The Maximum Colorful Arborescence problem parameterized by the structure of its color hierarchy graph

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Abstract. Let \( G = (V, A) \) be a vertex-colored arc-weighted directed acyclic graph (DAG) rooted in some vertex \( r \), and let \( H(G) \) be its color hierarchy graph, defined as follows: \( V(H(G)) \) is the color set \( C \) of \( G \), and an arc from \( c \) to \( c' \) exists in \( H(G) \) if there is an arc in \( G \) from a vertex of color \( c \) to a vertex of color \( c' \). In this paper, we study the Maximum Colorful Arborescence problem (or MCA), which takes as input a DAG \( G \) with the additional constraint that \( H(G) \) is also a DAG, and aims at finding in \( G \) an arborescence rooted in \( r \), of maximum weight, and in which no color appears more than once. The MCA problem is motivated by the inference of unknown metabolites from mass spectrometry experiments. However, whereas the problem has been studied for roughly ten years, the crucial property that \( H(G) \) is necessarily a DAG has only been pointed out and exploited very recently. In this paper, we further investigate MCA under this new light, by providing algorithmic results for the problem, with a specific focus on fixed-parameterized tractability (FPT) issues, and relatively to different structural parameters of \( H(G) \). In particular, we provide an \( O^*(3^{n_H}) \) time algorithm for solving MCA, where \( n_H \) is the number of vertices of indegree at least two in \( H(G) \), thereby improving the \( O^*(3^{|C|}) \) algorithm from Böcker et al. \[2\]. We also prove that MCA is \( \text{W}[2] \)-hard relatively to the treewidth \( H_t \) of \( H(G) \), and further show that it is FPT relatively to \( H_t + \ell_C \), where \( \ell_C = |V| - |C| \).

1 Introduction

Motivated by the inference de novo of metabolites from mass spectrometry experiments, Böcker et al. \[2\] introduced the Maximum Colorful Subtree problem, an optimization problem that takes as input a vertex-colored arc-weighted directed acyclic graph \( G = (V, A) \) rooted in \( r \), and asks for a maximum weighted arborescence from \( G \) that contains \( r \), and in which each color appears at most once. Under closer inspection, it appears, as recently pointed out in \[11\], that the definition of the Maximum Colorful Subtree problem omits a crucial property of \( G \), that directly derives from its motivation. Call \( H(G) \) the following
graph built from $G$: $V(\mathcal{H}(G))$ is the set $C$ of colors used to color $V(G)$, and there is an arc from $c$ to $c'$ in $\mathcal{H}(G)$ if there is an arc in $G$ from a vertex of color $c$ to a vertex of color $c'$. The abovementioned property can then be simply stated as follows: $\mathcal{H}(G)$ is necessarily a DAG. In other words, a partial order exists among the colors in $C$, and in the following $\mathcal{H}(G)$ will be called the hierarchy color graph of $G$. This led Fertin et al. [11] to introduce and study the Maximum Colorful Arborescence problem, formally stated as follows.

**Maximum Colorful Arborescence (MCA)**

**Input:** A DAG $G = (V, A)$ rooted in some vertex $r$, a set $C$ of colors, a coloring function $col : V \to C$ s.t. $\mathcal{H}(G)$ is a DAG and an arc weight function $w : A \to \mathbb{R}$.

**Output:** A colorful arborescence $T = (V_T, A_T)$ rooted in $r$ and of maximum weight $w(T) = \sum_{a \in A_T} w(a)$.

The study of MCA initiated in [11] was essentially focused on the particular case where $G$ is an arborescence (i.e. the underlying undirected graph obtained from $G$ is a tree), and showed among others that MCA is NP-hard even for very restricted instances. They also took advantage of the new problem definition and, in particular, proved the following result: MCA is in P when $\mathcal{H}(G)$ is an arborescence. This latter promising result is the starting point of the present paper, in which we aim at better understanding the structural parameters of $\mathcal{H}(G)$ that may lead to fixed-parameterized tractable (or FPT), algorithms, that are exact and moderately exponential.

**Related work and our contribution.** The MCA problem is known to be NP-hard even when every arc weight is equal to 1 [2], and is highly inapproximable even when $G$ is an arborescence with uniform weights [11]. The MCA problem is also W[1]-hard parameterized by $\ell_C = |V(G)| - |C|$ [11] (a consequence of Theorem 1 from [13]). On the positive side, and as previously mentioned, MCA is in P when $\mathcal{H}(G)$ is an arborescence [11]. Besides, it can be solved by dynamic programming in time $O^*(2^s)$ [11] where $s$ is the minimum number of arcs of $\mathcal{H}$ whose removal turns $\mathcal{H}$ into an arborescence, but also in $O^*(3^{|C|})$ time [2]. Finally, observe that a solution of MCA of order $k$ can be computed in $O^*(3^k)$ time using the coloring technique [1], by applying a new coloring function $col' : V \to \{1, 2, \ldots, k\}$ to the input graph $G$, and further using a dynamic programming algorithm similar to the one described in [6].

The goal of the present paper is to study two parameters from $\mathcal{H}(G)$, namely its number $n^*_t$ of vertices of indegree at least 2, and its treewidth $\mathcal{W}_t$—note that other parameters may also be considered, and see Table 1 for a summary of our results. This choice is motivated by the fact that when $\mathcal{H}(G)$ is an arborescence, each of these two parameters is constant (namely, $n^*_t = 0$ and $\mathcal{W}_t = 1$) and MCA is in P, a necessary condition for the problem to be FPT. Together with FPT issues, our study is also focused on the existence of polynomial kernels for these parameters.
Table 1. Overview of the results for the MCA problem presented in this paper. Here, $n^*_{\mathcal{H}}$ is the number of vertices of indegree at least 2 in $\mathcal{H}$, $\mathcal{H}_t$ is the treewidth of $\mathcal{H}$, $\ell_C = n_G - |C|$ and $\ell \geq \ell_C$ is the number of vertices that are not part of the solution.

| Parameter | FPT status | Kernel status |
|-----------|------------|---------------|
| $n^*_{\mathcal{H}}$ | $O^*(3^{n^*_{\mathcal{H}}})$ (Th. 1) | No poly. kernel (Prop. 1) |
| $\ell_C$ | W[1]-hard (from [13]) | |
| $n^*_{\mathcal{H}} + \ell_C$ | FPT (from Th. 1) | No poly. kernel (Prop. 2) |
| $n^*_{\mathcal{H}} + \ell$ | Poly. kernel (Th. 2) | |
| $\mathcal{H}_t$ | W[2]-hard (Prop. 3) | |
| $\mathcal{H}_t + \ell_C$ | $O^*(2^{\ell_C} \cdot 6^{\mathcal{H}_t})$ (Th. 3) | ? |

Preliminaries. In the following, let $G = (V, A)$ be the input graph of MCA, with $n_G = |V(G)|$ and $m_G = |A(G)|$, and let $U(G)$ be the underlying undirected graph of $G$. For any integer $p$, we let $[p] = \{1, \ldots, p\}$. For any vertex $v \in V$, $N^+(v)$ is the set of outneighbors of $v$. A set $S \subseteq V$ (resp. a graph $G$) is said colorful if no two vertices in $S$ (resp. in $G$) have the same color. Moreover, we say that a subgraph $G'$ of $G$ is fully-colorful if it contains exactly one occurrence of each color from $C$. The color hierarchy graph of $G$ is denoted $H(G) = (C, A_C)$, or, when clear from the context, simply $H$. For any instance of MCA, we denote by $\ell$ the number of vertices that are not part of the solution – thus $\ell \geq \ell_C$. We finally briefly recall the relevant notions of parameterized algorithmics (see e.g. [7]). Given $\Sigma^* \subseteq \Sigma$ for some finite alphabet $\Sigma$, a reduction to a problem kernel, or kernelization, is an algorithm that takes as input an instance $(x, k)$ of a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$ and produces in polynomial time an equivalent (i.e., having the same solution) instance $(x', k')$ such that (i) $|x'| \leq g(k)$, and (ii) $k' \leq k$. The instance $(x', k')$ is called problem kernel, and $g$ is called the size of the problem kernel. If $g$ is a polynomial function, then the problem admits a polynomial-size kernel. The classes $W[1]$ and $W[2]$ are classes of presumed fixed-parameter intractability: if a problem is $W[1]$-hard (resp. $W[2]$-hard) for parameter $k$, then we assume that it cannot be solved in $f(k) \cdot n^{O(1)}$ time.

This paper is organized as follows. In Section 2 we study in detail how $n^*_{\mathcal{H}}$ impacts on the parameterized complexity of the MCA problem, while in Section 3 the same type of study is realized with parameter $\mathcal{H}_t$.

2 Parameterizing the MCA Problem by $n^*_{\mathcal{H}}$

Two main reasons lead us to be particularly interested in $n^*_{\mathcal{H}}$. First, MCA is in $\mathcal{P}$ when $\mathcal{H}$ is an arborescence [11], thus when $n^*_{\mathcal{H}} = 0$. Second, MCA can be solved in $O^*(3^{|C|})$ time [2]. Since by definition $n^*_{\mathcal{H}} \leq |C|$, determining whether MCA is FPT w.r.t. $n^*_{\mathcal{H}}$ is of interest. Let us introduce some definitions. Let $X$
be the set of vertices of indegree at least two in $\mathcal{H}$ (thus $|X| = n^*_H$) and call it the set of difficult colors. Moreover, for any vertex $v \in V$ that has at least one outneighbor in $G$, assume that $\text{col}(N^+(v))$ has an arbitrary and fixed ordering of its vertices. Therefore, for any $i \in [|\text{col}(N^+(v))|]$, let $\text{col}^+(v, i)$ denote the $i$-th color in $\text{col}(N^+(v))$. Finally, for any arborescence $T$ in $G$, let $X(T)$ denote the set of difficult colors in $\text{col}(T)$. We have the following lemma.

**Lemma 1.** In $\mathcal{H}$, for any $c \in \mathcal{C}$, any pair of distinct colors $c_1, c_2 \in N^+(c)$ and any disjoint sets $X_1, X_2 \subseteq X$, any arborescence $T_1$ rooted in $c_1$ s.t. $X(T_1) \subseteq X_1$ is disjoint from any arborescence $T_2$ rooted in $c_2$ s.t. $X(T_2) \subseteq X_2$.

**Proof.** Assume wlog that $\mathcal{H}$ does not contain any path from $c_2$ to $c_1$. If $T_1$ and $T_2$ are not disjoint then there exists $c^* \in \mathcal{C}$ such that $c^*$ belongs both to $T_1$ and $T_2$. In order to prove that such a color $c^*$ cannot exist, let $\tau_1$ (resp. $\tau_2$) be the set of colors on the path from $c_1$ (resp. $c_2$) to $c^*$ including $c_1$ in $T_1$ (resp. $c_2$ in $T_2$). Then, either $\tau_2 \subseteq \tau_1$ or $c_2 \notin \tau_1$. First, if $\tau_2 \subseteq \tau_1$, then there exists a vertex $c' \in \tau_1$ such that $c' \neq c$ with an arc $(c', c_2)$. As there already exists an arc $(c, c_2)$, notice that $c_2$ is a difficult color and hence this contradicts the assumption that $X_1$ and $X_2$ are disjoint. Second, if $c_2 \notin \tau_1$, then notice that $|\tau_1 \cap \tau_2| \geq 1$ as $c^* \in \tau_1 \cap \tau_2$. Therefore, let $\bar{c} \in \tau_1 \cap \tau_2$ such that there exists a path from $\bar{c}$ to any other color of $\tau_1 \cap \tau_2$. By definition, the father of $\bar{c}$ in $\tau_1$ is different from the one in $\tau_2$ and $\bar{c}$ is a difficult color, which contradicts the assumption that $X_1$ and $X_2$ are disjoint. \hfill \Box

**Theorem 1.** MCA can be solved in $O^*(3^{n^*_H})$ time and $O^*(2^{n_H})$ space.

**Proof.** We propose a dynamic programming algorithm which makes use of two programming tables. The first one, $A[v, X', i]$, is computed for all $v \in V(G)$, $X' \subseteq X$ and $i \in \{0, \ldots, |\text{col}(N^+(v))|\}$ and stores the weight of the maximum colorful arborescence $T_A(v, X', i)$ such that

- $T_A(v, X', i)$ is rooted in $v$,
- $(X(T_A(v, X', i)) \setminus \{\text{col}(v)\}) \subseteq X'$, and
- $T_A(v, X', i)$ contains an arc $(v, u)$ only if $u \in N^+(v)$ and $\text{col}(u) \in \bigcup_{j \in [i-1]} \text{col}^+(v, j)$ in $H$.

The second one, $B[v, X', i]$, is computed for all $v \in V$, $X' \subseteq X$ and $i \in [|\text{col}(N^+(v))|]$ and stores the weight of the maximum colorful arborescence $T_B(v, X', i)$ such that

- $T_B(v, X', i)$ is rooted in $v$,
- $(X(T_B(v, X', i)) \setminus \{\text{col}(v)\}) \subseteq X'$, and
- $T_B(v, X', i)$ contains an arc $(v, u)$ only if $u \in N^+(v)$ and $\text{col}(u) = \text{col}^+(v, i)$.

In a nutshell, $T_A(v, X', i)$ and $T_B(v, X', i)$ share the same root $v$ and the same allowed set of difficult colors $X'$ (disregarding $\text{col}(v)$), but there cannot exist $u \in N^+(v)$ such that $(v, u) \in T_A(v, X', i)$ and $(v, u) \in T_B(v, X', i)$. We now explain the computation of the two programming tables.
For an entry of type $A[v, X', i]$, if $i = 0$ then recall that $T_A(v, X', i)$ can only contain $v$. Otherwise, observe that by definition there cannot exist any $u \in N^+(v)$ such that $u$ belongs both to $T_A(v, X'', i - 1)$ and $T_B(v, X' \setminus X'', i)$. Therefore, Lemma 1 proves that $v$ is the only possible common vertex between $T_A(v, X'', i - 1)$ and $T_B(v, X' \setminus X'', i)$ and thus that $T_A(v, X', i)$ is an arborescence. Moreover, for any $v \in V$ and any pair of vertices $\{u_1, u_2\} \subseteq N^+(v)$, Lemma 1 also proves that the color hierarchy graph of any arborescence rooted in $u_1$ is disjoint from the color hierarchy graph of any arborescence rooted in $u_2$, which proves that the combination of $A[v, X'', i - 1]$ and $B[v, X' \setminus X'', i]$ is colorful. Finally, testing every combination of $X'' \subseteq X'$ ensures the correctness of the formula.

\[
A[v, X', i] = \begin{cases} 
0 & \text{if } i = 0 \\
\max_{\forall X'' \subseteq X'} \{A[v, X'', i - 1] + B[v, X' \setminus X'', i]\} & \text{otherwise}
\end{cases}
\]

\[
B[v, X', i] = \begin{cases} 
\max_{\forall u : \col(u) = \col^+(v, i)} \{0, w(v, u) + A[u, X', |\col(N^+(u))|]\} & \text{if } \col^+(v, i) \notin X \\
\max_{\forall u : \col(u) = \col^+(v, i)} \{0, w(v, u) + A[u, X' \setminus \col(u), |\col(N^+(u))|]\} & \text{if } \col^+(v, i) \in X' \\
0 & \text{if } \col^+(v, i) \in X \setminus X'
\end{cases}
\]

For an entry of type $B[v, X', i]$, if $\col^+(v, i)$ is a difficult color which does not belong to $X'$, then notice that $V(T_B(v, X', i)) = \{v\}$, and hence $B[v, X', i] = 0$. Otherwise, recall that $B[v, X', i]$ stores the weight of the maximum colorful arborescence rooted in a vertex $u \in N^+(v)$ which has color $\col^+(v, i)$ in addition to the weight $w(v, u)$. Therefore, notice that computing the maximum colorful arborescences for any such $u$ and only keeping the best one if it is positive ensures the correctness of the formula. Finally, if $\col(u) \in X'$ then observe that $\col(u)$ cannot be contained a second time in $T_B(u, X', |\col(N^+(u))|)$ and must be removed from $X'$.

Recall that any DAG possesses a topological ordering of its vertices, i.e. it has a linear ordering of its vertices such that for every arc $(u, v)$, $u$ comes before $v$ in the ordering. We show how to compute the entries of the dynamic programming algorithm in Algorithm 1 and we use the notation “from last to first” relatively to any correct topological ordering. The total running time derives from the fact that the algorithm needs $3n^3$ steps to compute $A[v, X', i]$ since a difficult color can be in $X''$, $X' \setminus X''$ or in $X \setminus X'$.
Recall that a problem \( Q \) is FPT relatively to a parameter \( k \) if and only if there exists a kernelization algorithm for \( Q \) relatively to \( k \) \cite{fptbook}, but that such a kernel is not necessarily polynomial. In Proposition \ref{prop:kernelless} we prove that although MCA is FPT relatively to \( n_H^* \) (as proved by Theorem \ref{thm:main}), MCA is unlikely to admit a polynomial kernel relatively to \( n_H^* \). In order to prove that a parameterized problem \( Q \) does not admit a polynomial-size kernel, one can use or-cross composition techniques which, roughly speaking, are reductions that combine many instances of a problem into one instance of the problem \( Q \). Hence, if a NP-hard problem admits an or-cross composition into a parameterized problem \( Q \), then \( Q \) does not admit any polynomial-size problem kernel (unless \( \text{NP} \subseteq \text{coNP}/\text{Poly} \) \cite{or-cross}). We first present the formal definition of an or-cross composition and then use such a composition in order to show that MCA is unlikely to admit a polynomial kernel relatively to \(|C|\), and hence to \( n_H^* \).

\textbf{Definition 1.} \((A, k, \mathcal{B})\) A composition algorithm for a parameterized problem \( Q \in \Sigma^* \times \mathbb{N} \) is an algorithm that receives as input a sequence \((x_1, k), (x_2, k), \ldots, (x_t, k)\) with \((x_i, k) \in \Sigma^* \times \mathbb{N}\) for each \(1 \leq i \leq t\), uses polynomial time in \( \sum_{i=1}^t |x_i| + k\), and outputs \((y, k') \in \Sigma^* \times \mathbb{N}\) with \((y, k') \in Q\) iff \( \exists_{1 \leq i \leq t} (x_i, k) \in Q\) and \( k' \) is polynomial in \( k \). A parameterized problem is called compositional if there is a composition algorithm for it.

\textbf{Proposition 1.} Unless \( \text{NP} \in \text{coNP}/\text{Poly} \), MCA does not admit a polynomial kernel for parameter \(|C|\), and consequently for parameter \( n_H^* \), even if \( G \) is an arborescence.

\textbf{Proof.} In the following, let \( t \) be a positive integer. For any \( i \in [t] \), let \( G_i = (V_i, A_i) \) be the graph of an instance of MCA which is rooted in a vertex \( r_i \) and assume that the \( t \) instances are built on the same color set \( C' \). We now show a composition of these \( t \) instances of MCA into a new instance of MCA. Let \( G = (V, A) \) be the graph of such a new instance with \( V = \{r\} \cup \{r'_i : i \in [t]\} \cup \{V_i : i \in [t]\} \) and \( A = \{(r, r'_i) : i \in [t]\} \cup \{(r'_i, r_i) : i \in [t]\} \cup \{A_i : i \in [t]\} \). Here, \( r \) is a vertex not contained in any of the \( t \) MCA instances. Let \( C \) be the color set of \( G \), and let us define the coloring function on \( V(G) \) as follows: the root \( r \) is assigned a unique
show that $MCA$ polynomial parameter transformation is a reduction of $P_s$ parameterized by $Q$. Corollary 1. 

Moreover, if there does not exist $i \in [t]$ such that $G_i$ is not an arborescence, then $G$ is also an arborescence. We now prove that there exists $i \in [t]$ such that $G_i$ has a colorful arborescence $T = (V_T, A_T)$ rooted in $r_i$ of weight $W > 0$ if and only if $G$ has a colorful arborescence $T' = (V_{T'}, A_{T'})$ rooted in $r$ and of weight $W > 0$. 

$(\Rightarrow)$ If there exists $i \in [t]$ s.t. $G_i$ has a colorful arborescence $T = (V_T, A_T)$ rooted in $r_i$ and of weight $W > 0$, then let $T' = (V_{T'}, A_{T'})$ with $V_{T'} = V_T \cup \{r, r_i\}$ and $A_{T'} = A_T \cup \{(r, r_i), (r_i, r_i)\}$. Clearly, $T'$ is connected, colorful and of weight $W$.

$(\Leftarrow)$ Suppose there exists a colorful arborescence $T' = (V_{T'}, A_{T'})$ rooted in $r$ in $G$ of weight $W > 0$. As $T'$ is colorful and all vertices of type $r_i'$ share the same color, notice there cannot exist $i$ and $j$ in $[t]$, $v_i \in V_i$ and $v_j \in V_j$ such that both $v_i$ and $v_j$ belong to $T'$. Thus, let $i^*$ be the only index in $[t]$ such that $v_i \in V_T \neq \emptyset$ and let $T = (V_T, A_T)$ with $V_T = V_{T'} \setminus \{r, r_i'\}$ and $A_T = A_{T'} \setminus \{(r, r_i'), (r_i', r_i)\}$. Clearly, $T$ is connected, colorful and of weight $W$.

Now, notice that $|C| = |C'| + 2$ and thus that we made a correct composition of MCA into MCA. Moreover, recall that MCA is NP-hard [11] and that $n^*_H \leq |C|$. As a consequence, MCA does not admit a polynomial kernel relatively to $|C|$, and hence relatively to $n^*_H$, even in arborescences, unless NP \subseteq \text{coNP}/\text{Poly}. \hfill \Box

Let $s$ denote the minimum number of arcs of $H$ whose removal turns $H$ into an arborescence, and notice that $s \leq |C|^2$ as $V(H) = C$. Although MCA can be solved by dynamic programming in time $O^*(2^s)$ [11], we thus obtain the following corollary that directly derives from Proposition 1.

**Corollary 1.** Unless NP \subseteq \text{coNP}/\text{Poly}, MCA does not admit a polynomial kernel relatively to $s$, even in arborescences.

In Proposition 1 we made a composition of MCA into MCA in order to show that MCA is unlikely to admit a polynomial kernel relatively to $|C|$, and hence to $n^*_H$. In the following, we use a different technique, called polynomial parameter transformation, to show that MCA is also unlikely to admit such a kernel relatively to $n^*_H + \ell_C$, where $\ell_C = |V(G)| - |C|$. In the following, let $P$ be a NP-hard problem and $Q$ a problem which belongs to NP. Roughly speaking, a polynomial parameter transformation is a reduction of $P$ parameterized by $k$ into $Q$ parameterized by $k'$ such that $k' = p(k)$, where $p$ is a polynomial. Therefore, if $P$ parameterized by $k$ admits a polynomial parameter transformation into $Q$ parameterized by $k'$ and if $P$ admits a polynomial kernel relatively to $k$, then $Q$ admits a polynomial kernel relatively to $k'$ [8].

We first present the formal definition of a polynomial parameter transformation and then we use such a technique to prove the proposition.

**Definition 2.** ([8, 8, 12]) Let $P$ and $Q$ be two parameterized problems. We say that $P$ is polynomial parameter reducible to $Q$ if there exists a polynomial-time
computable function $f : \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ and a polynomial $p$, such that for all $(x, k) \in \Sigma^* \times \mathbb{N}$ the following holds: $(x, k) \in P$ iff $(x', k') = f(x, k) \in Q$, and $k' \leq p(k)$. The function $f$ is called a polynomial parameter transformation.

**Proposition 2.** unless NP $\subseteq$ coNP/Poly, MCA does not admit any polynomial kernel relatively to $n_H^* + \ell_C$.

Proof. We reduce from SET COVER, which is defined below.

**Set Cover**

**Input:** A universe $\mathcal{U} = \{u_1, u_2, ..., u_q\}$, a family $\mathcal{F} = \{S_1, S_2, ..., S_p\}$ of subsets of $\mathcal{U}$, an integer $k$.

**Output:** A $k$-sized subfamily $\mathcal{S} \subseteq \mathcal{F}$ of sets whose union is $\mathcal{U}$.

The reduction is as follows: for any instance of SET COVER, we create a three-levels DAG $G = (V = V_1 \cup V_2 \cup V_3, A)$ with $V_1 = \{r\}$, $V_2 = \{v_i : i \in [p]\}$ and $V_3 = \{z_j : j \in [q]\}$. We call $V_2$ the second level of $G$ and $V_3$ the third level of $G$. Informally, we associate one vertex at the second level with each set of $\mathcal{F}$ and one vertex at the third level with each element of $\mathcal{U}$. Then, we draw an arc of weight $-1$ from $r$ to each vertex at level 2 and an arc of weight $p$ from $v_i$ to $z_j$, for all $i \in [p]$ and $j \in [q]$ such that the element $u_j$ is contained in the set $S_i$. Now, we give a unique color to each vertex of $G$. Notice that $\mathcal{H}$ is also a three-levels DAG with col$(V_1)$, col$(V_2)$ and col$(V_3)$ at the first, second and third levels. Therefore, the above construction is an correct instance of MCA. We now prove that there exists a $k$-sized subfamily $\mathcal{S} \subseteq \mathcal{F}$ of sets whose union is $\mathcal{U}$ if

and only if there exists a colorful arborescence $T$ in $G$ of weight $w(T) = pq - k$.

$(\Rightarrow)$ Suppose there exists a $k$-sized subfamily $\mathcal{S} \subseteq \mathcal{F}$ of sets whose union is $\mathcal{U}$ and let $\textbf{True} = \{i \in [p] : S_i \in \mathcal{S}\}$. Then, we set $V_T = \{r\} \cup \{v_i : i \in \textbf{True}\} \cup \{z_j : j \in [q]\}$. Necessarily, $G[V_T]$ is connected: first, $r$ is connected to every level-2 vertex; second, a vertex $z_j$ corresponds to a element $u_j$ which is contained in some set $S_i \in \mathcal{S}$. Now, let $T$ be a spanning arborescence of $G[V_T]$. Clearly, $T$ is colorful and of weight $pq - k$.

$(\Leftarrow)$ Suppose there exists a colorful arborescence $T = (V_T, A_T)$ in $G$ of weight $w(T) = pq - k$. Notice that any arborescence $T'$ in $G$ which contains $r$ and at least one vertex from $V_3$ must contain at least one vertex of from $V_2$ in order to be connected. Therefore, if such an arborescence $T'$ does not contain one vertex of type $z_j$, then $w(T') < pq - p - 1$ and $w(T') < w(T)$. Hence, if $w(T) = pq - k$ then $T$ contains each vertex of the third level and $T$ contains exactly $k$ vertices at the second level. Now, let $\mathcal{S} = \{S_i : i \in [p]\}$ s.t. $v_i \in V_T$ and notice that $\mathcal{S}$ is a $k$-sized subfamily of $\mathcal{F}$ whose union is $\mathcal{U}$ as all vertices of the third level belong to $T$. Our reduction is thus correct.

Now, recall that $\mathcal{H}$ is a three-levels DAG with col$(V_1)$, col$(V_2)$ and col$(V_3)$ at the first, second and third levels. By construction of $G$, if there exists $c \in V(\mathcal{H})$ such that $d^-(c) \geq 1$, then notice that $c \in \text{col}(V_3)$. Moreover, recall that $|\text{col}(V_3)| = |\mathcal{U}|$ and observe that $\ell_C = 0$ as $G$ is colorful. Therefore, notice that $n_H^* + \ell_C \leq |\mathcal{U}|$ and thus that we made a correct polynomial parameter transformation from SET COVER parameterized by $|\mathcal{U}|$ to MCA parameterized
by $n_H^* + \ell_C$. Now, recall that SET COVER is unlikely to admit a polynomial kernel for $|\mathcal{U}|$ \[9\] and that the unparameterized version of SET COVER is NP-hard \[12\]. Moreover, notice that the decision version of MCA which asks for a solution of weight at least $k$ clearly belongs to NP. As a consequence, MCA does not admit any polynomial kernel for $n_H^* + \ell_C$ unless NP \(\subseteq\) coNP/Poly. 

If we let $occ_{max}$ be the maximum number of vertices of $G$ that are assigned the same color in $G$, we can see that $occ_{max} = 1$ in the reduction of Proposition\[2\] above. Hence the following corollary holds.

**Corollary 2.** MCA does not admit a polynomial kernel relatively to $n_H^* + \ell_C$, unless NP \(\subseteq\) coNP/Poly.

In Proposition\[2\] we showed that MCA is unlikely to admit a polynomial kernel relatively to $n_H^* + \ell_C$. As $\ell \geq \ell_C$, we now aim at determining whether a polynomial kernel exists for MCA relatively to $n_H^* + \ell$. Recall that $X$ is the set of **difficult colors**, i.e. colors of indegree at least 2 in $\mathcal{H}$, and hence $|X| = n_H^*$. We say that a vertex $v \in V(G)$ (resp. $c \in V(\mathcal{H})$) is reachable from an other vertex $v' \in V(G)$ (resp. $c' \in V(\mathcal{H})$) if there exists a path from $v'$ to $v$ in $G$ (resp. from $c'$ to $c$ in $\mathcal{H}$). For any vertex $v \in V(G)$, we define $G[v]$ as the induced subgraph of the set of vertices that are reachable from $v$ in $G$ (including $v$). Moreover, for any color $c \in V(\mathcal{H})$, we define $\mathcal{H}[c]$ as the induced subgraph of the set of vertices that are reachable from $c$ in $\mathcal{H}$ (including $c$). Such a color $c$ is said **autonomous** if (i) $\mathcal{H}[c]$ is an arborescence and (ii) there does not exist an arc from a color $c_1 \notin \mathcal{H}[c]$ to a color $c_2 \in \mathcal{H}[c]$ in $\mathcal{H}$. Finally, for any color $c \in \mathcal{C}$, we define $V_c = \{ v \in V : col(v) = c \}$ and we say that $c$ is **unique** if $|V_c| = 1$.

We now prove the two following lemmas, that will help us show that MCA admits a polynomial kernel relatively to $n_H^* + \ell$.

**Lemma 2.** For any instance $(G, \mathcal{C}, col, w, r)$ of MCA, if there exists a color $c \in \mathcal{C}$ such that $\mathcal{H}[c]$ is an arborescence, then there exists an equivalent instance $(G', \mathcal{C}', \text{col}', w', r)$ of MCA such that $\mathcal{C}' = \mathcal{C} \setminus V(\mathcal{H}[c]) \cup \{c\}$.

**Proof.** Let $c \in \mathcal{C}$ be an autonomous color in $\mathcal{H}$, let $(G', \mathcal{C}', \text{col}', w', r) = (G, \mathcal{C}, \text{col}, w, r)$ to begin with and, for each $v \in V_c$, let $T_v$ be the maximum colorful arborescence which is rooted in $v$ in $G$. For each such $v$, we are going to compute the value of $T_v$ in polynomial time, add it to each incoming arc of $v$ and then we will remove all vertices of $G$ that are reachable from a vertex in $V_c$, except $v$.

For any $v \in V_c$, notice that the color hierarchy graph of $G[v]$ is $\mathcal{H}[c]$. As $c$ is an autonomous color, notice that there does not exist any path from the root $r$ of $G$ to any vertex $v' \in G[v]$ which does not include a vertex from $V_c$. Therefore, for each $v \in V_c$, we locally use Theorem from \[11\] in order to compute $T_v$ in polynomial time. Now, for any such $v$ and any $u \in N^-(v)$, we set $w'(u,v) = w(u,v) + w(T_v)$. Moreover, for any color $c'$ which is below $c$ in $\mathcal{H}$, we remove $c'$ from $\mathcal{C}'$ as well as all vertices of color $c'$ and their incident arcs from $G'$. Finally, we define $\text{col}' : V' \rightarrow \mathcal{C}'$ and $w' : V' \rightarrow \mathbb{N}$ such that the rest of $(G', \mathcal{C}', \text{col}', w', r)$ does not change.
Now, we prove that there exists a colorful arborescence \( T = (V_T, A_T) \) of weight \( W \) in \( G \) if and only if there exists a colorful arborescence \( T' = (V_{T'}, A_{T'}) \) of weight \( W \) in \( G' \). First, recall that \( c \) is an autonomous color. Therefore, if \( T \) does not contain any vertex of color \( c \), then \( T \) does not contain any vertex whose color belongs to \( V(\mathcal{H}[c]) \) and we can trivially set \( T' = T \). Second, if \( T \) contains a vertex \( v_c \) of color \( c \) and whose inneighbor is called \( v^\rightarrow \) in \( T \), then we define a subset of vertices \( S_c \subseteq V_T \) such that any vertex \( v \in V_T \) belongs to \( S_c \) if \( v \) is reachable from \( v_c \). We thus state that \( V_{T'} = V_T \setminus S_c \cup \{v_c\} \) and that \( A_{T'} \) contains all the arcs from \( A_T \) that are not in \( \mathcal{H}[c] \). Now, recall that we computed the weight \( w(T_{v_c}) \) of the maximum colorful arborescence that was rooted in \( v_c \) in \( G \) and that \( w'(v^\rightarrow, v_c) = w(v^\rightarrow, v_c) + w(T_{v_c}) \), which ensures that \( w(T) = w(T') \).

Finally, notice that the same reasoning can be applied symmetrically in order to prove that if there exists a colorful arborescence of weight \( W \) in \( G' \) then there exists a colorful arborescence of weight \( W \) in \( G \).  

\[ \square \]

**Lemma 3.** For any instance \((G, C, \text{col}, w, r)\) of MCA, if there exists a triple \(\{c_1, c_2, c_3\} \subseteq C\) such that (i) \( c_1 \) is the unique inneighbor of \( c_2 \), (ii) \( c_2 \) is the unique inneighbor of \( c_3 \) and (iii) \( c_3 \) is the unique outneighbor of \( c_2 \), then there exists an equivalent instance \((G', C', \text{col}', w', r)\) with color set \(C' = C \setminus c_2\).

**Proof.** In the following, for any vertices \(\{v, v'\} \subseteq V(G)\) such that \(v'\) is reachable from \(v\) in \(G\), we denote \(\pi(v, v')\) as the length of the maximum weighted path from \(v\) to \(v'\) in \(G\). Now, let \((G', C', \text{col}', w', r) = (G, C, \text{col}, w, r)\) to begin with, and let \(\{c_1, c_2, c_3\} \subseteq C\) such that (i) \(c_1\) is the unique inneighbor of \(c_2\), (ii) \(c_2\) is the unique inneighbor of \(c_3\) and (iii) \(c_3\) is the unique outneighbor of \(c_2\). The goal here is to remove all vertices of \(V_{c_2}\) from \(G'\). First, for any vertices \(v_1 \in V_{c_1}\) and \(v_3 \in V_{c_3}\) such that there exists a path from \(v_1\) to \(v_3\) in \(G\), notice that such a path is necessarily of length 2 and must contain a vertex of color \(c_2\). For any such vertices \(v_1\) and \(v_3\), we create an arc \((v_1, v_3)\) and we set \(w'(v_1, v_3) = \pi(v_1, v_3)\).

Second, we create a vertex \(v^*\) of color \(c_3\) and, for any vertex \(v_1 \in V_{c_1}\), that has at least one outneighbor of color \(c_2\) in \(G\), we create an arc \((v_1, v^*)\) such that \(w'(v_1, v^*)\) corresponds to the highest weighted outgoing arc from \(v_1\) to any vertex of color \(c_2\) in \(G\). Third, we remove \(c_2\) from \(C'\) as well as all vertices of color \(c_2\) and their incident arcs from \(G'\). Finally, we define \(\text{col}' : V' \rightarrow C'\) and \(w' : V' \rightarrow \mathbb{N}\) such that the rest of \((G', C', \text{col}', w', r)\) does not change.

We now prove that our transformation is correct. Let \(T = (V_T, A_T)\) be a colorful arborescence of weight \(W\) in \(G\). First, if \(T\) does not contain a vertex of color \(c_2\) then let \(T' = T\) and notice that \(w(T) = w'(T')\) as \(w(v, u) = w'(v, u)\) for any arc of \(G\) such that \(\text{col}(v) \neq c_2\) and \(\text{col}(u) \neq c_2\). Second, if \(T\) contains a vertex \(v_2\) of color \(c_2\) whose inneighbor is \(v_1\) in \(T\) and if \(T\) does not contain any vertex of color \(c_3\), then let \(V_{T'} = V_T \setminus \{v_2\} \cup \{v^*\}\), let \(A_{T'} = A_T \setminus \{(v_1, v_2)\} \cup \{(v_1, v^*)\}\) and observe that \(w(T) = w'(T')\) as \(w(v_1, v_2) = w'(v_1, v^*)\). Third, if \(T\) contains a vertex \(v_2\) of color \(c_2\) whose inneighbor is \(v_1\) in \(T\) and if \(T\) contains a vertex \(v_3\) of color \(c_3\) (whose inneighbor is necessarily \(v_2\)), then let \(V_{T'} = V_T \setminus \{v_2, v_3\}\), let \(A_{T'} = A_T \setminus \{(v_1, v_2), (v_2, v_3)\} \cup \{(v_1, v_3)\}\) and observe that \(w(T) = w'(T')\) as \(w(v_1, v_2) + w(v_2, v_3) = w'(v_1, v_3)\). Finally, we claim that if there exists a
colorful arborescence of weight \( W \) in \( G' \) then there exists a colorful arborescence of weight \( W \) in \( G \) by using the same arguments in reverse order.

We are now ready to prove the following theorem.

**Theorem 2.** MCA does admit a polynomial kernel relatively to \( n_H^* + \ell \).

**Proof.** In the following, for any vertex \( v \in V \), we define \( N_u^-(v) \) as the set of unique colors in the inneighborhood of \( v \) in \( G \). We first explain the kernelization process and then show that the obtained instance is bounded relatively to \( n_H^* + \ell \).

First, we iteratively reduce \((G, C, col, w, r)\) via Lemmas 2 and 4. For sake of clarity, \((G, C, col, w, r)\) is now defined as the resulting instance when none of the two lemmas can be applied and we let \( T = (V_T, A_T) \) be a solution of MCA on such an instance. Second, recall that \( \ell \) is the maximum number of vertices that do not belong to \( T \) in \( G \). Therefore, if there exists \( v \in V \) such that \( |N_u^-(v)| \geq \ell + 2 \), then observe that \( T \) has to contain at least two vertices from \( N_u^-(v) \). Now, let \( v_1 \) be a vertex from \( N_u^-(v) \) such that \((v_1, v)\) is the lowest incoming arc from a unique color to such \( v \). Even if \( v_1 \) belongs to \( T \), there will always exist at least one other vertex \( v_2 \) that will also belong to \( T \) and such that \( w(v_1, v) \leq w(v_2, v) \). Thus, as \( T \) is an arborescence, the arc \((v_1, v)\) will never belong to \( T \), and hence we apply the following rule: for all vertices \( v \in V \) such that \( |N_u^-(v)| > \ell + 1 \), we delete the \(|N_u^-(v)| - \ell - 1\) lowest incoming arcs from vertices of unique colors in \( G \). Again, for sake of clarity, \((G, C, col, w, r)\) is now defined as the current and final instance after that the abovementioned rule has been applied.

It is clear that \((G, C, col, w, r)\) remains an instance of MCA after the kernelization - which is done in polynomial time. Now, we complete the proof in two times. First, we show that the indegree of any color in \( C \) is bounded by a function of \( \ell \). Then, we will use such a property to polynomially bound \( n_G \) by a function of \( n_H^* \) and \( \ell \).

Let us first show that the indegree of any color in \( C \) is bounded by a function of \( \ell \). As \( T \) must be colorful and such that \( |V_T| = n_G - \ell \), notice that there exists at most \( \ell \) non-unique colors in \( C \) and hence the inneighborhood of any color \( c \in V(H) \) cannot contain more than \( \ell \) non-unique colors in \( H \). Moreover, recall that the inneighborhood of any vertex \( v \in V(G) \) cannot contain more than \( \ell + 1 \) vertices of unique color in \( G \), and recall that \( T \) cannot be colorful if there exists more than \( \ell + 1 \) occurrences of any color in \( G \). As a consequence, for any color \( c \in V(H) \), the outneighborhood of \( c \) cannot contain more than \((\ell + 1)^2\) unique colors in \( H \), and hence \( d^- (c) \leq (\ell + 1)^2 + \ell \).

Now, let \( F \subseteq H \) be a forest whose vertex set is \( C_F = C \setminus X \) and which contains any arc \((c, c')\) of \( H \) such that \( \{c, c'\} \subseteq C_F \). In the following, we successively bound the maximum number of leaves of \( F \), the maximum number of vertices of \( F \), of \( V(H) \) and finally of \( V(G) \) relatively to \( \ell \) and \( n_H^* \). First, recall that there does not exist any color \( c \in C \) such that Lemma 2 can be applied on \( G \) after the kernelization process and thus notice that any leaf \( c \in V(H) \) is in fact a difficult color. Second, recall that the maximum indegree of any color in \( H \) is bounded by \((\ell + 1)^2 + \ell \). Hence, notice that the number of leaves of \( F \) is bounded by \( n_H^* (\ell + 1)^2 + \ell \). Now, notice via Lemma 3 that \( H \) does not contain any color...
which has a unique inneighbor and a unique outneighbor. As a consequence, as $F$ being a binary arborescence is the configuration that leads to the highest number of vertices in $F$, notice that $|V(F)| \leq 2 \cdot n_{H}^*((\ell+1)^2 + \ell)$. Hence, as $C_F = C \setminus X$, observe that $|C| \leq n_{H}^* + 2 \cdot n_{H}^*((\ell+1)^2 + \ell)$. Finally, as any color $c \in C$ occurs at most $\ell + 1$ times in $G$, notice that $|V(G)| \leq (\ell + 1) \cdot (n_{H}^* + 2 \cdot n_{H}^*((\ell+1)^2 + \ell))$ and hence the Maximum Colorful Arborescence problem does admit a kernel of size $O^*(\ell^3 + n_{H}^*)$. \hfill \Box

3 Parameterizing the MCA Problem by $H_t$

Recall that $U(H)$ is the underlying undirected graph of $H$. In this section, we are interested in parameter $H_t$, defined as the treewidth of $U(H)$. Roughly speaking, the treewidth is a parameter that measures how close from a tree is a graph. Therefore, as MCA is in $P$ whenever $H$ is an arborescence $[11]$, it is natural to ask whether or not there exist FPT algorithms for MCA parameterized by $H_t$.

To do so, we first introduce a few definitions.

**Definition 3.** Let $G = (V, E)$ be a undirected graph. A tree decomposition of $G$ is a pair $\langle\{X_i : i \in I\}, \mathcal{T}\rangle$, where each $X_i$ is a subset of $V$, called a bag, and $\mathcal{T}$ is a tree with the elements of $I$ as nodes. The following three properties must hold:

1) $\bigcup_{i \in I} X_i = V$;
2) for every edge $(u, v) \in E$, there is an $i \in I$ such that $u, v \in X_i$;
3) for all $i, j, k \in I$, if $j$ lies on the path between $i$ and $k$ in $\mathcal{T}$ then $X_i \cap X_k \subseteq X_j$.

The width of $\langle\{X_i : i \in I\}, \mathcal{T}\rangle$ equals $\max\{|X_i| : i \in I\} - 1$. The treewidth of $G$ is the minimum $k$ such that $G$ has a tree decomposition of width $k$.

**Definition 4.** A tree decomposition $\langle\{X_i : i \in I\}, \mathcal{T}\rangle$ is called a nice tree decomposition if the following conditions are satisfied:

1) Every node of $\mathcal{T}$ has at most two children.
2) If a node $i$ has two children $j$ and $k$, then $X_i = X_j = X_k$ (in this case, $X_i$ is called a Join Node).
3) If a node $i$ has one child $j$, then one of the following situations must hold:
   a) $|X_i| = |X_j| + 1$ and $X_j \subseteq X_i$ (in this case, $X_i$ is called an Introduce Node), or
   b) $|X_i| = |X_j| - 1$ and $X_i \subseteq X_j$ (in this case, $X_i$ is called a Forget Node).
4) If a node $i$ does not have any child, then $|X_i| = 1$ (in this case, $X_i$ is called a Leaf Node).

Unfortunately, we first show in the next proposition that MCA is unlikely to admit FPT algorithms for the parameter $H_t$.

**Proposition 3.** MCA is $W[2]$-hard relatively to $H_t$.

**Proof.** We reduce from the $k$-Multicolored Set Cover problem, which is defined below.
The reduction is as follows: for any instance of $k$-MULTICOLORED SET COVER, we create a three-levels DAG $G = (V = V_1 \cup V_2 \cup V_3, A)$ with $V_1 = \{r\}$, $V_2 = \{v_i : i \in [p]\}$ and $V_3 = \{z_j : j \in [q]\}$. Informally, we associate a vertex at the second level for each set of $F$ and a vertex at the third level for each element of $U$. Then, we draw an arc of weight $-1$ from $r$ to each vertex at level 2 and an arc of weight $p$ from $v_i$ to $z_j$, for all $i \in [p]$ and $j \in [q]$ such that the element $u_i$ is contained in the set $S_i$. Now, we give a unique color to $V_1 \cup V_3$. At the second level, two vertices of type $v_i$ have the same color if and only if the two associated sets have the same color according to $col'$. Notice that $\mathcal{H}$ is also a three-levels DAG with $\text{col}(V_1)$, $\text{col}(V_2)$ and $\text{col}(V_3)$ at the first, second and third levels. Therefore, $(G, C, \text{col}, w, r)$ is a correct instance of MCA.

We now prove that there exists a colorful set $S \subseteq F$ of size $k$ whose union is $U$ if and only if there exists a colorful arborescence $T$ in $G$ of weight $w(T) = pq - k$.

$(\Rightarrow)$ Suppose there exists a colorful set $S \subseteq F$ of size $k$ whose union is $U$ and let $\text{True} = \{i \in [p] : S_i \in S\}$. Let $V_T = \{r\} \cup \{v_i : i \in \text{True}\} \cup \{z_j : j \in [q]\}$. Necessarily, $G[V_T]$ is connected: first, $r$ is connected to every level-2 vertex; second, a vertex $z_j$ corresponds to a element $u_j$ which is contained in some set $S_j \in S$. Now, let $T$ be a spanning arborescence of $G[V_T]$. Clearly, $T$ is colorful and of weight $pq - k$.

$(\Leftarrow)$ Suppose there exists a colorful arborescence $T = (V_T, A_T)$ in $G$ of weight $w(T) = pq - k$. Notice that any arborescence $T'$ in $G$ which contains $r$ and at least one vertex from $V_3$ must contain at least one vertex of from $V_2$ in order to be connected. Therefore, if such an arborescence $T'$ does not contain one vertex of type $z_j$, then $w(T') < pq - p - 1$ and $w(T') < w(T)$. Hence, if $w(T) = pq - k$ then $T$ contains each vertex of the third level and $T$ contains exactly $k$ vertices at the second level. Now, let $S = \{S_i : i \in [p] \text{ s.t. } v_i \in V_T\}$ and notice that $S$ is a colorful subfamily of size $k$ whose union is $U$ as all vertices of the third level belong to $T$. Our reduction is thus correct.

Now, recall that $\mathcal{H}$ is a three-levels DAG with $\text{col}(V_1)$, $\text{col}(V_2)$ and $\text{col}(V_3)$ at the first, second and third levels. Therefore, notice that there exists a trivial tree decomposition $\{\{X_i : i \in [|\text{col}(V_3)| + 2]\}, T\}$ of $U(\mathcal{H})$ which is defined as follows: the bag $X_0 = \{\text{col}(r)\}$ has an arc towards the bag $X_1 = \{\{\text{col}(r)\} \cup \text{col}(V_2)\}$ and, for any $i \in [|\text{col}(V_3)|]$, there exists an arc from $X_i$ towards $X_j$, where each $X_i$ contains $\text{col}(V_2)$ and a different vertex of $\text{col}(V_3)$. As a conclusion, notice that the treewidth of $\{\{X_i : i \in [|\text{col}(V_3)| + 2]\}, T\}$ is $k$, and hence MCA is $\mathcal{W}[2]$-hard parameterized by $\mathcal{H}_t$ as $k$-MULTICOLORED SET COVER is $\mathcal{W}[2]$-hard parameterized by $k$ [14].

We now use the above proof to show that MCA is unlikely to admit FPT algorithms relatively to different parameters that are related to $\mathcal{H}$. First, recall
that the vertex-cover number of $U(H)$ is the size of the smallest subset $S \subseteq V(H)$ such that at least one incident vertex of any arc of $H$ belongs to $S$. Notice that $col(V_2)$ is a vertex cover of $U(H)$ and thus that the vertex cover number of $U(H)$ is at most $k$. Moreover, recall that the feedback vertex set parameter is the size of the smallest subset $S \subseteq H$ whose removal leaves $U(H)$ without cycles. The size of such a subset $S$ is an interesting parameter as $n^*_H = 0$ in $H[V(H) \setminus S]$ and any vertex cover of $U(H)$ is also a feedback vertex set of $U(H)$. Hence, $col(V_2)$ is also a feedback vertex set of $U(H)$. Altogether, we thus obtain the following corollary.

**Corollary 3.** MCA parameterized by the vertex-cover number of $U(H)$ or the feedback vertex set parameter of $U(H)$ is $W[2]$-hard.

Next, recall that each color from the third level of $H$ is a leaf in Proposition 3. Therefore, the number of colors of outdegree at least 2 in $H$ is $|col(V_1)| + |col(V_2)| = k + 1$. Although Theorem 1 showed an FPT algorithm for MCA relatively to $n^*_H$, we thus obtain the following corollary.

**Corollary 4.** MCA is $W[2]$-hard relatively to the number of colors of outdegree at least 2 in $H$.

We proved in Proposition 3 that MCA parameterized by $U(H)$ is $W[2]$-hard and we are thus looking for a parameter whose combination with $H_t$ may lead to an FPT algorithm for MCA. To do so, we focus on parameter $\ell_C = n_C - |C|$. Although MCA is $W[1]$-hard relatively to $\ell_C$, the problem can be solved in $O^*(2^{|C|})$ when $G$ is an arborescence [11]. Meanwhile, recall that MCA can be solved in polynomial time when $H$ is an arborescence [11], and hence when $H_t = 1$. Although $G$ being an arborescence does not necessarily imply that $H$ is also an arborescence, determining whether MCA is FPT relatively to $\ell_C + H_t$ remains of interest. We first recall that a fully-colorful subgraph of $G$ is a subgraph of $G$ that contains exactly one occurrence of each color $c \in C$ and show the following lemma.

**Lemma 4.** Given any graph $G$ with $|C|$ colors, there exist at most $2^{\ell_C}$ fully-colorful subgraphs of $G$.

**Proof.** Let $n_c$ be the number of vertices of color $c \in C$ and notice that $\prod_{c \in C} n_c$ is the number of fully-colorful subgraphs of $G$. Then, observe that $n_c \leq 2^{n_c} - 1$ for all $n_c \in \mathbb{N}$, which implies $\prod_{c \in C} n_c \leq 2^{\sum_{c \in C} n_c} - 1$ and thus $\prod_{c \in C} n_c \leq 2^{\ell_C}$. \Box

**Theorem 3.** MCA can be solved in $O^*(2^{\ell_C} \cdot 6^{|H_t|})$ time and $O^*(3^{|H_t|})$ space.

**Proof.** In the following, let $\langle \{X_i : i \in I\}, T \rangle$ be a nice tree decomposition of $U(H)$. In this proof, we provide a dynamic programming algorithm that makes use of $\langle \{X_i : i \in I\}, T \rangle$ in order to compute the solution of MCA in any fully-colorful subgraph $G' \subseteq G$. First, observe that $\langle \{X_i : i \in I\}, T \rangle$ is also a correct nice tree decomposition for the (undirected) color hierarchy graph of any subgraph of $G$. Second, as any colorful graph is equivalent to its color hierarchy
graph, notice that \( \{ \{ X_i : i \in I \}, T \} \) is also a correct nice tree decomposition of any fully-colorful subgraph \( G' \in G \). Therefore, we assume wlog that any bag \( X_i \) contains vertices of such \( G' \) instead of colors and that \( X_0 = \{ r \} \) is the root of \( \{ \{ X_i : i \in I \}, T \} \).

Now, for any \( i \in I \) and for any subsets \( L_1, L_2, L_3 \) that belong to \( X_i \) such that \( L_1 \oplus L_2 \oplus L_3 = X_i \), let \( T_i[L_1, L_2, L_3] \) store the weight of a partial solution of MCA in \( G' \), which is a collection of \( |L_1| \) disjoint arborescences such that:

- each \( v \in L_1 \) is the root of exactly one such arborescence,
- each \( v \in L_2 \) is contained in exactly one such arborescence,
- no vertex \( v \in L_3 \) belongs to any of these arborescences,
- any vertex \( v \in V \) whose color is forgotten below \( X_i \) can belong to any such arborescence,
- there does not exist another collection of arborescences with a larger sum of weights under the same constraints.

Besides, let us define an entry of type \( D_i[L_1, L_2, L_3] \) which stores the same partial solution than an entry of type \( T_i[L_1, L_2, L_3] \), except for the vertices \( v \in V \) whose colors are forgotten below \( X_i \) which cannot belong to any arborescence of the partial solution. We now explain how to fill each entry of \( T_i[L_1, L_2, L_3] \). We stress that each entry of \( D_i[L_1, L_2, L_3] \) is filled exactly as an entry of type \( T_i[L_1, L_2, L_3] \), apart from the case of forget nodes which we detail below.

- **If \( X_i \) is a leaf node**: \( T_i[L_1, L_2, L_3] = 0 \)
  Notice that leaf nodes are base cases of the dynamic programming algorithm as \( \{ \{ X_i : i \in I \}, T \} \) is a nice tree decomposition. Moreover, recall that leaf nodes have size 1 and thus that the only partial solution for such nodes has a null weight.

- **If \( X_i \) is an introduce node having a child \( X_j \) and if \( v^* \) is the introduced vertex**:

  \[
  T_i[L_1, L_2, L_3] = \begin{cases} 
  A) & \max_{\forall S \subseteq L_2} \left\{ \sum_{v \in S} w(v^*, v) + T_j[L_1 \cup S \setminus \{ v^* \}, L_2 \setminus S, L_3] \right\} \\
  & \quad \text{if } v^* \in L_1 \\
  B) & \max_{\forall u \in (L_1 \cup L_2)} \left\{ w(u, v^*) + \\
  & \quad \max_{\forall S \subseteq (L_2 \setminus \{ u \})} \left\{ \sum_{v \in S} w(v^*, v) + T_j[L_1 \cup S \setminus \{ v^* \}, L_2 \setminus S, L_3] \right\} \right\} \\
  & \quad \text{if } v^* \in L_2 \\
  C) & T_j[L_1, L_2, L_3 \setminus \{ v^* \}] \\
  & \quad \text{if } v^* \in L_3
  \end{cases}
  \]

There are three cases: \( v^* \) is a root of an arborescence in a partial solution (case \( A \)), an internal vertex of such a solution (case \( B \)) or \( v^* \) does not belong to such a solution (case \( C \)). In case \( A \), \( S \) corresponds to the set of outneighbors of \( v^* \) in the partial solution. The vertices of \( S \) thus do not have any other inneighbor in
the partial solution. Therefore, in the corresponding entry $T_j$, the vertices of $S$ are roots. Now, notice that $B_j$ is very similar to $A_j$. In addition to a given set $S$ of outneighbors, $v^*$ being in $L_2$ implies that $v^*$ has an inneighbor $u \in (L_1 \cup L_2)$ in the partial solution. Since the inneighbor $u$ cannot be an outneighbor at the same time, $u$ is not contained in $S$. Exhaustively trying all possibilities for both $S$ and $u$ ensures the correctness of the solution. Finally, observe that $V^*$ does not belong to the partial solution of $T_i[L_1, L_2, L_3]$ if $v^* \in L_3$.

- If $X_i$ is a forget node having a child $X_j$ and if $v^*$ is the forgotten vertex:
  \[ T_i[L_1, L_2, L_3] = \max\{T_j[L_1, L_2 \cup \{v^*\}, L_3], T_i[L_1, L_2, L_3 \cup \{v^*\}]\} \]

Informally, the formula determines whether the collection of arborescences that is stored in $T_i[L_1, L_2, L_3]$ had a higher weight with or without $v^*$ as an internal vertex. Observe that we do not consider the case where $v^*$ is the root of an arborescence as such an arborescence could not be connected to the rest of the partial solution via an introduced vertex afterwards. Besides, notice that $D_i[L_1, L_2, L_3] = D_j[L_1, L_2, L_3 \cup \{v^*\}]$ as the partial solution in $D_i[L_1, L_2, L_3]$ does not contain any forgotten vertex by definition.

- If $X_i$ is a join node having two children $X_j$ and $X_k$:
  \[ T_i[L_1, L_2, L_3] = T_j[L_1, L_2, L_3] + T_k[L_1, L_2, L_3] - D_i[L_1, L_2, L_3] \]

Informally, the partial solution in $T_i[L_1, L_2, L_3]$ can contain both the forgotten vertices of the partial solution in $T_j[L_1, L_2, L_3]$ and those of the partial solution in $T_k[L_1, L_2, L_3]$. Recall that the partial solution in $D_i[L_1, L_2, L_3]$ does not contain any forgotten vertices and therefore that any arc of the partial solution in $T_i[L_1, L_2, L_3]$ is only counted once. We fill the programming tables from the leaves to the root for all $i \in I$ until $T_0$ and any entry of type $T_i[L_1, L_2, L_3]$ is directly computed after the entry of type $D_i[L_1, L_2, L_3]$. If $T' = \langle V_{T'}, A_{T'} \rangle$ is a solution of MCA in a fully-colorful subgraph $G' \subseteq G$, then observe that $w(T') = T_0[\{r\}, \emptyset, \emptyset]$. Now, notice that the solution of MCA in $G$ is also the solution of at least one fully-colorful subgraph $G' \subseteq G$. Therefore, computing the solution of MCA for any such subgraph $G'$ ensures the correctness of the algorithm and hence, by Lemma 4 adding a factor $O(2^{|C|})$ to the complexity of the above algorithm proves our theorem. \hfill \(\square\)

## 4 Conclusion

In this paper, we obtained a $O^*(3^{|I|})$ time algorithm, which improves upon the $O^*(3^{|I|})$ of Böcker et al. 2. We also showed that MCA is unlikely to admit a polynomial kernel relatively to $n^{\ell} + \ell$ and then that the problem admits such a kernel relatively to $n^{\ell} + \ell$. Furthermore, we proposed a FPT algorithm for MCA relatively to $\mathcal{H}_f + \ell C$, although we showed that MCA is $\mathcal{W}[2]$-hard relatively to $\mathcal{H}_f$.

In light of these results, we ask the following question: does MCA admit a polynomial kernel relatively to $\ell C + \mathcal{H}_f$?
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