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Three-body continuum wave functions with a box boundary condition

Received: date / Accepted: date

Abstract. In this work we investigate the connection between discretized three-body continuum wave functions, in particular via a box boundary condition, and the wave functions computed with the correct asymptotics. The three-body wave functions are in both cases obtained by means of the adiabatic expansion method. The information concerning all the possible incoming and outgoing channels, which appears naturally when the continuum is not discretized, seems to be lost when the discretization is implemented. In this work we show that both methods are fully equivalent, and the full information contained in the three-body wave function is actually preserved in the discrete spectrum. Therefore, in those cases when the asymptotic behaviour is not known analytically, i.e., when the Coulomb interaction is involved, the discretization technique can be safely used.

1 Introduction

The description of an $N$-body system in the continuum requires knowledge of three different ingredients: How the $N$ particles in the system get close to each other, how they interact, and how the particles move far apart from each other. In other words, the description of a system in the continuum is equivalent to describing the collision between the particles involved in the system under investigation.

For two-body systems the problem is relatively simple. Let us consider two point-like particles approaching each other with a relative energy $E$ and some relative angular momentum $\ell$. If the two-body Hamiltonian $\mathcal{H}$ commutes with the $\ell^2$ operator, we then have that $\ell$ is a good quantum number, and the only possible outgoing channel is the one with the two particles moving far apart from each other with the same relative energy, and the same relative angular momentum. In this simple single-channel case the continuum two-body wave function is fully determined by one characteristic number, the phase shift. Let us now go a step beyond, and assume that $[\mathcal{H}, \ell^2] \neq 0$, which implies that the two-body potential couples different orbital angular momenta. In this case, when the two particles approach each other with relative energy $E$ and angular momentum $\ell$, several outgoing channels are possible, namely, all those with relative momentum $\ell'$ such that $\ell$ and $\ell'$ are coupled by the potential. Furthermore, in this multichannel case, full knowledge of all possible two-body processes requires knowledge of all the outgoing channels for all the possible incoming channels. In other words, in this case, the two-body reactions are fully determined, not by a number, but by a matrix (the $S$-matrix), whose $ij$-term, $S_{ij}$, is such that $|S_{ij}|^2$ gives the probability for an incoming channel $i$ to go out through channel $j$. Since given an incoming channel $i$ the system must necessarily go out through one of the available outgoing channels, it is trivial to conclude that the following condition, $\sum_j |S_{ij}|^2 = 1$, has to

This work was supported by funds provided by DGI of MINECO (Spain) under contract No. FIS2011-23565.

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be fulfilled for each $i$. This condition can be written in a more compact way as the unitarity condition $S^\dagger S = \mathbb{I}$, where $\mathbb{I}$ is the unity matrix.

For a three-particle system the continuum problem is equivalent to the multichannel problem described above. The total three-body energy $E$ is the sum of the relative energy between two of the particles and the relative energy between the third particle and the center of mass of the other two. These two relative energies are not fixed, and therefore there will be infinitely many channels consistent with one given total three-body energy. The same happens with the total angular momentum, which in general can be obtained by coupling different partial-wave components in the three-body system. Usually the partial-wave components are coupled by the interaction between the three particles. Furthermore, if some of the internal two-body subsystems can hold one or more two-body bound states, then additional available three-body channels appear. These are usually denoted as 1+2 channels, and they correspond to a bound two-body system and the third particle in the continuum. In general, the interactions between the particles are able to couple the different channels, in such a way that, for instance, three incoming particles in the continuum could go out, when available, in a 1+2 channel (recombination process) or go out keeping the three particles in the continuum (where infinitely many distributions of the total three-body energy is in principle possible). In the same way, if we consider an incoming 1+2 channel, provided that the incident energy is big enough, the system could go out through the same or another 1+2 channel (giving rise to elastic, inelastic, or transfer reactions) or lead to the three particles in the continuum (breakup process). The full information about all these possible reactions is contained in the three-body $S$ matrix, which connects all the possible incoming channels with all the possible outgoing channels.

The first difficulty when describing three-body continuum states is the identification of the different incoming and outgoing channels. In this connection the use of the adiabatic expansion method [1] is particularly useful [2]. In this method the three-body problem is reduced to a coupled set of radial equations where a family of effective adiabatic potentials enter. The wave function is expanded in an adiabatic basis such that each term in the expansion is associated to a single adiabatic potential. Therefore, in this basis the dimension of the $S$-matrix describing the full process is determined by the number of adiabatic terms included in the expansion, which is typically a rather modest number. The great advantage of the adiabatic expansion is that, asymptotically, the possible 1+2 channels and the breakup channels (the three-particles in the continuum) are clearly separated. As an illustration, we show in Fig.1 a typical set of adiabatic potentials entering in a three-body problem. They correspond to a three-body system where two of the two-body subsystems have a bound state. This is reflected in the fact that the two lowest effective adiabatic potentials go asymptotically to the binding energies $E_{2b}^{(1)}$ and $E_{2b}^{(2)}$ of each bound two-body system. As shown for instance in [3], the channels associated to these adiabatic terms correspond to asymptotic structures where two of the particles form a bound state (with energy $E_{2b}^{(1)}$ or $E_{2b}^{(2)}$) and the third particle moves in the continuum. In other words, these

![Fig. 1 Typical effective adiabatic potentials for a three-body system where two two-body bound states are present. The two lowest adiabatic potentials go asymptotically to the binding energies $E_{2b}^{(1)}$ and $E_{2b}^{(2)}$ of the two-body bound states. For a given three-body energy $E$, when $E_{2b}^{(1)} < E < E_{2b}^{(2)}$ only one channel is open, while when $E_{2b}^{(2)} < E < 0$ both channels are open.](image-url)
channels describe the 1+2 incoming or outgoing channels, which are separated from the infinitely many remaining ones, which describe the three particles in the continuum.

In this way we can distinguish different regions in Fig. 1, depending on the total three-body energy. All the three-body energies \( E \) such that \( E_{2b}^{(1)} < E < E_{2b}^{(2)} \) (like \( E^{(1)} \) in the figure) correspond to processes where only one channel is open. Only the elastic collision between the third particle and the bound two-body state with energy \( E_{2b}^{(1)} \) is possible. When the three-body energy increases up to the region \( E_{2b}^{(2)} < E < 0 \) (\( E^{(2)} \) in the figure) a second channel is open. Two different collisions are now possible, the one where a particle hits the bound state with binding energy \( E_{2b}^{(1)} \), and the one where a particle hits the state with binding energy \( E_{2b}^{(2)} \). In the same way, each of these reactions has two possible outgoing channels, corresponding to the two allowed bound two-body states and the third particle in the continuum. In particular, in this energy range the rearrangement process is open. The full process will be then described by just a \( 2 \times 2 \) \( S \)-matrix. When \( E > 0 \) the breakup channels are also open, and in principle infinitely many channels should be included in order to describe the reaction.

The second problem to be faced when describing three particles in the continuum is the normalization of the three-body wave function. When long-range interactions are not involved in the problem, the asymptotic behaviour is known analytically, and the continuum wave function can be normalized to match the expected asymptotic form. The \( S \) matrix can be extracted for each incoming and outgoing channel by direct comparison with the asymptotics, or more accurately, by the use of the integral relations recently derived \([3, 4]\). However, when at least two of the three particles are charged, the asymptotic form of the wave function is not known, and its normalization becomes a complicated task. One option to solve this problem is to obtain numerically the order and the Sommerfeld parameter of the regular and irregular Coulomb functions that determine the asymptotic outgoing wave for each of the open channels. The problem is that an inaccurate calculation of these parameters unavoidably leads to a wrong value of the computed \( S \)-matrix.

In this work we shall consider a different method in order to achieve proper asymptotic behaviour. This is the discretization of the continuum states. One of the simplest procedures to do so is to impose a box boundary condition. This procedure immediately leads to a set of discrete continuum states which are formally treated as bound states, and therefore they are just normalized to 1 inside the box. Playing with the size of the box it is possible to obtain the discrete continuum state for a particular value of the three-body energy. The apparent inconsistency of this method is that the discretization inside a box automatically leads to a set discrete energies, each of them corresponding to a single three-body wave function. No condition is given in order to determine the incoming and outgoing channels for each of the states, and therefore it is not clear to what incoming and outgoing channels each discretized state corresponds. In fact, for a given energy one should get as many different three-body wave functions as channels are open, and not only one. Therefore, apparently, a large part of the information has been lost when discretizing the spectrum. The purpose of this work is to show that this is not correct. The discrete continuum states keep the full information about the three-body state. All the possible incoming and outgoing channels are actually taken into account, and therefore the information contained in the \( S \)-matrix is fully preserved.

In the next two sections we briefly describe the adiabatic expansion method and the two different procedures used to compute the three-body wave functions. They are illustrated by means of a three-body system made of three identical spinless bosons coupled to spin and parity \( 0^- \). In the fourth section we discuss the details of the discretized spectrum, and show that it is equivalent to the results obtained without discretization. We close the paper with the summary and the conclusions.

### 2 Three-body continuum wave functions and the adiabatic expansion method.

The three-body wave functions will be written in terms of the usual \( x \) and \( y \) Jacobi coordinates. From them we can construct the hyperspherical coordinates \([4]\), which contain a radial one, the so-called hyperradius \( \rho (\rho^2 = x^2 + y^2) \) and the five hyperangles \( \Omega \), i.e., \( \alpha = \arctan(x/y), \Omega_x, \) and \( \Omega_y \).

In hyperspherical coordinates the Hamiltonian operator \( \hat{H} \) takes the form:

\[
\hat{H} = -\frac{\hbar^2}{2m} \hat{\rho}^2 + \hat{H}_\Omega,
\]  

(1)
where $\hat{T}_\rho$ is the hyperradial kinetic energy operator, and $\hat{H}_2$ contains the whole dependence on the hyperangles.

In the adiabatic expansion method the angular part of the Hamiltonian is first solved for fixed values of the hyperradius. This amounts to solve the eigenvalue problem

$$\hat{H}_2 \Phi_n(\rho, \Omega) = \frac{\hbar^2}{2m} \frac{1}{\rho^2} \lambda_n(\rho) \Phi_n(\rho, \Omega)$$

(2)

for individual values of $\rho$, which is treated as a parameter.

The angular functions $\{\Phi_n(\rho, \Omega)\}$ form an orthonormal basis for each value of $\rho$, and they are used to expand the total three-body wave function as:

$$\Psi_i(x, y) = \frac{1}{\rho^{n_0}} \sum_{n=1}^{\infty} f_{n}(\rho) \Phi_n(\rho, \Omega),$$

(3)

where $i = 1, \cdots, n_0$ labels the incoming channel and $n_0$ is the number of open channels. Obviously the summation above has to be truncated, and only a finite number $n_A$ of adiabatic terms will be in practice included in an actual calculation. Note that $n_0$ and $n_A$ are not necessarily equal. As an example, for the energy $E^{(2)}$ shown in Fig.1 we have that $n_0 = 2$, and $n_A$ will be as large a needed in order to get convergence in whatever observable is computed with the wave function $\Psi_i$.

In a second step, the radial wave functions $f_n(\rho)$ in the expansion in Eq. (3) are obtained after solving the following coupled set of radial equations:

$$\sum_{n'=1}^{n_A} (\hat{H}_{nn'} - E \delta_{nn'}) f_{n'}(\rho) = 0, \quad (n' = 1, 2, \cdots, n_A)$$

(4)

where the operator $\hat{H}_{nn'}$ acts on the radial functions and takes the form

$$\hat{H}_{nn'}(\rho) = \frac{\hbar^2}{2m} \left[ -\frac{d^2}{d\rho^2} - Q_{nn}(\rho) + \frac{1}{\rho^2} \left( \lambda_n(\rho) + \frac{15}{4} \right) \right]$$

(5)

$$\hat{H}_{nn'} = -\frac{\hbar^2}{2m} \left( 2P_{nn'}(\rho) \frac{d}{d\rho} + Q_{nn'}(\rho) \right)$$

(6)

for $n = n'$ and $n \neq n'$, respectively. The coupling terms $P_{nn'}$ and $Q_{nn'}$ are given for instance in Ref.[1].

The set of coupled differential equations given in Eq. (4) can be written in a matrix form as:

$$
\begin{pmatrix}
\hat{H}_{11} - E & \cdots & \hat{H}_{1n_A} \\
\hat{H}_{21} & \cdots & \hat{H}_{2n_A} \\
\vdots & \vdots & \vdots \\
\hat{H}_{n_A1} & \cdots & \hat{H}_{n_A n_A} - E
\end{pmatrix}
\begin{pmatrix}
f_{11} \\
f_{12} \\
\vdots \\
f_{n_A1} & \cdots & \hat{H}_{n_A n_A} - E
\end{pmatrix} = 0,
\tag{7}
$$

and the full three-body wave function is written as:

$$\Psi = \begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
\vdots \\
\Psi_{n_0}
\end{pmatrix} = \frac{1}{\rho^{n_0}} \begin{pmatrix}
f_{11} & f_{12} & \cdots & f_{1n_A} \\
f_{12} & f_{22} & \cdots & f_{2n_A} \\
\vdots & \vdots & \ddots & \vdots \\
f_{n_A1} & f_{2n_A} & \cdots & f_{n_A n_A}
\end{pmatrix}
\begin{pmatrix}
\Phi_1 \\
\Phi_2 \\
\vdots \\
\Phi_{n_0}
\end{pmatrix},$$

(8)

which contains all the possible incoming channels.

It is important to note that the diagonal operator $\hat{H}_{nn}$ in Eq. (5) contains the angular eigenvalues $\lambda_n(\rho)$ introduced in Eq. (2). They appear in the effective adiabatic potentials, which are given by:

$$V^{(n)}_{eff}(\rho) = \frac{\hbar^2}{2m} \left( \lambda_n(\rho) + \frac{15}{4} - Q_{nn}(\rho) \right),$$

(9)

and whose typical behaviour is shown in Fig.1.
Fig. 2 Continuum radial wave functions $f_{ni}(\rho)$ contained in Eq.(3) for incoming channels $i = 1$ (panel a), $i = 2$ (panel b), and $i = 3$ (panel c). The calculation corresponds to three spinless identical bosons with total spin and parity $0^+$. An arbitrary weakly attractive short-range potential has been used. The curves in the figure correspond to a total three-body energy of $E = 10$ MeV. The calculation has been done including 3 adiabatic terms in the expansion, and the corresponding functions for $n = 1$ are given by the solid curves, $n = 2$ by the dashed curves, and $n = 3$ by the dotted curves, respectively.

For short-range interactions the asymptotic behaviour of the wave function $\Psi$ is known to be \cite{5}:

$$\Psi \to F - K G,$$

where $K$ is the $K$-matrix, which is related to the $S$-matrix by the simple expression $S = (1 + iK)(1 - iK)^{-1}$. In the equation above $F$ and $G$ are column vectors whose $n^{th}$ term is given by \cite{3, 5}

$$F_n = \sqrt{\kappa j_{K+\frac{3}{2}}(\kappa \rho) \frac{1}{\rho^{j/2}}} \Phi_n(\rho, \Omega),$$

$$G_n = \sqrt{\kappa \eta_{K+\frac{3}{2}}(\kappa \rho) \frac{1}{\rho^{j/2}}} \Phi_n(\rho, \Omega),$$

where $n = 1, \cdots, n_0$, $j$ and $\eta$ are the regular and irregular spherical Bessel functions, respectively, $K$ is the hypermomentum associated asymptotically to the adiabatic potential $n$, and $\kappa = \sqrt{2mE/h}$ is the three-body momentum. When $n$ refers to a 1+2 channel Eqs.\cite{11} and \cite{12} have to be modified as given in Eqs.(12) and (13) in Ref.\cite{3}. This asymptotic behaviour determines how the wave function has to be normalized in order to extract from it the $n_0 \times n_0$ $K$-matrix (and therefore the $S$-matrix). When the Coulomb interaction is involved, the asymptotic behaviour of the wave function is still given by \cite{10}, but the Bessel functions contained in the vectors $F$ and $G$ have to be replaced by regular and irregular Coulomb functions whose order and Sommerfeld parameter have to be determined numerically.

As an example, we have considered a simple three-body system made of three identical spinless bosons with total spin and parity $0^+$. We have chosen an arbitrary simple weakly attractive short-range potential between the three particles. The three-body wave function in Eq.\cite{5} has then been computed and normalized as given in Eq.\cite{10}. The corresponding radial wave functions $f_{ni}(\rho)$ are shown in Fig.\cite{2} for a three-body energy of $E = 10$ MeV. The calculation has been performed including three adiabatic terms in the expansion, and for the energy chosen all the channels are open (therefore $n_0 = n_A = 3$ in Eq.\cite{8}). In the figure we show the radial wave functions when the incoming channels are the first adiabatic term ($i = 1$, panel a), the second adiabatic term ($i = 2$, panel b), and the third adiabatic term ($i = 3$, panel c). The functions corresponding to outgoing channels $n =1, 2,$ and $3$, are shown by the solid, dashed, and dotted curves, respectively. Therefore, Fig\cite{2} shows three different continuum wave function for the same system and the same total energy. As we can see in the figure, for each incoming channel $i$ the dominant term is the one with $n = i$. 

\begin{itemize}
  \item Fig. 2
  \item \textbf{Continuum radial wave functions $f_{ni}(\rho)$ contained in Eq.(3) for incoming channels $i = 1$ (panel a), $i = 2$ (panel b), and $i = 3$ (panel c). The calculation corresponds to three spinless identical bosons with total spin and parity $0^+$. An arbitrary weakly attractive short-range potential has been used. The curves in the figure correspond to a total three-body energy of $E = 10$ MeV. The calculation has been done including 3 adiabatic terms in the expansion, and the corresponding functions for $n = 1$ are given by the solid curves, $n = 2$ by the dashed curves, and $n = 3$ by the dotted curves, respectively.}
  \item For short-range interactions the asymptotic behaviour of the wave function $\Psi$ is known to be \cite{5}:
  \begin{equation}
  \Psi \to F - K G,
  \end{equation}
  \item where $K$ is the $K$-matrix, which is related to the $S$-matrix by the simple expression $S = (1 + iK)(1 - iK)^{-1}$. In the equation above $F$ and $G$ are column vectors whose $n^{th}$ term is given by \cite{3, 5}
  \begin{equation}
  F_n = \sqrt{\kappa j_{K+\frac{3}{2}}(\kappa \rho) \frac{1}{\rho^{j/2}}} \Phi_n(\rho, \Omega),\end{equation}
  \begin{equation}
  G_n = \sqrt{\kappa \eta_{K+\frac{3}{2}}(\kappa \rho) \frac{1}{\rho^{j/2}}} \Phi_n(\rho, \Omega),\end{equation}
  \item where $n = 1, \cdots, n_0$, $j$ and $\eta$ are the regular and irregular spherical Bessel functions, respectively, $K$ is the hypermomentum associated asymptotically to the adiabatic potential $n$, and $\kappa = \sqrt{2mE/h}$ is the three-body momentum. When $n$ refers to a 1+2 channel Eqs.\cite{11} and \cite{12} have to be modified as given in Eqs.(12) and (13) in Ref.\cite{3}. This asymptotic behaviour determines how the wave function has to be normalized in order to extract from it the $n_0 \times n_0$ $K$-matrix (and therefore the $S$-matrix). When the Coulomb interaction is involved, the asymptotic behaviour of the wave function is still given by \cite{10}, but the Bessel functions contained in the vectors $F$ and $G$ have to be replaced by regular and irregular Coulomb functions whose order and Sommerfeld parameter have to be determined numerically.
  \item As an example, we have considered a simple three-body system made of three identical spinless bosons with total spin and parity $0^+$. We have chosen an arbitrary simple weakly attractive short-range potential between the three particles. The three-body wave function in Eq.\cite{5} has then been computed and normalized as given in Eq.\cite{10}. The corresponding radial wave functions $f_{ni}(\rho)$ are shown in Fig.\cite{2} for a three-body energy of $E = 10$ MeV. The calculation has been performed including three adiabatic terms in the expansion, and for the energy chosen all the channels are open (therefore $n_0 = n_A = 3$ in Eq.\cite{8}). In the figure we show the radial wave functions when the incoming channels are the first adiabatic term ($i = 1$, panel a), the second adiabatic term ($i = 2$, panel b), and the third adiabatic term ($i = 3$, panel c). The functions corresponding to outgoing channels $n =1, 2,$ and $3$, are shown by the solid, dashed, and dotted curves, respectively. Therefore, Fig\cite{2} shows three different continuum wave function for the same system and the same total energy. As we can see in the figure, for each incoming channel $i$ the dominant term is the one with $n = i$. 
\end{itemize}
Fig. 3 Left: Discrete continuum states for the same system as in Fig. 2 for five different sizes of the normalization box. Right: Continuum radial wave functions $f^{(i)}(\rho)$ contained in Eq. (13) for the discrete states $i = 257$ with energy 9.99 MeV (panel a), $i = 258$ with energy 10.02 MeV (panel b), and $i = 259$ with energy 10.03 MeV (panel c) obtained with a discretization box with $L = 400$ fm. The calculation has been done including 3 adiabatic terms in the expansion, and the corresponding functions for $n = 1$, $n = 2$, and $n = 3$ are given by the solid, dashed, and dotted curves, respectively.

3 Discrete continuum states after imposing a box boundary condition.

As mentioned in the introduction, the continuum spectrum can be easily discretized by imposing a box boundary condition. This amounts to solving the coupled set of radial equations given in Eq. (4) imposing the radial wave functions to be zero at a given value $L$ of the hyperradius. This condition is satisfied only by a discrete set of values of the three-body energy $\{E_i\}$, each of them associated to a three-body state whose wave function can be written formally as in Eq. (3):

$$\Psi^{(i)}(x, y) = \frac{1}{\rho^{5/2}} \sum_{n=1}^{\infty} f^{(i)}_n(\rho) \Phi_n(\rho, \Omega),$$  \hspace{1cm} (13)

but with the important difference that the index $i$ in the expression above labels the wave function of the three-body state with discrete energy $E_i$. In Eq. (3) the index $i$ labels the incoming channel.

After discretization, it is simple to see that the energy separation between two consecutive discrete states is given by [6]:

$$\Delta E = \frac{2\pi}{L} \frac{E_i}{\kappa_i},$$  \hspace{1cm} (14)

where $\kappa_i = \sqrt{2mE_i/\hbar}$, and where we have assumed that $i$ is large enough. It is then clear that the larger the size of the box the smaller the separation between the discrete states, which decreases linearly with $L$. For the same reason, the number of discrete states with energy smaller than a given value increases linearly with $L$. This is illustrated in the left part of Fig. 3 where we show the discrete energies smaller than 15 MeV for the same three-body system as in Fig. 2. The result for box sizes of $L = 50$ fm, $100$ fm, $200$ fm, $400$ fm, and $800$ fm (from the top to the bottom in the figure) are shown. We can clearly see that, at the scale of the figure, the discrete energies for $L = 400$ fm and $800$ fm can hardly be distinguished from each other. The number of discrete states with energies smaller than 15 MeV are 33, 72, 153, 315, and 642 when we move from the smallest to the biggest box, which follow very closely the expected linear dependence on $L$.

4 A closer look into the discrete continuum states.

The interesting thing about the discrete energies shown in the left part of Fig. 3 is that they are distributed in a non-uniform way. They actually appear in groups (or bins) of states with rather similar energies. The number of states in each bin is equal to the number of adiabatic channels used in the expansion in Eq. (3), which has been taken equal to 3 in the calculation shown in the figure. This
grouping of the states appears no matter the value of $L$ chosen, as one could see by making a zoom of the figure for the cases with large $L$.

As also seen in the left part of Fig. 3, the bigger the box the closer the states in the same bin to degeneracy. In fact, the rule for the energy separation between states given in Eq. (14) is satisfied not by the individual states, but by the bins themselves. As an example, for a box of 400 fm and $E_i = 80$ MeV, the expected energy separation according to Eq. (14) is of $\sim 0.6$ MeV. When doing the discretization, the energy bins $\{79.17, 79.23, 79.27\}$ MeV and $\{79.81, 79.87, 79.89\}$ MeV are found.

From the discussion above, it is then clear that, first, for an infinitely big box, the bin structure has disappeared, in such a way that we have, for each energy, as many wave functions as possible incoming channels. And second, for a finite box the expected energy separation between the discrete states corresponds to the energy separation between the bins. From these facts, one can almost unavoidably conclude that the discrete states belonging to the same bin do actually describe the full wave function, as given in Eq. (8), at the energy of the bin (whose energy can be approximated by an average of the energies of all the states in the bin). In other words, the wave function contains all the terms associated to all the possible incoming channels. Also, it is important to keep in mind that the discrete continuum states are normalized to 1 inside the discretization box. Therefore, for sufficient large values of $\rho$ the continuum wave function will not in general behave as given in Eq. (10), but instead it will go as:

$$\Psi \rightarrow A(F - KG),$$

where $A$ is a normalization constant $n_0 \times n_0$ matrix, which can be computed as $A = F_b \cdot F_c^{-1}$, where $F_b$ and $F_c$ are, respectively, the matrices containing the radial functions, see Eq. (8), obtained with and without discretization of the continuum. This matrix is found to be independent of $\rho$.

The conclusion discussed above is confirmed when investigating the radial wave functions. Let us consider the same three-body system as the one used in Fig. 2, whose discrete energies have been shown in the left part of Fig. 3. In the right part of the figure, we show in panels (a), (b), and (c) the radial wave functions contained in Eq. (13) for the discrete states with labels $i = 257$, $i = 258$, and $i = 259$, obtained with a discretization box with $L = 400$ fm. The energies of these states are $9.99$ MeV, $10.02$ MeV, and $10.03$ MeV, respectively, which are very similar to the one used in the calculation shown in Fig. 2 (10 MeV). As we can see, the dominant term on each panel are the $n = 1$, $n = 2$, and $n = 3$ terms, respectively, exactly as in Fig. 2. In fact, a quick eye inspection reveals that the shape of the dominant term on each panel is very similar to the corresponding curve in Fig. 2. However, the same quick eye inspection also reveals that the non-dominant terms are different (this is particularly clear in panels b and c). Nevertheless, this fact does not imply that the wave functions in Figs. 2 and 3 are not consistent with each other. The reason is that the wave functions shown in Fig. 2 are normalized

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Same as in Fig. 2 where we have put on top the solid circles that show the discrete continuum wave functions given in Fig. 3 after making the transformation $\Psi \rightarrow A^{-1}\Psi$, where the matrix $A$ is defined according to Eq. (15).}
\end{figure}
according to Eq. (10), and the ones in Fig. 3 are normalized according Eq. (15). This means that a correct comparison between the two wave functions requires to multiply $\Psi$ in Eq. (15) from the left by $A^{-1}$ (or $\Psi$ in Eq. (10) from the left by $A$). When this is done we can really see that the wave functions shown in Fig. 2 and the discretized wave functions shown in Fig. 3 do really describe the same continuum states. This is shown in Fig. 4 where we show the same radial wave functions as in Fig. 2 (thin curves), and where the solid circles are the corresponding discretized functions shown in the right part of Fig. 3 but after multiplying $\Psi$ from the left by $A^{-1}$. The perfect agreement between the thin curves and the solid circles shows that after discretization of the continuum all the possible incoming and outgoing channels are still considered, and no information is actually lost in the process.

5 Summary and conclusions.

In this work we have described two different methods that can be used to compute continuum wave functions, namely, a full continuum treatment normalizing the wave functions according to Eq. (10), or a discretization of the continuum imposing a box boundary condition and normalizing the discrete wave functions to 1 inside the box. In particular, we have considered three-body systems and the wave function has been obtained using the adiabatic expansion method.

When using the first method, it is rather simple to identify the possible incoming and outgoing channels, in such a way that for a given total energy $E$ one can construct the three-body wave function for each of the possible incoming channels, and from it the $K$- (or $S$-) matrix can be obtained. However, when using the second method, just a family of discrete energies and wave functions appear. In principle, for each energy there is only one wave function associated. Since, according to the formal theory, we should have for each energy as many wave functions as open channels, one could easily conclude that in the process of discretization part of the information contained in the total wave function has been lost.

In this work we have shown that this is actually not true. When discretizing the spectrum the discrete states appear organized in bins of states. Each bin has as many elements as open channels are involved in the calculation. Furthermore, the bigger the discretization box the closer the states in each bin to degeneracy. Eventually, for an infinitely big box, all the discrete states in a bin would appear at precisely the same energy. We have seen that, in fact, the wave functions of a bin of discrete states perfectly agree with the ones corresponding to different incoming channels in a non-discretized calculation.

Therefore, after discretizing the spectrum by imposing a box boundary condition the full information about the continuum wave function is preserved. This is particularly important when dealing with systems where the Coulomb interaction is involved (as the three-alpha system, for instance). In this case, the asymptotic behaviour of the wave function is not known analytically, and therefore the correct normalization of the continuum wave function as given in Eq. (10) can become a quite difficult task. However, the discretization procedure does not require knowledge of the asymptotic form of the wave function, and the treatment of a system where the Coulomb interaction enters is formally identical to the one of a system where only short-range interactions are involved.

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