Polarization of the microwave background in inflationary cosmology

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ABSTRACT

We evaluate the large scale polarization of the cosmic microwave background induced, via Thomson scattering prior to decoupling, by nearly scale-invariant spectra of scalar and tensor metric perturbations, such as those predicted by most inflationary models of the early Universe. We solve the radiative transfer equation for the polarized photon distribution function analytically, for wavelengths of order or larger than the horizon size at decoupling, assuming no reionization. The induced polarization is proportional to the redshift induced by the perturbations at decoupling and to the duration of the decoupling transition. Normalizing the spectra to the quadrupole anisotropy measured by the COBE satellite, the expected degree of linear polarization is $P \lesssim 10^{-7}$, more than two orders of magnitude smaller than current bounds. The dependence of $P$ with the tilt in the spectrum away from scale invariance is too weak to be of observational relevance or to provide a sensitive discrimination between scalar and tensor fluctuations. The variation of $P$ with the angular scale is significant.

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The most plausible explanation of the quadrupole anisotropy in the temperature of the cosmic microwave background measured by the COBE satellite [1], is that it originates in redshifts induced by metric fluctuations around a Robertson-Walker background, through the Sachs-Wolfe effect [2].

The fluctuations in the metric may be scalar (energy-density) or tensor perturbations (gravitational waves). Inflationary cosmological models predict them both, as a result of quantum fluctuations at early stages. Exponential inflation predicts a scale-invariant, gaussian spectrum of scalar fluctuations [3], and a smaller amount of tensor fluctuations [4]. Other inflationary models, for instance power-law inflation [5], predict spectra slightly tilted away from scale invariance. It would be of great value, and would shed much light upon the Universe early evolution, to know what fraction of the quadrupole anisotropy measured by COBE is due to scalar fluctuations and what fraction is due to gravitational waves. Most inflationary models predict a ratio between scalar and tensor fluctuations proportional to the tilt in the spectrum away from scale invariance. In principle, independent measurements of the relative amplitudes of scalar and tensor fluctuations and of the spectral index, could thus provide significant evidence for inflation [3, 4, 5, 6, 11], if they happen to fit the predicted ratio. Put it another way, independent measurements of these parameters would allow a determination of the potential of the field driving inflation [11]. This program however, is not easy to carry out in practice, among other things because of the statistical nature of the predictions (the “cosmic variance” problem), and because discrimination between tensor and scalar sources of anisotropy requires measurements at small angular scales [7, 12], where the primordial spectrum is affected by many different processes that strongly depend on the details of the cosmological model.

Its degree of polarization is another (as yet unmeasured) property of the microwave background radiation, which determination would shed more light upon the Universe early evolution. The polarization of the microwave background radiation is actually tightly related to its anisotropy: Thomson scattering of anisotropic radiation by free electrons before recombination induces some degree of linear polarization [13, 14].

In this paper we evaluate the degree of linear polarization induced both by scalar as well as tensor fluctuations, with a gaussian, power law spectrum, not far away from the scale invariant (Harrison-Zeldovich) type. Our analysis is based on the analytic method and approximations developed by Basko and Polnarev to study the polarization induced by anisotropic expansion of the Universe [14] and by gravitational waves [15]. We extend their method to include the effect of scalar fluctuations. Closely following their approach to
the subject, we solve the radiation transfer equations for the polarized photon distribution function analytically, performing approximations which will prove valid for the large angular scale ($\gtrsim 1^\circ$) polarization. We assume that the Universe is already matter-dominated at recombination, and that there is no further reionization. Polarization measurements of the cosmic microwave background at large angular scales could very well serve as a test of the reionization history \cite{14, 16, 17}, but this is not the focus of the present paper. Instead, we concentrate (assuming no reionization) on the predictions of inflationary models, such as power law inflation. In other words, we normalize the scalar and tensor fluctuations spectra to the COBE measurements of the anisotropy in the microwave background temperature, and determine the expected large angular scale polarization. From Polnarev’s results \cite{15} one expects a rough estimate on the degree of polarization of the microwave background of around a few percent of its large scale anisotropy, and this is a rather small number, about two orders of magnitude smaller than current bounds. We feel, however, that the recent measurement of the quadrupole anisotropy justifies a more detailed analysis of the inflationary predictions for the large angular scale polarization. Our motivation is to find the dependence of the large scale polarization with the spectral index of the fluctuations, which if large could in principle provide an additional indirect test for inflation. Other dependences of the degree of polarization, such as upon the angular scale or the cosmological parameters, are also of interest.

Our work partially overlaps with a recent preprint by Crittenden, Davis and Steinhardt \cite{17}, where the same subject was treated numerically. Our approach, valid only for large angular scales under the assumption of no reionization, is of a more limited applicability than their full numerical treatment. On the other hand, our analytic results, based on the application of Basko and Polnarev’s method \cite{14, 15}, provide some interesting insights into the problem, such as the explicit dependence upon the spectral index, the angular scale and upon parameters of the cosmological model.

The polarization of the cosmic microwave background radiation will be determined from the Stokes parameters that result from solving the Boltzmann equation in a perturbed spatially-flat Robertson-Walker background. The metric reads

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^idx^j],$$  

where $\tau$ is the conformal time, related to proper time $t$ by $d\tau = dt/a(t)$, $|h_{ij}| \ll 1$, and $i, j = 1, 2, 3$. The polarized photon distribution function is represented, in a general case, by a column vector $\hat{n}(\nu, \theta, \phi) = (I_l, I_r, U, V)$, its components being the Stokes parameters of the radiation that arrives from a direction defined by the angles $\theta, \phi$ with frequency $\nu$ \cite{18}.
$I_l, I_r$ are the intensities in two orthogonal directions defined by the unit vectors $\hat{\theta}$ and $\hat{\phi}$ of spherical coordinates. The Boltzmann equation reads

$$
\left( \frac{\partial}{\partial \tau} + e^i \frac{\partial}{\partial x^i} \right) \hat{n} = - \frac{d\nu}{d\tau} \frac{\partial \hat{n}}{\partial \nu} - \sigma_T N_e a \left[ \hat{n} - \frac{1}{4\pi} \int_0^{2\pi} P(\mu, \mu', \phi', \phi') \hat{n} d\mu' d\phi' \right]
$$

Here $P(\mu, \phi, \mu', \phi')$ is a $4 \times 4$ matrix that characterizes the effect of Thomson scattering on polarization [18], $\mu = \arccos \theta$, $e^i$ is a unit vector in the direction of $(\theta, \phi)$, $\sigma_T$ is the Thomson scattering cross section and $N_e$ is the free electron density. The effect of the metric perturbations on the frequency of the photons is given by the Sachs-Wolfe formula, which in the synchronous gauge reads [2]

$$
\frac{1}{\nu} \frac{d\nu}{d\tau} = \frac{1}{2} \frac{\partial h_{ij}}{\partial \tau} e^i e^j
$$

The Stokes parameter $V$ actually decouples from the rest, and we will not take it into further consideration, since it is not necessary to determine the degree of linear polarization, given by $P = I_l + I_r$. $Q = I_l - I_r$.

We describe the growing mode of adiabatic scalar fluctuations in a matter-dominated, spatially-flat Robertson-Walker background, in terms of Bardeen’s gauge-invariant variable $\zeta$, as [19]

$$
h_{ij}(\vec{x}, \tau) = -\frac{1}{15} \tau^2 \frac{\partial^2 \zeta(\vec{x})}{\partial x^i \partial x^j} + \frac{1}{5} \zeta(\vec{x}) \delta_{ij}.
$$

$\zeta$ coincides with the energy-density fluctuations $\delta \rho / \rho$ in a matter-dominated regime for wavelengths inside the horizon. We decompose it in plane waves, $\zeta(\vec{x}) = \int d^3 k e^{i\vec{k} \cdot \vec{x}} \zeta(\vec{k})$, where $\zeta(\vec{k})$ is a gaussian, random variable with statistical average $\langle \zeta(\vec{k}) \zeta(\vec{q}) \rangle = P_\zeta(k) \delta(\vec{k} - \vec{q}) / 4\pi k^3$. $P_\zeta$ defines the spectrum of scalar fluctuations, which we will take to be of the form

$$
P_\zeta(k) = P_\zeta k^{n-1}
$$

where $P_\zeta$ is a constant, and the spectrum is scale invariant (constant amplitude on scales equal to the horizon at any given time) if $n = 1$.

Analogously, tensor fluctuations in the transverse traceless gauge, that enter the horizon during a matter-dominated regime, are described by

$$
h_{ij}(\vec{x}, \tau) = \sum_\lambda \int d^3 k e^{i\vec{k} \cdot \vec{x}} h_\lambda(\vec{k}) \epsilon_{ij}(\lambda, \vec{k}) \left( \frac{3 j_1(k\tau)}{k\tau} \right)
$$

where $\lambda = 1, 2$ denotes the gravitational wave polarization, characterized by $\epsilon_{ij}, j_1$ are spherical Bessel functions, and $h_\lambda$ are random variables with statistical average $\langle h_\lambda(\vec{k}, \tau) h_\lambda(\vec{q}, \tau) \rangle = P_h(k) \delta(\vec{k} - \vec{q}) \delta_{\lambda\lambda'} / 4\pi k^3$. We will consider a power spectrum of the form

$$
P_h(k) = P_h k^{n-1},
$$
which is scale invariant if \( n = 1 \).

We first discuss the polarization induced by one single scalar mode, of the form \( \zeta(\vec{x}) = \zeta(k)e^{i\vec{k} \cdot \vec{x}} \). Since the problem has rotational symmetry around the direction of \( \vec{k} \), only two Stokes parameters need to be considered (\( U = 0 \)). Now \( \hat{n} \) is a two-component column vector, which we expand to first order in the perturbation, as

\[
\hat{n} = \hat{n}_0 + \left( \frac{1}{2} \right) n_0 \hat{n}_1 (\vec{x}, \tau, \nu),
\]

with \( \hat{n}_0 = (1/2)n_0(1 1) \), where \( n_0(\nu) \) is the blackbody distribution. Boltzmann’s equation simplifies considerably:

\[
\left( \frac{\partial}{\partial \tau} + e^i \frac{\partial}{\partial x^i} \right) \hat{n}_1 = -pF \mu^2 \left( \frac{1}{1} \right) - q \left[ \hat{n}_1 - \frac{3}{8} \int_{-1}^1 \left( \begin{array}{c} 2(1 - \mu^2)(1 - \mu'^2) + \mu^2 \mu'^2 \\ \mu^2 \\ 1 \end{array} \right) \hat{n}_1 d\mu' \right]
\]

where \( q = \sigma_T N_0 a, p = \frac{d \ln n_0}{d \ln \nu} \approx 1 \) in the Rayleigh-Jeans zone of the blackbody spectrum, and \( F_S \equiv -\tau\zeta(k)e^{i\vec{k} \cdot \vec{x}}k^2/15 \) arises from the redshift induced by the perturbation, as given by the Sachs-Wolfe equation. We will solve equation (8) in the long wavelength limit, neglecting the spatial derivative in the l.h.s., but anyhow keeping the \( \vec{k} \cdot \vec{x} \)-dependence in the distribution function. This amounts to a quasi-stationary approximation, valid for perturbations with wavelength much longer than the photon mean-free path just prior to decoupling (\( k << q \)), and also much longer than the distance that a photon can travel during the decoupling transition, as we will discuss in more detail later. We normalize the present conformal time \( \tau_0 = 1 \), then \( a(\tau) = \frac{2\tau^2}{H_0} \), and \( q \approx 0.14\Omega_b h X_e (1 + z)^2 \), with \( \Omega_b \) the density in baryons in units of the critical energy density, \( X_e \) the ionization fraction, and \( H_0 = 100 \ h \ km \ sec^{-1} \ Mpc^{-1} \). Thus around decoupling, \( q \approx 10^4 \). Notice that with our normalization \( k \approx 2\pi \) corresponds to a wavelength comparable to the present horizon, and \( k \approx 30(2\pi) \) to the horizon at decoupling.

We solve eq. (8) through the following ansatz for \( \hat{n}_1 \):

\[
\hat{n}_1 = \alpha(\tau) \hat{a} + \beta(\tau) \hat{b} + \gamma(\tau) \hat{c}
\]

where

\[
\hat{a} = (\mu^2 - \frac{1}{3}) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) ; \quad \hat{b} = (1 - \mu^2) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) ; \quad \hat{c} = \left( \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

With this ansatz, and neglecting the spatial derivatives in the l.h.s. of the Boltzmann equation (8), we derive the following equations

\[
\frac{\partial \xi}{\partial \tau} + q \xi = F_S ; \quad \frac{\partial \beta}{\partial \tau} + \frac{3}{10} q \beta = -\frac{q}{10} \xi \quad ; \quad \frac{\partial \gamma}{\partial \tau} = \frac{1}{3} F_S
\]

where we have defined the variable \( \xi = \alpha + \beta \), which decouples from \( \beta \), and is a measure of the anisotropy in the photon distribution function. \( \beta \) instead is a measure of the polarization,
since $I_t - I_c = n_0\beta(1 - \mu^2)$, and the anisotropy, as measured by $\xi$, is its source. $\gamma$, on the other hand, decouples from the rest, and just reflects a local adjustment of the black body intensity, independent of the direction of propagation. The equations for $\xi$ and $\beta$ can be formally integrated as follows \[13\]

$$
\xi(\tau) = \int_0^\tau F_S(\tau') e^{-\kappa(\tau, \tau')} d\tau'
$$

(12)

$$
\beta(\tau) = -\frac{1}{10} \int_0^\tau q(\tau')\xi(\tau') e^{-\frac{3}{10}\kappa(\tau, \tau')} d\tau'
$$

(13)

where $\kappa(\tau, \tau') = \int_\tau^{\tau'} q(x') dx''$ is the optical depth. We wish to evaluate $\beta(\tau_0)$ from eq.(13).

This equation shows that the present polarization of the microwave background (assuming no reionization) is essentially that produced at the times when $qe^{-\frac{3}{10}\kappa(\tau_0, \tau')}$ is significantly different from zero, i.e. around the time of recombination, since much later the free electron density (and thus $q$) is negligible, while much earlier the optical depth is very large. To be more precise, the present polarization is the result of Thomson scattering around the time of decoupling of matter and radiation, which occurs slightly after the free electron density starts to drop significantly. Although these values depend on the particular cosmological model considered (through $\Omega, \Omega_b, H_0$, etc.), typically recombination occurs around $z \approx 1400$ while decoupling takes place between $z \approx 1200$ and $z \approx 900$. This is the interval during which the differential visibility differs significantly from zero, and is actually well approximated by a gaussian \[20\]. We will denote by $\tau_d$ the conformal time around which decoupling occurs, and $\Delta\tau_d$ the duration of the process. These quantities actually do not depend very strongly on the cosmological parameters, typical values being (with the convention $\tau_0 = 1$), $\tau_d \approx 3 \times 10^{-2}$ and $\Delta\tau_d \approx 3 \times 10^{-3}$ \[20, 21\].

As explained above, we only need to know $\xi(\tau)$ around $\tau_d$ in order to integrate eq.(13). If we define $\kappa(\tau) \equiv \kappa(\tau_0, \tau)$, then $\kappa(\tau_0, \tau') = \kappa(\tau') - \kappa(\tau)$, and we may approximate around $\tau_d$:

$$
\xi(\tau) \approx F_S(\tau_d)e^{\kappa(\tau)} \int_0^\tau e^{-\kappa(\tau')} d\tau' \approx F_S(\tau_d)\Delta\tau_d e^{\kappa(\tau)} E(\kappa(\tau))
$$

(14)

where in the last step we made the approximation $\frac{d\kappa}{d\tau'} \approx \frac{-\kappa(\tau')}{\Delta\tau_d}$, which is justified by the almost gaussian shape of the differential visibility around decoupling, and we defined $E(\kappa) = \int_1^{\infty} e^{-\kappa x} dx$. Then, using that $-qd\tau = dk$, we integrate $\beta$:

$$
\beta \approx -\frac{1}{10} F_S(\tau_d)\Delta\tau_d \int_0^{\infty} e^{-\frac{3}{10}\kappa} e^\kappa E(\kappa) d\kappa \approx -0.17 F_S(\tau_d)\Delta\tau_d
$$

(15)

Finally, the polarization produced by a single scalar mode is given by

$$
P_S^2 = |\beta|^2(1 - \mu^2)^2
$$

(16)
The polarization induced by a single tensor mode of a given polarization can be calculated in a similar way. Now, as opposed to the scalar case, there is no longer rotational invariance around the direction of $\vec{k}$, thus $U \neq 0$, and $\hat{n}$ is a three column vector $(I_l, I_r, U)$, which we expand as $\hat{n} = \hat{n}_0 + (1/2)n_0 \hat{n}_1(\vec{x}, \tau, \nu, \phi)$, with $\hat{n}_0 = (1/2)n_0(1 1 0)$. The angular dependence of the redshift produced by the gravitational wave due to the Sachs-Wolfe effect suggests an ansatz for $\hat{n}_1$ of the form: 

$$\hat{n}_1 = \alpha(\tau)\hat{a} + \beta(\tau)\hat{b}$$

where

$$\hat{a} = (1 - \mu^2) \cos(2\phi) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad \hat{b} = \begin{pmatrix} (1 + \mu^2) \cos(2\phi) \\ -(1 + \mu^2) \cos(2\phi) \\ 4\mu \sin(2\phi) \end{pmatrix}$$

This corresponds to a gravitational wave with polarization $h_\perp$. The formulae for a polarization $h_\times$ are identical to these after interchange of $\cos(2\phi)$ with $\sin(2\phi)$. With this ansatz, the equations for $\zeta = \alpha + \beta$ and $\beta$ are exactly the same as those in eq.(11) for the scalar case, but with $F_S$ replaced by $F_T \equiv \frac{1}{2}e^{i\vec{k} \cdot \vec{x}}h_\lambda(\vec{k}) \frac{d}{d\tau} \frac{3j_3(k\tau)}{kr}$. Under the same approximations assumed valid for the scalar case, the result for $\beta$ is then $\beta \approx 0.17F_T(\tau_d)\Delta \tau_d$. Finally, the polarization induced by a tensor mode is given by

$$P^2_T = |\beta|^2[(1 + \mu^2)^2 + 4\mu^2]$$

where we have already summed over the two polarizations of the gravitational wave.

Both for scalar as well as tensor fluctuations, $\beta \propto F(\tau_d)\Delta \tau_d$, with the appropriate $F$ in each case. This result for $\beta$, which is a measure of the induced polarization, has an intuitive explanation, after noticing that $F$ measures the anisotropy in the rate of change in the frequency of the cosmic background photons. Indeed, the Sachs-Wolfe formula for the scalar and tensor modes can be written as

$$\frac{1}{\nu} \frac{d\nu}{d\tau} = F_S \mu^2; \quad \frac{1}{\nu} \frac{d\nu}{d\tau} = F_T(1 - \mu^2) \cos(2\phi).$$

This anisotropy in the redshift translates into a difference in the number of photons of a given frequency traveling in different directions, and this is what makes possible a net polarization after Thomson scattering. Besides, the polarization builds up over a period $\Delta \tau_d$ only, since anisotropies in the photon distribution produced much before decoupling are erased by successive scatterings, while after decoupling scattering is negligible and thus polarization can no longer be induced.

Next we evaluate the rms total polarization that results from the superposition of all the
statistically independent contributions of scalar and tensor modes respectively:

\[
\langle P^2 \rangle_S = \int d^3k \left( 1 - \mu^2 \right)^2 |\beta|^2 = \frac{8}{15} \int_{k_{\text{max}}}^{k_{\text{max}}} \frac{dk}{k} \left( P_\zeta k^{n-1} \right) \left[ 0.17 \Delta \tau_d \frac{k^2 \tau_d}{15} \right]^2 \\
\langle P^2 \rangle_T = \int d^3k \left( (1 + \mu^2)^2 + 4 \mu^2 \right) |\beta|^2 = \frac{4}{5} \int_{k_{\text{max}}}^{k_{\text{max}}} \frac{dk}{k} \left( P_h k^{n-1} \right) \left[ 0.17 \Delta \tau_d \frac{d}{d\tau} \left( \frac{3 j_1(k\tau)}{k\tau} \right) \right]^2 
\]

We have introduced a cut-off, \( k_{\text{max}} \), determined by the minimum wavelength to which a given experimental setup is sensitive. In general, measurements result from an average over a certain angular scale, for instance the aperture of the antenna’s horn. With our conventions, a wavelength with comoving wavevector \( k \) subtends an angle \( \theta \) such that \( k \approx \pi / \sin(\theta/2) \). This relation determines \( k_{\text{max}} \) for measurements averaged over an angle \( \theta \).

Finally, for scalar perturbations

\[
\langle P^2 \rangle_S = \frac{8}{15} \frac{P_\zeta k_{\text{max}}^{n-1}}{(n+3)} \left[ 0.17 \Delta \tau_d \frac{k^2 \tau_d}{15} \right]^2 
\]

while the result for tensor perturbations can be written as

\[
\langle P^2 \rangle_T = D_p(n, k_{\text{max}}) \langle P^2 \rangle_S 
\]

where

\[
D_p \equiv 13.5 \ g(n, k_{\text{max}}) \frac{P_h}{P_\zeta} \quad ; \quad g(n, k_{\text{max}}) \equiv 25(n+3) \int_{0}^{k_{\text{max}} \tau_d} dx \ x^n \left( \frac{d}{dx} \left( \frac{3 j_1(x)}{x} \right) \right)^2 
\]

With this definition, \( g(n, k_{\text{max}}) \to 1 \) when \( k_{\text{max}} \tau_d \ll 1 \), i.e. for wavelengths well outside the horizon at decoupling. The total rms polarization is given by the sum of the contributions due to scalar and tensor modes, given their statistical independence:

\[
\sqrt{\langle P^2 \rangle} = \sqrt{\langle P^2 \rangle_S + \langle P^2 \rangle_T} 
\]

Now we assume that the scalar and tensor perturbations have as common origin a period of power law inflation, in which case the scalar and tensor power spectra are related by \( P_h / P_\zeta \approx 4(n-1)/(9(n-3)) \) \[7,8,9\]. The ratio of the tensor and scalar contribution to the quadrupole moment in the temperature anisotropy can also be approximated by a simple function of \( n \), \( \langle a^2 \rangle_T / \langle a^2 \rangle_S \equiv D_a \approx 13(n-1)/(n-3) \) \[9\]. Assuming that the anisotropy measured by COBE is due to these inflation-produced perturbations alone, then it is simple to express both \( P_\zeta \) and \( P_h \) in terms of the measured quadrupole and the spectral index. For instance:

\[
P_\zeta = \frac{45}{4\pi^2} \frac{2^{1-n}(3-n)}{(8-7n)} \frac{\Gamma((9-n)/2) \Gamma^2(2-n/2)}{\Gamma((3+n)/2) \Gamma(3-n)} \langle a^2 \rangle 
\]
with $\langle a^2 \rangle = \langle a^2 \rangle_S + \langle a^2 \rangle_T \approx 2 \times 10^{-5}$, the quadrupole measured by COBE. Finally, the total rms polarization predicted by power-law inflationary models, as a function of the spectral index $n$, is given by

$$P \approx 1.2 \times 10^{-7} \left( \frac{\Delta \tau_d}{3 \times 10^{-3}} \right)^2 \left( \frac{k_{\text{max}}}{50} \right)^2 \left( \frac{\tau_d}{3 \times 10^{-2}} \right) \left( \frac{\sqrt{\langle a^2 \rangle}}{2 \times 10^{-5}} \right) C(n, k_{\text{max}})$$

with

$$C^2 \equiv \frac{8}{3(n+3)\pi} k_{\text{max}}^{n-1} \frac{\Gamma((9-n)/2)}{2^{n-1}\Gamma((3+n)/2)} \frac{\Gamma(2-n/2)}{\Gamma(3-n)} \frac{(1+D_p)}{(1+D_a)}$$

(28)

The function $C$ has been defined such that $C = 1$ if $n = 1$. The value $k_{\text{max}} \approx 50$ corresponds to measurements that average over an angular scale of order $\theta \approx 7^\circ$.

In Fig. 1 we plot the total rms polarization for $k_{\text{max}} = 50$, as well as the individual tensor and scalar contributions, for values of $n$ between 0.5 and 1, those consistent with COBE measurements. The predicted degree of polarization is more than two orders of magnitude smaller than current bounds, $P < 6 \times 10^{-5}$ [22], obtained with a $7^\circ$ aperture horn at 33 GHz. The polarization induced by tensor modes vanishes if $n = 1$ and equals the scalar contribution when $n = 0.5$. The total polarization decreases by a factor about 0.6 as $n$ varies from 1 to 0.8 ($n \approx 0.8$ corresponds to equal contributions of scalar and tensor modes to the large scale anisotropy), and about 0.4 between $n = 1$ and $n = 0.5$. Notice that the dependence of the total polarization upon $n$ is mainly due to the tilt in the spectra, i.e. it comes mostly from the term $k_{\text{max}}^{(n-1)/2}$. This is because polarization is proportional to the redshift induced by the perturbations at decoupling, and is no longer produced afterwards, while anisotropy continues to build up. Consequently, the dominant wavelengths for polarization are much smaller than those that dominate the large scale anisotropy. Once the spectra are normalized to the large scale anisotropy measured by COBE, smaller $n$ implies less power on smaller scales.

A large class of inflationary models predict a definite ratio between the spectral index $n$ and the scalar and tensor contributions to the anisotropy in the microwave background [7, 8]. We have shown, applying the methods developed by Basko and Polnarev [14, 15] to the specific spectra predicted by inflationary models, that there is also a definite relation between the net polarization and $n$, at least assuming no reionization. Unfortunately, the rather weak dependence of $P$ with $n$ makes it unlikely that measurements of polarization could provide a sensitive confirmation of these predictions. Our conclusions in this regard are similar to those of Crittenden, Davis and Steinhardt [17], that in a recent preprint approached the same subject numerically. Our analytic results, on the other hand, are only
valid if \( k_{\text{max}} \Delta \tau_d << 1 \), which means \( k_{\text{max}} << 300 \), while the numerical treatment of Ref. [17] applies to a more general situation.

The large scale polarization depends upon cosmological parameters, such as \( \Omega, \Omega_b, H_0 \), etc., only through \( \tau_d \) and \( \Delta \tau_d \), \textit{i.e.} through the parameters that characterize the decoupling transition, and this is a very weak dependence [20].

Notice finally that \( P \) is proportional to \( k_{\text{max}}^2 \). This is a consequence of the fact that the polarization induced by a single mode is proportional to the redshift it causes at decoupling, irrespective of its wavelength, and because in the linear regime the perturbations grow as \((k\tau)^2\). Thus, the dependence of the degree of polarization with the angular scale over which polarization data is averaged is significant.

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**Figure Caption**

Fig. 1: Total rms large scale ($\approx 7^\circ$) polarization of the cosmic microwave background $P = \sqrt{P^2_T + P^2_S}$ and individual scalar and tensor contributions, $P_S$ and $P_T$, as a function of the spectral index $n$ in power-law inflationary models, assuming no reionization.