A note on homomorphisms between products of algebras

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Abstract. Let $\mathcal{K}$ be a congruence distributive variety and call an algebra hereditarily directly irreducible (HDI) if every of its subalgebras is directly irreducible. It is shown that every homomorphism from a finite direct product of arbitrary algebras from $\mathcal{K}$ to an HDI algebra from $\mathcal{K}$ is essentially unary. Hence, every homomorphism from a finite direct product of algebras $A_i$ ($i \in I$) from $\mathcal{K}$ to an arbitrary direct product of HDI algebras $C_j$ ($j \in J$) from $\mathcal{K}$ can be expressed as a product of homomorphisms from $A_{i_\sigma(j)}$ to $C_j$ for a certain mapping $\sigma$ from $J$ to $I$. A homomorphism from an infinite direct product of elements of $\mathcal{K}$ to an HDI algebra will in general not be essentially unary, but will always factor through a suitable ultraproduct.

Mathematics Subject Classification. 06B05.

Keywords. Direct product of chains, Homomorphism, Essentially unary mapping, Ultrafilter.

1. Introduction

Let $A_i$, $i \in I$, and $B_j$, $j \in J$, be algebras of the same type and $f$ a homomorphism from $\prod_{i \in I} A_i$ to $\prod_{j \in J} B_j$. For every $k \in J$ let $p_k$ denote the projection from $\prod_{j \in J} B_j$ onto $B_k$ and $f_k := p_k \circ f$. More generally, for any $J_0 \subseteq J$ we let $p_{J_0} : \prod_{j \in J} B_j \rightarrow \prod_{j \in J_0} B_j$ be the canonical projection map. It is evident that $f = (f_j : j \in J)$. Hence, the task of describing $f$ is reduced to the task of describing the homomorphisms $f_k$ from $\prod_{i \in I} A_i$ to $B_k$.
In [3] the authors solve this problem for the case that the algebras $A_i$ and $B_j$ are conservative median algebras and the index sets are finite. More generally, Couceiro et al. [2] considers the case that the $A_i$ are median algebras and the $B_j$ are tree-median algebras. It turns out that the method developed in [2, 3] can be further generalized to lattices. Let us note that every distributive lattice is a median algebra (but not conversely). We are even able to extend this result to arbitrary lattices $A_i$ provided the $B_j$ are chains. For lattice concepts used in the rest of the paper the reader is referred to the monographs [1, 5].

We call a mapping $f: \prod_{i \in I} A_i \to C$ essentially unary if there exists an $i_0 \in I$ and a mapping $g: A_{i_0} \to C$ with $g \circ p_{i_0} = f$. In this case we say that "$f$ depends only on the $i_0$-th coordinate", or that "$f$ factors through $p_{i_0}$".

From $f = g \circ p_{i_0}$ it easily follows that $g$ is a homomorphism if and only if $f$ is.

2. Fraser–Horn property and HDI algebras

**Definition 2.1.** A class $\mathcal{K}$ of algebras has the Fraser–Horn property if there are no skew congruences on any product $A_1 \times A_2$ with $A_1, A_2 \in \mathcal{K}$, or more explicitly:

For all $A_1, A_2 \in \mathcal{K}$, for every congruence $\theta \in \text{Con}(A_1 \times A_2)$ there are congruences $\theta_1 \in \text{Con}(A_1)$, $\theta_2 \in \text{Con}(A_2)$ such that $\theta = \theta_1 \times \theta_2$, i.e. $\theta = \{((x_1, x_2), (y_1, y_2)) \mid x_1 \theta_1 y_1, x_2 \theta_2 y_2\}$.

The following lemma is known from [4].

**Lemma 2.2.** Let $\mathcal{K}$ be a congruence distributive (CD) variety. Then $\mathcal{K}$ has the Fraser–Horn property.

For the rest of the paper we fix a variety $\mathcal{K}$ with the Fraser–Horn property.

We call an algebra non-trivial if its universe contains at least two elements.

**Definition 2.3.** We call an algebra $A$ hereditarily directly irreducible (HDI) if every subalgebra $B \leq A$ is directly irreducible, i.e., is not isomorphic to a direct product of two non-trivial factors.

**Fact 2.4.** (1) The variety of lattices is congruence distributive.

(2) A lattice is HDI if and only if it is a chain.

**Theorem 2.5.** Let $\mathcal{K}$ be a variety with the Fraser–Horn property. If $n$ is a positive integer, $A_1, \ldots, A_n$ are in $\mathcal{K}$ and $C \in \mathcal{K}$ is HDI, then every homomorphism $f$ from $A_1 \times \cdots \times A_n$ to $C$ is essentially unary, i.e., factors through one of the projections $p_i$.

*Proof.* Let $\theta = \ker(f)$. By a straightforward generalization of the Fraser–Horn property we know that $\theta = \theta_1 \times \cdots \times \theta_n$, where each $\theta_i$ is a congruence on $A_i$.

The homomorphism theorem tells us that $B' := f(A_1 \times \cdots \times A_n)$ is isomorphic to the direct product $(A_1/\theta_1) \times \cdots \times (A_n/\theta_n)$. By our assumption, $B'$ is directly irreducible, so at most one of these factors can be non-trivial,
so there is at most one \( i \) such that \( A_i/\theta_i \) has more than one element. So \( f \) depends only on the \( i \)-th coordinate. \( \square \)

**Remark 2.6.** If \( f: A_1 \times A_2 \to C \) is not constant, then there is at most one \( i \in \{1, 2\} \) such that \( f \) factors through \( p_i \).

As a consequence of the above theorem we obtain the following statement.

**Theorem 2.7.** If \( n \) is a positive integer, \( A_1, \ldots, A_n \in \mathcal{K} \), where \( \mathcal{K} \) has the Fraser–Horn property, \((C_j; j \in J)\) is a non-empty family of HDI algebras in \( \mathcal{K} \), and \( f \) is a homomorphism from \( A_1 \times \cdots \times A_n \) to \( \prod_{j \in J} C_j \) then there exists a mapping \( \sigma: J \to \{1, \ldots, n\} \) and for every \( j \in J \) a homomorphism \( g_j \) from \( A_{\sigma(j)} \) to \( C_j \) such that

\[
    f(x_1, \ldots, x_n) = (g_j(x_{\sigma(j)}); j \in J)
\]

for all \((x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n.\)

**Proof.** Apply Theorem 2.5 to the mappings \( f_j := p_j \circ f, j \in J. \) \( \square \)

**Theorem 2.8.** Let \( \mathcal{K} \) be a variety with the Fraser–Horn property. Let \( n, k \) be positive integers and let \( A_1, \ldots, A_n, C_1, \ldots, C_k \) be non-trivial HDI algebras in \( \mathcal{K} \) and assume \( A_1 \times \cdots \times A_n \cong C_1 \times \cdots \times C_k \). Then \( n = k \) and there exists a permutation \( \sigma \in S_n \) such that \( C_i \cong A_{\sigma(i)} \) for all \( i = 1, \ldots, n. \)

**Proof.** Let \( f \) denote an isomorphism from \( A_1 \times \cdots \times A_n \) to \( C_1 \times \cdots \times C_k \). According to Theorem 2.7, there exist mappings \( \sigma \) from \( \{1, \ldots, k\} \) to \( \{1, \ldots, n\} \) and \( \tau \) from \( \{1, \ldots, n\} \) to \( \{1, \ldots, k\} \), for every \( j \in \{1, \ldots, k\} \) a homomorphism \( g_j \) from \( A_{\sigma(j)} \) to \( C_j \) and for every \( i \in \{1, \ldots, n\} \) a homomorphism \( h_i \) from \( C_{\tau(i)} \) to \( A_i \) such that

\[
    f(x_1, \ldots, x_n) = (g_1(x_{\sigma(1)}), \ldots, g_k(x_{\sigma(k)}))
\]

for all \((x_1, \ldots, x_n) \in A_1 \times \cdots \times A_n\) and

\[
    f^{-1}(y_1, \ldots, y_k) = (h_1(y_{\tau(1)}), \ldots, h_n(y_{\tau(n)}))
\]

for all \((y_1, \ldots, y_k) \in C_1 \times \cdots \times C_k. \) The injectivity of \( f \) implies \( k \geq n \) and the injectivity of \( f^{-1} \) implies \( n \geq k. \) This shows \( n = k. \) Moreover, again since \( f \) is injective we have \( \sigma \in S_n. \) Finally, the injectivity of \( f \) implies the injectivity of \( g_1, \ldots, g_n \) and the surjectivity of \( f \) implies the surjectivity of \( g_1, \ldots, g_n. \) This shows that \( g_1, \ldots, g_n \) are isomorphisms, i.e. \( C_i \cong A_{\sigma(i)} \) for all \( i = 1, \ldots, n. \) \( \square \)

**Corollary 2.9.** If an algebra in \( \mathcal{K} \) is isomorphic to a finite product of non-trivial HDI algebras, then these factors are uniquely determined up to order and isomorphisms.

**Proof.** This follows from Theorem 2.8. \( \square \)

We can generalize this to infinite direct products as follows. Recall that an ultrafilter on a set \( I \) is a family \( U \) of subsets of \( I \) which is upwards closed and also closed under intersections such that for all \( I_0 \subseteq I \) exactly one of \( I_0, \)
$I \setminus I_0$ is in $U$. For any family $(A_i : i \in I)$ of sets and any ultrafilter $U$ on $I$ we define the equivalence relation $\sim_U$ on $\prod_i A_i$ by

$$(x_i : i \in I) \sim_U (y_i : i \in I) \iff \{i \in I \mid x_i = y_i\} \in U,$$

and we write $\prod_i A_i / U$ for the set of equivalence classes, the "ultraproduct of the $A_i$ modulo $U". The canonical map from $\prod_i A_i$ to $\prod_i A_i / U$ is denoted by $\kappa_U$. If $(A_i)_{i \in I}$ is a family of algebras of the same type, then the relation $\sim_U$ is a congruence relation on the product $\prod_i A_i$.

**Theorem 2.10.** Let $\mathcal{K}$ be a variety with the Fraser–Horn property. Let $I$ be a non-empty set, and for each $i \in I$ let $A_i$ be an algebra in $\mathcal{K}$. Let $C$ be an HDI algebra in $\mathcal{K}$, and let $h : \prod_{i \in I} A_i \to C$ be a homomorphism which is not constant. Then there is a unique ultrafilter $U$ on $I$ such that $h$ factors through $\kappa_U$, i.e., there is a homomorphism $h' : \prod_{i \in I} A_i / \sim_U \to C$ such that $h = h' \circ \kappa_U$.

In particular: If there is no $i \in I$ such that $h$ factors through $p_i$, then $U$ will be a non-principal ultrafilter.

**Proof.** Let $U$ be defined as the set of all $M \subseteq I$ such that $h$ factors through $p_M$, i.e., such that there exists $f_M : \prod_{i \in M} A_i \to C$ with $h = f_M \circ p_M$.

It is clear that $U$ is upwards closed, and from Theorem 2.5 and Remark 2.6 we get: If $M_1 \subseteq I$ and $M_2 := I \setminus M_1$, then $M_1 \in U$ and $M_2 \notin U$ or conversely. As $h$ is not constant, we have $\emptyset \notin U$.

We now show that $U$ is closed under intersections: Given $M_1, M_2 \in U$, then we can write $\prod_{i \in I} A_i$ as the direct product of four factors:

$$B_{11} = \prod_{i \in M_1 \cap M_2} A_i, B_{10} = \prod_{i \in M_1 \setminus M_2} A_i, B_{01} = \prod_{i \in M_2 \setminus M_1} A_i, B_{00} = \prod_{i \notin M_1 \cup M_2} A_i,$$

with corresponding projections $p_{11}, p_{10}, p_{01}, p_{00}$.

Since none of the sets $M_1 \setminus M_2, M_2 \setminus M_1$, and $I \setminus (M_1 \cup M_2)$ are in $U$, $h$ cannot factor through any of $p_{10}, p_{01},$ or $p_{00}$. Hence (by Theorem 2.5), $h$ must factor through $p_{11}$, so $M_1 \cap M_2 \in U$. So we have shown that $U$ is a filter, and even an ultrafilter.

We now check that $h$ factors through the canonical map $\kappa_U : \prod_i A_i \to \prod_i A_i / U$. All we have to show is that for all $x, y \in \prod_i A_i$ we have $h(x) = h(y)$. Now $x \sim_U y$ implies that the set $M := \{i \mid x(i) = y(i)\}$ is in $U$; by definition of $U$, there is some $f_M$ with $h = f_M \circ p_M$, so we get $h(x) = f(p_M(x)) = f(p_M(y)) = h(y)$.

Finally, we show that $U$ is unique. So let $U'$ be an ultrafilter such that $h$ factors through $\kappa_{U'}$. It is enough to show $U' \subseteq U$.

Let $M \in U'$, and let $U' \setminus M := \{N \cap M \mid N \in U'\}$ be the restriction of $U'$ to $M$. The map $\kappa_{U'}$ can be written as $\kappa_{U'} = \kappa_{U' \setminus M} \circ p_M$; as $h$ factors through $\kappa_{U'}$, $h$ also factors through $p_M$, so $M \in U$. $\square$

**Remark 2.11.** Theorem 2.5 was used in the proof of Theorem 2.10; but we can also view Theorem 2.5 as a special case of Theorem 2.10, as any ultrafilter on a finite index set must be principal.
3. Lattices

Theorem 2.10 is in some sense best possible, in the sense that homomorphisms from an infinite product $\prod_{i} A_i$ into an HDI algebra will in general not factor through any single projection $p_j$, as the following example shows.

Example 3.1. Let $U$ be an ultrafilter on the infinite index set $I$, and for each $i \in I$ let $A_i$ be the 2-element lattice $\{0, 1\}$. Then the ultraproduct $\prod_{i \in I} A_i/U$ is again the 2-element lattice.

Identifying $\prod_{i} A_i$ with the power set lattice $(P(I), \cup, \cap)$, the canonical map $\kappa_U: P(I) \to \{0, 1\}$ maps each element of $U$ to 1 and everything else to 0. If $U$ is a non-principal ultrafilter, then $h_U$ does not factor through any projection.

This example can be generalized to any Fraser–Horn variety where the class of HDI algebras is described by a set of first order formulas: If $\prod_{i} A_i$ is a product of algebras, and $(h_i : i \in I)$ is a family of homomorphisms $h_i: A_i \to C_i$, where each $C_i$ is HDI, then the family $(h_i : i \in I)$ naturally defines a homomorphism $h: \prod_{i} A_i \to \prod_{i} C_i$.

If $U$ is an ultrafilter on $I$, then the algebra $C := \prod_{i} C_i/U$ is again HDI (as $C$ satisfies all first order statements that are true in each $C_i$). Let $\kappa_U^C: \prod_{i} C_i \to \prod_{i} C_i/U$ and $\kappa_U^A: \prod_{i} A_i \to \prod_{i} A_i/U$ be the canonical maps. Then the map $\bar{h} := \kappa_U^C \circ h: \prod_{i} A_i \to C$ trivially factors through $\kappa_U^A$, i.e., there is $h': \prod_{i} A_i/U \to C$ with $\bar{h} = h' \circ \kappa_U^A$. By the uniqueness claim in Theorem 2.10, we see that $U$ is the set of all $M \subseteq I$ such that $\bar{h}$ factors through $p_M$. So if $U$ is non-principal, then $\bar{h}$ does not factor through any projection.

Fact 3.2. Let $A$ be a lattice. Then the following are equivalent:

- There is a non-constant homomorphism from $A$ into a chain.
- There is a non-constant homomorphism from $A$ into the 2-element chain.
- The lattice $A$ has a prime ideal.

The following corollary can be seen as a weak version of Theorem 2.5.

Corollary 3.3. The class of lattices without a prime ideal is closed under finite direct products.

The following example shows that even this weak version cannot be generalized to infinite products, not even if all factors are equal.

Example 3.4. (a) There are non-trivial lattices $M$ such that no (finite or infinite) direct power of $M$ has a prime ideal.
(b) On the other hand, there are lattices $A$ without a prime ideal such that any infinite direct power $A^I$ will contain a prime ideal.

Proof of (a). Let $M$ be the class of all lattices of height 3 with at least 5 elements, i.e., the class of all bounded lattices $M$ in which all elements except for $\sup M$ and $\inf M$ are incomparable. It is clear that no lattice in $M$ has a prime ideal. The class $M$ is closed under ultraproducts, since the property of being in $M$ can be expressed by a first order statement.
If $M = \prod_{i \in I} M_i$ is an arbitrary direct product with factors $M_i \in \mathcal{M}$, and $h : M \to C$ is a homomorphism into a chain, then $h$ factors through an ultraproduct $\prod_{i \in I} M_i \to \prod_{i \in I} M_i/U \to C$, $h = h' \circ \kappa_U$. The map $h'$ and therefore also $h$ must be constant. □

Proof of (b). Let $A$ be the lattice obtained from $\mathbb{N} = \{0, 1, 2, \ldots\}$ by replacing each odd number $2k + 1$ by a 3-element antichain $a_k, b_k, c_k$, and each even number $2k$ by a new element $d_k$. It is easy to see that $A$ has no prime ideal.

We will show that every infinite power $A^I$ contains a prime ideal. Clearly it is enough to show this for the case of countable $I$, say $I = \mathbb{N}$.

For any ultrafilter $U$ on $\mathbb{N}$ the following set $J_U$ is an ideal on $\prod_{i \in I} A_i$:

$$J_U := \{(x_n : n \in \mathbb{N}) \mid \exists k \exists C \in U \forall n \in C : x_n \leq d_k\}$$

We now show that $J_U$ is a prime ideal. If

$$\bar{x} = (x_i : i \in \mathbb{N}), \quad \bar{y} = (y_i : i \in \mathbb{N}), \quad \bar{z} = (z_i : i \in \mathbb{N}), \quad \bar{x} \wedge \bar{y} = \bar{z} \in J_U,$$

then there is some set $C \in U$ and some natural number $k \in \mathbb{N}$ such that $z_n \leq d_k$ holds for all $n \in C$. Now the two sets

$$\{n : x_n \geq d_{k+1}\}, \quad \{n : y_n \geq d_{k+1}\}$$

cannot both belong to $U$, as their intersection $D$ is disjoint to $C$. (Since $n \in D$ implies $x_n \wedge y_n \geq d_{k+1}$.)

Without loss of generality we have $\{n : x_n \leq d_{k+1}\} \in U$, so $\bar{x} \in J_U$. □

Acknowledgements

Open access funding provided by TU Wien (TUW). We are grateful to the referee of a previous version of this paper for alerting us to [4], and to Gábor Czédli for suggesting the definition of HDI algebras.

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Received: 3 April 2017.
Accepted: 15 January 2018.