GEOMETRY AND PHYSICS: AN OVERVIEW

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Abstract. We present some episodes from the history of interactions between geometry and physics over the past century.

I was asked by the conference organizers to survey the modern interactions between geometry, topology, and physics in a one-hour lecture. Rather than attempting to be comprehensive, I have chosen to provide some vignettes drawn from recent history which emphasize the impact each field has had upon the other. As part of the story, I will trace a particular circle of ideas from physics to math, back to physics, and on to math once again, and show the significant impacts at each stage along the way. My general theme is gauge theory, geometry, and topology.

1. Dirac quantization

The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities.

Paul Dirac (1931)

Paul Dirac laid out the manifesto quoted above for the proper interaction between mathematics and physics in the course of a beautiful 1931 paper [Dir31] devoted to the theoretical investigation of magnetic monopoles.

It had long been observed that Maxwell’s equations for electromagnetism can be made symmetric between electricity and magnetism by introducing particles carrying a net magnetic charge: the so-called magnetic monopoles. The quantum version of Maxwell’s theory, however, relies on the existence of an electromagnetic potential and prior to Dirac’s work it was believed that monopoles would prevent such a potential from being defined. Dirac showed that by using a careful interpretation of the role of the potential, quantum physics could be formulated using it (in spite of the definitional problem). Moreover, there was a surprise: the existence

2010 Mathematics Subject Classification. Primary 57R22, 81T13, 14D21, 81T50, 57R57, 81T45.

I am grateful to Arun Debray and Sean Pohorence for sharing the notes they took during the lecture, and to Andy Neitzke for a very useful remark. This research was partially supported by NSF grant PHY-1620842.

1I hasten to point out that magnetic monopoles have not (yet) been observed in nature.
of a magnetic monopole forces the “charge quantization” of electrically charged particles such as the electron.

There was some interesting topology in Dirac’s original argument, a variant of which I will now review. Consider the wave function of an electron in the presence of a magnetic monopole of magnetic charge $g$, located at the origin in $\mathbb{R}^3$, and consider a vector potential $A$ for the magnetic field. One might suppose that the vector potential could be defined throughout $\mathbb{R}^3 - \{\vec{0}\}$, but that is not possible for the following reason. If we consider a sphere of radius $r$ and a circle $\gamma_\theta$ at fixed spherical polar coordinate angle $\theta$ which bounds a spherical cap $\Sigma_\theta$ containing the north pole, then the line integral

\[
\int_{\gamma_\theta} A \, d\phi = \int_{\Sigma_\theta} \nabla \times A \, d\sigma = \int_{\Sigma_\theta} B \, d\sigma
\]

calculates the magnetic flux through $\Sigma_\theta$ (by Stokes’ theorem). That flux, which can be calculated to be $2\pi g (1 - \cos \theta)$ increases continuously as $\theta$ ranges from 0 to $\pi$. But at $\theta = \pi$ the circle has shrunk to zero size so that the integral must vanish, which is a contradiction!

Dirac’s interpretation was that the vector potential must become singular along some semi-infinite string anchored at the monopole. If we direct such a string from the origin through the south pole of the sphere, the contradiction is removed. The integral around a zero-size circle at the south pole cannot be computed directly due to the singularity in the vector potential, but takes the value $4\pi g$ thanks to the integral over the sphere.

The magnetic field causes a change in the phase of the wave function of an electric particle around a circle. For an electric particle of charge $e$, if the change in phase of the wave function around a circle is zero at the north pole, and it must take the value

\[
e \frac{e}{\hbar c} \int_{S^2} B \, d\sigma = \frac{4\pi g e}{\hbar c}
\]
at the south pole. In a quantum theory, a change of phase in the wave function is physically indetectable if it is a multiple of $2\pi$. The conclusion is that in order for the quantum theory to be well-defined, we must have

\[
\frac{4\pi g e}{\hbar c} = 2\pi N
\]

for some integer $N$. In other words, if there is a single monopole of magnetic charge $g$, then all electric charges must be integer multiples of $\hbar c/2e_0$. This is Dirac’s famous quantization condition.

Conversely, since we know experimentally the smallest possible electric charge $e_0$, all magnetic charges must be integer multiples of $\hbar c/2e_0$.

In modern topological terms, the vector potential $A$ should only be locally defined, with a change between northern hemisphere and southern hemisphere specified by a change of potential in a neighborhood of the equator, which amounts to a change by a variable element of $U(1)$ (known as an abelian gauge transformation). The vector potential can be regarded as a globally defined object if it is treated as a section of a bundle over $\mathbb{R}^3 - \{\vec{0}\}$ (or over $S^2$) whose transition functions are given by that variable element.

\[2\]This mathematical argument first appeared in the physics literature. [WY75].
The abelian gauge transformation is specified by a map $S^1 \to U(1)$. Since $\pi_1(U(1)) \cong \mathbb{Z}$, the topological type of the bundle is determined by an integer (the first Chern class of the bundle). Physically, this integer is identified with the multiple of $\hbar c/2e_0$ which gives the magnetic charge $g$ of the monopole in question.

Dirac did not formulate his result in terms of a bundle, but his treatment is surprisingly modern, given the relative novelty of topological concepts at that time. This result turns out to be only the first step in an important series of interactions between topology, geometry, and physics.

2. Missed opportunities

As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so fruitful in past centuries, has recently ended in divorce.

Freeman Dyson (1972)

In his 1972 Gibbs lecture [Dys72], Freeman Dyson lamented the divide between mathematics and physics which existed at that time, drawing a number of examples from history. One of his principal examples centered around Feynman’s approach to the study of relativistic quantum field theory.

The years immediately following World War II saw rapid advances in the study of relativistic quantum field theory, led by Dyson, Feynman, Schwinger, and Tomonaga. Feynman’s approach was based on his “sum over histories” idea. In the Lagrangian formulation of classical physics, the evolution of a physical system from a starting time to an ending time is the one which minimizes the physical action of the system. Such a minimum can be determined using the techniques of the calculus of variations to find the path through the configuration space which extremizes the action. In Feynman’s “sum over histories” approach to quantum physics, all paths must be considered, and the probability that a particular path is followed is proportional to the exponential of the negative of the action. Thus, the probability will be highest along the classical path, but the quantum theory requires consideration of contributions from other paths.

To determine physically measurable quantities, it is necessary to integrate over the space of all paths, and this is where the mathematical trouble arises: it is unknown how to carry out such integrals. One of the difficulties is a choice of measure on the space of paths which determines how different paths are to be weighted.

Nevertheless, by considering theories which are perturbations of “free” theories, Feynman was able to give a prescription for an asymptotic series describing the (purported) answer to the path integral, as an infinite sum indexed by the famous Feynman diagrams. The mathematical consistency of the path integral values as determined by Feynman diagrams has never been established, but in the hands of skilled practitioners unique answers are produced. Remarkably, when this approach is used to study quantum electrodynamics, it is incredibly precise: for example, the best theoretical and experimental values of the anomalous magnetic dipole moment of the electron agree to at least eight significant figures (see [PS95, Chapter 6]).
As Dyson remarks, Feynman himself was not concerned with mathematical rigor. The fact that physicists were obtaining such spectacular agreement between theory and experiment likely contributed to a similar attitude among many physicists from that era: they did not need modern mathematics to accomplish their goals. And so Dirac’s manifesto languished.

3. **Yang–Mills theory and connections on fiber bundles**

... we are concerned with the necessary concepts to describe the physics of gauge theories. It is remarkable that these concepts have already been studied as mathematical constructs.

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Tai Tsun Wu and Chen Ning Yang (1975)

In 1954, during the era of minimal communication between mathematics and theoretical physics, C. N. Yang and R. L. Mills [YM54] introduced *gauge transformations* consisting of locally varying symmetries taking values in a compact Lie group $\mathbb{G}$, and studied physical theories which are invariant under such gauge transformations. These generalized the already-familiar *abelian gauge transformations* from electromagnetism – the same ones we encountered in Section 1 – for which $\mathbb{G} = U(1)$. These gauge theories (or “Yang–Mills theories”) eventually became the basis of the Standard Model of particle physics, the formulation of which was finalized in the mid 1970s using the group $\mathbb{G} = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$.

In the late 1960s and early 1970s, Yang got acquainted with James Simons, then the mathematics department chair at SUNY Stony Brook where Yang was a professor of physics. In the course of their conversations, Yang and Simons came to recognize that there were important similarities between formulas which were showing up in Yang’s work, and formulas which appeared in parts of mathematics which Simons was familiar with. Simons identified the relevant mathematics as the mathematical theory of *connections on fiber bundles*, and recommended that Yang consult Steenrod’s foundational book on the subject [Ste51] (which coincidentally was published just a few years prior to the work of Yang and Mills). Yang found the book difficult to read, but through further discussions with Simons and other mathematicians (including S.-S. Chern) he came to appreciate the power of the mathematical tools which fiber bundle theory offered. By 1975, Yang had co-authored a paper with T. T. Wu [WY75] (quoted at the head of this section) which applied those methods to problems in physics. Within their paper, Wu and Yang provided a dictionary between the parallel concepts in physics and mathematics, allowing the application of topological and geometric techniques to the study of Yang–Mills theory.

Simons communicated these newly uncovered connections with physics to Isadore Singer at MIT who in turn discussed them with Michael Atiyah of Cambridge University. It is likely that similar observations were made independently by others.

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3 To be precise, [YM54] treats the case $\mathbb{G} = SU(2)$ but the notion was soon generalized to an arbitrary compact group.

4 For a mathematical account of the standard model gauge group and the connection between its representations and the elementary particles, see [BH10].

5 Yang and Simons shared memories of this period in a joint interview in 2008 [Zim08].
A new chapter in the interaction between mathematics and physics was about to open.

4. UNREASONABLE EFFECTIVENESS

... mathematical concepts turn up in entirely unexpected connections. Moreover, they often permit an unexpectedly close and accurate description of the phenomena in these connections.

Eugene Wigner (1960)

A key example of Wigner’s “unreasonable effectiveness of mathematics” principle occurred in the mid 1970s, not long after Dyson’s Gibbs lecture. Thanks to the opening of communication between mathematicians such as Atiyah and Singer on the one hand and the gauge theory community in physics on the other hand, when Polyakov proposed in the importance of studying instantons in Yang–Mills theory, mathematicians were ready to assist in finding such instantons.

A Yang–Mills instanton is a solution on $\mathbb{R}^4$ to the Euclidean version of the Yang–Mills equations for a compact Lie group $G$, which are the variational equations for the norm-squared

$$\|F\|^2 = \int_{\mathbb{R}^4} \text{tr}(F \wedge \star F)$$

of the curvature $F$ of a connection $A$ on a principal $G$-bundle, where the Hodge star operator is used to define the norm. Such a solution is only interesting to physicists if it has a suitably controlled behavior far from the origin, and the initial assumption made by mathematicians in studying the problem is that the solution extends to $S^4$.

After some progress had been made on the problem in the physics community, it was given a purely mathematical formulation by Atiyah, Hitchin, and Singer who established the dimension of the space of solutions. Not long thereafter the problem was solved in general by Atiyah, Drinfeld, Hitchin and Manin via what came to be known as the “ADHM construction.” These papers used techniques – the Penrose twistor transform, and the algebraic geometry of vector bundles – which were then unknown to physicists.

Let me briefly explain the topological setting of the Yang–Mills instanton problem (already described in one of the earliest papers), which can be viewed in two ways. From the perspective of a solution on $\mathbb{R}^4$, all $G$-bundles are trivial but there is an asymptotic behavior of an instanton which is determined by the behavior of the connection on the $S^3$ at infinity. There must be a gauge transformation on $S^3$, i.e., a map $S^3 \to G$, which trivializes the connection there. Thus, the topology

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6One of the important techniques for studying quantum field theory considers these Euclidean theories as a kind of analytic continuation from ordinary time $t$ to “imaginary time” $it$, via a procedure known as Wick rotation. It is beyond the scope of this lecture to explain why this is relevant to physics.

7It was later shown by Uhlenbeck that this is automatically true if $\|F\|^2$ is finite.
of an instanton is measured by $\pi_3(G)$, which is isomorphic to $\mathbb{Z}$ for any compact semisimple Lie group. The resulting integer $k$ is called the \textit{instanton number}.

From the perspective of a solution on $S^4$, the map $S^3 \to G$ specifies the bundle by giving gluing data along the equator. The topological classification of principal $G$-bundles on $S^4$ is via $\pi_3(G)$ and determines the instanton number as before. To see how this is related to the curvature of a connection, it is convenient to remember that the Hodge star operator $\star$ on a Riemannian four-manifold squares to the identity on 2-forms. Thus, $F$ can be decomposed into its self-dual and anti-self-dual parts:

\begin{equation}
F = F_{\text{sd}} + F_{\text{asd}}
\end{equation}

where $\star F_{\text{sd}} = F_{\text{sd}}$ and $\star F_{\text{asd}} = -F_{\text{asd}}$. We then have

\begin{equation}
\begin{aligned}
\|F\|^2 &= \int_{S^4} \text{tr}(F \wedge \star F) = \|F_{\text{sd}}\|^2 + \|F_{\text{asd}}\|^2 \\
8\pi^2 k &= \int_{S^4} \text{tr}(F \wedge F) = \|F_{\text{sd}}\|^2 - \|F_{\text{asd}}\|^2,
\end{aligned}
\end{equation}

reflecting the topology of the solution. We also see from this that the minimal action solutions must be either self-dual or anti-self-dual depending on the sign of $k$.

I was fortunate enough to attend the \textit{Loeb lectures} delivered at Harvard University by Michael Atiyah in the spring of 1978 in which he explained the ADHM construction. The lectures were held in the physics department, and probably constituted the largest meeting that had been held up until that time at Harvard between mathematicians and physicists. My memory is that the audience was roughly half and half: there were large numbers of mathematicians as well as large numbers of physicists. To an algebraic geometry graduate student such as myself, it was an amazing experience to see my quite abstract corner of mathematics applied to the “real world” of theoretical physics. The lectures were frustrating in one sense: Atiyah made the pedagogical choice of treating algebraic geometry as a “black box” for the purpose of the lectures, so we didn’t get to hear about the details of the algebraic geometry! (I believe that some of the physicists were frustrated by this as well, since they missed the opportunity to learn about the algebraic geometry.) However, we did learn about the twistor transform which was new material to many of us.

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\textsuperscript{8}Changing the orientation of the four-manifold changes the sign of the instanton number, and there is some ambiguity in the literature about how this is defined. We have attempted to be self-consistent in this paper.

\textsuperscript{9}Atiyah subsequently published notes based on the Loeb lectures as well as similar lectures delivered at two other places [Ati79].

\textsuperscript{10}Harvard was and is strong in the more traditional areas of mathematical physics, but the ADHM construction was something new and different and attracted a big audience from outside that community.
5. Anomalies and Index Theory

From a more mathematical standpoint, the study of anomalies has elicited very interesting applications of index theory in quantum field theory. [...] A powerful form of the Atiyah–Singer index theorem (the index theorem for families of elliptic operators) has been used to provide a global understanding of the non-Abelian anomaly as well as the gravitational anomalies.

Luis Alvarez-Gaumé (1986)

Let me now turn to another chapter in the math–physics dialogue of the 1970s and 1980s: the calculation of anomalies in quantum field theories and quantum theories of gravity.

Classical physical theories are determined by the equations of motion of the theory, but quantum theories require an understanding of the physical “action” on a broad configuration space of possible physical fields. Moreover, Feynman’s formulation requires an understanding of a measure on the space of paths through that configuration space as well.

In the 1960s, while attempting to understand how symmetries of classical theories of particle physics act on the associated quantum theory, it was discovered that they might not: there could an “anomaly” in the quantum theory which prevented the action from being well-defined. This phenomenon was originally expressed in terms of Feynman diagrams, and came to be regarded as the statement that, although the symmetry group preserves the Lagrangian, it fails to preserve the measure on the space of paths.

This formulation is unsatisfying to mathematicians, who know that the measure hasn’t been properly defined in mathematics and so who rightfully wonder how an ill-defined thing can fail to be preserved by a group action?

There is another interpretation of the anomaly, however, in terms of the Dirac operators of the quantum theory. The equations of motion for the bosonic fields in a (Wick-rotated) physical theory involve the Laplacian ∆ : \mathcal{V} → \mathcal{V} acting on a space of functions or differential forms on spacetimes (sometimes bundle-valued). However, for fermionic fields in the theory, the equations of motion involve a Dirac operator \nabla : \mathcal{V} → \mathcal{W} which is a “square root” of the Laplacian and typically does not map the classical space of spinors to itself. Quillen [Qui85] introduced a “determinant line bundle” associated to the Dirac operator \nabla, and Bismut and Freed [BF86] equipped it with a connection. The theory is anomaly-free if the line bundle is trivial and the connection is flat.

The beauty of this approach is the connection to the Atiyah–Singer index theorem, as mentioned in the quote from [AG86] at the head of this section. The original Atiyah–Singer theorem can be used to interpret anomalies of abelian group actions, but the anomalies of non-abelian group actions (and gravitational anomalies) generally require the families index theorem. In all cases one is calculating topological obstructions to the triviality of the determinant bundle and the connection on it.
6. Donaldson invariants

The surprise produced by Donaldson’s result was accentuated by the fact that his methods were completely new and were borrowed from theoretical physics, in the form of the Yang-Mills equations.

Michael Atiyah (1986)

As discussed in Section 4, the idea of Yang–Mills instantons was able to move from physics to mathematics thanks to renewed communication between mathematicians and physicists beginning in the mid 1970s. The next step, however, was truly remarkable. In the early 1980s, Simon Donaldson studied the Yang–Mills instanton equations on arbitrary compact four-manifolds, and using them, was able to make very unexpected progress in the study of differentiable four-manifolds [Don83].

For a fixed principal $G$-bundle $P$ with instanton number $k$ over a four-manifold $X$, where $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$, Donaldson considered the set $\mathcal{A}$ of connections on $P$ (which is an affine space acted upon by the space $\Omega^1(X, \mathfrak{g})$ of $\mathfrak{g}$-valued 1-forms on $X$), modulo the automorphism group $\mathcal{G}$ of $P$. On the orbit space $\mathcal{A}/\mathcal{G}$ the self-dual connections can be identified as the kernel of the operator $d_A^\perp : \Omega^1(X, \mathfrak{g}) \to \Omega^2(X, \mathfrak{g})$ which is the composition of covariant differentiation with projection to the anti-self-dual part. In his early work, Donaldson studied the kernel of $d_A^\perp$ on $\mathcal{A}/\mathcal{G}$ in the case $G = SU(2)$ and $k = 1$, obtaining a moduli space $\mathcal{M}_1 \subset \mathcal{A}/\mathcal{G}$ of self-dual connections. He found that if the intersection form on $X$ is negative definite, then the moduli space $\mathcal{M}_1$ is a 5-manifold away from a finite collection of singularities corresponding to reducible connections (i.e., connections compatible with a decomposition of the associated vector bundle $E = P \otimes_{SU(2)} \mathbb{C}^2$ into a sum of two line bundles $L \oplus L^{-1}$). Analyzing these singularities carefully led to restrictions on the intersection form on second cohomology, and in particular showed that if the intersection form is positive definite, then it is diagonalizable. This had many remarkable consequences including the failure of the smooth version of the h-cobordism conjecture in dimension four as well as the existence of an “exotic” differentiable structure on (topological) $\mathbb{R}^4$. In his proofs, Donaldson relied on earlier work of Taubes [Tan82] and Uhlenbeck [Uhl82a, Uhl82b].

This work led to the award of a Fields medal to Donaldson at the 1986 ICM, during which Donaldson’s work was presented by Atiyah [Ati87]. As Atiyah emphasized in the quote at the head of this section, the input from physics was one of the most remarkable aspects of the work. The conversation which had begun with the ADHM solution of a problem in physics had now become a two-way conversation!

Donaldson’s work on four-manifolds did not end with the awarding of the Fields medal. He extended the work in a number of directions, including a definition of polynomial invariants on the cohomology of $X$ of arbitrary degree, based on the moduli space $\mathcal{M}_k$ of instantons with second Chern class $k$ [Don90].

113 More precisely, the relevant space is given by the kernel of a small perturbation of the operator $d_A^\perp$, which makes the space more regular.
7. Topological quantum field theory

[Witten’s paper \[Wit88\], which introduces TQFT in the context of Donaldson’s theory of 4-manifolds and Floer’s theory of 3-manifolds, could well emerge as one of the most significant works in late 20th century topology.]

Daniel Freed (1988)

The problem which Donaldson solved was not really a problem from physics: it was a problem from mathematics whose techniques were inspired by physics. There were many developments stemming from Donaldson’s original work, including a related theory in three dimensions developed by Andreas Floer.

A good starting point for Floer’s theory is the Chern–Simons functional, which on a three-manifold $Y$ equipped with a connection $A$ on a $G$-bundle over $Y$ is the quantity

$$\text{CS}(A) = \frac{1}{4\pi} \int_Y \text{tr}(A \wedge F + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{R}/\mathbb{Z}. \quad (7.1)$$

Equivalently, if $A_0$ is the trivial connection on the trivial $G$-bundle over $Y$ and $A_t = (1 - t)A + tA_0$ then

$$\text{CS}(A) = \frac{1}{8\pi^2} \int_{Y \times [0,1]} \text{tr}(F \wedge *F). \quad (7.2)$$

In its interpolation between $A$ and $A_0$ at the two ends of $Y \times [0,1]$, the connection $A$ is an instanton in the sense the word is used in physics (see [Col85]), albeit an instanton of Euclidean signature.

Floer used a small perturbation of the function CS as a Morse function on the space of connections $\mathcal{A}$. Although $\mathcal{A}$ is infinite dimensional and the formal Hessian of CS at a critical point has infinite sets of both positive and negative eigenvalues, it is of Dirac type and essentially coincides with the operator $*d$ acting on $\Omega^1/d\Omega^0$, suitably extended to Lie algebra valued forms. What is well-defined is the index, the difference between the positive and negative eigenvalues. Moreover, as in Witten’s earlier interpretation of Morse theory in terms of quantum tunneling [Wit82], given two critical points $P$ and $Q$ the solutions to $dA/dt = -\text{grad} \text{CS}$ on $Y \times \mathbb{R}$ with connection $A_P$ as $t \to -\infty$ and connection $A_Q$ as $t \to \infty$ are identified with boundary operators in a chain complex. The resulting homology theory (which is only defined mod 8 for $G = SU(2)$ due to the index theorem in 4 dimensions) is Floer homology [Flo88].

The critical points of (slightly perturbed) CS are identified with irreducible representations $\pi_1(Y) \to G$ and account for the connection between Floer homology and the Casson invariant (which counts such representations, with appropriate signs). Moreover, the Floer theory provides a natural setting for Donaldson theory on four-manifolds with boundary, including Donaldson’s polynomial invariants.

Atiyah [Ati88] put Donaldson’s and Floer’s work together (also combining them with some ideas about Heegard splitting to extend the theory to dimension two), obtaining a non-relativistic quantum field theory. Witten [Wit88] went one step further, and found the proper physical setting for the work of Donaldson and Floer.
The relativistic quantum field theory which describes Donaldson’s and Floer’s results is a topological twist of the usual supersymmetric quantum field theory associated to the $SU(2)$ gauge group, a new notion which Witten introduced in order to provide the physical setting. Starting from a supersymmetric theory with certain supercharges, the action of those supercharges on the physical fields was modified in a way which made them independent of the choice of metric on spacetime. The correlation functions in the corresponding field theory turned out to precisely be the Donaldson polynomial invariants!

Witten used path integrals to motivate his construction, and the theory itself clearly belongs to physics, not mathematics. But as Dan Freed remarks in the review of [Wit88] quoted at the head of this section [Fre89], the implications for topology itself were profound. In fact, the intrinsic study of topological field theories (independent of the precise details of Donaldson theory) has become an important aspect of twenty-first century mathematics.

8. Seiberg–Witten theory

In the last three months of 1994 a remarkable thing happened: this research area was turned on its head by the introduction of a new kind of differential-geometric equation by Seiberg and Witten: in the space of a few weeks long-standing problems were solved, new and unexpected results were found, along with simpler new proofs of existing ones, and new vistas for research opened up.

Simon Donaldson (1995)

In 1994, Seiberg and Witten made some of the first progress in understanding quantum field theories from a non-perturbative perspective (that is, studying properties which are not closely tied to the path-integral formalism [SW94a, SW94b, SW94c]). The particular theory which they studied first – the $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions – was the same one which Witten had earlier shown could be topologically twisted to yield the Donaldson–Witten theory. Now, Seiberg and Witten were able to find a new description of the infrared behavior of the $SU(2)$ gauge theory which took the form of a $U(1)$ gauge theory coupled to a magnetic monopole.

This work in physics had an immediate consequence in mathematics (after twisting) in the form of new topological invariants analogous to the Donaldson invariants [Wit94]. These Seiberg–Witten invariants, as they came to be called, were substantially easier to compute than the Donaldson invariants and progress was quickly made on many difficult conjectures which had been left open by the original Donaldson theory. Donaldson himself expressed great astonishment at the speed of progress [Don96], as quoted at the head of this section.

As John Morgan said during a lecture at the 1995 Cornell Topology Festival [Moc]: “The physicists keep coming up with amazing equations for us to solve.

12More precisely, the supersymmetric quantum field theory with twice the minimal amount of supersymmetry.
Once we know the equations, we can get lots of mathematics out of them, but why can't we find the equations ourselves?" The exchange of ideas between physics and mathematics had now proceeded through at least four stages: from progress in solving the instanton equation in physics, to Donaldson's application of those ideas to the understanding of four-manifolds, to Witten's construction of topological quantum field theory, to the Seiberg–Witten study of the infrared properties of that theory and the spectacular mathematics which resulted!

9. Conclusions

I am sure that [the interaction of math and physics] is going to continue and I believe the reason it will continue is that quantum field theory and string theory . . . have rich mathematical secrets.

Edward Witten (2014)

The story I have presented about interactions between mathematics and physics did not end in 1994, but has continued to develop fruitfully in many directions. For lack of time, I did not mention string theory at all in this lecture, but many of the important interactions since the mid 1980s have involved string theory as well as quantum field theory. As Edward Witten predicted in a 2014 interview conducted by Hirosi Ooguri and quoted at the head of this section, this interaction is likely to continue for a long time to come! Witten went on to say: "When some of these secrets come to the surface, they often come as surprises to physicists because we do not really understand string theory properly as physics – we do not understand the core ideas behind it. At an even more basic level, the mathematicians are still not able to fully come to grips with quantum field theory and therefore things coming from it are surprises. So for both of those reasons, I think that the physics and math ideas generated are going to be surprising for a long time."

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