On the third level descendent fields in the Bullough-Dodd model and its reductions

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Abstract

Exact vacuum expectation values of the third level descendent fields \(\langle (\partial \varphi)^3 (\bar{\partial} \varphi)^3 e^{a \varphi} \rangle\) in the Bullough-Dodd model are proposed. By performing quantum group restrictions, we obtain \(\langle L_{-3} \bar{L}_{-3} \Phi_{lk} \rangle\) in perturbed minimal conformal field theories.

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1 Introduction

In 2-D integrable quantum field theories which can be considered as conformal field theories (CFTs) perturbed by a relevant operator, two-point correlation functions are complicated objects to study. However, using operator product expansion (OPE) in the short-distance limit one can reduce down their expression in terms of vacuum expectation values (VEVs) of local fields. Since four years, important progress has been made in this direction, as exact VEVs either of primary fields \(1, 2, 3, 4\) or their first descendent \(5, 6, 7, 8\) have been obtained explicitly. However it remains an open important problem to find all higher level VEVs of descendent fields and study their properties. Although a general method is still lacking, a case by case study based on CFT data provides a useful tool.

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in order to determine some of the simplest higher level VEVs. Beyond the technical aspects, the knowledge of any of such quantities improves the analytical prediction for short-distance expansion of two-point functions, which can be better compared with the results obtained from the numerical study of the model (see \cite{7} for instance).

Recently \cite{6}, we considered the Bullough-Dodd (BD) model and its quantum group restrictions, following the approach of \cite{5} concerning the sinh-Gordon or sine-Gordon models. In Euclidean space, the action associated with the BD model writes

\[ A_{BD} = \int d^2 x \left[ \frac{1}{16\pi} (\partial_\nu \varphi)^2 + \mu e^{b\varphi} + \mu' e^{-2b\varphi} \right]. \] (1)

Here, the parameters \( \mu \) and \( \mu' \) are introduced, as the two operators do not renormalize in the same way, on the contrary to any simply-laced affine Toda field theory. The purpose of this letter is to provide an exact expression for the VEV of the third level descendent fields (next to leading order in the UV limit of the two-point function) in the BD model, in order to complete the short-distance expansion of the two-point function calculated in \cite{6}. It should be stressed that differently from sine-Gordon (SG) model, this VEV is nonzero, due to the existence of a local conserved current of spin 3 in the BD model.

Finally, it is well-known that \( c < 1 \) minimal CFT with action

\[ \mathcal{A} = \mathcal{M}_{p/p'} + \lambda \int d^2 x \Phi_{\text{pert}} \] (2)

perturbed by the operator \( \Phi_{\text{pert}} \in \{ \Phi_{12}, \Phi_{21}, \Phi_{15} \} \) can be obtained by a quantum group (QG) restriction of imaginary Bullough-Dodd model \cite{8, 9, 10, 11} with special values of the coupling. Here we denote respectively \( \Phi_{12}, \Phi_{21} \) and \( \Phi_{15} \) as specific primary operators of the unperturbed minimal model \( \mathcal{M}_{p/p'} \) and introduce the parameter \( \lambda \) which characterizes the strength of the perturbation. Using this correspondence and the previous VEVs in the BD model, we will deduce \( \langle 0_s | L_{-3} \bar{L}_{-3} \Phi_{lk} | 0_s \rangle \) in the perturbed minimal model (2).

## 2 VEVs of the third level descendent fields

The BD model can be regarded as a relevant perturbation of a Gaussian CFT in which case the field is normalized such that \( \langle \varphi(z, \bar{z}) \varphi(0, 0) \rangle_{\text{Gauss}} = -2 \log(z\bar{z}) \). For imaginary coupling \( b = i\beta \), the perturbation is relevant for \( 0 < \beta^2 < 1 \). Although the model (11) for real coupling is very different from the one with imaginary coupling in its physical content (this latter model contains solitons and breathers), there are good reasons to believe that the expectation values obtained in the real coupling case provide also the expectation values for the imaginary coupling. Then, let us now consider the two-point function in the BD model with imaginary coupling \( G_{\alpha_1\alpha_2}(r) = \langle e^{i\alpha_1 \varphi(x)} e^{i\alpha_2 \varphi(y)} \rangle_{\text{BD}} \) with \( r = |x - y| \). It can be expanded in the short-distance limit \( (r \to 0) \) which, as mentioned above, contains a term corresponding to the third descendent contribution. The result reads (see \cite{5} for details)

\[
G_{\alpha_1\alpha_2}(r) = G_{\alpha_1+\alpha_2} r^{4\alpha_1\alpha_2} \left\{ 1 + \mathcal{F}_{12}(\alpha_1\beta, \alpha_2\beta, \beta^2) \mu(\mu')^2 r^{6-3\beta^2} + \frac{(\alpha_1\alpha_2)^2}{4} \mathcal{H}(\alpha_1 + \alpha_2) r^4 - \frac{\alpha_1^2\alpha_2^3(\alpha_1-\alpha_2)^2}{144} \mathcal{K}(\alpha_1 + \alpha_2) r^6 + O(\mu^2(\mu')^4 r^{12-6\beta^2}) \right\}
\]
\[ + \sum_{n=1}^{\infty} \mu^n r^{4\alpha_1\alpha_2+4n\beta(\alpha_1+\alpha_2)+2n(1-\beta^2)+2n^2\beta^2} \langle \langle \partial \varphi \rangle^2 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle = 1 + O(\mu^m) \]

\[ + \sum_{n=1}^{\infty} \mu^n r^{4\alpha_1\alpha_2-2n\beta(\alpha_1+\alpha_2)+2n(1-\beta^2)+2n^2\beta^2} \langle \langle \partial \varphi \rangle^2 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle = 1 + O(\mu^m) \]

\[ + \sum_{n=1}^{\infty} \mu^n \mu' r^{4\alpha_1\alpha_2+4(n-\frac{1}{2})\beta(\alpha_1+\alpha_2)+2n(1-2\beta^2)+2n^2\beta^2} \langle \langle \partial \varphi \rangle^2 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle = 1 + O(\mu^m) \]

where we defined \( \mathcal{H}(\alpha) \) and \( \mathcal{K}(\alpha) \) by the ratios

\[ \mathcal{H}(\alpha) = \frac{\langle \langle \partial \varphi \rangle^2 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle}{\langle \langle e^{i\alpha \varphi} \rangle B D \rangle} \quad \text{and} \quad \mathcal{K}(\alpha) = \frac{\langle \langle \partial \varphi \rangle^3 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle}{\langle \langle e^{i\alpha \varphi} \rangle B D \rangle} \]

and \( \mathcal{G}_\alpha = \langle e^{i\alpha \varphi} \rangle_B D \) is the VEV of the exponential field in the BD model. A closed analytic expression for \( \mathcal{G}_\alpha \) and \( \mathcal{H}(\alpha) \) has been proposed in ref. [2] and ref. [3], respectively. Their expression involves an integral representation which is well defined if

\[ -\frac{1}{2\beta} < \Re(\alpha) < \frac{1}{\beta} \]

and obtained by analytic continuation outside this domain. Here we used the notations of [4] for the Dotsenko-Fateev integrals \( j_n(a, b, \rho) \) and \( F_{n,m}(a, b, \rho) \). In particular, the integrals \( j_n(a, b, \rho) \) have been evaluated explicitly in [12] with the result

\[ j_n(a, b, \rho) = \pi^n \prod_{k=0}^{n-1} \frac{\gamma((k+1)\rho)}{\gamma(\rho)} \gamma(1+2a+k\rho)\gamma(1+2b+k\rho)\gamma(-1-2a-2b-(n-1+k)\rho) \]

where the notation \( \gamma(x) = \Gamma(x)/\Gamma(1-x) \) is used. Also, the integral \( F_{1,1}(a, b, \rho) \) can be obtained from the result of [13]. Instead, the integral \( F_{1,2}(a, b, \rho) \) is a quite complicated object, and its explicit calculation goes beyond the purposes of this letter.

In the (Gaussian) free field theory, the composite fields \( \langle \langle \partial \varphi \rangle^3 \rangle_B D \langle \langle e^{i\alpha \varphi} \rangle B D \rangle \) are spinless with scale dimension

\[ D \equiv \Delta + \Delta = 2\alpha^2 + 6 \]

For \( 0 < \beta^2 < 1 \) the perturbation is relevant and a finite number of lower scale dimension counterterms are sufficient to cancel the divergences arising in the VEVs of third level descendent fields. However, this procedure is regularization scheme dependent, i.e. one can always add finite counterterms. For generic values of \( \alpha \) this ambiguity in the definition of the renormalized expression for these fields can be eliminated by fixing their scale dimensions to be \( \Delta \). It exists however a set of values of \( \alpha \) for which the ambiguity still remains. In the BD model with imaginary coupling, this situation arises if two fields, say \( O_\alpha \) and \( O_{\alpha'} \), satisfy the resonance condition

\[ D_\alpha = D_{\alpha'} + 2n(1-\beta^2) + 2n'(1-\beta^2/4) \quad \text{with} \quad (n, n') \in \mathbb{N} \]
associated with the ambiguity

\[ O_\alpha \to O_\alpha + \mu^n \mu'^{n'} O_{\alpha'} . \]  

(9)

In this case one says that the renormalized field \( O_\alpha \) has an \((n|n')\)-th resonance \([5, 6]\) with the field \( O_{\alpha'} \). Due to the condition (8) we find immediately that a resonance can appear between the third level descendent field \((\partial \varphi)^3 (\partial \varphi)^3 e^{i\alpha \varphi}\) and the following primary fields:

\begin{align*}
(i) & \quad e^{i(\alpha-\beta)\varphi} \quad \text{i.e. } (n|n') = (1|4) \quad \text{for } \alpha = \frac{1}{\beta} - \frac{\beta}{2} ; \\
(ii) & \quad e^{i(\alpha+3\beta)\varphi} \quad \text{i.e. } (n|n') = (3|0) \quad \text{for } \alpha = -\beta ; \\
(iii) & \quad e^{i(\alpha-\beta^2)\varphi} \quad \text{i.e. } (n|n') = (0|1) \quad \text{for } \alpha = -\frac{2}{\beta} ; \\
(iv) & \quad e^{i(\alpha-\beta^2/2)\varphi} \quad \text{i.e. } (n|n') = (0|3) \quad \text{for } \alpha = \frac{\beta}{2} .
\end{align*}

(10)

If we now look at the expression (3), we notice that the contribution brought by the third level descendent field in (3), and that of any of the exponential fields in (i), (ii), (iii) and (iv), have the same power behavior in \( r \) \( (r^{4\alpha_1 \alpha_2 + 6}) \) at short-distance for the corresponding values of \( \alpha \). The integrals which appear in these contributions are, respectively:

\begin{align*}
(i) & \quad F_{1,4}(\alpha_1 \beta, \alpha_2 \beta, \beta^2) , \\
(ii) & \quad j_3(\alpha_1 \beta, \alpha_2 \beta, \beta^2) , \\
(iii) & \quad j_1\left(-\frac{\alpha_1 \beta}{2}, -\frac{\alpha_2 \beta}{2}, \frac{\beta^2}{2}ight) , \\
(iv) & \quad j_3\left(-\frac{\alpha_1 \beta}{2}, -\frac{\alpha_2 \beta}{2}, \frac{\beta^2}{2}ight) .
\end{align*}

As we will see, \( K(\alpha) \) (and similarly for the real coupling case) exhibits the same poles in order that the divergent contributions compensate each other. This last requirement leads for instance to a set of relations for \( K(\alpha) \). The third one reads

\[ \frac{\alpha_1^2 \alpha_2^2 \alpha - \alpha_2^2}{144} \text{Res}_{\alpha = -\frac{2}{3}} K(\alpha) = \mu' \frac{G_{\alpha-\beta/2}}{G_\alpha}|_{\alpha = -\frac{2}{3}} \text{Res}_{\alpha = -\frac{2}{3}} j_1\left(-\frac{\alpha_1 \beta}{2}, -\frac{\alpha_2 \beta}{2}, \frac{\beta^2}{2}\right) , \]

(11)

which is used to fix the \( \alpha \)-independent part (normalization) of \( K(\alpha) \).

On the other hand, to determine the explicit form of the \( \alpha \)-dependent part of \( K(\alpha) \), we use the reflection relations method. Indeed, the BD model \([\text{II}]\) can be regarded as two different perturbations of the Liouville field theory \([\text{II}]\). First, one can consider the Liouville action where the perturbation is identified with \( e^{-\frac{b}{2}\varphi} \). The holomorphic stress-energy tensor

\[ T(z) = -\frac{1}{4} (\partial \varphi)^2 + \frac{Q}{2} \partial^2 \varphi \]

(12)

which ensures the local conformal invariance of the Liouville field theory with coupling \( b \) can be written in terms of the standard Virasoro generators \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) and \( \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n \bar{z}^{-n-2} \). Then, using the OPE of the stress-energy tensor of the Liouville part with any primary field, we have the relation

\[ L_{-3} \bar{L}_{-3} e^{a \varphi} = \left[ \left( \frac{a + Q}{2} \right)^2 \partial^2 \varphi - \frac{1}{2} \partial^2 \varphi \partial \varphi \right] \left[ \left( \frac{a + Q}{2} \right)^2 \partial^2 \varphi - \frac{1}{2} \partial^2 \varphi \partial \varphi \right] e^{a \varphi} . \]

(13)
Furthermore, taking the expectation value of the combination above and using the (Gaussian) equations of motion $\partial\overline{\partial}\varphi = 0$ we obtain

$$\langle L_{-3}L_{-3}e^{a\varphi}\rangle_{BD} = \frac{a^2}{16}(a + 1/b)^2(a + b)^2\langle(\partial\varphi)^3(\overline{\partial}\varphi)^3e^{a\varphi}\rangle_{BD}. \quad (14)$$

Alternatively, we can consider $e^{b\varphi}$ as a perturbation. Using both pictures and CPT framework, we deduce reflection relations between operators with the same quantum numbers. We report the reader to [2, 3, 5, 6] for details about this approach. Consequently, if we denote

$$K(a) = \frac{\langle(\partial\varphi)^3(\overline{\partial}\varphi)^3e^{a\varphi}\rangle_{BD}}{\langle e^{a\varphi}\rangle_{BD}} , \quad (15)$$

then we obtain the following two functional relations

$$K(a) = \frac{[(b + 1/b - a)(b + 2/b - a)(2b + 1/b - a)]^2}{a(a + 1/b)(a + b)} K(Q - a) , \quad (16)$$

$$K(-a) = \frac{[(b/2 + 2/b - a)(b/2 + 4/b - a)(b + 2/b - a)]^2}{a(a + b/2)(a + b/2)} K(-Q' + a).$$

Notice that these equations are invariant with respect to the symmetry $b \rightarrow -2/b$ with $a \rightarrow -a$ in agreement with the well-known self-duality of the BD-model. Assuming that $K(a)$ is a meromorphic function in $a$, we find that the “minimal” solution which follows from (11), (16) is:

$$K(a) = \frac{1}{a^2} \left[ \frac{m \Gamma(\frac{k^2}{a}) \Gamma(\frac{k}{h})}{\Gamma(\frac{1}{h})} \right]^6 \gamma(\frac{2ba + b^2 + 2}{h}) \gamma(\frac{-2ba - 2}{h}) \gamma(\frac{2ba - b^2 + 4}{h}) \gamma(\frac{-2ba - 2b^2}{h})$$

$$\times \gamma(\frac{-2ba + 2b^2 - 2}{h}) \gamma(\frac{2ba - 4}{h}) \gamma(\frac{-2ba + b^2 + 2}{h}) \gamma(\frac{2ba - b^2}{h})$$

where $h = 6 + 3b^2$ is the “deformed” Coxeter number [14, 13]. Here we have used the exact relation between the parameters $\mu$ and $\mu'$ in the action (4) and the mass of the fundamental particle $m$ [3] :

$$m = \frac{2\sqrt{3} \Gamma(1/3)}{\Gamma(1 + b^2/h) \Gamma(2/h)} \left(-\mu \pi \gamma(1 + b^2)\right)^{1/h} \left(-2\mu' \pi \gamma(1 + b^2/4)\right)^{2/h}. \quad (17)$$

Notice that $K(a)$ is invariant under the duality transformation $b \rightarrow -2/b$ as expected, and contains all the expected poles. Accepting this conjecture and taking $a = 0$, we obtain for instance:

$$\langle L_{-3}L_{-3}\Pi\rangle_{BD} = -\frac{m^2}{2^{10/3}} \frac{\Gamma^2(1 + 2/h) \Gamma^2(1 + b^2/h) \Gamma^2(2/3)}{\gamma(1/2 + 2/h) \gamma(1/2 + b^2/h) \gamma(1/3 + 6/h) \gamma(1/3 + 3b^2/h) f_{BD}^2} \quad (18)$$

where $f_{BD}$ is the bulk free energy of the Bullough-Dodd model, obtained in [3].

### 3 Application to perturbed conformal field theories

For imaginary value of the coupling $b = i\beta$, with the substitutions $\mu \rightarrow -\mu$ and $\mu' \rightarrow -\mu'$ in (4) the BD model possesses quantum group symmetry $U_q(A_2^{(2)})$ with deformation parameter $q = e^{i\pi/\beta^2}$
Here we denote
\[ \xi > \frac{2}{p' - p}. \] (19)

For unitary minimal models \( \xi > 1 \) which, for \( \Im(m(\lambda)) = 0 \), corresponds to a massive phase \([3]\). Using the particle-breather identification \([2]\) \( m = 2M \sin(\frac{\pi \xi}{3\xi + 6}) \) and parameter \( a = i(\frac{1}{2\beta} - \frac{k-1}{2}) \) in \( K(a) \) it is then straightforward to get the VEV:

\[
\frac{\langle 0_s | L_{-3} T_{-3} \Phi_{lk} | 0_s \rangle}{\langle 0_s | \Phi_{lk} | 0_s \rangle} = -\left[\frac{2^{2/3} \pi \Gamma(\frac{2+2\xi}{3\xi+6})}{\sqrt{\Gamma(\frac{2}{3}) \Gamma(\xi \frac{3\xi+6}{3\xi+6})(1 + \xi)}} \right]^6 \frac{1}{\xi^2(1 + \xi)^2(3\xi + 6)^2} \times \frac{\gamma(\frac{n-4\xi-3}{3\xi+6}) \gamma(\frac{-n-4\xi-3}{3\xi+6}) \gamma(\frac{n+1+\xi}{3\xi+6}) \gamma(\frac{-n+1+\xi}{3\xi+6})}{\gamma(\frac{n+2\xi+3}{3\xi+6}) \gamma(\frac{-n+2\xi+3}{3\xi+6}) \gamma(\frac{n-2\xi+1}{3\xi+6}) \gamma(\frac{-n-2\xi+1}{3\xi+6})}. \] (20)

Here \( |0_s \rangle \) is one of the degenerate ground states of the QFT \([2]\) (see \([2]\) for a detailed discussion of the vacuum structure of the model).

For the second restriction \( \beta^2 = p'/p \) which leads to the action \([2]\) with \( \Phi_{\text{pert}} \equiv \Phi_{21} \), the exact relation between the parameter \( \lambda \) and the mass of the fundamental kink \( M \) has been obtained in \([3]\). The VEV of the third order descendent field immediately follows from (20) with the replacement \( \xi \rightarrow -1 - \xi \).

Another subalgebra of \( U_q(A_2^{(1)}) \) is the subalgebra \( U_q(sl_2) \). One can again restrict the phase space of the complex BD with respect to this subalgebra for a special value of the coupling \( \beta^2 = 4p/p' \) with \( 2p < p' \) relative prime integers in order to describe the third case, i.e. \( \Phi_{\text{pert}} = \Phi_{15} \). The calculations are straightforward so we will not report them here.

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