Abstract We study the quasinormal modes of the massless scalar field of Park black hole in the Hořava gravity using the third-order WKB approximation method and find that the black hole is stable against these perturbations. We compare and discuss the results with that of Schwarzschild–de Sitter black hole. Thermodynamic properties of Park black hole are investigated and the thermodynamic behavior of upper mass bound is also studied.

1 Introduction

Bardeen, Carter and Hawking formulated four laws of black hole mechanics [1] and Bekenstein introduced the idea of black hole entropy [2] in 1973 and in 1974 Hawking introduced the concept of black hole evaporation [3] and particle creation by black holes [4]. This led to the birth of black hole thermodynamics. From the birth itself, it became a source of hope and fascination because it provides a real connection between gravity and quantum mechanics. The recent Type Ia supernovae analysis [5] indicated that the expansion of the universe is accelerating and the positive cosmological constant could be made responsible for the acceleration of the universe [6, 7]. Due to the success of anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [8] much more attention has been given on studying gravity in the de Sitter (dS) space and asymptotically dS space [9]. This leads to an interesting proposal which is analogous to the AdS/CFT correspondence in de Sitter space, i.e., the dS/CFT correspondence [10, 11]. It has been suggested that there is a dual relation between quantum gravity on de Sitter (dS) space and Euclidean conformal field theory (CFT) on a boundary of the de Sitter space.

In 2009, Hořava proposed a field theoretic model of gravity as a complete theory in the UV limit [12–14]. This is a renormalizable theory of gravity in four dimensions. It is non-relativistic in the UV region whereas in the IR region it can be reduced to Einstein’s gravity theory with a cosmological constant. Many studies have been done regarding the cosmological and black hole solutions [15–28] of this theory. In [29], Park obtained a black hole and a cosmological solution by introducing two parameters $\omega$ and the cosmological constant $\Lambda_W$ and by choosing arbitrary values for these parameters. These solutions are analogous to the standard Schwarzschild (A)dS solutions which are absent in the original Hořava model.

Quasinormal modes (QNMs) of a black hole are defined as the solution of the perturbation equations belonging to a certain complex characteristic frequencies which satisfy the boundary conditions. They govern the decay of perturbations at intermediate times. Since they are important when studying the dynamics of black holes they have drawn much more attention in the past years. Vishveshwara first put forward the concept of QNMs in the calculations of scattering of gravitational radiation by a Schwarzschild black hole [30] and Press [31] proposed the term quasinormal frequencies. QNM frequencies are the characteristics of a black hole since they are independent of initial perturbations and depend only on black hole parameters, hence one can find information about mass, charge, and angular momentum [32–37]. In addition to this, the properties of QNMs have been well explored in the context of the AdS/CFT correspondence [38, 39] and loop quantum gravity [40, 41]. As a result of these findings, it is widely believed that QNMs carry a unique foot print to directly identify the existence of a black hole. For finding the QNMs, we use the third-order WKB approximation method [42, 43]. This method was developed by Schutz and Will [44] and later extended to sixth order by Konoplya [45]. In this paper we investigate the thermodynamics of Park black holes and also we investigate the QNMs of massless scalar field around Park black hole and compare with the results of Schwarzschild–de Sitter black hole.

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The rest of this paper is organized as follows. In Sect. 3, we investigate the thermodynamics of Park black hole in HL gravity and Schwarzschild–de Sitter black hole. The comparison of thermodynamics of these black holes and stability are discussed. In Sect. 4, the QNMs of Park black hole and SdS black hole are studied and compared the results. Finally, Sect. 5 ends up with a brief discussion and conclusion.

2 Park black hole in Hořava–Lifshitz gravity

Hořava considered the ADM decomposition of the metric as

\[ ds^2_d = -N^2 c^2 dt^2 + g_{ij}(dx^i + N_i dt)(dx^j + N^j dt), \]  

where \( N \) and \( N_i \) denote the lapse and shift function, respectively. By introducing an IR modification term \( \mu^4 R^{(3)} \) with an arbitrary cosmological constant in Hořava gravity, the modified action can be written as

\[
S = \int dt \, d^3x \, \sqrt{g} \left\{ \frac{2}{\kappa} (K_{ij} K^{ij} - \lambda K^2) - \frac{k^2}{2\nu^2} C_{ij} C^{ij} \right. \\
+ \frac{k^2}{2\nu^2} d_i K^{ij} R_{ij} R_{ij} - \frac{k^2}{8} R_{ij} R_{ij} - \frac{k^2}{8} \omega R^{(3)} C_{ij} C^{ij} \\
- \frac{\lambda}{8(3\lambda - 1)} R^{(3)} + \frac{\lambda}{4(3\lambda - 1)} \left( 4\lambda - 1 \right) \right. \\
- \frac{\lambda}{8(3\lambda - 1)} \left( R^{(3)} + 3A^2_W \right) \right\},
\]

where \( K_{ij} \) and \( C_{ij} \) are the extrinsic curvature and the Cotton tensor, respectively. In the action, \( \kappa, \nu, \mu, \lambda, A_W, \omega \) are constant parameters. The last term in (2) represents a soft violation of the detailed balance condition [12]. For static and spherically symmetric solution, substituting the metric ansatz as

\[ ds^2 = -N(r)^2 c^2 dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \]

in the action (2) and after angular integration, we obtain the Lagrangian as

\[
\mathcal{L} = \frac{k^2 \mu^2}{8(1 - 3\lambda)} \frac{N}{\sqrt{f}} \left\{ (2\lambda - 1) \frac{(f - 1)^2}{r^2} - 2\lambda \frac{f - 1}{r} f' + \frac{\lambda - 1}{2} f'^2 \right. \\
- 2(\omega - A_W)(1 - f - r f') - 3A^2_W r^2 \right\}.
\]

Kehagias and Sfetsos [17] obtained only the asymptotically flat solution (with \( A_W = 0 \)) while Mu-In Park [29] considered an arbitrary \( A_W \) and obtained a general solution. By varying the lapse function \( N \) we obtain

\[
(2\lambda - 1) \frac{(f - 1)^2}{r^2} - 2\lambda \frac{f - 1}{r} f' + \frac{\lambda - 1}{2} f'^2 \\
- 2(\omega - A_W)(1 - f - r f') - 3A^2_W r^2 = 0,
\]

and similarly by varying \( f \), we arrive at

\[
\left( \frac{N}{\sqrt{f}} \right)' \left( (\lambda - 1) f' - 2\lambda \frac{f - 1}{r} + 2(\omega - A_W) r \right) \\
+ (\lambda - 1) \frac{N}{\sqrt{f}} (f'' - 2(f - 1) \frac{1}{r^2}) = 0.
\]

These are the equations of motion. By giving \( \lambda = 1 \) and solving the field equations, we arrive at the Park solution [29],

\[
N^2 = f_{Park} = 1 + (\omega - A_W) r^2 - \sqrt{r \left[ \omega (\omega - 2A_W) r^3 + \beta \right]},
\]

where \( \beta \) is an integration constant related to the black hole mass. Park’s solution can easily be reduced to Lü, Mei, and Pope (LMP)’s solution [15] as well as Kehagias and Sfetsos (KS)’s solution [17]. By choosing \( \beta = -\alpha^2 / A_W \) and \( \omega = 0 \) one can obtain the LMP solution as

\[
f_{LMP} = 1 - A_W r^2 - \frac{\alpha}{\sqrt{A_W}} \sqrt{f}. \]

And if we choose \( \beta = 4\omega M \) and \( A_W = 0 \), KS solution is obtained as

\[
f_{KS} = 1 + \omega r^2 - \sqrt{r (\omega^2 r^3 + 4\omega M) r}. \]

Now let us consider the asymptotically dS case of Park solution. In this case, the action is given by an analytic continuation of the action given by (2) [15].

\[
\mu \rightarrow i\mu, \quad \nu \rightarrow -i\nu, \quad \omega \rightarrow -\omega.
\]

From (7), for \( r \gg [\beta / (\omega (\omega - 2A_W))]^{1/3} \), i.e., considering asymptotically the de Sitter case with \( A_W > 0 \) and \( \omega < 0 \), we can arrive at

\[
f = 1 - A_W \frac{1}{2} \frac{A_W}{\omega} \frac{2M}{\sqrt{1 + 2|A_W/\omega| r}} + O(r^{-4}).
\]

When we compare (11) with the Schwarzschild–dS solution

\[
f = 1 - \frac{A_W}{2} r^2 - \frac{2M}{r},
\]

we can see that it agrees up to some numerical factor corrections. In the coming sections we will investigate more about this agreement.
3 Thermodynamics of Park black hole

In order to explore the thermodynamics of a Park black hole, let us consider (11). In general the dS solution has two horizons. Larger one, \( r^{++} \) correspond to the cosmological horizon and the smaller one, \( r^+ \) for the black hole horizon. By considering the black hole horizon, we can arrive at a relation which connects mass and horizon radius of the black hole,

\[
M = \frac{1 + 2(\omega - \Lambda_W)r^2_+ + \Lambda_W^2 r^4_+}{4\omega r^+_+}.
\]  

(13)

Then from the usual definition of temperature in thermodynamics, we can arrive at temperature of the black hole with \( \Lambda_W > 0 \) and \( \omega < 0 \) as

\[
T = \frac{3\Lambda_W^2 r^4_+ + 2(\omega - \Lambda_W)r^2_+ - 1}{8\pi r_+(1 + (\omega - \Lambda_W)r^2_+)}.
\]  

(14)

We have plotted the variation of black hole temperature against the horizon radius in Fig. 1. From this plot it is evident that there is an infinite discontinuity in temperature. It occurs at

\[
\tilde{r}_+ = \frac{1}{\sqrt{\Lambda_W - \omega}}.
\]  

(15)

And for the region \( r^+ < \tilde{r}_+ \), interestingly the temperature is found to be negative. The heat capacity of the black hole is given by

\[
C = \frac{2\pi r^2_+ [r^2_+(\Lambda_W - \omega) - 1][r^2_+(\Lambda_W(3\Lambda_W r^2_+ - 2) + 2\omega) - 1]}{(\Lambda_W r^2_+ - 1)^2(3\Lambda_W r^2_+ - 1) - r^2_+(5 + \Lambda_W r^2_+(4 + 3\Lambda_W r^2_+))\omega + 2r^4_+ \omega^2}.
\]  

(16)

In Fig. 2, variation of heat capacity with respect to the black hole horizon \( r^+ \) for different values of the coupling parameters are plotted. From this figure we can see that the specific heat undergoes a transition from negative values to positive values or in other words the black hole changes from a thermodynamically unstable state to a thermodynamically stable state. By looking and comparing the two figures, Fig. 1 and Fig. 2, we can straightaway say that in the region where temperature shows the anomalous behavior due to its negative values, the black hole is found to be thermodynamically unstable as its heat capacity is negative. Now for a Schwarzschild–dS black hole, from (12) we can write

\[
f = 1 - \frac{\Lambda_W}{2} r^2 - \frac{2M}{r}.
\]  

(17)

The event horizon is defined by \( f(r^+) = 0 \),

\[
r^+ - \frac{\Lambda_W}{2} r^3 + 2M = 0.
\]  

(18)

So we can write the mass as

\[
M = \frac{r^+_+ - \Lambda_W r^3_+}{2}.
\]  

(19)

Using the Bekenstein–Hawking area law,

\[
S = \frac{A}{4} = \pi r^2_+.
\]  

(20)
we can rewrite the mass–horizon relation (19) as

\[ M = \frac{1}{2} \sqrt{\frac{S}{\pi}} - \Lambda_W \left( \frac{S}{\pi} \right)^{3/2}. \]

(21)

Using the definition of temperature as \( T = \frac{1}{4\sqrt{\pi S}} \) and that of the heat capacity as \( C = T \frac{\partial S}{\partial T} \), we can arrive at

\[ T = \frac{1}{4\sqrt{\pi S}} - \frac{3\Lambda_W \sqrt{S}}{2\pi^{3/2}}, \]

(22)

and

\[ C = \frac{\pi^2}{3\Lambda_W \pi + 18\Lambda_W^2 S} - \frac{3\Lambda_W \sqrt{S}}{2\pi^{3/2}} - \frac{\pi}{3\Lambda_W}. \]

(23)

Variation of temperature with respect to entropy is plotted in Fig. 3 while in Fig. 4 the variation of heat capacity with entropy is plotted. From Fig. 4 it is evident that Schwarzschild–dS black hole is thermodynamically unstable for all range of entropy values.

Now let us investigate the peculiar behavior of a Park black hole. As explained in [29], for the black hole horizon to exist and curvature singularity at \( r = 0 \) not to be naked, one has to consider another condition, which is given by

\[ M \leq \frac{(2\Lambda_W - \omega)}{4} r_+^3. \]

(24)

Or we can say that \( M < \frac{(2\Lambda_W - \omega)}{4} r_+^3 \) for all \( r_+ \) except for \( r_+ = r_+^* \). And at this point the mass of the black hole meets the upper bound.

Now let us investigate the thermodynamics of the black hole which has a mass given by the upper mass bound value given in (24) i.e.,

\[ M_{\text{bound}} = \frac{(2\Lambda_W - \omega)}{4} r_+^3. \]

(25)

Using the Bekenstein–Hawking area law, we can rewrite (25) as

\[ M_{\text{bound}} = \frac{(2\Lambda_W - \omega)}{4} \left( \frac{S}{\pi} \right)^{3/2}. \]

(26)

From the usual definition of temperature and that of heat capacity, we can arrive at

\[ T = \frac{3(2\Lambda_W - \omega) \sqrt{S}}{8\pi^{3/2}}, \]

(27)

and

\[ C = 2S. \]

(28)

Since the heat capacity is positive, the black hole having a mass given by (25) is thermodynamically stable. From this fact we can say that the occurrence of an infinite discontinuity as well as a negative temperature must be due to the existence of a restriction given by (24) for the mass parameter.

4 Quasinormal modes of Park black hole

In this part of our work, our main aim is to study the quasinormal modes of massless scalar field of the Park black hole in Hořava gravity. The Klein–Gordon equation for a massless scalar field in this space time is given by

\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right) \Phi = 0. \]

(29)

Separating the scalar field into spherical harmonics,

\[ \Phi = \frac{1}{r} \Psi(r) Y(\theta, \phi) e^{-i\omega t} \]

(30)

and using the tortoise coordinate defined by \( dr_* = \frac{1}{f} dr \), one can obtain the radial equation,

\[ \frac{d^2\Psi(r)}{dr_*^2} + \left[ \omega^2 - V(r) \right] \Psi(r) = 0, \]

(31)
in which the effective potential \( V(r) \) is given by
\[
V(r) = f(r) \left[ \frac{l(l+1)}{r} + \frac{1}{r} \frac{df(r)}{dr} \right],
\]
(32)
where (11) gives \( f(r) \).

The behavior of the effective potential for different values of the angular quantum number \( l \) is plotted in Fig. 5. From this figure we can say that as the \( l \) value increases the potential barrier also gets increased. Now we are evaluating the quasinormal frequencies of the scalar field around Park black hole using the third-order WKB approximation [42–44]. The formula for the complex quasinormal frequencies \( \omega_{\text{QNM}} \) in this approximation is given by
\[
\omega_{\text{QNM}}^2 = \left[ V_0 + (-2V_0')^{1/2} \Lambda \right] - i \left( n + \frac{1}{2} \right) (-2V_0')^{1/2} (1 + \Omega),
\]
(33)
where
\[
\Lambda = \frac{1}{(-2V_0')^{1/2}} \left\{ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0''} \right) \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0''} \right)^2 (7 + 60\alpha^2) \right\},
\]
(34)
\[
\Omega = \frac{1}{(-2V_0'')^{1/2}} \left\{ \frac{5}{6912} \left( \frac{V_0'''}{V_0''} \right)^4 \left( 77 + 188\alpha^2 \right) \right. \\
\times \frac{1}{384} \left( \frac{V_0'''V_0^{(4)}}{V_0''^3} \right) \\
- \left( 51 + 100\alpha^2 \right) + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0''} \right)^2 (67 + 68\alpha^2) \\
+ \frac{1}{288} \left( \frac{V_0^{(5)}}{V_0''} \right) (19 + 28\alpha^2) \\
- \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0''} \right) (5 + 4\alpha^2) \right\}
\]
(35)

and
\[
\alpha = n + \frac{1}{2}, \quad V_0^{(r)} = \frac{d^rV}{dr^r}\big|_{r=r_p},
\]
(36)
where \( n \) is the overtone and \( r_p \) is the value of polar coordinate \( r \) corresponding to the peak value of effective potential \( V(r) \) given in (32). Substituting the effective potential (32) into the above formula, we can obtain the quasinormal frequencies of scalar perturbation for the Park black hole.

![Fig. 5 Variation of effective potential with horizon radius for different \( l \) values](image1)

![Fig. 6 Massless scalar field QNMs of Park black hole for different values of \( l \)](image2)

| \( \omega \) | \( \omega_{\text{QNM}} (l = 1) \) | \( \omega_{\text{QNM}} (l = 2) \) | \( \omega_{\text{QNM}} (l = 3) \) |
|---|---|---|---|
| -0.5 | 0.2916911−0.0981963i | 0.4841694−0.096997150i | 0.676545−0.09670374i |
| -1.0 | 0.2914027−0.09809888i | 0.4836904−0.09690105i | 0.67587629−0.096607976 |
| -1.5 | 0.2913065−0.09806638i | 0.48353071−0.09686899i | 0.67565299−0.096576033i |
| -2.0 | 0.2912584−0.09805013i | 0.48345808−0.09685296i | 0.675541320−0.096560057i |
| -2.5 | 0.2912296−0.09804038i | 0.48340280−0.09684334i | 0.675474301−0.096550470i |
| -3.0 | 0.2912103−0.09803388i | 0.48337089−0.09683693i | 0.67542962−0.096544078i |
The quasinormal frequencies are shown in Table 1. Figure 6 shows that the real and imaginary parts of the quasinormal frequencies changes slowly as the parameter $\omega$ changes for a fixed value $\Lambda_W$. From Table 1, we can find that all frequencies have negative imaginary parts, which means that the black hole is stable against this perturbation.

Let us now consider the Schwarzschild–de Sitter case. In Fig. 7 we have plotted the effective potential for the Schwarzschild–de Sitter case along with the Park black hole case for a particular $l$ value. And from this figure we can see that the behavior is almost the same. Applying the third-order WKB approximation method to evaluate the fundamental quasinormal modes and from the obtained quasinormal mode frequencies of the SdS black hole and the Park black hole, we can compare them. In Fig. 8 we have plotted the fundamental quasinormal mode frequencies for fixed values of $\Lambda_W$ and $\omega$ for these two black holes. From this it is evident that both black holes show similar behavior against these massless scalar perturbations. So as in the case of the Park black hole, the SdS black hole is also stable against this perturbation.

5 Discussion and conclusion

In this paper we have investigated the quasinormal modes of the massless scalar field of the Park black hole. QNMs are studied using the third-order WKB approximation method. We have found that the frequencies all have negative imaginary parts. Hence the Park black hole is stable against this perturbation. These results are compared with those of the Schwarzschild–de Sitter case and find that they have almost the same behavior. We have also studied the thermodynamic aspects of the Park black hole and find that it shows negative values as well as an infinite discontinuity in temperature. This may be due to the existence of an upper bound of the mass parameter. In addition to this we have studied the thermodynamic properties of the Park black hole which has a mass equal to the upper bound value of the mass parameter and found that these black holes are thermodynamically stable. When we study the perturbative effects on the Park and SdS black holes, there is agreement between them up to some numerical factor corrections, while they have entirely different thermodynamic behaviors.

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