At the tunneling ionization of atoms by intense low-frequency laser radiation of linear polarization, the first (quantum) step is the penetration of an electron through the slowly oscillating potential barrier. The second (classical) step is the possible rescattering of this electron after some part of laser period on an atomic core. The possibility of this process is determined by the value of the laser phase when an electron leaves the potential barrier. During the second step, the electron achieves energy of the order of ponderomotive energy $U_p$ from the laser field that is much larger than the atomic ionization potential $I_p$. The third (semi-classical) step is the emission of a high-energy spontaneous photon when an electron returns to its initial ground atomic state. The maximum photon energy $E_{\text{max}} = N_{\text{max}} \omega = 3.173 U_p$ [1]. Of course, alternatively, at the third step this electron can be elastically scattered by the atomic core [2]. The energy distribution of electrons rescattered by their parent ions and emitted near the classical cut-off was considered analytically in [3].

The goal of this paper is to investigate the spontaneous photon yield near $N_{\text{max}}$ qualitatively. First, we consider the classical electron motion during the second step (below $\hbar = m = e = 1$). It is described by the Newton equation

$$\frac{d^2z}{dt^2} = -F \sin(\omega t).$$

(1)

Here $F$ and $\omega$ are the field strength amplitude and field frequency of the monochromatic laser radiation, respectively. The initial conditions at the time instance $t = t_i$ of the second step when an electron leaves the potential barrier are

$$z(t = t_i) = \frac{dz(t = t_i)}{dt} = 0.$$

(2)

In equation (2) we took into account that, at the tunneling regime, the excursion length $F/\omega^2$ is much larger than the width of the potential barrier $I_p/F$ and that the ponderomotive energy $U_p = F^2/4\omega^2$ is much larger than the typical longitudinal kinetic energy $E_k = F^3 / (\omega^2 I_p^{3/2})$ when an electron is ejected from the atom [4] (the transverse electron energy is small compared to the longitudinal energy).

With the solution of equation (1) under initial conditions, equation (2) is

$$\frac{dz(t)}{dt} = \frac{F}{\sqrt{2}} \left\{ \cos(\omega t) - \cos\varphi \right\}; \quad \varphi = \omega t;$$

$$z(t) = \frac{F}{\omega \sqrt{2}} \left\{ \sin(\omega t) - (\omega t - \varphi) \cos\varphi - \sin\varphi \right\}. 

(3)
The quantity \( \varphi \) is the initial field phase. The electron velocity and the kinetic energy are

\[
\begin{align*}
u(t) &= \frac{E}{m}[\cos(\omega t) - \cos \varphi] ; \\
E(t) &= 2U_p(\cos(\omega t) - \cos \varphi)^2. \tag{4}
\end{align*}
\]

Let \( t_0 \) be the time instance when an electron returns to the atomic core for the first time. i.e. \( \varphi(t_0) = 0 \). Then it follows from equation (3) that

\[
\sin(\tau + \varphi) - \tau \cos \varphi = \sin \varphi ; \quad \tau = \omega(t_0 - t). \tag{5}
\]

Hence,

\[
\varphi = \arctan \left( \frac{\sin \tau - \tau}{1 - \cos \tau} \right). \tag{6}
\]

The dependence of the initial field phase \( \varphi \) on the \( \tau \) according to equation (6) is shown in figure 1. It is seen that the return of an electron to the atomic core is possible only at phase values \(-\pi/2 < \varphi < 0\). It is forbidden when \( 0 < \varphi < \pi/2 \).

Substituting equation (6) into equation (4), one determines the electron kinetic energy as a function of the quantity \( \tau \) when \( t = t_0 \):

\[
E(\tau) = 2U_p \left[ \frac{\tau \sin \tau - 2(1 - \cos \tau)}{(\tau - \sin \tau)^2 + (1 - \cos \tau)^2} \right]^2. \tag{7}
\]

The dependence of the dimensionless ratio \( x = E(\tau)/U_p \) on the quantity \( \tau \) is shown in figure 2.

According to figure 2, the maximum value of the electron kinetic energy \( E_{\text{max}} = 3.173U_p \) is achieved when \( \tau_m = 4.08557, \varphi = \varphi_m = -1.257 \). Near \( \tau = \tau_m \) the Taylor expansion of the electron energy is of the form:

\[
x(\tau) = E(\tau)/U_p \approx 3.173 - 1.223(\tau - \tau_m)^2
= 3.173 - 1.223(\omega t_0 - \varphi_m)^2. \tag{8}
\]

Now we apply Keldysh’s quantum approach [5] to describe high harmonic generation near the cut-off \( E_{\text{max}} \). The exact time-dependent Schrödinger equation for an electron in the field of atomic potential \( V(r) \) and dipole potential \( rF(t) \) of the electromagnetic field is of the form

\[
i \frac{\partial \psi(r,t)}{\partial t} = \left\{ -\frac{1}{2} \Delta + V(r) + rF(t) \right\} \psi(r,t). \tag{9}
\]

The average semi-classical electron dipole moment is determined as

\[
r(t) = -\langle \psi(r,t) | r | \psi(r,t) \rangle. \tag{10}
\]

The photon energy spectrum is proportional to the square of the Fourier component of this quantity \( N\omega \) is the photon frequency, \( N \) is the harmonic number:

\[
r(N) = \left( \frac{\omega}{2\pi} \right) \langle \langle \psi(r,t) | r | \psi(r,t) \rangle \exp(iN\omega t) \rangle \]. \tag{11}
\]

A more exact quantum approach to spontaneous radiation at high harmonic generation is considered numerically in [6]. Our goal is to present a qualitative description only.

The final state is approximated by the unperturbed ground atomic state \( \phi_0(r,t) \). We neglect small field perturbation of this state since the laser field is assumed to be small compared to the atomic field value. It is also known (see [7]) that spontaneous transitions to the excited atomic states from the high-energy continuum states can be neglected in comparison to the transition into the ground state. Thus, one obtains from equation (11):

\[
r(N\omega) \approx -\int \langle \phi_0(r,t) | r | \psi(r,t) \rangle \exp(iN\omega t) \rangle \]. \tag{12}
\]

We assume that \( N\omega \gg I_p \). Because of the factorization of three steps originally proposed semiempirically in [8], we are not interested here in the first step, i.e. the tunneling process that results in standard tunneling expression \( \exp\left(-\frac{(2\mu)^{1/2}}{2p}\right) \) in the wave function. The second step is described by the Volkov wave function of an electron before collision with the atomic core [9]:
ψ(\textbf{r}, t) \rightarrow \Phi(t, t_i) = \exp\left\{-i \int_{t_i}^{t} E(t') \, dt'\right\}.

Here, \(E(t)\) is the electron energy when this electron moves back to the atomic core. It is seen from equation (12) that significant values of \(\textbf{r}\) are of the order of atomic size; therefore \(\textbf{r} \rightarrow 0\) in the Volkov function, equation (13).

Substituting equation (13) into equation (12), one obtains
\[
\mathbf{r}(N) \sim \int_{-\infty}^{\infty} \exp\left(-i \int_{t_i}^{t} E(t') \, dt' + iN\omega t\right) \, dt.
\]

(14)

It follows from equations (8) and (14) that near cut-off the high harmonic generation is determined by the dipole moment
\[
\mathbf{r}(N) \sim \int_{-\infty}^{\infty} \exp\left(i(N\omega - 3.173U_p) t + \frac{1}{4} 1.223U_p \omega^2 t^2\right) \, dt
\]

\[= 2\text{Ai}\left(\frac{hN\omega - 3.173U_p}{[1.223U_p \omega^2]^{1/3}}\right).
\]

(15)

Here \(\text{Ai}(x)\) is the Airy function. We restored the Planck constant in equation (15). Analogous dependence was obtained in [10] in the case of the 1D zero-range-potential model.

Instead of the sharp cut-off of the classical approach we have a quantum exponential decrease of high harmonic generation when \(hN\omega > 3.173U_p\):
\[
\mathbf{r}(N) \sim \exp\left\{-2 \frac{(hN\omega - 3.173U_p)^{3/2}}{3\omega [1.223U_p]^{1/2}}\right\}.
\]

(16)

Since \(hN\omega \gg I_p\), the cross section of atomic photo-recombination from high energy continuum states to the ground state can be estimated in the Wentzel–Kramers–Brillouin approximation using the Stobbe formula [7]
\[
\sigma \sim \frac{1}{(N\omega)^{3/2}}.
\]

(17)

Thus, the high harmonic spectrum is of the universal form which is independent of the atomic structure
\[
W(x_N) \sim \frac{1}{x_N^{3/2}} \text{Ai}^2\left(\frac{x_N - 1}{a}\right).
\]

(18)

We have \(a \ll 1\) since in the tunneling regime the Reiss parameter \(F^3/\omega^3 \gg 1\) [11]. It follows from the Keldysh condition \(F^3/\omega^3 \gg 1\) for tunneling and from the condition of low-frequency field \(I_p \gg \omega\). For example, at the tunneling ionization of atoms by the intense Nd:glass laser pulse with the field strength \(F = 0.1\) a.u. and frequency \(\omega = 1/27\) a.u. one finds that the parameter \(a = 0.25\).

The dependence, equation (18), is shown in figure 3 for the value of \(a = 0.25\). It is seen that high harmonic spectrum has a bell-shaped form. Of course, this is an envelope of the harmonic spectrum consisting of odd harmonic numbers \(N\). In addition, the maximum of this dependence is shifted remarkably to the lower value \(x_N = 0.65\), instead of \(x_N = 1\) (\(hN_{\text{max}}\omega = 3.173U_p\)). The reason is a strong decrease of the recombination cross section, equation (17), with the photon energy. These conclusions are in a good agreement with the experimental data [12].

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