Relativistic Brownian Motion

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Outline

- Assumptions
- Theoretical Derivation
- Numerical Verification
- Simulation
## Assumptions

| Classical Brownian Motion | Relativistic Brownian Motion |
|---------------------------|-----------------------------|
| Infinite Speed.           | Speed bounded by the speed of light: the *rapidity* is in the unit of speed of light. |
| The step is an independent identically distributed variable (i.i.d.). | The step is an independent identically distributed variable (i.i.d.). |
| The distribution of the step follows normal distribution. | The distribution of the step follows lognormal distribution. |

Note: $v = c \tanh(\theta)$ where $\theta$ is the rapidity and $v$ is velocity.
Theoretical Derivation

\( \mu_n(dx) \): probability measure -- log-normal probability density function. 

\[
\mu_n(dx) = \frac{1}{\sqrt{2\pi \sigma e^{\frac{\alpha^2 \sigma^2}{2}}}} \exp\left(-\frac{\log^2 x}{2\sigma^2}\right) x^{\alpha + 1} \, dx
\]

\( x \): step.

\( X_n \): displacement after \( n \) steps.

\( s \): time window.

\( \sigma^2 \): the variance of the rapidity.

\( -\alpha \sigma^2 \): the mean of the rapidity.
Theoretical Derivation

- Expected value for the exponential of displacement:

\[ \mathbb{E}(e^{-sX_n}) = \left[ \int_0^\infty e^{-sx} \mu_n(dx) \right]^n \]

Expected value for the exponential of step.

The sum becomes multiplication when taking the exponential.

\[ \mathbb{E}(e^{-sX_n}) = [1 + \frac{1}{n} \int_0^\infty (e^{-sx} - 1)n\mu_n(dx)]^n \]

\[ = \exp[n \log(1 + \frac{1}{n} \int_0^\infty (e^{-sx} - 1)n\mu_n(dx))] \]
Theoretical Derivation

First Approximation: Large number of steps. \( \frac{1}{n} \) is small, so \( \log \left(1 + \frac{k}{n}\right) \approx \frac{k}{n} \) where \( k \) is a constant.

\[
\mathbb{E}(e^{-sx_n}) = \exp\left[\int_0^\infty (e^{-sx} - 1)n\mu_n(dx)\right]
\]
Theoretical Derivation

- The log-normal probability density function.

\[
n\mu_n(dx) = \frac{n}{Z_{\sigma,\alpha}} \frac{\exp\left(-\frac{\log^2 x}{2\sigma^2}\right)}{x^{\alpha+1}} dx
\]

\[
Z_{\sigma,\alpha} = \sqrt{2\pi}\sigma e^{\frac{\alpha^2\sigma^2}{2}}
\]

- Normalization: Introduce \(\sigma_n\) such that

\[
Z_{\sigma,\alpha} = n \quad \Rightarrow \quad \sigma_n^2 = \frac{W\left(\frac{\alpha^2n^2}{2\pi}\right)}{\alpha^2} \quad (1)
\]

Lambert function: the principal branch solution of \(z = W(z)e^{W(z)}\).
Theoretical Derivation

- Second Approximation: Very large number of steps. \( \frac{1}{\log n} \) is small, so \( W \left( \frac{\alpha^2 n^2}{2\pi} \right) \approx \log \left( \frac{\alpha^2 n^2}{2\pi} \right) \).

\[
\mathbb{E}(e^{-sX_n}) = \exp\left[ \int_0^\infty (e^{-sx} - 1) \frac{\exp(-\frac{\log^2 x}{2\sigma^2})}{x^{\alpha+1}} \, dx \right]
\]  

(2)  

Approximated \( n\mu_n \, dx \).
Theoretical Derivation

- Third Approximation: \( \frac{\log^2 x}{2\sigma^2} \) is small, so \( \exp \left( -\frac{\log^2 x}{2\sigma^2} \right) \approx 1. \)

\[
\mathbb{E}(e^{-sX_n}) = \exp \left[ \int_0^\infty (e^{-sx} - 1) \frac{1}{x^{\alpha+1}} \, dx \right]
\]

Approximated \( n\mu_n \, dx \).
Theoretical Derivation

- **Fourth Approximation**: Large number of steps. \( n \to \infty \), so

\[
\int_0^\infty (e^{-sx} - 1) \frac{\exp\left(-\frac{\log^2 x}{2\sigma^2}\right)}{x^{\alpha+1}} dx \to \Gamma(-\alpha)s^\alpha - \frac{1}{2\sigma^2_n} \int_0^\infty \frac{\log^2 x}{x^{\alpha+1}} [e^{-sx} - 1] dx
\]

\[
\mathbb{E}(e^{-sX_n}) = \exp[\Gamma(-\alpha)s^\alpha]
\]  \hspace{1cm} (4)

The characteristic function of Levy distribution.
Theoretical Derivation

- Specifically, when $\alpha = \frac{1}{2}$

$$\mathbb{E}(e^{-sX_n}) = \exp(-2\pi^{\frac{1}{2}}s^{\frac{1}{2}})$$  \hspace{1cm} (5)
Numerical Verification

Equation 5
Equation 2 and 1
Equation 2
Equation 3

https://medlind.wordpress.com/cropped-images-snoopy-computer-jpg/
Simulation

- Example path

Large step: step larger than $\frac{10^{-2}}{n}$.
Simulation

- $N_j$ the number of large steps per trial follows Poisson Distribution.
Simulation

- Step distribution of large steps

![Diagram showing step distribution of large steps in a log-log scale.]
Acknowledgements

- Thank you to my research advisor Prof. Rajeev and to my academic advisor Prof. Iosevich.