Criterion of equationally Notherian property for posets

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Abstract. Many modern models of informational defence represents by graphs and partial orders (posets). It is very important to resolve such algorithmic problems as searching of elements that satisfies some conditions (or, another words, to resolve recognition problems) in this models. Algebraic geometry is resolving such problems and searches elements that is solution of system of equations. But systems could be infinite. In this article we’re formulating the condition for poset that every infinite system under such posets can be reduce to finite system and still be equivalent to origin system.

1. Introduction

Decision of equations and system of equations with various coefficients is one of the oldest problem in mathematics. Classical algebraic geometry studies solutions of system of equations (algebraic sets) under the real and complex fields. Classical algebraic geometry got great development in XIX century. But since the middle of XX century big development was received by universal algebraic geometry. Universal algebraic geometry studies algebraic sets under the arbitrary algebraic structures whether graph, group or universal algebra. For the present, a series of contributions by E.Yu. Daniyarova, A.G. Myasnikov and V.N. Remeslennikov \[1\], \[2\], \[3\], \[4\] and \[5\] is some kind of development’s result of universal algebraic geometry. In this article studies algebraic geometry under partial order. Inspite of theory of partial orders gained great development \[6\], many problems of algebraic geometry under partial orders doesn’t solve yet. In the works \[7\], \[8\], \[9\] some kind of partial orders such as lattices, semilattices and boolean algebras was studied from algebraic-geometric point of view. In \[7\] was observed coordinate algebras structure, algebraic sets, equationally Notherian property and compactness property for partial orders was studied. Equationally Notherian property of algebraic structure is very important property that allows to use variety of algebraic geometry results. In this article we’re looking at equationally Notherian property for arbitrary partial order and, as result, we’re formulate the criterion of it.

2. Preliminaries

In this section we reminds basic definitions from theory of partial orders and universal algebraic geometry under partial orders.
Partial ordered set (partial order or poset) is algebraic system $\mathcal{P} = (\mathcal{P}, \leq)$, where $\leq$ - binary order relation and $\mathcal{A}$ - constant symbols, and the structure satisfies 3 axioms:

1. $\forall p \in \mathcal{P} \ p \leq p$.
2. $\forall p_1, p_2 \in \mathcal{P} \ p_1 \leq p_2 \land p_2 \leq p_1 \rightarrow p_1 = p_2$.
3. $\forall p_1, p_2, p_3 \in \mathcal{P} \ p_1 \leq p_2 \land p_2 \leq p_3 \rightarrow p_1 \leq p_3$.

Language (signature) of posets with constants we’re note as $L_{\mathcal{A}}$.

Elements $x$ and $y$ of poset $\mathcal{P}$ are comparable if $x \leq y$ or $y \leq x$. Otherwise, elements are incomparable.

Define for every subset of elements $A$ of poset $\mathcal{P}$ sets $A^\uparrow = \{x \in \mathcal{P} \mid \forall a \in A \ a \leq x\}$ and $A^\downarrow = \{a \in \mathcal{P} \mid \forall a \in A \ x \leq a\}$. These sets calls upper set (upset) and lower set (down set) of $A$.

For one-element set $A = \{a\}$ notations are $a^\uparrow$ and $a^\downarrow$ correspondence.

The set of generators of up(down) set $A^\uparrow (A^\downarrow)$ is subset of poset’s elements $\mathcal{P}$ that $B^\uparrow = A^\uparrow$ ($B^\downarrow = A^\downarrow$). Up(down) set $A^\uparrow (A^\downarrow)$ is finitely generated if there exist finite set $B$ of generators of up(down) set.

Let’s separate some subset $A$ of elements of $\mathcal{P}$. Upper $A$-fan is pair $(A, A^\uparrow)$, where $A$ is base and $A^\uparrow$ is upset of $A$. Lower $A$-fan defines in a similar way. Upper $A$-fan is finitely generated if there exist finite subset $B \subseteq A$ that $B^\uparrow = A^\uparrow$.

We’re formulate important technical lemma.

**Lemma 1** For every poset $\mathcal{P} = (\mathcal{P}, \leq)$ there is true two properties: (a) $\forall A \subseteq \mathcal{P} \ (A^\uparrow)^\uparrow \supseteq A$ and (b) $\forall A \subseteq \mathcal{P} \ (A^\downarrow)^\downarrow \subseteq A$.

**Proof.** $(A^\uparrow)^\uparrow = \{x \mid x \geq b \ \forall b \in A^\uparrow\}$ by the definition. But if $\forall a \in A \ \forall b \in A^\uparrow \ a \geq b$ then $A \subseteq (A^\uparrow)^\uparrow$. Property (b) is proving the same way. ■

Next, instead of $(A^\uparrow)^\uparrow$ we are writing simple $A^\uparrow\uparrow$.

Let note $X = \{x_1, \ldots, x_n\}$ as a set of variables. Term of poset $\mathcal{P}$ in language $L_{\mathcal{A}}$ from variables $X$ is any constant from $\mathcal{A}$ or any variable from $X$. Atomic formula of poset $\mathcal{P}$ in language $L_{\mathcal{A}}$ from variables $X$ defines through next two statements:

1. $t_1 = t_2$ is atomic formula, where $t_1, t_2$ is a terms;
2. $t_1 \leq t_2$ is atomic formula, where $t_1, t_2$ is a terms.

Equation under poset $\mathcal{P}$ in language $L_{\mathcal{A}}$ from variables $X$ is any atomic formula under $\mathcal{P}$. It is easy to see all types of equations under $\mathcal{P}$:

1. $a_i = a_j$, where $a_i, a_j \in A$,
2. $a_i \leq a_j$, where $a_i, a_j \in A$,
3. $x_i = a_j$, where $x_i \in X, a_j \in A$,
4. $x_i = x_j$, where $x_i, x_j \in X$,
5. $a_i \leq x_j$, where $x_j \in X, a_i \in A$,
6. $x_i \leq a_j$, where $x_i \in X, a_j \in A$,
7. $x_i \leq x_j$, where $x_i, x_j \in X$.

System of equations $S(X)$ from variables $X = \{x_1, \ldots, x_n\}$ is any set of equations from variables $X$. Point $p = (p_1, \ldots, p_n) \in \mathbb{P}^n$ is a solution of system $S(X)$ if substitution of point’s coordinates in system $S(X)$ instead of correspondence variables is providing a true statement under $\mathcal{P}$: $\mathcal{P} \models \varphi(p_1, \ldots, p_n)$. Set of solution is defining in a natural way. Set system’s $S(X)$ solutions denotes as $V(S)$. Two systems $S_1$ and $S_2$ is equivalent if their sets of solutions are the same (denotes as $S_1 \sim S_2$).
Let’s reach agreement about notation of equation’s sets. Equations \( a_i = a_j \), where \( a_i, a_j \in A \) in system \( S(X_n) \) denote as \( S \subseteq a \). Set of the rest equations are similar: \( S \subseteq a, S \subseteq x, S \subseteq x \), \( S \subseteq x, S \subseteq x \), and \( S \subseteq x \) for system \( S(X_n) \).

It says that poset \( \mathcal{P} \) has **equationally Noetherian property** is every system of equations \( S(X_n) \) from \( n \) variables under \( \mathcal{P} \) has equivalent finite subsystem \( S'(X_n) \subseteq S(X_n) \). And if for every system \( S(X_n) \) under \( \mathcal{P} \) there is exist equivalent finite system \( S'(X_n) \) then \( \mathcal{P} \) has **weak equationaly Noetherian property**.

Class of posets \( K \) has (weak) equationaly Noetherian property if every poset \( \mathcal{P} \in K \) has (weak) equationally Noetherian property.

### 3. Main result

In this section we’re formulate the main result.

**Theorem 1** Poset \( \mathcal{P} \) has equationally Noetherian property iff for every subset \( A \) of \( \mathcal{P} \) upper and lower \( A \)-fans are finitely generated.

**Proof.** Firstly, we assume for \( \mathcal{P} \) every \( A \)-fans is finitely generated. There is system of equations \( S(X_n) \) under \( \mathcal{P} \) in language \( L_A \) from \( n \) variables. Also assume that the system contains every \( 7 \) equations and equations of every type are infinite.

Without loss generality we take an agreement that every constant in \( S(X_n) \) has separate interpretation in \( \mathcal{P} \). This condition isn’t necessary, but it makes further proof more easier and it hasn’t essential influence to proof.

It has to prove the system \( S(X_n) \) has finite subsystem \( S'(X_n) \) that equivalent to \( S(X_a) \). Note that system \( S(X_n) \) is divides on \( 7 \) noncrossing subsystems \( S \subseteq a, S \subseteq x, \) etc. It can define solution set of system \( S(X_n) \) as \( V(S) = V(S \subseteq a) \cap V(S \subseteq a) \cap V(S \subseteq x) \cap V(S \subseteq x) \cap V(S \subseteq x) \). Next we show that it couldn’t chose finite subsystem from every type of equations that equivalent to origin. For instance, for system \( S \subseteq x \subset S(X_n) \) there is exist finite subset \( S \subseteq x \sim S \subseteq x \).

Firstly, lets observe subsystem \( S \subseteq a \subset S(X_n) \). If this system if compatible under \( \mathcal{P} \) then it hasn’t influence on the rest system and subsystem \( S \subseteq a \). But if \( \mathcal{P} \neq S \subseteq a \), then there is exist equation \( a_i = a_j \in S \subseteq a \), that doesn’t compatible under \( \mathcal{P} \) and this equation is equivalent to \( S \subseteq a \). Subset \( S \subseteq a \subset S(X_n) \) observes the similar way.

There is exist finite subsystems \( S \subseteq x \) and \( S \subseteq x \) that equivalent to \( S \subseteq x \) and \( S \subseteq x \), because of finite power of variables. Note that subsystems \( S \subseteq x \) and \( S \subseteq x \) can’t be incompatible.

If subset \( S \subseteq x \subset S(X_n) \) is compatible then there is exist finite subsystem \( S \subseteq x \) that equivalent to origin, because of finite power of variables. Otherwise, \( S \subseteq x \) contains next pair of equations \( x_i = a_j, x_i = a_k, \) and \( a_j \neq a_k \). This pair of equations is equivalent to \( S \subseteq x \).

Note that we still didn’t use condition from criterion. The rest subsystems \( S \subseteq x \subset S(X_n) \) and \( S \subseteq x \subset S(X_n) \) use this condition for chose finite subsystems. Firstly, we observe \( S \subseteq x \).

Lets select subset \( S \subseteq x \subset S \subseteq x \) every equation of it depends from only one variable \( x_i \). This set could be represent as \( S \subseteq x \subseteq \{ x_i \leq a_j \mid j \in J, |J| = \infty \} \). It’s easy to see from noted representation that \( V(S \subseteq x) = A_i^j \), where \( A_i = \{ a_j \mid j \in J \} \). Using property of finite generation of every \( A \)-fan from poset \( \mathcal{P} \), chose subset \( J' \subset J \), that \( A_i = \{ a_j \mid j \in J' \} \) and \( A_i^j = A_i^j \). It means that there is exist finite subsystem \( S \subseteq x \sim S \subseteq x \).

Using this procedure for every variables from \( S \subseteq x \), we chose finite subsystem \( S \subseteq x \sim S \subseteq x \). Existing of finite subsystem \( S \subseteq x \sim S \subseteq x \) is proving the same way.

We showed the way to construct subsystems \( S \subseteq a, S \subseteq x, S \subseteq x \), \( S \subseteq x \), \( S \subseteq x \), and \( S \subseteq x \), equivalent to \( S \subseteq a, S \subseteq x, S \subseteq x \), \( S \subseteq x \), and \( S \subseteq x \), from system \( S(X_n) \). It means that \( V(S) = V(S \subseteq a) \cap V(S \subseteq a) \cap V(S \subseteq x) \cap V(S \subseteq x) \cap V(S \subseteq x) \). We know, that \( V(S) = V(S \subseteq a) \cap V(S \subseteq a) \cap V(S \subseteq x) \cap V(S \subseteq x) \). Next, we build system of equations \( S(x) = \{ x \leq a_i \mid a_i \in A \} \) under \( \mathcal{P} \). There is
exist finite subsystem \( S'(x) = \{ x \leq a_i \mid a_i \in A', |A'| < \infty \} \sim S(x) \) because of \( \mathcal{P} \) is equationally Noetherian. It means that \( A^\uparrow = A'^\uparrow \), and all lower \( A \)-fans is finitely generated. Finite generation of upper \( A \)-fans is proving the same way.

Note that if \( A \)-fan isn’t finitely generated it doesn’t mean \( A^\uparrow \) or \( A^\downarrow \) isn’t finitely generated.

Let’s observe the next example: poset \( \mathcal{P} = \mathbb{Z} \cup \{ q \} \), where \( \mathbb{Z} \) is linear order of integers and \( q \) is single element that is not comparable with all elements of \( \mathbb{Z} \). Scheme of this poset is shown on fig. 1.

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
i - 1 & i & \cdots & q \\
\downarrow & \downarrow & \downarrow & \downarrow \\
i & i & \cdots & i + 1 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Figure 1. Poset \( \mathcal{P} = \mathbb{Z} \cup \{ q \} \)

Observe \( \mathbb{Z} \)-fan. It’s easy to see that \( \mathbb{Z}^\downarrow = \emptyset \). Any finite subset of \( \mathbb{Z} \) generates nonempty lower set and it means that lower \( \mathbb{Z} \)-fan isn’t finitely generated. But if we take subset \( B = \{ 0, q \} \), then \( B^\downarrow = \emptyset = \mathbb{Z}^\downarrow \). And it means that \( \mathbb{Z}^\downarrow \) is finitely generated.

For this reason and with formulated equationally Noetherian property for posets it could be formulate next criterion.

**Consequence 1** Poset \( \mathcal{P} \) has weak equationally Noetherian property iff for every subset \( A \) of \( \mathcal{P} \) \( A^\uparrow \) and \( A^\downarrow \) is finitely generated.

It’s easy to produce criterion of equationally Noetherian property for the class of posets from criterion for single poset. Class of posets has an equationally Noetherian property iff every poset from this class has equationally Noetherian property.

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