Analytical Approximation of Brusselator Model via LADM

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In this paper, we have investigated the fractional-order Brusselator model for the approximate analytical solution. The concerned coupled system of nonlinear fractional-order partial differential equations (PDEs) has been considered in many research articles to describe various real-world problems. Nonlinear dynamical systems are increasingly used in autocatalytic chemical reactions. These types of autocatalytic chemical reaction models have a variety of applications in several physical systems. The Brusselator model is one of the intensively used autocatalytic models. We have obtained the approximate solution of the considered fractional nonlinear Brusselator model in the Caputo sense by using the Laplace–Adomian decomposition method (LADM). We have established a general scheme for the solution to the proposed model by applying the LADM. We then testified two examples to demonstrate our analysis. The results we have obtained ensure the accuracy and effectiveness of the proposed technique. The said technique does not require any kind of prior discretization or collocation. Also, with the help of Matlab, we have presented 2D and 3D plots for the approximate solution.

1. Introduction

The second law of thermodynamics states that “the total energy of an isolated system is constant” which leads to the largest mess. Here the following question arises: how then is it feasible for developmental activities to increase in complexity and order with the passage of time? The famous Nobel laureate Ilya Prigogine [1] wrote “from the simplest microorganism to macro organism, conservation of life needs a continuous exchange of matter and energy with the surrounding. Therefore, living organisms belong to the class of open system.” Thus, there is a large number of examples, when the out of equilibrium is the source of order. Describing such self-organization in physical systems is essential for the sake of understanding the physical and biological processes. In the last few decades, nonlinear chemical applications with diffusion reactions under PDEs have been studied analytically and numerically, for instance, the pellet model [2], the reversible Selkov model [3], the Brusselator model [4], the Schnackenberg model [5] and the Belousov–Zhabotinsky (BZ) reaction [6]. In addition, the diffusive nonlinear models have a large number of practical applications in several fields of applied mathematics. These models are using a continuous-flow stirred-tank reactor (CSTR) to discuss various oscillatory chemical phenomena. Some theoretical and experimental results devoted to CSTR steers for chemical oscillatory models (for details, see [7, 8]).

The major goal of examining the dynamical systems is to focus on the solution of the modeled equations that describe the real-world phenomenon of space and time. Such types of systems preserve numerous applications in various fields of chemistry, computer science, mathematics, physics, and economics. A particular application of dynamical systems is its use in chemical autocatalytic reactions which are nonlinear in nature. So, an important type of chemical reaction is known as the Brusselator model which was discovered in
1968 and presented in 1970 by physical chemist Ilya Prigogine and his collaborators. The term Brusselator is a portmanteau of Brussels and oscillator. In such types of reactions, the reactant substance shows interaction with others to step up its production rate. The considered Brusselator system comprises two kinds of BZ model that describes the contest of two chemical reactants in a chemical reaction. Also, it is the simplest reaction-diffusion system competent of creating complex spatial patterns. The discovery of diffusion and the contest between two reactors fulfill the basic essentials for pattern formation. The comprehensive study of pattern dynamics for the given model has been done by de Wit [9]. The considered model is actually a reaction-diffusion system that takes place in various physical problems, like the formation of ozone via triple collision of atomic oxygen and enzymatic reactions (see [10, 11]). It also plays an essential role in the analysis of chemical kinetics and biological systems (for readers, we refer [12, 13]). Significant investigations of solutions for the Brusselator model have been obtained earlier through various methods (for details, we refer [14, 15]).

Reaction-diffusion systems have been analyzed for the last few decades to investigate the far intuition of biological systems. In other words, these systems have been applied in various physical processes of life. In such types of systems, Schnakenberg, Brusselator, and Lengyel–Epstein models are prominent due to their reliable results and applicability.

One of the prominent reaction-diffusion systems is known as the Brusselator model. This model is manipulated to explain the mechanism of such systems with nonlinear oscillations [16]. The Brussels school developed and analyzed the behavior of a nonlinear oscillator associated with the chemical system as

\[
\begin{align*}
A & \rightarrow X, \\
2X + Y & \rightarrow 3X, \\
B + X & \rightarrow Y + D, \\
X & \rightarrow E,
\end{align*}
\]

where \(A\) and \(B\) are reactants and \(D\) and \(E\) are products while \(X\) and \(Y\) are intermediates. The nondimensional form of the nonlinear model is represented by

\[
\begin{align*}
\frac{\partial u}{\partial t} & = B + u^2 v - (A + 1)u + \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
\frac{\partial v}{\partial t} & = Au - u^2 v + \alpha \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).
\end{align*}
\]

The unknown \(u = u(x, y, t)\) and \(v = v(x, y, t)\) represent the chemical concentration of reactants, while \(A\) and \(B\) are constants and represent the concentrations of input reagents. \(\alpha\) is the diffusion coefficient. For detailed study, see [10–13].

At present, several real-world problems are modeled in terms of nonlinear fractional PDEs. Therefore, the researchers paid much attention to explore fractional nonlinear PDEs due to their extensive implementation in modern sciences and engineering. The class of fractional-order PDEs has superiority over the classical PDEs in that it possesses the hereditary property, nonlocality, and description memory effect for different problems with more chances of freedom.

The models containing such types of derivatives are more trustworthy and precise in contrast to classical derivatives. In some real-world problems, a mathematical model cannot produce reliable results due to the involvement of traditional derivatives. In this regard, PDEs of fractional order are more reliable to describe such type of real situations (see [17–24]). That is why different analytical and numerical techniques are applied to the proposed model to examine the desired solution (see [14, 25, 26]).

Among such differential operators, the Caputo operator is more precise and reliable in contrast to those which contain traditional derivatives. Due to this fact, the researchers used this operator for investigating different aspects like the existence, uniqueness, and stability of various mathematical models (see [27–32]). In addition, we have generalized the concept of integer-order derivatives for the considered model to fractional order for a good description of the solution by the Caputo fractional differential operator. Therefore, in the Caputo sense, the fractional-order Brusselator model is given by

\[
\begin{align*}
\frac{\partial^\alpha u}{\partial t^\alpha} & = B + u^2 v - (A + 1)u + \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\
\frac{\partial^\alpha v}{\partial t^\alpha} & = Au - u^2 v + \alpha \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),
\end{align*}
\]

subjected to the initial conditions (ICs):

\[u(x, y, 0) = g(x, y)\text{ and } v(x, y, 0) = h(x, y).\]

We have constructed a recursive numerical scheme for the considered model by applying the well-known numerical technique, i.e., LADM. This technique comprises a special polynomial called Adomian polynomial. The Adomian polynomial split the nonlinear term in the form of a series. The nonlinear term is decomposed as

\[
H(w(x, y, t)) = \sum_{m=0}^{\infty} A_m.
\]

The term \(A_m\) is known as Adomian polynomial and is defined as

\[
A_m = \frac{1}{\Gamma(m+1)} \frac{d^m}{dx^m} H^{\sum_{i=0}^{m} \lambda w_i(x, y, t)}.
\]

Among the available numerical techniques for the solution of nonlinear PDEs, LADM is one of the best, simple, and most efficient tools to handle the said problem. LADM has the superiority over other techniques in that it gives particular solutions directly without finding any general solution first. Furthermore, it requires no linearization and discretization and possesses fewer parameters. It also does
not requires predefined size declaration like the Runge–Kutta method. In comparison with other analytical techniques, the computation of results by this method confirm the potentiality and reliability of the applied method, especially for nonlinear fractional PDEs (for details, see [33, 34]). Few authors have investigated some fractional-order partial differential equations by using various numerical techniques [35, 38].

2. Preliminaries

This section provides fundamental definitions and results associated with fractional calculus that will be helpful in this research work.

Definition 1 (see [34]). Let \( h(x, y, t) \) be a function defined \( \forall t \geq 0 \); then, Laplace transform of \( h(x, y, t) \) is marked by \( H(x, y, s) \) and \( \mathcal{L}[h(x, y, t)] \) and is defined as

\[
H(x, y, s) = \mathcal{L}[h(x, y, t)] = \int_0^\infty e^{-st} h(x, y, t) dt.
\]

Definition 2 (see [34]). Let \( \phi(x, y, t) \) be a function defined on the interval \( (0, \infty) \times (0, \infty) \times (0, \infty) \). Then, the fractional-order Caputo derivative for the function \( \phi \) is defined as

\[
^cD^\alpha \phi(x, y, t) = \frac{1}{\Gamma(p - \alpha)} \int_0^t (t - s)^{p - \alpha - 1} \phi'(x, y, s) ds,
\]

\[ \alpha \in (p - 1, p), \quad p \in \mathbb{N}, \]

where the term \( \alpha \) denotes real number and \( p = 1 + [\alpha], \) in which \([\alpha]\) is the integral part of \( \alpha \).

Particularly, for \( \alpha \in (0, 1) \),

\[
^cD^\alpha \phi(x, y, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{1}{(t - s)^\alpha} \frac{\partial}{\partial s} \phi(x, y, s) ds.
\]

If \( \alpha \rightarrow p \), then \(^cD^\alpha \phi(x, y, t)D = \frac{\partial^\alpha}{\partial t^\alpha} \phi(x, y, t)\).

Definition 3 (see [34]). If \(^cD^\alpha \phi(x, y, t)\) represents the Caputo derivative of \( \phi(x, y, t) \), then

\[
\mathcal{L}\left[^cD^\alpha \phi(x, y, t)\right] = s^\alpha \phi(x, y, s) - \sum_{k=0}^{p-1} \frac{s^{\alpha-k-1} \phi^{(k)}(x, y, 0)}{\Gamma(\alpha-k-1)},
\]

\[ \alpha \in (p - 1, p), \quad p \in \mathbb{N}, \]

where \([\alpha]\) denotes the integral part of \( \alpha \) and \( p = 1 + [\alpha] \).

3. Scheme for the Approximate Solution of Brusselator Model

In this section, the authors have constructed a scheme for the approximate solution of the fractional nonlinear chemical Brusselator model through LADM. The nondimensional form of the considered model is given as

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = B + u^2 v - (A + 1)u + \alpha \left( \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right),
\]

\[
\frac{\partial^\alpha v}{\partial t^\alpha} = Au - u^2 v + \alpha \left( \frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha v}{\partial y^\alpha} \right),
\]

\[ 0 < \sigma \leq 1, \]

\[ 1 < \eta \leq 2, \]

subjected to ICs:

\[
u(x, y, 0) = g(x, y) \quad \text{and} \quad v(x, y, 0) = h(x, y),
\]

where \( u = u(x, y, t) \) and \( v = v(x, y, t) \) represent the chemical concentration of products and the terms \( A \) and \( B \) are constants that represent the concentrations of input reagents. Applying the Laplace transform on (11), we have

\[
\mathcal{L}\left[ \frac{\partial^\alpha u}{\partial t^\alpha} \right] = \mathcal{L}\left[ B + u^2 v - (A + 1)u + \alpha \left( \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right) \right],
\]

\[
\mathcal{L}\left[ \frac{\partial^\alpha v}{\partial t^\alpha} \right] = \mathcal{L}\left[ Au - u^2 v + \alpha \left( \frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha v}{\partial y^\alpha} \right) \right].
\]

By using properties of the Laplace transform, (13) becomes

\[
U(x, y, s) - \frac{1}{s} u(x, y, 0) = \frac{1}{s} \mathcal{L}\left[ B + u^2 v - (A + 1)u + \alpha \left( \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right) \right],
\]

\[
V(x, y, s) - \frac{1}{s} v(x, y, 0) = \frac{1}{s} \mathcal{L}\left[ Au - u^2 v + \alpha \left( \frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha v}{\partial y^\alpha} \right) \right].
\]

Now applying the inverse Laplace transform on (14) and using ICs, we have

\[
\mathcal{L}^{-1}\left[ \frac{1}{s} \mathcal{L}\left[ B + u^2 v - (A + 1)u + \alpha \left( \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial y^\alpha} \right) \right] \right],
\]

\[
\mathcal{L}^{-1}\left[ \frac{1}{s} \mathcal{L}\left[ Au - u^2 v + \alpha \left( \frac{\partial^\alpha v}{\partial x^\alpha} + \frac{\partial^\alpha v}{\partial y^\alpha} \right) \right] \right].
\]

The nonlinear term \( u^2 v \) present in (15) is decomposed as
\[ u^2v = \sum_{m=0}^{\infty} A_m, \quad \text{(16)} \]

where \( A_m \) is defined as
\[ A_m = \frac{1}{\Gamma(m+1)} \frac{d^m}{dx^m} \left[ \sum_{i=0}^{m} \lambda_i u_i(x, y, t) \right]^2 \left[ \sum_{i=0}^{m} \lambda_i v_i(x, y, t) \right] \]
\[ \text{for} \quad m = 0, \quad \text{one has} \quad A_0 = u_0(x, y, t)v_0(x, y, t). \quad \text{(18)} \]

For \( m = 1 \), we have
\[ A_1 = u_0(x, y, t)v_1(x, y, t) + 2u_0(x, y, t)u_1(x, y, t)v_0(x, y, t). \quad \text{(19)} \]

The assumed solutions \( u(x, y, t) \) and \( v_i(x, y, t) \) are in the form of
\[ u(x, y, t) = \sum_{i=0}^{\infty} u_i(x, y, t) \quad \text{and} \quad v(x, y, t) = \sum_{i=0}^{\infty} v_i(x, y, t). \quad \text{(20)} \]

Plugging these values in (15), we have
\[ \left\{ \begin{array}{l}
\sum_{i=0}^{\infty} u_i(x, y, t) = g(x, y) + \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left\{ B + \sum_{n=0}^{\infty} A_n - (A + 1) \sum_{i=0}^{\infty} u_i + a \left( \frac{\partial^n}{\partial x^n} u_0 + \frac{\partial^n}{\partial y^n} v_0 \right) \right\} \right], \\
\sum_{i=0}^{\infty} v_i(x, y, t) + h(x, y) = \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left\{ A \sum_{i=0}^{\infty} u_i - \sum_{n=0}^{\infty} A_n + a \left( \frac{\partial^n}{\partial x^n} \sum_{i=0}^{\infty} u_i + \frac{\partial^n}{\partial y^n} \sum_{i=0}^{\infty} v_i \right) \right\} \right].
\end{array} \right. \quad \text{(21)} \]

Comparing both sides of (21), we have
\[ \left\{ \begin{array}{l}
u_0 = g(x, y), \\
v_0 = h(x, y), \\
u_1 = g(x, y) + \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L} \left\{ B + A_0 - (A + 1)u_0 + a \left( \frac{\partial^n}{\partial x^n} u_0 + \frac{\partial^n}{\partial y^n} v_0 \right) \right\} \right], \\

\end{array} \right. \quad \text{(22)} \]

The desired solution is given by
\[ \left\{ \begin{array}{l}
u = \sum_{i=0}^{\infty} u_i(x, y, t), \\
v = \sum_{i=0}^{\infty} v_i(x, y, t).
\end{array} \right. \quad \text{(23)} \]
By computational work, we get

\[
\begin{align*}
\{ u_0 &= g(x, y), \\
v_0 &= h(x, y), \\
u_1 &= \left\{ B + u_0^1 v_0 - (A + 1)u_0 + \alpha \left( \frac{\partial^n}{\partial x^n} u_0 + \frac{\partial^n}{\partial y^n} u_0 \right) \right\} \frac{t^n}{\Gamma(\sigma + 1)}, \\
v_1 &= \left\{ u_1^1 v_1 + 2u_0 u_1 v_0 - (A + 1)\xi + \alpha \left( \frac{\partial^n}{\partial x^n} \xi + \frac{\partial^n}{\partial y^n} \xi \right) \right\} \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)}, \\
u_2 &= \left\{ A \xi - u_0^2 v_1 - 2u_0 u_1 v_0 + \alpha \left( \frac{\partial^n}{\partial x^n} \xi + \frac{\partial^n}{\partial y^n} \xi \right) \right\} \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)},
\end{align*}
\]

where

\[
\begin{align*}
\xi &= B + u_0^2 v_0 - (A + 1)u_0 + \alpha \left( \frac{\partial^n}{\partial x^n} u_0 + \frac{\partial^n}{\partial y^n} u_0 \right),
\end{align*}
\]

Thus, the three terms’ solution is given by

\[
\begin{align*}
\{ u &= g(x, y) + \left\{ 2B + u_0^2 v_0 - (A + 1)u_0 + \alpha \left( \frac{\partial^n}{\partial x^n} u_0 + \frac{\partial^n}{\partial y^n} u_0 \right) \right\} \frac{t^n}{\Gamma(\sigma + 1)} \\
&\quad + \left\{ u_1^1 v_1 + 2u_0 u_1 v_0 - (A + 1)\xi + \alpha \left( \frac{\partial^n}{\partial x^n} \xi + \frac{\partial^n}{\partial y^n} \xi \right) \right\} \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)}, \\
v &= h(x, y) + \left\{ u_0^2 v_1 + 2u_0 u_1 v_0 - (A + 1)\xi + \alpha \left( \frac{\partial^n}{\partial x^n} \xi + \frac{\partial^n}{\partial y^n} \xi \right) \right\} \frac{t^{\sigma}}{\Gamma(\sigma + 1)} \\
&\quad + \left\{ A \xi - u_0^2 v_1 - 2u_0 u_1 v_0 + \alpha \left( \frac{\partial^n}{\partial x^n} \xi + \frac{\partial^n}{\partial y^n} \xi \right) \right\} \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)},
\end{align*}
\]

4. Illustrative Applications

This section of the article is associated to some applications to justify the general scheme.

Example 1. If \( \alpha = \frac{1}{4}, \eta = 2, \ B = 0, \ and \ A = 1, \) then the proposed model becomes

\[
\frac{\partial^\sigma u}{\partial t^\sigma} = u^2 v - 2u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

\[
\frac{\partial^\sigma v}{\partial t} = u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\]
subjected to ICs:

\[ u(x, y, 0) = \exp(-x - y) \quad \text{and} \quad v(x, y, 0) = \exp(x + y). \tag{29} \]

Applying Laplace transform on (28), we have

\[
\begin{align*}
\mathcal{L}\left\{ \frac{\partial^2 u}{\partial x^2} \right\} &= \mathcal{L}\left\{ u^2 v - 2u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right\}, \\
\mathcal{L}\left\{ \frac{\partial^3 v}{\partial t \partial x} \right\} &= \mathcal{L}\left\{ u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right\}.
\end{align*}
\]

By using properties of Laplace transform, (30) becomes

\[
\begin{align*}
U(x, y, s) &= s e^{-s} - u(x, y, 0) + \frac{1}{s^2} \mathcal{L}\left\{ u^2 v - 2u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right\}, \\
V(x, y, s) &= s e^{-s} v(x, y, 0) + \frac{1}{s^2} \mathcal{L}\left\{ u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right\}.
\end{align*}
\]

Now applying the inverse Laplace transform on (31) and using ICs, we get

\[
\begin{align*}
\begin{cases}
\sum_{i=0}^{\infty} u_i = e^{-x-y} + \frac{1}{s} \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L}\left\{ u^2 v - 2u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right\} \right], \\
\sum_{i=0}^{\infty} v_i = e^{x+y} + \frac{1}{s} \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L}\left\{ u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right\} \right].
\end{cases}
\end{align*}
\]

The nonlinear term \( u^2 v \) present in (32) is decomposed as

\[ u^2 v = \sum_{n=0}^{\infty} A_n. \tag{33} \]

For \( m = 0 \), we have

\[ A_0 = e^{-x-y}. \tag{34} \]

For \( m = 1 \), one has

\[ A_1 = \frac{1}{2} e^{-x-y} \frac{t^2}{\Gamma(2\sigma + 1)}. \tag{35} \]

For \( m = 2 \), one gets

\[ A_2 = \frac{1}{2} e^{-x-y} \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)}. \tag{36} \]

The assumed solution is considered as

\[ u = \sum_{i=0}^{\infty} u_i \quad \text{and} \quad v = \sum_{i=0}^{\infty} v_i. \tag{37} \]

System (32) becomes

\[
\begin{align*}
\sum_{i=0}^{\infty} u_i &= e^{-x-y} + \frac{1}{s^2} \mathcal{L}^{-1} \left[ \frac{1}{s} \mathcal{L}\left\{ \sum_{i=0}^{\infty} A_i - 2 \sum_{i=0}^{\infty} u_i + \frac{1}{4} \left( \frac{\partial^2 u_i}{\partial x^2} \sum_{i=0}^{\infty} u_i + \frac{\partial^2 u_i}{\partial y^2} \sum_{i=0}^{\infty} u_i \right) \right\} \right], \\
\sum_{i=0}^{\infty} v_i &= e^{x+y} + \frac{1}{s} \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L}\left\{ \sum_{i=0}^{\infty} u_i - \sum_{i=0}^{\infty} A_i + \frac{1}{4} \left( \frac{\partial^2 v_i}{\partial x^2} \sum_{i=0}^{\infty} v_i + \frac{\partial^2 v_i}{\partial y^2} \sum_{i=0}^{\infty} v_i \right) \right\} \right].
\end{align*}
\]

Comparing both sides of (38), we have
\[
\begin{align*}
\{ u_0 &= e^{-x-y}, \\
v_0 &= e^{x+y}, \\
\{ u_1 &= \left( \frac{1}{2} e^{-x-y} \right) \frac{t^\sigma}{\Gamma(\sigma + 1)}, \\
v_1 &= \left( \frac{1}{2} e^{x+y} \right) \frac{t^\sigma}{\Gamma(\sigma + 1)}, \\
\{ u_2 &= \left( \frac{1}{4} e^{-x-y} \right) \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)}, \\
v_2 &= \left( \frac{1}{4} e^{x+y} \right) \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)}, \\
\{ u_3 &= \left( \frac{1}{8} e^{-x-y} \right) \frac{t^{3\sigma}}{\Gamma(3\sigma + 1)}, \\
v_3 &= \left( \frac{1}{4} e^{-x-y} + \frac{1}{8} e^{x+y} \right) \frac{t^{3\sigma}}{\Gamma(3\sigma + 1)}.
\end{align*}
\]

So, the truncated solution after four terms is

\[
\begin{align*}
\{ u &= e^{-x-y} + \left( \frac{1}{2} e^{-x-y} \right) \frac{t^\sigma}{\Gamma(\sigma + 1)} + \left( \frac{1}{4} e^{-x-y} \right) \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)} + \left( \frac{1}{8} e^{-x-y} + \frac{1}{8} e^{x+y} \right) \frac{t^{3\sigma}}{\Gamma(3\sigma + 1)}, \\
v &= e^{x+y} + \left( \frac{1}{2} e^{x+y} \right) \frac{t^\sigma}{\Gamma(\sigma + 1)} + \left( \frac{1}{4} e^{x+y} \right) \frac{t^{2\sigma}}{\Gamma(2\sigma + 1)} + \left( \frac{1}{4} e^{-x-y} + \frac{1}{8} e^{x+y} \right) \frac{t^{3\sigma}}{\Gamma(3\sigma + 1)}.
\end{align*}
\]  

(39)

The corresponding approximate solutions are plotted by using 2D and 3D graphs for the mentioned values of \( \sigma \) and \( t \) in Figures 1–4, respectively, for Example 1.

**Example 2.** Let \( \alpha = 1/4, \eta = 2, B = 1, \) and \( A = 1/2, \) so the proposed model becomes

\[
\begin{align*}
\left\{ \frac{\partial^\sigma u}{\partial t^\sigma} &= 1 + u^2 \nu - \frac{3}{4} u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} \right. + \left. \frac{\partial^2 u}{\partial y^2} \right), \\
\frac{\partial^\sigma v}{\partial t^\sigma} &= \frac{1}{2} u - u^2 \nu + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} \right. + \left. \frac{\partial^2 v}{\partial y^2} \right),
\end{align*}
\]

subjected to ICs:

\[
\begin{align*}
u(x, y, 0) = x^2 \text{ and } v(x, y, 0) = y^2.
\end{align*}
\]  

(42)

Applying Laplace transform on (41), we have

\[
\mathcal{L} \left[ \frac{\partial^\sigma u}{\partial t^\sigma} \right] = \mathcal{L} \left[ 1 + u^2 \nu - \frac{3}{4} u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right],
\]

(43)

By using properties of Laplace transform, (43) becomes
Figure 1: Representation of spatial approximate solution of $u$ and $v$ in 3D, respectively, for $\sigma = 1$ and $t = 0$.

Figure 2: Representation of spatial approximate solution of $u$ and $v$ in 3D, respectively, for $\sigma = 1$ and $t = 5$.

Figure 3: Representation of spatial approximate solution of $u$ and $v$ in 3D, respectively, for $\sigma = 1$ and $t = 10$. 
\[
\begin{align*}
U &= s^{-1}u(x, 0) + \frac{1}{s^2} \mathcal{L} \left[ 1 + u^2 v - \frac{3}{2} u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right], \\
V &= s^{-1}v(x, 0) + \frac{1}{s^2} \mathcal{L} \left[ \frac{1}{2} u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right].
\end{align*}
\]

Now applying the inverse Laplace transform on (44) and using ICs, we get

\[
\begin{align*}
\sigma &\leq \frac{1}{2} \\
&\rightarrow \\
\text{Plot of } u(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1 \\
\text{Plot of } v(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1
\end{align*}
\]

Figure 4: Representation of spatial approximate solution of \( u \) and \( v \) in 2D, respectively, for different values of \( \sigma \).

\[
\begin{align*}
U &= \mathcal{L}^{-1} \left[ s^{-1}u(x, 0) + \frac{1}{s^2} \mathcal{L} \left[ 1 + u^2 v - \frac{3}{2} u + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \right] \right], \\
V &= \mathcal{L}^{-1} \left[ s^{-1}v(x, 0) + \frac{1}{s^2} \mathcal{L} \left[ \frac{1}{2} u - u^2 v + \frac{1}{4} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right] \right].
\end{align*}
\]

Now applying the inverse Laplace transform on (44) and using ICs, we get

\[
\begin{align*}
\sigma &\leq \frac{1}{2} \\
&\rightarrow \\
\text{Plot of } u(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1 \\
\text{Plot of } v(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1
\end{align*}
\]

Figure 5: Representation of spatial approximate solution of \( u \) and \( v \) in 3D, respectively, for \( \sigma = 1 \) and \( t = 0 \).

\[
\begin{align*}
\sigma &\leq \frac{1}{2} \\
&\rightarrow \\
\text{Plot of } u(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1 \\
\text{Plot of } v(x, y, t) \text{ for } \sigma = 0.7, 0.8, 0.9, 1
\end{align*}
\]

The nonlinear term \( u^2 v \) present in (45) is decomposed as

\[
u^2 v = \sum_{n=0}^{\infty} A_n.
\]
For $m = 0$, we have
\[ A_0 = x^4 y^2. \]  
(47)

For $m = 1$, one has
\[ A_1 = \left( -3x^4 y^2 + 2x^6 y^4 + 3x^2 y^6 + \frac{1}{2} x^4 - x^2 y^2 + \frac{1}{2} x^4 \right) \frac{t^\sigma}{\Gamma(\sigma + 1)} \]  
(48)

Also, the required solution can be expressed as
\[ u = \sum_{i=0}^\infty u_i \quad \text{and} \quad v = \sum_{i=0}^\infty v_i. \]  
(49)

(45) becomes

Figure 6: Representation of spatial approximate solution of $u$ and $v$ in 3D, respectively, for $\sigma = 1$ and $t = 5$.

Figure 7: Representation of spatial approximate solution of $u$ and $v$ in 3D, respectively, for $\sigma = 1$ and $t = 10$. 
\begin{align*}
\sum_{i=0}^{\infty} u_i &= x^2 + \mathcal{L}^{-1} \left[ \frac{1}{\Gamma} \mathcal{L} \left\{ \frac{1}{2} \sum_{i=0}^{\infty} A_i - \frac{3}{2} \sum_{i=0}^{\infty} u_i + \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} \sum_{i=0}^{\infty} u_i + \frac{\partial^2}{\partial y^2} \sum_{i=0}^{\infty} u_i \right) \right\} \right], \\
\sum_{i=0}^{\infty} v_i &= y^2 + \mathcal{L}^{-1} \left[ \frac{1}{\Gamma} \mathcal{L} \left\{ \frac{1}{2} \sum_{i=0}^{\infty} u_i - \frac{3}{2} \sum_{i=0}^{\infty} A_i + \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} \sum_{i=0}^{\infty} v_i + \frac{\partial^2}{\partial y^2} \sum_{i=0}^{\infty} v_i \right) \right\} \right].
\end{align*}
\tag{50}

Comparing both sides of system (50), we have

\begin{align*}
&\begin{cases}
u_0 = x^2, \\
v_0 = y^2,
\end{cases} \\
&\begin{cases}
u_1 = \left( -\frac{3}{2} x^2 + \frac{3}{2} y^2 + \frac{9}{2} \right) \frac{t^\sigma}{\Gamma (\sigma + 1)}, \\
v_1 = \left( \frac{1}{2} x^2 - x^4 y^2 + \frac{1}{2} \right) \frac{t^\sigma}{\Gamma (\sigma + 1)},
\end{cases} \\
&\begin{cases}
u_2 = \left( -x^8 y^2 + 2x^6 y^4 - \frac{9}{2} x^4 y^2 + 6x^2 y^4 + \frac{1}{2} x^6 + x^4 + \frac{9}{4} x^4 - 2 \right) \frac{t^{2\sigma}}{\Gamma (2\sigma + 1)}, \\
v_2 = \left( x^8 y^2 - 2x^6 y^4 + \frac{7}{2} x^4 y^2 - 6x^2 y^4 - \frac{1}{2} x^6 - x^4 - \frac{3}{4} x^2 + 1 \right) \frac{t^{2\sigma}}{\Gamma (2\sigma + 1)}.
\end{cases}
\end{align*}
\tag{51}

So, the truncated solution after three terms is

\begin{align*}
&\begin{cases}
u = x^2 + \left( -\frac{3}{2} x^2 + \frac{3}{2} y^2 + \frac{9}{2} \right) \frac{t^\sigma}{\Gamma (\sigma + 1)} + \left( -x^8 y^2 + 2x^6 y^4 - \frac{9}{2} x^4 y^2 + 6x^2 y^4 + \frac{1}{2} x^6 + x^4 + \frac{9}{4} x^4 - 2 \right) \frac{t^{2\sigma}}{\Gamma (2\sigma + 1)}, \\
v = y^2 + \left( \frac{1}{2} x^2 - x^4 y^2 + \frac{1}{2} \right) \frac{t^\sigma}{\Gamma (\sigma + 1)} + \left( x^8 y^2 - 2x^6 y^4 + \frac{7}{2} x^4 y^2 - 6x^2 y^4 - \frac{1}{2} x^6 - x^4 - \frac{3}{4} x^2 + 1 \right) \frac{t^{2\sigma}}{\Gamma (2\sigma + 1)}.
\end{cases} 
\end{align*}
\tag{52}

The corresponding approximate solutions are plotted by using 2D and 3D graphs for the mentioned values of \( \sigma \) and \( t \) in Figures 5–8, respectively, for Example 2. The comparison between exact and approximate solutions is given in Tables 1 and 2, respectively.
5. Conclusion

An autocatalytic reaction has been used in several dynamical systems to understand the physical and biological processes. The Brusselator model is an autocatalytic chemical reaction model which has been extensively used in several dynamical systems. In this research work, we have constructed the scheme for the general approximate solution of the fractional-order chemical Brusselator model by using the well-known tool based on LADM. Consequently, a semi-analytic solution has been obtained for the considered Brusselator model. We have also provided numerical examples to justify the general scheme. In the end, the numerical solution has been presented graphically using 3D and 2D graphs by Matlab to elaborate the model’s dynamical behavior. We have plotted the corresponding solutions for different values of $t$ using $\sigma = 1$ in Figures 1–4 for Example 1 and Figures 5–8 for Example 2, respectively. LADM is a powerful tool that does not need any prior discretization or collocation of data. Moreover, the mentioned method is easy to implement for various fractional-order linear and nonlinear PDEs. The corresponding comparison between exact and approximate solutions is given in Tables 1 and 2 [35–38].

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Authors’ Contributions

Faiz Muhammad Khan, Amjad Ali, and Abdullah were responsible for the numerical part of the paper. Kamal Shah edited the manuscript. Aziz Khan included the literature. Ibrahim Mahariq revised the paper and updated the necessary changes.
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