UNIFORM ESTIMATES OF NONLINEAR SPECTRAL GAPS

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Abstract. We show that nonlinear spectral gaps of a finite connected graph are uniformly bounded from below by a positive constant which is independent of the target metric space by generalizing the path method. We apply our result to an $r$-ball $T_{d,r}$ in the $d$-regular tree, and observe that the asymptotic behavior of nonlinear spectral gaps of $T_{d,r}$ as $r \to \infty$ does not depend on the target metric space, which is in contrast to the case of a sequence of expanders.

1. Introduction

Let $G = (V, E)$ be a graph. We denote by $\bar{E}$ the set $\{(x, y) \in V \times V \mid \{x, y\} \in E\}$. A weight function on $G$ is a symmetric function $m : V \times V \to [0, \infty)$ whose support equals $\bar{E}$. The pair $(G, m)$ is called a weighted graph. A weight function $m$ induces a weight $m(x)$ of each vertex $x \in V$ by $m(x) = \sum_{y \in V} m(x, y)$. We use the convention that $m(\emptyset) = \sum_{x \in V} m(x)$. We call the following special weight function $m$ the uniform weight function on $G$:

$$m(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \bar{E}, \\ 0, & \text{otherwise}. \end{cases}$$

In this paper, graphs and metric spaces are always assumed to contain at least two distinct points.

Definition 1.1. Let $G = (V, E)$ be a finite connected graph with a weight function $m$, and let $(X, d)$ be a metric space. We define the nonlinear spectral gap $\lambda_{1}^{\text{Gro}}(G, X)$ of $G$ with respect to $X$ to be the reciprocal of the smallest constant $C > 0$ satisfying the following Poincaré inequality for any $f : V \to X$:

$$\frac{1}{m(\emptyset)} \sum_{x, y \in V} m(x)m(y)d(f(x), f(y))^2 \leq C \sum_{x, y \in V} m(x, y)d(f(x), f(y))^2.$$

We follow the notation used in [12] and [16] in the above definition. We will see later that the constant $C > 0$ in the definition always exists. Hence nonlinear spectral gaps are always positive real numbers. If $X = \mathbb{R}$, $\lambda_{1}^{\text{Gro}}(G, \mathbb{R})$ is no other than the first positive eigenvalue $\mu_1(G)$ of the combinatorial Laplacian $\Delta$ on $G$, which acts on a real-valued function $f$ on $V$ as

$$\Delta f(x) = f(x) - \sum_{y \in V} \frac{m(x, y)}{m(x)} f(y), \quad x \in V.$$
For a Hilbert space $\mathcal{H}$, by summing over the coordinates with respect to some orthonormal basis, it is straightforward to see that $\lambda_1^{\text{Gro}}(G, \mathcal{H}) = \lambda_1^{\text{Gro}}(G, \mathbb{R}) = \mu_1(G)$.

Nonlinear spectral gaps play important roles both in geometric group theory and metric geometry (see [5], [6], [8], [9], [10], [14], [15], [21]). For example, they relate to the fixed point property of discrete groups, measure concentration of metric measure spaces and coarse embeddability of a metric space into another metric space.

Suppose that a sequence $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ of (uniformly weighted) finite connected graphs satisfies the following properties for a metric space $X$:

1. The number of vertices of $G_n$ goes to infinity as $n$ goes to infinity.
2. There exists $d$ such that $\text{deg}(v) \leq d$ for all $v \in V_n$ and all $n$.
3. There exists $\lambda > 0$ such that $\lambda_1^{\text{Gro}}(G_n, X) \geq \lambda$ for all $n$.

It is known that such a sequence admits no coarse embedding into $X$ (see [14]). When $X$ is $\mathbb{R}$ (or a Hilbert space), such a sequence is called a sequence of expanders. The following proposition asserts that the asymptotic behavior of nonlinear spectral gaps of a sequence of expanders change drastically if the target metric space is changed.

In this paper, we write $A \lesssim B$ or $B \gtrsim A$ when there exists a universal constant $C > 0$ such that $A \leq CB$. We write $A \asymp B$ if both $A \lesssim B$ and $B \lesssim A$ hold. If we have $A \leq C_p B$ for a constant $C_p > 0$ which depends only on some parameter $p$, we write $A \lesssim_p B$ or $B \gtrsim_p A$. We write $A \asymp_p B$ if both $A \lesssim_p B$ and $B \gtrsim_p A$ hold.

**Proposition 1.2.** Suppose that a sequence $\{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ of (uniformly weighted) finite connected graphs satisfies the properties (1) and (2) above. Then there exists a metric space $X$ such that

$$\lambda_1^{\text{Gro}}(G_n, X) \lesssim_d \frac{1}{(\log|V_n|)^2}.$$

**Proof.** Let $(X, d)$ be a metric space containing all $G_n$ isometrically, and let $f_n : V_n \to X$ be an isometric embedding for each $n$. Since any $r$-ball in $G_n$ contains at most $\sum_{i=0}^{r}(d-1)^r \asymp_d (d-1)^r$ vertices, at least $|V_n|/2$ vertices $y \in V_n$ satisfy $d(x, y) \gtrsim_d \log|V_n|$ for any $x \in V_n$. Thus, at least half of the pairs $(x, y) \in V_n \times V_n$ satisfy

$$d(f_n(x), f_n(y)) \gtrsim_d \log|V_n|.$$

Hence, we have

$$\frac{\sum_{x,y \in V_n} m(x, y)d(f_n(x), f_n(y))^2}{m(\emptyset) \sum_{x,y \in V_n} m(x)m(y)d(f_n(x), f_n(y))^2} \leq \frac{d^2}{2} \frac{|V_n|^2}{\sum_{x,y \in V_n} d(f_n(x), f_n(y))^2} \lesssim_d \frac{1}{(\log|V_n|)^2},$$

which proves the proposition. \qed
For a complete CAT(0) space \(X\), Izeki and Nayatani \([10]\) introduced an invariant \(0 \leq \delta(X) \leq 1\) to estimate \(\lambda_1^{\text{Gro}}(G, X)\) from below, and showed that
\[
\frac{1}{2}(1 - \delta(X)) \mu_1(G) \leq \lambda_1^{\text{Gro}}(G, X) \leq \mu_1(G).
\]
The following estimates of \(\delta\) are known:

- If \(X\) is a Hilbert space, an Hadamard manifold, or a tree then \(\delta(X) = 0\) (see \([10]\)).
- If \(X\) is a complete CAT(0) cube complex then \(\delta(X) \leq \frac{1}{2}\) (see \([7]\)).
- For a prime \(p\), the Bruhat-Tits building \(I_p\) associated to \(PGL(3, \mathbb{Q}_p)\) satisfies \(\delta(I_p) \leq \frac{3}{4}\) (see \([9]\)). For \(p = 2\), the better estimate \(\delta(I_2) \leq \frac{37 - 18\sqrt{2}}{28}\) is known (see \([10]\)).

The inequality \(\delta(X) < 1\) guarantees that any sequence \(\{G_n\}_{n=1}^\infty\) of expanders also satisfies the property (3) with respect to \(X\). The second author \([19], [20]\) studied sufficient conditions of the supremum of \(\delta\) for a given family of CAT(0) spaces to be less than 1. For example, he proved that a geodesically complete CAT(0) space \(X\) having a cocompact isometric group action has \(\delta(X) < 1\). By using this, we see that if \(X\) is any Bruhat-Tits building associated to a semi-simple algebraic group then \(\delta(X) < 1\).

On the other hand, the first author \([12]\) proved the existence of a complete CAT(0) space \(X\) and a sequence \(\{G_n\}_{n=1}^\infty\) of expanders such that
\[
\lim_{n \to \infty} \lambda_1^{\text{Gro}}(G_n, X) = 0.
\]
In particular, such a CAT(0) space \(X\) satisfies \(\delta(X) = 1\). This result means that a drastic change in the nonlinear spectral gap may happen even within the class of CAT(0) spaces.

Let \(T_d\) be the \(d\)-regular tree, and let \(T_{d,r}\) be an \(r\)-ball in \(T_d\). We prove in this paper that in contrast to the case of a sequence of expanders described above, the asymptotic behavior of nonlinear spectral gaps of \(T_{d,r}\) as \(r \to \infty\) does not depend upon the target metric space. More precisely, we show in Corollary 3.3 that for any metric space \(X\) we have
\[
\lambda_1^{\text{Gro}}(T_{d,n}, X) \asymp_d \frac{1}{(d - 1)^n}.
\]
The asymptotic equation (1.2) is obtained by generalizing the well-known path method to the nonlinear setting (Theorem 1.3).

Let \(G = (V, E)\) be a graph. A path on \(G\) is a finite sequence
\[
(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)
\]
in \(\vec{E}\) such that \(y_i = x_{i+1}\) for each \(i = 1, \ldots, n - 1\). For \(x, y \in V\), we denote by \(\Gamma(x, y)\) the set of all paths from \(x\) to \(y\) without repeated edges. For a weight function \(w : V \times V \to [0, \infty)\) on \(G\), and for a path \(\gamma : (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)
on $G$ without repeated edges, set
$$|\gamma|_w = \sum_{i=1}^{n} \frac{1}{w(x_i, y_i)}.$$  

Our main technique to estimate nonlinear spectral gaps is the following.

**Theorem 1.3.** Let $(G, m)$ be a finite connected weighted graph. Let $w$ be another weight function on $G$. We assign one path $\gamma(x, y) \in \Gamma(x, y)$ to each $(x, y) \in V \times V$. We define
$$A(w) = \max_{e \in \vec{E}} \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(e)} \sum_{(x, y) \text{ s.t. } \gamma(x, y) \ni e} |\gamma(x, y)|_w m(x)m(y) \right\}.$$  

Then we have
$$\lambda_1^{\text{Gro}}(G, X) \geq \frac{1}{A(w)}$$  

for any metric space $X$.

We emphasize that the lower bound obtained in Theorem 1.3 is independent of the target metric space. For the case $X = \mathbb{R}$, Theorem 1.3 was proved in various forms by Jerrum and Sinclair [11], Diaconis and Stroock [3], and Quastel [17], and Diaconis and Saloff-Coste [4]. See also Saloff-Coste [18].

The paper is organized as follows. In Section 2, we develop techniques to estimate nonlinear spectral gaps uniformly from below. First, we present a simple method to estimate nonlinear spectral gaps by using estimates of Euclidean distortions. Then we prove Theorem 1.3. In Section 3 we prove (1.2) by using Theorem 1.3.

2. **Uniform lower bounds of nonlinear spectral gaps**

In this section, we prove Theorem 1.3. Before proving this, we present one simple argument to obtain a uniform lower bound on non-linear spectral gaps by using embeddings into a Hilbert space. This type of estimation is also found in Naor and Silberman [15].

**Definition 2.1.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. For an injective mapping $f : X \to Y$, the **distortion** of $f$ is defined to be the product
$$\sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} \cdot \sup_{x,y \in X, x \neq y} \frac{d_X(x, y)}{d_Y(f(x), f(y))}.$$  

If a mapping $f : X \to Y$ is not injective, the distortion of $f$ is defined to be $\infty$. The **distortion** $c_Y(X)$ is the infimum of the distortions of all mappings from $X$ to $Y$. We denote by $c_p(X)$ the distortion of $X$ into $L_p$.

**Proposition 2.2.** Let $(G, m)$ be a finite connected weighted graph with $n \geq 2$ vertices, and let $(X, d)$ be a metric space. We define
$$c_2(X, n) = \max\{c_2(X') \mid X' \subset X, |X'| \leq n\}.$$
Then we have

\[ \lambda_1^{\text{Gro}}(G, X) \geq \frac{1}{c_2(X, n)^2} \mu_1(G). \]

**Proof.** Let \( f : V \to X \) be an arbitrary mapping. Then, by the definition of \( c_2(X, n) \), for any \( \varepsilon > 0 \), there exists a mapping \( \varphi : f(V) \to \ell_2 \) such that

\[ d(f(x), f(y)) \leq \| \varphi \circ f(x) - \varphi \circ f(y) \| \leq (c_2(X, n) + \varepsilon)d(f(x), f(y)) \]

holds for any \( x, y \in V \). Hence, we have

\[
\frac{1}{m(\emptyset)} \sum_{x, y \in V} m(x)m(y)d(f(x), f(y))^2 \\
\leq \frac{1}{m(\emptyset)} \sum_{x, y \in V} m(x)m(y)\| \varphi \circ f(x) - \varphi \circ f(y) \|^2 \\
\leq \frac{1}{\mu_1(G)} \sum_{x, y \in V} m(x, y)\| \varphi \circ f(x) - \varphi \circ f(y) \|^2 \\
\leq \frac{(c_2(X, n) + \varepsilon)^2}{\mu_1(G)} \sum_{x, y \in V} m(x, y)d(f(x), f(y))^2.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves the proposition. \( \square \)

Combining this proposition with the following well-known Bourgain’s embedding theorem, we obtain a uniform estimate on nonlinear spectral gaps.

**Theorem 2.3** (Bourgain [1]). For every \( n \)-point metric space \((X, d)\), we have

\[ c_2(X) \lesssim \frac{1}{\log n}. \]

**Proposition 2.4.** Let \((G, m)\) be a connected weighted graph with \( n \) vertices, and let \((X, d)\) be a metric space. Then we have

\[ \frac{\lambda_1^{\text{Gro}}(G, X)}{\mu_1(G)} \gtrsim \frac{1}{(\log n)^2}. \]

Though we do not see the graph structure in the proof of Proposition 2.4, this is asymptotically sharp as is shown by a sequence \( \{G_n\}_{n=1}^\infty \) of expanders, for which we have \( \mu_1(G_n) \asymp 1 \) and \( \lambda_1^{\text{Gro}}(G_n, X) \lesssim_d (\log |G_n|)^{-2} \) for some metric space \( X \) by Proposition 1.2. However, as we will see later, for a specific sequence of graphs, we can obtain a more accurate estimate by generalizing the path method.
Proof of Theorem 1.3. Let $f : V \to X$ be a map. Then the triangle inequality and the Cauchy-Schwarz inequality yield the following.

$$d(f(x), f(y))^2 \leq \left\{ \sum_{(u,v) \in \gamma(x,y)} d(f(u), f(v)) \right\}^2$$

$$\leq \left( \sum_{(u,v) \in \gamma(x,y)} w(u,v)^{-1} \right) \left( \sum_{(u,v) \in \gamma(x,y)} d(f(u), f(v))^2 w(u,v) \right)$$

$$= |\gamma(x,y)|_w \sum_{(u,v) \in \gamma(x,y)} d(f(u), f(v))^2 w(u,v).$$

Multiplying $\frac{m(x)m(y)}{m(\emptyset)}$ both sides of the above inequality and summing over all $(x, y) \in V \times V$, we obtain the following.

$$\frac{1}{m(\emptyset)} \sum_{(x,y) \in V \times V} m(x)m(y)d(f(x), f(y))^2$$

$$\leq \sum_{(x,y) \in V \times V} \sum_{(u,v) \in \gamma(x,y)} \frac{1}{m(\emptyset)} |\gamma(x,y)|_w m(x)m(y)w(u,v)d(f(u), f(v))^2$$

$$= \sum_{(u,v) \in \bar{E}} \sum_{(x,y) \in \gamma(x,y) \cap (u,v)} \frac{1}{m(\emptyset)} |\gamma(x,y)|_w m(x)m(y)w(u,v)d(f(u), f(v))^2$$

$$= \sum_{(u,v) \in \bar{E}} \left[ m(u,v)d(f(u), f(v))^2 \right.$$

$$\left. \times \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(u,v)} w(u,v) \sum_{(x,y) \in \gamma(x,y) \cap (u,v)} |\gamma(x,y)|_w m(x)m(y) \right\} \right]$$

Thus,

$$\frac{1}{m(\emptyset)} \sum_{(x,y) \in V \times V} m(x)m(y)d(f(x), f(y))^2$$

$$\leq \max_{(u,v) \in \bar{E}} \left\{ \frac{1}{m(\emptyset)} \frac{1}{m(u,v)} w(u,v) \sum_{(x,y) \in \gamma(x,y) \cap (u,v)} |\gamma(x,y)|_w m(x)m(y) \right\}$$

$$\times \sum_{(u,v) \in \bar{E}} m(u,v)d(f(u), f(v))^2$$

$$= A(w) \sum_{(u,v) \in \bar{E}} m(u,v)d(f(u), f(v))^2,$$

which proves the theorem.
Example 2.5. Let $H_n$ be the $n$-dimensional Hamming cube equipped with the uniform weight. In [18], it was shown that

\[(2.1) \quad \mu_1(H_n) \gtrsim \frac{1}{n^2}\]

by using the path method. Since it is known that

\[(2.2) \quad \mu_1(H_n) = \frac{2}{n},\]

the above estimation (2.1) is not asymptotically sharp. Theorem 1.3 says that the estimation in [18] also works for nonlinear spectral gaps with respect to any target metric spaces. Thus we actually have

\[(2.3) \quad \lambda_1^{\text{Gro}}(H_n, X) \gtrsim \frac{1}{n^2}\]

for an arbitrary metric space $X$. This is the right order of magnitude for general metric spaces because if we take the identity mappings $\iota_n : H_n \to H_n$, we see that

\[\lambda_1^{\text{Gro}}(H_n, H_n) \leq \sum_{x,y \in V} m(x,y) d(\iota_n(x), \iota_n(y))^2 = 4 \frac{n(n+1)}{n(n+1)} = 4.\]

In fact, we will show that equality holds here in a forthcoming paper [13].

3. Nonlinear spectral gaps of trees

For an integer $d \geq 2$, let $T_d$ be the $d$-regular tree. For an integer $r \geq 0$, let $T_{d,r}$ be an $r$-ball in $T_d$. We consider the uniform weight function $m$ on $T_{d,r}$. In this section, we estimate $\lambda_1^{\text{Gro}}(T_{d,r}, X)$ for any metric space $X$ by applying Theorem 1.3.

Bourgain [2] estimated the Euclidean distortion $c_2(T)$ of an $n$-point tree $T$ as

\[c_2(T) \asymp \sqrt{\log \log n}.\]

Combining this with Proposition 2.2, it follows that for an $n$-point tree $T$ and an arbitrary metric space $X$, we have

\[\frac{\lambda_1^{\text{Gro}}(T, X)}{\mu_1(T)} \gtrsim \frac{1}{\log \log n}.\]

Thus, we have

\[(3.1) \quad \frac{\lambda_1^{\text{Gro}}(T_{d,r}, X)}{\mu_1(T_{d,r})} \gtrsim d \frac{1}{\log r}.\]

Unfortunately, the right hand side of (3.1) tends to 0 as $r$ goes to infinity. In what follows, we improve this estimate by using Theorem 1.3.

Proposition 3.1. Let $(X, d)$ be any metric space. Then we have

\[\lambda_1^{\text{Gro}}(T_{d,r}, X) \geq \frac{d - 2}{d^2(d - 1)} \frac{\{(d - 1)^r - 1\}}{(d - 1)^r} \times (d - 1)^{-r}\]

for any $d \geq 3$ and $r \geq 1$. 
Proof. Since $T_{d,r}$ is a tree, each $\Gamma(x, y)$ contains only one path $\gamma(x, y)$. We define a weight function $w : V \times V \to (0, \infty)$ on $T_{d,r}$ by setting

$$w(x, y) = (d - 1)^{i+1}, \quad (x, y) \in \vec{E},$$

where $i$ is the graph distance from the center vertex $o$ to $\{x, y\}$.

Let $e = (u, v) \in \vec{E}$ be an arbitrary ordered edge. Then $e$ separates $V$ into two connected components, $U$ containing $u$ and $W$ containing $v$. According to Theorem 1.3, we need to estimate

$$A(w, e) = \frac{1}{m(\emptyset)} \frac{1}{m(e)} \sum_{(x, y) \text{ s.t. } \gamma(x, y) \ni e} |\gamma(x, y)|_w m(x) m(y)$$

from above. We can assume that $o \in U$. Let $k$ be the graph distance between $o$ and $v$. Then we have

$$|W| = \sum_{l=0}^{r-k}(d - 1)^l = \frac{(d - 1)^{r-k+1} - 1}{d - 2} \leq (d - 1)^{r-k+1},$$

$$|U| \leq |V| = 1 + \sum_{l=1}^{r} d(d - 1)^{l-1} \leq \frac{d}{d - 2} (d - 1)^r,$$

$$m(\emptyset) = |\vec{E}| = 2 \sum_{l=1}^{r} d(d - 1)^{l-1} = \frac{2d}{d - 2} ((d - 1)^r - 1),$$

$$m(v) \leq d,$$

$$|\gamma(x, y)|_w \leq 2 \sum_{i=1}^{r} \frac{1}{(d - 1)^i} \leq \frac{2}{d - 2}$$

for every $v, x, y \in V$. Thus,

$$A(w, e) \leq \frac{d - 2}{2d \{(d - 1)^r - 1\}} \cdot \frac{1}{1} \cdot (d - 1)^k \cdot \sum_{(x, y) \text{ s.t. } \gamma(x, y) \ni e} \frac{2}{d - 2} d^2$$

$$\leq \frac{d - 2}{d \{(d - 1)^r - 1\}} \cdot (d - 1)^k \cdot (d - 1)^{r-k+1} \frac{d}{d - 2} (d - 1)^r \frac{1}{d - 2} d^2$$

$$= \frac{d^2(d - 1)^r}{d - 2} \{(d - 1)^r - 1\} \cdot (d - 1)^r,$$

which proves the proposition. \qed

Proposition 3.2. Let $(X, d)$ be any metric space. Then we have

$$\lambda_{1}^{\text{Gro}}(T_{d,r}, X) \leq 2 \left( \frac{d \{(d - 1)^{r-1} - 1\}}{d - 2} + (d - 1)^{r-1} \right. \left. + \frac{\{(d - 1)^{r-1} - 1\} (d - 1)^r}{(d - 1)^r - 1} + \frac{(d - 2) \{(d - 1)^{2r-1}\}}{d \{(d - 1)^r - 1\}} \right)^{-1}$$

for any $d \geq 3$ and $r \geq 1$. 
Proof. We take an edge \( e = \{o, v\} \in E \) containing the center. The edge \( e \) divides \( V \) into two components \( U \) containing \( o \) and \( W = U^c \). Let \( f : V \to X \) be a mapping sending \( U \) to \( p \) and \( W \) to \( q \), where \( p, q \in X \) are distinct points.

The weight of each vertex is either \( d \) or 1, and we have

\[
\begin{align*}
|\{u \in U \mid m(u) = d\}| &= \sum_{i=0}^{r-1} (d-1)^i = \frac{(d-1)^r - 1}{d-2}, \\
|\{u \in U \mid m(u) = 1\}| &= (d-1)^r, \\
|\{w \in W \mid m(w) = d\}| &= \sum_{i=0}^{r-2} (d-1)^i = \frac{(d-1)^{r-1} - 1}{d-2} + \frac{(d-1)^r - 1}{d-2}, \\
|\{w \in W \mid m(u) = 1\}| &= (d-1)^{r-1}.
\end{align*}
\]

Thus, we have

\[
\frac{1}{\mathcal{m}(\emptyset)} \sum_{x, y \in V} m(x)m(y)d(f(x), f(y))^2
= \frac{d - 2}{2d \{(d-1)^r - 1\}} \times 2d(p, q)^2 \left\{ d \frac{(d-1)^{r-1} - 1}{d-2} \frac{(d-1)^r - 1}{d-2} + d \frac{(d-1)^{r-1} - 1}{d-2} \frac{(d-1)^r - 1}{d-2} + (d-1)^{2r-1} \right\}
= d(p, q)^2 \left\{ \frac{d \{(d-1)^{r-1} - 1\}}{d-2} + (d-1)^{r-1} + \frac{(d-2) \{(d-1)^{2r-1}\}}{d \{(d-1)^r - 1\}} \right\}.
\]

On the other hand,

\[
\sum_{x, y \in V} m(x, y)d(f(x), f(y))^2 = 2d(p, q)^2.
\]

Hence,

\[
\frac{\sum_{x, y \in V} m(x, y)d(f(x), f(y))^2}{\mathcal{m}(\emptyset) \sum_{x, y \in V} m(x)m(y)d(f(x), f(y))^2}
= 2 \left\{ \frac{d \{(d-1)^{r-1} - 1\}}{d-2} + (d-1)^{r-1} + \frac{(d-2) \{(d-1)^{2r-1}\}}{d \{(d-1)^r - 1\}} \right\}^{-1},
\]

which proves the proposition. \(\square\)

The following corollary is straightforward from Proposition 3.1 and Proposition 3.2.
Corollary 3.3. Let $X$ be any metric space, and let $d \geq 3$. Then, we have 

$$\lambda_{1}^{\text{Gro}}(T_{d,r}, X) \simeq \frac{1}{(d-1)^{r}}.$$ 

Remark 3.4. When $d = 2$, $T_{2,r}$ is a path graph with $2r + 1$ vertices. We claim that in this case nonlinear spectral gaps are never less than the linear spectral gap. Let $P_n = (V, E)$ be a path graph with $n + 1$ vertices. More precisely, suppose that $V = \{v_0, \ldots, v_n\}$ and $(v_i, v_j) \in E$ if and only if $|i - j| = 1$. Let $m$ be an arbitrary weight function on $P_n$, and let $(X, d)$ be a metric space. For any $f : V \to X$, we define a map $\varphi_f : V \to \mathbb{R}$ by setting

$$\varphi_f(v_i) = \begin{cases} 0, & \text{if } i = 0, \\ \sum_{1 \leq l \leq i} d(f(v_l-1), f(v_l)), & \text{if } 2 \leq i \leq n. \end{cases}$$

Then we have

$$\frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)d(f(x), f(y))^2$$

$\leq \frac{1}{m(\emptyset)} \sum_{x,y \in V} m(x)m(y)|\varphi_f(x) - \varphi_f(y)|^2$

$\leq \frac{1}{\mu_1(P_n)} \sum_{x,y \in V} m(x,y)|\varphi_f(x) - \varphi_f(y)|^2$

$= \frac{1}{\mu_1(P_n)} \sum_{x,y \in V} m(x,y)d(f(x), f(y))^2,$

where the first inequality follows from the triangle inequality. Hence, we have

$$\lambda_{1}^{\text{Gro}}(P_n, X) \geq \mu_1(P_n).$$

We claim that if the weight function $m$ is uniform, it is known that

$$\mu_1(P_n) = 1 - \cos \frac{\pi}{n}.$$ 

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