Self-interference of a toroidal Bose–Einstein condensate

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Abstract
We demonstrate the self-interference of a single Bose–Einstein condensate on a non-simply connected geometry, focussing on a toroidally trapped ring-shaped condensate. First, we show how the opposite parts of the ring can interfere using the Wigner function representation. Then, using analytical expressions for the time-evolution of a freely expanding ring-shaped condensate with and without a persistent current, we show that the self-interference of the ring-shaped condensate is possible only in the absence of the persistent current. We conclude by proposing an experimental protocol for the creation of ring dark solitons using the toroidal self-interference.

Keywords: ring trap, toroidal geometry, self-interference, annular condensate, ring dark soliton

1. Introduction
Following the pioneering demonstration of interference between two freely expanding Bose–Einstein condensates (BECs) [1], also self-interference of a single condensate has been experimentally and numerically observed in hard-wall reflections [2, 3]. Later generalizations to optical lattices [4] have led to interference between as many as 30 separate BECs [5, 6].

In a conventional self-interference setting, the wavefunction splits up into two (or more) phase-correlated pieces that travel different paths; subsequently the self-interference arises
when the parts are later spatially recombined. This process can happen next to a hard-wall potential, as mentioned, whereby the condensate interferes with its reflection. However, a natural setting to consider the self-interference of a BEC in the absence of any boundary effects and splittings of the wavefunction is a non-simply connected geometry such as a torus. A BEC in a ring trap forms an extended quantum object that winds back to itself, and by controlling the size of the central forbidden region, one might expect the separate parts of the same condensate to show self-interference phenomena.

As the phase itself is not an observable, the appearance of interference fringes reflects the relative long-range coherence between the separate condensates, or different parts of the same condensate. Even in the case of well-defined atom numbers and fully uncertain phases, interference is produced by the action of the quantum measurement, which assigns instantaneous relative phase correlations [7]. In the case of self-interference, this action is equivalent to the condensate encountering itself. The phase coherence of BECs also leads to many other important properties such as quantized vortices [8] and dark solitons [9].

In general, ring traps [10–18] have attracted significant interest in the form of e.g. persistent currents [19, 20], atomic-phase interference devices [21], vortex dynamics [22, 23], and ring dark solitons (RDSs) [24–30], but their role in inducing self-interference is an open question. Also, given the high experimental relevance of ring traps, observation of toroidal self-interference offers one way to probe the spatial phase coherence over the extent of the condensate in the ring, and it can also serve as a mechanism for the creation of RDSs, as we show in section 3.3.

A RDS is a cylindrically symmetric circular dark soliton stripe that itself is a plane-wave extension of the one-dimensional dark soliton into two dimensions. The one-dimensional dark solitons are mathematically stable solutions of the Gross–Pitaevskii equation (GPE), which display a remarkable form-stability, e.g. they propagate without dispersion [31]. On the contrary, as a result of their two-dimensional nature, RDSs (and the two-dimensional planar dark solitons) are not in general stable, but instead decay into circular (or straight) arrays of vortex–antivortex pairs through the so-called snake instability [22, 25, 32–35].

The snake instability of a planar dark soliton can be avoided if the system is made quasi-one-dimensional [36], for example, by winding the planar dark soliton into a RDS of short enough circumference [37]. However, it has been recently shown that in a harmonic ring trap, the RDS can be made stable (i.e. sufficiently long-lived) also without restricting its length by instead making the transverse trapping tight enough [38] (specifically, $d \lesssim 3\xi$, where $\xi$ is the healing length, and $d$ is the radial transverse width of the ring trap). Therefore, this kind of a toroidal trap makes it possible to regain the characteristic natural stability (and the physics) of a one-dimensional dark soliton also in two dimensions provided that we consider a RDS instead, which makes the experimental study of these long-lived two-dimensional objects interesting. For example, we have performed numerical simulations showing that an initially displaced RDS is oscillating in the radial direction in a harmonic ring trap qualitatively like a classical particle, similarly to the physics of a one-dimensional dark soliton [39]. In other words, RDSs behave as particles with quasi-elastic collisions [40], and their dynamics is determined by conservation of energy [41] (see also [42–44] for other more detailed analytic treatments of the rich dynamics of RDSs).

Moreover, combining RDSs with bright counterparts [45] paves the way for other interesting physics such as constructing stable molecule-like bound states from these vector
solitons [46]. Given that the snake instability can be suppressed by focusing on RDSs as explained above, it seems plausible to consider two dark-bright ring solitons forming the molecules [47]. In addition, a pair of Josephson junctions (a cold atom SQUID) has been experimentally realized quite recently in a toroidal BEC [48]. In [28], a link between RDSs and ring fluxons [49] in large-area Josephson junctions [50] was established.

Furthermore, periodic oscillations between a vortex dipole (i.e. a single vortex–antivortex pair) and a short dark (grey) soliton have been experimentally observed [51], but a RDS is at least an order of magnitude longer in length, correspondingly producing several vortex–antivortex pairs in the snake instability. Eventually the decay leads to an example of quantum turbulence, but before that the vortex–antivortex arrays have been shown to recombine back into the RDSs (when \( d \lesssim 16 \xi \) in the case of a single RDS) [38, 26]. Experimental study of the coherent vortex dynamics before the onset of quantum turbulence (a type of a laminar-to-turbulent transition) forms another interesting topic regarding RDSs. To date, RDSs have not been experimentally observed in cold atomic BECs.

In the work presented here, we investigate in detail how the self-interference of a BEC can appear in a ring trap. We consider analytically the Wigner function representation of a Gaussian ring condensate, showing that if the radius of the ring trap is brought to zero, all the self-interference phenomena disappear, as expected. For non-zero radii, on the other hand, we demonstrate a possible experimental way to observe the self-interference by letting the ring expand freely and overlap with itself to produce circular fringes. Interestingly, we find that in the presence of a persistent current, there cannot be any self-interference because the central vortex is enforcing a density hole at \( r = 0 \), preventing the condensate from overlapping with itself. Insofar as the centrifugal potential barrier acts as a circular hard wall, we will not consider the separate case of self-interference arising from reflection of the condensate. Finally, based on the toroidal self-interference, we propose a protocol for the experimental creation of RDSs.

2. Theoretical background

We consider a scalar order parameter \( \psi \), representing the macroscopic wavefunction of a BEC trapped in a potential given by \( V_{\text{trap}} \), which is a solution to the GPE:

\[
i\psi = -\nabla^2 \psi + V_{\text{trap}} \psi + C_{2D} |\psi|^2 \psi.
\]

Here we have assumed a two-dimensional condensate whereby the z-direction is tightly trapped to the corresponding harmonic oscillator ground state (\( \omega_z \gg \omega_x = \omega_y \equiv \omega_r \)) and has been projected onto the xy-plane. Then \( C_{2D} = 4 \sqrt{\pi N a/a_r} \), where \( N, a, \) and \( a_r \) are the number of atoms in the cloud, the s-wave scattering length of the atoms, and the characteristic trap length in the z-direction respectively. We have obtained dimensionless quantities by measuring time, length and energy in terms of \( \omega_z^{-1}, a_r = \sqrt{\hbar/(2m \omega_z)} \) and \( \hbar \omega_z \) respectively, where \( \omega_z \) is the angular frequency of the trap in the r-direction. This basis is equivalent to setting \( \omega_z = \hbar = 2 \hbar = 1 \). All the units in this work are expressed in this dimensionless basis.

Using the experimental parameters of the toroidal BEC of [48] and \(^{87}\text{Rb} \), we obtain \( a_0 = 0.8 \mu \text{m}, C_{2D} \sim 200–2000 \), and one unit of time is \( \sim 1.5 \) ms. The experimental setup is quasi-two-dimensional with an approximative aspect ratio of 1:40 in the vertical and radial directions respectively (see also [52] for more details of the apparatus), with the quasi-two-
dimensional geometry restricting the atom numbers to \( N \sim 1 \times 10^3 \) in the absence of any painted two-dimensional potential. The actual number of atoms depends on the volume of the annulus that is printed on the free two-dimensional gas; the smaller the volume, the smaller the number of atoms. In the toroidal BEC of \([48]\), \( N = 1000–8000 \), while in the toroidal BEC of \([52]\), \( N = 5000 \).

We consider the toroidal condensate to be in the ground state of the harmonic ring trap given by

\[
V_{\text{trap}} = \frac{1}{4} \omega^2 (r - r_0)^2, \tag{2}
\]

where we take \( r_0 = 4.5 \), and \( \omega = 20 \). To a very good approximation, the ground state of the potential (2) with a weak nonlinear interaction of \( C_{2D} = 50 \) is given by a Gaussian,

\[
\psi_\omega(r) = \mathcal{N} \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right), \tag{3}
\]

where \( \mathcal{N} \) is the normalization, and \( \sigma = \sqrt{2/\omega} \).

### 3. Self-interference in the torus

#### 3.1. Wigner function of the ring

The effect of the ring trap in inducing self-interference can be seen in the Wigner function \([53, 54]\). There exists a one-to-one mapping between the vector \( \langle \mathbf{r} | \psi \rangle = \psi(\mathbf{r}) \) and the corresponding Wigner function \( W_\psi(\mathbf{r}, \mathbf{p}) \in \mathbb{R} \), defined for our two-dimensional states by

\[
W_\psi(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi)^2} \int \mathbf{u} \psi^*(\mathbf{r} - \frac{\mathbf{u}}{2})\psi\left(\mathbf{r} + \frac{\mathbf{u}}{2}\right)e^{-i\mathbf{u}\mathbf{p}}. \tag{4}
\]

The Wigner function can be thought of as being another name or representation for the state \( |\psi\rangle \), containing all the possible information of the state, with the important property that the marginals of the Wigner function recover the position and momentum distributions:

\[
\int_{\mathbb{R}^2} d\mathbf{p} \ W_\psi(\mathbf{r}, \mathbf{p}) = |\langle \mathbf{r} | \psi \rangle|^2, \tag{5a}
\]

\[
\int_{\mathbb{R}^2} d\mathbf{r} \ W_\psi(\mathbf{r}, \mathbf{p}) = |\langle \mathbf{p} | \psi \rangle|^2. \tag{5b}
\]

A necessary and sufficient condition for the Wigner function to be a true phase space density \([55]\) states that this happens only for coherent and squeezed vacuum states (Gaussian states). However, the Wigner function representation is often quite suitable for presenting non-classical states, and in our case, the negativity of the Wigner function is associated with the emergence of toroidal matter-wave interference.

Substituting equation (3) into equation (4), and setting \( r = 0 \) as we are interested in the neighbourhood of the origin, we obtain

\[
W_{\psi_\omega}(0, p) = \mathcal{N}^2 \int_0^\infty dp J_0(up) e^{-\frac{(ap)^2}{\sigma^2}}, \tag{6}
\]
where $J_0$ is the Bessel function of order 0. We can already see that $W_{\phi_0}(0, p)$ is manifestly positive, i.e. there are no fringes. The interference effects due to the toroidal geometry vanish when the radius of the torus is taken to zero. To consider the case of $r_0 > 0$, the integral in equation (6) can be evaluated analytically with a small correction that must be calculated numerically (see the appendix).

The interference effects induced by the toroidal geometry (of radius $r_0$) are shown in figure 1 in terms of the Wigner function. Considering the case of $r_0 = 4.5$ (left panel), the fringes in the middle of the torus show that the opposite parts of the ring are interfering with each other. As expected, for $r_0 = 0$ there can be no toroidal self-interference, and the Wigner function is manifestly positive (right panel). As $r_0$ is being increased from zero and therefore the ring gets a non-zero radius, we can see the appearance of interference fringes in $p$ (at the origin of the ring $r = 0$), with the fringe separation decreasing as $r_0$ increases.

The fringes of figure 1 do not appear in the position distribution as long as the trapping potential forbids occupation in the central region. Still, these results suggest that if we start with a toroidal condensate at some finite radius and let the ring expand towards the origin, we should observe circular fringes when the opposite and in general different parts of the ring overlap with each other.

### 3.2. Self-interference of an expanding ring

To gain preliminary understanding, we integrate the GPE (1) numerically in a time-dependent Mexican hat potential after the central region of the hat has been suddenly removed and the toroidal condensate allowed to expand freely towards the centre. The ground state is first found by propagating the GPE in imaginary time, and the propagation is done in real time. We use a split-operator Fourier method [56]. The potential we choose has the following form:

![Figure 1](image_url)
where $H$ is the Heaviside step function. The simulation reveals that once the opposite parts of the ring start to overlap around the origin at $t \approx 1.5$, circular fringes develop around a bright central spot (see figure 2). This is the self-interference of the ring induced by the toroidal geometry. We note that in the numerical simulation the abrupt removal of the central barrier only makes the condensate expand freely towards the centre, without creating other excitations.

$$V_{\text{trap}}(r, t) = \frac{1}{4} r^2 + 30H(-t)e^{-0.1r^2},$$  \hspace{1cm} (7)
We can also study more analytically the time-evolution of the wavefunction after the ring is allowed to expand inwards. We neglect the (weak) nonlinear interaction and also assume that the external potential vanishes after the toroidal trapping is released so that the time evolution is given by the free-particle linear Schrödinger equation.

To get the time evolution, we multiply the zeroth order Fourier–Bessel transform \( \tilde{\varphi}_G(k) \) of \( \varphi_r \) by \( e^{ikt} \) in \( k \)-space, where \( k \) is the magnitude of the (radially symmetric) Fourier space wavevector, and take the inverse transformation, viz.:

\[
\varphi_G(r, t) = \int_0^\infty dk J_0(kr) \tilde{\varphi}_G(k) e^{ikt} 
\]

\[
= N \int_0^\infty dk \, dr' \, k J_0(kr) J_0(kr') e^{-\frac{(r-r')^2}{2\sigma^2}} e^{ikt}.
\]

We note that while \( \tilde{\varphi}_G(k) \) can be evaluated exactly similarly to equation (6), it is a good approximation to consider only the \( m = 0 \) term (see the appendix):

\[
\varphi_G(r, t) = N \sigma^2 \int_0^\infty dk \, k J_0(kr) J_0(kr_0) e^{-\frac{k^2\sigma^2}{2}} e^{ikt}.
\]

If \( r_0 = 0 \), then equation (10) becomes exact and equivalent to equation (8), and can be evaluated in closed form to obtain an expanding radial Gaussian with no further dynamics;

\[
\varphi_G(r, t)\big|_{r_0=0} = \frac{N \sigma^2}{\sigma^2 - 2it} \exp \left( -\frac{r^2}{2\sigma^2 - 4it} \right).
\]

For \( r_0 > 0 \), on the other hand, the extra factor of \( J_0(kr_0) \) will give rise to oscillations, i.e. the origin of the self-interference fringes of a ring condensate in a cylindrically symmetric system is a Bessel function as opposed to the typical cosine modulation in planar systems. Evaluating equation (10) gives

\[
\varphi_G(r, t) = \frac{N \sigma^2}{\sigma^2 - 2it} \exp \left( -\frac{r^2}{2\sigma^2 - 4it} \right) \sum_{m=0}^\infty \frac{2(-r_0^2)^m}{m!(2\sigma^2 - 4it)^{m+1}} F_1 \left( 1 + m, 1, -\frac{r^2}{2\sigma^2 - 4it} \right)
\]

\[
= \frac{N \sigma^2}{\sigma^2 - 2it} \exp \left( -\frac{rr_0}{2\sigma^2 - 4it} \right) I_0 \left( \frac{rr_0}{\sigma^2 - 2it} \right),
\]

where the second equality follows after some algebra and has the form of a Skellam distribution [57] for complex arguments, and \( F_1 \) is the confluent hypergeometric function. In the limit as \( r_0 \to 0 \), equation (11) is a special case of equation (13) which in general describes an expanding ring condensate of initial radius \( r_0 \).

We note that equation (13) is general given that \( C_{2D} = 0 \), and it has the form of an exponential envelope modulated by \( I_0 \), the modified Bessel function of order 0. Very recently, matter-wave Bessel beams have been also experimentally produced by a similar free expansion of a toroidal condensate [58]. The self-interference fringes arise from \( I_0 \) as its argument is complex, but an exhaustive mapping of fringe periods, time scales, and contrasts as functions of \( r_0 \) and \( t \) is beyond the scope of this work.
Figures 3(a)–(c) show the time evolution given by equation (13) for various $r_0$. All the essential features of figure 2 are reproduced; the central bright peak and the circular self-interference fringes around it can be seen to emerge once the opposite parts of the ring overlap.

As an interesting generalization, we conclude this section by considering a vortex state (i.e. a persistent current) with winding number $\zeta$, given by $\varphi_\ell(r, \theta) = \varphi_G(r) r e^{i\zeta \theta}$. Taking the $\zeta = m$ term in a similar fashion as above, but setting $\zeta = 1$ for definiteness and assuming $r_0 \ll 1$ to avoid cumbersome expressions, we get

$$\varphi(r, \theta, t) = N \sigma^4 e^{i\theta} \int_0^\infty dk \, k^2 J_1(kr) J_0(kr_0) e^{-\frac{1}{2}k^2 r^2} e^{ik^3 t}.$$  

Figure 3. Density plot of a ring-shaped condensate of initial radius (a) $r_0 = 1.5$, (b) $r_0 = 4.5$, and (c) $r_0 = 20$ expanding freely as a function of time (see equation (13)). Self-interference fringes appear when opposite parts of the rings meet at $t \approx 0.1$, $t \approx 0.4$, and $t \approx 1.1$ respectively. Here $\sigma = \sqrt{2/20}$. (d) Density plot of a ring-shaped persistent current ($r_0 = 0.4$) expanding freely as a function of time (see equation (16)). There is no self-interference because the condensate cannot expand to $r = 0$ to overlap with itself. We have numerically confirmed that direct integration of the GPE with $C_{2D} = 0$ gives the same results as the analytical evolutions shown here. The colouring is not to scale.
The vortex at \( r = 0 \) must be accompanied by a hole in the density to avoid a diverging kinetic energy. This makes the vortex state unstable in an unrotated simply-connected condensate [59], but on toroidal geometry, the vortex line can exist in the central forbidden region with the result that the persistent current state is stable. The presence of the phase singularity means that the opposite parts of the ring condensate cannot overlap in the free expansion, and hence their self-interference is not possible (see figure 3 (d)). In [60], it was experimentally observed that this central hole in the atomic density is present even after a long period of free expansion.

3.3. Experimental creation of RDSs by toroidal self-interference

Earlier, we have proposed a protocol for the controlled creation of multiple concentric RDSs by means of a time-dependent double-well trap [26]. In [61], RDSs are identified with the nodes of numerically found radial solutions of the GPE (with cylindrical symmetry) that approach the Bessel functions in the linear limit, which, as we have shown, also give rise to the self-interference fringes. Here, we propose an alternative protocol that involves the use of the toroidal self-interference as a density imprinting mechanism for their creation. In other words, we let the BEC produce an interference pattern, which then evolves into RDSs. We note that density imprinting has been experimentally demonstrated for the production of planar dark solitons [51], and that interference fringes have been shown in general to evolve into stable soliton-like structures in the case of two colliding condensates if the kinetic energy of the condensates during the collision does not dominate over the nonlinear self-energy [62].

At least some of the fringes shown in figure 2 are soliton-like rings that are long-lived and oscillate back and forth in the harmonic trap repelling each other. When they are at the turning points of their oscillation, the associated phase step has a magnitude close to \( \pi \), which characterizes dark solitons in general. As a proof-of-concept demonstration for the creation of RDSs through toroidal self-interference, we consider the potential (7), but after the self-interference has taken place, we ramp the central region back up. Because the condensate obtains kinetic energy upon the removal of the central region, we must replace it by a higher barrier to prevent the condensate from sloshing back and forth over it. As an example potential, we consider a linear ramp starting at \( t = 1 \) and lasting until \( t = 8 \):

\[
V_{\text{trap}}(r, t) = \frac{1}{4} r^2 + 30 \left[ H(-t) + 3 H(t-1) H(-t+8) \frac{t-1}{7} + 3 H(t-8) \right] e^{-0.1 r^2} \tag{17}
\]

with \( C_{2D} = 400 \).

The time evolution given by the potential (17) above is shown in figure 4. Initially as the central region is removed, the condensate starts expanding and obtains kinetic energy. When the potential blocking the central region is later turned back on, the condensate is bound on the
toroidal geometry again, albeit with some kinetic energy. As is evident in figure 4, in this case the self-interference prints two clear long-lived RDSs. We have not observed the snake instability in the time scales associated here.

Apart from the resulting motion of the condensate, we have demonstrated that it is possible to generate RDSs in toroidal (and harmonic) traps by starting from a toroidal condensate that is let to interfere with itself. We note that such a protocol is easy to implement experimentally. For example, in [48], a condensate is prepared in a ring of radius $\mu_4m$ using time-averaged painted potentials, comparable to the first frame in figure 4. The spatial resolution (of details) of the painted potential is stated to be $\mu_\sim 1.5m$, which is significantly smaller than the size of the central barrier in our case. As stated in section 2, the physical scales corresponding to the experiment [48] of our dimensionless units are $a_0.8mr_\sim = C_2D^{2002000}$, and one unit of time is $\sim 1.5ms$. Our choice of $C_2D$ in figure 4 falls within this regime.

Since no experimental potential can be perfectly smooth and radially symmetric, a brief address about the sensitivity of this protocol to symmetry is needed. During the appearance of the self-interference fringes, the setup is quite symmetric as there is no central trapping. Later, when the central barrier is reintroduced, its position might not be perfectly in the middle, but direct numerical integration of the GPE ($C_2D = 400$) reveals that the protocol works and the RDSs do not decay into vortex–antivortex pairs if the location of the central barrier can be specified to an accuracy of $\sim 0.4 a_t = 0.3 \mu m$. We also note that a shallow (i.e. moving) RDS, which in general does not undergo the snaking instability, has been shown to be stable against small distortions of the radial symmetry [63]. For a discussion about the effect of the potential smoothness on the RDS stability, we refer to [25], and note that the snake instability is suppressed in general by making the healing length larger with respect to the trap size [38].

4. Conclusions

In conclusion, we have studied the self-interference of a toroidal condensate both with and without a persistent current. We investigated the Wigner function representation of the ring condensate to see how the self-interference arises from the toroidal geometry. In particular, for a
ring of zero radius, all interference phenomena vanished. We predicted both numerically and analytically the appearance of circular fringes if a toroidal condensate is allowed to freely expand towards the origin. In contrast, in the presence of a toroidal persistent current, we showed that there cannot be any ring-induced self-interference because the opposite parts of the condensate cannot overlap in the free expansion. Furthermore, we gave a proof-of-principle demonstration of how RDSs can be experimentally created using the toroidal self-interference.

The results are important because they open new possibilities for demonstrating the quantum wave-nature of matter on toroidal geometry. This enables probing the phase-coherence over the ring, for example, and also to experimentally create RDSs in ring traps. The rich dynamics of RDSs provides several possibilities for further work. One timely area of interest is the coherent recombination of circular vortex–antivortex arrays back into the RDSs after the snake instability, before the decay eventually results in an example of quantum turbulence [38, 26].

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**Appendix A. Evaluation of equation (6)**

Equation (6),

\[ W_{\psi_0}(0, p) = N^2 \int_0^{\infty} du \, u p J_0(u p) e^{-(u/2-r_0)^2/\sigma^2}, \quad \text{(A.1)} \]

is a non-standard integral involving the zeroth order Bessel function \( J_0 \). Let us first make the substitution \( t = u/2 - r_0 \) in equation (A.1):

\[ W_{\psi_0}(0, p) \approx N^2 \int_0^{\infty} dt \, 4(t + r_0) p J_0[2p(t + r_0)] e^{-(t^2/4r_0^2)}, \quad \text{(A.2)} \]

where we have assumed that \( r_0 \) is small enough that the lower limit of the integral can be set to 0. The error thus introduced could be corrected by numerically evaluating the integral in equation (A.2) from \( t = -r_0 \) to \( t = 0 \). Let us then make use of the Bessel function addition theorem

\[ J_n(y + z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z) \quad \text{(A.3)} \]

together with the symmetry property \( J_{-m}(z) = (-1)^m J_m(z) \) to obtain

\[ W_{\psi_0}(0, p) = \sum_{m=1}^{\infty} 8N^2 p(-1)^m J_m(2pr_0) \int_0^{\infty} dt \, (t + r_0) J_m(2pt) e^{-(t^2/4r_0^2)} \]

\[ + 4N^2 p J_0(2pr_0) \int_0^{\infty} dt \, (t + r_0) J_0(2pt) e^{-(t^2/4r_0^2)}. \quad \text{(A.4)} \]
After evaluating the integrals in equation (A.4), we obtain

\[
W_{\psi_0}(0, p) = \sum_{m=1}^{\infty} 4N^2 (-1)^m J_m(2p r_0) \\
\times p^{m+1} \sigma^{2+m} \left[ \Gamma \left( 1 + \frac{m}{2} \right) \frac{\text{\emph{F}}_1 \left( 1 + \frac{m}{2}, 1 + m, -p^2 \sigma^2 \right)}{\Gamma (1 + m)} + \frac{r_0}{\sigma} \Gamma \left( \frac{1 + m}{2} \right) \frac{\text{\emph{F}}_1 \left( \frac{1 + m}{2}, 1 + m, -p^2 \sigma^2 \right)}{\Gamma (1 + m)} \right] \\
+ 2N^2 p J_0(2p r_0) \sigma^2 e^{-p^2 \sigma^2} \left[ 1 + \frac{\sqrt{\pi} r_0 J_0 (p^2 / 2)}{\sigma} e^{p^2 \sigma^2} \right],
\]  
(A.5)

where \( I_0 \) is the modified Bessel function of order 0, \( \Gamma \) is the Gamma function, and \( \text{\emph{F}}_1 \) is the confluent hypergeometric function. If \( r_0 = 0 \), equation (A.5) reduces to \( 2N^2 p \sigma^2 e^{-p^2 \sigma^2} \), as required. In the right panel of figure 1, we have cut the infinite summation over \( m \) with \( m_{\text{max}} = 10 \).

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