Prismatic Species of Snelson's Tensegrity Structure

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Abstract. Snelson’s prisms are regular minimal tensegrity structures. They are symmetric according to the rotation axis and have two prisms in two parallel planes. These planes, separated by a certain height, have a finite number of vertices related to the number of bars. Evidently, by changing the position of the upper nodes of bars results in a new stable configuration. This paper demonstrates the possibility to obtain different species through one set and identify the structures’ class according to their species.

1. Introduction

The term tensegrity structure refers to a network of tensional and compressive members. The tensional members are designed to be cables, strings or simply elastic material and the compressive are bars or rigid elements. The use of these structures is extensive, as it has appeared in several fields such as aviation and robotics [1-3] because of the controllability, and plays a major role in engineering [4, 5]. However, despite all of these utilities, we only meet one aspect of tensegrity, which is the base of the revolution. Due to its simplicity, a prismatic tensegrity, “Snelson’s prism”, has two regular polygons in two parallel planes. They belong to the class $k$ when $k$ bars interconnect; otherwise, they are the first class. It is the base of many structures obtained by connecting several tensegrity units along their axis. These shapes result in domes [6], deployable cylinders [7] by substituting and increasing cables; towers and plates [8] are the result of additional nodes.

Referring to the literature, in [9] a configuration through a single layer and deriving different sets of bars is introduced. The method is general and allows to obtain for a define number of bars an exact stable configuration. Moreover, a mathematical concept was presented in [10] for the same structure within an analytical solution to obtain, for $v$ bars, a $v$-1 prismatic tensegrity structure. The method refers to planes formed by two bottom nodes and the center of the circle located on top.

In this paper, we show the possibility to obtain different species from an initial set, means $j=1$; and give out the general formulation of permutation matrix. The method consists in the multiplication and rotation operated on the first configuration.

This paper is organized as follow: firstly, we introduce the Snelson’s prims; following, the set of bars where the possibility to obtain different species is labeled. Next, a structure is simulated by variating the twist angle for vertical cables. The conclusion of this work is derived at the end.

2. Snelson’s Tensegrity Structure

A Snelson’s prism is a set of bars connected through cables. The $i$-th bar starts from the bottom vertex $n_i$ and its end is located on the top vertex $n_{v+i}$. The vertices are located on two parallel planes separated
by a defined height $h$ and each plane presents $v$ sides, receiving the name of *prismatic tensegrity*. Hence, the horizontal cables lying on the bottom and upper planes totalize $2v$. Likewise, $v$ vertical cables connect the bottom vertices and the top vertices to stabilize the configuration without any applied load. Therefore, we count $3v$ cables for any Snelson’s prism. For example, nine cables are enough to stabilize a 3-bar Snelson’s prism “*minimal regular tensegrity prism*” [11].

Table 1. bars and cables connecting vertices of a triangular regular prismatic.

|   | bars | horizontal cables | Vertical cables |
|---|------|-------------------|-----------------|
|   | $b_1$ | $b_2$ | $b_3$ | $S_{b1}$ | $S_{b2}$ | $S_{b3}$ | $S_{b4}$ | $S_{b5}$ | $S_{b6}$ | $S_{v1}$ | $S_{v2}$ | $S_{v3}$ |
| bottom vertex | $n_1$ | $n_2$ | $n_2$ | $n_1$ | $n_2$ | $n_3$ | $n_3$ | $n_3$ | $n_1$ | $n_2$ | $n_3$ |
| upper vertex | $n_4$ | $n_5$ | $n_6$ | $n_4$ | $n_5$ | $n_6$ |

Therefore, four parameters describe a set of a prismatic tensegrity structure, as seen on figure 1:
- $r_b$ and $r_u$ radii of bottom and top circles, respectively;
- $v$ (≥3) bars which define the polygons of lower and upper circle;
- a height $h$, which separates the bottom and top circles;
- a twist angle $\alpha$, relating the projection of the vertical cable lying the top of a bar to the bottom vertex.

Mathematically, the bottom node and the upper node of the $i$-th bar are

$$n_i = r_b \begin{bmatrix} \cos 2\pi v^{-1}(i-1) \\ \sin 2\pi v^{-1}(i-1) \end{bmatrix}^T \quad (1)$$

$$n_{v+i} = r_u \begin{bmatrix} \cos(2\pi v^{-1} + \alpha) \\ \sin(2\pi v^{-1} + \alpha) \end{bmatrix}^{hr_u^{-1}} \quad (2)$$

in which $i=1,2,3,\ldots,v$ and $\alpha=\pi(0.5-v^i)$. Then, the whole structure geometry matrix is

$$N = \begin{bmatrix} n_1 & n_2 & \cdots & n_i & \cdots & n_v \ \hline \text{bottom vertex} \end{bmatrix} \mid \begin{bmatrix} n_{v+1} & n_{v+2} & \cdots & n_{v+i} & \cdots & n_{2v} \ \hline \text{upper vertex} \end{bmatrix} \quad (3)$$
From table 1, the bar and cable matrices are

\[ B = [b_1 \ b_2 \ b_3] = NC_b^T \]

\[ S = \begin{bmatrix} s_{h1} & s_{h2} & s_{h3} & s_{h4} & s_{h5} & s_{h6} \\ s_{v1} & s_{v2} & s_{v3} \end{bmatrix} = NC_s^T \]

where \( C_b^T \) and \( C_s^T \) are bar and cable connectivity matrices given as

\[ C_b^T = \begin{bmatrix} -I_v \\ I_v \end{bmatrix} ; \quad C_s^T = \begin{bmatrix} E_1 \ O \ -I_v \\ \frac{c_b}{c_s} \quad \frac{E_2}{c_s} \end{bmatrix} \]

\( I_v \) and \( O \) are \( v \times v \) identity and zero matrix. \( E_1 \) and \( E_2 \) are

\[ E_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \cdots & 0 \\ 0 & 0 & \ddots & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}_{v \times v} \]

\[ E_2 = \begin{bmatrix} O_{v-1 \times 1} & I_v-1 \\ 1 & O_{1 \times v-1} \end{bmatrix}_{v \times v} \]

respectively. For any prismatic tensegrity structure, the sets of geometry and typology parameters remain the same.

3. Species of a Prismatic Tensegrity Structure

They are expressed in terms of the top nodes [10]. One bar can be connected to any bottom node except its corresponding node. Therefore, for \( v \) bars exist more than one structure. Due to the permutation of top nodes, we can reach another structure using multiplication operations. The bottom nodes are duplicated while the top nodes keep changing the position as seen on figure 2. Let’s consider matrix \( P_l \)

**Figure 2.** graphic representation of flux of top node.
and \( P_2 \) are divided into two parts. The first one is the duplicate of the bottom nodal coordinates and the second one is related to \( j \) different species.

\[
N_j = \begin{cases} 
N & \text{if } j = 1 \\
N_{j-1}P_1 & \text{if } j \neq 1 \text{ and } \nu \text{ odd} \\
& \text{for } \nu \text{ even} \\
[N_{[3\times \nu]}][R_z(\beta)N_{[3\times \nu+1, \ldots, 2\nu]}] & \text{if } j = 2 \\
N_{j-2}P_2 & \text{otherwise} 
\end{cases} \tag{8}
\]

in which

\[
[P_1]_{2\nu \times 2\nu} = \begin{bmatrix}
I_{\nu} & O_{\nu \times \nu} \\
O_{2\nu \times \nu} & -I_{\nu/2} & O_{\nu \times \nu/2} & \vdots & 0 \\
0 & O_{\nu \times \nu/2} & -I_{\nu/2} & \vdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & & 0 & I_{\nu/2} \\
\end{bmatrix} \\
[P_2]_{2\nu \times 2\nu} = \begin{bmatrix}
I_{\nu} & O_{\nu \times \nu} \\
O_{2\nu \times \nu} & I_{\nu/2} & 0 & \vdots & 0 \\
0 & 0 & I_{\nu/2} & \vdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& & & 0 & I_{\nu/2} \\
\end{bmatrix} \tag{9}
\]

The matrix \( R_z(\beta) \) gives a set of top nodes rotated about \( z \)-axis an angle \( \beta = \pi \nu^{-1} \).

4. Connectivity Matrix of Species

The related bar and cable matrix are

\[
B_j = N_jC_{\bar{b}}^T; \tag{10}
\]

\[
S_j = N_jC_{s_j}^T; \tag{11}
\]

where the connectivity matrix \( C_{s_j}^T \) has the following form

\[
C_{s_j}^T = \begin{bmatrix}
E_1 & 0 & -I_{\nu} \\
0 & E_1 & E_{2j} \\
0 & E_{2j}^T & C_{s_j}^T \\
\end{bmatrix} \tag{12}
\]

Then;

\[
E_{2j} = \begin{cases} 
E_2 & \text{if } j = 1 \\
E_{2j-1}E_2 & \text{otherwise} 
\end{cases} \tag{13}
\]

5. Simulated Results and Conclusion

The results obtained are in agreement with other literature [9, 10]. The success is the obtention of the twist angle of a vertical cable related to self-equilibrium. It defines also the class of tensegrity. The tensegrities with odd number of bars are class one. Differently to the structure constructed with even number of bars, when \( j = \nu/2 \), the structure is a \( \nu \)-th class tensegrity structure.
Figure 3. simulated prismatic tensegrity structure.

Table 2. twist angle of vertical cables.

| j  | 1 | 2  | 3  | 4  | 5  | 6  |
|----|---|----|----|----|----|----|
| 3  | 30 | -30 | -  | -  | -  | -  |
| 4  | 45 | 0  | -45 |  | -  | -  |
| 5  | 54 | 18 | -18 | -54 |  | -  |
| 6  | 60 | 30 | 0  | -30 | -60 | -  |
| 7  | 64.2857 | 38.5714 | 12.8571 | -12.8571 | -38.5714 | -64.2857 |

The method for constructing different species of prismatic tensegrity structures has been achieved. Different matrix, geometry matrix and typology matrix, are simply obtained from the first configuration. In addition, the bar and vertical cables connectivity matrix remains the same for any configuration while the vertical cables connectivity matrix are multiplied by itself for next configuration.

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**Acknowledgement**

This project is supported by National Natural Science Foundation of China (Grant No. 51605111, 51675114, 51875111).