Nearly Linear Row Sampling Algorithm for Quantile Regression

Yi Li  
School of Physical and Mathematical Sciences  
Nanyang Technological University, Singapore  
yili@ntu.edu.sg

Ruosong Wang  
Department of Computer Science  
Carnegie Mellon University, USA  
ruosongw@andrew.cmu.edu

Lin Yang  
Department of Electrical and Computer Engineering  
University of California Los Angeles, USA  
linyang@ee.ucla.edu

Hanrui Zhang  
Department of Computer Science  
Duke University, USA  
hrzhang@cs.duke.edu

Abstract
We give a row sampling algorithm for the quantile loss function with sample complexity nearly linear in the dimensionality of the data, improving upon the previous best algorithm whose sampling complexity has at least cubic dependence on the dimensionality. Based upon our row sampling algorithm, we give the fastest known algorithm for quantile regression and a graph sparsification algorithm for balanced directed graphs. Our main technical contribution is to show that Lewis weights sampling, which has been used in row sampling algorithms for $\ell_p$ norms, can also be applied in row sampling algorithms for a variety of loss functions. We complement our theoretical results by experiments to demonstrate the practicality of our approach.

1 Introduction
Linear regression is a fundamental problem in machine learning and statistics. For a data matrix $A \in \mathbb{R}^{n \times d}$ and a response vector $b \in \mathbb{R}^n$ with $n \gg d$, the overconstrained linear regression problem can be formulated as solving the optimization problem $\min_{x \in \mathbb{R}^d} \phi(Ax - b)$ where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a loss function. Via the technique of row sampling, we have obtained remarkable speedups for solving linear regression for a wide range of loss functions. Such techniques involve designing an importance sampling scheme and showing that if one samples the data matrix $A$ and the response vector $b$ accordingly, $\phi(Ax - b)$ is approximately preserved for all $x \in \mathbb{R}^d$. Thus, one can obtain an approximate solution to the original linear regression problem by solving a linear regression instance on the sampled data matrix and response vector, which is usually much smaller in size.

The technique of row sampling has received much attention in randomized numerical linear algebra and machine learning. It is known that if one samples $O(d \log d/\varepsilon^2)$ rows of $A$ according the to leverage scores, the resulting matrix $A'$ would satisfy

$$(1 - \varepsilon)\|Ax\|_2 \leq \|A'x\|_2 \leq (1 + \varepsilon)\|Ax\|_2$$

for all $x \in \mathbb{R}^d$ with high probability \[39\]. Moreover, such row sampling algorithm can be implemented in $\tilde{O}(\text{nnz}(A) + \text{poly}(d))$ time \[8, 28, 29, 26, 11\], where $\text{nnz}(A)$ is the number of non-zero entries of $A$. Building upon the result of Clarkson for $p = 1$ \[5\], Dasgupta et al. obtain the first row sampling algorithm for general $\ell_p$ norms \[14\]. They show that by sampling $\tilde{O}(d^{\max(p/2+1, p)+1}/\varepsilon^2)$ rows of $A$ according to the $\ell_p$ leverage scores, the resulting matrix $A'$ satisfies

$$(1 - \varepsilon)\|Ax\|_p \leq \|A'x\|_p \leq (1 + \varepsilon)\|Ax\|_p$$

Throughout the paper, we use $\tilde{O}(f)$ to denote $O(f \text{ polylog } f)$. 

arXiv:2006.08397v1 [cs.DS] 15 Jun 2020
for all \( x \in \mathbb{R}^d \) simultaneously with high probability. The algorithm in [14] runs in \( \tilde{O}(nd^2) \) time, and the running time is further improved to \( \tilde{O}(mnz(A) + \text{poly}(d)) \) in [31, 38, 28, 43, 40]. Later, based on results in Banach space theory [37, 38, 2], Cohen and Peng show that by sampling according to \( \ell_p \) Lewis weights [12], the number of sampled rows can be further reduced to \( \tilde{O}(d/\varepsilon^2) \) when \( p < 2 \) and \( \tilde{O}(d^{p/2}/\varepsilon^2) \) when \( p \geq 2 \), which is nearly optimal as recently shown by Li et al. [27]. Row sampling algorithms are also obtained for other loss functions, including \( M \)-estimators [10, 9], the Tukey loss function [7] and Orlicz norms [35].

Yang et al. study row sampling algorithms for the quantile loss function [44]. Let \( \tau \in (0, 1] \) be a parameter and \( y = Ax - b \). The quantile loss function is defined to be \( h_\tau(y) = \sum_{i=1}^n h_\tau(y_i) \), where

\[

h_\tau(y_i) = \begin{cases} 
\tau y_i & \text{if } y_i \geq 0 \\
(\tau - 1)y_i & \text{if } y_i \leq 0 .
\end{cases}
\]

(1)

The corresponding quantile regression problem is of particular interest, due to its wide applications in medicine [13, 32, 33], in hydrology [51], in economics [23, 21], in survival analysis [22], etc. See, e.g., [3, 4, 30, 46, 45] for recent developments. Using the notion of the \( \ell_1 \) leverage score [14] and known \( \ell_1 \) oblivious subspace embeddings [33, 28], Yang et al. obtain the first quantile regression algorithm that runs in \( \tilde{O}(\text{nnz} + \text{poly}(d/\varepsilon)) \) time [44]. More specifically, they show that by sampling \( \tilde{O}(\tau/(1-\tau) \cdot d^3/\varepsilon^2) \) rows according to the \( \ell_1 \) leverage scores, \( h_\tau(Ax) \) is preserved for all \( x \in \mathbb{R}^d \) up to a factor of \( 1 \pm \varepsilon \). Furthermore, quantile regression can be formulated as a linear program, and thus one can readily solve the smaller instance returned by the sampling process in \( \text{poly}(d/\varepsilon) \) time.

Although the results in [44] are interesting from a theoretical point of view, the \( d^2 \) factor in the number of sampled rows required by their algorithm is far from being practical due to the rise of large-scale datasets. Even for a moderate-sized dataset with \( d = 300 \), the algorithm in [44] requires at least \( d^2 = 2.7 \times 10^7 \) samples, which is too large to gain any speedup over the naive implementation without using row sampling. For modern large-scale datasets, e.g., the ImageNet dataset [16], \( d \) is much larger than \( 10^5 \) even after pre-processing. In that case, even quadratic number of samples is too large to be practical. Thus, one may naturally ask whether it is possible to obtain a row sampling algorithm for the quantile loss function which requires only nearly linear number of samples, similar to the case of \( \ell_p \) norms when \( 1 \leq p \leq 2 \). In this paper, we provide an affirmative answer to this question.

**Our Contributions.** We provide the first row sampling algorithm with nearly linear number of sampled rows for the quantile loss function. Note that we can write \( Ax - b = A'x' \) for \( A' = [A \ b] \in \mathbb{R}^{n \times (d+1)} \) (concatenation of \( A \) and \( b \)) and \( x' = [x^T \ 1] \in \mathbb{R}^{d+1} \), we overload the notations by renaming \( A' \) to \( A \) and \( x' \) to \( x \) and replacing \( d \) with \( d+1 \), and consider instead preserving loss functions on \( Ax \). For notational simplicity, we consider the equivalent problem of preserving

\[

\rho_\tau(Ax) = \sum_{i=1}^n \rho_\tau((Ax)_i),
\]

where \( \tau \in [0, 1] \) and \( \rho_\tau : \mathbb{R} \to \mathbb{R} \) is defined as

\[

\rho_\tau(t) = \begin{cases} 
t & t \geq 0 \\
-\tau t & t < 0 .
\end{cases}
\]

(3)

The problem of preserving \( h_\tau(\cdot) \) is equivalent to that of preserving \( \rho_\tau(\cdot) \), since \( h_\tau(t) = \tau \cdot \rho_\tau(1-\tau)/\tau(t) \) for \( \tau > 0 \). Now we formally state our sampling result for the loss function \( \rho_\tau(\cdot) \).

**Theorem 1.1.** There is an algorithm that receives a matrix \( A \in \mathbb{R}^{n \times d} \) and a parameter \( \tau \in (0, 1] \), and outputs a matrix \( \tilde{A} \in \mathbb{R}^{N \times d} \) with \( N = \tilde{O}(d/\varepsilon^2\tau^2) \) such that with probability at least \( 1 - 1/\text{poly}(d) \), for all \( x \in \mathbb{R}^d \),

\[

(1 - \varepsilon)\rho_\tau(Ax) \leq \rho_\tau(A\tilde{x}) \leq (1 + \varepsilon)\rho_\tau(Ax).
\]

The algorithm runs in \( \tilde{O}(\text{nnz}(A) + d^\omega) \) time\(^2\)

\(^2\)We use \( \omega \) to denote the matrix multiplication exponent. It is known that \( \omega \leq 2.373 \) [31].
Our assumption that $\tau \leq 1$ is without loss of generality, since $\rho_{1/\tau}(t) = \frac{1}{\tau}\rho_{\tau}(-t)$, and thus our results can be directly translated to the case where $\tau > 1$.

Compared with [44], the number of rows required by our row sampling algorithm has a nearly linear dependence on $d$, whereas the algorithm in [44] requires at least $O(d^1/(\varepsilon^2\tau^2))$ sampled rows, in our notation. Thus, our result is better by a factor of $d^2$ when $\tau$ is constant (which is usually the case for applications in practice). Furthermore, the running time of the row sampling algorithm in [44] is $\tilde{O}(\text{nnz}(A) + \text{poly}(d))$, whereas the exponent hidden in the poly($d$) term is much larger than 3 due to the use of the ellipsoid rounding routine in [6].

We then apply our sampling results to two problems: quantile regression and balanced directed graph sparsification.

Quantile Regression. For the quantile regression problem, notice that Theorem 1.1 already implies an algorithm that runs in input-sparsity time. Suppose that there is an algorithm that runs in time $\phi(s, d)$ for solving a quantile regression problem of size $s \times d$, then one can first use Theorem 1.1 to reduce the problem size from $n \times d$ to $\tilde{O}(d/\varepsilon^2\tau^2) \times d$, and thus the overall time complexity of the algorithm would be $\tilde{O}(\text{nnz}(A) + d\omega) + \phi\left(\tilde{O}(d/\varepsilon^2\tau^2), d\right)$. As a comparison, the algorithm in [44] runs in $\tilde{O}(\text{nnz}(A) + \text{poly}(d)) + \phi(\tilde{O}(d^3/\varepsilon^2\tau), d)$ time (see Theorem 1 in [44]). Again, the exponent hidden in the poly($d$) term is much larger than 3.

To further demonstrate the flexibility and versatility of Theorem 1.1, we develop an algorithm for solving quantile regression using accelerated first-order methods. Here, our goal is to optimize the lower-order term that polynomially depends on $d$ in the time complexity while keeping the high-order term to be $\tilde{O}(\text{nnz}(A))$. To achieve this goal, we adopt the celebrated sketch-and-solve paradigm [42], together with recently developed powerful tools for $\ell_1$ regression [18]. As a result, we give the fastest known input-sparsity time algorithm for the quantile regression problem.

**Theorem 1.2** (Informal version of Theorem 1.1). For a given $\tau \in (0, 1]$, the quantile regression problem can be solved up to a factor of $(1 + \varepsilon)$ in time $\tilde{O}(\text{nnz}(A) + d^{2.5}/(\varepsilon^2\tau^2))$.

Balanced Directed Graph Sparsification. Another unique advantage of our sampling results over previous works is that, for constant $\varepsilon$ and $\tau$, the number of sampled rows required by our algorithm is roughly linear in the dimensionality $d$. When $\tau = 0$, Theorem 1.1 does not apply; nevertheless, a similar sampling scheme, when applied to the edge-vertex incidence matrix of a balanced directed graph (see Definition 5.1), yields a graph sparsifier with nearly linear size. Recall Eqn. 2 for loss functions applied to a vector.

**Theorem 1.3** (Informal version of Corollary 5.1). For a given $\alpha$-balanced directed graph $G = (V, E, w)$ with $n$ vertices and $m$ edges, there is an algorithm that outputs $G' = (V, E', w')$ with $E' \subseteq E$ and $|E'| = O(n(\alpha/\varepsilon)^2 \log(n\alpha/\varepsilon))$ such that with probability at least $1 - 1/\text{poly}(n)$, for all $x \in \mathbb{R}^n$,

$$(1 - \varepsilon)\rho_0(Bx) \leq \rho_0(B'x) \leq (1 + \varepsilon)\rho_0(Bx).$$

Here $B$ is the edge-vertex incidence matrix of $G$ and $B'$ is the edge-vertex incidence matrix of $G'$.

We note that a sampling scheme gives meaningful sparsifiers for graphs of $n$ vertices only if the number of sampled rows is $o(n^2)$, since the largest possible number of edges in a graph is $O(n^2)$. Hence the sampling scheme by Yang et al. [44], for example, would sample $\Omega(n^3)$ rows and is not an option for graph sparsification. Our sparsification result yields sparsifiers of roughly the same size as the state-of-the-art sparsification algorithm for balanced graphs by Ikeda and Tanigawa [20]. However, the sparsifier produced by [20] is only guaranteed to be a cut sparsifier, meaning that [4] holds only for $x \in \{0, 1\}^n$ (see Remark 1), whereas for our sparsifier, [4] holds for all $x \in \mathbb{R}^n$, which is much stronger. Furthermore, the algorithm in [20] is specially tailored for balanced graphs, based on a number of highly combinatorial subroutines; in contrast, our algorithm follows a general-purpose sampling framework and utilizes the properties of balanced graphs in a rather black-box manner. We believe our algorithm provides an alternative view of sparsification of (balanced) directed graphs, which also illustrates the power of our sampling scheme.

---

*The two results are not directly comparable: for example, our bound has better dependence on the number of vertices $n$, while their bound has better dependence on the balance parameter $\alpha$. 

3
In Section 7, we conduct an empirical evaluation of our row sampling algorithm for the quantile loss function to demonstrate the practicality of our approach.

**Organization.** This paper is organized as follows. In Section 2, we introduce necessary notations and definitions and also review known results regarding Lewis weights. In Section 3, we present our main row sampling results. In Section 4, we present our algorithm for solving quantile regression. In Section 5, we give our results on balanced graph sparsification. In Section 6, we present our experimental results. We conclude and discuss future directions in Section 8. Most technical proofs are deferred to the appendix.

2 Preliminaries

**Notations.** We use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\). For a vector \(x \in \mathbb{R}^n\), we use \(\|x\|_p\) to denote its \(\ell_p\) norm i.e., \(\|x\|_p = (\sum_{i=1}^{n}|x_i|^p)^{1/p}\). For a matrix \(A \in \mathbb{R}^{n \times d}\), we use \(A^\dagger\) to denote its Moore-Penrose inverse. For a matrix \(A \in \mathbb{R}^{n \times d}\), we use \(A_i\) to denote its \(i\)-th row, viewed as a column vector. The leverage score of \(A_i\) is defined to be \(\tau_i(A) = A_i^\dagger (A^\dagger A)^\dagger A_i\). For two vectors \(u\) and \(v\), we use \(\langle u, v \rangle\) to denote their inner product. For a function \(\phi : \mathbb{R} \to \mathbb{R}\) and a vector \(y \in \mathbb{R}^n\), we use \(\phi(y)\) to denote \(\sum_{i=1}^{n} \phi(y_i)\).

**Graph Theory.** We give some background on graph theory, which provides the context for our graph sparsification result. In this paper, we focus on directed graphs \(G = (V, E, w)\) with edge set \(E \subseteq V \times V\), and edge weights \(w : E \to \mathbb{R}_+\). Throughout this paper, we assume that the directed graph \(G\) is strongly connected. For any two sets of vertices \(S\) and \(T\), we use \(w_G(S,T)\) to denote the total weight of edges from \(S\) to \(T\), i.e., \(w_G(S,T) = \sum_{(u,v) \in (S \times T) \cap E} w(u,v)\). We also say \(w_G(S,T)\) is the capacity of the directed cut from \(S\) to \(T\). We use \(w(S,T)\) to denote \(w_G(S,T)\) when \(G\) is clear from the context. For a directed graph \(G = (V, E, w)\) with \(n = |V|\) vertices and \(m = |E|\) edges, the edge-vertex incidence matrix \(B \in \mathbb{R}^{m \times n}\) is defined to be

\[
B_{e,u} = \begin{cases} w(e) & \text{if } e = (u, v) \\ -w(e) & \text{if } e = (v, u) \\ 0 & \text{otherwise} \end{cases}
\]

In words, each row of \(B\) corresponds to an edge \(e = (u, v)\) in \(G\), where the \(u\)-th entry is \(w(e)\), \(v\)-th entry is \(-w(e)\), and all other entries are 0.

**Lewis Weights.** We now briefly review the definition and some basic properties of Lewis weights \([12]\), which play a key role in our sampling results.

**Definition 2.1 (\(\ell_p\) Lewis weights \([12]\)).** For a given matrix \(A \in \mathbb{R}^{n \times d}\) and \(p \geq 1\), we say \(\{w_i\}_{i=1}^{n}\) are the \(\ell_p\) Lewis weights of \(A\) if they satisfy \(\tau_i(W^{1/2-1/p}A) = w_i\) for all \(i \in [n]\), where \(W = \text{diag}(w_1, \ldots, w_n)\) is the \(n \times n\) diagonal matrix whose \(i\)-th diagonal entry is \(w_i\).

We need the following fact regarding lewis weights proved in \([12]\).

**Lemma 2.1 (\([12]\)).** The \(\ell_p\) Lewis weights \(\{w_i\}_{i=1}^{n}\) always exist and are unique, and \(\sum_{i=1}^{n} w_i \leq d\).

The following theorem enables efficient (approximate) computation of Lewis weights, which we use as a subroutine in our sampling results.

**Theorem 2.2 (\([12]\)).** There is an algorithm that receives a matrix \(A \in \mathbb{R}^{n \times d}\) and \(p \geq 1\), and outputs \(\{w_i\}_{i=1}^{n}\), such that with probability at least \(\frac{1}{2} - 1/poly(d)\), for all \(i \in [n]\), \(w_i \leq w_i^\dagger \leq 2w_i\), where \(\{w_i\}_{i=1}^{n}\) are the \(\ell_p\) Lewis weights of \(A\). The algorithm runs in \(\tilde{O}(\text{nnz}(A) + d^p/p + O(1))\) time. When \(p < 4\), the algorithm runs in \(\tilde{O}(\text{nnz}(A) + d^p)\) time.

For graph sparsification specifically, we are interested in approximating Lewis weights for edge-vertex incidence matrices. The following theorem states that in such cases, the running time stated in Theorem 2.2 can be further improved.

**Theorem 2.3.** When the matrix \(A\) is the edge-vertex incidence matrix of a graph \(G\) and \(p < 4\), the algorithm in Theorem 2.2 runs in \(\tilde{O}(m)\) time, where \(m\) is the number of edges in \(G\).
3 Row Sampling Based on Lewis Weights

In this section, we present our row sampling results. Our results are built upon the Lewis weights sampling framework introduced in [12]. In particular, given a matrix $A \in \mathbb{R}^{n \times d}$, they show that if one samples $O(d \log(d/\varepsilon) / \varepsilon^2)$ rows of $A$ based on the $\ell_1$ Lewis weights (see Definition 2.1) and scales the rows properly, then one obtains a matrix $\tilde{A}$ such that with high probability, for all $x \in \mathbb{R}^d$, $(1 - \varepsilon)\|Ax\|_1 \leq \|\tilde{A}x\|_1 \leq (1 + \varepsilon)\|Ax\|_1$.

In the following theorem, we show that for any function $\phi(\cdot)$ of the form $\phi(x) = ax + bx$ for some $0 \leq b \leq a$, sampling according to $\ell_1$ Lewis weights still approximately preserves the value of $\phi(Ax)$ for all $x \in \mathbb{R}^d$. Our result generalizes that of [12], in the sense that if we set $a = 1$ and $b = 0$, we recover exactly their bound for the $\ell_1$ norm. Later, based on Theorem 3.1, we shall give a row sampling algorithm for the quantile loss function, and also provide an algorithm for producing sparsifiers for balanced directed graphs.

**Theorem 3.1.** For a given matrix $A \in \mathbb{R}^{n \times d}$, let $\{w_i\}_{i=1}^n$ be its $\ell_1$ Lewis weights, and suppose the function $\phi(x) = ax + bx$ (0 \leq b \leq a) satisfies that $\|Ax\|_1 \leq B \cdot \phi(Ax)$ for all $x \in \mathbb{R}^d$ for some $B > 0$. There exists an absolute constant $C$ such that the following holds.

For any values $p_1, \ldots, p_n \geq 0$ such that $\sum_{i \in [n]} p_i = N$ and $p_i \geq C(aB/\varepsilon)^2 w_i \log N$, if we generate a matrix $\tilde{A}$ with $N$ rows, where each row is chosen independently as $A_i / p_i$ with probability $p_i / N$, then with probability at least $1 - 1/\text{poly}(d)$, it holds for all $x \in \mathbb{R}^d$ simultaneously that

$$(1 - \varepsilon)\phi(Ax) \leq \phi(\tilde{A}x) \leq (1 + \varepsilon)\phi(Ax).$$

**Proof (sketch).** We follow the framework in [12]. Suppose that $i_1, i_2, \ldots, i_N$ are the indices chosen by the sampling process. Our goal is to derive an upper bound on

$$F = \sup_{x \in \phi(\tilde{A}x) = 1} |\phi(\tilde{A}x) - 1| = \sup_{x \in \phi(Ax) = 1} \left| \sum_{k=1}^N \frac{\phi(A_{i_k}x)}{p_{i_k}} - 1 \right|.$$

We shall show it holds for some $\ell$ that

$$M = \mathbb{E}_{i_1, \ldots, i_N} F^\ell \leq \frac{\varepsilon^\ell}{\text{poly}(d)},$$

and thus the result would follow from Markov’s inequality. To prove the moment bound above, we first apply the symmetrization technique and obtain that

$$M = \mathbb{E} \sup_{i \in [n]} \left| \sum_{k=1}^N \frac{\phi(A_{i_k}x)}{p_{i_k}} - 1 \right| \leq 2^\ell \mathbb{E} \left( \sup_{\phi(Ax) = 1} \left| \sum_{k=1}^N \frac{\phi(A_{i_k}x)}{p_{i_k}} - 1 \right| \right)^\ell.$$  

where $i = (i_1, i_2, \ldots, i_N)$, and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ is a Rademacher sequence. Using the comparison theorem of Rademacher processes, we can show that

$$\mathbb{E} \left( \sup_{\phi(Ax) = 1} \left| \sum_{k=1}^N \sigma_k \frac{\phi(A_{i_k}x)}{p_{i_k}} \right| \right)^\ell \leq 3d^\ell 2^{\ell - 1} \mathbb{E} \sup_{\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_N\}} \left| \sum_{k=1}^N \sigma_k \frac{\phi(A_{i_k}x)}{p_{i_k}} \right|.$$  

The supremum on the right-hand side is similar to the Rademacher process considered in [12]. Analogously we can show that, conditioned on the event that for all $x \in \mathbb{R}^d$, $\|\tilde{A}x\|_1 \leq C_1 \|Ax\|_1$, which holds with high probability for some absolute constant $C_1 > 0$, the expectation can be upper bounded as

$$\mathbb{E} \sup_{\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_N\}} \left| \sum_{k=1}^N \sigma_k \frac{\phi(A_{i_k}x)}{p_{i_k}} \right| \leq B \varepsilon^\ell \mathbb{E} \sup_{\sigma \in \{\sigma_1, \sigma_2, \ldots, \sigma_N\}} \left| \sum_{k=1}^N \sigma_k \frac{\phi(A_{i_k}x)}{p_{i_k}} \right| \leq B \varepsilon^\ell (\varepsilon/(C' aB))^{\ell} \text{poly}(d)$$

for some $\ell = \Theta(\log(N + d^2))$, where $C'$ is another absolute constant. Thus (5) holds if the expectation is conditioned on the event defined above. The failure probability of the event would be added to the overall failure probability, concluding the proof.
Now Theorem 4.1 immediately follows from Theorem 3.1. See Section B in the supplementary material for formal proofs.

We have successfully applied the Lewis weights sampling framework to provide row sampling algorithms for the quantile loss function. In general, one may wonder if it is possible to apply Lewis weights sampling to more general loss functions. Cohen and Peng show that the Lewis weights sampling works gracefully when the loss function is the $\ell_p$ norm. In Section 6, we show that when the loss function $\phi(x)$ can be approximated by $\|x\|_p^p$, one can preserve $\phi(Ax)$ approximately for all $x \in \mathbb{R}^d$, by sampling proportional to $\ell_p$ Lewis weights and using a weighted version of the loss function $\phi(\cdot)$.

4 Solving Quantile Regression

Based on the row sampling result in Theorem 4.1 and recent advances for $\ell_1$ regression \cite{18}, we obtain a faster algorithm for solving the quantile regression problem. The description of the algorithm is given in Algorithm 1, and we prove its running time and correctness in Theorem 4.4.

Algorithm 1. Algorithm for solving quantile regression

1: Obtain approximate $\ell_1$ Lewis weights for each row of $[A, b]$ using Theorem 2.2.
2: Sample $N = O\left(\frac{d}{\varepsilon^2 \log \frac{d}{\varepsilon}}\right)$ rows of $[A, b]$ according to $\ell_1$ Lewis weights and obtain $[\tilde{A}, \tilde{b}] \in \mathbb{R}^{N \times (d+1)}$.
3: Let $\tilde{A} = QR$ be the QR-decomposition, where $Q \in \mathbb{R}^{N \times d}$ and $R \in \mathbb{R}^{d \times d}$.
4: Run the Katyusha algorithm \cite{17} with $x_0 = (\tilde{A}R^{-1})^T b$ to obtain $\tilde{x}$ such that
   \[
   \rho_\tau \left( \tilde{A}R^{-1}\tilde{x} - \tilde{b} \right) \leq \left( 1 + \frac{\varepsilon}{3} \right) \min_{x \in \mathbb{R}^d} \rho_\tau \left( \tilde{A}R^{-1}x - \tilde{b} \right). \tag{6}
   \]
5: Return $x^* = R^{-1}\tilde{x}$.

The following lemma, which is an easy corollary of Theorem 4.1, shows that if $\tilde{x}$ satisfies the condition in (6), then it is indeed an approximate solution to $\min_{x \in \mathbb{R}^d} \rho_\tau (Ax - b)$.

Lemma 4.1. For sufficiently small $\varepsilon > 0$, if $\tilde{x}$ satisfies the condition in (6), with probability at least 0.9, the solution $x^* \in \mathbb{R}^d$ returned by Algorithm 1 satisfies
   \[
   \rho_\tau (Ax^* - b) \leq (1 + \varepsilon) \min_{x \in \mathbb{R}^d} \rho_\tau (Ax - b).
   \]

Thus, we focus on obtaining an approximate solution to $\min_{x \in \mathbb{R}^d} \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$ hereinafter. Our general strategy is to apply the accelerated SGD to $\min_{x \in \mathbb{R}^d} \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$. To this end, we formulate $\min_{x \in \mathbb{R}^d} \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$ as a finite-sum problem. The following lemma shows that after applying Lewis weights sampling, with constant probability, each summand in the finite-sum problem has a bounded Lipschitz constant (see Definition A.1 in the supplementary material for the definition), and $x_0 = (\tilde{A}R^{-1})^T b$ is close to the optimal solution.

Lemma 4.2. Let $x_0 = (\tilde{A}R^{-1})^T b$ and $\tilde{x}^{\text{opt}} = \arg\min_{x \in \mathbb{R}^d} \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$. With probability at least 0.9,

(i) the function $f(x) = \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$ can be written as $f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x)$, where each $f_i(\cdot)$ is $O(\sqrt{Nd})$-Lipschitz; and

(ii) it holds that
   \[
   \|x_0 - \tilde{x}^{\text{opt}}\|_2 \leq \sqrt{d/(N\tau^2)} \cdot \rho_\tau (\tilde{A}R^{-1}\tilde{x}^{\text{opt}} - \tilde{b})
   \]
   and
   \[
   \rho_\tau (\tilde{A}R^{-1}x_0 - \tilde{b}) \leq \sqrt{Nd} \cdot \rho_\tau (\tilde{A}R^{-1}\tilde{x}^{\text{opt}} - \tilde{b}).
   \]

\footnote{Katyusha is an accelerated stochastic gradient descent algorithm by Allen-Zhu \cite{17} for minimization of a convex function. Here the convex function is $x \mapsto \rho_\tau (\tilde{A}R^{-1}x - \tilde{b})$ and we use the variant in Corollary 3.7 of \cite{17}. The justification of applying this variant of Katyusha is given by Lemma 4.2.}
Now we apply Katyusha, an accelerated SGD algorithm, from [1]. The correctness and running time is summarized in the following lemma.

**Lemma 4.3.** In Algorithm 1, Step 4 obtains \( \vec{\pi} \) which satisfies that

\[
\rho_\tau(\tilde{A}R^{-1}\vec{x} - \vec{b}) \leq (1 + \varepsilon/3) \min_{\vec{x} \in \mathbb{R}^d} \rho_\tau(\tilde{A}R^{-1}\vec{x} - \vec{b})
\]

with probability at least 0.8, and Step 4 runs in \( \tilde{O}(d^{2.5}/(\varepsilon^2\tau^2)) \) time.

Now we give the efficiency and correctness of Algorithm 1.

**Theorem 4.4.** Algorithm 1 runs in time \( \tilde{O}(\text{nzz}(A) + d^{2.5}/(\varepsilon^2\tau^2)) \) and with probability at least 0.7 returns a vector \( \vec{x}^* \in \mathbb{R}^d \) such that

\[
\rho_\tau(A\vec{x}^* - \vec{b}) \leq (1 + \varepsilon) \cdot \min_{\vec{x} \in \mathbb{R}^d} \rho_\tau(A\vec{x} - \vec{b}).
\]

We note that the failure probability of Algorithm 1 can be reduced to an arbitrarily small constant by independent repetitions and taking the best solution found among all repetitions.

## 5 Balanced Graph Sparsification

In this section, we demonstrate how our sampling results in Section 4 can be applied to graph sparsification. In particular, we prove that our sampling scheme can be used to find a sparsifier of balanced directed graphs, a notion defined in [19] as follows.

**Definition 5.1.** For a strongly connected directed graph \( G = (V, E, w) \) and \( \alpha \geq 1 \), we say \( G \) is \( \alpha \)-balanced if for every subset \( S \subset V \), we have \( w(S, V \setminus S) \leq \alpha \cdot w(V \setminus S, S) \).

Recall that \( w(S, T) \) is the capacity of the directed cut between \( S \) and \( T \). Therefore, the preceding definition essentially says that a graph is balanced if the outgoing capacity from any set of vertices \( S \) is roughly the same (up to a factor of \( \alpha \)) as the incoming capacity to \( S \). The parameter \( \alpha \) measures how balanced the graph is. In particular, any undirected graph is 1-balanced.

Our plan is to apply our sampling results on the edge-vertex incidence matrix of the balanced graph. In order to do that, we need first to relate balanced graphs to the quantile loss function, and show that the incidence matrix indeed satisfies the conditions required by our sampling results. In particular, we shall need the following lemma regarding balanced directed graphs.

**Lemma 5.1.** For a given strongly connected directed graph \( G = (V, E, w) \) with \( n \) vertices and \( m \) edges, let \( B \in \mathbb{R}^{m \times n} \) be the edge-vertex incidence matrix of \( G \). If \( G \) is \( \alpha \)-balanced, then it holds for all \( \vec{x} \in \mathbb{R}^n \) that \( \|B\vec{x}\|_1 \leq (1 + \alpha)\rho_0(B\vec{x}) \).

Now we are ready to apply our sampling schemes to balanced graphs. As another application of Theorem 5.1, we give a randomized construction of a sparsifier of \( O(n(\alpha/\varepsilon)^2 \log(an/\varepsilon)) \) edges for \( \alpha \)-balanced graphs. The algorithm is given in Algorithm 2 and we prove its correctness and running time in Corollary 5.2.

**Algorithm 2** Algorithm for sparsifying balanced directed graphs.

1. Obtain approximate \( \ell_1 \) Lewis weights for each row of the edge-vertex incidence matrix \( B \) of the given \( \alpha \)-balanced graph \( G \) using Theorem 2.3.
2. Sample \( m' = O(n(\alpha/\varepsilon)^2 \log(an/\varepsilon)) \) rows of \( B \) according to approximate \( \ell_1 \) Lewis weights and obtain \( B' \in \mathbb{R}^{m' \times n} \).
3. Return the directed graph \( G' \) that corresponds to the edge-vertex incidence matrix \( B' \).

**Corollary 5.2.** For a given \( \alpha \)-balanced strongly connected directed graph \( G = (V, E, w) \) with \( n \) vertices and \( m \) edges, let \( B \in \mathbb{R}^{m \times n} \) be the edge-vertex incidence matrix of \( G \). Algorithm 2 outputs \( G' = (V, E', w') \) with \( E' \subseteq E \) and \( |E'| = m' = O(n(\alpha/\varepsilon)^2 \log(an/\varepsilon)) \) in \( \tilde{O}(m) \) time. Let \( B' \in \mathbb{R}^{m' \times n} \) be the edge-vertex incidence matrix of \( G' \). With probability at least \( 1 - 1/poly(n) \), for all \( \vec{x} \in \mathbb{R}^d \),

\[
(1 - \varepsilon)\rho_0(B\vec{x}) \leq \rho_0(B'\vec{x}) \leq (1 + \varepsilon)\rho_0(B\vec{x}).
\]
Remark 1. The sparsifier $G'$ obtained in Corollary 5.2 is a cut sparsifier, meaning that it approximately preserves the capacities of all cuts of $G$. To see this, for any given $S \subseteq V$, let $x_S \in \{0,1\}^n$ be the indicator vector of $S$, i.e., $(x_S)_i = \begin{cases} 1 & \text{if } i \notin S \\ 0 & \text{otherwise} \end{cases}$, then $\rho_0(Bx) = w_G(S, V \setminus S)$. Similarly, $\rho_0(B'x) = w_G(S, V \setminus S)$. Since for all $x \in \mathbb{R}^d$, $(1 - \varepsilon)\rho_0(Bx) \leq \rho_0(B'x) \leq (1 + \varepsilon)\rho_0(Bx)$, it follows that for all $S \subseteq V$, $(1 - \varepsilon)w_G(S, V \setminus S) \leq w_{G'}(S, V \setminus S) \leq (1 + \varepsilon)w_G(S, V \setminus S)$.

6 Row Sampling for Loss Functions Approximated by $\ell_p$ Norms

In Section 5, we have shown that $\ell_1$ Lewis weights sampling approximately preserves the quantile loss function $\rho_1(Ax)$ for all $x \in \mathbb{R}^d$. A natural question is whether it is possible to preserve other loss functions using $\ell_p$ Lewis weights sampling. The following theorem shows that it is indeed possible. However, owing to the lack of homogeneity in the loss function, we have to use a weighted version of the loss function as the estimator. The proof is similar to that of Theorem 5.1 and is provided in Section C in the supplementary material.

Theorem 6.1. Let $p \geq 1$. Suppose $\phi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $|\phi(x) - \phi(y)| \leq L \|x\|^p - |y|^p$ for all $x, y \in \mathbb{R}$ for some $L > 0$ and $\phi(x) \geq \gamma \|x\|^p$ for all $x \in \mathbb{R}$ for some $\gamma > 0$.

Let matrix $A \in \mathbb{R}^{n \times d}$ and $\{w_i\}_{i=1}^n$ be its $\ell_p$ Lewis weights. For any values $p_1, \ldots, p_n \geq 0$ such that $\sum_{i \in [n]} p_i = N$ and $p_i \geq f(p, d, \frac{\varepsilon}{\gamma}, \delta) w_i$, the following holds.

Let $i_1, \ldots, i_N$ be i.i.d. random variables such that $\Pr[i_k = j] = p_j/N$ for all $k \in [N]$. Define $w_i = 1/p_{i_k}$.

Construct $\tilde{A} \in \mathbb{R}^{N \times d}$ so that $\tilde{A}_{ik} = A_{ik}$ for each $k \in [N]$. With probability at least $1 - \delta$, it holds for all $x \in \mathbb{R}^d$ simultaneously that

$$(1 - \varepsilon)\phi(\tilde{A}x) \leq \phi^w(\tilde{A}x) \leq (1 + \varepsilon)\phi(\tilde{A}x),$$

where $\phi^w(y) = \sum_{i=1}^N w_i \phi(y_i)$ for $y \in \mathbb{R}^N$.

Bounds on $f(p, d, \varepsilon, \delta)$ are given in the table below, where $C_p$ is a constant depending only on $p$ and $C$ is an absolute constant.

| $p$ | $\delta$ | $f(p, d, \varepsilon, \delta)$ |
|-----|----------|-----------------------------|
| $p = 1$ | $1/\text{poly}(d)$ | $C(1/\varepsilon^2) \log(d/\varepsilon)$ |
| $1 < p < 2$ | constant | $C(1/\varepsilon^2) \log(d/\varepsilon) \log^2(d/\varepsilon)$ |
| $p \geq 2$ | $1/\text{poly}(d)$ | $C_p(1/\varepsilon^2) d^{p/2} \log^2(d/\varepsilon)$ |

Therefore, to obtain a row sampling algorithm for loss functions that satisfy the conditions in Theorem 6.1, we first invoke the algorithm in Theorem 2.2 to obtain approximate $\ell_p$ Lewis weights of the input matrix $A$ and then sample rows of $A$ according to Theorem 6.1. The total number of sampled rows $N = \sum_{i=1}^n p_i = O(d \cdot f(p, d, \varepsilon, \delta))$ by Lemma 2.1.

For solving linear regression, the problem is usually convex, in which case $\phi^w(\cdot)$ is also convex. Therefore, to obtain an approximate solution to the linear regression problem $\min_{x \in \mathbb{R}^d} \phi(Ax - b)$, one just needs to solve a much smaller instance $\min_{x \in \mathbb{R}^d} \phi^w(\tilde{A}x - b)$, which can still be formulated as a convex program and thus can be efficiently solved.

We remark that by using the notion of $\ell_p$ leverage score and techniques in [14], we may obtain a results similar to Theorem 6.1. However, the number of samples required by such an approach will be much larger than that of Theorem 6.1.

7 Empirical Evaluation of Row Sampling Algorithms

In this section, we conduct an empirical evaluation of our sampling scheme against the three algorithms proposed in [44], which are the current state of the art. The three algorithms, codenamed SPC1, SPC2 and SPC3, are all based on row sampling, but the weights are very different from Lewis weights. We also compare with uniform sampling as a baseline, which is widely used in practice for its simplicity. Throughout this section, we use all sampling algorithms only for preprocessing, and solve the reduced problem optimally using linear programming. This allows us to remove the possible bias introduced by different optimization methods and focus on the comparison of the sampling schemes.

All tests are run under MATLAB 2019b on a machine of Intel Core i7-6550U CPU@2.20GHz with 2 cores.
Figure 1: Mean of the relative errors of the solution vector (measured in three different norms, the $\ell_2$ norm, the $\ell_1$ norm and the $\ell_\infty$ norm) of 3 different methods on synthetic data. The plotted lines represent the mean of the errors of 50 independent trials.

Synthetic Data. We first evaluate our sampling scheme on a synthetic dataset using the same setup as in [44]. Here, we only compare with SPC2 and SPC3, which, as demonstrated in [44], significantly outperform all other sampling schemes on this dataset.

As described in [44], the data set is generated in the following manner. Each row of the matrix $A \in \mathbb{R}^{n \times d}$ is a canonical basis vector. Suppose the number of occurrences of $e_j$ is $c_j$, where $c_j = qc_{j-1}$, for $j = 2, \ldots, d$ and some $q \in (1, 2)$ — such distributions of rows introduce imbalanced measurements and therefore increase the difficulty of sampling. The true vector $x^* \in \mathbb{R}^d$ has i.i.d. standard normal coordinates. Let $b^* = Ax^*$. The noise vector $\nu$ is generated with independent Laplacian entries, scaled such that $\|\nu\|_1/\|b^*\|_1 = 0.2$. The response vector $b$ is given by $b_i = 500\nu_i$ with probability 0.001 and $b_i = b_i^* + \nu_i$ with probability 0.999. Here, we adhere to the same generation process so that our experimental results are comparable to those in [44].

In a similar manner to [44], rather than determining the sample size from a given distortion parameter $\varepsilon$, we vary the quantile parameter $\tau \in \{0.5, 0.75, 0.95\}$, and for each $\tau$, vary the sample size among $\{100, 200, \ldots, 1000\}$. Here the parameter $\tau$ corresponds to the parameter in the function $h_\tau(t)$ defined in Equation (1) rather than the renormalized function $\rho_\tau(t)$ defined in Equation (3). We choose $n = 10^5$ and $d = 50$.

See Figure 1 for results on the synthetic dataset and Figure 3 for the running time. When the same number of samples is at least 400, our sampling scheme (LEW) achieves the best approximation to $x$, outperforming both SPC2 and SPC3 in all three norms ($\ell_\infty$, $\ell_1$ and $\ell_2$) under the setting $\tau = 0.5$ and 0.75. When $\tau = 0.95$, our algorithm achieves a similar approximation error to that of SPC2, which are both better than SPC3. This suggests that when the input is highly imbalanced and therefore more difficult to process, our method is accurate and stable with a faster running time.

Census Data. We also evaluate our algorithm on a real-world dataset, the U.S. 2000 Census data\footnote{http://www.census.gov/census2000/PUMS5.html} which consists of salary and related information on people who met certain criteria. As in [44], we conduct an
experiment on the same 5% sample of the census data. In the corresponding regression problem, the matrix $A \in \mathbb{R}^{5048299 \times 11}$ is the feature matrix and the vector $b \in \mathbb{R}^{5048299}$ records the salary of the sampled people. Our goal is to solve the regression problem $\min_{x} h_{\tau}(Ax - b)$, whose minimizer is denoted by $x^\ast$. We conduct a similar experiment as above with same values of $\tau$ and varying sample sizes in $\{1000, 2000, \ldots, 10000\}$, which are at most 0.2% of the data size.

See Figure 2 for approximation errors to $x^\ast$ on the census data, where we consider all methods in [44]. Our method, quite naturally, outperforms uniform sampling and SCP1 by a considerable margin, and achieves almost the same relative error as, or a marginally smaller relative error in certain cases than, SCP2 and SPC3. The running time of different methods are compared in Figure 4. Our method runs significantly faster than SCP2 and SPC3, suggesting that even on relatively easy-to-process real-world data, our method is robust with a superior running time.

8 Conclusion

In this paper, we give the first row sampling algorithm for the quantile loss function with sample complexity nearly linear in the dimensionality of the data. Technically, we show that Lewis weights sampling can be applied in row sampling algorithms for a variety of loss functions.

There are a few interesting problems that remain open. Is it possible to prove lower bounds on sample complexity of row sampling algorithms for the quantile loss function using techniques in [27]? Furthermore, is it possible to use Lewis weights sampling to obtain nearly-linear sample complexity for other loss functions, e.g. the $M$-estimators studied in [10] [9]? Answering these problems would lead to a better understanding of the power and limitation of row sampling algorithms.
Figure 3: Mean running time of 50 independent trials on synthetic data.

Figure 4: Mean running time of 50 independent trials on census data.

Acknowledgements

Y. Li was supported in part by a Singapore Ministry of Education (AcRF) Tier 2 grant MOE2018-T2-1-013. H. Zhang was supported by an National Science Foundation (NSF) grant IIS-1814056. The authors would like to thank Tianyi Zhang for insightful discussions.

References

[1] Z. Allen-Zhu. Katyusha: The first direct acceleration of stochastic gradient methods. The Journal of Machine Learning Research, 18(1):8194–8244, 2017.

[2] J. Bourgain, J. Lindenstrauss, and V. Milman. Approximation of zonoids by zonotopes. Acta mathematica, 162(1):73–141, 1989.

[3] X. Chen, W. Liu, Y. Zhang, et al. Quantile regression under memory constraint. The Annals of Statistics, 47(6):3244–3273, 2019.

[4] J. Chowdhury, P. Chaudhuri, et al. Nonparametric depth and quantile regression for functional data. Bernoulli, 25(1):395–423, 2019.

[5] K. L. Clarkson. Subgradient and sampling algorithms for $\ell_1$ regression. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms, pages 257–266. Society for Industrial and Applied Mathematics, 2005.

[6] K. L. Clarkson, P. Drineas, M. Magdon-Ismail, M. W. Mahoney, X. Meng, and D. P. Woodruff. The fast cauchy transform and faster robust linear regression. SIAM Journal on Computing, 45(3):763–810, 2016.

[7] K. L. Clarkson, R. Wang, and D. P. Woodruff. Dimensionality reduction for tukey regression. In Proceedings of the Thirty-sixth International Conference on Machine Learning, 2019.

[8] K. L. Clarkson and D. P. Woodruff. Low rank approximation and regression in input sparsity time. In Proceedings of the forty-fifth annual ACM symposium on Theory of computing, pages 81–90. ACM, 2013.

[9] K. L. Clarkson and D. P. Woodruff. Input sparsity and hardness for robust subspace approximation. In 2015 IEEE 56th Annual Symposium on Foundations of Computer Science, pages 310–329. IEEE, 2015.

[10] K. L. Clarkson and D. P. Woodruff. Sketching for m-estimators: A unified approach to robust regression. In Proceedings of the twenty-sixth annual ACM-SIAM symposium on Discrete algorithms, pages 921–939. Society for Industrial and Applied Mathematics, 2015.
[11] M. B. Cohen, Y. T. Lee, C. Musco, C. Musco, R. Peng, and A. Sidford. Uniform sampling for matrix approximation. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science*, pages 181–190. ACM, 2015.

[12] M. B. Cohen and R. Peng. \( \ell_p \) row sampling by lewis weights. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pages 183–192. ACM, 2015.

[13] T. J. Cole and P. J. Green. Smoothing reference centile curves: the lms method and penalized likelihood. *Statistics in medicine*, 11(10):1305–1319, 1992.

[14] A. Dasgupta, P. Drineas, B. Harb, R. Kumar, and M. W. Mahoney. Sampling algorithms and coresets for \( \ell_p \) regression. *SIAM Journal on Computing*, 38(5):2060–2078, 2009.

[15] C. Davino, M. Furno, and D. Vistocco. *Quantile regression: theory and applications*, volume 988. John Wiley & Sons, 2013.

[16] J. Deng, W. Dong, R. Socher, L.-J. Li, K. Li, and L. Fei-Fei. ImageNet: A Large-Scale Hierarchical Image Database. In *CVPR09*, 2009.

[17] P. Drineas and M. W. Mahoney. Effective resistances, statistical leverage, and applications to linear equation solving. *arXiv preprint arXiv:1005.3097*, 2010.

[18] D. Durfee, K. A. Lai, and S. Sawlani. \( \ell_1 \) regression using Lewis weights preconditioning and stochastic gradient descent. In *Conference On Learning Theory*, pages 1626–1656, 2018.

[19] A. Ene, G. Miller, J. Pachocki, and A. Sidford. Routing under balance. In *Proceedings of the Forty-eighth Annual ACM Symposium on Theory of Computing*, STOC ’16, pages 598–611, New York, NY, USA, 2016. ACM.

[20] M. Ikeda and S.-i. Tanigawa. Cut sparsifiers for balanced digraphs. In L. Epstein and T. Erlebach, editors, *Approximation and Online Algorithms*, pages 277–294, Cham, 2018. Springer International Publishing.

[21] R. Koenker and G. Bassett Jr. Regression quantiles. *Econometrica: journal of the Econometric Society*, pages 33–50, 1978.

[22] R. Koenker and O. Geling. Reappraising medfly longevity: a quantile regression survival analysis. *Journal of the American Statistical Association*, 96(454):458–468, 2001.

[23] R. Koenker and K. F. Hallock. Quantile regression. *Journal of economic perspectives*, 15(4):143–156, 2001.

[24] M. Ledoux and M. Talagrand. Comparison theorems, random geometry and some limit theorems for empirical processes. *Ann. Probab.*, 17(2):596–631, 04 1989.

[25] M. Ledoux and M. Talagrand. *Probability in Banach Spaces: Isoperimetry and Processes*. Springer, 1991.

[26] M. Li, G. L. Miller, and R. Peng. Iterative row sampling. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 127–136. IEEE, 2013.

[27] Y. Li, R. Wang, and D. P. Woodruff. Tight bounds for the subspace sketch problem with applications. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1655–1674. SIAM, 2020.

[28] X. Meng and M. W. Mahoney. Low-distortion subspace embeddings in input-sparsity time and applications to robust linear regression. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 91–100. ACM, 2013.
[29] J. Nelson and H. L. Nguyen. OSNAP: Faster numerical linear algebra algorithms via sparser subspace embeddings. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on*, pages 117–126. IEEE, 2013.

[30] H. Ota, K. Kato, S. Hara, et al. Quantile regression approach to conditional mode estimation. *Electronic Journal of Statistics*, 13(2):3120–3160, 2019.

[31] G. R. Pandey and V.-T.-V. Nguyen. A comparative study of regression based methods in regional flood frequency analysis. *Journal of Hydrology*, 225(1-2):92–101, 1999.

[32] P. Royston and D. G. Altman. Regression using fractional polynomials of continuous covariates: parsimonious parametric modelling. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 43(3):429–453, 1994.

[33] P. Royston and E. Wright. Goodness-of-fit statistics for age-specific reference intervals. *Statistics in medicine*, 19(21):2943–2962, 2000.

[34] C. Sohler and D. P. Woodruff. Subspace embeddings for the $l_1$-norm with applications. In *Proceedings of the forty-third annual ACM symposium on Theory of computing*, pages 755–764. ACM, 2011.

[35] Z. Song, R. Wang, L. Yang, H. Zhang, and P. Zhong. Efficient symmetric norm regression via linear sketching. In *Advances in Neural Information Processing Systems*, pages 828–838, 2019.

[36] D. A. Spielman and N. Srivastava. Graph sparsification by effective resistances. *SIAM Journal on Computing*, 40(6):1913–1926, 2011.

[37] M. Talagrand. Embedding subspaces of $L^1$ into $\ell^q_1$. *Proceedings of the American Mathematical Society*, 108(2):363–369, 1990.

[38] M. Talagrand. Embedding subspaces of $L^p$ into $\ell^q_p$. In *Geometric aspects of functional analysis*, pages 311–326. Springer, 1995.

[39] J. A. Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of computational mathematics*, 12(4):389–434, 2012.

[40] R. Wang and D. P. Woodruff. Tight bounds for $\ell_p$ oblivious subspace embeddings. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1825–1843. SIAM, 2019.

[41] V. V. Williams. Multiplying matrices faster than coppersmith-winograd. In *STOC*, volume 12, pages 887–898. Citeseer, 2012.

[42] D. P. Woodruff. Sketching as a tool for numerical linear algebra. *Foundations and Trends® in Theoretical Computer Science*, 10(1–2):1–157, 2014.

[43] D. P. Woodruff and Q. Zhang. Subspace embeddings and $\ell_p$-regression using exponential random variables. In *Conference on Learning Theory*, pages 546–567, 2013.

[44] J. Yang, X. Meng, and M. Mahoney. Quantile regression for large-scale applications. In *International Conference on Machine Learning*, pages 881–887, 2013.

[45] W. Zhao, H. Lian, H. Liang, et al. Quantile regression for the single-index coefficient model. *Bernoulli*, 23(3):1997–2027, 2017.

[46] Q. Zheng, L. Peng, and X. He. High dimensional censored quantile regression. *Annals of statistics*, 46(1):308, 2018.
A Additional Background

A.1 Rademacher Process

We need the following classical results regarding comparison of Rademacher processes.

Lemma A.1 ([24 Proposition 1]). Let $F : [0, \infty) \to [0, \infty)$ be convex and increasing. Let $\sigma_1, \ldots, \sigma_n$ be a Rademacher sequence. Then for any bounded subset $T \subset \mathbb{R}^n$ it holds that

$$\mathbb{E} F \left( \sup_{t \in T} \left| \sum_{i=1}^{n} \sigma_i t_i \right| \right) \leq 2 \mathbb{E} F \left( \sup_{t \in T} \left| \sum_{i=1}^{n} \sigma_i \right| \right).$$

Lemma A.2 (Contraction Principle, Theorem 4.4 in [24]). Let $F : [0, \infty) \to [0, \infty)$ be convex. Let $\sigma_1, \ldots, \sigma_n$ be a Rademacher sequence. Then for any bounded subset $T \subset \mathbb{R}^n$ it holds that

$$\mathbb{E} F \left( \left| \sum_{i=1}^{n} \sigma_i a_i x_i \right| \right) \leq \mathbb{E} F \left( \left| \sum_{i=1}^{n} \sigma_i x_i \right| \right).$$

The next one is a consequence of the contraction principle (see, e.g., Theorem 4.12 in [25]).

Lemma A.3. Let $F : [0, \infty) \to [0, \infty)$ be convex and increasing. Suppose that $f(x), g(x)$ are nonnegative functions such that $|f(x) - f(y)| \leq L|g(x) - g(y)|$ for all $x, y \in \mathbb{R}$. Let $\sigma_1, \ldots, \sigma_n$ be a Rademacher sequence. Then for any bounded subset $T \subset \mathbb{R}^n$ it holds that

$$\mathbb{E} F \left( \frac{1}{2L} \sup_{t \in T} \left| \sum_{i=1}^{n} \sigma_i f(t_i) \right| \right) \leq \mathbb{E} F \left( \sup_{t \in T} \left| \sum_{i=1}^{n} \sigma_i g(t_i) \right| \right).$$

A.2 Convex Optimization

We recall some results in convex optimization.

Definition A.1. A function $f : \mathbb{R}^d \to \mathbb{R}$ is $G$-Lipschitz if for any $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \leq G \|x - y\|_2.$$

We need the following result in [1].

Theorem A.4 ([1 Corollary 3.7]). For a given function $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, let $x^* = \arg \min_{x \in \mathbb{R}^d} f(x)$. If each $f_i : \mathbb{R}^d \to \mathbb{R}$ is convex and $\sqrt{G}$-Lipschitz, then there is an algorithm that receives an initial solution $x_0 \in \mathbb{R}^d$, and outputs a solution $x \in \mathbb{R}^d$ satisfying $\mathbb{E} [f(x)] - f(x^*)$ in

$$T = O \left( n \log \frac{f(x_0) - f(x^*)}{\varepsilon} + \frac{\sqrt{nG} \|x_0 - x^*\|}{\varepsilon} \right)$$

stochastic subgradient iterations.

B Missing Proofs

B.1 Proof of Theorem 2.3

Proof. By Lemma 2.4 in [12], to calculate approximate $\ell_1$ Lewis weights of rows of a matrix $A$, it suffices to calculate approximate leverage scores of rows of matrices of the form $WA$, for $O(\log n)$ different diagonal matrices $W \in \mathbb{R}^n$.

When $A$ is the edge-vertex incidence matrix of a graph $G$, $WA$ is the edge-vertex incidence matrix of a reweighted graph $G'$. In this case, leverage scores of rows of $WA$ are the effective resistances of $G'$ (cf. [17]), which can be computed in $\tilde{O}(m)$ time using the algorithm in [36].

\end{document}
B.2 Proof of Theorem 3.1

Proof of Theorem 3.1 Suppose that $i_1, i_2, \ldots, i_N$ are the indices of $\{e_1, \ldots, e_m\}$ randomly chosen in the construction of $S$. Since $\phi(at) = \alpha \phi(t)$ for $\alpha > 0$, for each coordinate of $Ax$, we have

$$E \phi((\tilde{A}x)_k) = E \phi \left( \frac{1}{p_{i_k}} \langle A_{i_k}, x \rangle \right) = \sum_{j=1}^{n} \frac{1}{p_j} \phi(A_j^\top x) \cdot \frac{p_j}{N} = \frac{1}{N} \phi(Ax)$$

and

$$E \phi(\tilde{Ax}) = \sum_{k=1}^{N} E \phi((\tilde{A}x)_k) = \phi(Ax).$$

We just assume that $p_i \geq C_s \varepsilon \cdot \log N$ for now and shall rescale $\varepsilon$ in the end. The $\ell_1$ Lewis weight sampling result in [12] implies that with probability at least $1 - 1/\text{poly}(N)$ (over $i_1, \ldots, i_N$),

$$||\tilde{Ax}||_1 \leq C_1 ||Ax||_1, \quad \forall x \in \mathbb{R}^d.$$  \text{(7)}

We shall condition on this event in the rest of the proof. Our goal is to derive a tail inequality for

$$\sup_{x: \phi(Ax)=1} |\phi(\tilde{Ax}) - 1| = \sup_{x: \phi(Ax)=1} \left| \sum_{k=1}^{N} \frac{\phi((A_{i_k}, x))}{p_{i_k}} - 1 \right|.$$  \text{(8)}

We shall look at a higher moment of the deviation and apply Markov’s inequality. To this end, we investigate

$$M = E_S \left( \sup_{\phi(Ax)=1} |\phi(\tilde{Ax}) - 1| \right) = E_i \left( \sup_{\phi(Ax)=1} \left| \sum_{k=1}^{N} \frac{\phi((A_{i_k}, x))}{p_{i_k}} - 1 \right| \right)^\ell,$$

where $i = (i_1, i_2, \ldots, i_N)$.

A standard symmetrization argument gives that

$$M \leq 2^\ell \mathbb{E}_{i, \sigma} \left( \sup_{\phi(Ax)=1} \left| \sum_{k=1}^{N} \frac{\phi((A_{i_k}, x))}{p_{i_k}} \right| \right)^\ell,$$  \text{(9)}

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ is a Rademacher sequence. It follows that

$$\mathbb{E}_{i, \sigma} \left( \sup_{\phi(Ax)=1} \left| \sum_{k=1}^{N} \frac{\phi((A_{i_k}, x))}{p_{i_k}} \right| \right)^\ell$$

$$= \mathbb{E}_{i, \sigma} \sup_{\phi(Ax)=1} \left| a \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} + b \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right| \ell$$

$$\leq \mathbb{E}_{i, \sigma} \sup_{\phi(Ax)=1} \left( \left| a \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right| \ell + \left( b \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right) \ell \right)$$

$$\leq 2^{\ell-1} \cdot \mathbb{E}_{i, \sigma} \sup_{\phi(Ax)=1} \left[ \left| a \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right| + \left( b \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right) \right]$$

$$\leq (2a^\ell + b^\ell) \cdot 2^{\ell-1} \cdot \mathbb{E}_{i, \sigma} \sup_{\phi(Ax)=1} \left| \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right| \ell$$

$$\leq 3a^{\ell-1} \mathbb{E}_{i, \sigma} \sup_{\phi(Ax)=1} \left| \sum_{k=1}^{N} \sigma_k \frac{\langle A_{i_k}, x \rangle}{p_{i_k}} \right| \ell.$$
where the penultimate inequality follows from Lemma A.1.

For now, condition on the choices of \(i_1, i_2, \ldots, i_N\). By Lemma B.1 in [12], there exists a matrix \(A'\) of \(O(d^2)\) rows such that

\[
\|A'x\|_1 \leq C_1 \|Ax\|_1, \quad \forall x \in \mathbb{R}^d, \tag{10}
\]

and the Lewis weights of \(A'\) are \(O(1/d)\). Append \(A'\) to the matrix \(\tilde{A}\), obtaining a new matrix \(A''\) of \(N' = N + O(d^2)\) rows, and it is a direct consequence of the contraction principle (Lemma A.2) that

\[
\sum_{k=1}^{N'} \sigma_k(A''_k, x) \leq \varepsilon = \Theta(\log N), \tag{11}
\]

since the sum on the right-hand side has more terms.

Furthermore, it can be proved that the Lewis weights of \(A''\) are all \(O(\varepsilon^2 / \log N')\) (see [12]). In this case, for an appropriate choice of \(N\) (and thus \(N'\)), it is implicitly shown in the proof of Theorem 2.3 in [12] that

\[
\mathbb{E} \sup_{\|Ax\|_1 = 1} \left| \sum_{k=1}^{N'} \sigma_k(A''_k, x) \right|^\ell \leq \varepsilon^\ell / \text{poly}(N), \tag{12}
\]

for some \(\ell = \Theta(\log N').\) The next step is to relate the right-hand side of (11) to the left-hand side of (12). Note that

\[
\|A''x\|_1 = \|\tilde{A}x\|_1 + \|A'x\|_1 \leq C_1 \|Ax\|_1 + C_2 \|Ax\|_1 \quad \text{(by (7) and (10))}
\]

and thus

\[
\sup_{\|Ax\|_1 = 1} \left| \sum_{k=1}^{N'} \sigma_k(A''_k, x) \right|^\ell \leq (C_1 + C_2)^\ell B^\ell \sup_{\|Ax\|_1 = 1} \left| \sum_{k=1}^{N'} \sigma_k(A''_k, x) \right|^\ell.
\]

Taking expectation over \(\sigma\) on both sides, we obtain using (12) that

\[
\mathbb{E} \sup_{\|Ax\|_1 = 1} \left| \sum_{k=1}^{N'} \sigma_k(A''_k, x) \right|^\ell \leq ((C_1 + C_2)B)^\ell \varepsilon^\ell / \text{poly}(N).
\]

Taking expectation over \(i_1, \ldots, i_N\) on both sides subject to the conditioning (7) and combining with (6), (9) and (11), we obtain that

\[
M \leq C_3(C_4aB)^\ell \varepsilon^\ell / \text{poly}(N).
\]

The result follows from Markov’s inequality with a rescaling of \(\varepsilon\) by a factor of \(1/(C_5aB)\). \(\square\)

### B.3 Proof of Theorem 1.1

**Proof.** Recall that \(\rho_\tau(x) = -\tau x\) if \(x \leq 0\) and \(\rho_\tau(x) = x\) if \(x \geq 0\). We can rewrite \(\rho_\tau(x) = (1 + \tau)|x|/2 + (1 - \tau)x/2\). Also \(\rho_\tau(x) \leq \|x\|_1/\tau\), and thus we can take \(B = 1/\tau\) in Theorem 3.1. By Theorem 2.2, we can obtain \(\{w_i\}_{i=1}^n\), such that with probability \(1 - 1/\text{poly}(d)\), for all \(i \in [n], w_i \leq \overline{w}_1 \leq 2w_i\), where \(\{w_i\}_{i=1}^n\) are the \(\ell_1\) Lewis weights of \(A\). Now we invoke the row sampling algorithm in Theorem 3.1 and the fact in Lemma 2.1 to finish the proof. \(\square\)
B.4 Proof of Lemma 4.1

Proof. By Theorem 1.1 with probability at least 0.9, for all $x \in \mathbb{R}^d$,

$$\left(1 - \frac{\epsilon}{9}\right) \rho_r(Ax - b) \leq \rho_r(\tilde{A}x - \tilde{b}) \leq \left(1 + \frac{\epsilon}{9}\right) \rho_r(Ax - b).$$

Let $x^{\text{opt}} = \arg\min_{x \in \mathbb{R}^d} \rho_r(Ax - b)$, we have

$$\rho_r(Ax^{\text{opt}} - b) = \rho_r(AR^{-1}x - b)$$

$$\leq \frac{1}{1 - \epsilon/9} \rho_r(AR^{-1}x - \tilde{b})$$

$$\leq \frac{1 + \epsilon/3}{1 - \epsilon/9} \rho_r(AR^{-1}(Rx^{\text{opt}}) - \tilde{b})$$

$$= \frac{1 + \epsilon/3}{1 - \epsilon/9} \rho_r(AR^{-1}x - \tilde{b})$$

$$\leq \left(1 + \frac{\epsilon}{3}\right) \rho_r(Ax^{\text{opt}} - b)$$

$$\leq (1 + \epsilon) \rho_r(Ax^{\text{opt}} - b).$$

\[\square\]

B.5 Proof of Lemma 4.2

We need the following claim in our proof.

Claim 1. For a matrix $A \in \mathbb{R}^{n \times d}$ and a vector $y \in \mathbb{R}^n$,

$$\|A^T y\|_2 \leq \frac{1}{\tau} \rho_r(y) \max_{1 \leq i \leq N} \|A_i\|_2.$$

Proof. Observe that $\sum_{i=1}^n |y_i| \leq \frac{1}{\tau} \rho_r(y)$. It follows that

$$\|A^T y\|_2 = \left\| \sum_{i=1}^N y_i A_i \right\|_2 \leq \sum_{i=1}^N |y_i| \|A_i\|_2 \leq \max_{1 \leq i \leq N} \|A_i\|_2 \cdot \sum_{i=1}^N |y_i| \leq \frac{1}{\tau} \rho_r(y) \max_{1 \leq i \leq N} \|A_i\|_2.$$  \[\square\]

Now we are ready to prove Lemma 4.2.

Proof. By Lemma 30 in [18], with probability at least 0.9, the leverage score of each row of $[\bar{A}, \bar{b}]$ satisfies $\tau_i([\bar{A}, \bar{b}]) = O(d/N)$. We condition on this event in the remaining part of the proof. By Lemma 2 in [11],

$$\tau_i([\bar{A}, \bar{b}]) = \min_{[\bar{A}, \bar{b}]^T x = [\bar{A}, \bar{b}]} \|x\|_2^2,$$

and

$$\tau_i(\bar{A}) = \min_{\bar{A}^T x = A_i} \|x\|_2^2,$$

which implies $\tau_i(\bar{A}) \leq \tau_i([\bar{A}, \bar{b}])$. Thus, $\tau_i(\bar{A}) = O(d/N)$. Since $\bar{A}R^{-1} = Q$ has orthonormal columns, for each row $Q_i$ of $Q$,

$$\|Q_i\|_2^2 = \tau_i(\bar{A}) = O(d/N).$$

Here we used the standard fact of the relation between leverage scores and the QR decomposition. See Definition 2.6 in [13] for details.

Let $f(x) = \rho_r(AR^{-1}x - \tilde{b})$, we can write

$$f(x) = \sum_{i=1}^n \rho_r((\bar{A}R^{-1})_i x - b_i) = \frac{1}{N} \sum_{i=1}^n f_i(x).$$
Lemma 4.2. By Theorem A.4, with probability at least 

$$\|\nabla g_i(x)\|_2 \leq \|(\tilde{A}R^{-1})_i\|_2 = \|Q_i\|_2 = O(\sqrt{d/N}),$$

which implies that each $g_i(x)$ is $O(\sqrt{d/N})$-Lipschitz. Observe that $\rho_r(\cdot)$ is 1-Lipschitz, it follows that $f_i(x) = N\rho_r(g_i(x))$ is $O(\sqrt{Nd})$-Lipschitz.

By Claim 1, we have

$$\|x_0 - \tilde{x}_{\text{opt}}\|_2 = \|(\tilde{A}R^{-1})^\top b - \tilde{x}_{\text{opt}}\|_2 = \|(\tilde{A}R^{-1})^\top b - (\tilde{A}R^{-1})\tilde{x}_{\text{opt}}\|_2 \leq \sqrt{\frac{d}{N\tau^2}} \cdot \rho_r((\tilde{A}R^{-1})\tilde{x}_{\text{opt}} - b).$$

Finally, for any vector $y \in \mathbb{R}^N$, we have

$$\tau \|y\|_2 \leq \sqrt{\tau \|y\|_1} \leq \sqrt{N} \|y\|_2.$$ 

Since $\tilde{A}R^{-1} = Q$ has orthonormal columns, $x_0 = (\tilde{A}R^{-1})^\top b$ is the optimal solution to $\min_{x \in \mathbb{R}^d} \|\tilde{A}R^{-1}x_0 - \bar{b}\|_2$. Thus, we have $\|\tilde{A}R^{-1}x_0 - \bar{b}\|_2 \leq \|\tilde{A}R^{-1}\tilde{x}_{\text{opt}} - \bar{b}\|_2$. Therefore,

$$\rho_r(\tilde{A}R^{-1}x_0 - \bar{b}) \leq \sqrt{N} \|\tilde{A}R^{-1}x_0 - \bar{b}\|_2 \leq \sqrt{N} \|\tilde{A}R^{-1}\tilde{x}_{\text{opt}} - \bar{b}\|_2 \leq \sqrt{\frac{N}{\tau^2}} \rho_r(\tilde{A}R^{-1}\tilde{x}_{\text{opt}} - \bar{b}).$$

\[ \Box \]

B.6 Proof of Lemma 4.3

Proof: The initial solution $x_0 = (\tilde{A}R^{-1})^\top b$ can be calculated in $O(Nd)$ time. We condition on the event in Lemma 4.2 By Theorem A.4, with probability at least 0.9, after

$$T = O \left( N \log \frac{f(x_0) - f(x_{\text{opt}})}{\varepsilon \cdot f(x_{\text{opt}})} + \sqrt{N^2d\|x_0 - x^*\|_2^2} \right) = O \left( N \log \frac{N}{\varepsilon \tau^2} + \frac{dN^{1/2}}{\varepsilon \tau} \right) = \tilde{O} \left( \frac{d^{1.5}}{\tau^2 \varepsilon^2} \right)$$

stochastic subgradient iterations, we can find a solution $x$ such that

$$\mathbb{E} \left[ \rho_r(\tilde{A}R^{-1}x - \bar{b}) - \min_{x \in \mathbb{R}^d} \rho_r(\tilde{A}R^{-1}x - \bar{b}) \right] \leq \varepsilon/30 \cdot \rho_r(\tilde{A}R^{-1}x - \bar{b}).$$

By Markov’s inequality, with probability at least 0.9, we have

$$\rho_r(\tilde{A}R^{-1}x - \bar{b}) \leq (1 + \varepsilon/3) \cdot \min_{x \in \mathbb{R}^d} \rho_r(\tilde{A}R^{-1}x - \bar{b}).$$

Furthermore, each stochastic subgradient can be calculated in $O(d)$ time, since

$$\nabla f_i(x) = \begin{cases} \text{sign}(\langle A_i, x \rangle - b_i) \cdot A_i & \text{if } \langle A_i, x \rangle - b_i \geq 0, \\ \tau \cdot \text{sign}(\langle A_i, x \rangle - b_i) \cdot A_i & \text{otherwise} \end{cases}$$

where we choose a subgradient $\nabla f_i(x) = 0$ at the nondifferentiable points $x$. \[ \Box \]
B.7 Proof of Theorem 4.4

Proof. Finding the QR decomposition of the concatenated matrix $[\tilde{A}, \tilde{b}]$ can be done in $O(d^2/(\varepsilon^2 \tau^2))$ time (see Lemma 33 in [18]). By Lemma 4.3, we can obtain $\tau$ such that $\rho_\tau(\tilde{A}R^{-1}\tau - \tilde{b}) \leq 1 + \varepsilon/3 \cdot \min_{x \in \mathbb{R}^d} \rho_\tau(\tilde{A}R^{-1}x - \tilde{b})$ in $O(d^2 \tau^2)$ time and succeeds with probability at least $0.8$. By Lemma 4.1, with probability at least $0.9$, the obtained solution $x^* \in \mathbb{R}^d$ satisfies $\rho_\tau(\tilde{A}x^* - \tilde{b}) \leq (1 + \varepsilon) \min_{x \in \mathbb{R}^d} \rho_\tau(\tilde{A}x - \tilde{b})$. We complete the proof of the theorem by taking a union bound over the two events mentioned above. 

B.8 Proof of Lemma 5.1

Proof. By permuting the columns we may assume without loss of generality that $x_1 \leq x_2 \leq \cdots \leq x_n$. Suppose that $a = x_1 < x_2 = \cdots = x_{n-1} = x_n = b$, otherwise $\tilde{A}x = 0$ and the desired inequality holds automatically.

Let $I = \{i \in [n] : (Ax)_i \geq 0\}$. Let $u = \sum_{i \in I} A_i$ and $v = \sum_{i \notin I} A_i - \sum_{i \notin I} A_i$, then $\rho_0(Ax) = \langle u, x \rangle$ and $\rho_1(Ax) = \langle v, x \rangle$ for all $x \in P$, where $P = \{x \in [a, b]^n : a = x_1 \leq x_2 \leq \cdots \leq x_n = b\}$ is a polytope.

Observe that $\rho_1(Ax) \neq 0$ on $P$ and thus the function $f(x) = \langle u, x \rangle/\langle v, x \rangle$ attains its minimum value $\lambda$ on the compact set $P$. Suppose that $\langle u, x^* \rangle = \lambda \langle v, x^* \rangle$ for some $x^* \in P$. We claim that $x^*$ is the minimizer of $\langle u - \lambda v, x \rangle$ on $P$. Indeed, if $\langle u - \lambda v, x \rangle < \langle u - \lambda v, x^* \rangle$ for some $x' \in P$, then $\langle u, x' \rangle/\langle v, x' \rangle < \lambda$, contradicting the minimality of $x^*$.

Now, since $x^*$ is a minimizer of $\langle u - \lambda v, x \rangle$ on the polytope $P$, it must be some vertex of $P$, that is, there exists $k$ such that $x_k^* = \cdots = x_{k+1}^* = a$ and $x_{k+1}^* = \cdots = x_n^* = b$. Let $S = \{x_1, \ldots, x_k\}$, then

$$\frac{\rho_0(Ax^*)}{\rho_1(Ax^*)} = \frac{w(S, V \setminus S)}{w(S, V \setminus S) + w(V \setminus S, S)} \geq \frac{1}{\alpha + 1},$$

where the last step follows from the definition of the $\alpha$-balanced graph. We complete the proof by noticing that

$$\frac{\rho_0(Ax)}{\rho_1(Ax)} \geq \frac{\rho_0(Ax^*)}{\rho_1(Ax^*)} \geq \frac{1}{\alpha + 1}.$$ 

B.9 Proof of Corollary 5.2

Proof. Observe that $\rho_0(x) = \frac{1}{2}|x| + \frac{1}{2}x$. Moreover, by Lemma 5.1, we have $\|Bx\|_1 \leq (1 + \alpha)\rho_0(Bx)$ for all $x \in \mathbb{R}^n$. Now we invoke Theorem 3.1 with $a = b = \frac{1}{2}$ and $B = \alpha + 1$, which states that with probability at least $1 - 1/poly(n)$, for all $x \in \mathbb{R}^n$, $(1 - \varepsilon)\rho_0(Bx) \leq \rho_0(B'x) \leq (1 + \varepsilon)\rho_0(Bx)$. Moreover, the rows of $B'$ are reweighted rows of $B$, which implies $B'$ is the edge-vertex matrix of a graph $G'$, whose edges are reweighted edges of $G$. The running time of Algorithm 2 directly follows from Theorem 2.3.

C Proof of Theorem 6.1

Similar to the proof of Theorem 3.1 and below we shall only highlight the changes. Instead of the comparison result of Lemma A.3, we shall use Lemma A.3.

Note that we have here that

$$\mathbb{E} \phi^w(\widetilde{Ax}) = \phi(Ax)$$

and want to upper bound

$$M := \mathbb{E}_{S} \left[ \sup_{x \neq 0} \left| \frac{\phi^w(\widetilde{Ax})}{\phi(Ax)} - 1 \right|^\varepsilon \right].$$

Again by a standard symmetrization argument,

$$M = \mathbb{E}_{S} \left[ \sup_{x \neq 0} \left| \frac{\sum_{i=1}^N \phi(A_i x_i)}{\sum_{i=1}^N \phi(A_i x)} - 1 \right|^\varepsilon \right] \leq 2^\varepsilon \mathbb{E}_{i,\sigma} \left[ \sup_{x \neq 0} \left| \sum_{k=1}^N \sigma_k \phi(A_{i_k} x) \right|^\varepsilon \right],$$

19
where \( \sigma_1, \sigma_2, \ldots \) is a Rademacher sequence and \( i_1, i_2, \ldots \) are the indices of the rows chosen randomly during the construction of \( S \).

Invoking Lemma A.3 we have for fixed \( i_1, i_2, \ldots \) that

\[
\mathbb{E}_\sigma \left[ \sup_{x \neq 0} \left( \sum_{k=1}^{N} \sigma_k \frac{\phi(A_{i_k}, x)}{p_{i_k}} \phi(Ax) \right)^\ell \right] \\
\leq (2L)^\ell \mathbb{E}_\sigma \left[ \sup_{x \neq 0} \left( \sum_{k=1}^{N} \sigma_k \frac{|A_{i_k}, x|^p}{p_{i_k}} \phi(Ax) \right)^\ell \right] \\
\leq \left( \frac{2L}{\gamma} \right)^\ell \mathbb{E}_\sigma \left[ \left( \sup_{\|Ax\|_p=1} \left( \sum_{k=1}^{N} \sigma_k \frac{|A_{i_k}, x|^p}{p_{i_k}} \right) \right)^\ell \right] \\
\leq \left( \frac{2L}{\gamma} \right)^\ell \mathbb{E}_\sigma \left[ \left( \sup_{\|Ax\|_p=1} \left( \sum_{k=1}^{N} \sigma_k \frac{|A_{i_k}, x|^p}{p_{i_k}} \right) \right)^\ell \right]
\]

The problem is thus reduced to the analysis of \( \ell_p \) Lewis weight sampling and it has been proved implicitly in [12] that

\[
\mathbb{E}_{i, \sigma} \left[ \sup_{\|Ax\|_p=1} \left( \sum_{k=1}^{N} \sigma_k \frac{|A_{i_k}, x|^p}{p_{i_k}} \right) \right]^\ell \leq \left( \frac{C_1 \varepsilon}{\ell} \right)^\ell \frac{\text{poly}(N)}{\gamma \gamma/2L}.
\]

and hence

\[
M \leq 2^{2\ell} \cdot \left( \frac{2L}{\gamma} \right)^\ell \cdot \left( \frac{C_2 \varepsilon}{\ell} \right)^\ell \frac{\text{poly}(N)}{\gamma \gamma/2L}.
\]

The result follows from Markov’s inequality with a rescaling of \( \varepsilon \) by a factor of \( \gamma/(C_2 L) \).