Entropy in Poincaré gauge theory: Hamiltonian approach

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Abstract

The canonical generator of local symmetries in Poincaré gauge theory is constructed as an integral over a spatial section Σ of spacetime. Its regularity (differentiability) on the phase space is ensured by adding a suitable surface term, an integral over the boundary of Σ at infinity, which represents the asymptotic canonical charge. For black hole solutions, Σ has two boundaries, one at infinity and the other at horizon. It is shown that the canonical charge at horizon defines entropy, whereas the condition of regularity yields the first law of black hole thermodynamics.

1 Introduction

In the early 1960s, Kibble and Sciama [1] proposed a new theory of gravity, the Poincaré gauge theory (PG), based on gauging the Poincaré group of spacetime symmetries. By construction, PG is characterized by a Riemann-Cartan (RC) geometry of spacetime, in which both the torsion and the curvature are essential ingredients of the gravitational dynamics. Nowadays, PG is a well-established approach to gravity, representing a natural gauge-field-theoretic extension of general relativity (GR). For more details, see review articles by Hehl and coworkers [2], a reader with commentaries by Blagojević and Hehl [3], and monographs by Blagojević [4], Ponomariov et al. [5], and Mielke [6].

In the past half century, many investigations of PG have been aimed at clarifying different aspects of both geometric and dynamical role of torsion. In particular, successes in constructing exact solutions with torsion naturally raised the question of how their conserved charges are influenced by the presence of torsion; for a review, see Ref. [3]. Relying on these developments, we will reconsider the notion of conserved charge in the Hamiltonian formalism, as it represents the most natural basis for the main subject of the present paper, the influence of torsion on black hole entropy.

The expressions for the conserved charges in PG were first found for asymptotically flat solutions [7, 8]. The results obtained by Blagojević and Vasilić [8] are based on the

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Hamiltonian approach to PG \([9, 10]\) combined with the ideas of Regge and Teitelboim \([11]\). In this approach, the conserved charges are represented by a boundary term, defined by requiring the variation of the canonical gauge generator to be a well defined (differentiable) functional on the phase space. A covariant version of the Hamiltonian approach, introduced later by Nester \([12]\), turned out to be an important step in understanding the conservation laws. This was clearly demonstrated by Hecht and Nester \([13, 14]\), in their analysis of the conserved charges for asymptotically flat or (anti) de Sitter solutions. Further development of these ideas can be found in Nester and collaborators \([15]\), and a comprehensive exposition, incorporating the latest developments, is given by Chen et al. \([16]\).

In spite of such an intensive activity in exploring the notion of conserved charges in the generic four-dimensional (4D) PG, systematic studies of black hole entropy in the presence of torsion have been largely neglected in the literature. One should mention here an early and rather general proposal by Nester and collaborators \([15]\) which, however, did not prove to be quite successful. Investigations of black hole entropy in Refs. \([17]\) are restricted to a specific version of PG, the Einstein-Cartan theory, which is certainly not sufficient to justify any conclusion on the general relation between torsion and entropy. In 3D gravity, black hole entropy is well understood for solutions possessing the asymptotic conformal symmetry \([18, 19, 20]\), but for 4D black holes, such an approach is much less settled \([21]\).

The physics of black holes is an arena where thermodynamics, gravity and quantum theory are connected through the existence of entropy as an intrinsic dynamical aspect of black holes \([22]\). In the period around 1990s, understanding of the classical black hole entropy reached a level that can be best characterized by Wald’s words: “Black hole entropy is the Noether charge” \([23]\). The question that we wish to address in the present paper is whether such a challenging idea, transformed to a more natural canonical framework, can improve our understanding of black hole entropy in the generic PG.

The paper is organized as follows. In section 2, we give a short account of the Lagrangian formalism for PG, and discuss the notion of surface gravity. In section 3, we develop the canonical formulation of PG in the first order formalism and use it to construct the canonical gauge generator. In section 4, we use the improved form of the gauge generator to obtain the variational equation for the asymptotic canonical charge (energy and angular momentum), located at the spatial 2-boundary at infinity. Then, following the idea that “entropy is the canonical charge at horizon”, we are naturally led to define black hole entropy by the same variational equation, but located at black hole horizon. It is shown that the condition of differentiability of the gauge generator guarantees the validity of the first law of black hole thermodynamics. In sections 5 and 6, our results are tested on three illustrative examples, belonging to the family of spherically symmetric solutions of PG. Finally, section 7 is devoted to concluding remarks, and appendices contain important technical details.

Our conventions are as follows. The Latin indices \((i, j, \ldots)\) refer to the local Lorentz frame, the Greek indices \((\mu, \nu, \ldots)\) refer to the coordinate frame, \(b^i\) is the orthonormal tetrad (1-form), \(h_i\) is the dual basis (frame), so that \(h_i b^k = \delta^k_i\), and the Lorentz metric is \(\eta_{ij} = (1, -1, -1, -1)\). The volume 4-form is \(\hat{\epsilon} = b^0 b^1 b^2 b^3\), the Hodge dual of a form \(\alpha\) is \(\star \alpha\), with \(\star 1 = \hat{\epsilon}\), and totally antisymmetric tensor is defined by \(\star (b_i b_j b_m b_n) = \varepsilon_{ijmn}\), where \(\varepsilon_{0123} = +1\). The exterior product of forms is implicit, except in Appendix A.
2 Preliminaries: PG dynamics and surface gravity

As a preparation for discussing the notion of black hole entropy in PG, it is necessary to clarify to what extent is the existence of torsion compatible with the standard interpretation of surface gravity, introduced in GR.

2.1 A brief account of PG

Basic dynamical variables of PG are the tetrad field $b^i$ and the spin connection $\omega^{ij}$ (1-forms), the gauge potentials related to the translation and the Lorentz subgroups of the Poincaré group, respectively. The corresponding field strengths are the torsion $T^i = db^i + \omega^i{}_m b^m$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_m \omega^m{}^{ij}$ (2-forms), and the underlying spacetime continuum is characterized by a Riemann-Cartan (RC) geometry, see for instance [3, 4, 24].

Varying the gravitational Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (4-form) with respect to $b^i$ and $\omega^{ij}$ yields the gravitational field equations in vacuum. After introducing the covariant field momenta, $H_i := \frac{\partial L_G}{\partial T^i}$ and $H_{ij} := \frac{\partial L_G}{\partial R^{ij}}$, and the associated energy-momentum and spin currents, $E_i := \frac{\partial L_G}{\partial b^i}$ and $E_{ij} := \frac{\partial L_G}{\partial \omega^{ij}}$, these equations can be written in a compact form as

$$\delta b^i : \quad \nabla H_i + E_i = 0, \quad (2.1a)$$
$$\delta \omega^{ij} : \quad \nabla H_{ij} + E_{ij} = 0. \quad (2.1b)$$

Explicit expressions for the gravitational currents read

$$E_i = h_i \mathcal{L}_G - (h_i T^m) H_m - \frac{1}{2} (h_i R^{mn}) H_{mn},$$
$$E_{ij} = -(b_i H_j - b_j H_i). \quad (2.2)$$

Assuming the gravitational Lagrangian $L_G$ to be at most quadratic in the field strengths (quadratic PG) and parity invariant,

$$L_G = -^* (a_0 R + 2\Lambda) + T^i \sum_{n=1}^{3} * (a_n{}^{(n)} T_i) + \frac{1}{2} R^{ij} \sum_{n=1}^{6} * (a_n{}^{(n)} R_{ij}), \quad (2.3)$$

the gravitational field momenta take the form

$$H_i = 2 \sum_{m=1}^{3} * (a_m{}^{(m)} T_i), \quad (2.4a)$$
$$H_{ij} = -2 a_0 * (b^i b^j) + H'_{ij}, \quad H'_{ij} := 2 \sum_{n=1}^{6} * (b_n{}^{(n)} R_{ij}). \quad (2.4b)$$

Here, $(a_0, a_m, b_n)$ are the Lagrangian parameters, $\Lambda$ is a cosmological constant, and $^{(m)}T_i$ and $^{(n)}R_{ij}$ are the irreducible parts of the torsion and the curvature, see Appendix A.

In the presence of matter, the right-hand sides of (2.1a) and (2.1b) contain the corresponding matter currents.
2.2 Surface gravity

A black hole can be described as a region of spacetime which is causally disconnected from the rest of spacetime. The causal structure of spacetime is most naturally characterized with the help of null geodesics. The boundary of black hole is a null hypersurface, known as the event horizon.

To introduce surface gravity, consider a black hole characterized by the existence of a Killing vector field $\xi$. Then, a null hypersurface to which the Killing vector is normal, is called the Killing horizon ($\mathcal{K}$). As a consequence, $\xi^2 := g_{\mu\nu} \xi^\mu \xi^\nu = 0$ on $\mathcal{K}$. Then, since the gradient $\partial_\mu (\xi^2)$ is also normal to $\mathcal{K}$, it must be proportional to $\xi$, 

$$\partial_\mu (\xi^2) = -2\kappa \xi_\mu,$$  

(2.5)

where the scalar function $\kappa$ is known as surface gravity [22, 25]. Physical interpretation of surface gravity requires the Killing horizon of a black is to be its event horizon. One can show, without making use of any field equations, that indeed, for a wide class of stationary black holes (systems in “equilibrium”), Killing horizon coincides with event horizon.

The essential property of surface gravity is expressed by the zeroth law of black hole mechanics: for a wide class of stationary black holes, surface gravity is constant over the entire event horizon. For $\kappa \neq 0$, event horizon in the maximally extended spacetime is a branch of a bifurcate Killing horizon. Again, these results can be derived without using any field equations.

Since null geodesics and Killing vector fields are purely metric notions, they can be directly transferred to PG. Thus, the form of surface gravity (2.5) and the associated zeroth law of black mechanics, are not specific for GR, they are also valid in the framework of PG. This aspect of surface gravity was clearly recognized by Chen and Nester [15].

The calculation of $\kappa$ from (2.5) should be done in coordinates that are well defined on the outer horizon. In particular, the metric of static and spherically symmetric black holes in the ingoing Edington-Finkelstein coordinates reads

$$ds^2 = N^2 dv^2 - 2dvdr - r^2 d\Omega^2, \quad N = N(r),$$  

(2.6)

the Killing vector is $\xi = \partial_v$, and the definition (2.5) of surface gravity takes the form

$$\partial_r N^2 = 2\kappa.$$  

(2.7)

3 Hamiltonian analysis of PG

In GR, classical black hole entropy can be interpreted as the Noether charge on horizon [23, 26]. In order to examine this idea in the framework of PG, we find it natural to rely on the Hamiltonian approach, where the canonical charge is derived from the improved form of the gauge generator.

3.1 First order Lagrangian

In PG, the conserved charges (energy, momentum and angular momentum) are determined as the values of the (improved) canonical generators of spacetime symmetries, associated to
suitable asymptotic conditions [3][4]. The canonical procedure is simplified by transforming
the quadratic Lagrangian (2.3) into the “first order” form [12]
\[ L_G = T^i \tau_i + \frac{1}{2} R^{ij} \rho_{ij} - V(b^i, \tau_i, \rho_{ij}), \]
where both the gravitational potentials \((b^i, \omega^{ij})\) and the corresponding “covariant momenta”
\((\tau_i, \rho_{ij})\), are independent dynamical variables. The potential \(V\) is a quadratic function of
\((\tau_i, \rho_{ij})\) which ensures the on-shell relations \(\tau_i = H_i\) and \(\rho_{ij} = H_{ij}\), see Appendix [3].

In the tensor formalism, the Lagrangian density reads
\[ \tilde{L}_G = -\frac{1}{4} \varepsilon^{\mu \nu \lambda \rho} \left( T^{i \mu \nu} \tau_i^{\lambda \rho} + \frac{1}{2} R^{ij \mu \nu} \rho_{ij \mu \nu} \right) - \tilde{V}(b, \tau, \rho). \]
The gravitational field equations (in vacuum) are obtained by varying \(\tilde{L}_G\) with respect to the independent dynamical variables \(b^i, \omega^{ij}, \tau^{i \mu \nu}, \rho^{ij \mu \nu}\):
\[ \nabla_\mu \tau^{i \mu \nu} - \frac{\partial \tilde{V}}{\partial b^i_{\mu}} = 0, \tag{3.3a} \]
\[ 2b^j_{[j \mu} \tau^{i \mu \nu]} + \nabla_\mu \rho_{ij \mu \nu} = 0, \tag{3.3b} \]
\[ -T^{i \mu \nu} - \frac{\partial \tilde{V}}{\partial \tau_i^{\mu \nu}} = 0, \tag{3.3c} \]
\[ -R^{ij \mu \nu} - \frac{\partial \tilde{V}}{\partial \rho_{ij \mu \nu}} = 0. \tag{3.3d} \]
where we used the notation \(\tau^{i \mu \nu} := \frac{1}{2} \varepsilon^{\mu \nu \lambda \rho} \tau_i^{\lambda \rho}\), and similarly for \(\rho^{ij \mu \nu}\), \(T^{i \mu \nu}\) and \(R^{ij \mu \nu}\).

### 3.2 Primary constraints and Hamiltonians

Since the Lagrangian \(\tilde{L}_G\) describes a gauge invariant dynamical system, transition to the Hamiltonian formalism is characterized by the existence of constraints [27][3]. Starting with the field variables \(\varphi^A = (b^i, \omega^{ij}, \tau^{i \mu \nu}, \rho^{ij \mu \nu})\) and the corresponding canonical momenta \(\pi_A = (\pi^i, \pi^{ij \mu \nu}, P_i^{\mu \nu}, P_{ij}^{\mu \nu})\), one obtains the following primary constraints:
\[ \phi_i^0 := \pi_i^0 \approx 0, \quad \phi_i^\alpha := \pi_i^\alpha + \tau_i^{0 \alpha} \approx 0, \]
\[ \phi_{ij}^0 := \pi_{ij}^0 \approx 0, \quad \phi_{ij}^\alpha := \pi_{ij}^\alpha + \frac{1}{2} \rho_{ij}^{0 \alpha} \approx 0, \]
\[ P_i^{\mu \nu} \approx 0, \quad P_{ij}^{\mu \nu} \approx 0. \tag{3.4} \]
The canonical Hamiltonian is found to have the form
\[ H_c' = H_c + \partial^a D^a, \]
\[ H_c := b^i_0 \mathcal{H}_i + \frac{1}{2} \omega^{ij}_0 \mathcal{H}_{ij} + \tau^0_0 \mathcal{T}^{0 \alpha} + \frac{1}{2} \rho_{ij}^{0 \alpha} * R^{ij0 \alpha} + \tilde{V}, \tag{3.5a} \]
where
\[ \mathcal{H}_i := \nabla_\alpha \tau^{0 \alpha}_i, \]
\[ \mathcal{H}_{ij} := 2b_{[j \alpha} \tau^{0 \alpha}_{i]} + \nabla_\alpha \rho_{ij}^{0 \alpha}, \]
\[ D^\alpha := -b^i_0 \tau^{0 \alpha}_i - \frac{1}{2} \omega^{ij}_0 \rho_{ij}^{0 \alpha}. \tag{3.5b} \]
Time evolution of dynamical variables is determined by the total Hamiltonian

$$H_T := H_c + u^i_\mu \phi_i^\mu + \frac{1}{2} u^{ij}_\mu \phi_{ij}^\mu + \frac{1}{2} v^i_{\mu \nu} P_i^{\mu \nu} + \frac{1}{4} v^{ij}_{\mu \nu} P_{ij}^{\mu \nu}. \quad (3.6)$$

### 3.3 Consistency conditions

The dynamical evolution of the primary constraints $X_A = -(\pi_i^0, \pi_{ij}^0, P_i^{0\alpha}, P_{ij}^{0\alpha})$ is defined by the consistency conditions $\dot{X}_A = \{X_A, H_T\} \approx 0$. They produce the secondary constraints

$$\hat{\mathcal{H}}_i := \mathcal{H}_i + \frac{\partial \mathcal{V}}{\partial b_i^0} \approx 0,$$

$$\hat{\mathcal{H}}_{ij} := \mathcal{H}_{ij} \approx 0,$$

$$\hat{T}^{i0\alpha} := \Gamma^{i0\alpha} + \frac{\partial \mathcal{V}}{\partial \tau_{i0\alpha}} \approx 0,$$

$$\hat{R}^{ij0\alpha} := \mathcal{R}^{ij0\alpha} + \frac{\partial \mathcal{V}}{\partial \rho_{ij0\alpha}} \approx 0, \quad (3.7)$$

which correspond to certain components of the field equations (3.3).

The remaining primary constraints $Y_A = (\phi_i^\alpha, \phi_{ij}^\alpha, P_i^{\alpha \beta}, P_{ij}^{\alpha \beta})$ are second class, as follows from their Poisson brackets

$$\{\phi_i^\gamma, P_j^{\alpha \beta}\} = \eta_{ij}^{} \varepsilon^{\gamma \alpha \beta \delta}, \quad \{\phi_{ij}^\alpha, P_{kl}^{\beta \gamma}\} = \eta_{ik}^{} \eta_{jl}^{} \varepsilon^{\alpha \beta \gamma \delta}. \quad (3.8)$$

Their consistency conditions determine the canonical multipliers $(u_i^\alpha, u_{ij}^\alpha, v_i^{\alpha \beta}, v_{ij}^{\alpha \beta})$. However, we find it more convenient to construct the corresponding Dirac brackets and use them in the consistency procedure on the reduced phase space $\bar{R}$, defined by $Y_A = 0$. The only nontrivial Dirac brackets (different from their Poisson counterparts) are

$$\{b_i^\alpha, \tau_{j\beta\gamma}\}^* = \delta_i^j \varepsilon_{0\alpha \beta \gamma}, \quad \{\omega_{ij}^\alpha, \rho_{kl}^{\beta \gamma}\}^* = \delta_k^i \delta_l^j \varepsilon_{0\alpha \beta \gamma}. \quad (3.9)$$

The form of the total Hamiltonian on $\bar{R}$ is simplified:

$$H_T = H_c + u^i_0 \pi_i^0 + \frac{1}{2} u^{ij}_0 \pi_{ij}^0 + v^i_{0\beta} P_i^{0\beta} + \frac{1}{2} v^{ij}_{0\beta} P_{ij}^{0\beta}. \quad (3.10)$$

Note that $H_c$ can be expressed also in terms of the secondary constraints (3.7)

$$H_c = b^i_0 \mathcal{H}_i + \frac{1}{2} \omega^{ij}_0 \mathcal{H}_{ij} + \tau_{i0\alpha} \hat{T}^{i0\alpha} + \frac{1}{2} \rho_{ij0\alpha} \hat{R}^{ij0\alpha}. \quad (3.11)$$

which follows from the $\mu = 0$ component of the identity

$$\delta^0_{\mu} \mathcal{V} = b^i_{\mu} \frac{\partial \mathcal{V}}{\partial b_i^0} + \tau_{i\mu\alpha} \frac{\partial \mathcal{V}}{\partial \tau_{i0\alpha}} + \frac{1}{2} \rho_{ij\mu\alpha} \frac{\partial \mathcal{V}}{\partial \rho_{ij0\alpha}}. \quad (3.12)$$

Continuing this procedure, one would have to find the consistency conditions of the secondary constraints (3.7), and so on. However, since our main goal is to construct the gauge generator, we shall follow a simpler approach, described in Appendix C. Once the gauge generator is found, one can construct its improved form [4, 11], which defines not only the standard canonical charge, but also black hole entropy.
4 Entropy and torsion

The Hamiltonian formulation of gravity is based on the existence of a family of spacelike hypersurfaces $\Sigma$, labelled by the time parameter $t$. Each $\Sigma$ is bounded by a closed 2-surface at spatial infinity, which is used to define the asymptotic charge. When $\Sigma$ is a black hole manifold, it also possesses an “interior” boundary, the horizon, which serves to define black hole entropy [23, 26].

4.1 Canonical charge as a surface term at infinity

In PG, conserved charges (energy-momentum and angular momentum) are closely related to the canonical gauge generator $G$, the general form of which is given in Eq. (C.1). Since $G$ acts on dynamical variables via the Poisson (or Dirac) bracket operation, it should have well-defined functional derivatives on the phase space. In general, $G$ does not satisfy this requirement, but the problem can be solved by adding a suitable surface term $\Gamma_\infty$, located at the boundary of $\Sigma$ at infinity, such that $\tilde{G} = G + \Gamma_\infty$ is well defined. The value of $\Gamma_\infty$ is exactly the canonical charge of the system [3, 4].

Before continuing with the construction of $\Gamma_\infty$, one should clarify the importance of asymptotic conditions. Local symmetries of PG are characterized by a Killing-Lorentz pair $(\xi^\mu, \theta^{ij})$, where $\xi^\mu$ is a Killing vector that corresponds to local spacetime translations, and $\theta^{ij}$ describes local Lorentz rotations. Any particular solution of PG is characterized by a set of asymptotic conditions for basic dynamical variables. Demanding that local Poincaré transformations preserve these conditions, one obtains certain restrictions on the Killing-Lorentz parameters. The restricted parameters define the asymptotic symmetry, which is essential for the existence and type of conserved charges.

As we mentioned above, the boundary term $\Gamma_\infty$ is introduced so as to ensure the differentiability of the gauge generator $G = G[\xi, \theta]$. To see how this happens, consider the variation of the gauge generator (C.1)

$$\delta G = \int_\Sigma d^3x (\delta G_1 + \delta G_2),$$

$$\delta G_1 = \xi^\mu \left[ b_{i}^\mu \delta \hat{H}_i + \frac{1}{2} \omega^{ij}_{\mu} \delta H_{ij} + \tau_{ij\alpha} \delta \bar{T}^{ij0\alpha} + \frac{1}{2} \rho_{ij\mu\alpha} \delta \bar{R}_{ij0\alpha} \right] + R,$$

$$\delta G_2 = \frac{1}{2} \theta^{ij} \delta H_{ij} + R,$$  \hfill (4.1)

where $\delta$ is the variation over the set of asymptotic states, and $R$ denotes regular (differentiable) terms. Next, using the relations (3.7) one obtains

$$\delta G_1 = \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \xi^\mu \left[ b_{i}^\mu \nabla_\alpha \delta \tau_{ij\beta\gamma} + \frac{1}{2} \omega^{ij}_{\mu} \nabla_\alpha \delta \rho_{ij\beta\gamma} + 2 \tau_{ij\mu\gamma} \nabla_\alpha \delta b_{i}^{\mu} + \rho_{ij\mu\gamma} \nabla_\alpha \delta \omega^{ij}_{\beta\gamma} \right] + R,$$

$$\delta G_2 = \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \left[ \frac{1}{2} \theta^{ij} \nabla_\alpha \delta \rho_{ij\beta\gamma} \right] + R.$$

To get rid of the unwanted $\delta \partial_\mu \varphi$ terms which spoil the differentiability of $G$, one can perform
a partial integration, which yields
\[
\delta G_1 = \frac{1}{2} \varepsilon^{0\alpha\beta\gamma} \partial_\alpha \left\{ \xi^\mu \left[ b^i_\mu \delta \tau_{i\beta\gamma} + \frac{1}{2} \omega^{ij}_\mu \delta \rho_{ij\beta\gamma} + 2 \tau_{\mu\nu} \delta b^i_\beta + \rho_{ij\nu} \delta \omega^{ij}_\beta \right] \right\} + R ,
\]
\[
\delta G_2 = \frac{1}{2} \varepsilon^{0\alpha\beta\gamma} \partial_\alpha \left[ \frac{1}{2} \theta^{ij}_\delta \delta \rho_{ij\beta\gamma} \right].
\]
Integration with the help of Stockes' theorem transforms the above expressions to boundary integrals. Going over to the notation of differential forms \((\varepsilon^{0\alpha\beta\gamma} d^3x \to -dx^\alpha dx^\beta dx^\gamma)\), the result takes the form
\[
\delta G = -\delta \Gamma_\infty + R ,
\]
\[
\delta \Gamma_\infty := \oint_{S_\infty} \delta B ,
\]
\[
\delta B := (\xi \mathcal{J} b^i) \delta H_i + \delta b^i (\xi \mathcal{J} H_i) + \frac{1}{2} (\xi \mathcal{J} \omega^{ij}) \delta H_{ij} + \frac{1}{2} \delta \omega^{ij} (\xi \mathcal{J} H_{ij})
\]
\[
+ \frac{1}{2} \theta^{ij} \delta H_{ij} ,
\]
where \(S_\infty\) is the boundary of \(\Sigma\) at infinity.

If the adopted asymptotic conditions ensure \(\Gamma_\infty\) to be a finite solution of the variational equation (4.2), the improved gauge generator
\[
\tilde{G} := G + \Gamma_\infty
\]
has well-defined functional derivatives. Then, since \(G \approx 0\), the value of \(\tilde{G}\) is effectively given by the value of \(\Gamma_\infty\), which represents the canonical charge at infinity. In the particular case \(\xi = \partial_t, \theta^{ij} = 0\), the canonical charge is energy, \(E = \Gamma_\infty[\partial_t, \theta^{ij} = 0]\).

The variation of \(\Gamma_\infty\) is defined over the set of asymptotic states, leaving the background configuration fixed. In practical applications of this instruction, keeping the background configuration fixed might be sensitive to the choice of coordinates. Nester and collaborators \([15, 16]\) succeeded to explicitly construct a set of finite expressions \(\Gamma_\infty\), which have been tested on a rather wide class of asymptotic conditions. Although their approach yields highly reliable expressions for the conserved charges, we shall continue using the variational approach (4.2), as it can be naturally extended to a new definition of black hole entropy.

### 4.2 Entropy as the canonical charge on horizon

In order to interpret black hole entropy as the canonical charge on horizon, we assume that the boundary of \(\Sigma\) has two components, one at spatial infinity and the other at horizon, \(\partial \Sigma = S_\infty \cup S_H\). In an early application of this idea to PG, Nester and collaborators \([15]\) introduced entropy from a kind of the Hamiltonian conservation law, but with a limited success. The same idea was used later in \([19]\) to obtain entropy of the BTZ black hole.

Given the spatial hypersurface \(\Sigma\) with two boundaries, the condition of differentiability of the canonical generator \(G\) includes two boundary terms, the integrals of \(\delta B = \delta B(\xi, \theta)\) over \(S_\infty\) and \(S_H\):
\[
\delta G = -\oint_{S_\infty} \delta B + \oint_{S_H} \delta B + R .
\]
The sign change in the second term is due to a different orientation of $S_H$. Here, as we already know, the first term represents the asymptotic canonical charge,

$$\delta \Gamma_\infty = \int_{S_\infty} \delta B ,$$

whereas the second one defines entropy $S$ as the *canonical charge on horizon*,

$$\delta \Gamma_H := \oint_{S_H} \delta B .$$

The variation over horizon is performed by varying the parameters of a solution, but keeping surface gravity constant, in accordance with the zeroth law. Explicit form of entropy depends on two factors:

(f1) dynamical and geometric properties of a theory (Riemannian GR, Riemann-Cartan PG, teleparallel theory, etc.), and

(f2) specific structure of the black hole (static, stationary, etc.)

For stationary black holes in GR, the entropy formula (4.6) takes the well-known form

$$\delta \Gamma_H = T \delta S ,$$

where $T = \kappa / 2\pi$ represents the temperature and $S = \pi r_+^2$ is black hole entropy.

Returning to Eq. (4.4), one concludes that the gauge generator $G$ is regular if the sum of two boundary terms vanishes,

$$\delta \Gamma_\infty - \delta \Gamma_H = 0 ,$$

which is nothing but the *first law of black hole thermodynamics*. Thus, the validity of the first law directly follows from the regularity of the original gauge generator $G$.

In the framework of PG, the conserved charge is a well established concept which has been calculated for a number of exact solutions [8, 13, 14]. In contrast to that, much less is known about black hole entropy. In the next two sections, we will test our definition of black hole entropy (4.6) and the associated first law (4.8), on three illustrative examples from the family of Schwarzschild-AdS solutions.

5 **Riemannian Schwarzschild-AdS solution**

All exact solutions of GR are also solutions of PG, except for some degenerate cases [24]. However, certain properties of a solution may change when we go from GR to a new dynamical environment of PG. As the first test of our results for black hole entropy and the first law, we discuss the case of the Riemannian Schwarzschild-AdS black hole in PG.

We shall often use the notation $i = (A, c)$, where $A = 0, 1$ and $c = 2, 3$.  
5.1 Geometry

Riemannian geometry of the Schwarzschild-AdS spacetime is defined by the metric

\[ ds^2 = N^2 dt^2 - \frac{dr^2}{N^2} - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) , \quad N^2 := 1 - \frac{2m}{r} + \lambda r^2 , \quad (5.1) \]

where \( \lambda > 0 \). The zeros of \( N^2 \) determine the event horizon

\[ \lambda r^3 + r - 2m = 0 . \quad (5.2a) \]

Since the discriminant of this cubic equation is negative, \( \Delta = -4\lambda - 108\lambda^2 m^2 < 0 \), the equation has just one real root \( r_+ \). The relation

\[ 2m = r_+ (\lambda r_+^2 + 1) . \quad (5.2b) \]

implies that \( r_+ \) is positive iff \( m > 0 \), and \( N^2 \) is positive in the region \( r > r_+ \), where the Schwarzschild-like coordinates are well defined. The surface gravity and black hole temperature are obtained from Eq. (2.7) as

\[ \kappa = \frac{1}{2r_+} (3\lambda r_+^2 + 1) , \quad T = \frac{\kappa}{2\pi} . \quad (5.3) \]

The orthonormal tetrad is chosen to be

\[ b^0 = Ndt , \quad b^1 = \frac{dr}{N} , \quad b^2 = r d\theta , \quad b^3 = r \sin \theta d\varphi , \quad (5.4) \]

and the horizon area is

\[ A = \int_{S_{H}} b^2 b^3 = 4\pi r_+^2 . \quad (5.5) \]

The Riemannian connection reads

\[ \omega^{01} = -N' b^0 , \quad \omega^{1c} = \frac{N}{r} b^c , \quad \omega^{23} = \frac{\cos \theta}{r \sin \theta} b^3 , \quad (5.6) \]

and the corresponding curvature 2-form \( R^{ij} \) has two nonvanishing irreducible pieces

\[ \begin{align*}
(6) \ R^{ij} &= \lambda b^i b^j , \\
(1) \ R^{01} &= -\frac{2m}{r^3} b^0 b^1 , \\
(1) \ R^{23} &= -\frac{2m}{r^3} b^2 b^3 , \\
(1) \ R^A_c &= \frac{m}{r^3} b^A b^c . \end{align*} \quad (5.7) \]

The covariant momenta are \( H_i = 0 \) and

\[ \begin{align*}
H_{01} &= -2b^2 b^3 \left( a_0 + 2b_1 \frac{m}{r^3} - b_6 \lambda \right) , \\
H_{23} &= -2b^0 b^1 \left( a_0 + 2b_1 \frac{m}{r^3} - b_6 \lambda \right) , \\
H_{Ac} &= -\varepsilon_{Acmn} b^m b^n \left( a_0 - b_1 \frac{m}{r^3} - b_6 \lambda \right) . \end{align*} \quad (5.8) \]

By using the PG field equations (2.1), one can show that the Riemannian Schwarzschild-AdS spacetime is an exact solution of PG, provided that

\[ 3a_0 \lambda + \Lambda = 0 . \quad (5.9) \]
5.2 Entropy and the first law

Energy of the Riemannian Schwarzschild-AdS solution in PG can be calculated from the variational formula (4.2c) for $\xi = \partial_t$ and $\theta^{ij} = 0$. The result is given by (Appendix D)

$$E = 16\pi A_0 m, \quad A_0 := a_0 + \lambda(b_1 - b_0). \quad (5.10)$$

For energy at horizon, the variational formula (4.6) defines entropy as follows:

$$\delta \Gamma_H = \oint_{S_H} \omega^{01} \delta H_{01} = 8\kappa A_0 \delta (\pi r_+^2), \quad (5.11a)$$

$$\Rightarrow \quad \delta \Gamma_H = T \delta S, \quad S = 16\pi A_0 (\pi r_+^2). \quad (5.11b)$$

Since Eqs. (5.2b) and (5.3) imply $2\delta m = \kappa \delta r_+^2$, we have $\delta E = \delta \Gamma_H$, which confirms the validity of the first law

$$\delta E = T \delta S. \quad (5.12)$$

The presence of the multiplicative factor $A_0 \neq a_0$ shows that entropy of the Schwarzschild-AdS black hole in PG, as well as the first law, agrees with the corresponding result for diffeomorphism invariant Riemannian theories, see Wald [23] and Jacobson [26].

5.3 Reduction to GR

The GR limit is recovered for $b_1 = b_6 = 0, A_0 = a_0$ and $16\pi a_0 = 1$:

$$E = m, \quad S = \pi r_+^2. \quad (5.13)$$

6 Schwarzschild-AdS solutions with torsion

As is well known from our experience with GR, exact solutions have an essential role in revealing hidden aspects of the gravitational dynamics. The dynamics of the (parity invariant) PG is defined by a Lagrangian with ten coupling constants, which makes the search for exact solutions a rather complicated task. In spite of that, many of the known GR solutions have been successfully generalized to the corresponding solutions with torsion; for more details, see the reader [3]. In this section, we shall examine two spherically symmetric solutions.

6.1 Baekler solution

One of the first spherically symmetric solutions of PG has been constructed by Baekler [28]. We shall use this solution to verify our approach to entropy in the presence of torsion.

Formulation of the model

The metric of the Baekler solution is of the Schwarzschild-AdS form, with the tetrad field given as in Eq. (5.4). Ansatz for torsion is assumed to be $O(3)$ invariant (rotations and reflections) [29]. More specifically,

$$T^0 = T^1 = f b^0 b^1, \quad T^c = - f (b^0 - b^1) b^c. \quad (6.1a)$$
where $f$ is a function of $r$,

$$f := -\frac{m}{r^2 N}.$$  

(6.1b)

The third irreducible component of $T^i$ vanishes, $(^3)T^i = 0$.

Having adopted the anzatz for torsion, one can calculate the Riemann-Cartan connection

$$\omega^{01} = -(N' + f)b^0 + fb^1, \quad \omega^{0c} = -fb^c,$$

$$\omega^{1c} = \left(\frac{N}{r} - f\right)b^c, \quad \omega^{23} = \frac{\cos \theta}{r \sin \theta} b^3,$$

(6.2)

whereupon the curvature 2-form turns out to have only two nonvanishing irreducible parts,

$$R^{ij} = \lambda b^i b^j, \quad R^{Ac} = \frac{\lambda m}{r N^2} (b^0 - b^1) b^c.$$

Dynamics is determined by a two-parameter PG Lagrangian

$$L_G = a_1 T^i \ast (1^{(1)} T_i - 2^{(2)} T_i + ^3 T_i) + \frac{1}{2} b_1 R^{ij} R_{ij},$$

(6.3)

proposed by von der Heyde [30]. The field equations (2.1) produce the following restriction on the Lagrangian parameters:

$$2\lambda b_1 = -a_1.$$  

(6.4)

**Entropy and the first law**

The form of surface gravity is the same as in Eq. (5.3). Explicit expressions for the covariant momenta $H_{ij} = 2b_1 \ast R_{ij}$ and $H_i = 2a_1 \ast (1^{(1)} T_i - 2^{(2)} T_i)$ reads:

$$H_{01} = -a_1 b^2 b^3, \quad H_{23} = -a_1 b^0 b^1,$$

$$H_{02} = a_1 b^1 b^3 - a_1 \frac{m}{r N^2} (b^0 - b^1) b^3,$$

$$H_{03} = -a_1 b^1 b^2 + a_1 \frac{m}{r N^2} (b^0 - b^1) b^2,$$

$$H_{12} = -a_1 b^0 b^3 + a_1 \frac{m}{r N^2} (b^0 - b^1) b^3,$$

$$H_{13} = a_1 b^0 b^2 - a_1 \frac{m}{r N^2} (b^0 - b^1) b^2,$$

$$H_0 = -H_1 = 4a_1 \frac{m}{r^2 N} b^2 b^3.$$  

(6.5)

Energy of the solution is proportional to $m$ (Appendix D):

$$E = 16\pi a_1 m.$$  

(6.6)

Entropy is calculated from the variational equation (4.6) (integration implicitly understood):

$$b_i^t \delta H_i = -4 \left[N \delta (fr^2)\right]_{r^+} \cdot 4\pi a_1,$$

$$\frac{1}{2} \omega^{ij} t \delta H_{ij} = \omega^{01} t \delta H_{01} = (\kappa + N f_x) \delta r^2_+ \cdot 4\pi a_1,$$

$$\frac{1}{2} \delta \omega^{ij} H_{ij} = \left[-2fr^2 \delta N + 2N \delta (fr^2) - \frac{N f_x \delta r^2}{r^+}\right]_{r^+} \cdot 4\pi a_1.$$  

(6.7)
Summing up these terms, one can identify black hole entropy:

\[
\delta \Gamma_H = 8\pi a_1 \kappa \delta r_+^2 = T \delta S , \quad S := 16\pi a_1 \delta (\pi r_+^2) .
\] (6.8)

Then, the identity \(2 \delta m - \kappa \delta r_+^2 = 0\) implies the validity of the first law:

\[
\delta \Gamma_\infty = \delta \Gamma_H \quad \Rightarrow \quad \delta E = T \delta S .
\] (6.9)

With the normalization \(16\pi a_1 = 1\), both energy and entropy reduce to the corresponding GR expressions. However, since torsion sector gives a nontrivial contribution to entropy, see the first line in (6.7), *dynamical content* of the result is quite different. The approach of Refs. [15] yields a different expression for entropy, which is incompati ble with the first law.

### 6.2 Schwarzschild-AdS solution in teleparallel gravity

Teleparallel gravity (TG) is a subcase of PG, defined by the vanishing Riemann-Cartan curvature, \(R^{ij} = 0\). Choosing the related spin connection to vanish, \(\omega^{ij} = 0\), the tetrad field remains the only dynamical variable, and torsion takes the form \(T^i = db^i\). The general (parity invariant) TG Lagrangian has the form

\[
L_T := a_0 T^i \cdot \left( a_1^{(1)} T_i + a_2^{(2)} T_i + a_3^{(3)} T_i \right) .
\] (6.10a)

In physical considerations, a special role is played by a special *one-parameter family* of TG Lagrangians [3, 4], defined by a single parameter \(\gamma\) as

\[
a_1 = 1 , \quad a_2 = -2 , \quad a_3 = -1/2 + \gamma .
\] (6.10b)

This family represents a viable gravitational theory for macroscopic matter, empirically indistinguishable from GR.

Every spherically symmetric solution of GR is also a solution of the one-parameter TG. In particular, this is true for the Schwarzschild-AdS spacetime. Since \(^{(3)}T_i = 0\), the covariant momentum \(H^i\) does not depend on \(\gamma\):

\[
H^0 = \frac{2a_0}{r \sin(\theta)} \left[ \cos(\theta)b^1 b^3 - 2N \sin(\theta)b^2 b^3 \right] , \\
H^1 = \frac{2a_0}{r \sin(\theta)} b^0 b^3 , \\
H^2 = -\frac{2a_0}{r} (rN' + N)b^0 b^3 , \\
H^3 = \frac{2a_0}{r} (rN' + N)b^0 b^2 .
\] (6.11)

The energy of the Schwarzschild-AdS solution in TG is (Appendix D):

\[
E = m .
\] (6.12)

When the only nontrivial covariant momentum is \(H_i\), the entropy formula proposed in Refs. [15] does not work. Our approach to entropy yields (integration implicit)

\[
b^i \delta H_i = \left[ N \delta H_0 \right]_{r_+} = -16\pi a_0 \left[ N \delta (Nr) \right]_{r_+} = 0 , \\
b^i \delta H_{it} = \left[ b^2 \delta H_{2t} + b^2 \delta H_{3t} \right]_{r_+} = 8\pi a_0 \cdot \kappa \delta (r_+^2) ,
\] (6.13a)
where we used \( NN' = \kappa \) and \([N\delta N]_{r+} = 0\). Thus, with \( 16\pi a_0 = 1 \), one obtains

\[
\delta \Gamma_H = T \delta S, \quad S = \pi r_+^2.
\]  

(6.13b)

The identity \( 2\delta m = \kappa \delta r_+^2 \) confirms the validity of the first law

\[
\delta E = T \delta S.
\]  

(6.14)

7 Concluding remarks

In the present paper, we investigated the notion of entropy in the general (parity preserving) four-dimensional PG. Our approach is based on the idea that black hole entropy can be interpreted as the conserved charge on horizon, see Wald [23].

To examine this idea in the framework of the Hamiltonian formalism, we constructed the canonical generator \( G \) of gauge symmetries as an integral over the spatial section \( \Sigma \) of spacetime. Since \( G \) acts on the dynamical variables via the Poisson bracket operation, it has to be a regular (differentiable) functional on the phase space. An analysis along the lines proposed by Regge and Teitelboim [11] shows that regularity can be ensured by adding to \( G \) a suitable surface term \( \Gamma_\infty \), defined on the boundary of \( \Sigma \) at infinity. Combined with suitable boundary conditions, the form of \( \Gamma_\infty \) is determined by the variational equation (4.2) and its value defines the asymptotic charge of a PG solution.

For a black hole solution, \( \Sigma \) has two boundaries, one at infinity and the other at horizon, and the condition of regularity of \( G \) includes two boundary terms, \( \Gamma_\infty \) and \( \Gamma_H \). The new boundary term \( \Gamma_H \), determined by the variational equation (4.6), defines entropy as the canonical charge on horizon. Moreover, the regularity of \( G \) is expressed by the condition \( \Gamma_\infty - \Gamma_H = 0 \), which is just the first law of black hole thermodynamics.

We tested our results on three vacuum solutions of the Schwarzschild-AdS type. Treating the Riemannian Schwarzschild-AdS geometry as a solution of PG, we found that both energy and entropy differ from the GR expressions by a multiplicative factor, in agreement with earlier results [23, 26]. In the next step, we analyzed Baekler’s solution [28], one of the first exact solution found in PG. The result reveals new dynamical features of PG, the existence of nontrivial contributions to energy and entropy stemming from both the torsion and the curvature sectors. It is astonishing to see how these two sectors interfere to produce the final result which is exactly the same as in GR. In the last example we successfully applied our approach to the teleparallel gravity, where curvature vanishes and entropy is produced solely by torsion, respecting the first law.

An additional test of our approach to black hole entropy can be obtained from the analysis of Kerr black hole [32].

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A Irreducible decomposition of the field strengths

We present here formulas for the irreducible decomposition of the PGT field strengths in a 4D Riemann–Cartan spacetime. The torsion 2-form has three irreducible pieces:

\[(2) T_i = \frac{1}{3} b^i \wedge (h_m \wedge T^m), \]
\[(3) T_i = \frac{1}{3} h^i \wedge (T^m \wedge b_m), \]
\[(1) T_i = T_i - (2) T_i - (3) T_i. \]  \hspace{1cm} (A.1)

The RC curvature 2-form can be decomposed into six irreducible pieces:

\[(2) R^{ij} = *(b^i \wedge \Psi^j), \quad (4) R^{ij} = b^i \wedge \Phi^j, \]
\[(3) R^{ij} = \frac{1}{12} X^* (b^i \wedge b^j), \quad (6) R^{ij} = \frac{1}{12} F b^i \wedge b^j, \]
\[(5) R^{ij} = \frac{1}{2} b^i \wedge h^j \wedge (b^m \wedge F_m), \quad (1) R^{ij} = R^{ij} - \sum_{a=2}^{6} (a) R^{ij}. \] \hspace{1cm} (A.2a)

where

\[F^i := h_m \wedge R^{mi} = Ric^i, \quad F := h_i \wedge F^i = R, \]
\[X^i := *(R^{ik} \wedge b_k), \quad X := h_i \wedge X^i. \] \hspace{1cm} (A.2b)

and

\[\Phi_i := F_i - \frac{1}{4} h_i F - \frac{1}{2} h_i \wedge (b^m \wedge F_m), \]
\[\Psi_i := X_i - \frac{1}{4} b_i X - \frac{1}{2} h_i \wedge (b^m \wedge X_m). \] \hspace{1cm} (A.2c)

These formulas differ from those in Refs. [2] and [24] in two minor details: the definitions of \(F^i\) and \(X^i\) are taken with an additional minus sign, but simultaneously, the overall signs of all the irreducible curvature parts are also changed, leaving their final content unchanged.

B Explicit form of \(V\)

The torsion part of the potential \(V\) can be written in the form [20]

\[V_{\tau^2} := \tau^i \left( \mu_1 (1)^* \tau_i + \mu_2 (2)^* \tau_i + \mu_3 (3)^* \tau_i \right), \quad \mu_n = -\frac{1}{4a_n}. \] \hspace{1cm} (B.1)

Indeed, since the variation of the Lagrangian (3.1) with respect to \(\tau^i\) yields

\[T_i = 2(\mu_1 (1)^* \tau_i + \mu_2 (2)^* \tau_i + \mu_3 (3)^* \tau_i), \] \hspace{1cm} (B.2)

one obtains

\[a_n (n)^* T_i \Rightarrow -2 \sum_n a_n (n)^* T_i = * \tau_i \Rightarrow H_i = \tau_i, \] \hspace{1cm} (B.3)
as expected. Moreover, this result implies
\[ V_{\tau^2} = \frac{1}{2} \tau^i T_i \quad \Rightarrow \quad L_T = T^i \tau_i - \frac{1}{2} \tau^i T_i = \frac{1}{2} T^i H_i. \] (B.4)

Next, note that the duality properties
\[
\begin{align*}
(1)^* \tau_i &= \star (1)^* \tau_i, \\
(2)^* \tau_i &= \star (3) \tau_i, \\
(3)^* \tau_i &= \star (2) \tau_i,
\end{align*}
\] (B.5)
lead to an equivalent form of \( V_{\tau^2} \):
\[ V_{\tau^2} = \tau^i \star \left( \mu_1^{(1)} \tau_i + \mu_3^{(2)} \tau_i + \mu_2^{(3)} \tau_i \right), \] (B.6)

Transition to the tensor formalism yields
\[ \hat{V}_{\tau^2} = \frac{1}{2b} \tau^{imn} \left( \mu_1^{(1)} \tau_{imn} + \mu_3^{(2)} \tau_{imn} + \mu_2^{(3)} \tau_{imn} \right) \]
\[ = \frac{1}{b} \tau^{imn} \left( h_1 \tau_{imn} + h_2 \tau_{min} + h_3 \eta_{im} \tau_n \right), \quad \tau_n := \tau^k \tau_n, \] (B.7)
where \( h_1 = (2\mu_1 + \mu_2)/6, \) \( h_2 = (\mu_1 - \mu_2)/3, \) and \( h_3 = -(\mu_1 - \mu_3)/3. \)

An analogous construction holds also in the curvature sector.

C  Canonical gauge generator

The canonical generator of local Poincaré transformations can be constructed following Castellani’s algorithm [10], based on the knowledge of the Poisson bracket algebra of the first class constraints. However, there is an alternative procedure formulated in Ref. [10], according to which a phase-space functional \( G \) is a good gauge generator if it generates the correct gauge transformations of all phase-space variables. Relying on an explicit construction of \( G \) in the three-dimensional PG [20], we display here its generalization to 4D:

\[
G[\xi, \theta] = \int_{\Sigma} d^3x \left( G_1 + G_2 \right),
\]
\[
G_1 = \dot{\xi}^i \left( b^i_{\mu} \pi^0_{i\mu} + \frac{1}{2} \omega^{ij}_{\mu} \pi^0_{ij\mu} + \tau^i \mu_{\beta} P_i^{0\beta} + \frac{1}{2} \rho^{ij}_{\mu\beta} P_{ij}^{0\beta} \right) + \xi^\mu \mathcal{P}_\mu,
\]
\[
G_2 = \frac{1}{2} \theta^{ij} \pi^0_{ij} + \frac{1}{2} \theta^{ij} \mathcal{M}_{ij},
\] (C.1a)
where
\[
\mathcal{P}_\mu := b^i_{\mu} \mathcal{H}_i + \frac{1}{2} \omega^{ij}_{\mu} \mathcal{H}_{ij} + \tau^i \mu_{\beta} \mathcal{T}^{0\beta}_i + \frac{1}{2} \rho^{ij}_{\mu\beta} \mathcal{R}_{ij}^{0\beta}
\]
\[ + \left( \partial_\mu b^i_{0} \right) \pi^0_{i\mu} + \frac{1}{2} \left( \partial_\mu \omega^{ij}_{0} \right) \pi^0_{ij\mu} + \left( \partial_\mu \tau^i_{0\beta} \right) P_i^{0\beta} + \frac{1}{2} \left( \partial_\mu \rho^{ij}_{0\beta} \right) P_{ij}^{0\beta}
\]
\[ - \partial_\beta \left( \tau^i_{0\mu} P_i^{0\beta} + \frac{1}{2} \rho^{ij}_{0\mu} P_{ij}^{0\beta} \right),
\]
\[
\mathcal{M}_{ij} := \mathcal{H}_{ij} + 2b_{[i0} \pi_{j]}^0 + 2\omega^k_{[i0} \pi_{j]k}^0 + 2\tau_{[i0\gamma} P_{j]}^{0\gamma} + 2\rho^k_{[i0\gamma} P_{j]k}^{0\gamma},
\] (C.1b)
Now, we wish to show that the action of $G$ on the phase-space variables $\varphi$, defined by
\[ \delta_0 \varphi := \{ \varphi, G \}^*, \]
has the correct PG form. Note that the generator $P_0$ is just the total Hamiltonian, $P_0 = H_T$, as follows from (3.11); hence, $\{ \varphi, P_0 \}^* = \dot{\varphi}$.

To find the transformation laws generated by the action of $G = G_1 + G_2$, we start with the variables $(b^i_\alpha, \omega^{ij}_\alpha, \tau^i_{\alpha\beta}, \rho^{ij}_{\alpha\beta})$. For $b^i_\alpha$, the result has the form
\[ \delta_0 b^i_\alpha = -\theta^k i b^k_\alpha + (\partial_\alpha \xi^\mu) b^i_\mu + \xi^\mu \partial_\mu b^i_\alpha + \xi^\mu X^i_{\alpha\mu}, \]
whereas the transformation law for $\omega^{ij}_\alpha$ reads
\[ \delta_0 \omega^{ij}_\alpha = \nabla_\alpha \theta^{ij} + (\partial_\alpha \xi^\mu) \omega^{ij}_\mu + \xi^\mu \partial_\mu \omega^{ij}_\alpha + \xi^\mu Y^{ij}_{\alpha\mu}, \]
\[ Y^{ij}_{\alpha\mu} := R^{ij}_{\alpha\mu} - \frac{1}{2} \varepsilon_{\alpha\mu\lambda\rho} \frac{\partial \tilde{\mathcal{V}}}{\partial \tau^{ij}_{\lambda\rho}}. \]

An analogous calculation for $\tau^i_{\alpha\beta}$ yields
\[ \delta_0 \tau^i_{\alpha\beta} = -\theta^k i \tau^k_{\alpha\beta} + \partial_\beta \xi^\mu \tau^i_{\alpha\mu} + \partial_\alpha \xi^\mu \tau^i_{\alpha\mu} + \xi^\mu \partial_\mu \tau^i_{\alpha\beta} + \xi^\mu Z^i_{\alpha\beta\mu}, \]
\[ Z^i_{\alpha\beta\mu} := \nabla_\alpha \tau^i_{\beta\mu} + \nabla_\mu \tau^i_{\alpha\beta} + \nabla_\beta \tau^i_{\mu\alpha} + \varepsilon_{\alpha\beta\mu\nu} \frac{\partial \tilde{\mathcal{V}}}{\partial b^{i\nu}}. \]

Since the field equations (3.3) imply $X^i_{\alpha\mu}, Y^{ij}_{\alpha\mu}, Z^i_{\alpha\beta\mu} = 0$, one concludes that the above transformations have the correct on-shell form. The same conclusion holds for $\rho^{ij}_{\alpha\beta}$.

Next, the gauge transformations of the pair $(\tau_{i0\alpha}, P_{i0\alpha})$ are given by
\[ \delta_0 \tau_{i0\alpha} = -\theta^k k \tau_{k0\alpha} + (\partial_0 \xi^\mu) \tau_{i0\alpha} + (\partial_\alpha \xi^\beta) \tau_{0i\beta} + \xi^\mu \partial_\mu \tau_{i0\alpha}, \]
\[ \delta_0 P_{i0\alpha} = -\theta^k k P_{k0\alpha} - (\partial_0 \xi^\mu) P_{i0\alpha} - (\partial_\beta \xi^\alpha) P_{i0\beta} + \xi^\mu \partial_\mu (\xi^\mu P_{i0\alpha}). \]

These transformations, as well as those found for the pair $(\rho^{ij}_{0\beta}, P_{ij}^{0\beta})$ and the remaining momentum variables, are of the expected form (all the momenta are tensor densities).

The above results confirm that the expression (C.1) is the correct canonical generator of gauge transformations in PG.

## D Energy for Schwarzschild-AdS solutions

In this Appendix, we present the calculations of energy for the Schwarzschild-AdS solutions in vacuum, discussed in the main text. The results are found using the variational formula (4.2) with $\xi = \partial_t$ and $\theta^{ij} = 0$. The AdS asymptotics allows the variation of the mass parameter $m$, whereas $\lambda$ remains fixed.

Starting with the asymptotic formulas
\[ N' N = \lambda r + O_2, \quad N \delta N = -\frac{\delta m}{r}, \]
we conclude that the above expressions are of the correct form.
one finds that the energy of the Riemannian Schwarzscild-AdS solution in PG is determined by the following nonvanishing contributions to $\delta \Gamma_\infty$ (integration is implicit):

$$\frac{1}{2} \omega^{ij} \delta H_{ij} = \omega^{01} \delta H_{01} = 4\pi N' N \left( 4b_1 \frac{\delta m}{r} \right) = 16\pi \lambda b_1 \delta m ,$$

$$\frac{1}{2} \delta \omega^{ij} H_{ijt} = \delta \omega^{1c} H_{1ct} = -16\pi (\delta N) N r (a_0 - \lambda b_0) = 16\pi (a_0 - \lambda b_0) \delta m \, . \quad (D.2a)$$

Summing up, one obtains

$$E := \Gamma_\infty = 16\pi [a_0 + \lambda (b_1 - b_0)] m \, . \quad (D.2b)$$

Analogously, the calculation for Baekler’s solution with torsion yields

$$b^i t \delta H_i = N \delta H_0 = N \delta \left( 4a_1 \frac{m}{r^2 N} b^2 b^3 \right) = 16\pi a_1 \delta m \, ,$$

$$\Rightarrow \quad E = 16\pi a_1 m \, . \quad (D.3)$$

Finally, for the Schwarzscild-AdS solution in the one-parameter TG, the expression for energy takes the GR form

$$b^i t \delta H_i = N \delta H_0 = -16\pi a_0 r N \delta N = \delta m \, ,$$

$$\Rightarrow \quad E = m \, , \quad (D.4)$$

where we used the normalization $16\pi a_0 = 1$.

The above results for energy are identical to those obtained using the formulas of Nester and collaborators [13, 16].

References

[1] T. W. B. Kibble, Lorentz invariance and the gravitational field, J. Math. Phys. 2 (1961) 212-221;

D. W. Sciama, On the analogy between charge and spin in general relativity, in: Recent Developments in General Relativity, Festschrift for Infeld (Pergamon Press, Oxford; PWN, Warsaw, 1962) pp. 415439.

[2] F. W. Hehl, Four lectures on Poincaré gauge theory, in Proc. 6th Course of the Int. School of Cosmology and Gravitation on Spin Torsion and Supergravity, eds. P. G. Bergmann and V. de Sabbatta (Plenum, New York, 1980).

F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. 258 (1995) 1-177.

[3] M. Blagojević and F. W. Hehl (eds.), Gauge Theories of Gravitation, A Reader with Commentaries (Imperial College Press, London, 2013).

[4] M. Blagojević, Gravitation and Gauge Symmetries (IoP, Bristol, 2002).
[5] V. N. Ponomariov, A. O. Barvinsky and Yu. N. Obukhov, *Gauge Approach and Quantization Methods in Gravity Theory* (Moscow, Nauka, 2017). This book offers an impressive list of 3136 references on gauge theories of gravity.

[6] E. W. Mielke, *Geometrodynamics of Gauge Fields*, 2nd. edition (Springer, Switzerland, 2017).

[7] K. Hayashi and T. Shirafuji, Energy, momentum and angular momentum in Poincaré gauge theory, Prog. Theor. Phys. **73** (1985) 54-74.

[8] M. Blagojević and M. Vasić, Asymptotic symmetry and conserved quantities in the Poincaré gauge theory of gravity, Class. Quantum Grav. **5** (1988) 1241-1257.

[9] M. Blagojević and I. Nikolić, Hamiltonian dynamics of Poincaré gauge theory: General structure in the time gauge, Phys. Rev. D **28** (1983) 2455-2463; I. Nikolić, Dirac Hamiltonian structure of $R+R^2+T^2$ Poincaré gauge theory of gravity without gauge fixing, Phys. Rev. D **30** (1984) 2508-2520.

[10] M. Blagojević, I. Nikolić, and M. Vasić, Local Poincaré generators of the $R+T^2+R^2$ theory of gravity, Il Nuovo Cim. **101** B (1988) 439-451.

[11] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Ann. Phys. (N.Y.) **88** (1974) 286-318.

[12] J. M. Nester, A covariant Hamiltonian for gravity theories, Mod. Phys. Lett. **6** (1991) 2655-2661; in: *Directions in general relativity*, Vol. I, eds. B. L. Hu, M. P. Ryan and C. V. Vishveshwara (Cambridge Univ. Press, Cambridge, 1993) pp. 245-260.

[13] R. Hecht and J. M. Nester, A new evaluation of PGT mass and spin, Phys. Lett A **180** (1993) 324-331.

[14] R. Hecht, Mass and spin of Poincaré gauge theory, Gen. Rel. Grav. **27** (1995) 537-554.

[15] C.-M. Chen and J. M. Nester, Quasi local quantities for GR and other gravity theories, Class. Quant. Grav. **16** (1999) 1279-1301; C.-M. Chen, J. M. Nester, and R.-S. Tung, Quasy-local energy-momentum for geometric gravity theories, Phys. Lett. A **203** (1995) 5-11.

[16] Ch.-M. Chen, J. M. Nester, and R.-S. Tung, Gravitational energy for GR and Poincaré gauge theory: A covariant Hamiltonian approach, Int. J. Mod. Phys. D **24** (2015) 1530026 (73 pages).

[17] S. Chakraborty and R. Dey, Noether Current, black hole entropy and spacetime torsion, Phys. Lett. B **786** (2018) 432-441.

K. Prabhu, The first law of black hole mechanics for fields with internal gauge freedom, Class. Quantum Grav. **34** (2017) 035011 (56 pages).

[18] M. Blagojević and B. Cvetković, Conformally flat black holes in Poincaré gauge theory, Phys. Rev. D **93** (2016) 044018 (9 pages).
[19] M. Blagojević and B. Cvetković, Black hole entropy in 3D gravity with torsion, Class. Quantum Grav. 23 (2006) 4781-4795.

[20] M. Blagojević and B. Cvetković, 3D gravity with propagating torsion: the AdS sector, Phys. Rev. D 85 (2012) 104003 (10 pages).

[21] S. Carlip, Effective conformal descriptions of black hole entropy, Entropy 13 (2011) 1355-1379, [arXiv:1107.2678].

[22] R. M. Wald, The thermodynamics of black holes, Living Rev. in Rel. (2001) 4:6. https://doi.org/10.12942/lrr-2001-6 (44 pages) [gr-qc/9912119];

[23] R. M. Wald, Black hole entropy is the Noether charge, Phys. Rev. D 48 (1993) 3427-3431.

[24] Yu. N. Obukhov, Poincaré gauge gravity: Selected topics, Int. J. Geom. Methods Mod. Phys. 3 (2006) 95-138.

[25] E. Poisson, A Relativist’s Toolkit, The Mathematics of Black-Hole Mechanics (Cambridge University Press, Cambridge, 2004).

[26] T. Jacobson, Black hole entropy and Lorentz-diffeomorphism Noether charge, Phys. Rev. D 92 (2015) 124010 (8 pages).

[27] P. A. M. Dirac, Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York, 1964);

[28] P. Baekler, A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials, Phys. Lett. 99 B (1981) 329-332.

[29] R. Rauch and H. T. Nieh, Birkhoff’s theorem for general Riemann-Cartan-type $R + R^2$ theories of gravity, Phys. Rev. 24 (1981) 2019-2048;
R.-S. Tung, Ch.-H. Chang, D.-Ch. Chern, and J. M. Nester, Asymptotic anti-de Sitter conditions for Poincaré gauge theory, Prog. Theor. Phys. 88 (1992) 291-305.

[30] P. von der Heyde, Is gravitation mediated by the torsion of spacetime? Z. Naturforsch. 31 a (1976) 1725-1726.

[31] L. Castellani, Symmetries of constrained Hamiltonian systems, Ann. Phys. (N.Y.) 143 (1982) 357-371.

[32] M. Blagojević and B. Cvetković, Hamiltonian approach to black hole entropy: Kerr solution, in preparation.