded sequences. Therefore, $c'_n \partial_r \mathcal{Y}$ is uniformly bounded in $Z_{-3,R}$, implying $c'_n \to 0$ because of (4.24). Up to extracting a subsequence, $c_n$ converges to some limit $c_{\infty}$, and the uniform bound on $\tilde{u}_{0,n}$ then implies that at fixed $R > 0$,

$$\langle \log R \rangle^{-2} \| \partial_r u_0 - c_{\infty} \partial_r \Lambda W \|_{L^2(\{R < r < 2R\})} \lesssim \sum_{t \to \pm \infty} \int_{r \geq |t|} |\nabla_{t,x} u(t, x)|^2 dx,$$

and thus

$$\| \partial_r u_0 - c_{\infty} \partial_r \Lambda W \|_{Z_{-3}} \lesssim \sum_{t \to \pm \infty} \int_{r \geq |t|} |\nabla_{t,x} u(t, x)|^2 dx.$$ 

This implies Lemma 4.3 upon noticing that

$$\left\| \nabla (\Pi_{H^1}^{-1} u_0) \right\|_{Z_{-3}} = \left\| \nabla (\Pi_{H^1}^{-1} (u_0 - c_{\infty} \Lambda W)) \right\|_{Z_{-3}} \lesssim \| u_0 - c_{\infty} \Lambda W \|_{Z_{-3}},$$

where the last bound follows from explicit computation, using that $|\Lambda W| + |r \frac{d}{dr} \Lambda W| \lesssim \langle r \rangle^{-4}$.

The rest of this Subsection is devoted to the proof of Lemma 4.5. We decompose its proof in three intermediate lemmas and start with an elliptic result.

**Lemma 4.7 (Elliptic estimate close to a soliton).** There exists $C > 0$ such that the following holds true. Let $R > 0$ and assume that $u \in Z_{-2,R}(\mathbb{R}^6)$ solves for $r \geq R$:

$$- \Delta u + Vu = f + \partial_r g,$$

where $u$, $f$ and $g$ are radial and satisfy:

$$\| f \|_{Z_{-4,R}} + \| g \|_{Z_{-3,R}} < \infty. \tag{4.25}$$

Then, if $R \geq 1$:

$$d_{Z_{-2,R}}(u, \text{Span}(\Lambda W)) \leq C \left( \| f \|_{Z_{-4,R}} + \| g \|_{Z_{-3,R}} \right) \tag{4.26}$$

while if $0 < R < 1$:

$$d_{Z_{-2,R}}(u, \text{Span}(\Lambda W, \mathcal{Y})) \leq C \left( \| f \|_{Z_{-4,R}} + \| g \|_{Z_{-3,R}} \right). \tag{4.27}$$

**Proof.** We can assume without loss of generality that $f = 0$. This is because, defining $\tilde{g} = g + \tilde{f}$ with $\tilde{f}(r) = -\int_r^\infty f(s) ds$, then $f = \partial_r \tilde{f}$ so that $-\Delta u + Vu = \partial_r \tilde{g}$, and moreover $|f(r)| \lesssim \langle \log \frac{r}{|t|} \rangle^{-3} \| f \|_{Z_{-4,R}}$ for all $r > 0$ by elementary estimates, so that $\| \tilde{g} \|_{Z_{-3,R}} \lesssim \| f \|_{Z_{-4,R}} + \| g \|_{Z_{-3,R}}$.

**Step 1.** The case $0 < R < 1$. Note that in this case:

$$\| \cdot \|_{Z_{a,R}} \approx \sup_{\rho \geq R} \rho^{-3-a} \| \langle \log \rho \rangle \cdot \| L^2(\rho \leq r \leq 2\rho). \tag{4.28}$$
Solving (4.25) using (4.2), and $u \in Z_{-2,R}$, we find that there exists $c \in \mathbb{R}$ such that for all $r > R$:

$$u(r) = c \Lambda W(r) + \Lambda W(r) \int_1^r g(s) \partial_s (\Gamma(s)s^5)ds + \Gamma(r) \int_r^{\infty} g(s) \partial_s (\Lambda W(s)s^5)ds.$$

We recall that, using (4.23), there exists $\tilde{c} \neq 0$ such that

$$\tilde{c} \Upsilon(r) = \Gamma(r) + O(r^{-2}) \quad \text{as } r \to 0.$$

Introducing $\bar{c} = \tilde{c} \int_R^{\infty} g \partial_s (\Lambda W(s)s^5)ds$, we decompose $u = c \Lambda W + \bar{c} \Upsilon + \tilde{u}$ with:

$$\tilde{u}(r) = \Lambda W(r) \int_1^r g(s) \partial_s (\Gamma(s)s^5)ds + \Gamma(r) \int_r^{\infty} g(s) \partial_s (\Lambda W(s)s^5)ds - \bar{c} \Upsilon(r).$$

For $r \geq 1$ let $k(r) \in \mathbb{N}$ such that $2^{k(r)-1} \leq r < 2^{k(r)}$. Using dyadic partitioning, Cauchy-Schwarz, (1.3), (1.4) and (1.29) we estimate that:

$$\left| \int_1^r g(s) \partial_s (\Gamma(s)s^5)ds \right| \leq \sum_{k=1}^{k(r)-1} \int_2^{2^{k+1}} |g(s) \partial_s (\Gamma(s)s^5)|ds \leq \sum_{k=1}^{k(r)-1} \left( \int_2^{2^{k+1}} g^2(s)s^5ds \right)^{\frac{1}{2}} \left( \int_2^{2^{k+1}} s^3ds \right)^{\frac{1}{2}} \leq \sum_{k=1}^{k(r)-1} \langle k \rangle \|g\|_{Z_{-3,R}} 2^{2k} \leq r^2 (\log r) \|g\|_{Z_{-3,R}}.$$

Hence $|\Lambda W(1) \int_1^r g(s) \partial_s (\Gamma(s)s^5)ds| \lesssim r^{-2}(\log r) \|g\|_{Z_{-3,R}}$. Similarly, $|\Gamma(r) \int_r^{\infty} g(s) \partial_s (\Lambda W(s)s^5)ds| \lesssim r^{-2}(\log r) \|g\|_{Z_{-3,R}}$, and hence using $|\bar{c}| \lesssim \|g\|_{Z_{-3,R}}$ and (1.23):

$$\forall r \geq 1, \quad \tilde{u}(r) = O(r^{-2}(\log r) \|g\|_{Z_{-3,R}}).$$

Next, for $R \leq r \leq 1$ we decompose:

$$\Gamma(r) \int_r^{\infty} g(s) \partial_s (\Lambda W(s)s^5)ds - \bar{c} \Upsilon(r) = -\Gamma(r) \int_R^r g(s) \partial_s (\Lambda W(s)s^5)ds + \bar{c} \left( \frac{1}{c} \Gamma(r) - \Upsilon(r) \right).$$

With computations similar to (4.31), we obtain $|\Gamma(r) \int_R^r g(s) \partial_s (\Lambda W(s)s^5)ds| \lesssim r^{-2}(\log r) \|g\|_{Z_{-3,R}}$ and $|\Lambda W(r) \int_1^r g(s) \partial_s (\Gamma(s)s^5)ds| \lesssim r^{-2}(\log r) \|g\|_{Z_{-3,R}}$. Using this, (4.30) and $|\bar{c}| \lesssim \|g\|_{Z_{-3,R}}$ we infer:

$$\forall r \in [R,1], \quad \tilde{u}(r) = O(r^{-2}(\log r) \|g\|_{Z_{-3,R}}).$$

Combining (4.32) and (4.33) shows $\|\tilde{u}\|_{Z_{-2,R}} \lesssim \|g\|_{Z_{-3,R}}$, and hence (4.28) holds true and the Lemma is proved in the case $0 < R < 1$.

**Step 2. The case $R \geq 1$.** Note that in this case:

$$\|\cdot\|_{Z_{\alpha,R}} \approx \sup_{\rho \geq R} \rho^{-2-\alpha} \|\cdot\|_{L^2(\rho \leq r \leq 2\rho)}.$$

We solve (4.26) and get that for some $c \in \mathbb{R}$, for all $r > R$:

$$u(r) = c \Lambda W(r) + \Lambda W(r) \int_R^r g(s) \partial_s (\Gamma(s)s^5)ds + \Gamma(r) \int_r^{\infty} g(s) \partial_s (\Lambda W(s)s^5)ds.$$
For \( r \geq R \) let \( k(r) \in \mathbb{N} \) such that \( 2^{k(r)-1}R \leq r < 2^k R \). Using dyadic partitioning, Cauchy-Schwarz, (4.3), (4.4) and (4.31) we estimate that:

\[
\left| \int_R^r g(s) \partial_s (\Gamma(s)s^5) ds \right| \leq \sum_{k=1}^{k(r)-1} \int_{2^k R}^{2^{k+1} R} |g(s) \partial_s (\Gamma(s)s^5)| ds \\
\leq \sum_{k=1}^{k(r)-1} \| g \|_{Z_{3,R}} 2^{2k} R^2 \lesssim r^2 \log \frac{r}{R} \| g \|_{Z_{3,R}}.
\]

Similarly, \( |\Gamma(r) \int_1^\infty g(s) \partial_s (\Lambda W(s)s^5) ds| \lesssim r^{-2} \log \frac{r}{R} \| g \|_{Z_{3,R}} \). Combining, this proves (4.27) and the Lemma is proved in the case \( R \geq 1 \).

\[ \square \]

If \( u_L \) is a solution of (1.1) with initial data in \( \mathcal{H} \), we denote:

\[
\| \partial_t u_L \|_{Y_R} = \sup_{t \in \mathbb{R}} \left\| \Pi_{L^2} \partial_t u_L(t) \right\|_{L^2},
\]

where \( \Pi_{L^2} \) is defined in (4.6). For \( k \in \mathbb{Z}, R_k = 2^k \) and \( 0 \leq t \leq R_k \), we write for the remaining part of this Subsection:

\[
\partial_t u_L(t) = \begin{cases} 
\alpha_k(t) \Lambda W + g_k(t) & \text{for } R_k \geq 1, \ t \in \mathbb{R} \\
\alpha_k(t) \Lambda W + \bar{\alpha}_k(t) \bar{\Gamma} + g_k(t) & \text{for } R_k < 1, \ t \in \mathbb{R}.
\end{cases}
\]

where \( g_k(t) = \Pi_{L^2_{R_k}} \partial_t u_L(t) \), so that:

\[
\sup_{k \in \mathbb{Z}, R_k \geq R} \| g_k(t) \|_{L^2_{R_k}} \leq \| \partial_t u_L \|_{Y_R}.
\]

**Lemma 4.8** (Estimating \( u_L \) in weighted \( L^2 \) from \( \partial_t u_L \) for waves around a soliton). There exists \( C > 0 \) such that the following holds. Assume that \( u_L \) is a radial solution to (1.1) with \( \bar{w}(0) \in \mathcal{H} \). Then, for any \( R > 0 \),

\[
d_{Z_{-2,6R}}(u_0, \text{Span}(\Lambda W, \bar{\Upsilon})) \leq C \| \partial_t u_L \|_{Y_R}.
\]

**Proof.** Note first that it suffices to show the result for \( u_0 \) even in time. Indeed, for general \( u \), it suffices to apply the result to \( u_+ = \frac{1}{2}(u(t) + u(-t)) \), noticing that \( \partial_t u_+ = \frac{1}{2}(\partial_t u(t) - \partial_t u(-t)) \) satisfies \( \| \partial_t u_+ \|_Y \leq \| \partial_t u \|_Y \).

**Step 1.** An elliptic equation for \( u_0 \). Let for \( k \in \mathbb{Z}, R_k = 2^k \) and \( t_k = \frac{R_k}{2} \). Let \( k_0 \) be the unique integer such that \( R_{k_0-2} \leq R < R_{k_0-1} \). Note that for all \( k \geq k_0 \), there holds \( R_k \geq R + t_k \). Let \( \chi \) be a smooth cut-off function that is supported in [1, 3] such that for all \( r > 0 \), \( \sum_{k \in \mathbb{Z}} \chi(r/R_k) = 1 \). We write \( \chi_k(r) = \chi(r/R_k) \). Let \( \tilde{\chi} \) be a nonzero, smooth and nonnegative function with support inside [2, 3].

There exists a family of smooth functions \( \Phi_k \) for \( k \geq -1 \), and \( \bar{\Phi}_k \) for \( k \leq -1 \) supported in \([2R_k, 3R_k]\), such that \( |\partial^j \Phi_k| \lesssim R^{-j} \) and \( |\partial^j \bar{\Phi}_k| \lesssim R^{-j} \) for \( j = 0, 1, 2 \), and such that for all \( k \leq -1, \int \Gamma \Phi_k = R_k^2 \) and \( \int \Lambda W \bar{\Phi}_k = 0 \), and for all \( k \geq -1, \int \Lambda W \Phi_k = R_k^2 \) and \( \int \Gamma \Phi_{-1} = 0 \). The proof of this fact, a direct consequence of (4.3), and from the fact that \( \Lambda W(r) \approx 1/r^4 \) for large \( r \), is omitted. Note that we do not need to require \( \int \Gamma \Phi_k = 0 \) for \( k \geq 0 \).
Fix $k \in \mathbb{Z}$. Applying the fundamental Theorem of Calculus for $\partial_t u_L$ and then $u_L$ between $t = 0$ and $t = R_k$, using (4.11) and $\partial_t u(0) = 0$ as $u$ is even in time, gives:

$$(4.38) \quad -\Delta u_0 + V u_0 = -\frac{1}{t_k} \partial_t u_L(t_k) - (-\Delta + V) \left( \int_0^{t_k} (1 - \frac{t}{t_k}) \partial_t u_L(t) dt \right).$$

Let $k \geq k_0$. It follows from (4.11) and (4.35) that for all $r \geq R_k$ if $k \geq 0$, and for all $R_k \leq r \leq 2$ if $k \leq -1$:

$$(-\Delta + V) \left( \int_0^{t_k} (1 - \frac{t}{t_k}) \partial_t u_L(t) dt \right) = (-\Delta + V) \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right).$$

In (4.35), we introduce the notation $c_k = -t_k^{-1} \alpha_k(t_k)$ for all $k \geq k_0$, and $\tilde{c}_k = -t_k^{-1} \tilde{\alpha}_k(t_k)$ for $k_0 \leq k \leq -1$, and $\tilde{c}_k = 0$ for $k \geq 0$, and we obtain the following identity for all $r \geq R_k$ if $k \geq 0$, and for all $R_k \leq r \leq 2$ if $k \leq -1$:

$$(-\Delta + V)u_0 = c_k \Lambda \Gamma + \tilde{c}_k \Gamma - \frac{1}{t_k} g_k(t_k) - (-\Delta + V) \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right).$$

We equal the two above identities obtained for $k$ and $k + 1$ respectively on the interval $[2R_k, \infty)$ if $k \geq 0$, and on the interval $[2R_k, 2]$ if $k \leq -1$, giving:

$$(4.39) \quad (c_{k+1} - c_k) \Lambda \Gamma + (\tilde{c}_{k+1} - \tilde{c}_k) \Gamma = -\frac{1}{t_k} g_k(t_k) + \frac{1}{t_{k+1}} g_{k+1}(t_{k+1}) + (-\Delta + V) \left( \int_0^{t_{k+1}} (1 - \frac{t}{t_{k+1}}) g_{k+1}(t) dt - \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right).$$

If $k \geq -1$, we integrate (4.39) against $\Phi_k$, and find after using $\tilde{c}_k = 0$ for $k \geq 0$, integrating by parts and estimating using (4.36), that $(c_{k+1} - c_k) R_k^2 = O(R_k^2 \|\partial_t u_L\|_{Y_R})$, and hence:

$$|c_{k+1} - c_k| \lesssim \|\partial_t u_L\|_{Y_R} \quad \text{for all } k \geq -1.$$

If $k \leq -2$, we integrate (4.39) against $\Phi_{-1}$ and find after integrating by parts and estimating using (4.36), that $(c_{k+1} - c_k) \frac{1}{4} = O(R_k^{-1} \|\partial_t u_L\|_{Y_R})$, so that:

$$|c_{k+1} - c_k| \lesssim R_k^{-1} \|\partial_t u_L\|_{Y_R} \quad \text{for all } k \leq -2.$$

Let $k_1 = \max(k_0, 0)$. We deduce from the two inequalities above that for all $k \in \mathbb{Z}$:

$$(4.40) \quad c_k = c_k + c'_k,$$

with, if $k_0 \leq 0$,

$$c'_k = \begin{cases} O(k \|\partial_t u_L\|_{Y_R}) & \text{for } k \geq 0, \\ O(R_k^{-1} \|\partial_t u_L\|_{Y_R}) & \text{for } k_0 \leq k \leq -1, \end{cases}$$

or, if $k_0 > 0$

$$c'_k = O(\|k - k_0\| \|\partial_t u_L\|_{Y_R})) \quad \text{for } k \geq k_0.$$

If $k \leq -1$, we integrate (4.39) against $\tilde{\Phi}_k$ and find, after integrating by parts and estimating using (4.36), that $(\tilde{c}_{k+1} - \tilde{c}_k) R_k^2 = O(R_k^2 \|\partial_t u_L\|_{Y_R})$, and get that:

$$|\tilde{c}_{k+1} - \tilde{c}_k| \lesssim \|\partial_t u_L\|_{Y_R}.$$

We deduce from the inequality above and from the fact that $\tilde{c}_0 = 0$, that, when $k_0 \leq -1$, then for all $k_0 \leq k \leq -1$:

$$(4.41) \quad |\tilde{c}_k| \lesssim |k| \|\partial_t u_L\|_{Y_R}.$$

\footnote{If $k_0 \geq 0$ then $\tilde{c}_k = 0$ for all $k$ by convention.}
We then get the following identity for all \( k \geq k_0 \) and \( R_k \leq r \leq 3R_k \):

\[
-\Delta u_0 + V u_0 = c_k \Delta W + c_k' \Delta W + c_k \Gamma - \frac{1}{t_k} g_k(t_k) - (-\Delta + V) \left( \int_0^{t_k} \left( 1 - \frac{t}{t_k} \right) g_k(t) dt \right).
\]

We compute the commutator relation for \( k \in \mathbb{Z} \):

\[
(-\Delta + V) \left( \chi_k \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right) \right) - \chi_k(-\Delta + V) \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right)
= R_k^{-2} (2\chi'' - \Delta \chi) \left( \frac{r}{R_k} \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right) + \partial_r \left( -2R_k^{-1} \chi' \left( \frac{r}{R_k} \right) \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right).
\]

Using the \( \chi \) based partition of unity, we have that on \( [\frac{1}{2} R_k, \infty) \):

\[
(-\Delta + V) u_0 = \sum_{k \geq k_0} \chi_k (-\Delta + V) u_0
= c_k \Delta W + \sum_{k \geq k_0} \left( \chi_k f^1_k + f^2_k \right) + \partial_r \left( \sum_{k \geq k_0} \tilde{g}_k \right) - (-\Delta + V) \left( \sum_{k \geq k_0} \chi_k \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right) \right).
\]

We decompose:

(4.42) \quad u_0 = c_k \Upsilon + \tilde{u}_0 + \bar{u}_0, \quad \bar{u}_0 = - \sum_{k \geq k_0} \chi_k \left( \int_0^{t_k} (1 - \frac{t}{t_k}) g_k(t) dt \right).

The new unknown \( \bar{u}_0 \) solves for \( r \geq \frac{3}{2} R_k \):

(4.43) \quad (-\Delta + V) \bar{u}_0 = f + \partial_r g, \quad f = \sum_{k \geq k_0} \left( \chi_k f_k^1 + f_k^2 \right), \quad g = \sum_{k \geq k_0} \tilde{g}_k.

**Step 2. Solving the elliptic equation.** We estimate each term in (4.43). For the first one, using (4.40), (4.41) and (4.36), and the definition of \( k_0 \) and \( k_1 \), for all \( k \geq k_0 \):

\[
\|f_k^1\|_{L^2(\bar{r} \leq r \leq 3R_k)} \lesssim R_k^{-1} (\log \frac{R_k}{R}) \|\partial_t u\|_{Y_R}.
\]

For the second one, using (4.36) we estimate:

\[
\|f_k^2\|_{L^2(\bar{r} \leq r \leq 3R_k)} \lesssim R_k^{-2} \left( \int_0^{t_k} \|g_k(t)\|_{L^2(\bar{r} \leq r \leq 3R_k)} dt \right) \lesssim R_k^{-1} \|\partial_t u\|_{Y_R}.
\]

Hence

(4.44) \quad \|f\|_{Z_{-4,Rk_0}} \lesssim \|\partial_t u\|_{Y_R}.

We next estimate \( \tilde{g}_k \) using (4.36):

\[
\|\tilde{g}_k\|_{L^2(\bar{r} \leq r \leq 3R_k)} \lesssim R_k^{-1} \left( \int_0^{t_k} \|g_k(t)\|_{L^2(\bar{r} \leq r \leq 3R_k)} dt \right) \lesssim \|\partial_t u\|_{Y_R}.
\]

so that:

(4.45) \quad \|g\|_{Z_{-3,Rk_0}} \lesssim \|\partial_t u\|_{Y_R}.

We estimate similarly to (4.45) that:

(4.46) \quad \|\bar{u}_0\|_{Z_{-2,Rk_0}} \lesssim \|\partial_t u\|_{Y_R}.
We now consider the equation (4.43) for \( \tilde{u}_0 \), and apply Lemma 4.7 using the estimate (4.44) and (4.45) and get:

\[
d_{Z_{-2\frac{1}{2}R_k0}} (\tilde{u}_0, \text{Span}(\Lambda W, \Upsilon)) \lesssim \| \partial_t u \| Y_R.
\]

Injecting the above inequality, using \( 3R_{k_0} \leq 6R \) in the decomposition (4.42), using (4.46), gives the desired bound of Lemma 4.8.

\[\square\]

The next Lemma upgrades the weighted \( L^2 \) bound of Lemma 4.8 into a weighted \( L^2 \) bound for the gradient.

**Lemma 4.9** (Estimating \( \nabla u \) from \( u_0 \) and \( \partial_t u \) for waves around a soliton). There exists \( C > 0 \) such that the following holds true. Assume that \( u_L \) solves (1.1), with \( \tilde{u}(0) \in \mathcal{H} \). Then for any \( R > 0 \):

\[
d_{Z_{-3,24R}} (\partial_t u_0, \text{Span}(\partial_t \Lambda W, \partial_t \Upsilon)) \lesssim d_{Z_{-2,6R}} (u_0, \text{Span}(\Lambda W, \Upsilon)) + \| \partial_t u \| Y_R.
\]

**Proof.** We let \( q = d_{Z_{-2,6R}} (u_0, \text{Span}(\Lambda W, \Upsilon)) + \| \partial_t u \| Y_R \) to ease notations. There exist \( a_0, \tilde{a}_0 \in \mathbb{R} \) such that \( \| u_0 - a_0 \Lambda W - \tilde{a}_0 \Upsilon \|_{Z_{-2,6R}} \leq 2q \). Consider \( u' = u + (\tilde{a}_0 \mathbb{T} - a_0) \Lambda W - \tilde{a}_0 \Upsilon \). Then \( u' \) also solves (1.1), with \( \| u'_0 \|_{Z_{-2,6R}} + \| u'_t \|_{Y_R} \leq 3q \). Hence, it suffices to prove the result of the lemma for \( u'_0 \). Without loss of generality, we thus assume that \( a_0 = \tilde{a}_0 = 0 \) so that:

\[
\| u_0 \|_{Z_{-2,6R}} \lesssim q.
\]

We let for \( k \in \mathbb{Z}, R_k = 2^k \) and \( t_k = R_k/2 \). Let \( k_0 \) be such that \( R_{k_0-1} \leq 3R \leq R_{k_0} \). We recall the decomposition (4.35).

**Step 1.** An energy identity for a projection of \( u \). We have the following energy type equality for suitable radial functions, which is obtained by performing integration by parts:

\[
(4.48) \quad \int \int \psi (\partial_t v - \Delta v + V v) v dt dx = \int \int (|\nabla v|^2 - (\partial_t v)^2) \psi dt dx + \int \left( V + \frac{1}{2} (\partial_t - \Delta) \right) \psi v^2 dt dx.
\]

We let \( \chi \) be a smooth nonnegative radial cut-off function, with \( \chi(r) = 1 \) for \( r \in [3,9] \) and \( \chi(r) = 0 \) for \( r \leq 2 \) and \( r \geq 10 \). We let \( \tilde{\chi} \) be a smooth one-dimensional cut-off, \( \tilde{\chi}(t) = 1 \) for \( |t| \leq 1/2 \) and \( \tilde{\chi}(t) = 0 \) for \( |t| \geq 1 \). We define \( \chi_k(r) = \chi(r/R_k) \) and \( \tilde{\chi}_k(t) = \tilde{\chi}(t/R_k) \), and \( \psi_k(t,r) = \tilde{\chi}_k(t) \chi_k(r) \).

We now pick \( k \geq k_0 \). Observe that there holds \( \text{supp}(\chi_{R_k}) \subset \{ r \geq 6R \} \) and \( \text{supp}(\psi_k) \subset \{ r \geq |t| + R \} \). We introduce

\[
\langle u, v \rangle_{L^2_{\chi_k}} = \int u v \chi_k, \quad \| u \|_{L^2_{\chi_k}} = \sqrt{\langle u, u \rangle_{\chi_k}}.
\]

For each \( k \in \mathbb{Z}, |t| \leq R_k \), we let \( b_k(t) \) and \( \tilde{b}_k(t) \), with \( \tilde{b}_k = 0 \) for \( k \geq 0 \) by convention, be the unique parameters such that:

\[
(4.49) \quad u_L(t) = v_k(t) + b_k(t) \Lambda W + \tilde{b}_k(t) \tilde{\Gamma},
\]

where we recall that \( \tilde{\Gamma} \) is defined by (4.53), and \( v_k \) satisfies the orthogonality conditions:

\[
(4.50) \quad \int v_k(t) \Lambda W \chi_k dx = 0 \quad \text{for all } k \in \mathbb{Z}, \quad \text{and} \quad \int v_k(t) \tilde{\Gamma} \chi_k dx = 0 \quad \text{for all } k \leq -1.
\]

We do a slight abuse since \( u'_0 \) might not belong to \( \mathcal{H} \), but all the computations of the proof are nonetheless valid.
Then, using $(-\Delta + V)\Delta W = 0$, and $(-\Delta + V)\tilde{\Gamma} = 0$ for $r \leq 10$ from (4.53), we obtain that $v_k$ solves the following equation on the support of $\chi_k$, that is, on $\{2R_k \leq r \leq 10R_k\}$:

$$\partial_t v_k = \Delta v_k - V v_k - \partial_t b_k \Delta W - \partial_t \tilde{b}_k \tilde{\Gamma}.$$ 

Hence, from (4.43) and the orthogonality conditions (4.50), we get the energy identity for $v_k$:

$$\iint |\nabla v_k|^2 \psi_k dt dx = \iint (\partial_t v_k)^2 \psi_k dt dx - \iint \left(V + \frac{1}{2}(\partial_t - \Delta)\right) \psi_k v_k^2 dt dx.$$ \hspace{1cm} (4.51)

**Step 2. Estimates for $v_k$ and $\partial_t v_k$.** We claim that, for all $|t| \leq R_k$:

$$\|v_k(t)\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \langle \log \frac{R_k}{(R')} \rangle q \quad \text{and} \quad \|\partial_t v_k(t)\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim q.$$ \hspace{1cm} (4.52)

To show it, using the decompositions (4.35) and (4.39), we write:

$$u_0 = v_k(0) + \left[b_k(0)\Delta W + \tilde{b}_k(0)\tilde{\Gamma}\right],$$ \hspace{1cm} (4.53)

$$g_k(t) = \partial_t v_k(t) + \left[(\partial_t b_k(t) - \alpha R_k(t))\Delta W + (\partial_t \tilde{b}_k(t) - \tilde{\alpha} R_k(t))\tilde{\Gamma}\right]$$ \hspace{1cm} (4.54)

where the second equality is for $|t| \leq R_k$. Above, observe that:

$$\|u_0\|_{L^2_{\chi_k}} \lesssim \|u_0\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \langle \log \frac{R_k}{(R')} \rangle \|u_0\|_{Z_{-2,R}^2} \leq R_k \langle \log \frac{R_k}{(R')} \rangle q,$$ \hspace{1cm} (4.55)

$$\|g_k\|_{L^2_{\chi_k}} \lesssim \|g_k\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim \|g_k\|_{L^2_{2R_k}} \leq \|\partial_t u\|_{Y_R} \leq q.$$ \hspace{1cm} (4.56)

The two terms in the right-hand sides of (4.53), and of (4.54), respectively, are orthogonal for the bilinear form $\langle \cdot, \cdot \rangle_{L^2_{\chi_k}}$ from (4.50). Thus, by Pythagoras, (4.55) and (4.56):

$$\|b_k(0)\Delta W + \tilde{b}_k(0)\tilde{\Gamma}\|_{L^2_{\chi_k}} \leq \|u_0\|_{L^2_{\chi_k}} \lesssim R_k \langle \log \frac{R_k}{(R')} \rangle q,$$

$$\|\partial_t b_k(t) - \alpha R_k(t)\Delta W + (\partial_t \tilde{b}_k(t) - \tilde{\alpha} R_k(t))\tilde{\Gamma}\|_{L^2_{\chi_k}} \leq \|g_k(t)\|_{L^2_{\chi_k}} \lesssim q.$$ \hspace{1cm} (4.57)

Given the behaviours of $\Delta W$ and $\tilde{\Gamma}$, notice that for all general $c_1, c_2 \in \mathbb{R}$:

$$\|c_1\Delta W + c_2\tilde{\Gamma}\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim \|c_1\Delta W + c_2\tilde{\Gamma}\|_{L^2_{\chi_k}}.$$ 

Combining, we get:

$$\|b_k(0)\Delta W + \tilde{b}_k(0)\tilde{\Gamma}\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim R_k \langle \log \frac{R_k}{(R')} \rangle q,$$ \hspace{1cm} (4.58)

and:

$$\|\partial_t b_k(t) - \alpha R_k(t)\Delta W + (\partial_t \tilde{b}_k(t) - \tilde{\alpha} R_k(t))\tilde{\Gamma}\|_{L^2(2R_k \leq r \leq 10R_k)} \lesssim q.$$ \hspace{1cm} (4.59)

Injecting (4.55) and (4.57) in (4.53), and injecting (4.56) and (4.58) in (4.54), one obtains the first inequality in (4.52) at $t = 0$, and the second inequality in (4.52). Combined together, they imply the first inequality in (4.52) for all $|t| \leq R_k$. Hence (4.52) is established.

**Step 3. Estimate for $\nabla v_k$.** Using the bounds (4.52), the energy identity (4.51) gives:

$$\iint |\nabla v_k|^2 \psi_k dt dx \lesssim R_k \langle \log \frac{R_k}{(R')} \rangle q^2.$$ \hspace{1cm} (4.54)

By the mean value Theorem and the definition of $\psi$, there exists a time $|t_k| \leq R_k/2$ such that:

$$\int_{3R_k \leq |x| \leq 9R_k} |\nabla v(t_k)|^2 dx \lesssim \langle \log \frac{R_k}{(R')} \rangle q^2.$$ \hspace{1cm} (4.59)
Step 4. Estimate for $\nabla u_0$. For all $k \in \mathbb{Z}$, we have at the time $t_k$ of Step 3, using (4.35), (4.49):

$$u_L(t_k) = v_k(t_k) + b_k(t_k)\lambda W + \tilde{b}_k(t_k)\tilde{\Gamma},$$

$$\partial_t u_L(t_k) = g_k(t_k) + \alpha_k(t_k)\lambda W + \tilde{\alpha}_k(t_k)\tilde{\Gamma},$$

where we use the convention that $\tilde{\alpha}_k(t_k) = 0$ for $k \geq 0$. Hence, by finite speed of propagation, using (4.5), for $|t| \leq t_k$ and $4R_k \leq r \leq 8R_k$, we have

$$u_L(t) = w_k(t) + (b_k(t_k) + (t - t_k)\alpha_k(t_k))\lambda W + (\tilde{b}_k(t_k) + (t - t_k)\tilde{\alpha}_k(t_k))\tilde{\Gamma}$$

where $w_k$ is the solution to

$$\partial_{tt} w_k = \Delta w_k - Vw_k, \quad w_k(t_k) = v_k(t_k), \quad \partial_t w_k(t_k) = g_k(t_k).$$

Introducing $d_k = b_k(t_k) - t_k\alpha_R(t_k)$ and $e_k = \tilde{b}_k(t_k) - t_k\tilde{\alpha}_R(t_k)$ we obtain that for $4R_k \leq r \leq 8R_k$:

(4.60) $$u_0 = w_k(0) + d_k\lambda W + e_k\Gamma$$

Let an extension $(\tilde{w}_{k,0}, \tilde{w}_{k,1})$ be such that $(\tilde{w}_{k,0}, \tilde{w}_{k,1})(r) = (v_k, g_k)(t_k, r)$ for all $r \in [3R_k, 9R_k]$ with

(4.61) $$\|(\tilde{w}_{k,0}, \tilde{w}_{k,1})\|_H^2 \lesssim \int_{3R_k \leq |x| \leq 9R_k} (|\nabla v_k(t_k)|^2 + R_k^{-2}|v_k(t_k)|^2 + |g_k(t_k)|^2)dx,$$

and let $\tilde{w}_k$ solve $\partial_t^2 \tilde{w}_k - \Delta \tilde{w}_k + V\tilde{w}_k = 0$ with data $(\tilde{w}_k(t_k), \partial_t \tilde{w}_k(t_k)) = (\tilde{w}_{k,0}, \tilde{w}_{k,1})$. Then by finite speed of propagation, as $|t_k| \leq R_k/2$ we obtain $\tilde{w}_k(0) = w_k(0)$ for all $4R_k \leq r \leq 8R_k$. Applying standard energy estimates, and then using (4.61) with (4.59), (4.52) and (4.36):

$$\|\tilde{w}_k(0)\|_{L^1_{t_k}} \lesssim \|(\tilde{w}_{k,0}, \tilde{w}_{k,1})\|_H \lesssim (\log \frac{R_k}{R})q.$$ 

Hence, using Hardy, and then that $\tilde{w}_k(0) = w_k(0)$ for all $4R_k \leq r \leq 8R_k$:

(4.62) $$\|\nabla w_k(0)\|_{L^2(4R_k \leq r \leq 8R_k)} + R_k^{-1}\|w_k(0)\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim (\log \frac{R_k}{R})q.$$ 

Using the definition of the $Z_{-2}$ norm, (4.62) and (4.60) we obtain:

$$\|d_k\lambda W + e_k\Gamma\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim R_k(\log \frac{R_k}{R})q,$$

and so using (4.53): $\|d_k\lambda W\|_{L^2(4R_k \leq r \leq 8R_k)} + \|e_k\Gamma\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim R_k(\log \frac{R_k}{R})q$, so that, using (4.4):

$$\|\nabla d_k\lambda W\|_{L^2(4R_k \leq r \leq 8R_k)} + \|\nabla e_k\Gamma\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim (\log \frac{R_k}{R})q.$$ 

Injecting the above inequality and (4.62) in (4.60) one obtains:

$$\|\nabla u_0\|_{L^2(4R_k \leq r \leq 8R_k)} \lesssim (\log \frac{R_k}{R})q.$$ 

This implies the desired result of the Lemma, as $4R_{k_0} \leq 24R$. 

□

The proofs of the main results of this subsection are complete. Indeed, Lemma 4.3 follows from Lemmas 4.1, 4.8 and 4.9. Theorem 1.4 is a direct consequence of Lemmas 4.1 and 4.3.
5. Channels of energy close to a multisoliton

First, note that the solution \( u \) of (1.13) is globally well-posed in \( \mathcal{H} \): the local well-posedness can be proved by Strichartz estimates (3.2) and the fact that \( W \) is in \( L^4 \), and the global well-posedness follows from linearity. The exterior energy (1.3) is well-defined, as an application of Lemma 5.1 using finite speed of propagation.

We divide the proof of Proposition 5.1 into two parts. In §5.2 (see Lemma 5.5), we prove the result for odd in time solutions. In §5.3 we consider even in time solutions (see Lemma 5.7).

We start by some technical preliminaries on the equation (1.13) and on the spaces \( Z_{\alpha,\lambda} \) that will be needed in the proof.

5.1. Preliminaries.

**Lemma 5.1.** For all \( J \in \mathbb{N} \), a constant \( C > 0 \) exists such that for all \( \lambda \in \Lambda_J \) and \( R > 0 \), if \( u \) solves on \( \mathbb{R}^{1+6} \):

\[
\Box u + V_\lambda u = f, \quad \tilde{u}(0) \in \mathcal{H}
\]

where \( f \in L_t^1 L_x^{2}(\mathbb{R} \times \mathbb{R}^6) \) then:

\[
\sup_{t \in \mathbb{R}} \| \tilde{u}(t) \|_{\mathcal{H}_{R+|t|}} + \| u \|_{L_t^1 L_x^4(r \geq R+|t|)} \leq C \left( \| \tilde{u}(0) \|_{\mathcal{H}_R} + \| f \|_{L_t^1 L_x^2(r \geq R+|t|)} \right).
\]

**Proof.** The proof is the same as that of Lemma 2.8 in [DKM19], we omit it.

\[ \square \]

**Claim 5.2.** Let \( f \in \dot{H}^1 \), radial. Then \( |\nabla f| \in Z_{-3,\lambda} \) with

\[
\| \nabla f \|_{Z_{-3,\lambda}} \leq \| f \|_{\dot{H}^1},
\]

and

\[
|f(r)| \lesssim \frac{1}{r^2} \left( 1 + \min_{1 \leq j \leq J} \log \left( \frac{r}{\lambda_j} \right) \right) \| \nabla f \|_{Z_{-3,\lambda}}.
\]

Furthermore, assume that \( \chi^0 \) and \( \chi^1 \) are two cut-off functions, with \( \chi^0(r) = 1 \) for \( r \leq 1 \) and \( \chi^0(r) = 0 \) for \( r \geq 2 \), and \( \chi^1 \) that is compactly supported outside the origin, and denote \( \chi_R^i(r) = \chi^i(r/R) \) for \( R > 0 \) and \( i = 0,1 \), then we have:

\[
\| \chi_R^1 f \|_{\dot{H}^1} \leq C \left( 1 + \min_{1 \leq j \leq J} \log \left( \frac{R}{\lambda_j} \right) \right) \| \nabla f \|_{Z_{-3,\lambda}}
\]

and

\[
\| \nabla (\chi^0_R f) \|_{Z_{-3,\lambda}} \leq C \| \nabla f \|_{Z_{-3,\lambda}}, \quad \| \nabla ((1 - \chi^0_R) f) \|_{Z_{-3,\lambda}} \leq C \| \mathbb{I}_{\{|x| \geq R\}} \nabla f \|_{Z_{-3,\lambda}}
\]

for \( C \) depending on \( \chi^0 \) and \( \chi^1 \) but independent of \( R \).

**Proof.** The inequality (5.1) is a direct consequence of the inequality \( \| \nabla f \|_{Z_{-3,\lambda}} \leq \| \nabla f \|_{L^2} \).

Next, we have

\[
|f(r)| \lesssim \sum_{k \geq 0} \int_{2^k r}^{2^{k+1} r} \left| \frac{\partial f}{\partial \rho} \right| d\rho \lesssim \sum_{k \geq 0} \frac{1}{2^{2k} r^2} \left( \int_{2^k r}^{2^{k+1} r} \left| \frac{\partial f}{\partial \rho} \right|^2 \rho^5 d\rho \right)^{1/2}
\]

\[
\lesssim \sum_{k \geq 0} \frac{1}{(2^k r)^2} \| \nabla f \|_{Z_{-3,\lambda}} \left( 1 + \min_{j \in [1,J]} |\log(2^k r/\lambda_j)| \right)
\]

\[
\lesssim \sum_{k \geq 0} \frac{1}{(2^k r)^2} \| \nabla f \|_{Z_{-3,\lambda}} \left( k + \min_{j \in [1,J]} |\log(r/\lambda_j)| \right),
\]
which yields (5.2). Then, using Leibniz, (5.2), and the compact support of $\chi^1$ outside the origin, we infer:

$$
\|\chi^1_R f\|_{\dot{H}^1} \lesssim \|(\nabla \chi^1_R) f\|_{L^2} + \|\chi^1_R \nabla f\|_{L^2}
$$

$$
\lesssim \|r^{-2} \nabla \chi^1_R\|_{L^2} \left(1 + \left| \min_{1 \leq j \leq J} \log \left(\frac{R}{\lambda_j}\right) \right| \right) \|\nabla f\|_{Z_{-3,\lambda}} + \left(1 + \left| \min_{1 \leq j \leq J} \log \left(\frac{R}{\lambda_j}\right) \right| \right) \|\nabla f\|_{Z_{-3,\lambda}}
$$

which shows (5.3). The proof of (5.4) is similar and we omit it.

□

Lemma 5.3. There exists $C > 0$ such that, for any $\lambda > 0$, for any $u_0 \in \dot{H}^1$, the solution $u$ to $\partial_t^2 u - \Delta u + V(\lambda) u$ with initial data $(u(0), \partial_t u(0)) = (u_0, 0)$ satisfies:

$$
(5.5) \quad \|V u\|_{L^1 L^2(|t| \geq R)} \leq C \|\nabla u_0\|_{Z_{-3}}.
$$

If moreover $u_0(r) = 0$ for all $r \geq R$, for some $R \leq 1$, then:

$$
(5.6) \quad \|V u\|_{L^1 L^2(|t| \geq R)} \leq CR^\frac{1}{2} \|\nabla u_0\|_{Z_{-3}}.
$$

If moreover, $u_0(r) = 0$ for all $r \leq R$, for some $R \geq 1$, then:

$$
(5.7) \quad \|V u\|_{L^1 L^2(|t| \geq R)} \leq CR^{-1} \|\nabla u_0\|_{Z_{-3}}.
$$

Proof. We use a dyadic partition of unity $1 = \sum_{\ell \in \mathbb{Z}} \chi^\ell(r)$, with supp($\chi$) $\subset \left[\frac{1}{2}, 2\right]$. We decompose $u = \sum_{\ell \in \mathbb{Z}} u_\ell$, where $u_\ell$ is the solution to $\partial_t^2 u_\ell - \Delta u_\ell + V(\lambda) u_\ell = 0$ with initial data $(u_\ell(0), \partial_t u_\ell(0)) = (u_{\ell,0}, 0)$, where $u_{\ell,0} = \chi^\ell u_0$. By (5.3) (with $J = 1$, $\lambda_1 = 1$):

$$
\|u_{\ell,0}\|_{\dot{H}^1} \lesssim (1 + |\ell|) \|\nabla u_0\|_{Z_{-3}}.
$$

Since supp$(u_{\ell,0}) \subset [2^{\ell-1}, 2^{\ell+1}]$, we have by finite speed of propagation that supp$(u_{\ell,0}) \subset \left\{ \max(0, 2^{\ell-1} - |t| \leq r \leq 2^{\ell+1} + |t|) \right\}$. Using Hölder and Lemma 5.1, we deduce:

$$
\|V u_\ell\|_{L^1 L^2(|t| \geq R)} \lesssim (1 + |\ell|) \|V \|_{L^1 L^2(S_\ell)} \|\nabla u_0\|_{Z_{-3}}
$$

where we introduced the set $S_\ell = \left\{ \max(|t|, 2^{\ell-1} - |t|) \leq r \leq 2^{\ell+1} + |t| \right\}$. If $\ell \leq 0$ then:

$$
\|V\|_{L^1 L^2(S_\ell)} \leq \|V\|_{L^1 L^2(|t| \leq 2^{\ell+1} + |t|)} \lesssim 2^\frac{\ell}{2}
$$

by (B.1), while if $\ell \geq 1$ then, since $\max(|t|, 2^{\ell-1} - |t|) \geq \max(|t|, 2^{\ell-2})$:

$$
\|V\|_{L^1 L^2(S_\ell)} \lesssim \|V\|_{L^1 L^2(|t| \geq 2^{\ell-2})} \lesssim 2^{-2\ell}
$$

by (B.2). Summing, we get

$$
\|V u\|_{L^1 L^2(|t| \geq R)} \lesssim \sum_{\ell \in \mathbb{Z}} \|V u_\ell\|_{L^1 L^2(|t| \geq R)}
$$

$$
\lesssim \|\nabla u_0\|_{Z_{-3}} \left( \sum_{\ell \leq 0} (1 + |\ell|) 2^{\frac{\ell}{2}} + \sum_{\ell \geq 1} (1 + \ell) 2^{-2\ell} \right) \lesssim \|\nabla u_0\|_{Z_{-3}}.
$$

This is (5.5). If in addition $u_0(r) = 0$ for all $r \geq R$ for some $R \leq 1$, then, introducing $\ell_0$ the only integer such that $2^{\ell_0-1} < R \leq 2^{\ell_0}$, we have $u_\ell = 0$ for $\ell \geq \ell_0 + 1$ and so:

$$
\|V u\|_{L^1 L^2(|t| \geq R)} \lesssim \|\nabla u_0\|_{Z_{-3}} \sum_{\ell \leq \ell_0} (1 + |\ell|) 2^{\frac{\ell}{2}} \lesssim 2^{\ell_0} \|\nabla u_0\|_{Z_{-3}} \lesssim R^\frac{1}{4} \|\nabla u_0\|_{Z_{-3}}.
$$
which establishes (5.6). The proof of (5.7) is similar and we omit it. □

**Lemma 5.4.** For any $J \in \mathbb{N}$, there exists $\gamma^* > 0$ such that the following holds for all $\lambda \in \Lambda_J$ with $\gamma(\lambda) \leq \gamma^*$. For $1 \leq j \leq J$ denote by $\phi_j$ the solution to (1.13) with data $(0, (\Lambda W)_{\lambda_j})$, and by $\psi_j$ the solution to (1.13) with data $((\Lambda W)_{\lambda_j}, 0)$.

Then for any $1 \leq j \leq J$,

(5.8) \[ \sup_{t \in \mathbb{R}} \| \bar{\phi}_j(t) - (t \Lambda W_{\lambda_j}, \Lambda W_{\lambda_j}) \|_{H^1(t)} \lesssim \gamma(\lambda), \]

and

(5.9) \[ \sup_{t \in \mathbb{R}} \| \bar{\psi}_j(t) - (\Lambda W_{\lambda_j}, 0) \|_{H^1(t)} \lesssim \gamma(\lambda)^2 |\log(\gamma(\lambda))|. \]

**Proof.** By scaling invariance, we can assume $\lambda_j = 1$ and then $\lambda_{j+1} < 1 < \lambda_{j-1}$. We introduce $\bar{\phi}_j = \phi_j - t \Lambda W$ that solves:

(5.10) \[ \square \bar{\phi}_j + V_\lambda \bar{\phi}_j = - \sum_{i \neq j} t \Lambda W V_{\lambda_i}, \quad (\bar{\phi}_j(0), \partial_t \bar{\phi}_j(0)) = 0. \]

By Appendix B for $i > j$, $\| t \Lambda W V_{\lambda_i} \|_{L^2_t L^2_{x}([r \geq |t|])} \lesssim \lambda_i^2$, and for $i < j$, $\| t \Lambda W V_{\lambda_i} \|_{L^2_t L^2_{x}([r \geq |t|])} \lesssim \lambda_j^{-1}$, and Lemma 5.1 gives (5.8). The same proof gives (5.9), using that by Appendix B $\| \Lambda W V_{\lambda_i} \|_{L^2_t L^2_{x}([r \geq |t|])} \lesssim \lambda_i^2 |\log(\gamma(\lambda))|$ for $i > j$, while $\| \Lambda W V_{\lambda_i} \|_{L^2_t L^2_{x}([r \geq |t|])} \lesssim \lambda_j^{-2} |\log(\gamma(\lambda))|$ if $i < j$. □

5.2. **Channels of energy around a multisoliton for odd in time solutions.** In this subsection we prove the following result:

**Lemma 5.5** (Channels for odd in time solutions around a multisoliton). For any $J \in \mathbb{N}$, there exist $\gamma^*, C > 0$ such that for all $\lambda \in \Lambda_J$ with $\gamma(\lambda) \leq \gamma^*$, any radially symmetric solution $u$ of (1.13) satisfies:

(5.11) \[ \| \Pi_{L^2 x, \lambda} \partial_t u(0) \|_{L^2}^2 \leq C \left( \sum_{\pm} \lim_{t \to \pm \infty} \int_{|r| \geq |t|} |\nabla u|^2 + \gamma(\lambda)^2 \| \partial_t u(0) \|_{L^2}^2 \right). \]

For fixed $R \geq 0$ and $\mu > 0$, we will denote by $\Pi_{\mu, R}$ the orthogonal projection, in $L^2_R$ on the orthogonal of $\{(\Lambda W)_{\mu}\}$. We have

**Claim 5.6.** For all $\eta \geq 0$ there exists $C_\eta > 0$ such that for all $\mu > 0$ and $u$ a solution of

(5.12) \[ \partial_t^2 u - \Delta u + V(\mu) u = f, \quad |x| > \eta \mu + |t|, \]

with $u(0) \in \mathcal{H}$ and $f \in L^1 L^2$, there holds

\[ \left\| \Pi_{\mu, \eta \mu} \partial_t u(0) \right\|_{L^2_{\eta \mu}}^2 \leq C_\eta \| \mathbf{1}_{|x| > \eta \mu + |t|} f \|_{L^1 L^2}^2 + C_\eta \sum_{\pm} \lim_{t \to \pm \infty} \int_{|x| > \eta \mu + |t|} |\nabla u(t, x)|^2 dx. \]

**Proof.** By scaling, energy estimates, time symmetry and finite speed of propagation, it is sufficient to prove the result assuming that $u(0) \equiv 0$, $\mu = 1$ and $f \equiv 0$. If $\eta = 0$, this is Lemma 4.1 with $R = 0$.

If $\eta > 0$, using Lemma 4.1, one can decompose $u_1 = \partial_t u(0)$ as

(5.13) \[ u_1 = a \Gamma + \beta \Lambda W + \Pi_{L^2_q} u_1, \]

(5.14) \[ \left\| \Pi_{L^2_q} u_1 \right\|_{L^2_q}^2 \lesssim E_{\text{out}}(\eta) := \lim_{t \to +\infty} \int_{|x| > \eta + t} |\nabla u(t, x)|^2 dx. \]
Furthermore, writing $\alpha \Gamma = u_1 - \beta AW - \Pi_{\lambda_j}^+ u_1$, we obtain by (5.15),

\[
\alpha^2 \lesssim E_{\text{out}}(\eta).
\]

Combining:

\[
\|\alpha \Gamma + \Pi_{\lambda_j}^+ u_1\|_{L^2_\eta}^2 \leq C_\eta E_{\text{out}}(\eta).
\]

As $\Pi_{\lambda_j}^+ u_1 = \Pi_{\lambda_j}^+ (\alpha \Gamma + \Pi_{\lambda_j}^+ u_1)$, the above inequality implies the desired result. \hfill \square

**Proof of Lemma 5.5**

**Step 1:** reduction. We can assume

\[
(u, \partial_t u)|_{t=0} = (0, u_1).
\]

Furthermore, we can decompose

\[
u_1 = \Pi_{\lambda_j}^+ u_1 + v_1,
\]

where $v_1 \in \text{span} \{ (\lambda k) \}_{1 \leq k \leq J}$. Letting $v$ be the solution of (1.13) with initial data $(0, v_1)$, we see by (5.8) in Lemma 5.4 that

\[
\lim_{t \to +\infty} \int_{|x| > |t|} |\nabla t_x v(t, x)|^2 dx \lesssim \gamma(\lambda)^2 \|v_1 \|_{L^2}^2 \lesssim \gamma(\lambda)^2 \|u_1 \|_{L^2}^2.
\]

We are thus reduced to the case where

\[
\int u_1 (\lambda k, j) = 0, \ j \in [1, J].
\]

**Step 2:** induction. We prove by induction on $j \in [1, J]$ the following property:

**Induction Property for $j \in [1, J]$**: for any $\varepsilon > 0$ and any $\eta > 0$, there exists $\gamma_j > 0$ such that if $0 < \gamma(\lambda) \leq \gamma_j$ then:

\[
u_1 \|_{L^2_\eta} \leq \varepsilon \|u_1 \|_{L^2} + C_{\varepsilon, \eta} \sqrt{E_{\text{out}}},
\]

where $E_{\text{out}} = \lim_{t \to -\infty} \int_{|x| > |t|} |\nabla t_x (u(t, x))|^2 dx$ (the constant $C_{\varepsilon, \eta}$ depends also on $J$ but we omit this dependence).

We fix $j \in [1, J]$ and assume that $j = 1$, or that $j \geq 2$ and that the induction property holds true for $j - 1$, i.e. that for any $\varepsilon_{j-1} > 0$, for any $\eta_{j-1} > 0$, and for any $\gamma(\lambda)$ small enough we have

\[
u_1 \|_{L^2_{\eta_{j-1}}} \leq \varepsilon_{j-1} \|u_1 \|_{L^2} + C_{\varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{\text{out}}}.
\]

To prove the induction property for $j$, we fix $\varepsilon_j > 0$ and $\eta_j > 0$. Without loss of generality, we can choose $\eta_j > 0$ small enough (depending on $\varepsilon_j$), which we will do later on. Below we use the convention that $\lambda_{j-1} = \infty$ if $j = 1$. We let $\eta_{j-1} > 0$ and $\varepsilon_{j-1} > 0$ to be fixed later on depending on $\varepsilon_j$ and $\eta_j$. In what follows, the dependence of the constants on the parameters $\varepsilon_j$, $\varepsilon_{j-1}$, $\eta_j$ and $\eta_{j-1}$ is indicated by subscripts, but the dependence on $J$ is omitted. The notation $C$ thus stands for a generic constant only depending on $J$. We write

\[
\partial_t^2 u - \Delta u - 2W(\lambda_j) u = 2 \sum_{k \neq j} W(\lambda_k) u.
\]

By the Claim 5.6

\[
\Pi_{\lambda_j, \eta_j \lambda_j} u_1 \|_{L^2_{\eta_j \lambda_j}} \leq C_{\eta_j} \sum_{k \neq j} \|_{\text{span} \{|x| > \eta_j \lambda_j + |t|\}} W(\lambda_k) u \|_{L^2} + C_{\eta_j} \sqrt{E_{\text{out}}}
\]
If $k > j$,

$$
\left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) u \right\|_{L^1 L^2} \leq \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} u \right\|_{L^2 L^4}
$$

\[ \lesssim \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \left\| u_1 \right\|_{L^2 \eta_j \lambda_j} \]

by Lemma 5.1. We have by a direct computation

$$
\left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \lesssim \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} \frac{\lambda_k^2}{r_4^4} \right\|_{L^2 L^4} \lesssim \frac{\lambda_k^2}{\eta_j^2 \lambda_j^2} \lesssim \frac{\gamma(\lambda)^2}{\eta_j^2},
$$

and thus,

$$
(5.20) \quad \forall k \in [j + 1, J], \quad \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) u \right\|_{L^1 L^2} \lesssim \frac{\gamma(\lambda)^2}{\eta_j^2} \left\| u_1 \right\|_{L^2}.
$$

If $k < j$, by Hölder,

$$
(5.21) \quad \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) u \right\|_{L^1 L^2} \leq \left\| \Pi \{ |t| < |x| < \eta_{j-1} \lambda_{j-1} + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \left\| \Pi \{ |x| > |t| \} u \right\|_{L^2 L^4}
$$

\[ + \left\| \Pi \{ |x| > \eta_{j-1} \lambda_{j-1} + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \left\| \Pi \{ |x| > \eta_{j-1} \lambda_{j-1} + |t| \} u \right\|_{L^2 L^4}.
\]

We have since $\lambda_{j-1} \leq \lambda_k$

$$
\left\| \Pi \{ |t| < |x| < \eta_{j-1} \lambda_{j-1} + |t| \} W(\lambda_k) \right\|_{L^2 L^4} = \left\| \Pi \{ |t| < |x| < \eta_{j-1} \lambda_{j-1} + |t| \} W \right\|_{L^2 L^4} = o_{\eta_{j-1} \to 0}(1),
$$

uniformly for $1 \leq k \leq j - 1$, and, by Lemma 5.1

$$
\left\| \Pi \{ |x| > |t| \} u \right\|_{L^2 L^4} \lesssim \left\| u_1 \right\|_{L^2}.
$$

Combining, we obtain for all $1 \leq k \leq j - 1$:

$$
(5.22) \quad \left\| \Pi \{ |t| < |x| < \eta_{j-1} \lambda_{j-1} + |t| \} W(\lambda_k) \right\|_{L^2 L^4} \| u \|_{L^2 L^4} = o_{\eta_{j-1} \to 0}(\| u_1 \|_{L^2})
$$

Next, by Lemma 5.1, $\Pi \{ |x| > \eta_{j-1} \lambda_{j-1} + |t| \} u \|_{L^2 L^4} \leq C \| u_1 \|_{L^2 \eta_{j-1} \lambda_{j-1}}$. Combining with the induction hypothesis (5.18), we obtain

$$
(5.23) \quad \| \Pi \{ |x| > \eta_{j-1} \lambda_{j-1} + |t| \} u \|_{L^2 L^4} \leq C \left( \varepsilon_{j-1} \| u_1 \|_{L^2} + C_{\varepsilon_{j-1}, \eta_{j-1}} \sqrt{E_{out}} \right).
$$

Combining the inequalities (5.21), (5.22), and (5.23), we deduce that

$$
(5.24) \quad \forall k \in [1, j - 1], \quad \left\| \Pi \{ |x| > \eta_j \lambda_j + |t| \} W(\lambda_k) u \right\|_{L^1 L^2} \leq \left( C_{\varepsilon_j, \eta_j} (1) \right) \left\| u_1 \right\|_{L^2} + C_{\varepsilon_j, \eta_j} \sqrt{E_{out}},
$$

where $o_{\eta_{j-1}}(1) \to 0$ as $\eta_{j-1} \to 0$. By (5.19), (5.20) and (5.24), we obtain

$$
\left\| \Pi_{\lambda_j, \eta_j} u_1 \right\|_{L^2 L^2 \eta_j} \leq C_{\eta_j} \left( \varepsilon_{j-1} + o_{\eta_{j-1}}(1) + \frac{\gamma(\lambda)^2}{\eta_j^2} \right) \left\| u_1 \right\|_{L^2} + C_{\varepsilon_j, \eta_j} \sqrt{E_{out}}
$$

$$
(5.25) \quad \leq \frac{\varepsilon_j}{2} \left\| u_1 \right\|_{L^2} + C_{\varepsilon_j, \eta_j} \sqrt{E_{out}}
$$

where the last inequality holds for any $\eta_j, \varepsilon_j > 0$, provided $\varepsilon_{j-1}$ and $\eta_{j-1}$ are chosen small enough depending on $\varepsilon_j$ and $\eta_j$, and then $\gamma$ is taken small enough.
Next, we observe that (since \( u_1 \perp (\Lambda W)_{[\lambda_j]} \) in \( L^2 \)),

\[
\left| \int_{|x|>\eta_j \lambda_j} u_1(\Lambda W)_{[\lambda_j]} \right| = \int_{|x|<\eta_j \lambda_j} u_1(\Lambda W)_{[\lambda_j]} \leq \|u_1\|_{L^2} \left\| (\Lambda W)_{[\lambda_j]} \right\|_{L^2(|x|<\eta_j \lambda_j)} \\
\leq \|u_1\|_{L^2} \|\Lambda W\|_{L^2(|x|<\eta_j \lambda_j)} = o_{\eta_j}(1) \|u_1\|_{L^2},
\]

Together with (5.25) we obtain the desired conclusion (5.17).

**Step 3. Conclusion of the proof.**

Using the conclusion of Step 2, and the same argument as in Step 2 with \( j = J \) and \( \eta_J = 0 \), we obtain that for any \( \varepsilon > 0 \), there exists a constant \( C_{\varepsilon_J} \) such that for any radially symmetric solution \( u \)

\[
\|u_1\|_{L^2} \leq \varepsilon_J \|u_1\|_{L^2} + C_{\varepsilon_J} \sqrt{E_{out}}.
\]

Choosing \( 0 < \varepsilon < 1 \), this implies that \( \|u_1\|_{L^2} \leq \sqrt{E_{out}} \) which is the desired result.

\[\square\]

5.3. **Channels of energy around a multisoliton for even in time solutions.** In this subsection we prove:

**Lemma 5.7.** For any \( J \in \mathbb{N} \), there exist \( \gamma^*, C > 0 \) such that for all \( \lambda \in \Lambda_J \) with \( \gamma(\lambda) \leq \gamma^* \), any radially symmetric solution \( u \) of (1.13) satisfies:

\[
(5.26) \quad \|\nabla \Pi_{H^1,\lambda}^\perp u_0\|_{Z_{-3,\lambda}}^2 \leq C \left( \sum_{\pm} \lim_{t \to \pm \infty} \int_{|r| \geq |t|} |\nabla_{t,x} u|^2 dx + \gamma(\lambda)^4 \|\log \gamma(\lambda)\|_2 \|u_0\|_{Z_{-3,\lambda}}^2 \right).
\]

**Proof. Step 0. Preliminary reduction.**

In all the proof, to lighten notation, we denote

\[
\gamma = \gamma(\lambda) \quad \text{and} \quad E_{out} = \sum_{\pm} \lim_{t \to \pm \infty} \int_{|r| \geq |t|} |\nabla_{t,x} u|^2 dx.
\]

Replacing \( u(t, x) \) by \( u(t, x) + u(-t, x) \), we see that we can assume \( u_1 = 0 \).

We also claim that it suffices to prove the estimate

\[
(5.27) \quad \|\nabla \Pi_{H^1,\lambda}^\perp u_0\|_{Z_{-3,\lambda}} \lesssim \sqrt{E_{out}}
\]

when

\[
(5.28) \quad u_0 = \Pi_{H^1,\lambda}^\perp u_0
\]

Indeed, assume that we have proved (5.26) for all solutions of (1.13) with initial data \((u_0, 0)\) satisfying (5.28). Let then \( u \) be a solution of (1.13) with initial data \((u_0, 0)\) with \( u_0 \in H^1 \) and write \( u_0 = v_0 + w_0 \) with \( w_0 = \Pi_{H^1,\lambda}^\perp u_0 \). Let \( v \) and \( w \) be the corresponding solutions of (1.13).

By our assumption,

\[
(5.29) \quad \|\nabla v_0\|_{Z_{-3,\lambda}}^2 \lesssim \lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{t,x} w(t, x)|^2 dx.
\]

We claim

\[
(5.30) \quad \lim_{t \to \pm \infty} \int_{|x| > |t|} |\nabla_{t,x} v(t, x)|^2 dx \lesssim \gamma^4 \|\log \gamma\|_2 \|v_0\|_{Z_{-3,\lambda}}^2.
\]
By (5.29) and (5.30), we have
\[ \| \nabla u_0 \|_{Z_{-3, \lambda}}^2 \lesssim E_{\text{out}} + \lim_{t \to \infty} \int_{|x| > |t|} |\nabla_{t,x} v(t, x)|^2 \, dx \lesssim E_{\text{out}} + \gamma^4 |\nabla v_0|_{L^2_{-3, \lambda}}^2 \lesssim E_{\text{out}} + \gamma^4 |\nabla v_0|_{L^2_{-3, \lambda}}^2 \left( \| \nabla u_0 \|_{Z_{-3, \lambda}}^2 + \| \nabla v_0 \|_{Z_{-3, \lambda}}^2 \right), \]
which yields the desired conclusion, using the smallness of \( \gamma \).

To conclude Step 0, it remains to prove (5.30). Denoting \( v_0 = \sum_{j=1}^{J} \alpha_j (\Lambda W)_{(\Lambda_j)} \), we see that (5.30) is a direct consequence of
\[ \| \nabla v_0 \|_{Z_{-3, \lambda}} \gtrsim \sup_{1 \leq j \leq J} |\alpha_j| \]
(5.31)
and that if moreover \( \text{supp}(\cdot) \subset [R_j^+, \infty) \) then:
\[ \| f \|_{Z_{-3, \lambda}} \lesssim \| f \|_{Z_{-3, \lambda}} \]
(5.32)
The inequality (5.31) is an easy consequence of the definition of the norm \( Z_{-3, \lambda} \) and the fact that \( \gamma \) is small. The estimate (5.32) follows from (5.9) in Lemma 5.3.

**Step 1. Formulating the induction property.** We introduce the notation for \( 1 \leq j \leq J \):
\[ R_j^+ = \sqrt{\lambda_j \lambda_{j+1}}, \quad R_j^- = \frac{\lambda_{j+1}}{\gamma^2}. \]
Note that \( R_j^- \leq \gamma^2 R_j^+ \). For \( 1 \leq j \leq J \) we define \( \lambda_j = (\lambda_1, \ldots, \lambda_j) \) and, for any \( \lambda > 0 \):
\[ \| f \|_{Z_{-3, \lambda}} = \| f_{(\lambda)} \|_{Z_{-3}}. \]
Observe that for any \( 1 \leq j \leq J \):
\[ \| f \|_{Z_{-3, \lambda_j}} \lesssim \| f \|_{Z_{-3, \lambda_j}}, \]
and that if moreover \( \text{supp}(f) \subset [R_j^+, \infty) \) then:
\[ \| f \|_{Z_{-3, \lambda}} \lesssim \| f \|_{Z_{-3, \lambda_j}}. \]
We fix \( \alpha = 1/16 \). We will prove by induction on \( j = 1, \ldots, J \) the following property:

**Induction Property for \( j \in [1, J] \).** There holds:
\[ \| \Pi_{(r \geq R_j^+)} \nabla u_0 \|_{Z_{-3, \lambda_j}} \lesssim \sqrt{E_{\text{out}} + \gamma^6 \| \nabla u_0 \|_{Z_{-3, \lambda}}} \]
(5.35)
Then, for \( j = J \), (5.35) implies (5.27) as \( R_j^+ = 0 \) and the proof of the Lemma is over.

**Step 2. Proof of the induction property for \( 2 \leq j \leq J - 1 \).** We assume that the induction property (5.35) holds true for \( j - 1 \) for some \( 2 \leq j \leq J - 1 \) and prove in this step that it is true for \( j \) as well. We decompose the solution as:
\[ u = v_j + \tilde{v}_j, \]
(5.36)
where
\[ \partial_t^2 v_j - \Delta v_j + V(\lambda_j) v_j = 0, \quad (v_j(0), \partial_t v_j(0)) = (u_0, 0) \]
and
\[ \partial_t^2 \tilde{v}_j - \Delta \tilde{v}_j + V(\lambda_j) \tilde{v}_j = F(v_j) = -\sum_{i \neq j} V(\lambda_i) v_j, \quad (\tilde{v}_j(0), \partial_t \tilde{v}_j(0)) = (0, 0). \]
Let $\chi$ denote a smooth cut-off function with $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$. To estimate $F(v_j)$, we decompose in three pieces $v_j^m$ for $m = 1, 2, 3$ that each solve

$$\partial_t^2 v_j^m - \Delta v_j^m + V_{(\lambda_j)} v_j^m = 0,$$

with data

$$v_j^1(0) = \chi_{R_{j-1}^{-1}}(1 - \chi_{R_j^{-1}/48})u_0, \quad v_j^2(0) = (1 - \chi_{R_{j-1}^{-1}})u_0, \quad v_j^3(0) = \chi_{R_j^{-1}/48}u_0,$$

(where $\chi_R(r) = \chi(r/R)$), so that $v_j = v_j^1 + v_j^2 + v_j^3$, and hence

$$F(v_j) = F(v_j^1) + F(v_j^2) + F(v_j^3).$$

By finite speed of propagation, $v_j^3(t, r) = 0$ for $r \geq R_j^{-1}/24 + |t|$ and hence:

$$\left\| \mathbb{I}_{\{r \geq R_j^{-1}/24 + |t|\}} F(v_j^3) \right\|_{L^1 L^2} = 0.$$

We claim the following estimates, whose proofs are relegated to Step 4:

$$\left\| \mathbb{I}_{\{r \geq R_j^{-1}/24 + |t|\}} F(v_j^1) \right\|_{L^1 L^2} \lesssim \gamma^\alpha \| \nabla u_0 \|_{Z_{-3, \lambda}},$$

and

$$\left\| \mathbb{I}_{\{r \geq R_j^{-1}/24 + |t|\}} F(v_j^2) \right\|_{L^1 L^2} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{-3, \lambda}}.$$

Hence, applying Lemma 5.1 we infer:

$$\sup_t \| (\tilde{v}_j(t), \partial_t \tilde{v}_j(t)) \|_{H_{R_j^{-1}/24 + |t|}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{-3, \lambda}},$$

and hence because of (5.36):

$$\sum_{\pm} \lim_{t \to \infty} \int_{|x| \geq R_j^{-1}/24 + |t|} |\nabla_{t,x} v_j(t)|^2 dx \lesssim E_{\text{out}} + \gamma^{2\alpha} \| \nabla u_0 \|_{Z_{-3, \lambda}}^2.$$

Therefore, applying Lemma 5.5 to $v_j$, we infer that there exist two constants $c_j, d_j$ such that:

$$u_0 = c_j \Upsilon_{(\lambda_j)} + d_j (\Lambda W)_{(\lambda_j)} + \tilde{u}_0,$$

with

$$\left\| \mathbb{I}_{\{r \geq R_j^{-1}\}} \nabla \tilde{u}_0 \right\|_{Z_{-3, \lambda}} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{-3, \lambda}}.$$

As a consequence

$$c_j^2 \int_{R_j}^{2R_j} |\nabla \Upsilon_{(\lambda_j)}|^2 r^5 dr \lesssim \int_{R_j}^{2R_j} |\nabla u_0|^2 r^5 dr + \int_{R_j}^{2R_j} |\nabla \tilde{u}_0|^2 r^5 dr + d_j^2 \int_{R_j}^{2R_j} |\nabla (\Lambda W)_{(\lambda_j)}|^2 r^5 dr.$$

Using the fact that for $r \approx R_j$ we have $|\nabla (\Lambda W)_{(\lambda_j)}| \lesssim \frac{1}{(R_j)^{\lambda_j^4}}$ and $|\nabla \Upsilon_{(\lambda_j)}| \gtrsim \frac{\sqrt{\gamma}}{(R_j)^{\lambda_j^4}} \lambda_j^4$ we get:

$$c_j^2 \gamma^\frac{\lambda_j^4}{\lambda_j^4 + 1} \lesssim (\log \gamma)^2 \| \nabla u_0 \|_{Z_{-3, \lambda}} + \left( \log \frac{\lambda_j^4 + 1}{\gamma \lambda_j^{1/4}} \right)^2 \left\| \mathbb{I}_{\{r \geq R_j^{-1}\}} \nabla \tilde{u}_0 \right\|_{Z_{-3, \lambda}} + d_j^2 \left( \frac{\lambda_j^4 + 1}{\gamma \lambda_j^4} \right)^2.$$

Combining with the bound (5.41), we deduce

$$|c_j| \lesssim \frac{\lambda_j^4 + 1}{\sqrt{\gamma} \lambda_x^4} \left( \log \left( \frac{1}{\sqrt{\gamma}} \lambda_j^4 \right) \right)^2 \left( \frac{1}{\gamma} \lambda_j^4 \right) \left| d_j \right| + \| \nabla u_0 \|_{Z_{-3, \lambda}} + \sqrt{E_{\text{out}}}.$$
As \( u_0 = \Pi_{H^1,\lambda}^\perp u_0 \) we have \( \int \nabla u_0 \nabla (\Lambda W)_{(\lambda_j)} = 0 \), which, after truncation, using \( |\nabla \Lambda W(r)| \lesssim r \) and \( R_j^- = \frac{\lambda_{j+1}}{\lambda_j} \leq \frac{\gamma^2}{r} \lambda_j \), gives:

\[
\int_{R_j^- \leq |x|} \nabla u_0 \nabla (\Lambda W)_{(\lambda_j)} = \int_{|x| \leq R_j^-} \nabla u_0 \nabla (\Lambda W)_{(\lambda_j)} = O(\gamma^3 |\log \gamma| ||\nabla u_0||_{L^{\infty},x}).
\]

On the other hand, computing the left-hand side above using \((5.36)\), \( \int_{|x| \leq \frac{R_j^-}{\gamma}} |\nabla \Lambda W|^2 \lesssim \frac{R_j^-}{\gamma} \lesssim \gamma^6 \) and the bound \((5.41)\) shows:

\[
\int_{|x| \leq \frac{R_j^-}{\gamma}} |\nabla u_0 \nabla (\Lambda W)_{(\lambda_j)} = d_j \int |\nabla \Lambda W|^2 (1 + O(\gamma^6)) + O \left( |c_j| + \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x} \right).
\]

Combining the two identities above shows:

\[(5.43) \quad |d_j| \lesssim |c_j| + \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x},\]

Combining the two inequalities \((5.42)\) and \((5.43)\) shows:

\[
|d_j| \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x},
\]

and

\[
|c_j| \lesssim \frac{1}{\sqrt{\gamma}} \frac{\lambda_j^2}{\lambda_{j+1}^2} \log \left( \frac{1}{\sqrt{\gamma}} \frac{\lambda_j^2}{\lambda_{j+1}^2} \right) \left( ||\nabla u_0||_{L^{\infty},x} + \sqrt{E_{\text{out}}} \right) \lesssim \frac{1}{\gamma^\frac{3}{8}} \frac{\lambda_j^2}{\lambda_{j+1}^2} \left( ||\nabla u_0||_{L^{\infty},x} + \sqrt{E_{\text{out}}} \right).
\]

As a result, since \( ||\nabla (\Lambda W)_{(\lambda_j)}||_{L^{\infty},x} \lesssim 1 \):

\[(5.44) \quad ||d_j \nabla (\Lambda W)_{(\lambda_j)}||_{L^{\infty},x} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x},\]

and, since for \( r \geq R_j^+ \) we have \( |\nabla \gamma(\lambda_j)| \lesssim \frac{1}{\sqrt{\gamma}} \frac{\lambda_j^2}{R_j^+} \leq \frac{1}{r^2} \frac{\lambda_j^2}{\lambda_{j+1}^2} \), we infer:

\[(5.45) \quad ||\{ r \geq R_j^+ \} c_j \nabla (\gamma W)_{(\lambda_j)}||_{L^{\infty},x} \lesssim \frac{1}{\gamma^\frac{3}{8}} \frac{\lambda_j^2}{\lambda_{j+1}^2} \left( ||\nabla u_0||_{L^{\infty},x} + \sqrt{E_{\text{out}}} \right) \lesssim \gamma^\frac{3}{8} ||\nabla u_0||_{L^{\infty},x} + \gamma^\frac{3}{8} \sqrt{E_{\text{out}}}.
\]

Injecting \((5.41)\), \((5.44)\) and \((5.45)\) in \((5.40)\) shows:

\[
||\{ r \geq R_j^+ \} \nabla u_0||_{L^{\infty},x} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x}.
\]

Combined with the induction hypothesis \( ||\{ r \geq R_{j-1}^+ \} \nabla u_0||_{L^{\infty},x} \lesssim \sqrt{E_{\text{out}}} + \gamma^\alpha ||\nabla u_0||_{L^{\infty},x} \), and the inequality

\[
||\{ r > R_j^+ \} f||_{L^{\infty},x} \lesssim ||\{ r > R_j^+ \} f||_{L^{\infty},x} + ||\{ r > R_j^+ \} f||_{L^{\infty},x}
\]

we obtain \((5.35)\) for \( j \).

**Step 3.** *Proof of the induction property for \( j = 1 \) and \( j = J \).* For \( j = 1 \), the proof is the exact same one as in Step 2, but with the simplification that \( v_j^2 = 0 \) in the decomposition \((5.37)\), so that all computations and estimates of Step 2 are still valid for \( j = 1 \) with the convention that \( R_{j-1}^- = R_{j-1}^+ = \infty \), and hence the result holds true for \( j = 1 \).
For \( j = J \), the proof is again the exact same one, but with the simplification that \( c_j = 0 \) in (5.40) since Lemma 4.5 is applied for \( v_j \) for \( r \geq R_J^- = 0 \). All computations and estimates of Step 2 are then still valid for \( j = J \) with the convention that \( R_J^- = R_J^+ = 0 \), and hence the result holds true for \( j = J \).

**Step 4. Estimating the error terms.** In this step we prove (5.38) and (5.39).

We start with the proof of (5.38). Let \( 1 \leq i \leq J \) with \( i \neq j \). By (5.31) and (5.33) we have:
\[
\| \nabla v_j^i(0) \|_{Z_{3, \lambda_i}} \leq \| \nabla v_j^i(0) \|_{Z_{3, \lambda}} \lesssim \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

If \( i > j \), then notice that \( v_j^i(0, r) = 0 \) for \( r \leq \frac{\lambda}{48 \gamma^2} \leq \frac{R^-}{48} \). Applying (5.7) and a scaling argument, we obtain using the above inequality:
\[
\| \mathbb{I}_{\{r \geq |t|\}} V(\lambda_i) v_j^i \|_{L^1 L^2} \lesssim \gamma^\frac{1}{4} \| \nabla v_j^i \|_{Z_{3, \lambda}} \lesssim \gamma^\frac{1}{4} \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

If \( i < j \), then notice that \( v_j^i(0, r) = 0 \) for \( r \geq 2 \sqrt{\gamma \lambda_i} \geq 2 R_{j-1}^+ \). Applying (5.6), we obtain in this case:
\[
\| \mathbb{I}_{\{r \geq |t|\}} V(\lambda_i) v_j^i \|_{L^1 L^2} \lesssim \gamma^\frac{1}{4} \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

The two inequalities above imply (5.38).

We now prove (5.39). Let \( 1 \leq i \leq J \) with \( i \neq j \). By (5.41) and the induction hypothesis (5.35) at \( j - 1 \):
\[
\| \nabla v_j^2(0) \|_{Z_{3, \lambda_{j-1}}} \lesssim \| \mathbb{I}_{\{r \geq R_{j-1}^+\}} \nabla u_0 \|_{Z_{3, \lambda_{j-1}}} \lesssim \sqrt{E_{out}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

As \( v_j^2(0, r) = 0 \) for \( r \leq R_{j-1}^+ \), by (5.33) and (5.34):
\[
\| \nabla v_j^2(0) \|_{Z_{3, \lambda_j}} \leq \| \nabla v_j^2(0) \|_{Z_{3, \lambda}} \lesssim \| \nabla v_j^2(0) \|_{Z_{3, \lambda_{j-1}}}.
\]

Combining the two inequalities above,
\[
\| \nabla v_j^2(0) \|_{Z_{3, \lambda_j}} \lesssim \sqrt{E_{out}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

Applying (5.5), we obtain:
\[
\| \mathbb{I}_{\{r > |t|\}} V(\lambda_i) v_j^2 \|_{L^1 L^2} \lesssim \sqrt{E_{out}} + \gamma^\alpha \| \nabla u_0 \|_{Z_{3, \lambda}}.
\]

This implies (5.39), what ends the proof of the Lemma.

\( \square \)

6. Channels of energy estimate around the ground state in 8 dimensions

In this section we consider the linearised wave equation in eight dimensions
\[
(6.1) \quad \partial_t^2 u_L - \Delta u_L + V u_L = 0, \quad \bar{u}_L(0) \in \mathcal{H}(\mathbb{R}^8),
\]
where \( V = -\frac{5}{3} W^2 \) with \( W = \left(1 + \frac{|x|^2}{48}\right)^{-3} \). The function \( \Delta W = x \cdot \nabla W + 3W \) belongs to the kernel of \(-\Delta + V\). We claim an analogous channels of energy estimate to (1.10):

**Theorem 6.1.** There exists \( C > 0 \) such that any radial solution \( u_L \) of (6.1) satisfies:
\[
\| \Pi_{L^2} u_1 \|_{Z_{-4}} + \| \Pi_{H^1} u_0 \|_{H^1} \leq C \sqrt{E_{out}}.
\]

We believe this result can be extended to all dimensions \( N \equiv 4 \text{ mod } 4 \) up to a technical refinement of the proof. We only sketch the proof of Theorem 6.1 as a consequence of Lemmas 6.5, 6.6 and 6.7. We highlight only the main differences with that, similar, of Theorem 1.1. We believe an extension for an estimate around a multisoliton like (1.15) could be proved along the lines of Section 5.
6.1. Channels of energy for even in time solutions. The main difference between six and eight dimensions is that the analogue of Proposition 3.3 only holds true for even in time functions.

**Proposition 6.2** (Channels in 8d). Let \( u \in \mathbb{R} \times \mathbb{R}^8 \), solve \( \partial^2_t u - \Delta u = f \) with \( \bar{u}(0) \in \mathcal{H} \) and \( f \in L^1L^2 \). Fix \( R > 0 \) and write \( u_0 = \frac{\bar{u}}{r^2} + \frac{\bar{u}}{r^4} + u_0^\infty \), where \( \int_R^\infty \partial_r u_0^\infty r^{-7+2k}r^dr = 0 \) for \( k = 0, 1 \). Then,

\[
(6.2) \quad \left\| u_0^\infty \right\|_{H^1_R} \lesssim \sqrt{E_{\text{out}}} + \left\| f \right\|_{L^1_tL^2_r(c_{0,R}^\infty)}.
\]

**Proof.** The estimate for \( f = 0 \) is proved in [LSW21]. The extension for nonzero \( f \) is similar to the proof of Proposition 3.3.

\( \square \)

Counter examples to a channel of energy estimate for Equation (6.1) are built on the generalised kernel of \(- \Delta + V\). Namely, in eight dimensions another radial zero of \(- \Delta + V\) is given by a function \( \Gamma \) with

\[
\Gamma(r) \sim cr^{-6} \quad \text{as} \quad r \to 0, \quad \text{and} \quad \Gamma(r) \sim c' \quad \text{as} \quad r \to \infty.
\]

We define \( T_0^\infty = \Lambda W, \ T_0^0 = \Gamma, \) and \( T_1^\infty \) and \( T_1^0 \) by the following Lemma:

**Lemma 6.3.** There exist two functions \( T_1^\infty \) and \( T_1^0 \) solving:

\[
(-\Delta + V)T_1^\infty = -T_0^\infty \quad \text{and} \quad (-\Delta + V)T_1^0 = -T_0^0,
\]

that satisfy for \( c_{1,0}^\infty, c_{1,0}^0, c_{1,0}^0, e_0^0, c_{0,0}^0 \neq 0: \)

\[
\begin{align*}
T_1^\infty(r) & \sim c_{1,0}^\infty r^{-4}, \quad T_1^\infty(r) \sim c_{1,0}^\infty r^{-6}, \\
T_1^0(r) & \sim c_{1,0}^0 r^2, \quad T_1^0(r) \sim c_{1,0}^0 r^{-4}, \\
\tilde{T}_1^0(r) & = T_0^0 - e_0^0 T_1^\infty(r) \sim c_{0,0}^0 r^2.
\end{align*}
\]

**Proof.** It is a standard analysis of the underlying second order ODEs in the radial variable.

\( \square \)

Three explicit solutions of \( \partial^2_t u_L - \Delta u_L +Vu_L = 0 \) such that \( \lim_{t \to \pm \infty} \int_{|x|>R+|t|} \left| \nabla_t \cdot u_L(t,x) \right|^2 dx = 0 \) for any \( R > 0 \) are then given by:

\[
S_0^\infty = T_0^\infty, \quad S_1^\infty = tT_0^0, \quad S_2^\infty = T_1^\infty + \frac{t^2}{2} T_0^\infty.
\]

These are the only ones:

**Lemma 6.4.** Let \( R \geq 0 \). Assume that \( u_L \) is an even in time solution (6.1) such that

\[
\lim_{t \to \pm \infty} \int_{|x|>R+|t|} \left| \nabla_t \cdot u_L(t,x) \right|^2 dx = 0.
\]

Then:

(i) If \( R > 0 \), there exists \( c_0, c_2 \in \mathbb{R} \) such that \( u_L = c_0 S_0^\infty + c_2 S_2^\infty \) on \( \{|x|>R+|t|\} \).

(ii) If \( R = 0 \), there exists \( c_0 \in \mathbb{R} \) such that \( u_L = c_0 S_0^\infty \) for all \( x,t \).

**Proof.** The proof works exactly as that of Lemma 4.2 using the estimate (6.2) and the asymptotics of Lemma 6.3.

\( \square \)
We next introduce, for $\chi^0$ a cut-off with $\chi^0(r) = 1$ for $r \leq 10$ and $\chi^0(r) = 0$ for $r \geq 11$:

\begin{equation}
\Pi^\perp_{\dot{H}^1_{L}} = \begin{cases} 
\Pi_{\dot{H}^1_{L}}(\text{Span}(T_0^\infty, T_1^\infty)) & \text{for } R \geq 1, \\
\Pi_{\dot{H}^1_{L}}(\text{Span}(T_0^\infty, T_1^\infty, \chi^0 T_1^0)) & \text{for } 0 < R < 1, \\
\Pi_{\dot{H}^1_{L}}(\text{Span}(T_0^0)) & \text{for } R = 0.
\end{cases}
\end{equation}

Lemma 6.5. There exists a constant $C > 0$, such that for any even in time solution $u_L$ of (6.1) and any $R \geq 0$:

$$\|\pi^\perp_{\dot{H}^1_{L}} u_L(0)\|_{\dot{H}^1_{L}}^2 \leq C \lim_{t \to \infty} \int_{|x| > R + |t|} \|\nabla_{t,x} u_L\|^2 dx.$$  

Proof. The proof works exactly as that of Lemma 4.1, combining the rigidity result of Lemma 6.3, the channels of energy estimate (6.2) for $|x| > R + |t|$ with $R > 0$, and the following estimate for $R = 0$ established in [CKS14]:

$$\frac{1}{2} \|u_0\|_{\dot{H}^1}^2 \leq E_{\text{out}}^+.$$  

\hfill \square

6.2. Channels of energy for odd in time solutions. If $u_L$ is a solution of (6.1) with initial data in $\mathcal{H}$, we denote:

\begin{equation}
\|u_L\|_{\tilde{\mathcal{Y}}} = \sup_{t \in \mathbb{R}, R > |t|} \|\Pi^\perp_{\dot{H}^1_{L}} u_L(t)\|_{\dot{H}^1_{L}}.
\end{equation}

We introduce the following operators:

\begin{equation}
\mathcal{A}(r) = \int^\infty_r \rho f(\rho) d\rho, \quad \mathcal{A}^{-1}(r) = -\frac{1}{r} \partial_r f(r).
\end{equation}

They are used to state an averaged estimate in the following Lemma, and to lower the dimension in the proof of the next one.

Lemma 6.6 (Estimating $\partial_t u_L$ in average from $u_L$ in $\dot{H}^1$). Recall the notation (6.5). Assume that $u_L$ is a radial solution of (6.1). Then:

\begin{equation}
\|\mathcal{A}(\Pi^\perp_{L^2} \partial_t u(0))\|_{Z^{-2}} \lesssim \|u_L\|_{\tilde{\mathcal{Y}}}.
\end{equation}

Proof. The proof is similar to that of Lemma 4.8 it consists of a first step that we detail since slightly different, in which one expresses $\partial_t u(0)$ in terms of $u(t, x)$ for $|x| > |t|$ on a dyadic scale. For $k \in \mathbb{Z}$, applying the fundamental Theorem of calculus twice between $t = 0$ and $t = R_k = 2^k$, using (6.1) and that $u_L$ is even in time, one obtains the identity:

\begin{equation}
\partial_t u(0) = \frac{1}{R_k} u_L(R_k) + (-\Delta + V) \left( \int^R_0 (1 - \frac{t}{R_k}) u_L(t) dt \right).
\end{equation}

We let $\tilde{u}_k(t) = \Pi^\perp_{\dot{H}^1_{L_k}} u_L(t)$. We decompose for $|t| \leq R_k$:

\begin{equation}
\tilde{u}_k(t) = \alpha^\infty_{1,k} T_0^\infty + \alpha^\infty_{1,k} T_1^\infty + \alpha^0_{1,k} T_1^0 + \tilde{u}_k(t).
\end{equation}

with $\alpha^0_{1,k} = 0$ by convention for all $k \geq 0$. Using Lemma 6.3 and that $\chi^0(r) = 1$ for $r \leq 10$, we obtain that for all $r \geq R_k$ if $k \geq 0$, or for all $R_k \leq r \leq 10$ if $k \leq -1$:

$$(-\Delta + V) \left( \int^R_0 (1 - \frac{t}{R_k}) u_L(t) dt \right) = \int^R_0 \left( 1 - \frac{t}{R_k} \right) (-\alpha^\infty_{1,k} T_0^\infty - \alpha^0_{1,k} T_0^0 + (-\Delta + V) \tilde{u}_k(t) ) dt.$$
Injecting the two above identities in (6.8), recalling that $T_0^0 = \epsilon_0^0 T_1^\infty + \tilde{T}_0^0$ from Lemma 6.3, we obtain that there exist constants $c_{0,k}^\infty, c_{1,k}^\infty, c_{0,k}^0, c_{1,k}^0$ with $c_{0,k}^0 = 0$ and $c_{1,k}^0 = 0$ for $k \geq 0$ such that, for all $r \geq 0$ if $k \geq 0$, or for all $r \leq 10$ if $k \leq -1$:

$$\partial_t u(0) = c_{0,k}^\infty T_0^\infty + c_{1,k}^\infty T_1^\infty + c_{0,k}^0 \tilde{T}_0^0 + c_{1,k}^0 T_1^0 + \frac{1}{R_k} \tilde{u}_k(R_k) + (-\Delta + V) \left( \int_0^R_k (1 - \frac{t}{R_k}) \tilde{u}_k(t) dt \right).$$

The rest of the proof is now very similar to that of Lemma 6.8, one first estimate the same way the constants $c_{1,k}^\infty, c_{0,k}^\infty, c_{0,k}^0$ and $c_{1,k}^0$, and then inject these estimates directly in (6.10), showing (6.7). This last step is actually simpler since no elliptic equation is involved.

One then upgrades the averaged $L^2$ bound of Lemma 6.6 into a weighted $L^2$ bound.

**Lemma 6.7.** Assume that $u_L$ solves (6.1). Then:

$$\|\Pi_{L^2} \partial_t u(0)\|_{Z_4} \lesssim \|A(\Pi_{L^2} \partial_t u(0))\|_{Z_{-2}} + \|u\|_{Y}.$$  

**Proof.** The proof is very similar to that of Lemma 4.9, because one can perform a dimensional reduction leading to a wave equation in 6 dimensions. Indeed, without loss of generality, assume $\partial_t u(0) = \Pi_{L^2} \partial_t u_0$, and then consider the new unknown:

$$v = A \partial_t u.$$  

From the conjugation relation $\Delta_N = A^{-1} \Delta_{N-2} A$, where $\Delta_N = \partial_{rr} + \frac{N-2}{r} \partial_r$, $v$ is an even in time solution on $\mathbb{R}^{1+6}$ of

$$\left\{ \begin{array}{l}
\partial^2_r v - \Delta_6 v + AV A^{-1} v = 0, \\
v(0) = (A \partial_t u(0), 0).
\end{array} \right.$$  

The kernel of the above integro-differential operator is:

$$\left( -\Delta_6 + AV A^{-1} \right) f = 0, \quad \text{for } f \in \text{Span} \left( AT_0^\infty, \int_1^r \rho \Gamma(\rho) d\rho, 1 \right).$$

First, one notices that the initial data is estimated in $Z_{-3}$ by the right-hand of (6.11):

$$\|v(0)\|_{Z_{-2}} = \|A \partial_t u(0)\|_{Z_{-2}}.$$  

Second, one remarks that $\partial_t v$, outside the kernel, can be estimated by $u$. Indeed,

$$\partial_t v = -A(-\Delta_8 + V) u(t) = -r \partial_r u(t) - 6u(t) - AV u(t),$$

We rewrite (6.9) in a unified form for all $k \in \mathbb{Z}$:

$$u(t) = \alpha_{0,k}^\infty(t) T_0^\infty + \alpha_{1,k}^\infty(t) T_1^\infty + \alpha_{1,k}^0(t) T_1^0 \chi^0 + \tilde{u}_k(t),$$

where by convention $\alpha_{1,k}^0(t) = 0$ for $k \geq 0$. We introduce $\tilde{T}_0^0 = (-\Delta_8 + V)(T_1^0 \chi^0)$, and thus, from (6.14) and Lemma 6.3:

$$\partial_t v = \alpha_{1,k}^\infty AT_0^\infty + \alpha_{1,k}^0 \tilde{T}_0^0 - A(-\Delta_8 + V) \tilde{u}_k(t),$$

One can then apply the exact same gain of regularity technique for solutions to the wave equation in 6 dimensions as in the proof of Lemma 4.9.

□
APPENDIX A. A COUNTER-EXAMPLE TO A LINEAR CHANNEL OF ENERGY ESTIMATE

We make precise here the statement of Remark 1.3. This counter-example has similarities with that of [CKS14].

Lemma A.1. There exists a sequence of initial data $(u_{n,0}, 0)$ with $u_{n,0} \in \dot{H}^1(\mathbb{R}^6)$ and:
\[
\sup_{R>0} \|r^{-1} u_{n,0}\|_{L^2(\{R \leq r \leq 2R\})} \to \infty \quad \text{as } n \to \infty
\]
such that the corresponding sequence of solutions $u_{F,n}$ to (1.4) satisfies:
\[
\lim_{t \to \infty} \int_{|x| \geq t} |\nabla_{t,x} u_{F,n}|^2 \, dx \leq 1, \quad \forall n.
\]

Proof. Step 1. Proof assuming a technical claim. Let $\chi$ be a smooth cut-off with $\chi(r) = 0$ for $r \geq 3$ and $r \leq 1$ that generates a partition of unity:
\[
\sum_{k \in \mathbb{Z}} \chi_k(r) = 1, \quad \chi_k(r) = \chi(r/R_k), \quad R_k = 2^k.
\]
We introduce for $(c_k)_k \in \ell^2(\mathbb{Z})$:
\[
v[c_k](t, r) = \sum_k c_k \chi_k \frac{1}{r^2} p_1 \left( \frac{t}{r} \right), \quad p_1(\sigma) = (\sigma^2 - \frac{1}{2})
\]
and $\tilde{v}[c_k]$ the solution to
\[
(\partial_t^2 - \Delta)\tilde{v} = -(\partial_t^2 - \Delta)v, \quad (\tilde{v}(0), \partial_t \tilde{v}(0)) = (0, 0).
\]
We claim that for some $C > 0$ independent of $c_k$:
\[
(A.1) \quad \lim_{t \to \infty} \int_{r \geq t} |\nabla_{t,x} \tilde{v}(t)|^2 \, dx \leq C \sum_k |c_k - c_{k+1}|^2.
\]
Assuming the claim, the Lemma is proved by choosing $u_{F,n} = v[c_{n,k} + \tilde{v}[c_{n,k}]$ with $(c_{n,k})_k \in \ell^2(\mathbb{Z})$ such that $\max_{k \in \mathbb{Z}} |c_{n,k}| \to \infty$ as $n \to \infty$ while $\sum_{k \in \mathbb{Z}} |c_{n,k} - c_{n,k+1}|^2 \leq 1$ uniformly for all $n$. Such an example is given by, for any $0 < \delta < 1/2$:
\[
c_{n,k} = \begin{cases} n^{\frac{1}{2} - \delta} - \frac{1}{2} k - n^{\frac{1}{2} - \delta} & \text{for } 0 \leq k \leq 2n, \\ 0 & \text{for } |k - n| \geq n + 1. \end{cases}
\]

Step 2. Proof of the technical claim A.1. We have from the partition of unity and support properties of $\chi$ that $\chi_k(r) = 1$ for $\frac{3}{2} R_k \leq r \leq 2 R_k$ and $\chi_k(r) + \chi_{k+1}(r) = 1$ for $2 R_k \leq r \leq 3 R_k$. Hence $\Delta \chi_k(r) = -\Delta \chi_{k+1}(r)$ and $\partial_r \chi_k = -\partial_r \chi_{k+1}(r)$ for $2 R_k \leq r \leq 3 R_k$. Using this and $(\partial_t^2 - \Delta)(r^{-2} p_1(r/t)) = 0$ we obtain:
\[
(\partial_t^2 - \Delta)v = \sum_k (c_{k+1} - c_k) f_{\{R_k\}},
\]
\[
f\{R_k\}(t, r) = \frac{1}{R_k^3} f \left( \frac{t}{R_k}, \frac{r}{R_k} \right),
\]
\[
f(t, r) = \left( \Delta \chi \frac{1}{r^2} p_1 \left( \frac{t}{r} \right) + 2 \partial_r \chi \partial_r \left( \frac{1}{r^2} p_1 \left( \frac{t}{r} \right) \right) \right) 1(2 \leq r \leq 3).
\]
Therefore,
\[ \bar{v} = - \sum_k (c_{k+1} - c_k) w(R_k), \]
\[ w(R_k)(t, r) = \frac{1}{R_k^2} w \left( \frac{t}{R_k}, \frac{r}{R_k} \right), \]
\[ (\partial_t^2 - \Delta) w = f, \quad \text{with } (w(0), \partial_t w(0)) = (0, 0). \]

We recall that for any solution to \( \partial_t^2 u - \Delta u = F \), with \( \bar{u}(0) \in \mathcal{H} \), and \( F \in L^1 L^2(r > |t|) \), there exists a radiation profile \( G_+ \in L^2([0, \infty)) \) satisfying
\[ \lim_{t \to \infty} \int_t^{\infty} \left| \frac{1}{\sqrt{r}} \partial_t u(t, r) - G_+(r - |t|) \right|^2 dr = 0, \]
see [DKM20]. Let \( G_+[w] \) be the radiation profile of \( w \). Since \( \partial_t f \in L^1 L^2(r \geq |t|) \), then \( G_+[w] \in H^1([0, \infty)) \), see again [DKM20]. Due to support considerations and finite speed of propagation, it has compact support on the right: \( G_+[w](\rho) = 0 \) for all \( \rho \geq \rho_0 \), for some \( \rho_0 > 0 \). Moreover the radiation profile of \( w_k \) is:
\[ G_+[w_k] = G_+,R_k, \quad G_+,R_k(\rho) = \frac{1}{\sqrt{R_k}} G_+[w](\frac{\rho}{R_k}). \]
Hence the radiation emitted by \( \bar{v} \) as \( t \to \infty \) is:
\[ G_+[\bar{v}] = \sum_{k \in \mathbb{Z}} (c_{k+1} - c_k) G_+,R_k. \]
Applying \([A.2]\) to \( \bar{v} \), introducing \( d_k = c_{k+1} - c_k \) and using a scaling argument:
\[ \lim_{t \to \infty} \int_{r \geq |t|} \left| \nabla_{t,x} \bar{v} \right|^2 dx = 2 \left\| G_+[\bar{v}] \right\|_{L^2([0, \infty))}^2 = \sum_{k,l} d_k d_l \left\langle G_+,R_k, G_+,R_l \right\rangle_{L^2([0, \infty))} = \sum_{k,l} d_k d_l \left\langle G_+,R_0, G_+,R_{l-k} \right\rangle_{L^2([0, \infty))}. \]
Let \( e_k = \left\langle G_+,R_0, G_+,R_k \right\rangle_{L^2([0, \infty))} \). Then, since \( G_+[w] \) has compact support and is continuous,
\[ e_k \sim R_k^\frac{1}{2} G_+[w](0) \int_0^\infty G_+[w] d\rho \quad \text{as } k \to -\infty, \quad e_k \sim R_k^{-\frac{1}{2}} G_+[w](0) \int_0^\infty G_+[w] d\rho \quad \text{as } k \to \infty. \]
Hence \( e_k \in l^1 \), and, by Young inequality for convolution followed by Cauchy-Schwarz:
\[ \lim_{t \to \infty} \int_{r \geq |t|} \left| \nabla_{t,x} \bar{v} \right|^2 dx = \sum_k d_k \langle d_*, e_- \rangle_k \lesssim \sum_k |d_k|^2. \]
This proves the claim \([A.1]\).
\[ \square \]

Appendix B. A few estimates

Lemma B.1. If \( R < R' < \lambda \),
\[ \left\| W(\lambda) \mathbb{1}_{\{R+|t|<|x|<R'+|t|\}} \right\|_{L^2 L^4} \lesssim \left( \frac{R' - R}{\lambda} \right)^{1/4}. \]
If \( R \geq 1 \):
\[ \left\| W \mathbb{1}_{\{\max(|t|,R)<|x|\}} \right\|_{L^2 L^4} \lesssim R^{-2}. \]
Proof. The proof is by direct computations, using that $W$ and $\Lambda W$ are bounded and of order $1/|x|^4$ at infinity. We sketch the proof of (B.1). By scaling, we can assume $\lambda = 1$. Then

$$
\left\| W \mathbb{1}_{\{R+|t|<|x|<R'+|t|\}} \right\|_{L^2 L^4} \lesssim \left\| \mathbb{1}_{\{|t|<1\}} W \mathbb{1}_{\{R+|t|<|x|<R'+|t|\}} \right\|_{L^2 L^4}
$$

$$
+ \left\| \frac{1}{|x|^{\frac{4}{3}}} \mathbb{1}_{\{|t|>1\}} W \mathbb{1}_{\{R+|t|<|x|<R'+|t|\}} \right\|_{L^2 L^4} \lesssim \left( \frac{R'-R}{\lambda} \right)^{1/4}.
$$

To prove (B.2), we decompose:

$$
\left\| W \mathbb{1}_{\{\max(|t|,R)<|x|\}} \right\|_{L^2 L^4}^2 = \int_{|t| \leq R} \left\| W \mathbb{1}_{\{|x| \geq R\}} \right\|_{L^4}^2 dt + \int_{|t| \geq R} \left\| W \mathbb{1}_{\{|x| \geq |t|\}} \right\|_{L^4}^2 dt
$$

$$
\lesssim \int_{|t| \leq R} R^{-5} dt + \int_{|t| \geq R} |t|^{-5} dt \lesssim R^{-4}.
$$

□

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