Elliptic Gromov-Witten Invariants And Virasoro Conjecture

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The Virasoro conjecture predicts that the generating function of Gromov-Witten invariants is annihilated by infinitely many differential operators which form a half branch of the Virasoro algebra. This conjecture was proposed by Eguchi, Hori and Xiong [EHX2] and also by S. Katz [Ka] (see also [EJX]). It provides a powerful tool in the computation of Gromov-Witten invariants. In [LT], the author and Tian proved the genus-0 part of the Virasoro conjecture. The main purpose of this paper is to study the genus-1 part of this conjecture.

The system of Gromov-Witten invariants relevant to this paper are the so called descendant Gromov-Witten invariants. These invariants arose in the theory of topological sigma model coupled to gravity [W2]. Mathematical definition for such invariants was given in [RT2] for semipositive symplectic manifolds. Using virtual moduli cycles ([LiT1], [LiT2], and also [BF]), these invariants can also be defined for all compact symplectic manifolds and, in purely algebraic geometric setting, for smooth projective varieties. In this paper, we consider descendant Gromov-Witten invariants for a smooth projective variety $V$. For simplicity, we assume that $H^{\text{odd}}(V,\mathbb{C}) = 0$. Fix a basis $\{\gamma_1, \ldots, \gamma_N\}$ of $H^*(V,\mathbb{C})$ with $\gamma_1$ equal to the identity of the cohomology ring of $V$ and $\gamma_\alpha \in H^{p_\alpha, q_\alpha}(V,\mathbb{C})$ for every $\alpha$. For any non-negative integer $g$ and $A \in H_2(V,\mathbb{Z})$, let $\langle \tau_{n_1, \alpha_1} \cdots \tau_{n_k, \alpha_k} \rangle_{g,A}$ be the genus $g$ degree $A$ descendant Gromov-Witten invariants associated with cohomology classes $\gamma_{\alpha_1}, \ldots, \gamma_{\alpha_k}$ and non-negative integers $n_1, \ldots, n_k$ (See Section 1.1 for the definition of Gromov-Witten invariants). Summing up the Gromov-Witten invariants over all degrees, we obtain a quantity which is called the $k$-point correlators in the theory of topological sigma model:

$$\langle \tau_{n_1, \alpha_1} \cdots \tau_{n_k, \alpha_k} \rangle_g := \sum_{A \in H_2(V,\mathbb{Z})} q^A \langle \tau_{n_1, \alpha_1} \cdots \tau_{n_k, \alpha_k} \rangle_{g,A},$$

where $q^A$ belongs to the Novikov ring (i.e. the completion of the multiplicative ring generated by monomials $q^A = d_1^{a_1} \cdots d_r^{a_r}$ over the ring of rational numbers, where $\{d_1, \ldots, d_r\}$ is a fixed basis of $H_2(V,\mathbb{Z})$ and $A = \sum_{i=1}^r a_i d_i$). The generating function of genus-$g$ Gromov-Witten invariants is defined by

$$F_g(T) := \sum_{k \geq 0} \frac{1}{k!} \sum_{\substack{\alpha_1, \ldots, \alpha_k \in H^*(V,\mathbb{C}) \setminus \{0\} \atop n_1, \ldots, n_k \in \mathbb{N}}} t_{n_1}^{\alpha_1} \cdots t_{n_k}^{\alpha_k} \langle \tau_{n_1, \alpha_1} \cdots \tau_{n_k, \alpha_k} \rangle_g,$$
where \( T = \{ t_n^\alpha \mid n \in \mathbb{Z}_+, \alpha = 1, \cdots, N \} \) is an infinite set of parameters. The space of all parameters \( T \) is called the big phase space. This is an infinite dimensional space with coordinates \( \{ t_n^\alpha \} \). The finite dimensional subspace \( \{ T \mid t_n^\alpha = 0 \text{ if } n > 0 \} \) is called the small phase space. The function \( F_g \) is understood as a formal power series of \( T \). The generating function for Gromov-Witten invariants of all genera is defined to be

\[
Z(T; \lambda) := \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(T),
\]

where \( \lambda \) is another parameter which is used to separate information from different genera.

In topological sigma model, \( F_g \) is called the genus-\( g \) free energy function and \( Z \) is called the partition function.

In [EHX2], Eguchi, Hori, and Xiong constructed a sequence of linear differential operators, denoted by \( L_n \) with \( n \in \mathbb{Z} \), on the big phase space (See Section 1.5). They checked that these operators define a representation of the Virasoro algebra with the central charge equal to the Euler characteristic number of \( V \) under a condition which is equivalent to the vanishing of the Hodge number \( h^{p,q}(V) \) for \( p \neq q \). These operators were modified by S. Katz so that the last condition is not needed. They conjectured that \( L_n Z \equiv 0 \) for all \( n \geq -1 \). This conjecture is called the Virasoro conjecture and the equation \( L_n Z = 0 \) is called the \( L_n \) constraint. The \( L_{-1} \) constraint is the string equation (cf.[W2]). The \( L_0 \) constraint was discovered by Hori [H]. Both of these two constraints hold for all manifolds. When the underlying manifold is a point, the Virasoro conjecture is equivalent to a conjecture by Witten [W2] which predicted that the corresponding generating function is a \( \tau \)-function of the KdV hierarchy. Witten’s conjecture was proved by Kontsevich [Ko] and also by Witten [W3]. For arbitrary manifolds, we can write

\[
L_n Z(T; \lambda) = \left\{ \sum_{g \geq 0} \Psi_{g,n} \lambda^{2g-2} \right\} Z(T; \lambda).
\]

The \( L_n \) constraint is equivalent to \( \Psi_{g,n} = 0 \) for all \( g \geq 0 \). The equation \( \Psi_{g,n} = 0 \) is called genus-\( g \) \( L_n \)-constraint. It is, in general, a non-linear partial differential equation involving all free energy functions \( F_{g'} \) with \( 0 \leq g' \leq g \). The genus-\( g \) Virasoro conjecture predicts that for all \( n \geq -1 \), the genus-\( g \) \( L_n \) constraint is true. The genus-0 Virasoro conjecture was first proved in [L1]. Later, alternative proofs were given in [DZ2] and [G2]. We will give a brief review to the genus-0 case in Section 1.6. The genus-1 Virasoro conjecture for manifolds with semisimple quantum cohomology was proved in [DZ2]. There is a discussion of Virasoro conjecture for degree 0 Gromov-Witten invariants in [OP]. In this paper, we study the genus-1 Virasoro conjecture without assuming semisimplicity. The generating functions relevant to the genus-1 case are \( F_0 \) and \( F_1 \). As in the genus-0 case, most of our discussions only use basic properties of quantum cohomology, therefore could be extended to the setting of abstract Frobenius manifolds.

In most part of this paper (except Section 3) we will deal with the small phase space which can be identified with \( H^*(V, \mathbb{C}) \). We will write the coordinates \( t_0^\alpha \) simply as \( t^\alpha \) and identify the coordinate vector fields \( \frac{\partial}{\partial t^\alpha} \) with cohomology classes \( \gamma_\alpha \). The restriction of
$F_g$ to the small phase space is denoted by $F^s_g$. The third derivatives of $F^s_0$ defines a ring
structure, called the \textit{quantum cohomology ring}, on each tangent space of the small phase
space. This enables us to take product, called the \textit{quantum product}, of two vector fields on
the small phase space. There are two special vector fields on the small phase space. One
is $\gamma_1$, which is also the identity element with respect to the quantum product. Another
one is the so called \textit{Euler vector field}, which is defined by

$$E := c_1(V) + \sum_\alpha (b_1 + 1 - b_\alpha) e^\alpha \gamma_\alpha,$$

where $c_1(V)$ is the first Chern class of $V$ and $b_\alpha = p_\alpha - \frac{1}{2}(\dim CV - 1)$. Note that usually
the holomorphic dimension $p_\alpha$ is replaced by a half of the real dimension of $\gamma_\alpha$. This
modification is due to S. Katz. Let $E^k$ be the $k$-th quantum power of $E$, i.e.

$$E^k := E \cdot \cdots \cdot E, \quad \text{where } \cdot \text{ denotes the quantum product.}$$

Here we use the convention that $E^0 = \gamma_1$ and $E^1 = E$. It is perhaps well known that \{$E^k \mid k \geq 0$\} form a half branch of the Virasoro
algebra, i.e.

$$[E^k, E^m] = (m - k)E^{m+k-1}. \quad (1)$$

This fact was used in [DZ2] without giving a proof. A proof of this can be found in [HM] (see also the remark after equation (19)). It is also well known that $E^0 F^s_1 = 0$ and $EF^s_1 = \text{const.}$ In Section 2, we will prove the following

\textbf{Theorem 0.1} For any manifold $V$ and $k > 0$, the genus-1 data $E^k F^s_1 - (k/2)E^{k-1} E^2 F^s_1$
can be represented by derivatives of $F^s_0$.

See Theorem 2.4 for a more explicit form of this theorem. According to this theorem, if we know that $E^2 F^s_1$ can be represented by genus-0 data, so does $E^k F^s_1$ for all $k \geq 0$. When restricted to the small phase space, the genus-1 $L_1$ constraint is equivalent to say that $E^2 F^s_1$ is equal to the following function

$$\phi_2 := \sum_{\alpha, \beta} \eta^{\alpha, \beta} \left\{-\frac{1}{24} \nabla^2_{E, E} (\gamma_\alpha \gamma_\beta F^s_0) + \frac{1}{2} \left( b_\alpha b_\beta - \frac{b_1 + 1}{6} \right) \gamma_\alpha \gamma_\beta F^s_0 \right\}, \quad (2)$$

where $(\eta^{\alpha, \beta})$ is the inverse matrix of the intersection form on $V$, $\nabla$ is the flat connection
and $\nabla^2_{u, v} = \nabla_u \nabla_v - \nabla_{\nabla_u v}$ is the second covariant derivative. In Section 4, we will prove
that this last condition implies the genus-1 $L_n$ constraints for all $n \geq 1$.

\textbf{Theorem 0.2} For any manifold $V$, the genus-1 Virasoro conjecture holds if and only if $E^2 F^s_1 = \phi_2$.

Theorem 0.1 gives an expression for $E^k F^s_1$ in terms of $E^2 F^s_1$ and genus-0 data. If we replace $E^2 F^s_1$ in this expression by $\phi_2$ and denote the resulting expression by $\phi_k$ for $k > 2$, then $\phi_k$ is a function only involves genus-0 data (see formula (26) and Theorem 3.9 for
explicit forms of this function). We also define $\phi_0 = 0$ and $\phi_1$ equal to the constant in $EF^s$. Because of the relation (1), a necessary condition for $E^2F^s_1 = \phi_2$ is the following

$$E^k \phi_m - E^m \phi_k = (m - k)\phi_{k+m-1}, \quad (3)$$

for all $m, k \geq 0$. In Section 3 we will prove that this condition is always satisfied.

**Theorem 0.3** For any manifold $V$, equality (3) always holds.

At each point of the small phase space, the quantum powers of the Euler vector field span a subspace of the tangent space. The dimension of this subspace may vary as the base point changes. In an open subset, this dimension is constant and the quantum powers of the Euler vector field define an integrable distribution. Therefore one can talk about leaves of this distribution. In fact, in a proper sense, leaves of any collection of vector fields on a finite dimensional manifold are always well defined and are immersed submanifolds (see [Su]). Each leaf of $\{E^k \mid k \geq 0\}$ is a finite dimensional smooth submanifold, which may not be flat with respect to the intersection form, and therefore may not be a Frobenius manifold itself. On each leaf (restricting to an open subset if necessary), there exists a finite number $n$ such that

$$E^{n+1} = \sum_{k=0}^{n} f_k E^k, \quad (4)$$

where $f_k$’s are smooth functions on the leaf. Another necessary condition for $E^2F^s_1 = \phi_2$ is that, on each leaf,

$$\phi_{n+1} = \sum_{k=0}^{n} f_k \phi_k. \quad (5)$$

We conjecture that this condition is always satisfied. This can be verified easily for manifolds with semisimple quantum cohomology. In fact, equations (3) and (5) are equivalent to the existence of a local potential function whose derivative along $E^k$ is $\phi_k$ for all $k$. For manifolds with semisimple quantum cohomology, such a potential function exists globally and can be explicitly expressed in terms of the $\tau$-function of the isomonodromy deformation (c.f. [DZ2] proof of Proposition 4).

**Definition 0.4** We say that a manifold $V$ has non-degenerate quantum cohomology if at generic points of the small phase space, there exists an integer $m \geq 1$ such that $E^m$ is contained in the linear span of $\{E^0, Z_k \mid k \geq 0\}$, where $Z_k := \sum_{i=0}^{n}(E^k f_i)E^i$.

We have the following

**Theorem 0.5** For any manifold $V$ with non-degenerate quantum cohomology, if equality (5) is satisfied, then the genus-1 Virasoro conjecture holds.

This theorem will be proved in Section 4. Because of this theorem, it would be interesting to know which manifolds have non-degenerate quantum cohomology. We first note that vector fields $Z_k$ can also be defined by $f_i$’s without taking derivatives (see Remark 3.2 and Remark 3.3). Therefore the property of being non-degenerate can be checked pointwise.
In Section 6 we will give some sufficient conditions for the non-degeneracy. In particular, if the quantum cohomology of a manifold is semisimple, it must also be non-degenerate. As a corollary of Theorem 0.5, the genus-1 Virasoro conjecture holds for manifolds with semisimple quantum cohomology. This fact was proved before in [DZ2]. In the approach of [DZ2], the assumption of semisimplicity was needed from the very beginning since the canonical coordinates are used throughout all calculations. While in our approach, this is a corollary of a more general result and our assumption of non-degeneracy only comes at the last step. We would also like to make a comparison between the non-degeneracy condition and the semisimplicity condition. If a manifold has semisimple quantum cohomology, then at generic points, the powers of the Euler vector field span the entire tangent spaces. But for manifolds with non-degenerate quantum cohomology, this may not be the case. Even if we assume that the powers of the Euler vector field span tangent spaces, the non-degeneracy condition is still weaker than semisimplicity (we will see this through Lemma 6.6, its corollaries, and examples at the end of Section 6). Moreover, to verify semisimplicity, we need to know the quantum product of the Euler vector fields with tangent vectors in all directions. But, to verify non-degeneracy, we only need to know the quantum powers of the Euler vector field. Therefore it might be much easier to give a more geometric characterization. We recall a conjecture by Tian which predicts that all Fano varieties (which by definition have positive first Chern classes) have semisimple quantum cohomology [T]. This conjecture was verified for Grassmannians and complete intersections of low degrees (see [TX]). In general, it is still an open conjecture. A weaker version of this conjecture would be that all Fano varieties have non-degenerate quantum cohomology. Since the definition of the Euler vector field explicitly involves the first Chern class, it might be easier to verify this weaker version of Tian’s conjecture. We would like to study this in another paper.

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1 Preliminaries

In this section we recall the definition of Gromov-Witten invariants, Quantum cohomology, and some well known facts. We will also set up notation conventions used in this paper and define the Virasoro operators. In Section 1.6, we give a brief review of the genus-0 Virasoro conjecture.

1.1 Gromov-Witten invariants

Gromov-Witten invariants are defined via the intersection theory of moduli spaces of stable maps from Riemann surfaces to a fixed manifold $V$. For any element $A \in H_2(V, \mathbb{Z})$ and non-negative integers $g$ and $k$, the moduli space $\overline{M}_{g,k}(V, A)$ is defined to be the collection of all data $(C; x_1, \ldots, x_k; f)$ where $C$ is a genus-$g$ projective connected curve over $\mathbb{C}$ whose only possible singularities are simple double points, $x_1, \ldots, x_k$ are smooth
points on \( C \) (called marked points), and \( f \) is an algebraic map from \( C \) to \( V \) which is stable with respect to \( (C; x_1, \ldots, x_k) \), (i.e. there is no infinitesimal deformation for this data). Each marked point \( x_i \) defines a map, called the \( i \)-th evaluation map,

\[
ev_i : \overline{\mathcal{M}}_{g,k}(V, A) \rightarrow V \quad \text{(C; } x_1, \ldots, x_k; f) \mapsto f(x_i).
\]

It also defines a line bundle over \( \overline{\mathcal{M}}_{g,k}(V, A) \), denoted by \( E_i \), whose fiber over \( (C; x_1, \ldots, x_k; f) \) is \( T^*_x C \). For any cohomology classes \( \gamma_1, \ldots, \gamma_k \in H^*(V, \mathbb{C}) \) and non-negative integers \( n_1, \ldots, n_k \), the corresponding descendant Gromov-Witten invariants are defined by

\[
\langle \tau_{n_1}(\gamma_1) \cdots \tau_{n_k}(\gamma_k) \rangle_{g,A} = \int_{[\overline{\mathcal{M}}_{g,k}(V,A)]^{\text{virt}}} c_1(E_1)^{n_1} \cup \ev_1^* (\gamma_1) \cup \cdots \cup c_1(E_k)^{n_k} \cup \ev_k^* (\gamma_k),
\]

where \( [\overline{\mathcal{M}}_{g,k}(V,A)]^{\text{virt}} \) is the virtual fundamental class of \( \overline{\mathcal{M}}_{g,k}(V,A) \) (cf. \cite{LT}). When all \( n_i \)’s are zero, the corresponding invariants are called primary Gromov-Witten invariants. The notation \( \tau_{n,\alpha} \) which was used in the introduction will be explained in the next subsection.

### 1.2 Convention of notations

We will use \( d \) to denote the complex dimension of \( V \) and let \( N \) be the dimension of the space of cohomology classes \( H^*(V, \mathbb{C}) \). To define the generating functions, we need to fix a basis \( \{ \gamma_1, \ldots, \gamma_N \} \) of \( H^*(V, \mathbb{C}) \) with \( \gamma_1 \) equal to the identity of the cohomology ring of \( V \) and \( \gamma_\alpha \in H^{p_\alpha, q_\alpha}(V, \mathbb{C}) \) for every \( \alpha \). We also arrange the basis in such a way that the dimension of \( \gamma_\alpha \) is non-decreasing with respect to \( \alpha \) and if two cohomology classes have the same dimension, we also require that the holomorphic dimension \( p_\alpha \) is non-decreasing. We will abbreviate \( \tau_n(\gamma_\alpha) \) as \( \tau_{n,\alpha} \) and identify \( \tau_0,\alpha \) with \( \gamma_\alpha \). For each \( \tau_{n,\alpha} \), we associate a parameter \( t_n^\alpha \) and the collection of all such parameters is denoted by \( T = (t_n^\alpha \mid n \in \mathbb{Z}_+, \alpha = 1, \ldots, N) \), where \( \mathbb{Z}_+ \) is the set of non-negative integers. The space of all \( T \)’s is the big phase space and its subspace \( \{ T \mid t_n^\alpha = 0 \text{ if } n > 0 \} \) is the small phase space.

For convenience, we will always identify the symbol \( \tau_{n,\alpha} \) with the tangent vector field \( \frac{\partial}{\partial t_n^\alpha} \) on the big phase space. We also consider \( \tau_{n,\alpha} \) with \( n < 0 \) as a zero operator. On the small phase space, we write \( t_0^\alpha \) simply as \( t^\alpha \) and also identify the cohomology class \( \gamma_\alpha \) with the vector field \( \frac{\partial}{\partial t^\alpha} \).

As in the introduction, we can define the partition function \( Z \) and free energy function \( F_g \) on the big phase space. These are the generating functions of the corresponding classes of Gromov-Witten invariants. The restriction of \( F_g \) to the small phase space are denoted by \( F_g^s \). As in \cite{T}, we will denote the tensor defined by the \( k \)-th covariant derivative of \( F_g \) by \( \langle \cdots \rangle_k \). This is a symmetric \( k \)-tensors on the big phase space defined by

\[
\langle \langle \tau_{m_1,\alpha_1} \tau_{m_2,\alpha_2} \cdots \tau_{m_k,\alpha_k} \rangle \rangle_k := \frac{\partial^k}{\partial t_{m_1}^{\alpha_1} \partial t_{m_2}^{\alpha_2} \cdots \partial t_{m_k}^{\alpha_k}} F_g.
\]
This tensor is called the $k$-point (correlation) function. The corresponding tensor on the small phase space is denoted by $\langle\langle \cdots \rangle\rangle_{g,s}$.

Besides the above notations, we will also use the following convention throughout the paper unless otherwise stated. Lower case Greek letters, e.g. $\alpha, \beta, \mu, \nu, \sigma, \ldots$, etc., will be used to index the cohomology classes. The range of these indices is from 1 to $N$, where $N$ is the dimension of the space of cohomology classes. Lower case English letters, e.g. $i, j, k, m, n, \ldots$, etc., will be used to index the level of descendents. Their range is the set of all non-negative integers, i.e. $\mathbb{Z}_+$. All summations are over the entire ranges of the indices unless otherwise indicated. Let $\eta_{\alpha\beta} = \int_V \gamma_\alpha \cup \gamma_\beta$ be the intersection form on $H^*(V, \mathbb{C})$. We will use $\eta = (\eta_{\alpha\beta})$ and $\eta^{-1} = (\eta^{\alpha\beta})$ to lower and raise indices. Let $\mathcal{C} = (C_{\alpha\beta})$ be the matrix of multiplication by the first Chern class $c_1(V)$ in the ordinary cohomology ring, i.e. $c_1(V) \cup \gamma_\alpha = \sum \beta C_{\beta\alpha} \gamma_\beta$.

Since we are dealing with even dimensional cohomology classes only, both $\eta$ and $\mathcal{C}$ are symmetric matrices, where the entries of $\mathcal{C}$ are given by $C_{\alpha\beta} = \int_V c_1(V) \cup \gamma_\alpha \cup \gamma_\beta$.

Instead of coordinates $\{t^\alpha_m \mid m \in \mathbb{Z}_+, \alpha = 1, \ldots, N\}$, it is very convenient to use the following shifted coordinates on the big phase space

$$\tilde{t}^\alpha_m = t^\alpha_m - \delta_{m,1}\delta_{\alpha,1} = \begin{cases} t^\alpha_m - 1, & \text{if } m = \alpha = 1, \\ t^\alpha_m, & \text{otherwise}. \end{cases}$$

### 1.3 Topological recursion relation

Topological recursion relations reduce the levels of descendents in correlation functions. The genus-0 topological recursion relation has the following form (cf. [RT2] and [W2]):

$$\langle\langle \tau_{m,\mu} \rangle\rangle_0 = \sum_\sigma \langle\langle \tau_{m-1,\alpha} \gamma_\sigma \rangle\rangle_0 \langle\langle \gamma^\sigma \tau_{n,\beta} \rangle\rangle_0,$$

for $m > 0$. In this formula, we used the convention that the indices of cohomology classes are raised by $\eta^{-1}$. Therefore $\gamma^\sigma$ should be understood as $\sum_\rho \eta^{\rho\sigma} \gamma_\rho$. This recursion relation implies the following genus-0 constitutive relation [DW].

$$\langle\langle \tau_{m,\alpha} \tau_{n,\beta} \rangle\rangle_0 = \langle\langle \tau_{m,\alpha} \tau_{n,\beta} e^{\sum_\sigma u^\sigma \gamma_\sigma} \rangle\rangle_0,$$

where $u^\sigma = \langle\langle \gamma_1 \gamma^\sigma \rangle\rangle_0$. This relation is an important building block in defining the $\tau$-function for Frobenius manifolds [Du] (which corresponds to $F_0$ in the topological sigma model).
As noted by Witten [W2], the genus-0 topological recursion relation implies the generalized WDVV equation:

$$\sum_\sigma \langle\langle \tau_{m,\alpha} \tau_{n,\beta} \gamma_\sigma \rangle\rangle_0 \langle\langle \gamma_\sigma \tau_{k,\mu} \tau_{l,\nu} \rangle\rangle_0 = \sum_\sigma \langle\langle \tau_{m,\alpha} \tau_{k,\mu} \gamma_\sigma \rangle\rangle_0 \langle\langle \gamma_\sigma \tau_{n,\beta} \tau_{l,\nu} \rangle\rangle_0.$$ 

When restricted to the small phase space, this equation is usually called the WDVV equation. It gives the associativity for the quantum cohomology which is defined by the third derivatives of $F_0$ and $\eta^{-1}$ (see Section 1.7). On the big phase space, this equation is the key ingredient in the proof of the genus-0 Virasoro conjecture (cf. [LT]).

The genus-1 topological recursion relation is the following:

$$\langle\langle \tau_{m+1,\alpha} \rangle\rangle_1 = \sum_\sigma \langle\langle \tau_{m,\alpha} \gamma_\sigma \rangle\rangle_0 \langle\langle \gamma_\sigma \rangle\rangle_1 + \frac{1}{24} \sum_\sigma \langle\langle \tau_{m,\alpha} \gamma_\sigma \gamma_\sigma \rangle\rangle_0.$$ (9)

This formula implies the genus-1 constitutive relation [DW]

$$F_1 = \langle e \sum_\alpha u^\alpha \gamma_1 \rangle_1 + \frac{1}{24} \log \det \left( \frac{\partial u^\alpha}{\partial t_0^\beta} \right),$$ (10)

where $u^\alpha = \langle\langle \gamma_1 \gamma_\alpha \rangle\rangle_0$.

### 1.4 Some special vector fields on the big phase space

In [LT], we introduced several special vector fields on the big phase space. These vector fields played very important role in the proof of the genus-0 Virasoro conjecture. The first one is the string vector field:

$$\mathcal{S} := -\sum_{m,\alpha} \tilde{t}_m^{\alpha} \tau_{m-1,\alpha}.$$ 

The restriction of $\mathcal{S}$ to the small phase space is just $\gamma_1$. The famous string equation (cf. [RT2] and [W2]) can be expressed as

$$\langle\langle \mathcal{S} \rangle\rangle_g = \frac{1}{2} \delta_{g,0} \sum_{a,\beta} \eta_{a,\beta} t_0^{\alpha_0} t_0^{\beta_0}.$$ 

This equation is equivalent to Eguchi, Hori, and Xiong’s $L_{-1}$ constraint.

The second vector field is the Dilaton vector field:

$$\mathcal{D} := -\sum_{m,\alpha} \tilde{t}_m^{\alpha} \tau_{m,\alpha}.$$ 

When restricted to the small phase space, this vector field does not tangent to the small phase space. The so called dilaton equation is the following:

$$\langle\langle \mathcal{D} \rangle\rangle_g = (2g - 2) F_g + \frac{1}{24} \chi(V) \delta_{g,1},$$
where $\chi(V)$ is the Euler characteristic number of $V$. This equation implies the following (Lemma 1.2 in [LT]):

$$\langle \langle D \tau_{m,\alpha} \rangle \rangle_0 = - \langle \langle \tau_{m,\alpha} \rangle \rangle_0, \quad \text{and} \quad \langle \langle D \tau_{m,\alpha} \tau_{n,\beta} \rangle \rangle_0 \equiv 0. \quad (11)$$

Of particular importance is the following vector field:

$$\mathcal{X} := - \sum_{m,\alpha} (m + b_\alpha - b_1 - 1) \bar{r}_m \tau_{m,\alpha} - \sum_{m,\alpha,\beta} C^\beta_{m \alpha} \bar{r}_m \tau_{m-1,\beta}.$$

When restricted to the small phase space, this vector field is the Euler vector field $E$ mentioned in the introduction. Therefore we also call $\mathcal{X}$ itself the Euler vector field $E$ (on the big phase space). As noted in [EHX1], the divisor equation for the first Chern class $c_1(V)$ together with the selection rule implies the following quasi-homogeneity equation:

$$\langle \langle \mathcal{X} \rangle \rangle_g = 2(b_1 + 1)(1 - g)F_g + \frac{1}{2} \delta_{g,0} \sum_{\alpha,\beta} C_{\alpha \beta} t^\alpha_0 t^\beta_0 - \frac{1}{24} \delta_{g,1} \int_V c_1(V) \cup c_{d-1}(V),$$

where $d$ is the complex dimension of $V$ and $c_i$ is the $i$-th Chern class. This equation implies the following (Lemma 1.4 in [LT])

**Lemma 1.1**

(i) \quad \langle \langle \mathcal{X} \rangle \rangle_0 = 2(b_1 + 1)F_0 + \frac{1}{2} \sum_{\alpha,\beta} C_{\alpha \beta} t^\alpha_0 t^\beta_0.

(ii) \quad \langle \langle \mathcal{X} \tau_{m,\alpha} \rangle \rangle_0 = (m + b_\alpha + b_1 + 1) \langle \langle \tau_{m,\alpha} \rangle \rangle_0 + \sum_{\beta} C^\beta_{\alpha} \langle \langle \tau_{m-1,\beta} \rangle \rangle_0 + \delta_{m,0} \sum_{\beta} C_{\alpha \beta} t^\beta_0.

(iii) \quad \langle \langle \mathcal{X} \tau_{m,\alpha} \tau_{n,\beta} \rangle \rangle_0 = \delta_{m,0} \delta_{n,0} C_{\alpha \beta} + (m + n + b_\alpha + b_\beta) \langle \langle \tau_{m,\alpha} \tau_{n,\beta} \rangle \rangle_0

\quad \quad + \sum_{\mu} C^\mu_{\alpha} \langle \langle \tau_{m-1,\mu} \tau_{n,\beta} \rangle \rangle_0 + \sum_{\mu} C^\mu_{\beta} \langle \langle \tau_{m,\alpha} \tau_{n-1,\mu} \rangle \rangle_0.

In [LT], we also introduced a sequence of vector fields $\mathcal{L}_n$ which are the first derivative part of the Virasoro operators. The first four vector fields are

$$\mathcal{L}_{-1} := -S,$$

$$\mathcal{L}_0 := -\mathcal{X} - (b_1 + 1)D,$$

$$\mathcal{L}_1 := \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)\bar{r}_m \tau_{m+1,\alpha}$$

\quad \quad + \sum_{m,\alpha,\beta} (2m + 2b_\alpha + 1)C^\beta_{m \alpha} \bar{r}_m \tau_{m,\beta} + \sum_{m,\alpha,\beta} (C^2)^\beta_{m \alpha} \bar{r}_m \tau_{m-1,\beta}$$

$$\mathcal{L}_2 := \sum_{m,\alpha} (m + b_\alpha)(m + b_\alpha + 1)(m + b_\alpha + 2)\bar{r}_m \tau_{m+2,\alpha}$$

\quad \quad + \sum_{m,\alpha,\beta} \left\{3(m + b_\alpha)^2 + 6(m + b_\alpha) + 2\right\} C^\beta_{m \alpha} \bar{r}_m \tau_{m+1,\beta}

\quad \quad + \sum_{m,\alpha,\beta} 3(m + b_\alpha + 1)(C^2)^\beta_{m \alpha} \bar{r}_m \tau_{m,\beta} + \sum_{m,\alpha,\beta} (C^3)^\beta_{m \alpha} \bar{r}_m \tau_{m-1,\beta} \quad (12)$$
The following formulas were proved in [LT] and will be used later:

\[
\begin{align*}
\langle\langle \gamma_{\mu} L_0 \gamma_{\nu} \rangle\rangle_0 &= - \langle\langle \gamma_{\mu} X_{\gamma_{\nu}} \rangle\rangle_0 \\
\langle\langle \gamma_{\mu} L_1 \gamma_{\nu} \rangle\rangle_0 &= - \sum_{\alpha} \langle\langle \gamma_{\mu} X_{\gamma_{\alpha}} \rangle\rangle_0 \langle\langle \gamma^\alpha X_{\gamma_{\nu}} \rangle\rangle_0 \\
&+ \sum_{\alpha} b_{\alpha}(b_{\alpha} - 1) \langle\langle \gamma_{\alpha} \rangle\rangle_0 \langle\langle \gamma^\alpha \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
\langle\langle \gamma_{\mu} L_2 \gamma_{\nu} \rangle\rangle_0 &= - \sum_{\alpha,\beta} \langle\langle \gamma_{\mu} X_{\gamma_{\alpha}} \rangle\rangle_0 \langle\langle \gamma^\alpha X_{\gamma_{\beta}} \rangle\rangle_0 \langle\langle \gamma^\beta X_{\gamma_{\nu}} \rangle\rangle_0 \\
&+ \sum_{\alpha,\beta} b_{\alpha}(b_{\alpha} - 1) \langle\langle \gamma_{\alpha} \rangle\rangle_0 \langle\langle \gamma^\alpha \gamma^\beta \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
&+ \sum_{\alpha} (3b_{\beta}^2 - 1) c_{\beta} \langle\langle \gamma_{\alpha} \rangle\rangle_0 \langle\langle \gamma^\beta \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
&- \sum_{\alpha,\beta} (b_{\alpha} - 1)b_{\alpha}(b_{\beta} - 1) \langle\langle \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_{\alpha} \gamma_{\beta} \rangle\rangle_0 \langle\langle \gamma^\beta \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
&- \sum_{\alpha,\beta} b_{\beta}(b_{\beta} + 1) c_{\beta,\alpha} \langle\langle \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma^\beta \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0.
\end{align*}
\]

Due to Lemma 1.2 (3) in [LT], the first formula is just the definition of $L_0$. The second formula is a special case of the formula (19) in [LT] plus the generalized WDVV equation. The third formula is a special case of the formula (26) in [LT] plus the generalized WDVV equation and the second formula. Together with the obvious relation that the restriction of $L_{-1}$ to the small phase space is $-E^0$, these formulas reveal an interesting relationship between the Virasoro operators and the quantum powers of the Euler vector fields. In fact, when restricted to the small phase space, the first lines of the right hand sides of the above equations are respectively $-\langle\langle \gamma_{\mu} E_{\gamma_{\nu}} \rangle\rangle_0$, $-\langle\langle \gamma_{\mu} E^2_{\gamma_{\nu}} \rangle\rangle_0$, $-\langle\langle \gamma_{\mu} E^3_{\gamma_{\nu}} \rangle\rangle_0$. With a slight modification of $L_n$, the extra terms on the right hand sides of the above equations may disappear. This can be done by simply moving the extra terms to the left hand sides, expressing them as 3-point functions with two arguments equal to $\gamma_{\mu}$ and $\gamma_{\nu}$, then adding the third arguments (which are again vector fields) to the corresponding $L_n$’s (see also [G2]). We note here that for the second term on the right hand side of the third equation, we can interchange the position of $\gamma_{\mu}$ and $X$ (by the generalized WDVV equation), then using Lemma [LT] (iii) to remove $X$. The third equation can then be simplified as

\[
\begin{align*}
\langle\langle \gamma_{\mu} L_2 \gamma_{\nu} \rangle\rangle_0 &= - \sum_{\alpha,\beta} \langle\langle \gamma_{\mu} X_{\gamma_{\alpha}} \rangle\rangle_0 \langle\langle \gamma^\alpha X_{\gamma_{\beta}} \rangle\rangle_0 \langle\langle \gamma^\beta X_{\gamma_{\nu}} \rangle\rangle_0 \\
&+ \sum_{\alpha} (b_{\alpha} - 1)b_{\alpha}(b_{\alpha} + 1) \langle\langle \tau_{1,\alpha} \rangle\rangle_0 \langle\langle \gamma^\alpha \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
&+ \sum_{\alpha} (b_{\alpha} - 1)b_{\alpha}(b_{\alpha} + 1) \langle\langle \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_{\alpha} \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0 \\
&+ \sum_{\alpha,\beta} (3b_{\beta}^2 - 1) c_{\beta} \langle\langle \gamma_{\alpha} \rangle\rangle_0 \langle\langle \gamma^\beta \gamma_{\mu} \gamma_{\nu} \rangle\rangle_0.
\end{align*}
\]
1.5 Virasoro operators

The first four Virasoro operators constructed by Eguchi, Hori, and Xiong are the following

\[ L_{-1} := L_{-1} + \frac{1}{2\lambda^2} \sum_{a,\beta} \eta_{a\beta} t_0^a t_0^\beta, \]

\[ L_0 := L_0 + \frac{1}{2\lambda^2} \sum_{a,\beta} C_{a\beta} t_0^a t_0^\beta + \frac{1}{24} \left( (b_1 + 1) \chi(V) - \int_V c_1(V) \cup c_{d-1}(V) \right), \]

\[ L_1 := L_1 + \frac{\lambda^2}{2} \sum_{a} b_a (1 - b_a) \gamma_a \gamma^a + \frac{\lambda^2}{2\lambda^2} \sum_{a,\beta} (C^2)_{a\beta} t_0^a t_0^\beta \]

\[ L_2 := L_2 - \lambda^2 \sum_{a} (b_a - 1) b_a (b_a + 1) \tau_{1,a} \gamma^a - \frac{\lambda^2}{2} \sum_{a,\beta} (3b_a^2 - 1) C_{a\beta} \gamma^a \gamma^\alpha \]

\[ + \frac{1}{2\lambda^2} \sum_{a,\beta} (C^3)_{a\beta} t_0^a t_0^\beta. \] (15)

Because of the Virasoro relation

\[ [L_m, L_n] = (m - n)L_{m+n}, \]

for \( m, n \geq -1 \), the above operators generate all \( L_n \) operators with \( n \geq -1 \). We will not consider \( L_n \) operators with \( n < -1 \) in this paper. Recall that the \( L_{-1} \)-constraint is equivalent to the string equation, which is valid for all manifolds. Due to the above Virasoro bracket relation, to prove the Virasoro conjecture, it suffices to prove the \( L_2 \)-constraint.

1.6 Review of the genus-0 Virasoro conjecture

In [EHX2], a heuristic argument for deriving the genus-0 constraints for \( \mathbb{CP}^n \) was given. It was pointed out in [LT] that there is a serious gap in this derivation. This observation was confirmed by conversations between authors of [LT] and authors of [EHX2] both before and after the paper [LT] was written.

The first complete proof of the genus-0 Virasoro conjecture was given in [LT]. Actually the conjecture posed in [EHX2] was only for Fano varieties with vanishing Hodge numbers \( h^{p,q}(V, \mathbb{C}) \) for \( p \neq q \) (cf. [Bo1]). This was later improved to cover all compact smooth Kähler manifolds in [EJX] after the Virasoro operators were modified according to a suggestion of S. Katz. It was pointed out for the first time in [LT] that the genus-0 Virasoro conjecture does not need any assumption on manifolds, i.e. it is also valid for all compact symplectic manifolds (The modification suggested by Katz does not apply to general symplectic manifolds since it involves holomorphic dimensions of cohomology classes). Besides the two constraints known before, i.e. \( L_{-1} \) and \( L_0 \) constraints, the key ingredients used in [LT] were the genus-0 topological recursion relation and the generalized WDVV equation (those equations are also valid for all Frobenius manifolds [Du]). More precisely, we computed the following expression

\[ \sum_{\sigma} \langle \langle L_n (L_0 - (n + 1)D) \gamma_{\sigma} \rangle \rangle_0 \langle \langle \gamma^\sigma \tau_{k,\mu} \tau_{l,\nu} \rangle \rangle_0 = \langle \langle L_n \tau_{k,\mu} \gamma_{\sigma} \rangle \rangle_0 \langle \langle \gamma^\sigma (L_0 - (n + 1)D) \tau_{l,\nu} \rangle \rangle_0. \]
On the one hand, by the generalized WDVV equation, this expression is 0. On the other hand, if the $L_n$ constraint is correct, we can compute each 3-point function in the above expression separately, and when combining the results together and using the genus-0 topological recursion relation, we can show that this expression is just $\partial_{\nu_l} \partial_{\mu_k} \Psi_{0,n+1}$. Once we know that all second derivatives of $\Psi_{0,n+1}$ are zero, the dilaton equation then trivially implies that $\Psi_{0,n+1} = 0$. This gives an inductive proof to the genus-0 Virasoro conjecture. The key point in this proof is to observe the above relation between the generalized WDVV equation and the Virasoro conjecture. Once this relation is observed, the computation involved are quite straightforward, although it is a little tedious. In Section 3 and 4 of [LT], we gave the full details of the computations. The advantage for doing so instead of giving a more concise presentation is that one can see clearly how each term of these complicated operators evolves during this process. In particular, one can see how terms of $L_{n+1}$ emerge from the above manipulation of expressions involving only $L_n$ and $D$.

In [LT], the same method was also used to give the first proof to another sequence of genus-0 constraints, called $\tilde{L}_n$ constraints, which were also conjectured in [EHX2]. Note that in the derivation of [EHX2], the two sequences of constraints ($\{L_n\}$ and $\{\tilde{L}_n\}$ constraints) are always mingled together. It is not clear how to separate these two sequences using the original arguments in [EHX2]. To complete the proof along the original lines of [EHX2], one needs to prove $L_1$ and $L_2$ constraints first (In the presentation of [G2], it is not clear what kind of role the $\tilde{L}_2$ constraint plays, while in [EHX2] this constraint was mixed with the $L_2$ constraint.). It was noticed for the first time in [LT] that these two sequences can be treated completely independently. The method for proving them are the same. If one knows how to prove one sequence, one also knows how to handle another one.

After [LT] was submitted to journal and posted on the web, an alternative proof to the genus-0 Virasoro conjecture was given in [DZ2]. In fact, the genus-0 Virasoro constraints were extended in [DZ2] to the setting of abstract Frobenius manifolds, which are defined by solutions of the WDVV equation and also by axiomizing basic properties of the quantum cohomology. Since in genera bigger than 1, the corresponding constitutive relations do not exist yet, it is not clear how to define the analogue of $F_\beta$ for abstract Frobenius manifolds. Therefore it is not clear how to interpret the full Virasoro conjecture for this setting. The third proof to the genus-0 Virasoro conjecture was given in [G2] by combining arguments in [EHX2] and [DZ2].

### 1.7 Quantum cohomology

At each point of the small phase space, which is identified with $H^*(V, \mathbb{C})$, we can define a new product structure among cohomology classes, called the Quantum product, in the following way:

$$\gamma_\alpha \bullet \gamma_\beta = \sum_\sigma \langle \langle \gamma_\alpha \gamma_\beta \gamma_\sigma \rangle \rangle_{0,s} \gamma_\sigma.$$  

This product is commutative and associative (due to the WDVV equation). In this way, we obtain new ring structures on $H^*(V, \mathbb{C})$, which are called quantum cohomologies of $V$.  

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Since the restriction of the string vector field to the small phase space is $\gamma_1$, the string equation implies the following

**Lemma 1.2**

$$\langle\langle \gamma_1 \gamma_\alpha \rangle\rangle_{0,s} = \eta_{\alpha\beta}, \quad \text{and} \quad \langle\langle \gamma_1 \gamma_{\mu_1} \cdots \gamma_{\mu_k} \rangle\rangle_{0,s} = 0 \quad \text{if} \quad k \geq 3.$$  

Especially the first equation in the lemma tells us that $\gamma_1$ is always the identity of the quantum cohomology no matter which point in the small phase space is chosen.

Since $H^*(V, \mathbb{C})$ is a linear space, we can identify tangent spaces of $H^*(V, \mathbb{C})$ with $H^*(V, \mathbb{C})$ itself. Therefore we can take quantum product for any two vector fields on $H^*(V, \mathbb{C})$. The intersection form $\eta$ defines a flat metric (non-Riemannian) on $H^*(V, \mathbb{C})$.

Let $\nabla$ be the corresponding Levi-Civita connection. It is straightforward to verify the following

$$u \langle\langle v_1 \cdots v_k \rangle\rangle_{g,s} = \langle\langle uv_1 \cdots v_k \rangle\rangle_{g,s} + \sum_{i=1}^k \langle\langle v_1 \cdots (\nabla_u v_i) \cdots v_k \rangle\rangle_{g,s},$$  

for any vector fields $u$, and $v_1, \ldots, v_k$ on the small phase space. A simple application of this formula is the following

$$\nabla_u (v \cdot w) = (\nabla_u v) \cdot w + v \cdot (\nabla_u w) + \sum_\alpha \langle\langle uvw^\alpha \rangle\rangle_{0,s} \gamma_\alpha,$$  

for any vector fields $u$, $v$, and $w$ on the small phase space.

The most important vector field on the small phase space is the Euler vector field $E$ defined in the introduction. It is the restriction to the small phase space of the vector field $X$ defined in Section 1.4. Therefore Lemma 1.1 implies the following

**Lemma 1.3**

(i) $\langle\langle E \rangle\rangle_{0,s} = 2(b_1 + 1)F_0^s + \frac{1}{2} \sum_{\alpha,\beta} C_{\alpha\beta} t^\alpha t^\beta.$

(ii) $\langle\langle E \gamma_\alpha \rangle\rangle_{0,s} = (b_\alpha + b_1 + 1) \langle\langle \gamma_\alpha \rangle\rangle_{0,s} + \sum_{\beta} C_{\alpha\beta} t^\beta.$

(iii) $\langle\langle E \gamma_\alpha \gamma_\beta \rangle\rangle_{0,s} = C_{\alpha\beta} + (b_\alpha + b_\beta) \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_{0,s}.$

(iv) $\langle\langle E \gamma_\alpha \gamma_\beta \gamma_\mu \rangle\rangle_{0,s} = (b_\alpha + b_\beta + b_\mu - b_1 - 1) \langle\langle \gamma_\alpha \gamma_\beta \gamma_\mu \rangle\rangle_{0,s}.$

(v) $\langle\langle E \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \rangle\rangle_{0,s} = (b_\alpha + b_\beta + b_\mu + b_\nu - 2b_1 - 2) \langle\langle \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \rangle\rangle_{0,s}.$

A simple application of the third formula in this lemma is the following:

$$v \langle\langle E \gamma_\alpha \gamma_\beta \rangle\rangle_{0,s} = (b_\alpha + b_\beta) \langle\langle v \gamma_\alpha \gamma_\beta \rangle\rangle_{0,s},$$  

where $v$ is any vector field on the small phase space. Let $E^i$ be the $i$-th quantum power of $E$. Then

$$\langle\langle \gamma^\alpha E^i \gamma_\beta \rangle\rangle_{0,s} = \sum_{\mu_1, \ldots, \mu_{i-1}} \prod_{j=1}^i \langle\langle \gamma^{\mu_{j-1}} E \gamma_\mu \rangle\rangle_{0,s},$$  

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where $\mu_0 = \alpha$ and $\mu_i = \beta$. Applying equation (18) to each factor, we obtain

$$E^k \langle \langle \gamma^\alpha E^i \gamma^\beta \rangle \rangle_{0,s}$$

$$= (b_\beta - b_\alpha + i) \langle \langle \gamma^\alpha E^{k+i-1} \gamma^\beta \rangle \rangle_{0,s}$$

$$- \sum_{j=1}^{\min\{i,k\}-1} \sum_{\mu} b_\mu \langle \langle \gamma^\alpha E^j \gamma^\mu \rangle \rangle_{0,s} \langle \langle \gamma^\mu E^{k+i-1-j} \gamma^\beta \rangle \rangle_{0,s}$$

$$+ \sum_{j=1}^{\min\{i,k\}-1} \sum_{\mu} b_\mu \langle \langle \gamma^\alpha E^{k+i-1-j} \gamma^\mu \rangle \rangle_{0,s} \langle \langle \gamma^\mu E^j \gamma^\beta \rangle \rangle_{0,s},$$

for $i, k \geq 1$. For convenience, we will write

$$E^k = \sum_\alpha x^\alpha_k \gamma^\alpha, \quad \text{where} \quad x^\alpha_k = \langle \langle \gamma_1 E^k \gamma^\alpha \rangle \rangle_{0,s}.$$  

(20)

Since $[E^k, E^m] = \sum_\alpha (E^k x^\alpha_m - E^m x^\alpha_k) \gamma^\alpha$, a simple application of equation (19) proves

$$[E^k, E^m] = (m - k) E^{m+k-1}$$

for $m, k \geq 1$. If one of $m$ and $k$ is equal to 0, the corresponding formula follows from equation (17) and Lemma 1.2 since $[\gamma_1, E^k] = \nabla_{\gamma_1} E^k$. This gives a simple proof to equation (I).

2 Relations between genus-0 and genus-1 data

In this section, we will study how much genus-1 information can be obtained from genus-0 data. In particular, we will prove Theorem 0.1. We first define two symmetric 4-tensors $G_0$ and $G_1$ on the small phase space. Let $S_4$ be the permutation group of 4 elements which acts on the set $\{1, 2, 3, 4\}$. For any vector fields $v_1, \ldots v_4$ on the small phase space, we define

$$G_0(v_1, v_2, v_3, v_4) = \sum_{g \in S_4} \sum_\alpha, \beta \left\{ \frac{1}{6} \langle \langle v_{g(1)} v_{g(2)} v_{g(3)} \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma_\alpha v_{g(4)} \gamma^\beta \gamma^\beta \rangle \rangle_{0,s} \right. $$

$$+ \frac{1}{24} \langle \langle v_{g(1)} v_{g(2)} v_{g(3)} \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma_\alpha \gamma_\beta \gamma^\beta \rangle \rangle_{0,s}$$

$$- \frac{1}{4} \langle \langle v_{g(1)} v_{g(2)} \gamma^\alpha \gamma^\beta \rangle \rangle_{0,s} \langle \langle \gamma_\alpha \gamma_\beta v_{g(3)} v_{g(4)} \rangle \rangle_{0,s} \right\},$$

and

$$G_1(v_1, v_2, v_3, v_4) = \sum_{g \in S_4} 3 \langle \langle \{v_{g(1)} \cdot v_{g(2)}\} \{v_{g(3)} \cdot v_{g(4)}\} \rangle \rangle_{1,s}$$

$$- \sum_{g \in S_4} 4 \langle \langle \{v_{g(1)} \cdot v_{g(2)}\} \{v_{g(3)} \cdot v_{g(4)}\} \rangle \rangle_{1,s}.$$
\[-\sum_{g \in S_4} \sum_{\alpha} \left\langle \left\langle \{v_{g(1)} \cdot v_{g(2)}\} v_{g(3)} v_{g(4)}^\alpha \right\rangle \right\rangle_{0,s} \langle \langle \gamma_\alpha \rangle \rangle_{1,s} + \sum_{g \in S_4} \sum_{\alpha} 2 \left\langle \left\langle v_{g(1)} v_{g(2)} v_{g(3)}^\alpha \right\rangle \right\rangle_{0,s} \langle \langle \{\gamma_\alpha \cdot v_{g(4)}\} \rangle \rangle_{1,s} \cdot \]

Note that \( G_0 \) is determined solely by genus-0 data, while each term in \( G_1 \) contains genus-1 information. These two tensors are connected by the following equation:

\[ G_0 + G_1 = 0. \]

This equation was proved in \([G1]\) where it was written in a different form. The above formulation is a slight modification of the one given in \([DZ1]\). We first study the function \( G_1 \).

**Proposition 2.1**

\[ G_1(v_1, v_2, v_3, v_4) = \sum_{g \in S_4} \left\{ 3 \{v_{g(1)} \cdot v_{g(2)}\} \left\langle \left\langle \{v_{g(3)} \cdot v_{g(4)}\} \right\rangle \right\rangle_{1,s} - 4 v_{g(4)} \left\langle \left\langle \{v_{g(1)} \cdot v_{g(2)} \cdot v_{g(3)}\} \right\rangle \right\rangle_{1,s} - 6 \left\langle \left\langle \{v_{g(1)} \cdot v_{g(2)}, v_{g(3)} \cdot v_{g(4)}\} \right\rangle \right\rangle_{1,s} \right\}. \]

**Proof:** Using equation (16) and (17) to compute \( \sum_{g \in S_4} 3 \{v_{g(1)} \cdot v_{g(2)}\} \left\langle \left\langle \{v_{g(3)} \cdot v_{g(4)}\} \right\rangle \right\rangle_{1,s} \) and \( - \sum_{g \in S_4} 4 v_{g(4)} \left\langle \left\langle \{v_{g(1)} \cdot v_{g(2)} \cdot v_{g(3)}\} \right\rangle \right\rangle_{1,s} \), then combining the results together and using the symmetry of the tensors, we obtain the desired formula. \( \Box \)

Applying this proposition to quantum powers of the Euler vector field \( E \) and using equation (1), we obtain the following

**Corollary 2.2**

\[ \frac{1}{24} G_1(E^{m_1}, E^{m_2}, E^{m_3}, E^{m_4}) = (2m_1 + m) \left\langle \left\langle E^{m-1} \right\rangle \right\rangle_{1,s} + \sum_{i=2}^{4} E^{m_1 + m_i} \left\langle \left\langle E^{m-m_1-m_i} \right\rangle \right\rangle_{1,s} - \sum_{i=1}^{4} E^{m_i} \left\langle \left\langle E^{m-m_i} \right\rangle \right\rangle_{1,s}, \]

where \( m_1, \ldots, m_4 \) are arbitrary non-negative integers and \( m = m_1 + m_2 + m_3 + m_4 \).

In the rest of this paper, we will use the following simple formulas without mentioning:

**Lemma 2.3**

(i) \( \left\langle \left\langle E^0 \right\rangle \right\rangle_{1,s} = 0 \),

(ii) \( \left\langle \left\langle E \right\rangle \right\rangle_{1,s} = -\frac{1}{24} \int_V c_1(V) \cup c_{d-1}(V) \),

(iii) \( E^0 \left\langle \left\langle E^m \right\rangle \right\rangle_{1,s} = m \left\langle \left\langle E^{m-1} \right\rangle \right\rangle_{1,s} \),

(iv) \( E \left\langle \left\langle E^m \right\rangle \right\rangle_{1,s} = (m - 1) \left\langle \left\langle E^m \right\rangle \right\rangle_{1,s} \),

for any non-negative integer \( m \).
**Proof:** The first two equations are the restrictions of the genus-1 string equation and quasi-homogeneity equation to the small phase space respectively. The last two equations follows from the first two equations and equation (1). □

A special case of the Corollary 2.2 is the following

\[
\frac{1}{24} G_1(E^{m-2-i}, E^i, E, E) = E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - E^{m-i-2} \langle \langle E^{i+2} \rangle \rangle_{1,s} \\
+ 2E^{m-i-1} \langle \langle E^{i+1} \rangle \rangle_{1,s} - E^{m-i} \langle \langle E^i \rangle \rangle_{1,s}
\]

(22)

for \(1 \leq i \leq \left[ \frac{m}{2} \right] - 2\) where \(\left[ \frac{m}{2} \right]\) is the largest integer which is less than or equal to \(\frac{m}{2}\). If \(m\) is even, Corollary 2.2 implies

\[
\frac{1}{48} G_1(E^{m/2-1}, E^{m/2-1}, E, E) = \frac{1}{2} E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - E^{m/2+1} \langle \langle E^{m/2-1} \rangle \rangle_{1,s} \\
+ E^{m/2} \langle \langle E^{m/2} \rangle \rangle_{1,s} - E^{E^{m-1}} \langle \langle E^{m-1} \rangle \rangle_{1,s}
\]

(23)

Summing up equation (22) over \(1 \leq i \leq \left[ \frac{m}{2} \right] - 2\) and adding the above equation, we obtain

\[
\frac{1}{48} G_1(E^{m/2-1}, E^{m/2-1}, E, E) + \sum_{i=1}^{m/2-2} \frac{1}{24} G_1(E^{m-2-i}, E^i, E, E)
\]

\[
= \frac{m-1}{2} E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - \langle \langle E^{m-1} \rangle \rangle_{1,s}
\]

(23)

when \(m\) is an even integer. If \(m\) is odd, Corollary 2.2 implies

\[
\frac{1}{24} G_1(E^{(m-1)/2}, E^{(m-3)/2}, E, E) = E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - E^{(m+3)/2} \langle \langle E^{(m-3)/2} \rangle \rangle_{1,s} \\
+ E^{(m+1)/2} \langle \langle E^{(m-1)/2} \rangle \rangle_{1,s} - \langle \langle E^{m-1} \rangle \rangle_{1,s}
\]

(24)

Summing up equation (22) over \(1 \leq i \leq \left[ \frac{m}{2} \right] - 2\) and adding the above equation, we obtain

\[
\sum_{i=1}^{(m-3)/2} \frac{1}{24} G_1(E^{m-2-i}, E^i, E, E) = \frac{m-1}{2} E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - \langle \langle E^{m-1} \rangle \rangle_{1,s}
\]

(24)

when \(m\) is an odd integer. Using the symmetry of the tensor \(G_1\), we can express equation (23) and (24) in a unified form, which together with equation (21) implies the following

**Theorem 2.4** For an arbitrary manifold \(V\),

\[
\frac{m-1}{2} E^{m-2} \langle \langle E^2 \rangle \rangle_{1,s} - \langle \langle E^{m-1} \rangle \rangle_{1,s} = - \sum_{i=1}^{m-3} \frac{1}{48} G_0(E^{m-2-i}, E^i, E, E)
\]

for any integer \(m \geq 2\).

Since \(G_0\) is defined by derivatives of \(F_0^s\), this theorem in particular implies Theorem 0.1.
3 A sequence of genus-0 functions

Theorem 2.4 tells us that for \( k \geq 3 \), \( \langle \langle E^k \rangle \rangle_{1,s} \) can be computed in terms of \( \langle \langle E^2 \rangle \rangle_{1,s} \) and some genus-0 data. We will see later that the restriction of the genus-1 \( L_1 \) constraint to the small phase space is equivalent to \( \langle \langle E^2 \rangle \rangle_{1,s} = \phi_2 \) where \( \phi_2 \) is defined in (2). We can rewrite \( \phi_2 \) in the following form:

\[
\phi_2 = -\frac{1}{24} \sum_\alpha \langle \langle EE\gamma^\alpha \gamma^\alpha \rangle \rangle_{0,s} + \frac{1}{2} \sum_\alpha \left( b_\alpha (1 - b_\alpha) - \frac{b_1 + 1}{6} \right) \langle \langle \gamma^\alpha \gamma^\alpha \rangle \rangle_{0,s}.
\]

(25)

Motivated by Theorem 2.4, we define

\[
\phi_k := \frac{k}{2} E^{k-1} \phi_2 + \sum_{i=1}^{k-2} \frac{1}{48} G_0(E^{k-1-i}, E^i, E, E),
\]

(26)

for \( k \geq 3 \). For convenience, we also define

\[
\phi_0 := 0, \text{ and } \phi_1 := -\frac{1}{24} \int_V c_1(V) \cup c_{d-1}(V).
\]

(27)

The string equation and the quasi-homogeneity equation implies

\[
\phi_0 = \langle \langle E^0 \rangle \rangle_{1,s}, \text{ and } \phi_1 = \langle \langle E \rangle \rangle_{1,s}.
\]

An immediate consequence of Theorem 2.4 is the following

**Theorem 3.1** For any manifold \( V \), if \( \langle \langle E^2 \rangle \rangle_{1,s} = \phi_2 \), then \( \langle \langle E^k \rangle \rangle_{1,s} = \phi_k \) for every \( k \).

The definition of \( \phi_k \) given by (26) is hard to use. For the convenience of later applications, we will give another equivalent formulation in Theorem 3.9. Before proving Theorem 3.9, we need some preparations. First, taking derivatives of the WDVV equation twice and three times, we obtain the following

**Lemma 3.2** For any vector fields \( u, v, w_i \) on the small phase space, we have

(i) \[
\sum_\alpha \langle \langle uw_1 w_2 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha vw_3 \rangle \rangle_{0,s} + \sum_\alpha \langle \langle uw_1 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha vw_2 w_3 \rangle \rangle_{0,s} = \sum_\alpha \langle \langle vw_1 w_2 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha uw_3 \rangle \rangle_{0,s} + \sum_\alpha \langle \langle vw_1 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha uw_2 w_3 \rangle \rangle_{0,s},
\]

(ii) \[
\sum_\alpha \langle \langle uw_1 w_2 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha vw_3 w_4 \rangle \rangle_{0,s} + \sum_\alpha \langle \langle uw_1 w_3 \gamma^\alpha \rangle \rangle_{0,s} \langle \langle \gamma^\alpha vw_2 w_4 \rangle \rangle_{0,s}
\]
We can use these formulas and the WDVV equation to exchange positions of two vector fields in a product of two correlation functions. Using this lemma, we can prove the following.

**Lemma 3.3** For any \( \mu \) and \( \nu \),

\[
G_0(\gamma_\mu, \gamma_\nu, E, E) = \sum_\beta \left\langle \left\langle EE(\gamma_\mu \cdot \gamma_\nu)\gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} + \sum_\alpha (2b_\beta - b_\alpha + b_\mu - 1) \left\langle \left\langle \gamma_\mu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\nu \right\rangle \right\rangle_{0,s} + \sum_\alpha (2b_\beta - b_\alpha + b_\nu - 1) \left\langle \left\langle \gamma_\nu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s} + \sum_\alpha (b_\alpha - b_\beta)(-4b_\beta - 2b_\alpha + 2b_1 + 4) \left\langle \left\langle \gamma_\mu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s} + \sum_\alpha \{(-4b_\beta - 2b_\alpha + b_\mu - b_\nu + 2)(b_\mu - b_\nu - 2b_\alpha - 2b_\beta + 2) - 2(b_\nu + b_\alpha + b_\beta - b_1 - 1)\} \left\langle \left\langle \gamma_\mu \gamma_\alpha \gamma_\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\nu \right\rangle \right\rangle_{0,s}.
\]

**Proof:** Applying Lemma 3.2 (ii) with \( u = E, v = \gamma^\beta, w_1 = E, w_2 = \gamma_\mu, w_3 = \gamma_\nu \) and \( w_4 = \gamma_\beta \) to the expression

\[
\sum_\alpha \left\langle \left\langle \gamma_\mu EE \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} + \left\langle \left\langle \gamma_\mu EE \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\mu \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} + \left\langle \left\langle \gamma_\mu \gamma_\nu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s},
\]

then applying Lemma 3.2 (ii) again with \( u = E, v = \gamma_\mu, w_1 = E, w_2 = \gamma_\beta, w_3 = \gamma^\beta \) and \( w_4 = \gamma_\nu \) to the expression

\[
- \left\{ \sum_\alpha \left\langle \langle EE \gamma_\alpha \rangle \right\rangle_{0,s} \left\langle \langle \gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\beta \gamma_\beta \rangle \right\rangle_{0,s} + 2 \left\langle \langle EE \gamma_\beta \gamma_\beta \rangle \right\rangle_{0,s} \left\langle \langle \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \rangle \right\rangle_{0,s} \right\},
\]

18
after plugging the corresponding results into the definition of $G_0(\gamma_\mu, \gamma_\nu, E, E)$ and using Lemma 1.3 to 4-point and 5-point functions which involve only one $E$; we obtain

$$
G_0(\gamma_\mu, \gamma_\nu, E, E) = \sum_\beta \left\langle \left\langle EE(\gamma_\mu \bullet \gamma_\nu)\gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}
$$

$$
- \sum_{\alpha,\beta} (b_\alpha + b_\nu - 2b_1 - 1) \left\langle \left\langle \gamma_\mu E\gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma^\beta \gamma_\nu \right\rangle \right\rangle_{0,s}
$$

$$
- \sum_{\alpha,\beta} (b_\alpha + b_\mu - 2b_1 - 1) \left\langle \left\langle \gamma_\nu E\gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma^\beta \gamma_\mu \right\rangle \right\rangle_{0,s}
$$

$$
+ \sum_{\alpha,\beta} 2(2b_\beta + b_\mu + b_\nu - 2b_1 - 2) \left\langle \left\langle E\gamma^\beta \gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \right\rangle \right\rangle_{0,s}
$$

$$
+ \sum_{\alpha,\beta} 2(b_\alpha - b_1)(b_\mu + b_\nu - b_\alpha - b_1) \left\langle \left\langle \gamma_\mu \gamma_\nu \gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}
$$

$$
- \sum_{\alpha,\beta} 4(b_\mu - b_\alpha - b_\beta - b_1 + 1)(b_\nu + b_\alpha + b_\beta - b_1 - 1)
$$

$$
\left\langle \left\langle \gamma_\mu \gamma^\alpha \gamma^\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\nu \right\rangle \right\rangle_{0,s}.
$$

Applying Lemma 3.2 (i) with $u = \gamma_\nu$, $v = E$, $w_1 = \gamma_\mu$, $w_2 = \gamma_\beta$ and $w_3 = \gamma^\beta$ to the term

$$
\left\langle \left\langle E\gamma^\beta \gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\mu \gamma_\nu \right\rangle \right\rangle_{0,s},
$$

and using the symmetry of this expression with respect to $\gamma_\mu$ and $\gamma_\nu$, then using Lemma 1.3 to 4-point functions which involve only one $E$ and simplifying, we obtain the desired formula. □

We can simplify the formula in Lemma 3.3 by the following simple observation:

**Lemma 3.4** For any vector fields $v_1, \ldots, v_k$ on the small phase space,

$$
\sum_\alpha b_\alpha \left\langle \left\langle \gamma_\alpha \gamma^\alpha v_1 \cdots v_k \right\rangle \right\rangle_{g,s} = \frac{1}{2} \sum_\alpha \left\langle \left\langle \gamma_\alpha \gamma^\alpha v_1 \cdots v_k \right\rangle \right\rangle_{g,s}.
$$

**Proof:** Since for any $\alpha$ and $\beta$, $b_\alpha \eta^{\alpha\beta} \neq 0$ implies $b_\alpha = 1 - b_\beta$, we have

$$
\sum_\alpha b_\alpha \left\langle \left\langle \gamma_\alpha \gamma^\alpha v_1 \cdots v_k \right\rangle \right\rangle_{g,s} = \sum_{\alpha,\beta} b_\alpha \eta^{\alpha\beta} \left\langle \left\langle \gamma_\alpha \gamma_\beta v_1 \cdots v_k \right\rangle \right\rangle_{g,s}
$$

$$
= \sum_{\alpha,\beta} (1 - b_\beta) \eta^{\alpha\beta} \left\langle \left\langle \gamma_\alpha \gamma_\beta v_1 \cdots v_k \right\rangle \right\rangle_{g,s}
$$

$$
= \sum_\beta (1 - b_\beta) \left\langle \left\langle \gamma_\beta \gamma_\nu v_1 \cdots v_k \right\rangle \right\rangle_{g,s}.
$$

The lemma follows. □

Since

$$
G_0(E^m, E^k, E, E) = \sum_{\mu,\nu} x^\mu_m x^\nu_k G_0(\gamma_\mu, \gamma_\nu, E, E),
$$

where $x^\mu_m$ is defined by (20), an immediate consequence of Lemma 3.3 and Lemma 3.4 is the following...
Lemma 3.5
\[ G_0(E^m, E^k, E, E) = \sum_{\beta} \left\langle \left\langle EEE^{m+k} \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\alpha, \beta, \mu} (b_\mu - b_\alpha) x_m \left\langle \left\langle \gamma_\mu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta E^k \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\alpha, \beta, \nu} (b_\nu - b_\alpha) x_k \left\langle \left\langle \gamma_\mu E \gamma_\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha \gamma_\beta E^m \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\alpha, \beta} (b_\alpha - b_1)(2 - 2b_\alpha + 2b_1)x_{m+k} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\beta} (2b_1 - 6 + 12b_3^2) \left\langle \left\langle E^{m+k} \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\beta, \mu} (b_\mu^2 - b_\mu) x_m \left\langle \left\langle \gamma_\mu E^k \gamma_\beta \left( \gamma_\beta \cdot \gamma_\beta \right) \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{\beta, \nu} (b_\nu^2 - b_\nu) x_k \left\langle \left\langle \gamma_\mu E^m \gamma_\beta \left( \gamma_\beta \cdot \gamma_\beta \right) \right\rangle \right\rangle_{0,s} \]
\[ - \sum_{\beta, \mu, \nu} 2b_\mu b_\nu x_m x_k \left\langle \left\langle \gamma_\mu \gamma_\nu \left( \gamma_\beta \cdot \gamma_\beta \right) \right\rangle \right\rangle_{0,s} \]

To simplify this formula, we need to compute \( \sum_{\beta} \left\langle \left\langle \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\beta \right\rangle \right\rangle_{0,s} \). First, we have

Lemma 3.6 For any vector field \( v \) on the small phase space, let \( v^k \) be the \( k \)-th quantum power of \( v \). Then for any \( \alpha, \beta, \) and \( \mu \),
\[ \left\langle \left\langle v^k \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s} = - \sum_{i=1}^{k-1} \left\langle \left\langle v^{k-i} \left( \gamma_\alpha \cdot \gamma_\beta \cdot v^{i-1} \right) v \gamma_\mu \right\rangle \right\rangle_{0,s} \]
\[ + \sum_{i=1}^{k} \left\langle \left\langle v^{k-i} \left( \gamma_\alpha \right) \left( \gamma_\beta \cdot v^{i-1} \right) v \gamma_\mu \right\rangle \right\rangle_{0,s} \]

Proof: Since
\[ \left\langle \left\langle v^k \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s} = \sum_{\sigma} \left\langle \left\langle v^{k-1} v \gamma_\sigma \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\sigma \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s}, \]
using Lemma 3.2 (i) to exchange positions of \( v \) and \( \gamma_\alpha \), we obtain
\[ \left\langle \left\langle v^k \gamma_\alpha \gamma_\beta \gamma_\mu \right\rangle \right\rangle_{0,s} = - \left\langle \left\langle v^{k-1} \left( \gamma_\alpha \cdot \gamma_\beta \right) v \gamma_\mu \right\rangle \right\rangle_{0,s} \]
\[ + \left\langle \left\langle v^{k-1} \left( \gamma_\alpha \right) \gamma_\beta v \gamma_\mu \right\rangle \right\rangle_{0,s} \]
\[ + \left\langle \left\langle v^{k-1} \gamma_\alpha \left( \gamma_\beta \cdot v \right) \gamma_\mu \right\rangle \right\rangle_{0,s} \]
The lemma follows by repeatedly applying this formula to the last term to decrease the power of the first \( v \) and increase the power of the second \( v \). \( \square \)

In the special case when \( v = E \), Lemma 3.6 implies
Lemma 3.7 For any $\mu$ and $k \geq 1$,

$$
\sum_{\beta} \left\langle \left( E^k \gamma_\beta \gamma_\beta \gamma_\mu \right) \right\rangle_{0,s} = \sum_{\beta} (b_\mu - b_1 - 1 + k) \left\langle \left( E^{k-1} \gamma_\mu \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s} \\
- \sum_{i=1}^{k-1} \sum_{\sigma,\beta} b_\sigma x_{k-i}^\sigma \left\langle \left( E^{i-1} \left( \gamma_\sigma \bullet \gamma_\mu \right) \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s} \\
- \sum_{i=1}^{k-1} \sum_{\sigma,\beta} b_\sigma \left\langle \left( E^{k-1} \gamma_\mu \gamma_\sigma \right) \right\rangle_{0,s} \left\langle \left( \gamma_\rho \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s}.
$$

Proof: We first apply Lemma 3.6 to $\left\langle \left( E^k \gamma_\beta \gamma_\beta \gamma_\mu \right) \right\rangle_{0,s}$; then use Lemma 1.3 to remove $E$ from 4-point functions in the expressions

$$
\left\langle \left( E^{k-1} \left( \gamma_\beta \bullet \gamma_\beta \bullet E^{i-1} \right) \gamma_\mu \right) \right\rangle_{0,s} = \sum_{\sigma,\rho} x_{k-i}^\sigma \left\langle \left( E^{i-1} \left( \gamma_\beta \bullet \gamma_\rho \right) \right) \right\rangle_{0,s} \left\langle \left( \gamma_\sigma \gamma_\rho \gamma_\mu \right) \right\rangle_{0,s}
$$

and

$$
\left\langle \left( E^{k-1} \gamma_\beta \gamma_\beta \gamma_\rho \right) \right\rangle_{0,s} = \sum_{\sigma,\rho} \left\langle \left( E^{k-1} \gamma_\beta \gamma_\sigma \gamma_\rho \right) \right\rangle_{0,s} \left\langle \left( \gamma_\sigma \gamma_\rho \gamma_\mu \right) \right\rangle_{0,s}.
$$

The lemma is then obtained by using the fact

$$
\left\langle \left( (v_1 \bullet v_2) v_3 v_4 \right) \right\rangle_{0,s} = \left\langle \left( v_1 (v_2 \bullet v_3) v_4 \right) \right\rangle_{0,s}
$$

for any vector fields $v_1, \ldots, v_4$ on the small phase space, and the fact

$$
\sum_{\beta} b_\beta \left( \gamma_\beta \bullet \gamma_\beta \right) = \frac{1}{2} \sum_{\beta} \left( \gamma_\beta \bullet \gamma_\beta \right),
$$

which follows from Lemma 3.4.

We also need the following

Lemma 3.8

$$
\sum_{\beta} \left\langle \left( E^k E \gamma_\beta \gamma_\beta \right) \right\rangle_{0,s} = \sum_{\beta,\mu} (b_\mu - b_1) (b_\mu - 2b_1 - 1) x_{k}^{\mu} \left\langle \left( \gamma_\mu \gamma_\beta \gamma_\beta \right) \right\rangle_{0,s} \\
+ \sum_{\beta,\mu} b_\mu (b_\mu - b_1 + k - 1) x_{1}^{\mu} \left\langle \left( E^{k-1} \gamma_\mu \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s} \\
- \sum_{i=1}^{k-1} \sum_{\mu,\nu} b_\mu b_{\nu} x_{k-i}^{\mu} x_{1}^{\nu} \left\langle \left( E^{i-1} \left( \gamma_\mu \bullet \gamma_\nu \right) \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s} \\
- \sum_{i=1}^{k-1} \sum_{\mu,\nu} b_\mu b_{\nu} x_{1}^{\nu} \left\langle \left( E^{k-i} \gamma_\mu \gamma_\nu \right) \right\rangle_{0,s} \left\langle \left( \gamma_\rho \left( \gamma_\beta \bullet \gamma_\beta \right) \right) \right\rangle_{0,s}.
$$
Proof: Since
\[
\sum_{\beta} \left\langle \left\langle E^k E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} = \sum_{\beta, \mu, \nu} x^\mu_k x^\nu_1 \left\langle \left\langle \gamma_\mu \gamma_\nu E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s},
\]
using Lemma 1.3 to remove \(E\) in the 5-point function, we obtain
\[
\sum_{\beta} \left\langle \left\langle E^k E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} = \sum_{\beta, \mu} x^\mu_k (b_\mu - 2b_1 - 1) \left\langle \left\langle \gamma_\mu E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} + \sum_{\beta, \mu} x^\mu_k b_\nu \left\langle \left\langle E^k \gamma_\nu \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}.
\]
Using Lemma 1.3 again to remove \(E\) in the 4-point function in the first term and applying Lemma 3.7 to the second term, we obtain the desired formula. \(\square\)

Now we are ready to prove the following

**Theorem 3.9** For any manifold \(V\),

\[
\phi_m = -\frac{1}{24} \sum_{k=0}^{m-1} \sum_{\alpha, \beta, \sigma} b_\alpha \left\langle \left\langle \gamma_1 E^k \gamma^\alpha \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha E^{m-1-k} \gamma^\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\beta \gamma_\sigma \gamma^\sigma \right\rangle \right\rangle_{0,s}
\]

\[
-\frac{1}{4} \sum_{k=0}^{m-1} \sum_{\alpha, \beta} b_\alpha b_\beta \left\langle \left\langle \gamma_\alpha E^k \gamma^\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\beta E^{m-1-k} \gamma^\alpha \right\rangle \right\rangle_{0,s} + \frac{m}{12} \sum_{\sigma} \left\langle \left\langle \gamma_\sigma E^{m-1} \gamma^\sigma \right\rangle \right\rangle_{0,s}.
\]

Proof: Using formula (16), we obtain
\[
E^{m-1} \phi_2 = -\frac{1}{24} \sum_{\beta} \left\langle \left\langle E^{m-1} E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} - \frac{1}{12} \sum_{\beta, \mu} (b_1 + 1 - b_\mu) x^\mu_{m-1} \left\langle \left\langle \gamma_\mu E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}
\]

\[
+ \frac{1}{2} \sum_{\beta} \left\{ b_\beta (1 - b_\beta) - \frac{b_1 + 1}{6} \right\} \left\langle \left\langle E^{m-1} \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}.
\]

We can use Lemma 1.3 to remove \(E\) in the second term, then plugging the result and the formula in Lemma 3.3 into the definition of \(\phi_m\). To simplify the resulting expression of \(\phi_m\), we first use Lemma 3.8 to compute the 5-point functions. We can then use Lemma 3.7 to compute 4-point functions and obtain the following formula

\[
\sum_{k=1}^{m-2} \sum_{\alpha, \beta, \mu} (b_\mu - b_\alpha) x^\mu_{m-1-k} \left\langle \left\langle \gamma_\mu E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha E^k \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s}
\]

\[
= \sum_{k=1}^{m-2} \sum_{\alpha, \beta, \mu} b_\mu (b_\alpha - 1) x^\mu_{m-1-k} \left\langle \left\langle \gamma_\mu E \gamma_\beta \gamma^\beta \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\alpha E^{k-1} \left( \gamma_\beta \cdot \gamma^\beta \right) \right\rangle \right\rangle_{0,s}
\]

\[
+ \sum_{k=1}^{m-3} \sum_{\alpha, \beta, \mu} b_\alpha b_\mu \left\{ x^\alpha_{m-1-k} x^\mu_k \left\langle \left\langle \gamma_\alpha \gamma_\mu \left( \gamma_\beta \cdot \gamma^\beta \right) \right\rangle \right\rangle_{0,s} - x^\alpha_k x^\mu_{m-2-k} \left\langle \left\langle \left( \gamma_\alpha \cdot \gamma_\mu \right) E^{m-2-k} \left( \gamma_\beta \cdot \gamma^\beta \right) \right\rangle \right\rangle_{0,s} \right\}
\]

\[
+ \sum_{k=1}^{m-3} \sum_{\alpha, \beta, \mu} b_\alpha b_\mu \left\{ x^\alpha_{m-1-k} \left\langle \left\langle \gamma_\alpha E \gamma_\mu \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\mu E^{k-1} \left( \gamma_\beta \cdot \gamma^\beta \right) \right\rangle \right\rangle_{0,s}
\]

\[
+ \sum_{k=1}^{m-3} \sum_{\alpha, \beta, \mu} b_\alpha b_\mu \left\{ x^\alpha_{m-1-k} \left\langle \left\langle \gamma_\alpha E \gamma_\mu \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_\mu E^{k-1} \left( \gamma_\beta \cdot \gamma^\beta \right) \right\rangle \right\rangle_{0,s}
\]
\[-x_1^\alpha \langle \langle \gamma_\alpha E^{m-1-k} \gamma_\mu \rangle \rangle_{0,s} \langle \langle \gamma^\mu E^{k-1} (\gamma_\beta \bullet \gamma_\beta) \rangle \rangle_{0,s} \]
\[
- \sum_{k=1}^{m-2} \sum_{\alpha,\beta} b_\alpha^2 x_{m-k}^\alpha \langle \langle \gamma_\alpha E^{k-1} (\gamma_\beta \bullet \gamma_\beta) \rangle \rangle_{0,s} \\
+ \sum_{\alpha,\beta} b_\alpha b_\beta x_{m-1}^\alpha \langle \langle \gamma_\alpha \gamma_\beta \gamma_\beta \rangle \rangle_{0,s} + \sum_{\alpha,\beta} (m-2-b_1) b_\alpha x_1^\alpha \langle \langle \gamma_\alpha E^{m-2} (\gamma_\beta \bullet \gamma_\beta) \rangle \rangle_{0,s}.
\]

In this way, we obtain an expression for \( \phi_m \) which contains only 3-point functions. There are many cancellations among different terms in this expression. After simplifying it, we obtain the desired formula. \( \Box \)

**Remark:** The expression for \( \phi_m \) in Theorem 3.9 is the same as that for \( \langle \langle E^k \rangle \rangle_{1,s} \) obtained in [DZ2] (4.42) for the case where the quantum cohomology of \( V \) is semisimple.

### 4 A necessary and sufficient condition for the genus-1 Virasoro conjecture

The main purpose of this section is to prove Theorem 3.2. In this section, we will use \( \{u^1, \ldots, u^N\} \) to denote the coordinate on the small phase space in order to distinguish the one on the big phase space. In this coordinate, the vector field \( \frac{\partial}{\partial u^\alpha} \) is identified with \( \gamma_\alpha \). Let \( u^\alpha = \sum_\beta \eta_{\alpha\beta} u^\beta \). Then \( \frac{\partial}{\partial u^\alpha} \) is identified with \( \gamma_\alpha \). Let \( M \) be an \( N \times N \) matrix whose entries are \( u_{\alpha\beta} \). Temporarily, we think of each \( u_{\alpha\beta} \) as an independent variable. Define

\[
F_1(u_1, \ldots, u_N; M) := \langle \langle e^{\sum_\alpha u^\alpha \gamma^\alpha} \rangle \rangle_1 + \frac{1}{24} \log \det (\eta^{-1} M).
\]

Then

\[
\frac{\partial F_1}{\partial u^\alpha} = \langle \langle \gamma^\alpha \rangle \rangle_{0,s} \quad \text{and} \quad \frac{\partial F_1}{\partial u_{\alpha\beta}} = \frac{1}{24} (M^{-1})_{\alpha\beta}.
\]

The genus-1 constitutive relation says that \( F_1 \) is equal to \( F_1 \) after the transformation

\[
u_\alpha = \langle \langle \gamma_1 \gamma_\alpha \rangle \rangle_0 \quad \text{and} \quad u_{\alpha\beta} = \langle \langle \gamma_1 \gamma_\alpha \gamma_\beta \rangle \rangle_0.
\]

Taking derivative of the genus-1 constitutive relation once, we obtain

\[
\langle \langle \tau_{m,\alpha} \rangle \rangle_1 = \sum_\sigma \langle \langle \gamma_1 \tau_{m,\alpha} \gamma_\sigma \rangle \rangle_0 \frac{\partial F_1}{\partial u_\sigma} + \sum_{\sigma,\rho} \langle \langle \gamma_1 \tau_{m,\alpha} \gamma_\sigma \gamma_\rho \rangle \rangle_0 \frac{\partial F_1}{\partial u_{\sigma\rho}}
\]

for any \( m \) and \( \alpha \). On the other hand, the genus-0 constitutive relation says, in particular, that

\[
\langle \langle \gamma_\alpha \gamma_\beta \rangle \rangle_{0,s}\big|_{u_\alpha = \langle \langle \gamma_1 \gamma_\alpha \rangle \rangle_0} = \langle \langle \gamma_\alpha \gamma_\beta \rangle \rangle_0.
\]

Taking derivative of this relation once, we get

\[
\langle \langle \gamma_\alpha \gamma_\beta \gamma_\mu \rangle \rangle_{0,s}\big|_{u_\alpha = \langle \langle \gamma_1 \gamma_\alpha \rangle \rangle_0} = \sum_\nu \left( M^{-1} \eta \right)_{\nu\mu}\bigg|_{u_{\sigma\rho} = \langle \langle \gamma_1 \gamma_\sigma \gamma_\rho \rangle \rangle_0} \langle \langle \gamma_\alpha \gamma_\beta \gamma_\nu \rangle \rangle_0.
\]
Moreover combining equation (31) with Lemma 1.1 (iii) and Lemma 1.3 (iii), we obtain

\[ \langle \langle \gamma_\alpha E \gamma_\beta \rangle \rangle_{0,\sigma} \bigg|_{u_\sigma = \langle \langle \gamma_1 \gamma_\alpha \rangle \rangle_0} = \langle \langle \gamma_\alpha \mathcal{X} \gamma_\beta \rangle \rangle_0. \]  

(33)

The following lemma will be useful in the proof of Theorem 0.2.

**Lemma 4.1**

\[ \sum_{\alpha, \beta, \mu_1, \ldots, \mu_{k-1}} (M^{-1})_{\alpha \beta} \gamma_1 \left( \langle \langle \gamma_\alpha \mathcal{X} \gamma_{\mu_1} \rangle \rangle_0 \langle \langle \gamma_{\mu_1} \mathcal{X} \gamma_{\mu_2} \rangle \rangle_0 \cdots \langle \langle \gamma_{\mu_{k-1}} \mathcal{X} \gamma_\beta \rangle \rangle_0 \right) \]

\[ = k \sum_{\mu_1, \ldots, \mu_{k-1}} \langle \langle \gamma_{\mu_{k-1}} \mathcal{X} \gamma_{\mu_1} \rangle \rangle_0 \langle \langle \gamma_{\mu_1} \mathcal{X} \gamma_{\mu_2} \rangle \rangle_0 \cdots \langle \langle \gamma_{\mu_{k-2}} \mathcal{X} \gamma_{\mu_{k-1}} \rangle \rangle_0. \]

**Proof:** By Lemma 1.1 (iii),

\[ \gamma_1 \langle \langle \gamma_\alpha \mathcal{X} \gamma_\beta \rangle \rangle_0 = (b_\alpha + b_\beta) \langle \langle \gamma_1 \gamma_\alpha \gamma_\beta \rangle \rangle_0 = (b_\alpha + b_\beta) u_{\alpha \beta}. \]

Therefore

\[ \frac{1}{2} \sum_{\mu_1, \ldots, \mu_{k-1}} b_{\mu_1} \langle \langle \gamma_{\mu_{k-1}} \mathcal{X} \gamma_{\mu_1} \rangle \rangle_0 \langle \langle \gamma_{\mu_1} \mathcal{X} \gamma_{\mu_2} \rangle \rangle_0 \cdots \langle \langle \gamma_{\mu_{k-2}} \mathcal{X} \gamma_{\mu_{k-1}} \rangle \rangle_0. \]

In this calculation, one needs to switch the position of \( \gamma_1 \) and that of \( \mathcal{X} \) by using the generalized WDVV equation so that \( \gamma_1 \) can be pushed to the beginning or the end of the chain of the multiplications of 3-point functions. In this way we can always create entries of \( M \) which can be used to eliminate entries of \( M^{-1} \). Moreover, by interchanging all upper indices with the corresponding lower indices, we obtain

\[ \frac{1}{2} \sum_{\mu_1, \ldots, \mu_{k-1}} b_{\mu_1} \langle \langle \gamma_{\mu_{k-1}} \mathcal{X} \gamma_{\mu_1} \rangle \rangle_0 \langle \langle \gamma_{\mu_1} \mathcal{X} \gamma_{\mu_2} \rangle \rangle_0 \cdots \langle \langle \gamma_{\mu_{k-2}} \mathcal{X} \gamma_{\mu_{k-1}} \rangle \rangle_0. \]

The lemma then follows. \( \square \)

Recall that \( \mathcal{L}_1 \) is the vector field on the big phase space which is defined to be the first derivative part of the \( L_1 \) operator. The genus-1 \( L_1 \) constraint is \( \Psi_{1,1} = 0 \), where

\[ \Psi_{1,1} = \langle \langle \mathcal{L}_1 \rangle \rangle_1 + \frac{1}{2} \sum_{\alpha} b_\alpha (1 - b_\alpha) \{ \langle \langle \gamma_\alpha \gamma_\alpha \rangle \rangle_0 + 2 \langle \langle \gamma_\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \rangle \rangle_1 \}. \]

We have the following

**Proposition 4.2**

\[ \Psi_{1,1} = \left\{ -\langle \langle E^2 \rangle \rangle_{1,\sigma} + \phi_2 \right\} \bigg|_{u_\sigma = \langle \langle \gamma_1 \gamma_\sigma \rangle \rangle_0}. \]
Proof: Applying equation (28) and (30) to each genus-1 1-point function in $\Psi_{1,1}$, we obtain

$$\Psi_{1,1} = \sum_{\sigma} \left\{ \langle \gamma_1 L_1 \gamma_\sigma \rangle_0 + \sum_{\alpha} b_\alpha (1 - b_\alpha) \langle \gamma^\alpha \rangle_0 \langle \gamma_1 \gamma_\sigma \gamma_\alpha \rangle_0 \right\} \left\{ \langle (\gamma^\sigma) \rangle_0 \mid u_{\alpha,\beta} = \langle \gamma_1 \gamma_\sigma \rangle_0 \right\}$$

$$+ \frac{1}{24} \sum_{\sigma,\rho} \left\{ \langle \gamma_1 L_1 \gamma_\sigma \gamma_\rho \rangle_0 + \sum_{\alpha} b_\alpha (1 - b_\alpha) \langle \gamma^\alpha \rangle_0 \langle \gamma_1 \gamma_\sigma \gamma_\rho \gamma_\alpha \rangle_0 \right\} \left( M^{-1} \right)_{\sigma,\rho}$$

$$+ \frac{1}{2} \sum_{\alpha} b_\alpha (1 - b_\alpha) \langle \gamma_\alpha \gamma^\alpha \rangle_0,$$

(34)

where the entries of $M$ are $u_{\alpha,\beta} = \langle \gamma_1 \gamma_\alpha \gamma_\beta \rangle_0$. By the second equation of (13) and equation (33), the first line of the right hand side is equal to $- \langle (E^2) \rangle_{1,s} \mid u_{\alpha,\beta} = \langle \gamma_1 \gamma_\alpha \rangle_0$. Now we compute the second line. Since

$$\langle \gamma_1 L_1 \gamma_\sigma \gamma_\rho \rangle_0 = \gamma_1 \langle L_1 \gamma_\sigma \gamma_\rho \rangle_0 - b_1 (b_1 + 1) \langle \gamma_1 \gamma_\sigma \gamma_\rho \rangle_0 - (2b_1 + 1) \sum_\alpha C^\alpha_1 \langle \gamma_\alpha \gamma_\sigma \gamma_\rho \rangle_0,$$

by Lemma 1.1 and the second equation of (13), the second line of (34) is equal to

$$- \frac{1}{12} \sum_\alpha \langle \gamma_\alpha \gamma^\alpha \rangle_0 - \frac{1}{24} \sum_{\sigma,\rho,\alpha} \left( M^{-1} \right)_{\sigma,\rho} \left\{ (b_\alpha (1 - b_\alpha) + b_1 (b_1 + 1)) \langle \gamma_1 \gamma_\sigma \rangle_0 \langle \gamma^\alpha \gamma_\sigma \gamma_\rho \rangle_0 \right. \right.$$

$$\left. + (2b_1 + 1) C^\alpha_1 \langle \gamma_\alpha \gamma_\sigma \gamma_\rho \rangle_0 \right\}.$$

(35)

On the other hand, by Lemma 1.3 (iv)

$$\sum_\alpha \langle (EE \gamma_\alpha \gamma^\alpha) \rangle_{0,s} = \sum_{\alpha,\beta} \langle E \gamma_1 \gamma^\beta \rangle_{0,s} \langle \gamma_\beta E \gamma_\alpha \gamma^\alpha \rangle_{0,s}$$

$$= \sum_{\alpha,\beta} (b_\beta - b_1) \langle E \gamma_1 \gamma^\beta \rangle_{0,s} \langle \gamma_\beta \gamma_\alpha \gamma^\alpha \rangle_{0,s}.$$

By equation (32) and (33),

$$\sum_\alpha \langle (EE \gamma_\alpha \gamma^\alpha) \rangle_{0,s} \mid u_{\alpha,\beta} = \langle \gamma_1 \gamma_\alpha \rangle_0 = \sum_{\beta,\sigma,\rho} (b_\beta - b_1) \langle X \gamma_1 \gamma^\beta \rangle_0 \langle \gamma_\beta \gamma_\sigma \gamma_\rho \rangle_0 \left( M^{-1} \right)_{\sigma,\rho}.$$

By Lemma 1.1 (iii),

$$\sum_\alpha b_\alpha \langle \gamma_1 X \gamma^\alpha \rangle_0$$

$$= \sum_\alpha \left\{ -b_1 \langle \gamma_1 X \gamma^\alpha \rangle_0 + (2b_1 + 1) C^\alpha_1 + (b_\alpha (1 - b_\alpha) + b_1 (b_1 + 1)) \langle \gamma_1 \gamma^\alpha \rangle_0 \right\}.$$

Moreover

$$\sum_{\beta,\sigma,\rho} \langle X \gamma_1 \gamma^\beta \rangle_0 \langle \gamma_\beta \gamma_\sigma \gamma_\rho \rangle_0 \left( M^{-1} \right)_{\sigma,\rho} = \sum_{\beta,\sigma,\rho} \langle X \gamma_\sigma \gamma^\beta \rangle_0 \langle \gamma_\beta \gamma_1 \gamma_\rho \rangle_0 \left( M^{-1} \right)_{\sigma,\rho}$$

$$= \sum_{\sigma} \langle X \gamma_\sigma \gamma^\sigma \rangle_0$$

$$= \sum_{\sigma} \langle \gamma_\sigma \gamma^\sigma \rangle_0.$$
Therefore we have

\[
\sum_{\alpha} \langle \langle EE\gamma^\alpha \rangle \rangle_{0,\sigma} \bigg|_{u_\alpha = \langle \langle \gamma_\alpha \rangle \rangle_0} = \sum_{\alpha,\sigma,\rho} \left( (2b_1 + 1)C^\alpha_1 + (b_\alpha(1 - b_\alpha) + b_1(1 + b_1)) \langle \langle \gamma_\alpha \rangle \rangle_0 \right) \langle \langle \gamma_\alpha \sigma \gamma_\rho \rangle \rangle_0 \left( M^{-1} \right)_{\sigma \rho} - 2b_1 \sum_{\sigma} \langle \langle \gamma_\sigma \gamma^\sigma \rangle \rangle_0.
\]

Comparing this equation with (ii) and using (iii), we obtain the desired formula. □

We next prove the analogue of this proposition for the genus-1 $L_2$ constraint. We need the following

**Lemma 4.3**

(i) \[ \sum_{\alpha,\beta} b_\alpha \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \frac{1}{2} \sum_{\alpha,\beta} \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0, \]

(ii) \[ \sum_{\alpha,\beta} b_\alpha^2 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \sum_{\alpha,\beta} \left( -\frac{1}{4} + \frac{3}{2}b_\alpha^2 \right) \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0, \]

(iii) \[ \sum_{\alpha,\beta} (b_\alpha)^k C^\beta_\alpha \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = 0 \text{ if } k \text{ is odd.} \]

**Proof:** Interchanging the upper indices and lower indices in the expression \[ \sum_{\alpha,\beta} b_\alpha \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 \] and using the fact that $b_\alpha \eta^{\alpha \beta} \neq 0$ implies $b_\beta = 1 - b_\alpha$, we obtain \[ \sum_{\alpha,\beta} b_\alpha \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \sum_{\alpha,\beta} (1 - b_\alpha) \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0. \]

This implies (i). Similarly we have \[ \sum_{\alpha,\beta} b_\alpha^2 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \sum_{\alpha,\beta} (1 - b_\alpha)^3 \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 \langle \langle \gamma_\alpha \gamma^\beta \rangle \rangle_0. \]

Together with (i), this implies (ii). Using the fact $b_\alpha C_{\alpha \beta} \neq 0$ implies $b_\beta = -b_\alpha$, we have \[ \sum_{\alpha,\beta} (b_\alpha)^k C^\beta_\alpha \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \sum_{\alpha,\beta} (b_\alpha)^k C_{\alpha \beta} \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = \sum_{\alpha,\beta} (-b_\beta)^k C_{\alpha \beta} \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0. \]

Interchanging $\alpha$ with $\beta$, we have \[ \sum_{\alpha,\beta} (b_\alpha)^k C^\beta_\alpha \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0 = (-1)^k \sum_{\alpha,\beta} (b_\alpha)^k C^\beta_\alpha \langle \langle \gamma_\beta \gamma^\alpha \rangle \rangle_0. \]

This implies (iii). □

The genus-1 $L_2$ constraint is the equation $\Psi_{1,2} = 0$ where

\[
\Psi_{1,2} = \langle \langle L_2 \rangle \rangle_1 + \sum_{\alpha} b_\alpha(1 - b_\alpha^2) \left\{ \langle \langle \tau_1 \alpha \gamma^\alpha \rangle \rangle_0 + \langle \langle \tau_1 \alpha \rangle \rangle_0 \langle \langle \gamma^\alpha \rangle \rangle_1 + \langle \langle \tau_1 \alpha \rangle \rangle_1 \langle \langle \gamma^\alpha \rangle \rangle_0 \right\} - \frac{1}{2} \sum_{\alpha,\beta} (3b_\alpha^2 - 1) C^\beta_\alpha \left\{ \langle \langle \gamma^\alpha \gamma_\beta \rangle \rangle_0 + 2 \langle \langle \gamma^\alpha \rangle \rangle_1 \langle \langle \gamma_\beta \rangle \rangle_0 \right\}.
\]

We have the following
Proposition 4.4

\[
\Psi_{1,2} = \left\{ -\left\langle E^3 \right\rangle_{1,s} + \phi_3 \right\}_{\psi_\sigma = \langle \gamma_1 \gamma_\sigma \rangle_0}.
\]

**Proof:** Applying equation (28) and (33) to each genus-1 1-point function in \(\Psi_{1,2}\), using equation (14) and the fact

\[
\langle\langle \gamma_1 \mathcal{L}_2 \gamma_\sigma \gamma_\rho \rangle\rangle_0 = \gamma_1 \langle\langle \mathcal{L}_2 \gamma_\sigma \gamma_\rho \rangle\rangle_0 - b_1 (b_1 + 1) (b_1 + 2) \langle\langle \tau_{2,1} \gamma_\sigma \gamma_\rho \rangle\rangle_0
\]

\[
- \sum_\beta (3b_1^2 + 6b_1 + 2) C_1^\beta \langle\langle \tau_{1,\beta} \gamma_\sigma \gamma_\rho \rangle\rangle_0
\]

\[
- \sum_\beta 3(b_1 + 1) (C_1^2)^\beta \langle\langle \gamma_\beta \gamma_\sigma \gamma_\rho \rangle\rangle_0,
\]

then applying Lemma 4.4, equation (33) and the genus-0 topological recursion relation, we obtain

\[
\Psi_{1,2} = \left\{ -\left\langle E^3 \right\rangle_{1,s} - \frac{1}{8} \sum_\alpha \left\langle E^2 \gamma_\alpha \gamma^\alpha \right\rangle_{1,s} \right\}_{\psi_\sigma = \langle \gamma_1 \gamma_\sigma \rangle_0}
\]

\[
+ \frac{1}{24} \sum_{\mu, \nu, \beta} (M^{-1})_{\mu \nu} \left\langle \gamma_\mu \gamma_\nu \gamma^\beta \right\rangle_0
\]

\[
\left\{ \sum_\alpha (3b_1^2 - 1) C_{\alpha \beta} \langle\langle \gamma_1 \gamma^\alpha \rangle\rangle_0 + \sum_\alpha b_\alpha (b_\alpha^2 - 1) \langle\langle \gamma_1 \gamma^\alpha \rangle\rangle_0 \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_0
\]

\[
+ b_\beta (b_\beta^2 - 1) \langle\langle \gamma_1 \tau_{1,\beta} \rangle\rangle_0 - b_1 (b_1 + 1) (b_1 + 2) \langle\langle \tau_{1,1} \gamma_\beta \rangle\rangle_0
\]

\[
- \sum_\alpha (3b_1^2 + 6b_1 + 2) C_1^\alpha \langle\langle \gamma_\alpha \gamma_\beta \rangle\rangle_0 - 3(b_1 + 1) (C_1^2)^\beta \langle\langle \gamma_\beta \gamma_\sigma \gamma_\rho \rangle\rangle_0
\]

\[
- \sum_\beta b_\beta (b_\beta^2 - 1) \langle\langle \tau_{1,\beta} \gamma^\beta \rangle\rangle_0 - \frac{1}{2} \sum_{\alpha, \beta} (3b_1^2 - 1) C_{\alpha \beta}^\beta \langle\langle \gamma_\beta \gamma^\alpha \rangle\rangle_0 \right\}. \tag{36}
\]

A simple combination of Lemma 4.4 and the genus-0 topological recursion relation gives the following (cf. [EHX1] formula (8) and (9))

\[
(1 + b_\alpha + b_\beta) \langle\langle \tau_{1,\alpha} \gamma_\beta \rangle\rangle_0 = \sum_\sigma \langle\langle \gamma_\alpha \gamma^\sigma \rangle\rangle_0 \left\{ C_{\sigma \beta} + (b_\sigma + b_\beta) \langle\langle \gamma_\sigma \gamma_\beta \rangle\rangle_0 \right\} - \sum_\sigma C_{\alpha \sigma}^\beta \langle\langle \gamma_\sigma \gamma_\beta \rangle\rangle_0.
\]

This is a special case of the fundamental recursion relation of [EHX1]. Using this formula, we can express 2-point correlation functions of type \(\langle\langle \tau_{1,\alpha} \gamma_\beta \rangle\rangle_0\) in the right hand side of equation (33) in terms of correlation functions only involving \(\gamma_\sigma\), \(\sigma = 1, \ldots, N\). (In this procedure, first applying Lemma 3.2 in [LT] to shift the level of descendant in the term \(b_\beta (1 + b_\beta) \langle\langle \gamma_1 \tau_{1,\beta} \rangle\rangle_0\) may simplify the computation.) Then a straightforward computation using Lemma 4.4 and Lemma 4.3 shows that

\[
\Psi_{1,2} = \left\{ -\left\langle E^3 \right\rangle_{1,s} - \frac{1}{8} \sum_\alpha \left\langle E^2 \gamma_\alpha \gamma^\alpha \right\rangle_{1,s} \right\}_{\psi_\sigma = \langle \gamma_1 \gamma_\sigma \rangle_0}.
\]
\[+ \frac{1}{24} \sum_{\mu, \nu, \beta} (M^{-1})_{\mu \nu} \left\langle \gamma_{\mu} \gamma_{\nu} \gamma_{\beta} \right\rangle \langle b_1 + b_\alpha + 1 - b_\beta \rangle \left\langle \gamma_1 \mathcal{X} \gamma_\alpha \right\rangle_0 \left\langle \gamma_\alpha \mathcal{X} \gamma_\beta \right\rangle_0 \]

\[+ \sum_{\alpha, \beta} \left( \frac{3}{8} - \frac{1}{2} b_\beta^2 - \frac{1}{4} b_\alpha b_\beta \right) \left\langle \gamma_{\alpha} \mathcal{X} \gamma_{\beta} \right\rangle_0 \left\langle \gamma_\beta \mathcal{X} \gamma_\alpha \right\rangle_0.\]

The proposition then follows from equation (32), (33), and Theorem 3.9. □

Now we are ready to prove Theorem 0.2.

**Proof of Theorem 0.2:** The string equation implies that the transformation
\[u^\alpha = \left\langle \langle \gamma_1 \gamma^\alpha \rangle \right\rangle_0, s\] is an identity map when the right hand side of this equation is restricted to the small phase space. Therefore, by Proposition 4.2, the restriction of the genus-1 \( L_1 \) constraint to the small phase space is equivalent to the condition that \( \left\langle \langle E^2 \rangle \right\rangle_0, s = \phi_2. \)

Hence \( \left\langle \langle E^2 \rangle \right\rangle_0, s = \phi_2 \) is a necessary condition for the genus-1 Virasoro conjecture. On the other hand, if \( \left\langle \langle E^2 \rangle \right\rangle_0, s = \phi_2 \), Proposition 4.2 also implies that the genus-1 \( L_1 \) constraint is true. Moreover, Theorem 3.1 and Proposition 4.4 implies that the genus-1 \( L_2 \) constraint is also true. By the virasoro relation among the \( L_n \) operators, the genus-1 Virasoro conjecture holds. □

## 5 Virasoro type relation for \( \{ \phi_k \} \)

Because of Theorem 0.2, we are interested in when the equality \( \left\langle \langle E^2 \rangle \right\rangle_1, s = \phi_2 \) holds. The Virasoro relation (1) and Theorem 3.1 implies that a necessary condition for this equality to hold is that
\[E^k \phi_m - E^m \phi_k = (m - k) \phi_{k+m-1}.\]

In this section, we prove that this condition holds for all manifolds, i.e. Theorem 0.3 is true.

We begin with the following

**Lemma 5.1** Let \( H = \sum_{\beta} \gamma_{\beta} \bullet \gamma^{\beta} \). For any \( \alpha \), we have
\[E^2 \left\langle \langle \gamma_\alpha E^k H \right\rangle \right\rangle_0, s = (b_\alpha - b_1 + k) \left\langle \langle \gamma_\alpha E^{k+1} H \right\rangle \right\rangle_0, s + \sum_{\mu} b_\mu \left\langle \langle \gamma_\alpha E^{\gamma^\mu} \right\rangle \right\rangle_0, s \left\langle \langle \gamma_\mu E^k H \right\rangle \right\rangle_0, s - \sum_{\mu} b_\mu x_{\mu}^1 \left\langle \langle \gamma_\mu \left( \gamma_\alpha \bullet E^k H \right) \right\rangle \right\rangle_0, s.\]

**Proof:** Since
\[E^2 \left\langle \langle \gamma_\alpha E^k H \right\rangle \right\rangle_0, s = \sum_{\sigma, \beta} \left\{ E^2 \left\langle \langle \gamma_\alpha E^{k+1} \gamma^{\sigma} \right\rangle \right\rangle_0, s \right\} \left\{ \left\langle \langle \gamma_\sigma \gamma_{\beta} \gamma^{\beta} \right\rangle \right\rangle_0, s \right\} + \sum_{\sigma, \beta} \left\langle \langle \gamma_\alpha E^k \gamma^{\sigma} \right\rangle \right\rangle_0, s \left\{ E^2 \left\langle \langle \gamma_\sigma \gamma_{\beta} \gamma^{\beta} \right\rangle \right\rangle_0, s \right\},\]

the lemma follows by applying formula (19) to the first term and Lemma 3.7 to the second term and then simplifying the resulting expression. □

We also need the following
Lemma 5.2

\[
E^2 \left\{ \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} \right\}
\]

\[= k \sum_{\beta} b_{\beta}^2 \left\langle \left\langle E^k \gamma_{\beta} \gamma_{\beta} \right\rangle \right\rangle_{0,s} + (k - 2) \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} .
\]

Proof: First observe that

\[
E^2 \left\{ \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} \right\}
\]

\[= 2 \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta} \left( E^2 \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \right) \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} .
\]

After applying formula (13), we can simplify the expression by using identities

\[
\sum_{i=0}^{k-1} \sum_{\alpha,\beta,\mu} b_{\alpha}b_{\beta}b_{\mu} \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\mu} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} = \sum_{i=0}^{k-1} \sum_{\alpha,\beta,\mu} b_{\alpha}b_{\beta}b_{\mu} \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\mu} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s},
\]

and

\[
\sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta}^2 \left\langle \left\langle \gamma_{\alpha} E^{i+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} = \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta}^2 \left\langle \left\langle \gamma_{\alpha} E^{i+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} .
\]

These two identities are obtained by interchanging indices. We thus obtain

\[
E^2 \left\{ \sum_{i=0}^{k-1} \sum_{\alpha,\beta} b_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} \right\}
\]

\[= 2 \sum_{i=0}^{k-1} \sum_{\alpha,\beta} ib_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} .
\]

The lemma then follows from the identity

\[
\sum_{i=0}^{k-1} \sum_{\alpha,\beta} ib_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{i+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-i} \gamma_{\alpha} \right\rangle \right\rangle_{0,s}
\]

\[= k \sum_{\beta} b_{\beta}^2 \left\langle \left\langle E^k \gamma_{\beta} \gamma_{\beta} \right\rangle \right\rangle_{0,s} + \sum_{j=0}^{k-1} \sum_{\alpha,\beta} (k - 2 - j)b_{\alpha}b_{\beta} \left\langle \left\langle \gamma_{\alpha} E^{j+1} \gamma_{\beta} \right\rangle \right\rangle_{0,s} \left\langle \left\langle \gamma_{\beta} E^{k-1-j} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} .
\]

This identity is obtained by substituting \( k - 2 - j \) for \( i \) and interchanging indices. \( \square \)

We can now prove a special case of Theorem 1.3.
Proposition 5.3

\[ E^k \phi_2 - E^2 \phi_k = (2 - k) \phi_{k+1}. \]

**Proof:** By Lemma 1.3,

\[ \phi_2 = -\frac{1}{24} \sum_{\alpha, \beta} x_{\alpha}^1 (b_{\alpha} - b_1) \left\langle \left\langle \gamma_{\alpha} \gamma_{\beta} \gamma_{\alpha} \right\rangle \right\rangle_{0,s} + \frac{1}{2} \sum_{\beta} \left\{ b_{\beta} (1 - b_{\beta}) - \frac{b_1 + 1}{6} \right\} \left\langle \left\langle \gamma_{\beta} \gamma_{\beta} \right\rangle \right\rangle_{0,s}. \]

Using formula (16) and Lemma 3.7, we can express \( E^k \phi_2 \) in terms of products of 3-point functions. On the other hand, using Theorem 3.9, formula (19), Lemma 5.1 and Lemma 5.2, we can express \( E^2 \phi_k \) in terms of products of 3-point functions. Combine the two expressions together and simplifying it, we obtain the desired formula. \( \square \)

Now we are ready to prove Theorem 0.3.

**Proof of Theorem 0.3:** We prove this theorem by induction on \( \min\{m, k\} \). Without loss of generality, we may assume that \( m \leq k \).

If \( m = 0 \), equation (3) is equivalent to \( \gamma_1 \phi_k = k \phi_{k-1} \). This equality holds trivially when \( k = 0 \) or \( k = 1 \). When \( k = 2 \), it follows from formula (16), Lemma 1.2, and the following formula (cf. [Bor])

\[ \frac{1}{2} \sum_{\beta} b_{\beta} (1 - b_{\beta}) - \frac{b_1 + 1}{12} \chi(V) = -\frac{1}{12} \int_U c_1(V) \cup c_{d-1}(V). \]

Note that this is the reason why \( b_{\alpha} \) is defined in terms of the holomorphic dimension of \( \gamma_{\alpha} \) rather than a half of the real dimension of \( \gamma_{\alpha} \) as proposed in [EHX2]. For \( k > 2 \), the equality follows from Theorem 3.9, formula (16), the fact that \( \nabla_{\gamma_1} E^k = [\gamma_1, E^k] = k E^{k-1} \), and Lemma 1.2.

Assume that equality (3) holds for \( m \leq n \). We want to show that it also holds for \( m = n + 1 \). In fact for any \( k \), by equation (11) and Proposition 5.3, we have

\[ E^{n+1} \phi_k - E^k \phi_{n+1} = \frac{1}{n-2} \left\{ (E^2 E^n - E^n E^2) \phi_k - E^k \left( E^2 \phi_n - E^n \phi_2 \right) \right\}. \]

By the induction hypothesis, \( E^n \phi_k = E^k \phi_n + (k - n) \phi_{n+k-1} \), and by Proposition 5.3, \( E^2 \phi_k = E^k \phi_2 + (k - 2) \phi_{k+1} \). Therefore, by equation (11), we have

\[ E^{n+1} \phi_k - E^k \phi_{n+1} = \frac{1}{n-2} \left\{ (k - 2) \left( E^{k+1} \phi_n - E^n \phi_{k+1} \right) + (k - n) \left( E^2 \phi_{n+k-1} - E^{n+k-1} \phi_2 \right) \right\}. \]

Using the induction hypothesis and Proposition 5.3 again, we have

\[ E^{n+1} \phi_k - E^k \phi_{n+1} = (k - n - 1) \phi_{n+k}. \]

This proves the theorem. \( \square \)

We can use Theorem 1.3 to construct a representation of the Lie algebra spanned by \( \{E^k \mid k \geq 0\} \) in the following way. Let

\[ h_k := \left\langle \left\langle E^k \right\rangle \right\rangle_{1,s} - \phi_k. \]
By Theorem 2.4 and the definition of $\phi_k$, $h_0 = h_1 = 0$ and

$$h_k = \frac{k}{2} E^k h_2.$$  \hfill (38)

More generally, we have the following

**Lemma 5.4** For all $k \geq 0$ and $m > 0$,

$$E^k \frac{h_m}{m} = (m - 1) \frac{h_{m+k-1}}{m+k-1}.$$  \hfill (39)

**Proof:** Theorem 0.3 and formula (1) imply

$$E^k h_m - E^m h_k = (m - k) h_{k+m-1}$$

for all $m$ and $k$. Using this formula, one can show that the equation

$$E^k \frac{h_m}{m} = (m - 1) \frac{h_{m+k-1}}{m+k-1}$$

is equivalent to the equation

$$E^m \frac{h_k}{k} = (k - 1) \frac{h_{m+k-1}}{m+k-1}.$$  \hfill (40)

Formula (38) says that the lemma is true if $\min\{m, k\} = 2$. By formulas (38) and (1), we have

$$E^k h_m = \frac{m}{2} \left\{ E^{m-1} \frac{2}{k+1} h_{k+1} + (m - k - 1) \frac{2}{m+k-1} h_{m+k-1} \right\}.$$  \hfill (41)

The lemma then follows from induction on $\min\{m, k\}$. \hspace{1cm} \Box

Lemma 5.4 tells us that the linear span of $\{h_k \mid k \geq 2\}$ gives a representation of the Lie algebra spanned by $\{E^k \mid k \geq 0\}$. Theorem 0.2 means that the genus-1 Virasoro conjecture holds if and only if $h_2 = 0$, which is equivalent to say that this representation is trivial.

**6 Some sufficient conditions for genus-1 Virasoro conjecture**

In an open subset of the small phase space, $\{E^k \mid k \geq 0\}$ defines an integrable distribution. Each leaf of this distribution is a finite dimensional manifold. Fix one leaf of this distribution. There exists an integer $n$ such that $\{E^k \mid 0 \leq k \leq n\}$ are linearly independent and there are smooth functions $f_i$, $0 \leq i \leq n$, on the leaf such that

$$E^{n+1} = \sum_{i=0}^{n} f_i E^i.$$  \hfill (42)
Since $E^{k+n+1} = E^k \cdot E^{n+1}$, we have

$$E^{k+n+1} = \sum_{i=0}^{n} f_i E^{k+i}, \quad (39)$$

for every $k \geq 0$. For later applications, we need to compute $E^k f_i$.

**Lemma 6.1**

(i) $E^0 f_i = -(i + 1) f_{i+1}$ for $0 \leq i \leq n - 1$, and $E^0 f_n = n + 1$.

(ii) $E f_i = (n + 1 - i) f_i$ for $0 \leq i \leq n$.

(iii) $E^2 f_0 = f_n f_0$ and $E^2 f_i = (n - i + 2) f_{i-1} + f_n f_i$ for $1 \leq i \leq n$.

(iv) For $k > 0$,

$$\begin{cases} E^k f_0 = f_0 E^{k-1} f_n, \\ E^k f_i = f_i E^{k-1} f_n + E^{k-1} f_{i-1} \text{ for } 1 \leq i \leq n. \end{cases}$$

**Remark 6.2** Lemma 6.1 (i) and (iv) tell us that at each point, $E^m f_j$ is completely determined by the values of $f_0, \ldots, f_n$ at that point.

**Proof of Lemma 6.1**: We first prove formula (iv). By formulas (1) and (39), for $0 \leq m \leq k$,

$$\begin{align*}
(n + 1 + 2m - k) E^{n+k} &= \left[ E^{k-m}, E^{n+m+1} \right] = \left[ E^{k-m}, \sum_{i=0}^{n} f_i E^{i+m} \right] \\
&= \sum_{i=0}^{n} \left( E^{k-m} f_i \right) E^{i+m} + \sum_{i=0}^{n} f_i \left[ E^{k-m}, E^{i+m} \right] \\
&= \sum_{i=0}^{n} \left( E^{k-m} f_i \right) E^{i+m} + \sum_{i=0}^{n} f_i (i + 2m - k) E^{i+k-1}. \quad (40)
\end{align*}$$

Using the fact that $2m E^{k+n} = \sum_{i=0}^{n} 2m f_i E^{i+k-1}$, we obtain

$$\sum_{i=0}^{n} \left( E^{k-m} f_i \right) E^{i+m} = (n + 1 - k) E^{n+k} - \sum_{i=0}^{n} f_i (i - k) E^{i+k-1}.$$

Since the right hand side of this equation does not depend on $m$, so does the left hand side. Therefore we have

$$\sum_{i=0}^{n} \left( E^k f_i \right) E^i = \sum_{i=0}^{n} \left( E^{k-m} f_i \right) E^{i+m}, \quad (41)$$

for all $0 \leq m \leq k$. In the special case $m = 1$, we have

$$\sum_{i=0}^{n} \left( E^k f_i \right) E^i = \sum_{i=0}^{n} \left( E^{k-1} f_i \right) E^{i+1}.$$

Replacing $E^{n+1}$ on the right hand side of this equality by $\sum_{i=0}^{n} f_i E^i$ and using the fact that $\{E^0, \ldots, E^n\}$ are linearly independent, we obtain formula (iv).
Formula (i) is obtained from (40) by setting \( k = m = 0 \). Formula (ii) and (iii) are obtained by using (i) and the recursion formula (iv). \( \Box \)

Now we come back to the Virasoro conjecture. As pointed out in the introduction, a necessary condition for the genus-1 Virasoro conjecture to hold is the validity of formula (5), i.e.

\[
\phi_{n+1} = \sum_{k=0}^{n} f_k \phi_k.
\]

This condition implies the following

**Lemma 6.3** If formula (5) is correct, then

\[
\phi_{m+n+1} = \sum_{k=0}^{n} f_k \phi_{m+k},
\]

for all \( m \geq 0 \).

**Proof:** By formula (3),

\[
E^2 \left\{ \phi_{m+n+1} - \sum_{j=0}^{n} f_j \phi_{m+j} \right\} = E^{m+n+1} \phi_2 + (m + n - 1) \phi_{m+n+2} - \sum_{j=0}^{n} (E^2 f_j) \phi_{m+j} - \sum_{j=0}^{n} f_j \left\{ E^{m+j} \phi_2 + (m + j - 2) \phi_{m+j+1} \right\}.
\]

Using Lemma 6.1 (iii) and the formula (39), we obtain

\[
E^2 \left\{ \phi_{m+n+1} - \sum_{j=0}^{n} f_j \phi_{m+j} \right\} = (m + n - 1) \left\{ \phi_{m+n+2} - \sum_{j=0}^{n} f_j \phi_{m+j+1} \right\} + f_n \left\{ \phi_{m+n+1} - \sum_{j=0}^{n} f_j \phi_{m+j} \right\}.
\]

The lemma then follows by induction on \( m \). \( \Box \)

By Theorem 0.2, to prove the genus-1 Virasoro conjecture we only need to show that \( h_2 := \langle \langle E^2 \rangle \rangle_{1,s} - \phi_2 = 0 \). We first prove the following

**Proposition 6.4** Let

\[
Z_k := \sum_{i=0}^{n} \left( E^k f_i \right) E^i.
\]

If equality (5) holds, then \( Z_k h_2 = 0 \) for all \( k \geq 0 \).

**Remark 6.5** Formula (41) implies that \( Z_k = E^k \cdot Z_0 \).
Proof of Proposition 6.4: Setting $m = k$ in formula (30) and using formula (31), we obtain

$$Z_k = (n + k + 1)E^{n+k} - \sum_{i=0}^{n}(i + k)f_i E^{i+k-1}.$$ 

Therefore by Lemma 5.4,

$$Z_k h_2 = 2h_{n+k+1} - \sum_{i=0}^{n} 2f_i h_{i+k}.$$ 

The right hand side of this equality is equal to 0 because of formula (39) and Lemma 6.3. □

An immediate consequence of this proposition is Theorem 0.5.

Proof of Theorem 0.5: By Lemma 5.4, $E^0 h_2 = 0$. If for some positive integer $m$, $E^m$ is contained in the span of $\{E^0, Z_k \mid k \geq 0\}$, then by Proposition 6.4, $E^m h_2 = 0$. By Lemma 5.4,

$$h_{m+1} = \frac{m + 1}{2} E^m h_2 = 0.$$ 

If $m \geq 1$, then repeatedly taking derivatives (by $m - 1$ times) of $h_{m+1}$ along the direction $E^0$ and using Lemma 5.4, we obtain that $h_2 = 0$. The theorem then follows from Theorem 0.2. □

To apply Theorem 0.5, we need to know which manifolds have non-degenerate quantum cohomology. In the rest of this paper, we discuss some sufficient conditions for the non-degeneracy of the quantum cohomology. To this end, it is interesting to know how large is the vector space spanned by $\{Z_k \mid k \geq 0\}$. We first notice that by Remark 6.5 and formula (39),

$$Z_{n+1+k} = \sum_{i=0}^{n} f_i Z_{i+k}.$$ 

Therefore, at each point, $\{Z_k \mid k \geq 0\}$ and $\{Z_k \mid 0 \leq k \leq n\}$ span the same vector space. The following lemma gives us a sense on how large this vector space might be.

Lemma 6.6 At each point $t$,

$$\text{span}\{Z_k(t) \mid 0 \leq k \leq n\} = \text{span}\{E^k(t) \mid 0 \leq k \leq n\}$$

if and only if the polynomial in $x$

$$p_t(x) = x^{n+1} - \sum_{i=0}^{n} f_i(t)x^i$$

has no multiple roots.

Proof: The derivative of $p_t(x)$ with respect to $x$ is

$$p'_t(x) = (n + 1)x^n - \sum_{i=0}^{n-1} (i + 1)f_{i+1}(t)x^i.$$
Note that the coefficients of \( p_t'(x) \) are the same as the coefficients of \( Z_0(t) \). The resultant of polynomials \( p_t(x) \) and \( p_t'(x) \) is the determinant of the following \((2n + 1) \times (2n + 1)\) matrix

\[
\begin{pmatrix}
1, & -f_n, & -f_{n-1}, & \ldots, & -f_1, & -f_0, \\
1, & -f_n, & -f_{n-1}, & \ldots, & -f_1, & -f_0, \\
\vdots, & \vdots, & \ddots, & \vdots, & \vdots, & \vdots, \\
1, & -f_n, & -f_{n-1}, & \ldots, & -f_1, & -f_0, \\
-nf_n, & -(n-1)f_{n-1}, & \ldots, & -f_1, & \ldots, & \ldots, \\
-nf_n, & -(n-1)f_{n-1}, & \ldots, & -f_1, & \ldots, & \ldots,
\end{pmatrix}
\]

where non-zero entries of the first \( n \) rows are coefficients of \( p_t(x) \) and non-zero entries of the last \( n + 1 \) rows are coefficients of \( p_t'(x) \). Performing elementary row transformations, we can transform this matrix to the following form

\[
\begin{pmatrix}
B & C \\
0 & A
\end{pmatrix}
\]

where \( B \) is an \( n \times n \) upper triangular matrix whose diagonal entries are 1, and \( A = (a_{i,j}) \), \( 0 \leq i, j \leq n \), is an \((n+1) \times (n+1)\) matrix whose entries are given by the recursion formula

\[
a_{n,0} = n + 1, \quad a_{n,j} = -(n-j+1)f_{n-j+1} \text{ for } 1 \leq j \leq n;
\]

\[
\text{For } 1 \leq i \leq n,
\begin{align*}
& a_{n-i,n} = f_0a_{n-i+1,0}, \\
& a_{n-i,j} = f_{n-j}a_{n-i+1,0} + a_{n-i+1,j+1}, \quad \text{for } 0 \leq j \leq n - 1.
\end{align*}
\]

Comparing this recursion formula with the recursion formula in Lemma 6.6, we obtain that \( a_{i,j} = E^{n-i}f_{n-j} \) for all \( i \) and \( j \). Therefore \( A \) is the coefficient matrix of representing \( \{Z_n, Z_{n-1}, \ldots, Z_0\} \) in terms of \( \{E^n, E^{n-1}, \ldots, E^0\} \). Since the determinant of \( A \) is equal to the resultant of \( p_t(x) \) and \( p_t'(x) \), \( A \) is invertible if and only if \( p_t(x) \) has no multiple roots. This proves the lemma.

Recall that a manifold has non-degenerate quantum cohomology if there exists one \( m > 0 \) such that at generic points, \( E^m \) is contained in the span of \( \{E^0, Z_0, \ldots, Z_n\} \). Observe that if the first \( n \) columns of the matrix \( A \) in the proof of Lemma 6.6 has rank \( n \), then \( E^m \) is contained in the span of \( \{E^0, Z_0, \ldots, Z_n\} \) for all \( m \geq 0 \). Therefore such manifolds have non-degenerate quantum cohomology. However, to compare non-degeneracy with semisimplicity, we only need the following weaker result which corresponds to the case where \( A \) has rank \( n + 1 \).

**Corollary 6.7** If at generic points of the small phase space of a manifold \( V \), the polynomial

\[
p_t(x) = x^{n+1} - \sum_{i=0}^{n} f_i(t)x^i
\]

has no multiple roots, then the quantum cohomology of \( V \) is non-degenerate.
In the case that the quantum cohomology of $V$ is semisimple, at generic points of the small phase space, \{${E^k}$ | $0 \leq k \leq n$\} form a basis of the tangent space of the small phase space. With respect to this basis, the quantum multiplication by $E$ has the following matrix representation

$$
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & f_0 \\
1 & 0 & 0 & \cdots & 0 & f_1 \\
0 & 1 & 0 & \cdots & 0 & f_2 \\
0 & 0 & 1 & \cdots & 0 & f_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & f_n
\end{pmatrix}.
$$

The polynomial $p_t(x)$ in Corollary 6.7 is precisely the characteristic polynomial of this matrix, and therefore has no multiple roots at semisimple points. Hence we have

**Corollary 6.8** If the quantum cohomology of a manifold $V$ is semisimple, then it is also non-degenerate.

Another sufficient condition for the non-degeneracy is the following

**Lemma 6.9** If at every point of the small phase space, the dimension of the vector space spanned by $\{E^k | k \geq 0\}$ is less than or equal to 2, then the quantum cohomology is non-degenerate.

**Proof:** The case where the dimension of the vector space spanned by $\{E^k | k \geq 0\}$ is 1 is trivial since $E^k$ is proportional to $E^0$ for every $k > 1$. If the dimension of the vector space spanned by $\{E^k | k \geq 0\}$ is 2, then $E^2 = f_0 E^0 + f_1 E$ with $E^0 f_1 = 2$ (c.f. Lemma 6.1 (i)). Hence

$$E = \frac{1}{2} \left\{ Z_0 - \left( E^0 f_0 \right) E^0 \right\}.$$

By definition, the quantum cohomology is non-degenerate. \(\square\)

Now we give some examples where the quantum cohomologies are non-degenerate but not semisimple.

**Example 6.10** Let $C_g$ be a complex curve of genus $g$. We first consider only even dimensional cohomology classes $H^{\text{even}}(C_g; \mathbb{C})$. Since the dimension of the small phase space is 2, the quantum cohomology is non-degenerate. However except for $g = 0$, the quantum cohomology of $C_g$ is not semisimple since its first Chern class is either zero or negative. In the case of complex one dimensional tori, the Euler vector field is proportional to the identity element. Equality (5) is trivially satisfied since $\phi_1 = \phi_0 = 0$. However we can not apply Theorem 0.5 yet because $C_g$ has non-trivial odd dimensional cohomology classes. The reason is that if we do not consider all cohomology classes, we can not get the Euler characteristic number in Borisov’s formula (see the proof of Theorem 0.3). So Theorem 1.3, which is a necessary condition for genus-1 Virasoro conjecture, does not hold if we only consider even dimensional cohomology classes.

To prove the genus-1 Virasoro conjecture for $C_g$ with $g > 0$, we have to consider the space of all cohomology classes $H^*(C_g; \mathbb{C})$. All theorems stated in the introduction
can be extended to manifolds with non-trivial odd dimensional cohomologies without any difficulty. Therefore these theorems can be applied to \( C_g \). Since there is no non-constant holomorphic maps from a rational curve to \( C_g \), the quantum cohomology of \( C_g \) is the same as the ordinary cohomology. It follows that \( \cdot E^2 = -(t^1)^2 E^0 + 2t^1 E \). Therefore the quantum cohomology is again non-degenerate but not semisimple. Since the only genus-0 non-zero Gromov-Witten invariants for \( C_g \) are 3-point degree-0 invariants which can be computed via cup products, it is straightforward to check that \( \phi_2 = 2 \cdot \frac{t}{6} t^1 \). Therefore equality (5) holds. By theorem 0.5, the genus-1 Virasoro conjecture holds for all complex curves. To our knowledge, this result is not known before.

**Example 6.11** Let \( V \) be a K3 surface. Let \( \gamma_1 \) be the identity element of the cohomology ring of \( V \) and \( \gamma_N \) be a non-zero element of \( H^4(V) \). Since \( c_1(V) = 0 \),

\[
E = t^1 \gamma_1 - t^n \gamma_N.
\]

Because of the selection rule and the puncture equation, on the small phase space, any \( k \)-point function involving \( \gamma_N \) is zero if \( k \geq 4 \) and the only non-zero 3-point function involving \( \gamma_N \) is \( \langle \langle \gamma_N \gamma_1 \gamma_1 \rangle \rangle_{0,s} \). In particular, \( \gamma_N \cdot \gamma_N = 0 \). Therefore the vector space spanned by \( \{ E^k \mid k \geq 0 \} \) is of dimension 2. Hence the quantum cohomology of \( V \) is non-degenerate. It is not semisimple since \( c_1(V) = 0 \). Moreover \( \gamma_1 \phi_2 = E^0 \phi_2 = 2 \phi_1 = 0 \) by formula (3), and

\[
\gamma_N \phi_2 = -\frac{1}{24} \sum \langle \langle \gamma_N EE \gamma_1 \gamma_1 \rangle \rangle_{0,s} + \frac{1}{12} \sum \langle \langle \gamma_N E \gamma_0 \gamma_0 \rangle \rangle_{0,s} + \frac{1}{2} \sum \left( b_0 (1 - b_0) - \frac{b_1 + 1}{6} \right) \langle \langle \gamma_N \gamma_0 \gamma_0 \rangle \rangle_{0,s} = 0.
\]

Therefore, by formula (3) again, \( \phi_2 = E \phi_2 = 0 \). Consequently equality (5) holds trivially. Hence the genus-1 Virasoro conjecture holds for K3 surfaces. For Calabi-Yau manifolds with complex dimension bigger than 2, the Virasoro conjecture holds for dimension reasons (c.f. [G2]). Therefore we know that the genus-1 Virasoro conjecture holds for all Calabi-Yau manifolds.

After the first version of this paper had been posted on the web, the author was informed by Jim Bryan that the Virasoro conjecture for K3 surfaces follows from deformation invariance of the virtual moduli cycle and triviality of moduli space of stable maps for generic K3 surfaces except for the genus-1 case where the virtual moduli cycle of degree 0 maps is non-trivial. In the genus-1 case, one can use intersection theory on \( M_{1,n} \) to prove it. He also informed the author that F. Zahariev found a combinatorial proof to the genus-1 Virasoro conjecture for the 1-dimensional tori.

**Example 6.12** As for the semisimplicity, the non-degeneracy can also be defined for an abstract Frobenius manifold in the same way. We consider the Frobenius manifold \( M_n := \)
$H^*(\mathbb{C}P^n)$ where the Frobenius algebra structure is given by the ordinary cohomology ring structure at every point of $M_n$ (c.f. Example 1.5 in [Du]). It is not semisimple since it has nilpotent elements at each point. Let $\gamma$ be a non-zero element of $H^2(\mathbb{C}P^n)$. Then $\{\gamma^k \mid 0 \leq k \leq n\}$ form a basis of $M_n$, where $\gamma^k = \gamma \cup \cdots \cup \gamma$. We denote the corresponding coordinates by $t_k$, $0 \leq k \leq n$. (This notation is different from our convention before where superscripts were used instead of subscripts.) The Euler vector field on $M_n$ is given by $E = \sum_{k=0}^{n} (1 - k) t_k \gamma^k$. It is straightforward to verify that for $n \leq 3$, the dimension of the vector space spanned by $\{E^k \mid k \geq 0\}$ is less than or equal to 2. Therefore, in this case, $M_n$ is non-degenerate. For $n = 4$ or 5, $E^3 = t_0^3 E^0 - 3t_0^2 E + 3t_0 E^2$. Therefore $Z_0 = 3t_0^3 E^0 - 6t_0 E + 3E^2$ and $Z_k = t_k^3 Z_0$ for $k \geq 1$. Therefore $M_4$ and $M_5$ are degenerate. Notice that in this example, the polynomial in Corollary [DZ1] is of the form $p_t(x) = (x - t_0)^3$. In fact, it is not hard to show that in general, if the dimension of the vector space spanned by $\{E^k \mid k \geq 0\}$ is equal to 3, then the Frobenius manifold is non-degenerate unless $p_t(x) = (x - g(t))^3$ for some function $g$. For $M_6$, we have $E^4 = -t_0^4 E^0 + 4t_0^3 E - 6t_0^2 E^2 + 4t_0 E^3$. Therefore $Z_0 = -4t_0^3 E^0 + 12t_0^2 E - 12t_0 E^2 + 4E^3$ and $Z_k = t_k^3 Z_0$ for $k \geq 1$. Therefore $M_6$ is also degenerate.

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