On the Feynman-Kac Formula

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Abstract

In this article, given \( y : [0, \eta) \to H \) a continuous map into a Hilbert space \( H \) we study the equation

\[
\dot{y}(t) = e^{\int_0^t c(s, \dot{y})} y(t)
\]

where \( c(s, \cdot) \) is a given ‘potential’ on \( C([0, \eta), H) \). Applying the transformation \( y \to \dot{y} \) to the solutions of the SPDE and PDE underlying a diffusion, we study the Feynman-Kac formula.

Keywords: \( S' \) valued process, diffusion processes, Hermite-Sobolev space, path transformations, quasi linear SPDE, Feynman-Kac formula, Translation invariance

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1 Introduction

One of the well known formulas at the boundary of probability and analysis is the Feynman-Kac formula \( u(t, x) = E_x (f(X_t) e^{-\int_0^t V(X_s)ds}) \), which represents the solution \( u(t, x) \) of the evolution equation for the operator \( L - V \) where
\( L \) is the infinitesimal generator of a diffusion \((X_t, P_x), x \in \mathbb{R}^d, V(x) \geq 0\) the potential function and \( f \) the initial value \((5)\). We refer to [9], [6], [7] for basic material on this topic. It is also known that this formula defines a sub-Markovian semi-group whose underlying process \((\hat{X}_t)\) is obtained from \((X_t)\) by the operation known as ‘killing’ according to the multiplicative functional \( M_t := e^{-\int_0^t V(X_s)ds} \) \((13)\). It maybe of interest therefore to have an answer to the following natural question : is it possible to have a ‘pathwise’ construction of the process \( \hat{X} \). The special case when \((X_t)\) satisfies an Itô stochastic differential equation (SDE), is of interest. However, it turns out that it is the SPDE satisfied by the distribution valued process \((\delta_{X_t})\) \((10)\) rather than the SDE for \((X_t)\) that is more relevant for our purposes.

To motivate our ‘pathwise’ construction, we proceed as follows. Let \( H \) be a separable real Hilbert space and consider \( C([0, \eta), H) \), the space of continuous functions on \([0, \eta), 0 < \eta \leq \infty\), with values in \( H \). Let \( u(t) \in C([0, \eta), H) \) be the solution of the following evolution equation in \( H \) viz.

\[
\partial_t u(t) = Lu(t)
\]

with \( L : H \to H \) say, a linear operator. Consider \( \bar{u}(t) := u(t)e^{\int_0^t c(s, u)ds} \), where \( c(s, \cdot) : C([0, \eta), H) \to \mathbb{R} \) is a given function (the potential). Then, integrating by parts, it is easy to see that \( \bar{u} \) solves

\[
\partial \bar{u}(t) = L\bar{u}(t) + c(t, u)\bar{u}(t).
\]

We would have a good and proper evolution equation for \( \bar{u}(t) \) if we were able to write \( c(t, u) = \tilde{c}(t, \bar{u}) \). If the map \( S(u) := \bar{u} = u(t)e^{\int_0^t c(s, u)ds} \) were invertible, then we may define \( \tilde{c}(t, u) := c(t, R(u)) \) where \( R(u) := S^{-1}(u) \) so that \( \tilde{c}(t, \bar{u}) = c(t, u) \). It is easy to see that the inverse \( R \) is a path transformation \( R : C([0, \eta), H) \to C([0, \eta), H) \) induced by the ‘potential’ \( c : [0, \eta) \times C([0, \eta), H) \to \mathbb{R} \) as follows : For a given \( y \in C([0, \eta), H) \), \( R(y) \in C([0, \eta), H) \) is the solution \( \hat{y} \) of the equation

\[
\hat{y}(t) = y(t)e^{-\int_0^t \tilde{c}(s, \hat{y})ds}.
\]

In section 2, we prove existence and uniqueness to the above equation in Theorem (2.2), using a fixed point argument. Thus the map \( R \) is well defined and injective. Since \(-c\) satisfies the conditions of Theorem (2.2) whenever \( c \)
does, the map \( R \) is also onto. From a modeling point of view, \( R(y) \) maybe viewed as a perturbation, induced by the potential \( c(t, y) \), of the trajectory of a particle represented by \( y(.) \). We deal with real Hilbert spaces as we consider applications only to the theory of diffusions. However, complex Hilbert spaces and complex valued potentials (with the corresponding interpretation of ‘amplitude’ and ‘phase’) are also of interest.

Given a diffusion \((X_t, 0 \leq t < \eta, P_x, x \in \mathbb{R}^d)\), we try to realise the Feynman-Kac formula by applying the above transformation to the paths of the diffusion. We remark here that we could choose \( H = \mathbb{R}^d \) but this does not lead to the Feynman-Kac formula (see Remark (3.2)). However, if we look at the process \((Y_t) := (\delta_{X_t})\), up to time \( \eta \), then this is a semi-martingale in a Hilbert space \( S_p \)- the so called Hermite-Sobolev space - and indeed is the unique solution of a quasi linear stochastic partial differential equations (SPDE) \([10],[11]\); one may then look at the process \((\hat{Y}_t) := (e^{-\int_0^t V(X_s) \, ds} \delta_{X_t})\) and using the rules of stochastic calculus write an SPDE for \( \hat{Y} \). Note that we can write \( V(X_s) = \langle V, \delta_{X_s} \rangle =: c(s, \delta_X), 0 \leq s < \eta, \) if \( V \) belongs to a suitable class of test functions.

In section 3, we show that when \((Y_t)\) satisfies a quasi linear SPDE in \( S_p \), then \( \hat{Y}_t \) is the solution a new SPDE with a potential term viz. \( c(t, \hat{Y}) \) and whose coefficients are defined on the path space \( C([0, \eta), S_p) \) using the coefficients of the original equation and the transformation discussed above. This transformation works at both levels viz. the SPDE and the PDE underlying the diffusion, although the ‘Kac functional’ (we use the terminology from \([1]\)) induced by the potential function \( V(x) \) is necessarily different in the two cases (see the discussion on diffusions in section 5). In section 4, we allow \( c(.,.) \) to depend also on \( x \in \mathbb{R}^d \) and we show that the above transformation may also be applied directly to the solutions of a class of non-linear PDE’s. We conclude in Section 5 with a discussion on two classes of examples in both of which the functional \( c(t, x, y) \) depends on \( x \) albeit in different ways. The second example that we discuss in Section 5 concerns diffusion processes and shows also the connections that can arise between the transformations of the solutions to the SPDE and the associated PDE. In Sections 3,4 and 5, we work in the framework of \([11]\) to which we refer for results relating to SPDE’s, the related notations and references. See also Example 7 of \([11]\) where we had briefly indicated the results in Section 2.
2 A Transformation on path space

Let $H$ be a separable real Hilbert space with norm denoted by $\| . \|$. We consider for $0 \leq T < \infty$, the space $C([0, T], H)$ of continuous functions $y : [0, T] \rightarrow H$ with the sigma field $\mathcal{B}_t, 0 \leq t \leq T$ generated by the coordinate maps upto time $t$. For a continuous map $y : [0, T] \rightarrow H$ and $0 \leq s \leq T$, we denote its norm on $C([0, s], H)$ by $\| y \|_s := \sup_{u \leq s} \| y(u) \|$. Fix $T > 0$. Let $c : [0, T] \times C([0, T], H) \rightarrow \mathbb{R}$ satisfy

1. For $0 \leq t \leq T$, $|c(t, y_1) - c(t, y_2)| \leq \beta \| y_1 - y_2 \|_t$
   where $\beta = \beta(T)$ depends only on $T$.

   We note that as a consequence of the condition 1) we have the following : for $0 \leq s \leq T$ and $y_1, y_2 \in C([0, T], H)$, $y_1(u) = y_2(u), 0 \leq u \leq s$ implies $c(s, y_1) = c(s, y_2)$.

2. For $\alpha > 0$ and $T > 0$ there exists a constant $M(\alpha, T)$ such that

   $$|c(t, y)| \leq M(\alpha, T)$$

   for $0 \leq t \leq T$ and for all $y \in B(0, \alpha) \equiv B(0, \alpha, T) := \{ y \in C([0, T], H), \| y \|_T \leq \alpha \}$.

We note that if $c(t, y)$ satisfies conditions 1 and 2 then so does $-c(t, y)$.

Let $\alpha(t) \equiv \alpha(t, y) := e^{-\int_0^t c(s, y)ds}$ for $y \in C([0, T], H)$. Given a $\hat{y} \in C([0, \eta], H)$ for some $\eta > 0$, and $0 < T < \eta$ we consider the following equation in $C([0, T], H)$, viz.

$$\hat{y}(t) = y(t)\alpha(t, \hat{y}) = y(t)e^{-\int_0^t c(s, \hat{y})ds} \quad (2.1)$$

for $0 \leq t \leq T$. We first derive an apriori estimate for the distance between two solutions corresponding to two ‘inputs’ $y_1$ and $y_2$.

**Lemma 2.1** Let $y_1, y_2 \in C([0, \eta], H)$ and suppose $\hat{y}_1, \hat{y}_2$ are the corresponding solutions of (2.1). Then for every $0 < T < \eta$, we have the following
estimate viz.
\[
\|\hat{y}_1 - \hat{y}_2\|_T \leq M\|y_1 - y_2\|_T e^{M\|y_2\|_T \beta T e^\delta}.
\]

(2.2)

where \(\delta > \beta T\|\hat{y}_1 - \hat{y}_2\|_T\) and \(M := e^{\int_0^T |c(s, \tilde{y}_1)|} ds\).

**Proof** Let \(0 < T < \eta\) and \(\delta, M\) as above. Then
\[
\int_0^T |c(s, \hat{y}_1) - c(s, \hat{y}_2)| ds \leq \beta T\|\hat{y}_1 - \hat{y}_2\|_T < \delta
\]
and consequently, using the elementary estimate \(|1 - e^x| \leq e^\delta |x|, |x| < \delta\) we have for any \(0 \leq t \leq T\),
\[
|(1 - e^{\int_0^t c(s, \hat{y}_1) - c(s, \hat{y}_2) ds})| \leq e^{\delta T} \int_0^T \|\hat{y}_1 - \hat{y}_2\|_s ds.
\]

Then we have
\[
\|\hat{y}_1 - \hat{y}_2\|_T = \|(y_1 - y_2)e^{-\int_0^T c(s, \hat{y}_1) ds} + y_2 e^{-\int_0^T c(s, \hat{y}_1) ds} (1 - e^{\int_0^T c(s, \hat{y}_1) - c(s, \hat{y}_2) ds})\|_T
\]
\[
\leq \|(y_1 - y_2)\|_T e^{\int_0^T |c(s, \hat{y}_1)| ds} + \|y_2\|_T e^{\int_0^T |c(s, \hat{y}_1)| ds} \sup_{t \leq T} |(1 - e^{\int_0^t c(s, \hat{y}_1) - c(s, \hat{y}_2) ds})|
\]
\[
\leq M\|y_1 - y_2\|_T + M\|y_2\|_T e^{\delta T} \int_0^T \|\hat{y}_1 - \hat{y}_2\|_s ds
\]
\[
\leq M\|y_1 - y_2\|_T e^{T M \|y_2\|_T \beta T e^\delta}
\]
where the last step follows from Gronwal’s inequality.

Let \(y_1 \in C([0, t_1], H), y_2 \in C([0, t_2], H)\) where \(0 \leq t_1 < t_1 + t_2 < T\). In the proof of the following theorem we need the following construction of ‘concatenation’ \(y_1 \circ y_2 \in C([0, T], H)\) of the paths \(y_1\) and \(y_2\):
\[
y_1 \circ y_2(s) := y_1(s)|_{[0, t_1]}(s) + (y_2(s - t_1) - y_2(0) + y_1(t_1))|_{(t_1, t_1 + t_2]}(s) + (y_2(S) - y_2(0) + y_1(t_1))|_{(t_1 + t_2, T]}(s),
\]
where \(|_A\) is the indicator of the set \(A\). The following theorem is our main result.
Theorem 2.2 Let $\eta > 0$ and let $c(t, y)$ satisfy conditions 1 and 2 above for every $T, 0 \leq T < \eta$. Then for a given $y \in C([0, \eta), H)$ there exists a unique $\hat{y} \in C([0, \eta), H)$ satisfying equation (2.1) for every $T, 0 < T < \eta$.

Proof: It suffices to show existence and uniqueness of equation (2.1) on $[0, T]$ for every $T < \eta$. Using uniqueness, we can then patch up the solutions on overlapping intervals to get the required solution. So let $0 < T < \eta$. Uniqueness is immediate from (2.2).

To show existence on $[0, T]$, suppose $(0, T] = \bigcup_{n=0}^{m-1} (T_n, T_{n+1}]$. Fix $n, 0 \leq n \leq m - 1$. Define $\hat{y}(0) := y(0)$. Suppose that $\hat{y}(t), t \in [0, T_n]$ has been defined. Then we extend $\hat{y}$ to the interval $(T_n, T_{n+1}]$ as follows: We first solve the following equation on $[0, T_{n+1} - T_n]$ viz.

$$
\hat{y}_n(t) = y(t + T_n)\alpha(T_n, \hat{y})\alpha_n(t, \hat{y}_n) - y(T_n)\alpha(T_n, y) + a_n
$$

(2.3)

where for $y \in C([0, T_{n+1} - T_n], H)$ and $t \in [0, T_{n+1} - T_n]$,

$$
\alpha_n(t, y) := e^{-\int_0^t c(s + T_n, \hat{y}) ds} \quad y_n(t) := y(T_n + t)\alpha(T_n, \hat{y}),
$$

and $a_n := -y(T_n)\alpha(T_n, \hat{y})$.

We extend $\hat{y}$ as follows:

$$
\hat{y}(t) := \hat{y}_n(t - T_n) + \hat{y}(T_n), \quad t \in (T_n, T_{n+1}].
$$

Then provided $\hat{y}$ satisfies equation (2.1) in $[0, T_n]$, we have for $t \in (T_n, T_{n+1}]$,

$$
\hat{y}(t) := \hat{y}_n(t - T_n) + \hat{y}(T_n)
$$

$$
= y(t)\alpha(T_n, \hat{y})\alpha_n(t - T_n, \hat{y}_n) - y(T_n)\alpha(T_n, \hat{y}) + \hat{y}(T_n)
$$

$$
= y(t)\alpha(T_n, \hat{y})\alpha_n(t - T_n, \hat{y}_n)
$$

$$
= y(t)\alpha(t, \hat{y})
$$

where in the third equality we have used the assumption that $\hat{y}$ satisfies equation (2.1) in $[0, T_n]$. As for the fourth equality, we use the fact that $\hat{y}$ on the interval $(T_n, T_{n+1}]$ is the concatenation of $\hat{y}$ in $C([0, T_n], H)$ and $\hat{y}_n$ in $C([0, T_{n+1} - T_n], H)$ i.e. $\hat{y}(t) = \hat{y} \circ \hat{y}_n(t), t \in (T_n, T_{n+1}]$. 

Thus it suffices to solve (2.3) on \([0, T_{n+1} - T_n]\) for a suitable choice of \(\{0 = T_0 < T_1 < \cdots < T_n < \cdots < T_m = T\}\).

Let \(\alpha > \sup_{s \leq T} \|y(s)\|\), and \(c(.,.)\) satisfy conditions 1 and 2 on \([0, T]\) for some \(M(\alpha) := M(\alpha, T)\) and \(\beta\). Let \(\epsilon > 0\) be such that \(\epsilon e^{M(3\alpha)T} < \frac{\alpha}{2}\). By uniform continuity of \(y\) on \([0, T]\) we can divide \([0, T]\) into a finite number (say \(m\)) of subintervals \([T_n, T_{n+1}]\), with \(T_m = T\) such that

\[
\|y(t_1) - y(t_2)\| \leq \epsilon \quad \forall t_1, t_2 \in [T_n, T_{n+1}], \ n = 0, \cdots, m - 1.
\]

Next we choose \(\delta > 0\) such that \(|e^x - 1| < e^\delta |x|\) for \(|x| < \delta\).

By refining the partition if necessary we may assume without loss of generality that

\[
\alpha M(3\alpha)e^{M(\alpha)T + \delta}(T_{n+1} - T_n) < \frac{\alpha}{2};
\]

\(K_n := 2\alpha \beta e^{M(3\alpha)T + \delta}(T_{n+1} - T_n) < 1, \ n = 0, \cdots, m - 1\)

and,

\(2M(3\alpha))(T_{n+1} - T_n) < \delta\).

With this choice of the partition \(\{T_n\}\) we now solve equation (2.3) on \([0, T_{n+1} - T_n]\) by a fixed point argument. Let \(\alpha\) be as above. Recall the definition of \(B(0, \alpha)\) from condition 2 above, with \(T\) there replaced with \(T_{n+1} - T_n\). For \(z \in B(0, \alpha), t \in [0, T_{n+1} - T_n]\) let

\[
S_n(z)(t) := y(t + T_n)\alpha(T_n, \hat{y})\alpha_n(t, z) - y(T_n)\alpha(T_n, \hat{y}).
\]

Note that \(\alpha_n(t, z)\) depends on \(\hat{y} \circ z\) where \(\hat{y}\) is the solution of (2.1) on \([0, T_n]\). Assume that \(\hat{y} \in B(0, \alpha)\). Then we claim that

\(S_n : B(0, \alpha) \subset C([0, T_{n+1} - T_n], H) \to B(0, \alpha)\).

To see this we write \(S_n(z)(t)\) as

\[
S_n(z)(t) = (y(t + T_n) - y(T_n))\alpha(T_n, \hat{y})\alpha_n(t, z) + y(T_n)\alpha(T_n, \hat{y})(\alpha_n(t, z) - 1).
\]

Let \(t \in [0, T_{n+1} - T_n]\). Then from the triangle inequality and the choice of \(\epsilon\) and \(\{T_n\}\) we have
\[
\|S_n(z)(t)\| \leq \|(y(t + T_n) - y(T_n))\| \alpha(T_n, \hat{y}) \alpha_n(t, z) \\
+ \|y(T_n)\| \alpha(T_n, \hat{y}) |\alpha_n(t, z) - 1| \\
\leq e^{M(3\alpha)T}\|(y(t + T_n) - y(T_n))\| \\
+ \alpha e^{M(\alpha)T + \delta} |\int_0^t c(u + T_n, \hat{y} \circ z)du| \\
\leq e^{M(3\alpha)T} + \alpha M(3\alpha)e^{M(\alpha)T + \delta}(T_{n+1} - T_n) \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
\]

Note that in the second equality we have used the fact that condition 2) implies \(|\int_0^t c(s + T_n, \hat{y} \circ z)ds| < M(3\alpha)(T_{n+1} - T_n) < \delta, t \in [0, T_{n+1} - T_n]\) and in the second and third inequality above we have used the fact that \(|\hat{y} \circ z(t)|| \leq 3\alpha, t \in [0, T]|.

We now show that the map \(S_n : B(0, \alpha) \to B(0, \alpha)\) is a contraction. Let \(y_1, y_2 \in B(0, \alpha)\) and \(S_n(\cdot)\) be as defined above. For \(t \in [0, T_{n+1} - T_n]\),
\[
\|S_n(y_1)(t) - S_n(y_2)(t)\| = \|y(t + T_n)\| \alpha(T_n, \hat{y}) \alpha_n(t, y_2) \\
\times \left| e^{\int_0^t \frac{c(s + T_n, \hat{y} \circ y_1) - c(s + T_n, \hat{y} \circ y_2)}{c(s + T_n, \hat{y} \circ y_1)}ds} - 1 \right| \\
\leq \|y(t + T_n)\| \alpha(T_n, \hat{y}) \alpha_n(t, y_2) \\
\times e^\delta \left| \int_0^t (c(s + T_n, \hat{y} \circ y_1) - c(s + T_n, \hat{y} \circ y_2))ds \right| \\
\leq e^{TM(3\alpha) + \delta} \alpha \beta 2(T_{n+1} - T_n) \|y_1 - y_2\|_{T_{n+1} - T_n}
\]

and by definition of the constant \(K_n\) we have
\[
\|S(y_1) - S(y_2)\|_{T_{n+1} - T_n} \leq K_n \|y_1 - y_2\|_{T_{n+1} - T_n}.
\]

Since \(K_n < 1\) by our choice, the map
\[
y \to S_n(y) : C([0, T_{n+1} - T_n], B(0, \alpha)) \to C([0, T_{n+1} - T_n], B(0, \alpha))
\]
is a contraction on a complete metric space and has a unique fixed point. Thus equation (2.3) has a unique solution. This completes the proof of the Theorem. \(\square\)
Corollary 2.3 For \( y \in C([0, \eta), H) \), let \( R(y) := \hat{y} \) where \( \hat{y} \) is the solution of (2.1). Then \( R \) is one to one and onto. Further for every \( t > 0 \), \( R : C([0, t], H) \rightarrow C([0, t], H) \) is a homeomorphism. In particular, for every \( t > 0 \), the map \( R : (C([0, t], H), B_t) \rightarrow (C([0, t], H), B_t) \) is a measurable isomorphism.

Proof: To see that \( R \) is one-one, suppose that \( R(y_1) = R(y_2) \). Then since this implies \( \hat{y}_1 = \hat{y}_2 \), we also have \( y_1 = y_2 \). That \( R \) is onto follows from the observation that if \( \hat{y} \in C([0, \eta), H) \) is given and if we define \( y(t) := \hat{y}(t)e^{\int_0^t c(s, \hat{y})ds} \) then clearly \( R(y) = \hat{y} \).

Note that for a given \( y \in C([0, \eta), H) \), \( R^{-1}(y) = \tilde{y} := ye^{\int_0^t c(s, y)ds} \) follows since \( R(\tilde{y}) = y \). Since \( R^{-1} \) has the same form as \( R \) it suffices to show that \( R \) is continuous. But this is clear from (2.2). The last statement follows from the continuity of \( R \) and the fact that the Borel sigma field on \( C([0, t], H) \) is the same as \( B_t \).

3 Application to Stochastic PDE’s

Let \( S_p, p \in \mathbb{R} \) be the family of Hermite-Sobolev spaces; \( S, S' \) respectively the Schwartz space of rapidly decreasing smooth functions and its dual. We refer to \([11],[4],[8]\) for the results and notations related to these spaces that we use. We refer to \([2],[3]\) for results on stochastic calculus in Hilbert spaces. We work on a probability space \((\Omega, F, P)\) on which is given an \( r \)-dimensional standard Brownian motion \((B_t)\). Let \((F^B_t)_{t \geq 0}\) be the filtration of \((B_t)\). We now consider solutions of the SPDE

\[
\begin{align*}
    dY_t &= L(t, Y)dt + A(t, Y) \cdot dB_t \\
    Y_0 &= Y.
\end{align*}
\]

where \( L, A_i, i = 1, \ldots r \) are second order quasi-linear partial differential operators with coefficients \( \sigma_{ij}, b_i : S_p \rightarrow \mathbb{R}^d, i = 1, \ldots d, j = 1, \ldots r \) defined as
follows
\[
L(y) := \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^t)_{ij}(y) \partial^2_{ij}y - \sum_{i=1}^{d} b_i(y) \partial_i y
\]
\[
A_i(y) := - \sum_{j=1}^{d} \sigma_{ji}(y) \partial_j y
\]

In \[11\] we have proved existence and uniqueness of solutions to equation (3.4) and shown that for a given \(Y : \Omega \rightarrow \mathcal{S}_p\) a unique solution \((Y_t, \eta)\) exists under a Lipschitz condition on the coefficients \(\sigma_{ij}\) and \(b_i\). Here \(0 < \eta \leq \infty\) is the lifetime of the process and if \(\sigma_{ij}, b_i\) are uniformly bounded on \(\mathcal{S}_p\) then \(\eta = \infty\) almost surely (see \[11\], Proposition (5.2)).

Let \(c(., .) : [0, \infty) \times C([0, \infty), \mathcal{S}_p) \rightarrow \mathbb{R}\) satisfy conditions 1 and 2 of Section 2 on bounded intervals \([0, T]\). Let \(\eta > 0\). Given \(y \in C([0, \eta), \mathcal{S}_p)\) let \(\hat{y}\) be the solution of equation (2.1) given by theorem (2.1) with \(H = \mathcal{S}_p\).

Suppose now that we are given \(\sigma_{ij}, b_i : \mathcal{S}_p \rightarrow \mathbb{R}\) and \(L, A^i, i = 1, \ldots r\) as above. The transformation \(y \rightarrow \hat{y}\) induced by the map \(c(., .)\) and equation (2.1) induces a corresponding transformation of maps \(\sigma_{ij}(.), b_i(., .) \rightarrow \hat{\sigma}_{ij}(., .), \hat{b}_i(., .)\) as follows: \(\hat{\sigma}_{ij}, \hat{b}_i : [0, \eta) \times C([0, \eta), \mathcal{S}_p) \rightarrow \mathbb{R}\) by \(\hat{\sigma}_{ij}(s, y) := \sigma_{ij}(\hat{y}(s)), \hat{b}_i(s, y) := b_i(\hat{y}(s))\). Define \(\hat{c}(s, y) := c(s, \hat{y}), 0 \leq s < \eta, y \in C([0, \eta), \mathcal{S}_p)\). Let \(\hat{L}(t, y)\) and \(\hat{A}_i(t, y)\) be maps from \([0, \eta) \times C([0, \eta), \mathcal{S}_p)\) to \(\mathcal{S}_p\) for fixed \(\eta > 0\) defined as follows:
\[
\hat{L}(s, y) := \frac{1}{2} \sum_{i,j=1}^{d} (\hat{\sigma} \hat{\sigma}^t)_{ij}(s, y) \partial^2_{ij}y_s - \sum_{i=1}^{d} \hat{b}_i(s, y) \partial_i y_s + \hat{c}(s, y)y_s
\]
\[
\hat{A}_i(s, y) := - \sum_{j=1}^{d} \hat{\sigma}_{ji}(s, y) \partial_j y_s
\]

Let \((Y_t, \eta)\) be a pathwise unique strong solution of equation (3.4) with initial value \(Y\). Then for each \(\omega \in \Omega\), the trajectory \(Y(\omega) \in C([0, \eta(\omega)), \mathcal{S}_p)\). Define for \(0 \leq t < \eta(\omega)\)
\[
\hat{Y}_t(\omega) := Y_t(\omega)e^{\int_0^t c(s,Y(\omega))ds}
\]
Let $\hat{\sigma}_{ij}, \hat{b}_i, \hat{c}, \hat{L}, \hat{A}_i$ be as above. We take $\hat{Y}_t(\omega) := \delta, t \geq \eta$ where $\delta$ is the coffin state. By the continuity of $c(\cdot, \cdot)$ and the definition of a strong solution (see [11]), $(\hat{Y}_t)$ is a continuous $\mathcal{F}_t^B$-adapted, $\hat{S}_p := S_p \cup \{\delta\}$ valued process.

**Theorem 3.1** Let $(Y_t)_{0 \leq t < \eta}$ be a strong solution of equation (3.4) and let $c(\cdot, \cdot)$ satisfy conditions 1 and 2 of Section 2. Then $(\hat{Y}_t)_{0 \leq t < \eta}$ is a strong solution of the equation

$$
\begin{align*}
\hat{d}Y_t &= \hat{L}(t, \hat{Y}) \, dt + \hat{A}(t, \hat{Y}) \cdot dB_t \\
\hat{Y}_0 &= Y.
\end{align*}
$$

If equation (3.4) has a unique strong solution, then so has equation (3.5).

**Proof:** Let $M_t := e^\int_0^t c(s, Y)ds$. To prove existence, we use integration by parts. Indeed one can verify the following equation by acting on it with a test function. We have in differential form

$$
\begin{align*}
\hat{d}Y_t &= d(M_t Y_t) = Y_t \, dM_t + M_t dY_t \\
&= \hat{Y}_t c(t, Y) \, dt + L(Y_t) M_t \, dt + M_t A(Y_t) \cdot dB_t.
\end{align*}
$$

Now from the definition of $\hat{Y}_t(\omega)$, we have that for each fixed $\omega$, $Y_t(\omega), 0 \leq t < \eta(\omega)$ is the unique solution $\hat{y}$ of equation (2.1) with $y(t) := \hat{Y}(t), 0 \leq t < \eta$, viz.

$$
\hat{y}(t) = \hat{Y}_t(\omega) e^{-\int_0^t c(s, \hat{y})ds}.
$$

It follows that $\sigma_{ij}(Y_t) = \hat{\sigma}_{ij}(t, \hat{Y}), c(t, Y) = \hat{c}(t, \hat{Y})$ etc. and hence from above,

$$
\begin{align*}
\hat{d}Y_t &= \hat{L}(t, \hat{Y}) \, dt + \hat{A}(t, \hat{Y}) \cdot dB_t.
\end{align*}
$$

The uniqueness of solutions of equation (3.5) follows from the uniqueness of equation (3.4). Indeed if $\hat{Y}^1, \hat{Y}^2$ are solutions of equation (3.5), then if $(Y^i_t)$ solves $Y^i(t) = \hat{Y}^i e^{-\int_0^t c(s, Y^i)ds}$ it is easy to check using the integration by parts formula for the product $\hat{Y}^i e^{-\int_0^t c(s, Y^i)ds}$ and the definition of the ‘hat’ functionals that $Y^i, i = 1, 2$ both solve equation (3.4) and hence $Y^1 = Y^2$ which in turn implies $\hat{Y}^1 = \hat{Y}^2$. \qed
Remark 3.2 Let \((X^x, \eta)\) be the solution of an Itô SDE with diffusion and drift coefficients \(\tilde{\sigma}_{ij}\) and \(\tilde{b}_i\) respectively and initial value \(x \in \mathbb{R}^d\). Let \(\tilde{V} : \mathbb{R}^d \to \mathbb{R}\) be a locally bounded function. We can apply Theorem 2.2, with \(H = \mathbb{R}^d\) to transform \(X\) into \(\hat{X} := T(X)\) with \(T : C([0, \eta], \mathbb{R}^d) \to C([0, \eta], \mathbb{R}^d)\) given by
\[
T(X)(s) := e^{\int_0^s c(s, X) \, ds} \, X_s, \quad 0 \leq t < \eta
\]
with \(c(\cdot, y) := \tilde{V}(y_s)\) and \(y \in C([0, \infty), \mathbb{R}^d)\). Then \(\hat{X}\) will satisfy an SDE with path dependent coefficients which can be determined as in the case of \(\hat{Y}\) in Theorem (3.1). In general, the transformation \(T\) applied to an Itô process \((X_t)\) changes the drift term by adding a term like \(c(t, \hat{X}) \hat{X}_t\).

On the other hand, let \(Y_t := \delta_{X_t}, t < \eta\) with \(Y_0 = \delta_x\) and suppose that the coefficients \(\tilde{\sigma}_{ij}, \tilde{b}_i, \tilde{V} \in \mathbb{S}_p, p > \frac{d}{2}\). Let \(\sigma_{ij}, b_i, V\) be the linear functionals on \(\mathcal{S}_p\) given by \(\sigma_{ij}(y) := \langle \tilde{\sigma}_{ij}, y \rangle\) etc. Then \((Y_t)\) is the unique solution of the SPDE (3.4) with \(Y_0 = \delta_x\) and with \(c : [0, \infty) \times C([0, \infty), \mathbb{S}_{-p}) \to \mathbb{R}\) given by \(c(s, y) := \langle \tilde{V}, y_s \rangle\), \(y \in C([0, \infty), \mathbb{S}_{-p})\), \(\hat{Y}\) is the unique solution of (3.5) up to the lifetime \(\eta\).

4 Application to PDE’s

We now apply the transformation \(y \to \hat{y}\) developed in Section 2, to solutions of partial differential equations of the form
\[
\begin{align*}
\partial_t u(t, x) &= L(x, u(t, x)) \\
&= L(x, u(x)), \\
u(0, x) &= u(x).
\end{align*}
\tag{4.6}
\]
with \(u : \mathbb{R}^d \to \mathbb{S}_p\). Here the operator \(L(x) : \mathbb{S}_p \to \mathbb{S}_{p-1}\) is defined by
\[
L(x)(y) \equiv L(x, y) := \frac{1}{2} \sum_{i,j=1}^d (\sigma^{ij})_{ij}(x, y) \partial^2_{ij} y - \sum_{j=1}^d b_i(x, y) \partial_i y
\]
where \(\sigma_{ij}, b_i : \mathbb{R}^d \times \mathbb{S}_p \to \mathbb{R}, i, j = 1, \cdots, d\) are assumed to satisfy a Lipschitz condition as follows: Let \(f : \mathbb{R}^d \times \mathbb{S}' \to \mathbb{R}\). We say that \(f\) satisfies a \((p, q)\) local Lipschitz condition, uniformly in \(x \in \mathbb{R}^d\) if for all \(\lambda > 0\) there exists
\[ C = C(\lambda, p, q) \text{ such that} \]
\[ |f(x, \varphi) - f(x, \psi)| \leq C\|\varphi - \psi\|^q_p \]
for all \( \varphi, \psi \in B_p(0, \lambda) \) and \( x \in \mathbb{R}^d \).

Under the above condition, we can show the existence and uniqueness of solutions of the above equation \([12]\). Here, given a measurable map \( u : \mathbb{R}^d \to \mathcal{S}_p \) we will assume the existence of a unique solution to the above PDE i.e. for each \( x \in \mathbb{R}^d \), the existence of a unique map \( u(., x) : [0, T] \to \mathcal{S}_p \) which is continuous and satisfies
\[
 u(t, x) = u(x) + \int_0^t L(x, u(s, x))ds.
\]
where the equation holds in \( \mathcal{S}_q, q \leq p - 1 \). Suppose now we are given a potential function i.e. a real valued function of the form \( c(t, x, y), 0 \leq t \leq T, x \in \mathbb{R}^d, y \in C([0, T], \mathcal{S}_p), \) satisfying for each \( x \), conditions 1 and 2 of Section 2 for \( H = \mathcal{S}_p \). Let \( \hat{\sigma}_{ij}(t, x, .), \hat{b}_i(t, x, .), \hat{c}(t, x, .) \) be as defined in Section 3. For \( t \in [0, T], x \in \mathbb{R}^d, y \in C([0, T], \mathcal{S}_p) \) define the operator
\[
 \hat{L}(s, x, y) := \frac{1}{2} \sum_{i,j=1}^d (\hat{\sigma}^{ij})_{ij}(s, x, y)\partial_{ij}^2 y_s - \sum_{j=1}^d \hat{b}_i(s, x, y)\partial_i y_s \\
+ \hat{c}(s, x, y)y_s.
\]
The following theorem can be proved in the same manner as Theorem (3.1).

**Theorem 4.1** Let \( (u(t,x)) \) be a solution of equation (4.6) for a given \( u : \mathbb{R}^d \to \mathcal{S}_p \) and let \( c(., .) \) satisfy conditions 1) and 2). Then,
\[
 \hat{u}(t, x) := u(t,x)e^{\int c(s,x,u(.,x))ds}
\]
satisfies
\[
 \partial_t \hat{u}(t, x) = \hat{L}(t, x, u(., x)) \\
\hat{u}(0, x) = u(x).
\]
If equation (4.6) has a unique solution so has equation (4.7).
For a given r-dimensional Brownian motion \((B_t)\) and \(u(t, x)\) satisfying (4.6) let
\[ Z_t^x := \int_0^t \sigma(x, u(s, x)) \cdot dB_s + \int_0^t b(x, u(s, x)) ds \]
Let \(Y_t^x := \tau_{Z_t^x} u(x)\), where \(\tau_x : S_p \to S_p\) are the translation operators. Note that for each \(x\), \(E\|\tau_{Z_t^x} u(x)\|_p < \infty\). Then as in the proof of Theorem 6.3, [11], we have \(u(t, x) = EY_t^x\). Let \(\hat{Y}_t^x := (\tau_{Z_t^x} u(x)) e^{\int_0^t c(s, x, u(s, x)) ds}\). Then we have the following

**Corollary 4.2** For each \(x \in \mathbb{R}^d\), we have
\[ \hat{u}(t, x) = E(Y_t^x)e^{\int_0^t c(s, x, EY_s^x))ds}, \quad 0 \leq t \leq T. \]

**Remark 4.3** We note that for fixed \((t, x)\), \(\hat{u}(t, x) \in S_p\) whenever \(u(t, x) \in S_p\). Further, Theorem (4.1) implies that the degree of smoothness of \(\hat{u}(t, x)\) in the backward variable \(x\) is the minimum of the degree of smoothness of the maps \(x \to u(t, x)\) and that of \(x \to c(t, x, u(t, x))\).

## 5 Conclusion

In this section we make a few remarks on the applications of Theorem 2.2. We first consider the PDE (4.6) and its interplay with the \(S_p\) valued processes considered in Section 3. The existence and uniqueness of solutions of (4.6) in the non-linear case will be considered in a separate paper ([12]). Here we will consider two separate classes of equation (4.6), in remarks 1 and 2 below, corresponding to different classes of coefficients \(\sigma_{ij}, b_i\) and the corresponding classes of linear operators \(L(x, \phi)\) in (4.6).

1. We assume that the coefficients depend only on \(x \in \mathbb{R}^d\) i.e \(\sigma_{ij}(x, \phi) = \sigma_{ij}(x), b_i(x, \phi) = b_i(x), i = 1, \cdots, d, j = 1, \cdots, r\) and the initial condition \(u(x)\) is arbitrary. In this case \(L(x) : S_p \to S_q, q \leq p - 1\) is a linear
operator. The solution $u(t, x)$ exists uniquely - because of the monotonicity inequality satisfied by $L(x)$ and is given by $u(t, x) := E_{\tau Z_t^x}u(x)$ where for each $x \in \mathbb{R}^d$,

$$Z_t^x := \sigma(x) \cdot B_t + b(x)t$$

In particular, $\hat{u}(t, x)$ is the unique solution to (4.7) for any given potential function $c(t, x, y)$ satisfying conditions 1 and 2. In this example, the role of the variable $x \in \mathbb{R}^d$ in the coefficients of the equation is that of an ‘external parameter’ and as a consequence $(Z_t^x)$ is a Gaussian process, for each $x$.

2. Suppose that $\bar{\sigma}_{ij}, \bar{b}_i \in \mathcal{S}_{-p}, p > \frac{d}{4}$ and $\sigma_{ij}, b_i : \mathcal{S}_p \to \mathbb{R}$ are defined by $\sigma_{ij}(\phi) := \langle \bar{\sigma}_{ij}, \phi \rangle$ etc. In particular $\bar{\sigma}_{ij}, \bar{b}_i$ do not depend on $\phi \in \mathcal{S}_p$ and $\sigma_{ij}, b_i$ do not depend on $x \in \mathbb{R}^d$. Consider the operators $L, \bar{L}$, respectively non-linear and linear, associated, respectively with (3.4) and (4.6). In the following computations we will show the connection between solutions of (3.4)-(3.5) associated with the non-linear operator $L$ and the solutions of (4.6)-(4.7) associated with the linear operator $L(x)$ which we here denote by $\bar{L}(x)$, acting on $\phi \in \mathcal{S}_p, p > \frac{d}{4}$ as follows:

$$\bar{L}\phi(x) := \frac{1}{2} \sum_{i,j=1}^{d} (\bar{\sigma}\bar{\sigma})_{ij}(x) \partial_{ij}^2 \phi(x) + \sum_{j=1}^{d} \bar{b}_i(x) \partial_i \phi(x).$$

If $\sigma_{ij}, b_i$ are bounded measurable functions and $p > 1$ then $\bar{L} : \mathcal{S}_p \to \mathcal{S}_0 = L^2$. Associated (as above) with the coefficients $\bar{\sigma}_{ij}, \bar{b}_i \in \mathcal{S}_p$ is the non-linear operator $L : \mathcal{S}_p \to \mathcal{S}_{p-1}$

$$L(\phi) := \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^t)_{ij}(\phi) \partial_{ij}^2 \phi + \sum_{j=1}^{d} b_i(\phi) \partial_i \phi.$$  

where $\sigma_{ij}, b_i : \mathcal{S}_p \to \mathbb{R}$ are defined above. Let $(X_t^x)$ denote the unique solution of

$$dX_t = \bar{\sigma}(X_t) \cdot dB_t + \bar{b}(X_t)dt$$

$$X_0 = x.$$
Let \( Y_t := \delta_{X^t_x} \in \mathcal{S}_{-p}, t \geq 0 \), and for simplicity we assume the associated lifetime \( \eta^x = \infty \) almost surely. We denote by \((P_t)\) the transition semi-group corresponding to \((X^t_x)\) and by \( P^*_t(x) := EY^x_t = E\delta_{X^t_x}\) which is just the transition probability measure of \((X^t_x)\) represented as an element of \( \mathcal{S}_{-p} \). Taking expected values in (3.4) we see that \( P^*_t(x) \) satisfies

\[
\partial_t u(t, x) = \bar{L}^* u(t, x) \tag{5.8}
\]

where \( \bar{L}^*(x) \) is the formal adjoint of \( \bar{L} \) satisfying :

\[
\bar{L}^* P^*_t(x) = E \delta_{X^t_x}.
\]

This maybe verified by acting with a test function \( u \in \mathcal{S} \). Equation (5.8) is the same as equation (4.6) for the linear operator \( L(x) = \bar{L}^*(x) \).

When \( \sigma_{ij}, b_i \) are twice continuously differentiable with bounded derivatives then (5.8) has a unique solution (see [14], Theorem 2.2.9). On the other hand, the operator \( \hat{L} \) in (4.7) when \( \bar{L}^*(x) = L^*(x) \) is just \( L^*(x) + V^*(x) = \bar{L}^* + V(x) \) and hence the solution of (4.6) viz. \( (P^*_t(x)) \) transforms into the solution of (4.7) viz. \( P^*_t(x)e^{tV(x)} \). Hence by the uniqueness result quoted above and the uniqueness result in Theorem (4.1), the evolution equation

\[
\partial_t u(t, x) = (\bar{L}^* + V) u(t, x) \tag{5.9}
\]

has a unique solution given by \( \hat{u}(t, x) := P^*_t(x)e^{tV(x)}, \) when \( \sigma_{ij}, b_i \) are twice continuously differentiable with bounded derivatives.

For \( V \in \mathcal{S}_p \) define \( c(.,.) : [0, \infty) \times C([0, \infty), \mathcal{S}_{-p}) \to \mathbb{R} \) as \( c(t, y) := \langle V, y(t) \rangle, \) \( y \in C([0, \infty), \mathcal{S}_{-p}) \). Let

\[
\hat{Y}_t := Y_t \int_0^t c(s, Y_s) ds
\]

where \( c(s, Y) := \langle V, Y_s \rangle = V(X^s_x) \). Since \( |c(t, Y)| \leq \|V\|pK \) where \( \|\delta_z\|_{-p} \leq K \), we have

\[
E\|\hat{Y}_t\|_p \leq e^{K\|V\|_p t}E\|Y_t\|_p < \infty.
\]
Let for $f \in S_p$,

$$P^V_t f(x) := E(e^{\int_0^t V(X^t_s)ds} f(X^t_t))$$

and let

$$P^{V*}_t(x) := E\hat{Y}_t = E(e^{\int_0^t V(X^t_s)ds} \delta_{X^t_t}) \in S_{-p}.$$ 

Then the following calculations show that $P^{V*}_t(x)$ satisfies (5.9). Let $f \in S$. From the definition of $\hat{L}(t,y)$ we have

$$\langle f, \hat{L}(t,\hat{Y}) \rangle = \int_0^t e^{\int_0^u V(X^u_s)ds} \langle f, L(\delta_{X^u_t}) \rangle + e^{\int_0^t V(X^t_s)ds} V(X^t_t)f(\delta_{X^t_t})$$

$$= \int_0^t e^{\int_0^u V(X^u_s)ds} (\bar{L} + V)f(X^t_t).$$

Hence from the equation satisfied by $\hat{Y}_t$ we get

$$\langle f, P^{V*}_t(x) \rangle = P^V_t f(x) = \langle f, E\hat{Y}_t \rangle = f(x) + \int_0^t \langle f, \hat{L}(s,\hat{Y}) \rangle ds$$

$$= f(x) + \int_0^t E[\int_0^u e^{\int_0^v V(X^v_s)ds} (\bar{L} + V)f(X^v_s)]ds$$

$$= f(x) + \int_0^t P^V_s ((\bar{L} + V)f)(x)ds$$

$$= \langle f, \delta_x \rangle + \int_0^t \langle f, \bar{L}^* + V \rangle (P^{V*}_s(x))ds.$$

It follows by uniqueness of solutions of (5.9) that with the coefficients $\sigma_{ij}, b_i$ as above, we have the following special case of Corollary (4.2) with $c(t, x, y) := V(x)$ and with equality in $S_{-p}$, for each $x \in \mathbb{R}^d$:

$$P^{V*}_t(x) = P^*_t(x) e^{tV(x)} = e^{tV(x)} E\delta_{X^t_t}.$$

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