To the problem of transverse anisotropy effect on the distribution of contact stresses in a beam-strip loaded with a rigid stamp

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Abstract. The contact interaction of an orthotropic bar-beam loaded with a rigid stamp. The nature of contact stresses distribution along contact surface of beam and the size of the contact region depending on the beam material orthotropy degree is studied. Analytical expressions for contact pressure under the stamp, dimensionless force, size of contact region and stamp settlement depending on contact region size are derived. Numerical results of calculations are given.

1. Introduction. The problem of forced bending of a thin rod over a predetermined rigid surface and the determination of maximum stresses in it were first considered by S.P. Timoshenko [1] in the formulation of hypotheses of the classical Bernoulli-Euler bending theory of rods. In a refined formulation, taking into account transverse shear deformations, the problem was first solved by M.M. Filonenko-Borodich [2] (1949 г.). On the example of thin strip which contacts the rigid template it is shown that at relatively large contact region the contact pressure is concentrated practically on the ends of contact region and gets the highest values there.

For the first time, most consistently and quite thoroughly, the periodic problem of bending an infinite strip by stamps based on the equations of the plane theory of elasticity was considered by Cyrus and Silva [3], where the magnitude of the contact region could be both large and relatively small. The solution was constructed in trigonometric series and was reduced to paired integral equations, which were solved numerically. The authors [3] showed that for small areas of contact the distribution of contact pressure under the stamp is close to the distribution according to G. Hertz, where the maximum pressure is concentrated in the center of the strip, and for large - close to its edge. The same problem, but already in the formulation of the cylindrical bend of the plate, was considered by B.L. Pelekh and V.I. Shvabiuk [4], but according to a refined theory, the equation of which took into account the transverse compression. E.I. Grigolyuk and V.M. Tolkachev [5], as well as S.O. Sargsyan [6] in the formulation of a plane deformation reduced this problem to the solution of the
integral equation of Fredholm of the second kind, which was solved numerically. The reduction of the contact problem for an isotropic strip to a singular integral equation, with the subsequent use of the method of mechanical quadratures, was also carried out by the authors [7].

A similar problem of contact stress distribution arose in the mixed problem of bending an orthotropic beam lying on a rigid base [2,8]. In this case, the nature of the contact pressure distribution also very much depends on the calculation equations used to solve this problem. As well as for the parabolic stamp, found by V.I. Feodosiev [2], the contact pressure is expressed through hyperbolic functions with a maximum at the boundary of the contact of the plate with the rigid base, but a new fictitious concentrated force appears at the left end of half of the plate. This distribution of contact pressure corresponds to the solutions of contact problems for beams and plates, the bending equations of which take into account only the deformations of the transverse shear. Therefore, most of the works, the solutions of which are found on the basis of "shear" theories such as S.P. Tymoshenko [2], lead to the fact that the contact pressure at the boundary of the contact region is different from zero, even for hard surfaces without angular points. This fact contradicts the physical content of the problem, as well as the results obtained on the basis of the equations of the plane problem of the theory of elasticity of beam theories that take into account the deformations of the transverse shear and compression [3,5,8]. With the use of such models, these inconsistencies disappear and the distribution of contact pressure becomes similar to the distribution as in the case of a parabolic stamp.

To eliminate this shortcoming, in solving this problem, the authors [9,10] previously used refined theories, which in addition to the deformations of the transverse shear also took into account the transverse compression. Therefore, it is necessary to investigate in more detail the effect of this clarification on the nature of the contact stresses under the stamp.

2. Problem formulation. Let the strip of rectangular section is under the influence of rigid parabolic stamp, which is pressed to the upper surface of strip \((z = -h)\) by force \(P\), directed downwards. On its ends \(x = \pm l\) the strip is simply supported. Assume that the surface of the base of the stamp is perfectly smooth and is described by the equation \(z = -\varphi \cdot f(x)\). Contact pressure occurs between the strip and the stamp \(q^- (x) = q(x)\). We will neglect the forces of friction. The basic bending equation of the system is reduced to the form:

\[
EIw'''' = q(x) - \varepsilon_1h^2q'' - \varepsilon_2h^4q''''E / 4E', \quad (|x| \leq a);
\]

\[
w''''(x) = 0, \quad (|x| > a),
\]
where \( \epsilon_1 = 0,1 \left( \frac{4E}{G'} - 3v^* \right) \); \( \epsilon_2 = 0,2 \left( 1 - v^* \frac{G'}{E} \right) \); \( 2a \) – contact region width, outside which the pressure on the upper face of the strip is zero \( \left( q^- (x) = 0 \right) \).

Let’s find the solution of equation (1) separately in the contact region \( \left( |x| \leq a \right) \), where there is an unknown contact pressure, and in the region, which is free of contact interaction \( \left( a \leq |x| \leq l \right) \). Assume that when \( z = -h \) the top surface of the strip \( \left( |x| \leq a \right) \) tightly adjacent to the lower base of the stamp, i.e. —

\[
W(x, -h) = \delta - \omega f(x),
\]

where \( \delta \) – stamp settlement; \( \omega \) – arbitrary constant, which can be taken as a multiple of the curvature of the base of the stamp.

The left side of the equation for \( W(x, -h) \) we obtain from the corresponding ratio for \( W \).

Then, we get another relationship between the pressure under the stamp \( q(x) \) and the deflection of the middle line of the strip \( w(x) \):

\[
A_0 q(x) = \delta - \omega f(x) - w(x) - \frac{6}{13} v^* h^2 \frac{d^2 w}{dx^2}.
\]

Substituting expression (3) in equation (1), accepting \( f(x) = x^2 \), we obtain the following differential equation with respect to the deflection of the midline of the strip:

\[
w'''' - 2g^2 w'' + \lambda_4^4 w = \lambda_4^4 \left( \delta + 2\omega \xi h^2 - \omega x^2 \right),
\]

where

\[
\lambda_4^4 = \left[ \frac{5,9 E}{E'} + 0,3v^* \left( \frac{E}{G'} + 2 \frac{G'}{E'} \right) \right]^4 h^4;
\]

\[
g^2 = 0,2 \left( \frac{E}{G'} - 1,9v^* \right) h^2 \lambda_4^4.
\]

\[
\alpha = \sqrt{\lambda_4^2 + g^2} / 2; \quad \beta = \sqrt{\lambda_4^2 - g^2} / 2.
\]

The choice of one or another solution of equation (4) depends on the relationship \( E / G' \), \( E / E' \), values of Poisson's ratios \( v \) and \( v^* \), that is, of the characteristics of the strip material.

Для даної задачі співвідношення \( g^2 \leq \lambda^2 \) означають наступні залежності:

\[
E \frac{E}{G'} - 2v^* \leq 2,5 \sqrt{\frac{1}{3} \left( \frac{5,9 - 0,3v' \nu^* + 0,3v' \left( \frac{E}{G'} + 2 \frac{G'}{E'} \right)}{E'} \right) E'}.\]

We will accept further, without loss of generality that \( g^2 < \lambda^2 \). Then, given the symmetry of the problem, the solution can be reduced to the form:

\[
w' = A_1 K_1(x) + A_2 K_2(x) + w^*,
\]

where \( K_1(x) = chax \cdot cos \beta x; \quad K_2(x) = shax \cdot sin \beta x; \quad K_3(x) = shax \cdot cos \beta x; \)
4 $K_4(x) = ch\alpha x \sin \beta x$ – O.M. Krylov – V.Z. Vlasov fundamental functions.

3. Solution of the problem. Expression for the contact pressure under the stamp, the relationship between the dimensionless force $\bar{P}$, the size of the contact region and the stamp settlement can be written as follows:

$$q(x) = -A_1 \left( C_1 K_1(x) - C_2 K_2(x) \right), \quad \frac{1}{\bar{P}} = (1 - \theta) + \frac{C_3 K_1(a) - C_4 K_2(a)}{l(C_5 K_3(a) - C_6 K_4(a))},$$

$$\delta = \theta(2 - \theta) + \frac{2}{3} \bar{P}(1 - \theta) \left[ (1 - \theta)^2 + 3(\varepsilon' - \nu') \right] - \frac{(1 - \theta) \bar{P} C_7 K_3(a) - C_8 K_4(a)}{2\omega l C_5 K_3(a) - C_6 K_4(a)},$$

where $\bar{P} = \frac{Pl}{4\omega E l}$, $\delta = \frac{\delta}{\beta \omega l^2}$, $\theta = \frac{a}{l} C_i$ – known constants.

The solution of a similar problem in the exact formulation of the plane problem of the theory of elasticity for an isotropic material was obtained in [3,5,8].

On the basis of the given formulas it is easy to show that at the decision of this type of contact problems it is necessary to consider cross compression even in case of very thin rods ($l/h = 200$) and at large contact regions. So, M.M. Filonenko-Borodych [2] gives a numerical example for a steel strip $\left( E = 2.1 \cdot 10^6 \text{k}\sigma / \text{cm}^2, \nu = 1/3 \right)$ of length $2l = 20 \text{cm}$, width $t = 1 \text{cm}$ and thickness $2h = 0.1 \text{cm}$, which is pressed against the absolutely rigid base of the pattern by concentrated forces $P'$, attached to its ends. The radius of curvature of the base of the pattern was equal to $R = 100 \text{cm}$. It was necessary to determine the magnitude of the forces applied to the strip to bend it along a given pattern, as well as to investigate the nature of the pressure distribution on the strip. M.M. Filonenko-Borodych finds the solution of the problem with the help of a theory that takes into account only the influence of the deformation of the transverse shear. According to the solution given in the paper, the magnitude of these forces is equal to $P' = 33.9 \text{k}\sigma$ each. The maximum pressure is reached at the ends of the strip and is equal to $q_{\text{max}}$. The results obtained using the corresponding formulas (6) give the following values for maximum forces and maximum pressure:

$$P' = 0.5P = 25.2\text{k}\sigma, \quad q_{\text{max}} = 237\text{k}\sigma / \text{cm}.$$
to the solution of M. Filonenko-Borodych [2] according to the theory of Tymoshenko-type rods. The solid line is constructed by formula (6).

Analysis of the nature of the curves in Fig.1 shows that the reaction of the pattern to the strip is concentrated at its ends in regions the size of $2\delta cm$. The other part of the pattern is almost free of load. Also, from the numerical data for $P'$ and $q_{max}$ it is seen that the solution of MM Filonenko-Borodych gives inflated (by 34.5%) values both for the forces required to keep the strip in a given state, and for the contact pressure (almost twice) under the pattern. These results once again confirm the conclusion that solutions obtained on the basis of equations that do not take into account the compression over the thickness of the rod for this type of contact problems can lead to significant errors.

![Fig.1. Distribution of contact pressure in a long rod](image1)

![Fig.2 Dependence of the force applied to the stamp on the size of the contact region](image2)

In more detail, on the basis of the developed formulas (6), the study of the stress-strain state of the strip supported on two supports and loaded with a stamp of parabolic shape. The basis of the stamp is described by the function $f(x) = \delta + x^2 / 2R$. The supports are placed symmetrically relative to the stamp at a distance from each other. Here it is marked $\delta$ – stamp settlement, $R$ – stamp base curvature radius.

Fig. 2 shows graphs characterizing the change in force $\bar{P} = PL / 4\omega EI$ from the size of the contact area for the short beam ($h/l = 0.5$) according to the formula (6). The dashed curve is found according to the solution of M. Filonenko-Borodych [2] according to the formulas of the theory of Tymoshenko's type, and the dashed curve - the exact solution [3]. The double dashed line indicates the curve corresponding to the Bernoulli-Euler model of thin rods.
Curves 1 correspond to the isotropic case, and curve 2 to the transversely isotropic case (fiberglass 27-63C), with the following anisotropy parameters: $\frac{E}{E'} = 5$, $\frac{G}{G'} = 3$, $\nu = 0.25$, $\nu'' = 0$.

From the analysis of the graphical data presented in Figure 2, it is seen that the relationship (6) between the magnitude of the contact area and the force applied to the stamp in the case of isotropic material is very close to exact solutions [3,8] and in the case of thin strips the curves almost coincide. At the same time, the double dash-dotted curve shown in Fig. 2 shows that the Bernoulli-Euler theory in these dependences leads to a number of physical inconsistencies. For example, according to this theory to make full contact of the rod with the stamp ($\theta = 1$) it is necessary to apply infinitely great force $P$ to the latter, which even at zero contact ($\theta = 0$) and the weightless stamp must take values other than zero.

Analysis of the nature of the curves in Fig. 2 also shows that to achieve the same value of the contact region to the isotropic band it is necessary to apply a much greater (approximately three times) force $P$, than to the corresponding orthotropic strip with relations $\frac{E}{E'}$, $\frac{G}{G'} > 1$.

Figures 3 and 4 show graphs of changes in contact pressure $\bar{q} = q / (Gl\omega)$ for isotropic and transtropic strips, respectively, for different values of the contact area and material, as well as the
calculation model of the problem. In Fig. 3 curves 1, 3 are constructed for the size of the contact region \( \theta = a/l = 0.1 \), and curves 2, 4 for \( \theta = a/l = 0.8 \). In particular, curves 1,2 (solid) are constructed by formula (6), and curves 3,4 (dashed) are found according to the solution of the shift theory of Tymoshenko-type beams [2]. The dashed-dotted curves are taken from [3], which are constructed according to the formulas of the exact solution of the plane problem of the theory of elasticity. Curve 1, with circles, corresponds to the previous solution of the authors when the parameter \( \varepsilon_0 = 0 \). From the graphs of changes in the reduced contact pressure \( \bar{q} \) (curves 1,2) it is seen that the proposed model of bending of beams gives results quite close to the results of the plane problem of the theory of elasticity: for small regions of contact \( \theta < 0,2 \) the nature of the distribution of contact stresses is close to the Hertz distribution - the value \( q_{\text{max}} \) is in the center of the contact region (curve 1); for large regions - , the maximum contact stress approaches the edge of the contact region (curves 2).

At the same time, solutions of the problem with the help of theories that take into account only the influence of the deformation of the transverse shear (dashed curves 3,4), give results that do not correspond to the real distribution of contact pressure under the stamp.

In fig. 4 it is studied the influence of material anisotropy on the distribution of contact pressure in the transtropic beam-strip depending on the values of the contact region and the characteristics of the material. In particular, curves 1 (solid, dashed and dashed) are constructed for isotropic material. Curves 2,3,4 are constructed for transverse-isotropic material with characteristics: \( \frac{E}{E'} = 1, \frac{G}{G'} = 0; \)
\[
\frac{E}{E'} = 2, \frac{G}{G'} = 1, \nu = \nu^* = 0,25 \quad \frac{E}{E'} = 5, \frac{G}{G'} = 3, \nu = 0,25 \text{ and } \nu^* = 0 \quad \text{respectively.} \]
The dashed-dotted curve is taken from [3], and the dashed one is constructed by the formulas of the shear theory of Tymoshenko-type beams [2].

From the analysis of the nature of the contact pressure distribution in Fig. 3,4 we can conclude that the ratio of the characteristics of the transverse anisotropy - \( E/E' \) and \( G/G' \) differently affect the magnitude and distribution of contact pressure in the beam. Thus, the increase in the transverse shear modulus \( G' \) leads to an increase in contact pressure (curves 1,2) and vice versa - a decrease in the shear modulus \( G' \) leads to a decrease in contact pressure (curves 1, 4). At the same time, increase the ratio \( E/E' \) leads to the spread of the magnitude of the contact pressure region towards its center.
5. Conclusion. The character of the contact pressure distribution under the rigid stamp along the surface of the contact region of the transverse isotropic beam is investigated. Based on the analysis of graphical data presented in Fig. 1-4, it is concluded that obtaining the real distribution of contact pressure on the surface of the beam-strip is impossible without the use of computational equations of nonclassical models (or the plane problem of the theory of elasticity), which take into account the deformation of transverse compression. It is shown that the values of the transverse shear modulus $G'$ and modulus of elasticity $E'$ differently affect the value (modulo) of the contact pressure and its distribution relative to the contact region: increase the transverse shear modulus $G'$ leads to an increase in contact pressure and a decrease in the shear modulus $G'$ leads to a decrease in contact pressure. Modulus of elasticity $E'$ change has a greater effect on the very nature of the distribution curves and its reduction leads to a decrease in its maximum and the spread of the magnitude of the contact pressure region towards its center.

References

1. Timoshenko S.P., Lessels Dzh. Prikladnaya teoriya uprugosti. M.-L.: Gos. nauchno-tehn. izd.-vo, 1931. 391 s.
2. Filonenko-Borodich M.M. Igib tonkogo sterzhnya po za - dannoy krivoy // Trudy Mosk.el.-mech. in-ta inzh. transporta, 1949. V.38. S. 3-10.
3. Keer L.M., Silva M.A.G. Bending of a cantilever brought gradually into contact with a cylindrical supporting surface. – Int.J. Mech. Sci. Pergamon Press. 1970. Vol.12, №9, p.751-760.
4. Peleh B.L., Shvab'yuk V.I. Ob odnom obobschenii teorii uprugih transversalno-izotropnyh plit primenitelno k nekotoryim kontaktnym zadacham // Soprotivlenie materialov i teoriya sooruzheniy. K.: BudIvelnik, 1975. V.26. S.40-45.
5. Grigolyuk E.I., Tolkachev V.M. Kontaktnyie zadachi teorii plastin i obolochek. - M.: Mashinostroenie, 1980. 416 s.
6. Sarkisyan S.O. O tsilindricheskom izgibe plastinki zhestkimi shtampami // Dokladyi AN Arm.SSR, 1977. T.69. №4. S.216-223.
7. Shvab'yuk V.I., Maksimovich Ya.V. Kontaktna zadacha teorIYi pruzhnosti dlja smugi, scho rozmschena na oporah // Vilsnik Lviv. un-tu. Serlya meh.- mat., 2000. Vip. 57. S.195-198.
8. Shvab'yuk V.I., Pasternak Y.M., Rotko S.V. Refined solution of the Timoshenko problem for an orthotropic beam on a rigid base // Materials Science. – 2010. – Vol. 46. – No. 1. – P. 56–63.
9. Shvab'yuk V.I., Rotko S.V., Uzhegova O.A. Bending of a Composite Beam with a Longitudinal Section. // Strenght of Materials. – 2014. – Vol. 46. – No. 4. – P. 558–566.
10. Goryk A.V. and S.B. Koval’chuk Solution of a Transverse Plane Bending Problem of a Laminated Cantilever Beam Under the Action of a Normal Uniform Load // Strength of Materials. – 2018. – Vol. 50, Iss. 3. – pp.406-418.