CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN
MODULES OF $\mathfrak{sl}(3)$

VYACHESLAV FUTORNY, DIMITAR GRANTCHAROV, AND LUIS ENRIQUE RAMIREZ

Abstract. We provide a classification and an explicit realization of all irreducible Gelfand-Tsetlin modules of the complex Lie algebra $\mathfrak{sl}(3)$. The realization of these modules uses regular and derivative Gelfand-Tsetlin tableaux. In particular, we list all simple Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules with infinite-dimensional weight spaces. Also, we express all simple Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules as subquotientets of localized Gelfand-Tsetlin $E_{21}$-injective modules.

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Contents

1. Introduction 2
2. Preliminaries 4
   2.1. Index of notations 4
3. Gelfand-Tsetlin modules for $\mathfrak{gl}(n)$ 6
   3.1. Gelfand-Tsetlin modules for $\mathfrak{sl}(n)$ 6
4. Families of Gelfand-Tsetlin modules for $\mathfrak{gl}(n)$ 6
   4.1. Irreducible finite dimensional modules 6
4.2. Generic modules 10
   4.3. Singular Gelfand-Tsetlin modules 14
5. Gelfand-Tsetlin modules for $\mathfrak{sl}(n)$ 16
6. Further properties of Gelfand-Tsetlin modules for $\mathfrak{sl}(n)$ 21
7. Realizations of all irreducible Gelfand-Tsetlin modules for $\mathfrak{sl}(n)$ 22
   7.1. Structure of generic $\mathfrak{sl}(n)$-modules 22
7.2. Realizations of all irreducible generic Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules 33
   7.3. Structure of singular $\mathfrak{sl}(3)$-modules $V(T(\bar{v}))$ 53
7.4. Realizations of all irreducible singular Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules 65
8. Localization functors for Gelfand-Tsetlin modules 87
   8.1. Localization and twisted localization functors 87
   8.2. Injectivity and surjectivity of the Gelfand-Tsetlin operators 88
   8.3. Localization functors on $\mathfrak{sl}(3)$-case 89
   8.4. Irreducible Gelfand-Tsetlin modules and localization functors 95
References 95

Date: 1
1. Introduction

Let \( \mathfrak{g} \) be a finite-dimensional simple Lie algebra over the complex numbers, and let \( \mathfrak{h} \) be a Cartan subalgebra of \( \mathfrak{g} \). A \( \mathfrak{g} \)-module \( M \) is called a weight module if \( M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda \), where \( M_\lambda = \{ v \in M \mid hv = \lambda(h)v, \forall h \in \mathfrak{h} \} \). The space \( M_\lambda \) is called a weight space, the set \( \{ \lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0 \} \) is called the weight support of \( M \) and the dimension of \( M_\lambda \) is called the weight multiplicity of \( \lambda \). If \( \mathfrak{h} \) does not necessarily act diagonally but acts locally finitely on \( M \), then we say that \( M \) is a generalized weight module. For irreducible modules the two notions of weight modules coincide.

A weight module \( M \) is torsion free provided that all root vectors of \( \mathfrak{g} \) act injectively on \( M \). If \( M \) is a torsion free module then all weight multiplicities of \( M \) (finite or infinite) are equal. This invariant of \( M \) is called the weight degree of \( M \). Furthermore, the weight support of a torsion free module \( M \) is a full coset \( \lambda + Q \) of \( \mathfrak{h}^*/Q \), where \( Q \) is the root lattice of \( \mathfrak{g} \) and \( \lambda \) is in the weight support of \( M \). On the other hand, an irreducible weight module may have “full support” without being torsion free, in which the weight multiplicities are necessarily infinite. The first examples of such modules were given in [F86b]. Irreducible modules with full support are called dense.

A breakthrough in the theory of weight modules with finite weight multiplicities was made by Fernando, [Fe90], in 1990 who reduced the classification of all such irreducible modules to determining the irreducible torsion free modules and showed that the only simple Lie algebras admitting torsion free modules are those of type \( A \) or \( C \). The next breakthrough was made in 2000 by Mathieu, [M00], who classified and provided a realization of all irreducible torsion free weight modules of finite degree. Previously, the case of degree 1 was worked out in [BL87]. Important properties of the annihilators of the torsion free modules were established in [Jos92].

For weight modules with infinite multiplicities there is a result similar to the one of Fernando reduces the classification of all such irreducible modules to the one of all irreducible dense modules (present for all simple Lie algebras). For classical simple Lie algebras this reduction was obtained first in [F86a] and for all exceptional simple Lie algebras except \( E_8 \) in [FOT95]. Finally, the paper [DMP00] completed the reduction in all cases, including all important classes of finite-dimensional Lie superalgebras.

In contrast to the case of finite multiplicities, the classification of all irreducible dense modules is out of reach, except for the case \( sl(2) \). In the rank-one case all simple modules are classified, [BS1], and in particular, the irreducible dense modules have always weight degree 1.

One natural category of weight modules is the category of Gelfand-Tsetlin modules. More precisely, this is the full subcategory of the category of generalized weight modules consisting of modules that admit a generalized eigenbasis for the Gelfand-Tsetlin subalgebra, a maximal commutative subalgebra of the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \). Gelfand-Tsetlin modules were introduced in [DFO89], [DFO92], [DFO94] as an attempt to generalize the celebrated tableau construction of Gelfand-Tsetlin bases of irreducible finite-dimensional representations of simple classical Lie algebras, [GT50], [Mol06], [Zh74].

Gelfand-Tsetlin subalgebras have applications that extend beyond the study of Gelfand-Tsetlin modules. For example, these subalgebras were related to the solutions of the Euler equation in [FoM78], and to the subalgebras of \( U(\mathfrak{g}) \) of
maximal Gelfand-Kirillov dimension in [V91]. Gelfand-Tsetlin subalgebras were studied in [KW06a], [KW06b] in connection with classical mechanics, and also in [Gr04], [Gr07] in connection with general hypergeometric functions on the Lie group $GL(n, \mathbb{C})$.

A general theory of Gelfand-Tsetlin modules for a class of Galois algebras (for a definition see [FO10]) was developed in [FO14]. The results for these Galois algebras can be applied to the universal enveloping algebras of $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$ and provide structural properties of the corresponding irreducible Gelfand-Tsetlin modules. In the generic case the characters of the Gelfand-Tsetlin subalgebra parametrize such irreducible modules. However, in the nongeneric case, i.e. in the singular case and $n > 2$, we may have more than one isomorphism class of irreducible Gelfand-Tsetlin modules with a fixed character of the Gelfand-Tsetlin subalgebra. The theory of singular Gelfand-Tsetlin modules was initiated in [FGR16] where 1-singular modules were constructed and studied in detail. Immediately after the construction of the 1-singular modules, there was an abundance of successful attempts to construct irreducible Gelfand-Tsetlin modules with a given singular character. For more details, we refer the reader to the following papers [EMV, FGR16, FGR16a, FGR17, FGRZ, FGRZ1, FK17, FRZ16, HL17, RZ17, V17a, V17b, Za17].

A classification of the irreducible 1-singular Gelfand-Tsetlin modules was obtained in [FGR17] and leads to the classification of all irreducible Gelfand-Tsetlin modules of the Lie algebra $\mathfrak{sl}(3)$ (and of $\mathfrak{gl}(3)$). The latter classification is the main purpose of the present paper and it is provided via very explicit tableaux construction.

Our classification result relies on many old partial results on Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules obtained in [BFL95, F86b, F89, F86a, F91, FGR14, R13], among the others. We remark that some technical statements in the paper on the properties of Gelfand-Tsetlin modules can be simplified using the theory recently developed, for example in [FGR17, FGRZ1, RZ17]. However, for the sake of completeness and for reader’s convenience, we opted to keep the original manuscript containing detailed and explicit proofs.

The structure of the paper is as follows. In Section 2 we set up the notation and state basic definitions and results needed in the rest of the paper. In Section 3 we prove some general results about the Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$. Section 4 is devoted to the description of certain “easier to study” classes of Gelfand-Tsetlin modules of $\mathfrak{gl}(n)$, namely finite-dimensional modules, generic modules, and 1-singular modules. In Section 5 we collect some important definitions and preliminary results that relate Gelfand-Tsetlin modules to their Gelfand-Tsetlin character. In Section 6 we prove the main results about existence and uniqueness of irreducible Gelfand-Tsetlin modules of $\mathfrak{sl}(3)$. The explicit description of all irreducible Gelfand-Tsetlin modules for $\mathfrak{sl}(3)$ is included in Section 7. Finally, in Section 8, we study localization functors on the category of Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules and prove that any irreducible module in this category can be obtain from an $E_{21}$-injective module using a $E_{21}$-localization functors.

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2. Preliminaries

The ground field will be \( \mathbb{C} \). In the first part of the paper we fix an integer \( n \geq 2 \). For \( a \in \mathbb{Z} \), we write \( \mathbb{Z}_{\geq a} \) for the set of all integers \( m \) such that \( m \geq a \). Similarly, we define \( \mathbb{Z}_{< a} \), etc. For a Lie algebra \( \mathfrak{a} \) by \( U(\mathfrak{a}) \) we denote the universal enveloping algebra of \( \mathfrak{a} \). For a commutative ring \( R \), \( \text{Spec} R \) will stand for the set of maximal ideals of \( R \).

By \( \mathfrak{gl}(n) \) we denote the general linear Lie algebra consisting of all \( n \times n \) complex matrices, and by \( \{ E_{i,j} \mid 1 \leq i, j \leq n \} \) - the standard basis of \( \mathfrak{gl}(n) \) of elementary matrices. We fix the standard Cartan subalgebra of \( \mathfrak{gl}(n) \), the standard triangular decomposition and the corresponding basis of simple roots of \( \mathfrak{gl}(n) \). The weights of \( \mathfrak{gl}(n) \) will be written as \( n \)-tuples \( (\lambda_1, ..., \lambda_n) \) through the identification \( \mathfrak{h}^* \to \mathbb{C}^n \).

The Lie subalgebra \( \mathfrak{g} \) by \( \mathfrak{gl} \) \( \mathfrak{c} \) matrices, and by \( \{ \mathfrak{g} \} \) simple roots of the root system \( \Delta \) of \( \mathfrak{g} \).

In this section we will collect some general results about Gelfand-Tsetlin modules for \( \mathfrak{gl}(n) \). Let for \( \Lambda \) the polynomial algebra in variables \( \{ a_i \} \).

3. Gelfand-Tsetlin modules for \( \mathfrak{gl}(n) \)

In this section we will collect some general results about Gelfand-Tsetlin modules for \( \mathfrak{gl}(n) \). Let for \( m \leq n \), \( \mathfrak{gl}_m \) be the Lie subalgebra of \( \mathfrak{gl}(n) \) spanned by \( \{ E_{i,j} \mid i, j = 1, \ldots, m \} \). We have the following chain

\[ \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \ldots \subset \mathfrak{gl}_n. \]

It induces the chain \( U_1 \subset U_2 \subset \ldots \subset U_n \) for the universal enveloping algebras \( U_m = U(\mathfrak{gl}_m), 1 \leq m \leq n \). Let \( Z_m \) be the center of \( U_m \). Then \( Z_m \) is the polynomial algebra in the \( m \) variables \( \{ c_{mk} \mid k = 1, \ldots, m \} \),

\[ c_{mk} = \sum_{(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1}. \]

The (standard) Gelfand-Tsetlin subalgebra \( \Gamma \) in \( U \) \( \text{[DF08]} \) \( \text{[CT50]} \) is generated by \( \bigcup_{i=1}^n Z_i \).

The algebra \( \Gamma \) is a polynomial algebra in \( \frac{n(n+1)}{2} \) variables \( \{ c_{ij} \mid 1 \leq j \leq i \leq n \} \). For \( i = 1, \ldots, n \) denote by \( S_i \) the \( i \)-th symmetric group and set \( G = S_n \times \cdots \times S_1 \). Let \( \Lambda \) be the polynomial algebra in variables \( \{ l_{ij} \mid 1 \leq j \leq i \leq n \} \).
Let \( \iota : \Gamma \rightarrow \Lambda \) be the embedding given by \( \iota(c_{mk}) = \gamma_{mk}(l) \) where

\[
\gamma_{mk}(l) = \sum_{i=1}^{m}(l_{mi} + m - 1)^{k} \prod_{j \neq i}(1 - \frac{1}{l_{mi} - l_{mj}}),
\]

The image of \( \iota \) coincides with the subalgebra of \( G \)-invariant polynomials in \( \Lambda \) \((\text{Zh74})\) which we identify with \( \Gamma \).

**Remark 3.1.** Note that \( \Gamma \) contains the standard Cartan subalgebra of \( \mathfrak{gl}(n) \) spanned by \( E_{ii} \), \( i = 1, \ldots, n \). Indeed, \( c_{m1} = \sum_{i=1}^{m} E_{ii} \) for each \( 1 \leq m \leq n \). Therefore, \( E_{ii} \) belong to \( \Gamma \) for each \( 1 \leq i \leq n \).

**Remark 3.2.** We should note that the polynomials \( \gamma_{mk}(l) \) are symmetric of degree \( k \) in variables \( l_{m1}, \ldots, l_{mm} \), and \( \{\gamma_{m1}(l), \ldots, \gamma_{mm}(l)\} \) generate the algebra of \( S_{m} \)-invariant polynomials in variables \( l_{m1}, \ldots, l_{mm} \) \((\text{see Zh74})\).

**Example 3.3.** The polynomials \( \gamma_{mk}(l) \) for \( m \leq 4 \) can be written explicitly as follows:
\[
\begin{align*}
\gamma_{11}(l) &= l_{11}, \\
\gamma_{21}(l) &= (l_{21} + l_{22}) + 1, \\
\gamma_{22}(l) &= (l_{21}^2 + l_{22}^2) + (l_{21} + l_{22}), \\
\gamma_{31}(l) &= (l_{31} + l_{32} + l_{33}) + 3, \\
\gamma_{32}(l) &= (l_{31}^2 + l_{32}^2 + l_{33}^2) + 2(l_{31} + l_{32} + l_{33}) + 1, \\
\gamma_{33}(l) &= (l_{31}^3 + l_{32}^3 + l_{33}^3) + 4(l_{31}^2 + l_{32}^2 + l_{33}^2) - (l_{31}l_{32} + l_{31}l_{33} + l_{32}l_{33}) - 6 \\
&\quad + (l_{31} + l_{32} + l_{33}). \\
\gamma_{41}(l) &= (l_{1} + l_{2} + l_{3} + l_{4}) + 6, \\
\gamma_{42}(l) &= (l_{1}^2 + l_{2}^2 + l_{3}^2 + l_{4}^2) + 3(l_{1} + l_{2} + l_{3} + l_{4}) + 4, \\
\gamma_{43}(l) &= (l_{1}^3 + l_{2}^3 + l_{3}^3 + l_{4}^3) - (l_{1}l_{2} + l_{1}l_{3} + l_{1}l_{4} + l_{2}l_{3} + l_{2}l_{4} + l_{3}l_{4}) + \\
&\quad + 6(l_{1}^2 + l_{2}^2 + l_{3}^2 + l_{4}^2) + 3(l_{1} + l_{2} + l_{3} + l_{4}) - 19, \\
\gamma_{44}(l) &= (l_{1}^4 + l_{2}^4 + l_{3}^4 + l_{4}^4) + 9(l_{1}^3 + l_{2}^3 + l_{3}^3 + l_{4}^3) + 21(l_{1}^2 + l_{2}^2 + l_{3}^2 + l_{4}^2) - \\
&\quad (l_{1}l_{2} + l_{1}l_{3} + l_{1}l_{4} + l_{2}l_{3} + l_{2}l_{4} + l_{3}l_{4} + l_{4}l_{1} + l_{4}l_{2} + l_{4}l_{3}) + \\
&\quad - (l_{1}^2l_{2} + l_{1}^2l_{3} + l_{1}^2l_{4} + l_{2}^2l_{3} + l_{2}^2l_{4} + l_{3}^2l_{4} + l_{4}^2l_{1} + l_{4}^2l_{2} + l_{4}^2l_{3} + l_{4}^2l_{4}) \\
&\quad - 10(l_{1}l_{2} + l_{1}l_{3} + l_{1}l_{4} + l_{2}l_{3} + l_{2}l_{4} + l_{3}l_{4} + l_{4}l_{1} + l_{4}l_{2} + l_{4}l_{3}) - 19(l_{1} + l_{2} + l_{3} + l_{4}) - 120.
\end{align*}
\]

**Definition 3.4.** A finitely generated \( U \)-module \( M \) is called a Gelfand-Tsetlin module (with respect to \( \Gamma \)) provided that the restriction \( M|_{\Gamma} \) is a direct sum of \( \Gamma \)-modules:

\[
M|_{\Gamma} = \bigoplus_{m \in \text{Specm} \Gamma} M(m),
\]

where \( M(m) = \{v \in M \mid m^{k}v = 0 \text{ for some } k \geq 0\} \).

**Definition 3.5.** Any algebra homomorphism \( \chi : \Gamma \rightarrow \mathbb{C} \) will be called Gelfand-Tsetlin character.

**Remark 3.6.** For each \( m \in \text{Specm} \Gamma \) we have associated a character \( \chi_{m} : \Gamma \rightarrow \mathbb{C} \). In the same way, for each non-zero character \( \chi : \Gamma \rightarrow \mathbb{C} \), \( \text{Ker}(\chi) \) is a
maximal ideal of $\Gamma$. So, we have a natural identification between characters of $\Gamma$ and elements of $\text{Specm} \Gamma$. So, using Gelfand-Tsetlin characters, a Gelfand-Tsetlin module (with respect to $\Gamma$) $M$ can be decomposed as $M = \bigoplus_{\chi \in \Gamma} M(\chi)$, where $M(\chi) = \{ v \in M \mid \text{for each } \gamma \in \Gamma, \exists k \in \mathbb{Z}_{\geq 0} \text{ such that } (\gamma - \chi(\gamma))^k v = 0 \}$.

**Definition 3.7.** Given a Gelfand-Tsetlin module $M$, the Gelfand-Tsetlin support of $M$ is the set

$$\text{Supp}_{GT}(M) := \{ \chi \in \Gamma^* \mid M(\chi) \neq 0 \}.$$ 

Note that any irreducible Gelfand-Tsetlin module over $\mathfrak{gl}(n)$ is a weight module with respect to a standard Cartan subalgebra spanned by $E_{ii}$, $i = 1, \ldots, n$ (see Remark 3.1). Moreover, $\Gamma$ is diagonalizable on any finite dimensional irreducible module. The converse is not true in general, that is an irreducible weight module need not to be Gelfand-Tsetlin. But it is the case when the weight subspaces with respect to a standard Cartan subalgebra are finite dimensional, since in this case $\Gamma$ has a common eigenvector in every non-zero weight space. In particular, every highest weight module or, more general, every module from the category $\mathcal{O}$ is Gelfand-Tsetlin. Moreover, when $n = 2$ every irreducible weight module is Gelfand-Tsetlin.

In a similar manner as above one can define a series of Gelfand-Tsetlin subalgebras of $\mathfrak{gl}(n)$. Let $\pi = \{ \beta_1, \ldots, \beta_n \}$ be a basis of the root system of $\mathfrak{gl}(n)$. Denote by $\mathfrak{gl}_i \simeq \mathfrak{gl}(i)$ a subalgebra of $\mathfrak{gl}(n)$ corresponding to simple roots $\beta_1, \ldots, \beta_i$, $i = 1, \ldots, n$. Then we have a chain of embeddings

$$\mathfrak{gl}_1 \subset \ldots \subset \mathfrak{gl}_n.$$ 

Let $Z_i$ be the center of $U(\mathfrak{gl}_i)$ and $\Gamma(\pi)$ is the subalgebra generated by $Z_i$, $i = 1, \ldots, n$. We will call $\Gamma(\pi)$ a Gelfand-Tsetlin subalgebra associated with $\pi$.

Each subalgebra $\Gamma(\pi)$ gives rise to a category of Gelfand-Tsetlin modules which we denote by $\mathcal{GT}_{\pi}(n)$. Let $\pi$ and $\pi'$ be different bases of the root system. Then $\pi$ and $\pi'$ are conjugated by the Weyl group of $\mathfrak{gl}(n)$ it extends to the conjugation of $\Gamma(\pi)$ and $\Gamma(\pi')$. We immediately obtain that the categories $\mathcal{GT}_{\pi}(n)$ and $\mathcal{GT}_{\pi'}(n)$ are equivalent.

**Example 3.8.** Let $\Delta = \{ \pm \alpha, \pm \beta, \pm (\alpha + \beta) \}$ be the root system of $\mathfrak{gl}(3)$. Then we have three Gelfand-Tsetlin subalgebras depending on the choice of $\mathfrak{gl}(2)$ subalgebra which starts a chain of embeddings.

### 3.1. Gelfand-Tsetlin modules for $\mathfrak{sl}(n)$

Let $\Gamma$ be a Gelfand-Tsetlin subalgebra of $\mathfrak{gl}(n)$. Consider a natural projection $\tau : \mathfrak{gl}(n) \to \mathfrak{sl}(n)$ which extends to an epimorphism $\bar{\tau} : U(\mathfrak{gl}(n)) \to U(\mathfrak{sl}(n))$. Then the image $\bar{\tau}(\Gamma)$ of $\Gamma$ is called a Gelfand-Tsetlin subalgebra of $\mathfrak{sl}(n)$.

It is a maximal commutative subalgebra of $U(\mathfrak{sl}(n))$ isomorphic to a polynomial ring in $\frac{n(n+1)}{2} - 1$ generators.

If $\Gamma$ is the standard Gelfand-Tsetlin subalgebra of $\mathfrak{sl}(n)$ then we denote by $\mathcal{GT}(n)$ the category of all Gelfand-Tsetlin $\mathfrak{sl}(n)$-modules with respect to $\Gamma$.

### 4. Families of Gelfand-Tsetlin modules for $\mathfrak{gl}(n)$

As examples of Gelfand-Tsetlin modules we have irreducible finite dimensional modules. In this case, we have a basis of tableaux which is an eigenbasis for the
action of the generators of $\Gamma$ and also, $\dim(M(\chi)) \leq 1$ for each $\chi \in \Gamma^*$. In order to describe such tableaux realization we will fix some notations first.

**Definition 4.1.** Fix a vector $v = (v_{ij})_{1 \leq j \leq n} \in \mathbb{C}^{n(n+1)/2}$

(i) By $T(v)$ we will denote the following array with complex entries $\{v_{ij}\}$

\[
\begin{array}{cccc}
  v_{n1} & v_{n2} & \cdots & v_{n,n-1} \\
  v_{n-1,1} & & \cdots & v_{n-1,n-1} \\
  & & \ddots & \vdots \\
  & & & v_{21} \\
  & & & v_{22} \\
  & & & \vdots \\
  v_{11} & & & \\
\end{array}
\]

Such an array will be called a Gelfand-Tsetlin tableau of height $n$.

(ii) Throughout the paper, for any ring $R$, $T_n(R)$ will stand for the space of the Gelfand-Tsetlin tableaux of height $n$ with entries in $R$. We will identify $T_n(\mathbb{C})$ with the set $\mathbb{C}^{n(n+1)/2}$ in the following way: to

$$v = (v_{11}, \ldots, v_{nn}, |v_{n-1,1}, \ldots, v_{n-1,n-1}| \cdots |v_{21}, v_{22}||v_{11}) \in \mathbb{C}^{n(n+1)/2}$$

we associate a tableau $T(v) \in T_n(\mathbb{C})$ as above.

**Remark 4.2.** There is a natural correspondence between the set $\Gamma^*$ of characters $\chi : \Gamma \to \mathbb{C}$ and the set of Gelfand-Tsetlin tableaux of height $n$. In fact, to obtain a Gelfand-Tsetlin tableau from a character $\chi$ we find a solution $(l_{ij})$ of the system of equations

$$\{\gamma_{mk}(l) = \chi(c_{mk})\}_{1 \leq k \leq m \leq n}$$

Conversely, for every Gelfand-Tsetlin tableau with entries $(l_{ij} \mid 1 \leq j \leq i \leq n)$ we associate $\chi \in \Gamma^*$ by defining $\chi(c_{mk})$ via the above equations. It is clear that each tableau defines such a character uniquely. On the other hand, tableau is defined by a character up to permutations in rows.

### 4.1. Irreducible finite dimensional modules

In this section we recall a classical result of I. Gelfand and M. Tsetlin which gives an explicit basis for each irreducible finite dimensional $\mathfrak{gl}(n)$-module.

**Definition 4.3.** A Gelfand-Tsetlin tableau of height $n$ is called standard if $v_{ki} - v_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $v_{k-1,i} - v_{k,i+1} \in \mathbb{Z}_{> 0}$ for all $1 \leq i \leq k \leq n-1$.

Note that, for sake of convenience, the second condition above is slightly different from the original condition in [GT50].

**Theorem 4.4** (Gelfand-Tsetlin, [GT50]). Let $L(\lambda)$ be the irreducible finite dimensional module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exist a basis of $L(\lambda)$ parametrized by the set of all standard tableaux $T(v) = T(v_{ij})$ with fixed top row $v_{nj} = \lambda_j - j + 1$, $j = 1, \ldots, n$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the Gelfand-Tsetlin formulas:

$$E_{k,k+1}(T(v)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1}(v_{ki} - v_{k+1,j})}{\prod_{j \neq i}^{k}(v_{ki} - v_{kj})} \right) T(v + \delta_{ki}),$$
The formulas above are called Gelfand-Tsetlin formulas for \( \mathfrak{gl}(n) \).

\[ E_{k+1,k}(T(v)) = \sum_{i=1}^{k} \left( \prod_{j \neq i}^{k} (v_{ki} - v_{k-1,j}) \right) T(v - \delta^{ki}), \]

\[ E_{kk}(T(v)) = \left( k - 1 + \sum_{i=1}^{k} v_{ki} - \sum_{i=1}^{k-1} v_{k-1,i} \right) T(v), \]

where \( \delta^{ij} \in T_n(\mathbb{Z}) \) is defined by \( (\delta^{ij})_{ij} = 1 \) and all other \( (\delta^{ij})_{k\ell} \) are zero. If the new tableau \( T(v \pm \delta^{ki}) \) is not standard, then the corresponding summand of \( E_{k,k+1}(T(v)) \) or \( E_{k+1,k}(T(v)) \) is zero by definition.

Theorem 4.5 (Zhelobenko, [Zh74]). Let \( L(\lambda) \) be the irreducible finite dimensional module over \( \mathfrak{gl}(n) \) of highest weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \), with basis as described in Theorem 4.4. The action of the generators \( c_{rs} \) of \( \Gamma \) (see (1)) is given by

\[ c_{rs}(T(v)) = \gamma_{rs}(v)T(v), \]

where \( \gamma_{rs}(v) \) are the symmetric polynomials defined in (2).

As a direct consequence of Theorem 4.4 and Theorem 4.5, any irreducible finite dimensional \( \mathfrak{gl}(n) \)-module is a Gelfand-Tsetlin module with one dimensional Gelfand-Tsetlin subspaces.

Remark 4.6. Whenever we refer to finite dimensional \( \mathfrak{sl}(n) \)-modules we will use the same vector space and the Gelfand-Tsetlin formulas for generators \( E_{r,r+1} \) or \( E_{r+1,r} \), for the action of the generators of a Cartan subalgebra \( \{h_1, \ldots, h_{n-1}\} \) we define \( h_i(T(v)) := (E_{ii} - E_{i+1,i+1})(T(v)) \). We also fix the action of the central element \( E_{11} + \ldots + E_{nn} \) as zero.

Example 4.7. Let us to denote by \( M \) the simple highest weight \( \mathfrak{gl}(3) \)-module with highest weight \( (1,0,-1) \). \( M \) is a finite dimensional module of dimension 8. The tableaux realization guaranteed by Theorem 4.4 consist of a vector space spanned by the set of all standard tableaux of height 3 with top row \((1,-1,-3)\).
By Theorem 4.4, the module $M$ is isomorphic to $\text{span}_\mathbb{C}\{T_i \mid i = 1, \ldots, 8\}$ endowed with the action of $\mathfrak{gl}(3)$ given by the Gelfand-Tsetlin formulas.

When we fix the action of $E_{11} + E_{22} + E_{33}$ to be zero and consider $\mathfrak{h} = \text{span}_\mathbb{C}\{h_1 = E_{11} - E_{22}, h_2 = E_{22} - E_{33}\}$, $M$ becomes an $\mathfrak{sl}(3)$-module with weight support $\text{Supp}(M) = \{(1, 1), (-1, 2), (2, -1), (0, 0), (-2, 1), (1, -2), (-1, -1)\}$.

Then as an $\mathfrak{sl}(3)$-module, $M$ is isomorphic to $L(1, 1)$ (the irreducible finite dimensional $\mathfrak{sl}(3)$-module of highest weight $(1, 1)$). The following picture shows the weights lattice of the $\mathfrak{sl}(3)$-module $M$. Note that $M_{(0,0)}$ is 2-dimensional with basis $\{T_4, T_8\}$.

In particular, the basis elements of $M_{(0,0)}$ cannot be distinguished by the action of the Cartan subalgebra. However, using $\Gamma$ the module decomposes as a direct sum of 1-dimensional $\Gamma$-submodules.

The following theorem will give us information about the dimension of Gelfand-Tsetlin subspaces for irreducible Gelfand-Tsetlin modules and the possible number of non-isomorphic Gelfand-Tsetlin modules with a given Gelfand-Tsetlin character in its support.

**Theorem 4.8** ([FO14], Theorem 6.1; [Ov02]). Let $U = U(\mathfrak{gl}(n))$, $\Gamma \subset U$ the Gelfand-Tsetlin subalgebra, $m \in \text{Specm} \Gamma$ and set $Q_n = 1!2! \cdots (n-1)!$. Then

(i) For a $U$-module $M$, such that $M(m) \neq 0$ and $M$ is generated by some $x \in M(m)$ (in particular for an irreducible module), one has

$$\dim_\mathbb{C} M(m) \leq Q_n.$$
(ii) The number of isomorphism classes of irreducible $U$-modules $N$ such that $N(m) \neq 0$ is always nonzero and does not exceed $Q_n$.

Theorem above shows that elements of $\text{Specm} \, \Gamma$ classify irreducible $\mathfrak{gl}(n)$-modules (and, hence, irreducible $\mathfrak{sl}(n)$-modules) up to some finiteness and up to an isomorphism of Gelfand-Tsetlin modules which contain two different Gelfand-Tsetlin characters.

In [Maz98], Gelfand-Tsetlin modules with a tableaux realization with respect to a Gelfand-Tsetlin subalgebra $\Gamma$ are those with a tableaux basis satisfying $\dim(V(\chi)) \leq 1$ for all $\chi \in \Gamma^*$ and the action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand-Tsetlin formulas. In what follows we will consider a more general definition of tableaux realization, which will allow to consider certain classes of modules with $\dim(V(\chi))$ greater than 1.

For any Gelfand-Tsetlin tableau $T(v) \in T_n(\mathbb{C})$ we consider the set
\begin{equation}
B(T(v)) := \{T(v + z) \mid z \in T_n(\mathbb{Z}), z_{nk} = 0, 1 \leq k \leq n\}.
\end{equation}

If an indecomposable Gelfand-Tsetlin module $V$ has a tableaux realization and $T(v)$ is one of the basis tableaux then it has a basis which is a subset of $B(T(v))$. On the other hand, we might have a module with a basis consisting of a subset of tableau from $B(T(v))$ but without a tableaux realization. This may happen, for example, when $V$ has a Gelfand-Tsetlin character of multiplicity more than 1. For this reason we extend the notion of modules with a tableaux realization.

**Definition 4.9.** We say that a Gelfand-Tsetlin module admits a generalized tableaux realization with respect to a Gelfand-Tsetlin subalgebra $\Gamma$ if it has a basis labelled by a certain subset of $B(T(v))$ for some tableau $T(v)$ (on which generators of $\Gamma$ have as generalized eigenvalues elementary symmetric polynomials on shifted entries of the corresponding rows) with some action of the generators of $\mathfrak{gl}(n)$.

Denote by $\mathcal{G}T_{T(v)}(n)$ the full subcategory of the category of Gelfand-Tsetlin modules $\mathcal{G}T(n)$ which consists of modules with a generalized tableaux realization with respect to $\Gamma$ whose basis contains $T(v)$. The subcategory $\mathcal{G}T_{T(v)}(n)$ is closed under the operations of taking submodules and quotients. Moreover, as our main result will imply, irreducible modules of subcategories $\mathcal{G}T_{T(v)}(3)$ for all $T(v)$ exhaust all irreducible Gelfand-Tsetlin modules for $\mathfrak{sl}(3)$.

**Remark 4.10.** There exist indecomposable modules in $\mathcal{G}T(n)$ which do not belong to $\mathcal{G}T_{T(v)}(n)$ for any $T(v)$. To get an example consider $n = 2$ and an irreducible weight module $V(\lambda, \gamma)$ with a weight $\lambda \in \mathbb{C}$ on which a Casimir element acts by a scalar $\gamma \in \mathbb{C}$ such that $\gamma \neq (\lambda + k)^2 + 2(\lambda + 2k)$ for any integer $k$. Then $V(\lambda, \gamma)$ has a self-extension of any length which remains a weight module but the Casimir element acts as a Jordan matrix with the same eigenvalue $\gamma$. This module will be indecomposable Gelfand-Tsetlin module but not a module with a generalized tableaux realization.

**Definition 4.11.** We will call the subcategory $\mathcal{G}T_{T(v)}(n)$ a block of $\mathcal{G}T(n)$ generated by $T(v)$. A module $V \in \mathcal{G}T_{T(v)}(n)$ will be called a universal module in $\mathcal{G}T_{T(v)}(n)$ if every irreducible module of $\mathcal{G}T_{T(v)}(n)$ is isomorphic to a subquotient of $V$.

4.2. **Generic modules.** Observing that the coefficients in the Gelfand-Tsetlin formulas in Theorem 4.4 are rational functions on the entries of the tableaux, Y. Drozd,
V. Futorny and S. Ovsienko [DFO94] extended the Gelfand-Tsetlin construction to
more general modules. In the case when all denominators nonzero for all possible
integral shifts, one can use the same formulas and define a new class of infinite
dimensional $\mathfrak{gl}(n)$-modules: generic Gelfand-Tsetlin modules (cf. [DFO94], section
2.3.).

**Definition 4.12.** A Gelfand-Tsetlin tableau $T(v)$ (equivalently, $v \in T_n(\mathbb{C})$) is
called generic if $v_{ki} - v_{kj} \notin \mathbb{Z}$ for all $1 \leq i \neq j \leq k \leq n - 1$.

Remember that $\mathcal{B}(T(v)) = \{ T(v + z) \mid z \in T_{n-1}(\mathbb{Z}) \}$ for any Gelfand-Tableau $T(v)$ (see (5)).

**Theorem 4.13** ([DFO94], Section 2.3). Let $T(v)$ be a generic Gelfand-Tsetlin
tableau of height $n$.

(i) The vector space $V(T(v)) = \text{span}_C \mathcal{B}(T(v))$ has a structure of a $\mathfrak{gl}(n)$-
module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand-Tsetlin
formulas.

(ii) The action of the generators of $\Gamma$ on the elements of $\mathcal{B}(T(v))$ is given by
(6).

(iii) The $\mathfrak{gl}(n)$-module $V(T(v))$ is a Gelfand-Tsetlin module with $\mathfrak{gl}(n)$
multiplicities equal to 1.

By a slight abuse of notation we will denote the module constructed in Theorem
4.13 by $V(T(v))$ and will call it the generic Gelfand-Tsetlin module associated with
$T(v)$.

**Remark 4.14.** By Theorem 4.13(iii), given two different tableaux $T(w)$ and $T(w')$
in $\mathcal{B}(T(v))$, there exist an element of $\Gamma$ with different eigenvalues for $T(w)$ and
$T(w')$. Whenever we say that $\Gamma$ “separates” tableaux of $V(T(v))$ we will refer to
this property.

In general $V(T(v))$ need not to be irreducible. Because $\Gamma$ has simple spectrum
on $V(T(v))$ for $T(w)$ in $\mathcal{B}(T(v))$ we may define the irreducible $U$-module in $V(T(v))$
containing $T(w)$ to be the irreducible subquotient of $V(T(v))$ containing $T(w)$.

4.2.1. Gelfand-Tsetlin formulas in terms of permutations. In this section we will
rewrite the Gelfand-Tsetlin formulas in terms of permutations, that will be useful
in order to simplify the expressions whenever we use the Gelfand-Tsetlin formulas.

Let $\tilde{S}_m$ denotes the subset of $S_m$ consisting of the transpositions $(1,i)$, $i =
1, \ldots, m$. For $\ell < m$, set $\Phi_{\ell m} = \tilde{S}_{m-1} \times \cdots \times \tilde{S}_\ell$. For $\ell > m$ we set $\Phi_{\ell m} = \Phi_{m \ell}$.
Finally we define, $\Phi_{\ell \ell} = \{ \text{Id} \}$. Every $\sigma$ in $\Phi_{\ell m}$ will be written as an $|\ell - m|$-tuple
of transpositions and by $\sigma[i]$ we will denote the $t$-th component of the tuple.

**Remark 4.15.** Recall that in order to have well defined action of $\sigma \in \Phi_{\ell m}$ on
$T_{n-1}(\mathbb{C})$, for $w \in T_{n-1}(\mathbb{C})$ and $\sigma \in \Phi_{\ell m}$ on $w$ we set

$$\sigma(w) := (w_{n-1,1}^{-1}[n-1][1], \ldots, w_{n-1,\sigma^{-1}[n-1][1]} \ldots |w_{1,\sigma^{-1}[1][1]}|).$$

**Definition 4.16.** Let $1 \leq r < s \leq n$. Define

$$\varepsilon_{rs} := \delta^{r+1,1} + \delta^{s+1,1} + \ldots + \delta^{s-1,1} \in T_n(\mathbb{Z}).$$
Furthermore, define $\varepsilon_{rr} = 0$ and $\varepsilon_{sr} = -\varepsilon_{rs}$.
Definition 4.17. For each generic vector \( w \) and any \( 1 \leq t \leq n - 1 \) define
\[
e_t^{(+)}(w) := \prod_{j \neq t}^{t+1} (w_{t+1} - w_{t+j}) / \prod_{j \neq t}^{t} (w_{t+1} - w_{t+j});
\]
\[
e_t^{(-)}(w) := \prod_{j \neq t}^{t-1} (w_{t-1} - w_{t+j}) / \prod_{j \neq t}^{t} (w_{t-1} - w_{t+j});
\]
\[
e_k,k+1(w) := -\prod_{j \neq k}^{k+1} (w_{k+1} - w_{k+j}) / \prod_{j \neq k}^{k} (w_{k+1} - w_{k+j});
\]
\[
e_{k+1,k}(w) := \prod_{j \neq k}^{k} (w_{k} - w_{k+j}) / \prod_{j \neq k}^{k} (w_{k} - w_{k+j}).
\]

Lemma 4.18. For each \( m > k \) the action of \( E_{mk} \) is given by the expression:
\[
E_{mk}(T(v)) = \sum_{\sigma \in \Phi_{mk}} e_{k+1,k}(\sigma(v)) \left( \prod_{j=k+2}^{m} (e_j^{(+)}(\sigma(v))) \right) T(v + \sum_{i=k}^{m-1} \sigma(\epsilon_{i+1,i+1})).
\]

Proof. The case \( k = m + 1 \) follows from the Gelfand-Tsetlin formulas. The general case follows by induction in \( m - k \) using the relation
\[
E_{m,k+1}(E_{k+1,k}(T(v))) - E_{k+1,k}(E_{m,k+1}(T(v))) = E_{m,k}(T(v))
\]
for any generic vector \( v \).

Lemma 4.19. For each \( r < s \) the action of \( E_{rs} \) is given by the expression:
\[
E_{rs}(T(v)) = \sum_{\sigma \in \Phi_{rs}} \left( \prod_{j=r+1}^{s-2} (e_j^{(+)}(\sigma(v))) \right) e_{s-1,s}(\sigma(v)) T(v + \sum_{i=r}^{s-1} \sigma(\epsilon_{i+1,i+1})).
\]

Proof. The case \( s = r + 1 \) follows from the Gelfand-Tsetlin formulas. The general case follows by induction in \( s - r \) using the relation
\[
E_{r,r+1}(E_{r+1,s}(T(v))) - E_{r+1,s}(E_{r,r+1}(T(v))) = E_{rs}(T(v))
\]
for any generic vector \( v \).

Definition 4.20. For each generic vector \( w \in T_n(\mathbb{C}) \) and any \( 1 \leq r, s \leq n \) we define
\[
e_{rs}(w) := \begin{cases} 
\left( \prod_{j=r}^{s-2} (e_j^{(+)}(w)) \right) e_{s-1,s}(w), & \text{if } r < s \\
e_{s+1,s}(w) \left( \prod_{j=s+2}^{r} (e_j^{(-)}(w)) \right), & \text{if } r > s \\
r - 1 + \sum_{i=1}^{r} w_{ri} - \sum_{i=1}^{r-1} w_{r-1,i}, & \text{if } r = s.
\end{cases}
\]

Proposition 4.21. Let \( v \in T_n(\mathbb{C}) \) be any generic vector and \( z \in T_{n-1}(\mathbb{Z}) \). The Gelfand-Tsetlin formulas for the \( U \)-module \( V(T(v)) \) can be written as follows:
\[
E_{tm}(T(v + z)) = \sum_{\sigma \in \Phi_{tm}} e_{tm}(\sigma(v + z)) T(v + z + \sigma(\epsilon_{tm})).
\]

Proof. Follows from Lemmas 4.18 and 4.19 and the fact that \( \sum_{i=m}^{m-1} \sigma(\epsilon_{i+1,i+1}) = \sigma(\epsilon_{m,m}) \) when \( m > \ell \) and \( \sum_{i=m}^{\ell-1} \sigma(\epsilon_{i+1,i+1}) = \sigma(\epsilon_{m,\ell}) \) when \( m < \ell \).

Example 4.22. Let us write explicitly the expressions in Definition 4.21 and write the Gelfand-Tsetlin formulas as in Proposition 4.21 in the case of \( \mathfrak{gl}(3) \). Let \( v \in T_3(\mathbb{C}) \) be any generic vector, \( z \in T_2(\mathbb{Z}) \) and \( w = v + z \). Set also \( \tau \) to be the permutation that interchanges entries in positions \((2,1)\) and \((2,2)\). Considering
The action of $\mathfrak{gl}(3)$ on any tableau is given by:

\[
\begin{align*}
E_{11}(T(w)) &= e_{11}(w)T(w) \\
E_{22}(T(w)) &= e_{22}(w)T(w) \\
E_{33}(T(w)) &= e_{33}(w)T(w) \\
E_{12}(T(w)) &= e_{12}(w)T(w + \varepsilon_{12}) \\
E_{21}(T(w)) &= e_{21}(w)T(w + \varepsilon_{21}) \\
E_{23}(T(w)) &= e_{32}(w)T(w + \varepsilon_{32}) + e_{32}(\tau(w))T(w + \tau(\varepsilon_{32})) \\
E_{23}(T(w)) &= e_{23}(w)T(w + \varepsilon_{23}) + e_{23}(\tau(w))T(w + \tau(\varepsilon_{23})) \\
E_{31}(T(w)) &= e_{31}(w)T(w + \varepsilon_{31}) + e_{31}(\tau(w))T(w + \tau(\varepsilon_{31})).
\end{align*}
\]

The explicit description of all irreducible generic modules for $\mathfrak{gl}(3)$ was obtained first in [R12]. The classification of irreducible generic modules $\mathfrak{gl}(n)$ was completed in [FGR15]. Let us discuss briefly the main details of such classification.

**Definition 4.23.** Let $T(v)$ be a fixed Gelfand-Tsetlin tableau. For any $T(w) \in \mathcal{B}(T(v))$, and for any $1 < r \leq n$, $1 \leq s \leq r$ and $1 \leq u \leq r - 1$ we define:

\[
\Omega(T(w)) := \{(r, s, u) \mid w_{rs} - w_{r-1,u} \in \mathbb{Z}\}
\]

\[
\Omega^+(T(w)) := \{(r, s, u) \mid w_{rs} - w_{r-1,u} \in \mathbb{Z}_{\geq 0}\}
\]

Basis for the irreducible subquotients of $V(T(v))$ can be described as follows.

**Theorem 4.24** ([FGR15], Theorems 6.8 and 6.14). Let $T(v)$ be a fixed generic Gelfand-Tsetlin tableau and $T(w)$ any tableau in $\mathcal{B}(T(v))$.

(i) The module $U \cdot T(w)$ has a basis of tableaux

\[
\mathcal{N}(T(w)) = \{T(w') \in \mathcal{B}(T(w)) \mid \Omega^+(T(w)) \subseteq \Omega^+(T(w'))\}.
\]

(ii) The irreducible module containing $T(w)$ has a basis of tableaux

\[
\mathcal{I}(T(w)) = \{T(w') \in \mathcal{B}(T(w)) \mid \Omega^+(T(w)) = \Omega^+(T(w'))\}.
\]

The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{N}(T(w))$ is given by the Gelfand-Tsetlin formulas. The action of $\mathfrak{gl}(n)$ on $T(w') \in \mathcal{I}(T(w))$ is given by the Gelfand-Tsetlin formulas with the correction that all tableau $T(w' + \delta^k)$ for which $\Omega^+(T(w' + \delta^k)) \neq \Omega^+(T(w))$ are omitted in the sums for $E_{k,k+1}(T(w'))$ and $E_{k+1,k}(T(w'))$.

**Corollary 4.25.** Let $T(v)$ be a generic Gelfand-Tsetlin tableau. The module $V(T(v))$ is irreducible if and only if $\Omega(T(v)) = \emptyset$. 

\begin{tabular}{|c|c|}
\hline
$\varepsilon_{11} = (0,0,0)$ & $e_{11}(w) = w_{11}$ \\
$\varepsilon_{22} = (0,0,0)$ & $e_{22}(w) = w_{21} + w_{22} - w_{11} + 1$ \\
$\varepsilon_{33} = (0,0,0)$ & $e_{33}(w) = w_{31} + w_{32} + w_{33} - w_{21} - w_{22} + 2$ \\
$\varepsilon_{12} = (0,0,1)$ & $e_{12}(w) = -(w_{11} - w_{21})(w_{11} - w_{22})$ \\
$\varepsilon_{21} = (0,0,-1)$ & $e_{21}(w) = 1$ \\
$\varepsilon_{23} = (1,0,0)$ & $e_{23}(w) = -(w_{21} - w_{31})(w_{21} - w_{32})(w_{21} - w_{33})$ \\
$\varepsilon_{32} = (-1,0,0)$ & $e_{32}(w) = \frac{w_{11} - w_{21}}{w_{21} - w_{22}}$ \\
$\varepsilon_{31} = (-1,0,-1)$ & $e_{31}(w) = \frac{w_{11} - w_{21}}{w_{21} - w_{22}}$ \\
$\varepsilon_{13} = (1,0,1)$ & $e_{13}(w) = -(w_{11} - w_{21})(w_{11} - w_{22})(w_{11} - w_{33})(w_{11} - w_{32})$ \\
\hline
\end{tabular}
Example 4.26. Consider $a, b, c \in \mathbb{C}$ such that \{a - b, a - c, b - c\} \cap \mathbb{Z} = \emptyset and $v = (a, b, c, a, b + 2, a)$, then

\[
T(v) = \begin{array}{ccc}
 a & b & c \\
 a & b+2 & \\
 a & \\
\end{array}
\]

then $\Omega(T(v)) = \{(3, 1, 1), (3, 2, 2), (2, 1, 1)\}$, $\Omega^+(T(v)) = \{(3, 1, 1), (2, 1, 1)\}$. So, by Theorem 4.24 the irreducible subquotient of $V(T(v))$ containing $T(v)$ as a basis element is generated by

$I(T(v)) = \{T(v + (m, n, k)) \mid m \leq 0, k \leq m, \text{ and } n > -2\}$.

Example 4.27. (See also [Maz98], Section 4.3) Set $a_1, \ldots, a_n$ complex numbers such that $a_i - a_j \notin \mathbb{Z}$ for any $i \neq j$. Denote by $T(v)$ the Gelfand-Tsetlin tableau of height $n$ with entries $v_{ij}$, such that $v_{rs} = a_s$ for $1 \leq s \leq r \leq n$. The tableau $T(v)$ is a generic Gelfand-Tsetlin tableau and by Theorem 4.24 a basis for an irreducible $\mathfrak{gl}(n)$-module containing $T(v)$ as a basis element is parameterized by the set of tableaux

$I(T(v)) = \{T(v + z) \mid z_{rs} - z_{r-1,s} \in \mathbb{Z}_{\geq 0} \text{ for any } r, s\}$.

Moreover, we can easily check that $\text{span}_\mathbb{C}(I(T(v)))$ is a submodule of $V(T(v))$ isomorphic to the irreducible Verma module $M(a_1, a_2 + 1, \ldots, a_n + n - 1)$.

4.3. Singular Gelfand-Tsetlin modules. The construction of irreducible finite dimensional modules and generic modules presented in Sections 4.1 and 4.2 have as a main common ingredient an explicit basis parameterized by certain sets of Gelfand-Tsetlin tableaux. In the finite dimensional case all the entries of the tableaux involved satisfy that $v_{ki} - v_{kj} \in \mathbb{Z}$ and for the generic case all the entries of the tableaux involved satisfy that $v_{ki} - v_{kj} \notin \mathbb{Z}$ for any $k \neq n$.

Definition 4.28. A vector $v \in T_n(\mathbb{C})$ will be called singular if there exist $1 \leq s < t \leq r \leq n - 1$ such that $v_{rs} - v_{rt} \in \mathbb{Z}$. The vector $v$ will be called 1-singular if for there exist $k, i, j$ with $1 \leq i < j \leq k \leq n - 1$ such that $v_{ki} - v_{kj} \in \mathbb{Z}$ and $v_{rs} - v_{rt} \notin \mathbb{Z}$ for all $(r, s, t) \neq (k, i, j)$, $r \neq n$. If $v$ is 1-singular, the tableau $T(v)$ will be called 1-singular tableau.

We can think about standard and generic tableaux as extreme cases of singular Gelfand-Tsetlin tableaux.

4.3.1. Construction of 1-singular Gelfand-Tsetlin modules. In [FGR16] the authors gave an explicit construction of modules with a generalized tableaux realization (see Definition 4.29) associated with any 1-singular Gelfand-Tsetlin tableau. In this section we will describe the main details of this construction.

For any $w \in T_n(\mathbb{C})$ we define the Gelfand-Tsetlin character $\chi_w : \Gamma \rightarrow \mathbb{C}$ on generators as follows: $\chi_w(c_{rs}) = \gamma_{rs}(w)$. If $w$ is 1-singular, the character $\chi_w$ will be called 1-singular Gelfand-Tsetlin character.

Set $v$ a vector of variables with $\frac{n(n+1)}{2}$ entries indexed by $(r, s)$ such that $1 \leq s \leq r \leq n$. By $\mathcal{F}$ we will denote the space of rational functions on $v_{ij}$, $1 \leq j \leq i \leq n$, with poles on the hyperplanes $v_{rs} - v_{rt} = 0$. Note that $V(T(v))$ is defined for all generic $v$ and that $V(T(v)) = V(T(v'))$ whenever $v - \sigma(v') \in T_{n-1}(\mathbb{Z})$ for some
σ ∈ G. Thus V(T(v)) is defined for elements v in the (generic) complex torus
T = Tn(C)/Tn−1(Z). Let us to denote by Tn(C)gen the set of all generic vectors v
in Tn(C) such that V(T(v)) is irreducible (i.e. vr,s − vr−1,t /∈ Z for any r, s, t). From
now on we fix (i, j, k) such that 1 ≤ i < j ≤ k ≤ n − 1.

By H we denote the hyperplane vki − vkj = 0 in Tn(C), also by τ ∈ Sn−1×⋯×S1
we denote the transposition on the kth row interchanging the ith and jth entries.
H stands for the subset of all w in Tn(C) such that wτr = wτs for all triples (t, r, s)
except for (t, r, s) = (k, i, j). Finally, by Fij denote the subspace of F consisting
of all functions that are smooth on H.

Let us fix ⃗v in H such that ⃗vk,i = δkj and all other differences ⃗vmr − ⃗vms are
noninteger. In other words, ⃗v ∈ H and ⃗v + Tn−1(Z) < H.

Remark 4.29. For any generic vector w we can choose a representative of the
class w + Tn−1(Z) of w in T = Tn(C)gen/Tn−1(Z) as “close” as possible to ⃗v as
follows. Let mr,s := |Re(⃗vr,s − vr τs)| (the integer part of the real part of ⃗vr,s − vr τs),
and m be the vector in Tn−1(Z) with components mr,s, and set ⃗vw := w + m. Let
S be the set ⃗vw | w ∈ Tn(C)gen \} a fixed set of representatives of T.

We will construct a module with Gelfand-Tsetlin support \{χv+mr | m ∈ Tn−1(Z)\},
and refer to this module as the 1-singular Gelfand-Tsetlin module associated with
⃗v.

We formally introduce the complex vector space V(T(⃗v)) freely spanned by vectors
\{T(⃗v + z) | z ∈ Tn−1(Z)\} satisfying T(⃗v + z) − T(⃗v + τ(z)) = 0 and vectors
\{DT(⃗v + z) | z ∈ Tn−1(Z)\} subject to the relations DT(⃗v + z) + DT(⃗v + τ(z)) = 0.
We will refer to T(u) as the regular Gelfand-Tsetlin tableau associated with u and
to DT(u) as the derivative Gelfand-Tsetlin tableau associated with u.

Remark 4.30. Note that \{T(⃗v + z), DT(⃗v + z) | z ∈ Tn−1(Z)\} is not a basis since
T(⃗v + z) − T(⃗v + τ(z)) = 0, DT(⃗v + z) + DT(⃗v + τ(z)) = 0. From now on we will
fix a basis of V(T(⃗v)) being

B(T(⃗v)) := \{T(⃗v + z), DT(⃗v + w) | zk,i ≤ zk,j, wk,i > wk,j\}.

Set Ygen = ⊕ v∈S V(T(v)) and Y′ = V(T(⃗v)) ⊕ Ygen. Then F ⊗ Ygen is a gl(n)-module
with the trivial action on F. Let us introduce a gl(n)-module structure on
V(T(⃗v)).

The evaluation map, ev(⃗v) : Fij ⊗ Y′ → Y′ is the linear map which is defined
for generators by ev(⃗v)(f(T(⃗v + z))) = f(⃗v)T(⃗v + z), ev(⃗v)(fDT(⃗v + z)) =
f(⃗v)DT(⃗v + z). Finally, D⃗v : Fij ⊗ V(T(v)) → V(T(⃗v)) will denote the linear
map defined by D⃗v(f(T(⃗v + z))) = D⃗v(f)T(⃗v + z) + f(⃗v)D⃗vT(⃗v + z), where

D⃗v(f) = 1 2 (∂f ∂vk,i − ∂f ∂vk,j ) (v), z ∈ Tn−1(Z), f ∈ Fij and v ∈ S. In other words, D⃗v
is the map

D⃗v ⊗ ev(⃗v) + ev(⃗v) ⊗ D⃗v.

This map extends to a linear map Fij ⊗ Ygen → V(T(⃗v)) which we will also denote
by D⃗v.

Theorem 4.31 ([EGR10] Theorems 4.9 and 5.6). V(T(⃗v)) has structure of Gelfand-
Tsetlin module over gl(n) with action of the generators of gl(n) given by

(6) Ers(T(⃗v + z)) = D⃗v((vk,i − vk,j)Ers(T(v + z))),
(7) Ers(D(T(⃗v + w))) = D⃗v(Ers(T(v + w))),
and action of the generators of $\Gamma$ given by
\begin{align}
&c_{rs}(T(v + z)) = D^v((v_{ki} - v_{kj})c_{rs}(T(v + z))), \\
&c_{rs}(D^v(T(v + w))) = D^v(c_{rs}(T(v + w))),
\end{align}

where $v$ is a generic vector in the set of representatives $S$, and $z, w \in T_{n-1}(\mathbb{Z})$ with $w \neq \tau(w)$.

**Remark 4.32.** In the case of $\mathfrak{gl}(3)$ we can give the following interpretation of the basis elements of the module $V(T(\bar{v}))$. Let $T(v)$ represent any generic tableau such that $V(T(v))$ is irreducible, and let $T(\bar{v})$ be a critical tableau as follows,

$T(v) :=
\begin{array}{ccc}
v_{31} & v_{32} & v_{33} \\
v_{21} & v_{22} \\
v_{11}
\end{array}
$,

$T(\bar{v}) :=
\begin{array}{ccc}
v_{\bar{3}1} & v_{\bar{3}2} & v_{\bar{3}3} \\
v_{\bar{2}1} & v_{\bar{2}2} \\
v_{\bar{1}1}
\end{array}
$

Then $T(\bar{v} + (m, n, k))$ and $DT(\bar{v} + (m, n, k))$ can be considered as formal limits in the following way:

\begin{align}
T(\bar{v} + (m, n, k)) &:= \lim_{v \to \bar{v}} T(v + (m, n, k)), \\
DT(\bar{v} + (m, n, k)) &:= \lim_{v \to \bar{v}} \left( \frac{T(v + (m, n, k)) - T(v + (n, m, k))}{v_{21} - v_{22}} \right).
\end{align}

One essential property of generic Gelfand-Tsetlin modules described in Theorem 4.13 is that $\Gamma$ “separates” tableaux basis elements in $B(T(v))$, that is, for any two different tableaux $T_1, T_2$ in $B(T(v))$ there exists an element $\gamma \in \Gamma$ such that $\gamma \cdot T_1 = T_2$ and $\gamma \cdot T_2 = 0$ (see Remark 4.14). In the case of 1-singular modules $V(T(\bar{v}))$ this is also true and follows from the fact that no derivative tableau $D(T(\bar{v} + w))$ is an eigenvector for the action of $c_{k2} \in \Gamma$. A detailed proof can be found in [GoR18], Section 5.

**Theorem 4.33.** Let $\bar{v}$ be any 1-singular vector and $B(T(\bar{v}))$ the fixed basis as before, then $\Gamma$ separates tableaux in $B(T(\bar{v}))$.

In the case of $T_3(\mathbb{C})$ (equivalently, Gelfand-Tsetlin tableaux of height 3) every singular vector is a 1-singular vector. Therefore, in the case of $\mathfrak{gl}(3)$ the construction of 1-singular modules will give us all the necessary information about singular Gelfand-Tsetlin modules.

**Example 4.34.** The irreducible Verma $\mathfrak{sl}(3)$-module $M(-1, -1)$ admits a tableau realization as a subquotient of the module $V(T(\bar{v}))$, where $\bar{v} = (-1, -1, -1, -1, -1)$. In this module we have Gelfand-Tsetlin characters of dimension 2. Namely, if $\chi$ is the Gelfand-Tsetlin character associated with the tableaux $T(\bar{v} + (-1, 0, -1))$ and $DT(\bar{v} + (0, -1, -1))$ then $\dim(M_\chi) = 2$ (see §7.4 (C13) for details).

5. **Gelfand-Tsetlin modules for $\mathfrak{sl}(3)$**

From now on we will concentrate on the case $n = 3$. We fix the standard Gelfand-Tsetlin subalgebra $\Gamma$ of $\mathfrak{g} = \mathfrak{sl}(3)$, that is the one corresponding to a chain which contains $\mathfrak{gl}(2)$ generated by $E_{12}$ and $E_{21}$. The corresponding category of Gelfand-Tsetlin modules $GT(3)$ we will simply denote by $GT$. 
In this section we recall some basic properties of modules in $\mathcal{G}T$ obtained in [F86b], [F89]. Let $C(\mathfrak{h})$ be the centralizer of the Cartan subalgebra

$$\mathfrak{h} = \text{span}_\mathbb{C}\{H_1 := E_{11} - E_{22}, \ H_2 := E_{22} - E_{33}\}.$$

Recall the following basic fact (see for example [F86a]):

**Lemma 5.1.** For any irreducible $C(\mathfrak{h})$-module $W$ there exists an irreducible weight $\mathfrak{g}$-module $M$ such that $M_\lambda \simeq W$ for some $\lambda \in \mathfrak{h}^*$. Conversely, if $M$ is an irreducible weight $\mathfrak{g}$-module then $M_\lambda$ is an irreducible $C(\mathfrak{h})$-module.

Denote $A := E_{12}E_{21}$, $B := E_{23}E_{32}$.

**Lemma 5.2** ([BFL95], Lemma 1.1). The centralizer $C(\mathfrak{h})$ is an associative algebra generated by $H_1$, $H_2$, $A$, $B$ and the center of $U(\mathfrak{gl}(3))$.

The generators of $C(\mathfrak{h})$ satisfy the following identities which were found in [F86b]. Let $W$ be any module over $C(\mathfrak{h})$ having identity operator $I$ such that $H_1 = h_1I$, $H_2 = h_2I$, $c_1 = \gamma_1I$, $c_2 = \gamma_2I$, where $c_1$ and $c_2$ are the following generators of the center of $U(\mathfrak{gl}(3))$:

$$c_1 = \frac{1}{18}(H_1^2 + H_2^2 + H_1H_2) + \frac{1}{6}(H_1 + H_2) + \frac{1}{6}(E_{21}E_{12} + E_{32}E_{23} + E_{31}E_{13}),$$

$$c_2 = -\frac{2}{9}H_1^3 + \frac{2}{9}H_2^3 - \frac{1}{3}H_1^2H_2 + \frac{1}{3}H_1H_2^2 - H_1^2H_2 - H_2^2 - H_1 + H_2 -$$

$$E_{21}E_{12}H_1 - E_{31}E_{13}H_1 + 2E_{32}E_{23}H_1 - 2E_{21}E_{12}H_2 + E_{31}E_{13}H_2 +$$

$$E_{32}E_{23}H_2 - 3E_{32}E_{21}E_{13} - 3E_{31}E_{12}E_{23} +$$

$$3E_{32}E_{23} - 3E_{21}E_{12} + 3E_{31}E_{13}.$$

**Remark 5.3.** We can express elements $c_1$ and $c_2$ in terms of the generators $\{c_{mk}\}$ of the center of $\mathfrak{gl}(3)$ (see [4]) as follows:

$$c_1 = \frac{1}{12}c_{32} \quad \text{and} \quad c_2 = \frac{3}{2}c_{32} - c_{33}.$$

Then the following identities are satisfied in $W$:

(12) \quad aA = A^2 + AB + BA + ABA - \frac{1}{2}A^2B - \frac{1}{2}BA^2 + rB + \tau I,$

(13) \quad aB = B^2 + AB + BA + BAB - \frac{1}{2}B^2A - \frac{1}{2}AB^2 + r_1B + \tau_1 I,$

(14) \quad \frac{1}{4}(AB - BA)^2 = ABA + BAB + \frac{1}{2}rB^2 + \frac{1}{2}r_1A^2 -$$

(\frac{a}{2} - 1)(AB + BA) + (\tau + r)B + (\tau_1 + r_1)A + \eta I,$

where

\begin{align*}
r &= \frac{1}{2}(h_1^2 - 2h_1) ; \quad r_1 = \frac{1}{2}(h_2^2 - 2h_2), \\
a &= 6\gamma_1 + h_1 + h_2 - \frac{1}{3}h_1^2 - \frac{1}{3}h_2^2 + \frac{1}{6}h_1h_2, \\
\eta &= \frac{1}{4}p^2 + \frac{1}{6}p(h_1h_2 + h_1^2 + h_2^2 - 18\gamma_1) + \frac{1}{4}h_1h_2(h_1h_2 + 4 - 2a),
\end{align*}
By solving both quadratic equations in terms of $i$, the following relation is satisfied for any $\mu$:

$$ p = \frac{1}{3} \left[ \frac{1}{9} (h_1 - h_2)^3 - \gamma_2 + 6\gamma_1 (h_2 - h_1 + 3) - h_1^2 - h_2^2 - h_1 h_2 + 2h_1 - 2h_2 \right], $$

$$ \tau = \frac{1}{2} h_1 p + h_1 h_2 : \tau_1 = -\frac{1}{3} p h_2^2 + \frac{1}{3} h_2^2 \left[ 18 \gamma_1 - h_1^2 - h_2^2 - h_1 h_2 + 3h_1 \right]. $$

Let $M$ be an irreducible module in $\mathcal{G}T$. Then it is a weight module in particular. Consider any $\lambda \in \mathfrak{h}^*$ from the weight support of $M$. Central elements $c_1$ and $c_2$ act on $M$, and hence on $M_\lambda$, as multiplication for some complex scalars $\gamma_1$ and $\gamma_2$. If $\lambda(M_1) = h_1$ and $\lambda(M_2) = h_2$ then $H_1 = h_2 I$ and $H_2 = h_2 I$ on $M_\lambda$. Since $M$ is a Gelfand-Tsetlin module, then each component $M(\chi)$ is finite dimensional. Hence, one can choose a basis of $M_\lambda$ with respect to which $A|M_\lambda$ has a Jordan canonical form $A_\lambda$.

Consider the following polynomial in 2 variables:

$$ g_\lambda(x, y) = (x - y)^2 - 2(x + y) - (h_1^2 - 2h_1). $$

Note that the polynomial $g_\lambda(x, y)$ depends only on $h_1 = \lambda(H_1)$ due to the fixed choice of the embedding $\mathfrak{sl}(2) \subseteq \mathfrak{sl}(3)$ and that $A \in U(\mathfrak{sl}(2))$.

We say that the eigenvalues of $A_\lambda$ form a connected chain if all distinct eigenvalues can be ordered in such a way that $g_\lambda(\mu_i, \mu_{i+1}) = 0$ for all $i$.

**Lemma 5.4.** ([BFL95], Lemma 2.2) Let $M$ be an irreducible Gelfand-Tsetlin module and $\lambda \in \mathfrak{h}^*$ a weight of $M$. Then the eigenvalues of $A_\lambda = A |_{M_\lambda}$ form a connected chain.

**Proof.** Indeed, if $[A_\lambda] = \text{diag}(A_i)$ a Jordan form of $A_\lambda$, where $A_i$ is the generalized eigenspace with eigenvalue $\lambda_i$, then with respect to the same basis of $M_\lambda$ we have $[B_\lambda] = (B_{ij})$ is a block of matrix $B$ which corresponds to a division of this matrix with respect to the block decomposition of $A$, i.e. the block between $B_{ii}$ and $B_{ij}$.

Substituting matrices $[A_\lambda]$ and $[B_\lambda]$ into the first relation on $A$ and $B$ we obtain for each pair $i \neq j$:

$$ 0 = A_i B_{ij} + B_{ij} A_i + A_i B_{ij} A_i - \frac{1}{2} B_{ij} A_i^2 - \frac{1}{2} A_i^2 B_{ij} + \tau B_{ij}. $$

Replacing in this relation $A_i$ by $\lambda_i I$ (where $I$ is the identity matrix) one immediately sees that $B_{ij} = 0$ if $(\lambda_i, \lambda_j) - \frac{1}{4}(\lambda_i - \lambda_j)^2 + \tau \neq 0$ (cf. [BFL95] Lemma 2.2).

Since $M$ is irreducible, the statement follows. \hfill $\square$

**Lemma 5.5.** If $\{\mu_i\}$ is a connected chain of eigenvalues of $A_\lambda$ for some $\lambda$, then the following relation is satisfied for any $i$:

$$ \mu_{i+1} + \mu_{i-1} = 2\mu_i + 2 $$

(15)

**Proof.** As $\{\mu_i\}$ is a connected chain, we have $g_\lambda(\mu_i, \mu_{i-1}) = 0$ and $g_\lambda(\mu_i, \mu_{i+1}) = 0$.

By solving both quadratic equations in terms of $\mu_i$ we have:

$$ \{\mu_{i-1}, \mu_{i+1}\} = \{\mu_i + 1 \pm \sqrt{1 + 4\mu_i + h_1^2 - 2h_1}\} $$

therefore, $\mu_{i+1} + \mu_{i-1} = 2\mu_i + 2$. \hfill $\square$

**Definition 5.6** ([FGS65]). Let $\{\mu_j\}$ be the connected chain of eigenvalues of $A_\lambda$. Also set $r = \frac{1}{2}(h_1^2 - 2h_1)$. We say that $\{\mu_j\}$ is:

(i) Degenerate if $\mu_i = -\frac{1}{2} r$ for some $i$.

(ii) Critical if $\mu_i = -1/4 - \frac{1}{2} r$ for some $i$.

(iii) Singular if it is a subchain of a degenerate or critical chain.
Lemma 5.7. Let \( \{\mu_i\} \) be the connected chain of all distinct eigenvalues of \( A_\lambda \).

(i) If \( \{\mu_i\} \) is degenerate, then \( \{\mu_i\} \subseteq \{n(n+1) - \frac{r}{2} \mid n \geq 0\} \).

(ii) If \( \{\mu_i\} \) is critical, then \( \{\mu_i\} \subseteq \{n^2 - \frac{1}{4} - \frac{r}{2} \mid n \geq 0\} \).

(iii) If \( \{\mu_i\} \) is generic, then \( \{\mu_i\} \subseteq \{n^2 + n\sqrt{1+4\mu_0 + 2r} + \mu_0 \mid n \in \mathbb{Z}\} \).

Proof. By Lemma 5.5 in order to known completely a connected chain is enough to know explicitly two connected eigenvalues. In the case of a degenerate chain we have \( \mu_i = -\frac{1}{2}r \) for some \( i \), which implies \( \mu_{i+1} = \mu_i \) or \( \mu_{i-1} = \mu_i \). In any case we have an explicit solution for the Equation (15), namely \( \{n(n+1) - \frac{r}{2} \mid n \geq 0\} \). For critical chain we have \( \mu_i = -\frac{1}{4} - \frac{r}{2} \) for some \( i \), which implies \( \mu_{i+1} = \mu_i + 1 \) or \( \mu_{i-1} = \mu_i + 1 \). In any case the solution for Equation (12) is \( \{n^2 - \frac{1}{4} - \frac{r}{2} \mid n \geq 0\} \). In the generic case, given \( \mu_i \) in the connected chain, we have \( \mu_{i+1} = \mu_i + 1 \pm \sqrt{1+4\mu_0 + 2r} \). In any case the solution for Equation (15) is given by \( \{n^2 + n\sqrt{1+4\mu_0 + 2r} + \mu_0 \mid n \in \mathbb{Z}\} \).

The properties of \( A_\lambda \) are described in the following theorem.

Theorem 5.8. If \( M \) is an irreducible Gelfand-Tsetlin module then:

(i) For every \( \lambda \) in the support of \( M \) the multiplicity of any eigenvalue \( \mu \) of \( A_\lambda \) is not greater than 2.

(ii) One can order the eigenvalues of \( A_\lambda \) in such a way that the set of all distinct eigenvalues form a connected chain.

Moreover, one of the following possibilities holds:

(a) If the chain is generic then all eigenvalues of \( A_\lambda \) are distinct;

(b) If the chain is degenerate then all distinct eigenvalues can be ordered in the following way:

\[ \{\mu_1, \mu_2, \ldots\} , \]

where \( \mu_1 = -\frac{1}{4}(h_1^2 - 2h_1) \), and if the multiplicity of \( \mu_i \) equals 1 then the multiplicity of \( \mu_{i+1} \) is also 1;

(c) If the chain is critical then all distinct eigenvalues can be ordered in the following way:

\[ \{\mu_1, \mu_2, \ldots\} , \]

where \( \mu_1 + 1 = \mu_2 \), multiplicity of \( \mu_1 \) is 1, and if the multiplicity of \( \mu_i \) equals 1 for \( i > 1 \), then the multiplicity of \( \mu_{i+1} \) is also 1;

(d) If the chain is singular but not degenerate or critical, then all eigenvalues of \( A_\lambda \) are distinct.

Proof. The proof of all items was obtained in [F86a] (see also [F89] and [BFL95, Theorem 2.7 for items (b) and (d)] by applying the relations (12)-(14) to a Jordan form of \( A_\lambda \).

As a consequence of Theorem 5.8 we have the following statement.

Corollary 5.9. Let \( M \) be an irreducible weight \( \mathfrak{sl}(3) \)-module. Then for any \( \lambda \in \mathfrak{h}^* \) and any \( i \neq j \) we have \( \dim(\ker(E_{ij}|_{M_{\lambda}})) \leq 1 \).
Proof. Suppose that \( \dim(\ker(E_{ij}|_{M\lambda})) \neq 0 \) for some \( \lambda \in \mathfrak{h}^* \). Consider a Lie subalgebra of \( \mathfrak{a} \subset \mathfrak{sl}(3) \) isomorphic to \( \mathfrak{sl}(2) \) and generated by \( E_{ij} \) and \( E_{ji} \). Then \( M \) is a Gelfand-Tsetlin module with respect to the Gelfand-Tsetlin subalgebra generated by \( \mathfrak{h} \), the center of \( U(\mathfrak{sl}(3)) \) and the center of \( U(\mathfrak{a}) \). The statement follows then from Theorem 5.13. \( \square \)

Definition 5.10. We say that a Gelfand-Tsetlin character \( \chi \) belongs to a connected chain \( \{\mu_1, \mu_2, \ldots\} \) of eigenvalues of \( A_\lambda \) if \( \chi(h) = \lambda(h) \) for each \( h \in \mathfrak{h} \) and \( \chi(A) = \mu_i \) for some \( i \).

(i) \( \chi \) is called generic if it belongs to a generic connected chain.
(ii) \( \chi \) is called singular if it belongs to a singular connected chain.
(iii) \( \chi \) is called critical if \( \chi(A) = -\frac{1}{4} - \frac{r}{2} \).
(iv) \( \chi \) is called degenerate if \( \chi(A) = -\frac{r}{2} \).

Since \( A \in \Gamma \), we can extend the concepts of generic and singular chains to Gelfand-Tsetlin modules. A Gelfand-Tsetlin module \( M \) is called generic if every Gelfand-Tsetlin character of \( M \) is generic. A Gelfand-Tsetlin module \( M \) is called singular if it has a singular Gelfand-Tsetlin character.

Note that any finite dimensional module is a singular Gelfand-Tsetlin module, moreover, any 1-singular module as defined in Section 4.2 is a singular Gelfand-Tsetlin module. Also, generic modules as defined in 4.3 are generic Gelfand-Tsetlin modules.

Proposition 5.11. If an irreducible Gelfand-Tsetlin module \( M \) is singular, then each Gelfand-Tsetlin character of \( M \) is singular.

Proof. The statement can be checked by a direct computation. Let \( \chi \) be a singular Gelfand-Tsetlin character of \( M \), \( v \in M(\chi) \), \( v \neq 0 \). If \( E_{12}v \neq 0 \) then \( E_{12}v \in M(\chi') \), where \( \chi' \) is singular. Similarly for \( E_{21}v \). Suppose \( E_{23}v \neq 0 \), then \( E_{23}v \in M(\chi') \oplus M(\chi'') \), where \( \chi' \) and \( \chi'' \) are both singular (one of the subspaces can be zero). Moreover, if \( \chi \) belongs to a critical (respectively, degenerate) connected chain, then \( \chi' \) and \( \chi'' \) belong to a degenerate (respectively, critical) connected chain. Similarly for \( E_{32} \). \( \square \)

Definition 5.12. Given a Gelfand-Tsetlin character \( \chi \) and \( M \) a Gelfand-Tsetlin module, we say that \( M \) is an irreducible extension of the character \( \chi \) if \( M \) is irreducible and \( \chi \in \text{Supp}_{GT}(M) \) (i.e. \( M(\chi) \neq 0 \)).

It follows that generic Gelfand-Tsetlin modules are completely determined by \( \Gamma^* \). Namely,

Theorem 5.13 ([F89], [F86a]). If \( M \) is a generic irreducible Gelfand-Tsetlin module then for any \( \chi \in \text{Supp}_{GT}(M) \) the subspace \( M(\chi) \) is one dimensional and \( M \) is a unique irreducible extension of \( \chi \).

Proof. Let \( \chi \in \text{Supp}_{GT}(M) \) be a character associated with a weight \( \lambda \), that is \( \chi|_{\mathfrak{h}} = \lambda \). Let \( \mu_0 \in \mathbb{C} \) be an eigenvalue of the operator \( A_\lambda = A |_{M(\chi)} \). Then all eigenvalues of \( A_\lambda \) form a connected chain, i.e. belong to a sequence \( \mu_i = i^2 + i(1 + 4\mu_0 + 2r)^2 + \mu_0 \), \( i \in \mathbb{Z} \) for some choice of the square root (see Lemma 5.7(iii)).
Using relations between $A$ and $B$ we can choose a basis \{\(w_i \mid i \in \mathbb{Z}\}\) (this set can be finite or bounded from one side or unbounded) of \(V_\lambda\) such that

\[
A_\lambda w_i = \mu_i w_i; \quad \text{and} \quad B_\lambda w_i = \begin{cases} 
  w_{i-1} + b_i w_i + d_{i+1} w_{i+1}, & i < 0 \\
  w_{-1} + b_0 w_0 + w_1, & i = 0 \\
  d_i w_{i-1} + b_i w_i + w_{i+1}, & i > 0
\end{cases}
\]

where

\[
\eta = \frac{1}{4} \mu^2 + \frac{1}{6} \rho (h_1 h_2 + h_1^2 + h_2^2 - 18 \gamma_1) + \frac{1}{4} h_1 h_2 (h_1 h_2 + 4 - 2 \rho),
\]

\[
p = \frac{1}{3} \left[ \frac{1}{9} (h_1 - h_2)^3 - \gamma_2 + 6 \gamma_1 (h_2 - h_1 + 3) - h_1^2 - h_2^2 - h_1 h_2 + 2 h_1 - 2 h_2 \right],
\]

\[
\tau = \frac{1}{2} h_1 p + h_1 h_2; \quad \tau_1 = -\frac{1}{2} \rho h_2^2 + \frac{1}{3} h_2 \left[ 18 \gamma_1 - h_1^2 - h_2^2 - h_1 h_2 + 3 h_1 \right],
\]

and

\[
b_i = \frac{(\alpha \mu_i - \mu_i^2 - \tau)}{2 \mu_i + r},
\]

\[
d_i = \frac{\xi(\mu_i - 1)}{4 (\mu_i - 1 - \mu_i + 1)} \left( \frac{\gamma(\mu_i - 1 - \mu_i + 1)}{\mu_i - 1} \right),
\]

\[
\xi(\mu_i) = \frac{1}{2} (2 \mu_i + r) b_i^2 - (2 \mu_i + r) b_i - \frac{1}{2} r_1 \mu_i^2 - (r_1 + \tau_1) \mu_i - \eta,
\]

\[
\theta(\mu_i) = (a - 2 \mu_i) b_i - b_i^2 - r_1 \mu_i - \tau_1.
\]

Hence, in this case \(B_\lambda\) is completely determined by \(\chi\) and \(A_\lambda\). On the other hand, since \(M_\lambda\) is irreducible as an \(C(\mathfrak{h})\)-module, the knowledge of any eigenvalue of \(A_\lambda\) determines completely \(A_\lambda\) and, hence, \(B_\lambda\). The uniqueness follows.

\[\Box\]

Note that in singular cases the subspace \(M(\chi)\) can be 2-dimensional (see Example \ref{example:2-dimensional}). Also, in those cases for a given \(\chi \in \Gamma^*\) there can exist two non-isomorphic irreducible extensions of \(\chi\). Such examples were first constructed in \cite{FS65}.

6. FURTHER PROPERTIES OF GELFAND-TSETLIN MODULES FOR \(s(3)\)

In this section we discuss the structure of Gelfand-Tsetlin modules with critical characters.

**Theorem 6.1.** If \(\chi\) is a critical Gelfand-Tsetlin character then \(\chi\) admits a unique irreducible extension.

**Proof.** By Theorem \ref{theorem:existance} there exist at most 2 irreducible modules \(M^{(1)}\) and \(M^{(2)}\) such that \(\chi \in \text{Supp}_{C\ell} (M^{(i)})\) for \(i = 1, 2\). Let \(\lambda \in \text{Supp}(M^{(1)}) \cap \text{Supp}(M^{(2)})\) such that \(\lambda(h) = \chi(h)\) for all \(h \in \mathfrak{h}\). For \(i = 1, 2\), consider the restriction \(A_\lambda^{(i)}\) of \(A = E_{12} E_{21}\) on \(M_\lambda^{(i)}\). Since \(M^{(i)}\) are Gelfand-Tsetlin modules, then we can choose bases \(\varepsilon_1 = \{w_0, \ldots, w_m\}, 0 \leq m \leq \infty\) and \(\varepsilon_2 = \{w'_0, \ldots, w'_k\}, 0 \leq k \leq \infty\) of \(M^{(1)}_\lambda\) and \(M^{(2)}_\lambda\) such that the matrices \([A_\lambda]_{\varepsilon_1}\) of \(A_\lambda^{(i)}\) with respect to \(\varepsilon_1\) has a Jordan normal form and each eigenvalue of \(A_\lambda^{(i)}\) has algebraic multiplicity at most 2.

By Theorem \ref{theo:2} \((ii)(c)\), the eigenvalue \(\chi(A)\) of \(A_\lambda^{(1)}\) and \(A_\lambda^{(2)}\) has multiplicity 1. Suppose first that all eigenvalues of both \(A_\lambda^{(1)}\) and \(A_\lambda^{(2)}\) have multiplicity 1. Then...
they can be ordered in connected chains \( \{\lambda_n \mid 0 \leq n \leq m\} \) and \( \{\lambda_n \mid 0 \leq n \leq k\} \) with
\[
\lambda_0 = \chi(A) = -\frac{1}{4} - \frac{1}{4}(h_1^2 - 2h_1),
\]
and \( \lambda_i = i^2 - \frac{1}{4} - \frac{1}{r} r \) for \( i \geq 1 \) (see Lemma 5.7(ii)).

Applying the relations (12), (13), (14) we obtain
\[
Aw_i = \lambda_i w_i, \quad i \geq 0, \quad Bw_i = \begin{cases}
\begin{align*}
b_0 w_0 + w_1, & \quad i = 0 \\
c_i w_{i-1} + b_i w_i + w_{i+1}, & \quad 0 < i \leq m
\end{align*}
\end{cases}
\]
\[
Aw_i' = \lambda_i w_i', \quad i \geq 0, \quad Bw_i' = \begin{cases}
\begin{align*}
b_0 w_0' + w_1', & \quad i = 0 \\
c_i w_{i-1} + b_i w_i' + w_{i+1}', & \quad 0 < i \leq k
\end{align*}
\end{cases}
\]
where
\[
c_1 = \frac{\theta(\lambda_0)}{2},
\]
\[
c_2 = \begin{cases}
\frac{\theta(\lambda_1)}{(1 + \lambda_2 - \lambda_1)}, & \quad n = 1 \\
d_2, & \quad n > 1
\end{cases},
\]
\[
c_i = d_i, \quad i > 2
\]
and
\[
2\xi(\lambda_0) = \left(\frac{3}{2} + 2\lambda_0 + r\right) \theta(\lambda_0).
\]

If \( m < k \), then \( c_{m+1} = 0 \) implying that \( M^{(2)} \) is reducible. Similarly, if \( m > k \), then \( c_{k+1} = 0 \) and \( M^{(1)} \) is reducible. It follows that \( k = m \). But then formulas (10) define uniquely an irreducible \( C(\mathfrak{g}) \)-module \( M^{(1)}_\lambda \simeq M^{(2)}_\lambda \). Therefore, \( M^{(1)} \simeq M^{(2)} \).

Suppose now that the algebraic multiplicity of some \( \lambda_i \) is two in \( M^{(1)}_\lambda \). For simplicity assume that
\[
[A_\lambda]_{e_1} = \begin{pmatrix}
\lambda_0 & 0 & 0 \\
0 & \lambda_1 & 1 \\
0 & 0 & \lambda_1
\end{pmatrix}
\quad \text{and} \quad
[B_\lambda]_{e_1} = \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix}.
\]

Applying the relation (12) we obtain \( b_{32} = 0 \). Note that due to the irreducibility of \( M^{(1)}_\lambda \) as an \( C(\mathfrak{g}) \)-module we have \( b_{12} \neq 0 \) and \( b_{31} \neq 0 \). Hence, one can change the basis \( e_1 \) in such a way that \( b_{13} \) becomes 0 (performing row-column operations and using \( b_{12} \)).

Now, applying the relation (13) we obtain
\[
\begin{align*}
b_{22} &= b_{33}, \\
\frac{1}{b_{11}} + \frac{1}{b_{22}} &= a - 2\lambda_0 - 1, \\
\frac{1}{b_{12}b_{31}} &= -ab_{22} + b_{22}^2 + 2\lambda_1 b_{22} + r_1 \lambda_1 + \tau_1, \\
\frac{2}{b_{23}} &= -1 - b_{11} - 2b_{22}, \\
2b_{12}b_{21} &= -2(2\lambda_0 + 1 - a)(4\lambda_0 + 1) - 3r_1 \lambda_1 + r_1 - 3\tau_1, \\
\frac{1}{b_{31}}b_{12} &= (4\lambda_0 + 3)(2\lambda_0 + 1 - a) + r_1 \lambda_1 + \tau_1, \\
(b_{11} + b_{22})b_{31} &= 0.
\end{align*}
\]

We can make \( b_{12} = 1 \). Hence, the values of \( b_{31} \) and \( b_{21} \) can be computed. Since \( b_{31} \neq 0 \) we also have \( b_{11} = -b_{22} \). Therefore, the matrix \( [B_\lambda]_{e_1} \) is completely
determined by $\chi$ and matrix $[A_\lambda]_{e_1}$. Certainly the same holds for an arbitrary matrix $[A_\lambda]_{e_1}$. Consider now the matrix $[A_\lambda^{(2)}]_{e_2}$. If this matrix has a different shape and size than $[A_\lambda]_{e_1}$, then one of the modules will be reducible (this can be immediately seen from the form of matrices $[B_\lambda]_{e_1}$ and $[B_\lambda]_{e_2}$). On the other hand, if $[A_\lambda^{(1)}]_{e_1} = [A_\lambda^{(2)}]_{e_2}$ then $M_\lambda^{(1)} \simeq M_\lambda^{(2)}$ as $C(\mathfrak{h})$-modules and $M^{(1)} \simeq M^{(2)}$. The general form of $[A_\lambda]_{e}$ is treated analogously.

If $\chi$ is a singular Gelfand-Tsetlin character in a critical connected chain and $\chi$ is not critical then there might exist two irreducible extensions of $\chi$ (see [F86b] for examples). On the other hand we have

**Corollary 6.2.** Suppose that $\chi$ is a singular character in a critical connected chain and $\chi$ is not critical. Let $\lambda \in \mathfrak{h}^*$ be a weight associated with $\chi$. Then there exists a unique irreducible extension $M$ of $\chi$ with diagonalizable $A_\lambda$.

**Proof.** Indeed, if $A_\lambda$ is diagonalizable then $B_\lambda$ is determined uniquely. As it was shown in the proof of Theorem 6.1, it is sufficient to know one eigenvalue of $A_\lambda$ to reconstruct the whole $A_\lambda$ in an irreducible module. Hence, the statement follows.

**Lemma 6.3.** Let $M$ be an irreducible Gelfand-Tsetlin module, $\chi$ a degenerated character of $M$ associated with the weight $\lambda \in \mathfrak{h}^*$, $\chi(A) = \mu_1 = -\frac{1}{2}(h_1^2 - 2h_1)$, where $h_1 = \chi(H_1)$, $\mu_2 = 2 - \frac{1}{4}(h_1^2 - 2h_1)$ an eigenvalue of $A_\lambda = A|_{M_\lambda}$ connected with $\mu_1$. Suppose that both $\mu_1$ and $\mu_2$ have multiplicity 2. Then the Gelfand-Tsetlin support of $M$ contains a critical character $\chi'$.

**Proof.** Let $u_1, u_2$ be non-zero elements of $M$ such that

$$A_\lambda u_1 = \mu_1 u_1, \ A_\lambda u_2 = u_1 + \mu_1 u_2,$$

and let $u_3, u_4$ be non-zero elements such that

$$A_\lambda u_3 = \mu_2 u_3, \ A_\lambda u_4 = u_3 + \mu_2 u_4.$$

Set $\phi := E_{23}$. Suppose that $M_{\lambda+e_2-e_3}$ does not contain a critical character. Then the eigenvalues $\{\lambda_k, \lambda_{k+1}, \ldots, \lambda_m\}$, $k > 1$, of $A_{\lambda+e_2-e_3}$ form a part of a critical connected chain but without the critical character $\lambda_1$. By Theorem 5.8 these eigenvalues are of multiplicity 1. Let $v_1, v_2$ be eigenvectors of $A_\lambda$ with eigenvalues $\lambda_2$ and $\lambda_3$ respectively. Then we have:

$$\phi(u_1) = a_1 v_1,$$

$$\phi(u_2) = a_2 v_1,$$

$$\phi(u_3) = a_3 v_1 + a_4 v_2,$$

$$\phi(u_4) = a_5 v_1 + a_6 v_2.$$  

Since $u_1, u_2, u_3$ and $u_4$ are linearly independent and their images span at most two dimensional space we have that

$\dim(\ker(\phi)) \geq 2$ that is impossible for irreducible modules by Corollary 5.9.

From Lemma 6.3 we immediately obtain.

**Corollary 6.4.** Let $\chi$ be a degenerate Gelfand-Tsetlin character such that $\chi(A) = \mu_1$ and $\chi|_{H} = \lambda \in \mathfrak{h}^*$. If $\mu_1, \mu_2 \in \mathbb{C}$ are connected with respect to $\lambda$, then there exist at most one irreducible extension of $\chi$ such that both $\mu_1$ and $\mu_2$ have multiplicity 2.
Proof. Indeed, any such irreducible module \( M \) will contain a critical character \( \chi' \) determined by a condition \( E_{23}(\chi) \subset M(\chi') + M(\chi'') \). But, by Theorem 6.1, \( \chi' \) defines \( M \) uniquely. \( \square \)

**Lemma 6.5.** Let \( M \) be an irreducible Gelfand-Tsetlin module such that \( M \) is singular but has no critical characters. Then \( A \) is diagonalizable on \( M \).

**Proof.** Fix \( \lambda \in \mathfrak{h}^* \) from the weight support of \( M \) and consider \( A_\lambda = A|_{M_\lambda} \). Then the distinct eigenvalues of \( A_\lambda \) form a singular connected chain with respect to \( \lambda \). If this chain is critical then \( A_\lambda \) is diagonalizable since there is no critical eigenvalue (see 5.8(ii)(d)). Suppose that the chain is degenerate. Then by Theorem 5.8(ii)(b) one can order the distinct eigenvalues of \( A_\lambda \) in the following way: \( \{\mu_1, \mu_2, \ldots, \mu_m\} \), where \( \mu_1 = -\frac{1}{2}(h_1^2 - 2h_3) \), and if the multiplicity of \( \mu_i \) equals 1 then the multiplicity of \( \mu_{i+1} \) is also 1. Suppose that \( M \) has a character \( \tilde{\chi} \) such that \( \tilde{\chi}(A) = \mu_1 \) and \( \mu_1 \) has multiplicity 2. If \( \mu_2 \) has multiplicity 2, then by Lemma 6.3 there exists a critical character \( \chi' \) in the Gelfand-Tsetlin support of every irreducible extension of \( \tilde{\chi} \), and we obtain a contradiction. Assume now that \( \mu_1 \) has multiplicity 2 but \( \mu_2 \) has multiplicity 1. Consider the weight subspace \( M' = M_{\lambda + \epsilon_2 - \epsilon_3} \) and \( A' = A|_{M'} \). Observe that \( M' \neq 0 \), otherwise \( \dim(\ker E_{23}|_{M'}) \geq 2 \) which is a contradiction by Corollary 5.9. If \( A' \) has no critical eigenvalue then neither does \( A|_{M_{\lambda + \epsilon_2 - \epsilon_3 + k(\epsilon_1 - \epsilon_2)}} \), for all integer \( k \). In this case all these subspaces \( M_{\lambda + \epsilon_2 - \epsilon_3 + k(\epsilon_1 - \epsilon_2)} \), \( k \in \mathbb{Z} \) can generate only one eigenvector of \( A' \) with eigenvalue \( \mu_1 \), and, hence, produce only multiplicity 1 eigenvalue \( \mu_1 \) of \( A' \). But this contradicts to the irreducibility of \( M \). Therefore, \( A' \) must contain a critical eigenvalue giving a contradiction again. Therefore, \( A_\lambda \) is diagonalizable, which completes the proof. \( \square \)

The proof of Lemma 6.5 implies immediately the following statement.

**Corollary 6.6.** Let \( M \) be an irreducible Gelfand-Tsetlin module and \( \chi \) a Gelfand-Tsetlin character of \( M \) associated with \( \lambda \in \mathfrak{h}^* \) and such that \( \dim(M(\chi)) = 2 \). Then \( M \) has a critical character \( \chi' \) associated with the weight \( \lambda + \epsilon_2 - \epsilon_3 \).

**Theorem 6.7.** Let \( M \) be an irreducible Gelfand-Tsetlin module and \( \chi \) a Gelfand-Tsetlin character such that \( \dim(M(\chi)) = 2 \). Then \( M \) is the unique irreducible extension of \( \chi \).

**Proof.** Let \( \chi|_H = \lambda \in \mathfrak{h}^* \). Since \( \dim(M(\chi)) = 2 \), the distinct eigenvalues of \( A_\lambda \) form a singular connected chain \( \{\mu_1, \ldots, \mu_m\}, m \leq \infty \). Moreover, there exists \( \mu_i \) of multiplicity 2. We will consider different cases.

(i) If the connected chain is critical, by Theorem 5.8 all distinct eigenvalues can be ordered in the following way: \( \{\mu_1, \mu_2, \ldots, \mu_m\} \), where \( \mu_1 + 1 = \mu_2 \), multiplicity of \( \mu_1 \) is 1, and if the multiplicity of \( \mu_i \) equals 1 for \( i > 1 \) then the multiplicity of \( \mu_{i+1} \) is also 1. Therefore, the module \( M \) has a critical character \( \chi' \) such that \( \chi'(A) = \mu_1 \). Thus, every irreducible extension of \( \chi \) contains \( \chi' \) in its Gelfand-Tsetlin support. Applying Theorem 6.1 we conclude that \( M \) is unique.

(ii) If the connected chain is degenerate, then by Theorem 5.8 one can order the distinct eigenvalues of \( A_\lambda \) in the following way: \( \{\mu_1, \mu_2, \ldots, \mu_m\} \), where \( \mu_1 = -\frac{1}{2}(h_1^2 - 2h_3) \), and if the multiplicity of \( \mu_i \) equals 1 then the multiplicity of \( \mu_{i+1} \) is also 1. Therefore \( M \) has a character \( \tilde{\chi} \) such that \( \tilde{\chi}(A) = \mu_1 \).
and $\mu_1$ has multiplicity 2. Then $A|_{M_{\lambda+1^2}}$ must contain a critical eigenvalue by Corollary 6.6. Thus, $M$ is unique by Theorem 6.1. This completes the proof.

\[ \square \]

**Theorem 6.8.** Let $M$ be an irreducible Gelfand-Tsetlin module such that $M$ is singular but has no critical characters. Then for each character $\chi \in \text{Supp}_{GT}(M)$, $M$ is the unique irreducible extension of $\chi$ without critical characters.

**Proof.** Let $\chi \in \text{Supp}_{GT}(M)$ associated with a weight $\lambda$. Let $A_\lambda = A|_{M_\lambda}$. It follows from Lemma 6.5 that $A_\lambda$ is diagonalizable.

(i) Suppose first that $\chi$ belongs to a critical connected chain. As $M$ does not have critical characters, $\chi(A)$ is in a generic part of a critical connected chain $\lambda_i = i^2 - \frac{1}{4} - \frac{1}{r}$, $1 \leq n \leq i \leq m \leq \infty$ for some integers $n$ and $m$. Then, as in Theorem 6.1, there exists a basis $\{w_i, n \leq i \leq m\}$ such that

\[
Aw_i = \lambda_i w_i, Bw_i = \begin{cases} 
  b_n w_n + w_{n+1}, & i = n \\
  d_i w_{i-1} + b_i w_i + w_{i+1}, & i > n \\
  d_m w_{m-1} + b_m w_m, & i = m.
\end{cases}
\]

Hence, $B_\lambda = B|_{M_\lambda}$ is determined uniquely by $\chi$ and $A_\lambda$. Since $M_\lambda$ is irreducible as a $C(\mathfrak{h})$-module, then $\chi(A)$ determines completely $A_\lambda$ and $B_\lambda$, implying the uniqueness. Therefore $M$ is the unique irreducible extension of $\chi$.

(ii) Consider now $\chi$ such that $\chi(A)$ belongs to a degenerate chain $\{\mu_1, \mu_2, \ldots\}$ where $\mu_n = n(n-1) - \frac{1}{2} r$, $n \geq 1$ (see Lemma 6.7(1)).

**Case 1. Assume that this chain does not contain a degenerate character, that is, the eigenvalues of $A|_{M_\lambda}$ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_s\}$ for some $k > 1$ and some $s < \infty$.** Applying relations (12) - (14) one can choose a basis $\{w_i, k \leq i \leq s\}$ of $M_\lambda$ such that the matrix of $A_\lambda$ is diagonal and the matrix of $B_\lambda$ has a tridiagonal form as in the generic case. Suppose there exists another irreducible extension $W$ of $\chi$ satisfying the conditions of the theorem such that the eigenvalues of $A|_{W_\lambda}$ are $\{\mu_d, \mu_{d+1}, \ldots, \mu_t\}$ for some $d > 1$ and some $t \leq \infty$. If $d = k$ and $s = t$, then the diagonal matrices $[A|_{M_\lambda}] = [A|_{W_\lambda}]$ will give the same matrix of $B_\lambda$, which implies $M \simeq W$.

Suppose $d < k$ (note that in this case $k \leq t$). Then applying relations (12) - (14) we obtain that $W_\lambda$ has a $C(\mathfrak{h})$-submodule $U$ such that the eigenvalues of $A|_U$ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_t\}$. Hence $M$ has a nontrivial proper submodule $\tilde{M}$ such that the eigenvalues of $A|_{\tilde{M}_\lambda}$ are $\{\mu_k, \mu_{k+1}, \ldots, \mu_t\}$. This contradicts the irreducibility of $M$. The case $d > k$ is treated analogously.

**Case 2. Suppose finally that the eigenvalues of $A|_{M_\lambda}$ are $\{\mu_1, \mu_2, \ldots, \mu_s\}$, and $\chi'$ is the character associated with $\mu_1$.** Using relations (12) - (14) we see that $r^2 + 2ar + 4\tau = 0$ and there exists a basis $\{w_i, 1 \leq i \leq s\}$ of $M_\lambda$ such that

\[
Aw_i = \lambda_i w_i, 1 \leq i \leq s, Bw_i = \begin{cases} 
  Tw_1 + w_2, & i = 1 \\
  q_i w_{i-1} + b_i w_i + w_{i+1}, & 1 < i < s \\
  q_s w_{s-1} + b_s w_s, & i = s.
\end{cases}
\]
where

\[ q_2 = \frac{1}{3} \left( -\frac{1}{8}r_1r^2 + \frac{1}{2}(r_1 + \tau_1)r - \eta \right) \]

\[ q_i = d_i, \quad i > 2 \]

and \( T \) is a root of the equation

\[ x^2 - (r + a)x + \tau_1 - \frac{1}{8}r_1r^2 + \frac{1}{2}r\tau_1 - \eta = 0. \]

Let \( W \) be another irreducible extension of \( \chi \). Then \( \mu_1 \) must be an eigenvalue of \( A \mid_{W_\chi} \), otherwise \( M \) is not irreducible. In fact, the quadratic equation on \( T \) shows that there might exist two non isomorphic irreducible modules with the same degenerate chain \( \mu_1, \ldots, \mu_\ell \). We will show that only one such module will satisfy the conditions of the theorem.

The hypothesis that there is no critical characters in all connected chains implies that \( E_{23}(M(\chi')) \subset M(\check{\chi}) \), where \( \check{\chi} \) is a character such that \( \check{\chi}(A) \) belongs to a generic part of a critical connected chain. Also \( E_{23}(W(\chi')) \subset W(\check{\chi}) \). If both \( E_{23}(M(\chi')) \) and \( E_{23}(W(\chi')) \) are non-zero then \( M(\check{\chi}) \neq 0 \) and \( W(\check{\chi}) \neq 0 \). We immediately conclude that \( M \cong W \) by Theorem 6.1. Suppose \( E_{23}(M(\chi')) = E_{23}(W(\chi')) = 0 \). Apply the same arguments for \( E_{13} \). If both \( E_{13}(M(\chi')) \) and \( E_{13}(W(\chi')) \) are non-zero, then \( M \cong W \) as above. On the other hand, if \( E_{13}(M(\chi')) = E_{13}(W(\chi')) = 0 \), then \( M \) and \( W \) are irreducible quotients of the same generalized Verma module generated by a weight vector \( v \) such that \( E_{23}v = E_{13}v = 0 \). But such generalized Verma module has a unique irreducible quotient implying \( M \cong W \). Hence, it remains to consider mixed cases. Consider the following cases.

(a) Suppose, \( E_{23}(M(\chi')) \neq 0, \ E_{23}(W(\chi')) = 0, \ E_{13}(M(\chi')) \neq 0, \) and \( E_{13}(W(\chi')) = 0 \). Then \( W \) is a quotient of the generalized Verma module \( M_1 \) generated by an element \( v \in M_1 \) such that \( E_{23}v = E_{13}v = 0 \). Suppose \( E_{32}(W(\chi')) \neq 0 \) and \( E_{31}(W(\chi')) \neq 0 \). If one of \( E_{32}(M(\chi')) \) or \( E_{31}(M(\chi')) \) is non-zero then we are done. Suppose \( E_{32}(M(\chi')) = E_{31}(M(\chi')) = 0 \) and thus \( M \) is a quotient of generalized Verma module \( M_2 \) generated by \( \nu' \) such that \( E_{32}\nu' = E_{31}\nu' = 0 \). In order not to have critical characters both \( M \) and \( W \) must be completely pointed modules, that is all weight spaces have dimension 1. Let \( \chi'(H_1) = h_1, \ \chi'(H_2) = h_2 \). Comparing the values of Casimir elements on \( M \) and \( W \) we obtain \( h_1 = -2h_2 \). This condition guarantees that \( M \) and \( W \) have common degenerate character \( \chi' \). Let us find the condition when \( W \) is a completely pointed module. It is sufficiently to check when the following system has a non-trivial solution:

\[
\begin{cases}
E_{23}(\alpha E_{32}v + \beta E_{31}E_{12}v) = 0; \\
E_{13}(\alpha E_{32}v + \beta E_{31}E_{12}v) = 0.
\end{cases}
\]

We have

\[
\begin{cases}
h_2\alpha v + \beta E_{21}E_{12}v = 0, \\
\alpha E_{12}v + \beta (h_1 + h_2 + 1)E_{12}v = 0.
\end{cases}
\]
Assume \( E_{12}v \neq 0 \). Then we have
\[
\begin{align*}
  h_2\alpha - \beta h_1 - \frac{1}{2}\beta(h_1^2 - 2h_1) &= 0, \\
  \alpha + \beta(h_1 + h_2 + 1) &= 0.
\end{align*}
\]

If \( h_1 = 0 \), then \( M \simeq W \simeq \mathbb{C} \). If \( h_1 \neq 0 \) then \( h_2 = 2 \), \( h_2 = -1 \) and \( \chi'(A) = 0 \). It follows that \( E_{23}v = E_{13}v = E_{21}v = 0 \) with \( h_1 = 2 \), \( h_2 = -1 \) or \( E_{23}v = E_{13}v = E_{12}v = 0 \) with \( h_1 = -2 \), \( h_2 = 1 \).

Consider first the case \( h_1 = -2 \). Let \( \mu \in \mathfrak{h}^* \) be such that \( \mu(H_1) = \mu(H_2) = -1 \). Then \( W_\mu \) is 1-dimensional and \( E_{12}W_\mu = 0 \). Hence, \( W_\mu \) is a Gelfand-Tsetlin subspace and \( A|_{W_\mu} = -I \). But, this is a critical value and, thus, \( W \) contains critical characters, which is a contradiction.

Suppose now that \( h_1 = 2 \) and consider \( \mu \in \mathfrak{h}^* \) such that \( \mu(H_1) = 1 \), \( \mu(H_2) = -2 \). Then \( W_\mu \) is 1-dimensional and \( E_{21}W_\mu = 0 \). Hence, \( W_\mu \) is a Gelfand-Tsetlin subspace and \( A|_{W_\mu} = 0 \). Again, this is a critical value which is a contradiction.

Suppose now that \( E_{23}(W(\chi')) = 0 \). Then \( E_{12}(W(\chi')) = 0 \) and \( W \) is a highest weight module of highest weight \( \chi'|_\mu \) and \( \chi'(H_2) = h_2 = 0 \).

Since \( \chi' \) is degenerate we have \( h_1 = 0 \) or \( h_1 = -2 \). In the case \( h_1 = 0 \) we obtain \( M \simeq W \simeq \mathbb{C} \). Now suppose \( h_1 = -2 \) and \( A|_{W(\chi')} = -2I \). This highest weight module has no critical characters. Since \( E_{31}W(\chi') \neq 0 \) we have \( E_{31}M(\chi') = 0 \), otherwise \( M \simeq W \) as before.

Since \( A|_{M(\chi')} = -2I \) and all characters have multiplicity 1, we have \( E_{12}M(\chi') = 0 \). Thus, in addition we have \( E_{32}M(\chi') = 0 \). Consider a weight \( \mu \in \mathfrak{h}^* \) such that \( \mu(H_1) = -1 \), \( \mu(H_2) = 1 \). The subspace \( M_\mu \) is 1-dimensional. In fact, this is a critical Gelfand-Tsetlin subspace, since \( A|_{M(\chi')} = -I \). Hence, \( M \) does not satisfy the conditions of the theorem and \( W \) again is a unique required module.

(b) Suppose \( E_{23}(M(\chi')) \neq 0 \), \( E_{32}(W(\chi')) = 0 \), \( E_{13}(M(\chi')) = 0 \) and \( E_{12}(W(\chi')) \neq 0 \). Now we act by \( E_{32} \) and \( E_{31} \). Suppose first that \( E_{32}(M(\chi')) = 0 \) and \( E_{31}(W(\chi')) = 0 \). Hence, \( M \) contains a non-zero vector \( v \) such that \( E_{13}v = E_{32}v = E_{12}v = 0 \). On the other hand, \( W \) contains a non-zero vector \( v' \) such that \( E_{31}v' = E_{23}v' = E_{21}v' = 0 \).

Moreover, \( H_1v = H_1v' = 0 \). But, since \( A \) is diagonalizable on \( M \) and on \( W \), we have \( E_{21}v = E_{12}v' = 0 \). Thus \( M \simeq W \simeq \mathbb{C} \).

Suppose now \( E_{31}(M(\chi')) = 0 \) and \( E_{32}(W(\chi')) = 0 \). Hence, \( M \) contains a non-zero vector \( v \) such that \( E_{13}v = E_{31}v = 0 \), and \( W \) contains a non-zero vector \( v' \) such that \( E_{32}v' = E_{23}v' = 0 \). We obtain that \( H_1v = H_1v' = 0 \) and \( H_2v = H_2v' = 0 \). Moreover, \( Av = Av' = 0 \).

Since \( A \) is diagonalizable on \( M \) and \( W \) we have \( E_{12}v = E_{12}v' = 0 \) and \( E_{21}v = E_{21}v' = 0 \). Thus \( M \simeq W \simeq \mathbb{C} \).

Finally, let \( E_{31}(M(\chi')) = 0 \) and \( E_{32}(M(\chi')) = 0 \). Hence, \( M \) contains a non-zero vector \( v \) such that \( E_{13}v = E_{31}v = E_{32}v = 0 \), implying \( E_{12}v = 0 \). So, either \( H_1v = H_2v = 0 \) and \( M \simeq W \simeq \mathbb{C} \), or \( H_1v = -2 \) and \( H_2v = 2 \). In the latter case \( W \) contains a non-zero vector \( v' \) such that \( E_{23}v' = 0 \), \( H_1v' = -2v' \) and \( Av' = -2v' \). Hence, \( E_{12}v' = 0 \) and \( E_{23}v' = 0 \) since \( A \) is diagonalizable. We have \( c_1v = c_1v' \) implying \( H_2v = H_2v' = 0 \) which is a contradiction.
(c) Suppose $E_{23}(M(\chi')) \neq 0$, $E_{23}(W(\chi')) = 0$, $E_{13}(M(\chi')) = 0$ and $E_{13}(W(\chi')) = 0$. Now we act by $E_{32}$ and $E_{31}$ on $M(\chi')$ and $W(\chi')$. Without loss of generality we may assume that $E_{32}(M(\chi')) = 0$ and $E_{33}(W(\chi')) = 0$. Therefore $E_{21}W(\chi') = [E_{23}, E_{31}]W(\chi') = 0$ and $AW(\chi') = 0$. Hence, we have either $H_1w = H_2w = 0$ or $H_1w = 2w$, $H_2w = -2w$ for any $w \in W(\chi')$. In the first case we obtain $M \simeq W \simeq \mathbb{C}$. Consider the second case. Since $AM(\chi') = 0$, we must have $E_{21}(M(\chi')) = 0$ (otherwise $A$ will not be diagonalizable in the weight subspace $M_\mu$, where $E_{21}(M(\chi')) \subset M_\mu$. Thus $E_{23}(M(\chi')) = [E_{21}, E_{13}]M(\chi) = 0$. Which is a contradiction. All other cases are considered analogously.

\[ \square \]

We summarize the obtained results in the following

**Theorem 6.9.** Let $M$ be an irreducible Gelfand-Tsetlin module and $\chi$ a Gelfand-Tsetlin character in $\text{Supp}_{\text{GT}}(M)$. Then

(i) If $\chi$ is a non critical character and for any character $\chi' \in \Gamma^*$ the dimension of $M(\chi')$ is less or equal to 1, then $M$ is the unique irreducible extension of $\chi$ with such property;

(ii) If $\chi$ is a non critical character and $\dim(M(\chi)) = 2$, then $M$ is a unique irreducible extension of $\chi$;

(iii) If $\chi$ is a critical character, then $M$ is its unique irreducible extension;

(iv) If $M$ has no critical characters, then $M$ is a unique irreducible extension of $\chi$.

**Proof.** Let $\chi$ be a non critical character of $M$. Suppose first that $M$ is generic. Then $M$ is the only irreducible extension of $\chi$ by Theorem 5.13. Suppose now that $M$ is singular and satisfies the conditions of item (i). Assume first that $\chi(A)$ belongs to a critical chain but $\chi$ itself is non critical. Then statement (i) follows immediately from Corollary 6.2. Assume now that $\chi(A)$ belongs to a degenerate chain $\{\mu_1, \mu_2, \ldots\}$ where $\mu_n = n(n-1) - \frac{1}{2}r$, $n \geq 1$ (see Lemma 5.7(i)). Suppose that $M^1$ and $M^2$ are two irreducible extensions of $\chi$ satisfying the conditions of (i). If $\mu_1$ is not an eigenvalue of $A_{\lambda}^1 = A|_{M_{\lambda}^1}$, where $\lambda = \chi|_H$, then $M^1 \simeq M^2$ since $A_{\lambda}^1$ (and thus $M^1$) is uniquely determined by $\chi(A)$ in this case. Assume that $\mu_1$ is an eigenvalue of both $A_{\lambda}^1$, $i = 1, 2$. Consider the eigenvalues of $A_{\lambda+\epsilon_2-\epsilon_3}^1$. They belong to the same critical chain. If both $A_{\lambda+\epsilon_2-\epsilon_3}^1$ have a common non critical eigenvalue then $M^1 \simeq M^2$ by Corollary 6.2. Hence, suppose that $A_{\lambda+\epsilon_2-\epsilon_3}^1$ have distinct eigenvalues. This is only possible if $\dim M_{\lambda+\epsilon_2-\epsilon_3}^1 = 1$.

**Case 1.** $\chi(A) - k\chi(H_2) \neq 0$ for all integer $k$ and $b_i - m\chi(H_2) \neq 0$ for all integer $m$. In this case both $M^1$ and $M^2$ are pointed torsion free modules with all multiplicities equal to 1. Consider the weight $\mu = \lambda + \epsilon_2 - \epsilon_3$ and let $\mu(H_i) = h_i$, $i = 1, 2$. Then we have $A_{\mu}^1 = \lambda_1 \cdot I$ and $A_{\mu}^2 = \lambda_1 \cdot I$ (up to a permutation), where $\lambda_0 = -\frac{1}{2} - \frac{1}{2}r$ and $\lambda_1 = \frac{1}{2} - \frac{1}{2}r$, $r = \frac{1}{2}(h_2^2 - 2h_1)$. We also have $\theta(\lambda_0) = \theta(\lambda_1) = 0$.

Substitute $H_1$ and $H_2$ instead of $h_1$ and $h_2$ in $f(h_1, h_2) = \theta(\lambda_0)$. Then $f(H_1, H_2)$ is a polynomial in $H_1$ and $H_2$ which has degree 8 in $H_1$ and has a pair $(h_1, h_2)$ as one of
its solutions. An easy calculation shows that $A_{\mu+2c_2-2c_3}^{\lambda_1'} = \lambda_1' \cdot I$ and $A_{\mu+2c_2-2c_3}^{\lambda_1'} = \lambda_1' \cdot I$, where $\lambda_1' = \frac{2}{3} - \frac{1}{3}r'$ and $\lambda_2' = -\frac{2}{3} + \frac{1}{3}r'$, $r' = \frac{1}{2}((h_1 - 2)^2 - 2(h_1 - 2))$. Then $f(h_1 - 2, h_2 + 4) = \theta(\lambda_1') = 0$. If we now move to the weight $\mu' = \mu + 4\xi_1 - 2c_2 - 2c_3$ (by applying $E_{12}^l$) and repeat the argument then we will obtain $f(h_1 - 6, h_2) = 0$. Now moving vertically by $E_{12}^l$ and then by $E_{12}^l$ we obtain $f(h_1 + 12, h_2) = 0$ and so on. Consider the polynomial $g(H_1) = f(H_1, h_2)$. Then $g$ is a polynomial of degree 8 which has infinitely many roots. We have a contradiction which shows that $M^1$ and $M^2$ can not be pointed under our assumptions.

Case 2. Suppose now that only one of the assumptions of Case 1 holds. Without loss of generality we may assume that $\chi(A) = 0$ and $b_i - m\chi(H_2) \neq 0$ for all integer $m$, $i = 1, 2$. Since $A$ is diagonalizable on both modules, then this immediately implies that both $M^1$ and $M^2$ are quotients of generalized Verma modules (induced from infinite dimensional irreducible sl(2)-module $W$) that have the same central character and the same weight support. Since the generalized Harish-Chandra homomorphism defines $W$ uniquely we conclude that $M^1 \simeq M^2$. For the same reason the statement follows when $\chi(A) = 0$ and $b_i - m\chi(H_2) \neq 0$ for all integer $m$ just for one $i$.

Case 3. Finally consider the case when $\chi(A) = 0$ and $b_i = m_i\chi(H_2)$ for some integer $m_i$, $i = 1, 2$. Therefore, as in Case 2, both $M^1$ and $M^2$ are quotients of the same generalized Verma module. Hence, $m_1 = m_2$ and $M^1 \simeq M^2$. This completes the proof of (i).

If $\chi$ is a non critical character such that $\dim(M(\chi)) = 2$, then $M$ is a unique irreducible extension of $\chi$ by Theorem 6.1 implying item (ii). Item (iii) follows immediately from Theorem 6.1. Now we prove item (iv). Suppose $M$ is generic. Then uniqueness of irreducible extension for any character of $M$ again follows from Theorem 5.13. If $M$ is singular but without critical characters, then item (iv) follows from Theorem 6.8. □

7. Realizations of all irreducible Gelfand-Tsetlin modules for $\mathfrak{sl}(3)$

In this section we will give an explicit realization of any irreducible Gelfand-Tsetlin module for $\mathfrak{sl}(3)$. To do that we will consider any Gelfand-Tsetlin character $\chi \in \Gamma^*$ and construct a Gelfand-Tsetlin module $M$ such that any irreducible extension of $\chi$ is isomorphic to some subquotient of $M$ (remember that by Theorem 4.8 the number of non-isomorphic irreducible extensions is at least one and at most two). By Remark 4.2 there exists a natural correspondence between Gelfand-Tsetlin characters and Gelfand-Tsetlin tableaux; by using this correspondence, given a character $\chi$ we can associate a tableau $T(\chi)$ and the problem of construct an irreducible extension for $\chi$ becomes a problem of find irreducible modules with tableaux realization containing $T(\chi)$ as a basis element.

Definition 7.1. Let $T(w)$ be any Gelfand-Tsetlin tableau with entries $w_{ij}$.

(i) $T(w)$ is called generic, if $w_{21} - w_{22} \notin \mathbb{Z}$.

(ii) $T(w)$ is called singular, if $w_{21} - w_{22} \in \mathbb{Z}$.

(iii) $T(w)$ is called critical, if $w_{21} - w_{22} = 0$.

Remember that any Gelfand-Tsetlin tableau $T(\chi)$ of high 3, the tableau is generic $(v_{21} - v_{22} \notin \mathbb{Z})$ or 1-singular $(v_{21} - v_{22} \in \mathbb{Z})$, then the constructions in 4.1 allow us to describe an explicit Gelfand-Tsetlin module $V(T(\chi))$ for any $T(\chi)$. 

In this section we will describe explicitly all irreducible subquotients of modules of the form $V(T(v))$ (with $T(v)$ generic or 1-singular); which by Theorem 6.9 will be enough for the complete classification.

### 7.1. Structure of generic $sl(3)$-modules.

In this subsection we will consider all possible generic Gelfand-Tsetlin tableaux $T(v)$ of height 3 and describe all irreducible subquotients of the $sl(3)$-module $V(T(v))$. The description will include explicit basis for each irreducible module, weight support and Loewy decomposition. Let us fix some notation and remember the definition of the module $V(T(v))$ and the Gelfand-Tsetlin formulas for $sl(3)$.

As we want to work with $sl(3)$, we will assume that the action of $E_{11} + E_{22} + E_{33}$ is zero. Therefore, we will consider Gelfand-Tsetlin tableaux $T(w)$ such that $w_{31} + w_{32} + w_{33} + 3 = 0$. Let us rewrite Theorem 4.13 in the case of $sl(3)$.

**Definition 7.2.** For any Gelfand-Tsetlin tableau $T(v)$ we will denote by $B(T(v))$ the following set \( \{ T(v + z) \mid z \in T_2(\mathbb{Z}) \} \). Also, for any $T(w) \in B(T(v))$ set:

\[
\Omega(T(w)) := \{ (r, s, t) \mid w_{rs} - w_{r-1,t} \in \mathbb{Z} \},
\]

\[
\Omega^+(T(w)) := \{ (r, s, t) \mid w_{rs} - w_{r-1,t} \in \mathbb{Z}_{\geq 0} \}.
\]

**Theorem 7.3.** If $T(v)$ is a generic Gelfand-Tsetlin tableau of height 3, then the vector space $V(T(v))$ spanned by the set of tableaux $B(T(v))$ has a structure of a Gelfand-Tsetlin $sl(3)$-module with the action of $sl(3)$ on $V(T(v))$ given by the Gelfand-Tsetlin formulas:

\[
\begin{align*}
H_1(T(w)) &= (2w_{11} - (w_{21} + w_{22} + 1))T(w), \\
H_2(T(w)) &= (2(w_{21} + w_{22} + 1) - w_{11})T(w), \\
E_{12}(T(w)) &= -(w_{21} - w_{11})(w_{22} - w_{11})T(w + \delta^{11}), \\
E_{21}(T(w)) &= T(w - \delta^{11}), \\
E_{32}(T(w)) &= \frac{(w_{21} - w_{22})}{(w_{21} - w_{22})}T(w - \delta^{21}) - \frac{(w_{22} - w_{11})}{(w_{21} - w_{22})}T(w - \delta^{22}), \\
E_{23}(T(w)) &= \frac{(w_{31} - w_{21})(w_{32} - w_{21})(w_{33} - w_{21})}{(w_{21} - w_{22})}T(w + \delta^{21}) - \frac{(w_{31} - w_{22})(w_{32} - w_{22})(w_{33} - w_{22})}{(w_{21} - w_{22})}T(w + \delta^{22}).
\end{align*}
\]

By direct computation we have also the explicit action of $E_{13}$ and $E_{31}$ on any tableau.

\[
\begin{align*}
E_{13}(T(w)) &= \frac{(w_{21} - w_{31})(w_{21} - w_{32})(w_{21} - w_{33})(w_{22} - w_{11})}{(w_{21} - w_{22})}T(w + \delta^{21} + \delta^{11}) \\
&\quad + \frac{(w_{31} - w_{22})(w_{32} - w_{22})(w_{33} - w_{22})(w_{21} - w_{11})}{(w_{21} - w_{22})}T(w + \delta^{22} + \delta^{11}), \\
E_{31}(T(w)) &= \frac{1}{(w_{21} - w_{22})}T(w - \delta^{21} - \delta^{11}) - \frac{1}{(w_{21} - w_{22})}T(w - \delta^{22} - \delta^{11}).
\end{align*}
\]

By Theorem 5.13 a generic character admits a unique irreducible extension. In order to describe such irreducible extension, given a tableau $T(w) \in B(T(v))$ we
will describe explicit basis of tableaux for the irreducible subquotient of $V(T(v))$ that contains $T(w)$.

By the Gelfand-Tsetlin formulas, it is clear that $\mathcal{B}(T(v))$ is an eigenbasis for the action of the generators of the Cartan subalgebra $\mathfrak{h}$. In particular, any subquotient of $V(T(v))$ is a weight module. The following proposition describes explicitly the weight subspaces of subquotients of $V(T(v))$.

**Proposition 7.4.** Let $M$ be a Gelfand-Tsetlin module with basis of tableaux $\mathcal{B}_M \subset \mathcal{B}(T(v))$ for some generic tableau $T(v)$. If $T(w) \in \mathcal{B}_M$ is a tableau of weight $\lambda$, then the weight space $M_\lambda$ is spanned by the set of tableaux $\{T(w + (i, -i, 0)) : i \in \mathbb{Z}\} \cap \mathcal{B}_M$.

**Proof.** As $\mathcal{B}(T(v)) = \mathcal{B}(T(w))$, we just need to characterize tableaux of the form $T(w + (m, n, k))$ in $\mathcal{B}_M$ with the same weight $\lambda$ of $T(w)$. By the Gelfand-Tsetlin formulas we have

$$H_1(T(w + (m, n, k))) = (2(w_{11} + k) - (w_{21} + m + w_{22} + n + 1))T(w + (m, n, k)),$$

$$H_2(T(w + (m, n, k))) = (2(w_{21} + m + w_{22} + n + 1) - (w_{11} + k))T(w + (m, n, k))$$

In particular, $T(w)$ is vector of weight

$$\lambda = (2w_{11} - (w_{21} + w_{22} + 1), 2(w_{21} + w_{22} + 1) - w_{11}).$$

So, a tableau $T(w + (m, n, k))$ in $\mathcal{B}_M$ has weight $\lambda$ if $m, n, k$ satisfy the following linear system

$$2w_{11} - (w_{21} + w_{22} + 1) = 2(w_{11} + k) - (w_{21} + m + w_{22} + n + 1),$$

$$2(w_{21} + w_{22} + 1) - w_{11} = 2(w_{21} + m + w_{22} + n + 1) - (w_{11} + k),$$

which is, $m + n = 0$ and $k = 0$.

Remember that for any Gelfand-Tsetlin tableau $T(w)$ the character $\chi_w \in \Gamma^*$ is given by $\chi_w(c_{rs}) = \gamma_{rs}(w)$.

**Definition 7.5.** For each Gelfand-Tsetlin tableau $T(v)$ we denote $\mathcal{C}(v)$ the set of characters $\{\chi_w \mid T(w) \in \mathcal{B}(T(v))\}$. The block of $\mathcal{G}\mathcal{T}$ associated with $T(v)$ will be

$$\Theta(T(v)) := \{V \in \mathcal{G}\mathcal{T} \mid \text{Supp}_{\mathcal{G}\mathcal{T}}(V) \subset \mathcal{C}(v)\}.$$

Theorem [42] gives an explicit description of a basis of any irreducible subquotient of the generic modules $V(T(v))$. Let us remember this result for $\mathfrak{sl}(3)$-modules.

**Theorem 7.6.** If $T(v)$ is a generic Gelfand-Tsetlin tableau of high 3 and $T(w) \in \mathcal{B}(T(v))$ is a tableau associated with a Gelfand-Tsetlin character $\chi$, then

(i) The complex vector space generated by the set of tableaux

$$\mathcal{N}(T(w)) := \{T(w') \in \mathcal{B}(T(v)) \mid \Omega^+(T(w')) \subseteq \Omega^+(T(w'))\}$$

is a submodule of $V(T(v))$ containing $T(w)$.

(ii) The unique irreducible extension of $\chi$ has the following basis of tableaux:

$$\mathcal{I}(T(w)) := \{T(w') \in \mathcal{B}(T(v)) \mid \Omega^+(T(w)) = \Omega^+(T(w'))\}.$$
**Definition 7.7.** Let $T(w)$ be a tableau and $B$ be a subset of $\mathbb{Z}^3$ defined by a set of inequalities of the form $a \leq b$ or $a < b$ where $a, b$ are among $m, n, k, 0, -t, -s$ for some $s, t \in \mathbb{Z}_{\geq 0}$. Assume that $M$ is a Gelfand-Tsetlin module with basis $\{T(w + (m, n, k)) \mid (m, n, k) \in B\}$. Then we will denote $M$ by $M(B, T(w))$, or simply by $M(B)$ if $T(w)$ is fixed. If $M(B)$ is irreducible, then we will write $L(B)$ for $M(B)$.

**Remark 7.8.** Using the notation in Definition 7.7 we can write the basis of the irreducible module in Example 4.20 as follows:

$$L = \begin{pmatrix} m \leq 0 \\ k \leq m \\ n > -2 \end{pmatrix} = M \begin{pmatrix} m \leq 0 \\ k \leq m; T(v) \\ n > -2 \end{pmatrix}$$

where $v = (a, b, c, a, b + 2, a)$.

Using the description of irreducible subquotients, we will also be able to describe the Loewy series decomposition for $V(T(v))$ (the first module in the list being the socle) of $V(T(v))$.

**Theorem 7.9.** Let $T(v)$ be any generic tableau and set $t := |\Omega(T(v))|$. The Loewy series decomposition of the Gelfand-Tsetlin module $V(T(v))$ is given by

$$D_t, D_{t-1}, \ldots, D_0$$

where, $D_i = \text{Span}_C \{T(w) \in B(T(v)) \mid |\Omega^+(T(w))| = i\}$ and $0 \leq i \leq t$. If $D_i = \emptyset$ for some $1 < i < t$ we omit this term in the Loewy decomposition.

**Proof.** Let us show first that $D_i$ is an irreducible submodule of $V(T(v))$. By Theorem 7.6(i), if $|\Omega^+(T(w))| = t$, the module generated by $T(w)$ is irreducible and hence, equal to $\text{Span}_C \{T(w') \mid |\Omega^+(T(w'))| = t\}$. That is, $V(T(v))$ has a unique irreducible submodule $M_i$, namely $M = D_i$.

Set $M_{i+1} := V(T(v))$ and define $M_i := M_{i+1}/D_i$. Note that

$$M_{i+1} = \text{Span}_C \{T(w) \in B(T(v)) \mid |\Omega^+(T(w))| \leq i\}$$

So, by Theorem 7.6(ii) any element basis of $D_i$ is a basis element of an irreducible submodule of $M_{i+1}$ and, then $D_i$ is the sum of all irreducible submodules of $M_{i+1}$.

**Remark 7.10.** If $\{D_{ij}\}_{j=1}^r$ is the set of all non isomorphic irreducible subquotients of $V(T(v))$ generated by some tableau $T(w_j)$ such that $|\Omega^+(T(w_j))| = i$. The modules $D_i$ in the previous theorem can be decompose as follows:

$$D_i = D_{i1} \oplus \ldots \oplus D_{ir}$$

**Example 7.11.** Theorem 7.6 may suggest that category $\mathcal{G}T_{T(v)}$ is completely determined by $\Omega(T(v))$. In this example we present tableaux $T(v)$ and $T(v')$ such that $\Omega(T(v)) = \Omega(T(v'))$ however $\mathcal{G}T_{T(v)}$ is not equivalent to $\mathcal{G}T_{T(v')}$. Set

$$T(v) := \begin{array}{ccc} a & a & a \\ a & y & z \end{array} \quad T(v') := \begin{array}{ccc} a & a+1 & a+2 \\ a & y & z \end{array}$$

In this case we have $\Omega(T(v)) = \Omega(T(v'))$. The Loewy series for $T(v)$ is given by $D_3, D_0$ however, the Loewy series for $T(v')$ is given by $D_3, D_2, D_1, D_0$. 


7.2. Realizations of all irreducible generic Gelfand-Tsetlin $\mathfrak{sl}(3)$-modules.

In this section we will describe all possible blocks for generic Gelfand-Tsetlin characters (see Definitions 4.11 and 7.5). Such description will be by presentation of explicit basis for each irreducible subquotient in the blocks $\Theta(T(v))$, weight lattice description and components of the Loewy series for the universal module $V(T(v))$. A rigorous proof based in Theorem 7.6, Proposition 7.4 and Theorem 7.9 is given just for case (G6), for the other cases we use the same arguments.

From now on and until the end of this section we will use the letters $a, b, c, x, y, z$ to denote complex numbers. We will assume also that if any two different letters appear in the same or consecutive rows of a Gelfand-Tsetlin tableau then its difference is not an integer.

(G1) Consider the following Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
  a & b & c \\
  x & y \\
  z
\end{array}
\]

\[T(v):= x \ y \ z\]

In this case the module $V(T(v))$ is irreducible and, then the block $\Theta(T(v))$ consist of a single irreducible module with infinite dimensional Cartan weight multiplicities.

| Module | Basis       |
|--------|-------------|
| $L_1$  | $L(\mathbb{Z}^3)$ |

(G2) Let $T(v)$ be the tableau

\[
\begin{array}{ccc}
  a & b & c \\
  x & y \\
  x
\end{array}
\]

I. Irreducible subquotients.

In this case the module $V(T(v))$ has 2 irreducible subquotients with infinite dimensional weight multiplicities:

| Module | Basis       |
|--------|-------------|
| $L_1$  | $L(k \leq m)$ |
| $L_2$  | $L(m < k)$ |

II. Loewy series.

$L_1$, $L_2$.

(G3) Consider the tableau:

\[
\begin{array}{ccc}
  a & b & c \\
  a & y \\
  z
\end{array}
\]

\[T(v):= a \ y \ z\]
I. Irreducible subquotients.
In this case the module $V(T(v))$ has 2 irreducible subquotients, both of them with infinite dimensional weight multiplicities.

| Module | Basis |
|--------|-------|
| $L_1$  | $L(m \leq 0)$ |
| $L_2$  | $L(0 < m)$ |

II. Loewy series.
$L_1; L_2$.

(G4) Consider the tableau:

\begin{tabular}{ccc}
  a & b & c \\
  a & y &  \\
  y &  &  \\
\end{tabular}

I. Irreducible subquotients.
In this case the module $V(T(v))$ has 4 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c}
  m \leq 0 \\
  k \leq n
\end{array} \right)$ |
| $L_4$  | $L \left( \begin{array}{c}
  0 < m \\
  n < k
\end{array} \right)$ |

(ii) Modules with unbounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_2$  | $L \left( \begin{array}{c}
  0 < m \\
  k \leq n
\end{array} \right)$ |
| $L_3$  | $L \left( \begin{array}{c}
  m \leq 0 \\
  n < k
\end{array} \right)$ |
II. Loewy series.

\[ L_1; \ L_2 \oplus L_3; \ L_4. \]

\((G5)\) Let \( T(v) \) be the generic tableau

\[
\begin{array}{ccc}
  & a & b & c \\
  & a & y \\
  & a \\
\end{array}
\]

I. Irreducible subquotients.

In this case the module \( V(T(v)) \) has 4 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight multiplicities:

| Module \( L_2 \) | Basis \( L \) |
|------------------|------------------|
| \( m \leq 0 \) \( (m < k) \) |

| Module \( L_3 \) | Basis \( L \) |
|------------------|------------------|
| \( 0 < m \) \( (k \leq m) \) |

(ii) Two modules with unbounded finite weight multiplicities:

| Module \( L_1 \) | Basis \( L \) |
|------------------|------------------|
| \( m \leq 0 \) \( (k \leq m) \) |

| Module \( L_4 \) | Basis \( L \) |
|------------------|------------------|
| \( 0 < m \) \( (m < k) \) |

\[
\begin{array}{c}
\rightarrow L_1 \\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow L_4 \\
\end{array}
\]

II. Loewy series.

\[
L_1; \ L_2 \oplus L_3; \ L_4 
\]

\((G6)\) Consider the tableau:
I. Irreducible subquotients.

In this case the module $V(T(v))$ has 8 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight spaces

| Module | Basis |
|--------|-------|
| $L_4$  | $L \left( \begin{array}{c} 0 < m \\ n \leq 0 \\ k \leq m \end{array} \right)$ |
| $L_5$  | $L \left( \begin{array}{c} m \leq 0 \\ 0 < n \\ m < k \end{array} \right)$ |

(ii) Six modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \\ k \leq m \end{array} \right)$ |
| $L_2$  | $L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \\ m < k \end{array} \right)$ |
| $L_3$  | $L \left( \begin{array}{c} m \leq 0 \\ 0 < n \\ k \leq m \end{array} \right)$ |
| $L_6$  | $L \left( \begin{array}{c} 0 < m \\ n \leq 0 \\ m < k \end{array} \right)$ |
| $L_7$  | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ k \leq m \end{array} \right)$ |
| $L_8$  | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ m < k \end{array} \right)$ |
II. Loewy series.

\[ L_1; L_2 \oplus L_3 \oplus L_4; L_5 \oplus L_6 \oplus L_7; L_8. \]

**Proof.** If \( M \) denotes the universal module \( V(T(v)) \), we will prove that \( L_1 \) is an irreducible submodule of \( M \), then that \( M_1 := M/L_1 \) has irreducible submodules isomorphic to \( L_2 \), \( L_3 \) and \( L_4 \); then we prove that \( M_2 := M_1/(L_2 \oplus L_3 \oplus L_4) \) has irreducible submodules isomorphic to \( L_5 \), \( L_6 \) and \( L_7 \). And finally we prove that \( L_8 = M_2/(L_5 \oplus L_6 \oplus L_7) \) is an irreducible module.

By Theorem 4.24 we see that, \( L_1 \) (respectively \( L_2 \), \( L_3 \), \( L_4 \), \( L_5 \), \( L_6 \), \( L_7 \) and \( L_8 \)) is an irreducible subquotient of \( V(T(v)) \) containing the tableau \( T(v) \) (respectively \( T(v+(0, 0, 1)), T(v+(0, 1, 0)), T(v+(1, 0, 1)), T(v+(0, 1, 1)), T(v+(1, 0, 2)), T(v+(1, 1, 0)), T(v+(1, 1, 1)) \) and \( T(v+(1,1,2)) \).
Let us see that \(L_1\) is a submodule of \(M\). By Proposition 7.6(i) the module generated by \(N(T(v)) = \{T(w) \mid \Omega^+(T(v)) \subseteq \Omega^+(T(w))\}\) is a submodule of \(M\); but \(\Omega^+(T(v)) = \{(3, 1, 1), (3, 2, 2), (2, 1, 1)\} = \Omega(T(v))\) so, \(N(T(v)) = \mathcal{I}(T(v))\). Hence, \(L_1\) is an irreducible submodule of \(M\).

Now we prove that \(M_1 := M/L_1\) has irreducible submodules isomorphic to \(L_2\), \(L_3\) and \(L_4\). For these modules we have that

\[
\begin{align*}
\Omega^+(T(v + (1, 0, 1))) & = \{(3, 2, 2), (2, 1, 1)\}, \\
\Omega^+(T(v + (0, 1, 0))) & = \{(3, 1, 1), (3, 2, 2)\} \\
\Omega^+(T(v + (0, 1, 0))) & = \{(3, 1, 1), (2, 1, 1)\}.
\end{align*}
\]

Hence, by Proposition 7.6(i) the modules generated by \(\mathcal{I}(T(v + (1, 0, 1))) \cup \mathcal{I}(T(v))\), \(\mathcal{I}(T(v + (0, 1, 0))) \cup \mathcal{I}(T(v))\) and \(\mathcal{I}(T(v + (0, 1, 0))) \cup \mathcal{I}(T(v))\) are submodules of \(M\). Therefore, the modules generated by \(\mathcal{I}(T(v + (1, 0, 1)))\), \(\mathcal{I}(T(v + (0, 1, 0)))\) and \(\mathcal{I}(T(v + (0, 1, 0)))\) are submodules of \(M_1 := M/L_1\) since \(L_1\) is generated by \(\mathcal{I}(T(v))\).

Now we prove that \(M_2 := M_1/(L_2 \oplus L_3 \oplus L_4)\) has irreducible submodules isomorphic to \(L_5\), \(L_6\) and \(L_7\). For these modules we have

\[
\begin{align*}
\Omega^+(T(v + (0, 1, 1))) & = \{(3, 1, 1)\}, \\
\Omega^+(T(v + (1, 1, 1))) & = \{(2, 1, 1)\} \\
\Omega^+(T(v + (1, 0, 2))) & = \{(3, 2, 2)\}.
\end{align*}
\]

Hence, by Proposition 7.6(i) the modules generated by \(\mathcal{I}(T(v + (1, 0, 1))) \cup \mathcal{I}(T(v))\), \(\mathcal{I}(T(v + (0, 1, 0))) \cup \mathcal{I}(T(v))\) and \(\mathcal{I}(T(v + (0, 1, 0))) \cup \mathcal{I}(T(v))\) are submodules of \(M\). Therefore, the modules generated by \(\mathcal{I}(T(v + (1, 0, 1)))\), \(\mathcal{I}(T(v + (0, 1, 0)))\) and \(\mathcal{I}(T(v + (0, 1, 0)))\) are submodules of \(M_2 := M_1/(L_2 \oplus L_3 \oplus L_4)\) because \(L_1\) is generated by \(\mathcal{I}(T(v))\).

Finally we prove that \(M_3/(L_5 \oplus L_6 \oplus L_7) = L_8\). In fact, \(\Omega^+(T(v + (1, 1, 2))) = \emptyset\) so the submodule of \(V(T(v))\) generated by \(T(v + (1, 1, 2))\) is \(V(T(v))\) and the irreducible subquotient containing \(T(v + (1, 1, 2))\) has the same basis as \(M_3/(L_5 \oplus L_6 \oplus L_7)\) so, we have \(M_3/(L_5 \oplus L_6 \oplus L_7) = L_8\).

\(\square\)

\((G7)\) Consider the tableau:

\[
\begin{array}{c|c|c}
\hline
a & b & c \\
\hline
T(v)= & a & b \\
& z & \\end{array}
\]

I. Irreducible subquotients.

In this case the module \(V(T(v))\) has 4 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:
II. Loewy series.

$L_1; L_2 \oplus L_3; L_4$.

(G8) Set $t \in \mathbb{Z}_{>0}$ and consider the following Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
\text{a} & \text{a} - t & \text{c} \\
\text{a} & \text{y} \\
\text{z} \\
\end{array}
\]

I. Irreducible subquotients.

In this case the module $V(T(v))$ has 3 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:

\[
\begin{array}{|c|c|}
\hline
\text{Module} & \text{Basis} \\
\hline
L_1 & L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \end{array} \right) \\
L_3 & L \left( \begin{array}{c} m \leq 0 \\ 0 < n \end{array} \right) \\
\hline
\end{array}
\]

(ii) A cuspidal module with $t$-dimensional weight spaces:
II. Loewy series.

$L_1$, $L_2$, $L_3$.

\((G9)\) For each \(t \in \mathbb{Z}_{>0}\), let consider the following generic tableau:

\[
\begin{array}{ccc}
  a & a-t & c \\
  a & y & \\
  a & \\
\end{array}
\]

I. Irreducible subquotients.

In this case the module \(V(T(v))\) has 6 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| \(L_2\) | \(L\left(\begin{array}{c}
  m \leq -t \\
  m < k
\end{array}\right)\) |
| \(L_5\) | \(L\left(\begin{array}{c}
  0 < m \\
  k \leq m
\end{array}\right)\) |

(ii) Two modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| \(L_1\) | \(L\left(\begin{array}{c}
  m \leq -t \\
  k \leq m
\end{array}\right)\) |
| \(L_6\) | \(L\left(\begin{array}{c}
  0 < m \\
  m < k
\end{array}\right)\) |

(iii) Two modules with bounded weight multiplicities:
II. Loewy series.

\[ L_1; \ L_2 \oplus L_3; \ L_4 \oplus L_5; \ L_6. \]

\((G10)\) For each \( t \in \mathbb{Z}_{>0} \), let \( T(v) \) be the following Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
   a & a-t & c \\
   a & y & \\
   y & \\
\end{array}
\]

I. Irreducible subquotients.
In this case the module \( V(T(v)) \) has 6 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_1 \) | \( L \begin{cases} m \leq -t \\ k \leq n \end{cases} \) |
| \( L_6 \) | \( L \begin{cases} 0 < m \\ n < k \end{cases} \) |

(ii) Two modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_3 \) | \( L \begin{cases} m \leq -t \\ n < k \end{cases} \) |
| \( L_4 \) | \( L \begin{cases} 0 < m \\ k \leq n \end{cases} \) |
(iii) Two modules with bounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_2$  | $L \begin{cases} -t < m \leq 0 \\ k \leq n \end{cases}$ |
| $L_5$  | $L \begin{cases} -t < m \leq 0 \\ n < k \end{cases}$ |

II. Loewy series.

$L_1; \ L_2 \oplus L_3; \ L_4 \oplus L_5; \ L_6$.

(G11) For any $t \in \mathbb{Z}_{>0}$, let $T(v)$ be the tableau:

I. Irreducible subquotients.
In this case the module $V(T(v))$ has 12 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:
(ii) Six modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_5$  | $L \begin{cases} m \leq -t \\ 0 < n \\ m < k \end{cases}$ |
| $L_8$  | $L \begin{cases} 0 < m \\ n \leq 0 \\ k \leq m \end{cases}$ |

| Module | Basis |
|--------|-------|
| $L_1$  | $L \begin{cases} m \leq -t \\ n \leq 0 \\ k \leq m \end{cases}$ |
| $L_2$  | $L \begin{cases} m \leq -t \\ n \leq 0 \\ m < k \end{cases}$ |
| $L_3$  | $L \begin{cases} m \leq -t \\ 0 < n \\ k \leq m \end{cases}$ |
| $L_{10}$ | $L \begin{cases} 0 < m \\ n \leq 0 \\ m < k \end{cases}$ |
| $L_{11}$ | $L \begin{cases} 0 < m \\ 0 < n \\ k \leq m \end{cases}$ |
| $L_{12}$ | $L \begin{cases} 0 < m \\ 0 < n \\ m < k \end{cases}$ |
(iii) Four modules with bounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_4$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ n \leq 0 \\ k \leq m \end{array} \right)$ |
| $L_6$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ 0 < n \\ k \leq m \end{array} \right)$ |
| $L_7$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ n \leq 0 \\ m < k \end{array} \right)$ |
| $L_9$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ 0 < n \\ m < k \end{array} \right)$ |
II. Loewy series.

\[ L_1, L_2 \oplus L_3 \oplus L_4, L_5 \oplus L_6 \oplus L_7 \oplus L_8, L_9 \oplus L_{10} \oplus L_{11}, L_{12}. \]

(G12) Consider \( t \in \mathbb{Z}_{>0} \), and \( T(v) \) to be the Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
  a & b & b - t \\
  a & b \\
  a
\end{array}
\]

I. Irreducible subquotients.

In this case the module \( V(T(v)) \) has 12 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_4 \) | \( L \left( \begin{array}{c}
  0 < m \\
  n \leq -t \\
  k \leq m
\end{array} \right) \) |
| \( L_9 \) | \( L \left( \begin{array}{c}
  m \leq 0 \\
  0 < n \\
  m < k
\end{array} \right) \) |

(ii) Six modules with unbounded weight multiplicities:
| Module | Basis |
|-------|-------|
| $L_1$ | $L$ \begin{align*} m & \leq 0 \\ n & \leq -t \\ k & \leq m \end{align*} |
| $L_2$ | $L$ \begin{align*} m & \leq 0 \\ n & \leq -t \\ m & < k \end{align*} |
| $L_5$ | $L$ \begin{align*} m & \leq 0 \\ 0 & < n \\ k & \leq m \end{align*} |
| $L_6$ | $L$ \begin{align*} 0 & < m \\ n & \leq -t \\ m & < k \end{align*} |
| $L_{11}$ | $L$ \begin{align*} 0 & < m \\ 0 & < n \\ k & \leq m \end{align*} |
| $L_{12}$ | $L$ \begin{align*} 0 & < m \\ 0 & < n \\ m & < k \end{align*} |
(iii) Four modules with bounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_3$  | $L$ \( \begin{cases} -t < n \leq 0 \\ m \leq 0 \\ k \leq m \end{cases} \) |
| $L_7$  | $L$ \( \begin{cases} m \leq 0 \\ -t < n \leq 0 \\ m < k \end{cases} \) |
| $L_8$  | $L$ \( \begin{cases} 0 < m \\ -t < n \leq 0 \\ k \leq m \end{cases} \) |
| $L_{10}$ | $L$ \( \begin{cases} 0 < m \\ -t < n \leq 0 \\ m < k \end{cases} \) |
II. Loewy series.

\[ L_1, L_2 \oplus L_3 \oplus L_4, L_5 \oplus L_6 \oplus L_7 \oplus L_8, L_9 \oplus L_{10} \oplus L_{11}, L_{12}. \]

\[ (G13) \text{ Consider } t \in \mathbb{Z}_{\geq 0}. \text{ Set } T(v) \text{ to be the following Gelfand-Tsetlin tableau:} \]

\[ \begin{array}{ccc}
  a & a-t & c \\
  a & c & z \\
\end{array} \]

I. Irreducible subquotients.

In this case the module \( V(T(v)) \) has 6 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:

\[
\begin{array}{|c|c|}
\hline
\text{Module} & \text{Basis} \\
\hline
L_3 & L \left\{ \begin{array}{l}
m \leq -t \\
0 < n
\end{array} \right. \\
\hline
L_4 & L \left\{ \begin{array}{l}
0 < m \\
n \leq 0
\end{array} \right. \\
\hline
\end{array}
\]

(ii) Two modules with unbounded finite weight multiplicities:

\[
\begin{array}{|c|c|}
\hline
\text{Module} & \text{Basis} \\
\hline
L_1 & L \left\{ \begin{array}{l}
m \leq -t \\
n \leq 0
\end{array} \right. \\
\hline
L_6 & L \left\{ \begin{array}{l}
0 < m \\
0 < n
\end{array} \right. \\
\hline
\end{array}
\]
(iii) Two modules with bounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_2$  | $L \begin{cases} -t < m \leq 0 \\ n \leq 0 \end{cases}$ |
| $L_5$  | $L \begin{cases} -t < m \leq 0 \\ 0 < n \end{cases}$ |

II. Loewy series.

$L_1; L_2 \oplus L_3; L_4 \oplus L_5; L_6$.

(G14) Set $t, s \in \mathbb{Z}_{\geq 0}$ with $t < s$ and let $T(v)$ be the following Gelfand-Tsetlin tableau:

```
T(v) = | a       | a-t     | a-s     |
        | a       | y       |
        |         | a       |
```

I. Irreducible subquotients.

In this case the module $V(T(v))$ has 8 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:
(ii) Two modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left( \begin{array}{c} m \leq -s \\ m < k \end{array} \right)$ |
| $L_6$  | $L \left( \begin{array}{c} 0 < m \\ k \leq m \end{array} \right)$ |

(iii) Four modules with bounded weight multiplicities; two of them with multiplicities bounded by $t$ and two of them with multiplicities bounded by $r := s - t$:

| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c} m \leq -s \\ k \leq m \end{array} \right)$ |
| $L_8$  | $L \left( \begin{array}{c} 0 < m \\ m < k \end{array} \right)$ |
| $L_2$  | $L \left( \begin{array}{c} -s < m \leq -t \\ k \leq m \end{array} \right)$ |
| $L_4$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ k \leq m \end{array} \right)$ |
| $L_5$  | $L \left( \begin{array}{c} -s < m \leq -t \\ m < k \end{array} \right)$ |
| $L_7$  | $L \left( \begin{array}{c} -t < m \leq 0 \\ m < k \end{array} \right)$ |
II. Loewy series.

\[ L_1; L_2 \oplus L_3; L_4 \oplus L_5; L_6 \oplus L_7; L_8. \]

\((G15)\) Set \( t, s \in \mathbb{Z}_{>0} \) with \( t < s \) and let \( T(v) \) be the following tableau:

\[
\begin{array}{ccc}
  a & a-t & a-s \\
  \hline
  a & y \\
  y
\end{array}
\]

I. Irreducible subquotients.

In this case the module \( V(T(v)) \) has 8 irreducible subquotients.

(i) Two modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_1 \) | \( L \) \( \begin{cases} m \leq -s \\ k \leq n \end{cases} \) |
| \( L_8 \) | \( L \) \( \begin{cases} 0 < m \\ n < k \end{cases} \) |

(ii) Two modules with unbounded finite weight multiplicities:
(iii) Four modules with bounded weight multiplicities; two of them with multiplicities bounded by $t$ and the other two bounded by $r := s - t$:
CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN MODULES OF $\mathfrak{sl}(3)$

I. Irreducible subquotients.

In this case the module $V(T(\bar{v}))$ has 4 irreducible subquotients. The bases and corresponding weight lattices are given by:

(i) Two modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_1$  | $L\left\{m \leq -s \right\}$ |
| $L_4$  | $L\left\{0 < m \right\}$ |

(ii) Two cuspidal modules; $L_2$ with weight multiplicities $s - t$ and $L_3$ with weight multiplicities $t$:

| Module | Basis |
|--------|-------|
| $L_2$  | $L\left\{-s < m \leq -t \right\}$ |
| $L_3$  | $L\left\{-t < m \leq 0 \right\}$ |

II. Loewy series.

$L_1; L_2 \oplus L_3; L_4 \oplus L_5; L_6 \oplus L_7; L_8$.

(G16) For any $t, s \in \mathbb{Z}_{>0}$ with $t < s$ let $T(v)$ be the following tableau:

\[
T(v) = \begin{array}{ccc}
 a & a - t & a - s \\
 a & y & \\
 & z & \\
\end{array}
\]

II. Loewy series.

$L_1; L_2; L_3; L_4$.

7.3. Structure of singular $\mathfrak{sl}(3)$-modules $V(T(\bar{v}))$. In this section we will use the construction in §4.3 to describe singular $\mathfrak{sl}(3)$-modules. As in the generic case will be enough to present explicitly all irreducible subquotients of singular modules of the form $V(T(\bar{v}))$ for any 1-singular vector $\bar{v} \in T_3(\mathbb{C})$.

7.3.1. Singular Gelfand-Tsetlin formulas. In this section we remember the construction of 1-singular modules (see §4.3) for the lie algebra $\mathfrak{sl}(3)$. In this particular case the singularity should appear in row 2. Let as fix $\bar{v} \in T_3(\mathbb{C})$ such that
For any rational function \( f \) a fixed critical tableau, in \([R13]\) (22)

\[
B(\bar{v}) := \{ T(\bar{v} + z), DT(\bar{v} + w) | z_{21} \leq z_{22} \text{ and } w_{21} > w_{22} \}.
\]

**Definition 7.13.** From now on we will fix a basis \( B(T(\bar{v})) \) of \( V(T(\bar{v})) \) defined by

\[
B(T(\bar{v})) := \{ T(\bar{v} + z), DT(\bar{v} + w) | z_{21} \leq z_{22} \text{ and } w_{21} > w_{22} \}.
\]

Given \( w \in \mathbb{Z}^3 \), the tableau associated to \( w \) with respect to \( B(T(\bar{v})) \) is defined by

\[
Tab(w) := \begin{cases} 
T(\bar{v} + w), & \text{if } w_{21} \leq w_{22} \\
DT(\bar{v} + w), & \text{if } w_{21} > w_{22}.
\end{cases}
\]

So, we can write \( B(T(\bar{v})) = \{ Tab(w) | w \in \mathbb{Z}^3 \} \).

Explicit formulas for the action of \( gl(3) \) on singular modules were obtained first in \([R13]\) §4 and generalized to 1-singular Gelfand-Tsetlin modules in \([CGR10]\). Let us write explicitly the formulas for singular \( gl(3) \)-modules.

Set \( v = (v_{31}, v_{32}, v_{31}, v_{21}, v_{22}, v_{11}) \) a vector with variable entries, \( \bar{v} = (a, b, c, x, x, z) \) a fixed critical tableau, \( w = (0, 0, 0, m, n, k) \) any integral shift, and \( \bar{w} = \bar{v} + w \). Also, for any rational function \( f \) on variables \( \{ v_{ij} \}_{1 \leq j \leq 3} \), we by \( \bar{v}_{ij}(f) \) will stand for \( \frac{\partial f}{\partial v_{ij}}(\bar{v}) \).

By Remark 7.12, it is enough to describe the action of \( gl(3) \) on the basis elements

\[
\{ T(\bar{v} + z), DT(\bar{v} + w) | z_{21} \leq z_{22} \text{ and } w_{21} > w_{22} \}.
\]

(23)

\[
E_{21}(T(\bar{w})) = T(\bar{w} - \delta^{11}).
\]

\[
E_{12}(T(\bar{w})) = -(x + m - z - k)(x + n - z - k)T(\bar{w} - \delta^{11}).
\]

\[
E_{32}(T(\bar{w})) = \begin{cases} 
T(\bar{w} - \delta^{21}) + 2(x + m - z - k)DT(\bar{w} - \delta^{21}), & \text{if } m = n \\
\frac{a + n - z - b}{m - n}T(\bar{w} - \delta^{21}) - \frac{a + m - z - b}{m - n}T(\bar{w} - \delta^{22}), & \text{if } m \neq n.
\end{cases}
\]

\[
E_{23}(T(\bar{w})) = \begin{cases} 
\delta_{ij}^{\bar{v}_{21}}((v_{31} - v_{21} - m)(v_{32} - v_{21} - m)(v_{33} - v_{21} - m))T(\bar{w} + \delta^{22}) - 2(a - x - m)(b - x - m)(c - x - m)DT(\bar{w} + \delta^{22}), & \text{if } m = n \\
\frac{(a - x - m)(b - x - m)(c - x - m)}{(m - n)}T(\bar{w} - \delta^{21}) - \frac{(a - x - n)(b - x - n)(c - x - n)}{(m - n)}T(\bar{w} + \delta^{22}), & \text{if } m \neq n.
\end{cases}
\]

Recall that by definition we have \( DT(\bar{w}) = 0 \) if \( m = n \), therefore, we just compute the action of \( gl(3) \) on derivative tableaux \( DT(\bar{w}) \) such that \( m > n \).
(24)
\[ E_{21}(DT(\bar{w})) = DT(\bar{w} - \delta^{11}). \]
\[ E_{12}(DT(\bar{w})) = -(x + m - z - k)(x + n - z - k)DT(\bar{w} + \delta^{11}) + \frac{(m - n)}{2} T(\bar{w} + \delta^{11}). \]
\[ E_{32}(DT(\bar{w})) = \frac{x + m - z - k}{m - n} DT(\bar{w} - \delta^{21}) - \frac{x + n - z - k}{m - n} DT(\bar{w} - \delta^{22}) + \]
\[ \frac{1}{2} \left( \frac{1}{(m - n)} - \frac{2(x + m - z - k)}{(m - n)^2} \right) T(\bar{w} - \delta^{21}) - \]
\[ \frac{1}{2} \left( \frac{1}{(m - n)} - \frac{2(x + n - z - k)}{(n - m)^2} \right) T(\bar{w} - \delta^{22}). \]
\[ E_{23}(DT(\bar{w})) = \frac{(a - x - n)(b - x - n)(c - x - n)}{m - n} DT(\bar{w} + \delta^{21}) - \]
\[ \frac{1}{n - m} (a - x - n)(b - x - n)(c - x - n) DT(\bar{w} + \delta^{22}) + \]
\[ \frac{1}{2} \frac{x(v_{31} - v_{22} - m)(v_{32} - v_{22} - n)(v_{33} - v_{22} - n)}{(m - n)^2} T(\bar{w} + \delta^{22}) + \]
\[ - \frac{1}{2} \frac{x(v_{31} - v_{21} - m)(v_{32} - v_{21} - m)(v_{33} - v_{21} - m)}{(m - n)^2} T(\bar{w} + \delta^{21}) - \]
\[ 2 \frac{(a - x - n)(b - x - n)(c - x - n)}{m - n} DT(\bar{w} - \delta^{22}) + \]
\[ - \frac{1}{2} \frac{x(v_{31} - v_{21} - m)(v_{32} - v_{21} - m)(v_{33} - v_{21} - m)}{(m - n)^2} T(\bar{w} - \delta^{21}). \]

Remember that by Proposition 4.21, the Gelfand-Tsetlin formulas for generic modules can be written as follows:

\[ E_{rs} T(v + z) = \sum_{\sigma \in \Phi_{rs}} e_{rs}(\sigma(v + z)) T(v + z + \epsilon_{rs}) \]

where
\[ e_{11}(w) = w_{11}, \]
\[ e_{22}(w) = w_{21} + w_{22} - w_{11} + 1, \]
\[ e_{33}(w) = w_{31} + w_{32} + w_{33} - w_{21} - w_{22} + 2, \]
\[ e_{21}(w) = 1, \]
\[ e_{12}(w) = -(w_{11} - w_{21})(w_{11} - w_{22}), \]
\[ e_{23}(w) = -\frac{(w_{21} - w_{31})(w_{21} - w_{32})(w_{21} - w_{33})}{w_{21} - w_{22}}, \]
\[ e_{32}(w) = \frac{w_{21} - w_{11}}{w_{21} - w_{22}}, \]
\[ e_{31}(w) = \frac{1}{w_{21} - w_{22}}, \]
\[ e_{13}(w) = -\frac{(w_{21} - w_{31})(w_{21} - w_{32})(w_{21} - w_{33})(w_{11} - w_{22})}{w_{21} - w_{22}}. \]

Remember that we have some identifications of tableaux, namely: \( T(\bar{v} + z) \simeq T(\bar{v} + \tau(z)) \) and \( DT(\bar{v} + z) \simeq -DT(\bar{v} + \tau(z)) \) (see Section 4.3). Let \( w \in \mathbb{Z}^3 \) and \( \bar{w} = \bar{v} + w \). The action of the generators of \( gl(3) \) on any tableau \( Tab(w) \) can be written as follows:

\[
(25) \quad E_{ii}(Tab(w)) = e_{ii}(\bar{v} + w)Tab(w), \quad \text{for } i = 1, 2, 3.
\]
\[
(26) \quad E_{21}(Tab(w)) = Tab(w + \epsilon_{21}).
\]
\[
(27) \quad E_{12}(Tab(w)) = e_{12}(\bar{v} + w)Tab(w + \epsilon_{12}).
\]

If \( E \in \{E_{32}, E_{23}, E_{31}, E_{13}\} \) and \( e(w) := (v_{21} - v_{22})e(w) \). The action of \( E \) on \( Tab(w) \) is given by:

\[
(28) \quad 2(\bar{e}(\bar{w})DT(\bar{w} + \epsilon) + D\bar{v}(\bar{e}(\bar{v} + w))T(\bar{w} + \epsilon)), \quad \text{if } w_{21} = w_{22}.
\]
\[
(29) \quad e(\bar{w})T(\bar{w} + \epsilon) + e(\tau(\bar{w}))T(\bar{w} + \tau(\epsilon)), \quad \text{if } w_{21} < w_{22}.
\]
\[
(30) \quad \begin{cases} D\bar{v}(e(\bar{v} + w))T(\bar{w} + \epsilon) + D\bar{v}(e(\tau(\bar{v} + w)))T(\bar{w} + \tau(\epsilon)) \\ + e(\bar{w})DT(\bar{w} + \epsilon) + e(\tau(\bar{w}))DT(\bar{w} + \tau(\epsilon)). \end{cases} \quad \text{if } w_{21} > w_{22}.
\]

**Remark 7.14.** Note that the Gelfand-Tsetlin formulas for singular tableaux have the same coefficients as in the classical formulas for tableaux of the same type (see formulas (25), (26), (27), (28), (29), (30)), that is, applying Gelfand-Tsetlin formulas for regular tableaux will give us a linear combination of regular tableaux with the same coefficients as in the classical case and applying Gelfand-Tsetlin formulas to a derivative tableau will give us a linear combination of regular tableaux and derivative tableaux and the coefficients for the derivative tableaux that appear are the same as in the classical formulas.

**Example 7.15.** Let us do some explicit computations. For any \( a \in \mathbb{C} \) and variables \( v_{ij}, \ 1 \leq j \leq i \leq 3 \) consider the following Gelfand-Tsetlin tableaux.

\[
T(v) := \begin{array}{ccc} v_{31} & v_{32} & v_{33} \\ v_{21} & v_{22} \\ v_{11} \end{array} \quad T(\bar{v}) := \begin{array}{ccc} a & a & a \\ a & a \\ a \end{array}
\]
Consider \( w = (0,1,0) = \delta^{22} \) and \( w' = (1,0,0) = \delta^{21} \) in \( \mathbb{Z}^3 \). In this case we have:

(i) \( E_{32}(T(\bar{v} + w)) \) is equal to:

\[
= D^6 ((v_{21} - v_{22}) E_{32}(T(v + w)))
= D^6 \left( \frac{(v_{21} - v_{11})}{(v_{21} - (v_{22} + 1))} T(v + w - \delta^{21}) + \left( \frac{(v_{22} + 1) - v_{11}}{(v_{22} + 1) - v_{21}} \right) T(v + w - \delta^{22}) \right)
\]

(ii) By definition we have \( E_{32}(D^6 T(\bar{v} + w')) = D^6 (E_{32}(T(v + w'))) \) equal to:

\[
= D^6 \left( \frac{v_{21} + 1 - v_{11}}{v_{22} + 1 - v_{22}} \right) T(\bar{v} + \delta^{21}) + \frac{v_{22} - v_{11}}{v_{22} - (v_{21} + 1)} T(\bar{v} + \delta^{21} - \delta^{22})
\]

Lemma 7.16. The action of \( \Gamma \) on \( V(T(\bar{v})) \) is given by the formulas:

\[
(31) \quad c_{rs}(T(\bar{v} + w)) = \gamma_{rs}(\bar{v} + w)T(\bar{v} + w),
\]

\[
(32) \quad c_{rs}(D^6 T(\bar{v} + w)) = \gamma_{rs}(\bar{v} + w)D^6 T(\bar{v} + w) + D^6 (\gamma_{rs}(v + w))T(\bar{v} + w).
\]

Proof. Follows directly from Theorem [4.31] In fact,

\[
c_{rs}(T(\bar{v} + z)) = D^7 ((v_{21} - v_{22}) c_{rs} T(v + z))
= D^7 ((v_{21} - v_{22}) \gamma_{rs} (v + z) T(v + z))
= D^7 ((v_{21} - v_{22}) \gamma_{rs} (v + z)) T(\bar{v} + z) + ev(\bar{v}) ((v_{21} - v_{22}) \gamma_{rs} (v + z)) D^6 T(\bar{v} + z)
= \gamma_{rs} (\bar{v} + z) T(\bar{v} + z),
\]

\[
c_{rs}(D^6 T(\bar{v} + z)) = D^7 (c_{rs} (T(v + z)))
= D^7 (\gamma_{rs} (v + z) T(v + z))
= D^7 (\gamma_{rs} (v + z)) T(\bar{v} + z) + ev(\bar{v}) (\gamma_{rs} (v + z)) D^6 T(\bar{v} + z)
= D^7 (\gamma_{rs} (v + z)) T(\bar{v} + z) + (\gamma_{rs} (v + z)) D^6 T(\bar{v} + z).
\]
This page contains a selection of mathematical text, likely from a research paper or a book. The text is written in English and includes mathematical notation and theorems. Here is a representation of the text as it appears:

**Submodule generated by a singular tableau.** In this section we will obtain an analogous of Theorem 7.21(i) for 1-singular tableaux.

Remember that $B(T(\bar{v}))$ denotes the basis $\{\text{Tab}(z) \mid z \in \mathbb{Z}^3\}$ of $V(T(\bar{v}))$.

**Definition 7.17.** Set $\bar{v}$ be the fixed critical vector as before, $\text{Tab}(w) \in B(T(\bar{v}))$ and $\bar{w} = \bar{v} + w$, we define:

$$\Lambda^+(\text{Tab}(w)) = \begin{cases} \Omega^+(\text{Tab}(w)), & \text{if } w_{21} \leq w_{22} \\ \Omega^+(\text{Tab}(\tau(w))), & \text{if } w_{21} > w_{22}. \end{cases}$$

**Lemma 7.18.** Assume that $w_{21} \neq w_{22}$. Then $T(\bar{v} + w)$ belongs to $U \cdot DT(\bar{v} + w)$.

**Proof.** The action of $c_{22} - \gamma_{22}(\bar{v} + w)$ on $DT(\bar{v} + w)$ is given by the formula (32) and can be easily check that is a nonzero multiple of $(w_{21} - w_{22})T(\bar{v} + w)$. □

**Lemma 7.19.** Suppose $\text{Tab}(w)$ is a critical tableau. If $\text{Tab}(w')$ is a derivative tableau such that $\Lambda^+(\text{Tab}(w)) \subseteq \Lambda^+(\text{Tab}(w'))$, then $\text{Tab}(w') \in U \cdot \text{Tab}(w)$. In particular, the irreducible subquotient $M$ of $V(T(\bar{v}))$ containing $\text{Tab}(w)$ satisfies $\dim(M_{\chi_w}) = 2$.

**Proof.** Follows directly from Remark 7.14 and formulas (26, 27, 28), in fact, we can use the same arguments as in the proof of Lemma 4.24(i). □

**Definition 7.20.** For any tableau $\text{Tab}(w) \in B(T(\bar{v}))$ define

$$\mathcal{A}(\text{Tab}(w)) = \{\text{Tab}(w') \in B(T(\bar{v})) \mid \Lambda^+(\text{Tab}(w)) \subseteq \Lambda^+(\text{Tab}(w'))\}.$$  

By $\mathcal{C}(w)$ we will denote the set of all critical tableaux in $\mathcal{A}(\text{Tab}(w))$ and by $\mathcal{R}(w)$ we will denote the set of all regular tableaux in $\mathcal{A}(\text{Tab}(w))$. Finally, we define:

$$\mathcal{N}(\text{Tab}(w)) = \begin{cases} \mathcal{R}(w) \cup \left( \bigcup_{w' \in \mathcal{C}(w)} \mathcal{A}(\text{Tab}(w')) \right), & \text{if } w_{21} < w_{22} \\ \mathcal{A}(\text{Tab}(w)), & \text{if } w_{21} \geq w_{22}. \end{cases}$$

**Lemma 7.21.** For any tableau $\text{Tab}(w)$ we have $\mathcal{N}(\text{Tab}(w)) \subseteq U \cdot \text{Tab}(w)$.

**Proof.** Follows from Lemmas 7.19, 7.18 and Remark 7.14, in fact, we can use the same argument as in the generic case that allows us to get Theorem 7.6(i) (see the proof of Proposition 6.2 in [FGR15]). □

The following lemma together with Lemma 7.18 will give us a sufficient condition in order to have modules with Gelfand-Tsetlin multiplicity 2.

**Lemma 7.22.** Suppose that $\text{Tab}(w)$ is a regular tableau such that $\Omega^+(\text{Tab}(w)) = \Omega^+(\text{Tab}(w'))$ for some critical tableau $\text{Tab}(w')$, then $\text{Tab}(\tau(w)) \in U \cdot \text{Tab}(w)$.

**Proof.** Follows directly from Lemma 7.19 □

**Corollary 7.23.** Let $\text{Tab}(w)$ be a regular tableau associated with a Gelfand-Tsetlin character $\chi$. If $\{\text{Tab}(w') \mid \Omega^+(\text{Tab}(w')) = \Omega^+(\text{Tab}(w))\}$ does not contains critical tableaux, then any irreducible subquotient $N$ of $V(T(\bar{v}))$ satisfies $\dim(N_{\chi}) \leq 1$. 
Proof. The condition in $\text{Tab}(w)$ implies that $w_{21} < w_{22}$ and $\text{Tab}(\tau(w))$ is a derivative tableau such that $\text{Tab}(w) \in U \cdot \text{Tab}(\tau(w))$ (see Lemma 7.18). Therefore, it is enough to prove that $\text{Tab}(\tau(w)) \not\in U \cdot \text{Tab}(w)$, but this is clear by Theorem 4.31 and the fact that we cannot obtain critical tableaux from $\text{Tab}(w)$ with the same $\Omega^+(\text{Tab}(w))$, in particular we cannot obtain derivative tableaux $\text{Tab}(w')$ such that $\Omega^+(\text{Tab}(w')) = \Omega^+(\text{Tab}(w))$. Therefore, $\text{Tab}(\tau(w)) \not\in U \cdot \text{Tab}(w)$.

Remark 7.24. By definition of the set $N(\text{Tab}(w))$, any tableau $\text{Tab}(w')$ in $N(\text{Tab}(w))$ satisfies the relation $|\Omega^+(\text{Tab}(w))| \leq |\Omega^+(\text{Tab}(w'))|$, however it is possible to have $\text{Tab}(w') \in U \cdot \text{Tab}(w)$ with $|\Omega^+(\text{Tab}(w'))| = |\Omega^+(\text{Tab}(w))| - 1$. For instance, consider $\tau = (a, b, c, x, x, x)$ such that $\{a - x, b - x, c - x\} \cap \mathbb{Z} = \emptyset$ and $w = (0, 0, 0)$, then $|\Omega^+(\text{Tab}(w))| = 2$ while $E_{32} \text{Tab}(w) = \text{Tab}(w - \delta^{21})$ and $|\Omega^+(\text{Tab}(w - \delta^{21}))| = 1$.

Let us write $\text{Tab}(w') \prec_g \text{Tab}(w)$ if $\text{Tab}(w')$ appears with no zero coefficient in the decomposition of $g \cdot \text{Tab}(w)$ for some generator $g \in \mathfrak{gl}(n)$.

The following Lemma is proved in [GoR18] for any $1$-singular Gelfand-Tsetlin tableau.

Lemma 7.25 ([GoR18 Lemma 7.4]). Suppose that $\text{Tab}(w') \prec_g \text{Tab}(w)$ with $g \in \mathfrak{gl}(n)$ of the form $E_{k,k+1}$ or $E_{k+1,k}$, then $|\Omega^+(\text{Tab}(w'))| \geq |\Omega^+(\text{Tab}(w))| - 1$. Moreover, the complete list of Gelfand-Tsetlin tableaux $\text{Tab}(w)$ and $\text{Tab}(w')$ such that $\text{Tab}(w') \prec_g \text{Tab}(w)$ and $|\Omega^+(\text{Tab}(w'))| = |\Omega^+(\text{Tab}(w))| - 1$ is the following:

(i) Set $t \in \mathbb{Z}_{>0}$.

\[
\begin{array}{ccc}
\text{Tab}(w) := & x & x - t \\
& x & \\
\text{Tab}(w') := & x - t & x \\
& x + 1 & \\
\end{array}
\]

(ii) Set $t \in \mathbb{Z}_{>0}$.

\[
\begin{array}{ccc}
\text{Tab}(w) := & x & x - t \\
& x & \\
\text{Tab}(w') := & x - t & x - 1 \\
& x & \\
\end{array}
\]

(iii) Set $t \in \mathbb{Z}_{>0}$.

\[
\begin{array}{ccc}
\text{Tab}(w) := & x & x \\
& x & \\
\text{Tab}(w') := & x - 1 & x \\
& x & \\
\end{array}
\]

(iv) Set $t \in \mathbb{Z}_{>0}$ and assume and $b \neq x$, $c \neq x$.

\[
\begin{array}{ccc}
\text{Tab}(w) := & x & x - t \\
& z & \\
\text{Tab}(w') := & x - t & x + 1 \\
& z & \\
\end{array}
\]

□
Definition 7.27. We will say that a tableau $\text{Tab}(w)$ is of type (I) if can be written in the form of one of the tableaux of items (i), (ii) or (iii) of Lemma 7.25 for some $x,a,b,c,t,z$. We also say that the tableau is of type (II), if can be written in the form of one of the tableaux of items (iv) or (v) of Lemma 7.25 for some $x,b,c,t,z$ and $x$ appear in the top row in position $3i$.

Remark 7.28. In Lemma 7.25, for tableaux of type (I) we have $\Omega^+(\text{Tab}(w')) = \Omega^+(\text{Tab}(w)) \setminus \{(2,1,1)\}$ and for tableaux of type (II), we have $\Omega^+(\text{Tab}(w')) = \Omega^+(\text{Tab}(w)) \setminus \{(3,i,2)\}$.

Definition 7.29. Let $\text{Tab}(w) \in \mathcal{B}(T(\bar{v}))$ be any tableau and $(r,s,t) \in \Omega^+(\text{Tab}(w))$, we define

$$N_{(r,s,t)}(\text{Tab}(w)) = N(\text{Tab}(w)) \cup N(\text{Tab}(w')),$$

where $\text{Tab}(w')$ is any tableau such that $\Omega^+(\text{Tab}(w)) \setminus \{(r,s,t)\} = \Omega^+(\text{Tab}(w'))$.

Finally, we define:

$$N^{(1)}(\text{Tab}(w)) = \begin{cases} N_{(2,1,1)}(\text{Tab}(w)), & \text{if } \text{Tab}(w) \text{ is of type (I)} \\ N(\text{Tab}(w)), & \text{otherwise} \end{cases}$$

$$N^{(2)}(\text{Tab}(w)) = \begin{cases} N_{(3,i,2)}(\text{Tab}(w)), & \text{if } \text{Tab}(w) \text{ is of type (II)} \\ N(\text{Tab}(w)), & \text{otherwise} \end{cases}$$

Lemma 7.30. Let $\text{Tab}(w)$ be any Gelfand-Tsetlin tableau. For any $\text{Tab}(w') \in N(\text{Tab}(w))$ we have $N^{(1)}(\text{Tab}(w')) \subseteq U \cdot \text{Tab}(w)$ for $i = 1,2$.

Proof. Follows directly from Lemmas 7.24 and 7.25 and Remark 7.28.

The following proposition summarize the results of this section.

Theorem 7.31. Let $\text{Tab}(w)$ be any tableau in $\mathcal{B}(T(\bar{v}))$. The submodule $U \cdot \text{Tab}(w)$ is generated by the set of tableaux:

$$\hat{N}(\text{Tab}(w)) := \bigcup_{N(\text{Tab}(w))} N^{(1)}(\text{Tab}(w')) \cup N^{(2)}(\text{Tab}(w'))$$

Proof. Follows from Lemmas 7.24 and 7.30.

Definition 7.32. Let $M$ be a Gelfand-Tsetlin module with basis $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$ for some 1-singular vector $\bar{v}$. We say that $\text{Tab}(w) \in \mathcal{B}_M$ is $\Omega^+$-maximal in $M$ if $|\Omega^+(\text{Tab}(w))|$ is maximal between all tableaux in $\mathcal{B}_M$. Also, denote by $U \cdot M \text{Tab}(w)$ the submodule of $M$ generated by $\text{Tab}(w)$.
The following corollaries follow from Theorem 7.31 and will be useful in order to describe irreducible subquotients of the module \( V(T(\vec{v})) \).

**Corollary 7.33.** Let \( M \) be a Gelfand-Tsetlin module with basis \( \mathcal{B}_M \subseteq \mathcal{B}(T(\vec{v})) \) for some 1-singular vector \( \vec{v} \). If \( \text{Tab}(w) \in \mathcal{B}_M \) is a regular tableau that is \( \Omega^+ \)-maximal in \( M \), then \( U \cdot_M \text{Tab}(w) \) is an irreducible submodule of \( M \).

**Proof.** It is enough to proof that \( \text{Tab}(w) \) belongs to \( U \cdot_M \text{Tab}(w') \) for any \( \text{Tab}(w') \) in \( U \cdot_M \text{Tab}(w) \). As \( \text{Tab}(w') \) in \( U \cdot_M \text{Tab}(w) \) and \( \text{Tab}(w) \) is a regular tableau, we have \( \Omega^+(\text{Tab}(w)) \subseteq \Omega^+(\text{Tab}(w')) \cup \{(r,s,t)\} \) for some \( (r,s,t) \). As \( \text{Tab}(w) \) is \( \Omega^+ \)-maximal, we should have \( \Omega^+(\text{Tab}(w)) = \Omega^+(\text{Tab}(w')) \cup \{(r,s,t)\} \) for some \( (r,s,t) \). Therefore, \( U \cdot_M \text{Tab}(w) \subseteq U \cdot_M \text{Tab}(w') \) and, then we have \( \text{Tab}(w) \in U \cdot_M \text{Tab}(w') \). \( \square \)

**Corollary 7.34.** Let \( M \) be a Gelfand-Tsetlin module with basis \( \mathcal{B}_M \subseteq \mathcal{B}(T(\vec{v})) \) for some 1-singular vector \( \vec{v} \). If \( \{ \text{Tab}(w) \in \mathcal{B}_M \mid \text{Tab}(w) \text{ is } \Omega^+ \text{-maximal} \} \) does not contain regular tableaux, then for any \( \Omega^+ \)-maximal tableau \( \text{Tab}(w) \) the submodule \( U \cdot_M \text{Tab}(w) \) is an irreducible submodule of \( M \).

**Proof.** Analogous to the proof of Corollary 7.33. \( \square \)

**Notation 7.35.** Let \( \vec{v} \) be any 1-singular vector and \( D \) be a subset of \( \mathbb{Z}^3 \) defined by a set of inequalities of the form \( a \leq b \) or \( a < b \). By \( \mathcal{B}(D) \) we will denote the set of tableaux \( \{ \text{Tab}(m,n,k) \mid (m,n,k) \in D \} \). Assume that \( M \) is a Gelfand-Tsetlin module with basis \( \mathcal{B}(D) \). Then we will denote \( M \) by \( M(D,\vec{v}) \), or simply by \( M(D) \) if \( \vec{v} \) is fixed. If \( M(D) \) is irreducible then we will write \( L(D) \) for \( M(D) \).

**Example 7.36.** Let us consider the following critical tableau:

\[
\begin{array}{ccc}
a & b & c \\
a & a & \\
 & z & \\
\end{array}
\]

\( T(\vec{v}) = \)

We will describe a basis for the submodule of \( V(T(\vec{v})) \) generated by \( \text{Tab}(0,0,0) \).

(i) In this case \( \mathcal{B}(T(\vec{v})) \) does not contain tableaux of type (I), (II)2 or (II)3.

However, the set of all tableau of type (II)1 is

\[
\{ \text{Tab}(0,n,k) \mid n \in \mathbb{Z}_{\leq 0} \}.
\]

(ii) \( \hat{A}(\text{Tab}(0,0,0)) = A(\text{Tab}(0,0,0)) = \{ \text{Tab}(m,n,k) \mid m \leq 0, n \leq 0 \} \).

(iii) \( \hat{A}(\text{Tab}(0,-1,0)) = A^{(2)}(\text{Tab}(0,-1,0)) = \{ \text{Tab}(m,n,k) \mid m \leq 0 \} \).

(iv) The submodule of \( V(T(\vec{v})) \) generated by \( \text{Tab}(0,0,0) \) is generated by:

\[
\bigcup_{\mathcal{N}(\text{Tab}(0,0,0))} \hat{A}(\text{Tab}(w')) = \hat{A}(\text{Tab}(0,0,0)) \cup \hat{A}(\text{Tab}(0,-1,0))
\]

\[
= \mathcal{B}(m \leq 0).
\]

The following example shows how to construct explicit basis for any irreducible subquotient of the module \( V(T(\vec{v})) \), where \( \vec{v} = (a,a,a|a,a|a) \).

**Example 7.37.** Let us consider the following critical tableau:
Note that none of the tableaux in $B(T(\bar{v}))$ can be of type $(\text{II})$. Therefore $A^{(2)}(\text{Tab}(w)) = A(\text{Tab}(w))$ for any $\text{Tab}(w)$.

(ii) The set of all tableaux of type $(\text{I})$ is $\{\text{Tab}(m,n,k) \mid n \leq m = k\}$.

(iii) Now we will use Theorem 7.31, and Corollaries 7.33 and 7.34 to describe one by one all irreducible subquotients of the module $V(T(\bar{v}))$. The procedure will be the following:

Step 1. Given a module $M$ with basis $B_{M} \subseteq B(T(\bar{v}))$ we choose a $\Omega^{+}$-maximal tableau $\text{Tab}(w)$ in $B_{M}$.

Step 2. If $\text{Tab}(w)$ is regular, by Corollary 7.33 the module $U \cdot M \text{Tab}(w)$ will be an irreducible submodule of $M$.

Step 3. If there are not $\Omega^{+}$-maximal regular tableaux in $B_{M}$, consider any $\Omega^{+}$-maximal regular tableaux $\text{Tab}(w)$. By Corollary 7.33 the module $U \cdot M \text{Tab}(w)$ will be an irreducible submodule of $M$.

Step 4. As we have explicit basis for $M$ and $U \cdot \text{Tab}(w)$ (see Theorem 7.31), the quotient module $M/(U \cdot M \text{Tab}(w))$ will have an explicit basis contained in $B(T(\bar{v}))$.

Step 5. Start over the procedure with the module $M' := M/(U \cdot M \text{Tab}(w))$.

1. The tableau $\text{Tab}(0,0,0)$ is $\Omega^{+}$-maximal on $V(T(\bar{v}))$. By Corollary 7.33, $U \cdot \text{Tab}(0,0,0)$ is an irreducible submodule of $V(T(\bar{v}))$ and by Theorem 7.31 the submodule $U \cdot \text{Tab}(0,0,0)$ is generated by:

$$\hat{N}(\text{Tab}(0,0,0)) = \hat{A}(\text{Tab}(0,0,0)) = A^{(1)}(\text{Tab}(0,0,0)) = B \left( \begin{cases} m \leq 0 \\ n \leq 0 \\ k \leq n \end{cases} \right).$$

Denote this module by $L_1$ and $M_1 := V(T(\bar{v}))/L_1$.

2. Now, the derivative tableau $\text{Tab}(0,-2,-1)$ is $\Omega^{+}$-maximal in $M_1$. By Theorem 7.31, $U \cdot \text{Tab}(0,-2,-1)$ is generated by:

$$\hat{N}(\text{Tab}(0,-2,-1)) = \hat{A}(\text{Tab}(0,0,1)) = A^{(1)}(\text{Tab}(0,-2,-1)) = B \left( \begin{cases} m \leq 0 \\ n \leq 0 \end{cases} \right).$$
Moreover, by Corollary 7.34 \( \tilde{U} \cdot M, \tilde{\text{Tab}}(0, -2, -1) \) is an irreducible submodule of \( M_1 \) and is generated by
\[
B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0
\end{array} \right) \setminus B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0 \\
k \leq n
\end{array} \right)
\]
\[
\setminus B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0 \\
n < k
\end{array} \right) .
\]

Denote by \( L_3 \) this module and \( M_2 = M_1 / L_3 \).

(3) The tableau \( \text{Tab}(0, 1, 0) \) is \( \Omega^+ \)-maximal in \( M_2 \) and \( U \cdot \text{Tab}(0, 1, 0) \) is generated by \( \tilde{A}(\text{Tab}(0, 1, 0)) \cup \tilde{A}(\text{Tab}(0, 0, 0)) \) which is equal to
\[
B \left( \begin{array}{c}
m \leq n \\
m \leq 0 \\
k \leq m \\
k \leq n
\end{array} \right) \cup B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0 \\
k \leq n
\end{array} \right)
\]

Therefore, \( U \cdot M_2 \text{Tab}(0, 1, 0) \) is generated by
\[
B \left( \begin{array}{c}
m \leq n \\
m \leq 0 \\
k \leq m \\
k \leq n
\end{array} \right) \setminus B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0 \\
k \leq n
\end{array} \right) \cup B \left( \begin{array}{c}
m \leq 0 \\
n \leq 0 \\
n < k
\end{array} \right)
\]
\[
= B (k \leq m \leq 0 < n),
\]
call this module \( L_2 \) and \( M_3 := M_2 / L_2 \).

(4) There are not \( \Omega^+ \)-maximal regular tableaux in \( M_3 \), so we choose the derivative tableau \( \text{Tab}(1, 0, 0) \) which is \( \Omega^+ \)-maximal in \( M_3 \). By Corollary 7.34 the module \( U \cdot M_3 \text{Tab}(1, 0, 0) \) is an irreducible submodule of \( M_3 \) with basis
\[
B (k \leq n \leq 0 < m)
\]
call this module \( L_5 \) and \( M_4 := M_3 / L_5 \).

(5) The tableau \( \text{Tab}(0, 1, 1) \) is \( \Omega^+ \)-maximal in \( M_4 \) and \( U \cdot M_4 \text{Tab}(0, 1, 1) \) is an irreducible submodule of \( M_4 \) with basis
\[
B \left( \begin{array}{c}
m \leq 0 \\
m < k \leq n
\end{array} \right)
\]
call this module \( L_4 \) and \( M_5 := M_4 / L_4 \).

(6) The derivative tableau \( \text{Tab}(1, 0, 1) \) is \( \Omega^+ \)-maximal in \( M_5 \). Therefore, by Corollary 7.34 \( U \cdot M_5 \text{Tab}(1, 0, 1) \) an irreducible submodule of \( M_5 \) with basis
\[
B \left( \begin{array}{c}
n \leq 0 < m \\
n < k \leq m
\end{array} \right)
\]
call this module \( L_7 \) and \( M_6 := M_5 / L_7 \).
(7) The tableau $\text{Tab}(0, 1, 2)$ is $\Omega^+$-maximal in $M_6$ so, $U \cdot M_6 \text{Tab}(0, 1, 2)$ is an irreducible submodule of $M_6$ generated by
\[
\mathcal{B}(m \leq 0 < n < k)
\]
call this module $L_6$ and $M_7 := M_6/L_6$.

(8) The tableau $\text{Tab}(1, 0, 2)$ is $\Omega^+$-maximal in $M_7$ and $U \cdot M_7 \text{Tab}(1, 0, 2)$ is an irreducible submodule of $M_7$ generated by
\[
\mathcal{B}(n \leq 0 < m < k)
\]
call this module $L_9$ and $M_8 := M_7/L_9$.

(9) The tableau $\text{Tab}(1, 1, 0)$ is $\Omega^+$-maximal in $M_8$. The module $U \cdot M_8 \text{Tab}(1, 1, 0)$ is an irreducible submodule of $M_8$ generated by
\[
\mathcal{B} \left( \begin{array}{c}
0 < m \\
0 < n \\
k \leq n
\end{array} \right)
\]
call this module $L_8$ and $M_9 := M_8/L_8$.

(10) The tableau $\text{Tab}(1, 1, 2)$ is $\Omega^+$-maximal in $M_9$ and $U \cdot \text{Tab}(1, 1, 2)$ is generated by $\mathcal{B}(\mathbb{Z}^3)$. Therefore, $U \cdot M_9 \text{Tab}(1, 1, 2)$ is generated by
\[
\mathcal{B} \left( \begin{array}{c}
0 < m \\
0 < n \\
n < k
\end{array} \right)
\]
call this module $L_{10}$.

Remark 7.38. The previous procedure for description of all irreducible subquotients of $V(T(\bar{v}))$ also give us an effective method to describe the Loewy series decomposition of $V(T(\bar{v}))$. In Example 7.37 the Loewy series decomposition is:
\[
L_1; \ L_2 \oplus L_3; \ L_4; \ L_5 \oplus L_6; \ L_7; \ L_8 \oplus L_9; \ L_{10}.
\]

7.3.3. Weight spaces for subquotients of $V(T(\bar{v}))$. As we did for the description of generic blocks we will characterize the weight spaces for subquotients of the singular module $V(T(\bar{v}))$.

Proposition 7.39. Let $M$ be a singular Gelfand-Tsetlin module with basis of tableaux $\mathcal{B}_M \subseteq \mathcal{B}(T(\bar{v}))$ for some 1-singular vector $\bar{v}$. If $\text{Tab}(z) \in \mathcal{B}_M$ is a tableau of weight $\lambda$, then the weight space $M_{\lambda}$ is spanned by the set of tableaux $\{\text{Tab}(z + (i, -i, 0)) \mid i \in \mathbb{Z}\} \cap \mathcal{B}_M$.

Proof. We characterize tableaux of the form $\text{Tab}(z + (m, n, k))$ with the same weight $\lambda$ of $\text{Tab}(z)$. Let $\bar{v}$ be the vector $\bar{v} + z$. By the Gelfand-Tsetlin formulas we have
\[
h_1(\text{Tab}(z + (m, n, k))) = (2(w_{11} + k) - (w_{21} + m + w_{22} + n + 1))\text{Tab}(z + (m, n, k))
\]
\[
h_2(\text{Tab}(z + (m, n, k))) = (2(w_{21} + m + w_{22} + n + 1) - (w_{11} + k))\text{Tab}(z + (m, n, k))
\]

In particular, $\text{Tab}(z)$ is a tableau with weight $\lambda = (2w_{11} - (w_{21} + w_{22} + 1), 2(w_{21} + w_{22} + 1) - w_{11})$. So, a tableau $\text{Tab}(z + (m, n, k))$ in $\mathcal{B}_M$ has weight $\lambda$ if $m, n, k$ satisfy the following linear system
\[
2w_{11} - (w_{21} + w_{22} + 1) = 2(w_{11} + k) - (w_{21} + m + w_{22} + n + 1)
\]
\[
2(w_{21} + w_{22} + 1) - w_{11} = 2(w_{21} + m + w_{22} + n + 1) - (w_{11} + k)
\]
which is, $m + n = 0$ and $k = 0$. \hfill \square
7.4. Realizations of all irreducible singular Gelfand-Tsetlin \( \mathfrak{sl}(3) \)-modules. Now we will describe all the possible blocks for 1-singular Gelfand-Tsetlin characters. Such description will be (as in the case of generic modules) by presentation of explicit basis for each irreducible module in the block, weight lattice description and components of the Loewy series for the module \( V(T(\bar{v})) \). In order to describe all possible blocks for 1-singular modules we will consider all possible critical Gelfand-Tsetlin tableau \( T(\bar{v}) \) and describe the basis for any irreducible subquotient of the module \( V(T(\bar{v})) \) as we did in Example 7.37.

Recall that for \( n = 3 \), we have that 
\[
B(T(\bar{v})) = \{ T(\bar{v} + (m, n, k)), DT(\bar{v} + (m', n', k')) | m \leq n \text{ and } n' > m' \}.
\]
Also, recall that for \( w \in \mathbb{Z}^3 \), the tableau associated to \( w \) is 
\[
Tab(w) := \begin{cases} 
T(\bar{v} + w), & \text{if } w_{21} \leq w_{22} \\
DT(\bar{v} + w), & \text{if } w_{21} > w_{22}.
\end{cases}
\]
In particular, the map \( w \mapsto Tab(w) \) defines a bijection \( Tab : \mathbb{Z}^3 \to B(T(\bar{v})) \). Henceforth the irreducible subquotients will be defined by their corresponding sets in \( \mathbb{Z}^3 \), equivalently, by their bases in \( B(T(\bar{v})) \). We should note that all subsets of \( \mathbb{Z}^3 \) that define an irreducible subquotient are defined by a set of inequalities of the form \( a \leq b \) or \( a < b \) where \( a, b \) are elements in the set \( \{m, n, k, 0, -t, -s\} \). With this in mind we have the following definition.

**Definition 7.40.** Let \( B \) be a convex subset of \( \mathbb{Z}^3 \) such that there is a subquotient of \( V(T(\bar{v})) \) whose basis is \( Tab(B) \). Then this subquotient is denoted by \( M(B) \), if \( M(B) \) is simple, it will be denoted by \( L(B) \).

We now describe the sets \( B \subseteq \mathbb{Z}^3 \) that define all irreducible subquotients of \( V(T(\bar{v})) \). For convenience, the modules listed in one row are isomorphic. It is worth noting that all isomorphisms between simple subquotients of \( V(T(\bar{v})) \) are \( \tau \)-induced, that is all isomorphisms between irreducible subquotients are given by \( L(B) \mapsto L(\tau(B)) \).

**Remark 7.41.** In general it is not true that if \( B \subseteq \mathbb{Z}^3 \) defines a subquotient of \( V(T(\bar{v})) \) then \( \tau(B) \) defines also a subquotient of \( V(T(\bar{v})) \).

From now on and until the end of this section we will use the letters \( a, b, c, x, z \) to denote complex numbers. We will assume also that if any two different letters appear in the same or consecutive rows of a Gelfand-Tsetlin tableau then its difference is not an integer.

(C1) Consider the Gelfand-Tsetlin tableau:

\[
T(\bar{v}) = \begin{array}{ccc}
  a & b & c \\
  x & x & z \\
\end{array}
\]

In this case the module \( V(T(\bar{v})) \) is irreducible and all its weight spaces are infinite dimensional, therefore:

| Module | Basis |
|--------|-------|
| \( L_1 \) | \( L(\mathbb{Z}^3) \) |

(C2) Consider the following Gelfand-Tsetlin tableau:
I. Irreducible subquotients.
This block consist of 6 irreducible modules with infinite dimensional weight spaces.

\[
\begin{array}{cc}
\text{Module} & \text{Basis} \\
L_1 & L(m \leq 0) \\
L_2 & L(0 < m)
\end{array}
\]

II. Loewy series.

\[L_1; L_2\]

(C4) Consider the Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
a & b & c \\
a & a \\
a
\end{array}
\]

I. Irreducible subquotients.
This block consist of 6 irreducible modules; two of them are isomorphic modules with infinite dimensional weight spaces.

(i) Modules with unbounded finite weight multiplicities:
(ii) Two isomorphic modules with infinite dimensional weight spaces:

| Module | Basis |
|--------|-------|
| $L_2$  | $L \left( \begin{array}{c} m \leq 0 \\ 0 < n \\ m < k \leq n \end{array} \right)$ |
| $L_5$  | $L \left( \begin{array}{c} n \leq 0 < m \\ n < k \leq m \end{array} \right)$ |

II. Loewy series.

$L_1, L_2, L_3 \oplus L_4, L_5, L_6$. 
(C5) For any $t \in \mathbb{Z}_{>0}$, consider the following Gelfand-Tsetlin tableau:

$$T(\bar{v}) = \begin{array}{c}
\begin{array}{c}
a \\
a - t \\
a
\end{array}
\end{array}$$

I. Irreducible subquotients.

This module $V(T(\bar{v}))$ has 16 irreducible subquotients; two of them are isomorphic with infinite dimensional weight spaces. The bases are given by:

(i) Six modules with weight multiplicities of dimension at most $t$,
   two pairs of isomorphic modules and two more modules:

| Module | Basis | Module | Basis |
|--------|-------|--------|-------|
| $L_2$  | $L\left(\begin{array}{c}m \leq -t < n \\
                           n \leq 0 \\
                           m < k \leq n\end{array}\right)$ | $L_8$  | $L\left(\begin{array}{c}n \leq -t < m \\
                           m \leq 0 \\
                           n < k \leq m\end{array}\right)$ |
| $L_7$  | $L\left(\begin{array}{c}-t < m \leq 0 \\
                           0 < n \\
                           m < k \leq n\end{array}\right)$ | $L_{15}$ | $L\left(\begin{array}{c}-t < n \leq 0 \\
                           0 < m \\
                           n < k \leq m\end{array}\right)$ |
| $L_4$  | $L\left(\begin{array}{c}-t < m \leq 0 \\
                           0 < n \\
                           k \leq m\end{array}\right)$ $\cup$ $L\left(\begin{array}{c}-t < m \leq 0 \\
                           n \leq 0 \\
                           k \leq n\end{array}\right)$ | $L_{13}$ | $L\left(\begin{array}{c}m \leq n \\
                           -t < m \leq 0 \\
                           n < k \leq m\end{array}\right)$ $\cup$ $L\left(\begin{array}{c}-t < m \leq 0 \\
                           n \leq -t \\
                           m < k\end{array}\right)$ $\cup$ $L\left(\begin{array}{c}m \leq 0 \\
                           -t < n \\
                           n < k\end{array}\right)$ |

(ii) Modules with unbounded weight multiplicities:

$L_7 \rightarrow$ 
$L_2 \rightarrow$ 
$L_4 \rightarrow$ 
$L_{13} \leftarrow$
CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN MODULES OF $\mathfrak{sl}(3)$

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left( \begin{array}{l} m \leq -t \\ 0 < n \\ k \leq m \end{array} \right)$ |
| $L_9$  | $L \left( \begin{array}{l} m \leq -t \\ 0 < n \\ n < k \end{array} \right)$ |
| $L_{10}$ | $L \left( \begin{array}{l} n \leq -t \\ 0 < m \\ k \leq n \end{array} \right)$ |
| $L_{14}$ | $L \left( \begin{array}{l} n \leq -t \\ 0 < m \\ m < k \end{array} \right)$ |

| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{l} m \leq -t < n \leq 0 \\ k \leq m \end{array} \right) \cup \left( \begin{array}{l} m \leq -t \\ n \leq -t \\ k \leq n \end{array} \right)$ |
| $L_6$  | $L \left( \begin{array}{l} m \leq -t < n \leq 0 \\ n < k \end{array} \right) \cup \left( \begin{array}{l} m \leq -t \\ n \leq -t \\ n < k \end{array} \right)$ |
| $L_{11}$ | $L \left( \begin{array}{l} m \leq n \\ 0 < m \end{array} \right) \cup \left( \begin{array}{l} -t < n \leq 0 \\ k \leq n \end{array} \right)$ |
| $L_{16}$ | $L \left( \begin{array}{l} m \leq n \\ 0 < m \end{array} \right) \cup \left( \begin{array}{l} -t < n \leq 0 \\ n < k \end{array} \right) \cup \left( \begin{array}{l} 0 < n \\ m < k \end{array} \right)$ |
(iii) Two isomorphic modules with infinite dimensional weight spaces:

\[
\begin{array}{c|c|c|c}
\text{Module} & \text{Basis} & \text{Module} & \text{Basis} \\
L_5 & L \left( \begin{array}{c}
m \leq -t \\
0 < n \\
m < k \leq n 
\end{array} \right) & L_{12} & L \left( \begin{array}{c}
n \leq -t \\
0 < m \\
n < k \leq m 
\end{array} \right)
\end{array}
\]

II. Loewy series.

\[
L_1; L_2 \oplus L_3; L_4 \oplus L_5 \oplus L_6; L_7 \oplus L_8 \oplus L_9 \oplus L_{10}; L_{11} \oplus L_{12} \oplus L_{13}; L_{14} \oplus L_{15}; L_{16}.
\]

(C6) Set \( t \in \mathbb{Z}_{>0} \) and consider the following Gelfand-Tsetlin tableau:

\[
T(\bar{\nu}) = \begin{array}{ccc}
  a & a - t & c \\
  a & a & \end{array}
\]

\[
\begin{array}{ccc}
  a & a - t & c \\
  a & a & z \\
\end{array}
\]

I. Irreducible subquotients.

The module \( V(T(\bar{\nu})) \) has 5 irreducible subquotients.

(i) Two modules with unbounded weight multiplicities:

\[
\begin{array}{c|c|c}
\text{Module} & \text{Basis} \\
L_1 & L \left( \begin{array}{c}
m \leq n \\
m \leq -t \\
n \leq 0 
\end{array} \cup \begin{array}{c}
n < m \\
m \leq -t 
\end{array} \right) \\
L_5 & L \left( \begin{array}{c}
m \leq n \\
0 < m \\
-t < n 
\end{array} \right)
\end{array}
\]
(ii) A cuspidal module with $t$-dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left\{ -t < m \leq 0 \right\}$ |

(iii) Two isomorphic modules with infinite dimensional weight spaces:

| Module | Basis |
|--------|-------|
| $L_2$  | $L \left\{ m \leq -t \right\}$ |
| $L_4$  | $L \left\{ n \leq -t \right\}$ |

II. Loewy series.

$L_1; L_2; L_3; L_4; L_5$

(C7) Consider the following Gelfand-Tsetlin tableau:

$I$. Irreducible subquotients.

The module $V(T(\bar{v}))$ has 10 irreducible subquotients; two isomorphic with infinite dimensional weight spaces. The bases of such modules are given by:

(i) Eight modules with unbounded weight multiplicities:
| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \\ k \leq n \end{array} \right)$ |
| $L_2$  | $L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \\ n < k \end{array} \right)$ |
| $L_9$  | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ k \leq n \end{array} \right)$ |
| $L_{10}$ | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ n < k \end{array} \right)$ |

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left( \begin{array}{c} k \leq m \leq 0 < n \end{array} \right)$ |
| $L_5$  | $L \left( \begin{array}{c} k \leq n \leq 0 < m \end{array} \right)$ |
| $L_6$  | $L \left( \begin{array}{c} m \leq 0 < n < k \end{array} \right)$ |
| $L_8$  | $L \left( \begin{array}{c} n \leq 0 < m < k \end{array} \right)$ |

![Diagram of modules and bases](attachment:image.png)
I. Irreducible subquotients.

The module $V(T(\bar{v}))$ has 4 irreducible subquotients; two of them are isomorphic with infinite dimensional weight spaces.

(i) Modules with unbounded weight multiplicities:

| Module | Basis               | Module | Basis               |
|--------|---------------------|--------|---------------------|
| $L_1$  | $L \left\{ \begin{array}{l} m \leq 0 < n \\ m < k \leq n \end{array} \right\}$ | $L_4$  | $L \left\{ \begin{array}{l} n \leq 0 < m \\ n < k \leq m \end{array} \right\}$ |

(C8) Consider the following Gelfand-Tsetlin tableau:

$$T(\bar{v}) = \begin{bmatrix} a & a & c \\ a & a & z \end{bmatrix}$$

II. Loewy series.

$L_1$; $L_2 \oplus L_3$; $L_4$; $L_5 \oplus L_6$; $L_7$; $L_8 \oplus L_9$; $L_{10}$.

(ii) Two isomorphic modules with infinite dimensional weight spaces:
(ii) Two isomorphic modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_2$  | $L\{m \leq 0 < n\}$ |
| $L_3$  | $L\{n \leq 0 < m\}$ |

II. Loewy series.

$L_1; L_2 \oplus L_3; L_4$.

(C9) Let $s, t \in \mathbb{Z}_{>0}$ be such that $t < s$. Consider the following Gelfand-Tsetlin tableau:

$T(\bar{v}) = \begin{array}{ccc} a & a-t & a-s \\ a & a & \\ z \end{array}$

I. Irreducible subquotients.

The module $V(T(\bar{v}))$ has 10 irreducible subquotients; two of them are isomorphic with infinite dimensional weight spaces.

(i) Two modules with unbounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_1$  | $L\left(\begin{array}{ccc} m \leq n \\ m \leq -s \\ n \leq -t \end{array}\right) \cup \left(\begin{array}{c} n < m \\ m \leq -s \\ n \leq -t \end{array}\right)$ |
| $L_{10}$ | $L\left(\begin{array}{ccc} m \leq n \\ 0 < m \end{array}\right) \cup \left(\begin{array}{c} n < m \\ 0 < m \end{array}\right)$ |
(ii) Three modules with weight multiplicities bounded by $t$:

| Module | Basis                                      |
|--------|--------------------------------------------|
| $L_2$  | $L \begin{cases} \ m \leq -s \\ \ -t < n \leq 0 \end{cases}$ |
| $L_6$  | $L \begin{cases} \ n \leq -s \\ \ -t < m \leq 0 \end{cases}$ |

(iii) Three modules with weight multiplicities bounded by $s - t$:

| Module | Basis                                      |
|--------|--------------------------------------------|
| $L_5$  | $L \left( \begin{cases} -s < m \leq -t \\ 0 < n \end{cases} \right)$ |
| $L_9$  | $L \left( \begin{cases} -s < n \leq -t \\ 0 < m \end{cases} \right)$ |

| Module | Basis                                      |
|--------|--------------------------------------------|
| $L_3$  | $L \left( \begin{cases} m \leq n \\ -s < m \leq -t \\ n \leq 0 \end{cases} \right)$ |
(iv) Two isomorphic modules with infinite dimensional weight spaces

| Module | Basis | Module | Basis |
|--------|-------|--------|-------|
| \( L_4 \) | \( L \left\{ \begin{array}{l} m \leq -s \\ 0 < n \end{array} \right\} \) | \( L_7 \) | \( L \left\{ \begin{array}{l} n \leq -s \\ 0 < m \end{array} \right\} \) |

II. Loewy series.

\( L_1; L_2; L_3 \oplus L_4; L_5 \oplus L_6; L_7 \oplus L_8; L_9; L_{10}. \)

(C10) Let \( t, s \in \mathbb{Z}_{>0} \) be such that \( t < s \). Consider the following Gelfand-Tsetlin tableau:

\[
\begin{array}{ccc}
  & a & a-t \\
  a & a & a-s \\
  & a \\
\end{array}
\]

T(\( \overline{v} \)) =

I. Irreducible subquotients.

The module \( V(T(\overline{v})) \) has 32 irreducible subquotients, two of them are isomorphic to the irreducible finite dimensional module with highest weight \( \lambda = (t-1, s-t-1) \). Also, there are two isomorphic modules with infinite dimensional weight spaces.

(i) Two isomorphic finite dimensional modules with weight multiplicities of degree \( \min\{t, s-t\} \) and highest weight \( \lambda = (t-1, s-t-1) \).

| Module | Basis | Module | Basis |
|--------|-------|--------|-------|
| \( L_8 \) | \( L \left\{ \begin{array}{l} -s < m \leq -t \\ -t < n \leq 0 \\ m < k \leq n \end{array} \right\} \) | \( L_{20} \) | \( L \left\{ \begin{array}{l} -s < n \leq -t \\ -t < m \leq 0 \\ n < k \leq m \end{array} \right\} \) |

(ii) Twenty weight modules with weight multiplicities bounded by \( t \) or \( s-t \). There are eight pairs of isomorphic modules and four more modules in the list:
| Module | Basis | Module | Basis |
|--------|-------|--------|-------|
| $L_2$  | $L \left\{ \begin{array}{l} m \leq -s \\ -t < n \leq 0 \\ k \leq m \end{array} \right. $ | $L_9$  | $L \left\{ \begin{array}{l} n \leq -s \\ -t < m \leq 0 \\ k \leq n \end{array} \right. $ |
| $L_3$  | $L \left\{ \begin{array}{l} m \leq -s < n \\ n \leq -t \\ m < k \leq n \end{array} \right. $ | $L_{12}$ | $L \left\{ \begin{array}{l} n \leq -s < m \\ m \leq -t \\ n < k \leq m \end{array} \right. $ |
| $L_6$  | $L \left\{ \begin{array}{l} m \leq -s \\ -t < n \leq 0 \\ m < k \leq n \end{array} \right. $ | $L_{15}$ | $L \left\{ \begin{array}{l} n \leq -s \\ -t < m \leq 0 \\ n < k \leq m \end{array} \right. $ |
| $L_{10}$ | $L \left\{ \begin{array}{l} m \leq -s \\ -t < n \leq 0 \\ n < k \end{array} \right. $ | $L_{22}$ | $L \left\{ \begin{array}{l} n \leq -s \\ -t < m \leq 0 \\ m < k \end{array} \right. $ |
| $L_{11}$ | $L \left\{ \begin{array}{l} -s < m \leq -t \\ 0 < n \\ k \leq m \end{array} \right. $ | $L_{23}$ | $L \left\{ \begin{array}{l} -s < n \leq -t \\ 0 < m \\ k \leq n \end{array} \right. $ |
| $L_{16}$ | $L \left\{ \begin{array}{l} -s < m \leq -t \\ 0 < n \\ m < k \leq n \end{array} \right. $ | $L_{27}$ | $L \left\{ \begin{array}{l} -s < n \leq -t \\ 0 < m \\ n < k \leq m \end{array} \right. $ |
| $L_{21}$ | $L \left\{ \begin{array}{l} -t < m \leq 0 \\ 0 < n \\ m < k \leq n \end{array} \right. $ | $L_{30}$ | $L \left\{ \begin{array}{l} -t < n \leq 0 \\ 0 < m \\ n < k \leq m \end{array} \right. $ |
| $L_{24}$ | $L \left\{ \begin{array}{l} -s < m \leq -t \\ 0 < n \\ n < k \end{array} \right. $ | $L_{31}$ | $L \left\{ \begin{array}{l} -s < n \leq -t \\ 0 < m \\ m < k \end{array} \right. $ |
The following picture describes the weight support of the above listed modules in the case of $t = 1$ and $s = 2$ (i.e. the principal block). Recall that modules in the same row are isomorphic, moreover for the modules whose support is the picture to the left, the weight multiplicities are bounded by $t$ and for the modules corresponding to the picture to the right, the multiplicities are bounded by $s - t$.

Note that the action of the Cartan subalgebra on each $L_i$ can be recovered from the Gelfand-Tsetlin formulas. In particular, $(2k - (m + n + 1), 2(m + n + 1) - k)$ is the weight corresponding to the tableau $Tab(m, n, k)$. In the pictures above, the point in the middle is $(0,0)$ and corresponds to the trivial (i.e. the finite-dimensional) module.
(iii) Eight simple Verma modules (with unbounded set of weight multiplicities). There are two pairs of isomorphic modules and four more modules in the list.

| Module | Basis |
|--------|-------|
| $L_5$  | $L \left\{ \begin{array}{l} k \leq m \leq -s \\ 0 < n \end{array} \right\}$ |
| $L_{19}$ | $L \left\{ \begin{array}{l} m \leq -s \\ 0 < n < k \end{array} \right\}$ |
| $L_{18}$ | $L \left\{ \begin{array}{l} k \leq n \leq -s \\ 0 < m \end{array} \right\}$ |
| $L_{29}$ | $L \left\{ \begin{array}{l} n \leq -s \\ 0 < m < k \end{array} \right\}$ |

The weight supports are listed below.

The weight supports are listed below.
(iv) Two weight modules with infinite weight multiplicities. In this case we have one pair of isomorphic modules:

\[
\begin{array}{c|c|c|c}
\text{Module} & \text{Basis} & \text{Module} & \text{Basis} \\
L_{13} & L \left( \begin{array}{c}
  m \leq -s \\
  0 < n \\
  m < k \leq n
\end{array} \right) \\
L_{25} & L \left( \begin{array}{c}
  n \leq -s \\
  0 < m \\
  n < k \leq m
\end{array} \right)
\end{array}
\]

II. Loewy series.

\begin{align*}
L_1, \; L_2 \oplus L_3, \; L_4 \oplus L_5 \oplus L_6 \oplus L_7, \; L_8 \oplus L_9 \oplus L_{10} \oplus L_{11} \oplus L_{12} \oplus L_{13}, \\
L_{14} \oplus L_{15} \oplus L_{16} \oplus L_{17} \oplus L_{18} \oplus L_{19}, \; L_{20} \oplus L_{21} \oplus L_{22} \oplus L_{23} \oplus L_{24} \oplus L_{25}, \\
L_{26} \oplus L_{27} \oplus L_{28} \oplus L_{29}, \; L_{30} \oplus L_{31}, \; L_{32}.
\end{align*}

(C11) Set \( t \in \mathbb{Z}_{>0} \) and consider the following Gelfand-Tsetlin tableau:

\[
T(\bar{v}) = \begin{array}{c|c|c|c}
  a & a & a - t \\
  a & a & \\
  & a
\end{array}
\]

I. Irreducible subquotients.

In this case the module \( V(T(\bar{v})) \) has 20 irreducible subquotients; two of them are isomorphic with infinite dimensional weight spaces.

(i) Six modules with finite dimensional weight spaces of dimension at most \( t \), given by:
(ii) Six modules with unbounded finite weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left( \begin{array}{l} k \leq m \leq -t \\ 0 < n \end{array} \right)$ |
| $L_{11}$ | $L \left( \begin{array}{l} m \leq -t \\ 0 < n < k \end{array} \right)$ |
| $L_{13}$ | $L \left( \begin{array}{l} k \leq n \leq -t \\ 0 < m \end{array} \right)$ |
| $L_{18}$ | $L \left( \begin{array}{l} n \leq -t \\ 0 < m < k \end{array} \right)$ |
| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c} m \leq n \\ m \leq -t \\ -t < n \leq 0 \\ k \leq m \end{array} \right) \bigcup \left( \begin{array}{c} m \leq n \\ n \leq -t \\ k \leq n \end{array} \right) \bigcup \left( \begin{array}{c} n < m \\ m \leq -t \\ k \leq n \end{array} \right) \bigcup \left( \begin{array}{c} n < m \\ m \leq -t \\ k \leq n \end{array} \right)$ |
| $L_5$  | $L \left( \begin{array}{c} m \leq n \\ m \leq -t \\ n \leq 0 \\ n < k \end{array} \right) \bigcup \left( \begin{array}{c} n < m \\ m \leq -t \\ n < k \end{array} \right)$ |
| $L_{16}$ | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ k \leq n \end{array} \right)$ |
| $L_{20}$ | $L \left( \begin{array}{c} 0 < m \\ 0 < n \\ n < k \end{array} \right)$ |
(iii) Two isomorphic modules with infinite dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_6 \) | \( L \left( \begin{array}{c} m \leq -t \\ 0 < n \\ m < k \leq n \end{array} \right) \) |
| \( L_{15} \) | \( L \left( \begin{array}{c} n \leq -t \\ 0 < m \\ n < k \leq m \end{array} \right) \) |

II. Loewy series.

\[ L_1: L_2 \oplus L_3 \oplus L_4; \ L_5 \oplus L_6 \oplus L_7; \ L_8 \oplus L_9 \oplus L_{10} \oplus L_{11} \oplus L_{12} \oplus L_{13}; \ L_{14} \oplus L_{15} \oplus L_{16}; \ L_{17} \oplus L_{18} \oplus L_{19}; \ L_{20}. \]

(C12) For any \( t \in \mathbb{Z}_{>0} \) consider the tableau:

\[
T(\bar{v}) = \begin{array}{c|c|c}
\bullet & \bullet & \bullet \\
\hline
\bullet & \bullet & T
\end{array}
\]

I. Irreducible subquotients.

This block consist of 5 irreducible modules.

(i) Two modules with unbounded weight multiplicities:

| Module | Basis |
|--------|-------|
| \( L_1 \) | \( L \left( \begin{array}{c} m \leq n \\ m \leq -t \\ n \leq 0 \end{array} \right) \cup \left( \begin{array}{c} n < m \\ m \leq -t \end{array} \right) \) |
| \( L_5 \) | \( L \left( \begin{array}{c} m \leq n \\ 0 < m \end{array} \right) \cup \left( \begin{array}{c} 0 < m \\ -t < n \end{array} \right) \) |
(ii) A cuspidal module with $t$-dimensional weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_3$  | $L \left( -t < m \leq 0 \right)$ |

(iii) Two isomorphic modules with infinite dimensional weight spaces:

| Module | Basis          | Module | Basis          |
|--------|----------------|--------|----------------|
| $L_2$  | $L \left( \begin{array}{c} m \leq -t \\ 0 < n \end{array} \right)$ | $L_4$  | $L \left( \begin{array}{c} n \leq -t \\ 0 < m \end{array} \right)$ |

II. Loewy series.

$L_1; L_2; L_3; L_4; L_5$.

(C13) Consider the following Gelfand-Tsetlin tableau:

```
T(\bar{v})=
```

```
a   a   a
a   a
a
```

I. Irreducible subquotients.

The module $V(T(\bar{v}))$ has 10 irreducible subquotients; two isomorphic with infinite dimensional weight spaces.

(i) Modules with unbounded weight multiplicities:
CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN MODULES OF $\mathfrak{sl}(3)$

| Module | Basis |
|--------|-------|
| $L_1$  | $L\left(\begin{array}{c} m \leq 0 \\ n \leq 0 \\ k \leq n \end{array}\right)$ |
| $L_3$  | $L\left(\begin{array}{c} m \leq 0 \\ n \leq 0 \\ n < k \end{array}\right)$ |
| $L_8$  | $L\left(\begin{array}{c} 0 < m \\ 0 < n \\ k \leq n \end{array}\right)$ |
| $L_{10}$ | $L\left(\begin{array}{c} 0 < m \\ 0 < n \\ n < k \end{array}\right)$ |

| Module | Basis |
|--------|-------|
| $L_2$  | $L\left(\begin{array}{c} k \leq m \leq 0 \\ 0 < n \end{array}\right)$ |
| $L_6$  | $L\left(\begin{array}{c} m \leq 0 < n \\ n < k \end{array}\right)$ |
| $L_5$  | $L\left(\begin{array}{c} k \leq n \leq 0 \\ 0 < m \end{array}\right)$ |
| $L_9$  | $L\left(\begin{array}{c} n \leq 0 < m \\ m < k \end{array}\right)$ |

\[ \rightarrow L_1 \]
\[ \leftarrow L_2 \]
(ii) Two isomorphic modules with infinite dimensional weight spaces:

| Module | Basis |
|--------|-------|
| $L_4$  | $L \left( \begin{array}{c} m \leq 0 < n \\ m < k \leq n \end{array} \right)$ |
| $L_7$  | $L \left( \begin{array}{c} n \leq 0 < m \\ n < k \leq m \end{array} \right)$ |

II. Loewy series.

$L_1; L_2 \oplus L_3; L_4; L_5 \oplus L_6; L_7; L_8 \oplus L_9; L_{10}$.

(C14) Consider the following Gelfand-Tsetlin tableau:

\[ T(\bar{v}) = \begin{array}{ccc} a & a & a \\ a & a & z \end{array} \]

I. Irreducible subquotients.

The module $V(T(\bar{v}))$ has 4 irreducible subquotients; two of them isomorphic with infinite dimensional weight spaces.

(i) Modules with unbounded weight multiplicities:

| Module | Basis |
|--------|-------|
| $L_1$  | $L \left( \begin{array}{c} m \leq 0 \\ n \leq 0 \end{array} \right)$ |
| $L_4$  | $L \left( \begin{array}{c} 0 < n \\ 0 < m \end{array} \right)$ |
Let Lemma 8.1. $u$ where $x$ of $\Phi$ is actually finite. Note that for details we refer the reader to [De80] and [M00].

For every simple root $\alpha \in \Delta$ the multiplicative set $F_\alpha := \{f_n^\alpha | n \in \mathbb{Z}_{\geq 0}\} \subset U$ satisfies Ore’s localizability conditions because $\text{ad} f_\alpha$ acts locally nilpotent on $U$. Let $D_\alpha U$ be the localization of $U$ relative to $F_\alpha$. For every weight module $M$ we denote by $D_\alpha M$ the $\alpha$-localization of $M$, defined as $D_\alpha M = D_\alpha U \otimes_U M$. If $f_\alpha$ is injective on $M$, then $M$ can be naturally viewed as a submodule of $D_\alpha M$. Furthermore, if $f_\alpha$ is injective on $M$, then it is bijective on $M$ if and only if $D_\alpha M = M$. Finally, if $[f_\alpha, f_\beta] = 0$ and both $f_\alpha$ and $f_\beta$ are injective on $M$, then $D_\alpha D_\beta M = D_\beta D_\alpha M$.

For $x \in \mathbb{C}$ and $u \in D_\alpha U$ we set

\[
\Theta_x(u) := \sum_{i \geq 0} \binom{x}{i} (\text{ad} f_\alpha)^i (u) f_\alpha^{-i},
\]

where $\binom{x}{i} = \frac{x(x-1) \ldots (x-i+1)}{i!}$. Since $\text{ad} f_\alpha$ is locally nilpotent on $U_\alpha$, the sum above is actually finite. Note that for $x \in \mathbb{Z}$ we have $\Theta_x(u) = f_n^\alpha u f_n^{-\alpha}$. For a $D_\alpha U$-module $M$ by $\Phi^x_\alpha M$ we denote the $D_\alpha U$-module $M$ twisted by the action

\[
u \cdot v^x := (\Theta_x(u) \cdot v)^x,
\]

where $u \in D_\alpha U$, $v \in M$, and $v^x$ stands for the element $v$ considered as an element of $\Phi^x_\alpha M$. Since $v^n = f_n^\alpha \cdot v$ whenever $n \in \mathbb{Z}$ it is convenient to set $f_n^x \cdot v := v^{x^n}$ in $\Phi^x_\alpha M$ for $x \in \mathbb{C}$.

The following lemma is straightforward.

**Lemma 8.1.** Let $M$ be a $D_\alpha U$-module, $v \in M$, $u \in D_\alpha U$ and $x, y \in \mathbb{C}$. Then

(i) $\Phi^x_\alpha M \simeq M$ whenever $x \in \mathbb{Z}$.

(ii) $\Phi^x_\alpha (\Phi^y_\alpha M) \simeq \Phi^{x+y}_\alpha M$ and, consequently, $\Phi^x_\alpha \circ \Phi^{-x}_\alpha M \simeq \Phi^{-x}_\alpha \circ \Phi^x_\alpha M \simeq M$.

\[
(ii) \text{ Two isomorphic modules with infinite dimensional weight multiplicities:}
\]

| Module | Basis |
|--------|-------|
| $L_2$  | $L \ (m \leq 0 < n)$ |
| $L_3$  | $L \ (n \leq 0 < m)$ |

II. Loewy series.

$L_1; L_2 \oplus L_3; L_4$.

8. Localization functors for Gelfand-Tsetlin modules

8.1. Localization and twisted localization functors. In this subsection $U = U(\mathfrak{sl}(3))$. We recall the definition of the localization functor on $U$-modules. For details we refer the reader to [De80] and [M00].

\[

\text{CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN MODULES OF } \mathfrak{sl}(3) \hfill 87
\]
rules. In particular, suppose that whenever we have a weight space.

Remark 8.3. As a proof when we restrict the action of the operator to weight spaces. Moreover, we give conditions for injectivity or surjectivity of the operator \( E_{21} \) (i.e. we will give conditions for injectivity or surjectivity of \( E_{21} \)).

8.2. Injectivity and surjectivity of the Gelfand-Tsetlin operators. Our goal is to apply localization functors to \( \mathfrak{sl}(3) \) Gelfand-Tsetlin modules and also, we want to have a criteria for verifying if a certain module can be obtained by localization or as a subquotient of a localized module.

With this in mind, our first step will be to obtain conditions on the basis of the modules that guarantee injectivity or surjectivity of the operator \( f_\alpha \).

8.2.1. Injectivity and surjectivity of the operator \( E_{21} \). In this section, by \( V \) we will mean the generic module \( V(T(v)) \) (or the singular module \( V(T(\bar{v})) \), and \( B \) will stands for the lattice of tableaux \( B(T(v)) \) (or \( B(T(\bar{v})) \)). We also consider a Gelfand-Tsetlin module \( M \) that is a subquotient of \( V \) with a basis \( B_M \subseteq B \).

**Remark 8.3.** As \( V \) is a weight module, every subquotient is a weight module and, then in order to prove injectivity or surjectivity of an operator, it is enough to make a proof when we restrict the action of the operator to weight spaces. Moreover, we should remember that whenever we have a weight \( \mathfrak{sl}(3) \)-module \( M \) and a weight \( \lambda = (\lambda_1, \lambda_2) \), \( E_{21}(M_\lambda) \subseteq M_{(\lambda_1-2, \lambda_2+1)} \).

In the case of a generic tableau \( T(v) \) will be convenient to write \( Tab(w) := T(v+w) \). Having this in mind, we can write the action of the operator \( E_{21} \) on \( Tab(w) \) (generic or singular) in \( B \) follows:

\[
E_{21}(Tab(w)) = Tab(w - \delta^{11})
\]

From Propositions 7.4 and 7.39 if \( Tab(w) \in B_M \) is a weight vector of weight \( \lambda \), then the weight space \( M_\lambda \) is spanned by \( \{ Tab(w + (i, -i, 0)) \mid i \in \mathbb{Z} \} \cap B_M \). If \( w_i \) denotes the vector \( w + (i, -i, 0) \in \mathbb{Z}^3 \), any element of \( M_\lambda \) will be of the form \( u = \sum_{i \in I} a_i T(w_i) \) with \( I \subseteq \mathbb{Z} \) finite.

**Lemma 8.4** (\( E_{21} \) injective). The operator \( E_{21} \) acts injective on \( M \) if and only if, \( Tab(w) \in B_M \) implies \( Tab(w - \delta^{11}) \in B_M \).

**Proof.** Suppose first that there exists \( Tab(w) \in B_M \) such that \( Tab(w - \delta^{11}) \notin B_M \), then \( E_{21}(Tab(w)) = Tab(w - \delta^{11}) = 0 \) (on \( M \)), which implies that \( E_{21} \) is not injective. On the other hand, suppose \( u := \sum_{i \in I} a_i Tab(w_i) \in M_\lambda \) with \( Tab(w_i) \in

\[ B_M \] for any \( i \). If \( u \) is such that \( 0 = E_{21}(u) = \sum_{i \in I} a_i \text{Tab}(w_i - \delta^{11}) \), then \( a_i \text{Tab}(w_i - \delta^{11}) = 0 \) for any \( i \in I \). Therefore, as by hypothesis \( \text{Tab}(w_i - \delta^{11}) \in B_M \) we should have \( a_i = 0 \) for any \( i \in I \).

\[ \square \]

**Lemma 8.5** (\( E_{21} \) surjective). The operator \( E_{21} \) acts surjective on \( M \) if and only if, \( T(w) \in B_M \) implies \( T(w + \delta^{11}) \in B_M \).

**Proof.** Any element of \( M(\lambda_1 - 2, \lambda_2 + 1) \) is of the form \( u' = \sum_{i \in I} a_i \text{Tab}(w_i - \delta^{11}) \) and a direct computation using formula (36) shows that \( E_{21} \left( \sum_{i \in I} a_i \text{Tab}(w_i) \right) = u' \). \[ \square \]

8.3. **Localization functors on \( \mathfrak{so}(3) \)-case.** In this section we see when the localization of a module on \( \mathcal{G}T \) is again a module on \( \mathcal{G}T \) and in this case, we will see the relation between tableaux defining an irreducible module \( M \) and tableaux defining the module \( \mathcal{D}_\alpha(M) \), where \( \mathcal{D}_\alpha(M) \) denotes the localization of \( M \) with respect to \( X_\alpha \). In what follows we set \( \mathcal{D}_\alpha^z := \Phi^z(\mathcal{D}_\alpha M) \) and refer to it as a twisted localization of \( M \). In the case when \( f_\alpha \) acts injectively on \( M \), set \( \mathcal{QD}_\alpha M := \mathcal{D}_\alpha M/M \). Also, if \( \alpha = \epsilon_i - \epsilon_j \) we will write \( \mathcal{D}_{ij}, \mathcal{D}_{ij}^x, \) and \( \mathcal{QD}_{ij} \) for \( \mathcal{D}_\alpha, \mathcal{D}_\alpha^z, \) and \( \mathcal{QD}_\alpha \), respectively.

In the following sections we will describe the action of \( \Theta_x \) on the generators of \( \Gamma \) and see how this action can be related with Gelfand-Tsetlin modules with tableaux realization.

8.3.1. **Localization with respect to \( E_{21} \).** Recall that for \( x \in \mathcal{C} \) and \( u \in \mathcal{D}_{12}U \) we have

\[ \Theta_x(u) := \sum_{i \geq 0} \binom{x}{i} (\text{ad } E_{21})^i(u) E_{21}^{-i}, \]

and that for \( x \in \mathbb{Z} \) we have \( \Theta_x(u) = E_{21}^x u E_{21}^{-x} \). Moreover, for a \( \mathcal{D}_{12}U \)-module \( M \) by \( \mathcal{D}_{12}^x M \) we denote the \( \mathcal{D}_{12}U \)-module \( M \) twisted by the action

\[ u \cdot v^x := (\Theta_x(u) \cdot v)^x, \]

where \( u \in \mathcal{D}_{12}U, v \in M, \) and \( v^x \) stands for the element \( v \) considered as an element of \( \mathcal{D}_{12}^x M \).

**Lemma 8.6.** Let \( \{c_{ij}\}_{1 \leq j \leq i \leq 3} \) be the generators of \( \Gamma \) defined in (7). Then

\[ \Theta_x(c_{ij}) = \begin{cases} c_{ij}, & \text{if } (i,j) \neq (1,1) \\ c_{11} + x, & \text{if } (i,j) = (1,1). \end{cases} \]

**Proof.** First of all we should note that whenever \( u \) commutes with \( E_{21} \) we have \( \Theta_x(u) = u \). Now, as the generators \( \{c_{ij}\}_{2 \leq j \leq i \leq 3} \) commute with \( E_{21} \) we have the first part of the lemma. For the second part we should note that \( c_{11} = E_{11} \) and that \( (\text{ad } E_{21})^2(1_{11}) = 0 \). \[ \square \]

As an immediate consequence of Lemma 8.6 we have the following important corollary.

**Corollary 8.7.** Let \( M \) be any Gelfand-Tsetlin module such that \( E_{21} \) acts injectively. Then

(i) The twisted localized module \( \mathcal{D}_{12}^x M \) is also a Gelfand-Tsetlin module.
(ii) If \( v \in M \) has Gelfand-Tsetlin character \( \chi = (a_1, a_2, a_3, a_4, a_5, a_6) \), then the element \( v^x \) on the localized module has Gelfand-Tsetlin character \( \tilde{\chi} = (a_1, a_2, a_3, a_4, a_5, a_6 + x) \).

In the case of a Gelfand-Tsetlin module \( M \) with tableaux realization and basis \( B_M \), it is possible to describe explicitly the basis of the twisted localized modules, for that the following notation will be useful:

For a region \( B \subseteq \mathbb{Z}^3 \), denote by \( B + t \delta^{11} \) the region \( \{(m, n, k) \mid (m, n, k - 1) \in B\} \). Set \( B + t \delta^{11} = (B + (t - 1) \delta^{11}) + \delta^{11} \) for \( t \in \mathbb{N} \) and \( B + \mathbb{N} \delta^{11} = \bigcup_{t=0}^{\infty} (B + t \delta^{11}) \).

**Proposition 8.8.** Let \( B \subseteq \mathbb{Z}^3 \) and \( L(B) \) be an irreducible Gelfand-Tsetlin module. Assume that \( E_{21} \) acts injectively on \( L(v; B) \). Then \( \mathcal{D}_{12} L(v; B) \simeq M(v + x \delta^{11}, B + \mathbb{N} \delta^{11}) \) and \( \mathcal{Q} \mathcal{D}_{12} L(v; B) \simeq M(v + x \delta^{11}, (B + \mathbb{N} \delta^{11}) \setminus B) \). In particular, if \( x \) is an integer we have \( \mathcal{P}_{12} L(B) \simeq M(B + \mathbb{N} \delta^{11}) \) and \( \mathcal{Q} \mathcal{P}_{12} L(B) \simeq M((B + \mathbb{N} \delta^{11}) \setminus B) \).

**Corollary 8.9.** If \( M \) is an irreducible module in \( \mathcal{G} \mathcal{T} \), generated by a tableau \( \text{Tab}(w) \) such that \( E_{21} \) acts injective on \( M \), then for any \( x \in \mathbb{C} \) the twisted localized module \( \mathcal{D}_{12} M \) contains a subquotient isomorphic to a irreducible \( \mathfrak{sl}(3) \)-module in \( \mathcal{G} \mathcal{T} \) generated by the tableau \( \text{Tab}(v + x \delta^{11}) \).

### 8.4. Irreducible Gelfand-Tsetlin modules and localization functors

In this section we will describe the irreducible Gelfand-Tsetlin \( \mathfrak{sl}(3) \)-modules via localization functors and subquotients starting with some irreducible \( E_{21} \)-injective Gelfand-Tsetlin module. In order to give such description we are using strongly Lemmas 7.4, 7.5 and Proposition 8.8. In fact, in order to use Proposition 8.8 we have to check if the corresponding module defined in such region is bijective.

Let us denote by \( L_i^{(G_3)} \) the irreducible module \( L_i \) in the \( i \)-th block for generic modules described in Table 7.2 and \( L_i^{(G_3)} \) the irreducible module \( L_i \) in the \( i \)-th block for singular modules described in Table 7.4.

The list of all irreducible modules which are \( E_{21} \)-injective is the following:

1. **Irreducible \( E_{21} \)-injective generic modules:**

2. **Irreducible \( E_{21} \)-injective singular modules:**

Now will use localization functors and subquotient on each block in order to obtain all the irreducible modules as show the following tables:
| Block | Module | Subquotient of $E_{21}$-localization |
|-------|--------|-------------------------------------|
| G1    | $L_1^{(G1)}$, $L_2^{(G1)}$ | $D_{12}^{(z-2)}(L_1^{(G2)})$ |
| G2    | $L_3^{(G2)}$ | $QD_{12}(L_1^{(G2)})$ |
| G3    | $L_1^{(G3)}$, $L_2^{(G3)}$, $L_3^{(G3)}$ | $D_{12}^{(y-z)}(L_1^{(G4)}) \simeq D_{12}^{(z-a)}(L_1^{(G5)})$ $D_{12}^{(y-z)}(L_2^{(G4)}) \simeq D_{12}^{(z-a)}(L_3^{(G5)})$ |
| G4    | $L_3^{(G4)}$, $L_4^{(G4)}$ | $QD_{12}(L_1^{(G4)})$ $QD_{12}(L_2^{(G4)})$ |
| G5    | $L_2^{(G5)}$, $L_3^{(G5)}$ | $QD_{12}(L_1^{(G5)})$ $QD_{12}(L_2^{(G5)})$ |
| G6    | $L_2^{(G6)}$, $L_5^{(G6)}$, $L_4^{(G6)}$, $L_8^{(G6)}$ | $QD_{12}(L_1^{(G6)})$ $QD_{12}(L_3^{(G6)})$ $QD_{12}(L_4^{(G6)})$ $QD_{12}(L_7^{(G6)})$ |
| G7    | $L_1^{(G7)}$, $L_2^{(G7)}$, $L_3^{(G7)}$, $L_4^{(G7)}$, $L_5^{(G7)}$ | $D_{12}^{(z-a)}(L_1^{(G6)})$ $D_{12}^{(z-a)}(L_2^{(G6)})$ $D_{12}^{(z-a)}(L_3^{(G6)})$ $D_{12}^{(z-a)}(L_4^{(G6)})$ $D_{12}^{(z-a)}(L_5^{(G6)})$ |
| G8    | $L_1^{(G8)}$, $L_2^{(G8)}$, $L_3^{(G8)}$ | $D_{12}^{(z-a)}(L_1^{(G9)}) \simeq D_{12}^{(y-z)}(L_1^{(G10)})$ $D_{12}^{(z-a)}(L_3^{(G9)}) \simeq D_{12}^{(y-z)}(L_2^{(G10)})$ $D_{12}^{(z-a)}(L_5^{(G9)}) \simeq D_{12}^{(y-z)}(L_4^{(G10)})$ |
| G9    | $L_2^{(G9)}$, $L_4^{(G9)}$, $L_6^{(G9)}$ | $QD_{12}(L_1^{(G9)})$ $QD_{12}(L_3^{(G9)})$ $QD_{12}(L_5^{(G9)})$ |
| G10   | $L_1^{(G10)}$, $L_5^{(G10)}$, $L_6^{(G10)}$ | $QD_{12}(L_1^{(G10)})$ $QD_{12}(L_2^{(G10)})$ $QD_{12}(L_4^{(G10)})$ |
| Block | Module | Subquotient of $E_{21}$-localization |
|-------|--------|-------------------------------------|
| G11   | $L_1^{(G11)}$, $L_4^{(G11)}$, $L_7^{(G11)}$, $L_9^{(G11)}$, $L_{10}^{(G11)}$, $L_{12}^{(G11)}$ | $\mathcal{D}_{12}(L_1^{(G11)})$, $\mathcal{D}_{12}(L_4^{(G11)})$, $\mathcal{D}_{12}(L_6^{(G11)})$, $\mathcal{D}_{12}(L_8^{(G11)})$, $\mathcal{D}_{12}(L_{G11})$ |
| G12   | $L_1^{(G12)}$, $L_6^{(G12)}$, $L_7^{(G12)}$, $L_9^{(G12)}$, $L_{10}^{(G12)}$, $L_{12}^{(G12)}$ | $\mathcal{D}_{12}(L_1^{(G12)})$, $\mathcal{D}_{12}(L_4^{(G12)})$, $\mathcal{D}_{12}(L_6^{(G12)})$, $\mathcal{D}_{12}(L_8^{(G12)})$, $\mathcal{D}_{12}(L_{G12})$ |
| G13   | $L_1^{(G13)}$, $L_2^{(G13)}$, $L_4^{(G13)}$, $L_5^{(G13)}$, $L_6^{(G13)}$, $L_{10}^{(G13)}$ | $\mathcal{D}_{12}^{z-a}(L_1^{(G11)})$, $\mathcal{D}_{12}^{z-a}(L_4^{(G11)})$, $\mathcal{D}_{12}^{z-a}(L_6^{(G11)})$, $\mathcal{D}_{12}^{z-a}(L_8^{(G11)})$, $\mathcal{D}_{12}^{z-a}(L_{G11})$ |
| G14   | $L_3^{(G14)}$, $L_5^{(G14)}$, $L_7^{(G14)}$, $L_8^{(G14)}$ | $\mathcal{D}_{12}(L_1^{(G14)})$, $\mathcal{D}_{12}(L_2^{(G14)})$, $\mathcal{D}_{12}(L_4^{(G14)})$, $\mathcal{D}_{12}(L_{G14})$ |
| G15   | $L_1^{(G15)}$, $L_2^{(G15)}$, $L_4^{(G15)}$, $L_5^{(G15)}$, $L_6^{(G15)}$, $L_{12}^{(G15)}$ | $\mathcal{D}_{12}(L_1^{(G15)})$, $\mathcal{D}_{12}(L_2^{(G15)})$, $\mathcal{D}_{12}(L_4^{(G15)})$, $\mathcal{D}_{12}(L_6^{(G15)})$ |
| G16   | $L_1^{(G16)}$, $L_2^{(G16)}$, $L_4^{(G16)}$ | $\mathcal{D}_{12}^{z-a}(L_1^{(G14)}) \simeq \mathcal{D}_{12}^{y-z}(L_1^{(G15)})$, $\mathcal{D}_{12}^{z-a}(L_2^{(G14)}) \simeq \mathcal{D}_{12}^{y-z}(L_2^{(G15)})$, $\mathcal{D}_{12}^{z-a}(L_4^{(G14)}) \simeq \mathcal{D}_{12}^{y-z}(L_6^{(G15)})$ |
| Block | Module | Subquotient of $E_{21}$-localization |
|-------|--------|-------------------------------------|
| C1    | $L_1^{(C1)}$ | $QD_{12}^{(z-a)}(L_1^{(C2)})$ |
| C2    | $L_2^{(C2)}$ | $QD_{12}(L_1^{(C2)})$ |
| C3    | $L_1^{(C3)}$ | $QD_{12}^{(z-a)}(L_1^{(C4)})$ |
|       | $L_2^{(C3)}$ | $QD_{12}^{(z-a)}(L_4^{(C4)})$ |
| C4    | $L_2^{(C4)} \simeq L_5^{(C4)}$ | $soc(QD_{12}(L_1^{(C4)})) \simeq QD_{12}(L_1^{(C4)})$ |
|       | $L_3^{(C4)}$ | $QD_{12}(L_1^{(C4)})/L_2^{(C4)}$ |
|       | $L_5^{(C4)}$ | $QD_{12}(L_1^{(C4)})/L_5^{(C4)}$ |
| C5    | $L_2^{(C5)} \simeq L_8^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_5^{(C5)} \simeq L_7^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_6^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_3^{(C5)} \simeq L_9^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_7^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_4^{(C5)} \simeq L_8^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_{10}^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
|       | $L_{11}^{(C5)}$ | $soc(QD_{12}(L_1^{(C5)})) \simeq QD_{12}(L_1^{(C5)})$ |
| C6    | $L_1^{(C6)} \simeq L_4^{(C6)}$ | $QD_{12}^{(z-a)}(L_1^{(C6)})$ |
|       | $L_2^{(C6)} \simeq L_4^{(C6)}$ | $QD_{12}^{(z-a)}(L_4^{(C6)})$ |
|       | $L_3^{(C6)} \simeq L_6^{(C6)}$ | $QD_{12}^{(z-a)}(L_4^{(C6)})$ |
|       | $L_5^{(C6)} \simeq L_6^{(C6)}$ | $QD_{12}^{(z-a)}(L_4^{(C6)})$ |
| C7    | $L_{2}^{(C7)} \simeq L_{7}^{(C7)}$ | $QD_{12}(L_1^{(C7)})$ |
|       | $L_{3}^{(C7)} \simeq L_{7}^{(C7)}$ | $QD_{12}(L_1^{(C7)})$ |
|       | $L_{4}^{(C7)} \simeq L_{7}^{(C7)}$ | $QD_{12}(L_1^{(C7)})$ |
|       | $L_{5}^{(C7)} \simeq L_{7}^{(C7)}$ | $QD_{12}(L_1^{(C7)})$ |
| C8    | $L_1^{(C8)} \simeq L_3^{(C8)}$ | $QD_{12}^{(z-a)}(L_3^{(C8)})$ |
|       | $L_2^{(C8)} \simeq L_3^{(C8)}$ | $QD_{12}^{(z-a)}(L_3^{(C8)})$ |
|       | $L_3^{(C8)} \simeq L_9^{(C8)}$ | $QD_{12}^{(z-a)}(L_3^{(C8)})$ |
| C9    | $L_1^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_1^{(C10)})$ |
|       | $L_2^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_3^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_4^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_5^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_6^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_8^{(C9)} \simeq L_9^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
|       | $L_{10}^{(C9)}$ | $QD_{12}^{(z-a)}(L_9^{(C10)})$ |
| Block | Module | Subquotient of $E_{21}$-localization |
|-------|--------|-----------------------------------|
| C10   | $L_{12}^{(C10)} \simeq L_{20}^{(C10)}$ | soc($\mathcal{Q}D_{12}(L_1^{(C10)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_9^{(C10)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_7^{(C10)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_5^{(C10)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_3^{(C10)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_1^{(C10)})$) |
|       | $L_4^{(C10)} \simeq L_{10}^{(C10)}$ | $\mathcal{Q}D_{12}(L_4^{(C10)})/L_2^{(C10)}$ |
|       | $L_5^{(C10)} \simeq L_{15}^{(C10)}$ | $\mathcal{Q}D_{12}(L_5^{(C10)})/L_4^{(C10)}$ |
|       | $L_6^{(C10)} \simeq L_{20}^{(C10)}$ | $\mathcal{Q}D_{12}(L_6^{(C10)})/L_5^{(C10)}$ |
|       | $L_7^{(C10)} \simeq L_{21}^{(C10)}$ | $\mathcal{Q}D_{12}(L_7^{(C10)})/L_6^{(C10)}$ |
|       | $L_8^{(C10)} \simeq L_{26}^{(C10)}$ | $\mathcal{Q}D_{12}(L_8^{(C10)})/L_7^{(C10)}$ |
|       | $L_9^{(C10)} \simeq L_{27}^{(C10)}$ | $\mathcal{Q}D_{12}(L_9^{(C10)})/L_8^{(C10)}$ |
|       | $L_{10}^{(C10)} \simeq L_{30}^{(C10)}$ | $\mathcal{Q}D_{12}(L_{10}^{(C10)})/L_9^{(C10)}$ |
| C11   | $L_2^{(C11)} \simeq L_7^{(C11)}$ | soc($\mathcal{Q}D_{12}(L_1^{(C11)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_4^{(C11)})$) |
|       | $L_5^{(C11)} \simeq L_{10}^{(C11)}$ | $\mathcal{Q}D_{12}(L_5^{(C11)})/L_4^{(C11)}$ |
|       | $L_6^{(C11)} \simeq L_{15}^{(C11)}$ | $\mathcal{Q}D_{12}(L_6^{(C11)})/L_5^{(C11)}$ |
|       | $L_7^{(C11)} \simeq L_{19}^{(C11)}$ | $\mathcal{Q}D_{12}(L_7^{(C11)})/L_6^{(C11)}$ |
|       | $L_8^{(C11)} \simeq L_{24}^{(C11)}$ | $\mathcal{Q}D_{12}(L_8^{(C11)})/L_7^{(C11)}$ |
|       | $L_9^{(C11)} \simeq L_{27}^{(C11)}$ | $\mathcal{Q}D_{12}(L_9^{(C11)})/L_8^{(C11)}$ |
|       | $L_{10}^{(C11)} \simeq L_{32}^{(C11)}$ | $\mathcal{Q}D_{12}(L_{10}^{(C11)})/L_9^{(C11)}$ |
| C12   | $L_2^{(C12)} \simeq L_4^{(C12)}$ | $\mathcal{Q}D_{12}^{(z-a)}(L_1^{(C12)})$ $\simeq$ $\mathcal{Q}D_{12}^{(z-a)}(L_3^{(C12)})$ $\simeq$ $\mathcal{Q}D_{12}^{(z-a)}(L_5^{(C12)})$ $\simeq$ $\mathcal{Q}D_{12}^{(z-a)}(L_7^{(C12)})$ |
|       | $L_3^{(C12)} \simeq L_5^{(C12)}$ | $\mathcal{Q}D_{12}^{(z-a)}(L_2^{(C12)})$ $\simeq$ $\mathcal{Q}D_{12}^{(z-a)}(L_4^{(C12)})$ $\simeq$ $\mathcal{Q}D_{12}^{(z-a)}(L_6^{(C12)})$ |
| C13   | $L_3^{(C13)} \simeq L_3^{(C13)}$ | $\mathcal{Q}D_{12}(L_1^{(C13)})$ $\simeq$ soc($\mathcal{Q}D_{12}(L_2^{(C13)})$) $\simeq$ soc($\mathcal{Q}D_{12}(L_3^{(C13)})$) $\simeq$ $\mathcal{Q}D_{12}(L_4^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_5^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_6^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_7^{(C13)})$ |
|       | $L_4^{(C13)} \simeq L_4^{(C13)}$ | $\mathcal{Q}D_{12}(L_3^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_5^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_6^{(C13)})$ $\simeq$ $\mathcal{Q}D_{12}(L_7^{(C13)})$ |
| C14   | $L_4^{(C14)} \simeq L_4^{(C14)}$ | $\mathcal{Q}D_{12}(L_3^{(C14)})$ $\simeq$ $\mathcal{Q}D_{12}(L_5^{(C14)})$ $\simeq$ $\mathcal{Q}D_{12}(L_6^{(C14)})$ $\simeq$ $\mathcal{Q}D_{12}(L_7^{(C14)})$ |

**Corollary 8.10.** Every irreducible Gelfand-Tsetlin module can be obtained via a composition of twisted localization functors and taking subquotients from an irreducible $E_{21}$-injective Gelfand-Tsetlin module.
CLASSIFICATION OF IRREDUCIBLE GELFAND-TSETLIN MODULES OF \( \mathfrak{sl}(3) \)

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Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo SP, Brasil
E-mail address: futorny@ime.usp.br

University of Texas at Arlington, Arlington, TX 76019, USA
E-mail address: grandim@uta.edu

Universidade Federal do ABC, Santo André SP, Brasil
E-mail address: luis.enrique@ufabc.edu.br