SOME ELEMENTARY EXAMPLES OF NON-LIFTABLE VARIETIES

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Abstract. We present some simple examples of smooth projective varieties in positive characteristic, arising from linear algebra, which do not admit a lifting neither to characteristic zero, nor to the ring of second Witt vectors. Our first construction is the blow-up of the graph of the Frobenius morphism of a homogeneous space. The second example is a blow-up of $\mathbb{P}^3$ in a ‘purely characteristic-$p$’ configuration of points and lines.

1. Introduction

Various theorems in modern algebraic geometry are proved using characteristic $p$ methods along the following lines. Given a complex algebraic variety $X$, one reduces the variety mod $p$, exploits the Frobenius morphism on the reduction $X_p$, and deduces statements about the original $X$. Similarly, characteristic zero (particularly complex analytic) methods are employed to study varieties in positive characteristic. The main technical obstacle is that, while every characteristic zero variety can be reduced mod $p$, not every variety in positive characteristic arises as the reduction mod $p$ of a variety in characteristic zero. The first example of such a variety was given by Serre [Ser61].

It turns out that for many purposes, one does not need to lift a given variety all the way to characteristic zero, and it suffices to have a lifting modulo $p^2$. For example, Deligne and Illusie [DI87] showed that for a smooth variety $X$ over a perfect field $k$ of characteristic $p > \dim X$ admitting a lifting to $W_2(k)$ (the ring of Witt vectors of length 2), the Hodge–de Rham spectral sequence degenerates, and the Kodaira vanishing theorem holds. More recently, Langer [Lan16] showed that the logarithmic Bogomolov–Miyaoka–Yau inequality holds for surfaces liftable to $W_2(k)$ (as long as $p > 2$). Counterexamples to Kodaira vanishing in positive characteristic given by Raynaud [Ray78] give the first example of varieties which do not lift to $W_2(k)$. Subsequently, a rational example was given by Lauritzen and Rao [LR97]. In the positive direction, it is known that every Frobenius split variety lifts to $W_2(k)$ [Lan15, Proposition 8.4].

In this paper, we construct new examples of smooth projective varieties that do not admit lifts neither to characteristic zero, nor to $W_2(k)$ (some of them do not even lift to any ring $A$ with $pA \neq 0$). However it turns out that they avoid standard characteristic $p$ pathologies, in particular they satisfy the following

**Good properties:**

1. they are smooth, projective, rational, and simply connected,
2. their classes in the Grothendieck ring of varieties are polynomials in the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$ with non-negative integer coefficients,
3. their $\ell$-adic integral cohomology rings are generated by algebraic cycles,
4. their integral crystalline cohomology groups are torsion-free $F$-crystals,
5. their Hodge–de Rham and conjugate spectral sequences degenerate, they are ordinary in the sense of Bloch–Kato, and of Hodge–Witt type (cf. [15] for the relevant definitions).

Since our constructions are very simple, we try to aim our exposition at non-experts, and go for elementary arguments whenever possible.

The first construction is given by the blow-up of the two-fold self product of a suitable projective homogeneous space $\neq \mathbb{P}^n$ along the graph of its Frobenius morphism. The easiest examples of such homogeneous spaces being the three-dimensional complete flag variety $\text{SL}_3/B$ (isomorphic to the incidence variety $\{x_0y_0 + x_1y_1 + x_2y_2 = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2$) and the three-dimensional smooth quadric hypersurface $Q = \{x_0^2 + x_1x_2 + x_3x_4 = 0\} \subset \mathbb{P}^4$, the smallest non-liftable examples given by the construction are six-dimensional with Picard numbers five and three, respectively. The proof of the above good properties and non-liftability is given in Theorem 2.1.
The second construction is the following. Let $X$ be the variety obtained from $\mathbb{P}^3$ by (1) blowing up all $\mathbb{F}_p$-rational points, and (2) blowing up the strict transforms of all lines connecting $\mathbb{F}_p$-rational points. Then $X$ satisfies (1)--(5) above, but does not admit a lift to any ring $A$ with $pA \neq 0$. The proofs are presented in Theorem 3.1.

Both in Theorem 2.1 and Theorem 5.1, the proofs of non-liftability use the key observation (cf. [LS14], and Proposition 1.3 below) that if the blow-up of a smooth variety $X$ along a smooth subvariety $Z$ lifts, then both $X$ and $Z$ lift. In Theorem 2.1, if $X$ was liftable, the homogeneous space $Y$ would be liftable together with Frobenius, which is known to be impossible by the work of Paranjape–Srinivas [PSS00] (for lifts to characteristic zero) and Buch–Thomsen–Lauritzen–Mehta [BTLM97] (for lifts to $W_2(\mathbb{k})$). In Theorem 5.1, we show that the liftability of $X$ would imply the liftability of the arrangement of all $\mathbb{F}_p$-rational points in $\mathbb{P}^2$ preserving the incidence relations; thus non-liftability is established by means of elementary linear algebra. The properties (1)--(5) in both theorems are established quite easily using standard formulas expressing the cohomology of a blow-up which we recall in 4.3.

1.1. Notation. Throughout $k$ denotes a perfect field of characteristic $p > 0$. For any $k$-scheme $X$ by $X^{(1)}$ we denote the Frobenius pullback $X^{(1)} \overset{\text{def}}{=} X \times_{\text{Spec}(k), \mathbb{F}_p} \text{Spec}(k)$ and by $F_{X/k} : X \to X^{(1)}$ the relative Frobenius of $X$ over $k$. We say that a scheme $X/k$ admits a $W_2(\mathbb{k})$-lifting if there exists a flat $W_2(k)$-scheme $\tilde{X}$ such that $\tilde{X} \times_{\text{Spec}(W_2(k))} \text{Spec}(k) \simeq X$. Finally, we say that a scheme $X$ lifts to $W_2(k)$ compatibly with Frobenius if there exists a $W_2(k)$-lifting $\tilde{X}$ of $X$ together with a morphism $\tilde{F}_{X/k} : \tilde{X} \to \tilde{X}^{(1)} \overset{\text{def}}{=} \tilde{X} \times_{\text{Spec}(W_2(k)), \sigma} \text{Spec}(W_2(k))$ restricting to the relative Frobenius morphism $F_{X/k} : X \to X^{(1)}$. For schemes defined over the field $\mathbb{F}_p$, the absolute Frobenius morphism is in fact $\mathbb{F}_p$-linear and therefore the relative Frobenius morphism can be interpreted as an endomorphism $\tilde{F}_{X/k} : X \to X^{(1)} \simeq X$.

By $L_{X/k}$ we denote the cotangent complex of a scheme $X$ over $k$. Moreover, by $\text{Def}_X$ we mean the deformation functor of $X$, that is, a covariant functor from the category $\text{Art}_{W(k)}(k)$ of Artinian local $W(k)$-algebras with residue field $k$ to the category of sets defined by the formula:

$$\text{Art}_{W(k)}(k) \ni (A, m_A) \mapsto \text{Def}_X(A) \overset{\text{def}}{=} \left\{ \text{isomorphism classes of flat deformations of } X \text{ over } \text{Spec}(A) \right\}.$$ 

Similarly, if $Z = \{Z_i\}_{i \in I}$ is a family of closed subschemes of $X$ indexed by a preorder $I$ (i.e., a set with a reflexive and transitive binary relation), such that $Z_i$ is a closed subscheme of $Z_j$ whenever $i \leq j$ (in other words, $I$ is a small category whose morphism sets have at most one element, and $Z$ is a functor from $I$ to the category of closed subschemes of $X$), we denote by $\text{Def}_{f, Z}$ the functor of flat deformations of $X$ together with compatible embedded deformations of the $Z_i$, preserving the inclusion relations given by the relation $\leq$. If $f : X \to Y$ is a map of $k$-schemes, we denote by $\text{Def}_f$ the functor of flat deformations of $X$, and $Y$ along with a deformation of $f$.

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2. The first construction

We fix a semisimple algebraic group $G$ over $k = \mathbb{F}_p$, a reduced parabolic subgroup $P \subseteq G$, and set $Y = G/P$. We assume that either $G$ is of type $A$ and $Y$ is not a projective space, or that $P$ is contained in a maximal parabolic subgroup as listed in [BTLM97] 4.3.1–4.3.7 (these are the cases in which we know that $Y$ does not lift to $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$ together with Frobenius). For example, $Y$ could be the Grassmannian $\text{Gr}(n, k)$ ($1 < k < n - 1$) or the full flag variety $\text{SL}_n/B$ ($n \geq 3$, $B$ = upper-triangular matrices), or a smooth quadric hypersurface in $\mathbb{P}^n$, $n \geq 4$. Presumably all homogeneous spaces which are not toric (i.e., not a product of projective spaces) do not admit a lift to $W_2(k)$ together with Frobenius.
Theorem 2.1. Let $\Gamma_F \subseteq X \times Y$ be the graph of the Frobenius morphism $F_Y : Y \to Y$. Let $X = \text{Bl}_{\Gamma_F}(Y \times Y)$ be the blow-up of $Y \times Y$ along $\Gamma_F$, and $X' = \text{Bl}_Y(Y \times Y)$ the blow-up of $Y \times Y$ along the diagonal. Then $X$ and $X'$ share the good properties of Section 4 and moreover:

a) they are étale homeomorphic, i.e., their étale sites are equivalent;
b) their $t$-adic integral cohomology rings are isomorphic as Galois representations
c) their integral crystalline cohomology groups are isomorphic torsion-free $F$-crystals.

However, $X'$ admits a projective lift to $W(k)$, while $X$ lifts neither to characteristic zero (even formally), nor to $W_2(k)$.

Proof. Good property (1) follows from Bruhat decomposition and the birational invariance of the étale fundamental group of smooth varieties. Properties (2)–(5) follow from the results of sections 4.4–4.5. Property (a) follows from the existence of the following cartesian diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{u} & X' \\
\downarrow f & & \downarrow f' \\
Y \times Y & \xrightarrow{id \times F_Y} & Y \times Y,
\end{array}
$$

where $f$ and $f'$ are the respective blow-up maps. Indeed, the Frobenius map $F_Y : X' \to X'$ and the composition $(F_Y \times id) \circ f' : X' \to Y \times Y$ yield a map $v : X' \to X$ making the diagram

$$
\begin{array}{ccc}
& & \\
& & \\
& & \\
& & \\
F_Y & & F_Y \\
\downarrow & & \downarrow \\
Y \times Y & \xrightarrow{id \times F_Y} & Y \times Y,
\end{array}
$$

commute. In particular, $v \circ u = F_X$ and $u \circ v = F_{X'}$. Since $F_X$ and $F_{X'}$ are étale homeomorphisms by [Gro77, XIV=XV §1 n° 2, Pr. 2(c)], $u$ and $v$ are étale homeomorphisms as well. Property (b) follows from (a). Finally, property (c) follows from the blow-up formula [4.3]. We remark that the crystalline cohomology algebras $H^{n}_{\text{cris}}(X/W)$ and $H^{n}_{\text{cris}}(X'/W)$ are not isomorphic, but become so after inverting $p$.

We now prove that $X'$ lifts to $W(k)$ projectively and that $X$ does not lift either to $W_2(k)$ or any ramified extension of $W(k)$. For the first claim, we observe that $Y$ lifts to a projective scheme $\mathcal{Y}$ over $W(k)$ and consequently $X' = \text{Bl}_{\mathcal{Y}}(Y \times W(k) \mathcal{Y})$ is a projective lifting of $X'$. We now proceed to the second claim. We begin with a proposition addressing Frobenius liftability of homogeneous spaces and describing their cohomological properties necessary to apply deformation theoretic results stated in 4.4.

Proposition 2.2. Let $Y$ be a homogeneous space over $k$ of a semisimple algebraic group $G$ not isomorphic to any projective space. Then, $Y$ does not admit a $W_2(k)$-lifting compatible with Frobenius. Moreover, it satisfies $H^1(Y, T_Y) = 0$ and $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$.

Proof. For the part of the proof concerning Frobenius liftability see [BTLM97, Theorem 6]. Vanishing of $H^1(Y, T_Y)$ follows from [Dem77, Théorème 2]. Finally, $H^i(Y, \mathcal{O}_Y) = 0$ is the consequence of Kempf vanishing (i.e., a characteristic $p$ analogue of the Borel–Weil–Bott theorem) as 0 is a dominant weight for the parabolic subgroup of $G$ corresponding to $Y$.

We show $X$ does not lift to $W_2(k)$. Assume the contrary, i.e., that there exists a $W_2(k)$-lifting of $\text{Bl}_{\Gamma_{F_Y/k}}(Y \times Y)$. By Proposition 4.4 there exist two liftings $\tilde{Y}$ and $\tilde{Y}'$ of $Y$ together with a lifting $\tilde{F}_G/k : \tilde{Y} \to \tilde{Y}'$ of $F_Y/k : Y \to Y$. However, by the property $H^1(Y, T_Y) = 0$ the homogeneous space $Y$ is rigid, which implies that the lifting $\tilde{Y}'$ is isomorphic to $\tilde{Y}$. This implies that $Y$ is $W_2(k)$-liftable compatibly with Frobenius, which contradicts Proposition 2.2.

Finally, we address characteristic $0$ non-liftability of $X$. Again, we reason by contradiction. Any characteristic $0$ lifting of $X$ induces a formal lifting of $X$ which be Proposition 4.3 and rigidity of $Y$ gives a formal lifting of a non-trivial endomorphism $F_Y : Y \to Y$. By the Grothendieck
algebraization theorem the formal lifting of the finite morphism $F_Y$ extends to an algebraic lifting which contradicts the final result of [PSS89] stating that homogeneous spaces in characteristic 0 not isomorphic to products of projective spaces admit no non-trivial endomorphisms.

3. Second Construction

We work over an algebraically closed field $k$ of characteristic $p$. Let $P = \mathbb{P}^3(\mathbb{F}_p) \subseteq \mathbb{P}^3$ be the set of all $\#P^4(\mathbb{F}_p) = 1 + p + p^2 + p^3$ $\mathbb{F}_p$-rational points, let $Y = Bl_p \mathbb{P}^1$, and let $L$ be the set of $\#G(2,4)(\mathbb{F}_p) = \left(\frac{1 + p + p^2 + p^3}{2}\right) = 1 + p + 2p^2 + p^3 + p^4$ lines in $\mathbb{P}^3$ meeting $P$ at least twice. Finally, let $\hat{L} \subseteq Y$ be the set of the strict transforms of all elements of $L$, and let $X = Bl_{\hat{L}} Y$.

**Theorem 3.1.** The threefold $X$ has the good properties from the introduction, but does not admit a lift to any ring $A$ with $pA \neq 0$.

For the good properties (1)-(5), we argue exactly as in the previous section. The proof that $X$ does not deform to any algebra $A$ with $pA \neq 0$ consists of the following three propositions.

**Proposition 3.2.** Let $A$ be an object of $\textbf{Art}_{W(k)}(k)$, and suppose that $X$ lifts to $A$. Then $\mathbb{P}^3_3$ lifts to $A$ together with all $\mathbb{F}_p$-rational points and lines, preserving the incidence relations.

**Proof.** Let $E$ be the set consisting of the preimages in $Y$ of the elements of $P$, $F$ the set of preimages in $X$ of the elements of $L$. Finally, let $Q = (\bigcup \hat{L}) \cap (\bigcup E)$ (treated as a set of points). We have the following chain of natural transformations between various deformation functors:

$$
\text{Def}_X \rightarrow \text{Def}_{X,F} \rightarrow \text{Def}_{Y,L} \rightarrow \text{Def}_{Y,L,E} \rightarrow \text{Def}_{Y,L,E,Q} \rightarrow \text{Def}_{\mathbb{F}_3^4,L,P}.
$$

We remind the reader of our convention (cf. [1]) that for a family of closed subschemes $Z = \{Z_t\}_{t \in I}$ of a scheme $X$ indexed by a preorder $I$, $\text{Def}_{X,Z}$ is the functor of deformations of $X$, together with embedded deformations of $Z_i$, preserving the inclusion relations $Z_i \subseteq Z_{i'}$ for $i \leq i'$. Above, we give the families $F, L, \hat{L}, E$ the trivial order, and order $L \cup E \cup Q$ and $L \cup P$ by inclusion. In particular, the functor $\text{Def}_{Y,L,E,Q}$ parametrizes deformations of $Y$ together with the strict transforms of the $\mathbb{F}_p$-rational lines (i.e., $\hat{L}$) and the preimages of the $\mathbb{F}_p$-rational points (i.e., $E$) in $\mathbb{P}^3_3$ such that their mutual intersections are flat over the base (i.e., induce a compatible embedded deformation of $Q$). Similarly, $\text{Def}_{\mathbb{F}_3^4,L,P}$ is the functor of deformations of $\mathbb{P}^3_3$ together with all the $\mathbb{F}_p$-rational points and lines, preserving the incidence relations. We discuss the maps in this chain below.

The maps $\text{Def}_{X,F} \rightarrow \text{Def}_X$, $\text{Def}_{Y,L,E} \rightarrow \text{Def}_{Y,L}$, and $\text{Def}_{Y,L,E,Q} \rightarrow \text{Def}_{Y,L,E}$ are the forgetful transformations. The first two are isomorphisms by Proposition 4.3(2), and the last map is an isomorphism by Corollary 4.3 of Lemma 4.3 applied to the local equations of $E$ and $\hat{L}$.

The maps $\text{Def}_{X,F} \rightarrow \text{Def}_{Y,L}$ and $\text{Def}_{Y,L,E,Q} \rightarrow \text{Def}_{\mathbb{F}_3^4,L,P}$ are the maps of Proposition 4.3(1). For the latter, strictly speaking, Proposition 4.3(1) yields a map $\text{Def}_{Y,L,E,Q} \rightarrow \text{Def}_{\mathbb{F}_3^4,Z}$, where $Z = \{Z_s\}_{s \in S}$ is the ‘image’ of $\hat{L} \cup E \cup Q$, defined as follows. Let $S = L \cup P \cup K$ where $K = \{(x, \ell) \in P \times L : x \in \ell\}$, given the ordering whose nontrivial relations are $(\ell, x) \leq \ell$ and $(x, \ell) \leq x$ for $x \in P, \ell \in L, (x, \ell) \in K$. Then set $Z_\ell = \ell$ for $\ell \in L, Z_x = x$ for $x \in P$, and $Z_{(x,\ell)} = x$ for $(x,\ell) \in K$. For an algebra $A$, an element of $\text{Def}_{\mathbb{F}_3^4,Z}$ is thus given by a deformation of $\mathbb{P}_3^4$ together with deformations of $L \cup E$, preserving the relations of $(x,\ell) \subseteq \hat{x}$ and $Z_{(x,\ell)} \subseteq \hat{\ell}$ for $(x,\ell) \in K$ (here the tildes mean the corresponding deformations over $A$). But each $x$ is a point, so $Z_{(x,\ell)} \subseteq \hat{x}$ implies $\tilde{Z}_{(x,\ell)} = \hat{x}$, and the deformation of $(\mathbb{P}_3^4, Z)$ simplifies to a deformation of $(\mathbb{P}_3^4, L \cup P)$ preserving the incidence relations. Thus $\text{Def}_{\mathbb{F}_3^4,Z}$ can be identified with $\text{Def}_{\mathbb{F}_3^4,L,P}$.

**Remark 3.3.** Since we will have to deal with a little bit of elementary projective geometry and matroid representability over arbitrary rings, let us fix some conventions. Let $A$ be a local ring with residue field $k$. A projective $n$-space $\mathbb{P}$ over $A$ is an $A$-scheme isomorphic to $\mathbb{P}^n_A$, and a $d$-dimensional linear subspace $L$ of $\mathbb{P}$ is a closed subscheme of $\mathbb{P}$ which is flat over $A$ and such that $L \otimes k$ is a linear subspace of $\mathbb{P} \otimes k$. Zero-dimensional linear subspaces of $\mathbb{P}$ can be identified with the set $\mathbb{P}(A)$. If $x, y \in \mathbb{P}(A)$ are points whose images in $\mathbb{P}(k)$ are distinct, there exists a unique line (i.e., a one-dimensional linear subspace) $\ell(x,y)$ containing both $x$ and $y$. We say that
points $x, y, z$ are collinear (resp. coplanar) if they lie on one line (resp. 2-dimensional subspace). If $x_0, \ldots, x_n, z$ are points whose images in $\mathbb{P}(k)$ are in general position, there exists a unique isomorphism $\phi : \mathbb{P} \to \mathbb{P}^n_k$ such that $\phi(x_i) = e_i := (0 : \ldots : 0 : 1 : 0 : \ldots : 0)$ (with 1 on the $i$-th coordinate) and $\phi(z) = f := (1 : \ldots : 1)$. In particular, if $A \in \text{Art}_{W(k)}(k)$, and $S$ is a configuration of linear subspaces of $\mathbb{P}^n_k$ ordered by inclusion, containing the points $e'_i = (0 : \ldots : 0 : 1 : 0 : \ldots : 0)$ and $f' = (1 : \ldots : 1)$, we can identify the deformation functor $\text{Def}^\mathbb{P}_k S(A)$ with the set of all families of linear subspaces $\tilde{S}$ in $\mathbb{P}^n_A$ which yield the given $S$ upon restriction to $k$, and such that $e_i' = e_i$ and $\tilde{f}' = f$.

**Proposition 3.4.** Suppose that $\mathbb{P}^3_k$ lifts to an Artinian $W(k)$-algebra $A$ together with all $\mathbb{F}_p$-rational points, preserving collinearity. Then the same holds for $\mathbb{P}^2_k$.

**Proof.** The key observation is that coplanarity is also preserved, i.e., that $\text{Def}^\mathbb{P}_k F_{L,P} = \text{Def}^\mathbb{P}_k F_{L,\cup L,P}$, where $H$ denotes the set of all $\mathbb{F}_p$-rational hyperplanes in $\mathbb{P}^n_k$ (with $H \cup L \cup P$ ordered by inclusion). Indeed, let $A$ be an object of $\text{Art}_{W(k)}(k)$, and suppose we are given an element of $\text{Def}^\mathbb{P}_k F_{L,\cup L,P}(A)$, which by simple rigidification (e.g., the requirement that the points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and $(1 : 1 : 1)$ do not deform) can be identified with a configuration of points $\tilde{x}$ and lines $\tilde{\ell}$ in $\mathbb{P}^3_A$, indexed by $P$ and $L$ respectively, such that $\tilde{x} \subseteq \tilde{\ell}$ whenever $x \in \ell$. To get an element of $\text{Def}^\mathbb{P}_k F_{L,\cup L,P}$, it suffices to show that whenever $x_1, x_2, x_3, x_4 \in P$ is a quadruple of coplanar points, the points $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \in \mathbb{P}^3(A)$ are coplanar. If two of the points $x_i$ coincide, there is nothing to show, and similarly if all four lie on a line. Otherwise, let $\ell_{12} = \ell(x_1, x_2)$ and $\ell_{34} = \ell(x_3, x_4)$, then $\ell_{12} \cap \ell_{34} = \ell(\tilde{x}_1, \tilde{x}_2)$ and $\ell_{12} \cap \ell_{34} = \ell(\tilde{x}_3, \tilde{x}_4)$. Since the $x_i$ are coplanar, the lines $\ell_{12}$ and $\ell_{34}$ intersect in a unique point $y \in P$. Then $y \in \ell_{12} \cap \ell_{34} = \ell(\tilde{x}_1, \tilde{x}_2) \cap \ell(\tilde{x}_3, \tilde{x}_4)$. Thus the hyperplane through $y, \tilde{x}_1, \tilde{x}_2$ yields a lift of the hyperplane through $x_1, x_2, x_3, x_4$.

Since coplanarity is preserved, we can forget everything except for the plane $x_0 = 0$ (say) and get a desired lifting of $\mathbb{P}^2_k$. Equivalently, we could have used a projection from one of the $\mathbb{F}_p$-rational points. 

To finish, we prove that the matroid $\mathbb{P}^2(\mathbb{F}_p)$ does not admit a projective representation over any ring $A$ with $pA \neq 0$. For $A$ a field, this is well-known (cf. e.g. [Gor88, §2]), but we need to make sure that the proof works for arbitrary rings.

**Proposition 3.5.** Let $A$ be a ring, $\rho : \mathbb{P}^2(\mathbb{F}_p) \to \mathbb{P}^2(A)$ a map taking triples of collinear points to triples of coplanar points. Then $pA = 0$.

**Proof.** Changing coordinates in $\mathbb{P}^2(A)$, we can assume that

$$\rho(1 : 0 : 0) = (1 : 0 : 0), \quad \rho(0 : 1 : 0) = (0 : 1 : 0), \quad \rho(0 : 0 : 1) = (0 : 0 : 1),$$

and $\rho(1 : 1 : 1) = (1 : 1 : 1)$. Thus

$$\rho(1 : 1 : 0) = \rho(\ell((0 : 0 : 1),(1 : 1 : 0)) \cap \ell((0 : 1 : 0),(0 : 1 : 0))) = (1 : 1 : 0)$$

as well. For $n \in \mathbb{Z}$, let $P_n = (n : 0 : 1), Q_n = (n + 1 : 1 : 1) \in \mathbb{P}^2(\mathbb{F}_p)$, and let $P_n', Q_n' \in \mathbb{P}^2(A)$ be the points with the same coordinates as $P_n, Q_n$. We check by induction on $n \geq 0$ that $\rho(P_n) = P_n'$ and $\rho(Q_n) = Q_n'$; the base case is ok, and for the induction step we note that $P_n = \ell(Q_{n-1}, (0 : 1 : 0)) \cap \ell(P_0, (1 : 0 : 0), Q_n = \ell(P_n, (1 : 0)) \cap \ell(Q_0, (1 : 0 : 0))$, and that the same statements hold with the primes (see Figure 3). Thus $(p : 0 : 1) = \rho(p : 0 : 1) = (0 : 0 : 1)$, and hence $p = 0$ in $A$. 

**Remark 3.6.** Note that the proof exhibits a sub-matroid (denoted $M_\rho$) in [Gor88] consisting of $2p + 3$ points sharing the desired property of $\mathbb{P}^2(\mathbb{F}_p)$. This means that in our second non-liftable example we could have blown up a smaller configuration of $2p + 4$ points and (strict transforms of) $4p + 7$ lines between them.

**Remark 3.7.** With the same proof, one can construct similar examples in higher dimensions: blow up $\mathbb{P}^n_k$ ($n \geq 3$) in all $\mathbb{F}_p$-rational points, (strict transforms of) lines, planes, and so on. Such varieties were studied in [RTW13, Definition 1.2] in relation to automorphisms of the Drinfeld half-space.
Remark 3.8. We also remark that in [Lan16, Proposition 8.4] it is proved that a pair \((X, D)\) where \(X\) is the blow-up of \(\mathbb{P}^2_k\) in all \(\mathbb{F}_p\)-rational points and \(D\) is a union of strict transforms of at least \(4p - 3\) \(\mathbb{F}_p\)-rational lines does not lift to \(W_2(k)\). The argument above proves that the matroid \(M_p\) leads to a non-liftable example with a fewer number of lines equal to \(2p + 3\). We do not know whether \(2p + 3\) is the minimal number of lines necessary to exhibit \(W_2(k)\) non-liftability.

4. Technical background

Here we review the necessary technical results regarding deformation theory of products (§4.1), descending deformations along morphisms (§4.2), cohomology of blowing up (§4.4), and Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property of blow-ups (§4.5).

4.1. Deformations of products. Our goal is to show that given two \(k\)-schemes \(X\) and \(Y\) such that \(H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0\), then every deformation of \(X \times Y\) comes from a pair of deformations of \(X\) and \(Y\) (Proposition 4.2). We begin with a few remarks concerning deformation obstruction classes. Firstly, observe that by [Ill71] we know that for any \(k\)-scheme \(Z\) the obstruction class to lifting an element \((\tilde{f} : \tilde{Z} \to \text{Spec}(A)) \in \text{Def}_Z(A)\) to a thickening \(\eta : 0 \to I \to (B, m_B) \to (A, m_A) \to 0\), satisfying \(m_BI = 0\), is given by a class in \(\text{Ext}^2(L_{Z/k}, \mathcal{O}_Z) \otimes_k I\) defined as the Yoneda composition of Kodaira–Spencer class \(K_{\tilde{Z}/A/Z} \in \text{Ext}^1(L_{\tilde{Z}/A}, LF^*\mathbb{L}_{A/Z}[1])\) and the pullback \(f^*\eta\) of the extension class \(\eta \in \text{Ext}^1(L_{A/Z}, I)\).

Moreover, by a simple diagram chase based on the properties of the cotangent complex we obtain:

**Lemma 4.1** (Additivity of Kodaira–Spencer). Let \(f : X \to Z\) and \(g : Y \to Z\) be morphisms of \(S\)-schemes. Let \(p_X : X \times_Z Y \to X\) and \(p_Y : X \times_Z Y \to Y\) denote the projections, \(h : X \times_Z Y \to Z\) the composition \(h = f \circ p_X = g \circ p_Y\). Then, Kodaira–Spencer class:

\[K_{X \times_Z Y/Z/S} \in \text{Ext}^1(L_{X \times_Z Y/Z}, Lh^*\mathbb{L}_{Z/S})\]

equals the direct sum of pullbacks of Kodaira–Spencer classes:

\[K_X/Z/S \in \text{Ext}^1(L_{X/Z}, Lf^*\mathbb{L}_{Z/S})\]
\[K_Y/Z/S \in \text{Ext}^1(L_{Y/Z}, Lg^*\mathbb{L}_{Z/S}).\]

Equipped with the above description, we are ready to prove:
Proposition 4.2. The morphism of deformation functors

\[ \prod_{X,Y} : \text{Def}_X \times \text{Def}_Y \to \text{Def}_{X \times Y}, \quad (\tilde{X}, \tilde{Y}) \mapsto \tilde{X} \times_{\text{Spec}(A)} \tilde{Y} \]

is smooth (in particular levelwise surjective) if \( H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0 \).

Proof. By the above general considerations and the additivity of Kodaira–Spencer class we see that the morphisms on tangent and obstruction space

\[
\begin{align*}
T_{\prod_{X,Y}} : \text{Ext}^1(\mathbb{L}_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^1(\mathbb{L}_{Y/k}, \mathcal{O}_Y) & \to \text{Ext}^1(\mathcal{L}p_{X*}\mathbb{L}_{X/k} \oplus \mathcal{L}p_{Y*}\mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}), \\
\text{Ob}_{\prod_{X,Y}} : \text{Ext}^2(\mathbb{L}_{X/k}, \mathcal{O}_X) \oplus \text{Ext}^2(\mathbb{L}_{Y/k}, \mathcal{O}_Y) & \to \text{Ext}^2(\mathcal{L}p_{X*}\mathbb{L}_{X/k} \oplus \mathcal{L}p_{Y*}\mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}),
\end{align*}
\]

are given as direct sums of morphisms:

\[
\begin{align*}
\text{Ext}^*(\mathbb{L}_{X/k}, \mathcal{O}_X) & \to \text{Ext}^*(\mathbb{L}_{X/k}, \mathcal{L}p_{X*}\mathbb{L}_{X/k}, \mathcal{O}_{X \times Y}) \simeq \text{Ext}^*(\mathcal{L}p_{X*}\mathbb{L}_{X/k}, \mathcal{O}_{X \times Y}); \\
\text{Ext}^*(\mathbb{L}_{Y/k}, \mathcal{O}_Y) & \to \text{Ext}^*(\mathbb{L}_{Y/k}, \mathcal{L}p_{Y*}\mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}) \simeq \text{Ext}^*(\mathcal{L}p_{Y*}\mathbb{L}_{Y/k}, \mathcal{O}_{X \times Y}),
\end{align*}
\]

which arise from the natural distinguished triangles

\[
\begin{align*}
\mathcal{O}_X & \to \mathcal{R}p_{X*}\mathcal{O}_{X \times Y} \to \mathcal{C}_{px} \quad \text{and} \quad \mathcal{O}_Y \to \mathcal{R}p_{Y*}\mathcal{O}_{X \times Y} \to \mathcal{C}_{pv},
\end{align*}
\]

induced by the structure morphisms \( p_X^\# \) and \( p_Y^\# \) of the projections.

By the assumptions and the spectral sequence:

\[
E_2^{ij} = \text{Ext}^i(\mathbb{L}_{X/k}, \mathcal{H}^j(C_{px})) \Rightarrow \text{Ext}^{i+j}(\mathbb{L}_{X/k}, C_{px})
\]

we see that \( \text{Ext}^1(\mathbb{L}_{X/k}, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) \otimes \text{Ext}^0(\mathbb{L}_{X/k}, \mathcal{O}_X) = 0 \). Analogously we obtain \( \text{Ext}^1(\mathbb{L}_{Y/k}, \mathcal{O}_X) = 0 \). Therefore \( T_{\prod_{X,Y}} \) is surjective and \( \text{Ob}_{\prod_{X,Y}} \) is injective, which by [FM98, Lemma 6.1] implies that \( \prod_{X,Y} \) is a smooth morphism of deformation functors. \( \square \)

4.2. Descending deformations along morphisms. One of our main tools is the following proposition, which can prove along the same lines as [LS14, Proposition 2.2]. See also [CvS09] and [Wah79], where this idea appeared previously.

Proposition 4.3. (1) Let \( f : Y \to X \) be a map satisfying \( Rf_*\mathcal{O}_Y = \mathcal{O}_X \). Then there exists a natural transformation \( \text{Def}_Y \to \text{Def}_X \). More generally, if \( W = \{W_i\}_{i \in I} \) (resp. \( Z = \{Z_i\}_{i \in I} \)) is a family of closed subschemes of \( Y \) (resp. \( X \)) parametrized by a preorder \( I \) (cf. [12]), and if \( Rf_*\mathcal{O}_{W_i} = \mathcal{O}_{Z_i} \) (in particular, \( Z_i = f(W_i) \)), then there exists a natural transformation \( \text{Def}_{Y,W} \to \text{Def}_{X,Z} \).

(2) Let \( X \) be a smooth scheme, \( Z \subseteq X \) a smooth closed subscheme of codimension \( \geq 2 \), \( f : Y = \text{Bl}_Z X \to X \) the blow-up of \( X \) along \( Z \), \( E = \{E_j\}_{j \in J} \) the set of connected components of \( f^{-1}(Z) \). Then the forgetful transformation \( \text{Def}_{Y,E} \to \text{Def}_Y \) is an isomorphism (here the index set \( J \) is given the trivial order). More generally, if \( W = \{W_i\}_{i \in I} \) is a family of closed subschemes of \( Y \), then the forgetful transformation \( \text{Def}_{Y,W,E} \to \text{Def}_{Y,W} \) is an isomorphism. Here by \( W \sqcup E \) we mean the family \( \{W_i\}_{i \in I} \sqcup \{E_j\}_{j \in J} \) parametrized by \( I \sqcup J \) with no nontrivial relations between \( I \) and \( J \).

As a simple corollary we obtain:

Proposition 4.4. Let \( f : X \to Y \) be a morphism of schemes over a field \( k \) satisfying \( H^1(X, \mathcal{O}_X) = H^1(Y, \mathcal{O}_Y) = 0 \). If \( \text{Bl}_{\Gamma_f}(X \times Y) \) lifts to \( A \in \text{Art}_{W(k)}(k) \), then there exist \( A \)-liftings of \( X \) and \( Y \) together with a lifting of \( f \).

Proof. Assume \( \text{Bl}_{\Gamma_f}(X \times Y) \) lifts to \( A \). By Proposition 4.3 there exists a deformation \( \tilde{X} \times \tilde{Y} \) of the product \( X \times Y \) together with an embedded deformation \( \Gamma_{\tilde{f}} \) of \( \Gamma_f \). By Proposition 4.2, the \( A \)-scheme \( \tilde{X} \times \tilde{Y} \) is isomorphic to \( \tilde{X} \times_{\text{Spec}(A)} \tilde{Y} \) for some deformations of \( X \) and \( Y \). The restriction of the projection \( \tilde{p}_Y : \tilde{X} \times_{\text{Spec}(A)} \tilde{Y} \to \tilde{X} \) to \( \Gamma_{\tilde{f}} \) is an isomorphism (as its restricting to \( \text{Spec}(k) \) is an isomorphism) and therefore the tuple \( (\tilde{X}, \tilde{Y}, \tilde{p}_Y \circ (\tilde{p}_X|_{\Gamma_{\tilde{f}}})^{-1}) \) gives the desired pair of liftings of \( X \) and \( Y \) together with a lifting of \( f \). \( \square \)
4.3. Regular sequences and flatness. In the proof of Theorem 3.1 we need the following simple claim: if \( R \) is a 3-dimensional regular local \( k \)-algebra with residue field \( k \), \( L \) and \( H \) a smooth curve and a smooth hypersurface in \( X = \text{Spec} R \) intersecting transversally at the closed point \( P \), then any embedded deformation of \((X, L, H)\) induces a deformation of \( P \) inside \( L \) and \( H \). This claim is implied by the following general results regarding deformations and regular sequences.

Lemma 4.5. Suppose \((A, \mathfrak{m}_A)\) is an element of \( \text{Art}_{W(k)}(k) \) and \( R \) is a local \( k \)-algebra. Moreover let \( S \) be an \( A \)-flat local ring such that \( S \otimes_A k = R \). Then for any element \( f \in S \) such that \( \mathfrak{f} \in R \) (we denote by \( \mathfrak{f} \) the image of \( f \) under the natural map \( S \to R \)) is a non-zero divisor the following assertions hold true:

1. the element \( f \) is a non-zero divisor in \( S \),
2. the quotient ring \( S/(f) \) is \( A \)-flat.

Proof. The proof of the first claim follows by induction with respect to the length of \( A \). For the case \( \text{len}(A) = 1 \) we know that \( A = k \) and therefore the claim is clear. For \( \text{len}(A) > 1 \), we observe that \((A, \mathfrak{m}_A)\) is an extension of \((A', \mathfrak{m}_A)\) of \( \text{len}(A') = \text{len}(A) - 1 \) by a principal ideal \( I = (s) \) satisfying \( A'I = 0 \). Now, take an element \( g \in S \) such that \( gf = 0 \). By the induction hypothesis applied for \( A' \), the element \( [f] \in S' \) is a non-zero divisor in \( S' \) which implies that \( [g] = 0 \) and there therefore exists an element \( g' \in S \) such that \( g = sg' \). Consequently from the relation \( sg'f = 0 \) we infer that \( g'f \in \mathfrak{m}_A S \), that is \( g'f \mathfrak{f} = 0 \). By the induction hypothesis applied for \( A = k \) we see that \( g' \in \mathfrak{m}_A S \) yields that \( g \in \mathfrak{m}_A I \cdot S = (0) \). This implies that \( f \) is a non-zero divisor and thus proves the first part of the lemma.

The proof of the second claim is a standard application of local criterion of flatness applied to an \( A \)-flat resolution:

\[
\begin{array}{c}
0 \rightarrow S \xrightarrow{f} S \xrightarrow{g} S/(f) \xrightarrow{h} 0
\end{array}
\]

implied by the first claim.

Corollary 4.6. Let the rings \((A, \mathfrak{m}_A)\), \( R \) and \( S \) be as above. Moreover, let \( f_1, \ldots, f_k \in S \) for \( k \geq 1 \) be a sequence of elements such that their reductions \( \overline{f}_1, \ldots, \overline{f}_k \in R \) form a regular sequence in \( R \). Then \( f_1, \ldots, f_k \) is a regular sequence in \( S \) and \( S/(f_1, \ldots, f_k) \) is an \( A \)-flat lifting of \( R/(\overline{f}_1, \ldots, \overline{f}_k) \).

Proof. The proof follows from Lemma 4.5 by induction with respect to the parameter \( k \).

4.4. Blow-up formulas. In this section, we review formulas for the cohomology of the blow-up of a smooth proper scheme \( X \) along a smooth subscheme \( Z \), and deduce statements regarding Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property. It is best to deduce the blow-up formulas for different cohomology theories from a single motivic statement.

Proposition 4.7 (cf. [Voe00, 3.5.3]). Suppose that \( X \) is a smooth proper scheme over a field \( k \), \( Z \subseteq X \) a smooth closed subscheme of codimension \( c \). Then there is a decomposition of Chow motives

\[
M(\text{Bl}_Z X) = M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(i)[2i]
\]

In particular, \([\text{Bl}_Z X] = [X] + ([L_1 + L_2 + \ldots + L^{c-1}])[Z]\) in the Grothendieck ring of varieties, where \( L = [A_1^1] \).

Corollary 4.8. Suppose that \( X \) is a smooth proper scheme over a field \( k \), \( Z \subseteq X \) a smooth closed subscheme of codimension \( c \). Let \( H^n \) denote one of the following families of functors of smooth projective varieties \( X \):

1. \( H^n(X \otimes k, Z_\ell) \) for some \( \ell \) invertible in \( k \), treated as a \( \text{Gal}(k/k) \)-module,
2. (if \( k \) is perfect of characteristic \( p > 0 \)) \( H^n(X/W(k)) \), the integral crystalline cohomology, a \( W(k) \)-module with a \( \sigma \)-linear endomorphism induced by the Frobenius,
3. \( H^n_{dR}(X) = H^n(X, \Omega^\bullet_{X/k}) \), the algebraic de Rham cohomology, endowed with the Hodge filtration,
4. \( H^n_{dR}(X) = \bigoplus_{p+q=n} H^q(X, \Omega^p_{X/k}) \), Hodge cohomology, a graded \( k \)-vector space,
Proposition 7.3) for several equivalent criteria). It is called Hodge–Witt if it is ordinary, and that \( p = \text{char} k \).

Moreover, let \( -(n) \) denote the Tate twist, i.e., the tensor product with \( \mathbb{H}^{2}(\mathbb{P}^{1}) \otimes n \) in the appropriate tensor category. Then there is a natural isomorphism of objects in the appropriate category as listed above:

\[
H^{n}(\mathcal{B} \mathcal{I}_{Z}(X)) = H^{n}(X) \oplus \bigoplus_{i=1}^{c-1} H^{n-2i}(Z)\langle i \rangle.
\]

Proof. This follows from Proposition 4.8 and the fact that the cohomology theories \( H \) above all admit cycle class maps and actions by correspondences. For \( \ell \)-adic and crystalline cohomology this is well-known, and for Hodge and Hodge–Witt cohomology it follows from the work of Chatzistamatiou and Rülling [CR11].

4.5. Hodge–de Rham degeneration, ordinarity, and the Hodge–Witt property. Let \( X \) be a smooth proper scheme over \( k \). The first hypercohomology spectral sequence of the de Rham complex \( \Omega^{\bullet}_{X/k} \),

\[
E^{ij}_{1} = H^{i}(X, \Omega^{j}_{X/k}) \Rightarrow H^{i+j}_{dR}(X/k) := H^{i+j}(X, \Omega^{\bullet}_{X/k}),
\]

is called the Hodge–de Rham spectral sequence of \( X \). We say that it degenerates if it degenerates on the first page, i.e., there are no nonzero differentials. As \( X \) is proper, the cohomology groups are finite dimensional, and hence the degeneration is equivalent to the condition that

\[
\sum_{n} \dim H^{n}_{dR}(X/k) = \sum_{p,q} \dim H^{p}(X, \Omega^{q}_{X/k}).
\]

The Hodge–de Rham spectral sequence of \( X \) degenerates if \( k \) is of characteristic zero, or if \( \dim X < p = \text{char} k \) and \( X \) lifts to \( W_{2}(k) \) [DI87, Corollaire 2.4].

The scheme \( X \) is called ordinary (in the sense of Bloch and Kato) if the Frobenius \( F : H^{q}(X, \omega^{p}_{X/k}) \to H^{q}(X, \omega^{p}_{X/k}) \) on Hodge–Witt cohomology is bijective for all \( p \) and \( q \) (cf. [BK86, Proposition 7.3]) for several equivalent criteria. It is called Hodge–Witt if the Hodge–Witt groups \( H^{q}(X, \omega^{p}_{X/k}) \) are finitely generated \( W(k) \)-modules. It follows from [IR83, IV 4] that \( X \) is Hodge–Witt if it is ordinary, and that \( X \times Y \) is ordinary if \( X \) and \( Y \) are.

Corollary 4.9. Suppose that \( X \) is a smooth proper scheme over a field \( k \), \( Z \subseteq X \) a smooth closed subscheme of codimension \( > 1 \). Then

1. The Hodge–de Rham spectral sequences of \( Z \) and \( X \) degenerate if and only if the Hodge–de Rham sequence of \( \mathcal{B} \mathcal{I}_{Z} X \) degenerates.
2. The scheme \( \mathcal{B} \mathcal{I}_{Z} X \) is ordinary (resp. Hodge–Witt) if and only if both \( X \) and \( Y \) are ordinary (resp. Hodge–Witt).

Proof. The first assertion follows from Corollary 4.8 for \( H^{n}_{dR} \) and \( H^{n}_{dR} \) and [II]. For the latter, use Corollary 4.8 for \( H^{n}_{HW} \) and the characterizations given above.

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