The variational 1-capacity and BV functions with zero boundary values on metric spaces

Panu Lahti
August 31, 2017

Abstract
In the setting of a metric space that is equipped with a doubling measure and supports a Poincaré inequality, we define and study a class of BV functions with zero boundary values. In particular, we show that the class is the closure of compactly supported BV functions in the BV norm. Utilizing this theory, we then study the variational 1-capacity and its Lipschitz and BV analogs. We show that each of these is an outer capacity, and that the different capacities are equal for certain sets.

1 Introduction
Spaces of Sobolev functions with zero boundary values are essential in specifying boundary values in various Dirichlet problems. This is true also in the setting of a metric measure space \((X, d, \mu)\), where \(\mu\) is a doubling Radon measure and the space supports a Poincaré inequality; see Section 2 for definitions and notation. In this setting, given an open set \(\Omega \subset X\), the space of Newton-Sobolev functions with zero boundary values is defined for \(1 \leq p < \infty\) by

\[
N_0^{1,p}(\Omega) := \{u|_{\Omega} : u \in N^{1,p}(X) \text{ with } u = 0 \text{ on } X \setminus \Omega\}.
\]

2010 Mathematics Subject Classification: 30L99, 31E05, 26B30.
Keywords: metric measure space, bounded variation, zero boundary values, variational capacity, outer capacity, quasi-continuity
Dirichlet problems for minimizers of the $p$-energy, and Newton-Sobolev functions with zero boundary values have been studied in the metric setting in [6, 10, 11, 41].

In the case $p = 1$, instead of the $p$-energy it is natural to minimize the total variation of a function. Local minimizers of the total variation are called functions of least gradient, see e.g. [12, 21, 36, 43, 45]. To study these, or alternatively solutions to Dirichlet problems that minimize the total variation globally, we need a class of functions of bounded variation (BV functions) with zero boundary values. Such a notion has been considered in the Euclidean setting in e.g. [4] and in the metric setting in [19, 29, 34]. However, unlike in the case $p > 1$, for BV functions there seem to be several natural ways to define the notion of zero boundary values, depending for example on whether one considers local or global minimizers. In this paper we define the class $\text{BV}_0(\Omega)$ in a way that mimics the definition of the classes $N^{1,p}_0(\Omega)$ as closely as possible; we expect such a definition to be useful when extending results of fine potential theory from the case $p > 1$ to the case $p = 1$, see Remark 3.13. Then we show that various properties that are known to hold for $N^{1,p}_0(\Omega)$ hold also for $\text{BV}_0(\Omega)$.

Classically, the space of Sobolev functions with zero boundary values is usually defined as the closure of $C^\infty_0(\Omega)$ in the Sobolev norm. In the metric setting, it can be shown that the space $N^{1,p}_0(\Omega)$ is the closure of the space of Lipschitz functions with compact support in $\Omega$, see [41, Theorem 4.8] or [5, Theorem 5.46]. In this paper we show that the class $\text{BV}_0(\Omega)$ is, analogously, the closure of BV functions with compact support in $\Omega$. This is Theorem 3.16.

Newton-Sobolev classes with zero boundary values are needed in defining the variational capacity $\text{cap}_{p}(A, \Omega)$, which is an essential concept in nonlinear potential theory, see e.g. the monographs [24, 35] for the Euclidean case and [5] for the metric setting. The properties of the variational capacity $\text{cap}_{p}(A, D)$, also for nonopen $D$, have been studied systematically in the metric setting in [7]. In this paper, we extend some of these results from the case $1 < p < \infty$ to the case $p = 1$. In particular, in Theorem 4.6 we show that the variational 1-capacity $\text{cap}_1$ is an outer capacity.

Moreover, the BV analog of the variational 1-capacity, denoted by $\text{cap}_{\text{BV}}$, has been studied in the metric setting in [22, 29]. Again, there are several different possible definitions available, depending on the definition of the class of BV functions with zero boundary values, and usually the BV-capacity is defined in a way that automatically makes it an outer capacity. In this paper
we instead give a definition that is closely analogous to the definition of \( \text{cap}_{1} \). Then we show, in Theorem 4.13, that \( \text{cap}_{\text{BV}} \) is in fact an outer capacity. Moreover, we show that when \( K \) is a compact subset of an open set \( \Omega \), \( \text{cap}_{\text{BV}}(K, \Omega) \) is equal to the Lipschitz version of the 1-capacity \( \text{cap}_{\text{lip}}(K, \Omega) \). This is Theorem 4.22.

In the literature, proving that compactly supported Lipschitz functions are dense in \( N_{0}^{1,p}(\Omega) \), as well as proving that \( \text{cap}_{p} \) is an outer capacity, relies on the quasicontinuity of Newton-Sobolev functions, see [9]. In this paper, our main tool is a partially analogous quasi-semicontinuity property of BV functions proved in [33] in the metric setting, and previously in [13, Theorem 2.5] in the Euclidean setting.

## 2 Definitions and notation

In this section we introduce the notation, definitions, and assumptions used in the paper.

Throughout this paper, \((X, d, \mu)\) is a complete metric space that is equipped with a doubling Borel regular outer measure \( \mu \) and satisfies a Poincaré inequality defined below. The doubling condition means that there is a constant \( C_{d} \geq 1 \) such that

\[
0 < \mu(B(x,2r)) \leq C_{d}\mu(B(x,r)) < \infty
\]

for every ball \( B = B(x,r) \) with center \( x \in X \) and radius \( r > 0 \). By iterating the doubling condition, we obtain that for any \( x \in X \) and \( y \in B(x,R) \) with \( 0 < r \leq R < \infty \), we have

\[
\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq \frac{1}{C_{d}^{Q}} \left( \frac{r}{R} \right)^{Q}, \tag{2.1}
\]

where \( Q > 1 \) only depends on the doubling constant \( C_{d} \). When we want to state that a constant \( C \) depends on the parameters \( a, b, \ldots \), we write \( C = C(a, b, \ldots) \). When a property holds outside a set of \( \mu \)-measure zero, we say that it holds almost everywhere, abbreviated a.e.

All functions defined on \( X \) or its subsets will take values in \([-\infty, \infty]\). Since \( X \) is complete and equipped with a doubling measure, it is proper, meaning that closed and bounded sets are compact. Since \( X \) is proper, given an open set \( \Omega \subset X \) we define \( \text{Lip}_{\text{loc}}(\Omega) \) to be the space of functions that are
in the Lipschitz class Lip(Ω′) for every open set Ω′ whose closure is a compact subset of Ω. Other local spaces of functions are defined analogously.

The measure-theoretic boundary ∂∗E of a set E ⊂ X is the set of points x ∈ X at which both E and its complement have strictly positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0. \tag{2.2}
\]

For any A ⊂ X and 0 < R < ∞, the restricted spherical Hausdorff content of codimension one is defined by

\[
\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.
\]

The codimension one Hausdorff measure of A ⊂ X is then defined by

\[
\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).
\]

By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval into X. We say that a nonnegative Borel function g on X is an upper gradient of a function u on X if for every curve γ, we have

\[
|u(x) - u(y)| \leq \int_{\gamma} g \, ds, \tag{2.3}
\]

where x and y are the end points of γ, and the curve integral is defined by using an arc-length parametrization, see [25, Section 2] where upper gradients were originally introduced. We interpret |u(x) - u(y)| = ∞ whenever at least one of |u(x)|, |u(y)| is infinite.

In the following, let 1 ≤ p < ∞ (later we will almost exclusively consider the case p = 1). We say that a family Γ of curves is of zero p-modulus if there is a nonnegative Borel function ρ ∈ Lp(X) such that for all curves γ ∈ Γ, the curve integral \( \int_{\gamma} \rho \, ds \) is infinite. A property is said to hold for p-almost every curve if it fails only for a curve family with zero p-modulus. If g is a nonnegative μ-measurable function on X and (2.3) holds for p-almost every curve, we say that g is a p-weak upper gradient of u. By only considering curves γ in a set A ⊂ X, we can talk about a function g being a (p-weak) upper gradient of u in A.
Given a $\mu$-measurable set $D \subset X$, we define
\[ \|u\|_{N^{1,p}(D)} := \left( \int_D |u|^p \, d\mu + \inf \int_D g^p \, d\mu \right)^{1/p}, \]
where the infimum is taken over all $p$-weak upper gradients $g$ of $u$ in $D$. The substitute for the Sobolev space $W^{1,p}$ in the metric setting is the Newton-Sobolev space
\[ N^{1,p}(D) := \{ u : \|u\|_{N^{1,p}(D)} < \infty \}, \]
which was introduced in [12]. We understand a Newton-Sobolev function to be defined at every $x \in D$ (even though $\| \cdot \|_{N^{1,p}(D)}$ is then only a seminorm). For any $D \subset X$, the space of Newton-Sobolev functions with zero boundary values is defined as
\[ N^{1,p}_0(D) := \{ u_D : u \in N^{1,p}(X) \text{ and } u = 0 \text{ on } X \setminus D \}. \]
The space is a subspace of $N^{1,p}(D)$ when $D$ is $\mu$-measurable, and it can always be understood to be a subspace of $N^{1,p}(X)$.

It is known that for any $u \in N^{1,p}_{loc}(X)$, there exists a minimal $p$-weak upper gradient of $u$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. for any $p$-weak upper gradient $g \in L^p_{loc}(X)$ of $u$, see [5, Theorem 2.25].

The $p$-capacity of a set $A \subset X$ is given by
\[ \text{Cap}_p(A) := \inf \|u\|^p_{N^{1,p}(X)}, \]
where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \geq 1$ on $A$. By truncation we see that we can additionally require $0 \leq u \leq 1$. We know that $\text{Cap}_p$ is an outer capacity, meaning that
\[ \text{Cap}_p(A) = \inf \{ \text{Cap}_p(U) : U \supset A \text{ is open} \}, \]
for any $A \subset X$, see [5, Theorem 5.31]. If a property holds outside a set $A \subset X$ with $\text{Cap}_p(A) = 0$, we say that it holds $p$-quasieverywhere, or $p$-q.e. If $u \in N^{1,p}(X)$, then
\[ \|u - v\|_{N^{1,p}(X)} = 0 \quad \text{if and only if} \quad u = v \ p\text{-q.e.}, \quad (2.4) \]
see [5, Proposition 1.61]. Thus in the definition of $N^{1,p}_0(D)$, we can equivalently require that $u = 0 \ p\text{-q.e.}$ on $X \setminus D$. The variational $p$-capacity of a set $A \subset D$ with respect to a set $D \subset X$ is
\[ \text{cap}_p(A, D) := \inf \int_X g_u^p \, d\mu, \quad (2.5) \]
where the infimum is taken over functions \( u \in N_0^{1,p}(D) \) such that \( u \geq 1 \) on \( A \) (equivalently, \( p\)-q.e. on \( A \)). For basic properties satisfied by the \( p \)-capacity and the variational \( p \)-capacity, such as monotonicity and countable subadditivity, see [5,7].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, essentially following [39]. See also e.g. [2,14,15,17,44] for the classical theory in the Euclidean setting. Let \( \Omega \subset X \) be an open set. For \( u \in L^1_{\text{loc}}(\Omega) \), we define the total variation of \( u \) in \( \Omega \) by

\[
\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \). Note that in [39], local Lipschitz constants were used instead of upper gradients, but the properties of the total variation can be proved similarly with either definition. We say that \( u \in L^1(\Omega) \) is a function of bounded variation, and denote \( u \in \text{BV}(\Omega) \), if \( \|Du\|(\Omega) < \infty \). For an arbitrary set \( A \subset X \), we define

\[
\|Du\|(A) := \inf \{ \|Du\|(U) : A \subset U, U \subset X \text{ is open} \}.
\]

If \( \|Du\|(\Omega) < \infty \), then \( \|Du\| (\cdot) \) is a finite Radon measure on \( \Omega \) by [39, Theorem 3.4]. A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in \( \Omega \) is also denoted by

\[
P(E, \Omega) := \|D\chi_E\|(\Omega).
\]

The BV norm is defined by

\[
\|u\|_{\text{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).
\]

The BV-capacity of a set \( A \subset X \) is defined by

\[
\text{Cap}_{\text{BV}}(A) := \inf \|u\|_{\text{BV}(X)},
\]

where the infimum is taken over all \( u \in \text{BV}(X) \) with \( u \geq 1 \) in a neighborhood of \( A \).

The following coarea formula is given in [39, Proposition 4.2]: if \( \Omega \subset X \) is an open set and \( u \in L^1_{\text{loc}}(\Omega) \), then

\[
\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt.
\]
If $\|Du\|(\Omega) < \infty$, the above holds with $\Omega$ replaced by any Borel set $A \subset \Omega$.

We will assume throughout the paper that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P \geq 1$ and $\lambda \geq 1$ such that for every ball $B(x,r)$, every locally integrable function $u$ on $X$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \int_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

The $(1,1)$-Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see e.g. [5, Theorem 4.21], and by applying the latter to approximating locally Lipschitz functions in the definition of the total variation, we get the following Sobolev-Poincaré inequality for BV functions. For every ball $B(x,r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\left( \int_{B(x,r)} |u - u_{B(x,r)}|^{Q/(Q-1)} \, d\mu \right)^{Q/(Q-1)} \leq C_{SP} \frac{\|Du\|(B(x,2\lambda r))}{\mu(B(x,2\lambda r))},$$

where $Q > 1$ is the exponent from (2.1) and $C_{SP} = C_{SP}(C_d, C_P, \lambda) \geq 1$ is a constant.

Given an open set $\Omega \subset X$ and a $\mu$-measurable set $E \subset X$ with $P(E, \Omega) < \infty$, for any $A \subset \Omega$ we have

$$\alpha \mathcal{H}(\partial^* E \cap A) \leq P(E, A) \leq C_d \mathcal{H}(\partial^* E \cap A),$$

where $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6].

The lower and upper approximate limits of a function $u$ on $X$ are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u < t\} \cap B(x,r))}{\mu(B(x,r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(\{u > t\} \cap B(x,r))}{\mu(B(x,r))} = 0 \right\}.$$

It is straightforward to show that $u^\wedge$ and $u^\vee$ are Borel functions.
We understand BV functions to be $\mu$-equivalence classes. For example, in the coarea formula (2.6), each $\{u > t\}$ is precisely speaking not a set but a $\mu$-equivalence class of sets. On the other hand, the pointwise representatives $u^\wedge$ and $u^\vee$ are defined at every point. From Lebesgue’s differentiation theorem (see e.g. [23, Chapter 1]) it follows that $u^\wedge = u^\vee = u$ a.e.

3 BV functions with zero boundary values

In this section we define and study a class of BV functions with zero boundary values.

First we gather some results that we will need. The following result is well known and proved for sets of finite perimeter in [39], but we recite the proof of the more general case here.

Lemma 3.1. Let $\Omega \subset X$ be an open set and let $u, v \in L^1_{\text{loc}}(\Omega)$. Then

$$
\| D \min\{u, v\}\|(\Omega) + \| D \max\{u, v\}\|(\Omega) \leq \| Du\|(\Omega) + \| Dv\|(\Omega).
$$

Proof. We can assume that the right-hand side is finite. Take sequences of functions $(u_i), (v_i) \subset \text{Lip}_{\text{loc}}(\Omega)$ such that $u_i \rightarrow u$ and $v_i \rightarrow v$ in $L^1_{\text{loc}}(\Omega)$, and

$$
\lim_{i \rightarrow \infty} \int_{\Omega} g_{u_i} \, d\mu = \| Du\|(\Omega) \quad \text{and} \quad \lim_{i \rightarrow \infty} \int_{\Omega} g_{v_i} \, d\mu = \| Dv\|(\Omega).
$$

By [5, Corollary 2.20], we have

$$
g_{\min\{u_i, v_i\}} = g_{u_i} \chi_{\{u_i \leq v_i\}} + g_{v_i} \chi_{\{u_i > v_i\}}, \quad g_{\max\{u_i, v_i\}} = g_{u_i} \chi_{\{u_i > v_i\}} + g_{v_i} \chi_{\{u_i \leq v_i\}}
$$
in $\Omega$. Since also $\min\{u_i, v_i\} \rightarrow \min\{u, v\}$ and $\max\{u_i, v_i\} \rightarrow \max\{u, v\}$ in $L^1_{\text{loc}}(\Omega)$, we get

$$
\| D \min\{u, v\}\|(\Omega) + \| D \max\{u, v\}\|(\Omega)
\leq \liminf_{i \rightarrow \infty} \int_{\Omega} g_{\min\{u_i, v_i\}} \, d\mu + \liminf_{i \rightarrow \infty} \int_{\Omega} g_{\max\{u_i, v_i\}} \, d\mu
\leq \liminf_{i \rightarrow \infty} \left( \int_{\Omega} g_{u_i} \, d\mu + \int_{\Omega} g_{v_i} \, d\mu \right)
= \| Du\|(\Omega) + \| Dv\|(\Omega).
$$

\[ \square \]
Moreover, for any $u, v \in L^1_{\text{loc}}(\Omega)$, it is straightforward to show that
\[
\|D(u + v)\|(\Omega) \leq \|Du\|(\Omega) + \|Dv\|(\Omega).
\] (3.2)

Since Lipschitz functions are dense in $N^{1,1}(X)$, see [5, Theorem 5.1], it follows that
\[
N^{1,1}(X) \subset \text{BV}(X) \quad \text{with} \quad \|Du\|(X) \leq \int_X g_u \, d\mu \quad \text{for every } u \in N^{1,1}(X).
\] (3.3)

Recall that we interpret Newton-Sobolev functions to be pointwise defined, whereas BV functions are $\mu$-equivalence classes, but nonetheless the inclusion $N^{1,1}(X) \subset \text{BV}(X)$ has a natural interpretation.

The BV-capacity is often convenient due to the following property not satisfied by the 1-capacity: if $A_1 \subset A_2 \subset \ldots \subset X$, then
\[
\text{Cap}_{\text{BV}}\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j \to \infty} \text{Cap}_{\text{BV}}(A_j),
\] (3.4)
see [18, Theorem 3.4]. On the other hand, by [18, Theorem 4.3] we know that for some constant $C(C_d, C_P, \lambda) \geq 1$ and any $A \subset X$, we have
\[
\text{Cap}_{\text{BV}}(A) \leq \text{Cap}_1(A) \leq C \text{Cap}_{\text{BV}}(A).
\] (3.5)

By [18, Theorem 4.3, Theorem 5.1] we know that for $A \subset X$,
\[
\text{Cap}_1(A) = 0 \quad \text{if and only if} \quad \mathcal{H}(A) = 0.
\] (3.6)

The following lemma states that a sequence converging in the BV norm has a subsequence converging pointwise $\mathcal{H}$-almost everywhere.

**Lemma 3.7.** Let $u_i, u \in \text{BV}(X)$ with $u_i \to u$ in $\text{BV}(X)$. By passing to a subsequence (not relabeled), we have $u_i^\wedge \to u^\wedge$ and $u_i^\vee \to u^\vee$ $\mathcal{H}$-a.e.

**Proof.** By [33, Lemma 4.2], for every $\varepsilon > 0$ there exists $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that by passing to a subsequence, if necessary (not relabeled), $u_i^\wedge \to u^\wedge$ and $u_i^\vee \to u^\vee$ uniformly in $X \setminus G$. From this it easily follows that we find a subsequence (not relabeled) such that $u_i^\wedge \to u^\wedge$ and $u_i^\vee \to u^\vee$ 1-q.e., and then (3.6) completes the proof.

It is a well-known fact that Newton-Sobolev functions are quasicontinuous; for a proof see [9, Theorem 1.1] or [5, Theorem 5.29].
**Theorem 3.8.** Let $1 \leq p < \infty$, let $u \in N^{1,p}(X)$, and let $\varepsilon > 0$. Then there exists an open set $G \subset X$ with $\text{Cap}_p(G) < \varepsilon$ such that $u|_{X \setminus G}$ is real-valued continuous.

In this paper we will rely heavily on the fact that BV functions have the following quasi-semicontinuity property, which was first proved in the Euclidean setting in [13, Theorem 2.5]. Since we understand BV functions to be $\mu$-equivalence classes, we need to consider the representatives $u^\land$ and $u^\lor$ when studying continuity properties.

**Proposition 3.9.** Let $u \in \text{BV}(X)$ and let $\varepsilon > 0$. Then there exists an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that $u^\land|_{X \setminus G}$ is real-valued lower semicontinuous and $u^\lor|_{X \setminus G}$ is real-valued upper semicontinuous.

**Proof.** This follows from [33, Theorem 1.1].

The following fact clarifies the relationship between the different pointwise representatives.

**Proposition 3.10.** Let $u \in N^{1,1}(X)$. Then $u = u^\land = u^\lor \; \mathcal{H}$-a.e.

**Proof.** We know that $u$ has Lebesgue points $1$-q.e., that is,

$$\lim_{r \to 0} \int_{B(x,r)} |u - u(x)| \, d\mu = 0$$

for $1$-q.e. $x \in X$, see [26, Theorem 4.1, Remark 4.2] (note that in [26] it is assumed that $\mu(X) = \infty$, but this assumption can be avoided by using [40, Lemma 3.1] instead of [26, Theorem 3.1] in the proof of the Lebesgue point theorem). It follows that $u(x) = u^\land(x) = u^\lor(x)$ for such $x$, and then (3.6) completes the proof.

Now we turn our attention to defining the class of BV functions with zero boundary values. We recall that the Newton-Sobolev class with zero boundary values $N_0^{1,1}(D)$ consists of the restrictions to $D$ of those functions $u \in N^{1,1}(X)$ with $u = 0$ $1$-q.e. on $X \setminus D$, or equivalently $\mathcal{H}$-a.e. on $X \setminus D$. When dealing with BV functions, we need to consider both representatives $u^\land$ and $u^\lor$, and thus we give the following definition.

**Definition 3.11.** Let $D \subset X$. We let

$$\text{BV}_0(D) := \{ u|_D : u \in \text{BV}(X), \; u^\land(x) = u^\lor(x) = 0 \; \text{for} \; \mathcal{H}\text{-a.e.} \; x \in X \setminus D \}.$$
Since $\text{BV}(X)$ consists of $\mu$-equivalence classes of functions on $X$, $\text{BV}_0(D)$ consists of $\mu$-equivalence classes of functions on $D$. For an open set $\Omega \subset X$, the class $\text{BV}_0(\Omega)$ is a subclass of $\text{BV}(\Omega)$ (note that we have defined the class $\text{BV}(\Omega)$ only for open $\Omega \subset X$). If $u \in \text{BV}_0(D)$ and $u = v|_D = w|_D$ for $v, w \in \text{BV}(X)$ with $v^\wedge = v^\vee = w^\wedge = w^\vee = 0$ $\mathcal{H}$-a.e. on $X \setminus D$, then by Lebesgue’s differentiation theorem, necessarily $v = 0 = w$ $\mu$-a.e. on $X \setminus D$, and so $v$ and $w$ are the same BV function. Thus for any $D \subset X$, the class $\text{BV}_0(D)$ can also be understood to be a subclass of $\text{BV}(X)$. Most of the time we will in fact, without further notice, understand functions in $\text{BV}_0(D)$ to be defined on the whole space.

Note that for $u \in N^{1,1}(X)$, requiring that $u = 0$ 1-q.e. on $X \setminus D$ is equivalent to requiring that $u^\wedge = u^\vee = 0$ $\mathcal{H}$-a.e. on $X \setminus D$, due to (3.6) and Proposition 3.10. Thus our definition of $\text{BV}_0(D)$ is a close analog of the definition of $N^{1,1}_0(D)$. In fact, since $N^{1,1}(X) \subset \text{BV}(X)$ (recall (3.3)), we always have

$$N^{1,1}_0(D) \subset \text{BV}_0(D).$$

(3.12)

Remark 3.13. Other definitions of $\text{BV}_0(\Omega)$ have been given in previous works, always for open $\Omega \subset X$. In [19] the class was defined by requiring that $u = 0$ on $X \setminus \Omega$ (that is, $u = 0$ $\mu$-a.e. on $X \setminus \Omega$). This definition is convenient when solving Dirichlet problems, because the condition persists under $L^1$-limits.

By contrast, when considering functions of least gradient, i.e. local minimizers of the total variation in an open set $\Omega$, it is natural to consider test functions $\varphi \in \text{BV}(\Omega)$ satisfying

$$\lim_{r \to 0} \int_{B(x,r) \cap \Omega} |\varphi| \, d\mu = 0 \quad \text{for } \mathcal{H}\text{-a.e } x \in \partial \Omega$$

or

$$\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r) \cap \Omega} |\varphi| \, d\mu = 0 \quad \text{for } \mathcal{H}\text{-a.e } x \in \partial \Omega,$$

see [29, Section 9]. The latter condition is very close to that of Definition 3.11 but here we are not assuming the function $\varphi$ to be in the class $\text{BV}(X)$, only in $\text{BV}(\Omega)$.

For $1 < p < \infty$, there seems to be no ambiguity in how the class of Newton-Sobolev functions with zero boundary values ought to be defined, because the class $N_0^{1,p}(D)$ is closed under $L^p$-limits of sequences that are bounded in the $\| \cdot \|_{N^{1,p}(X)}$-norm (up to a choice of $\mu$-representative), which
is what one needs in the calculus of variations. Our current definition of $BV_0(D)$ does not have the same property, but it is motivated by the fact that it is otherwise a close analog of the definition of $N_0^{1,p}(D)$. In the case $p > 1$, it is fruitful to consider the class $N_0^{1,p}(D)$ for finely open sets $D$, e.g. when constructing $p$-strict subsets with the help of a Cartan property, see [8, Lemma 3.3]. We expect the class $BV_0(D)$ to be similarly useful when extending these concepts to the case $p = 1$ in future work, see [30, 31, 32] for results so far.

**Proposition 3.14.** Let $D \subset X$ and let $u \in BV_0(D)$. Then $\|Du\|(X \setminus D) = 0$.

**Proof.** Let $x \in X \setminus D$ with $u^\wedge(x) = u^\vee(x) = 0$. It follows in a straightforward manner from the definitions that $x \notin \partial^* \{u > t\}$ for all $t \neq 0$; recall the definition of the measure-theoretic boundary from (2.2). By combining the coarea formula (2.6) and (2.8), it is easy to show that $\|Du\|$ is absolutely continuous with respect to $\mathcal{H}$. By using this fact, the coarea formula in the Borel set $\{u^\wedge = 0\} \cap \{u^\vee = 0\}$, and again (2.8), we get

$$
\|Du\|(X \setminus D) \leq \|Du\|(\{u^\wedge = 0\} \cap \{u^\vee = 0\})
= \int_{-\infty}^{\infty} P(\{u > t\}, \{u^\wedge = 0\} \cap \{u^\vee = 0\}) \, dt
\leq C_d \int_{-\infty}^{\infty} \mathcal{H}(\partial^* \{u > t\} \cap \{u^\wedge = 0\} \cap \{u^\vee = 0\}) \, dt
= 0.
$$

By the above proposition, it is natural to equip the space $BV_0(D)$ with the norm $\| \cdot \|_{BV(X)}$.

It is well known that $BV(X)$ is a Banach space. The following proposition states that so is $BV_0(D)$.

**Proposition 3.15.** Let $D \subset X$. Then $BV_0(D)$ is a closed subspace of $BV(X)$.

**Proof.** It is easy to check that $BV_0(D)$ is a vector space. Consider a sequence $(u_i) \subset BV_0(D)$ with $u_i \to u$ in $BV(X)$. Then it follows from Lemma 3.7 that also $u \in BV_0(D)$. ⊠
Besides BV functions with zero boundary values, we wish to consider compactly supported BV functions. The support of a function \( u \) on \( X \) is the closed set
\[
\text{spt} \; u := \{ x \in X : \mu(B(x, r) \cap \{ u \neq 0 \}) > 0 \text{ for all } r > 0 \}.
\]
Moreover, the positive and negative parts of a function \( u \) are \( u^+ := \max\{u, 0\} \) and \( u^- := -\min\{u, 0\} \).

In the following theorem, we show that \( BV_0(D) \) is the closure of compactly supported functions in the BV norm. A similar result has been given previously in \([34, \text{Theorem 6.9}]\), but only for open \( D \subset X \), and with additional assumptions either on the space or on the boundary of \( D \).

**Theorem 3.16.** Let \( D \subset X \) and let \( u \in BV(X) \). Then the following are equivalent:

1. \( u \in BV_0(D) \).

2. There exists a sequence \( (u_k) \subset BV(X) \) such that \( \text{spt} \; u_k \) is a compact subset of \( D \) for each \( k \in \mathbb{N} \), and \( u_k \to u \) in \( BV(X) \).

**Proof.**

(1) \( \implies \) (2): Fix \( x_0 \in X \) and let \( \eta_j(x) := (1 - \text{dist}(x, B(x_0, j)))_+ \), \( j \in \mathbb{N} \), so that each \( \eta_j \) is a 1-Lipschitz function with \( \eta_j = 1 \) on \( B(x_0, j) \) and \( \eta_j = 0 \) outside \( B(x_0, j + 1) \). By a suitable Leibniz rule, see \([21, \text{Lemma 3.2}]\), we have
\[
\|D(\eta_j u - u)\|(X) \leq \|Du\|(X \setminus B(x_0, j)) + \int_{X \setminus B(x_0, j)} |u| \, d\mu \to 0
\]
as \( j \to \infty \). Thus we can assume that \( u \) is compactly supported (in \( X \)). Since \( u^+ \) and \( u^- \) both belong to \( BV_0(D) \) and \( u = u^+ - u^- \), we can assume that \( u \geq 0 \). Finally, by using the coarea formula \([2.6]\) it is easy to check that \( \| \min\{u, j\} - u \|_{BV(X)} \to 0 \) as \( j \to \infty \), and so we can also assume that \( u \) is bounded.

Note that for \( \varepsilon > 0 \), by the coarea formula
\[
\|D(u - (u - \varepsilon)_+)\|(X) = \|D\min\{u, \varepsilon\}\|(X)
\]
\[
= \int_{-\infty}^{\infty} P(\min\{u, \varepsilon\} > t), X \, dt
\]
\[
= \int_0^\infty P(\{u > t\}, X) \, dt
\]
\[
\to 0 \quad \text{as } \varepsilon \to 0.
\]
Clearly also \((u - \varepsilon)_+ \to u\) in \(L^1(X)\) as \(\varepsilon \to 0\). Fix \(\varepsilon > 0\). By Proposition \(3.9\) there exist open sets \(G_j \subset X\) such that \(\text{Cap}_1(G_j) \to 0\) and \(u^\vee|_{X \setminus G_j}\) is upper semicontinuous for each \(j \in \mathbb{N}\). Since \(H(\{u^\vee > 0\} \setminus D) = 0\) and thus \(\text{Cap}_1(\{u^\vee > 0\} \setminus D) = 0\) by \(3.6\), we can assume that \(\{u^\vee > 0\} \setminus D \subset G_j\) for each \(j \in \mathbb{N}\) (recall also that \(\text{Cap}_1\) is an outer capacity). For any fixed \(j \in \mathbb{N}\), since \(\{u^\vee < \varepsilon\}\) is an open set in the subspace topology of \(X \setminus G_j\), there exists an open set \(W \subset X\) such that
\[
W \setminus G_j = \{u^\vee < \varepsilon\} \setminus G_j,
\]
and thus \(V_j := \{u^\vee < \varepsilon\} \cup G_j = W \cup G_j\) is an open set. Note that \(X \setminus D \subset V_j\), and \(X \setminus V_j\) is bounded by the fact that \(u\) has compact support in \(X\), so in conclusion \(X \setminus V_j\) is a bounded subset of \(D\) (in fact, compact).

There exist functions \(w_j \in \mathbb{N}^{1,1}(X)\) such that \(0 \leq w_j \leq 1\) on \(X\), \(w_j = 1\) on \(G_j\), and \(\|w_j\|_{\mathbb{N}^{1,1}(X)} \to 0\). Then also \(\|w_j\|_{\text{BV}(X)} \to 0\) by \(3.3\). By Proposition \(3.7\) we can extract a subsequence (not relabeled) such that \(w_j^\vee(x) \to 0\) for \(H\text{-a.e. } x \in X\). Let \(u_{\varepsilon,j} := (1 - w_j)(u - \varepsilon)_+, j \in \mathbb{N}\). Note that \(u_{\varepsilon,j} \geq 0\). Clearly \(u_{\varepsilon,j} = 0\) on \(G_j\) and on \(\{u^\vee < \varepsilon\}\). Thus \(u_{\varepsilon,j} = 0\) in the open set \(V_j\), and it follows that \(\text{spt } u_{\varepsilon,j} \subset X \setminus V_j\). Since \(X \setminus V_j\) is a bounded subset of \(D\), \(\text{spt } u_{\varepsilon,j}\) is a compact subset of \(D\), as desired.

Using the Leibniz rule for bounded BV functions, see \(27\), Proposition \(4.2\), we get for some constant \(C = C(C_d, C_P, \lambda) \geq 1\)
\[
\|D(u_{\varepsilon,j} - (u - \varepsilon)_+)\|(X) = \|D(w_j(u - \varepsilon)_+)\|(X)
\leq C \int_X (u - \varepsilon)_+^\vee d\|Dw_j\| + C \int_X w_j^\vee d\|D(u - \varepsilon)_+\|
\leq C\|u\|_{L^\infty(X)}\|Dw_j\|(X) + C \int_X w_j^\vee d\|D(u - \varepsilon)_+\|.
\]
Here the first term goes to zero since \(\|w_j\|_{\text{BV}(X)} \to 0\), and the second term goes to zero by Lebesgue’s dominated convergence theorem, since \(\|D(u - \varepsilon)_+\|\) is absolutely continuous with respect to \(H\). Clearly also
\[
u_{\varepsilon,j} \to (u - \varepsilon)_+ \quad \text{in } L^1(X) \quad \text{as } j \to \infty.
\]
Since we had \((u - \varepsilon)_+ \to u\) in \(\text{BV}(X)\) as \(\varepsilon \to 0\), by a diagonal argument we can choose numbers \(\varepsilon_k \downarrow 0\) and indices \(j_k \to \infty\) to obtain a sequence \(u_k := u_{\varepsilon_k, j_k}\) such that \(u_k \to u\) in \(\text{BV}(X)\).
(2) $\implies$ (1): Take a sequence of functions $u_k \in BV(X)$ such that $\text{spt } u_k$ are compact subsets of $D$ and $u_k \to u$ in $BV(X)$. By Lemma 3.7 and by passing to a subsequence (not relabeled), we have $u_k^\wedge(x) \to u^\wedge(x)$ and $u_k^\vee(x) \to u^\vee(x)$ for $\mathcal{H}$-a.e. $x \in X$. Since clearly $u_k^\wedge = u_k^\vee = 0$ on $X \setminus \text{spt } u_k \supset X \setminus D$, also $u^\wedge(x) = u^\vee(x) = 0$ for $\mathcal{H}$-a.e. $x \in X \setminus D$, and so $u \in BV_0(D)$. 

Remark 3.17. The proof of the implication (1) $\implies$ (2) essentially follows along the lines of the proof of an analogous result for Newton-Sobolev functions given in [5, Section 5.4], but instead of the quasicontinuity of Newton-Sobolev functions we use the quasi-semicontinuity of the representative $u^\vee$.

Lemma 3.18. Let $\Omega \subset X$ be an open set and let $u \in BV(\Omega)$ such that $\text{spt } u$ is a compact subset of $\Omega$. Then there exists a sequence $(u_i) \subset Lip_c(\Omega)$ such that $u_i \to u$ in $L^1(\Omega)$ and

$$\|Du\|(\Omega) = \lim_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu,$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $\Omega$. It follows that $u \in BV_0(\Omega)$, and then the above holds also with $\Omega$ replaced by $X$.

Proof. The first claim is proved in [22, Lemma 2.6]. To prove the second claim, denote by $u, u_i$ also the zero extensions of these functions. Note that the minimal 1-weak upper gradient $g_{u_i}$ (now as a function defined on $X$) is clearly the zero extension of $g_{u_i}$ (as a function defined only on $\Omega$), and so we have

$$\|Du\|(X) \leq \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu = \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu = \|Du\|(\Omega).$$

Thus $u \in BV(X)$ and then clearly $u \in BV_0(\Omega)$. 

Suppose $(u_i) \subset Lip_{\text{loc}}(X)$ with $u_i \to u$ in $BV(X)$. Then $(u_i)$ is a Cauchy sequence in $BV(X)$, and by [20, Remark 4.7], $(u_i)$ is a Cauchy sequence also in $N^{1,1}(X)$. Since $N^{1,1}(X)/\sim$ is a Banach space with the equivalence relation $u \sim v$ if $\|u - v\|_{N^{1,1}(X)} = 0$, see [5, Theorem 1.71], we conclude that $u \in N^{1,1}(X)$. Thus, for an open set $\Omega \subset X$, $Lip_c(\Omega)$ cannot be dense in $BV_0(\Omega) \supset N^{1,1}_0(\Omega)$. On the other hand, compactly supported Lipschitz functions are dense in $BV_0(\Omega)$ in the following weak sense.
Proposition 3.19. Let \( \Omega \subset X \) be an open set and let \( u \in BV_0(\Omega) \). Then there exists a sequence \((u_i) \subset Lip_c(\Omega)\) with \( u_i \to u \) in \( L^1(X) \) (with the understanding that the functions \( u_i \) are extended outside \( \Omega \) by zero) and

\[
\int_X g_{u_i} \, d\mu \to \|Du\|(X) \quad \text{as } i \to \infty.
\]

Proof. By Theorem 3.16 we find a sequence \((v_i) \subset BV(X)\) of functions with compact support in \( \Omega \) and \( \|v_i - u\|_{BV(X)} < 1/i \) for each \( i \in \mathbb{N} \). Then by Lemma 3.18 for each \( i \in \mathbb{N} \) we find \( u_i \in Lip_c(\Omega) \) with \( \|u_i - v_i\|_{L^1(X)} < 1/i \) and

\[
\left| \int_X g_{u_i} \, d\mu - \|Dv_i\|(X) \right| < 1/i.
\]

We conclude that \( \|u_i - u\|_{L^1(X)} < 2/i \) and

\[
\left| \int_X g_{u_i} \, d\mu - \|Du\|(X) \right| < 2/i.
\]

\( \square \)

Now we can also show the following result, the analog of which is well known for Newton-Sobolev functions, see [5, Lemma 2.37].

Proposition 3.20. Let \( \Omega \subset X \) be an open set and let \( u \in BV(\Omega) \) and \( v, w \in BV_0(\Omega) \) such that \( v \leq u \leq w \) in \( \Omega \). Then \( u \in BV_0(\Omega) \).

Proof. By subtracting \( v \) from all terms and observing that \( u \in BV_0(\Omega) \) if and only if \( u - v \in BV_0(\Omega) \), we can assume that \( v \equiv 0 \). Denote the zero extension of \( u \) outside \( \Omega \) by \( u_0 \). By Theorem 3.16 we find a sequence of nonnegative functions \((w_k) \subset BV(X)\) compactly supported in \( \Omega \) with \( w_k \to w \) in \( BV(X) \) (the nonnegativity actually follows from the proof, or alternatively by truncation). Then \( \varphi_k := \min\{w_k, u_0\} \in BV(\Omega) \) by Lemma 3.1 and \( \varphi_k \in BV(X) \) by Lemma 3.18 for each \( k \in \mathbb{N} \). Moreover, \( \varphi_k \to u_0 \) in \( L^1(X) \), and since each \( \varphi_k \) has compact support in \( \Omega \),

\[
\liminf_{k \to \infty} \|D\varphi_k\|(X) = \liminf_{k \to \infty} \|D\varphi_k\|(\Omega)
\]

\[
\leq \liminf_{k \to \infty} \|Dw_k\|(\Omega) + \|Du_0\|(\Omega) \quad \text{by Lemma 3.1}
\]

\[
= \|Dw\|(\Omega) + \|Du\|(\Omega).
\]
Thus by the lower semicontinuity of the total variation with respect to $L^1$-convergence, $u_0 \in BV(X)$. Moreover, $u_0^\vee(x) \leq w^\vee(x) = 0$ for $\mathcal{H}$-a.e. $x \in X \setminus \Omega$, and obviously $u_0^\wedge(x) \geq 0$ for all $x \in X \setminus \Omega$, guaranteeing that $u_0^\wedge = u_0^\vee = 0$ $\mathcal{H}$-a.e. in $X \setminus \Omega$.

4 The variational 1-capacity

In this section we study the variational (Newton-Sobolev) 1-capacity and its Lipschitz and BV analogs. Utilizing the results of the previous section, we show that each of these is an outer capacity, and that the capacities are equal for certain sets.

**Definition 4.1.** Let $A \subset D \subset X$ be arbitrary sets. We define the variational (Newton-Sobolev) 1-capacity by

$$\text{cap}_1(A, D) := \inf \int_X g_u d\mu,$$

where the infimum is taken over functions $u \in N_0^{1,1}(D)$ such that $u \geq 1$ on $A$.

We define the variational Lipschitz 1-capacity by

$$\text{cap}_{\text{lip}}(A, D) := \inf \int_X g_u d\mu,$$

where the infimum is taken over functions $u \in N_0^{1,1}(D) \cap \text{Lip}_{\text{loc}}(X)$ such that $u \geq 1$ on $A$.

Finally, we define the variational BV-capacity by

$$\text{cap}_{\text{BV}}(A, D) := \inf \|Du\|(X),$$

where the infimum is taken over functions $u \in BV_0(D)$ such that $u^\wedge \geq 1$ $\mathcal{H}$-almost everywhere on $A$.

In each case, we say that the functions $u$ over which we take the infimum are admissible (test) functions for the capacity in question.

Again, $g_u$ always denotes the minimal 1-weak upper gradient of $u$. Recall that we understand Newton-Sobolev functions to be defined at every point, but in the definition of $\text{cap}_1(A, D)$ we can equivalently require $u \geq 1$ 1-q.e.
on $A$, by (2.4). On the other hand, perturbing the representatives $u^\wedge$ and $u^\vee$ even at a single point requires perturbing the function $u$ in a set of positive $\mu$-measure. In each definition, we see by truncation that it is enough to consider test functions $0 \leq u \leq 1$, and then the conditions $u \geq 1$ and $u^\wedge \geq 1$ are replaced by $u = 1$ and $u^\wedge = 1$, respectively.

Our definition of $\text{cap}_1$ is the same as the one given in [7], where the variational $p$-capacity was studied for all $1 \leq p < \infty$. In our definition of $\text{cap}_\text{BV}$, we have then mimicked the definition of $\text{cap}_1$ as closely as possible — note that for $u \in N_0^{1,1}(D)$, requiring that $u = 1$ 1-q.e. on $A$ is equivalent to requiring that $u^\wedge = 1$ $\mathcal{H}$-a.e. on $A$, due to (3.6) and Proposition 3.10. Since $N_0^{1,1}(D) \subset \text{BV}_0(D)$ with $\|Du\|(X) \leq \int_X g u d\mu$ for every $u \in N_0^{1,1}(D)$ (recall (3.3) and (3.12)), we conclude that always $\text{cap}_\text{BV}(A, D) \leq \text{cap}_1(A, D)$. Clearly we also always have $\text{cap}_1(A, D) \leq \text{cap}_{\text{lips}}(A, D)$. In Theorem 4.22 and Example 4.25 below we investigate when equalities hold.

Also other definitions of $\text{cap}_\text{BV}$ have been given in the literature. In [22], given an open set $\Omega \subset X$ and a compact set $K \subset \Omega$, $\text{cap}_\text{BV}(K, \Omega)$ was defined by considering BV test functions that are compactly supported in $\Omega$ and take the value 1 in a neighborhood of $K$. By Theorem 4.22, this turns out to agree with our current definition of $\text{cap}_\text{BV}(K, \Omega)$. For more general sets, however, the definitions can give different results.

**Example 4.2.** Let $X = \mathbb{R}^2$ (unweighted), and let $D = [0, 1] \times [0, 1]$ and $A = \{(0,0)\}$. Since $\text{Cap}_1(A) = 0$, also $\text{cap}_1(A, D) = 0$ (as $u \equiv 0$ satisfies $u = 1$ 1-q.e. on $A$). On the other hand, if we defined $\text{cap}_\text{BV}(A, D)$ by requiring the test functions to take the value 1 in a neighborhood of $A$, as in e.g. [22, 28], then we would have $\text{cap}_\text{BV}(A, D) = \infty$, since there are no admissible functions. The same would already happen if we required that $u^\wedge = 1$ at every point in $A$, instead of $\mathcal{H}$-a.e. point. Our current definition of the variational BV-capacity has the advantage that the natural inequality $\text{cap}_\text{BV}(A, D) \leq \text{cap}_1(A, D)$ always holds. From this example we also see that it is possible to have $\text{cap}_1(A, D) < \text{cap}_{\text{lips}}(A, D)$ (the latter being $\infty$).

In [30], for an open set $\Omega \subset X$ and an arbitrary set $A \subset \Omega$, $\text{cap}_\text{BV}(A, \Omega)$ was defined otherwise similarly as here, but the condition $u \in \text{BV}_0(\Omega)$ was replaced by the condition $u = 0$ on $X \setminus \Omega$ (meaning that $u = 0$ $\mu$-a.e. on $X \setminus \Omega$). This corresponds to the different possible ways of defining the class of BV functions with zero boundary values, as discussed earlier. The advantage of the definition in [30] is that in some cases it is possible to prove
the existence of capacitary potentials, i.e. admissible functions $u$ that yield the infimum in the definition of $\text{cap}_\text{BV}$.

**Example 4.3.** Let $X = \mathbb{R}$, let

$$w(x) := \begin{cases} 
1 & \text{for } x < 0, \\
1 + x & \text{for } 0 \leq x \leq 1, \\
2 & \text{for } 1 < x,
\end{cases}$$

and let $d\mu := w \, d\mathcal{L}^1$, where $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure. Let $A = (1, 2)$ and $D = (0, 3)$. Defining $u_i := \chi_{(1/i, 2)}$, we get

$$\text{cap}_\text{BV}(A, D) \leq \|Du_i\|(X) = 3 + 1/i, \quad i \in \mathbb{N}.$$  

Thus $\text{cap}_\text{BV}(A, D) \leq 3$. Conversely, let $0 \leq u \leq 1$ be an admissible function. Note that $\mathcal{H}$ is now comparable to the counting measure, and so necessarily $u^\vee(0) = 0$. Thus for some $0 < r < 1$ we have

$$\frac{\mathcal{L}^1(B(0, r) \cap \{u < 1/2\})}{\mathcal{L}^1(B(0, r))} > \frac{1}{2}.$$  

Let $(v_i) \subset \text{Lip}_{\text{loc}}(\mathbb{R})$ with $v_i \to u$ in $L^1_{\text{loc}}(\mathbb{R})$ and $\int_{\mathbb{R}} g_{v_i} \, d\mu \to \|Du\|(\mathbb{R})$. By passing to a subsequence (not relabeled) we have $v_i(x) \to u(x)$ for a.e. $x \in \mathbb{R}$. Then we have $v_i(x_1) \to 0$ for some $x_1 < 0$, $v_i(x_2) \to u^\vee(x_2) < 1/2$ for some $0 < x_2 < 1$, $v_i(x_3) \to 1$ for some $1 < x_3 < 2$, and $v_i(x_4) \to 0$ for some $x_4 > 3$. Thus

$$\int_{\mathbb{R}} g_{v_i} \, d\mu \geq \int_{x_1}^{x_2} g_{v_i} \, d\mu + \int_{x_2}^{x_3} g_{v_i} \, d\mu + \int_{x_3}^{x_4} g_{v_i} \, d\mu$$

$$\geq |v_i(x_1) - v_i(x_2)| + w(x_2)|v_i(x_2) - v_i(x_3)| + 2|v_i(x_3) - v_i(x_4)|$$

$$\to u^\vee(x_2) + w(x_2)(1 - u^\vee(x_2)) + 2 \quad \text{as } i \to \infty$$

$$> 3$$  

since $w(x_2) > 1$. Thus $\|Du\|(\mathbb{R}) > 3$, that is, $\text{cap}_\text{BV}(A, D) = 3$ but no admissible function gives this infimum. On the other hand, if we defined $\text{cap}_\text{BV}(A, D)$ by only requiring that $u = 0$ on $\mathbb{R} \setminus D$, then the function $\chi_D$ would be admissible and $\text{cap}_\text{BV}(A, D) = 3 = \|D\chi_D\|(\mathbb{R})$. The drawback of such a definition is that Theorem 4.22 below would no longer hold, see Example 4.24.
Now we prove a few simple properties of the variational BV-capacity, the analogs of which are known for the 1-capacity $\text{cap}_1$, see [7, Theorem 3.4].

**Proposition 4.4.** The following hold:

1. For any $D \subset X$, $\text{cap}_{BV}(\emptyset, D) = 0$.
2. If $A_1 \subset A_2 \subset D$, then $\text{cap}_{BV}(A_1, D) \leq \text{cap}_{BV}(A_2, D)$.
3. If $A \subset D_1 \subset D_2$, then $\text{cap}_{BV}(A, D_2) \leq \text{cap}_{BV}(A, D_1)$.
4. If $A_1, A_2 \subset D \subset X$, 
   $$\text{cap}_{BV}(A_1 \cap A_2, D) + \text{cap}_{BV}(A_1 \cup A_2, D) \leq \text{cap}_{BV}(A_1, D) + \text{cap}_{BV}(A_2, D).$$

**Proof.**

(1)–(3): These statements are trivial.

(4): We can assume that the right-hand side is finite. Fix $\varepsilon > 0$. Take $u_j \in \text{BV}_0(D)$ with $0 \leq u_j \leq 1$, $u_j^\wedge = 1$ on $A_j$, and $\|Du_j\|(X) < \text{cap}_{BV}(A_j, D) + \varepsilon$, $j = 1, 2$. Let $v := \min\{u_1, u_2\}$ and $w := \max\{u_1, u_2\}$. By Lemma [3.1] we have

$$\|Dv\|(X) + \|Dw\|(X) \leq \|Du_1\|(X) + \|Du_2\|(X),$$

and so $v, w \in \text{BV}(X)$. Clearly $w^\wedge = 1$ on $A_1 \cup A_2$. To verify that $v^\wedge = 1$ on $A_1 \cap A_2$, we note that for any $x \in A_1 \cap A_2$ and any $\delta > 0$,

$$\limsup_{r \to 0} \frac{\mu(\{v < 1 - \delta\} \cap B(x, r))}{\mu(B(x, r))} \leq \limsup_{r \to 0} \frac{\mu(\{u_1 < 1 - \delta\} \cap B(x, r))}{\mu(B(x, r))} + \limsup_{r \to 0} \frac{\mu(\{u_2 < 1 - \delta\} \cap B(x, r))}{\mu(B(x, r))} = 0$$

by the fact that $u_1^\wedge(x) = u_2^\wedge(x) = 1$. Thus $v^\wedge(x) \geq 1 - \delta$, and by letting $\delta \to 0$ we get $v^\wedge(x) = 1$. Similarly, $v^\vee = 0 = w^\vee$ on $X \setminus D$, so that $v, w \in \text{BV}_0(D)$. Thus

$$\text{cap}_{BV}(A_1 \cap A_2, D) + \text{cap}_{BV}(A_1 \cup A_2, D) \leq \|Dv\|(X) + \|Dw\|(X) \leq \text{cap}_{BV}(A_1, D) + \text{cap}_{BV}(A_2, D) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ completes the proof. \qed
Next we show that each of the three capacities we have defined is an outer capacity, in a suitable sense. First we prove this for the variational (Newton-Sobolev) 1-capacity. This gives a positive answer to a question posed in [7], where the analogous result for $1 < p < \infty$ was proved. In fact, using methods similar to those in [7], we give a proof that covers all the cases $1 \leq p < \infty$; recall the definition of $\text{cap}_p$ from (2.5).

We need the following lemma, which is a special case of [5, Lemma 1.52].

**Lemma 4.5.** Let $u_i \leq 1$, $i \in \mathbb{N}$, be functions on $X$ with $p$-weak upper gradients $g_i$. Let $u := \sup_{i \in \mathbb{N}} u_i$ and $g := \sup_{i \in \mathbb{N}} g_i$. Then $g$ is a $p$-weak upper gradient of $u$.

**Theorem 4.6.** Let $1 \leq p < \infty$ and let $D \subset X$ and $A \subset \text{int} D$. Then

$$\text{cap}_p(A, D) = \inf_{V \text{ open}} \text{cap}_p(V, D).$$

**Proof.** One inequality is clear. To prove the opposite inequality, we can assume that $\text{cap}_p(A, D) < \infty$. Fix $\varepsilon > 0$. Take $u \in N^1_p(D)$ with $0 \leq u \leq 1$, $u = 1$ on $A$, and $\int_X g^{p} \, d\mu < \text{cap}_p(A, D) + \varepsilon$. For each $j \in \mathbb{N}$, let

$$D_j := \{x \in D : \text{dist}(x, X \setminus D) > 1/j\}$$

and $A_j := A \cap D_j$. Take $j$-Lipschitz functions $\eta_j := (1 - j \text{dist}(\cdot, D_j))_+$, so that $0 \leq \eta_j \leq 1$ on $X$ and $\eta_j = 1$ on $D_j$. Fix $j \in \mathbb{N}$. By the quasicontinuity of Newton-Sobolev functions (recall Theorem 3.8), there exists an open set $G_j \subset X$ with $\text{Cap}_p(G_j)^{1/p} < 2^{-j}\varepsilon/j$ such that $u|_{X \setminus G_j}$ is continuous. Thus there exists an open set $W \subset X$ such that

$$W \setminus G_j = \{u > 1 - \varepsilon\} \setminus G_j.$$

Thus the set $\{u > 1 - \varepsilon\} \cup G_j = W \cup G_j$ is open, and then so is

$$V_j := (\{u > 1 - \varepsilon\} \cup G_j) \cap D_j.$$

Since $u = 1$ on $A_j$, we conclude that $A_j \subset V_j$. Take a function $v_j \in N^1_p(X)$ with $0 \leq v_j \leq 1$ on $X$, $v_j = 1$ on $G_j$, and $\|v_j\|_{N^1_p(X)} < 2^{-j}\varepsilon/j$. Let $w_j := v_j \eta_j$. Then

$$\left(\int_X w^{p}_j \, d\mu\right)^{1/p} \leq \left(\int_X v^{p}_j \, d\mu\right)^{1/p} < 2^{-j}\varepsilon/j, \quad (4.7)$$

21
and by the Leibniz rule [5, Theorem 2.15],
\[
\left( \int_X g_{w_j}^p \, d\mu \right)^{1/p} \leq \left( \int_X (v_j g_{\eta_j})^p \, d\mu \right)^{1/p} + \left( \int_X (\eta_j g_{v_j})^p \, d\mu \right)^{1/p} \\
\leq j \left( \int_X v_j^p \, d\mu \right)^{1/p} + \left( \int_X g_{v_j}^p \, d\mu \right)^{1/p} \\
\leq j 2^{-j} \varepsilon / j + 2^{-j} \varepsilon / j \\
\leq 2^{-j+1} \varepsilon.
\]

Now \( w_j = 1 \) on \( G_j \cap D_j \), and so \( u + w_j > 1 - \varepsilon \) on \( V_j \). Let \( w := \sup_{j \in \mathbb{N}} w_j \). Then \( u + w > 1 - \varepsilon \) in the open set \( V := \bigcup_{j=1}^\infty V_j \). Note that \( A = \bigcup_{j=1}^\infty A_j \) since \( A \subset \text{int} \, D \), and so \( A \subset V \). We have \( w \in L^p(X) \) by (4.7), and by Lemma 4.5 we know that \( g_w \leq \sup_{j \in \mathbb{N}} g_{w_j} \), so that
\[
\left( \int_X g_w^p \, d\mu \right)^{1/p} \leq \sum_{j=1}^\infty \left( \int_X g_{w_j}^p \, d\mu \right)^{1/p} \leq \sum_{j=1}^\infty 2^{-j+1} \varepsilon = 2 \varepsilon.
\]

Clearly \( w = 0 \) on \( X \setminus D \), and so we conclude \( w \in N_0^{1,p}(D) \). Then \( (u+w)/(1-\varepsilon) \in N_0^{1,p}(D) \) is an admissible function for the set \( V \), whence
\[
\text{cap}_p(V, D)^{1/p} \leq \frac{1}{1-\varepsilon} \left( \int_X g_{u+w}^p \, d\mu \right)^{1/p} \\
\leq \frac{1}{1-\varepsilon} \left( \left( \int_X g_u^p \, d\mu \right)^{1/p} + \left( \int_X g_w^p \, d\mu \right)^{1/p} \right) \\
\leq \frac{1}{1-\varepsilon} \left( (\text{cap}_p(A, D) + \varepsilon)^{1/p} + 2 \varepsilon \right).
\]

Since \( \varepsilon > 0 \) was arbitrary, we have the result. \( \square \)

Now we can extend a few other results of [7] to the case \( p = 1 \).

**Proposition 4.8.** Let \( D \subset X \) and let \( K_1 \supset K_2 \supset \ldots \supset K := \bigcap_{j=1}^\infty K_j \) be compact subsets of \( \text{int} \, D \). Then
\[
\text{cap}_1(K, D) = \lim_{j \to \infty} \text{cap}_1(K_j, D).
\]

**Proof.** Follow verbatim the proof of [7, Theorem 4.8], except that instead of [7, Theorem 4.1], refer to Theorem 4.0 \( \square \)
Example 4.9. Let $X = \mathbb{R}$ (unweighted), let $\Omega := (0,2)$, and let $A_j := (1/j,1)$, $j \in \mathbb{N}$. Then it is easy to check that
\[ \text{cap}_1(A_j, \Omega) = 2 = \text{cap}_{BV}(A_j, \Omega) \]
for all $j \in \mathbb{N}$, but
\[ \text{cap}_1\left(\bigcup_{j=1}^{\infty} A_j, \Omega\right) = \infty = \text{cap}_{BV}\left(\bigcup_{j=1}^{\infty} A_j, \Omega\right), \]
since there are no admissible functions. This shows that
\[ \text{cap}_1\left(\bigcup_{j=1}^{\infty} A_j, \Omega\right) \neq \sup_{K \text{ compact}} \text{cap}_1(K, \Omega) \]
(and similarly for $\text{cap}_{BV}$). Thus neither $\text{cap}_1(\cdot, \Omega)$ nor $\text{cap}_{BV}(\cdot, \Omega)$ is a Choquet capacity, see e.g. [7] for more discussion on Choquet capacities. Note that by contrast, the BV-capacity $\text{Cap}_{BV}$ is continuous with respect to increasing sequences of sets, recall (3.4), and a Choquet capacity, see [18, Corollary 3.8].

As a small digression, following [7], let us define for bounded $D \subset X$ and $A \subset D$
\[ \tilde{\text{cap}}_1(A, D) := \inf_{V \text{ relatively open in } D} \text{cap}_1(V, D) = \inf_{V \text{ open}} \text{cap}_1(V \cap D, D). \]
By Theorem 4.6 clearly
\[ \text{cap}_1(A, D) = \tilde{\text{cap}}_1(A, D) \quad \text{for any } A \subset \text{int } D. \quad (4.10) \]

Proposition 4.11. Let $A \subset D$ be bounded sets. Then $\text{cap}_1(A, D) = \tilde{\text{cap}}_1(A, D)$ or $\tilde{\text{cap}}_1(A, D) = \infty$.

Proof. Follow verbatim the proof of [7, Proposition 6.5], except that instead of [7, Remark 6.4], refer to (4.10). \hfill \Box

Remark 4.12. In Theorem 4.6 for $p = 1$, Proposition 4.8 and Proposition 4.11 our standing assumptions that $X$ is complete, $\mu$ is doubling and the space supports a $(1,1)$-Poincaré inequality can be weakened to the assumption that all functions in $N^{1,1}(X)$ are quasicontinuous, and additionally that $X$ has the zero 1-weak upper gradient property in the case of Proposition 4.11, see [7].
For the variational Lipschitz 1-capacity, we obviously have for any $A \subset D \subset X$ that
\[
\text{cap}_{\text{lip}}(A, D) = \inf_{V \text{ open}} \text{cap}_{\text{lip}}(V, D).
\]
(Of course, both sides may be $+\infty$.) Next we show that also the variational BV-capacity is an outer capacity, in the same sense as the variational (Newton-Sobolev) 1-capacity. The proof is almost the same, but instead of the quasicontinuity of Newton-Sobolev functions we again rely on quasicontinuity, this time of the lower representative $u^\wedge$.

**Theorem 4.13.** Let $D \subset X$ and $A \subset \text{int } D$. Then
\[
\text{cap}_{\text{BV}}(A, D) = \inf_{V \text{ open}} \text{cap}_{\text{BV}}(V, D).
\]

**Proof.** One inequality is clear. To prove the opposite inequality, we can assume that $\text{cap}_{\text{BV}}(A, D) < \infty$. Fix $\varepsilon > 0$. Take $u \in \text{BV}_0(D)$ with $0 \leq u \leq 1$, $u^\wedge = 1$ on $A \setminus N$ for some $\mathcal{H}$-negligible set $N$, and $\|Du\|(X) < \text{cap}_{\text{BV}}(A, D) + \varepsilon$. For each $j \in \mathbb{N}$, let
\[
D_j := \{x \in D : \text{dist}(x, X \setminus D) > 1/j\}
\]
and $A_j := A \cap D_j$. Take $j$-Lipschitz functions $\eta_j := (1 - j \text{dist}(\cdot, D_j))_+$, so that $0 \leq \eta_j \leq 1$ on $X$ and $\eta_j = 1$ on $D_j$. Fix $j \in \mathbb{N}$. By Proposition [3.9], there exists an open set $G_j \subset X$ with $\text{Cap}_1(G_j) < 2^{-j \varepsilon/j}$ such that $u^\wedge_{|X \setminus G_j}$ is lower semicontinuous. We can assume that $N \subset G_j$ (recall that $\text{Cap}_1$ is an outer capacity). Thus the set $\{u^\wedge > 1 - \varepsilon\} \cup G_j$ is open, and then so is
\[
V_j := (\{u^\wedge > 1 - \varepsilon\} \cup G_j) \cap D_j.
\]
Since $u^\wedge(x) = 1$ for every $x \in A_j \setminus G_j$, we conclude that $A_j \subset V_j$.

Take $v_j \in N^{1,1}(X)$ with $0 \leq v_j \leq 1$ on $X$, $v_j = 1$ on $G_j$, and $\|v_j\|_{N^{1,1}(X)} < 2^{-j \varepsilon/j}$. Let $w_j := v_j \eta_j$. Now $w_j^\wedge = 1$ on $G_j \cap D_j$, and so $(u + w_j)^\wedge > 1 - \varepsilon$ on $V_j$. Let $w := \sup_{j \in \mathbb{N}} w_j$. Then $(u + w)^\wedge > 1 - \varepsilon$ on $V := \bigcup_{j=1}^\infty V_j$. Note that $A = \bigcup_{j=1}^\infty A_j$, and so $A \subset V$. The function $w$ is the same function as in the proof of Theorem [4.6] (for $p = 1$), and so $w \in N^{1,1}_0(D) \subset \text{BV}_0(D)$, and by (3.3),
\[
\|Dw\|(X) \leq \int_X g_w \, d\mu \leq 2\varepsilon.
\]
Hence $u + w \in \text{BV}_0(D)$, and

\[
\text{cap}_{\text{BV}}(V, D) \leq \frac{1}{1 - \varepsilon} \| D(u + w) \|(X)
\]
\[
\leq \frac{1}{1 - \varepsilon} (\| Du \|(X) + \| Dw \|(X)) \quad \text{by (3.2)}
\]
\[
\leq \frac{1}{1 - \varepsilon} (\| Du \|(X) + 2\varepsilon)
\]
\[
\leq \frac{1}{1 - \varepsilon} (\text{cap}_{\text{BV}}(A, D) + 3\varepsilon).
\]

Since $\varepsilon > 0$ was arbitrary, we have the result. \hfill \Box

For the BV-capacity $\text{Cap}_{\text{BV}}$, which we have essentially defined as an outer capacity, we can analogously (and much more easily) show the following; see also [13, Section 2] (which uses [16, Section 4]) for a corresponding result in the Euclidean setting.

**Proposition 4.14.** For any $A \subset X$,

\[
\text{Cap}_{\text{BV}}(A) = \inf \| u \|_{\text{BV}(X)},
\]

where the infimum is taken over all $u \in \text{BV}(X)$ with $u^\wedge(x) \geq 1$ for $\mathcal{H}$-a.e. $x \in A$.

**Proof.** One inequality is clear. To prove the opposite inequality, fix $A \subset X$ and denote the infimum on the right-hand side by $\beta$; we can assume that $\beta < \infty$. Fix $\varepsilon > 0$ and take $u \in \text{BV}(X)$ such that $u^\wedge(x) \geq 1$ for every $x \in A \setminus N$ for some $\mathcal{H}$-negligible set $N$, and $\| u \|_{\text{BV}(X)} < \beta + \varepsilon$. By Proposition 3.9 we find an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $u^\wedge|_{X \setminus G}$ is lower semicontinuous, and we can assume that $N \subset G$. Take $w \in N^{1,1}(X)$ such that $w \geq 1$ on $G$ and $\| w \|_{N^{1,1}(X)} < \varepsilon$. By (3.3), $w \in \text{BV}(X)$ with $\| w \|_{\text{BV}(X)} < \varepsilon$. Now $u + w > 1 - \varepsilon$ on $\{ u^\wedge > 1 - \varepsilon \} \cup G$, which is an open set containing $A$, and so

\[
\text{Cap}_{\text{BV}}(A) \leq \frac{\| u + w \|_{\text{BV}(X)}}{1 - \varepsilon} \leq \frac{\| u \|_{\text{BV}(X)} + \| w \|_{\text{BV}(X)}}{1 - \varepsilon}
\]
\[
\leq \frac{\| u \|_{\text{BV}(X)} + \varepsilon}{1 - \varepsilon} \leq \frac{\beta + 2\varepsilon}{1 - \varepsilon}.
\]

Letting $\varepsilon \to 0$, we get the result. \hfill \Box
Now we can prove Maz’ya-type inequalities for BV functions. We adapt the proof of [5, Theorem 5.53], where such inequalities are given for Newton-Sobolev functions (the inequalities were originally proven in the Euclidean setting in [38]; see also [37, Theorem 10.1.2]). In the following, given a ball \( B = B(x, r) \) and \( \beta > 0 \), we use the abbreviation \( \beta B := B(x, \beta r) \). Moreover, recall the definition of the exponent \( Q > 1 \) from (2.1), and the constants \( C_P \) and \( C_{SP} \) from the Poincaré and Sobolev-Poincaré inequalities.

**Theorem 4.15.** Let \( u \in \text{BV}(X) \), let \( S := \{ u^\wedge = u^\vee = 0 \} \), and let \( B = B(x, r) \) for some \( x \in X \) and \( r > 0 \). Then we have

\[
\left( \int_{2B} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{\text{BV}}(B \cap S)} \| Du \|(4\lambda B) \tag{4.16}
\]

and

\[
\left( \int_{2B} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq \frac{3(C_P + C_{SP})}{\text{cap}_{\text{BV}}(B \cap S, 2B)} \| Du \|(4\lambda B),
\]

if the denominators are nonzero.

**Proof.** Let \( q := Q/(Q - 1) \). First assume that \( u \) is nonnegative. Let

\[
a := \left( \int_{2B} u^q \, d\mu \right)^{1/q}.
\]

We can clearly assume that \( a > 0 \). Take a \( 1/r \)-Lipschitz function \( 0 \leq \eta \leq 1 \) with \( \eta = 1 \) on \( B \) and \( \eta = 0 \) on \( X \setminus 2B \), and then let \( v := \eta(1 - u/a) \). Now \( v \in \text{BV}(X) \) (this actually follows from (4.17) below) with \( v^\wedge \leq 0 \), \( v^\vee \leq 0 \) on \( X \setminus 2B \) and \( v^\wedge = v^\vee = 1 \) on \( B \cap S \). By Proposition 4.14 and a suitable Leibniz rule, see [21, Lemma 3.2], we get

\[
\text{Cap}_{\text{BV}}(B \cap S) \leq \int_X |v| \, d\mu + \| Du \|(X)
\]

\[
\leq \int_X |v| \, d\mu + \frac{1}{a} \left( \int_X \eta \| Du \| + \int_X g_\eta |u - a| \, d\mu \right)
\]

\[
\leq \frac{1}{a} \int_{2B} |u - a| \, d\mu + \frac{1}{a} \left( \| Du \|(2B) + \int_{2B} g_\eta |u - a| \, d\mu \right)
\]

\[
\leq \frac{1 + r^{-1}}{a} \int_{2B} |u - a| \, d\mu + \frac{1}{a} \| Du \|(2B).
\]

(4.17)
To estimate the first term, we write
\[
\int_{2B} |u - a| \, d\mu \leq \int_{2B} |u - u_{2B}| \, d\mu + |u_{2B} - a| \mu(2B)
\]
by the Poincaré inequality. Here the second term can be estimated by
\[
|a - u_{2B}| \mu(2B)^{1/q} = \left( \int_{2B} |u - u_{2B}|^q \, d\mu \right)^{1/q} \mu(2B)^{1/q}
\]
by the Sobolev-Poincaré inequality (2.7). Inserting this into (4.18), we get
\[
\int_{2B} |u - a| \, d\mu \leq 2(C_P + C_{SP}) \|Du\|(4\lambda B).
\]
Inserting this into (4.17), we then get
\[
\text{Cap}_{BV}(B \cap S) \leq 3(C_P + C_{SP}) \frac{r + 1}{a} \|Du\|(4\lambda B).
\]
Recalling the definition of \(a\), this implies \[ \left( \int_{2B} u^q \, d\mu \right)^{1/q} \leq 3(C_P + C_{SP}) \frac{r + 1}{\text{Cap}_{BV}(B \cap S)} \|Du\|(4\lambda B) \]
provided that \(\text{Cap}_{BV}(B \cap S) > 0\). Next we drop the nonnegativity assumption of \(u \in BV(X)\). We have \(u = u_+ - u_-\) with \(u_+, u_- \in BV(X)\). Letting \(S_+ := \{u_+^\wedge = u_+^\vee = 0\}\) and \(S_- := \{u_-^\wedge = u_-^\vee = 0\}\), we clearly have \(S \subset S_+\) and \(S \subset S_-\), and so
\[
\left( \int_{2B} |u|^q \, d\mu \right)^{1/q} \leq \left( \int_{2B} u_+^q \, d\mu \right)^{1/q} + \left( \int_{2B} u_-^q \, d\mu \right)^{1/q}
\]
\[
\leq \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{BV}(B \cap S_+)} \|Du_+\|(4\lambda B) + \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{BV}(B \cap S_-)} \|Du_-\|(4\lambda B)
\]
\[
\leq \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{BV}(B \cap S)} \|Du\|(4\lambda B)
\]
provided that \( \text{Cap}_{\text{BV}}(B \cap S) > 0 \), as by the coarea formula (2.6) it is easy to check that \( \| Du \|(4\lambda B) = \| Du_+\|(4\lambda B) + \| Du_-\|(4\lambda B) \). This completes the proof of the first inequality of the theorem. The second is proved similarly; we just need to drop the term \( \int_X |v| \, d\mu \) from (4.17) and proceed as above. \( \square \)

Using the above Maz’ya-type inequalities, we can now show the following Poincaré inequality for BV functions with zero boundary values. The proof is again similar to the one for Newton-Sobolev functions, see [5, Corollary 5.54].

**Corollary 4.19.** Let \( D \subset X \) be a bounded set with \( \text{Cap}_1(X \setminus D) > 0 \). Then there is \( C_D = C_D(C_d, C_P, \lambda, D) > 0 \) such that for all \( u \in \text{BV}_0(D) \),

\[
\int_X |u| \, d\mu \leq C_D \| Du \|(D).
\]

If \( D \) is \( \mu \)-measurable, the integral on the left-hand side can be taken with respect to \( D \).

**Proof.** Since \( D \) is bounded, we can take a ball \( B(x, r) \supset D \). By (3.5) we know that \( \text{Cap}_{\text{BV}}(X \setminus D) > 0 \), and by (3.4) we can conclude that \( \text{Cap}_{\text{BV}}(B(x, r) \setminus D) > 0 \) by making \( r \) larger, if necessary. Take \( u \in \text{BV}_0(D) \). By (3.6) and (3.5),

\[
\text{Cap}_{\text{BV}}(B(x, r) \setminus D) = \text{Cap}_{\text{BV}}(B(x, r) \cap \{ u^\wedge = u^\vee = 0 \} \setminus D) \\
\leq \text{Cap}_{\text{BV}}(B(x, r) \cap \{ u^\wedge = u^\vee = 0 \}).
\]

By Hölder’s inequality and the Maz’ya-type inequality (4.16),

\[
\frac{1}{\mu(B(x, 2r))} \int_X |u| \, d\mu = \int_{B(x, 2r)} |u| \, d\mu \\
\leq \left( \int_{B(x, 2r)} |u|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \\
\leq \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{\text{BV}}(B(x, r) \cap \{ u^\wedge = u^\vee = 0 \})} \| Du \|(B(x, 4\lambda r)) \\
\leq \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{\text{BV}}(B(x, r) \setminus D)} \| Du \|(B(x, 4\lambda r)) \quad \text{by (4.20)} \\
= \frac{3(C_P + C_{SP})(r + 1)}{\text{Cap}_{\text{BV}}(B(x, r) \setminus D)} \| Du \|(D)
\]
by Proposition 3.14. Thus we can choose

\[ C_D = \frac{3(C_P + C_{SP})(r + 1)\mu(B(x, 2r))}{\text{Cap}_{\text{BV}}(B(x, r) \setminus D)}. \]

Now we can prove the following property of the variational BV-capacity. Combined with Proposition 4.4, this shows that \( \text{cap}_{\text{BV}}(\cdot, D) \) is an outer measure on the subsets of \( D \).

**Proposition 4.21.** If \( D \subset X \) is bounded and \( A_1, A_2, \ldots \subset D \), then

\[ \text{cap}_{\text{BV}}\left( \bigcup_{j=1}^{\infty} A_j, D \right) \leq \sum_{j=1}^{\infty} \text{cap}_{\text{BV}}(A_j, D). \]

**Proof.** We can assume that the right-hand side is finite. Fix \( \varepsilon > 0 \). For each \( j \in \mathbb{N} \), choose \( u_j \in \text{BV}_0(D) \) such that \( 0 \leq u_j \leq 1 \), \( u_j^\wedge = 1 \) on \( A_j \), and

\[ \|Du_j\|(X) \leq \text{cap}_{\text{BV}}(A_j, D) + 2^{-j}\varepsilon. \]

Consider first the case \( \text{Cap}_1(X \setminus D) = 0 \). Let \( u := \min \left\{ 1, \sum_{j=1}^{\infty} u_j \right\} \), so that \( u^\wedge = 1 \) on \( \bigcup_{j=1}^{\infty} A_j \). By Lebesgue’s dominated convergence theorem, \( \min \left\{ 1, \sum_{j=1}^{N} u_j \right\} \to u \) in \( L^1(X) \) as \( N \to \infty \). Thus by lower semicontinuity of the total variation with respect to \( L^1 \)-convergence, we get

\[ \|Du\|(X) \leq \liminf_{N \to \infty} \|D\left( \min \left\{ 1, \sum_{j=1}^{N} u_j \right\} \right)\|(X) \]

\[ \leq \liminf_{N \to \infty} \|D\left( \sum_{j=1}^{N} u_j \right)\|(X) \]

\[ \leq \sum_{j=1}^{\infty} \|Du_j\|(X) \quad \text{by (3.2)} \]

\[ \leq \sum_{j=1}^{\infty} \text{cap}_{\text{BV}}(A_j, D) + \varepsilon. \]
Thus \( u \in \text{BV}(X) \) and then obviously \( u \in \text{BV}_0(D) \), since \( \text{Cap}_1(X \setminus D) = 0 \). Thus we get

\[
\text{cap}_{\text{BV}} \left( \bigcup_{j=1}^{\infty} A_j, D \right) \leq \| Du \| (X) \leq \sum_{j=1}^{\infty} \text{cap}_{\text{BV}}(A_j, D) + \varepsilon.
\]

Letting \( \varepsilon \to 0 \), we obtain the result.

Then consider the case \( \text{Cap}_1(X \setminus D) > 0 \). By Corollary 4.19, there is a constant \( C_D > 0 \) such that \( \| u_j \|_{L^1(X)} \leq C_D \| Du_j \| (X) \) for all \( j \in \mathbb{N} \). Thus \( \sum_{j=1}^{\infty} \| u_j \|_{\text{BV}(X)} < \infty \), so that defining \( u := \sum_{j=1}^{\infty} u_j \), by Proposition 3.15 we have that \( u \in \text{BV}_0(D) \). Clearly \( u^\wedge \geq 1 \) on \( \bigcup_{j=1}^{\infty} A_j \). Thus

\[
\text{cap}_{\text{BV}} \left( \bigcup_{j=1}^{\infty} A_j, D \right) \leq \| Du \| (X) \leq \sum_{j=1}^{\infty} \| Du_j \| (X) \leq \sum_{j=1}^{\infty} \text{cap}_{\text{BV}}(A_j, D) + \varepsilon.
\]

Again letting \( \varepsilon \to 0 \), we obtain the result.

The result given in the following theorem is perhaps unexpected, since the class of admissible test functions for \( \text{cap}_{\text{BV}} \) is so much larger than the class of admissible test functions for \( \text{cap}_{\text{lip}} \). Previously, a similar result was given in [22, Theorem 4.3], but there the variational BV-capacity \( \text{cap}_{\text{BV}}(K, \Omega) \) was defined by requiring the test functions to be compactly supported in \( \Omega \) and to take the value 1 in a neighborhood of \( K \). We need to obtain these two properties by using our previous results, but after that we employ similar methods as in [22].

**Theorem 4.22.** Let \( \Omega \subset X \) be open and let \( K \subset \Omega \) be compact. Then

\[
\text{cap}_{\text{lip}}(K, \Omega) = \text{cap}_{\text{BV}}(K, \Omega).
\]

**Proof.** One inequality is clear. To prove the opposite inequality, we can assume that \( \text{cap}_{\text{BV}}(K, \Omega) < \infty \). Fix \( \varepsilon > 0 \). By Theorem 4.13, we find an open set \( V \) such that \( K \subset V \subset \Omega \) and \( \text{cap}_{\text{BV}}(V, \Omega) < \text{cap}_{\text{BV}}(K, \Omega) + \varepsilon \). Then we find \( u \in \text{BV}_0(\Omega) \) such that \( 0 \leq u \leq 1 \), \( u^\wedge = 1 \) on \( V \), and \( \| Du \| (X) < \text{cap}_{\text{BV}}(K, \Omega) + \varepsilon \). By Proposition 3.19 we find functions \( u_i \in \text{Lip}_c(\Omega) \) such that

\[
\| u_i - u \|_{L^1(X)} < 1/i \quad \text{and} \quad \int_X g_{u_i} \, d\mu < \| Du \| (X) + 1/i, \quad i \in \mathbb{N}.
\]

(4.23)
Take \( \eta \in \text{Lip}_c(V) \) with \( 0 \leq \eta \leq 1 \) on \( X \) and \( \eta = 1 \) on \( K \). Then let
\[
v_i := \eta + (1 - \eta)u_i, \quad i \in \mathbb{N}.
\]
We have \( v_i \in \text{Lip}_c(\Omega) \) and \( v_i = 1 \) on \( K \), so that each \( v_i \) is admissible for \( \text{cap}_{\text{lip}}(K, \Omega) \). By a Leibniz rule, see [5, Lemma 2.18], we have
\[
g_{v_i} \leq (1 - \eta)g_{u_i} + |1 - u_i|g_\eta,
\]
and so by (4.23),
\[
\int_X g_{v_i} \, d\mu \leq \int_X g_{u_i} \, d\mu + \sup_X g_\eta \int_V |1 - u_i| \, d\mu \\
\leq \|Du\|(X) + 1/i + \sup_X g_\eta \int_V |1 - u_i| \, d\mu \\
= \|Du\|(X) + 1/i + \sup_X g_\eta \int_V |u - u_i| \, d\mu \\
\to \|Du\|(X) \quad \text{as } i \to \infty.
\]
Thus for some sufficiently large index \( i \in \mathbb{N} \), we have
\[
\text{cap}_{\text{lip}}(K, \Omega) \leq \int_X g_{v_i} \, d\mu \leq \|Du\|(X) + \varepsilon < \text{cap}_{\text{BV}}(K, \Omega) + 2\varepsilon.
\]
Letting \( \varepsilon \to 0 \), we conclude the proof.

**Example 4.24.** Let \( X = \mathbb{R} \) (unweighted) and choose
\[
K = [-2, -1] \cup [1, 2] \quad \text{and} \quad \Omega = (-3, 0) \cup (0, 3).
\]
Then it is straightforward to show that
\[
\text{cap}_{\text{lip}}(K, \Omega) = \text{cap}_1(K, \Omega) = \text{cap}_{\text{BV}}(K, \Omega) = 4.
\]
On the other hand, if we defined \( \text{cap}_{\text{BV}}(K, \Omega) \) by only requiring the test functions to satisfy \( u = 0 \) on \( \mathbb{R} \setminus \Omega \) (almost everywhere), then we would have \( \text{cap}_{\text{BV}}(K, \Omega) = 2 \). In this sense, our current definition of \( \text{cap}_{\text{BV}} \) can be considered to be the natural one.

Moreover, if we defined \( \text{cap}_{\text{BV}}(K, \Omega) \) by requiring the test functions to satisfy \( u^\vee \geq 1 \) on \( K \) instead of \( u^\wedge \geq 1 \), we would get a generally smaller but comparable quantity, see [22, Example 4.4, Theorem 4.5, Theorem 4.6].
More generally, consider \( \text{cap}_{\text{lip}}(A, D) \) for \( A \subset D \subset X \). Recall from Example 4.2 that it is possible to have \( \text{cap}_1(A, D) = 0 \) but \( \text{cap}_{\text{lip}}(A, D) = \infty \). According to [7, Example 6.1], when \( 1 < p < \infty \) it is also possible to have
\[
0 < \text{cap}_p(K, D) < \text{cap}_{\text{lip}, p}(K, D) < \infty
\]
for a compact set \( K \subset \text{int} \ D \), where \( \text{cap}_{\text{lip}, p} \) is defined by requiring the test functions in the definition of \( \text{cap}_p \) to be Lipschitz.

**Open Problem.** Do we have \( \text{cap}_1(K, D) = \text{cap}_{\text{lip}}(K, D) \) for every \( D \subset X \) and compact \( K \subset \text{int} \ D \)?

The following example shows that \( \text{cap}_{\text{BV}}(K, D) \) and \( \text{cap}_1(K, D) \) can differ even for a compact \( K \subset \text{int} \ D \).

**Example 4.25.** Let \( X = \mathbb{R} \), define a weight function \( w := 1 + \chi_{[0,3]} \), and let \( d\mu := w \, d\mathcal{L}^1 \). Choose \( D = [0,3], \, K = [1,2], \) and \( u := \chi_{[0,3]} \), so that \( u \in \text{BV}_0(D) \). It is easy to check that \( \|Du\|(X) = 2 \), and so \( \text{cap}_{\text{BV}}(K, D) \leq 2 \). On the other hand, clearly
\[
\liminf_{i \to \infty} \int_{\mathbb{R}} g_{u_i} \, d\mu = 2 \liminf_{i \to \infty} \int_{\mathbb{R}} g_{u_i} \, d\mathcal{L}^1 \geq 4
\]
for every sequence of functions \( (u_i) \subset \text{Lip}_{\text{loc}}(X) \) with \( u_i \to u \) in \( L^1(X) \) and \( \text{spt} \ u_i \subset D \). Thus Proposition 3.19 can fail if \( \Omega \) is not open.

Similarly, \( \int_{\mathbb{R}} g_v \, d\mu \geq 4 \) for every \( v \in N_0^{1,1}(D) \) with \( v = 1 \) on \( K \). Thus \( \text{cap}_1(K, D) \geq 4 \) (in fact, equality holds). Thus \( \text{cap}_{\text{BV}}(K, D) < \text{cap}_1(K, D) \).

**Acknowledgments.** The research was funded by a grant from the Finnish Cultural Foundation. Most of the research for this paper was conducted during the author’s visit to the University of Cincinnati, whose hospitality the author wishes to acknowledge. The author also wishes to thank Nageswari Shanmugalingam for helping to derive Maz’ya-type inequalities for BV functions, and Anders and Jana Björn for discussions on variational \( p \)-capacities.

**References**

[1] L. Ambrosio, *Fine properties of sets of finite perimeter in doubling metric measure spaces*, Calculus of variations, nonsmooth analysis and related topics. Set-Valued Anal. 10 (2002), no. 2-3, 111–128.
[2] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] L. Ambrosio, M. Miranda, Jr., and D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.

[4] L. Beck and T. Schmidt, *Convex duality and uniqueness for BV-minimizers*, J. Funct. Anal. 268 (2015), no. 10, 3061–3107.

[5] A. Björn and J. Björn, *Nonlinear potential theory on metric spaces*, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.

[6] A. Björn and J. Björn, *Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology*, Rev. Mat. Iberoam. 31 (2015), no. 1, 161–214.

[7] A. Björn and J. Björn, *The variational capacity with respect to nonopen sets in metric spaces*, Potential Anal. 40 (2014), no. 1, 57–80.

[8] A. Björn, J. Björn, and V. Latvala, *Sobolev spaces, fine gradients and quasicontinuity on quasiopen sets*, Ann. Acad. Sci. Fenn. Math. 41 (2016), no. 2, 551–560.

[9] A. Björn, J. Björn, and N. Shanmugalingam, *Quasicontinuity of Newton-Sobolev functions and density of Lipschitz functions on metric spaces*, Houston J. Math. 34 (2008), no. 4, 1197–1211.

[10] A. Björn, J. Björn, and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions on metric spaces*, J. Reine Angew. Math. 556 (2003), 173–203.

[11] A. Björn, J. Björn, and N. Shanmugalingam, *The Dirichlet problem for p-harmonic functions with respect to the Mazurkiewicz boundary, and new capacities*, J. Differential Equations 259 (2015), no. 7, 3078–3114.
[12] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. 7 1969 243–268.

[13] M. Carriero, G. Dal Maso, A. Leaci, and E. Pascali, *Relaxation of the nonparametric plateau problem with an obstacle*, J. Math. Pures Appl. (9) 67 (1988), no. 4, 359–396.

[14] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics series, CRC Press, Boca Raton, 1992.

[15] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969 xiv+676 pp.

[16] H. Federer and W. P. Ziemer, *The Lebesgue set of a function whose distribution derivatives are p-th power summable*, Indiana Univ. Math. J. 22 (1972/73), 139–158.

[17] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984. xii+240 pp.

[18] H. Hakkarainen and J. Kinnunen, *The BV-capacity in metric spaces*, Manuscripta Math. 132 (2010), no. 1-2, 51–73.

[19] H. Hakkarainen, J. Kinnunen, and P. Lahti, *Regularity of minimizers of the area functional in metric spaces*, Adv. Calc. Var. 8 (2015), no. 1, 55–68.

[20] H. Hakkarainen, J. Kinnunen, P. Lahti, and P. Lehtelä, *Relaxation and integral representation for functionals of linear growth on metric measure spaces*, Anal. Geom. Metr. Spaces 4 (2016), 288–313.

[21] H. Hakkarainen, R. Korte, P. Lahti, and N. Shanmugalingam, *Stability and continuity of functions of least gradient*, Anal. Geom. Metr. Spaces 3 (2015), 123–139.

[22] H. Hakkarainen and N. Shanmugalingam, *Comparisons of relative BV-capacities and Sobolev capacity in metric spaces*, Nonlinear Anal. 74 (2011), no. 16, 5525–5543.
[23] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext. Springer-Verlag, New York, 2001. x+140 pp.

[24] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006. xii+404 pp.

[25] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. 181 (1998), no. 1, 1–61.

[26] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *Lebesgue points and capacities via the boxing inequality in metric spaces*, Indiana Univ. Math. J. 57 (2008), no. 1, 401–430.

[27] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *Pointwise properties of functions of bounded variation in metric spaces*, Rev. Mat. Complut. 27 (2014), no. 1, 41–67.

[28] J. Kinnunen, R. Korte, N. Shanmugalingam, and H. Tuominen, *The De Giorgi measure and an obstacle problem related to minimal surfaces in metric spaces*, J. Math. Pures Appl. (9) 93 (2010), no. 6, 599–622.

[29] R. Korte, P. Lahti, X. Li, and N. Shanmugalingam, *Notions of Dirichlet problem for functions of least gradient in metric measure spaces*, preprint 2016. https://arxiv.org/abs/1612.06078

[30] P. Lahti, *A Federer-style characterization of sets of finite perimeter on metric spaces*, preprint 2016. https://arxiv.org/abs/1612.06286

[31] P. Lahti, *A notion of fine continuity for BV functions on metric spaces*, Potential Anal. 46 (2017), no. 2, 279–294.

[32] P. Lahti, *Superminimizers and a weak Cartan property for p = 1 in metric spaces*, preprint 2017. https://arxiv.org/abs/1706.01873

[33] P. Lahti and N. Shanmugalingam, *Fine properties and a notion of quasicontinuity for BV functions on metric spaces*, J. Math. Pures Appl. (9) 107 (2017), no. 2, 150–182.
[34] P. Lahti and N. Shanmugalingam, *Trace theorems for functions of bounded variation in metric spaces*, preprint 2015. https://arxiv.org/abs/1507.07006

[35] J. Malý and W. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp.

[36] J. M. Mazón, J. D. Rossi, and S. S. De León, *Functions of least gradient and 1-harmonic functions*, Indiana Univ. Math. J. 63 No. 4 (2014), 1067–1084.

[37] V. G. Maz’ya, *Sobolev spaces*, Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. xix+486 pp.

[38] V. G. Maz’ya, *The Dirichlet problem for elliptic equations of arbitrary order in unbounded domains*, Dokl. Akad. Nauk SSSR 150 1963 1221–1224.

[39] M. Miranda, Jr., *Functions of bounded variation on “good” metric spaces*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[40] T. Mäkäläinen, *Adams inequality on metric measure spaces*, Rev. Mat. Iberoam. 25 (2009), no. 2, 533–558.

[41] N. Shanmugalingam, *Harmonic functions on metric spaces*, Illinois J. Math. 45 (2001), no. 3, 1021–1050.

[42] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana 16(2) (2000), 243–279.

[43] P. Sternberg, G. Williams, and W. Ziemer, *Existence, uniqueness, and regularity for functions of least gradient*, J. Reine Angew. Math. 430 (1992), 35–60.

[44] W. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Graduate Texts in Mathematics, 120. Springer-Verlag, New York, 1989.
[45] W. Ziemer and K. Zumbrun, *The obstacle problem for functions of least gradient*, Math. Bohem. 124 (1999), no. 2-3, 193–219.

Address:

Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden
E-mail: panu.lahti@aalto.fi