Liouville-type theorems for the fourth order nonlinear elliptic equation*

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Abstract: In this paper, we are concerned with Liouville-type theorems for the nonlinear elliptic equation

\[ \Delta^2 u = |x|^a |u|^{p-1} u \text{ in } \Omega, \]

where \( a \geq 0, \ p > 1 \) and \( \Omega \subset \mathbb{R}^n \) is an unbounded domain of \( \mathbb{R}^n, \ n \geq 5. \) We prove Liouville-type theorems for solutions belonging to one of the following classes: stable solutions and finite Morse index solutions (whether positive or sign-changing). Our proof is based on a combination of the Pohozaev-type identity, monotonicity formula of solutions and a blowing down sequence, which is used to obtain sharper results.

Keywords: Liouville-type theorem; stable or finite Morse index solutions; monotonicity formula; blowing down sequence

1 Introduction

The paper is devoted to the study of the following nonlinear fourth order elliptic equation

\[ \Delta^2 u = |x|^a |u|^{p-1} u \text{ in } \Omega \]  

(1.1)

where \( a \geq 0, \ p > 1 \) and \( \Omega \subset \mathbb{R}^n \) is an unbounded domain of \( \mathbb{R}^n, \ n \geq 5. \) We are interested in the Liouville-type theorems—i.e., the nonexistence of the solution \( u \) which is stable or finite Morse index, and the underlying domain \( \Omega \) is an arbitrarily unbounded domain of \( \mathbb{R}^n. \)

The idea of using the Morse index of a solution of a semilinear elliptic equation was first explored by Bahri and Lions [1] to obtain further qualitative properties of the

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solution. In 2007, Farina [6] made significant progress, and considered the Lane-Emden equation
\[- \Delta u = |u|^{p-1}u \text{ in } \Omega, \quad (1.2)\]
on bounded and unbounded domains of \( \Omega \subset \mathbb{R}^n \), with \( n \geq 2 \) and \( p > 1 \). Farina completely classified finite Morse index solutions (positive or sign-changing) in his seminal paper [6]. His proof makes a delicate application of the classical Moser iteration method. There exist many excellent papers to use the generalization of Moser’s iteration technique to discuss the harmonic and fourth-order elliptic equation. We refer to [3, 9, 16–18] and the reference therein.

However, the classical Moser’s iterative technique may fail to obtain the similarly complete classification for the biharmonic equation
\[ \Delta^2 u = |u|^{p-1}u \text{ in } \Omega \subset \mathbb{R}^n. \quad (1.3) \]
Recently, Dávila, Dupaigne, Wang and Wei [4] have derived a monotonicity formula for solutions of (1.3) to reduce the nonexistence of nontrivial entire solutions for the problem (1.3), to that of nontrivial homogeneous solutions, and gave a complete classification of stable solutions and those of finite Morse index solutions. We note that Pacard [10, 11] studied the partial regularity results for stationary weak solution of \(-\Delta u = u^p\) by the use of monotonicity formula.

Let us recall that for the Liouville-type theorems and properties of the subcritical case has been extensively studied by many authors. Gidas and Spruck have been investigated the optimal Liouville-type theorems in the celebrated paper [7]. Thus, the equation (1.2) has no positive solution if and only if
\[ p < \frac{n+2}{n-2} = +\infty, \text{ if } n \leq 2. \]
The supercritical case \( p > \frac{n+2}{n-2} \) is much less complete understood. Bidaut-Véron and Véron [2] proved the asymptotic behavior of positive solution of (1.2) by the use of the Bochner-Lichnerowicz-Weitzenböck formula in \( \mathbb{R}^n \).

On the other hand, that the understanding of the case \( a \neq 0 \) is less complete and is more delicate to handle than the case \( a = 0 \). In [7], Gidas and Spruck concluded that for \( a \leq -2 \), the equation
\[- \Delta u = |x|^a u^p, \text{ in } \Omega \quad (1.4)\]
has no positive solution in any domain $\Omega$ containing the origin. Recently, Dancer, Du and Guo have researched the asymptotical behavior of stable and finite Morse index solutions of (1.4), where $a > -2$ and $p < p(a^-)$, $p^- = \min\{0, a\}$.

The case $a > 0$ seems some difficult. Since the classical techniques and many properties may fail to deal with the corresponding equations. In 2012, Phan and Souplet used the delicate method in [15] to prove that if $n \geq 2$, $a > 0$, $1 < p < \frac{n + 2 + 2a}{n - 2}$ and $n = 3$, then the equation (1.4) has no positive bounded solution in $\Omega = \mathbb{R}^n$.

Meanwhile, adopting the similar method, Fazly and Ghoussoub proved the following result:

**Theorem A.** ([5, Theorem 3]) Let $n \geq 5$, $a \geq 0$ and $p > 1$. Then for any Sobolev subcritical exponent, i.e.,

$$1 < p < \frac{n + 4 + 2a}{n - 4},$$

the equation (1.1) has no positive solution with finite Morse index.

Inspired by the ideas in [4, 10], our purpose in this paper is to prove the Liouville-type theorems in the class of stable solution and finite Morse index solution. Thus for any fixed $a \geq 0$ and $n \geq 3$, we get

**Theorem 1.1.** If $u$ is a smooth stable solution of (1.1) in $\mathbb{R}^n$ and $1 < p < p_a(n)$, then $u \equiv 0$.

**Theorem 1.2.** Let $u$ be a smooth solution to (1.1) with finite Morse index.

- If $p \in (1, p_a(n))$, $p \neq \frac{n + 4 + 2a}{n - 4}$, then $u \equiv 0$.

- If $p = \frac{n + 4 + 2a}{n - 4}$, then $u$ has finite energy, i.e.,

$$\int_{\mathbb{R}^n} (\Delta u)^2 = \int_{\mathbb{R}^n} |x|^\alpha |u|^{p+1} < +\infty.$$

Here the representation of $p_a(n)$ in Theorem 1.1 and 1.2 is given by (2.1) below.

**Remark 1.1.**

1. Let us note that for any fixed $a \geq 0$ and $\frac{n + 4 + 4a}{n - 4} < p < p_a(n)$, we adopt a new method—a combination of monotonicity formula and blowing down sequence—to deal with the case and get Liouville-type theorem.

2. For the subcritical and critical cases, the proof is based on the combination of the Pohozaev identity with some integral and pointwise estimates obtained by the doubling lemma in [13, Lemma 5.1].
In contrast with the results of [5], our result is extended to the larger interval $(1, p_a(n))$ and the proof of method is different and independent interesting. For equation (1.1), we do not impose any sign condition for $u$ and extra restrictions on $n$, $a$ and $p$.

To describe our results more accurately, we need to make precise several terminologies.

- **Definition.** We recall that a critical point $u \in C^2(\Omega)$ of the energy functions

$$
\mathcal{L}(u) = \int_{\Omega} \frac{1}{2} |\Delta u|^2 dx - \frac{1}{p+1} \int_{\Omega} |x|^a |u|^{p+1} dx
$$

is said to be

(i) a stable solution of (1.1), if for any $\psi \in C^4_0(\Omega)$, we have

$$
\mathcal{L}_{uu}(\psi) := \int_{\Omega} |\Delta \psi|^2 dx - p \int_{\Omega} |x|^a |u|^{p-1} \psi^2 dx \geq 0.
$$

(ii) a solution $u$ of (1.1) with a Morse index equal to $l \geq 0$, if $l$ is the maximal dimension of a subspace $X_l$ of $C^l_0(\Omega)$ such that $\mathcal{L}_{uu}(\psi) < 0$ for all $\psi \in X_l \setminus \{0\}$. Therefore, $u$ is stable if and only if its Morse index is equal to zero.

(iii) a stable solution $u$ of (1.1) outside a compact set $\Gamma \subset \Omega$, if $\mathcal{L}_{uu}(\psi) \geq 0$ for any $\psi \in C^l_0(\Omega \setminus \Gamma)$. It follows that any finite Morse index solution $u$ is stable outside some compact set $\Gamma \subset \Omega$.

- **Notation.** Here and in the following, we use $B_r(x)$ to denote the open ball on $\mathbb{R}^n$ central at $x$ with radius $r$. we also write $B_r = B_r(0)$. $C$ denotes various irrelevant positive constants.

The organization of rest of the paper is as follows. In section 2, we construct a monotonicity formula which is a crucial tool to handle the supercritical case, and derive various integral estimates. Then we prove Liouville-type theorem for stable solutions of (1.1), this is Theorem 1.1 in Section 3. To prove the result, we first obtain the nonexistence of homogeneous, stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, where $p$ belongs to

$$
\left(\frac{n+4+2a}{n-4}, p_a(n)\right)
$$

(the representation of $p_a(n)$ in the below (2.1)). Secondly, we obtain some estimates of solutions, and show that the limit of blowing down sequence $u^\infty(x) = \lim_{\tau \to \infty} \tau^{\frac{n+4}{p-1}} u(\tau x)$ satisfies $E(r; 0, u) \equiv \text{const.}$ Here, we use the monotonicity formula of Theorem 2.1. In Section 4, we study Liouville-type theorem of finite Morse index solutions by the use of the Pohozaev-type identity, monotonicity formula and blowing down sequence.
2 A Monotonicity formula and some estimates

In this section, we construct a monotonicity formula which play an important role in dealing with the supercritical case, and obtain various integral estimates of stable solutions.

To explore the main results in this paper, we need to define a critical power of (1.1). For any fixed $a \geq 0$ and $n \geq 5$, we define the functions by

$$g(p) = p \left( \frac{4 + a}{p - 1} + 2 \right) \left( n - 4 - \frac{4 + a}{p - 1} \right) + p \frac{4 + a}{p - 1} \left( n - 2 - \frac{4 + a}{p - 1} \right),$$

$$f(p) = p \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} + 2 \right) \left( n - 4 - \frac{4 + a}{p - 1} \right) \left( n - 2 - \frac{4 + a}{p - 1} \right).$$

A direct computation finds

$$g \left( \frac{n + 4 + 2a}{n - 4} \right) = \frac{n + 4 + 2a}{n - 4} \times \frac{n(n - 4)}{2} > \frac{n(n - 4)}{2},$$

$$f \left( \frac{n + 4 + 2a}{n - 4} \right) = \frac{n + 4 + 2a}{n - 4} \times \frac{n^2(n - 4)^2}{16} > \frac{n^2(n - 4)^2}{16}.$$

and differentiating the function $f(p)$ in $p$, we get

$$f'(p) = 2p \frac{(4 + a)^2}{(p - 1)^3} \left[ n - 3 - \frac{4 + a}{p - 1} \right] - \frac{4 + a}{(p - 1)^2} \left( 6 + a + \frac{8 + 2a}{p - 1} \right) \left( n - 4 - \frac{4 + a}{p - 1} \right) \left( n - 2 - \frac{4 + a}{p - 1} \right).$$

It is easy to check that

$$f' \left( \frac{n + 4 + 2a}{n - 4} \right) = \frac{n^2(n - 4)^2}{16} > 0.$$

Let $n(a)$ be the integer part of the largest real root of the algebra equation

$$x^3 - 4x^2 - 32(a + 4)x + 64a + 256 = 0,$$

and $p(n, a)$ be the largest real root of the algebra equation

$$\left[ n^4 - 8n^3 - 16(2a + 7)n^2 + 192(a + 4)n - 256(a + 4) \right] x^4$$

$$- 4 \left[ n^4 + 8n^3 + 4(a^2 + 2a - 4)n^2 - 8(5a^2 + 22a + 8)n + 16(5a^2 + 28a + 32) \right] x^3$$

$$+ 2 \left[ 3n^4 - 24n^3 + 16(a^2 + 5a + 7)n^2 + 16(a^3 + 2a^2 - 14a - 24)n - 64(a^3 + 7a^2 + 14a + 8) \right] x^2$$

$$- 4 \left[ n^4 - 8n^3 + 4(a^2 + 6a + 12)n^2 + 8(a^3 + 7a^2 + 14a + 8)n + 4a(a^3 + 8a^2 + 20a + 16) \right] x$$

$$+ n^4 - 8n^3 + 16n^2 = 0.$$
For any fixed $a \geq 0$ and $n \geq 5$, we define
\[
p_a(n) = \begin{cases} 
+\infty, & \text{if } n \leq n(a), \\
p(n,a), & \text{if } n \geq n(a) + 1.
\end{cases}
\] (2.1)

Therefore, we find
\[
f(p) > \frac{n^2(n-4)^2}{16},
\]
for any $\frac{n + 4 + 2a}{n - 4} < p < p_a(n)$.

In particular, if $a = 0$, then $p_0(n)$ in (2.1) is the fourth order Joseph-Lundgren exponent which is computed by Gazzola and Grunau \[8\].

Furthermore, using the inequality $x + y \geq 2\sqrt{xy}$, for all $x, y \geq 0$, and combining with the definition of the functions $g(p)$ and $f(p)$, we obtain
\[
g(p) > \frac{n(n-4)}{2},
\]
for any $\frac{n + 4 + 2a}{n - 4} < p < p_a(n)$.

For any given $x \in \Omega$, let $0 < r < R$ and $B_r(x) \subset B_R(x) \subset \Omega$, we choose $u \in W^{4,2}_\text{loc}(\Omega)$ and $|x|^{a}|u|^{p+1} \in L^1_{\text{loc}}(\Omega)$ and define
\[
E(r; x, u) := r^{4(p+1)+2a} \int_{B_r(x)} \frac{1}{2} (\Delta u)^2 - \frac{1}{p+1} |x|^{a}|u|^{p+1}
\]
\[
+ \frac{4 + a}{2(p-1)} \left( n - 2 - \frac{4 + a}{p-1} \right) \frac{d}{dr} \left( r^{\frac{8 + 2a}{p-1} + 1-n} \int_{\partial B_r(x)} u^2 \right)
\]
\[
+ \frac{4 + a}{2(p-1)} \left( n - 2 - \frac{4 + a}{p-1} \right) \frac{d}{dr} \left( r^{\frac{8 + 2a}{p-1} + 2-n} \int_{\partial B_r(x)} u^2 \right)
\]
\[
+ \frac{r^3}{2} \frac{d}{dr} \left[ r^{\frac{8 + 2a}{p-1} + 1-n} \int_{\partial B_r(x)} \left( \frac{4 + a}{p-1} r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right]
\]
\[
+ \frac{1}{2} \frac{d}{dr} \left[ r^{\frac{8 + 2a}{p-1} + 1-n} \int_{\partial B_r(x)} \left( \nabla u^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right]
\]
\[
+ \frac{1}{2} r^{\frac{8 + 2a}{p-1} + 3-n} \int_{\partial B_r(x)} \left( \nabla u^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right). \tag{2.2}
\]

Then, we can investigate a monotonicity formula.

**Theorem 2.1.** Suppose that $n \geq 5$, $a \geq 0$ and $p > \frac{n + 4 + 2a}{n - 4}$, $u \in W^{4,2}_\text{loc}(\Omega)$ and $|x|^{a}|u|^{p+1} \in L^1_{\text{loc}}(\Omega)$ is a weak solution of (1.1). Then $E(r; x, u)$ is non-decreasing in $r \in (0, R)$. Furthermore, we have
\[
\frac{d}{dr} E(r; 0, u) \geq c(n, p, a) r^{-n+2+\frac{8 + 2a}{p-1}} \int_{\partial B_r} \left( \frac{4 + a}{p-1} r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \, dS, \tag{2.3}
\]
where the constant $c(n, p, a) > 0$ is only relevant to $n$, $p$ and $a$. 


Proof. We follow the lines of analysis process in [4] to prove the conclusion. From the variational of the equation (1.1), we define the rescaled energy function

\[ \hat{E}(\tau) := \tau^{\frac{4(p+1)\pm 2a}{p-1} - n} \int_{B_{\tau}} \frac{1}{2} (\Delta u)^2 - \frac{1}{p+1} |x|^a |u|^{p+1}. \]  

(2.4)

Denote

\[ v := \Delta u \]

and

\[ u^\tau (x) := \tau^{\frac{4+a}{p-1}} u(\tau x), \quad v^\tau (x) := \tau^{\frac{4+a}{p-1}+2} v(\tau x). \]

It is easy to check that

\[ v^\tau = \Delta u^\tau \quad \text{and} \quad \Delta v^\tau = |x|^a |u^\tau (x)|^{p-1} u(\tau x), \]

and taking the derivative of the first equality in \( \tau \) to get

\[ \frac{dv^\tau}{d\tau} = \Delta \frac{du^\tau}{d\tau}. \]

We observe that differentiation in \( \tau \) exchanges with differentiation and integration in \( x \).

Rescaling in (2.4) to yield

\[ \hat{E}(\tau) = \int_{B_1} \frac{1}{2} (v^\tau)^2 - \frac{1}{p+1} |x|^a |u^\tau|^{p+1}. \]

Differentiating the function \( \hat{E}(\tau) \) in \( \tau \), we obtain

\[
\begin{align*}
\frac{d\hat{E}(\tau)}{d\tau} & = \int_{B_1} \frac{d}{d\tau} \left( v^\tau \frac{dv^\tau}{d\tau} - |x|^a |u^\tau|^{p-1} u^\tau \frac{du^\tau}{d\tau} \right) \\
& = \int_{B_1} v^\tau \frac{\Delta u^\tau}{d\tau} - \Delta v^\tau \frac{du^\tau}{d\tau} \\
& = \int_{\partial B_1} v^\tau \frac{\partial u^\tau}{\partial r} \frac{du^\tau}{d\tau} - \frac{\partial v^\tau}{\partial r} \frac{du^\tau}{d\tau}. \quad (2.5)
\end{align*}
\]

In the following, all derivatives of \( u^\tau \) in the \( r = |x| \) variable will be expressed by the derivations in the \( \tau \) variable.

From the definition of \( u^\tau \) and \( v^\tau \), differentiating in \( \tau \) implies

\[ \frac{du^\tau}{d\tau}(x) = \frac{1}{\tau} \left[ \frac{4+a}{p-1} u^\tau(x) + r \frac{\partial u^\tau}{\partial r}(x) \right] \]

(2.6)

and

\[ \frac{dv^\tau}{d\tau}(x) = \frac{1}{\tau} \left[ \frac{2(p+1)+a}{p-1} v^\tau(x) + r \frac{\partial v^\tau}{\partial r}(x) \right]. \]

In (2.6), differentiating in \( \tau \) once again yields

\[ \frac{d^2 u^\tau}{d\tau^2} + \frac{du^\tau}{d\tau} = \frac{4+a}{p-1} \frac{du^\tau}{d\tau} + r \frac{\partial u^\tau}{\partial r} \frac{du^\tau}{d\tau}. \]
So we get
\[ r \frac{\partial}{\partial r} du^\tau = \tau \frac{d^2 u^\tau}{d\tau^2} + \frac{p - 5 - a}{p - 1} \frac{du^\tau}{d\tau}, \]
\[ r \frac{\partial v^\tau}{\partial r} = \tau \frac{dv^\tau}{d\tau} - \frac{2(p + 1) + a}{p - 1} v^\tau. \]

Inserting the above two equalities into (2.5), we find
\[
\frac{d\hat{E}}{d\tau} = \int_{\partial B_1} v^\tau \left( \tau \frac{d^2 u^\tau}{d\tau^2} + \frac{p - 5 - a}{p - 1} \frac{du^\tau}{d\tau} \right)
- \frac{du^\tau}{d\tau} \left( \tau \frac{dv^\tau}{d\tau} - \frac{2(p + 1) + a}{p - 1} v^\tau \right)
= \int_{\partial B_1} \tau v^\tau \frac{d^2 u^\tau}{d\tau^2} + 3v^\tau \frac{du^\tau}{d\tau} - \tau \frac{du^\tau}{d\tau} \frac{dv^\tau}{d\tau}.
\]

(2.7)

Now, we need to represent the function \( v^\tau \) by the use of a combination of \( u^\tau \) and the derivation of \( u^\tau \) in \( \tau \). Taking derivative of (2.6) in \( r \), we obtain on \( \partial B_1 \)
\[
\frac{\partial^2 u^\tau}{\partial r^2} = \tau \frac{\partial}{\partial r} \frac{du^\tau}{d\tau} - \frac{p + 3 + a}{p - 1} \frac{\partial u^\tau}{\partial r}
= \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \frac{p - 5 - a}{p - 1} \frac{du^\tau}{d\tau}
- \frac{p + 3 + a}{p - 1} \left( \frac{d^2 u^\tau}{d\tau^2} - \frac{4 + a}{p - 1} u^\tau \right)
= \tau^2 \frac{d^2 u^\tau}{d\tau^2} - \frac{8 + 2a}{p - 1} \frac{du^\tau}{d\tau} + \frac{(4 + a)(p + 3 + a)}{(p - 1)^2} u^\tau.
\]

Using spherical coordinates to write \( u^\tau(x) = u^\tau(r, \theta) \) with \( r = |x| \) and \( \theta = \frac{x}{|x|} \in S^{n-1} \), then on \( \partial B_1 \), we get
\[
v^\tau = \frac{\partial^2 u^\tau}{\partial r^2} + \frac{n - 1}{r} \frac{\partial u^\tau}{\partial r} + \frac{1}{r^2} \Delta_\theta u^\tau
= \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \left( n - 1 - \frac{8 + 2a}{p - 1} \right) \tau \frac{du^\tau}{d\tau}
+ \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right) u^\tau + \Delta_\theta u^\tau
= \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \tau \frac{du^\tau}{d\tau} + \gamma u^\tau + \Delta_\theta u^\tau,
\]

(2.8)

where \( \Delta_\theta \) is the Laplace-Beltrami operator on \( \partial B_1 \) and \( \nabla_\theta \) (see section 3) is the tangential derivative on \( \partial B_1 \), and
\[
\rho := n - 1 - \frac{8 + 2a}{p - 1}, \quad \gamma := \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right).
\]

(2.9)
Substituting (2.8) into (2.7), we have

\[
\frac{d}{d\tau} \hat{E}(\tau) = \int_{\partial B_1} \tau \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \frac{d^2 u^\tau}{d\tau^2} \\
+ 3 \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \frac{du^\tau}{d\tau} \\
- \tau \frac{du^\tau}{d\tau} \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \\
+ \int_{\partial B_1} \tau \Delta \theta u^\tau \frac{d^2 u^\tau}{d\tau^2} + 3 \Delta \theta u^\tau \frac{du^\tau}{d\tau} - \tau \frac{du^\tau}{d\tau} \Delta \theta \frac{du^\tau}{d\tau} \\
=: T_1 + T_2. \tag{2.10}
\]

The calculation for \(T_1\) is processed as follows

\[
T_1 = \int_{\partial B_1} \tau \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \frac{d^2 u^\tau}{d\tau^2} \\
+ 3 \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \frac{du^\tau}{d\tau} \\
- \tau \frac{du^\tau}{d\tau} \left( \tau^2 \frac{d^2 u^\tau}{d\tau^2} + \rho \frac{du^\tau}{d\tau} + \gamma u^\tau \right) \\
= \int_{\partial B_1} \tau^3 \left( \frac{d^2 u^\tau}{d\tau^2} \right)^2 + \tau^2 \frac{d^2 u^\tau}{d\tau^2} \frac{du^\tau}{d\tau} + \gamma \tau u^\tau \frac{d^2 u^\tau}{d\tau^2} + 3 \gamma u^\tau \frac{du^\tau}{d\tau} \\
+ (2\rho - \gamma) \tau \left( \frac{du^\tau}{d\tau} \right)^2 - \gamma \frac{d u^\tau}{d\tau} \frac{d^3 u^\tau}{d\tau^3} \\
= \int_{\partial B_1} 2\tau^3 \left( \frac{d^2 u^\tau}{d\tau^2} \right)^2 + 4\tau^2 \frac{d^2 u^\tau}{d\tau^2} \frac{du^\tau}{d\tau} + 2(\rho - \gamma) \tau \left( \frac{du^\tau}{d\tau} \right)^2 \\
+ \frac{\gamma}{2} \frac{d^2}{d\tau^2} \left[ (u^\tau)^2 \right] - \frac{1}{2} \frac{d}{d\tau} \left[ \tau^3 \frac{d}{d\tau} \left( \frac{du^\tau}{d\tau} \right)^2 \right] + \frac{\gamma}{2} \frac{d(u^\tau)^2}{d\tau} \\
\geq \int_{\partial B_1} \frac{\gamma}{2} \frac{d^2}{d\tau^2} \left[ (u^\tau)^2 \right] - \frac{1}{2} \frac{d}{d\tau} \left[ \tau^3 \frac{d}{d\tau} \left( \frac{du^\tau}{d\tau} \right)^2 \right] + \gamma \frac{d(u^\tau)^2}{d\tau}. \tag{2.11}
\]

Here choosing \(p > \frac{n + 4 + 2a}{n - 4}\), it implies that

\[
\rho - \gamma = \left( n - 1 - \frac{8 + 2a}{p - 1} \right) - \frac{4 + a}{p - 1} \left( \frac{4 + a}{p - 1} - n + 2 \right) > 1
\]

and

\[
2\tau^3 \left( \frac{d^2 u^\tau}{d\tau^2} \right)^2 + 4\tau^2 \frac{d^2 u^\tau}{d\tau^2} \frac{du^\tau}{d\tau} + 2(\rho - \gamma) \tau \left( \frac{du^\tau}{d\tau} \right)^2 \\
= 2\tau \left( \frac{d^2 u^\tau}{d\tau^2} + \frac{du^\tau}{d\tau} \right)^2 + 2(\rho - \gamma - 1) \tau \left( \frac{du^\tau}{d\tau} \right)^2 \\
\geq 0. \tag{2.12}
\]
Integrating by parts on $\partial B_1$, we get

\[
T_2 = \int_{\partial B_1} -\tau \nabla u^\tau \nabla \theta \frac{d^2 u^\tau}{d\tau^2} - 3 \nabla \theta u^\tau \nabla \theta \frac{du^\tau}{d\tau} + \tau \left| \frac{du^\tau}{d\tau} \right|^2 
\]

\[
= -\frac{\tau}{2} \frac{d^2}{d\tau^2} \int_{\partial B_1} |\nabla \theta u^\tau|^2 - 3 \frac{d}{d\tau} \int_{\partial B_1} |\nabla \theta u^\tau|^2 + 2\tau \int_{\partial B_1} \left| \frac{du^\tau}{d\tau} \right|^2 
\]

\[
= -\frac{1}{2} \frac{d^2}{d\tau^2} \left( \tau \int_{\partial B_1} |\nabla \theta u^\tau|^2 \right) - \frac{1}{2} \frac{d}{d\tau} \int_{\partial B_1} |\nabla \theta u^\tau|^2 + 2\tau \int_{\partial B_1} \left| \frac{du^\tau}{d\tau} \right|^2 
\]

\[
\geq -\frac{1}{2} \frac{d^2}{d\tau^2} \left( \tau \int_{\partial B_1} |\nabla \theta u^\tau|^2 \right) - \frac{1}{2} \frac{d}{d\tau} \int_{\partial B_1} |\nabla \theta u^\tau|^2 
\]

(2.13)

On the other hand, we observe that all terms in (2.11) and (2.13) by the use of the rescaling can be expressed as follows:

\[
\int_{\partial B_1} \frac{d}{d\tau} (\tau u^\tau)^2 = \frac{d}{d\tau} \left( \tau^{\frac{n+2a}{p-1}+1-n} \int_{\partial B_r} u^2 \right),
\]

\[
\int_{\partial B_1} \frac{d^2}{d\tau^2} [\tau (u^\tau)^2] = \frac{d^2}{d\tau^2} \left( \tau^{\frac{n+2a}{p-1}+2-n} \int_{\partial B_r} u^2 \right),
\]

\[
\int_{\partial B_1} \frac{d}{d\tau} \left[ \tau^3 \frac{d}{d\tau} (du^\tau)^2 \right] = \frac{d}{d\tau} \left[ \tau^3 \frac{d}{d\tau} \left( \tau^{\frac{n+2a}{p-1}+1-n} \int_{\partial B_r} \left( \frac{4 + a}{p - 1} \tau^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right) \right],
\]

\[
\frac{d}{d\tau} \left( \int_{\partial B_1} |\nabla \theta u^\tau|^2 \right) = \frac{d}{d\tau} \left[ \tau^{\frac{n+2a}{p-1}+3-n} \int_{\partial B_r} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right].
\]

Combining with (2.10)–(2.13), we obtain

\[
\frac{dE(\tau)}{d\tau} \geq \frac{\gamma}{2} \frac{d}{d\tau} \left( \tau^{\frac{n+2a}{p-1}+1-n} \int_{\partial B_r} u^2 \right) + \frac{\gamma}{2} \frac{d^2}{d\tau^2} \left( \tau^{\frac{n+2a}{p-1}+2-n} \int_{\partial B_r} u^2 \right)
\]

\[
- \frac{1}{2} \frac{d}{d\tau} \left[ \tau^3 \frac{d}{d\tau} \left( \tau^{\frac{n+2a}{p-1}+1-n} \int_{\partial B_r} \left( \frac{4 + a}{p - 1} \tau^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right) \right]
\]

\[
- \frac{1}{2} \frac{d}{d\tau} \left[ \tau^{\frac{n+2a}{p-1}+3-n} \int_{\partial B_r} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right]
\]

\[
- \frac{1}{2} \frac{d^2}{d\tau^2} \left[ \tau^{\frac{n+2a}{p-1}+4-n} \int_{\partial B_r} \left( |\nabla u|^2 - \left| \frac{\partial u}{\partial r} \right|^2 \right) \right].
\]

Therefore, combining the inequality with (2.2) and (2.4), we get the inequality (2.3).

From the properties of integration, we conclude that $E(r; x, u)$ is nonincreasing in $r \in (0, R)$. \qed

**Remark 2.1.** From (2.11)–(2.13), it implies that we can take

\[
c(n, p, a) = 2(\rho - \gamma - 1).
\]
In particular, if \( p = \frac{n + 4 + 2a}{n - 4} \), then \( c(n, p, a) = \frac{n^2 - 4n + 8}{2} > 0 \). Therefore, Theorem 2.1 also holds for \( p = \frac{n + 4 + 2a}{n - 4} \).

Corollary 2.1. If the hypotheses of Theorem 2.1 hold and \( E(\sigma; 0, u) \equiv \text{const} \), for all \( \sigma \in (0, R) \), then \( u \) is homogeneous in \( B_R \setminus \{0\} \), i.e.,

\[
    u(\tau x) = \tau^{-\frac{4 + a}{p - 4}} u(x), \quad \forall \tau \in (0, 1], \quad x \in B_R \setminus \{0\}. \tag{2.14}
\]

Proof. Taking arbitrarily \( r_1, r_2 \in (0, R) \) with \( r_1 < r_2 \), we obtain from Theorem 2.1 that

\[
    0 = E(r_2; 0, u) - E(r_1; 0, u) = \int_{r_1}^{r_2} \frac{d}{d\sigma} E(\sigma; 0, u) d\sigma \geq c(n, p, a) \int_{B_{r_2} \setminus B_{r_1}} \left( \frac{4 + a}{p - 4} \sigma^{-1} u + \frac{\partial u}{\partial \sigma} \right)^2 \frac{1}{|x|^{n - 2 - \frac{8 + 2a}{p - 4}}} dx.
\]

This implies

\[
    4 + a \sigma^{-1} u + \frac{\partial u}{\partial \sigma} = 0, \quad \text{a.e. in } B_R \setminus \{0\}.
\]

Hence for any fixed \( x \in B_R \setminus \{0\} \), we have

\[
    \frac{d}{d\tau} \left( \tau^{-\frac{4 + a}{p - 4}} u(\tau x) \right) \equiv 0, \quad \forall \tau \in (0, 1].
\]

Obviously, the equality (2.14) holds. \( \square \)

The following basic integral estimates for solutions (whether positive or sign-changing) of (1.1) follows from the rescaled test function method.

Lemma 2.1. \([18, \text{Lemma 2.3}]\) For any \( \zeta \in C^4(\mathbb{R}^n) \) and \( \eta \in C_0^\infty(\mathbb{R}^n) \), we have the following identities

\[
    \int_{\mathbb{R}^n} (\Delta^2 \zeta) \zeta \eta^2 dx = \int_{\mathbb{R}^n} [\Delta(\zeta \eta)]^2 dx + \int_{\mathbb{R}^n} [-4(\nabla \zeta \cdot \nabla \eta)^2 + 2\zeta \Delta \zeta |\nabla \eta|^2] dx \\
    + \int_{\mathbb{R}^n} \zeta^2 [2(\nabla (\Delta \eta) \cdot \nabla \eta + (\Delta \eta)^2)] dx \tag{2.15}
\]

and

\[
    2 \int_{\mathbb{R}^n} |\nabla \zeta|^2 |\nabla \eta|^2 dx = \int_{\mathbb{R}^n} [2\zeta (-\Delta \zeta) |\nabla \eta|^2 + \zeta^2 \Delta (|\nabla \eta|^2)] dx. \tag{2.16}
\]

Lemma 2.2. Let \( u \in C^4(\mathbb{R}^n) \) be a stable solution of (1.1). Then for large enough \( m \), we get that for all \( \psi \in C_0^4(\mathbb{R}^n) \) with \( 0 \leq \psi \leq 1 \)

\[
    \int_{\mathbb{R}^n} (|\Delta u|^2 + |x|^a |u|^{p+1}) \psi^{2m} \leq C \int_{\mathbb{R}^n} \left| x \right|^{-\frac{2a}{p - 4}} |G(\psi^m)|^{\frac{p + 1}{p - 4}},
\]

11
where $G(\psi^m) = |\nabla \psi|^4 + \psi^{2(2-m)} \left[ |\nabla(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2 | \right]$. 

Furthermore, we find 

$$\int_{B_R(x)} (|\Delta u|^2 + |x|^a |u|^{p+1}) \leq CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} u^2 + CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} |u \Delta u| \tag{2.17}$$ 

and 

$$\int_{B_R(x)} (|\Delta u|^2 + |x|^a |u|^{p+1}) \leq CR^{n-\frac{4(p+1)+2a}{p-1}} \tag{2.18}$$ 

for all $B_R(x)$. Here the constant $C$ does not depend on $R$ and $u$.

**Proof.** Since $u$ is a stable solution of (1.1), we choose arbitrarily $\zeta \in C_0^4(\mathbb{R}^n)$ and find 

$$\int_{\mathbb{R}^n} |x|^a |u|^{p-1} u \zeta = \int_{\mathbb{R}^n} \Delta u \Delta \zeta \tag{2.19}$$ 

and 

$$p \int_{\mathbb{R}^n} |x|^a |u|^{p-1} \zeta^2 \leq \int_{\mathbb{R}^n} |\Delta \zeta|^2. \tag{2.20}$$ 

Testing (2.19) on $\zeta = u \psi^2$ for $\psi \in C_0^4(\mathbb{R}^n)$, we obtain 

$$\int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^2 = \int_{\mathbb{R}^n} \Delta u \Delta (u \psi^2).$$ 

Testing (2.20) on $\zeta = u \psi$ to yield 

$$p \int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^2 \leq \int_{\mathbb{R}^n} [\Delta (u \psi)^2].$$ 

Combining the above two results with (2.15), we get 

$$(p - 1) \int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^2 \leq \int_{\mathbb{R}^n} [4(\nabla u \cdot \nabla \psi)^2 - 2u \Delta u |\nabla \psi|^2] \, dx 
+ \int_{\mathbb{R}^n} u^2 \left[2|\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 \right] \, dx. \tag{2.21}$$ 

Again a direct application of the identity (2.16) leads to 

$$\int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^2 \, dx \leq C \int_{\mathbb{R}^n} |uv||\nabla \psi|^2 \, dx 
+ C \int_{\mathbb{R}^n} u^2 \left[|\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 + |\Delta |\nabla \psi|^2 | \right] \, dx. \tag{2.21}$$ 

Since $\Delta(u \psi) = v \psi + 2 \nabla u \cdot \nabla \psi + u \Delta \psi$, we obtain 

$$\int_{\mathbb{R}^n} v^2 \psi^2 \, dx \leq C \int_{\mathbb{R}^n} |uv||\nabla \psi|^2 \, dx 
+ C \int_{\mathbb{R}^n} u^2 \left[|\nabla(\Delta \psi) \cdot \nabla \psi| + |\Delta \psi|^2 + |\Delta |\nabla \psi|^2 | \right] \, dx. \tag{2.22}$$
We take a cut-off function $\psi \in C^\infty_0(B_{2R}(x))$ such that $\psi \equiv 1$ in $B_R(x)$ and for $k \leq 3$, $|\nabla^k \psi| \leq \frac{C}{R^k}$. Combining (2.21) with (2.22), we get

$$\int_{B_{2R}(x)} (u^2 + |x|^a |u|^{p+1}) dx \leq CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} u^2 + CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} |uv|.$$

Noting that the constant $C$ does not depend on $R$ and $u$.

Next, the functions $\psi$ in (2.21) and (2.22) are replaced by $\psi^m$, where $m$ is a large integer. Then

$$\int_{\mathbb{R}^n} [ |x|^a |u|^{p+1} + v^2 ] \psi^{2m} dx \leq C \int_{\mathbb{R}^n} |uv| \psi^{2(m-1)} |\nabla \psi|^2 + C \int_{\mathbb{R}^n} u^2 \left[ |(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2 \right] dx.$$

A simple application of Young’s inequality yields

$$\int_{\mathbb{R}^n} |uv| \psi^{2(m-1)} |\nabla \psi|^2 \leq \frac{1}{2C} \int_{\mathbb{R}^n} v^2 \psi^{2m} + C \int_{\mathbb{R}^n} u^2 \psi^{2(m-2)} |\nabla \psi|^4.$$

Therefore we get

$$\int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^{2m} + v^2 \psi^{2m} dx \leq C \int_{\mathbb{R}^n} u^2 \psi^{2(m-2)} \psi^m G(\psi^m), \quad (2.23)$$

where $G(\psi^m) = |\nabla \psi|^4 + \psi^{2(2-m)} \left[ |(\Delta \psi^m) \cdot \nabla \psi^m| + |\Delta \psi^m|^2 + |\Delta |\nabla \psi^m|^2 \right]$. Utilizing Hölder’s inequality to find

$$\int_{\mathbb{R}^n} u^2 \psi^{2(m-2)} G(\psi^m) = \int_{\mathbb{R}^n} |x|^\frac{2a}{p+1} u^2 \psi^{2(m-2)} |x|^{-\frac{2a}{p+1}} G(\psi^m)$$

$$\leq \left( \int_{\mathbb{R}^n} |x|^a |u|^{p+1} \psi^{(m-2)(p+1)} \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^n} |x|^{-\frac{2a}{p+1}} G(\psi^m)^{\frac{p+1}{p+1}} \right)^{\frac{p+1}{p+1}}.$$

Choosing $m$ large enough such that $(m-2)(p+1) \geq 2m$, and combining with (2.23), we get

$$\int_{\mathbb{R}^n} (|\Delta u|^2 + |x|^a |u|^{p+1}) \psi^{2m} dx \leq C \int_{\mathbb{R}^n} |x|^{-\frac{2a}{p+1}} G(\psi^m)^{\frac{p+1}{p+1}},$$

where the constant $C$ only depends on $n, p, a, m$ and $\psi$. In the above inequality, we take a cut-off function $\psi \in C^\infty_0(B_{2R}(x))$ such that $\psi \equiv 1$ in $B_R(x)$ and $|\nabla^i \psi| \leq \frac{C}{R^i}$ for $i = 1, 2, 3$ once again. Then

$$\int_{B_R(x)} (|\Delta u|^2 + |x|^a |u|^{p+1}) dx \leq C \int_{B_{2R}(x) \setminus B_R(x)} |x|^{-\frac{2a}{p+1}} R^{-4} R^{\frac{p+1}{p+1}} dx$$

$$\leq CR^{n-\frac{4(p+1) + 2a}{p-1}}.$$

The proof is completed. \qed
3 Proof of Theorem 1.1

We first obtain a nonexistence result for homogeneous stable solution of (1.1).

**Theorem 3.1.** For any \( p \in \left( \frac{n+4+2a}{n-4}, p_a(n) \right) \), assume that \( u \in W^{2,2}_{\text{loc}}(\mathbb{R}^n\setminus\{0\}) \) is a homogeneous, stable solution of (1.1), and \( |x|^a |u|^{p+1} \in L^1_{\text{loc}}(\mathbb{R}^n\setminus\{0\}) \), where \( p_a(n) \) is given by (2.1). Then \( u \equiv 0 \).

**Proof.** From the conditions of Theorem, we can assume that there exists a \( w \in W^{2,2}(S^{n-1}) \) such that in polar coordinates

\[
u(r, \theta) = r^{\frac{4+a}{p-1}} w(\theta). \]

Since \( u \in W^{2,2}(B_2\setminus B_1) \) and \( |x|^a |u|^{p+1} \in L^1(B_2\setminus B_1) \), it implies that \( w \in W^{2,2}(S^{n-1}) \cap L^{p+1}(S^{n-1}) \). A straightforward calculation of (1.1) to get

\[
\Delta^2 w - \ell_1 \Delta w + \ell_2 w = |w|^{p-1} w, \tag{3.1}
\]

where

\[
\ell_1 = \left( \frac{4+a}{p-1} + 2 \right) \left( n-4 - \frac{4+a}{p-1} \right) + \frac{4+a}{p-1} \left( n-2 - \frac{4+a}{p-1} \right),
\]

\[
\ell_2 = \frac{4+a}{p-1} \left( \frac{4+a}{p-1} + 2 \right) \left( n-4 - \frac{4+a}{p-1} \right) \left( n-2 - \frac{4+a}{p-1} \right).
\]

From \( w \in W^{2,2}(S^{n-1}) \), multiplying (3.1) by \( w \) and integrating by parts imply

\[
\int_{S^{n-1}} |\Delta w|^2 + \ell_1 |\nabla w|^2 + \ell_2 w^2 = \int_{S^{n-1}} |w|^{p+1}. \tag{3.2}
\]

On the other hand, for any \( \epsilon > 0 \), we choose an \( \zeta_\epsilon \in C_0^\infty \left( (\frac{\epsilon}{2}, \frac{\epsilon}{\gamma}) \right) \) such that \( \zeta_\epsilon \equiv 1 \) in \( (\epsilon, \frac{1}{\gamma}) \) and

\[
|w|\zeta_\epsilon'(r) + r^2|\zeta_\epsilon''(r)| \leq C
\]

for all \( r > 0 \). Since \( u \) is a stable solution, we can choose a test function \( r^{-\frac{n-4}{2}} w(\theta) \zeta_\epsilon(r) \) and get

\[
p \int_{\mathbb{R}^n} |x|^a |u|^{p-1} \left( r^{-\frac{n-4}{2}} w(\theta) \zeta_\epsilon(r) \right)^2 dx \leq \int_{\mathbb{R}^n} \left| \Delta \left( r^{-\frac{n-4}{2}} w(\theta) \zeta_\epsilon(r) \right) \right|^2 dx.
\]

A simple calculation implies

\[
\Delta \left( r^{-\frac{n-4}{2}} w(\theta) \zeta_\epsilon(r) \right) = - \frac{n(n-4)}{4} r^{-\frac{n}{2}} \zeta_\epsilon(r) w(\theta) + r^{-\frac{n}{2}} \zeta_\epsilon(r) \Delta w(\theta) + 3r^{-\frac{n}{2}+1} \zeta_\epsilon'(r) w(\theta) + r^{-\frac{n}{2}+2} \zeta_\epsilon''(r) w(\theta).
\]
and
\[
p \int_0^\infty \int_{S^{n-1}} r^a |u|^{p-1} \left( r^{-\frac{n-4}{2}} w(\theta) \zeta_e(r) \right)^2 r^{n-1} dr d\theta = p \left( \int_{S^{n-1}} |w|^{p+1} d\theta \right) \left( \int_0^\infty r^{-1} \zeta_e^2(r) dr \right)
\leq \left( \int_{S^{n-1}} |\Delta_\theta w|^2 + \frac{n(n-4)}{2} |\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16} w^2 d\theta \right) \left( \int_0^\infty r^{-1} \zeta_e^2(r) dr \right) \quad (3.3)
+ O \left\{ \left( \int_0^\infty [r |\zeta_e'(r)|^2 + r^3 |\zeta_e''(r)|^2 + |\zeta_e'(r)| |\zeta_e(r) + r \zeta_e(r)| |\zeta_e''(r)|] dr \right) \times \int_{S^{n-1}} [w(\theta)^2 + |\nabla_\theta w(\theta)|^2] d\theta \right\}.
\]

From the definition of \( \zeta_e \), one can easily estimate that
\[
\int_0^\infty r^{-1} \zeta_e^2(r) dr \geq \int_\epsilon^1 r^{-1} dr \geq |\ln \epsilon|
\]
and
\[
\int_0^\infty \left[ r |\zeta_e'(r)|^2 + r^3 |\zeta_e''(r)|^2 + |\zeta_e'(r)| |\zeta_e(r) + r \zeta_e(r)| |\zeta_e''(r)| \right] dr \leq C.
\]

Letting \( \epsilon \to 0 \), it implies from (3.3) that
\[
p \int_{S^{n-1}} |w|^{p+1} d\theta \leq \int_{S^{n-1}} \left( |\Delta_\theta w|^2 + \frac{n(n-4)}{2} |\nabla_\theta w|^2 + \frac{n^2(n-4)^2}{16} w^2 \right) d\theta. \quad (3.4)
\]

Now, combining (3.2) with (3.4), we obtain
\[
\int_{S^{n-1}} (p-1)|\Delta_\theta w|^2 + \left( p\ell_1 - \frac{n(n-4)}{2} \right) |\nabla_\theta w|^2 + \left( p\ell_2 - \frac{n^2(n-4)^2}{16} \right) w^2 \leq 0.
\]

Since \( \frac{n+4+2a}{n-4} < p < p_a(n) \), we get from the definition of \( p_a(n) \) that
\[
p\ell_1 - \frac{n(n-4)}{2} > 0 \quad \text{and} \quad p\ell_2 - \frac{n^2(n-4)^2}{16} > 0.
\]

Therefore we have
\[
w \equiv 0.
\]

Thus \( u \equiv 0 \).

Remark 3.1. One can easily check that
\[
u_a(r) = \ell_2^{\frac{1}{p-1}} r^{-\frac{4+a}{p-1}}
\]
is a singular solution of (1.1) in \( \mathbb{R}^n \setminus \{0\} \), where
\[
\beta = \frac{4+a}{p-1}, \quad \ell_2 = \beta(\beta+2)(\beta+4-n)(\beta+2-n).
\]
Using the well-known Hardy-Rellich inequality \([14]\) with the best constant
\[
\int_{\mathbb{R}^n} |\Delta \psi|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{\psi^2}{|x|^4} \, dx, \quad \forall \psi \in H^2(\mathbb{R}^n),
\]
we conclude that the singular solution \(u_*\) is stable in \(\mathbb{R}^n \setminus \{0\}\) if and only if
\[
p^{\frac{1}{2}} \leq \frac{n^2(n-4)^2}{16}.
\]

In what follows, we assume that \(u\) is a smooth stable solution of (1.1) in \(\mathbb{R}^n\) and \(\frac{n + 4 + 2a}{n - 4} < p < p_a(n)\). Then we obtain the following three lemmas which play an important role in dealing with the supercritical case.

For all \(\tau > 0\), we define blowing down sequences
\[
\begin{align*}
u^\tau(x) &:= \tau^{\frac{4+a}{p-1}} u(\tau x), & v^\tau(x) &:= \tau^{\frac{4+a}{p-1} + 2} v(\tau x).
\end{align*}
\]
It is easy to check that \(u^\tau\) is also a smooth stable solution of (1.1) and for all ball \(B_r(x) \subset \mathbb{R}^n\), the following estimate holds
\[
\int_{B_r(x)} \left[ (u^\tau)^2 + |x|^a |u^\tau|^{p+1} \right] \, dx
\leq C r^{-\frac{4(p+1)+2a}{p-1} - n} \left( \int_{B_r(x)} \left[ v(x)^2 + |x|^a |u(x)|^{p+1} \right] \, dx \right)^{\frac{p-1}{p}}.
\]

Moreover, using Hölder’s inequality to lead to
\[
\int_{B_r(x)} (u^\tau)^2 \, dx \leq \left( \int_{B_r(x)} |x|^a |(u^\tau)|^{p+1} \, dx \right)^{\frac{2}{p+1}} \left( \int_{B_r(x)} \left( |x| - \frac{2a}{p+1} \right)^{\frac{p+1}{p-1}} \, dx \right)^{\frac{p-1}{p+1}}
\leq C r^{-\frac{8+2a}{p-1}}.
\]

We note that \(u^\tau\) are uniformly bounded in \(L^{p+1}_{loc}(\mathbb{R}^n)\). From elliptic regularity theory, it implies that \(u^\tau\) are also uniformly bounded in \(W^{2,2}_{loc}(\mathbb{R}^n)\). Hence, we can suppose that \(u^\tau \rightharpoonup u^\infty\) weakly in \(W^{2,2}_{loc}(\mathbb{R}^n) \cap L^{p+1}_{loc}(\mathbb{R}^n)\) (if necessary, we can extract a subsequence).

Utilizing standard embeddings, we get \(u^\tau \rightharpoonup u^\infty\) strongly in \(W^{1,2}_{loc}(\mathbb{R}^n)\). Then for any ball \(B_R(0)\), applying interpolation between \(L^q\) spaces and noting the above two inequalities, for any \(q \in (1, p + 1)\), we get
\[
\|u^\tau - u^\infty\|_{L^q(B_R(0))} \leq \|u^\tau - u^\infty\|_{L^1(B_R(0))}^\mu \|u^\tau - u^\infty\|_{L^{p+1}(B_R(0))}^{1-\mu} \to 0, \quad (3.5)
\]
Thus, Lemma 3.1. From Theorem 2.1, we see that the integral yields

$$\int_{\mathbb{R}^n} \Delta u^\tau \Delta \zeta - |x|^a |u^\tau|^{p-1} u^\tau \zeta = \lim_{\tau \to \infty} \int_{\mathbb{R}^n} \Delta u^\tau \Delta \zeta - |x|^a |u^\tau|^{p-1} u^\tau \zeta,$$

$$\int_{\mathbb{R}^n} (\Delta \zeta)^2 - p|x|^a |u^\tau|^{p-1} \zeta^2 = \lim_{\tau \to \infty} \int_{\mathbb{R}^n} (\Delta \zeta)^2 - p|x|^a |u^\tau|^{p-1} \zeta^2 \geq 0.$$ 

Thus, \(u^\infty \in W_{loc}^{2,2}(\mathbb{R}^n) \cap L_{loc}^{p+1}(\mathbb{R}^n)\) is a stable solution of (1.1) in \(\mathbb{R}^n\).

Lemma 3.1. \(\lim_{r \to +\infty} E(r; 0, u) < +\infty\).

Proof. From Theorem 2.1, we see that \(E(r; 0, u)\) is non-decreasing in \(r\). Properties of the integral yields

$$E(r; 0, u) \leq \frac{1}{r} \int_r^{2r} E(\sigma; 0, u) d\sigma \leq \frac{1}{r^2} \int_r^{2r} \int_t^{t+r} E(\sigma; 0, u) d\sigma dt.$$ 

From (2.18), we have

$$\frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \left( \frac{4(p+1)+2a}{p+1} - n \right) \int_{B_\sigma} \left( \frac{1}{2} |\Delta u|^2 - \frac{1}{p+1} |x|^a |u|^{p+1} \right) d\sigma dt$$

$$\leq C \frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \left( \frac{4(p+1)+2a}{p+1} - n \right) \sigma^{\frac{4(p+1)+2a}{p+1} - n} \sigma^{\frac{4(p+1)+2a}{p+1}} d\sigma dt$$

$$\leq C.$$ 

A simple application of Hölder’s inequality and (2.18) to get

$$\frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \left( \frac{8+2a}{p+1} + 1 - n \right) \int_{\partial B_r} u^2 d\sigma dt$$

$$= \frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \left( \frac{8+2a}{p+1} + 1 - n \right) \int_{\partial B_r} u^2 d\sigma dt$$

$$\leq \frac{1}{r^2} \int_r^{2r} \left( \int_{B_t+r} \left( |x| \frac{8+2a}{p+1} + 1 - n \right) \frac{p+1}{p-1} \right) \left( \int_{B_{r+r}} |x|^{a+1} \right) \frac{2}{p+1}$$

$$\leq C \frac{1}{r^2} \int_r^{2r} \left( |x| \frac{8+2a}{p+1} + 1 - n \right) \frac{p+1}{p-1} \left( \int_{B_r} |x|^{a+1} \right) \frac{2}{p+1}$$

$$\leq C.$$ 

Again applying Hölder’s inequality, we find

$$\int_{B_r} |\nabla u|^2 \leq C r^2 \int_{B_r} |\Delta u|^2 + C r^{-2} \int_{B_r} u^2$$

$$\leq C r^2 \int_{B_r} |\Delta u|^2 + C r^{-2} \left( \int_{B_r} |x|^{a+1} \right) \frac{p+1}{p-1} \left( \int_{B_r} |x|^{a+1} \right) \frac{2}{p+1}$$

$$\leq C r^{-\frac{8+2a}{p+1} - 2}.$$
Then from the above inequality, it implies that

\[
\frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \frac{d}{d\sigma} \left( \frac{\sigma^{\frac{8+2a}{p-1}+4-n}}{r^{\frac{8+2a}{p-1}+4-n}} \int_{\partial B_{t+r}} |\nabla u|^2 \right) d\sigma dt \\
= \frac{1}{r^2} \int_r^{2r} \left\{ (t+r)^{\frac{8+2a}{p-1}+4-n} \int_{\partial B_{t+r}} |\nabla u|^2 - t^{\frac{8+2a}{p-1}+4-n} \int_{\partial B_t} |\nabla u|^2 \right\} d\sigma dt \\
\leq C \frac{1}{r^2} \int_{B_{2r}\setminus B_r} |x|^{\frac{8+2a}{p-1}+4-n} |\nabla u|^2 \\
\leq C
\]

and

\[
\frac{1}{r^2} \int_r^{2r} \int_t^{t+r} \frac{\sigma}{2} d\sigma \left[ \int_{\partial B_{r}} \left( \frac{4+a}{p-1} u + \frac{\partial u}{\partial r} \right)^2 \right] d\sigma dt \\
= \frac{1}{2r^2} \int_r^{2r} \left\{ (t+r)^{\frac{8+2a}{p-1}+4-n} \int_{\partial B_{t+r}} \left( \frac{4+a}{p-1} (t+r)^{-1} u + \frac{\partial u}{\partial r} \right)^2 \right\} d\sigma dt \\
- \frac{3}{2r^2} \int_r^{2r} \int_t^{t+r} \sigma^{\frac{8+2a}{p-1}+3-n} \int_{\partial B_{r}} \left( \frac{4+a}{p-1} \sigma^{-1} u + \frac{\partial u}{\partial r} \right)^2 d\sigma dt \\
\leq C \frac{1}{r^2} \int_{B_{2r}\setminus B_r} |x|^{\frac{8+2a}{p-1}+2-n} \left( u^2 + |x|^2 \left( \frac{\partial u}{\partial r} \right)^2 \right) \\
\leq C.
\]

Similarly, we can discuss the boundedness of the remaining terms in \( E(r;0,u) \) and obtain the desired result. \( \square \)

**Lemma 3.2.** \( u^\infty \) is homogeneous.

**Proof.** From the monotonicity of \( E(r;0,u) \) and Lemma 3.1, it implies that for any \( 0 < r_1 < r_2 < +\infty \),

\[
\lim_{\tau \to \infty} [E(\tau r_2;0,u) - E(\tau r_1;0,u)] = 0.
\]

Then applying Corollary 2.1 and the scaling invariance of \( E \), we get

\[
0 = \lim_{\tau \to \infty} [E(r_2;0,u^\tau) - E(r_1;0,u^\tau)] \\
= \lim_{\tau \to \infty} \int_{r_1}^{r_2} \frac{d}{d\sigma} E(\sigma;0,u^\tau) d\sigma \\
\geq \lim_{\tau \to \infty} c(n,p,a) \int_{B_{r_2}\setminus B_{r_1}} \left( \frac{4+a}{p-1} \sigma^{-1} u^\tau + \frac{\partial u^\tau}{\partial \sigma} \right)^2 dx \\
= c(n,p,a) \int_{B_{r_2}\setminus B_{r_1}} \left( \frac{4+a}{p-1} \sigma^{-1} u^\infty + \frac{\partial u^\infty}{\partial \sigma} \right)^2 dx,
\]

18
where \( \sigma = |x| \). Therefore, we obtain
\[
\frac{4 + a}{p - 1} \sigma^{-1} u^\infty + \frac{\partial u^\infty}{\partial \sigma} = 0, \quad \text{a.e.}
\]
A simple computation finds
\[
u^\infty(x) = |x|^{-\frac{4 + a}{p - 1}} u^\infty \left( \frac{x}{|x|} \right), \quad x \in \mathbb{R}^n \setminus \{0\},
\]
i.e., \( u^\infty \) is homogeneous.

**Lemma 3.3.** \( \lim_{r \to \infty} E(r; 0, u) = 0 \).

**Proof.** From Lemma 3.2 it implies that \( u^\infty \) is a homogeneous, stable solution of (1.1). Therefore, from Theorem 3.1 we have
\[
\nu^\infty \equiv 0.
\]
Combining with (3.5), we find that
\[
\lim_{\tau \to +\infty} \nu^\tau = 0, \quad \text{strongly in } L^2(B_5(0))
\]
implies
\[
\lim_{\tau \to +\infty} \int_{B_5(0)} (u^\tau)^2 = 0.
\]
Combining with the uniformly bounded of \( v^\tau \) in \( L^2(B_5(0)) \), we get
\[
\lim_{\tau \to +\infty} \int_{B_5(0)} |u^\tau v^\tau| \leq \lim_{\tau \to +\infty} \left( \int_{B_5(0)} (u^\tau)^2 \right)^{\frac{1}{2}} \left( \int_{B_5(0)} (v^\tau)^2 \right)^{\frac{1}{2}} = 0.
\]
Then, it implies from (2.17) that
\[
\lim_{\tau \to +\infty} \int_{B_5(0)} (\Delta u^\tau)^2 + |x|^a |u^\tau|^{p+1} \leq C \lim_{\tau \to +\infty} \int_{B_5(0)} (u^\tau)^2 + |u^\tau v^\tau| = 0. \tag{3.6}
\]
Applying the interior \( L^p \)-estimates yields
\[
\lim_{\tau \to +\infty} \int_{B_5(0)} \sum_{k \leq 2} |\nabla^k u^\tau| = 0.
\]
Then, we obtain
\[
\int_1^2 \sum_{i=1}^\infty \int_{\partial B_r} \sum_{k \leq 2} |\nabla^k u^\tau_i|^2 \, dr \leq \sum_{i=1}^\infty \int_{B_{2r} \setminus B_r} \sum_{k \leq 2} |\nabla^k u^\tau_i|^2 \leq 1.
\]
Therefore, there exists a \( \iota \in (1, 2) \) such that
\[
\lim_{\tau \to +\infty} \|u^\tau\|_{W^{\iota, 2}(\partial B_1)} = 0.
\]
Combining with (3.6) and the scaling invariance of $E(r; 0, u)$, we get

$$\lim_{i \to \infty} E(\tau_i; 0, u) = \lim_{i \to \infty} E(i; 0, u^\tau) = 0.$$  

Again since $\tau_i \to +\infty$ and $E(r; 0, u)$ is non-decreasing in $r$, we have

$$\lim_{r \to \infty} E(r; 0, u) = 0.$$  

The proof is completed. \qed

Proof of Theorem 1.1 We divide the proof into three cases.

Case I. The subcritical $1 < p < \frac{n + 4 + 2a}{n - 4}$.

Since $p < \frac{n + 4 + 2a}{n - 4}$ implies $n < \frac{4(p + 1) + 2a}{p - 1}$, and combining with (2.18), we find

$$\int_{B_R(x)} (|\Delta u|^2 + |x|^a|u|^{p+1}) \, dx \leq CR^{n - \frac{4(p+1)+2a}{p-1}} \to 0, \quad \text{as } R \to +\infty.$$  

Consequently, we obtain

$$u \equiv 0.$$

Case II. The critical $p = \frac{n + 4 + 2a}{n - 4}$.

Utilizing the inequality (2.18) once again to find

$$\int_{R^n} (v^2 + |x|^a|u|^{p+1}) \, dx < +\infty.$$  

Then, it implies that

$$\lim_{R \to +\infty} \int_{B_{2R}(x) \setminus B_R(x)} (v^2 + |x|^a|u|^{p+1}) \, dx = 0.$$  

From (2.17), a direct application of H"older’s inequality leads to

$$\int_{B_R(x)} (v^2 + |x|^a|u|^{p+1}) \, dx \leq CR^{-4} \int_{B_{2R}(x) \setminus B_R(x)} u^2 dx + CR^{-2} \int_{B_{2R}(x) \setminus B_R(x)} |uv| dx$$

$$\leq CR^{-4} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{4}{p+1}} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^{\frac{2a}{p-1}} \, dx \right)^{\frac{p-1}{2(p+1)}}$$

$$+ CKR^{-2} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{1}{p+1}} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^{\frac{2a}{p-1}} \, dx \right)^{\frac{p-1}{2(p+1)}}$$

$$\leq CR^{-4} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{4}{p+1}} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{1}{p+1}}$$

$$+ CKR^{-2} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{1}{2(p+1)}} \left( \int_{B_{2R}(x) \setminus B_R(x)} |x|^a|u|^{p+1} \, dx \right)^{\frac{2(p-1)}{2(p+1)}}.$$
where \( K = \left( \int_{B_{2R}(x) \backslash B_{R}(x)} v^2 dx \right)^{\frac{1}{2}} \). Since \( p = \frac{n + 4 + 2a}{n - 4} \), the right side of the above inequality tends to 0 as \( R \to +\infty \). So we get

\[ u \equiv 0. \]

**Case III.** The supercritical \( \frac{n + 4 + 2a}{n - 4} < p < p_a(n) \).

The smoothness of \( u \) implies that

\[ \lim_{r \to 0} E(r; 0, u) = 0. \]

From the monotonicity of \( E(r; 0, u) \) and Lemma 3.3 it implies that

\[ E(r; 0, u) = 0, \quad \text{for all } r > 0. \]

Then, from Corollary 2.1 \( u \) is homogeneous. Therefore from Theorem 3.1 we obtain

\[ u \equiv 0. \]

\[ \square \]

### 4 Proof of Theorem 1.2

In this section, we study the finite Morse index solutions of (1.1) by the use of the Pohozaev-type identity, monotonicity formula and blowing down sequence.

A basic ingredient of the proof of the subcritical case in Theorem 1.2 is the following Pohozaev-type identity.

**Lemma 4.1.** we have the equality

\[ \int_{B_R} \left( \frac{n-4}{2} |\Delta u|^2 - \frac{n+a}{p+1} |x|^a |u|^{p+1} \right) dx = \int_{\partial B_R} \left( \frac{R}{2} (\Delta u)^2 - \frac{1}{p+1} R^{1+a} |u|^{p+1} + R \frac{\partial u \partial \Delta u}{\partial r} - \Delta u \frac{\partial (x \cdot \nabla u)}{\partial r} \right) dS. \quad (4.1) \]

Applying the doubling lemma in [13, Lemma 5.1], we get the following estimates.

**Lemma 4.2.** Let \( u \) be a finite Morse index solution of (1.1). Then there exist constants \( C \) and \( R^* \) such that

\[ |u(x)| \leq C |x|^{-\frac{4+a}{p-1}}, \quad \text{for all } x \in B_{R^*}^c, \quad (4.2) \]

and

\[ \sum_{k \leq 3} |x|^{\frac{4+a}{p-1}+k} |\nabla^k u(x)| \leq C, \quad \text{for all } x \in B_{3R^*}^c. \quad (4.3) \]
Proof. The inequality (4.2) can be deduced as in [4, Lemma 5.1].

Next, we only prove the inequality (4.3). Take arbitrarily \( \tilde{x} \) with \(|\tilde{x}| > 3R^* \) and \( \tau = \frac{|\tilde{x}|}{2} \), and denote
\[
\omega(x) := \tau^{\frac{4+a}{p-1}} u(\tau x).
\]
From (4.2), it implies that for any \( x \in B_1(0) \)
\[
|\omega(x)| \leq C \tau^{\frac{4+a}{p-1}} (|\tilde{x}| + \tau x)^{-\frac{4+a}{p-1}} \leq C_1.
\]
Then we get from the standard elliptic estimates that
\[
\sum_{k \leq 3} |\nabla^k \omega(0)| \leq C_2.
\]
Noting that \( \nabla^k \omega(x) = \tau^{\frac{4+a}{p-1} + k} \nabla^k u(\tau x) \). Therefore we conclude that
\[
\sum_{k \leq 3} |x|^{\frac{4+a}{p-1} + k} |\nabla^k u(x)| \leq C_2,
\]
for all \( x \in B_{3R^*}(0) \).

Proof of Theorem 1.2. The proof consists of three cases.

Case I. The subcritical \( 1 < p < \frac{n+4+2a}{n-4} \).

From (4.2) and (4.3), we get the estimate of the right side in (4.1),
\[
\int_{\partial B_{R + 2}} \frac{R}{2} \left( \Delta u \right)^2 + \frac{R^{1+a}}{p+1} |u|^{p+1} + R^2 \frac{Du}{\partial \nu} \partial u \partial \Delta u + \left| \Delta u \frac{Du}{\partial r} \partial u \right| \partial r \rightarrow 0, \text{ as } R \rightarrow +\infty.
\]
On the other hand, since \( u \) is stable outside a compact set \( \Omega \subset \mathbb{R}^n \), we can take a test function \( \zeta_R \in C^4_0(\mathbb{R}^n \setminus \Omega) \) for \( R > R^* + 4 \) and \( \Omega \subset B_{R^*} \),
\[
\zeta_R(x) = \begin{cases} 0, & \text{if } |x| < R^* + 1 \text{ or } |x| > 2R, \\ 1, & \text{if } R^* + 2 < |x| < R. \end{cases}
\]
which satisfies \( 0 \leq \zeta_R \leq 1, \| \nabla^i \zeta_R \|_{L^{\infty}(B_{2R} \setminus B_R)} \leq \frac{C}{R^*} \) and \( \| \nabla^i \zeta_R \|_{L^{\infty}(B_{R^*+2} \setminus B_{R^*+1})} \leq C_{R^*} \), for \( i = 1, 2, 3, 4 \). Then from Lemma 2.2 we have
\[
\int_{R^*+2 < |x| < R} \left( |\Delta u|^2 + |x|^a |u|^{p+1} \right) dx \leq C_{R^*} + CR^{n-\frac{k(p+1)+2a}{p-1}}.
\]
Again since \( n < \frac{4(p+1)+2a}{p-1} \), we obtain
\[
\int_{\mathbb{R}^n} \left[ (\Delta u)^2 + |x|^a |u|^{p+1} \right] dx < +\infty.
\]
Taking limit in \((1.1)\), we obtain
\[
\int_{\mathbb{R}^n} \left[ \frac{n - 4}{2} |\Delta u|^2 - \frac{n + a}{p + 1} |x|^a |u|^{p+1} \right] dx = 0. \tag{4.4}
\]

Now, we claim that
\[
\int_{\mathbb{R}^n} |\Delta u|^2 dx = \int_{\mathbb{R}^n} |x|^a |u|^{p+1} dx. \tag{4.5}
\]

Indeed, multiply the equation \((1.1)\) with \(u_R\) for \(\zeta_R \in C_0^1(B_{2R})\) which satisfies
\[
0 \leq \zeta_R \leq 1, \quad \|\nabla^i \zeta_R\|_{L^\infty} \leq \frac{C}{R^i}, \quad \text{for } i = 1, 2, 3, 4, \text{ and }
\]
\[
\zeta_R(x) = \begin{cases} 
1, & \text{if } |x| < R, \\
0, & \text{if } |x| > 2R.
\end{cases}
\]

A simple computation implies
\[
\int_{B_R} \left( |x|^a |u|^{p+1} - (\Delta u)^2 \right) \zeta_R dx = \int_{B_R} (u \Delta u \Delta \zeta_R + 2 \Delta u \nabla u \cdot \nabla \zeta_R) dx := S_1(R) + S_2(R).
\]

We may use Hölder’s inequality in \(S_1(R)\) and \(S_2(R)\) to obtain
\[
|S_1(R)| \leq R^{-2} \int_{B_R} |\Delta u| \left( |x|^{\frac{n}{p+1}} |u| \right) |x|^{-\frac{n}{p+1}}
\]
\[
\leq R^{-2} \left( \int_{B_{2R}} (\Delta u)^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^a |u|^{p+1} \right)^{\frac{1}{p+1}} \left( \int_{B_{2R}} |x|^{-\frac{2a}{p-1}} \right)^{p-1 \over 2(p+1)}
\]
\[
\leq CR^{n \left( \frac{p-1}{2(p+1)} \right) - \frac{2a}{p-1} - 2} \left( \int_{B_{2R}} (\Delta u)^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^a |u|^{p+1} \right)^{\frac{1}{p+1}}
\]
\[
\leq CR^{n \left( \frac{p-1}{2(p+1)} \right) - 4(p+1) + \frac{2a}{p-1}}
\]

and
\[
|S_2(R)| = \int_{B_{2R}} |\Delta u| \cdot |\nabla u| \cdot |\nabla \zeta_R| \leq \left( \int_{B_{2R}} (\Delta u)^2 \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |\nabla u|^2 |\nabla \zeta_R|^2 \right)^{\frac{1}{2}}
\]
\[
= \left( \int_{B_{2R}} (\Delta u)^2 dx \right)^{\frac{1}{2}} \left( \int_{B_{2R}} u(-\Delta u)|\nabla \zeta_R|^2 + \frac{1}{2} \int_{B_{2R}} u^2 \Delta (|\nabla \zeta_R|^2) \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{B_{2R}} (\Delta u)^2 dx \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |u||\Delta u||\nabla \zeta_R|^2 \right)^{\frac{1}{2}} + C \left( \int_{B_{2R}} (\Delta u)^2 dx \right)^{\frac{1}{2}} \left( \int_{B_{2R}} |x|^a |u|^{p+1} \right)^{\frac{1}{p+1}} \left( \int_{B_{2R}} |x|^{-\frac{2a}{p-1}} \right)^{p-1 \over 2(p+1)}
\]
\[
\leq CR^{n \left( \frac{p-1}{2(p+1)} \right) - 4(p+1) + \frac{2a}{p-1}}
\]

In the above, we use the results in \((2.16), (4.2)\) and \((4.3)\). Since \(n < \frac{4(p + 1) + 2a}{p - 1}\), we get
\[
\lim_{R \to +\infty} S_1(R) = 0 \text{ and } \lim_{R \to \infty} S_2(R) = 0.
\]
Thus, the claim (4.5) holds.

Combining (4.4) with (4.5), this leads to

\[
\left( \frac{n - 4}{2} - \frac{n + a}{p + 1} \right) \int_{\mathbb{R}^n} |u|^{p+1} \, dx = 0.
\]

Thus we get that

\[ u \equiv 0. \]

**Case II.** The critical \( n = \frac{4(p + 1) + 2a}{p - 1} \).

Since \( u \) is stable outside \( B_{R^*} \), we adopt the similar argument as in the subcritical case and find

\[
\int_{B_R \setminus B_{3R^*}} [(\Delta u)^2 + |x|^a|u|^{p+1}] \, dx \leq C, \quad \text{for } R > 3R^*
\]

and

\[
\int_{\mathbb{R}^n} [(\Delta u)^2 + |x|^a|u|^{p+1}] \, dx < +\infty.
\]

The elliptic regularity theory implies

\[
\lim_{R \to \infty} \int_{B_{2R} \setminus B_R} R^{-1} |\nabla u| + R^{-2} |u| = 0.
\]

Therefore, it is easy to verify that

\[
\int_{\mathbb{R}^n} (\Delta u)^2 - |x|^a|u|^{p+1} = 0.
\]

**Case III.** The supercritical \( \frac{n + 4 + 2a}{n - 4} < p < p_a(n) \).

**Claim I.** There exists a constant \( C \) such that for all \( r > 3R^* \), \( E(r; 0, u) \leq C \).

Indeed, applying the inequality (4.2) and (4.3), we obtain

\[
E(r; 0, u) \leq \begin{align*}
& Cr^{\frac{4(p+1)+2a}{p-1} - n} \int_{B_r} (\Delta u)^2 + |x|^a|u|^{p+1} \\
& + Cr^{\frac{8+2a}{p-1} + 1-n} \int_{\partial B_r} u^2 \quad + Cr^{\frac{8+2a}{p-1} + 3-n} \int_{\partial B_r} |\nabla u|^2 \\
& + Cr^{\frac{8+2a}{p-1} + 2-n} \int_{\partial B_r} |u||\nabla u| \quad + Cr^{\frac{8+2a}{p-1} + 4-n} \int_{\partial B_r} |\nabla u||\nabla^2 u| \\
& \leq C,
\end{align*}
\]

for all \( r > 3R^* \), where \( C \) does not depend on \( r \).
Now, we can apply Theorem 2.1 to get
\[
\frac{d}{dr} E(r; 0, u) \geq c(n, p, a) r^{-n + 2 + \frac{8 + 2a}{p - 1}} \int_{\partial B_r} \left( \frac{4 + a}{p - 1} r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \\
= c(n, p, a) \int_{\partial B_r} \left( \frac{4 + a}{p - 1} r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \frac{r^{n - 2 - \frac{8 + 2a}{p - 1}}}{r^{n - 2 - \frac{8 + 2a}{p - 1}}}. 
\]
Integrating the above inequality from $3R^*$ to $+\infty$ in both sides and combining with Claim I, we find
\[
\int_{B_{3R^*}} \frac{(4 + a)}{p - 1} r^{-1} u + \frac{\partial u}{\partial r} \right)^2 \frac{r^{n - 2 - \frac{8 + 2a}{p - 1}}}{r^{n - 2 - \frac{8 + 2a}{p - 1}}} < +\infty. \tag{4.6}
\]

Claim II. $\lim_{r \to +\infty} E(r; 0, u) = 0.$

Indeed, for $\tau > 0$, we define a blowing down sequence
\[
u(x) := \tau^{\frac{4 + a}{p - 1}} u(\tau x).
\]

It implies from Lemma 4.2 that $u^\tau$ is uniformly bounded in $C^5 (B_r(0) \setminus B_1(0))$ for any fixed $r > 1$, and $u^\tau$ is stable outside $B_r(0)$. Then there exists a function $u^\infty$ in $C^4 (\mathbb{R}^n \setminus \{0\})$ such that $u^\infty$ is a stable solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. For any $r > 1$, we get from (4.6) that
\[
\int_{B_r \setminus B_1} \frac{(4 + a)}{p - 1} r^{-1} u^\infty + \frac{\partial u^\infty}{\partial r} \right)^2 \frac{r^{n - 2 - \frac{8 + 2a}{p - 1}}}{r^{n - 2 - \frac{8 + 2a}{p - 1}}} \\
= \lim_{\tau \to \infty} \int_{B_r \setminus B_1} \frac{(4 + a)}{p - 1} r^{-1} u^\tau + \frac{\partial u^\tau}{\partial r} \right)^2 \frac{r^{n - 2 - \frac{8 + 2a}{p - 1}}}{r^{n - 2 - \frac{8 + 2a}{p - 1}}} \\
= \lim_{\tau \to \infty} \int_{B_{r\tau} \setminus B_{r\tau/2}} \frac{(4 + a)}{p - 1} r^{-1} u^\tau + \frac{\partial u^\tau}{\partial r} \right)^2 \frac{r^{n - 2 - \frac{8 + 2a}{p - 1}}}{r^{n - 2 - \frac{8 + 2a}{p - 1}}} \\
= 0.
\]
From Corollary 2.1 we conclude that $u^\infty$ is a homogeneous, stable solution of (1.1). Then we get from Theorem 3.1 that
\[
u(x) := \tau^{\frac{4 + a}{p - 1}} u(\tau x).
\]

Consequently, from the definition of the blowing down sequence $u^\tau(x)$ and the argument as in (4.3), we get
\[
limit_{|x| \to +\infty} \frac{4 + a}{p - 1} |u(x)| = 0,
\]
and
\[
\lim_{|x| \to \infty} \sum_{k \leq 4} |x|^{\frac{4+a}{p-1} + k} |\nabla^k u(x)| = 0.
\]

For any \( \epsilon > 0 \) and \( R_0 > 0 \), we find
\[
\sum_{k \leq 4} |x|^{\frac{4+a}{p-1} + k} |\nabla^k u(x)| \leq \epsilon
\]
for all \( |x| > R_0 \). Then for \( r \gg R_0 \), we have
\[
E(r; 0, u) \leq C r^{\frac{4(p+1)+2a}{p-1} - n} \left\{ \begin{array}{c}
\int_{B_{R_0}(0) \cup [B_r(0) \setminus B_{R_0}(0)]} \left[ (\Delta u)^2 + |x|^2 |u|^{p+1} \right] \\
+ C \epsilon r^{\frac{8+2a}{p-1} - 1-n} \int_{\partial B_r(0)} |x|^{-\frac{8+2a}{p-1}}
\end{array} \right\} 
\]
\[
\leq C(R_0) \left( r^{\frac{4(p+1)+2a}{p-1} - n} + \epsilon \right).
\]

From the inequality \( n > \frac{4(p+1)+2a}{p-1} \) and the arbitrary of \( \epsilon \), we conclude that Claim II holds.

From the smoothness of \( u \), it is easy to see that
\[
\lim_{r \to 0} E(r; 0, u) = 0.
\]

Hence, from Claim II and the monotonicity of \( E \), we get
\[
u \equiv 0. \]}

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