Generalized para-Kähler manifolds

by

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ABSTRACT. We define a generalized almost para-Hermitian structure to be a commuting pair \((\mathcal{F}, \mathcal{J})\) of a generalized almost para-complex structure and a generalized almost complex structure with an adequate non-degeneracy condition. If the two structures are integrable the pair is called a generalized para-Kähler structure. This class of structures contains both the classical para-Kähler structure and the classical Kähler structure. We show that a generalized almost para-Hermitian structure is equivalent to a triple \((\gamma, \psi, F)\), where \(\gamma\) is a \((\text{pseudo})\) Riemannian metric, \(\psi\) is a 2-form and \(F\) is a complex \((1, 1)\)-tensor field such that \(F^2 = \text{Id}, \gamma(FX, Y) + \gamma(X, FY) = 0\). We deduce integrability conditions similar to those of the generalized Kähler structures and give several examples of generalized para-Kähler manifolds. We discuss submanifolds that bear induced para-Kähler structures and, on the other hand, we define a reduction process of para-Kähler structures.

1 Introduction

The framework of this paper is the \(C^\infty\)-category and the notation follows [9], with the exception of the wedge product evaluation, where we follow Cartan (i.e., \(\alpha \wedge \beta(X, Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)\) for 1-forms, etc.).

Generalized geometry on a manifold \(M\), as defined by Hitchin, is the geometry of structures of the big tangent bundle \(TM = TM \oplus T^*M\) endowed with the pairing metric

\[
g(X, Y) = \frac{1}{2}(\alpha(Y) + \beta(X)), \ X = (X, \alpha), Y = (Y, \beta) \in TM \oplus T^*M
\]

(e.g., see [5], which we also follow for terminology and notation).

The interpretation of the notion of a classical almost para-Hermitian structure in terms of generalized geometry leads us to define a generalized almost para-Hermitian structure to be a commuting pair \((\mathcal{F}, \mathcal{J})\), where \(\mathcal{J}\) is a generalized almost complex structure and \(\mathcal{F}\) is a generalized almost para-complex structure [5] [12], with an adequate non-degeneracy condition. Furthermore, if
the two structures \((\mathcal{F}, \mathcal{J})\) are integrable, the structure will be called a generalized para-Kähler structure. The integrability of \(\mathcal{F}\) and \(\mathcal{J}\) means that the eigenbundles of each of these endomorphisms are closed under the Courant bracket

\[
[(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - L_Y \alpha + \frac{1}{2} d(\alpha(Y) - \beta(X)).
\]

The Courant bracket \([6]\) replaces the Lie bracket in integrability conditions of generalized geometry.

In this paper, we apply the techniques of generalized Kähler geometry \([5]\) and obtain corresponding results for generalized para-Kähler structures.

We show that a generalized almost para-Hermitian structure is equivalent to a triple \((\gamma, \psi, F)\), where \(\gamma\) is a (pseudo) Riemannian metric on \(M\), \(\psi\) is a 2-form and \(F \in \text{End} T^c M\) (the index \(c\) denotes complexification) is a complex \((1,1)\)-tensor field such that \(F^2 = Id, \gamma(FX, Y) + \gamma(X, FY) = 0\). As a consequence, we show that the classical almost para-Hermitian and almost (pseudo) Hermitian manifolds have a generalized almost para-Hermitian structure. We also show that a generalized almost para-Hermitian structure is equivalent to a decomposition \(T^c M = H_+ \oplus \overline{H}_- \oplus H_+ \oplus \overline{H}_-\), where \(H_+ \oplus \overline{H}_+, H_- \oplus \overline{H}_-\) are maximal \(g\)-isotropic subbundles (the bar denotes complex conjugation).

We show that the integrability of the structure is equivalent to the closure of the subbundles \(H_+\) under the Courant bracket. Then, we prove that a generalized para-Kähler structure is characterized by the involutivity of the eigenbundles of the endomorphism \(F\) together with either a certain expression of the covariant derivative \(\nabla^\gamma F\) in terms of \(d\psi\) or the equality \(d\psi(X, Y, Z) = i d\omega(FX, FY, FZ)\) (\(\nabla^\gamma\) is the Levi-Civita connection of \(\gamma\) and \(i\) is the complex unit). In particular, if \(d\psi = 0\), we remain with the single condition \(\nabla^\gamma F = 0\).

Among others, we construct a generalized para-Kähler structure on \(\mathbb{C}^2\) that projects to a structure of the complex 2-torus and a generalized para-Kähler structure on a circle bundle over the classical paracomplex projective model \([7]\). Furthermore, we show that the complete lift of a generalized para-Kähler structure of a manifold \(M\) to the tangent manifold \(TM\) is a generalized para-Kähler structure of \(TM\). Then, we briefly discuss a cohomology of the Dolbeault type.

In the last section, we discuss submanifolds that get an induced structure. These are the submanifolds \(N\) of \(M\) such that the tensor induced by \(\gamma\) is non degenerate and \(T^c N\) is invariant by \(F\). Another characterization of these submanifolds is that the pullbacks of the subbundles \(H_+\) decompose \(T^c N\) in the way required by a generalized almost para-Hermitian structure. If the structure of \(M\) is integrable, the induced structure of \(N\) is integrable too.

Finally, we show that, if a structure \((\mathcal{F}, \mathcal{J})\) is invariant by a Lie group \(G\), a reduction process may exist and we define a momentum map for the complete lift of the \(G\)-invariant structure of \(M\) to \(TM\).
2 Generalized almost para-Hermitian structures

In classical differential geometry an almost paracomplex structure on a $2n$-dimensional manifold $M$ is $F \in \text{End}TM$ with $F^2 = Id$ and with $\pm 1$-eigenbundles $S_\pm$ of dimension $n$. A pseudo-Riemannian metric $\gamma$ of $M$ is $F$-compatible if $\gamma(FX,Y) + \gamma(X,FY) = 0$; then $S_\pm$ are isotropic and $\gamma$ has signature zero. The pair $(\gamma,F)$ is called an almost para-Hermitian structure. Furthermore, if the 2-form $\omega(X,Y) = \gamma(FX,Y)$ is closed, the structure is almost para-Kähler and if, in addition, $S_\pm$ are involutive, the structure is para-Kähler.

With the musical isomorphisms $\flat, \sharp : \gamma, \omega \mapsto \gamma^{\sharp -1}, \omega^{\flat -1}$, we have

\begin{align*}
\flat \gamma \circ F &= -F^* \circ \flat \gamma, F \circ \sharp \gamma = -\sharp \gamma \circ F^*, \\
\flat \omega &= F^* \circ \flat \gamma = -\flat \gamma \circ F, \; \sharp \omega = -\sharp \gamma \circ F^* = F \circ \sharp \gamma.
\end{align*}

These objects may be encoded in the endomorphisms $F, J, H \in \text{End}TM$ given by the action of the matrices

$$
F = \begin{pmatrix} F & 0 \\ 0 & -F^* \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \sharp \omega \\ \flat \omega & 0 \end{pmatrix}, \quad H = F \circ J = \begin{pmatrix} 0 & \sharp \gamma \\ -\flat \gamma & 0 \end{pmatrix} \tag{2.2}
$$

on columns $\begin{pmatrix} X \\ \alpha \end{pmatrix}$, $X \in TM, \alpha \in T^* M$. Then, $F, J$ are skew-symmetric with respect to the pairing metric $g$ and $J^2 = -Id, F^2 = Id$, which means that $J$ is a generalized almost complex structure [5] and $F$ is a generalized almost paracomplex structure [12]. Notice also the relations

$$
F \circ J = J \circ F = H, \; F \circ H = H \circ F = J, \; J \circ H = H \circ J = -F. \tag{2.3}
$$

The endomorphism $H$ satisfies the properties

$$
H^2 = -Id, \; g(HX,Y) = g(X,HY), \tag{2.4}
$$

hence, it is not a generalized almost complex structure. We will say that $H$ is a $g$-symmetric big almost complex structure. The $\pm i$-eigenbundles $H, \overline{H}$ of $H$ are not $g$-isotropic, therefore, they cannot be closed under the Courant bracket and integrability of $H$ is a non-sense condition. Using $H = \text{im}(Id - i\mathcal{H})$, it follows that the non-degeneracy of $\gamma$ is equivalent to $H \cap T^c M = 0$.

This situation suggests the following definition.

**Definition 2.1.** A generalized almost para-Hermitian structure is a commuting pair $(F, J)$, where $F$ is a generalized almost paracomplex structure and $J$ is a generalized almost complex structure, such that the symmetric bivector

$$
\gamma(\alpha, \beta) = -2g(F(0, \alpha), J(0, \beta)) \tag{2.5}
$$

is non-degenerate. If $J$ is integrable, the structure is generalized para-Hermitian. If $F$ is integrable, the structure is generalized almost para-Kähler. If both $F$ and $J$ are integrable, the structure is generalized para-Kähler.
For any generalized almost para-Hermitian structure, the operator $H = J \circ F$ is a $g$-symmetric big almost complex structure that satisfies the relations (2.3), (2.4) and we denote by $H$ the i-eigenbundle of $H$.

**Proposition 2.1.** The tensor $\gamma$ defined by (2.5) is non-degenerate iff $H \cap T^c M = 0$.

**Proof.** We write down the following matrix representation of $F, J, H$:

$F = \begin{pmatrix} P & \sharp \phi \\ b_\theta & -P^* \end{pmatrix}$, $J = \begin{pmatrix} A & \sharp \pi \\ b_\sigma & -A^* \end{pmatrix}$, $H = \begin{pmatrix} Q & \sharp \gamma \\ b_\nu & Q^* \end{pmatrix}$, (2.6)

where $\phi, \theta, \pi, \sigma$ are skew-symmetric, $\gamma$ is the tensor (2.5), $\nu \in \odot^2 T^* M$ is a second symmetric tensor and

\begin{align*}
P^2 &= Id - \sharp \phi \circ b_\theta, \sharp \phi \circ P^* = P \circ \sharp \phi, b_\theta \circ P = P^* \circ b_\theta, \\
A^2 &= -Id - \sharp \pi \circ b_\sigma, \sharp \pi \circ A^* = A \circ \sharp \pi, b_\sigma \circ A = A^* \circ b_\sigma, \\
Q^2 &= -Id - \sharp \gamma \circ b_\nu, \sharp \gamma \circ Q^* = -Q \circ \sharp \gamma, b_\nu \circ Q = -Q^* \circ b_\nu, \\
Q &= P \circ A + \sharp \phi \circ b_\sigma = A \circ P + \sharp \pi \circ b_\theta, \\
\sharp \gamma &= A \circ \sharp \pi - \sharp \phi \circ A^* = A \circ \sharp \phi + \sharp \pi \circ P^*, \\
b_\nu &= b_\theta \circ A - P^* \circ b_\sigma = b_\sigma \circ P - A^* \circ b_\theta; \\
\end{align*}

(2.7)

these relations ensure that $F, J, H$ are structures of the required type and $H = F \circ J$.

The condition $H \cap T^c M = 0$ is equivalent to $T^c M = T^c M \oplus H$, so that the corresponding projection onto $T^c M$ has an inverse $\tau : T^c M \to H$. The latter must be of the form

$$X' + iX'' \mapsto (X' + iX'', \alpha' + i\alpha''),$$

where the right hand side is an i-eigenvector of $H$, equivalently:

\begin{align*}
(a) & \quad QX' + \sharp \gamma \alpha' = -X'', QX'' + \sharp \gamma \alpha'' = X', \\
(b) & \quad b_\nu X' + Q^* \alpha' = -\alpha'', b_\nu X'' + Q^* \alpha'' = \alpha'.
\end{align*}

If the isomorphism $\tau$ exists, equations $(a)$ must define $\alpha', \alpha''$ uniquely, for any $X', X''$ and this happens iff $\gamma$ is non-degenerate.

If $\gamma$ is non-degenerate and if we solve $(a)$, we get

$$\alpha' = -b_\gamma(QX' + X''), \quad \alpha'' = b_\gamma(X' - QX''),$$

and we can check that the solutions also satisfy $(b)$ by using properties (2.7). This shows that the obtained $\alpha', \alpha''$ yield an isomorphism $\tau$ as required. \qed
Furthermore, put $b_\psi = -b_\gamma \circ Q$. Then, $\psi$ is a 2-form and the values of $\alpha', \alpha''$ obtained above give the expression
$$
\tau(X) = (X, b_\psi + i_\gamma X), \forall X = X' + iX'' \in T^c M.
$$
(2.8)
The image of $\tau$ is $H$, the image of the conjugate operator $\bar{\tau}$ is the $-i$-eigenbundle $\bar{H}$ of $H$ and (2.8) is, in fact, one more way to express the operator $H$. Another interesting consequence of formula (2.8) is
$$
\gamma(X, Y) = -\frac{i}{2}g(\tau X, \tau Y),
$$
(2.9)
where $X, Y \in T^c M$ and $\gamma$ is extended by complex linearity (we should have written $\gamma^{-1}$, but, we follow the custom of Riemannian geometry).

**Proposition 2.2.** The generalized almost para-Hermitian structures $(F, J)$ are in a one-to-one correspondence with triples $(\gamma, \psi, F)$, where $\gamma$ is a pseudo-Riemannian metric of $M$, $\psi$ is a 2-form and $F \in \text{End} T^c M$ is a complex $(1, 1)$-tensor field such that
$$
F^2 = \text{Id}, \quad \gamma(FX, Y) = -\gamma(X, FY).
$$
(2.10)
The same structures $(F, J)$ are in a one-to-one correspondence with triples $(\gamma, \psi, J \in \text{End} T^c M)$, where
$$
J^2 = -\text{Id}, \quad \gamma(JX, Y) = -\gamma(X, JY).
$$
(2.11)
The tensors $F, J$ of a given structure $(F, J)$ are related by the equality $J = iF$.

**Proof.** Since $F, H$ commute, $F$ preserves $H$ and leads to a tensor $F \in \text{End}(T^c M)$ given by
$$
FX = pr_{T^c M} F(\tau X).
$$
(2.12)
Formulas (2.12), (2.9) imply (2.10). Similarly, $J$ preserves $H$ and produces the complex operator $JX = pr_{T^c M} J(\tau X)$ with the properties (2.11).

Generally, $F$ and $J$ are not real and, if we put
$$
F = F_1 + iF_2, \quad J = J_1 + iJ_2
$$
and use (2.10), (2.14), we get
$$
F_1 = P - z_\phi \circ b_\gamma \circ Q, \quad F_2 = z_\phi \circ b_\gamma,
$$
$$
J_1 = A - z_\pi \circ b_\gamma \circ Q, \quad J_2 = z_\pi \circ b_\gamma.
$$
(2.13)
Thus, $F$ is real iff $\phi = 0$ and $J$ is real iff $\pi = 0$. The classical structure $(\gamma, F)$ is the case $\psi = 0, \phi = 0, \theta = 0$ and, then, the operators on $T^c M$ are $F$ and $J = iF_\omega \circ b_\gamma = iF$.

In real terms conditions (2.10) become
$$
F_1^2 - F_2^2 = \text{Id}, \quad F_1 \circ F_2 + F_2 \circ F_1 = 0,
$$
$$
\gamma(F_1 X, Y) = -\gamma(X, F_1 Y), \quad \gamma(F_2 X, Y) = -\gamma(X, F_2 Y)
$$
(2.14)
and the same for the pair \((J_2, J_1)\) instead of \((F_1, F_2)\).

Conversely, the pair \((\gamma \in \odot^2 T^*M, \psi \in \wedge^2 T^*M)\), where \(\gamma\) is non degenerate, allows us to reconstruct the big almost complex structure \(\mathcal{H}\) by taking its eigenbundle \(\mathcal{H}\) to be the image of \(\tau\) given by \(\mathcal{H}(\gamma)\). If we add the endomorphism \(F\) of \(T^cM\) that satisfies \((2.10)\), we are also able to reconstruct \(F\) on \(\mathcal{H}, \mathcal{H}\) by lifting \(F\) and its complex conjugation. The resulting \(\mathcal{F}\) commutes with \(\mathcal{H}\) (check on \(\mathcal{H}, \mathcal{H}\)). Finally, we will take \(\mathcal{J} = \mathcal{H} \circ \mathcal{F}\) and \(\mathcal{J}\) may be reconstructed from \(\mathcal{J}\) as \(\mathcal{F}\) was from \(F\). Therefore, we have

\[
\begin{pmatrix}
JX \\
\flat_{\psi+\gamma} JX
\end{pmatrix} =
\begin{pmatrix}
Q & \flat_{\nu} \\
\flat_{\nu} & Q^*
\end{pmatrix}
\begin{pmatrix}
FX \\
\flat_{\psi+\gamma} FX
\end{pmatrix} (X \in T^cM),
\]

whence we deduce the required relation \(J = iF\). This relation is equivalent to \(J_1 = -F_2, J_2 = F_1\).

The entries of the respective matrices \((2.10)\) are determined by the real and imaginary part of the equalities

\[
\begin{pmatrix}
P & \sharp_\phi \\
\flat_\theta & -P^*
\end{pmatrix}
\begin{pmatrix}
X \\
\flat_{\psi+\gamma} X
\end{pmatrix} =
\begin{pmatrix}
F_1X + iF_2X \\
\flat_{\psi+\gamma} (F_1X + iF_2)X
\end{pmatrix},
\]

\[
\begin{pmatrix}
A & \sharp_\pi \\
\flat_\sigma & -A^*
\end{pmatrix}
\begin{pmatrix}
X \\
\flat_{\psi+\gamma} X
\end{pmatrix} =
\begin{pmatrix}
J_1X + iJ_2X \\
\flat_{\psi+\gamma} (J_1X + iJ_2)X
\end{pmatrix},
\]

where we assume that \(X \in TM\) is a real vector. We shall also use \(Q = -\sharp_\gamma \circ \flat_\psi, \flat_\nu = -\flat_\gamma \circ (Id + Q^2)\), which follows from the definition of \(\psi\) and \((2.7)\). The results are

\[
P = F_1 + F_2 \circ Q, \quad \sharp_\phi = F_2 \circ \sharp_\gamma,
\]

\[
\flat_\theta = \flat_\psi \circ F_1 - \flat_\gamma \circ F_2 + (F_1^* + Q^* \circ F_2^*) \circ \flat_\psi,
\]

\[
A = J_1 + J_2 \circ Q, \quad \sharp_\pi = J_2 \circ \sharp_\gamma,
\]

\[
\flat_\sigma = \flat_\psi \circ J_1 - \flat_\gamma \circ J_2 + (J_1^* + Q^* \circ J_2^*) \circ \flat_\psi.
\]

(2.15)

Manifolds \(M\) endowed with a pair \((\gamma \in \odot^2 T^*M, F \in \text{End}(T^cM))\), where \(\gamma\) is non-degenerate and \((2.10)\) holds, belong to a class of manifolds discussed in [8] under the name of almost-Hermitian manifolds in the enlarged sense. In [8], \(\gamma\) too was allowed to be a complex tensor. In our case (\(\gamma\) real) we call them almost complex-para-Hermitian manifolds. They have two complementary fields of complex planes defined by the \(\pm 1\)-eigenbundles of \(F, S_{\pm} = \text{im}(Id \pm F)\), that are \(\gamma\)-isotropic, hence, both must have the same complex dimension \(n = \dim M/2\) and the non-degeneracy of \(\gamma\) implies that \(X \mapsto \flat_\gamma X\) defines isomorphisms \(S_{\pm} \approx S^*_{\pm}, S_{\pm} = S^*_{\pm}\). It follows that, if we fix the metric \(\gamma\), there exists a one-to-one correspondence between the complex tensor fields \(F\) that satisfy \((2.10)\) and the decompositions \(T^cM = S_+ \oplus S_-\), where the terms are maximally \(\gamma\)-isotropic subbundles.
For another remark, consider a B-field transformation \((B \in \wedge^2 TM)\) of the generalized almost para-Hermitian structure \((\mathcal{F}, \mathcal{J})\), i.e., the transformation 
\[
\mathcal{F} \mapsto \mathcal{F}' = B^{-1} \mathcal{F} B, \quad \mathcal{J} \mapsto \mathcal{J}' = B^{-1} \mathcal{J} B,
\]
where \(B = \begin{pmatrix} \text{Id} & 0 \\ b & \text{Id} \end{pmatrix}\). The result is again a generalized almost para-Hermitian structure. If the original structure is integrable and \(B\) is closed, the transformed structure is integrable too. It follows easily that the B-field transformation preserves the tensors \(\phi, \pi, \gamma\), while,
\[
P \mapsto P' = P + \Zeta_\phi \circ b_B, \quad A \mapsto A' = A + \Zeta_\pi \circ b_B, \quad Q \mapsto Q' = Q + \Zeta_\gamma \circ b_B.
\]
Then, formula (2.13) shows that the tensors \(F, J\) are preserved, while, the definition of \(\psi\) shows that \(\psi \mapsto \psi' = \psi - B\).

**Example 2.1.** Let \((\gamma, J)\) be a classical almost (pseudo) Hermitian structure and take \(\psi = 0\). Since (2.11) holds, we get a corresponding generalized almost para-Hermitian structure given by
\[
\mathcal{F} = \begin{pmatrix} 0 & -J \circ \Zeta_\gamma \\ b \circ J & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 & \Zeta_\gamma \\ -b \circ J & 0 \end{pmatrix}.
\]
Hence, we may see the generalized almost para-Hermitian structures as a bridge between the almost Hermitian and almost para-Hermitian structures.

**Example 2.2.** Assume that \(M\) has two (pseudo) Hermitian structures \((\gamma, J_\pm)\) with the same metric \(\gamma\), such that the complex structures \(J_+, J_-\) anti-commute and take any \(\psi \in \wedge^2 T M\). Then, the addition of a tensor field \(F = \alpha J_+ + \beta J_-\), where \(\alpha, \beta \in \mathbb{R}, \beta^2 - \alpha^2 = 1\), produces a generalized almost para-Hermitian structure on \(M\). Similarly, if \((\gamma, K), (\gamma, L)\) are classical, almost pseudo-Hermitian and para-Hermitian, respectively, and if \(K \circ L + L \circ K = 0\), then, \(\gamma\) together with a 2-form \(\psi\) and with \(F = \alpha K + \beta L\) where \(\alpha^2 + \beta^2 = 1\), \(\alpha, \beta \in \mathbb{R}\), define a generalized, almost para-Hermitian structure with \(F_1 = \alpha K, F_2 = \beta L\).

**Example 2.3.** The direct product \(\tilde{M} = M \times M'\), where \(M, M'\) are generalized almost para-Hermitian manifolds with the structures \((\mathcal{F}, \mathcal{J}), (\mathcal{F}', \mathcal{J}')\), equivalently \((\gamma, \psi, F), (\gamma', \psi', F')\), has the generalized almost para-Hermitian structure \((\mathcal{F} \oplus \mathcal{F}', \mathcal{J} \oplus \mathcal{J}')\). This structure corresponds to the triple \((\gamma \oplus \gamma', \psi \oplus \psi', F \oplus F')\). The direct sum is just a notation to indicate that the terms act on corresponding components of the tangent bundle of \(M\). For instance, if \(\gamma\) is a pseudo-Euclidean metric of \(\mathbb{R}^{2n}\), of positive-negative inertia indices \((p, q), p + q = 2n, p \geq q\), we may write \(\mathbb{R}^{2n} = \mathbb{R}^q \times \mathbb{R}^{2(n-q)}\), where the first factor has the standard para-Hermitian structure and the second factor has the standard Hermitian structure. The product structure is a generalized almost para-Hermitian structure on \(\mathbb{R}^{2n}\) (2-forms \(\psi\) as needed may be added).

Consider a product \(M \times M'\), where \((M, \gamma, F)\) is almost para-Hermitian and \((M', \gamma', iF')\) is generalized almost para-Hermitian with a real tensor \(F'\) (i.e.,
$(M', \gamma', F')$ is classical almost Hermitian. In this case the real and imaginary part of the product structure are $F_1 = F \oplus \{0\}, F_2 = \{0\} \oplus F'$, which commute and we get $F_1 \circ F_2 = 0$. Generally, a generalized almost para-Hermitian structure of a manifold $M$ such that $F_1 \circ F_2 = 0$ will be called a split structure. The name is taken from [1] and it is motivated as follows. If $F \circ F = 0$, then, (2.14) implies $F_2 \circ F_1 = 0$. $F_1^3 = F_1, F_2^3 = -F_2$. Furthermore, if $\Phi = F_1^2 + F_2^2$, $\Phi^2 = Id$ and we get a decomposition $TM = P \oplus Q$, where $P, Q$ are the self-eigenbundles of $\Phi$ and $F_1^2|P = Id, F_2^2|Q = -Id$. $P = im F_2^2 = ker F_1$, $Q = im F_2^2 = im F_2 = ker F_1$.

**Example 2.4.** Let $M = G/H$, where $G$ is a connected Lie group that acts transitively and effectively on $M$ and $H$ is a closed subgroup, be a homogeneous, (pseudo) Riemannian space with the invariant metric $\gamma$. Let $x_0$ be the point defined by the unit $e$ of $G$. Using the construction indicated at the end of Example 2.3, we get a generalized almost para-Hermitian structure on $T_{x_0}M$ with a corresponding triple $(\gamma_{x_0}, \psi_0, F_0)$. Then, acting by $G$ on $F_0, \psi_0$ via the derived transformations we get invariant tensor fields $F, \psi$ of $M$ such that $F$ satisfies (2.10) everywhere. The triple $(\gamma, \psi, F)$ produces an invariant, generalized almost para-Hermitian structure on $M$. It is known that the invariant tensors $\gamma, \psi, F$ may be identified with ad $h$-invariant tensors $\tilde{\gamma}, \tilde{\psi}, \tilde{F}$ ($\tilde{F}$ is complex) on the vector space $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}, \mathfrak{h}$ are the Lie algebras of $G, H$, respectively. Accordingly, the invariant, generalized almost para-Hermitian structures of $M = G/H$ may be identified with generalized almost para-Hermitian structures of the vector space $\mathfrak{g}/\mathfrak{h}$ (the structures live on $(\mathfrak{g}/\mathfrak{h}) \oplus (\mathfrak{g}/\mathfrak{h})^\ast$).

We end this section by proving the following result.

**Proposition 2.3.** A generalized almost para-Hermitian structure on $M$ is equivalent with a decomposition

$$T^c M = H_+ \oplus H_- \oplus \bar{H}_+ \oplus \bar{H}_-,$$

(2.17)

where $H_+ \oplus \bar{H}_+, H_- \oplus \bar{H}_-$ are maximal $g$-isotropic subbundles.

**Proof.** Start with a structure $(\mathcal{F}, J)$ and denote by $F_{\pm}$ the self-eigenbundles of $\mathcal{F}$ and by $J, \bar{J}$ the $\pm i$-eigenbundles of $\mathcal{F}$. The commutation properties ensure that the projections of a vector of $F_{\pm}, J$ on $H, \bar{H}$ belong to $F_{\pm}, J$, respectively, and the projections of a vector of $H$ on $F_{\pm}, J$ belong to $H$. This leads to the existence of the following decompositions

$$F_{\pm} = (F_{\pm} \cap H) \oplus (F_{\pm} \cap \bar{H}),$$

$$H = (F_+ \cap H) \oplus (F_- \cap H),$$

(2.18)

Moreover, by looking at the properties of a vector in the corresponding intersection, we get

$$F_+ \cap H = F_+ \cap J = H \cap J, F_- \cap H = F_- \cap \bar{J} = H \cap \bar{J}.$$

(2.19)
Then, if we define $H_{\pm} = F_{\pm} \cap H$, equalities (2.18) give us (2.17) and the subbundles of the conclusion are exactly $F_{\pm}, J$, hence, the conclusion holds.

Conversely, a decomposition (2.17) produces endomorphisms $F, J$ such that

$$F_{H_{+}} = F_{H_{+}} = 1d, F_{H_{-}} = F_{H_{-}} = -1d,$$

$$J_{H_{+}} = J_{H_{+}} = i1d, J_{H_{-}} = J_{H_{-}} = -i1d.$$ 

These endomorphisms $F, J$ are real, commute and satisfy the conditions $F^2 = 1d, J^2 = -1d$. Since the subbundles $F_{\pm} = H_{\pm} \oplus \bar{H}_{\mp}, F_{\mp} = H_{\pm} \oplus \bar{H}_{\mp}, J = H_{\pm} \oplus \bar{H}_{\mp}$. were required to be maximal $g$-isotropic (which also implies $dim H_{\pm} = dim H_{\mp}$), the pair $(F, J)$ is a generalized almost para-Hermitian structure of $M.$

3 Generalized para-Kähler manifolds

In this section we investigate the integrability conditions of a generalized almost para-Hermitian structure $(F, J)$. These conditions can be obtained in the same way as for generalized Kähler manifolds [5, 13].

**Theorem 3.1.** A generalized almost para-Hermitian structure $(F, J)$ is integrable iff the corresponding subbundles $H_{\pm}$ are closed under Courant brackets. Furthermore, if $(\gamma, \psi, F)$ is the equivalent triple of tensors of the structure, $(F, J)$ is a generalized para-Kähler structure iff the (complex) eigenbundles of $F$ are involutive and

$$(\nabla^h F)(Y) = \frac{1}{2} \gamma^{-1}[i(X)i(FY)d\psi + (i(X)i(\bar{Y})d\psi) \circ F],$$

(3.1)

where $\nabla^h$ is the Levi-Civita connection of the metric $\gamma$.

**Proof.** We will use decomposition (2.17). Equalities (2.19) show that, if $F, J$ are both integrable, the subbundles $H_{\pm}$ are closed under the Courant bracket. Conversely, assume that $H_{\pm}$ are closed under the Courant bracket. Consider the following general property [6]

$$(pr_{TM} \mathcal{Z})(g(\mathcal{X}, \mathcal{Y})) = g([\mathcal{X}, \mathcal{Y}], \mathcal{Z}) + g(\mathcal{X}, [\mathcal{Z}, \mathcal{Y}]) + \frac{1}{2}(pr_{TM} \mathcal{X})(g(\mathcal{Z}, \mathcal{Y})) + \frac{1}{2}(pr_{TM} \mathcal{Y})(g(\mathcal{Z}, \mathcal{X})).$$

For $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \mapsto (\mathcal{Z}, \bar{\mathcal{Y}}, \mathcal{X})$, where $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in H_{\pm}$, this property and the $g$-isotropy of $F_{\pm} = H_{\pm} \oplus \bar{H}_{\pm}$ yield $[\mathcal{X}, \bar{\mathcal{Y}}] \perp g H_{\pm}$. By conjugation, and changing the role of $\mathcal{X}, \mathcal{Y}$, we also get $[\mathcal{Y}, \bar{\mathcal{X}}] \perp g H_{\pm}$, therefore $[\mathcal{X}, \bar{\mathcal{Y}}] \perp g F_{\pm}$ and, since $F_{\pm}$ is maximally isotropic, $[\mathcal{X}, \bar{\mathcal{Y}}] \in F_{\pm}$. This proves the closure of $F_{\pm}$ under Courant brackets, i.e., the integrability of $F$. The closure of $J$ under Courant brackets follows by the same computations for arguments $\mathcal{X}, \mathcal{Z} \in H_{\pm}, \mathcal{Y} \in H_{\mp}$ and $\mathcal{X} \in H_{\pm}, \mathcal{Z}, \mathcal{Y} \in H_{\mp}$ using the maximal isotropy of $J$ and the already proven result for $F_{\pm}$. Therefore $J$ is integrable too.
For the second part of the theorem, straightforward computations (as in [5]) yield the Courant bracket

\[ [\tau X, \tau Y] = \tau [X, Y] + (0, i(Y)i(X)\psi + i(L_Xi(Y) - i(X)L_Y)(\gamma)), \]

(3.2)

where \( X, Y \in T^cM \).

Therefore, \( pr_{T^cM}[\tau X, \tau Y] = [X, Y] \) and, since \( H_\pm = \tau(S_\pm) \), where \( S_\pm \) are the (complex) \( \pm 1 \)-eigenbundles of \( F \), involutivity of \( S_\pm \subseteq T^cM \) is a necessary condition for the integrability of the structure \((F, J)\). Moreover, (3.2) shows that the necessary and sufficient conditions are involutivity of \( S_\pm \) plus the two equalities

\[ (L_Xi(Y) - i(X)L_Y)(\gamma)) = ii(X)i(Y)\psi, \quad X, Y \in S_\pm. \]

(3.3)

We shall express these conditions in terms of the Levi-Civita connection \( \nabla^\gamma \) of \( \gamma \). Involutivity of \( S_\pm \) is equivalent to the vanishing of the (complex) Nijenhuis tensor

\[ N_F(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] = 0, \quad X, Y \in T^cM, \]

where the brackets are Lie brackets. Thus, we can express them by the (torsionless) Levi-Civita connection \( \nabla^\gamma \) of the metric \( \gamma \). The result is

\[ (\nabla^\gamma_{FX}F)(Y) - (\nabla^\gamma_{FY}F)(X) = F((\nabla^\gamma_XF)(Y) - (\nabla^\gamma_YF)(X)). \]

(3.4)

Then, if we evaluate (3.3) on \( Z \in T^cM \) and express the Lie derivatives using \( \nabla^\gamma \), (3.3) becomes

\[ \gamma(X, \nabla^\gamma_XY) - \gamma(Y, \nabla^\gamma_YX) = id\psi(X, Y, Z), \quad X, Y \in S_\pm. \]

(3.5)

Furthermore, let \( \nabla \) be a connection on \( T^cM \) such that \( \nabla^\gamma = 0, \nabla F = 0 \). Such connections exist. For instance, we may take

\[ \nabla_XY = pr_{S_+}\nabla_X^\gamma(pr_{S_+}Y) + pr_{S_-}\nabla_X^\gamma(pr_{S_-}Y). \]

\( \nabla \) preserves \( S_\pm \), which is equivalent to \( \nabla F = 0 \), and \( \nabla^\gamma = 0 \) follows by taking into account \( \gamma|_{S_\pm} = 0, \nabla^\gamma = 0 \). If \( \Theta = \nabla - \nabla^\gamma \), \( \Theta \) satisfies the conditions

\[ \gamma(\Theta(X, Y), Z) + \gamma(Y, \Theta(X, Z)) = 0, \]

\[ (\nabla^\gamma_XF)(Y) = F\Theta(X, Y) - \Theta(X, FY). \]

(3.6)

Using (3.6) and the \( \gamma \)-isotropy of \( S_\pm \), condition (3.5) becomes

\[ \gamma(Y, \Theta(Z, X)) = \frac{1}{2}d\psi(X, Y, Z), \quad X, Y \in S_\pm. \]

(3.7)

In (3.7) we may replace \( X, Y \) by \( X \pm FX, Y \pm FY \), where the new arguments \( X, Y \in T^cM \) are arbitrary vector fields. Then, using (3.6) again, we get

\[ \gamma(X, (\nabla^\gamma_XF)(FY)) + \gamma(X, (\nabla^\gamma_XF)(Y)) \]

\[ = \frac{1}{2}[d\psi(X, Y, Z) + d\psi(FX, FY, Z) + d\psi(FX, FY, Z) + d\psi(FX, Y, Z)], \]

\[ \gamma(X, (\nabla^\gamma_XF)(FY)) - \gamma(X, (\nabla^\gamma_XF)(Y)) \]

\[ = \frac{1}{2}[d\psi(X, Y, Z) + d\psi(FX, FY, Z) - d\psi(X, FY, Z) - d\psi(FX, Y, Z)]. \]

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These conditions may be replaced by their sum and difference, which turn out to be the same condition applied to \(Y\) and \(FY\), respectively. The single remaining condition is

\[
\gamma(X, (\nabla^F_Z F)(Y)) = \frac{i}{2}[d\psi(X, FY, Z) + d\psi(FX, Y, Z)],
\]

which is equivalent to (3.11).

Notice that (2.10) implies

\[
\gamma(X, (\nabla^F_Z F)(Y)) = (\nabla^F_Z \omega)(X, Y),
\]

where \(\omega(X, Y) = \gamma(X, FY)\). This yields one more expression of the integrability condition (3.8).

**Corollary 3.1.** A generalized almost para-Hermitian structure associated to a triple \((\gamma, \psi, F)\) where \(d\psi = 0\) is generalized para-Kähler iff \(\nabla^F F = 0\).

**Proof.** For \(d\psi = 0\), (3.11) becomes \(\nabla^F F = 0\). Furthermore, \(\nabla^F F = 0\) and (3.8) imply the involutivity of \(S_{\pm}\). □

**Remark 3.1.** In the general case, formula \(\nabla_X Y = \nabla^\gamma_X Y - \frac{i}{2}\gamma([X,Y]d\psi)\) defines a new metric connection and (3.11) is equivalent to \(\nabla F = 0\).

Another form of the integrability conditions is given by the following theorem.

**Theorem 3.2.** The generalized almost para-Hermitian structure associated to a triple \((\gamma, \psi, F)\) is integrable iff the eigenspaces \(S_{\pm}\) of \(F\) are involutive and \(\psi\) satisfies the condition

\[
d\omega(X, Y, Z) = \frac{i}{2}d\omega(FX, FY, FZ).
\]

**Proof.** If we evaluate the 1-form \(\flat_{\gamma} \omega\) on the left hand side of (3.9) and take into account (3.9), the involutivity condition of the subbundles \(S_{\pm}\) becomes

\[
(\nabla^\gamma_{FX} \omega)(Z, Y) - (\nabla^\gamma_{FY} \omega)(Z, X) = (\nabla^\gamma_X \omega)(FZ, X) - (\nabla^\gamma_Y \omega)(FZ, Y).
\]

(3.11)

Changing \(Z\) to \(FZ\) in (3.11) and using the formula

\[
d\omega(X, Y, Z) = \sum_{Cycl(X,Y,Z)} (\nabla^\gamma_X \omega)(Y, Z),
\]

we see that the involutivity of \(S_{\pm}\) is equivalent to

\[
d\omega(X, Y, Z) = (\nabla^F_Z \omega)(X, Y) - (\nabla^F_Y \omega)(Z, FY) + (\nabla^F_X \omega)(Z, FX).
\]

(3.12)

Notice that (2.10) and \(\nabla^\gamma(F^2) = 0\) imply

\[
(\nabla^F_Z \omega)(FX, Y) = \gamma(FX, (\nabla^F_Z F)Y) = -\gamma(X, F(\nabla^F_Z F)Y)
\]

\[
= \gamma(X, (\nabla^F_Z FY)) = (\nabla^F_Z \omega)(X, FY),
\]

(3.13)

(3.14)
whence, also, \((\nabla^2_\gamma \omega)(FX, FY) = (\nabla^2_\gamma \omega)(X, Y)\).

Furthermore, changing the arguments to \(FX, FY, Z\) in (3.13), the result becomes
\[
2(\nabla^2_\gamma \omega)(X, Y) = d\omega(X, Y, Z) + d\omega(FX, FY, Z),
\]
which, therefore, is another expression of the involutivity of \(S_\pm\). This condition has the following consequences. Take \(X, Y, Z\) either all in \(S_+\) or all in \(S_-\). Then (3.15) yields \(d\omega(X, Y, Z) = (\nabla^2_\gamma \omega)(X, Y)\) and, if we add the two cyclic permutation, the result is \(d\omega(X, Y, Z) = 0\). This allows us to check the following equality for general arguments \(X, Y, Z \in T^c M\):
\[
d\omega(FX, FY, Z) + d\omega(X, Y, Z) = -d\omega(FX, FY, FZ).
\]
(3.16)

(check for each possible combination of arguments that are eigenvectors of \(F\)).

Now, modulo (3.9) and (3.15), the integrability condition (3.8) becomes
\[
d\omega(X, Y, Z) + d\omega(FX, FY, Z) = i[d\psi(X, Y, Z) + d\psi(FX, FY, Z)].
\]
(3.17)

Furthermore, notice the following direct consequence of (3.8), (3.12):
\[
d\omega(X, Y, Z) = i[d\psi(FX, FY, Z) + d\psi(X, FY, Z) + d\psi(X, Y, FZ)].
\]
(3.18)

If (3.18) is inserted into (3.17) and \(Z\) is replaced by \(FZ\), we get the equality (3.10), which, therefore, is a necessary condition of integrability. We will show that condition (3.10) is also sufficient for integrability. Firstly, we notice that (3.10) implies \(d\psi(X, Y, Z) = 0\), \(\forall X, Y, Z \in S_\pm\) and, checking on arguments that are eigenvectors of \(F\), we get
\[
d\psi(FX, FY, Z) + d\psi(X, FY, Z) + d\psi(X, Y, FZ) = -d\psi(FX, FY, FZ).
\]
(3.19)

If we calculate the left hand side of (3.17) modulo (3.10) and take into account (3.19), the result is exactly the right hand side of (3.17) and we are done. □

**Corollary 3.2.** A generalized almost para-Hermitian structure associated to a triple \((\gamma, \psi, F)\) where \(d\psi = 0\) is generalized para-Kähler iff the eigenbundles \(S_\pm\) of \(F\) are involutive and the 2-form \(\omega\) is closed.

**Proof.** This is a straightforward consequence of Theorem 3.2. □

**Example 3.1.** If \((\gamma, F), (\gamma, J)\) are a classical para-Kähler and classical Kähler structure of a manifold \(M\), the corresponding generalized structures (2.2), (2.16) are integrable. Similarly, if the quadruple \((\gamma, \psi, J_\pm)\), with an anti-commuting pair \(J_\pm\) and a closed form \(\psi\), defines a generalized Kähler structure, the triple \((\gamma, \psi, F)\), where \(F = \alpha J_+ + i\beta J_-\) \((\alpha, \beta \in \mathbb{R}, \beta^2 - \alpha^2 = 1)\) (see Example 2.2) defines a generalized para-Kähler structure. Indeed, in the considered case, the generalized Kähler condition is \(\nabla^\gamma J_\pm = 0\) [5, 12], which implies \(\nabla^\gamma F = 0\). The previous conclusion also holds for the second case of Example 2.2 i.e., \(F = \alpha K + i\beta L\).
We shall give the following concrete example. Take $M = \mathbb{C}^2$ with the coordinates

\[ z^1 = x^1 + iy^1, \quad z^2 = x^2 + iy^2, \quad x^1, x^2, y^1, y^2 \in \mathbb{R}. \]

Put

\[ \gamma = i(dx^1 \otimes dz^2 - dz^2 \otimes dx^1) = 2(dx^1 \otimes dy^2 - dx^2 \otimes dy^1),\]

where $\otimes$ is the symmetrized tensor product, and consider the real $(1,1)$-tensor fields

\[ K \frac{\partial}{\partial z^a} = i \frac{\partial}{\partial x^a}, \quad L \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^a}, \quad L \frac{\partial}{\partial y^a} = - \frac{\partial}{\partial x^a}, \quad a = 1, 2.\]

It is easy to check that $K, L$ anti-commute, $(\gamma, K)$ is a Kähler structure and $(\gamma, L)$ is a para-Kähler structure. The Kähler form of $(\gamma, K)$ and the para-Kähler (fundamental) form of $(\gamma, K)$ are

\[ \omega_K = dx^1 \wedge dx^2 + dy^1 \wedge dy^2, \quad \omega_L = dx^2 \wedge dy^1 + dy^2 \wedge dx^1.\]

Hence, $(\gamma, \psi, F = \alpha K + i\beta L)$, with $d\psi = 0$ and real coefficients $\alpha, \beta$ such that $\alpha^2 + \beta^2 = 1$, defines a generalized para-Kähler structure. Furthermore, if we quotientize by a lattice, the structure descends to the 4-dimensional torus. On the other hand, we can obtain a generalized almost para-Hermitian structure on the Hopf manifold $(\mathbb{C}^2 \setminus \{0\})/\Delta_\lambda$, where $\Delta_\lambda$ is the group generated by the transformation $(z^1, z^2) \mapsto (\lambda z^1, \lambda z^2)$ with a real constant $\lambda \neq 0, 1$. Namely, consider the same $F$ as above and the metric $\gamma/(z^1 \overline{z}^1 + z^2 \overline{z}^2)$ on $(\mathbb{C}^2 \setminus \{0\})$ and go down to the quotient manifold. This structure is not integrable even though the $\pm 1$-eigenbundles of $F$ are involutive.

**Example 3.2.** The product $M \times M'$ of a classical para-Kähler manifold $M$ and a classical Kähler manifold $M'$ with the structure defined in Example 2.3, where we assume that $\psi, \psi'$ are closed, is a generalized para-Kähler manifold. Indeed, it satisfies the condition $\nabla^\gamma F = 0$.

More generally, assume that $p : M \to N$ is a locally trivial fiber bundle with fiber $S$ and structural group $G$. Assume that $N$ has a classical para-Kähler structure $(\gamma_N, F_N)$ and $S$ has a $G$-invariant Kähler structure $(\gamma_S, F_S)$. Then, $M$ has a generalized para-Kähler structure obtained by gluing-up the product structures of the domains $U_\alpha \times S$, where $\{U_\alpha\}$ is a covering of $N$ by local trivializing neighborhoods. The same holds if $N$ is Kähler and $S$ has a $G$-invariant para-Kähler structure.

A concrete example is that of the manifold $M = (E \times (\mathbb{C}^m \setminus \{0\}))/\mathbb{R}$, where $E = \{(x^k, y^k) / \sum_{k=1}^n x^k y^k = 1\} \subset \mathbb{R}^{2n}$ and the additive group $\mathbb{R}$ acts by

\[ (x^k, y^k, z^u) \mapsto (e^t x^k, e^{-t} y^k, e^{it} z^u), \quad t \in \mathbb{R}, \tag{3.20} \]

where $(z^u)$ are the natural coordinates of $\mathbb{C}^m$. Then, $(x^k, y^k, z^u) \mapsto (x^k, y^k)$ defines a submersion $p : M \to \mathbb{B}P^{n-1}$, where $\mathbb{B}P^{n-1}$ is the paracomplex projective model, a known example of a para-Kähler manifold [7, 15]. Hence, $M$ is a locally trivial bundle over $\mathbb{B}P^{n-1}$ with fiber $\mathbb{C}^m \setminus \{0\}$ and structural group $S^1$. Since the latter preserves the classical Kähler structure of $\mathbb{C}^m \setminus \{0\}$, $M$ has
a generalized para-Kähler structure. Formula (3.20) shows that the pair $(\gamma, F)$ of this structure is induced by the tensor fields

$$\gamma = \sum_{k=1}^{n} dx^k \otimes dy^k + \sum_{u=1}^{m} dz^u \otimes d\bar{z}^u,$$

$$dx^k \circ F = dx^k, \quad dy^k \circ F = -dy^k, \quad d\bar{z}^u \circ F = dz^u, \quad d\bar{z}^u \circ F = d\bar{z}^u$$

and any closed 2-form $\psi$ may be added.

Notice that $\dim M = 2n + 2m - 2$ and the differentiable structure of $M$ is given by the following, local, non-homogeneous coordinates. Cover $M$ by the domains $U(\ell)_{\ell=1}^{n} = pr_M \{(x^k, y^k, z^u) \in E \times (C^{2m} \setminus \{0\}) / x^k \neq 0\}$ (obviously, $x^k$ cannot all vanish at the same point) and define the coordinates on $U(\ell)$ by

$$(X^h_{(\ell)} = x^h/|x^h|, Y^h_{(\ell)} = y^h/|x^h|, Z^u_{(\ell)} = e^{-i\delta_{n}|x^h|}z^u) (h \neq k)$$

$(y^k$ is determined by the equation $\sum_{h=1}^{n} x^h y^h = 1$ where the term $x^h y^h$ becomes $\text{sgn}(x^h) y^h$). $M$ also has a second foliation, which is defined by the quotient of the $2n$-planes $\text{span} (\partial/\partial x^k)$ under the action of $\mathbb{R}$ and it is $\gamma$-orthogonal to the fibers. Thus, $M$ is a locally product manifold.

**Remark 3.2.** The previous example belongs to the class of split generalized para-Kähler manifolds. Indeed, in the representation $F = F_1 + iF_2$, $F_1 = 0$ on the tangent spaces of the fibers and $F_2 = 0$ on the orthogonal spaces, therefore, $F_1 \circ F_2 = F_2 \circ F_1 = 0$. It would be interesting to get more information about this class of manifolds. Here, we just notice the following simple fact. Recall the decomposition $TM = P \oplus Q$ (Section 2). Since $F_2^3 = F_1, F_2^3 = -F_2$, we get $P = P_+ \oplus P_-$, where the terms are the $\pm$-eigenbundles of $F_1$, and $Q^c = E \oplus \bar{E}$, where $E$ is the $E$-eigenbundle of $F_2$. In the generalized para-Kähler case, the evaluation of $N_F = 0$ on arguments in $P_\pm$ shows that the real subbundles $P_\pm$ are foliations. Similarly, the evaluation on $E$ shows that the complex bundles $E, \bar{E}$ are involutive, hence, if $Q$ is a foliation, the leaves are complex manifolds.

**Example 3.3.** The invariant structure of a homogeneous space $G/H$ defined in Example [2.3] is integrable iff, with the notation of the latter, for any $ad_h$-invariant vectors $X_1, X_2, X_3 \in \mathfrak{g}/h$, the following two relations hold:

$$[\tilde{F} X_1, \tilde{F} X_2]_\mathfrak{g} - \tilde{F}[\tilde{F} X_1, X_2]_\mathfrak{g} - \tilde{F} [X_1, \tilde{F} X_2]_\mathfrak{g} + [X_1, X_2]_\mathfrak{g} = 0,$$

$$\sum_{Cycl(1,2,3)} \psi((\hat{X}_1, [\hat{X}_2, \hat{X}_3]) = i \sum_{Cycl(1,2,3)} \tilde{\omega}(\hat{X}_1, [\hat{X}_2, \hat{X}_3]).$$

In these relations the bracket $[,]_\mathfrak{g}$ is naturally induced by the bracket of $\mathfrak{g}$ and, using the expression of the Nijenhuis tensor and of the exterior differential, the result follows from Theorem 3.2 since $[,]_\mathfrak{g}$ corresponds to the Lie bracket of $ad_h$-invariant vector fields on $M$.

**Example 3.4.** Let $p : TM \to M$ be a tangent bundle with local coordinates $(x^i, y^i)$ on the total space, where $(x^i)$ are coordinates on $M$ and $(y^i)$ are corresponding vector coordinates. We will denote the vertical and complete lifts of
tensors from $M$ to $TM$ by upper indices $v, c$ and recall the following definitions.

\[ f^p = p^sf, \quad f^c = W(f^\gamma) (W = [y^i(\partial/\partial x^i)]_\mathfrak{M} \in T(TM)/\mathfrak{M}), \]
\[ X^p f^p = 0, \quad X^c f^c = X^c f^p = (X f)^p, \quad X^c f^c = (X f)^c, \]
\[ \alpha^p(X^p) = 0, \quad \alpha^c(X^c) = \alpha^c(X^p) = (\alpha(X))^p, \quad \alpha^c(X^c) = (\alpha(X))^c, \]
\[ (P \otimes Q)^p = P^p \otimes Q^p, \quad (P \otimes Q)^c = P^c \otimes Q^p + P^p \otimes Q^c, \]

where $\mathfrak{M}$ is the vertical bundle, tangent to the fibers, $f \in C^\infty(M)$, $X \in \chi(M), \alpha \in \Omega^1(M)$ and $P, Q$ are arbitrary tensor fields on $M$. On the other hand, any subbundle $\mathcal{S} \subseteq TM$ produces a subbundle $\mathcal{S}^t \subseteq TTM$, which we call the tangent lift of $\mathcal{S}$ and has $\text{rank} \mathcal{S}^t = 2\text{rank} \mathcal{S}$. Namely, if we denote by underlining the sheaf of germs of sections of a bundle, then, $\mathcal{S}^t = \text{span}_{M \times \mathbb{R}} \{X^c, X^\gamma / X \in \mathcal{S}\}$.

Now, assume that $M$ has a generalized almost para-Hermitian structure defined by a triple $(\gamma, \psi, F)$ and consider the triple $(\gamma^c, \psi^c, F^c)$. Using (3.22) we get
\[ \gamma^c(X^\gamma, Y^\gamma) = 0, \quad \gamma^c(X^c, Y^\gamma) = (\gamma(X, Y))^p, \]
\[ \gamma^c(X^c, Y^c) = (\gamma(X, Y))^c, \]
\[ F^c X^\gamma = (FX)^c, \quad F^c X^c = (FX)^c, \]
\[ \omega^c(X^\gamma, Y^\gamma) = 0, \quad \omega^c(X^c, Y^\gamma) = (\omega(X, Y))^p, \]
\[ \omega^c(X^c, Y^c) = (\omega(X, Y))^c, \]

where $\omega$ is the 2-form associated to $(\gamma, F)$.

Using (3.23), it follows easily that $(F^c)^2 = Id$, $F^c$ is $\gamma^c$-skew-symmetric and the $\pm 1$-eigenbundles of $F^c$ are the tangent lift $S^t_\pm$ of the $\pm 1$-eigenbundles $S_\pm$ of $F$. Therefore, the triple $(\gamma^c, \psi^c, F^c)$ defines a generalized almost para-Hermitian structure on the tangent manifold $TM$. Moreover, since one has $[X^c, Y^\gamma] = [X, Y]^c, [X^c, Y^c] = [X, Y]^\gamma, [X^\gamma, Y^\gamma] = 0$, if the subbundles $S_\pm$ are involutive the subbundles $S^t_\pm$ are involutive too. Finally, (3.23) also implies that the 2-form $\omega$ of the lifted structure is just the complete lift $\omega^c$. Furthermore, for any form $\lambda \in \Omega^k(M)$, if we evaluate $\lambda^c$ on vertical and complete lifts, we get zero, except for
\[ \lambda^c(X^1, ..., X_{k-1}^c, X_k^\gamma) = (\lambda(X_1, ..., X_k))^\gamma, \quad \lambda^c(X^1, ..., X_{k-1}^c, X_k^c) = (\lambda(X_1, ..., X_k))^c \]
\[ d\lambda^c = (d\lambda)^c. \]

This allows us to check that, if integrability condition (3.10) holds for the structure of $M$, it also holds for the lifted structure of $TM$. Therefore, if $M$ is a para-Kähler manifold the tangent manifold $TM$ is also para-Kähler.

We end this section by a few remarks on manifolds $(M, F)$, where $F$ is a complex $(1, 1)$-tensor field with involutive eigenbundles $S_\pm$ and $F^2 = Id$ (including the generalized para-Kähler manifolds), in the footsteps of complex geometry.

The functions and forms below are complex valued even though this is not apparent in notation. The decomposition $T^cM = S_+ \oplus S_-$ yields a bi-grading
\( \Omega^* = \oplus_{p,q \geq 0} \Omega_{p,q} \), where the terms are the spaces of forms of type \((p, q)\), defined by the condition that they take the value 0 unless evaluated on \(p\) arguments in \(S_+\) and \(q\) arguments in \(S_-\). Furthermore, one has \(d = d_+ + d_-\), where \(d_+: \Omega_{p,q} \to \Omega_{p+1,q}\), \(d_-: \Omega_{p,q} \to \Omega_{p,q+1}\) and \(d_+^2 = 0, d_+d_- + d_-d_+ = 0\). The bi-grading defines a double complex structure on the complex-valued de Rham complex of \(M\) and spectral sequence theory may give information about de cohomology \(H^\bullet(M, \mathbb{C})\). In particular, we have corresponding \(d_\pm\)-cohomology spaces \(H^\bullet_+ (M, F)\), \(H^\bullet_- (M, F)\) defined by the complexes

\[
\begin{align*}
\Omega_{0,q}(M) &\xrightarrow{d_1} \Omega_{1,q}(M) \xrightarrow{d_2} \Omega_{2,q}(M) \xrightarrow{d_3} \ldots, \\
\Omega_{p,0}(M) &\xrightarrow{d_1} \Omega_{p,1}(M) \xrightarrow{d_2} \Omega_{p,2}(M) \xrightarrow{d_3} \ldots.
\end{align*}
\] (3.24)

For functions, we have \(\langle d_{\pm}f, X_{\pm}\rangle = X_{\pm}f, \langle d_{\pm}f, X_{\pm}\rangle = 0\) and we will denote \(C^\infty_{\pm}(M) = \{f \in C^\infty(M, \mathbb{C}) \mid d_{\pm}f = 0\}\) the algebras of \(F\)-positive and \(F\)-negative functions, respectively. Furthermore, we may define an \(F\)-positive (\(F\)-negative) vector bundle as a vector bundle \(V\) endowed with an atlas of local trivializations that has \(F\)-positive (\(F\)-negative) transition functions. Then, acting on components, the operators \(d_{\pm}\) extend to \(V\)-valued forms and we may define \(F\)-positive, respectively \(F\)-negative cross sections \(s \in IV\) by the condition \(d_{\pm}s = 0\). The bundles \(S_{\pm}\) may not be \(F\)-positive (\(F\)-negative). We can define \(F\)-positive (\(F\)-negative) vector fields \(X\) on \(M\) by the condition \([X^\pm, X] \in S_{\mp}\), \(\forall X^\pm \in S_{\mp}\). Then, the field \(fX\) is also \(F\)-positive (\(F\)-negative) iff \(f \in C^\infty_{\pm}(M)\). Therefore, if \(S_{\pm}\) has local bases of \(F\)-positive (\(F\)-negative) sections, it will be an \(F\)-positive (\(F\)-negative) bundle.

We shall indicate an interesting case where the complexes of sheaves associated to \(3.24\) are exact resolutions of the sheaves of germs of \(F\)-negative (\(F\)-positive) functions, i.e., where \(d_{\pm}\) satisfies a local Poincaré lemma. Recall that a complex subbundle \(S \subseteq T^cM\) of rank \(r\) is Nirenberg integrable if: (1) \(S\) is involutive, (2) \(k = \dim S \cap \bar{S} = \text{const.}\) and \(S + \bar{S}\) is involutive. Nirenberg integrability is characterized by the local existence of real, differentiable functions \(y^a, t^a, a = 1, \ldots, k, u = 1, \ldots, \dim M - 2r + k\) and complex, differentiable functions \(z^\alpha, \alpha = 1, \ldots, r - k\) such that \((y^a, t^a, z^\alpha, \bar{z}^\alpha)\) are functionally independent and \(S\) has the local equations \(dt^u = 0, dz^\alpha = 0\) \([10]\).

**Proposition 3.1.** (Poincaré lemma) Assume that the complex tensor field \(F\) \((F^2 = 1d)\) has the following properties: (i) the two eigenbundles \(S_{\pm}\) are Nirenberg integrable, (ii) \(S_- \cap \bar{S}_+ = 0\). If \(\lambda_{p,q}\), where \(q \geq 1\) satisfies the condition \(d_-\lambda = 0\), then, there exists a local form \(\mu_{p,q-1}\) such that \(\lambda = d_-\mu\). A similar result holds if we change the role of the indices \(+, -\).

**Proof.** As recalled above, we have the local functions \(y^a_{\pm}, t^a_{\pm}, z^\alpha_{\pm}\) such that \(\text{ann} S_{\pm}\) is generated by \((dt^u_{\pm}, dz^\alpha_{\pm})\) (the signs distinguish between the two bundles and must be used concordantly). We have \(r = \text{rank} S_{\pm} = n\) and, since \(S_- \cap \bar{S}_+ = 0\), the 1-forms \((dt^u_- - dz^\alpha_-, dt^u_+, dz^\alpha_+)\) are a local basis of \(T^cM\), where \((dt^u_-, dz^\alpha_-)\) is a basis of the \((1, 0)\)-forms and \((dt^u_+, dz^\alpha_+)\) is a basis of the
(0, 1)-forms. We will denote the elements of these bases by $e^i, \kappa^j$, respectively, therefore, $de^i = 0, d\kappa^j = 0$. Accordingly, we have the local expression

$$\lambda = \frac{1}{pq!} \lambda_{i_1, i_2, \ldots, i_q} e^{i_1} \wedge \ldots \wedge e^{i_p} \wedge \kappa^{j_1} \wedge \ldots \wedge \kappa^{j_q}. \quad (3.25)$$

We will continue by using the same induction as in the case of foliations \cite{11}, induction on the index $h$ defined as the largest among the indices of $e^i, \ldots, e^n$ that is actually contained in (3.25). We shall refer to the induction start $h = 0$ at the end and, first, we show that, if the result holds for largest indices $l < h$, it also holds for $h$. If $h$ is the indicated largest index of $\lambda$, we have $\lambda = e^h \wedge \lambda'_{p-1,q} + \lambda''_{p,q}$, where the largest index of $\lambda', \lambda''$ is $< h$. Since $de^h = 0$, $d_\lambda = 0$ is equivalent to $d_-\lambda' = 0, d_+\lambda'' = 0$, which is assumed to imply $d_-\lambda' = d_-\mu', d_+\lambda'' = d_-\mu''$, whence $\lambda = -d_-(e^h \wedge \mu' + \mu'')$.

For the case $h = 0$, $\lambda$ is of type $(0, q)$ and we shall need hypothesis (ii). The latter implies that there is no relation between the functions $z^-_{q-1}, z^+_{q-1}$. In this situation, if $d_-\lambda_{0,q} = 0$, the consequence $\lambda_{0,q} = d_-\mu_{0,q-1}$ follows from the usual Poincaré lemma in the space of the coordinates $(t^a_+, z^a_+)$, with $(t^a_+, z^a_+)$ seen as parameters. Note that, if $F = F_1 + iF_2$, then (ii) holds if either $F_1$ is non degenerate or $F_2$ does not have eigenvalue $i$. The result for $d_+$ will be obtained similarly.

The type decomposition of $\xi \in T^* M$ is $\xi = \xi_+ + \xi_-$, where $\xi_+ = (1/2)(\xi \circ (Id + F)), \xi_- = (1/2)(\xi \circ (Id - F))$ and the symbols of $d_\pm$ are

$$\sigma(d_\pm)(\xi) = \xi_+ \wedge \lambda, \quad \sigma(d_-)(\xi) = \xi_+ \wedge \lambda, \quad \xi \in T^* M, \forall \lambda \in \Omega^*(M).$$

For $\xi_\pm \neq 0$, we have $\sigma(d_\pm)(\xi) = 0$ iff $\exists \lambda' \in \Omega^*(M)$ such that $\lambda = \xi_\pm \wedge \lambda'$. Therefore, the first, respectively the second, complex (3.24) is elliptic iff for any real covector $\xi$, $\xi_+ = 0$ implies $\xi = 0$, respectively, $\xi_- = 0$ implies $\xi = 0$. We can write this condition as $(\text{ann } S_\pm) \cap (T^* M) = 0$, where the terms of the intersection are real subspaces of the real dual of the 4n-dimensional space $T^* M = TM \oplus (iTM)$ defined by the real structure of the complexification (iTM is also seen as a real vector space). In terms of the annihilated spaces, this condition becomes $S_\pm + iTM = T^* M$, which, because the terms have real dimension 2n, is the same as $S_\pm \cap (iTM) = 0$. In a different way, for a real $\xi$, $\xi_\pm = (1/2)[\xi \circ F_1 + i\xi \circ F_2]$ and the result is zero iff $\xi \circ F_1 = 0, \xi \circ F_2 = 0$. These conditions imply $\xi = 0$ for $\xi \in (\ker F_2^* \cap (\text{ann } (Id \pm F_1)^*))$. In particular, if $F_2$ is an isomorphism both complexes are elliptic.

If there exists a real tangent vector $Z \neq 0$ such that $iZ \in S_\pm$, then $iz = -iz \in S_\pm$ and $iz \in S_\pm \cap S_\pm$. Therefore, if we ask $S_\pm \cap S_\pm = 0$, we get an elliptic complex. $S_\pm \cap S_\pm = 0$ means that $S_\pm$ is the i-eigenbundle of some almost complex structure $\Phi (\Phi^2 = -Id)$ on $M$, i.e.,

$$F(X - i\Phi X) = \pm(X - i\Phi X), \quad \Phi(X \pm FX) = i(X \pm FX).$$

With $F = F_1 + iF_2$, these conditions reduce to

$$F_2 = (F_1 \mp Id) \circ \Phi \Leftrightarrow F_1 + F_2\Phi = \pm Id,$n

$$F_2 = \mp\Phi \circ (Id \pm F_1) \Leftrightarrow F_1 = \Phi \circ F_2 \mp Id, \quad (3.26)$$

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which implies $F_1 \circ \Phi \circ \Phi \circ F_1 = 0$. Using that, (3.26) implies
\[
0 = F_1 \circ \Phi \circ \Phi \circ F_1 = \pm F_1^2 \circ (Id + \Phi),
\]
whence, composing by $(Id - \Phi)$, we get $F_1^2 = 0$. Furthermore, since $F_1^2 - F_2^2 = Id$, $F_2^2 = -Id$. Finally, if $\gamma$ is a compatible metric, i.e., $\gamma(F_1 X, Y) + \gamma(X, F_1 Y) = 0$, then, $F_2$ is $\gamma$-skew-symmetric iff $\Phi$ is $\gamma$-skew-symmetric (with $F_2^2 = -Id$, (3.26) implies $\Phi = -F_1 F_2 \pm F_2$, which has to be used to go from $F_2$ to $\Phi$).

For the structure $(\gamma, F)$ and if the complexes (3.24) are elliptic, one may continue to a corresponding Hodge theory.

4 Submanifolds and reduction

Let $(M, F, \mathcal{J})$ be a generalized almost para-Hermitian manifold with the corresponding triple $(\gamma, \psi, F)$ and let $\iota : N \subseteq M$ be a submanifold of $M$. We will say that $N$ is a regular invariant submanifold if $\iota^* \gamma$ is non degenerate and $T^c N$ is invariant by $F$. Then, $N$ has a naturally induced generalized almost para-Hermitian structure $(F_N, \mathcal{J}_N)$ defined by the triple $(\gamma_N = \iota^* \gamma, \psi_N = \iota^* \psi, F_N = F|_{T^c N})$. All the objects related to the induced structure will be denoted by the index $N$.

We recall the general pullback operation (e.g., [4])
\[
\bar{f}^*U = \{(X, f^* \eta) / (f, X, \eta) \in U \} \subseteq V \oplus V^* , \tag{4.1}
\]
where $f : V \to W$ is a linear mapping of vector spaces, $f^*$ is the transposed mapping and $U$ is a subspace of $W \oplus W^*$.

For any submanifold, taking $f_* : T_x N \subseteq T_x M$ ($x \in N$), formula (2.8) gives
\[
H_N = \{(X, b_{x, \psi + \iota, \gamma} X) / X \in TN \} = \{(X, \iota^*(b_{\psi + \gamma} X)) / X \in TN \} = \mathcal{T}^* H.
\]
In the same way, since we know that $H_{\pm} = \tau(S_{\pm})$, we get $H_{N \pm} = \mathcal{T}^* H_{\pm}$, which leads to the equalities $F_{N \pm} = \mathcal{T}^* F_{\pm}$, $J_N = \mathcal{T}^* J$. Hence, $\mathcal{F}$ induces a generalized almost para-complex structure and $\mathcal{J}$ induces a generalized almost complex structure of $N$ in the sense of [3, 12].

**Proposition 4.1.** $\iota : N \subseteq M$ is a regular invariant submanifold iff
\[
T^c N = \mathcal{T}^* H_+ + \mathcal{T}^* H_- + \mathcal{T}^* H_+ + \mathcal{T}^* H_-, \tag{4.2}
\]
defines a generalized almost para-Hermitian structure on $N$. In this case, the structure defined by (4.2) is the induced structure $(F_N, \mathcal{J}_N)$ defined above.

**Proof.** Since, if the induced structure exists, $H_{N \pm} = \mathcal{T}^* H_{\pm}$, (4.2) is the decomposition (2.17) defined by the induced structure. For the converse, let us denote by a tilde the elements of the generalized almost para-Hermitian structure defined by decomposition (4.2). Definition (4.1) of the pullback operation gives
\[
\mathcal{H} = \mathcal{T}^* H_+ + \mathcal{T}^* H_- = \tau(TN^c \cap S_+) \oplus \tau(TN^c \cap S_-),
\]
where $\tau$ is defined on $N$ by $\tau(X) = (X, \tilde{\gamma}_X \tilde{\psi} + \iota_X \gamma)$. This equality and the definition of $\tilde{\gamma}$ yield $\psi = \iota^* \psi, \tilde{\gamma} = \iota^* \gamma$. In particular, $\iota^* \gamma$ is non degenerate because $\tilde{\gamma}$ is non degenerate.

Furthermore, we have

$$F_{\pm} = \tilde{\tau}^* H_{\pm} \oplus \tilde{\tau}^* H_{\pm} = \tau(T^c N \cap S_{\pm}) \oplus \tau(T^c N \cap \bar{S}_{\pm}).$$

Then, since $\dim F_{\pm} = \dim N$, we must have $T^c N = (T^c N \cap S_{\pm}) \oplus (T^c N \cap \bar{S}_{\pm})$, whence $T^c N$ is invariant by $F$ and $\bar{F} = F|_{T^c N}$.

**Corollary 4.1.** If $M, F, J$ is a generalized para-Kähler manifold and $N$ is a regular invariant submanifold, $N$ with the induced structure is again a generalized para-Kähler manifold.

**Proof.** The integrability of the structure of $M$ is equivalent with the fact that $H_{\pm}$ are closed under the Courant bracket. The corresponding bundles of the induced structure are $\tilde{\tau}^* H_{\pm}$ and a result of [6] tells that the pullback bundles are also closed under the Courant bracket. \[\square\]

In order to give another result, we recall a way to see the pullback $\tilde{\tau}^*$ used in [2]. Consider the bundle $B_N = T N \oplus T^* M|_N \subseteq T M$ and the projection

$$s : B_N \to B_N/(\text{ann} T N) \approx T N.$$  

Then, for any subbundle $U \subseteq T^c M$, we get $\tilde{\tau}^* U = s(U \cap B^c)$ at each point of $N$ (the field of subspaces $\tilde{\tau}^* U$ may not be a smooth subbundle). Now, as in the generalized almost Hermitian case [2] we prove

**Proposition 4.2.** The submanifold $N$ is a regular invariant submanifold iff, $\forall x \in N$, the structures $\mathcal{F}_x, J_x$ induce a generalized paracomplex, respectively complex, structure on $T_x N$ and $B = B \cap (F B \cap (J B) \cap (H B) + \text{ann} T N)$.

**Proof.** For any submanifold $N$ of $M$, let us denote $C = B \cap (F B \cap (J B) \cap (H B)$, equivalently, $\mathcal{X} \in T M$ belongs to $C$ iff $\mathcal{X}, \mathcal{F} \mathcal{X}, J \mathcal{X}, H \mathcal{X} \in B$. Since $H_{\pm} = F_{\pm} \cap H, \mathcal{X} \in T^c M$ has the projections

$$pr_{H_{\pm}} \mathcal{X} = \frac{1}{4}(Id \pm F)(Id - iH) \mathcal{X}$$

and we see that $\mathcal{X} \in C$ (\( \mathcal{X} \) is real) iff $pr_{H_{\pm}} \mathcal{X} \in B^c$. This result implies the equality

$$C^c = (B^c \cap H^c) \oplus (B^c \cap H^c) \oplus (B^c \cap H^c) \oplus (B^c \cap H^c),$$

hence,

$$s(C^c) = \tilde{\tau}^* H^c \oplus \tilde{\tau}^* H^c \oplus \tilde{\tau}^* H^c \oplus \tilde{\tau}^* H^c. \quad (4.3)$$

If $N$ is a regular invariant submanifold, Proposition [4.1] tells us that the right hand side of (4.3) is $T^c N$ and $B = C + \text{ann} T N$. On the other hand, we already know that a regular invariant submanifold has structures induced by $\mathcal{F}, \mathcal{J}$.  

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Conversely, if $B = C + \text{ann} TN$, $s(C^c) = T^c N$ and (4.3) is a decomposition of the form (4.2). Moreover, if structures induced by $F, J$ exist, $\hat{\xi}^* H_\pm \oplus \hat{\xi}^* H_\mp, \hat{\xi}^* H_\pm \oplus \hat{\xi}^* H_-$ are the maximal isotropic subbundles that define the induced structures. Accordingly, (4.3) defines an induced, generalized almost para-Hermitian structure on $N$.

**Corollary 4.2.** Any submanifold $\iota : N \subseteq (M, F, J)$ such that $FB = B$ is a regular invariant submanifold.

**Proof.** A submanifold with the indicated property will be called strongly $F$-invariant and we will see that $F$ induces a generalized almost para-complex structure on $N$. Indeed, the necessary and sufficient condition for that are (12)

$$TN \cap \sharp \phi (\text{ann} TN) = 0, \ P(TN) \subseteq TN + \text{im} \sharp \phi,$$

where $P, \phi$ are entries of the matrix (2.6) of $F$. Since strong invariance is equivalent to $P(TN) \subseteq TN, \text{im} \sharp \phi \subseteq TN$ the second condition (4.4) obviously holds. On the other hand, we have

$$<\nu, \sharp \phi \lambda > = -<\lambda, \sharp \phi \nu > = 0, \forall \nu \in \text{ann} TN, \lambda \in T^* M|_N,$$

whence $\sharp \phi (\text{ann} TN) = 0$ and the first condition (4.4) holds too.

Furthermore, we prove that every submanifold satisfies the property $B = B \cap (HB) + \text{ann} TN$. Indeed, by looking at the projections $pr_H X = (1/2)(X - iHX), pr_H X = (1/2)(X + iHX), X \in B \cap (HB)$, we see that

$$s(B \cap (HB))^c = s(H \cap B^c) \oplus s(H \cap B^c) = \check{T}^* H \oplus \check{T}^* H = T^c N = s(B^c).$$

Then, from $FB = B$, we deduce

$$B = B \cap (HB) + \text{ann} TN = B \cap (J(FB)) + \text{ann} TN$$

$$= B \cap (JB) + \text{ann} TN,$$

which characterizes the fact that the tangent spaces $T_x N, x \in N$ have a generalized almost complex structure induced by $J$ (this was proven in [12]). On the other hand, we also have $B \cap (FB) \cap (JB) \cap (HB) = B \cap (JB)$. Together with (4.5), this shows that the hypotheses of Proposition 4.2 hold.

The second subject of this section is reduction. Let $(M, F, J)$ be a generalized almost para-Hermitian manifold with the corresponding triple $(\gamma, \psi, F)$. Assume that a Lie group $G$ acts on $M$ such that the structure is preserved. The last assertion has the following meaning: if we see $g \in G$ as a diffeomorphism of $M$ and define its action on $TM$ by

$$g \star (X, \alpha) = (g_* X, g^{-1}_* \alpha),$$

then, $F$ and $J$ commute with $g \star$. This is easily seen to be equivalent with the fact that $g \star$ preserves the subbundles $H_\pm$, which, further, is equivalent with
the preservation of the triple \((\gamma, \psi, F)\) (preservation of \(F\) means commutation with the differentials \(g_a\)).

Reduction is a process originating in symplectic geometry and, in the present case, we shall define it with reference to the complex 2-form \(\omega(X, Y) = \gamma(X, F(Y))\) as in the similar, para-Hermitian case \([15]\). A submanifold \(\iota : N \subset M\) that is invariant by a subgroup \(G' \subset G\) is said to be a reducing submanifold if (i) the action of \(G'\) on \(N\) is free and proper, (ii) \(\forall x \in N\), the orbit \(G(x)\) cleanly intersects \(N\) and \(T^cG(x) = (T^cN)^{\perp}\), (iii) the submanifold \(G(x) \cap N\) is equal to the orbit \(G'(x)\).

**Proposition 4.3.** Let \(N\) be a reducing submanifold of the generalized almost para-Hermitian manifold \((M, \gamma, \psi, F)\). Assume that the metric \(\gamma\) is non degenerate on \(N\) and on the orbits \(G'(x)\) and that \(\text{ann} \psi_x \subset T_xG'(x)\) \((x \in N)\). Then, the quotient manifold \(Q = N/G'\) has a generalized almost para-Hermitian structure defined by the structure of \(M\).

**Proof.** Since the intersection \(G(x) \cap N\) is clean, if we denote \(K = \text{ann}(\iota^*\omega_x)\), we get

\[
T^c(G'(x)) = T^cG(x) \cap T^cN = (T^cN)^{\perp} \cap T^cN = K \quad (\forall x \in N) \quad (4.6)
\]

and \(\text{rank}(\iota^*\omega) = \text{const.}\). We will denote by \(K\) the manifold given by the sum of the orbits of \(G'\). Condition (i) ensures the existence of the quotient manifold \(Q\) and of the natural submersion \(p : N \to Q\) that sends \(x\) to the orbit \(G'(x)\). By \([14.10]\), we have \(i(X)(\iota^*\omega_x) = 0, \forall X \in T^c(G'(x))\). Furthermore, any local vector field \(X \in TK\) is a finite linear combination of infinitesimal transformations given by elements \(\xi\) of the Lie algebra \(g'\) of \(G'\) and, since \(G'\) preserves \(\omega\) and \(N\), we must have \(L_X(\iota^*\omega) = 0\). The three facts: \(p\) submersion, \(i(X)(\iota^*\omega) = 0, L_X(\iota^*\omega) = 0\) ensure the existence of a non degenerate complex 2-form \(\omega'\) on \(Q\) such that \(\iota^*\omega' = \iota^*\omega\) and \(\omega'\) is called the reduction of \(\omega\).

Then, the hypotheses on \(\gamma\) imply the existence of the subbundle \(V = (TK)^{\perp}\), with the non degenerate metric \(\iota^*\gamma|_V\), which is invariant by the infinitesimal transformations of \(g'\). Accordingly, we must have \(\iota^*\gamma|_V = p^*\gamma'\) where \(\gamma'\) is a non degenerate metric of \(Q\). Furthermore, by (ii), for \(x \in N\), we have \(T^cN = (T^cG(x))^{\perp}\) and, since \(F\) is compatible with \(\omega\) and commutes with the infinitesimal transformation of \(G\), we see that \(F|_N\) preserves \(T^cN\). Moreover, along \(N\), \(F\) preserves the bundle \(TK\), therefore, it induces the endomorphism \(F' = -\partial_\gamma \circ \partial_\psi\) of \(V^c\). We may see \(F'\) as an endomorphism of \(T^cQ\) that satisfies \((2.10)\). Finally, the hypotheses on \(\psi\) ensure that \(i(X)^{\psi*}\psi = 0, L_X\iota^*\psi = 0, \forall X \in TN\) (same explanation as for \(\omega\)), whence, the existence of a 2-form \(\psi'\) of \(Q\) such that \(\iota^*\psi' = p^*\psi'\). The triple \((\gamma', \psi', F')\) defines the required structure of \(Q\), also called the reduction of the original structure of \(M\) via the reducing submanifold \(N\).

**Proposition 4.4.** If \(M\) is a generalized para-Kähler manifold and \(N\) is a reducing submanifold that satisfies the hypotheses of Proposition 4.3, the corresponding reduced manifold \(Q\) is also a generalized para-Kähler manifold.
Proof. By Theorem 3.2, the integrability conditions of the structure of $M$ are given by $\mathcal{N}_F = 0$ together with the equality (3.10). The Lie bracket on $Q$ is given by $[p_\ast X_1, p_\ast X_2] = p_\ast [X_1, X_2]$, where $X_1, X_2$ are vector field on $N$ that admit a projection on $Q$. Thus, for the same $X_1, X_2$, the definition of the reduction $F'$ gives

$$N_{F'}(p_\ast X_1, p_\ast X_2) = p_\ast N_F(X_1, X_2) = 0.$$ 

Similarly, equality (3.10) on $Q$ may be identified with its pullback by $p^\ast$ to $N$, hence it is implied by (3.10) on $M$. 

Proposition 4.5. Under the hypotheses of Proposition 4.3 the subbundles $H'_\pm$ of the reduced manifold $Q$ are given by $H'_\pm = \overline{\theta}_\ast H_\pm$, where $H_\pm$ define the generalized almost para-Hermitian structure of $M$.

Proof. In the proposition, $\overline{\theta}_\ast$ denotes the push-forward operation defined by (e.g., [4])

$$\overline{\theta}_\ast U = \{ (p_\ast X, \lambda) \in T^c Q / (X, p^\ast \lambda) \in U \} \quad (U \subseteq T^c N).$$

Since $F(T^c N) \subseteq T^c N$ (see the proof of Proposition 4.3), the eigenbundles of $F|_{T^c N}$ are $S_\pm \cap T^c N$, where $S_\pm$ are the eigenbundles of $F$. Consequently, the eigenbundles of $F'$ on $Q$ are $p_\ast (S_\pm \cap T^c N)$. From $H_\pm = \tau S_\pm$ and the definition (4.7), we get

$$\overline{\theta}_\ast H_\pm = \{ (Y, \bar{\eta} \ast \psi + i\bar{\gamma}, \gamma) / Y \in S_\pm \cap T^c N \}.$$

Then, (4.7) and the definition of the mapping $\tau$ on $Q$ lead to the required result. 

In particular, we can define a version of the Marsden-Weinstein reduction. The real vector field $X$ of $M$ will be called a Hamiltonian vector field if $L_X \omega = 0$ and there exists a complex valued function $f$ such that $df = i(X)\omega$. If $d\omega = 0$, the first condition follows from the second. Otherwise, it may be very restrictive ([4].) The structure-preserving action of $G$ on $(M, \mathcal{F}, \mathcal{J})$ is a Hamiltonian action if the infinitesimal transformations $\xi_M$ are Hamiltonian for all $\xi \in \mathfrak{g}$. Furthermore, an equivariant momentum map is an equivariant mapping $\mu : M \rightarrow \mathfrak{g}^c$ (i.e., $\forall g \in G$, $\mu(g(x)) = (\text{coad } g)(\mu(x))$) such that, $\forall \xi \in \mathfrak{g}$,

$$L_{\xi_M} \omega = 0, \quad i(\xi_M) \omega = d\mu \xi, \quad \mu \xi(x) = \langle \mu(x), \xi \rangle, \quad x \in M.$$ 

If $\mu$ is an equivariant momentum map, a level set $N = \mu^{-1}(\theta)$, where $\theta \in \mathfrak{g}^c$ is a non-critical value of $\mu$ may be a reducing submanifold of $M$. Indeed, take the subgroup $G' = G_\theta$, where $G_\theta$ is the isotropy subgroup of $\theta$ for the coadjoint action of $G$. By the equivariance of $\mu$, $N$ is $G'$-invariant and, for all $x \in N$, the orbits satisfy condition (iii) of a reducing submanifold, $G_\theta(x) = G(x) \cap N$. Furthermore, at $x \in N$, take the tangent vector $\xi_M(x)$ of $G(x)$ ($\xi \in \mathfrak{g}$) and the complex tangent vector $X$ of $N$. Then,

$$\omega(\xi_M(x), X) = \langle i(\xi_M(x)) \omega, X \rangle = \langle d_\xi \mu \xi, X \rangle = \langle d_\xi \mu(X), \xi \rangle = 0,$$
whence we see that $X \perp_\omega T_x G(x)$ is equivalent to $d_x \mu(X) = 0$. Accordingly, $T^c G(x) = (T^c N)^\perp_\omega$ and condition (ii) of a reducing submanifold holds. If we ask the action of $G_\theta$ on $N$ to be proper and free (condition (i)), then, $N$ is a reducing submanifold. Furthermore, if the other hypotheses of Proposition 4.3 are added, we get a reduction that may be seen as a Marsden-Weinstein reduction.

**Example 4.1.** Let us come back to Example 3.4 and assume that the connected Lie group $G$ acts on $M$ and preserves $(\gamma, \psi, F)$. The differential of this action defines an action of $G$ on the manifold $TM$. Since for any tensor field $P$ the Lie derivative has the property $L_X^c P^c = (L_X^c P)^c$, we have $L_{\xi_M}^c \gamma^c = 0, L_{\xi_M}^c \psi = 0, L_{\xi_M}^c F = 0$ and the action on $TM$ preserves the complete lift of the structure. We define a momentum map $\mu : TM \to g^\ast$ in the following way

$$<\mu(x, y), \xi> = \mu^c(x, y) = \omega^c(\xi_M^c, E)((x, y) \in TM, \xi \in g^c),$$

(4.8)

where $E = y^i(\partial/\partial y^i)$ is the so-called Euler vector field. Putting $\alpha = i(\xi_M)\omega = \alpha_i dx^i$ and taking into account the equality $i(\xi_M^c)\omega^c = (i(\xi_M)\omega)^c$ (which may be checked on vertical and complete lifts), we get

$$\mu^c = \alpha^c(E) = y^k \alpha_k, \quad d\mu^c = y^k d\alpha_k + \alpha_k dy^k = \alpha^c = i(\xi_M)\omega^c$$

as required for a momentum map. The coad $g$-equivariance of $\mu$ is a straightforward consequence of the fact that the action of $G$ on $TM$ preserves $\omega^c$ and, also, preserves $E$ (for any vector field $X^c$ one has $[X^c, E] = 0$, hence $L_{\xi^c} E = 0, \forall \xi \in g$).

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