CAUSAL THERMODYNAMICS
IN RELATIVITY

Lectures given at the
Hanno Rund Workshop on Relativity and Thermodynamics
Natal University, South Africa, June 1996

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Chapter 1

Relativistic Fluid Dynamics

In this chapter I review the basic theory of fluid kinematics and dynamics (without dissipation) in relativistic spacetime. The classic paper in this field is Ellis’ 1971 review [1]. That paper is at a more advanced level than these lectures. For a basic introduction to tensors, relativity and fluids, see for example [2].

I use units in which the speed of light in vacuum, Einstein’s gravitational constant and Boltzmann’s constant are all one:

\[ c = 8\pi G = k = 1 \]

I use \( A \equiv B \) to denote equality of \( A \) and \( B \) in an instantaneous orthonormal frame at a point (defined below).

1.1 Brief Review of Relativity

The observed universe is a 4 dimensional spacetime. Physical laws should be expressible as equations in spacetime that are independent of the observer. Together with experimental and observational evidence, and further principles, this leads to Einstein’s relativity theory - special relativity in the case where the gravitational field may be neglected, and general relativity when gravity is incorporated.

Local coordinates, which are typically based on observers, are usually chosen so that \( x^0 \) is a time parameter and \( x^i \) are space coordinates. A change of coordinates (or of observers) is

\[ x^\alpha = (x^0, x^i) = (t, \vec{x}) \quad \rightarrow \quad x'^\alpha = (x'^0, x'^i) = (t', \vec{x}') \quad (1.1) \]

Physical laws should then be invariant under such transformations. This means that these laws are expressible in terms of tensor fields and tensor–derivatives. Tensors have different types \((r, s)\), but they all transform linearly under \((1.1)\). The simplest example is a scalar, which is invariant. Using the chain rule, the transformation of the coordinate differentials is seen to be linear:

\[ dx'^\alpha = \sum_\alpha \frac{\partial x'^\alpha}{\partial x^\alpha} dx^\alpha \equiv \frac{\partial x'^\alpha}{\partial x^\alpha} dx^\alpha \]
Extending this to partial derivatives of scalars and generalising, we are led to the transformation properties of tensors in general:

\begin{align*}
(0, 0) \text{ scalar} & \quad f \rightarrow f \\
(1, 0) \text{ vector} & \quad u^\alpha \rightarrow u^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} u^\alpha \\
(0, 1) \text{ covector} & \quad k_\alpha \rightarrow k_{\alpha'} = \frac{\partial x^\alpha}{\partial x'^\alpha} k_\alpha \\
(1, 1) \text{ tensor} & \quad T^{\alpha}_{\beta} \rightarrow T^{\alpha'}_{\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\beta} T^{\alpha}_{\beta} \\
\cdots & \quad \cdots \\
(r, s) \text{ tensor} & \quad J^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \rightarrow J^{\alpha'_1 \cdots \alpha'_r}_{\beta'_1 \cdots \beta'_s} = \frac{\partial x^{\alpha'_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\alpha'_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta'_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x^{\beta'_s}}{\partial x^{\beta_s}} J^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} \quad (1.2)
\end{align*}

It follows that if a tensor vanishes in one coordinate frame, it vanishes in all frames. Consequently, if two tensors are equal in one frame, they are equal in all frames.

Fields and equations that transform according to \((1.2)\) are called tensorial or covariant. Restricted covariance arises when the class of allowable coordinate systems is restricted. In special relativity (flat spacetime), one can choose orthonormal coordinates \(x^\alpha\) which correspond to inertial observers, and if \(x^{\alpha'}\) is required to be also orthonormal, then

\[
\frac{\partial x^{\alpha'}}{\partial x^\alpha} = \Lambda_{\alpha}^{\alpha'} \quad \iff \quad x^{\alpha'} = \Lambda_{\alpha}^{\alpha'} x^\alpha + C^\alpha \quad (1.3)
\]

where \(\Lambda, C\) are constants and \(\Lambda\) is a Lorentz matrix. In other words, special relativity says that the laws of physics (leaving aside gravity) are invariant under Lorentz transformations that connect any inertial observers in relative motion. Under this restriction, the partial derivatives of tensors transform according to \((1.2)\), i.e. they are Lorentz covariant. We use the notation

\[
J^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s} = \frac{\partial x^{\alpha_1}}{\partial x^{\alpha'_1}} \cdots \frac{\partial x^{\alpha_r}}{\partial x^{\alpha'_r}} \frac{\partial x^{\beta_1}}{\partial x^{\beta'_1}} \cdots \frac{\partial x^{\beta_s}}{\partial x^{\beta'_s}} J^{\alpha'_1 \cdots \alpha'_r}_{\beta'_1 \cdots \beta'_s} \quad (1.9)
\]

for partial derivatives. Thus in special relativity, physical laws are expressed in orthonormal coordinates as PDE’s; for example the Klein–Gordon equation for a massless scalar field is

\[
\Box \Psi \equiv \eta^{\alpha\beta} \partial_\alpha \partial_\beta \Psi = 0 \quad (1.5)
\]

where

\[
\eta_{\alpha\beta} = \text{diag} (-1, 1, 1, 1) = \eta^{\alpha\beta} \quad (1.6)
\]

are the orthonormal components of the metric tensor.

The metric \(g_{\alpha\beta}\) of any (in general curved) spacetime determines the spacetime interval between events, the scalar product of vectors, and the raising and lowering of indices on general tensors:

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (1.7)
\]
\[
u \cdot v = g_{\alpha\beta} u^\alpha v^\beta = u^\alpha v_\alpha = u_\alpha v^\alpha \quad (1.8)
\]
\[
J^{\alpha}_{\beta\mu} = g^{\alpha\nu} g_{\beta\sigma} J^{\sigma}_{\nu \mu}, \quad \text{etc.} \quad (1.9)
\]
where the inverse metric is defined by $g^{\alpha \mu} g_{\mu \beta} = \delta^{\alpha \beta}$.

The metric is a symmetric tensor. For any rank–2 tensor, we can define its covariant symmetric and skew parts:

$$V_{(\alpha \beta)} = \frac{1}{2} (V_{\alpha \beta} + V_{\beta \alpha}), \quad V_{[\alpha \beta]} = \frac{1}{2} (V_{\alpha \beta} - V_{\beta \alpha})$$

so that $g_{\alpha \beta} = g_{(\alpha \beta)}$.

At any point (or event) $P$, an observer can choose coordinates $x^\alpha$ that bring $g_{\alpha \beta}(P)$ into orthonormal form. I will call such a coordinate system an instantaneous orthonormal frame (IOF), characterised by

$$g_{\alpha \beta} \doteq \eta_{\alpha \beta} \Leftrightarrow g_{\alpha \beta}(P) \big|_{\text{iof}} = \eta_{\alpha \beta}$$

At each event along the observer’s worldline, the IOF is in general different. In fact an IOF is orthonormal in a neighbourhood of the original point $P$ if and only if the spacetime is locally flat.

In curved spacetime, the partial derivative (1.3) is not covariant (except when $J$ is a scalar). The metric defines a connection that ‘corrects’ for the variations in the coordinate basis (equivalently, that provides a rule for parallel transport of vectors):

$$\Gamma^\alpha_{\beta \gamma} = \frac{1}{2}g^{\alpha \mu} (g_{\mu \beta, \gamma} + g_{\mu \gamma, \beta} - g_{\beta \gamma, \mu}) = \Gamma^\alpha_{(\beta \gamma)}$$

The connection, which is not a tensor since it corrects for non–tensorial variations, defines a covariant derivative

$$f_{\alpha} = f_{\alpha}, \quad u^\alpha_{;\beta} = u^\alpha_{,\beta} + \Gamma^\alpha_{\mu \beta} u^\mu, \quad k_{\alpha;\beta} = k_{\alpha,\beta} - \Gamma^\mu_{\alpha \beta} k_\mu, \quad \ldots$$

$$J^{\alpha \cdots;\beta,\sigma} = J^{\alpha \cdots;\beta,\sigma} + \Gamma^\alpha_{\mu \sigma} J^{\mu \cdots;\beta} + \cdots - \Gamma^\mu_{\beta \sigma} J^{\alpha \cdots;\mu}$$

We also write $\nabla_{\sigma} J^{\alpha \cdots;\beta}$ for the covariant derivative. One can always find an IOF at any event $P$ such that the connection vanishes at $P$:

$$\Gamma^\alpha_{\beta \gamma} \doteq 0 \Rightarrow J^{\alpha \cdots;\beta,\mu} \doteq J^{\alpha \cdots;\beta,\mu}$$

From now on, any IOF will be assumed to have this property.

The connection also defines a covariant measure of spacetime curvature – the Riemann tensor:

$$R^\alpha_{\beta \mu \nu} = -\Gamma^\alpha_{\beta \mu, \nu} + \Gamma^\alpha_{\beta \nu, \mu} + \Gamma^\alpha_{\sigma \mu} \Gamma^\sigma_{\beta \nu} - \Gamma^\alpha_{\sigma \nu} \Gamma^\sigma_{\beta \mu}$$

Curvature is fundamentally reflected in the non–commutation of covariant derivatives\footnote{except for scalars: $f_{[\alpha \beta]} = 0.$} as given by the Ricci identity

$$u_{\alpha;\beta \gamma} - u_{\alpha;\gamma \beta} = R^\mu_{\alpha \beta \gamma} u_\mu$$

\begin{align*}
\end{align*}
and its generalisations for higher rank tensors. The trace–free part of the Riemann tensor is the Weyl tensor $C_{\alpha \beta \mu \nu}$, which represents the ‘free’ gravitational field and describes gravity waves, while the trace gives the Ricci tensor and Ricci scalar $R_{\alpha \beta} = R^\mu_{\alpha \mu \beta}$, $R = R^\alpha_{\alpha}$, which are determined by the mass–energy–momentum distribution via Einstein’s field equations

\[ R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta} = T_{\alpha \beta} \]

where $T_{\alpha \beta}$ is the energy–momentum tensor, discussed below. The Ricci tensor obeys the contracted Bianchi identity

\[ (R_{\alpha \beta} - \frac{1}{2} R g_{\alpha \beta})_{;\beta} = 0 \]

### 1.2 Fluid Kinematics

Consider the motion of a particle with rest mass $m$. An observer records the particle’s history – its worldline – as $x^\alpha = (t, x^i(t))$. We need a covariant (observer–independent) description of the worldline and velocity of the particle. If $m > 0$, then along the worldline $ds^2 < 0$ (the particle moves slower than light). If $\tau$ is the time recorded by a clock comoving with the particle, the worldline is given by $x^\alpha = x^\alpha(\tau)$, independently of any observer. The covariant comoving time is called the proper time. In an IOF $ds^2 = -d\tau^2$. Since both sides of this equation are tensors (scalars), the equation holds in any frame, and at all points along the worldline, i.e. $ds^2 = -d\tau^2$. The kinematics of the particle are covariantly described by the 4–velocity

\[ u^\alpha = \frac{dx^\alpha}{d\tau} \Rightarrow u^\alpha u_\alpha = -1 \]

and the 4–acceleration

\[ \dot{u}^\alpha = u^\alpha;\beta u^\beta \]

where $\dot{u}^\alpha u_\alpha = 0$. The particle moves in free–fall, subject to no non–gravitational forces, if and only if $\dot{u}_\alpha = 0$, in which case its worldline is a (timelike) geodesic. In the observer’s IOF

\[ u^\alpha \doteq \gamma(v)(1, \frac{d\vec{x}}{dt}) = \gamma(1, \vec{v}), \quad \gamma(v) = (1 - v^2)^{-1/2} = \frac{dt}{d\tau} \]

where $t$ is the observer’s proper time at that point, and $\vec{v}$ is the measured velocity of the particle.

If $m = 0$, the particle (photon or massless neutrino or graviton) moves at the speed of light, and along its worldline $ds^2 = 0$, so that proper time cannot parametrise the worldline. In the IOF of an observer $u^\alpha$, the light ray has angular frequency $\omega$ and wave vector $\vec{k}$ (where $|\vec{k}| = \omega$), with phase $\phi \doteq \vec{k} \cdot \vec{x} - \omega t$, so that

\[ \phi_{,\alpha} \doteq (-\omega, \vec{k}) \quad \text{and} \quad \phi_{,\alpha} \phi^\alpha \doteq 0 \]

Now the phase is a covariant scalar, and its gradient is a covariant null vector, which we call the 4–wave vector, and which is geodesic:

\[ k_\alpha = \phi_{,\alpha} \quad \text{and} \quad k_\alpha k^\alpha = 0 \Rightarrow k^\alpha;_\beta k^\beta = 0 \]
From the above, in the observer’s IOF, \( \omega = -k_\alpha u^\alpha = \dot{\phi} \). This gives a covariant expression for the redshift between events \( E \) (‘emitter’) and \( R \) (‘receiver’) along a ray:

\[
1 + z \equiv \frac{\omega_E}{\omega_R} = \frac{(u_\alpha k^\alpha)_E}{(u_\alpha k^\alpha)_R}
\]  

A fluid is modelled as a continuum with a well–defined average 4–velocity field \( u^\alpha \), where \( u^\alpha u_\alpha = -1 \). This hydrodynamic description requires that the mean collision time is much less than any macroscopic characteristic time (such as the expansion time in an expanding universe); equivalently, the mean free path must be much less than any macroscopic characteristic length. For a perfect fluid, \( u^\alpha \) is uniquely defined\(^2\) as the 4–velocity relative to which there is no particle current, i.e.

\[
n^\alpha = nu^\alpha
\]

where \( n \) is the number density.

The field of comoving observers \( u^\alpha \) defines a covariant splitting of spacetime into time + space \((1 + 3)\) via the projection tensor

\[
h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta = h_{\beta\alpha} \Rightarrow \begin{cases} h_{\alpha\beta} u^\beta = 0, & h_{\alpha\mu} h_{\mu\beta} = h_{\alpha\beta} \\ h_{\alpha\alpha} = 3, & h_{\alpha\beta} q^\beta = q_\alpha \text{ if } q_\alpha u^\alpha = 0 \end{cases}
\]

which projects at each point into the instantaneous rest space of the fluid/ observer, and provides a 3–metric in the rest space. In the comoving IOF

\[
u^\alpha \doteq (1, \vec{0}), \quad h_{\alpha\beta} \doteq \text{diag} (0, 1, 1, 1), \quad h_{\alpha\beta} q^\beta = \vec{q} \cdot \vec{q}
\]

where \( q_\alpha u^\alpha = 0 \). This allows us to compare relativistic fluid kinematics and dynamics with its Newtonian limit.

The covariant time derivative along \( u^\alpha \) is

\[
\dot{A}^{\alpha \beta \cdots \gamma} = A^{\alpha \beta \cdots \gamma} u^\mu
\]

and describes the rate–of–change relative to comoving observers. In the comoving IOF

\[
\dot{A}^{\alpha \beta \cdots \gamma} \doteq \frac{d}{d\tau} A^{\alpha \beta \cdots \gamma}
\]

The covariant spatial derivative is

\[
D_\alpha f = h_\alpha^\beta f_{\beta} 
\]

\[
D_\alpha q_\beta = h_\alpha^{\mu} h_\beta^{\nu} \nabla_\mu q_\nu
\]

\[
D_\alpha \sigma_{\beta\gamma} = h_\alpha^{\mu} h_\beta^{\nu} h_\gamma^{\kappa} \nabla_\mu \sigma_{\nu\kappa}, \quad \text{etc.}
\]

\(^2\)If the fluid is out of equilibrium as a result of dissipative effects, then there is no unique average 4–velocity
and describes spatial variations relative to comoving observers. In the comoving IOF, with $q^\alpha \equiv (q^0, \vec{q})$

$$D_\alpha f \equiv (0, \vec{\nabla} f), \quad D^\alpha q_\alpha \equiv \vec{\nabla} \cdot \vec{q}, \quad \varepsilon^{ijk} D_j q_k \equiv (\vec{\nabla} \times \vec{q})^i$$ (1.31)

Any spacetime vector can be covariantly split as

$$V^\alpha = Au^\alpha + B^\alpha, \quad \text{where} \quad A = -u_\alpha V^\alpha, \quad B^\alpha = h^\alpha_\beta V^\beta \Leftrightarrow B^\alpha u_\alpha = 0$$ (1.32)

For a rank–2 tensor:

$$V_{\alpha\beta} = Au_\alpha u_\beta + B_\alpha u_\beta + u_\alpha C_\beta + F_{\alpha\beta}$$ (1.33)

where $A = V_{\alpha\beta} u^\alpha u^\beta$, $B_\alpha u^\alpha = 0 = C_\alpha u^\alpha$ and

$$F_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu V_{\mu\nu} \Leftrightarrow F_{\alpha\beta} u^\alpha = 0 = F_{\alpha\beta} u^\beta$$

For example, if $V_{\alpha\beta} = W_{\alpha\beta}$, then $F_{\alpha\beta} = D_\beta W_\alpha$. Now $F_{\alpha\beta}$ may be further decomposed into symmetric and skew parts:

$$F_{\alpha\beta} = F_{(\alpha\beta)} + F_{[\alpha\beta]}$$

In the comoving IOF, the corresponding decomposition of the matrix of components $F_{ij}$ is simply

$$F \doteq (F) + [F] = \frac{1}{2} \left( F + F^T \right) + \frac{1}{2} \left( F - F^T \right)$$

and $(F)$ may be further split into its trace and trace–free parts:

$$(F) \doteq \left\{ \frac{1}{3} \text{tr} F \right\} I + \langle F \rangle$$

The covariant expression of this is

$$F_{(\alpha\beta)} = \left\{ \frac{1}{3} F^{\gamma\gamma} \right\} h_{\alpha\beta} + F_{<\alpha\beta>}$$

where the symmetric, spatial trace–free part of any tensor is defined by

$$V_{<\alpha\beta>} = h_\alpha^{\mu} h_\beta^{\nu} \left\{ V_{(\mu\nu)} - \frac{1}{3} V_{\alpha\kappa} h^{\sigma\kappa} h_{\mu\nu} \right\}$$ (1.34)

Thus we can rewrite the decomposition (1.33) in the covariant irreducible form

$$V_{\alpha\beta} = Au_\alpha u_\beta + B_\alpha u_\beta + u_\alpha C_\beta + \frac{1}{3} V_{\mu\nu} h^{\mu\nu} h_{\alpha\beta} + V_{<\alpha\beta>} + V_{[\mu\nu]} h^{\mu}_{\alpha} h^{\nu}_{\beta}$$ (1.35)

Now we are ready to define the quantities that covariantly describe the fluid kinematics. These quantities are simply the irreducible parts of the covariant derivative of the fluid
4-velocity. With $V_{\alpha\beta} = u_{\alpha\beta}$, we have $A = 0 = C_{\alpha}$ since $u_{\alpha\beta} u^\alpha = 0$, and then $B_{\alpha} = -u_{\alpha\beta} u^\beta = -\dot{u}_\alpha$. Thus (1.33) gives

$$
\begin{align*}
3H &= u_{\alpha\beta} = H h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \dot{u}_\alpha u_\beta \quad \text{where} \quad 3H = u^\alpha_{\ ;\alpha} = D^\alpha u_\alpha, \\
\sigma_{\alpha\beta} &= u_{<\alpha;\beta>} = D_{<\beta} u_{>\alpha>,} \quad \omega_{\alpha\beta} = h_{\alpha\mu} h_{\beta\nu} u_{[\mu;\nu]} = D_{[\beta} u_{\alpha]}.
\end{align*}$$  

(1.36)

In a comoving IOF at a point $P$, $\vec{v}$ is zero at $P$, but its derivatives are not, and we find using (1.31) that

$$3H \div \vec{\nabla} \cdot \vec{v}, \quad \varepsilon^{ijk} \omega_{jk} \div -\left(\vec{\nabla} \times \vec{v}\right)^i$$

so that $H$ generalises the Newtonian expansion rate and $\omega_{\alpha\beta}$ generalises the Newtonian vorticity. Similarly, it can be seen that $\sigma_{\alpha\beta}$ is the relativistic generalisation of the Newtonian shear. These kinematic quantities therefore have the same physical interpretation as in Newtonian fluids. A small sphere of fluid defined in the IOF of a comoving observer at $t = 0$, and then measured in the observer’s IOF a short time later, undergoes the following changes:

- due to $H$, its volume changes but not its spherical shape;
- due to $\sigma_{\alpha\beta}$, its volume is unchanged but its shape is distorted in a way defined by the eigenvectors (principal axes) of the shear;
- due to $\omega_{\alpha\beta}$, its volume and shape are unchanged, but it is rotated about the direction $\vec{\nabla} \times \vec{v}$.

The expansion rate defines a comoving scale factor $a$ that determines completely the volume evolution:

$$H = \frac{\dot{a}}{a}$$  

(1.37)

### 1.3 Conservation Laws - Perfect Fluids

Assuming there are no unbalanced creation/annihilation processes, particle number is conserved in the fluid. In an IOF, this is expressed via the continuity equation

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \vec{v}) = 0$$

By (1.23), the covariant form of particle conservation is

$$n^\alpha_{\ ;\alpha} = 0 \quad \Leftrightarrow \quad \dot{n} + 3H n = 0 \quad \Leftrightarrow \quad na^3 = \text{comoving const}$$  

(1.38)

where (1.37) was used to show that the comoving particle number $N \propto na^3$ is constant.

A perfect fluid is described by its 4-velocity $u^\alpha$, number density $n$, energy (or mass-energy) density $\rho$, pressure $p$ and specific entropy $S$. In a comoving IOF, the pressure is isotropic and given by the Newtonian stress tensor $\tau_{ij} = p \delta_{ij}$. This can be covariantly combined with the energy density into the symmetric energy-momentum tensor

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + ph_{\alpha\beta}$$  

(1.39)

\[\text{3The form of the energy–momentum tensor may be justified via relativistic kinetic theory}\]
so that \( T_{00} \equiv \rho = \text{energy density}, \ T_{ij} \equiv \tau_{ij} = \text{momentum density}, \ T_{0i} \equiv 0. \) Just as the divergence of \( n^{\alpha} \) produces a conservation law (1.38), so too does the divergence of \( T^{\alpha\beta}. \)

\[
T^{\alpha\beta} ;_{\beta} = 0 \ \Rightarrow \ \dot{\rho} + 3H(\rho + p) = 0 \quad (1.40)
\]

(\( \rho + p \)) \( \dot{u}_{\alpha} + D_{\alpha} p = 0 \quad (1.41) \)

In a comoving IOF these become

\[
\frac{\partial \rho}{\partial t} + (\rho + p) \nabla \cdot \vec{v} = 0, \quad (\rho + p) \frac{\partial \vec{v}}{\partial t} = -\nabla p
\]

so that (1.40) is an energy conservation equation, generalising the mass conservation equation of Newtonian fluid theory, while (1.41) is a momentum conservation equation, generalising the Euler equation. (In relativity, the pressure contributes to the effective energy density.) The energy–momentum conservation equation also follows from Einstein’s field equations (1.18) and the contracted Bianchi identity (1.19). Equivalently, the conservation equation ensures that the identity holds, i.e. that this integrability condition of the field equations is satisfied.

Finally, the entropy is also conserved. In a comoving IOF, there is no entropy flux, and the specific entropy \( S \) is constant for each fluid particle. The covariant expression of this statement is

\[
S^{\alpha} ;_{\alpha} = 0 \quad \text{where} \quad S^{\alpha} = Sn^{\alpha} \ \Rightarrow \ \dot{S} = 0 \quad (1.42)
\]

where (1.38) was used. Note that \( S \) is constant along fluid particle worldlines, and not throughout the fluid in general. If \( S \) is the same constant on each worldline – i.e. if \( D_{\alpha} S = 0 \) as well as \( \dot{S} = 0 \), so that \( S,_{\alpha} = 0 \) – then the fluid is called isentropic.

### 1.4 Equilibrium Thermodynamics

A perfect fluid is characterised by \((n^{\alpha}, S^{\alpha}, T^{\alpha\beta})\), or equivalently by \((n, \rho, p, S, u^{\alpha})\), subject to the conservation laws above. What are the further relations amongst the thermodynamic scalars \( n, \rho, p, S \) and \( T \), the temperature? Firstly, the temperature is defined via the Gibbs equation

\[
TdS = d \left( \frac{\rho}{n} \right) + pd \left( \frac{1}{n} \right) \quad (1.43)
\]

where \( df = f^{\alpha}_{,\alpha} dx^{\alpha} \). Secondly, thermodynamical equations of state are needed in order to close the system of equations. Equations of state are dependent on the particular physical properties of the fluid, and are deduced from microscopic physics (i.e. kinetic theory and statistical mechanics), or from phenomenological arguments. In fact, assuming the metric is known (and so leaving aside Einstein’s field equations), there are 7 equations – i.e. (1.38), (1.40), (1.41), (1.42), (1.43) – for 8 variables – i.e. \( n, \rho, p, u_{i}, S, T \). Thus a single scalar equation of state will close the system.

The Gibbs equation shows that in general two of the thermodynamical scalars are needed as independent variables. For example, taking \( n, \rho \) as independent, the remaining thermodynamical scalars are \( p(n, \rho), S(n, \rho), T(n, \rho) \), and given any one of these, say \( p = p(n, \rho) \), the others will be determined. Often a barotropic equation of state for the pressure
is assumed, i.e. \( p = p(\rho) \). By the Gibbs equation, this implies \( S \) is constant (see below), i.e. the fluid is isentropic.

The adiabatic speed of sound \( c_s \) in a fluid is given in general by

\[
c_s^2 = \left( \frac{\partial p}{\partial \rho} \right)_S
\]

(1.44)

For a perfect fluid, this becomes

\[
c_s^2 = \frac{\dot{p}}{\dot{\rho}}
\]

(1.45)

as can be seen by choosing \( \rho, S \) as independent variables, and using the fact that \( \dot{S} = 0 \):

\[
\dot{p} = \left( \frac{\partial p}{\partial \rho} \right)_S \dot{\rho} + \left( \frac{\partial p}{\partial S} \right)_{\rho} \dot{S}
\]

The preceding considerations are phenomenological and mathematical. If the fluid model is based on microscopic physics, further conditions are imposed. For example, if the fluid is a collision–dominated gas in equilibrium, then relativistic kinetic theory, based essentially on imposing energy–momentum conservation at a microscopic level, leads to stringent conditions\(^4\). If \( m > 0 \) is the rest mass of the particles and

\[
\beta_\mu = \frac{\beta}{m} u_\mu, \quad \beta = \frac{m}{T}
\]

then the following conditions hold:\(^5\)

\[
\beta_{(\mu,\nu)} = 0 \quad (1.46)
\]

\[
mn = c_0 \frac{K_2(\beta)}{\beta}, \quad p = nT \quad (1.47)
\]

\[
\rho = c_0 \left[ \frac{K_1(\beta)}{\beta} + 3 \frac{K_2(\beta)}{\beta^2} \right] \quad (1.48)
\]

where \( c_0 \) is a constant and \( K_n \) are modified Bessel functions of the second kind. Furthermore, (1.46) shows that \( \beta_\mu \) is a Killing vector field, so that the spacetime is stationary. In particular, (1.38) implies

\[
H = 0, \quad \dot{u}_\alpha = -D_\alpha \ln T, \quad \sigma_{\alpha\beta} = 0 \quad (1.49)
\]

and then (1.38), (1.40) lead to

\[
\dot{n} = \dot{\rho} = \dot{p} = \dot{T} = 0 \quad (1.50)
\]

Thus if the perfect fluid is a relativistic Maxwell–Boltzmann gas in equilibrium, severe restrictions are imposed not only on the fluid dynamics but also on the spacetime geometry.

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\(^4\)Note that kinetic theory incorporates assumptions about the interactions of particles, in particular that the interactions are described by the Boltzmann collision integral.

\(^5\)See [3]. In standard units, \( \beta = mc^2/kT \).
In the case of a gas of massless particles in collisional equilibrium, the conditions are
less severe:
\[ \beta_{(\mu;\nu)} = -\frac{T}{T^2} g_{\mu\nu} \Rightarrow H = -\frac{T}{T}, \quad \sigma_{\alpha\beta} = 0 \] (1.51)
\[ n = b_0 T^3, \quad \rho = 3p = 3nT \] (1.52)
Thus \( \beta_\mu \) is a conformal Killing vector field, so that expansion is possible in equilibrium.

Kinetic theory shows that a purely phenomenological approach to fluid thermodynamics holds potential problems in the form of hidden or unknown consistency conditions that may be violated. Any phenomenological model needs to be applied with caution.

The best motivated barotropic perfect fluid model is that for incoherent radiation or massless particles, for which \( p = \frac{1}{3} \rho \), as in (1.52). The energy conservation equation (1.40) integrates, on using (1.37):
\[ \rho = (\text{comoving const}) a^{-4} \] (1.53)
Cold, non–relativistic matter is often modelled as pressure–free ‘dust’, so that
\[ p = 0 \Rightarrow \rho = (\text{comoving const}) a^{-3} = mn \] (1.54)
A kinetic theory motivation for the dust model arises from (1.47), (1.48) in the limit \( \beta \gg 1 \):
\[ p = nT, \quad \rho \approx mn + \frac{3}{2} nT \quad \text{where} \quad T \ll m \] (1.55)
The energy density is \( \rho \approx n(m c^2 + \varepsilon) \), where \( m c^2 \) is the rest mass energy per particle, and \( \varepsilon = \frac{3}{2} kT \) is the thermal energy per particle. While (1.55) is still reasonable at high temperatures (e.g. for the electron, \( m \approx 10^9 \)K, and (1.55) should be very accurate for \( T \) up to about \( 10^6 \)K), the exact limiting dust case is only reasonable at low temperatures, when random velocities are negligible. Of course the hydrodynamic description is no longer valid in this limit.

We can find the evolution of the temperature easily in the case of radiation. Comparing (1.52) and (1.53), we get
\[ \text{radiation:} \quad T \propto \frac{1}{a} \] (1.56)
In the general case, the Gibbs equation (1.43) can be written as
\[ dS = -\left( \frac{\rho + p}{T n^2} \right) dn + \frac{1}{T n} d\rho \]
and the integrability condition
\[ \frac{\partial^2 S}{\partial T \partial n} = \frac{\partial^2 S}{\partial n \partial T} \]
becomes
\[ n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial p}{\partial \rho} \] (1.57)
Furthermore, since \( T = T(n, \rho) \), it follows on using number and energy conservation (1.38) and (1.40) that
\[ \dot{T} = -3H \left[ n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} \right] \]
and then (1.57) implies
\[ \frac{\dot{T}}{T} = -3H \left( \frac{\partial p}{\partial \rho} \right)_n \] (1.58)

From the derivation of (1.58), we see that it will hold identically if the Gibbs integrability condition, number conservation and energy conservation are satisfied.

This equation holds for any perfect fluid. For non–relativistic matter (1.55) gives
\[ p = \frac{2}{3} \left( \rho - mn \right) \]
so that (1.58) implies:
non–relativistic matter: \[ T \propto \frac{1}{a^2} \] (1.59)

This shows that the mean particle speed decays like \( a^{-1} \), since the thermal energy per particle is \( \varepsilon \approx \frac{1}{2} kT \approx \frac{1}{2} m \bar{v}^2 \). Strictly, the limiting case of dust has \( T = 0 \), but if dust is understood as negligible pressure and temperature rather than exactly zero pressure, then (1.59) holds.

Note that the Gibbs integrability condition shows explicitly that one cannot independently specify equations of state for the pressure and temperature. This is clearly illustrated in the barotropic case.

**Barotropic Perfect Fluids**

With \( \rho, p \) as the independent variables in the Gibbs equation (1.43) in the general perfect fluid case, we find:
\[ \frac{n^2 T}{(\rho + p)} dS = - \left[ \frac{\partial n}{\partial \rho} d\rho + \frac{\partial n}{\partial p} dp \right] + \frac{n}{(\rho + p)} d\rho 
= \left[ \frac{n}{\rho + p} - \frac{\partial n}{\partial \rho} \right] d\rho - \frac{\partial n}{\partial p} dp 
= \left[ \frac{n}{\rho + p} - \frac{\dot{\rho}}{\dot{\rho}} + \frac{\dot{\rho}}{\dot{\rho}} \frac{\partial n}{\partial \rho} \right] d\rho - \frac{\partial n}{\partial p} dp 
= \frac{\dot{\rho}}{\dot{\rho}} \frac{\partial n}{\partial \rho} - \frac{\partial n}{\partial p} dp 
\]

where we used the conservation equations (1.38) and (1.40). Thus, for any perfect fluid
\[ n^2 T dS = (\rho + p) \frac{\partial n}{\partial \rho} \left[ \frac{\dot{\rho}}{\dot{\rho}} d\rho - dp \right] \] (1.60)

Suppose now that the pressure is barotropic: \( p = p(\rho) \). It follows immediately from (1.60) that \( dS = 0 \), i.e. the fluid is isentropic.

The same conclusion follows in the case of barotropic temperature. Choosing \( \rho, T \) as the independent variables, we find
\[ n^2 T dS = (\rho + p) \frac{\partial n}{\partial T} \left[ \frac{\dot{T}}{\dot{\rho}} d\rho - dT \right] \]
so that $T = T(\rho)$ implies $dS = 0$.

If the pressure and temperature are barotropic, then the Gibbs integrability condition (1.57) strongly restricts the form of $T(\rho)$:

$$\begin{align*}
p &= p(\rho) \quad \text{and} \quad T &= T(\rho) \quad \Rightarrow \quad T &\propto \exp \int \frac{dp}{\rho(p) + p} \quad (1.61)
\end{align*}$$

The radiation and dust models are cases of a linear barotropic equation of state that is often used for convenience

$$p = (\gamma - 1)\rho \quad \Rightarrow \quad \rho = \text{(comoving const)}a^{-3\gamma} \quad (1.62)$$

By (1.45), the speed of sound is $c_s = \sqrt{\gamma - 1}$. For fluids which have some basis in kinetic theory, one can impose the restriction $1 \leq \gamma \leq \frac{4}{3}$. In principle $\frac{4}{3} < \gamma \leq 2$ still leads to an allowable speed of sound ($\gamma = 2$ is known as ‘stiff matter’). The false vacuum of inflationary cosmology may be formally described by the case $\gamma = 0$.

If (1.62) holds then the Gibbs integrability condition (1.57) becomes

$$n\frac{\partial T}{\partial n} + \gamma \rho \frac{\partial T}{\partial \rho} = (\gamma - 1)T$$

whose solution by the method of characteristics yields

$$T = \rho^{(\gamma - 1)/\gamma} F \left( \frac{\rho^{1/\gamma}}{n} \right) \quad (1.63)$$

where $F$ is an arbitrary function. By (1.38) and (1.62), $F$ is a comoving constant, i.e. $\dot{F} = 0$. If $T$ is also barotropic, then $F$ is constant and we have a power–law form with fixed exponent for the temperature:

$$T \propto \rho^{(\gamma - 1)/\gamma} \quad (1.64)$$

The same result follows directly from (1.61).

Note that (1.62) and (1.64) are consistent with the ideal gas law $p = nT$. For dissipative fluids, this is no longer true.

### 1.5 Example: Cosmological Fluids

The Ricci identity (1.16) for the fluid 4–velocity, appropriately projected and contracted, together with the field equations (1.18), leads to an evolution equation for the expansion rate

$$3\dot{H} + 3H^2 - \ddot{u}_\alpha^\alpha + \sigma_{\alpha\beta} \sigma^{\alpha\beta} - \omega_{\alpha\beta} \omega^{\alpha\beta} = -\frac{1}{2}(\rho + 3p) \quad (1.65)$$

known as Raychaudhuri’s equation.

In the standard FRW cosmological models, the rest spaces of comoving observers mesh together to form spacelike 3–surfaces $\{ t = \text{const} \}$, where $t$ is proper time for comoving observers. Each comoving observer sees that there are no preferred spatial directions - i.e.
the cosmic 3–surfaces are spatially isotropic and homogeneous. Thus for any covariant scalar \( f \) and vector \( v_\alpha \)
\[
D_\alpha f = 0 \quad [\Leftrightarrow f = f(t)] , \quad h_\alpha^\beta v_\beta = 0 \quad [\Leftrightarrow v_\alpha = V(t)u_\alpha]
\]
and
\[
u_{\alpha;\beta} \propto h_{\alpha\beta} \Leftrightarrow \dot{u}_\alpha = 0 , \quad \sigma_{\alpha\beta} = 0 = \omega_{\alpha\beta}
\]
Raychaudhuri’s equation (1.65) reduces to
\[
3\dot{H} + 3H^2 = -\frac{1}{2}(\rho + 3p) \tag{1.66}
\]
The momentum conservation equation (1.41) is identically satisfied. Since \( \rho = \rho(t), \ p = p(t) \), it follows that \( p = p(\rho) \), i.e. one may assume a barotropic equation of state (for a single–component fluid). Then (1.66) and the energy conservation equation (1.40) are coupled equations in the 2 variables \( H, \rho \), and can be solved for a given \( p(\rho) \). However it is more convenient to use the Friedmann equation, the \((0, 0)\) field equation, which is a first integral of the Raychaudhuri equation:
\[
H^2 = \frac{1}{3}\rho - \frac{k}{a^2} \tag{1.67}
\]
where \( a(t) \) is the scale factor defined by (1.37) and \( k = 0, \pm 1 \) is the curvature index for the cosmic 3–surfaces, which by symmetry are spaces of constant curvature. In comoving spherical coordinates, the FRW metric and 4–velocity are
\[
ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] , \quad u^\alpha = \delta^\alpha_0 \tag{1.68}
\]
where \( d\Omega^2 \) is the metric of the unit sphere.

The expansion of the universe \( (H > 0) \) is confirmed by the systematic redshift in electromagnetic radiation that reaches us from distant galaxies. By (1.24) and (1.68)
\[
1 + z = \frac{a(t_R)}{a(t_E)} \tag{1.69}
\]
showing that \( a \) is increasing, so that by (1.62) \( \rho \) is decreasing. The early universe is very hot, as confirmed by the after–glow we observe in the form of the cosmic microwave background radiation. The early universe is modelled by a radiation fluid (1.53), while the late universe is cold and the dust model (1.54) is appropriate. The transition from radiation–to matter–domination requires a careful analysis, and has to deal with the interaction between radiation and matter. This covers the recombination era of the universe, and involves dissipative processes which I will discuss later.

Leaving aside this transition (which occupies a very short time in the evolution of the universe), the matter and radiation are effectively non–interacting. In the super–hot conditions of the early universe, matter particles are ultra–relativistic and effectively massless, so that a radiation fluid in equilibrium is a good approximation. In the late universe, (1.53) and (1.54) show that the energy density of radiation is negligible compared
to that of matter, and the dust model becomes reasonable. For the flat universe case 
\(k = 0\), (1.53), (1.54) and (1.67) lead to the solutions:

\[
\begin{align*}
\text{radiation: } & a \propto t^{1/2}, \\
\text{matter: } & a \propto t^{2/3}
\end{align*}
\] (1.70)

Einstein’s theory predicts that a radiation FRW universe will begin at \(t = 0\) with
infinite energy density and temperature. However, for times less than the Planck time
\(t_P \approx 10^{-43}\) sec, quantum gravity effects are expected to become dominant, and Einstein’s
theory will no longer hold. As yet, no satisfactory quantum gravity theory has been
developed, and models of the very early universe are necessarily speculative. One fairly
successful model, which applies during the semi–classical period between the quantum era
and the classical Einstein era, is inflation. Inflationary models aim to answer some of the
problems that arise in the standard classical cosmology (the ‘big bang’ model).

In these models, the energy density of the universe is dominated by a scalar field at
around \(10^{-34} \sim 10^{-32}\) sec. The pressure of the scalar field is negative, which acts like an
effective repulsive force, leading to accelerated expansion, or inflation, during which the
scale factor \(a\) increases by around \(10^{30}\). Although the scalar field is not a fluid, it has an
energy–momentum tensor of the perfect fluid form (1.39). The condition for accelerated
expansion is \(\ddot{a} > 0\), so that, by (1.60)

\[
\text{inflation } \iff \ddot{a} > 0 \iff p < -\frac{1}{3}\rho
\] (1.71)

Particular forms of inflation are exponential inflation in a flat FRW universe, for which

\[
a \propto \exp(HI t) \quad \text{and} \quad p = -\rho
\] (1.72)

and power–law inflation, for which \(a \propto t^N, \ N > 1\).
Chapter 2

Dissipative Relativistic Fluids

Perfect fluids in equilibrium generate no entropy and no ‘frictional’ type heating, because their dynamics is reversible and without dissipation. For many processes in cosmology and astrophysics, a perfect fluid model is adequate. However, real fluids behave irreversibly, and some processes in cosmology and astrophysics cannot be understood except as dissipative processes, requiring a relativistic theory of dissipative fluids.

In order to model such processes, we need non–equilibrium or irreversible thermodynamics. Perhaps the most satisfactory approach to irreversible thermodynamics is via non–equilibrium kinetic theory. However, this is very complicated, and I will take instead a standard phenomenological approach, pointing out how kinetic theory supports many of the results. A comprehensive, modern and accessible discussion of irreversible thermodynamics is given in [4]. This text includes relativistic thermodynamics, but most of the theory and applications are non–relativistic. A relativistic, but more advanced, treatment may be found in [3] (see also [5], [6]).

Standard, or classical, irreversible thermodynamics was first extended from Newtonian to relativistic fluids by Eckart in 1940. However, the Eckart theory, and a variation of it due to Landau and Lifshitz in the 1950’s, shares with its Newtonian counterpart the problem that dissipative perturbations propagate at infinite speeds. This non–causal feature is unacceptable in a relativistic theory – and worse still, the equilibrium states in the theory are unstable.

The problem is rooted in the way that non–equilibrium states are described – i.e. via the local equilibrium variables alone. Extended irreversible thermodynamics takes its name from the fact that the set needed to describe non–equilibrium states is extended to include the dissipative variables. This feature leads to causal and stable behaviour under a wide range of conditions. A non–relativistic extended theory was developed by Muller in the 1960’s, and independently a relativistic version was developed by Israel and Stewart in the 1970’s. The extended theory is also known as causal thermodynamics, second–order thermodynamics (because the entropy includes terms of second order in the dissipative variables), and transient thermodynamics (because the theory incorporates transient phenomena on the scale of the mean free path/ time, outside the quasi–stationary regime of the classical theory).

In this chapter I will give a simple introduction to these features, leading up to a formulation of relativistic causal thermodynamics that can be used for applications in
2.1 Basic Features of Irreversible Thermodynamics

For a dissipative fluid, the particle 4–current will be taken to be of the same form as (1.38). This corresponds to choosing an average 4–velocity in which there is no particle flux – known as the particle frame. At any event in spacetime, the thermodynamic state of the fluid is close to a fictitious equilibrium state at that event. The local equilibrium scalars are denoted $\bar{n}$, $\bar{\rho}$, $\bar{p}$, $\bar{S}$, $\bar{T}$, and the local equilibrium 4–velocity is $\bar{u}^\mu$. In the particle frame, it is possible to choose $\bar{u}^\mu$ such that the number and energy densities coincide with the local equilibrium values, while the pressure in general deviates from the local equilibrium pressure:

$$ n = \bar{n} , \quad \rho = \bar{\rho} , \quad p = \bar{p} + \Pi $$

(2.1)

where $\Pi = p - \bar{p}$ is the bulk viscous pressure. From now on I will drop the bar on the equilibrium pressure and write $p + \Pi$ for the effective non–equilibrium pressure:

$$ p_{\text{eff}} = p + \Pi \quad (p \rightarrow p_{\text{eff}}, \quad \bar{p} \rightarrow p) $$

The form of the energy–momentum tensor may be deduced from the equilibrium form (1.39) and the general covariant decomposition (1.35), given that $T_{\alpha\beta}$ is symmetric:

$$ T_{\alpha\beta} = \rho u_\alpha u_\beta + (p + \Pi) h_{\alpha\beta} + q_\alpha u_\beta + q_\beta u_\alpha + \pi_{\alpha\beta} $$

(2.2)

where

$$ q_\alpha u^\alpha = 0 , \quad \pi_{\alpha\beta} = \pi_{<\alpha\beta>} \Rightarrow \pi_{\alpha\beta} u^\beta = \pi_{[\alpha\beta]} = \pi^\alpha{}_{\alpha} = 0 $$

In a comoving IOF, $q_\alpha \equiv (0, \vec{q})$ and $\pi_{\alpha\beta} \equiv \pi_{ij} \delta^i_\alpha \delta^j_\beta$, so that $\vec{q}$ is an energy flux (due to heat flow in the particle frame) relative to the particle frame, while $\pi_{ij}$ is the anisotropic stress.

Both the standard and extended theories impose conservation of particle number and energy–momentum:

$$ n^{\alpha}{}_{;\alpha} = 0 , \quad T^{\alpha\beta}{}_{;\beta} = 0 $$

Particle number conservation leads to the same equation (1.38) that holds in the equilibrium case. However the equilibrium energy and momentum conservation equations (1.40) and (1.41) are changed by the dissipative terms in (2.2):

$$ \dot{\rho} + 3H(\rho + p + \Pi) + D^\alpha q_\alpha + 2\dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^\alpha{}^\beta = 0 $$

(2.3)

$$ (\rho + p + \Pi) \dot{u}_\alpha + D_\alpha(p + \Pi) + D^\beta \pi_{\alpha\beta} + \dot{u}^\beta \pi_{\alpha\beta} + h_\alpha{}^\beta \ddot{q}_\beta + (4H h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}) q^\beta = 0 $$

(2.4)

In irreversible thermodynamics, the entropy is no longer conserved, but grows, according to the second law of thermodynamics. The rate of entropy production is given by

1Note that the equilibrium states are different at different events, and therefore not subject to differential conditions such as (1.40) – (1.52)
the divergence of the entropy 4-current, so that the covariant form of the second law of thermodynamics is

\[ S_{\alpha} \geq 0 \]  

(2.5)

\( S^\alpha \) no longer has the simple form in (1.42), but has a dissipative term:

\[ S^\alpha = S n u^\alpha + \frac{R^\alpha}{T} \]  

(2.6)

where \( S = \bar{S} \) and \( T = \bar{T} \) are still related via the Gibbs equation (1.43).

The dissipative part \( R^\alpha \) of \( S^\alpha \) is assumed to be an algebraic function (i.e. not containing derivatives) of \( n^\alpha \) and \( T_{\alpha\beta} \), that vanishes in equilibrium:

\[ R^\alpha = R^\alpha(n^\beta, T^{\mu\nu}) \quad \text{and} \quad \bar{R}^\alpha = 0 \]

This assumption is part of the hydrodynamical description, in the sense that non-equilibrium states are assumed to be adequately specified by the hydrodynamical tensors \( n^\alpha, T_{\alpha\beta} \) alone. The standard and extended theories of irreversible thermodynamics differ in the form of this function.

### 2.2 Standard Irreversible Thermodynamics

The standard Eckart theory makes the simplest possible assumption about \( R^\alpha \) - i.e. that it is linear in the dissipative quantities. The only such vector that can be algebraically constructed from \( (\Pi, q^\alpha, \pi_{\alpha\beta}) \) and \( u^\alpha \) is

\[ f(n, \rho) \Pi u^\alpha + g(\rho, n) q^\alpha \]

Now the entropy density \(-u^\alpha S^\alpha\) should be a maximum in equilibrium, i.e.

\[ \left[ \frac{\partial}{\partial \Pi} (-u^\alpha S^\alpha) \right]_{\text{eqm}} = 0 \]

This implies \( f = 0 \). In a comoving IOF, \( q^\alpha/T \equiv (0, \bar{q}/T) \), which is the entropy flux due to heat flow. Thus \( g = 1 \) and (2.6) becomes

\[ S^\alpha = S n u^\alpha + \frac{q^\alpha}{T} \]  

(2.7)

Using the Gibbs equation (1.43) and the conservation equations (1.38) and (2.3), the divergence of (2.7) becomes

\[ T S_{\alpha}^{\alpha} = -3 H \Pi + (D_{\alpha} \ln T + \dot{u}_{\alpha}) q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta} \]  

(2.8)

\[ ^2 \text{In extended thermodynamics, this is the Israel-Stewart approach. An alternative approach is to extend the Gibbs equation by including dissipative terms, and to use a generalised temperature, specific entropy and pressure. The two approaches agree near equilibrium.} \]

\[ ^3 \text{In kinetic theory, this corresponds to truncating the non-equilibrium distribution function – via the Grad 14–moment approximation.} \]
Notice that the equilibrium conditions (1.49) from kinetic theory lead to the vanishing of each factor multiplying the dissipative terms on the right, and therefore to \( S^\alpha_{\alpha} = 0 \).

From (2.8), we see that the simplest way to satisfy (2.5) is to impose the following linear relationships between the thermodynamic ‘fluxes’ \( \Pi, q_\alpha, \pi_{\alpha\beta} \) and the corresponding thermodynamic ‘forces’ \( H, \dot{u}_\alpha + D_\alpha \ln T, \sigma_{\alpha\beta} \):

\[
\begin{align*}
\Pi &= -3\zeta H \\
q_\alpha &= -\lambda (D_\alpha T + T \dot{u}_\alpha) \\
\pi_{\alpha\beta} &= -2\eta \sigma_{\alpha\beta}
\end{align*}
\]

These are the constitutive equations for dissipative quantities in the standard Eckart theory of relativistic irreversible thermodynamics. They are relativistic generalisations of the corresponding Newtonian laws:

\[
\begin{align*}
\Pi &= -3\zeta \vec{\nabla} \cdot \vec{v} \quad \text{(Stokes)} \\
\vec{q} &= -\lambda \vec{\nabla} T \quad \text{(Fourier)} \\
\pi_{ij} &= -2\eta \sigma_{ij} \quad \text{(Newton)}
\end{align*}
\]

This is confirmed by using a comoving IOF in (2.9) – (2.11) – except that in the relativistic case, as discovered by Eckart, there is an acceleration term in (2.10) arising from the inertia of heat energy. Effectively, a heat flux will arise from accelerated matter even in the absence of a temperature gradient.

The Newtonian laws allow us to identify the thermodynamic coefficients:

- \( \zeta(\rho, n) \) is the bulk viscosity
- \( \lambda(\rho, n) \) is the thermal conductivity
- \( \eta(\rho, n) \) is the shear viscosity

Given the linear constitutive equations (2.9) – (2.11), the entropy production rate (2.8) becomes

\[
S^\alpha_{\alpha} = \frac{\Pi^2}{\zeta T} + \frac{q_\alpha q^\alpha}{\lambda T^2} + \frac{\pi_{\alpha\beta} \pi^{\alpha\beta}}{2\eta T}
\]

which is guaranteed to be non-negative provided that

\[
\zeta \geq 0, \quad \lambda \geq 0, \quad \eta \geq 0
\]

Note that the Gibbs equation (1.43) together with number and energy conservation (1.38) and (2.3), leads to an evolution equation for the entropy:

\[
T n \dot{S} = -3H \Pi - q^\alpha_{\alpha} - \dot{u}_\alpha q^\alpha - \sigma_{\alpha\beta} \pi^{\alpha\beta}
\]

Many, probably most, of the applications of irreversible thermodynamics in relativity have used this Eckart theory. However the algebraic nature of the Eckart constitutive equations leads to severe problems. Qualitatively, it can be seen that if a thermodynamic force is suddenly switched off, then the corresponding thermodynamic flux instantaneously vanishes. This indicates that a signal propagates through the fluid at infinite speed, violating relativistic causality.

\[
\text{[4] Even in the Newtonian case, infinite signal speeds present a problem, since physically we expect the signal speed to be limited by the maximum molecular speed.}
\]
2.3 Simple Example: Heat Flow

For a quantitative demonstration, consider the flow of heat in a non–accelerating, non–
expanding and vorticity–free fluid in flat spacetime, where the comoving IOF may be
chosen as a global orthonormal frame. In the non–relativistic regime the fluid energy
density is given by (1.55), and then the energy conservation equation (2.3) gives

$$\frac{3}{2n} \frac{\partial T}{\partial t} = -\mathbf{\nabla} \cdot \mathbf{q}$$

since $\partial n/\partial t = 0$ by (1.38). The Eckart law (2.10) reduces to

$$\mathbf{q} = -\lambda \mathbf{\nabla} T$$

Assuming that $\lambda$ is constant, these two equations lead to

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T \quad \text{where} \quad \chi = \frac{2\lambda}{3n}$$

which is the heat conduction equation. This equation is parabolic, corresponding to
infinite speed of propagation.

Apart from causality violation, the Eckart theory has in addition the pathology of
unstable equilibrium states. It can be argued that a dissipative fluid will very rapidly
tend towards a quasi–stationary state that is adequately described by the Eckart theory.
However, there are many processes in which non–stationary relaxational effects dominate.
Furthermore, even if the Eckart theory can describe the asymptotic states, it is clearly
unable to deal with the evolution towards these states, or with the overall dynamics of
the fluid, in a satisfactory way.

Qualitatively, one expects that if a thermodynamic force is switched off, the corre-
sponding thermodynamic flux should die away over a finite time period. Referring to the
heat flow example above, if $\nabla T$ is set to zero at time $t = 0$, then instead of $\mathbf{q}(t) = 0$ for
$t \geq 0$, as predicted by the Eckart law, we expect that

$$\mathbf{q}(t) = q_0 \exp \left(-\frac{t}{\tau}\right)$$

where $\tau$ is a characteristic relaxational time for transient heat flow effects. Such a relax-
ational feature would arise if the Eckart–Fourier law were modified as

$$\tau \dot{q} + q = -\lambda \mathbf{\nabla} T$$

(2.15)

This is the Maxwell–Cattaneo modification of the Fourier law, and it is in fact qualitatively
what arises in the extended theory.

With the Maxwell–Cattaneo form (2.15), the heat conduction equation (2.14) is mod-
ified as

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \chi \nabla^2 T = 0$$

(2.16)
which is a damped wave equation. A thermal plane–wave solution

\[ T \propto \exp \left[ i \left( \vec{k} \cdot \vec{x} - \omega t \right) \right] \]

leads to the dispersion relation

\[ k^2 = \frac{\tau \omega^2}{\chi} + i\omega \]

so that the phase velocity is

\[ V = \frac{\omega}{\text{Re}(k)} = \left[ \frac{2\chi\omega}{\tau \omega + \sqrt{1 + \tau^2 \omega^2}} \right]^{1/2} \]

In the high frequency limit, i.e. \( \omega \gg \tau^{-1} \), we see that

\[ V \approx \sqrt{\frac{\chi}{\tau}} \]

The high–frequency limit gives the speed of thermal pulses – known as second sound – and it follows that this speed is finite for \( \tau > 0 \). Thus the introduction of a relaxational term removes the problem of infinite propagation speeds.

The intuitive arguments of this section form an introduction to the development of the extended theory of Israel and Stewart.

### 2.4 Causal Thermodynamics

Clearly the Eckart postulate (2.7) for \( R^\alpha \) is too simple. Kinetic theory indicates that in fact \( R^\alpha \) is second–order in the dissipative fluxes. The Eckart assumption, by truncating at first order, removes the terms that are necessary to provide causality and stability. The most general algebraic form for \( R^\alpha \) that is at most second–order in the dissipative fluxes is

\[
S^\mu = S \nu u^\mu + \frac{q^\mu}{T} - \left( \beta_0 \Pi^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\kappa} \pi^{\nu\kappa} \right) \frac{u^\mu}{2T} + \frac{\alpha_0 \Pi q^\mu}{T} + \frac{\alpha_1 \pi^{\mu\nu} q_\nu}{T}
\]  

(2.17)

where \( \beta_\alpha(\rho, n) \geq 0 \) are thermodynamic coefficients for scalar, vector and tensor dissipative contributions to the entropy density, and \( \alpha_\alpha(\rho, n) \) are thermodynamic viscous/ heat coupling coefficients. It follows from (2.17) that the effective entropy density (measured by comoving observers) is

\[
- u_\mu S^\mu = S n - \frac{1}{2T} \left( \beta_0 \Pi^2 + \beta_1 q_\nu q^\nu + \beta_2 \pi_{\mu\nu} \pi^{\mu\nu} \right)
\]

(2.18)

independent of \( \alpha_0, \alpha_1 \). (Note that the entropy density is a maximum in equilibrium.)

For simplicity, I will assume

\[ \alpha_0 = 0 = \alpha_1 \quad \text{i.e. no viscous/ heat coupling} \]  

(2.19)
This assumption is consistent with linearisation in a perturbed FRW universe, since the coupling terms lead to non-linear deviations from the FRW background. However, the assumption (2.19) may not be reasonable for non-uniform stellar models and other situations where the background solution is inhomogeneous.

The divergence of the extended current (2.17) – with (2.19) – follows from the Gibbs equation and the conservation equations (1.38), (2.3) and (2.4):

\[ T S^\alpha_{, \alpha} = - \Pi \left[ 3H + \beta_0 \dot{H} + \frac{1}{2} T \left( \frac{\beta_0}{T} u^\alpha \right) \right] \Pi \]

\[ -q^\alpha \left[ D_\alpha \ln T + \dot{u}_\alpha + \beta_1 \dot{q}_\alpha + \frac{1}{2} T \left( \frac{\beta_1}{T} u^\mu \right) q_\alpha \right] \]

\[ -\pi^{\alpha \mu} \left[ \sigma_{\alpha \mu} + \beta_2 \pi_{\alpha \mu} + \frac{1}{2} T \left( \frac{\beta_2}{T} u^\nu \right) \pi_{\alpha \mu} \right] \]

(2.20)

The simplest way to satisfy the second law of thermodynamics (2.5), is to impose, as in the standard theory, linear relationships between the thermodynamical fluxes and forces (extended), leading to the following constitutive or transport equations:

\[ \tau_0 \dot{H} + \Pi = - 3 \zeta H - \left[ \frac{\zeta}{3} T \left( \frac{\tau_0}{T} u^\alpha \right) \right] \Pi \]

(2.21)

\[ \tau_1 h_\alpha^\beta \dot{q}_\beta + q_\alpha = - \lambda \left( D_\alpha T + T \dot{u}_\alpha \right) - \left[ \frac{1}{2} \tau_1 \left( \frac{T}{\lambda T^2} u^\beta \right) q_\alpha \right] \]

(2.22)

\[ \tau_2 h_\alpha^\mu h_\beta^\nu \dot{\pi}_{\mu \nu} + \pi_{\alpha \beta} = - 2 \eta \sigma_{\alpha \beta} - \left[ \eta T \left( \frac{\tau_2}{2 \eta T} u^\nu \right) \pi_{\alpha \beta} \right] \]

(2.23)

where the relaxational times \( \tau_A(\rho, n) \) are given by

\[ \tau_0 = \zeta \beta_0, \quad \tau_1 = \lambda T \beta_1, \quad \tau_2 = 2 \eta \beta_2 \]

(2.24)

With these transport equations, the entropy production rate has the same non-negative form (2.12) as in the standard theory.

Because of the simplifying assumption (2.19), there are no couplings of scalar/ vector/ tensor dissipative fluxes. As well as these viscous/ heat couplings, kinetic theory shows that in general there will also be couplings of heat flux and anisotropic pressure to the vorticity – which, unlike the shear, does not vanish in general in equilibrium (see (1.49)). These couplings give rise to the following additions to the right hand sides of (2.22) and (2.23) respectively:

\[ + \lambda T \gamma_1 \omega_{\alpha \beta} q^\beta \quad \text{and} \quad + 2 \eta \gamma_2 \pi^\mu <_{\alpha \omega_{\beta}} >_\mu \]

where \( \gamma_1(\rho, n), \gamma_2(\rho, n) \) are the thermodynamic coupling coefficients. In a comoving IOF, (1.31) shows that the addition to (2.22) has the form

\[ \lambda T \gamma_1 \vec{\omega} \times \vec{q} \quad \text{where} \quad \vec{\omega} = \vec{\nabla} \times \vec{v} \]

6This linear assumption is in fact justified by kinetic theory, which leads to the same form of the transport equations [3].
If the background solution has zero vorticity, as is the case in a perturbed FRW universe, then these vorticity coupling terms will vanish in linearised theory. However, they would be important in rotating stellar models, where the background equilibrium solution has $\omega_{\alpha\beta} \neq 0$.

The terms in square brackets on the right of equations (2.21) – (2.23) are often omitted. This amounts to the implicit assumption that these terms are negligible compared with the other terms in the equations. I will call the simplified equations the truncated Israel–Stewart equations. One needs to investigate carefully the conditions under which the truncated equations are reasonable. This will be further discussed in the next chapter. The truncated equations, together with the no–coupling assumption (2.19), are of covariant relativistic Maxwell–Cattaneo form:

\[
\tau_0 \dot{\Pi} + \Pi = -3\zeta H \tag{2.25}
\]
\[
\tau_1 h_{\alpha}^{\beta} \dot{q}_\beta + q_\alpha = -\lambda (D_\alpha T + T \dot{u}_\alpha) \tag{2.26}
\]
\[
\tau_2 h_{\alpha}^{\mu} h_{\beta}^{\nu} \dot{\pi}_{\mu\nu} + \pi_{\alpha\beta} = -2\eta \sigma_{\alpha\beta} \tag{2.27}
\]

The crucial difference between the standard Eckart and the extended Israel–Stewart transport equations is that the latter are differential evolution equations, while the former are algebraic relations. As we saw in the previous section, the evolution terms, with the relaxational time coefficients $\tau_A$, are needed for causality – as well as for modelling high–frequency or transient phenomena, where ‘fast’ variables and relaxation effects are important. The price paid for the improvements that the extended causal thermodynamics brings is that new thermodynamic coefficients are introduced. However, as is the case with the coefficients $\zeta, \lambda, \eta$ that occur also in standard theory, these new coefficients may be evaluated or at least estimated via kinetic theory. The relaxation times $\tau_A$ involve complicated collision integrals. In fact, they are usually estimated as mean collision times, of the form

\[
\tau \approx \frac{1}{n\sigma v} \tag{2.28}
\]

where $\sigma$ is a collision cross section and $v$ the mean particle speed.

It is important to remember that the derivation of the causal transport equations is based on the assumption that the fluid is close to equilibrium. Thus the dissipative fluxes are small:

\[
|\Pi| \ll p, \quad (\pi_{\alpha\beta} \pi^{\alpha\beta})^{1/2} \ll p, \quad (q_\alpha q^\alpha)^{1/2} \ll p \tag{2.29}
\]

Consider the evolution of entropy in the Israel–Stewart theory. The equation (2.13) still holds in the extended case:

\[
T n \dot{S} = -3H \Pi - q_\alpha q^\alpha - \dot{u}_\alpha q^\alpha - \sigma_{\alpha\beta} \pi^{\alpha\beta} \tag{2.30}
\]

Consider a comoving volume of fluid, initially of size $a_0^3$, where $a$ is the scale factor defined in general by (1.37). The entropy in this comoving volume is given by

\[
\Sigma = a^3 n S \tag{2.31}
\]

Then, by virtue of number conservation (1.38) and (2.30), it follows that the growth in comoving entropy over a proper time interval $t_0 \rightarrow t$ is

\[
\Sigma(t) = \Sigma_0 - \int_{t_0}^t \frac{a^3}{T} (3H \Pi + q_\alpha q^\alpha + \dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta}) \, dt \tag{2.32}
\]
The second law, which is built into the theory, guarantees that \( \Sigma(t) \geq \Sigma_0 \). However, it is possible that the local equilibrium specific entropy \( S \) is not increasing at all times – but the effective, non-equilibrium specific entropy \(-u_\alpha S^\alpha/n\) is monotonically increasing [4].

Next we look at the temperature behaviour in causal thermodynamics. The Gibbs integrability condition (1.57) still holds:

\[
n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial p}{\partial \rho} \tag{2.33}
\]

However, the change in the energy conservation equation (2.3) leads to a generalisation of the temperature evolution (1.58):

\[
\frac{\dot{T}}{T} = -3H \left( \frac{\partial p}{\partial \rho} \right)_n - \frac{1}{T} \left( \frac{\partial T}{\partial \rho} \right)_n \left[ 3H \Pi + q^\alpha;\alpha + \dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta} \right] \tag{2.34}
\]

\[
= -3H \left( \frac{\partial p}{\partial \rho} \right)_n + n\dot{S} \left( \frac{\partial T}{\partial \rho} \right)_n
\]

Note that if the Gibbs integrability condition, number conservation and energy conservation are satisfied, then the evolution equation (2.34) will be an identity. This evolution equation shows quantitatively how the relation of temperature to expansion is affected by dissipation. The first term on the right of (2.34) represents the cooling due to expansion. In the second, dissipative term, viscosity in general contributes to heating effects, while the contribution of heat flow depends on whether heat is being transported into or out of a comoving volume.

If instead of \((n, \rho)\) we choose \((n, T)\) as independent variables, then the Gibbs integrability condition (2.33) becomes

\[
T \frac{\partial p}{\partial T} + n \frac{\partial \rho}{\partial n} = \rho + p \tag{2.35}
\]

and the temperature evolution equation (2.34) becomes

\[
\frac{\dot{T}}{T} = -3H \left( \frac{\partial p/\partial T}{\partial \rho/\partial T} \right)_n - \frac{1}{T(\partial \rho/\partial T)_n} \left[ 3H \Pi + q^\alpha;\alpha + \dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta} \right] \tag{2.36}
\]

\[
= -3H \left( \frac{\partial p/\partial T}{\partial \rho/\partial T} \right)_n + n\dot{S} \left( \frac{1}{\partial \rho/\partial T} \right)_n
\]

Finally, we consider briefly the issue of equations of state for the pressure and temperature in dissipative fluids. Using the energy conservation equation (2.3), the Gibbs equation in the form (1.60) generalises to

\[
n^2 T dS = \left[ \frac{nD}{3H(\rho + p) + D} \right] d\rho + (\rho + p) \frac{\partial n}{\partial \rho} \left[ \frac{\dot{\rho}}{\rho} d\rho - dp \right]
\]

where the dissipative term is

\[
D = 3H \Pi + q^\alpha;\alpha + \dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta} \tag{2.37}
\]
It follows that in the presence of dissipation, barotropic pressure no longer forces \( dS \) to vanish:

\[
dS = \frac{1}{nT} \left[ \frac{\mathcal{D}}{3H(\rho + p) + \mathcal{D}} \right] d\rho
\]  

(2.38)

As in the equilibrium case, it remains true, via the Gibbs integrability condition, that barotropic \( T = T(\rho) \) together with \( p = (\gamma - 1)\rho \) leads to the power–law form \((1.64)\) for the temperature. However, in the dissipative case, these relations are not in general compatible with the ideal gas law \( p = nT \):

\[
p = nT, \quad p = (\gamma - 1)\rho, \quad T \propto \rho^{(\gamma-1)/\gamma} \Rightarrow n \propto \rho^{1/\gamma}
\]

\[
\Rightarrow \frac{\dot{n}}{n} = \frac{1}{\gamma \rho} \dot{\rho}
\]

Then number and energy conservation imply \( \mathcal{D} = 0 \).

However, it is possible to impose the \( \gamma \)–law and the ideal gas law simultaneously, provided the temperature is not barotropic. The temperature evolution equation \((2.36)\) and energy conservation \((2.3)\) give

\[
\frac{\dot{T}}{T} = \left[ \left( \frac{\gamma - 1}{\gamma} \right) \frac{\dot{\rho}}{\rho} + \frac{\mathcal{D}}{\gamma \rho} \right] \left[ 1 + \frac{\mathcal{D}}{nT} \right]
\]

(2.39)

These results have interesting implications for a dissipative fluid which is close to a thermalised radiation fluid, i.e. \( p = \frac{1}{3} \rho \). If we insist that \( p = nT \), then the Stefan–Boltzmann law \( \rho \propto T^4 \) cannot hold out of equilibrium. Alternatively, if we impose the Stefan–Boltzmann law, then the ideal gas law cannot hold unless the fluid returns to equilibrium.
Chapter 3

Applications to Cosmology and Astrophysics

The evolution of the universe contains a sequence of important dissipative processes, including:

- GUT (Grand Unified Theory) phase transition ($t \approx 10^{-34}$ sec, $T \approx 10^{27}$ K), when gauge bosons acquire mass (spontaneous symmetry breaking).

- Reheating of the universe at the end of inflation (at about $10^{-32}$ sec), when the scalar field decays into particles.

- Decoupling of neutrinos from the cosmic plasma ($t \approx 1$ sec, $T \approx 10^{10}$ K), when the temperature falls below the threshold for interactions that keep the neutrinos in thermal contact. The growing neutrino mean free path leads to heat and momentum transport by neutrinos and thus damping of perturbations. Shortly after decoupling, electrons and positrons annihilate, heating up the photons in a non-equilibrium process.

- Nucleosynthesis (formation of light nuclei) ($t \approx 100$ sec).

- Decoupling of photons from matter during the recombination era ($t \approx 10^{12}$ sec, $T \approx 10^{3}$ K), when electrons combine with protons and so no longer scatter the photons. The growing photon mean free path leads to heat and momentum transport and thus damping.

Some astrophysical dissipative processes are:

- Gravitational collapse of local inhomogeneities to form galactic structure, when viscosity and heating lead to dissipation.

- Collapse of a radiating star to a neutron star or black hole, when neutrino emission is responsible for dissipative heat flow and viscosity.

- Accretion of matter around a neutron star or black hole.
Further discussion of such processes can be found in [9], [10] (but not from a causal thermodynamics standpoint). The application of causal thermodynamics to cosmology and astrophysics remains relatively undeveloped – partly because of the complexity of the transport equations, partly because all of the important dissipative processes have been thoroughly analysed using the standard theory or kinetic theory or numerical methods.

Causal bulk viscosity in cosmology has been fairly comprehensively investigated – see [11] – [18]. Shear viscosity in anisotropic cosmologies has been considered in [11], [19], while heat flow in inhomogeneous cosmologies has been discussed in [20]. Causal dissipation in astrophysics has been investigated in [8], [21] – [23]. In all of these papers, it is found that causal thermodynamic effects can have a significant impact and can lead to predictions very different from those in the standard Eckart theory.

In this chapter I will briefly discuss some overall features of causal thermodynamics in a cosmological/ astrophysical setting, and then conclude with a more detailed discussion of bulk viscosity in an FRW universe, which is the most accessible problem.

3.1 General Features of Cosmic Dissipation

The expanding universe defines a natural time–scale – the expansion time $H^{-1} = a/\dot{a}$. Any particle species will remain in thermal equilibrium with the cosmic fluid so long as the interaction rate is high enough to allow rapid adjustment to the falling temperature. If the mean interaction time is $t_c$, then a necessary condition for maintaining thermal equilibrium is

$$t_c < H^{-1}$$

(3.1)

Now $t_c$ is determined by

$$t_c = \frac{1}{n\sigma v}$$

(3.2)

where $\sigma$ is the interaction cross–section, $n$ is the number density of the target particles with which the given species is interacting, and $v$ is the mean relative speed of interacting particles.

As an example, consider neutrinos in the early universe. At high enough temperatures, the neutrinos are kept in thermal equilibrium with photons and electrons via interactions with electrons that are governed by the weak interaction. The cross–section is

$$\sigma_w = g_0 T^2$$

(3.3)

where $g_0$ is a constant. The number density of electrons is $n \propto T^3$, by (1.52), since the electrons are effectively massless at these very high temperatures. Since $v = 1$, (3.2) gives $t_c \propto T^{-5}$. By (1.56), we can see that $H \propto T^2$. Thus

$$t_c H = \left(\frac{T_c}{T}\right)^3$$

(3.4)

and using (3.1) and the numerical values of the various constants, it follows that the neutrinos will decouple for

$$T < T_* \approx 10^{10} \text{K}$$
Other cosmic decoupling processes may be analysed by a similar approach. The differences arise from the particular forms of $\sigma(T)$, $n(T)$ and $H(T)$. For example, in the case of photons interacting with electrons via Thompson scattering, the Thompson cross-section is constant, while the number density of free electrons is given by a complicated equation (the Saha equation), which takes account of the process of recombination. The expansion rate $H$ is also fairly complicated, since the universe is no longer radiation-dominated. One finds that the decoupling temperature is about $10^3$ K.

In the case of a collapsing star, similar arguments are applied – except that the characteristic time in this case is determined by the rate of collapse, which is governed by stellar dynamics. For example, for neutrinos in the core of a neutron star, interactions with electrons and nucleons determine an interaction time that must be compared with the collapse time to estimate the decoupling conditions for the neutrinos – after which they transport heat and momentum away from the core.

The entropy generated in a dissipative process that begins at $t_0$ and ends at $t_0 + \Delta t$ is given by (2.32):

$$
\Delta \Sigma = - \int_{t_0}^{t_0+\Delta t} \frac{a^3}{T} \left( 3H\Pi + q^{\alpha,\alpha} + \dot{u}_\alpha q^\alpha + \sigma_{\alpha\beta} \pi^{\alpha\beta} \right) dt
$$

(3.5)

For example, $\Delta t$ could be the time taken for a decoupling process in the universe or a star.

The observed universe has a high entropy, as indicated by the high number of photons per baryon, about $10^8$. This gives a total entropy in the observable universe of about $10^{88}$. Inflationary cosmology predicts that nearly all of this entropy is generated by the reheating process at the end of inflation – i.e. that all other dissipative processes in the evolution of the universe make a negligible contribution to entropy production by comparison. In this model, the formula (3.5) would have to be modified to include the dissipation not just from the fluid effects that we have been discussing, but also from particle production. Particle production, at a rate $\nu$, leads to non-conservation of particle number, so that (1.38) is replaced by

$$
n^{\alpha,\alpha} = \dot{n} + 3Hn = \nu n
$$

(3.6)

Then it is found that $\nu$ contributes to entropy production. The contribution from particle production may be modelled as an effective bulk viscosity.

Many dissipative processes are well described by a radiative fluid – i.e. a fluid consisting of interacting massless and massive particles. The radiative fluid is dissipative, and kinetic theory or fluctuation theory arguments may be used to derive the dissipative coefficients in terms of the relaxation times $\tau_A$ (which are usually assumed equal to the appropriate interaction time $t_c$). The results are collected in the table below. The table also includes the case of a relativistic Maxwell–Boltzmann gas – i.e. a dilute monatomic gas with high collision rate – in both the ultra-relativistic and non-relativistic limits. The local equilibrium energy density and pressure are given by the equations of state (1.47), (1.48) (but not subject to the global equilibrium conditions (1.49), (1.50)).
In the table, $\beta$ is given in standard units by

$$\beta = \frac{mc^2}{kT}$$

where $m$ is the mass of the matter particles (usually electrons); $r_0$ is the radiation constant for photons, and $\frac{7}{8}$ times the radiation constant for massless neutrinos; $\Gamma$ is effectively the deviation of $p/\rho$ from its pure-radiation value:

$$\Gamma = \frac{1}{3} - \left(\frac{\partial p}{\partial \rho}\right)_n = \frac{1}{3} - \frac{\left(\partial p/\partial T\right)_n}{\left(\partial \rho/\partial T\right)_n}$$

where $p, \rho$ refer to the pressure and energy density of the radiation/matter mixture as a whole. For example, when the matter is non-relativistic, (1.52) and (1.53) show that in standard units

$$p \approx nkT + \frac{1}{3}r_0T^4, \quad \rho \approx mc^2n + \frac{3}{2}nkT + r_0T^4$$

where $n$ is the number density of matter.

Note that for both the radiative fluid and the Maxwell–Boltzmann gas, the bulk viscosity tends to zero in the ultra-relativistic and non-relativistic limits. Bulk viscous effects are greatest in the mildly relativistic intermediate regime, $\beta \approx 1$. This discussed further in the next section.

The radiative fluid and Maxwell–Boltzmann gas are perhaps the best motivated dissipative fluid models. However, their equations of state and thermodynamic coefficients are very complicated, and for the purposes of analytical rather than numerical investigations, simplified equations are often assumed. These are usually barotropic:

$$p = p(\rho), \quad T = T(\rho), \quad \zeta = \zeta(\rho), \quad \lambda = \lambda(\rho), \quad \eta = \eta(\rho), \quad \tau_A = \tau_A(\rho)$$
However these assumptions are subject to consistency conditions (as shown earlier in the case of \( p \) and \( T \)), and may correspond to unphysical behaviour. Whenever such assumptions are made in a model, the consequences should be carefully checked. An example is given in the next section.

### 3.2 Causal Bulk Viscosity in Cosmology

I will use the simplest case of scalar dissipation due to bulk viscosity in order to illustrate some of the issues that arise in modelling cosmological dissipation via Israel–Stewart theory. Furthermore, this case covers the standard cosmological models. If one assumes that the universe is exactly isotropic and homogeneous – i.e. an FRW universe (1.68) – then the symmetries show that only scalar dissipation is possible – i.e. \( q_\alpha = 0 = \pi_{\alpha\beta} \). In this event, the no–coupling assumption (2.19) is automatically fulfilled.

Bulk viscosity arises typically in mixtures – either of different species, as in a radiative fluid, or of the same species but with different energies, as in a Maxwell–Boltzmann gas. Physically, we can think of bulk viscosity as the internal ‘friction’ that sets in due to the different cooling rates in the expanding mixture. The dissipation due to bulk viscosity converts kinetic energy of the particles into heat, and thus we expect it to reduce the effective pressure in an expanding fluid – i.e. we expect \( \Pi \leq 0 \) for \( H \geq 0 \). This is consistent with \( \dot{S} \geq 0 \) by (2.13):

\[
T \dot{\Pi} = -3H \Pi \tag{3.10}
\]

Any dissipation in an exact FRW universe is scalar, and therefore may be modelled as a bulk viscosity within a thermodynamical approach. As I have argued in the previous chapter, the Israel–Stewart thermodynamics is causal and stable under a wide range of conditions, unlike the standard Eckart theory. Therefore, in order to obtain the best thermo–hydrodynamic model with the available physical theories, one should use the causal Israel–Stewart theory of bulk viscosity.

Writing out the full Israel–Stewart transport equation (2.21) (using \( \tau \equiv \tau_0 \)), we get

\[
\tau \dot{\Pi} + \Pi = -3\zeta H - \frac{1}{2} \tau \Pi \left[ 3H + \frac{\dot{T}}{\tau} - \frac{\dot{\zeta}}{\zeta} - \frac{T}{\tau} \right] \tag{3.11}
\]

A natural question is – what are the conditions under which the truncated form (2.25) is a reasonable approximation of the full Israel–Stewart transport equation? It follows from (3.11) that if

\[
\frac{T}{a^3H} \left| \Pi \left( \frac{a^3 \tau}{\zeta T} \right) \right| \ll 1 \tag{3.12}
\]

holds, then the additional terms in (3.11) are negligible in comparison with \( 3\zeta H \). The condition (3.12) is clearly very sensitive to the particular forms of the functions \( p(n, \rho) \), \( \zeta(n, \rho) \) and \( \tau(n, \rho) \). The temperature is determined on the basis of these particular forms by the Gibbs integrability condition (2.33) and the evolution equation (2.34):

\[
\frac{\dot{T}}{T} = -3H \left[ \left( \frac{\partial p}{\partial \rho} \right)_n + \frac{\Pi}{T} \left( \frac{\partial T}{\partial \rho} \right)_n \right] \tag{3.13}
\]

\[\text{or equivalently by (2.35) and (2.36)}\]
The second term on the right shows that bulk stress tends to counteract the cooling due to expansion.

For simplicity, suppose that the pressure and temperature are barotropic, with \( p \) linear:

\[
p = (\gamma - 1) \rho \tag{3.14}
\]

This pressure equation is not unreasonable if the local equilibrium state is radiation or cold matter. Since the temperature is also barotropic, it then follows from the Gibbs integrability condition \( (2.33) \) that as in the perfect fluid case, \( T \) must have the power–law form \( (1.64) \):

\[
T \propto \rho^{(\gamma - 1)/\gamma} \tag{3.15}
\]

Thus there is no freedom to choose the form of \( T(\rho) \) – it is a power–law, with index fixed by \( \gamma \). With these forms of \( p(\rho) \) and \( T(\rho) \), we can see that the temperature evolution equation \( (3.13) \) is identically satisfied by virtue of the energy conservation equation \( (2.3) \):

\[
\dot{\rho} + 3H(\rho + p + \Pi) = 0 \tag{3.16}
\]

A simple relation between \( \tau \) and \( \zeta \) is found as follows. It is shown in the appendix to this chapter that

\[
\frac{\zeta}{\zeta(\rho + p)} = c_b^2 \tag{3.17}
\]

where \( c_b \) is the speed of bulk viscous perturbations – i.e. the non–adiabatic contribution to the speed of sound \( v \) in a dissipative fluid without heat flux or shear viscosity. The dissipative speed of sound is given by

\[
v^2 = c_s^2 + c_b^2 \leq 1 \tag{3.18}
\]

where \( c_s \) is the adiabatic contribution \( (1.44) \), and the limit ensures causality. When \( (3.14) \) holds, \( c_s^2 = \gamma - 1 \), so that

\[
c_b^2 \leq 2 - \gamma
\]

We will assume that \( c_b \) is constant, like \( c_s \).

Putting together the thermodynamic relationships \( (3.14), (3.15) \) and \( (3.17) \), the full transport equation \( (3.11) \) becomes

\[
\tau_\ast \dot{\Pi} + \Pi = -3\zeta_\ast H \left[ 1 + \frac{1}{\gamma c_b^2} \left( \frac{\Pi}{\rho} \right)^2 \right] \tag{3.19}
\]

where the effective relaxation time and bulk viscosity are

\[
\tau_\ast = \frac{\tau}{1 + 3\gamma \tau H} , \quad \zeta_\ast = \frac{\zeta}{1 + 3\gamma \tau H} = c_b^2 \gamma \rho \tau_\ast \tag{3.20}
\]

Now the near–equilibrium condition \( (2.29) \) with \( (3.14) \) implies

\[
|\Pi| \ll \rho
\]
and shows that the second term in square brackets in (3.19) is negligible. Thus the full equation leads to a truncated equation with reduced relaxation time and reduced bulk viscosity:

$$\tau \dot{\Pi} + \Pi = -3\zeta_* H$$

(3.21)

The amount of reduction depends on the size of $\tau$ relative to $H$. If $\tau$ is of the order of the mean interaction time, then the hydrodynamical description requires $\tau H < 1$. If $\tau H \ll 1$, then $\tau_* \approx \tau$ and $\zeta_* \approx \zeta$. But if $\tau H$ is close to 1, the reduction could be significant.

Although this reduction is based on the simplified thermodynamical relations assumed above, it indicates that the validity of the truncated Israel–Stewart equation can impose significant conditions. More realistic thermodynamical relations will require numerical calculations. In the case of a Maxwell–Boltzmann gas, such calculations show that the behaviour of the truncated and full theories can be very different. The conclusion seems to be that the full theory should be used, unless one is able to derive explicitly – and satisfy – the conditions under which the truncated version is adequate.

Assuming that the FRW universe is flat, the Friedmann equation (1.67) is

$$\rho = 3H^2$$

(3.22)

By (3.16) and (3.22) we get

$$\Pi = -2\dot{H} - 3\gamma H^2$$

(3.23)

and together with (3.21) and (3.20), this leads to the evolution equation for $H$:

$$\ddot{H} + (6\gamma + N)H\dot{H} + \frac{3}{2}\gamma \left[3(\gamma - c_b^2) + N\right] H^3 = 0$$

(3.24)

where

$$N = (\tau H)^{-1}$$

(3.25)

is of the order of the number of interactions in an expansion time. Intuitively, when $N \gg 1$, the fluid is almost perfect, while when $N$ is close to 1, the dissipative effects are significant. This is confirmed by (3.24). For $N \gg 1$, the equation reduces to

$$\dot{H} + \frac{3}{2}\gamma H^2 \approx 0$$

with the well–known perfect fluid solution:

$$H \approx \frac{2}{3\gamma(t - t_0)}$$

On the other hand, for $N$ close to 1, the second derivative in (3.24) cannot be neglected, and the solutions will show a range of behaviour very different from the perfect fluid – and the standard Eckart – solutions. (Note that the Eckart limit $\tau \to 0$ is $c_b \to \infty$ by (3.17); the causality condition (3.18) does not hold.)

Of course, a complete model requires the specification of $N$. Consider the ultra–relativistic fluid of the early universe, with a particle species whose growing mean free path is giving rise to dissipation, such as the neutrino. Suppose that $\tau \approx t_c$, where $t_c$ is
the mean interaction time, and that the interaction cross-section is proportional to $T^2$, like the neutrino’s. Then by (3.4) we get

$$N = \left( \frac{T}{T_*} \right)^3 = \left( \frac{H}{H_*} \right)^{3/2}$$

(3.26)

For $T \gg T_*$, we have $N \gg 1$, and dissipation is negligible. But for $T$ close to $T_*$, dissipation effects become significant. The evolution equation (3.24) becomes

$$\ddot{H} + \left[ 8 + \left( \frac{H}{H_*} \right)^{3/2} \right] \dot{H} + 2 \left[ 4 - 3c_s^2 + \left( \frac{H}{H_*} \right)^{3/2} \right] H^3 = 0$$

(3.27)

One could try to solve this equation perturbatively, by the ansatz

$$H = \frac{1}{2(t - t_0)} + \varepsilon H_1 + O(\varepsilon^2)$$

I will briefly discuss the question of bulk viscous inflation. Suppose dissipation in the cosmic fluid produced sufficiently large bulk viscous stress to drive the effective pressure negative and thus initiate inflationary expansion. By (1.71), using the effective pressure, the condition for inflationary expansion is

$$-\Pi > p + \frac{1}{3} \rho$$

(3.28)

For a fluid, this violates the near-equilibrium condition

$$|\Pi| < p$$

Thus viscous fluid inflation, if it were physically possible, would involve non-linear thermodynamics, far from equilibrium. The Israel–Stewart theory, as well as other versions of extended thermodynamics and also Eckart’s standard thermodynamics, are all based on near-equilibrium conditions, and cannot be applied to inflationary expansion – unless one makes the drastic assumption that the linear theory applies in the strongly non-linear regime.

Furthermore, there are serious physical problems with hydrodynamic inflation (without particle production). The point is that under conditions of super-rapid expansion – i.e. very small expansion time – the hydrodynamic regime requires even smaller interaction time. It is hard to see how the fluid interaction rate could increase to stay above the expansion rate under conditions where fluid particles are expanding apart from each other extremely rapidly.

For a satisfactory model of bulk viscous inflation, one needs: (a) a non-linear generalisation of the Israel–Stewart transport equation (3.11); (b) a consistent model of fluid behaviour under super-rapid expansion and strongly non-linear conditions.

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2 See [16], [24] for particle production models.
3 One possible generalisation is developed in [25].
On the other hand, the reheating period at the end of inflation can be modelled by near–equilibrium theory, and the expansion rate is no longer inflationary. However, a thermodynamic model needs to incorporate particle production.\footnote{See \[26\].}

Finally, for those who like to analyse and solve differential equations\footnote{See also \[27\] – \[30\].} more than they like physical analysis, I will give the evolution equation of $H$ with mathematically more general (but physically no more satisfactory) thermodynamic equations of state. Suppose $p$ and $T$ are given as above, by (3.14) and (3.15), but instead of the relation (3.17), with constant $c_b$, linking $\tau$ and $\zeta$, we assume the barotropic forms

$$\zeta \propto \rho^r, \quad \tau \propto \rho^q$$

(3.29)

where $r$ and $q$ are constants.

Then with (3.29), the (non–truncated) evolution equation (3.11) becomes

$$\dot{H} + 3 \left[ 1 + \frac{1}{2} (1 + q - r) \gamma \right] H \dot{H} + \alpha_1 H^{-2q} \dot{H} + \left( q - r - 1 + \gamma^{-1} \right) H^{-1} \dot{H}^2$$

$$+ \frac{9}{4} \gamma H^3 + \frac{3}{2} \gamma \alpha_1 H^{2(1-q)} + \frac{3}{2} \alpha_2 H^{2r-2q+1} = 0$$

(3.30)

where $\alpha_1$ and $\alpha_2$ are constants. One can find special exact solutions, including exponential and power–law inflation, and perform a qualitative dynamical analysis of (3.30), or of similar equations arising from different forms for the equations of state and thermodynamic coefficients.
3.3 Appendix: Bulk Viscous Perturbations

A comprehensive analysis of the causality and stability properties of the full Israel–Stewart theory has been performed by Hiscock and Lindblom \[6\]. They consider general perturbations – i.e. Π, q_α, π_αβ all nonzero – about a (global) equilibrium in flat spacetime, but the results are valid in cosmology for short wavelength perturbations. In this appendix, I will extract from their complicated general results the special case of scalar perturbations (only Π ≠ 0), when remarkably simple expressions can be obtained.

The characteristic velocities for general dissipative perturbations are given by equations (110) – (128) in \[6\]. The purely bulk viscous case is

\[ \alpha_0 = 0 = \alpha_1; \quad \frac{1}{\beta_1}, \quad \frac{1}{\beta_2} \to 0; \quad \beta_0 \equiv \frac{\tau}{\zeta} \]  

(See (2.17) and (2.21) – (2.24).)

Equation (127) of \[6\] gives the speed of the propagating transverse modes:

\[ v_T^2 = \frac{(\rho + p)\alpha_2^2 + 2\alpha_1 + \beta_1}{2\beta_2 [\beta_1 (\rho + p) - 1]} \to 0 \]

on using (3.31). This is as expected for scalar sound–wave perturbations. Equation (128) governing the speed \( v = v_L \) of propagating longitudinal modes becomes, on dividing by \( \beta_0 \beta_2 \) and setting \( \alpha_0 = 0 = \alpha_1 \):

\[
\begin{align*}
[\beta_1 (\rho + p) - 1] v^4 + & \left\{ \frac{2nT}{T} \left( \frac{\partial T}{\partial n} \right)_S - \frac{(\rho + p)}{nT^2} \left( \frac{\partial T}{\partial S} \right)_n - \beta_1 \left\{ (\rho + p) \left( \frac{\partial p}{\partial \rho} \right)_S + \frac{1}{\beta_0} \right\} \right\} v^2 \\
& + \frac{1}{nT^2} \left( \frac{\partial T}{\partial S} \right)_n \left\{ (\rho + p) \left( \frac{\partial p}{\partial \rho} \right)_S + \frac{1}{\beta_0} \right\} - \left[ \frac{n}{T} \left( \frac{\partial T}{\partial n} \right)_S \right]^2 = 0
\end{align*}
\]

Dividing by \( \beta_1 \) and taking the limit \( \beta_1 \to \infty \), this gives

\[ v^2 = \left( \frac{\partial p}{\partial \rho} \right)_S + \frac{1}{(\rho + p)\beta_0} \]  

(3.32)

The first term on the right is the adiabatic contribution \( \epsilon_a^2 \) to \( v^2 \), and the second term is the dissipative contribution \( \epsilon_b^2 \), as in (3.17).

It is also shown in \[6\] (pp 478–480) that causality and stability require

\[ \Omega_3(\lambda) \equiv (\rho + p) \left\{ 1 - \lambda^2 \left[ \left( \frac{\partial p}{\partial \rho} \right)_S + \frac{1}{(\rho + p)\beta_0} \right] \right\} \geq 0 \]

for all \( \lambda \) such that \( 0 \leq \lambda \leq 1 \). This condition is shown to hold for all \( \lambda \) if it holds for \( \lambda = 1 \), leading to the requirement

\[ \epsilon_b^2 \equiv \frac{\zeta}{(\rho + p)\tau} \leq 1 - \epsilon_a^2 \]  

(3.33)

i.e. \( v^2 \leq 1 \), as expected. This establishes (3.18).
These results refine and correct the widely-quoted statement in [1] that $\zeta/\rho \tau = 1$ is required by causality.

Acknowledgements

Thanks to Sunil Maharaj for organising the Workshop so well and for his wonderful hospitality. The participants at the Workshop helped improve these notes by their questions and comments. I was supported by a Hanno Rund Research Fellowship. I have had many useful and inspiring discussions with Diego Pavon, Winfried Zimdahl, David Jou, Josep Triginer, David Matravers and others.
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