A simple Efimov space with sequentially-nice space of probability measures

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Abstract
Under Jensen’s diamond principle ♦, we construct a simple Efimov space $K$ whose space of nonatomic probability measures $P_{na}(K)$ is first-countable and sequentially compact. These two properties of $P_{na}(K)$ imply that the space of probability measures $P(K)$ on $K$ is selectively sequentially pseudocompact. We show that any sequence of probability measures on $K$ that converges to a purely atomic measure converges in norm, and any sequence of probability measures on $K$ converging to zero in sup-norm has a subsequence converging to a nonatomic probability measure. We show also that the Banach space $C(K)$ of continuous functions on $K$ has the Gelfand–Phillips property but it does not have the Grothendieck property.

Keywords Efimov space · Selective sequential pseudocompactness · Space of probability measures · Nonatomic probability measure · Gelfand–Phillips property · Grothendieck property

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1 Introduction
For a Tychonoff space $K$, we denote by $P(K)$ the space of all regular probability measures on $K$ and by $C(K)$ the space of all real-valued continuous functions on $K$. Then the space $P(K)$ endowed with the weak$^*$ topology induced from the dual space $C(K)'$ of the Banach space $C(K)$ is a compact Hausdorff space.

The study of spaces of probability measures is an important direction of research in Measure Theory and General Topology. For numerous results, open questions and historical remarks we refer the reader to the survey of Fedorchuk [19] and the recent monograph of
Bogachev [4]. Many results in this area describe the interplay between properties of the space \( P(K) \) and topological properties of the compact space \( K \). For example, according to a classical result of Lindenstrauss [28], a compact space \( K \) is Eberlein if and only if so is its space of probability measures \( P(K) \). However, the question of whether \( P(K) \) is Corson compact for any Corson compact \( K \) does not depend on ZFC.

The most important topological properties of compact spaces are properties of sequential type such as first-countability, sequential compactness, countable tightness or selective sequential pseudocompactness. Following [8], we call a topological space \( X \) selectively sequentially pseudocompact if for any sequence \((U_n)_{n \in \omega}\) of nonempty open sets in \( X \) there exists a sequence \((x_n)_{n \in \omega} \subseteq \prod_{n \in \omega} U_n\) containing a convergent subsequence. Clearly, every infinite compact selectively sequentially pseudocompact space contains many nontrivial convergent sequences. For any compact space \( K \) we have the following implications.

\[
\text{metrizable} \quad \Rightarrow \quad \text{sequentially compact} \quad \Rightarrow \quad \text{selectively sequentially pseudocompact}
\]

\[
\text{first countable} \quad \Rightarrow \quad \text{sequential} \quad \Rightarrow \quad \text{countably tight}
\]

It is easy to show that the space \( P(K) \) is metrizable if and only if so is \( K \). However, already the first-countability of \( K \) may not imply even the countable tightness of \( P(K) \). Using the Continuum Hypothesis, Haydon [23] constructed a first-countable non-metrizable compact space \( K \) which is the support of some nonatomic probability measure. As it was noticed by Fedorchuk [19], for the Haydon compact \( K \) the tightness of \( P(K) \) is uncountable. Under CH, Talagrand showed in [33] that there is a Corson compact space \( K \) such that \( P(K) \) contains a copy of \( \beta\omega \). These results motivate the inverse question: Assume that \( P(K) \) has one of the sequential properties from the diagram. What can be said about sequential properties of the compact space \( K \)? Clearly, if \( P(K) \) is countably tight (sequentially compact or first countable) then so is \( K \). But what about the case when the compact space \( P(K) \) is selectively sequentially pseudocompact? Is then also \( K \) selectively sequentially pseudocompact? In our main result, under Jensen’s diamond principle \( \Box \), we answer this question in the negative in a very strong form. More precisely, under \( \Box \) we construct a fully simple Efimov space \( K \) whose space of probability measures \( P(K) \) contains a dense first-countable sequentially compact subspace and hence is selectively sequentially pseudocompact. Let us recall that a compact space is called Efimov if it contains neither nontrivial convergent sequences nor a topological copy of the Stone–Čech compactification \( \beta\omega \) of the discrete space \( \omega \) (we recall the notion of a fully simple compact space below in Sect. 2). It should be mentioned that the existence of a Efimov space in ZFC is a major unsolved problem of General and Set-Theoretic Topology and all known examples of Efimov spaces are constructed only under some additional set-theoretic assumptions: \( \mathfrak{s} = \omega_1 < 2^{\omega_1} = \mathfrak{c} \) by Fedorchuk [17], \( \Box \) by Fedorchuk [16] and Džamonja and Plebanek [13], or Martin’s Axiom by Dow and Shelah [11]. Our example of a Efimov space \( K \) is based on a classical construction of Efimov spaces under \( \Diamond \).

Our interest to the selective sequential pseudocompactness is motivated by a result of Drewnowski and Emmanuele [12, Theorem 4.1] which states that if a compact space \( K \) is selectively sequentially pseudocompact (that is, \( K \) belongs to the class \( \mathcal{K}'' \) in the terminology of [12]), then the Banach space \( C(K) \) has the Gelfand–Phillips property. Recall that a Banach space \( E \) has the Gelfand–Phillips property if every limited set in \( E \) is precompact, where a bounded subset \( B \) of \( E \) is called limited if each weak* null sequence \((x_n)_{n \in \omega}\) in the dual space \( E' \) converges uniformly on \( B \), that is \( \lim_n \sup \{|x_n(x)| : x \in B\} = 0 \). By
a classical result of Phillips [29], the Banach space $\ell_\infty = C(\beta\omega)$ is not Gelfand–Phillips, where $\beta\omega$ is the Stone–Cech compactification of the discrete space $\omega$ of nonnegative integers. In [3], we prove that the selective sequential pseudocompactness of $K$ in this theorem of Drewnowski and Emmanuele can be weakened to the selective sequential pseudocompactness of any subspace $A \subseteq P(K)$ that contains $K$. In particular, if $P(K)$ is selectively sequentially pseudocompact, then the Banach space $C(K)$ has the Gelfand–Phillips property. Since any compact selectively sequentially pseudocompact space has many nontrivial convergent sequences, Drewnowski and Emmanuele ask in [12] whether there exists a compact space $K$ without nontrivial convergent sequences such that the Banach space $C(K)$ still has the Gelfand–Phillips property. Under CH, this question was answered negatively by Schlumprecht [31]. These results motivate a problem posed in [3, Problem 2.7] of finding a Efimov space $K$ whose probability measure space $P(K)$ has in addition a quite strong sequential type property of being a selectively sequentially pseudocompact space.

Let $K$ be a compact space. The Hahn decomposition theorem states that any probability measure $\mu \in P(K)$ can be represented as a convex combination

$$\mu = t\mu_a + (1 - t)\mu_{na},$$

where $t \in [0, 1]$, $\mu_a$ is purely atomic and $\mu_{na}$ is nonatomic. Denote by $P_a(K)$ and $P_{na}(K)$ subspaces of $P(K)$ consisting of all purely atomic and all nonatomic measures, respectively. It is clear that $\delta[K] \subseteq P_a(K)$ and $P_a(K) \cap P_{na}(K) = \emptyset$. This observation motivates the problem of study of (topological) properties of the spaces $P_a(K)$ and $P_{na}(K)$ and their relationships to (topological) properties of $K$ and $P(K)$.

The set $P_a(K)$ is dense in $P(K)$ and can be considered also as a subset of the Banach space $\ell_1(K)$ as well as a subset of the Banach space $c_0(K)$. Since $\ell_1(K)$ has the Schur property, one can naturally ask: Can also $P_a(K)$ have the same property in the sense that every convergent sequence in $P_a(K)$ is also norm convergent? On the other hand, it is interesting to understand what happen with sequences $(\mu_n)_{n \in \omega}$ converging to zero in the sup-norm of $c_0(K)$. This motivates the following notion. A sequence $(\mu_n)_{n \in \omega}$ of measures on $K$ (not necessary purely atomic) is defined to be $c_0$-vanishing if

$$\lim_{n \to \infty} \sup_{x \in K} \mu_n(\{x\}) = 0.$$ 

It turns out that our Efimov space $K$ satisfies very strong conditions: (1) every sequence in $P(K)$ that converges to a purely atomic measure is norm convergent, and (2) every $c_0$-vanishing sequence in $P(K)$ has a subsequence converging to a nonatomic measure. The latter property of $K$ implies that the space $P_{na}(K)$ is sequentially compact. In fact, our Efimov space $K$ is crowded (i.e., it has no isolated points), and therefore the set $P_{na}(K)$ is dense in $P(K)$. Moreover, $P_{na}(K)$ is a first-countable sequentially compact space. The space $P_{na}(K)$ is not metrizable (and cannot be metrizable as every metrizable sequentially compact space is compact and $P_{na}(X)$ is not compact for a crowded compact space $X$). On the other hand, in Example 2.3 we construct a non-metrizable crowded compact space $C$ whose space of nonatomic probability measures $P_{na}(C)$ is metrizable.

The Gelfand–Phillips property of the Banach space $C(K)$ naturally motivates the study of other geometrical properties of $C(K)$. One of the most important properties of Banach spaces is the Grothendieck property. A Banach space $E$ has the Grothendieck property if every weak* null-sequence in $E'$ is also weakly null. We refer the reader to the books [6, 7] and the very recent survey [22].

Now we formulate our main result in which (under $\diamondsuit$) the clause (iv) solves Problem 2.7 from [3].
Theorem 1.1 Under Jensen’s Diamond Principle \(\diamond\), there exists a fully simple crowded separable Efimov space \(K\) such that

(i) every \(c_0\)-vanishing sequence \((\mu_n)_{n<\omega}\) in \(P(K)\) has a subsequence converging to a nonatomic measure \(\mu \in P_{na}(K)\);

(ii) every sequence in \(P(K)\) that converges to a purely atomic measure converges in norm, so the weak* convergence and the norm convergence coincide on \(P_0(K)\);

(iii) the space \(P_{na}(K)\) is non-metrizable, first-countable, \(\check{C}\)ech-complete, sequentially compact, and the set \(P_{na}(K)\) is dense in \(P(K)\);

(iv) the space \(P(K)\) is selectively sequentially pseudocompact but not sequentially compact;

(v) the Banach space \(C(K)\) has the Gelfand–Phillips property and hence it does not contain an isomorphic copy of \(\ell_\infty\);

(vi) \(C(K)\) does not have the Grothendieck property and hence \(K\) is zero-dimensional but not extremally disconnected.

It should be mentioned that consistently there are Efimov spaces \(K\) such that \(C(K)\) have the Grothendieck property, see for example the articles of Talagrand [32] and Brech [5]. We also note that, by Corollary 2.10 proved below, those Banach spaces \(C(K)\) do not have the Gelfand–Phillips property.

Theorem 1.1 will be proved in Sect. 6 after some preliminary work done in Sects. 2–5.

2 Preliminaries

All topological spaces in the paper are Tychonoff. A subset of a topological space is clopen if it is both closed and open. A topological space \(X\) is called crowded if it has no isolated points. An indexed family of subsets \((F_i)_{i \in I}\) of a set is called disjoint if \(F_i \cap F_j = \emptyset\) for any distinct indices \(i, j \in I\). For two sets \(A, B\) we write \(A \subseteq^* B\) if \(A \setminus B\) is finite. If \(f : X \to Y\) is a map and \(F\) is a subset of \(X\), we denote by \(f[F]\) the image \(\{f(x) : x \in F\}\) of the set \(F\). For a set \(A\) and an indexed family \((x_\alpha)_{\alpha \in A}\) of points of a topological space \(X\), we say that \((x_\alpha)_{\alpha \in A}\) converges to a point \(x \in X\) and write \(\lim_{\alpha \in A} x_\alpha = x\) if for every neighborhood \(O_x \subseteq X\) the set \(\{\alpha \in A : x_\alpha \notin O_x\}\) is finite.

The following lemma is Corollary 3.4 in [8]. We give its direct and short proof for the sake of completeness.

Lemma 2.1 If some dense subspace \(Y\) of a topological space \(X\) is selectively sequentially pseudocompact, then \(X\) itself is selectively sequentially pseudocompact.

Proof Let \(\{U_n\}_{n<\omega}\) be a sequence of nonempty open subsets of \(X\). Then \(\{Y \cap U_n\}_{n<\omega}\) is a sequence of nonempty open subsets of \(Y\). Since \(Y\) is selectively sequentially pseudocompact, there exists a sequence \((x_n)_{n<\omega}\) in \(\prod_{n<\omega} Y \cap U_n\) containing a convergent subsequence \((x_{n_k})_{k<\omega}\). It is clear that \((x_n)_{n<\omega}\) in \(\prod_{n<\omega} U_n\) and \((x_{n_k})_{k<\omega}\) converges in \(X\). Thus \(X\) is selectively sequentially pseudocompact. \(\square\)

By a measure on a compact space \(X\) we understand any regular \(\sigma\)-additive measure \(\mu : B(X) \to [0, \infty)\) on the \(\sigma\)-algebra \(B(X)\) of Borel subsets of \(X\). The regularity of \(\mu\) means that for every Borel set \(B \subseteq X\) and every \(\varepsilon > 0\) there exists a closed subset \(F \subseteq B\) of \(X\) such that \(\mu(B \setminus F) < \varepsilon\). A measure \(\mu\) is a probability measure if \(\mu(X) = 1\). The space \(P(X)\) of probability measures on \(X\) is endowed with the topology generated by the subbase consisting of the sets \(\{\mu \in P(X) : \mu(U) > a\}\) where \(a \in \mathbb{R}\) and \(U\) runs over open subsets of \(X\).
By the Riesz Representation Theorem [24, 7.6.1], the space $P(X)$ can be identified with the subspace 

$$\{\mu \in C(X)' : \|\mu\| = 1 = \mu(1_X)\}$$

of the dual Banach space $C(X)'$, endowed with the weak* topology. Besides the weak* topology we shall also use the topology generated by the norm on $P(X)$. For two measures $\mu, \lambda$ on $X$ their norm-distance can be found by the formula 

$$\|\mu - \lambda\| = \sup_{B \in B(X)} |\mu(B) - \lambda(B)|.$$

Each element $x \in X$ can be identified with the Dirac measure 

$$\delta_x : B(X) \to [0, 1], \quad \delta_x : B \mapsto \begin{cases} 1 & \text{if } x \in B; \\ 0 & \text{otherwise.} \end{cases}$$

So, $X$ can be identified with the closed subspace $\{\delta_x : x \in X\}$ of $P(X)$.

The convex hull of $X$ in $P(X)$ is denoted by $P_\omega(X)$; elements of $P_\omega(X)$ are called finitely supported measures on $X$. The support of a measure $\mu \in P(X)$ denoted by $\text{supp}(\mu)$ is the smallest closed subset $F \subseteq K$ such that $\mu(X \setminus F) = 0$. The support of a finitely supported measure $\mu \in P_\omega(X)$ is finite and coincides with the set 

$$\text{atom}(\mu) = \{x \in X : \mu([x]) > 0\}$$

of atoms of $\mu$. The additivity of a probability measure $\mu \in P(X)$ implies that the set $\text{atom}(\mu)$ is at most countable and $\mu(\text{atom}(\mu)) \leq 1$. A measure $\mu \in P(X)$ is called

- purely atomic if $\mu(\text{atom}(\mu)) = 1$;
- nonatomic if $\text{atom}(\mu) = \emptyset$.

By $P_a(X)$ and $P_{na}(X)$ we denote the subspaces of $P(X)$ consisting of all purely atomic measures and all nonatomic measures on $X$, respectively. Note that the set $P_a(X)$ of purely atomic measures contains $P_\omega(X)$ and hence is dense in $P(X)$.

Every continuous map $f : X \to Y$ between compact spaces induces a continuous map $Pf : P(X) \to P(Y)$ assigning to every $\mu \in P(X)$ the measure $Pf(\mu) : B(Y) \to [0, 1]$, $Pf(\mu) : B \mapsto \mu(f^{-1}[B])$.

**Lemma 2.2** For every (crowded) compact space $X$, the set $P_\omega(X)$ is dense in $P(X)$ and the set $P_{na}(X)$ is a (dense) $G_\delta$-set in $P(X)$.

**Proof** The density of $P_\omega(X)$ in $P(X)$ is a well-known fact, but for the convenience of the reader, we present here a simple proof. Fix any measure $\mu \in P(X)$ and a neighborhood $O_\mu$ of $\mu$ in $P(X)$. By the definition of the topology on $P(X)$, there exist $\varepsilon > 0$ and open sets $U_1, \ldots, U_n$ in $X$ such that

$$\bigcap_{i=1}^n [\lambda \in P(X) : \lambda(U_i) > \mu(U_i) - \varepsilon] \subseteq O_\mu.$$

Let $B$ be the smallest Boolean algebra of subsets of $X$ containing the family $\{U_1, \ldots, U_n\}$, and let $A$ be the set of atoms of $B$. In each set $A \in A$ choose a point $x_A \in A$ and observe that the finitely supported measure $\sum_{A \in A} \mu(A) \cdot \delta_{x_A}$ belongs to the intersection $O_\mu \cap P_\omega(X)$, witnessing that the set $P_\omega(X)$ is dense in $P(X)$. 
Next, we show that the set \( P_{na}(X) \) is (dense) \( G_\delta \) in \( P(X) \). For every natural number \( n > 0 \), consider the continuous map

\[
\Xi_n : P(X) \times X \times \left[ \frac{1}{n}, 1 \right] \to P(X), \quad \Xi_n : (\mu, x, t) \mapsto (1 - t)\mu + t\delta_x,
\]

and observe that the image

\[
A_n = \Xi \left( P(X) \times X \times \left[ \frac{1}{n}, 1 \right] \right) = \left\{ \mu \in P(X) : \exists x \in X \mu((x)) \geq \frac{1}{n} \right\}
\]

is compact and hence closed in \( P(X) \). Since \( P_{na}(X) = \bigcap_{n=1}^{\infty} (P(X) \setminus A_n) \), the set \( P_{na}(X) \) is of type \( G_\delta \) in \( P(X) \).

If the space \( X \) is crowded, then every set \( A_n \) is nowhere dense in \( P(X) \). Indeed, given any nonempty open set \( U \subseteq P(X) \) we can use the density of \( P_{\omega}(X) \) in \( P(X) \) and find a finitely supported measure \( \mu \in U \cap P_{\omega}(X) \). By the definition of the topology on \( P(X) \), there exist a positive real number \( \varepsilon > 0 \) and a family of open sets \( (O_x)_{x \in \text{supp}(\mu)} \) in \( X \) such that \( x \in O_x \) for every \( x \in \text{supp}(\mu) \) and

\[
\bigcap_{x \in \text{supp}(\mu)} \left\{ \lambda \in P(X) : \lambda(O_x) > \mu(O_x) - \varepsilon \right\} \subseteq U.
\]

Since the space \( X \) is Hausdorff, we can additionally assume that the indexed family \( (O_x)_{x \in \text{supp}(\mu)} \) is disjoint. The absence of isolated points in the crowded space \( X \) ensures that for every \( x \in \text{supp}(\mu) \), the set \( O_x \) is infinite and hence contains a finite subset \( F_x \) of cardinality \( |F_x| > n \). Then the finitely supported measure

\[
\sum_{x \in \text{supp}(\mu)} \sum_{y \in F_x} \frac{\mu([x])}{|F_x|} \delta_y
\]

belongs to the set \( U \setminus A_n \), witnessing that the closed set \( A_n \) is nowhere dense in \( P(X) \).

By the Baire Theorem, the \( G_\delta \)-set \( P_{na}(X) = \bigcap_{n \in \mathbb{N}} (P(X) \setminus A_n) \) is dense in \( P(X) \), being a countable intersection of open dense sets.

\( \square \)

**Example 2.3** There is a crowded compact nonmetrizable space \( C \) such that the space \( P_{na}(C) \) is metrizable.

**Proof** Let \( D \) be an uncountable discrete space. Denote by \( C = \{\infty\} \cup (D \times 2^\omega) \) the one point compactification of the space \( D \times 2^\omega \). It is clear that \( C \) is crowded and non-metrizable, and \( D \times 2^\omega \) is a metrizable open dense subspace in \( C \). By Theorem 1.4 in [1] and Theorem 4.4 in [2], the subspace \( \hat{P}(D \times 2^\omega) = \{ \mu \in P(C) : \mu(D \times 2^\omega) = 1 \} \) of \( P(C) \) is metrizable and so is its subspace \( P_{na}(C) \subset \hat{P}(D \times 2^\omega) \). \( \square \)

Let us recall some classical notions, we follow [10].

A map \( f : X \to Y \) between topological spaces \( X \) and \( Y \) is called (fully) simple if there exists a unique point \( z \in Y \) such that the set \( f^{-1}[z] \) (is nowhere dense in \( X \) and) contains exactly two points, and \( f^{-1}[y] \) is a singleton for all \( y \in Y \setminus \{z\} \). In this case \( X \) is called a (fully) simple extension of \( Y \).

An inverse system is a family \( (X_\alpha, \pi^\beta_\alpha : \alpha \leq \beta < \kappa) \), where (1) all \( X_\alpha \) are topological spaces, (2) \( \pi^\beta_\alpha : X_\alpha \to X_\beta \) is a continuous map such that the equation \( \pi^\gamma_\alpha = \pi^\beta_\alpha \circ \pi^\gamma_\beta \) holds for all \( \alpha \leq \beta \leq \gamma < \kappa \), and (3) \( \pi^\alpha_\alpha \) is the identity map for every \( \alpha < \kappa \). The inverse limit of the system is the appropriate subspace of the topological product of the family \( \{ X_\alpha : \alpha < \kappa \} \) as described in §2.5 of [14].
Definition 2.4 An inverse system $(X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \kappa)$ is

(i) continuous if $X_\alpha$ is the inverse limit of the system $(X_\gamma, \pi_\gamma^\beta : \gamma \leq \beta < \alpha)$ for any limit ordinal $\alpha < \kappa$;

(ii) based on (fully) simple extensions if for every $\alpha < \kappa$, the bonding map $\pi_\alpha^{\alpha+1} : X_{\alpha+1} \to X_\alpha$ is (fully) simple.

We shall use the following important result due to Koppelberg [27]; a topological proof was given by Dow [9].

Proposition 2.5 If $X$ is the limit of the inverse system $(X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \kappa)$ of metrizable compact spaces based on simple extensions, then $X$ does not map onto $[0, 1]^{\omega_1}$, unless $X_0$ does.

A compact space $X$ is called (fully) simple if it is the limit of a continuous inverse system of length $\omega_1$ consisting of zero-dimensional compact metrizable spaces based on (fully) simple extensions. It is clear that every fully simple compact space is simple. Fully simple compact spaces are the simplest examples of compact spaces which are limits of continuous inverse systems with fully closed bonding maps. For more information on such compacta, see the survey papers of Fedorchuk [16, 18, 20].

The following characterization of fully simple compact spaces will be used in the proof of the main Theorem 1.1.

Lemma 2.6 A non-metrizable compact space $X$ is fully simple if and only if $X$ is the limit of a continuous inverse system $(X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \omega_1)$ consisting of zero-dimensional compact metrizable spaces $X_\alpha$ such that for every $\alpha < \omega_1$ there exists a point $z \in X_\alpha$ such that the set $(\pi_\alpha^{\alpha+1})^{-1}(z)$ is nowhere dense in $X_{\alpha+1}$, contains one or two points, and for every point $y \in X_\alpha \setminus \{z\}$ the preimage $(\pi_\alpha^{\alpha+1})^{-1}(y)$ is a singleton.

Proof The necessity is clear. To prove the sufficiency, let $X$ be the limit of a continuous inverse system $(X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \omega_1)$ satisfying the assumption of the lemma. Let $\Omega \subseteq \omega_1$ be the set containing all ordinals $\alpha \in \omega_1$ such that for every $\beta < \alpha$ the bonding projection $\pi_\alpha^\beta$ is not a homeomorphism. It is easy to see that the set $\Omega$ is closed in $\omega_1$. The non-metrizability of $X$ ensures that the set $\Omega$ is unbounded in $\omega_1$ and hence $X$ can be identifies with the limit of the inverse system $(X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \alpha, \beta \in \Omega)$, witnessing that the compact space $X$ is fully simple.

Lemma 2.7 Every fully simple compact space $X$ is separable.

Proof Write $X$ as the limit of a well-ordered continuous spectrum $X = (X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \omega_1)$ consisting of separable metrizable compact spaces $X_\alpha$ and fully simple bonding maps $\pi_\alpha^{\alpha+1}$. For every $\alpha \in \omega_1$, let $\pi_\alpha : X \to X_\alpha$ be the limit projection.

To show that the space $X$ is separable, fix any countable set $D \subseteq X$ whose image $\pi_0[D]$ is dense in the compact space $X_0$. The separability of $X$ will be proved as soon as we shall check that for every ordinal $\alpha < \omega_1$, the set $D_\alpha := \pi_\alpha[D]$ is dense in $X_\alpha$. This fact will be proved by induction on $\alpha$. Assume that for some ordinal $\alpha \in \omega_1$ we have proved that the set $D_\beta$ is dense in $X_\beta$ for every ordinal $\beta < \alpha$. Given a nonempty open set $U_\alpha \subseteq X_\alpha$, we shall prove that $U_\alpha \cap D_\alpha \neq \emptyset$ distinguishing between three possible cases.

1. If $\alpha = 0$, then $U_\alpha \cap D_\alpha \neq \emptyset$ by the density of the set $D_0 = \pi_0[D]$ in $X_0$. 

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2. If \( \alpha \) is a nonzero limit ordinal, then by the continuity of the spectrum \( \mathcal{X} \) and Proposition 2.5.5 of [14], there is an ordinal \( \beta < \alpha \) such that \( U_\alpha \) contains a subset of the form \((\pi_\alpha^\beta)^{-1}[U_\beta] \) for some nonempty open set \( U_\beta \) in \( X_\beta \). Since, by assumption, \( D_\beta \) is dense in \( X_\beta \) there is \( d_\beta \in D_\beta \cap U_\beta \). Therefore, if \( d \in D \) is such that \( \pi_\beta^\alpha(d) = d_\beta \), then \( \pi_\alpha(d) \in (\pi_\alpha^\beta)^{-1}[U_\beta] \subseteq U_\alpha \), as desired.

3. Finally, assume that \( \alpha = \beta + 1 \) is a successor ordinal. Since the bonding map \( \pi_\beta^\alpha : X_\alpha \to X_\beta \) is fully simple, there exists a point \( y_\beta \in X_\beta \) such that

(a) \( (\pi_\beta^\alpha)^{-1}(y) \) is a singleton for all \( y \in X_\beta \setminus \{y_\beta\} \), and

(b) \( (\pi_\beta^\alpha)^{-1}(y_\beta) \) is nowhere dense in \( X_\alpha \).

Let \( Y_\beta = X_\beta \setminus \{y_\beta\} \) and \( Y_\alpha = X_\alpha \setminus (\pi_\alpha^\beta)^{-1}(y_\beta) = (\pi_\beta^\alpha)^{-1}[Y_\beta] \). The condition (a) implies that the map \( \pi_\beta^\alpha|_{Y_\alpha} : Y_\alpha \to Y_\beta \) is a homeomorphism. The nowhere density of the set \( (\pi_\beta^\alpha)^{-1}(y_\beta) \) in \( X_\alpha \) implies that the set \( U_\alpha \cap Y_\alpha \) is nonempty. Then \( V_\beta = \pi_\beta^\alpha(U_\alpha \cap Y_\alpha) \) is a nonempty open set in \( Y_\beta \) and also in \( X_\beta \). By the inductive assumption, the set \( D_\beta \) is dense in \( X_\beta \), so there exists a point \( d_\beta \in D_\alpha \cap V_\beta \). Let \( d \in D \) be such that \( \pi_\beta(d) = d_\beta \). It follows from (a) that \( \pi_\alpha(d) \in D_\alpha \cap U_\alpha \cap Y_\alpha \subseteq D_\alpha \cap U_\alpha \), as desired. \( \square \)

According to [27], simple compact spaces do not contain topological copies of \( \beta \omega \), so constructing a simple Efimov space \( X \) one should only care about killing all convergent sequences in \( X \). Another helpful property of simple compact spaces is the uniform regularity of nonatomic probability measures on such spaces, see [13].

A measure \( \mu \in P(X) \) on a compact space \( X \) is called uniformly regular if there exists a continuous map \( f : X \to Y \) to a compact metrizable space \( Y \) such that \( \mu(F) = \mu(f^{-1}[f(F)]) \) for every closed set \( F \subseteq X \).

**Lemma 2.8** For every simple compact space \( X \), the space \( P(X) \) is first-countable at points of the set \( P_{na}(X) \).

**Proof** By Lemma 4.1 of [13], any nonatomic measure on a simple compact space is uniformly regular. By (the proof of) Proposition 2 in [30] (see also the statement (v) in [13, p.2605]) the space \( P(X) \) is first countable at each uniformly regular measure. Consequently, \( P(X) \) is first-countable at points of the set \( P_{na}(X) \). \( \square \)

A Banach space \( E \) is called Shur if each weakly convergent sequence in \( E \) is norm convergent. A Banach space has the Dunford–Pettis property if for any weakly-null sequences \( \{x_n\}_{n \in \omega} \subseteq E \) and \( \{x'_n\}_{n \in \omega} \subseteq E' \), the sequence \( \{x'_n(x_n)\}_{n \in \omega} \) converges to zero. If \( A \) is a bounded subset of a Banach space \( E \) and \( \chi \in E' \) we set \( \|\chi\|_A := \sup\{|\langle \chi, a \rangle| : a \in A \cup \{0\}\} \). To prove the clause (vi) of the main result we need the following assertion which is of independent interest.

**Proposition 2.9** Let \( E \) be a Banach space with the Gelfand–Phillips property and the Dunford–Pettis property. If \( E \) is not Shur, then \( E \) does not have the Grothendieck property.

**Proof** Suppose for a contradiction that \( E \) has the Grothendieck property. Since \( E \) is not Shur, there exists a weakly null-sequence \( S = \{x_n\}_{n \in \omega} \) in \( E \) such that \( \|x_n\| = 1 \) for all \( n \in \omega \). We claim that \( S \) is not precompact in \( E \). Indeed, otherwise, \( S \) contains a subsequence \( \{x_{n_k}\}_{k \in \omega} \) converging to some element \( x \in E \). As \( x_n \to 0 \) weakly it follows that \( x = 0 \). Therefore \( 1 = \|x_{n_k}\| \to 0 \), a contradiction.
Since $E$ is Gelfand–Phillips, Theorem 2.1 of [3] implies that there is a weak* null-sequence $T = \{\chi_n\}_{n \in \omega}$ in the dual space $E'$ such that

$$\varepsilon := \frac{1}{2} \inf_{n \in \omega} \|\chi_n\| > 0.$$ 

By the Grothendieck property of $E$, the weak* null-sequence $T$ is weakly null.

For every $k \in \omega$, choose numbers $n_k, m_k \in \omega$ such that

(a) $|\chi_{n_k}(x_{m_i})| < \varepsilon$ for all $i < k$;

(b) $|\chi_{n_k}(x_{m_j})| > \varepsilon$.

These two conditions imply that $n_i \neq n_j$ and $m_i \neq m_j$ for all $i \neq j$ and hence the sequences $(\chi_{n_i})_{i \in \omega}$ and $(\chi_{m_i})_{i \in \omega}$ are weakly-null.

Now the Dunford–Pettis property implies that $\chi_{n_i}(x_{m_i}) \to 0$ which contradicts (b). \hfill \Box

**Corollary 2.10** If $K$ is a compact space such that the Banach space $C(K)$ has the Gelfand–Phillips property, then $C(K)$ does not have the Grothendieck property.

**Proof** It is well known that $C(K)$ has the Dunford–Pettis property (see Theorem 13.43 of [15]) and does not have the Schur property (see for example Proposition 2.14 of [21]). Now Proposition 2.9 applies. \hfill \Box

### 3 Measures on compact spaces without non-trivial convergent sequences

We shall use the following probabilistic characterization of compact spaces without non-trivial convergent sequences which can have an independent value.

**Proposition 3.1** For a compact space $K$ the following assertions are equivalent:

(i) $K$ has no non-trivial convergent sequences;

(ii) if a sequence $(\mu_n)_{n \in \omega} \in P(K)^\omega$ converges to a measure $\mu \in P(K)$, then

$$\limsup_{n \to \infty} \sup_{x \in K} (\mu_n(\{x\}) - \mu(\{x\})) \leq 0.$$ 

**Proof** (i)\(\Rightarrow\)(ii) Suppose for a contradiction that there is a sequence $(\mu_n)_{n \in \omega} \in P(K)^\omega$ converging to some measure $\mu \in P(K)$ such that

$$\limsup_{n \to \infty} \sup_{x \in K} (\mu_n(\{x\}) - \mu(\{x\})) > 0.$$ 

Then, passing to a subsequence of $(\mu_n)_{n \in \omega}$ if needed, we can assume that there are $\varepsilon > 0$ and a sequence $\{x_n\}_{n \in \omega}$ in $K$ such that

$$\mu_n(\{x_n\}) - \mu(\{x_n\}) \geq \varepsilon. \tag{1}$$

Since $\mu_n \to \mu$, Theorem 4.3.2 of [4] implies

$$\mu(F) \geq \limsup_{n \to \infty} \mu_n(F) \quad \text{for every closed } F \subseteq K. \tag{2}$$

Therefore, by (1) and (2), the sequence $S = \{x_n\}_{n \in \omega}$ must contain infinitely many distinct points. Hence, passing to a subsequence we can assume additionally that all $x_n$ are distinct. Observe that the set of cluster points of $S$ is infinite. Indeed, otherwise, we can find a closed
(hence compact) neighborhood $W$ of one of the cluster points such that the sequence $S \cap W$ has a unique cluster point and hence $S \cap W$ is a non-trivial convergent sequence that contradicts the assumption of (i). Fix $m \in \omega$ such that $(m - 1)\varepsilon > 1$. Let $\{z_1, \ldots, z_m\}$ be distinct cluster points of $S$. Choose a pairwise disjoint closed subsets $\{F_1, \ldots, F_m\}$ of $K$ such that $F_i$ is a neighborhood of $z_i$ for every $1 \leq i \leq m$. By (2), for every $1 \leq i \leq m$, we can find a point $x_{n_i} \in F_i$ such that $\mu(F_i) \geq \mu_{n_i}(\{x_{n_i}\}) - \frac{\varepsilon}{m}$. Then (1) implies

$$\mu(K) \geq \sum_{i=1}^{m} \mu(F_i) \geq \sum_{i=1}^{m} \left(\mu_{n_i}(\{x_{n_i}\}) - \frac{\varepsilon}{m}\right) \geq m\varepsilon - \varepsilon > 1$$

which is impossible, a contradiction.

(ii)$\Rightarrow$(i) Suppose for a contradiction that $K$ contains a sequence of points $(z_n)_{n \in \omega}$ that converge to some point $z \in K \setminus \{z_n\}_{n \in \omega}$. Then the sequence of Dirac measures $(\delta_{z_n})_{n \in \omega}$ converges to $\delta_z$ in the space $P(K)$, however

$$\lim_{n \to \infty} \sup_{x \in K} \left(\delta_{z_n}(\{x\}) - \delta_z(\{x\})\right) \geq \lim_{n \to \infty} \left(\delta_{z_n}(\{z_n\}) - \delta_z(\{z_n\})\right) = 1 \not= 0.$$ 

We recall that a sequence of measures $(\mu_n)_{n \in \omega}$ on a Tychonoff space $X$ is $c_0$-vanishing if

$$\lim_{n \to \infty} \sup_{x \in X} \mu_n(\{x\}) = 0.$$

The following lemma will be used in the proof of our main theorem.

**Lemma 3.2** Let $X$ be a compact space without non-trivial convergent sequences such that every $c_0$-vanishing sequence $(\lambda_n)_{n \in \omega}$ of probability measures on $X$ has a subsequence that converges to a nonatomic measure on $X$. Then for any sequence $(\mu_n)_{n \in \omega} \in P(X)^\omega$ that converges to a measure $\mu \in P(X)$, we have $\lim_{n \to \infty} \sup_{x \in X} |\mu_n(\{x\}) - \mu(\{x\})| = 0$.

**Proof** To derive a contradiction, assume that $\lim_{n \in \omega} \sup_{x \in X} |\mu_n(\{x\}) - \mu(\{x\})| \not= 0$ for some sequence $(\mu_n)_{n \in \omega} \in P(X)^\omega$ that converges to a measure $\mu \in P(X)$. Since the probability measure $\mu$ has at most countably many atoms, we can replace $(\mu_n)_{n \in \omega}$ by a suitable subsequence, and assume additionally that

- for every $x \in \text{atom}(\mu)$, the limit $\lim_{n \in \omega} \mu_n(\{x\})$ exists, and
- the limit $a = \lim_{n \in \omega} \sup_{x \in X} |\mu_n(\{x\}) - \mu(\{x\})|$ exists and hence is strictly positive.

For every $x \in X$, consider the number

$$\mu_{\infty}(x) := \lim_{n \to \infty} \sup_{x \in X} \mu_n(x)$$

and observe that $\mu_{\infty}(x) = \lim_{n \to \infty} \mu_n(x)$ for every atom $x \in \text{atom}(\mu)$ of the measure $\mu$. $\square$

**Claim 3.3** For every $x \in X$ we have $\mu_{\infty}(x) = \lim_{n \to \infty} \mu_n(\{x\}) \leq \mu(\{x\})$.

**Proof** Suppose for a contradiction that $\mu_{\infty}(x) > \mu(\{x\})$ for some $x \in X$. Choose any real number $a$ with $\mu_{\infty}(x) > a > \mu(\{x\})$. Then $\mu(X \setminus \{x\}) = 1 - \mu(\{x\}) > 1 - a$, and hence the set

$$U = \{\lambda \in P(X) : \lambda(X \setminus \{x\}) > 1 - a\}$$
is an open neighborhood of the measure \( \mu \) in \( P(X) \). Since \( \lim_{n \in \omega} \mu_n = \mu \) and \( \limsup_{n \in \omega} \mu_n(\{x\}) = \mu_\infty(\{x\}) > a \), there exists \( n \in \omega \) such that \( \mu_n(\{x\}) > a \) and \( \mu_n \in U \). The latter inclusion implies that \( \mu_n(X \setminus \{x\}) > 1 - a \) and hence \( \mu_n(\{x\}) = 1 - \mu_n(X \setminus \{x\}) < 1 - (1 - a) = a \), which is a desired contradiction completing the proof of the inequality \( \mu_\infty(\{x\}) \leq \mu(\{x\}) \) for all \( x \in X \).

If \( x \in \text{atom}(\mu) \), then \( \mu_\infty(\{x\}) = \limsup_{n \to \infty} \mu_n(\{x\}) = \lim_{n \to \infty} \mu_n(\{x\}) \), by the choice of the sequence \((\mu_n)_{n \in \omega} \). If \( x \notin \text{atom}(\mu) \), then

\[
0 \leq \lim_{n \to \infty} \sup \mu_n(\{x\}) = \mu_\infty(\{x\}) \leq \mu(\{x\}) = 0
\]

implies that \( \lim_{n \to \infty} \mu_n(x) \) exists and is equal to \( 0 = \mu_\infty(x) \).

\[\square\]

Claim 3.3 ensures that \( \sum_{x \in X} \mu_\infty(\{x\}) \leq \sum_{x \in X} \mu(\{x\}) \leq 1 \) which implies that

\[
\mu_\infty = \sum_{x \in X} \mu_\infty(\{x\}) \delta_x
\]

is a well-defined purely atomic measure on \( X \). For every \( n \in \omega \), consider the measures

\[
\mu^+_n = \max\{\mu_n - \mu_\infty, 0\} \quad \text{and} \quad \mu^-_n = \max\{\mu_\infty - \mu_n, 0\}
\]

and observe that \( \mu_n - \mu_\infty = \mu^+_n - \mu^-_n \).

By Claim 3.3 and Proposition 3.1, we have

\[
0 \leq \lim_{n \in \omega} \sup_{x \in X} \mu^+_n(\{x\}) = \lim_{n \in \omega} \sup_{x \in X} \max\{\mu_n(\{x\}) - \mu_\infty(\{x\}), 0\}
\]

\[
\leq \lim_{n \in \omega} \sup_{x \in X} \max\{\mu_n(\{x\}) - \mu(\{x\}), 0\} = 0,
\]

which means that the sequence of measures \((\mu^+_n)_{n \in \omega}\) in \( c_0\)-vanishing.

Claim 3.4 The sequence \((\mu^-_n)_{n \in \omega}\) is \( c_0\)-vanishing.

**Proof** Since \( \sum_{x \in X} \mu_\infty(\{x\}) \leq 1 \), for every \( \varepsilon > 0 \), there exists a finite set \( F \subseteq X \) such that \( \sup_{x \in X \setminus F} \mu_\infty(\{x\}) < \varepsilon \). Since, for every \( x \in F \), the sequence \((\mu_n(\{x\}))_{n \in \omega}\) converges to \( \mu_\infty(\{x\}) \), there exists \( m \in \omega \) such that

\[
\sup_{n \geq m} \sup_{x \in F} \{\mu_n(\{x\}) - \mu_\infty(\{x\})\} < \varepsilon.
\]

Then for every \( n \geq m \) and every \( x \in F \), we have

\[
\mu^-_n(\{x\}) = \max\{0, \mu_\infty(\{x\}) - \mu_n(\{x\})\} \leq |\mu_n(\{x\}) - \mu_\infty(\{x\})| < \varepsilon.
\]

Also for every \( x \in X \setminus F \) we have

\[
\mu^-_n(\{x\}) = \max\{0, \mu_\infty(\{x\}) - \mu_n(\{x\})\} \leq \mu_\infty(\{x\}) < \varepsilon.
\]

Therefore, \( \sup_{n \geq m} \sup_{x \in X} \mu^-_n(\{x\}) \leq \varepsilon \) and the sequence \((\mu^-_n)_{n \in \omega}\) is \( c_0\)-vanishing.

\[\square\]

Claim 3.5 Each norm-bounded \( c_0\)-vanishing sequence of measures \((\lambda_n)_{n \in \omega}\) on \( X \) contains a subsequence that converges to a nonatomic measure.

**Proof** We lose no generality assuming that \( \lambda_n(X) \neq 0 \) for every \( n \in \omega \). Since \( \sup_{n \in \omega} \lambda_n(X) = \sup_{n \in \omega} \|\lambda_n\| < \infty \), we can find an infinite subset \( I \subseteq \omega \) such that the sequence \((\lambda_n(X))_{n \in I}\) converges to some real number \( a \geq 0 \). If \( a = 0 \), then \((\lambda_n(X))_{n \in I}\) converges to the zero measure, which is nonatomic.
If \( a > 0 \), then \( \left( \frac{\lambda_n}{|\lambda_n|} \right)_{n \in I} \) is a \( c_0 \)-vanishing sequence of purely atomic probability measures. By the assumption of the lemma, there exists an infinite set \( J \subseteq I \) such that the subsequence \( \left( \frac{\lambda_n}{|\lambda_n|} \right)_{n \in J} \) converges to some nonatomic probability measure \( \lambda_\infty \). Since \( \|\lambda_n\| \to a \neq 0 \), it follows that the sequence \( (\lambda_n)_{n \in J} \) converges to the nonatomic measure \( a\lambda_\infty \). \( \Box \)

By Claim 3.5, for the \( c_0 \)-vanishing sequence \( \left( \mu_n^+ \right)_{n \in \omega} \) there exists an infinite set \( I^+ \subseteq \omega \) such that the subsequence \( \left( \mu_n^+ \right)_{n \in I^+} \) converges to some nonatomic probability measure \( \mu^+ \). Since the sequence \( (\mu_n^-)_{n \in I^+} \) is \( c_0 \)-vanishing, there exists an infinite set \( I \subseteq I^+ \) such that the sequence \( (\mu_n^-)_{n \in I} \) converges to some nonatomic measure \( \mu^- \) on \( X \). Then the sequence \( (\mu_n - \mu_\infty)_{n \in I} = (\mu_n^+ - \mu_n^-)_{n \in I} \) converges to the nonatomic sign-measure \( \mu^+ - \mu^- \). On the other hand, this sequence converges to the measure \( \mu - \mu_\infty \). Therefore, the measure \( \mu - \mu_\infty = \mu^+ - \mu^- \) is nonatomic and hence

\[
\mu(\{x\}) = \mu_\infty(\{x\}) \quad \text{for every } x \in X.
\]

Now the \( c_0 \)-vanishing property of the sequences \( \left( \mu_n^+ \right)_{n \in \omega} \) and \( \left( \mu_n^- \right)_{n \in \omega} \) guarantees that

\[
\lim_{n \to \omega} \sup_{x \in X} |\mu_n(\{x\}) - \mu(\{x\})| = \lim_{n \to \omega} \sup_{x \in X} \left| \mu_n(\{x\}) - \mu_\infty(\{x\}) \right|
\]

\[
= \lim_{n \to \omega} \sup_{x \in X} \max\{\mu_n^+(\{x\}), \mu_n^-(\{x\})\} = 0.
\]

\( \Box \)

**Lemma 3.6** Let \( X \) be a compact space without non-trivial convergent sequences such that every \( c_0 \)-vanishing sequence \( (\lambda_n)_{n \in \omega} \) of probability measures on \( X \) has a subsequence that converges to a nonatomic measure on \( X \). Then any sequence \( (\mu_n)_{n \in \omega} \in P(X)^\omega \) that converges to a purely atomic measure \( \mu \in P_a(X) \) converges to \( \mu \) in norm.

**Proof** To show that \( \lim_{n \to \omega} \|\mu_n - \mu\| = 0 \), fix any \( \varepsilon > 0 \). Since

\[
\|\mu\| = \sum_{x \in \text{atom}(\mu)} \mu(\{x\}) = 1,
\]

there exists a finite set \( F \subseteq \text{atom}(\mu) \) such that \( \sum_{x \in F} \mu(\{x\}) > 1 - \frac{1}{4}\varepsilon \). Let \( \mu = \lambda + \nu \) where \( \lambda = \sum_{x \in F} \mu(\{x\})\delta_x \) and \( \nu = \mu - \lambda \). It follows that \( \|\lambda\| = \|\mu(F)\| = \sum_{x \in F} \mu(\{x\}) > 1 - \frac{1}{4}\varepsilon \) and hence \( \|\nu\| = \|\mu(X)\setminus F\| < \frac{1}{4}\varepsilon \).

By Lemma 3.2, \( \lim_{n \to \omega} \sup_{x \in X} |\mu_n(\{x\}) - \mu(\{x\})| = 0 \), and hence there exists \( m \in \omega \) such that

\[
\sup_{n \geq m} \sup_{x \in F} |\mu_n(\{x\}) - \mu(\{x\})| < \frac{\varepsilon}{4|F|} \quad \text{for every } n \geq m.
\]

For every \( n \geq m \), write the measure \( \mu_n \) as \( \mu_n = \lambda_n + \nu_n \) where \( \lambda_n = \sum_{x \in F} \mu_n(\{x\})\delta_x \) and \( \nu_n = \mu_n - \lambda_n \). Observe that

\[
\|\lambda_n\| = \|\mu_n(F)\| = \sum_{x \in F} \mu_n(\{x\}) > \sum_{x \in F} (\mu(\{x\}) - \frac{\varepsilon}{4|F|}) = \mu(F) - \frac{1}{4}\varepsilon > 1 - \frac{1}{4}\varepsilon - \frac{1}{4}\varepsilon = 1 - \frac{1}{2}\varepsilon
\]

and hence \( \|\nu_n\| = \nu_n(\mu(X)\setminus F) < \frac{1}{2}\varepsilon \). Then for every \( n \geq m \), we have

\[
\|\mu_n - \mu\| = \|\lambda_n + \nu_n - \lambda - \nu\| \leq \|\nu_n\| + \|\nu\| + \|\lambda_n - \lambda\|
\]

\[
= \|\nu_n\| + \|\nu\| + \sum_{x \in F} |\mu_n(\{x\}) - \mu(\{x\})| < \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon.
\]

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which means that \( \lim_{n \to \omega} \| \mu_n - \mu \| = 0. \)

In the proof of Theorem 1.1 we shall need some results on the preservation of the \( c_0 \)-vanishing property by projections \( \pi^\omega_\alpha : 2^\omega \to 2^\alpha, \pi^\omega_\alpha : x \mapsto x|_\alpha. \) For every \( \alpha \in \omega_1, \) the projection \( \pi^\omega_\alpha \) induces a continuous map \( P\pi^\omega_\alpha : P(2^\omega) \to P(2^\alpha) \) between the corresponding spaces of probability measures.

**Lemma 3.7** For any nonatomic measure \( \mu \in P_{\omega_0}(2^\omega) \) there exists an ordinal \( \alpha \in \omega_1 \) such that for any ordinal \( \beta \in [\alpha, \omega_1) \) the measure \( P\pi^\omega_\beta(\mu) \in P(2^\beta) \) is nonatomic.

**Proof** Since \( \mu \) is nonatomic and regular, for every \( n \in \omega \) and \( x \in 2^\omega \), there exists a neighborhood \( O_{x,n} \subseteq 2^\omega \) of \( x \) such that \( \mu(O_{x,n}) < \frac{1}{2^n} \). We lose no generality assuming that \( O_{x,n} \) is of basic form \( \{ y \in 2^\omega : y|_{F_{x,n}} = x|_{F_{x,n}} \} \) for some finite set \( F_{x,n} \subseteq \omega_1 \). By the compactness of \( 2^\omega \) for every \( n \in \omega \), there exists a finite set \( E_n \subseteq 2^\omega \) such that \( 2^\omega = \bigcup_{x \in E_n} O_{x,n} \). We claim that any countable ordinal \( \alpha \) containing the countable set \( \bigcup_{n \in \omega} \bigcup_{x \in E_n} F_{x,n} \) has the desired property: for any ordinal \( \beta \in [\alpha, \omega_1) \) the measure \( \lambda = P\pi^\omega_\beta(\mu) \) is nonatomic. To derive a contradiction, assume that \( \lambda(\{a\}) > 0 \) for some \( a \in 2^\beta \).

Find \( n \in \omega \) such that \( \frac{1}{2^n} < \lambda(\{a\}) \). Choose any point \( b \in 2^\omega \) such that \( \pi^\omega_\beta(b) = a \) and find \( x \in E_n \) such that \( b \in O_{x,n} \). Then

\[
(\pi^\omega_\beta)^{-1}(\{a\}) = \{ z \in 2^\omega : z|_\beta = b|_\beta \} \subseteq \{ z \in 2^\omega : z|_{F_{x,n}} = b|_{F_{x,n}} \} = O_{x,n}
\]

and hence

\[
\lambda(\{a\}) = \mu((\pi^\omega_\beta)^{-1}(\{a\})) \subseteq \mu(O_{x,n}) < \frac{1}{2^n},
\]

which contradicts the choice of \( n \).

**Lemma 3.8** For every \( c_0 \)-vanishing sequence of measures \( (\mu_n)_{n \in \omega} \in \left( P(2^\omega) \right)^{\omega} \) there exists a countable ordinal \( \alpha \) such that for any ordinal \( \beta \in [\alpha, \omega_1) \) the sequence of measures \( (P\pi^\omega_\beta(\mu_n))_{n \in \omega} \) is \( c_0 \)-vanishing.

**Proof** Write each measure \( \mu_n \) as \( \mu_n = (1 - t_n)\lambda_n + t_n v_n \) for some \( t_n \in [0,1] \) and measures \( \lambda_n \in P_\alpha(2^\omega_\alpha) \) and \( v_n \in P_{\omega_0}(2^\omega_\alpha) \). By Lemma 3.7, there exists a countable ordinal \( \alpha \in \omega_1 \) such that for every \( \beta \in [\alpha, \omega_1) \) and every \( n \in \omega \) the measure \( P\pi^\omega_\beta(\mu_n) \) is nonatomic. Replacing \( \alpha \) by a larger countable ordinal, if necessary, we can additionally assume that the projection \( \pi^\omega_\alpha \) is injective on the countable set \( A = \bigcup_{n \in \omega} \text{atom}(\lambda_n) \).

We claim that the ordinal \( \alpha \) has the required property. Indeed, fix any ordinal \( \beta \in [\alpha, \omega_1) \) and for every \( n \in \omega \) denote the measure \( P\pi^\omega_\beta(\mu_n) \) by \( \mu_n^\beta \). It is clear that \( \mu_n^\beta = (1 - t_n)\lambda_n^\beta + t_n v_n^\beta \) where \( \lambda_n^\beta = P\pi^\omega_\alpha(\lambda_n) \) and \( v_n^\beta = P\pi^\omega_\beta(v_n) \). The choice of the ordinal \( \alpha \) ensures that the measure \( v_n^\beta \) is nonatomic. The injectivity of the projection \( \pi^\omega_\alpha \) on \( A \) implies the injectivity of the projection \( \pi^\omega_\beta \) on \( A \). Then we can choose a function \( f : 2^\beta \to 2^\omega_\alpha \) such that \( \pi^\omega_\beta \circ f(y) = y \) for every \( y \in 2^\beta \) and \( f \circ \pi^\omega_\alpha(x) = x \) for every \( x \in A \).

Observe that for every \( y \in \pi^\omega_\beta[A] \), we have

\[
\mu_n^\beta([y]) = (1 - t_n)\lambda_n^\beta([y]) + t_n v_n^\beta([y]) = (1 - t_n)\lambda_n([f(y)]) + 0 = \mu_n([f(y)])
\]

and for every \( y \in 2^\beta \setminus \pi^\omega_\beta[A] \), we have

\[
\mu_n^\beta([y]) = (1 - t_n)\lambda_n^\beta([y]) + t_n v_n^\beta([y]) = 0 + 0.
\]
This implies
\[ \sup_{y \in 2^\omega} \mu_n^\beta(y) = \sup_{x \in 2^{\omega_1}} \mu_n(x) \]
and hence
\[ \lim_{n \to \infty} \sup_{y \in 2^\omega} \mu_n^\beta(y) = \lim_{n \to \infty} \sup_{x \in 2^{\omega_1}} \mu_n(x) = 0, \]
which means that the sequence \((P \pi_\beta^\omega(\mu_n))_{n \in \omega} = (\mu_n^\beta)_{n \in \omega}\) is c_0-vanishing. \qed

4 A key lemma

In the following lemma we use some notations related to the binary tree \(2^{<\omega} := \bigcup_{n \in \omega} 2^n\). Here \(2\) stands for the ordinal \(2 = \{0, 1\}\). The unique element \(\emptyset\) of \(2\) is the root of the tree \(2^{<\omega}\). For an element \(s \in 2^{<\omega}\), we denote by \(|s|\) the unique number \(n \in \omega\) such that \(s \in 2^n\). The number \(|s|\) is called the length of \(s\). For two elements \(s, t \in 2^{<\omega}\), we write \(s \preceq t\) if \(|s| \leq |t|\) and \(s = t\,|s|\). Also we write \(s < t\) if \(s \preceq t\) and \(s \neq t\). Let \(\uparrow s := \{t \in 2^{<\omega} : s \preceq t\}\) be the upper set of \(s\) in the tree \(2^{<\omega}\) (so \(\uparrow s\) consists of all possible extensions of the function \(s\)). For \(i \in \omega\), let \(s^i\) be the unique element of \(\uparrow s\) such that \(|s^i| = |s| + 1\) and \(s^i(|s|) = i\). So, \(s^0\) and \(s^1\) are immediate successors of \(s\) in the tree \(2^{<\omega}\).

Lemma 4.1 Let \(X\) be a zero-dimensional compact metrizable space, \((\mu_n)_{n \in \omega} \in P(X)^\omega\) be a c_0-vanishing sequence of probability measures that converge to a measure \(\mu \in P(X)\). For every \(z \in \text{atom}(\mu)\) there exist a subsequence \((\mu_{n_k})_{k \in \omega}\) of the sequence \((\mu_n)_{n \in \omega}\) and a family \((X_s)_{s \in 2^{<\omega}}\) of clopen subsets of \(X \setminus \{z\}\) such \(X_\emptyset = X \setminus \{z\}\) and for every \(s \in 2^{<\omega}\) the following conditions are satisfied:

(i) \(X_s^0 \cup X_s^1 = X_s\) and \(X_s^0 \cap X_s^1 = \emptyset\);

(ii) for any clopen neighborhood \(U \subseteq X\) of \(z\) and any \(\varepsilon > 0\) there exists \(q \in \omega\) such that
\[ \mu_{n_k}(U \cap X_s) \leq \mu(U \setminus \{z\}) + \frac{1}{2^{|s|}} + \varepsilon \quad \text{for any } k \geq q. \]

Proof Fix a neighborhood base \((U_n)_{n \in \omega}\) at the point \(z\) consisting of clopen subsets of \(X\) such that \(U_{n+1} \subseteq U_n\) for every \(n \in \omega\).

We shall inductively construct increasing number sequences \((n_k)_{k \in \omega}, (m_k)_{k \in \omega}\) and a sequence \((S_k)_{k \in \omega}\) of compact subsets of \(X \setminus \{z\}\) such that for every \(k \in \mathbb{N}\) the following inductive conditions are satisfied:

(a_k) \(S_{k-1} \cap U_{m_k} = \emptyset\);

(b_k) \(\sup_{n \geq n_k} \max\{|\mu_n(U_{m_k}) - \mu(U_{m_k})|, \sup_{x \in X} \mu_n((x))\} < \frac{1}{6^k} \mu(|\{z\}|) \leq \frac{1}{6^k} \mu(|\{z\}|)\);

(c_k) \(S_k \subseteq U_{m_k}\) and \(\mu_{n_k}(S_k) > \mu_{n_k}(U_{m_k} \setminus \{z\}) - \frac{1}{6^k} \mu(|\{z\}|)\).

We start the inductive construction letting \(m_0 = 0\) and \(S_0 = \emptyset\) and choosing \(n_0 \in \omega\) such that the condition \((b_0)\) is satisfied. Assume that for some \(k \in \mathbb{N}\), we have constructed a compact subset \(S_{k-1}\) of \(X \setminus \{z\}\) and numbers \(n_{k-1}, m_{k-1}\). As \(S_{k-1}\) is a compact subset of \(X \setminus \{z\}\), there is a number \(m_k > m_{k-1}\) such that \(U_{m_k} \cap S_{k-1} = \emptyset\). Since \(\lim_{n \in \omega} \mu_n = \mu\) and \(\lim_{n \in \omega} \sup_{x \in X} \mu_n((x)) = 0\), there exists a number \(n_k > n_{k-1}\) satisfying the inductive condition \((b_k)\). Using the regularity of the measure \(\mu_{n_k}\), choose a compact set \(S_k \subseteq U_{m_k} \setminus \{z\}\) satisfying the condition \((c_k)\).

This completes the inductive step.
Observe that for every $k \in \mathbb{N}$, the inductive conditions $(b_k)$ and $(c_k)$ imply
\[
\mu_{n_k}(S_k) > \mu_{n_k}(U_{n_k \setminus \{z\}}) - \frac{1}{6^k} \mu(\{z\}) = \mu_{n_k}(U_{n_k}) - \mu_{n_k}(\{z\}) - \frac{1}{6^k} \mu(\{z\})
\geq (\mu(U_{n_k}) - \frac{1}{6^k} \mu(\{z\}) - \frac{1}{6^k} \mu(\{z\}) - \frac{1}{6^k} \mu(\{z\})) = (1 - \frac{3}{6^k}) \mu(\{z\}).
\] (3)

**Claim 4.2** For every $k \in \mathbb{N}$, the compact set $S_k$ can be represented as the union $\bigcup_{t \in 2^k} S_{k,t}$ of $2^k$ pairwise disjoint compact sets such that
\[
\max_{t \in 2^k} \left| \mu_{n_k}(S_{n,k,t}) - \frac{1}{2^k} \mu_{n_k}(S_k) \right| < \frac{4}{6^k}.
\] (4)

**Proof** Identify $2^k$ with the finite ordinal $|2^k| = \{0, \ldots, |2^k| - 1\}$ via the bijective map $2^k \to \{0, \ldots, |2^k| - 1\}, t \mapsto \sum_{i=0}^{k-1} t(i) 2^i$. Since the compact space $S_k$ is zero-dimensional, it is homeomorphic to a subspace of the real line. This allows us to identify $S_k$ with a compact subset of the real line. For a real number $r$, let $\downarrow r = S_k \cap (-\infty, r] \subseteq S_k \subseteq \mathbb{R}$. For every $t \in 2^k$, let
\[
s_t = \inf \left\{ s \in S_k : \mu_{n_k}(\downarrow s) \geq \frac{t+1}{2^k} \mu_{n_k}(S_k) \right\}.
\]

The regularity of the measure $\mu_{n_k}$ ensures that
\[
\frac{t+1}{2^k} \mu_{n_k}(S_k) \leq \mu_{n_k}(\downarrow s_t) \leq \frac{t+1}{2^k} \mu_{n_k}(S_k) + \mu_{n_k}((s_t]) < \frac{t+1}{2^k} \mu_{n_k}(S_k) + \frac{1}{6^k} \mu(\{z\})
\leq \frac{t+1}{2^k} \mu_{n_k}(S_k) + \frac{\mu_{n_k}(S_k)}{6^k (1 - 3/6^k)} < \frac{t+2}{2^k} \mu_{n_k}(S_k),
\]
which implies $s_t < s_{t+1}$ if $t < 2^k$. Let $r_{-1}, r_{2^k-1} \in \mathbb{R} \setminus S_k$ be any real numbers such that $\downarrow r_{-1} = \emptyset$ and $\downarrow r_{2^k-1} = S_k$. Since the space $S_k \subseteq \mathbb{R}$ is zero-dimensional, for every $t \in 2^k - 1$ we can choose a real number $r_t \in \mathbb{R} \setminus S_k$ such that $s_t < r_t < s_{t+1}$ and $\mu_{n_k}(\downarrow r_t) < \mu_{n_k}(\downarrow s_t) + \frac{1}{6^k}$. Then
\[
|\mu_{n_k}(\downarrow r_t) - \frac{t+1}{2^k} \mu_{n_k}(S_k)| \leq |\mu_{n_k}(\downarrow r_t) - \mu_{n_k}(\downarrow s_t)| + |\mu_{n_k}(\downarrow s_t) - \frac{t+1}{2^k} \mu_{n_k}(S_k)|
\leq \frac{1}{6^k} + \frac{1}{6^k} = \frac{2}{6^k}.
\]

It follows from $r_t \in \mathbb{R} \setminus S_k$ that for every $t \in 2^k$ the subspace $S_{k,t} = S_k \cap (\downarrow r_t) \setminus (\downarrow r_{t-1})$ is compact and $S_k = \bigcup_{t \in 2^k} S_{k,t}$. It is clear that the family $(S_{k,t})_{t \in 2^k}$ consists of pairwise disjoint sets.

Finally, observe that
\[
|\mu_{n_k}(S_{k,t}) - \frac{t+1}{2^k} \mu_{n_k}(S_k)| = |\mu_{n_k}(\downarrow r_t) - \mu_{n_k}(\downarrow r_{t-1}) - \frac{t+1}{2^k} \mu_{n_k}(S_k) + \frac{t}{2^k} \mu_{n_k}(S_k)|
\leq |\mu_{n_k}(\downarrow r_t) - \frac{t+1}{2^k} \mu_{n_k}(S_k)| + |\mu_{n_k}(\downarrow r_{t-1}) - \frac{t}{2^k} \mu_{n_k}(S_k)| < \frac{2}{6^k} + \frac{2}{6^k} = \frac{4}{6^k}.
\]

Now we continue the proof of the lemma. For every $s \in 2^{<\omega}$, consider the set
\[
T_s = \bigcup_{k \geq \|s\|} \bigcup \{S_{k,t} : t \in 2^k, t|_{\|s\|} = s\},
\]
where $S_{0,0} = S_0 = \emptyset$. Taking into account that the subsets $S_{k,t}$ of $U_{n_{k-1}}$ are compact and the sequence $(U_k)_{k \in \omega}$ is a neighborhood base at $z$, we conclude that the set $T_s$ is closed in $X \setminus \{z\}$. The definition of the sets $T_s$ implies $T_{s^0_0} \cup T_{s^1_0} \subseteq T_s$. To show that $T_{s^0_0} \cap T_{s^1_0} = \emptyset$ for any $s \in 2^{<\omega}$, it suffices to note that the sequence $(S_k)_{k \in \omega}$ is disjoint by the inductive conditions $(a_k)$ and $(c_k)$ and, for every $k \in \omega$, the family $(S_{k,t})_{t \in 2^k}$ is disjoint by the construction.
Now, by induction on the binary tree $2^{<\omega}$, we shall find a family $(X_s)_{s \in 2^{<\omega}}$ of clopen subsets of $X \setminus \{z\}$ such that $X_\emptyset = X \setminus \{z\}$ (the base of induction) and

$$T_s \subseteq X_s, \quad X_{s^0} \cup X_{s^1} = X_s, \quad X_{s^0} \cap X_{s^1} = \emptyset \quad \text{for every } s \in 2^{<\omega}.$$  

Assume that for some $s \in 2^{<\omega}$ the set $X_s$ has been constructed. Since $X$ is a zero-dimensional compact metrizable space, the space $X_s$ is zero-dimensional and being Lindelöf it is strongly zero-dimensional in the sense that any disjoint closed sets in $X_s$ can be separated by clopen neighborhoods, see [14, Theorem 6.2.7]. Since $X_s$ is strongly zero-dimensional in the sense that any disjoint closed sets in $X_s$ can be separated by clopen neighborhoods, see [14, Theorem 6.2.7]. Since $T_s \subseteq X_s$, there are clopen sets $X_{s^0}$ and $X_{s^1}$ such that $T_s \subseteq X_{s^0}, T_s \subseteq X_{s^1}$, $X_{s^0} \cup X_{s^1} = X_s$, and $X_{s^0} \cap X_{s^1} = \emptyset$. This completes the inductive step.

By the inductive construction, the family $(X_s)_{s \in 2^{<\omega}}$ satisfies the condition (i) of the lemma. To verify the condition (ii), fix $s \in 2^{<\omega}$, $\varepsilon > 0$ and a clopen neighborhood $U$ of $z$ in $X$. Find $q \geq |s|$ such that

$$U_{m_k-1} \subseteq U, \quad \frac{3}{6^k} + \frac{4}{3^k} < \varepsilon \quad \text{and} \quad |\mu_{m_k}(U) - \mu(U)| < \frac{\varepsilon}{2} \quad \text{for every } k \geq q. \quad (5)$$

We claim that the number $q$ has the property required in (ii). Indeed, if $t \in \uparrow s$, then $S_{k,t} \subseteq T_s \subseteq X_s$, and if $t \notin \uparrow s$, then the disjointness of $T_{|s}$ and $T_s$ and the construction of $X_s$ imply $S_{k,t} \cap X_s = \emptyset$ and therefore

$$S_k \cap X_s = \bigcup_{t \in 2^k} (S_{k,t} \cap X_s) = \bigcup_{t \in 2^k \cap \uparrow s} (S_{k,t} \cap X_s) = \bigcup_{t \in 2^k \cap \uparrow s} S_{k,t}. \quad (6)$$

Note also that by (c_k) and (5), $S_k \subseteq U_{m_k} \subseteq U$. Now, for any $k \geq q$, we have

$$\mu_{m_k}(U \cap X_s) \leq \mu_{m_k}(U \setminus S_k) + \mu_{m_k}(S_k \cap X_s) \quad (6)$$

$$= \mu_{m_k}(U \setminus S_k) + \sum_{t \in 2^k \cap \uparrow s} \mu_{m_k}(S_{k,t})$$

$$= \mu_{m_k}(U \setminus S_k) + \mu_{m_k}(S_k) - \sum_{t \in 2^k \setminus \uparrow s} \mu_{m_k}(S_{k,t})$$

$$\leq \mu_{m_k}(U) - (2^k - 2^{|s|}) \left( \frac{1}{2^k} \mu_{m_k}(S_k) - \frac{4}{3^k} \right) \quad (4)$$

$$\leq \mu_{m_k}(U) - \left( 1 - \frac{1}{2^{|s|}} \right) \left( \mu(U_{m_k}) - \frac{3}{6^k} - \frac{4}{3^k} \right) \quad (3)$$

$$\leq (\mu(U) + \frac{\varepsilon}{2}) - \mu(U_{m_k}) + \frac{1}{2^{|s|}} + \frac{3}{6^k} + \frac{4}{3^k} \quad (5)$$

$$\leq \mu(U \setminus U_{m_k}) + \varepsilon + \frac{1}{2^{|s|}} \leq \mu(U \setminus \{z\}) + \frac{1}{2^{|s|}} + \varepsilon,$$

as desired. \hfill \Box

5 An implication of ♦

In this section we apply Jensen’s diamond principle ♦ to producing a special enumeration of the set $(P (2^{\omega_1}))^{\omega}$ by countable ordinals.

Jensen’s diamond principle is the following statement (see [25]):

(♦) there exists a transfinite sequence $(x_\alpha)_{\alpha \in \omega_1} \in \prod_{\alpha \in \omega_1} 2^\alpha$ such that for every $x \in 2^{\omega_1}$, the set $\{ \alpha \in \omega_1 : x|_\alpha = x_\alpha \}$ is stationary in $\omega_1$.  

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A set $S \subseteq \omega_1$ is stationary if $S$ has nonempty intersection with any closed unbounded subset of $\omega_1$. By [25], $\diamondsuit$ implies CH and follows from the Gödel’s Constructibility Axiom $V = L$ (which is consistent with ZFC).

We shall apply $\diamondsuit$ to produce a nice enumerations of elements of the limit spaces of subcontinuous $\omega_1$-spectra.

Let $\lambda$ be any ordinal. A $\lambda$-spectrum if an indexed family $\Sigma = (X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \lambda)$ consisting of sets $X_\alpha$ and functions $\pi_\alpha^\beta : X_\beta \to X_\alpha$ such that $\pi_\alpha^\gamma = \pi_\beta^\gamma \circ \pi_\alpha^\beta$ for any ordinals $\alpha \leq \beta < \gamma$ in $\lambda$. The spectrum $\Sigma$ is defined to be subcontinuous if for every limit ordinal $\beta < \lambda$ and any distinct elements $x, x' \in X_\beta$ there exists an ordinal $\alpha < \beta$ such that $\pi_\alpha^\beta(x) \neq \pi_\alpha^\beta(x')$.

**Proposition 5.1** Let $\Sigma = (X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta \leq \omega_1)$ be a subcontinuous $(\omega_1 + 1)$-spectrum such that for every $\alpha < \omega_1$ the set $X_\alpha$ has cardinality $|X_\alpha| \leq \aleph$. Under $\diamondsuit$ there exists a transfinite sequence $(x_\alpha)_{\alpha \in \omega_1}$ such that for every $\alpha \in X_\omega$, the set $\{ \alpha \in X_\omega : x_\alpha = \pi_\alpha^\omega(x) \}$ is stationary in $\omega_1$.

**Proof** Assuming $\diamondsuit$, we obtain a transfinite sequence $(Y_\alpha)_{\alpha \in \omega_1} \subseteq \prod_{\alpha \in \omega_1} 2^\alpha$ such that for every $\alpha < \omega_1$, the set $\{ \alpha \in X_\omega : y_\alpha \in Y_\alpha \}$ is stationary in $\omega_1$.

Identifying $\omega_1$ with the ordinal $\omega \cdot \omega_1$, we conclude that $\diamondsuit$ implies the existence of a transfinite sequence $(z_\alpha)_{\alpha \in \omega_1} \subseteq \prod_{\alpha \in \omega_1} (2^{\omega_1})^\alpha$ such that for any $\alpha \in \omega_1$ the set $\{ \alpha \in \omega_1 : z_\alpha = z|_\alpha \}$ is stationary in $\omega_1$.

For every $\alpha \in \omega_1$, the set $X_\alpha$ has cardinality $|X_\alpha| \leq |2^{\omega_1}|$ and hence admits an injective function $f_\alpha : X_\alpha \to 2^{\omega_1}$. For every $\beta \in \omega_1$, consider the function $g_\beta : X_\beta \to (2^{\omega_1})^\beta$, $y_\alpha \mapsto (f_\alpha \circ \pi_\alpha^\beta)_{\alpha \in \beta}$.

**Claim 5.2** For every limit ordinal $\beta \in \omega_1$ the function $g_\beta : X_\beta \to (2^{\omega_1})^\beta$ is injective.

**Proof** Given any distinct elements $x, x' \in X_\beta$, use the subcontinuity of the spectrum $\Sigma$ and find an ordinal $\alpha < \beta$ such that $\pi_\alpha^\beta(x) \neq \pi_\alpha^\beta(x')$. The injectivity of the function $f_\alpha : X_\alpha \to 2^{\omega_1}$ implies that $f_\alpha(\pi_\alpha^\omega(x)) \neq f_\alpha(\pi_\alpha^\omega(x'))$. Let $\pi^\beta_{\alpha} : (2^{\omega_1})^\beta \to 2^{\omega_1}$, $x \mapsto (x|_\alpha)$, be the projection of $(2^{\omega_1})^\beta$ onto the $\alpha$-th coordinate. It follows from $\pi^\beta_{\alpha} \circ g_\beta(x) = f_\alpha \circ \pi_\alpha^\beta(x) \neq f_\alpha \circ \pi_\alpha^\beta(x') = \pi^\beta_{\alpha} \circ g_\beta(x')$ that $g_\beta(x) \neq g_\beta(x')$, which means that the function $g_\beta : X_\beta \to (2^{\omega_1})^\beta$ is injective.

Let $(x_\alpha)_{\alpha \in \omega_1} \subseteq \prod_{\alpha \in \omega_1} X_\alpha$ be any sequence such that for every $\alpha \in \omega_1$, we have $g_\alpha(x_\alpha) = z_\alpha$. We claim that the transfinite sequence $(x_\alpha)_{\alpha \in \omega_1}$ has the required property.

Fix any $x \in X_{\omega_1}$ and consider the element $z = g_{\omega_1}(x) \in (2^{\omega_1})^{\omega_1}$. The choice of the transfinite sequence $(z_\alpha)_{\alpha \in \omega_1}$ ensures that the set $S = \{ \alpha \in \omega_1 : z|_\alpha = z_\alpha \}$ is stationary in $\omega_1$.

To see that the set $\{ \alpha \in \omega_1 : \pi_\alpha^{\omega_1}(x) = x_\alpha \}$ is stationary, it suffices to show that it contains the stationary set $S \cap \omega_1$, where $\omega_1$ is the closed unbounded set of all limit countable ordinals.

Given any limit ordinal $\beta \in S$, observe that $z_\beta = z|_\beta = g_{\omega_1}(x)|_\beta = (f_\alpha \circ \pi_\alpha^{\omega_1}(x))_{\alpha \in \beta} = g_\beta(\pi_\beta^{\omega_1}(x)) \in g_\beta[x_\beta]$ and hence $z_\beta = g_\beta(x_\beta)$. On the other hand,

$z_\beta = z|_\beta = g_{\omega_1}(x)|_\beta = g_\beta(\pi_\beta^{\omega_1}(x))$. 
Since \( g_\beta(x_\beta) = z_\beta = g_\beta(\pi_\beta^\omega(x)) \), the injectivity of \( g_\beta \) implies \( x_\beta = \pi_\beta^\omega(x) \). Therefore, \( S \cap \omega_\beta^\omega \subseteq \{ \alpha \in \omega_1 : \pi_\alpha^\omega(x) = x_\alpha \} \) and the set \( \{ \alpha \in \omega_1 : \pi_\alpha^\omega(x) = x_\alpha \} \) is stationary. \( \square \)

Now we apply Proposition 5.1 to subcontinuous \( \lambda \)-spectra in the category \( \text{Comp} \) of compact Hausdorff spaces and their continuous measures. A \( \lambda \)-spectrum in the category \( \text{Comp} \) is a \( \lambda \)-spectrum \( (X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \lambda) \) consisting of compact Hausdorff spaces \( X_\alpha \) and continuous bounding maps \( \pi_\alpha^\beta \). An example of such spectrum is the \((\omega + 1)\)-spectrum \((2^\omega, \pi_\alpha^\beta : \alpha \leq \beta \leq \omega_1)\), where the cubes \( 2^\omega \) are endowed with the Tychonoff product topology and the projection maps \( \pi_\alpha^\beta : 2^\beta \rightarrow 2^\alpha, \pi_\alpha^\beta : x \mapsto x|_\alpha \) are continuous.

A functor \( F : \text{Comp} \rightarrow \text{Comp} \) is defined to be subcontinuous if for any ordinal \( \lambda \) and any subcontinuous \( \lambda \)-spectrum \( (X_\alpha, \pi_\alpha^\beta : \alpha \leq \beta < \lambda) \) in the category \( \text{Comp} \), the spectrum \((FX_\alpha, F\pi_\alpha^\beta : \alpha \leq \beta < \lambda)\) is subcontinuous. By [34, 2.3.3], the functor of probability measures \( \pi : \text{Comp} \rightarrow \text{Comp} \) is subcontinuous and so is the functor \((\cdot)^\omega : \text{Comp} \rightarrow \text{Comp} \) of taking the countable power. Then the composition \( P^{\omega} : \text{Comp} \rightarrow \text{Comp} \) of the functors \( P \) and \((\cdot)^\omega \) is a subcontinuous functor, too. This functor assigns to each compact Hausdorff space \( X \) the countable power \( P^{\omega}(X) \) of the space \( P(X) \). To any continuous function \( f : X \rightarrow Y \) between compact Hausdorff space the function \( P^{\omega}f : P^{\omega}(X) \rightarrow P^{\omega}(Y) \), \( P^{\omega} : (\mu_\alpha)_{\alpha \in \omega_1} \mapsto (f(\mu_\alpha))_{\alpha \in \omega_1} \).

Corollary 5.3 Under \( \diamond \) there exists a transfinite sequence \( (\mu_\alpha)_{\alpha \in \omega_1} \in \prod_{\alpha \in \omega_1} P^{\omega}(2^\omega) \) such that for every \( \mu \in P^{\omega}(2^\omega) \), the set \( \{ \alpha \in \omega_1 : \mu_\alpha = P^{\omega}\pi_\alpha^{\omega_1}(\mu) \} \) is stationary in \( \omega_1 \).

Proof By the subcontinuity of the functor \( P^{\omega} \), the \((\omega_1 + 1)\)-spectrum \((P^{\omega}(2^\omega), P^{\omega}\pi_\alpha^{\beta}) : \alpha \leq \beta \leq \omega_1)\) is subcontinuous. For every countable ordinal \( \alpha \), the compact space \( P^{\omega}(2^\omega) \) is metrizable and hence has cardinality \( \leq \mathfrak{c} \). Applying Proposition 5.1, we obtain a transfinite sequence \( (\mu_\alpha)_{\alpha \in \omega_1} \in \prod_{\alpha \in \omega_1} P^{\omega}(2^\omega) \) with the desired property. \( \square \)

6 Proof of Theorem 1.1

Consider the \((\omega_1 + 1)\)-spectrum \((2^\omega, \pi_\alpha^\beta : \alpha \leq \beta \leq \omega_1)\) consisting of the Cantor cubes \( 2^\alpha \) and projections \( \pi_\alpha^\beta : 2^\alpha \rightarrow 2^\alpha, \pi_\alpha^{\alpha_\beta} : x \mapsto x|_\alpha \).

Assume \( \diamond \). By Corollary 5.3, there exists a transfinite sequence

\[
(\mu^\alpha_n)_{n \in \omega} \in \prod_{\alpha \in \omega_1} (P(2^\omega))^\omega
\]

such that for any sequence of measures \( (\mu_n)_{n \in \omega} \in (P(2^{\omega_1}))^\omega \) the set

\[
\{ \alpha \in \omega_1 : \mu^\alpha_n = P^{\omega}\pi_\alpha^{\omega_1}(\mu_n) \quad \text{for every} \quad n \in \omega \}
\]

is stationary in \( \omega_1 \).

Let \( \Lambda \) be the set of all nonzero limit ordinals \( \alpha \in \omega_1 \) such that \( (\mu^\alpha_n)_{n \in \omega} \) is a sequence of pairwise distinct Dirac measures that converges to the Dirac measure \( \mu^\alpha_0 \in X_\alpha \subseteq P(X_\alpha) \).

Let \( \Omega \) be the set of all nonzero limit ordinals \( \alpha \in \omega_1 \) such that \( \lim_{n \in \omega} \sup_{x \in 2^\omega} \mu^\alpha_n([x]) = 0 \).

Observe that the sets \( \Lambda \) and \( \Omega \) are disjoint.

Now we shall inductively construct families of sets \( (I_\alpha)_{\alpha \in \omega_1} \), \( (X_\alpha)_{\alpha \in \omega_1} \), \( (A_\alpha)_{\alpha \in \omega_1} \), \( (B_\alpha)_{\alpha \in \omega_1} \) such that for every \( \alpha \in \omega_1 \), the following conditions are satisfied:

1. if \( \alpha \leq \omega \), then \( X_\alpha = A_\alpha = 2^\omega, B_\alpha = \emptyset \) and \( I_\alpha = \omega \);
2. if the ordinal \( \alpha \) is limit, then \( X_\alpha = \bigcap_{\gamma < \alpha} (\pi_\gamma^\alpha)^{-1}[X_\gamma] \);
It is easy to see that the families

\[ \text{Case 1: Assume that } J \text{ exists an infinite set } I \text{ such that they are closed crowded subspaces of } 2^\alpha \text{ and } X_\alpha \neq \emptyset; \]

\[ \text{Case 2: Assume that } A_\alpha \cup B_\alpha \text{ and } |A_\alpha \cap B_\alpha| \leq 1; \]

\[ \text{Case 3: Assume that } \exists \text{ an infinite set } I \text{ with } |I| = 1 \text{ and } \text{supp}(\mu_0^\alpha) \subset X_\alpha, \text{ then } \left( P(X_{\alpha+\omega}) \right)^\omega \text{ with } P_{\alpha_{\alpha+\omega}}(\mu_n) = \mu_\alpha \text{ for all } n \in \omega, \text{ every accumulation point of the sequence } (\lambda_n)_{n \in I_\alpha} \text{ in } P(X_{\alpha+\omega}) \text{ is a nonatomic measure on the space } X_{\alpha+\omega}. \]

To start the inductive construction, for every \( \alpha \leq \omega \), put \( X_\alpha = A_\alpha = 2^\alpha, B_\alpha = \emptyset \) and \( I_\alpha = \omega \). Assume that for some limit ordinal \( \alpha \) and all \( \gamma < \alpha \) we have constructed nonzero closed subspaces \( X_\gamma \subseteq 2^\gamma \) that have no isolated points and satisfy the inductive condition (7). Put \( X_\alpha = \bigcap_{\gamma < \alpha} (\pi_\gamma)^{-1}[X_\gamma] \) and observe that \( X_\alpha \) is crowded and satisfies the inductive condition (7).

To define the other sets we consider the following three cases.

**Case 1:** Assume that \( \alpha \neq \emptyset \cup \Omega \) or \( \bigcup_{n \in \omega} \text{supp}(\mu_n^\alpha) \nsubseteq X_\alpha \). In this case put

\[ I_{\alpha+n} = \omega, \quad X_{\alpha+n} = A_{\alpha+n} = X_\alpha \times \{0\}^n \quad \text{and} \quad B_{\alpha+n} = \emptyset \]

for every \( n \in \omega \). It is easy to see that the families \( (I_\beta)_{\beta < \alpha + \omega}, (X_\beta)_{\beta < \alpha + \omega}, (A_\beta)_{\beta < \alpha + \omega}, (B_\beta)_{\beta < \alpha + \omega} \) satisfy the inductive conditions (1)–(9).

**Case 2:** Assume that \( \alpha \in \Lambda \) and \( \bigcup_{n \in \omega} \text{supp}(\mu_n^\alpha) \subseteq X_\alpha \). In this case the definition of the set \( \Lambda \) ensures that the sequence \( (\mu_n^\alpha)_{n \in \omega} \) consists of pairwise distinct Dirac measures that converge to the Dirac measure \( \mu_0^\alpha \) in \( 2^\alpha \). Since the sets \( \bigcup_{n \in \omega} \text{supp}(\mu_{2n+1}^\alpha) \) and \( \bigcup_{n \in \omega} \text{supp}(\mu_{2n+2}^\alpha) \) are closed and disjoint in the zero-dimensional metric space \( X_\alpha \setminus \text{supp}(\mu_0^\alpha) \), we can use [14, Theorem 6.2.7] (as in the proof of Lemma 4.1) and find two disjoint clopen sets \( V_\alpha, W_\alpha \) in \( X_\alpha \setminus \text{supp}(\mu_0^\alpha) \) such that

\[ \bigcup_{n \in \omega} \text{supp}(\mu_{2n+1}^\alpha) \subseteq V_\alpha, \quad \bigcup_{n \in \omega} \text{supp}(\mu_{2n+2}^\alpha) \subseteq W_\alpha, \quad V_\alpha \cap W_\alpha = \emptyset, \quad V_\alpha \cup W_\alpha = X_\alpha \setminus \text{supp}(\mu_0^\alpha). \]

Since the space \( X_\alpha \setminus \text{supp}(\mu_0^\alpha) \) is crowded, so are its clopen subspaces \( V_\alpha \) and \( W_\alpha \).

Consider the subsets \( A_\alpha := V_\alpha \cup \text{supp}(\mu_0^\alpha), B_\alpha := W_\alpha \cup \text{supp}(\mu_0^\alpha) \) of \( 2^\alpha \), and observe that they are closed in \( 2^\alpha \) and have no isolated points. Define

\[ X_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\}) \subseteq X_\alpha \times 2. \]

Set \( I_\alpha := \omega \). For every ordinal \( n \in [1, \omega) \), put

\[ I_{\alpha+n} := \omega, \quad X_{\alpha+n} = A_{\alpha+n} := X_{\alpha+1} \times \{0\}^{n-1} \subseteq X_\alpha \times 2^n \quad \text{and} \quad B_{\alpha+n} := \emptyset. \]

It is easy to see that the families \( (I_\beta)_{\beta < \alpha + \omega}, (X_\beta)_{\beta < \alpha + \omega}, (A_\beta)_{\beta < \alpha + \omega}, (B_\beta)_{\beta < \alpha + \omega} \) satisfy the inductive conditions (1)–(9).

**Case 3:** Assume that \( \alpha \in \Omega \) and \( \bigcup_{n \in \omega} \text{supp}(\mu_n^\alpha) \subseteq X_\alpha \). In this case the definition of the set \( \Omega \) ensures that \( \lim_{n \in \omega} \text{supp}(\mu_n^\alpha) = 0 \). Since \( P(X_\alpha) \) is a compact metrizable space, there exists an infinite set \( J_\alpha \subseteq \omega \) such that the sequence \( (\mu_n^\alpha)_{n \in J_\alpha} \) converges to some measure \( \mu \in P(X_\alpha) \). If the measure \( \mu \) is nonatomic, then, for every ordinal \( n < \omega \), we put

\[ I_{\alpha+n} = J_\alpha, \quad X_{\alpha+n} = A_{\alpha+n} = X_\alpha \times \{0\}^n \quad \text{and} \quad B_{\alpha+n} = \emptyset. \]
and observe that the families \((I_\beta)_{\beta<\alpha+\omega}, (X_\beta)_{\beta<\alpha+\omega}, (A_\beta)_{\beta<\alpha+\omega}\) and \((B_\beta)_{\beta<\alpha+\omega}\) satisfy the inductive conditions (1)–(9).

Now we assume that the measure \(\mu\) has atoms. Let \(T_\mu = \{ x \in X_\alpha : \mu(\{x\}) > 0 \}\) be the set of all atoms of the measure \(\mu\). The additivity of the probability measure \(\mu\) ensures that the set \(T_\mu\) is at most countable. Since the compact metrizable space \(X_\alpha\) is crowded, the \(G_\delta\)-subset \(G_\alpha := X_\alpha \setminus T_\mu\) of \(X_\alpha\) is dense in \(X_\alpha\) (by the Baire Theorem) and also crowded.

The set \(T_\mu\) is at most countable and hence admits a well-order \(\leq\) such that for every \(a \in T_\mu\) the set \(\downarrow a = \{ x \in T_\mu : x \leq a \}\) is finite.

Using Lemma 4.1 inductively, for every \(a \in T_\mu\) we can choose an infinite subset \(J_{a,a}\) of \(J_a\) and a family \((Z_{(a,s)})_{s \in 2^{<\omega}}\) of clopen subsets of \(X_\alpha\) \(\setminus\{a\}\) such that \(Z_{(a,s)} = X_\alpha \setminus \{a\}\), \(J_{a,a} \subseteq J_{a,b}\) for any \(b \in T_\mu\) with \(b \leq a\) and for every \(s \in 2^{<\omega}\) the following conditions are satisfied:

\[(\dagger)\] \(Z_{(a,s),0} \cup Z_{(a,s),1} = Z_{(a,s)}\) and \(Z_{(a,s),0} \cap Z_{(a,s),1} = \emptyset;\)

\[(\ddagger)\] for any clopen neighborhood \(U \subseteq X_\alpha\) of \(a\) and each \(e > 0\) there exists an \(m \in \omega\) such that for every \(n \in J_{a,a}\) with \(n \geq m\) we have \(\mu^n(U \cap Z_{(a,s)}(n)) < \mu(U \setminus \{a\}) + \frac{1}{2^n} + e.\)

Choose any infinite set \(I_a \subseteq J_a\) such that \(I_a \subseteq^\ast J_{a,a}\) for every \(a \in T_\mu\).

Fix a bijective function \(\xi : [a) \to T_\mu \times 2^{<\omega}\) such that for any \(a \in T_\mu\) and \(s \leq t \in 2^{<\omega}\) we have \(\xi^{-1}(a, s) < \xi^{-1}(a, t)\) (to construct such bijection it is sufficient to consider a partition of \(\omega\) onto \(|T_\mu|\) infinite sets, and on every set of the partition to consider a copy of the bijection \(2^{<\omega} \to \omega, t \mapsto \sum_{i=0}^{t-1} i(t)2^i\)). For every \(n \in \omega\), let \(\gamma_n : G_\alpha \to 2\) be the characteristic function of the clopen subset \(Z_{\xi(n)} \cap G_\alpha\) in \(G_\alpha\).

Now, for every \(n \in \omega\), we define the sets \(X_{a+n}, A_{a+n}\) and \(B_{a+n}\) as follows:

- \(X_{a+n}\) is the closure of the set \(\{ (z, \gamma_0(z), \ldots, \gamma_{n-1}(z)) : z \in G_\alpha \}\) in \(X_\alpha \times 2^n;\)
- \(A_{a+n}\) is the closure of the set \(\{ (z, \gamma_0(z), \ldots, \gamma_{n-1}(z)) : z \in G_\alpha \setminus Z_{\xi(n)} \}\) in \(X_\alpha \times 2^n;\)
- \(B_{a+n}\) is the closure of the set \(\{ (z, \gamma_0(z), \ldots, \gamma_{n-1}(z)) : z \in G_\alpha \cap Z_{\xi(n)} \}\) in \(X_\alpha \times 2^n.\)

Since \(Z_{\xi(n)} \cap G_\alpha\) is a clopen set in \(G_\alpha\), the spaces \(X_{a+n}, A_{a+n}, B_{a+n}\) are crowded, being closures of the topological copies of the crowded spaces \(G_\alpha, G_\alpha \setminus Z_{\xi(n)}\) and \(G_\alpha \cap Z_{\xi(n)}\). This means that the condition (4) is satisfied.

Next, we show that the condition (5) is satisfied. For every \(n \in \omega\), the equality \(X_{a+n} = A_{a+n} \cup B_{a+n}\) holds by the definitions of the sets \(X_{a+n}, A_{a+n}, B_{a+n}\). To see that \(|A_{a+n} \cap B_{a+n}| \leq 1\), take any points \((x_0, 0), (x_1, t) \in A_{a+n} \cap B_{a+n} \subseteq X_{a+n} \subseteq X_\alpha \times 2^n\), where \(x_0, x_1 \in X_\alpha\) and \(t_0, t_1 \in 2^n\). Find \(a \in T_\mu\) and \(s \in 2^{<\omega}\) such that \((a, s) = \xi(n)\). Taking into account that all \(Z_{(a,s)}\) are clopen in \(X_\alpha \setminus \{a\}\), the condition \((\dagger)\) implies

\[x_0, x_1 \in \pi_{a+n}^{-1}(A_{a+n}) \cap \pi_{a+n}^{-1}(B_{a+n}) \subseteq G_\alpha \setminus Z_{\xi(n)} \cap G_\alpha \cap Z_{\xi(n)} \subseteq \{a\},\]

where the closure is taken in the compact space \(X_\alpha\). Therefore \(x_0 = a = x_1\). Assuming that \(t_0 \neq t_1\), we can find an ordinal \(k \in n\) such that \(t_0(k) \neq t_1(k)\). Let \(\xi(k) = (b, t)\) for some \(b \in T_\mu\) and \(t \in 2^{<\omega}\). If \(b \neq a\), then either \(Z_{\xi(k)}\) or \(X_\alpha \setminus Z_{\xi(k)}\) is a neighborhood of \(a\) in \(X_\alpha\). By the definition of \(\gamma_k\), in the first case we get \(t_0(k) = t_1(k) = 1\), and in the second case we get \(t_0(k) = t_1(k) = 0\). But this contradicts the inequality \(t_0(k) \neq t_1(k)\), and hence \(b = a\).

By the choice of the function \(\xi\), either \(t \leq s\) or \(t \geq s\) in the tree \(2^{<\omega}\). In the case \(t \leq s\), then \(Z_{\xi(n)} \subseteq Z_{\xi(k)}\) and hence \(\gamma_k(G_\alpha \cap Z_{\xi(n)}) = \{1\}\) and \(t_0(k) = t_1(k) = 1\). If \(s \geq t\) and \(t \geq s\) are incomparable, then \(Z_{\xi(n)} \cap Z_{\xi(k)} = \emptyset\) and \(\gamma_k(G_\alpha \cap Z_{\xi(n)}) = \{0\}\), which implies \(t_0(k) = t_1(k) = 0\). In both cases we get a contradiction with the choice of \(k\). Therefore, \(|A_{a+n} \cap B_{a+n}| \leq 1\) and the condition (5) holds.

The conditions (6) and (7) follow from the definition of the spaces \(X_{a+n+1}, A_{a+n}, B_{a+n}\) and the inclusions \(\gamma_n(G_\alpha \setminus Z_{\xi(n)}) \subseteq \{0\}\) and \(\gamma_n(G_\alpha \cap Z_{\xi(n)}) \subseteq \{1\}\) holding for all \(n \in \omega\).
Finally, we check the inductive condition (9). Choose a sequence of measures \( (\lambda_n)_{n \in \omega} \subseteq (P(X_{\alpha+\omega}))^{\omega} \) such that \( P\pi_\alpha^{\alpha+\omega}(\lambda_n) = \mu^\alpha_n \) for all \( n \in \omega \). Let \( \lambda \in P(X_{\alpha+\omega}) \) be an accumulation point of the sequence \( (\lambda_n)_{n \in I_\alpha} \). The convergence of the sequence \( (\mu^\alpha_n)_{n \in I_\alpha} \) to the measure \( \mu \) and the continuity of the map \( \pi_\alpha^{\alpha+\omega} : P(X_{\alpha+\omega}) \to P(X_{\alpha}) \) imply that \( P\pi_\alpha^{\alpha+\omega}(\lambda) = \mu \).

We should prove that the measure \( \lambda \) is nonatomic. To derive a contradiction, assume that \( \lambda((b)) > 0 \) for some \( b \in X_{\alpha+\omega} \). Consider the point \( a = \pi_\alpha^{\alpha+\omega}(b) \) and conclude that \( \mu((a)) = \lambda((\pi_\alpha^{\alpha+\omega})^{-1}(a)) \geq \frac{1}{4}\lambda((b)) > 0 \) and hence \( a \in T_{\mu} \). By the regularity of the measure \( \mu \), there exists a clopen neighborhood \( U \subseteq X_{\alpha} \) of \( a \) such that \( \mu(U\setminus\{a\}) < \frac{1}{4}\lambda((b)) \).

Find \( l \in \omega \) such that \( \frac{2^l}{2^l} < \frac{1}{4}\lambda((b)) \). By the property \((\xi)\) and almost inclusion \( I_\alpha \subseteq J_{a,a} \), there exists an \( m \in \omega \) such that for every \( s \in 2^l \) and every \( n \in I_\alpha \) with \( n \geq m \) we have

\[
\mu^\alpha_n(U \cap Z_{\alpha,s}) < \mu(U\setminus\{a\}) + \frac{1}{2^m} + \frac{1}{2^l} < \frac{2^l}{4}\lambda((b)).
\]

Since \( \lim_{n \to \omega} \sup_{x \in 2^\omega} \mu^\alpha_n((x)) = 0 \), we can replace \( m \) by a larger number, if necessary, and assume additionally that \( \mu^\alpha_n((a)) < \frac{1}{4}\lambda((b)) \) for all \( n \geq m \).

Consider the continuous map

\[
\gamma : G_\alpha \to X_{\alpha+\omega} \subseteq X_\alpha \times 2^{\omega}, \quad \gamma : x \mapsto (x, (\gamma_n(x))_{n \in \omega}),
\]

and observe that the image \( \gamma[G_\alpha] \) is dense in \( X_{\alpha+\omega} \). For every \( s \in 2^l \), consider the closed set \( W_s = \gamma[G_\alpha \cap Z_{\alpha,s}] \) in \( X_{\alpha+\omega} \). We claim that the family \( \{W_s\}_{s \in 2^l} \) is disjoint. Indeed, suppose for a contradiction that for some \( s_0, s_1 \in 2^l \), the intersection \( W_{s_0} \cap W_{s_1} \) contains a point, say \( x \). Since \( X_{\alpha+\omega} \) is metrizable, there are two sequences \( (z_n^0)_{n \in \omega} \) and \( (z_n^1)_{n \in \omega} \) in \( G_\alpha \cap Z_{\alpha,s_0} \) and \( G_\alpha \cap Z_{\alpha,s_1} \), respectively, such that \( \lim_n \gamma(z_n^0) = x = \lim_n \gamma(z_n^1) \). Set \( m := \xi^{-1}(a, s_0) \). Since, by \((\xi)\), \( Z_{\alpha,s_0} \cap Z_{\alpha,s_1} = \emptyset \), the definition of \( \gamma_m \) implies that \( \gamma_m(z_n^0) = 1 \) and \( \gamma_m(z_n^1) = 0 \) for every \( n \in \omega \), and hence \( \lim_n \gamma(z_n^0) \neq \lim_n \gamma(z_n^1) \). This contradiction proves the claim. Then the density of \( \gamma[G_\alpha] \) implies that the finite family \( \{W_s\}_{s \in 2^l} \) covers the compact space \( X_{\alpha+\omega} \). This fact and the claim imply that \( \{W_s\}_{s \in 2^l} \) is a disjoint clopen cover of \( X_{\alpha+\omega} \).

Find a unique \( s \in 2^l \) such that the clopen set \( W_s := \gamma(G_\alpha \cap Z_{\alpha,s}) \) contains the point \( b \). Then the set \( W := W_s \cap (\pi_\alpha^{\alpha+\omega})^{-1}(U) \) is a clopen neighborhood of the point \( b \) in \( X_{\alpha+\omega} \).

Since \( \lambda \) is an accumulating point of the sequence \( (\lambda_n)_{n \in I_\alpha} \), there exists \( n \in I_\alpha \) with \( n \geq m \) such that \( \lambda_n(W) > \frac{3}{4}\lambda((b)) \).

Taking into account that \( Z_{\alpha,s} \) is a closed subset of \( X_\alpha \setminus \{a\} \), we conclude that

\[
W_s \subseteq (\pi_\alpha^{\alpha+\omega})^{-1}(\{a\} \cup Z_{\alpha,s}) \quad \text{and} \quad W \subseteq (\pi_\alpha^{\alpha+\omega})^{-1}(\{a\} \cup (U \cap Z_{\alpha,s})).
\]

Now observe that

\[
\mu^\alpha_n(U \cap Z_{\alpha,s}) = \mu^\alpha_n((U \cap Z_{\alpha,s}) \cup \{a\}) - \mu^\alpha_n(\{a\})
\]

\[
> \lambda_n((\pi_\alpha^{\alpha+\omega})^{-1}((U \cap Z_{\alpha,s}) \cup \{a\})) - \frac{1}{4}\lambda((b))
\]

\[
\geq \lambda_n(W) - \frac{1}{4}\lambda((b)) > \frac{3}{4}\lambda((b)),
\]

which contradicts (7). This finishes the inductive construction.

After completing the inductive construction, consider the closed subspace

\[
K = \bigcap_{\alpha \in \omega_1} \{x \in 2^{\omega_1} : x|_\alpha \in X_\alpha\}
\]

of the compact space \( 2^{\omega_1} \). Below we show that the compact space \( K \) satisfies all conditions of Theorem 1.1.

Claim 6.1 The compact space \( K \) is fully simple, separable and crowded.
\textbf{Proof} By (4)–(7) and Lemma 2.6, the compact space \( K \) is fully simple. By Lemma 2.7, \( K \) is separable. To show that \( K \) is crowded, fix a point \( z \in K \) and an arbitrary basic neighborhood \( U = \{ x \in K : x \mid_F = z \mid_F \} \) of \( z \), where \( F \subseteq \omega_1 \). Choose a countable ordinal \( \alpha \in \omega_1 \) such that \( F \subseteq \alpha \). The space \( \mathcal{X}_\alpha \) is crowded (by (4)) and hence contains a point \( x_\alpha \in \mathcal{X}_\alpha \) such that \( x_\alpha \in \pi_{\alpha}^{\omega_1}(U) \setminus \pi_{\alpha}^{\omega_1}(z) \). Then any \( x \in (\pi_{\alpha}^{\omega_1})^{-1}(x_\alpha) \cap K \) belongs to \( U \setminus \{ z \} \), witnessing that \( K \) is crowded. \( \square \)

\textbf{Claim 6.2} The space \( K \) has no non-trivial convergent sequences.

\textbf{Proof} To derive a contradiction, assume that \( K \) contains a nontrivial convergent sequence. Then there exists a sequence \( \{ \mu_n \}_{n \in \omega} \subseteq K \subseteq P(K) \) of pairwise distinct Dirac measures, converging to the Dirac measure \( \mu_0 \). The choice of the transfinite sequence \( (\mu_n^{\alpha})_{n \in \omega} \alpha \in \omega_1 \) guarantees that the set

\[ S := \{ \alpha \in \omega_1 : \mu_n^{\alpha} = P\pi_{\alpha}^{\omega_1}(\mu_n) \text{ for all } n \in \omega \} \]

is stationary in \( \omega_1 \), and hence it contains an infinite limit ordinal \( \alpha \in S \) such that the function \( \omega \rightarrow 2^{\alpha} \subseteq P(2^{\alpha}) \), \( n \mapsto P\pi_{\alpha}^{\omega_1}(\mu_n) \), is injective. Then the ordinal \( \alpha \) belongs to the set \( \Lambda \) and the inductive condition (8) guarantees that \( \bigcup_{n \in \omega} \text{supp}(\mu_n^{\alpha+1}) \subseteq A_\alpha \) and \( \bigcup_{n \in \omega} \text{supp}(\mu_n^{\omega+2}) \subseteq B_\alpha \). Since the sets \( A_\alpha \times \{ 0 \} \) and \( B_\alpha \times \{ 1 \} \) have disjoint closures in \( \mathcal{X}_{\alpha+1} \), the sequence \( (P\pi_{\alpha+1}^{\omega_1}(\mu_n))_{n \in \omega} \) is divergent, which contradicts the convergence of the sequence \( (\mu_n^{\alpha})_{n \in \omega} \). \( \square \)

\textbf{Claim 6.3} The space \( K \) is a fully simple crowded Efimov space.

\textbf{Proof} By Proposition 2.5, the compact space \( K \), being simple, does not admit continuous surjective maps onto \([0, 1]^{\omega_1}\). Since \( \beta\omega \) admits a continuous surjective map onto \([0, 1]^{\omega_1}\) (because \([0, 1]^{\omega_1}\) is separable), the Tietze–Urysohn extension theorem implies that \( K \) contains no copies of \( \beta\omega \). This result and Claims 6.1 and 6.2 imply that \( K \) is a fully simple crowded Efimov space. \( \square \)

\textbf{Claim 6.4} Every \( c_0 \)-vanishing sequence \( (\mu_n)_{n \in \omega} \) in \( P(K) \) has a subsequence that converges to some nonatomic measure.

\textbf{Proof} By Lemma 3.8, the \( c_0 \)-vanishing property of \( (\mu_n)_{n \in \omega} \) implies the existence of an ordinal \( \upsilon \in \omega_1 \) such that for every countable ordinal \( \alpha \geq \upsilon \) the sequence of measures \( (P\pi_{\alpha}^{\omega_1}(\mu_n))_{n \in \omega} \) is \( c_0 \)-vanishing. By the choice of the transfinite sequence \( (\mu_n^{\alpha})_{n \in \omega} \alpha \in \omega_1 \), the set

\[ S = \{ \alpha \in \omega_1 : \mu_n^{\alpha} = P\pi_{\alpha}^{\omega_1}(\mu_n) \text{ for all } n \in \omega \} \]

is stationary and hence contains a limit ordinal \( \alpha \in S \) such that \( \alpha \geq \upsilon \). By the choice of \( \upsilon \), the sequence of measures \( (\mu_n^{\alpha})_{n \in \omega} = (P\pi_{\alpha}^{\omega_1}(\mu_n))_{n \in \omega} \) is \( c_0 \)-vanishing and hence \( \alpha \in \Omega \). Let \( \mu \in P(K) \) be any accumulation point of the sequence \( (\mu_n)_{n \in I_\alpha} \). Then \( P\pi_{\alpha+\omega}^{\omega_1}(\mu) \) is an accumulation point of the sequence \( (P\pi_{\alpha+\omega}^{\omega_1}(\mu_n))_{n \in I_\alpha} \). Since the ordinal \( \alpha \) belongs to the set \( \Omega \), the inductive condition (9) guarantees that the measure \( P\pi_{\alpha+\omega}^{\omega_1}(\mu) \) is nonatomic and so is the measure \( \mu \). By Lemma 2.8, the space \( P(K) \) is first-countable at \( \mu \). Then there exists an infinite set \( J \subseteq I_\alpha \) such that the sequence \( (\mu_k)_{k \in J} \) converges to \( \mu \). \( \square \)

Claim 6.4 and Lemma 3.6 imply

\textbf{Claim 6.5} Every sequence in \( P(K) \) that converges to a purely atomic measure \( \mu \in P_\alpha(K) \) converges to \( \mu \) in norm.
Claim 6.6 The space $P_{na}(K)$ is non-metrizable, first-countable, Čech-complete, sequentially compact, and the set $P_{na}(K)$ is dense in $P(K)$.

Proof The first-countability, Čech-completeness, and density of $P_{na}(K)$ in $P(K)$ follows from Lemmas 2.8 and 2.2, respectively. Claim 6.4 implies that the space $P_{na}(K)$ is sequentially compact.

To show that $P_{na}(K)$ is non-metrizable, suppose for a contradiction that $P_{na}(K)$ is metrizable. Then $P_{na}(K)$ being also sequentially compact must be compact. Since, by Lemma 2.2, $P_{na}(K)$ is dense in $P(K)$ we obtain that $P_{na}(K) = P(K)$, a contradiction. Thus $P_{na}(K)$ is not metrizable. □

Claim 6.7 The space $P(K)$ is selectively sequentially pseudocompact but not sequentially compact.

Proof The space $P(K)$ is selectively sequentially pseudocompact by Lemma 2.1 and Claim 6.6. Since $K$ is a closed non-sequentially compact subspace of $P(K)$, the space $P(K)$ is not sequentially compact as well. □

Claim 6.8 The Banach space $C(K)$ has the Gelfand–Phillips property and hence it does not contain an isomorphic copy of $\ell_\infty$.

Proof Corollary 2.2 of [3] states that if $P(K)$ is selectively sequentially pseudocompact, then $C(K)$ has the Gelfand–Phillips property. It remains to note that $P(K)$ is selectively sequentially pseudocompact by Claim 6.7. To prove the last assertion of the claim it suffices to note that the property of being a Gelfand–Phillips space is hereditary and the space $\ell_\infty$ does not have the Gelfand–Phillips property by the classical result of Phillips [29]. □

Claim 6.9 The Banach space $C(K)$ does not have the Grothendieck property and hence $K$ is a zero-dimensional but not extremally disconnected space.

Proof The space $C(K)$ does not have the Grothendieck property by Claim 6.8 and Corollary 2.10. By construction the compact space $K$ is a subspace of the zero-dimensional space $2^{\omega_1}$. Therefore $K$ is zero-dimensional as well. Since $C(K)$ is not a Grothendieck space, Theorem 4.5.6 of [6] implies that the compact space $K$ is not extremally disconnected. □

Remark 6.10 It immediately follows from Theorems 4.1 and 6.10 of [26] that the Banach space $C(K)$ does not have even the $\ell_1$-Grothendieck property. This property introduced very recently in [26] is much weaker than the Grothendieck property. □

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