A brief introduction to Gromov’s notion of hyperbolic groups

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dedicated to Leon Ehrenpreis and Mitchel Taibleson

The basic reference is [13], and see also [6, 7, 11, 12, 14, 16, 24], for instance.

Basic concepts

Let $\Gamma$ be a group, and let $F$ be a finite set of elements of $\Gamma$. By a word over $F$ we mean a formal product of elements of $F$ and their inverses. Every word over $F$ determines an element of the group $\Gamma$, simply using the group operations. The “empty word” is considered a word over $F$, which corresponds to the identity element of $\Gamma$.

If $z$ is a word over $F$, then the length of $z$ is denoted $L(z)$ and is the number of elements of $F$ such that they or their inverses are used in $z$, counting multiplicities. A word $z$ is said to be irreducible if it does not contain an $\alpha \in F$ next to $\alpha^{-1}$, i.e., so that all obvious cancellations have been made. If a word $z$ over $F$ corresponds to the identity element of $\Gamma$, then $z$ is said to be trivial.

A finite subset $F$ of a group $\Gamma$ is a set of generators of $\Gamma$ if every element of $\Gamma$ corresponds to a word over $F$. A group is said to be finitely-generated if it has a finite set of generators. Let us make the convention that a generating set $F$ of a group $\Gamma$ should not contain the identity element of $\Gamma$.

Suppose that $\Gamma$ is a group and that $F$ is a finite set of generators of $\Gamma$. The Cayley graph associated to $\Gamma$ and $F$ is the graph consisting of the elements of $\Gamma$ as vertices with the provision that $\gamma_1, \gamma_2$ in $\Gamma$ are adjacent if $\gamma_2 = \gamma_1 \alpha$, where $\alpha$ is an element of $F$ or its inverse. Thus this relation is symmetric in $\gamma_1$ and $\gamma_2$. 

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A finite sequence \( \theta_0, \theta_1, \ldots, \theta_k \) of elements of \( \Gamma \) is said to define a path if \( \theta_j, \theta_{j+1} \) are adjacent in the Cayley graph for each \( j, 0 \leq j \leq k - 1 \). The length of this path is defined to be \( k \). We include the degenerate case where \( k = 0 \), so that a single element of \( \Gamma \) is viewed as a path of length 0.

If \( \phi, \psi \) are elements of \( \Gamma \), then the distance between \( \phi \) and \( \psi \) is defined to be the shortest length of a path that connects \( \phi \) to \( \psi \). In particular, note that for any two elements \( \phi, \psi \) in \( \Gamma \) there is a path which starts at \( \phi \) and ends at \( \psi \). To see this, one can write \( \psi \) as \( \phi \beta \) for some \( \beta \) in \( \Gamma \), and then use the assumption that \( \Gamma \) is generated by \( F \) to obtain a path from \( \phi \) to \( \psi \) one step at a time.

These definitions are invariant under left translations in \( \Gamma \). In other words, if \( \delta \) is any fixed element of \( \Gamma \), then the transformation \( \gamma \mapsto \delta \gamma \) on \( \Gamma \) defines an automorphism of the Cayley graph, and it also preserves distances between elements of \( \Gamma \). This follows from the definitions, since the Cayley graph was defined in terms of right-multiplication by generators and their inverses.

A basic fact is that this definition of distance does not depend too strongly on the choice of generating set \( F \), in the sense that if one has another finite generating set, then the two distance functions associated to these generating sets are each bounded by a constant multiple of the other. This is not difficult to check, by expressing each generator in one set as a finite word over the other set of generators. There are only a finite number of these expressions, so that their maximal length is a finite number.

Let us continue with the assumption that we have a fixed generating set \( F \) for the group \( \Gamma \). Suppose that \( R \) is a finite set of words over \( F \). We say that \( R \) is a set of relations for \( \Gamma \) if every element of \( R \) is a trivial word. The inverses of elements of \( R \) are also then trivial words, as well as conjugates of elements of \( R \). That is, if \( r \) is an element of \( R \) and \( u \) is any word over \( F \), then \( u r u^{-1} \) is the conjugate of \( r \) by \( u \), and it is a trivial word since \( r \) is. Products of conjugates of elements of \( R \) and their inverses are trivial words too, as well as words obtained from these through cancellations, i.e., by cancelling \( \alpha \alpha^{-1} \) and \( \alpha^{-1} \alpha \) whenever \( \alpha \) is an element of \( F \). The combination of \( F \) and a set \( R \) of relations defines a presentation of \( \Gamma \) if every word over \( F \) which corresponds to the identity element of \( \Gamma \) can be obtained in this manner. The empty word is viewed as being equal to the empty product of relations, so that it is automatically included. A group \( \Gamma \) is said to be finitely-presented if there is a presentation with a finite set of generators and a finite set of relations. For instance, if \( \Gamma \) is the free group with generators in \( F \), then one
can take \( R \) to be the set consisting of the empty word, and this defines a presentation for \( \Gamma \).

Let us call a word over \( F \) trivial if it corresponds to the identity element of \( \Gamma \). Suppose that \( w \) is a trivial word, with

\[
(1) \quad w = \beta_1 \beta_2 \cdots \beta_n,
\]

where each \( \beta_i \) is an element of \( F \) or an inverse of an element of \( F \). This leads to a path \( \theta_0, \theta_1, \ldots, \theta_n \), where \( \theta_0 \) is the identity element of \( \Gamma \) and \( \theta_j \) is equal to \( \beta_1 \beta_2 \cdots \beta_j \) when \( j \geq 1 \). Because \( w \) is a trivial word, \( \theta_n \) is also the identity element in \( \Gamma \), which is to say that this path is a loop that begins and ends at the identity element.

Fix a finite set \( R \) of relations, so that \( F \) and \( R \) give a presentation for \( \Gamma \). Let \( w \) be a trivial word over \( F \) which is also irreducible. Define \( A(w) \) to be the smallest nonnegative integer \( A \) for which there exist relations \( r_1, r_2, \ldots, r_k \) in \( R \), integers \( b_1, b_2, \ldots, b_k \), and words \( u_1, u_2, \ldots, u_k \) over \( F \) such that the expression

\[
(2) \quad u_1 r_1^{b_1} u_1^{-1} u_2 r_2^{b_2} u_2^{-1} \cdots u_k r_k^{b_k} u_k^{-1}
\]

can be reduced to \( w \) after cancellations as before,

\[
(3) \quad \sum_{j=1}^{k} L(u_j) \leq A,
\]

and

\[
(4) \quad \sum_{j=1}^{k} |b_j| L(r_j)^2 \leq A.
\]

Here if \( z \) is a word over \( F \) and \( b \) is an integer, then \( z^b \) is defined in the obvious manner, by simply repeating \( z \) \( b \) times when \( b \geq 0 \), or repeating \( z^{-1} \) \(-b\) times when \( b < 0 \). A representation of this type for \( w \) necessarily exists, since \( F \) and \( R \) give a presentation for \( \Gamma \).

The group \( \Gamma \) is said to be hyperbolic if there is a nonnegative real number \( C_0 \geq 0 \) so that

\[
(5) \quad A(w) \leq C_0 L(w)
\]

for all irreducible trivial words \( w \). The property of hyperbolicity does not depend on the choice of finite presentation for \( \Gamma \), and in fact there are other definitions for which one only needs to assume that \( \Gamma \) is finitely generated,
and the existence of a finite presentation is then a consequence. This characterization of hyperbolicity is discussed in Section 2.3 of [13]. Some examples of hyperbolic groups are finitely-generated free groups and the fundamental groups of compact connected Riemannian manifolds without boundary and strictly negative curvature. In particular, this includes the fundamental group of a closed Riemann surface with genus at least 2.

**Spaces of homogeneous type**

Let us digress now a bit and review some notions from real-variable harmonic analysis. Let $M$ be a nonempty set. A nonnegative real-valued function $d(x, y)$ on the Cartesian product $M \times M$ is said to be a *quasimetric* if $d(x, y) = 0$ exactly when $x = y$, $d(x, y) = d(y, x)$ for all $x, y \in M$, and

$$d(x, z) \leq C \left( d(x, y) + d(y, z) \right)$$

(6)

for some positive real number $C$ and all $x, y, z \in M$. If this last condition holds with $C = 1$, then $d(x, y)$ is said to be a *metric* on $M$.

If $d(x, y)$ is a quasimetric on $M$ and $a$ is a positive real number, then $d(x, y)^a$ is also a quasimetric on $M$. If $d(x, y)$ is a metric on $M$ and $a$ is a positive real number such that $a \leq 1$, then $d(x, y)^a$ is a metric on $M$ too. These statements are not difficult to verify. There is a very nice result going in the other direction, which states that if $d(x, y)$ is a quasimetric on $M$, then there are positive real numbers $C', \delta$ and a metric $\rho(x, y)$ on $M$ such that

$$C'^{-1} \rho(x, y)^\delta \leq d(x, y) \leq C' \rho(x, y)^\delta$$

(7)

for all $x, y \in M$. See [20].

If $d(x, y)$ is a quasimetric on $M$ and $f$ is a real-valued function on $M$, then $f$ is said to be *Lipschitz* if there is a nonnegative real number $L$ such that

$$|f(x) - f(y)| \leq L d(x, y)$$

(8)

for all $x, y \in M$. In general, for a quasimetric, there may not be any nonconstant Lipschitz functions. This is the case when $M = \mathbb{R}^n$ equipped with the quasimetric $d(x, y)^a$ with $a > 1$, for which any Lipschitz function would have to have first derivatives equal to 0 everywhere. However, if $d(x, y)$ is a metric, then $f_p(x) = d(x, p)$ satisfies the Lipschitz condition with $L = 1$ for all $p \in M$. This can be checked using the triangle inequality.
If $d(x, y)$ is a quasimetric on $M$, then one has many of the same basic notions as for a metric, such as convergence of sequences, open and closed sets, dense subsets, and so on. One should be a bit careful with some of the standard results, since for instance it is not so clear that an open ball defined using a quasimetric is an open set, as in the situation of ordinary metrics. At any rate, it still makes sense to say that $M$ is separable with respect to a quasimetric if it has a subset which is at most countable and also dense, and one can define the topological dimension for $M$ as in [17]. The diameter of a subset can be defined in the usual manner using the quasimetric, and this permits one to define the Hausdorff dimension of a nonempty subset of $M$. A famous result about metric spaces is that the topological dimension is always less than or equal to the Hausdorff dimension. See Chapter VII of [17]. This does not work for quasimetrics in general, and it cannot possibly work. For if $(M, d(x, y))$ is a quasimetric space with Hausdorff dimension $s$ and $a$ is a positive real number, then $(M, d(x, y)^a)$ has Hausdorff dimension $s/a$, while the topological dimension of $(M, d(x, y)^a)$ is the same as that of $(M, d(x, y))$.

A quasimetric space $(M, d(x, y))$ is said to have the doubling property if there is a positive real number $C_1$ so that every open ball $B(x, r) = \{ y \in M : d(x, y) < r \}$ in $M$ of radius $r$ can be covered by a family of at most $C_1$ open balls of radius $r/2$. By iterating this condition one obtains that for each positive integer $l$ and each open ball $B(x, r)$ there is a family of at most $C_1^l$ open balls of radius $2^{-l}r$ which covers $B(x, r)$. This is a kind of condition of polynomial growth; if one chooses $\alpha \geq 0$ so that $2^{\alpha} = C_1$, then we can say that each open ball $B(x, r)$ can be covered by a family of at most $(2^l)^\alpha$ balls of radius $2^{-l}r$. Note that if $(M, d(x, y))$ has the doubling property, then so does $(M, d(x, y)^a)$ for any positive real number $a$.

Suppose that $(M, d(x, y))$ is a quasimetric space, and that $\mu$ is a nonnegative Borel measure on $M$. Let us assume that open balls in $M$ are Borel sets. Of course, if $d(x, y)$ is a metric, then open balls are open sets, and hence are Borel sets. In practice, the quasimetrics that one would consider do have this property, and anyway one could make adjustments if necessary. One says that $\mu$ is a doubling measure if the $\mu$-measure of open balls is positive and finite, and if there is a positive real number $C_2$ such that

$$\mu(B(x, 2r)) \leq C_2 \mu(B(x, r))$$

for all $x \in M$ and $r > 0$. A basic fact is that if there is a doubling measure on $(M, d(x, y))$, then $(M, d(x, y))$ is doubling as a quasimetric space.
A quasimetric space \((M, d(x, y))\) equipped with a doubling measure \(\mu\) is often called a space of homogeneous type. As in [3, 4], a lot of real-variable methods in harmonic analysis carry over to spaces of homogeneous type. See [28, 30] for the classical setting of harmonic analysis on Euclidean spaces, and see [19, 18, 20, 20, 29] for more information related to real-variable methods, doubling measures, spaces of homogeneous type, etc.

**Spaces at infinity of hyperbolic groups**

Let \(\Gamma\) be a finitely-presented group which is hyperbolic. Associated to \(\Gamma\) is a space \(\Sigma\) which is a kind of “space at infinity” or ideal boundary of \(\Gamma\), consisting of equivalence classes of asymptotic directions in \(\Gamma\). This space is a compact Hausdorff topological space of finite dimension, as on p110-1 of [13], and it contains a copy of the Cantor set as soon as it has at least three elements. If \(\Sigma\) has at most two elements, then \(\Gamma\) is said to be elementary. For a free group with at least two generators the space at infinity is homeomorphic to a Cantor set, while \(\mathbb{Z}\), a free group with one generator, is elementary and has two points in the space at infinity. If \(\Gamma\) is the fundamental group of a closed Riemann surface of genus at least 2, then \(\Sigma\) is homeomorphic to the unit circle in \(\mathbb{R}^2\). More generally, if \(\Gamma\) is the fundamental group of a compact \(n\)-dimensional Riemannian manifold without boundary with strictly negative curvature, then \(\Sigma\) is homeomorphic to the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\).

Actually, the space at infinity is defined for any hyperbolic metric space in [13], and this can be specialized to a hyperbolic group. It is often preferable to work with metric spaces which are “geodesic”, in the sense that any pair of points can be connected by a curve whose length is equal to the distance between the two points. It is often useful to think of a hyperbolic group as acting on a geodesic hyperbolic metric space by isometries, and to use that to study the space at infinity.

It does not customarily seem to be said this way, but I think it is fair to say that what are basically defined on the space at infinity are quasimetrics, at least initially. More precisely, it is more like the logarithm of a quasimetric, or, in other words, there is a one-parameter family of quasimetrics which are powers of each other. In Section 7.2 of [13] one takes a different route, in effect compactifying a geodesic hyperbolic metric space by looking at different measurements of lengths of curves which take densities into account, densities which decay suitably at infinity. For a parameter in the density in an appropriate range, this measurement of lengths of curves leads
to a measurement of distance on the compactification with nice properties, including upper and lower bounds by positive constant multiples of quantities defined more directly. By defining distance in terms of lengths of curves in the compactification one gets an actual metric in particular, i.e., with the usual triangle inequality.

It should perhaps be emphasized that in measuring distances between points at infinity through weighted lengths of curves, the curves are going through the hyperbolic metric space; curves in the space at infinity are another matter, especially with about the correct length.

In nice situations, such as hyperbolic groups, and universal coverings of compact Riemannian manifolds without boundary and strictly negative curvature in particular, there are doubling conditions on the space at infinity. Compare with [25]. There are also interesting measures around, as in [5].

A well-known result of Borel [1, 27] says that simply-connected symmetric spaces can be realized as the universal covering of a compact manifold. If the symmetric space is of noncompact type and rank 1, it has negative curvature, and thus the fundamental group of the compact quotient, which is a uniform lattice in the group of isometries of the symmetric space, is a hyperbolic group. If the symmetric space is a classical hyperbolic space of dimension \( n \), with constant negative curvature, then the space at infinity can be identified with a Euclidean sphere of dimension \( n - 1 \). If the symmetric space is a complex hyperbolic space of complex dimension \( m \), then the space at infinity can be identified topologically with a Euclidean sphere of real dimension \( 2m - 1 \), but the geometry corresponds to a sub-Riemannian or Carnot–Carathéodory space when \( m \geq 2 \), associated to a distribution of hyperplanes in the tangent bundle of the sphere. One can think of the sphere as being the unit sphere in \( \mathbb{C}^m \), and the hyperplanes in the tangent bundle are the maximal complex subspaces. For other symmetric spaces of noncompact type and rank 1, one again obtains topological spheres of dimension 1 less than the real dimension of the symmetric space, and with sub-Riemannian structures coming from distributions of planes of larger codimension in the tangent bundle.

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