Perturbative Determination of
Mass Dependent $O(a)$ Improvement Coefficients
for the Vector and Axial Vector Currents
with a Relativistic Heavy Quark Action

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Abstract

We carry out a perturbative determination of mass dependent renormalization factors and $O(a)$ improvement coefficients for the vector and axial vector currents with a relativistic heavy quark action, which we have designed to control $m_Qa$ errors by extending the on-shell $O(a)$ improvement program to the case of $m_Q \gg \Lambda_{\text{QCD}}$ with $m_Q$ the heavy quark mass. We discuss what kind of improvement operators are required for the heavy-heavy and the heavy-light cases under the condition that the Euclidean rotational symmetry is not retained anymore because of the $m_Qa$ corrections. Our calculation is performed employing the ordinary perturbation theory with the fictitious gluon mass as an infrared regulator. We show that all the improvement coefficients are determined free from infrared divergences. Results of the renormalization factors and the improvement coefficients are presented as a function of $m_Qa$ for various improved gauge actions as well as the plaquette action.
I. INTRODUCTION

This paper is the third in a series of publications [1, 2] on a new relativistic approach which was recently proposed from the viewpoint of the on-shell $O(a)$ improvement program. The generic quark action, proposed first in Ref. [3], is given by

$$S_q = \sum_x \left[ m_0 \bar{q}(x)q(x) + \bar{q}(x)\gamma_0 D_0 q(x) + \nu \sum_i \bar{q}(x)\gamma_i D_i q(x) \right. $$

$$\left. - \frac{r_t a}{2} \bar{q}(x)D_0^2 q(x) - \frac{r_s a}{2} \sum_i \bar{q}(x)D_i^2 q(x) \right. $$

$$\left. - \frac{iga}{2} c_E \sum_i \bar{q}(x)\sigma_{0i} F_{0i} q(x) - \frac{iga}{4} c_B \sum_{i,j} \bar{q}(x)\sigma_{ij} F_{ij} q(x) \right].$$

(1)

While we are allowed to choose $r_t = 1$, other four parameters $\nu$, $r_s$, $c_E$ and $c_B$ should be properly adjusted as functions of $m_Q a$ and the gauge coupling constant $g$, in order to achieve the $O(a)$ improvement for all on-shell matrix elements. In Ref. [2] we determine $\nu$, $r_s$, $c_E$ and $c_B$ up to the one-loop level for various improved gauge actions. We now report on the $O(a)$ improvement of the vector and axial vector currents at the one-loop level for the relativistic heavy quark action.

In this paper we first make a general discussion about what kind of improvement operators are required from the symmetries allowed on the lattice, in which the Euclidean rotational symmetry is violated because of $m_Q a$ corrections. We consider both the heavy-heavy and heavy-light cases, where the light quark is massless for the latter. Following Ref. [2] we evaluate one-loop diagrams employing the conventional perturbative method with the use of the fictitious gluon mass to regularize the infrared divergence. In the massless case this method was successfully applied to the calculation of the renormalization constants and the improvement coefficients for the bilinear operators [4, 5].

This paper is organized as follows. In Secs. II and III we determine the renormalization constants and the improvement coefficients for the vector and axial vector currents up to one-loop level. The results are presented both for the heavy-heavy and heavy-light cases as a function of $m_Q a$ with various improved gauge actions in addition to the ordinary plaquette action. In Sec.IV we explain how to implement the mean field improvement for the renormalization factors. Our conclusions are summarized in Sec. V. Some preliminary results are presented in Ref. [6].

The quark and gluon actions and their Feynman rules are already presented in Sec. II.
We employ the notations introduced there without further notice throughout this paper. As for the numerical evaluation of the one-loop diagrams relevant for the vertex functions of the vector and axial vector currents, we employ the same method used in the perturbative determination of the improvement parameters for the relativistic heavy quark action, whose technical details are described in Sec. III of Ref. [2]. The physical quantities are expressed in lattice units and the lattice spacing $a$ is suppressed unless necessary. We take SU($N_c$) gauge group with the gauge coupling constant $g$.

II. $O(a)$ IMPROVEMENT OF THE VECTOR CURRENTS

We consider the on-shell $O(a)$ improvement of the vector currents both for the heavy-heavy and heavy-light cases. Without Euclidean space-time rotational symmetry, the renormalized operators with the $O(a)$ improvement is written as

$$\begin{align*}
V^{\text{latt},R}_\mu(x) &= Z^{\text{latt}}_{V\mu} \left[ \bar{q}(x) \gamma_\mu Q(x) - g^2 c^+_\nu \partial^-_\mu \{ \bar{q}(x) Q(x) \} - g^2 c^-_\nu \partial^+_\mu \{ \bar{q}(x) Q(x) \} ight. \\
&\quad \left. - g^2 c^L_\nu \{ \bar{q}(x) \} \gamma_\mu \gamma_5 Q(x) - g^2 c^H_\nu \bar{q}(x) \gamma_\mu \gamma_5 \{ \bar{q}(x) Q(x) \} + O(g^4) \right]
\end{align*}$$

(2)

where $Z^{\text{latt}}_{V\mu}$ and $c^{(+,-,H,L)}_\nu$ depend on the quark masses $m_Q$ and $m_q$. Here $\partial^\pm = \partial^-_\mu + \partial^\pm_\mu$ and $\partial^-_\mu = \partial^-_\mu - \partial^\pm_\mu$. For the time component of the vector currents we can choose $c^H_\nu = c^L_\nu = 0$ with the aid of equation of motion. In the case of $m_Q = m_q$ we find $c^H_{\mu} = 0$ and $c^L_{\mu} = c^L_{\mu}$ from the charge conjugation symmetry. Once the both quark masses are massless, all the improvement coefficients except $c^+_\mu$ vanishes.

In this section $Z^{\text{latt}}_{V\mu}$ and $c^{(+,-,H,L)}_{\nu}$ are determined at the one-loop level as a function of $m_Q a$ both for the heavy-heavy and heavy-light cases. We employ the relativistic heavy quark action proposed by the authors [1] both for the heavy and light quarks.

A. Determination of the improvement coefficients for the vector currents

We consider the general form of the off-shell vertex functions of the vector currents on the lattice at the one-loop level:

$$\Lambda^{(1)}_k(p, q, m_{p1}, m_{p2}) = \gamma_k F^k_1 + \gamma_k \{ \bar{q} F^k_2 + \bar{q} s F^k_3 \} + \{ \bar{q} F^k_4 + \bar{q} s F^k_5 \} \gamma_k$$

$$\quad + \bar{q} \gamma_k \bar{q} F^k_6 + \bar{q} \gamma_k \bar{q} s F^k_7 + \bar{q} s \gamma_k \bar{q} F^k_8 + \gamma_k \bar{q} \bar{q} s F^k_9 + \bar{q} \bar{q} s \gamma_k F^k_{10}$$

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The vertex functions (3) and (4) are defined for the process depicted in Fig. 1. The coefficients and 

\[ \Lambda_0^{(1)}(p, q, m_{p_1}, m_{p_2}) = \gamma_0 F_1^0 + \gamma_0 \phi F_2^0 + \phi \gamma_0 F_3^0 + \phi \gamma_0 F_4^0 \]

\[ + (p_0 + q_0) \left[ G_1^0 + \phi G_2^0 + \phi G_3^0 + \phi G_4^0 \right] \]

\[ + (p_0 - q_0) \left[ H_1^0 + \phi H_2^0 + \phi H_3^0 + \phi H_4^0 \right] + O(a^2) \]  

(4)

where we assume that the off-shell vertex functions are perturbatively expanded as

\[ \Lambda_\mu(p, q, m_{p_1}, m_{p_2}) = \gamma_\mu + \sum_{i=1}^{3} (g^2)^i \Lambda^{(i)}_\mu(p, q, m_{p_1}, m_{p_2}) \]  

(6)

The vertex functions (3) and (4) are defined for the process depicted in Fig. 1. The coefficients \( F_\mu, G_\mu, H_\mu \) are functions of \( p^2, q^2, p \cdot q, m_{p_1} \) and \( m_{p_2} \).

Sandwiching (3) and (4) by the on-shell quark states \( u(p) \) and \( \bar{u}(q) \), which satisfy \( \phi u(p) = \text{im} p_1 u(p) \) and \( \bar{u}(q) \phi = \text{im} p_2 \bar{u}(q) \), the matrix elements are reduced to

\[ \bar{u}(q) \Lambda^{(1)}_k(p, q, m_{p_1}, m_{p_2}) u(p) \]

\[ = \sum_{a=0}^{3} F_{a1} + \text{im} p_1 F_{a2} + \text{im} p_2 F_{a4} - m_{p_1} m_{p_2} F_{a4} \]

\[ + \sum_{a=0}^{3} G_{a1} + \text{im} p_1 G_{a2} + \text{im} p_2 G_{a3} - m_{p_1} m_{p_2} G_{a3} \]

\[ + \sum_{a=0}^{3} H_{a1} + \text{im} p_1 H_{a2} + \text{im} p_2 H_{a3} - m_{p_1} m_{p_2} H_{a3} \]

\[ + O(a^2). \]  

(7)

and

\[ \bar{u}(q) \Lambda^{(1)}_0(p, q, m_{p_1}, m_{p_2}) u(p) \]

\[ = \sum_{a=0}^{3} F_{a0} + \text{im} p_1 F_{a2} + \text{im} p_2 F_{a3} - m_{p_1} m_{p_2} F_{a3} \]

\[ + \sum_{a=0}^{3} G_{a0} + \text{im} p_1 G_{a2} + \text{im} p_2 G_{a3} - m_{p_1} m_{p_2} G_{a3} \]

\[ + \sum_{a=0}^{3} H_{a0} + \text{im} p_1 H_{a2} + \text{im} p_2 H_{a3} - m_{p_1} m_{p_2} H_{a3} \]

\[ + O(a^2). \]  

(8)
For convenience we express the coefficients as

\[
X_k = F_1^k + im_{p_1}F_2^k + im_{p_2}F_4^k - m_{p_1}m_{p_2}F_{6}^k, \tag{9}
\]

\[
Y_k = G_1^k + im_{p_1}G_2^k + im_{p_2}G_3^k - m_{p_1}m_{p_2}G_{4}^k, \tag{10}
\]

\[
Z_k = H_1^k + im_{p_1}H_2^k + im_{p_2}H_3^k - m_{p_1}m_{p_2}H_{4}^k, \tag{11}
\]

\[
R_k = F_3^k + im_{p_2}F_7^k + im_{p_1}F_9^k, \tag{12}
\]

\[
S_k = F_5^k + im_{p_1}F_8^k + im_{p_2}F_{10}^k \tag{13}
\]

and

\[
X_0 = F_1^0 + im_{p_1}F_2^0 + im_{p_2}F_3^0 - m_{p_1}m_{p_2}F_{4}^0, \tag{14}
\]

\[
Y_0 = G_1^0 + im_{p_1}G_2^0 + im_{p_2}G_3^0 - m_{p_1}m_{p_2}G_{4}^0, \tag{15}
\]

\[
Z_0 = H_1^0 + im_{p_1}H_2^0 + im_{p_2}H_3^0 - m_{p_1}m_{p_2}H_{4}^0. \tag{16}
\]

Since the above coefficients contain both the lattice artifacts and the physical contributions which remains in the continuum, we have to isolate the lattice artifacts in order to determine the improvement coefficients in eq.(2). The improvement coefficients are given by

\[
\Delta_\gamma_k = (X_k)^{\text{latt}} - (X_k)^{\text{cont}}, \tag{17}
\]

\[
i c_{Y_k}^+ = (Y_k)^{\text{latt}} - (Y_k)^{\text{cont}}, \tag{18}
\]

\[
i c_{Y_k}^- = (Z_k)^{\text{latt}} - (Z_k)^{\text{cont}}, \tag{19}
\]

\[-ic_{Y_k}^L = (S_k)^{\text{latt}}, \tag{20}\]

\[ic_{Y_k}^H = (R_k)^{\text{latt}}, \tag{21}\]

\[
\Delta_{V_0} = (X_0)^{\text{latt}} - (X_0)^{\text{cont}}, \tag{22}
\]

\[ic_{V_0}^+ = (Y_0)^{\text{latt}} - (Y_0)^{\text{cont}}, \tag{23}\]

\[ic_{V_0}^- = (Z_0)^{\text{latt}} - (Z_0)^{\text{cont}}. \tag{24}\]

where the continuum contributions are obtained by employing the naive dimensional regularization (NDR) with the modified minimal subtraction scheme (\MS). We have $R_k = S_k = 0$ in the continuum from the space-time rotational symmetry. Here it is reminded that in case of $m_{p_2} = m_{p_1}$ we obtain $c_{Y_k}^L = c_{Y_k}^H$ and $c_{V_0}^- = c_{V_0}^- = 0$ from the charge conjugation symmetry.

The renormalization factor of the vector currents is obtained by

\[
\frac{Z_{\nu}^{\text{latt}}}{Z_{\nu}^{\text{cont}}} = \sqrt{Z_{Q,\text{latt}}^{(0)}(m_{p_1}^{(0)})} \sqrt{Z_{Q,\text{latt}}^{(0)}(m_{p_2}^{(0)})} \left(1 - g^2 \Delta_{V_0} \right) \tag{25}\]
with

\[ Z_{Q, \text{latt}}^{(0)}(m_{p_1}^0) = \cosh(m_{p_1}^0) + r_t \sinh(m_{p_1}^0) \]  
\[ Z_{q, \text{latt}}^{(0)}(m_{p_2}^0) = \cosh(m_{p_2}^0) + r_t \sinh(m_{p_2}^0) \] 
\[ \Delta V_{\mu} = \Delta_{\gamma_{\mu}} - \frac{\Delta Q}{2} - \frac{\Delta q}{2}, \]  

where \( \Delta_{Q,q} \) denote the wave function renormalization factors, which are already given in Ref. [2]. Although we evaluate \( Z_{\nu}^{\text{cont}} \) in \( \overline{\text{MS}} \) scheme with NDR in this paper, the reader may be interested in the value defined in \( \overline{\text{MS}} \) scheme with DRED. The conversion between these two definitions is easily done by the relation

\[ Z_{\nu}^{\text{cont}}(\text{NDR}) = Z_{\nu}^{\text{cont}}(\text{DRED}) - \frac{1}{2} g^2. \]  

Employing a set of special momentum assignments \( p = p^* \equiv (p_0 = i m_{p_1}, p_i = 0) \) and \( q = q^*_s \equiv (q_0 = i m_{p_2}, q_i = 0) \) or \( q = q^*_d \equiv (q_0 = -i m_{p_2}, q_i = 0) \), where subscripts \( s \) and \( d \) represent the scattering and the decay respectively, we extract the relevant coefficients \( X_k, Y_k, Z_k, R_k, S_k \) for \( V_k \) from the off-shell vertex function

\[ X_k^d = \frac{1}{4} \text{Tr} \left[ \Lambda_k^{(1)}(1 + \gamma_0)\gamma_k \right]_{p=p^*,q=q^*_d}, \]  
\[ Y_k^s + Z_k^s = \frac{1}{4} \text{Tr} \left[ \frac{\partial}{\partial p_k} \Lambda_k^{(1)}(1 + \gamma_0) - \frac{\partial}{\partial q_i} \Lambda_k^{(1)}(1 + \gamma_0) \gamma_i \gamma_k \right]_{p=p^*,q=q^*_s}, \]  
\[ Y_k^s - Z_k^s = \frac{1}{4} \text{Tr} \left[ \frac{\partial}{\partial q_k} \Lambda_k^{(1)}(1 + \gamma_0) - \frac{\partial}{\partial q_i} \Lambda_k^{(1)}(1 + \gamma_0) \gamma_i \gamma_k \right]_{p=p^*,q=q^*_s}, \]  
\[ X_k^s + 2i m_{p_1} R_k^s = \frac{1}{4} \text{Tr} \left[ \Lambda_k^{(1)}(1 + \gamma_0) + 2i m_{p_1} \frac{\partial}{\partial p_i} \Lambda_k^{(1)}(1 + \gamma_0) \gamma_i \gamma_k \right]_{p=p^*,q=q^*_s}, \]  
\[ X_k^s + 2i m_{p_2} S_k^s = \frac{1}{4} \text{Tr} \left[ \Lambda_k^{(1)}(1 + \gamma_0) \gamma_k + 2i m_{p_2} \frac{\partial}{\partial q_i} \Lambda_k^{(1)}(1 + \gamma_0) \gamma_k \gamma_i \right]_{p=p^*,q=q^*_s}, \]

where superscripts \( s \) and \( d \) in \( X_k, Y_k, Z_k, R_k, S_k \) represent their momentum assignments. On the other hand, we employ \[ \Box \] to determine \( X_0, Y_0, Z_0 \) for \( V_0 \):

\[ X_0^d = \frac{1}{4} \text{Tr} \left[ \Lambda_0^{(1)}(1 + \gamma_0) \gamma_k + im_{p_1} \frac{\partial}{\partial p_k} \Lambda_0^{(1)}(1 + \gamma_0) \gamma_k + im_{p_2} \frac{\partial}{\partial q_k} \Lambda_0^{(1)}(1 + \gamma_0) \gamma_k \right]_{p=p^*,q=q^*_d}, \]
\[ X_0^d + im_{p_1}(Y_0^d + Z_0^d) - im_{p_2}(Y_0^d - Z_0^d) \]
\[ = \frac{1}{4} \text{Tr} \left[ \Lambda_0^{(1)}(1 + \gamma_0) + 2i m_{p_2} \frac{\partial}{\partial q_k} \Lambda_0^{(1)}(1 + \gamma_0) \gamma_k \right]_{p=p^*,q=q^*_d}, \]
\[ X_0^s + im_{p_1}(Y_0^s + Z_0^s) + im_{p_2}(Y_0^s - Z_0^s) = \frac{1}{4} \text{Tr} \left[ \Lambda_0^{(1)}(1 + \gamma_0) \gamma_k \right]_{p=p^*,q=q^*_s}, \]
where we have used the fact that $F^k, G^k$ and $H^k$ are functions of $p^2, q^2$ and $p \cdot q$, so that
\[
\frac{\partial F^\mu}{\partial p_i} \bigg|_{p=p^*, q=q^*} = \frac{\partial F^\mu}{\partial q_i} \bigg|_{p=p^*, q=q^*} = 0, \tag{38}
\]
\[
\frac{\partial H^\mu}{\partial p_i} \bigg|_{p=p^*, q=q^*} = \frac{\partial H^\mu}{\partial q_i} \bigg|_{p=p^*, q=q^*} = 0, \tag{39}
\]
\[
\frac{\partial G^\mu}{\partial p_i} \bigg|_{p=p^*, q=q^*} = \frac{\partial G^\mu}{\partial q_i} \bigg|_{p=p^*, q=q^*} = 0 \tag{40}
\]
with $i = 1, 2, 3$.

Here we briefly explain how to deal with the infrared divergence in the above coefficients at the one-loop level. We basically follow the method employed in Refs. [2, 7]. Suppose the vertex function $\Lambda_{k,0}$ at the one-loop level is written as
\[
\Lambda_{k,0}^{(1)} = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} I_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda), \tag{41}
\]
where $\lambda$ is the fictitious gluon mass introduced to regularize the infrared divergence. We extract the infrared divergent term as
\[
\int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} I_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda)
= \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \left[ I_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda) - \tilde{I}_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda) \right] \bigg|_{\lambda \to 0} + \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \tilde{I}_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda), \tag{42}
\]
where $\tilde{I}_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda)$ should have an analytically integrable expression, whose infrared behavior is the same as $I_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda)$. For $\tilde{I}_{k,0}(k, p, q, m_{p_1}, m_{p_2}, \lambda)$ we employ
\[
\tilde{I}_{\mu}(k, p, q, m_{p_1}^{(0)}, m_{p_2}^{(0)}, \lambda)
= \theta(\Lambda^2 - k^2) C_F i\gamma^\alpha \frac{1}{i(\not{q} + m_{p_2}^{(0)})} \gamma^\mu \frac{1}{i(\not{p} + m_{p_1}^{(0)})} i\gamma^\alpha \frac{1}{k^2 + \lambda^2}, \tag{43}
\]
where $C_F = (N_c^2 - 1)/(2N_c)$ denotes the second Casimir of SU($N_c$) group. A domain of integration is restricted to a hypersphere of radius $\Lambda (\leq \pi)$ for convenience of an analytical integration.
B. Results for the improvement coefficients of the vector currents

1. Heavy-heavy case

We obtain $\Delta_{\gamma_k}^{(+,H)}$, $c_{V_k}^+$ and $\Delta_{\gamma_0}$ from eqs. (17), (18) and (21-23) choosing $m_{p1} = m_{p2}$, where the charge conjugation symmetry demands $c_{V_k}^c = c_{V_k}^H$ and $c_{V_0}^c = c_{V_0}^c = 0$. In Figs. 2 and 3 we plot $\Delta_{\gamma_k}^{(+,H)}$, $c_{V_k}^+$ and $\Delta_{\gamma_0}$, respectively, as a function of $m_{p1}^{(0)}$ for the plaquette and the Iwasaki gauge actions. The $m_{p1}^{(0)}$ dependence of $\Delta_{V_\mu}$ in eq. (25) is also plotted in Fig. 4. The solid lines denote the fitting results of the interpolation:

\[
\Delta_{\gamma_k} = \Delta_{\gamma_k}^{(+,H)} m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(44)

\[
c_{V_k}^+ = c_{V_k}^+ m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(45)

\[
c_{V_k}^H = \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(46)

\[
\Delta_{\gamma_k} = \Delta_{\gamma_k}^{(+,H)} m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(47)

\[
\Delta_{\gamma_0} = \Delta_{\gamma_0} m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(48)

\[
c_{V_0}^+ = c_{V_0}^+ m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(49)

\[
\Delta_{V_0} = \Delta_{V_0} m_{p1}^{(0)} = 0 + \frac{\sum_{i=1}^{5} v_{ni} k^i \{ m_{p1}^{(0)} \}^i}{1 + \sum_{i=1}^{5} v_{di} k^i \{ m_{p1}^{(0)} \}^i},
\]

(50)

where we assume $c_{V_k}^H = 0$ at $m_{p1}^{(0)} = 0$. We employ $\Delta_{\gamma_k} = \Delta_{\gamma_0} = 0.05169$ (plaquette), 0.04802 (Iwasaki), 0.04595 (DBW2), $c_{V_k}^c = c_{V_0}^+ = 0.01633$ (plaquette), 0.009728 (Iwasaki), 0.004884 (DBW2) and $\Delta_{V_k} = \Delta_{V_0} = 0.1294$ (plaquette), 0.06279 (Iwasaki), 0.02566 (DBW2) at $m_{p1}^{(0)} = 0$. The values of the parameters $v_{ni} k^i$ and $v_{di} k^i$ $(i = 1, \ldots, 5)$ are summarized in Tables II and III. The relative errors of these interpolations to the data are less than a few % over the range $0 \leq m_{p1}^{(0)} \leq 10$. 

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2. Heavy-light case

We use eqs. (17)–(24) to determine $\Delta_{\gamma_k}$, $c_{V_k}^{(+,-,H,L)}$ and $\Delta_{\gamma_0}$, $c_{V_0}^{(+,-)}$. Assuming that the $m_{p2}a$ corrections are negligible, we evaluate the improvement coefficients, except $c_{V_k}^L$ and $c_{V_0}^{(+,-)}$, as a function of $m_{p1}^{(0)}$ with $m_{p2}^{(0)} = 0$. In eqs. (34), (36), (37) we find that $S^s_k$ and $Y^d_0 - Z^d_0$ are not determined if we set $m_{p2}^{(0)} = 0$. Therefore one should extrapolate data at non-zero $m_{p2}^{(0)}$ to $m_{p2}^{(0)} = 0$. We however keep $m_{p2}^{(0)} = 0.0001$ in our calculation to determine $c_{V_k}^L$ and $c_{V_0}^{(+,-)}$ since the difference between the value at $m_{p2}^{(0)} = 0.0001$ and the one extrapolated to $m_{p2}^{(0)} = 0$ is less than 1%. Figures 5 and 6 show the $m_{p1}^{(0)}$ dependence of $\Delta_{\gamma_k}$, $c_{V_k}^{(+,-,H,L)}$ and $\Delta_{\gamma_0}$, $c_{V_0}^{(+,-)}$, respectively, for the plaquette and the Iwasaki gauge actions. We also plot $\Delta_{V_p}$ in Fig. 7. The interpolation denoted by the solid lines are expressed as

$$\Delta_{\gamma_k} = \Delta_{\gamma_k}\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{k\gamma} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{k\gamma} \{m_{p1}^{(0)}\}^i};$$

$$c_{V_k}^+ = c_{V_k}^+\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{k+} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{k+} \{m_{p1}^{(0)}\}^i};$$

$$c_{V_k}^{(-,H,L)} = \frac{\sum_{i=1}^{5} v_{ni}^{k(-,H,L)} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{k(-,H,L)} \{m_{p1}^{(0)}\}^i};$$

$$\Delta_{V_k} = \Delta_{V_k}\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{k\gamma} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{k\gamma} \{m_{p1}^{(0)}\}^i};$$

$$\Delta_{\gamma_0} = \Delta_{\gamma_0}\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{0\gamma} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{0\gamma} \{m_{p1}^{(0)}\}^i};$$

$$c_{V_0}^+ = c_{V_0}^+\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{0+} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{0+} \{m_{p1}^{(0)}\}^i};$$

$$c_{V_0}^- = \frac{\sum_{i=1}^{5} v_{ni}^{0-} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{0-} \{m_{p1}^{(0)}\}^i};$$

$$\Delta_{V_0} = \Delta_{V_0}\big|_{m_{p1}^{(0)} = 0} + \frac{\sum_{i=1}^{5} v_{ni}^{0\gamma} \{m_{p1}^{(0)}\}^i}{1 + \sum_{i=1}^{5} v_{ni}^{0\gamma} \{m_{p1}^{(0)}\}^i};$$

with $v_{ni}^{k,0}$ and $v_{di}^{k,0}$ ($i = 1, \ldots, 5$) given in Tables III and IV. Here we use the constraint that $c_{V_k}^{(-,H,L)} = 0$ and $c_{V_0}^- = 0$ at $m_{p1}^{(0)} = 0$. The data are well described by these interpolations within a few % errors over the range $0 \leq m_{p1}^{(0)} \leq 10$. 

9
III. $O(a)$ IMPROVEMENT OF THE AXIAL VECTOR CURRENTS

Let us turn to the axial vector currents. The discussion is in parallel with the case of the vector currents. The renormalized operators with the $O(a)$ improvement is given by

$$A_{\mu}^{\text{latt},R}(x) = Z_{A}^{\text{latt}} \left[ \bar{q}(x)\gamma_{\mu}\gamma_{5}Q(x) - g^{2}c_{A_{\mu}}^{+}\partial_{\mu}\{\bar{q}(x)\gamma_{5}Q(x)\} - g^{2}c_{A_{\mu}}^{-}\partial_{\mu}\{\bar{q}(x)\gamma_{5}Q(x)\} \right] - g^{2}c_{A_{\mu}}^{+}\{\bar{q}(x)\gamma_{\mu}\gamma_{5}Q(x) - g^{2}c_{A_{\mu}}^{H}\bar{q}(x)\gamma_{\mu}\gamma_{5}\gamma_{4}\{\bar{q}(x)\} + O(g^{4}) \}$$

(58)

where we assume that the Euclidean space-time rotational symmetry is not retained on the lattice. The coefficients $Z_{A_{\mu}}^{\text{latt}}$ and $c_{A_{\mu}}^{(+,-,H,L)}$ are functions of the quark masses $m_{Q}$ and $m_{q}$. With the aid of equation of motion we are allowed to set $c_{A_{\mu}}^{H} = c_{A_{\mu}}^{L} = 0$. In the special case of $m_{Q} = m_{q}$, $c_{A_{\mu}}^{-} = 0$ and $c_{A_{\mu}}^{H} = c_{A_{\mu}}^{L}$ is derived from the charge conjugation symmetry. We also note that all the improvement coefficients except $c_{A_{\mu}}^{+}$ vanishes in the limit of $m_{Q} = m_{q} = 0$.

We determine $Z_{A_{\mu}}^{\text{latt}}$ and $c_{A_{\mu}}^{(+,-,H,L)}$ at the one-loop level for both heavy-heavy and heavy-light cases.

A. Determination of the improvement coefficients for the axial vector currents

The general form of the off-shell vertex functions at the one-loop level on the lattice are given by

$$A_{k_{5}}^{(1)}(p, q, m_{p1}, m_{q2}) = \gamma_{k_{5}}\gamma_{5}F_{1}^{k_{5}} + \gamma_{k_{5}}\gamma_{5}\{\bar{q}F_{k_{5}}^{b} + \bar{q}F_{5}^{k_{5}}\} + \{\bar{q}F_{k_{5}}^{a} + \bar{q}F_{k_{5}}^{b}\}$$

(59)

$$A_{0_{5}}^{(1)}(p, q, m_{p1}, m_{q2}) = \gamma_{0}\gamma_{5}F_{1}^{0_{5}} + \gamma_{0}\gamma_{5}\{\bar{q}F_{2}^{0_{5}} + \bar{q}F_{3}^{0_{5}}\} + \{\bar{q}F_{4}^{a} + \bar{q}F_{4}^{b}\}$$

(60)

where the coefficients $F_{k_{5}}^{a}$, $G_{k_{5}}^{b}$, $H_{k_{5}}^{0_{5}}$ are functions of $p^{2}$, $q^{2}$, $p \cdot q$, $m_{p1}$ and $m_{q2}$.

Sandwiching (59) and (60) by the on-shell quark states $u(p)$ and $\bar{u}(q)$, which satisfy $\bar{q}u(p) = im_{p1}u(p)$ and $\bar{q}\bar{u}(q) = im_{q2}\bar{u}(q)$, the matrix elements are reduced to

$$\bar{u}(q)A_{k_{5}}^{(1)}(p, q, m_{p1}, m_{q2})u(p)$$
\[
\begin{align*}
\bar{u}(q) \gamma_5 u(p) & \left\{ F_1^{k5} + \text{i} m_p F_2^{k5} + \text{i} m_p F_4^{k5} - m_p m_p F_6^{k5} \right\} \\
+\bar{u}(q) \gamma_5 \not p u(p) & \left\{ F_3^{k5} + \text{i} m_p F_7^{k5} + \text{i} m_p F_9^{k5} \right\} \\
+\bar{u}(q) \not q \gamma_5 u(p) & \left\{ F_5^{k5} + \text{i} m_p F_8^{k5} + \text{i} m_p F_{10}^{k5} \right\} \\
+(p_k - q_k) \bar{u}(q) \gamma_5 u(p) & \left\{ G_1^{k5} + \text{i} m_p G_2^{k5} + \text{i} m_p G_3^{k5} - m_p m_p G_4^{k5} \right\} \\
+(p_k + q_k) \bar{u}(q) \gamma_5 u(p) & \left\{ H_1^{k5} + \text{i} m_p H_2^{k5} + \text{i} m_p H_3^{k5} - m_p m_p H_4^{k5} \right\} + O(a^2), \quad (61)
\end{align*}
\]

and

\[
\begin{align*}
\bar{u}(q) \Lambda^{(1)}_{05}(p, q, m_p, m_p) u(p) & = \bar{u}(q) \gamma_0 \gamma_5 u(p) \left\{ F_1^{05} \text{ + } \text{i} m_p F_2^{05} + \text{i} m_p F_3^{05} - m_p m_p F_4^{05} \right\} \\
+\bar{u}(q) u(p) & \left\{ G_1^{05} + \text{i} m_p G_2^{05} + \text{i} m_p G_3^{05} - m_p m_p G_4^{05} \right\} \\
+(p_0 - q_0) \bar{u}(q) \gamma_5 u(p) & \left\{ H_1^{05} + \text{i} m_p H_2^{05} + \text{i} m_p H_3^{05} - m_p m_p H_4^{05} \right\} + O(a^2), \quad (62)
\end{align*}
\]

where the coefficients are summarized as

\[
\begin{align*}
X_{k5} & = F_1^{k5} + \text{i} m_p F_2^{k5} + \text{i} m_p F_4^{k5} - m_p m_p F_6^{k5} ; \\
Y_{k5} & = G_1^{k5} + \text{i} m_p G_2^{k5} + \text{i} m_p G_3^{k5} - m_p m_p G_4^{k5} ; \\
Z_{k5} & = H_1^{k5} + \text{i} m_p H_2^{k5} + \text{i} m_p H_3^{k5} - m_p m_p H_4^{k5} ; \\
R_{k5} & = F_3^{k5} + \text{i} m_p F_7^{k5} + \text{i} m_p F_9^{k5} ; \\
S_{k5} & = F_5^{k5} + \text{i} m_p F_8^{k5} + \text{i} m_p F_{10}^{k5} 
\end{align*}
\]

and

\[
\begin{align*}
X_{05} & = F_1^{05} + \text{i} m_p F_2^{05} + \text{i} m_p F_3^{05} - m_p m_p F_4^{05} ; \\
Y_{05} & = G_1^{05} + \text{i} m_p G_2^{05} + \text{i} m_p G_3^{05} - m_p m_p G_4^{05} ; \\
Z_{05} & = H_1^{05} + \text{i} m_p H_2^{05} + \text{i} m_p H_3^{05} - m_p m_p H_4^{05} .
\end{align*}
\]

In terms of these coefficients the improvement coefficients in eq. (63) are given by

\[
\begin{align*}
\Delta_{\gamma \gamma} & = (X_{k5})^{\text{latt}} - (X_{k5})^{\text{cont}} , \\
\text{i} c^+_{A_k} & = (Y_{k5})^{\text{latt}} - (Y_{k5})^{\text{cont}} , \\
\text{i} c^-_{A_k} & = (Z_{k5})^{\text{latt}} - (Z_{k5})^{\text{cont}} ,
\end{align*}
\]

\[
\begin{align*}
-\text{i} c^L_{A_k} & = (S_{k5})^{\text{latt}} ,
\end{align*}
\]
\[ \Delta_{\gamma_{05}} = (X_{05})^{\text{latt}} - (X_{05})^{\text{cont}}, \] (76)

\[ i e_{A_0}^+ = (Y_{05})^{\text{latt}} - (Y_{05})^{\text{cont}}, \] (77)

\[ i e_{A_0}^- = (Z_{05})^{\text{latt}} - (Z_{05})^{\text{cont}}, \] (78)

where we calculate the continuum contributions employing the \( \overline{\text{MS}} \) scheme with NDR. It should be noted that \( R_{k5} = S_{k5} = 0 \) in the continuum from the space-time rotational symmetry and \( c_{A_k}^L = c_{A_k}^H \) and \( c_{A_k}^- = c_{A_0}^- = 0 \) for \( m_{p1} = m_{p2} \) from the charge conjugation symmetry.

Combining \( \Delta_{\gamma_{05}} \) and the wave function renormalization factors, we obtain the renormalization factor of the axial vector currents:

\[
\frac{Z_{A_\mu}^{\text{latt}}}{Z_{A_\mu}^{\text{cont}}} = \sqrt{\frac{Z_{Q,\text{latt}}^{(0)}(m_{p1}^{(0)})}{Z_{q,\text{latt}}^{(0)}(m_{p2}^{(0)})}} \left( 1 - g^2 \Delta_{A_\mu} \right)
\] (79)

with

\[
Z_{Q,\text{latt}}^{(0)}(m_{p1}^{(0)}) = \cosh(m_{p1}^{(0)}) + r_t \sinh(m_{p1}^{(0)})
\] (80)

\[
Z_{q,\text{latt}}^{(0)}(m_{p2}^{(0)}) = \cosh(m_{p2}^{(0)}) + r_t \sinh(m_{p2}^{(0)})
\] (81)

\[
\Delta_{A_\mu} = \Delta_{\gamma_{05}} - \Delta_\mu - 2 \frac{\Delta_\mu}{2}, \tag{82}
\]

where \( \Delta_{Q,q} \) are found in Ref. [2]. For convenience we give the relation for \( Z_{A_\mu}^{\text{cont}} \) between NDR and DRED in \( \overline{\text{MS}} \) scheme:

\[
Z_{A_\mu}^{\text{cont}}(\text{NDR}) = Z_{A_\mu}^{\text{cont}}(\text{DRED}) - \frac{1}{2} g^2.
\] (83)

The relevant coefficients \( X_{k5}, Y_{k5}, Z_{k5}, R_{k5}, S_{k5} \) for \( A_k \) are determined from the off-shell vertex function [39]:

\[
X_{k5}^s = \frac{1}{4} \text{Tr} \left[ \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_5 \gamma_k \right]_{p=p^*, q=q^*}, \tag{84}
\]

\[
Y_{k5}^d + Z_{k5}^d = \frac{1}{4} \text{Tr} \left[ \frac{\partial}{\partial p_k} \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_5 - \frac{\partial}{\partial p_i} \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_i \gamma_5 \right]_{p=p^*, q=q^*_i}, \tag{85}
\]

\[
Y_{k5}^d - Z_{k5}^d = \frac{1}{4} \text{Tr} \left[ - \frac{\partial}{\partial q_k} \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_5 + \frac{\partial}{\partial q_i} \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_i \gamma_5 \right]_{p=p^*, q=q^*_d}, \tag{86}
\]

\[
X_{k5}^d + 2im_{p1}R_{k5}^d = \frac{1}{4} \text{Tr} \left[ \Lambda^{(1)}_{k5} \gamma_5 \gamma_k (1 - \gamma_0) + 2im_{p1} \frac{\partial}{\partial p_i} \Lambda^{(1)}_{k5} (1 + \gamma_0) \gamma_i \gamma_5 \gamma_k \right]_{p=p^*, q=q^*_i}, \tag{87}
\]

\[
X_{k5}^d + 2im_{p2}S_{k5}^d = \frac{1}{4} \text{Tr} \left[ \Lambda^{(1)}_{k} (1 + \gamma_0) \gamma_5 \gamma_k + 2im_{p2} \frac{\partial}{\partial q_i} \Lambda^{(1)}_{k} (1 + \gamma_0) \gamma_i \gamma_5 \gamma_k \right]_{p=p^*, q=q^*_d}, \tag{88}
\]
where \( p^* \equiv (p_0 = i m_{p1}, p_i = 0) \) and \( q_s^* \equiv (q_0 = i m_{p2}, q_i = 0) \) or \( q_d^* \equiv (q_0 = -i m_{p2}, q_i = 0) \). We obtain \( X_{05}, Y_{05}, Z_{05} \) for \( A_0 \) from eq. (60):

\[
X_{05}^s = \frac{1}{4} \text{Tr} \left[ \Lambda_{05}^{(1)} (1 + \gamma \gamma) \right] \gamma_5 \gamma_0 + 2 \frac{\partial F^{p_0}}{\partial p_i} \frac{\partial F^{q_0}}{\partial q_i} \left( \Lambda_{05}^{(1)} (1 + \gamma \gamma) \right) \gamma_5 \gamma_0 , \tag{89}
\]

\[
X_{05}^s + i m_{p1} (Y_{05}^s - Z_{05}^s) - i m_{p2} (Y_{05}^s - Z_{05}^s) = \frac{1}{4} \text{Tr} \left[ \Lambda_{05}^{(1)} (1 + \gamma \gamma) \right] \gamma_5 \gamma_0 , \tag{90}
\]

\[
X_{05}^d - i m_{p1} (Y_{05}^d - Z_{05}^d) - i m_{p2} (Y_{05}^d - Z_{05}^d) = -\frac{1}{4} \text{Tr} \left[ \Lambda_{05}^{(1)} (1 + \gamma \gamma) \right] p^*, q = q^* , \tag{91}
\]

where \( F^k, G^k \) and \( H^k \) are functions of \( p^2, q^2 \) and \( p \cdot q \) resulting in

\[
\frac{\partial F^{p_0}}{\partial p_i} \bigg|_{p^*, q^*} = \frac{\partial F^{q_0}}{\partial q_i} \bigg|_{p^*, q^*} = 0 , \tag{92}
\]

\[
\frac{\partial H^{p_0}}{\partial p_i} \bigg|_{p^*, q^*} = \frac{\partial H^{q_0}}{\partial q_i} \bigg|_{p^*, q^*} = 0 , \tag{93}
\]

\[
\frac{\partial C^{p_0}}{\partial p_i} \bigg|_{p^*, q^*} = \frac{\partial C^{q_0}}{\partial q_i} \bigg|_{p^*, q^*} = 0 , \tag{94}
\]

with \( i = 1, 2, 3 \).

As for the counterterm to isolate the infrared divergence in the above coefficients at the one-loop level we employ

\[
\bar{A}_{\mu 5}(k, p, q, m_{p1}^{(0)}, m_{p2}^{(0)}, \lambda) = \theta(\Lambda^2 - k^2) C_F i \gamma_\alpha \frac{1}{i(\not{\mathbf{q}} + \ell)} \frac{1}{i(\not{\mathbf{q}} + \ell')} \frac{1}{m_{p1}^{(0)} i \gamma_\alpha \not{\mathbf{l}} + m_{p2}^{(0)} i \gamma_\alpha} \frac{1}{k^2 + \lambda^2} , \tag{95}
\]

with a cutoff \( \Lambda (\leq \pi) \).

### B. Results for the improvement coefficients of the axial vector current

#### 1. Heavy-heavy case

With the choice of \( m_{p1} = m_{p2} \) the improvement coefficients \( \Delta_{\gamma_k \gamma_\alpha}, c_{A_k}^{(+H)} \) and \( \Delta_{\gamma_5 \gamma_5}, c_{A_0}^{+} \) are determined from eqs. (71), (72) and (75-77). It is noted that \( c_{A_k}^{L} = c_{A_k}^{H} \) and \( c_{A_k}^{-} = c_{A_0}^{-} = 0 \) from the charge conjugation symmetry. The quark mass dependences of \( \Delta_{\gamma_k \gamma_\alpha}, c_{A_k}^{(+H)} \) and \( \Delta_{\gamma_5 \gamma_5}, c_{A_0}^{+} \) are shown in Figs. 8 and 9 respectively, employing the plaquette and the Iwasaki
gauge actions. We also give the $m_{p1}^{(0)}$ dependence of $\Delta_{A\mu}$ in Fig. 10. The solid lines denote the interpolation with the use of the following functions:

$$\Delta_{\gamma\gamma_5} = \Delta_{\gamma\gamma_5}^{(0)} + \sum_{i=1}^{5} a_{ni}^{k\gamma} (m_{p1}^{(0)})_i / (1 + \sum_{i=1}^{5} a_{di}^{k\gamma} (m_{p1}^{(0)})_i),$$  \hspace{1cm} (96)

$$c_{Ak}^+ = c_{Ak}^+ - \frac{\sum_{i=1}^{5} a_{ni}^{k+} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{k+} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (97)

$$c_{Ak}^H = \frac{\sum_{i=1}^{5} a_{ni}^{kH} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{kH} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (98)

$$\Delta_{Ak} = \Delta_{Ak}^{(0)} + \frac{\sum_{i=1}^{5} a_{ni}^{0} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{0} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (99)

$$\Delta_{\gamma\gamma_5} = \Delta_{\gamma\gamma_5}^{(0)} + \frac{\sum_{i=1}^{5} a_{ni}^{0\gamma} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{0\gamma} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (100)

$$c_{Ak}^0 = c_{Ak}^0 + \frac{\sum_{i=1}^{5} a_{ni}^{0+} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{0+} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (101)

$$\Delta_{Ak} = \Delta_{Ak}^{(0)} + \frac{\sum_{i=1}^{5} a_{ni}^{0Z} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{0Z} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (102)

with $a_{ni}^{k,0}$ and $a_{di}^{k,0}$ ($i = 1, \ldots, 5$) given in Tables III and IV. The relative errors of these interpolations are a few % over the range $0 \leq m_{p1}^{(0)} \leq 10$. The massless values are $\Delta_{\gamma\gamma_5}^{(0)}$ = 0.03873 (plaquette), 0.04184 (Iwasaki), 0.04359 (DBW2), $c_{Ak}^+ = c_{Ak}^0 = 0.007574$ (plaquette), 0.003801 (Iwasaki), 0.001492 (DBW2) and $\Delta_{Ak} = \Delta_{Ak}^{(0)} = 0.1165$ (plaquette), 0.05663 (Iwasaki), 0.02330 (DBW2). The coefficient $c_{Ak}^H$ should vanish at $m_{p1}^{(0)} = 0$.

2. Heavy-light case

The set of improvement coefficients are determined from eqs.(111–118). For the same reason before we keep $m_{p2}^{(0)} = 0.0001$ for the determination of $c_{Ak}^+$ and $c_{Ak}^0$, while the vanishing light quark mass $m_{p1}^{(0)} = 0$ is employed for other coefficients.

Numerical results for $\Delta_{\gamma\gamma_5}, c_{Ak}^{(+,-,H,L)}$ and $\Delta_{\gamma\gamma_5}, c_{Ak}^{(+,-)}$ are presented in Figs. 11 and 12 respectively, in the case of the plaquette and the Iwasaki gauge actions. We also plot the $m_{p1}^{(0)}$ dependence of $\Delta_{A\mu}$ in Fig. 13. The solid lines represent the interpolation expressed as

$$\Delta_{\gamma\gamma_5} = \Delta_{\gamma\gamma_5}^{(0)} + \frac{\sum_{i=1}^{5} a_{ni}^{k\gamma} (m_{p1}^{(0)})_i}{1 + \sum_{i=1}^{5} a_{di}^{k\gamma} (m_{p1}^{(0)})_i},$$  \hspace{1cm} (103)
where the relative errors to the data are a few % over the range $0 \leq m_{p1}^{(0)} \leq 10$. The values of the parameters $a_{ni}^{k,0}$ and $a_{di}^{k,0}$ ($i = 1, \ldots, 5$) are listed in Table VII and VIII. Here we use the constraint that $c_{A_k}^{(-H,L)}$ and $c_{A_0}^{(-)}$ vanish at $m_{p1}^{(0)} = 0$.

### IV. MEAN FIELD IMPROVEMENT

Let us explain the mean-field improvement on the renormalization factors of the vector and the axial vector currents in eqs. (26) and (29). We first rewritten their expressions as follows:

$$
\frac{Z_{V_\mu}^{\text{latt}}}{Z_{V_\mu}^{\text{cont}}} = \sqrt{Z_{Q,\text{latt}}^{(0)}(\tilde{m}_{p1}^{(0)})} \sqrt{Z_{q,\text{latt}}^{(0)}(\tilde{m}_{p2}^{(0)})} u \left( 1 - g^2 \Delta_{V_\mu} \right) \\
+ g^2 \frac{C_F}{2} T_{MF} + \frac{g^2}{2} \frac{\partial Z_{Q,\text{latt}}^{(0)}}{\partial m_{p1}^{(0)}} \Delta m_{p1} + \frac{g^2}{2} \frac{\partial Z_{q,\text{latt}}^{(0)}}{\partial m_{p2}^{(0)}} \Delta m_{p2} \\
$$

$$
\frac{Z_{A_\mu}^{\text{latt}}}{Z_{A_\mu}^{\text{cont}}} = \sqrt{Z_{Q,\text{latt}}^{(0)}(\tilde{m}_{p1}^{(0)})} \sqrt{Z_{q,\text{latt}}^{(0)}(\tilde{m}_{p2}^{(0)})} u \left( 1 - g^2 \Delta_{A_\mu} \right) \\
+ g^2 \frac{C_F}{2} T_{MF} + \frac{g^2}{2} \frac{\partial Z_{Q,\text{latt}}^{(0)}}{\partial m_{p1}^{(0)}} \Delta m_{p1} + \frac{g^2}{2} \frac{\partial Z_{q,\text{latt}}^{(0)}}{\partial m_{p2}^{(0)}} \Delta m_{p2} 
$$
where $\tilde{m}_{p1,2}^{(0)}$ and $\Delta m_{p1,2}$ are defined in Ref. [2] and $T_{MF}$ is the one-loop correction to the mean-field factor defined by

$$u = P^{1/4} = 1 - g^2 C_F T_{MF}$$  \hspace{1cm} (112)$$

with $P$ the plaquette. We find a detailed description on the derivation of $T_{MF}$ in Sec. III of Ref. [10]. We then replace $u$ by $P^{1/4}$ measured by Monte Carlo simulation.

The mean-field improved MS coupling $g_{\text{MS}}^2(\mu)$ at the scale $\mu$ is obtained using the lattice bare coupling $g_0^2$ and $P$:

$$\frac{1}{g_{\text{MS}}^2(\mu)} = \frac{P}{g_0^2} + d_g + c_p + \frac{22}{16\pi^2} \log(\mu a) + N_f \left( d_f - \frac{4}{48\pi^2} \log(\mu a) \right)$$  \hspace{1cm} (113)$$

with $N_f$ the number of quark flavor. The values of $c_p$, $d_g$ and $d_f$ for various gauge and quark actions are summarized in Ref. [11]. For the improved gauge action one may use an alternative formula [12]

$$\frac{1}{g_{\text{MS}}^2(\mu)} = \frac{c_0 P + 8c_1 R_1 + 16c_2 R_2 + 8c_3 R_3}{g_0^2}$$

$$+ d_g + (c_0 \cdot c_p + 8c_1 \cdot c_{R1} + 16c_2 \cdot c_{R2} + 8c_3 \cdot c_{R3}) + \frac{22}{16\pi^2} \log(\mu a)$$

$$+ N_f \left( d_f - \frac{4}{48\pi^2} \log(\mu a) \right),$$  \hspace{1cm} (114)$$

where

$$P = \frac{1}{3} \text{Tr} U_{pl} = 1 - c_p g_0^2 + O(g_0^4),$$  \hspace{1cm} (115)$$

$$R_1 = \frac{1}{3} \text{Tr} U_{rtg} = 1 - c_{R1} g_0^2 + O(g_0^4),$$  \hspace{1cm} (116)$$

$$R_2 = \frac{1}{3} \text{Tr} U_{chr} = 1 - c_{R2} g_0^2 + O(g_0^4),$$  \hspace{1cm} (117)$$

$$R_3 = \frac{1}{3} \text{Tr} U_{plg} = 1 - c_{R3} g_0^2 + O(g_0^4),$$  \hspace{1cm} (118)$$

and the measured values are employed for $P$, $R_1$, $R_2$ and $R_3$. We also find the values of $c_{R1}$, $c_{R2}$ and $c_{R3}$ for various gauge actions in Ref. [11].

V. CONCLUSION

In this paper we have determined the $O(a)$ improvement coefficients of the vector and the axial vector currents in a mass dependent way at the one-loop level. Our calculation
is made employing the relativistic heavy quark action, which we have recently proposed, with the various gauge actions. The results are presented both for the heavy-heavy and the heavy-light cases as a function of heavy quark mass.

For convenience we have given a brief description about the implementation of the mean field improvement for the renormalization factors. We are now performing a numerical simulation for the heavy-heavy and heavy-light meson systems using the relativistic heavy quark action with the $O(a)$ improved vector and axial vector currents, whose parameters are mean-field improved at the one-loop level. This work would reveal to what extent our relativistic heavy quark formulation is quantitatively efficient to study the heavy quark physics.

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TABLE I: Values of parameters $v_{ni}^{k(\gamma, H, Z)}$ and $v_{di}^{k(\gamma, H, Z)}$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta_{\gamma_k}, c_{V_k}^{(+H)}$ for heavy-heavy case with eqs. (44–47), respectively.

| gauge action $v_{n1}^{k}$ | $v_{n2}^{k}$ | $v_{n3}^{k}$ | $v_{n4}^{k}$ | $v_{n5}^{k}$ | $v_{d1}^{k}$ | $v_{d2}^{k}$ | $v_{d3}^{k}$ | $v_{d4}^{k}$ | $v_{d5}^{k}$ |
|---------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| plaquette 0.014684 & 0.13219 & 0.89527 & -0.11174 & 0.31131 & 7.3765 & 34.893 & 2.4179 & 14.916 & 0.070066 |
| $\Delta_{\gamma_k}$ Iwasaki | -0.015146 & 1.8778 & -0.49846 & -1.5767 & -0.31618 & 169.33 & 117.23 & 127.59 & 13.898 & 0.0027264 |
| DBW2 0.010395 & -0.047110 & -4.5545 & 0.89861 & -1.0730 & 63.029 & 44.429 & 3.2276 & 13.901 & -0.022777 |
| plaquette 25.076 | -181.54 | -195.34 | 2.4509 | -4.1924 | 22843. | 7808.2 | 13344. | -324.79 | 260.72 |
| $c_{V_k}^{+}$ Iwasaki | 0.94447 | -8.8867 | 2.5958 | 23.741 | -1.9005 | 1351.1 | -2317.0 | 361.82 | -2370.9 | 187.06 |
| DBW2 -0.0022351 | -0.00014326 | -0.034665 | 0.047341 | -5.2267 | 20.895 | -17.228 | 16.897 | -9.3839 |
| plaquette 0.037455 | 0.14443 | 0.32618 | -0.029729 | 0.056502 | 12.703 | 14.177 | 41.911 | 3.7738 | 6.8538 |
| $c_{V_k}^{H}$ Iwasaki | -0.0029594 | -0.011657 | 0.0030631 | -0.25435 | -0.37900 | 1.3206 | 9.8642 | 51.400 | 29.745 | 31.961 |
| DBW2 -0.088565 | -0.18583 | -0.22715 | -0.019377 | -0.19151 | 6.4502 | 6.1322 | 7.0131 | 2.7602 | 4.6325 |
| plaquette -0.16445 | -7.8793 | 0.59083 | -0.45059 | 0.058239 | 42.738 | 36.668 | 2.0995 | 3.5218 | 0.0079208 |
| $\Delta_{V_k}$ Iwasaki | -0.057093 | -10.029 | -5.0701 | -0.97000 | -0.56750 | 116.04 | 153.83 | 4.2651 | 24.187 | -0.027538 |
| DBW2 -0.026515 | -1.6401 | -5.5463 | -1.0198 | -2.0489 | 43.104 | 62.938 | 17.548 | 24.974 | -0.013591 |
TABLE II: Values of parameters $v_{ni}^{0(\gamma,+Z)}$ and $v_{di}^{0(\gamma,+Z)}$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta_{\gamma_0}, c_{V_0}^+, \Delta V_0$ for heavy-heavy case with eqs. [48-50], respectively.

| gauge action | $v_{n1}^0$ | $v_{n2}^0$ | $v_{n3}^0$ | $v_{n4}^0$ | $v_{n5}^0$ | $v_{d1}^0$ | $v_{d2}^0$ | $v_{d3}^0$ | $v_{d4}^0$ | $v_{d5}^0$ |
|--------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| plaquette    | 0.00019720 | 2.7579     | -4.4172    | -0.95205   | -0.047970  | 90.897     | 50.506     | 33.105     | 0.57994    | 0.021081   |
| $\Delta_{\gamma_0}$ Iwasaki | 0.015673   | -1.1138    | 5.9122     | -6.4339    | -3.4423    | -64.281    | 99.711     | 73.380     | 66.707     | -0.35366   |
| DBW2         | 0.012159   | -0.021725  | -4.5857    | -0.90786   | -0.89745   | 38.421     | 46.673     | 14.201     | 9.0663     | -0.0099410 |
| plaquette    | 0.0071689  | -0.34738   | 0.011308   | 0.042259   | -0.17224   | 15.931     | -6.9566    | 12.093     | -2.5858    | 4.7195     |
| $c_{V_0}^+$  | 0.019308   | -4.4339    | -1.9827    | 0.50641    | -0.19121   | 148.80     | 57.869     | 62.367     | -13.529    | 4.7863     |
| Iwasaki      | 0.019308   | -4.4339    | -1.9827    | 0.50641    | -0.19121   | 148.80     | 57.869     | 62.367     | -13.529    | 4.7863     |
| DBW2         | -0.098707  | -0.67466   | -1.2994    | -0.58528   | -0.047428  | 9.5348     | 25.681     | 22.818     | 10.537     | 0.82699    |
| plaquette    | -0.16426   | -9.6984    | -3.3551    | -1.0849    | -0.26984   | 56.744     | 44.807     | 5.8125     | 6.4396     | 0.0055241  |
| $\Delta V_0$ Iwasaki | -0.077730  | -1.6754    | -6.3923    | -1.1045    | -0.99597   | 21.914     | 79.398     | 4.6312     | 16.091     | 0.0065184  |
| DBW2         | -0.026338  | -0.76017   | -3.1685    | -1.5062    | -0.78087   | 18.143     | 29.877     | 13.287     | 7.5750     | -0.0039958 |
TABLE III: Values of parameters $v_{ni}^k(\gamma,+,H,L,Z)$ and $v_{di}^k(\gamma,+H,L,Z)$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta \gamma_k, c_{V_k}^{(+,-H,L)}, \Delta V_k$ for heavy-light case with eqs.(51–54), respectively.

| gauge action | $v_{n1}^k$ | $v_{n2}^k$ | $v_{n3}^k$ | $v_{n4}^k$ | $v_{n5}^k$ | $v_{d1}^k$ | $v_{d2}^k$ | $v_{d3}^k$ | $v_{d4}^k$ | $v_{d5}^k$ |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| plaquette    | 0.0062511   | 0.052107    | 0.55038     | 0.12282     | 0.081216    | 8.3279      | 94.574      | 36.477      | 15.865      | 2.8063      |
| $\Delta \gamma_k$ Iwasaki | 0.0031637 | 0.26026     | 0.55065     | $-0.20725$  | 0.060822    | 64.787      | 116.19      | $-8.9718$   | 0.47615     | 3.3216      |
| DBW2         | 0.0021629   | 0.025944    | $-0.039416$ | 0.14083     | 0.045331    | 4.9552      | $-3.1981$   | 26.268      | 25.127      | 4.2530      |
| plaquette    | 0.0015687   | $-0.011647$ | $-0.56066$  | 0.12164     | $-0.21315$  | 14.953      | 39.726      | 29.119      | 5.8968      | 12.722      |
| $c_{V_k}^+$  | 0.70482     | $-8.0596$   | $-9.4795$   | $-0.071982$ | 0.023359    | 1422.3      | 920.72      | 921.97      | 5.9010      | $-2.2079$   |
| DBW2         | $-0.0053483$| 0.030398    | $-0.1720$   | 0.20455     | 0.077260    | $-8.4836$   | 23.308      | $-21.287$   | $-12.747$   | $-14.273$   |
| plaquette    | $-0.0022183$| 0.091343    | $-0.81318$  | 1.0896      | 0.025931    | $-23.899$   | 120.15      | 188.77      | 242.08      | 75.209      |
| $c_{V_k}^-$  | 0.0072349   | 0.033385    | 0.090949    | 0.14881     | 0.0022120   | 3.5137      | 32.527      | 29.132      | 42.756      | 8.4052      |
| DBW2         | 0.062422    | 0.40457     | $-0.18140$  | 0.15221     | 0.0031810   | 15.041      | 49.001      | $-2.6721$   | 11.963      | 7.7052      |
| plaquette    | 0.0054262   | $-0.11193$  | 0.67635     | $-1.0049$   | $-0.028206$ | $-25.354$   | 249.89      | $-910.86$   | $-443.70$   | $-147.73$   |
| $c_{V_k}^H$  | $-0.00071764$| 0.00044474  | 0.020620    | 0.0011663   | $-0.00070314$| $-1.3698$   | 16.036      | 3.7954      | 3.5247      | $-0.088466$ |
| DBW2         | 0.025927    | $-0.65927$  | 2.7483      | $-9.1542$   | $-0.11381$  | $-371.59$   | 1369.9      | 3734.8      | 1534.4      | $-1256.5$   |
| plaquette    | 0.039461    | 0.16335     | 0.17557     | 0.53702     | 0.31432     | 7.9009      | 6.4773      | 24.665      | 19.259      | 3.7410      |
| $c_{V_k}^L$  | 0.0035068   | 0.013396    | $-0.0074426$| 0.092313    | 0.10210     | 3.5571      | $-2.5954$   | 24.803      | 16.574      | 3.6739      |
| DBW2         | $-0.10580$  | $-2.3541$   | $-3.0241$   | $-0.75090$  | $-0.28242$  | 41.820      | 151.64      | 105.43      | 34.058      | 9.0980      |
| plaquette    | $-0.092742$ | 0.18493     | $-1.7660$   | 0.073087    | $-0.044986$ | $-1.2430$   | 16.954      | 13.695      | 1.3841      | 0.61047     |
| $\Delta V_k$ Iwasaki | $-0.043329$ | 2.6134      | $-2.8159$   | 0.29050     | $-0.061514$ | $-59.887$   | 18.652      | 40.112      | $-0.18607$  | 1.0233      |
| DBW2         | 0.021307    | 2.2956      | 0.36278     | $-1.9582$   | $-0.26512$  | $-171.10$   | 10.537      | 37.125      | 83.567      | 2.1427      |
TABLE IV: Values of parameters $v_{ni}^0(\gamma,+,−,Z)$ and $v_{di}^0(\gamma,+,−,Z)$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta_{\gamma_0}, c_{v_0}^{(+,−)}, \Delta_{\gamma_0}$ for heavy-light case with eqs.(54−57), respectively.

| gauge action | $v_{n1}^0$ | $v_{n2}^0$ | $v_{n3}^0$ | $v_{n4}^0$ | $v_{n5}^0$ | $v_{d1}^0$ | $v_{d2}^0$ | $v_{d3}^0$ | $v_{d4}^0$ | $v_{d5}^0$ |
|--------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| plaquette    | 0.012802   | 0.49100    | 0.33452    | 0.23128    | 0.040371   | 38.22      | 37.290     | 48.752     | 16.627     | 1.3978     |
| $\Delta_{\gamma_0}$ Iwasaki | 0.0077753 | 0.061860   | 0.024810   | 0.14301    | 0.069308   | 6.9955     | 7.3898     | 13.856     | 22.055     | 2.9415     |
| DBW2         | 0.0063022  | −0.083313  | 0.56176    | −0.17488   | 0.017591   | −11.399    | 87.354     | 7.4716     | −8.3999    | 1.1371     |
| plaquette    | 0.0048159  | −0.022115  | −0.65883   | 0.0040211  | −0.17926   | 23.777     | 68.425     | 36.502     | 19.322     | 9.7039     |
| $c_{v_0}^{+}$ Iwasaki | −0.0094956 | 0.12375    | 0.53923    | 0.36455    | 0.29355    | −14.207    | −47.045    | −49.638    | −38.991    | −14.935    |
| DBW2         | −0.047989  | −0.42990   | −0.37339   | −0.55380   | −0.33072   | 13.176     | 23.031     | 24.888     | 25.931     | 10.361     |
| plaquette    | −0.014388  | 0.00032644 | 0.13076    | 0.046446   | 0.0015886  | 7.5778     | 21.438     | 13.630     | 9.7527     | 0.74691    |
| $c_{v_0}^{−}$ Iwasaki | 0.015770   | −0.085430  | 0.55703    | 0.30343    | 0.022609   | −4.5521    | 32.649     | 32.280     | 25.481     | 2.2470     |
| DBW2         | 0.12368    | 1.3600     | −0.071861  | 0.088198   | 0.14040    | 24.213     | 35.781     | 4.4921     | 1.9685     | 5.1811     |
| plaquette    | −0.092162  | 0.56122    | −0.53593   | 0.082763   | −0.054690  | −5.1290    | −0.12544   | 5.3433     | −1.1662    | 0.88503    |
| $\Delta_{v_0}$ Iwasaki | 0.18102    | 44.773     | −167.94    | 16.819     | −4.2353    | −1236.9    | 3747.4     | 2472.3     | 43.654     | 76.008     |
| DBW2         | 0.051388   | 4.0357     | 2.3373     | −1.5950    | −0.26105   | −335.95    | −109.32    | −56.028    | 89.233     | 2.1341     |
TABLE V: Values of parameters $a_{n_i}^{k(\gamma,+;H,Z)}$ and $a_{d_i}^{k(\gamma,+;H,Z)}$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta_{\gamma_k\gamma_5}, c_{A_k}^{(+;H)}$, $\Delta_{A_k}$ for heavy-heavy case with eqs. (96–99), respectively.

|                      | $a_{n_1}^k$ | $a_{n_2}^k$ | $a_{n_3}^k$ | $a_{n_4}^k$ | $a_{n_5}^k$ | $a_{d_1}^k$ | $a_{d_2}^k$ | $a_{d_3}^k$ | $a_{d_4}^k$ | $a_{d_5}^k$ |
|----------------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| **gauge action**     |            |            |            |            |            |            |            |            |            |            |
| Iwasaki             | −0.019142  | −0.047713  | −0.0062276 | −0.017168  | 0.035701   | 1.7330     | 5.6259     | 1.2548     | 4.7507     | 0.27302    |
| DBW2                | −0.0045598 | 0.11534    | 0.022512   | 0.11025    | −0.00049818| 5.1866     | 0.97387    | 4.8579     | 0.79928    | −0.012343  |
| **$c_{A_k}^+$**      |            |            |            |            |            |            |            |            |            |            |
| Iwasaki             | −0.029644  | 0.010150   | −0.27996   | −0.30399   | −0.12102   | 0.51603    | 9.4466     | 18.071     | 11.317     | 5.4941     |
| DBW2                | −0.15917   | −1.5545    | −2.5459    | −1.1665    | −0.045035  | 18.164     | 67.405     | 67.406     | 33.382     | 1.2761     |
| **$c_{A_k}^H$**      |            |            |            |            |            |            |            |            |            |            |
| Iwasaki             | 0.00045423 | 0.21074    | 0.23023    | 0.10905    | 0.061551   | 15.102     | 14.308     | 23.396     | 4.7100     | 5.0676     |
| DBW2                | 0.072677   | 0.62160    | 0.53520    | 0.19668    | 0.019528   | 17.971     | 38.152     | 24.444     | 10.407     | 1.0515     |
| **$\Delta_{\gamma_k\gamma_5}$** |            |            |            |            |            |            |            |            |            |            |
| Iwasaki             | −0.10682   | −2.6283    | −4.7827    | −0.20855   | −0.23399   | 26.635     | 58.249     | 71.704     | 0.47394    | 5.7885     |
| DBW2                | −0.040023  | −0.033273  | −0.28351   | −0.0094526 | −0.010229  | 3.7131     | 6.7739     | 20.057     | −1.3448    | 1.6373     |

plaquette: $c_{A_k}$
TABLE VI: Values of parameters $a_{n1}^{0(\gamma,+)}$ and $a_{d1}^{0(\gamma,+)}$ ($i = 1, \ldots, 5$) in the interpolation of $\Delta_{\gamma_0}, c_{A_0}^+, \Delta_{A_0}$ for heavy-heavy case with eqs. [100]-[102], respectively.

| Plaquette | $\Delta_{\gamma_0}$ Iwasaki | $c_{A_0}^+$ Iwasaki | DBW2 | $\Delta_{A_0}$ Iwasaki | DBW2 |
|-----------|----------------------------|---------------------|------|------------------------|------|
| $a_{n1}^0$ | 0.052844 | -0.020399 | 0.021817 | -0.19696 | -0.094508 | 0.035180 |
| $a_{n2}^0$ | 0.098447 | 0.041856 | -4.0728 | -8.7519 | -2.4786 | 0.032360 |
| $a_{n3}^0$ | -0.095352 | 0.49428 | 1.8993 | -5.3415 | 0.35337 | 0.029443 |
| $a_{n4}^0$ | 0.0000049485 | -0.078257 | -2.2319 | -0.48001 | -0.032386 | 0.00021265 |
| $a_{n5}^0$ | 0.00013520 | 0.24277 | -0.061938 | -0.10054 | -0.00029766 | -0.00029766 |
| $a_{d1}^0$ | 13.949 | 12.589 | 33.245 | 44.563 | 28.057 | 2.8063 |
| $a_{d2}^0$ | 9.2995 | 29.569 | 78.786 | 55.001 | 21.317 | -2.8195 |
| $a_{d3}^0$ | 7.5620 | -4.1770 | -31.755 | 31.792 | 4.6126 | 3.0682 |
| $a_{d4}^0$ | 4.0175 | 12.693 | 49.858 | 5.2293 | -0.66769 | -0.27495 |
| $a_{d5}^0$ | -0.050684 | 1.4849 | 1.5508 | 1.3789 | 0.11312 | 0.033404 |
TABLE VII: Values of parameters \( a_n^k(\gamma,+,H,L,Z) \) and \( a_d^k(\gamma,+,H,L,Z) \) \((i = 1, \ldots, 5)\) in the interpolation of \( \Delta_{\kappa\gamma}, c_{A_k}^{(+,-,H,L)}, \Delta_A \) for heavy-light case with eqs. (103–106), respectively.

| gauge action | \( a_n^1 \)       | \( a_n^2 \)       | \( a_n^3 \)       | \( a_n^4 \)       | \( a_n^5 \)       | \( a_d^1 \)       | \( a_d^2 \)       | \( a_d^3 \)       | \( a_d^4 \)       | \( a_d^5 \)       |
|--------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| plaqette     | -0.0077994         | -0.41731          | 1.1354            | 0.094127          | 0.29482           | 45.381            | -95.269           | -62.935           | -98.936           | -29.162           |
| \(\Delta_{\kappa\gamma}\) | Iwasaki           | 0.010623          | -8.7336           | -0.14090          | -1.3650           | -0.040416         | 1596.5            | 1554.8            | 1388.6            | 375.05            | 9.6218            |
| DBW2         | -0.0027159         | -0.023333         | 0.11288           | -0.091528         | -0.0081287        | 18.209            | -16.459           | -67.226           | -42.775           | -14.363           |
| \(c_{A_k}^{+}\) | Iwasaki           | -0.0085895        | 0.10146           | -0.15900          | 0.32234           | 0.081455          | -9.2200           | -11.177           | -1.5297           | -89.575           | -19.944           |
| DBW2         | -0.061215          | 0.080878          | -0.031488         | 0.019396          | 0.0023197         | 6.6362            | -7.8813           | 1.1771            | -1.5133           | -1.2521           |
| \(c_{A_k}^{-}\) | Iwasaki           | -0.0015928        | 0.010270          | 0.057064          | 0.0053304         | -0.00012449       | 1.8033            | 15.634            | 4.7995            | 3.5412            | 0.0063867         |
| DBW2         | 0.0036783          | 0.00027955        | 0.016457          | 0.0096355         | 0.000079022       | -1.0696           | 5.2927            | 1.5712            | 2.3740            | 0.31191           |
| \(c_{A_k}^{H}\) | Iwasaki           | 0.0017201         | -0.0031462        | 0.053022          | 0.0013268         | -0.0000044448     | -1.1603           | 14.397            | 4.2428            | 3.3529            | -0.040156         |
| DBW2         | 0.0010232          | -0.000089135      | 0.030173          | 0.00049781        | -0.000051947      | -1.2988           | 11.412            | 2.6448            | 2.7244            | -0.067972         |
| \(c_{A_k}^{L}\) | Iwasaki           | 0.0010612         | 0.0021013         | 0.014049          | -0.00012538       | -0.000040511      | -1.6824           | 8.7455            | 0.27197           | 2.1129            | -0.11394          |
| DBW2         | -0.028427          | -0.12517          | -0.025615         | -0.15291          | 0.010900          | 10.057            | 23.991            | 10.071            | 34.909            | 3.2330            |
| \(\Delta_A\)  | Iwasaki           | 0.017833          | -0.64125          | -3.1710           | 4.4169            | 0.64578           | -242.31           | -174.96           | 214.38            | 450.52            | 33.050            |
| DBW2         | 0.092039           | 0.66663           | -0.51025          | 1.1278            | 0.22717           | 16.138            | 5.1945            | 9.1509            | 32.237            | 4.3118            |
| plaqette     | -0.10998           | 0.30354           | -2.9348           | 0.12711           | -0.084194         | -1.8757           | 23.675            | 19.521            | 1.4942            | 0.75518           |
| DBW2         | -0.11610           | -12.061           | -3.5918           | -1.7161           | 0.0095736         | 621.73            | 304.15            | 192.74            | 23.838            | -0.43073          |
TABLE VIII: Values of parameters $a_{n_{i}}^{0(\gamma,+,\gamma Z)}$ and $a_{d_{i}}^{0(\gamma,+,\gamma Z)}$ ($i = 1, \ldots , 5$) in the interpolation of $\Delta_{\gamma 0\gamma_{5}}, c_{A_{0}}^{(+,-)} , \Delta_{A_{0}}$ for heavy-light case with eqs. (106-109), respectively.

| gauge action | $a_{n_{1}}^{0}$ | $a_{n_{2}}^{0}$ | $a_{n_{3}}^{0}$ | $a_{n_{4}}^{0}$ | $a_{n_{5}}^{0}$ | $a_{d_{1}}^{0}$ | $a_{d_{2}}^{0}$ | $a_{d_{3}}^{0}$ | $a_{d_{4}}^{0}$ | $a_{d_{5}}^{0}$ |
|--------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| plaquette    | 0.000068956    | 0.00047409     | 0.037244       | -0.015986      | 0.013863       | 4.6946         | 21.769         | -6.2023        | 6.2455         | 0.67199 |
| $\Delta_{\gamma 0\gamma_{5}}$ Iwasaki | -0.0032698 | 0.046507 | 0.085071 | -0.098217 | -0.0091805 | 54.923 | -14.421 | -17.932 | -10.622 | -0.41183 |
| DBW2         | 0.0016117     | -0.018932      | 0.27891        | -0.16669       | 0.060548       | 20.906         | 58.558         | -19.071        | 3.4284         | 4.5131 |
| plaquette    | 0.016913      | 0.098293       | 0.13800        | 0.11672        | 0.058833       | 9.2871         | 49.110         | 25.152         | 12.626         | 1.7409 |
| $c_{A_{0}}^{+}$ Iwasaki | -0.012639 | 0.12162 | -0.20061 | -0.49574 | 0.094685 | -8.5494 | 5.8278 | 52.240 | 44.235 | 10.945 |
| DBW2         | -0.10453      | -0.42038       | 0.25130        | -0.44821       | -0.11844       | 10.887         | 7.7564         | 2.5891         | 13.146         | 6.4704 |
| plaquette    | -0.011811     | -0.035056      | -0.57156       | -0.17770       | -0.36685       | 5.9258         | 76.946         | 38.435         | 42.946         | 8.8915 |
| $c_{A_{0}}^{-}$ Iwasaki | 0.0071466 | -0.24087 | 0.15744 | -0.21666 | 0.13313 | -47.112 | -65.362 | -38.220 | -50.492 | -10.153 |
| DBW2         | 0.046117      | 0.28507        | -0.026074      | 0.24895        | 0.045710       | 10.757         | 10.777         | 6.7615         | 11.691         | 2.7523 |
| plaquette    | -0.10088      | 0.73718        | -4.0136        | 0.12455        | -0.11617       | -6.4483        | 33.715         | 29.058         | 2.8742         | 1.3900 |
| $\Delta_{A_{0}}$ Iwasaki | -0.024699 | -6.5964 | -0.47808 | -0.45893 | -0.018953 | 136.12 | 130.02 | 14.889 | 13.667 | 0.091530 |
| DBW2         | -0.011150     | 1.7357         | -1.4097        | -0.44819       | -0.13594       | -98.330        | 54.859         | 32.635         | 28.748         | 1.5363 |
FIG. 1: One-loop diagrams for the vertex functions. $q$ denotes the outgoing quark momentum and $p$ denotes the incoming quark momentum.
FIG. 2: $\Delta_{\gamma_k}, c_{\gamma_k}^{(+,H)}$ for heavy-heavy case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 3: $\Delta_{\gamma_0}, c_{V_0}^+$ for heavy-heavy case as a function of $m_{p_1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 4: $\Delta V_\mu$ for heavy-heavy case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 5: $\Delta \gamma_k, c_{V_k}^{(+,−,H,L)}$ for heavy-light case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 6: $\Delta \gamma_0, c_{V_0}^{(\pm)}$ for heavy-light case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 7: $\Delta_{V_{\mu}}$ for heavy-light case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 8: $\Delta_{\gamma_5}$, $c^{(+,H)}_{Ak}$ for heavy-heavy case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 9: $\Delta_{\gamma_{\omega}}, c_{A_0}^+$ for heavy-heavy case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 10: $\Delta_{A_\mu}$ for heavy-heavy case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 11: $\Delta_{\gamma_5,\gamma_5}, c_{A_k}^{(+, -, H, L)}$ for heavy-light case as a function of $m_{p1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 12: $\Delta_{\gamma_0 \gamma_5}, c_{A_0}^{(+,-)}$ for heavy-light case as a function of $m_{p_1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.
FIG. 13: $\Delta_A^\mu$ for heavy-light case as a function of $m_{p_1}^{(0)}$. Solid symbols denote the plaquette gauge action and open ones for the Iwasaki gauge action.