Critical and Tricritical Points for the Massless 2d Gross-Neveu Model Beyond large-\(N\)

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Using optimized perturbation theory, we evaluate the effective potential for the massless two-dimensional Gross-Neveu model at finite temperature and density containing corrections beyond the leading large-\(N\) contribution. For large-\(N\), our results exactly reproduce the well known \(1/N\) leading order results for the critical temperature, chemical potential and tricritical points. For finite \(N\), our critical values are smaller than the ones predicted by the large-\(N\) approximation and seem to observe Landau’s theorem for phase transitions in one space dimension. New analytical results are presented for the tricritical points that include \(1/N\) corrections. The easiness with which the calculations and renormalization are carried out allied to the seemingly convergent optimized results displayed, in this particular application, show the robustness of this method and allows us to obtain neat analytical expressions for the critical as well as tricritical values beyond the results currently known.

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\section{I. INTRODUCTION}

The study of symmetry breaking/restoration in quantum field theories is an important issue of relevance in many areas of physics. For example, today, problems regarding phase transitions in Bose-Einstein condensates (BEC) or in Quantum Chromodynamics (QCD) concentrate a lot of theoretical as well as experimental efforts.

Of topical importance regarding studies of phase transitions in quantum field theory is the reliability of perturbation theory and its eventual breakdown. For instance perturbation theory at high temperatures breaks down due to the appearance of large infrared divergences, happening for example in massless field theories, like in QCD \(^1\), close to critical temperatures (in field theories displaying a second order phase transition or a weakly first order transition \(^2\) \(^3\)), or just because that at high temperatures there are parameter regimes where conventional perturbation schemes become unreliable when powers of the coupling constants become surmounted by powers of the temperature.

In these cases, a nontrivial problem arises since non-perturbative methods must be used. Various non-perturbative techniques have been used to deal with these problems. Among them we will be particularly interested in the \(1/N\) approximation \(^4\) which, here, will be considered mainly for comparison. Though a powerful resummation method, the \(1/N\) approximation can quickly become cumbersome after the resummation of the first leading contributions, like for example in \(N = 2\) (e.g., BEC and polyacetylene) or \(N = 3\) (e.g., QCD) finite \(N\) problems. This is due to technical difficulties such as the formal resummation of infinite subsets of Feynman graphs and their subsequent renormalization.

An alternative non-perturbative analytical method that we will make use in this work is the optimized perturbation theory, or linear \(\delta\) expansion (LDE) \(^5\). In calculational terms (including renormalization) its appeal regards the fact that one remains within the framework of perturbation theory. Then, non-perturbative results are obtained by optimizing the perturbatively evaluated quantities. This procedure amounts to eliminate, variationally, mass parameters used to deform (interpolate) the original action. Recently, the LDE has been successful in treating scalar field theories at finite temperature and/or density. Relativistic scalar \(\lambda \phi^4\) theories have been treated at finite temperature \(^6\) as well as at finite temperature and density \(^7\). At the same time, their nonrelativistic counter part, which is relevant for BEC, has been studied in connection with the problem of the dependence with interactions of the critical temperature shift \(\Delta T_c\), given by the difference between the interacting and ideal gas critical temperatures (for a recent review see for instance \(^8\) and references therein). It suffices to say that the latest LDE results \(^9\) are in excellent agreement with the Monte Carlo results, considered the most accurate prediction for \(\Delta T_c\), and it performs much better than the \(1/N\) expansion used at leading and at next to the leading order (for a review on the different

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results and methods used to study the $\Delta T_c$, in the BEC problem, see [11]). Moreover, the convergence properties for this critical theory have been proved by the present and other authors [9,10,12].

The present work will focus on the LDE applications to problems involving phase transitions in asymptotically free models at finite temperatures and densities. Here, we shall consider the 1+1 dimensional Gross-Neveu model (GN), which is extensively used as a prototype model in studies related to phase transitions in particle physics as well as condensed matter physics. One recent application concerning particle physics is the next-to-leading order in $1/N$ evaluation of the effective potential at finite temperature performed in Ref. [13]. In the condensed matter domain, a very interesting recent application of the GN model was to the study of polymers [14], where a massive version of the model was used to show the appearance of a kink-antikink crystal phase that was missed in a previous work [15].

In Ref. [14] the authors work only within the leading order of the $1/N$ expansion, so their results still show a phase transition at finite temperature $T$ and density $\mu$. On the other hand they consider the case of inhomogeneous background fields, which should be physically more relevant in the 1+1 dimensional GN model. This is so because, in one space dimension, the Landau-Mermin-Wagner-Coleman theorem [16,17,18], no phase transition related to a discrete symmetry breaking (in this case a discrete chiral symmetry in the massless GN model considered in this work) is expected at any finite temperature. In the GN model this can be explained by the role played by kink-like inhomogeneous configurations [19] that come to dominate the action functional, instead of just homogeneous, constant field configurations. By accounting for kink-like configurations in the large-$N$ approximation, the authors of Ref. [14] (see also Ref. [20] for a review) find evidence for a crystal phase that shows up in the extreme $T \sim 0$ and large $\mu$ part of the phase diagram. The other extreme of the phase diagram, large $T$ and $\mu \sim 0$, remains identical to the usual large-$N$ results for the critical temperature and tricritical points, which are well known results [21,22,23].

In the study performed here, we only consider homogeneous backgrounds, but go beyond large-$N$, so the phase diagram changes as a whole. However, we cannot see any crystal phase at small $\mu$, that should be a consequence of kink-like configurations dominating the action functional at that extreme of the phase diagram. Despite of that, we are still improving the calculations for the GN model even though we are not considering inhomogeneous fields. And, as explained above, since inhomogeneous backgrounds do not seem to change appreciably the large $T$ and small $\mu$ region, up to the tricritical point of the phase diagram, we are certainly improving the knowledge in that part of the phase diagram. At the same time, we believe that our results are not faithful in the small $T$ and large $\mu$ part of the phase diagram, which gets affected at large by inhomogeneous backgrounds as shown by the results obtained for the GN model in the large-$N$ approximation [14], though no results beyond large-$N$ are currently available. Eventually, in the future it would be opportune to contrast the results found by the authors of Ref. [14] who considered an inhomogeneous background field to evaluate the effective action with the ones provided by the LDE in the same context so that it could generate the effects of kink configurations beyond the large $N$ limit considered in Ref. [14].

Another purpose of the work done here is to show the advantages and reliability of an alternative non-perturbative method like the LDE in the understanding of the phase diagram of the massless GN model when considered beyond the large-$N$ limit. In this case, the massless GN model provides an excellent testing framework for the following reasons. As mentioned in the previous paragraph, large-$N$ results for the critical temperature ($T_c$) [21], critical chemical potential ($\mu_c$) [22], and tricritical points [15] are well known [24]. At the same time, Landau’s theorem [17,19,21] for phase transitions states that they cannot occur in one space dimension, so that rigorously $T_c = 0$, meaning that the large-$N$ approximation behaves poorly in this case. These two extreme results allow us to gauge the LDE performance in connection with the problem since we know that for $N \to \infty$ our results should converge to the “exact”, although wrong, large-$N$ result. For finite $N$, on the other hand, our results should predict smaller values for the critical temperatures, in accordance with Landau’s theorem. At zero temperature and density, the LDE has been applied to this model with some success [21,25], since in this simpler case the LDE could even be summed to all perturbative orders (at least in the $1/N$ approximation), so that the large order behavior of the LDE could be investigated. In fact, its convergence properties have been proved for a particular perturbation series in this context [25]. At finite temperature, an early application to the GN model [21] showed the potentiality of this method. However, the renormalization program has not been addressed in Ref. [21]. Here, our aim is to use all the latest LDE improvements to evaluate the GN effective potential at finite temperature and density for any value of $N$.

This work is organized as follows. In the next section we present the model. In Sec. III we review the well established $1/N$ results to leading order considering the following four situations: (a) $T = \mu = 0$, (b) $T \neq 0$, $\mu = 0$, (c) $T = 0$, $\mu \neq 0$ and (d) $T \neq 0$, $\mu \neq 0$. In Sec. IV we present the LDE method and the interpolated GN model, evaluating the effective potential for $N \to \infty$. We show, in accordance with Ref. [27], that when correctly applied, the LDE exactly reproduces, already at first order, the large-$N$ results. The situation is unchanged, at any order in $\delta$, provided that one stays within the $N \to \infty$ limit. This nice result is valid for any parameter values. In the same section, we explicitly evaluate the $1/N$ correction that also appears at the first LDE nontrivial order. In Sec. V the LDE order-$\delta$ results are presented and compared to the large-$N$ results for the four situations described above. Our major result is the production of analytical relations for the fermionic mass, critical temperature, critical chemical potential as well as tricritical points containing a finite $N$ correction. We show that all these quantities depend on
an optimized mass scale set by the LDE. All the analytical expressions have been cross checked numerically. In Sec. VI we contrast the LDE and the $1/N$ approximation results to leading order and to the next to the leading order. Our conclusions are presented in Sec. VII. Two appendices are included to show some technical details and the renormalization for the interpolated model.

II. THE GROSS-NEVEU MODEL

The Gross-Neveu model is described by the Lagrangian density for a fermion field $\psi_k$ ($k = 1, \ldots, N$) given by

$$\mathcal{L} = \bar{\psi}_k (i \not\partial) \psi_k + m_F \bar{\psi}_k \psi_k + \frac{g^2}{2} (\bar{\psi}_k \psi_k)^2 ,$$  \hspace{1cm} (2.1)

where the summation over flavors is implicit in the above equation, with e.g. $\bar{\psi}_k \psi_k = \sum_{k=1}^N \bar{\psi}_k \psi_k$. Since we restrict to two-dimensional space-time dimension, $\psi_k$ represents a two-component Dirac spinor for each value of the flavor index $k$. When $m_F = 0$ the theory is invariant under the discrete transformation

$$\psi \rightarrow \gamma_5 \psi ,$$  \hspace{1cm} (2.2)

displaying a discrete chiral symmetry (CS). In addition, Eq. (2.1) has a global $SU(N)$ flavor symmetry.

For the studies of the model Eq. (2.1) in the large-$N$ limit it is convenient to define the four-fermion interaction as $g^2 N = \lambda$. Since $g^2$ vanishes like $1/N$ we study the theory in the large-$N$ limit with fixed $\lambda$ (see e.g. [29]). At finite temperature and density, we can study the model Eq. (2.1) in terms of the grand partition function given by

$$Z(\beta, \mu) = \text{Tr} \exp \left\{ -\beta (H - \mu Q) \right\} ,$$  \hspace{1cm} (2.3)

where $\beta$ is the inverse of the temperature, $\mu$ is the chemical potential, $H$ is the Hamiltonian corresponding to Eq. (2.1) and $Q = \int dx \bar{\psi}_k \gamma_0 \psi_k$ is the conserved charge. Transforming Eq. (2.3) to the form of a path integral in the imaginary-time (Euclidean) formalism of finite temperature field theory [30], we then have

$$Z(\beta, \mu) = \int \Pi_{k=1}^N D\bar{\psi}_k D\psi_k \exp \left\{ -S_E[\bar{\psi}_k, \psi_k] \right\} ,$$  \hspace{1cm} (2.4)

where the Euclidean action reads

$$S_E[\bar{\psi}_k, \psi_k] = \int_0^\beta d\tau \int dx \left[ \bar{\psi}_k (i \not\partial + \mu \gamma_0 - m_F) \psi_k - \frac{\lambda}{2N} (\bar{\psi}_k \psi_k)^2 \right] ,$$  \hspace{1cm} (2.5)

and the functional integration in Eq. (2.4) is performed over the fermion fields satisfying the anti-periodic boundary condition in Euclidean time: $\psi_k(x, \tau) = -\psi_k(x, \tau + \beta)$.

III. REVIEW OF BASIC 1/N RESULTS TO LEADING ORDER

Let us briefly review some of the standard large-$N$ results for the GN model. Considering $m_F = 0$ in Eq. (2.1) we start by looking for a fermionic mass that can be generated dynamically via radiative corrections. This exercise will also allow us to set up notation, conventions as well as reviewing useful formulae to be used within the formalism of finite temperature and finite density. As usual, it is useful to rewrite Eq. (2.1) expressing it in terms of an auxiliary (composite) field $\sigma$, so that

$$\mathcal{L} = \bar{\psi}_k (i \not\partial) \psi_k - \sigma \bar{\psi}_k \psi_k - \frac{\sigma^2 N}{2\lambda} .$$  \hspace{1cm} (3.1)
By using the solution of the equation of motion for \( \sigma \) in Eq. (3.1) we recover the original model. For renormalization purposes, we can also add to Eq. (3.1) a counterterm Lagrangian density, \( \mathcal{L}_{ct} \), whose most general form can be expressed as

\[
\mathcal{L}_{ct} = \overline{\psi}_k (iA \gamma) \psi_k - B \sigma \overline{\psi}_k \psi_k - C \frac{\sigma^2 N}{2\lambda} + D \overline{\psi}_k \psi_k + E \sigma + X,
\]

where \( A, B, C, D, E \) and \( X \) are renormalization counterterms with the latter representing the zero point energy subtraction.

The appearance of a non-vanishing vacuum expectation value for \( \sigma \), \( \langle \sigma \rangle = \bar{\sigma}_c \neq 0 \) can be associated to a mass term for the fermion field. This is better studied in terms of the effective potential for \( \sigma \), \( V_{\text{eff}}(\sigma_c) \). As it is well known, using the \( 1/N \) approximation, the large-\( N \) expression for the effective potential is

\[
\frac{V_{\text{eff}}^N}{N}(\sigma_c) = \frac{\sigma_c^2}{2\lambda} + i \int \frac{d^2p}{(2\pi)^2} \ln \left(p^2 - \sigma_c^2\right) \tag{3.3}
\]

The above equation can be extended to finite temperature \( T \) and density \( \mu \) using the usual associations and replacements (see Appendix A for details and notation), with the result

\[
\frac{V_{\text{eff}}^N}{N}(\sigma_c, T, \mu) = \frac{\sigma_c^2}{2\lambda} - \int_p \omega_p(\sigma_c) + T \ln \left(1 + \exp \left(-\frac{\omega_p(\sigma_c) + \mu}{T}\right)\right) + T \ln \left(1 + \exp \left(-\frac{\omega_p(\sigma_c) - \mu}{T}\right)\right), \tag{3.4}
\]

where \( \omega_p^2(\sigma_c) = p^2 + \sigma_c^2 \) and \( \int_p \) denotes integration over space momentum. The \( T = 0, \mu = 0 \) term in Eq. (3.4) gives

\[
\frac{V_{\text{eff}}^N}{N}(\sigma_c, T = 0, \mu = 0) = \frac{\sigma_c^2}{2\lambda} - \int_p \omega_p(\sigma_c) = \frac{\sigma_c^2}{2\lambda} + \frac{1}{4\pi} \sigma_c^2 \left[ \frac{2}{\epsilon} + 1 + \ln \left(\frac{M^2}{\sigma_c^2}\right) + \mathcal{O}(\epsilon) \right], \tag{3.5}
\]

where \( M \) is the arbitrary mass scale introduced by dimensional regularization. The divergent term in Eq. (3.5) can either be rendered finite, by imposing a renormalization condition directly on the effective potential by defining a renormalized coupling constant \( \lambda_R \), via

\[
\frac{d^2 V_{\text{eff}}^N}{d\sigma_c^2} = \frac{N}{\lambda_R}, \tag{3.6}
\]

or by direct use of the counterterms in Eq. (3.2), in which case we only require a mass counterterm for the \( \sigma \) field that exactly cancels the divergent term in Eq. (3.5). The two renormalizations are of course equivalent, only differing by a different choice of mass scale in (3.5). Choosing the latter form of renormalization, we immediately get the renormalized relation

\[
\frac{V_{\text{eff}}^N}{N}(\sigma_c, T = 0, \mu = 0) = \frac{\sigma_c^2}{2\lambda} - \frac{1}{4\pi} \sigma_c^2 \left[ 1 + \ln \left(\frac{M^2}{\sigma_c^2}\right) \right], \tag{3.7}
\]

From the \( T, \mu \) dependent term of Eq. (3.4),

\[
\frac{V_{\text{eff}}^N}{N}(\sigma_c, T, \mu) = - \int_p \left[T \ln \left(1 + \exp \left(-\frac{\omega_p(\sigma_c) + \mu}{T}\right)\right) + T \ln \left(1 + \exp \left(-\frac{\omega_p(\sigma_c) - \mu}{T}\right)\right)\right], \tag{3.8}
\]

we can take the limit \( T \to 0 \) and perform the momentum integral to obtain the finite result
The general expression Eq. (3.8) for \( T \neq 0 \) and \( \mu \neq 0 \), however, does not have a close analytical expression, but we can express it in terms of a high temperature expansion, in powers of \( \mu/T \) and \( \sigma_c/T \), in analogous form as it is done for the bosonic like temperature dependent momentum integrals [31]. Defining the function \( I_1(a, b) \) by

\[
I_1(a, b) = \int_0^\infty dx \left[ \ln \left( 1 + e^{-\sqrt{x^2 + a^2 - b}} \right) + \ln \left( 1 + e^{-\sqrt{x^2 + a^2 + b}} \right) \right],
\]

(3.10)

Eq. (3.8) can be written as

\[
\frac{V_{\text{eff}}}{N}(\sigma_c, T, \mu) = -\frac{T^2}{(2\pi)^2 \Gamma \left( \frac{1}{2} \right)} I_1(\sigma_c/T, \mu/T).
\]

(3.11)

We now take \( \sigma_c/T = a \ll 1 \) and \( \mu/T = b \ll 1 \). Expanding Eq. (3.10) in powers of \( a \) and \( b \), the result is finite and given by [32]

\[
I_1(a \ll 1, b \ll 1) = \frac{\pi^2}{6} + \frac{b^2}{2} - \frac{a^2}{2} \ln \left( \frac{\pi}{a} \right) - \frac{a^2}{4} (1 - 2\gamma_E) - \frac{7\zeta(3)}{8\pi^2} a^2 \left( b^2 + \frac{a^2}{4} \right) + O(a^3 b^4, a^4 b^2),
\]

(3.12)

where \( \zeta(3) \simeq 1.202 \). Using Eq. (3.12) in Eq. (3.11), the high temperature expansion for \( V_{\text{eff}}(\sigma_c, T, \mu) \) becomes

\[
\frac{V_{\text{eff}}}{N}(\sigma_c, T \neq 0, \mu \neq 0) = \frac{\sigma_c^2}{2\pi} \ln \left( \frac{\pi T}{\sigma_c} \right) - \frac{\sigma_c^2 \gamma_E}{2\pi} + \frac{\sigma_c^2}{4\pi} + \frac{7\zeta(3)}{8\pi^2 T^2} \sigma_c^2 \left( \mu^2 + \frac{\sigma_c^2}{4} \right) + O \left( \frac{\sigma_c^2 \mu}{T^4}, \frac{\sigma_c^4 \mu^2}{T^4} \right),
\]

(3.13)

where we have dropped terms that do not depend on \( \sigma_c \). Let us now review four important situations

A. The \( T = \mu = 0 \) case: the fermionic mass at large-\( N \)

At \( T = \mu = 0 \) the large-\( N \) effective potential is given by Eq. (3.7),

\[
\frac{V_{\text{eff}}}{N}(\sigma_c, T = 0, \mu = 0) = \frac{\sigma_c^2}{2\lambda} - \frac{1}{4\pi} \sigma_c \left[ 1 + \ln \left( \frac{M^2}{\sigma_c^2} \right) \right].
\]

(3.14)

We can already check here that, contrary to the classical (tree-level) potential where the minimum occurs at \( \bar{\sigma}_c = 0 \), one now has symmetry breaking that is quantum generated since the minimum of the effective potential occurs at a non-vanishing value given by

\[
\frac{\partial V_{\text{eff}}}{\partial \sigma_c}(\sigma_c, T = 0, \mu = 0) \bigg|_{\sigma_c = \bar{\sigma}_c} = 0,
\]

(3.15)

where \( \bar{\sigma}_c \) sets the large-\( N \) result for the fermionic mass, at \( T = \mu = 0 \), as

\[
m_F(0) = \bar{\sigma}_c = M \exp \left( -\frac{\pi}{\lambda} \right).
\]

(3.16)
B. The $T \neq 0$ and $\mu = 0$ case: the critical temperature $T_c$

Physically, the case $\mu = 0$ means that the number of fermions and anti-fermions are the same. For this case, from Eq. (3.17) and using the result (3.13) for the finite temperature term for the effective potential, we have that

\[ \frac{V_{\text{eff}}^N}{N}(\sigma_c, T \neq 0, \mu = 0) = \frac{\sigma_c^2}{2\lambda} - \frac{1}{2\pi} \sigma_c^2 \ln \left( \frac{M}{\pi T} \right) - \frac{\sigma_c^2 \gamma_E}{2\pi} + \frac{7\zeta(3)}{32\pi^3 T^2} \sigma_c^4, \]  

(3.17)

which shows chiral symmetry restoration at a critical temperature $T_c$ given by

\[ T_c = m_F(0) \frac{e^{\gamma_E}}{\pi} \simeq 0.567 m_F(0), \]  

(3.18)
in accordance with Ref. [21]. This transition is second order, as can be easily checked by writing Eq. (3.17) in the form

\[ \frac{V_{\text{eff}}^N}{N}(\sigma_c, T \neq 0, \mu = 0) = \frac{1}{2\pi} \ln \left( \frac{T}{T_c} \right) \sigma_c^2 + \frac{7\zeta(3)}{32\pi^3 T^2} \sigma_c^4, \]  

(3.19)

which for $T$ close to $T_c$ gives a temperature dependent vacuum expectation for $\sigma_c(T) \equiv m_F(T)$ in the form

\[ m_F(T) \simeq \sqrt{\frac{8\pi^2}{7\zeta(3)}} T_c \sqrt{\frac{T_c - T}{T_c}}, \]  

(3.20)

that shows a continuous transition at the critical point $T = T_c$. For $T \geq T_c$ the fermion mass $m_F(T)$ vanishes, restoring the chiral symmetry.

C. The $T = 0$ and $\mu \neq 0$ case: the critical density $\mu_c$

For the case of zero temperature but unequal number of fermions and anti-fermions, $\mu \neq 0$, one considers Eqs. (3.7) and (3.9), which give the expression for the effective potential at finite density but zero temperature. Then,

\[ \frac{V_{\text{eff}}^N}{N}(\sigma_c, T = 0, \mu \neq 0) = \frac{\sigma_c^2}{2\lambda} - \frac{1}{4\pi} \sigma_c^2 \left[ 1 + \ln \left( \frac{M^2}{\sigma_c^2} \right) \right] + \frac{1}{2\pi} \theta(\mu - \sigma_c) \left[ \sigma_c^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - \sigma_c^2}}{\sigma_c} \right) - \mu \sqrt{\mu^2 - \sigma_c^2} \right]. \]  

(3.21)

For $\mu \leq \sigma_c$ the minimum of the potential reproduces the same result as Eq. (3.16). For $\mu > \sigma_c$ we have from Eq. (3.21) that

\[ \bar{\sigma}_c(\mu) = \sqrt{m_F(0) \left[ 2\mu - m_F(0) \right]}, \]  

(3.22)

which is valid for $m_F(0)/2 \leq \mu < m_F(0)$. In the case $\bar{\sigma}_c = m_F(0)$ and from Eq. (3.21) we obtain

\[ \frac{V_{\text{eff}}^N}{N}(\sigma_c, T = 0, \mu < \sigma_c) = -\frac{1}{4\pi} m_F^2(0). \]  

(3.23)

For $\mu < m_F(0)/2, V_{\text{eff}}^N(\sigma_c, T = 0, \mu \neq 0)$ will have a minimum for $\bar{\sigma}_c = m_F(0)$ and a maximum for $\bar{\sigma}_c = 0$. For $m_F(0)/2 \leq \mu < m_F(0), V_{\text{eff}}^N(\sigma_c, T = 0, \mu \neq 0)$ has minima at $\bar{\sigma}_c = 0$ and $\bar{\sigma}_c = m_F(0)$ and a maximum at $\bar{\sigma}_c(\mu)$. There is a value of $\mu$ where both minima satisfy

\[ V_{\text{eff}}^N(\bar{\sigma}_c = m_F(0), T = 0, \mu_c) = V_{\text{eff}}^N(\bar{\sigma}_c = 0, T = 0, \mu_c), \]  

(3.24)

where
\[ \mu_c = \frac{m_F(0)}{\sqrt{2}} \]  

(3.25)  

in agreement with Refs. [22, 23]. This result indicates a first order phase transition: for values of \( \mu > \mu_c \), \( \bar{\sigma}_c = m_F(0) \) is a local minimum (false vacuum), while \( \bar{\sigma}_c = 0 \) is a global minimum (true vacuum). For \( \mu < \mu_c \) the two vacua get interchanged.

### D. The \( T \neq 0 \) and \( \mu \neq 0 \) case

This situation can be analyzed numerically only. On the \( (T, \mu) \) plane one can identify a tricritical point \( P_{tc} \) where the first and second order transition lines meet. In units of \( m_F(0) \), the numerical result is \( P_{tc} = (T_{tc}, \mu_{tc}) = (0.318, 0.608) \) [17]. The large-\( N \) phase diagram will be presented in subsection V.D together with the LDE result.

### IV. THE EFFECTIVE POTENTIAL FOR THE INTERPOLATED THEORY

Let us now turn our attention to the implementation of the LDE procedure within the GN model. According to the usual LDE interpolation prescription [5] (for a long, but far from complete list of references on the method, see [33]) the deformed original four fermion theory displaying CS reads

\[
\mathcal{L}_\delta(\psi, \bar{\psi}) = \bar{\psi}_k (i \not{\partial}) \psi_k + \eta (1 - \delta) \bar{\psi}_k \psi_k + \delta \frac{\lambda}{2N} (\bar{\psi}_k \psi_k)^2.
\]  

(4.1)

So, that at \( \delta = 0 \) we have a theory of free fermions. Now, the introduction of an auxiliary scalar field \( \sigma \) can be achieved by adding the quadratic term,

\[
-\frac{\delta N}{2\lambda} \left( \sigma + \frac{\lambda}{N} \bar{\psi}_k \psi_k \right)^2,
\]  

(4.2)

to \( \mathcal{L}_\delta(\psi, \bar{\psi}) \). We are then lead to the interpolated model

\[
\mathcal{L}_\delta = \bar{\psi}_k (i \not{\partial}) \psi_k - \delta \sigma \bar{\psi}_k \psi_k - \eta (1 - \delta) \bar{\psi}_k \psi_k - \frac{\delta N}{2\lambda} \sigma^2 + \mathcal{L}_{ct,\delta},
\]  

(4.3)

where the counterterm Lagrangian density, \( \mathcal{L}_{ct,\delta} \), has the same polynomial form as in the original theory, Eq. (3.2), while the coefficients are allowed to be \( \delta \) and \( \eta \) dependent [6, 24]. Details of the renormalization process for the interpolated model are given in the appendix. Note that the same interpolation of the form (4.3) was also used in Ref. [34], but it is different from the ones used in Refs. [24, 26, 27, 35]. In those references the interpolation was not carried out in the original four fermion theory but on its bosonized version. However, we argue that the present choice is more adequate because at \( \delta = 0 \) one has only free fermions. Otherwise, the quadratic bosonic term \( \sigma^2/(2\lambda) \) survives at \( \delta = 0 \) and the theory looks to be composed by free fermions and bosons but this is misleading since, by the equations of motion, \( \sigma = (\lambda/N) \bar{\psi}_k \psi_k \). From the Lagrangian density in the interpolated form, Eq. (4.3), we can immediately read the corresponding new Feynman rules in Minkowski space. Each Yukawa vertex carries a factor \( -i\delta \) while the (free) \( \sigma \) propagator is now \( -i\lambda/(N\delta) \). The LDE dressed fermion propagator is

\[
S_F(p) = \frac{i}{\not{p} - \eta* + i\epsilon},
\]  

(4.4)

where \( \eta* = \eta - (\eta - \sigma_c)\delta \).
A. The LDE Effective Potential in the Large-$N$ Limit

In the following we start our analysis of the LDE effective potential in the large-$N$ limit, i.e., the LDE is applied directly to the effective potential after the large-$N$ limit was taken. In practice, as already emphasized, within the LDE one perturbs in powers of $\delta$ only, without having to notice the actual powers of $N$ as we do here. However, in this subsection, we do the $\delta$ expansion within the $1/N$ leading order only to be able to compare both methods. In other words, we first want to establish the reliability of the LDE-PMS by considering $N \to \infty$ in order to further compare our results with the "exact" ones furnished by large-$N$ approximation. The reader should not be confused by this apparent mixed type of expansion. Using the new Feynman rules from the interpolated model (4.3), in the large-$N$ limit, one can write the effective potential, analogous to Eq. (3.3), to arbitrary orders in $\delta$ as \[ \text{Eq. (4.5)} \]

\[
V_{\text{eff}}^{N}(\sigma_{c}, \eta_{*}, N \to \infty) = \delta \frac{\sigma_{c}^{2}}{2\lambda} + i \int \frac{d^{d}p}{(2\pi)^{d}} \text{tr} \ln \left( \not{p} - \eta_{*} \right),
\]

which can be seen as obtained directly from the series containing all perturbative terms, Eq. (3.3), simply by performing the replacement $\sigma_{c} \to \eta_{*}$. Developing Eq. (4.5) to first order in $\delta$ one obtains \[ \text{Eq. (4.6)} \]

\[
V_{\text{eff}, \delta}^{1} N(\sigma_{c}, \eta, N \to \infty) = \delta \frac{\sigma_{c}^{2}}{2\lambda} + i \int \frac{d^{d}p}{(2\pi)^{d}} \text{tr} \ln \left( \not{p} - \eta \right) + \delta i \int \frac{d^{d}p}{(2\pi)^{d}} \text{tr} \frac{\eta - \sigma_{c}}{\not{p} - \eta + i\epsilon}.
\]

Fixing $\delta = 1$, we now optimize $V_{\text{eff}}$ for $\eta$ as in most of the previous references on the LDE method by using the principle of minimal sensitivity (PMS). In the PMS procedure one requires that a physical quantity $\Phi^{(k)}(k)$, that is calculated perturbatively to some $k$-th order in $\delta$, be evaluated at the point where it is less sensitive to this parameter. This criterion then translates into the variational relation \[ \text{Eq. (4.7)} \]

\[
\left. \frac{d\Phi^{(k)}}{d\eta} \right|_{\eta, \delta = 1} = 0.
\]

The optimum value $\bar{\eta}$ that satisfies Eq. (4.7) must be a function of the original parameters, including the couplings, thus generating non-perturbative results. In the following we will apply Eq. (4.7) directly to the effective potential. Applying the PMS to Eq. (4.6) immediately gives the result $\bar{\eta} = \sigma_{c}$, which then recovers exactly the large-$N$ result, Eq. (3.3). This same trend holds at any temperature and/or density, value of the coupling $\lambda$, as well as number of space-time dimensions since there is no need to perform the integrals explicitly. Also, as shown in Ref. [27], the inclusion of higher order terms does not spoil this nice result, provided we stay within the large-$N$ limit, since they are all of the form $\delta^{k} f_{k}(\eta - \sigma_{c})^{k}$, where $k \geq 2$. Note that the PMS admits another solution given by

\[
\left. \frac{d}{d\eta} \left[ \int \frac{d^{d}p}{(2\pi)^{d}} \text{tr} \frac{i}{\not{p} - \eta + i\epsilon} \right] \right|_{\bar{\eta}} = 0.
\]

However, this solution, which depends only on scales introduced by the regularization process, and thus is not proportional to the basic scale of the model after dimensional transmutation: $M e^{-\pi/\lambda}$, can be taken as unphysical since it brings no information about the theory being studied.

---

\[ \text{Eq. (4.8)} \]

\[ \text{Diagram} \]

1 In this subsection we shall work in an arbitrary number of space-time dimensions to emphasize the generality of our optimization procedure.
FIG. 1: Diagrammatic relation between Eqs. (4.5) and (4.6). The thick continuous line on the LHS represents the LDE dressed fermionic propagator which is \( \eta^* \) dependent. The first diagram on the RHS is an order-\( \delta \) vacuum graph. The dashed line represents the auxiliary field, \( \sigma_c \), while the black dot stands for the \( \delta \eta \) vertex. The last two graphs on the RHS are of order-\( \delta \).

One can spot a subtle point associated to the LDE evaluation of the effective potential by looking at the diagrams considered to order-\( \delta \) in Eq. (4.6). These graphs are displayed in Fig. 1 which shows the diagrammatic relation between Eqs. (4.5) and (4.6). Note that Eq. (4.6) contains two \( \sigma_c \) independent (vacuum) terms that would be neglected in most evaluations since they are irrelevant as far as CS breaking/restoration are concerned. However, as pointed out in Ref. [27], they are \( \eta \) dependent and so must be considered until the theory is optimized. One can also easily see that, in fact, those contributions are responsible for the quick LDE convergence towards the exact large-\( N \) result already at the first non-trivial order. The explicit form of the LDE large-\( N \) effective potential is quickly obtained by applying the finite \( T \) and \( \mu \) rules described in the previous section to Eq. (4.5). In particular, from Eqs. (3.7) and (3.11) for \( d = 2 \), one then obtains the renormalized result

\[
V_{\text{eff}, \delta} \left( \sigma_c, \eta^*, N \to \infty \right) = \delta \frac{\sigma_c^2}{2 \lambda} - \frac{1}{2 \pi} \left( \eta^* \right)^2 \left[ \frac{1}{2} + \ln \left( \frac{M}{\eta^*} \right) \right] + 2T^2 I_1 \left( \frac{\eta^*}{T}, \frac{\mu}{T} \right).
\]  

(4.9)

Expanding Eq. (4.9) to order \( \delta \) at \( T = \mu = 0 \) one retrieves, as before, the PMS result \( \bar{\eta} = \sigma_c \) that reproduces the large-\( N \) result. The same optimum value for \( \eta \) appears when we express Eq. (4.9) in the other limits studied in the previous section. At the same time, as discussed above, the PMS gives another solution, Eq. (4.8), which is \( \bar{\eta} = M/e \).

Rather curiously, this result coincides with \( \bar{\sigma}_c = m_F(0) \) evaluated at \( \lambda = \pi \).

B. The Effective Potential to Order \( \delta \): going beyond the large-\( N \) limit

In the following we show how the optimization procedure implemented by the LDE improves over the large-\( N \) results. Here we revert to the usual LDE procedure by expanding the effective potential in powers of \( \delta \) only. This quantity can be expressed in terms of the full fermionic self-energy, whose terms contributing up to order \( \delta^2 \) are displayed in Fig. 2. The corresponding contributions to the effective potential up to order \( \delta^2 \) are shown in Fig. 3. This nicely illustrates how the LDE incorporates, at the same perturbative order, graphs that have different \( N \) dependence.

FIG. 2: Contributions to the fermion self-energy up to order \( \delta^2 \). The continuous thick line represents the LDE, \( \eta^* \) dependent, dressed fermion propagator while the thin lines represent the \( \eta \) dependent propagators. The dashed line represents the scalar auxiliary field, \( \sigma \). Tadpole diagrams are not shown since they do not contribute to \( V_{\text{eff}} \).

The second graph represents a correction to the \( \sigma \) propagator while the third has corrections to the Yukawa vertex. The first and fourth are exchange (rainbow) type of graphs.

FIG. 3: Feynman diagrams contributing to the effective potential at order \( \delta^2 \). Note that the first and second have a \( 1/N \) dependence while the last two have a \( 1/N^2 \) dependence. The thick lines in the first two graphs represent the LDE, \( \eta^* \), propagators which, as discussed in the text must be further expanded to order-\( \delta^2 \).

Let us now consider the second graph of Fig. 3 which, for comparison purposes, has been neglected in the previous subsection. It reads
\[
\frac{V_{\text{eff},a}^{(a)}}{N}(\eta_\ast) = -\frac{i}{2} \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left[ \Sigma_a(\eta_\ast) \right],
\]

where the trace is over Dirac’s matrices only \(^2\). The term \(\Sigma_a\) represents the first contribution of Fig. 2 to the fermion self-energy,

\[
\Sigma_a(\eta_\ast) = -\delta \left( \frac{\lambda}{N} \right) i \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q - \eta_\ast + i\epsilon}.
\]

Originally, Root \(^{37}\) has summed up all \(1/N\) corrections to next to the leading order and by further perturbatively expanding his result one easily retrieves Eq. (4.10).

Then, using the rules described previously, at finite temperature and density, one finds, after taking the trace and renormalizing with the fermionic mass counterterm (see the appendix), the result for Eq. (4.10)

\[
\frac{V_{\text{eff},a}^{(a)}}{N}(\sigma_c, \eta_\ast, T, \mu) = \frac{\lambda}{4\pi^2 N} \left\{ \eta_\ast^2 \left[ \ln \left( \frac{M}{\eta_\ast} \right) - I_2(\eta_\ast/T, \mu/T) \right]^2 + T^2 J_2^2(\eta_\ast/T, \mu/T) \right\},
\]

where

\[
I_2(a, b) = -\frac{\partial I_1(a, b)}{\partial a^2} = \int_0^\infty dx \frac{1}{\sqrt{x^2 + a^2}} \left( \frac{1}{e^{\sqrt{x^2 + a^2} + b} + 1} + \frac{1}{e^{-\sqrt{x^2 + a^2} - b} + 1} \right),
\]

with \(I_1(a, b)\) defined by Eq. (4.10) where, now, \(a = \eta_\ast/T\), and similarly

\[
J_2(a, b) = \sinh(b) \int_0^\infty dx \frac{1}{\cosh(\sqrt{x^2 + a^2}) + \cosh(b)}.
\]

The (finite) contributions given by the term proportional to \(I_2(a, b)\) and \(J_2(a, b)\) originate from the summation over Matsubara frequencies of the \(T\) and \(\mu\) dependent contributions, more precisely from Eqs. (A2) and (A3) respectively (see Appendix A for details). Note that the divergence is only contained, at this order, in the \(T = 0, \mu = 0\) part, which is renormalized by standard counterterms in the \(\overline{\text{MS}}\) scheme (see Appendix B for details).

Finally, by summing the contribution Eq. (4.12) to the effective potential expression to leading order in \(N\), Eq. (4.9), and expanding the resulting expression one obtains the complete LDE expression to order-\(\delta\)

\[
\frac{V_{\text{eff},a}^{(a)}}{N}(\sigma_c, \eta_\ast, T, \mu) = \frac{\sigma_c^2}{2\lambda} - \frac{1}{2\pi} \left\{ \eta_\ast^2 \left[ \ln \left( \frac{M}{\eta_\ast} \right) + \frac{1}{2} + \ln \left( \frac{M}{\eta} \right) \right] + 2T^2 I_1(\eta/T, \mu/T) \right\} + \delta \frac{\eta(\eta - \sigma_c)}{\pi} \left[ \ln \left( \frac{M}{\eta} \right) - I_2(\eta/T, \mu/T) \right] + \delta \frac{\lambda}{4\pi^2 N} \left\{ \eta_\ast^2 \left[ \ln \left( \frac{M}{\eta_\ast} \right) - I_2(\eta/T, \mu/T) \right]^2 + T^2 J_2^2(\eta/T, \mu/T) \right\},
\]

Notice once more, from Eq. (4.15), that our first order takes into account the first next to leading order correction to the large-\(N\) result.

When considering the case \(T = 0, \mu \neq 0\) in Eq. (4.15) one can take the limit \(T \to 0\), which then gives for the functions \(I_1, I_2\) and \(J_2\) the results

\(^2\) The factor \(-1\) corresponding to a closed fermionic has already been taken into account \(^{37}\).
\[
\lim_{T \to 0} T^2 I_1(\eta/T, \mu/T) = -\frac{1}{2} \theta(\mu - \eta) \left[ \eta^2 \ln \left( \frac{\mu + \sqrt{\mu^2 - \eta^2}}{\eta} \right) - \mu \sqrt{\mu^2 - \eta^2} \right],
\]
\[
\lim_{T \to 0} I_2(\eta/T, \mu/T) = \theta(\mu - \eta) \ln \left( \frac{\mu + \sqrt{\mu^2 - \eta^2}}{\eta} \right),
\]
\[
\lim_{T \to 0} T J_2(\eta/T, \mu/T) = \text{sgn}(\mu) \theta(\mu - \eta) \sqrt{\mu^2 - \eta^2}.
\]

Note that all the above results vanish for \( \mu < \eta \). When considering the case \( T \neq 0 \) and \( \mu = 0 \) the high temperature limit, \( \eta/T \ll 1 \) and \( \mu/T \ll 1 \) will prove to be useful. For \( I_1 \), this approximation follows from Eq. (5.12), while for \( I_2 \) it can be obtained using Eq. (4.13) which yields
\[
I_2(a, b) = \ln \left( \frac{\pi}{a} \right) - \gamma_E + \frac{7}{4} \left( b^2 + \frac{a^2}{2} \right) + \mathcal{O}(a^4, b^4).
\]

In the case \( T \neq 0 \) and \( \mu \neq 0 \) the integrals \( I_1, I_2 \) and \( J_2 \) will be handled numerically.

V. OPTIMIZATION AND NUMERICAL RESULTS BEYOND LARGE-\( N \)

Before proceeding to the specific \( d = 2 \) case, considered in this work, let us apply the PMS to the most general order-\( \delta \) effective potential which is given by
\[
\frac{V_{\text{eff.}, \delta}}{N}(\sigma_c, \eta) = \frac{\delta \sigma_c^2}{2\lambda} + i \int \frac{d^d p}{(2\pi)^d} \text{tr} \ln (p^2 - \eta^2) + \delta i \int \frac{d^d p}{(2\pi)^d} \text{tr} \frac{\eta - \sigma_c}{p^2 - \eta^2 + i\epsilon} + \frac{\delta \lambda}{2N} \text{tr} \left[ i \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right]^2.
\]

This exercise will help the reader to visualize the way the LDE-PMS resums the perturbative series. After taking the traces in Eq. (5.1) one obtains
\[
\frac{V_{\text{eff.}, \delta}}{N}(\sigma_c, \eta) = \frac{\delta \sigma_c^2}{2\lambda} + i \int \frac{d^d p}{(2\pi)^d} \ln (p^2 - \eta^2) \delta + 2i \int \frac{d^d p}{(2\pi)^d} \frac{\eta(\eta - \sigma_c)}{p^2 - \eta^2 + i\epsilon}
\]
\[+ \frac{\delta \lambda}{N} \left[ \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right]^2 + \frac{\delta \lambda}{N} \left[ \int \frac{d^d p}{(2\pi)^d} \frac{p_0}{p^2 - \eta^2 + i\epsilon} \right]^2.
\]

Then, setting \( \delta = 1 \) and applying the PMS one gets
\[
0 = \left\{ \left[ \eta - \sigma_c + \eta \frac{\lambda}{N} \left[ \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right] \left( 1 + \eta \frac{d}{d\eta} \right) \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right] \right\} \left. \left[ \eta - \sigma_c + \frac{\lambda}{N} \left( \int \frac{d^d p}{(2\pi)^d} (p^2 - \eta^2 + i\epsilon) \right) \frac{d}{d\eta} \left( \int \frac{d^d p}{(2\pi)^d} (p^2 - \eta^2 + i\epsilon) \right) \right] \right|_{\eta = \bar{\eta}}.
\]

As one can see in Appendix A (Eq. (A3)) the last term of the equation above only survives when \( \mu \neq 0 \). In the case \( \mu = 0 \), Eq. (5.3) factorizes in a nice way which allows us to understand the way the LDE-PMS procedure resums the series producing non-perturbative results. With this aim one can easily check that (at \( \delta = 1 \))
\[
\Sigma_c(\mu = 0, \eta, T) = -\frac{\lambda}{N} \left[ i \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right].
\]

Then, when \( \mu = 0 \) the PMS equation factorizes to
\[
[\bar{\eta} - \sigma_c - \Sigma_c(\mu = 0, \eta, T)] \left( 1 + \eta \frac{d}{d\eta} \right) \left[ i \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - \eta^2 + i\epsilon} \right] = 0.
\]
which leads to the self consistent relation

\[ \bar{\eta} = \sigma_c + \Sigma_a(\mu = 0, \eta, T) \, , \]  

(5.6)

which is valid for any temperature and number of space-time dimensions. In this way the LDE fermionic loops get dressed by \( \sigma_c \) as well as rainbow (exchange) type of self energy terms like the first graph of figure 3. Typical optimized Feynman graphs are shown in Fig. 4. The graphs in part (a) of that figure represent the order-\( \delta \) contributions prior to optimization while the infinite set shown in part (b) represents the non-perturbative optimized result. Note how the graphs are dressed by rainbow type of self energy contributions. This was expected since at order-\( \delta \) the perturbative LDE effective potential receives information about this type of topology only. If one proceeds to order-\( \delta^2 \) information about corrections to the scalar propagator as well as Yukawa vertex (see figure 2) will enter the perturbative effective potential. Then the PMS will dress up these perturbative contributions and so on. In other words, the simple evaluation of a first topologically distinct graph will bring non-perturbative information concerning that type of contribution. Figure 4 clearly shows that the LDE-PMS re-sums all powers of \( 1/N \) corresponding to the (rainbow) class of graph.

Note that the mathematical possibility

\[ i \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - \eta^2 + i\epsilon)} = 0 \, , \]  

(5.7)

corresponds to the unphysical solution discussed in Sec. IVA (Eq. (4.8)).

Having illustrated the way the LDE-PMS resummation works, let us concentrate in the \( d = 2 \) case by collecting our results for the complete order-\( \delta \) effective potential, Eq. (4.15), once the optimization equation is applied to it. Using the PMS procedure we then obtain, from Eq. (4.15) at \( \eta = \bar{\eta} \), the general result

\[ \left\{ \left[ \mathcal{V}(\eta, T, \mu) + \eta \frac{d}{d\eta} \mathcal{V}(\eta, T, \mu) \right] \left[ \eta - \sigma_c + \eta \frac{\lambda}{2\pi N} \mathcal{V}(\eta, T, \mu) \right] + \frac{\lambda T^2}{2\pi N} J_2(\eta/T, \mu/T) \frac{d}{d\eta} J_2(\eta/T, \mu/T) \right\} \bigg|_{\eta = \bar{\eta}} = 0 \, , \]  

(5.8)

where we have defined the function

\[ \mathcal{Y}(\eta, T, \mu) = \ln \left( \frac{M}{\eta} \right) - J_2(\eta/T, \mu/T) \, . \]  

(5.9)
Eq. (5.8) expresses our general PMS result, Eq. (5.3), for the specific \( d = 2 \) case. This can be easily seen by recalling that in this number of space dimensions (and \( \delta = 1 \)) the exchange (Fock) type of self energy is given by

\[
\Sigma_a(\eta, T, \mu) = -\eta \frac{\lambda}{2\pi N^2} \mathcal{V}(\eta, T, \mu) \tag{5.10}
\]

In the following we will present the analytical (whenever possible) and numerical results for Eq. (5.8). For convenience, the results will be presented in units of \( M \) for different values of \( \lambda \) and \( N \). We start by analyzing the simplest case, of zero temperature and density.

### A. The \( T = \mu = 0 \) case

Taking Eq. (5.8) at \( T = \mu = 0 \) [that is, \( \mathcal{V}(\eta, T = 0, \mu = 0) = \ln(M/\eta) \)] one gets

\[
\left[ \ln \left( \frac{M}{\tilde{\eta}} \right) - 1 \right] \left[ \tilde{\eta} - \sigma_c - \tilde{\eta} \frac{\lambda}{2\pi N} \ln \left( \frac{\tilde{\eta}}{M} \right) \right] = 0 . \tag{5.11}
\]

As discussed previously, the first factor leads to the coupling independent result, \( \tilde{\eta} = M/e \), which we shall neglect. At the same time the second factor in (5.11) leads to a self-consistent gap equation for \( \tilde{\eta} \), given by

\[
\tilde{\eta}_{\delta^1}(\sigma_c) = \sigma_c \left[ 1 - \frac{\lambda}{2\pi N} \ln \left( \frac{\tilde{\eta}_{\delta^1}}{M} \right) \right]^{-1} . \tag{5.12}
\]

The solution for \( \tilde{\eta}_{\delta^1} \) obtained from Eq. (5.12) is

\[
\tilde{\eta}_{\delta^1}(\sigma_c) = -\frac{2\pi N}{\lambda} W^{-1} \left[ \frac{2\pi N \sigma_c}{\lambda} \exp \left( -\frac{2\pi N}{\lambda} \right) \right] \sigma , \tag{5.13}
\]

where \( W(x) \) is the Lambert \( W \) (implicit) function \[35\], which satisfies \( W(x) \exp[W(x)] = x \).

To analyze CS breaking we then replace \( \eta \) by Eq. (5.13) in Eq. (4.15), which is taken at \( T = 0 \) and \( \mu = 0 \). As usual, CS breaking appears when the effective potential displays minima at some particular value \( \tilde{\sigma}_c \neq 0 \) which is obtained by minimizing the resulting effective potential with respect to \( \sigma_c \),

\[
\left. \frac{\partial V_{\text{eff}, \delta^1}(\sigma_c, \eta = \tilde{\eta}_{\delta^1})}{\partial \sigma_c} \right|_{\delta = 1, \sigma_c = \tilde{\sigma}_c} = \tilde{\sigma}_c - \frac{1}{\lambda} \tilde{\eta} \ln \frac{\tilde{\eta}}{M} = 0 . \tag{5.14}
\]

Since \( m_F = \tilde{\sigma}_c \), after some algebraic manipulation of Eq. (5.14) using the definition of the \( W(x) \) function, we find

\[
m_F(0) = \tilde{\sigma}_c(T = 0, \mu = 0) = M \mathcal{F}(\lambda, N) \left( 1 - \frac{1}{2N} \right)^{-1} , \tag{5.15}
\]

where we have defined the quantity \( \mathcal{F}(\lambda, N) \) as

\[
\mathcal{F}(\lambda, N) = \exp \left\{ -\frac{\pi}{\lambda [1 - 1/(2N)]} \right\} . \tag{5.16}
\]

Eq. (5.15) is our result for the fermionic mass at first order in \( \delta \) which goes beyond the large-N result, Eq. (3.16). Note also that in the \( N \to \infty \) limit, since \( \mathcal{F}(\lambda, N \to \infty) = \exp(-\pi/\lambda) \), Eq. (5.15) correctly reproduces, within the LDE non-perturbative resummation, the large-N result. In Fig. 5 we compare the order-\( \delta \) LDE-PMS results for \( \tilde{\sigma}_c \) with the one provided by the large-N approximation.

At the same time, \( \tilde{\eta} \) evaluated at \( \tilde{\sigma}_c(T = 0, \mu = 0) \), to first order in \( \delta \), also follows analytically from Eqs. (5.13) and (5.15),

\[
\tilde{\eta}_{\delta^1}(\tilde{\sigma}_c) = \left( 1 - \frac{1}{2N} \right) \tilde{\sigma}_c = M \mathcal{F}(\lambda, N) . \tag{5.17}
\]
The analytical results Eqs. (5.15) and (5.17) are exact expressions (of course, at this first order of the LDE) following only from the minimization condition Eq. (5.14) after some algebra. In particular, we emphasize the simple scaling relation obtained between $\bar{\eta}$ and $\bar{\sigma}_c$ in Eq. (5.17), that only depends on $N$, leading to the result (5.15), and that will prove to be a very good approximation even for the more general case, $T \neq 0$, as we shall see next. Note that this relation is the explicit form, for this case $T = \mu = 0$, of the more general relation Eq. (5.10) above, namely: $\Sigma_a(\mu = 0, \eta, T = 0) = -\sigma_c/(2N)$.

Actually, there is an alternative simpler way of deriving the results (5.15)–(5.17) without using the solution for $\bar{\eta}$ in Eq. (5.13), that will moreover prove useful later on when we shall consider the more complicate situations with $\mu \neq 0$, or typically also if we would consider higher LDE order contributions. That is, instead of using Eq. (5.14) as giving $\bar{\sigma}_c(\bar{\eta})$, we can use it to substitute the rather complicated logarithmic dependence $\ln \bar{\eta}/M$ in Eq. (5.12), canceling out at the same time the $\lambda$ dependence, and obtaining thus a very simple linear equation for $\bar{\sigma}_c/\bar{\eta}$ which only depends on $N$:

$$\frac{\bar{\eta}}{\bar{\sigma}_c} = \left(1 + \frac{1}{2N} \frac{\bar{\sigma}_c}{\bar{\eta}} \right)^{-1},$$

(5.18)

which immediately leads to the results (5.15)–(5.17). The solution in Eq. (5.13) is useful however, as it gives $\bar{\eta}$ for any $\sigma_c$ values, away from the minimum in Eq. (5.14).

B. The $T \neq 0, \mu = 0$ Case

For the finite temperature but zero chemical potential case, still using Eq. (5.8), the optimized $\bar{\eta}$ is now determined by the solution of the expression

$$\left\{ \left[ \gamma(\eta, T, \mu = 0) + \eta \frac{d}{d\eta} \gamma(\eta, T, \mu = 0) \right] \left[ \eta - \sigma_c + \eta \frac{\lambda}{2\pi N} \gamma(\eta, T, \mu = 0) \right] \right\}_{\eta = \bar{\eta}} = 0.$$

(5.19)

The solution coming from the first term in Eq. (5.19), corresponds to the unphysical optimized result, whereas the solution obtained from the second term gives the equivalent of the self-consistent gap equation, (5.12).
\[ \bar{\eta}\delta (\sigma_c, T) = \sigma_c \left[ 1 + \frac{\lambda}{2\pi N} \frac{\lambda}{2(2\pi)^2} \eta \right]^{-1}. \] (5.20)

We could next proceed by numerically solving Eq. (5.20) for \( \bar{\eta} \) and substituting it in place of \( \eta \) in Eq. (4.15) (evaluated at \( \mu = 0 \)). Then, by minimizing the effective potential we would obtain the general behavior of \( \bar{\sigma}_c \) as a function of the temperature. The critical temperature for CS restoration would then be determined by the solution of \( \bar{\sigma}_c(T = T_c) = 0 \) as usual. However, here an explicit analytical result for \( T_c \) can also be obtained if we apply the high temperature approximation to Eq. (4.15), with \( \eta/T \ll 1 \), and then optimize the resulting expression. The validity of using the high-\( T \) approximation before the optimization procedure could be questioned in principle since \( \eta \) at the level of Eq. (4.15), is arbitrary. However, one may easily perform a cross check by performing a numerical PMS application, as described above, without using the high-\( T \) expansion. The results we found for \( T_c \) from both approaches agree very well with each other, showing that the high-\( T \) expansion is valid in a large range, though actually \( \bar{\eta} \) and \( T_c \) are numerically of the same order of magnitude. A simple reason for this is that the true high temperature expansion parameter is not \( \eta^2/T^2 \), but rather \( 7/4[\zeta(3)/(2\pi)^2] \eta^2/T^2 \sim 0.05 \eta^2/T^2 \), as it is clear from Eq. (5.21) below, so that even if \( T_c \sim \eta \) the expansion parameter is still small enough. If we then expand Eq. (4.15) at high temperatures, up to order \( \eta^2/T^2 \), we obtain

\[ V_{\text{eff,4t}}(\sigma_c, \eta, T, \mu = 0) = \frac{\delta^{\sigma_c^2}}{2} - T^2 \frac{\eta^2}{2} \frac{\ln \left( \frac{M e^{\gamma E}}{T\pi} \right)}{4(2\pi)^2} - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \]

\[ + \frac{\eta(\eta - \sigma_c)}{N} \left[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \right] \]

\[ + \frac{\delta \frac{\lambda}{N} \eta^2}{(2\pi)^2} \left[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \right] + O(\eta^4/T^4). \] (5.21)

Then, setting \( \delta = 1 \) and applying the PMS one gets

\[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{21\zeta(3) \eta^2}{2(2\pi)^2 T^2} \left\{ \bar{\eta} - \sigma_c + \frac{\lambda}{N(2\pi)} \left[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \right] + O(\eta^4/T^4) \right\} = 0, \] (5.22)

where, once more, the first term in Eq. (5.22) corresponds to the unphysical (\( \lambda \) independent) solution, while the second term gives

\[ \bar{\eta}(\sigma_c, T) = \sigma_c \left[ 1 + \frac{\lambda}{N(2\pi)} \left[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \right] \right]^{-1}, \] (5.23)

whose solution can be expressed in the form

\[ \bar{\eta}(\sigma_c, T) = \sigma_c \left[ 1 + \frac{\lambda}{N(2\pi)} \left[ \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) - \frac{7\zeta(3) \eta^2}{2(2\pi)^2 T^2} \right] \right]^{-1} + O(\sigma_c^4/T^4). \] (5.24)

The result given by Eq. (5.24) is then plugged back into Eq. (5.21) for \( \delta = 1 \) and the resulting expression is again expanded for \( \sigma_c/T \ll 1 \) in the high-\( T \) approximation. The order of the transition can easily be checked numerically simply by plotting the effective potential for different values of \( T \) as shown in Fig. 7.

The result shows that the first order LDE result for finite \( N \) predicts a continuous phase transition as in the large-\( N \) case. By extremizing the effective potential, at high-\( T \), we obtain a maximum at \( \bar{\sigma}_c = 0 \) and two minima at:

\[ \bar{\sigma}_c(T) = \pm \frac{T}{N^2 \sqrt{14\pi(3)\lambda}} \left[ 2N\pi + \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) \right]^{3/2} \left[ -2N\pi + (2N - 1)\lambda \ln \left( \frac{M e^{\gamma E}}{T\pi} \right) \right]^{1/2}. \] (5.25)

Figure 7 shows \( \bar{\sigma}_c/M \) given by Eq. (5.24) as a function of \( T/M \), again showing a continuous (second order) phase transition for CS breaking/restoration.
FIG. 6: The effective potential, $V_{\text{eff}}/N$ as a function of $\sigma_c$ for $\mu = 0$ and $T = 0.170757 M$. The parameter values are $\lambda = \pi$ and $N = 3$. The continuous curve, which shows CS restoration, was plotted using the LDE optimized results. The dashed curve, which signals CS breaking, corresponds to the large-$N$ predictions. Both $V_{\text{eff}}/N$ and $\sigma_c$ are in units of $M$.

FIG. 7: The non-trivial minimum $\bar{\sigma}_c$ as a function of $T$ for the parameter values $\lambda = \pi$ and $N = 3$. The first order LDE curve (continuous line) displays a continuous phase transition occurring at the critical temperature $T_c = 0.170 M$ while the large-$N$ result is $T_c = 0.208 M$. Both $\bar{\sigma}_c$ and $T$ are in units of $M$.

The numerical results illustrated by Figs. 6 and 7 show that the transition is of the second kind and an analytical equation for the critical temperature can be obtained by requiring that the minima vanish at $T_c$. From Eq. (5.25), one sees that $\bar{\sigma}_c(T = T_c) = 0$ can lead to two possible solutions for $T_c$. The one coming from

$$\left[2N\pi + \ln\left(\frac{Me^{\gamma E}}{T_c\pi}\right)\right] = 0,$$

(5.26)

can easily be seen as not been able to reproduce the known large-$N$ result, when $N \to \infty$, $T_c = M \exp(\gamma_E - \pi/\lambda)/\pi$.  


However, the other possible solution coming from

\[
-2N\pi + (2N - 1)\lambda \ln \left( \frac{Me^{\gamma_E}}{\pi T_c} \right) = 0 ,
\]

(5.27)
gives for the critical temperature, evaluated at first order in \( \delta \), the result

\[
T_{c,\delta} = M \frac{e^{\gamma_E}}{\pi} \exp \left\{ - \frac{x}{\lambda(1 - 1/(2N))} \right\} = M \frac{e^{\gamma_E}}{\pi} F(\lambda, N) ,
\]

(5.28)
with \( F_\lambda(N) \) as given before, by Eq. (5.19). Thus, Eq. (5.28) exactly reproduces the large-\( N \) result for \( N \rightarrow \infty \). The results given by this equation are plotted in Fig. 8 as a function of \( \lambda \) with the one observed in the numerical results starting from Eqs. (4.15) and (5.19) without the use of the high-\( T \) approximation. We note in Eq. (5.28) the very same scaling relation than the one for \( m_F(0) \) at zero temperature, in Eq. (5.17), thus involving a relation that only depends on \( N \). This simple scaling derives in fact from the equation for the optimized solution \( \tilde{\eta}(T) \), where to obtain the factorized form Eq. (5.22), we have neglected \( \mathcal{O}(1/T^4) \) small terms. Plugging back these neglected terms, and manipulating Eq. (5.22), one easily obtains the relation:

\[
\sigma_c = \tilde{\eta}(T) \left( 1 - \frac{1}{2N} \right)^{-1} + x^2 \frac{\lambda}{\pi} H(\tilde{\eta}, N) ,
\]

(5.29)
where we defined for convenience the high-temperature expansion parameter \( x = 7/4[\zeta(3)/(2\pi)^2] \tilde{\eta}^2/T^2 \sim 0.05 \tilde{\eta}^2/T^2 \), so that \( x^2(\lambda/\pi) \tilde{\eta} H(\tilde{\eta}, N) \) defines the remnant part of order \( 1/T^4 \), whose explicit form we do not need to specify. Thus, neglecting these \( \mathcal{O}(x^2) \) terms in Eq. (5.29) gives the very same relation between \( \tilde{\eta} \) and \( \sigma_c \) than at \( T = 0 \) in Eq. (5.17), and furthermore, Eq. (5.29) is directly related, after some algebra, to the result (5.28) for \( T_c \) above. Accordingly, though the simple scaling relation is generally not expected to hold at arbitrary \( T \neq 0 \), the deviation from this relation is essentially negligible due to the smallness of those \( \mathcal{O}(x^2) \) corrections. We shall argue later on, in section V.E, that this universal scaling property is expected to remain a good approximation as well at higher orders of the LDE approximation.

The (non-perturbative) LDE result shows that \( T_c \) is always smaller (for the realistic finite \( N \) case) than the value predicted by the large-\( N \) approximation. In the light of Landau’s theorem for phase transitions in one space dimensions, which predicts \( T_c = 0 \), our LDE results, including the first \( 1/N \) corrections, seem to converge to the right direction.

C. The \( T = 0, \mu \neq 0 \) Case

As discussed in Ref. 14 this extremum of the phase diagram is the very sensitive to the role played by kink like configurations. However, in the present work only homogeneous background fields are considered and we are not in position to compare our results for this part of the phase diagram with the ones provided in that reference. Nevertheless, we are in position to contribute by computing finite \( N \) corrections so that one may, eventually, use the LDE-PMS in conjunction with inhomogeneous background fields to further improve the phase diagram found by the authors of Ref. 14. The case of zero temperature but finite chemical potential (density) also follows from Eq. (5.8).

Using Eqs. (4.10) and (4.17) in Eq. (5.28), we find two situations. In the first, for \( \eta > \mu \), the optimized \( \tilde{\eta} \) is found from the solution of

\[
\left\{ \left[ \ln \left( \frac{M}{\eta} \right) - 1 \right] \eta - \sigma_c + \frac{\lambda\eta}{2\pi N} \ln \left( \frac{M}{\eta} \right) \right\} \bigg|_{\eta=\tilde{\eta}} = 0 .
\]

(5.30)
For \( \eta < \mu \), using the relations (4.10), (4.17) and (4.18) in Eq. (4.15) we obtain that the optimized \( \tilde{\eta} \) is the solution of

\[
\left\{ \eta - \sigma_c - \frac{\lambda\eta}{2\pi N} \ln \left( \frac{\mu + \sqrt{\mu^2 - \eta^2}}{M} \right) \right\} \left[ - \ln \left( \frac{\mu + \sqrt{\mu^2 - \eta^2}}{M} \right) - \frac{\eta^2}{(\eta^2 - \mu^2 - \mu \sqrt{\mu^2 - \eta^2})} \right] - \frac{\lambda\eta}{2\pi N} \bigg|_{\eta=\tilde{\eta}} = 0 .
\]

(5.31)
FIG. 8: The dimensionless critical temperature, $T_c$ (in units of $M$), as a function of $\lambda$ for $T \neq 0$ and $\mu = 0$. The dashed line represents the $N \to \infty$ result while the continuous lines were produced by the LDE-PMS at order-$\delta$. The numbers next to the curves identify the value of $N$ for each case.

The solution for Eq. (5.30) is exactly the one obtained previously, given by Eq. (5.13). Concerning Eq. (5.31) one can again obtain an analytical solution, by following a reasoning similar to the one done for $T = \mu = 0$ leading directly to Eq. (5.18), but with slightly more involved algebra in the present case. Thus, consider the non-trivial minimum $\bar{\sigma}_c$, obtained from $\partial V_{\text{eff}} / \partial \sigma_c = 0$, now for $\mu \neq 0$. From Eq. (5.14) it follows that $\bar{\sigma}_c$ is given by

$$\bar{\sigma}_c = -\frac{\lambda}{\pi} \bar{\eta} \ln \left[ \frac{\mu + \sqrt{\mu^2 - \bar{\eta}^2}}{M} \right], \quad (5.32)$$

which replaces Eq. (5.14) for $\mu \neq 0$. Now again we can use this to eliminate simply the complicated logarithmic dependence $\ln[(\mu + \sqrt{\mu^2 - \bar{\eta}^2})/M]$ in Eq. (5.31), thus obtaining after straightforward algebra a second order equation for $\bar{\sigma}_c$ as a function of $\bar{\eta}$ and $\mu$, whose explicit solution reads:

$$\bar{\sigma}_c = \frac{1}{2} \frac{\bar{\eta}}{1 - 1/(2N)} \left[ 1 + G(\lambda, \bar{\eta}, \mu, N) + \sqrt{\left[ 1 - G(\lambda, \bar{\eta}, \mu, N) \right]^2 - \frac{2}{N} \left( 1 - \frac{1}{2N} \right) \frac{\lambda^2}{\pi^2}} \right], \quad (5.33)$$

where

$$G(\lambda, \bar{\eta}, \mu, N) = \frac{\lambda}{\pi} \left( 1 - \frac{1}{2N} \right) \left( 1 - \frac{\mu}{\sqrt{\mu^2 - \bar{\eta}^2}} \right), \quad (5.34)$$

contains the $\mu$ dependence. The relation (5.33) is the appropriate generalization, for $\mu \neq 0$, of the simple scaling relation obtained at $T = \mu = 0$ in Eq. (5.17). We have eliminated the other possible solution for $\bar{\sigma}_c$ (namely with $\sqrt{\cdots} \to -\sqrt{\cdots}$ in Eq. (5.33)), by noting that for $\lambda \to 0$, Eq. (5.33) correctly reproduces the simpler scaling relation in (5.17), while the other solution would give $\bar{\sigma}_c \to 0$. It will prove also useful to expand (5.33) in powers of $\lambda$:

\[3\] The leading order large $N$ relation $\bar{\sigma}_c = \bar{\eta}$ is also consistently reproduced for $N \to \infty$ in Eq. (5.33).
\[ \bar{\sigma}_c \sim \frac{\bar{\eta}}{1 - 1/(2N)} \left[ 1 - \frac{1}{2N} \left( 1 - \frac{1}{2N} \right) \left( \frac{\lambda}{\pi} \right)^2 + \frac{1}{2N} \left( 1 - \frac{1}{2N} \right)^2 \left( 1 - \frac{\mu}{\sqrt{\mu^2 - \bar{\eta}^2}} \right) \left( \frac{\lambda}{\pi} \right)^3 + \mathcal{O} \left( \frac{\lambda}{\pi} \right)^4 \right], \] (5.35)

which should be valid thus for moderate values of \( \lambda/\pi \). This immediately shows that the corrections due to \( \mu \neq 0 \) to the simple scaling obtained previously for \( T = \mu = 0 \) in Eq. [5.17] are actually suppressed by \( \mathcal{O}(\lambda^2/(\pi^2 N)) \); moreover the \( \mu \) dependence enters only at the next order, \( (\lambda/\pi)^3 \). These properties are somewhat analogous to the case of the \( T \neq 0 \) (\( \mu = 0 \)) corrections to the scaling in Eq. [5.29] which are also suppressed by the small high-temperature expansion parameter. We shall come back in the next sub-sections to the important consequences of these relations for the general case \( T, \mu \neq 0 \).

Next, we can extract the critical chemical potential, \( \mu_c \), in an analogous way as in the large-\( N \) problem shown in Sub-sec. III C and, as in the large-\( N \) problem, we can check that a first order transition also occurs. To this aim we first calculate the effective potential at the value \( \sigma_c \) given by Eq. [5.15], where the relevant expression is given by \( V_{\text{eff}}(\sigma_c) \) for \( \mu < \eta \). After some algebra many terms cancel out so that we obtain simply

\[ V^{\delta(1)}_{\text{eff}}(\sigma_c = \bar{\sigma}_c, \eta = \bar{\eta}) = -\frac{\bar{\eta}_\delta^2}{4\pi}, \] (5.36)

i.e., the same expression as the leading order \( N \to \infty \) one, but with the appropriate fermion mass at first order in \( \delta, \bar{\eta}_\delta \) given by Eq. [5.17]. This has now to be compared with the value of \( V_{\text{eff}} \) for \( \mu \neq 0 \) but with \( \eta = 0 \), which is simply obtained from Eq. [5.15] as

\[ V^{\delta(1)}_{\text{eff}}(\sigma_c = \eta = 0, \mu) = -\frac{\mu^2}{2\pi} \left( 1 - \frac{\lambda}{2\pi N} \right), \] (5.37)

so that we can deduce an analytic expression for the critical density \( \mu_c \),

\[ \mu_c,\delta^1 = \frac{\bar{\eta}_\delta^1}{\sqrt{2}} \left( 1 - \frac{\lambda}{2\pi N} \right)^{-1/2}, \] (5.38)

valid at this first order in \( \delta \). The appearance of a pole in Eq. [5.38] for \( \lambda = 2\pi N \) is an artifact of our first order in \( \delta \) approximation and thus probably not physically relevant. The point is that going to higher orders in the expansion in \( \delta \) it will also bring different corrections of same order in powers of \( 1/N \), if expanded. Thus, we could for example limit ourselves in this analysis to the first \( 1/N \) order in this expansion, since complete \( 1/N^2 \) corrections are not included at the order in \( \delta \) we are considering. Seen this way, we could also expand Eq. [5.38] in powers of \( 1/N \),

\[ \mu_c,\delta^1,1/N = \frac{M}{\sqrt{2}} \exp(-\pi/\lambda) \left[ 1 - \frac{\pi}{2N\lambda} + \frac{\lambda}{4\pi N} + \mathcal{O}(1/N^2) \right], \] (5.39)

which then exhibits no pole.

We also proceeded numerically to obtain the solution from [5.34] and subsequent values of \( \mu_c \), in order to have a useful crosscheck of the more complicated most general case with both \( T \) and \( \mu \) finite, where analytical expressions are not available. An example for fixed values of \( N \) and \( \lambda \) is presented in Fig. 4 which shows the effective potential as a function of \( \sigma_c \). The analogous condition to Eq. [5.24] leads to the result \( \mu_{c,\delta^1} \approx 0.232 M \), in the first order of the LDE, that should be compared with the large-\( N \) result, \( \mu_{c,N \to \infty} \approx 0.260 M \).

D. The \( T \neq 0, \mu \neq 0 \) Case

We now arrive at the point of analyzing the complete LDE phase diagram, at order-\( \delta \). In the lack of possible fully analytical solutions in this general case, we use numerical routines to determine both curves of second order and first order phase transitions and their point of intersection which then defines the tricritical point. In Fig. 10 we show the large-\( N \) result compared to the LDE first order result. It can be seen that in the LDE non-perturbative approach the region for CSB is diminished in an appreciable way. In units of \( M \), the LDE result for the tricritical
**FIG. 9:** Large-$N$ (dashed line) and first order LDE (continuous line) results for the effective potential, $V_{\text{eff}}/N$. The parameter values are $N = 3$, $\lambda = \pi$ and $T = 0$. The effective potential has been evaluated at the (first order) LDE critical value $\mu_c = 0.232 M$ for which the large-$N$ approximation still predicts CSB. Both $V_{\text{eff}}/N$ and $\mu$ are in units of $M$.

**FIG. 10:** Large-$N$ (outmost line) and LDE (innermost line) phase diagrams. The parameter values are $N = 3$ and $\lambda = \pi$. The dots represent the tricritical points. The large-$N$ result for the tricritical point is $P_{tc,N \to \infty} = (T_{tc}, \mu_{tc}) = (0.117, 0.224)$ while the LDE approximation gives $P_{tc,\delta 1} \simeq (T_{tc}, \mu_{tc})_{\delta 1} = (0.091, 0.192)$. In both cases, the lines above the tricritical point represent second order phase transitions while the ones situated below represent first order transitions. All quantities are given in units of $M$.

point, with $N = 3$ and $\lambda = \pi$, is $P_{tc,\delta 1} = (T_{tc}, \mu_{tc})_{\delta 1} = (0.091, 0.192)$ while the large-$N$ approximation gives $P_{tc,N \to \infty} = (T_{tc}, \mu_{tc}) = (0.117, 0.224)$.

It is worth remarking how the LDE tricritical point falls, approximately, over a line joining the large-$N$ result and the origin. This remarkable result is shown in Fig. 11 that shows $P_{tc}$ for $\lambda = \pi/2, \pi, 2\pi$ and $N = 3, 10$, and $N \to \infty$ (LN). A deviation of about 9% is observed only for large values of the ratio $\lambda/N$ (e.g., $2\pi/3$). For ratios close to the unity the deviation is very small (about 3%).
FIG. 11: The tricritical points \( P_t = (T_{tc}, \mu_{tc}) \), in units of \( M \), for different values of \( \lambda \) and \( N \). The stars represent \( \lambda = \pi/2 \), the dots \( \lambda = \pi \) and the diamonds \( \lambda = 2\pi \). The corresponding values of \( N \) are directly specified by the numbers on the graph. The angle \( \gamma \) represents the quantity \( T_{tc}/\mu_{tc} \) as discussed in the text.

E. Generality of Scaling for the LDE Results

For moderate values of \( \lambda/N \), we thus note the almost invariance, up to very small corrections in the LDE finite \( N \) case, of the ratio \( T_{tc}/\mu_{tc} \) that defines the angle \( \gamma \),

\[
\tan \gamma = \frac{T_{tc}}{\mu_{tc}} \simeq 0.523. \tag{5.40}
\]

All our results show that this ratio remains to a good approximation largely independent of both \( N \) and \( \lambda \). As already discussed, this strongly suggests to postulate a very simple (but approximate) result for the predictions of the tricritical points and for all other dimensionful quantities obtained for the GN model within the LDE. The result (5.40) indicates that \( T_{tc} \) and \( \mu_{tc} \), as a function of \( N \) and \( \lambda \), to a good approximation in the relevant range, scale according to

\[
T_{tc}(\lambda, N) \simeq c g(\lambda, N) M, \tag{5.41}
\]

\[
\mu_{tc}(\lambda, N) \simeq g(\lambda, N) M, \tag{5.42}
\]

where \( c = \tan \gamma \) and \( g(\lambda, N) \) is a function of the parameters \( \lambda \) and \( N \). At the first LDE order, we have obtained explicitly \( g(\lambda, N) = \mathcal{F}(\lambda, N) \), where \( \mathcal{F}(N) \) is given by Eq. (5.16), up to very small corrections in the \( T \neq 0 \) (\( \mu = 0 \)) case. For the case \( \mu \neq 0 \) (\( T = 0 \)) we saw that the approximation is valid only for \( \lambda \leq \pi \) approximately, which is easily understood by examining the approximate expanded form Eq. (5.35), of the relation Eq. (5.33). As anticipated, one can see that the corrections for \( \mu \neq 0 \) to the simple scaling relation (5.17), strictly valid for \( T = \mu = 0 \) only, are moderate as long as \( \lambda \) is not too large, which remain essentially true also for the more general case \( T \neq 0 \) and \( \mu \neq 0 \).

Interestingly, the actual result for the tricritical points, if compared with the large-\( N \) result shown in Subsec. III D, is of the form

\[
P_{tc} = (T_{tc}, \mu_{tc}) \simeq (0.318, 0.608) \tilde{\eta}_{\delta^1}(\tilde{\sigma}_c), \tag{5.43}
\]

where \( \tilde{\eta}_{\delta^1} = \mathcal{F}(\lambda, N)M \). The numerical deviations from the above result are very small (less than 5%) for \( \lambda/N < 1.3 \).

The result (5.43) together with our previous results obtained within the first order in the LDE, Eqs. (5.14), (5.28) and (5.38), respectively for the fermion mass (\( \tilde{\sigma}_c \)), critical temperature (\( T_c \)) and critical chemical potential (\( \mu_c \)), show
the same approximate scaling as given in terms of the optimized LDE quantity $\tilde{\eta}_4$. Also, as we have already seen previously, in the large-$N$ limit, $\tilde{\eta}_4(\tilde{\sigma}_c) \rightarrow m_F(0)$ and all our LDE results correctly reproduce those of Sec. III.

The reason why most of our results (except perhaps for the somewhat extreme case $T = 0, \mu \neq 0$) exhibit approximately universal scaling properties as a function of $\tilde{\eta}$, to a very good approximation, are well understood at the first LDE order, as explained in the previous subsections. On more general grounds, in writing the interpolated Lagrangian density Eq. (11), a explicitly dimensionful (mass) parameter $\eta$, is introduced in the originally massless model where there is no other dimensionful parameter. Thus, from simple dimensional analysis, clearly all physical quantities should scale with $\eta$, or actually $\tilde{\eta}$ derived from the optimization procedure. More precisely, at first LDE order, and $T = \mu = 0$, we obtained the exact simple relation Eq. (5.17) between $\tilde{\eta}$ and $\sigma_c$, which only depends on $N$. More generally, in terms of the basic scale of the model, say $\lambda \equiv M e^{-\pi/\lambda}$ in the $\overline{MS}$ renormalization scheme, truly non-perturbative results are expected to give for the ratio of the relevant quantities $m_F/\lambda$, $T_c/\lambda$ and $\mu_c/\lambda$, some specific dimensionless coefficients depending only on $N$. A priori, there is no reason why these coefficients should be exactly the same for the three quantities. In contrast, on purely perturbative grounds, for $T \neq 0$ and $\mu \neq 0$ we expect at finite LDE order, to obtain after optimization more complicated scaling relations, i.e., not only with different coefficients for the three relevant quantities, but with such coefficients being some non trivial (dimensionless) functions of $T$, $\mu$, $\sigma_c$, $\lambda$, and $N$. Indeed, an explicit example that this is the case already at first LDE order is illustrated by the relations Eq. (6.24) for $T \neq 0$, and Eq. (6.33) for $\mu \neq 0$. Both relations strictly depend on $\lambda$, and $T$ (or, respectively: $\mu$). However, we have seen that, quite remarkably, this extra dependence upon the coupling is quite suppressed, in such a way that the fermion mass $m_F$, critical temperature $T_c$, and even $\mu_c$, have to a very good approximation identical scaling factors which only depend on $N$. Moreover, we have shown that in the $T \neq 0$ ($\mu = 0$) case, this result is not a numerical accident but can be well understood analytically by noting that the corrections to this universal scaling are intrinsically quite negligible. Even in the most general case when both $T \neq 0$ and $\mu \neq 0$, the results for the tricritical point happens in a regime where the simple scaling relation appears to be still a good approximation. The only exception is the somewhat extreme case when $T = 0, \mu \neq 0$, in this case, if $\lambda$ is sufficiently large, the appropriate generalization given analytically by Eq. (6.39), leads to substantial deviations from the simple scaling relation for $T = \mu = 0$ in Eq. (5.17).

We believe in fact that this trend generalizes as well to arbitrary higher orders of the LDE expansion, as we shall argue heuristically next. To examine what is happening at higher LDE orders, let us first consider the $T = \mu = 0$ case. It is simpler to generalize the reasoning used to obtain Eq. (5.18) at first order, i.e. using the solution $\sigma_c$ of Eq. (5.14) to replace the logarithmic dependence $\ln(\tilde{\eta}/M)$ into the (optimization) equation defining $\tilde{\eta}$, that thus generalizes Eq. (5.12) at higher orders. In this way the latter will be a simpler, purely algebraic equation, for the (only) parameter $\sigma_c/\tilde{\eta}$, in terms of the remaining parameters, the coupling $\lambda$ and $N$. More precisely, at higher LDE orders, from general arguments the effective potential will have additional perturbative terms of the generic form

$$\sim \sum_{k \geq 2} (\eta_\pi)^2 \frac{\lambda}{2\pi} \left( \frac{\lambda}{M} \right)^k \left[ c_{\text{LL}}(N) \ln^{k+1} \frac{\eta_\pi}{M} + c_{\text{NLL}}(N) \ln^k \frac{\eta_\pi}{M} + \cdots \right]$$

with leading, next-to-leading, etc different powers of $\ln(\eta_\pi/M)$, and $c_{\text{LL}}, c_{\text{NLL}}, \ldots$, the corresponding coefficients to be calculated from the relevant Feynman graphs. Next, performing the LDE to some given order and taking the limit $\delta \rightarrow 1$ etc, the resulting dependence upon $\sigma_c$ and $\eta$ is much more involved than at first order, so that one obtains from the CS breaking condition a non-linear relation for $\sigma_c$, generalizing Eq. (5.17), and as well for the optimized $\tilde{\eta}$ solution, Eq. (5.12). Nevertheless, it is clear that these two relations can be used e.g. to eliminate the logarithmic dependence $\ln(\eta/M)$, thus obtaining a polynomial equation for $\sigma_c/\tilde{\eta}$, which only depends on $\lambda$ and $N$. Actually, if we were able to re-sum this LDE series to all orders, we would certainly expect that the only dependence on the coupling $\lambda$ in all physical quantities would be entirely included in terms of the basic dimensionful scale of the model e.g. in the $\overline{MS}$ scheme: $\Lambda_{\overline{MS}} \equiv M e^{-\pi/\lambda}$. However, at finite LDE orders, the optimization generally defines a rather complicated dependence on $\lambda$ for $\tilde{\eta}$. But we have checked explicitly at the next ($\delta^2$) order that this induces relatively small deviations from the first order scaling relation in Eq. (5.17) (upon assuming obviously that the unknown coefficients $c_i$ appearing in Eq. (5.14) take generic values of $O(1)$ ). Next, we saw that the $T \neq 0$ result gives, at this LDE order,

Note that though at first the interpolation procedure done in Eq. (5.18) seems to explicitly break the discrete chiral symmetry of the model, $\eta$ is initially an undetermined parameter. The introduction of the auxiliary field $\sigma$ just makes the optimized $\eta$ a function of the background scalar field $\sigma_c$. After optimization $\tilde{\eta}(\tilde{\sigma}_c) \sim \sigma_c$ and so, chiral symmetry breaking and restoration have the same interpretation as in the original (non-interpolated) GN model. It is a non-vanishing value for the minimum of the $\sigma$ field effective potential, $(\sigma_c) = \bar{\sigma}_c \neq 0$ that still signals the breakdown of chiral symmetry, and $\tilde{\eta}(\bar{\sigma}_c) \sim \bar{\sigma}_c$ provides the scale for all physical quantities derived.
a very small correction to the scaling relation existing for \( T = 0 \). For \( \mu \neq 0 \), a deviation to this simple scaling is predicted by Eq. (5.33), but it remains, however, numerically moderate up to relatively large values of the coupling \( \lambda \). Now, since as we just examined the \( T = 0 \) scaling properties are expected to generalize at higher LDE orders, and the LDE generally converges quite rapidly, we can expect that the higher order modifications for \( T, \mu \neq 0 \) to these first LDE order scaling properties should remain small corrections.

VI. COMPARING THE LDE AND THE 1/N RESULTS

Let us compare, in this section, the LDE leading order results with the ones given by the 1/N approximation at leading order (LO) as well as next to the leading order (NLO). As already pointed out, the effective potential, at \( T = 0 \) and \( \mu = 0 \), for the 2d Gross-Neveu model to the next to leading order (NLO) was first evaluated by Root \[37\]. The NLO correction to the fermionic mass, at \( T = \mu = 0 \), was explicitly evaluated by Forgács, Niedermayer and Weisz \[39\]. Using a combination of the thermodynamic Bethe ansatz and the 1/N effective potential at \( T = \mu = 0 \), Chodos and Minakata \[40\] were able to obtain the NLO correction for \( \mu_c \). Recently, the authors in Ref. \[13\] computed the complete NLO in the 1/N expansion for the effective potential at \( T \neq 0 \) and \( \mu = 0 \) performing a detailed numerical analysis of their results. They also exhibit in details a number of non-trivial properties, in particular for the expected behavior at high temperature, but because of the appearance of a Landau pole near the \( T \sim T_c \) regime, they do not conclude on a well-defined value of \( T_c \) from the full 1/N calculation. It is thus difficult to compare our numerical LDE estimates of \( T_c \) with their numerical results.

Next, we emphasize that, as far as we know, there are no 1/N NLO results for the case \( T \neq 0 \) and \( \mu \neq 0 \), thus no results for the tricritical points beyond the large-N approximation. For comparison purposes, let us use our notations and conventions to present the only two available analytical results.

For the case \( T = \mu = 0 \) Ref. \[39\] gives

\[
\bar{\sigma}_c^{1/N,NLO}(0) = M \exp\left(-\frac{\pi}{\Lambda}\right) \left[ 1 + \frac{1}{N} (2 \ln 2 - \gamma_E) \right].
\] (6.1)

Figure 12 shows the LDE fermionic mass at \( T = \mu \), \( \bar{\sigma}_c = \frac{\mathcal{F}(\lambda, N)}{[1 - 1/(2N)]} \) as a function of \( \lambda \) for \( N = 3 \). The same figure shows the 1/N results at LO, given by Eq. (6.10), and at NLO, given by Eq. (6.11).

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5 In our comparison with the exact 1/N result of the fermion mass as given in Ref. \[37\], we should be cautious to remove from their expression a term: \( M \exp(-\pi/\Lambda)/N \), due our scheme being such that our reference scale is \( M \exp(-\pi/\Lambda) \), rather than the full \( \Lambda_{\text{MS}} \) expression of \[37\].
FIG. 12: The fermionic mass $m_F(0) = \bar{\sigma}_c$, in units of $M$, plotted as a function of $\lambda$ for $N = 3$, $T = 0$ and $\mu = 0$. The dashed line represents the $1/N$ result at leading order, the dot-dashed line represents the $1/N$ result at next to leading order and the continuous line is the first order LDE result.

For the case $T = 0, \mu \neq 0$ Ref. [40] gives

$$\mu_{c,1/N, NLO} = \frac{M}{\sqrt{2}} \exp(-\pi/\lambda) \left(1 - \frac{0.47}{N}\right).$$ (6.2)

FIG. 13: The critical chemical potential $\mu_c$ in units of $M$, plotted as a function of $\lambda$ for $N = 3$ and $T = 0$. The dashed line represents the $1/N$ result at leading order, the dot-dashed line represents the $1/N$ result at next to leading order and the continuous line is the first order LDE result.

Figure 13 shows the LDE critical chemical potential at $T = 0$, $\mu_c = M/\sqrt{2} \mathcal{F}(\lambda, N)$, as a function of $\lambda$ for $N = 3$. The same figure shows the $1/N$ results at LO, given by Eq. (6.23), and at NLO, given by Eq. (6.2). As we have emphasized in the text, the LDE results for this case are valid only up to $\lambda \sim \pi$. Apart from that, this is the case where kink like configurations start to play a major role so any quantitative results must be interpreted with the due care [14].

VII. CONCLUSIONS

The analytical non-perturbative technique known as the LDE has been applied to the two dimensional Gross-Neveu model effective potential at finite temperature and chemical potential. Following the prescription suggested in Ref. [27] we have shown that, within the large-$N$ limit, the LDE exactly agrees with $1/N$ LO “exact” result for any values of $T$ or $\mu$. Having established this reliability we have considered the first finite $N$ correction that already appears at the first non-trivial order. The variational optimization procedure has produced interesting results that turned out to be possible to be cast into analytical form. A careful analysis of our numerical and analytical optimized results has led us to write down five simple relations, that take into account finite $N$ corrections, concerning the $T = 0 = \mu$ fermionic mass $(\bar{\sigma}_c)$, $\mu_c$ (at $T = 0$), $T_c$ (at $\mu = 0$), and the tricritical points, $(T_{t,c}, \mu_{t,c})$. As we have discussed, four of these quantities essentially scale, to a very good approximation, with the LDE optimized mass scale at $T = 0 = \mu$ given by $\bar{\eta}/M = \mathcal{F}(\lambda, N)$, with $\mathcal{F}(\lambda, N)$ defined by Eq. (5.10), with $\bar{\eta}$ being the only dimensionful quantity present in the interpolated Lagrangian density. The only exception is the case of $\mu_c$ for $T = 0$ where, for large enough coupling $\lambda$, we obtained substantial deviation from the simple scaling, with an explicit expression of its dependence on $\lambda$. However, as already mentioned, we believe our results are not faithful in the small $T$ and large $\mu$ part of the phase diagram, which is the most substantially affected by inhomogeneous backgrounds as shown at leading $1/N$ order in Ref. [14]. The
basic reasons for the approximately very good validity of the universal scaling relations, at the first LDE order, were identified and understood, as explained in some detail in Sec. V. Moreover, we argued that this quasi-universality of scaling is expected to hold also at arbitrary higher orders of the LDE. At $T \neq 0$ and $\mu \neq 0$ our main results concern the evaluation of the phase diagram, containing finite $N$ corrections that, as far as we know, has not been carried out before. Comparing our perturbative type of evaluation with the ones performed in Refs. [13, 38, 40], for example, one may notice some of the LDE advantages. Namely, it automatically introduces an infrared cutoff that makes possible completely perturbative evaluations. At the same time, at each (perturbative) order one has just a few Feynman graphs to evaluate as compared to the traditional non-perturbative methods, such as the $1/N$ approximation. In particular, this advantage of the LDE procedure means that the renormalization program can be easily implemented.

Now, to proceed beyond the first LDE order, as far as the particular, this advantage of the LDE procedure means that the renormalization program can be easily implemented. Moreover, even at first order, it is very difficult to compare our results with theirs, due to the different momenta routing used, resulting in their much more involved expressions of the $T$-dependent contributions in particular (note however that this alternative routing is the only possible one beyond the first order). More generally, at the first LDE order here investigated, we cannot expect to get very close to the exact $1/N$ results, but at the same time we expect the LDE to converge faster and it will incorporates terms beyond the $1/N$ at higher orders, as discussed before.

Speculating on the expected behavior at higher LDE orders, the occurrence of a Landau pole in the complete $1/N$ results found by the authors of Ref. [13] therefore invalidating a priori a unambiguous determination of a $T_c$ value, deserves some general comments. Accordingly, it is useful to recall how the Landau pole emerges in the construction of Ref. [13]: in very rough terms, at finite $T$ a pole can occur in the dressed propagator of the $\sigma$ field for some value of the $T$-dependent effective fermion mass, $m_F(T)$, because the latter decreases as $T$ increases from zero, reaching eventually $m_F(T_c) = 0$ at critical temperature\(^6\).

Now, it is interesting to note that the LDE will avoid this Landau pole problem, in a rather trivial way: since by construction the LDE stops at finite orders, where there cannot be a pole, while a Landau pole is relevant only when considering a resummed perturbative series. Moreover, let us assume that one would manage to re-sum the LDE perturbative series to all orders, in some approximation (which can be done explicitly e.g. for the GN mass gap [21, 22]) at order-$1/N$, using the LDE together with renormalization group properties: one would thus obtain a resummed expression exhibiting possibly a pole at a critical mass value, but even in such a case, the optimization prescription used in the LDE construction will escape this pole, i.e. the optimization “freezes out” the mass at a value which generally cannot coincide with a possible Landau pole value.

Although convergence properties cannot be accessed in the first non-trivial order, the fact that our $T_c$ is, in accordance with Landau’s theorem, smaller than the LO large $N$ result gives further support to the method. However, a deeper discussion about convergence is beyond the scope of the present work due to the technical difficulties in evaluating, at $T \neq 0$ and $\mu \neq 0$, some of the three loop graphs shown in Fig. 5. We recall that the LDE-PMS convergence properties in critical theories have been analyzed, by the present authors, in connection with homogeneous Bose-Einstein condensates [8]. We intend to extend the present work by considering the order-$\delta^2$ three loop graphs that will bring the first $1/N^2$ corrections. This will help us to gauge convergence properties, the effects of the $1/N^2$ terms, as well as the eventual generalization of the universal scaling relations between the critical and tricritical quantities with the LDE optimized mass scale.

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\(^6\) Note in contrast that this Landau pole is harmless at $T = 0$, because in the corresponding exact $1/N$ calculation of the mass gap (see e.g. [33, 41]), this pole actually cannot be reached for any consistent value of the mass gap.
APPENDIX A: SUMMING MATSUBARA FREQUENCIES AND RELATED FORMULAS

The integrals encountered in Feynman’s graphs are performed, as usual, at finite temperature and density with the substitution rules \( p_0 \to i(\omega_n - i\mu) \) where \( \mu \) is the chemical potential and \( \omega_n = (2n + 1)\pi T \), \( n = 0, \pm 1, \pm 2, \ldots \), are the Matsubara frequencies for fermions. We sum over the Matsubara frequencies with standard contour integration techniques, and regularize the remaining (Euclidean) momentum integrals with dimensional regularization and carry the renormalization in the \( \overline{\text{MS}} \) scheme. E.g., momentum integrals of functions \( f(p_0, |p|) \) are replaced by (see Ref.

\[
\int \frac{d^2p}{(2\pi)^2} f(p_0, |p|) \to iT \sum_n f[i(\omega_n - i\mu), |p|] = iT \sum_n \int \frac{dp}{(2\pi)^2} f[i(\omega_n - i\mu), |p|],
\]

For the divergent, zero temperature contributions, dimensional regularization is carried out in dimensions \( d = 1 - 2\epsilon \) and in the \( \overline{\text{MS}} \) scheme, in which case the momentum integrals are written as

\[
\int \frac{dp}{(2\pi)^2} \to \int_p = \frac{(e^{\gamma_E} M^2)}{4\pi} \epsilon \int \frac{d^d p}{(2\pi)^d},
\]

where \( M \) is an arbitrary mass scale and \( \gamma_E \approx 0.5772 \) is the Euler-Mascheroni constant.

The Matsubara sums which are relevant for the different integrals considered in section IV can be derived as (see e.g. (31)):

\[
T \sum_{n=\infty}^{+\infty} \ln[(\omega_n - i\mu)^2 + \omega^2(p)] = \omega(p) + T \ln\{1 + e^{-[\omega(p) + \mu]/T} \} + T \ln\{1 + e^{-[\omega(p) - \mu]/T} \}, \quad (A1)
\]

\[
T \sum_{n=-\infty}^{+\infty} \frac{1}{(\omega_n - i\mu)^2 + \omega^2(p)} = \frac{1}{2\omega(p)} \left\{ \frac{1}{e^{\omega(p) + \mu}/T + 1} - \frac{1}{e^{\omega(p) - \mu}/T + 1} \right\}, \quad (A2)
\]

\[
T \sum_{n=-\infty}^{+\infty} \frac{\omega_n - i\mu}{(\omega_n - i\mu)^2 + \omega^2(p)} = \frac{i}{2} \frac{\sinh(\mu/T)}{\cosh(\mu/T) + \cosh(\omega(p)/T)} \quad (A3)
\]

where \( \omega^2(p) = p^2 + \eta^2 \).

APPENDIX B: RENORMALIZATION OF THE LDE EFFECTIVE POTENTIAL

We give here some details on the renormalization procedure for the effective potential of the interpolated model \( (4,3) \). First, let us consider the non-renormalized result for the effective potential in the large-\( N \) approximation, which from Eq. \( (4.5) \) is given by

\[
\frac{V_{\text{eff}, \delta}}{N} \sigma_c, \eta_s, N \to \infty = e^2 \sigma^2 \frac{2}{2\lambda} - \frac{1}{2\pi} \left\{ \eta_s^2 \left[ \frac{1}{\epsilon} + \frac{1}{2} + \ln \left( \frac{M}{\eta_s} \right) \right] + 2T^2 I_1 \left( \frac{\eta_s}{T} \frac{\mu}{T} \right) \right\}. \quad (B1)
\]

Going beyond large-\( N \) one must add the order-\( \delta \) term, Eq. \( (4.10) \),

\[
\frac{V_{\text{eff}, \delta}}{N} \sigma_c, \eta_s, N \to \infty = -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \text{tr} \left[ \frac{\Sigma_a(\eta_s)}{\hat{p} - \eta_s + i\epsilon} \right], \quad (B2)
\]

where \( \Sigma_a \) is the term 3c in Fig. \( (4) \)

\[
\Sigma_a(\eta_s) = -\delta \left( \frac{\lambda}{N} \right) i \int \frac{d^d q}{(2\pi)^d} \frac{1}{q - \eta_s + i\epsilon}. \quad (B3)
\]
It is worth noting that the relative simplicity of our final result, Eq. (14.12), for this two-loop graph is largely due to the appropriate choice of momenta routing, allowing to factorize the result into (squared) one-loop contributions. We checked explicitly that it is strictly equivalent to another momenta routing choice in the literature [13, 37], more appropriate when considering higher order contributions but which at this two-loop level would result into much more involved intermediate expressions.

As it has been shown [6, 24], the standard \( \overline{\text{MS}} \) renormalization procedure and the LDE commute with each other, so that one may perform the LDE before renormalization, introducing thus extra \( \delta \)-dependent counterterms, or alternatively directly on renormalized quantities. We shall rather follow here the first approach which may be more illustrative for our purpose. An explicit evaluation of Eqs. (B2) and (B3) with the rules given in appendix A gives

\[
\frac{V^{(a)}_{\text{eff},\delta}}{N}(\sigma_c, \eta_*, T, \mu) = \delta \frac{\lambda}{4\pi^2 N} \left\{ \eta_*^2 \left\{ \frac{1}{4 \epsilon^2} + \frac{1}{\epsilon} \left[ \ln \left( \frac{M}{\eta_*} \right) - I_2(\eta_*/T, \mu/T) \right] \right\} + \ln \left( \frac{M}{\eta_*} \right) - I_2(\eta_*/T, \mu/T) \right\} + J_2^2(\eta_*/T, \mu/T) \right\}. \tag{B4}
\]

Eqs. (B11) and (B12) give the total contributions from diagrams 1a, 1b, 1c and 1d in figure 14.

To evaluate the contribution given by graph 2b in Fig. 14 we need to define the mass counterterm \( D_{\text{exc}}^\delta \psi \bar{\psi} \) used to renormalize the exchange self energy given by 3c. Note that the divergences are only associated to the terms proportional to the mass. This sets the Feynman rule to \( iD_{\text{exc}}^\delta \psi \bar{\psi} \) where

\[
D_{\text{exc}}^\delta = -\delta \frac{\lambda}{4\pi N} \eta_* \frac{1}{\epsilon}. \tag{B5}
\]

Then, one can evaluate the one loop contribution corresponding to 2b in Fig. 14.

\[
\frac{V^{(a,\text{term})}_{\text{eff},\delta}}{N}(\sigma_c, \eta_*, T, \mu) = -i \int \frac{d^dp}{(2\pi)^d} \text{tr} \left[ \frac{D_{\text{exc}}^\delta}{p - \eta_* + i\epsilon} \right], \tag{B6}
\]

obtaining

\[
\frac{V^{(a,\text{term})}_{\text{eff},\delta}}{N}(\sigma_c, \eta_*, T, \mu) = \delta \frac{\lambda}{4\pi^2 N} \eta_*^2 \left\{ \frac{1}{2 \epsilon^2} - \frac{1}{\epsilon} \left[ \ln \left( \frac{M}{\eta_*} \right) - I_2(\eta_*/T, \mu/T) \right] - \ln \left( \frac{M}{\eta_*} \right) - \frac{\pi^2}{24} \right\}. \tag{B7}
\]
so that when adding the two contributions (B7) and (B4) the $1/\epsilon$ divergence as well as some finite terms cancel out, and there only remains the $1/\epsilon^2$ divergence to be renormalized by zero point (vacuum energy) subtraction counterterms.

The Feynman renormalization coefficient corresponding to the linear counterterm $E^3\sigma$ is obtained from the divergent part of the fermionic loop contained in the graph 3a of Fig. 14, namely

$$E^3 = \delta \frac{\eta}{\pi \epsilon}.$$ \hfill (B8)

Then, adding all contributions one has

$$\frac{V_{\text{eff},\delta^1}(\sigma_c, \eta, T, \mu)}{N} = \delta \frac{\sigma_c^2}{2\lambda} - \frac{1}{2\pi} \left\{ \eta^2 \left[ \frac{1}{2} + \ln \left( \frac{M}{\eta} \right) \right] + 2T^2I_1(\eta/T, \mu/T) \right\} + \delta \frac{\eta(\eta - \sigma_c)}{\pi} \left[ \ln \left( \frac{M}{\eta} \right) - I_2(\eta/T, \mu/T) \right] + \delta \frac{\lambda}{4\pi^2} \eta^2 \left[ \ln \left( \frac{M}{\eta} \right) - I_2(\eta/T, \mu/T) \right]^2 - \frac{\eta^2}{2\pi \epsilon} + \delta \frac{\eta^2}{\pi \epsilon} \left[ \eta^2 - \left( \frac{\eta^2}{4\pi} \right) \frac{\lambda}{4\pi N} \right] + X^3(\eta, \mu) + \frac{T^2}{4\pi^2}J_2^2(\eta/T, \mu/T).$$ \hfill (B9)

The remaining divergent contributions come from purely vacuum (field independent) graphs and can be absorbed by the zero point energy subtraction counterterm

$$X^3(\eta, \mu) = \eta^2 \frac{\pi}{2\pi \epsilon} - \frac{\delta \eta^2}{\pi \epsilon} \left[ 1 - \frac{\lambda}{4\pi N} \left( \frac{1}{4\epsilon} \right) \right],$$ \hfill (B10)

finally giving the finite effective potential

$$\frac{V_{\text{eff},\delta^1}(\sigma_c, \eta, T, \mu)}{N} = \delta \frac{\sigma_c^2}{2\lambda} - \frac{1}{2\pi} \left\{ \eta^2 \left[ \frac{1}{2} + \ln \left( \frac{M}{\eta} \right) \right] + 2T^2I_1(\eta/T, \mu/T) \right\} + \delta \frac{\eta(\eta - \sigma_c)}{\pi} \left[ \ln \left( \frac{M}{\eta} \right) - I_2(\eta/T, \mu/T) \right] + \delta \frac{\lambda}{4\pi^2} \eta^2 \left[ \ln \left( \frac{M}{\eta} \right) - I_2(\eta/T, \mu/T) \right]^2 + T^2J_2^2(\eta/T, \mu/T),$$ \hfill (B11)

which is the result shown in Eq. \((4.15)\). More details about renormalization within the LDE can be found in the first work of Ref. \(6\) (see also \(24, 27\) for the GN model).

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