Spin Aharonov-Bohm effect and topological spin transistor

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Ever since its discovery, the electron spin has only been measured or manipulated through the application of an electromagnetic force acting on the associated magnetic moment. In this work, we propose a spin Aharonov-Bohm effect in which the electron spin is controlled by a magnetic flux while no electromagnetic field is acting on the electron. Such a nonlocal spin manipulation is realized in an Aharonov-Bohm ring made from the recently discovered quantum spin Hall insulator, by taking advantage of the defining property of the quantum spin Hall edge states: the one-to-one correspondence between spin polarization and direction of propagation. The proposed setup can be used to realize a new spintronics device, the topological spin transistor, in which the spin rotation is completely controlled by a magnetic flux of $hc/2e$, independently of the details of the sample.

I. INTRODUCTION

The spin of the electron is one of the most fundamental quantum mechanical degrees of freedom in Nature. Historically, the discovery of the electron spin helped to lay the foundation of relativistic quantum mechanics. In recent years, the electron spin has been proposed as a possible alternate state variable for the next generation of computers, which led to extensive efforts towards achieving control and manipulation of the electron spin, a field known as spintronics\textsuperscript{1}. Despite the great variety of currently used or theoretically proposed means of manipulating the electron spin, a feature common to all of them is that they all make use of the \textit{classical} electromagnetic force or torque acting \textit{locally} on the magnetic moment associated with the spin.

On the other hand, it is known that due to the Aharonov-Bohm (AB) effect\textsuperscript{2}, electrons in a ring can be affected in a purely \textit{quantum mechanical} and \textit{nonlocal} way by the flux enclosed by the ring even though no magnetic field — hence no classical force — is acting on them. This effect could be termed ‘charge AB effect’, as it relies only on the electron carrying an electric charge. This observation leads naturally to the question of whether it is possible to observe a ‘spin AB effect’ which would enable one to manipulate the electron spin in a purely nonlocal and quantum mechanical way, without any classical force or torque acting locally on the spin magnetic moment.

In this work, we show that the spin AB effect is indeed possible by making use of the edge states of the recently discovered quantum spin Hall (QSH) insulators. In recent years, the QSH insulator state has been proposed in several different materials\textsuperscript{3,4,5,6,7,8}. In particular, this topologically non-trivial state of matter has been recently predicted\textsuperscript{3} and realized experimentally\textsuperscript{4,5,6,7,8} in HgTe quantum wells (QWs). The QSH insulator is invariant under time reversal (TR), has a charge excitation gap in the bulk, but has topologically protected gapless edge states that lie inside the bulk insulating gap. These edge states have a distinct helical property: two states with opposite spin polarization counterpropagate at a given edge barrier\textsuperscript{4,5,6,7,8}. The edge states come in Kramers doublets, and TR symmetry ensures the crossing of their energy levels at TR invariant points in the Brillouin zone. Because of this level crossing, the spectrum of a QSH insulator cannot be adiabatically deformed into that of a topologically trivial insulator without closing the bulk gap. The helicity of the QSH edge states is the decisive property which allows the spin AB effect to exist: the perfect correlation between spin orientation and direction of propagation allows the transmutation of a usual charge AB effect into a spin AB effect, as will be explained in detail below.

The mechanism we propose to realize the spin AB effect is illustrated in Fig. 1. Consider a two-terminal device consisting of a bounded QSH insulator region pierced by a hole which is threaded by a magnetic flux $\phi$. If the edge electrons propagating clockwise have their spin point-
The AB effect is described by the matrix for the central QSH region. Inside the QSH region, combining the similar submatrices for \( S \), the electromagnetic fields are zero in the region plane. The magnetic flux being confined to the hole in the device (Fig. 1), the electromagnetic fields are zero in the region \( \phi = 0 \) (mod \( \phi_0 \)) and minimal for \( \phi = 2\phi_0 \) (mod \( \phi_0 \)), thus realizing a “topological” spin transistor (see Fig. 3 c). This effect is topological in the sense that the spin is always rotated by one cycle for each period of flux \( \phi_0 \), regardless of the details of the device, such as the size of the system or the shape of the ring.

II. PHENOMENOLOGICAL SCATTERING MATRIX ANALYSIS

Before considering any microscopic model of transport in a QSH system, generic features of two-terminal transport in the device of Fig. 1 that depend only on symmetry considerations can be extracted from a simple phenomenological scattering matrix or \( S \)-matrix analysis. The left and right junctions are each described by a scattering matrix \( S_L \) and \( S_R \), respectively (e.g. Fig. 2a for the left junction). Considering the left junction first, \( S_L \) consists of four submatrices \( t_L, t_L^1, r_L, r_L^1 \), which correspond respectively to transmission from left to right, transmission from right to left, reflection from the left, and reflection from the right. One can define similar submatrices for \( S_R \). We wish to obtain an effective \( S \)-matrix \( S \) for the whole device, by combining the \( S \)-matrices of the junctions together with the \( S \)-matrix for the central QSH region. Inside the QSH region, the AB effect is described by the matrix \( \Phi = e^{-i\sigma_x/2} \), where \( \sigma_x, \sigma_y, \sigma_z \) are the three Pauli matrices. In addition to the geometric phase, the edge electrons also acquire a dynamical phase \( \lambda = 2k_F \ell \) identical for both spin polarizations, where \( \ell \) is the distance travelled by the edge electrons from left to right junction and \( k_F \) is the edge state Fermi wave vector. Details of the analysis are presented in Appendix A; here we discuss only the main results. We obtain the effective \( 2 \times 2 \) device scattering matrix \( S \),

\[
S(\phi, \theta) = (1 - e^{i\lambda} \Phi t_L^1(0) \Phi t_R(\theta))^{-1} e^{i\lambda/2} \Phi,
\]

where the junction reflection matrices \( t_L^1 \) and \( t_R \) depend on the angles \( \theta_L, \theta_R \) of the magnetization \( M_L, M_R \) in the left and right leads. For simplicity we consider \( \theta_L = 0 \) and define \( \theta \equiv \theta_R \) (Fig. 1).

The two-terminal conductance \( G \) of the device can be written as

\[
G = \frac{e^2}{h} \text{tr} \rho_R S_L S_R^\dagger, \tag{2}
\]

using equation (A5) of Appendix A. Here \( \rho_L, \rho_R \) are \( 2 \times 2 \) effective spin density matrices for the FM leads, and have the form

\[
\rho_\alpha(\theta_\alpha) = \frac{1}{2} \mathbf{T}_\alpha(\theta_\alpha) \left( 1 + \mathbf{P}_\alpha(\theta_\alpha) \cdot \sigma \right), \tag{3}
\]

with \( \alpha = L, R \), where \( T_\alpha = \text{tr} \rho_\alpha \) is the transmission coefficient of the junction and \( \mathbf{P}_\alpha \) is a polarization vector. For simplicity, we can assume the device to have a \( \pi \)-rotation symmetry, which together with TR symmetry restricts the generic form of the reflection matrices \( r_L^1 \) and \( r_R \) in equation (1) to be

\[
r_L^1(\theta) = \begin{pmatrix} \alpha_\theta & -\beta_\theta \\ \gamma_\theta & \alpha_{\theta+\pi} \end{pmatrix}, \quad r_R(\theta) = \begin{pmatrix} \alpha_{\theta+\pi} & -\beta_\theta \\ \gamma_\theta & \alpha_\theta \end{pmatrix}. \tag{4}
\]

Physically, \( \alpha_{\theta} \) is a non-spin-flip reflection amplitude whereas \( \beta_\theta, \gamma_\theta \) are spin-flip reflection amplitudes, with \( \beta_\theta \) corresponding to a \( |\uparrow\rangle \rightarrow |\downarrow\rangle \) and \( \gamma_\theta \) to a \( |\downarrow\rangle \rightarrow |\uparrow\rangle \) reflection. These amplitudes are generally different due to the breaking of TR symmetry at the junctions by the nearby FM leads.

III. MINIMAL MODEL DESCRIPTION

These expressions being so far very general, to make further progress it is useful to consider a simple continuum Hamiltonian model for the FM/QSH junctions in which the reflection matrices \( r_L^1 \) and \( r_R \) can be calculated explicitly. This model satisfies the symmetries invoked earlier and will be seen to be a good description of the realistic HgTe system in spite of its simplicity. We model the FM leads as 1D spin-\( \frac{1}{2} \) fermions with a term which explicitly breaks the \( SU(2) \) spin rotation symmetry:

\[
H_{\text{FM}} = \int dx \Psi^\dagger \left( \frac{1}{2m} \frac{\partial^2}{\partial x^2} - \mathbf{M}(\theta) \cdot \sigma \right) \Psi,
\]

where \( \mathbf{M}(\theta) = \mathbf{M} \mathbf{n}, \) with \( \mathbf{n} = \hat{x} \cos \theta + \hat{y} \sin \theta, \) is an in-plane magnetization vector and \( \Psi \) is a two-component spinor \( \Psi = (\psi_\uparrow, \psi_\downarrow)^T. \) In the absence of AB flux, the QSH edge liquid consists of 1D massless helical fermions. When the spins of the edge states are polarized along the \( z \) direction, the Hamiltonian is given by

\[
H_{\text{QSH}} = -iv \sum_{\alpha = t, b} \eta_\alpha \int dx \left( \psi_{\alpha \uparrow}^\dagger \partial_x \psi_{\alpha \downarrow} - \psi_{\alpha \downarrow}^\dagger \partial_x \psi_{\alpha \uparrow} \right),
\]

where \( v \) is the edge state velocity and \( \alpha = t, b \) refers to the top and bottom edge, respectively, with \( \eta_t = 1 \) and \( \eta_b = -1 \).

In this simple model, the junction is described as a sharp interface between the FM region and the QSH region, from
FIG. 2: Illustration of the minimal model describing a FM/QSH junction. a, Schematic picture of the junction between the left FM lead and the QSH insulator. Incoming channels $a_{l,1}, \ldots, a_{l,p_{L}}$ from the left lead scatter at the junction into transmitted QSH edge channels $b_{l',1}, b_{l',1}$ and reflected lead channels $b_{l,1}, \ldots, b_{l,1,p_{L}}$. This scattering process is described by a scattering matrix $S_{l,1}$. b, Minimal model description of the junction. The FM lead is described by 1D parabolic bands with a spin splitting $2M$, while the QSH edge states are linearly dispersing and TR invariant, with opposite spin states counter-propagating.

which the reflection matrix $r'_{L}$ in equation (1) and the spin density matrix $\rho_{L}$ in equation (3) can be obtained. The calculation yields the reflection matrices precisely in the form of equation (4) with $\alpha_{\theta} = a$ and $\beta_{\theta} = \gamma_{\theta}^{*} = be^{-i\theta}$. In the limit of small spin splitting $M/\varepsilon_{F} \ll 1$ where $\varepsilon_{F}$ is the Fermi energy in the leads (Fig. 2b), the reflection amplitudes $a$ and $b$ are given by

$$a \simeq \frac{v - v_{F}}{v + v_{F}}, \quad b \simeq \frac{M}{2\varepsilon_{F}} \frac{v_{F}^{3}}{v(v + v_{F})^{2}},$$

where $v_{F} = \sqrt{2\varepsilon_{F}/m}$ is the Fermi velocity in the FM leads. The off-diagonal spin-flip reflection amplitude $b$ is proportional to the magnetization $M$ and along with its accompanying scattering phase shift $e^{i\theta}$ is an explicit signature of TR symmetry breaking at the junction. The diagonal non-spin-flip reflection amplitude $a$ does not break TR symmetry and is the same as would be obtained in the scattering from a nonmagnetic metal with $M = 0$. The lead spin density matrices $\rho_{L}, \rho_{R}$ can also be calculated explicitly and are found to follow the form of equation (3) as expected from the general S-matrix analysis. In the limit $M/\varepsilon_{F} \ll 1$, we obtain $T_{L} = T_{R} = 8v_{F}/(v + v_{F})^{2}$ and

$$P_{L}(\theta) = P_{R}(\theta) = P(\theta) = \frac{M(\theta)}{4\varepsilon_{F}} \frac{v_{F}^{2}}{v(v + v_{F})},$$

i.e. the spin polarization vector is directly proportional to the magnetization $M$.

From the results obtained above, we can readily evaluate the conductance $G$, which has the following expression in the limit $M/\varepsilon_{F}, P \equiv |P(\theta)| \ll 1$ and $\lambda = 0$:

$$G(\varphi, \theta; \lambda = 0) = \frac{e^{2}}{h} \frac{T_{L}T_{R}/2}{1 - 2a^{2}\cos\varphi + a^{4}} \times \left[1 + \cos(\theta - \varphi) + (1 - t^{2})^{2}\cos(\theta + \varphi) + C(\varphi, \theta)\right] p^{2} + O(P^{4}),$$

where $t = 1 - a$ and $C(\varphi, \theta) \equiv \gamma \cos\varphi + \delta \cos\theta$ with $\gamma, \delta$ some constants depending only on $a$. The effect of a finite $\lambda$ will be addressed in the next section, where we study numerically a more realistic model of the QSH state in HgTe QWs. Physically, $a$ and $t$ can be interpreted as reflection and transmission coefficients for the $S_{z}$ spin current. The generic behavior of equation (7) is illustrated in Fig. 3. The term $C(\varphi, \theta)$ is an interesting background term which manifests no correlation between AB phase $\varphi$ and rotation angle of the electron spin $\theta$. The term $\propto \cos(\theta - \varphi)$ corresponds to a rotation of the electron spin by $\varphi$, and the term $\propto \cos(\theta + \varphi)$ corresponds to a rotation by $-\varphi$. The conductance is thus maximal for $\varphi_{\text{max}} = \pm\theta$ (Fig. 3b), manifesting the desired flux-induced spin rotation effect. Physically, the $\varphi_{\text{max}} = \theta$ term corresponds to a process in which electrons traverse the device without undergoing spin flips (Fig. 3a, purple trajectory), while the $\varphi_{\text{max}} = -\theta$ term corresponds to a process involving at least one TR breaking spin-flip reflection (Fig. 3a, orange trajectory). As can be seen from equation (7), the
differential intensity of the two contributions to the conductance is $I_{on}/I_{off} = (1 - I^2)$ which can be close to unity for strongly reflecting junctions $t \ll 1$. As both contributions are minimal for $\varphi = \pi$ at $\theta = 0$, one can consider $\varphi = \pi, \theta = 0$ as the ‘off’ state of a spin transistor (Fig. 3 c, right) where the rotation of the spin is provided by a purely quantum mechanical Berry phase effect. This is in contrast with the famous Datta-Das spin transistor\textsuperscript{12} where the rotation of the spin is achieved through the classical spin-orbit force. The ‘on’ state corresponds to the absence of spin rotation for $\varphi = 0$ (Fig. 3 c, left).

IV. EXPERIMENTAL REALIZATION IN HgTe QUANTUM WELLS

We now show that this proposal can in principle be realized experimentally in HgTe QWs. We model the device of Fig. 1 as a rectangular QSH region threaded by a magnetic AB flux through a single plaquette in the center, and connected to semi-infinite metallic leads on both sides by rectangular QSH constrictions modeling quantum point contacts (QPC) (Fig. 4a). The QSH region is described by an effective $4 \times 4$ tight-binding Hamiltonian\textsuperscript{6,18} with the chemical potential in the bulk gap, while the metallic leads are described by the same model with the chemical potential in the conduction band. The detailed form of the model is given in Appendix C. The injection of spin-polarized carriers by the FM layers of Fig. 1 is mimicked by the inclusion of an effective Zeeman term in the Hamiltonian of the semi-infinite leads. We calculate numerically the two-terminal conductance through the device of Fig. 4a for a QW thickness $d = 80 \text{ Å}$. We use the standard lattice Green function Landauer-Büttiker approach\textsuperscript{19} in which the conductance is obtained from the Green function of the whole device, the latter being calculated recursively\textsuperscript{20}.

The results of the numerical calculation are plotted in Fig. 4 b, c, d. In the absence of phase-breaking scattering processes, one distinguishes two temperatures regimes $T \ll T_\lambda$ and $T \gg T_\lambda$ separated by a crossover temperature $T_\lambda = \pi \hbar v/k_B \ell$ with $\ell$ the edge state velocity, defined as the temperature for which a thermal spread $\Delta \mu \sim k_B T$ in the energy distribution of injected electrons corresponds to a spread in the distribution of dynamical phases $\lambda = 2k_F \ell$ of $\Delta \lambda \sim 2\pi$. In the low temperature regime $T \ll T_\lambda$, $\Delta \lambda \ll 2\pi$ and the dynamical phase is essentially fixed such that $G(T \ll T_\lambda) \simeq G(T = 0)$. In this regime, $G(T = 0, \mu)$ is approximately periodic in $\mu$ for $\mu$ within the bulk gap, with period $\Delta \mu \sim k_B T_\lambda$. A crossing pattern (Fig. 4b, top) occurs periodically and can be obtained by tuning the chemical potential. It corresponds to the flux-induced spin rotation effect (Fig. 3). In the high temperature regime $T \gg T_\lambda$, one could expect that the crossing pattern, and thus the spin rotation effect, would be washed out by thermal self-averaging of the dynamical phase. Surprisingly, the pattern remains (Fig. 4b, bottom), and actually acquires a more symmetric structure through the self-averaging procedure. In both temperature regimes, the conductance pattern agrees qualitatively with the result of the simple 1D Hamiltonian model (Fig. 3 b).

In the presence of phase-breaking scattering processes, the upper bound for the system size $\ell$ is given by the phase coherence length $\ell_\varphi$, which defines a minimum crossover temperature $T \approx \pi \hbar v/k_B \ell_\varphi$. The self-averaging regime can in principle be reached for $T \gg T_\lambda$; however, at too high temperatures one expects phase-breaking processes to reduce $\ell_\varphi$ below the system size and the QSH state can be destroyed. In HgTe QWs one estimates\textsuperscript{20} $\ell_\varphi \sim 1 \text{ μm}$, such that for a typical edge state velocity $\hbar v \sim 3.5 \text{ eV}\cdot\text{Å}$ one obtains a minimum crossover temperature $T_\approx \sim 13$ K. Therefore, for a device of size $\ell \lesssim 1 \text{ μm}$ one should be able to tune the spin rotation crossing pattern with the chemical potential for measurement temperatures $T \lessgtr 13$ K. At high temperatures $T \gtrsim 13$ K, the QSH state is presumably destroyed due to increased amounts of inelastic phase-breaking processes and the self-averaging regime cannot be reached. However, in type-II QW\textsuperscript{21} the edge state velocity is about one order of magnitude smaller, hence $T_\lambda \sim 1$ K and it might be possible to reach the self-averaging high temperature regime $T \gtrsim 1$ K without destroying the phase coherence of the sample.

In our calculations, for simplicity we have assumed that electrons on both the top and bottom edges acquire the same
dynamical phase $\lambda$. In a real system, the two arms of the ring are not perfectly symmetric and the electrons propagating on different arms can certainly acquire different dynamical phases $\lambda_{\text{bottom}} \neq \lambda_{\text{top}}$. However, the dynamical phase difference $\delta \equiv \lambda_{\text{bottom}} - \lambda_{\text{top}}$ only leads to an additional flux-independent rotation of the spin of the outgoing electrons, which leads to a shift of the conductance pattern in the angle $\theta$ by an amount $\delta$ (see equation (A6) of Appendix A). Thus the transistor remains effective if one uses $\theta = \delta$ instead of $\theta = 0$ in the right FM lead. If one prefers to use $\theta = 0$, one can cancel out the phase asymmetry by patterning an electrostatic gate on top of one given arm. By tuning the potential of this gate, one can adjust the Fermi wave vector locally and introduce a dynamical phase offset which cancels out the phase asymmetry $\delta$.

In Fig. 4c,d we plot the on/off ratio $G_{\text{on}}/G_{\text{off}}$ of the topological spin transistor, which can be taken as the figure of merit of the device. We define $G_{\text{on}} \equiv G(\phi = 0, \theta = 0)$ and $G_{\text{off}} \equiv G(\phi = \frac{1}{2} \varphi_0, \theta = 0)$ (see Fig. 3 c). We use two parameters, the junction spin polarization $P$ and the bulk spin splitting $\Delta_s$, to quantify the degree of spin polarization of the injected carriers. An actual experimental implementation of the transistor concept described here will require optimization of these or similar parameters. The junction spin polarization $P$ is obtained for a given junction geometry, i.e. a given choice of QPC width $W$ and length $L$ (Fig. 4a), by calculating the transfer matrix $S(A)$ of the junction directly from the TB model and using equation (3) with $P \equiv |P|$. The spin splitting $\Delta_s$ is obtained from the continuum $k \cdot p$ HgTe QW Hamiltonian mentioned earlier, and is defined as the energy difference between ‘spin up’ ($E_{1+}$) and ‘spin down’ ($E_{1-}$) energy level at the $\Gamma$ point. The on/off ratio increases rapidly for a polarization $P$ of order unity (Fig. 4c). It is reasonable to expect that optimized junction designs, better that the simplistic proof-of-concept geometry used here, would yield even higher on/off ratios. There is also an optimal width $W_{\text{opt}} \simeq 0.29 L_y$ for the junction QPC (Fig. 4d). For $W < W_{\text{opt}}$, interedge tunneling strongly backscatters the incoming electrons and reduces $G_{\text{on}}$, which suppresses the on/off ratio. For $W > W_{\text{opt}}$, the edge states on opposite edges are too far apart to recombine coherently and to produce the desired spin rotation effect, which increases $G_{\text{off}}$ and also suppresses the on/off ratio.

V. CONCLUSION AND OUTLOOK

In this work, we have shown the possibility of using a topologically nontrivial state of matter, the QSH insulator state, to manipulate the spin of the electron by purely nonlocal, quantum mechanical means, without recourse to local interactions with classical electromagnetic fields. This spin AB effect, which is a spin analog of the usual charge AB effect, relies on the helical and topological nature of the QSH edge states which is peculiar to that state of matter, combined with a Berry phase effect. In addition, we have shown that the spin AB effect can be used to design a new kind of spin transistor which is fundamentally different from the previous proposals, in that there is no classical force or torque acting on the spin of the electron. Furthermore, edge transport in the QSH regime being dissipationless, the proposed topological spin transistor would have the advantage of a lower power consumption in comparison to previous proposals for spin transistors. More generally, such a quantum manipulation of the electron spin, if observed, could open new directions in spintronics research and applications, and would at the same time demonstrate the practical usefulness of topological states of quantum matter.

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Note added.—After the completion of this work, we became aware of a recent preprint by Usaj [20] which discusses a similar effect in a different physical system.

APPENDIX A: S-MATRIX ANALYSIS

We wish to obtain an expression for the $S$-matrix $S$ relating outgoing $b$ to incoming $a$ currents amplitudes,

$$
\begin{pmatrix}
  b_l \\
  b_r
\end{pmatrix}
= S
\begin{pmatrix}
  a_l \\
  a_r
\end{pmatrix}
$$

with $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$,

(A1)

where $a_l$ and $b_l$ ($a_r$ and $b_r$) are $p_L \times (p_R \times 1)$ column vectors of the current amplitudes outside the QSH region in the left (right) lead (see Fig. 1), and $p_L$ ($p_R$) is the number of propagating channels at the Fermi energy in the left (right) lead. The matrix $S$ therefore has dimensions $(p_L + p_R) \times (p_L + p_R)$ and the submatrices $r, r', t, t'$ are reflection and transmission matrices, respectively. The two-terminal conductance $G$ from left to right is given by the Landauer formula $G = \frac{2 e^2}{h} \text{tr} tt'$. We assume that phase coherence is preserved throughout the sample so that $S$ can be obtained by combining $S$-matrices for different portions of the device coherently [21]. We define the $(p_{L,R} + 2) \times (p_{L,R} + 2)$ scattering matrices $S_L, S_R$ for the left ($L$) and right ($R$) FM/QSH junctions (e.g. see Fig. 2a for the left junction),

$$
\begin{pmatrix}
  b_l \\
  b_r
\end{pmatrix}
= S_L \begin{pmatrix} a_l \\ a_r \end{pmatrix},
\begin{pmatrix}
  b_{l'} \\
  b_{r'}
\end{pmatrix}
= S_R \begin{pmatrix} a_{l'} \\ a_{r'} \end{pmatrix},
$$

(A2)

where $l'$ ($r'$) is the QSH region immediately to the right (left) of the left (right) junction, such that $a_{l'}, a_{r'}$ and $b_{l'}, b_{r'}$ are the 2-component spinors of edge state current amplitudes. They are related through the geometric AB phase $\varphi$ (different for

Note added.—After the completion of this work, we became aware of a recent preprint by Usaj [20] which discusses a similar effect in a different physical system.
each spin polarization) and the dynamical phase \( \lambda = 2k_F \ell \)
(identical for both spin polarizations) where \( \ell \) is the distance travelled by the edge electrons from left to right junction and \( k_F \) is the edge state Fermi wavevector,

\[
\begin{align*}
\{ a_{r,l}^{\dagger} \} &= e^{i\lambda/2} e^{i\varphi_{r,l}/2} \{ b_{l,r}^{\dagger} \},
\end{align*}
\]

where the upper sign for \( \varphi \) corresponds to spin up. Using equations (A2) and (1), we can write

\[
\begin{align*}
\left( e^{-i\lambda/2} \Phi a_L \right) &= S_R \left( e^{i\lambda/2} \Phi b_R \right),
\end{align*}
\]

where we define \( \Phi \equiv e^{-i\varphi_{r,l}/2} \). Using the first equality in equation (A2) together with equation (A4), we can eliminate the intermediate amplitudes \( a_L, b_R \) and obtain relations between the left lead amplitudes \( a_r, b_l \) and the right lead amplitudes \( a_r, b_r \), which gives us \( \Sigma \) (equation (A1)). The \( 2 \times 2 \) transmission matrix \( t \), i.e. the lower left block of \( \Sigma \), is then obtained in the form

\[
t = t_R S t_L, \quad (A5)
\]

where \( t_L \) and \( t_R \) are the \( 2 \times p_L \) and \( p_R \times 2 \) transmission matrices for the left and right junctions, respectively (i.e. the lower left blocks of \( S_L, S_R \) following the notation of equation (A1)), and \( S \) is a \( 2 \times 2 \) matrix defined in equation (1). The effective spin density matrices \( \rho_L, \rho_R \) of the FM leads used in equation (2) are defined as \( \rho_L = t_L t_L^\dagger \) and \( \rho_R = t_R^\dagger t_R \). If the arms of the ring are asymmetric, the dynamical phase \( \lambda \) is generally different for each arm and we have \( \lambda_{\text{bottom}} = \lambda_{\text{top}} \equiv \delta \neq 0 \). In this case, one can show that equation (2) still holds, but with the substitutions

\[
\begin{align*}
\rho_L(\theta_L) &\rightarrow R_\delta \rho_L(\theta_L) R_\delta^{-1} = \rho_L(\theta_L + \delta), \\
n \rightarrow R_\delta n(\theta_L) R_\delta^{-1} = n(\theta_L + \delta),
\end{align*}
\]

where \( R_\delta \equiv e^{-i\varphi_{r,l}/2} \) rotates the spin about the \( z \) axis by an angle \( \delta \). In other words, a phase asymmetry is equivalent to a rigid flux-independent rotation of the electron spin, and simply shifts the conductance pattern by a constant angle \( \delta \):

\[
G(\phi, \theta \equiv \theta_R - \theta_L) \rightarrow G(\phi, \theta_R - (\theta_L + \delta)) = G(\phi, \theta - \delta). \quad (A6)
\]

**APPENDIX B: SCATTERING AT THE JUNCTION**

In order to solve the 1D scattering problem at the FM/QSH interface, we first observe that the number of degrees of freedom is equal on either side of the junction. If the Fermi level \( \varepsilon_F \) is chosen such that both spin subbands in the FM leads are occupied, there are four propagating modes on each side of the junction (two spins and two chiralities). The QSH spin states \( \phi_{\text{QSH}}(\pm) \) are \( \sigma_z \) eigenstates while the FM spin states \( \phi_{\text{FM}}(\pm)(\theta) \) are eigenstates of \( \mathbf{n} \cdot \sigma \) and depend explicitly on \( \theta \). The Schrödinger equation for the junction is then solved by the following scattering ansatz,

\[
\psi_\sigma^\dagger(x) = \begin{cases}
\phi_\sigma^\dagger(+) e^{ik_\sigma x} + \sum_{\sigma'} r_{\sigma\sigma'}^\dagger \phi_{\sigma'}^\dagger(-) e^{-ik_{\sigma'} x}, & x < 0, \\
\sum_{\sigma'} t_{\sigma\sigma'} \phi_{\sigma'}^\dagger(+) e^{ik_{\sigma'} x}, & x > 0,
\end{cases}
\]

for a right-moving scattering state, and with similar expressions for a left-moving scattering state \( \psi_\sigma^- \). Spin is denoted by \( \sigma \), chirality by \( \pm \) and side of the junction by \( \langle, \rangle \). The propagating modes are explicitly normalized to unit flux such that \( r_{\sigma\sigma} \) and \( t_{\sigma\sigma} \) are the desired reflection and transmission matrices. Requiring the continuity of \( \psi_\sigma^\dagger(x) \) and \( \psi(x) \) at the interface \( x = 0 \) (with \( \psi(x) \equiv \partial H / \partial k_x \) the velocity operator), we obtain a system of linear equations for the sixteen matrix elements \( r_L, t_L, r_R, t_R \) constituting \( S \). As illustrated in Fig. 1, the magnetization angle is set to zero in the left lead and to \( \theta \) in the right lead and we obtain \( r'_L(0) \) and \( r_R(\theta) \) in equation (1).

**APPENDIX C: TIGHT-BINDING MODEL**

The effective tight-binding model describing HgTe QWs is defined as \( H = \sum_i c_i^\dagger V_i c_i + \sum_{ij} \left( c_i^\dagger T_{ij} e^{iA_{ij}} c_j + \text{h.c.} \right) \),

\[
H = \sum_i c_i^\dagger V_i c_i + \sum_{ij} \left( c_i^\dagger T_{ij} e^{iA_{ij}} c_j + \text{h.c.} \right), \quad (C1)
\]

where \( T_{ij} = T_{ij}^2 \delta_j, i + x + y = T_{ij}^2 \delta_j, i + y \) is the nearest-neighbor hopping matrix, \( A_{ij} = \int_{\mathbb{B}} \mathbf{J} \cdot \mathbf{A} \) is the Peierls phase with \( \mathbf{A} \) the electromagnetic vector potential, and \( V_i, T_{ij} \) and \( T_{ij}^2 \) are \( 4 \times 4 \) matrices containing the \( \mathbf{k} \cdot \mathbf{p} \) parameters and the effective Zeeman term. The \( 4 \times 4 \) matrices \( T_{ij}, T_{ij}^2 \) and \( V_i \) used in the tight-binding Hamiltonian (C1) are given by

\[
T_{ij} = \begin{pmatrix}
D_+ & D - & -iA_2 & -iA_1 \\
-\Delta_4 & 2D - & 0 & -iA_2 \\
0 & -iA_2 & 2D + & 0 \\
iA_1 -iA_2 & 0 & 0 & 2D -
\end{pmatrix},
\]

\[
T_{ij}^2 = \begin{pmatrix}
D_+ & A + & \Delta_4 & 0 \\
-\Delta_4 & 2D - & 0 & -iA_2 \\
0 & -iA_2 & 2D + & 0 \\
iA_1 -iA_2 & 0 & 0 & 2D -
\end{pmatrix},
\]

and

\[
V_i = (C - 4D - \varepsilon_F + E_g(i)) \mathbb{1}_{4 \times 4}
\]

\[
+ (M - 4B) \mathbb{1}_{2 \times 2} \otimes \sigma_z + H_{\text{eff}}^{\parallel} + H_{\text{eff}}^{\perp}. \quad (C3)
\]

where \( D_\pm \equiv D \pm B \) and \( A, B, C, D, M, \Delta_4, \Delta_2 \) are \( \mathbf{k} \cdot \mathbf{p} \) parameters [13] and \( \mathbb{1}_{n \times n} \) denotes the \( n \times n \) unit matrix. The Fermi energy \( \varepsilon_F \) is uniform throughout the device. The gate potential \( E_g(i) \) is different in the QSH and lead regions (Fig. 4a), and is used to tune the central region into the QSH insulating regime. The in-plane \( H_{\text{eff}}^{\parallel} \) and out-of-plane \( H_{\text{eff}}^{\perp} \) effective Zeeman terms, which are used to mimick the injection
of spin-polarized carriers from a FM layer (Fig. 1), are given by

\[
H_{Z\parallel}^{\text{eff}} = g_{\parallel} \mu_B \begin{pmatrix}
0 & 0 & B_{\text{eff}} \\
0 & 0 & 0 \\
B_{\text{eff}} & 0 & 0
\end{pmatrix},
\]

\[
H_{Z\perp}^{\text{eff}} = \mu_B B_{\text{eff}} \begin{pmatrix}
g_{E\perp} & 0 & 0 \\
0 & g_{H\perp} & 0 \\
0 & 0 & -g_{E\perp}
\end{pmatrix},
\]

(C4)

where \(B_{\text{eff}} = B_{\text{ex}} \pm i B_{\text{ey}}\), \(B_{\text{eff}} = (B_{\text{ex}}, B_{\text{ey}}, B_{\text{ez}})\) is some effective magnetic field whose role is to induce a spin polarization in the leads, \(\mu_B\) is the Bohr magneton, and \(g_{\parallel}\) and \(g_{E\perp}, g_{H\perp}\) are the in-plane and out-of-plane \(g\)-factors, respectively.

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