FROM TWO SPECIES VLASOV-MAXWELL-BOLTZMANN SYSTEM TO MAGNETOHYDRODYNAMICS SYSTEM

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Abstract. In this work, the magnetohydrodynamics system is formally derived from two species Vlasov-Maxwell-Boltzmann system. By employing the hypocoercivity of the linear Boltzmann operator and overcoming the difficulties resulting from the singular Lorentz term, we first obtain the uniform estimates of solutions with respect to the Knudsen number and then derive the magnetohydrodynamics system from the dimensionless Vlasov-Maxwell-Boltzmann system.

Keywords: Vlasov-Maxwell-Boltzmann equation; diffusive limit; magnetohydrodynamics

1. Introduction and Motivation

The plasma where the charged particles in dilute gas move under the influence of the self-consistent electromagnetic field and collisions can be described by the Vlasov-Maxwell-Boltzmann (VMB) system. For two species case with equal mass and charge (e.g. anion and cation), the VMB system on the torus in three dimension is

\[
\begin{aligned}
\partial_t f^+ + v \cdot \nabla_v f^+ + (E + v \times B) \cdot \nabla_x f^+ &= \mathcal{Q}(f^+, f^+) + \mathcal{Q}(f^+, f^-), \\
\partial_t f^- + v \cdot \nabla_v f^- - (E + v \times B) \cdot \nabla_x f^- &= \mathcal{Q}(f^-, f^+) + \mathcal{Q}(f^-, f^-), \\
\varepsilon_0 \mu_0 \partial_t E - \nabla \times B &= -\mu_0 \int_{\mathbb{R}^3} (f^+ - f^-) \, dv, \\
\partial_t B + \nabla \times E &= 0, \\
\text{div} B &= 0, \\
\text{div} E &= \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} (f^+ - f^-) \, Mdv.
\end{aligned}
\]

In the above system, \( f^\pm = f^\pm(t, x, v)(x \in T^3, v \in \mathbb{R}^3) \) are the number density of anion or cation respectively. \( B \) and \( E \) are electric and magnetic field. The \( \mathcal{Q}(f^\pm, f^\pm) \) describe the collision between \( f^\pm \) and \( f^\pm \) and will be detailed later. In (1), the first two equations are kinetic equations and describe that the anions or cations move along their trajectories under the Lorentz force \((E + v \times B) \cdot \nabla_x f^\pm)\) and collisions. The third and forth equations are Maxwell’s system which models how electromagnetic field are self-generated. Specially, \( \varepsilon_0 \) and \( \mu_0 \) are the electric permittivity and magnetic permeability respectively. Furthermore, \( \varepsilon_0 \) and \( \mu_0 \) satisfy

\[ c^2 \varepsilon_0 \mu_0 = 1, \quad c \text{ is the speed of light}. \]

The coupling of kinetic system and Maxwell system shows that the moving charged particles in electromagnetic field change the status of the current which in turn affect the creation of the field.

The collision operator \( \mathcal{Q} \)

\[ \mathcal{Q}(f, g) = \int_{\mathbb{R}^3 \times S^2} (f'(g'_*) - fg_*)b(v - v_*, \omega)dv_*d\omega, \]

with \( f' = f(v'), \) \( g'_* = g(v'_*), \) \( g = g(v_*). \) Here, \( v \) and \( v_* \) are the velocities of two particles before the elastic collision and \( v' \) and \( v'_* \) denotes velocities after collision. Moreover, the velocities of particles satisfy

\[ \begin{aligned}
\begin{cases}
&v + v_* = v' + v'_*, \\
&|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2,
\end{cases}
\end{aligned} \]

and

\[ \begin{aligned}
\begin{cases}
&v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \omega, \\
v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \omega, \quad \omega \in S^2.
\end{cases}
\end{aligned} \]

In this work, the cross section \( b \) is assumed to be hard potential, i.e, the exists constant \( C_b \) such that

\[ b(v - v_*, \omega) = C_b |v - v_*| \cos \hat{\theta}, \quad \cos \hat{\theta} = \langle \frac{v - v_*}{|v - v_*|}, \omega \rangle. \]

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\[ b(v - v_*, \omega) = C_b |v - v_*| \cos \hat{\theta}, \quad \cos \hat{\theta} = \langle \frac{v - v_*}{|v - v_*|}, \omega \rangle. \]
Moreover, there exists some positive constant $C_b$ such that
\[ \forall z \in [-1, 1], \ |m(z)| \leq C_b, \ |m'(z)| \leq C_b. \]

Furthermore, $Q(f, f)$ enjoys the following properties:
\[ \int_{\mathbb{R}^3} Q(f, f) dv = 0, \quad \int_{\mathbb{R}^3} Q(f, f) u dv = 0, \quad \int_{\mathbb{R}^3} Q(f, f) \frac{|u|^2 - 3}{2} dv = 0. \]

From the point view of physics, the small parameters (such as Debye number and Knudsen number) play a key role in describing the transition phenomenon. The Knudsen number which is defined as the ratio of the mean free path length of the molecular to representative physical length scale determines whether model of statistical mechanics or continuum mechanics is used. From the point view of physics, while the Knudsen number is very small, the dilute gas under consideration are fluid regimes. In this work, we only consider the incompressible fluid regimes where the dynamics behavior of the charged particles are described by Navier-Stokes type equations. According to different kinds of scalings of the electromagnetic field in the VMB system, the electric and magnetic field may vanish or preserve in the limiting fluid system. The limiting fluid system could be Navier-Stokes (NS) equations (the affect of electromagnetic is negligible), Navier-Stokes-Poisson (NSP) system (the affect of the magnetic field is negligible), resistive magnetohydrodynamics (MHD) system (the effect of the electric field is negligible) and Navier-Stokes-Maxwell (NSW) system (both fields are important). The formal derivation of NS, NSP and NSW system from VMB system can be found in [2]. The dimensionless VMB system to MHD system can be found in [15].

In this work, we concern with the rigorous justification work. Based on our knowledge, the Navier-Stokes-Maxwell, Navier-Stokes-Poisson and Navier-Stokes’ limits of VMB system have been rigorously verified in the current literature (see [17, 16, 30, 14] for instance). The Navier-Stokes-Poisson limit of two species VMB system was verified in [14] where the magnetic field in VMB system disappears in the limiting system. The Navier-Stokes-Maxwell limit can be found in [17, 16] where both electric field and magnetic field preserve in the limiting system. The resistive magnetohydrodynamics system is also a very important model in describing the status of plasma and has not been rigorously derived from VMB system. Besides, there exist some technical difficulties during the justification. The concern of this work is to justify the transition of system (4) to the following MHD system:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= (\nabla \times B) \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} u &= \text{div} B = 0, \quad \rho + \theta = 0, \\
\partial_t B - \frac{1}{\sigma} \Delta B - \nabla \times (u \times B) &= 0.
\end{align*}
\]

In (3), $u$ is the velocity of fluid, $\rho$ and $\theta$ are density and temperature respectively. $P$ is the pressure. The positive constants $\nu$, $\kappa$ and $\sigma$ have an explanation from the point view of the kinetic system (see (22)). Compared to the Maxwell system in (4) where there exist both electric and magnetic field, only magnetic field preserves as the Knudsen go to zero.

The formal derivation of MHD equations from VMB system was performed in [15]. But the Lorentz force term in their dimensionless system is extremely singular and hard to be controlled. In what follows, we try to derive a new dimensionless VMB system to overcome this difficulty. To deduce the MHD system from the VMB system, we need to perform the dimensionless analysis to system (1). The dimensionless Boltzmann equation (see [27] and [2] for instance) is
\[
\text{St} \partial_t f + v \cdot \nabla_x f = \frac{1}{\text{Kn}} Q(f, f),
\]
where St is the Strouhal number and Kn is the Knudsen number. In the incompressible regimes,
\[
\text{St} = \text{Kn} = \epsilon.
\]

For the dimensionless analysis, we first introduce the new variables $\hat{t}$, $\hat{x}$, $\hat{v}$ with
\[
\hat{t} = \frac{t}{\epsilon}, \quad \hat{x} = \frac{x}{\epsilon}, \quad \hat{v} = \frac{v}{\epsilon}.
\]

Inspired by [15], to get the MHD system, the electric permittivity $\epsilon_0$ is assumed to be very small and magnetic permeability $\mu_0$ be $O(1)$ (in fact, $\mu_0$ is a small constant, without loss of generality, we set it be one), i.e.,
\[
\epsilon_0 = \epsilon, \quad \mu_0 = 1.
\]
Under the above setting, the speed of light tends to infinity as $\epsilon$ goes to zero. Since we are performing the dimensionless analysis, this unrealistic discrepancy should be understood as asymptotic regimes where proper physical approximations are still valid.

Then, the dimensionless functions for (1) are defined as follows:

$$f^\pm(t, x, v) = \epsilon^3 f^\pm(\tilde{t}, \tilde{x}, \tilde{v}), \quad E(t, x) = E_\epsilon(\tilde{t}, \tilde{x}), \quad B(t, x) = B_\epsilon(\tilde{t}, \tilde{x}),$$

$$\tilde{Q}(\tilde{f}, \tilde{f}) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} (\tilde{f}^* f^* - \tilde{f} f) \tilde{b}(\tilde{v}, \omega) d\tilde{v} d\omega, \quad b(v, \omega) = \frac{1}{2} \tilde{b}(\tilde{v}, \omega).$$

Plugging the above relations into (1) and dropping the tilde, we can obtain that

$$\begin{align*}
&\epsilon \partial_t f^+_{\epsilon} + v \cdot \nabla_x f^+_{\epsilon} + (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v f^+_{\epsilon} = \frac{1}{\epsilon} Q(f^+_{\epsilon}, f^+_{\epsilon}) + \frac{1}{\epsilon} Q(f^+_{\epsilon}, f^-_{\epsilon}), \\
&\epsilon \partial_t f^-_{\epsilon} + v \cdot \nabla_x f^-_{\epsilon} - (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v f^-_{\epsilon} = \frac{1}{\epsilon} Q(f^-_{\epsilon}, f^+_{\epsilon}) + \frac{1}{\epsilon} Q(f^-_{\epsilon}, f^-_{\epsilon}), \\
&\epsilon \partial_t E_\epsilon - \nabla \times B_\epsilon = -\frac{1}{\epsilon \tau} \int_{\mathbb{R}^3} (f^+_{\epsilon} - f^-_{\epsilon}) v d\nu, \\
&\partial_t B_\epsilon + \nabla \times E_\epsilon = 0, \\
&\text{div } B_\epsilon = 0, \quad \text{div } E_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^3} (f^+_{\epsilon} - f^-_{\epsilon}) M d\nu.
\end{align*}$$

Before comparing the difference between (4) and the dimensionless VMB system in [15, Equ.(3.10)], denoting

$$F_\epsilon = f^+_{\epsilon} + f^-_{\epsilon}, \quad H_\epsilon = f^+_{\epsilon} - f^-_{\epsilon},$$

then (4) can be rewritten as

$$\begin{align*}
&\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon + (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v F_\epsilon = \frac{1}{\epsilon} Q(F_\epsilon, F_\epsilon), \\
&\epsilon \partial_t H_\epsilon + v \cdot \nabla_x H_\epsilon + (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v H_\epsilon = \frac{1}{\epsilon} Q(H_\epsilon, F_\epsilon), \\
&\epsilon \partial_t E_\epsilon - \nabla \times B_\epsilon = -\frac{1}{\epsilon \tau} \int_{\mathbb{R}^3} H_\epsilon v d\nu, \\
&\partial_t B_\epsilon + \nabla \times E_\epsilon = 0, \\
&\text{div } B_\epsilon = 0, \quad \text{div } E_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^3} H_\epsilon d\nu.
\end{align*}$$

From [15], the counterpart of (5) in their work is as follows

$$\begin{align*}
&\epsilon \partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon + (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v G_\epsilon = \frac{1}{\epsilon} Q(F_\epsilon, F_\epsilon), \\
&\epsilon \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon + \frac{1}{\epsilon} (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v F_\epsilon = \frac{1}{\epsilon} Q(G_\epsilon, F_\epsilon), \\
&\epsilon \mu_0 \partial_t E_\epsilon - \nabla \times B_\epsilon = -\mu_0 \int_{\mathbb{R}^3} G_\epsilon v d\nu, \\
&\partial_t B_\epsilon + \nabla \times E_\epsilon = 0, \\
&\text{div } B_\epsilon = 0, \quad \text{div } E_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^3} G_\epsilon d\nu.
\end{align*}$$

In the above system, $G_\epsilon$ is the difference of $f^+_{\epsilon}$ and $f^-_{\epsilon}$. Compared to the equations of $G_\epsilon$ in (6) and $H_\epsilon$ in (5), there exists one more $\frac{1}{\epsilon}$ before the Lorentz force in the equation of $G_\epsilon$. In Sec.3, we shall show the Lorentz force in system (4) is also very singular and barely controlled. With one more $\frac{1}{\epsilon}$, the method of this work completely fails. The rigorous justification of (6) remains open.

The goal of this work is to derive the MHD system (3) from the dimensionless system (4) in the classic solution framework. The key ingredient of this work is the uniform estimates of solutions to (4) with respect to the Knudsen number. This is the first rigorous derivation work of MHD system from VMB equations. This work can be seen as kinetic approach to [1] where the MHD system is derived rigorously from the two fluid Navier-Stokes-Maxwell system in weak solution framework. We follows the “mixed norm” strategies used in used in [7, 26] to obtain the uniform estimates. The advantage of this framework is to recover the dissipative estimates of macroscopic part within less pages. But their strategies can not directly used to this work. Indeed, the works [7, 26] are just for Boltzmann equation. In this work, we consider the VMB system. There exists very singular Lorentz force term. Furthermore, the norm used in [7] is anisotropic. It makes the Lorentz term harder to be controlled. More details will be explained in Remark 3.3 and Sec. 3.1.

The rest part of this work is arranged as follows. Section 2 is devoted to the notations, the assumption of the linear Boltzmann operator and the assumption of the initial data. The main results and the difficulties of this work can be found in Section 3. The Section 4 consists in deducing the uniform estimates of solutions. The MHD limit of (4) is verified in Sec.5.
2. Preliminaries

Firstly, we try to deduce the fluctuations system of (4). Letting $M$ be the global Maxwellian, under the Navier-Stokes scalings for $G_ε$ and $F_ε$,

$$F_ε = M(1 + εf_ε), \quad H_ε = εMh_ε,$$

and denoting

$$j_ε = \frac{1}{ε} \int_{\mathbb{R}^3} h_ε vMdv, \quad n_ε = \int_{\mathbb{R}^3} h_ε MdV,$$

we can infer the fluctuation system:

$$\begin{aligned}
\partial_t f_ε + \frac{1}{ε} v \cdot \nabla x f_ε + \frac{1}{ε} L(f_ε) &= E_ε \cdot v \cdot h_ε - (E_ε + \frac{1}{ε} v \times B_ε) \cdot \nabla x h_ε + \frac{1}{ε} \Gamma(f_ε, f_ε), \\
\partial_t h_ε + \frac{1}{ε} v \cdot \nabla x h_ε - \frac{1}{ε} E_ε \cdot v + \frac{1}{ε} L(h_ε) &= E_ε \cdot v \cdot f_ε - (E_ε + \frac{1}{ε} v \times B_ε) \cdot \nabla x f_ε + \frac{1}{ε} \Gamma(h_ε, f_ε), \\
\varepsilon \partial_t E_ε - \nabla \times B_ε &= -j_ε, \\
\varepsilon \partial_t B_ε + \nabla \times E_ε &= 0, \\
\text{div} B_ε &= 0, \quad \varepsilon \cdot \text{div} E_ε = n_ε,
\end{aligned}$$

(7)

where the linear Boltzmann operator $L$ and $\Lambda$ are defined as follows

$$-L(w) = (M)^{-1} Q(Mw, M), \quad -\Lambda(w) = (M)^{-1} (Q(Mw, M) + Q(M, Mw)) .$$

By simple computation, (7) satisfies the following global conservation laws:

$$\frac{d}{dt} \int_{\mathbb{R}^3} (u_ε + \varepsilon E_ε \times B_ε)(t)dx = 0, $$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \left(\theta_ε + \varepsilon \frac{|E_ε|^2 + |B_ε|^2}{3}\right)(t)dx = 0, $$

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho_ε(t)dx = \frac{d}{dt} \int_{\mathbb{R}^3} n_ε(t)dx = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^3} B_ε(t)dx = 0,$$

where

$$\rho_ε = \int_{\mathbb{R}^3} f_ε MdV, \quad u_ε = \int_{\mathbb{R}^3} f_ε vMdV, \quad \theta_ε = \int_{\mathbb{R}^3} f_ε \left(\frac{|v|^2 - 3}{3}\right) MdV, \quad n_ε = \int_{\mathbb{R}^3} h_ε MdV.$$  

2.1. Notations and Terms. $\nabla^i f = \partial^i_{x_1} \partial^i_{x_2} \partial^i_{x_3} f \left(\sum_{k=1}^{3} i_k = i\right)$ is the $i$-th derivative of $f$ with respect to $x$. We denote by $\nabla f$ the gradient of scalar function $f$. $\nabla^i f$ and $\nabla_v f$ can be defined in the same way. The Sobolev norm of $f$ are defined like this:

$$\|f\|^2_{L^2} = \int_{\mathbb{R}^3} f^2 Mdv, \quad \|f\|^2_{H^2} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^2 Mdvdx, $$

$$\|f\|^2_{H^k} = \sum_{k=0}^{s} \|\nabla^k f\|^2_{L^2}, \quad \|f\|^2_{H^k} = \sum_{k=0}^{s} \|\nabla^k f\|^2_{L^2},$$

Denoting $\hat{v} = \sqrt{1 + |v|}$, the norms with weight on $v$ are defined as follows:

$$\|f\|^2_{L^2} = \|f\hat{v}\|^2_{L^2}, \quad \|f\|^2_{H^k} = \sum_{k=0}^{s} \|\nabla^k f\hat{v}\|^2_{L^2},$$

Thus, the positive constant $C$ is independent of $\varepsilon$ and different from line to line. $a \lesssim b$ means that there exists some positive constant $C$ such that $a \leq Cb$. We also use $C_0$ to indicate that the constant is dependent of the initial data.
2.2. **Assumption on the linear operators.** This section is on hypocoercivity theory of the linear Boltzmann operator. The assumptions in this subsection are the same to those in [26] and [7]. The verification of the assumption of those assumptions can be found in [25, Sec.5.4].

**H1 (Coercivity and general controls.)** The Boltzmann operator \( \mathcal{L} \) and \( \mathcal{L} \) are self-adjoint operator from \( L^2_v \) to \( L^2_v \) with the following decomposition

\[
\mathcal{L} = -K + \Lambda, \quad \mathcal{L} = -\Phi + \Lambda,
\]

where \( \Lambda \) is a coercive operator. Furthermore, \( \Lambda \) satisfies the following properties.

- For any \( h, g \in L^2_v \), there exists some \( \lambda_0 > 0 \) such that
  \[
  \lambda_0 \| h \|^2_{L^2_v, \Lambda} \leq \int_{\mathbb{R}^3} \Lambda(h) \cdot h \, Mdv \leq C \| h \|^2_{L^2_v, \Lambda},
  \]

and

\[
| \int_{\mathbb{R}^3} \Lambda(h) \cdot g \, Mdv | \leq C \| h \|_{L^2_v, \Lambda} \| g \|_{L^2_v, \Lambda}.
\]

- With respect to the derivative of \( v \), the operator \( \Lambda \) admits “a defect of coercivity”, i.e., there exist some strictly positive constant \( 1 > \delta > 0 \) and \( C_\delta \) such that

\[
\int_{\mathbb{R}^3} | \int_{\mathbb{R}^3} \nabla^i_v \nabla^j_x \Lambda(h) \cdot \nabla^i_v \nabla^j_x h \, Mdv | \, dx \geq \delta \| \nabla^i_v \nabla^j_x h \|^2_{L^2_\Lambda} - C_\delta \| h \|^2_{H^i+j-1}, \quad i \geq 1.
\]

**H2 (Mixing property in velocity.)** This assumptions are about \( \mathcal{L} \) and \( \mathcal{L} \). For any \( 1 > \delta > 0 \), there is some constant \( C_\delta > 0 \) such that

\[
| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^i_v \nabla^j_x K(h) \cdot \nabla^i_v \nabla^j_x h \, Mdv | + | \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^i_v \nabla^j_x \Phi(h) \cdot \nabla^i_v \nabla^j_x h \, Mdv | \leq C_\delta \| h \|^2_{H^i+j-1} + \delta \| \nabla^i_v \nabla^j_x h \|^2_{L^2_\Lambda}, \quad i \geq 1.
\]

**H3 (Relaxation to the local equilibrium.)** The operators \( \mathcal{L} \) and \( \mathcal{L} \) are closed and self-adjoint operators in \( L^2_v \) space. Moreover,

\[
\text{Ker} \mathcal{L} = \text{Span}\{1, v_1, v_2, v_3, \frac{|v|^2-3}{2}\}, \quad \text{Ker} \mathcal{L} = \text{Span}\{1\}.
\]

Furthermore, \( \mathcal{L} \) and \( \mathcal{L} \) satisfy “local coercivity assumption”:

\[
\int_{\mathbb{R}^3} \mathcal{L}(g) \cdot g \, Mdv \geq \| g - \mathcal{P} g \|^2_{L^2_v, \Lambda}, \quad \int_{\mathbb{R}^3} \mathcal{L}(h) \cdot h \, Mdv \geq \| h - \mathcal{P} h \|^2_{L^2_v, \Lambda},
\]

where \( \mathcal{P} \) is the projection operator of \( \mathcal{L} \) and \( \mathcal{L} \) onto their kernel space respectively, i.e.,

\[
\mathcal{P} g = \int_{\mathbb{R}^3} g \, Mdv + v \cdot \int_{\mathbb{R}^3} g v \, Mdv + \frac{|v|^2-3}{2} \int_{\mathbb{R}^3} g \, Mdv, \quad \mathcal{P} h = \int_{\mathbb{R}^3} h \, Mdv.
\]

Besides, we also assume that

\[
| \int_{\mathbb{R}^3} f \cdot \mathcal{L}(g) \, Mdv | \leq C \| f \|_{L^2_v, \Lambda} \| g \|_{L^2_v, \Lambda}, \quad | \int_{\mathbb{R}^3} f \cdot \mathcal{L}(g) \, Mdv | \leq C \| f \|_{L^2_v, \Lambda} \| g \|_{L^2_v, \Lambda}, \quad \forall f, \ g \in L^2_v.
\]

**H4 (Control on the second order operator.)** This assumption is on \( \Gamma(g, g) \) and \( \Gamma(g, h) \).

- For any \( g, h \in L^2_v \), \( \Gamma(g, g) \in \text{Ker}(\mathcal{L}) \), \( \Gamma(g, h) \in \text{Ker}(\mathcal{L}) \).

- For the non-linear operator and \( s \geq 3 \)

\[
| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^i_x \Gamma(g, h) \cdot f \, Mdv dx | \lesssim \| (g, h) \|_{H^i} \| (g, h) \|_{H^j \Lambda} \| f \|_{L^2_\Lambda},
\]

\[
| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^i_x \nabla^j_x \Gamma(g, h) \cdot f \, Mdv dx | \lesssim \| (g, h) \|_{H^i} \| (g, h) \|_{H^j \Lambda} \| f \|_{L^2_\Lambda}, \quad i \geq 1, \ s = i + j.
\]

**Remark 2.1.** In the general case, the lower bound \( \lambda_0 \) is determined by the collision frequency in (10). To avoid for using too many notations and without loss of generality, we assume the lower bound \( \lambda_0 \) to be one in the rest part of this paper.
2.3. Assumption on the initial data. Recalling
\[ \rho_\epsilon = \int_{\mathbb{R}^3} f_\epsilon \rho_\epsilon, \quad u_\epsilon = \int_{\mathbb{R}^3} f_\epsilon u_\epsilon, \quad \theta_\epsilon = \int_{\mathbb{R}^3} \left( \frac{|u_\epsilon|^2}{2} - 1 \right) f_\epsilon, \quad n_\epsilon = \int_{\mathbb{R}^3} h_\epsilon, \]
similar to \([7, 12]\), we can assume the initial data
\[ \int_{\mathbb{R}^3} (u_\epsilon + \epsilon E_\epsilon \times B_\epsilon)(0) dx = 0, \]
\[ \int_{\mathbb{R}^3} (\theta_\epsilon + \epsilon \frac{|E_\epsilon|^2 + |B_\epsilon|^2}{3})(0) dx = 0, \]
\[ \int_{\mathbb{R}^3} \rho_\epsilon(0) dx = \int_{\mathbb{R}^3} n_\epsilon(0) dx = 0, \quad \int_{\mathbb{R}^3} B_\epsilon(0) dx = 0. \]
The assumption (18) means that the initial data of (4) are with the same mass, total momentum and energy to the steady state \((M, M, 0, 0)\). Furthermore, from (8), we can infer that the solution preserves these properties all the time.

3. Main results

Define
\[ \mathcal{H}_\epsilon^s(t) := \|(f_\epsilon, g_\epsilon, B_\epsilon, \sqrt{\epsilon} E_\epsilon)\|_{\mathcal{H}_\epsilon^s}^2 + \epsilon^2 \|(\nabla_v f_\epsilon, \nabla_v g_\epsilon)\|_{\mathcal{H}_\epsilon^{-1}}^2, \]
\[ \mathcal{D}_\epsilon^s(t) := \|(f_\epsilon, h_\epsilon)\|_{\mathcal{H}_\epsilon^s}^2 + \|(E_\epsilon, B_\epsilon)\|_{\mathcal{H}_\epsilon^{-1}}^2 + \frac{1}{\epsilon^2} \|(f_\epsilon^\perp, h_\epsilon^\perp)\|_{\mathcal{H}_{0, \epsilon}}^2 \]
\[ + \frac{1}{\epsilon^2} \|n_\epsilon\|_{\mathcal{H}_{-1, \epsilon}}^2. \]

**Theorem 3.1.** Under the assumption in the section 2.2 and the assumption (18) on the initial data, there exists some small enough constant \(c_0\) such that if the initial data \((f_\epsilon(0), g_\epsilon(0), E_\epsilon(0), B_\epsilon(0))\) satisfy
\[ \mathcal{H}_\epsilon^s(0) \leq c_0, \quad s \geq 3, \]
then system (7) admit a unique global classic solution \((f_\epsilon, h_\epsilon, B_\epsilon, E_\epsilon)\) satisfying for any \(t > 0\)
\[ \sup_{0 \leq s \leq t} \mathcal{H}_\epsilon^s(t) + \frac{1}{\epsilon^2} \int_0^t \mathcal{D}_\epsilon^s(\tau) d\tau \leq \frac{c_0}{c_1} \mathcal{H}_\epsilon^s(0), \quad \forall \epsilon \in (0, 1], \]
where \(c_1\) and \(c_0\) are positive constants only dependent of the Sobolev embedding constant.

**Remark 3.2.** We use a equivalent norm \(\tilde{\mathcal{H}}_\epsilon^s\) (86) instead of \(\mathcal{H}_\epsilon^s\) to obtain the prior estimate (19). The constants \(c_1\) and \(c_0\) come from the equivalent relation of \(\tilde{\mathcal{H}}_\epsilon^s\) and \(\mathcal{H}_\epsilon^s\), see (90). Furthermore, since we have set the lower bound in (10) to be one (see Remark 2.1), in the general case, \(c_0\) is dependent of the lower bound in (10).

**Remark 3.3.** Noticing in the definition of \(\mathcal{D}_\epsilon^s\), we lose order one derivative (with \(x\)) of electromagnetic field. But in the Lorentz term, the exists one extra order one derivative (with \(v\)). Owing to these two facts, the Lorentz force are not easy to bound. On the other hand, the key point of the proof is to obtain energy estimates like:
\[ \frac{d}{dt} \mathcal{H}_\epsilon^s + \mathcal{D}_\epsilon^s \leq \mathcal{D}_\epsilon^s(t) \mathcal{H}_\epsilon^s(t). \]
Noticing that there exists \(\sqrt{\epsilon}\) before \(E_\epsilon\) in \(\mathcal{H}_\epsilon^s\), it brings difficulties during the proof. Taking the first equation of (4) for example, during the energy estimates, there will exists terms like \(-\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^s_x E_\epsilon \cdot \nabla_v (Mh_\epsilon) \nabla^s_x f_\epsilon dx dv\) which is not easy to bound by \(\mathcal{D}_\epsilon^s(t) \sqrt{\mathcal{H}_\epsilon^s(t)}\). We must split \(\epsilon\) from \(-\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla^s_x E_\epsilon \cdot \nabla_v (Mh_\epsilon) \nabla^s_x f_\epsilon dx dv\) to close the energy estimates. See Sec. 3.1 for more details and strategies.

Furthermore, from \([2, \text{pp.}19]\)(where \(\alpha = \epsilon, \beta = \gamma = 1\)), the dimensionless VMB system(for strong interaction \(\delta = 1\)) to Navier-Stokes-Maxwell system is as follows:
\[ \begin{aligned}
&\partial_t f_\epsilon^+ + v \cdot \nabla_x f_\epsilon^+ + (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v f_\epsilon^+ = \frac{1}{\epsilon} Q(f_\epsilon^+, f_\epsilon^-) + \frac{1}{\epsilon^2} Q(f_\epsilon^+, f_\epsilon^-), \\
&\partial_t f_\epsilon^- + v \cdot \nabla_x f_\epsilon^- - (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v f_\epsilon^- = \frac{1}{\epsilon} Q(f_\epsilon^-, f_\epsilon^+) + \frac{1}{\epsilon^2} Q(f_\epsilon^-, f_\epsilon^+), \\
&\partial_t E_\epsilon - \nabla \times B_\epsilon = -\frac{1}{\epsilon^2} \int_{\mathbb{R}^3} (f_\epsilon^+ - f_\epsilon^-) dv, \\
&\partial_t B_\epsilon + \nabla \times E_\epsilon = 0, \\
&\text{div} B_\epsilon = 0, \quad \text{div} E_\epsilon = \frac{1}{\epsilon^2} \int_{\mathbb{R}^3} (f_\epsilon^+ - f_\epsilon^-) dv.
\end{aligned} \]
Although the kinetic parts of (20) and (4) are the same, the Lorentz force in (4) are harder to be bounded than that in (20). Indeed, there exists $\epsilon$ before $\partial_t E_\epsilon$ in the third equation of (4). Due to this extra coefficient $\epsilon$, there is no useful estimate of electric field. This makes the Lorentz term are harder to be bounded. See Sec. 3.1 for more details.

Before stating the theorem on fluid limit, we introduce the following $A(v)$ and $B(v)$, vector $\tilde{v}$

$$A(v) = v \otimes v - \frac{|v|^2}{2} I, \quad B(v) = v(\frac{|v|^2}{2} - \frac{2}{5}), \quad \mathcal{L} \hat{A}(v) = A(v), \quad \mathcal{L} \hat{B}(v) = B(v), \quad L \tilde{v} = v. \quad (21)$$

Then, denoting

$$\nu = \frac{1}{15} \sum \int_{|s|+|\xi|+|\eta|+|\zeta| \leq 3} A_{ij} \hat{A}_{ij} \mathrm{Md}v, \quad \kappa = \frac{2}{15} \sum \int_{|s|+|\xi|+|\eta|+|\zeta| \leq 3} B_{ij} \hat{B}_{ij} \mathrm{Md}v, \quad \sigma = \frac{1}{3} \int \hat{v} \cdot v \mathrm{Md}v, \quad (22)$$

these three strictly positive constants will appear in the MHD system. Furthermore, let $u_0, \theta_0, n_0, E_0, B_0 \in H^s_x$ and satisfy (up to a subsequence)

$$\mathbf{P} u_\epsilon(0) \to u_0, \quad \frac{3}{2} \theta_\epsilon(0) - \frac{2}{5} \rho_\epsilon(0) \to \theta_0, \quad B_\epsilon(0) \to B_0, \text{ in } H^{s-1}_x.$$

where $\mathbf{P}$ is the Leray projector.

**Theorem 3.4 (Fluid limit).** Under the assumption in the section 2.2 and the assumption (18) on the initial data, for the solutions $f_\epsilon, \, h_\epsilon, \, E_\epsilon, \, B_\epsilon$ constructed in Theorem 3.1, it follows that for any $T > 0$

$$f_\epsilon \to \rho + u \cdot v + \frac{|v|^2}{2} \theta, \quad h_\epsilon \to 0, \quad B_\epsilon \to B, \text{ in } L^2(0, T; H^s_x), \quad \text{(23)}$$

where $\rho, \, u, \, \theta, \, B$ (belonging to $L^\infty((0, \infty); H^s_x)$) are strong solutions to the following MHD system:

$$\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P = (\nabla \times B) \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0, \\
\text{div} u = \text{div} B = 0, \quad \rho + \theta = 0, \\
\partial_t B - \frac{1}{\sigma} \Delta B = \nabla \times (u \times B), \\
u(0) = u_0, \quad \theta(0) = \theta_0, \quad B(0) = B_0.
\end{aligned}$$

Furthermore, for any $\tau > 0$, we can infer

$$\mathbf{P} u_\epsilon \to u, \quad \frac{3}{2} \theta_\epsilon - \frac{2}{5} \rho_\epsilon \to \theta, \text{ in } C([\tau, +\infty); H^{s-1}_x). \quad (24)$$

**Remark 3.5.** The derivation of system (3) is based on the approximate conservation laws. Here, we comments on the Ohm’s law. From Remark 3.3, the kinetic parts of (4) and (20) are the same, in [30] [2, pp.61], the Ohm’s law derived from (20) is

$$j = \sigma(E + u \times B + \nabla n) - nu.$$

Noticing that there exists a coefficient $\frac{1}{\epsilon}$ before the dissipative energy estimates of $n_\epsilon$ in the definition of $D^\epsilon_*$, for (4), we can infer that

$$n = 0.$$

From the third equation of (4), as Knudsen number goes to zero, we can infer that

$$j = \nabla \times B.$$

All together, we can verify the Ohm’s law for MHD system

$$j_\epsilon \to j = \nabla \times B = \sigma(E + u \times B), \text{ in the distributional sense.}$$

This is how we can recover the magnetic field equation in (3).

**Remark 3.6.** If the initial data are well-prepared, i.e.,

$$f_\epsilon(0) = \rho_0 + u_0 \cdot \frac{|v|^2}{2} \theta_0, \quad \text{div} u_0 = 0, \quad \rho_0 + \theta_0 = 0,$$

then (24) can be improved to

$$\mathbf{P} u_\epsilon \to u, \quad \frac{3}{2} \theta_\epsilon - \frac{2}{5} \rho_\epsilon \to \theta, \text{ in } C([0, +\infty); H^{s-1}_x).$$

### 3.1. Historical background, difficulties and novelty.
3.1.1. Historical background strategies. There are two ways of justifying the hydrodynamics limit of the Boltzmann equation and its coupled system. One is based on the renormalized solution. We refer to [4, 8, 10, 20, 22, 23, 24] for the work on the existence of renormalized solutions and fluid limit in renormalized solutions work. The other is in the classic solution framework. The existence of classic solution to VMB system can be found in [9, 12, 28]. Basically, there are three strategies of verifying rigorously the fluid limits of the Boltzmann equation and its coupled system: spectral analysis of the semi-group(see [5, 21]) , Hilbert expansion methods (see [13, 29, 14, 17]) and convergence method based on uniform estimates (see [18, 7, 16, 11]). The Navier-Stokes limit of the Boltzmann equation can be found in [5, 7, 13, 18]. The diffusive limit of the Valsov-Poisson-Boltzmann(VPB) equation was inv estigated in [11, 19, 21, 29].

As mentioned before in the Introduction, the key step towards to the justification is the uniform estimates. From the local coercivity properties of the linear Boltzmann operators ((14)), there is only dissipative estimates for the microscopic parts ( \( f_\varepsilon - \mathcal{P} f_\varepsilon \) and \( h_\varepsilon - \mathcal{P} h_\varepsilon \)). To get inequalities like

\[
\frac{d}{dt}\mathcal{H}_\varepsilon^s(t) + \mathcal{D}_\varepsilon^s(t) \leq \mathcal{H}_\varepsilon^s(t)\mathcal{D}_\varepsilon^s(t),
\]

we must obtain the dissipative estimates of the macroscopic parts \( \mathcal{P} f_\varepsilon \) and \( \mathcal{P} h_\varepsilon \). To achieve this, one idea is to employ the Grad’s 13 moment equations (see \( P \) we must obtain the dissipative estimates of the macroscopic parts

\[
\frac{d}{dt}\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x f_\varepsilon \cdot \nabla_v f_\varepsilon M dv dx + \frac{1}{\varepsilon}\|\nabla_x f_\varepsilon\|^2_{L^2} \leq \cdots .
\] (25)

In the above equation, there exists dissipative estimates for the macroscopic part. The advantage of this framework is that we can recover the macroscopic parts in one simple inequality to avoid using the Grad’s 13 moment equations which is quite involved and hard to bound for the dimensionless VMB system (see the almost one hundred pages work [16] on Navier-Stokes-Maxwell limit of VMB system). But for the dimensionless system, the norm used in [7] is anisotropic. It brings new difficulties in the process of bounding the singular Lorentz term.

3.1.2. difficulties. The difficulties in obtaining the uniform estimates come from the very singular Lorentz force term and the hyperbolicity of Maxwell’s system. Firstly, recalling the instant energy norm \( \mathcal{H}_\varepsilon^s \)

\[
\mathcal{H}_\varepsilon^s(t) = \|(f_\varepsilon, g_\varepsilon, B_\varepsilon, \sqrt{\varepsilon} E_\varepsilon)\|_{\mathcal{H}_\varepsilon^2}^2 + \varepsilon^2\|(\nabla_v f_\varepsilon, \nabla_v g_\varepsilon)\|_{\mathcal{H}_\varepsilon^{-1}}^2,
\]

there is no \( L^\infty \) bound of \( f_\varepsilon \) and \( h_\varepsilon \) on the phase space. Besides, there also no useful \( L^2 \) estimates of \( (\nabla_v f_\varepsilon, \nabla_v h_\varepsilon) (i \geq 1) \) with respect to time \( t \). Furthermore, since there exists \( \sqrt{\varepsilon} \) before \( E_\varepsilon \), there is no useful estimate of the electric field can be derived from the anisotropic norm \( \mathcal{H}_\varepsilon^s \) too.

Due to this anisotropic norm \( \mathcal{H}_\varepsilon^s \) and the singular Lorentz force term, there exist difficulties in the process of obtaining uniform estimates. Indeed, recalling the equations of \( f_\varepsilon \) and \( h_\varepsilon \)

\[
\begin{align*}
\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{L}(f_\varepsilon) &= E_\varepsilon \cdot (v \cdot h_\varepsilon - \nabla_v h_\varepsilon) - \frac{1}{\varepsilon}(v \times B_\varepsilon) \cdot \nabla_v h_\varepsilon + \frac{1}{\varepsilon} \Gamma(f_\varepsilon, f_\varepsilon), \\
\partial_t h_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x h_\varepsilon - \frac{1}{\varepsilon^2} \mathcal{L}(h_\varepsilon) &= E_\varepsilon \cdot (v \cdot f_\varepsilon - \nabla_v f_\varepsilon) - \frac{1}{\varepsilon}(v \times B_\varepsilon) \cdot \nabla_v f_\varepsilon + \frac{1}{\varepsilon} \Gamma(h_\varepsilon, f_\varepsilon), \\
\varepsilon \partial_t E_\varepsilon - \nabla \times B_\varepsilon &= 0, \\
\partial_t B_\varepsilon + \nabla \times E_\varepsilon &= 0, \\
\text{div} B_\varepsilon &= 0, \\
\text{div} E_\varepsilon &= n_\varepsilon,
\end{align*}
\] (26)

the first two terms on the right hand of the first two equations in (26) are generated by the Lorentz force term. Since there exists extra \( \frac{1}{\varepsilon^2} \) before \( (v \times B_\varepsilon) \cdot \nabla_v h_\varepsilon \), there exists difficulty in bounding this term with magnetic field. Formally, the term \( E_\varepsilon \cdot \nabla_v h_\varepsilon \) is not singular. But it is very hard to bound. Indeed, as mentioned before, no \( L^\infty \) bound of \( f_\varepsilon \) and \( h_\varepsilon \), no \( L^2 \) bound of \( \nabla_v f_\varepsilon \) and \( \nabla_v h_\varepsilon \) and no useful bound of \( E_\varepsilon \) are at our disposal.

The idea of dealing with these difficulties goes like this. By decomposing \( f_\varepsilon \) and \( h_\varepsilon \) into macroscopic part and microscopic part, we can obtain the \( L^\infty \) bound of the macroscopic part \( \langle \rho_\varepsilon, u_\varepsilon, \theta_\varepsilon, n_\varepsilon \rangle \) from \( \mathcal{H}_\varepsilon^s \). For the microscopic part, since there exists coefficient \( \frac{1}{\varepsilon^2} \) before the microscopic part in the definition of \( \mathcal{D}_\varepsilon^s \) (only with derivation to \( x \)), the singular Lorentz force term can be bounded by virtue of integration by part and the structure of the Maxwell’s system.
Since the Maxwell’s equations are hyperbolic, to close the energy estimates, the dissipative estimates of the electromagnetic field are needed. This difficulty can be overcome by employing the idea used in [30]. The idea is to employ the equation of \( g_e \) to obtain a new equation containing a damping term of \( E_e \). In details, multiplying the equation of \( g_e \) by \( \tilde{v}(\text{see } (21)) \) and \( \tilde{J}_e = \frac{1}{\epsilon} \int_{\mathbb{R}^3} h_e \tilde{v} \text{d}v \) and then integrating over \( \mathbb{R}^3 \), we can obtain that

\[
- \partial_t \tilde{J}_e + \cdots + \frac{1}{\epsilon} E_e = \cdots. 
\]  
From this equation, the dissipative energy estimate of the electromagnetic field can be obtained.

Based on the uniform estimates, the MHD system can be obtained by employing the local conservation laws of system (7). The idea of recovering the equation of \( B \) in (3) is to employ the Ohm’s law derived from the dimensionless VMB system. Indeed, from (4), we can finally obtain that

\[
j = \nabla \times B = \sigma(E + u \times B).
\]

Based on the above relation, the last equation in (3) can be obtained (see Remark 3.5 for more details).

4. A priori estimates

This section is devoted to proving the existence of solutions to (7), i.e., Theorem 3.1. The key ingredient is the uniform prior estimate of solutions. The proof is quite involved. We split the whole proof into four lemmas.

\[
\begin{align*}
\begin{cases}
\partial_t f_e + \frac{1}{\epsilon} v \cdot \nabla_x f_e - \frac{1}{\epsilon} L(f_e) = - \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v (M h_e) + \frac{1}{\epsilon} \Gamma(f_e, f_e) = N_1, \\
\partial_t h_e + \frac{1}{\epsilon} v \cdot \nabla_x h_e - \frac{1}{\epsilon} E_e \cdot v - \frac{1}{\epsilon} L(h_e) = - \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v (M f_e) + \frac{1}{\epsilon} \Gamma(h_e, f_e) = N_2, \\
e \partial_t E_e - \nabla \times B_e = - J_e, \\
\partial_t B_e + \nabla \times E_e = 0, \\
d \text{div} B_e = 0, \quad e \cdot \text{div} E_e = n_e.
\end{cases}
\end{align*}
\]  

4.1. The dissipative estimates of the microscopic part.

**Lemma 4.1** (only related to \( \nabla^k_x \)). Under the assumptions of Theorem 3.1, if \((f_e, h_e, B_e, E_e)\) are strong solutions to (7), then

\[
\frac{d}{dt} \| (f_e, h_e, \sqrt{\epsilon} E_e, B_e) \|_{H_x^k}^2 + \frac{1}{\epsilon} \| (f^k_e, h^k_e) \|_{H_x^k}^2 \lesssim D_1(t) \sqrt{H_x^k(t)}.
\]  

**Proof.** Applying \( \nabla^k_x \) to the first four equations of (28) and then multiplying the resulting equations by \( \nabla^k_x f_e, \nabla^k_x h_e, \nabla^k_x E_e \) and \( \nabla^k_x B_e \), the integration over the phase space leads to

\[
\begin{align*}
\frac{d}{dt} \| & (f_e, h_e, \sqrt{\epsilon} E_e, B_e) \|^2_{H_x^k} + \frac{1}{\epsilon} \| (f^k_e, h^k_e) \|^2_{H_x^k} \\
= & - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v (M h_e) \right) \nabla^k_x f_e \cdot \nabla^k_x f_e + \nabla^k_x \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v (M f_e) \right) \nabla^k_x h_e \cdot \nabla^k_x h_e \, \text{d}v \, \text{d}x \\
& + \frac{1}{\epsilon} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla^k_x \Gamma(f_e, f_e) \cdot \nabla^k_x f_e + \nabla^k_x \Gamma(h_e, h_e) \cdot \nabla^k_x h_e \right) \, \text{d}v \, \text{d}x \\
=: & D_1 + D_2.
\end{align*}
\]  

Thus we only need to pay attention to the term with coefficient \( \frac{1}{\epsilon} \). Secondly, while \( k = s \) and all the derivative acts on \( h_e \) and \( f_e \), that is to say,

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v \nabla^k_x (M h_e) \right) \nabla^k_x f_e + \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v \nabla^k_x (M f_e) \right) \nabla^k_x h_e \right) \, \text{d}v \, \text{d}x,
\]  

the above term can not be directly controlled. To overcome these difficulties, we first split

\[
D_1 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v \nabla^k_x (M h_e) \right) \nabla^k_x f_e \right) \, \text{d}v \, \text{d}x \\
+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \nabla^k_x \left( \frac{E_e + v \times B_e}{\epsilon M} \cdot \nabla_v \nabla^k_x (M f_e) \right) \nabla^k_x h_e \right) \, \text{d}v \, \text{d}x + D_r
\]  

\[
:= D_2 + D_r.
\]
$D_2$ is simple and can be bounded by integrating by parts over the phase space. Indeed, by simple computation, we can conclude that

$$D_2 = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot E_v \nabla_x^k f \nabla_x^k (h_e) \text{Mdvd}x$$

$$= - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot E_v \nabla_x^k f \nabla_x^k (n_e) \text{Mdvd}x - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot E_v \nabla_x^k f \nabla_x^k (h_e^\perp) \text{Mdvd}x. \quad (32)$$

Noticing that while $s \geq 3$

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot E_v \nabla_x^k f \nabla_x^k (n_e) \text{Mdvd}x \right| \lesssim \|E_v\|_{H^{s-2}} \|f_s\|_{H^s_{\perp}} \|h_s\|_{H^s_{\perp}} \lesssim D^s_Y(t) \sqrt{H^s_Y(t)},$$

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot E_v \nabla_x^k f \nabla_x^k (h_e^\perp) \text{Mdvd}x \right| \lesssim \|E_v\|_{H^{s-2}} \|f_s\|_{H^s_{\perp}} \|h_s^\perp\|_{H^s_{\perp}} \lesssim D^s_Y(t) \sqrt{H^s_Y(t)},$$

combining (32), it follows that

$$D_2 \lesssim D^s_Y(t) \sqrt{H^s_Y(t)} \quad (33)$$

For $D_2$, denoting

$$D_r = \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x^i \left( \frac{E_v \cdot v + B}{\epsilon M} \right) \cdot \nabla_v \nabla_x^j (M h_e) \nabla_x^k f_e \text{Mdvd}x$$

$$+ \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x^i \left( \frac{E_v \cdot v + B}{\epsilon M} \right) \cdot \nabla_v \nabla_x^j (M f_e) \nabla_x^k h_e \text{Mdvd}x \quad (34)$$

$$:= D_{r1} + D_{r2},$$

and noticing that

$$h_e(t, x, v) = n_e(t, x) + h_e^\perp(t, x, v), \quad \nabla_v n(t, x) = 0,$$

we have that

$$D_{r2} = \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x^i \left( E_v \cdot v + B \right) \cdot \nabla_v \nabla_x^j (M f_e) \nabla_x^k h_e \text{Mdvd}x$$

$$= \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x^i \left( E_v \cdot v + B \right) \cdot \nabla_v \nabla_x^j (M f_e) \nabla_x^k h_e \text{Mdvd}x$$

$$= - \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot \nabla_x^i \left( E_v \cdot v + B \right) \cdot \nabla_v \nabla_x^j f_e \nabla_x^k (h_e^\perp) \text{Mdvd}x$$

$$+ \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot \nabla_x^i \left( E_v \cdot v + B \right) \cdot \nabla_v \nabla_x^j f_e \cdot \nabla_x^k (h_e^\perp) \text{Mdvd}x$$

$$+ \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} v \cdot \nabla_x^i \left( E_v \cdot v + B \right) \cdot \nabla_v \nabla_x^j f_e \cdot \nabla_x^k (h_e^\perp) \text{Mdvd}x$$

$$= D_{r21} + D_{r22} + D_{r23}. \quad (35)$$

The three $D_{r21}, D_{r22}$ and $D_{r23}$ can be bounded in the similar way. Taking $D_{r23}$ for example,

$$D_{r23} \leq \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} |\nabla_x^i B_e| \int_{\mathbb{R}^3} |v| |\nabla_v \nabla_x^j f_e| \cdot |\nabla_x^k h_e^\perp| \text{Mdvd}x$$

$$\leq \sum_{i \geq 1, \, i+j=k} \int_{\mathbb{R}^3} |\nabla_x^i B_e(t, x)| |\nabla_v \nabla_x^j f_e(t, x)| L_{3, v}^2 \|\nabla_x^k h_e^\perp(t, x)\| L_{3, v}^2 dx$$
\[ + \sum_{\frac{1}{2} \leq \frac{i+j}{k} \leq 1} \int_{\mathbb{R}^3} |\nabla_x^i B_\epsilon(t,x)| \|\nabla_v \nabla_x^j f_\epsilon(t,x)\|_{L^2_{\Lambda_v}} \|\nabla_x^{k-1} h_\epsilon^I(t,x)\|_{L^2_{\Lambda_v}} \, dx \]

\[ \leq \|\nabla_v \nabla_x^j f_\epsilon(t,x)\|_{L^2_{\Lambda_v}} \|L^2_{\Lambda_v} \| \sum_{\frac{1}{2} \leq \frac{i+j}{k} \leq 1} \int_{\mathbb{R}^3} |\nabla_x^i B_\epsilon(t,x)| \|\nabla_x^{k-1} h_\epsilon^I(t,x)\|_{L^2_{\Lambda_v}} \, dx \]

\[ + \|\nabla_x^i B_\epsilon(t,x)\|_{L^2_{\Lambda_v}} \sum_{\frac{1}{2} \leq \frac{i+j}{k} \leq 1} \int_{\mathbb{R}^3} \|\nabla_v \nabla_x^j f_\epsilon(t,x)\|_{L^2_{\Lambda_v}} \|\nabla_x^{k-1} h_\epsilon^I(t,x)\|_{L^2_{\Lambda_v}} \, dx \]

\[ \lesssim \|B_\epsilon\|_{H^s} \|f_\epsilon\|_{H^s_{\Lambda_v}} \|\nabla_x^{k-1} h_\epsilon^I\|_{H^s_{\Lambda_v}} \lesssim D^s_\epsilon(t) \sqrt{H^s_{\Lambda_v}(t)}. \]
Based on the assumptions in Sec. 4.2. the dissipative energy estimates of \((f, h_e)\). Denoting

\[
H^m_{v, e}(t) = \sum_{1 \leq k \leq m} \left( \sum_{j \geq 1, i = j+1} \bigg( 8^j \|(\nabla^i_v \nabla^j_v f, \nabla^i_v \nabla^j_v h_e)\|_{L^2}^2 + \|(\nabla^k_v f, \nabla^k_v h_e)\|_{L^2}^2 \bigg) \right), \quad m \geq 1,
\]

the following lemma is to bound the derivative of \(f_e\) and \(g_e\) with respect to \(v\).

**Lemma 4.2.** Under the assumptions of Theorem 3.1, if \((f_e, h_e, B_e, E_e)\) are solutions to (7), then

\[
e^2 \frac{d}{dt} \sum_{m=1}^a 8^m H^m_{v, e}(t) + \frac{3}{4} \|(\nabla_v f, \nabla_v h_e)\|_{H^{k-1}}^2 \leq \|(f, h_e)\|_{H^k}^2 + \|E_e\|_{H^{k-1}}^2 + D^e(t) \sqrt{\mathcal{H}^e(t)}.
\]

where \(c_1\) comes from the computation and is only dependent of the Sobolev embedding constants.

**Proof.** Applying \(\nabla^j_v \nabla^i_v\) to equation of \(f_e\) and \(g_e\) in (28), based on the resulting equations, we can infer that

\[
e^2 \frac{d}{dt} \sum_{m=1}^a 8^m H^m_{v, e}(t) + \frac{3}{4} \|(\nabla_v f, \nabla_v h_e)\|_{H^{k-1}}^2 \leq \|(f, h_e)\|_{H^k}^2 + \|E_e\|_{H^{k-1}}^2 + D^e(t) \sqrt{\mathcal{H}^e(t)}.
\]

By the Hölder inequality, we can infer

\[
T_1 \leq 4 \|(\nabla^{i+1}_v f_e, h_e)\|_{L^2} \|\nabla^i_v f_e\|_{L^2} + \epsilon \|\nabla^{i+1}_v h_e\|_{L^2} \|\nabla^i_v h_e\|_{L^2} \leq 4 \|\nabla^{i+1}_v f_e\|_{L^2} + 4 \epsilon^2 \|\nabla^i_v h_e\|_{L^2} + \frac{1}{\epsilon^2} \|\nabla^i_v \nabla^j_v f_e, h_e\|_{L^2}.
\]

For the second term \(T_2\) in (45), recalling that

\[
N_1 = E_e \cdot v \cdot h_e - (E_e + \frac{1}{\epsilon} v \times B_e) \cdot \nabla_v h_e + \frac{1}{\epsilon} \Gamma(f, f_e),
\]

\[
N_2 = E_e \cdot v \cdot f_e - (E_e + \frac{1}{\epsilon} v \times B_e) \cdot \nabla_v f_e + \frac{1}{\epsilon} \Gamma(h, f_e).
\]
The structure of the first two terms on the right hand of (48) is similar to $D_1$ in (42). The third line in (48) can be estimated by the assumption on the quadratic collision operator in Sec. 2.2. Thus, we can infer that

$$T_2 \leq \epsilon \|(E_k, B_k)\|_{H^2}^2 \|\epsilon(f_k, h_k)\|_{H^2}^2 + \epsilon \|(f_k, h_k)\|_{H^2} \|\epsilon(f_k, h_k)\|_{H^2}^2 \lesssim \mathcal{D}_e^*(t) \sqrt{\mathcal{H}_e^*(t)}. \quad (49)$$

Combining (46), (47) and (49), we can infer that

$$\epsilon^2 \frac{d}{dt} \left( \sum_{i+j+k=2} \frac{8^i}{3} \|\nabla v \nabla_x f \nabla v \nabla_x h\|^2_{L^2} + \|\nabla v f \nabla v h\|^2_{L^2} \right) + \frac{3}{4} \sum_{i+j=k} \|\nabla v \nabla^j f \nabla v \nabla^j h\|^2_{L^2} \lesssim \epsilon^2 \|E_\epsilon\|_{H^k_{\epsilon-1}}^2 + \|\epsilon(f_k, h_k)\|_{H^k_{\epsilon-1}}^2 + \mathcal{D}_e^*(t) \sqrt{\mathcal{H}_e^*(t)}. \quad (51)$$

From (51), there exists some $c_1 \geq 1$ independent of $k$ such that

$$\epsilon^2 \frac{d}{dt} H^1_{v, \epsilon}(t) + \frac{3}{4} \|\nabla v (f_k, h_k)\|_{L^2}^2 \lesssim \epsilon^2 \|E_\epsilon\|_{L^2}^2 + \|\epsilon(f_k, h_k)\|_{H^2_{\epsilon}}^2 + \frac{1}{4} \|\nabla v (f_k, h_k)\|_{H^2_{\epsilon}}^2 + \mathcal{D}_e^*(t) \sqrt{\mathcal{H}_e^*(t)}. \quad (52)$$

$$\epsilon^2 \frac{d}{dt} H^k_{v, \epsilon}(t) + \frac{3}{4} \|\nabla v (f_k, h_k)\|_{L^2}^2 - c_1 \|\nabla v (f_k, h_k)\|_{H^k_{\epsilon}}^2 \lesssim \epsilon^2 \|E_\epsilon\|_{H^k-2}^2 + \|\epsilon f_k\|_{H^k_{\epsilon}}^2 + \mathcal{D}_e^*(t) \sqrt{\mathcal{H}_e^*(t)}, \quad (52)$$

where $k \geq 2$ and $\|h\|_{H^0} = \|h\|_{L^2}$.

By the similar method of deducing (51), we can infer that

$$\epsilon^2 \frac{d}{dt} \sum_{m=1}^s \frac{8c_{-m}}{3} H^m_{v, \epsilon}(t) + \frac{3}{4} \|\nabla v f_k \nabla v h_k\|_{H^k_{\epsilon}}^2 \lesssim \|\epsilon(f_k, h_k)\|_{H^k_{\epsilon}}^2 + \|\epsilon f_k\|_{H^k_{\epsilon}}^2 + \mathcal{D}_e^*(t) \sqrt{\mathcal{H}_e^*(t)}. \quad (53)$$

4.3. The dissipative estimates of the macroscopic parts. Denoting

$$H^s_{v, \epsilon} := \sum_{k=1}^s \int_{\mathbb{R}^3} \left( \nabla_x \nabla^k f \nabla_x h + \nabla_x \nabla^k h \nabla_x f \right) \text{M} \text{d}x,$$

the following lemma is to provide the dissipative energy estimates of $f_k, g_k$ with derivative to $x$.\[\Box\]
Lemma 4.3. Under the assumptions of Theorem 3.1, if \((f_\epsilon, h_\epsilon, B_\epsilon, E_\epsilon)\) are solutions to (7), then
\[
\frac{d}{dt} H_{s,\epsilon}^2 + ||\nabla (f_\epsilon, h_\epsilon)||^2_{H^{-1}} - \delta_1 ||\nabla (f_\epsilon, h_\epsilon)||^2_{H^{-1}} - \delta_2 ||E_\epsilon||^2_{H^{-1}} + \frac{1}{2} ||h_\epsilon||^2_{H^{-1}} \lesssim \frac{1}{2} ||(f_\epsilon^+, h_\epsilon^+)||^2_{H^{-1}} + D_\epsilon^*(t) \sqrt{H_\epsilon^*(t)}.
\]

**Proof.** According to the definition of \(H_{s,\epsilon}^2\), we can infer that
\[
\frac{d}{dt} H_{s,\epsilon}^2 + \sum_{k=1}^{s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla v \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_x \nabla_x^{k-1} f_\epsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon) \, Mdvdx
\]
\begin{align*}
+ \sum_{k=1}^{s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla v \nabla_x^{k-1} (v \cdot \nabla_x h_\epsilon) \cdot \nabla_x \nabla_x^{k-1} h_\epsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x h_\epsilon) \cdot \nabla_v \nabla_x^{k-1} h_\epsilon) \, Mdvdx \\
- \frac{1}{4} \sum_{k=1}^{s} \int_{\mathbb{R}^3} (\nabla_x \nabla_x^{k-1} \mathcal{L}(f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon + \nabla_x \nabla_x^{k-1} \mathcal{L}(h_\epsilon) \cdot \nabla_v \nabla_x^{k-1} h_\epsilon) \, Mdvdx \\
- \frac{1}{4} \sum_{k=1}^{s} \int_{\mathbb{R}^3} (\nabla v \nabla_x^{k-1} \mathcal{L}(f_\epsilon) \cdot \nabla_x \nabla_x^{k-1} f_\epsilon + \nabla_v \nabla_x^{k-1} \mathcal{L}(h_\epsilon) \cdot \nabla_x \nabla_x^{k-1} h_\epsilon) \, Mdvdx \\
= \sum_{k=1}^{s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla v \nabla_x^{k-1} (v \cdot \nabla_x E_\epsilon) \cdot \nabla_x \nabla_x^{k-1} h_\epsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla x E_\epsilon) \cdot \nabla_v \nabla_x^{k-1} h_\epsilon) \, Mdvdx \\
+ \sum_{k=1}^{s} \epsilon \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla_x \nabla_x^{k-1} N_1 \cdot \nabla_x \nabla_x^{k-1} f_\epsilon + \nabla_v \nabla_x^{k-1} N_1 \cdot \nabla_x \nabla_v^{k-1} h_\epsilon) \, Mdvdx \\
+ \sum_{k=1}^{s} \epsilon \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla_x \nabla_x^{k-1} N_2 \cdot \nabla_x \nabla_x^{k-1} f_\epsilon + \nabla_v \nabla_x^{k-1} N_2 \cdot \nabla_x \nabla_v^{k-1} h_\epsilon) \, Mdvdx \\
= M_1 + M_2 + M_3.
\end{align*}

In what follows, we try to estimate each term in (55) for each \(k\) first and then sum them up. For the second term in the first line of (55), denoting
\[
I := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon \, Mdvdx,
\]
by integration by parts (three times), we can infer that
\[
I = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (v' \cdot \nabla_x^{2k} f_\epsilon) \partial_{v_i} \nabla_x^{k-1} f_\epsilon \, Mdvdx = ||\nabla_x \nabla_x^{k-1} f_\epsilon||^2_{L^2} - ||v' \cdot \nabla_x \nabla_x^{k-1} f_\epsilon||^2_{L^2} - I.
\]

On the other hand, we can infer that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla x \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_x \nabla_x^{k-1} f_\epsilon \, Mdvdx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon \, Mdvdx + ||v' \cdot \nabla_x \nabla_x^{k-1} f_\epsilon||^2_{L^2}.
\]

All together, it follows that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla v \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_x \nabla_x^{k-1} f_\epsilon + \nabla_x \nabla_x^{k-1} (v \cdot \nabla_x f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon) \, Mdvdx = ||\nabla_x \nabla_x^{k-1} f_\epsilon||^2_{L^2}.
\]

For the terms in the third and forth line of (55), there exists coefficient \(\frac{1}{7}\). The ideal is to use the microscopic part (see (29)) to deal with this difficulty. Indeed, for the second line, noticing that
\[
\nabla_x \nabla_x^{k-1} \mathcal{L}(f_\epsilon) = \mathcal{L}(\nabla_x \nabla_x^{k-1} f_\epsilon) = \mathcal{L}(\nabla_x \nabla_x^{k-1} f_\epsilon),
\]
by Hölder’s inequality, we can infer that
\[
|\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nabla x \nabla_x^{k-1} \mathcal{L}(f_\epsilon) \cdot \nabla_v \nabla_x^{k-1} f_\epsilon + \nabla_x \nabla_x^{k-1} \mathcal{L}(h_\epsilon) \cdot \nabla_v \nabla_x^{k-1} h_\epsilon) \, Mdvdx| \\
\leq \frac{1}{2} \bar{\omega} \bar{\omega} ||\nabla_x (f_\epsilon^+, h_\epsilon^+)||^2_{L^4} + \frac{1}{2} \delta \bar{\omega} ||\nabla_v \nabla_x^{k-1} (f_\epsilon, g_\epsilon)||^2_{L^4},
\]
where \(\delta\) is a positive constant to be chosen later.
The dissipative estimates of the electromagnetic parts. Denoting 
\[ H_{e,c}^s(t) = \frac{d}{dt} \left[ \sum_{k=0}^{s-1} \int_{\mathbb{R}^3} \frac{c^2}{4} \nabla_{x} \nabla_{x}^{k-1} E_c \cdot \nabla_{x} \nabla_{x}^{k-1} E_c \, dx + \sum_{k=0}^{s-2} \int_{\mathbb{R}^3} \left( c^2 \nabla \times \nabla_{x}^{k} E_c \cdot \nabla \times \nabla_{x}^{k} E_c + \epsilon \cdot \frac{3\epsilon}{4} \nabla_{x}^{k} E_c \cdot \nabla \times \nabla_{x}^{k} B_c \right) \, dx \right], \]
the following lemma provides the dissipative energy estimates of \( B_e \) and \( E_e \).

**Lemma 4.4.** Under the assumptions of Theorem 3.1, if \( (f_e, h_e, B_e, E_e) \) are solutions to (7), then
\[
-\frac{d}{dt} H_{e,c}^s + \frac{3\epsilon}{4} \sum_{k=0}^{s-1} \| \nabla_{x}^{k} E_c \|_{L^2}^2 + \frac{\epsilon}{4} \sum_{k=1}^{s-2} \| \nabla \times \nabla_{x}^{k-1} B_e \|_{L^2}^2 \lesssim \frac{1}{\tau} \| h_e^+ \|_{H_{e,c}^s}^2 + D_e^s(t) \sqrt{H_{e,c}^s(t)}. \]
Proof. Recalling that
\[ \tilde{j}_e = \frac{1}{\epsilon} \int_{\mathbb{R}^3} h_e \tilde{v}, \quad \sigma = \frac{1}{\epsilon} \int_{\mathbb{R}^3} \tilde{v} \cdot \nabla M \, dv, \quad \mathcal{L}(\tilde{v}) = v, \]
from the second equation of (28), we can infer that \( \tilde{j}_e \) satisfies the following equation:
\[ \epsilon^2 \partial_t \tilde{j}_e + \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v h_e M \, dv - \sigma E_e - j_e = \epsilon \int_{\mathbb{R}^3} \tilde{v} N_1 M \, dv. \tag{67} \]
There exists a "damping" term of \( E_e \) in equation (67). On the other hand, the equations of \( E_e \) and \( B_e \) are
\[ \epsilon \partial_t E_e - \nabla \times B_e = - j_e, \]
\[ \partial_t B_e + \nabla \times E_e = 0. \tag{68} \]
The dissipative energy estimates of \( E_e \) can be deduced by employing structure of (67) and (68). Indeed, after applying \( \nabla \times \nabla^k \) to (67) and (68), we can infer
\[ - \epsilon^2 \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k E_e \, dx - \int_{\mathbb{R}^3} \nabla \times \left( \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v \nabla^k h_e M \, dv \right) \cdot \nabla \times \nabla^k E_e \, dx \\
+ \sigma ||\nabla \times \nabla^k E_e||^2_{L^2} - \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k E_e \, dx = \epsilon \int_{\mathbb{R}^3} \nabla \times \left( \tilde{v} \nabla^k N_2 M \right) \cdot \nabla \times \nabla^k E_e \, dx, \tag{69} \]
and
\[ - \epsilon^2 \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \partial_t \nabla \times \nabla^k E_e \, dx + \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla \times \nabla^k B_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx - \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx = 0. \tag{70} \]
With the help of (69) and (70), we can infer that
\[ - \epsilon^2 \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k E_e \, dx + \sigma ||\nabla \times \nabla^k E_e||^2_{L^2} = \int_{\mathbb{R}^3} \nabla \times \left( \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v \nabla^k h_e M \, dv \right) \cdot \nabla \times \nabla^k E_e \, dx \\
+ \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx - \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla^k B_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx \\
- \epsilon \int_{\mathbb{R}^3} \nabla \times \left( \tilde{v} \nabla^k N_2 M \right) \cdot \nabla \times \nabla^k E_e \, dx + \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k E_e \, dx. \tag{71} \]
Noticing that
\[ \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v \nabla^k h_e M \, dv = \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v \nabla^k h_e^+ M \, dv + \sigma \nabla n_e, \]
then by Hölder’s inequality, we can infer that
\[ \left| \int_{\mathbb{R}^3} \nabla \times \left( \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v \nabla^k h_e M \, dv \right) \cdot \nabla \times \nabla^k E_e \, dx \right| \leq C ||\nabla^k+1 h_e^+||^2_{L^2} + \frac{\sigma}{16} ||\nabla \times \nabla^k E_e||^2_{L^2}. \]
Recalling that
\[ j_e = \frac{1}{\epsilon} \int_{\mathbb{R}^3} h_e^+ v M \, dv, \quad \tilde{j}_e = \int_{\mathbb{R}^3} \tilde{v} h_e^+ M \, dv, \]
we can infer that
\[ \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx - \epsilon \int_{\mathbb{R}^3} \nabla \times \nabla \times \nabla^k B_e \cdot \nabla \times \nabla^k \tilde{j}_e \, dx + \int_{\mathbb{R}^3} \nabla \times \nabla^k \tilde{j}_e \cdot \nabla \times \nabla^k E_e \, dx \leq C \frac{1}{\epsilon^2} ||\nabla^k+1 h_e^+||^2_{L^2} + \frac{\sigma}{16} ||\nabla \times \nabla^k (B_e, E_e)||^2_{L^2}. \tag{72} \]
For the left term on the right hand of (71), noticing that \( k \leq s - 2 \), it follows that
\[
\epsilon \int_{\mathbb{R}^3} \nabla \cdot \int_{\mathbb{R}^3} \hat{\mu} \nabla^k \mathbf{N} \cdot \nabla \nabla^k \mathbf{E}_d \, dx
\]
\[= \epsilon \int_{\mathbb{R}^3} \nabla \cdot \int_{\mathbb{R}^3} \hat{\mu} \nabla^k (\mathbf{E}_d \cdot \mathbf{v} \cdot f_s) \nabla \nabla^k \mathbf{E}_d \, dx
\]
\[= \int_{\mathbb{R}^3} \nabla \cdot \int_{\mathbb{R}^3} \hat{\mu} \nabla^k ((\epsilon \mathbf{E}_d + \mathbf{v} \cdot \nabla \mathbf{B}_s) \cdot \nabla \mathbf{E}_s + \Gamma (h_{E}, f_s)) \nabla \nabla^k \mathbf{E}_d \, dx
\]
\[\leq C \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)} + \frac{\nu}{c} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2.
\]  
(73)

All together, we can infer that
\[
- \epsilon^2 \frac{d}{dt} \int_{\mathbb{R}^3} \nabla \nabla^k_j \mathbf{E} \cdot \nabla \nabla^k \mathbf{E}_d \, dx + \frac{3\pi}{4} \int_{\mathbb{R}^3} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 - \frac{\sigma}{16} \int_{\mathbb{R}^3} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2
\]
\[\lesssim \frac{1}{c^2} \| \nabla \nabla^k_j \mathbf{E}_d \|_{L^2}^2 + \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)},
\]  
(74)

and
\[
- \epsilon^2 \frac{d}{dt} \sum_{k=0}^{s-2} \int_{\mathbb{R}^3} \nabla \nabla^k_j \mathbf{E} \cdot \nabla \nabla^k \mathbf{E}_d \, dx + \frac{3\pi}{4} \sum_{k=0}^{s-2} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 - \frac{\sigma}{8} \sum_{k=1}^{s-2} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2
\]
\[\lesssim \frac{1}{c^2} \| \nabla \nabla^k_j \mathbf{E}_d \|_{L^2}^2 + \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)}.
\]  
(75)

By the same way of deducing (71) and (75), we can infer that
\[
- \epsilon^2 \frac{d}{dt} \sum_{k=0}^{s-1} \int_{\mathbb{R}^3} \nabla^k \mathbf{E}_d \nabla \nabla^k \mathbf{E}_d \, dx + \frac{3\pi}{4} \sum_{k=0}^{s-1} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 - \frac{\sigma}{8} \sum_{k=1}^{s-1} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2
\]
\[\lesssim \frac{1}{c^2} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 + \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)}.
\]  
(76)

Combing (75) and (76), we can infer that
\[
- \epsilon^2 \frac{d}{dt} \left( \sum_{k=0}^{s-1} \int_{\mathbb{R}^3} \nabla^k \mathbf{E}_d \nabla \nabla^k \mathbf{E}_d \, dx + \int_{\mathbb{R}^3} \nabla \nabla^k_j \mathbf{E}_d \cdot \nabla \nabla^k \mathbf{E}_d \, dx \right)
\]
\[+ \frac{3\pi}{4} \sum_{k=0}^{s-2} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 + \frac{3\pi}{4} \sum_{k=0}^{s-2} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2 - \frac{\sigma}{8} \sum_{k=1}^{s-2} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2
\]
\[\lesssim \frac{1}{c^2} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 + \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)}.
\]  
(77)

To finish the proof of this lemma, we still need to obtain the dissipative energy estimates of magnetic field. From the equations of \( \mathbf{B}_s \) and \( \mathbf{E}_d \), we can infer that
\[
- \epsilon \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^k \mathbf{E}_d \cdot \nabla \nabla^k \mathbf{B}_s \mathbf{dx} + \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2 - \epsilon \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 = \int_{\mathbb{R}^3} \nabla^k \mathbf{E}_d \cdot \nabla \nabla^k \mathbf{B}_s \mathbf{dx}.
\]

By the Hőlder’s inequality, we obtain that
\[
- \epsilon \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^k \mathbf{E}_d \cdot \nabla \nabla^k \mathbf{B}_s \mathbf{dx} + \frac{3}{4} \| \nabla \nabla^k \mathbf{B}_s \|_{L^2}^2 - \epsilon \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2 \lesssim \frac{1}{c^2} \| \nabla \nabla^k \mathbf{E}_d \|_{L^2}^2.
\]  
(78)

Combining (77) and (78), we complete the proof. 
\[\Box\]

4.5. The whole estimates. In the left of this section, based on (29), (44), (54), (77) and (78), we can obtain estimates like this
\[
\frac{d}{dt} \mathcal{H}_\epsilon^s + \mathcal{D}_\epsilon^s \lesssim \mathcal{D}_\epsilon^s (t) \sqrt{\mathcal{H}_{\epsilon}^s (t)}.
\]

Then, if the initial data is some enough, we can obtain uniform estimates of solutions with respect to the Knudsen number.
Lemma 4.5. Under the assumptions of Theorem 3.1, if \((f_\epsilon, h_\epsilon, B_\epsilon, E_\epsilon)\) are solutions to (7), then there exists some small enough constant \(c_0\) such that

\[
\sup_{0 \leq s \leq t} H^s_v(t) + \frac{1}{2} \int_0^t \left( \|(f_\epsilon, h_\epsilon)\|_{H^2}^2 + \frac{2}{s^2} \|(E_\epsilon, B_\epsilon)\|_{H^{s - 1}}^2 + \|\left( f_\epsilon^\perp, h_\epsilon^\perp \right) \|_{H^{s - 1}}^2 \right) (s) \mathrm{d}s \leq \frac{c_1}{c_0} H^s_v(0),
\]

(79)

where \(c_1\) and \(c_0\) are positive constants only dependent of the Sobolev embedding constant.

Proof. This Lemma can be proved by employing the Poincare’s inequality and choosing proper constants. Combining (64) and (66) up and setting the \(\delta_2 = \frac{\varphi}{2}\) (in (64)), then we can infer that

\[
\frac{d}{dt} \left( H^s_{v, \epsilon} - H^s_{v, e} \right) (t) + \frac{4}{\epsilon} \left( \|
abla_x f_\epsilon \nabla_x h_\epsilon\|_{H^{s - 1}}^2 + \frac{1}{\epsilon} \|n_\epsilon\|_{H^{s - 1}}^2 \right) + \frac{1}{\epsilon} \|
abla_x (f_\epsilon, h_\epsilon)\|_{H^{s - 1}}^2
\]

\[+ \frac{\varphi}{2} \sum_{k=0}^{s-2} \|
abla \times \nabla_x^{k-1} B_\epsilon\|_{L^2}^2 + \frac{\varphi}{2} \sum_{k=1}^{s-2} \|
abla \times \nabla_x^{k-1} B_\epsilon\|_{L^2}^2 \leq \frac{1}{\epsilon} \left( \|f_\epsilon^\perp, h_\epsilon^\perp\|_{H^s_t}^2 + \mathcal{D}_v^s (t) \sqrt{\mathcal{H}_v^s (t)} \right).
\]

(80)

According to the definition of \(\mathcal{D}_v^s\), the dissipative energy estimates of \(f_\epsilon, h_\epsilon\) and \(B_\epsilon\) is not complete. We need to recover the \(L^2\) estimates of \(f_\epsilon, h_\epsilon\) and \(B_\epsilon\). For the magnetic field \(B_\epsilon\), from (18),

\[
\int_{\mathbb{R}^3} B_\epsilon(t) \mathrm{d}x = 0, \quad \forall t \geq 0,
\]

on the other hand,

\[\mathrm{div} B_\epsilon = 0,\]

then by Poincare’s inequality, we can infer that

\[
\sum_{k=1}^{s-2} \|
abla \times \nabla_x^{k-1} B_\epsilon\|_{L^2}^2 \approx \|B_\epsilon\|_{H^{s - 1}}^2.
\]

(81)

For \(f_\epsilon\) and \(h_\epsilon\), recalling that we can decompose \(f_\epsilon\) and \(h_\epsilon\) like this

\[
f_\epsilon = \mathcal{P} f_\epsilon + f_\epsilon^\perp, \quad h_\epsilon = n_\epsilon + h_\epsilon^\perp,
\]

from (18), we can infer that

\[
\int_{\mathbb{R}^3} \mathcal{P} f_\epsilon (t) \mathrm{d}x = -c_\epsilon \cdot \int_{\mathbb{R}^3} E_\epsilon \times B_\epsilon (t) \mathrm{d}x - c_\epsilon \|u\|_{L^6}^2 \|E_\epsilon (t)\|_{L^2}^2, \quad \int_{\mathbb{R}^3} n_\epsilon (t) \mathrm{d}x = 0.
\]

Based on the above mean value, it follow that

\[
\|(f_\epsilon, h_\epsilon)\|_{H^s_t}^2 \lesssim \|
abla_x (f_\epsilon, h_\epsilon)\|_{H^{s - 1}}^2 + \|(f_\epsilon^\perp, h_\epsilon^\perp)\|_{H^{s - 1}}^2 + \mathcal{D}_v^s (t) \sqrt{\mathcal{H}_v^s (t)}.
\]

(82)

Combining (80), (81) and (82), there exists some \(c_0 > 0\) such that

\[
\frac{d}{dt} \left( H^s_{v, \epsilon} - H^s_{v, e} \right) (t) + c_0 \|(f_\epsilon, h_\epsilon)\|_{H^s_t}^2 + \|(E_\epsilon, B_\epsilon)\|_{H^{s - 1}}^2 + \frac{1}{\epsilon} \|n_\epsilon\|_{H^{s - 1}}^2 - \delta_2 \|
abla_v (f_\epsilon, h_\epsilon)\|_{H^{s - 1}}^2
\]

\[\lesssim \frac{1}{\epsilon} \left( \|f_\epsilon^\perp, h_\epsilon^\perp\|_{H^s_t}^2 + \mathcal{D}_v^s (t) \sqrt{\mathcal{H}_v^s (t)} \right).
\]

(83)

To get the whole \(\mathcal{D}_v^s\), (83) and (44) should be put together. From (44), there exists some \(c_7\) such that

\[
c_7 \|f_\epsilon, h_\epsilon\|_{H^s_t}^2 \leq c_7 \|
abla_x (f_\epsilon, h_\epsilon)\|_{H^{s - 1}}^2 - c_7 \|(f_\epsilon^\perp, h_\epsilon^\perp)\|_{H^{s - 1}}^2 - c_7 \epsilon \|E_\epsilon\|_{H^{s - 1}}^2 \lesssim \mathcal{D}_v^s (t) \sqrt{\mathcal{H}_v^s (t)}.
\]

(84)

From (83) and (84), choosing \(c_8\) and \(\delta_2\) such that

\[c_8 c_0 \geq c_7 + \frac{\varphi}{2}, \quad c_8 \delta_2 = \frac{1}{2}, \quad c_8 \geq 1,
\]

then there exists some \(d_1 > 0\) such that

\[
\frac{d}{dt} \left( 2c_8 H^s_{v, \epsilon} - 2c_8 H^s_{v, e} + c_2 \sum_{m=1}^s \frac{8c_1^{s - m}}{3} H^m_{v, \epsilon} (t) \right) (t)
\]

\[+ \|(f_\epsilon, h_\epsilon)\|_{H^s_t}^2 + \|(E_\epsilon, B_\epsilon)\|_{H^{s - 1}}^2 + \frac{1}{\epsilon} \|n_\epsilon\|_{H^{s - 1}}^2
\]

\[+ \|
abla_v (f_\epsilon, h_\epsilon)\|_{H^{s - 1}}^2 - \frac{\delta_2}{\epsilon} \|(f_\epsilon^\perp, h_\epsilon^\perp)\|_{H^s_t}^2 \lesssim \mathcal{D}_v^s (t) \sqrt{\mathcal{H}_v^s (t)}.
\]

(85)
Finally, from (29), denoting
\[ \tilde{H}_\epsilon^s := 2c_8 \epsilon H_{\epsilon,x}^s - 2c_8 H_{\epsilon,x}^s + 2\epsilon^2 \sum_{n=1}^{s} \frac{8c_7^m F_n}{3} H_{\epsilon,x}^m + d_2 \|(f_n, h_n, \sqrt{\epsilon} E_n, B_n)\|^2_{H_2}, \]

where \( d_2 \) are chosen to satisfy
\[ d_2 \geq d_1 + 1, \quad \tilde{H}_\epsilon^s \approx H_\epsilon^s, \]

then it follows that there exists some positive constant \( d_3 \) such that
\[ \frac{d}{dt} \tilde{H}_\epsilon^s + \mathcal{D}_\epsilon^s \leq d_3 \mathcal{D}_\epsilon^s(t) \sqrt{\tilde{H}_\epsilon^s(t)}. \]

If the initial data satisfy
\[ H_\epsilon^s(0) \leq c_0 := \frac{1}{4d_3}, \]

then we can infer that for any \( t > 0 \)
\[ \sup_{0 \leq s \leq t} \tilde{H}_\epsilon^s(t) + \frac{1}{2} \int_0^t \mathcal{D}_\epsilon^s(\tau)d\tau \leq \tilde{H}_\epsilon^s(0). \]

On the other hand, \( \tilde{H}_\epsilon^s \) is equivalent to \( H_\epsilon^s \), i.e., there exist \( 0 < c_l < 1 \) and \( c_u > 0 \) such that
\[ c_l \|H_\epsilon^s \leq \tilde{H}_\epsilon^s \leq c_u H_\epsilon^s. \]

Thus, we can infer that for any \( t > 0 \)
\[ \sup_{0 \leq s \leq t} H_\epsilon^s(t) + \frac{1}{2} \int_0^t \mathcal{D}_\epsilon^s(\tau)d\tau \leq \frac{c_u}{c_l} H_\epsilon^s(0), \quad \forall \epsilon \in (0, 1]. \]

We complete the proof of this lemma. \( \square \)

4.6. The existence of system 7. For each fixed \( \epsilon \), the existence of solutions to system 7 can be found in [12]. But it also can be obtained by employing the following iteration system \( (n \geq 1) \):
\[
\begin{align*}
\partial_t f_\epsilon^n + \frac{1}{\epsilon} v \cdot \nabla_{x} f_\epsilon^n &- \frac{1}{\epsilon} L(f_\epsilon^n) + \frac{\epsilon E_{\epsilon}^{n-1} \times B_{\epsilon}^{n-1}}{M_{\epsilon}} \cdot \nabla_{v}(M_{\epsilon} f_\epsilon^n) = \frac{1}{\epsilon} \Gamma(f_{\epsilon}^{n-1}, f_{\epsilon}^{n-1}), \\
\partial_t h_\epsilon^n + \frac{1}{\epsilon} v \cdot \nabla_{x} h_\epsilon^n &- \frac{1}{\epsilon} L(h_\epsilon^n) + \frac{\epsilon E_{\epsilon}^{n-1} \times B_{\epsilon}^{n-1}}{M_{\epsilon}} \cdot \nabla_{v}(M_{\epsilon} f_\epsilon^n) = \frac{1}{\epsilon} \Gamma(h_{\epsilon}^{n-1}, f_{\epsilon}^{n-1}), \\
\partial_t E_\epsilon^n - \nabla \times B_\epsilon^n &= -f_\epsilon^n, \\
\partial_t B_\epsilon^n + \nabla \times E_\epsilon^n &= 0, \\
div B_\epsilon^n &= 0, \quad cdiv E_\epsilon^n = \int_{\mathbb{R}^3} h_\epsilon^n Mdv, \\
(f_\epsilon^{n}, h_\epsilon^{n}, B_\epsilon^{n}, E_\epsilon^{n})(0) &= (f_\epsilon, h_\epsilon, B_\epsilon, E_\epsilon)(0),
\end{align*}
\]

with
\[ f_\epsilon^0 = g_0 = 0, \quad E_\epsilon^0 = B_\epsilon^0 = 0. \]

The approximate solutions can be constructed by iteration method. Then based on the uniform estimates (obtained by induction method), the solutions can be obtained by employing Rellich-Kondrachov compactness theorem.

5. The proof of Theorem 3.4

In Sec. 4, for any \( t > 0 \), the solution \((f_\epsilon, h_\epsilon, B_\epsilon, E_\epsilon)\) satisfy the following uniform estimate:
\[
\begin{align*}
&\sup_{0 \leq s \leq t} \|(f_\epsilon, h_\epsilon, B_\epsilon, \sqrt{\epsilon} E_\epsilon)(s)\|^2_{H_2} + \int_0^t \|(f_\epsilon, h_\epsilon)(\tau)\|^2_{H_2} d\tau \\
&+ \int_0^t \left( \frac{1}{2} \|(f_\epsilon^+ h_\epsilon^+)(\tau)\|^2_{H_2} + \frac{1}{2} \|(n_\epsilon, \sqrt{\epsilon} B_\epsilon, \sqrt{\epsilon} E_\epsilon)\|^2_{H_{\epsilon}-1} \right)(\tau)d\tau \leq C_0
\end{align*}
\]

Based on (92), we can verify the MHD limit of VMB system.

Step 1: the limit of \( f_\epsilon \) and \( h_\epsilon \).

First, noticing that
\[
\int_0^t \left( \|(f_\epsilon, h_\epsilon)(\tau)\|^2_{H_2} + \frac{1}{2} \|(f_\epsilon^+ h_\epsilon^+)(\tau)\|^2_{H_2} + \frac{1}{2} \|(n_\epsilon(\tau)\|^2_{H_{\epsilon}-1} \right)d\tau \leq C_0, \quad \forall t > 0,
\]
and recalling
\[ f_\epsilon = \rho_\epsilon + u_\epsilon \cdot v + \frac{|v|^2 - 3}{2} \theta_\epsilon + f_\epsilon^1, \quad h_\epsilon = n_\epsilon + h_\epsilon^1, \]
then we can infer that there exist \( \rho, u, \theta, n \) belonging to \( H_x^2 \) space such that
\[ f_\epsilon \rightarrow f = \rho(t, x) + u(t, x) \cdot v + \frac{|v|^2 - 3}{2} \theta(t, x), \quad h_\epsilon \rightarrow n = 0, \text{ in } L^2 ((0, +\infty); H_x^{s-1}). \] (93)
Furthermore, there exist \( \{B, E\} \subset H_x^s \) such that
\[ \rho_\epsilon \rightarrow \rho, \quad u_\epsilon \rightarrow u, \quad \theta_\epsilon \rightarrow \theta, \quad n_\epsilon \rightarrow 0, \quad B_\epsilon \rightarrow B, \quad E_\epsilon \rightarrow E, \text{ in } L^2 ((0, +\infty); H_x^{s-1}). \] (94)
The next step is to verify that \( (\rho, u, \theta, B, E) \) satisfies the MHD system.

**Step 2: the limiting equation**

From (28), based on the local conservation laws, we can find that \( \rho_\epsilon, u_\epsilon, \theta_\epsilon, B_\epsilon, E_\epsilon \) satisfy the following system:
\[
\begin{aligned}
\partial_t \rho_\epsilon + \frac{1}{\epsilon} \text{div} u_\epsilon &= 0, \\
\partial_t u_\epsilon + \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} A(\epsilon) f_\epsilon \cdot Mdv + \frac{1}{\epsilon} \nabla_x (\rho_\epsilon + \theta_\epsilon) &= n_\epsilon \cdot E_\epsilon + j_\epsilon \times B_\epsilon, \\
\partial_t \theta_\epsilon + \frac{2}{3\epsilon} \text{div} \int_{\mathbb{R}^3} B(\epsilon) f_\epsilon \cdot Mdv + \frac{2}{3\epsilon} \text{div} u_\epsilon &= \epsilon \frac{2}{3} j_\epsilon \cdot E_\epsilon, \\
\epsilon \partial_t E_\epsilon - \nabla \times B_\epsilon &= -j_\epsilon, \\
\partial_t B_\epsilon + \nabla \times E_\epsilon &= 0, \\
\text{div} B_\epsilon &= 0, \\
\epsilon \text{div} E_\epsilon &= n_\epsilon.
\end{aligned}
\] (95)
where
\[ A(v) = v \otimes v - \frac{|v|^2}{2}, \quad B(v) = v (\frac{|v|^2}{2} - \frac{v}{2}), \quad \mathcal{L}A(v) = A(v), \quad \mathcal{L}B(v) = B(v). \] (96)
By the first equation of (95), in the distributional sense
\[ \text{div} u_\epsilon \rightarrow \text{div} u = 0. \] (97)
Furthermore, recalling that
\[ j_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^3} h_\epsilon^1 v Mdv, \quad \tilde{j}_\epsilon = \frac{1}{\epsilon} \int_{\mathbb{R}^3} h_\epsilon^1 \tilde{v} Mdv, \]
from (92), we can infer that
\[ \int_0^\infty \|(j_\epsilon, \tilde{j}_\epsilon)(\tau)\|_{H_x^2}^2 d\tau \lesssim \epsilon^2. \] (98)
Then from the forth equation of (95), we can infer that
\[ j_\epsilon \rightarrow j = \nabla \times B, \text{ in } L^2 ((0, +\infty); H_x^{s-1}). \] (99)

**Step 3, the limiting equation of \( u \).**

Furthermore, for velocity and temperature equation in (95), we can infer that
\[
\begin{aligned}
\partial_t u_\epsilon + \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{A}(\epsilon) f_\epsilon \cdot Mdv + \frac{1}{\epsilon} \nabla_x (\rho_\epsilon + \theta_\epsilon) &= n_\epsilon \cdot E_\epsilon + j_\epsilon \times B_\epsilon, \\
\partial_t \left( \frac{2}{3\epsilon} \theta_\epsilon - \frac{2}{9\epsilon} \rho_\epsilon \right) + \frac{2}{3\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{B}(\epsilon) f_\epsilon \cdot Mdv &= \epsilon \frac{2}{3} j_\epsilon \cdot E_\epsilon.
\end{aligned}
\] (100)
For the integration term in (100), recalling that
\[ \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{A}(\epsilon) f_\epsilon \cdot Mdv = \frac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{A}(\epsilon) f_\epsilon \cdot Mdv, \]
Based on (92), we can infer that
\[ \int_0^\infty \|rac{1}{\epsilon} \text{div} \int_{\mathbb{R}^3} \hat{A}(\epsilon)(\tau) Mdv\|_{H_x^{s-1}}^2 d\tau \leq C_0. \] (101)
With the help of (98) and (101), we can infer that
\[ \nabla_x (\rho_\epsilon + \theta_\epsilon) \rightarrow \rho + \theta = 0, \text{ in the distribution sense.} \] (102)
Now we can try to deduce the equation of $u$. Let $P$ be the Leray projection operator on torus, from (95), it follows that
\begin{equation}
\partial_t P u_\epsilon + \frac{1}{\epsilon} P \left( \text{div} \int_{\mathbb{R}^3} \hat{A} \mathcal{L} f_\epsilon Mdv \right) = P \left( n_\epsilon E_\epsilon + j_\epsilon \times B_\epsilon \right).
\end{equation}

Based on the first equation of (28), we can represent $\mathcal{L}(f_\epsilon)$ like this:
\begin{equation}
\frac{1}{\epsilon} \mathcal{L}(f_\epsilon) = -v \cdot \nabla_x f_\epsilon - \epsilon \partial_t f_\epsilon + \Gamma(f_\epsilon, f_\epsilon) + \epsilon E_\epsilon \cdot v \cdot h_\epsilon - (\epsilon E_\epsilon + v \times B_\epsilon) \cdot \nabla_v h_\epsilon
= \Gamma(f_\epsilon, f_\epsilon) - v \cdot \nabla_x f_\epsilon + R_1(\epsilon).
\end{equation}

By simple calculation (see [2, 4, 3]), it follows that
\begin{equation}
\int_{\mathbb{R}^3} \hat{A} \cdot \frac{1}{\epsilon} \mathcal{L} f_\epsilon Mdv = u_\epsilon \otimes u_\epsilon - \frac{\ln\epsilon^2}{\epsilon^2} I - \mu \left( \nabla_x u_\epsilon + \nabla^T_\epsilon u_\epsilon - \frac{4}{3} \text{div} u_\epsilon I \right) - R_f(\epsilon)
\end{equation}
with
\begin{equation}
R_f(\epsilon) := \int_{\mathbb{R}^3} \hat{A} \cdot (R_1(\epsilon) - v \cdot \nabla_x f_\epsilon^1 + \Gamma(f_\epsilon^1, f_\epsilon) + \Gamma(f_\epsilon, f_\epsilon^1)) Mdv,
\end{equation}

and
\begin{equation}
\mu = \frac{15}{13} \sum_{1 \leq i \leq 3} \int_{\mathbb{R}^3} A_{ij} \hat{A}_{ij} Mdv.
\end{equation}

According to (92), the microscopic part is $O(\epsilon)$ in $H_x^s$ sense. Thus, in the distributional sense
\begin{equation}
R_f(\epsilon) \to 0.
\end{equation}

Based on (92), (98), (101) and (103), it follows that
\begin{equation}
\partial_t P u_\epsilon \in H_x^{s-1}.
\end{equation}

By Aubin-Lions-Simon theorem (see [6]), we can infer the following strong convergence with time:
\begin{equation}
P u_\epsilon \in C((0, +\infty; H_x^{s-1}); P u_\epsilon \to u, \text{ in } C((0, +\infty; H_x^{s-1}).
\end{equation}

Then according to (92) and (104), in the distributional sense,
\begin{equation}
P u_\epsilon \to u, \quad \frac{1}{\epsilon} P \left( \text{div} \int_{\mathbb{R}^3} \hat{A} \mathcal{L} h_\epsilon Mdv \right) \to u \cdot \nabla u - \mu \Delta u.
\end{equation}

Finally, from (94), (99) and (93), we can finally deduce that
\begin{equation}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = \nabla \times B \times B.
\end{equation}

**Step 4, the limiting equation of $\theta$.**

By the similar way of deducing (104), for the approximate temperature equation, we can infer that
\begin{equation}
\partial_t \left( \frac{2}{5} \theta_\epsilon - \frac{2}{5} \rho_\epsilon \right) + \text{div}(u_\epsilon \theta) - \kappa \Delta \theta_\epsilon = \frac{2}{5} j_\epsilon \cdot E_\epsilon + \text{div} R_\theta(\epsilon),
\end{equation}

with
\begin{equation}
\frac{2}{5} \int_{\mathbb{R}^3} \hat{B} \cdot \frac{1}{\epsilon} \mathcal{L} h_\epsilon Mdv = u_\epsilon \cdot \theta_\epsilon - \kappa \nabla \theta_\epsilon - R_\theta(\epsilon)
\end{equation}

and
\begin{equation}
R_\theta(\epsilon) := \frac{5}{2} \int_{\mathbb{R}^3} \hat{B} \cdot (R_1(\epsilon) - v \cdot \nabla_x f_\epsilon^1 + \Gamma(f_\epsilon^1, f_\epsilon) + \Gamma(f_\epsilon, f_\epsilon^1)) Mdv, \quad \kappa = \frac{2}{15} \sum_{1 \leq i \leq 3} \int_{\mathbb{R}^3} B_i \hat{B}_i Mdv.
\end{equation}

By the similar way of deducing (106), we can infer that
\begin{equation}
\left( \frac{2}{5} \theta_\epsilon - \frac{2}{5} \rho_\epsilon \right) \in C((0, +\infty; H_x^{s-1}); \left( \frac{2}{5} \theta_\epsilon - \frac{2}{5} \rho_\epsilon \right) \to \theta, \text{ in } C((0, +\infty; H_x^{s-1}).
\end{equation}

Based on (94) and (98), the right hand of (109) will go to zero in the distributional sense. Finally, we have
\begin{equation}
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0.
\end{equation}

**Step 5, the Ohm’s law.**
Based on the previous analysis in this section, (95) turns to
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= (\nabla \times B) \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} u &= \text{div} B = 0, \quad \rho + \theta = 0, \\
\partial_t B + \nabla \times E &= 0.
\end{aligned}
\tag{113}
\]

We need to represent $E$ in another way. This useful relation hides in the Ohm’s law. From (67),
\[
j_e = -\epsilon^2 \partial_t j_e + \sigma E_e - \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v h_e \text{Md}v + \epsilon \int_{\mathbb{R}^3} \tilde{v} N_1 \text{Md}v.
\]

For the last two terms in the above equation, recalling the microscopic parts of $f_e$ and $h_e$ are $O(\epsilon)$, it follows that
\[
\text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v h_e \text{Md}v = \text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v h_e^+ \text{Md}v + \sigma \nabla \cdot n_e = \sigma \nabla \cdot n_e + O(\epsilon),
\]
\[
\epsilon \int_{\mathbb{R}^3} \tilde{v} N_1 \text{Md}v = - \int_{\mathbb{R}^3} (v \times B \cdot \nabla \nu f_e) \tilde{v} \text{Md}v + n_e \int_{\mathbb{R}^3} \Gamma(1, P f_e) \tilde{v} \text{Md}v + R_4(\epsilon),
\tag{114}
\]
with
\[
R_4(\epsilon) = \epsilon \int_{\mathbb{R}^3} (E_e, \nabla \nu (M f_e)) \tilde{v} \text{d}v + \int_{\mathbb{R}^3} \Gamma(\hat{h}_e^+, f_e) \tilde{v} \text{Md}v + \int_{\mathbb{R}^3} \Gamma(h_e, f_e^+) \tilde{v} \text{Md}v.
\]

According to (94), (92) and (93), in the distributional sense:
\[
\text{div} \int_{\mathbb{R}^3} \tilde{v} \otimes v h_e \text{Md}v \rightarrow 0, \quad \epsilon \int_{\mathbb{R}^3} \tilde{v} N_1 \text{Md}v \rightarrow \sigma u \times B.
\tag{115}
\]

Based on (99) and (98), in the distributional sense, we have
\[
j_e \rightarrow j = \nabla \times B = \sigma (E + u \times B).
\tag{116}
\]

Then (113) becomes to MHD system
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla P &= (\nabla \times B) \times B, \\
\partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} u &= \text{div} B = 0, \quad \rho + \theta = 0, \\
\partial_t B - \frac{1}{\sigma} \Delta B &= \nabla \times (u \times B).
\end{aligned}
\tag{117}
\]

### APPENDIX A. Formal derivation

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