Orthogonal polynomials and the deformed Jordan plane

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Abstract

We consider the unital associative algebra $A$ with two generators $X$, $Z$ obeying the defining relation $[Z, X] = Z^2 + \Delta$. We construct irreducible tridiagonal representations of $A$. Depending on the value of the parameter $\Delta$, these representations are associated to the Jacobi matrices of the para-Krawtchouk, continuous Hahn, Hahn or Jacobi polynomials.

Keywords: Para-Krawtchouk polynomials, deformed Jordan plane, tridiagonal representations.

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1 Introduction

This paper is devoted to the study of irreducible tridiagonal representations of the two-generated algebra $A$ which is a deformation of the Jordan plane. It is shown how the para-Krawtchouk polynomials appear quite naturally in this context, along with the other families of classical orthogonal polynomials (OPs) of the Jacobi, continuous Hahn and Hahn type.

The algebra $A$ over $\mathbb{R}$, with generators $X$, $Z$ and satisfying

$$[Z, X] = Z^2 + \Delta$$

with $\Delta$ a parameter, is a special case of the most general two-generated quadratic algebra $Q$ with defining relation

$$\alpha_1 X^2 + \alpha_2 X Z + \alpha_3 Z X + \alpha_4 Z^2 + \alpha_5 X + \alpha_6 Z + \alpha_7 = 0.$$  

This algebra has been of interest to various communities. Ring theorists have provided classifications \cite{1} of the special cases it entails and studied their properties. The algebra $Q$ has also been related to non-commutative probability theory \cite{2} and is related to martingale polynomials associated to quadratic harnesses \cite{3}. On the physics side, $Q$ describes various 1D asymmetric exclusion models \cite{4}.

Recently, the last two authors have begun connecting $Q$ and its various isomorphism classes to families of special functions. In \cite{5}, by studying tridiagonal representations of the $q$-oscillator algebra $XZ - qZX = 1$, \cite{6}. 


they have identified how they encompass the recurrence relations of the big \( q \)-Jacobi, the \( q \)-Hahn and the \( q \)-para-Krawtchouk polynomials. The case of the \( q \)-Weyl algebra \( XZ - qZX = 0 \) has also been studied in [9]. The present paper will add to this program by considering an interesting special case of (1.2) and identifying the orthogonal polynomials that can be interpreted from this algebra.

Since their introduction in [10], para-polynomials have been the object of growing interest. Four families have been defined and studied offering para-versions of the polynomials of Krawtchouk, \( q \)-Krawtchouk, Racah and \( q \)-Racah type. While they do not fall in the category of classical orthogonal polynomials \footnote{They obey a three term recurrence relation but a higher order difference equation.} they are understood as non-standard truncations of infinite-dimensional families of classical OPs [11][13]. In addition to their natural occurrence in the study of perfect state transfer and fractional revival in quantum spin chains [10][14][15], recent advances have identified these para-polynomials as the basis for finite-dimensional representations of degenerations of the Sklyanin algebra [16][18]. They have also appeared in the study of the Dunkl oscillator in the plane [19]. The main goal of this paper is to show that these para-Krawtchouk polynomials as well as the Jacobi, continuous Hahn and Hahn polynomials arise in representations of the two-generated algebra \( A \).

When \( \Delta \neq 0, A \) as defined in (1.1) is a deformation of the Jordan plane \( (X,Z) \) viewed as noncommutative coordinates). Three cases will be distinguished depending on whether \( \Delta = 0, \Delta > 0 \) or \( \Delta < 0 \). These three cases will be studied separately and provide a complete picture of the connection between the algebra (1.1) and orthogonal polynomials.

The presentation is organized as follows. Section 2 will introduce the tridiagonal representations of the algebra \( A \) and the non-degeneracy condition. Standardized versions of \( A \) corresponding to \( \Delta = 0, \Delta < 0, \Delta > 0 \) will then be examined in the following sections. The case \( \Delta = 0 \) will be studied in section 3 and the Jacobi OPs will appear, while the case \( \Delta > 0 \) and the continuous Hahn polynomials will be the object of section 4. Section 5 will focus on the case \( \Delta < 0 \) and will feature both the Hahn and the para-Krawtchouk polynomials. Some concluding remarks and perspectives will close the paper.

## 2 Tridiagonal representations of the algebra \( A \)

Consider a tridiagonal representation of \( A \) where \( X \mapsto X \) and \( Z \mapsto Z \). The actions of \( X, Z \) on a semi-infinite orthonormal basis \( |n\rangle, n = 0, 1, 2, \ldots \) are taken to be of the form

\[
X |n\rangle = c_n |n - 1\rangle + b_n |n\rangle + a_n |n + 1\rangle, \quad (2.1a)
\]

\[
Z |n\rangle = u_n |n - 1\rangle + v_n |n\rangle + w_n |n + 1\rangle, \quad (2.1b)
\]

with \( c_0 = u_0 = 0 \). To ensure that such a representation is irreducible we shall assume that the off-diagonal coefficients are non-zero for \( n > 0 \). Acting with (1.1) on the basis \( |n\rangle \) and using the above definitions, one obtains

\[
(ZX - XZ - Z^2 - \Delta) |n\rangle = (c_n u_{n-1} - c_{n-1} u_n - u_{n-1} u_n) |n - 2\rangle
+ (b_n u_{n} - b_{n-1} u_{n-1} + c_n v_{n-1} - c_{n-1} v_n - u_{n} v_{n-1}) |n - 1\rangle
+ (-\Delta - a_{n-1} u_{n} + a_n u_{n+1} - v_{n}^2 + c_n w_{n-1} - u_{n} w_{n-1} - c_{n-1} w_n - u_{n-1} w_n) |n\rangle
+ (a_n v_{n+1} - a_{n+1} v_n + b_n w_{n} - b_{n+1} w_{n-1} - v_{n} w_{n-1} - v_{n+1} w_n) |n + 1\rangle
+ (a_n w_{n+1} - a_{n+1} w_n - w_{n} w_{n+1}) |n + 2\rangle. \quad (2.2)
\]

For the actions \( (2.1) \) to define a representation of \( A \), each side of the above equation must vanish. As the basis vectors are orthonormal, one obtains the following conditions on the coefficients of (2.1) that define the representations:

\[
0 = c_n u_{n-1} - c_{n-1} u_n - u_{n-1} u_n, \quad (2.3)
\]

\[
0 = b_n u_{n} - b_{n-1} u_{n-1} + c_n v_{n-1} - c_{n-1} v_n - u_{n} v_{n-1}, \quad (2.4)
\]

\[
0 = -\Delta - a_{n-1} u_{n} + a_n u_{n+1} - v_{n}^2 + c_n w_{n-1} - u_{n} w_{n-1} - c_{n-1} w_n - u_{n-1} w_n, \quad (2.5)
\]

\[
0 = a_n v_{n+1} - a_{n+1} v_n + b_n w_{n} - b_{n+1} w_{n-1} - v_{n} w_{n-1} - v_{n+1} w_n, \quad (2.6)
\]

\[
0 = a_n w_{n+1} - a_{n+1} w_n - w_{n} w_{n+1}. \quad (2.7)
\]
2.1 General solutions to the recurrence relations

We now determine the general solutions to the above system of recurrence equations. Dividing (2.3) by \( u_n u_{n-1} \), one obtains

\[
\frac{c_n}{u_n} - \frac{c_{n-1}}{u_{n-1}} = 1.
\]

This implies

\[
\phi_n = \phi_0 + n, \quad \phi_n \equiv \frac{c_n}{u_n}.
\]  (2.8)

Equation (2.7) can be solved similarly. Dividing by \( w_n w_{n+1} \), one has

\[
\delta_n = \delta_0 - n, \quad \delta_n \equiv \frac{a_n}{w_n}.
\]  (2.9)

Rewriting (2.4) and (2.6) in terms of \( \phi_n \) and \( \delta_n \) and dividing by \( u_n \) or \( w_n \), respectively, one obtains

\[
b_{n-1} - b_n = (\phi_n - 1)v_{n-1} - (\phi_n + 1)v_n, \quad (2.10)
\]

\[
b_{n+1} - b_n = (\delta_n - 1)v_{n+1} - (\delta_n + 1)v_n. \quad (2.11)
\]

To solve for \( v_n \), shift the index of (2.10) and add (2.11) to find

\[
0 = (\delta_n - \phi_n + 1)v_{n+1} - (\delta_n - \phi_n + 2)v_n. \quad (2.12)
\]

Substituting the solutions (2.8) and (2.9) in (2.12) leads to

\[
0 = (\delta_0 - \phi_0 - 2(n + 1))v_{n+1} - (\delta_0 - \phi_0 - 2n + 1)v_n
\]

\[
= \mu_{n+2}v_{n+1} - \mu_nv_n, \quad (2.13)
\]

\[
= \mu_{n+2}v_{n+1} - \mu_nv_n, \quad (2.14)
\]

with \( \mu_n \equiv (\delta_0 - \phi_0 - 2n + 1) \). Multiplying the above by \( \mu_{n+1} \) as an integrating factor, one can solve the recurrence to obtain

\[
v_n = \frac{(\delta_0 - \phi_0 - 1)(\delta_0 - \phi_0 + 1)v_0}{(\delta_0 - \phi_0 - 2n + 1)(\delta_0 - \phi_0 - 2n - 1)}. \quad (2.15)
\]

To find \( b_n \), subtract instead (2.10) with shifted index from (2.11) and get

\[
b_{n+1} - b_n = \frac{1}{2}(\delta_n + \phi_n + 1)(v_{n+1} - v_n), \quad (2.16)
\]

which, upon using (2.8) and (2.9), can be solved immediately and yields

\[
b_n = \frac{1}{2}(\delta_0 + \phi_0 + 1)(v_n - v_0) + b_0. \quad (2.17)
\]

Finally, (2.5) is written as follows in terms of \( \phi_n \) and \( \delta_n \) using (2.8) and (2.9), as

\[
\Delta + v_n^2 = (\delta_0 - \phi_0 - 2(n + 1))\kappa_{n+1} - (\delta_0 - \phi_0 - 2(n - 1))\kappa_n, \quad (2.18)
\]

with

\[
\kappa_n \equiv u_n w_{n-1}. \quad (2.19)
\]

Multiplying both sides by \( (\delta_0 - \phi_0 - 2n) \) as an integrating factor, one can reduce the above to

\[
(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)\kappa_n = (\delta_0 - \phi_0)(\delta_0 - \phi_0 + 2)\kappa_0 + \sum_{k=0}^{n-1}(\Delta + v_k^2)(\delta_0 - \phi_0 - 2k). \quad (2.20)
\]
The sum over \( k \) in (2.20) can be reexpressed\(^2\) as
\[
\sum_{k=0}^{n-1} (\Delta + v_k^2)(\delta_0 - \phi_0 - 2k) = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2}.
\]
(2.21)

From (2.20) and (2.21), recalling that \( u_0 \) was required to vanish so that \( \kappa_0 = u_0w_1 = 0 \), one has
\[
\kappa_n = \frac{n(\delta_0 - \phi_0 - n + 1)(\Delta(\delta_0 - \phi_0 - 2n + 1)^2 + v_0^2(\delta_0 - \phi_0 - 1)^2)}{(\delta_0 - \phi_0 - 2n + 1)^2(\delta_0 - \phi_0 - 2n)(\delta_0 - \phi_0 - 2n + 2)}.
\]
(2.22)

### 2.2 The linear pencil \( \mathcal{X} + \mu \mathcal{Z} \)

The algebra \( \mathcal{A} \) is invariant under the affine transformation
\[
\mathcal{X} \mapsto \mathcal{X} + \mu \mathcal{Z}, \quad \mu \in \mathbb{R}.
\]
As a result, one expects the transformed solutions for the coefficients in (2.1) to be given by (2.8), (2.9) and (2.17) with modified parameters. Indeed one finds the parameters to be replaced by
\[
\phi_0 \mapsto \phi_0 - \mu, \quad \delta_0 \mapsto \delta_0 - \mu, \quad b_0 \mapsto b_0 - \mu v_0.
\]
Thus, the diagonalization of the linear pencil \( \mathcal{X} + \mu \mathcal{Z} \) amounts to the diagonalization of \( \mathcal{X} \) up to a shift in the parameters.

### 2.3 Representations on polynomials

Denoting by \( \langle x \rangle \) the dual eigenvectors:
\[
\langle x \rangle X = x \langle x \rangle,
\]
one can look for the polynomials \( q_n(x) \equiv \langle x | n \rangle \) that diagonalize \( X \)
\[
Xq_n(x) = xq_n(x) = c_nq_{n-1}(x) + b_nq_n(x) + a_nq_{n+1}(x).
\]
(2.23)

By appropriate renormalization, one obtains a monic recurrence relation
\[
Xp_n(x) \equiv xp_n(x) = a_{n-1}c_np_{n-1}(x) + b_np_n(x) + p_{n+1}(x), \quad p_n(x) = \left( \prod_{i=0}^{n-1} a_i \right) q_n(x).
\]
(2.24)

The families of polynomials \( p_n(x) \) that diagonalize \( X \) can be determined by identifying the coefficients \( a_{n-1}c_n \) and \( b_n \).

From (2.8), (2.9), (2.19) and (2.22), one has that
\[
a_{n-1}c_n = (n + \phi_0)(n - \delta_0 - 1)\frac{n(n + \phi_0 - \delta_0 - 1)(\Delta(2n + \phi_0 - \delta_0 - 1)^2 + v_0^2(\phi_0 - \delta_0 + 1)^2)}{(2n + \phi_0 - \delta_0 - 1)^2(2n + \phi_0 - \delta_0)(2n + \phi_0 - \delta_0 - 2)}
\]
(2.25)

and from (2.17) and (2.15), that
\[
b_n = \frac{1}{2} \frac{(\delta_0 + \phi_0 + 1)(\phi_0 - \delta_0 + 1)(\phi_0 - \delta_0 - 1)v_0}{(2n + \phi_0 - \delta_0 - 1)(2n + \phi_0 - \delta_0 + 1)} + \tilde{b}_0, \quad \tilde{b}_0 \equiv b_0 - \frac{1}{2} (\delta_0 + \phi_0 + 1)v_0.
\]
(2.26)

Finite-dimensional representations of dimension \( N + 1 \) are obtained if \( w_N = 0 \) since it follows that \( a_N = 0 \) from (2.7). This implies that \( \kappa_{N+1} = 0 \). From (2.22), we see that this is achieved for any value of \( \Delta \) by
\[
N = (\delta_0 - \phi_0).
\]
(2.27)

If \( \Delta \neq 0 \), one finds an additional pair of solutions given by
\[
N + 1 = \frac{1}{2} \left[ \phi_0 - \delta_0 - 1 \pm (\phi_0 - \delta_0 + 1)v_0 \sqrt{-\Delta} \right].
\]
(2.28)

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\(^2\)This is done by noticing the sum to be telescopic or via the polygamma function of the first order.
3  The case $\Delta = 0$: Jacobi polynomials

With $\Delta$ vanishing, the coefficient $a_{n-1}c_n$ simplifies to

$$a_{n-1}c_n = \frac{n(n + \phi_0)(n - \delta_0 - 1)(n + \phi_0 - \delta_0 - 1)(\phi_0 - \delta_0 + 1)^2 v_0^2}{(2n + \phi_0 - \delta_0 - 1)^2(2n + \phi_0 - \delta_0)(2n + \phi_0 - \delta_0 - 2)}. \quad (3.1)$$

Setting $v_0 = 2(\phi_0 - \delta_0 + 1)^{-1}$, one identifies the basis vector to be proportional to the Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ with parameters

$$\alpha = -\delta_0 - 1, \quad \beta = \phi_0. \quad (3.2)$$

With $b_0 = 0$, the coefficient $b_n$ of (2.26) is given by

$$b_n = \frac{(\beta^2 + \alpha^2)}{(2n + \beta + \alpha)(2n + \beta + \alpha + 2)}. \quad (3.3)$$

Comparing the expressions (3.1) and (3.3) for the coefficients using for instance [20], we conclude:

**Proposition 1.** In the case $\Delta = 0$, the eigenfunctions $p_n(x)$ of $X$ are the monic Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ with parameters $\alpha, \beta$ given in (3.2).

The only truncation condition possible is (2.27). However, it yields singular expressions in (3.1) and (3.3) for $n \leq N$.

4  The case $\Delta > 0$: Continuous Hahn polynomials

If $\Delta \neq 0$, upon scaling the generators of the algebra according to

$$\hat{\mathcal{X}} = \Omega \mathcal{X}, \quad \hat{\mathcal{Z}} = \Omega \mathcal{Z},$$

we obtain

$$[\hat{\mathcal{Z}}, \hat{\mathcal{X}}] = \hat{\mathcal{Z}}^2 + \Omega^2 \Delta. \quad (4.1)$$

In view of (4.1), one can choose $\Omega$ so that $\Delta = \pm \frac{1}{4}$. In this section, we shall consider the case $\Delta = +\frac{1}{4}$. The coefficient $a_{n-1}c_n$ is then given by

$$a_{n-1}c_n = (n + \phi_0)(n - \delta_0 - 1)\frac{n(n + \phi_0 - \delta_0 - 1)((2n + \phi_0 - \delta_0 - 1)^2/4 + v_0^2(\phi_0 - \delta_0 + 1)^2)}{(2n + \phi_0 - \delta_0 - 1)^2(2n + \phi_0 - \delta_0)(2n + \phi_0 - \delta_0 - 2)}. \quad (4.2)$$

Writing

$$\phi_0 + 1 = a + c, \quad -\delta_0 = b + d, \quad v_0 = -\frac{(a - b - c + d)}{2(a + b + c + d)} \quad (4.3)$$

one can factorize the term with $v_0$:

$$\frac{1}{4}(2n + \phi_0 - \delta_0 - 1)^2 + v_0^2(\phi_0 - \delta_0 + 1)^2 = (n + a + d - 1)(n + b + c - 1).$$

With (4.3) and the above, (4.2) becomes

$$a_{n-1}c_n = (n + a + c - 1)(n + b + d - 1) \times \frac{n(n + a + b + c + d - 2)(n + a + d - 1)(n + b + c - 1)}{(2n + a + b + c + d - 1)(2n + a + b + c + d - 2)^2(2n + a + b + c + d - 3)}. \quad (4.4)$$
Using (4.3) and taking \( \tilde{b}_n = \frac{1}{2}(a + b - c - d) \), the coefficient \( b_n \) (2.26) is found to be

\[
b_n = i \left[ -\frac{(n + a + b + c + d - 1)(n + a + c)(n + a + d)}{(2n + a + b + c + d - 1)(2n + a + b + c + d - 1)} + a \right]. \tag{4.5}
\]

The coefficients (4.4) and (4.5) can be identified in [20] and one arrives at:

**Proposition 2.** In the case \( \Delta > 0 \), the eigenfunctions \( p_n(x) \) of \( X \) (2.24) are the monic continuous Hahn polynomials \( F_n^{(a,b,c,d)}(x) \) with parameters given in (4.3).

### 4.1 Finite-dimensional representations and orthogonal polynomials

Using (4.3), condition (2.27) becomes

\[
N + 1 = -a - c - b - d, \tag{4.6}
\]

which leads to expressions for (4.2) and (4.5) that are ill-defined for \( n < N \). However, this can be resolved using limits and one thus obtains the para-Krawtchouk polynomials [10].

Condition (2.28) reads

\[
N + 1 = -\frac{1}{2} \left[ (a + b + c + d - 2) \pm (a - b - c + d) \right] = \begin{cases} -a - d + 1 \\ -b - c + 1 \end{cases} \tag{4.7}
\]

and corresponds to the truncation of the continuous Hahn polynomials to Hahn polynomials.

However, for each of these truncations (4.6) and (4.7) to define real polynomials, the operator \( X \) has to be scaled by an imaginary number; this is equivalent to setting \( \Delta \to -\Delta \) which corresponds to the situation \( \Delta < 0 \) that is the object of the next section.

### 5 The case \( \Delta < 0 \): Hahn and para-Krawtchouk polynomials

When \( \Delta < 0 \), polynomials of a real variable are obtained only if (2.27) or (2.28) are satisfied. We begin by treating the latter case.

#### 5.1 Hahn polynomials

In view of (4.1), we may take \( \Delta = -\frac{1}{4} \) without loss of generality. Expressing the parameters as follows

\[
\phi_0 = \beta, \quad -\delta_0 = \alpha + 1, \quad \nu_0 = -\frac{(\alpha + \beta + 2N + 2)}{2(\alpha + \beta + 2)}, \quad \tilde{b}_0 = \frac{1}{4}(2N - \alpha + \beta), \tag{5.1}
\]

so that (2.28) is satisfied, one obtains

\[
a_{n-1}c_n = \frac{n(n + \alpha)(n + \beta)(n + \alpha + \beta)(n + \alpha + \beta + N + 1)(N - n + 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta + 1)^2(2n + \alpha + \beta + 1)}, \tag{5.2}
\]

as well as

\[
b_n = \frac{(n + \alpha + \beta + 1)(n + \alpha + 1)(N - n)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)} + \frac{n(n + \alpha + \beta + N + 1)(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}. \tag{5.3}
\]

The coefficients given by (5.2) and (5.3) are found in [20].

**Proposition 3.** In the case \( \Delta < 0 \), the eigenfunctions \( p_n(x) \) of \( X \) (2.24) related to the finite-dimensional representation condition (2.28) are given in terms of the monic Hahn polynomials \( Q_n^{(\alpha,\beta)}(x) \) for the choice of parameters given in (5.1).
As previously mentioned, these polynomials can also be obtained as a truncation of the recurrence defined by (4.2) and (4.5). Indeed, setting

\[ \alpha = a + c - 1, \quad \beta = b + d - 1, \]  

(5.4)

with one of (4.7), the coefficient (4.2) and (4.5) become proportional to (5.2) and (5.3), respectively. Hence, the action of \( iX \) when \( \Delta = + \frac{1}{4} \) also leads to the recurrence relation of the monic Hahn polynomials.

5.2 Para-Krawtchouk polynomials

We shall finally indicate how a family of finite-dimensional representations of \( \mathcal{A} \) relates to para-Krawtchouk polynomials. Consider the condition (2.27). Although leading to singular expressions for certain values of \( n \), well-defined polynomials are obtained by carefully taking limits. Mindful of (4.1), it is convenient in this case to take \( \Delta = -1 \). Let \( N = 2j + p \) with \( j \) an integer and \( p = 0, 1 \) depending on the parity of \( N \), and set

\[ \phi_0 + 1 = -j + e_1 t, \quad -\delta_0 = -j + e_2 t + 1 - p, \quad v_0 = -\frac{(\gamma + p - 1)}{(-2j + e_1 t + e_2 t - p + 1)}, \quad e_1 = e_2 = 1. \]  

(5.5)

The parameters \( e_1 \) and \( e_2 \) are chosen equal in order to simplify the expressions. The more general solutions can be recovered using isospectral deformations [12, 21]. With the above parametrization, it can be seen that (2.27) is verified in the limit where \( t \to 0 \). With the parameters as in (5.5), the coefficient \( a_{n-1}c_n \) (2.25) becomes

\[ a_{n-1}c_n = (n - j + t - 1)(n - j + t - p) \times \frac{n(n - 2j + 2t - p - 1)(N - 2n + p + \gamma)(N - 2n - p + 2 - \gamma)}{(2n - 2j + 2t - p - 1)^2(2n - 2j + 2t - p)(2n - 2j + 2t - p - 2)}. \]  

(5.6)

Taking the limit \( t \to 0 \) and treating the cases for \( p = 0, 1 \) separately, one finds that the results can be combined as follows

\[ \lim_{t \to 0} a_{n-1}c_n = \frac{n(N + 1 - n)(N - 2n + p + \gamma)(N - 2n - p + 2 - \gamma)}{4(2n - N + p - 1)(2n - N - p - 1)}. \]  

(5.7)

For the coefficient \( b_n \), setting \( \hat{b}_0 = \frac{1}{2}(N + \gamma - 1) \) and inserting (5.5) in (2.26), one finds

\[ b_n = \frac{1}{2} \frac{(p - 1)(-2j + p + 2t - 1)(\gamma + p - 1)}{2n - 2j + 2t - 1)(2n - 2j + 2t + 1) + \frac{1}{2}(N + \gamma - 1). \]  

(5.8)

Treating the cases \( p = 0 \) or \( p = 1 \) separately and taking the limit \( t \to 0 \), one sees that the results can be written jointly as

\[ \lim_{t \to 0} b_n = \frac{- (N - n)(N - 2n - 2 + p + \gamma)}{2(2n - N + p + 1)} - \frac{n(N - 2n + 2 - p - \gamma)}{2(2n - N + p - 1)}. \]  

(5.9)

The coefficients given by (5.7) and (5.9) are recognized in [12] as the coefficients for the recurrence relation of the monic para-Krawtchouk polynomials.

**Proposition 4.** In the case \( \Delta < 0 \), the eigenfunctions \( p_n(x) \) of \( X \) (2.24) in the finite-dimensional representation (2.27) of \( \mathcal{A} \) are the monic para-Krawtchouk polynomials.

6 Conclusion

We have studied tridiagonal representations of the algebra \( \mathcal{A} \) with defining relation \([\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \Delta \). Depending on the value of \( \Delta \), in these representations, the linear pencil \( X + \mu \mathcal{Z} \) entailed the recurrence relations of the Jacobi (\( \Delta = 0 \)), continuous Hahn (\( \Delta > 0 \)), Hahn and para-Krawtchouk (\( \Delta < 0 \)) polynomials.

In the wake of this work, two research avenues present themselves. One is the exploration of the tridiagonal representations of the algebra \([\mathcal{Z}, \mathcal{X}] = \mathcal{Z}^2 + \alpha \mathcal{X} \), another class of the general quadratic algebra [12]. It is
expected that the tridiagonal representations will lead to the Wilson, Racah and para-Racah polynomials in
a similar fashion.

Another related direction is the study of the so-called meta algebras, poised to describe both polynomial
and rational functions of a given type, as shown in [22] for functions of the Hahn type. The meta-Hahn
algebra is in fact obtained by adjoining to \( A \) an additional generator. As it turns out, the extra generator
offers a rationale for considering tridiagonal representations. This suggests in particular that the work on
the \( q \)-oscillator algebra [8] should be revisited in order to bring to the fore the associated rational functions.

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