Gravity as Lorentz Force

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Abstract. The main object of the proposed theory is not a pseudometric, but a symmetric affine connection on the Minkowski space. The coefficients of this connection have one upper and two lower indices. These coefficients are symmetric with respect to the permutation of the lower indices. We identify the convolution of the connection coefficients with the vector – potential of the electromagnetic field. Then the gravity is the Lorentz force of this electromagnetic field.

1 Introduction

The idea of the relativistic gravitation theory was proposed by Poincaré [1]:

"In the paper cited Lorentz found it necessary to supplement his hypothesis so that the relativity postulate could be valid for other forces besides the electromagnetic ones. According to his idea, owing to Lorentz transformation (and therefore owing to the translational movement) all forces behave like electromagnetic.

It turned out to be necessary to consider this hypothesis more attentively and to study the changes it makes in the gravity laws in particular. First of all, it enables us to suppose that the gravity forces propagate not instantly, but at the light velocity. One could think that it is enough to reject such a hypothesis, for Laplace has shown that it can not take place. But in fact the effect of this propagation is largely balanced by some other circumstance, so there is no any contradiction between the law proposed and the astronomical observations.

Is it possible to find a law satisfying the condition stated by Lorentz and at the same time coming to the Newton law in all the cases when the velocities of the celestial bodies are small enough to neglect their squares (and also the products of the accelerations and the distance) with respect to the square of the velocity of light?"

Poincaré found that the mathematical solution of the problem is not unique. It is easy to solve the Poincaré problem by making use of the physical reasons. The form of the Newton gravity law coincides with the form of the Coulomb law describing the interaction between two oppositely charged bodies. The relativistic form of the Coulomb law is well – known. It is the Lorentz force. Thus the relativistic form of the gravity law must be the same.

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It is possible to solve the derived equations by using a computer. We consider a simple problem of the interaction of two bodies when the mass of one body is equal to zero. It is a problem of the light propagation in the gravity field of one body. The received exact solution of this problem describes all effects predicted by the general relativity: the distortion of the light beams in the gravity field, the light motion along the closed trajectory in the gravity field, etc. If the Mercury mass is considered small, it is possible to calculate the shift of Mercury perihelion for a hundred years. It turns out to be 45”. The general relativity predicts 43”.

In this paper we do not consider the radiation effects. It seems that these effects lead to the non-stability of the orbits of the celestial bodies. These effects need computer calculations. It seems that the study of these effects requires not electrodynamics, but quantum electrodynamics.

In the general relativity the equation of motion in the gravity field is the geodesic equation where the parameter is the proper time corresponding to the pseudometric defining the gravity field. It is possible to obtain the Newton gravity law from this equation only when one of two bodies is at rest. For an arbitrary case it is needed the geodesic equation with the parameter not connected to the pseudometric and with the connection coefficients which are not the Christoffel’s symbols (the connection is not Riemannian and it is not compatible with any pseudometric). Therefore it seems natural to consider the generalization of general relativity when not a pseudometric but a symmetric affine connection on the Minkowski space is given. As the equations of motion we consider all possible equations of geodesic type. The parameter of these equations is the proper time corresponding to the Minkowski pseudometric. It turns out that the unique Lagrangian equation among these equations is the equation of motion under the action of Lorentz force. We come back to Poincaré idea.

Let us consider the equation of field. In the book ([2], Chapter 7) all theories are examined where gravitation potential is taken in the form of scalar, vector and at last symmetric tensor fields. All such theories have their defects. We consider the gravitation potential as the coefficient of the symmetric affine connection on the Minkowski space. This coefficient has three indices and it is not a tensor. It is possible to consider this coefficient as Yang–Mills field. The curvature tensor corresponding to this connection is the Yang–Mills field strength tensor. Following Einstein, we study the convolutions of the curvature tensor. The curvature tensor has one upper and three lower indices. It is antisymmetric in respect of the permutation of two lower indices. Therefore two convolutions of the curvature tensor are possible. One of these convolutions is the trace of the Yang–Mills field strength tensor. It coincides with the electromagnetic field strength tensor. The vector–potential of this electromagnetic field is the convolution of the symmetric affine connection coefficients. The symmetric affine connection coefficient has one upper and two lower indices. It is symmetric with respect to the permutation of the lower indices. Hence the convolution of the coefficients of the symmetric affine connection is unique. The second convolution of the curvature tensor is the Ricci tensor. The antisymmetric part of Ricci tensor is proportional to above electromagnetic field strength tensor. If the symmetric affine connection is Riemannian (compatible with some pseudometric), then the electromagnetic field strength tensor is equal to zero and the Ricci tensor is symmetric with respect to the permutation of its indices. The symmetric affine connection coefficients satisfy the wave equations. The substitution of the first convolution of the curvature tensor into the equation of motion under the action of the Lorentz force yields the relativistic Newton gravity law.

In the second section we study the relativistic Newton gravity law. The third section
is devoted to the relation between the geodesic equation and the Newton gravity law. In the fourth section the curvature tensor of the symmetric affine connection on the Minkowski space is considered.

2 Relativistic Newton Gravity Law

Let two bodies have the masses \(m_k\), \(k = 1, 2\), and move along the world lines \(x^\mu_k(t)\), \(k = 1, 2\), \(\mu = 0, ..., 3\), \(x^0_k(t) = ct\). The gravitation interaction of these bodies is given by the Newton gravity law

\[
\frac{d^2 x^i_k}{dt^2} = -\frac{\partial U(x_k; \dot{x}_k)}{\partial x^i_k},
\]  

\[\sum_{j=1}^{3} \left( \frac{\partial}{\partial x^j} \right)^2 U(x; y) = 4\pi m_g G \delta(x - y(t))
\]

where \(i = 1, 2, 3; k = 1, 2; U(x_1; \dot{x}_1) = U(x_1; x_2), U(x_2; \dot{x}_2) = U(x_2; x_1)\) and the gravitation constant

\[G = (6,673 \pm 0,003) \cdot 10^{-11} \ m^3 \cdot kg^{-1} \cdot s^{-2}.
\]

The form of the law (2.4), (2.2) coincides with the form of the Coulomb law for the electric interaction of two oppositely charged bodies. The relativistic form of the Coulomb law is the movement under the action of the Lorentz force. Therefore the relativistic form of the Newton gravity law is the movement under the action of the Lorentz force

\[
c \frac{d^2 x^\mu_k}{ds_k^2} = -\frac{1}{c} \eta^\mu\nu \sum_{\nu = 0}^{3} F_{\mu
u}(x_k; \dot{x}_k) \frac{dx^\nu_k}{ds_k},
\]

\[F_{\mu\nu}(x; y) = \frac{\partial A_\nu(x; y)}{\partial x^\mu} - \frac{\partial A_\mu(x; y)}{\partial x^\nu},
\]

\[\sum_{\nu = 0}^{3} \eta^\nu\nu \left( \frac{\partial}{\partial x^\nu} \right)^2 A_\mu(x; y) = \eta_{\mu\nu} 4\pi m_g G \frac{dy^\nu(x^0)}{dx^0} \delta(x - y(x^0/c)),
\]

\[\frac{dx^\mu}{ds} = \frac{dt}{ds} \frac{dx^\mu}{dt}, \quad \frac{d^2 x^\mu}{ds^2} = \frac{dt}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\mu}{dt} \right),
\]

\[\frac{dt}{ds} = c^{-1}(1 - \frac{1}{c^2} |\dot{x}|^2)^{-1/2}
\]

where \(\mu = 0, ..., 3; k = 1, 2; |\dot{v}|^2 = \sum_{i=1}^{3} (v^i)^2\); the 4 \times 4 matrix \(\eta_{\mu\nu}\) is diagonal and \(\eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1\); the 4 \times 4 matrix \(\eta^\mu\nu\) is the inverse matrix of \(\eta_{\mu\nu}\).

Let the second body be at rest: \(x^i_2(t) = 0\), \(i = 1, 2, 3\). Let \(A_i(x) = 0\), \(i = 1, 2, 3\), and \(A_0(x)\) be the solution of the equation (2.6) for \(\mu = 0\), \(y^i(t) = 0\), \(i = 1, 2, 3\). \(A_0(x)\) does not depend on the variable \(x^0\). Then the equations (2.4) for \(k = 1\) have the following form

\[c \frac{d}{dt} \left( (1 - \frac{1}{c^2} |\dot{x}_1|^2)^{-1/2} \right) = \frac{1}{c} \sum_{i=1}^{3} \frac{\partial A_0(x_1)}{\partial x^i_1} \frac{dx^i_1}{dt},
\]

\[\frac{d}{dt} \left( (1 - \frac{1}{c^2} |\dot{x}_1|^2)^{-1/2} \frac{dx^i_1}{dt} \right) = \frac{\partial A_0(x_1)}{\partial x^i_1}
\]
where \( i = 1, 2, 3 \). The following identity is valid
\[
\frac{d}{dt} \left( 1 - \frac{1}{c^2} \frac{d}{dt} \left( \frac{dx^i}{dt} \right)^2 \right)^{-1/2} = \frac{1}{c} \sum_{i=1}^{3} \frac{dx^i}{dt} \frac{d}{dt} \left( 1 - \frac{1}{c^2} \frac{d}{dt} \left( \frac{dx^i}{dt} \right)^2 \right)^{-1/2} \frac{dx^i}{dt}.
\] (2.11)

This identity implies that the equation (2.9) is the consequence of the equations (2.10). The comparison of the equations (2.2) and (2.6) for \( y'(t) = 0 \) gives \( A_0(x) = -U(x) \). Hence if \( \left| \frac{dy_1}{dt} \right|^2 \) is small with respect to \( c^2 \), the equations (2.10) coincide with the equations (2.1) for \( k = 1 \).

Let us consider the general case when two bodies move. By using the identity (2.11) it is easy to prove that the equations (2.4) for \( \mu = 0 \) are the consequence of the equations (2.4) for \( \mu = 1, 2, 3 \). Since \( x_k^0(t) = ct \) the equations (2.4) – (2.6) may be written in following form
\[
\frac{d}{dt} \left( 1 - \frac{1}{c^2} \frac{d}{dt} \left( \frac{dx_k^i}{dt} \right)^2 \right)^{-1/2} \frac{dx_k^i}{dt} = \frac{\partial A_0(x_k(t); \hat{x}_k)}{\partial x_k^i} - \frac{1}{c} \frac{\partial A_1(x_k(t); \hat{x}_k)}{\partial t} + \frac{1}{c} \sum_{j=1}^{3} \left( \frac{\partial A_j(x_k(t); \hat{x}_k)}{\partial x_k^i} - \frac{\partial A_1(x_k(t); \hat{x}_k)}{\partial x_k^i} \right) \frac{dx_j^j}{dt},
\] (2.12)
\[
\left( \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 - \sum_{j=1}^{3} \left( \frac{\partial}{\partial x_j} \right)^2 \right) A_0(x; y) = 4\pi m_y G \delta(x - y(t)),
\] (2.13)
\[
\left( \frac{1}{c^2} \left( \frac{\partial}{\partial t} \right)^2 - \sum_{j=1}^{3} \left( \frac{\partial}{\partial x_j} \right)^2 \right) A_i(x; y) = -\frac{4\pi}{c} m_y G \frac{dy^0}{dt} \delta(x - y(t)).
\] (2.14)

If we formally allow \( c \) to tend to the infinity, the equations (2.12) – (2.14) coincide with the equations (2.1), (2.3). In order to study the limit \( c \to \infty \) we need to consider the behaviour of the solutions of the equations (2.12) – (2.14) when \( c \to \infty \). It will be done below for the particular case.

Let us verify that the equations (2.4) – (2.6) are Lorentz covariant. For the world line \( y^\mu(t), y^0(t) = ct \), we define the current
\[
j^\mu(x; y) = c \frac{dy^\mu}{dx^0} \delta(x - y^0(t)).
\] (2.15)

The world line \( y^\mu(t), y^0(t) = ct \), is called timelike if the vector \( \frac{dy^\mu(t)}{dt} \) lies in the upper light cone
\[
\left| \frac{dy(t)}{dt} \right| < c.
\] (2.16)

**Lemma 2.1.** If the world line \( y^\mu(t), y^0(t) = ct \), is timelike, then the current (2.15) satisfies the covariance relation
\[
j^\mu(\Lambda x; \Lambda y) = \sum_{\nu=0}^{3} \Lambda^\nu_\mu j^\nu(x; y)
\] (2.17)
for an arbitrary matrix \( \Lambda \) from Lorentz group.

**Proof.** The definition (2.15) may be rewritten as
\[
j^\mu(x; y) = \int \delta(x - y(t)) \frac{dy^\mu(t)}{dt} dt,
\] (2.18)
\[\delta(x - y(t)) = \delta(x^0 - y^0(t)) \delta(x - y(t)). \quad (2.19)\]

Let \(4 \times 4\) matrix \(\Lambda^\nu_{\mu}\) belong to the Lorentz group. By making use of the relation (2.18) we have

\[\sum_{\nu=0}^{3} \Lambda^\nu_{\mu} j^\nu(\Lambda^{-1} x; y) = \int \delta(x - \Lambda y(t)) \frac{d}{dt}(\Lambda y(t))^\mu dt, \quad (2.20)\]

\[(\Lambda y(t))^\mu = \sum_{\nu=0}^{3} \Lambda^\nu_{\mu} y^\nu(t). \quad (2.21)\]

Since the vector \(\frac{dy^\mu(t)}{dt}\) lies in the upper light cone, the vector \((\Lambda \frac{dy(t)}{dt})^\mu\) lies in the upper light cone too.

Let the relation

\[ct_1 = \sum_{\nu=0}^{3} \Lambda^0_{\nu} y^\nu(t(t_1)) \quad (2.22)\]

define the mapping \(t(t_1)\). The mapping \(t(t_1)\) has positive derivative

\[\frac{dt}{dt_1} = c((\Lambda \frac{dy(t)}{dt}(t(t_1)))^0)^{-1} \quad (2.23)\]

as the vector \((\Lambda \frac{dy(t)}{dt})^\mu\) lies in the upper light cone.

Taking the integration variable \(t_1\) in the right hand side of the relation (2.20) we get

\[\sum_{\nu=0}^{3} \Lambda^\nu_{\mu} j^\nu(\Lambda^{-1} x; y) = \int \delta(x - \Lambda y(t(t_1))) \frac{d}{dt_1}(\Lambda y(t(t_1)))^\mu dt_1. \quad (2.24)\]

The world line \((\Lambda y(t(t_1)))^\mu\) has the property (2.22). In virtue of the world line \(y^\mu(t)\) is timelike, the world line \((\Lambda y(t(t_1)))^\mu\) is timelike too. The relations (2.18), (2.24) imply

\[\sum_{\nu=0}^{3} \Lambda^\nu_{\mu} j^\nu(\Lambda^{-1} x; y) = j^\mu(x; \Lambda y). \quad (2.25)\]

The substitution \(\Lambda x\) for \(x\) in the relation (2.25) gives the relation (2.17). The lemma is proved.

Lemma 2.1 implies the Lorentz covariance of the equations (2.4) – (2.6). Any solution of the equation (2.6) has the following form

\[A^\mu(x; y) = \eta_{\mu\nu} 4 \pi m y G \int E(x - x_1) \frac{dy^\mu(x_1^0)}{dx_1^0} \delta(x_1 - y(x_1^0/c))d^4 x_1 \quad (2.26)\]

where the distribution \(E(x) \in S'(\mathbb{R}^4)\) satisfies the equation

\[\sum_{\nu=0}^{3} \eta^\mu_{\nu} \left( \frac{\partial}{\partial x^\nu} \right)^2 E(x) = \delta(x). \quad (2.27)\]

The solution (2.26) is Lorentz covariant if the distribution \(E(x)\) is Lorentz invariant.

**Lemma 2.2.** Any Lorentz invariant solution of the equation (2.27) has the following form

\[E(x) = (2\pi)^{-1} \theta(x^0) \delta((x^0)^2 - |x|^2) + c_1 \epsilon(x^0) \delta((x^0)^2 - |x|^2) + c_2 \left( ((x^0 + i0)^2 - |x|^2)^{-1} + ((x^0 - i0)^2 - |x|^2)^{-1} \right) + c_3 \quad (2.28)\]
where \( c_1, c_2, c_3 \) are the arbitrary constants; the step function
\[
\theta(x^0) = \begin{cases} 
1, & x^0 > 0 \\
0, & x^0 < 0 
\end{cases}
\]  \hspace{1cm} (2.29)

the sign function \( \epsilon(x^0) = \theta(x^0) - \theta(x^0) \) and \( ((x^0 \pm i0)^2 - |x|^2)^{-1} \) are the distribution boundary values of the holomorphic functions \( ((x^0 \pm i\epsilon)^2 - |x|^2)^{-1} \).

Proof. Due to \((3), \text{Chapter 5, Section 30}\) the distribution \((2.28)\) satisfies the equation \((2.27)\). It is Lorentz invariant. Let us prove that the equation \((2.27)\) has no other Lorentz invariant solutions. It is sufficient to prove that any Lorentz invariant distribution satisfying the wave equation
\[
\sum_{\nu=0}^{3} \eta^{\nu\nu} \left( \frac{\partial}{\partial x^\nu} \right)^2 f(x) = 0
\]  \hspace{1cm} (2.30)

has the following form
\[
f(x) = c_1 \epsilon(x^0) \delta((x^0)^2 - |x|^2) + c_2 \left( ((x^0 + i0)^2 - |x|^2)^{-1} + ((x^0 - i0)^2 - |x|^2)^{-1} \right) + c_3
\]  \hspace{1cm} (2.31)

where \( c_i, i = 1, 2, 3 \) are the arbitrary constants.

The first distribution in the right hand side of the equality \((2.31)\) is odd with respect to the reflection: \( f(-x) = -f(x) \). The two last distributions in the right hand side of the equality \((2.31)\) are even with respect to the reflection: \( f(-x) = f(x) \).

Let the distribution \( f(x) \) be Lorentz invariant and odd with respect to the reflection. Its Fourier transform \( F[f](x) \) has the same properties and satisfies the equation
\[
((x^0)^2 - |x|^2) F[f](x) = 0.
\]  \hspace{1cm} (2.32)

Due to \([4]\) for any Lorentz invariant odd with respect to the reflection distribution \( F[f](x) \in S'(\mathbb{R}^4) \) there is the distribution \( g(t) \in S'(\mathbb{R}) \) such that
\[
\int F[f](x) \phi(x) \, d^4x = \int dt \, g(t) \int \epsilon(x^0) \delta((x^0)^2 - |x|^2 - t) \phi(x) \, d^4x
\]  \hspace{1cm} (2.33)

for any test function \( \phi(x) \) vanishing with its every derivative at the point \( x = 0 \). If the distribution \( F[f](x) \) satisfies the equation \((2.32)\), then the distribution \( g(t) \) satisfies the equation \( t \phi(x) \) at \( x = 0 \). Therefore \( g(t) = c'_1 \delta(t) \) where \( c'_1 \) is an arbitrary constant. The substitution of this solution into the right hand side of the equality \((2.33)\) gives
\[
\int F[f](x) \phi(x) \, d^4x = c'_1 \int \epsilon(x^0) \delta((x^0)^2 - |x|^2) \phi(x) \, d^4x
\]  \hspace{1cm} (2.34)

for any test function \( \phi(x) \) vanishing with its every derivative at the point \( x = 0 \). The difference between the distribution in the left hand side of the equality \((2.34)\) and the distribution in the right hand side of the equality \((2.34)\) is Lorentz invariant distribution with the support at the point \( x = 0 \)
\[
F[f](x) - c'_1 \epsilon(x^0) \delta((x^0)^2 - |x|^2) = \sum_{k=0}^{N} a_k \left( \sum_{\nu=0}^{3} \eta^{\nu\nu} \left( \frac{\partial}{\partial x^\nu} \right)^2 \right)^k \delta(x).
\]  \hspace{1cm} (2.35)

The left hand side of the equality \((2.33)\) is odd with respect to the reflection and the right hand side of the equality \((2.35)\) is even with respect to the reflection. Hence both sides of the equality \((2.33)\) are equal to zero. Then due to \((3), \text{Chapter 5, Section 30}\) we have
\[
f(x) = i(2\pi)^2 c'_1 \epsilon(x^0) \delta((x^0)^2 - |x|^2).
\]  \hspace{1cm} (2.36)
Thus any Lorentz invariant odd with respect to the reflection satisfying the wave equation (2.30) distribution has the form of the first distribution in the right hand side of the equality (2.31).

Let the distribution \( f(x) \) be Lorentz invariant even with respect to the reflection and satisfy the wave equation (2.30). Then its Fourier transform \( F[f](x) \) is Lorentz invariant even with respect to the reflection and satisfies the equation (2.32). Due to [4] there is the distribution \( g(t) \in S'(\mathbb{R}) \), such that

\[
\int F[f](x)\phi(x)d^4x = \int dtg(t) \int \delta((x^0)^2 - |x|^2 - t)\phi(x)d^4x \tag{2.37}
\]

for any test function \( \phi(x) \) vanishing with its every derivative at the point \( x = 0 \). If the distribution \( F[f](x) \) satisfies the equation (2.32), then the distribution \( g(t) \) satisfies the equation \( tg(t) = 0 \). Therefore \( g(t) = c'_2\delta(t) \) where \( c'_2 \) is an arbitrary constant. The substitution of this solution into the right hand side of the equality (2.37) gives

\[
\int F[f](x)\phi(x)d^4x = c'_2 \int \delta((x^0)^2 - |x|^2)\phi(x)d^4x \tag{2.38}
\]

for any test function \( \phi(x) \) vanishing with its every derivative at the point \( x = 0 \). The difference between the distribution in the left hand side of the equality (2.38) and the distribution in the right hand side of the equality (2.38) is Lorentz invariant distribution with the support at the point \( x = 0 \)

\[
F[f](x) - c'_2\delta((x^0)^2 - |x|^2) = \sum_{k=0}^{N} a_k \left( \sum_{\nu} \eta^\nu \left( \frac{\partial}{\partial x^\nu} \right)^2 \right)^k \delta(x). \tag{2.39}
\]

The distributions in the left hand of the equality (2.39) satisfy the equation (2.32). The distribution in the right hand side of the equality (2.39) satisfies the equation (2.32) only if \( a_k = 0 \) for \( k > 0 \). Then due to \([3]\), Chapter 5, Section 30) we have

\[
f(x) = -2\pi c'_2 \left( (x^0 + i0)^2 - |x|^2 \right)^{-1} + (2\pi)^{-4}a_0. \tag{2.40}
\]

Thus any Lorentz invariant even with respect to the reflection satisfying the wave equation (2.30) distribution has the form of the sum of two last distributions in the right hand side of the equality (2.31). The lemma is proved.

If we substitute the expression (2.28) into the right hand side of the relation (2.26), the electromagnetic field strength will depend on the whole world line \( y^\mu(t) \). The equations (2.4) are causal if the electromagnetic field strength depends only on the points of the world line \( y^\mu(t) \) lying in the lower light cone with the origin at the point \( x \). Then the support of the distribution \( E(x) \) must lie in the upper light cone. It is possible only for \( c_i = 0, i = 1, 2, 3 \), in the expression (2.28).

**Lemma 2.3.** Any Lorentz covariant solution of the equations (2.3), (2.6) which depends only on the points of the world line \( y^\mu(t) \) lying in the lower light cone with the origin at the point \( x \) has the following form

\[
F_{\mu\nu}(x; y) = \eta_{\mu\rho}\eta_{\nu\sigma}m_g G \left( c|x - y(t')| - \sum_{i=1}^{3} (x^i - y^i(t')) \frac{dy^i(t')}{dt} \right)^{-2} \times \left( (x^\mu - y^\mu(t')) \frac{d^2 y^\mu(t')}{dt'^2} - (x^\nu - y^\nu(t')) \frac{d^2 y^\nu(t')}{dt'^2} \right).
\]
\( \eta_{\mu\nu} \eta_{\alpha\beta} m y G \left( c|x - y(t')| - \sum_{i=1}^{3} (x^i - y^i(t')) \frac{dy^i(t')}{dt'} \right)^{-3} \times \)
\( \left( \sum_{\alpha=0}^{3} \eta_{\alpha\alpha} \left( \frac{dy^\alpha(t')}{dt'} \right)^2 - \sum_{\alpha=0}^{3} \eta_{\alpha \alpha} (x^\alpha - y^\alpha(t')) \frac{d^2 y^\alpha(t')}{dt'^2} \right) \times \)
\( \left( (x^\mu - y^\mu(t')) \frac{dy^\mu(t')}{dt'} - (x^\nu - y^\nu(t')) \frac{dy^\nu(t')}{dt'} \right) \) \hspace{1cm} (2.41)

where the time \( t' \) satisfies the equation
\( x^0 - c t' = |x - y(t')| \) \hspace{1cm} (2.42)

**Proof.** The substitution of the expression (2.28) with \( c_i = 0, i = 1, 2, 3 \), into the relation (2.26) gives
\[
A_\mu(x; y) = \eta_{\mu\nu} m y G \left( \sum_{\alpha=0}^{3} \eta_{\alpha\alpha} (x^\alpha - y^\alpha(t')) \frac{dy^\alpha(t')}{dt'} \right)^{-1} \frac{dy^\mu(t')}{dt'}
\] \hspace{1cm} (2.43)

where the time \( t' \) satisfies the equation (2.42). By making use of this equation we have
\[
\sum_{\alpha=0}^{3} \eta_{\alpha\alpha} (x^\alpha - y^\alpha(t')) \frac{dy^\alpha(t')}{dt'} = c|x - y(t')| - \sum_{i=1}^{3} (x^i - y^i(t')) \frac{dy^i(t')}{dt'}.
\] \hspace{1cm} (2.44)

The timelike world line \( y^\mu(t) \) satisfies the inequality (2.16). Therefore the expression (2.44) is positive.

The equation (2.42) implies
\[
\sum_{\alpha=0}^{3} \eta_{\alpha\alpha} (x^\alpha - y^\alpha(t'))^2 = 0
\]
\[
\frac{dt'}{dx^\nu} = \eta_{\nu\nu} (x^\nu - y^\nu(t')) \left( \sum_{\alpha=0}^{3} \eta_{\alpha\alpha} (x^\alpha - y^\alpha(t')) \frac{dy^\alpha(t')}{dt'} \right)^{-1}
\] \hspace{1cm} (2.45)

Now by using the definition (2.3) and the relations (2.43) – (2.45) it is easy to obtain the relation (2.41). The lemma is proved.

The formulae (2.43) are called the Liénard – Wiechert potentials ([3], Chapter VIII, Section 63). The equations (2.4), (2.41), (2.42) are the relativistic Newton gravity law. If we allow \( c \) to tend to the infinity, the equations (2.4), (2.41), (2.42) coincide with the Newton gravity law (2.1), (2.2).

It is easy to generalize the equations (2.4) – (2.6) to the gravitation interaction of many bodies
\[
c \frac{d^2 x^\mu_k}{ds^2_k} = -\frac{1}{c^2} \eta^{\mu\nu} \sum_{j=0}^{n} \sum_{\nu=0}^{3} F_{\nu \nu}(x_k; x_j) \frac{dx^\nu_j}{ds^2_j},
\] \hspace{1cm} (2.46)

where \( k = 1, ..., n, \mu = 0, ..., 3 \), the vectors \( \frac{dx^\mu_k}{ds_k}, \frac{d^2 x^\mu_k}{ds^2_k} \) are given by the relations (2.7), (2.8) and the electromagnetic field strengths are given by the relations (2.44), (2.42).

Let us consider the light propagation in the gravity field. It corresponds to the equations (2.4), (2.41), (2.42) for \( m_1 = 0 \). Then the second equation (2.4) is the following equation
\[
\frac{d}{dt} \left( 1 - \frac{1}{c^2} \left| \frac{dx_2}{dt} \right|^2 \right)^{-1/2} \frac{dx_2^\mu}{ds_2} = 0.
\]
Hence
\[
(1 - \frac{1}{c^2} \left| \frac{dx_2}{dt} \right|^2)^{-1/2} \frac{dx_2^i}{dt} = \text{const}
\]
and
\[
\frac{dx_2^i}{dt} = \text{const}.
\]

Therefore the second body moves uniformly and rectilinearly. The equations (2.4), (2.41), (2.42) are Lorentz covariant. Let us choose the coordinate system where the second body is at rest: \( x_2^0(t) = ct, x_2^i(t) = 0, i = 1, 2, 3 \). Then the equations (2.4) and (2.41) imply

\[
\frac{d}{dt} \left( (1 - \frac{1}{c^2} \left| \frac{dx_1^1}{dt} \right|^2)^{-1/2} \frac{dx_1^1}{dt} \right) = -m_2 G |x_1|^3 x_1^i
\]

(2.47)

where \( i = 1, 2, 3 \).

Let us find the conservation laws of the equation (2.47). Let the tensor \( \epsilon_{ijk} \) be antisymmetric for all indices and \( \epsilon_{123} = 1 \). We define the vector

\[
M_k = \sum_{i,j=1}^3 \epsilon_{ijk} (x_1^i \frac{dx_1^j}{dt} - x_1^j \frac{dx_1^i}{dt})(1 - \frac{1}{c^2} \left| \frac{dx_1^1}{dt} \right|^2)^{-1/2}
\]

(2.48)

where \( k = 1, 2, 3 \). It follows from the equations (2.47) that

\[
\frac{dM_k}{dt} = 0.
\]

The identity (2.11) and the equations (2.47) imply that the value

\[
E = c^2 (1 - \frac{1}{c^2} \left| \frac{dx_1^1}{dt} \right|^2)^{-1/2} - m_2 G |x_1|^{-1} = \text{const}
\]

(2.49)

is constant:

\[
\frac{dE}{dt} = 0.
\]

In view of the relation (2.48) the vector \( x_1 \) is orthogonal to the constant vector \( M \).

By making use of the rotation we choose the coordinate system such that \( M_1 = M_2 = 0 \). Therefore \( x_1^3 = 0 \). Let us choose the polar coordinates

\[
x_1^1(t) = r(t) \cos \phi(t),
\]
\[
x_1^2(t) = r(t) \sin \phi(t),
\]
\[
x_1^3(t) = 0.
\]

The relations (2.48), (2.49) imply

\[
M_3 = r^2 \frac{d\phi}{dt} \left( 1 - \frac{1}{c^2} \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} \right)^2 \right) \right)^{-1/2},
\]

(2.50)

\[
E = c^2 \left( 1 - \frac{1}{c^2} \left( \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} \right)^2 \right) \right)^{-1/2} - m_2 Gr^{-1},
\]

(2.51)
\[
d\phi \over dt = r^{-2}c^2 M_3(E + m_2 Gr^{-1})^{-1},
\]
(2.52)

\[
\left(\frac{dr}{dt}\right)^2 = c^2 - c^6(E + m_2 Gr^{-1})^{-2}(1 + r^{-2}c^{-2}M_3^2).
\]
(2.53)

It follows from the equation (2.53) that

\[
\epsilon \frac{dr}{dt} = c(E + m_2 Gr^{-1})^{-1}((m_2^2 G^2 - c^2 M_3^2) \left(r^{-1} + m_2 GE(m_2^2 G^2 - c^2 M_3^2)^{-1}\right)^2 - c^2 M_3^2 E^2(m_2^2 G^2 - c^2 M_3^2)^{-1} - c^4)^{1/2}
\]
(2.54)

where \(\epsilon = \pm 1\). The equations (2.52), (2.54) imply

\[
- \epsilon \left(\frac{d\phi}{dt}\right)^{-1} \frac{d}{dt} r^{-1} = (cM_3)^{-1}((m_2^2 G^2 - c^2 M_3^2) \left(r^{-1} + m_2 GE(m_2^2 G^2 - c^2 M_3^2)^{-1}\right)^2 - c^2 M_3^2 E^2(m_2^2 G^2 - c^2 M_3^2)^{-1} - c^4)^{1/2}
\]
(2.55)

If

\[
c^2 M_3^2 - m_2^2 G^2 > 0,
\]
(2.56)

\[
c^2 M_3^2 (E^2 - c^4) + m_2^2 G^2 c^4 > 0,
\]
(2.57)

then the solution of the equation (2.55) is the following

\[
\frac{p}{r} = 1 - \epsilon e \sin \left((c^2 M_3^2 - m_2^2 G^2)^{1/2}(cM_3)^{-1}(\phi - \phi_0)\right)
\]
(2.58)

where \(\phi_0\) is a constant and

\[
p = (c^2 M_3^2 - m_2^2 G^2)(m_2 GE)^{-1},
\]
(2.59)

\[
e = (c^2 M_3^2 (E^2 - c^4) + m_2^2 G^2 c^4)^{1/2}(m_2 GE)^{-1}.
\]
(2.60)

The equation (2.58) is the equation of the conic section with the focus at the origin of coordinates. For \(e^2 < 1\) this conic section is an ellipse and the light does not leave the gravitating body.

If the inequality (2.56) holds and

\[
c^2 M_3^2 (E^2 - c^4) + m_2^2 G^2 c^4 < 0,
\]
(2.61)

the right hand side of the equation (2.53) is negative and the equation (2.53) has no solutions.

If

\[
m_2^2 G^2 - c^2 M_3^2 > 0,
\]
(2.62)

then the inequality (2.57) holds and the equation (2.53) has the following solution

\[
\frac{p}{r} = 1 + \epsilon e \cosh \left((m_2^2 G^2 - c^2 M_3^2)^{1/2}(cM_3)^{-1}(\phi - \phi_0)\right)
\]
(2.63)

where \(\phi_0\) is a constant.

If

\[
c^2 M_3^2 - m_2^2 G^2 = 0,
\]
(2.64)
the equation \( (2.53) \) has the following form
\[
-\epsilon \left( \frac{d\phi}{dt} \right)^{-1} \frac{d}{dt} r^{-1} = (cM_3)^{-1} (2m_2GEr^{-1} + E^2 - c^4)^{1/2}
\] (2.65)

The solution of the equation \( (2.63) \) is the following
\[
\phi = -\epsilon M_3(|M_3|E)^{-1} (2m_2GEr^{-1} + E^2 - c^4)^{1/2} + \phi_0
\] (2.66)

where \( \phi_0 \) is a constant.

Let us consider the equation \( (2.54) \). Let \( r \) and \( t \) be the function of the parameter \( \xi \). If the inequality \( (2.57) \) holds and
\[
E^2 - c^4 < 0,
\] (2.67)
then the equation \( (2.54) \) has the following solution
\[
r(\xi) = m_2GE(c^4 - E^2)^{-1}(1 + \epsilon \sin \left( (c^4 - E^2)^{1/2} c^{-1}(\xi - \xi_0) \right)),
\]
\[
t(\xi) = \epsilon m_2GE^2 c^{-1} (c^4 - E^2)^{-3/2} (c^3E^{-2} (c^4 - E^2)^{1/2} (\xi - \xi_1) - \epsilon \cos \left( (c^4 - E^2)^{1/2} c^{-1}(\xi - \xi_0) \right))
\] (2.68)

where \( \xi_0, \xi_1 \) are the constants.

If the inequalities \( (2.61) \) and \( (2.67) \) hold, the right hand side of the equation \( (2.53) \) is negative and the equation \( (2.53) \) has no solutions.

If
\[
E^2 - c^4 > 0,
\] (2.69)
then the equation \( (2.54) \) has the following solution
\[
r(\xi) = m_2GE(E^2 - c^4)^{-1}(e \cosh \left( (E^2 - c^4)^{1/2} c^{-1}(\xi - \xi_0) \right) - 1),
\]
\[
t(\xi) = -\epsilon m_2GE^2 c^{-1} (E^2 - c^4)^{-3/2} (c^3E^{-2} (E^2 - c^4)^{1/2} (\xi - \xi_1) - \epsilon \sinh \left( (E^2 - c^4)^{1/2} c^{-1}(\xi - \xi_0) \right))
\] (2.70)

where \( \xi_0, \xi_1 \) are the constants.

If
\[
E^2 - c^4 = 0,
\] (2.71)
then the equation \( (2.54) \) has the following form
\[
\epsilon \frac{dr}{dt} = c(c^2r + m_2G)^{-1} (2m_2Gc^2r + m_2^2G^2 - c^2M_3^2)^{1/2}.
\] (2.72)

The equation \( (2.72) \) has the following solution
\[
t = \epsilon (2m_2Gc^2r + m_2^2G^2 - c^2M_3^2)^{1/2} (r(3m_2Gc)^{-1} + 2/3c^{-3} + 1/3M_3^2(m_2^2G^2c)^{-1}) + t_0
\] (2.73)

where \( t_0 \) is a constant.

Let us study the limit of the solutions of the equations \( (2.54), (2.55) \) when \( c \to \infty \). It follows from the expressions \( (2.48), (2.49) \) that for \( c \to \infty \)
\[
M_3 \to x_1 \frac{dx_1}{dt} - x_2 \frac{dx_2}{dt} = \bar{M}_3,
\] (2.74)
\[ c^{-2} (E^2 - c^4) \rightarrow \left| \frac{dx_1}{dt} \right|^2 - 2m_2 G |x_1|^{-1} = 2 \bar{E}. \] (2.75)

For \( c \to \infty \) only the inequalities (2.54), (2.57) are valid. For \( c \to \infty \) the limit of the solution (2.58) is the following function

\[ \frac{M_3^2}{m_2 Gr} = 1 - \epsilon (2\bar{E}M_3^2 + m_3^2G^2)^{1/2} (m_2G)^{-1} \sin(\phi - \phi_0). \] (2.76)

Due to ([8], Chapter III, Section 15) the function (2.76) is the solution of the equations (2.1) for \( m_1 = 0, x_i = 0, i = 1, 2, 3. \)

For \( c \to \infty \) the functions (2.68), (2.71) tend to following functions

\[ r(\xi) = \frac{m_2 G}{2|\bar{E}|} (1 + (1 + 2\bar{E} \frac{M_3^2}{m_3^2G^2})^{1/2} \sin((2|\bar{E}|)^{1/2}(\xi - \xi_0))), \]

\[ t(\xi) = \frac{m_2 G}{(2|\bar{E}|)^{3/2}} ((2|\bar{E}|)^{1/2}(\xi - \xi_1) - (1 + 2\bar{E} \frac{M_3^2}{m_3^2G^2})^{1/2} \cos((2|\bar{E}|)^{1/2}(\xi - \xi_0))) \] (2.77)

and

\[ r(\xi) = -\epsilon \frac{m_2 G}{(2|\bar{E}|)^{3/2}} ((2|\bar{E}|)^{1/2}(\xi - \xi_1) - (1 + 2\bar{E} \frac{M_3^2}{m_3^2G^2})^{1/2} \sin((2|\bar{E}|)^{1/2}(\xi - \xi_0))) - 1, \]

\[ t(\xi) = -\epsilon \frac{m_2 G}{(2|\bar{E}|)^{3/2}} ((2|\bar{E}|)^{1/2}(\xi - \xi_1) - (1 + 2\bar{E} \frac{M_3^2}{m_3^2G^2})^{1/2} \sin((2|\bar{E}|)^{1/2}(\xi - \xi_0))). \] (2.78)

The formulae (2.77), (2.78) coincide with the formulae from ([8], Chapter III, Section 15) for the equations (2.1) when \( m_1 = 0, x_i = 0, i = 1, 2, 3. \)

For \( c \to \infty \) the function (2.73) tends to the following function

\[ t = \epsilon (2m_2 Gr - \bar{M}_2^2)^{1/2} (r(3m_2G)^{-1} + 1/3\bar{M}_2^2 m_2^{-2}G^{-2}) + t_0. \] (2.79)

This case is not considered in ([8], Chapter III, Section 15).

Let us study the equations (2.4), (2.41), (2.42) for the case when the first body is Mercury and the second body is the Sun. Let us estimate the value of the electromagnetic field strength \( F_{\mu\nu}(x_2; x_1) \). We consider that Mercury moves along the circle of the radius \( a \) with the angular frequency \( \omega \). Then the expression (2.41) implies

\[ |F_{00}(x_2; x_1)| \leq m_1 G (a - a^2 \omega c^{-1})^{-2} a^2 (\omega c^{-1})^2 + m_1 G (a - a^2 \omega c^{-1})^{-3} (1 + 2a^2 (\omega c^{-1})^2) (a + a^2 \omega c^{-1}), \] (2.80)

\[ |F_{ij}(x_2; x_1)| \leq m_1 G (a - a^2 \omega c^{-1})^{-2} 2a^2 (\omega c^{-1})^2 + m_1 G (a - a^2 \omega c^{-1})^{-3} (1 + 2a^2 (\omega c^{-1})^2) 2a^2 \omega c^{-1}, \] (2.81)

where \( i, j = 1, 2, 3. \) Due to ([8], Chapter 25, Section 25.1, Appendix 25.1) \( m_1 = 3, 28 \cdot 10^{23} \) kg, \( \omega c^{-1} = 275, 8 \cdot 10^{-17} m^{-1}, a = 0.5791 \cdot 10^{11} m. \) Then \( |F_{00}(x_2; x_1)| \leq 6.52 \cdot 10^{-9} m \cdot s^{-2}, |F_{ij}(x_2; x_1)| \leq 2.08 \cdot 10^{-12} m \cdot s^{-2}. \) Therefore the case \( m_1 = 0 \) is a good approximation for our problem.

Let us consider the equation (2.58). If \( \epsilon^2 \geq 1, \) there is the angle \( \phi \) for which \( r = \infty. \) Thus the orbit is not closed. If \( \epsilon^2 < 1 \) and \( E < 0, \) then \( p < 0 \) and the equation (2.58) has no
solutions. Let \( e^2 < 1 \) and \( E > 0 \). The substitution \( \epsilon \rightarrow -\epsilon \) corresponds with the substitution \( \phi_0 \rightarrow \phi_0 + \pi M_3(c^2 M_3^2 - m_2^2 G^2)^{-1/2} \). We choose \( \epsilon = 1 \), \( \phi_0 = \pi / 2 c M_3(c^2 M_3^2 - m_2^2 G^2)^{-1/2} \). The curve (2.58) is an ellipse with the focus at the origin of coordinates. Its big and small semiaxes are

\[
a = p(1 - e^2)^{-1} = m_2 G E (c^4 - E^2)^{-1}, \tag{2.82}
\]

\[
b = a(1 - e^2)^{1/2} = (c^2 M_3^3 - m_2^2 G^2)^{1/2}(c^4 - E^2)^{-1/2}. \tag{2.83}
\]

The inequalities (2.56) and \( e^2 < 1 \) imply the inequality (2.67). Thus the equations (2.68) are valid. For the parameter \( \xi_2 = \xi_0 - \pi / 2 c (c^4 - E^2)^{-1/2} \)

\[
r(\xi_2) = r_{min} = m_2 G E (c^4 - E^2)^{-1}(1 - e). \tag{2.84}
\]

For the parameter \( \xi_3 = \xi_0 + \pi / 2 c (c^4 - E^2)^{-1/2} \)

\[
r(\xi_3) = r_{max} = m_2 G E (c^4 - E^2)^{-1}(1 + e). \tag{2.85}
\]

Hence the period of the motion along the ellipse (2.58) is equal to

\[
T = 2|t(\xi_3) - t(\xi_2)| = 2 \pi m_2 G c^3 (c^4 - E^2)^{-3/2}. \tag{2.86}
\]

We define the mean angular frequency \( \omega = 2 \pi T^{-1} \). The relation (2.80) implies

\[
\omega = (c^4 - E^2)^{3/2}(m_2 G c^3)^{-1}. \tag{2.87}
\]

Therefore

\[
E^2 = c^2 (c^2 - (\omega m_2 G)^{2/3}). \tag{2.88}
\]

The substitution of the expression (2.88) into the equality (2.82) yields

\[
m_2 G = \omega^2 a^3 \left( 1/2 (1 + \sigma (1 - (2 aw c^{-1})^2)^{1/2} ) \right)^{-3/2} \tag{2.89}
\]

where \( \sigma = \pm 1 \). Let \( c \rightarrow \infty \). Then \( m_2 G = \omega^2 a^3 (1/2 (1 + \sigma ))^{-3/2} \). For \( \sigma = 1 \) this expression is in accordance with the expression obtained in the Kepler problem ([4], Chapter III, Sections 13, 15). Hence we choose \( \sigma = 1 \) in the relation (2.89). The substitution of the expression (2.89) with \( \sigma = 1 \) into the equality (2.88) gives

\[
E^2 = c^4 - 2 c^2 \omega^2 a^3 (1 + (1 - (2 aw c^{-1})^2)^{1/2})^{-1}. \tag{2.90}
\]

By making use of the relations (2.59), (2.60), (2.82), (2.83), (2.89), (2.90) we get

\[
(c^2 M_3^2 - m_2^2 G^2)^{1/2}(c |M_3|)^{-1} = (1 + 4 a^2 (\omega c^{-1})^2 (1 - e^2)^{-1} (1 + (1 - (2 aw c^{-1})^2)^{1/2})^{-2})^{-1/2}. \tag{2.91}
\]

The relativistic orbit (2.58) differs from the non-relativistic orbit (2.76) by the multiplier (2.91). Let us calculate the shift of Mercury perihelion for a hundred years in the seconds of arc. Due to ([2], Chapter 25, Section 25.1, Appendix 25.1) the period of Mercury circulation is 87,9686 days and the period of circulation of the Earth is 365,257 days. Hence

\[
\Delta \phi = 2 \pi (1 - (c^2 M_3^2 - m_2^2 G^2)^{1/2}(c |M_3|)^{-1}) \cdot 100 \cdot 365,257 \cdot (87,9686)^{-1} \cdot 360 \cdot 3600 \tag{2.92}
\]

For Mercury \( \omega c^{-1} = 275, 8 \cdot 10^{-17} m^{-1}, a = 0, 5791 \cdot 10^{11} m, e = 0, 21 \). Therefore the relations (2.91), (2.92) imply that \( \Delta \phi = 45^\circ, 09 \). Due to ([2], Chapter 40, Section 40.5, Appendix 40.3) the shift for a hundred years which must be attached to the general relativity and to the Sun flattening is \( \Delta \phi = 42^\circ, 56 \pm 0^\circ, 94 \).
3  Lorentz Force

Let the world line \( x^\mu(\lambda) \), \( \mu = 0, ..., 3 \), be given. It depends on the parameter \( \lambda \). Let on \( \mathbb{R}^4 \) the pseudometric \( g_{\mu\nu} = g_{\nu\mu} \) be given. We suppose that the matrix \( g_{\mu\nu} \) has the inverse matrix \( g^{\mu\nu} \). We define the proper time \( \tau \) in the following way

\[
d\lambda = \left( \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{-1/2}.
\]  (3.1)

Due to \([2]\), Chapter 13, Section 13.4) the world line \( x^\mu(\lambda) \) is called geodesic if the functional

\[
\int_{\lambda_1}^{\lambda_2} \left( \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda
\]  (3.2)

is extremal. Then the world line \( x^\mu(\lambda) \) satisfies the geodesic equation with the affine parameter \( \tau \)

\[
\frac{d\lambda}{d\tau} \frac{d}{d\lambda} \left( \frac{d\lambda}{d\tau} \frac{dx^\mu}{d\lambda} \right) + \sum_{\alpha,\beta=0}^{3} \Gamma^\mu_{\alpha\beta} \frac{d\lambda}{d\tau} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0
\]  (3.3)

where the Christoffel’s symbol

\[
\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \sum_{\sigma=0}^{3} \left( \frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right) g^{\sigma\mu}.
\]  (3.4)

The equation (3.3) does not depend on a choice of a parameter \( \lambda \).

The gravitation interaction of two bodies is given by the Newton gravity law \((2.1), (2.2)\). Following \([2]\), Chapter 17, Section 17.4) we shall represent the equations \((2.1), (2.2)\) as a particular case of the equations \((3.3), (3.4)\). Let \( U(x; y) \) be the solution of the equation \((2.2)\).

It depends on the variables \( x^0 = ct, x^i, i = 1, 2, 3 \). Let us consider the pseudometric

\[
g_{00} = 1 + 2c^{-2}U(x; y),
g_{0i} = g_{i0} = 0,
g_{ij} = -\delta_{ij}
\]  (3.5)

where \( i, j = 1, 2, 3 \). Then the definition \((3.1)\) implies

\[
\frac{dt}{d\tau} = c^{-1}(1 + 2c^{-2}U) - c^{-2} \sum_{i=1}^{3} \left( \frac{dx^i}{dt} \right)^2)^{1/2}.
\]  (3.6)

It follows from the relations \((3.4), (3.3)\) that

\[
\Gamma^\mu_{ij} = 0,
\Gamma^i_{\mu j} = \Gamma^i_{j\mu} = 0,
\Gamma^0_{00} = c^{-2}(1 + 2c^{-2}U)^{-1} \frac{\partial U}{\partial x^0},
\Gamma^i_{00} = c^{-2} \frac{\partial U}{\partial x^i},
\Gamma^0_{0i} = \Gamma^0_{i0} = c^{-2}(1 + 2c^{-2}U)^{-1} i \frac{\partial U}{\partial x^i}
\]  (3.7)
where \( i, j = 1, 2, 3; \mu = 0, \ldots, 3 \).

The substitution of the expressions (3.6), (3.7) into the equation (3.3) for \( \mu = 0 \) yields

\[
c(1 + 2c^{-2}U - c^{-2} \sum_{j=1}^{3} \left( \frac{dx^j}{dt} \right)^2 )^{1/2} \frac{d}{dt} (1 + 2c^{-2}U - c^{-2} \sum_{j=1}^{3} \left( \frac{dx^j}{dt} \right)^2 )^{-1/2} +
\]

\[
(1 + 2c^{-2}U)^{-1} \left( \frac{\partial U}{\partial x^0} + 2c^{-1} \sum_{j=1}^{3} \frac{dx^j}{dt} \frac{\partial U}{\partial x^j} \right) = 0. \tag{3.8}
\]

The substitution of the expressions (3.6), (3.7) into the equations (3.3) for \( \mu = 1, 2, 3 \) gives

\[
(1 + 2c^{-2}U - c^{-2} \sum_{j=1}^{3} \left( \frac{dx^j}{dt} \right)^2 )^{1/2} \frac{d}{dt} (1 + 2c^{-2}U - c^{-2} \sum_{j=1}^{3} \left( \frac{dx^j}{dt} \right)^2 )^{-1/2} \frac{d^2 x^i}{dt^2} + \frac{\partial U}{\partial x^i} = 0 \tag{3.9}
\]

where \( i = 1, 2, 3 \).

We suppose that the velocity \( \frac{dx^i}{dt} \) is small with respect to \( c \) and the acceleration \( \frac{d^2 x^i}{dt^2} \) is bounded with respect to \( c \). We also suppose that the potential \( U \) is small with respect to \( c \) and its derivatives are bounded with respect to \( c \). Then for \( c \to \infty \) the equations (3.8), (3.9) get the following form

\[
\frac{\partial U}{\partial x^0} = 0, \quad \frac{d^2 x^i}{dt^2} + \frac{\partial U}{\partial x^i} = 0. \tag{3.10}
\]

The first equation (3.10) shows that the potential \( U(x; y) \) is independent of the variable \( x^0 = ct \), i.e. the body with the world line \( y^\mu \) is at rest.

Following (2), Chapter 12, Section 12.1) we show that the equations (2.1), (2.2) are the particular case not of the geodesic equation (3.3) with the affine parameter but of the geodesic equation with the parameter \( t \)

\[
\frac{d^2 x^\mu}{dt^2} + \sum_{\alpha, \beta = 0}^{3} \Gamma^\mu_{\alpha \beta}(x_k; \hat{x}_k) \frac{dx^\alpha_k}{dt} \frac{dx^\beta_k}{dt} = 0 \tag{3.11}
\]

where \( \mu = 0, \ldots, 3 \), the world lines \( x^0_k = ct, x^i_k(t), k = 1, 2, i = 1, 2, 3 \), and the coefficients

\[
\Gamma^i_{00}(x_k; \hat{x}_k) = c^{-2} \frac{\partial U(x_k; \hat{x}_k)}{\partial x^i_k}. \tag{3.12}
\]

All the other coefficients \( \Gamma^\mu_{\alpha \beta}(x_k; \hat{x}_k) \) are equal to zero.

Let us prove that it is impossible to represent the coefficients (3.12) in the form of the Christoffel’s symbols (3.4). Let these coefficients have the form (3.4). Then

\[
\frac{1}{2} \left( \frac{\partial g_{\beta \gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha \gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha \beta}}{\partial x^\gamma} \right) = \sum_{\sigma = 0}^{3} g_{\gamma \sigma} \Gamma^\sigma_{\alpha \beta}. \tag{3.13}
\]

The cyclic permutation \( \alpha \to \beta \to \gamma \to \alpha \) in the equation (3.13) yields

\[
\frac{1}{2} \left( \frac{\partial g_{\beta \alpha}}{\partial x^\gamma} + \frac{\partial g_{\alpha \beta}}{\partial x^\gamma} - \frac{\partial g_{\beta \gamma}}{\partial x^\alpha} \right) = \sum_{\sigma = 0}^{3} g_{\alpha \sigma} \Gamma^\sigma_{\beta \gamma}. \tag{3.14}
\]
In virtue of the symmetry $g_{\nu \mu} = g_{\mu \nu}$ the sum of the equations (3.13), (3.14) is

$$\frac{\partial g_{\alpha \gamma}}{\partial x^\beta} = \sum_{\sigma=0}^{3} (g_{\gamma \sigma} \Gamma_{\alpha \beta}^{\sigma} + g_{\alpha \sigma} \Gamma_{\beta \gamma}^{\sigma}).$$ (3.15)

The equation (3.15) and the definition (3.12) imply

$$\frac{\partial g_{\alpha \gamma}}{\partial x^i} = 0 \quad (3.16)$$

where $i = 1, 2, 3$. Hence the pseudometric $g_{\mu \nu}$ does not depend on the variables $x^i$, $i = 1, 2, 3$, and it is impossible to represent the coefficients (3.12) in the form (3.4).

The Newton gravity law seems to be very important. Therefore we propose to consider the equations of the type (3.3) where the coefficients $\Gamma_{\alpha \beta}^{\mu}$ do not have a form of the Christoffel’s symbols (3.4). Thus we propose to refuse the pseudometric $g_{\mu \nu}$. We want our equation not to depend on the parametrization of the world line. Thus we change the pseudometric $g_{\mu \nu}$ in the expression (3.1) for the fixed Minkowski pseudometric $\eta_{\mu \nu}$

$$\frac{d\lambda}{ds} = \left( \sum_{\mu, \nu=0}^{3} \eta_{\mu \nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{-1/2}.$$ (3.17)

By changing the expression (3.1) in the equation (3.3) for the expression (3.17) we get

$$\frac{d\lambda}{ds} \frac{d}{d\lambda} \left( \frac{d\lambda}{ds} \frac{dx^\mu}{ds} \right) + \sum_{\alpha, \beta=0}^{3} \eta_{\alpha \beta} \frac{d\lambda}{ds} \frac{dx^\alpha}{ds} = 0 \quad (3.18)$$

The equation (3.18) does not depend on a choice of a parameter $\lambda$. We choose the time $t$ as the parameter. We want our equation to be Lagrangian as the equation (3.3). Thus we consider the whole class of the equations of the type (3.18)

$$mc \frac{d}{ds} \frac{d}{dt} \left( \frac{dt}{ds} \frac{dx^\mu}{ds} \right) + \frac{e}{c} \sum_{k=0}^{N} \sum_{\alpha_1, \ldots, \alpha_k=0}^{3} \eta_{\mu \nu} F_{\mu \alpha_1 \cdots \alpha_k}(x) \frac{dt}{ds} \frac{dx^{\alpha_1}}{ds} \cdots \frac{dt}{ds} \frac{dx^{\alpha_k}}{ds} = 0 \quad (3.19)$$

and choose the Lagrangian equations. We consider the world line $x^\mu(t)$: $x^0(t) = ct$, $x^i(t)$, $i = 1, 2, 3$. The definitions (2.7), (2.8) imply

$$\sum_{\alpha=0}^{3} \eta_{\alpha \alpha} \left( \frac{dx^\alpha}{ds} \right)^2 = 1,$$ (3.20)

$$\sum_{\alpha=0}^{3} \eta_{\alpha \alpha} \frac{d^2 x^\alpha}{ds^2} = 0.$$ (3.21)

The identity (3.21) coincides with the identity (2.11).

The equation (3.19) and the identity (3.21) imply

$$\sum_{k=0}^{N} \sum_{\mu, \alpha_1, \ldots, \alpha_k=0}^{3} F_{\mu \alpha_1 \cdots \alpha_k}(x) \frac{dx^\mu}{ds} \frac{dx^{\alpha_1}}{ds} \cdots \frac{dx^{\alpha_k}}{ds} = 0 \quad (3.22)$$
Conversely, it follows from the equations (3.21), (3.22) that the equation (3.19) for $\mu = 0$ is the consequence of the equations (3.19) for $\mu = 1, 2, 3$. Let us denote $v^i = \frac{dx^i}{dt}$, $i = 1, 2, 3$.

The substitution of the expression (2.8) into the equation (3.19) gives

$$m \frac{d}{dt} \left( (1 - \frac{1}{c^2} |v|^2)^{-1/2} v^i \right) - e \sum_{k=0}^{N} c^{-k} (1 - \frac{1}{c^2} |v|^2)^{-\frac{k-1}{2}} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} F_{i \alpha_1 ... \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_k}}{dt} = 0. \quad (3.23)$$

**Lemma 3.1.** Let there exist the Lagrange function $L(x, v, t)$ such that for any world line $x^\mu(t)$, $x^0(t) = ct$, and for any $i = 1, 2, 3$

$$\frac{d}{dt} \frac{dL}{dv^i} - \frac{\partial L}{\partial x^i} = m \frac{d}{dt} \left( (1 - \frac{1}{c^2} |v|^2)^{-1/2} v^i \right) - e \sum_{k=0}^{N} c^{-k} (1 - \frac{1}{c^2} |v|^2)^{-\frac{k-1}{2}} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} F_{i \alpha_1 ... \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_k}}{dt}. \quad (3.24)$$

Then the Lagrange function has the form

$$L(x, v, t) = -mc^2(1 - \frac{1}{c^2} |v|^2)^{1/2} + \frac{e}{c} \sum_{i=1}^{3} A_i(x, t) v^i + eA_0(x, t), \quad (3.25)$$

the coefficients in the equation (3.19) are

$$F_{i \alpha_1 ... \alpha_k}(x) = 0, \ k \neq 1 \quad (3.26)$$

and

$$F_{ij}(x) = \frac{\partial A_j(x, t)}{\partial x^i} - \frac{\partial A_i(x, t)}{\partial x^j},$$

$$F_{i0}(x) = \frac{\partial A_0(x, t)}{\partial x^i} - \frac{1}{c} \frac{\partial A_i(x, t)}{\partial t} \quad (3.27)$$

where $i, j = 1, 2, 3$; $\alpha_1, ..., \alpha_k = 0, ..., 3$.

**Proof.** We look for the Lagrange function in the following form

$$L(x, v, t) = -mc^2(1 - \frac{1}{c^2} |v|^2)^{1/2} + L_1(x, v, t). \quad (3.28)$$

The substitution of the expression (3.28) into the equality (3.24) yields

$$d \frac{\partial L_1}{dv^i} - \frac{\partial L_1}{\partial x^i} = -e \sum_{k=0}^{N} c^{-k} (1 - \frac{1}{c^2} |v|^2)^{-\frac{k-1}{2}} \sum_{\alpha_1, ..., \alpha_k = 0}^{3} F_{i \alpha_1 ... \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \cdots \frac{dx^{\alpha_k}}{dt}. \quad (3.29)$$

The right hand side of the equality (3.29) has not the vector $\frac{dx^i}{dt} = \frac{d^2x^i}{dt^2}$. Since the world line $x^i(t)$, $i = 1, 2, 3$, is arbitrary, the left hand side of the equality (3.29) can not have the vector $\frac{dx^i}{dt}$, i.e. the function $L_1(x, v, t)$ is linear for the variables $v^i$

$$L_1(x, v, t) = e \sum_{i=1}^{3} \frac{A_i(x, t)}{c} v^i + eA_0(x, t). \quad (3.30)$$
By definition
\[ \frac{dA_\mu(x, t)}{dt} = \sum_{j=1}^{3} \frac{\partial A_\mu(x, t)}{\partial x^j} v^j + \frac{\partial A_\mu(x, t)}{\partial t} \] (3.31)
where \( \mu = 0, \ldots, 3 \). The substitution of the expression (3.31) into the left hand side of the equality (3.29) gives
\[ \frac{1}{c} \sum_{j=1}^{3} \left( \frac{\partial A_j(x, t)}{\partial x^i} - \frac{\partial A_i(x, t)}{\partial x^j} \right) v^j + \frac{\partial A_0(x, t)}{\partial t} - \frac{1}{c} \frac{\partial A_i(x, t)}{\partial t} = \sum_{k=0}^{N} c^{-k}(1 - \frac{1}{c} |v|^2)^{-\frac{k-1}{2}} \sum_{\alpha_1, \ldots, \alpha_k = 0}^{3} F_{i\alpha_1 \ldots \alpha_k}(x) \frac{dx^{\alpha_1}}{dt} \ldots \frac{dx^{\alpha_k}}{dt} . \] (3.32)

We note that \( \frac{dx^0}{dt} = c, \frac{dx^i}{dt} = v^i, i = 1, 2, 3 \). Since the world line \( x^i(t), i = 1, 2, 3 \), is arbitrary, the equation (3.32) implies the equalities (3.26), (3.27). The lemma is proved. Let us define
\[ F_{00} = 0, \] (3.33)
\[ F_{0i} = -F_{i0} \] (3.34)
where \( i = 1, 2, 3 \). Then the following identity is valid
\[ \sum_{\alpha, \beta = 0}^{3} F_{\alpha \beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \] (3.35)
by virtue of the antisymmetry \( F_{\beta \alpha} = -F_{\alpha \beta} \). The equation (3.33) is a particular case of the equation (3.22). Thus we obtain the equation
\[ mc \frac{d^2x^\mu}{ds^2} = -c \frac{\eta^{\mu \nu}}{c} \sum_{\nu = 0}^{3} F_{\mu \nu}(x) \frac{dx^\nu}{ds} \] (3.36)
where \( F_{\mu \nu}(x) \) are given by the equalities (3.27), (3.33), (3.34).

The equation (3.36) is well – known ([2], Chapter 3, Section 3.1; [5], Chapter III, Section 23). The right hand side of the equation (3.36) divided by \( \frac{dt}{ds} \) is called the Lorentz force.

4 Connection

The affine connection at the point \( x \in \mathbb{R}^4 \) is the function establishing a correspondence between the tangent vector \( X \in T\mathbb{R}^4_x \), the vector field \( Y \) and a new tangent vector \( \nabla_X Y \in T\mathbb{R}^4_x \) called the covariant derivative of \( Y \) in the direction \( X \). This vector is required to be a bilinear function of \( X \) and \( Y \). Moreover, if \( f(x) \) is a real function and if \( fY \) is the vector field \( (fY)_y = f(y)Y_y \), then the operation \( \nabla \) is required to satisfy the condition
\[ \nabla_X(fY) = (Xf)Y_x + f(x)\nabla_X Y. \] (4.1)

The affine connection (or simply connection) on \( \mathbb{R}^4 \) is the function establishing a correspondence between the point \( x \in \mathbb{R}^4 \) and the affine connection at the point \( x \in \mathbb{R}^4 \) such that the following smooth condition is fulfilled: If \( X \) and \( Y \) are the smooth vector fields on \( \mathbb{R}^4 \), then the vector field \( (\nabla_X Y)_x = \nabla_{X_x} Y \) must be smooth.
Let us denote $\partial_\mu$ the vector field $\frac{\partial}{\partial x_\mu}$ on $\mathbb{R}^4$ where $x^\mu$ is a coordinate of the point $x \in \mathbb{R}^4$. Any vector field $U$ on $\mathbb{R}^4$ has the form

$$U = \sum_{\mu=0}^{3} u^\mu(x) \frac{\partial}{\partial x^\mu}. \quad (4.2)$$

In particular

$$\nabla_{\partial_\nu} \partial_\lambda = \sum_{\lambda=0}^{3} A^\lambda_{\mu\nu}(x) \frac{\partial}{\partial x^\lambda}. \quad (4.3)$$

The relation (4.1) and the bilinearity of $\nabla U V$ imply

$$\nabla U V = \sum_{\lambda=0}^{3} \left( \sum_{\mu=0}^{3} u^\mu v^\lambda \right) \frac{\partial}{\partial x^\lambda},$$

$$v^\lambda_{\mu} = \frac{\partial v^\lambda}{\partial x^\mu} + \sum_{\nu=0}^{3} A^\lambda_{\mu\nu} v^\nu. \quad (4.4)$$

Let us introduce the new coordinates $y^\mu(x)$

$$\frac{\partial}{\partial x^\mu} = \sum_{\nu=0}^{3} C^\nu_{\mu} \frac{\partial}{\partial y^\nu},$$

$$C^\nu_{\mu} = \frac{\partial y^\nu(x)}{\partial x^\mu}. \quad (4.5)$$

The relations (4.3) – (4.5) imply

$$\sum_{\lambda,\kappa=0}^{3} A^\lambda_{\mu\nu}(x) C^\kappa_{\lambda} \frac{\partial}{\partial y^\kappa} = \nabla_{(C^\rho_{\mu\nu})}(C_{\lambda} \frac{\partial}{\partial y^\nu}) = \sum_{\lambda,\kappa=0}^{3} C^\lambda_{\mu} \left( \frac{\partial C^\kappa_{\nu}}{\partial y^\lambda} + \sum_{\sigma=0}^{3} A^\kappa_{\nu\sigma}(y) C^\sigma_{\nu} \right) \frac{\partial}{\partial y^\kappa}, \quad (4.6)$$

$$A^\lambda_{\mu\nu}(x) = \sum_{\kappa,\sigma,\tau=0}^{3} C^\tau_{\mu} C^\nu_{\kappa} (C^{-1})^\rho_{\lambda} A^\kappa_{\nu\sigma}(y) + \sum_{\kappa=0}^{3} \frac{\partial C^\kappa_{\nu}}{\partial x^\mu} (C^{-1})^\lambda_{\kappa}. \quad (4.7)$$

Thus, if $y^\mu(x)$ is a linear function of the variables $x^\nu$, then due to the relation (4.7) the connection coefficients $A^\lambda_{\mu\nu}(x)$ transform as a tensor.

Let us define the torsion

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (4.8)$$

$$[X, Y] f = X(Y f) - Y(X f). \quad (4.9)$$

A torsion is a bilinear operation with respect to the multiplication by a function: If the vector fields $U, V$ have the form (4.2), then

$$T(U, V) = \sum_{\lambda,\mu,\nu=0}^{3} T^\lambda_{\mu\nu} u^\mu v^\nu \frac{\partial}{\partial x^\lambda}, \quad (4.10)$$

$$T^\lambda_{\mu\nu}(x) = A^\lambda_{\mu\nu}(x) - A^\lambda_{\nu\mu}(x). \quad (4.11)$$

A connection with zero torsion is called symmetric. Due to the relation (4.11) for a symmetric connection

$$A^\lambda_{\mu\nu}(x) = A^\lambda_{\nu\mu}(x). \quad (4.12)$$
Let us define the curvature

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z. \quad (4.13)$$

A curvature is a trilinear operation with respect to the multiplication by a function: If the vector fields $U, V, W$ have the form (4.2), then

$$R^\nu_{\mu\kappa\lambda}(x) = \frac{\partial A^\nu_{\kappa\mu}(x)}{\partial x^\kappa} - \frac{\partial A^\nu_{\kappa\mu}(x)}{\partial x^\lambda} + \sum_{\sigma=0}^{3} (A^\sigma_{\lambda\mu}(x)A^\nu_{\kappa\sigma}(x) - A^\sigma_{\kappa\mu}(x)A^\nu_{\lambda\sigma}(x)). \quad (4.15)$$

The equality (4.14) is proved in ([7], Lemma 9.1).

Let us introduce the new coordinates $y^\mu(x)$ and define the matrix (4.5). The relation (4.14) implies

$$R^\nu_{\mu\kappa\lambda}(x) = \sum_{\alpha,\beta,\gamma,\delta=0}^{3} C^\alpha_{\mu\nu} C^\beta_{\kappa\lambda} (C^{-1})^\gamma_{\delta}(x) R^\delta_{\alpha\beta\gamma}(y). \quad (4.16)$$

Thus the curvature (4.15) is a tensor for any invertible smooth functions $y^\mu(x)$.

Let us introduce $4 \times 4$ matrices

$$(A^\nu_{\mu\kappa}(x), (\hat{R}^\nu_{\mu\kappa\lambda}(x). \quad (4.17)$$

For a symmetric connection the equality (4.15) may be rewritten as

$$\hat{R}^\nu_{\mu\kappa\lambda}(x) = \frac{\partial A^\nu_{\kappa\mu}(x)}{\partial x^\kappa} - \frac{\partial A^\nu_{\kappa\mu}(x)}{\partial x^\lambda} + \sum_{\sigma=0}^{3} (A^\sigma_{\lambda\mu}(x)A^\nu_{\kappa\sigma}(x) - A^\sigma_{\kappa\mu}(x)A^\nu_{\lambda\sigma}(x)). \quad (4.19)$$

where the commutator of two matrices

$$([\hat{A}, \hat{B}])^\nu_{\mu} = (\hat{A}\hat{B} - \hat{B}\hat{A})^\nu_{\mu} = \sum_{\sigma=0}^{3} (A^\nu_{\mu\sigma}\hat{B}^\sigma_{\nu} - \hat{B}^\nu_{\mu\sigma}A^\sigma_{\nu}). \quad (4.20)$$

Thus the curvature (4.15) looks like a field strength for the Yang–Mills field.

Let us define

$$\nabla_Z R(X, Y) = [\nabla_Z, R(X, Y)] - R(\nabla_Z X, Y) - R(X, \nabla_Z Y). \quad (4.21)$$

For a symmetric connection the following identities are valid:

the first Bianchi’s identity

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0, \quad (4.22)$$

the second Bianchi’s identity

$$(\nabla_Z R)(X, Y) + (\nabla_Y R)(Z, X) + (\nabla_X R)(Y, Z) = 0. \quad (4.23)$$

The identities (4.22), (4.23) are proved in ([8], Chapter 3, Section 6, Theorems 1,2).
Let on $\mathbb{R}^4$ the pseudometric be given. We denote by $\langle X, Y \rangle$ the bilinear form of the two vectors $X, Y \in T\mathbb{R}^4_0$. We suppose that $\langle Y, X \rangle = \langle X, Y \rangle$ and the matrix $g_{\mu\nu} = \left( \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right)$ is invertible. Its inverse matrix is denoted by $g^{\mu\nu}$. A connection is called metric if

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \quad (4.24)$$

for any vector fields $X, Y, Z$.

$\mathbb{R}^4$ with pseudometric $g_{\mu\nu}$ has the unique metric connection with zero torsion

$$A^\lambda_{\mu\nu} = \frac{1}{2} \sum_{\kappa=0}^3 \left( \frac{\partial g_{\kappa\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\mu} \right) g^{\kappa\lambda}. \quad (4.25)$$

This proposition is called the basic lemma of Riemann geometry ([7], Lemma 8.6). The expression (4.25) coincides with the Christoffel’s symbol (3.4). The metric connection with zero torsion is called the Riemannian connection (sometimes it is given the name of Levi–Civita connection). It was shown in Section 3 that an arbitrary symmetric connection is not a Riemannian connection.

**Lemma 4.1.** Let on $\mathbb{R}^4$ the symmetric connection with the coefficients $A^\lambda_{\mu\nu}$ be given. By using the curvature coefficients (4.15) we define two tensors: the Ricci tensor

$$R_{\mu\nu} = \sum_{\kappa=0}^3 R^\kappa_{\nu\kappa\mu} = - \sum_{\kappa=0}^3 R^\kappa_{\nu\kappa\mu}, \quad (4.26)$$

and the tensor

$$F_{\mu\nu} = \sum_{\kappa=0}^3 R^\kappa_{\kappa\mu\nu}. \quad (4.27)$$

Then

$$R_{\mu\nu} = \sum_{\kappa=0}^3 \frac{\partial A^\kappa_{\mu\nu}}{\partial x^\kappa} - \frac{\partial}{\partial x^\mu} \left( \sum_{\kappa=0}^3 A^\kappa_{\nu\kappa} \right) + \sum_{\kappa=0}^3 \sum_{\kappa=0}^3 \left( A^\sigma_{\mu\kappa} A^\kappa_{\sigma\nu} - A^\sigma_{\nu\kappa} A^\kappa_{\sigma\mu} \right), \quad (4.28)$$

$$F_{\mu\nu} = \frac{\partial}{\partial x^\mu} \left( \sum_{\kappa=0}^3 A^\kappa_{\nu\kappa} \right) - \frac{\partial}{\partial x^\nu} \left( \sum_{\kappa=0}^3 A^\kappa_{\kappa\mu} \right), \quad (4.29)$$

$$R_{\nu\mu} - R_{\mu\nu} = F_{\mu\nu}. \quad (4.30)$$

If the symmetric connection with the coefficients $A^\lambda_{\mu\nu}$ is Riemannian, then

$$F_{\mu\nu} = 0, \quad (4.31)$$

$$R_{\nu\mu} = R_{\mu\nu}. \quad (4.32)$$

**Proof.** The relation (4.15) implies the relations (4.28) – (4.31).

If the connection is Riemannian, then the coefficients $A^\lambda_{\mu\nu}$ have the form (4.25) and

$$\sum_{\kappa=0}^3 A^\kappa_{\kappa\mu} = \frac{1}{2} \sum_{\kappa,\lambda=0}^3 g^{\kappa\lambda} \frac{\partial g_{\lambda\kappa}}{\partial x^\mu}. \quad (4.33)$$

The definition of the inverse matrix $g^{\kappa\lambda}$ implies

$$\frac{\partial g^{\kappa\lambda}}{\partial x^\mu} = - \sum_{\sigma,\tau=0}^3 g^{\kappa\sigma} g^{\lambda\tau} \frac{\partial g_{\sigma\tau}}{\partial x^\mu}. \quad (4.34)$$
It follows from the relations (4.33), (4.34) that
\[
\frac{\partial}{\partial x^\nu} \left( \sum_{\kappa=0}^3 A_{\kappa\mu}^\kappa \right) = \frac{1}{2} \sum_{\kappa,\lambda=0}^3 g^{\kappa\lambda} \frac{\partial^2 g_{\kappa\lambda}}{\partial x^\mu \partial x^\nu} - \frac{1}{2} \sum_{\kappa,\lambda,\sigma,\tau=0}^3 g^{\kappa\sigma} g^{\lambda\tau} \frac{\partial g_{\kappa\lambda}}{\partial x^\mu} \frac{\partial g_{\sigma\tau}}{\partial x^\nu}.
\]
(4.35)

The substitution of the relation (4.35) into the relation (4.29) yields the relation (4.31). The relations (4.30), (4.31) imply the relation (4.32). The lemma is proved.

The tensor (4.29) has the form of the electromagnetic field strength tensor. Let the connection coefficients satisfy the equation
\[
\sum_{\alpha=0}^3 \eta^{\alpha\alpha} \left( \frac{\partial}{\partial x^\alpha} \right)^2 \lambda_{\mu\nu}^\alpha(x, y) = 4\pi m y \eta_{\mu\nu} \eta_{\mu\nu} \frac{dy^\mu(x^0)}{dx^0} \frac{dy^\nu(x^0)}{dx^0} \times \\
\left( \sum_{\alpha=0}^3 \eta^{\alpha\alpha} \left( \frac{dy^\alpha(x^0)}{dx^0} \right)^2 \right)^{-1} \delta(x - y(x^0/c)).
\]
(4.36)

Then the equations (4.29), (4.36) imply the equations (2.5), (2.6).

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