STOCHASTIC ANALYSIS METHODS IN WISHART THEORY

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Part I. Yamada-Watanabe Theorem for Matrix Stochastic Differential Equations. Moments of Wishart Processes. By P. Graczyk

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Part II

WISHART PROCESSES AND WISHART DISTRIBUTIONS:
AN AFFINE PROCESSES POINT OF VIEW

by E. Mayerhofer

1. Introduction

In his 1928 Biometrika contribution [30], Wishart introduced the distribution of covariance matrices of samples from a normally distributed random variable. His contribution triggered a lot of research on theory and application of these and other related multivariate distributions in e.g., multivariate statistics, probability theory and most recently in finance and financial mathematics (for an account of the literature, see [4] and the references therein). An important subclass of Wishart distributions arise as pushforwards of normal distributions under certain quadratic forms: Let $\xi_1, \xi_2, \ldots, \xi_k$ be a sequence of $\mathbb{R}^d$-valued independent, normally distributed random variables with mean vectors $\mu_i \in \mathbb{R}^d$ and covariance matrix $\Sigma$. Then

$$\Xi := \xi_1^\top \xi_1 + \cdots + \xi_k^\top \xi_k$$
is Wishart distributed with scale parameter \( p := k/2 \), shape parameter \( \sigma := 2\Sigma \) and parameter of non-centrality \( \omega := \sum_{i=1}^{k} \mu_i \mu_i^\top \). We use the notation \( \Gamma(p, \omega; \sigma) \) from [19] for the distribution of \( \Xi \) which in turn is motivated by Letac and Massam’s [16] family of Wishart distributions. Note that \( \Xi \) is positive semidefinite almost surely, and also \( \omega \) and \( \sigma \) are positive semidefinite matrices. Considering first the one-dimensional case \( d = 1 \) and assuming the non-trivial case \( \Sigma \neq 0 \), we can calculate the Laplace transform (or quite similarly the characteristic function) of \( \xi_j \xi_j^\top = \xi_j^2 \), by mere completion of a square,

\[
\mathbb{E}[e^{-u\xi_j^2}] = \frac{1}{\sqrt{2\pi\Sigma}} \int_{\mathbb{R}} e^{-\frac{-(\eta-\mu_j)^2}{2\Sigma}} d\eta = \frac{1}{\sqrt{2\pi\Sigma}} \int_{\mathbb{R}} \exp \left( -\frac{1 + 2\Sigma u}{2\Sigma} (\eta - \frac{\mu_j}{1 + 2\Sigma u})^2 - \frac{1}{2\Sigma} \mu_j^2 \left( \frac{1 - \frac{1}{1 + 2\Sigma u}}{1 + 2\Sigma u} \right) \right) d\eta
\]

\[
= \frac{1}{\sqrt{1 + 2\pi\Sigma}} \int_{\mathbb{R}} \exp \left( -\frac{1 + 2\Sigma u}{2\Sigma} (\eta - \frac{\mu_j}{1 + 2\Sigma u})^2 \right) d\eta \times e^{-u\mu_j^2/(1+2\Sigma u)}
\]

\[
= \frac{e^{-u(1+2\Sigma u)^{-1}\mu_j^2}}{(1 + 2\Sigma u)^{1/2}}, \quad u \geq 0.
\]

Here \( \mathbb{E}[\cdot] \) denotes the expectation operator on the respective probability space which supports the random variables \( \xi_j \). By the independence of \( \xi_j \), \( j = 1, \ldots, k \), we obtain

\[
\mathbb{E}[e^{-u\Xi}] = \prod_{j=1}^{k} \mathbb{E}[e^{-u\xi_j^2}] = \frac{e^{-u(1+\sigma u)^{-1}\omega}}{(1 + \sigma u)^p}, \quad u \geq 0.
\]

We see that the family of distributions \( \Gamma(p, \omega; \sigma) \) is a natural extension of non-central chi-square distributions on the one hand, where \( \Sigma = 1 \), and of gamma distributions on the other hand, where \( \omega = 0 \). Let now \( d > 1 \). In the following we denote by \( \mathcal{S}_d^+ \) the cone of positive semidefinite matrices, by \( \text{tr}(A) \) the trace of a \( d \times d \) matrix \( A \), and by \( \det(A) \) the matrix determinant. The positive definite matrices are abbreviated as \( \mathcal{S}_d^+ \).

With this notation, a similar calculation as above (taking into account the non-commutativity of the matrix multiplication) yields the formula for the Laplace transform

\[
\mathbb{E}[e^{-u\text{tr}(\Xi)}] = \frac{e^{-u\text{tr}(I+\sigma u)^{-1}\omega)}}{\det(I + \sigma u)^p}, \quad u \in \mathcal{S}_d^+.
\]

Here \( I \) denotes the unit \( d \times d \) matrix, \( ab \) denotes the matrix product of matrices \( a, b \) and \( a^{-1} \) is the inverse of a non-degenerate matrix \( a \).

It is well known that chi-square distributions and gamma distributions exist for all shape parameters \( p \geq 0 \). It therefore may be conjectured that the same holds true in dimensions \( d \geq 2 \). However, this is not the case. A number of authors from different scientific communities (see the references given in [22]) proved that for invertible \( \sigma \), the
central Wishart distributions $\Gamma(p; \sigma) := \Gamma(p, \omega = 0; \sigma)$ exist if and only if $p$ belongs to the Gindikin ensemble, which equals the set

$$\Lambda_d := \{0, 1/2, \ldots, (d-1)/2\} \cup (d - 1/2, \infty).$$

In other words, the right side of eq. (1.1) is the Laplace transform of a distribution on $S_d^+$ if and only if $p \in \Lambda_d$.

Note that for $d \geq 3$ the set consists of a discrete part and a continuous part. Of course, $\Gamma(p = 0; \sigma)$ is trivial (the unit mass at the origin). Some authors therefore exclude 0 from the Gindikin set. The non-central case $\omega \neq 0$ is more complicated, but also more interesting. Peddada and Richards [21] show by using technically complicated but elementary arguments involving special functions that if $\text{rank}(\omega) = 1$, then $p \in \Lambda_d$. The general case (arbitrary rank) has been understood completely very recently. While [16, Proposition 2.3] conjectures the same characterization holds for non-central Wishart distributions, the author of the present note has shown in [19] that for $p < d - 1/2$, also the parameter of non-centrality must be of lower rank, namely $\text{rank}(\omega) \leq 2p + 1$ (see Theorem 6.1 below). Furthermore, a preliminary version of that paper, [18], conjectured that in this case $\text{rank}(\omega) \leq 2p$ (subsequently it turned out that the method I use only implies the weaker rank condition $\text{rank}(\omega) \leq 2p + 1$). Very recently, Letac and Massam [17] confirm my conjecture, while they falsify theirs (see Theorem 7.5).

There is a dynamic way to generate noncentral chi-square distributions. Namely, by taking a $k$–dimensional standard Brownian motion $(B^1_t, \ldots, B^k_t)^\top$ and some initial value $y = (y_i)_{i=1}^k \in \mathbb{R}^k$, we see that the non-negative stochastic process $X_t := (y + \sqrt{\Sigma}B_t)^\top(y + \sqrt{\Sigma}B_t)$ is distributed according to $\Gamma(k/2, x; 2\Sigma)$, with initial value $X_0 = x = y^\top y \geq 0$. This follows from the fact that $y_i + \sqrt{\Sigma}B_t$ are independent, normally distributed random variables with mean $y_i$ and variance $\Sigma t$. Processes constructed this way are termed “Square Bessel Processes”, and it can be shown that they are well defined also for any non-negative parameter $p \geq 0$. For $\Sigma = 1$ and $\delta = 2p$, Pitman and Yor [26] denote this class as $W^\delta_x$.

The matrix-variate generalization of these Square Bessel Processes are the so-called Wishart processes introduced by Bru [1]. Their crucial feature is the affine property: Their Laplace transform is exponentially affine in the initial state, $X_0 = x$. A modern way of looking at Wishart processes is by considering them as a subclass of affine processes on positive semi-definite matrices or subsets thereof, while the traditional way originating from Bru and followed by others is of solving certain stochastic differential equations (in these notes they will be called Wishart SDEs). These lecture notes try to explain the connections between the two viewpoints. See also section 2.

We also discuss the existence of Lebesgue densities for Wishart distributions as well as the existence of transition densities for Wishart processes. Final remarks are on the existence of Wishart processes on state-spaces different from the positive semi-definite matrices.
Wishart semimartingales, Wishart distributions and Wishart processes

In this section we introduce and comment on the three main objects of this article: Wishart semimartingale, their marginal distributions, which are Wishart distributions, and the Wishart processes, which in these notes are Markov processes having so-called Wishart transition laws. We will show that Wishart semimartingales can be realized as solutions to Wishart SDEs, and that Wishart processes are actually Wishart semimartingales.

**Wishart semimartingales.**

**Definition 2.1.** Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a filtered probability space satisfying the usual conditions. Let \(p \geq 0, \beta\) be a \(d \times d\) matrix and let further \(\alpha \in \mathbb{S}^+_d\). A continuous semi-martingale \(X_t\) is called Wishart semimartingale with parameters \((\alpha, p, \beta)\), if we can write \(X_t = x + D_t + M_t\), where \(X_0 = x, D_t = \int_0^t (2p\alpha + \beta X_s + X_s\beta^\top)ds\) and \(M_t\) is a local martingale with quadratic variation

\[
[M_{t,ij}, M_{t,kl}] = \int_0^t ((X_s)_{ik}\alpha_{jl} + (X_s)_{il}\alpha_{jk} + (X_s)_{jk}\alpha_{il} + (X_s)_{jl}\alpha_{ik}) \, ds,
\]

(2.1)

It follows immediately that \(M_t\) is continuous with \(M_0 = 0\) a.s., and \(D_0 = 0\) a.s., as well.

An important class of Wishart semimartingales are those obtained by certain squares of Ornstein-Uhlenbeck processes. These correspond to drift parameters \(2p \in \mathbb{N}\):

**Example 2.2.** Let \(p \in \mathbb{N}/2, m := 2p, \alpha \in \mathbb{S}^+_d\) and \(\beta\) a \(d \times d\) matrix. Choose a \(d \times d\) matrix \(Q\) for which \(Q^\top Q = \alpha\) (there are, in general, arbitrary many choices for \(Q\)). For \(i = 1,\ldots,m\) we define

\[
Y_{i,t} := e^{\beta t} \left( y_i + \int_0^t e^{-\beta s} Q^\top dW^i_s \right), \quad t \geq 0,
\]

where \(W = (W^1,\ldots,W^m)\) is a \(d \times m\) standard Brownian motion, and \(W^i\) is an \(d\)-dimensional column vector of standard Brownian motions, and \(y_i \in \mathbb{R}^d\).

Then \(Y_i\) is a Gaussian process for every \(i = 1,\ldots,m\). In fact, \(Y_i\) is an OU-process solving the stochastic differential equation

\[
dY_{i,t} = \beta Y_{i,t} + Q^\top dW^i_t, \quad Y_{i,0} = y_i.
\]

We define the continuous semimartingale \(X_t := \sum_{i=1}^m Y_{i,t}Y_{i,t}^\top\). Then \(X_t\) starts at \(X_0 = \sum_{i=1}^m y_iy_i^\top\), and we have that

\[
dX_t = (2pQ^\top Q + \beta X_t + X_t\beta^\top)dt + dM_t,
\]

where \(M_t\) consists of Brownian terms only. A straightforward calculation yields that \(M_t\) has quadratic variation \((2.1)\), where we have to define \(\alpha = Q^\top Q\). Hence \(X_t\) is a Wishart semimartingale with parameters \((\alpha, p, \beta)\).

**Example 2.3.** The following is a particular case of the preceding example \((Q = I, \beta = 0)\), but written in matrix form. Let \(W\) be a \(d \times n\) matrix valued Brownian motion, where \(n \geq d\). That is, the entries of \(W\) consist of \(d \times n\) independent standard Brownian motions.
Let further $x = yy^\top$. Then, as can be calculated using Itô-calculus, the process $X_t := (y + W)(y + W)^\top$ is an $S_d^+$-valued Wishart semimartingale with $2p = n$, $\alpha = I$, $\beta = 0$, starting at $X_0 = x$.

The following note aims at those readers, who are already accustomed to Wishart processes in the sense of Bru:

**Remark 2.4.** It is not so trivial to write $X_t$ as solution of a Wishart SDEs, which are defined below in equation eq. (2.2). The main technical problem is to derive from $W$ and $Y$ a new Brownian motion $B$, for which $X$ satisfies the stochastic differential equation (2.2). For $2p \geq d + 1$, Pfaffel [25, Theorem 4.19] succeeds by using Lévy’s characterization of Brownian motion. For $2p < d + 1$ one can show this by an appropriate enlargement of the underlying probability space. See statement and proof of Lemma 2.5. This technical problem supports our decision to introduce the notion of Wishart semimartingale through these notes. A further and independent motivation is coming from the recent affine processes literature, where the notion of affine semimartingale appears [15]. In our case, the affine character of Wishart semimartingales is reflected by the instantaneous drift $dD_t/dt$ and the instantaneous quadratic variation $dM_t/dt$, which are both affine function in the state $X_t$. The second and important affine character of Wishart semimartingales is constituted by their exponentially affine Laplace transform, see Lemma 2.9 below.

**Wishart semimartingales are solutions to Wishart SDEs.** We now relate this class of semimartingales to solutions of certain stochastic differential equations (SDEs).

Let $\sqrt{A}$ denote the unique square root of $A \in S_d^+$, let $Q, \beta$ be real valued $d \times d$ matrices and $p \geq 0$. As **Wishart SDE** we define the stochastic differential equation

$$dX_t = \sqrt{X_t} dB_t Q + Q^\top dB_t^\top \sqrt{X_t} + (2p Q^\top Q + \beta X_t + X_t \beta^\top)dt, \quad X_0 = x \in S^+_d, (2.2)$$

where $B$ is a standard $d \times d$ Brownian motion.

We understand any solution of (2.2) as **weak solution**, which means that for given $Q$, $\beta$ and $p$, there exists a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which supports a pair of $\mathcal{F}_t$ adapted processes $(X, B)$, which satisfy the Itô-integral equation (2.2) (which, as is common, is written in differentials). However, if the probability space as well as $B$ are given in advance, then any solution of (2.2) is called a **strong** one. While every strong solution yields a weak solution, the converse does not hold in general. An example of a stochastic differential equation (SDE) which admits weak\(^1\) but not strong solution is Tanaka’s one-dimensional equation [20, Example 5.3.2]

$$dY_t = \operatorname{sgn}(Y_t) dW_t, \quad Y_0 = 0.$$\

While the symmetrization in equation (2.2) is necessary to guarantee a stochastic evolution on the space of symmetric $d \times d$ matrices, the existence of solutions for (2.2) is far from trivial. In fact, it is not quite straightforward to ensure that for positive semidefinite

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\(^1\)Let $X = B$ be a standard Brownian motion, then $dW_t = \operatorname{sgn}(B_t) dB_t$ is a new Brownian motion in virtue of Lévy’s characterization of Brownian motion, and we have $dX_t = \operatorname{sgn}(X_t) dW_t$ and clearly $X_0 = B_0 = 0$ because $B$ is standard.
starting values $X_0 = x \in \mathcal{S}_d^+$, a local solution exists: If we assume $x \in S_d^+$, then a solution exists at least for an almost surely strictly positive stopping time $T_x$. This is the first hitting time of the boundary, for $X$. The solution is a strong one, and its existence follows from the fact that the matrix square root is locally Lipschitz on $S_d^+$. However, in general, the interval $[0, T_x]$ is purely stochastic, i.e., we might have $\inf_{\omega \in \Omega} \{ T_x(\omega) \mid T_x(\omega) > 0 \} = 0$. This definitely happens when $p < \frac{d-1}{2}$ and under the premise that $Q$ is non-degenerate, see Lemma 3.4 and Lemma 7.4. For $p \geq \frac{d+1}{2}$, it has been shown in [23] that $T_x = \infty$ almost surely. In our terminology this means in particular that for each $(\alpha, \beta, p)$ with $p \geq \frac{d-1}{2}$ a Wishart semimartingale exists. When $p \in \left(\frac{d-1}{2}, \frac{d+1}{2}\right)$ and in special cases, particularly when $\beta = 0$ and $Q$ is invertible, Graczyk and Malecki [12] provide the existence of strong solutions with different methods than this note (see also the first part of these lecture notes written by P. Graczyk).

We allow for starting values $x \in \mathcal{S}_d^+$, hence in particular we allow that the process $X_t$ starts at the boundary $\partial \mathcal{S}_d^+$ of $\mathcal{S}_d^+$ which are precisely the positive semidefinite matrices with rank strictly smaller than $d$. As the square root is not Lipschitz on $\partial \mathcal{S}_d^+$, standard existence results do not apply. We will see however in a moment, how to infer weak solutions from the existence of Wishart semimartingales:

**Lemma 2.5.** Any solution of the Wishart SDE (2.2) is a Wishart semimartingale. Conversely, suppose $X_t$ is a Wishart semimartingale. Then there exists an enlargement of $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which supports a $d \times d$ Brownian motion and some $d \times d$ matrix $Q$ for which $Q^\top Q = \alpha$ such that $X_t$ is a solution of the Wishart SDE (2.2).

**Proof.** Clearly, the solution of the Wishart SDE is an Itô-process with instantaneous drift $b(X_t) = (2pQ^\top Q + \beta X_t + X_t\beta^\top)$. By definition, $X_t$ is the sum of a local martingale (the Brownian terms) and a process of finite variation (the integrated drift) plus initial value, $X_t = x + D_t + M_t$. Furthermore, writing out the Brownian terms of (2.2) in coordinates (and using Einstein’s summation convention, where summation is performed over all indices which appear twice), we have

$$dM_{t,ij} = (\sqrt{X_t})_{ir} dB_{t,rs} Q_{sj} + Q_{ri} dB_{t,sr} (\sqrt{X_t})_{sj}.$$

Hence, using the formal rules $d[B_{t,ab}, B_{t,cd}] = 0$ if $(a, b) \neq (c, d)$ and $d[B_{t,ab}, B_{t,cd}] = dt$ if $(a, b) = (c, d)$, we have

$$d[X_{t,ij}, X_{t,kl}] = d[M_{t,ij}, M_{t,kl}] = ((X_t)_{ik} \alpha_{jl} + (X_t)_{il} \alpha_{jk} + (X_t)_{jk} \alpha_{il} + (X_t)_{jl} \alpha_{ik}) dt,$$

where we have set $\alpha = Q^\top Q$. Hence we see that $X_t$ is a Wishart semimartingale.

The converse direction is proved in full generality in [1]. To avoid technicalities (which only arise in view of the multivariate character of the problem), and to see the essence of the problem, we just consider the case $d = 1$ here. This is also in some way a prelude foreplay for what is demonstrated in more generality in section 3.

We have $X_t = X_0 + D_t + M_t$, where $X_0 = x$, $dD_t = (b + \beta X_t) dt$, $b \geq 0$ and $d[M, M]_t = \sigma^2 X_t dt$. 


Suppose first $x > 0$, and $b \geq \sigma^2/2$. Using Itô-calculus we see that $Y_t = \log(e^{-\beta t}X_t)$ satisfies
\[
dY_t = -\beta dt + \frac{1}{X_t}(dX_t + dM_t) - \frac{1}{2X_t^2}\sigma^2 X_t dt = \frac{b - \sigma^2/2}{X_t} dt + dM_t/X_t,
\]
which equals the differentials of a non-negative process plus a continuous local martingale. If $X_t$ would hit zero in finite time, then $Y_t$ would go to $-\infty$ in finite time. Because the first summand above is non-negative, this carries over to the second one. But $\int_0^t X_s^{-1}dM_s$ is actually just a time changed Brownian motion, hence oscillates infinitely often (and a.s.) between $-\infty$ and $+\infty$. It can not go to $-\infty$ in finite time! So we see that $X_t$ is strictly positive a.s., and for all $t \geq 0$. Now we can invert $X_t$, and therefore the process $B_t$ defined by
\[
 dB_t := \frac{dX_t - dD_t}{\sigma \sqrt{X_s}} = \frac{dX_t - (b + \beta X_t) dt}{\sigma \sqrt{X_t}} = \frac{dM_t}{\sigma \sqrt{X_t}}, \quad B_0 := 0,
\]
is a well defined continuous local martingale and by construction, $[B_t, B_t] = t$ a.s., for all $t \geq 0$. Lévy’s continuity theorem applies and yields that $B_t$ is a standard Brownian motion. Rewriting the definition of $B_t$ yields that $X_t$ is a solution of the Wishart SDE
\[
 dX_t = (b + \beta X_t) dt + \sigma \sqrt{X_t} dB_t, \quad X_0 = x.
\]
In the general case (where $X_t$ may hit zero in finite time, or even start there), one must in general enlarge the probability space to obtain $X_t$ as solution of a corresponding Wishart SDE. To this end we use the arguments of [27, Theorem V.20.1], which are much simpler in the case $d = 1$. Let $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{P})$ be an enlargement of the current probability space which supports a standard Brownian motion $W$ independent of $X$. We define the process
\[
 \tilde{B}_t := \int_0^t \theta_s dM_s + \int_0^t \rho_s dW_s,
\]
where $\theta$ and $\rho$ are the predictable processes
\[
 \theta_t := \frac{1}{\sigma \sqrt{X_s}} 1_{X_s > 0}, \quad \rho_t := 1_{X_t = 0}.
\]
Then by construction $[\tilde{B}, \tilde{B}]_t = t$ and $\tilde{B}$ is a continuous local martingale starting at 0. Hence, by Lévy’s characterization, it is standard Brownian motion. Furthermore
\[
 dX_t = dM_t + dD_t = \sigma \sqrt{X_t} d\tilde{B}_t + (b + \beta X_t) dt,
\]
which is just the Wishart SDE in the one-dimensional situation. In the multivariate case, $\theta$ and $\rho$ are vectors, whose construction is due to [27, Lemma V.20.7].

One can show with very little effort that $M$ is an $L^2$ martingale, for instance by using the fact that $X_t$ is Wishart distributed (see Lemma [27]), since the Wishart distribution exhibits exponential moments.
**Wishart semimartingales are Wishart distributed.** First, we define the family of Wishart distributions, which is motivated by the derivation of (1.1):

**Definition 2.6.** We define the non-central Wishart distribution $\Gamma(p, \omega; \sigma)$ on the space of symmetric $d \times d$ matrices $S_d$—whenever it exists—by its Laplace transform

$$
\mathcal{L}(\Gamma(p, \omega; \sigma))(u) = (\det(I + \sigma u))^{-p} e^{-\tr(\omega(I + \sigma u)^{-1} \omega)}, \quad u \in S_d^+,
$$

where $p \geq 0$ denotes its shape parameter, $\sigma \in S_d^+$ is the scale parameter and the parameter of non-centrality equals $\omega \in S_d^+$.

**Lemma 2.7.** Any Wishart distribution $\Gamma(p, \omega; \sigma)$ is supported on $S_d^+$.

**Proof.** It suffices to show that for any $v \in \mathbb{R}^d$, we have that the push forward $\Pi_*$ of $\Gamma(p, \omega; \sigma)$ under the map $\Pi : S_d \to \mathbb{R}$, $x \mapsto v^\top x v$ is supported on $\mathbb{R}_+$. This, in turn, follows from the fact that $\Pi_*(\Gamma(p, \omega; \sigma)(d\xi))$ is non-centrally gamma distributed: By Proposition 6.2 (i), we may assume $\sigma = 2I$ without loss of generality. In the following we use $\lambda$ as the Laplace variable, and we let $U$ be an orthogonal matrix and $\mu \geq 0$ such that $v = \mu U e_1$, where $e_1$ is the first canonical basis vector of $\mathbb{R}^d$. Accordingly, $\omega' = U^\top \omega U$. The Laplace transform of $\Pi_*(\Gamma(p, \omega; \sigma))(d\eta)$ equals

$$
det(I + 2\lambda v u^\top) - p e^{-\tr(\lambda v u^\top + 2\lambda v u^\top - \lambda \omega u^\top \omega^{-1} \omega')} = (1 + 2\lambda \mu^2)^{-2p} p e^{-\lambda \mu^2 (1 + 2\lambda \mu^2)^{-1} \omega'},
$$

which is the Laplace transform of $\mu^2 X$, where $X$ is a non-central chi-square distributed random variable with shape parameter $2p$ and parameter of non-centrality $w'_{11}$. □

Suppose $\beta$ is again a $d \times d$ matrix with real entries, and let $\alpha \in S_d^+$. In the following we denote by $\omega^\beta_t$ the flow of the vector field $\beta x + x \beta^\top$, that is,

$$
\omega^\beta : \mathbb{R} \times S_d^+ \to S_d^+, \quad \omega^\beta_t(x) := e^{\beta_t x} e^{\beta^\top_t}. 
$$

Its twofold integral is denoted by

$$
\sigma^\beta : \mathbb{R}^+ \times S_d^+ \to S_d^+, \quad \sigma^\beta_t(y) = 2 \int_0^t \omega^\beta_s(y) ds.
$$

Using these two functions, we define a curve $\phi(t, \cdot)$ and a matrix-valued curve $\psi(t, \cdot)$

$$
\dot{\phi}(t, u) = 2p \tr(\alpha \psi(t, u)), \quad \phi(0, u) = 0, \quad \dot{\psi}(t, u) = -2u \alpha u + \psi(t, u) \beta + \beta^\top \psi(t, u), \quad \psi(0, u) = u.
$$

We show now the elementary fact:

**Proposition 2.8.** $\phi$ and $\psi$ satisfy a system of generalized Riccati equations, namely,

$$
\dot{\phi}(t, u) = 2p \tr(\alpha \psi(t, u)), \quad \phi(0, u) = 0,
$$

$$
\dot{\psi}(t, u) = -2u \alpha u + \psi(t, u) \beta + \beta^\top \psi(t, u), \quad \psi(0, u) = u.
$$
Proof. In order to obtain the generalized Riccati equations (2.6)–(2.7), we differentiate the formula (2.4) for \( \psi \) by using the fact that for any differentiable matrix-valued curve \( t \mapsto a(t) \) we have \( \frac{d}{dt} a^{-1}(t) = -a^{-1}(t) \frac{da(t)}{dt} a^{-1}(t) \), see for instance [9, Proposition III.4.2 (iii)]. Formula (2.4) is obtained by using the rule \( \frac{d}{dt} \log(\det(a(t))) = \text{tr}(a^{-1}(t) \frac{da(t)}{dt}) \), see [9, Proposition II.3.3 (i)]. □

The following is proved in [1], but with different notation, and for solutions to Wishart SDEs. The statement, however, is in fact a result concerning the law of Wishart semimartingales:

**Lemma 2.9.** Suppose \( X_t \) is a Wishart semimartingale with parameters \( (\alpha, p, \beta) \) starting at \( X_0 = x \). Then for each \( t \geq 0 \), \( X_t \sim \Gamma(p, \omega_\alpha(x); \sigma_\beta^2(\alpha)) \).

**Proof.** Let \( t > 0 \) and \( u \in \mathbb{S}_d^+ \). Let \( (\phi, \psi) \) be the functions defined by eqs. (2.4)–(2.5). Applying the Itô-formula to the process

\[
J_s := e^{-\phi(t-s,u) - \text{tr}(\psi(t-s,u)X_s)}, \quad 0 \leq s \leq t,
\]

and using thereby Proposition 2.8 we obtain

\[
\frac{dJ_s}{J_s} = \left( \partial_t \phi(t-s,u) + \text{tr}(\partial_s \psi(t-s,u)X_s) \right) ds - \text{tr}(\psi(t-s,u)((\beta X_s + X_s \beta^\top + 2pQ^\top Q)ds + dM_s))
\]

\[
\quad + \frac{1}{2} \text{tr}(\psi(t-s,u)\alpha \psi(t-s,u)X_s)
\]

\[
= \left( \partial_s \phi(t-s,u) - 2p \text{tr}(Q^\top Q \psi(t-s,u)) \right) ds - \text{tr}(\psi(t-s,u)dM_s)
\]

\[
\quad + \text{tr}(X_s \partial_s \psi(t-s,u) + 2\psi(t-s,u)\alpha \psi(t-s,u) - \psi(t-s,u)\beta + \beta^\top \psi(t-s,u))
\]

\[
= - \text{tr}(\psi(t-s,u)dM_s),
\]

where the first two brackets vanish because of equations (2.6)–(2.7). We conclude that \( (J_s)_s \) is a local martingale on \([0, t]\). Furthermore, since \( \phi(t,u) \geq 0 \) for all \( t \geq 0 \) and \( \psi(t,u) \in \mathbb{S}_d^+ \) for all \( t \geq 0 \), we have that \( J \) is uniformly bounded on \([0, t]\). Hence \( J \) is a true martingale, and therefore

\[
E[e^{-uX_t} \mid X_0 = x] = E[J_t \mid X_0 = x] = J(0) = e^{-\phi(t,u) - \text{tr}(\psi(t,u)x)},
\]

where we have used that \( J_t = e^{-\text{tr}(\psi(t,X_t))} \) (which follows from \( \phi(0,u) = 0 \) and \( \psi(0,u) = u \)). The assertion concerning the distribution of \( X_t \) now follows from the explicit formulas (2.6)–(2.7) and the very definition of the Wishart distribution (2.3).

For the derivation of the exponentially affine characteristic function on general state-spaces, see the proof of [10, Theorem 2.2], which uses similar arguments. □

**Wishart processes from the Markovian viewpoint.**

**Definition 2.10.** A family of distributions \( \{p_t(x,d\xi) \mid t \geq 0, x \in \mathbb{S}_d^+\} \) which is noncentrally Wishart distributed according to

\[
p_t(x,d\xi) = \Gamma(p, \omega_\alpha^\beta(x); \sigma_\beta^2(\alpha))(d\xi) \tag{2.8}
\]
is termed Wishart transition function with constant drift parameter \( p \geq 0 \), linear drift parameter \( \beta \) and diffusion coefficient \( \alpha \in \mathbb{S}^+_d \).

By using the Laplace transform of the Wishart distribution, we obtain that the Laplace transform of the laws \( p_t(x, d\xi) \) is given by

\[
\int_{\mathbb{S}^+_d} e^{-\text{tr}(u\xi)} p_t(x, d\xi) = \left( \det(I + \sigma_t^\beta(\alpha)u) \right)^{-p} e^{-\text{tr}(u(I+\sigma_t^\beta(\alpha))^{-1}\omega_t^\beta(x))} = e^{-\phi(t, u) - \text{tr}(\psi(t, u)x)}, \quad u \in \mathbb{S}^+_d,
\]

where \((\phi, \psi)\) are of the same form as in (2.4)–(2.5).

We start with the following observation.

**Lemma 2.11.** Any Wishart transition function is a Markovian transition function supported on \( \mathbb{S}^+_d \). The associated Markovian semigroup \((P_t)_{t \geq 0}\) defined by

\[
f \mapsto P_t f(x) := \int_{\mathbb{S}^+_d} f(\xi) p_t(x, d\xi)
\]

is a Feller semigroup, that is, \(P_t\) reduces to a strongly continuous contraction semigroup acting on \( C_0(\mathbb{S}^+_d) \), the continuous functions on \( \mathbb{S}^+_d \) vanishing at infinity.

We use the terminology **Wishart process** for Markov processes with Wishart transition function.

**Proof.** First, by Lemma 2.7 we know that for all \( t \geq 0, x \in \mathbb{S}^+_d \), the laws \( p_t(x, d\xi) \) are supported on \( \mathbb{S}^+_d \). For any Borel set \( B \), measurability of \( p_t(x, B) \) in \((t, x)\) holds by construction (and in view of the continuity of the maps \( \sigma_t^\beta(\alpha), \omega_t^\beta(x) \) in \((t, x)\)) . So the family \((P_t)_t\) of linear maps defined by (2.11) is well defined on \( B_0(\mathbb{S}^+_d) \), the set of bounded, Borel measurable functions on \( \mathbb{S}^+_d \). We only need to show that it gives rise to a semigroup on \( B_0(\mathbb{S}^+_d) \).

Since the linear hull of the family of exponentials \( \{f_u(\xi) := \exp(-\text{tr}(u\xi)) \mid u \in \mathbb{S}^+_d\} \) is dense in the space of continuous functions vanishing at infinity (and therefore ultimately in \( B(\mathbb{S}^+_d) \)), it suffices to show the semigroup property for the exponential functions \( \xi \mapsto f_u(\xi), u \in \mathbb{S}^+_d \). Now, since by Proposition 2.8 we have that \((\phi, \psi)\) are the unique solutions to a system of ordinary differential equations, it follows (from their specific form) that they satisfy the so-called semiflow equations

\[
\phi(t + s, u) = \phi(t, u) + \phi(s, \psi(t, u)) \quad \psi(t + s, u) = \psi(s, \psi(t, u)).
\]
Lemma 2.13. The generator of a Wishart process.

Hence we can write

\[ P_{t+s} f_u(x) = \int_{\mathbb{S}_d^+} f_u(\xi) p_{t+s}(x, d\xi) = e^{-\phi(t+s,u) - \text{tr}(\psi(t+s,u)x)} \]

\[ = e^{-\phi(t,u)} e^{-\phi(s,\psi(t,u)) - \text{tr}(\psi(s,\psi(t,u))x)} = e^{-\phi(t,u)} \int_{\mathbb{S}_d^+} f_{\psi(t,u)}(\eta) p_s(x, d\eta) \]

\[ = \int_{\mathbb{S}_d^+} e^{-\phi(t,u) - \text{tr}(\psi(t,u),\eta)} p_s(x, d\eta) = P_s(P_t f_u(x)). \]

It remains to prove the Feller property. By [28, Proposition III.2.4] and using some density argument, it suffices to show that

- \( P_t f_u(x) \in C_0(\mathbb{S}_d^+) \) for all \( t \geq 0 \), and \( u \in \mathbb{S}_d^+ \) and this can be seen by inspection of \( \psi(t, u) \), which is strictly positive definite, as well.
- \( P_t f_u(x) \) converges pointwise to \( f_u(x) \) as \( t \to 0 \), which follows immediately from the continuity of \( \phi(t, u) \) and \( \psi(t, u) \) in \( t \).

\[ \square \]

A Markov process with transition laws \( p_t(x, d\xi) \) on \( \mathbb{S}_d^+ \) is called affine [4], if eq. (2.10) holds. Hence it is obvious that

**Lemma 2.12.** Wishart processes are affine processes.

**The generator of a Wishart process.**

**Lemma 2.13.** Let \( X \) be a Wishart process on \( \mathbb{S}_d^+ \) with admissible parameters \((p, \beta, \alpha)\). Then the associated semigroup \((P_t)_{t \geq 0}\) has infinitesimal generator \( \mathcal{A} \) acting on \( C_0^\infty \subset \mathcal{D}(\mathcal{A}) \) as

\[ \mathcal{A} f(x) = \frac{1}{2} \sum_{1 \leq i,j,k,l \leq d} A_{ijkl}(x) \frac{\partial^2 f(x)}{\partial x_{ij} \partial x_{kl}} + \text{tr}((\beta x + x \beta^\top + 2pQ^\top Q)\nabla f(x)), \quad (2.12) \]

where \( \nabla f(x) = (f_x(x))_{ij} \) and \( A_{ijkl}(x) = (x_{ik} \alpha_{jl} + x_{il} \alpha_{jk} + x_{jk} \alpha_{il} + x_{jl} \alpha_{ik}) \).

There are different possible proofs of this fact. By using the fact that \( X \) can be realized as solution of a corresponding Wishart SDE \( X_t \) starting at \( X_0 = x \), one could just determine the generator of \( X \) by applying the It\'o-formula or using general results on It\'o-diffusions. Another, maybe more elegant way is the following. By the very definitions of the Wishart process, we can calculate the pointwise limit

\[ \lim_{t \downarrow 0} \frac{P_t f_u(x) - f_u(x)}{t} = f_u(x) \lim_{t \downarrow 0} e^{-\phi(t,u) - \text{tr}(\psi(t,u)x)} - e^{-\phi(t,u)} \frac{-\phi(t,u) - \text{tr}(\psi(t,u)x)}{t} = f_u(x) \left( \frac{\partial \phi(t,u)}{\partial t} - \text{tr}(\partial_t \psi(t,u)x) \right)_{|t=0} = -f_u(x) \left( 2p \text{tr}(Q^\top Q u - \text{tr}(-(2u \alpha u + u \beta + \beta^\top u)x)) \right). \quad (2.13) \]
The convergence actually holds in sup-norm; this essentially follows from the fact that the pointwise limit lies in \( C^0_b(S^+_d) \) (see [4, Proof of Proposition 4.12]). Furthermore, a density argument proves that elements of \( C^\infty_b \) can be suitably approximated by the linear hull of exponentials \( f_u(x), u \in S^+_d \). On the other hand it is readily checked that (2.12) evaluated at \( f_u(x), u \in S^+_d \) equals eq. (2.13).

### The drift condition.

**Theorem 2.14.** Let \( X \) be a Wishart process on \( S^+_d \) with parameters \( (p, \beta, \alpha) \), and suppose \( \alpha \neq 0 \). Then we must have \( p \geq \frac{d-1}{2} \).

**Proof.** Any positive Feller semigroup has an infinitesimal generator \( A \) which satisfies the strong maximum principle. That is, let \( f \in \mathcal{D}(A) \) and \( f(x) \geq f(x_0) \) for all \( x \in S^+_d \). Then \( A f(x_0) \leq 0 \). (here the following analogy from calculus helps to remember the sign: Let \( g \) be a twice differentiable function on an interval \( I \subseteq \mathbb{R} \) which has a local maximum at \( x_0 \). Then \( f''(x_0) \leq 0 \). If, in addition, \( x_0 \) lies in the interior of \( I \), then \( f'(x_0) = 0 \). The analogy comes from the fact that the generator of a Feller semigroup has a principal symbol which is differential operator of second order). In [4] we used the determinant \( f(x) = \det(x) \) and (diagonal) boundary points \( x_0 \in \partial S^+_d \), because \( f \) vanishes precisely there. The theory of [4] is more general than these notes, so it is enough to use [4, Lemma 4.16 and Lemma 4.17] to prove the assertion. \( \square \)

**Remark 2.15.** Note that when \( \alpha = 0 \) then we have a deterministic motion, because then we have that
\[ \mathbb{E}[e^{-\text{tr}(uX_t)} \mid X_0 = x] = e^{-\text{tr}(u\omega_t(x))}, \]
i.e., \( X_t = \omega_t^\beta(x) \). From the Wishart SDE point of view, we clearly have
\[ \dot{X}(t) = \beta X + X\beta^\top, \quad X_0 = x. \]
In that case, \( p \) can be anything but is superfluous.

**Wishart processes are Wishart semimartingales.** So far we did not need to be specific about the realization of Wishart processes as stochastic processes; we only looked at the Markovian transition function. In order to relate Wishart processes and Wishart semimartingales, we consider for each initial state \( x \), an associated (to the Wishart transition function) Markov process \( X \) on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x) \). Such realizations exist and are well known. We repeat in the following a little the definitions for Markov transition functions and a canonical construction of the associated stochastic process, which is then called Markov. In the end of the section we prove that every Wishart process is a Wishart semimartingale.

A (suitably measurable) family of probability laws \( t \mapsto (p_t(x, d\xi)) \) on \( S^+_d \), indexed by \( t \geq 0, x \in S^+_d \), is called Markovian transition function, if \( p_0(x, d\xi) = \delta_x(d\xi) \) (the unit mass
\[ ^2 \text{In [4] the rapidly decreasing smooth functions } \mathcal{S}(S^+_d) \text{ are used. For the corresponding Stone-Weierstrass Theorem, see [4, Theorem B.3]} \]
at \( x \) and it satisfies the Chapman-Kolmogorov equations
\[
p_{t+s}(x, A) = \int p_s(\xi, A)p_t(x, d\xi), \quad s, t \geq 0.
\] (2.14)
Note that using the function \( f_A(x) = 1_A(x) \) (the indicator function on the Borel set \( A \)), we can write (2.14) equivalently in semigroup form
\[
P_{t+s}f_A(x) = P_t(P_s f_A)(x),
\]
where the action \( P_t \) is defined above in eq. (2.11). We have therefore shown the Chapman-Kolmogorov equations for the continuous functions \( f_u \) in Lemma 2.11 and that’s enough by some density argument to ensure (2.14).

Now by [8, Theorem 1.1], for any initial distribution \( \nu(d\xi) \) on \( S^+_d \) there exists a stochastic process \( X \) on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) whose finite-dimensional distributions fulfil
\[
\mathbb{P}[X(0) \in A_0, X(t_1) \in A_1, \ldots, X(t_n) \in A_n] = \int_{A_0} \int_{A_1} \cdots \int_{A_n} p_{t_{n-1}t_{n-2}}(y_{n-1}, dy_{n-1}) p_{t_{n-2}t_{n-3}}(y_{n-2}, dy_{n-2}) \cdots p_{t_1t_0}(y_0, dy_0) \nu(dy_0)
\]
This construction is “canonical” in that \( \Omega = (S^+_d)^{[0,\infty)} \) (i.e. the space of all possible paths with values in \( S^+_d \)), the process is just given by the projections onto the \( t \)-th coordinate, that is
\[
X_t(\omega) = \omega(t), \quad \omega \in \Omega
\]
and the sigma algebra is given by the product sigma algebra
\[
\mathcal{F} = \mathcal{B}(S^+_d)^{[0,\infty)}.
\]
The filtration is generated by the projections \( X_t \):
\[
\mathcal{F}_t = \sigma(X_s \mid 0 \leq s \leq t).
\]
Starting at \( \nu(x) = \delta_x(d\xi) \), where \( x \in S^+_d \), we denote the associated probability measure \( \mathbb{P} \) by \( \mathbb{P}^x \). Since \( \Omega, \mathcal{F} \) and \( \mathcal{F}_t \) independent of the initial law, we have constructed a family of stochastic processes \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x) \) which satisfy the Markov property for all bounded Borel measurable functions \( f \),
\[
\mathbb{E}^x[ f(X_{t+s}) \mid \mathcal{F}_s] = \int_{S^+_d} f(\xi)p_t(x, d\xi) = \mathbb{E}^{X_t}[f(X_t)]
\]
which holds \( \mathbb{P}^x \) a.s., and for all \( t, s \geq 0 \). Here we use \( \mathbb{E}^x \) to denote the expectation operator with respect to \( \mathbb{P}^x \).

This is equivalent to the more intuitive statement
\[
\mathbb{P}^x[X_{t+s} \in A \mid \mathcal{F}_t] = \mathbb{P}^{X_t}[X_{t+s} \in A],
\]
By Lemma 2.11 we also know that \( X \) is a Feller process (this is a Markov process with a Feller semigroup), which implies in view of [28, Theorem III. 2.7] that \( X \) admits a cadlag modification. That means for each \( x \in S^+_d \), we have that the probability law \( \mathbb{P}^x \) is actually
concentrated on the space of paths which are continuous from the right and have limits from the left. Our aim is to show that for each \( x \in \mathbb{S}_d^+ \), the process \( X_t \) is a Wishart semimartingale. That is continuous, as we know, which will follow a little in direct:

**Proposition 2.16.** Let \( X \) be a Wishart process. Then for each \( x \), \( X \) is a Wishart semimartingale on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\).

**Proof.** Since \( X \) is a Feller process, we have by [28, Proposition VII. 1.6] that for any \( f_u(x) = \exp(-\text{tr}(ux)) \)

\[
M^u_t := f_u(X_t) - f_u(x) - \int_0^t A f_u(X_s) ds
\]

is an \((\mathcal{F}_t, \mathbb{P})\)-martingale. Hence by [14, Theorem II.2.42] we have that \( X \) is a \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\)-semimartingale, associated to the generator \( A \). The continuity of \( X \) follows from the lack of a jump-component in the generator (that is the compensator of the jumps of \( X \) vanishes). As quadratic variation and drift component are evident from the specific form of the generator, we see that \( X \) is a Wishart semimartingale on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). □

### 3. Boundary non-attainment

Suppose now that \( X_0 = x \) is positive definite in (2.2). In view of the standard existence and uniqueness result for SDEs—the square root is analytic, hence locally Lipschitz on \( \mathbb{S}_d^+ \)—there exists a unique strong solution of the Wishart SDE as long as \( X_t \) does not hit the boundary. We call this time

\[
T_x := \inf\{t > 0 \mid \det(X) = 0\}
\]

the first hitting time of the boundary. Of course when \( T_x = \infty \), unique strong solutions of the Wishart SDE are guaranteed. This is particular the case, when \( p \) is large enough.

**Theorem 3.1.** Suppose \( p \geq \frac{d+1}{2} \). Then \( T_x = \infty \) almost surely.

This is a special case of [23, Corollary 3.2] but written in the notation of [1, chapter 6]. The motivation for this result has been the introductory work of [1] for Wishart processes with \( Q = I \) and \( \beta = 0 \) (for more detailed comparison with Bru’s work, see [23, Proposition 3.1]). It should be noted that this is a result concerning the support of Wishart semimartingales, and the existence of strong solutions is a mere by-product of the latter. For affine jump-diffusions on symmetric cones, the corresponding result is [5, Proposition 6.1].

A random time \( T : \Omega \rightarrow \mathbb{R}_+ \) is a random variable taking non-negative values. \( T \) is called a stopping time, if the sets \( \{T \leq t\} \) are measurable with respect to \( \mathcal{F}_t \). In our context \( T = T_x \) will always be the first hitting time of solutions to Wishart SDEs of the boundary \( \partial \mathbb{S}_d^+ \). \([0, T)\) is called stochastic interval. A local martingale \( M_t \) on the stochastic interval \([0, T)\) is a stochastic process for which there exists an a.s. strictly increasing sequence \( T_n \uparrow T \) such that for each \( n \), the stopped process \( M_{t \wedge T_n} \) is an \( \mathcal{F}_t \)-martingale.
MCKean’s argument. This result on continuous semimartingales is fundamental for the derivation of Theorem 3.1. To simplify the setting, we shall from now on assume that $T > 0$ almost surely. This assumption actually holds for $T_x$, because any diffusion started in the interior of some domain needs a strictly positive time to reach its boundary.

Lemma 3.2. For a continuous local martingale on $[0, T]$ almost surely either $\lim_{t \uparrow T} [M_t, M_t]$ exists or we have $\limsup_{t \uparrow T} M_t = -\liminf_{t \uparrow T} M_t = \infty$.

One way of obtaining this result is by performing a time-change $T_t$ on $A := \{\lim_{t \uparrow T} [M_t, M_t] = \infty\}$ such that $M_{T_t}$ becomes a continuous local martingale on $A$ with quadratic variation $t$. Then by Lévy’s characterization of Brownian motion $M_{T_t}$ is a Brownian motion on $A$, hence we just need to use the pathwise properties of Brownian motion—that a.s. oscillates infinitely often between $-\infty$ and $\infty$, as $t \to \infty$. The appropriate time change is $T_t := \inf\{s > 0 \mid [M_s, M_s] > t\}$.

A stripped-down version of MCKean’s argument is the following. A more general formulation may be found in [23, Proposition 4.3]:

Proposition 3.3. Let $Z$ be a continuous adapted stochastic process on a stochastic interval $[0, T)$ such that $Z_0 > 0$ a.s., and $T := \inf\{t > 0 \mid Z_s = 0\}$. Suppose $h : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}$ satisfies the following

(i) for $t < T$ we have $h(Z_t) = h(Z_0) + M_t + P_t$, where $P$ is a non-negative process and $M$ is a continuous local martingale on $[0, T)$.

(ii) $\lim_{z \downarrow 0} h(z) = -\infty$

Then $T = \infty$ a.s.

Proof. As a consequence of the assumptions $h(Z_t) \downarrow -\infty$ as $t \uparrow T$. Since $P$ is non-negative, we have that $M_t \downarrow -\infty$ as $t \uparrow T$. But $M$ is a continuous local martingale on $[0, T)$. In view of the preceding lemma this is only possible, when $T = \infty$. \qed

Now we shortly sketch the proof of Theorem 3.1. All the details can be found in an old (and unpublished) version of the paper [23] on [24, pp. 5–7]. They base on a few more Lemmas.

Proof. We define for $t \in [0, T_x)$

$$Z_t := \det(e^{-\beta T} X_t e^{-\beta t}), \quad h(z) = \log(z),$$

then after application of Itô’s formula [24, Lemma 4.7] and some lines of calculations we obtain

$$h(Z_t) = h(Z_0) + M_t + P_t,$$

where

$$M_t = 2 \int_0^t \sqrt{\text{tr}(X_s^{-1} Q^T Q)} dW_s$$

and

$$P_t = \int_0^t \text{tr}((2p - (d+1)Q^T Q) X_s^{-1}) ds,$$
where \( W \) is a one-dimensional standard Brownian motion. Hence \( M \) is a continuous local martingale on \([0, T_x]\) and \( P \) is non-negative. Proposition 3.3 can be applied and yields \( T_x = \infty \). \( \square \)

**Hitting the boundary.** The following shows that Theorem 3.1 does not hold under weaker conditions:

**Lemma 3.4.** Let \( \beta, Q \) be \( d \times d \) matrices, and suppose \( Q \neq 0 \). When \( p < \frac{d-1}{2} \), there exists \( x \in S_d^+ \) such that \( T_x < \infty \) with positive probability.

*Proof.* Assume, for a contradiction, that for all \( x \in S_d^+ \) we have that \( T_x = \infty \). By Lemma 2.2 any solution \( X \) of the Wishart SDE is a Wishart semimartingale. And by Lemma 2.9 we have that \( X_t \sim \Gamma(p, \omega^\beta_t(X_t); \sigma^\beta_t(x)) =: p_t(x, d\xi) \) where \( \alpha = Q^\top Q \). By definition \((p_t(x, d\xi))_{t \geq 0, x \in S_d^+}\) is a Wishart process. By Theorem 2.14 we must have \( p \geq \frac{d-1}{2} \), a contradiction. \( \square \)

For a similar and partially stronger result see Lemma 7.4 below.

In the case that \( \beta = 0 \) and \( Q = I \), \([7, \text{Theorem 1.4}]\) asserts that the boundary is hit in finite time, when \( p \in (\frac{d-1}{2}, \frac{d+1}{2}) \). A similar result including general \( \beta \) or \( Q \neq 0 \) seems not to be known yet. However, we conjecture

**Conjecture 3.5.** Let \( \beta, Q \) be arbitrary \( d \times d \) matrices, and let \( p < \frac{d+1}{2} \). Any solution of the Wishart SDE with initial condition \( X_0 = x \in S_d^+ \) hits the boundary in finite time, that is \( \mathbb{P}(T_x < \infty) > 0 \).

We further conjecture

**Conjecture 3.6.** Let \( \beta, Q \) be arbitrary \( d \times d \) matrices, and let \( p \leq \frac{d-1}{2} \). Any solution of the Wishart SDE with initial condition \( X_0 = x \in S_d^+ \) hits the boundary almost surely, that is \( \mathbb{P}(T_x < \infty) = 1 \).

4. **Changing the drift**

Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a filtered probability space satisfying the usual conditions, which supports an \( \mathcal{F}_t \)-Brownian motion. Girsanov transformations are tools to derive solutions to SDEs as follows. We consider for a moment the one-dimensional case. Let \( X_t \) be a solution of

\[
\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x.
\]

Suppose we actually seek to solve such an equation for an alternative drift \( \tilde{b}(\cdot) \). If \( \sigma \) is invertible, we can rewrite the above equation as

\[
\mathrm{d}X_t = \tilde{b}(X_t)\mathrm{d}t + (b(X_t) - \tilde{b}(X_t))\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t = \tilde{b}(X_t)\mathrm{d}t + \sigma(X_t)\left(\frac{b(X_t) - \tilde{b}(X_t)}{\sigma(X_t)}\mathrm{d}t + \mathrm{d}W_t\right).
\]

In the following we abbreviate \( \gamma_t := \frac{b(X_t) - \tilde{b}(X_t)}{\sigma(X_t)} \). If we can show that under a new probability measure \( Q \), the process \( \int_0^t \gamma_s\mathrm{d}s + W_t \) is a Brownian motion, then we have
achieved our goal. Note the weak character of this solution: The Brownian motion is not
given in advance, but constructed from the pair \((X_t, W_t)\).

What we have outlined above is indeed possible; it is a consequence of Girsanov’s theo-
rem, which asserts that if

\[
Z_t := \mathcal{E}\left(-\int_0^t \gamma_s dW_s\right) = e^{-\int_0^t \gamma_s dW_s - \frac{1}{2} \int_0^t |\gamma_s|^2 ds}
\]
is a martingale on \([0, T]\), then \(\mathbb{Q}\) defined as

\[
d\mathbb{Q} = \mathcal{E}\left(-\int_0^T \gamma_s dW_s\right)d\mathbb{P}
\]
is a probability measure equivalent to \(\mathbb{P}\), and \(\int_0^t \gamma_s ds + W_t\) is a \(\mathbb{Q}\)-Brownian motion
on \([0, T]\). The essential problem therefore is to show the martingale proper-
ty of \((Z_t)_{t \leq T}\). That can be quite tricky.

Bru [1] used the Girsanov theorem to derive solutions of Wishart SDEs with nonzero
linear drift \(\beta\) from SDEs with constant drift only. In special cases she derives solutions
until the first time the eigenvalues of the process collide. The respective time of collision is
not dealt with in her work when \(\beta \neq 0\); recent work elaborates on this issue, see Graczyk
and Malecki [12].

We have already shown the existence of solutions in the preceding chapter when
\(\beta \neq 0\) under the more stringent condition \(p \geq \frac{d+1}{2}\). So we do not need the Girsanov theorem
to create new solutions, and we also never had to care about the collision of eigenvalues.
But what we can do is to relate solutions with respect to different drift parameters to each
other:

**Theorem 4.1.** Suppose \(X_t\) is a solution of a Wishart SDE with parameters \((p, \beta, Q)\),
where \(Q\) is invertible, and let \(X_0 = x \in \mathbb{R}_d^+\). For \(p^Q \in \mathbb{R}\) and a \(d \times d\) matrix \(\beta^Q\) we set

\[
\gamma_t := \sqrt{X_t((\beta^T - (\beta^Q)^T)Q^{-1} + (p - p^Q)\sqrt{X_t}^{-1}Q^T} \tag{4.1}
\]

and

\[
Z_t := \mathcal{E}\left(-\int_0^t \text{tr}(\gamma_t dB_t)\right). \tag{4.2}
\]

If \(\min(p, p^Q) \geq \frac{d+1}{2}\), then \(Z_t\) is a martingale on \([0, T]\), and \(dB_t^Q := \gamma_t + B_t\) is a \(\mathbb{Q}\)-Brownian
motion on \([0, T]\). Furthermore \(X_t\) satisfies the Wishart SDE with parameters \((p^Q, \beta^Q, Q)\)
under \(\mathbb{Q}\).

**Remark 4.2.**
- On the level of Wishart semimartingales, the result translates in a
  statement concerning their absolute continuity, see [3].
- The theorem bases on the fact that under the condition \(\min(p, p^Q) \geq \frac{d+1}{2}\) the
  respective Wishart semimartingales do not attain the boundary \(\partial \mathbb{S}_d^+\) in finite time,
  see Theorem [3.1]. Note also: it is impossible to define \(\gamma_t\) unless \(X_t \in \mathbb{S}_d^+\) for all
  \(t \leq T\).
• The best known sufficient criterium for \( E( - \int_0^t \gamma_s dB_s ) \) to be a martingale is provided by Novikov’s condition,

\[
E[ e^{\frac{1}{2} \int_0^T \| \gamma_t \|^2 dt} ] < \infty.
\]

This condition is hard to check in our context. Furthermore, it fails, in general. In fact, for the particular case \( Q = I, \beta = 0, \beta^Q = I, p = p^Q \) we have that

\[
E[ e^{\frac{1}{2} \int_0^T \| \gamma_t \|^2 dt} ] = E[ e^{\frac{1}{2} \int_0^T \tr(X_t) dt} ]
\]

which is infinite for sufficiently large \( T \). To see this we interpret it as the exponential moment of a new stochastic process (a so-called affine process) \( (X_t, Y_t = \tr(X_t)) \) on \( \mathbb{S}_d^+ \times \mathbb{R}_+ \) whose moment generating equals

\[
E[e^{\frac{1}{2} \int_0^T Y_t dt}] = e^{\phi(t) + \tr(x\psi(t))}, \quad (4.3)
\]

where \( \psi(t) \) satisfies the ODE

\[
\partial_t \psi(t) = 2\psi(t)^2 + \frac{I}{2}, \quad \psi(0) = 0.
\]

This is a matrix Riccati differential equation which has explosion in finite time (say at \( t_+ > 0 \), and by the positivity of \( x \) we have that \( \tr(x\psi(t)) \uparrow \infty \) as \( t \uparrow t_+ \). It follows that the moment \( (4.3) \) explodes.

For more information on the technique of enlargement of the state space and calculation of the moment generating function of affine processes, see for instance \[10, Proof of Theorem 4.1\].

Proof. We start with the second part. Under the premise that \( Z \) is a true martingale, the conclusion of Girsanov’s theorem holds and we obtain

\[
dX_t = \sqrt{X_t} dB_t Q + Q^T dB_t^\top \sqrt{X_t} = (2p^Q^T Q + \beta X_t + X_t\beta^\top) dt
\]

\[
= \sqrt{X_t} (\gamma_t dt + dB_t) Q + \gamma_t^T (\gamma_t dt + dB_t) \sqrt{X_t} = (2p^Q^T Q + \beta Q X_t + X_t(\beta^Q)^\top) dt
\]

\[
= \sqrt{X_t} dB_t^Q dQ + Q^T dB_t^Q (\beta^Q)^\top \sqrt{X_t} = (2p^Q^T Q + \beta Q X_t + X_t(\beta^Q)^\top) dt
\]

and therefore \( X_t \) is a solution of the Wishart SDE on \([0, T]\) with new parameters \((p^Q, \beta^Q, Q)\) under the measure \( Q \).

It remains to show that \( Z_t \) is a true martingale. We use the exact arguments as provided by the proof of \[2, Theorem 1\] but adapted to our matrix-valued setting.

Since \( p \geq \frac{d+1}{2} \), we have a well defined positive definite solution \( X_t \) of the Wishart SDE \((4.6)\) (subject to \( X_0 = x \)) in view of Theorem \[3,1\] and therefore the process \( \gamma(X_t) \) of eq. \((4.1)\) is well defined on \( 0 \leq t \leq T \). The stochastic exponential \( Z_t \) given by \((4.2)\) is a strictly positive local martingale, hence it is a supermartingale. To show that it is a true
martingale, it suffices to prove that
\[ \mathbb{E}[Z_T] = 1. \] (4.7)

Quite similarly, there also exists a solution \( \tilde{X}_t \) of the Wishart SDE
\[ d\tilde{X}_t = \sqrt{\tilde{X}_t}dB_tQ + Q^\top dB_t^\top \sqrt{\tilde{X}_t} + (2p^Q Q^\top Q + \beta Q \tilde{X}_t + \tilde{X}_t(\beta^Q)^\top)dt \]
subject to the same initial condition \( \tilde{X}_t = x \) (note here: we use the desired new drift parameters with \( Q \) superscripts, but the SDE is driven by the original Brownian motion \( B \)). This process serves as auxiliary process to show condition (4.7). We also can define \( \gamma_t(\tilde{X}_t) \) exactly as in (4.1), but using \( \tilde{X}_t \) instead of \( X_t \).

We introduce the two sequences of stopping times
\[ \tau_n = \inf\{t > 0 \mid \|\gamma(X_t)\| \geq n\} \wedge T \]
and
\[ \tilde{\tau}_n = \inf\{t > 0 \mid \|\gamma(\tilde{X}_t)\| \geq n\} \wedge T. \]
These are increasing sequences satisfying
\[ \lim_{n \to \infty} P(\tau_n = T) = \lim_{n \to \infty} P(\tilde{\tau}_n = T) \] (4.8)
because we use the convention that the infimum of an empty set is \( +\infty \). For each \( n \geq 1 \) we define the process
\[ \gamma^n_t := \gamma(X_t)1_{\tau_n \leq t}, \quad t \in [0, T]. \]
By construction \( \int_0^t \|\gamma^n_s(X)\|^2 ds \leq n^2 t \), and therefore Novikov’s condition
\[ \mathbb{E}[e^{\frac{1}{2} \int_0^T \|\gamma^n_s(X)\|^2 ds}] < \exp(n^2 T/2) \]
which let us conclude that
\[ Z^n_t = \mathcal{E} \left( -\int_0^t \text{tr}(\gamma^n_s dB_s) \right) \]
is a martingale and \( dQ^n := Z^n_T d\mathbb{P} \) defines a probability measure equivalent to \( \mathbb{P} \) for which \( B^n_t = \int_0^t \gamma^n_s(X) ds \) is a \( d \times d \) standard Brownian motion on \([0, T] \). Furthermore, for each \( n \), the stopped process \( X^n_t := X_{t \wedge \tau_n} \) have the same law under \( Q^n \) as the stopped processes \( \tilde{X}^n_t = \tilde{X}_{t \wedge \tilde{\tau}_n} \) under \( \mathbb{P} \). We therefore have
\[
\mathbb{E}[Z_T] = \lim_{n \to \infty} \mathbb{E}[Z^n_T 1_{\tau_n = T}]
\]
\[ = \lim_{n \to \infty} \mathbb{E}^{Q^n}[1_{\tau_n = T}] = \lim_{n \to \infty} Q^n(\{\tau_n = T\}) \]
\[ = \lim_{n \to \infty} \mathbb{P}(\{\tilde{\tau}_n = T\}) = 1, \]
where the first identity follows from monotone convergence (which is applicable because the sets \( \tau_n = T \) are increasing in \( n \), and \( Z^n_T \) is a constant sequence along this sequence; hence the sequence \( Z^n_T 1_{\tau_n = T} \) is a monotonically increasing one). □

Every positive local martingale is a supermartingale and every supermartingale with constant expectation is a martingale. See [28, p.123].
5. ON THE EXISTENCE OF WISHART DISTRIBUTIONS

In this section we provide some results concerning the existence of Wishart distributions and their densities. To this end, we introduce some further notation. Let $a \in \mathbb{R}$ and $k \in \mathbb{N}_0$. The hypergeometric coefficient $(a)_k$ is defined as

$$(a)_k := \begin{cases} 
1, & \text{if } k = 0 \\
a(a+1) \cdots (a+k-1), & \text{otherwise}
\end{cases}$$

Let $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}_0^d$ be a multi-index with length $|\kappa| = \kappa_1 + \cdots + \kappa_d$. The generalized ($d$–dimensional) hypergeometric coefficient $(p)_\kappa$ is given by

$$(p)_\kappa = \prod_{j=1}^{d} \left( p - \frac{1}{2}(j-1) \right)^{\kappa_j},$$

see for instance [13, p. 30]. $C_\kappa : S_d \to \mathbb{R}$ shall denote the zonal polynomial of order $\kappa$, where $\kappa \in \mathbb{N}_0^d$. There are several equivalent definitions, for instance $C_\kappa(\xi)$ equals the $\kappa$th component of $(\text{tr}(\xi))^k$, (see [13, Definition 1.5.1]). Hence

$$(\text{tr}(\xi))^k = \sum_{|\kappa| = k} C_\kappa(\xi).$$

A more abstract definition [9, p. 234] is that

$$C_\kappa(\xi) := \omega_{\kappa} \int_{k \in SO(d)} \Delta_\kappa(k : \xi) dk$$

where $dk$ is the normalized unique Haar measure on the special orthogonal group $SO(d)$, and $\omega_{\kappa}$ is some normalizing constant.

**Proposition 5.1.** Let $p \in \Lambda_d$, $\sigma \in \overline{S}_d^+$ and $\omega \in \overline{S}_d^+$. We have:

(i) If $2p \in \mathbb{N}$ and if $\text{rank}(\omega) \leq 2p$, then $\Gamma(p, \omega; \sigma)$ exists.

(ii) If $p \geq \frac{d-1}{2}$, then the right side of (2.3) is the Laplace transform of a probability measure $\Gamma(p, \omega; \sigma)$ on $\overline{S}_d^+$.

(iii) In particular, if $p > \frac{d-1}{2}$ and if $\sigma$ is invertible, then the density of $\Gamma(p, \omega; \sigma)$ exists and is given by

$$F(p, \omega; \sigma)(\xi) := (\det \sigma)^{-p} e^{-\text{tr}(\sigma^{-1} \xi + \sigma a)} (\det \xi)^{p - \frac{d+1}{2}} \times \left( \sum_{m=0}^{\infty} \sum_{|\kappa| = m} C_\kappa(\sqrt{a} \xi \sqrt{a}) \frac{\Gamma_k(p)}{m!} \right) \frac{1}{\sigma} \frac{1}{\Gamma_k(p)},$$

where we have set $a = a(\omega) := \sigma^{-1} \omega \sigma^{-1}$, $\sigma := q(\sigma) = \sqrt{\sigma}$.

(iv) If $\sigma$ is degenerate, $\Gamma(p, \omega; \sigma)$ is not absolutely continuous with respect to the Lebesgue measure on $\overline{S}_d^+$. 

Proof. Statement is proved by summing up squares of normally distributed $\mathbb{R}^d$-valued random variables, see section 1.

Note that if $\sigma \in S^+_d$, our definition of non-central Wishart distribution is related to the one of [16] in that $\Gamma(p, \omega; \sigma) = \gamma(p, \sigma^{-1}\omega\sigma^{-1}; \sigma)$, the latter being called ”general non-central Wishart distribution” in [16]. Hence statement [iii] is a consequence of [16, p. 1400].

Now for each $\varepsilon > 0$ we regularize $\sigma$, $a$ and $p$ by setting
\[
\sigma_\varepsilon := \sigma + \varepsilon I, \quad a_\varepsilon := (\sigma + \varepsilon I)^{-1}\omega(\sigma + \varepsilon I)^{-1}, \quad p_\varepsilon = p + \varepsilon.
\]
Then for each $\varepsilon > 0$, we pick $X_\varepsilon$, an $S^+_d$-valued random variable according to [16, Proposition 2.3] such that
\[
X_\varepsilon \sim \Gamma(p_\varepsilon, \omega; \sigma_\varepsilon)(= \gamma(p_\varepsilon, a_\varepsilon; \sigma_\varepsilon)).
\]
Letting $\varepsilon \to 0$ and using Lévy’s continuity theorem, we figure that
\[
\lim_{\varepsilon \to 0} (\det(I + \sigma u))^{-p} e^{-\text{tr}(u(I + \sigma u)^{-1})\omega} = \lim_{\varepsilon \to 0} (\det(I + \sigma_\varepsilon u))^{-p_\varepsilon} e^{-\text{tr}(u(I + \sigma_\varepsilon u)^{-1})\omega}
\]
must be the Laplace transform of some random variable $X \sim \Gamma(p, \omega; \sigma)$, to which $X_\varepsilon$ converges in distribution as $\varepsilon \to 0$. This settles part [ii]. Finally, we consider assertion [iv] Assume, by contradiction, that $\Gamma(p, \omega; \sigma)$ has a Lebesgue density, for some $\sigma$ of rank $r < d$. Let $X$ be an $S^+_d$-valued random variable distributed according to $\Gamma(p, \omega; \sigma)$. Since linear transformations do not affect the property of having a density and since the non-central Wishart family is invariant under linear transformations (this is easy to check), we may without loss of generality assume that $\sigma = \text{diag}(0, I_r)$, where $I_r$ is the $r \times r$ unit matrix. Consider the projection
\[
\pi_r : x = (x_{ij})_{1 \leq i,j \leq d} \mapsto \pi_r(x) := (x_{ij})_{1 \leq i,j \leq r}.
\]
A simple algebraic manipulation yields that the Laplace transform of $\pi_r(X)$ equals
\[
e^{-\text{tr}(\pi_r(\omega)v)}, \quad v \in S^+_r,
\]
which is the Laplace transform of the unit mass concentrated at $\pi_r(\omega)$. But the pushforward of a measure with density under a projection must have a density again. This yields the desired contradiction. □

A very important consequence of this statement in combination of the results of section 2 is the following existence result

**Corollary 5.2.** For all $p \geq \frac{d+1}{2}$, $\alpha \in S^+_d$ and $d \times d$ matrices $\beta$, Wishart processes with Wishart transition function with parameters $(\alpha, p, \beta)$ exist. Therefore all Wishart semi-martingales with the same parameters, starting at $x \in S^+_d$ exist. Similarly, for all $Q$ with $Q^\top Q = \alpha$ and for all $x \in S^+_d$, the Wishart SDE admits global weak solutions.

**Proof.** Proposition 5.1 allows a well defined Wishart transition function of the form (2.8). This transition function is Markovian by Lemma 2.11. Now we can combine Proposition 2.10 and Lemma 2.5 to obtain the remaining assertions. □
6. A Rank Condition for Non-Central Wishart Distributions

Not for all triples \((p, \omega, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_d^+ \times \mathbb{S}_d^+\) Wishart distributions \(\Gamma(p, \omega; \sigma)\) exist. [19] shows the following:

**Theorem 6.1.** Let \(d \in \mathbb{N}, p > 0, \omega \in \mathbb{S}_d^+\). Suppose \(\sigma \in \mathbb{S}_d^+\) is invertible. If the right side of (2.3) is the Laplace transform of a non-trivial probability measure \(\Gamma(p, \omega; \sigma)\) on \(\mathbb{S}_d^+\), then \(p \in \Lambda_d\) and \(\text{rank}(\omega) \leq 2p + 1\).

This result contradicts the preceding characterization of Letac and Massam [16], where no constraint on the non-centrality parameter had been imposed, which we call here rank-condition. Motivated by [19], Letac and Massam [17] deliver very recently an even stronger result which uses different methods, and fully characterizes the existence and non-existence of the non-central Wishart family (see Theorem 7.3 below).

Theorem 6.1 uses very nicely the construction and properties of Wishart processes, but also elementary arguments, such as Lévy’s continuity theorem. The latter allows to conclude, by using the characterization of central Wishart distributions, that \(p \in \Lambda_d\). The proof for the rank condition is indirect; we assume, for a contradiction, the existence of a single Wishart distribution which violates the rank condition. We then use the exponential family of the latter to construct a whole family of Wishart laws, which determine a Wishart process on \(\mathbb{S}_d^+\). That, in turn, ultimately violates the drift condition of Theorem 2.14. We start with a few lemmas.

Let \((p, \omega, \sigma) \in \mathbb{R}_+ \times \mathbb{S}_d^+ \times \mathbb{S}_d^+\) such that \(\mu := \Gamma(p, \omega; \sigma)\) is a probability measure, that is, eq. (2.3) holds. The domain of its moment generating function is defined as

\[
D(\mu) := \{ u \in \mathbb{S}_d \mid \mathcal{L}_\mu(u) := \int_{\mathbb{S}_d^+} e^{-\langle u, \xi \rangle} \mu(d\xi) < \infty \},
\]

which is the maximal domain to which the Laplace transform, originally defined for \(u \in \mathbb{S}_d^+\) only, can be extended. It is well known that \(D(\mu)\) is a convex (hence connected) set, and we also know that \(\mathbb{S}_d^+ \subset D(\mu)\). Clearly \((I + \sigma u)\) is invertible if and only if the (symmetric) matrix \((I + \sqrt{\sigma} u \sqrt{\sigma})\) is non-degenerate. Using these facts and the defining equation (2.3) we infer that

\[
D(\mu) := \{ u \in \mathbb{S}_d \mid (I + \sqrt{\sigma} u \sqrt{\sigma}) \in \mathbb{S}_d^+ \} = -\sigma^{-1} + \mathbb{S}_d^+,
\]

and therefore \(D(\mu)\) is even open. Accordingly, the natural exponential family of \(\mu\) is the family of probability measures \(\mathcal{F}(\mu) = \left\{ \frac{\exp(v\xi)\mu(d\xi)}{\mathcal{L}_\mu(v)} \mid v \in -\sigma^{-1} + \mathbb{S}_d^+ \right\}\).

We start by stating some key properties of Wishart distributions.

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4 In order to avoid confusions with calculations in the proof of the upcoming proposition, we change here from \(u\) notation to \(v\), because \(v\) denotes the Fourier-Laplace variable in this paper.

5 Some related properties can be found in Letac and Massam [16], but in a different notation. Letac and Massam use instead of \(\Gamma(p, \omega; \sigma)\) the parameterized family \(\gamma(p, a; \sigma)\), where \(\omega\) is replaced by \(a := \sigma^{-1} \omega \sigma^{-1}\).
Proposition 6.2.  

(i) Let \( p \geq 0, \omega \in S_d^+ \). Suppose \( X \) is an \( S_d^+ \)-valued random variable distributed according to \( \Gamma(p, \omega; I) \). Let \( q \in S_d^+ \) and set \( \sigma := q^2 \). Then \( qXq \sim \Gamma(p, q\omega; q\sigma) \) in particular, \( \Gamma(p, \omega; I) \) exists if and only if \( \Gamma(p, q\omega; q\sigma) \) exists.

(ii) Let \( p \geq 0, \sigma \in S_d^+ \) and \( \omega \in S_d^+ \) such that \( \mu := \Gamma(p, \omega; I) \) is a probability measure. For \( v = \sigma^{-1} - I \) we have that
\[
\frac{\exp(v\xi)\mu(d\xi)}{L_\mu(v)} \sim \Gamma(p, \sigma\omega\sigma; \sigma).
\] (6.3)
Conversely, if \( \Gamma(p, \sigma\omega\sigma; \sigma) \) is a well defined probability measure, so is \( \mu \), and (6.3) holds. In particular, we have that the exponential family generated by \( \mu \) is a Wishart family and equals
\[
F(\mu) = \{ \Gamma(p, \sigma\omega\sigma, \sigma) \mid \sigma \in S_d^+ \}, \quad \sigma^{-1} - I \in D(\mu) \}.
\]

(iii) Suppose that \( \Gamma(p, \omega_0; \sigma_0) \) is a probability measure, for \( p \geq 0 \) and \( \omega_0, \sigma_0 \in S_d^+ \). Then we have:

(a) \( \Gamma(p, t\omega_0; \sigma_0) \) is a probability measure for each \( t > 0 \).

(b) If, in addition, \( \omega_0 \) is invertible, then \( \Gamma(p, \omega; \sigma) \) is a probability measure for each \( \omega \in S_d^+, \sigma \in S_d^+ \).

Proof. Let \( \mathbb{E} \) be the corresponding expectation operator. By repeated use of the cyclic property of the trace and by the product formula for the determinant, we have
\[
\mathbb{E}[e^{-(u,qXq)}] = \mathbb{E}[e^{-(quq,X)}] = \det(I + quq)^{-1} \exp(-\text{tr}(quq(I + quq)^{-1}q))
\]
\[
= \det(I + \sigma u)^{-1} \exp(-\text{tr}(uq(I + quq)^{-1}q\omega))
\]
\[
= \det(I + \sigma u)^{-1} \exp(-\text{tr}(u(I + \sigma u)^{-1}q\omega)),
\]
which proves assertion \( (i) \). Next we show \( (ii) \). We note first, that by (6.1) we have that \( v = \sigma^{-1} - I \in D(\mu) \). Hence exponential tilting is admissible. Furthermore, we have
\[
\int_{S_d^+} e^{-(u+v,\xi)} \Gamma(p, \omega; I)(d\xi) = \det(I + (u + v))^{-p} \exp(-\text{tr}((u + v)(I + u + v)^{-1}q))
\] (6.4)
and setting \( v = \sigma^{-1} - I \) we obtain
\[
I + u + v = \sigma^{-1}(I + \sigma u).
\]
Hence the first factor on the right side of eq. (6.4) is proportional to \( \det(I + \sigma u)^{-p} \). It remains to show that
\[
-\text{tr}((u + v)(I + u + v)^{-1}q\omega) = c + \text{tr}(u(I + \sigma u)^{-1}q\omega)
\] (6.5)
Accordingly (2.3) can be written in the form
\[
\mathcal{L}(\gamma(p, a; \sigma))(u) = (\det(I + \sigma u))^{-p} e^{-\text{tr}(u(t + \sigma u)^{-1}q\omega)\sigma}, \quad u \in S_d^+.
\] (6.2)
Note that this requires \( \sigma \) to be invertible.

6Expressed in geometric language, we say that the pushforward of \( \Gamma(p, \omega; I) \) under the map \( \xi \mapsto q\xi q \) equals \( \Gamma(p, q\omega; q\sigma) \)
for some real constant $c$, because then the right side of (6.4) is proportional to the Laplace transform of $\Gamma(p, \sigma \omega \sigma; \sigma)$. To this end, we do some elementary algebraic manipulations:

$$-(u + v)(I + u + v)^{-1}\omega = -(u - I + \sigma^{-1}u[I + \sigma u])^{-1}\omega$$

$$= -(\sigma^{-1}u + u)(\sigma^{-1}u + u)^{-1}\omega$$

$$= -\omega + (\sigma - \sigma)\omega + (\sigma^{-1}u + u)^{-1}\omega$$

$$= (\sigma - I)\omega - (\sigma^{-1}u + u)(\sigma^{-1}u + u)^{-1}\omega + (\sigma^{-1}u + u)^{-1}\omega$$

$$= (\sigma - I)\omega - \sigma u(\sigma^{-1}u + u)^{-1}\omega$$

$$= (\sigma - I)\omega - \sigma u(I + \sigma u)^{-1}\sigma.$$

We set now $c := \text{tr}((\sigma - I)\omega)$ which is the real number we talked about before. Taking trace and performing cyclic permutation inside, we obtain (6.3), and therefore the identity (6.3) is shown. The assertion concerning the exponential family follows by the very definition of the latter.

We may therefore proceed to (iii) which is proved by repeatedly applying (i) and (ii). Let $\Gamma(p, \omega_0; \sigma_0)$ be a probability measure. Then by (ii), also $\Gamma(p, \sigma_0^{-1}\omega_0\sigma_0^{-1}; I)$ is one. Let $q_1$ such that $q_1^2 = \sigma_1 \in S_d^+$. We may write $\Gamma(p, \sigma_0^{-1}\omega_0\sigma_0^{-1}; I) = \Gamma(p, q_1^{-1}(q_1\sigma_0^{-1}\omega_0\sigma_0^{-1}q_1^{-1}); I)$, and by applying (i) we obtain the pushforward measure $\Gamma(p, q_1\sigma_0^{-1}\omega_0\sigma_0^{-1}q_1^{-1}; I)$. By (iii) we have that $\Gamma(p, q_1^{-1}\sigma_0^{-1}\omega_0\sigma_0^{-1}q_1^{-1}; I)$ is a probability measure as well, and once again by (ii) we infer that for all $\sigma \in S_d^+$, $\Gamma(p, \sigma_0^{-1}\omega_0\sigma_0^{-1}q_1^{-1}\sigma, \sigma)$ is a probability. We use this fact to prove both parts of the assertion. Without loss of generality we assume that $\sigma$ is non-degenerate, because in the case $\sigma \in \partial S_d^+$ we may invoking Lévy’s continuity theorem.

Setting $q_1 = 1/\sqrt{I}$ and $\sigma = \sigma_0$, we see that (iii) holds. For $\omega_0 \in S_d^+$ we choose $q_1 \in S_d^+$ such that $q_1^{-1}\sigma_0^{-1}\omega_0\sigma_0^{-1}q_1^{-1} = \sigma^{-1}\omega\sigma^{-1}$, which allows to conclude (iii). □

Next, we restate the characterization of the central Wishart laws by using [21]:

**Theorem 6.3.** Let $d \geq 2$, $\sigma \in S_d^+$ and $p \geq 0$. The following are equivalent:

(i) $\det(I + \sigma u)^{-p}$ is the Laplace transform of a probability measure $\Gamma(p, \omega; \sigma)$ on $S_d^+$.

(ii) $p \in \Lambda_d$.

We are prepared to deliver our proof of Theorem 6.1.

**Proof.** Let $p > 0$ such that for some $\omega_0 \in \overline{S_d^+}$, $\sigma \in S_d^+$, the right side of (2.3) is the Laplace transform of a non-trivial probability measure $\Gamma(p, \omega_0; \sigma)$. By Proposition 6.2 (iii), we have that $\Gamma(p, \omega_0/n; \sigma)$ is a probability measure for each $n \in \mathbb{N}$. Letting $n \to \infty$ and invoking Lévy’s continuity theorem, we obtain that $\Gamma(p; \sigma)$ is a probability measure. But then by the characterization of central Wishart laws, Theorem 6.3 (ii) we have that $p \in \Lambda_d$.

Let now $p_0 \in \Lambda_d \setminus \left\{\frac{d-1}{2}, \infty\right\}$, and let us assume, by contradiction, that there exist $(\omega_0, \sigma) \in \overline{S_d^+} \times S_d^+$, $\text{rank}(\omega_0) > 2p_0 + 1$ such that $\Gamma(p_0, \omega_0; \sigma)$ is a probability measure. Pick now

7Strictly speaking, Lévy’s continuity theorem applies to characteristic functions. However, in the Wishart case, the right side of (2.3) can even be extended to even the Fourier-Laplace transform with ease, and by preserving its functional form.
ω_1 ∈ \overline{\mathbb{S}^+}\_d such that ω^* := ω_1 + ω_0 has rank(ω^*) := rank(ω_1) + rank(ω_0) = d, and set p_1 := \frac{d – rank(ω_0)}{2}. By construction 2p_1 = rank(ω_1), and p_1 ∈ \Lambda_d \setminus \left[\frac{d-1}{2}, \infty\right). Hence Proposition 5.16 implies the existence of a non-central Wishart distribution Γ(p_1, ω_1, σ). Note that p^* := p_0 + p_1 ∈ \Lambda_d \setminus \left[\frac{d-1}{2}, \infty\right) and that by convolution

Γ(p^*, ω^*, σ) := Γ(p_0, ω_0, σ) * Γ(p_1, ω_1, σ)

is a probability measure as well. Since ω^* is of full rank, we have by Proposition 6.2 (iii)b that Γ(p^*, ω; σ) exists for all (ω, σ) ∈ (\mathbb{S}^+_d)^2. Hence Γ(p^*, ω; tσ) exists for all (t, ω, σ) ∈ \mathbb{R}_+ × (\mathbb{S}^+_d)^2.

We may now construct a Wishart process by picking some α ∈ \mathbb{S}^+_d \setminus \{0\} and declaring a Markovian transition function by setting for each (t, x) ∈ \mathbb{R}_+ × (\mathbb{S}^+_d)^2, p_t(x, dξ) the probability measure given by the Laplace transform

\int_{\mathbb{S}^+_d} e^{-(u, ξ)} p_t(x, dξ) = (\det(I + 2tαu))^{-p^*} e^{tr(-u(I+2tαu)^{-1}x)} (6.6)

(cf. (2.9) for \beta = 0). Hence X is a Wishart process with constant drift parameter 2p^*, diffusion coefficient α and zero drift \beta = 0. But 2p^* \geq (d – 1), which contradicts Theorem 2.14. This shows that we indeed must have rank(ω_0) ≤ 2p_0 + 1.

7. Existence of Wishart transition densities

The aim of this section is to fully characterize the existence of transition densities for Wishart processes. That is, we investigate whether the transition laws of Wishart processes admit a Lebesgue density.

**Theorem 7.1.** Let p > \frac{d-1}{2}. The following are equivalent

(i) p_t(x, dξ) has a Lebesgue density \mathcal{F}_{t,x}(ξ), for one (hence all) t > 0.

(ii) The d × d^2 matrix

\begin{bmatrix} Q^\top \\
βQ^\top \\
\vdots \\
β^{d-1}Q^\top 
\end{bmatrix}

(7.1)

has maximal rank.

Furthermore, if any of the above conditions are satisfied, then \mathcal{F}_{t,x} is C^p(\mathbb{S}^+_d), for any t > 0.

**Remark 7.2.**

• Note that in (7.1) the matrix Q may be replaced by any matrix K for which K^\top K = Q^\top Q = α. This is obvious from the proof of Proposition 7.3 below.

• By an inspection of the (Gaussian) transition law of Ornstein-Uhlenbeck processes of the form

Y_t := e^{βt} \left( y + \int_0^t e^{-βs} Q^\top dW_s \right),

where W is a d-dimensional standard Brownian motion, one can infer the well known result that (8) characterizes the existence of Lebesgue densities for Y_t. In fact, by 7.3 the covariance matrix of Y_t is non-degenerate, for each t > 0, so the result holds because Y is a Gaussian process.
Condition (ii) is well known in linear control theory, and characterizes the controllability of the linear system
\[
\partial_t x(t) = \beta x(t) + Q^\top u(t), \quad x(0) = x_0.
\]
That is, let \( T > 0 \). Then for each \( x^* \in \mathbb{R}^d \) there exists a control \( u \) such that \( x(T) = x^* \). For more details, see [29, Chapter 3].

The following proposition is a well known ingredient in the characterization of controllability of linear systems. For the sake of completeness and as service for the reader, we also prove it here. See, for instance the statements [29, 3.1 to 3.4] and their proofs.

**Proposition 7.3.** The following are equivalent:

(i) For one (hence any) \( t > 0 \), the matrix \( \sigma_t^\beta(\alpha) \) is positive definite.

(ii) The \( d \times d \) matrix (7.1) has maximal rank.

**Proof.** By additivity of the integral, it is clear that if \( \sigma_t^\beta(\alpha) \) is positive definite for some \( t > 0 \), it is for all \( s \geq t > 0 \).

By Cayley–Hamilton, for each \( t \geq 0 \) there exist numbers \( a_j(t), j = 1, \ldots, d - 1 \) such that
\[
e^{tA} = \sum_{j=1}^{d-1} a_j(t)A^j.
\]
Hence
\[
\int_0^t e^{s\beta^\top}Qe^{\beta^\top}sds = \sum_{j,k} g_{jk}(\beta_j^\top Q^\top)(\beta_k^\top Q^\top)
\]
with
\[
g_{jk} := \int_0^t a_j(s)a_k(s)ds,
\]
which by construction yields a positive semidefinite matrix \( g := (g_{jk})_{jk} \).

Proof of (i) \( \Rightarrow \) (ii) Since \( \sigma_t^\beta(Q^\top Q) \) is positive definite, we have by using eq. (7.3) that for each \( z \in \mathbb{R}^d \setminus \{0\} \) it holds
\[
\sum_{j,k} g_{jk}(z^\top \beta_j^\top Q^\top)(z^\top \beta_k^\top Q^\top) > 0.
\]
But \( g \) is positive semidefinite, hence the vector with \( (z^\top \beta_j^\top Q^\top)_{j=1}^d \) must be nonzero. Since \( z \) was an arbitrary nonzero element of \( \mathbb{R}^d \), we have proved the rank condition (ii).

For the reverse implication, we proceed by an indirect argument. Suppose, there exists \( z \neq 0 \) such that for all \( t > 0 \), we have that \( z^\top \sigma_t^\beta(Q^\top Q)z = 0 \). Due to the positivity of the integrand
\[
Qe^{\beta^\top t}z = 0
\]
for all \( t > 0 \), or equivalently,
\[
w^\top Qe^{\beta^\top t}z = 0
\]
for all $w \in \mathbb{R}^d$, $t > 0$. Since the $j$-th derivative of $e^{\beta t}Q^T$ at $t = 0$ equals $(-1)^j \beta^j Q^T$, we have by differentiation of eq. (7.4) that $w^\top \beta^j Q^T z = 0$ for all $w$ and therefore (ii) cannot hold.

**Proof of Theorem 7.1.**

Proof. We start with the implication (ii) $\Rightarrow$ (i). By Proposition 7.3, we have that for any $t > 0$, the matrix $\sigma^{(\beta)}(\alpha)$ is positive definite. By assumption we have that $p > \frac{d-1}{2}$, and comparing the Laplace transform (2.9) of $p_t(x,d\xi)$ with the right side of (2.3) we realize that $p_t(x,d\xi) \sim \Gamma(p,\omega^\beta(x);\sigma^{(\beta)}(\alpha))$. Hence by Proposition 5.1 (iii) we have that $p_t(x,d\xi)$ has a Lebesgue density $F_{t,x}(\xi)$, for each $t > 0$.

For the converse direction, we proceed by an indirect argument. Assume, for a contradiction, that the Kalman matrix (7.1) has rank strictly smaller than $d$. Then by Proposition 7.3, $\sigma^{(\beta)}(\alpha)$ is degenerate for some $t > 0$. But then by Proposition 5.1 (iv) $p_t(x,d\xi) \sim \Gamma(p,\omega^\beta(x);\sigma^{(\beta)}(\alpha))$ is not absolutely continuous with respect to the Lebesgue density.

So we have shown the equivalence of (i) and (ii). The claim concerning the regularity of the densities $F_{t,x}(\xi)$ is an immediate consequence of the second part of Proposition 5.1 (iii).

**Hitting the boundary revisited.** As application of this section, we prove the following which is stronger to some extent than the assertion of Lemma 3.4:

**Lemma 7.4.** Let $\beta, Q$ be $d \times d$ matrices, and suppose that the Kalman matrix (7.1) has maximal rank. When $p < \frac{d-1}{2}$, then for any $x \in S^+_d$ we have for the solution of the Wishart SDE that not only $T_x < \infty$ with positive probability but also the stochastic interval $[0,T_x]$ does not contain a deterministic time interval $[0,T]$, $T > 0$.

**Proof.** Assume, for a contradiction, the existence of $x \in S^+_d$ for which $T_x \geq T > 0$, where $T$ is a positive quantity. An adaption of Lemma 2.5 shows that any solution $X_t$ of the Wishart SDE on $[0,T]$ is a Wishart semimartingale on $[0,T]$. And also from the proof of Lemma 2.9 we see that $X_t \sim \Gamma(p,\omega^\beta(x);\sigma^{(\beta)}(\alpha))$ for all $t \leq T$, where $\alpha = Q^T Q$. But that means that a Wishart distribution exists with non-centrality parameter of full rank, and–in view of Proposition 7.3–also with scale parameter of full rank, but the shape parameter satisfies $p < \frac{d-1}{2}$. This is impossible in view of the subsequent statement.

We cite a special case of Letac and Massam’s very recent result [17] on necessary conditions for the parameters of the Wishart distributions. Translated into our notation it reads:

**Theorem 7.5.** Suppose $\sigma$ and $\omega$ are invertible. $\Gamma(p,\omega;\sigma)$ can only exist, if $p \geq \frac{d-1}{2}$.

**8. Wishart processes on new state spaces**

In a recent work with Cuchiero, Keller-Ressel and Teichmann [10], the class of affine processes on finite-dimensional symmetric cones [8] have been completely characterized.

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8These are closed convex selfdual cones on which the linear automorphism group acts transitively.
Those also contain affine diffusion processes such as the Wishart processes. Symmetric cones are classified completely [9], therefore one could try to find SDE realizations as the Wishart SDEs [22] on $S_d^+$. However, only in the case of Hermitian matrices the literature provides such realizations. In the latter case we let $W_1, W_2$ be two jointly independent $d \times n$ Brownian motions ($n \geq d$), and $y$ be a complex $d \times n$ matrix. Then $X_t := (y + W_1 + iW_2)(\bar{y} + W_1 - iW_2)^\top$ satisfies

$$dX_t = \sqrt{X_t}dB_t + dB_t^\top \sqrt{X_t} + 2pIdt, \quad X_0 = yy^\top,$$

with $B_t$ some $d \times d$ complex Brownian matrix, i.e. $B = B_1 + iB_2$, where $B_1, B_2$ are two independent $d \times d$ standard Brownian motions. Here $\bar{c}$ denotes the complex conjugate of a complex number $c$ and $p = d$. Demni [9, chapter 2] discusses this case, calling these processes Laguerre process of integer index. For general drift parameters $p > d - 1$ see the Laguerre processes of [6, chapter 4].

[5] delivers for the first time a Wishart process on a non-symmetric cone, namely the dual Vinberg cone. This cone is given by the five-dimensional subset $K \subset S_3^+$

$$K = \left\{ u = \begin{pmatrix} a & b_1 & b_2 \\ b_1 & c_1 & 0 \\ b_2 & 0 & c_2 \end{pmatrix} : \text{u is positive semi-definite} \right\}$$

and any element $x \in K$ can be written as $x = \sum_{i=1}^3 y_i y_i^\top$, where

$$y_1 = \begin{pmatrix} y_{1,1} \\ 0 \\ 0 \end{pmatrix}, \quad y_2 = \begin{pmatrix} y_{2,1} \\ y_{2,2} \\ 0 \end{pmatrix}, \quad y_3 = \begin{pmatrix} y_{3,1} \\ 0 \\ y_{3,3} \end{pmatrix}.$$

We give here a slightly different, yet fully equivalent construction. Let $(B_1, B_2, B_3)$ be a three dimensional standard Brownian motion. We introduce the process $X_t := \sum_{i=1}^3 Y_{t,i} Y_{t,i}^\top$, where for $i = 1, 2, 3$ we set

$$Y_{t,i} := y_i + B_{t,i} e_1$$

and $e_1 = (1, 0, 0)^\top$ denotes the first canonical basis vector. By Example 2.2 (using $\beta = 0, Q = \text{diag}(1, 0, 0)$ and extending $B_t$ to vector-valued Brownian motions) we know that $X_t$ is a Wishart semimartingale on $S_3^+$, and by construction $X_t$ is supported on $K$. Hence by the second part of Lemma 2.5 there exists an enlargement of the original probability space which supports a $3 \times 3$ standard Brownian $W$ motion such that $X_t$ is a weak solution of the Wishart SDE

$$dX_t = (\sqrt{X_t}dW_t Q + Q^\top dW_t^\top \sqrt{X_t}) + 2pQ^\top Qdt, \quad X_0 = x \in K,$$

where $p = 3/2$.

A full understanding of Wishart processes (leave alone general affine processes) on general homogenous cones is not available at the date this manuscript is printed.
A final note might be of interest. By using Example 2.2 one obtains Wishart processes with non-convex cone state-space

$$D_m := \{ u \in \mathbf{S}_d^+ \mid \text{rank}(u) \leq m \}$$

and these have drift parameter $p = m/2$, possibly smaller then $(d - 1)/2$, thus violating the enigmatic drift condition established in Theorem 2.14 which Wishart processes with support on $\mathbf{S}_d^+$ must satisfy. Hence there is no way to extend the so constructed affine processes on $D_m$ to its convex hull $\overline{S}_d^+$. This also shows that there are more Wishart semimartingales on $\mathbf{S}_d^+$ than those which naturally arise from Wishart processes on $\mathbf{S}_d^+$. But these are supported on the strict submanifolds $D_m$ of $\mathbf{S}_d^+$, $m < d$.

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