SQUARE ROOT OF THE MONODROMY MAP FOR THE EQUATION OF RSJ MODEL OF JOSEPHSON JUNCTION

SERGEY I. TERTYCHNY

Abstract. Several noteworthy properties of the differential equation utilized for the modeling of Josephson junctions are considered. The explicit representation of the monodromy transform of the space of its solutions is given. In case of positive integer order, the transformation interpreted as the square root of the monodromy transformation noted is derived making use of a symmetry the associated linear second order differential equation possesses.

The present notes are devoted to discussion of some noteworthy properties of the differential equation

\[ \dot{\varphi} + \sin \varphi = B + A \cos \omega t, \]

in which \( \varphi = \varphi(t) \) is the unknown function, the symbols \( A, B, \omega \) stand for some real constants, and \( t \) is a free real variable, the dot denoting derivation with respect to \( t \). Eq. (1) and its generalizations are of interest, in particular, in view of their application in a number of models in physics, mechanics, and geometry \cite{1,2}. Most widely Eq. (1) is known as the equation utilized in the so called RSJ model of Josephson junction \cite{3,4,5,6,7} which applies if the effect of the junction electric capacitance is negligible (the case of so called overdamped Josephson junctions).

Eq. (1) is equivalent to the Riccati equation

\[ \Phi' = (2i \omega z)^{-1}(1 - \Phi^2) + (\ell z^{-1} + \mu(1 + z^{-2}))\Phi, \]

where \( \Phi = \Phi(z) \) is a holomorphic function of the free complex variable \( z \), the prime denoting the derivative with respect to the latter, and \( \ell, \mu, \omega \) are the constant parameters. Indeed, the formal substitutions

\[ z \Leftarrow e^{i\omega t}, \quad \Phi(z) \Leftarrow e^{i\varphi(t)} \]

convert Eq. (2) to Eq. (1) get with the parameters related to the parameters involved in the former equation by the transformation

\[ A = 2\omega \mu, \quad B = \omega \ell. \]

The nonlinear equation (2) has the only singular point \( z = 0 \) but its solutions may diverge at of some other values of argument. Nevertheless, one can claim the following \cite{13}:

---

Supported in part by RFBR grant N 17-01-00192.
Proposition 1. Let the constants $\ell, \mu$, and $\omega > 0$ be real and let $\Phi(z)$ be a solution to Eq. (2) holomorphic at $z = 1$ such that $|\Phi(1)| = 1$. Then $\Phi(z)$ is also holomorphic in some vicinity of the curve $|z| = 1$; moreover, if $|z| = 1$ then $|\Phi(z)| = 1$.

Indeed, any solution to Eq. (1) is real analytic and can be extended to the whole real axis $\mathbb{R}$. Carrying out analytic continuation of $\varphi(t)$ from $\mathbb{R}$ to some vicinity in $\mathbb{C}$, one obtains the function $\Phi = \exp(i\varphi(t))$ which is holomorphic in $t$ and which can also be considered as a holomorphic function of the variable $z = \exp(i\omega t)$ varying in some open set embodying the curve $|z| = 1$. The function $\Phi(z)$ possesses the properties asserted above, obviously.

It is also obvious that if we are given the function $\Phi(z)$ obeying Eq. (2) and unimodular on the curve $|z| = 1$, then the real-valued smooth function $\varphi(t)$ such that $e^{i\varphi(t)} = \Phi(e^{i\omega t})$ can be constructed. It verifies Eq. (1), evidently.

It is in order now to comment on the term “the curve $|z| = 1$” used above instead of something like “the unit circle $S^1 \subset \mathbb{C}$” which one could argue to be more customary. The point is that, strictly speaking, apart of the very special conditions, a solution to Eq. (2) can not be holomorphic on $S^1$. The rationale is here fairly simple: indeed, there is no reason why a generic solution $\varphi(t)$ to Eq. (1) should obey the constraint $\varphi(\frac{T}{2}) - \varphi(\frac{T}{2}) \equiv 0$, where

$$T = 2\pi \omega^{-1}$$

is the period of the right hand side expression in Eq. (1). Accordingly, following the way of constructing of the function $\Phi$ via the function $\varphi$ utilized above, one obtains $\Phi(-1 + 0) - \Phi(-1 - 0) = e^{i\varphi(\frac{T}{2})} - e^{i\varphi(\frac{T}{2})} \neq 0$, meaning that $\Phi$, when considered on $S^1$, proves to be not continuous at $-1$.

There is, definitely, no singularity of $\Phi$ at $-1$ and the deficiency in the above construction originates in the improper selection of the $\Phi$ domain. Generally speaking, it can not be the complex plane or any its subset; instead, the universal cover $\mathcal{C}$ of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus 0$ (with the subset of the isolated singular points of $\Phi$ removed) has to be utilized. In this setting, the image of the “$t$-axis”, produced by the map extending the transformation (3) to $\mathcal{C}$, is not $S^1$ but the non-compact curve covering $S^1$. It is this curve which, admitting some abuse of notations, was referred to as “the curve $|z| = 1$”.

Here we shall not, however, consider solutions to Eq. (2) on their whole domains but only on the sub-domain $\mathcal{C}$ which is in bijective correspondence with (projects to) the subset $\mathcal{C}^*$ of $\mathbb{C}^*$ obtained by removal of the ray of negative real numbers, $\mathcal{C}^* = \mathbb{C} \setminus \mathbb{R}_{<0}$, (and removal, for each $\Phi$, its singular points, if any). In most cases, $\mathcal{C}$ can be (and will be) considered to be undistinguished from $\mathcal{C}^*$. However, there are two boundaries of $\mathcal{C}$, which projects to the same (removed) ray $\mathbb{R}_{<0}$, on which the values of $\Phi$ do not coincide. To mirror such a difference, we may identify these boundaries with the two edges of the corresponding cut in $\mathbb{C}^*$.
It is then convenient to refer to the points of the cut edge contacting the half-plane \( \{ z \in \mathbb{C}^*, \Im z > 0 \} \) by the symbol \( \rho e^{i\pi} \), where \( \rho \) stands for a positive real number, \( \rho \in \mathbb{R}_{>0} \), and by the symbol \( \rho e^{-i\pi} \) for a point belonging to the cut edge contacting the half-plane \( \{ z \in \mathbb{C}^*, \Im z < 0 \} \). If \( \rho = 1 \) the factor \( \rho \) is omitted. In such a framework, the portion of \( S^1 \) coming to be in \( \mathring{\mathbb{C}}^* \) is “the punctured circle”

\[
\mathring{S}^1 = \{ z \in \mathbb{C}, |z| = 1, z \neq -1 \}.
\]

It approaches near the ends the (distinct) boundary points denoted \( e^{i\pi} \) and \( e^{-i\pi} \).

Each solution to Eq. (2) is holomorphic and non-zero in some vicinity of \( \mathring{S}^1 \). Besides, at the boundary points of \( \mathring{S}^1 \), it holds

\[
\Phi(e^{i\pi}) = e^{i\pi(\frac{1}{2}T)} \quad (\pm) \quad \Phi(e^{-i\pi}) = e^{i\pi(\frac{1}{2}T)}.
\]

The following statement is an obvious consequence of the periodicity of the right hand side of Eq. (1):

**Proposition 2.** Let the function \( \varphi(t) \) verify Eq. (1). Then the function \( \varphi_M(t) := \varphi(t + T) \) is also a solution to Eq. (1).

Let the solution \( \Phi(z) \) to Eq. (2) be constructed in accordance with the algorithm specified above. The above statement and definitions imply the following.

**Proposition 3.** There exists the solution \( \Phi_M(z) \) to Eq. (2) such that

\[
\Phi_M(e^{-i\pi}) = \Phi(e^{i\pi}).
\]

Moreover, if (7) holds true for some solutions \( \Phi, \Phi_M \) then

\[
\Phi_M(\rho e^{-i\pi}) = \Phi(\rho e^{i\pi})
\]

for all \( \rho > 0 \) excluding ones at which \( \Phi(\rho e^{i\pi}) \) is not analytic (i.e. is undefined).

Indeed, the function \( \Phi_M(z) \) wanted can be constructed from the function \( \varphi_M(t) \) in the same way as the function \( \Phi(z) \) is constructed from \( \varphi(t) \). The function \( \varphi_M(t) \) is actually some “portion” of the maximally extended solution \( \varphi(t) \) get from the segment \( (-\frac{1}{2}T, T, \frac{1}{2}T + T) \) and “put down” to the segment \( (-\frac{1}{2}T, \frac{1}{2}T) \) considered as the common domain with \( \varphi(t) \); similarly, the function \( \Phi_M \) is, essentially, the function \( \Phi \) get on the sub-domain adjacent via the common boundary with \( \mathring{\mathbb{C}} \) and considered on the same domain with \( \Phi \) trough their projections to the common area \( \mathring{\mathbb{C}}^* \) which can be considered equivalent to \( \mathring{\mathbb{C}} \).

Alternatively, the function \( \Phi_M \) can also be defined as the result of point-wise analytic continuations in \( \mathbb{C}^* \) of the function \( \Phi \) defined on \( \mathring{\mathbb{C}}^* \) along the full circles with centers situated at zero which are passed in the counter-clockwise direction (or along the curves avoiding \( \Phi \) singularities and homotopic to such circles). The latter interpretation allows one to refer to transformation \( M: \Phi \mapsto \Phi_M \) as the **monodromy map** which acts on the space of solutions to Eq. (2).

We are now ready to formulate the first non-evident result the present notes are devoted to.
Theorem 4. Let a solution \( \Phi \) to Eq. \((2)\) holomorphic in some vicinity of \( S^1 \) be given. Let also \( \Psi = \Psi(z) \) be a solution of the linear homogeneous first order ordinary differential equation
\[
2i\omega z\Psi' = (\Phi + \Phi^{-1})\Psi.
\]
Let, finally,
\[
\Psi(1) = 1 \text{ and } |\Phi(1)| = 1.
\]
Then the formula
\[
\Phi_M(z) = \left( e^{\frac{i}{2}P(T)} \cos \frac{i}{2}\varphi\left(\frac{T}{2}\right) \cdot \Psi(z) \frac{1}{2}\Phi(z) \frac{1}{2} \right. \\
\left. + ie^{\frac{i}{2}P(-\frac{T}{2})} \sin \frac{i}{2}(\varphi(\frac{T}{2}) - \varphi(-\frac{T}{2})) \cdot \Psi(1/z)\frac{1}{2}\Phi(1/z)\frac{1}{2} \right) \times \\
\left( e^{\frac{i}{2}P(T)} \cos \frac{i}{2}\varphi\left(\frac{T}{2}\right) \cdot \Psi(z) \frac{1}{2}\Phi(z) \frac{1}{2} \\
- ie^{\frac{i}{2}P(-T)} \sin \frac{i}{2}(\varphi(\frac{T}{2}) - \varphi(-\frac{T}{2})) \cdot \Psi(1/z)\frac{1}{2}\Phi(1/z)\frac{1}{2} \right)^{-1}
\]
in which the continuous (and then necessarily real analytic) function \( \varphi \) is determined by the equation \( \Phi(e^{i\varphi(t)}) = e^{P(t)} t \in (-\frac{T}{2}, \frac{T}{2}) \), yields the explicit representation of the result of the monodromy transformation of the function \( \Phi \).

The above assertion means that any solution to Eq. \((2)\) can be extended from its sub-domain with the closure equal to \( S^1 \) to the whole domain with the closure equal to \( C \) by means of certain algebraic transformations (provided the function \( \Psi \) had once only been computed on \( S^1 \)).

Before proving these, it is worth commenting on the existence of \( \Psi \). Given \( \Phi \), it is determined on the base of the equality
\[
\Psi(e^{i\varphi(t)}) = e^{P(t)}, t \in (-\frac{T}{2}, \frac{T}{2}), \text{ where } P(t) = \int_0^t \cos \varphi(\tilde{t}) d\tilde{t},
\]
reducing, therefore, to the quadrature and subsequent analytic continuation of its result from an arc of the curve \(|z| = 1\). Notice also that for such \( \Psi \) the square root \( \Psi^{1/2} \) is uniquely defined via analytic continuation of \( e^{P(t)/2} \). The functions \( \Phi^{1/2} \) are endowed with unique values in a similar way.

The proof of the formula \((11)\) splits into two steps. First, its right hand side is evaluated for the argument \( z = e^{-i\pi} \). Performing substitutions in accord with definitions, one obtains \( \Phi_M(e^{-i\pi}) = e^{P(T)} = \Phi(e^{i\pi}) \). Second, the expression \((11)\) is substituted into Eq. \((2)\). Then, upon elimination of the derivatives \( \Phi' \) and \( \Psi' \) with the help of Eq. \((2)\) and Eq. \((9)\), respectively, the identical equality follows. Thus, the expression \((11)\) verifies the first order differential equation \((2)\) and obeys the initial condition \((7)\) which distinguishes the solution representing the monodromy transformation of \( \Phi \). The identical coincidence follows and we are done.

It will be further assumed throughout that the parameter \( \ell \) is a positive integer, \( \ell \in \mathbb{N} \).
We set up the following definition \[9\].

**Definition 1.** Let the four sequences \(p_k, q_k, r_k, s_k, \ k = 0, 1, 2\ldots\) of functions of the complex variable \(z\) and the constant parameters \(\ell, \mu, \lambda = (2\omega)^{-2} - \mu^2\) be defined by means of the following recurrent scheme

\[
\begin{aligned}
  p_0 &= 0, \ q_0 = 1, \ r_0 = z^{-2}, \ s_0 = -\mu; \\
  p_k &= (1 - \ell)z^{k-1} + q_{k-1} + z^2 p_{k-1}'; \\
  q_k &= z^2(-\lambda + (\ell + 1)\mu z)p_{k-1} + \mu(1 - z^2)q_{k-1} + z^2 q_{k-1}'; \\
  r_k &= 2(k - 2)z r_{k-1} - s_{k-1} - z^2 r_{k-1}', \\
  s_k &= z^2 (\lambda - (\ell + 1)\mu z) r_{k-1} + ((2k - \ell - 3) z + \mu(z^2 - 1)) s_{k-1} - z^2 s_{k-1}'.
\end{aligned}
\]

We pick up their “diagonal” representatives denoting them \(p, q, r, s\), i.e. define

\[
\begin{aligned}
  p &= p_\ell, \ q = q_\ell, \ r = r_\ell, \ s = s_\ell.
\end{aligned}
\]

It can be shown that the functions \(p, q, r, s\) are the polynomials in \(z\) of the degrees \(2\ell - 2, 2\ell, 2\ell - 2, 2\ell\), respectively \[9\]; they are polynomial in the parameters \(\lambda\) and \(\mu\) as well.

We define now the following four holomorphic functions \(\Phi_B, \Psi_B, \Theta_B, \tilde{\Theta}_B\) representing them in terms of the two other holomorphic functions \(\Phi\) and \(\Psi\) of a complex variable and the functions \(\varphi, P\) of a real variable, the latter pair being evaluated for several fixed values of their argument alone:

\[
\begin{aligned}
  \Phi_B(z) &= - \left( 2ie^{\frac{1}{2}P(\frac{1}{4}T)}(\mathcal{D}_+w_-u_- + \mathcal{D}_-w_+u_+) \cdot \Psi(z)\frac{i}{4}\Phi(z)\frac{i}{4} \\
  &+ \left( - \mathcal{D}_+w_+(e^{\frac{1}{2}P(\frac{1}{4}T)}u_+ + e^{\frac{1}{4}P(\frac{1}{4}T)}v_+) \\
  &+ \mathcal{D}_-w_+(e^{\frac{1}{4}P(\frac{1}{4}T)}u_+ + e^{\frac{1}{2}P(\frac{1}{4}T)}v_+) \right) \cdot \Psi(1/z)\frac{i}{4}\Phi(1/z)\frac{i}{4} \right) \times \\
  \left( -2ie^{\frac{1}{2}P(\frac{1}{4}T)}(\mathcal{D}_+w_-u_- + \mathcal{D}_-w_+u_+) \cdot \Psi(z)\frac{i}{4}\Phi(z)\frac{i}{4} \\
  + \left( - \mathcal{D}_+w_+(e^{\frac{1}{2}P(\frac{1}{4}T)}u_+ + e^{\frac{1}{4}P(\frac{1}{4}T)}v_+) \\
  + \mathcal{D}_-w_+(e^{\frac{1}{4}P(\frac{1}{4}T)}u_+ + e^{\frac{1}{2}P(\frac{1}{4}T)}v_+) \right) \cdot \Psi(1/z)\frac{i}{4}\Phi(1/z)\frac{i}{4} \right)^{-1},
\end{aligned}
\]

\[
\begin{aligned}
  \Psi_B(z) &= (2i)^{-1}(\Theta_B(z) - \tilde{\Theta}_B(z)), \text{ where}
\end{aligned}
\]
\( \Theta_B(z) = (\cos \varphi(0))^{-1} \times \\
\left( -i (\mathcal{D}_+ w_+((2 \sin \varphi(0) - 1) e^{\frac{i}{2}P(\frac{1}{2}T)} u_+ - e^{\frac{i}{2}P(-\frac{1}{2}T)} v_+) \\
+ \mathcal{D}_- w_-((2 \sin \varphi(0) + 1) e^{\frac{i}{2}P(\frac{1}{2}T)} u_- + e^{\frac{i}{2}P(-\frac{1}{2}T)} v_-) \right) \cdot \Psi(z) \frac{i}{2} \Phi(z) \frac{i}{2} \\
+ \left( - \mathcal{D}_+ w_+((2 \sin \varphi(0) - 2) e^{\frac{i}{2}P(\frac{1}{2}T)} u_+ + \sin \varphi(0) e^{\frac{i}{2}P(\frac{1}{2}T)} v_+) \\
+ \mathcal{D}_- w_-((2 \sin \varphi(0) + 2) e^{\frac{i}{2}P(\frac{1}{2}T)} u_- + \sin \varphi(0) e^{\frac{i}{2}P(-\frac{1}{2}T)} v_-) \right) \times \\
\Psi(1/z) \frac{i}{2} \Phi(1/z) \frac{i}{2} \right)^{-1}, \\
\right)

\( \tilde{\Theta}_B(z) = (\cos \varphi(0))^{-1} \times \\
\left( i ( - \mathcal{D}_- w_-((2 \sin \varphi(0) - 1) e^{\frac{i}{2}P(\frac{1}{2}T)} u_- + e^{\frac{i}{2}P(-\frac{1}{2}T)} v_-) \\
+ \mathcal{D}_+ w_+((2 \sin \varphi(0) + 1) e^{\frac{i}{2}P(\frac{1}{2}T)} u_+ + e^{\frac{i}{2}P(-\frac{1}{2}T)} v_+) \right) \cdot \Psi(z) \frac{i}{2} \Phi(z) \frac{i}{2} \\
+ \left( - \mathcal{D}_- w_-((2 \sin \varphi(0) - 2) e^{\frac{i}{2}P(\frac{1}{2}T)} u_- + \sin \varphi(0) e^{\frac{i}{2}P(\frac{1}{2}T)} v_-) \\
+ \mathcal{D}_+ w_+((2 \sin \varphi(0) + 2) e^{\frac{i}{2}P(\frac{1}{2}T)} u_+ + \sin \varphi(0) e^{\frac{i}{2}P(-\frac{1}{2}T)} v_+) \right) \times \\
\Psi(1/z) \frac{i}{2} \Phi(1/z) \frac{i}{2} \right)^{-1}, \\
\right)

Above, the following coefficient shortcuts
\[
\begin{align*}
  u_\pm & := (-1)^\ell e^{\frac{i}{2} \varphi(\frac{1}{2}T)} \pm i e^{\frac{i}{2} \varphi(\frac{1}{2}T)} \\
  v_\pm & := e^{\frac{i}{2} \varphi(\frac{1}{2}T)} \pm i (-1)^\ell e^{\frac{i}{2} \varphi(\frac{1}{2}T)}, \\
  w_\pm & := e^{\frac{i}{2} \varphi(0)} \pm i e^{-\frac{i}{2} \varphi(0)}, \\
  \mathcal{D}_\pm & := p(1) \pm 2 \omega \tau(1),
\end{align*}
\]

are utilized.

It is worth noting that the involvement of the functions \( \varphi, P \) in Eqs \( 15 \)-\( 18 \) is not obligatory. Their values utilized there can be expressed in terms of the functions \( \Phi, \Psi \) alone, provided the following identifications are taken into
account (cf Eq.s (6)):

\[ e^{i\varphi(T/2)} = \Phi(e^{i\pi}), \quad e^{i\varphi(0)} = \Phi(1), \quad e^{i\varphi(T/2)} = \Phi(e^{-i\pi}); \]

\[ e^{P(T/2)} = \Psi(e^{i\pi}), \quad e^{P(T/2)} = \Psi(e^{-i\pi}). \]

The (second) non-obvious result to be here reported is as follows:

**Theorem 5.** Let the functions \( \Phi \) and \( \Psi \) verify the equations (2) and (9), respectively, obeying also the constraints (10). Then

- the functions \( \Phi_B \) and \( \Psi_B \) verify the same equations and constraints as \( \Phi \) and \( \Psi \), respectively;
- the transformation \( B : (\Phi, \Psi) \mapsto (\Phi_B, \Psi_B) \) repeated twice coincides with the monodromy transformation \( M \).

Thus, \( B \) can be considered as a square root of \( M \).

The first assertion is proven by straightforward computation. With regard to the second one, we replace here its formal proof with outline of derivation of the very formulas (15)-(18) demonstrating how they had been arisen. Besides, along the way, a profound relationship of the equation (2) (and (1)) with another family of differential equations is demonstrated.

To that end, let us consider the two holomorphic functions \( E_{\pm}(z) \) defined through the functions \( \Phi, \Psi \) as follows:

\[ E_{\pm}(z) := 2^{-1}e^{\mu(z+1/z-2)/2}z^{-\ell/2} \times \left( \frac{1 \pm i}{\sqrt{2}}(\Psi(z)\Phi(z))^{1/2} + \frac{1 \mp i}{\sqrt{2}}(\Psi(1/z)\Phi(1/z))^{1/2} \right). \]

Straightforward calculation proves the following equalities

\[ E'_{\pm}(z) = \pm(2\omega)^{-1}z^{-\ell-1}E_{\pm}(1/z) + \mu E_{\pm}(z), \]

taking place provided the functions \( \Phi, \Psi \) obey the equations (2) and (9), respectively. Eq.s (23) imply, in turn, the fulfillment of the equation

\[ z^{2}E''(z) + (\ell + 1)z + \mu(1 - z^{2})E'(z) + \mu(\ell + 1)z + \lambda)E(z) = 0 \]

by the both functions \( E = E_{\ell} \) and \( E = E_{\ell} \).

The equations of the form (24) with arbitrary constant parameters \( \ell, \lambda, \mu \) constitute a subfamily of the family of so called double confluent Heun equations, see Refs. [10, 11, 12].

Eq. (21) is a linear homogeneous differential equation with coefficients holomorphic everywhere except zero. Hence their solutions, including \( E_{\pm} \), are holomorphic everywhere except zero including the points of divergence and roots of the solution \( \Phi \) of the non-linear equation (9) connected with \( E_{\pm} \) via Eq.s (22). At the same time, the common singular point \( z = 0 \) for all the functions \( E_{\ell}, E_{\ell} \) and \( \Phi, \Psi \) is actually the branching point of their common domain, the universal cover \( \mathbb{C} \) of the punctured complex plane \( \mathbb{C}^* \) (for \( \Phi \) and \( \Psi \), with their singular points removed). Being defined on \( \mathbb{C} \), the functions \( E_{\ell} \) behave like multi-valued
functions on \( \mathbb{C}^* \) and may thus undergone the monodromy transformation without violation of fulfillment of Eq. \((24)\). Similarly to the case of solutions to Eq. \((2)\), the monodromy transformation of \( E_{(\ell)} \) can be understood as point-wise analytic continuations along the arcs projected to full circles with centers situated at zero which are passed in the counter-clockwise direction (as opposed to the case of \( \Phi \), no singular points can now be encountered on such arcs).

Substituting \( z = 1 \) into \((22)\), one gets
\[
E_{(\pm)}(1) = \mp \sin(\frac{i}{2}(\varphi(0) \mp \frac{i}{2}\pi))
\]
Thus, if \( \varphi(0) \mp \frac{i}{2}\pi \mod \pi \) (i.e. if
\[
\Phi(1)^2 = -1,
\]
see Eqs \((21)\) then \( E_{(\ell)}(z) \neq 0 \neq E_{(-\ell)}(z) \) and the functions \( E_{(\ell)} \) and \( E_{(-\ell)} \) are linear independent. Moreover, since the linear space of solutions to Eq. \((24)\) is two-dimensional, the functions \( E_{(\ell)} \) constitute its basis and any solution to Eq. \((24)\) can be represented as their linear combination with constant coefficients. Thus, the two formulas \((22)\) ensure, in fact, the explicit representation of all the solutions to Eq. \((2)\) in terms of any generic solution \( \Phi \) to Eq. \((2)\) and some related quadrature (the function \( \Psi \)).

Conversely, the formula
\[
\Phi^{(\alpha)}(z) := -iz^\ell \frac{\cos(\frac{i}{2}\alpha)E_{(\ell)}(z) + i\sin(\frac{i}{2}\alpha)E_{(-\ell)}(z)}{\cos(\frac{i}{2}\alpha)E_{(\ell)}(1/z) - i\sin(\frac{i}{2}\alpha)E_{(-\ell)}(1/z)}
\]
in which \( \alpha \) stands for an arbitrary real number, represent a solution to Eq. \((2)\) obeying the constraint \( |\Phi(e^{i\omega})| = 1 \), provided the functions \( E_{(\ell)} \) obey Eqs \((23)\) and \( \Im E_{(\ell)}(1) = 0 \). The composition of the transformations Eq. \((22)\) and Eq. \((27)\) takes a solution to Eq. \((2)\) to the function verifying the same equation. If \( \alpha = \frac{i}{2}\pi \) then this map of the space of solutions to Eq. \((2)\) into itself reduces to the identical map.

On the other hand, \textit{in case of integer} \( \ell \), there exist two additional (as compared to the case of generic \( \ell \)) transformations preserving the space of solutions to Eq. \((24)\). One of them, which we denote \( \mathcal{L}_B \), can be represented as follows:
\[
\mathcal{L}_B: E(z) \mapsto \mathcal{L}_B[E](z) := (-1)^\ell 2\omega z^{-\ell+1}e^{\mu(z+z^{-1})}(z^2\tau(-z)E'(-z) + s(-z)E(-z))
\]
The invariance of the space of solutions to Eq. \((24)\) with respect to \( \mathcal{L}_B \) can be established by straightforward computations, provided the following property of the polynomials \( p, q, r, s \)
\[
p(-z) = (-1)^{\ell+1}(\lambda + \mu^2)^{-1}(\mu z^2\tau(z) + s(z))
\]
\[
q(-z) = \mu z^2 p(z) + q(z) + (-1)^\ell(\lambda + \mu^2)^{-1}\mu z^2 (\mu z^2\tau(z) + s(z))
\]
\[
r(-z) = r(z)
\]
\[
s(-z) = (-1)^{\ell+1}(\lambda + \mu^2)p(z) - \mu z^2\tau(z)
\]
and the differential equations

\[
\begin{align*}
    z^2 p' &= (\mu + (\ell - 1)z)p - q + (-1)^\ell z^2 r, \\
    q' &= (\lambda - (\ell + 1)\mu z)p + \mu q + (-1)^\ell s, \\
    z^2 r' &= (-1)^{\ell+1}(\lambda + \mu^2)p + z(2(\ell - 1) - \mu z)r - s, \\
    z^2 s' &= (-1)^{\ell+1}(\lambda + \mu^2)q + z^2 (\lambda - (\ell + 1)\mu z)r + ((\ell - 1)z - \mu)s,
\end{align*}
\]

which they obey \[9\] are taken into account.

Moreover, applying the operator \(L_B\) twice and utilizing the same reductions ensured by Eq.s (29) and Eq.s (30), one finds that on solutions to Eq. (24), the function-argument \(E\) is finally restored up to some constant factor and up to modification of its argument which undergoes, ultimately, the transformation looking like a full revolution around zero yielding no ultimate effect in projection to \(\mathbb{C}^*\) but identical to the monodromy transformation on the actual domain \(\mathcal{E}\) of \(E\). This result can be captured by means of the following equality:

\[
L_B \circ L_B = \mathcal{D} \cdot \mathcal{M}.
\]

In computation of the operator composition \(L_B \circ L_B\), the factor \(\mathcal{D}\) appears originally in the following form

\[
\mathcal{D} = z^{2(1-\ell)}(p(z)s(z) - q(z)r(z)).
\]

However, a straightforward computation shows that \(\mathcal{D}\) is the first integral of the system of differential equations (30) which the polynomials involved in its definition obey. Thus \(\mathcal{D}\) does not actually depend on the variable \(z\) and can be determined setting any value of the latter. Substituting, in particular, \(z = 1\), one obtains

\[
\mathcal{D} = (2\omega)^{-2} \mathcal{D}_+ \mathcal{D}_-,
\]

where the factors on the right are defined in (19).

The equality (31) and formulas (19) now say us the following.

**Proposition 6.** If

\[
\mathcal{D}_+ \neq 0 \neq \mathcal{D}_- \text{ or, equivalently, } p(1)^2 \neq (2\omega)^2 r(1)^2
\]

then the linear operator \(L_B\) (28) determines the automorphism of the space of solutions to Eq. (24).

The violation of the condition (34) would impose severe restrictions on the constant parameters involved in Eq. (24). We assume to consider a generic case claiming (34) to be fulfilled throughout.

The linear operator \(L_B\) acting on the two-dimensional linear space of solutions to Eq. (24) can be presented with respect to any basis of this space as some \(2 \times 2\)
matrix. In particular, it can be shown that in the basis \((E_{\uparrow}), E_{\downarrow}\) introduced above the matrix form of the operator \(L_B\) reads
\[
B = i^\ell (2\omega)^{-1} e^{\frac{2i}{\pi} B(0)} \times \\
\begin{pmatrix}
- e^{\frac{2i}{\pi}\mathcal{D}_+} \cos \left(\frac{1}{2}(\varphi(0) - \frac{1}{2}\pi)\right)^{-1}, e^{\frac{2i}{\pi}\mathcal{D}_-} \cos \left(\frac{1}{2}(\varphi(0) + \frac{1}{2}\pi)\right)^{-1} \\
\left( e^{\frac{1}{2} P(\frac{1}{2} T) u_-} - e^{\frac{1}{2} P(-\frac{1}{2} T) u_-} \right) \times \left( e^{\frac{1}{2} P(\frac{1}{2} T) u_+} + e^{\frac{1}{2} P(-\frac{1}{2} T) u_+} \right), \\
i\left( e^{\frac{1}{2} P(\frac{1}{2} T) u_-} + e^{\frac{1}{2} P(-\frac{1}{2} T) u_-} \right) \times \left( e^{\frac{1}{2} P(\frac{1}{2} T) u_+} + e^{\frac{1}{2} P(-\frac{1}{2} T) u_+} \right).
\end{pmatrix}
\]  

Let us note now that the numerator of the fraction in Eq. (27) is a solution to Eq. (24) and the denominator is also a solution with the modified argument \(1/z\) substituted in place of \(z\), the former and the later being mutually conjugated since \(E_{\uparrow}(\bar{z}) = E_{\downarrow}(z)\) under the conditions assumed.

If one replaces in (27), formally, the functions \(E_{\uparrow}\) in the numerator with the functions \(L_B E_{\downarrow}\) expanding them further as linear combinations of the original \(E_{\downarrow}\) derived making use of the matrix (35), and carry out the corresponding transformation of the denominator preserving its complex conjugacy with the numerator on the unit circle, then the formula similar to (27) but with distinct parameter \(\alpha\) results. Repeating such a transformation twice, one comes, in view of (31), to the original functions \(E_{\downarrow}\) with the original coefficients \(\cos(\frac{1}{2}\alpha), \sin(\frac{1}{2}\alpha)\) (times the constant factor of \(\mathcal{D}\) which cancels out) but with arguments undergone the monodromy transformation. In other words, the transformation of (27) induced by the map (28), repeated twice, results in the monodromy transformation of the function \(\Phi^{(\alpha)}\). Setting \(\alpha = \frac{1}{2}\pi\), performing such a transformation of \(\Phi^{(\alpha)}\) once, and eliminating the functions \(E_{\downarrow}\) by means of their expansions (22), the formula (15) results.

The functions \(\Theta\) and \(\tilde{\Theta}\) (see Eqs. (17), (18)) which have been used for determination of the function \(\Psi\) alone (see Eq. (16)) are actually of notable interest in their own rights. Such functions are closely related to solutions to (2). They can be defined as solutions to the linear differential equations
\[
2i \omega z \Theta' = -\Phi(\Theta - \tilde{\Theta}), \quad 2i \omega z \tilde{\Theta}' = \Phi^{-1}(\Theta - \tilde{\Theta})
\]

obeying the initial conditions
\[
\Theta(1) = i, \quad \tilde{\Theta}(1) = -i.
\]
If these are fulfilled, then the difference \(\Psi = (2i)^{-1}(\Theta - \tilde{\Theta})\) (see Eq. (16)) obeys Eq. (9) and represents the analytic continuation of the function \(e^{\Phi(t)} = \exp \int_0^t \cos \varphi(\tilde{t}) \, d\tilde{t}\), where \(2 \cos \varphi(t) = \Phi(e^{\omega t}) + \Phi(e^{\omega t})^{-1}\), from an arc of the unit circle to its vicinity in \(\mathbb{C}^*\).

Concerning the very formulas (17), (18), the fulfillment of Eqs. (36), (37) by the functions the former define for the corresponding right hand side factors \(\Phi^{\pm 1}_B\) defined by Eq. (15) is verified by straightforward computations.
In conclusion, it should be emphasized that the transformation (17), taking solutions to the nonlinear equation (2) to solutions of the same equation (and, then, determining the associated transformation on the space of solutions to Eq. (1)), arises here as a byproduct of the specific symmetry of the space of solutions to the linear equation (24). Such a symmetry has been shown to exist in case of integer value of the parameter \( \ell \) (sometimes called the order). The existence of an analogue of the above \( B \)-transformation under less restrictive conditions remains an open problem.

References

[1] R. L. Foote. *Geometry of the Prytz planimeter*. Reports Math. Physics 42 (1998), 249–271.
[2] R. L. Foote, M. Levi, S. Tabachnikov. *Tractrices, Bicycle Tire Tracks, Hatchet Planimeters, and a 100-year-old Conjecture*. arXiv:1207.0834v1 (2012).
[3] W. C. Stewart. *Current-voltage characteristics of Josephson junctions*. Appl. Phys. Lett., 12, 277-280(1968).
[4] D. E. McCumber. *Effect of ac impedance on dc voltage-current characteristics of superconductor weak-link junctions*. J. Appl. Phys., 39, 3113-3118 (1968).
[5] A. Barone, G. Paterno *Physics and applications of the Josephson effect* John Wiley and Sons Inc. 1982
[6] P. Mangin, R. Kahn. *Superconductivity An introduction*, Springer, 2017
[7] B. V. Шмидт. Введение в физику сверхпроводников. Изд. 2-е, М.: МСХМО, 2000 — В. V. Schmidt. *Introduction to physics of superconductors*, 2000 (in Russian).
[8] J. Guckenheimer, Yu. S. Ilyashenko. *The duck and the devil: canards on the staircase*. Mosc. Math. J., 2001, vol 1, 1 pp. 27–47
[9] V. M. Buchstaber, S. I. Tertychnyi. *Automorphisms of the solution spaces of special double-confluent Heun equations*. Funct. Anal. Appl., 50:3 (2016), 176–192
[10] D. Schmidt, G. Wolf. *Double confluent Heun equation, in: Heun’s differential equations*, Ronveaux (Ed.) Oxford Univ. Press, Oxford, N.Y., (1995), Part C.
[11] S. Yu. Slavyanov, W. Lay. *Special Function: A Unified Theory Based on Singularities I* Foreword by A. Seeger. Oxford; New York: Oxford University Press, 2000. — ISBN 0-19-850573-6
[12] *Heun functions, their generalizations and applications*, http://theheunproject.org/bibliography.html
[13] Tertychniy S.I *The interrelation of the special double confluent Heun equation and the equation of RSJ model of Josephson junction revisited*, arXiv, math-ph/1811.03971, (2018)