Functional derivation of Casimir energy at non-zero temperature

Avijit K. Ganguly
Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012, INDIA

Palash B. Pal
Saha Institute of Nuclear Physics, 1/AF Bidhan-Nagar, Calcutta 700064, INDIA

Abstract
Performing functional integration of the free Lagrangian, we find the vacuum energy of a field. The functional integration is performed in a way which easily generalizes to systems at non-zero temperature. We use this technique to obtain the Casimir energy density and pressure at arbitrary temperatures.

1 Introduction
If we have electromagnetic field in a restricted region, the vacuum energy of the electromagnetic field shows up as an effective zero point energy, which is known as Casimir energy. For a rectangular region bounded by two perfectly conducting plates separated by a distance $\zeta$ along the $z$-axis, the magnitude of this energy is given by

$$E = -\frac{\pi^2 L_x L_y}{720 \zeta^3}, \quad (1.1)$$

where $L_x$ and $L_y$ are the dimensions of the plates (assumed large) along the other two axes. The energy density in the region between the plates is therefore given by

$$\rho = -\frac{\pi^2}{720 \zeta^4}. \quad (1.2)$$

There are various ways that the magnitude of this energy density can be derived. Here, we outline a method using functional integration of the Lagrangian. We will show that this method easily extends to the case when the region between the two plates is at a finite temperature. Using this extension, we will calculate the Casimir energy at finite temperature.

It is well known that the functional formulation of the electromagnetic field is complicated because of gauge invariance. The gauge volume has to be divided out by a Fadeev-Popov procedure. In order to avoid these complications, we perform all our calculation using a complex scalar field of mass $M$. In other words, we will calculate the zero-point energy of a complex scalar field in the specified geometry, assuming that this field satisfies the same boundary conditions as the electromagnetic field at the boundaries of the region. Once this is done, the result can be applied for the electromagnetic field as well, since it has the same number of degrees of freedom. We only need to put the mass of the field to be equal to zero.
2 The functional formulation

Let us then start from the Lagrangian of a free scalar field:

\[ \mathcal{L} = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - M^2 \phi^\dagger \phi. \]  

(2.1)

The generating functional obtained from this Lagrangian is given by

\[ Z = \int [\mathcal{D} \phi][\mathcal{D} \phi^\dagger] \exp \left( i \int d^4 x \mathcal{L} \right). \]

(2.2)

Since the free Lagrangian is quadratic, this integration can be performed formally, and one obtains

\[ Z = \det (\partial^2 + M^2)^{-1}. \]

(2.3)

Thus,

\[ \ln Z = - \text{tr} \ln (\partial^2 + M^2), \]

(2.4)

using a well-known identity that for any operator, the logarithm of its determinant is the same as the trace of its logarithm.

The trace mentioned above runs not only over any internal degrees of freedom that may be present (in the particular case at hand, there isn’t any), but also over the space-time points. Thus, more explicitly, we can write

\[ \ln Z = - \int d^4 x \langle x | \ln (\partial^2 + M^2) | x \rangle. \]

(2.5)

Going over to the momentum representation, this can be written as

\[ \ln Z = - \int \frac{d^4 k}{(2\pi)^4} \ln (-k^2 + M^2) = \int \frac{d^4 k}{(2\pi)^4} \int dM^2 \frac{1}{k^2 - M^2}, \]

(2.6)

where the integration over \( M^2 \) is indefinite. The integral on the right side is, of course, infinite. This is no surprise. For a general system, the quantity \( Z \) can be interpreted as

\[ \ln Z = - i f_{\text{vac}}, \]

(2.7)

where \( f_{\text{vac}} \) is the free energy density of the vacuum, so that we can write

\[ f_{\text{vac}} = i \int dM^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - M^2}. \]

(2.8)

For a system at zero temperature, the free energy density is equal to the internal energy density, \( \rho_{\text{vac}} \). The infinity obtained for the integral on the right side then just corresponds to the well known result in quantum field theory that the vacuum energy of a field is formally infinite. Once normal ordered fields are employed, this infinite contribution vanishes. Effectively, this amounts to setting the zero of energy at the vacuum energy of the field.

In the derivation above, we have implicitly assumed that the space-time region is infinite in all directions. For a more general situation, let us denote the obvious generalization of Eq. (2.8) by

\[ f_{\text{vac}} = i \int dM^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - M^2}, \]

(2.9)

where the square brackets around the momentum integration measure now indicate that the measure has to be taken as appropriate in a particular situation. In fact, if the momentum values happen to be discrete, we should sum rather than integrate over the momenta.

In order to calculate the vacuum energy at a non-zero temperature \( 1/\beta \), we can employ the imaginary time formalism, where the \( k_0 \) values for a bosonic field are quantized in the form

\[ k_0 = 2\pi n/\beta, \]

(2.10)
for arbitrary integers \( n \). In this case, the integration over \( k_0 \) is to be replaced by a sum over the imaginary energies. Thus,

\[
\int \frac{[d^4k]}{(2\pi)^4} = \frac{i}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{[d^4k]}{(2\pi)^4},
\]

(2.11)

We are still keeping the square brackets around the integration measure for the spatial components of momentum in view of the fact that in a restricted geometry as is relevant for the Casimir problem, the spatial components of momenta will also not be continuous. Thus, for a scalar field at finite temperature, we can write

\[
f_\beta = -\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{[d^4k]}{(2\pi)^4} \int dM^2 \frac{1}{(2\pi/\beta)^2 - k^2 - M^2}.
\]

(2.12)

The sum over \( n \) present in Eq. (2.12) can be performed by using the identity

\[
\sum_{l=-\infty}^{\infty} \frac{1}{l^2 + a^2} = \frac{\pi}{a} \coth \pi a.
\]

(2.13)

Writing

\[
\coth z = 1 + \frac{2}{e^{2z} - 1},
\]

(2.14)

we can rewrite Eq. (2.12) as

\[
f_\beta = \int \frac{[d^3k]}{(2\pi)^3} \int d\omega_k \left( 1 + \frac{2}{e^{\beta \omega_k} - 1} \right),
\]

(2.15)

using the shorthand

\[
\omega_k \equiv \sqrt{k^2 + M^2}.
\]

(2.16)

The integral over \( M^2 \) present in the resulting form is an indefinite integral. Instead of using \( k \) and \( M \) as independent parameters, we can use \( k \) and \( \omega_k \). This change can be effected by making the substitution

\[
dM^2 = 2\omega_k \, d\omega_k,
\]

(2.17)

which gives

\[
f_\beta = \int \frac{[d^3k]}{(2\pi)^3} \int d\omega_k \left( 1 + \frac{2}{e^{\beta \omega_k} - 1} \right) = f_\infty + f'_\beta.
\]

(2.18)

Clearly, the first term in the integrand gives a temperature independent part, which is the contribution at zero temperature (i.e., at \( \beta \to \infty \)). As indicated, this term will be denoted by \( f_\infty \). The second term is the temperature dependent contribution, which we have denoted by \( f'_\beta \). We now discuss these contributions one by one.

### 3 The temperature independent contribution

For the first term, which we denoted by \( f_\infty \), we can straightaway perform the indefinite integral over \( \omega_k \) to write

\[
f_\infty = \int \frac{[d^3k]}{(2\pi)^3} \omega_k
\]

(3.1)

To proceed, we now use the identity

\[
\omega_k = \frac{1}{\Gamma(-1/2)} \int_{0}^{\infty} ds \frac{s^{3/2}}{\sqrt{s/3}} \exp[-s(k^2 + M^2)]
\]

(3.2)
Substituting this form into the first term of Eq. (2.18), we can perform the \( k \) integration if the space were infinite. This would give

\[
f_\infty = - \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{3/2}} e^{-sM^2}. \tag{3.3}
\]

This is essentially the form obtained by Schwinger through his proper-time formalism. If the integration is performed now, we would get a result proportional to \( \Gamma(-2) \), which would be infinite, as remarked earlier.

But we also said that this energy is not really relevant for us. In the Casimir geometry, i.e., in the region between two infinite conducting plates at \( z = 0 \) and \( z = \zeta \), the component of momentum perpendicular to the plates will be quantized. For an electromagnetic field, these quantized values will be given by

\[
k_z = \frac{\pi l}{\zeta} \tag{3.4}
\]

with arbitrary integers \( l \) so that the potential can vanish at both the plates. As we said in the Introduction, we will take this same boundary condition for the complex scalar field as well. Thus, for the Casimir geometry, we should write

\[
\int \frac{[d^3k]}{(2\pi)^3} = \frac{1}{2\zeta} \sum_{l=-\infty}^{\infty} \int \frac{d^2k_\perp}{(2\pi)^2}, \tag{3.5}
\]

where \( k_\perp \) indicates momentum values in the plane perpendicular to the \( z \)-axis. Using Eqs. (3.1) and (3.2) as before, we can now perform the integration over \( k_\perp \) to obtain

\[
f_\infty = - \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{3/2}} e^{-sM^2} \frac{1}{2\zeta} \sum_{l=-\infty}^{\infty} \exp\left(-\frac{\pi^2 s}{\zeta^2} l^2\right)
\]

\[
= - \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^{3/2}} e^{-sM^2} \frac{1}{2\zeta} \vartheta_3\left(0, \frac{i\pi s}{\zeta^2}\right), \tag{3.6}
\]

using the standard definition of the Jacobi \( \vartheta \)-function:

\[
\vartheta(u, \tau) = \sum_{l=-\infty}^{\infty} \exp\left(2lu + i\pi \tau l^2\right). \tag{3.7}
\]

The expression for \( f_\infty \) given above can be simplified, noting that mathematically, the problem of Casimir geometry at zero temperature is equivalent to the problem of infinite geometry at non-zero temperature, tackled through the imaginary time formalism. We therefore follow a procedure enumerated by Dittrich for non-zero temperatures, adapting it to the present case. For this, one notes the following property of the Jacobi \( \vartheta \)-function:

\[
\vartheta(0, i\tau) = \frac{1}{\sqrt{\tau}} \vartheta(0, \frac{i}{\tau}), \tag{3.8}
\]

which enables us to write

\[
\frac{1}{2\zeta} \vartheta_3\left(0, \frac{i\pi s}{\zeta^2}\right) = \frac{1}{\sqrt{4\pi s}} \vartheta_3\left(0, \frac{i\pi^2}{\zeta^2}\right)
\]

\[
= \frac{1}{\sqrt{4\pi s}} \sum_{l=-\infty}^{\infty} \exp\left(-\frac{\zeta^2 l^2}{s}\right). \tag{3.9}
\]

Putting this form back in Eq. (3.6), we obtain

\[
f_\infty = - \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^2} \sum_{l=-\infty}^{\infty} \exp\left(-\frac{\zeta^2 l^2}{s} - sM^2\right). \tag{3.10}
\]

There is a summation over \( l \). Notice that, the \( l = 0 \) term in this sum is exactly same as the result obtained in Eq. (3.3) for infinite space. Thus, this result is also infinite. However, the physically important quantity is the difference of this quantity and the corresponding result for the infinite space, which is:

\[
f'_\infty = f_\infty - f_{\text{vac}} = -2 \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^2} \sum_{l=-\infty}^{\infty} \exp\left(-\frac{\zeta^2 l^2}{s} - sM^2\right). \tag{3.11}
\]

\(^1\)See, e.g., Ref. [4], §8.180, eq. 4 and the comment below.
The integral over $s$ can be expressed in terms of a modified Bessel function, defined by

$$
K_\nu(z) = \frac{1}{2} \left( \frac{z}{2} \right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} \exp \left( -t - \frac{z^2}{4t} \right).
$$

(3.12)

Writing $sM^2 = t$ in Eq. (3.11), the expression can be rewritten in the form

$$
f'_\infty = -\frac{2M^4}{(4\pi)^2} \sum_{l=1}^\infty \int_0^\infty \frac{dt}{t^2} \exp \left( -\frac{\zeta^2 l^2 M^2}{t} - t \right) = -\frac{M^2}{4\pi^2 \zeta^2} \sum_{l=1}^\infty \frac{1}{l^2} K_2(2\zeta l M).
$$

(3.13)

This result is twice that of Ambjorn and Wolfram [7], who calculated the same quantity for a real scalar field.

For the $M \to 0$ limit which is relevant for finding the result for the electromagnetic field, we can use the form

$$
K_2(z) \approx 2z^{-2},
$$

(3.14)

which is valid for small $z$. This gives

$$
f'_\infty = -\frac{1}{8\pi^2 \zeta^4} \sum_{l=1}^\infty \frac{1}{l^4} = \frac{\pi^2}{720\zeta^2}.
$$

(3.15)

This is the correct result for the boundary conditions explained above. As remarked in the Introduction, this result is applicable for the electromagnetic field which has the same number of degrees of freedom as the complex scalar field. Indeed, this result agrees with Eq. (1.3), since the free energy density in this case is equal to the internal energy density.

## 4 The temperature dependence

We now consider the temperature dependent part of the generating functional, which was given by the second term in Eq. (2.18). Holstein and Pal [6] suggested that this term can be easily tackled if we rewrite it in the form

$$
f'_\beta = 2 \int \frac{[d^3 k]}{(2\pi)^3} \int d\omega_k \sum_{r=1}^\infty e^{-r\beta\omega_k}
$$

$$
= -\sum_{r=1}^\infty \frac{2}{r^2} \int \frac{[d^3 k]}{(2\pi)^3} e^{-r\beta\omega_k}.
$$

(4.1)

For infinite 3-dimensional space, the integral can directly be written in the form of a modified Bessel function [6]. To modify this procedure for the Casimir geometry, we first note that the exponent, written in terms of $k$, has a square root owing to the definition of $\omega_k$ given in Eq. (2.16). This can be eliminated by using the identity

$$
e^{-z} = \frac{1}{\sqrt{\pi}} \int_0^\infty dx x^{-1/2} \exp \left( -x - \frac{z^2}{4x} \right).
$$

(4.2)

Using this, for the Casimir geometry, we can utilize Eq. (3.33) to write

$$
f'_\beta = -\sum_{r=1}^\infty \frac{1}{\sqrt{\pi^2 r^2 \beta^2 \zeta^4}} \int_0^\infty dx x^{-1/2} e^{-x} \sum_{l=-\infty}^\infty \int \frac{d^2 k_\perp}{(2\pi)^2} \exp \left( -\frac{r^2 \beta^2 \omega_k^2}{4x} \right).
$$

(4.3)

Using

$$
\omega_k^2 = k_\perp^2 (\pi l/\zeta)^2 + M^2
$$

(4.4)

as is appropriate for this case, we can perform the integration over $k_\perp$ in a straightforward manner. This gives

$$
f'_\beta = -\sum_{r=1}^\infty \frac{1}{(\pi^2 r^2 \beta^2 \zeta^2)^{3/2}} \int_0^\infty dx x^{+1/2} \exp \left( -x - \frac{r^2 \beta^2 M^2}{4x} \right) \sum_{l=-\infty}^\infty \exp \left( -\frac{r^2 \beta^2 \pi^2 l^2}{4x \zeta^2} \right)
$$

$$
= -\sum_{r=1}^\infty \frac{1}{(\pi^2 r^2 \beta^2 \zeta^2)^{3/2}} \int_0^\infty dx x^{+1/2} \exp \left( -x - \frac{r^2 \beta^2 M^2}{4x} \right) 0 \int \frac{d^2 k_\perp}{(2\pi)^2} \exp \left( \frac{i\pi r^2 \beta^2}{4x \zeta^2} \right).
$$

(4.5)

---

2 See, e.g., Ref. [4], §8.432, eq. 6.
Using the property of the $\vartheta$-function given in Eq. (3.8), this can be rewritten in the form

\[
f'_\beta = -\sum_{r=1}^{\infty} \frac{2}{(\pi r \beta)^2} \int_0^\infty dx \ x \exp \left( -x - \frac{r^2 \beta^2 M^2}{4x} \right) \vartheta_3 \left( 0, \frac{4ix\zeta^2}{\pi r \beta^2} \right)
\]

\[
= -\sum_{r=1}^{\infty} \frac{2}{(\pi r \beta)^2} \sum_{l=-\infty}^{\infty} \int_0^\infty dx \ x \exp \left( -xA^2_{rl} - \frac{r^2 \beta^2 M^2}{4x} \right),
\]

where, for the sake of notational simplicity, we have written

\[
1 + \frac{4\zeta^2}{r^2 \beta^2} l^2 \equiv A^2_{rl}.
\]

The integral over $x$ can be represented in terms of the modified Bessel function defined in Eq. (3.12), and we obtain

\[
f'_\beta = -\sum_{r=1}^{\infty} \frac{M^2}{(\pi r \beta)^2} \sum_{l=-\infty}^{\infty} \frac{1}{A^2_{rl}} K_{-2}(r \beta MA_{rl}).
\]

For arbitrary values of $M$, this cannot be reduced further analytically. However, we are interested in the limit $M \to 0$. From the general expression for modified Bessel functions given in Eq. (3.12), one can check that $K_\nu(z) = K_{-\nu}(z)$. Thus, we can write $K_2$ in place of $K_{-2}$ in the last equation. Using now the limiting form of this function for small arguments which was given in Eq. (3.14), we obtain

\[
f'_\beta = -\sum_{r=1}^{\infty} \frac{1}{8\pi^2 \zeta^4} \sum_{l=-\infty}^{\infty} \frac{1}{[l^2 + (r \beta/2 \zeta)^2]^2}.
\]

The sum over $l$ can be performed by using the formula

\[
\sum_{l=-\infty}^{\infty} \frac{1}{[l^2 + a^2]^2} = \frac{\pi}{2a^3} \left[ \coth \pi a + \pi a \csch^2 \pi a \right],
\]

which can be obtained by differentiating Eq. (2.13) with respect to the parameter $a$. Using this, we obtain

\[
f'_\beta = -\sum_{r=1}^{\infty} \frac{1}{(\pi r \beta)^2} \left[ \frac{\pi r \beta}{2\zeta} \coth \frac{\pi r \beta}{2\zeta} + \left( \frac{\pi r \beta}{2\zeta} \right)^2 \csch^2 \frac{\pi r \beta}{2\zeta} \right].
\]

This result was obtained by various authors earlier, using different techniques [8, 9, 10, 11].

One interesting check of this result is that, for $\zeta \to \infty$, the expression in the square bracket has a limiting value of 2, so that

\[
f'_\beta(\zeta \to \infty) = -\sum_{r=1}^{\infty} \frac{2}{(\pi r \beta)^2} = -\frac{\pi^2}{45 \beta^4}.
\]

The internal energy density can be calculated from here, which gives

\[
\rho'_\beta = f'_\beta + \beta \frac{\partial f'_\beta}{\partial \beta} = \frac{\pi^2}{15 \beta^4},
\]

which is the well-known result for the energy density of a Planck distribution.

Experimentally, however, we are interested about small values of $\zeta$, for which the remaining sum in Eq. (4.11) has to be performed numerically. The results are summarized in the next section.

5 Discussion of the results

Adding up the contributions given in Eqs. (3.15) and (4.11), we can write the total free energy density in the form

\[
f_\beta = -\frac{\pi^2}{720 \zeta^4} G(z),
\]
where $z$ is a dimensionless parameter defined as 

\[ z = \frac{2\zeta}{\pi\beta}, \quad (5.2) \]

and

\[ G(z) = 1 + 45 \sum_{r=1}^{\infty} \frac{z^4}{r^4} \left( \frac{r}{z} \coth \frac{r}{z} + \left( \frac{r}{z} \right)^2 \csch^2 \frac{r}{z} \right). \quad (5.3) \]

The corresponding density of the total internal energy will be given by

\[ \rho_\beta \equiv -\frac{\pi^2}{720\zeta^4} K(z), \quad (5.4) \]

where

\[ K(z) = G(z) - zG'(z), \quad (5.5) \]

the prime on the function $G$ implying differentiation with respect to its argument. This gives

\[ K(z) = 1 - 90 \sum_{r=1}^{\infty} \frac{z^4}{r^4} \left( \frac{r}{z} \coth \frac{r}{z} + \left( \frac{r}{z} \right)^2 \csch^2 \frac{r}{z} + \left( \frac{r}{z} \right)^3 \csch^2 \frac{r}{z} \coth \frac{r}{z} \right). \quad (5.6) \]

Experimentally what is measured is the force on the condenser plates which define the boundaries of the region in the $z$-direction. The force per unit area, or the pressure $p$, is related to the total free energy $F$ of a system through the thermodynamic relation

\[ p = -\left( \frac{\partial F}{\partial V} \right)_\beta = -f_\beta - V \left( \frac{\partial f_\beta}{\partial V} \right)_\beta, \quad (5.7) \]

where $V$ is the volume of the region, and $\beta$, or the temperature, is to be kept constant while taking the partial derivative. In the present situation, we can write this as

\[ p = -f_\beta - \zeta \left( \frac{\partial f_\beta}{\partial \zeta} \right)_\beta. \quad (5.8) \]

Substituting the expression for $f_\beta$ obtained above, we obtain

\[ p = -\frac{\pi^2}{240\zeta^4} H(z), \quad (5.9) \]

where $z$ is the variable defined in Eq. (5.2), and

\[ H(z) = G(z) - \frac{1}{3}zG'(z) \]

\[ = 1 - 30 \sum_{r=1}^{\infty} \frac{z^4}{r^4} \csch^2 \frac{r}{z} \coth \frac{r}{z}. \quad (5.10) \]

Notice again that, in the limit $\zeta \to \infty$, this gives the familiar formula for the radiation pressure, which is one-third the internal energy density given in Eq. (4.13).

The expressions for internal energy and pressure given above are valid if the region inside the capacitor plates are at a temperature $1/\beta$ and the outside region is at zero temperature. In an experiment, this is a difficult situation to realize, specially since the distance $\zeta$ between the plates is very small. Rather, a more practical scenario is when the region between the plates, as well as the ambient region, are both kept at a temperature $1/\beta$. In this case, the quantities of physical interest are the differences in the internal energy and pressure between the two regions — the one inside the plates, and the one outside. We denote these quantities by essentially the same notation used in Eqs. (5.4) and (5.9, with two new functions $\tilde{K}(z)$ and $\tilde{H}(z)$ replacing $K(z)$ and $H(z)$ respectively. These new functions are given by

\[ \tilde{K}(z) = 1 - 90 \sum_{r=1}^{\infty} \frac{z^4}{r^4} \left( \frac{r}{z} \coth \frac{r}{z} + \left( \frac{r}{z} \right)^2 \csch^2 \frac{r}{z} + \left( \frac{r}{z} \right)^3 \csch^2 \frac{r}{z} \coth \frac{r}{z} - 3 \right), \quad (5.11) \]
Figure 1: The solid line is a plot of the function $\tilde{K}(z)$, and the dotted line is of $\tilde{H}(z)$. These functions give the difference between the internal energy density and pressure of the inside and outside regions. Both are normalized to have the value unity at zero temperature.

and

$$\tilde{H}(z) = 1 - 30 \sum_{r=1}^{\infty} \left[ \frac{z}{r} \text{csch}^2 \frac{r}{z} \coth \frac{r}{z} - \left( \frac{z}{r} \right)^4 \right].$$

(5.12)

In Fig. 1 we show these functions $\tilde{K}(z)$ and $\tilde{H}(z)$ as functions of the dimensionless variable $z$. The left end of the plots correspond to small $z$, i.e., large $\beta$ or vanishing temperature. In this case, the results should reduce to the ordinary Casimir results, so that the gains should equal unity, as seen in the figure. As $z$ increases, we see that the energy density quickly approaches the result appropriate for infinite volume.

As far as the difference between the external and internal pressures is concerned, we see that the relevant function $\tilde{H}(z)$ increases roughly linearly with $z$ when $z$ is large. Now, $z$ can increase either because $\zeta$ increases, or because the temperature increases. In the case when $\zeta$ grows, the pressure difference, despite the increase of the function $\tilde{H}(z)$, decreases owing to the factor of $1/\zeta^4$ in Eq. (5.9), and vanishes in the infinite volume limit as expected. On the other hand, if we keep $\zeta$ fixed and increase the temperature, we see that although the difference in energy density between the external and the internal regions goes to zero, the pressure difference in fact increases with the temperature. This increase in the pressure difference is actually offset by an increase in entropy difference to keep the energy difference zero.

**Acknowledgments:** We thank P. Majumdar for discussions and S. Sinha for enlightening us on various aspects of earlier work done in the field. After a first version of this paper was written and submitted to the electronic archive, S. Sinha, S. Odintsov, and F. Ravndal have brought various earlier references to our attention. We thank all of them. AKG wants to thank the hospitality of the Saha Institute of Nuclear Physics where this work was performed.

**References**

[1] See, e.g., L. Dolan and R. Jackiw: Phys. Rev. D9 (1974) 3320; S. Weinberg: Phys. Rev. D9 (1974) 3357.

[2] J. Schwinger: Phys. Rev. 82 (1951) 664.

[3] I. S. Gradshteyn, I. M. Ryzhik: *Tables of integrals, series and products*, 5th edition, Academic Press 1994.

[4] D. J. Toms: Phys. Rev. D21 (1980) 928.
[5] W. Dittrich: Phys. Rev. D19 (1979) 2385.

[6] B. R. Holstein, P. B. Pal: University of Massachusetts report UMHEP-280 (1987) unpublished.

[7] J. Ambjorn, S. Wolfram: Ann. Phys. 147 (1983) 1 and 33.

[8] J. Mehra: Physica 37 (1967) 145.

[9] L. S. Brown, G. J. Maclay: Phys. Rev. 184 (1969) 1272.

[10] F. Ravndal, D. Tollefsen: Phys. Rev. D40 (1989) 4191.

[11] For a review of earlier work, see e.g. G. Plunien, B. Müller, W. Greiner: Phys. Rep. 134 (1986) 87; E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, S. Zerbini: Zeta Regularization Technique with applications (World Scientific, Singapore, 1994).