Critical Fluctuation of Wind Reversals in Convective Turbulence

Rudolph C. Hwa\textsuperscript{1}, C. B. Yang\textsuperscript{2}, S. Bershadskii\textsuperscript{3}, J. J. Niemela\textsuperscript{3}, and K. R. Sreenivasan\textsuperscript{3}

\textsuperscript{1}Institute of Theoretical Science and Department of Physics
University of Oregon, Eugene, OR 97403-5203, USA
\textsuperscript{2}Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430079, P. R. China and
\textsuperscript{3}International Center of Theoretical Physics, Strada Costiera 11, I-34100 Trieste, Italy

(Dated: November 11, 2018)

The irregular reversals of wind direction in convective turbulence are found to have fluctuating intervals that can be related to critical behavior. It is shown that the power law scaling of the wind reversal phenomenon exhibits signs of self-organized criticality.

In turbulent thermal convection at high Rayleigh numbers (Ra) it has recently been observed that there exists not only large-scale circulating motion, called mean wind, but also abrupt reversals of the wind direction, whose physical origin is still largely unknown. For our purposes, Ra is simply a non-dimensional measure of the temperature difference between the bottom and the top plates of the container within which the convective motion occurs. Metastable states have been suggested to describe the two opposite directions of the wind, and the reversal of its direction is to be understood in terms of the imbalance between buoyancy effects and friction. Instead of searching for the origin of the wind reversals in the framework of hydrodynamical considerations, we investigate in this paper the possibility of understanding the phenomenon in a totally different context, namely: critical phenomenon. We shall find a measure to quantify the fluctuations in the wind direction, and then demonstrate that its behavior corresponds to one exhibited by a system undergoing a second-order phase transition. We then perform a detrended fluctuation analysis to determine the detailed properties of the fluctuations of the wind, more specifically its scaling behavior.

The experimental data that we analyze are the same as those reported in\textsuperscript{7} and studied in\textsuperscript{8}. By varying the pressure and lowering the temperature of the gas, the Rayleigh number could be varied between $10^6$ and $10^{16}$. Further details of the apparatus can be found in\textsuperscript{6}. We focus on the data that give the wind speed and direction for a continuous period of up to one week at Ra = $1.5 \times 10^{11}$. Figure 1 shows a small segment of the wind velocity data for 6.5 hr, starting at an arbitrary time. The wind changes direction suddenly in the time scale of that figure. We proceed directly to an interpretation of the fluctuations of the wind velocity.

In a fully developed turbulent convection at high Ra there are two opposing dynamical features. One is the emission of plumes from the top and bottom boundary layers; they occur at random locations and at random times in varying sizes. The other is the existence of mean wind that rotates in one direction or another, making rapid reversals at seemingly random intervals. We regard the former as the disordered motion of the components of a complex system, and the latter as the ordered motion of the whole of the system. For low Ra (say below 10\textsuperscript{6}), the ordered motion is not sufficiently impeded by the disordered motion to cause reversals of the wind direction. At high Ra, the cumulated effect of the many plumes that is strong enough to reverse the wind direction. The system then proceeds as before except that the wind rotates in the opposite direction with varying magnitude until another reversal occurs due to the collective action of the disorganized plumes. If the system is at a critical state, whether self-organized or not, the competition between the ordered and disordered motions leads to the wind switching directions at irregular intervals of all scale. The probability of occurrence of the wind duration $\tau$ between reversals should satisfy a power law

$$p(\tau) \sim \tau^{-\gamma}$$

as a manifestation of criticality. Such a power law has been found in the data\textsuperscript{7}. The discussion above describes our view of the origin of such a scaling behavior.

We now advance the idea that the above description of the wind and plumes in convective turbulence in terms of ordered and disordered motions has its corresponding counterparts in the 2D Ising model of critical behavior. In the Ising system of near-neighbor interactions without external magnetic field the lattice spins tend to align in the same direction except for the random disorientation due to thermal fluctuation. For a finite lattice the net magnetization, $M$, is non-vanishing. For $T < T_c$, the critical temperature, $M$ is likely to persist in the same direction for longer time in lattice-spin updating than at higher $T$. At $T > T_c$ the thermal interaction dominates, and $M$ is more likely to flip sign more frequently upon updating. The fluctuation of the signs of $M$ is therefore a property that reflects the tension between the ordered and disordered interactions of the whole system.

Since the mean wind is a global phenomenon in a vessel of finite volume, it is sensible for us to associate the wind direction with the sign of $M$ of the Ising lattice of finite size. We can then map wind reversal to the reversal of $M$ upon updating the lattice spins in a simulation. The plumes are the disordered fluctuations that correspond to the spin fluctuations due to thermal agitation, and the wind is the ordered motion that can change direction just
Near the critical point, durations of all lengths can occur. Is analogous to the mean wind rotating in one direction. Divide the whole series into 30 segments. The values of L with the lattice at the critical temperature.

versal problem corresponds to the Ising problem of finite infinite lattice. Our task is to show that the wind re-

sults were placed outside the boundary layer on the sidewall of the container near the middle section of the container, and were separated vertically by a distance of 1.27 cm.

as the magnetization can change sign when enough lattice spins change directions. The key connection between the two problems is the mapping of the real time in turbulence to the time of updating the Ising configurations. It is therefore crucial that each configuration has some memory of the previous configuration before updating; hence we employ the Metropolis algorithm, which does precisely this. It should be noted that we are entering into a rather unexplored territory where the process of computer simulation itself is endowed with some physical significance, quite unrelated to the large body of analytical work that has been devoted to the Ising model of infinite lattice. Our task is to show that the wind reversal problem corresponds to the Ising problem of finite lattice at the critical temperature.

To be more specific, we consider a square lattice of size $L^2$, where $L$ is taken to be 255, an odd number. We start with the $L^2$ site spins having a random distribution of ±1 values. We then visit each site and determine from the usual near-neighbor interaction whether its spin should be reversed: yes, if the energy is lowered by the flip; if not lowered, the flip can still take place according to a thermal distribution specified by temperature $T$. One time step is taken by the whole system when all sites are updated. We take $3 \times 10^5$ time steps in total, and divide the whole series into 30 segments. The values of $M$ at each of the $10^4$ time points in each segment are discretized to $\pm 1$, according to $M \geq 0$. A continuous string of $M$ of one sign, either +1 or −1, forms a duration that is analogous to the mean wind rotating in one direction. The reversals of $M$ correspond to the reversals of wind. Near the critical point, durations of all lengths can occur.

Before considering the issue of criticality for a finite lattice, let us discuss the measure that we shall use for quantifying the duration fluctuations appropriate for both the wind and Ising problems. The experimental data on wind consist of 8 segments, each having $T = 10,282$ time points. For the Ising case we have 30 segments, each having $T = 10^4$, roughly the same as wind data. Let $N$ denote the number of reversals in a segment. With the locations of the reversals denoted by $t_i$, $i = 1, \cdots, N$, define $\tau_i = t_{i+1} - t_i$ to be the $i$th duration (or gap), where $t_0$ and $t_{N+1}$ are assigned to be the left and right ends of the segment, respectively. Now, define the moment

$$G_q = \frac{1}{N+1} \sum_{i=0}^{N} \left( \frac{\tau_i}{T} \right)^q ,$$

where $q$ is any positive integer. Clearly, $G_0 = 1$ and $G_1 = 1/(N+1)$. $G_q$ is a measure that quantifies the pattern of reversals in each segment. For large $q$, $G_q$ is a small number, since $\tau_i/T$ is small. Its value can be dominated by a few large gaps, as when $T < T_c$, or may become significant from the sum over many small contributions due to many small gaps, as when $T > T_c$. For a measure of the fluctuations of $G_q$ from segment to segment, we define an entropy-like quantity

$$S_q = - \langle G_q \ln G_q \rangle ,$$

where $\langle \cdots \rangle$ implies an average over all segments. For brevity we shall refer to the study of the time series in terms of $S_q$ as the gap analysis. In Fig. 2 we show by filled circles the result of the gap analysis on the wind data at $Ra= 1.5 \times 10^{11}$. It is evident that for $q \geq 2$ the points can be well fitted by a straight line, shown by the solid line, exhibiting an exponential behavior for $S_q$

$$\ln S_q = - \lambda q + \lambda_0 , \quad \lambda = 0.264 .$$

For the Ising simulation we must first decide on the proper value of the critical temperature $T_c$ for a finite lattice. For an infinite lattice its value has been determined analytically to be 2.269 in units of $J/\hbar \nu$, where $J$ is the coupling strength of near-neighbor interaction and $\hbar \nu$ the Boltzmann constant [12]. For a finite lattice the value of $T_c$ should be higher. We have performed the simulation of our Ising system at three values of $T$, and determined the properties of $M$ reversal. In Fig. 2 we show the results of our calculated values of $S_q$ at $T = 2.305, 2.310$ and 2.315. Only the one at $T = 2.310$ (lowered by a factor of 2 for clarity) shows a nearly linear dependence in the plot. The dashed line is a linear fit of the points in open circle, giving a slope of $\lambda = 0.261$. At the two neighboring values of $T$, the $q$ dependencies of $\log S_q$ (shown by triangles and squares) are not linear, the values at high $q$ being higher than at $T = 2.310$. The linear behavior at $T = 2.310$ is almost the same as in the wind reversal problem, as can be seen visually by the dash-dot line, which is a parallel transport of the solid line for comparison, but displaced slightly from the
dashed line to avoid overlap. We regard $T = 2.31$ as the critical temperature $T_c$ in our Ising system, since it has the unique property of being different from those of the neighboring $T$ on both sides. When $T < T_c$, the gaps are longer and $G_q$ is larger at large $q$ (but still $\ll 1$) with the consequence that $S_q$ is larger. When $T > T_c$, the gaps are shorter, but many gaps can contribute in the sum in Eq. (2), resulting in $G_q$ still being larger at large $q$ with the consequence that $S_q$ is also larger. It is only at the critical point that gaps of all sizes can occur, resulting in $G_q$ to be smaller and therefore $S_q$ also smaller at large $q$. Thus the exponential decrease of $S_q$ is a signature of criticality. The value of $T_c$ obtained here is in accord with the result of another calculation, in which the normalized factorial moments are found to exhibit non-scaling behaviors at neighboring $T$ [13]. In that calculation the measure studied quantifies the fluctuation of the cluster sizes in an Ising system on a square lattice of size $L = 288$, for which $T_c$ is found to be 2.315.

![Graph](image)

**FIG. 2**: Moments in the gap analysis of wind reversal (filled circles) and magnetization reversal in Ising lattice (open symbols) for different temperatures. The open circles are lowered by a factor of 2 to give space for clarity. The solid line is a linear fit of filled circles, and the dashed line is a linear fit of open circles. The dash-dot line is parallel to the solid line, placed near the dashed line for comparison.

The normalizations of $S_q$ for the wind and Ising problems are not the same, since the average numbers $N$ of reversals are different. However, the exponential behaviors are remarkably identical. The $q$ dependence of $S_q$ is a quantitative measure of the fluctuation behavior of the reversals. The fact that the slope $\lambda$ is the same for both the wind and magnetization problems suggests strongly that the wind reversal in convective turbulence at high $Ra$ is a critical phenomenon. Moreover, since we have not tuned any adjustable parameter in the wind problem to bring the system to the critical point, as we have done for the Ising system by varying $T$, we conclude that the wind reversal phenomenon is a manifestation of self-organized criticality (SOC) [14].

We now search for a power-law behavior that characterizes changes in the wind direction. (For other such efforts, see [7]). Our method is the detrended fluctuation analysis (DFA), which has been found to reveal the scale-independent nature of time series in a variety of problems, ranging from heartbeat irregularity [15] and EEG [16] to economics [17]. In that analysis we look for scaling behavior in the RMS deviation of the wind velocity from local linear trends. Given the time series of the wind velocity $V(t)$ over a total range of $T_{max}$, we divide it into $B$ equal bins of width $k$, discarding the remainder $T_{max} - Bk$. Let $V_b(t)$ denote the linear fit of $V(t)$ in the $b$th bin. The variance of the deviation of $V(t)$ from the local trend, $\bar{V}_b(t)$, in bins of size $k$ is defined by

$$F^2(k) = \frac{1}{B} \sum_{b=1}^{B} \frac{1}{k} \sum_{t=t_1}^{t_2} [V(t) - \bar{V}_b(t)]^2,$$  \hspace{1cm} (5)

where $t_1 = 1 + (b-1)k$ and $t_2 = bk$, measured in units of $\Delta t = 5$ sec, so that the values of $t$ are dimensionless integers that count the time points in the data. The goal is to study the behavior of the RMS fluctuations $F(k)$, as $k$ is varied. If there is no characteristic scale in the problem, then $F(k)$ should have a scaling behavior

$$F(k) \propto k^\alpha.$$  \hspace{1cm} (6)

This power law cannot be valid for arbitrarily large $k$ because the series $V(t)$ is bounded, so for very large $k$ the linear trend is just the $V(t) = 0$ line, and the RMS fluctuation $F(k)$ must become independent of $k$. Thus we expect $\ln F(k)$ to saturate and deviate from $\ln F$ at some large $k$. We note parenthetically that we have applied DFA to the unintegrated time series $V(t)$, which is a departure from the usual practice.

![Graph](image)

**FIG. 3**: Scaling behaviors of $F(k)$ in DFA of wind reversal (filled symbols) and magnetization reversal in Ising lattice at the critical temperature (open circles). Lines are linear fits.

In Fig. 3 we show $F(k)$ in a log-log plot for four equal segments of the complete wind data in solid symbols.
The segment seg1 is for time running from 0 to 116,435 s, corresponding to $t_{\text{max}} = 23,287$; other segments all have the same length. We have limited the maximum bin size to 2,580, so that even for the largest bin the fluctuations can be averaged over 9 bins. Evidently, there is a good scaling for each segment. The points for seg3 and seg4 are shifted upwards by the quantities indicated in order to give clarity without overlap. Note that the seg1 data do not have the same magnitude of $F(k)$ as the other segments; yet the scaling exponents are essentially the same. The deviation from the straight lines at the upper end is the saturation effect already discussed. There is another short region of scaling with a higher slope at low $k$. It is a consequence of fluctuations of the wind velocity within one direction of the wind, whose presence is evident in Fig. 1. Since the critical behavior identified here refers to wind reversals, and not to fluctuations of the wind velocity within one direction, we should ignore the lower short scaling region.

In the scaling region to which we pay attention here, the slopes are $\alpha = 1.20, 1.20, 1.21$ and 1.22, for seg1 to seg4, respectively. The deviations among the segments are obviously small. The average value is

$$\alpha = 1.21. \quad (7)$$

This large value of $\alpha$ implies a smoother landscape compared to the rough time series of white noise that is characterized by complete unpredictability [14]. Indeed, the fluctuations of the wind reversal time series has gaps of all sizes, the signature of critical behavior that is characterized by $1/f$ noise [14]. It is interesting to compare our result with the properties of the power spectral density for the velocity found in Ref. [18], where a scaling behavior is shown to exist with a slope roughly $-7/5$ (not by fitting) in the region $-3 < \log f < -1.8$. That range of frequency corresponds to $4.1 < \ln(1/f) < 6.9$. If we identify the values of $k$ in DFA to the time scale $1/f$, then that range of $\ln(1/f)$ corresponds to the range of $\ln k$ in Fig. 3, in which we find the scaling behavior with the exponent $\alpha$ given in Eq. (4). That value of $\alpha$ is not too different from $7/5$. The scaling behavior found in DFA uses shorter segments of the whole data and exhibits the power law more precisely, from which the value of $\alpha$ can be more accurately determined.

We now apply DFA to the Ising problem. We consider 10 segments of the $M$ reversal time series of the Ising lattice set at $T_c$, each segment having $10^4$ time points. From the $F(k)$ determined in each segment, we average over all segments and show the resultant dependence on $k$ in Fig. 3 by the open circles. Clearly, the points can be well fitted by a straight line. The slope is

$$\alpha_M = 1.22, \quad (8)$$

which is essentially the same as that in Eq. (4) for wind reversal. With the equivalence of these two scaling behaviors established, we have found stronger evidence that the wind reversal problem is a critical phenomenon.

To summarize, we have studied the time series of wind reversal in convective turbulence by two methods (gap analysis and detrended fluctuation analysis) and applied the same methods to the time series of the reversal of the net magnetization of a 2D Ising lattice. The same results are obtained for both problems. We therefore can assert that wind reversal exhibits all the essential properties characteristic of a critical behavior; apparently requiring no tuning, it can be regarded as self-organized.

This work was supported, in part, by the U. S. Department of Energy under Grant No. DE-FG03-96ER40972, and by the National Science Foundation under Grant No. DMR-95-29609.

[1] R. Krishnamurti and L. N. Howard, Proc. Natl. Acad. Sci. U.S.A. 78. 1981 (1981).
[2] M. Sano, X.-Z. Wu, and A. Libchaber, Phys. Rev. A 40, 6421 (1989).
[3] S. Ciliberto, S. Cioni, and C. Laroche, Phys. Rev. E 54, R5901 (1996).
[4] X.-L. Qiu, S.-H. Yao, and P. Tong, Phys. Rev. E 61, R6075 (2000).
[5] L. Kadanoff, Phys. Today 54(8), 34 (2001).
[6] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, J. Fluid Mech. 449, 169 (2001).
[7] K. R. Sreenivasan, A. Bershadskii and J. J. Niemela, Phys. Rev. E 65, 056306 (2002).
[8] J. J. Niemela and K. R. Sreenivasan, Physica A 315, 203 (2002).
[9] J. J. Niemela and K. R. Sreenivasan, Europhys. Lett. 62, 829 (2003).
[10] R. C. Hwa and Q. H. Zhang, Phys. Rev. D 62, 014003 (2000).
[11] R. C. Hwa and Q. H. Zhang, Phys. Rev. C 66, 014904 (2002).
[12] K. Huang, Statistical Mechanics (Wiley, New York, 1963).
[13] Z. Cao, Y. Gao, and R. C. Hwa, Z. Phys. C 72, 661 (1996).
[14] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A 38, 364 (1988).
[15] C.K. Peng, S. Havlin, H. E. Stanley, and A. L. Goldberger, Chaos 5, 82 (1995).
[16] R. C. Hwa and T. C. Ferree, Phys. Rev. E 66, 021901 (2002).
[17] K. Hu, P. C. Ivanov, Z. Chen, P. Carpena and H. E. Stanley, Phys. Rev. E 64, 011114 (2001).
[18] J. J. Niemela, L. Skrbek, K. R. Sreenivasan and R. J. Donnelly, J. Low Temp. Phys. 126, 297 (2002).