An Introduction to $L_\infty$-Algebras and their Homotopy Theory

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Abstract
In this review we give a detailed introduction to the theory of (curved) $L_\infty$-algebras and $L_\infty$-morphisms. In particular, we recall the notion of (curved) Maurer-Cartan elements, their equivalence classes and the twisting procedure. The main focus is then the study of the homotopy theory of $L_\infty$-algebras and $L_\infty$-modules. In particular, one can interpret $L_\infty$-morphisms and morphisms of $L_\infty$-modules as Maurer-Cartan elements in certain $L_\infty$-algebras, and we show that twisting the morphisms with equivalent Maurer-Cartan elements yields homotopic morphisms.

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1 Introduction

$L_\infty$-algebras (also called strong homotopy Lie algebras or SH Lie algebras) are a generalization of differential graded Lie algebras (DGLAs), where the Jacobi identity only holds up to compatible higher homotopies. They were introduced in [35,47,49], and they appeared at first in a supporting role in deformation theory.

The idea is the following: In a usual DGLA $(\mathfrak{g},d,[\cdot,\cdot])$ one has a cochain differential $d$ on a $\mathbb{Z}$-graded vector space $\mathfrak{g}$ and a compatible graded Lie bracket $[\cdot,\cdot]$, i.e. $d$ satisfies the graded Leibniz rule. In an $L_\infty$-algebra $(L,\{l_n\}_{n\in\mathbb{N}})$ one has instead a whole collection of antisymmetric maps $l_n: \Lambda^n L \to L$ of degrees $2-n$, where $n \geq 1$, satisfying certain compatibilities that generalize those of DGLAs. These maps are called Taylor coefficients or structure maps. In particular, the binary map $l_2$ that corresponds to the Lie bracket $[\cdot,\cdot]$ satisfies the Jacobi identity only up to terms depending on the homotopy $l_3$, and so on. Using the Koszul duality between DGLAs and graded cocommutative coalgebras, it is shown in [34] that $L_\infty$-algebra structures $\{l_n\}_{n\in\mathbb{N}}$ on a graded vector space $L$ are in one-to-one correspondence with codifferentials $Q$ of degree 1 on the cofree cocommutative conilpotent coalgebra $S(L[1])$ generated by the $L[1]$, where the degree is shifted by one. This dual coalgebraic formulation goes actually back to the BV-BRST formalism [1,2], see also [50].

As mentioned above, $L_\infty$-algebras play an important role in deformation theory. Here the basic philosophy is that, over a field of characteristic zero, every deformation problem is governed by a DGLA or more generally an $L_\infty$-algebra via solutions of the Maurer-Cartan equation modulo equivalences, see e.g. [31,39]. Staying for the moment for simplicity in the context of DGLAs, a Maurer-Cartan element $\pi$ in $(\mathfrak{g},d,[\cdot,\cdot])$ is an element of degree one satisfying the Maurer-Cartan equation

$$0 = d\pi + \frac{1}{2}[\pi,\pi].$$

This can be generalized to $L_\infty$-algebras, and there are suitable notions of gauge and homotopy equivalences of Maurer-Cartan elements, and in deformation theory one is interested in the transformation groupoid of the gauge action, the also called Goldman-Millson groupoid or Deligne groupoid [25]. Moreover, one can use Maurer-Cartan elements to twist the $L_\infty$-structure, i.e. change the $L_\infty$-structure in a certain way. For example, for a DGLA $(\mathfrak{g},d,[\cdot,\cdot])$ with Maurer-Cartan element $\pi$, the twisted DGLA takes the following form $(\mathfrak{g},d+[\pi,\cdot],[\cdot,\cdot])$, where the Maurer-Cartan equation implies that $d+[\pi,\cdot]$ squares to zero.

One nice feature of $L_\infty$-algebras is that they allow a more general notion of morphisms, so-called $L_\infty$-morphisms: an $L_\infty$-morphism from an $L_\infty$-algebra $(L,Q)$ to $(L',Q')$ is just a coalgebra
morphism between the corresponding coassociative co-Lie algebras $S(L[1])$ and $S(L'[1])$ that commutes with the coderivatives. In particular, $L_\infty$-morphisms are generalizations of Lie algebra morphisms and they are still compatible with Maurer-Cartan elements. Moreover, there is a notion of $L_\infty$-quasi-isomorphism generalizing the notion of quasi-isomorphisms between DGLAs. One can prove that such $L_\infty$-quasi-isomorphisms admit quasi-inverses, and that $L_\infty$-quasi-isomorphisms induce bijections on the equivalence classes of Maurer-Cartan elements, which is important for deformation theory. One can also twist $L_\infty$-morphisms by Maurer-Cartan elements, which gives $L_\infty$-morphisms between the twisted $L_\infty$-algebras. Note that the notion of $L_\infty$-algebras and $L_\infty$-morphisms can be generalized to algebras over general Koszul operads, see e.g. [37]. In addition, as one would expect, the generalization from DGLAs to $L_\infty$-algebras also leads to a generalization of the representation theory, i.e. from DG Lie modules to $L_\infty$-modules.

One famous deformation problem solved by $L_\infty$-algebraic techniques is the deformation quantization problem of Poisson manifolds, which was solved by Kontsevich’s celebrated formality theorem [30], see also [12,13] for the globalization of this result and the invariant setting of Lie group actions. More explicitly, the formality theorem provides an $L_\infty$-quasi-isomorphism between the differential graded Lie algebra of polyvector fields $\mathcal{T}_{poly}(M)$ and the polydifferential operators $\mathcal{D}_{poly}(M)$ on a smooth manifold $M$. As such, it induces a one-to-one correspondence between equivalence classes of Maurer-Cartan elements, i.e. between equivalence classes of (formal) Poisson structures and equivalence classes of star products. Using the language of $L_\infty$-modules, there has also been proven a formality theorem for Hochschild chains [14,48]. Moreover, in the last years many additional developments have taken place, see e.g. [6,7,36].

Apart from that, $L_\infty$-algebras can be used to describe many more geometric deformation problems, e.g. deformations of complex manifolds [38], deformations of foliations [52], deformations of Dirac structures [26], and many more. In addition, $L_\infty$-algebras are also an important tool in homological reduction theory, following the BV-BRST spirit, see e.g. [9,20,21,46], and in physics, where they occur for example in string theory and in quantum field theory.

Another important observation from [15] is that $L_\infty$-morphisms themselves correspond to Maurer-Cartan elements in a certain convolution-like $L_\infty$-algebra, which gives a way to speak of homotopic $L_\infty$-morphisms in the case of equivalent Maurer-Cartan elements. Homotopic $L_\infty$-morphisms share many features: for example, an $L_\infty$-morphism that is homotopic to an $L_\infty$-quasi-isomorphism is automatically an $L_\infty$-quasi-isomorphism itself, and homotopic $L_\infty$-morphisms induce the same maps on the equivalence classes of Maurer-Cartan elements. The second observation was used in [3] to prove that the globalization of the Kontsevich formality by Dolgushev [12,13] is, at the level of equivalence classes, independent of the chosen connection.

Finally, note that there is a generalization of $L_\infty$-algebras to curved $L_\infty$-algebras $(L, \{l_n\}_{n \in \mathbb{N}_0})$, where one allows an additional zero-th structure map $l_0: \mathbb{K} \to L[2]$, where $\mathbb{K}$ denotes the ground field. This corresponds to the curvature $l_0(1) \in L^2$. In this way, one can generalize the notion of curved Lie algebras $(\mathfrak{g}, R, d, [\cdot, \cdot])$, where the curvature $R \in \mathfrak{g}^2$ is closed, and where $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}]$. In this curved setting, one can speak of curved Maurer-Cartan elements, and all the notions from the above (flat) $L_\infty$-algebras generalize to this setting. For example, the curved Maurer-Cartan equation in a curved Lie algebras reads

$$0 = R + d\pi + \frac{1}{2}[\pi, \pi].$$

In particular, one can now twist with general elements of degree one, not just with Maurer-Cartan elements, without leaving the setting of curved $L_\infty$-algebras. Moreover, one also obtains a more general notion of curved $L_\infty$-morphisms [24], that are no longer coalgebra morphisms as in the above (flat) setting, but only coalgebra morphisms up to a twist. Curved $L_\infty$-algebras play for example an important role in Tsygan’s conjecture of an equivariant formality in deformation quantization [51], where the fundamental vector fields of the Lie group action play the role of the curvature.

The aim of this paper is to give a review over the theory of curved $L_\infty$-algebra, Maurer-Cartan elements, $L_\infty$-modules, and their homotopy theories. We mainly collect known results from the literature, but also include some results that are to our knowledge new or at least folklore knowledge, but not yet written properly down. For example, following [22] we show that $L_\infty$-algebras that are twisted with equivalent Maurer-Cartan elements are $L_\infty$-isomorphic, which
we could only find in the literature for the case of DGLAs. Moreover, we study the homotopy theory of curved \( L_\infty \)-morphisms, where we show in particular that curved \( L_\infty \)-morphisms that are twisted with equivalent Maurer-Cartan elements are homotopic. This result for DGLAs allowed us in \[33\] to prove that Dolgushev’s globalization procedure \[12, 13\] of the Kontsevich formality \[30\] with respect to different covariant derivatives yields homotopic \( L_\infty \)-quasi-isomorphisms.

The paper is organized as follows: We start in Section 2 with the construction of cocommutative cofree conilpotent coalgebras and coderivations on them. This allows a compact definition of (curved) \( L_\infty \)-algebras and \( L_\infty \)-morphisms in Section 3 in terms of codifferentials \( Q \) on \( S(L[1]) \) and coalgebra morphisms commuting with the codifferentials. In Section 4 we give explicit formulas for the homotopy transfer theorem, which provides a way to transfer \( L_\infty \)-structures along deformation retracts. There are many formulations and proofs for it, and in our special cases we use the symmetric tensor trick. In Section 5 we introduce the notion of Maurer-Cartan elements and compare the gauge equivalence of Maurer-Cartan elements in the setting of DGLAs and the more general notion of homotopy equivalence in \( L_\infty \)-algebras. Moreover, we recall the twisting procedure of curved \( L_\infty \)-algebras and of \( L_\infty \)-morphisms. In Section 6 we introduce the interpretation of \( L_\infty \)-morphisms as Maurer-Cartan elements and the notion of homotopic \( L_\infty \)-morphisms. We recall the homotopy classification of flat \( L_\infty \)-algebras and show that \( L_\infty \)-morphisms that are twisted with equivalent Maurer-Cartan elements are homotopic. In particular, we study curved \( L_\infty \)-morphisms, which are no longer coalgebra morphisms, but in some sense coalgebra morphisms up to twist. They are still compatible with curved Maurer-Cartan elements and have an analogue homotopy theory as the strict \( L_\infty \)-morphisms in the flat case. Finally, we recall in Section 7 the notion of \( L_\infty \)-modules over \( L_\infty \)-algebras, \( L_\infty \)-module morphisms between them, and the corresponding notion of homotopy equivalence.

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2 Introduction to Coalgebras

In this first section we want to recall the basic properties related to graded coalgebras. In view of \( L_\infty \)-algebras we are particularly interested in cocommutative conilpotent graded coalgebras. We mainly follow \[18, 37, 54\].

2.1 Reminder on (Free) Graded Algebras

Before starting with coalgebras we recall some notions connected to graded coalgebras over some commutative unital ring \( \mathbb{K} \), where we always assume \( \mathbb{Q} \subseteq \mathbb{K} \). By grading we mean \( \mathbb{Z} \)-grading and we always apply the Koszul sign rule, e.g. for tensor products of homogeneous morphisms \( \phi: V^* \to V^{*+|v|} \), \( \psi: W^* \to W^{*+|w|} \) between graded \( \mathbb{K} \)-modules \( V^*, V^*, W^*, W^* \):

\[
(\phi \otimes \psi)(v \otimes w) = (-1)^{|v||w|}\phi(v) \otimes \psi(w),
\]

where \( v \in V^{[v]}, w \in W \). Note that graded \( \mathbb{K} \)-modules with degree zero morphisms become a symmetric monoidal category with the graded tensor product and the graded switching map \( \tau \). Recall that on homogeneous elements \( v \in V, w \in W \) one has \( \tau(v \otimes w) = (-1)^{|v||w|} w \otimes v \). However, this monoidal structure is not compatible with the degree shift functor, where we write \( V^{[i]} = V^{i+k} \) and \( V^{[i]'} = (\mathbb{K}[i] \otimes V)^{[i]} \).

By \( (A^*, \mu) \) we usually denote a graded algebra with associative product \( \mu \) and by \( 1_A = 1: \mathbb{K} \to A \) we denote a unit. Let \( \phi: A^* \to B^* \) be a morphism of graded algebras, which is necessarily of degree zero. Then a graded derivation \( D: A^* \to B^{*+k} \) of degree \( k \in \mathbb{Z} \) along \( \phi \) is a \( \mathbb{K} \)-linear homogeneous map \( D \) such that

\[
D \circ \mu_A = \mu_B \circ (\phi \otimes D + D \otimes \phi).
\]

The tensor product \( A \otimes B \) becomes a graded algebra with product

\[
\mu_{A \otimes B} = (\mu_A \otimes \mu_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B),
\]
where \( \tau \) is again the graded switching map. The notion of free objects will be used frequently in the following, whence we recall the free associative algebra generated by \( \mathbb{K} \)-modules, as well as the commutative analogues. We write \( T^k(V) = V \otimes^k \) for \( k > 0 \) and \( T^0(V) = \mathbb{K} \) and obtain:

**Proposition 2.1** Let \( V^\bullet \) be a graded \( \mathbb{K} \)-module. Then the tensor algebra \( T^\bullet(V) = \bigoplus_{k=0}^\infty T^k(V) \) with induced grading from \( V^\bullet \) and canonical inclusion \( \iota : V = T^1(V) \to T(V) \) is the free graded associative unital algebra generated by \( V^\bullet \). More precisely, for every homogeneous \( \mathbb{K} \)-linear map \( \phi : V \to A \) from \( V \) to a unital graded associative algebra \( A \), there exists a unique algebra morphism \( \Phi : T(V) \to A \) such that the following diagram

\[
\begin{array}{ccc}
T(V) & \xrightarrow{\exists! \Phi} & A \\
\iota \downarrow & & \downarrow \phi \\
V & \xrightarrow{\sigma} & \\
\end{array}
\]

commutes.

Explicitly, the map \( \Phi \) is given by

\[ \Phi(1) = 1_A \quad \text{and} \quad \Phi(v_1 \otimes \cdots \otimes v_n) = \phi(v_1) \cdots \phi(v_n). \]

In order to construct the free commutative algebra generated by \( V^\bullet \), i.e. the analogue of the above proposition in the commutative setting, we can consider the (graded) symmetric algebra

\[ S(V) = T(V) / \langle x \otimes y - (-1)^{|x||y|} y \otimes x \rangle \quad (2.1) \]

with product denoted by \( \forall \). For later use we also recall the definition of the (graded) exterior algebra

\[ A(V) = T(V) / \langle x \otimes y + (-1)^{|x||y|} y \otimes x \rangle \quad (2.2) \]

with product denoted by \( \wedge \). The following sign conventions are also needed:

**Definition 2.2 (Graded signature, Koszul sign)** Let \( \sigma \in S_n \) be a permutation, \( V^\bullet \) a graded \( \mathbb{K} \)-module, and \( x_1, \ldots, x_n \) homogeneous elements of degree \( |x_i| \) for \( i = 1, \ldots, n \). The Koszul sign \( \epsilon(\sigma) \) is defined by the relation

\[ \epsilon(\sigma)x_{\sigma(1)} \land \cdots \land x_{\sigma(n)} = x_1 \land \cdots \land x_n. \quad (2.3) \]

By \( \chi(\sigma) = \text{sign}(\sigma)\epsilon(\sigma) \) we denote the antisymmetric Koszul sign, i.e.

\[ \chi(\sigma)x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)} = x_1 \wedge \cdots \wedge x_n. \quad (2.4) \]

Note that both signs depend on the degrees of the homogeneous elements, i.e. we should actually write \( \epsilon(\sigma) = \epsilon(\sigma, x_1, \ldots, x_n) \), analogously for \( \chi(\sigma) \). With these signs one can consider a right action of \( S_n \) on \( V^{\otimes n} \) given by the symmetrization

\[ \text{Sym}_n(v) = \frac{1}{n!} \sum_{\sigma \in S_n} v \circ \sigma, \]

where

\[ (v_1 \otimes \cdots \otimes v_n) \circ \sigma = \epsilon(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \]

It turns out that \( \bigoplus_n \text{im Sym}_n \cong S(V) \) is indeed the free associative graded commutative unital algebra generated by \( V \).

These free algebras also satisfy a universal property with respect to derivations:

**Proposition 2.3** Let \( V^\bullet \) be a graded \( \mathbb{K} \)-module.

i.) Let \( \phi : V^\bullet \to A^\bullet \) be a homogeneous map of degree zero into a graded associative unital algebra \( A \) and let \( d : V^\bullet \to A^{\bullet+k} \) be a linear map of degree \( k \in \mathbb{Z} \). Then there exists a unique graded derivation

\[ D : T(V)^\bullet \to A^{\bullet+k} \]

along \( \Phi : T(V) \to A \) such that \( D|_V = d \).
ii.) If $A$ is in addition graded commutative, then there exists a unique graded derivation

$$D: S(V)^* \to A^{*+\mathbb{K}}$$

along $\Phi: S(V) \to A$ such that $D|_V = d$.

The proof is straightforward, one defines $D$ on homogenous factors by

$$D(v_1 \otimes \cdots \otimes v_n) = \sum_{r=1}^{n} (-1)^{k(|v_1|+\cdots+|v_{r-1}|)}\phi(v_1) \cdots d(v_r) \cdots \phi(v_n)$$

and $D(1) = 0$ and notes that in the commutative case

$$I(V) = \text{span}\{x \otimes y - (-1)^{|x||y|} y \otimes x\}$$

is in the kernel, whence $D$ passes to the quotient $S(V)$.

### 2.2 Definition and First Properties of Graded Coalgebras

Now we want to recall the analogous constructions for graded coalgebras. We start with the definition of graded coalgebras and some basic properties.

**Definition 2.4 (Graded Coalgebra)** A graded coassociative coalgebra over $\mathbb{K}$ is a graded $\mathbb{K}$-module $C^*$ equipped with a binary co-operation, i.e. a linear map

$$\Delta: C^* \rightarrow C^* \otimes C^* \quad (2.6)$$

of degree zero that is coassociative, i.e.

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta. \quad (2.7)$$

The map $\Delta$ is called coproduct and we have

$$\Delta(C^i) \subset \bigoplus_{j+k=i} C^j \otimes C^k. \quad (2.8)$$

To simplify the notation we use Sweedler’s notation

$$\Delta^n(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(n+1)} = x_{(1)} \otimes \cdots \otimes x_{(n+1)} \in C^{\otimes n+1},$$

where $\Delta^n = (\Delta \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \Delta^{n-1}$ denotes the iterated coproduct $\Delta^n: C \rightarrow C^{\otimes n+1}$ with $\Delta^1 = \Delta$ and $\Delta^0 = \text{id}$. Note that by the coassociativity we have

$$\Delta^n = (\text{id} \otimes \cdots \otimes \text{id} \Delta \otimes \text{id} \otimes \cdots \otimes \text{id}) \circ \Delta^{n-1}, \quad (2.9)$$

whence the notation makes sense. The coassociative coalgebra $C$ is called counital if it is equipped with a counit, i.e. a linear map of degree zero with

$$\epsilon: C \rightarrow \mathbb{K}, \quad \text{satisfying} \quad (\epsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \epsilon) \circ \Delta. \quad (2.10)$$

For example, $\mathbb{K}$ itself is a coassociative coalgebra with $\Delta(1) = 1 \otimes 1$. A morphism of graded coalgebras $f: C \rightarrow C'$ is a linear map of degree zero that commutes with the coproducts and in the counital case also with the counits, i.e.

$$(f \otimes f) \circ \Delta_C = \Delta_{C'} \circ f, \quad \epsilon_{C'} \circ f = \epsilon_C. \quad (2.11)$$

The graded coalgebra $(C, \Delta)$ is called cocommutative if $\tau \circ \Delta = \Delta$, where $\tau$ denotes again the graded switching map. Denoting the graded dual by $\langle C^* \rangle^* = \text{Hom}_C^*(C, \mathbb{K})$, one can check that the inclusion $C^* \otimes C^* \rightarrow \langle C \otimes C \rangle^*$ induces always an algebra structure on $(C^*)^*$. The converse is generally not true; e.g., if $\mathbb{K}$ is a field, then the converse is only true in the case of finite dimensional vector spaces over $\mathbb{K}$.

Another useful interplay of coalgebras and algebras is the convolution algebra:
Definition 2.5 (Convolution product) Let \((A^*, \mu, 1)\) be a graded associative unital algebra over \(\mathbb{K}\) and let \((C^*, \Delta, \epsilon)\) be a graded coassociative counital algebra over \(\mathbb{K}\). Then one defines for \(\phi, \psi \in \text{Hom}_{\mathbb{K}}(C, A)\) their convolution by

\[
\phi \star \psi = \mu \circ (\phi \otimes \psi) \circ \Delta.
\] (2.12)

One can directly check that \((\text{Hom}_{\mathbb{K}}(C, A), \star, 1)\) is a graded associative unital algebra, see e.g. [37, Proposition 1.6.1]. Dually to ideals and subalgebras of algebras one can consider coideals and subcoalgebras.

Definition 2.6 (Coideal and Subcoalgebra) Let \((C^*, \Delta)\) be a graded coalgebra over \(\mathbb{K}\).

i.) A graded subspace \(I^* \subseteq C^*\) is called coideal if

\[
\Delta(I) \subseteq I \otimes C + C \otimes I \quad \text{and} \quad I \subseteq \ker \epsilon.
\] (2.13)

ii.) A graded subspace \(U^* \subseteq C^*\) is called subcoalgebra if

\[
\Delta(U) \subseteq U \otimes U.
\] (2.14)

Note that if \(\mathbb{K}\) is just a ring and not a field, then \(I \otimes C\) might not be mapped injectively into \(C \otimes C\) due to some torsion effects. We always assume that our modules have enough flatness properties and ignore these subtleties. As expected, the image of a coalgebra morphism is a subcoalgebra and the kernel is a coideal. We can quotient by coideals and every subcoalgebra \(U\) with \(U \subseteq \ker \epsilon\) is automatically a coideal.

There are special elements in coalgebras:

Definition 2.7 (Group-like elements) Let \((C^*, \Delta, \epsilon)\) be a graded coalgebra. An element \(g \in C\) is called group-like if

\[
\Delta(g) = g \otimes g \quad \text{and} \quad \epsilon(g) = 1.
\] (2.15)

The second condition is just to exclude the trivial case \(g = 0\), since \(\Delta(g) = g \otimes g\) implies \(g = \epsilon(g)g\) and since we assume torsion-freeness. Note that a morphism of coalgebras maps grouplike elements to grouplike elements. In view of \(L_\infty\)-algebras, coderivations will play an important role.

Definition 2.8 (Coderivation) Let \(\Phi: C^* \rightarrow E^*\) be a morphism of graded coalgebras. A graded coderivation along \(\Phi\) is a linear map \(D: C^* \rightarrow E^{*+k}\) of degree \(k\) such that

\[
\Delta \circ D = (D \otimes \Phi + \Phi \otimes D) \circ \Delta.
\] (2.16)

One can check that the set of coderivations of \(C\) along the identity is a graded Lie subalgebra of \(\text{End}_{\mathbb{K}}^*(C)\).

Proposition 2.9 A coderivation \(D: C^* \rightarrow E^{*+k}\) along \(\Phi\) satisfies \(\epsilon \circ D = 0\).

Proof: We compute

\[
D(x) = (\text{id} \otimes \epsilon) \circ \Delta \circ D(x) = D(x) \pm \Phi(x(1))\epsilon(D(x(2)))
\]

and thus applying \(\epsilon\) gives the result. □

There are examples of coalgebras with many grouplike elements, e.g. group coalgebras. However, we are mainly interested in coalgebras with one specific group-like element 1, called coaugmented.

Definition 2.10 A counital graded coalgebra \((C, \Delta, \epsilon)\) is called coaugmented if there exists a coalgebra morphism \(u: \mathbb{K} \rightarrow C\).
The element $1 = u(1)$ is indeed group-like since we know $\Delta \circ u = (u \otimes u) \circ \Delta$ and we obtain a non-full subcategory of coaugmented coalgebras, where the morphisms satisfy $\Phi(1) = 1$. Moreover, the definition implies $\text{id}_K = \epsilon \circ u$ and we get

$$C = C \oplus K1 = \ker \epsilon \oplus K1$$

(2.17)

via $c \mapsto (c - \epsilon(c)1) + \epsilon(c)1$. In this case one can define a reduced coproduct $\overline{\Delta} : C \rightarrow C \otimes C$ by

$$\overline{\Delta}(x) = \Delta(x) - 1 \otimes x - x \otimes 1$$

(2.18)

and we directly check for $x \in C$

$$(\epsilon \otimes \text{id})(\overline{\Delta}(x)) = x - \epsilon(1)x - \epsilon(x)1 = 0 = (\text{id} \otimes \epsilon)(\overline{\Delta}(x)).$$

An element $x \in C$ is called primitive if

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

(2.19)

or equivalently $x \in C$ with $\overline{\Delta}(x) = 0$. The set of primitive elements is denoted by $\text{Prim}(C)$.

**Proposition 2.11** The map $\overline{\Delta} : C \rightarrow C \otimes C$ is coassociative and restricts to $\overline{\Delta} : C \rightarrow C \otimes C$. Thus $(C, \overline{\Delta})$ becomes a coalgebra without counit, the so-called reduced coalgebra.

In particular, a map $\Phi : (C, \Delta, \epsilon, 1) \rightarrow (E, \Delta, \epsilon, 1)$ is a morphism of coaugmented coalgebras, i.e. a coalgebra morphism with $\Phi(1) = 1$, if and only if

$$(\Phi \otimes \Phi) \circ \overline{\Delta} = \overline{\Delta} \circ \Phi,$$

i.e. if and only if $\Phi$ induces a coalgebra morphism $\Phi : (C, \overline{\Delta}) \rightarrow (E, \overline{\Delta})$. The analogue holds for coderivations: a linear map $D : C^* \rightarrow E^{*+k}$ with $D(1) = 0$ is a coderivation along $\Phi$ if and only if

$$(D \otimes \Phi + \Phi \otimes D) \circ \overline{\Delta} = \overline{\Delta} \circ D.$$

A coaugmented graded coalgebra $(C, \Delta, \epsilon, 1)$ is said to be conilpotent if for each $x \in C$ there exists $n \in \mathbb{N}$ such that $\overline{\Delta}^n(x) = 0$ for all $m \geq n$. This is equivalent to the coradical filtration being exhaustive, i.e.

$$C^* = \bigcup_{k \in \mathbb{N}} F_k C^*.$$ 

Here $F_0 C = K1$ and

$$F_k C = K1 \oplus \{ c \in C | \overline{\Delta}^k c = 0 \}.$$ 

In particular, $F_1 C = K1 \oplus \text{Prim}(C)$. Note that this filtration is canonical and that morphisms of coaugmented graded coalgebras are automatically compatible with this filtration. Moreover, one can check that in the case of conilpotent coalgebras the group-like element is unique.

**Proposition 2.12** Let $(C^*, \Delta, \epsilon, 1)$ be a conilpotent coaugmented graded coalgebra over a field $K$. Then $1$ is the only group-like element.

**Proof:** Let $g \in C$ be a group-like element and set $c = g - 1 \in C$ since $\epsilon(g) = 1$. Then

$$\Delta(g) = \Delta(1 + c) = 1 \otimes 1 + \Delta(c) \quad \text{and} \quad \Delta(g) = (1 + c) \otimes (1 + c)$$

imply

$$\overline{\Delta}^k(c) = c \otimes \cdots \otimes c$$

for any $k$. By the conilpotency $\overline{\Delta}^k(c) = 0$ for some $k$, and thus $c = 0$. \qed

We want to list some other immediate features of coaugmented coalgebras.
Lemma 2.13 Let \((C^\bullet, \Delta, \epsilon, 1)\) be a coaugmented coalgebra and let \(V^\bullet\) be a graded \(\mathbb{K}\)-module. Then
\[
(\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta^{n-1} = (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta^{n-1}
\]
for all \(\mathbb{K}\)-linear maps \(\phi_i : C \to V\) with \(\phi_i(1) = 0\).

This implies for the convolution the following simplification:

Lemma 2.14 Let \((C^\bullet, \Delta, \epsilon, 1)\) be a coaugmented coalgebra and let \((A^\bullet, \mu, 1)\) be an algebra. Then one has
\[
\phi_1 \ast \cdots \ast \phi_n = \mu^{n-1} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta^{n-1}
\]
for all \(\mathbb{K}\)-linear maps \(\phi_i : C \to A\) with \(\phi_i(1) = 0\).

Remark 2.15 This lemma motivates the following convention. Whenever we have a \(\mathbb{K}\)-linear map \(\phi : C \to V\) we understand it to be extended to \(C\) by zero.

These observations allow us to define power series in \(\text{Hom}_{\mathbb{K}}(C, A)\):

Lemma 2.16 Let \((C^\bullet, \Delta, \epsilon, 1)\) be a conilpotent coalgebra and let \(A^\bullet\) be an algebra. For \(\phi \in \text{Hom}_{\mathbb{K}}(C, A)\) the map
\[
\mathbb{K}[\epsilon] \ni a = \sum_{n=0}^{\infty} a_n x^n \mapsto a_\ast(\phi) = \sum_{n=0}^{\infty} a_n \phi^\ast n \in \text{Hom}_{\mathbb{K}}(C, A)
\]
is a well-defined unital algebra morphism by the conilpotency, where \(\phi^\ast 0 = 1\epsilon\).

Example 2.17 This allows us to define for \(\phi \in \text{Hom}_{\mathbb{K}}(C, A)\)
\[
\log_\ast (1\epsilon + \phi) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \phi^\ast n \quad \text{and} \quad \exp_\ast (\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^\ast n
\]
and they are inverse to each other by the usual combinatorial formulas.

A nice consequence is that in the setting of conilpotent coaugmented coalgebras we can easily construct the cofree objects.

2.3 Cofree Coalgebras

We state at first the universal property that the cofree conilpotent coalgebra should satisfy.

Definition 2.18 (Cofree conilpotent coalgebra) Let \(V\) be a graded \(\mathbb{K}\)-module and let \(C\) be a conilpotent graded coassociative coalgebra. The cofree conilpotent coassociative coalgebra over \(V\) is a conilpotent coassociative coalgebra \(\mathcal{F}^c(V)\) equipped with a linear map \(p : \mathcal{F}^c(V) \to V\) such that \(1 \mapsto 0\) and such that the following universal condition holds for all \(C\): Any linear map \(\phi : C \to V\) with \(\phi(1) = 0\) extends uniquely to a coaugmented coalgebra morphism \(\Phi : C \to \mathcal{F}^c(V)\) such that the following diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\Phi} & \mathcal{F}^c(V) \\
\Phi \downarrow & \searrow \phi & \downarrow p \\
& V & \\
\end{array}
\]
commutes.

In other words, \(\mathcal{F}^c\) is right adjoint to the forgetful functor. Moreover, recall that conilpotent coalgebras are by definition coaugmented. Now we want to construct the cofree coalgebra, where the uniqueness up to isomorphism follows as usual for universal properties.
Let $V^\bullet$ be a graded $\mathbb{K}$-module, then the \textit{tensor coalgebra} $(T^c(V), \Delta)$ is given by the tensor module $T^c(V) = T(V) = \mathbb{K}1 \oplus V \oplus V^\otimes 2 \oplus \cdots$ with deconcatenation coproduct

\[
\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} (v_1 \cdots v_i) \otimes (v_{i+1} \cdots v_n) \in T(V) \otimes T(V) \quad \text{and} \quad \Delta(1) = 1 \otimes 1, \tag{2.23}
\]

where $v_1, \ldots, v_n \in V$. The counit $\epsilon: T^c(V) \to \mathbb{K}$ is the identity on $\mathbb{K}$ and zero otherwise, and the coalgebra turns out to be coassociative and counital. Moreover, it is coaugmented via the inclusion $\iota: \mathbb{K} \to T^c(V)$ and thus

\[
T^c(V) \cong T^c(\mathbb{K}) \oplus \mathbb{K}1 \tag{2.24}
\]

with reduced tensor module $T^c(\mathbb{K}) = \bigoplus_{i \geq 1} V^\otimes i$. The reduced coproduct is given by

\[
\Delta(v_1 \cdots v_n) = \sum_{i=1}^{n-1} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n, \quad \text{in particular} \quad \Delta(v) = 0. \tag{2.25}
\]

Thus $(T^c(V), \Delta, \epsilon, 1)$ is conilpotent and the construction is functorial in the following sense:

**Lemma 2.19** Let $\phi: V^\bullet \to W^\bullet$ be a homogeneous $\mathbb{K}$-linear map of degree zero. Then the map $\Phi: T(V) \to T(W)$ is a morphism of coaugmented coalgebras.

We also have the projection $\text{pr}_{T^c(V)}: T^c(V) \to V$ that is the identity on $V$ and zero elsewhere, giving the following useful property:

**Lemma 2.20** Let $V^\bullet$ be a graded $\mathbb{K}$-module. Then one has for all $n \in \mathbb{N}$

\[
\text{pr}_{T^n(V)} = \mu^{n-1} \circ (\text{pr}_V \otimes \cdots \otimes \text{pr}_V) \circ \Delta^{n-1} = \mu^{n-1} \circ (\text{pr}_V \otimes \cdots \otimes \text{pr}_V) \circ \overline{\Delta}^{n-1}. \tag{2.26}
\]

where $\mu = \otimes$ denotes the tensor product.

Thus we can write the projection $\text{pr}_{T^n(V)} = \text{pr}_V^n$ as convolution, which also holds for $n = 0$ since $\text{pr}_{T^0(V)} = 1\epsilon$, and we write $x_n = \text{pr}_{T^n(V)} x$ for $x \in T(V)$. We can show that we constructed indeed the cofree conilpotent coalgebra, compare \cite{37} Proposition 1.2.1:

**Theorem 2.21 (Cofree conilpotent coalgebra)** Let $V^\bullet$ be a graded module over $\mathbb{K}$ and let $(C^\bullet, \Delta, \epsilon, 1)$ be a conilpotent graded coassociative coalgebra over $\mathbb{K}$.

i.) The tensor coalgebra $(T^c(V), \Delta, \epsilon, 1)$ is cofree and cogenerated by $V$ in the category of conilpotent coalgebras. Explicitly, for every homogeneous $\mathbb{K}$-linear map $\phi: C \to V$ of degree zero extended to $C$ by $\phi(1) = 0$ there exists a unique counital coalgebra morphism $\Phi: C \to T^c(V)$ such that $\text{pr}_V \circ \Phi = \phi$.

ii.) If in addition $d: C \to V^\otimes k$ is a homogeneous $\mathbb{K}$-linear map of degree $k$ extended to $C$ by $d(1) = 0$, then there exists a unique coderivation $D: C^\bullet \to T^c(V)^{\otimes k}$ along $\Phi$ vanishing on $1$ such that $\text{pr}_V \circ D = d$.

**Proof:** The morphism $\Phi: C \to T^c(V)$ has to satisfy for $x \in C$

- $\Phi(1) = 1$ since $\Phi$ maps the group-like element to the group-like element,
- $\Phi(x)_0 = 0$ by counitality,
- $\Phi(x)_1 = \phi(x)$ by the universal property,
- $\Phi(x)_n = \sum \phi(x_{(1)}) \otimes \cdots \otimes \phi(x_{(n)})$ by the coalgebra morphism property, since

\[
\text{pr}_{T^n(V)} \Phi(x) = \mu^{n-1} \circ (\phi \otimes \cdots \otimes \phi) \circ \overline{\Delta}^{n-1}(x).
\]
As $C$ is conilpotent, there is only a finite number of nontrivial $\Phi(x)_n$ and $\Phi(x) = \sum_n \Phi(x)_n$ gives a well-defined linear map of degree zero. This shows the uniqueness and a direct computation shows that $\Phi$ is indeed a well-defined coalgebra morphism. For the second part we show the uniqueness by the same arguments: Suppose $D$ is such a coderivation, then since $\epsilon \circ D = 0$ by Proposition 2.9 we know

$$D(c) = \sum_{n=1}^{\infty} D(c)_n$$

with $D(c)_1 = d(c)$ for $c \in C$. For $n > 1$ we get with the Leibniz rule

$$D(c)_n = \mu^{n-1} \circ (\text{pr}_V \otimes \cdots \otimes \text{pr}_V) \circ \left( \sum_{r=0}^{n-1} \Phi \otimes \cdots \otimes \Phi \otimes D \otimes \Phi \otimes \Phi \right) \circ \Delta^{n-1}(c)$$

$$= \mu^{n-1} \circ \left( \sum_{r=0}^{n-1} \phi \otimes \cdots \otimes \phi \otimes d \otimes \phi \otimes \phi \right) \circ \Delta^{n-1}(c)$$

This shows that necessarily $D(1) = 0$ and a straightforward computation shows that this is indeed a well-defined coderivation.

**Corollary 2.22** For the coalgebra morphism $\Phi$ and the coderivation $D$ along $\Phi$ from the above theorem one has

$$\Phi = 1 + \phi \ast \phi + \cdots = \frac{1}{1 - \phi} \ast$$

and

$$D = d + \phi \ast d + d \ast \phi + \phi \ast d \ast \phi + \cdots = \Phi \ast d \ast \Phi.$$
Now we want to conclude this introductory section with the construction of the cofree cocommutative conilpotent coalgebras. From the point of \( L_\infty \)-algebras they are of great interest since they are the structures which one uses to define \( L_\infty \)-structures. The abstract reason for this is that the Koszul dual of the Lie operad is the cooperad encoding cocommutative coalgebras, see e.g. [27] Chapter 10 for the general setting.

Consider again a graded \( \mathbb{K} \)-module \( V^* \). We can use the tensor algebra \( T(V) \) of \( V \) to define a new coproduct, different from the deconcatenation coproduct \( \Delta \) from above. Explicitly, we equip \( T(V) \) with the structure of a bialgebra, i.e. we construct the new coproduct as an algebra morphism \( \Delta_{\text{sh}} : T(V) \to T(V) \otimes T(V) \). Since \( T(V) \) is the free algebra, we specify \( \Delta_{\text{sh}} \) on the generators by

\[
\Delta_{\text{sh}}(v) = v \otimes 1 + 1 \otimes v
\]

for \( v \in V \) and \( \Delta_{\text{sh}}(1) = 1 \otimes 1 \). One can check with the signs from Definition \[2.22\] that we get

\[
\Delta_{\text{sh}}(v_1 \cdots v_n) = \sum_{k=0}^{n} \sum_{\sigma \in Sh(k,n-k)} \epsilon(\sigma) (v_{\sigma(1)} \cdots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \cdots v_{\sigma(n)}). \tag{2.27}
\]

Here \( Sh(k,n-k) \subset S_n \) denotes the set of \((k,n-k)\)-shuffles, i.e. \( \sigma(1) < \cdots < \sigma(k) \) and \( \sigma(k+1) < \cdots < \sigma(n) \). We set \( Sh(0,n) = Sh(n,0) = \{1d\} \), and the above coproduct is well-defined because of \( S_n \cong Sh(k,n-k) \circ (S_k \times S_{n-k}) \). Moreover, \( \Delta_{\text{sh}} \) is coassociative, counital and graded cocommutative with respect to the usual counit \( \epsilon : T(V) \to \mathbb{K} \) and we call it shuffle coproduct. Since \( \Delta_{\text{sh}} \) is an algebra morphism, it is sufficient to show all these claims on generators. In particular, we can again consider the reduced coproduct

\[
\overline{\Delta}_{\text{sh}}(v_1 \cdots v_n) = \sum_{k=1}^{n-1} \sum_{\sigma \in Sh(k,n-k)} \epsilon(\sigma) (v_{\sigma(1)} \cdots v_{\sigma(k)}) \otimes (v_{\sigma(k+1)} \cdots v_{\sigma(n)}) \tag{2.28}
\]

and as it splits the tensor factors without using the unit 1 we get again

\[
\overline{\Delta}_{\text{sh}}(v_1 \cdots v_n) = 0.
\]

**Proposition 2.25** The tensor algebra \( (T(V), \Delta_{\text{sh}}, \epsilon, 1) \) is a conilpotent cocommutative coalgebra. In fact, it is even a bialgebra with respect to the usual tensor product \( \mu = \otimes \) and the usual unit 1.

In analogy to Lemma \[2.20\] we get the following result:

**Lemma 2.26** Let \( V^* \) be a graded \( \mathbb{K} \)-module. For \( n \in \mathbb{N} \) one has

\[
(pr_V \otimes \cdots \otimes pr_V) \circ \overline{\Delta}_{\text{sh}}^{n-1}(v_1 \cdots v_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{2.29}
\]

This is not exactly the property we would like to have since the right hand side is not the tensor we started with. Recall that this property was exactly the statement of Lemma \[2.20\] that we used extensively in Theorem \[2.24\] to show that \( (T(V), \Delta) \) is the cofree conilpotent coalgebra. However, we see that the right hand side of \[2.29\] is \( n! \text{Sym}_n(v_1 \otimes \cdots \otimes v_n) \), whence we would get the desired equation if we pass from \( T(V) \) to \( S(V) \). This is indeed possible.

**Lemma 2.27** Let \( V^* \) be a graded \( \mathbb{K} \)-module. Then the ideal \( I(V) \subseteq T(V) \) from \[2.3\] is a coideal with respect to \( \Delta_{\text{sh}} \) and \( \epsilon \).

**Proof:** The property \( I(V) \subseteq \ker \epsilon \) is obvious, and for \( v, w \in V^* \) we have

\[
\Delta_{\text{sh}}(vw - (-1)^{|v||w|} wv) = 1 \otimes (vw - (-1)^{|v||w|} wv) + (vw - (-1)^{|v||w|} wv) \otimes 1 \in I(V) \otimes T(V) + T(V) \otimes I(V)
\]

and the algebra morphism property shows the result.

This gives us a bialgebra structure on the quotient.
Proposition 2.28 Let $V^\bullet$ be a graded $\mathbb{K}$-module.

i.) The shuffle coproduct and the counit pass to the quotient by $I(V)$ and yield a bialgebra structure $(S(V), \mu_S, \epsilon)$, where $\mu_S = \circ$.

ii.) The bialgebra $(S(V), \mu_S, \Delta, 1, \epsilon)$ is graded commutative and graded cocommutative.

iii.) The coalgebra $(S(V), \Delta, \epsilon, 1)$ is coaugmented and conilpotent with coradical filtration

$$F_n(S(V)) = \bigoplus_{k=0}^n S^k(V).$$  \hspace{1cm} (2.30)

In particular, Lemma \ref{lem:2.26} implies now immediately the desired identity:

Lemma 2.29 Let $V^\bullet$ be a graded $\mathbb{K}$-module. For $n \in \mathbb{N}$ one has

$$pr_{S^n(V)} = \frac{1}{n!} \mu_S^{n-1} \circ (pr_V \otimes \cdots \otimes pr_V) \circ \Delta \otimes \cdots \otimes \Delta = \frac{1}{n!} \mu_S^{n-1} \circ (pr_V \otimes \cdots \otimes pr_V) \circ \Delta \otimes \cdots \otimes \Delta.$$ \hspace{1cm} (2.31)

This allows us to show that $S(V)$ is the cofree cocommutative conilpotent coalgebra:

Theorem 2.30 (Cofree cocommutative conilpotent coalgebra) Let $V^\bullet$ be a graded module over $\mathbb{K}$ and let $(C^\bullet, \Delta, \epsilon, 1)$ be a conilpotent graded cocommutative coalgebra over $\mathbb{K}$.

i.) The conilpotent cocommutative coalgebra $(S(V), \Delta, \epsilon, 1)$ is the cofree cocommutative conilpotent coalgebra cogenerated by $V$, i.e. the free object in the category of conilpotent cocommutative coalgebras. Explicitly, for every homogeneous $\mathbb{K}$-linear map $\phi: C \to V$ of degree zero with $\phi(1) = 0$ there exists a unique coalgebra morphism $\Phi: C \to S(V)$ such that

$$\Phi$$

commutes, i.e. $pr_V \circ \Phi = \phi$. Explicitly, one has $\Phi = \exp_\phi$ for the convolution product of $\text{Hom}^\bullet(C, S(V))$ with respect to $\mu_S$.

ii.) Let $d: C^\bullet \to V^{\bullet+k}$ be a homogeneous $\mathbb{K}$-linear map of degree $k$, then there exists a unique coderivation $D: C^\bullet \to S(V)^{\bullet+k}$ along $\Phi$ such that

$$\Phi$$

commutes. Explicitly, one has $D = \Phi \circ d = d \circ \Phi$ and $D(1) = 0$ if and only if $d(1) = 0$.

Proof: The proof is completely analogue to the proof of Theorem \ref{thm:2.21} \hfill \Box

For $C = S(V)$ itself and $\phi = pr_V$, i.e. $\Phi = \text{id}$, we get analogously to Corollary \ref{cor:2.24}

Corollary 2.31 Coderearations of $S(V)$ form a Lie subalgebra of the endomorphisms. This Lie algebra is in bijection to $\text{Hom}^\bullet_K(S(V), V)$ via

$$\text{Hom}^\bullet_K(S(V), V) \ni D \mapsto D = d \circ \text{id} \in \text{CoDer}^\bullet(S(V)).$$

Let now $D \in \text{CoDer}^\bullet(S(V))$, then we know again that it is completely determined by its projection $pr_V \circ D = d$. Here

$$d = \sum_{n=0}^{\infty} d_n \quad \text{with} \quad d_n = pr_V \circ D \circ pr_{T^n(V)}$$

and the maps $d_n: T^n(V) \to V$ are again called Taylor coefficients of $D$. By the above corollary we have the isomorphism

$$\text{CoDer}^\bullet(S(V)) \cong \text{Hom}^\bullet_K(S(V), V)$$
3 Introduction to $L_{\infty}$-algebras

3.1 Definition and First Properties

After this introduction to coalgebras we are now ready to study $L_{\infty}$-algebras. We start with (flat) $L_{\infty}$-structures, i.e. those corresponding to coderivations $Q$ with $Q(1) = 0$. For simplicity, we assume from now on that $K$ is a field of characteristic zero and we follow mainly [8, 13].

Definition 3.1 ($L_{\infty}$-algebra) A (flat) $L_{\infty}$-algebra is a graded vector space $L$ over $K$ endowed with a degree one codifferential $Q$ on the reduced symmetric coalgebra $(\mathcal{S}(L[1]), \Delta_{sh})$. An $L_{\infty}$-morphism between two $L_{\infty}$-algebras $F: (L, Q) \rightarrow (L', Q')$ is a morphism of graded coalgebras

$$F: \mathcal{S}(L[1]) \rightarrow \mathcal{S}(L'[1])$$

(3.1)
such that $F \circ Q = Q' \circ F$.

By Corollary 2.31 we can characterize the coderivation by its Taylor coefficients, also called structure maps.

Proposition 3.2 An $L_{\infty}$-algebra $(L, Q)$ is a graded vector space $L$ endowed with a sequence of maps

$$Q^n : \mathcal{S}^n(L[1]) \rightarrow L[2]$$

(3.2)
for $n > 0$. The coderivation $Q$ is given by

$$Q(x_1 \vee \cdots \vee x_n) = \sum_{k=1}^{n} \sum_{\sigma \in Sh(k,n-k)} \epsilon(\sigma)Q_k^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k)} \vee x_{\sigma(k+1)} \vee \cdots \vee x_{\sigma(n)})$$

(3.3)
for homogeneous vectors $x_1, \ldots, x_n \in L$, and $Q^2 = 0$ is equivalent to

$$\sum_{k=1}^{n} \sum_{\sigma \in Sh(k,n-k)} \epsilon(\sigma)Q_k^1Q_{k+1}^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k)} \vee x_{\sigma(k+1)} \vee \cdots \vee x_{\sigma(n)}) = 0.$$  

(3.4)

In particular, (3.4) implies $(Q^1)^2 = 0$, i.e. that $Q_1^1$ defines a differential on $L$ of degree one. The next order shows that $Q_1^2$ satisfies a Leibniz rule with respect to $Q_1^1$, and the third one can be interpreted as $Q_2^2$ satisfying a Jacobi identity up to terms depending on $Q_3^3$. We also write $Q_k = Q_k^1$ and following [8] we denote by $Q_n^i$ the component

$$Q_n^i = \text{pr}_{\mathcal{S}^i(L[1])} \circ Q|_{\mathcal{S}^n(L[1])} : \mathcal{S}^n(L[1]) \rightarrow \mathcal{S}^i(L[2])$$

of $Q$. It is given by

$$Q_n^i(x_1 \vee \cdots \vee x_n) = \sum_{\sigma \in Sh(n+1-i, i-1)} \epsilon(\sigma)Q_{n+1-i}^1(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(n+1-i)} \vee x_{\sigma(n+2-i)} \vee \cdots \vee x_{\sigma(n)})$$

(3.5)
where $Q_{n+1-i}^1$ are the usual structure maps. $L_{\infty}$-algebras have also been called strong homotopy Lie algebras as they are DGLAs up to higher homotopies.
Example 3.3 (DGLA) A differential graded Lie algebra (DGLA) \((\mathfrak{g}, d, [\cdot, \cdot])\) is an \(L_\infty\)-algebra with \(Q_1 = -d\) and \(Q_2(\gamma \vee \mu) = -(-1)^{|\gamma|}|\gamma, \mu|\) and \(Q_n = 0\) for all \(n > 2\), where \(|\gamma|\) denotes the degree in \(\mathfrak{g}[1]\).

This means that every DGLA is an \(L_\infty\)-algebra and for later use we consider the following example:

Example 3.4 Let \((M^\bullet, d)\) be a cochain complex over \(\mathbb{K}\). Then we define
\[
\text{End}^k(M) = \{\phi: M^\bullet \to M^\bullet + k \mid \phi \text{ linear}\}
\]
and \(\text{End}^k(M) = \bigoplus_{k \in \mathbb{Z}} \text{End}^k(M)\). For elements \(A_i \in \text{End}^{|A_i|}(M)\), we define
\[
[A_1, A_2] := A_1 \circ A_2 - (-1)^{|A_i||A_j|} A_2 \circ A_1,
\]
which is a graded Lie bracket. Finally, setting \(D = [d, \cdot]\), one can show that \((\text{End}^k(M), D, [\cdot, \cdot])\) is a DGLA and hence an \(L_\infty\)-algebra.

Remark 3.5 Note that \((Q_1^1)^2 = 0\) allows us to study the cohomology \(H(L)\) of the cochain complex \((L, Q_1^1)\). In particular, [40, Proposition 3.6] implies that \(H(L)\) inherits a Lie algebra structure since the bracket induced by \(Q_1^1\) satisfies the usual Jacobi identity.

Remark 3.6 (Antisymmetric formulation) Using the décalage-isomorphism
\[
dec^n: S^n(L) \to \Lambda^n(L[-1])[n]
\]
one can show that an \(L_\infty\)-algebra structure on \(L\) is equivalently given by a sequence of maps
\[
Q_n^\alpha: \Lambda^n L \to L[2 - n]
\]
for \(n > 0\) with
\[
\sum_{k=1}^{n} (-1)^{n-k} \sum_{\sigma \in \text{Sh}(k, n-k)} \chi(\sigma) Q_{n-k+1}^\alpha (Q_k^\alpha(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}) \wedge x_{\sigma(k+1)} \wedge \cdots \wedge x_{\sigma(n)}) = 0 \quad (3.7)
\]
for \(n \geq 1\), compare e.g. [40, Theorem 1.3.2]. Because of the grading shift from \(S(L[1])\) to \(\Lambda L\) one sometimes also speaks of \(L_\infty[1]\)-algebra structures on \(L[1]\) in the symmetric setting, and of \(L_\infty\)-algebra structures on \(L\) in the antisymmetric setting. But since they are equivalent we refer to them just as \(L_\infty\)-algebras. Moreover, note that the symmetric interpretation of the structure maps of \(L_\infty\)-morphisms is more natural in the sense of the definition via the cocommutative cofree coalgebra. However, if one sees \(L_\infty\)-algebras as a generalization of DGLAs the antisymmetric interpretation is the most natural one.

Similarly, we know from Theorem 2.30 that every \(L_\infty\)-morphism is characterized by its Taylor coefficients, i.e. by a sequence of maps.

Proposition 3.7 An \(L_\infty\)-morphism \(F:\ (L, Q) \to (L', Q')\) is uniquely determined by the collection of multilinear graded maps
\[
F_{n}^1 = \text{pr}_{L[1]} \circ F\big|_{S^n(L[1])}: S^n(L[1]) \to L'[1]
\]
for \(n \geq 1\). Setting \(F_0^1 = 0\), it follows that \(F\) is given by
\[
F(x_1 \vee \cdots \vee x_n) = \exp(F_1(x_1 \vee \cdots \vee x_n)) = \sum_{p \geq 1} \frac{e(\sigma)}{p!} \sum_{k_1 + \cdots + k_p = n, k_i \geq 1} \sum_{\sigma \in \text{Sh}(k_1, \ldots, k_p)} F_{k_1}^1 (x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k_1)}) \vee \cdots \vee F_{k_p}^1 (x_{\sigma(n-k_p+1)} \vee \cdots \vee x_{\sigma(n)})
\]
and the compatibility with the coderivations leads to further constraints. In particular, one has in lowest order \(F_1^1 \circ Q_1^1 = (Q')_1^1 \circ F_1^1\).
We also write $F_k = F_k^1$ and we get coefficients $F^1_n = \text{pr}_{S(L[1])} \circ F|_{S^n(L[1])} : S^n(L[1]) \to S^j(L'[1])$ of $F$. Note that $F^1_n$ depends only on $F^1_k = F_k$ for $k \leq n - j + 1$.

**Remark 3.8** Analogously to the case of coderivations, one can interpret an $L_\infty$-morphism as a sequence of multilinear maps

$$F_n : \Lambda^n L \longrightarrow L'[1 - n],$$

satisfying certain compatibility relations. In the case of DGLAs $(g_1, d_1, [\cdot, \cdot])$ and $(g_2, d_2, [\cdot, \cdot], 2)$ one can show that the compatibility with the differentials takes the following form

$$d_2 F_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} (-1)^{k_1 + \cdots + k_{n-1} + 1 + n} F_n(x_1, \ldots, d_1 x_i, \ldots, x_n)$$

$$\quad + \frac{1}{2} \sum_{k=1}^{n-1} \sum_{\sigma \in S(k, n-k)} \pm [F_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), F_{n-k}(x_{\sigma(k+1)}, \ldots, x_{\sigma(n)})]$$

$$\quad - \sum_{i \neq j} \pm F_{n-1}([x_i, x_j]_1, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$$

for $x_i \in g_1^{k_i}$, compare [12].

In order to gain a better understanding, we consider at first $L_\infty$-isomorphisms and their inverses. We follow [5, Section 2].

**Proposition 3.9** An $L_\infty$-morphism $F$ between $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ is an isomorphism if and only if $F^1_1$ is an isomorphism.

**Proof:** We recall the proof from [5, Proposition 2.2]. We only have to show that $F$ is invertible as coalgebra morphism if and only if $F^1_1$ is invertible with inverse $(F^1_1)^{-1}$. The fact that $F^{-1}$ is an $L_\infty$-morphism follows then directly:

$$F^{-1}Q' = F^{-1}Q'FF^{-1} = F^{-1}QF^{-1} = QF^{-1}.$$

Let therefore $F$ be an isomorphism with inverse $F^{-1}$. Then

$$\text{id}_{L[1]} = (\text{id}_{S(L[1])})^1_1 = (F^{-1}F)^1_1 = (F^{-1})^1_1 F^1_1,$$

analogously $\text{id}_{L'[1]} = F^1_1(F^{-1})^1_1$, whence $F^1_1$ is an isomorphism.

Suppose now that $F^1_1$ is an isomorphism. We construct a left inverse $G$ of $F$ by defining recursively its coefficient functions $G_n$. Starting with $G_1 = (F^1_1)^{-1}$ we want for $n > 1$

$$(GF)_n = \sum_{i=1}^{n} G^1_i F^1_n = (\text{id}_{S(L[1])})^1_n = 0,$$

which is fulfilled for

$$G_n = G^1_n = -\left(\sum_{i=1}^{n-1} G^1_i F^1_n\right) (F^1_n)^{-1}.$$

Note that $F^1_n$ is invertible since $F^1_1$ is invertible. By the same argument there exists a coalgebra morphism $F'$ with $F'G = \text{id}$ and thus $F' = F'GF = F$. 

We saw in Proposition 3.7 that the first Taylor coefficient resp. structure map $F_1$ of an $L_\infty$-morphism is a morphism of cochain complexes $(L, Q_1)$ and $(L, Q'_1)$, which leads us to the following definition.

**Definition 3.10** ($L_\infty$-quasi-isomorphism) An $L_\infty$-quasi-isomorphism between two $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ is a morphism $F$ of $L_\infty$-algebras such that the first structure map $F_1$ is a quasi-isomorphism of cochain complexes.
Example 3.11 (DGLA II) A DGLA quasi-isomorphism $\phi: g \to g'$ is an $L_\infty$-quasi-isomorphism $F$ with only non-vanishing structure map $F_1 = \phi$.

The notion of $L_\infty$-quasi-isomorphisms allows us to introduce the notion of formal $L_\infty$-algebras.

Definition 3.12 An $L_\infty$-algebra $(L, Q)$ is called formal, if there exists $L_\infty$-quasi-isomorphism $(L, Q) \to (\text{H}(L), Q_\text{H})$, where $Q_\text{H}$ denotes the Lie algebra structure on the cohomology $H(L)$ from Remark 3.5.

Example 3.13 (Kontsevich’s Formality Theorem) In [10] Kontsevich proved that there exists an $L_\infty$-quasi-isomorphism from the DGLA of polyvector fields $T_{\text{poly}}(\mathbb{R}^d)$ to the DGLA of polydifferential operators $D_{\text{poly}}(\mathbb{R}^d)$, compare Example 5.3 and Example 5.5. This implies that the DGLA of polydifferential operators is formal, explaining the name "formality theorem".

One reason why $L_\infty$-algebras are particularly useful is that all $L_\infty$-quasi-isomorphisms admit $L_\infty$-quasi-inverses, i.e. $L_\infty$-quasi-isomorphisms in the other direction that induce the inverse isomorphism in cohomology, see Theorem 4.13 below. Moreover, $L_\infty$-quasi-isomorphisms are important for the homotopy classification of $L_\infty$-algebras, compare Section 6.2, for which also another construction is needed: the direct sum resp. direct product of $L_\infty$-algebras.

Lemma 3.14 Let $(L, Q)$ and $(L', Q')$ be two $L_\infty$-algebras. Then

$$\hat{Q} = i \circ Q \circ p + i' \circ Q' \circ p'$$

(3.12)

defines a codifferential on $S(\hat{L}[1])$, where $\hat{L} = L \oplus L'$, i: $S(L[1]) \to S(\hat{L}[1])$ is the inclusion and $p: S(\hat{L}[1]) \to S(L[1])$ is the projection, analogously for $i', p'$. Moreover, $(\hat{L}, \hat{Q})$ is the direct product in the category of conilpotent cocommutative differential graded coalgebras without counit.

Proof: Explicitly, one has for $x_1, \ldots, x_m \in L$ and $x_{m+1}, \ldots, x_n \in L'$

$$\hat{Q}^1(x_1 \vee \cdots \vee x_n) = \begin{cases} Q^1(x_1 \vee \cdots \vee x_n) & \text{if } m = n \\ Q^1(x_1 \vee \cdots \vee x_n) & \text{if } m = 0 \\ 0 & \text{if } 0 < m < n, \end{cases}$$

and one can directly check $\hat{Q} \hat{Q} = 0$. In order to show that $(\hat{L}, \hat{Q})$ is the direct product of $(L, Q)$ and $(L', Q')$, consider two morphisms $F: C \to S(L[1])$ and $F': C \to S(L'[1])$, where $(C, D, \nabla)$ is a conilpotent cocommutative DG coalgebra without counit. Then $\hat{F}: C \to S(\hat{L}[1])$ defined by $\hat{F}^1 = F^1 \oplus F'^1$ is the only coalgebra morphism with $p\hat{F} = F$ and $p'\hat{F} = F'$. We only have to check $\hat{Q}\hat{F} = \hat{F} D$, where we get with Lemma 2.29

$$\hat{Q}^1 \hat{F} = \sum_{n>0} \frac{1}{n!} \hat{Q}^1_n \mu_{S}^{n-1}(F^1 \oplus F'^1)^{\otimes n} \Delta^{(n-1)}$$

$$= \sum_{n>0} \frac{1}{n!} \left( Q^1_n \mu_S^{n-1}(F^1)^{\otimes n} \Delta^{(n-1)} \oplus Q'^1_n \mu_S^{n-1}(F'^1)^{\otimes n} \Delta^{(n-1)} \right)$$

$$= Q^1 F \oplus Q'^1 F' = F^1 D \oplus F'^1 D = \hat{F}^1 D.$$

This shows the desired equality.

3.2 Curved $L_\infty$-algebras

As mentioned above, we can use the whole power of Theorem 2.30 by considering coderivations of $S(L[1])$ that do not vanish on the unit, yielding the notion of curved $L_\infty$-algebras.

Definition 3.15 (Curved $L_\infty$-algebra) A curved $L_\infty$-algebra is a graded vector space $L$ over $\mathbb{K}$ endowed with a degree one codifferential $Q$ on the cofree conilpotent coalgebra $(S(L[1]), \Delta_{sh})$ cogenerated by $L[1]$. 

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This codifferential $Q$ is equivalent to a sequence of maps $Q_n$ with $n = 0, 1, \ldots$, where the sum \[ Q \] starts now at $k = 0$. In particular, $Q^2 = 0$ implies
\[ Q_1(Q_0(1)) = 0 \quad \text{and} \quad Q_1(Q_1(x)) = -Q_2(Q_0(1) \lor x), \tag{3.13} \]
i.e. $Q_0(1)$ is always closed with respect to $Q_1$, but $Q_1$ in general no longer a coboundary operator. However, if $Q_0(1)$ is central, i.e.
\[ Q_{n+1}(Q_0(1) \lor x_1 \lor \cdots \lor x_n) = 0 \]
for all $n \geq 1$, then we have again $(Q_1)^2 = 0$. Morphisms of curved $L_\infty$-algebras are degree 0 counital coalgebra morphisms $F$ such that $F \circ Q = Q' \circ F$. As in the flat setting they are characterized by a sequence of maps $F_n$ with $n \geq 1$ satisfying the properties of Proposition \[ 3.7 \]
and the fact that $F'(1) = 1$. Note that this last property is clear since we consider a morphism between conilpotent counital coalgebras. These have unique grouplike elements 1 and $F$ has to map one to the other. Finally, note that a curved $L_\infty$-algebra with $Q_0 = 0$ is just a flat $L_\infty$-algebra as expected.

Example 3.16 (Curved Lie algebra) The basic example is a curved Lie algebra $(g, R, d, [\cdot, \cdot])$, i.e. a graded Lie algebra with derivation $d$ of degree +1 and $d^2 = \text{ad}(R) = [R, \cdot]$ as well as $dR = 0$. The element $R \in g^0$ is also called curvature. By setting $Q_0(1) = -R$ and with higher orders as in Example \[ 3.3 \] we obtain a curved $L_\infty$-algebra and $d$ is a differential, i.e. $d^2 = 0$, if and only if $R$ is central. Morphisms of curved Lie algebras are Lie algebra morphisms $f : g \to g'$ such that $f \circ d' = d' \circ f$ and $f(R) = R'$.

Remark 3.17 (Curved morphisms of curved Lie algebras) Note that there exists a more general notion of curved morphisms $f, \alpha : (g, R, d) \to (g', R', d')$ of curved Lie algebras where $\alpha \in g^1$ and where for all $x \in g$
\[ d'f(x) = f(dx) + [\alpha, f(x)] \quad \text{and} \quad R' = f(R) + d'\alpha - \frac{1}{2}[\alpha, \alpha], \]
see \[ 42 \] Definition 4.3. The usual case with $\alpha = 0$ is called strict and for a curved Maurer-Cartan element $x \in g^1$, see Section \[ 5.1 \] for the definition, one gets a curved Maurer-Cartan element $(f, \alpha)(x) = f(x) - \alpha \in g'^1$. In particular, $(\text{id}, \alpha)$ corresponds to twisting with $\alpha$ since
\[ R' = R + \alpha + [\alpha, \alpha] - \frac{1}{2}[\alpha, \alpha] = R^{\alpha}, \]
and $(f, \alpha)$ can be seen as strict morphism into the twisted curved Lie algebra $(g', R'^{\alpha}, d'^{\alpha}) = (g', R' - d'\alpha + \frac{1}{2}[\alpha, \alpha], d' - [\alpha, \cdot])$, see again Section \[ 5.1 \] for more details on the twisting procedure.

Remark 3.18 (Curved morphisms of curved $L_\infty$-algebras) The above curved morphisms of curved Lie algebras can be generalized to curved $L_\infty$-algebras by allowing zero-th Taylor coefficients $F_0^1 : K \to L'[1]$ with $F_0^1(1) = \alpha \in L'[1, 0]$. These curved morphisms of $L_\infty$-algebras are no longer coalgebra morphisms, but they still have some nice properties. In order to fully understand them we need to introduce Maurer-Cartan elements and the concept of twisting, which is done in the next sections. Afterwards, we will investigate the curved morphisms of curved $L_\infty$-algebras in Remark \[ 5.3 \] see also \[ 15 \] for the case of curved $A_\infty$-algebras.

Concerning the compatibility of flat $L_\infty$-morphisms with the curvature one has the following relation:

**Proposition 3.19** Let $F$ be an $L_\infty$-morphism of flat $L_\infty$-algebras $(L, Q)$ and $(L', Q')$. In addition, let $Q_0 \in L'[1], Q'_0 \in L'[1, 1]$ be closed and central with respect to the $L_\infty$-structures $Q$ and $Q'$, with induced curved $L_\infty$-structures $\tilde{Q}$ on $L$ and $\tilde{Q}'$ on $L'$. Then the structure maps of $F$ induces a morphism of curved $L_\infty$-algebras if and only if
\[ F_k^1(Q_0) = Q'_0 \quad \text{and} \quad F_k^1(Q_0 \lor \cdot) = 0 \quad \forall k > 1. \tag{3.14} \]
Proof: One only has to check if $F$ is compatible with the codifferentials $\tilde{Q}$ and $\tilde{Q}'$, where one gets for $v_1 \vee \cdots \vee v_n$ with $v_i \in L[1]^{k_i}$ and $n > 0$
\[
F^i \circ \tilde{Q}(v_1 \vee \cdots \vee v_n) = F^i(Q_0 \vee v_1 \vee \cdots \vee v_n + Q(v_1 \vee \cdots \vee v_n)) \\
= F^i_{n+1}(Q_0 \vee v_1 \vee \cdots \vee v_n) + (Q')^1 \circ F(v_1 \vee \cdots \vee v_n) \\
= \tilde{Q}' \circ F(v_1 \vee \cdots \vee v_n).
\]
In addition, one has
\[
Q_0' = \tilde{Q}' \circ F(1) = F \circ \tilde{Q}(1) = F^i_1(Q_0),
\]
which directly yields the above identities. \qed

In the following, if we speak of $L_\infty$-algebras we allow curved $L_\infty$-algebras, in cases where flatness is required we speak of flat $L_\infty$-algebras.

## 4 The Homotopy Transfer Theorem and the Minimal Model of a $L_\infty$-algebra

It is well-known that given a homotopy retract one can transfer $L_\infty$-structures, see e.g. [37, Section 10.3]. Explicitly, a homotopy retract (also called homotopy equivalence data) consists of two cochain complexes $(A, d_A)$ and $(B, d_B)$ with chain maps $i, p$ and homotopy $h$ such that
\[
(A, d_A) \xymatrix{ \ar[r]^i & (B, d_B) \ar[l]_p } \xymatrix@C=1cm{ \ar[r]_h } \quad (4.1)
\]
with $h \circ d_B + d_B \circ h = \text{id} - i \circ p$, and such that $i$ and $p$ are quasi-isomorphisms. Then the homotopy transfer theorem states that if there exists a flat $L_\infty$-structure on $B$, then one can transfer it to $A$ in such a way that $i$ extends to an $L_\infty$-quasi-isomorphism. By the invertibility of $L_\infty$-quasi-isomorphisms, a statement that we prove in Theorem 4.13 below, there also exists an $L_\infty$-quasi-isomorphism into $A$ denoted by $P$, see e.g. [37, Proposition 10.3.9].

### 4.1 Homotopy Transfer Theorem via Symmetric Tensor Trick

We want to state different versions of this statement. For simplicity, we assume that we have a deformation retract, i.e. we are in the situation $(4.1)$ with additionally
\[
p \circ i = \text{id}_A.
\]
By [28, Remark 2.1] we can assume that we have even a special deformation retract, also called contraction, where
\[
h^2 = 0, \quad h \circ i = 0 \quad \text{and} \quad p \circ h = 0.
\]
Assume now that $(B, Q_B)$ is an $L_\infty$-algebra with $(Q_B)_1 = -d_B$. In the following we give a more explicit description of the transferred $L_\infty$-structure $Q_A$ on $A$ and of the $L_\infty$-projection $P: (B, Q_B) \rightarrow (A, Q_A)$ inspired by the symmetric tensor trick [14, 27, 28, 41], see also [13] for the case of $A_\infty$-algebras. The map $h$ extends to a homotopy $H_n: S^n(B[1]) \rightarrow S^n(B[1])[-1]$ with respect to $Q_{B,n}^n: S^n(B[1]) \rightarrow S^n(B[1])[1]$, see e.g. [37, p. 383] for the construction on the tensor algebra, which adapted to our setting works as follows: we define the operator
\[
K_n: S^n(B[1]) \rightarrow S^n(B[1])
\]
by
\[
K_n(x_1 \vee \cdots \vee x_n) = \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\epsilon(\sigma)}{n-i} ipX_{\sigma(1)} \vee \cdots \vee ipX_{\sigma(i)} \vee X_{\sigma(i+1)} \vee X_{\sigma(n)}.
\]
where $Q$ With the definition we have, together with equations (4.3), that

$$
\hat{H}_n(x_1 \vee \cdots \vee x_n) := - \sum_{\sigma \in \text{Sh}(1,n-1)} \epsilon(\sigma) \ circ h x_{\sigma(1)} \vee x_{\sigma(2)} \vee \cdots \vee x_{\sigma(n)}.
$$

**Lemma 4.1** With the definition above, we have

$$K_n \circ Q_{B,n}^n = Q_{B,n}^n \circ K_n \text{ and } K_n \circ \hat{H}_n = \hat{H}_n \circ K_n,$$

(4.2)

where $Q_{B,n}^n$ is the extension of the differential $Q_{B,1}^1 = - d_B$ to $S^n(B[1])$ as a coderivation.

**Proof:** Since $i$ and $p$ are chain maps, it is clear that $K_n \circ Q_{B,n}^n = Q_{B,n}^n \circ K_n$. The second equation follows from the fact that we have $h \circ i = 0$ and $p \circ h = 0$. \hfill \Box

With the definition

$$H_n := K_n \circ \hat{H}_n = \hat{H}_n \circ K_n$$

we have, together with equations (4.3), that

$$Q_{B,n}^n H_n + H_n Q_{B,n}^n = (n \cdot \text{id} - ip) \circ K_n,$$

where $ip$ is extended as a coderivation to $S(B[1])$.

**Proposition 4.2** In the above setting one has

$$Q_{B,n}^n H_n + H_n Q_{B,n}^n = \text{id} - (ip)^n.$$

(4.3)

**Proof:** Because of the previous results, it is enough to show that $\text{id} - (ip)^n = (n \cdot \text{id} - ip) \circ K_n$. Let $X_1 \vee \cdots \vee X_n \in S^n(B[1])$, then we have

$$(n \cdot \text{id} - ip) \circ K_n(X_1 \vee \cdots \vee X_n)$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$- \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} i \epsilon(\sigma) \ circ n - i \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$- \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee ip x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$- \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\epsilon(\sigma \ circ (i \ circ \tau))}{n - i} \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee ip x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$- \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \frac{\epsilon(\sigma)}{n - i} \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee ip x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

$$= \frac{1}{n!} \sum_{i=0}^{n-1} \sum_{\sigma \in S_n} \epsilon(\sigma) \ circ ip x_{\sigma(1)} \vee \cdots \vee ip x_{\sigma(i)} \vee x_{\sigma(i+1)} \vee x_{\sigma(n)}$$

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The finalization of the proof is just a comparison of the summands.

Suppose now that we already have constructed a codifferential $Q_A$ and a morphism of coalgebras $P$ with structure maps $P^i_k : S^k(B[1]) \to A[1]$ such that $P$ is an $L_{\infty}$-morphism up to order $k$, i.e.

$$\sum_{\ell=1}^m P^1_{\ell} \circ Q^i_{B,m} = \sum_{\ell=1}^m Q^1_{A,\ell} \circ P^i_{m}$$

for all $m \leq k$. Then we have the following statement.

**Lemma 4.3** Let $P : S(B[1]) \to S(A[1])$ be an $L_{\infty}$-morphism up to order $k \geq 1$. Then

$$L_{\infty,k+1} = \sum_{\ell=2}^{k+1} Q^1_{A,\ell} \circ P^i_{k+1} - \sum_{\ell=1}^k P^1_{\ell} \circ Q^i_{B,k+1}$$

satisfies

$$L_{\infty,k+1} \circ Q^i_{B,k+1} = -Q^1_{A,1} \circ L_{\infty,k+1}.$$  

**Proof:** The statement follows from a straightforward computation. For convenience we omit the index of the differential:

$$L_{\infty,k+1} Q^i_{k+1} = \sum_{\ell=2}^{k+1} Q^i_{A,\ell} (P \circ Q)_{k+1} = \sum_{\ell=2}^{k+1} \sum_{i=1}^k Q^i_{A,\ell} P^i_1 Q^i_{i,k+1} + \sum_{\ell=1}^k P^i_{\ell} Q^i_{i,k+1}$$

$$= \sum_{\ell=2}^{k+1} Q^i_{A,\ell} (P \circ Q)_{k+1} - \sum_{\ell=2}^{k+1} \sum_{i=1}^k Q^i_{A,\ell} P^i_1 Q^i_{i,k+1} + \sum_{\ell=1}^k P^i_{\ell} Q^i_{i,k+1}$$

$$= -Q^i_{A,1} (P \circ Q)_{k+1} + Q^i_{1} \sum_{i=1}^k P^i_1 Q^i_{i,k+1} = -Q^i_{A,1} L_{\infty,k+1},$$

where the last equality follows from $Q^i_{A,1} = 0$.

This allows us to prove one version of the homotopy transfer theorem as formulated in [21] Theorem B.2.

**Theorem 4.4 (Homotopy transfer theorem)** Let $(B, Q_B)$ be a flat $L_{\infty}$-algebra with $(Q_B)_1 = -d_B$ and contraction

$$(A, d_A) \xleftarrow{i} P \xrightarrow{h} (B, d_B)$$  

(4.6)

Then

$$(Q_A)_1 = -d_A, \quad (Q_A)_{k+1} = \sum_{i=1}^k P^i_{1} \circ (Q_B)_k \circ i^r(k+1)$$

(4.7)

turns $(A, Q_A)$ into an $L_{\infty}$-algebra with $L_{\infty}$-quasi-isomorphism $P : (B, Q_B) \to (A, Q_A)$. Moreover, one has $P^i_{k} \circ i^r_k = 0$ for $k \neq 1$.

**Proof:** We observe $P^i_{k+1} (ix_1 \vee \cdots \vee ix_{k+1}) = 0$ for all $k \geq 1$ and $x_i \in A$, which directly follows from $h \circ i = 0$ and thus $H^k_{i+1} \circ i^r(k+1) = 0$. Suppose that $Q_A$ is a codifferential up to order $k \geq 1$, i.e. $\sum_{i=1}^m (Q_A)_{1} (Q_A)_m = 0$ for all $m \leq k$, and that $P$ is an $L_{\infty}$-morphism up to order $k \geq 1$. We
know that these conditions are satisfied for \( k = 1 \) and we show that they hold for \( k + 1 \). Starting with \( Q_A \) we compute

\[
(Q_A Q_A)_{k+1}^1 = (Q_A Q_A)_{k+1}^1 \circ P_{k+1}^+ \circ i^{\cdot(k+1)} = \sum_{\ell=1}^{k+1} (Q_A Q_A)_{\ell}^1 P_{k+1}^\ell \circ i^{\cdot(k+1)} = (Q_A Q_A)_{k+1}^1 \circ i^{\cdot(k+1)}
\]

\[
= \sum_{\ell=2}^{k+1} (Q_A)_{\ell}^1 (Q_A P)_{k+1}^\ell \circ i^{\cdot(k+1)} + (Q_A)_{k+1}^1 \circ i^{\cdot(k+1)}
\]

\[
= \sum_{\ell=2}^{k+1} (Q_A P)_{k+1}^\ell \circ i^{\cdot(k+1)} + (Q_A)_{k+1}^1 \circ i^{\cdot(k+1)}
\]

\[
= \sum_{\ell=2}^{k+1} (Q_A P)_{k+1}^\ell \circ i^{\cdot(k+1)} + (Q_A)_{k+1}^1 \circ i^{\cdot(k+1)}
\]

By the same computation as in Lemma 4.3, where one in fact only needs that

\[
\begin{align*}
Q_B &\equiv 0, \\
Q_B^+ &\equiv \sum_{\ell=1}^{k+1} Q_{A,\ell}^1 P_{k+1}^\ell - \sum_{\ell=1}^{k} P_{k+1}^\ell Q_{B,\ell}^1 \circ \overline{(i \circ p)}^{\cdot(k+1)}
\end{align*}
\]

Therefore

\[
P_{k+1}^1 \circ (Q_B)_{k+1}^1 - (Q_A)_{k+1}^1 \circ P_{k+1}^1 = L_{k,1},
\]

i.e. \( P \) is an \( L_{k,1} \)-morphism up to order \( k + 1 \), and the statement follows inductively.

\[
\text{Remark 4.5 (HTT vs. HPL)}
\]

In view of Lemma 4.2, we can define \( H : S(B[1]) \to S(B[1]) \) by

\[
H_{|S^n(B[1])} = H_n
\]

to obtain

\[
(S(A[1]), \hat{Q}_A) \xleftarrow{I} (S(B[1]), \hat{Q}_B) \xrightarrow{H} (S(A[1]), \hat{Q}_A)
\]

where \( I \) (resp. \( P \)) is the map \( i \) (resp. \( p \)) extended as a coalgebra morphism and \( \hat{Q}_A \) (resp. \( \hat{Q}_B \)) is the differential \(- d_A \) (resp. \(- d_B \)) extended as a coderivation. Note that if \( h^2 = 0 \) we also have \( H^2 = 0 \). Now we can see the higher brackets \( Q_B - \hat{Q}_B \) as a perturbation of the coderivation \( \hat{Q}_B \). Since \( (Q_B - \hat{Q}_B) \circ H \) decreases the symmetric degree, it is locally nilpotent, and we can apply the homological perturbation lemma, see e.g. [13] and references therein. Note, that it is not clear why the resulting deformation retract gives maps which are compatible with the coalgebra structure.
With the homotopy transfer theorem we can show that every contraction induces a splitting of the $L_\infty$-algebra in the following sense, compare Lemma \ref{lem:splitting} for the notion of a direct sum of $L_\infty$-algebras.

**Theorem 4.6** In the above setting one has an $L_\infty$-isomorphism

$$L: B \longrightarrow A \oplus \text{im}[d_B, h],$$

where the $L_\infty$-structure on $\text{im}[d_B, h]$ is given by just the differential $Q^1_1 = -d_B$ and the $L_\infty$-structure on $A \oplus \text{im}[d_B, h]$ is the product $L_\infty$-structure of the transferred one on $A$ and the differential on $\text{im}[d_B, h]$.

**Proof:** By Theorem \ref{thm:homotopy-transfer} we already have an $L_\infty$-morphism $P: B \rightarrow A$ with first structure map $p$. Now we construct an $L_\infty$-morphism $F: B \rightarrow \text{im}[d_B, h] = C$ by setting

$$F^1_1 = [d_B, h] \quad \text{and} \quad F^1_n = -h \circ \sum_{i=1}^{n-1} F^i_1(Q_B)_n^i \quad \text{for } n > 1.$$ 

It is an $L_\infty$-morphism up to order one. Suppose it is one up to order $n \geq 1$, then we get

$$(Q_C)^n_1 F_{n+1}^1 = d_B|_C \circ h \circ \sum_{i=1}^n F^i_1(Q_B)^{i}_{n+1} = (\text{id} - ip - h \circ d_B|_C) \sum_{i=1}^n F^i_1(Q_B)^{i}_{n+1}$$

$$= \sum_{i=1}^n F^i_1(Q_B)^{i}_{n+1} + h \circ \sum_{i=1}^n (FQ_B)^i_1(Q_B)^{i}_{n+1} = \sum_{i=1}^{n+1} F^i_1(Q_B)^{i}_{n+1},$$

thus by induction $F$ is an $L_\infty$-morphism. The universal property of the product gives the desired $L_\infty$-morphism $L = P \oplus F$ which is even an $L_\infty$-isomorphism since its first structure map $p \oplus (d_Bh + h d_B)$ is an isomorphism with inverse $\oplus \text{id}$, see Theorem \ref{thm:isomorphism}.

We also want to give an explicit formula for a $L_\infty$-quasi-inverse of $P$, where we follow \cite[Proposition B.3]{21}.

**Proposition 4.7** The coalgebra map $I: S^*(A[1]) \rightarrow S^*(B[1])$ recursively defined by the maps $I^1_1 = i$ and $I^1_{k+1} = h \circ L_\infty,k+1$ for $k \geq 1$ is an $L_\infty$-quasi inverse of $P$. Since $h^2 = 0 = h \circ i$, one even has $I^1_{k+1} = h \circ \sum_{\ell=2}^{k+1} Q_B^\ell \circ I^\ell_{k+1}$ and $P \circ I = \text{id}_A$.

**Proof:** We proceed by induction: assume that $I$ is an $L_\infty$-morphism up to order $k$, then we have

$$I^1_{k+1}Q_A^{k+1} - Q_B^{k+1}I^1_{k+1} = -Q_B^{k+1} \circ h \circ L_\infty,k+1 \circ h \circ L_\infty,k+1 \circ Q_A^{k+1}$$

$$= -Q_B^{k+1} \circ h \circ L_\infty,k+1 - h \circ Q_B^{k+1} \circ L_\infty,k+1$$

$$= (\text{id} - i \circ p)L_\infty,k+1.$$ 

We used that $Q_B^{1} = -d_B$ and the homotopy equation of $h$. Moreover, we get with $p \circ h = 0$

$$p \circ L_\infty,k+1 = p \circ \left( \sum_{\ell=2}^{k+1} Q_B^{\ell} \circ I^\ell_{k+1} + \sum_{\ell=1}^{k} I^\ell_{k+1} \circ Q_A^{\ell} \right)$$

$$= \sum_{\ell=2}^{k+1} (P \circ Q_B)^\ell_1 \circ I^\ell_{k+1} + \sum_{\ell=2}^{k} \sum_{i=2}^{\ell} P_i^1 \circ Q_B^{i} \circ I^i_{k+1} - Q_A^{i}$$

$$= \sum_{\ell=2}^{k+1} (Q_A \circ P)^\ell_1 \circ I^\ell_{k+1} + \sum_{\ell=2}^{k} \sum_{i=2}^{\ell} P_i^1 \circ Q_B^{i} \circ I^i_{k+1} - Q_A^{i}$$

$$= Q_A^{k+1} \sum_{\ell=2}^{k+1} \sum_{i=2}^{\ell} P_i^1 \circ I^i_{k+1} \circ Q_A^{\ell} - Q_A^{k+1} = 0,$$

and therefore $I$ is an $L_\infty$-morphism. \hfill $\square$
Remark 4.8 Note that in the homotopy transfer theorem the property \( h^2 = 0 \) is not needed, and that one can also adapt the above construction of \( I \) to this more general case.

Following [20], let us now consider the special case of contractions of DGLAs. More explicitly, let now \( A, B \) be two DGLAs and assume in addition that \( i \) is a DGLA morphism. Then the homotopy transfer theorem immediately yields:

**Proposition 4.9** Defining \( P_1^i = p \) and \( P_{k+1}^i = L_{\infty,k+1} \circ H_{k+1} \) for \( k \geq 1 \) yields an \( L_{\infty} \)-quasi-isomorphism \( P: (B,Q_B) \rightarrow (A,Q_A) \) that is quasi-inverse to \( i \). Here the codifferentials are induced by the respective DGLA structures.

**Proof:** The transferred \( L_{\infty} \)-structure on \( A \) from (4.7) is indeed just the DGLA structure since \( i \) is a DGLA morphism.

Let us now assume that \( p: B \rightarrow A \) in the contraction (4.11) is a DGLA morphism and that \( i \) is just a chain map. Then we can analogously give a formula for the extension \( I \) of \( i \) to an \( L_{\infty} \)-quasi-isomorphism.

**Proposition 4.10** The coalgebra map \( I: S^\bullet(A[1]) \rightarrow S^\bullet(B[1]) \) recursively defined by the maps \( I_1^i = i \) and \( I_k^i = h \circ L_{\infty,k} \) for \( k \geq 2 \) is an \( L_{\infty} \)-quasi inverse of \( i \). Since \( i^2 = 0 = h \circ i \), one even has \( I_k^i = h \circ Q_2^i \circ I_k^2 \).

**Proof:** This is just a special case of Proposition [4.7] □

### 4.2 The Minimal Model and the Existence of \( L_{\infty} \)-Quasi-Inverses

As a first application of the homotopy transfer theorem we want to show that \( L_{\infty} \)-algebras split into the direct product of two special ones, whence we recall some definitions.

**Definition 4.11** An \( L_{\infty} \)-algebra \((L,Q)\) is called minimal if \( Q_1^L = 0 \) and linear contractible if \( Q_n^L = 0 \) for \( n > 1 \) with acyclic \( Q_1^L \).

An \( L_{\infty} \)-algebra is called contractible if it is isomorphic to a linear contractible one, and we can show that every \( L_{\infty} \)-algebra is isomorphic to the direct sum of these two types [8] Proposition 2.8:

**Proposition 4.12** Any \( L_{\infty} \)-algebra is isomorphic to the direct sum of a minimal and of a linear contractible one.

**Proof:** Let \((L,Q)\) be an \( L_{\infty} \)-algebra and let \((L,d = -Q_1^L)\) be the underlying cochain complex. Since we are working over a field \( \mathbb{K} \) of characteristic zero, we can find a deformation retract

\[
(H(L),0) \xleftarrow{i} (L,d) \xrightarrow{h} (H(L),0)
\]

where \( H(L) \) denotes the cohomology of \((L,d)\). Then we apply Theorem 4.10 and get the result. □

Denoting the transferred minimal \( L_{\infty} \)-structure on \( H(L) \) by \( Q_H \), the above proposition gives in particular an \( L_{\infty} \)-quasi-isomorphism \((L,Q) \rightarrow (H(L),Q_H)\). For this reason, \((H(L),Q_H)\) is also called minimal model of \((L,Q)\). This result allows us to explicitly invert \( L_{\infty} \)-quasi-isomorphisms.

**Theorem 4.13** If \( F \) is an \( L_{\infty} \)-quasi-isomorphism from \((L,Q)\) to \((L',Q')\), then there exists an \( L_{\infty} \)-morphism \( G \) in the other direction, inducing the inverse isomorphism in cohomology.

**Proof:** We know that both \( L \) and \( L' \) are isomorphic to direct sums of minimal \( L_{\infty} \)-algebras \( L_{\text{min}} \) and \( L'_{\text{min}} \) and linear contractible ones. In particular, the inclusions and projections

\[
i: (L_{\text{min}},Q_{\text{min}}) \rightarrow (L,Q), \quad p: (L,Q) \rightarrow (L_{\text{min}},Q_{\text{min}})
\]

are \( L_{\infty} \)-quasi-isomorphisms, analogously for \( i' \) and \( p' \). In particular,

\[
F_{\text{min}} = p'Fi: (L_{\text{min}},Q_{\text{min}}) \rightarrow (L'_{\text{min}},Q'_{\text{min}})
\]

is an \( L_{\infty} \)-quasi-isomorphism. But since \( (Q_{\text{min}})_1 = 0 = (Q_{\text{min}})_1' \) we know that \( (F_{\text{min}})_1 \) is an isomorphism, thus \( F_{\text{min}} \) is an \( L_{\infty} \)-isomorphism by Proposition 3.9 and we can set \( G = i(F_{\text{min}})^{-1}p' \) for the \( L_{\infty} \)-quasi-isomorphism in the other direction. □
5 Maurer-Cartan Elements, their Equivalence Classes and Twisting

In this section we recall the notion of Maurer-Cartan elements and different notions of equivalences between them. One important application is the twisting of $L_\infty$-algebras and $L_\infty$-morphisms.

5.1 Maurer-Cartan Elements in DGLAs

We start with the case of Maurer-Cartan elements in DGLAs.

**Definition 5.1 (Maurer-Cartan elements)** Let $(g, d, [\cdot, \cdot])$ be a DGLA. Then $\pi \in g^1$ is called **Maurer-Cartan element** if it satisfies the Maurer-Cartan equation

\[ d\pi + \frac{1}{2}[\pi, \pi] = 0. \]  

(5.1)

The set of Maurer-Cartan elements is denoted by $MC(g)$ and we directly see that for a Maurer-Cartan element $\pi$ the map $d + [\pi, \cdot]$ is again a differential, the so-called **twisted differential**. Taking a general element $x \in g^1$ the derivation $d + [x, \cdot]$ yields a curved Lie algebra with curvature $R_x = dx + \frac{1}{2}[x, x]$.

Starting with a curved Lie algebra $(g, R, d, [\cdot, \cdot])$, the twisting yields the new curvature $R^x = R + dx + \frac{1}{2}[x, x]$ and one calls $\pi \in g^1$ **curved Maurer-Cartan element** if one has $R^\pi = R + d\pi + \frac{1}{2}[\pi, \pi] = 0$.

In this case the twisted DGLA $(g, R^\pi = 0, d + [\pi, \cdot], [\cdot, \cdot])$ is a flat DGLA. One example for curved Maurer-Cartan elements are principal connections on principal bundles, see e.g. [29, Section 11] for more details.

**Example 5.2 (Principal Connection)** Let $G$ be a Lie group and $\pi: P \to M$ be a smooth principal $G$-bundle. Then one way to define a principal $G$-connection on $P$ is as an equivariant $g$-valued differential $1$-form $\omega \in \Omega^1(P, g)^G$, where the equivariance is taken with respect to the product of the action on $P$ and the adjoint action on the Lie algebra $g$ of $G$, satisfying $\omega(\xi_P) = \xi$ for all $\xi \in g$, where $\xi_P$ denotes the fundamental vector field. The curvature form $\Omega \in \Omega^2(P, g)^G$ is given by

\[ \Omega = d\omega + \frac{1}{2}[\omega, \omega], \]  

(5.2)

where $d$ denotes the de Rham differential, and where the Lie bracket $[\cdot, \cdot]$ is induced by the $\wedge$ product of ordinary differential forms and the Lie bracket on $g$. In other words, the principal connection $\omega$ satisfies the Maurer-Cartan equation in the curved Lie algebra $(\Omega(P, g)^G, -\Omega, d, [\cdot, \cdot])$.

Two other examples of DGLAs and Maurer-Cartan elements that we want to mention are the DGLAs of polyvector fields and the DGLA of polydifferential operators. They play an important role in (formal) deformation quantization [3, 30, 53].

**Example 5.3 (Polyvector fields)** Let $M$ be a smooth manifold. The **polyvector fields** are the sections $T^{\bullet}_{\text{poly}}(M) = \Gamma^\infty(\Lambda^{\bullet+1}TM)$. Together with the Schouten bracket $[\cdot, \cdot]$ they form a graded Lie algebra, and together with the zero differential a DGLA

\[ T^{\bullet}_{\text{poly}}(M) = (\Gamma^\infty(\Lambda^{\bullet+1}TM), 0, [\cdot, \cdot]). \]  

(5.3)

The Maurer-Cartan elements are given by bivectors $\pi \in \Gamma^\infty(\Lambda^2 TM)$ satisfying the Maurer-Cartan equation $[\pi, \pi]_S = 0$, i.e. by Poisson structures. We denote by $\{\cdot, \cdot\}$ the corresponding Poisson brackets on $C^\infty(M)$. In deformation quantization one is interested in formal deformations, whence
one considers formal polyvector fields $T_{\text{poly}}^\bullet(M)[[h]] = \Gamma^\infty(\Lambda^\bullet^+TM)[[h]]$ in the real formal parameter $h$. Formal Poisson structures are then formal power series

$$\pi_h = \pi_0 + h\pi_1 + \cdots \in \Gamma^\infty(\Lambda^2 TM)[[h]]$$

with $[\pi_h, \pi_h]_S = 0$. In lowest order this implies in particular $[\pi_0, \pi_0]_S = 0$. Two such formal Poisson structures $\pi_h$ and $\tilde{\pi}_h$ are called equivalent if there exists a formal diffeomorphism such that

$$\pi_h = \exp(h[X, \cdot]_S)\tilde{\pi}_h,$$

where $X \in \Gamma^\infty(TM)[[h]]$. In particular, $\pi_h$ and $\tilde{\pi}_h$ deform the same $\pi_0$. In view of later applications, it turns out to be useful to consider formal Poisson structures $\pi_h = h\pi_1 + \cdots$ that start in the first order of $h$, i.e. that deform the zero Poisson structure. Consequently, one considers formal Maurer-Cartan elements $MC^h = MC \cap \h T_{\text{poly}}^\bullet(M)[[h]]$. We denote the equivalence classes by

$$\text{Def}(T_{\text{poly}}^\bullet(M)[[h]]) = \frac{MC^h(T_{\text{poly}}^\bullet(M))}{G^0(T_{\text{poly}}^\bullet(M)[[h]])},$$

where $G^0(T_{\text{poly}}^\bullet(M)[[h]]) = \{\exp([X, \cdot]_S) | X \in \h T_{\text{poly}}^0\}$ is called gauge group.

In deformation quantization, the polyvector fields with their formal Poisson structures as Maurer-Cartan elements corresponds to the classical side. From the quantum point of view one looks for star products on $C^\infty(M)[[h]]$ that can be interpreted as formal power series of bidifferential operators. Before we can introduce the corresponding DGLA of polydifferential operators we have to recall the Hochschild cochains:

**Example 5.4 (Hochschild cochains).** For a unital associative algebra $(A, \mu_0, 1)$ we recall the Hochschild cochains with a shifted grading

$$C^n(A) = \text{Hom}(A^\otimes(n+1), A)$$

for $n \geq 0$ and $C^{-1}(A) = A$. There are different operations for cochains $D$ and $E$, the cup-product

$$D \cup E(a_0, \ldots, a_{d+e+1}) = D(a_0, \ldots, a_d)E(a_{d+1}, \ldots, a_{d+e+1}),$$

where $a_0, \ldots, a_{d+e+1} \in A$, the concatenation

$$D \circ E(a_0, \ldots, a_{d+e}) = \sum_{i=0}^{\lvert D \rvert} (-1)^{\mid E \mid i} D(a_0, \ldots, a_{i-1}, E(a_i, \ldots, a_{i+e}), a_{i+e+1}, \ldots, a_{d+e})$$

and the Gerstenhaber bracket

$$[D, E]_G = (-1)^{\mid E \mid \mid D \mid} \left( D \circ E - (-1)^{\mid D \mid \mid E \mid} E \circ D \right).$$

Note that we use the sign convention from [5], not the original one from [22].

This yields a graded Lie algebra $(C^\bullet(A), [\cdot, \cdot]_G)$ and a graded associative algebra $(C^{\bullet-1}(A), \cup)$. The product $\mu_0$ is an element of $C^1(A)$ and one notices that the associativity of $\mu_0$ is equivalent to $[\mu_0, \mu_0]_G = 0$, compare Remark [22]. In particular, we get an induced differential $\partial: C^\bullet(A) \to C^{\bullet+1}(A)$ via $\partial = [\mu_0, \cdot]_G$, the so-called Hochschild differential, and thus a DLGA structure. One can check that $\pi \in C^1(A)$ is a Maurer-Cartan element if and only if $\mu_0 + \pi$ is an associative product. The cohomology of $(C^\bullet(A), \partial)$ inherits even the structure of a Gerstenhaber algebra: More explicitly, in cohomology the $\cup$-product is graded commutative and one has the following Leibniz rule:

$$[F, D \cup E]_G = [F, D]_G \cup E + (-1)^{\mid F \mid [\mid D \mid -1]} D \cup [F, E]_G,$$

compare [22] and [53] Satz 6.2.18.

In the context of deformation quantization [3] we are interested in star products. They are Maurer-Cartan elements in the DGLA of polydifferential operators, i.e. in the differential Hochschild cochain complex.
Thus star products correspond to Maurer-Cartan elements where
\[ T = \log \mu_0(f, g) \] and where all \( C_r \) are bidifferential operators vanishing on constants. Two star products \( \star \) and \( \star' \) are equivalent if there exists an \( h \)-linear isomorphism \( S = \text{id} + \sum_{r=1}^{\infty} h^r S_r \) with differential operators \( S_r \) and

\[ S(f \star g) = Sf \star' Sg. \] (5.12)

To describe the notion of star products in terms of a DGLA we consider the associated Hochschild DGLA \( (C^\bullet(\mathcal{E}^\infty(M)), \partial, [\cdot, \cdot]_G) \). To incorporate the bidifferentiability we restrict ourselves to the polydifferential operators

\[ D^k_{\text{poly}}(M)[[h]] = \bigoplus_{n=-1}^{\infty} D^n_{\text{poly}}(M)[[h]], \] (5.13)

were \( D^n_{\text{poly}}(M)[[h]] = \text{Hom}_{\text{diff}}(\mathcal{E}^\infty(M) \otimes \mathcal{E}^{\infty}(M))[[h]] \) are polydifferential operators vanishing on constants. We know that star products can be interpreted as \( \star = \mu_0 + \sum_{r=1}^{\infty} h^r C_r = \mu_0 + h m_* \in D^1_{\text{poly}}(M)[[h]] \) and the associativity leads to

\[ 0 = [\star, \star]_G = 2[\mu_0, h m_*] + [h m_*, h m_*]. \] (5.14)

Thus star products correspond to Maurer-Cartan elements \( h m_* \in h D^1_{\text{poly}}(M)[[h]] \), and the equivalence of \( \star \) and \( \star' \) is equivalent to

\[ \exp(hT, \cdot)_G \star = \star', \] (5.15)

where \( hT = \log S \in h D^0_{\text{poly}}(M)[[h]] \), see [53, Proposition 6.2.20]. Consequently, the equivalence classes of star products are given by

\[ \text{Def}(D_{\text{poly}}(M)[[h]] = \frac{MC^h(D_{\text{poly}}(M)[[h]])}{G^0(D_{\text{poly}}(M)[[h]])}, \] (5.16)

where the gauge group action of \( G^0(D_{\text{poly}}(M)[[h]]) = \{ \exp([hT, \cdot]_G) \mid hT \in h D^0_{\text{poly}}(M)[[h]] \} \) is given by

\[ h m_* = \exp([hT, \cdot]_G) h m_* = \exp([hT, \cdot]_G)(\mu_0 + h m_*) - \mu_0, \]

and where we consider again only formal Maurer-Cartan elements \( h m_* \in MC^h = MC \cap h D^1_{\text{poly}}(M)[[h]] \), i.e. starting in order one of \( h \).

Example 5.5 (Polydifferential operators) Recall that a star product \( \ast \) on a Poisson manifold \((M, \pi)\) is an associative product on \( \mathcal{E}^\infty(M)[[h]] \) of the form

\[ f \ast g = \mu_0(f, g) + \sum_{r=1}^{\infty} h^r C_r(f, g) \in \mathcal{E}^\infty(M)[[h]], \] (5.11)

where \( C_1(f, g) - C_1(g, f) = \{ f, g \} \), and all \( C_r \) are bidifferential operators vanishing on constants. Two star products \( \ast \) and \( \ast' \) are equivalent if there exists an \( h \)-linear isomorphism \( S = \text{id} + \sum_{r=1}^{\infty} h^r S_r \) with differential operators \( S_r \) and

\[ S(f \ast g) = Sf \ast' Sg. \] (5.12)

In the above examples we have seen that it is useful to identify 'equivalent' Maurer-Cartan elements by actions of elements of degree 0. In both settings the exponential maps were well-defined because of the complete filtration induced by the formal power series. Therefore, we restrict ourselves to (curved) Lie algebras with complete descending filtrations \( F^\ast \mathfrak{g} \) satisfying

\[ \cdots \supseteq F^{-2} \mathfrak{g} \supseteq F^{-1} \mathfrak{g} \supseteq F^0 \mathfrak{g} \supseteq F^1 \mathfrak{g} \supseteq \cdots, \quad \mathfrak{g} \cong \varprojlim_{\ell} F^\ell \mathfrak{g}, \] (5.17)

and

\[ d(F^k \mathfrak{g}) \subseteq F^{k+1} \mathfrak{g} \quad \text{and} \quad [F^k \mathfrak{g}, F^\ell \mathfrak{g}] \subseteq F^{k+\ell} \mathfrak{g}. \] (5.18)

In most cases the filtration will be bounded below, i.e. bounded from the left with \( \mathfrak{g} = F^k \mathfrak{g} \) for some \( k \in \mathbb{Z} \), preferably \( k = 0 \) or \( k = 1 \). If the filtration is unbounded, then we assume in addition that it is exhaustive, i.e. that

\[ \mathfrak{g} = \bigcup_{\ell} F^\ell \mathfrak{g}, \] (5.19)

even if we do not mention it explicitly. Note that instead of considering filtered DGLAs one can tensorize the DGLAs with nilpotent algebras.
**Remark 5.6 (Nilpotent DGLA)** Alternatively to the filtration, one can tensorize the DGLA \((\mathfrak{g}, d, [\cdot, \cdot])\) by a graded commutative associative \(\mathbb{K}\)-algebra \(\mathfrak{m}\), compare [S39, Section 2.4]:

\[
(g \otimes m)^n = \bigoplus_{i=1}^{n} (g^i \otimes m^{n-i})
\]

\[d(x \otimes m) = dx \otimes m \]

\[[x \otimes m, y \otimes n] = (-1)^{|m||y|}[x, y] \otimes mn.
\]

In particular, if \(\mathfrak{m}\) is nilpotent, the DGLA \(g \otimes m\) is nilpotent, too. Thus under the assumption that \(g^0 \otimes m\) is nilpotent, the above exponential maps are also in the non-filtered setting well-defined. For further details on this approach and more details on the deformation functor see [S39].

**Example 5.7 (Formal power series)** If we consider formal power series \(\mathfrak{g} = \mathfrak{g}[\hbar]\) of a DGLA \(g\), where all maps are \(h\)-linearly extended, then we can choose the filtration \(\mathcal{F}^h \mathfrak{g} = h^k \mathfrak{g}\) and the completeness is trivially fulfilled.

As in the above examples we set for the curved Maurer-Cartan elements in \((\mathfrak{g}, R, d, [\cdot, \cdot])\)

\[\mathcal{MC}^1(\mathfrak{g}) = \{ \pi \in \mathcal{F}^1 \mathfrak{g}^1 \mid R + d\pi + \frac{1}{2}[\pi, \pi] = 0 \} \quad (5.20)
\]

and we want to define a group action by the gauge group

\[G^0(\mathfrak{g}) = \{ \Phi = e^{[a, \cdot]} : \mathfrak{g} \to \mathfrak{g} \mid a \in \mathcal{F}^1 \mathfrak{g}^0 \}, \quad (5.21)
\]

where we consider again only elements of filtration order 1. In order to define the gauge action, we consider as in [S39] the DGLA \((\mathfrak{g}_d, [\cdot, \cdot]_d, 0)\) with \(\mathfrak{g}_d = \mathfrak{g} \oplus \mathbb{K} d\) and

\[[x + r d, y + s d]_d = [x, y] + r d(y) - (-1)^{|x|} s d(x) + 2rsR. \quad (5.22)
\]

Then \(\pi\) is a Maurer-Cartan element in \(g\) if and only if \(\phi(\pi) = \pi + d\pi\) is a Maurer-Cartan element in \(g_d\) with zero differential. But in \((\mathfrak{g}_d, [\cdot, \cdot]_d, 0)\) we already have an action of \(G^0(\mathfrak{g})\) on the Maurer-Cartan elements by the adjoint representation. Pulling this action back on \(g\) yields the following:

**Proposition 5.8 (Gauge action)** Let \((\mathfrak{g}, R, d, [\cdot, \cdot])\) be a curved Lie algebra with complete descending filtration. The gauge group \(G^0(\mathfrak{g})\) acts on \(\mathcal{MC}^1(\mathfrak{g})\) via

\[
\exp([g, \cdot]) \circ \pi = \sum_{n=0}^{\infty} \frac{([g, \cdot])^n}{n!}(\pi) - \sum_{n=0}^{\infty} \frac{([g, \cdot])^n}{(n+1)!}(d\pi) = \pi + \sum_{n=0}^{\infty} \frac{([g, \cdot])^n}{(n+1)!}(g, \pi) - d\pi. \quad (5.23)
\]

The equivalence classes of Maurer-Cartan elements are denoted by

\[
\text{Def}(\mathfrak{g}) = \frac{\mathcal{MC}^1(\mathfrak{g})}{G^0(\mathfrak{g})}. \quad (5.24)
\]

\(\text{Def}(\mathfrak{g})\) is the orbit space of the transformation groupoid \(G^0(\mathfrak{g}) \times \mathcal{MC}^1(\mathfrak{g})\) of the gauge action and \(G^0(\mathfrak{g}) \ltimes \mathcal{MC}^1(\mathfrak{g})\) is also called Goldman-Millson groupoid or Deligne groupoid [25]. It plays an important role in deformation theory [39].

An additional motivation for this gauge action comes from the following general consideration:

**Lemma 5.9** Let \((\mathfrak{g}, R, d, [\cdot, \cdot])\) be a curved Lie algebra with complete descending filtration. For all \(g \in \mathcal{F}^1 \mathfrak{g}^0\) and all derivations \(D\) of degree +1 one has

\[
\exp([g, \cdot]) \circ D \circ \exp([-g, \cdot]) = D - \left[ \frac{\exp([g, \cdot]) - \text{id}}{[g, \cdot]} \right] (Dg), \cdot. \quad (5.25)
\]

**Proof:** The proof is the same as [H4, Lemma 1.3.20]. It follows directly from \(\text{Ad}(\exp([g, \cdot]))D = e^{\text{ad}([g, \cdot])}D\) and \(\text{ad}([g, \cdot])D = -[Dg, \cdot]\).
This immediately implies that twisting with gauge equivalent Maurer-Cartan elements leads to isomorphic DGLAs.

**Corollary 5.10** Let \((g, R, d, [\cdot, \cdot])\) be a curved Lie algebra with complete descending filtration and let \(\pi, \tilde{\pi}\) be two gauge equivalent Maurer-Cartan elements via \(\tilde{\pi} = \exp([g, \cdot]) \circ \pi\). Then one has

\[
d + [\tilde{\pi}, \cdot] = \exp([g, \cdot]) \circ (d + [\pi, \cdot]) \circ \exp([-g, \cdot]),
\]

i.e. \(\exp([g, \cdot]): (g, d + [\pi, \cdot], [\cdot, \cdot]) \to (g, d + [\tilde{\pi}, \cdot], [\cdot, \cdot])\) is an isomorphism of DGLAs. Moreover, one also has in the coalgebra setting

\[
Q^{\tilde{\pi}} = \exp([g, \cdot]) \circ Q^\pi \circ \exp([-g, \cdot]),
\]

where \(Q^\pi, Q^{\tilde{\pi}}\) are the codifferentials with \(Q^\pi_1 = -d - [\pi, \cdot]\) and \(Q^{\tilde{\pi}}_1 = -d - [\tilde{\pi}, \cdot]\) and second structure map given by the bracket.

**Proof:** The first equation is clear by (5.26). The second one is clear since \(\exp([g, \cdot])\) is a Lie algebra automorphism of degree zero intertwining the differentials. \(\square\)

Finally, note that we recover indeed the equivalence notions for the polyvector fields and polydifferential operators.

**Example 5.11 (\(T_{\text{poly}}(M)\) and \(D_{\text{poly}}(M)\))** It is easy to see that the action of \(G^0\) on \(MC^0\) as defined in (5.23) coincides with the actions in the above examples \(T_{\text{poly}}(M)[[h]]\) and \(D_{\text{poly}}(M)[[h]]\). For \(T_{\text{poly}}(M)\) in Example 5.3 the differential of the DGLA is zero, i.e. the gauge action \(\exp(hX, \cdot)\) coincides with the usual action by formal diffeomorphisms. In the case of \(D_{\text{poly}}(M)\) in Example 5.2 the differential is given by \(\partial = [\mu_0, \cdot]\), where \(\mu_0\) denotes the pointwise product on functions. Therefore, two formal star products \(\star = \mu_0 + h\mu_0\) and \(\star' = \mu_0 + h\mu_0\) are equivalent via \(\exp(h[T, \cdot])\star = \star'\) with \(hT \in T_{\text{poly}}(M)[[h]]\) if and only if

\[
\star' = \mu_0 + h\mu_0 = \exp(h[T, \cdot])\mu_0 + h\mu_0 = \mu_0 + \exp(h[T, \cdot])\mu_0 + h\mu_0.
\]

### 5.2 Maurer-Cartan Elements in \(L_\infty\)-algebras

The notion of Maurer-Cartan elements can be transferred to general \(L_\infty\)-algebras. There are again different possibilities for conditions on the \(L_\infty\)-algebra. One possibility is to require the \(L_\infty\)-algebra \((L, Q)\) to be nilpotent [23, 10]; for example, one requires \(L^{[n]} = 0\) for \(n > 0\), where

\[
L^{[n]} = \text{span}\{Q_k(x_1 \vee \cdots \vee x_k) \mid k \geq 2, \ x_i \in L^{[n_i]}, \ 0 < n_i < n, \sum n_i > n\}
\]

and \(L^{[1]} = L\). In particular, this implies \(Q_n = 0\) for \(n > 0\). However, we consider again \(L_\infty\)-algebras with complete descending and exhaustive filtrations, where we implicitly include exhaustive when we say 'complete filtration'. Moreover, we require from now on that the codifferentials and the \(L_\infty\)-morphisms are compatible with the filtrations.

**Remark 5.12** There are again different conventions about the filtrations. Sometimes one considers complete \(L_\infty\)-algebras \(L\), i.e. \(L_\infty\)-algebras with complete descending filtrations where \(L = \mathcal{F}^1 L\), see e.g. [13].

**Definition 5.13 (Maurer-Cartan elements II)** Let \((L, Q)\) be a (curved) \(L_\infty\)-algebra with complete descending filtration. Then \(\pi \in \mathcal{F}^1 L[1]^0 = \mathcal{F}^1 L^1\) is called (curved) Maurer-Cartan element if it satisfies the Maurer-Cartan equation

\[
Q^1(\exp(\pi)) = \sum_{n \geq 0} \frac{1}{n!} Q_n(\pi \vee \cdots \vee \pi) = 0.
\]

The set of (curved) Maurer-Cartan elements is denoted by \(MC^1(L)\).
Note that the sum in (5.29) is well-defined for \( \pi \in \mathcal{F}^1 L^1 \) because of the completeness. From now on we assume to be in this setting and we collect some useful properties:

**Lemma 5.14** Let \( F : (L,Q) \to (L',Q') \) be an \( L_\infty \)-morphism of (curved) \( L_\infty \)-algebras and \( \pi \in \mathcal{F}^1 L^1 \).

i.) \( \pi \) is a (curved) Maurer-Cartan element if and only if \( Q(\exp(\pi)) = 0 \).

ii.) \( F(\exp(\pi)) = \exp(S) \) with \( S = F_{MC}(\pi) = F^1(\exp(\pi)) \), where \( \exp(\pi) = \sum_{k=1}^{\infty} \frac{1}{k!} \pi^k \).

iii.) If \( \pi \) is a (curved) Maurer-Cartan element, then so is \( F_{MC}(\pi) \).

**Proof:** The proof for the case of flat DGLAs can be found in [13, Proposition 1]. Note that in the flat case it suffices to consider the completion as the space of equivalence classes of Cauchy sequences.

Remark 5.15 Let \( L \) be a \( L_\infty \)-algebra with complete descending filtration and consider \( L[t] = L \otimes \mathbb{K}[t] \) which has again a descending filtration

\[
\mathcal{F}^k L[t] = \mathcal{F}^k L \otimes \mathbb{K}[t].
\]

We denote its completion by \( \hat{L}[t] \) and note that since \( Q \) is compatible with the filtration it extends to \( \hat{L}[t] \). Similarly, \( L_\infty \)-morphisms extend to these completed spaces.

**Remark 5.15** Note that one can define the completion as space of equivalence classes of Cauchy sequences with respect to the filtration topology. Alternatively, the completion can be identified with

\[
\lim_{n\to\infty} L[t]/\mathcal{F}^n L[t] \subset \prod_n L[t]/\mathcal{F}^n L[t] \cong \prod_n L/\mathcal{F}^n L \otimes \mathbb{K}[t]
\]

consisting of all coherent tuples \( X = (x_n) \in \prod_n L[t]/\mathcal{F}^n L[t] \), where

\[
L[t]/\mathcal{F}^{n+1} L[t] \ni x_{n+1} \mapsto x_n \in L[t]/\mathcal{F}^n L[t]
\]

under the obvious surjections. Moreover, \( \mathcal{F}^n \hat{L}[t] \) corresponds to the kernel of \( \lim_{n\to\infty} L[t]/\mathcal{F}^n L[t] \to L[t]/\mathcal{F}^n L[t] \) and thus

\[
\hat{L}[t]/\mathcal{F}^n \hat{L}[t] \cong L[t]/\mathcal{F}^n L[t].
\]

Since \( L \) is complete, we can also interpret \( \hat{L}[t] \) as the subspace of \( L[[t]] \) such that \( X \mod \mathcal{F}^n L[[t]] \) is polynomial in \( t \). In particular, \( \mathcal{F}^n \hat{L}[t] \) is the subspace of elements in \( \mathcal{F}^n L[[t]] \) that are polynomial in \( t \) modulo \( \mathcal{F}^m L[[t]] \) for all \( m > n \).
By the above construction of \( \hat{L}[t] \) it is clear that differentiation \( \frac{d}{dt} \) and integration with respect to \( t \) extend to it since they do not change the filtration. Moreover,

\[
\delta_t : \hat{L}[t] \ni X(t) \mapsto X(s) \in L
\]
is well-defined for all \( s \in \mathbb{K} \) since \( L \) is complete.

**Example 5.16** In the case that the filtration of \( L \) comes from a grading \( L^\bullet \), the completion is given by \( \hat{L}[t] \cong \prod L^i[t] \), i.e. by polynomials in each degree. A special case is here the case of formal power series \( L = V[[h]] \) with \( \hat{L}[t] \cong (V[t])[h] \) as in [5, Appendix A].

Now we can introduce a general equivalence relation between Maurer-Cartan elements of \( L_\infty \)-algebras. We write \( \pi_0 \sim \pi_1 \) if there exist \( \pi(t) \in \mathcal{F}^1 \hat{L}[t] \) and \( \lambda(t) \in \mathcal{F}^1 \hat{L}[t] \) such that

\[
\frac{d}{dt} \pi(t) = Q^1(\lambda(t) \lor \exp(\pi(t))) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^1_{n+1}(\lambda(t) \lor \pi(t) \lor \cdots \lor \pi(t)),
\]

(5.30)

\[
\pi(0) = \pi_0 \quad \text{and} \quad \pi(1) = \pi_1.
\]

We directly see that \( \sim \) is reflexive and symmetric and one can check that it is also transitive. We write \([\pi_0]_\sim \) for the homotopy class of \( \pi_0 \) and define:

**Definition 5.17 (Homotopy equivalence)** Let \((L,Q)\) be a (curved) \( L_\infty \)-algebra with complete descending filtration. The homotopy equivalence relation on the set \( \text{MC}^1(L) \) is given by the relation \( \sim \) from (5.30). The set of equivalence classes of Maurer-Cartan elements is denoted by \( \text{Def}(L) = \text{MC}^1(L)/\sim \).

**Remark 5.18** This definition can be reformulated: two Maurer-Cartan elements \( \pi_0 \) and \( \pi_1 \) in \( L \) are homotopy equivalent if and only if there exists a Maurer-Cartan element \( \pi(t) - \lambda(t) \, dt \) in \( \hat{L}[t,dt] \) with \( \pi(0) = \pi_0 \) and \( \pi(1) = \pi_1 \), see e.g. [16] for \( L_\infty \)-algebras and [39] for DGLAs.

Note that in the case of nilpotent \( L_\infty \)-algebras it suffices to consider polynomials in \( t \) as there is no need to complete \( L[t] \), compare [23]. We check now that this is well-defined and even yields a curve \( \pi(t) \) of Maurer-Cartan elements, see [5, Proposition 4.8].

**Proposition 5.19** For every \( \pi_0 \in \mathcal{F}^1 L^1 \) and \( \lambda(t) \in \mathcal{F}^1 \hat{L}[t] \) there exists a unique \( \pi(t) \in \mathcal{F}^1 \hat{L}[t] \) such that \( \frac{d}{dt} \pi(t) = Q^1(\lambda(t) \lor \exp(\pi(t))) \) and \( \pi(0) = \pi_0 \). If \( \pi_0 \in \text{MC}^1(L) \), then \( \pi(t) \in \text{MC}^1(L) \) for all \( t \in \mathbb{K} \).

**Proof:** At first we show that there exists a unique solution \( \pi(t) = \sum_{k=0}^{\infty} \pi_k t^k \) in the formal power series \( \mathcal{F}^1 L^1 \otimes \mathbb{K}[[t]] \). On one hand one has

\[
\frac{d}{dt} \pi(t) = \sum_{k=0}^{\infty} (k+1) \pi_{k+1} t^k,
\]
on the other hand there exist \( \phi_k \in \mathcal{F}^1 L^1 \) such that

\[
Q^1(\lambda(t) \lor \exp(\pi(t))) = \sum_{k=0}^{\infty} \phi_k t^k.
\]

Here the \( \phi_k \) depend only on the \( \lambda_j \) of \( \lambda(t) = \sum_{j=0}^{\infty} \lambda_j t^j \) and the \( \pi_i \) for \( i \leq k \), hence they can be defined inductively. It remains to check that one has even \( \pi(t) \in \mathcal{F}^2 \hat{L}[t] \), i.e. by Remark 5.15 that \( \pi(t) \text{ mod } \mathcal{F}^n L_1[[t]] \in L^1[t] \) for all \( n \). Indeed, we have inductively

\[
\frac{d}{dt} \pi(t) \text{ mod } \mathcal{F}^2 L^1[[t]] = Q^1(\lambda(t)) \text{ mod } \mathcal{F}^2 L^1[[t]] \in L^1[t].
\]
For the higher orders we get
\[ \frac{d}{dt} \pi(t) = \sum_{k=0}^{n-2} \frac{1}{k!} Q_{k+1}(\lambda(t) \vee (\pi(t) \mod \mathcal{F}^{n-1}) \vee \cdots \vee (\pi(t) \mod \mathcal{F}^{n-1})) \mod \mathcal{F}^n L^1[[t]] \]
and thus \( \pi(t) \mod \mathcal{F}^n L^1[[t]] \in L^1[t] \).

Let now \( \pi_0 \) be a curved Maurer-Cartan element, the flat case follows directly from the curved one. We have to show \( g(t) = Q^1(\exp(\pi(t))) = 0 \) for all \( t \), so it suffices to show \( g^{(n)}(0) = \frac{d^n}{dt^n} g(0) = 0 \) for all \( n \geq 0 \). The case \( n = 0 \) is clear, for \( n = 1 \) we get
\[ g^{(1)}(t) = Q^1(\exp(\pi(t)) \vee Q^1(\lambda(t) \vee \exp(\pi(t)))) = Q^1(Q(\lambda(t) \vee \exp(\pi(t))) + \lambda(t) \vee \exp(\pi(t)) \vee Q^1(\exp(\pi(t)))) = Q^1(\lambda(t) \vee \exp(\pi(t)) \vee g^{(0)}(t)). \]
The statement follows by induction.

In the case of a curved Lie algebra \((\mathfrak{g}, R, d, [\cdot, \cdot])\) this recovers the gauge action from Proposition 5.8. Going from gauge equivalence to homotopy equivalence is easy: Explicitly, let \( \pi_1 = \exp([g, \cdot]) \triangleright \pi_0 \), then setting \( \lambda(t) = g \) and \( \pi(t) = \exp([tg, \cdot]) \triangleright \pi \) satisfies
\[ \frac{d}{dt} \pi(t) = \exp([tg, \cdot])[g, \pi_0] - \exp([tg, \cdot])(dg) = -dg + \exp([tg, \cdot])[g, \pi_0] - \sum_{n=0}^{\infty} \frac{([tg, \cdot])^{n+1}}{(n+1)!} (dg) = Q^1(\lambda(t)) + [\lambda(t), \exp([tg, \cdot]) \triangleright \pi_0]. \]
For the flat setting, the other direction from the homotopy equivalence to the gauge equivalence is contained in the following theorem, see e.g. [39, Theorem 5.5].

**Theorem 5.20** Two Maurer-Cartan elements in \((\mathfrak{g}, d, [\cdot, \cdot])\) are homotopy equivalent if and only if they are gauge equivalent.

This theorem can be rephrased in a more explicit manner in the following proposition, see [33, Proposition 2.13].

**Proposition 5.21** Let \((\mathfrak{g}, R, d, [\cdot, \cdot])\) be a curved Lie algebra equipped with a complete descending filtration. Consider \( \pi_0 \sim \pi_1 \) with homotopy equivalence given by \( \pi(t) \in \mathcal{F}^1 \mathfrak{g}^0[[t]] \) and \( \lambda(t) \in \mathcal{F}^1 \mathfrak{g}^0[[t]] \). The formal solution of
\[ \lambda(t) = \frac{\exp([A(t), \cdot]) - \id}{[A(t), \cdot]} \left( \frac{d}{dt} A(t) \right), \quad A(0) = 0 \tag{5.31} \]
is an element \( A(t) \in \mathcal{F}^1 \mathfrak{g}^0[[t]] \) and satisfies
\[ \pi(t) = e^{[A(t), \cdot]} \pi_0 - \frac{\exp([A(t), \cdot]) - \id}{[A(t), \cdot]} \, dA(t). \tag{5.32} \]
In particular, one has for \( g = A(1) \in \mathcal{F}^1 \mathfrak{g}^0 \)
\[ \pi_1 = \exp([g, \cdot]) \triangleright \pi_0. \tag{5.33} \]

**Proof:** As formal power series in \( t \) Equation (5.31) has a unique solution \( A(t) \in \mathcal{F}^1 \mathfrak{g}^0 \otimes \mathbb{K}[[t]] \). But one has even \( A(t) \in \mathcal{F}^1 \mathfrak{g}^0[[t]] \) since
\[ \frac{dA(t)}{dt} = \lambda(t) - \sum_{k=1}^{n-2} \frac{1}{(k+1)!} [A(t), \cdot] \frac{dA(t)}{dt} \mod \mathcal{F}^n \mathfrak{g}[[t]] \]
Then we know \( \tilde{\lambda}(t) \) and \( \tilde{\lambda}(t) \) are also called \( \lambda \)-morphisms map equivalence classes of Maurer-Cartan elements in \( \mathcal{L}_{\infty} \)-algebras: for example the above definition, sometimes called \textit{Quillen homotopy}, the \textit{gauge homotopy} where one requires \( \lambda(t) = \lambda \) to be constant, compare [13], and the \textit{cylinder homotopy}. In [16] it is shown that these notions are also equivalent for flat \( \mathcal{L}_{\infty} \)-algebras with complete descending filtration and compatible higher brackets, extending the result for DGLAs from [39].

Now we can finally show that \( \mathcal{L}_{\infty} \)-morphisms map equivalence classes of Maurer-Cartan elements to equivalence classes.

**Proposition 5.23** Let \( F: (L, Q) \to (L', Q') \) be an \( \mathcal{L}_{\infty} \)-morphism between (curved) \( \mathcal{L}_{\infty} \)-algebras, and \( \pi_0, \pi_1 \in \text{MC}^1(L) \) with \( [\pi_0] = [\pi_1] \). Then \( F \) is compatible with the homotopy equivalence relation, i.e. one has \( [F^1(\exp \pi_0)] = [F^1(\exp \pi_1)] \). In particular, one has an induced map \( F_{\text{MC}}: \text{Def}(L) \to \text{Def}(L') \).

**Proof:** Let \( \pi(t) \) and \( \lambda(t) \) encode the equivalence between \( \pi_0 \) and \( \pi_1 \). We set \( \tilde{\pi}(t) = F^1(\exp \pi(t)) \) and \( \tilde{\lambda}(t) = F^1(\lambda(t) \lor \exp(\pi(t))) \). We compute

\[
\frac{d}{dt} \tilde{\pi}(t) = F^1(\exp(\pi(t)) \lor Q^1(\lambda(t) \lor \exp(\pi(t))))
\]

\[
= F^1(Q^1(\lambda(t) \lor \exp(\pi(t))) + \lambda(t) \lor \exp(\pi(t))) \lor Q^1(\exp(\pi(t)))
\]

\[
= Q^1 \circ F(\lambda(t) \lor \exp(\pi(t)))
\]

\[
= Q^1(F^1(\lambda(t) \lor \exp(\pi(t))) \lor \exp(F^1(\exp(\pi(t)))))
\]

\[
= Q^1(\tilde{\lambda}(t) \lor \exp(\tilde{\pi}(t))
\]

and thus the desired \( [F^1(\exp \pi_0)] = [F^1(\exp \pi_1)] \).
If one does not want to restrict to $L_\infty$-algebras with complete filtrations, one can tensorize general $L_\infty$-algebras by nilpotent algebras, compare Remark 5.6 for the case of DGLAs. In this setting the deformations are not a set but a functor: For a (curved) $L_\infty$-algebra $L$ the deformation functor $\text{Def}_L$: $\text{Art}_K \rightarrow \text{Set}$ maps a local Artinian ring $A$ to the set

$$\text{Def}_L(A) = \text{Def}(L \otimes m_A).$$

Here $m_A$ is the maximal ideal of $A$ and thus $L \otimes m_A$ is a nilpotent $L_\infty$-algebra and the above is well-defined. In this case it can be shown that one even has the following statement, see [30, Theorem 4.6] and [8, Theorem 4.12] and also [12, Proposition 4] for the filtered setting.

**Theorem 5.24** Let $F: (L, Q) \rightarrow (L', Q')$ be an $L_\infty$-quasi-isomorphism of flat $L_\infty$-algebras. Then the map

$$F_{\text{MC}}: \pi \mapsto F_{\text{MC}}(\pi) = \sum_{n>0} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi)$$

induces an isomorphism between the deformation functors $\text{Def}_L$ and $\text{Def}_{L'}$.

**Proof (Sketch):** In Proposition 5.12 we have shown that every $L_\infty$-algebra $L$ is isomorphic to the direct product of a minimal one $L_{\text{min}}$, i.e. one with $Q_1 = 0$, and a linear contractible one $L_{\text{lc}}$, see also [8, Section 4]. On linear contractible $L_\infty$-algebras the deformation functor is trivial, i.e. for all $A$ the set $\text{Def}_{L_{\text{lc}}}(A)$ contains just one element: A Maurer-Cartan element $\pi$ is just a closed element. But since the cohomology is trivial, it is exact and there exists $\lambda \in L_{\text{lc}}^0 \otimes m_A$ with $Q_1(\lambda) = \pi$, and $t(\pi) = t\pi$ shows that $\pi$ is homotopy equivalent to zero. In addition, one easily sees that the deformation functor is compatible with direct products, i.e.

$$\text{Def}_{L \oplus L'} \cong \text{Def}_L \times \text{Def}_{L'}.$$ 

Summarizing, this yields

$$\text{Def}_L \cong \text{Def}_{L_{\text{min}}} \cong \text{Def}_{L'_{\text{min}}} \cong \text{Def}_{L'},$$

since $L_{\text{min}}$ and $L'_{\text{min}}$ are $L_\infty$-isomorphic, see also [8, Section 4].

**Remark 5.25** The above result can be further generalized: In fact, a morphism $L: \mathfrak{g} \rightarrow \mathfrak{g}'$ of DGLAs induces an isomorphism on the deformation functors if the induced map in cohomology is bijective in degree one, injective in degree two and surjective in degree zero, compare [39, Theorem 3.1].

We are mainly interested in the setting of formal power series, where we directly get the following statement.

**Corollary 5.26** Let $F$ be an $L_\infty$-quasi-isomorphism between two flat DGLAs $\mathfrak{g}$ and $\mathfrak{g}'$. Then it induces a bijection $F_{\text{MC}}$ between $\text{Def}(\mathfrak{g}[[h]])$ and $\text{Def}(\mathfrak{g}'[[h]])$.

**Proof:** The above statement follows from Theorem 5.24 since $\mathfrak{K}[[h]] = \lim K[h]/h^kK[h]$ is pro-Artinian with the pro-nilpotent $h\mathfrak{K}[[h]]$ as maximal ideal.

In the curved setting the situation is more complicated since the proof of Theorem 5.24 does not generalize: if the curvature is not central one does not even have a differential and there is no obvious notion of $L_\infty$-quasi-isomorphisms between curved $L_\infty$-algebras. Therefore, we postpone these considerations until we understand the twisting procedure, which is a way to obtain a flat $L_\infty$-algebra out of a curved one, see Lemma 5.29 below.

### 5.3 Twisting of (Curved) $L_\infty$-Algebras

Recall that for a Maurer-Cartan element $\pi$ of a DGLA $(\mathfrak{g}, d, [\cdot, \cdot])$ the map $d + [\pi, \cdot]$ is a differential on $\mathfrak{g}$, the *twisted* differential by $\pi$. This can be generalized to $L_\infty$-algebras, see e.g. [12,13,17,19].
Lemma 5.27 Let \((L, Q)\) be a (curved) \(L_\infty\)-algebra and \(\pi \in \mathcal{F}^1 L[1]^0\). Then the map \(Q^\pi\) given by
\[
Q^\pi(X) = \exp(-\pi \lor) Q(\exp(\pi \lor) X), \quad X \in S(L[1]),
\] (5.35)
defines a codifferential on \(S(L[1])\). If \(\pi\) is in addition a (curved) Maurer-Cartan element, then \((L, Q^\pi)\) is a flat \(L_\infty\)-algebra.

**Proof:** At first we have to show that \(Q^\pi\) is a well-defined map into \(S(L[1])\) and not into its completion with respect to the symmetric degree. This is clear since its structure maps \((Q^\pi)_k^1\) are well-defined maps into \(L[1]\) by the completeness of the filtration and since \(Q^\pi\) defines a coderivation on \(S(L[1])\) by
\[
\Delta_n Q^\pi(X) = \Delta_n \exp(-\pi \lor) Q(\exp(\pi \lor) X)
= \exp(-\pi \lor) \otimes \exp(-\pi \lor)(Q \otimes \text{id} \otimes \pi)(\exp(\pi \lor) \otimes \exp(\pi \lor))(\Delta_n X)
= (Q^\pi \otimes \text{id} + \text{id} \otimes Q^\pi)(\Delta_n X).
\]
The property \((Q^\pi)^2 = 0\) is clear. If \(\pi\) is in addition a (curved) Maurer-Cartan element, then one obtains \(Q^\pi(1) = \exp(-\pi \lor) Q(\exp(\pi)) = 0\) and thus after twisting a flat \(L_\infty\)-algebra.

The \(L_\infty\)-algebra \((L, Q^\pi)\) is again called twisted and it turns out that one can also twist the \(L_\infty\)-morphism, see [13, Proposition 1] for the flat setting and [19, Lemma 2.7] for the curved setting.

**Proposition 5.28** Let \(F: (L, Q) \to (L', Q')\) be an \(L_\infty\)-morphism between (curved) \(L_\infty\)-algebras, \(\pi \in \mathcal{F}^1 L[1]\) and \(S = F^1(\pi \exp \pi) \in \mathcal{F}^1 (L')^1\).

i.) The map
\[
F^\pi = \exp(-S \lor) F \exp(\pi \lor): S(L[1]) \to S(L'[1])
\]
defines an \(L_\infty\)-morphism between the (curved) \(L_\infty\)-algebras \((L, Q^\pi)\) and \((L', (Q')^S)\).

ii.) The structure maps of \(F^\pi\) are given by
\[
F^\pi_n(x_1, \ldots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F^\pi_{n+k}(\pi, \ldots, \pi, x_1, \ldots, x_n)
\] (5.36)
and \(F^\pi\) is called twisted by \(\pi\).

iii.) If \(\pi\) is a (curved) Maurer-Cartan element, then \(F^\pi\) is an \(L_\infty\)-morphism between flat \(L_\infty\)-algebras.

iv.) Let \(F\) be an \(L_\infty\)-quasi-isomorphism between flat \(L_\infty\)-algebras such that \(F^1\) is not only a quasi-isomorphism of filtered complexes \(L \to L'\) but even induces a quasi-isomorphism
\[
F^1: \mathcal{F}^k L \to \mathcal{F}^k L'
\]
for each \(k\). If \(\pi\) is a flat Maurer-Cartan element, then \(F^\pi\) is also an \(L_\infty\)-quasi-isomorphism.

**Proof:** \(F^\pi\) is well-defined since its structure maps are well-defined maps into \(L'[1]\) by the completeness of the filtration and since \(F^\pi\) is a coalgebra morphism
\[
\Delta_n \exp(-S \lor) F \exp(\pi \lor) X = \exp(-S \lor) \otimes \exp(-S \lor)(F \otimes F) \Delta_n \exp(\pi \lor) X
= (F^\pi \otimes F^\pi) \Delta_n(X).
\]
The compatibility of \(F^\pi\) with the coderivations \(Q^\pi\) and \((Q')^S\) follows directly with the definitions. The third point follows directly from Lemma 5.27.

The last claim follows by a standard argument of spectral sequences. Since \(F^1\) is a quasi-isomorphism w.r.t. \(Q^1\) and \((Q')^1\) that is compatible with the filtrations, the map \(F^\pi\) induces a quasi-isomorphism on the zeroth level of the corresponding spectral sequence, and therefore also on the terminal \(E_\infty\)-level, compare [13, Proposition 1].
It follows directly that the twisting procedure is functorial in the sense that
\[(G \circ F)^\pi = G^S \circ F^\pi\] (5.37)
for an \(L_\infty\)-morphism \(G: L' \to L''\), see [13] Proposition 4], as well as
\[(Q^\pi)^B = Q^{\pi+B}, \quad (F^\pi)^B = F^{\pi+B}.\]

Now we can come back to the correspondences of curved Maurer-Cartan elements under \(L_\infty\)-morphisms. Let \((L, Q)\) be a curved \(L_\infty\)-algebra with curved Maurer-Cartan element \(m \in F^1 L^1\). Then we know from Lemma 5.27 that the twisted codifferential \(\pi\) is flat and we get the following, compare [42, Proposition 4.6] for the case of curved Lie algebras.

**Lemma 5.29** Let \((L, Q)\) be a curved \(L_\infty\)-algebra with curved Maurer-Cartan element \(m \in F^1 L^1\). Then the curved Maurer-Cartan elements in \((L, Q)\) are in one-to-one correspondence with flat Maurer-Cartan elements in \((L, Q^m)\) via \(\pi \mapsto \pi - m\). The correspondence is compatible with equivalences.

**Proof:** We have for \(\pi \in F^1 L[1]^0\)
\[Q^1(\exp \pi) = Q^1(\exp(\pi) \lor \exp(m) \lor \exp(-m)) = (Q^m)^1(\exp(\pi - m)),\]
so the first part follows. Suppose that \(\pi_0\) and \(\pi_1\) are equivalent, i.e. there exists \(\pi(t)\) with \(\pi(0) = \pi_0, \pi(1) = \pi_1\) and
\[\frac{d}{dt}\pi(t) = Q^1(\lambda(t) \lor \exp(\pi(t))).\]
Then \(\pi'(t) = \pi(t) - m\) induces the equivalence between \(\pi_0 - m\) and \(\pi_1 - m\) in the flat setting since
\[\frac{d}{dt}\pi'(t) = \frac{d}{dt}\pi(t) = Q^1(\exp(m) \lor \lambda(t) \lor \exp(\pi(t) - m)) = Q^m(\lambda(t) \lor \exp(\pi'(t))),\]
and the statement is shown. \(\square\)

This directly implies for the equivalence classes of curved Maurer-Cartan elements:

**Corollary 5.30** Let \(F: (L, Q) \to (L', Q')\) be an \(L_\infty\)-morphism between two curved \(L_\infty\)-algebras with complete filtrations. If \(L\) has a curved Maurer-Cartan element \(m \in F^1 L^1\) such that \(F^m\) induces bijection on the equivalence classes of flat Maurer-Cartan elements, then \(F\) induces a bijection \(F_{\text{MC}}\) on the equivalence classes of curved Maurer-Cartan elements.

**Proof:** We know that \(F\) maps \(m\) to a curved Maurer-Cartan element \(m' = F_{\text{MC}}(m) = F^1(\exp m) \in F^1 L'[1]\) and by Lemma 5.29 we know that
\[\text{Def}(L, Q) \cong \text{Def}(L, Q^m), \quad \text{Def}(L', Q') \cong \text{Def}(L', (Q')^m).\]
But by assumption we have
\[\text{Def}(L, Q^m) \cong \text{Def}(L', (Q')^m)\]
and the statement is clear. \(\square\)

## 6 Homotopic \(L_\infty\)-morphisms

One of our aims is to investigate the relation between twisted \(L_\infty\)-morphisms. More explicitly, let \(F: (g, Q) \to (g', Q')\) be an \(L_\infty\)-morphism between DGLAs and let \(\pi \in F^1 g^1\) be a Maurer-Cartan element equivalent to zero. Then we want to take a look at the relation between \(F^\pi\) and \(F\).

To this end we need to recall the definition of homotopic \(L_\infty\)-morphisms, which also has consequences for the homotopy classification of \(L_\infty\)-algebras. At first, let us recall from Corollary 2.31 that an \(L_\infty\)-structure \(Q\) on the vector space \(L\) is equivalent to a Maurer-Cartan element in the DGLA \((\text{Hom}^* g(S(L[1]), L[1]), 0, [\cdot, \cdot],_{NR})\). Analogously, we show now that we can also interpret \(L_\infty\)-morphisms as Maurer-Cartan elements in a convolution \(L_\infty\)-algebra.
6.1 $L_\infty$-morphisms as Maurer-Cartan Elements

Let $(L, Q), (L', Q')$ be two flat $L_\infty$-algebras and consider the space $\text{Hom}(S(L[1]), L')$ of graded linear maps. If $L$ and $L'$ are equipped with complete descending filtrations, then we require the maps to be compatible with the filtrations. We can interpret elements $F^1, G^1 \in \text{Hom}(S(L[1]), L'[1])$ as maps in $\text{Hom}(S(L[1]), S(L'[1]))$, where we have a convolution product $\ast$, compare Definition 2.5.

For example, one has

$$F^1 \ast G^1 = \lor (F^1 \otimes G^1) \circ \overline{\Delta}_{\text{sh}} : S(L[1]) \longrightarrow S^2(L[1]),$$

and since $\overline{\Delta}_{\text{sh}}$ and $\lor$ are (co-)commutative, one directly sees that $\ast$ is graded commutative. We can use this and the $L_\infty$-structures on $L$ and $L'$ to define an $L_\infty$-structure on this vector space of maps, see [15, Proposition 1 and Proposition 2] and also [5] for the case of DGLAs.

**Proposition 6.1** The coalgebra $\overline{S}(\text{Hom}(S(L[1]), L'[1]))$ can be equipped with a codifferential $\overline{Q}$ with structure maps

$$\overline{Q}^1 F = Q^1 F = (-1)^{|F|} F \circ Q$$

and

$$\overline{Q}^1_n (F_1 \lor \cdots \lor F_n) = (Q^1)^n_n \circ (F_1 \ast F_2 \cdots \ast F_n).$$

It is called convolution $L_\infty$-algebra and its Maurer-Cartan elements can be identified with $L_\infty$-morphisms. Here $|F|$ denotes the degree in $\text{Hom}(S(L[1]), L'[1])$.

**Proof:** The fact that this yields a well-defined $L_\infty$-structure follows directly from the fact that $L$ and $L'$ are $L_\infty$-algebras, and in particular from the cocommutativity and coassociativity of $\overline{\Delta}_{\text{sh}}$.

Now we want to show that the Maurer-Cartan elements are indeed in one-to-one correspondence with the $L_\infty$-morphisms. At first we recall that a coalgebra morphism $F$ from $\overline{S}(L[1])$ into $\overline{S}(L'[1])$ is uniquely determined by its projection $F^1$ to $L'$ via $F = \exp_1(F^1)$, compare Theorem 2.30. Consequently, we can identify it with a degree one element in $\text{Hom}(S(L[1]), L')$. It remains to show that the Maurer-Cartan equation is equivalent to the fact that $F$ commutes with the codifferentials. But again by Theorem 2.30 one sees that $Q^1 F = F Q$ is equivalent to $\text{pr}_{L'[1]}(Q^1 F - F Q) = 0$ which is just the Maurer-Cartan equation for $F^1$. \hfill $\square$

**Example 6.2 (Convolution DGLA)** Let $\mathfrak{g}, \mathfrak{g}'$ be two DGLAs. Then $\text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}')$ is in fact a DGLA, the so-called convolution DGLA with differential

$$\partial F = d' \circ F + (-1)^{|F|} F \circ Q$$

and bracket

$$[F, G] = -(-1)^{|F|}(Q^1)^1 \circ (F \ast G).$$

Here $|F|$ denotes again the degree in $\text{Hom}(S(\mathfrak{g}[1]), \mathfrak{g}'[1])$ and the induced codifferential is also denoted by $\overline{Q}$.

In order to obtain a notion of equivalent Maurer-Cartan elements we need a complete filtration. Note that the convolution $L_\infty$-algebra $\mathcal{H} = \text{Hom}(S(L[1]), L')$ is indeed equipped with the following complete descending filtration:

$$\mathcal{F} = F^1 \mathcal{H} \supset F^2 \mathcal{H} \supset \cdots \supset F^k \mathcal{H} \supset \cdots$$

$$\mathcal{F}^k \mathcal{H} = \left\{ f \in \text{Hom}(S(L[1]), L') \mid f \big|_{S^{<k}(L[1])} = 0 \right\}.$$  

(6.5)

Thus all twisting procedures are well-defined and one can define a notion of homotopic $L_\infty$-morphisms.

**Definition 6.3** Two $L_\infty$-morphisms $F, F'$ between flat $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ are called homotopic if they are homotopy equivalent Maurer-Cartan elements in the convolution $L_\infty$-algebra $(\text{Hom}(S(L[1]), L'), Q)$.  

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However, we are mainly interested in $L_\infty$-morphisms between $L_\infty$-algebras resp. DGLAs with complete filtrations, whence we introduce a new filtration on the convolution $L_\infty$-algebra $H = \text{Hom}(S(L[1]), L')$ that takes into account the filtrations on $S(L[1])$ and $L'$:

$$H = \hat{S}^1H \supset \hat{S}^2H \supset \cdots \supset \hat{S}^kH \supset \cdots$$

$$\hat{S}^kH = \sum_{n+m=k} \left\{ f \in \text{Hom}(S(L[1]), L') \mid f|_{S^n(L[1])} = 0 \quad \text{and} \quad f : \mathcal{F}^* \to \mathcal{F}^{*+m} \right\}. \quad (6.6)$$

Here the filtration on $S(L[1])$ is the product filtration induced by

$$\mathcal{F}^k(L[1] \otimes L[1]) = \sum_{n+m=k} \text{im}(\mathcal{F}^n L[1] \otimes \mathcal{F}^m L[1] \to L[1] \otimes L[1]),$$

see e.g. [17] Section 1.

**Proposition 6.4** The above filtration (6.6) is a complete descending filtration on the convolution $L_\infty$-algebra $\text{Hom}(S(L[1]), L')$.

**Proof**: The filtration is obviously descending and $H = \hat{S}^1H$ since we consider in the convolution $L_\infty$-algebra only maps that are compatible with the filtration. It is compatible with the convolution $L_\infty$-algebra structure and complete since $L'$ is complete. $\square$

Recall that we introduced in Remark 3.17 the definition of curved morphisms between curved Lie algebras from [12] Definition 4.3. There exists a similar generalizations of curved morphisms between curved $A_\infty$-algebras [15]. Considering now the convolution $L_\infty$-algebra between two curved $L_\infty$-algebras we get directly the analogue generalization of curved morphisms for the $L_\infty$-setting:

**Remark 6.5 (Curved convolution $L_\infty$-algebra)** Let us consider now two curved $L_\infty$-algebras $(L, Q)$ and $(L', Q')$. Here we use the counital coaugmented coalgebra $S(\text{Hom}(S(L[1]), L')([1]))$ with coproduct $\Delta_\text{sh}$ and codifferential $\hat{Q}$ with Taylor components (6.1) and (6.2) as in the flat case plus curvature component

$$\hat{Q}_0^1 = (1 \mapsto Q_0^1(1)) \in (\text{Hom}(K, L')([1]))^1.$$

This gives indeed a curved convolution $L_\infty$-algebra and we want to interpret its Maurer-Cartan elements. We restrict ourselves to the filtered setting where we set $\mathcal{F}^0 K = K$, and where we assume $Q_0^1 \in \mathcal{F}^1 L'$. Then Maurer-Cartan elements $F \in \hat{S}^1(\text{Hom}(S(L[1]), L')([1]))^0$ with the filtration from (6.6) are given by Taylor components $F_0^n : S^n(L[1]) \to L'[1]$ for $n \geq 0$. The only difference to the flat setting is the zero component $F_0^n(1) = \alpha \in \mathcal{F}^n L^0$. Then the Maurer-Cartan equation implies

$$0 = \hat{Q}_0^1 + Q_0^1 \circ F^1 - F^1 \circ Q + \sum_{n=2}^{\infty} \frac{1}{n!} Q_n^1 \circ (F^1)^n. \quad (*)$$

If $F_0^1(1) = 0$ the evaluation at 1 yields $Q_0^1(1) = F_0^1(Q_0^1(1))$ and thus $F$ is a $L_\infty$-morphism of curved $L_\infty$-algebras in the usual sense, i.e. a coalgebra morphism commuting with the codifferentials. But for $F_0^1(1) = \alpha \neq 0$ we no longer get an induced coalgebra morphism since we no longer have $F(1) = 1$. In analogy to [12] Definition 4.3 we call such a general $F$ curved morphism of $L_\infty$-algebras and for $\alpha = 0$ the morphism $F$ is called strict. Note that in [24] those curved $L_\infty$-morphisms are just called $L_\infty$-morphism. We will study the curved setting in more details in Section 6.4 where we show that curved $L_\infty$-morphisms still satisfy some nice properties; in particular they are compatible with Maurer-Cartan elements, see e.g. Proposition 6.23.

For now we restrict us to the simpler flat case. Therefore, from now on, unless stated otherwise, all DGLAs and $L_\infty$-algebras are assumed to be flat. We collect a few immediate consequences about homotopic $L_\infty$-morphisms, see e.g. [12] Proposition 1.4.6:

**Proposition 6.6** Let $F, F'$ be two homotopic $L_\infty$-morphisms between the flat $L_\infty$-algebras $(L, Q)$ and $(L', Q')$. 38
i.) $F^1$ and $(F')^1$ are chain homotopic.
ii.) If $F$ is an $L_\infty$-quasi-isomorphism, then so is $F'$.

iii.) $F$ and $F'$ induce the same maps from $\text{Def}(L)$ to $\text{Def}(L')$, i.e. $F_{\text{mc}} = F'_{\text{mc}}$.
iv.) In the case of DGLAs $g, g'$, compositions of homotopic $L_\infty$-morphisms with a DGLA morphism of degree zero are again homotopic.

**Proof:** Concerning the first two points, let $F^1(t)$ and $\lambda^1(t)$ be the paths encoding the homotopy equivalence, i.e.

$$\frac{d}{dt} F^1(t) = \hat{Q}^1(\lambda^1(t) \lor \exp(F^1(t)))$$

with $F^1(0) = F^1$ and $F^1(1) = (F')^1$. In particular, this implies $\frac{d}{dt} F^1(t) = (Q')^1 \circ \lambda^1(t) + \lambda^1(t) \circ Q^1$ which gives the statement with $F^1(0) = F^1$.

For DGLAs the third point is proven in [3, Lemma B.5]. In our general setting we consider a Maurer-Cartan element $\pi \in \mathcal{F}^1 L^1$ and recall that $\exp(\pi) = \sum_{k=1}^\infty \frac{1}{k!} \pi^k$ satisfies $\Delta_{ab} \exp(\pi) = \exp(\pi) \land \exp(\pi)$ and $\hat{Q} \exp(\pi) = 0$ by Lemma [5.1]. Applying now [3] on $\exp$ gives

$$\frac{d}{dt} F^1(t)(\exp(\pi)) = \hat{Q}^1(\lambda^1(t) \lor \exp(F^1(t)))(\exp(\pi))$$

$$= (Q')^1(\lambda^1(t)(\exp(\pi)) \lor \exp(F^1(t))(\exp(\pi)),$$

i.e. $\pi(t) = F^1(t)(\exp(\pi))$ and $\lambda(t) = \lambda^1(t)(\exp(\pi))$ encode the homotopy equivalence between $F_{\text{mc}}(\pi) = F^1(\exp(\pi))$ and $F'_{\text{mc}}(\pi) = (F')^1(\exp(\pi))$.

The last point follows directly since DGLA morphisms commute with brackets and differentials. $\square$

We want to generalize the last point to compositions with $L_\infty$-morphisms. Since we could not find a reference we prove the statements in detail. We start with the post-composition as in [3] Proposition 3.5).

**Proposition 6.7** Let $F_0, F_1$ be two homotopic $L_\infty$-morphisms from $(L, Q)$ to $(L', Q')$. Let $H$ be an $L_\infty$-morphisms from $(L', Q')$ to $(L'', Q'')$, then $HF_0 \sim HF_1$.

**Proof:** For $F^1 \in \text{Hom}(\mathcal{S}(L[1]), L')$ we write $H^1 F = H^1 \circ \exp_*(F^1)$, where $*$ denotes again the convolution product with respect to $\lor$ and $\Delta_{ab}$, resp. $\Delta_{ab}$. Let us denote by $F^1(t) \in (\text{Hom}(\mathcal{S}(L[1]), L'[1])[1])^0[t]$ and $\lambda(t) \in (\text{Hom}(\mathcal{S}(L[1]), L'[1])[1])^{-1}[t]$ the paths encoding the homotopy equivalence between $F_0$ and $F_1$. Then $H^1 F(t) \in (\text{Hom}(\mathcal{S}(L[1]), L''[1])[1])^{-1}[t]$ satisfies

$$\frac{d}{dt} H^1 F(t) = H^1 \circ (\hat{Q}^1(\lambda^1(t) \lor \exp(F^1(t))) \star \exp_*(F^1)).$$

Since $F^1(t)$ is a path of Maurer-Cartan elements, we get

$$\frac{d}{dt} H^1 F(t) = H^1 \circ Q' \circ (\lambda^1(t) \star \exp_*(F^1(t))) + H^1 \circ (\lambda^1(t) \star \exp_*(F^1(t))) \circ Q$$

$$= (Q')^1 \circ H \circ (\lambda^1(t) \star \exp_*(F^1(t))) + H^1 \circ (\lambda^1(t) \star \exp_*(F^1(t))) \circ Q$$

$$= (\hat{Q})^1 \{H^1 \circ (\lambda^1(t) \star \exp_*(F^1(t)))\} + \sum_{\ell=2}^\infty (Q')^1 \circ H^\ell \circ (\lambda^1(t) \star \exp_*(F^1(t))).$$

Finally, we know from Theorem 2.30 that $\lambda(t) \star \exp_*(F^1(t))$ is a coderivation along $F(t)$ and we get for the second term

$$H^\ell \circ (\lambda^1(t) \star \exp_*(F^1(t))) = \left( H^1 \circ (\lambda^1(t) \star \exp_*(F^1)) \star \frac{1}{(\ell-1)!} (H^1 F)^{(t-1)} \right).$$

Summarizing, we have

$$\frac{d}{dt} H^1 F(t) = (\hat{Q})^1 \{(H^1 \circ (\lambda^1(t) \star \exp_*(F^1)) \lor \exp(H^1 F)\}$$

and the statement is shown. $\square$
Proposition 6.8 Let $F_0, F_1$ be two homotopic $L_\infty$-morphisms from $(L, Q)$ to $(L', Q')$. Let $H$ be an $L_\infty$-morphism from $(L'', Q'')$ to $(L, Q)$, then $F_0H \sim F_1H$.

**Proof:** Let $F^1(t) \in (\text{Hom}(\mathbb{L}(1), L'))[t]$ and $\lambda^1(t) \in (\text{Hom}(\mathbb{L}(1), L'))[t]$ describe the homotopy equivalence between $F_0$ and $F_1$. Then we consider
\[
F^1(t)H = F^1(t) \circ \exp(H) \in (\text{Hom}(\mathbb{L}(1), L'))[t]
\]
in the notation of the above proposition. We compute
\[
\frac{d}{dt}(F^1(t)H) = \tilde{Q}^1(\lambda^1(t) \lor \exp(F^1(t))) \circ H
\]
\[
= (Q')^1 \circ \lambda^1 \circ H + \lambda^1 \circ Q \circ H + \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!}(Q')^1 \circ (\lambda^1 \star F^1 \star \cdots \star F^1) \circ H
\]
\[
= (Q')^1 \circ \lambda^1 \circ H + \lambda^1 \circ H \circ Q'' + \sum_{\ell=2}^{\infty} \frac{1}{(\ell-1)!}(Q')^1 \circ (\lambda^1 H \star F^1 H \star \cdots \star F^1) H
\]
\[
= \tilde{Q}^1(\lambda^1(t) \lor \exp(F^1(t)H))
\]
since $H$ is a coalgebra morphism intertwining $Q''$ and $Q$ and of degree zero. Finally, since $\lambda^1(t)H \in (\text{Hom}(\mathbb{L}(1), L'))[t]$ the statement follows. \(\square\)

Corollary 6.9 Let $F_0, F_1$ be two homotopic $L_\infty$-morphisms from $(L, Q)$ to $(L', Q')$, and let $H_0, H_1$ be two homotopic $L_\infty$-morphisms from $(L', Q')$ to $(L'', Q'')$, then $H_0F_0 \sim H_1F_1$.

We want to end this section with an example of homotopic $L_\infty$-morphisms: Let $(\mathfrak{g}, d, [\cdot, \cdot])$ be a DGLA with complete descending filtration, and assume that $h \in \mathcal{F}^1\mathfrak{g}^0$.

**Proposition 6.10** If $g$ is closed, then
\[
e^{[g, \cdot]}: (\mathfrak{g}, d, [\cdot, \cdot]) \rightarrow (\mathfrak{g}, d, [\cdot, \cdot])
\]
is an DGLA automorphism that maps equivalent Maurer-Cartan elements to equivalent ones. If $g$ is exact, then it is even homotopic to the identity $\text{id}_1$.

**Proof:** If $g$ is closed, then $d \circ e^{[g, \cdot]} = e^{[g, \cdot]} \circ d$, see e.g. Lemma 5.9. Let now $g \in \mathcal{F}^1\mathfrak{g}^1$ be a Maurer-Cartan element. Then we know from the formula for the gauge action from Proposition 5.8 that we have
\[
\exp([g, \cdot]) \circ \pi = e^{[g, \cdot]} \pi,
\]
i.e. $e^{[g, \cdot]}$ maps a Maurer-Cartan element to an equivalent one. Suppose now that $g = -d\alpha$ with $\alpha \in \mathcal{F}^1\mathfrak{g}^{-1}$. Then we set
\[
\pi(t) = e^{[tg, \cdot]}, \quad \lambda(t) = e^{[tg, \cdot]} \circ [\alpha, \cdot].
\]
We have $\pi(0) = \text{id}_1$ and $\pi(1) = e^{[g, \cdot]}$, and we want to show that $\pi(t)$ and $\lambda(t)$ encode the homotopy equivalence relation as in Definition 5.17. Using the formulas for the convolution $L_\infty$-structure, we have to show
\[
e^{[tg, \cdot]} \circ [g, \cdot] = \frac{d}{dt}\pi(t) = \tilde{Q}^1(\lambda(t) \lor \exp(\pi(t))).
\]
For the right hand side we compute
\[
\tilde{Q}^1(\lambda(t) \lor \exp(\pi(t))) = -d \circ \lambda(t) - \lambda(t) \circ d + \lambda(t) \circ Q_2^1 \circ (\lambda(t) \star \pi(t)),
\]
where $Q_2^1(x \lor y) = (-1)^{|x||y|}[x, y]$ for $x, y \in \mathfrak{g}$ with $x \in (\mathfrak{g}[1])^{|x|}$. For the first two terms we get
\[
-d \circ \lambda(t) - \lambda(t) \circ d = -d \circ e^{[tg, \cdot]} \circ [\alpha, \cdot] - e^{[tg, \cdot]} \circ [\alpha, \cdot] \circ d = -e^{[tg, \cdot]} \circ [\alpha, \cdot] = e^{[tg, \cdot]} \circ [g, \cdot].
\]
Thus we only have to show 
\[ \lambda(t) \circ Q^1_2 + Q^1_1 \circ (\lambda(t) \ast \pi(t)) = 0. \]
For homogeneous \( x \in (\mathfrak{g}[1])^{[x]}, y \in (\mathfrak{g}[1])^{[y]} \) we compute
\[
\lambda(t) \circ Q^1_2(x \vee y) + Q^1_1 \circ (\lambda(t) \ast \pi(t))(x \vee y) = -(-1)^{|x|}e^{[\mathfrak{g} \cdot \cdot \cdot]_1} \circ [\alpha, \cdot \cdot \cdot](x, y)
- (-1)^{|x|-1}e^{[\mathfrak{g} \cdot \cdot \cdot]_1}[\alpha, x], e^{[\mathfrak{g} \cdot \cdot \cdot]_1} - (-1)^{|y|-1}e^{[\mathfrak{g} \cdot \cdot \cdot]_1}[\alpha, y], e^{[\mathfrak{g} \cdot \cdot \cdot]_1} = 0,
\]
and the proposition is shown. □

### 6.2 Homotopy Classification of Flat \( L_\infty \)-algebras

The above considerations allow us to understand the homotopy classification of flat \( L_\infty \)-algebras in a better way.

**Definition 6.11** Two flat \( L_\infty \)-algebras \((L, Q)\) and \((L', Q')\) are said to be homotopy equivalent if there are \( L_\infty \)-morphisms \( F: (L, Q) \to (L', Q') \) and \( G: (L', Q') \to (L, Q) \) such that \( F \circ G \sim \text{id}_{L'} \) and \( G \circ F \sim \text{id}_L \). In that case \( F \) and \( G \) are said to be quasi-inverse to each other.

As in \([33, \text{Lemma 3.8}]\) we can immediately show that this definition coincides indeed with the definition of homotopy equivalence via \( L_\infty \)-quasi-isomorphisms from \([8]\).

**Lemma 6.12** Two flat \( L_\infty \)-algebras \((L, Q)\) and \((L', Q')\) are homotopy equivalent if and only if there exists an \( L_\infty \)-quasi-isomorphism between them.

**Proof:** Due to Proposition \([4.1.2]\) every \( L_\infty \)-algebra \( L \) is isomorphic to the product of a linear contractible one and a minimal one \( L[1] \cong V \oplus W \). This means \( L[1] \cong V \oplus W \) as vector spaces, such that \( V \) is an acyclic cochain complex with differential \( d_V \) and \( W \) is an \( L_\infty \)-algebra with codifferential \( Q_W \). \( Q_W = 0 \). The codifferential \( Q \) on \( \mathfrak{g}[1] \oplus W \) is given on \( v_1 \vee \cdots \vee v_m \) with \( v_1, \ldots, v_k \in V \) and \( v_{k+1}, \ldots, v_m \in W \) by
\[
Q^1(v_1 \vee \cdots \vee v_m) = \begin{cases} 
- d_V(v_1), & \text{for } k = m = 1 \\
Q^1_W(v_1 \vee \cdots \vee v_m), & \text{for } k = 0 \\
0, & \text{else.}
\end{cases}
\]
This implies in particular that the canonical maps
\[
i: W \to V \oplus W \text{ and } p: V \oplus W \to W
\]
are \( L_\infty \)-quasi-isomorphisms. We want to show now that \( i \circ p \sim \text{id} \) and therefore choose a contracting homotopy \( h_V: V \to V \) with \( h_V d_V + d_V h_V = \text{id}_V \) and define the maps
\[
P(t): V \oplus W \ni (v, w) \mapsto (tv, w) \in V \oplus W
\]
and
\[
H(t) = H: V \oplus W \ni (v, w) \mapsto (-h_V(v), 0) \in V \oplus W.
\]
Note that \( P(t) \) is a path of \( L_\infty \) morphisms because of the explicit form of the codifferential. We clearly have
\[
\frac{d}{dt} P^1(t) = d_{\text{pr}_V}Q^1_1 = Q^1_1 \circ H(t) + H(t) \circ Q^1_1 = \hat{Q}^1_1(H(t))
\]
since \( h_V \) is a contracting homotopy. This implies
\[
\frac{d}{dt} P^1(t) = \hat{Q}^1_1(H(t)) \vee \exp(P(t))
\]
as \( \text{im}(H(t)) \subseteq V \) and as the higher brackets \( Q \) vanish on \( V \). From \( P(0) = i \circ p \) and \( P(1) = \text{id} \) we conclude that \( i \circ p \sim \text{id} \). We choose a similar splitting for \( L'[1] = V' \oplus W' \) with the same
properties and consider an $L_\infty$-quasi-isomorphism $F: L \to L'$. In Theorem 4.13 we constructed an $L_\infty$-quasi-inverse $G = i \circ (F_{\text{min}})^{-1} \circ p'$. Since by Proposition 6.7 and Proposition 6.8 compositions of homotopic $L_\infty$-morphisms are again homotopic, we get

$$F \circ G = F \circ i \circ (F_{\text{min}})^{-1} \circ p' \sim i' \circ p' \circ F \circ i \circ (F_{\text{min}})^{-1} \circ p'$$

and similarly $G \circ F \sim \text{id}$.

The other direction follows from Proposition 6.7. Suppose $F \circ G \sim \text{id}$ and $G \circ F \sim \text{id}$, then we know that $F^1 \circ G^1$ and $G^1 \circ F^1$ are both chain homotopic to the identity. Therefore, $F$ and $G$ are $L_\infty$-quasi-isomorphisms.

\[\□\]

**Corollary 6.13** Let $F: (L, Q) \to (L', Q')$ be an $L_\infty$-quasi-isomorphism with two given quasi-inverses $G, G': (L', Q') \to (L, Q)$ in the sense of Definition 6.11. Then one has $G \sim G'$.

**Proof:** One has

$$G \sim G \circ (F \circ G') = (G \circ F) \circ G' \sim G'$$

and the statement is shown. \[\□\]

As a first application, we want to show that the construction of the morphisms for the homotopy transfer theorem 4.4 are natural with respect to homotopy equivalences.

**Corollary 6.14** In the setting of Theorem 4.4 one has $P \circ I = \text{id}_A$ and $I \circ P \sim \text{id}_B$.

**Proof:** By Lemma 6.12 $P$ admits a quasi-inverse $I'$ such that $P \circ I' \sim \text{id}_A$ and $I' \circ P \sim \text{id}_B$, which implies

$$I \circ P = \text{id}_B \circ I \circ P \sim I' \circ P \circ I \circ P = I' \circ P \sim \text{id}_B,$$

and the statement is shown. \[\□\]

### 6.3 Homotopy Equivalence between Twisted Morphisms

Let now $F: (\mathfrak{g}, d, [\cdot, \cdot]) \to (\mathfrak{g}', d', [\cdot, \cdot])$ be an $L_\infty$-morphism between (flat) DGLAs with complete descending and exhaustive filtrations. Instead of comparing the twisted morphisms $F^\pi$ and $F'^\pi$ with respect to two equivalent Maurer-Cartan elements $\pi$ and $\pi'$, we consider for simplicity just a Maurer-Cartan element $\pi \in \mathcal{F}^1 \mathfrak{g}^1$ equivalent to zero via $\pi = \exp([g, \cdot]) \circ 0$, i.e. $\lambda(t) = g = A(t) \in \mathcal{F}^1 \mathfrak{g}^0[t]$. Then we know that $0$ and $S = F_{\text{MC}}(\pi) = F^1(\exp(\pi)) \in \mathcal{F}^1 (\mathfrak{g}')^1$ are equivalent Maurer-Cartan elements in $(\mathfrak{g}', d')$. Let the equivalence be implemented by an $A'(t) \in \mathcal{F}^1 (\mathfrak{g}')^0[t]$ as in Proposition 5.21. Then we have the diagram of $L_\infty$-morphisms between (flat) DGLAs

\[
\begin{array}{ccc}
(\mathfrak{g}, d) & \xrightarrow{e_{[A(\cdot), \cdot]}} & (\mathfrak{g}, d + [\pi, \cdot]) \\
F & \downarrow & \downarrow F^\pi \\
(\mathfrak{g}', d') & \xrightarrow{e_{[A'(\cdot), \cdot]}} & (\mathfrak{g}', d' + [S, \cdot])
\end{array}
\]

(6.8)

where $e_{[A(\cdot), \cdot]}$ and $e_{[A'(\cdot), \cdot]}$ are well-defined by the completeness of the filtrations. Following Proposition 3.10], we show that it commutes up to homotopy, which is indicated by the vertical arrow.

**Proposition 6.15** The $L_\infty$-morphisms $F$ and $e_{[\cdot, \cdot]} \circ F^\pi \circ e_{[A(\cdot), \cdot]}$ are homotopic, i.e. homotopy equivalent Maurer-Cartan elements in $(\text{Hom}(\mathcal{S}(\mathfrak{g}[1])), \mathfrak{g}', \hat{Q})$.  

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The candidate for the path between $F$ and $e^{-A(t) \cdot \cdot} \circ F^x \circ e^{[A(t) \cdot \cdot]}$ is

$$F(t) = e^{-A(t) \cdot \cdot} \circ F^x(t) \circ e^{[A(t) \cdot \cdot]}.$$ 

However, $F(t)$ is not necessarily in the completion $\text{Hom}(\hat{S}[\mathfrak{g}(1)], \mathfrak{g}^0)[t]$ with respect to the filtration from \([33, 36]\) since for example

$$F(t) \mod \mathcal{F}^2 \text{Hom}(\hat{S}[\mathfrak{g}(1)], \mathfrak{g}^0)[[t]] = e^{-A(t) \cdot \cdot} \circ F^x(t) \circ e^{[A(t) \cdot \cdot]}$$

is in general not polynomial in $t$. But using the filtration from \([36]\), we can prove Proposition \([6, 15]\).

**Proof (of Proposition \([6, 15]\):** The path $F(t) = e^{-A(t) \cdot \cdot} \circ F^x(t) \circ e^{[A(t) \cdot \cdot]}$ is an element in the completion $(\text{Hom}(\hat{S}[\mathfrak{g}(1)], \mathfrak{g}^0)[1])[t]$ with respect to the filtration from \([6, 6]\). This is clear since $A(t) \in \mathcal{F}_1 \mathfrak{g}^0[t]$, $A'(t) \in \mathcal{F}_1 \mathfrak{g}^0[t]$ and $\pi(t) \in \mathcal{F}_1 \mathfrak{g}^1[t]$ imply that

$$\sum_{i=1}^{n-1} e^{-A(t) \cdot \cdot} \circ F^x(t) \circ e^{[A(t) \cdot \cdot]} \mod \mathfrak{g}^n(\text{Hom}(\hat{S}[\mathfrak{g}(1)], \mathfrak{g}^0)[1])[t]$$

is polynomial in $t$. Moreover, $F(t)$ satisfies by \([5, 31]\)

$$\frac{dF(t)}{dt} = -e^{-A(t) \cdot \cdot} \circ [\lambda'(t), \cdot \cdot] \circ F^x(t) \circ e^{[A(t) \cdot \cdot]} + e^{-A(t) \cdot \cdot} \circ F^x(t) \circ [\lambda(t), \cdot \cdot] \circ e^{[A(t) \cdot \cdot]}$$

But we have

$$\frac{dF^x(t)}{dt}(X_1 \vee \cdots \vee X_k) = F^x_{k+1}(Q_1^{(t), 1}(\lambda(t)) \vee X_1 \vee \cdots \vee X_k)$$

$$= F^x_{k+1}(Q_1^{(t), k+1}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) + F^x_{k+1}(\lambda(t) \vee Q_1^{(t), k}(X_1 \vee \cdots \vee X_k))$$

$$= Q_1^{(t), 1} F^x_{k+1}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) + Q_1^{(t), k} F^x_{k+1}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)$$

$$= -F^x_{k+1} Q_1^{(t), k} \lambda(t) \vee X_1 \vee \cdots \vee X_k + F^x_{k+1}(\lambda(t) \vee Q_1^{(t), 1}(X_1 \vee \cdots \vee X_k)).$$

Setting now $\lambda^F(t) = F^x_{k+1}(\lambda(t) \vee \cdots \cdot)$ we get

$$\frac{dF^x(t)}{dt} = \tilde{Q}_1^{(t)}(\lambda^F(t)) + \tilde{Q}_2^{(t)}(\lambda^F(t) \vee F^x(t)) - F^x_{k+1}(\lambda(t), \cdot \cdot) + [\lambda'(t), \cdot \cdot] F^x_{k+1}.$$ 

Thus we get

$$\frac{dF(t)}{dt} = e^{-A(t) \cdot \cdot} \circ \left( \tilde{Q}_1^{(t)}(\lambda^F(t)) + \tilde{Q}_2^{(t)}(\lambda^F(t) \vee F^x(t)) \right) \circ e^{[A(t) \cdot \cdot]}$$

$$= \tilde{Q}_1^{(t)}(e^{-A(t) \cdot \cdot} \lambda^F(t)) \circ e^{[A(t) \cdot \cdot]} + \tilde{Q}_2^{(t)}(e^{-A(t) \cdot \cdot} \lambda^F(t) \circ [\lambda(t), \cdot \cdot]) \circ F^x(t)$$

since the $\exp([\lambda(t), \cdot \cdot])$ and $\exp([\lambda'(t), \cdot \cdot])$ commute with the brackets and intertwine the differentials. Thus $F(0) = F$ and $F(1)$ are homotopy equivalent. \(\square\)

**Remark 6.16 (Application to Deformation Quantization)** This result allowed us in \([33]\) to prove that Dolgushev’s globalizations \([12, 13]\) of the Kontsevich formality \([30]\) with respect to different covariant derivatives are homotopic.

Now we want to generalize the results from the above section to twisted morphisms between general $L_{\infty}$-algebras. As a first step, we have to generalize Lemma \([5, 9]\) and Corollary \([5, 10]\) i.e. we have to show that $L_{\infty}$-algebras that are twisted with equivalent Maurer-Cartan elements are $L_{\infty}$-isomorphic.
Lemma 6.17 Let \( (L, Q) \) be a flat \( L_\infty \)-algebra with complete descending filtration, and let \( \pi(t) \) and \( \lambda(t) \) encode a homotopy equivalence between two Maurer-Cartan elements as in Definition 5.14. For \( a \in S^i(L[1]) \) with \( i \geq 0 \) the recursively defined system of differential equations

\[
\frac{d}{dt} (\Phi_t)^i_j(a) = \sum_{k=1}^i \left( (Q^{\pi(t)}, \lambda(t) \vee \cdot) - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) (\Phi_t)^j_k(a) \\
= \sum_{k=1}^i (Q^{\pi(t)})^j_{k+1}(\lambda(t) \vee (\Phi_t)^j_k(a)), \quad (\Phi_0)^i_j(a) = pr_{L[1]}(a)
\]

has unique solutions \( (\Phi_t)^i_j : S^i(L[1]) \to L[1][t] \), where \( (\Phi_t)^j_k(a) \) depends indeed only on \( (\Phi_t)^j_1 \) for \( j = i - k + 1 \) as for \( L_\infty \)-morphisms. In fact, one has \( \Phi_t^i \in \text{Hom}(\mathcal{S}(L[1]), L)[1][t] \).

Proof: The right hand side of the differential equation \((6.9)\) depends only on \( (\Phi_t)^j_1 \) with \( j \leq i \). Thus it has a unique solution \( (\Phi_t)^i_j \) defined on \( \mathcal{S}(L[1]) \) since \([Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot : \mathcal{S}(L[1]) \to L[1][t]\). Similarly, we see that has \( \Phi_t^i \in \text{Hom}(\mathcal{S}(L[1]), L)[1][t] \): With the filtration \( \mathfrak{g}^* \) of the convolution algebra from (6.6) we have

\[
\frac{d}{dt} (\Phi_t)^i_j(\Phi_t)^j_k(a) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=1}^i Q_{k+\ell+1}^j(\pi(t)^{\vee \ell} \vee (\Phi_t)^j_1(\cdot)) \mod \mathfrak{g}^n \\
= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k=1}^i Q_{k+\ell+1}^j(\pi(t) \mod \mathfrak{g}^{n-1})^{\vee \ell} \vee (\lambda(t) \mod \mathfrak{g}^n) \vee (\Phi_t)^j_k(\Phi_t)^j_1 \mod \mathfrak{g}^n
\]

and thus \( (\Phi_t)^i_j \) mod \( \mathfrak{g}^n \in L[t] \) by induction on \( i \) and \( n \).

Thus \( \Phi_t^i \) induces for all evaluations of \( t \) a coalgebra morphism \( \Phi_t \) and we have

\[
\frac{d}{dt} \Phi_t(a) = \left( [Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) \Phi_t(a), \quad \Phi_0(a) = a
\]

since \( \frac{d}{dt} \) and \([Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \) are coderivations with respect to \( \Delta_{\mathfrak{g}} \) vanishing on \( 1 \), i.e. also with respect to \( \Delta_{\mathfrak{g}} \). But one can even show that \( \Phi_t \) is an \( L_\infty \)-morphism, i.e. compatible with the coderivations:

Lemma 6.18 One has

\[
\Phi_t \circ Q^\pi = Q^\pi \circ \Phi_t.
\]

In particular, \( \Phi_t \) induces an \( L_\infty \)-isomorphism from \( (L, Q^\pi) \) to \( (L, Q^\pi) \).

Proof: We compute for \( a \in S(L[1]) \)

\[
\frac{d}{dt} (Q^{\pi(t)} \circ \Phi_t(a)) = \left[ Q^{\pi(t)}, \frac{d}{dt} \pi(t) \vee \cdot \right] \Phi_t(a) + Q^{\pi(t)} \circ \left( [Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) \circ \Phi_t(a)
\]

\[
= \left( [Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) \circ Q^{\pi(t)} \circ \Phi_t(a).
\]

Thus \( Q^{\pi(t)} \circ \Phi_t(a) \) is at \( t = 0 \) just \( Q^\pi(a) \) and satisfies the differential equation \((6.9)\). Since the solution is unique, it follows \( \Phi_t \circ Q^\pi(a) = Q^\pi \circ \Phi_t(a) \).

In order to show that \( \Phi_t \) is an \( L_\infty \)-isomorphism it suffices by Proposition 3.9 to show that \( (\Phi_t)^1_1 \) is an isomorphism. But this is clear since \((\Phi_t)^1_1 - \text{id} \equiv 0 \mod \mathfrak{g}^2 \) and the completeness of the filtration. Moreover, by the construction of the \( L_\infty \)-inverse we even have \((\Phi_t)^{-1})^1_1 \in \text{Hom}(\mathcal{S}(L[1]), L)[1][t] \).

Example 6.19 If \( (L, Q) \) is just a DGLA \( (\mathfrak{g}, \{ \cdot, \cdot \}) \), then \((6.9)\) simplifies to

\[
\frac{d}{dt} (\Phi_t)^1_1(x) = \left( \lambda(t), (\Phi_t)^1_1(x) \right), \quad (\Phi_0)^1_1(x) = x, \quad \forall x \in \mathfrak{g}.
\]

For \( \lambda(t) = g \) this has the solution \( (\Phi_t)^1_1(x) = e^{(g \cdot \cdot)} x \) and \( (\Phi_t)^1_1 = 0 \) for \( n \neq 1 \), i.e. we recover the setting of Lemma 5.9 and Corollary 5.10.
Finally, we can use this $\Phi_t$ in order to generalize Proposition 6.15 to $L_\infty$-algebras.

**Proposition 6.20** Let $(L, Q)$ be a flat $L_\infty$-algebra with complete descending filtration and let $\pi \in \mathcal{F}^1L$ be a Maurer-Cartan element that is homotopy equivalent to 0 via $\pi(t), \lambda(t)$. Moreover, let $F : (L, Q) \to (L', Q')$ be an $L_\infty$-morphism. Then the $L_\infty$-morphisms $F$ and $(\Phi'_t)^{-1} \circ F^{\pi(t)} \circ \Phi_t$ are homotopic.

**Proof:** The candidate for the homotopy is $F(t) = (\Phi'_t)^{-1} \circ F^{\pi(t)} \circ \Phi_t$. In the proof of Lemma 6.15 we saw that $((\Phi'_t)^{-1})^1 \in \text{Hom}(\mathcal{S}(L[1]), L[1])$. Thus it directly follows that $F^1(t)$ is indeed in the completion $\text{Hom}(\mathcal{S}(L[1]), L[1])$. We compute

$$
\frac{d}{dt} F(t) = - (\Phi'_t)^{-1} \circ \left( [Q^{\pi(t)}, \lambda'(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) \circ F^{\pi(t)} \circ \Phi_t
$$

$$
+ (\Phi'_t)^{-1} \circ F^{\pi(t)} \circ \left( [Q^{\pi(t)}, \lambda(t) \vee \cdot] - Q^{\pi(t)}(\lambda(t)) \vee \cdot \right) \circ \Phi_t + (\Phi'_t)^{-1} \circ \frac{d}{dt} F^{\pi(t)} \circ \Phi_t.
$$

With $\frac{d}{dt} F^{\pi(t)} = F^{\pi(t)} \circ (\frac{d}{dt} \pi(t) \vee \cdot) - (\frac{d}{dt} \pi(t) \vee \cdot) \circ F^{\pi(t)}$ we get

$$
\frac{d}{dt} F(t) = Q' \circ (\Phi'_t)^{-1} \circ (-\lambda'(t) \vee \cdot) \circ F^{\pi(t)} \circ \Phi_t + (\Phi'_t)^{-1} \circ F^{\pi(t)} \circ (\lambda(t) \vee \cdot) \circ \Phi_t
$$

$$
+ (\Phi'_t)^{-1} \circ (-\lambda'(t) \vee \cdot) \circ F^{\pi(t)} \circ \Phi_t + (\Phi'_t)^{-1} \circ F^{\pi(t)} \circ (\lambda(t) \vee \cdot) \circ \Phi_t \circ Q.
$$

Projecting to $L[1]$ yields indeed

$$
\frac{d}{dt} F^1(t) = \hat{Q}^1(\lambda_F(t) \vee \exp F^1(t))
$$

for $\lambda_F(t) = \text{pr}_{L[1]}((\Phi'_t)^{-1} \circ (-\lambda'(t) \vee \cdot) \circ F^{\pi(t)} \circ \Phi_t + (\Phi'_t)^{-1} \circ F^{\pi(t)} \circ (\lambda(t) \vee \cdot) \circ \Phi_t)$. Note that this is true since the term in the bracket is a coderivation with respect to $\Delta_{\infty}$ along $F^1(t)$ that vanishes on 1, therefore also a coderivation along $\Delta_{\infty}$. Moreover, $\lambda_F$ is indeed in the completion $\text{Hom}(\mathcal{S}(L[1]), L[1])$ since $F^1(t)$ is.

**6.4 Homotopy Theory of Curved $L_\infty$-Algebras**

As mentioned above, we want to generalize now the above homotopy theory of flat $L_\infty$-algebras to the curved setting. Therefore, we want to interpret $L_\infty$-morphisms again as Maurer-Cartan elements, as we did in the flat case from 6.11. From Remark 6.3 we recall the following result:

**Proposition 6.21** Let $(L, Q)$ and $(L', Q')$ be (curved) $L_\infty$-algebras with complete descending filtrations such that one has $Q^0_0 \in \mathcal{F}^1 L$ and $Q^0_0 \in \mathcal{F}^1 L'$ for the curvatures. Then the coalgebra $S(\text{Hom}(S(L[1]), L'[1]))$ can be equipped with a codifferential $\hat{Q}$ with structure maps

$$
\hat{Q}^1_0 = (1 \mapsto Q^0_0(1)) \in \text{Hom}(\mathcal{S}(L'), L'[1])^1,
$$

(6.12)

$$
\hat{Q}^1_1 F = Q^0_1 \circ F - (-1)^{|F|} F \circ Q
$$

(6.13)

and

$$
\hat{Q}^1_{\infty} (F_1 \vee \cdots \vee F_n) = (Q^0)^{\infty}_n \circ (F_1 \ast F_2 \ast \cdots \ast F_n),
$$

(6.14)

where $|F|$ denotes the degree in $\text{Hom}(S(L[1]), L'[1])$. Moreover, (6.10) generalizes to a complete descending filtration $\mathfrak{F} \text{Hom}(S(L[1]), L'[1])$. The curved $L_\infty$-algebra $(\text{Hom}(S(L[1]), L'), \hat{Q})$ is called convolution $L_\infty$-algebra.

**Proof:** The fact that this defines an $L_\infty$-structure follows as in Proposition 6.1 the fact that we get a complete descending filtration follows as in Proposition 6.3.

From now on we always assume that our curved $L_\infty$-algebras $(L, Q)$ have a complete descending filtration and that $Q^0_0 \in \mathcal{F}^1 L$. In this case, the above proposition immediately leads us to the following definition of curved $L_\infty$-morphisms and their homotopy equivalence relation, generalizing the observations for curved Lie algebras from Remark 3.11.
Definition 6.22 (Curved $L_\infty$-morphism) Let $(L, Q)$ and $(L', Q')$ be (curved) $L_\infty$-algebras with complete descending filtrations such that one has $Q_0^1 \in \mathfrak{F}^1 L$ and $Q_0^1 \in \mathfrak{F}^1 L'$ for the curvatures. The Maurer-Cartan elements in $\mathfrak{F}^1 \text{Hom}(S(L[1]), L')$ are called curved morphisms between $(L, Q)$ and $(L', Q')$ or curved $L_\infty$-morphisms. Two curved $L_\infty$-morphisms are called homotopic if they are homotopy equivalent Maurer-Cartan elements.

We write for a curved $L_\infty$-morphism $F^1$: $(L, Q) \sim (L', Q')$ and $F^1 \sim F'^1$ if $F^1$ and $F'^1$ are homotopic.

Remark 6.23 (Difficulties) Note that this generalization to the curved setting yields two main difficulties compared to the flat case:

- As mentioned above, in the case of curved $L_\infty$-algebras $(L, Q)$ the first structure map $Q_1^1$ does in general not square to zero. Thus we do not have the notion of an $L_\infty$-quasi-isomorphism as in Definition 3.10 and we can not use Proposition 4.12, i.e. that every flat $L_\infty$-algebra is isomorphic to the direct sum of a minimal one and a linear contractible one. Using the homotopy classification of flat $L_\infty$-algebras from Lemma 6.12 we propose below a notion of curved quasi-isomorphisms.

- Curved morphisms of $L_\infty$-algebras $F^1 \in \mathfrak{F}^1 \text{Hom}(S(L[1]), L')[1]^0$ are given by Taylor components $F_n^1: S^n(L[1]) \to L'[1]$ for $n \geq 0$, where one has in particular a zero component $F_0^1 = F_0^1(1) = \alpha \in \mathfrak{F}^1 L'$. By definition, they satisfy the Maurer-Cartan equation

$$0 = \tilde{Q}_n^1 + Q_n^1 \circ F^1 - F^1 \circ Q + \sum_{n=2}^{\infty} \frac{1}{n!} Q_n^1 \circ (F^1)^n. \quad (6.15)$$

If $F_0^1(1) = 0$ the evaluation at 1 yields $Q_0^1(1) = F_1^1(0)$ and thus $F$ is a $L_\infty$-morphism of curved $L_\infty$-algebras in the usual sense, from now on called strict $L_\infty$-morphism. However, for $F_0^1(1) = \alpha \neq 0$ we no longer get an induced coalgebra morphism on the symmetric coalgebras, compare Theorem 2.50.

The second point is in fact no big problem as we explain now: At first, note that we can still extend $F^1$ to all symmetric orders via $F^1 = \frac{1}{n!} (F^1)^n$ and $F^0 = \text{pr}_X$. Writing $F = \exp F^1$ we get a coalgebra morphism

$$F: S(L[1]) \to \tilde{S}(L'[1]), \quad X \mapsto F(X) = \exp \alpha \lor \tilde{F}(X) \quad (6.16)$$

into the completed symmetric coalgebra, where we complete with respect to the induced filtration. Here $\tilde{F}: S(L[1]) \to S(L'[1])$ is the extension of $F_n^1$ with $n \geq 1$ to a coalgebra morphism, i.e. in particular $\tilde{F}(1) = 1$. We sometimes identify $F^1$ with its extension $F$. Note that since $Q^1$ is compatible with the filtration, it extends to the completion $\tilde{S}(L'[1])$ and (6.15) implies $F^1 \circ Q = Q^1 \circ F$ as expected.

Thus we see that curved $L_\infty$-morphisms $F^1$ are still well-behaved and we collect some useful properties:

Proposition 6.24 Let $(L, Q)$ and $(L', Q')$ be (curved) $L_\infty$-algebras with complete descending filtrations such that one has $Q_0^1 \in \mathfrak{F}^1 L$ and $Q_0^1 \in \mathfrak{F}^1 L'$ for the curvatures. Moreover, let $F^1 \in \mathfrak{F}^1 \text{Hom}(S(L[1]), L')[1]^0$ be a curved $L_\infty$-morphism.

i.) Extending $F_n^1$ with $n \geq 1$ to a coalgebra morphism $\tilde{F}$, one gets for all $X \in S(L[1])$

$$\tilde{F}^1 \circ Q(X) = Q^1 \circ (\exp(\alpha) \lor \tilde{F}(X)), \quad (6.17)$$

i.e. $\tilde{F}$ is a strict morphism into the twisted $L_\infty$-algebra $(L, Q^\alpha)$.

ii.) Conversely, every strict morphism $\tilde{F}: (L, Q) \to (L', Q'^\alpha)$ corresponds to a curved $L_\infty$-morphism $F^1$ with $F_0^1(1) = \alpha$ and $F^1 = \tilde{F}^1$ for $i > 0$. 

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iii.) $F^1$ induces a map at the level of Maurer-Cartan elements. If $\pi \in T^1 L^1$ is a curved Maurer-Cartan element in $(L, Q)$, then

$$F_{MC}(\pi) = F^1(\exp \pi) = \alpha + \sum_{k=1}^{\infty} \frac{1}{k!} F^1_k (\pi \cdots \pi)$$

(6.18)

is a curved Maurer-Cartan element in $(L', Q')$.

iv.) The map from (6.13) is compatible with homotopy equivalences, i.e. $F^1$ induces a map $F_{MC}: \text{Def}(L) \to \text{Def}(L')$.

v.) The case $F^1 = 1d$ yields $F^1 = \alpha$ and $F^1_n = 0$ for all $n \geq 2$ corresponds again to twisting by $\alpha$. We denote it by

$$\text{tw}_\alpha: (L, Q) \rightsquigarrow (L, Q^{-\alpha}).$$

(6.19)

vi.) The curved $L_\infty$-morphisms from the curved $L_\infty$-algebra $0$ to $L$ are just the set of curved Maurer-Cartan elements of $L$.

**Proof:** The first and second point follow directly from the Maurer-Cartan equation (6.15) and (6.19). Then the third and fourth point follow from Lemma 5.29 since $\tilde{F}$ is compatible with equivalences by Proposition 5.23. The other statements are clear.

**Corollary 6.25** Curved $L_\infty$-morphisms $F^1$ from $(L, Q)$ to $(L', Q')$ are in one-to-one correspondence with strict $L_\infty$-morphisms $\tilde{F}$ from $(L, Q)$ into $(L', Q'^\alpha)$ with $F^1_0(1) = \alpha \in T^1 L^1$ and $F^1_i = \tilde{F}_i$ for $i > 0$.

As expected, we can compose curved $L_\infty$-morphisms between curved $L_\infty$-algebras:

**Proposition 6.26** Let $F^1: (L, Q) \rightsquigarrow (L', Q')$ and $G^1: (L', Q') \rightsquigarrow (L'', Q''\alpha)$ be curved $L_\infty$-morphisms with $F^1_0 = \alpha \in T^1 L^1$ and $G^1_0 = \beta \in T^1 L'^{\alpha}$. Then there exists a curved $L_\infty$-morphism

$$(G \circ F)^1 := G^1 \circ F: (L, Q) \rightsquigarrow (L'', Q'')$$

(6.20)

with $(G \circ F)_0^1 = G^1(\exp \alpha) = \beta + \tilde{G}^1(\exp \alpha)$ and $G \circ \tilde{F} = \tilde{G}^\alpha \circ \tilde{F}$, and the composition is associative. Moreover, one has

$$(G \circ F)^{MC}_{MC} = G^1 \circ F^1_{MC}: \text{Def}(L) \to \text{Def}(L'').$$

(6.21)

at the level of equivalence classes of Maurer-Cartan elements.

**Proof:** We saw in Proposition 6.24 that $F$ and $G$ correspond to strict $L_\infty$-morphisms

$$\tilde{F}: (L, Q) \rightarrow (L', Q'^\alpha) \quad \text{and} \quad \tilde{G}: (L', Q') \rightarrow (L'', (Q''\beta)).$$

In particular, we can twist $\tilde{G}$ with $\alpha$ and obtain

$$(L, Q) \xrightarrow{\tilde{F}} (L', Q'^\alpha) \xrightarrow{\tilde{G}^\alpha} (L'', (Q''\beta + \tilde{G}^1(\exp \alpha))).$$

Thus $(G \circ F)_i^1 = (\tilde{G}^\alpha \circ \tilde{F})_i^1$ for $i > 0$ and $(G \circ F)_0^1 = \beta + \tilde{G}^1(\exp \alpha)$ defines indeed a curved $L_\infty$-morphism. Moreover, we have $G^1 \circ F = G^1 \circ \exp \circ F^1 = G^1 \circ \exp(\alpha) \vee \tilde{F} = (G \circ F)^1$. It is easy to check that the composition is associative. Therefore, let us now look at the induced map at the level of Maurer-Cartan elements: by (6.13) $(G \circ F)^{MC}_{MC}$ maps $\pi \in T^1 L^1$ to

$$(G \circ F)_{MC}(\pi) = \beta + \tilde{G}^1(\exp \alpha) + G \circ \tilde{F}(\exp \pi) = \beta + \tilde{G}^1(\exp \alpha) + \tilde{G}^1(\exp \alpha \vee \tilde{F}(\exp \pi)) = G^1(\exp \alpha \vee \exp \tilde{F}(\exp \pi)) = G^1 \circ F^1_{MC}(\pi)$$

as desired.

We see that if both $F, G$ are strict morphisms, then $F = \tilde{F}$ and $G = \tilde{G}$ and the above composition is just the usual one. Moreover, we have the following observation:

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Corollary 6.27 Let $F^1: (L, Q) \rightsquigarrow (L', Q')$ be a curved $L_\infty$-morphism with $F_0^1 = \alpha \in \mathcal{F}^1 L'$. Then one has for $\beta \in \mathcal{F}^1 L'$
\[
tw_\alpha \circ tw_\beta = tw_{\alpha + \beta}, \quad F^1 = tw_\alpha \circ \tilde{F}, \quad \text{and} \quad tw_{-\alpha} \circ F = \tilde{F}^1 \tag{6.22}
\]

Proof: By the fifth point of Proposition 6.24 we know that $tw_\alpha$ corresponds to twisting with $-\alpha$, i.e.
\[
tw_\alpha: (L', Q'^{\alpha}) \rightsquigarrow (L', Q'^{\alpha-\alpha}) = (L', Q'), \quad (tw_\alpha)^1_0 = \alpha, \quad (tw_\alpha)^1_1 = \text{id}.
\]
We compute $(tw_\alpha \circ tw_\beta)^1_0 = \alpha + \beta$ and $(tw_\alpha \circ tw_\beta)^1_1 = \delta^1_i \text{id}$ for $i \geq 0$, thus $tw_\alpha \circ tw_\beta = tw_{\alpha + \beta}$ is shown. Then Proposition 6.28 gives the desired $F^1 = tw_\alpha \circ \tilde{F}$, and the last identity follows with the first. \qed

In a next step, we want to investigate if the (curved) homotopy equivalence is compatible with the interpretation of curved morphisms as strict $L_\infty$-morphisms into a twisted codomain. At first, let $F^1$ and $F'^1$ be homotopic via $F^1(t)$ and $\lambda^1(t)$, i.e.
\[
\frac{d}{dt} F^1(t) = \tilde{Q}^1(\lambda^1(t) \lor \exp(F^1(t))) = \tilde{Q}^1(\lambda^1(t) \lor \exp(F_0^1(t)(1)) \lor \exp(\tilde{F}^1(t))) \tag{6.23}
\]
with $F^1(0) = F^1$ and $F^1(1) = F'^1$. This implies for the component $F^1_0(t) = F_0^1(t)(1)$:
\[
\frac{d}{dt} F^1_0(t) = \lambda^1(t) \circ Q^1_0 + Q^1_1 \circ (\lambda^1_0(t) \lor \exp(F^1_0(t))).
\]
Thus we see that $\alpha_0 \neq \alpha_1$ is possible, and we can in general not directly compare $\tilde{F}^1_0$ and $\tilde{F}^1_1$ since they can have different codomains. However, we directly see:

Proposition 6.28 Let $F^1, F'^1: (L, Q) \rightsquigarrow (L', Q')$ be two curved $L_\infty$-morphisms with $\alpha = F_0^1 = F'^1_0$. Then $F^1 \sim F'^1$ if and only if $\tilde{F}^1 \sim \tilde{F}^1$. Proof: By Corollary 6.27 we only have to show that $F^1 \sim F'^1$ implies $(tw_\beta \circ F) \sim (tw_\beta \circ F')$ for all $\beta \in \mathcal{F}^1 L'$ and all curved $L_\infty$-morphisms with $F_0^1 = F'^1_0$, where
\[
(tw_\beta \circ F), (tw_\beta \circ F'): (L, Q) \rightsquigarrow (L', Q'^{-\beta}).
\]
Thus assume that $F^1(t), \lambda^1(t)$ encode the equivalence between $F^1$ and $F'^1$, then we have by (6.24)
\[
\frac{d}{dt} F^1(t) = \tilde{Q}^1(\lambda^1(t) \lor \exp(F^1(t))) \quad \text{with} \quad F^1(0) = F^1 \quad \text{and} \quad F^1(1) = F'^1.
\]
With $G^1(t) = tw_\beta \circ F^1(t)$ we get
\[
\frac{d}{dt} G^1(t) = \frac{d}{dt} F^1(t) = \tilde{Q}^1(\lambda^1(t) \lor \exp(F^1(t)))
\]
\[
= \tilde{Q}^1(\lambda^1(t) \lor \exp(-\beta) \lor \exp(\beta) \lor \exp(F^1(t)))
\]
\[
= \tilde{Q}^{-\beta}(\lambda^1(t) \lor \exp(G^1(t))),
\]
where $\tilde{Q}^{-\beta}$ is the codifferential on the curved convolution $L_\infty$-algebra induced by $(L, Q)$ and $(L', Q'^{-\beta})$. Thus we get the homotopy equivalence between $tw_\beta \circ F$ and $tw_\beta \circ F'$. \qed

Moreover, (6.23) implies:

Corollary 6.29 Let $F^1$ and $F'^1$ be homotopic curved $L_\infty$-morphisms from $(L, Q)$ to $(L', Q')$ with $\alpha_0 = F_0^1$ and $\alpha' = F'^1_0$. If $Q^1_0 = 0$ if and if $\alpha$ is a Maurer-Cartan element, then so is $\alpha'$ and both are equivalent.

Analogously to the flat case in Proposition 6.6 we can show that homotopic curved $L_\infty$-morphisms induce the same maps on the equivalence classes of Maurer-Cartan elements.
Proposition 6.30 Let $F^1, F'^1: (L, Q) \rightsquigarrow (L', Q')$ be two curved $L_\infty$-morphisms. If $F^1$ and $F'^1$ are homotopic, then they induce the same maps from $\text{Def}(L)$ to $\text{Def}(L')$, i.e. $F_{MC} = F'^1_{MC}$.

Proof: Let us assume that $F^1(1), \lambda^1(t)$ encode the equivalence between $F^1$ and $F'^1$, then we have by (6.28)
\[
\frac{d}{dt} F^1(t) = \bar{Q}^1(\lambda^1(t) \vee \exp(F^1(t))) \quad \text{with } F^1(0) = F^1 \quad \text{and } F^1(1) = F'^1.
\]
Applying this to $\exp(\pi)$ for $\pi \in MC^1(L)$ gives
\[
\frac{d}{dt} F^1(t) \exp(\pi) = \bar{Q}^1(\lambda^1(t) \exp(\pi) \vee \exp(F^1(t) \exp(\pi))),
\]
i.e. $\pi(t) = F^1(t)(\exp\pi)$ and $\lambda(t) = \lambda^1(t)(\exp\pi)$ encode the homotopy equivalence between $F_{MC}(\pi) = F^1(\exp\pi)$ and $F'_{MC}(\pi') = (F'^1)^1(\exp\pi)$. Thus $F^1$ and $F'^1$ map indeed Maurer-Cartan elements to equivalent ones.

In the proof of Proposition 6.28 we saw that $\text{tw}_\alpha \circ \tilde{F}$ is homotopic to $\text{tw}_\alpha \circ \tilde{F}'$ if $\tilde{F}$ and $\tilde{F}'$ are homotopic. We want to generalize this in the spirit of Proposition 6.7 and Proposition 6.8 and show that general pre- and post-compositions of homotopic curved $L_\infty$-morphisms with a curved $L_\infty$-morphism are again homotopic.

Proposition 6.31 Let $F^1, F'^1: (L, Q) \rightsquigarrow (L', Q')$ be two curved $L_\infty$-morphisms and assume $F^1 \sim F'^1$.

i.) If $H^1: (L', Q') \rightsquigarrow (L'', Q'')$ is a curved $L_\infty$-morphism, then $(H^1 \circ F) \sim (H^1 \circ F')$.

ii.) If $H^1: (L', Q') \rightsquigarrow (L, Q)$ is a curved $L_\infty$-morphism, then $(F^1 \circ H) \sim (F'^1 \circ H)$.

Proof: The statements follow as in the flat case in Proposition 6.7 and Proposition 6.8 since we used there only bialgebraic properties that still hold in our setting by the completeness of the filtration.

Corollary 6.32 Let $F^1, F'^1$ be two homotopic curved $L_\infty$-morphisms from $(L, Q)$ to $(L', Q')$, and let $H^1, H'^1$ be two homotopic curved $L_\infty$-morphisms from $(L', Q')$ to $(L'', Q'')$, then $(H^1 \circ F) \sim (H'^1 \circ F')$.

Let us now address the first difficulty from Remark 6.24, namely the fact that we do not have an obvious notion of curved $L_\infty$-quasi-isomorphism since curved $L_\infty$-algebras $(L, Q)$ do not induce a cochain complex $(L, Q)^1$. Recall that we showed for the flat case in Lemma 6.12 that there exists an $L_\infty$-quasi-isomorphism between two flat $L_\infty$-algebras $(L, Q)$ and $(L', Q')$ if and only if there are $L_\infty$-morphisms $F: (L, Q) \rightarrow (L', Q')$ and $G: (L', Q') \rightarrow (L, Q)$ such that $F \circ G \sim id_{L'}$ and $G \circ F \sim id_L$. Thus we define:

Definition 6.33 (Curved $L_\infty$-quasi-isomorphism) Let $F^1: (L, Q) \rightsquigarrow (L', Q')$ be a curved $L_\infty$-morphism between curved $L_\infty$-algebras. One calls $F$ curved $L_\infty$-quasi-isomorphism if and only if there exists a curved $L_\infty$-morphism $G^1: (L', Q') \rightsquigarrow (L, Q)$ such that $F^1 \circ G \sim id_{L'}$ and $G^1 \circ F \sim id_L$. In this case, $F^1$ and $G^1$ are said to be quasi-inverse to each other.

Concerning the behaviour on Maurer-Cartan elements we directly get analogue of Theorem 6.24.

Corollary 6.34 Let $F^1: (L, Q) \rightsquigarrow (L', Q')$ be a curved $L_\infty$-quasi-isomorphism. Then the induced map $F_{MC}: \text{Def}(L) \rightarrow \text{Def}(L')$ is a bijection.

Proof: Let $G^1$ be the quasi-inverse of $F^1$, then we know by Proposition 6.30 that $(F \circ G)_{MC}$ and $(G \circ F)_{MC}$ are the identity maps on $\text{Def}(L')$ resp. $\text{Def}(L)$. But Proposition 6.28 implies that $(F \circ G)_{MC} = F_{MC} \circ G_{MC}$ and $(G \circ F)_{MC} = G_{MC} \circ F_{MC}$, which implies that $F_{MC}$ and $G_{MC}$ are bijections.
We can also generalize the twisting procedure for $L_\infty$-morphisms from the strict case in Proposition 5.28 to our curved $L_\infty$-morphisms, which is straightforward:

**Proposition 6.35** Let $F^1: (L, Q) \sim (L', Q')$ be a curved $L_\infty$-morphism with $F^1_0 = \alpha$ and let $\pi \in \mathfrak{F}L^1$. Then the curved $L_\infty$-morphism

$$(F^\pi)_1 = \text{tw}_{-F^1(\exp \pi)} \circ F \circ \text{tw}_\pi: (L, Q^\pi) \sim \sim \sim \sim \sim \sim (L', Q^{F^1(\exp \pi)})$$ (6.24)

has structure maps $(F^\pi)_1 = \sum_{k=0}^{\infty} \frac{1}{k!} F^1_{k+1}(\pi^\wedge k \vee \cdot)$ for $i > 0$ and $(F^\pi)_0 = \alpha$.

**Proof:** At first, it is clear that the composition $(F^\pi)_1 = \text{tw}_{-F^1(\exp \pi)} \circ F \circ \text{tw}_\pi$ is a curved $L_\infty$-morphism

$$(L, Q^\pi) \sim \sim \sim \sim (L, Q) \sim \sim \sim \sim (L', Q') \sim \sim \sim \sim (L', Q^{F^1(\exp \pi)}).$$

For the structure maps, we see that $(F^\pi)_1 = -\tilde{F}^1(\exp \pi) + \alpha + \tilde{F}^1(\exp \pi) = \alpha$ and for $i > 0$ we get

$$(F^\pi)_i^1 = (F \circ \text{tw}_\pi)_i^1 = (\tilde{F}^\pi)_i^1,$$

which implies in particular $\tilde{F}^\pi = \tilde{F}^\pi$.

**Remark 6.36** We collect a few immediate observations:

i.) We directly see that $(F^0)_1 = F^1$, i.e. twisting by zero does not change the morphism.

ii.) The above definition for the twisted curved $L_\infty$-morphism $(F^\pi)_1$ recovers the results for the strict case from Proposition 5.28. If $F^1$ is strict, i.e. $F^1_0 = 0$ and $F = \tilde{F}$, then we get indeed $F^\pi = \tilde{F}^\pi = \tilde{F}^\pi$.

iii.) If $F^1_0 \neq 0$ and if we twist with a Maurer-Cartan element, then the image does no longer have to be a flat $L_\infty$-algebra, since $\tilde{F}^1(\exp \pi)$ does not have to be a Maurer-Cartan element.

iv.) The twist of a curved $L_\infty$-morphism is always a curved $L_\infty$-morphism, i.e. we can not obtain a strict one by twisting.

Finally, note that the only thing we did not generalize yet are the results from Section 6.3 where we showed that strict $L_\infty$-morphisms between flat $L_\infty$-algebras that are twisted with equivalent Maurer-Cartan elements are homotopic. Note that in Proposition 6.15 and Proposition 6.20 we considered only the case of a Maurer-Cartan element $\pi$ equivalent to zero, which was sufficient for the flat case since it implies the statement for all generic equivalent Maurer-Cartan elements $\pi$ and $\pi'$.

- Let us consider at first the context of strict $L_\infty$-morphisms of curved $L_\infty$-algebras. Let $\pi \sim \pi'$ be two equivalent curved Maurer-Cartan elements in $(L, Q)$. Then we know from Lemma 5.27 that $(L, Q^\pi)$ and $(L, Q^{\pi'})$ are flat $L_\infty$-algebras, and from Lemma 5.29 that $\pi' - \pi \sim 0$ in $(L, Q^\pi)$. Thus we can directly apply Proposition 6.20.

- In the context of curved $L_\infty$-morphisms it is more difficult to generalize these results since we saw in Proposition 6.35 that if we twist a curved $L_\infty$-morphism with a Maurer-Cartan element, then the codomain does not need to be a flat $L_\infty$-algebra. More explicitly, let $F^1: (L, Q) \sim (L', Q')$ be a curved $L_\infty$-morphism and let $\pi, \pi' \in \mathfrak{F}L^1$ be two equivalent Maurer-Cartan elements. Then we end up with the following diagram:

\[
\begin{array}{ccc}
(L, Q^\pi) & \xrightarrow{(F^\pi)_1} & (L', Q^{F^1(\exp \pi)}) \\
\Phi_1 \downarrow & & \downarrow \text{tw}_\alpha \circ \Phi_1' \circ \text{tw}_{-\alpha} \\
(L, Q'^\pi) & \xrightarrow{(F'^\pi)_1} & (L', Q'^{F^1(\exp \pi)})
\end{array}
\] (6.25)
where $\Phi_1$ and $\Phi'_1$ are strict $L_\infty$-isomorphisms by Lemma 6.18. Note that in particular

$$\Phi'_1 : (L', Q^{\alpha+F^1(\exp \pi)}) \longrightarrow (L', Q^{\alpha+F^1(\exp \pi')})$$

is a well-defined strict $L_\infty$-isomorphism since $\alpha + F^1(\exp \pi) = F_{MC}(\pi)$ and $\alpha + F^1(\exp \pi') = F_{MC}(\pi')$ are equivalent Maurer-Cartan elements.

We can show that Diagram (6.25) commutes up to homotopy:

**Proposition 6.37** The curved $L_\infty$-morphisms $(F^\pi)^1$ and $tw_\alpha \circ (\Phi'_1)^{-1} \circ tw_{-\alpha} \circ F^{\pi'} \circ \Phi_1$ are homotopic.

**Proof:** We know from Corollary 6.27 that we have

$$(F^\pi)^1 = tw_\alpha \circ \tilde{F}^\pi,$$

and

$$(F^{\pi'})^1 = tw_\alpha \circ \tilde{F}^\pi'.$$

which implies

$$tw_\alpha \circ (\Phi'_1)^{-1} \circ tw_{-\alpha} \circ F^{\pi'} \circ \Phi_1 = tw_\alpha \circ (\Phi'_1)^{-1} \circ F^{\pi'} \circ \Phi_1.$$ 

Moreover, Proposition 6.28 implies

$$\left((F^\pi)^1 \sim tw_\alpha \circ (\Phi'_1)^{-1} \circ tw_{-\alpha} \circ F^{\pi'} \circ \Phi_1\right) \iff \left(\tilde{F}^\pi \sim (\Phi'_1)^{-1} \circ \tilde{F}^{\pi'} \circ \Phi_1\right).$$

But the right hand side is exactly Proposition 6.20 since we know $\tilde{F}^\pi = \tilde{F}^\pi$ and $\tilde{F}^{\pi'} = \tilde{F}^{\pi'}$. \qed

7 $L_\infty$-modules

After having introduced the notion of $L_\infty$-algebras, we want to understand the basics of their representation theory, i.e. the notion of $L_\infty$-modules, see e.g. [13, 14, 19]. For example, they play an important role in the formality theorem for Hochschild chains [14, 48].

7.1 Definition and First Properties

**Definition 7.1 (L_\infty-module)** Let $(L, Q)$ be a (curved) $L_\infty$-algebra. An $L_\infty$-module over $(L, Q)$ is a graded vector space $M$ over $\mathbb{K}$ equipped with a codifferential $\phi$ of degree 1 on the cofreely cogenerated comodule $S(L[1]) \otimes M$ over $(S(L[1]), Q)$.

On the total space of the comodule $S(L[1]) \otimes M$ one has the coaction

$$a : S(L[1]) \otimes M \longrightarrow S(L[1]) \otimes (S(L[1]) \otimes M)$$

that is defined by

$$a(\gamma_1 \vee \cdots \vee \gamma_n \otimes m) = (\Delta_m \otimes \text{id})(\gamma_1 \vee \cdots \vee \gamma_n \otimes m)$$

$$= \sum_{k=0}^n \sum_{\sigma \in S(n,k,n-k)} \epsilon(\sigma) \gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)} \otimes (\gamma_{\sigma(k+1)} \vee \cdots \vee \gamma_{\sigma(n)} \otimes m),$$

where $\gamma_i \in L$ and $m \in M$ are homogeneous. For example, we have $a(1 \otimes m) = 1 \otimes 1 \otimes m$ and $a(\gamma \otimes m) = \gamma \otimes 1 \otimes m + 1 \otimes \gamma \otimes m$. By the coassociativity of $\Delta_m$ it follows directly $(\text{id} \otimes a) a(X) = (\Delta_m \otimes \text{id}) a(X)$ for all $X \in S(L[1]) \otimes M$, thus the well-definedness of the coaction.

**Remark 7.2** Note that in the flat setting one can restrict to modules over $\bar{S}(L[1])$ instead of $S(L[1])$. In this case the sum in the definition of $a$ starts at $k = 1$. For example, one has then in the flat case $\ker a = M$, which is not true in the curved setting, see [14 Section 2.2].
By definition, an $L_{\infty}$-module structure is a codifferential $\phi$ of $S(L[1]) \otimes M$, which means
\[ a \circ \phi(X) = (\text{id} \otimes \phi)(aX) + (Q \otimes \text{id})((aX). \] (7.2)
In terms of homogeneous elements this takes the form
\[ \phi(\gamma_1 \vee \cdots \vee \gamma_m) = Q(\gamma_1 \vee \cdots \vee \gamma_m) \otimes m \]
\[ + \sum_{k=0}^{n} \sum_{\sigma \in \text{Sh}(k,n-k)} (-1)^{\sum_{i=1}^{k} \gamma_{\sigma(i)}} \epsilon(\sigma) \gamma_{\sigma(1)} \vee \cdots \vee \gamma_{\sigma(k)} \vee \phi_{n-k}(\gamma_{\sigma(k+1)} \vee \cdots \vee \gamma_{\sigma(n)} \otimes m) \]
with
\[ \phi_n = \phi_n : S^n(L[1]) \otimes M \rightarrow M[1], \quad n \geq 0. \] (7.3)
For example, this implies
\[ \phi(\gamma \otimes m) = Q(\gamma) \otimes m + 1 \otimes \phi_1(\gamma \otimes m) + (-1)^{|\gamma|} \phi_0(\gamma \otimes m). \]
In particular, this gives
\[ \phi_0(1 \otimes \phi_0(1 \otimes m)) + \phi_1((Q_0(1 \otimes m) = 0 \]
and
\[ \phi_0(1 \otimes \phi_1(\gamma \otimes m)) + \phi_1(Q_0(1 \otimes \phi_0(1 \otimes m) + \phi_2(Q_0(1 \otimes m) \otimes m) + (-1)^{|\gamma|} \phi_1(\gamma \otimes \phi_0(1 \otimes m) = 0 \]
for $\gamma \in L[1]^{\gamma}$. In the flat setting, i.e. if $Q_0(1) = 0$, the map $\phi_0$ is indeed a differential on $M$ and $\phi_1$ is closed with respect to the induced differential on $\text{Hom}(L[1] \otimes M, M)$.

**Example 7.3 (DGLA module)** The simplest example is as expected a DG module $(M, b, \rho)$ over a DGLA $(g, d, [\cdot, \cdot])$. The only non-vanishing structure maps of $\phi$ are $\phi_0 = -b$ and $\phi_1(\gamma \otimes m) = (-1)^{|\gamma|} \rho(\gamma)m$, where $\rho$ is the action of $g$ on $M$.

**Example 7.4** Another basic example comes from $L_{\infty}$-morphisms. Let $F : (L, Q) \rightarrow (L', Q')$ be an $L_{\infty}$-morphism, then it induces on $L'$ the structure of an $L_{\infty}$-module over $L$ via
\[ \phi_k(\gamma_1 \vee \cdots \vee \gamma_k \otimes m) = Q^k(F(\gamma_1 \vee \cdots \vee \gamma_k) \otimes m) \]
for $\gamma_i \in L, m \in L'$. As in the case of $L_{\infty}$-algebras, $L_{\infty}$-module structures can be interpreted as Maurer-Cartan elements in a convolution-like algebra:

**Proposition 7.5** Let $M$ be a graded vector space over $\mathbb{K}$ and $(L, Q)$ an $L_{\infty}$-algebra. Then the vector space $\mathfrak{h}_{L,M} = \text{Hom}(S(L[1]) \otimes M, M)$ of graded linear maps can be equipped with the structure of a DGLA with differential
\[ \partial \phi = (-1)^{|\phi|} \phi \circ (Q \otimes \text{id}), \] (7.6)
where $|\cdot|$ denotes the degree in $\text{Hom}(S(L[1]) \otimes M, M)$, and bracket induced by the product
\[ \phi \bullet \psi = \phi \circ (\text{id} \otimes \psi) \circ (\Delta_{ab} \otimes \text{id}). \] (7.7)

The Maurer-Cartan elements $\phi$ of this DGLA can be identified with $L_{\infty}$-module structures $\hat{\phi}$ on $M$.\[ ]
Proof: The product \((7.7)\) is associative and thus induces a Lie bracket:
\[
\phi \cdot (\psi \cdot \eta) = \phi \circ (\text{id} \otimes (\psi \circ (\text{id} \otimes (\Delta_{ab} \otimes \text{id}))) \circ (\Delta_{ab} \otimes \text{id}) \\
= \phi \circ (\text{id} \otimes \psi) \circ (\text{id} \otimes \text{id} \otimes \eta)(\Delta_{ab} \otimes \text{id}) = (\phi \cdot \psi) \cdot \eta.
\]
The identity \(\partial^2 = 0\) is clear. The compatibility of \(\partial\) with the product follows from
\[
\partial(\phi \cdot \psi) = -(\Delta_{ab} \otimes \text{id}) \circ (\Delta_{ab} \otimes \text{id})(Q \otimes \text{id}) \\
= -(\partial \phi \cdot \psi + (-1)^{|\phi||\psi|}\phi \cdot \partial \psi).
\]
Let now \(\phi\) be a Maurer-Cartan element, i.e.,
\[
0 = \partial \phi + \phi \cdot \phi = \phi \circ (Q \otimes \text{id}) + \phi \circ (\text{id} \otimes \phi) \circ (\Delta_{ab} \otimes \text{id}).
\]
But by \((7.5)\) this is equivalent to \(\phi\) inducing a codifferential \(\hat{\phi} = Q \otimes \text{id} + (\text{id} \otimes \phi) \circ (\Delta_{ab} \otimes \text{id}). \)

As usual, if \(L\) and \(M\) are equipped with filtrations, we require the maps in \(\mathfrak{h}_{M,L}\) to be compatible with the filtrations.

Remark 7.6 For a flat \(L_\infty\)-algebra \((L, Q)\) the DGLA \(\mathfrak{h}_{M,L}\) itself has again complete descending filtration
\[
\mathfrak{h}_{M,L} = \mathcal{F}^0 \subseteq \mathcal{F}^1 \subseteq \mathcal{F}^2 \subseteq \cdots \subseteq \mathcal{F}^k \subseteq \cdots
\]
\[
\mathcal{F}^k \mathfrak{h}_{M,L} = \left\{ f \in \text{Hom}(S(L[1]) \otimes M, M) \mid f|_{S^{<k}(L[1]) \otimes M} = 0 \right\},
\]
 analogous to \((6.6)\). In the curved setting we have to assume that \(L\) and \(M\) have complete descending filtrations and that \(Q_0(1) \in \mathcal{F}^1 L\). This induces a filtration on \(\mathfrak{h}_{M,L}\) analogously to \((6.6)\), taking the filtrations on \(L\) and \(M\) into account.

This allows us to give another equivalent definition of \(L_\infty\)-modules, now in terms of \(L_\infty\)-morphisms.

Lemma 7.7 Let \(\phi\) be a coderivation of degree one on the comodule \(S(L[1]) \otimes M\) over \((S(L[1]), Q)\) with Taylor coefficients \(\phi_n, n \geq 0\). Then \(\phi\) is a codifferential, i.e., defines an \(L_\infty\)-module structure on \(M\), if and only if the maps \(\Phi_k : S(L[1]) \rightarrow \text{End}(M)\) defined by
\[
\Phi(X_1 \vee \cdots \vee X_k)(m) := \phi_k(X_1 \vee \cdots \vee X_k \otimes m)
\]
for \(X_1 \in L[1], m \in M\) are the Taylor coefficients of an \(L_\infty\)-morphism, where the \(L_\infty\)-structure on \(\text{End}(M)\) is the one induced by the Lie bracket from Example \((3.4)\) and zero differential.

Proof: First we notice that if we consider the graded tensor-hom adjunction, we get
\[
\text{Hom}(S(L[1]) \otimes M, M) \simeq \text{Hom}(S(L[1]), \text{End}(M))
\]
and the isomorphism is given by equation \((7.9)\). One can check that the induced \(L_\infty\)-algebra structure on \(\text{Hom}(S(L[1]), \text{End}(M))\) coincides up to a sign with the convolution algebra structure of the two \(L_\infty\)-algebras.

Next we want to recall the definition of morphisms between \(L_\infty\)-modules.

Definition 7.8 (Morphism of \(L_\infty\)-modules) Let \((L, Q)\) be an \(L_\infty\)-algebra with two \(L_\infty\)-modules \((M, \phi^M)\) and \((N, \phi^N)\) over \(L\). Then a morphism between \(L_\infty\)-modules is a morphism \(\kappa\) between the comodules \(S(L[1]) \otimes M\) and \(S(L[1]) \otimes N\), i.e.,
\[
(id \otimes \kappa) \circ a^M = a^N \circ \kappa,
\]
such that \(\kappa \circ \phi^M = \phi^N \circ \kappa. \)
One can again show that $\kappa$ is uniquely determined by its structure maps

$$\kappa_n: S^n(L[1]) \otimes M \rightarrow N$$

via

$$\kappa(\gamma_1 \cdots \gamma_n \otimes m) = \sum_{k=0}^{n} \sum_{\sigma \in S(n,k-n)} \epsilon(\sigma)\gamma_{\sigma(1)} \cdots \gamma_{\sigma(k)} \otimes \kappa_{k-n}(\gamma_{\sigma(k+1)} \cdots \gamma_{\sigma(n)} \otimes m).$$

(7.12)

The compatibility with the coderivations in the flat case implies in particular $\kappa_0 \circ \phi^M_0 = \phi^N_0 \circ \kappa_0$, see [14, Formula (2.29)] for the general relation.

**Definition 7.9** A quasi-isomorphism $\kappa$ of $L_\infty$-modules over flat $L_\infty$-algebras is a morphism with the zeroth structure map $\kappa_0$ being a quasi-isomorphism of complexes $(M, \phi^M_0)$ and $(N, \phi^N_0)$.

Moreover, as for $L_\infty$-morphisms between $L_\infty$-algebras in Proposition 3.9 we see that an $L_\infty$-module morphism is an isomorphism if and only if the first structure map $\kappa_0: M \rightarrow N$ is an isomorphism:

**Proposition 7.10** Let $\kappa: (M, \phi^M) \rightarrow (N, \phi^N)$ be a morphism of $L_\infty$-modules over $(L, Q)$. Then $\kappa$ is an $L_\infty$-module isomorphism if and only if $\kappa_0: M \rightarrow N$ is an isomorphism.

**Proof:** The proof is completely analogue to the $L_\infty$-algebra case in Proposition 3.9.

As expected, morphisms of $L_\infty$-modules can be again interpreted as Maurer-Cartan elements of a convolution DGLA, now even a commutative one, i.e. morphisms of $L_\infty$-modules are just closed elements of degree one in a cochain complex:

**Proposition 7.11** Let $(M, \phi^M)$ and $(N, \phi^N)$ be two $L_\infty$-modules over $(L, Q)$. The vector space $\text{Hom}(S(L[1]) \otimes M, N)[-1]$ can be equipped with the structure of an abelian DGLA $(\text{Hom}(S(L[1]) \otimes M, N)[-1], \partial, 0)$ with differential

$$\partial X = \phi^{N,1} \circ (\text{id} \otimes X) \circ (\Delta_{ab} \otimes \text{id}) + (-1)^{|X|} X \circ \phi^M$$

(7.13)

and zero bracket. Maurer-Cartan elements, i.e. closed elements $\kappa^1$ of degree one, can be identified with morphisms of $L_\infty$-modules via

$$\kappa = (\text{id} \otimes \kappa^1) \circ (\Delta_{ab} \otimes \text{id}): S(L[1]) \otimes M \rightarrow S(L[1]) \otimes N.$$  

(7.14)

**Proof:** The fact that $\partial^2 = 0$ follows since $\phi^M$ and $\phi^N$ are codifferentials. Explicitly, we have

$$\partial^2 X = \phi^{N,1} \circ (\text{id} \otimes \phi^{N,1}) \circ (\text{id} \otimes X) \circ (\Delta_{ab} \otimes \text{id}) \circ (\Delta_{ab} \otimes \text{id}) + (-1)^{|X|+|X|+1} X \circ \phi^M \circ \phi^M$$

$$+ (-1)^{|X|} \phi^{N,1} \circ (\text{id} \otimes X) \circ (\Delta_{ab} \otimes \text{id}) \circ \phi^M$$

$$+ (-1)^{|X|} \phi^{N,1} \circ (\text{id} \otimes X \circ \phi^M) \circ (\Delta_{ab} \otimes \text{id})$$

$$= \phi^{N,1} \circ (\text{id} \otimes \phi^{N,1}) \circ (\text{id} \otimes X) \circ (\Delta_{ab} \otimes \text{id}) \circ (\Delta_{ab} \otimes \text{id})$$

$$+ (-1)^{|X|} \phi^{N,1} \circ (\text{id} \otimes \phi^M) \circ (\Delta_{ab} \otimes \text{id})$$

$$+ (-1)^{|X|} \phi^{N,1} \circ (\text{id} \otimes \phi^M) \circ (\Delta_{ab} \otimes \text{id})$$

$$= \phi^{N,1} \circ (\text{id} \otimes \phi^{N,1}) \circ (\text{id} \otimes X) \circ (\Delta_{ab} \otimes \text{id}) \circ (\Delta_{ab} \otimes \text{id}) + \phi^{N,1} \circ (Q \otimes X) \circ (\Delta_{ab} \otimes \text{id})$$

$$= \phi^{N,1} \circ (\text{id} \otimes \phi^{N,1}) \circ (\Delta_{ab} \otimes \text{id}) \circ (\Delta_{ab} \otimes \text{id}) = 0.$$

Moreover, (7.14) yields a comodule morphism since

$$(\text{id} \otimes \kappa) \circ a = (\text{id} \otimes ((\text{id} \otimes \kappa^1) \circ (\Delta_{ab} \otimes \text{id}))) \circ (\Delta_{ab} \otimes \text{id}) = (\text{id} \otimes \Delta_{ab} \otimes \text{id}) \circ (\text{id} \otimes \kappa^1) \circ (\Delta_{ab} \otimes \text{id})$$

$$= (\Delta_{ab} \otimes \text{id}) \circ (\text{id} \otimes \kappa^1) \circ (\Delta_{ab} \otimes \text{id}) = a \circ \kappa.$$

If $\kappa^1$ is of degree one in the shifted complex, then the induced comodule morphism is of degree zero and $\partial \kappa^1 = 0$ is equivalent to $\phi^N \circ \kappa = \kappa \circ \phi^M$. 

□
7.2 Homotopy Equivalence of Morphisms of $L_\infty$-Modules

This allows us to define a homotopy equivalence relation between morphisms of $L_\infty$-modules. Analogously to the case of $L_\infty$-algebras, it is defined via the gauge equivalence in the convolution DGLA. Since the convolution DGLA has only the differential as non-vanishing structure map, there is no filtration needed for the well-definedness of the gauge action.

**Definition 7.12** Let $(M,\phi^M)$ and $(N,\phi^N)$ be two $L_\infty$-modules over $(L,Q)$. Two morphisms $\kappa_1, \kappa_2$ of $L_\infty$-modules are called homotopic if they are gauge equivalent Maurer-Cartan elements in $(\text{Hom}(S(L[1]) \otimes M,N)[-1],\partial,0)$, i.e. if

$$\kappa_2 = \kappa_1 - \partial h$$

(7.15)

for some $h$ of degree zero. In other words, $[\kappa_1] = [\kappa_2]$ in $H^1(\text{Hom}(S(L[1]) \otimes M,N)[-1],\partial)$.

The twisting procedures from Section 5.3 can be transferred to $L_\infty$-modules, see [14, Proposition 3] for the flat case and [19] for the curved setting.

**Proposition 7.13** Let $(L,Q)$ be an $L_\infty$-algebra with $L_\infty$-module $(M,\phi)$ and let $\pi \in \text{MC}^1(L)$.

i.) For any $X \in S(L[1]) \otimes M$ 

$$a(\exp(\pi \vee)X) = \exp(X \vee) \otimes \exp(\pi \vee)(aX).$$

(7.16)

ii.) The map 

$$\phi^\pi = \exp(-\pi \vee) \phi \exp(\pi \vee)$$

(7.17)

is a coderivation of $S(L[1]) \otimes M$ along $Q^\pi$ squaring to zero.

iii.) If $\kappa: M \to N$ is an $L_\infty$-morphism of $L_\infty$-modules over $L$, then 

$$\kappa^\pi = \exp(-\pi \vee) \kappa \exp(\pi \vee)$$

(7.18)

is an $L_\infty$-morphism of the twisted modules over the twisted $L_\infty$-algebra.

**Proof:** The first claim is clear. The second and the third claim follow directly. \hfill \square

The twisted structure maps are as expected given by

$$\phi^\pi_n(\gamma_1 \vee \cdots \vee \gamma_n \otimes m) = \sum_{k=0}^{\infty} {\frac{1}{k!}} \phi_{n+k}(\pi \vee \cdots \vee \pi \vee \gamma_1 \vee \cdots \vee \gamma_n \otimes m)$$

(7.19)

and

$$\kappa^\pi_n(\gamma_1 \vee \cdots \vee \gamma_n \otimes m) = \sum_{k=0}^{\infty} {\frac{1}{k!}} \kappa_{n+k}(\pi \vee \cdots \vee \pi \vee \gamma_1 \vee \cdots \vee \gamma_n \otimes m)$$

(7.20)

We want to investigate the relation between equivalent Maurer-Cartan elements, i.e. equivalent $L_\infty$-module structures. To this end, assume that we have a complete descending filtration $F^\bullet$ on the convolution DGLA $h_{M,L} = \text{Hom}(S(L[1]) \otimes M,M)$, compare Remark 7.6.

**Proposition 7.14** Let $\phi_0, \phi_1 \in F^1h^1_{M,L}$ be two equivalent Maurer-Cartan elements with equivalence described by

$$\phi_1 = e^{[h,\cdot]} \phi_0 - \frac{e^{[h,\cdot]} - \text{id}}{[h,\cdot]} \partial h,$$

(7.21)

Then $A_h = (\text{id} \otimes e^h) \circ (\Delta^+_h \otimes \text{id})$ is an $L_\infty$-isomorphism between the $L_\infty$-modules $(S(L[1]) \otimes M,\phi_0)$ and $(S(L[1]) \otimes M,\phi_1)$.
Proof: $A_h$ is a comodule morphism by Proposition 7.11. Concerning the compatibility with the codifferentials we compute

\[
A_h \hat{\phi}_0 = (\id \otimes \phi^h) \circ (\Delta_h \otimes \id) \circ (Q \otimes \id) + (\id \otimes \phi^h) \circ (\Delta_h \otimes \id) \circ (\id \otimes \phi_0) \circ (\Delta_h \otimes \id) = (Q \otimes \id) \circ A_h + (\id \otimes (-\partial e^h + e^h \bullet \phi_0)) \circ (\Delta_h \otimes \id)
\]

\[
= (Q \otimes \id) \circ A_h + \left( \id \otimes \left( -\frac{e^{[h, \cdot]} - \id}{[h, \cdot]} \partial h \bullet e^h + e^{[h, \cdot]} \phi_0 \bullet e^h \right) \right) \circ (\Delta_h \otimes \id)
\]

\[
= (Q \otimes \id) \circ A_h + (\id \otimes (\phi_1 \bullet e^h)) \circ (\Delta_h \otimes \id) = \hat{\phi}_1 A_h
\]

and the statement is shown. \qed

Note that an $L_\infty$-morphism $F : (L, Q) \to (L', Q')$ induces a map

\[
F^* : \mathfrak{h}_{M,L'} \ni \phi \mapsto F^* \phi = \phi \circ (F \otimes \id) \in \mathfrak{h}_{M,L}
\]

via the pull-back. It is compatible with the DGLA structure:

**Lemma 7.15** The map $F^* : \mathfrak{h}_{M,L'} \to \mathfrak{h}_{M,L}$ is a DGLA morphism.

**Proof:** Concerning the differential we get

\[
F^* \partial' \phi = -(-1)^{|\phi|} \phi \circ (Q' \otimes \id) \circ (F \otimes \id) = -(-1)^{|\phi|} \phi \circ (F \otimes \id) \circ (Q \otimes \id) = \partial F^* \phi.
\]

Similarly, we get for the product

\[
F^* (\phi \bullet \psi) = F^* (\phi \circ (\id \otimes \psi) \circ (\Delta_h \otimes \id)) = (\phi \circ (\id \otimes \psi) \circ (F \otimes F \otimes \id) \circ (\Delta_h \otimes \id) = F^* \phi \bullet F^* \psi
\]

and the statement is shown. \qed

**Remark 7.16** Using Lemma 7.14 we can identify the DGLA $\mathfrak{h}_{M,L} = \Hom(S(L[1]) \otimes M, M)$ with $\Hom(S(L[1]), \End(M))$ equipped with the convolution $L_\infty$-structure. Under this identification the above pull-back is indeed just the usual pull-back. All the constructions below work in both points of view, however, we formulate them in terms of $\mathfrak{h}_{M,L}$.

We want to investigate the relation between pull-backs with homotopic $L_\infty$-morphisms, where we assume for simplicity that our $L_\infty$-algebras are flat. Let therefore $F(t)$ and $\lambda(t)$ encode the homotopy equivalence between two $L_\infty$-morphisms from $L$ to $L'$ in the convolution $L_\infty$-algebra $(\Hom(S(L[1]), L'), Q)$ with

\[
\frac{d}{dt} F^1(t) = \hat{Q} (\lambda^1(t) \lor \exp(F^1(t))),
\]

compare Definition 6.9. This can be rewritten in the following way:

**Proposition 7.17** The map $\Gamma = \lambda^1(t) \circ F(t) = \lor \circ \left( \lambda^1(t) \otimes F(t) \right) \circ \Delta_h$ satisfies

\[
\frac{d}{dt} F(t) = Q' \circ \Gamma + \Gamma \circ Q
\]

and

\[
\Delta_h \circ \Gamma = (\Gamma \otimes F + F \otimes \Gamma) \circ \Delta_h.
\]

**Proof:** By Theorem 2.30 we know that $\Gamma$ is a coderivation along $F$ and (7.24) is clear. Moreover, both $\frac{d}{dt} F$ and $(Q' \circ \Gamma + \Gamma \circ Q)$ are coderivations along $F$ and thus it suffices to show

\[
\frac{d}{dt} F^1(t) = Q'^1 \circ \Gamma + \lambda^1 \circ Q.
\]

But we know

\[
\frac{d}{dt} F^1(t) = \hat{Q}' (\lambda^1(t) \lor \exp(F^1(t))) = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} Q'^1_i \circ (\lambda^1(t) \star F^1(t) \cdots \star F^1(t)) + \lambda^1(t) \circ Q
\]

\[
= Q'^1 \circ \Gamma + \lambda^1 \circ Q
\]

and the proposition is shown. \qed

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This allows us to show that homotopic $L_\infty$-morphisms give homotopic pull-back morphisms.

**Theorem 7.18** Let $F_0$ and $F_1$ be two homotopic $L_\infty$-morphisms between the flat $L_\infty$-algebras $(L, Q)$ to $(L', Q')$ and assume that $h_{M,L'}$ and $h_{M,L}$ are equipped with complete filtrations as described in Remark 7.6. Then the induced DGLA morphisms

$$F_0^*, F_1^*: h_{M,L'} \longrightarrow h_{M,L}$$

are homotopic.

**Proof:** Let $F(t)$ and $\lambda^1(t)$ encode the homotopy equivalence between $F_0$ and $F_1$. We show that

$$\frac{d}{dt} F_t^* = \hat{Q}_1^1 (\Gamma^*_t \vee \exp F_t^*).$$

Here $\Gamma_t^* \phi = (-1)^{|\phi|} \phi \circ (\Gamma(t) \otimes \text{id})$, $F_t^* \phi = \phi \circ (F(t) \otimes \text{id})$, where $|\phi|$ denotes now the degree in $h_{M,L'}$. Moreover, $\hat{Q}_1$ is the codifferential on the convolution DGLA $S(\text{Hom}(\mathbb{S}(h_{M,L'}[1]), h_{M,L}[1]))$. It is given by

$$\hat{Q}_1^1 X = - \partial X = (-1)^{|X|} X \circ Q h_{M,L'},$$

and the bracket is induced by the bracket on $h_{M,L}$. Using Proposition 7.14 we have for $\phi \in h_{M,L'}$

$$\frac{d}{dt} F_t^* \phi = \frac{d}{dt} \phi \circ (F(t) \otimes \text{id}) = \phi \circ ((Q' \circ \Gamma + \Gamma \circ Q) \otimes \text{id}).$$

Moreover, we get

$$(\hat{Q}_1^1 \Gamma_t^*)_1 \phi = - \partial \Gamma_t^* \phi + \Gamma_t^* - \partial \phi = \phi \circ ((\Gamma \circ Q + Q' \circ \Gamma) \otimes \text{id}).$$

The higher orders of $\hat{Q}_1^1 (\Gamma_t^* \vee \exp F_t^*)$ are given by

$$\Gamma_t^* \circ \hat{Q}_{b_{M,L'}} \cdot 2 + \hat{Q}_{b_{M,L}} \cdot 2 \circ (\Gamma_t^* \vee F_t^*) \circ \Delta_{ab}.$$ 

We can compute

$$\Gamma_t^* (\phi, \psi) = (-1)^{|\phi| + |\psi| + 1} \phi \circ (\text{id} \otimes \psi) \circ (\Delta_{ab} \otimes \text{id}) \circ (\Gamma \otimes \text{id})$$

$$= (-1)^{|\phi| + |\psi| + 1} \phi \circ (\text{id} \otimes \psi) \circ ((\Gamma \otimes F + F \otimes \Gamma) \otimes \text{id})(\Delta_{ab} \otimes \text{id})$$

$$= \Gamma_t^* \phi \cdot F_t^* \psi - (-1)^{|\phi|} F_t^* \phi \cdot \Gamma_t^* \psi$$

which yields

$$\Gamma_t^* \circ \hat{Q}_{b_{M,L'}} \cdot 2 (\phi \vee \psi) = \hat{Q}_{b_{M,L'}} \cdot 2 (\phi, \psi) = - (-1)^{|\phi|} \Gamma_t^* (\phi \cdot \psi - (-1)^{|\phi| - 1} F_t^* \phi \cdot \Gamma_t^* \psi$$

$$= (-1)^{|\phi|} \Gamma_t^* \phi \cdot F_t^* \psi - (-1)^{|\phi| - 1} F_t^* \phi \cdot \Gamma_t^* \psi$$

$$= (-1)^{|\phi| - 1} (\Gamma_t^* \phi \cdot F_t^* \psi - (-1)^{|\phi|} F_t^* \phi \cdot \Gamma_t^* \psi$$

$$= \hat{Q}_{b_{M,L}} \circ (\Gamma_t^* \vee F_t^*) \circ \Delta_{ab} (\phi \vee \psi).$$

Thus the higher terms cancel each other. We only have to check that $F_t^*$ and $\Gamma_t^*$ are in the completion of the polynomials in $t$ with respect to the filtration. But this is clear since $F(t)$ and $\Gamma(t)$ have these properties.

With Proposition 7.14 this immediately yields:

**Corollary 7.19** The pull-backs of an $L_\infty$-module structure $\phi \in T^1 h_{M,L'}$ on $M$ over $L'$ via two homotopic $L_\infty$-morphisms $F_0, F_1: (L, Q) \rightarrow (L', Q')$ yield two isomorphic $L_\infty$-module structures on $M$ over $L$.
Remark 7.20 This statement is similar to the following theorem from algebraic topology, see e.g. [10] Theorem 2.1: Let \( p: E \to B \) be a fiber bundle and let \( f_0, f_1 : X \to B \) be two homotopic maps. Then the pull-back bundles are isomorphic. It would be interesting to see if there are other statements that can be transferred to our algebraic setting.

Analogously to our computations in Section 6.3 for the case of flat \( L_\infty \)-algebras and Section 6.4 for curved \( L_\infty \)-algebras, we want to show now that \( L_\infty \)-module morphisms that are twisted with equivalent Maurer-Cartan elements are homotopic.

At first, one can check that one has a version of Proposition 6.15 for DGLAs \((g,d,[\cdot,\cdot])\) and DGLA-modules \((M,b,\rho)\) and \((\mathcal{N},b',\rho')\) with complete filtrations. Let \( \pi \in \mathbb{F}^L \) be equivalent Maurer-Cartan elements.

Analogously, one can pull-back the \( \kappa \)-module morphism \( \pi \) and \( \kappa \)-module morphisms that are twisted with \( \pi \).

This can be generalized to general \( L_\infty \)-modules over \( L_\infty \)-algebras. Let \( (L,Q) \) be a (flat) \( L_\infty \)-algebra and \( \pi \in \mathbb{F}^L \) be a Maurer-Cartan element equivalent to zero via \( \pi(\lambda),\lambda(t) \). Let us assume for simplicity that \( \lambda(t) = \lambda \) is constant, which is possible by Remark 5.22. Let moreover \( \kappa: (M,\phi) \to (\mathcal{N},\phi') \) be an \( L_\infty \)-module morphism of \( L_\infty \)-modules over \( (L,Q) \). Then we know from Proposition 7.20 that \( \kappa^* : (M,\phi^* ) \to (\mathcal{N},\phi'^* ) \) is an \( L_\infty \)-module morphism of \( L_\infty \)-modules over \( (L,Q^\pi) \).

By Proposition 6.18 we have an \( L_\infty \)-isomorphism \( \Phi_{\lambda} : (L,Q) \to (L,Q^\pi(\lambda)) \) satisfying

\[
\frac{d}{dt} \Phi_{\lambda} = \left( [Q^\pi(\lambda), \lambda \vee \cdot ] - Q^\pi(\lambda) \vee \cdot \right) \circ \Phi_{\lambda}
\]

By Lemma 7.19 we can use \( \Phi_{\lambda} \) to pull-back the \( L_\infty \)-module structures \( \phi^* \) and \( \phi'^* \) on \( M \) resp. \( N \). Analogously, one can pull-back \( \kappa^* \) and one obtains an \( L_\infty \)-module morphism

\[
\Phi_{\lambda}^* \kappa^* : (M,\Phi_{\lambda}^* \phi^* ) \longrightarrow (\mathcal{N},\Phi_{\lambda}^* \phi'^* )
\]

over \( (L,Q) \), where \((\Phi_{\lambda}^* \kappa^*)^1 = (\kappa^*)^1 \circ (\Phi_{\lambda} \otimes \text{id})\), analogously to the case 7.22 for the codifferential.

Proposition 7.21 In the above setting there exists an \( L_\infty \)-module isomorphism \( \Psi_{\lambda}: (M,\phi) \to (\mathcal{M},\Phi_{\lambda}^* \phi'^* ) \) over \( L_\infty \) with structure maps \( (\Psi_{\lambda})^1 : \mathcal{F}(L[1]) \otimes M \to \mathcal{M} \) for \( i \geq 0 \) defined by

\[
\frac{d}{dt} (\Psi_{\lambda})^1_i = \left( (\Phi_{\lambda}^* \phi^* ) \circ (\lambda \vee \cdot ) \right)^1 \circ (\Psi_{\lambda})^1_i, \quad (\Psi_{\lambda})^1_0 = \text{pr}_M.
\]

PROOF: The well-definedness of \( \Psi_{\lambda} \) follows analogously to the well-definedness of \( \Phi_{\lambda} \) in Lemma 6.17.

It remains to show that it is indeed a \( L_\infty \)-module morphism. To this end, we compute for \( X \in \mathcal{F}(L[1]) \), \( m \in M \)

\[
\frac{d}{dt} (\Phi_{\lambda}^* \phi^* )^i \circ \Psi_{\lambda}(X \otimes m) = \frac{d}{dt} \phi^1 \circ (\exp(\pi(t))) \circ (\Psi_{\lambda}(X^1) \otimes \Psi_{\lambda}(X^2) \otimes m))
\]

\[
= (\phi^1)^1 \circ (Q^\pi(\lambda)) \vee (\Phi_{\lambda} \otimes \text{id}_M) \circ (X^1 \otimes \Psi_{\lambda}(X^2) \otimes m))
\]

\[
+ (\phi^1)^1 \circ (\left( [Q^\pi(\lambda), \lambda \vee \cdot ] - Q^\pi(\lambda) \vee \cdot \right) \circ \Phi_{\lambda} \otimes \text{id}_M) \circ (X^1 \otimes \Psi_{\lambda}(X^2) \otimes m))
\]

\[
+ (\phi^1)^1 \circ (\Phi_{\lambda} \otimes \text{id}_M) \circ (X^1 \otimes (\Phi_{\lambda}^* \phi'^* ) \circ (\lambda \vee \cdot ))^1 \circ (X^2 \otimes \Psi_{\lambda}(X^3) \otimes m))
\]

\[
= (\phi^1)^1 \circ (\left( [Q^\pi(\lambda), \lambda \vee \cdot ] \Phi_{\lambda}(X^1) \right) \otimes \Psi_{\lambda}(X^2) \otimes m))
\]
\[ + (\phi^{\pi(t)})^1 \circ \left( \Phi_t X^{(1)} \otimes \left( (\phi^{\pi(t)})^1 \circ \Phi_t (\lambda \lor X^{(2)}) \otimes \psi_t^1 (X^{(3)} \otimes m) \right) \right). \]

Using \((\Phi_t^* \phi^{\pi(t)})^2(\lambda \lor X^{(1)} \otimes \psi_t^1 (X^{(2)} \otimes m)) = 0\) and the fact that \(\Phi_t \circ (\lambda \lor \cdot) = (\lambda \lor \cdot) \circ \Phi_t\), we see that this coincides with

\[ \frac{d}{dt} (\Phi_t^* \phi^{\pi(t)})^1 \circ \psi_t (X \otimes m) = (\phi^{\pi(t)})^1 \circ (\Phi_t \otimes \text{id}) \circ (\lambda \lor \cdot) \circ (Q X^{(1)} \otimes \psi_t^1 (X^{(2)} \otimes m)) \]

\[ + X^{(1)} \otimes (\phi^{\pi(t)})^1 (\Phi_t X^{(2)} \otimes \psi_t^1 (X^{(3)} \otimes m)) \]

\[ = \left( (\Phi_t^* \phi^{\pi(t)}) \circ (\lambda \lor \cdot) \right)^1 \circ (\Phi_t^* \phi^{\pi(t)}) \circ \psi_t (X \otimes m). \]

Therefore, \((\Phi_t^* \phi^{\pi(t)})^1 \circ \psi_t (X \otimes m)\) coincides with \(\psi_t^1 \circ \phi (X \otimes m)\) since

\[ \frac{d}{dt} \psi_t^1 \circ \phi (X \otimes m) = \left( (\Phi_t^* \phi^{\pi(t)}) \circ (\lambda \lor \cdot) \right)^1 \circ (\psi_t \circ \phi (X \otimes m)), \]

i.e. both expressions satisfy the same differential equation, and both coincide at \(t = 0\). The fact that \(\psi_t\) is even an \(L_\infty\)-module isomorphism follows since \((\psi_t)_0 : M \to M\) is an isomorphism compared to Proposition 7.10.

This allows us to generalize the case of DGLA modules from (7.26) and we can show:

**Proposition 7.22** The \(L_\infty\)-module morphisms \(\kappa\) and \(\psi_{N,1}^{-1} \circ (\Phi_t^* \kappa^\pi) \circ \psi_{M,1}\) from \((M, \phi)\) to \((N, \phi')\) over \((L, Q)\) are homotopic.

**Proof:** We set \(\kappa(t) = \psi_{N,1}^{-1} \circ (\Phi_t^* \kappa^\pi) \circ \psi_{M,1}\) and compute

\[ \frac{d}{dt} \psi_{M,t} = \left( \text{id} \otimes (\Phi_t^* \phi^{\pi(t)})^1 \right) \circ (\text{id} \otimes (\lambda \lor \cdot) \otimes \text{id}) \circ (\text{id} \otimes \psi_{M,t}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \]

\[ = \left( \text{id} \otimes (\Phi_t^* \phi^{\pi(t)})^1 \right) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \circ \lambda \lor \psi_{M,t} + \left( \text{id} \otimes (\Phi_t^* \phi^{\pi(t)})^1 \right) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \circ \psi_{M,t} \]

\[ = (\Phi_t^* \phi^{\pi(t)}) \circ \lambda \lor \psi_{M,t} - Q \circ \lambda \lor \psi_{M,t} + (\lambda \lor \cdot) \circ (\Phi_t^* \phi^{\pi(t)}) \circ \psi_{M,t} - \lambda \lor Q \circ \psi_{M,t} \]

\[ = (\Phi_t^* \phi^{\pi(t)}) \circ \lambda \lor \psi_{M,t} + (\lambda \lor \cdot) \circ (\Phi_t^* \phi^{\pi(t)}) \circ \psi_{M,t} - [Q, \lambda \lor \cdot] \circ \psi_{M,t} \]

and similarly

\[ \frac{d}{dt} \psi_{N,t}^{-1} = -\psi_{N,t}^{-1} \circ (\Phi_t^* \phi^{\pi(t)}) \circ (\lambda \lor \cdot) - \psi_{N,t}^{-1} \circ (\Phi_t^* \phi^{\pi(t)}) + \psi_{N,t}^{-1} \circ [Q, \lambda \lor \cdot]. \]

In addition, we have

\[ \frac{d}{dt} \Phi_t^* \kappa^\pi(t) = \frac{d}{dt} (\text{id} \otimes \kappa^\pi(t)) \circ (\text{id} \otimes \exp(\pi(t)) \lor \Phi_t \otimes \text{id}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \]

\[ = (\text{id} \otimes (\kappa^\pi(t))^1) \circ (\text{id} \otimes Q^\pi(t)(\lambda) \lor \Phi_t \otimes \text{id}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \]

\[ + (\text{id} \otimes (\kappa^\pi(t))^1) \circ (\text{id} \otimes [Q^\pi(t), \lambda \lor \cdot] - Q^\pi(t)(\lambda) \lor \cdot) \circ \Phi_t \otimes \text{id}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \]

\[ = (\text{id} \otimes (\kappa^\pi(t))^1) \circ (\text{id} \otimes [Q^\pi(t), \lambda \lor \cdot] \circ \Phi_t \otimes \text{id}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}). \]

With \(\Phi_t \circ (\lambda \lor \cdot) = (\lambda \lor \cdot) \circ \Phi_t\) and \(Q^\pi(t) \circ \Phi_t = \Phi_t \circ Q\), this gives

\[ \frac{d}{dt} \Phi_t^* \kappa^\pi(t) = (\text{id} \otimes (\kappa^\pi(t))^1) \circ (\text{id} \otimes \Phi_t \otimes [Q, \lambda \lor \cdot] \otimes \text{id}) \circ (\Delta_{\mathbb{A}} \otimes \text{id}) \]

\[ = (\Phi_t^* \kappa^\pi(t)) \circ [Q, \lambda \lor \cdot] - [Q, \lambda \lor \cdot] \circ (\Phi_t^* \kappa^\pi(t)). \]

Summarizing, we have shown

\[ \frac{d}{dt} \kappa(t) = \phi' \circ \left( -\psi_{N,t}^{-1} \circ (\lambda \lor \cdot) \circ (\Phi_t^* \kappa^\pi(t)) \circ \psi_{M,t} + \psi_{N,t}^{-1} \circ (\Phi_t^* \kappa^\pi(t)) \circ (\lambda \lor \cdot) \circ \psi_{M,t} \right) \]
which implies
\[
\frac{d}{dt} \kappa_1(t) = \partial \left( -\Psi_{N,t}^{-1} \circ (\lambda \lor \cdot) \circ (\Phi_1^*r(t)) \circ \Psi_{M,t}^{-1} + \Psi_{N,t}^{-1} \circ (\Phi_1^*r(t)) \circ (\lambda \lor \cdot) \circ \Psi_{M,t} \right) \circ \phi
\]
as desired.

**Remark 7.23 (Application to Deformation Quantization II)** These results imply that Dolgushev's globalizations [14] of Shoikhet's formality for Hochschild chains [48] with respect to different covariant derivatives are homotopic, completely analogously to the cochain case in [33].
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