LOCAL REGULARITY CRITERIA OF THE NAVIER-STOKES EQUATIONS WITH SLIP BOUNDARY CONDITIONS

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Abstract. We present regularity conditions for suitable weak solutions of the Navier-Stokes equations with slip boundary data near the curved boundary. To be more precise, we prove that suitable weak solutions become regular in a neighborhood boundary points, provided the scaled mixed norm $L^{p,q}_{x,t}$ with $3/p + 2/q = 2$, $1 \leq q < \infty$ is sufficiently small in the neighborhood.

1. Introduction

We study the regularity problem for suitable weak solutions $(u, p) : \Omega \times I \to \mathbb{R}^3 \times \mathbb{R}$ to the Navier-Stokes equations in three dimensions,

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \text{div} \, u = 0 \quad \text{in} \quad Q_T = \Omega \times I,$$

where $u$ is the velocity field and $p$ is the pressure. Here $f$ is an external force and $\Omega$ is a bounded domain with $C^2$ boundary. After the existence of weak solutions was proved by Leray [18] and Hopf [11], regularity problem has remained open. It has been known that weak solutions become unique and regular in $\Omega \times [0,T)$ if the following additional conditions are imposed on weak solutions:

$$\|v\|_{L^{p,q}_{x,t}(\Omega \times [0,T))} := \left\| \|v(\cdot, t)\|_{L^p_x(\Omega)} \right\|_{L^q_t([0,T))} < \infty, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad 3 \leq p \leq \infty.$$ 

In this direction, lots of significant contributions have been made so far (refer to e.g. [6, 7, 8, 9, 13, 15, 21, 22, 30, 32, 33, 35, 36]).

For the partial regularity theory, after Scheffer’s works in a series of papers [23, 24, 25, 26], Caffarelli, Kohn and Nirenberg [4] proved that the one-dimensional parabolic Hausdorff measure of possible singular set is zero for suitable weak solutions of the Navier-Stokes equations. The extension up to boundary was shown in [28] (see also [29]). In [5], the estimate of size of a
possible singular set was improved by a logarithmic factor. The following local regularity criterion was proved in [4] and crucially used for partial regularity: there exists $\epsilon > 0$ such that if suitable weak solution $u$ satisfies
\[
\limsup_{r \to 0} \frac{1}{r} \int_{Q_{z,r}} |\nabla u(y,s)|^2 \, dyds \leq \epsilon,
\]
then $u$ is regular in a neighborhood of $z$ (refer to [27] for flat boundary and [29] for curved boundary). This regularity criterion was improved in terms of scaled mixed norm regarding velocity field in [10, Theorem 1.1]. On the other hand, in [9], the following regularity criteria was proved near the flat boundary:
\[
\limsup_{r \to 0} \frac{1}{r} \left\|u\right\|_{L^p(B^+(t-r^2,t))} \leq \epsilon, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 2 < q < \infty.
\]
In [14], the following local regularity criteria was proved near the curved boundary in case of homogeneous boundary conditions:
\[
\limsup_{r \to 0} r^{-\left(\frac{3}{p} + \frac{2}{q} - 1\right)} \left\|u\right\|_{L^p(\Omega_{x,r})} \left\|\right\|_{L^q(t-r^2,t)} \leq \epsilon, \quad 1 \leq \frac{3}{p} + \frac{2}{q} \leq 2, \quad 2 < q \leq \infty, \quad (p,q) \neq \left(\frac{3}{2},\infty\right).
\]
For the case of slip boundary conditions, the existence of the weak or strong solutions was studied by Solonnikov, Šcadilov [34], Maremonti [20] and Itoh, Tani [12]. Some regularity results for weak solutions were showed in [3] for the stationary case. Bae, Choe and Jin [2] proved the following: Suppose $(u,p)$ is a suitable weak solution. There exists a positive constant $\sigma$ such that if $u \in L^{p,q}(Q^+_T)$ with $\|u\|_{L^{3,\infty}(Q^+_T)} \leq \epsilon_0$ for some small $\epsilon_0$, then
\[
\sup_{Q^+_T} |u| \leq N \left(\int_{Q^+_T} |u|^3 \, dxdt\right)^{\frac{3}{3+q}} + N
\]
for some positive constant $N$ depending on $\epsilon_0$.

The main objective of this paper is to establish the regularity criteria (1) for the Navier-Stokes equations with ship boundary conditions near the curved boundary. To be more precise, we study suitable weak solutions of the following Navier-Stokes equations in three dimensions
\[
\left\{
\begin{array}{ll}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, & \text{in } Q_T = \Omega \times I, \\
u \cdot n = 0, & u \cdot (u, p) \cdot \tau = 0 & \text{on } \partial \Omega \times I,
\end{array}
\right.
\]
where $u$ is the velocity field, $p$ is the pressure, $n$ is the outer unit normal vector, $\tau$ is the unit tangent vector and $T(u,p)$ is a stress tensor, which is given as
\[
T(u, p) = \frac{1}{2} \left(\nabla u + (\nabla u)^\top\right) - p\delta_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i)_{i,j=1,2,3} - p\delta_{ij}.
\]
Here $f$ is an external force and $\Omega$ is a bounded domain with $C^2$ boundary. Suitable weak solution will be defined in Definition 2.1 in next section. The existence of suitable weak solutions with slip boundary conditions was proved in [2] for the case of half space. In Appendix, we provide the existence of suitable weak solutions for the bounded domains as in [4].

We prove that suitable weak solution $u$ becomes Hölder continuous near regular curved boundary, provided that the scaled mixed $L^{p,q}$-norm of the velocity field $u$ is sufficiently small (the proof will be given in Section 3). More precisely, our main result reads as follows:

**Theorem 1.1.** Let $u$ be a suitable weak solution of the Navier-Stokes equations in $\Omega$ with extra force $f \in M_{2,\gamma}$ for some $\gamma > 0$, $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$ and $B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}$. Assume further that $\Omega$ is any domain with $C^2$ boundary satisfying Assumption 2.1. Suppose that $(x,t) \in \partial \Omega \times I$. For every pair $p,q$ satisfying

$$\frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq q < \infty,$$

there exists a constant $\epsilon > 0$ depending on $p$, $q$, $\gamma$ and $\|f\|_{M_{2,\gamma}}$ such that, if the pair $u$, $p$ is a suitable weak solution of the Navier-Stokes equations (2) satisfying Definition 2.1 and

$$\limsup_{r \to 0} r^{-1} \left\| u \right\|_{L^p(\Omega_{x,r})} \left\| u \right\|_{L^q(t-r^2,t)} < \epsilon,$$

then $u$ is regular at $z = (x,t)$.

## 2. Preliminaries

In this section, we introduce notations, define suitable weak solutions, and derive equations (5) changed by flatting the boundary. For notational convenience, we denote for a point $x = (x',x_3) \in \mathbb{R}^3$ with $x' \in \mathbb{R}^2$

$$B_{x,r} = \{y \in \mathbb{R}^3 : |y - x| < r\}, \quad D_{x',r} = \{y' \in \mathbb{R}^2 : |y' - x'| < r\}.$$

For $x \in \overline{\Omega}$, we use the notation $\Omega_{x,r} = \Omega \cap B_{x,r}$ for some $r > 0$. If $x = 0$, we drop $x$ in the above notations, for instance $\Omega_{x,r}$ is abbreviated to $\Omega_r$. A solution $u$ to (2) is said to be regular at $z = (x,t) \in \overline{\Omega} \times I$ if $u \in L^\infty(\Omega_{x,r} \times (t-r^2,t))$ for some $r > 0$. In such case, $z$ is called a regular point. Otherwise we say that $u$ is singular at $z$ and $z$ is a singular point. We begin with some notations. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$. We denote by $N = N(\alpha,\beta,\ldots)$ a constant depending on the prescribed quantities $\alpha,\beta,\ldots$, which may change from line to line. For $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ denotes the usual Sobolev space, i.e., $W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), 0 \leq |\alpha| \leq k\}$. We write the average of $f$ on $E$ as $\overline{f}_E$, that is $\overline{f}_E = \frac{1}{|E|} \int_E f$. We suppose that $f$ belongs to a parabolic Morrey space $M_{2,\gamma}(Q_T)$ for some $0 < \gamma \leq 2$ equipped with the
norm

\[ \|f\|_{M_2,\gamma(Q_T)} = \sup \left\{ \left( \frac{1}{r^{1+2\gamma}} \int_{Q_{z,r}} |f|^2 \, dx \right)^{\frac{1}{2}} : z = (x,t) \in \overline{Q}_T, r > 0 \right\}, \]

where \( Q_{z,r} = (\Omega_{x,r} \times (t-r^2,t)) \cap Q_T \). We note that \( M_{2,\gamma}(Q_T) \) contains \( L^{\frac{2}{\gamma}}(Q_T) \). We make some assumptions on the boundary of \( \Omega \).

**Assumption 2.1.** Suppose that \( \Omega \) be a domain with \( C^2 \) boundary such that the following is satisfied: For each point \( x = (x',x_3) \in \partial \Omega \), there exist absolute constant \( N \) and \( r_0 \) independent of \( x \) such that we can find a Cartesian coordinate system \( \{y_i\}_{i=1}^3 \) with the origin at \( x \) and a \( C^2 \) function \( \varphi : D_{r_0} \to \mathbb{R} \) satisfying

\[ \Omega_{r_0} = \Omega \cap B_{x,r_0} = \{ y = (y',y_3) \in B_{x,r_1} : y_3 > \varphi(y') \} \]

and

\[ \varphi(0) = 0, \quad \nabla_y \varphi(0) = 0, \quad \sup_{D_{r_0}} |\nabla_y^2 \varphi| \leq N. \]

**Remark 2.1.** The main condition on Assumption 2.1 is the uniform estimate of the \( C^2 \)-norms of the function \( \varphi \) for each \( x \in \partial \Omega \). More precisely, there exists a sufficiently small \( r_1 \) with \( r_1 < r_0 \), where \( r_0 \) is the number in Assumption 2.1 such that for any \( r < r_1 \)

\[ \sup_{x \in \partial \Omega} \|\varphi\|_{C^2(D_r)} \leq N(1 + r + r^2). \]

Next lemma is related with Gagliardo-Nirenberg in [1, 17]:

**Lemma 2.2.** Let \( \Omega \) be a domain of \( \mathbb{R}^3 \) satisfying Assumption 2.1 and \( \int_{\Omega} u = 0 \). For every fixed number \( r \geq 1 \) there exists a constant \( N \) such that

\[ \|u\|_{L_r^p(\Omega)} \leq N \|\nabla u\|_{L_r^p(\Omega)}^{\frac{\theta}{p}} \|
abla u\|_{L_r^q(\Omega)}^{1-\theta}, \]

where \( \theta \in [0,1], p, q \geq 1 \), are linked by \( \theta = \left( \frac{1}{p} - \frac{1}{q} \right) \left( \frac{1}{p} - \frac{1}{r} + \frac{1}{q} \right)^{-1} \).

Next we recall suitable weak solutions for the Navier-Stokes equations (2) in three dimensions.

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain satisfying Assumption 2.1 and \( I = [0,T] \). We denote \( Q_T = \Omega \times I \). Suppose that \( f \) belongs to the Morrey space \( M_{2,\gamma}(Q_T) \) for some \( \gamma > 0 \). A pair of \((u,p)\) is a suitable weak solution to (2) if the following conditions are satisfied:

(a) The functions \( u : Q_T \to \mathbb{R}^3 \) and \( p : Q_T \to \mathbb{R} \) satisfy

\[ u \in L^\infty(I; L^2(\Omega)) \cap L^2(I; W^{1,2}(\Omega)), \quad p \in L^\frac{2}{\gamma}(\Omega \times I), \]

\[ \nabla^2 u, \nabla p \in L^\frac{2}{\gamma}(\Omega \times I). \]

(b) \( u \) and \( p \) solve the Navier-Stokes equations in \( Q_T \) in the sense of distributions and \( u \) satisfies slip boundary conditions on \( \partial \Omega \times I \).
(c) \(u\) and \(p\) satisfy the local energy inequality
\[
\int_{\Omega} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{t_0}^t \int_{\Omega} |\nabla u(x, t')|^2 \phi(x, t') dx dt' \\
\leq \int_{t_0}^t \int_{\Omega} \left(|u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) \cdot \nabla \phi + 2f \cdot \phi \right) dx dt'
\]
for all \(t \in I = (0, T)\) and for all non-negative functions \(\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})\), vanishing in a neighborhood of the set \(\Omega \times \{t = 0\}\).

Let \(x_0 \in \partial \Omega\). Under Assumption 2.1, we can represent \(\Omega_{x_0, r_0} = \Omega \cap \mathcal{B}_{x_0, r_0} = \{y = (y', y_3) \in B_{x_0, r_0} : y_3 > \varphi(y')\}\) where \(\varphi\) is the graph of \(C^2\) in Assumption 2.1. Flattening the boundary near \(x_0\), we introduce new coordinates \(x = \psi(y)\) by formulas
\[
\psi(y) = \psi_1(y), \quad \psi_2(y), \quad \psi_3(y) = y_3 - \varphi(y_1, y_2),
\]
where \(\psi\) is a bijection whose Jacobian is equal to 1. We note that the mapping \(y \mapsto \psi(y)\) straightens out \(\partial \Omega\) near \(x_0\) such that \(\Omega_{x_0, r}\) is transformed onto a subdomain \(\psi(\Omega_{x_0, r})\) of \(\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}\). We define \(v = u \circ \psi^{-1}\), \(\pi = p \circ \psi^{-1}\), and \(g = f \circ \psi^{-1}\) in \(\psi(\Omega_{x_0, r})\). Then using the change of variables (4), in this case, the outer unit normal vector is \((0, 0, -1)\) and unit tangent vectors are \((1, 0, 0)\), \((0, 1, 0)\). The equations (2) result in the following equations for \(v\) and \(\pi\):
\[
\begin{cases}
 v_t - \Delta v + (v \cdot \nabla) v + \nabla \pi = g, \\
v_3 = 0, \quad (\partial_3 v_1 = \varphi_{x_3}, \partial_3 v_3, \partial_3 v_3 = \varphi_{x_3})
\end{cases}
\]
in \(\psi(\Omega_{x_0, r})\),
\[
\begin{cases}
 \Delta = a_{ij}(x) \partial^2_{x_i, x_j} + b_i(x) \partial_{x_i}, \\
(\partial_{x_1} - \varphi_{x_1}, \partial_{x_3}, \partial_{x_2} = \varphi_{x_2}, \partial_{x_3}, \partial_{x_3}),
\end{cases}
\]
where \(a_{ij}\) and \(b_i\) are given as
\[
a_{ij}(x) = \delta_{ij}, \quad a_{i3}(x) = a_{3i}(x) = -\varphi_{x_i}, \quad b_i(x) = 0, \quad i = 1, 2,
\]
and
\[
a_{33}(x) = 1 + \sum_{i=1}^2 (\varphi_{x_i})^2, \quad b_3(x) = -\sum_{i=1}^2 \varphi_{x_i}.
\]
As mentioned in Remark 2.1, if we take a sufficiently small \(r_1\) with \(r_1 < r_0\), then (3) holds for any \(r < r_1\). In addition, the followings are satisfied:
\[
\begin{align*}
\frac{1}{2} |\nabla v(x, t)| &\leq |\nabla \psi(x, t)| \leq 2 |\nabla v(x, t)| \quad \text{for all} \ x \in \psi(\Omega_{x_0, 2r}), \\
B^+_{\psi(x_0, \frac{r}{2})} &\subset \psi(\Omega_{x_0, r}) \subset B^+_{\psi(x_0, 2r)}, \\
\psi^{-1}(B^+_{\psi(x_0, \frac{r}{2})}) &\subset \Omega_{x_0, r} \subset \psi^{-1}(B^+_{\psi(x_0, 2r)}).
\end{align*}
\]
From now on, we fix \( x_0 = 0 \) without loss of generality. We suppose that, as above, \( \psi \) is a coordinate transformation so that \( v, \pi \) satisfies (5) in \( \psi(\Omega_{x_0}) \).

**Remark 2.3.** Due to the suitability of \( u, p \) (see Definition 2.1), \( (v, \pi) \) solve (5) in a weak sense and satisfies the following local energy inequality: There exists \( r_2 \) with \( r_2 < r_0 \) where \( r_0 \) is the number in Assumption 2.1 such that

\[
\int_{\Omega_{x_0}} |v(x, t)|^2 \xi \xi dx + 2 \int_{t_0}^t \int_{\Omega_{x_0}} \left| \nabla \nabla v(x, t') \right|^2 \xi(x, t') dx dt' \\
\leq \int_{t_0}^t \int_{\Omega_{x_0}} \left( |v|^2 (\partial_t \xi + \Delta \xi) + (|v|^2 + 2\pi) v \cdot \nabla \xi + 2g \cdot v \xi \right) dx dt',
\]

where \( \xi \in C^{\infty}_0(B_r) \) with \( r < r_2 \) and \( \xi \geq 0 \), and \( \nabla \) and \( \Delta \) are differential operators in (6).

Next we define some scaling invariant functionals, which are useful for our purpose. Let \( B^+_r = B_r \cap \{ x \in \mathbb{R}^3 : x_3 > 0 \} \) and \( Q^+_r = B^+_r \times (-r^2, 0) \). As defined earlier, we also denote \( \Omega_r = \Omega \cap B_r \) and \( Q_r = \Omega_r \times (-r^2, 0) \). Let \( r_0 \) and \( r_1 \) be the numbers in Assumption 2.1 and Remark 2.1, respectively. For any \( r < r_1 \) and a suitable weak solution \((u, p)\) of (2) we introduce

\[
A(r) := \frac{1}{r^2} \int_{Q^+_r} |u(y, s)|^3 dy ds,
\]

\[
D(r) := \sup_{-r^2 \leq t \leq 0} \frac{1}{r} \int_{Q^+_r} |u(y, s)|^2 dy,
\]

\[
E(r) := \frac{1}{r} \int_{Q^+_r} |\nabla u(y, s)|^2 dy ds,
\]

\[
K(r) := \frac{1}{r} \left( \int_{-r^2}^t \left( \int_{Q^+_r} |u(y, s)|^p dy \right)^{\frac{2}{p}} ds \right)^{\frac{1}{2}}, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq q < \infty,
\]

\[
C(r) := \frac{1}{r^2} \int_{Q^+_r} |p(y, s)|^\frac{3}{2} dy ds.
\]

For a suitable weak solution \((v, \pi)\) and \( B^+_r \subset \psi(\Omega_{x_0}) \), we introduce

\[
\hat{A}(r) := \frac{1}{r^2} \int_{Q^+_r} |v(y, s)|^3 dy ds, \quad \hat{A}_a(r) := \frac{1}{r^2} \int_{Q^+_r} |v - (v)_r|^3 dy ds,
\]

\[
\hat{D}(r) := \sup_{-r^2 \leq t \leq 0} \frac{1}{r} \int_{B^+_r} |v(y, s)|^2 dy, \quad \hat{E}(r) := \frac{1}{r} \int_{Q^+_r} |\nabla v(y, s)|^2 dy ds,
\]

\[
\hat{K}(r) := \frac{1}{r} \left( \int_{-r^2}^t \left( \int_{B^+_r} |v(y, s)|^p dy \right)^{\frac{2}{p}} ds \right)^{\frac{1}{2}},
\]

\[
\hat{C}(r) := \frac{1}{r^2} \int_{Q^+_r} |\pi(y, s)|^\frac{3}{2} dy ds, \quad \hat{C}_a(r) := \frac{1}{r^2} \int_{Q^+_r} |\pi - (\pi)_r|^\frac{3}{2} dy ds,
\]

where \((v)_r = \int_{B^+_r} v(y, s) dy\). Next lemma shows relations between scaling invariant quantities above.
Lemma 2.4. Let $\Omega$ be a bounded domain satisfying Assumption 2.1 and $x_0 \in \partial \Omega$. Suppose that $(u, p)$ and $(v, \pi)$ are suitable weak solutions of (3) in $\Omega_3$ and (5) in $\psi(\Omega_{x_0}) \times I$, respectively, where $\psi$ is the mapping flattening the boundary in Assumption 2.1. Let $x = \psi(x_0)$. Then there exist sufficiently small $r_1$ and an absolute constant $N$ such that for any $4r < r_1$ the following inequalities hold:

\[
\frac{1}{N} E(r) \leq \hat{E}(2r) \leq NE(4r), \quad \frac{1}{N} A(r) \leq \hat{A}(2r) \leq NA(4r),
\]

\[
\frac{1}{N} K(r) \leq \hat{K}(2r) \leq NK(4r), \quad \frac{1}{N} C(r) \leq \hat{C}(2r) \leq NS(4r),
\]

\[
\frac{1}{N} D(r) \leq \hat{D}(2r) \leq ND(4r).
\]

Proof. We just show one of above estimates, since others follows similar arguments. For convenience, we denote $\Pi_r = \psi(\Omega_r) \times (-r^2, 0)$ and $\Pi_r^{-1} = \psi^{-1}(\Omega_r) \times (-r^2, 0)$. As indicated earlier, we take a sufficiently small $r_1$ such that (3), (7) and (8) hold. Then

\[
E(r) \leq \frac{N}{r} \int_{\Pi_r} |\nabla v|^2 \leq \frac{N}{r} \int_{\Pi_r} \hat{\nabla} v \leq \frac{N}{2r} \int_{Q_{2r}^+} |\hat{\nabla} v|^2 = N \hat{E}(2r).
\]

On the other hand,

\[
\hat{E}(2r) \leq \frac{1}{2r} \int_{Q_{2r}^+} |\nabla v|^2 \leq \frac{N}{2r} \int_{\Pi_{r}^{-1}} |\nabla u|^2 \leq \frac{N}{4r} \int_{Q_{4r}} |\nabla u|^2 = NE(4r).
\]

This completes the proof. \qed

Remark 2.5. We note that $f$ and $g$ have relations as in Lemma 2.4. To be more precise,

\[
\int_{Q_r} |f|^2 \leq N \int_{\Pi_r} |g|^2 \leq N \int_{Q_{2r}^+} |g|^2 \leq N \int_{\Pi_{r}^{-1}} |f|^2 \leq N \int_{Q_{4r}} |f|^2.
\]

Therefore, it is direct that $\|g\|_{M_{2, \gamma}((\Pi_r))} \leq N \|f\|_{M_{2, \gamma}(Q_r)}$.

In the sequel, for simplicity, we denote $\|f\|_{M_{2, \gamma}} = m_\gamma$.

3. Local regularity near boundary

In this section, we present the proof of Theorem 1.1. We first show a local regularity criterion for $v$ near the boundary.

Lemma 3.1. Let $\Omega$ be a bounded domain satisfying Assumption 2.1 and $x_0 \in \partial \Omega$. Suppose that $(v, \pi)$ is a suitable weak solution of (5) in $\psi(\Omega_{x_0}) \subset \mathbb{R}_+^3$, where $\psi$ is the mapping flattening the boundary in Assumption 2.1. Let $w = (y, t)$ with $y = \psi(x_0)$. Assume further that $g \in M_{2, \gamma}$ for some $\gamma \in (0, 2]$. Then there exist $\epsilon > 0$ and $r_1$ depending on $\gamma$, $\|g\|_{M_{2, \gamma}}$, such that if $\hat{A}^+(r) + \hat{C}^+(r) < \epsilon$ for some $r < r_1$, then $w$ is a regular point.
The proof of Lemma 3.1 is based on the following, which shows a decay property of \( v \) in a Lebesgue spaces. From now on, we denote \( \|g\|_{M_{2, \gamma}} = m_\gamma \), unless any confusion is expected.

**Lemma 3.2.** Let \( 0 < \theta < \frac{1}{2} \) and \( \beta \in (0, \gamma) \). Under the same assumption as in Lemma 3.1, there exist \( \varepsilon_1 > 0 \) and \( r_1 \) depending on \( \theta, \gamma, \beta \) and \( m_\gamma \) such that if \( \mathcal{A}(r) + \mathcal{C}(r) + m_\gamma r^\beta < \varepsilon_1 \) for some \( r \in (0, r_1) \), then

\[
\mathcal{A}(\theta r) + \mathcal{C}(\theta r) < N\theta^\alpha \left( \mathcal{A}(r) + \mathcal{C}(r) + m_\gamma r^\beta \right),
\]

where \( 0 < \alpha < 1 \) and \( N \) is a constant.

**Proof.** For convenience we denote \( \tau(r) := \mathcal{A}(r) + \mathcal{C}(r) + m_\gamma r^\beta \). Suppose the statement is not true. Then for any \( \alpha \in (0, 1) \) and \( N > 0 \), there exist \( z_n = (x_n, t_n) \), \( r_n \searrow 0 \) and \( \varepsilon_n \searrow 0 \) such that

\[
\tau(r_n) = \varepsilon_n, \quad \mathcal{A}(\theta r_n) + \mathcal{C}(\theta r_n) > N\theta^\alpha \varepsilon_n.
\]

Let \( w = (y, s) \) where \( y = \frac{1}{r_n}(x - x_n) \), \( s = \frac{1}{r_n^2}(t - t_n) \) and we define \( \hat{v}_n, \hat{\pi}_n \) and \( \hat{g}_n \) by \( \hat{v}_n(w) = \frac{1}{r_n}(v(z) - (v(z))_{x_n}) \), \( \hat{\pi}_n(w) = \frac{1}{r_n^2}(v(z) - (v(z))_{x_n}) \) and \( \hat{g}_n(w) = g(z) \), respectively. We also introduce scaling invariant functionals \( \mathcal{A}_n(\hat{v}_n, \theta) \) and \( \mathcal{C}_n(\hat{\pi}_n, \theta) \) as follows:

\[
\mathcal{A}_n(\hat{v}_n, \theta) := \frac{1}{\theta^2} \int_{Q_3^+} |\hat{v}_n - (\hat{v}_n)_{x_n}|^2 dw, \quad \mathcal{C}_n(\hat{\pi}_n, \theta) := \frac{1}{\theta^2} \int_{Q_3^+} |\hat{\pi}_n - (\hat{\pi}_n)_{x_n}|^2 dw.
\]

The change of variables lead to

\[
\begin{align*}
\varepsilon_n \hat{v}_n(w) &= r_n \hat{v}_n, \quad \varepsilon_n \hat{v}_n^2 = r_n^2 \hat{v}_n^2, \\
\varepsilon_n \partial_s \hat{v}_n(w) &= r_n \partial_s \hat{v}_n, \quad \varepsilon_n \hat{v}_n^2 \partial_s = r_n \hat{v}_n^2 \partial_s, \\
(\hat{v}_n)_{B_1^+}(s) &= 0, \quad (\hat{\pi}_n)_{B_1^+}(s) = 0, \quad s \in (-1, 0), \\
\tau_n(1) &= ||\hat{v}_n||_{L^2(Q_1^+)} + ||\hat{\pi}_n||_{L^2(Q_1^+)} + m_\gamma \frac{r^\beta_n}{\varepsilon_n} = 1, \\
\tau_n(\theta) := \mathcal{A}_n(\hat{v}_n, \theta) + \mathcal{C}_n(\hat{\pi}_n, \theta) \geq C\theta^\alpha,
\end{align*}
\]

where \( m_\gamma := ||g_n||_{M_{2, \gamma}} \). On the other hand, \( \hat{v}_n, \hat{\pi}_n \) solve the following system in a weak sense

\[
\begin{align}
\partial_t \hat{v}_n - \Delta \hat{v}_n + \varepsilon_n r_n (\hat{v}_n \cdot \nabla) \hat{v}_n + (\hat{v}_n \cdot \nabla) r_n a_n + \nabla \hat{\pi}_n = \frac{r_n^2}{\varepsilon_n} \hat{g}_n, & \quad \text{in } Q_1^+, \\
\hat{v}_n \cdot \nabla \hat{v}_n = 0, & \quad \text{on } Q_1^+ \cap \{x_3 = 0\} \times (-1, 0), \\
\hat{v}_3, n = 0, & \quad \text{on } B_{1/n} \cap \{x_3 = 0\} \times (-1, 0), \\
\partial_3 \hat{v}_n = \varphi_{x_1} \partial_3 \hat{v}_n, & \quad \text{on } B_{1/n} \cap \{x_3 = 0\} \times (-1, 0),
\end{align}
\]

where \( a_n = \langle v(z) \rangle_{r_n} = \int_{B_{1/n}} \langle v(y, \cdot) \rangle_{t_n} dy \).

Since \( \tau_n(1) = 1 \), we have following weak convergence:

\[
\begin{align}
\hat{v}_n \rightharpoonup \hat{v}, & \quad \text{in } L^3(Q_1^+), \\
\hat{\pi}_n \rightharpoonup \hat{\pi}, & \quad \text{in } L^2(Q_1^+), \\
(\hat{v})_{B_1^+}(s) = 0, & \quad (\hat{\pi})_{B_1^+}(s) = 0.
\end{align}
\]
Then, from (10) and (12),
\[ \tau(1) = \hat{\lambda}^2(1) + \hat{C}^2(1) \leq 1. \]

According to the definition of \( m_r \), we have
\[
\frac{r_n^2}{\varepsilon_n} \| \hat{g}_n \|_{L^2(Q_t^+)} \leq \frac{r_n^2}{\varepsilon_n} m_r r_n^{\gamma - 2} \leq \frac{m_r r_n^\beta}{\varepsilon_n} r_n^{\gamma - \beta} \leq r_n^{\gamma - \beta} \to 0 \quad \text{as } n \to \infty.
\]

Since \( |r_n a_n| \) be a bound, without loss of generality it may be assumed that:
\[ r_n a_n \to b \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad |b| \leq M. \]

Using (10) and (13), we take
\[
\int_{Q_t^+} (-\hat{v}_n \cdot \partial_s X) dw = \int_{Q_t^+} \left\{ \hat{\nu}_n \cdot \hat{\Delta} X + \hat{\nu}_n \cdot (\varepsilon_n r_n \hat{v}_n) \hat{\nabla} X + \hat{\nu}_n \cdot (r_n a_n) \hat{\nabla} X + \frac{r_n^2}{\varepsilon_n} \hat{g}_n \cdot X \right\} dw
\]
\[
\leq N(M) \| X \|_{L^2((-1,0);W^{2,2}(B_t^+))}
\]
for all \( X \in C^1((-1,0);W^{2,2}(B_t^+)) \).

Therefore, \( \partial_s \hat{v}_n \) is uniformly bounded in \( L^2((-1,0);(W^{2,2}(B_t^+))^\prime) \) and we also have
\[ \partial_s \hat{v}_n \to \partial_s \hat{v} \quad \text{in } L^2((-1,0);(W^{2,2}(B_t^+))^\prime). \]

From the local energy inequality (9), we obtain for every \( \sigma \in (-1,0) \)
\[
\int_{B_t^+} |\hat{v}_n(y, \sigma)|^2 \xi(y, \sigma) dy + 2 \int_{-1}^\sigma \int_{B_t^+} |\hat{\nabla} \hat{v}_n|^2 \xi dy ds
\]
\[
\leq \int_{-1}^\sigma \int_{B_t^+} \left\{ |\hat{v}_n|^2 (\partial_s \xi + \hat{\Delta} \xi) + r_n |\hat{v}_n|^2 (\varepsilon_n \hat{v}_n + a_n) \cdot \hat{\nabla} \xi \right. 
\]
\[
\left. + \frac{r_n}{\varepsilon_n} \hat{g}_n \cdot \hat{v}_n \right\} dy ds
\]
for all \( \xi \in C_0^\infty(B_r) \). Recalling (10), (13) and (14), we deduce from (16) the bound
\[ \text{ess sup}_{s \in (-3/4,0)} \| \hat{v}_n(s) \|_{L^2(B(3/4))} + \| \hat{\nabla} \hat{v}_n \|^2_{L^2(Q_{3/4}^+)} \leq N(M). \]

The Gagliardo-Nirenberg inequality and (17) yield estimate
\[ \| \hat{v}_n \|_{L^\infty(Q_{3/4}^+)} \leq N(M). \]

Using the standard compactness arguments and (15), (17) and (18), we conclude following convergence:
\[ \hat{v}_n \to \hat{v} \quad \text{in } L^3(Q_{3/4}^+). \]
Next we observe that ĭ and ĭ̂ solve the following perturbed Stokes system
\[ \partial_t \hat{\nu} - \hat{\Delta} \hat{\nu} + \hat{\nabla} \hat{\pi} = 0, \quad \text{div} \hat{\nu} = 0 \quad \text{in} \quad Q^+_1 \]
with
\[ \hat{\nu}_3 = 0, \quad \partial_3 \hat{v}_1 = \varphi_{x_3} \partial_3 \hat{v}_3 \quad \text{and} \quad \partial_3 \hat{v}_2 = \varphi_{x_3} \partial_3 \hat{v}_3 \quad \text{on} \quad (B_1 \cap \{ x_3 = 0 \}) \times (-1, 0). \]

Indeed, by the Hölder’s inequality, we have
\[ \left\| (\hat{\nu}_n \cdot \hat{\nabla}) \hat{v}_n \right\|_{L^\infty(B_{7/8}^+)} \leq N \left\| \hat{\nu}_n \right\|_{L^2(B_{7/8}^+)} \left\| \hat{v}_n \right\|_{L^2(B_{7/8}^+)} \]
\[ \leq N \left\| \hat{\nu}_n \right\|_{L^2(B_{7/8}^+)} \left\| \hat{v}_n \right\|_{L^2(B_{7/8}^+)} \]
\[ \leq N \left\| \hat{\nu}_n \right\|_{L^2(B_{7/8}^+)} \]

Therefore,
\[ (20) \]
\[ \left\| (\hat{\nu}_n \cdot \hat{\nabla}) \hat{v}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} \leq N. \]

Moreover, ĭ̂ and ĭ̂̂ solve the following problem:
\[ \partial_t \hat{\nu}_n - \hat{\Delta} \hat{\nu}_n + \hat{\nabla} \hat{\pi}_n = -\varepsilon_n r_n (\hat{\nu}_n \cdot \hat{\nabla}) \hat{v}_n - (\hat{\nu}_n \cdot \hat{\nabla}) r_n a_n + \frac{\hat{a}_n}{\varepsilon_n} \hat{g}_n \quad \text{in} \quad Q_{5/6}^+ \]
with
\[ \hat{v}_{3,n} = 0, \quad \partial_3 \hat{v}_{1,n} = \varphi_{x_3} \partial_3 \hat{v}_{3,n} \quad \text{and} \quad \partial_3 \hat{v}_{2,n} = \varphi_{x_3} \partial_3 \hat{v}_{3,n} \quad \text{on} \quad (B_{5/6} \cap \{ x_3 = 0 \}) \times \left(-\left(\frac{5}{6}\right), 0\right). \]

Due to the local boundary estimate for the Stokes system in Lemma 4.2, we have the following estimate for ĭ̂̂ and ĭ̂̂̂:
\[ \left\| \partial_3 \hat{v}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} + \left\| \hat{\nabla}^2 \hat{v}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} + \left\| \hat{\nabla} \hat{\pi}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} \]
\[ \leq N \left( \varepsilon_n r_n \left\| \hat{v}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} + \frac{r_n^2}{\varepsilon_n} \left\| \hat{g}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} \right) \]
\[ + \left\| \hat{\nu}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} \left\| \hat{\nabla} \hat{v}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} + \left\| \hat{\nabla} \hat{\pi}_n \right\|_{L^{\frac{2}{3}}(Q_{5/6}^+)} \] 
\[ \leq N (1 + \varepsilon_n r_n), \]
where we used (10), (13), (17) and (20). Thus, we get
\[ \hat{\Delta} \hat{v}_n, \hat{\nabla} \hat{\pi}_n \in L^{\frac{2}{3}}(Q_{4/5}^+). \]

According to estimates of the perturbed stokes system near boundary in [29], ĭ̂ is Hölder continuous in \( Q_{1/2}^+ \) with the exponent \( \alpha \). Then, by Hölder continuity
of \( \hat{v} \) and strong convergence of the \( L^3 \)-norm of \( \hat{v}_n \), we obtain

\[
\hat{A}(\hat{v}_n, \theta) \to \hat{A}(\hat{v}, \theta), \quad \hat{A}^\circ(\hat{v}, \theta) \leq N_1 \theta^\alpha,
\]

where \( N_1 \) is an arbitrary constant.

Let \( \overline{B}^+ \) be a domain with smooth boundary such that \( B_{4/5}^+ \subset \overline{B}^+ \subset B_{6/5}^+ \), and \( Q^+_r := B^+ \times (-5/6)^2, 0) \). Now we consider the following initial and boundary problem of \( \tau_n, \pi_n \)

\[
\partial_0 \tilde{\tau}_n - \Delta \tilde{\tau}_n + \tilde{\nabla} \tilde{\pi}_n = -\varepsilon_n r_n (\tilde{v}_n \cdot \tilde{\nabla}) \tilde{v}_n - (\tilde{v}_n \cdot \tilde{\nabla}) r_n a_n + \tilde{g}_n \quad \text{in} \; Q_r^+,
\]

\[
(\tilde{\tau}_n)(\overline{\tau}^+) = 0, \quad (\tilde{\pi}_n)(\overline{\tau}^+) = 0, \quad s \in \left( -\left(\frac{5}{6}\right)^2, 0 \right),
\]

\[
\tilde{\tau}_{3,n} = 0, \quad \partial_0 \tilde{\tau}_{1,n} = \varphi_{x_{1}} \partial_{1} \tilde{\tau}_{4,n} \quad \text{on} \; \partial B^+ \times \left( -\left(\frac{5}{6}\right)^2, 0 \right),
\]

\[
\tilde{\tau}_n = 0 \quad \text{on} \; \overline{B}^+ \times \left\{ s = -\left(\frac{5}{6}\right)^2 \right\}.
\]

Using the global estimate of perturbed Stokes system (see [29, Lemma 3.1]), we get

\[
\begin{aligned}
&\| \partial_0 \tilde{\tau}_n \|_{L^\infty \left( Q_r^+ \right)} + \| \tilde{\tau}_n \|_{L^2 \left( (-5/6)^2, 0; W^{1, 2}_0 \right)} \\
&+ \| \tilde{\pi}_n \|_{L^2 \left( (-5/6)^2, 0; W^{1, 2}_0 \right)} \\
\leq & \; N \varepsilon_n r_n \left\| \left( v_n \cdot \tilde{\nabla} \right) v_n \right\|_{L^\infty \left( Q_r^+ \right)} + N \left\| \left( v_n \cdot \tilde{\nabla} \right) r_n a_n \right\|_{L^2 \left( Q_r^+ \right)} \\
&+ N r_n \varepsilon_n \left\| \tilde{g}_n \right\|_{L^2 \left( \overline{Q}_r^+ \right)} \\
& \leq \; N(1 + \varepsilon_n r_n + r_n \gamma^{-\beta}).
\end{aligned}
\]

Next, we define \( \tilde{\tau}_n = \tilde{v}_n - \tau_n, \tilde{\pi}_n = \tilde{\pi}_n - \pi_n \). Then it is straightforward that \( \tilde{\tau}_n \) and \( \tilde{\pi}_n \) solve

\[
\partial_0 \tilde{\tau}_n - \Delta \tilde{\tau}_n + \tilde{\nabla} \tilde{\pi}_n = 0, \quad \text{div} \; \tilde{v}_n = 0 \quad \text{in} \; Q_r^+,
\]

\[
\tilde{\tau}_{3,n} = 0, \quad \partial_0 \tilde{\tau}_{1,n} = \varphi_{x_{1}} \partial_{1} \tilde{\tau}_{4,n} \quad \text{on} \; (B^+ \cap \{x_3 = 0\}) \times \left( -\left(\frac{4}{5}\right)^2, 0 \right),
\]

\[
\left\| \tilde{\nabla} \tilde{\tau}_n \right\|_{L^\infty \left( Q_r^+ \right)} + \left\| \tilde{\nabla} \tilde{\pi}_n \right\|_{L^2 \left( Q_r^+ \right)} \leq N(1 + \varepsilon_n r_n + r_n \gamma^{-\beta}),
\]

and we obtain

\[
\left\| \tilde{\nabla} \tilde{\pi}_n \right\|_{L^2 \left( Q_r^+ \right)} \leq N(1 + \varepsilon_n r_n + r_n \gamma^{-\beta}).
\]
Next, let $\tilde{C}_1(\tau_n, \theta) = \frac{1}{\theta^2} \left( \int_{-\theta^2}^0 \left( \int_{B^*_\theta} \left| \nabla \tau \right|^9 dy \right) ds \right)^{\frac{1}{9}} \cdot$ By the Poincaré inequality, we have

$$\tilde{C}_a^2(\tau_n, \theta) \leq N_2 \left( \tilde{C}_1(\tau_n, \theta) + \tilde{C}_1(\bar{\tau}_n, \theta) \right).$$

We note that $\tilde{C}_1(\tau_n, \theta)$ goes to zero as $n \to \infty$ because of (22). On the other hand, using the H"{o}lder inequality, we have

$$\tilde{C}_1 \leq \theta^2 \left( \int_{-\theta^2}^0 \left( \int_{B^*_\theta} \left| \nabla \tau \right|^9 dy \right) ds \right)^{\frac{1}{9}} \leq N\theta^\alpha(1 + \epsilon_n r_n + r_n^{\gamma - \beta}).$$

Summing up, we obtain

$$(23) \lim_{n \to \infty} \tilde{C}_a^2(\tau_n, \theta) \leq \lim_{n \to \infty} N\theta^\alpha(1 + \epsilon_n r_n + r_n^{\gamma - \beta}) \leq N_2\theta^\alpha.$$  

Thus, we obtain from (10) that

$$N\theta^\alpha \leq N_1\theta^\alpha + \lim_{n \to \infty} \tilde{C}_a^2(\theta).$$

Consequently, if we take a constant $N$ in (10) bigger than $2(N_1 + N_2)$ in (21) and (23), this leads to a contradiction, since

$$2(N_1 + N_2)\theta^\alpha \leq N\theta^\alpha \leq \lim_{n \to \infty} \tau_n(\theta) \leq (N_1 + N_2)\theta^\alpha.$$  

This deduces the lemma. \( \Box \)

Since Lemma 3.2 is the crucial part of the proof of Lemma 3.1, we present only a brief sketch of the streamline of Lemma 3.1.

**Proof of Lemma 3.1.** We note that due to Lemma 3.2 there exists a positive constant $\alpha < 1$ such that

$$\tilde{A}^+(r) + \tilde{C}_a^2(r) < N\theta^\alpha \left( \tilde{A}^+(\rho) + \tilde{C}_a^2(\rho) + m_\gamma r^\alpha \right), \quad r < \rho < r_1,$$

where $r_1$ is the number in Lemma 3.1. For any $x \in B_{r_1/2}^+$ and for any $r < r_1/4$, let $\tilde{B}(r) := \tilde{A}^+(r) + \tilde{C}_a^2(r)$. By Lemma 3.2, we obtain

$$\tilde{B}(\theta r) \leq N\theta^\alpha \tilde{B}(r) \leq N\theta^{1+\alpha} \tilde{B}(r).$$

Thus, we have

$$\tilde{B}(\theta^k r) \leq N \left( \theta^{1+\alpha} \right)^k \tilde{B}(r).$$

In case of $\rho = \theta^k r$, we get $\tilde{A}^+(\rho) \leq \tilde{B}(\rho) \leq N\rho^{1+\alpha}$. Next we consider the case that $\theta^k r < \rho < \theta^{k-1} r$. For the scaled $L^\beta$ norm of $v$,

$$\tilde{A}^+(\theta^k r) = \left( \frac{1}{(\theta^k r)^2} \int_{Q^+_{\theta^k r}} |v|^3 \right)^{\frac{1}{3}} \leq \theta^{-\frac{1}{3}} \left( \frac{1}{\rho^2} \int_{Q^+_{\rho}} |v|^3 \right)^{\frac{1}{3}} = \theta^{-\frac{1}{3}} \tilde{A}^+(\rho).$$
In the same way, we get \( \hat{C}^\sharp(\theta^k r) \leq \theta^{-\frac{1}{r}} \hat{C}^\sharp(\rho) \) and therefore
\[
\hat{B}(\rho) \leq \theta^\sharp \hat{B}(\theta^k r) \leq N \theta^\sharp (\theta^k)^{1+\alpha} \hat{B}(r) \leq N \theta^\sharp \hat{B}(r) \left( \frac{r}{\rho} \right)^{1+\alpha} \leq N \rho^{1+\alpha}.
\]
Thus, we can show that \( \hat{A}^\sharp_c(r) \leq N r^{1+\alpha} \), where \( N \) is an absolute constant independent of \( v \). Hölder continuity of \( v \) is a direct consequence of this estimate, which immediately implies that \( v \) is also Hölder continuous locally near boundary by the Morrey & Campanato lemma. This completes the proof. \( \square \)

Next lemma is an estimate of the pressure.

**Lemma 3.3.** Suppose \( 0 < 2r \leq \rho \). Then
\[
\hat{C}(r) \leq N \left( \frac{r}{\rho} \right) \left( \hat{A}_u(\rho) + \rho^\sharp (r^{(r+1)} m(x)) \right) + N \left( \frac{r}{\rho} \right) \hat{C}(\rho).
\]

**Proof.** Define \( v^* = (v_1^*, v_2^*, v_3^*) \) by
\[
e^*_1(x,t) = \begin{cases} v_1(x,t) & \text{if } x_3 \geq 0, \\ v_1(x^*,t) & \text{if } x_3 < 0, \end{cases}
\]
\[
e^*_2(x,t) = \begin{cases} v_2(x,t) & \text{if } x_3 \geq 0, \\ v_2(x^*,t) & \text{if } x_3 < 0, \end{cases}
\]
\[
e^*_3(x,t) = \begin{cases} v_3(x,t) & \text{if } x_3 \geq 0, \\ -v_3(x^*,t) & \text{if } x_3 < 0, \end{cases}
\]
where \( x^* = (x_1, x_2, -x_3) = (y_1, y_2, -y_3 + \varphi(y_1, y_2)) \). We consider \( \pi^*, -(v^* \cdot \nabla)v^* \), \( g^* \) as the even-even-odd extension. Then, we construct \( (v^*, \pi^*) \) as the solution of the Stokes system in \( \mathbb{R}^3 \times (0, T) \):
\[
v^* - \Delta v^* + \nabla \pi^* = -(v \cdot \nabla)v^* + g^*
\]
with initial data \( v^*(x, 0) = v_0^*(x) \).

Let \( \phi(x) \geq 0 \) be standard cut-off function such that \( 0 \leq \phi \leq 1 \), \( \phi \equiv 1 \) in \( B_\rho \), \( \phi = 0 \) outside on \( B_\frac{1}{2} \). The divergence \( \langle \hat{\nabla} \rangle \) of (25) gives in \( \mathbb{R}^3 \times (0, T) \)
\[
-\Delta \pi^* = \hat{\nabla} \cdot \hat{\nabla} (v^* \otimes v^*) - \hat{\nabla} \cdot g^*
\]
in the sense of distribution. Let
\[
\pi_1(x,t) = \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|} \left\{ \hat{\nabla} \cdot \hat{\nabla} [(v^* - (v^*)_\rho) \otimes (v^* - (v^*)_\rho)] \phi - \hat{\nabla} \cdot (g^* \phi) \right\} (y,t) dy.
\]
Then, by Calderon-Zygmund and potential estimates,
\[
\frac{r}{\rho^2} \int_{B_\rho} |\pi_1|^2 dx \leq \frac{1}{r} \int_{B_\rho} |\pi_1|^2 dx
\]
\[
\leq \frac{N}{r^2} \int_{B_r} |v^* - (v^*)_\rho|^3 \, dx + \frac{N}{r^2} \rho^\frac{7}{6} \left( \int_{B_r} |g^*|^2 \, dx \right)^\frac{3}{4}.
\]

We set \( \pi_2(x,t) := \pi^*(x,t) - \pi_1(x,t) \). It is direct that \( \hat{\Delta} \pi_2 = 0 \), \( \hat{\nabla} \cdot v^* = 0 \) in \( B_2 \), and thus we get
\[
\frac{r}{r^2} \int_{B_r} |\pi_2|^2 \, dx \leq \frac{N}{r^2} \rho^\frac{7}{6} \int_{B_2} |\pi_2|^2 \, dx
\]
\[
\leq \frac{N}{r^2} \rho^\frac{7}{6} \int_{B_r} |\pi|^2 \, dx + \frac{N}{r^2} \rho^\frac{7}{6} \int_{B_r} |\pi_1|^2 \, dx.
\]

Integrating the first term of the right side in (26) in time, and using \( \hat{\Delta} \pi_2 = 0 \), \( \hat{\nabla} \cdot v^* = 0 \) in \( B_2 \) and thus we get
\[
\int_{t_0}^0 \frac{r}{r^2} \rho^\frac{7}{6} \left( \int_{B_r} |g^*|^2 \, dx \right)^\frac{3}{4} \, dt \leq Nr^{-\frac{2}{3}} \rho^{\frac{7}{6}} \frac{7}{6} m^2_\gamma,
\]

we obtain
\[
\frac{1}{r^2} \int_{Q_r} |\pi|^2 \, dx dt \leq \frac{1}{r^2} \int_{Q_r} |\pi_1|^2 + |\pi_2|^2 \, dx dt
\]
\[
\leq N \left( \frac{r}{r^2} \right)^\frac{7}{6} \left( \int_{B_r} |v^* - (v^*)_\rho|^3 \, dx + \rho^\frac{7}{6} (\gamma + 1) m^2_\gamma \right)
\]
\[
+ N \left( \frac{r}{r^2} \right)^\frac{7}{6} \int_{B_r} |\pi|^2 \, dx dt.
\]

This completes the proof. \( \square \)

We estimate the scaled \( L^3 \)-norm of suitable weak solutions.

**Lemma 3.4.** Under the same assumption as in Lemma 3.1. Let \( p, q \) be satisfied \( \frac{3}{p} + \frac{2}{q} = 2 \) and \( 1 \leq q < \infty \), there exists \( r_1 \) such that for any \( r < r_1 \)
\[
\hat{A}_a(r) \leq N \left( \hat{D}(r) + \hat{E}(r) \right) \hat{K}(r).
\]

**Proof.** Using the Hölder inequality, we obtain
\[
\int_{B^+_r} |v - (v)_r|^3 \, dy
\]
\[
\leq N \left( \int_{B^+_r} |v|^2 \, dy \right)^\frac{1}{3} \left( \int_{B^+_r} |v - (v)_r|^6 \, dy \right)^\frac{1}{6} \left( \int_{B^+_r} |v|^p \, dy \right)^\frac{1}{p}
\]
\[
\leq N \left( \int_{B^+_r} |v|^2 \, dy \right)^\frac{1}{3} \left( \int_{B^+_r} |\hat{\nabla} v|^2 \, dy \right)^{-\frac{1}{3}} \left( \int_{B^+_r} |v|^2 \, dy \right)^{-\frac{1}{4}} \left( \int_{B^+_r} |v|^p \, dy \right)^\frac{1}{4}
\]
\[
= N \left( \int_{B^+_r} |v|^2 \, dy \right)^\frac{1}{3} \left( \int_{B^+_r} |\hat{\nabla} v|^2 \, dy \right)^{-\frac{1}{3}} \left( \int_{B^+_r} |v|^p \, dy \right)^\frac{1}{4}.
\]
Indeed, it is straightforward via the Hölder inequality that obtain

$$\|v\|_{L^2} \leq N \left( \int_{B^+} |v|^2 dy \right) \left( \int_{B^+} |v|^p dy \right)^{\frac{1}{p}}$$,

where general Sobolev imbedding is used. Integrating in time, we get

$$\int_{S^+} |v - (v)_r|^3 dy dt$$

$$\leq N \left( \sup_{t \leq 0} \int_{B^+} |v|^2 dy \right)^{\frac{1}{2}} \int_{-r^2}^0 \left( \int_{B^+} |\nabla v|^2 dy \right)^{1-\frac{1}{2}} \left( \int_{B^+} |v|^p dy \right)^{\frac{1}{2}} \dt$$

$$+ N \left( \sup_{t \leq 0} \int_{B^+} |v|^2 dy \right)^{\frac{1}{2}} \int_{B^+} |v|^p dy \right)^{\frac{1}{2}} \dt$$

$$\leq N \left( \sup_{t \leq 0} \int_{B^+} |v|^2 dy \right)^{\frac{1}{2}} \left( \int_{Q^+} |\nabla v|^2 dy dt \right)^{1-\frac{1}{2}} \left( \int_{B^+} |v|^p dy \right)^{\frac{1}{2}} \dt$$

$$+ N \left( \sup_{t \leq 0} \int_{B^+} |v|^2 dy \right)^{\frac{1}{2}} \left( \int_{B^+} |v|^p dy \right)^{\frac{1}{2}} \dt$$,

where Hölder inequality is used. Dividing both sides by $r^2$, we have

$$\tilde{A}_a(r) \leq N \left( \tilde{D}^\frac{1}{2} (r) \tilde{E}^{1-\frac{1}{2}}(r) \tilde{K}(r) + \tilde{D}(r) \tilde{K}(r) \right).$$

For the first term, applying Young’s inequality, we deduce the lemma. $$\square$$

Next we observe that for $0 < 2r \leq \rho$

$$\tilde{A}(r) \leq N \left( \frac{\rho}{r} \right)^2 \tilde{A}_a(\rho) + N \left( \frac{\rho}{\rho} \right) \tilde{A}(\rho).$$

Indeed, it is straightforward via the Hölder inequality that obtain

$$\tilde{A}(r) \leq N \frac{1}{r^2} \int_{Q^+} |v - (v)_r|^3 + |(v)_r|^3 dy ds \leq N \left( \frac{\rho}{r} \right)^2 \tilde{A}_a(\rho) + N \left( \frac{\rho}{\rho} \right) \tilde{A}(\rho).$$

**Remark 3.5.** From local energy inequality (9), we obtain

$$\tilde{D} \left( \frac{r}{2} \right) + \tilde{E} \left( \frac{r}{2} \right) \leq N \left( \tilde{A}^\frac{1}{2}(r) + \tilde{A}(r) + \tilde{A}^\frac{1}{2}(r) \tilde{C}(r) + r \int_{S^+} |g|^2 dw \right),$$

$$\leq N \left( \tilde{A}^\frac{1}{2}(r) + \tilde{A}(r) + \tilde{A}^\frac{1}{2}(r) \tilde{C}(r) + r^{2}m_2^2 \right),$$

$$\leq N \left( 1 + \tilde{A}(r) + \tilde{C}(r) + r^{2}m_2^2 \right).$$

Now we are ready to present the proof of Theorem 1.1.
Proof of Theorem 1.1. Let $4r < \rho$. We consider $\hat{A}(r) + \hat{C}(r)$. Due to (28), (24), (27) and (29), we obtain
\[
\hat{A}(r) + \hat{C}(r) \leq N \left( \frac{r}{\rho} \right)^2 \hat{K}(\rho) (\hat{A}(\rho) + \hat{C}(\rho)) + N \left( \frac{r}{\rho} \right)^2 (1 + \rho^{2\gamma+2}m_2^2) \hat{K}(\rho) + N \left( \frac{r}{\rho} \right)^2 \rho^{2(\gamma+1)m_2^2}.
\]
We choose $\theta \in (0,1/4)$ such that $C \theta < 1/4$ where $N$ is an absolute constant in the above inequality. Now we fix $r_0 < \min \left\{ 1, \frac{m_2}{m_1} \left( \frac{m_1}{\rho} \right)^{2/3} \right\}$ such that $\hat{K}(r) < \frac{\theta^2}{1+8\theta} \min\{1, \varepsilon\}$ for all $r \leq r_0$. By replacing $r$, $\rho$ by $\theta r$ and $r$, respectively, we obtain
\[
\hat{A}(\theta r) + \hat{C}(\theta r) \leq \frac{1}{2} \left( \hat{A}(r) + \hat{C}(r) \right) + \frac{\varepsilon}{2}, \quad \forall r \leq r_0.
\]
By iterating, we have
\[
\hat{A}(\theta^k r) + \hat{C}(\theta^k r) \leq \left( \frac{1}{2} \right)^k \left( \hat{A}(r) + \hat{C}(r) \right) + \frac{\varepsilon}{2}, \quad \forall r \leq r_0.
\]
Thus, for $k$ sufficiently large, $\hat{A}(\theta^k r) + \hat{C}(\theta^k r) \leq \varepsilon$. By Lemma 3.1, this completes the proof.

4. Appendix

In this section, we provide the existence of suitable weak solutions and Stokes estimates of the Stokes system with slip boundary conditions.

4.1. Existence of suitable weak solutions

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $I = (0,T)$. We consider the Stokes system with slip boundary conditions:
\[
\begin{cases}
  u_t - \Delta u + \nabla p = f - (w \cdot \nabla)v, & \text{div} \ u = 0 \quad \text{in} \ Q_T = \Omega \times I, \\
  u \cdot n = 0, & n \cdot (u, p, \tau) = 0 \quad \text{on} \ \partial \Omega \times I, \\
  u = u_0 & \text{at} \ t = 0,
\end{cases}
\]
where $w \in C^{\infty}(Q_T)$, $f \in L^2(Q_T)$ and $u_0 \in H^2(\Omega)$, $v \in W^{2,1}_0(Q_T) = L^2(I : H^2(\Omega)) \cap H^1(I : L^2(\Omega))$. The Banach space $L^2(\Omega)^3$ admits the Helmholtz decomposition:
\[
L^2(\Omega)^3 = J^2(\Omega) \oplus G^2(\Omega),
\]
where
\[
J^2(\Omega) = \overline{C^{\infty}_{0,\sigma}(\Omega)}^{L^2(\Omega)}, \quad G^2(\Omega) = \{ \nabla p \mid p \in \dot{W}^{1,2}(\Omega) \},
\]
\[
C^{\infty}_{0,\sigma}(\Omega) = \{ u \in C^{\infty}(\Omega)^3 \mid \nabla \cdot u = 0 \text{ in } \Omega \},
\]
\[
\dot{W}^{1,2}(\Omega) = \{ p \in L^2_{loc}(\Omega) \mid \nabla p \in L^2(\Omega)^3 \}.
\]
It should be noted that since boundary is $C^{2,1}$-hypersurface, $J^2(\Omega)$ is characterized as

$$J^2(\Omega) = \{ u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0 \mbox{ in } \Omega, \ u \cdot n = 0 \mbox{ on } \partial \Omega \}. $$

Let $P$ be a continuous projection from $L^2(\Omega)^3$ onto $J^2(\Omega)$. By using $P$ we shall define the Stokes operator with slip boundary conditions $A$ by

$$ Au = -P \Delta u \quad \mbox{for } u \in D(A), $$

$$ D(A) = J^2(\Omega) \cap \{ u \in W^{2,2}(\Omega)^3 \mid u \cdot T(u, p) \cdot \tau = 0 \}. $$

Now, we consider operator form of system:

$$ A \hat{u} = P(f - (w \cdot \nabla)v), \quad u(0) = u_0. $$

Since $A$ is the generator of an analytic semigroup in $L^2(\Omega)$, solving (31) is equivalent to show that mapping

$$ F(v) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}P(f - (w \cdot \nabla)v)ds $$

has a unique fixed point.

**Lemma 4.1.** Let $T \in (0, \infty)$. There exists a unique solution

$$ u \in L^2((0, \infty) : H^2(\Omega)) \cap H^1((0, \infty) : L^2(\Omega)) $$

satisfies

$$ u_t + Au = P(f - (w \cdot \nabla)v), \quad u(0) = u_0. $$

**Proof.** Let $F$ is mapping such that $F(v) = u$. Then

$$ \| u \|_{W^{2,1}_2(Q_T)} = \| F(v) \|_{W^{2,1}_2(Q_T)} $$

$$ \leq N \left\{ \| u_0 \|_{W^{2,1}_2(Q_T)} + \| f - (w \cdot \nabla)v \|_{L^2(Q_T)} \right\} $$

$$ \leq N \left\{ \| u_0 \|_{W^{2,1}_2(Q_T)} + \| f \|_{L^2(Q_T)} + \| w \|_{L^\infty(Q_T)} \| \nabla v \|_{L^2(Q_T)} \right\}. $$

Thus, $F$ is well-defined on $W^{2,1}_2(Q_T)$. For $v_1, v_2 \in W^{2,1}_2(Q_T)$,

$$ \| F(v_1) - F(v_2) \|_{H^2(\Omega)} \leq \int_0^t \left\| \nabla e^{-(t-s)A} \right\|_{L^2(\Omega)} ds $$

$$ \leq \int_0^t N(t - s)^{-\frac{1}{2}} \| \nabla P((w \cdot \nabla)(v_2 - v_1)) \|_{L^2(\Omega)} ds $$

$$ = N t^{-\frac{1}{2}} \| \nabla P((w \cdot \nabla)(v_2 - v_1)) \|_{L^2(\Omega)}. $$

Taking integral on $[0, t]$ for small $t$,

$$ \| F(v_1) - F(v_2) \|_{L^2(0,t)H^2(\Omega)} $$

$$ \leq N \| t^{-\frac{1}{2}} * \| \nabla P((w \cdot \nabla)(v_2 - v_1)) \|_{L^2(\Omega)} \|_{L^2(0,t)} $$

$$ \leq N \| t^{-\frac{1}{2}} \|_{L^1(0,t)} \| \nabla P((w \cdot \nabla)(v_2 - v_1)) \|_{L^2(0,t)L^2(\Omega)}.$$
Similarly, taking integral on $[0,t]$, therefore, we have

$$\|(F(v_1) - F(v_2))\|_{L^2(\Omega)} \leq \|P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(\Omega)}$$

and thus, taking $L^2$-norm, we have

$$\|(F(v_1) - F(v_2))\|_{L^2(\Omega)} \leq \|P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(\Omega)}$$

$$+ \int_0^t C(t-s)^{-\frac{1}{2}} \|\nabla P((w \cdot \nabla)(v_2 - v_1))\|_{L^2(\Omega)}ds.$$ 

Similarly taking integral on $[0,t]$ for small $t$,

$$\|F(v_1) - F(v_2)\|_{W^{2,1}(Q_T)} \leq N\sqrt{t}\|v_2 - v_1\|_{L^2(0,t;H^2(\Omega))}.$$ 

Therefore,

$$\|F(v_1) - F(v_2)\|_{W^{2,1}(Q_T)} \leq N\sqrt{t}\|v_2 - v_1\|_{W^{2,1}(Q_T)}.$$ 

Hence, the contraction mapping principle then yields a unique solution $u \in W^{2,1}(Q_T)$ for small $T > 0$.

Next, let $T^* < \infty$ be a maximal time. For $T < T^*$, a solution $u \in W^{2,1}(Q_T)$ of

$$u_t + Au = P(f - (w \cdot \nabla)u), \quad u(0) = u_0$$

satisfies the following inequality:

$$\|u_t\|_{L^2(0,T;L^2(\Omega))} + \|\nabla^2 u\|_{L^2(0,T;L^2(\Omega))} \leq N \left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|(w \cdot \nabla)u\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{W^{2,1}(Q_T)}\right).$$

Let $T \to T^*$. Then, left-hand side of (32) is infinity. But, since $\|(w \cdot \nabla)u\|_{L^2(0,T;L^2(\Omega))} \leq \|w\|_{L^\infty(Q_T)}\|f\|_{L^2(0,T;L^2(\Omega))}$, right-hand side of (32) is uniformly finite. Thus, the contraction mapping principle then yields a unique solution $u \in W^{2,1}(Q_T)$ for all time.

For fixed $T > 0$, we consider a suitable weak solution $u$ to Navier-Stokes equations:

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0$$

in $Q_T$ with the initial condition $u(x,0) = u_0 \in L^2$ satisfying $\nabla \cdot u_0 = 0$ in a weak sense. For the existence we follow the steps in [4]. For fixed $N > 0$, we set $\delta = T/N$. Then we find a sequences $(u_N, p_N)$ such that

$$u_N \in C(0,T;J^2(\Omega)) \cap L^2(0,T;J(\Omega)),$$

$$\partial_t u_N + \Psi_\delta(u_N) \cdot \nabla u_N - \Delta u_N + \nabla p_N = f,$$

$$\nabla \cdot u_N = 0, \quad u_N(0) = u_0.$$
Here, the *retarded mollifier* $\Psi_\delta$ is defined by
\[
\Psi_\delta(v)(x,t) \equiv \delta^{-4} \int_{\mathbb{R}^4} \psi\left(\frac{y}{\delta},\tau\right) v^*(x-y,t-\tau)dydt,
\]
where $\psi(x,t) \in C^\infty$ satisfies
\[
\psi \geq 0, \quad \int \psi dx dt = 1, \quad \text{and} \quad \text{supp} \psi \subset \{(x,t) : |x|^2 < t, 1 < t < 2\},
\]
and $v^* : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is defined by
\[
v^*(x,t) = \begin{cases} v(x,t) & \text{if } (x,t) \in \Omega \times \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases}
\]
The values of $\Psi_\delta(v)$ at time $t$ clearly depend only on the values of $v$ at times $\tau \in (t-2\delta, t-\delta)$. For $v \in L^\infty(0,T; J^2(\Omega)) \cap L^2(0,T; J(\Omega))$, it is clear that \( \nabla \cdot \Psi_\delta(v) = 0 \) a.e. $x \in \Omega$,
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |\Psi_\delta(v)|^2(x,t)dx \leq N \text{ ess sup}_{0 < t < T} \int_{\Omega} |v|^2 dx,
\]
\[
\int_{\Omega} |\nabla \Psi_\delta(v)|^2 dx \leq N \int_{\Omega} |\nabla v|^2 dx.
\]
Such $(u_N, p_N)$ exist by Lemma 4.1 inductively on each time interval $(m\delta, (m+1)\delta)$, $0 \leq m \leq N - 1$.

By $\frac{d}{dt} \int_{\Omega} |u|^2 dx = 2 \int_{\Omega} (u_t, u) dx$, we have
\[
\int_{\Omega \times \{t\}} |u_N|^2 dx ds + 2 \int_0^t \int_{\Omega} |\nabla u_N|^2 dx ds = \int_{\Omega} |u_0|^2 dx + 2 \int_0^t \int_{\Omega} f \cdot u_N dx ds
\]
for $0 < t < T$. Therefore, we have
\[
\int_{\Omega \times \{t\}} |u_N|^2 dx ds + \int_0^t \int_{\Omega} |\nabla u_N|^2 dx ds \leq \int_{\Omega} |u_0|^2 dx + \int_0^t \|f\|_{H^{-1}}^2 d\tau ds.
\]
In particular,
\[
\text{\emph{\textbf{u}}}_N \text{ stays bounded in } L^\infty(0,T; L^2) \cap L^2(0,T; H^1),
\]
\[
\frac{d}{dt} u_N \text{ stays bounded in } L^2(0,T; H_0^{-2})
\]
and hence, \( \{u_N\} \) stays bounded in $L^2(Q_T)$.

From Stokes estimate,
\[
\{p_N\} \text{ stays bounded in } L^2(Q_T).
\]
Thus, there exist their limits \((u_\ast, p_\ast)\) such that

\[
\begin{align*}
u_N &\to u_\ast \left\{ \begin{array}{ll}
\text{Strongly in } L^q(Q_T), & 2 \leq q < \frac{10}{3}, \\
\text{weakly in } L^2(0, T; J(\Omega)), & \\
\text{weak-star in } L^\infty(0, T; J^2(\Omega)), & 
\end{array} \right. \\
p_N &\to p_\ast \text{ weakly in } L^\dot{2}(0, T; J(\Omega)).
\end{align*}
\]

We note that \((u_\ast, p_\ast)\) is a suitable weak solution of the Navier-Stokes equations (33). The remaining parts of the proof are similar to that of [4].

### 4.2. Stokes estimates

Here we sketch the local boundary estimate for the Stokes system with slip boundary conditions in [31]. Let,

\[
\begin{align*}
&D_t^{1/2}u(t) = \mathcal{F}_\xi^{-1}[(1 + s^2)\tilde{F}_\xi u(s)](t), \\
&H^{1/2}_q(R, X) = \{ u \in L_q(R, X) \mid (D_t)^{1/2}u(t) \in L_q(R, X) \}, \\
&\|u\|_{H^{1/2}_q(R, X)} = \|u\|_{L_q(R, X)} + \|(D_t)^{1/2}u\|_{L_q(R, X)},
\end{align*}
\]

where \(\mathcal{F}_\xi\) and \(\mathcal{F}_\xi^{-1}\) denote the Fourier transform and its inverse formula, respectively.

Set

\[
H^{1/2}_{p,q}(\mathbb{R}_+^3 \times \mathbb{R}_+) = H^{1/2}_q(\mathbb{R}_+^3; L^p(\mathbb{R}_+^3)) \cap L^q(\mathbb{R}_+^3; W^1_p(\mathbb{R}_+^3)),
\]

\[
\|u\|_{H^{1/2}_{p,q}(\mathbb{R}_+^3 \times \mathbb{R}_+^3)} = \|u\|_{H^{1/2}_q(\mathbb{R}_+^3; L^p(\mathbb{R}_+^3))} + \|u\|_{L^q(\mathbb{R}_+^3; W^1_p(\mathbb{R}_+^3))}.
\]

Moreover,

\[
\begin{align*}
\mathcal{W}^{+}_q(x) &= \left\{ u \in W^1_q(X) \mid \int_X u(x)dx = 0 \right\}, \\
\mathcal{W}^{-1}_q(x) &= (\mathcal{W}^{+}_q(x))^* \quad q' = q/q - 1, \quad 1 < q < \infty,
\end{align*}
\]

\[
\|u\|_{\mathcal{W}^{-1}_q(x)} = \sup_{0 \neq v \in \mathcal{W}^{+}_q(x)} \frac{\|\nabla v\|_{L^q_q(X)}}{\|v\|_{L^q_q(X)}},
\]

where \([\cdot, \cdot]\) denotes the duality of \(\mathcal{W}^{-1}_q(x)\) and \(\mathcal{W}^{+}_q(x)\).

**Lemma 4.2.** Let \(1 < p, q < \infty\), \(2r < \rho\) and \(Q^+_\rho = B^+_\rho \times (-\rho^2, 0)\). Suppose that \(v \in L^2_p W^{1,p}_\omega(Q^+_\rho), \) \(v_i \in L^1_T L^p(Q^+_\rho)\) and \(\pi \in L^2_p W^{1,p}_\omega(Q^+_\rho)\) such that \((v, \pi)\) solves the following Stokes system:

\[
\begin{align*}
&v_t - \Delta v + \nabla \pi = \mathbf{g}, \quad \nabla \cdot v = 0 \quad \text{in } Q^+_\rho, \\
&v_3 = 0, \quad \partial_3 v_1 = \varphi_x, \partial_3 v_3, \quad \partial_3 v_2 = \varphi_x, \partial_3 v_3 \quad \text{on } \partial Q^+_\rho \cap \{x_3 = 0\},
\end{align*}
\]

where \(\varphi\) is given in Assumption 2.1, and \(\tilde{\Delta}, \tilde{\nabla}\) are differential operators in Section 2. Then \((v, \pi)\) satisfies

\[
\begin{align*}
&\|v_t\|_{L^{p,s}(Q^+_\rho)} + \|v\|_{L^s((-r^2, 0), W^2_p(B^+_\rho))} + \|\nabla \pi\|_{L^{p,s}(Q^+_\rho)} \\
&\leq N \left( (\|g\|_{L^{p,s}(Q^+_\rho)} + \|v\|_{L^{p,s}(Q^+_\rho)} + \|\nabla v\|_{L^{p,s}(Q^+_\rho)} + \|\pi\|_{L^{p,s}(Q^+_\rho)} \right).
\end{align*}
\]
Proof. Let \( \xi \) be a standard cut-off function satisfying:
\[
\xi \in C_0^\infty(\mathbb{R}^3), \quad 0 \leq \xi \leq 1 \text{ in } \mathbb{R}^3, \\
\xi \equiv 1 \text{ in } B_r, \quad \xi = 0 \text{ outside on } B_r,
\]
\[
|\hat{\nabla}\xi| < \frac{c}{\rho - r}, \quad |\hat{\nabla}^2\xi| < \frac{c}{(\rho - r)^2}.
\]
Take \( \nu = v\xi, \Pi = \pi\xi \). Then,
\[
(35) \quad \begin{cases}
\nu_t + \nu - \Delta\nu + \hat{\nabla}\Pi = G, \\
\nu_3 = 0, \\
\partial_3\nu_1 = h_1, \\
\partial_3\nu_2 = h_2
\end{cases}
\text{ in } \mathbb{R}_+^3 \times \mathbb{R}^3,
\]
where
\[
G = \nu - 2\hat{\nabla}v\hat{\nabla}\xi - v\Delta\xi + \pi\hat{\nabla}\xi + \xi g, \\
h_1 = v_1\partial_3\xi + \xi \varphi_1\partial_3v_3, \\
h_2 = v_2\partial_3\xi + \xi \varphi_2\partial_3v_3.
\]
Then (35) can be expressed:
\[
(36) \quad \begin{cases}
\nu_t + \nu - \Delta\nu + \hat{\nabla}\Pi = G^*, \\
\nu_3 = 0, \\
\partial_3\nu_1 = h_1, \\
\partial_3\nu_2 = h_2
\end{cases}
\text{ in } \mathbb{R}_+^3 \times \mathbb{R}^3,
\]
where
\[
G^* = G + \Delta'\nu - \hat{\nabla}\Pi, \\
d^* = d - \nabla'\nu,
\]
\[
\Delta' = \hat{\Delta} - \Delta = -2\varphi_2\partial_1\xi + 2\varphi_2\partial_2\xi + (\varphi_1)^2\partial_3\xi + \Delta = \hat{\nabla}' - \nabla = (-\varphi_1\partial_3, -\varphi_2\partial_3, 0).
\]
Using the maximal estimate for Stokes system with slip boundary [31, Theorem 5.1], we get
\[
||\nu_t||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} + ||\nu||_{L^q(\mathbb{R}_+^3, W^{2,1}_p(\mathbb{R}_+^3))} + ||\hat{\nabla}\Pi||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)}
\leq N \left(||G^*||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} + ||d^*||_{L^q(\mathbb{R}_+^3, W^{1,1}_p(\mathbb{R}_+^3))} + ||d^*_1||_{L^q(\mathbb{R}_+^3, W^{1,1}_p(\mathbb{R}_+^3))} + ||h||_{H^{1/2,1/2}_p(\mathbb{R}_+^3 \times \mathbb{R}^3)}) \right).
\]
Then, the following estimates hold:
\[
||\Delta'\nu||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} \leq c\epsilon ||\nu||_{L^q(\mathbb{R}_+^3, W^{2,1}_p(\mathbb{R}_+^3))},
\]
\[
||\nabla'\Pi||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} \leq c ||\nabla\Pi||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)}.
\]
Thus, choosing \( \epsilon \) small enough, we have
\[
||\nu_t||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} + ||\nu||_{L^q(\mathbb{R}_+^3, W^{2,1}_p(\mathbb{R}_+^3))} + ||\hat{\nabla}\Pi||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)}
\leq N \left(||G||_{L^p(\mathbb{R}_+^3 \times \mathbb{R}^3)} + ||d||_{L^q(\mathbb{R}_+^3, W^{1,1}_p(\mathbb{R}_+^3))} + ||d^*_1||_{L^q(\mathbb{R}_+^3, W^{1,1}_p(\mathbb{R}_+^3))} + ||h||_{H^{1/2,1/2}_p(\mathbb{R}_+^3 \times \mathbb{R}^3)}) \right).
From [31], we get the following estimate:

\[
\|d\|_{L^2(B^+)} + \|dt\|_{L^2(B^+)} \\
\leq N \left( \|v \cdot \tilde{\nabla} \xi\|_{L^p(B^+ \times R)} + \|\varepsilon \cdot \tilde{\nabla} \xi\|_{L^p(B^+ \times R)} \right) \\
\leq N \left( \|v \cdot \tilde{\nabla} \xi\|_{L^p(B^+ \times R)} + \|\varepsilon\|_{L^2(B^+ \times R)} \right)
\]

Therefore, choosing

\[
B \leq h \leq \varepsilon + \|
\nabla^2 (v \cdot \tilde{\nabla} \xi)\|_{L^p(B^+ \times R)}
\]

and

\[
\| \varepsilon \|_{L^2(B^+ \times R)} + \|\varepsilon \|_{L^2(B^+ \times R)}
\]

Thus, we obtain

\[
\|v\|_{L^p(B^+ \times R)} - R^{-\frac{1}{2}} \|v\|_{L^p(B^+ \times R)} = \varepsilon \|\nabla (\pi \xi)\|_{L^p(B^+ \times R)} + \|\nabla (\pi \xi)\|_{L^p(B^+ \times R)}
\]

Therefore, choosing \( R \) large enough, \( \varepsilon \) and \( \varepsilon_0 \) small enough, recalling that \( \xi \equiv 1 \) on \( B_r \) and \( \xi = 0 \) outside on \( B_{\rho} \), and \( G = \nu - 2 \varepsilon \nabla^2 \xi = -v \nabla \xi + \nabla \nabla^2 \xi = \xi \nabla \xi \xi \),

\[
\|v\|_{L^p(B^+ \times R)} + \|g\|_{L^p(B^+ \times R)} \\
\leq N \left( \|g\|_{L^p(B^+ \times R)} + \|v\|_{L^p(B^+ \times R)} + \|\nabla \xi\|_{L^p(B^+ \times R)} \right)
\]
holds.

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