THE \( r \)TH MOMENT OF THE DIVISOR FUNCTION:
AN ELEMENTARY APPROACH

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Abstract. Let \( \tau(n) \) be the number of divisors of \( n \). We give an elementary proof of the fact that
\[
\sum_{n \leq x} \tau(n)^r = xC_r (\log x)^{2r-1} + O(x(\log x)^{2r-2}),
\]
for any integer \( r \geq 2 \). Here,
\[
C_r = \frac{1}{(2^r - 1)!} \prod_{p \geq 2} \left( 1 - \frac{1}{p} \right)^{2r} \left( \sum_{\alpha \geq 0} \frac{(\alpha + 1)^r}{p^{\alpha}} \right).
\]

1. Introduction

Let \( \tau(n) \) be the number of divisors of \( n \). Ramanujan \([2]\) stated without proof that, given any real number \( \varepsilon > 0 \), the estimate
\[
\sum_{n \leq x} \tau(n)^2 = x(A(\log x)^3 + B(\log x)^2 + C \log x + D) + O(x^{3/5+\varepsilon})
\]
holds with \( A = \pi^{-2} \). An elementary proof of the asymptotic formula
\[
\sum_{n \leq x} \tau(n)^2 \sim Ax(\log x)^3,
\]
as \( x \to \infty \), appears in several places (see, for example, \([1, \text{Thm. 7.8}]\)). Wilson \([3]\) proved Ramanujan’s claim and generalized it by showing that for any integer \( r \geq 2 \) one has
\[
\sum_{n \leq x} \tau(n)^r = x(C_{r,1}(\log x)^{2r-1} + C_{r,2}(\log x)^{2r-2} + \cdots + C_{r,2^r}) + O(x^{2r-2 + \varepsilon}).
\]
Note that when \( r = 2 \), Wilson’s error term is better than the one claimed by Ramanujan. We are not aware even of elementary proofs for the asymptotic formula
\[
\sum_{n \leq x} \tau(n)^r \sim C_r x(\log x)^{2r-1}
\]
as \( x \to \infty \) for any \( r \geq 2 \). In this note, we give an elementary proof of the following more general result.

**Theorem 1.** Let \( k \) be a positive integer and \( f(n) \) be a multiplicative function which on prime powers \( p^\alpha \) satisfies

\[
f(p) = k \quad \text{and} \quad f(p^\alpha) = \alpha^{O(1)} \quad \text{for all primes } p \text{ and integers } \alpha \geq 2,
\]

where the constant implied by the above \( O \) is uniform in \( p \). Then

\[
\sum_{n \leq x} f(n) = xC_f(\log x)^{k-1} + O(x(\log x)^{k-2})
\]

where

\[
C_f = \frac{1}{(k-1)!} \left( \prod_{p \geq 2} \left( 1 - \frac{1}{p} \right)^k \left( \sum_{\alpha \geq 0} f(p^\alpha) \frac{1}{p^\alpha} \right) \right).
\]

In the case \( f(n) = \tau(n)^r \) for integer \( r \geq 1 \), Theorem 1 applies with \( k = 2^r \).

The only facts that we use are Abel’s summation formula, the M"obius inversion formula, the elementary estimate

\[
\sum_{n \leq t} \frac{1}{n} = \log t + \gamma + O(1/t)
\]
valid for all real \( t \geq 1 \), and the fact that the counting function of the squarefull numbers \( s \leq t \) is \( O(t^{1/2}) \), where \( s \) is squarefull if and only if \( p^2 \mid s \) for all prime factors \( p \) of \( s \), all provable by elementary means.

## 2. A lemma

**Lemma 2.** Assume that \( r \) is a positive integer and \( f(n) \) is some arithmetic function such that

\[
\sum_{n \leq x} f(n) = \sum_{j=0}^{r} c_j (\log x)^j + O(x^{-1/2+o(1)}),
\]

for some constants \( c_j, j = 0, \ldots, r \). Then

\[
\sum_{n \leq x} f(n)(\log(x/n))^k = \sum_{\ell=0}^{k+r} C_\ell(\log x)^\ell + O(x^{-1/2+o(1)}),
\]

holds for all positive integers \( k \) with some constants \( C_0, \ldots, C_{k+r} \). Here, if \( \ell \in \{k, k+1, \ldots, k+r\} \), then

\[
C_\ell := c_{\ell-k} \left( 1 + (\ell-k) \sum_{i=1}^{k} \frac{(-1)^i}{\ell-k+i} \binom{k}{i} \right).
\]
Furthermore, if \( r \geq t \geq 1 \) are positive integers and
\[
\sum_{n \leq x} f(n) = \sum_{j=t}^{r} c_j (\log x)^j + O((\log x)^{t-1}),
\]
then
\[
\sum_{n \leq x} f(n)(\log(x/n))^k = \sum_{j=k+t}^{k+r} C_j (\log x)^j + O((\log x)^{t+k-1}).
\]

**Proof.** We show how to deduce (3) out of (2) with the leading coefficients given by (4). Let
\[
A(x) = \sum_{n \leq x} f(n).
\]
Then
\[
A(x) = \sum_{j=0}^{r} c_j (\log x)^j + R(x),
\]
where \(|R(x)| = x^{-1/2+o(1)}\) as \( x \to \infty \). Let \( i \geq 1 \). Put
\[
B_i(x) := \sum_{n \leq x} f(n)(\log n)^i.
\]
Then, by the Abel summation formula and by interchanging the order between the summation and the integration, we get
\[
B_i(x) = A(x)(\log x)^i - i \int_1^x A(t) \left( \frac{(\log t)^{i-1}}{t} \right) dt
\]
\[
= \sum_{j=0}^{r} \left( c_j (\log x)^{j+i} - i \int_1^x \left( c_j (\log t)^{j+i-1} \right) dt \right)
\]
\[
- i \int_1^x \frac{(\log t)^{i-1} R(t)}{t} dt + R(x)(\log x)^i
\]
\[
= \sum_{j=0}^{r} \left( c_j (\log x)^{j+i} - \frac{c_j i}{j+i}(\log t)^{j+i} \big|_1 \right) +
\]
\[
- i \int_1^x \frac{(\log t)^{i-1} R(t)}{t} dt + i \int_x^\infty \frac{(\log t)^{i-1} R(t)}{t} dt + R(x)(\log x)^i
\]
\[
= \sum_{j=0}^{r} \frac{c_j j}{j+i} (\log x)^{j+i} + D_i + O(x^{-1/2+o(1)}),
\]
where

\[ D_i := -i \int_1^\infty \frac{(\log t)^{i-1} R(t)}{t} dt \]

In the above, we used the fact that \(|R(t)| \leq t^{-1/2+o(1)}\) as \(t \to \infty\) to deduce that the above integral converges and that its tail from \(x\) to infinity as well as the other errors are \(O(x^{-1/2+o(1)})\) as \(x \to \infty\). Using the binomial formula and the above arguments, we have

\[
C_k(x) := \sum_{n \leq x} f(n)(\log(x/n))^k
\]

\[
= \sum_{i=0}^k (-1)^i \binom{k}{i} (\log x)^{k-i} \sum_{n \leq x} f(n)(\log n)^i
\]

\[
= \sum_{n \leq x} f(n) + \sum_{i=1}^k (-1)^i \binom{k}{i} (\log x)^{k-i} B_i(x)
\]

\[
= \sum_{\ell=0}^{k+r} C_\ell (\log x)^\ell + O(x^{-1/2+o(1)}),
\]

where \(C_\ell\) are given by formula (4) for \(\ell \geq k\). For \(\ell = 1, \ldots, k - 1\), the coefficient \(C_\ell\) involves the expression \(D_\ell\). The deduction of (6) out of (5) is immediate by similar arguments. \(\square\)

3. The proof of Theorem 1

Let \(f_0(n) := f(n)\). Recursively define \(f_j(n)\) such that

\[
f_{j-1}(n) = \sum_{d|n} f_j(d), \quad j = 1, 2, \ldots.
\]

By Möbius inversion,

\[
f_j(n) = \sum_{d|n} \mu(d) f_{j-1}(n/d).
\]

On primes

\[
f_j(p) = f_{j-1}(p) - 1, \quad j = 1, 2, \ldots.
\]

Since \(f_0(p) = k\), we get that \(f_j(p) = k - j\). In particular, \(f_k(p) = 0\). Further, for \(\alpha \geq 2\), we have that

\[
f_j(p^\alpha) = f_{j-1}(p^\alpha) - f_{j-1}(p^{\alpha-1}).
\]
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Since $f_0(p^\alpha) = \alpha^{O(1)}$ it follows that $f_j(p^\alpha) = \alpha^{O(1)}$ for all $j \geq 2$. The constant in $O(1)$ might depend on $j$. Further,

$$\sum_{\alpha \geq 0} f_j(p^\alpha) \frac{1}{p^\alpha} = \left(1 - \frac{1}{p}\right) \sum_{\alpha \geq 0} f_{j-1}(p^\alpha) \frac{1}{p^\alpha}, \quad j = 1, 2, \ldots,$$

therefore

$$\sum_{\alpha \geq 0} f_j(p^\alpha) \frac{1}{p^\alpha} = \left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} f(p^\alpha) \frac{1}{p^\alpha}, \quad j = 0, 1, \ldots$$

Put

$$E_j := \prod_{p \geq 2} \left(\sum_{\alpha \geq 0} f_j(p^\alpha) \frac{1}{p^\alpha}\right) = \prod_{p \geq 2} \left(\left(1 - \frac{1}{p}\right)^j \sum_{\alpha \geq 0} f(p^\alpha) \frac{1}{p^\alpha}\right).$$

Fix $j \geq 1$. Then

$$F_{j-1}(x) := \sum_{n \leq x} \frac{f_{j-1}(n)}{n} = \sum_{n \leq x} \frac{1}{n} \sum_{d|n} f_j(d) = \sum_{d \leq x} f_j(d) \sum_{n \leq x \atop d|n} \frac{1}{n}.\quad (7)$$

In the inner sum, we write an $n \leq x$ which is a multiple of $d$ as $n = dm$ for some integer $m \leq x$. We get

$$F_{j-1}(x) = \sum_{d \leq x} \frac{f_j(d)}{d} \sum_{m \leq x/d} \frac{1}{m} = \sum_{d \leq x} \frac{f_j(d)}{d} \left(\log(x/d) + \gamma + O(d/x)\right)$$

$$= \sum_{d \leq x} \frac{f_j(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_j(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} |f_j(d)|\right)$$

for $j = 1, 2, \ldots$. When $j = k$, since $f_k(p) = 0$, it follows that $f_k(d) = 0$ if $d$ is not squarefull. Thus, when $j = k$ in the right-hand side of (7), we have

$$\sum_{d \leq x} \frac{f_k(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_k(d)}{d} + O\left(\frac{1}{x} \sum_{d \leq x} |f_k(d)|\right).$$

Note that

$$\sum_{d \leq x} \frac{f_k(d)}{d} = \sum_{d \geq 1} \frac{f_k(d)}{d} + O\left(\sum_{d > x} \frac{|f_k(d)|}{d}\right) = E_k + O\left(\sum_{d \geq x \atop d \text{ squarefull}} \frac{1}{d^{1+o(1)}}\right)$$

$$= E_k + O(x^{-1/2+o(1)}),$$

(8)
where for the error term we used the fact that $|f_k(d)| = |\tau(d)|^{O(1)} = d^{o(1)}$ as $d \to \infty$ and the Abel summation formula to conclude that
\[
\sum_{d > x} \frac{1}{d^{1+o(1)}} \leq x^{-1/2+o(1)} \quad \text{as} \quad x \to \infty.
\]

Further, we have
\[
\sum_{d \leq x} \frac{f_k(d)}{d} (-\log d + \gamma) = \sum_{d \geq 1} \frac{f_k(d)}{d} (-\log d + \gamma) + O\left( \sum_{d > x} \frac{|f_k(d)| \log d}{d} \right)
\]
\[
(9) \quad := F_k + O(x^{-1/2+o(1)})
\]
as $x \to \infty$, by a similar argument since $|f_k(d)| \log d \leq d^{o(1)}$ as $d \to \infty$. Finally
\[
(10) \quad \sum_{d \leq x} |f_k(d)| \leq x^{1/2+o(1)},
\]
again since $f_k(d) = 0$ if $d$ is not squarefull. Collecting (8), (9) and (10) and putting them into (7) with $j = k$, we get
\[
F_{k-1}(x) = \sum_{n \leq x} \frac{f_{k-1}(n)}{n} = E_k \log x + F_k + O(x^{-1/2+o(1)}).
\]

In a similar way,
\[
G_{k-1}(x) := \sum_{n \leq x} \frac{|f_{k-1}(n)|}{n} = E'_k \log x + F'_k + O(x^{-1/2+o(1)}).
\]

for some (maybe different) constants $E'_k$ and $F'_k$. We now apply Lemma 2 in order to find recursively $F_{k-2}(x), F_{k-3}(x), \ldots, F_0(x)$. We claim, by induction on $j$, that
\[
(11) \quad F_{k-j}(x) = A_j (\log x)^j + B_j (\log x)^{j-1} + O((\log x)^{j-2})
\]
for $j = 2, \ldots, k$. At $j = 1$, this is so with $A_1 = E_k$, $B_1 = F_k$ and the error term is better, namely $O(x^{-1/2+o(1)})$. In order to realize the induction step from $j = 1$ to $j = 2$, we use the first part of Lemma 1 with $r = 1$, whereas for the induction step from $j \geq 2$ to $j + 1$ we use the second part of Lemma 2 with $r = j$ and $t = j - 1$. Assuming that
(11) holds for $j \geq 1$, we have, by (7),

$$F_{k-j-1}(x) = \sum_{d \leq x} \frac{f_{k-j-1}(d)}{d} = \sum_{d \leq x} \frac{f_{k-j}(d)}{d} \log(x/d) + \gamma \sum_{d \leq x} \frac{f_{k-j}(d)}{d}$$

$$= O \left( \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right).$$

By Lemma 2, we get that the right hand side is

$$\frac{A_j}{j+1}(\log x)^{j+1} + \left( \frac{B_j}{j} + \gamma A_j \right) (\log x)^j$$

$$+ O \left( (\log x)^{j-1} + \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right)$$

$$:= A_{j+1}(\log x)^{j+1} + B_{j+1}(\log x)^j + O \left( (\log x)^{j-1} + \frac{1}{x} \sum_{d \leq x} |f_{k-j}(d)| \right),$$

where

$$A_{j+1} = \frac{A_j}{j+1}, \quad \text{and} \quad B_{j+1} = \gamma A_j + \frac{B_j}{j}.$$

Thus, we note that $A_j = E_k/j!$. It remains to deal with the sum in the error term. But the exact same approach applies to $|f_{k-j}(n)|$. That is $g_0(n) = |f_{k-j}(n)|$ satisfies the same conditions as our initial $f_0(n)$ with $k$ replaced by $k - j$. Thus,

$$\sum_{d \leq x} \frac{|f_{k-j}(d)|}{d} = C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2}),$$

where for $j = 1$, the error term is $O(x^{-1/2+o(1)})$ as $x \to \infty$. By Abel summation, we get that

$$\sum_{d \leq x} |f_{k-j}(d)| = x(C_j(\log x)^j + D_j(\log x)^{j-1} + O((\log x)^{j-2}))$$

$$- \int_1^x (C_j(\log t)^j + D_j(\log t)^{j-1} + O((\log t)^{j-2})) dt$$

$$= O(x(\log x)^{j-1}),$$

which is sufficient for us. This completes the induction procedure and shows that at $j = k$ we have

$$\sum_{n \leq x} \frac{f(n)}{n} = \frac{1}{k!} E_k(\log x)^k + B_k(\log x)^{k-1} + O((\log x)^{k-2}).$$
Abel summation formula once again gives
\[
\sum_{n \leq x} f(n) = \left( \frac{E_k}{k!} (\log x)^k + B_k (\log x)^{k-1} + O((\log x)^{k-2}) \right) x
\]
\[- \int_1^x \left( \frac{E_k}{k!} (\log t)^k + B_k (\log t)^{k-1} + O((\log t)^{k-2}) \right) dt
= \frac{E_k}{(k-1)!} x(\log x)^{k-1} + O(x(\log x)^{k-2}),
\]
which is what we wanted.

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