Gauss-Bonnet dark energy

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We propose the Gauss-Bonnet dark energy model inspired by string/M-theory where standard gravity with scalar contains additional scalar-dependent coupling with Gauss-Bonnet invariant. It is demonstrated that effective phantom (or quintessence) phase of late universe may occur in the presence of such term when the scalar is phantom or for non-zero potential (for canonical scalar). However, with the increase of the curvature the GB term may become dominant so that phantom phase is transient and $w = -1$ barrier may be passed. Hence, the current acceleration of the universe may be caused by mixture of scalar phantom and (or) potential/stringy effects. It is remarkable that scalar-Gauss-Bonnet coupling acts against the Big Rip occurrence in phantom cosmology.

I. INTRODUCTION

It became clear recently that late-time dynamics of the current accelerated universe is governed by the mysterious dark energy. The interpretation of the astrophysical observations indicates that such dark energy fluid (if it is fluid!) is characterized by the negative pressure and its equation of state parameter $w$ lies very close to $-1$ (most probably below of it). Quite possible that it may be oscillating around $-1$. It is extremely difficult to present the completely satisfactory theory of the dark energy (also due to lack of all required astrophysical data), especially in the case of (oscillating) $w$ less than $-1$. (For instance, thermodynamics is quite strange there with possible negative entropy). The successful dark energy theory may be searched in string/M-theory. Indeed, it is quite possible that some unusual gravity-matter couplings predicted by the fundamental theory may become important at current, low-curvature universe (being not essential in intermediate epoch from strong to low curvature). For instance, in the study of string-induced gravity near to initial singularity the role of Gauss-Bonnet (GB) coupling with scalar was quite important for occurrence of non-singular cosmology (for account of dilaton and higher order corrections near to initial singularity, see also ). The present paper is devoted to the study of the role of GB coupling with the scalar field to the late-time universe. It is explicitly demonstrated that such term itself can not induce the effective phantom late-time universe if the scalar is canonical in the absence of potential term. It may produce the effective quintessence (or phantom) era, explaining the current acceleration only when the scalar is phantom or when the scalar is canonical with non-zero potential. It is interesting that it may also have the important impact to the Big Rip singularity, similarly to quantum effects, preventing it in the standard phantom cosmology. Note that we concentrate mainly on the exponential scalar-GB coupling and exponential scalar potential, while the consideration of other types of such functions and their role in late time cosmology will be considered elsewhere.

II. THE ACCELERATED UNIVERSE FROM SCALAR-GB GRAVITY

We consider a model of the scalar field $\phi$ coupled with gravity. As a stringy correction, the term proportional to the GB invariant $G$ is added:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$ (1)
The starting action is given by
\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{\gamma}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) + f(\phi)G \right\} . \] (2)

Here \( \gamma = \pm 1 \). For the canonical scalar, \( \gamma = 1 \) but at least when GB term is not included, the scalar behaves as phantom only when \( \gamma = -1 \) showing in this case the properties similar to quantum field [3]. In analogy with model [10] where also non-trivial coupling of scalar Lagrangian with some power of curvature was considered, one may expect that such GB coupling term may be relevant for the explanation of dark energy dominance.

By the variation over \( \phi \), we obtain
\[ 0 = \gamma \nabla^2 \phi - V'(\phi) + f'(\phi)G . \] (3)

On the other hand, the variation over the metric \( g_{\mu\nu} \) gives
\[ 0 = \frac{1}{\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + \gamma \left( \frac{1}{2} \partial^\mu \phi \partial^\nu \phi - \frac{1}{4} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) + \frac{1}{2} g^{\mu\nu} (-V(\phi) + f(\phi)G) \\
-2f(\phi) R^{\rho\mu\nu\rho} - 2\nabla^\mu \nabla^\nu (f(\phi) R) - 2g^{\mu\nu} \nabla^2 (f(\phi) R) \\
+8f(\phi) R^\mu_{\rho\nu\rho} - 4\nabla_\rho \nabla^\mu (f(\phi) R^{\nu\rho}) - 4\nabla_\rho \nabla^\nu (f(\phi) R^{\mu\rho}) \\
+4\nabla^2 (f(\phi) R^{\nu\rho}) + 4g^{\mu\nu} \nabla_\rho \nabla_\sigma (f(\phi) R^{\nu\rho\sigma}) - 2f(\phi) R^{\mu\nu\rho\sigma} R^{\nu\rho\sigma} + 4\nabla_\rho \nabla_\sigma (f(\phi) R^{\mu\rho\sigma}) . \] (4)

By using the identities obtained from the Bianchi identity
\[ \nabla^\rho R_{\rho\tau\mu\nu} = \nabla_\mu R_{\nu\tau} - \nabla_\nu R_{\mu\tau} , \]
\[ \nabla^\rho R_{\rho\mu} = \frac{1}{2} \nabla_\mu R , \]
\[ \nabla_\rho \nabla_\sigma R^{\mu\rho\sigma} = \nabla^2 R^{\mu\nu} - \frac{1}{2} \nabla^\mu \nabla^\nu R + R^{\mu\nu\rho\sigma} R_{\rho\sigma} - R^{\mu\nu} R^{\rho\sigma} , \]
\[ \nabla_\rho \nabla^\mu R^{\nu\rho} + \nabla_\nu \nabla^\rho R^{\rho\mu} = \frac{1}{2} (\nabla^\mu \nabla^\nu R + \nabla^\nu \nabla^\mu R) - 2R^{\mu\nu\rho\sigma} R_{\rho\sigma} + 2R^\mu_{\rho\nu} R^{\rho\nu} , \]
\[ \nabla_\rho \nabla_\sigma R^{\rho\sigma} = \frac{1}{2} \Box R , \] (5)

one can rewrite (4) as
\[ 0 = \frac{1}{\kappa^2} \left( -R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + \gamma \left( \frac{1}{2} \partial^\mu \phi \partial^\nu \phi - \frac{1}{4} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) + \frac{1}{2} g^{\mu\nu} (-V(\phi) + f(\phi)G) \\
-2f(\phi) R^{\rho\mu\nu\rho} + 4f(\phi) R^\mu_{\rho\nu\rho} R^{\rho\nu} - 2f(\phi) R^{\mu\nu\rho\sigma} R^{\rho\sigma} + 4f(\phi) R^{\mu\nu\rho\sigma} R_{\rho\sigma} \\
+2 (\nabla^\mu \nabla^\nu f(\phi)) R - 2g^{\mu\nu} \left( \nabla^2 f(\phi) R - 4(\nabla_\rho \nabla^\mu f(\phi)) R^{\rho\mu} - 4(\nabla_\rho \nabla^\nu f(\phi)) R^{\rho\nu} \right) \\
+4 \left( \nabla^2 f(\phi) R \right) + 4g^{\mu\nu} (\nabla_\rho \nabla_\sigma f(\phi)) R^{\rho\sigma} - 4(\nabla_\rho \nabla_\sigma f(\phi)) R^{\rho\sigma} . \] (6)

The above expression is valid in arbitrary spacetime dimensions. In four dimensions, the terms proportional to \( f(\phi) \) without derivatives, are cancelled with each other and vanish since the GB invariant is a total derivative in four dimensions.

The starting Friedmann-Robertson-Walker (FRW) universe metric is:
\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 , \] (7)

where
\[ \Gamma^i_{ij} = a^2 H \delta_{ij} , \quad \Gamma^i_{jt} = H \delta^i_j , \quad R_{tj} = - \left( \dot{H} + H^2 \right) \delta_{ij} , \quad R_{ijkl} = a^4 H^2 (\delta_{ik} \delta_{lj} - \delta_{il} \delta_{kj}) , \]
\[ R_{tt} = -3 \left( \dot{H} + H^2 \right) , \quad R_{ij} = a^2 \left( \dot{H} + 3H^2 \right) \delta_{ij} , \quad R = 6 \dot{H} + 12H^2 , \quad \text{other components} = 0 , \] (8)

(here the Hubble rate \( H \) is defined by \( H = \dot{a}/a \)). Assuming \( \phi \) only depends on time, the \((\mu, \nu) = (t, t)\)-component in (4) has the following simple form:
\[ 0 = -\frac{3}{\kappa^2} H^2 + \frac{\gamma}{2} \dot{\phi}^2 + V(\phi) - 2\dot{\phi} f'(\phi) H^3 . \] (9)
On the other hand, Eq. (2) becomes:

$$0 = -\gamma \left( \dot{\phi} + 3H\dot{\phi} \right) - V'(\phi) + 24f'(\phi) \left( HH^2 + H^4 \right).$$

(10)

We now consider the case that $V(\phi)$ and $f(\phi)$ are given as exponents with the constant parameters $V_0$, $f_0$, and $\phi_0$

$$V = V_0 e^{-\frac{2\phi}{\kappa}}, \quad f(\phi) = f_0 e^{\frac{2\phi}{\kappa}}.$$  

(11)

Assume that the scale factor behaves as $a = a_0 t^{h_0}$ (power law). In case that $h_0$ is negative, this scale factor does not correspond to expanding universe but it corresponds to shrinking one. If one changes the direction of time as $t \to -t$, the expanding universe whose scale factor is given by $a = a_0(-t)^{h_0}$ emerges. In this expression, however, since $h_0$ is not always an integer, $t$ should be negative so that the scale factor should be real. To avoid the apparent difficulty, we may further shift the origin of the time as $t \to -t \to t_s - t$. Then the time $t$ can be positive as long as $t < t_s$. Hence, we can propose

$$H = \frac{h_0}{t}, \quad \phi = \phi_0 \ln \frac{t}{t_1},$$

(12)

when $h_0 > 0$ or

$$H = -\frac{h_0}{t_s - t}, \quad \phi = \phi_0 \ln \frac{t_s - t}{t_1},$$

(13)

when $h_0 < 0$, with an undetermined constant $t_1$. By the assumption (12) or (13), one obtains

$$0 = -\frac{3h_0^2}{\kappa^2} + \frac{\gamma \phi_0^2}{2} + V_0 t_1^2 - \frac{48f_0 h_0^3}{t_1^2},$$

(14)

from (12) and

$$0 = \gamma (1 - 3h_0) \phi_0^2 + 2V_0 t_1^2 + \frac{48f_0 h_0^3}{t_1^2} (h_0 - 1),$$

(15)

from (13). Using (12) and (13), it follows

$$V_0 t_1^2 = -\frac{1}{\kappa^2 (1 + h_0)} \left[ 3h_0^2 (1 - h_0) + \frac{\gamma \phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right],$$

$$\frac{48f_0 h_0^3}{t_1^2} = -\frac{6}{\kappa^2 (1 + h_0)} \left( h_0 - \frac{\gamma \phi_0^2 \kappa^2}{2} \right).$$

(16)

The second Eq. (16) shows that if $-1 < h_0 < 0$ and $\gamma = 1$, $f_0$ should be negative. Without the GB term, that is, $f_0 = 0$, a well known result follows:

$$h_0 = \frac{\gamma \phi_0^2 \kappa^2}{2}.$$  

(17)

Since the equation of state parameter $w$ is given by

$$w = -1 + \frac{2}{3h_0},$$

(18)

if $h_0 < 0$ ($h_0 > 0$), $w < -1$ ($w > -1$). Eqs.(16) indicate that even if $\gamma = 1$, with the proper choice of parameters $h_0$ can be negative or $w < -1$. Even if $\gamma > 0$, when $h_0 < -1$, $V_0$ is positive, which means that the potential $V(\phi)$ is bounded below. As a special case we consider

$$\phi_0^2 = \frac{-6h_0^2 (1 - h_0)}{\gamma (1 - 5h_0) \kappa^2},$$

(19)

which gives $V(\phi) = 0$. In order that $\phi_0$ could be real, one has

$$\frac{1}{5} < h_0 < 1, \quad \text{when} \gamma = 1, \quad \text{or} \quad h_0 > \frac{1}{5} \text{ or } h_0 \geq 1.$$  

(20)
In the case (19), the scalar field \( \phi \) is canonical (\( \gamma = 1 \)), and there is no potential \( V(\phi) = 0 \), even if we include the term proportional to the GB invariant, we cannot obtain the effective phantom cosmological solution with \( h_0 < 0 \) or \( w < -1 \). Eq. (16) tells, however, when \( \gamma = 1 \) and \( V_0 > 0 \) even if \( V_0 \) is arbitrary small, if we choose \( f_0 \) properly, we may obtain the effective phantom. The qualitative behavior of \( \gamma \phi_0^2 \) versus \( h_0 \) when \( V_0 = 0 \) is given in Figure 1. There is one positive solution, which may mimic the effective matter with \( 1/5 < h_0 < 1 \) when \( \gamma = 1 \). We also find, when \( \gamma = -1 \), there are always three solutions for \( h_0 \) from (19), one is given by \( h_0 < 0 \) and describes the phantom cosmology, one is \( h_0 > 1 \) describing the quintessence cosmology, and another corresponds to the matter with \( 0 < h_0 < 1/5 \). Then even if \( \gamma = -1 \), there appear the solutions describing non-phantom cosmology corresponding the quintessence or matter.

As an example, we consider the case that

\[
  h_0 = -\frac{80}{3} < -1 ,
\]

(21)

which gives, from (15),

\[
  w = -1.025 ,
\]

(22)

This is consistent with the observational bounds for effective \( w \) (for recent discussion and complete list of refs., see [11]). Then from (16), one obtains

\[
  V_0 t_1^2 = \frac{1}{\kappa^2} \left( \frac{531200}{231} + \frac{403}{154} \gamma \phi_0 K^2 \right) , \\
  f_0 t_1^2 = -\frac{1}{\kappa^2} \left( \frac{9}{49280} + \frac{27}{7884800} \gamma \phi_0 K^2 \right) .
\]

(23)

Therefore even starting from the canonical scalar theory with positive potential before introducing the term proportional to the GB invariant, we may obtain a solution which reproduces the observed value of \( w \) as in (22).

In case of the model induced from the string theory [2], we have \( V_0 = 0, (\dot{V}(\phi) = 0) \) and

\[
  \phi_0^2 = \frac{2}{\kappa^2} ,
\]

(24)

in (11). Then Eq. (19) reduces as

\[
  3h_0^3 - 3h_0^2 + 5h_0 - 1 = 0 ,
\]

(25)

which has only one real solution as

\[
  h_0 = 0.223223 .
\]

(26)

The solution gives

\[
  w = 1.98654 .
\]

(27)

There is another solution of (9) and (10) with (11). In the solution, \( \phi \) and \( H \) are constants,

\[
  \phi = \varphi_0 , \quad H = H_0 ,
\]

(28)
what corresponds to deSitter space. Using (9) and (10) with (11), one finds

$$H_0^2 = -\frac{e^{-\frac{2\phi_0}{\kappa}}}{8f_0\kappa^2}.$$  

Therefore in order for the solution to exist, we may require $f_0 < 0$. In (20), $\phi_0$ can be arbitrary. Hence, the Hubble rate $H = H_0$ might be determined by an initial condition.

In case of the model (11), the term including the GB invariant always gives the contribution in the same order with those from other terms even if the curvature is small. This is due to the factor $f(\phi)$, which enhances the contribution when the curvature is small.

### III. LATE-TIME ASYMPTOTIC COSMOLOGY IN SCALAR-GB GRAVITY AND BIG RIP AVOIDANCE

In the following, another model, which is slightly different from (11), may be considered:

$$V(\phi) = V_0 e^{-\frac{2\phi_0}{\kappa}}, \quad f(\phi) = f_0 e^{-\frac{2\phi_0}{\kappa}}, \quad (\alpha > 1),$$  

Different from the model (11), the model (20) will not be solved exactly. We can only find the asymptotic qualitative behavior of the solutions. Nevertheless, the asymptotic behavior suggests the existence of the cosmological solution, where the value of $w$ could vary with time (oscillation) and/or could depend on the curvature.

Assuming the solution behaves as (12) or (13), when the curvature is small, that is $t$ in (12) or $t_s-t$ in (13) is large, the GB term becomes small and could be neglected since it behaves like $1/(t_s-t)^{-\frac{2}{\alpha}+4}$ or $1/(t_s-t)^{-\frac{2}{\alpha}+4}$. When the curvature is small, the solution could be given by (17), then the effective phantom with $w < -1$ could appear only in case $\gamma = -1 < 0$. On the other hand, when the curvature is large, that is $t$ in (12) or $t_s-t$ in (13) is small, the classical potential could be neglected. Without the classical potential, by assuming, instead of (12)

$$H = \frac{h_0}{t}, \quad \phi = \alpha \phi_0 \ln \frac{t}{t_1},$$  

when $h_0 > 0$ or

$$H = -\frac{h_0}{t_s-t}, \quad \phi = \alpha \phi_0 \ln \frac{t_s-t}{t_1},$$  

when $h_0 < 0$, the following equations replace (14) and (15)

$$0 = -\frac{3h_0^2}{\kappa^2} + \frac{\gamma^2\phi_0^2}{2} - \frac{48f_0h_0^3}{t_1^4},$$  
$$0 = \gamma (1-3h_0) \alpha^2 \phi_0^2 + \frac{48f_0h_0^3}{t_1^4} (h_0-1).$$

By deleting $f_0$ in the above two equations, one gets

$$\phi_0^2 = -\frac{6h_0^2 (1-h_0)}{\gamma \alpha^2 (1-5h_0) \kappa^2},$$  

which corresponds to (19). Then when $\gamma = 1$, the solutions of (34) are not qualitatively changed from those of (19), and there is only one solution $1/5 < h_0 < 1$. On the other hand, when $\gamma = -1$, since the sign of the r.h.s. in (14) is changed from $\gamma = 1$ case, as clear from FIG(11) there are three solutions, corresponding to the phantom $h_0 < 0$ or $w < -1$, the quintessence $h_0 > 1$ or $-1 < w < -1/3$, and the matter with $0 < h_0 < 1/5$ or $w > 7/3$. Then if the term proportional to the GB invariant in case $\gamma < 0$ (which corresponds to a scalar phantom solution without GB term) is included the effective $w$ can become larger than $-1$ and the Big Rip singularity might be avoided (see (3) for quantum effects account to escape of Big Rip). That is, in case $\gamma < 0$, when the curvature is small as in the current universe, the GB term is negligible and the potential term dominates, which gives the cosmic acceleration with $w < -1$. Then the curvature increases gradually and the universe seems to tend to the Big Rip singularity (3). However, when the curvature is large, the GB term becomes dominant and might prevent the singularity. Hence, in case $\gamma < 0$, the GB term may work against the Big Rip singularity occurrence, like quantum effects (3). After the GB term dominates
when $\gamma < 0$, the curvature turns to become smaller. Then the potential term dominates again. This might tell that the behavior of the universe might approach to the de Sitter space with $w = -1$ by the damped oscillation. In fact even in the model \ref{eq:30}, if $\alpha > 1$ is assumed, there is a de Sitter solution corresponding to \ref{eq:28}:

$$
H_0^2 = -e^{-2\phi_0/8f_0\kappa^2}, \quad \phi_0 = \frac{\alpha\phi_0}{2(1 - \alpha)} \ln \left( \frac{8V_0 f_0\kappa^2}{3} \right).
$$

(35)

In \ref{eq:30}, we have assumed $\alpha > 1$. If we consider the case that $V(\phi) = V_0 e^{-2\phi/2w}$ and $f(\phi) = f_0 e^{2\phi/2w}$ as in \ref{eq:30} but $0 < \alpha < 1$, there appears a solution where the term including GB invariant becomes dominant even if the curvature is small. By assuming \ref{eq:31} or \ref{eq:32}, Eq.\ref{eq:33} is obtained again. Hence, when $0 < \alpha < 1$, the solution where $h_0$ can be positive or $w > -1$ even if $\gamma < 0$ (scalar phantom) appears.

On the other hand, if $\gamma > 0$ there is no accelerated universe solution with $w < -1$. The parameter $w$ may change with time but $w$ is larger than $-1/3$. curvature due to the GB term. effective equation of occurring eventually.

IV. DISCUSSION

We considered essentially two models with exponential couplings given by \ref{eq:11} and \ref{eq:30}. The model \ref{eq:11} may be considered as the special case corresponding to $\alpha = 1$. The main results can be summarized as follows:

1. $\alpha = 1$ case: exactly solvable
   
   (a) $V = 0$ case: When $\gamma = 1$, there is only one solution $-1/3 < w < 7/3$. On the other hand, when $\gamma = -1$, there are three solutions, corresponding to $w < -1$, $-1 < w < -1/3$, and $w > 7/3$.

2. $\alpha > 1$ case: the potential term dominates for small curvature and the GB term for large one.
   
   (a) $\gamma > 0$: The value of $w$ may be time dependent but there is no solution describing acceleration of the universe.

   (b) $\gamma < 0$: There might appear the Big Rip singularity but there might be a solution asymptotically approaching to the de Sitter space.

3. $0 < \alpha < 1$ case: the potential term dominates for large curvature and the GB term for small one.

   (a) $\gamma > 0$: There is no solution describing acceleration of the universe.

   (b) $\gamma < 0$: There appears the Big Rips singularity.

For the models \ref{eq:11} and \ref{eq:30}, in case $V_0 = 0$ (that is, when the potential vanishes), by replacing $\alpha\phi_0$ with $\phi_0$, it follows the two models are equivalent. Especially $\gamma = -1$ case has been well studied and it has been shown that there are always three effective cosmological phases corresponding to the phantom with $h_0 < 0$ or $w < -1$, the quintessence with $h_0 > 1$ or $-1 < w < -1/3$, and the matter with $0 < h_0 < 1/5$ or $w > 7/3$. Even if $V_0 \neq 0$, the model \ref{eq:11} can be solved exactly and the solutions where $h_0$ and therefore $w$ are constants may be found. On the other hand, if $V_0 \neq 0$, in the model \ref{eq:30} there exist the solutions where the values of $h_0$ and therefore of $w$ are time-dependent. There could emerge a cosmology, which behaves as phantom one with $w < -1$ when the curvature is small and as a usual matter dominated universe with $w > -1$ when the curvature is large. Moreover, Big Rip singularity does not occur.

Our study indicates that current acceleration may be significantly influenced by stringy/M-theory effects (terms) which somehow became relevant quite recently (in cosmological sense). It remains a challenge to construct the consistent dark energy universe model from string/M-theory.

Note added: After the first version of this paper (with some error) appeared in hep-th, the related study, for instance, of the influence of scalar-GB term to Big Rip has appeared in ref.\cite{12}.

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APPENDIX A: STABILITY OF PHANTOM COSMOLOGY

In this appendix, we check the stability of the above solutions. The following quantities are convenient to introduce:

\[ X = \frac{\dot{\phi}}{H}, \quad Z = H^2 f'(\phi), \quad \frac{d}{dN} = \frac{a}{H} \frac{d}{dt}. \] \hspace{1cm} (A1)

For simplicity, we also put \( \kappa^2 \) to be unity. Then by using (9) and (10) with (11), one finds

\[ \frac{dX}{N} = \frac{\gamma^2 \phi_0 X^3 + 2\gamma X \left(8X^2Z - \phi_0 (3 + 52XZ)\right) + 4 \left(\frac{2V_0 f_0}{\phi_0 Z} + \frac{24V_0 f_0 X}{\phi_0} + 12Z \left(\phi_0 + 16\phi_0 XZ - 8X^2Z\right)\right)}{2\phi_0 \left(\gamma + 6\gamma XZ + 96Z^2\right)}, \] \hspace{1cm} (A2)

\[ \frac{dZ}{N} = \frac{Z \left(-\gamma^2 \phi_0 X^2 - 16Z \left(\frac{2V_0 f_0}{\phi_0 Z} + 12 \left(\phi_0 - X\right)Z\right) + 2\gamma \left(X + 16\phi_0 XZ\right)\right)}{\phi_0 \left(\gamma + 6\gamma XZ + 96Z^2\right)}. \] \hspace{1cm} (A3)

For the solution (12) or (13), it follows

\[ X = X_0 = \frac{\phi_0}{h_0}, \quad Z = Z_0 = \frac{2f_0 h_0^2}{\phi_0 h_1^2}. \] \hspace{1cm} (A4)

In terms of \( X_0 \) and \( Z_0 \), Eqs. (A2) and (A3) can be rewritten as

\[ 0 = -\frac{3}{\kappa^2} + \frac{\gamma X_0^2}{2} + \frac{2V_0 f_0}{\phi_0 Z_0} - 24Z_0 X_0, \] \hspace{1cm} (A5)

\[ 0 = \gamma X_0^3 - 3\phi_0 \gamma X_0 + \frac{4V_0 f_0}{\phi_0 Z_0} + 24\phi_0 Z_0 - 24Z_0 X_0. \] \hspace{1cm} (A6)

For the solution (A3), by using (A5) and (A6), the right hand sides of Eqs. (A2) and (A3) vanish consistently. We now consider the perturbation around the solution (A4):

\[ X = X_0 + \delta X, \quad Y = Y_0 + \delta Y. \] \hspace{1cm} (A7)

We now only check the stability for \( V = 0 \) (\( V_0 = 0 \)) case.

Using (A2) and (A3), one obtains

\[ \frac{d}{dN} \left(\frac{\delta X}{\delta Y}\right) = M \left(\frac{\delta X}{\delta Y}\right), \quad M = \begin{pmatrix} \check{A} & \check{B} \\ \check{C} & \check{D} \end{pmatrix}. \] \hspace{1cm} (A8)

Here

\[ \check{A} = \frac{3\gamma^2 \phi_0 X_0^2 + 48\gamma X_0^2 Z_0 - 6\gamma \phi_0 - 208\gamma \phi_0 X_0 Z_0 + 768 \left(\phi_0 - X_0\right) Z_0^2}{2\phi_0 \left(\gamma + 8\gamma X_0 Z_0 + 96Z_0^2\right)}, \]

\[ \check{B} = \frac{16\gamma X_0^3 - 104\gamma \phi_0 X_0^2 + 48\phi_0 + 1536\phi_0 X_0 Z_0 - 768X_0^2 Z_0}{2\phi_0 \left(\gamma + 8\gamma X_0 Z_0 + 96Z_0^2\right)}, \]

\[ \check{C} = \frac{2Z_0 \left(-\gamma^2 \phi_0 X_0 + 96Z_0^2 + \gamma + 16\gamma \phi_0 Z_0\right)}{\phi_0 \left(\gamma + 8\gamma X_0 Z_0 + 96Z_0^2\right)}, \]

\[ \check{D} = \frac{32Z_0 \left(-12 \left(\phi_0 - X_0\right) Z_0 + \gamma \phi_0 X_0\right)}{\phi_0 \left(\gamma + 8\gamma X_0 Z_0 + 96Z_0^2\right)}. \] \hspace{1cm} (A9)

If the real parts of all the eigenvalues of the matrix \( M \) are negative, the perturbation becomes small and the system is stable. Then the condition of the stability is given by

\[ \check{A} + \check{D} < 0, \quad \check{A} \check{D} - \check{B} \check{C} > 0. \] \hspace{1cm} (A10)

First, the check of the stability for \( V = 0 \) case is in order. By using (A4), (A5), and (A6), we find

\[ X_0^2 = \frac{\phi_0^2}{h_0^2} = -\frac{6 (h_0 - 1)}{\gamma (5h_0 - 1)}, \quad Z_0^2 = -\frac{\gamma (3h_0 - 1)^2}{96 (h_0 - 1) (5h_0 - 1)}, \quad X_0 Z_0 = -\frac{3h_0 - 1}{4 (5h_0 - 1)}. \] \hspace{1cm} (A11)
In order that $X_0^2$ and $Z_0^2$ are positive, it follows

$$\frac{1}{\gamma} < h_0 < 1, \quad \text{when } \gamma > 0$$

or $h_0 < \frac{1}{\gamma}$ or $h_0 > 1, \quad \text{when } \gamma < 0.$  \hspace{1cm} (A12)

By using (A11), $\tilde{A}, \tilde{B}, \tilde{C},$ and $\tilde{D}$ in (A10) can be expressed in terms of $h_0$:

$$\tilde{A} = \frac{(h_0 - 1) (9h_0^2 - 4h_0 + 1)}{h_0 (5h_0^2 - 4h_0 + 1)}, \quad \tilde{B} = \frac{24 (h_0 - 1) (3h_0^2 - 2h_0 + 1)}{\gamma h_0 (5h_0^2 - 4h_0 + 1)},$$

$$\tilde{C} = \frac{\gamma (3h_0 - 1) (3h_0^2 - 1)}{12 (h_0 - 1) (5h_0^2 - 4h_0 + 1)}, \quad \tilde{D} = -\frac{2 (9h_0^2 - 9h_0 + 1)}{h_0 (5h_0^2 - 4h_0 + 1)}.  \hspace{1cm} (A13)$$

Then we obtain very simple results:

$$\tilde{A} + \tilde{D} = -\frac{3 (h_0 - 1)}{h_0},  \hspace{1cm} (A14)$$

$$\tilde{A} \tilde{D} - \tilde{B} \tilde{C} = \frac{2 (1 - 3h_0)}{h_0^2}.  \hspace{1cm} (A15)$$

Therefore Eq. (A10) is satisfied and the system is stable if and only if

$$h_0 < 0.  \hspace{1cm} (A16)$$

Then the case corresponding to phantom cosmology with $h_0 < 0$ is always stable.

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