Study on Einstein Warped Product space with Quarter Symmetric Connection

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Abstract
We study Einstein warped space with a quarter symmetric connection. As a result, first, we find basic results on curvature, Ricci and scalar tensors with respect to the quarter symmetric connection. Moreover, we prove some results corresponding to second order quarter symmetric connection. Finally, we prove that if \( M \) is an Einstein warped space with nonpositive scalar curvature and compact base with respect to quarter symmetric connection and the warping function satisfy some condition then \( M \) is simply a Riemannian product space.

Key words: Einstein manifold, quarter-symmetric connection, Warped product, Ricci tensor, Hessian tensor, Ricci identity.

1. Introduction
Throughout this paper one considers \( B \) a compact manifold. The notion of Warped product was first introduced by Bishop and O’Neil ([2]) to construct examples of Riemannian Manifolds with negative curvature. It generalizes that of a surface of revolution. Let \((B^n, g_B)\) and \((F^n, g_F)\) be two Riemannian manifolds with a positive smooth function \( f \) on \( B \). We define a metric on the product space \( B \times F \) is given by
\[
g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),
\]
where \( \pi : B \times F \to B \) and \( \sigma : B \times F \to F \) are the projections on the manifold \( B \) and \( F \) respectively and \( * \) denotes pull-back operator on tensors. The product space \( B \times F \) equipped with metric tensor \( g \) is called warped product. It is denote by \( M = B \times_f F \). The function \( f \) is called warping function on \( B \), \( B \) and \( F \) are called the base and fiber of \( M \) respectively. If the warping function \( f \) is constant then the warped product \( B \times_f F \) is a direct product.

A Riemannian manifold \((M^n, g)\) with dimension \((n \geq 2)\) an Einstein manifold provided
\[
Ric = \lambda g,
\]
for some constant \( \lambda \). After taking trace on both sides of (1.2) we have \( \lambda = \frac{r}{n} \), where \( Ric \) and \( r \) denote the Ricci tensor and the scalar curvature of \((M^n, g)\) respectively. Einstein manifold play a crucial role in Riemannian geometry, as well as in general theory of
relativity.

In ([1].p.265), there was an open problem asked by A.L.Besse that is, "Does there exists a compact Einstein warped product space with nonconstant warping function?" and some authors provided the partial answer. In 2002, D. S. Kim ([5]) proved that there does not exist a compact Einstein warped product space with non-constant warping function. In 2003, Kim and Kim [4] gave the negative partial answer of the open problem. He prove that an Einstein warped product space with compact base and non-positive scalar curvature is just Riemannian product. In 2005, Mustafa [17] construct a result for Einstein warped space with no condition on scalar curvature that is the extension of the theorem in [4]. In [18], D. Dumitru gave some obstructions to the existence of compact Einstein warped products. In 2017, F. E. S. Feitosa, A. A. F. Filho and J. N. V. Gomes [19] proved that if warping function on gradient Ricci soliton warped product attains maximum and minimum then it must be Riemannian product.

Now, we will give a brief history and importance of semi-symmetric and quarter-symmetric linear connection on a Riemannian manifolds. In 1924, Friedmann and Schouten [8] introduced the concept of a semi-symmetric linear connection on a differential manifold. In 1932, Hayden gave the definition of a semi-symmetric metric connection with torsion on Riemannian manifold [7]. After that in 1970, K.Yano studied some properties of semi-symmetric metric connection. He prove that a Riemannian manifold is conformally flat iff its curvature tensor with respect to semi-symmetric metric vanishes identically. He also established that a Riemannian manifold is of constant curvature iff it considers a semi-symmetric metric connection for which the manifold becomes a group manifold. A group manifold is a differential manifold if its curvature tensor \( R \) vanishes and torsion tensor \( T \) is covariantly constant with respect to linear connection \( \nabla \). In 1992, N. S. Agashe and M. R. Chaflle [14] studied properties of semi-symmetric non-metric connection on Riemannian manifold.

In 1975, Gloab [10] introduced the concept of quarter symmetric linear connection on a differential manifold with affine connection, which is a generalization of semi-symmetric connection. Later S.C. Rastogi [15] studied some properties of quarter-symmetric metric connection. In 1980, R. S. Mishra and S. N. Pardey [16] studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. They prove that if the curvature tensor of an Einstein manifold \( M \) with respect to a quarter symmetric metric connection vanishes, then \( M \) is projectively flat. They also introduced the necessary and sufficient condition for an Einstein manifold \( M \) with quarter symmetric metric connection to be a group manifold. In 2015, Q. Qu, Y. Wang [6] established some results on multiply warped product space with respect to quarter symmetric connection. In 2016 S. Pahan, B. Pal and A. Bhattacharyya [11] computed the warping functions for a Ricci flat Einstein multiply warped product \( M \) with a quarter symmetric connection for different dimension of \( M \). In [12] they have studied warped products and multiply warped product on quasi-Einstein manifolds with a quarter symmetric connection. After that in 2017 [13], they have also characterized the warping function for a multiply generalized Robertson-Walker space-time to get an Einstein space \( M \) with a quarter symmetric connection for different dimension of \( M \).

Motivated from the above papers arrangement of this paper follows as, In section 2, we give the definition of semi-symmetric, quarter symmetric connection after that give the formula for curvature, Ricci and scalar tensor with respect to quarter symmetric connection. We establishes necessary and sufficient conditions for warped product space to be an Einstein manifold with respect to quarter symmetric connection. In
In section 3, we discuss some results on curvature, Ricci and scalar tensor with respect to quarter symmetric connection and adopt a condition to be symmetric of Ricci tensor with respect to quarter symmetric connection. In section 4, First we define second order quarter symmetric tensor derivative then compute some preliminaries results with respect to second order quarter symmetric connection and last, of this section prove the Ricci identity with respect to quarter symmetric connection. In section 5, we define Hessian of a function $H_f$ on Riemannian manifold with respect to quarter symmetric connection and adopt a condition to be symmetric of $H_f$. After that calculate some results on $H_f$. In last section, firstly we prove a lemma that gives the relation between Hessian and Ricci tensor, secondly we construct a compact Einstein warped space with quarter symmetric connection, finally we prove that if $M$ is an Einstein warped space with nonpositive scalar curvature and compact base with respect to quarter symmetric connection and the warping function satisfy some condition them $M$ is simply a Riemannian product space.

2. Preliminaries

A linear connection $\nabla$ on a Riemannian manifold $(M^n, g)$ is called metric connection if $\nabla g = 0$, otherwise, it is called non-metric connection. A linear connection is symmetric metric connection if and only if it is Levi-Civita connection. A linear connection $\nabla$ on $(M^n, g)$ is said to be semi-symmetric connection if its torsion tensor $T$ of the form

$$T(X,Y) = \pi(Y)X - \pi(X)Y,$$

where $\pi$ is a 1-form with the allied vector field $P$ defined by $\pi(X) = g(X,P)$, for all vector fields $X$ on $M$.

A linear connection $\nabla$ on Riemannian manifold $(M^n, g)$ with Levi-Civita connection $\nabla$ is referred to as quarter-symmetric connection if the torsion tensor $T$ of the linear connection is

$$T(X,Y) = \pi(Y)X - \pi(X)Y - [X,Y],$$

where $\pi$ is a 1-form associated to the vector field $P$ defined by $\pi(X) = g(X,P)$, for all vector fields $X$ on $M$. A quarter-symmetric connection $\nabla$ is called quarter-symmetric metric connection if $\nabla g = 0$. $\nabla$ is called quarter-symmetric non-metric connection if $\nabla g \neq 0$.

The relation between quarter-symmetric metric connection $\nabla$ and the Levi-Civita connection $\nabla$ of $M^n$ and it is given by [10],

$$\nabla_X Y = \nabla_X Y + \lambda_1 \pi(Y)X - \lambda_2 g(Y,P),$$

where $g(X,P) = \pi(X)$ and $\lambda_1 \neq 0, \lambda_2 \neq 0$ are scalar functions.

The equation (2.3) imply that:

when $\lambda_1 = \lambda_2 = 1$, $\nabla$ is a semi-symmetric metric connection,

when $\lambda_1 = \lambda_2 \neq 1$, $\nabla$ is a quarter-symmetric metric connection,
when \( \lambda_1 \neq \lambda_2 \), \( \nabla \) is a quarter-symmetric non-metric connection.

Further, a relation between the curvature tensors \( R \) and \( \overline{R} \) of type (1,3) of the connections \( \nabla \) and \( \overline{\nabla} \) respectively is given by [10],

\[
\overline{R}(X, Y)Z = R(X, Y)Z + \lambda_1 g(Z, \nabla_X P)Y - \lambda_1 g(Z, \nabla_Y P)X,
+ \lambda_2 [g(X, Z)\nabla_Y P - g(Y, Z)\nabla_X P]
+ \lambda_1 \lambda_2 \pi(P)[g(X, Z)\nabla Y - g(Y, Z)\nabla X]
+ \lambda_2^2 [g(Y, Z)\pi(X) - g(X, Z)\pi(Y)]P
+ \lambda_1^2 \pi(Z)[\pi(Y)X - \pi(X)Y],
\]

for any vector field \( X, Y, Z \) on \( M \).

Ricci tensor and scalar tensor with respect to the quarter symmetric connection as follows:

\[
\overline{Ric}(X, Y) = \sum_{i=1}^{n} g(\overline{R}(E_i, X)Y, E_i)
\]

and

\[
\overline{S} = \sum_{i=1}^{n} \overline{Ric}(E_i, E_i)
\]

where \( \{E_1, ..., E_n\} \) is an frame field of \( M \).

**Notation:** Thorough this paper, we will consider the following:

(1) \( M = B \times_f F \), \( dim(B) = n_1 \), \( dim(F) = n_2 \), \( dim(M) = n_1 + n_2 = \pi \).

(2) \( Ric_B \) and \( Ric^F \) denotes the lifts to \( M \) of the Ricci curvature of \( B \) and \( F \) respectively, \( \overline{Ric}^B \) and \( \overline{Ric}^F \) denotes the lifts to \( M \) of the Ricci curvature of \( B \) and \( F \) with respect to the quarter symmetric connection \( \overline{\nabla} \) respectively.

(4) \( Ric_B \) and \( Ric_F \) denotes the Ricci curvature of \( B \) and \( F \) respectively, \( \overline{Ric}_B \) and \( \overline{Ric}_F \) denotes the Ricci curvature of \( B \) and \( F \) with respect to the quarter symmetric connection \( \overline{\nabla} \) respectively.

(3) \( \text{div}, \overline{Ric} \) and \( \overline{\nabla}^f \) represent divergent, Ricci and Hessian of \( f \) with respect to \( \nabla \) respectively.

(4) \( |\nabla_B f|^2_B = g(\nabla_B f, \nabla_B f) \).

Now from Proposition 3.5 [6], we have

**Theorem 2.1.** Let \( M = B \times_f F \) be a warped product, \( dim(B) = n_1 \), \( dim(F) = n_2 \), \( dim(M) = n_1 + n_2 = \pi \). If \( X, Y, Z \in \Gamma(TB), V, W \in \Gamma(TF) \) and \( P \in \Gamma(TB) \), then:

\[
\overline{Ric}(X, Y) = \overline{Ric}^B(X, Y) - n_2 \left[ \frac{B\pi(X, Y)}{f} + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1 \lambda_2 \pi(P) g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_2^2 \pi(X) \pi(Y) \right],
\]

\[
\overline{Ric}(X, V) = \overline{Ric}(V, X) = 0
\]

\[
\overline{Ric}(V, W) = \overline{Ric}^F(V, W) - \left\{ \frac{\Delta_P}{f} + (n_2 - 1) \frac{|\nabla_B f|^2_B}{f} + [(\pi - 1) \lambda_1 \lambda_2 - \lambda_2^3 \pi(P) \right\}
\]
\[ + \lambda_2 \text{div}_B P + \left\{ (\overline{\pi} - 1) \lambda_1 + (n_2 - 1) \lambda_2 \right\} \frac{P}{f} \right\} g(V, W), \]

where \( \text{div}(P) = \sum_{1}^{n_1} \langle \nabla E_k P, E_k \rangle \), and \( E_k, 1 \leq k \leq n_1 \) is an orthonormal base of \( B \).

Then, we get necessary and sufficient condition to be an Einstein manifold with respect to quarter symmetric connection \( \nabla \).

**Corollary 2.2.** The warped product \( M = B \times_f F \) with \( \text{Ric} = \lambda g \) is Einstein if and only if the following conditions hold:

\[
(2.2.1) \quad \text{Ric}_B(X, Y) = \left[ \lambda + n_2 \lambda_2 \frac{P}{f} + n_2 \lambda_1 \lambda_2 \pi(P) \right] g_B(X, Y) + n_2 \left[ \frac{H_f(X, Y)}{f} + \lambda_2 \pi(Y) \right].
\]

\[
(2.2.2) \quad (F, g_f) \text{ is Einstein with } \text{Ric}_F(X, Y) = \lambda g_f(X, Y).
\]

for every \( X, Y \in \Gamma(TB) \) and \( V, W \in \Gamma(\Gamma(TF)) \),

\[
(2.2.3) \quad \lambda' = \lambda f^2 + f \Delta_B f + (n_2 - 1) |\nabla_B f|^2 + f^2 |(\overline{\pi} - 1) \lambda_1 \lambda_2 - \lambda_2^2| \pi(P) + f^2 \lambda_2 \text{div}_B P + f \lambda_2 \pi(P) = 0.
\]

3. Curvature, Ricci and scalar tensor with respect to the quarter symmetric connection

In (2.4), (2.5) and (2.6) we have seen the expression for the curvature, Ricci and scalar tensor with respect to the quarter symmetric connection respectively.

We know that, if \( (M, g) \) be a Riemannian manifold with Levi-Civita connection \( \nabla \) and \( Z \) is a gradient vector field on \( M \) then

\[
(3.1) \quad g(X, \nabla_Y Z) = g(Y, \nabla_X Z),
\]

for any smooth vector field \( X \) and \( Y \) on \( M \).

Now using this result we prove the following lemma.

**Lemma 3.1.** Let \( (M, g) \) be a Riemannian manifold with Levi-Civita connection \( \nabla \) and \( P \) is a gradient vector field on \( (M, g) \) then

\[
(3.2) \quad g(X, \nabla_P P) = (\nabla_{P X}) = \frac{1}{2} \pi(P)(X),
\]

for every smooth vector field \( X \) on \( M \).

Write \( g(\overline{R}(X, Y)Z, W) = \overline{R}(X, Y, Z, W) \) and prove the following proposition.

**Proposition 3.2.** Let \( (M, g) \) is a Riemannian manifold with quarter symmetric connection \( \nabla \), then

\[
(3.2.1) \quad \overline{R}(X, Y)Z = -\overline{R}(X, Z)Y.
\]

\[
(3.2.2) \quad \overline{R}(X, Y, Z, W) = -\overline{R}(Y, X, Z, W),
\]

\[
(3.2.3) \quad \text{If } P \text{ is gradient vector field then,}
\]

\[
\overline{R}(X, Y)Z + \overline{R}(X, Z)Y + \overline{R}(X, Y) = 0.
\]

and

\[
\overline{R}(X, Y, Z, W) + \overline{R}(Y, X, Z, W) + \overline{R}(Z, X, Y) = 0.
\]

**Proof.** The first part is straightforward from the definition of curvature tensor in (2.4) and the second part will follow immediate from the first part. Next, we prove the first part of (3.2.3)

\[
\overline{R}(X, Y)Z + \overline{R}(Y, Z)X + \overline{R}(Z, X)Y = R(X, Y)Z + R(Y, Z)X + R(Z, X)Y
\]
\[ +\lambda_1[g(X, \nabla_Y P)Z + -g(X, \nabla_Z P)Y + g(Y, \nabla_Z P)X \]

\[ -g(Y, \nabla_X P)Z + g(Z, \nabla_X P)Y - g(Z, \nabla_Y P)X] \]

From the Bianchi’s first identity and equation (3.1) the above equation will be zero. The second part of (3.2.3) follows from the first part of (3.2.3). \( \square \)

**Proposition 3.3.** Let \((B^n, g_B)\) is a Riemannian manifold and \(\{E_1, \ldots, E_{n_1}\}\) be frame field on \(B\) then we have,

(3.3.1) \(\overline{\text{Ric}}_B(X, Y) = \text{Ric}_B(X, Y) + [(1 - n_1)\lambda_1 + \lambda_2]g(Y, \nabla_X P) + [-\lambda_2 \text{div} P \]

\[ + \{(1 - n_1)\lambda_1 \lambda_2 + \lambda_2^2\} \pi(P)\] \[ + [(n_1 - 1)\lambda_1^2 - \lambda_2^2]\pi(X)\pi(Y). \]

(3.3.2) \(\overline{S}_B = S_B + (n_1 - 1)[(\lambda_1^2 + \lambda_2^2 - n_1\lambda_1 \lambda_2)\pi(P) - (\lambda_1 + \lambda_2)\text{div} P].\)

**Proof.** Proof of both results is straightforward from the definition of Ricci and scalar curvature. \( \square \)

**Remark:** \(\overline{\text{Ric}}_B\) is symmetric (0,2)-type tensor if and only if \(P\) is gradient vector field.

**Proposition 3.4.** Let \((B^n, g_B)\) be a Riemannian manifold and \(\overline{\text{Ric}}_B\) is the symmetric (0,2)—type tensor then for every smooth vector field \(X\) on \(B\) we have,

(3.4.1) \(\overline{\text{div}}(\overline{\text{Ric}}_B)(X) = \text{div}(\overline{\text{Ric}}_B)(X) - \lambda_1 \pi(X)S_B - \lambda_2 d(\text{div} P)(X) + [-\lambda_1 \]

\[ + (n_1 + 1)\lambda_2]\text{Ric}_B(X, P) + [(1 - n_1)\lambda_1 + \lambda_2]g(X, \sum_{i=1}^{n_1} \nabla^2_{E_i, E_i} P) \]

\[ + \frac{[2(n_1 - 1)\lambda_1^2 + (n_1 + 3)\lambda_2^2 + (1 - n_1 - n_1^2)]}{2} \text{d}(\pi(P))(X) \]

\[ + [2(n_1 - 1)\lambda_1^2 + (n_1 - 2)\lambda_2^2 + (n_1 - 2)\lambda_1 \lambda_2] d(\text{div} P)\pi(X) \]

\[ + (1 - n_1)[2\lambda_1^2 - 2(n_1 + 1)\lambda_1 \lambda_2 + (n_1 + 2)\lambda_1 \lambda_2^2] \pi(P)\pi(X). \]

(3.4.2) \(\overline{\text{div}}(\overline{\text{Ric}}_B)(X) = \text{div}(\overline{\text{Ric}}_B)(X) - \lambda_2 d(\text{div} P)(X) + [(1 - n_1)\lambda_1 + \lambda_2] \]

\[ g(X, \sum_{i=1}^{n_1} \nabla^2_{E_i, E_i} P) + \frac{[(n_1 - 1)(\lambda_1^2 - 2\lambda_1 \lambda_2) + \lambda_2^2]}{2} \]

\[ \text{d}(\pi(P))(X) + [(n_1 - 1)\lambda_1^2 - \lambda_2^2]\text{div}(\text{P})\pi(X). \]

**Proof.** Let \(\{E_1, \ldots, E_{n_1}\}\) be a frame field on \(B\). \(\overline{\text{Ric}}_B\) is symmetric (0,2)—type tensor therefore,

\(\overline{\text{div}}(\overline{\text{Ric}}_B)(X) = \sum_{i=1}^{n_1} (\nabla_{E_i} \overline{\text{Ric}}_B)(E_i, X)\)

\[ = \sum_{i=1}^{n_1} \nabla_{E_i}(\overline{\text{Ric}}_B(E_i, X)) - \sum_{i=1}^{n_1} \overline{\text{Ric}}_B(\nabla_{E_i} E_i, X) - \sum_{i=1}^{n_1} \overline{\text{Ric}}_B(E_i, \nabla_{E_i} X). \)
We calculate the value of all term of the R.H.S of the above equation,
\[
\sum_{i=1}^{n_1} \nabla_E_i (\overline{\text{Ric}}_B(E_i, X)) = \sum_{i=1}^{n_1} \nabla_E_i (\text{Ric}(E_i, X)) - \lambda_2 d(div P)(X) + [(1 - n_1)\lambda_1 + \lambda_2]
\]
\[
\sum_{i=1}^{n_1} \nabla_E_i (g(X, \nabla_E_i P)) + \{(1 - n_1)\lambda_1 \lambda_2 + \lambda_2^2\} \pi(P) - \lambda_2 \text{div} P
\]
\[
\sum_{i=1}^{n_1} \nabla_E_i (g(E_i, X)) + [(1 - n_1)\lambda_1 \lambda_2 + \lambda_2^2 \text{d}(\pi(P))(X)
\]
\[
(3.3)
\]
\[
+ [(1 - n_1)\lambda_1^2 - \lambda_2^2] \{\text{d}(\pi(X))(P) + \pi(X) \sum_{i=1}^{n_1} \nabla_E_i (\pi(E_i))\},
\]
\[
\sum_{i=1}^{n_1} \overline{\text{Ric}}_B(N_E_i E_i, X) = \sum_{i=1}^{n_1} \text{Ric}(N_E_i E_i, X) + [(1 - n_1)\lambda_1 + \lambda_2] \sum_{i=1}^{n_1} g(X, \nabla N_E_i E_i, P)
\]
\[
+ \{(1 - n_1)\lambda_1 \lambda_2 + \lambda_2^2\} \pi(P) - \lambda_2 \text{div} P \sum_{i=1}^{n_1} g(N_E_i E_i, X)
\]
\[
+ [(n_1 - 1)\lambda_1^2 - \lambda_2^2] \pi(X) \sum_{i=1}^{n_1} \pi(N_E_i E_i) + (\lambda_1 - n_1 \lambda_2)[\text{Ric}_B(X, P)
\]
\[
+ \lambda_2 \text{div} P \pi(X)] + [(1 - n_1)\lambda_1^2 - n_1 \lambda_2^2 + (1 - n_1 + n_1^2)\lambda_1 \lambda_2]
\]
\[
g(X, \nabla P P) + (n_1 - 1)[\lambda_1^2 - (n_1 + 1)\lambda_2^2 + n_1 \lambda_1 \lambda_2^2] \pi(P) \pi(X),
\]
\[
(3.4)
\]
\[
\sum_{i=1}^{n_1} \overline{\text{Ric}}_B(E_i, \nabla E_i X) = \sum_{i=1}^{n_1} \text{Ric}(E_i, \nabla E_i X) + \lambda_1 \pi(X) S - \lambda_2 \text{Ric}_B(X, P) + [(1 - n_1)\lambda_1
\]
\]
\[
+ \lambda_2 \sum_{i=1}^{n_1} g(N_E_i E_i, \nabla E_i P) + \{(1 - n_1)\lambda_1 \lambda_2 + \lambda_2^2\} \pi(P) - \lambda_2 \text{div} P
\]
\[
\sum_{i=1}^{n_1} g(E_i, \nabla E_i X) + [(1 - n_1)\lambda_1 (\lambda_1 + \lambda_2) + \lambda_2^2] \text{div} P \pi(X)
\]
\[
+ [(n_1 - 1)\lambda_1^2 - \lambda_2^2] \pi(\nabla P X) + [(n_1 - 1)\lambda_1 \lambda_2 - \lambda_2^2] \pi(\nabla X P)
\]
\[
(3.5)
\]
\[
+ (n_1 - 1)[\lambda_1^2 + 2\lambda_1 \lambda_2^2 - (n_1 + 1)\lambda_1^2 \lambda_2] \pi(P) \pi(X),
\]

after substituting the value of (3.4), (3.5), and (3.6) in (3.3) and using (3.2) we proved
the first part. Proof of the second part is same as above. \(\square\)

4. Second order quarter symmetric connection

Let \((M, g)\) be a Riemannian manifold with quarter symmetric connection \(\nabla\). Let \(T\) be a \((r, s)\) - type tensor field. The second order quarter symmetric tensor derivative of \(T\) denoted by \(\nabla^2 T\) is a \((r, s + 2)\) - type tensor field and
\[
(\nabla^2 T)(\theta^1, ..., \theta^r, Z_1, ..., Z_s) = (\nabla_X (\nabla Y T))(\theta^1, ..., \theta^r, Z_1, ..., Z_s)
\]
\[
= (\nabla_X (\nabla Y T))(\theta^1, ..., \theta^r, Z_1, ..., Z_s)
\]
\[
- (\nabla_X Y T)(\theta^1, ..., \theta^r, Z_1, ..., Z_s).
\]

From the above we have,

(i). If \(f : M \to \mathbb{R}\) is a smooth function then the second order quarter symmetric tensor
Lemma 4.1. Let
\[ (4.3) \]

\[ \nabla^2_{X,Y} f - \lambda_1 \pi(Y) X f + \lambda_2 g(X,Y) P f, \]
where we use the fact that \( \nabla f = df \) in the second line.

(ii). If \( X \), \( Y \) and \( Z \) are smooth vector fields on \( (M,g) \) then second order quarter symmetric tensor derivative of \( Z \) with respect to \( X \) and \( Y \) is

\[ \nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \]
\[ = \nabla^2_{X,Y} Z + \lambda_1 [(\nabla_X \pi(Z)) Y + \pi(\nabla_Y Z) X - \pi(Y) \nabla_X Z]\]
\[ + \lambda_2 [g(X,Y) \nabla_P Z - g(Y,Z) \nabla_X P - g(X,\nabla_Y Z) P - g(Y,\nabla_X Z) P] \]
\[ + \lambda_1 \lambda_2 [g(X,Y) \pi(Y) P - g(Y,Z) \pi(P) Y] \]
\[ = \lambda_2^2 [-g(X,Y) \pi(Z) + g(Y,Z) \pi(X)] P. \]

(iii). \( \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = R(X,Y) Z + \lambda_1 [g(Z,\nabla_X P) - g(Z,\nabla_Y P)] X \]
\[ + \pi(X) \nabla_Y Z - \pi(Y) \nabla_X Z + \lambda_2 [g(X,Z) \nabla_Y P - g(Y,Z) \nabla_X P] + \lambda_1 \lambda_2 \{g(X,Z) \{\pi(Y) P + \pi(P) Y\} - g(Y,Z) \{\pi(X) P + \pi(P) X\} \}
\[ + \lambda_2^2 [-g(X,Z) \pi(Y) + g(Y,Z) \pi(X)] P. \]

Lemma 4.1. Let \( X \) and \( Y \) are vector fields on Riemannian manifold \( M \). If \( w \) is a first form and \( \nabla \) be a quarter symmetric connection on \( M \) then:

\begin{enumerate}
\item \( (\nabla_X w)^\# = \nabla_X w^\# - \lambda_1 w(X) P + \lambda_2 w(P) X, \)
\item \( (\nabla_X^2 w)^\# = \nabla^2_{X,Y} w^\# + \lambda_1 [-\nabla_X (w(Y) P) - ((\nabla_Y w)(X)) P - \pi(Y) \nabla_X w^\# + w(\nabla_Y w)(X) P + \lambda_2 [g(X,Z) \nabla_Y P - g(Y,Z) \nabla_X P] + \lambda_1 g(X,Z) \{\pi(Y) P + \pi(P) Y\} \]
\end{enumerate}
\[ + \lambda_2^2 [-g(X,Z) \pi(Y) + g(Y,Z) \pi(X)] P. \]

\begin{enumerate}
\item \( (\nabla^2_{X,Y} w)(Z) = (\nabla^2_{X,Y} w)(Z) + \lambda_1 [-\nabla_X (w(Y) P) - ((\nabla_Y w)(X)) P - \pi(Y) \nabla_X w^\# + \lambda_2 g(X,Y) \]
\[ (\nabla_P w)(Z) + g(X,Z) (\nabla_Y w)(P) + g(Y,Z) (\nabla_X w)(P) \]
\[ + \lambda_1 \lambda_2 [-g(X,Z) \pi(Z) w(P) - g(Y,Z) \{\pi(Y) w(P) \}
\[ - \pi(P) w(Y) \} + \lambda_2^2 [g(X,Y) \pi(Z) + \pi(\nabla_X P) \}\]
\[ + \lambda_2^2 [-g(X,Z) \pi(Y) - g(Y,Z) \pi(X)] w(P), \]
\item \( (\nabla^2_{X,Y} w)(Z) - (\nabla^2_{Y,X} w)(Z) = -w(R(X,Y) Z) + \lambda_1 [\pi(X) (\nabla_Y w)(Z) - \pi(Y) \]
\[ (\nabla_X w)(Z) + w(X)(\nabla_Y \pi)(Z) - w(Y)(\nabla_X \pi)(Z) \]
\[ + \lambda_2 [-g(X,Z) \pi(P) w(X) + g(Y,Z) \pi(P) w(X) \]
\[ + \lambda_1 \lambda_2 [-g(X,Z) \pi(Y) w(P) + \pi(P) w(Y) \} \]
\[ + g(Y,Z) \{\pi(Y) w(P) + \pi(P) w(X) \} \]
\[ + \lambda_2^2 [-g(X,Z) \pi(Y) - g(Y,Z) \pi(X)] w(P), \]
\end{enumerate}
where \( \# \) is musically operator and \( \nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \) denote the second order quarter symmetric connection on \( M \).

**Proof.** Let \( w \) stands for the dual 1-form associated to \( w^\# \) that is,

We know that if \( w \) is a first form then \( w^\# \) is a vector field corresponding to \( w \) and we can define

\[
(4.4) \quad w(X) = g(w^\#, X),
\]

for every vector field \( X \) on \( B \)

From (4.4) we deduced the following results,

\[
(4.5) \quad (\nabla_X w)^\# = \nabla_X w^\#,
\]

i.e,

\[
(4.6) \quad g(\nabla_X w^\#, Y) = (\nabla_X w)(Z),
\]

and

\[
(4.7) \quad (\nabla_{X,Y} w)^\# = \nabla_{X,Y} w^\#.
\]

Recall that,

\[
(4.8) \quad (\nabla_X \pi)(Z) = g(\nabla_X P, Z).
\]

The proof of (4.1.1) goes as follows:

\[
(\nabla_X w)(Y) = \nabla_X w(Y) - w(\nabla_X Y)
\]

\[
= (\nabla_X w)(Y) - \lambda_1 \pi(Y)w(X) + \lambda_2 g(X,Y)w(P)
\]

\[
= g(\nabla_X w^\#, Y) - g(\lambda_1 w(X)P, Y) + g(\lambda_2 w(P)X, Y)
\]

\[
= g(\nabla_X w^\# - \lambda_1 w(X)P + \lambda_2 w(P)X, Y),
\]

where we used (4.6) in the third line. Therefore from (4.5) the last equation imply that,

\[
(\nabla_X w)^\# = \nabla_X w^\# - \lambda_1 w(X)P + \lambda_2 w(P)X.
\]

Hence (4.1.1) is proved.

Now, we proof the second part of the lemma

\[
(4.9) \quad (\nabla_{X,Y} w)^\# = (\nabla_X (\nabla_Y w))^\# - (\nabla_{\nabla_X Y} w)^\#,
\]

as \( \# \) is a linear operator. Apply (4.1.1) in the right side both term of (4.9) we have,

\[
(\nabla_X (\nabla_Y w))^\# = \nabla_X (\nabla_Y w)^\# - \lambda_1 [\nabla_X (w(Y)P) + ((\nabla_Y w)(X))P]
\]

\[
+ \lambda_2 [\nabla_X (w(p)Y) + ((\nabla_Y w)(P))X] - \lambda_1 \lambda_2 [g(X,Y)]
\]

\[
(4.10) \quad w(P)P + \pi(P)w(Y)X + \lambda_1^2 \pi(X)w(Y)P + \lambda_2^2 \pi(Y)w(P)X,
\]

and

\[
(\nabla_{\nabla_X Y} w)^\# = \nabla_{\nabla_X Y} w^\# + \lambda_1 [\pi(Y)\nabla_X w^\# - w(\nabla_X Y)P] + \lambda_2 [g(X,Y)]
\]

\[
\nabla_P w^\# + w(P)\nabla_X Y + \lambda_1 \lambda_2 w(P)[g(X,Y)P + \pi(Y)X]
\]

\[
- \lambda_1^2 \pi(Y)w(X)P - \lambda_2^2 w(P)g(X,Y)P.
\]

After putting (4.10) and (4.11) in (4.9) we get the result (4.1.2). Henc second part of the lemma is proved.

Next, we proof the third part of the lemma

\[
(\nabla_{X,Y}^2 w)(Z) = g((\nabla_{X,Y} w)^\#, Z).
\]
After using the value of $(\nabla^2_{X,Y} w)^\#$ from (4.1.2) and applying (4.6), (4.8) in the last equation we get the result (4.1.3). Hence third part of lemma is proved. Proof of the last part of the lemma immediate from (4.1.3). □

**Remark :** If we put $\lambda_1 = \lambda_2 = 0$ then quarter symmetric connection will become into simple connection and the result (4.1.4) of lemma (4.1) converted into Ricci identity ([9], p. 159).

5. **Hessian of $f$ with respect to quarter symmetric connection**

**Definition 5.1.** Let $(M, g)$ be Riemannian manifold of $\dim n$ then Hessian of a smooth function $f : M \to \mathbb{R}$ with w.r.t to quarter symmetric connection $\nabla$ is denoted by $H_f$ and $H_f := \nabla(\nabla f)$

**Lemma 5.2.** The $\overline{H}_f$ of $f$ is $(0,2)$-type tensor such that

\[(5.1) \quad \overline{H}_f(X,Y) = H_f(X,Y) - \lambda_1 \pi(Y)Xf + \lambda_2 g(X,Y)Pf,\]

for every smooth vector field $X, Y$ on $M$.

*Proof.* Since $\nabla f = df$,

$$\overline{H}_f(X,Y) = \nabla(df)(X,Y) = (\nabla_X df)(Y).$$

Then from (4.2) follows the proof. □

**Remark :** $\overline{H}_f$ of $f$ is symmetric $(0,2)$-type tensor if and only if

\[(5.2) \quad \pi(Y)Xf = \pi(X)Yf\]

for any smooth vector field $X, Y$ on $M$.

**Lemma 5.3.** If $\overline{H}_f$ is symmetric $(0,2)$-type tensor then

\[(5.3) \quad \pi(X)|\nabla f|^2 = df(P)df(X),\]

for every smooth vector field $X$ on $M$.

**Lemma 5.4.** Let $(M, g)$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a smooth function. If $\overline{H}_f$ is symmetric then show that,

\[(5.4.1) \quad (\nabla_X df)(Y) = (\nabla_Y df)(X),\]

\[(5.4.2) \quad (\nabla_{X,Y} df)(Z) = (\nabla_{X,Z} df)(Y),\]

for every smooth vector field $X, Y$ and $Z$ on $M$.

*Proof.* From definition of $\overline{H}_f$ we have,

$$\overline{H}_f(X,Y) = \overline{H}_f(X,Y).$$

Hence symmetry of $\overline{H}_f$ prove the first part. Now, we prove the second part

$$L.H.S = \overline{\nabla}_X(\overline{\nabla}_Y df(Z)) - (\overline{\nabla}_Y df)(\overline{\nabla}_X Z) - (\overline{\nabla}_{\overline{\nabla}_{X,Y} df})(Z)$$

after using (5.4.1) in this equation we get,

$$= \overline{\nabla}_X(\overline{\nabla}_Z df(Y)) - (\overline{\nabla}_{\overline{\nabla}_{X,Z} df})(Y) - (\overline{\nabla}_Z df)(\overline{\nabla}_X Y) = R.H.S$$

□
Proposition 5.5. Let \((B^{n_1},g_B)\) be a Riemannian manifold. If \(\mathbf{H}^f\) is the symmetric \((0,2)\)--type tensor then we have,

\[
\begin{align*}
\text{div}(\mathbf{H}^f)(X) &= \text{div}(H^f)(X) - \lambda_1(\nabla \nabla f \pi)(X) + \lambda_2 d(P f)(X) - 2\lambda_1 \Delta f \pi(X) \\
&\quad + [(n_1 + 1)\lambda_2 - \lambda_1]H^f(X, P) + [2\lambda_1^2 + (n_1 + 1)\lambda_2^2 - 2(n_1 + 1)\lambda_1 \lambda_2] \pi(P) d f(X).
\end{align*}
\]

\(
\text{div}(\mathbf{H}^f)(X) = \text{div}(H^f)(X) + \lambda_1(\nabla \nabla \pi)(X) - \Delta f \pi(X) + \lambda_2 d(P f)(X).
\)

\[
\begin{align*}
\text{div}(\mathbf{H}^f)(X) &= \text{div}(H^f)(X) - \frac{\lambda_1}{f} (\nabla \nabla f \pi)(X) - 2\frac{\lambda_1}{f} \Delta f \pi(X) + \frac{\lambda_2}{f} d(P f)(X) \\
&\quad + \frac{(\lambda_1 - \lambda_2)}{f^2} d f(p) d f(X) + \left[\frac{\lambda_2(n_1 + 1) - \lambda_1}{f} \right] H^f(X, P) + \left[\frac{2\lambda_1^2 + (n_1 + 1)\lambda_2^2 - 2(n_1 + 1)\lambda_1 \lambda_1}{f} \right] \pi(P) d f(X).
\end{align*}
\]

\[
\begin{align*}
\text{div}(\mathbf{H}^f)(X, Y) &= \text{div}(H^f)(f) - \frac{\lambda_1}{f} \text{div}_B P d f(X) - \frac{\lambda_2}{f} H^f(X, P) \\
&\quad + \frac{\lambda_2}{f} d(P f)(X) + \frac{(\lambda_1 - \lambda_2)}{f^2} d f(P) d f(x).
\end{align*}
\]

Proof. Let \(\{E_1,\ldots, E_{n_1}\}\) be a frame field on \(B\). \(\mathbf{H}^f\) is symmetric \((0,2)\)--type tensor therefore,

\[
\begin{align*}
\text{div}(\mathbf{H}^f)(X) &= \sum_{i=1}^{n_1} (\nabla_{E_i} \mathbf{H}^f)(E_i, X) \\
&= \sum_{i=1}^{n_1} \nabla_{E_i}(\mathbf{H}^f(E_i, X)) - \sum_{i=1}^{n_1} \mathbf{H}^f(\nabla_{E_i} E_i, X) - \sum_{i=1}^{n_1} \mathbf{H}^f(E_i, \nabla_{E_i} X).
\end{align*}
\]

Now we have to calculate the value of all term of the R.H.S of the above equation,

\[
\begin{align*}
\sum_{i=1}^{n_1} \nabla_{E_i}(\mathbf{H}^f(E_i, X)) &= \sum_{i=1}^{n_1} \nabla_{E_i}(H^f(E_i, X)) - \lambda_1 \nabla \nabla f(\pi(X)) - \lambda_1 \pi(X) \\
&\quad + \frac{\lambda_1}{f} \sum_{i=1}^{n_1} \nabla_{E_i}(H^f(E_i, X)) + \lambda_2(P f)(X),
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{n_1} \mathbf{H}^f(\nabla_{E_i} E_i, X) &= \sum_{i=1}^{n_1} \mathbf{H}^f(\nabla_{E_i} E_i, X) - \lambda_1 \pi(X) \sum_{i=1}^{n_1} \nabla \nabla f E_i f + \lambda_2(P f) \\
&\quad + \sum_{i=1}^{n_1} \nabla f(\nabla_{E_i} E_i, X) + (\lambda_1 - n_1 \lambda_2) \{ H^f(X, P) - (\lambda_1 - \lambda_2) \pi(X) (P f) \},
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{n_1} \mathbf{H}^f(E_i, \nabla_{E_i} X) &= \sum_{i=1}^{n_1} H^f(E_i, \nabla_{E_i} X) + \lambda_1 \pi(X) \Delta f - \lambda_1 \pi(\nabla \nabla f X) \\
&\quad + \lambda_2(P f) \sum_{i=1}^{n_1} g(E_i, \nabla_{E_i} X) - \lambda_2 H^f(X, P) \\
&\quad - \lambda_1^2 + \lambda_2^2 - (n_1 + 1) \lambda_1 \lambda_2] \pi(P) d f(X),
\end{align*}
\]
after putting the value of (5.5), (5.6) and (5.7) in (5.4) we get the result (5.5.1).
Proof of (5.5.2) is same as above. Next we proof (5.5.3),
\[
\frac{1}{f}H^f(X) = \sum_{i=1}^{n_1}(\nabla_{E_i}(\frac{1}{f}H^f))(E_i, X)
\]
\[
= \sum_{i=1}^{n_1} \nabla_{E_i}(\frac{1}{f}\Pi^f(E_i, X)) - \frac{1}{f} \sum_{i=1}^{n_1} \Pi^f(\nabla_{E_i}E_i, X) - \frac{1}{f} \sum_{i=1}^{n_1} \Pi^f(E_i, \nabla_{E_i}X).
\]
Now,
\[
\sum_{i=1}^{n_1} \nabla_{E_i}(\frac{1}{f}\Pi^f(E_i, X)) = \sum_{i=1}^{n_1} \nabla_{E_i}(\frac{1}{f}H^f(E_i, X)) - \frac{\lambda_1}{f} \nabla_{\nabla f}(\pi(X))
\]
\[- \frac{\lambda_1}{f} \pi(X) \sum_{i=1}^{n_1} \nabla_{E_i} \nabla f + \frac{\lambda_2}{f} \nabla_X \nabla P f
\]
\[+ \frac{\lambda_2}{f} (P f) \sum_{i=1}^{n_1} \nabla_{E_i}(g(E_i, P)) + \frac{(\lambda_1 - \lambda_2)}{f^2} (P f) df(X),
\]

after substituting (5.9), (5.6) and (5.7) in (5.8) implies the result (5.6.3). Proof of fourth part is same as above.

6. Einstein warped product space with non positive scalar curvature with respect to quarter symmetric connection

Lemma 6.1. Let \((B^{n_1}, g_B)\) be a Riemannian manifold with quarter symmetric connection \(\nabla\) and \(f\) be a smooth function on \(B\). If \(\Pi\) is symmetric then for any smooth vector field \(X\) on \(B\) we have,
\[
\text{div}(H^f)(X) = d(\Delta f)(X) + Ric(\nabla f, X)
\]
\[+ \lambda_1[(\Delta f)\pi(X) - d(P f)(X)].
\]

Proof. Substituting \(w = df\) in (4.1.4) and using the result (5.4.2) we get,
\[
(\nabla^2_{X, Z} df)(Y) - (\nabla^2_{Y, X} df)(Z) = -d f(R(X, Y)Z) + \lambda_1(\pi(X)\nabla_Y df)(Z) - \pi(Y)
\]
\[\nabla_X df(Z) + df(X)(\nabla_Y \pi)(Z) - df(Y)(\nabla_X \pi)(Z)]
\[+ \lambda_1[-g(X, Z)df(\nabla_Y P) + g(Y, Z)df(\nabla_X P)]
\[+ \lambda_1\lambda_2[-g(X, Z)\{\pi(Y)df(P) + \pi(P)df(Y)]
\[+ g(Y, Z)\{\pi(X)df(P) + \pi(P)df(X)]\}
\[+ \lambda_3^2d g(X, Z)\pi(Y) - g(Y, Z)\pi(X)]df(P),
\]
for every smooth vector field \(X, Y\) and \(Z\) on \(B\).
For a fixed \(p \in B\), we can find a local orthonormal frame \(\{E_1, ... E_{n_1}\}\) of the space \(B\) such that \(\nabla_{E_i}E_i(p) = 0\). We can also choose \(\nabla_{E_i}Y(p) = \nabla_{E_i}P(p) = 0\). Taking trace of (6.2) with respect to \(X, Z\) and using the symmetry of \(\Pi^f\) we have,
\[
\sum_{i=1}^{n_1}(\nabla^2_{E_i, E_i} df)(Y) = \sum_{i=1}^{n_1} (\nabla^2_{Y, E_i} df)(E_i) - \sum_{i=1}^{n_1} \{df(R(E_i, Y)E_i) + \lambda_1[d(P f)(Y)]
\]
\[- (\Delta f)\pi(P)\} + (1 - n_1)(2\lambda_1 - \lambda_2)\lambda_2\pi(P) df(Y).
\]
Since,
\[
\sum_{i=1}^{n_1} (\nabla^2_{E_i, E_i} df)(Y) = \text{div}(H^f)(Y) - 2\lambda_1(\Delta f)\pi(Y) + [(n_1 + 2)\lambda_2 - \lambda_1]d(df)(Y)
\]
(6.4) 
\[+ (2\lambda_1^2 + (n_1 + 1)\lambda_2^2 - 2(n_1 + 1)\lambda_1\lambda_2]\pi(P)df(X).
\]

and
\[
\sum_{i=1}^{n_1} (\nabla^2_{Y, E_i} df)(E_i) = d(\Delta f)(Y) + [(n_1 + 2)\lambda_2 - 3\lambda_1]d(Pf)(X)
\]
(6.5) 
\[+ 2(\lambda_1 - \lambda_2)^2\pi(P)df(X).
\]

Therefore after using (6.4), (6.5) in (6.3), we get the result (6.1). Hence proved the lemma. □

**Proposition 6.2.** Let \((B^{n_1}, g_B)\) is a compact Riemannian manifold with quarter symmetric connection \(\nabla\) of dimension \(n_1 \geq 2\) and \(\overline{\nabla}, \text{Ric}\) both are symmetric tensor. Let \(f\) be a non-constant smooth function on \(B\) satisfying (2.2.1) for a constant \(\lambda \in \mathbb{R}\) and a natural number \(n_2 \in \mathbb{N}\). Then show that \(f\) satisfies (2.2.3) for a constant \(\lambda'\) if
\[
a_1 Pf Xf + f\{a_2 Pf f(X) + a_3 df(\nabla_X P)
\]
\[+ a_4 Pf Xf f + 2\lambda_1 \Delta f \pi(X)\} + f^2\{a_5 df(divP)(X)
\]
(6.6) 
\[+ a_6(df\pi(P))(X) + a_7 g(X, \sum_{i=1}^{n_1} \nabla^2_{E_i, E_i} P) + a_8 divP \pi(X)\} = 0,
\]

for every smooth vector field \(X\) on \(B\), where,
\[
a_1 = \[(1 - \pi)\lambda_1] + (n_2 - 1)\lambda_2\}, a_2 = \[-(\pi + 1)\lambda_1] + (2n_1 - 3)\lambda_2\]
\[
a_3 = 2(\pi - 1)\lambda_1 - \lambda_2, a_4 = 2(1 - \pi)\lambda_1^2 - \lambda_2^2, a_5 = \[(1 - \pi)\lambda_1 + (3 - \pi)\lambda_2\]
\[
a_6 = \[(\pi - 2)(\pi - 1)\lambda_1 - \lambda_2\}, a_7 = \frac{a_3}{n_2}, a_8 = \frac{[(1 - \pi)\lambda_1^2 + \lambda_2^2]}{n_2}.
\]

Moreover, we can construct a compact Einstein warped product space \(M = B \times_f F\) with \(\text{Ric} = \lambda' g\) for a compact Einstein space \((F, g_F)\) of dimension \(n_2\) with \(\text{Ric}_F = \lambda' g_F\).

**Proof.** On contracting both sides of (2.1.1), we have
\[
S = n_1 \left\{ \lambda + n_2 \lambda_2 \frac{Pf}{f} + n_2 \lambda_1 \lambda_2 \pi(P) \right\} + n_2 \left\{ \frac{\Delta f}{f} + \lambda_1 \text{div} P - \lambda_2^2 \pi(P) \right\},
\]

The above equation imply that,
\[
d\overline{S}_B(X) = n_2 \left\{ \lambda_1 (n_1 \lambda_2 - \lambda_1) d(\pi(P))(X) + \lambda_1 d(\text{div} P)(X) + \frac{1}{f} d(\Delta f)(X)
\]
(6.7) 
\[+ \frac{n_1 \lambda_2}{f} d(Pf)(X) - \frac{[n_1 \lambda_2 Pf + \Delta f]}{f^2} df(X)\}.
\]

From equation (3.3.2) we have
\[
d\overline{S}_B(X) = 2 \text{div}(\text{Ric}_B)(X) + (n_1 - 1) [(\lambda_1^2 + \lambda_2^2 - n_1 \lambda_1 \lambda_2)] d(\pi(P))(X)
\]
(6.8) 
\[-(\lambda_1 + \lambda_2) d(\text{div} P)(X)].\]
From equation (6.7) and (6.8), we have
\[ 2 \text{div}(\text{Ric}_B)(X) = \frac{n_2}{f} d(\Delta f)(X) + \frac{n_1 n_2 \lambda_2}{f} d(Pf)(X) - \frac{n_2}{f^2} \Delta fd\chi(X) \]
\[ - \frac{n_1 n_2 \lambda_2}{f^2} Pf \chi(X) + [(\pi - 1) \lambda_1 + (n_1 - 1) \lambda_2] d(\text{div}P)(X) \]
\[ + [(1 - \pi)(\lambda_1^2 - n_1 \lambda_1 \lambda_2) + (1 - n_1) \lambda_2^2] d(\pi(P))(X). \]
(6.9)

The equation (2.1.1) can be written as,
\[ \overline{\text{Ric}}_B(X, Y) = [\lambda + n_2 \lambda_2 \pi(P)] g(X, Y) + n_2 \lambda_1 g(Y, \nabla_X P) \]
\[ - n_2 \lambda_1^2 \pi(X) \pi(Y) + \frac{n_2 \lambda_1}{f} \pi(Y) df(X) + \frac{n_2}{f} \overline{H}^f(X, Y) \]
(6.10)

On taking divergent on the both side of (6.10) and using (3.2) we get,
\[ \text{div}(\overline{\text{Ric}}_B)(X) = n_2 \text{div}(\frac{1}{f} \overline{H}^f)(X) + n_2 \lambda_1 g(X, \sum_{i=1}^{n_1} \nabla_{E_i, E_i} P) - n_2 \lambda_1^2 \text{div}P \pi(X) \]
\[ + \frac{n_2(2\lambda_2 - \lambda_1)\lambda_1}{2} d(\pi(P))(X) + \frac{n_2 \lambda_1}{f} (\nabla_{\nabla f} \pi)(X) \]
\[ + \frac{n_2 \lambda_1}{f} \Delta f \pi(X) - \frac{n_2 \lambda_1}{f^2} |\nabla f|^2 \pi(X) \]
(6.11)

Using (3.4.2) and (5.6.4) in (6.11), we have
\[ \text{div}(\text{Ric}_B)(X) - n_2 \text{div}(\frac{1}{f} \overline{H}^f)(X) - \lambda_2 d(\text{div}P)(X) + [(\pi - 1) \lambda_1^2 - \lambda_2^2] \text{div}(P) \pi(X) \]
\[ + [(1 - \pi) \lambda_1 + \lambda_2] g(X, \sum_{i=1}^{n_1} \nabla_{E_i, E_i} P) + \frac{[(\pi - 1)(\lambda_1^2 - 2 \lambda_1 \lambda_2) + \lambda_2^2]}{2} \]
\[ d(\pi(P))(X) - \frac{n_2 \lambda_2}{f} d(Pf)(X) + \frac{n_2 \lambda_2}{f^2} Pf df(X) = 0. \]
(6.12)

Multiplying on both sides by \(- \frac{2f^2}{n_2}\) in the above equation, we have
\[ 2f^2 \text{div}(\frac{1}{f} \overline{H}^f)(X) - \frac{2}{n_2} f^2 \text{div}(\text{Ric}_B)(X) + \frac{2 \lambda_2}{n_2} f^2 d(\text{div}P)(X) + \frac{2[(\pi - 1) \lambda_1^2 + \lambda_2^2]}{n_2} \]
\[ f^2 d(P) \pi(X) + \frac{2[(\pi - 1) \lambda_1 - \lambda_2]}{n_2} f^2 g(X, \sum_{i=1}^{n_1} \nabla_{E_i, E_i} P) \]
\[ + \frac{[(1 - \pi)(\lambda_1^2 - 2 \lambda_1 \lambda_2) - \lambda_2^2]}{n_2} f^3 d(\pi(P))(X) \]
\[ + 2f \lambda_2 d(Pf)(X) - 2n_2 Pf df(X) = 0. \]
(6.13)

From equation (2.2.1) and (3.3.1), we have
\[ \text{Ric}_B(X, Y) = [\lambda + \lambda_2 \text{div}P + \frac{n_2 \lambda_2}{f} Pf + (\pi - 1) \lambda_1 \lambda_2] \pi(P)] g(X, Y) + \frac{n_2}{f} \]
\[ H^f(X, Y) + [(\pi - 1) \lambda_1 - \lambda_2] g(Y, \nabla_X P) + [(1 - \pi) \lambda_1^2 + \lambda_2^2] \pi(X) \pi(Y). \]
(6.14)

we know that,
\[ \text{div}(\frac{1}{f} \overline{H}^f)(X) = - \frac{1}{2f^2} d(\overline{\nabla}_B f^2_B) + \frac{1}{f} \text{div}(H^f)(X). \]
Putting the value of (6.15) and using the condition (6.16), we get

\[ \text{div}(\frac{1}{f}Hf)(X) = (n_2 - 1)\frac{d(|\nabla_B f|^2)}{2f^2} + \frac{1}{f}d(\Delta f)(X) + \frac{\lambda_2}{f}dPf(X) + \frac{\lambda_2}{f}dPfd(X) \]

\[ + \frac{n_2\lambda_2}{f}Pfd(X) + \left[ (1 - \frac{2n}{2})(\lambda_2^2 - \lambda_2^2) - 2\lambda_2^2 \right] \frac{\pi}{f^2}(P)df(X). \]

(6.15)

\[ - d(Pf)(X)] + \left[ \frac{1 - \frac{2n}{2}}{(n_2 - 1)\lambda_2 - \lambda_2^2} \frac{2\lambda_2^2}{f^2} \pi(P)df(X). \]

Putting the value of \( \text{div}(\frac{1}{f}Hf)(X) \) and \( \text{div}(Ric_B)(X) \) from (6.15) and (6.9) respectively and using the condition (6.6), we get

\[ d\{ \lambda f^2 + f\Delta f + (n_2 - 1)d(|\nabla_B f|^2) \} + [(\frac{2n}{2} - 1)\lambda_1 - \lambda_2^2] \{ 2f\pi(P)df(X) \]

\[ + f^2d(\pi(P))(X) \} + \lambda_2 \{ 2f\text{div}Pfd(X) + f^2d(\text{div})(X) \]

\[ + [(\frac{2n}{2} - 1)\lambda_1 + (n_2 - 1)\lambda_2] \{ Pfdf(X) + fdf(Pf)(X) \} = 0. \]

This equation can be written as

\[ d\{ \lambda f^2 + f\Delta f + (n_2 - 1)|\nabla_B f|^2 + [(\frac{2n}{2} - 1)\lambda_1 - \lambda_2^2] f^2\pi(P) \]

\[ + f^2\lambda_2 \text{div}Bf + [(\frac{2n}{2} - 1)\lambda_1 + (n_2 - 1)\lambda_2] Pf \} = 0. \]

Hence first part of the proposition is prove. The second part of the proposition is hold by using the sufficiencies of Corollary (2.2).

\[ \square \]

**Theorem 6.3.** Let \( M = B \times_f F \) be a Einstein warped product space with quarter symmetric connection \( \nabla \) and compact base \( B \). If has non-positive scalar curvature, and warping function \( f \) satisfies

\[ \frac{\lambda_2}{V(B)} \int_B \left\{ f^2 \text{div} - f(x)^2 \text{div}(x) \right\} + \frac{b_1}{V(B)} \int_B \left\{ f^2 \pi(P) - f(x)^2 \pi(P)(x) \right\} \]

\[ + \frac{b_2}{V(B)} \int_B f(Pf) = 0, \]

(6.16)

where \( x \) may be minimum or maximum points of \( f \) on \( B \), \( V(B) \) denotes the volume of \( B \), \( b_1 = [(\frac{2n}{2} - 1)\lambda_1 - \lambda_2^2] \), \( b_2 = [(\frac{2n}{2} - 1)\lambda_1 - (n_2 - 1)\lambda_2] \), then it \( M \) simply a Riemannian product.

**Proof.** Equation (2.2.3) can be written as

\[ \chi' = \lambda f^2 + \text{div}(f \nabla f) + (n_2 - 2)|\nabla_B f|^2 + [(\frac{2n}{2} - 1)\lambda_1 - \lambda_2^2] f^2\pi(P) \]

\[ + \lambda_2 f^2 \text{div}Bf + [(\frac{2n}{2} - 1)\lambda_1 + (n_2 - 1)\lambda_2] Pf. \]

(6.17)

By taking integration of (6.17) over \( B \), we have

\[ \chi' = \frac{\lambda}{V(B)} \int_B f^2 + \frac{(n_2 - 2)}{V(B)} \int_B |\nabla_B f|^2 + \frac{\lambda_2}{V(B)} \int_B f^2 \text{div}Bf \]

\[ + \frac{b_1}{V(B)} \int_B f^2 \pi(P) + \frac{b_2}{V(B)} \int_B f(Pf). \]

(6.18)
**Case.1** Let \( n_2 \geq 3 \) and \( l \) be the maximum point of \( f \) on \( B \) then we have,
\[
0 > f(l) \Delta f(l)
\]
\[
= \lambda' - \lambda f(l)^2 - \lambda_2 f(l)^2 \text{div} P(l) - b_1 f(l)^2 \pi(P)(l)
\]
\[
= \frac{\lambda}{V(B)} \int_B \{ f^2 - f(l)^2 \} + \frac{(n_2 - 2)}{V(B)} \int_B |\nabla_B f|^2 + \frac{\lambda_2}{V(B)} \int_B \{ f^2 \text{div} B P
\]
\[- f(l)^2 \text{div} P(l) \} + \frac{b_1}{V(B)} \int_B \{ f^2 \pi(P) - f(l)^2 \pi(P)(l) \} + \frac{b_2}{V(B)} \int_B f(P f)
\]
\[
\geq 0.
\]
The last inequality holds from the properties of scalar curvature \( \lambda \) and by the condition (6.16). Hence \( f \) is constant.

**Case.2** Let \( n_2 = 1, 2 \) and we consider \( m \) as a minimum point of \( f \) on \( B \) then, we have \( f(m) > 0, \nabla f(m) = 0 \) and \( \Delta f(m) \geq 0 \). Therefore from (2.2.3) and (6.18) we have
\[
0 \leq f(m) \Delta f(m)
\]
\[
= \lambda' - \lambda f(l)^2 - \lambda_2 f(m)^2 \text{div} P(m) - b_1 f(l)^2 \pi(P)(m)
\]
\[
= \frac{\lambda}{V(B)} \int_B \{ f^2 - f(l)^2 \} + \frac{(n_2 - 2)}{V(B)} \int_B |\nabla_B f|^2 + \frac{\lambda_2}{V(B)} \int_B \{ f^2 \text{div} B P
\]
\[- f(m)^2 \text{div} P(m) \} + \frac{b_1}{V(B)} \int_B \{ f^2 \pi(P) - f(m)^2 \pi(P)(m) \} + \frac{b_2}{V(B)} \int_B f(P f)
\]
\[
\leq 0.
\]
The last inequality holds same as case 1. Hence \( f \) is constant. \( \square \)

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