ON COHOMOLOGY AND SUPPORT VARIETIES FOR LIE SUPERALGEBRAS

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ABSTRACT. Support varieties for Lie superalgebras over the complex numbers were introduced in [BKN1] using the relative cohomology. In this paper we discuss finite generation of the relative cohomology rings for Lie superalgebras, we formulate a definition for subalgebras which detect the cohomology, also discuss realizability of support varieties. In the last section as an application we compute the relative cohomology ring of the Lie superalgebra $S(n)$ relative to the graded zero component $S(n)_0$ and show that this ring is finitely generated. We also compute support varieties of all simple modules in the category of finite dimensional $S(n)$-modules which are completely reducible over $S(n)_0$.

1. Introduction

1.1. Throughout the present article we work with the complex numbers $\mathbb{C}$ as the ground field. All vector spaces are assumed to be finite dimensional unless otherwise is noted. For a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$, $\overline{v}$ will denote the $\mathbb{Z}_2$ degree of homogeneous element $v \in V$. A Lie superalgebra is a finite dimensional $\mathbb{Z}_2$-graded vector spaces $g = g_0 \oplus g_1$ with a bracket $[,] : g \otimes g \to g$ which preserves the $\mathbb{Z}_2$-grading and satisfies graded versions of the operations used to define Lie algebras. The even part $g_0$ is a Lie algebra under the bracket. We view Lie algebras as a Lie superalgebra concentrated in degree $\overline{0}$. A Lie superalgebra $g$ will be called classical if there is a connected reductive algebraic group $G_0$ such that Lie$(G_0) = g_0$ and an action of $G_0$ on $g_1$ which differentiates to the adjoint action of $g_0$ on $g_1$.

Given a Lie superalgebra $g$, $U(g)$ will denote the universal enveloping algebra of $g$. $U(g)$ is an associative $\mathbb{Z}_2$-graded algebra. The category of $g$-modules has objects that are left $U(g)$-modules which are $\mathbb{Z}_2$-graded and the action of $U(g)$ preserves the $\mathbb{Z}_2$-grading. Morphisms are described in [BKN1], §2.1. The category of $g$-modules is not an abelian category. However the category of graded modules consisting of the same objects but with $\mathbb{Z}_2$-graded morphisms is an abelian category. The parity change functor, $\Pi$, which interchanges the $\mathbb{Z}_2$-grading of a module, allows one to make use of the standard tools of homological algebra. Since $U(g)$ is a Hopf algebra, one can use the antipode and coproduct of $U(g)$ to define a $g$-module structure on the dual of a module and the tensor product of two modules. All submodules are assumed to be graded. Superdimension of a $g$-module $M = M_0 \oplus M_1$ is defined to be $\dim M_0 - \dim M_1$.

A $g$-module $M$ is finitely semisimple if it is isomorphic to a direct sum of finite dimensional simple $g$ submodules. Let $t$ be a Lie subsuperalgebra of $g$. Let $C(g,t)$ be the full subcategory of the category of all $g$-modules which are finitely semisimple as $t$-modules. The category

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\(\mathcal{C}_{(g,t)}\) is closed under arbitrary direct sums, quotients, and finite tensor products (cf. \cite{Kum}, 3.1.6). For \(M, N\) in \(\mathcal{C}_{(g,t)}\), \(\text{Ext}_{\mathcal{C}_{(g,t)}}^d(M, N)\) will denote the degree \(d\) extensions between \(N\) and \(M\) in \(\mathcal{C}_{(g,t)}\).

For background on Lie superalgebras and relative cohomology see \cite{Kac} and \cite{BKN1}, respectively.

1.2. Let \(g\) be a Lie superalgebra and \(t\) be a Lie subsuperalgebra of \(g\). In Section 2 we review the relative cohomology for Lie superalgebras and record the properties we are going to need in the rest. After that we give necessary and sufficient conditions for identify the relative cohomology \(H^\bullet(g, t; \mathbb{C})\) with the cochains and discuss when the ring of invariants will be finitely generated. We also introduce the notion of a detecting subalgebra for a Lie superalgebra.

In section 3 under the assumption of the finite generation of \(H^\bullet(g, t; \mathbb{C})\) we define support varieties. Here we associate varieties to modules for Lie superalgebras. These varieties are affine conical varieties. We prove a theorem which is called realization theorem in the theory of support varieties. This theorem basically says that we can realize any conical subvariety as variety of some module.

In Section 3 we give an application. Here using the work done for the Cartan type Lie superalgebra \(W(n)\) in \cite{BAKN}, we compute the relative cohomology ring for the Lie superalgebra \(\overline{S}(n)\) relative to the degree zero component \(\overline{S}(n)_0\) which is isomorphic to \(\mathfrak{g}(n)\) as a Lie algebra. In particular we show that the cohomology ring is a polynomial ring and \(\overline{S}(n)\) admits a detecting subalgebra. By using finite generation theorem we defined support varieties for \(g\)-modules. By using the fact that a simple module of \(W(n)\) is typical if and only if its superdimension is zero and results of Serganova on representations of \(g\), we were able to compute support varieties of all finite dimensional simple modules which are completely reducible over \(g_0\). We also show that the realization theorem proven in Section 1 holds for \(g\).

2. Cohomology

2.1. Relative cohomology for Lie superalgebras. Let us recall the definition of relative cohomoloy for Lie superalgebras. Let \(g\) be a Lie superalgebra and let \(t \subseteq g\) be a Lie subsuperalgebra. Let \(M\) be a \(g\)-module. The cochain complex \(C^\bullet(g, M)\) is defined by

\[
C^\bullet(g, M) = \bigoplus_{p \in \mathbb{Z}_{\geq 0}} C^p(g, M) \quad \text{with} \quad C^p(g, M) = \text{Hom}(\Lambda^p_s(g), M).
\]

Here \(\Lambda^p_s(g)\) denotes the super wedge product, i.e, the space of \(\mathbb{Z}_2\)-graded \(p\)- alternating tensors on \(g\). \((\Lambda^p_s(g))_r, r \in \mathbb{Z}_{2},\) is spanned by

\[
x_1 \wedge x_2 \wedge \cdots \wedge x_p \quad (x_j \in g)
\]
satisfying \(\Sigma_{j=1}^{p} x_j = r\) and

\[
x_1 \wedge x_2 \wedge \cdots \wedge x_j \wedge x_{j+1} \wedge \cdots \wedge x_p = (-1)^{rj} x_{j+1} x_1 \wedge x_2 \wedge \cdots \wedge x_{j+1} \wedge x_j \wedge \cdots \wedge x_p.
\]
Thus $x_j, x_{j+1}$ skew commute unless both are odd in which case they commute.

The differential (or coboundray operator) $d$ is the even linear map of $\mathbb{Z}$ degree one satisfying

$$d(\phi)(x_1 \wedge \ldots \wedge x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j+((\overline{j}+\overline{i})}k_i+j+\overline{k}_{i,j}\phi([x_i, x_j] \wedge x_1 \wedge \ldots \hat{x}_i \wedge \ldots \hat{x}_j \wedge \ldots \wedge x_{p+1})$$

$$+ \sum_i (-1)^{i+1+\overline{k}_j+\overline{0}}x_i\phi(x_1 \wedge \ldots \hat{x}_i \wedge \ldots \wedge x_{p+1}),$$

where $x_1, \ldots, x_{p+1}$ and $\phi$ are assumed to be homogeneous, and $k_i := \Sigma_{j=1}^{i-1} \overline{x}_j$, $k_{i,j} := \Sigma_{s=i+1}^{j-1} \overline{x}_s$.

We denote the restriction of $d$ to $C^p(\mathfrak{g}, M)$ by $d^p$. Since $d^2 = 0$, i.e., $d^p \circ d^{p-1} = 0$ for all $p \in \mathbb{Z}_{\geq 0}$, $(C^\bullet(\mathfrak{g}, M), d)$ is a chain complex.

Then we define

$$H^p(\mathfrak{g}; M) = \text{Ker } d^p / \text{Im } d^{p-1}.$$

The relative version of the above construction is defined as follows: Let $\mathfrak{g}$, $\mathfrak{t}$, and $M$ be as above. Define

$$C^p(\mathfrak{g}, \mathfrak{t}; M) = \text{Hom}_t(\Lambda^p(\mathfrak{g}/\mathfrak{t}), M).$$

Then the map $d^p$ gives a map $d^p : C^p(\mathfrak{g}, \mathfrak{t}; M) \rightarrow C^{p+1}(\mathfrak{g}, \mathfrak{t}; M)$ and we define

$$H^p(\mathfrak{g}, \mathfrak{t}; M) = \text{Ker } d^p / \text{Im } d^{p-1}.$$

For $U(\mathfrak{g})$ modules $M, N$ we can define cohomology for the pair $(U(\mathfrak{g}), U(\mathfrak{t}))$ which we denote by $\text{Ext}^\bullet_{(U(\mathfrak{g}), U(\mathfrak{t}))}(M, N)$ (cf. [BKN1, Section 2.2]). If $\mathfrak{g}$ is finitely semisimple as a $\mathfrak{t}$ module under the adjoint action then for $M, N \in C(\mathfrak{g}, \mathfrak{t})$ we have the following important isomorphisms

$$\text{Ext}^\bullet_{C(\mathfrak{g}, \mathfrak{t})}(M, N) \cong \text{Ext}^\bullet_{(U(\mathfrak{g}), U(\mathfrak{t}))}(\mathbb{C}, M^* \otimes N) \cong H^\bullet(\mathfrak{g}, \mathfrak{t}; M^* \otimes N).$$

2.2. Relating cohomology rings to invariants. In this subsection we give necessary and sufficient conditions to identify the relative cohomology ring $H^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ with the cochains and discuss finite generation of the relative cohomology rings.

**Proposition 2.2.1.** Let $\mathfrak{t}$ be a Lie subsuperalgebra of $\mathfrak{g}$. Then $H^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C})$ is isomorphic to the ring of invariants $\Lambda^\bullet(\mathfrak{t}^\bullet)$ if and only if $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{t}$.

**Proof.** Recall that the cochains are defined by

$$C^p(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) = \text{Hom}_t(\Lambda^p(\mathfrak{g}/\mathfrak{t}), \mathbb{C}) \cong \Lambda^p(\mathfrak{t}^\bullet).$$

If $H^\bullet(\mathfrak{g}, \mathfrak{t}; \mathbb{C}) \cong \Lambda^\bullet(\mathfrak{t}^\bullet)$, then since cohomology is a subquotient of cochains all differentials has to be zero. If $[\mathfrak{g}, \mathfrak{g}]$ is not contained in $\mathfrak{t}$ one can always construct a linear functional $\phi : \mathfrak{g}/\mathfrak{t} \rightarrow \mathbb{C}$ which does not go to zero under the differential $d$. 

Suppose conversely that $[g, g] \subseteq t$. Now note that in this case the differential $d^p$ in (2.1.1) is identically zero. The second sum of (2.1.1) is zero since here $M = \mathbb{C}$ and since each $[x_i, x_j]$ is zero in the quotient $g/t$ the first sum of (2.1.1) is zero as well. Therefore

$$H^*(g, t; C) \cong \Lambda^*_s((g/t)^*)^t.$$ 

□

**Corollary 2.2.2.** Assume that $H^*(g, t; C) \cong \Lambda^*_s((g/t)^*)^t$ and assume in addition that $t$ is a reductive Lie algebra. Let $G$ be the connected reductive algebraic group such that $\text{Lie}(G) = t$. Let $M$ be a finite dimensional $g$-module. Then,

(a) The superalgebra $H^*(g, t; \mathbb{C})$ is finitely generated as a ring.

(b) $H^*(g, t; M)$ is finitely generated as an $H^*(g, t; C)$-module.

**Proof.**

(a) Since $H^*(g, t; \mathbb{C}) \cong \Lambda^*_s((g/t)^*)^t = \Lambda^*_s((g/t)^*)^G$ and $G$ is reductive this statement follows from the classical invariant theory result of Hilbert [PV, Theorem 3.6].

(b) Since $M$ is finite dimensional

$$\text{Hom}_C(\Lambda^*_s(g/t), M) \cong \Lambda^*_s((g/t)^*) \otimes M$$

is finitely generated as a $\Lambda^*_s((g/t)^*)$-module and by [PV, Theorem 3.25]

$$\text{Hom}_C(\Lambda^*_s(g/t), M)^G = \text{Hom}_C(\Lambda^*_s(g/t), M) = C^*(g, t; M)$$

is finitely generated as a $\Lambda^*_s((g/t)^*)^G \cong H^*(g, t; C)$-module. Now we can argue as in [BKN1, Theorem 2.5.3] to deduce that $H^*(g, t; M)$ is finitely generated as a $H^*(g, t; C)$-module.

□

**Example 2.2.3.** Suppose that $t = g_0$. Since super wedge product is symmetric product on odd spaces and $[g_1, g_1] \subseteq g_0$ we have the following important isomorphism

$$H^*(g, g_0; \mathbb{C}) \cong \Lambda^*_s((g/g_0)^*)^{g_0} \cong S(g_1^*)^{g_0}.$$ 

Recall that a Lie superalgebra $g = g_0 \oplus g_1$ is called classical if there is a connected reductive algebraic group $G_0$ such that $\text{Lie}(G_0) = g_0$ and an action of $G_0$ on $g_1$ which differentiates to the adjoint action of $g_0$ on $g_1$. Note that if $g$ is a classical Lie superalgebra then $H^*(g, g_0; \mathbb{C})$ is always finitely generated.

**Hypothesis 2.2.4.** Throughout the rest of this section and next section we fix a pair $(g, t)$ and assume that $H^*(g, t; C)$ is finitely generated.

### 2.3. Detecting subalgebras.

In [BKN1, BKN2] by applying invariant theory results in [LR] and [DK] the authors showed that under suitable conditions a classical Lie superalgebra $g = g_0 \oplus g_1$ admits a subalgebra $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1$ such that the restriction map in cohomology induces an isomorphism

$$H^*(g, g_0; \mathbb{C}) \cong H^*(\mathfrak{e}, \mathfrak{e}_0; \mathbb{C})^W,$$

where $W$ is a finite pseudoreflection group.
Similar subalgebras were constructed for the Cartan type Lie superalgebra $W(n)$ in [BAKN]. The goal of all this work is to construct subalgebras that can play the role of elementary abelian groups in the theory of support varieties for finite groups.

**Definition 2.3.1.** Let $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1$ be a subalgebra of $\mathfrak{g}$ such that

(a) $\mathfrak{e}$ is a classical Lie superalgebra, and

(b) the inclusion map $\mathfrak{e} \hookrightarrow \mathfrak{g}$ induces an isomorphism

$$H^\bullet(\mathfrak{g}, t; \mathbb{C}) \cong H^\bullet(\mathfrak{e}, \mathfrak{e}_0; \mathbb{C})^W$$

for some group $W$ such that $(-)^W$ is exact.

A subalgebra with these properties will be called a *detecting subalgebra* for the pair $(\mathfrak{g}, t)$.

**Remark 2.3.2.** Observe that in the definition above we are implicitly assuming that $\mathfrak{e}_0 \subseteq t_0$.

**Example 2.3.3.** Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ be the Lie superalgebra of $(m + n) \times (m + n)$ complex matrices with $\mathbb{Z}_2$-grading given by

$$E_{i,j} = \overline{0} \text{ if } 1 \leq i, j \leq m \text{ or } m + 1 \leq i, j \leq m + n \text{ and } E_{i,j} = \overline{1} \text{ otherwise},$$

where $E_{i,j}$ denotes the $(i, j)$ matrix unit. Let $t = \mathfrak{g}_0$. As in [BAKN] Section 8.10, one can take $\mathfrak{e}_1 \subseteq \mathfrak{g}_1$ to be the subspace spanned by $E_{m+1-s, m+s} + E_{m+s, m+1-s}$ for $s = 1, \ldots, r$.

Let $\mathfrak{e}_0$ be the stabilizer of $\mathfrak{e}_1$ in $\mathfrak{g}_0$. Then $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1$ is a detecting subalgebra for the pair $(\mathfrak{g}, t)$.

Since a pair $(\mathfrak{g}, t)$ may not have a detecting subalgebra we make the following technical assumption about our fixed pair $(\mathfrak{g}, t)$.

**Hypothesis 2.3.4.** Throughout the next section we assume that our fixed pair $(\mathfrak{g}, t)$ has a detecting subalgebra $\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_1$.

### 3. Support Varieties

Recall that throughout this section we assume that $H^\bullet(\mathfrak{g}, t; \mathbb{C})$ is finitely generated.

3.1. Let $M$ and $N$ be $\mathfrak{g}$-modules such that $H^\bullet(\mathfrak{g}, t; \text{Hom}_\mathbb{C}(M, N))$ is finitely generated as an $H^\bullet(\mathfrak{g}, t; \mathbb{C})$-module.

Let

$$I_{(\mathfrak{g}, t)}(M, N) = \text{Ann}_{H^\bullet(\mathfrak{g}, t; \mathbb{C})}(H^\bullet(\mathfrak{g}, t; \text{Hom}_\mathbb{C}(M, N)))$$

be the annihilator ideal of this module. We define the *relative support variety* of the pair $(M, N)$ to be

$$\mathcal{V}_{(\mathfrak{g}, t)}(M, N) = \text{MaxSpec}(H^\bullet(\mathfrak{g}, t; \mathbb{C})/I_{(\mathfrak{g}, t)}(M, N)),$$

the maximal ideal spectrum of the quotient of $H^\bullet(\mathfrak{g}, t; \mathbb{C})$ by $I_{(\mathfrak{g}, t)}(M, N)$. For short when $M = N$, write

$$I_{(\mathfrak{g}, t)}(M) = I_{(\mathfrak{g}, t)}(M, M),$$
$$\mathcal{V}_{(g, t)}(M) = \mathcal{V}_{(g, t)}(M, M).$$

We call $\mathcal{V}_{(g, t)}(M)$ the support variety of $M$.

3.2. As the detecting subalgebra $e = e_0 \oplus e_1$ is classical $H^\bullet(e, e_0; \mathbb{C}) \cong S(e_1^\ast)^{e_0}$ is finitely generated. Therefore one can define support varieties for $e$-modules as in (3.1).

The canonical restriction map

$$\text{res} : H^\bullet(g, t; \mathbb{C}) \to H^\bullet(e, e_0; \mathbb{C})$$

induces a map of varieties

$$\text{res}^* : \mathcal{V}_{(e, e_0)}(\mathbb{C}) \to \mathcal{V}_{(g, t)}(\mathbb{C}).$$

By the isomorphism $H^\bullet(g, t; \mathbb{C}) \cong H^\bullet(e, e_0; \mathbb{C})^W$ one then has

$$\mathcal{V}_{(e, e_0)}(\mathbb{C})/W \cong \mathcal{V}_{(g, t)}(\mathbb{C}).$$

In particular for any finite dimensional $g$-module $M$, where $H^\bullet(g, t; \text{Hom}_\mathbb{C}(M, M))$ is finitely generated, $\mathcal{V}_{(e, e_0)}(M)/W$ and $\mathcal{V}_{(g, t)}(M)$ can naturally be viewed as affine subvarieties of $\mathcal{V}_{(g, t)}(\mathbb{C})$.

Furthermore, $\text{res}^*$ restricts to give a map,

$$\mathcal{V}_{(e, e_0)}(M) \to \mathcal{V}_{(g, t)}(M).$$

Since $\mathcal{V}_{(e, e_0)}(M)$ is stable under the action of $W$ we have the following embedding induced by $\text{res}^*$,

$$(3.2.1) \quad \mathcal{V}_{(e, e_0)}(M)/W \cong \text{res}^*(\mathcal{V}_{(e, e_0)}(M)) \hookrightarrow \mathcal{V}_{(g, t)}(M).$$

3.3. For a homogeneous $x \in e$, let $< x >$ denote the Lie subsuperalgebra generated by $x$. If $M$ is a $e$-module then define the rank variety of $M$ to be

$$\mathcal{V}_e^{\text{rank}} = \{x \in g_1 \mid M \text{ is not projective as a } U(< x >) - \text{module}\} \cup \{0\}$$

**Hypothesis 3.3.1.** We assume that the detecting subalgebra $e$ has a rank variety description; i.e., for any $e$-module $M$, $\mathcal{V}_e^{\text{rank}}(M) \cong \mathcal{V}_{(e, e_0)}(M)$.

By using the rank variety description one can prove a number of properties of $e$-support varieties. We record some of these properties and for the proofs and other properties we refer the reader to [BKN1, Section 6].

**Lemma 3.3.2.** Assume that the Hypotheses [2.2.4], [2.3.4] and [3.3.1] hold for our fixed pair $(g, t)$. Let $M$ and $N$ be finite dimensional $e$-supermodules. Then

1. $\mathcal{V}_{(e, e_0)}(M \otimes N) = \mathcal{V}_{(e, e_0)}(M) \cap \mathcal{V}_{(e, e_0)}(N)$.
2. $\mathcal{V}_{(e, e_0)}(M \oplus N) = \mathcal{V}_{(e, e_0)}(M) \cup \mathcal{V}_{(e, e_0)}(N)$.
3. $M$ is projective if and only if $\mathcal{V}_{(e, e_0)}(M) = \{0\}$.
4. If superdimension of $M$ is nonzero, i.e., $\dim M_0 \neq \dim M_1$, then

$$\mathcal{V}_{(e, e_0)}(M) = \mathcal{V}_{(e, e_0)}(\mathbb{C}).$$
3.4. Let 0 ≠ ζ ∈ H^n(g, t; C). Since H^n(g, t; C) ≅ Hom_g(Ω^n(C), C), where Ω^n(C) denotes the nth syzygy, ζ corresponds to a surjective map ˆζ : Ω^n(C) → C. We set
\[ L_ζ = \text{Ker}(\hat{\zeta} : \Omega^n(C) \to \mathbb{C}) \subseteq \Omega^n(C). \]
The modules L_ζ are often called “Carlson modules”. The importance of the modules L_ζ is that their support variety can be explicitly computed.

Lemma 3.4.1. Assume that the Hypotheses 2.2.4, 2.3.4 and 3.3.1 hold for the pair (g, t). Let ζ ∈ H^n(g, t; C) and L_ζ be as above, then
\[ V_{(\zeta, t_0)}(L_ζ) = V_{(\zeta, t_0)}(L_{\text{res}(\zeta)}) = Z(\text{res}(\zeta)). \]

Proof. This is argued as in [BKN1, Theorem 6.4.3]. □

3.5. Realization Theorem. An important property in the theory of support varieties is the realization of any homogeneous variety as the support variety of a module. Realizability of support varieties was first proven in [BKN1, Theorem 6.7] for the detecting subalgebras of classical Lie superalgebras. This result was later extended to g support varieties for classical Lie superalgebras and Cartan type Lie superalgebra W(n) in [BKN] Theorem 8.8.1.

Proposition 3.5.1. Assume that the Hypothesis 2.2.4 holds for our fixed pair (g, t). Let ζ_1, . . . , ζ_n ∈ H^*(g, t; C) be homogeneous elements with corresponding Carlson modules L_ζ_1, . . . , L_ζ_n. Then
\[ (1) \ H^*(g, t; L_ζ_1^* \otimes \cdots \otimes L_ζ_n^*) \text{ is finitely generated over } H^*(g, t; C). \]
\[ (2) \ V_{(g, t)}(L_ζ_1 \otimes \cdots \otimes L_ζ_n, C) \subseteq V_{(g, t)}(L_ζ_1, C) \cap \cdots \cap V_{(g, t)}(L_ζ_n, C). \]

Proof. The proof is the same as in [BKN] Proposition 8.6.1 and will be skipped. □

Lemma 3.5.2. Suppose that the pair (g, t) satisfies the Hypotheses 2.2.4, 2.3.4 and 3.3.1. Let ζ ∈ H^n(g, t; C) and let L_ζ be the corresponding Carlson module.

(1) If L_ζ is finite dimensional, then
\[ V_{(g, t)}(L_ζ) = \text{res}^*(V_{(\zeta, t_0)}(L_ζ)) = Z(\zeta). \]

(2) If L_ζ is infinite dimensional, then
\[ V_{(g, t)}(L_ζ, C) = \text{res}^*(V_{(\zeta, t_0)}(L_ζ)) = Z(\zeta). \]

Proof. (1) Since \( \text{res}^*(V_{(\zeta, t_0)}(L_ζ)) = Z(\zeta) \) by Lemma 3.4.1, we have
\[ Z(\zeta) = \text{res}^*(V_{(\zeta, t_0)}(L_ζ)) \subseteq V_{(g, t)}(L_ζ). \]
For the other containment \( V_{(g, t)}(L_ζ) \subseteq Z(\zeta) \) it is enough to show that some power of \( \zeta \) annihilates \( H^*(g, t; \mathbb{C}) \). One can argue exactly as in [Ben, Proposition 6.13] to show that \( \zeta^2 \) annihilates \( H^*(g, t; \mathbb{C}) \).
(2) We have
\[ \text{res}^*: \mathcal{V}_{(\epsilon, \xi)}(L_\xi, \mathbb{C}) \rightarrow \mathcal{V}_{(\mu, t)}(L_\mu). \]

As in the case of finite groups \( L_\xi = L_{\text{res}(\xi)} \oplus P \) as \( \epsilon \)-modules where \( P \) is some projective \( \epsilon \)-module (cf. [Ben] Section 5.9). Observe that by definition \( L_{\text{res}(\xi)} \) can be assumed to lie in the principal block of \( \epsilon \). By [BKN] Proposition 5.2.2 we also know that there are no simple modules other than trivial module in the principal block of \( \epsilon \). These observations with [Ben] Proposition 5.7.1 imply that
\[ \mathcal{V}_{(\epsilon, \xi)}(L_\xi) = \mathcal{V}_{(\mu, t)}(L_\xi, \mathbb{C}). \]

Since \( \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_\xi)) = Z(\xi) \) by Lemma 3.4.1,
\[ Z(\xi) = \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_\xi, \mathbb{C}) \subseteq \mathcal{V}_{(\mu, t)}(L_\xi, \mathbb{C}). \]

For the reverse containment once again we can use the proof given in [Ben, Proposition 6.13] to show that \( \zeta^2 \) annihilates \( H^*(\mu, t; L_\xi \otimes L_\xi) \). This also implies that \( \zeta^2 \) annihilates \( H^*(\mu, t; L_\xi) \).

\[ \square \]

We can now prove the realization theorem.

**Theorem 3.5.3.** Suppose that the pair \((\mathfrak{g}, t)\) satisfies the Hypotheses [2.2.4, 2.3.4 and 3.3.1]. Let \( X \) be a conical subvariety of \( \mathcal{V}_{(\mu, t)}(\mathbb{C}) \).

(a) If the Carlson modules are finite dimensional for \( \mathfrak{g} \) then there exists a finite dimensional \( \mathfrak{g} \)-module \( M \) such that
\[ \mathcal{V}_{(\mu, t)}(M) = X. \]

(b) There exists a \( \mathfrak{g} \)-module \( M \) such that
\[ \mathcal{V}_{(\mu, t)}(M, \mathbb{C}) = X. \]

**Proof.** Let \( J = (\xi_1, \ldots, \xi_n) \subseteq H^*(\mu, t; \mathbb{C}) \) be the homogeneous ideal which defines the homogeneous variety \( X \). That is,
\[ X = Z(\xi_1) \cap \cdots \cap Z(\xi_n). \]

Let \( M = L_{\xi_1} \otimes \cdots \otimes L_{\xi_n} \).

Let us also observe that since by the Hypothesis 2.3.4 the group \( W \) is exact by arguing as in [BKN] Lemma 8.3.1 we can show that
\[ \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_\xi \otimes L_\mu)) = \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_\xi)) \cap \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_\mu)). \]

for Carlson modules \( L_\xi \) and \( L_\mu \).

(a) Combining (3.5.1), Lemma 3.5.2 (1) and using the fact that \( \mathcal{V}_{(\mu, t)}(M_1 \otimes M_2) \subseteq \mathcal{V}_{(\mu, t)}(M_1) \cap \mathcal{V}_{(\mu, t)}(M_2) \) we have
\[
X = Z(\xi_1) \cap \cdots \cap Z(\xi_n) = \mathcal{V}_{(\mu, t)}(L_{\xi_1}) \cap \cdots \cap \mathcal{V}_{(\mu, t)}(L_{\xi_n})
= \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_{\xi_1})) \cap \cdots \cap \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(L_{\xi_n})) = \text{res}^*(\mathcal{V}_{(\epsilon, \xi)}(M)) \subseteq \mathcal{V}_{(\mu, t)}(M)
= \mathcal{V}_{(\mu, t)}(L_{\xi_1}) \cap \cdots \cap \mathcal{V}_{(\mu, t)}(L_{\xi_n}) = X.
\]
It then follows that \( \gamma_{(g,t)}(M) = X \).

(b) Argued as in (a) by using (3.5.1), Lemma 3.5.2 (2) and Proposition 3.5.1 (2).  

\[ \text{□} \]

4. AN APPLICATION

4.1. The Lie superalgebra \( \mathcal{S}(n) \). We begin by recalling the definition of finite dimensional Lie superalgebras of type \( S(n) \). As a background source we refer the reader to the pioneering paper of Kac [Kac] or the book of M. Scheunert [Sch].

Let \( n \) be a positive integer and assume that \( n \geq 2 \). Let \( V \) be an \( n \)-dimensional complex vector space and let \( \Lambda(n) \) denote the exterior algebra of \( V \). The exterior algebra \( \Lambda(n) = \bigoplus_{l=0}^{n} \Lambda(l) \) is an associative \( \mathbb{Z} \)-graded superalgebra. The \( \mathbb{Z} \)-grading is inherited from \( \mathbb{Z} \)-grading by setting \( \Lambda(n)_0 = \Lambda(n)_0 \) and \( \Lambda(n)_1 = \bigoplus \Lambda(n)_{2l+1} \). Fix an ordered basis \( \xi_1, \ldots, \xi_n \) for \( V \). For each ordered subset \( I = \{i_1, i_2, \ldots, i_l\} \) of \( N = \{1, 2, \ldots, n\} \) with \( i_1 < i_2 < \cdots < i_l \), let \( \xi_I \) denote the product \( \xi_1 \xi_2 \cdots \xi_l \). The set of all such \( \xi_I \) forms a basis of \( \Lambda(n) \).

Then as a super space \( W(n) \) is the space of super derivations of \( \Lambda(n) \). \( W(n) \) is a Lie superalgebra via supercommutator bracket. The \( \mathbb{Z} \)-grading on \( \Lambda(n) \) induces a \( \mathbb{Z} \)-grading on \( W(n) \)

\[
W(n) = W(n)_1 \oplus W(n)_0 \oplus \cdots \oplus W(n)_{n-1},
\]

where \( W(n)_l \) consists of derivations that increase the degree of a homogeneous element by \( l \) and this \( \mathbb{Z} \)-grading is consistent with the \( \mathbb{Z} \)-grading, i.e., \( W(n)_0 = \bigoplus W(n)_{2l} \) and \( W(n)_1 = \bigoplus W(n)_{2l+1} \). The supercommutator bracket preserves the \( \mathbb{Z} \)-grading on \( W(n) \) and thus \( W(n)_0 \) is a Lie algebra and each \( W(n)_l \) is a \( W(n)_0 \)-module under the bracket action.

Every element of \( W(n) \) maps \( V \) into \( \Lambda(n) \) and since it is a superderivation it is completely determined by its action on \( V \). Thus \( W(n) \) can be identified with \( \Lambda(n) \otimes V^* \) as a vector space.

Denote by \( \partial_i \), \( 1 \leq i \leq n \), the derivation of \( \Lambda(n) \) defined by

\[
\partial_i(\xi_j) = \delta_{ij}.
\]

Then the set of all \( \xi_I \otimes \partial_i \) forms a basis of \( \Lambda(n) \otimes V^* \). We will write \( \xi_I \partial_i \) instead of \( \xi_I \otimes \partial_i \). We use the identification above to identify \( W(n) \) with \( \Lambda(n) \otimes V^* \). Under this identification the derivation \( \partial_i \) corresponds to the dual of \( \xi_i \) and every element \( D \) of \( W(n) \) can be uniquely written in the form

\[
\sum_{i=1}^{n} f_i \partial_i,
\]

where \( f_i \in \Lambda(n) \).

The superalgebra \( S(n) \) is the subalgebra of \( W(n) \) consisting of all elements \( D \in W(n) \) such that \( \text{div}(D) = 0 \), where

\[
\text{div}(\sum_{i=1}^{n} f_i \partial_i) = \sum_{i=1}^{n} \partial_i(f_i).
\]
The superalgebra \( S(n) \) has a \( \mathbb{Z} \)-grading induced by the grading of \( W(n) \)
\[
S(n) = S(n)_{-1} \oplus S(n)_{0} \oplus \cdots \oplus S(n)_{n-2}
\]
and \( S(n)_{0} \) is isomorphic to \( \mathfrak{sl}(n) \).

Let \( \mathcal{E} = \Sigma_{i=1}^{n} \xi_{i} \partial_{i} \in W(n) \). Note that \( \mathcal{E} \not\in S(n) \). We shall attach \( \mathcal{E} \) to \( S(n) \) and consider the subsuperalgebra \( \overline{S}(n) = S(n) \oplus \mathbb{C} \mathcal{E} \) of \( W(n) \). The superalgebra \( \overline{S}(n) \) admits a \( \mathbb{Z} \)-grading
\[
\overline{S}(n) = \overline{S}(n)_{-1} \oplus \overline{S}(n)_{0} \oplus \cdots \oplus \overline{S}(n)_{n-2},
\]
where \( \overline{S}(n)_{0} \cong \mathfrak{gl}(n) \) as a Lie algebra and \( \overline{S}(n)_{k} = S(n)_{k} \) for \( k \neq 0 \).

4.2. Notation. We fix the following notations for the rest of the paper. Set \( \mathfrak{g} = \overline{S}(n) \) with \( \mathfrak{g}_{i} = \overline{S}(n)_{i}, i \in \mathbb{Z} \) and \( \mathfrak{g}_{\tau} = \overline{S}(n)_{\tau}, \tau \in \mathbb{Z}_{2} \). Furthermore, let \( \mathfrak{g}^{+} = \mathfrak{g}_{1} \oplus \cdots \oplus \mathfrak{g}_{n-2} \). Then
\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}^{+}.
\]
All \( \mathfrak{g} \)-modules will be in the category \( \mathcal{C}(\mathfrak{g}, \mathfrak{g}_{0}) \).

4.3. Basis for \( \mathfrak{g} = \overline{S}(n) \). In this subsection we fix a basis for \( \mathfrak{g} \) and use this basis in the rest. The \( \mathfrak{g}_{-1} \) has basis \( \{ \partial_{i} \mid 1 \leq i \leq n \} \) and \( \mathfrak{g}_{0} \) has basis \( \{ \xi_{i} \partial_{j} \mid 1 \leq i, j \leq n \} \).

Let \( N = \{ 1, \ldots, n \} \) and let \( I \) be an ordered subset of \( N \). A spanning set for each \( \mathfrak{g}_{k} \) for \( k \neq 0, -1 \) can be defined as follows and contains two distinct types of elements. The elements of type \((I, k)\) are all those of the form \( \xi_{I} \partial_{i} \) with \( i \not\in I \) and \( |I| = k + 1 \). Those of type \((II, k)\) are of the form \( \xi_{A} h_{ij} \) where \( i, j \not\in A \) and \( |A| = k \). Here by definition \( h_{ij} = \xi_{i} \partial_{i} - \xi_{j} \partial_{j} \).

The type I elements are all linearly independent, and their span \( \mathfrak{g}_{k}^{I} \) is independent of the span \( \mathfrak{g}_{k}^{II} \) of the type II elements. The type II elements are not independent however, since \( h_{ij} + h_{jk} = h_{ik} \) we reduce the set of type II elements to a basis for \( \mathfrak{g}_{k}^{II} \) as follows. For each \( A \) with \( |A| = k \), order the complement \( B = N - A \) in the natural way as a subset of \( N \) and let \( i \) be the first element of \( B \). Select those element of the form \( \xi_{A} h_{ij} \) where \( i < j \in B \). These are easily seen to be independent and span \( \mathfrak{g}_{k}^{II} \).

4.4. Representation theory and atypicality for \( \mathfrak{g} = \overline{S}(n) \). Let \( \mathfrak{g} = W(n) \) or \( \overline{S}(n) \). A Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) coincides with a Cartan subalgebra of \( \mathfrak{g}_{0} \). We fix a maximal torus \( \mathfrak{h} \subseteq \mathfrak{g}_{0} \) and a Borel subalgebra \( \mathfrak{b}_{0} \) of \( \mathfrak{g}_{0} \). We will denote by \( X_{0}^{+} \) the parametrizing set of highest weights for the simple finite dimensional \( \mathfrak{g}_{0} \)-supermodules with respect to our fixed pair \((\mathfrak{h}, \mathfrak{b}_{0})\). Let \( L_{0}(\lambda) \) denote the simple finite dimensional \( \mathfrak{g}_{0} \)-supermodule with highest weight \( \lambda \in X_{0}^{+} \). We view \( L_{0}(\lambda) \) as a \( \mathfrak{g}_{0} \)-supermodule concentrated in degree 0.

The Kac supermodule \( K(\lambda) \) is the induced representation of \( \mathfrak{g} \),
\[
K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{0} \oplus \mathfrak{g}^{+})} L_{0}(\lambda),
\]
where \( L_{0}(\lambda) \) is viewed as a \( \mathfrak{g}_{0} \oplus \mathfrak{g}^{+} \) by letting \( \mathfrak{g}^{+} \) act trivially. \( K(\lambda) \) is finite dimensional and with respect to the choice of Borel subalgebra \( \mathfrak{b}_{0} \oplus \mathfrak{g}^{+} \subseteq \mathfrak{g} \) one has a dominance order on weights. With respect to this ordering \( K(\lambda) \) has highest weight \( \lambda \) and a unique simple quotient which we denote by \( L(\lambda) \). Conversely, every finite dimensional simple supermodule appears as the head of some Kac supermodule (cf. [Ser, Theorem 3.1]).

From our discussion above we see that the set
\[
\{ L(\lambda) \mid \lambda \in X_{0}^{+} \}
\]
is a complete irredundant collection of simple finite dimensional $g$-supermodules.

Following Serganova, we call $\lambda \in \mathfrak{h}^*$ typical if $K(\lambda)$ is a simple module atypical otherwise. Choose the standard basis $\varepsilon_1, \ldots, \varepsilon_n$ of $\mathfrak{h}^*$ where $\varepsilon_i(\xi_j \partial_j) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. Serganova determines a necessary and sufficient combinatorial condition for $\lambda$ to be typical. Namely, by [Ser, Lemma 5.3] one has that the set of atypical weights $\Omega_W$ for $W(n)$ is

$$\Omega_W = \{ a\varepsilon_i + \varepsilon_{i+1} + \cdots + \varepsilon_n \mid a \in \mathbb{C}, \ 1 \leq i \leq n \}.$$ 

and the set of atypical weights $\Omega$ for $g = \overline{S}(n)$ is

$$\Omega = \{ a\varepsilon_1 + \cdots + a\varepsilon_{i-1} + b\varepsilon_i + (a+1)\varepsilon_{i+1} + \cdots + (a+1)\varepsilon_n \mid a, b \in \mathbb{C}, 1 \leq i \leq n \}.$$ 

Let $\sigma = \varepsilon_1 + \cdots + \varepsilon_n$. For each $\lambda \in \Omega, \lambda \neq a\sigma$ there exists a unique $\overline{\lambda} = \lambda - a\sigma$ such that $\overline{\lambda}$ is atypical for $W(n)$. Since $\dim L(a\sigma) = 1$, we have

$$L(\lambda) \cong L(\overline{\lambda}) \otimes L(a\sigma).$$

**Theorem 4.4.1.** Let $g = \overline{S}(n)$ and let $p \geq 0$. Then, $\Lambda_p((g/g_0)^*)^{g_0} \cong S_p(g_{-1}^* \oplus g_1^*)^{g_0}$. 

**Proof.** Since $g$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$ and $g_0 \cong W(n)_0$, the result follows from [BAKN, Theorem 4.2.1]. \qed

Now we can use Theorem 4.4.1 to show that the cohomology ring $\bigwedge^*(g, g_0; \mathbb{C})$ is identified with a ring of invariants.

**Lemma 4.4.2.** Let $g = \overline{S}(n)$. Let $G_0 \cong \text{GL}(n)$ be the connected reductive group such that $\text{Lie}(G_0) = g_0$ and the adjoint action of $G_0$ on $g$ differentiates to the adjoint action of $g_0$ on $g$. Then,

$$\bigwedge^*(g, g_0; \mathbb{C}) \cong S(g_{-1}^* \oplus g_1^*)^{g_0} = S(g_{-1}^* \oplus g_1^*)^{G_0}.$$ 

**Proof.** Recall that cochains are defined as follows

\begin{equation}
C^p(g, g_0; \mathbb{C}) = \text{Hom}_{g_0}(\Lambda^p_g((g/g_0)^*), \mathbb{C}) \cong \Lambda^p_g((g/g_0)^*)^{g_0}.
\end{equation}

By Theorem 4.4.1 we have

\begin{equation}
\Lambda^p(g,g_0)^* \cong \Lambda^p((g/g_0)^*)^{g_0}.
\end{equation}

Combining (4.4.1) and (4.4.2), we have

\begin{equation}
C^p(g, g_0; \mathbb{C}) \cong \Lambda^p(g_{-1}^* \oplus g_1^*)^{g_0}.
\end{equation}

Next step is to show that differentials are identically zero and this can be done by arguing as in Proposition 2.2.1 by using Theorem 4.4.1. \qed

**Theorem 4.4.3.** Let $M$ be a finite dimensional $g = \overline{S}(n)$-supermodule.

(a) The superalgebra $\bigwedge^*(g, g_0; \mathbb{C})$ is a finitely generated commutative ring.

(b) The cohomology $H^*(g, g_0; M)$ is finitely generated as an $\bigwedge^*(g, g_0; \mathbb{C})$-supermodule.

**Proof.** This follows from Lemma 4.4.2 and Corollary 2.2.2. \qed
4.5. Recall that the Lie superalgebra $\mathfrak{g} = \mathfrak{s}(n)$ admits a $\mathbb{Z}$-grading and $\mathfrak{g}_0 \cong \mathfrak{s}(n) \oplus \mathbb{C} \cong \mathfrak{gl}(n)$ as a Lie algebra. Following [BAKN], we are interested in the natural problem of computing the relative cohomology for the pair $(\mathfrak{g}, \mathfrak{g}_0)$.

Since by Lemma [4.4.1]

(4.5.1) \[ H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S^\bullet((\mathfrak{g}_- \oplus \mathfrak{g}_1)^* \otimes \mathfrak{g}_0), \]

it is enough to compute $S^\bullet((\mathfrak{g}_- \oplus \mathfrak{g}_1)^* \otimes \mathfrak{g}_0)$. Since $\mathfrak{g}$ is a $\mathbb{Z}$-graded subalgebra of $W(n)$, we will benefit from calculations done in [BAKN Section 5].

4.6. Let $T \subseteq G_0$ denote the maximal torus consisting set of all diagonal matrices and $\mathfrak{h} \subseteq \mathfrak{g}_0$ to be $\mathfrak{h} = \text{Lie}(T)$, the Cartan subalgebra of $\mathfrak{g}_0$.

Recall that $\mathfrak{g}_-$ has basis $\{\delta_i | 1 \leq i \leq n\}$, $\mathfrak{g}_0$ has basis $\{\xi_1 \delta_j | 1 \leq i, j \leq n\}$ and $\mathfrak{g}_1$ with basis $\{\xi_1 \xi_2 \delta_k | 1 \leq i \neq j \leq k \leq n, i < j\} \cup \{\xi_1 h_{2j} | 3 \leq j \leq n\} \cup \{\xi_1 h_{1k} | 2 \leq i \neq k \leq n\}$.

Let $f_1$ be as in [BAKN Lemma 5.5.2], i.e., the $\mathbb{C}$-span of the vectors

\[ \{\partial_1, \xi_1 \xi_2 \partial_1, \cdots, \xi_1 \partial_1 | i = 2, \ldots, n\}. \]

The intersection of $f_1$ with $\mathfrak{g}_- \oplus \mathfrak{g}_1$ is spanned by the vectors

(4.6.1) \[ a^*_1 := \{\partial_1, \xi_1 \xi_2 \partial_2 - \xi_1 \xi_1 \partial_1 | i = 3, \ldots, n\}. \]

Let $N$ denote the normalizer of $a^*_1$ in $G_0$. By arguing as in [?, Lemma] one can show that

\[ N = T \Sigma_{n-2}. \]

4.7. We can now explicitly describe the cohomology ring $H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$. Let $Z_k \in a^*_1$ be given by $Z_k(\xi_1 \xi_2 \partial_2 - \xi_1 \xi_1 \partial_1) = \delta_{i,k}$ for $i, k = 3, \ldots, n$ and $Z_k(\partial_1) = 0$. Let $\partial^*_1$ be given by $\partial^*_1(\xi_1 \xi_2 \partial_2 - \xi_1 \xi_1 \partial_1) = 0$ for all $i = 3, \ldots, n$ and $\partial_1^*(\partial_1) = 1$.

**Theorem 4.7.1.** Restriction of functions defines an isomorphism,

\[ H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C}) \cong S^\bullet((\mathfrak{g}_- \oplus \mathfrak{g}_1)^* \otimes \mathfrak{g}_0)^N \cong S^\bullet(a_1^*)^N = \mathbb{C}[Z_3 \partial_1^*, \ldots, Z_n \partial_1^*]_{\Sigma_{n-2}}, \]

where $\Sigma_{n-2}$ acts on $Z_3 \partial_1^*, \ldots, Z_n \partial_1^*$ by permutations. Therefore, $R$ is a polynomial ring in $n - 2$ variables of degree $2, 4, \ldots, 2n - 4$.

**Proof.** The first isomorphism is by Lemma [4.4.1] and the second isomorphism follows from [LR Corollary 4.4]. Since $T$ is a normal subgroup of $N$, we can first compute the $S^\bullet(a_1^*)^T = \mathbb{C}[Z_3 \partial_1^*, \ldots, Z_n \partial_1^*]$. It’s straightforward to check that $\Sigma_{n-2}$ acts by permuting the variables $Z_3 \partial_1^*, \ldots, Z_n \partial_1^*$. By a well known result on invariants under a symmetric group, it follows that $H^\bullet(\mathfrak{g}, \mathfrak{g}_0; \mathbb{C})$ is a polynomial ring generated by elementary symmetric polynomials in the $Z_i \partial_1^*$, where degree of $Z_i \partial_1^*$ is two.

\[ \square \]

4.8. Let $a_1 \subset \mathfrak{g}_1$ be as in (4.6.1) and let $a_0 = \text{Lie}(T) = \mathfrak{h} \subseteq \mathfrak{g}_0$. One can easily verify that $a = a_0 \oplus a_1$ is a Lie subsuperalgebra of $\mathfrak{g}$.

\[ H^\bullet(a, a_0; \mathbb{C}) \cong S^\bullet(a_1^*)^a_0 \cong S^\bullet(a_1^*)^T \]
4.9. **Detecting subalgebra for** \( g = \mathcal{S}(n) \). Let \( e_1 = a_1 \) and let \( e_0 = \text{Lie}(T_{n-1}) \), where
\[
T_{n-1} := \{ \text{diag}(t_1, \ldots, t_n) \in T \mid t_1 = 1 \}.
\]
Set \( e = e_0 \oplus e_1 \). One can directly verify that \( e \) is a Lie subsuperalgebra of \( g \). The classical Lie superalgebra \( e \) has the following important property

\[
[e_0, e_0] = [e_0, e_1] = 0.
\]

Since \( H^\bullet(a, a_0; \mathbb{C}) \cong H^\bullet(e, e_0; \mathbb{C})^T \) and \( H^\bullet(g, g_0; \mathbb{C}) \cong H^\bullet(f, f_0; \mathbb{C})^{\Sigma_{n-2}} \) we have the following isomorphism
\[
H^\bullet(g, g_0; \mathbb{C}) \cong H^\bullet(e, e_0; \mathbb{C})^{T \Sigma_{n-2}}.
\]

From the discussion above one sees that \( e \) is a detecting subalgebra for the pair \( (g, g_0) \).

Since by (4.9.1) the structure of \( e \) is of the type considered in [BKN1, Sections 5, 6], Theorem 6.3.2] one proves that
\[
\mathcal{V}(e, e_0)(M) \cong \mathcal{V}_{\text{rank}}(M)
\]
for any finite dimensional \( e \)-supermodule \( M \) which is an object of \( \mathcal{C}(e, e_0) \). We identify the rank and support varieties of \( e \) via this isomorphism.

4.10. **Support varieties of simple modules.** Since \( H^\bullet(e, e_0; \mathbb{C}) \cong S(e_1^*)^{a_0} = S(e_1^*) \cong \mathbb{C}[\theta_1^*, Z_2, \ldots, Z_n] \), we can identify the support variety of any finite dimensional module \( M \in \mathcal{C}(e, e_0) \) with the conical affine subvariety of the affine \((n-1)\)-space
\[
\text{MaxSpec}(H^\bullet(e, e_0; \mathbb{C})) = \mathcal{V}(e, e_0)(\mathbb{C}) \cong \mathbb{A}^{n-1}
\]
defined by the ideal \( I_{(e, e_0)}(M) \).

The inclusion \( e \hookrightarrow g \) induces a restriction map on cohomology which, in turn, induces maps of support varieties. That is, given supermodules \( M \) and \( N \) in \( \mathcal{C}(g, g_0) \) one has \( M \in \mathcal{C}(e, e_0) \) by restriction to \( e \) and one has maps of varieties
\[
\text{res}^* : \mathcal{V}(e, e_0)(M, N) \to \mathcal{V}(g, g_0)(M, N),
\]
\[
\text{res}^* : \mathcal{V}(e, e_0)(M) \to \mathcal{V}(g, g_0)(M).
\]

Similarly the inclusion \( e \hookrightarrow f \) induces the following maps of varieties
\[
\text{res}^* : \mathcal{V}(e, e_0)(M, N) \to \mathcal{V}(a, a_0)(M, N),
\]
\[
\text{res}^* : \mathcal{V}(e, e_0)(M) \to \mathcal{V}(a, a_0)(M).
\]

for finite dimensional modules \( M, N \in \mathcal{C}(a, a_0) \).

By arguing exactly as in [BKN1, Theorem 6.4.1] one proves that
\[
\mathcal{V}(a, a_0)(M) \cong \mathcal{V}(e, e_0)(M)/T
\]
for any finite dimensional module \( M \in \mathcal{C}(a, a_0) \).
4.11. In this subsection we compute support varieties of all simple finite dimensional g-modules. The first step in this calculation is to compute support variety of Kac modules. Our results show that $L(\lambda)$ is typical if and only if the support variety of $L(\lambda)$ is zero.

**Proposition 4.11.1.** Let $\lambda \in X_0^+$, $K(\lambda)$ be the associated Kac supermodule. Then

$$\mathcal{V}_{(g_0)}(K(\lambda)) = \{0\}.$$

*Proof.* It is enough to show that $\text{Ext}^n_{i,j} C_{g_0,0}(K(\lambda), K(\lambda)) = 0$, for $n >> 0$.

By Frobenius reciprocity, for all $n$ we have

$$\text{Ext}^n_{i,j} C_{g_0,0}(K(\lambda), K(\lambda)) \cong \text{Ext}^n_{C_{g_0,0} \oplus g^+}(L_0(\lambda), K(\lambda)).$$

Since $g^+$ is an ideal in $g_0 \oplus g^+$ one can apply the Lyndon-Hochschild-Serre spectral sequence to $(g^+, \{0\}) \subseteq (g_0 \oplus g^+, g_0)$:

$$E^{i,j}_2 = \text{Ext}^i_{C_{g_0,0}}(L_0(\lambda), \text{Ext}^j_{C_{g_0,0} \oplus g^+}(C, K(\lambda))) \Rightarrow \text{Ext}^{i+j}_{C_{g_0,0} \oplus g^+}(L_0(\lambda), K(\lambda)).$$

Since $C_{g_0,0}$ consists of $g_0$-supermodules which are finitely semisimple over $g_0$, this spectral sequence is zero for $i > 0$. That is, it collapses at the $E_2$ page and yields

$$(4.11.1) \quad \text{Hom}_{g_0}(L_0(\lambda), \text{Ext}^n_{C_{g_0,0} \oplus g^+}(C, K(\lambda))) \cong \text{Ext}^n_{C_{g_0,0} \oplus g^+}(L_0(\lambda), K(\lambda)).$$

Now one can argue as in the proof of the [BAKN Proposition 7.1.1] □

Recall that superdimension of a module $M$ is defined to be $\dim M_0 - \dim M_1$ and the set of atypical weights for $g$ is given by

$$\Omega = \{a \varepsilon_1 + \ldots + a \varepsilon_{i-1} + b \varepsilon_i + (a + 1) \varepsilon_{i+1} + \ldots + (a + 1) \varepsilon_n \mid a, b \in \mathbb{C}, 1 \leq i \leq n\}.$$

**Theorem 4.11.2.** Let $\lambda \in X_0^+$ and let $L(\lambda)$ be the finite dimensional simple $g$-supermodule with highest weight $\lambda$. Then

(a) If $\lambda \notin \Omega$ then $\mathcal{V}_{(g_0)}(L(\lambda)) = \{0\}$.

(b) If $\lambda \in \Omega$ then $\mathcal{V}_{(g_0)}(L(\lambda)) = \mathcal{V}_{(g_0)}(C)$. In this case the support variety has dimension $n - 2$.

*Proof.* (a) If $\lambda \notin \Omega$, i.e., $\lambda$ is typical, then by [Ser] Theorem 6.3 $L(\lambda) = K(\lambda)$. Since $\mathcal{V}_{(g_0)}(K(\lambda)) = \{0\}$ by Proposition 4.11.1 the result follows.

(b) First observe that

$$\Omega \cap X_0^+ = \{a \varepsilon_1 + a \varepsilon_2 \cdot \cdot \cdot + a \varepsilon_{n-1} + b \varepsilon_n \mid a, b \in \mathbb{Z}, b \leq a\}.$$

Furthermore, it is enough to prove that if $\lambda \in \Omega \cap X_0^+$ then $\mathcal{V}_{(g_0)}(L(\lambda)) = \mathcal{V}_{(g_0)}(C)$. Then by (4.10.1) we have

$$\mathcal{V}_{(g_0)}(L(\lambda)) \cong \mathcal{V}_{(g_0)}(L(\lambda))/T = \mathcal{V}_{(g_0)}(C)/T \cong \mathcal{V}_{(g_0)}(C)$$

this in turn gives

$$\mathcal{V}_{(g_0)}(C) = \text{res}^* \mathcal{V}_{(g_0)}(C) = \text{res}^* \mathcal{V}_{(g_0)}(L(\lambda)) \subseteq \mathcal{V}_{(g_0)}(L(\lambda)) \subseteq \mathcal{V}_{(g_0)}(C),$$

which implies the result for $g$. 
There are two cases we need to consider: \( \lambda = a\sigma \) and \( \lambda \neq a\sigma \), where \( \sigma = \varepsilon_1 + \ldots + \varepsilon_n \).

If \( \lambda = a\sigma \), since \( \dim L(a\sigma) = 1 \), super dimension of \( L(\lambda) = \dim L(\lambda)_0 - \dim L(\lambda)_1 \) is nonzero, then by Lemma 3.3.2 (4) we have

\[
V_{(\varepsilon,\varepsilon_0)}(L(\lambda)) = V_{(\varepsilon,\varepsilon_0)}(\mathbb{C}).
\]

Now if \( \lambda \neq a\sigma \), then there exists an atypical weight \( \bar{\lambda} \) for \( W(n) \) such that \( L(\lambda) \cong L(\bar{\lambda}) \otimes L(a\sigma) \). We also know from the work of Serganova [Ser] that a simple finite dimensional module for \( W(n) \) is typical if and only if its superdimension is zero. Since \( \bar{\lambda} \) is atypical for \( W(n) \), super dimension of \( L(\bar{\lambda}) \) is nonzero and thus super dimension of \( L(\lambda) \) which is equal to superdimension of \( L(\bar{\lambda}) \) is nonzero. Now again from Lemma 3.3.2 (4) it follows that

\[
V_{(\varepsilon,\varepsilon_0)}(L(\lambda)) = V_{(\varepsilon,\varepsilon_0)}(\mathbb{C}).
\]

This completes the proof.

4.12. In this subsection we show that a realization theorem holds for \( g \). Recall that \( C_{(g, g_0)} \) denotes the category of \( g \)-modules which are finitely semisimple as \( g_0 \)-module and \( L_\zeta \) denotes the Carlson module corresponding to the homogeneous element \( \zeta \in H^*_{(g, g_0); \mathbb{C}} \). Since the Carlson modules may not be finite dimensional we are going to work with relative support varieties.

**Theorem 4.12.1.** Let \( X \) be a conical subvariety of \( V_{(g, g_0)}(\mathbb{C}) \). Then there exists a \( g \)-module \( M \in C_{(g, g_0)} \) such that

\[
V_{(g, g_0)}(M, \mathbb{C}) = X.
\]

**Proof.** By Theorem 4.4.3 we know that \( H^*_{(g, g_0); \mathbb{C}} \) is finitely generated. We have seen in subsection 4.9 that the pair \( (g, g_0) \) admits a detecting subalgebra \( e = e_0 \oplus e_1 \) and \( e \) — support varieties have rank variety description. Thus the pair \( (g, g_0) \) satisfies all assumptions of Theorem 3.5.3 and the result follows.

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