A Conditional Linear Combination Test with Many Weak Instruments∗

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Abstract

We consider a linear combination of jackknife Anderson-Rubin (AR), jackknife Lagrangian multiplier (LM), and orthogonalized jackknife LM tests for inference in IV regressions with many weak instruments and heteroskedasticity. We choose the weights in the linear combination based on a decision-theoretic rule that is adaptive to the identification strength. Under both weak and strong identification, the proposed linear combination test controls asymptotic size and is admissible among certain class of tests. Under strong identification, we further show that our linear combination test has optimal power against local alternatives. Simulations and an empirical application to Angrist and Krueger’s (1991) dataset confirm the good power properties of our test.

Keywords: Many instruments, power, size, weak identification

JEL codes: C12, C36, C55

1 Introduction

Various recent surveys in leading economics journals suggest that weak instruments remain important concerns for empirical practice. For instance, I. Andrews, Stock, and Sun (2019) survey 230 instrumental variable (IV) regressions from 17 papers published in the American Economic Review (AER). They find that many of the first-stage F-statistics (and non-homoskedastic generalizations) are in a range that raises such concerns, and virtually all of these papers report at least one first-stage F with a value smaller than 10. Similarly, in Lee, McCrary, Moreira, and Porter’s (2022)

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survey of 123 AER articles involving IV regressions, 105 out of 847 specifications have first-stage F's smaller than 10. Moreover, many IV applications involve a large number of instruments. For example, in their seminal paper, Angrist and Krueger (1991) study the effect of schooling on wages by interacting three base instruments (dummies for the quarter of birth) with state and year of birth, resulting in 180 instruments. Hansen, Hausman, and Newey (2008) show that using the 180 instruments gives tighter confidence intervals than using the base instruments even after adjusting for the effect of many instruments. In addition, as pointed out by Mikusheva and Sun (2022), in empirical papers that employ the “judge design” (e.g., see Maestas, Mullen, and Strand (2013), Sampat and Williams (2019), and Dobbie, Goldin, and Yang (2018)), the number of instruments (the number of judges) is typically proportional to the sample size, and the famous Fama-MacBeth two-pass regression in empirical asset pricing (e.g., see Fama and MacBeth (1973), Shanken (1992), and Anatolyev and Mikusheva (2022)) is equivalent to IV estimation with the number of instruments proportional to the number of assets. Similarly, Belloni, Chen, Chernozhukov, and Hansen (2012) consider an IV application involving more than one hundred instruments for the study of the effect of judicial eminent domain decisions on economic outcomes. Carrasco and Tchuente (2015) used many instruments in the estimation of the elasticity of intertemporal substitution in consumption. Furthermore, as pointed out by Goldsmith-Pinkham, Sorkin, and Swift (2020), the shift-share or Bartik instrument (e.g., see Bartik (1991) and Blanchard, Katz, Hall, and Eichen-green (1992)), which has been widely applied in many fields such as labor, public, development, macroeconomics, international trade, and finance, can be considered as a particular way of combining many instruments. For example, in the canonical setting of estimating the labor supply elasticity, the corresponding number of instruments is equal to the number of industries, which is also typically proportional to the sample size.

In this paper, we propose a jackknife conditional linear combination (CLC) test, which is robust to weak identification, many instruments, and heteroskedasticity. The proposed test also achieves efficiency under strong identification against local alternatives. The starting point of our analysis is an observation that, under strong identification, an orthogonalized jackknife Lagrangian multiplier (LM) test is the uniformly most powerful test against local alternatives among the class of tests that are invariant to sign changes and constructed based on jackknife LM and Anderson-Rubin (AR) tests only. However, the orthogonalized LM test may not have good power under weak identification or against certain fixed alternatives. We therefore consider a linear combination of jackknife AR, jackknife LM, and orthogonalized LM tests. Specifically, we follow I. Andrews (2016) and determine the linear combination weights by minimizing the maximum power loss, which can be viewed as a maximum regret and is further calibrated based on the limit experiment of interest and a sufficient statistic for the identification strength under many instruments. We show such a jackknife CLC test is adaptive to the identification strength in the sense that (1) it achieves exact asymptotic size under both weak and strong identification, (2) it is asymptotically and conditionally
admissible under weak identification among some class of tests, (3) it converges to the uniformly most powerful test mentioned above under strong identification against local alternatives, and (4) it has asymptotic power equal to 1 under strong identification against fixed alternatives. Simulations based on the limit experiment as well as calibrated data confirm the good power properties of our test. Then, we apply the new jackknife CLC test to Angrist and Krueger’s (1991) dataset with the specifications of 180 and 1,530 instruments. We find that in both specifications, our confidence intervals (CIs) are the shortest among those constructed by weak identification robust tests, namely, the jackknife AR, LM, and CLC tests, and the two-step procedure. Furthermore, our CIs are found to be even shorter than the non-robust Wald test CIs based on the jackknife IV estimator (JIVE) proposed by Angrist, Imbens, and Krueger (1999), which is in line with the theoretical result that the jackknife CLC test is adaptive to the identification strength and is efficient under strong identification.

Relation to the literature. The contributions in the present paper relate to two strands of literature. First, it is related to the literature on many instruments; see, for example, Kunitomo (1980), Morimune (1983), Bekker (1994), Donald and Newey (2001), Chamberlain and Imbens (2004), Chao and Swanson (2005), Stock and Yogo (2005a), Han and Phillips (2006), D. Andrews and Stock (2007), Hansen et al. (2008), Newey and Windmeijer (2009), Anderson, Kunitomo, and Matsushita (2010), Kuersteiner and Okui (2010), Anatolyev and Gospodinov (2011), Belloni, Chernozhukov, and Hansen (2011), Okui (2011), Belloni et al. (2012), Carrasco (2012), Chao, Swanson, Hausman, Newey, and Woutersen (2012), Hausman, Newey, Woutersen, Chao, and Swanson (2012), Hansen and Kozbur (2014), Carrasco and Tchuente (2015), Wang and Kaffo (2016), Kolesár (2018), Matsushita and Otsu (2020), Sølvsten (2020), Crudu, Mellace, and Sándor (2021), and Mikusheva and Sun (2022), among others. For implementing inferences in the context of many instruments and heteroskedasticity, Chao et al. (2012) and Hausman et al. (2012) provide standard errors for Wald-type inferences that are based on JIVE and a jackknifed version of the limited information maximum likelihood (LIML) estimator and the Fuller’s (1977) estimator. These estimators are more robust to many instruments than the commonly used two-stage least squares (TSLS) estimator as they are able to correct the bias due to the high dimension of IVs. In simulations derived from the data in Angrist and Krueger (1991), which is representative of empirical labor studies with a many-instrument concern, Angrist and Frandsen (2022, Section IV) show that such bias-corrected estimators outperform the TSLS that is based on the instruments selected by the least absolute shrinkage and selection operator (LASSO) introduced in Belloni et al. (2012) or the random forest-fitted first stage introduced in Athey, Tibshirani, and Wager (2019).

However, the Wald inference methods are not valid under weak identification, a situation in which the ratio of the so-called concentration parameter, a measure of the overall instrument strength, over the square root of the number of instruments stays bounded as the sample size diverges to infinity. In this case, even the aforementioned bias-corrected estimators are inconsistent,
and there is no consistent test for the structural parameter of interest (see the discussions in Section 3 of Mikusheva and Sun (2022)). For weak identification robust inference under many instruments, D. Andrews and Stock (2007) consider the AR test, the score test introduced in Kleibergen (2002), and the conditional likelihood ratio test introduced in Moreira (2003). Their IV model is homoskedastic and requires the number of instruments to diverge slower than the cube root of the sample size ($K^3/n \to 0$, where $K$ and $n$ denote the number of instruments and the sample size, respectively). Anatolyev and Gospodinov (2011) propose a modified AR test, which allows for the number of instruments to be proportional to the sample size but also require homoskedastic errors. Recently, Crudu et al. (2021) and Mikusheva and Sun (2022) propose jackknifed versions of the AR test in a model with many instruments and heteroskedasticity. Both tests are robust toward weak identification, whereas Mikusheva and Sun’s (2022) jackknife AR test has better power properties because of the usage of a cross-fit variance estimator. However, the jackknife AR tests may be inefficient under strong identification. Mikusheva and Sun (2022) also propose a new pre-test for weak identification under many instruments and apply it to form a two-stage testing procedure with a Wald test based on the JIVE introduced in Angrist et al. (1999). The JIVE-Wald test is more efficient than the jackknife AR under strong identification. An empirical researcher can therefore employ the jackknife AR if the pre-test suggests weak identification or the JIVE-Wald if the pre-test suggests strong identification. Furthermore, Matsushita and Otsu (2020) propose a jackknife LM test, which is also robust to weak identification, many instruments, and heteroskedastic errors. Under strong identification and local alternatives, our jackknife CLC test proves to be more efficient than the jackknife AR, the jackknife LM, and the two-step test.

Second, our paper is related to the literature on weak identification under the framework of a fixed number of instruments or moment conditions, in which various robust inference methods are available for non-homoskedastic errors; see, for example, Stock and Wright (2000), Kleibergen (2005), D. Andrews and Cheng (2012), I. Andrews (2016), I. Andrews and Mikusheva (2016), I. Andrews (2018), Moreira and Moreira (2019), D. Andrews and Guggenberger (2019), and Lee et al. (2022). In particular, our jackknife CLC test extends I. Andrews (2016) to the framework with many weak instruments. I. Andrews (2016) considers the convex combination between the generalized AR statistic (S statistic) introduced by Stock and Wright (2000) and the score statistic (K statistic) introduced by Kleibergen (2005). We find that under many weak instruments, the orthogonalized jackknife LM statistic plays a role similar to the K statistic. However, the trade-off between the jackknife AR and orthogonalized LM statistics turns out to be rather different from that between the S and K statistics. As pointed out by I. Andrews (2016), in the case with a fixed number of weak instruments (or moment conditions), the K statistic picks out a particular (random) direction corresponding to the span of a conditioning statistic that measures the identification strength and restricts attention to deviations from the null along this specific direction. In contrast to the K statistic, the S statistic treats all deviations from the null equally. Therefore, the trade-off
between the K and S statistics is mainly from the difference in attention to deviation directions. We find that with many weak instruments, the jackknife AR and orthogonalized LM tests do not have such difference in deviation directions. Instead, their trade-off is mostly between local and non-local alternatives. Furthermore, although the standard LM test (without orthogonalization) is not weak identification robust under A. Andrews (2016)’s framework, the jackknife LM test is robust under many instruments. Therefore, we consider a linear combination of jackknife AR, jackknife LM, and orthogonalized jackknife LM tests, and we find that the resulting CLC test has good power properties in a variety of scenarios.

**Notation.** We denote \( Z(\mu) \) as the normal random variable with unit variance and expectation \( \mu \) and \( [n] = \{1, 2, \cdots , n\} \). We further simplify \( Z(0) \) as \( Z \), which is just a standard normal random variable. We denote \( z_\alpha \) as the \((1 - \alpha)\) quantile of a standard normal random variable and \( C_\alpha(a_1, a_2; \rho) \) as the \((1 - \alpha)\) quantile of random variable \( a_1 Z_1^2 + a_2 (\rho Z_1 + (1 - \rho^2)^{1/2} Z_2) + (1 - a_1 - a_2) Z_2^2 \) where \( Z_1 \) and \( Z_2 \) are two independent standard normal random variables. Furthermore, we let \( C_\alpha = z_{\alpha/2}^2 \) and \( C_{\alpha, \text{max}}(\rho) = \sup_{(a_1, a_2) \in \mathbb{A}_0} C_\alpha(a_1, a_2; \rho), \) where \( \mathbb{A}_0 = \{(a_1, a_2) \in [0, 1] \times [0, 1], a_1 + a_2 \leq \overline{\sigma}\} \) for some \( \overline{\sigma} < 1 \). The operators \( \mathbb{E}^* \) and \( \mathbb{P}^* \) are expectation and probability taken conditionally on data, respectively. For example, \( \mathbb{E}^*1\{ Z^2(\hat{\mu}) \geq C_\alpha \} \), in which \( \hat{\mu} \) is some estimator of the expectation \( \mu \) based on data, means the expectation is taken over the normal random variable by treating \( \hat{\mu} \) as deterministic. We use \( \rightsquigarrow \) to denote convergence in distribution and \( U \overset{d}{=} V \) to denote that \( U \) and \( V \) share the same distribution.

## 2 Setup and Limit Problems

We consider the linear IV regression with a scalar outcome \( Y_i \), a scalar endogenous variable \( X_i \), and a \( K \times 1 \) vector of instruments \( Z_i \) such that

\[
Y_i = X_i \beta + e_i, \quad X_i = \Pi_i + V_i, \quad \forall i \in [n],
\]

(2.1)

where \( \Pi_i = \mathbb{E}(X_i | Z_i) \). We focus on the model with a single endogenous variable, which is prevalent in empirical research. We let \( K \) diverge with sample size \( n \), allowing for the case that \( K \) is of the same order of magnitude as \( n \). For the rest of the paper, we follow the many-instrument literature and treat \( \{ Z_i \}_{i \in [n]} \) as fixed so that \( \Pi_i \) can also be written as \( \mathbb{E}X_i | Z_i | \) which is non-random, \( \mathbb{E}V_i = 0 \) by construction, and \( \mathbb{E}e_i = 0 \) by IV exogeneity. We allow \( (e_i, V_i) \) to be heteroskedastic across \( i \).

Also, following the literature on many instruments, we assume without loss of generality that there are no controls included in our model as they can be partialled out from \( (Y_i, X_i, Z_i) \).

We are interested in testing \( \beta = \beta_0 \). Let \( e_i(\beta_0) = Y_i - X_i \beta_0 = e_i + X_i \Delta \), where \( \Delta = \beta - \beta_0 \).

\(^1\)When there are control variables and we partial them out from both \( Y \) and \( X \), the residuals for \( Y \) and \( X \) are not exactly independent. However, all the analyses in this paper are still valid because they only require \( (2.2) \), which still holds for the residuals of \( Y \) and \( X \).
We collect the transpose of \( Z_i \) in each row of \( Z \), an \( n \times K \) matrix of instruments, and denote \( P = Z(Z^\top Z)^{-1}Z^\top \). In addition, Let \( Q_{ab} = \sum_{i \in [n]} \sum_{j \neq i} a_i b_j P_{ij} \) and \( C = Q_{III} \). Then, as pointed out by Mikusheva and Sun (2022) point, the (rescaled) \( C \) is the concentration parameter that measures the strength of identification in the heteroskedastic IV model with many instruments. Specifically, the parameter \( \beta \) is weakly identified if \( C \) is bounded and strongly identified if \( |C| \to \infty \). We consider drifting sequence asymptotics so that all quantities are indexed by the sample size \( n \). We omit such dependence for notation simplicity.

Throughout the paper, we consider three scenarios: (1) weak identification and fixed alternatives in which both \( C \) and \( \Delta \) are fixed and bounded, (2) strong identification and local alternatives in which \( C = \tilde{C}/d_n \), \( \Delta = \tilde{\Delta} d_n \), \( \tilde{C} \) and \( \tilde{\Delta} \) are bounded constants independent of \( n \), and \( d_n \to 0 \) is a deterministic sequence, and (3) strong identification and fixed alternatives in which \( C = \tilde{C}/d_n \) and \( \Delta \) is fixed and bounded. All the weak identification robust tests proposed in the literature (namely, the jackknife AR tests in Crudu et al. (2021) and Mikusheva and Sun (2022) and the jackknife LM test in Matsushita and Otsu (2020)) depend on a subset of the following three quantities: \( (Q_{ee}(\beta_0), Q_{Xe}(\beta_0), Q_{XX}) \). Throughout the paper, we maintain the following high-level assumption.

**Assumption 1.** Under both weak and strong identification, the following weak convergence holds:

\[
\begin{pmatrix}
Q_{ee} \\
Q_{Xe} \\
Q_{XX} - C
\end{pmatrix} \rightsquigarrow N\left(\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\Phi_1 & \Phi_12 & \Phi_13 \\
\Phi_12 & \Psi & \tau \\
\Phi_13 & \tau & \Upsilon
\end{pmatrix}\right),
\]

for some \( (\Phi_1, \Phi_12, \Phi_13, \Psi, \tau, \Upsilon) \).

Assumption 1 has already been verified by Chao et al. (2012) and Mikusheva and Sun (2022) under regularity conditions. It implies that, under both strong and weak identification,

\[
\begin{pmatrix}
Q_{ee}(\beta_0) - \Delta^2 C \\
Q_{Xe}(\beta_0) - \Delta C \\
Q_{XX} - C
\end{pmatrix} \overset{d}{\to} N\left(\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\Phi_1(\beta_0) & \Phi_12(\beta_0) & \Phi_13(\beta_0) \\
\Phi_12(\beta_0) & \Psi(\beta_0) & \tau(\beta_0) \\
\Phi_13(\beta_0) & \tau(\beta_0) & \Upsilon
\end{pmatrix}\right) + o_p(1),
\]

where

\[
\begin{align*}
\Phi_1(\beta_0) &= \Delta^4 \Upsilon + 4\Delta^3 \tau + \Delta^2 (4\Psi + 2\Phi_13) + 4\Delta \Phi_12 + \Phi_1, \\
\Phi_12(\beta_0) &= \Delta^3 \Upsilon + 3\Delta^2 \tau + \Delta (2\Psi + \Phi_13) + \Phi_12, \\
\Phi_13(\beta_0) &= \Delta^2 \Upsilon + 2\Delta \tau + \Phi_13, \\
\Psi(\beta_0) &= \Delta^2 \Upsilon + 2\Delta \tau + \Psi, \\
\tau(\beta_0) &= \Delta \Upsilon + \tau.
\end{align*}
\]
Specifically, under strong identification, we have $Q_{XX}d_n \xrightarrow{p} \bar{C}$, which has a degenerate distribution. Also, under local alternatives, we have $\Delta = o(1)$ so that

$$(\Phi_1(\beta_0), \Phi_{12}(\beta_0), \Phi_{13}(\beta_0), \Psi(\beta_0), \tau(\beta_0)) \rightarrow (\Phi_1, \Phi_{12}, \Phi_{13}, \Psi, \tau).$$

To describe a feasible version of the test, we assume we have consistent estimates for all the variance components.

**Assumption 2.** Let $\rho(\beta_0) = \frac{\Phi_{12}(\beta_0)}{\sqrt{\Phi_{1}(\beta_0)\Psi(\beta_0)}}$, $\hat{\gamma}(\beta_0) = (\hat{\Phi}_1(\beta_0), \hat{\Phi}_{12}(\beta_0), \hat{\Phi}_{13}(\beta_0), \hat{\Psi}(\beta_0), \hat{\tau}(\beta_0), \hat{\Upsilon}, \hat{\rho}(\beta_0))$ be an estimator, and $B \in \mathbb{R}$ be a compact parameter space. Then, we have $\inf_{\beta_0 \in B} \Psi(\beta_0) > 0$, $\inf_{\beta_0 \in B} \Psi(\beta_0) > 0$, $\Upsilon > 0$, and for $\beta_0 \in B$,

$$||\hat{\gamma}(\beta_0) - \gamma(\beta_0)||_2 = o_p(1),$$

where $\gamma(\beta_0) \equiv (\Phi_1(\beta_0), \Phi_{12}(\beta_0), \Phi_{13}(\beta_0), \Psi(\beta_0), \tau(\beta_0), \Upsilon, \rho(\beta_0))$.

Several remarks on Assumption 2 are in order. First, Chao et al. (2012) propose a consistent estimator of $\Psi$ under strong identification and many instruments. It is possible to compute $\hat{\gamma}(\beta_0)$ based on Chao et al.’s (2012) argument with their JIVE-based residuals $e_i$ from the structure equation replaced by $e_i(\beta_0)$. Under weak identification and $\beta_0 = \beta$, Crudu et al. (2021) and Matsushita and Otsu (2021) establish the consistency of such estimators for $\Phi_1(\beta_0)$ and $\Psi(\beta_0)$, respectively. Similar arguments can be used to show the consistency of the rest of the elements in $\hat{\gamma}(\beta_0)$ under both weak and strong identification. In addition, the consistency can be established under both local and fixed alternatives. We provide more details in Section A.1 in the Online Supplement. Second, motivated by Kline, Saggio, and Sølvsten (2020), Mikusheva and Sun (2022) propose cross-fit estimators $\hat{\Phi}_1(\beta_0)$ and $\hat{\Upsilon}$, which are consistent under both weak and strong identification and lead to better power properties. Following their lead, one can write down the cross-fit estimators for the rest of the elements in $\gamma(\beta_0)$ and show they are consistent.\(^2\) We provide more details in Section A.2 in the Online Supplement. Note that both Crudu et al.’s (2021) and Mikusheva and Sun’s (2022) estimators are consistent under heteroskedasticity and allow for $K$ to be of the same order of $n$. Third, in order for our jackknife CLC test proposed below to control size under both weak and strong identification, it suffices to require $\hat{\gamma}(\beta_0)$ to be consistent under the null only. Fourth, in the following, we study the power properties of jackknife test statistics against local or fixed alternatives under different identification scenarios. The power analysis in Lemmas 2.1 and 2.4 below, and subsequently, Theorems 4.1 and 4.2, only requires the consistency of $\hat{\gamma}(\beta_0)$ under strong identification with local alternatives and weak identification with fixed alternatives, respectively.

\(^2\)For example, Mikusheva and Sun (2022, p.22) establish the limit of their cross-fit estimator $\hat{\Psi}$ under weak identification and many instruments when the residual $e_i$ from the structure equation is computed based on the JIVE estimator. We can construct $\hat{\Psi}(\beta_0)$ by replacing $e_i$ by $e_i(\beta_0)$. Then, the argument, as theirs with $-Q_{Xe}/Q_{XX}$ replaced by $\Delta$, establishes that $\hat{\Psi}(\beta_0) \xrightarrow{p} \Psi(\beta_0)$. 

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Under this framework, Crudu et al. (2021) and Mikusheva and Sun (2022) consider the jackknife AR test

\[ 1\{AR(\beta_0) \geq z_\alpha\}, \quad AR(\beta_0) = \frac{Qe(\beta_0)e(\beta_0)}{\Phi^{1/2}(\beta_0)}, \quad (2.5) \]

and Matsushita and Otsu (2020) consider the jackknife LM test

\[ 1\{LM^2(\beta_0) \geq C_\alpha\}, \quad LM(\beta_0) = \frac{QXe(\beta_0)}{\Psi^{1/2}(\beta_0)}. \quad (2.6) \]

Both tests are robust to weak identification, many instruments, and heteroskedasticity. Lemma 2.1 below characterizes the joint limit distribution of \((AR(\beta_0), LM(\beta_0))^\top\) under strong identification and local alternatives.

**Lemma 2.1.** Suppose Assumptions 1 and 2 hold and we are under strong identification with local alternatives, that is, there exists a deterministic sequence \(d_n \to 0\) such that \(C = \tilde{C}/d_n\) and \(\Delta = \tilde{\Delta}d_n\), where \(\tilde{C}\) and \(\tilde{\Delta}\) are bounded constants independent of \(n\). Then, we have

\[ \left( \begin{array}{c} AR(\beta_0) \\ LM(\beta_0) \end{array} \right) \overset{d}{\to} \mathcal{N}\left( \begin{array}{c} 0 \\ \frac{\tilde{C}}{\Phi^{1/2}} \end{array}, \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right) \]

where \(\rho = \Phi_{12}/\sqrt{\Phi_1\Psi}\).

Two remarks are in order. First, under strong identification, we consider local alternatives so that \(\beta - \beta_0 \to 0\). This is why we have \((\Psi(\beta_0), \Phi_1(\beta_0), \Phi_{12}(\beta_0))\) converge to \((\Psi, \Phi_1, \Phi_{12})\), which are just the counterparts of \((\Psi(\beta_0), \Phi_1(\beta_0), \Phi_{12}(\beta_0))\) when \(\beta_0\) is replaced by \(\beta\). Second, although \(AR(\beta_0)\) has zero mean, and hence, no power, it is correlated with \(LM(\beta_0)\) in the current context of many instruments. It is therefore possible to use \(AR(\beta_0)\) to reduce the variance of \(LM(\beta_0)\) and obtain a test that is more powerful than the LM test.

**Lemma 2.2.** Consider the limit experiment in which researchers observe \((N_1, N_2)\) with

\[ \left( \begin{array}{c} N_1 \\ N_2 \end{array} \right) \overset{d}{\to} \mathcal{N}\left( \begin{array}{c} 0 \\ \frac{\tilde{C}}{\Psi^{1/2}} \end{array}, \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right), \]

know the value of \(\rho\) and that \(E(N_1) = 0\), and want to test for \(\theta = 0\) versus the two-sided alternative. In this case, the uniformly most powerful level-\(\alpha\) test that is invariant to sign changes is \(1\{N_2^* \geq C_\alpha\}\), where

\[ N_2^* = (1 - \rho^2)^{-1/2}(N_2 - \rho N_1) \]

is the normalized residual from the projection of \(N_2\) on \(N_1\).
Let the orthogonalized jackknife LM statistic be $LM^*(\beta_0) = (1 - \hat{\rho}(\beta_0)^2)^{-1/2}(LM(\beta_0) - \hat{\rho}(\beta_0)AR(\beta_0))$. Then, Lemma 2.1 implies, under strong identification and local alternatives,

$$\begin{pmatrix} AR(\beta_0) \\ LM^*(\beta_0) \end{pmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ \Delta \hat{c} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right). \quad (2.7)$$

Lemma 2.2 with $\theta = \Delta \hat{c} \Psi^{-1/2}$ implies, in this case, that the test $1\{LM^2(\beta_0) \geq C_\alpha\}$ is asymptotically strictly more powerful than the jackknife AR and LM tests based on $AR(\beta_0)$ and $LM(\beta_0)$ against local alternatives as long as $\rho \neq 0$. In addition, under strong identification and local alternatives, Mikusheva and Sun’s (2022) two-step test statistic is asymptotically equivalent to $LM^*(\beta_0)$, and thus, is less powerful than $LM^*(\beta_0)$ too.

Next, we compare the behaviors of $AR(\beta_0)$, $LM(\beta_0)$, and $LM^*(\beta_0)$ under strong identification and fixed alternatives.

**Lemma 2.3.** Suppose Assumption 2 holds, $(Q_{e(\beta_0)v(\beta_0)} - \Delta^2 \mathcal{C}, Q_{Xe(\beta_0)} - \Delta \mathcal{C}, Q_{XX} - \mathcal{C})^\top = O_p(1)$, and we are under strong identification so that $d_n \mathcal{C} \rightarrow \tilde{\mathcal{C}}$ for some $d_n \rightarrow 0$. Then, we have, for any fixed $\Delta \neq 0$,

$$d_n^2 \begin{pmatrix} AR^2(\beta_0) \\ LM^2(\beta_0) \\ LM^*^2(\beta_0) \end{pmatrix} \xrightarrow{p} \begin{pmatrix} \Phi_1^{-1}(\beta_0) \Delta^4 \hat{C}^2 \\ \Psi^{-1}(\beta_0) \Delta^2 \hat{C}^2 \\ (1 - \rho^2(\beta_0))^{-1}(\Psi^{-1/2}(\beta_0) - \rho(\beta_0)\Phi_1^{-1/2}(\beta_0)\Delta)^2 \Delta^2 \hat{C}^2 \end{pmatrix}.$$

Given $d_n \rightarrow 0$ and $\Phi_1^{-1}(\beta_0) \Delta^4 \hat{C}^2 > 0$, $AR^2(\beta_0)$ has power 1 against fixed alternatives asymptotically. By contrast, $LM^*^2(\beta_0)$ may not have power if $\Delta = \Delta*(\beta_0) \equiv \Phi_1^{1/2}(\beta_0)\Psi^{-1/2}(\beta_0)\rho^{-1}(\beta_0)$.

Next, we compare the performance of $AR(\beta_0)$ and $LM^*(\beta_0)$ under weak identification and fixed alternatives.

**Lemma 2.4.** Suppose Assumptions 1 and 2 hold and we are under weak identification so that $\mathcal{C}$ is fixed. Then, we have, for any fixed $\Delta \neq 0$,

$$\begin{pmatrix} AR(\beta_0) \\ LM^*(\beta_0) \end{pmatrix} \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} m_1(\Delta) \\ m_2(\Delta) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad (2.8)$$

where $\rho(\beta_0) = \frac{\Phi_{12}(\beta_0)}{\sqrt{\Psi(\beta_0)\Phi_1(\beta_0)}}$ and

$$\begin{pmatrix} m_1(\Delta) \\ m_2(\Delta) \end{pmatrix} = \begin{pmatrix} \Phi_1^{-1/2}(\beta_0)\Delta^2 \mathcal{C} \\ (1 - \rho^2(\beta_0))^{-1/2}\Psi^{-1/2}(\beta_0)\Delta \mathcal{C} - \rho(\beta_0)(1 - \rho^2(\beta_0))^{-1/2}\Phi_1^{-1/2}(\beta_0)\Delta^2 \mathcal{C} \end{pmatrix}.$$
In particular, as $\Delta \to \infty$, we have
\[ m_1(\Delta) \to \frac{C}{\gamma^{1/2}} \quad \text{and} \quad m_2(\Delta) \to \frac{C}{\gamma^{1/2}} \left( 1 - \rho_{23}^2 \right)^{1/2}, \]
where $\rho_{23} = \frac{\tau}{\sqrt{\Psi}}$ is the correlation between $Q_{Xe}$ and $Q_{XX}$.\(^3\)

By comparing the means of the normal limit distribution in (2.8), we notice that under weak identification and fixed alternatives, neither $LM^*(\beta_0)$ dominates $AR(\beta_0)$ or vice versa. We also notice from Lemma 2.4 that for testing distant alternatives, the power of $LM^*(\beta_0)$ is different from $AR(\beta_0)$ by a factor of $\rho_{23}/\sqrt{1 - \rho_{23}^2}$, so that it will be lower when $|\rho_{23}| \leq 1/\sqrt{2}$. Under weak identification and homoskedasticity,\(^4\) we have $\rho_{23} = \rho = \Phi_{12}/\sqrt{\Psi \Phi_1}$. Therefore, although the test $1\{LM^{*2}(\beta_0) \geq C_\alpha\}$ has a power advantage under strong identification against local alternatives, it may lack power under weak identification against distant alternatives if the degree of endogeneity is low. Furthermore, $LM^*(\beta_0)$ may not have power if $\Delta = \Delta_\ast(\beta_0)$. We notice that such a power issue of $LM^*(\beta_0)$ is similar to that of the tests based on the K statistic introduced by Kleibergen (2002, 2005) under the framework of a fixed number of instruments. Under such a framework, the K statistic is efficient under strong identification against local alternatives but may have a non-monotonic power function under weak identification (e.g., see the discussions in Section 3.1 of I.Andrews (2016)).

To achieve the advantages of $AR(\beta_0)$, $LM(\beta_0)$, and $LM^*(\beta_0)$ in all three scenarios above, we need to combine them in a way that is adaptive to the identification strength. Following I.Andrews (2016), we consider the linear combination of $AR^2(\beta_0)$, $LM^2(\beta_0)$, and $LM^{*2}(\beta_0)$. Recall that $(\mathcal{N}_1, \mathcal{N}_2)$ are the limits of $(AR(\beta_0), LM^*(\beta_0))$ in either strong or weak identification. See (2.7) and (2.8) for their expressions in these two cases. Then, in the limit experiment, the linear combination test can be written as
\[ \phi_{a_1, a_2, \infty} = 1\{a_1 \mathcal{N}_1^2 + a_2 (\tilde{\rho}\mathcal{N}_1 + (1 - \tilde{\rho}^2)^{1/2}\mathcal{N}_2^2)^2 + (1 - a_1 - a_2)\mathcal{N}_2^2 \geq C_\alpha(a_1, a_2; \tilde{\rho})\}, \tag{2.9} \]
where $(a_1, a_2) \in \mathcal{A}_0$ are the combination weights, $\mathcal{N}_1 \sim \mathcal{Z}(\theta_1)$, and $\mathcal{N}_2 \sim \mathcal{Z}(\theta_2)$; the mean parameters $\theta_1$ and $\theta_2$ are defined in Lemmas 2.1 and 2.4 for strong and weak identification, respectively; and $\tilde{\rho}$ is the limit of $\tilde{\rho}(\beta_0)$.\(^5\) Let the eigenvalue decomposition of the matrix
\[ \begin{pmatrix} a_1 + a_2\tilde{\rho}^2 & a_2\tilde{\rho}(1 - \tilde{\rho}^2)^{1/2} \\ a_2\tilde{\rho}(1 - \tilde{\rho}^2)^{1/2} & 1 - a_1 - a_2\tilde{\rho}^2 \end{pmatrix} \]
be
\[ \begin{pmatrix} a_1 + a_2\tilde{\rho}^2 & a_2\tilde{\rho}(1 - \tilde{\rho}^2)^{1/2} \\ a_2\tilde{\rho}(1 - \tilde{\rho}^2)^{1/2} & 1 - a_1 - a_2\tilde{\rho}^2 \end{pmatrix} = \mathbf{U} \begin{pmatrix} s_1(a_1, a_2) & 0 \\ 0 & s_2(a_1, a_2) \end{pmatrix} \mathbf{U}^\top, \tag{2.10} \]

\(^3\)We suppress the dependence of $m_1(\Delta)$ and $m_2(\Delta)$ on $\gamma(\beta_0)$ and $C$ for notation simplicity.

\(^4\)Specifically, we say the data are homoskedastic if $(\sigma_i, \gamma_i, \eta_i)$ defined after (A.1) in Section A of the Online Supplement are constant across $i$.

\(^5\)Under fixed alternatives, $\tilde{\rho} = \rho(\beta_0)$; under local alternatives, $\tilde{\rho} = \rho$. 

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where, by construction, \(s_1(a_1, a_2) \geq s_2(a_1, a_2) \geq 0\) and \(U\) is a 2 \(\times\) 2 unitary matrix. We highlight the dependence of eigenvalues \((s_1, s_2)\) on the weights \((a_1, a_2)\). The dependence of \(U\) on \((a_1, a_2)\) is suppressed for notation simplicity. Then, we have

\[
a_1N_1^2 + a_2(\tilde{\rho}N_1 + (1 - \tilde{\rho}^2)^{1/2}N_2^\ast)^2 + (1 - a_1 - a_2)N_2^2 = s_1(a_1, a_2)\tilde{N}_1^2 + s_2(a_1, a_2)\tilde{N}_2^2
\]

and \(\phi_{a_1, a_2, \infty} = \{s_1(a_1, a_2)\tilde{N}_1^2 + s_2(a_1, a_2)\tilde{N}_2^2 \geq \mathcal{C}_\alpha(a_1, a_2; \tilde{\rho})\}\), where

\[
\begin{pmatrix}
\tilde{N}_1 \\
\tilde{N}_2
\end{pmatrix} = U^\top
\begin{pmatrix}
N_1^\ast \\
N_2^\ast
\end{pmatrix}
\]

and \(\tilde{N}_1\) and \(\tilde{N}_2\) are independent normal random variables with unit variance. This implies that \(\phi_{a_1, a_2, \infty}\) can be viewed as a linear combination test of two independent chi-squares random variables with one degree of freedom, and those two chi-squares random variables are obtained by properly rotating \(N_1\) and \(N_2^\ast\) (i.e., the limits of \(AR(\beta_0)\) and \(LM^*(\beta_0)\)).

Theorem 2.1 states the key properties of \(\phi_{a_1, a_2, \infty}\) under the limit experiment. Theorems 4.1-4.3 further establish that these properties hold asymptotically for our linear combination test.

**Theorem 2.1.**  
(i) Under weak identification and fixed alternatives, \(N_1 \sim Z(\theta_1), N_2^\ast \sim Z(\theta_2)\), and they are independent, where \(\theta_1 = m_1(\Delta)\) and \(\theta_2 = m_2(\Delta)\) as in (2.8). We consider the test of \(H_0: \theta_1 = \theta_2 = 0\) against \(H_1: \theta_1 \neq 0\) or \(\theta_2 \neq 0\). Then, for any \((a_1, a_2) \in \mathcal{A}_0\), \(\phi_{a_1, a_2, \infty}\) defined in (2.9) is admissible among the level-\(\alpha\) tests based on test statistics \(\tilde{s}_1\tilde{N}_1^2 + \tilde{s}_2\tilde{N}_2^2\) for \((\tilde{s}_1, \tilde{s}_2) \in \mathbb{R}^+ \times \mathbb{R}^+\).

(ii) Under strong identification and local alternatives, \(N_1^2 \sim Z^2, N_2^2 \sim Z^2(\theta)\), where \(\theta = \frac{\tilde{\Delta}^\top}{(1 - \tilde{\rho}^2)^{1/2}}\) as in (2.7). We consider the test of \(H_0: \theta = 0\) against \(H_1: \theta \neq 0\). Then, \(\phi_{a_1, a_2, \infty}\) defined in (2.9) is the uniformly most powerful test in the class of tests that depend on \((N_1, N_2^\ast)\) and are invariant to sign changes if and only if \(a_1 = 0\) and \(a_2 = 0\).

(iii) Suppose Assumption 2 holds, \((Q_{\ell(\beta_0)\ell(\beta_0)} - \Delta^2C, Q_{X\ell(\beta_0)} - \Delta C, Q_{XX} - C)^\top = O_p(1)\), and we are under strong identification with fixed alternatives. If \(1 \geq a_{1,n} \geq \frac{\tilde{\varphi}_{1}(\beta_0)}{C^2\Delta^2(\beta_0)}\) for some constant \(\tilde{q} > C_{\alpha, \max}(\rho(\beta_0))\) and \((a_{1,n}, a_{2,n}) \in \mathcal{A}_0\), where \(\Delta_+(\beta_0) = \Phi_1^{1/2}(\beta_0)\Psi^{-1/2}(\beta_0)\rho^{-1}(\beta_0)\), then

\[
1\{a_{1,n}AR^2(\beta_0) + a_{2,n}LM^2(\beta_0) + (1 - a_{1,n} - a_{2,n})LM^*2(\beta_0) > C_\alpha(a_{1,n}, a_{2,n}; \tilde{\rho}(\beta_0))\} \xrightarrow{P} 1.
\]

Five remarks are in order. First, in the case with a fixed number of weak instruments (or moment conditions), I. Andrews (2016) consider the linear combination of \(K\) and \(S\) statistics. The trade-off between \(K\) and \(S\) statistics is from the difference in attention to deviation directions (see the discussions in Section 3 of I. Andrews (2016)). We notice from Theorem 2.1 that \(\phi_{a_1, a_2, \infty}\) is
constructed based on a quadratic function of $AR(\beta_0)$ and $LM^*(\beta_0)$, which play roles similar to $S$ and $K$, respectively. However, $AR(\beta_0)$ and $LM^*(\beta_0)$ do not have such a difference in deviation directions. Instead, the trade-off between $AR(\beta_0)$ and $LM^*(\beta_0)$ is between local and non-local alternatives. Additionally, although the standard score test is not weak identification robust under a fixed number of instruments, $LM(\beta_0)$ is robust under many instruments. Therefore, we consider the linear combination of $AR(\beta_0)$, $LM(\beta_0)$, and $LM^*(\beta_0)$ to take advantage of the power properties of all three tests.

Second, unlike the one-sided jackknife AR test proposed by Mikusheva and Sun (2022), we construct the jackknife CLC test based on $AR^2(\beta_0)$ for several reasons. First, under weak identification, when the concentration parameter $C$ and, thus, $m_1(\Delta)$ defined in Lemma 2.4 is nonnegative, the one-sided test has good power. However, even in this case, the power curves simulation in Section 5.1 shows that our jackknife CLC test is more powerful than the one-sided AR test in most scenarios. Second, our jackknife CLC test will have good power even when $C$ is negative. Third, we show below that under strong identification and local alternatives, our jackknife CLC test converges to the uniformly most powerful test $1\{N_2^2 > C_\alpha\}$ whereas both the one- and two-sided tests based on $AR(\beta_0)$ have no power, as shown in Lemma 2.1. Fourth, under strong identification and fixed alternatives, our jackknife CLC test has asymptotic power equal to 1, as shown in Lemma 2.3 and Theorem 4.3 below. In this case, using the one-sided jackknife AR test cannot further improve the power. Fifth, combining $LM^{*2}(\beta_0)$ with $AR^2(\beta_0)$ (and $LM^2(\beta_0)$), rather than $AR(\beta_0)$, can substantially mitigate the impact of power loss of $LM^*(\beta_0)$ at $\Delta_*(\beta_0)$, as shown in the numerical investigation in Section 5.

Third, Theorem 2.1(i) implies that $\phi_{a_1,a_2,\infty}$ is admissible among tests that are also quadratic functions of $N_1$ and $N_2$ with the same rotation $U$ but different eigenvalues $(\tilde{s}_1, \tilde{s}_2)$; that is,

$$\langle N_1, N_2 \rangle U \begin{pmatrix} \tilde{s}_1 \\ 0 \\ 0 \end{pmatrix} U^\top \begin{pmatrix} N_1 \\ 0 \\ N_2 \end{pmatrix}.$$  

Specifically, in the special case with $a_2 = 0$, the rotation matrix $U = I_2$ and $\phi_{a_1,0,\infty}$ is admissible among level-$\alpha$ tests based on the test statistics of the form $a_1N_1^2 + (1 - a_1)N_2^2$ for $a_1 \in [0, 1]$, which is similar to the result for the linear combination of $S$ and $K$ statistics in I.Andrews (2016).

Fourth, under strong identification and local alternatives, $a_1 = 0$ and $a_2 \rho = 0$ imply that $\phi_{a_1,a_2,\infty} = 1\{N_2^{*2} \geq C_\alpha\}$, which is the uniformly most powerful invariant test. When $\rho = 0$ and under local alternatives, $a_2N_2^{*2}$ in the second and third terms of $\phi_{a_1,a_2,\infty}$ cancels out, which implies that $\phi_{a_1,a_2,\infty} = 1\{N_2^{*2} \geq C_\alpha\}$ as long as $a_1 = 0$.

Fifth, we note that both the rotation matrix $U$ and the eigenvalues $s_1$ and $s_2$ in (2.10) are func-

\textsuperscript{6}We note that $C = \sum_{i \in [n]} \sum_{j \in [n]} \Pi_i \Pi_j \sum_{i \in [n]} (1 - \Pi_i \Pi_j) \Pi_i \Pi_j^{\top} M \Pi_j$, where $M = I - P$. If $M^{\top} M$ and $\sum_{i \in [n]} \Pi_i \Pi_i^{\top}$ are sufficiently large, $C$ can be negative. Mikusheva and Sun (2022) further assume that $M^{\top} M \leq Cn^{\top} \Pi^{\top} \Pi$ for some constant $C > 0$, which implies that $C > 0$. 

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E overcame this issue, we follow I. Andrews (2016) and calibrate the power of the CLC test will typically remain unknown as the true parameter \( \delta \) ranges over all possible values that \( \Delta \) can potentially take; we define the range of potential values of \( \Delta \) below.

In this section, we determine the weights \( \{ a \} \) in the jackknife CLC test via a minimax procedure. In the following, we use a minimax procedure to select the weights \( \{ a \} \) in our jackknife CLC test \( \phi_{a_1, a_2, \infty} \). We can do the same to select \( a \) and \( \zeta \) for the new parametrization in (2.11). Under strong identification and local alternatives, Lemma 2.2 shows that \( L \{ LM^{i2}(\beta_0) \geq C_\alpha \} \) is the most powerful test against local alternatives, which is achieved by our jackknife CLC test \( \phi_{a_1, a_2, \infty} \) with \( a_1 = 0 \) and \( a_2 \rho = 0 \). In this setting, the new parametrization does not bring any additional power.

3 A Conditional Linear Combination Test

In this section, we determine the weights \( \{ a_1, a_2 \} \) in the jackknife CLC test via a minimax procedure. Under weak identification, the limit power of the jackknife CLC test with weights \( \{ a_1, a_2 \} \) is

\[
\mathbb{E} \phi_{a_1, a_2, \infty} = \mathbb{E} \{ a_1 Z_1^2(m_1(\Delta)) + a_2(\rho(\beta_0) Z_1(m_1(\Delta)) + (1 - \rho^2(\beta_0))^{1/2} Z_2(m_2(\Delta)) \} \geq C_\alpha(a_1, a_2; \rho(\beta_0))
\]

where \( m_1(\Delta) \) and \( m_2(\Delta) \) are defined in Lemma 2.4, and \( Z_1(\cdot) \) and \( Z_2(\cdot) \) are independent. In this case, we can be explicit and write \( \phi_{a_1, a_2, \infty} = \phi_{a_1, a_2, \infty}(\Delta) \). However, the limit power of the jackknife CLC test will typically remain unknown as the true parameter \( \beta \) (and hence \( \Delta \)) is unknown. To overcome this issue, we follow I. Andrews (2016) and calibrate the power of \( \mathbb{E} \phi_{a_1, a_2, \infty}(\delta) \), where \( \delta \) ranges over all possible values that \( \Delta \) can potentially take; we define \( \phi_{a_1, a_2, \infty}(\delta) \) as well as the range of potential values of \( \Delta \) below.

Let \( \hat{D} = Q_{XX} - (Q_{e(\beta_0)e(\beta_0)}, Q_{Xe(\beta_0)}) \left( \begin{array}{c} \hat{\Phi}_1(\beta_0) \\ \hat{\Phi}_1(\beta_0) \\ \hat{\Phi}_2(\beta_0) \\ \hat{\Phi}_2(\beta_0) \\ \hat{\Phi}_{13}(\beta_0) \\ \hat{\Phi}_{13}(\beta_0) \\ \hat{\Phi}_{13}(\beta_0) \\ \hat{\Phi}_{13}(\beta_0) \end{array} \right) \) be the residual from the projection of \( Q_{XX} \) on \( (Q_{e(\beta_0)e(\beta_0)}, Q_{Xe(\beta_0)}) \). By (2.3), under weak identification, \( \hat{D} = D + a_p(1), \quad D \overset{d}{=} N(\mu_D, \sigma_D^2) \).
where

\[
\mu_D = C \left[ 1 - (\Delta^2, \Delta) \left( \begin{pmatrix} \Phi_1(\beta_0) & \Phi_12(\beta_0) \\ \Phi_12(\beta_0) & \Psi(\beta_0) \end{pmatrix}^{-1} \begin{pmatrix} \Phi_{13}(\beta_0) \\ \tau(\beta_0) \end{pmatrix} \right) \right] \quad \text{and}
\]

\[
\sigma_D^2 = \gamma - \left( \begin{pmatrix} \Phi_{13}(\beta_0), \tau(\beta_0) \end{pmatrix} \begin{pmatrix} \Phi_1(\beta_0) & \Phi_12(\beta_0) \\ \Phi_12(\beta_0) & \Psi(\beta_0) \end{pmatrix}^{-1} \begin{pmatrix} \Phi_{13}(\beta_0) \\ \tau(\beta_0) \end{pmatrix} \right).
\]

We note that \( \bar{D} \) is a sufficient statistic for \( \mu_D \), which contains information about the concentration parameter \( C \) and is asymptotically independent of \( AR(\beta_0) \), \( LM(\beta_0) \), and hence \( LM^*(\beta_0) \).

Under weak identification, we observe that \( m_1(\Delta) \) and \( m_2(\Delta) \) in Lemma 2.4 can be written as

\[
\begin{pmatrix} m_1(\Delta) \\ m_2(\Delta) \end{pmatrix} = \begin{pmatrix} C_1(\Delta) \\ C_2(\Delta) \end{pmatrix} \mu_D,
\]

where

\[
\begin{pmatrix} C_1(\Delta) \\ C_2(\Delta) \end{pmatrix} = \begin{pmatrix} \Phi_1^{-1/2}(\beta_0) \Delta^2 \\ (1 - \rho^2(\beta_0))^{-1/2} (\Psi^{-1/2}(\beta_0) \Delta - \rho(\beta_0) \Phi_1^{-1/2}(\beta_0) \Delta^2) \end{pmatrix} \times \begin{pmatrix} \Phi_1(\beta_0) & \Phi_12(\beta_0) \\ \Phi_12(\beta_0) & \Psi(\beta_0) \end{pmatrix}^{-1} \begin{pmatrix} \Phi_{13}(\beta_0) \\ \tau(\beta_0) \end{pmatrix} \] \quad \text{(3.2)}

By (3.1), we see that \( \phi_{a_1,a_2,\infty} = \phi_{a_1,a_2,\infty}(\Delta) \) defined in (2.9) can be written as

\[
\begin{cases}
1 \times a_1 Z_1^2(C_1(\Delta) \mu_D) + a_2(\rho(\beta_0)) Z_1(C_1(\Delta) \mu_D) + (1 - \rho^2(\beta_0))^{1/2} Z_2(C_2(\Delta) \mu_D) \geq C_0(a_1,a_2; \rho(\beta_0)) \\
+ (1 - a_1 - a_2) Z_2^2(C_2(\Delta) \mu_D) \geq C_0(a_1,a_2; \rho(\beta_0))
\end{cases}
\]

This motivates the definition that

\[
\phi_{a_1,a_2,\infty}(\delta) = \begin{cases}
1 \times a_1 Z_1^2(C_1(\delta) \mu_D) + a_2(\rho(\beta_0)) Z_1(C_1(\delta) \mu_D) + (1 - \rho^2(\beta_0))^{1/2} Z_2(C_2(\delta) \mu_D) \geq C_0(a_1,a_2; \rho(\beta_0)) \\
+ (1 - a_1 - a_2) Z_2^2(C_2(\delta) \mu_D) \geq C_0(a_1,a_2; \rho(\beta_0))
\end{cases}
\]

To emphasize the dependence of \( \phi_{a_1,a_2,\infty}(\delta) \) on \( \mu_D \) and \( \gamma(\beta_0) \), we further write \( \phi_{a_1,a_2,\infty}(\delta) \) as \( \phi_{a_1,a_2,\infty}(\delta, \mu_D, \gamma(\beta_0)) \).

The range of values that \( \Delta \) can take is defined as \( D(\beta_0) = \{ \delta : \delta + \beta_0 \in \mathcal{B} \} \), where \( \mathcal{B} \) is the parameter space. For example, in their empirical application of returns to education, Mikusheva and Sun (2022) posit that the value of \( \beta \) (i.e., the return to education) is from -0.5 to 0.5 (i.e., \( \mathcal{B} = [-0.5, 0.5] \)). We follow the same practice in the simulation based on calibrated data in Section 5.2 and the empirical application in Section 6.
Following the lead of I. Andrews (2016), we define the highest attainable power for each $\delta \in \mathcal{D}(\beta_0)$ as $P_{\delta, \mu_D} = \sup_{(a_1, a_2) \in \mathcal{A}(\mu_D, \gamma(\beta_0))} \mathbb{E} \phi_{a_1, a_2, \infty}(\delta, \mu_D, \gamma(\beta_0))$, which means that

$$P_{\delta, \mu_D} - \mathbb{E} \phi_{a_1, a_2, \infty}(\delta, \mu_D, \gamma(\beta_0))$$

is the power loss when the weights are set as $(a_1, a_2)$. Here we denote the domain of $(a_1, a_2)$ as $\mathcal{A}(\mu_D, \gamma(\beta_0))$ and define it as $\mathcal{A}(\mu_D, \gamma(\beta_0)) = \{(a_1, a_2) \in \mathcal{A}_0, a_1 \in [\underline{a}(\mu_D, \gamma(\beta_0)), 1]\}$ where $\mathcal{A}_0 = \{(a_1, a_2) \in [0, 1] \times [0, 1], a_1 + a_2 \leq \pi\}$ for some $\pi < 1$.

$$\underline{a}(\mu_D, \gamma(\beta_0)) = \min \left(0.01, \frac{1.1 C_{\beta_0, \text{max}}(\rho(\beta_0)) \Phi_1(\beta_0) c_B(\beta_0)}{\Delta_2^*(\beta_0) \mu_D^2} \right),$$

$$\Delta_2^*(\beta_0) = \Phi_1^{1/2}(\beta_0) \Psi^{-1/2}(\beta_0) \rho^{-1}(\beta_0)$$

as defined after Lemma 2.3, and

$$c_B(\beta_0) = \sup_{\delta \in \mathcal{D}(\beta_0)} \left[ 1 - (\delta^2, \delta) \left( \begin{pmatrix} \Phi_1(\beta_0) & \Phi_{12}(\beta_0) \\ \Phi_{12}(\beta_0) & \Psi(\beta_0) \end{pmatrix}^{-1} \begin{pmatrix} \Phi_{13}(\beta_0) \\ \tau(\beta_0) \end{pmatrix} \right) \right]^2.$$

The maximum power loss over $\delta \in \mathcal{D}(\beta_0)$ can be viewed as a maximum regret. Then, we choose $(a_1, a_2)$ that minimizes the maximum regret; that is,

$$(a_1(\mu_D, \gamma(\beta_0)), a_2(\mu_D, \gamma(\beta_0))) \in \arg \min_{(a_1, a_2) \in \mathcal{A}(\mu_D, \gamma(\beta_0))} \sup_{\delta \in \mathcal{D}(\beta_0)} (P_{\delta, \mu_D} - \mathbb{E} \phi_{a_1, a_2, \infty}(\delta, \mu_D, \gamma(\beta_0))). \quad (3.4)$$

Four remarks on the domain of $(a_1, a_2)$ (i.e., $\mathcal{A}(\mu_D, \gamma(\beta_0))$) are in order. First, the lower bound $\underline{a}(\mu_D, \gamma(\beta_0))$ is motivated by Theorem 2.1(iii). Second, under weak identification, $\mu_D$ is fixed, and $\frac{1.1 C_{\beta_0, \text{max}}(\rho(\beta_0)) \Phi_1(\beta_0) c_B(\beta_0)}{\Delta_2^*(\beta_0) \mu_D^2}$ may be larger than 0.01. In this case, we have $\mathcal{A}(\mu_D, \gamma(\beta_0)) = \{(a_1, a_2) \in \mathcal{A}_0, a_1 \in [0.01, 1]\}$. In our simulations, the minimax $a_1$ never hits the lower bound so that setting the lower bound to be 0.01 or 0 does not make any numerical difference. Third, under strong identification and local alternatives, $\frac{1.1 C_{\beta_0, \text{max}}(\rho(\beta_0)) \Phi_1(\beta_0) c_B(\beta_0)}{\Delta_2^*(\beta_0) \mu_D^2}$ will converge to zero so that

$$\mathcal{A}(\mu_D, \gamma(\beta_0)) = \left\{(a_1, a_2) \in \mathcal{A}_0, a_1 \in \left[ \frac{1.1 C_{\beta_0, \text{max}}(\rho(\beta_0)) \Phi_1(\beta_0) c_B(\beta_0)}{\Delta_2^*(\beta_0) \mu_D^2}, 1 \right] \right\}.$$

We show in Theorem 4.2 below that in this case, the minimax jackknife CLC test converges to $1\{N^2 \geq C_\alpha\}$ defined in Lemma 2.2, which is the uniformly most powerful invariant test. Furthermore, the minimax $a_1$ satisfies the requirement in Theorem 2.1(iii) with $\bar{q} = 1.1 C_{\beta_0, \text{max}}(\rho(\beta_0))$ so that under strong identification, our CLC test has asymptotic power 1 against fixed alternatives, as shown in Theorem 4.3. Fourth, we require $\pi < 1$ for some technical reason. Again, in our simulations, we never observe the minimax $a_1 + a_2$ hitting the upper bound so that setting the upper bound to be $\pi$ or 1 does not make any numerical difference.
In practice, we do not observe $\mu_D$ and $\gamma(\beta_0)$. Therefore, we follow I. Andrews (2016, Section 6) and consider the plug-in method. We can replace $\gamma(\beta_0)$ by its consistent estimator $\hat{\gamma}(\beta_0)$ introduced in Assumption 2. To obtain a proxy of $\mu_D$,\footnote{In fact, as $\phi_{a_1,a_2,\infty}(\delta,\mu_D,\gamma(\beta_0))$ only depends on $\mu_D$, we aim to find a good estimator of $\mu_D^\ast$.} we define

$$
\hat{\sigma}_D = \left( \hat{Y} - (\hat{\Phi}_{13}(\beta_0), \hat{\tau}(\beta_0)) \left( \begin{array}{cc} \hat{\Phi}_1(\beta_0) & \hat{\Phi}_{12}(\beta_0) \\ \hat{\Phi}_{12}(\beta_0) & \hat{\psi}(\beta_0) \end{array} \right)^{-1} \left( \begin{array}{c} \hat{\Phi}_{13}(\beta_0) \\ \hat{\tau}(\beta_0) \end{array} \right) \right)^{1/2},
$$

which is a function of $\hat{\gamma}(\beta_0)$ and a consistent estimator of $\sigma_D$ by Assumption 2. Then, under weak identification, we have $\hat{D}^2/\hat{\sigma}_D^2 = D^2/\sigma_D^2 + o_p(1) \overset{d}{=} \chi_1^2(\mu_D^2/\sigma_D^2) + o_p(1)$ and $D^2/\sigma_D^2$ is a sufficient statistic for $\mu_D^2$. Let $\hat{r} = \hat{D}^2/\hat{\sigma}_D^2$. We consider two estimators for $\mu_D$ as functions of $\hat{D}$ and $\hat{\sigma}_D$, namely, $f_{pp}(\hat{D}, \hat{\gamma}(\beta_0)) = \hat{\sigma}_D\sqrt{\hat{r}_{pp}}$ and $f_{krs}(\hat{D}, \hat{\gamma}(\beta_0)) = \hat{\sigma}_D\sqrt{\hat{r}_{krs}}$, where $\hat{r}_{pp} = \max(\hat{r} - 1, 0)$ and

$$
\hat{r}_{krs} = \hat{r} - 1 + \exp \left( -\frac{\hat{r}}{2} \right) \left( \sum_{j=0}^{\infty} \left( -\frac{\hat{r}}{2} \right)^j \frac{1}{j!(1+2j)} \right)^{-1}.
$$

Specifically, Kubokawa, Robert, and Saleh (1993) show that $\hat{r}_{krs}$ is positive as long as $\hat{r} > 0$ and $\hat{r} \geq \hat{r}_{krs} \geq \hat{r} - 1$. It is also possible to consider the MLE based on a single observation $\hat{D}^2/\hat{\sigma}_D^2$. However, such an estimator is harder to use because it does not have a closed-form expression.

In practice, we estimate $E\phi_{a_1,a_2,\infty}(\delta, \mu_D, \gamma(\beta_0))$ by $E^*\phi_{a_1,a_2,s}(\delta, \hat{D}, \hat{\gamma}(\beta_0))$ for $s \in \{pp, krs\}$, where

$$
\phi_{a_1,a_2,s}(\delta, \hat{D}, \hat{\gamma}(\beta_0)) = 1 \left\{ a_1 Z_1^2(\hat{C}_1(\delta) f_s(\hat{D}, \hat{\gamma}(\beta_0))) \right. + a_2 \left[ \hat{\rho}(\beta_0) Z_1(\hat{C}_1(\delta) f_s(\hat{D}, \hat{\gamma}(\beta_0))) + (1 - \hat{\rho}^2(\beta_0))^{1/2} Z_2(\hat{C}_2(\delta) f_s(\hat{D}, \hat{\gamma}(\beta_0))) \right] ^2 \right\},
$$

and $(\hat{C}_1(\delta), \hat{C}_2(\delta))$ are similarly defined as $(C_1(\delta), C_2(\delta))$ in (3.2) with $\gamma(\beta_0)$ replaced by $\hat{\gamma}(\beta_0)$; that is,

$$
\left( \begin{array}{c} \hat{C}_1(\delta) \\ \hat{C}_2(\delta) \end{array} \right) \equiv \left( \begin{array}{c} \hat{\Phi}_1^{-1/2}(\beta_0) \delta^2 \\ (1 - \hat{\rho}(\beta_0))^{-1/2} (\hat{\psi}^{-1/2}(\beta_0) \delta - \hat{\rho}(\beta_0) \hat{\Phi}_1^{-1/2}(\beta_0) \delta^2) \end{array} \right) \times \left[ 1 - (\delta^2, \delta) \left( \begin{array}{cc} \hat{\Phi}_1(\beta_0) & \hat{\Phi}_{12}(\beta_0) \\ \hat{\Phi}_{12}(\beta_0) & \hat{\psi}(\beta_0) \end{array} \right)^{-1} \left( \begin{array}{c} \hat{\Phi}_{13}(\beta_0) \\ \hat{\tau}(\beta_0) \end{array} \right) \right]^{-1}.
$$

Let $P_{\delta,s}(\hat{D}, \hat{\gamma}(\beta_0)) = \sup_{(a_1,a_2) \in A(f_s(\hat{D}, \hat{\gamma}(\beta_0)), \hat{\gamma}(\beta_0))} E^*\phi_{a_1,a_2,s}(\delta, \hat{D}, \hat{\gamma}(\beta_0))$. Then, for $s \in \{pp, krs\}$,
we can estimate $a(\mu_D, \gamma(\beta))$ in (3.4) by $A_s(\tilde{D}, \tilde{\gamma}(\beta)) = (A_{1,s}(\tilde{D}, \tilde{\gamma}(\beta)), A_{2,s}(\tilde{D}, \tilde{\gamma}(\beta)))$ defined as

$$A_s(\tilde{D}, \tilde{\gamma}(\beta)) \in \arg \min_{(a_1, a_2) \in \Delta(\hat{f}(\tilde{D}, \tilde{\gamma}(\beta)), \hat{\gamma}(\beta))} \sup_{\delta \in \mathcal{D}(\beta)} (P_{\delta,s}(\tilde{D}, \tilde{\gamma}(\beta)) - E^\ast \phi_{a_1,a_2,s}(\delta, \tilde{D}, \tilde{\gamma}(\beta))), \quad (3.6)$$

where $\phi_{a_1,a_2,s}(\delta, \tilde{D}, \tilde{\gamma}(\beta))$ is defined in (3.5),

$$a(\hat{f}(\tilde{D}, \tilde{\gamma}(\beta)), \tilde{\gamma}(\beta)) = \min \left( 0, 1, \frac{1.1C_{a,\max}(\hat{\rho}(\beta))\hat{\Phi}_1(\beta)\hat{c}_B(\beta.now)}{\Delta^2(\beta)f_s^2(\tilde{D}, \tilde{\gamma}(\beta))} \right),$$

$$\hat{c}_B(\beta) = \sup_{\delta \in \mathcal{D}(\beta)} \left[ 1 - (\delta^2, \delta) \begin{pmatrix} \hat{\Phi}_1(\beta) & \hat{\Phi}_{12}(\beta) \\ \hat{\Phi}_{12}(\beta) & \hat{\Psi}(\beta) \end{pmatrix}^{-1} \begin{pmatrix} \hat{\Phi}_{13}(\beta) \\ \hat{\tau}(\beta) \end{pmatrix} \right]^2,$$

and $\Delta_s(\beta) = \hat{\Phi}_1^{1/2}(\beta)\hat{\Psi}^{-1/2}(\beta)\hat{\rho}^{-1}(\beta)$. Then, the feasible jackknife CLC test is, for $s \in \{pp, krs\}$,

$$\hat{\phi}_{A_s(\tilde{D}, \tilde{\gamma}(\beta))} = \begin{cases} A_{1,s}(\tilde{D}, \tilde{\gamma}(\beta))AR^2(\beta) + A_{2,s}(\tilde{D}, \tilde{\gamma}(\beta))LM^2(\beta) \\ +(1 - A_{1,s}(\tilde{D}, \tilde{\gamma}(\beta))) - A_{2,s}(\tilde{D}, \tilde{\gamma}(\beta))LM^*\rho^2(\beta) \geq C_{a}(A_s(\tilde{D}, \tilde{\gamma}(\beta)); \hat{\rho}(\beta)) \end{cases},$$

$$\hat{D} \sim D \stackrel{d}{=} \mathcal{N}(\mu_D, \sigma_D^2).$$

We see from (3.4) and (3.6) that $A_s(d, r) = (a_1(f_s(d, r), r), a_2(f_s(d, r), r))$ for $(d, r) \in \mathbb{R} \times \Gamma$, where $\Gamma$ is the parameter space for $\gamma(\beta)$ and $s \in \{pp, krs\}$. We make the following assumption on $A_s(\cdot)$.

**Assumption 3.** Suppose we are under weak identification with a fixed $\beta_0$. Let $S_s$ be the set of discontinuities of $A_s(\cdot, \gamma(\beta)) : \mathbb{R} \mapsto [0, 1] \times [0, 1]$. Then, we assume $A_s(d, r)$ is continuous in $r$ at $r = \gamma(\beta_0)$ for any $d \in \mathbb{R}/S_s$, and the Lebesgue measure of $S_s$ is zero for $s \in \{pp, krs\}$.

Assumption 3 is a technical condition that allows us to apply the continuous mapping theorem. It is mild because $A_s(\cdot)$ is allowed to be discontinuous in its first argument. In practice, we can approximate $A_s(\cdot)$ by a step function defined over a grid of $d$ so that there is a finite number of discontinuities. The continuity of $A_s(\cdot)$ in its second argument is due to the smoothness of the
bivariate normal PDF with respect to the covariance matrix. Therefore, in this case, Assumption 3 holds automatically.

**Theorem 4.1.** Suppose we are under weak identification and fixed alternatives and that Assumptions 1, 2, and 3 hold. Then, for \( s \in \{pp, krs\} \),

\[
A_s(\hat{D}, \hat{\gamma}(\beta_0)) \rightsquigarrow A_s(D, \gamma(\beta_0)) = (a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)))
\]

and

\[
E_{\phi_{A_s(D, \gamma(\beta_0))}} \rightarrow E_{\phi_{a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), \infty}}(\Delta, \mu_D, \gamma(\beta_0)),
\]

where \( \phi_{a_1,a_2,\infty} \) is defined in (3.3) and \( a_l(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)) \) is interpreted as \( a_l(\mu_D, \gamma(\beta_0)) \) defined in (3.4) with \( \mu_D \) replaced by \( f_s(D, \gamma(\beta_0)) \) for \( l = 1, 2 \).

In addition, let \( BL_1 \) be the class of functions \( f(\cdot) \) of \( D \) that is bounded and Lipschitz with Lipschitz constant 1. Then, if the null hypothesis holds such that \( \Delta = 0 \), we have

\[
E(\hat{\phi}_{A_s(D, \gamma(\beta_0))} - e) f(\hat{D}) \rightarrow 0, \quad \forall f \in BL_1.
\]

Several remarks on Theorem 4.1 are in order. First, Theorem 4.1 shows that the asymptotic power of the CLC test with the weights \((a_1, a_2)\) selected by the minimax procedure is the same as that in the limit experiment when the weights equal \( A_s(D, \gamma(\beta_0)) \), which is a function of \( D \). Given that \( D \) is independent of both normal random variables in \( \phi_{a_1,a_2,\infty}(\delta) \) in (3.3), the jackknife CLC test is asymptotically admissible conditional on \( \hat{D} \) among the tests specified in Theorem 2.1(i). Second, we see that the power of our jackknife CLC test is \( E_{\phi_{A_s(D, \gamma(\beta_0))}}(\Delta, \mu_D, \gamma(\beta_0)) \), which does not exactly match the minimax power

\[
E_{\phi_{a_1(\mu_D, \gamma(\beta_0)), a_2(\mu_D, \gamma(\beta_0))}}(\Delta, \mu_D, \gamma(\beta_0))
\]

in the limit problem. This is because under weak identification, it is impossible to consistently estimate \( \mu_D \), or equivalently, the concentration parameter. A similar result holds under weak identification with a fixed number of moment conditions in I. Andrews (2016). The best we can do is to approximate \( \mu_D \) by reasonable estimators based on \( D \) such as \( f_{pp}(D, \gamma(\beta_0)) \) and \( f_{krs}(D, \gamma(\beta_0)) \). Last, Theorem 4.1 implies that our jackknife CLC test controls size asymptotically conditionally on \( \hat{D} \), and thus, unconditionally.

Next, we consider the performance of \( \hat{\phi}_{A_s(D, \gamma(\beta_0))} \) defined in (3.7) under strong identification and local alternatives.

---

\( ^8 \)We assume that \( \frac{C}{0} = +\infty \) if \( C > 0 \) and \( \min(C, +\infty) = C. \)
Theorem 4.2. Suppose that Assumptions 1 and 2 hold. Further suppose that we are under strong identification and local alternatives as described in Lemma 2.1. Then, for \( s \in \{pp, krs\} \), we have

\[
A_{1,s}(\hat{D}, \hat{\gamma}(\beta_0)) \xrightarrow{p} 0, \quad A_{2,s}(\hat{D}, \hat{\gamma}(\beta_0)) \rho \xrightarrow{p} 0, \quad \text{and} \quad \hat{\phi}_{A_s}(\hat{D}, \hat{\gamma}(\beta_0)) \sim 1\{N^*_2 \geq C_\alpha\},
\]

where \( N^*_2 \overset{d}{=} N\left(\frac{\tilde{\Delta} \hat{C}}{(1-\rho^2)^{1/2}} / \Psi, 1\right) \).

Three remarks are in order. First, Theorem 4.2 shows that under strong identification and local alternatives, our jackknife CLC test converges to the uniformly most powerful level-\( \alpha \) test characterized in Lemma 2.2. Therefore, it is more powerful than the jackknife AR and LM tests. Second, under strong identification and local alternatives, the JIVE-based Wald test proposed by Chao et al. (2012) is asymptotically equivalent to the jackknife LM test, which implies that the jackknife AR and JIVE-Wald-based two-step test in Mikusheva and Sun (2022) is also dominated by the jackknife CLC test. Third, Theorem 4.2 shows that our jackknife CLC test is adaptive. In practice, econometricians do not know whether or not the alternative \( \beta_0 \) is close to the null \( \beta \). Therefore, our jackknife CLC test calibrates the power over all of the values \( \delta \) can take (i.e., \( \delta \in D(\beta_0) \)), which includes both local and fixed alternatives. Yet, Theorem 4.2 shows that the minimax procedure can produce the most powerful test as if it is known that \( \beta_0 \) is under local alternatives.

Last, we show that, under strong identification, the jackknife CLC test \( \hat{\phi}_{A_s}(\hat{D}, \hat{\gamma}(\beta_0)) \) defined in (3.7) has asymptotic power 1 against fixed alternatives.

Theorem 4.3. Suppose Assumption 2 holds, and \((Q_{\epsilon(\beta_0)} e(\beta_0) - \Delta^2 C, Q_{Xe(\beta_0)} - \Delta C, Q_{XX} - C)^\top = O_p(1)\). Further suppose that we are under strong identification with fixed alternatives so that \( \Delta = \beta - \beta_0 \) is nonzero and fixed. Then, we have

\[
\hat{\phi}_{A_s}(\hat{D}, \hat{\gamma}(\beta_0)) \xrightarrow{p} 1.
\]

5 Simulation

5.1 Power Curve Simulation for the Limit Problem

In this section, we simulate the power behavior of tests under the limit problem described in Section 2. We compare the following tests with a nominal rate of 5\%: our jackknife CLC test in which \( \mu_D \) is estimated by the methods \( pp \) and \( krs \), respectively, the one-sided jackknife AR test defined in (2.5), the jackknife LM test defined in (2.6), and the test that is based on the orthogonalized jackknife LM statistic \( LM^*_2(\beta_0) \) defined in this paper. The results below are based on 5,000 simulation replications.
We set the parameter space for $\beta$ as $B = [-6/C, 6/C]$, where $C = 3$ and 6 represent weak and strong identification, respectively. The choice of parameter space follows that in I. Andrews (2016, Section 7.2). We set $\beta_0 = 0$, and the values of the covariance matrix in (2.2) are set as follows: $\Phi_1 = \Psi = \Upsilon = 1$, and $\Phi_{12} = \Phi_{13} = \tau = \rho$, where $\rho \in \{0.2, 0.4, 0.7, 0.9\}$. We then compute $\gamma(\beta_0)$ based on (2.4) as $\beta$ ranges over $B$ and generate $AR(\beta_0)$ and $LM(\beta_0)$ based on (2.3). Last, we implement our CLC test purely based on $AR(\beta_0)$, $LM(\beta_0)$, $\gamma(\beta_0)$, and $B$ without assuming the knowledge of $(C, \beta, \Phi_1, \Psi, \Upsilon, \Phi_{12}, \Phi_{13}, \tau)$. We have tried to simulate under alternative settings of the covariance matrix, and the obtained patterns of the power behavior are very similar.

Figures 1–4 plot the power curves for $\rho = 0.2, 0.4, 0.7,$ and 0.9. In each figure, we report the results under both weak and strong identification ($C = 3$ and 6, respectively). We observe that overall, the two jackknife CLC tests have the best power properties in terms of maximum regret. Especially when the identification is strong ($C = 6$) and/or the degree of endogeneity is not very low ($\rho = 0.4, 0.7,$ or 0.9), the jackknife CLC tests outperform their AR and LM counterparts by a large margin. In addition, we notice that when $C = 3$, for some parameter values $LM^*(\beta_0)$ can suffer from substantial declines in power relative to the other tests, which is in line with our theoretical predictions. By contrast, our jackknife CLC tests are able to guard against such substantial power loss because of the adaptive nature of their minimax procedure.

![Figure 1: Power Curve for $\rho = 0.2$](image)

### 5.2 Simulation Based on Calibrated Data

We follow Angrist and Frandsen (2022) and Mikusheva and Sun (2022) and calibrate a data generating process (DGP) based on the 1980 census dataset from Angrist and Krueger (1991). Let the instruments be

$$Z_i = \{(1\{Q_i = q, C_i = c\})_{q \in \{2,3,4\}, c \in \{31,\ldots,39\}}, (1\{Q_i = q, P_i = p\})_{q \in \{2,3,4\}, p \in \{51 \text{ states}\}}\),$$
where $Q_i, C_i, P_i$ are individual $i$’s quarter of birth (QOB), year of birth (YOB) and place of birth (POB), respectively, so that there are 180 instruments. Note that the dummy with $q = 1$ and $c = 30$ is omitted in $Z_i$. We denote $\tilde{Y}_i$ as income, $\tilde{X}_i$ as the highest grade completed, and $\tilde{W}_i$ as the full set of YOB-POB interactions; that is,

$$\tilde{W}_i = \{1\{C_i = c, P_i = p\} \}_{c \in \{30, \ldots, 39\}, p \in \{51 \text{ states} \}},$$

which is a $510 \times 1$ matrix.

As in Angrist and Frandsen (2022), using the full 1980 sample (consisting of 329,509 individuals), we first obtain the average $\tilde{X}_i$ for each QOB-YOB-POB cell; we call this $\bar{s}(q, c, p)$. Next we use
LIML to estimate the structural parameters in the following linear IV regression:

\[ \tilde{Y}_i = \tilde{X}_i \beta + \tilde{W}_i^\top \beta_w + e_i, \]

\[ \tilde{X}_i = Z_i^\top \Gamma Z_i + \tilde{W}_i^\top \Gamma W + V_i, \]

where \( \tilde{X} \) is endogenous and is instrumented by \( Z_i \) and \( \tilde{W}_i \) is the exogenous control variable. Denote the LIML estimate for \( \beta_{X,W} \equiv (\beta_{X}^\top, \beta_{W}^\top)^\top \) as \( \hat{\beta}_{LIML} = (\hat{\beta}_{LIML,X}, \hat{\beta}_{LIML,W}) \). We let \( \tilde{y}(C_i, P_i) = \tilde{W}_i^\top \hat{\beta}_{LIML,W} \) and \( \omega(Q_i, C_i, P_i) = \tilde{Y}_i - \tilde{X}_i \hat{\beta}_{LIML,X} - \tilde{W}_i^\top \hat{\beta}_{LIML,W} \).

Based on the LIML estimate and the calibrated \( \omega(Q_i, C_i, P_i) \), we simulate the following two DGPs:

1. DGP 1:

\[ \tilde{y}_i = \bar{y} + \beta \tilde{s}_i + \omega(Q_i, C_i, P_i)(\nu_i + \kappa_2 \xi_i) \]

\[ \tilde{s}_i \sim Poisson(\mu_i), \]

where \( \beta \) is the parameter of interest, \( \nu_i \) and \( \xi_i \) are independent standard normal, \( \bar{y} = \frac{1}{n} \sum_{i=1}^{n} \tilde{y}(C_i, P_i) \), \( \mu_i \equiv max\{1, \gamma_0 + \gamma_Z^\top Z_i + \kappa_1 \nu_i\} \), and \( \gamma_0 + \gamma_Z^\top Z_i \) is the projection of \( \tilde{s}_i(q,c,p) \) onto a constant and \( Z_i \). We set \( \kappa_1 = 1.7 \) and \( \kappa_2 = 0.1 \) as in Mikusheva and Sun (2022).

2. DGP 2: Same as DGP 1 except that \( \kappa_1 = 2.7 \) and

\[ \tilde{s}_i \sim \lfloor Poisson(2\mu_i)/2 \rfloor. \]
We consider varying sample size $n$ based on 0.5%, 1%, and 1.5% of the full sample size. Upon obtaining $n$ observations, we exclude instruments with $\sum_{i=1}^{n} Z_{ij} < 5$. This results in small, medium, and large samples with 1,648, 3,296, and 4,943 observations and 119, 142, and 150 numbers of IVs, respectively. Our DGP 1 is exactly the same as that in Mikusheva and Sun (2022), which has $\rho = 0.41$. Our DGP 2 has $\rho = 0.7$. The concentration parameters (defined as $C/Y^{1/2}$) for small, medium, and large samples are 2.15, 3.62, and 4.85, respectively, for DGP 1, and 2.38, 3.97, 5.28, respectively, for DGP 2.

We emphasize that following Angrist and Frandsen (2022) and Mikusheva and Sun (2022), we only use $\tilde{W}_i$ to compute the LIML estimator and calibrate $\omega(Q_i, C_i, P_i)$, but do not use it to generate new data. Therefore, for the simulated data, the outcome variable is $\tilde{y}_i$, the endogenous variable is $\tilde{s}_i$, the IV $Z_i$ is viewed to be fixed, and the exogenous control variable is just an intercept. We then denote the demeaned versions of $\tilde{y}_i$ and $\tilde{s}_i$ as $Y_i$ and $X_i$, respectively, in (2.1) and implement various inference methods described below. Following Mikusheva and Sun (2022), we test the null hypothesis that $\beta = \beta_0$ for $\beta_0 = 0.1$ while varying the true value $\beta \in B$. The parameter space is set as $B = [-0.5, 0.5]$, which is consistent with the choice of parameter space for the empirical application below. The results below are based on 1,000 simulation repetitions. We provide more details about the implementation in Section B in the Online Supplement.

We compare the following tests with a nominal rate of 5%:

1. pp: our jackknife CLC test when $\mu_D$ is estimated by the method $pp$.
2. krs: our jackknife CLC test when $\mu_D$ is estimated by the method $krs$.
3. AR: the one-sided jackknife AR test with the cross-fit variance estimator proposed by Mikusheva and Sun (2022).
4. LM_CF: Matsushita and Otsu’s (2021) jackknife LM test, but with a cross-fit variance estimator (details are given in Section A.2 in the Online Supplement).
5. 2-step: Mikusheva and Sun’s (2022) two-step estimator in which the overall size is set at 5%.
6. LM*: LM* test defined in this paper.
7. LM_MO: Matsushita and Otsu’s (2021) original jackknife LM test.

Figures 5 and 6 plot the power curves of the aforementioned tests. We can make six observations. First, all methods control size well because they are all weak identification robust. Second, the performance of the jackknife CLC test with $krs$ is slightly better than that with $pp$, which is consistent with the power curve simulation in Section 5.1. Third, in DGP 1 with a small sample size, the power of the jackknife AR test is about 9.2% higher than that of the $krs$ test when $\beta$ is around -0.3. However, for alternatives close to the null (e.g., when $\beta$ is around 0), the power of
the krs test is 24% higher, which implies that the power of the krs test is still better than that for the jackknife AR test in the minimax sense. The power of the jackknife LM tests is similar to that of the krs test in DGP 1 with a small sample size. Fourth, for the rest of the scenarios, the power of the krs test is the highest in most regions of the parameter space. The power of the jackknife AR and LM is at most 0.7% higher than that of the krs test at some point. For DGP 1 with medium and large sample sizes, the maximum power gaps between our krs test and the jackknife LM are about 8.6% and 5.6%, and about 43.2% and 50% compared with the jackknife AR. Furthermore, they are 23.3%, 19.5%, and 18.5% compared with the jackknife LM for DGP 2 with small, medium, and large sample sizes, respectively, and about 41.5%, 55.3%, and 55.5% compared with the jackknife AR. Fifth, Figures 7 and 8 show the average values of \((a_1, a_2)\), the weights of the jackknife AR and LM for our CLC tests, under DGPs 1 and 2, respectively. We observe that the minimax procedure does not put all the weights on the LM* test. Furthermore, because the jackknife AR is more powerful on the left side of the parameter space relative to the right, the minimax weights for AR\(^2(\beta_0)\) \((a_1)\) are higher on the left than on the right. The summation of \(a_1\) and \(a_2\) is the lowest for alternatives that are close to the null, which is consistent with our theory that LM* is most powerful for local alternatives. Compared with those for DGP 1, the weights for DGP 2 are lower in general because the identification is slightly stronger in this case. Last, although the power of LM*\(^2(\beta_0)\) drops at both ends of the parameter space, the power of the jackknife CLC tests remains stable. From Figures 7 and 8, we see that in those regions, more weights are put on AR\(^2(\beta_0)\) and LM\(^2(\beta_0)\).
6 Empirical Application

In this section, we consider the linear IV regressions with the specification underlying Angrist and Krueger (1991, Table VII, column (6)), using the full original dataset. The outcome variable \( Y \) and endogenous variable \( X \) are log weekly wages and schooling, respectively. We follow Angrist and Krueger (1991) and focus on two specifications with 180 and 1,530 instruments. The 180 instruments include 30 quarter and year of birth interactions (QOB-YOB) and 150 quarter and place of birth interactions (QOB-POB). For the second specification with 1,530 instruments, we also include full interactions among QOB-YOB-POB. The exogenous control variables have been partialled out from the outcome and endogenous variables. More details of the empirical application are given in Section C in the Online Supplement. The considered tests are similar to those in the previous section. The jackknife AR test is defined in (2.5) with \( \hat{\Phi}_1 \) being the cross-fit estimator in Mikusheva and Sun (2022). The jackknife LM test is defined in (2.6) with the cross-fit estimator for \( \Psi(\beta_0) \). The \( pp \) and \( krs \) tests are our jackknife CLC tests. The two-step procedure is given by Mikusheva and Sun (2022, Section 5). Specifically, the researcher accepts the null if \( \tilde{F} > 9.98 \) and \( Wald(\beta_0) < C_{0.02} \) or if \( \tilde{F} \leq 9.98 \) and \( AR(\beta_0) < z_{0.02} \). In the case of 180 instruments, because \( \tilde{F} = 13.42 > 9.98 \), the lower and upper bounds of the 95% confidence interval (CI) for the two-step procedure correspond respectively to the minimum and maximum of the set \{\( \beta_0 \in \mathbb{R} : Wald(\beta_0) < C_{0.02} \)\}; similarly, for the 1,530 instruments, as \( \tilde{F} = 6.32 \leq 9.98 \), the lower and

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\[ \tilde{F} = Q_{XX}/\hat{V}, \] where \( \hat{T} \) is the cross-fit estimator. \( Wald(\beta_0) \) is defined as \( \left( \hat{\beta} - \beta_0 \right)^2 \hat{V} \), where \( \hat{\beta} \) is the JIVE estimator and \( \hat{V} \) is a cross-fit estimator of the asymptotic variance of \( \hat{\beta} \). We refer interested readers to Mikusheva and Sun (2022, Section 5) for more details.
upper bounds of the CI for the two-step procedure correspond respectively to the minimum and maximum of the set \( \{ \beta_0 \in \mathbb{R} : AR(\beta_0) < z_{0.02} \} \). We also report the 95% Wald test CI based on the JIVE estimator, denoted as JIVE-t. Table 1 reports the 95% CIs by inverting the corresponding 5% tests mentioned above for the parameter space \( B = [-0.5, 0.5] \). Note all CIs except JIVE-t are robust to weak identification. As \( \tilde{F} \)'s are higher than 4.14 in both cases, the JIVE-t (5%) has the Stock and Yogo (2005b)-type guarantee with at most a 5% size distortion (i.e., the overall size is less than 10%).

|         | jackknife AR (5%) | jackknife LM (5%) | JIVE-t (5%) | Two-step (5%) | pp (5%) | krs (5%) |
|---------|-------------------|-------------------|-------------|---------------|---------|---------|
| 180 IVs | 0.008,0.201       | 0.067,0.135       | 0.066,0.132 | 0.059,0.139   | 0.067,0.128 | 0.067,0.128 |
| 1530 IVs| -0.035,0.22       | 0.036,0.138       | 0.035,0.133 | -0.051,0.242  | 0.037,0.133 | 0.037,0.133 |

Table 1: **Confidence Intervals**

Notes: The \( \tilde{F} \)'s for 180 and 1,530 instruments are 13.42 and 6.32, respectively. The grid-search used for our confidence interval was over 10,000 equidistant grid-points for \( \beta_0 \in [-0.5, 0.5] \). Our jackknife AR confidence interval for 1530 instruments differs from that in Mikusheva and Sun (2022) because they used year-of-birth 1930-1938 dummies for the QOB-YOB-POB interactions, whereas we used 1930-1939 dummies. More details are provided in Section C in the Online Supplement.

Table 1 highlights that the CIs generated by our jackknife CLC tests are the shortest among all the weak identification robust CIs (i.e., pp, krs, jackknife AR, jackknife LM, and two-step). Furthermore, the jackknife CLC CIs are 7.6% and 2.0% shorter than the non-robust JIVE-t CIs with 180 and 1,530 instruments, respectively, which is in line with our theoretical result that the CLC tests are adaptive to the identification strength and efficient under strong identification.
Figure 8: Average Values of $a$ for DGP 2

7 Conclusion

In this paper, we consider a jackknife CLC test that is adaptive to the identification strength in IV regressions with many weak instruments. We show that the proposed test is (i) robust to weak identification, many instruments, and heteroskedasticity, (2) admissible under weak identification among some class of tests, and (3) uniformly most powerful among sign-invariant tests under strong identification against local alternatives. Simulation experiments confirm the good power properties of the jackknife CLC test.
A Verifying Assumption 2

A.1 Standard Estimators

In this section, we maintain Assumption 4, which is stated below and just Mikusheva and Sun (2022, Assumption 1).

Assumption 4. The observations \((Y_i, X_i, Z_i)_{i \in [n]}\) are i.i.d. Suppose \(P\) is an \(n \times n\) projection matrix of rank \(K\), \(K \to \infty\) as \(n \to \infty\) and there exists a constant \(\delta\) such that \(P_{ii} \leq \delta < 1\).

Following the results in Chao et al. (2012) and Mikusheva and Sun (2022), we can show that under either weak or strong identification, Assumption 1 in the paper holds:

\[
\begin{pmatrix}
Q_{ee} \\
Q_{Xe} \\
Q_{XX} - C
\end{pmatrix} \Rightarrow_N \begin{pmatrix}
0 \\
0 \\
\Phi_1 & \Phi_{12} & \Phi_{13} \\
\Phi_{12} & \Psi & \tau \\
\Phi_{13} & \tau & \Upsilon
\end{pmatrix},
\]

(A.1)

where \(\sigma_i^2 = \mathbb{E} e_i^2, \eta_i^2 = \mathbb{E} V_i^2, \gamma_i = \mathbb{E} e_i V_i, \omega_i = \sum_{j \neq i} P_{ij} \Pi_j, \)

\[
\Phi_1 = \lim_{n \to \infty} \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \sigma_i^2 \sigma_j^2,
\]

\[
\Phi_{12} = \lim_{n \to \infty} \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\gamma_j \sigma_i^2 + \gamma_i \sigma_j^2),
\]

\[
\Phi_{13} = \lim_{n \to \infty} \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \gamma_i \gamma_j,
\]

\[
\Psi = \lim_{n \to \infty} \left[ \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\eta_i^2 \sigma_j^2 + \gamma_i \gamma_j) + \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \sigma_i^2 \right],
\]

\[
\tau = \lim_{n \to \infty} \left[ \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \gamma_j + \frac{2}{K} \sum_{i \in [n]} \omega_i^2 \gamma_i \right], \text{ and}
\]

\[
\Upsilon = \lim_{n \to \infty} \left[ \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \eta_j^2 + \frac{4}{K} \sum_{i \in [n]} \omega_i^2 \eta_i^2 \right].
\]

We note that the standard estimators of the above variance components proposed by Crudu et al. (2021) are equal to Chao et al.’s (2012) estimators with their residual \(\hat{e}_i\) replaced by \(e_i(\beta_0)\). Specifically, let

\[
\tilde{\Phi}_1(\beta_0) = \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 e_i^2(\beta_0) e_j^2(\beta_0),
\]
Then, Mikusheva and Sun (2022) consider the cross-fit estimators for $\Phi$ as

$$
\hat{\Phi}_{12}(\beta_0) = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 (X_j e_j(\beta_0)) e_i^2(\beta_0) + X_i e_i(\beta_0) e_j^2(\beta_0),
$$

$$
\hat{\Phi}_{13}(\beta_0) = \frac{2}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 X_i e_i(\beta_0) X_j e_j(\beta_0),
$$

$$
\hat{\Psi}(\beta_0) = \frac{1}{K} \sum_{i\in[n]} (\sum_{j\neq i} P_{ij} X_j)^2 e_i^2(\beta_0) + \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 X_i e_i(\beta_0) X_j e_j(\beta_0)),
$$

$$
\hat{\tau}(\beta_0) = \frac{1}{K} \sum_{i\in[n]} (\sum_{j\neq i} P_{ij} X_j)^2 X_i e_i(\beta_0) + \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 X_i^2 X_j e_j(\beta_0), \quad \text{and}
$$

$$
\hat{\Upsilon} = \frac{2}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 X_i^2 X_j^2.
$$

**Assumption 5.** Suppose $\max_{i\in[n]} |\Pi_i| \leq C$, $p_n^{1/4} = o(1)$, and $\mathbb{E}(e_i^6 + V_i^6) < \infty$, where $p_n = \max_{i\in[n]} P_{ii}$.

Two remarks on Assumption 5 are in order. First, $\max_{i\in[n]} |\Pi_i| \leq C$ is mild because $\Pi_i = \mathbb{E}X_i$. Second, Assumption 5 allows for weak identification when $\Pi^\top \Pi / \sqrt{K} \to c$ for a constant $c$. It also allows for strong identification when $\Pi^\top \Pi / \sqrt{K} \to \infty$. In this case, if $K/n \to 0$ so that $p_n = o(1)$, we allow $\Pi^\top \Pi / K \to c$ for a positive constant $c$. Otherwise, if $K$ is proportional to $n$, Assumption 5 requires $\Pi^\top \Pi / K \to 0$. Such a restriction is needed because Assumption 2 includes the case of fixed alternatives (i.e., fixed $\Delta \neq 0$), which is not considered in Crudu et al. (2021) and Chao et al. (2012).

**Theorem A.1.** Suppose Assumptions 4 and 5 hold. Then Assumption 2 holds for Crudu et al.’s (2021) estimators defined above.

### A.2 Cross-Fit Estimators

Let $M = I - P$, $M_{ij}$ be the $(i, j)$ element of $M$, $M_i$ be the $i$th row of $M$, and $\tilde{P}_{ij}^2 = \frac{P_{ij}^2}{M_i M_{ij} + M_{ij}^2}$. Then, Mikusheva and Sun (2022) consider the cross-fit estimators for $\Phi(\beta_0)$, $\Psi(\beta_0)$, and $\Upsilon$ defined as

$$
\tilde{\Phi}_1(\beta_0) = \frac{2}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 [e_i(\beta_0) M_i e(\beta_0)] e_j(\beta_0) M_j e(\beta_0)],
$$

$$
\tilde{\Psi}(\beta_0) = \frac{1}{K} \left[ \sum_{i\in[n]} (\sum_{j\neq i} P_{ij} X_j)^2 e_i(\beta_0) M_i e(\beta_0) \right] M_{ii} + \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_i X_i e_i(\beta_0) M_j X_j e_j(\beta_0) \right], \quad \text{and}
$$

$$
\tilde{\Upsilon} = \frac{2}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 X_i(\beta_0) M_i X_j(\beta_0) X_j(\beta_0) M_j X_j.
$$
where \( X \) and \( e(\beta_0) \) are the column vectors that collect all \( X_i \) and \( e_i(\beta_0) \), respectively. Following their lead, we can construct the cross-fit estimators for the rest three elements in \( \gamma(\beta) \) as follows:

\[
\hat{\Phi}_{12}(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 (M_j X e_j(\beta_0) e_i(\beta_0) M_i e(\beta_0) + M_i X e_i(\beta_0) e_j(\beta_0) M_j e(\beta_0)),
\]

\[
\hat{\Phi}_{13}(\beta_0) = \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_i X e_i(\beta_0) M_j X e_j(\beta_0), \quad \text{and}
\]

\[
\hat{\tau}(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 (X_i M_i X) (M_j X e_j(\beta_0)) + \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 \left( \frac{e_i(\beta_0) M_i X}{2M_{ii}} + \frac{X_i M_i e(\beta_0)}{2M_{ii}} \right),
\]

**Assumption 6.** Suppose Assumption 5 holds. Further suppose that \( \Pi^\top \Pi / K \leq C n^2 / K \) for some constant \( C > 0 \).

Compared with the assumptions in Mikusheva and Sun (2022), Assumption 6 further requires that \( \max_{i \in [n]} |\Pi_i| \leq C \). However, it allows for the case that \( \Pi^\top \Pi / K \to c \), where \( c \) is a nonzero constant, as long as \( p_n = o(1) \), which is weaker than those in Mikusheva and Sun (2022) (e.g., Theorems 3 and 5 in their paper require \( \Pi^\top \Pi / K \to 0 \) and \( \Pi^\top \Pi / K^{2/3} \to 0 \), respectively, for the consistency of the cross-fit variance estimators).

**Lemma A.1.** Suppose Assumptions 4 and 6 hold. Then, Lemmas 2, 3, S3.1, S3.2 in Mikusheva and Sun (2022) hold.

**Theorem A.2.** Suppose Assumptions 4 and 6 hold. Then, Assumption 2 holds for Mikusheva and Sun’s (2022) cross-fit estimators defined above.

## B Details for Simulations Based on Calibrated Data

The DGP contains only the intercept as the control variable. Therefore, we implement our jackknife CLC test on the demeaned version of \((\bar{y}_i, \bar{s}_i, Z_i)\). The parameter space is \( B = [-0.5, 0.5] \). We test the null hypothesis that \( \beta = \beta_0 \) for \( \beta_0 = 0.1 \) while varying the true value \( \beta \) over 30 equal-spaced grids over \( B \). The grids for \( \delta \) is the grid for \( \beta \) minus \( \beta_0 \). We generate grids of \((a_1, a_2)\) as \( a_1 = \sin^2(t_1) \) and \( a_2 = \cos^2(t_1) \sin^2(t_2) \) with \( t_1 \) taking values over 15 equal-spaced grids over \([a^{1/2}, a^{1/2}] / \pi / 2\] and \( t_2 \) taking values over 15 equal-spaced grids over \([0, \pi / 2]\). We gauge \( E^* \phi_{a_1, a_2, s}(\delta, \bar{D}, \hat{\gamma}(\beta_0)) \) via a Monte Carlo integration with \( N = 2000 \) draws of independent standard normal random variables. In practice, it is rare but possible that \( A_s(\bar{D}, \hat{\gamma}(\beta_0)) \) defined in (3.6) is not unique. To increase numerical stability, we follow I.Andrews (2016) and allow for some slackness in the minimization. Let \( G_a \) be the grid of \((a_1, a_2)\) mentioned above, \( \tilde{Q}(a_1, a_2) = \sup_{\delta \in \mathcal{A}(\beta_0)} (P_{\delta, s}(\bar{D}, \hat{\gamma}(\beta_0)) - E^* \phi_{a_1, a_2, s}(\delta, \bar{D}, \hat{\gamma}(\beta_0))) \), \( \tilde{Q}_{\min} = \min_{(a_1, a_2) \in G_a} \tilde{Q}(a_1, a_2) + 1/n \), where \( n \) is the sample size, and

\[
\Xi = \{(a_1, a_2) \in G_a : \tilde{Q}(a) \leq \tilde{Q}_{\min} + (\tilde{Q}_{\min}(1 - \tilde{Q}_{\min}))^{1/2} (2 \log(\log(N)))^{1/2} N^{-1/2} \}.
\]
The slackness term in the definition of Ξ is due to the law of the iterated logarithm for sum of
Bernoulli random variables and captures the randomness of the Monte Carlo integration. Suppose
there are \( L \) elements in Ξ, which are denoted as \( \{(a_{1,l}, a_{2,l})\}_{l=1}^{L} \). We then define \( \mathcal{A}_k(\hat{D}, \hat{\gamma}(\beta_0)) \) as \( (a_{1,[L/2]}, a_{2,[L/2]}) \). We use the cross-fit estimators defined in Section A.2 throughout the simulation.

C Details for Empirical Application

We consider the 1980s census of 329,509 men born in 1930-1939 based on Angrist and Krueger’s
(1991) dataset. The model for 180 instruments follows Mikusheva and Sun (2022), which can be
written explicitly as

\[
\ln W_i = \text{Constant} + H_i^\top \zeta + \sum_{c=30}^{38} YOB_{i,c} \xi_c + \sum_{s \neq 56} POB_{i,s} \eta_s + \beta E_i + \gamma_i
\]

\[
E_i = \text{Constant} + H_i^\top \lambda + \sum_{c=30}^{38} YOB_{i,c} \mu_c + \sum_{s \neq 56} POB_{i,s} \alpha_s
\]

\[
+ \sum_{j=1}^{3} \sum_{s \neq 56} QOB_{i,j} POB_{i,s} \delta_{c,s} + \sum_{j=1}^{39} \sum_{c=30}^{39} QOB_{i,j} YOB_{i,c} \theta_{j,c} + \epsilon_i,
\]

where \( W_i \) is the weekly wage, \( E_i \) is the education of the \( i \)-th individual, \( H_i \) is a vector of covariates,\(^{11}\) \( YOB_{i,c} \) is a dummy variable indicating whether the individual was born in year \( c = \{30,31,\ldots,39\} \),
while \( QOB_{i,j} \) is a dummy variable indicating whether the individual was born in quarter-of-birth \( j \in \{1,2,3,4\} \). \( POB_{i,s} \) is the dummy variable indicating whether the individual was born in state \( s \in \{51 \text{ states}\}.\(^{12}\) The coefficient \( \beta \) is the return to education. We vary this \( \beta \) across 10,000 equidistant grid-points from -0.5 to 0.5 (i.e., \( \beta \in \{-0.5, -4.9999, -4.9998, \ldots, 0, \ldots, 4.9999, 0.5\} \))
and solve for the range of \( \beta \) where the null hypothesis cannot be rejected. Specifically, we can write the
above model as

\[
\ln W_i = C_i^\top \Gamma + \beta E_i + \gamma_i
\]

\[
E_i = C_i^\top + Z_i \Theta + \epsilon_i,
\]

where \( C_i \) is a (329,509×71)-matrix of controls containing the first four terms on the right-hand
of the first equation, while \( Z_i \) is the (329,509×180)-matrix of instruments containing the first two
terms in the third line. We can then partial out the controls \( C_i \) by multiplying each equation by

---

\(^{11}\)The covariates we consider are: RACE, MARRIED, SMSA, NEWENG, MIDATL, ENOCENT, WNOCENT, SOATL, ESOCENT, WSOCENT, and MT.

\(^{12}\)The state numbers are from 1 to 56, excluding (3,7,14,43,52), corresponding to U.S. state codes.
the residual matrix $I - C(C^\top C)^{-1}C^\top$ to obtain a form analogous to that in the main text:

$$Y_i = X_i \beta + e_i,$$

$$X_i = \Pi_i + v_i.$$  

Then, at each grid-point we take $\beta_0 = \beta$ and compute $AR(\beta_0)$, $LM(\beta_0)$, $Wald(\beta_0)$, $\hat{\phi}_{App}(\hat{D}, \hat{\gamma}(\beta_0))$ and $\hat{\phi}_{Akr}(\hat{D}, \hat{\gamma}(\beta_0))$. We reject the chosen value of $\beta_0$ for $AR(\beta_0)$ if it exceeds the one-sided 5%-quantile of the standard normal (i.e., reject if $AR(\beta_0) > z_{0.05}$). If $LM(\beta_0)^2 > C_{0.05}$, we reject the chosen $\beta_0$ for Jackknife LM. If $Wald(\beta_0) > C_{0.05}$, we reject for JIVE-t. If $Wald(\beta_0) > C_{0.05}$, we reject for JIVE-t. If $\hat{\phi}_{Akr}(\hat{D}, \hat{\gamma}(\beta_0)) > C_{0.05}$, we reject accordingly. The two-step procedure depends on the value of $\tilde{F}$. If $\tilde{F} > 9.98$, we reject if $Wald(\beta_0) > C_{0.02}$; otherwise if $\tilde{F} \leq 9.98$, we reject if $AR(\beta_0) > z_{0.02}$.

The model for 1,530 instruments can be written explicitly as

$$\ln W_i = \text{Constant} + H_i^\top \zeta + \sum_{c=30}^{38} YOB_{i,c} \xi_c + \sum_{s \neq 56} POB_{i,s} \eta_s + \beta E_i + \gamma_i.$$  

$$E_i = \text{Constant} + H_i^\top \lambda + \sum_{c=30}^{38} YOB_{i,c} \mu_c + \sum_{s \neq 56} POB_{i,s} \alpha_s$$  

$$+ \sum_{j=1}^{3} \sum_{c=30}^{39} \sum_{s \in \{51 \text{ states}\}} QOB_{i,j} \ YOB_{i,c} \ POB_{i,s} \delta_{j,c,s}.$$  

The main difference between this 1,530-instrument specification and the 180-instrument one is that we now have QOB-YOB-POB interactions as our instruments, compared with QOB-YOB and QOB-POB interactions in the case of 180 instruments. Note that in both cases, only quarter-of-birth 1–3 are used; quarter 4 is omitted in order to avoid multicollinearity.

D Proof of Lemma 2.1

Under strong identification, by (2.3) and Assumption 2, we have

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & d_n
\end{pmatrix}
\begin{pmatrix}
Q_{ee} \\
Q_{Xe} \\
Q_{XX}
\end{pmatrix} \to \mathcal{N}
\begin{pmatrix}
0 \\
0 \\
\tilde{c}
\end{pmatrix},
\begin{pmatrix}
\Phi_1 & \Phi_{12} & 0 \\
\Phi_{12} & \Psi & 0 \\
0 & 0 & 0
\end{pmatrix},$$

In addition, we note that $e_i(\beta_0) = e_i + X_i \Delta$ with $\Delta = d_n \tilde{\Delta} \to 0$. Therefore, Under strong
identification, we have \( \mathcal{C} \Delta = \mathcal{C} \Delta \),

\[
Q_{e(\beta_0)e(\beta_0)} = Q_{ee} + 2\Delta Q_{Xe} + \Delta^2 Q_{XX} = Q_{ee} + o_p(1),
\]

\[
Q_{Xe(\beta_0)} = Q_{Xe} + \Delta Q_{XX} = Q_{Xe} + \mathcal{C} \Delta + o_p(1).
\]

This implies

\[
\begin{pmatrix}
AR(\beta_0) \\
LM(\beta_0)
\end{pmatrix} = \begin{pmatrix}
Q_{e(\beta_0)e(\beta_0)}/\hat{\Phi}_1^{1/2} \\
Q_{Xe(\beta_0)}/\hat{\Psi}_1^{1/2}
\end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix}
0 \\
\mathcal{C} \Delta
\end{pmatrix}, \begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix} \right).
\]

\section*{E Proof of Lemma 2.2}

Recall \( N_2^* = (1 - \rho^2)^{-1/2}(N_2 - \rho N_1) \) and

\[
\begin{pmatrix}
N_1 \\
N_2^*
\end{pmatrix} \overset{d}{=} \mathcal{N} \left( \begin{pmatrix}
0 \\
\frac{\theta}{(1-\rho^2)^{1/2}}
\end{pmatrix}, \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \right).
\]

Because \( \rho \) is known, it suffices to construct the uniformly most powerful invariant test based on observations \( (N_1, N_2^*) \). As the null and alternative are invariant to sign changes, the maximum invariant is \( (N_1, N_2^*) \). Then, \textit{Lehmann and Romano (2006, Theorem 6.2.1)} implies the invariant test should be based on the maximum invariant. Note \( (N_1, N_2^*) \) are independent, \( N_1 \) follows a standard normal distribution, and \( N_2^* \) follows a noncentral chi-square distribution with one degree of freedom and noncentrality parameter \( \lambda = \frac{\theta^2}{1-\rho^2} \). Therefore, by the Neyman-Pearson’s Lemma (\textit{Lehmann and Romano (2006, Theorem 3.2.1)}), the most powerful test based on observations \( (N_1, N_2^*) \) is the likelihood ratio test where the likelihood ratio function evaluated at \( (N_1 = \ell_1, N_2^* = \ell_2) \) depends on \( \ell_2 \) only and can be written as

\[
LR(\ell_2; \lambda) = -\frac{\lambda}{2} + \log \left( \frac{\exp(\sqrt{\lambda \ell_2}) + \exp(-\sqrt{\lambda \ell_2})}{2} \right).
\]

In addition, we note that \( LR(\ell_2; \lambda) \) is monotone increasing in \( \ell_2 \) for any \( \lambda \geq 0 \) and \( \ell_2 \geq 0 \). Therefore, \textit{Lehmann and Romano (2006, Theorem 3.4.1)} implies the likelihood ratio test is equivalent to \( 1\{N_2^* \geq \mathcal{C}_\alpha\} \), which is uniformly most powerful among tests for \( \lambda = 0 \) v.s. \( \lambda > 0 \) and based on observations \( (N_1, N_2^*) \) only. This means it is also the uniformly most powerful test that is invariant to sign changes.
F Proof of Lemma 2.3

Under strong identification and fixed alternatives, because 
\((Q_{e(\beta_0)e(\beta_0)} - \Delta^2 C, Q_{Xe(\beta_0)} - \Delta C, Q_{XX} - C)^\top = O_p(1)\), we have

\[
\left( \frac{d_nAR(\beta_0)}{d_nLM(\beta_0)} \right) \xrightarrow{p} \left( \frac{\Delta^2 \tilde{C}}{\Phi_{1/2}(\beta_0) \Delta C}, \frac{\Delta C}{\Psi^{1/2}(\beta_0)} \right).
\]

This implies

\[
d_{n}LM^*(\beta_0) \xrightarrow{p} \frac{1}{(1 - \rho^2(\beta_0))^{1/2}} \left( \frac{\Delta \tilde{C}}{\Psi^{1/2}(\beta_0)} - \frac{\rho(\beta_0)\Delta^2 \tilde{C}}{\Phi_{1/2}^{1/2}(\beta_0)} \right),
\]

which leads to the desired result.

G Proof of Lemma 2.4

Under weak identification, (2.3) implies

\[
\left( \frac{Q_{e(\beta_0)e(\beta_0)}}{Q_{Xe(\beta_0)}} \right) = \left( \frac{Q_{ee} + 2\Delta Q_{Xe} + \Delta^2 Q_{XX}}{Q_{Xe} + \Delta Q_{XX}} \right) \sim \mathcal{N} \left( \left( \frac{\Delta^2 C}{\Delta C} \right) ; \left( \frac{\Phi_1(\beta_0)}{\Phi_{12}(\beta_0)} \right) \right),
\]

which leads to the first result.

For the second result, it is obvious that \(m_1(\Delta) \to C \Upsilon^{-1/2}\). In addition, we have

\[
m_2(\Delta) = \frac{C (\Delta \Phi_1(\beta_0) - \Delta^2 \Phi_{12}(\beta_0))}{\Phi_1(\beta_0)(\Phi_1(\beta_0)\Psi(\beta_0) - \Phi_{12}^2(\beta_0)))^{1/2}} \to \frac{C}{\Upsilon^{1/2} (1 - \rho_{23}^2)^{1/2}},
\]

where we use the fact that

\[
\Phi_1(\beta_0)/\Delta^4 \to \Upsilon,
(\Phi_1(\beta_0)\Psi(\beta_0) - \Phi_{12}^2(\beta_0))/\Delta^4 \to \Upsilon \Psi - \tau^2,
\Phi_1(\beta_0) - \Delta \Phi_{12}(\beta_0)/\Delta^3 \to \tau.
\]

H Proof of Theorem 2.1

Theorem 2.1(i) is a direct consequence of Marden (1982, Theorem 2.1) because the acceptance region \(\mathcal{A} = \{(A, B) : s_1A^2 + s_2B^2 \leq C_\alpha(a_1, a_2; \rho(\beta_0))\} \) is closed, convex, and monotone decreasing.
in the sense that if \((A, B) \in \mathcal{A}\) and \(A' \leq A, B' \leq B\), then \((A', B') \in \mathcal{A}\).

For Theorem 2.1(ii), we note that \(\tilde{\rho} = \rho\) under local alternatives and

\[
\phi_{a_1, a_2, \infty} = 1 \left\{ (a_1 + a_2 \rho^2) N_1^2 \geq \mathbb{C}_1(a_1, a_2; \rho) \right\}.
\]

The “if” part of Theorem 2.1(ii) is a direct consequence of Lemma 2.2. The “only if” part of Theorem 2.1(ii) is a direct consequence of the necessary part of Lehmann and Romano (2006, Theorem 3.2.1). Specifically, given \(\mathcal{N}_1\) and \(\mathcal{N}_2\) are independent, the “only if” part requires \(a_1 + a_2 \rho^2 = 0\), which implies \(a_1 = 0\) and \(a_2 \rho = 0\).

For Theorem 2.1(iii), we consider two cases of fixed alternatives: (1) \(\Delta \neq \Phi_{\beta}(\beta_0)\Psi^{-1/2}(\beta_0)\rho^{-1}(\beta_0)\) and (2) \(\Delta = \Phi_{\beta}(\beta_0)\Psi^{-1/2}(\beta_0)\rho^{-1}(\beta_0)\). In Case (1), by Lemma 2.3, the limits of \(d_n^2 AR^2(\beta_0), d_n^2 LM^2(\beta_0), d_n^2 LM^*^2(\beta_0)\) are all positive, implies that for all \((a_{1, n}, a_{2, n}) \in \mathcal{A}_0\)

\[
1 \{ a_{1, n} AR^2(\beta_0) + a_{2, n} LM^2(\beta_0) + (1 - a_{1, n} - a_{2, n}) LM^*^2(\beta_0) \geq \mathbb{C}_1(a_{1, n}, a_{2, n}; \tilde{\rho}(\beta_0)) \} \xrightarrow{p} 1.
\]

In Case (2), we have

\[
\mathbb{P} \left( a_{1, n} AR^2(\beta_0) + a_{2, n} LM^2(\beta_0) + (1 - a_{1, n} - a_{2, n}) LM^*^2(\beta_0) \geq \mathbb{C}_1(a_{1, n}, a_{2, n}; \tilde{\rho}(\beta_0)) \right)
\geq \mathbb{P} \left( \frac{\tilde{\rho}^2(\beta_0) \Phi_{\beta}(\beta_0)}{C^2 \tilde{\Phi}_{\beta}(\beta_0)} d_n^2 AR^2(\beta_0) \geq \mathbb{C}_1(a_{1, n}, a_{2, n}; \tilde{\rho}(\beta_0)) \right)
= \mathbb{P} \left( \tilde{q} + o_p(1) \geq \mathbb{C}_{\alpha, \max}(\rho(\beta_0)) \right) \xrightarrow{p} 1,
\]

where the first inequality follows from the restriction on \(a_{1, n}\) and the facts that \(LM^2(\beta_0) \geq 0\) and \(LM^*^2(\beta_0) \geq 0\), the first equality follows from \(d_n^2 AR^2(\beta_0) \xrightarrow{p} \Phi_{\beta}(\beta_0)\Delta^2(\beta_0)\Phi_{\beta}(\beta_0)\) (by Lemma 2.3) and \(\tilde{\rho}(\beta_0) \xrightarrow{p} \rho(\beta_0)\), and the last convergence follows from the fact that \(\tilde{q} > \mathbb{C}_{\alpha, \max}(\rho(\beta_0))\). This concludes the proof.

## I Proof of Theorem 4.1

We are under weak identification. By Lemma 2.4 and Assumption 2, we have

\[
\begin{pmatrix}
AR(\beta_0) \\
LM^*(\beta_0) \\
\hat{D}
\end{pmatrix}
\sim N
\begin{pmatrix}
\left( \begin{array}{c}
m_1(\Delta) \\
m_2(\Delta) \\
\mu_D
\end{array} \right) & \left( \begin{array}{c}
0 & 0 & 0 \\
0 & 0 & \sigma_D^2
\end{array} \right)
\end{pmatrix}.
\]

This implies \((AR(\beta_0), LM^*(\beta_0), \hat{D})\) are asymptotically independent. In addition, by Assumption 3, we have

\[
(AR^2(\beta_0), LM^*^2(\beta_0), A_s(\hat{D}, \hat{\gamma}(\beta_0))) \sim (\mathbb{Z}^2(m_1(\Delta)), \mathbb{Z}^2(m_2(\Delta)), A_s(\hat{D}, \gamma(\beta_0)))
\]

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where the two normal random variables are independent and independent of $D$, and by definition, 
$A_s(D, \gamma(\beta_0)) = (a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)))$. In addition, we have $\hat{\rho}(\beta_0) \xrightarrow{p} \rho(\beta_0)$. By the bounded convergence theorem, this further implies

$$
\mathbb{E}\hat{\phi}_{A_s(D, \gamma(\beta_0))} \rightarrow \mathbb{E}(\phi_{a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))}, (\Delta, \mu_D, \gamma(\beta_0)).
$$

(I.1)

In addition, suppose the null holds so that $\Delta = 0$. This implies $m_1(\Delta) = m_2(\Delta) = 0$. Then, we have

$$(\hat{\phi}_{A_s(D, \gamma(\beta_0))} - \alpha)f(\hat{D}) \sim (\phi_{a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))}, (0, \mu_D, \gamma(\beta_0)) - \alpha)f(D),$$

where

$$
\phi_{a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))}, (0, \mu_D, \gamma(\beta_0))
= \begin{cases}
  a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))Z_1^2 + a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))(\rho(\beta_0)Z_1 + (1 - \rho^2(\beta_0))^{1/2}Z_2) \\
  (1 - a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)) - a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0) ))Z_2^2 \\
  \geq C_\alpha(a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)), a_2(f_s(D, \gamma(\beta_0)), \gamma(\beta_0)); \rho(\beta_0))
\end{cases},
$$

$Z_1$ and $Z_2$ are independent standard normals, and they are independent of $D$. Then, by the definition of $C_\alpha(\cdot)$, we have

$$
\mathbb{E}(\hat{\phi}_{A_s(D, \gamma(\beta_0))} - \alpha)f(\hat{D}) \rightarrow \mathbb{E}\left[\mathbb{E}(\phi_{a_1(f_s(D, \gamma(\beta_0)), \gamma(\beta_0))}, (0, \mu_D, \gamma(\beta_0)) - \alpha|D) f(D)\right] = 0.
$$

**J Proof of Theorem 4.2**

Denote $c_B = c_B(\beta)$ and $\Delta_\ast = \Delta_\ast(\beta)$. By Assumption 2, $\Phi_1 > 0$, which implies $|\Delta_\ast| > 0$. Under strong identification and local alternatives, we have $\Delta \rightarrow 0$, $c_B(\beta_0) \rightarrow c_B$, $\Delta_\ast(\beta_0) \rightarrow \Delta_\ast$, $C_{\alpha, \max}(\rho(\beta_0)) \rightarrow C_{\alpha, \max}(\rho)$, and

$$
\begin{pmatrix}
  AR(\beta_0) \\
  LM^r(\beta_0) \\
  d_n\hat{D}
\end{pmatrix} 
\sim \mathcal{N}\left(\begin{pmatrix}
  0 \\
  \frac{\Delta^2}{(1-\rho^2)^{1/2}} \\
  \frac{\bar{C}}{C}
\end{pmatrix}, \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 0
\end{pmatrix}\right).
$$

This implies $d_n\hat{\sigma}_D\sqrt{\hat{r}} = d_n\hat{D} \xrightarrow{p} \bar{C}$, which further implies $d_nf_{pp}(\hat{D}, \gamma(\beta_0)) \xrightarrow{p} \bar{C}$. For $f_{krs}(\hat{D}, \gamma(\beta_0))$, we note that

$$
\max(\hat{r} - 1, 0) \leq \hat{r}_{krs} \leq \hat{r}.
$$
Therefore, we also have $f_{k,s}(\tilde{D},\tilde{\gamma}(\beta_0))d_n \overset{p}{\to} \tilde{C}$. Let $\mathcal{E}_n(\varepsilon) = \{|\tilde{\gamma}(\beta_0) - \gamma(\beta_0)| + |\delta_n\tilde{D} - \tilde{C}| \leq \varepsilon\}$. Then, for an arbitrary $\varepsilon > 0$, we have $\mathbb{P}(\mathcal{E}_n(\varepsilon)) \geq 1 - \varepsilon$ when $n$ is sufficiently large.

Denote $\delta = d_n\tilde{\delta}$. We have

$$A_s(\tilde{D},\tilde{\gamma}(\beta_0)) \in \arg\min_{(a_1,a_2) \in \mathbb{A}(f_s(\tilde{D},\tilde{\gamma}(\beta_0)),\tilde{\gamma}(\beta_0))} \sup_{\delta \in \tilde{D}_n} \left( p_{a_1,a_2,s}(\tilde{D},\tilde{\gamma}(\beta_0)) - E^*\phi_{a_1,a_2,s}(d_n\tilde{\delta},\tilde{D},\tilde{\gamma}(\beta_0)) \right),$$

where $\tilde{D}_n = \{\tilde{\delta} : d_n\tilde{\delta} \in D(\beta_0)\}$. Let

$$Q_n(a_1,a_2,\tilde{\delta}) = p_{a_1,a_2,s}(\tilde{D},\tilde{\gamma}(\beta_0)) - E^*\phi_{a_1,a_2,s}(d_n\tilde{\delta},\tilde{D},\tilde{\gamma}(\beta_0)) \quad \text{and} \quad Q(a_1,a_2,\tilde{\delta}) = \mathbb{E}\{Z_2^2(1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \geq C_\alpha \}
$$

$$-\mathbb{E}\left\{ a_1Z_1^2 + a_2\left(\rho Z_1 + (1 - \rho^2)^{1/2}Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \right)^2 \right\},$$

where $Z_1$ is standard normal, $Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C})$ is normal with mean $(1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}$ and unit variance, and $Z_1$ and $Z_2(\cdot)$ are independent. Then, we aim to show that

$$\sup_{(a_1,a_2) \in \mathbb{A}(f_s(\tilde{D},\tilde{\gamma}(\beta_0)),\tilde{\gamma}(\beta_0)),\tilde{\delta} \in \tilde{D}_n} |Q_n(a_1,a_2,\tilde{\delta}) - Q(a_1,a_2,\tilde{\delta})| \overset{p}{\to} 0. \quad (J.1)$$

We divide $\tilde{D}_n$ into three parts:

$$\tilde{D}_{n,1}(\varepsilon) = \{\tilde{\delta} \in \tilde{D}_n, |\tilde{\delta}| \leq M_1(\varepsilon)\},$$

$$\tilde{D}_{n,2}(\varepsilon) = \left\{\tilde{\delta} \in \tilde{D}_n, \frac{d_n\tilde{\delta}}{\Delta_s(\beta_0)} - 1 \leq \varepsilon \right\}, \quad \text{and}$$

$$\tilde{D}_{n,3}(\varepsilon) = \tilde{D}_n \cap \tilde{D}_{n,1}(\varepsilon) \cap \tilde{D}_{n,2}(\varepsilon),$$

where $M_1(\varepsilon)$ is a large constant so that

$$\mathbb{P}\left((1 - \bar{\alpha})Z^2 \left(\frac{M_1^2(\varepsilon)\varepsilon^2\tilde{C}^2}{2(1 - \rho^2)\Psi c_B} \right) \geq C_{\alpha,\max}(\rho) + 1\right) = 1 - \varepsilon. \quad (J.2)$$

When $n$ is sufficiently large and $\varepsilon$ is sufficiently small, on $\mathcal{E}_n(\varepsilon)$, there exists a constant $c$ such that

$$|\hat{\Delta}_s(\beta_0) - \Delta_s| \leq c\varepsilon, \quad \inf_{\tilde{\delta} \in \tilde{D}_{n,2}(\varepsilon)} |d_n\tilde{\delta}| \geq (1 - \varepsilon)(|\Delta_s| - c\varepsilon),$$

$$|\hat{\Phi}_1(\beta_0) - \Phi_1| \leq c\varepsilon, \quad |d_n^2f_s^2(\tilde{D},\tilde{\gamma}(\beta_0)) - \tilde{C}^2| \leq c\varepsilon,$$
\[
\sup_{\delta \in \bar{D}_{n,2}(\varepsilon)} \left[ 1 - (d_n^2 \delta^2, d_n \delta) \left( \begin{pmatrix} \Phi_1(\beta_0) & \Phi_1(\beta_0) \\ \Phi_1(\beta_0) & \Phi(\beta_0) \end{pmatrix}^{-1} \begin{pmatrix} \Phi_1(\beta_0) \\ \Phi(\beta_0) \end{pmatrix} \right) \right]^2 \\
\leq \left[ 1 - (\Delta_s^2, \Delta_s) \left( \begin{pmatrix} \Phi_1 & \Phi_1 \\ \Phi_1 & \Phi \end{pmatrix}^{-1} \begin{pmatrix} \Phi \\ \tau(\beta_0) \end{pmatrix} \right) \right]^2 + c\varepsilon \leq c_B + c\varepsilon,
\]

This further implies

\[
\bar{D}_{n,1}(\varepsilon) \cap \bar{D}_{n,2}(\varepsilon) = \emptyset.
\]

Recall \( \phi_{a_1,a_2,s}(\delta, \hat{D}, \hat{\gamma}(\beta_0)) \) defined in (3.5). With \( \delta \) replaced by \( d_n \delta \) and when \( \tilde{\delta} \in \bar{D}_{n,1}(\varepsilon) \), we have

\[
\begin{pmatrix} d_n^{-1} \hat{C}_1(d_n \delta) \\ d_n^{-1} \hat{C}_2(d_n \delta) \end{pmatrix} (d_n f_s(\hat{D}, \hat{\gamma}(\beta_0))) \to 0 \begin{pmatrix} 1 - \rho^2 - 1/2 \Psi^{-1/2} \delta \hat{C} \end{pmatrix},
\]

Therefore, uniformly over \((a_1, a_2) \in A_0 \) and \( \tilde{\delta} \in \bar{D}_{n,1}(\varepsilon) \) and conditional on data, we have

\[
\phi_{a_1,a_2,s}(d_n \delta, \hat{D}, \hat{\gamma}(\beta_0)) \to 1 \begin{cases} a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \delta \hat{C}) \right)^2 \\
+ (1 - a_1 - a_2) Z_2^2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \delta \hat{C}) \geq C_o(a_1, a_2; \rho) \end{cases}.
\]

This implies

\[
\sup_{(a_1, a_2) \in A_0, \tilde{\delta} \in \bar{D}_{n,1}(\varepsilon)} \left| \mathbb{E}^* \phi_{a_1,a_2,s}(d_n \delta, \hat{D}, \hat{\gamma}(\beta_0)) \right| \to 0.
\]

In addition, by Lemma 2.2, for any \( \tilde{\delta} \), \( \mathbb{E} \left\{ a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \delta \hat{C}) \right)^2 \\
+ (1 - a_1 - a_2) Z_2^2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \delta \hat{C}) \geq C_o(a_1, a_2; \rho) \right\} \]

is maximized at \( a_1 = 0 \) and \( a_2\rho = 0 \). This implies

\[
\sup_{\delta \in \bar{D}_{n,1}(\varepsilon)} \left| P_{d_n \delta, s}(\hat{D}, \hat{\gamma}(\beta_0)) - \mathbb{E}(Z_2^2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \delta \hat{C}) \geq C_o) \right| = \sup_{\delta \in \bar{D}_{n,1}(\varepsilon)} \left| \sup_{(a_1, a_2) \in A_0} \mathbb{E}^* \phi_{a_1,a_2,s}(d_n \delta, \hat{D}, \hat{\gamma}(\beta_0)) \right| = \sup_{\delta \in \bar{D}_{n,1}(\varepsilon)} \left| \sup_{(a_1, a_2) \in A_0} \mathbb{E}^* \phi_{a_1,a_2,s}(d_n \delta, \hat{D}, \hat{\gamma}(\beta_0)) \right|.
\]

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\[
\begin{align*}
\leq & \sup_{\delta \in \tilde{D}_{n,1}(\varepsilon)} \left( \sup_{(a_1,a_2) \in \mathcal{A}(f_s(\tilde{D},\tilde{\gamma}(\beta_0)),\tilde{\gamma}(\beta_0))} \mathbb{E}_1 \left\{ a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2} \Psi - 1/2 \tilde{\delta} C) \right)^2 \right\} \\
&+ (1 - a_1 - a_2) Z_2^2((1 - \rho^2)^{-1/2} \Psi - 1/2 \tilde{\delta} C) \geq C_1(a_1,a_2; \rho) \right\} \\
&- \mathbb{E}_1 \left\{ Z_2^2((1 - \rho^2)^{-1/2} \Psi - 1/2 \tilde{\delta} C) \geq C_1 \right\} + o_p(1),
\end{align*}
\]

where the second inequality is due to the facts that \( a(f_s(\tilde{D},\tilde{\gamma}(\beta_0)),\tilde{\gamma}(\beta_0)) = o_p(1) \) under strong identification and \( \mathbb{E}_1 \left\{ a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2} \Psi - 1/2 \tilde{\delta} C) \right)^2 \right\} \) is continuous at \( a_1 = 0 \) uniformly over \( |\tilde{\delta}| \leq M_1(\varepsilon) \). Therefore, we have

\[
\sup_{(a_1,a_2) \in \mathcal{A}(f_s(\tilde{D},\tilde{\gamma}(\beta_0)),\tilde{\gamma}(\beta_0))} \mathbb{E}_1 \left\{ Q_n(a_1,a_2,\tilde{\delta}) - Q(a_1,a_2,\tilde{\delta}) \right\} \overset{p}{\rightarrow} 0. \quad (J.4)
\]

Next, we consider the case when \( \tilde{\delta} \in \tilde{D}_{n,2}(\varepsilon) \). We have

\[
\phi_{a_1,a_2,s}(d_n\tilde{\delta},\tilde{D},\tilde{\gamma}(\beta_0)) = 1 \left\{ a_1 Z_1^2(\tilde{C}_1^2(d_n\tilde{\delta})f_s^2(\tilde{D},\tilde{\gamma}(\beta_0))) + a_2 \left( \tilde{\rho}(\beta_0) Z_1(\tilde{C}_1^2(d_n\tilde{\delta})f_s^2(\tilde{D},\tilde{\gamma}(\beta_0))) + (1 - \tilde{\rho}^2(\beta_0))^{1/2} Z_2(\tilde{C}_2^2(d_n\tilde{\delta})f_s^2(\tilde{D},\tilde{\gamma}(\beta_0))) \right)^2 \right\}
\]

and

\[
\begin{align*}
\phi_{a_1,a_2,s}(d_n\tilde{\delta},\tilde{D},\tilde{\gamma}(\beta_0)) &\leq \frac{\tilde{C}_1^2(d_n\tilde{\delta})(d_n f_s(\tilde{D},\tilde{\gamma}(\beta_0)))^2}{\tilde{\Phi}_1^{-1}(\beta_0)(d_n\tilde{\delta})^4(d_n f_s(\tilde{D},\tilde{\gamma}(\beta_0)))^2} \\
&\geq \frac{(\Phi_1(\beta_0) + c_\varepsilon)^{-1}(1 - \varepsilon)^{1/4}(|\Delta_2| - c_\varepsilon)^4(\tilde{C}_2^2 - c_\varepsilon)}{c_\varepsilon + c_\varepsilon} \geq c
\end{align*}
\]

By (J.3), on \( \mathcal{E}_n(\varepsilon) \), there exists a constant \( c > 0 \) such that

\[
\begin{align*}
\tilde{C}_1^2(d_n\tilde{\delta})(d_n f_s(\tilde{D},\tilde{\gamma}(\beta_0)))^2 &\geq \frac{\tilde{\Phi}_1^{-1}(\beta_0)(d_n\tilde{\delta})^4(d_n f_s(\tilde{D},\tilde{\gamma}(\beta_0)))^2}{\tilde{\Phi}_1(\beta_0) + c_\varepsilon)^{-1}(1 - \varepsilon)^{1/4}(|\Delta_2| - c_\varepsilon)^4(\tilde{C}_2^2 - c_\varepsilon)} \geq c
\end{align*}
\]

and

\[
\phi_{a_1,a_2,s}(d_n\tilde{\delta},\tilde{D},\tilde{\gamma}(\beta_0)) \tilde{C}_1^2(d_n\tilde{\delta})f_s^2(\tilde{D},\tilde{\gamma}(\beta_0))
\]
where the last inequality holds because \( \varepsilon \) can be arbitrarily small. This means, on \( \mathcal{E}_n(\varepsilon) \) and when \( \tilde{\delta} \in \tilde{D}_{n,2}(\varepsilon) \),

\[
\mathbb{E}^* \phi_{a_1,a_2,s}(d_n\tilde{\delta}, \hat{D}, \hat{\gamma}(\beta_0)) \geq \mathbb{P}^*(a_p(1) + (1 - c\varepsilon)C_{\alpha,\max}(\hat{\rho}(\beta_0)) \geq C_{\alpha,\max}(\hat{\rho}(\beta_0))) \to 1.
\]

As \( \mathbb{P}(\mathcal{E}_n(\varepsilon)) \to 1 \), we have

\[
\sup_{(a_1,a_2)\in\mathcal{A}(f_s(\hat{D},\hat{\gamma}(\beta_0)),\hat{\gamma}(\beta_0),\tilde{\delta}\in\tilde{D}_{n,2}(\varepsilon))} \left[ 1 - \mathbb{E}^* \phi_{a_1,a_2,s}(d_n\tilde{\delta}, \hat{D}, \hat{\gamma}(\beta_0)) \right] \xrightarrow{p} 0,
\]

and thus,

\[
\sup_{(a_1,a_2)\in\mathcal{A}(f_s(\hat{D},\hat{\gamma}(\beta_0)),\hat{\gamma}(\beta_0),\tilde{\delta}\in\tilde{D}_{n,2}(\varepsilon))} \left[ \mathcal{P}_{d_n\tilde{\delta},s}(\hat{D}, \hat{\gamma}(\beta_0)) - \mathbb{E}^* \phi_{a_1,a_2,s}(d_n\tilde{\delta}, \hat{D}, \hat{\gamma}(\beta_0)) \right] \leq \sup_{(a_1,a_2)\in\mathcal{A}(f_s(\hat{D},\hat{\gamma}(\beta_0)),\hat{\gamma}(\beta_0),\tilde{\delta}\in\tilde{D}_{n,2}(\varepsilon))} \left[ 1 - \mathbb{E}^* \phi_{a_1,a_2,s}(d_n\tilde{\delta}, \hat{D}, \hat{\gamma}(\beta_0)) \right] \xrightarrow{p} 0. \quad (J.5)
\]

Furthermore, note that \( a_1 + a_2 \leq \bar{a} < 1 \) and when \( \tilde{\delta} \in \tilde{D}_{n,2}(\varepsilon) \), on \( \mathcal{E}_n(\varepsilon) \), (J.3) implies \( \delta^2 \to \infty \). Therefore, we have

\[
a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \right)^2 + (1 - a_1 - a_2) Z_2^2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \geq (1 - \bar{a}) \delta^2 c^2 = \frac{(1 - \bar{a}) \delta^2 c^2}{(1 - \rho^2)\Psi} (1 + o_p(1)) \to \infty,
\]

which further implies

\[
\sup_{(a_1,a_2)\in\mathcal{A}(f_s(\hat{D},\hat{\gamma}(\beta_0)),\hat{\gamma}(\beta_0),\tilde{\delta}\in\tilde{D}_{n,2}(\varepsilon))} \left[ 1 - \mathbb{E} \left\{ a_1 Z_1^2 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \right)^2 \right\} + (1 - a_1 - a_2) Z_2^2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \geq C_{\alpha}(a_1,a_2;\rho) \right] \xrightarrow{p} 0
\]

and

\[
\sup_{(a_1,a_2)\in\mathcal{A}(f_s(\hat{D},\hat{\gamma}(\beta_0)),\hat{\gamma}(\beta_0),\tilde{\delta}\in\tilde{D}_{n,2}(\varepsilon))} \left[ \mathbb{E} \left\{ Z_2^2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\tilde{\delta}\tilde{C}) \geq C_{\alpha} \right\} \right]
\]
Combining (J.5) and (J.6), we have

\[ \sup_{(a_1, a_2) \in \mathcal{A}(f, D, \tilde{\gamma}(\beta_0)), \tilde{\delta} \in \tilde{D}_{n,2}(\varepsilon)} |Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta})| \to 0. \quad (J.7) \]

Last, we consider the case in which \( \tilde{\delta} \in \tilde{D}_{n,3}(\varepsilon) \). On \( \mathcal{E}_n(\varepsilon), (J.3) \) implies

\[
\tilde{C}_2^2(d_n \tilde{\delta}) f^2_s(\tilde{D}, \tilde{\gamma}(\beta_0)) = \frac{\tilde{\delta}^2(1 - \frac{d_n \tilde{\delta}}{\Delta_+ (\beta_0)})^2}{(1 - \tilde{\rho}^2(\beta_0)) \tilde{\Psi}(\beta_0)} \left[ 1 - (d_n^2 \tilde{\delta}^2, d_n \tilde{\delta}) \left( \begin{array}{cc} \tilde{\Phi}_1(\beta_0) & \tilde{\Phi}_12(\beta_0) \\ \tilde{\Phi}_12(\beta_0) & \tilde{\Psi}(\beta_0) \end{array} \right)^{-1} \left( \begin{array}{c} \tilde{\Phi}_13(\beta_0) \\ \tilde{\gamma}(\beta_0) \end{array} \right) \right] \geq \frac{(1 - c\varepsilon) M_1^2(\varepsilon) \varepsilon^2 \tilde{C}_2^2}{(1 - \rho^2) \Psi \Phi} \geq \frac{M_1^2(\varepsilon) \varepsilon^2 \tilde{C}_2^2}{2(1 - \rho^2) \Psi},
\]

where the second inequality holds when \( \varepsilon \) is sufficiently small. In this case,

\[
\mathbb{E}^{*} \phi_{a_1, a_2, s}(d_n \tilde{\delta}, \tilde{D}, \tilde{\gamma}(\beta_0)) \geq \mathbb{P}^{*}((1 - \tilde{\rho}) Z_2^2(\tilde{C}_2^2(d_n \tilde{\delta}) f^2_s(\tilde{D}, \tilde{\gamma}(\beta_0))) \geq C_{\alpha, \max}(\tilde{\rho}(\beta_0))) \geq \mathbb{P}^{*} \left( \frac{(1 - \tilde{\rho}) Z_2^2}{2(1 - \rho^2) \Psi} \geq C_{\alpha, \max}(\tilde{\rho}(\beta_0)) \right) \geq \mathbb{P}^{*} \left( \frac{(1 - \rho)}{Z_2^2} \geq \frac{M_1^2(\varepsilon) \varepsilon^2 \tilde{C}_2^2}{2(1 - \rho^2) \Psi} \geq C_{\alpha, \max}(\rho) + c\varepsilon \right) - \varepsilon \geq 1 - 2\varepsilon,
\]

where the second inequality is by the fact that the CDF (survival function) of \( Z_2^2(\lambda) \) is monotone decreasing (increasing) in \( |\lambda| \) and the last equality is by the definition of \( M_1(\varepsilon) \) in (J.2) and the fact that \( C_{\alpha, \max}(\tilde{\rho}(\beta_0)) \to \mathcal{C}_{\alpha, \max}(\rho) \). This implies, on \( \mathcal{E}_n(\varepsilon) \),

\[
\sup_{(a_1, a_2) \in \mathcal{A}(f, D, \tilde{\gamma}(\beta_0)), \tilde{\delta} \in \tilde{D}_{n,3}(\varepsilon)} \left[ \mathcal{P}_{d_n \tilde{\delta}, s}((\tilde{D}, \tilde{\gamma}(\beta_0)) - \mathbb{E}^{*} \phi_{a_1, a_2, s}(d_n \tilde{\delta}, \tilde{D}, \tilde{\gamma}(\beta_0))) \right] \leq 2\varepsilon. \quad (J.8)
\]

In addition, we note that \((1 - \rho^2)^{-1} \Psi^{-1} \tilde{\delta}^2 \tilde{C}_2^2\) satisfies

\[
(1 - \rho^2)^{-1} \Psi^{-1} \tilde{\delta}^2 \tilde{C}_2^2 \geq \frac{M_1^2(\varepsilon) \varepsilon^2 \tilde{C}_2^2}{2(1 - \rho^2) \Psi}.
\]
where we use the facts that $\bar{\delta}^2 \geq M^2_f(\varepsilon), c_B \geq 1$, and $\varepsilon < 1$. Therefore, by the same argument, we have

$$
\mathbb{E}_1 \left\{ a_1 Z^2_1 + a_2 \left( \rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2} \Psi^{-1/2} \bar{\delta} C) \right)^2 \right\} + (1 - a_1 - a_2) Z^2_2 \left( (1 - \rho^2)^{-1/2} \Psi^{-1/2} \bar{\delta} C \right) \geq C_\alpha(a_1, a_2; \rho)
$$

and

$$
\sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left[ \mathbb{E}_1 \{ Z^2_2 ((1 - \rho^2)^{-1/2} \Psi^{-1/2} \bar{\delta} C) \} \geq C_\alpha \right]
$$

Combining (J.8) and (J.9), we have, on $\mathcal{E}_n(\varepsilon)$,

$$
\sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| \leq 3\varepsilon.
$$

Combining (J.4), (J.7), and (J.10), we have

$$
\mathbb{P} \left( \sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| > 5\varepsilon \right)
\leq \mathbb{P} \left( \sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| > \varepsilon, \mathcal{E}_n(\varepsilon) \right)
+ \mathbb{P} \left( \sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| > \varepsilon, \mathcal{E}_n(\varepsilon) \right)
+ \mathbb{P} \left( \sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| > 3\varepsilon, \mathcal{E}_n(\varepsilon) \right) + \mathbb{P} \left( \mathcal{E}_n^c(\varepsilon) \right)
\leq o(1) + \varepsilon.
$$

Since $\varepsilon$ is arbitrary, we have

$$
\omega_n \equiv \sup_{(a_1, a_2) \in A(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), \tilde{\delta} \in \tilde{D}_n} \left| Q_n(a_1, a_2, \tilde{\delta}) - Q(a_1, a_2, \tilde{\delta}) \right| \xrightarrow{p} 0.
$$

Then we have

$$
0 \leq \sup_{\tilde{\delta} \in \tilde{D}_n} Q_n(a(f_s(\mathcal{D}, \mathcal{G}(\beta_0)), \mathcal{G}(\beta_0)), 0, \tilde{\delta}) - \sup_{\tilde{\delta} \in \tilde{D}_n} Q_n(a, \mathcal{G}(\beta_0), \tilde{\delta})
$$
\[
\begin{align*}
\leq \sup_{\tilde{\delta} \in \mathcal{D}_n} Q(a(f_s(\hat{D}, \hat{\gamma}(\beta_0)), \tilde{\gamma}(\beta_0)), 0, \tilde{\delta}) - \sup_{\tilde{\delta} \in \mathcal{D}_n} Q(A_s(\hat{D}, \hat{\gamma}(\beta_0)), \tilde{\delta}) + 2\omega_n \\
= o_p(1) - \sup_{\tilde{\delta} \in \mathcal{D}_n} Q(A_s(\hat{D}, \hat{\gamma}(\beta_0)), \tilde{\delta}) + 2\omega_n,
\end{align*}
\]

where the equality holds because (1) \(\sup_{\tilde{\delta} \in \mathbb{R}} Q(a_1, 0, \tilde{\delta})\) is continuous at \(a_1 = 0\) as shown in the proof of I. Andrews (2016, Theorem 5), (2) \(q(f_s(\hat{D}, \hat{\gamma}(\beta_0)), \tilde{\gamma}(\beta_0)) = o_p(1)\) under strong identification, and (3) \(\sup_{\tilde{\delta} \in \mathbb{R}} Q(0, 0, \tilde{\delta}) = 0\) by construction.

On the other hand, we have

\[
Q(a_1, a_2, \tilde{\delta}) = \mathbb{E}\{Z_2^2(1 - \rho^2)^{-1/2}\Psi^{-1/2}\delta\tilde{C}) \geq C_\alpha \}
-
\mathbb{E}\left\{a_1 Z_1^2 + a_2 \left(\rho Z_1 + (1 - \rho^2)^{1/2} Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\delta\tilde{C})\right)^2 \right\}
-
\mathbb{E}\{Z_2^2(1 - \rho^2)^{-1/2}\Psi^{-1/2}\delta\tilde{C}) \geq C_\alpha \}
-
\mathbb{E}\left\{(a_1 + a_2 \rho^2) Z_1^2 + a_2 \rho (1 - \rho^2)^{1/2} Z_1 Z_2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\delta\tilde{C}) \right\}
-
\mathbb{E}\left\{(1 - a_1 - a_2 \rho^2) Z_2^2((1 - \rho^2)^{-1/2}\Psi^{-1/2}\delta\tilde{C}) \geq C_\alpha (a_1, a_2; \rho) \right\}
\]

Note that \(a_1 = 0\) and \(a_2\rho = 0\) if and only if \(a_1 + a_2\rho^2 = 0\), given that \(a_1\) and \(a_2\) are nonnegative. Therefore, Theorem 2.1(ii) implies, for any constant \(C > 0\), there exists a constant \(c > 0\) such that

\[
\inf_{(a_1, a_2) \in \mathbb{R}_+, a_1 + a_2 \rho^2 \geq C} \sup_{\tilde{\delta} \in \mathcal{D}_n} Q(a_1, a_2, \tilde{\delta}) \geq c > 0.
\]

Therefore,

\[
\mathbb{P}\left(A_{1,s}(\hat{D}, \hat{\gamma}(\beta_0)) + A_{2,s}(\hat{D}, \hat{\gamma}(\beta_0))\rho^2 \geq C > 0\right) \leq \mathbb{P}(c \leq o_p(1) + 2\omega_n) \to 0.
\]

This implies \(A_{1,s}(\hat{D}, \hat{\gamma}(\beta_0)) \overset{p}{\to} 0\) and \(A_{2,s}(\hat{D}, \hat{\gamma}(\beta_0))\rho \overset{p}{\to} 0\).

**K Proof of Theorem 4.3**

We consider strong identification with fixed alternatives. By construction, we have \(A_{1,s}(\hat{D}, \hat{\gamma}(\beta_0)) \geq \frac{1.1\Delta_{\text{max}}(\hat{\rho}(\beta_0))\hat{\Phi}_1(\beta_0)\hat{\gamma}_E(\beta_0)}{\Delta^*_s(\beta_0) f^2_s(\hat{D}, \hat{\gamma}(\beta_0))}\). By Theorem 2.1(iii), it suffices to show that, w.p.a.1,

\[
\frac{1.1\Delta_{\text{max}}(\hat{\rho}(\beta_0))\hat{\Phi}_1(\beta_0)\hat{\gamma}_E(\beta_0)}{\Delta^*_s(\beta_0) f^2_s(\hat{D}, \hat{\gamma}(\beta_0))} \geq \frac{\tilde{q} \Psi^2(\beta_0)\rho^4(\beta_0)}{C^2\Phi_1(\beta_0)},
\]

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or equivalently,
\[
\frac{1.1C_{a,\text{max}}(\tilde{\rho}(\beta_0))\hat{\Phi}_1(\beta_0)\tilde{c}_B(\beta_0)}{\Delta_4^4(\beta_0)d_n^2f_n^2(\hat{D},\tilde{\gamma}(\beta_0))} \geq \frac{\hat{q}\Psi^2(\beta_0)\rho^4(\beta_0)}{C^2\Phi_1(\beta_0)} = \frac{\hat{q}\Phi_1(\beta_0)}{C^2\Delta_4^4(\beta_0)},
\]
for some constant \(\hat{q} > C_{a,\text{max}}(\rho(\beta_0))\). Under strong identification and fixed alternatives, we have
\[
d_n\hat{D} = nq\left(Q_{XX} - (Q_{\epsilon(\beta_0)c(\beta_0)}, Q_{\epsilon(\beta_0)})\right)\left(\begin{array}{cc}
\hat{\Phi}_1(\beta_0) & \hat{\Phi}_1(\beta_0) \\
\hat{\Phi}_1(\beta_0) & \hat{\Psi}(\beta_0)
\end{array}\right)^{-1}\left(\begin{array}{c}
\hat{\Phi}_1(\beta_0) \\
\hat{\Psi}(\beta_0)
\end{array}\right) \xrightarrow{p} 1 - (\Delta^2, \Delta)\left(\begin{array}{cc}
\hat{\Phi}_1(\beta_0) & \hat{\Phi}_1(\beta_0) \\
\hat{\Phi}_1(\beta_0) & \hat{\Psi}(\beta_0)
\end{array}\right)^{-1}\left(\begin{array}{c}
\hat{\Phi}_1(\beta_0) \\
\tau(\beta_0)
\end{array}\right).
\]
Therefore, we have
\[
d_nf_s(\hat{D}, \tilde{\gamma}(\beta_0)) = d_n\hat{D} + o_p(1) \xrightarrow{p} 1 - (\Delta^2, \Delta)\left(\begin{array}{cc}
\hat{\Phi}_1(\beta_0) & \hat{\Phi}_1(\beta_0) \\
\hat{\Phi}_1(\beta_0) & \hat{\Psi}(\beta_0)
\end{array}\right)^{-1}\left(\begin{array}{c}
\hat{\Phi}_1(\beta_0) \\
\tau(\beta_0)
\end{array}\right),
\]
for \(s \in \{pp, krs\}\). This means for any \(\varepsilon > 0\), w.p.a.1,
\[
d_n^2f_n^2(\hat{D}, \tilde{\gamma}(\beta_0)) \leq (c_B(\beta_0) + \varepsilon)C^2.
\]
In addition, we have \(c_B(\beta_0) \xrightarrow{p} c_B(\beta_0) \geq 1, \Delta_4(\beta_0) \xrightarrow{p} \Delta_4(\beta_0), C_{a,\text{max}}(\tilde{\rho}(\beta_0)) \xrightarrow{p} C_{a,\text{max}}(\rho(\beta_0))\), and \(\hat{\Phi}_1(\beta_0) \xrightarrow{p} \Phi_1(\beta_0) > 0\), which imply \(c_B(\beta_0) \geq c_B(\beta_0) - \varepsilon, \hat{\Phi}_1(\beta_0) \geq \Phi_1(\beta_0) - \varepsilon, C_{a,\text{max}}(\tilde{\rho}(\beta_0)) \geq C_{a,\text{max}}(\rho(\beta_0)) - \varepsilon, \text{ and } \Delta_4^4(\beta_0) \leq \Delta_4^4(\beta_0) + \varepsilon, \text{ w.p.a.1. Therefore, we have, w.p.a.1,}
\[
\frac{1.1C_{a,\text{max}}(\tilde{\rho}(\beta_0))\hat{\Phi}_1(\beta_0)\tilde{c}_B(\beta_0)}{\Delta_4^4(\beta_0)d_n^2f_n^2(\hat{D},\tilde{\gamma}(\beta_0))} \geq \frac{1.1(C_{a,\text{max}}(\rho(\beta_0)) - \varepsilon)(c_B(\beta_0) - \varepsilon)(\Phi_1(\beta_0) - \varepsilon)}{(\Delta_4^4(\beta_0) + \varepsilon)(c_B(\beta_0) + \varepsilon)C^2}
\geq \frac{(1.1 - \varepsilon)C_{a,\text{max}}(\rho(\beta_0))\Phi_1(\beta_0)}{\Delta_4^4(\beta_0)C^2},
\]
where the second inequality holds because \(\varepsilon\) can be arbitrarily small. Then, we can let \(\hat{q}\) in (K.1) be \((1.1 - \varepsilon)C_{a,\text{max}}(\rho(\beta_0))\) which is greater than \(C_{a,\text{max}}(\rho(\beta_0))\). This concludes the proof.

## L Proof of Theorem A.1

We focus on the consistency of \(\hat{\Phi}_1(\beta_0)\) and \(\hat{\Psi}(\beta_0)\). The consistency of the rest four estimators can be established in the same manner. We have \(e_i(\beta_0) = e_i + \Delta X_i = U_i(\Delta) + \Delta \Pi_i\), where
\[ U_i(\Delta) = e_i + \Delta V_i. \] Therefore,
\[
\hat{\Phi}_1(\beta_0) = \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 e_i^2(\beta_0)c_j^2(\beta_0)
\]
\[
= \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\Delta^2 \Pi_i^2 + 2 \Delta \Pi_i U_i(\Delta) + U_i^2(\Delta))(\Delta^2 \Pi_j^2 + 2 \Delta \Pi_j U_j(\Delta) + U_j^2(\Delta))
\]
\[
= \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 U_i^2(\Delta)U_j^2(\Delta) + \Delta^4 \frac{4}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\Pi_i U_i(\Delta)U_j^2(\Delta) + \Pi_j U_j(\Delta)U_i^2(\Delta))
\]
\[
+ \Delta^2 \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\Pi_i^2 U_j^2(\Delta) + \Pi_j^2 U_i^2(\Delta) + 4 \Pi_i \Pi_j U_i(\Delta)U_j(\Delta))
\]
\[
+ \Delta^3 \frac{4}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\Pi_i^2 \Pi_j U_j(\Delta) + \Pi_j^2 \Pi_i U_i(\Delta)) + \Delta^4 \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i^2 \Pi_j^2
\]
\[
\equiv 4 \sum_{l=0} \Delta^l T_l.
\]

We first note that \( \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \sigma_i^2 = o(1), \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \gamma_i = o(1), \) and \( \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \eta_i^2 = o(1). \) To see this, note that

\[
\frac{1}{K} \sum_{i \in [n]} \omega_i^2 \sigma_i^2 \leq \frac{C}{K} \sum_{i \in [n]} \omega_i^2
\]
\[
\leq \frac{C}{K} (2 \Pi^T P \Pi + 2 \sum_{i \in [n]} P_{ii}^2 \Pi_i^2)
\]
\[
\leq \frac{C}{K} (\sum_{i,j \in [n]} ||\Pi_i|||P_{ij}||\Pi_j| + \Pi^T \Pi p_n)
\]
\[
\leq C p_n^{1/2} \Pi^T \Pi \frac{1}{K} = o(1),
\]

where the second inequality is shown is the Proof of Mikusheva and Sun (2022, Lemma S1.4) and last inequality is by the fact that \( P_{ij}^2 \leq \sum_{j \in [n]} P_{ij}^2 = P_{ii}. \) The results for \( \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \gamma_i = o(1) \) and \( \frac{1}{K} \sum_{i \in [n]} \omega_i^2 \eta_i^2 = o(1) \) can be established in the same manner.

We first consider \( T_0. \) Denote \( \xi_{ij} = U_i^2(\Delta)U_j^2(\Delta) - EU_i^2(\Delta)U_j^2(\Delta). \) We want to show that
\[
\frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \xi_{ij} = o_p(1).
\]
Note that

\[ \mathbb{E} \left[ \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \xi_{ij} \right]^2 = \frac{1}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^4 \mathbb{E} \xi_{ij}^2 + \frac{4}{K^2} \sum_{i \in [n]} \sum_{j \neq i} \sum_{i' \neq i, j} P_{i'j}^2 P_{ij}^2 \mathbb{E} \xi_{ij} \xi_{i'j}. \]

As both \( \mathbb{E} \xi_{ij}^2 \) and \( |\mathbb{E} \xi_{ij} \xi_{i'j}| \) are bounded, we have

\[ \frac{1}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^4 \mathbb{E} \xi_{ij}^2 \leq \frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \leq \frac{C}{K} = o(1) \]

and

\[ \left| \frac{1}{K^2} \sum_{i \in [n]} \sum_{j \neq i} \sum_{i' \neq i, j} P_{i'j}^2 P_{ij}^2 \mathbb{E} \xi_{ij} \xi_{i'j} \right| \leq \frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} \sum_{i' \neq i, j} P_{i'j}^2 P_{ij}^2 \leq \frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{i'j}^2 P_{ij} = o(1). \]

Therefore, we have

\[ T_0 = \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \mathbb{E}(U_i^2(\Delta)U_j^2(\Delta)) + o_p(1) \]

\[ = \Delta^4 \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \eta_j^2 + \Delta^3 \frac{4}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\eta_i^2 \gamma_j + \gamma_i \eta_i^2) + \Delta^2 \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\eta_i^2 \sigma_j^2 + \eta_j^2 \sigma_i^2 + 4 \gamma_i \gamma_j) \]

\[ + \Delta \frac{4}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\gamma_i \sigma_j^2 + \gamma_j \sigma_i^2) + \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \sigma_i^2 \sigma_j^2 + o_p(1) \]

\[ = \Phi_1(\beta_0) + o_p(1). \]

By the same argument above, we have

\[ T_1 = \mathbb{E}T_1 + o_p(1) = o_p(1) \]

because \( \mathbb{E}T_1 = 0 \). Similarly, we have \( \mathbb{E}T_3 = 0 \) and \( T_3 = o_p(1) \). Next, we have

\[ p_nT_2 = \mathbb{E}T_2 + o_p(1) \leq \frac{C}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i^2 + o_p(1) \leq \frac{Cp_n \Pi^\top \Pi}{K} + o_p(1) = o_p(1). \]

Last, we have

\[ T_4 \leq \frac{C}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i^2 = o(1), \]

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where the first inequality is by \( \max_{i \in [n]} |\Pi_i| < C \). This implies

\[
\Phi_1(\beta_0) - \Phi_1(\beta_0) = o_p(1).
\]

Next, we consider the consistency of \( \hat{\Psi}(\beta_0) \). By the similar argument above, we have

\[
\frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 X_i e_i(\beta_0) X_j e_j(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i e_i(\beta_0) \Pi_j e_j(\beta_0) + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i e_i(\beta_0) V_j e_j(\beta_0)
\]

\[
= \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \Pi_i e_i(\beta_0) \Pi_j e_j(\beta_0) + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 V_i e_i(\beta_0) V_j e_j(\beta_0)
\]

\[
= \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\gamma_i + \Delta \eta_i^2)(\gamma_j + \Delta \eta_j^2) + o_p(1). \tag{L.1}
\]

In addition, we have

\[
\frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 e_i^2(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} V_j \right)^2 e_i^2(\beta_0)
\]

\[
= \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} \right)^2 e_i^2(\beta_0) + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_j e_i^2(\beta_0) + o_p(1)
\]

\[
= \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_j^2 (\sigma_i^2 + 2 \gamma_i \Delta + \Delta^2 \eta_i^2) + o_p(1), \tag{L.2}
\]

where the second equality is due to Mikusheva and Sun (2022, Lemma S3.2). In the next section, we show the same results hold under Assumption 5. Combining (L.1) and (L.2), we have

\[
\hat{\Psi}(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\gamma_i + \Delta \eta_i^2)(\gamma_j + \Delta \eta_j^2) + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_j^2 (\sigma_i^2 + 2 \gamma_i \Delta + \Delta^2 \eta_i^2) + o_p(1)
\]

\[
= \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\gamma_i \gamma_j + \eta_i^2 \eta_j^2) + \frac{2 \Delta^2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_j^2 \gamma_j + o_p(1)
\]

\[
= \Psi(\beta_0) + o_p(1).
\]
M Proof of Theorem A.2

Given Lemma A.1, Lemmas 2 and 3 in Mikusheva and Sun (2022) hold under Assumptions 4 and 6. Therefore, Mikusheva and Sun (2022, Theorem 3) shows that

\[ \hat{\Phi}_1(\beta_0) = \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \mathbb{E} U_i^2(\Delta) \mathbb{E} U_j^2(\Delta) = o_p(1). \]

In addition, the proof of Theorem A.1 shows that

\[ \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \mathbb{E} U_i^2(\Delta) \mathbb{E} U_j^2(\Delta) = \Phi_1(\beta_0) + o(1), \]

which implies the consistency of \( \hat{\Phi}_1(\beta_0) \).

Similarly, given Lemma A.1, Lemma S3.1 in Mikusheva and Sun (2022) holds under Assumptions 4 and 6, so that the consistency of \( \hat{\Upsilon} \) to \( \Upsilon \) is also shown by using their argument. In addition, we use the same argument in the proof of Mikusheva and Sun (2022, Theorem 5) to show that

\[
\hat{\Psi}(\beta_0) = \left\{ \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 e_i M_i e + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_i X_i e_i M_j X_j \right\}
+ \Delta \left\{ \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 \left( \frac{e_i M_i X_i}{M_i e} + \frac{X_i M_i e}{M_i e} \right) \right. \\
+ \left. \frac{2}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_i X_i e_i M_j X_j \right\}
+ \Delta^2 \left\{ \frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 \frac{X_i M_i X_i}{M_i e} + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_i X_i X_i M_j X_j \right\}
= \Psi + 2\Delta \tau + \Delta^2 \Upsilon + o_p(1) = \Psi(\beta_0) + o_p(1),
\]

where the second equality also follows from Lemma S3.1 in Mikusheva and Sun (2022).

Next for \( \hat{\Phi}_{12}(\beta_0) \), we have

\[
\frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_j X_j e_j(\beta_0) e_i(\beta_0) M_i e(\beta_0)
= \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 M_j X_j e_j e_i M_i e
+ \Delta \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 (M_j X_j e_i M_i e + M_j X_j e_j M_i e + M_j X_j e_i M_i X)
+ \Delta^2 \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 (M_j X_j X_i M_i e + M_j X_j X_j e_i M_i X + M_j X_j e_j X_i M_i X)
\]

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By using similar arguments, we find that 
\[ \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X_j X_i M_i X. \]

Note that \( \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X e_j e_i M_i e = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 (M_j V + \lambda_i) e_j e_i M_i e, \) where \( \lambda_i = M_i \Pi. \) Then, by Lemma A.1 and Lemma 3 of Mikusheva and Sun (2022),
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X e_j e_i M_i e = o_p(1). \]

Furthermore, by Lemma A.1 and Lemma 2 of Mikusheva and Sun (2022),
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j V e_j e_i M_i e = o_p(1). \]

By using similar arguments, we find that
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X_j X_i e_i M_i e = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \eta_j^2 \sigma_i^2 + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X e_j e_i M_i e = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \gamma_j \gamma_i + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X e_j e_i M_i X = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \gamma_j \gamma_i + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X_j X_i e_i M_i e = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \eta_j^2 \gamma_i + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X_j e_i M_i X = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \gamma_j \eta_i^2 + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X_j X_i e_i M_i X = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \gamma_j \eta_i^2 + o_p(1), \]
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_j X e_j e_i M_i X = \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} P_{ij}^2 \eta_j^2 \eta_i^2 + o_p(1). \]

Putting these results together, we obtain
\[ \hat{\Phi}_{12}(\beta_0) = \Phi_{12} + \Delta(2\Psi + \Phi_{13}) + 3\Delta^2 \tau + \Delta^3 \Upsilon + o_p(1) = \Phi_{12}(\beta_0) + o_p(1). \]

We use similar arguments to prove the results for \( \hat{\Psi}_{13}(\beta_0) \) and \( \hat{\tau}(\beta_0) \). For \( \hat{\Phi}_{13}(\beta_0) \), notice that
\[ \frac{1}{K} \sum_{i\in[n]} \sum_{j\neq i} \tilde{P}_{ij}^2 M_i X e_i(\beta_0) M_j X e_j(\beta_0) \]
which implies that

\[ \tilde{\Phi}_{13}(\beta_0) = \Phi_{13} + 2\Delta \tau + \Delta^2 \Upsilon + o_p(1) = \Phi_{13}(\beta_0) + o_p(1). \]

Finally, for \( \tilde{\tau}(\beta_0) \), notice that

\[
\frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} \tilde{P}_{ij}^2 X_i M_i X M_j X_i \epsilon_j(\beta_0) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \gamma_j + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \eta_j^2 \Delta + o_p(1),
\]

\[
\frac{1}{K} \sum_{i \in [n]} \left( \sum_{j \neq i} P_{ij} X_j \right)^2 \left( \frac{e_i(\beta_0) M_i X}{2M_{ii}} + \frac{X_i M_i e(\beta_0)}{2M_{ii}} \right) = \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \gamma_j + \frac{1}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 \eta_i^2 \eta_j^2 \Delta + o_p(1),
\]

which implies that

\[ \tilde{\tau}(\beta_0) = \tau + \Delta \Upsilon + o_p(1) = \tau(\beta_0) + o_p(1). \]

This completes the proof of the theorem.

\section{N Proof of Lemma A.1}

Let \( p_n = \max_i P_{ii} \). We first give some useful bounds:

\[
\sum_{i \in [n]} \omega_i^2 \leq C \max_i P_{ii}^{1/2} \Pi^\top \Pi = C p_n^{1/2} \Pi^\top \Pi,
\]

\[
\max_{i \in [n]} \omega_i^2 \leq \max_{i \in [n]} \left( \sum_{j \neq i} P_{ij} \Pi_j \right)^2 \leq \max_{i \in [n]} \left( \sum_{j \neq i} P_{ij}^2 \right) \Pi^\top \Pi \leq p_n \Pi^\top \Pi,
\]

which imply

\[
\sum_{i \in [n]} \omega_i^4 \leq \max_{i \in [n]} \omega_i^2 \left( \sum_{i \in [n]} \omega_i^2 \right) \leq C p_n (\Pi^\top \Pi)^2.
\]
First, we show that Mikusheva and Sun (2022, Lemma S2.1) hold under our conditions following the lines of argument in their proof. More specifically, we notice that to show $\Delta^2|E A_2| = o(1)$, where $A_2$ is defined in the proof of Mikusheva and Sun (2022, Lemma S2.1), it suffices to show the following terms are $o(1)$:

$$
\frac{C \Delta^2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 |\lambda_i||\Pi_j| \leq \frac{C \Delta^2}{K} \left( \sum_{i \in [n]} P_{ii}^2 \right)^{1/2} \left( \sum_{j \in [n]} P_{jj}^2 \right)^{1/2} \leq \frac{C \Delta^2}{K} p_n \left( \lambda^\top \lambda \right)^{1/2} \left( \Pi^\top \Pi \right)^{1/2}
$$

$$
\leq \frac{C \Delta^2}{K^{3/2}} p_n \left( \Pi^\top \Pi \right) = o(1) \text{ by } \lambda^\top \lambda \leq C \frac{\Pi^\top \Pi}{K},
$$

$$
\frac{C \Delta^2}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 |\Pi_i||\Pi_j| \leq \frac{C \Delta^2}{K} \left( \sum_{i \in [n]} P_{ii}^2 \right)^{1/2} \left( \sum_{j \in [n]} P_{jj}^2 \right)^{1/2} \leq \frac{C \Delta^2}{K} p_n \left( \Pi^\top \Pi \right) = o(1).
$$

Then, we prove the variance of $\Delta^2 A_2 = o(1)$ by showing that

$$
\frac{C \Delta^4}{K^2} \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^4 \lambda_i^2 \lambda_j^2 \leq \frac{C \Delta^4}{K^2} p_n^2 \left( \lambda^\top \lambda \right)^2 \leq \frac{C \Delta^4}{K^2} p_n^2 \left( \Pi^\top \Pi \right)^2 = o(1) \text{ by } P_{ij}^2 \leq P_{ii},
$$

$$
\frac{C \Delta^4}{K^2} \left( \sum_{i \in [n]} \lambda_i^2 \left( \sum_{j \in [n]} P_{ij}^2 \right) \Pi^\top \Pi + \lambda^\top \lambda \left( \sum_{j \in [n]} P_{jj}^2 |\Pi_j| \right) \right)^2 \leq \frac{C \Delta^4}{K^2} \left( p_n (\lambda^\top \lambda) (\Pi^\top \Pi) + (\lambda^\top \lambda) (p_n K)(\Pi^\top \Pi) \right)
$$

$$
\leq \frac{C \Delta^4}{K^3} \left( p_n (\Pi^\top \Pi)^2 + p_n K (\Pi^\top \Pi)^2 \right) = o(1) \text{ by } \sum_{j \in [n]} P_{jj}^2 \leq p_n K,
$$

$$
\frac{C \Delta^4}{K^2} \left( \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 |\Pi_i||\Pi_j| \right)^2 \leq \frac{C \Delta^4}{K^2} \left( \sum_{i \in [n]} P_{ii}^2 \right)^{1/2} \left( \sum_{j \in [n]} P_{jj}^2 \right)^{1/2} \leq \frac{C \Delta^4}{K^2} p_n^2 (\Pi^\top \Pi)^2 = o(1),
$$

and

$$
\frac{C \Delta^4}{K^2} \sum_{j \in [n]} \sum_{k \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\lambda_i||\Pi_i||M_{jk}| \right)^2 = \frac{C \Delta^4}{K^2} \sum_{j \in [n]} \sum_{k \neq j} \left( \sum_{i \in [n]} P_{ij}^2 |\lambda_i||\Pi_i||M_{jk}| \right)^2 + \frac{C \Delta^4}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\lambda_i||\Pi_i||M_{jj}| \right)^2
$$

$$
\leq \frac{C \Delta^4}{K^2} \sum_{j \in [n]} \sum_{k \neq j} M_{jk}^2 \left( \sum_{i \in [n]} P_{ii}^2 \lambda_i^2 \right) \left( \sum_{i \in [n]} P_{ii}^2 \Pi_i^2 \right) + \frac{C \Delta^4}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\lambda_i| \right)^2
$$

$$
\leq \frac{C \Delta^4}{K^2} K p_n^2 (\lambda^\top \lambda) (\Pi^\top \Pi) + \frac{C \Delta^4}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 \lambda_i^2 \right) \Pi^\top \Pi
$$

$$
\leq \frac{C \Delta^4}{K^2} K p_n^2 (\Pi^\top \Pi)^2 + \frac{C \Delta^4}{K^2} p_n K \Pi^\top \Pi = o(1) \text{ by } \sum_{j \in [n]} \sum_{k \neq j} M_{jk}^2 = \sum_{j \in [n]} \sum_{k \neq j} P_{jk}^2 \leq K \text{ and } P_{ij}^2 \leq P_{ii} \leq p_n.
$$
Second, we show that Mikusheva and Sun (2022, Lemma S2.2) holds under our conditions. Notice that \(|\Delta E A_1| = o(1)| by

\[
\frac{C|\Delta|}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 |\Pi_i| \leq \frac{C|\Delta|}{K} \left( \sum_{i \in [n]} P_{ii}^2 \right)^{1/2} (\Pi^\top \Pi)^{1/2} \leq \frac{C|\Delta|}{K} (p_n K)^{1/2} (\Pi^\top \Pi)^{1/2} = o(1),
\]

Then, we show that the variance of \(\Delta A_1\) is \(o(1)\) by showing the following terms are \(o(1)\):

\[
\frac{C^2}{K^2} \sum_{i \in [n]} \left( \sum_{j \in [n]} P_{ij}^2 \right) \lambda_i^2 + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 |\lambda_i||\lambda_j| \leq \frac{C^2}{K^2} \left( p_n (\lambda^\top \lambda) + p_n (\lambda^\top \lambda) \right) = o(1),
\]

\[
\frac{C^2}{K^2} \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^4 (\lambda_i^2 + |\lambda_i||\lambda_j|) + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 \lambda_i^2 \leq \frac{C^2}{K^2} \left( p_n^2 (\lambda^\top \lambda) + p_n^2 (\lambda^\top \lambda) + p_n (\lambda^\top \lambda) \right) = o(1),
\]

\[
\frac{C^2}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\lambda_i| \right) + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 |\lambda_i||\lambda_j| \leq \frac{C^2}{K^2} \left( p_n (\lambda^\top \lambda) + p_n (\lambda^\top \lambda) \right) = o(1),
\]

\[
\frac{C^2}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\Pi_i| \right) + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^4 \Pi^\top \Pi \leq \frac{C^2}{K^2} \left( p_n K (\Pi^\top \Pi) \right) = o(1),
\]

Then, to show that Mikusheva and Sun (2022, Lemma 3) holds under our conditions, we show
the following terms are $o(1)$:

$$
\frac{C}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 |\Pi_i \lambda_i \Pi_j \lambda_j| \leq \frac{C}{K} \left( \sum_{i \in [n]} \sum_{j \in [n]} P_{ij} \Pi_i^2 \Pi_j^2 \right)^{1/2} \left( \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 \lambda_i^2 \lambda_j^2 \right)^{1/2} \\
\leq \frac{C}{K} p_n \left( \Pi^\top \Pi \right) \left( \lambda^\top \lambda \right)^2 \leq \frac{C}{K} p_n \left( \Pi^\top \Pi \right)^2 = o(1),
$$

$$
\frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} P_{ij}^2 |\Pi_i \lambda_i| \right)^2 \leq \frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \in [n]} |\Pi_i \lambda_i| \right)^2 \lambda_j^2 \leq \frac{C}{K^2} p_n^2 \left( \Pi^\top \Pi \right) \left( \lambda^\top \lambda \right)^2 = o(1),
$$

$$
\frac{C}{K^2} \sum_{i \in [n]} \sum_{j \in [n]} \sum_{i' \in [n]} \sum_{j' \in [n]} \sum_{k \in [n]} P_{ij}^2 |\Pi_i \lambda_i \Pi_j \lambda_j| P_{i'j'}^2 |\Pi_{i'} \lambda_{i'} \Pi_{j'}| |M_{jk} M_{j'k}| \\
\leq \frac{C}{K^2} \left( \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 \lambda_i^2 \lambda_j^2 \right) \left( \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 \lambda_i^2 \lambda_j^2 \right) \leq \frac{C}{K^2} p_n^2 \left( \Pi^\top \Pi \right) \left( \lambda^\top \lambda \right)^2 \leq \frac{C}{K^2} p_n^2 \left( \Pi^\top \Pi \right)^2 = o(1),
$$

where $\sum_{k \in [n]} |M_{jk} M_{j'k}| \leq 1$ by Mikusheva and Sun (2022, Lemma S1.1(ii)).

Now we show that Mikusheva and Sun (2022, Lemma S3.2) holds under our conditions, i.e.,

$$(a) \quad \frac{1}{K} \sum_{i=1}^{n} (\omega_i + \sum_{j \neq i} P_{ij} V_j)^2 U_i - \left( \frac{1}{K} \sum_{i=1}^{n} \omega_i^2 \mathbb{E}[U_i] + \frac{1}{K} \sum_{i,j \neq i} P_{ij}^2 \mathbb{E}[U_i U_j^2] \right) \xrightarrow{p} 0,$$

$$(b) \quad \frac{1}{K} \sum_{i=1}^{n} (\omega_i + \sum_{j \neq i} P_{ij} V_j)^2 \xi_{1,i} \sum_{k \neq j} P_{ik} \xi_{2,k} \xrightarrow{p} 0,$$

$$(c) \quad \frac{1}{K} \sum_{i=1}^{n} (\omega_i + \sum_{j \neq i} P_{ij} V_j)^2 a_i \xi_{1,i} \xrightarrow{p} 0,$$

$$(d) \quad \frac{1}{K} \sum_{i=1}^{n} (\omega_i + \sum_{j \neq i} P_{ij} V_j)^2 a_i \xi_{1,i} \sum_{k \neq i} P_{ik} \xi_{1,k} - \frac{2}{K} \sum_{i=1}^{n} \sum_{j \neq i} P_{ij} \omega_i a_i \mathbb{E}[V_j \xi_{1,j}] \xrightarrow{p} 0,$$

$$(e) \quad \frac{1}{K} \sum_{i=1}^{n} (\omega_i + \sum_{j \neq i} P_{ij} V_j)^2 \Pi_i \lambda_i \frac{a_i}{M_{ii}} \xrightarrow{p} 0,$$

where $\xi_{1,i}, \xi_{2,i}$ stay for either $e_i$ or $V_i$, $U_i$ stay for $e_i^2, e_i V_i$, or $V_i^2$, and $a_i$ stay for either $\Pi_i$ or $\frac{a_i}{M_{ii}}$.

To prove statement (a), following the arguments in Mikusheva and Sun (2022), we just need to show the following terms are $o(1)$:

$$
\mathbb{E} \left[ \frac{1}{K} \sum_{i \in [n]} \omega_i^2 U_i \right]^2 \leq \frac{C}{K^2} \sum_{i \in [n]} \omega_i^4 \leq \frac{C}{K^2} \max_{i \in [n]} \omega_i^2 \left( \sum_{i \in [n]} \omega_i^2 \right) \leq \frac{C}{K^2} p_n^{3/2} \left( \Pi^\top \Pi \right)^2 = o(1),
$$
\[
\frac{C}{K} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\omega_i^2 + |\omega_i||\omega_j|) \leq \frac{C}{K} \left( \sum_{i \in [n]} P_{ii} \omega_i^2 + \left( \sum_{i \in [n]} P_{ii} \omega_i^2 \right)^{1/2} \left( \sum_{j \in [n]} P_{jj} \omega_j^2 \right)^{1/2} \right) \leq \frac{C}{K} P_n^{3/2} \Pi^\top \Pi = o(1),
\]

where we have used \( \max_{i \in [n]} \omega_i^2 \leq p_n \Pi^\top \Pi \), \( \sum_{i \in [n]} \omega_i^2 \leq C p_n^{1/2} \Pi^\top \Pi \), and Mikusheva and Sun (2022, Lemma S1.3(b)).

To prove statement (b), we show that
\[
\frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} \left( P_{ij}^2 \omega_i^4 + P_{ij}^2 \omega_i^2 \omega_j^2 + P_{ij}^4 \omega_i^2 |\omega_j| \right)
\leq \frac{C}{K^2} \left( p_n^2 (\Pi^\top \Pi)^2 + p_n^2 (\Pi^\top \Pi)^2 + p_n^{5/2} (\Pi^\top \Pi) + p_n^{5/2} (\Pi^\top \Pi) \right) = o(1),
\]
\[
\frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^2 + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 |\omega_i| |\omega_j| \right) \leq \frac{C}{K^2} \left( p_n^{1/2} \Pi^\top \Pi + p_n^{3/2} \Pi^\top \Pi \right) = o(1),
\]

where we have used \( \sum_{i \in [n]} \omega_i^2 \leq C p_n^{1/2} \Pi^\top \Pi \) and \( \sum_{i \in [n]} \omega_i^4 \leq C p_n (\Pi^\top \Pi)^2 \).

To prove statement (c), we show that, for \( a_i = \Pi_i \) or \( \lambda_i / M_{ii} \),
\[
\frac{C}{K^2} \left( \sum_{i \in [n]} P_{ii}^2 a_i^2 + \sum_{i \in [n]} \sum_{j \in [n]} P_{ij}^2 |a_i| |a_j| \right) \leq \frac{C}{K^2} \left( p_n^2 a^\top a + p_n a^\top a \right) = o(1),
\]
\[
\frac{C}{K^2} \sum_{i \in [n]} \omega_i^4 \leq \frac{C}{K^2} \left( \max_{i \in [n]} \omega_i^2 \right)^2 \lambda_i^2 \leq C p_n^2 \left( \frac{\Pi^\top \Pi}{K} \right)^3 = o(1),
\]
\[
\frac{C}{K^2} \sum_{i \in [n]} \omega_i^4 \Pi_i \leq \frac{C}{K^2} \sum_{i \in [n]} \omega_i^4 \leq \frac{C}{K^2} p_n \left( \Pi^\top \Pi \right)^2 = o(1), \text{ where we have used } \max_{i \in [n]} |\Pi_i| \leq C,
\]
\[
\frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (a_i^2 + |a_i| |a_j|) \leq \frac{C}{K^2} \left( p_n^2 a^\top a + p_n^2 a^\top a \right) = o(1),
\]
\[
\frac{C}{K^2} \sum_{i \in [n]} \sum_{j \neq i} P_{ij}^2 (\omega_i^2 a_i^2 + |\omega_i a_i| |\omega_j a_j|) \leq \frac{C}{K^2} \left( p_n^2 (\Pi^\top \Pi) (a^\top a) + p_n^2 (\Pi^\top \Pi) (a^\top a) \right) = o(1).
\]

To prove statement (d), we first show that
\[
\frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^2 |a_i| \right)^2 = o(1).
\]
In particular, when \( a_i = \Pi_i \), we have

\[
\frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^2 |\Pi_i| \right) + \left( \sum_{i \in [n]} |\omega_i \Pi_i| \right)^2 \leq \frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^2 \right)^2 + \left( \sum_{i \in [n]} |\omega_i \Pi_i| \right)^2
\]

\[
\leq \frac{C}{K^2} \left( p_n^{1/2} \Pi^T \Pi \right)^2 + \left( \sum_{i \in [n]} \omega_i^2 \right) \left( \Pi^T \Pi \right) \leq \frac{C}{K^2} \left( p_n (\Pi^T \Pi)^2 + p_n^{1/2} (\Pi^T \Pi)^2 \right) = O(1),
\]

When \( a_i = \frac{\lambda_i}{M_{ii}} \), we have

\[
\frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^2 \left| \frac{\lambda_i}{M_{ii}} \right| \right)^2 + \left( \sum_{i \in [n]} \left| \omega_i \frac{\lambda_i}{M_{ii}} \right| \right)^2 \leq \frac{C}{K^2} \left( \sum_{i \in [n]} \omega_i^4 \right) \left( \lambda^T \lambda \right) + \left( \sum_{i \in [n]} \omega_i^2 \right) \left( \lambda^T \lambda \right)
\]

\[
\leq \frac{C}{K^2} \left( p_n (\Pi^T \Pi)^2 (\lambda^T \lambda) + p_n^{1/2} (\Pi^T \Pi)(\lambda^T \lambda) \right) = O(1).
\]

Furthermore, we can show that

\[
\frac{C}{K^2} \left( \sum_{i \in [n]} |\omega_i a_i| \right)^2 \leq \frac{C}{K^2} p_n^{1/2} (\Pi^T \Pi)(a^T a) = o(1),
\]

\[
\frac{C}{K} \sum_{i \in [n]} P_{ii} |a_i| \leq \frac{C}{K} \left( \sum_{i \in [n]} P_{ii}^2 \right)^{1/2} \left( a^T a \right)^{1/2} \leq \frac{C}{K} \left( p_n K \right)^{1/2} \left( a^T a \right)^{1/2} = o(1),
\]

\[
\frac{C}{K^2} \left( \sum_{i \in [n]} P_{ii} |a_i| \right)^2 \leq \frac{C}{K^2} \left( \sum_{i \in [n]} P_{ii}^2 \right) \left( a^T a \right) \leq \frac{C}{K^2} p_n K \left( a^T a \right) = o(1).
\]

To prove statement (e), we show that

\[
\left| \frac{C}{K} \sum_{i \in [n]} \omega_i^2 \Pi_i \frac{\lambda_i}{M_{ii}} \right| \leq \frac{C}{K} \sum_{i \in [n]} \omega_i^2 \left| \frac{\lambda_i}{M_{ii}} \right| \leq \frac{C}{K} \left( \sum_{i \in [n]} \omega_i^4 \right)^{1/2} \left( \lambda^T \lambda \right)^{1/2} \leq \frac{C}{K^2} p_n^{1/2} (\Pi^T \Pi)(\lambda^T \lambda)^{1/2} = o(1),
\]

\[
\frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \neq j} P_{ij} |\omega_i| |\lambda_i| \right) \leq \frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \neq j} P_{ij}^2 \omega_i^2 \right) \leq \frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \neq j} \omega_i^2 \right) \left( \sum_{i \neq j} P_{ij}^2 \lambda_i^2 \right)
\]

\[
\leq \frac{C K p_n^{1/2} \Pi^T \Pi \lambda^T \lambda}{K^2} = o(1),
\]

\[
\frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \neq j} P_{ij}^2 \frac{\lambda_i}{M_{ii}} \right)^2 \leq \frac{C}{K^2} \sum_{j \in [n]} \left( \sum_{i \neq j} P_{ij}^2 \right)^2 \leq \frac{C K p_n \lambda^T \lambda}{K^2} = o(1),
\]
where we have used Mikusheva and Sun (2022, Lemma S1.1(ii)).

Finally, we can show that Mikusheva and Sun (2022, Lemma S3.1) also holds under our conditions by using similar arguments. We omit the details for brevity.

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