New conditionally exactly solvable potentials of exponential type

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Abstract
Based on a method that produces the solutions to the Schrödinger equations of partner potentials, we give two conditionally exactly solvable partner potentials of exponential type defined on the half line. These potentials are multiplicative shape invariant and each of their linearly independent solution includes a sum of two hypergeometric functions. Furthermore we calculate the scattering amplitudes and study some of their properties.

KEYWORDS: Exactly solvable potentials; Shape invariance; Hypergeometric function; Scattering amplitude.
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1 Introduction
We know that in physics the exactly solvable problems are useful in the analysis of physical systems since they allow us to study in detail their properties or they are suitable approximations to more complex systems. In non relativistic quantum mechanics the potentials for which we can solve exactly the Schrödinger equation are used in the analysis of several phenomena. Therefore the search of new solvable potentials and the study of their properties is thoroughly investigated [1]–[14]. At present time there are several methods to find exact solutions to the Schrödinger equation. We know the factorization method [4]–[6], the methods based on supersymmetric quantum mechanics [1]–[3], [7]–[11], on the point canonical transformations [12], and on the Darboux transformations [13], [14].
Recently in Ref. [15] it is shown that for $x \in (0, +\infty)$ we can solve exactly the Schrödinger equations of the partner potentials

$$V_\pm(x) = \frac{m^2}{x} \pm \frac{m}{2} \frac{1}{x^{3/2}},$$

where $m$ is a constant. We also find that these potentials are multiplicative shape invariant and each linearly independent solution includes the sum of two confluent hypergeometric functions.

For the inverse square root potential

$$V_{SR} = \frac{\tilde{V}_0}{x^{1/2}},$$

where $\tilde{V}_0$ is a constant, in Ref. [16] Ishkhanyan finds that the Schrödinger equation is exactly solvable. Furthermore in Ref. [17] it is shown that another exactly solvable potential is the Lambert $W$-function step potential

$$V_W = \frac{\tilde{V}_0}{1 + W(e^{-u/\sigma})},$$

where $u \in (-\infty, +\infty)$, $W$ is the Lambert function, and $\sigma$ is a constant. We notice that each linearly independent solution found in Refs. [16], [17] includes the sum of two confluent hypergeometric functions with non-constant coefficients as the exact solutions previously studied in Ref. [15]. Furthermore, in Ref. [18] it is shown that for the sum of the potentials (1) and (2) the Schrödinger equation can be exactly solved and the linearly independent solutions include a sum of two confluent hypergeometric functions.

Based on the method of Ref. [15], for the partner potentials

$$V_{II}^\pm = m^2 e^u \frac{e^{u/2}}{e^u + 1} \pm \frac{m}{2} \frac{e^{u/2}}{(e^u + 1)^{3/2}},$$

where $m$ is a constant (as for the potentials (1)), in Ref. [19] we showed that the Schrödinger equation is exactly solvable and each linearly independent solution involves a sum of two hypergeometric functions with non-constant coefficients. Thus the results of Ref. [19] complement those of Refs. [15]–[18]. Recently these results are extended in Ref. [20] where an extensive study is carried out of the potentials whose solutions involve Heun functions and a list of known potentials with linearly independent solutions expanded as a sum of (confluent) hypergeometric functions is given.

Our purpose in this work is to extend the results on the potentials (1) and (4). Here we study the properties of two partner potentials defined on the half line and possessing the property that each of their linearly independent solutions includes two hypergeometric functions as those of Ref. [19] and in contrast to the potentials of Refs. [15]–[18] whose linearly independent solutions include two confluent hypergeometric functions. The expressions of the potentials that we
The potentials (5) are algebraic modifications of the Hulthen potential [2]

\[ V_H = \frac{Q}{e^x - 1}, \]  

where \( Q \) is a constant. We notice that the Hulthen potential near \( x = 0 \) behaves as \( 1/x \) and decays exponentially as \( x \to +\infty \). It is convenient to notice that we can not obtain the Hulthen potential as a limit of the potentials (5). From the shape of the potentials (5) we think that they can be useful to study scattering or tunneling phenomena (see Figs. 2-5) [22]. For some values of the parameters, in an interval, one of our potentials reminds us the shape of the effective potentials that govern the propagation of the Dirac field in a Schwarzschild black hole [23], [24] and therefore they can be used as a model to understand its dynamics in this spacetime. Furthermore we do not find the potentials (5) in the list of Ref. [20] that enumerates the known examples of potentials with linearly independent solutions involving a sum of (confluent) hypergeometric functions (see Table 3 of Ref. [20]). Thus we believe that these potentials are first studied in this work.

We think that the partner potentials (5) may be useful in supersymmetric quantum mechanics as a basis to generate new exactly solvable potentials of the Schrödinger equation [1]–[3], [9], [10]. Also the mathematical form of the exact solutions is not common (see the expressions (29), (30), (35)) and they can be used as a model to search new exact solutions of the Schrödinger equation, since exact solutions of this mathematical form appear previously in Refs. [15]–[19].

For several potentials we can find exact solutions to their Schrödinger equations in terms of special functions only when the parameters of the potentials satisfy some restrictions [3], [18], [25]–[27]. These potentials are known as conditionally exactly solvable potentials (CES potentials in what follows). For the partner potentials (5) that we study in this work we show that their parameters satisfy an algebraic constriction and therefore they are CES in the sense of Ref. [18], that is, we call a potential as CES when its parameters can not be varied independently, that is, they satisfy a constriction [18]. Notice that this definition of CES potential does not impose that some parameter takes a fixed value [18].

We organize this paper as follows. In Sect. [2] we study the properties of the partner potentials (5) that we analyze in this work. Using the method of Ref. [15] we solve exactly the Schrödinger equations of the studied potentials. We
also expound some facts on these partner potentials and verify the solutions that we previously found. In Sect. 3, we calculate the scattering amplitudes of the potentials (5). We study some additional characteristics of the potentials that we analyze in this paper in Sect. 4. Finally, for the method used in this work, in Appendix we verify that it produces the solutions to the Schrödinger equations of partner potentials.

2 Solution method

In a similar way to Refs. [15], [19], here we show that in the interval \( x \in (0, +\infty) \), for the partner potentials (5), we can solve exactly their Schrödinger equations in terms of hypergeometric functions. For these partner potentials the superpotential \( W \) is equal to

\[ W(x, m) = -\frac{m}{\sqrt{e^x - 1}}. \]  

(7)

We notice that in Refs. [1]–[3], [20], [28]–[32] that enumerate the solvable potentials already known, a search for the partner potentials (5) shows that they have not been previously discussed.

![Figure 1: Plots of the superpotential W for m = 1 (solid line) and for m = -1 (broken line).](image)

Since for the potentials (5), the constants multiplying to the factors \( 1/(e^x - 1) \) \((m^2)\) and \( e^x/(e^x - 1)^{3/2} \) \((\pm m/2)\) fulfill the expression \(-m^2/4 + (\pm m/2)^2 = 0\) these are CES potentials, as those previously studied in Refs. [15], [18], [19], [25]–[27]. It is convenient to notice that we classify the partner potentials (5) as

\[ W(x, m) = -m(Ae^x - B)^{-1/2} \]  

(with the constants \( A > B > 0 \)), by making the change of variable \( y = x + \ln(A/\sqrt{B}) \) and redefining the constant \( m \) by \( \tilde{m} = m/\sqrt{B} \), we simplify the Schrödinger equations of their partner potentials to those of the potentials for the superpotential (7).
CES since its parameters cannot be varied independently [15]. Some previously found CES potentials are \[3, 26\].

\[
\begin{align*}
\hat{V}_1(x) &= \frac{\hat{a}_1}{1 + e^{-2u}} - \frac{\hat{b}_1}{(1 + e^{-2u})^{1/2}} - \frac{3}{4(1 + e^{-2u})^2}, \\
\hat{V}_2(x) &= \frac{\hat{a}_2}{1 + e^{-2u}} - \frac{\hat{b}_2e^{-u}}{(1 + e^{-2u})^{1/2}} - \frac{3}{4(1 + e^{-2u})^2},
\end{align*}
\]

where the constants \(\hat{a}_1\) and \(\hat{b}_1\) (\(\hat{a}_2\) and \(\hat{b}_2\)) satisfy some constraints \[3, 26\]. We point out that the CES potentials (8) remind us to our potentials (5), but notice that we can not get these as a limit of the CES potentials (8) of Refs. \[3, 26\]. Moreover the interval where they are defined is different for the CES potentials (5) and (8).

In what follows we assume that \(m > 0\), since for \(m < 0\) we get the same results with the potentials \(V_+\) and \(V_-\) interchanged. Since for \(x > 0\) it is true that \(1/\sqrt{e^x - 1} > 0\), we note that the superpotential (7) does not cross the \(x\)-axis and therefore the supersymmetry is broken \[1, 2\]. We also find

\[
W_+ = \lim_{x \to +\infty} W = 0^-, \quad \lim_{x \to 0^+} W = -\infty,
\]

where \(0^+ (0^-)\) means that the quantity goes to zero taking positive (negative) values. Notice that as \(x \to +\infty\) the superpotential \(W\) decays exponentially, whereas near \(x = 0\) it behaves as \(1/\sqrt{x}\). Since for \(m > 0\) the derivative of \(W\) satisfies \(dW/dx > 0\), we obtain that the superpotential is an increasing function for \(x \in (0, +\infty)\). To illustrate these facts we plot the superpotential (7) in Fig. 1.

For the potentials \(V_\pm\) we get the following limits

\[
\begin{align*}
\lim_{x \to +\infty} V_+ &= 0^+, & \lim_{x \to 0^+} V_+ &= +\infty, \\
\lim_{x \to +\infty} V_- &= 0^-, & \lim_{x \to 0^+} V_- &= -\infty.
\end{align*}
\]
Furthermore these potentials decay exponentially to zero as $x \to +\infty$ (as the Hulthen potential (6)) and near $x = 0$ they diverge as $1/x^{3/2}$ (in a different way than the Hulthen potential (6)). We point out that near $x = 0$ the potential $V_+$ diverges to $+\infty$, whereas the potential $V_-$ diverges to $-\infty$. The potential $V_+$ does not cross the $x$-axis and it is strictly positive, but for $m > 1$ the potential $V_-$ crosses the $x$-axis at the two points $s_\pm = 2m^2 \pm 2m\sqrt{m^2 - 1}$, where $s = e^x$. Notice that $s_+ > s_- > 0$. For $m < 1$ the potential $V_-$ does not cross the $x$-axis and it is strictly negative.

Owing to the derivative of the potential $V_+$ satisfies $dV_+/dx < 0$, for $x \in (0, +\infty)$ we obtain that the potential $V_+$ decreases in this interval. For the potential $V_-$ we find that its derivative

$$
\frac{dV_-}{dx} = -\frac{e^x}{(e^x - 1)^2} \left( m^2 - \frac{m}{2} \frac{1 + e^x/2}{(e^x - 1)^{1/2}} \right), \tag{11}
$$

has critical points at $s_{1,2} = 8m^2 - 2 \pm 4m\sqrt{4m^2 - 3}$. Hence for $m > \sqrt{3}/2$ the potential $V_-$ has two real critical points, whereas for $m < \sqrt{3}/2$ it does not have real critical points. We note that $s_1 > s_2 > 0$, and we also point out that the critical point $s_2$ is a maximum and $s_1$ is a minimum. Furthermore we notice that for $m > 1$ the quantities $s_\pm$ and $s_{1,2}$ satisfy $s_1 > s_+ > s_2 > s_-$, that is, the maximum of the potential $V_-$ is located between the points $s_\pm$ where $V_-$ intersects the $x$ axis and its minimum has a coordinate greater than the intersections of the potential $V_-$ with the $x$ axis. We illustrate these facts in Figs. 2–5.

In what follows, using the method of Ref. [15], (see also Ref. [19]) we solve exactly the Schrödinger equations of the partner potentials [15]. With this objective we write these equations as

$$
\frac{d^2Z_-}{dx^2} + \omega^2Z_- = \left( W^2 - \frac{dW}{dx} \right) Z_-,
\tag{12}
$$

$$
\frac{d^2Z_+}{dx^2} + \omega^2Z_+ = \left( W^2 + \frac{dW}{dx} \right) Z_+,
$$
and as in Ref. [15], to simplify the equations that follow, we denote the energy $E$ as $\omega^2$. In Ref. [15] it is shown that for $\omega \neq 0$ the Schrödinger equations (12) can be written as

\[
\left( \frac{d}{dx} - W \right) \frac{1}{i\omega} \left( \frac{d}{dx} + W \right) Z_- = i\omega Z_- ,
\]

(13)

\[
\left( \frac{d}{dx} + W \right) \frac{1}{i\omega} \left( \frac{d}{dx} - W \right) Z_+ = i\omega Z_+ ,
\]

from which we obtain that the functions $Z_+$ and $Z_-$ satisfy the coupled system

\[
\left( \frac{d}{dx} + W \right) Z_- = i\omega Z_+ , \quad \left( \frac{d}{dx} - W \right) Z_+ = i\omega Z_- .
\]

(14)

Defining $Z_{\pm} = R_1 \pm R_2$, we get that Eqs. (14) transform into the coupled system

\[
\frac{dR_1}{dx} - i\omega R_1 = WR_2 , \quad \frac{dR_2}{dx} + i\omega R_2 = WR_1 ,
\]

(15)

(see Eqs. (11) of Ref. [15]). In Appendix we show that the solutions of these coupled equations produce the solutions to the Schrödinger equations of partner potentials.

As in Ref. [15] we take $R_1 = e^{-i\pi/4} \tilde{R}_1$, $R_2 = e^{i\pi/4} \tilde{R}_2$, and defining the variable $z$ by

\[
z = e^{-x} ,
\]

(16)

we find that the coupled system of differential equations (16) transforms into

\[
\frac{z}{d^2} \frac{d\tilde{R}_1}{dz} + i\omega \tilde{R}_1 = im \frac{z^{1/2}}{(1 - z)^{1/2}} \tilde{R}_2 ,
\]

\[
\frac{z}{d^2} \frac{d\tilde{R}_2}{dz} - i\omega \tilde{R}_2 = -im \frac{z^{1/2}}{(1 - z)^{1/2}} \tilde{R}_1 .
\]

(17)

From this coupled system we obtain that the functions $\tilde{R}_1$ and $\tilde{R}_2$ must be solutions of the decoupled differential equations

\[
\frac{d^2 \tilde{R}_k}{dz^2} + \left( \frac{1/2}{z} - \frac{1/2}{1 - z} \right) \frac{d\tilde{R}_k}{dz} + \frac{\omega^2 - i\omega \epsilon / 2}{z^2} \tilde{R}_k - \frac{m^2 + i\omega \epsilon / 2}{z(1 - z)} \tilde{R}_k = 0 ,
\]

(18)

where $k = 1, 2$, and $\epsilon = 1$ ($\epsilon = -1$) for $\tilde{R}_1$ ($\tilde{R}_2$).

If the functions $\tilde{R}_1$ and $\tilde{R}_2$ take the form $\tilde{R}_k = z^{A_k} \tilde{R}_k$, with the quantities $A_k$ being solutions of the algebraic equations

\[
A_k^2 - \frac{A_k}{2} - \frac{i\omega \epsilon}{2} + \omega^2 = 0 ,
\]

(19)

\[\text{Notice that for } x \in (0, +\infty) \text{ the variable } z \text{ varies over the range } 0 < z < 1.\]
we find that the functions $\tilde{R}_k$ satisfy the differential equations
\[
\frac{d^2 \tilde{R}_k}{dz^2} + \left( \frac{2A_k + 1/2}{z} - \frac{1/2}{1-z} \right) \frac{d\tilde{R}_k}{dz} - \frac{m^2 + i\omega/2 + A_k/2}{z(1-z)} \tilde{R}_k = 0. \tag{20}
\]
These equations are of hypergeometric type \[33\]–\[36\]
\[
z(1-z)\frac{d^2 F}{dz^2} + (c - (a + b + 1)z)\frac{dF}{dz} - abF = 0, \tag{21}
\]
with the parameters $a_k$, $b_k$, $c_k$ equal to
\[
a_k = A_k + i(m^2 + \omega^2)^{1/2}, \quad b_k = A_k - i(m^2 + \omega^2)^{1/2}, \quad c_k = 2A_k + 1/2. \tag{22}
\]
If the parameters $c_k$ are not integers then the functions $\tilde{R}_k$ are
\[
\tilde{R}_k = z^{A_k} [G_k \, _2F_1(a_k, b_k; c_k; z) + H_k \, z^{1-c_k} \, _2F_1(a_k - c_k + 1, b_k - c_k + 1; 2 - c_k; z)], \tag{23}
\]
where $_2F_1(a, b; c; z)$ denotes the hypergeometric function \[33\]–\[36\], and the quantities $G_k$, $H_k$ are constants.

In a straightforward way we find that Eqs. \[15\] impose conditions on the constants $G_k$ and $H_k$. To discuss this fact we take the quantities $A_1$ and $A_2$ as $A_1 = i\omega + 1/2 = A_2 + 1/2$. Therefore the constants $a_k$, $b_k$, $c_k$ are equal to
\[
a_1 = a_2 + 1/2 = 1/2 + i\omega + i(m^2 + \omega^2)^{1/2}, \quad b_1 = b_2 + 1/2 = 1/2 + i\omega - i(m^2 + \omega^2)^{1/2}, \quad c_1 = c_2 + 1 = 2i\omega + 3/2. \tag{24}
\]

From Eqs. \[15\] and the contiguous relations of the hypergeometric function \[34\] we obtain:

a) If we choose the function $\tilde{R}_1$ as
\[
\tilde{R}_1 = G_1 \, z^{A_1} \, _2F_1(a_1, b_1; c_1; z), \tag{25}
\]
then from Eqs. \[15\] we get that the function $\tilde{R}_2$ must be equal to
\[
\tilde{R}_2 = G_1 \, \frac{c_1 - 1}{im} \, z^{A_2} \, _2F_1(a_2, b_2; c_2; z), \tag{26}
\]
and the constants $G_1$ and $G_2$ are related by $G_2 = G_1 (c_1 - 1)/(im)$.

b) If we select the function $\tilde{R}_1$ in the form
\[
\tilde{R}_1 = H_1 \, z^{A_1+1-c_1} \, _2F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; z), \tag{27}
\]
then from Eqs. \[15\] we obtain that the function $\tilde{R}_2$ must be equal to
\[
\tilde{R}_2 = H_1 \, \frac{(a_1 - c_1 + 1)(b_1 - c_1 + 1)}{im(2 - c_1)} \, z^{A_2+1-c_2} \times _2F_1(a_2 - c_2 + 1, b_2 - c_2 + 1; 2 - c_2; z), \tag{28}
\]

\[3\]We take the constants $c_k$ different from integral numbers to discard the solutions of the hypergeometric equation \[21\] that include logarithmic terms \[33\]–\[36\].
and the constants $H_1$ and $H_2$ satisfy $H_2 = H_1(a_1 - c_1 + 1)(b_1 - c_1 + 1)/(im(2 - c_1))$. Hence we find that Eqs. (15) impose the previous constrictions on the constants $G_k$ and $H_k$.

Considering the previous definitions we get that as function of $\tilde{R}_k$ the solutions $Z_\pm$ take the form

$$Z_\pm = e^{-i\pi/4}(\tilde{R}_1 \pm i\tilde{R}_2).$$

Thus from our results we get that the linearly independent solutions to the Schrödinger equations of the potentials $V_\pm$ are

$$Z_\pm^I = G_1 e^{-i\pi/4}\left(z^{A_1} F_1(a_1, b_1; c_1; z) \pm \frac{m - 1}{m} z^{A_2} F_1(a_2, b_2; c_2; z)\right), \quad (29)$$

and

$$Z_\pm^{II} = H_1 e^{-i\pi/4}\left(z^{A_1} F_1(a_1 - c_1 + 1, b_1 - c_1 + 1; 2 - c_1; z)\right) \quad (30)$$

$$\pm z^{A_2} (a_1 - c_1 + 1)(b_1 - c_1 + 1)(2 - c_1)m \times 2 F_1(a_2 - c_2 + 1, b_2 - c_2 + 1; 2 - c_2; z).$$

Using that for the linearly independent solutions to the hypergeometric differential equation (21) its Wronskian is

$$W_z[Z_\pm^I, Z_\pm^{II}] = \frac{(1 - c)z^{c-1}}{(1 - z)^{a+b+1-c}}, \quad (31)$$

in a straightforward way we find that the Wronskian of the solutions $Z_\pm^I$ and $Z_\pm^{II}$ is equal to (for $G_1 = H_1 = 1$)

$$W_z[Z_\pm^I, Z_\pm^{II}] = \pm \frac{2\omega(c_1 - 1)}{m}. \quad (32)$$

Furthermore, taking into account Eqs. (20), we get that the functions $\tilde{R}_k$ satisfy

$$\frac{d}{dz}\left(z \frac{d\tilde{R}_k}{dz}\right) = \frac{1}{2} \frac{d\tilde{R}_k}{dz} - \frac{A_k}{2(1 - z)} \tilde{R}_k + \frac{A_k^2}{z} \tilde{R}_k + \frac{m^2 + A_k/2 + i\omega/2}{1 - z} \tilde{R}_k. \quad (33)$$

From these equations we get that the functions $Z_\pm$ fulfill

$$\frac{d}{dz}\left(z \frac{dZ_\pm}{dz}\right) + \left(\frac{\omega^2 - m^2}{z} \pm \frac{m}{2} \frac{1}{z^{1/2}(1 - z)^{3/2}}\right) Z_\pm = 0, \quad (34)$$

that are the Schrödinger equations (12) in the variable $z$ defined in the expression (16). Hence, if the functions $\tilde{R}_k$ are solutions of Eqs. (20), then the functions $Z_\pm$ solve the Schrödinger equations (12) with the potentials (5).
3 Scattering amplitude

In what follows we determine the scattering amplitudes for the potentials $V_{\pm}$. To calculate the scattering amplitude it is convenient to use the variable $v = 1 - z$ to write the solutions of the Schrödinger equations. We find that in this variable, the linearly independent solutions of the Schrödinger equations for the potentials (5) take the form

$$\tilde{Z}_I^\pm = \tilde{C}_1 \left( (1-v)^{B_1} \right)_2 F_1(\alpha_1, \beta_1; \gamma_1; v)$$
$$- \frac{m}{\gamma_1} (1-v)^{B_1} v^{1-2\gamma_2} F_1(\alpha_2 - \gamma_2 + 1, \beta_2 - \gamma_2 + 1; 2 - \gamma_2; v),$$
$$\tilde{Z}_{II}^\pm = \tilde{C}_2 \left( (1-v)^{B_1} v^{1-2\gamma_2} F_1(\alpha_1 - \gamma_1 + 1, \beta_1 - \gamma_1 + 1; 2 - \gamma_1; v)$$
$$- \frac{\gamma_1}{m} (1-v)^{B_2} F_1(\alpha_2, \beta_2; \gamma_2; v) \right),$$

where $\tilde{C}_1, \tilde{C}_2$ are constants and

$$B_1 = \frac{1}{2} + i\omega, \quad B_2 = i\omega,$$
$$\alpha_k = B_k + i(m^2 + \omega^2)^{1/2}, \quad \beta_k = B_k - i(m^2 + \omega^2)^{1/2}, \quad \gamma_k = 1/2. \quad (36)$$

In what follows we study in detail the potential $V_+$ since a similar calculation produces the result for the potential $V_-$. We see that near $x = 0$ ($v = 0$) the solutions $\tilde{Z}_\pm$ behave as

$$\tilde{Z}_I^\pm \approx 1 - \frac{m}{\gamma_1} v^{1/2}, \quad \tilde{Z}_{II}^\pm \approx -\frac{\gamma_1}{m} + v^{1/2}. \quad (37)$$

Since we like to impose as boundary condition that the solution is equal to zero at $x = 0$ it is convenient to define the new solutions

$$Y_I^+ = \tilde{Z}_I^+, \quad Y_{II}^+ = \tilde{Z}_{II}^+, \quad Y_I^- = \tilde{Z}_I^-, \quad Y_{II}^- = \tilde{Z}_{II}^-,$$

that near $x = 0$ behave as

$$Y_I^+ \approx 2 - \frac{2m}{\gamma_1} v^{1/2}, \quad Y_{II}^+ \approx 0. \quad (39)$$

Therefore to satisfy the boundary condition we choose the solution $Y_{II}^+$ that is equal to zero at $x = 0$.

Taking into account the Kummer property of the hypergeometric function $^{33} - ^{39}$

$$2 F_1(a, b; c; v) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} 2 F_1(a, b; a + b + 1 - c; 1 - v)$$
$$+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}(1 - v)^{c-a-b} 2 F_1(c - a, c - b; c + 1 - a - b; 1 - v),$$

\footnote{We point out that the variable $v$ varies over the range $0 < v < 1$.}
we obtain that as $x \to \infty$ ($v \to 1$) the solution $Y_{II}^+$ behaves as

$$Y_{II}^+ \approx \frac{\Gamma(1/2 + 2i\omega)\Gamma(-2\alpha_2)\Gamma(-2\beta_2)}{\Gamma(1/2 - 2i\omega)\Gamma(2\alpha_2)\Gamma(2\beta_2)}$$

$$\times \frac{m\Gamma(\alpha_2)\Gamma(\beta_2) + \Gamma(1/2 + \alpha_2)\Gamma(1/2 + \beta_2)}{m\Gamma(-\alpha_2)\Gamma(-\beta_2) + \Gamma(1/2 - \alpha_2)\Gamma(1/2 - \beta_2)} e^{i\omega x} - e^{-i\omega x},$$

that is, the scattering amplitude of the potential $V_+$ is equal to

$$S_+ = \frac{\Gamma(1/2 + 2i\omega)\Gamma(-2\alpha_2)\Gamma(-2\beta_2)}{\Gamma(1/2 - 2i\omega)\Gamma(2\alpha_2)\Gamma(2\beta_2)}$$

$$\times \frac{m\Gamma(\alpha_2)\Gamma(\beta_2) + \Gamma(1/2 + \alpha_2)\Gamma(1/2 + \beta_2)}{m\Gamma(-\alpha_2)\Gamma(-\beta_2) + \Gamma(1/2 - \alpha_2)\Gamma(1/2 - \beta_2)} e^{i\omega x} - e^{-i\omega x},$$

(42)

Notice that the previous scattering amplitude satisfies $S_+ S_+^* = 1$. A similar result is valid for the scattering amplitude of the potential $V_-$.  

4 Discussion

Here we show that each of the exact solutions (29) and (30) of the Schrödinger equations for the partner potentials (5) includes a sum with non-constant coefficients of two hypergeometric functions (see also the exact solutions (35)). We note that this form of the solutions is not common in the previous references [1]–[3], [10]. The potentials that we study in this work may be suitable to analyze scattering and tunneling phenomena and they can be taken as a basis to search new exactly solvable potentials, since the mathematical form of their solutions is not widely explored.

To finish this work we notice the following facts on the potentials (5).

- Considering that $e^x = e^{x/2}/e^{-x/2}$ and employing hyperbolic functions we obtain that the superpotential (7) and the potentials (5) take the form

$$W = -\mu(\coth(x/2) - 1)^{1/2},$$

$$V_\pm = \mu^2(\coth(x/2) - 1) \pm \frac{\mu}{4} \frac{(1 + \coth(x/2))^{1/2}}{\sinh(x/2)},$$

(43)

with $\mu = m/\sqrt{2}$.

- We notice that near $x = 0$, the potentials $V_\pm$ behave as (preserving the leading and subleading terms)

$$\frac{m^2}{x} \pm \frac{m}{2} \frac{1}{x^{3/2}},$$

(44)

that are the potentials (1) previously studied in Ref. [15], that is, near $x = 0$ our potentials $V_\pm$ yield the behavior analyzed in Ref. [15]. In contrast to the potentials of Ref. [15], the potentials $V_\pm$ decay exponentially as $x \to +\infty$ (the potentials (1) decay as $1/x$ as $x \to +\infty$). Thus we can consider to the potentials (5) as a generalization of the potentials (1).
• The partner potentials are shape invariant if they satisfy $V_+(x, \alpha) = V_-(x, \alpha_1) + R(\alpha_0)$, where the parameters $\alpha_0, \alpha_1$ are independent of the coordinate $x$, with $\alpha_1 = f(\alpha_0)$, and $R(\alpha_0)$ is also a function of $\alpha_0$. From the expressions (37) we notice that the potentials $V_\pm$ fulfill $V_-(x, -m) = V_+(x, m)$, and therefore they are multiplicative shape invariant, since $\alpha_0 = m, \alpha_1 = -\alpha_0 = q\alpha_0$ with $q = -1$ and $R(\alpha_0) = 0$. For several multiplicative shape invariant potentials that are already found [1], [10], we know them in series form, but we get the potentials (5) in closed form, in a similar way to the multiplicative shape invariant potentials of Refs. [15], [19].

Recently in Ref. [38] is studied the concept of shape invariance with reflection transformations. The analyzed transformations include reflections of the coordinates and translations of the parameters. For the potentials (5) the formula $\alpha_1 = -\alpha_0$ remind us a reflection, but for the parameters of the potential and it is different from the mathematical operations considered in Ref. [38].

• As previously noted, the superpotential $W$ does not cross the $x$-axis and therefore the supersymmetry is broken [1], [2]. Thus the functions $\psi_0^\mp$ that are solutions of the differential equations

$$
\left(\frac{d}{dx} + W\right)\psi_0^- = 0, \quad \left(-\frac{d}{dx} + W\right)\psi_0^+ = 0,
$$

are equal to

$$
\psi_0^\mp = \exp\left(\mp \int W(x')dx'\right) = \left[\sqrt{1 - e^{-x} + ie^{-x/2}}\right]^{\pm 2im},
$$

and they are not normalizable.

Taking into account the formulas (15.4.11) and (15.4.15) of Ref. [36], we obtain that the functions $\psi_0^\mp$ can be written in the form

$$
\psi_0^\mp = 2F_1(\mp im, \pm im; 1/2; z) \mp 2mz^{1/2}2F_1(1/2 \mp im, 1/2 \pm im; 3/2; z).
$$

From the values of the parameters $A_k$ and $a_k, b_k, c_k$ with $\omega = 0$ we find that the functions $\tilde{R}_k$ of the expressions (25) and (26) simplify to

$$
\tilde{R}_1 = z^{1/2}2F_1(1/2 + im, 1/2 - im; 3/2; z),
\tilde{R}_2 = \frac{1}{2m_1^2}2F_1(im, -im; 1/2; z).
$$

Thus from the expressions (47) and (48) we obtain

$$
\psi_0^\mp = \mp 2m(\tilde{R}_1 \mp i\tilde{R}_2) = \mp 2me^{i\pi/4}Z_\mp.
$$

Hence the functions $\psi_0^\mp$ are proportional to $Z_\mp$, as we expect from the previous analysis.
Finally we notice that for the potentials (5) each linearly independent solution of the Schrödinger equations (12) includes a sum with non-constant coefficients of two hypergeometric functions (see the expressions (29), (30), and (35)), and we have not been able to simplify this sum to a single hypergeometric function, but this fact must be studied carefully. Therefore, as those of Refs. [15]–[19], the potentials (5) are examples of the potentials analyzed, but not given in explicit form in Ref. [21] whose linearly independent solutions include a sum of (confluent) hypergeometric functions.

From the results of Ref. [20] we think that the solutions of the Schrödinger equations for the potentials (5) also can be expanded in terms of Heun functions. As far as we can see the advantage of writing the solutions (29) and (30) (see also (35)) as a sum of hypergeometric functions (instead of Heun functions) is that we can use the well developed techniques involving (confluent) hypergeometric functions (as Kummer's property (40)) in the study of the characteristics for the potentials (5), as illustrated in Sect. 3 (and in Refs. [17], [19]). Thus we think that it is convenient to search and study the potentials whose linearly independent solutions have this property.

To generalize the results of Refs. [15]–[19] and this paper, a problem to analyze in detail is the search of potentials with the property that each linearly independent solution includes a sum of three or more (confluent) hypergeometric functions.

5 Appendix

In this Appendix we show that the solutions to the system of coupled equations (15) produce the solutions to the Schrödinger equations of the partner potentials

\[ \tilde{V}_\pm = W^2 \pm \frac{dW}{dx}. \]

(50)

First, from Eqs. (12) and the definition of \( Z_\pm \) we notice that the Schrödinger equations for the partner potentials can be written in the form

\[ \frac{d}{dx} \left( \frac{d}{dx} (R_1 \pm R_2) + \omega^2 (R_1 \pm R_2) - \left( W^2 \pm \frac{dW}{dx} \right) (R_1 \pm R_2) \right) = 0. \]

(51)

Using Eqs. (15) we get that the left hand sides of the previous equations transform into

\[ \frac{d}{dx} \left( i\omega R_1 \mp i\omega R_2 + WR_2 \pm WR_1 \right) + \omega^2 (R_1 \pm R_2) \]

\[ \quad - \left( W^2 \pm \frac{dW}{dx} \right) (R_1 \pm R_2). \]

(52)
Expanding the first four factors of the previous expressions and considering Eqs. (15) we find that the formulas (52) become

\[
\begin{align*}
- \omega^2 R_1 \mp \omega^2 R_2 + \frac{dW}{dx}(R_2 \pm R_1) + W^2 R_1 \pm W^2 R_2 \\
+ \omega^2 (R_1 \pm R_2) - \left( W^2 \pm \frac{dW}{dx} \right) (R_1 \pm R_2). \tag{53}
\end{align*}
\]

Simplifying the previous expressions we obtain

\[
\frac{dW}{dx}(R_2 \pm R_1) - \frac{dW}{dx}(R_2 \pm R_1) = 0. \tag{54}
\]

Therefore from the solutions of the coupled system (15) we obtain the solutions to the Schrödinger equations of the partner potentials (50).

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