On the higher derivatives of $Z(t)$ associated with the Riemann Zeta-Function

Kaneaki Matsuoka

1 INTRODUCTION

Let $s = \sigma + it$ be a complex variable and $\zeta(s)$ the Riemann zeta-function. The functional equation for the Riemann zeta-function is

$$h(s)\zeta(s) = h(1-s)\zeta(1-s),$$

where $h(s) = \pi^{-s/2}\Gamma(s/2)$. Define

$$Z(t) = e^{i\theta(t)}\zeta(1/2 + it),$$

where $\theta(t) = \arg h(1/2 + it)$. From the functional equation it follows that $Z(t)$ is real and we can easily see that zeros of $Z(t)$ coincide with those of $\zeta(1/2 + it)$. These properties make it possible to investigate the zeros of the Riemann zeta-function on the critical line. Using a function similar to $Z(t)$ Hardy [4] first proved that there are infinitely many zeros on the critical line and Hardy and Littlewood [5] showed that the number of zeros on the line segment from $1/2$ to $1/2 + iT$ is $\gg T$. Siegel [8] showed that the number of those is $> 3e^{-3/2}T/8\pi + o(T)$. He defined $Z(t)$ and derived the Riemann-Siegel formula from the manuscript of Riemann, which was the essential part of his proof. Later A. Selberg [7] improved the bounds to $\gg T \log T$ and recently H. Bui, B. Conrey and M. Young [2] showed that more than 41 % of the zeros of the Riemann zeta-function are on the critical line. The Riemann-Siegel formula plays an important role in the investigation of the behavior of the Riemann zeta-function on the critical line as well as the calculation of the number of the complex zeros of the Riemann zeta-function. We sometimes call the function $Z(t)$ the Hardy function or the Riemann-Siegel function because of the above reason.
It is well known that under the assumption of the Riemann hypothesis \(Z'(t)\) has exactly one zero between consecutive zeros of \(Z(t)\) (see Edwards [3, p.176]). R. J. Anderson [1] showed the same relationship between zeros of \(Z'(t)\) and those of \(Z''(t)\). K. Matsumoto and Y. Tanigawa [6] studied the number of zeros of the higher derivatives of \(Z(t)\). They showed that under the assumption of the Riemann hypothesis the number of zeros of \(Z^{(n)}(t)\) in the interval \((0, T)\) is \(T/2\pi \log T/2\pi - T/2\pi + O(\log T)\), where \(n\) is any positive integer and the implied constant depends on \(n\). From this result we find that the same type of relationship as above is valid between \(Z^{(n)}(t)\) and \(Z^{(n+1)}(t)\) in almost cases except for \(O(\log t)\) terms. In this paper we will prove the following theorem.

**Theorem 1.1.** If the Riemann hypothesis is true then for any positive integer \(n\) there exists a \(t_n > 0\) such that for \(t > t_n\) the function \(Z^{(n+1)}(t)\) has exactly one zero between consecutive zeros of \(Z^{(n)}(t)\).

The case \(n = 1\) is the result of Anderson [1, Theorem 3]. R. J. Anderson [1] constructed and studied the meromorphic function \(\eta(s)\). K. Matsumoto and Y. Tanigawa [6] introduced a function \(\eta_n(s)\) which is a generalization of Anderson’s \(\eta(s)\). These functions played an important role to show their results. We will define the function \(g_n(s)\) which has properties similar to those of Anderson’s \(\eta(s)\) in Section 2 and this section will be the most essential part of our proof. Theorem 1.1 will be proved in the last section after preparing some auxiliary results in Sections 3-5. These have been inspired by the proof of Anderson [1].

### 2 THE DEFINITION OF FUNCTIONS

Let \(\chi(s) = h(1 - s)/h(s)\) and \(\omega(s) = (\chi'/\chi)(s)\). We see that

\[
\omega(1/2 + it) = -2\theta'(t),
\]

and

\[
\omega(1 - s) = \omega(s).
\]

Now let \(f_0(s) = \zeta(s)\), and we define \(f_n(s)\) for \(n \geq 1\) recursively by

\[
f_{n+1}(s) = f'_n(s) - \frac{1}{2} \omega(s)f_n(s) \quad (n \geq 0).
\]
Let \( h_0(s) = 1 \), and we define \( h_n(s) \) for \( n \geq 1 \) recursively by

\[
h_{n+1}(s) = h_n'(s) - \frac{1}{2} \omega(s) h_n(s) \quad (n \geq 0).
\]

(2.4)

We denote \( g_n(s) \) by \( f_n(s)/h_n(s) \).

**Proposition 2.1.** For any non-negative integer \( n \), we have

\[
Z^{(n)}(t) = i^n f_n \left( \frac{1}{2} + it \right) e^{i\theta(t)}.
\]

(2.5)

**Proof.** The case \( n = 0 \) is the definition of \( Z(t) \). Assume that (2.5) is valid for \( n \). Then

\[
Z^{(n+1)}(t) = \left( i^{n+1} f_n'(\frac{1}{2} + it) + i^{n+1} \theta'(t) f_n \left( \frac{1}{2} + it \right) \right) e^{i\theta(t)}.
\]

From (2.1), we find that (2.5) is valid for \( n + 1 \). Hence the result follows. \( \square \)

Matsumoto and Tanigawa [6] defined a meromorphic function \( \eta_n(s) \) which has the property

\[
Z^{(n)}(t) = i^n \theta'(t) \eta_n(1/2 + it) e^{i\theta(t)}.
\]

From (2.1) and (2.5) we have \( f_n(s) = -\omega(s) \eta_n(s)/2 \).

**Proposition 2.2.** For any non-negative integer \( n \), we have

\[
\chi(s) f_n(1 - s) = (-1)^n f_n(s).
\]

(2.6)

**Proof.** The case \( n = 0 \) is nothing but the functional equation for the Riemann zeta-function. From (2.2) and the functional equation for \( \eta_n(s) \) (see Matsumoto and Tanigawa [6, Proposition 2]), we obtain the result. \( \square \)

**Remark 2.3.** From (2.2) and the definition of \( g_n(s) \), if \( n = 1 \) the formula which is of the same form as (2.6) but with replacing \( f_1(s) \) by \( g_1(s) \) is also valid. This is the functional equation for \( \eta(s) \) (see Anderson [1]). But for \( n \geq 2 \) we can not replace \( f_n(s) \) by \( g_n(s) \) in (2.6).

From (2.3) we see that \( f_n(s) \) can be expressed as

\[
f_n(s) = \sum_{k=0}^{n} a_{n,k} \zeta^{(k)}(s),
\]

(2.7)
where $a_{n,k}(s)$ is a polynomial in the variables $\omega(s), \omega'(s), \cdots, \omega^{(n)}(s)$ with constant coefficients and we denote

$$a_{n,k}(s) = \sum_{h=0}^{n} c_{n,k,h}(s) \omega^h(s), \quad \text{(2.8)}$$

where $c_{n,k,h}(s)$ is a polynomial in the variables $\omega'(s), \omega''(s), \cdots, \omega^{(n)}(s)$ with constant coefficients. It is easy to see that $a_{n,0}(s) = h_n(s)$ and hence we have

$$g_n(s) = \zeta(s) + \sum_{k=1}^{n} \frac{a_{n,k}(s)}{h_n(s)} \zeta^{(k)}(s).$$

We stress that the coefficient of $\zeta(s)$ is 1, which enables us to make use of a method in the theory of the Riemann zeta-function.

### 3 BASIC PROPERTIES OF $f_n(s), g_n(s)$ AND $h_n(s)$

We denote by $n$ a positive integer and Landau’s symbol $O$ depends on $n$. It is well known that

$$\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s), \quad \text{(3.1)}$$

and it follows that

$$\omega(s) = \log(2\pi) + \frac{\pi}{2} \tan\left(\frac{\pi s}{2}\right) - \frac{\Gamma'}{\Gamma}(s). \quad \text{(3.2)}$$

It is known that

$$\log \Gamma(s) = \left( s - \frac{1}{2} \right) \log s - s + \frac{1}{12s} - \int_{0}^{\infty} \frac{P(x)}{(s+x)^3} dx, \quad \text{(3.3)}$$

where $P(x)$ is a certain periodic function (see Edwards [3, p.109]). Hence we have

$$\frac{\Gamma'}{\Gamma}(s) = \log |s| + O(1) \quad (\sigma > 1/4), \quad \text{(3.4)}$$

and

$$\frac{d^n}{ds^n} \frac{\Gamma'}{\Gamma}(s) = O(|s|^{-n}) \quad (n \geq 1, \sigma > 1/4). \quad \text{(3.5)}$$
Define the set $D$ by removing all small circles whose centers are odd positive integers and even non-positive integers from the complex plane. We denote $D_1$ by $\mathbb{C} - D$.

**Lemma 3.1.** In the region $\{ s \in D | \sigma > 1/4, t > 0 \}$ we have

$$\tan s = i + O(e^{-2t}), \quad (3.6)$$

and

$$\frac{d^n}{ds^n} \tan s = O(e^{-2t}) \quad (n \geq 1). \quad (3.7)$$

**Proof.** Since

$$\tan s = -i\left(-1 + \frac{2e^{2i\sigma-2t}}{e^{2i\sigma-2t} + 1}\right), \quad (3.6)$$

immediately follows. Next we show (3.7). Since $\tan'(x) = \tan^2(x) + 1$, we have $\tan'(s) = O(e^{-2t})$ and

$$\tan''(x) = 2 \tan(x) \tan'(x). \quad (3.8)$$

So the case $n = 2$ follows. Assume that (3.7) is valid for $n+2$. Differentiating (3.8) $n+1$ times we see that

$$\tan^{(n+3)}(x) = 2 \sum_{k=0}^{n+1} \binom{n+1}{k} \tan^{(k)}(x) \tan^{(n-k+2)}(x)$$

$$= 2 \tan^{(n+2)}(x) \tan(x) + 2 \sum_{k=1}^{n+1} \binom{n+1}{k} \tan^{(k)}(x) \tan^{(n-k+2)}(x).$$

We find that (3.7) is valid for $n + 3$. This proves the lemma. \qed

**Lemma 3.2.** For $s \in D$, we have

$$\omega(s) = -\log |s| + O(1), \quad (3.9)$$

and

$$\omega^{(n)}(s) = O(1) \quad (n \geq 1). \quad (3.10)$$

**Proof.** From the previous lemma and (3.2), we can prove the lemma if $s$ is in $D \cap \{ s| \sigma > 1/4 \}$. But considering equation (2.2) the lemma follows. \qed
If \( k \neq 0 \) and \( n \geq 1 \) then \( c_{n,k,n}(s) = 0 \), hence with Lemma 3.2 if \( s \in D \) we have
\[
a_{n,k}(s) = O((\log |s|)^{n-1}) \quad (k \neq 0, n \geq 1). \tag{3.11}
\]
From (2.4) if \( n \geq 1 \) then \( c_{n,0,n}(s) = (-1)^n / 2^n \), hence with Lemma 3.2 and (2.8) if \( s \in D \) we have
\[
h_n(s) = a_{n,0}(s) = \left( \frac{\log |s|}{2} \right)^n + O((\log |s|)^{n-1}) \quad (n \geq 1). \tag{3.12}
\]

**Lemma 3.3.** We have
\[
\zeta(s) = 1 + O(2^{-\sigma}) \quad (\sigma > 2),
\]
and
\[
\zeta^{(n)}(s) = O(2^{-\sigma}) \quad (n \geq 1, \sigma > 2). \tag{3.13}
\]

**Proof.** From the definition of the Riemann zeta-function we have
\[
\zeta(s) = 1 + \sum_{k=2}^{\infty} \frac{1}{k^s},
\]
and differentiating it \( n \) times we get
\[
|\zeta^{(n)}(s)| \leq \sum_{k=2}^{\infty} \frac{(\log k)^n}{k^\sigma}.
\]
Let \( f(x) = x^{-\sigma}(\log x)^n \). Since \( f'(x) = x^{-\sigma-1}(\log x)^{n-1}(n - \sigma \log x) \), if \( \sigma \geq n / \log 2 \) then \( f(x) \) is decreasing on \( x \geq 2 \). Hence with
\[
\int_2^\infty x^{-\sigma}(\log x)^n dx = \frac{2^{-\sigma+1}}{\sigma - 1}(\log 2)^n + \frac{n}{\sigma - 1} \int_2^\infty x^{-\sigma}(\log x)^{n-1} dx,
\]
we obtain the result. \(\square\)

From (2.7), (2.8), (3.11), (3.12) and Lemma 3.3, if \( s \in D \cap \{ s | \sigma > 2 \} \) we have
\[
f_n(s) = \left( \left( \frac{\log |s|}{2} \right)^n + O((\log |s|)^{n-1}) \right) (1 + O(2^{-\sigma})) + \sum_{k=1}^{n} O \left( \frac{(\log |s|)^{n-1}}{2^\sigma} \right)
\]
\[
= (1 + O(2^{-\sigma})) \left( \frac{\log |s|}{2} \right)^n + O((\log |s|)^{n-1}). \tag{3.13}
\]
4 ZEROS AND POLES OF $f_n(s), g_n(s), h_n(s)$

From (3.1) we have

**Lemma 4.1.** (Matsumoto and Tanigawa [6] Lemma 1) The poles of $\omega(s)$ are all simple, and are located at $1, 3, 5, \cdots$ (with residue $-1$) and at $0, -2, -4, \cdots$ (with residue $1$).

First we investigate the poles and zeros of $h_n(s)$.

**Lemma 4.2.** The function $h_n(s)$ has poles of order $n$ which are located only at $1, 3, 5, \cdots$, $0, -2, -4, \cdots$.

**Proof.** The case $n = 1$ is the previous lemma. Assume that this lemma is valid for $n$. Let $a$ be a pole of $h_n(s)$. We expand $h_n(s)$ in a Laurent series of powers of $s - a$. Thus $h_n(s) = c_n/(s - a)^n + \cdots$, where $c_n$ does not vanish. From Lemma 4.1 and (2.1) we have

$$h_{n+1}(s) = \frac{-nc_n \pm \frac{1}{2}c_n}{(s - a)^{n+1}} + \cdots.$$  

If $a$ is positive we take plus and $a$ is not positive we take minus. Since $-nc_n \pm c_n/2$ does not vanish, this lemma is valid for $n + 1$. This proves the lemma.

It is difficult to determine the location of zeros of $h_n(s)$ exactly but for large $|s|$ we roughly know the location.

**Lemma 4.3.** Let $2m$ be a sufficiently large even integer. In the region \{s|σ ≥ 2m\} ∪ \{s|σ ≤ 1 - 2m\} zeros of $h_n(s)$ are all located in $D_1$ and the number of those in a circle is $n$. Let $T$ be sufficiently large. In the region \{s|1 - 2m < σ < 2m, |t| > T\} there exists no zero of $h_n(s)$.

**Proof.** From (3.12) if $|s|$ is sufficiently large $\Re h_n(s)$ is positive in $D$. By the argument principle and the previous lemma the result follows.

Next we investigate the poles and zeros of $f_n(s)$.

**Lemma 4.4.** The function $f_n(s)$ has poles of order $n$ located at $0, 3, 5, 7, \cdots$ and those of order $n + 1$ located at $1$ and those of order $n - 1$ located at $-2, -4, -6 \cdots$.  

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Proof. From (2.4) the case \( k = 1, 3, 5 \cdots \) is proved in the same way as in Lemma 4.2. The case \( k = 0, -2, -4 \cdots \) is proved from (2.6). \hfill \Box

**Lemma 4.5.** Let \( 2m \) be a sufficiently large even integer. In the region \( \{ s | \sigma \geq 2m \} \cup \{ s | \sigma \leq 1 - 2m \} \) zeros of \( f_n(s) \) are located in \( D_1 \) and the number of those in a circle is \( n \).  

**Proof.** From (3.13) if \( m \) is sufficiently large \( \Re f_n(s) \) is positive in the region \( \{ s | \sigma \geq 2m \} \cap D \) hence the lemma is proved in the same way as in Lemma 4.1. From (2.6) the lemma is also proved in the region \( \{ s | \sigma \leq 1 - 2m \} \cap D \). \hfill \Box

From the previous lemmas we know the location of the zeros and poles of \( g_n(s) \). For each circle \( B \) included in \( D_1 \), let \( N_0(B) \) (resp. \( N_\infty(B) \)) be the number of zeros (resp. poles) in \( B \).

**Lemma 4.6.** Let \( T \) and \( m \) be large. In the region \( \{ s | 1 - 2m < \sigma < 2m, |t| > T \} \) there exists no pole of \( g_n(s) \). In the region \( \{ s | \sigma \geq 2m \} \) zeros and poles of \( g_n(s) \) are all located in \( D_1 \) and the number of zeros in a circle \( B \) is at most \( n \) and \( N_0(B) = N_\infty(B) \). In the region \( \{ s | \sigma \leq 1 - 2m \} \) zeros and poles of \( g_n(s) \) are located in \( D_1 \) and the number of zeros in a circle \( B \) is at most \( n + 1 \) and \( N_0(B) = N_\infty(B) + 1 \).

Let \( m = m(n) \) be a sufficiently large positive integer. We define \( N_{g_n}(T) \) by the number of zeros of \( g_n(s) \) with \( -2m + 1 < \sigma < 2m \) and \( 0 < t < T \). From Lemma 4.3 and the proof of Theorem 1 and Theorem 2 in Matsumoto and Tanigawa \[6\] we have the following results.

**Proposition 4.7.** We have 

\[
N_{g_n}(T) = T \log T - T/2 + O(\log T).
\]

**Proposition 4.8.** On the Riemann hypothesis if \( T \) is large, zeros of \( g_n(s) \) in the region \( \{ s | 1 - 2m < \sigma < 2m, |t| > T \} \) are on the critical line \( \sigma = 1/2 \).

## 5 LEMMAS FOR THE PROOF OF THE THEOREM

**Lemma 5.1.** Let \( T \) be large. There exists a positive number \( A \) such that in the region \( \{ s | -2m + 1 < \sigma < 2m, t > T \} \)

\[
g_n(s) = O(t^A).
\]
Proof. Let \( M \geq 2 \) and \( K \) be positive integers. It is known that

\[
\zeta(s) = \sum_{n=1}^{M} \frac{1}{n^s} + \frac{M^{1-s}}{s-1} - \frac{M^{-s}}{2} + \sum_{k=1}^{K} \frac{B_{2k}}{(2k)!} s(s+1) \cdots (s+2k-2) M^{1-s-2k} s(s+1) \cdots (s+2k) \int_{M}^{\infty} B_{2K+1}(x-[x]) x^{-s-2K-1} \, dx,
\]

where \( B_n \) is the \( n \)-th Bernoulli number and \( B_n(x) \) is the \( n \)-th Bernoulli polynomial (see Edwards [3, p.114]). Hence with Lemma 3.2 the result follows.

**Lemma 5.2.** For \( s \in \mathcal{D} \cap \{ s | \sigma > 2m \} \) we have

\[
\frac{g'}{g_n}(\sigma + it) = O(1).
\]

Proof. From (2.7), (2.8), (3.11), (3.12) and Lemma 3.3 for the region \( \{ s | \sigma \geq 2 \} \cap \mathcal{D} \) we have

\[
g_n(s) = \zeta(s) + \sum_{k=1}^{n} \frac{a_{n,k}(s)}{h_n(s)} \zeta^{(k)}(s)
= 1 + O(2^{-\sigma}) + \sum_{k=1}^{n} O((\log |s| n^{-1} 2^{-\sigma}) \left( \frac{\log |s|}{2} \right)^n + O((\log |s|)^{n-1})
= 1 + O(2^{-\sigma}).
\]

Differentiating (5.1) we have

\[
g'_n(s) = \zeta'(s) + \sum_{k=1}^{n} \left( \frac{a_{n,k}'(s)}{h_n(s)} - \frac{a_{n,k}(s) h_n'(s)}{h_n(s)^2} \right) \zeta^{(k)}(s) + \frac{a_{n,k}(s)}{h_n(s)} \zeta^{(k+1)}(s)
\]

(5.3)

From (2.8), (3.9), (3.10), if \( k \neq 0 \) we have \( a_{n,k}^{(l)}(s) = O((\log |s|)^{n-1}) \) for any positive integer \( l \). Hence with (3.11), (3.12), (5.3) and Lemma 3.3 we have

\[
g_n(s) = O(2^{-\sigma}) + \frac{O(2^{-\sigma} (\log |s|)^{n-1})}{(\log |s|)^n + O((\log |s|)^{n-1})} + \frac{O(2^{-\sigma} (\log |s|)^{2n-2})}{(\log |s|)^{2n} + O((\log |s|)^{2n-1})}
= O(2^{-\sigma}).
\]

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Hence with (5.2) we have
\[ \frac{g'_n(\sigma + it)}{g_n} = \frac{O(2^{-\sigma})}{1 + O(2^{-\sigma})} = O(1). \]

\[ \square \]

**Lemma 5.3.** We have
\[ N_{g_n}(T + 1) - N_{g_n}(T) = O(\log T). \]

**Proof.** From Proposition 4.7 we have
\[ N_{g_n}(T + 1) - N_{g_n}(T) = \left( \frac{T + 1}{2\pi} \right) \left( \log \frac{T}{2\pi} + O\left( \frac{1}{T} \right) \right) - \frac{T}{2\pi} \log \frac{T}{2\pi} + O(\log T) = O(\log T). \]

\[ \square \]

**Lemma 5.4.** (Titchmarsh [9, p.56] LEMMA \( \alpha \)) If \( f(s) \) is regular, and
\[ \left| \frac{f(s)}{f(s_0)} \right| < e^M \quad (M > 1) \]
in the circle \( |s - s_0| \leq r \), then
\[ \left| \frac{f'(s)}{f(s)} - \sum \frac{1}{s - \rho} \right| < \frac{AM}{r} \quad (|s - s_0| \leq \frac{1}{4}r), \]
where \( \rho \) runs through the zeros of \( f(s) \) such that \( |\rho - s_0| \leq \frac{1}{2}r \) and \( A \) is a positive constant.

**Lemma 5.5.** For large \( t \) and \( 1 - 2m \leq \sigma \leq 2m \) we have
\[ \frac{g'_n(\sigma + it)}{g_n} = \sum_{|s - \rho| < 1} \frac{1}{s - \rho} + O(\log t), \]
where \( \rho = \beta + i\gamma \) runs through the zeros of \( g_n(s) \) such that \( |t - \gamma| < 1 \).
Proof. Let $T$ be sufficiently large and $f(s) = g_n(s)$, $s_0 = 2m + 1 + iT$, $r = 16m + 4$ in the previous lemma. If $|s - s_0| \leq 16m + 4$ then $-14m - 3 \leq \sigma \leq 18m + 5$, $T - 16m - 4 \leq t \leq T + 16m + 4$. There exists a constant $A$ such that $g_n(\sigma + it) = O(t^A)$ uniformly in the region $\{s| -14m - 3 \leq \sigma \leq 18m + 5, t > T - 16m - 4\}$ by Lemma 5.1 and we have $g_n(2m + 1 + it) = 1 + O(4^{-m})$ by (5.2). Hence with the previous lemma in the region $|s - s_0| \leq 4m + 1$ we have

$$
\frac{g'_n}{g_n}(\sigma + it) = \sum_{\rho} \frac{1}{s - \rho} + O(\log T),
$$

where the summation is over the zeros of $g_n(s)$ such that $|\rho - s_0| \leq 8m + 2$. In particular if $t = T$ we have

$$
\frac{g'_n}{g_n}(\sigma + it) = \sum_{\rho} \frac{1}{s - \rho} + O(\log t).
$$

If $|\rho - s_0| \leq 8m + 2$ but $|\gamma - t| \geq 1$ then $|s - \rho| \geq 1$ hence with Lemma 5.3 we have

$$
\frac{g'_n}{g_n}(\sigma + it) = \sum_{|\tau - \gamma| < 1} \frac{1}{s - \rho} + O(\log t),
$$

uniformly for $|\sigma - 2m - 1| \leq 4m + 1$. \hfill \Box

**Lemma 5.6.** There exists a sequence $\{T_j\}$ tending to infinity, such that if $g_n(\beta + i\gamma) = 0$ then $|\gamma - T_j|^{-1} = O(\log T_j)$.

**Proof.** Let $j$ be a large positive integer. Suppose the rectangle defined by $-2m + 1 \leq \sigma \leq 2m$, $j \leq t \leq j + 1$ contains $N$ zeros of $g_n(s)$. Divide it to $N + 1$ rectangles of width $1/N + 1$. At least one of these contains no zero of $g_n(s)$. There is a $T_j$ with $j < T_j < j + 1$ such that $|\gamma - T_j| > 1/2(N + 1)$. From Lemma 5.3 we have $|\gamma - T_j|^{-1} = O(N) = O(\log T_j)$. \hfill \Box

**Lemma 5.7.** There exists a sequence $\{T_j\}$ tending to infinity, such that

$$
\frac{g'_n}{g_n}(\sigma + iT_j) = O(\log^2 T_j),
$$

uniformly for $-2m + 1 \leq \sigma \leq 2m$.
Proof. Let \( \{T_j\} \) be as in Lemma 5.6. If \( s = \sigma + iT_j \) then \( |s - \rho|^{-1} \leq |T_j - \gamma|^{-1} = O(\log T_j) \). Since the number of zeros with \( |\gamma - T_j| < 1 \) is \( O(\log T_j) \) so Lemma 5.5 implies
\[
\frac{g_n'}{g_n}(\sigma + iT_j) = O(\log^2 T_j) + O(\log T_j) = O(\log^2 T_j).
\]

6 PROOF OF THEOREM 1.1

It is sufficient to prove that \( Z^{(n+1)}/Z^{(n)}(t) \) is decreasing for large \( t \). Write \( T_j \) as in Lemma 5.7, for \( k \geq m \) let \( R_k \) be the rectangle with corners at \( 1 - 2k \pm iT_j, 2k \pm iT_j \). Let \( G_n(w) = h(w)g_n(w) \) and \( s = \sigma + it \), where \( s \) is in \( R_k \) and \( t \) is not the ordinate of some zero or pole of \( G_n(s) \). We define \( I \) by
\[
I = \frac{1}{2\pi i} \int_{\partial R_k} \frac{G_n'(w)}{G_n(w)} \frac{s}{w(s-w)} dw. \tag{6.1}
\]

From (2.6) we have
\[
\chi(s)h_n(1-s)g_n(1-s) = (-1)^n h_n(s)g_n(s),
\]
hence we get
\[
\frac{h_n'}{h_n}(s) + \frac{g_n'}{g_n}(s) = -\frac{h_n'}{h_n}(1-s) - \frac{g_n'}{g_n}(1-s) + \omega(s). \tag{6.2}
\]
Write \( h_n(s) \) as
\[
\begin{align*}
    h_n(s) &= \sum_{k=0}^{n} b_{n,k}(s)\omega^k(s) = (-1)^n \frac{\omega^n(s)}{2^n} + \sum_{k=0}^{n-1} b_{n,k}(s)\omega^k(s), \tag{6.3}
\end{align*}
\]
where \( b_{n,k}(s) = c_{n,0,k}(s) \). Differentiating it we have
\[
\begin{align*}
    h_n'(s) &= \frac{(-1)^n}{2^n} n\omega'(s)\omega^{n-1}(s) + \sum_{k=0}^{n-1} b_{n,k}(s)k\omega^{k-1}(s)\omega'(s) + \sum_{k=0}^{n-1} b_{n,k}'(s)\omega^k(s). \tag{6.4}
\end{align*}
\]
Since $b_{n,k}(s)$ is a polynomial in the variables $\omega'(s), \omega''(s), \ldots, \omega^{(n)}(s)$ with constant coefficients, $b'_{n,k}(s)$ is also a polynomial in the variables $\omega'(s), \omega''(s), \ldots, \omega^{(n+1)}(s)$ with constant coefficients. Therefore we have
\[ b_{n,k}(s) = O(1), \quad (6.5) \]
and $b'_{n,k}(s) = O(1)$ in $D$, hence with (3.9), (3.10), (6.4) it follows that $h'_n(s) = O((\log |s|)^{n-1})$ in $D$. With (3.12) if $|s|$ is large we obtain
\[ h'_n(s) = O(1) \quad (6.6) \]
in $D$. From Lemma 5.2 and Lemma 5.7 we have
\[ \frac{g'_n}{g_n}(\sigma + iT_j) = O(\log^2 T_j) \quad (6.7) \]
for $1 - 2m \leq \sigma \leq 2k$. Hence with (3.9), (6.2), (6.6), we have
\[ \frac{g'_n}{g_n}(\sigma + iT_j) = O(\log^2 (k + T_j)) \quad (6.8) \]
for $1 - 2k \leq \sigma \leq 2k$. From the definition of $G_n(s)$ we have
\[ \frac{G'_n}{G_n}(s) = \frac{h'}{h}(s) + \frac{g'_n}{g_n}(s), \quad (6.9) \]
hence with (3.4) and (6.8) we have
\[ \frac{G'_n}{G_n}(\sigma + iT_j) = O(\log^2 (k + T_j)) \quad (6.10) \]
for $1/4 < \sigma \leq 2k$. From (6.2) and (6.9) we have
\[ \frac{G'_n}{G_n}(1 - s) = -\frac{h'_n}{h_n}(s) - \frac{G'_n}{G_n}(s) + \frac{h'}{h}(s) - \frac{h'_n}{h_n}(1 - s) + \frac{h'}{h}(1 - s) + \omega(s) \]
\[ = -\frac{h'_n}{h_n}(s) - \frac{G'_n}{G_n}(s) - \frac{h'_n}{h_n}(1 - s), \quad (6.11) \]
hence with (6.6), (6.10) we have
\[ \frac{G'_n}{G_n}(\sigma + iT_j) = O(\log^2 (k + T_j)) \quad (6.12) \]
for \(1 - 2k \leq \sigma \leq 2k\). If \(w\) is on the horizontal sides of \(\partial R_k\) and \(T_j\) is sufficiently larger than \(t\), then \(|w(s - w)| > T_j^2/2\). Hence with (6.12) we have

\[
\frac{1}{2\pi i} \int \frac{G_n'(w)}{G_n(w)} \frac{s}{w(s - w)} \, dw = O\left(\frac{k \log^2(k + T_j)}{T_j^2}\right),
\]

where the path of integration is the horizontal sides of \(\partial R_k\) and the implied constant depends on \(s\). Let us consider the vertical sides of \(I\). We can write the integrals on the vertical sides as

\[
\frac{1}{2\pi i} \int_{2k - iT_j}^{2k + iT_j} \frac{G_n'(w)}{G_n(w)} \frac{s}{w(s - w)} \, dw = \frac{1}{2\pi i} \int_{2k - iT_j}^{2k + iT_j} \frac{G_n'(w)}{G_n(1-w)} \frac{s}{(1-w)(s-1+w)} \, dw.
\]

From (3.4), (5.2), (6.9), we have

\[
\frac{G_n'(w)}{G_n(w)}(2k + iy) = O(\log(k + |y|)).
\]

Hence with (6.6), (6.11) we have

\[
\frac{G_n'(w)}{G_n(w)}(1 - 2k + iy) = O(\log(k + |y|)).
\]

From (6.15), (6.16) if we write \(w = 2k + iy\) the integral in (6.14) are

\[
O\left(\frac{\log(k + |y|)}{k^2 + y^2}\right),
\]

where the implied constant depends on \(s\). Since

\[
\int_{-\infty}^{\infty} \frac{\log(k + |y|)}{k^2 + y^2} \, dy = \frac{2}{k} \int_{0}^{\infty} \frac{\log(k + kz)}{1 + z^2} \, dz = O(k^{-1} \log k),
\]

we can see that (6.14) is \(O(k^{-1} \log k)\) and with (6.13) we have

\[
I = O(kT_j^{-2} \log^2(k + T_j)) + O(k^{-1} \log k),
\]

where the implied constant depends on \(s\). One can evaluate \(I\) by the residue theorem. By Lemma 4.2 and Lemma 4.4 \(w = 0\) is a simple pole of \(G_n(w)\), hence we have

\[
I = -\frac{G_n'(s)}{G_n(s)} + r_0 + \sum_r \frac{s}{a_r(s - a_r)} - \sum_r \frac{s}{b_r(s - b_r)},
\]
where \(a_r\) runs through the zeros of \(G_n(w)\) in \(R_k\), \(b_r\) runs through the poles of \(G_n(w)\) in \(R_k\) except at \(w = 0\) and \(r_0\) is the residue of the integral in (6.1) at \(w = 0\). If we expand \(s/w(s - w)\) in a Laurent series of powers of \(w\) then the constant term is \(1/s\). Hence with (6.17) as \(j \to \infty\) and \(k \to \infty\) in (6.18) we have

\[
\frac{G_n'(s)}{G_n(s)} = -\frac{1}{s} + A + \sum_{r_1} \frac{s}{a_{r_1}(s - a_{r_1})} - \sum_{r_1} \frac{s}{b_{r_1}(s - b_{r_1})},
\]

where \(A\) is a constant, \(a_{r_1}\) runs through all zeros of \(G_n(w)\) and \(b_{r_1}\) runs through the poles of \(G_n(w)\) except at \(w = 0\). From Lemma 4.6 and Lemma 5.3 these sums are locally uniformly absolutely convergent since \(\sum_{n=1}^{\infty} \log n/n^2\) is convergent. Let \(s = 1/2 + it\) in (6.19) and assume the Riemann hypothesis hereafter. From Proposition 4.8 if we differentiate (6.19) with respect to \(t\) then we have

\[
i \frac{d}{dt} \frac{G_n'(1/2 + it)}{G_n(1/2 + it)} = O(|t|^{-2}) - \sum_{r_1} \frac{1}{(t - \gamma)^2} + \sum_{r_2} \frac{1}{(\frac{1}{2} + it - a_{r_2})^2} - \sum_{r_2} \frac{1}{(\frac{1}{2} + it - b_{r_2})^2},
\]

where \(\gamma\) runs through the zeros of \(G_n(w)\) on the critical line, \(a_{r_2}\) runs through those in the region \(D_1 = \{w|1 - 2m < \Re w < 2m\}\) and \(b_{r_2}\) runs through the poles of \(G_n(w)\) in the same region. From Lemma 4.6 we have

\[
\sum_{r} \frac{1}{(\frac{1}{2} + it - a_r)^2} \ll \sum_{k=m}^{\infty} \frac{n}{(t + 2k + 1)^2} \ll \int_{2m+1}^{\infty} \frac{dx}{(t + x)^2} \ll \frac{1}{|t|},
\]

similarly we have

\[
\sum_{r} \frac{1}{(\frac{1}{2} + it - b_r)^2} = O(|t|^{-1}),
\]

hence with (6.20) we have

\[
i \frac{d}{dt} \frac{G_n'(1/2 + it)}{G_n(1/2 + it)} = O(|t|^{-1}) - \sum_{\gamma} \frac{1}{(t - \gamma)^2},
\]

(6.21)

Let

\[F(t) = \left| h\left(\frac{1}{2} + it\right) \right|.
\]
From Proposition 2.1 we have
\[ Z^{(n)}(t) h_n^{-1} \left( \frac{1}{2} + it \right) h \left( \frac{1}{2} + it \right) = i^n G_n \left( \frac{1}{2} + it \right) e^{i\theta(t)}, \]
hence with the definition of $F(t)$ we get
\[ Z^{(n+1)}(t) = i G' n G_n \left( \frac{1}{2} + it \right) + i h'_n h_n \left( \frac{1}{2} + it \right) - \frac{F'(t)}{F(t)}. \] (6.22)

From (3.2), (3.5) and Lemma 3.1 we have
\[ \omega^{(j)}(s) = O(|t|^{-j}), \] (6.23)
for $D \cap \{ s | -2m + 1 < \sigma < 2m \}$ where $j$ is a positive integer. Since $b_{n,k}^{(j)}(s)$ is a polynomial in the variables $\omega'(s), \omega''(s), \ldots, \omega^{(n+j)}(s)$ with constant coefficients whose constant term vanishes, we have
\[ b_{n,k}^{(j)}(s) = O(|t|^{-1}) \] (6.24)
in the same region. From (3.12) we have
\[ h_n(s) = \frac{(\log |t|)^n}{2^n} + O((\log |t|)^{n-1}), \]
and from (3.9), (6.3), (6.5), (6.23), (6.24) we have
\[ h_n^{(j)}(s) = O(|t|^{-1}(\log |t|)^{n-1}) \]
in the same region. It follows that if $t$ is large then
\[ \frac{d}{ds} \left( \frac{h'_n(s)}{h_n(s)} \right) = \frac{h''_n(s)}{h_n(s)} - \frac{(h'_n(s))^2}{h_n(s)^2} = O \left( (|t| \log |t|)^{-1} \right) \] (6.25)
for $D \cap \{ s | -2m + 1 < \sigma < 2m \}$. It is known (see Edwards [3, p.177]) that
\[ \frac{d}{dt} F'(t) = O(|t|^{-2}). \] (6.26)
If $0 < \gamma < t$ then $0 < t - \gamma < t$ so $(t - \gamma)^2 < t^2$. Hence if $t$ is large then by (6.21), (6.22), (6.25), (6.26) we have
\[
\frac{d}{dt} Z^{(n+1)}(t) = -\sum_{\gamma} \frac{1}{(t - \gamma)^2} + O(t^{-1}) + O(t^{-1}(\log t)^{-1}) + O(t^{-2})
\]
\[
< -\sum_{0<\gamma<t} \frac{1}{(t - \gamma)^2} + A t^{-1}
\]
\[
< -t^{-2} N'_{g_n}(t) + A t^{-1}
\]
\[
= t^{-1}(A - t^{-1} N'_{g_n}(t)), \quad (6.27)
\]
where $N'_{g_n}(T)$ is the number of zeros of $g_n(1/2 + it)$ with $0 < t < T$ and $A$ is a positive constant. From Proposition 4.7 and Proposition 4.8, (6.27) is negative for large $t$. This completes the proof.

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Address: Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya 464-8602, Japan

Email address: m10041v@math.nagoya-u.ac.jp