The orientation and magnitude of the orbital precession velocity of a binary pulsar system with double spins

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The measurability of the spin–orbit (S–L) coupling induced orbital effect is dependent on the orientation and magnitude of the orbital precession velocity, \( \Omega_0 \). This paper derives \( \Omega_0 \), in the case that both spins in the binary system contribute to the spin–orbit (S–L) coupling, which is suitable for the most popular binary pulsars, Neutron star–White Dwarf star (NS–WD) binaries (as well as for NS–NS binaries). This paper shows that from two constraints, the conservation of the total angular momentum and the triangle formed by the orbital angular momentum, \( \mathbf{L} \), the sum the spin angular momenta of the two stars, \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \), the total angular momentum, \( \mathbf{J} \), the orbital precession velocity, \( \Omega_0 \), along \( \mathbf{J} \) is inevitable. Moreover, by the relation, \( S/L \ll 1 \), which is satisfied for a general binary pulsar, a significant \( \Omega_0 \) (in magnitude) is inevitable. 1.5 Post Newtonian order (PN). Which are similar to the case of one spin as discussed by many authors. However unlike the one spin case, the magnitude of the precession velocity of \( \Omega_0 \) varies significantly due to the variation of the sum the spin angular momenta of the two stars, \( \mathbf{S} \), which can lead to significant secular variabilities in binary pulsars.

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I. INTRODUCTION

Barker & O’Connell\[1\] derived the first gravitational two-body equation with spins. In which the precession of the angular momentum vector, \( \mathbf{L} \), responsible for the S–L coupling is expressed as around a combined vector of the spin angular momenta of the pulsar, \( \mathbf{S}_1 \), and its companion star, \( \mathbf{S}_2 \) (\( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 \)). Which is insignificant, 2PN, typically \( 10^{-4} \text{deg per year} \), and therefore ignorable in pulsar timing measurement. But notice that this ignorable orbital precession velocity, \( \Omega_0 \), is expressed as along a combined vector by \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \), which is not static to the line of sight (LOS). Therefore, the measurability of \( \Omega_0 \) need to be further investigated.

Since the precession of the pulsar spins (\( \mathbf{S}_1 \) and \( \mathbf{S}_2 \)) around \( \mathbf{L} \) (and \( \mathbf{J} \)) is significant, 1.5 PN (typically 1deg per year) by Barker & O’Connell two–body equation, and the precession of \( \mathbf{S}_1 \) relative to the line of sight (LOS) has been measured through the structure parameters of pulsar profile\[2, 3\]. Therefore the small precession velocity of \( \mathbf{L} \) with respect to a combined vector by \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) doesn’t mean that it is also small relative to LOS.

For binary pulsars the change of the orbital period due to the gravitational radiation is 2.5PN; whereas the geodetic precession of the pulsar is 1.5PN. Therefore, the influence of gravitational radiation on the motion of a binary system can be ignored. And the total angular momentum, \( \mathbf{J} (\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2) \), can be treated as invariable both in magnitude and direction. Therefore after counting out the proper motion effect, \( \mathbf{J} \) is at rest to LOS. So to make sense in observation, the precession velocity, i.e., \( \Omega_0 \), should be expressed as around \( \mathbf{J} \).

In the study of observational effect of the orbital precession due to the coupling of the spin induced quadruple moment of the companion star with the orbital angular momentum\[4, 5, 6\], the precession of \( \mathbf{L} \) is expressed as relative the total angular momentum vector, \( \mathbf{J} \). And the orbital velocity obtained can explain some secular variabilities measured. However this model might not suitable for neutron star-white dwarf (NS-WD) and neutron star-neutron star (NS-NS) binaries which have much smaller quadruple moment.

On the other hand, Apostolatos et al\[7\], Kidder\[8\] as well as Wex and Kopeikin\[6\] studied the precession of \( \mathbf{L} \) as relative to \( \mathbf{J} \) in discussing the spin-orbit (S–L) coupling in the NS-BH binary systems. Similar to the former quadruple-orbit coupling, this S–L coupling only considers the spin (quadruple) of the companion star (BH), the spin of the pulsar is ignored. The difference is that the former Q–L coupling corresponds to 2PN along and \( \mathbf{J} \), whereas the latter S-L coupling corresponds to 1.5 PN along \( \mathbf{J} \). Therefore, the latter is easier to explain the measurements of NS-WD and NS-NS binaries, which have much smaller quadruple moments.

The S–L coupling in the one spin case predicts a static precession of \( \mathbf{L} \), or \( \Omega_0 \) is unchanged both in magnitude and direction. Therefore, the orbital effect induced in such case can be completely absorbed the orbital parameters such as orbital period, \( P_b \), and the precession of periastron, \( \omega \). However, this model cannot explain the significant secular variabilities measured in many binary pulsars. Therefore, it actually has not been used in pulsar timing measurement.

This paper studies the precession velocity of \( \mathbf{L} \) around the static direction, \( \mathbf{J} \), in the a general case, two spins. It comes to the following conclusion: (a) as that of one spin case, the three vectors, \( \mathbf{L} \), \( \mathbf{J} \) and \( \mathbf{S} \) are still in the same plane at any instant; (b) as that of one spin case, the precession velocity of \( \mathbf{L} \) around \( \mathbf{J} \) is significant (measurable for many binary pulsars); (c) unlike the one spin case,
the magnitude of the precession velocity of $\Omega_0$ varies significantly due to the variation of the sum the spin angular momenta of the two stars, $S$, which can naturally explain the significant secular variabilities measured in binary pulsars.$^3$

This paper is arranged as follows. Section II introduces the orbital velocity of Barker and O’Connell$^1$ and that of one spin equation by Kidder$^5$, Wex and Kopeikin$^6$. The measurability of them are also discussed and compared. Section III derived the orbital velocity as around $J$ in the general (two spins) case. Section IV gives the derivatives of $\Omega_0$, responsible for the significant secular variabilities.

II. ORBITAL PRECESSION VELOCITIES BY BARKER & O’CONNELL AND OTHER AUTHORS

According to the gravitational two-body equations$^1$, the motion of the a binary system can be described by three vectors, the spin angular momentum of the pulsar, $S_1$, its companion star, $S_2$, and the orbital angular momentum, $L$. The secular result for $S_1$ is

$$\Omega_1 = \frac{GL(4 + 3m_2/m_1)}{2c^2a^3R(1 - e^2)^{3/2}} n_0 + \frac{GS_2}{2c^2a^3R(1 - e^2)^{3/2}} [n_2 - 3(n_0 \cdot n_2) n_0] ,$$

(1)

where $n_0$, $n_1$, and $n_2$ are unit vectors of the orbital angular momentum, the spin angular momentum of the pulsar and its companion star, respectively, $m_1$, $m_2$ are the masses of pulsar and the companion star, respectively, $e$ is the eccentricity of the orbit, $a_2$ is the semi-major axis. The first term corresponds to the geodetic (de Sitter) precession which represents the precession of $S_1$ around $L$, since $L \propto a^{1/2}$, the first term actually corresponds to $a^{-5/2}_R$ (1.5 PN); and the second term represents the Lense-Thirring precession, $S_1$ around $S_2$, which corresponds to $a^{-3}_R$ (2 PN). $\Omega_2$ can be obtained by exchanging the subscript 1 and 2 at the right side of Eq(1).

The precession of the orbital angular momentum, $\Omega_0$, can be given as follows$^1$:

$$\Omega_0 = \Omega_{PN} + \Omega_L + \Omega_Q ,$$

(2)

where $\Omega_{PN}$ is the relativistic periastron advance, $\Omega_L$ is the geodetic precession caused by $S$–$L$ coupling, and $\Omega_Q$ is the precession due to the Newtonian coupling of the orbital angular momentum vector to the quadruple moment of the two bodies. For a general binary pulsar system, like NS–NS and NS–WD binaries, $\Omega_Q$ is ignorable, 2.5PN, $\Omega_{PN}$ is significant, 1.5PN, but it has no contribution to the $S$–$L$ coupling, due to it is along $L$. Therefore, only $\Omega_L$ (2 PN) contributes to the $S$–$L$ coupling. By Barker and O’Connell$^1$,

$$\Omega_L = \frac{2}{\alpha + 1} \frac{GS_2(4 + 3m_2/m_1)}{2c^2a^3R(1 - e^2)^{3/2}} [n_2 - 3(n_0 \cdot n_2) n_0] ,$$

(3)

where $\alpha + 1$ meant modulo 2 ($2 + \alpha = 1$), Eq(3) means that $L$ precesses around a vector combined by $S_1$ and $S_2$ with a very small velocity (2PN), and the precession of $L$ due to geodetic precession is thus considered negligible to pulsar timing measurement.

However since $S_1$ and $S_2$ and thus the combined vector by $S_1$ and $S_2$ precess rapidly (1.5PN) with respect to LOS, therefore from the small precession velocity $L$ to the combined vector of $S_1$ and $S_2$ (2PN) given by Eq(3), we still cannot conclude that $L$ also precesses with respect to LOS slowly.

Actually in the study of the observational effect of the $Q$–$L$ coupling, $\Omega_Q$ is transformed to the direction of $J$. The value of $\Omega_Q$ along $J$ is about $L/S$ times larger than that of the previous one, thus $\Omega_Q$ along $J$ corresponds to 2PN instead of the previous 2.5PN.

Similarly if orbital velocity corresponding to the $S$–$L$ coupling is also expressed as along $J$, then it would also be $L/S$ times the velocity $\Omega_L$ given by Eq(3), which would be 1.5PN instead of 2PN. Then it should be comparable to $\Omega_{PN}$ in magnitude and measurable.

Actually this work has been done by many authors$^6, 7, 8$, in the special case of one spin. From the conversation of the total angular momentum, we have $J = 0$, which can be written,

$$\Omega_0 \times L = -\Omega_1 \times S_1 - \Omega_2 \times S_2 .$$

(4)

Notice that as defined by Barker and O’Connell$^1$ and Apostolatos et al$^9$, $L = \mu M^{1/2}r^{1/2}n_0$, where $M = m_1 + m_2$, and $\mu = m_1 m_2 / M$.

In the special case that one spin angular momentum is ignorable, i.e., $S_2 = 0$, as discussed by Kidder$^5$, Wex and Kopeikin$^6$, (Apostolatos et al$^9$ discussed the special case that $S_1 \cdot S_2 = 0$, $S$ is a constant in magnitude. And thus $L$ and $S$ precess about the fixed vector $J$ at the same rate with a precession frequency approximately$^6$

$$\Omega_0 = \frac{|J|}{r^3} \frac{(4 + 3m_2/m_1)}{m_1} .$$

(5)

Note that Eq(5) uses $G = c = 1$ and $r = a_R(1 - e^2)^{1/2}$. Since $L$, $J$ and $S$ forms a triangle, therefore, the three vectors are in the same plane. Or more clearly, $L$ and $S$ are at the opposite side of $J$ at any instant ($J$ is invariable).

Notice the velocity given by Eq(5) is relative to the static direction $J$ (which makes sense in observation), and the magnitude of Eq(5) is 1.5PN, which means the precession velocity of $L$ around $J$ is much larger than that of $L$ around a combined vector by $S_1$ and $S_2$ given by Eq(3).

If Eq(3) is applied to pulsar timing measurement, the static precession of $L$ around $J$ can be completely absorbed by parameters like the orbital period, $P_b$, and the
advance of precession of periastron, \( \dot{\omega} \). Therefore, the effects of Eq (3) might be unmeasurable, although it is significant.

However, if both spins are considered, then the precession of \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) will lead to a variable \( \mathbf{S} \) (both in direction and magnitude), and therefore, through S–L coupling, the orbital velocity, \( \mathbf{\Omega}_0 \) varies in magnitude. Which cannot be completely absorbed by parameters, like, \( P_b \) and \( \dot{\omega} \). And it is this variation that automatically explains the secular variabilities measured in many binary pulsars 4.

Therefore, it is necessary to derive an orbital precession velocity that is relative to \( \mathbf{J} \) and including two spins.

### III. ORBITAL PRECESSION VELOCITY IN GENERAL CASE

#### A. The scenario of the motion of \( \mathbf{L} \), \( \mathbf{J} \) and \( \mathbf{S} \)

This section derives orbital precession velocity as around \( \mathbf{J} \) in the case that \( \mathbf{S}_1 \neq 0 \) and \( \mathbf{S}_2 \neq 0 \), so that it can be applied to general NS–WD and NS–NS binaries. Eq (4) can be rewritten as

\[
\mathbf{\Omega}_0 \times \mathbf{L} = -\mathbf{\Omega}_2 \times \mathbf{S} - (\mathbf{\Omega}_1 - \mathbf{\Omega}_2) \times \mathbf{S}_1 \ .
\]

(6)

Notice that when ignoring the terms over 2PN, \( \mathbf{\Omega}_1 \) and \( \mathbf{\Omega}_2 \) are along \( \mathbf{n}_0 \). The first term at right side of Eq (6) represents a torque that is perpendicular to the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \). And the second term at right side of Eq (6) also has a component in the same direction as the first one, which can be written as, \( (\mathbf{\Omega}_1 - \mathbf{\Omega}_2) \mathbf{S}_1 \sin \lambda_{LS1} \), in magnitude. Where \( \mathbf{S}_1 \) corresponds the component of \( \mathbf{S}_1 \) in the plane determined by \( \mathbf{S} \) and \( \mathbf{J} \), which satisfies \( \mathbf{S}_1^\parallel = \mathbf{S}_1 \cos \eta_{SS1} \), and \( \lambda_{LS1} \) is the misalignment angle between \( \mathbf{S}_1^\parallel \) and \( \mathbf{J} \). \( \eta_{SS1} = \eta_{S1} - \eta_S \) (\( \eta_{S1} \) and \( \eta_S \) are precession phases of \( \mathbf{S}_1 \) and \( \mathbf{S} \) respectively).

Obviously \( \mathbf{S} \) should precess at the velocity which is equal to that \( \mathbf{L} \) precess around \( \mathbf{J} \), whereas, Eq (4) uses \( \mathbf{\Omega}_2 \) to replace this velocity, therefore, an extra term \( (\mathbf{\Omega}_1 - \mathbf{\Omega}_2) \mathbf{S}_1^\parallel \sin \lambda_{LS1} \) is needed to make up the difference. The torques made by these two terms can be given,

\[
(\tau_1)_R = [\mathbf{\Omega}_2 \sin \lambda_{LS} + (\mathbf{\Omega}_1 - \mathbf{\Omega}_2) \mathbf{S}_1^\parallel \sin \lambda_{LS1}] \ .
\]

(7)

(\( \tau_1 \))_R is equivalent to a combined one, \( \mathbf{\Omega}_S \mathbf{S}_1 \sin \lambda_{LS} \). The second term at right hand side of Eq (6) can produce a torque that is perpendicular to that of the first term. Which can be written,

\[
(\tau_2)_R = (\mathbf{\Omega}_1 - \mathbf{\Omega}_2) \mathbf{S}_1^\perp \sin \lambda_{LS1} \ .
\]

(8)

where \( \mathbf{S}_1^\perp \) is the component of the \( \mathbf{S}_1 \), which is at the plane perpendicular to the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \) instantaneously. Which produces a torque, \( (\tau_2)_R \), that is perpendicular to \( (\tau_1)_R \), obviously in the one spin case, \( (\tau_2)_R = 0 \). Actually Eq (8) can also be regarded as \( \mathbf{S} \) precesses around a vector that is perpendicular to the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \) with velocity \( (\tau_2)_R / S \) instantaneously.

\( (\tau_2)_R \neq 0 \) (in the general case) indicates that the left hand side of Eq (1) and Eq (5) must make a corresponding torque to balance with \( (\tau_2)_R \), or the orbital precession velocity, \( \mathbf{\Omega}_0 \), much have a component that is perpendicular to the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \) to balance with \( (\tau_2)_R \). Then it seems that the three vectors may not be in the same plane (or \( \mathbf{J} \) is not in the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \)), due to \( (\tau_2)_R \neq 0 \).

However this situation will never happen because the constraint, \( \mathbf{J} = \mathbf{L} + \mathbf{S} \), must be satisfied at any instant in the two spins case also. Which demands that \( \mathbf{J} \), \( \mathbf{L} \) and \( \mathbf{S} \) must be in the same plane at any instant. Simultaneously, the constraint, \( \mathbf{J} = 0 \), demands that \( \mathbf{J} \) is unchanged both in magnitude and direction. Therefore, \( \mathbf{L} \) and \( \mathbf{S} \) have to be at the opposite side of \( \mathbf{J} \) at any instant to satisfy to the two constraints at the same time. In other words, the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \) rotates around a fixed vector, \( \mathbf{J} \).

Then what is the role of the torque, \( (\tau_2)_R \), in the motion of the three vectors? It can cause a precession of \( \mathbf{S} \) around a vector perpendicular to the plane determined by \( \mathbf{L} \), \( \mathbf{J} \) and \( \mathbf{S} \) (plane LJS) instantaneously, which can change the angle between \( \mathbf{L} \) and \( \mathbf{S} \). Similarly \( \mathbf{L} \) also precesses around a direction perpendicular to the plane LJS instantaneously, which changes the angle between \( \mathbf{L} \) and \( \mathbf{J} \).

Thus \( (\tau_2)_R \) can only changes the misalignment angles of \( \mathbf{L} \) and \( \mathbf{S} \) with respect to \( \mathbf{J} \), it cannot influence the scenario that \( \mathbf{L} \) and \( \mathbf{S} \) precess rapidly at opposite side of \( \mathbf{J} \) at any moment. In the one spin case \( (\tau_2)_R = 0 \), the misalignment angles between \( \mathbf{S} \) and \( \mathbf{L} \) as well as \( \mathbf{L} \) and \( \mathbf{J} \) are unchanged.

Therefore the difference between one spin and two spins is that the former corresponds to a simple precession of \( \mathbf{L} \) and \( \mathbf{S} \) around \( \mathbf{J} \), but the latter corresponds to a nutation superimposed on an overall precession. While the common point is that at any instant, the three vectors are in the same plane, and moreover, \( \mathbf{L} \) and \( \mathbf{S} \) are at the opposite side of \( \mathbf{J} \).

As discussed above from the two constraints, \( \mathbf{J} = \mathbf{L} + \mathbf{S} \) and \( \mathbf{J} = 0 \), we can conclude that the plane determined by \( \mathbf{L} \) and \( \mathbf{S} \) precesses around a fixed vector, \( \mathbf{J} \), rapidly (1.5PN). In other words, the precession velocity of \( \mathbf{L} \) and \( \mathbf{S} \) (\( \mathbf{\Omega}_0 \) and \( \mathbf{\Omega}_S \)) is along \( \mathbf{J} \). One can also choose vectors other than \( \mathbf{J} \) as axis that \( \mathbf{L} \) and \( \mathbf{S} \) precess around. However that would be like using a planet instead of the Sun as the center of a reference frame to explain the motion of the solar system.

Actually the scenario of the motion of vectors in a binary system have been discussed by authors, Smarr and Blandford 4, Hamilton and Sarazin 10. From which one can easily deduce that the small velocity of \( \mathbf{L} \) around a combined vector by \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) doesn’t influence that the three vectors, \( \mathbf{L} \), \( \mathbf{S}_1 \) and \( \mathbf{S}_2 \) precess rapidly around \( \mathbf{J} \). In
other words, the precession velocity of $\mathbf{L}$ around $\mathbf{J}$ can be significant and therefore measurable.

Since Barker and O’Connell’s two-body equation satisfies the two constraints, $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and $\mathbf{J} = 0$. Therefore, Barker and O’Connell’s two-body equation actually means that the orbital precession velocity, $\Omega_0$, is along $\mathbf{J}$. However the misalignment angle between $\mathbf{L}$ and $\mathbf{J}$ is arbitrary in Barker and O’Connell’s two-body equation, therefore, it is impossible to obtain $\Omega_0$ along $\mathbf{J}$ without further constraint.

**B. The magnitude of $\Omega_0$**

Fortunately there is a constraint, $S/L \ll 1$, that must be satisfied for a general binary pulsar. $S/L \ll 1$ can impose strong constraint on $\lambda_{LJ}$ (misalignment angle between $\mathbf{J}$ and $\mathbf{S}$), and therefore, the magnitude (also direction) of $\Omega_0$.

By the triangle $\mathbf{J} = \mathbf{L} + \mathbf{S}$ and the relation $S/L \ll 1$, one can know that the misalignment angle between $\mathbf{L}$ and $\mathbf{J}$ must be very small, $\lambda_{LJ} \ll 1$. Thus no matter $\mathbf{L}$ participates in what ever motions, precession and nutation, $\mathbf{L}$ must always be very close to $\mathbf{J}$.

As discussed above, $(\tau_1)_R$ and $(\tau_2)_R$ correspond to torques caused by components of $\mathbf{S}_1$ and $\mathbf{S}_2$ that is in the plane $\mathbf{LJS}$ and perpendicular to it respectively. Then $\mathbf{L}$ must make a torque that equals $(\tau_1)_R$ in magnitude. Then $\mathbf{L}$ can react to $(\tau_1)_R$ in two ways:

(a) $\mathbf{L}$ can precession around a vector combined by $\mathbf{S}_1$ and $\mathbf{S}_2$ (which is the previous treatment of Eq(8));

(b) $\mathbf{L}$ precesses around $\mathbf{J}$, with the opening angle of the precession cone $\lambda_{LJ} \approx S/(\rho L)$ ($\rho = 1/\sin \lambda_{JS}$).

Mathematically, both (a) and (b) are allowed. (a) doesn’t contradictory with (b), since that when $\mathbf{L}$ has certain angle with respect to the vector combined by $\mathbf{S}_1$ and $\mathbf{S}_2$ (usually this angle is large), the angle between $\mathbf{L}$ and $\mathbf{J}$ can be arbitrary. Because the magnitude of $\mathbf{L}$ and $\mathbf{S}$ can be both comparable and very different ($S/L \ll 1$) in Barker and O’Connell’s equation.

In (b), although the angle between $\mathbf{L}$ and $\mathbf{J}$ is very small, the angle between $\mathbf{L}$ and the vector combined by $\mathbf{S}_1$ and $\mathbf{S}_2$ ($\mathbf{S}$) can still be arbitrary. Therefore, (b) doesn’t contradictory with (a). The difference is that (b) uses one more constraint (which must be satisfied by a general binary pulsar) than (a), that is $\lambda_{LJ} \ll 1$. Thus for binary pulsars, the orbital velocity obtained from (b) is more closer to the true one than that of (a). Moreover, the result from (a) has no observational means (not at rest to LOS), whereas the result from (b) make sense in observation (at rest to LOS).

Therefore, for a general binary pulsar system, especially NS-WD and NS-NS binaries, only (b) is considered. The left hand side of Eq(10) and Eq(11) can be written as

$$(\tau_1)_L = |\Omega_0 \mathbf{n}_J \times L \mathbf{n}_o| = \Omega_0 L \sin \lambda_{LJ}.$$  \hspace{1cm} (9)

in which $\Omega_0$ denotes the velocity of $\mathbf{L}$ around $\mathbf{J}$. By Eq(8) and Eq(9), we have the precession rate,

$$\Omega_0 = \frac{(\tau_1)_R}{L \sin \lambda_{LJ}} = \frac{\rho \Omega_2 \sin \lambda_{LS} + \rho (\Omega_1 - \Omega_2) S^{\parallel}/S \sin \lambda_{LS_1}}{L}.$$  \hspace{1cm} (10)

Note $\rho L \sin \lambda_{LS} \approx S$. In reaction to the torque, $(\tau_2)_R$, of Eq(11), $\mathbf{L}$ should precess around a vector, $\mathbf{n}_J^\perp$ (perpendicular to the plane $\mathbf{LJS}$ instantaneously), which corresponds to an opening angle of the precession cone $\pi/2$ at any moment.

$$(\tau_2)_L = |\Omega_0^{nu} \mathbf{n}_J^\perp \times L \mathbf{n}_o| = \Omega_0^{nu} L \sin \frac{\pi}{2}.$$  \hspace{1cm} (11)

This precession is usually called nutation, superimposed on an overall precession of the system about the axis of total angular momentum. By Eq(11) and Eq(12), we have

$$\Omega_0^{nu} = \frac{(\tau_2)_R}{L} = \frac{(\Omega_1 - \Omega_2) S^{\perp} \sin \lambda_{LS_1}^\perp}{L}.$$  \hspace{1cm} (12)

Eq(10) corresponds to a rapid precession of $\mathbf{L}$ around $\mathbf{J}$ (1.5PN); whereas Eq(12) corresponds to a slow precession (nutation) of $\mathbf{L}$ around a vector that is perpendicular to the plane $\mathbf{LJS}$ at any instant (2PN). In other words, the velocity of Eq(12) derived from the torque, $(\tau_2)_R$, can only change the misalignment angles, $\lambda_{LJ}$ and $\lambda_{LS}$, it cannot influence the direction of $\Omega_0$.

In summary, from the constraints, $\mathbf{L} + \mathbf{S} = \mathbf{J}$, and $\mathbf{J} = 0$, the precession velocity $\Omega_0$ along $\mathbf{J}$ is inevitable. Further by the constraint, $S/L \ll 1$, which is correct for a general binary pulsar, a significant $\Omega_0$ (comparable to $\omega$ the precession of periastron) is inevitable.

Therefore, the one spin case and the general case are similar in serval aspects. The orbital precession velocities of a binary pulsar system are both significant, 1.5PN and both around $\mathbf{J}$. Moreover, the three vectors, $\mathbf{L}$, $\mathbf{J}$ and $\mathbf{S}$ are in the same plane at any instance in both cases. The one spin case corresponds to a zero nutation velocity, while the two spins case corresponds to an ignorable nutation velocity of the orbit plane, 2PN.

However, there is significant difference between one spin and two spins cases, the former corresponds to a constant $S$, which means a static precession of $\mathbf{S}$ and $\mathbf{L}$ around $\mathbf{J}$, $\Omega_0 = \text{const}$; whereas the latter corresponds to a variable $S$, and therefore, a variable $\Omega_0$, which is responsible for the significant secular variabilities measured in many binary pulsars. On the other hand, the significant secular variabilities measured in many binary pulsar seems to support the two spins case.

**IV. DERIVATIVES OF $\Omega_0$**

By Eq(11), if $S$ is unchanged then $\mathbf{L}$ will precession with a static velocity, $\Omega_0$, around $\mathbf{J}$. However, since $\mathbf{S}_1$ and $\mathbf{S}_2$ precess with different velocities, $\Omega_1$ and $\Omega_2$, respectively. Thus $\mathbf{S}$ varies both in magnitude and direction ($\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}$ form a triangle), and in turn $\lambda_{LJ}$
and $\lambda_{LS}$ also vary with time ($S$, $L$ and $J$ form a triangle, Fig.1).

The motion of the vectors $S_1$, $S_2$ and $S$ can be studied in the coordinate system of the total angular momentum, in which the z-axis directs to $J$, and the x- and y-axes are in the invariance plane. $S$ can be represented by $S^P$ and $S^V$, the components parallel and vertical to the z-axis, respectively:

$$S = (S^V + S^P)^{1/2}.$$  \hspace{1cm} (13)

$S^P$ and $S^V$ can be expressed as

$$S^P = S_1 \cos \lambda_{JS_1} + S_2 \cos \lambda_{JS_2},$$

$$S^V = [(S_1^V)^2 + (S_2^V)^2 - 2S_1^VS_2^V \cos \eta_{S_1 S_2}]^{1/2},$$ \hspace{1cm} (14)

$S_1^V = S_1 \sin \lambda_{JS_1}$ and $S_2^V = S_2 \sin \lambda_{JS_2}$ represent components of $S_1$ and $S_2$ that are vertical to the $J$. $S_1^V$, $S_2^V$ and $S^V$ form a triangle. $\eta_{S_1 S_2}$, the misalignment angle between $S_1^V$ and $S_2^V$ can be written

$$\eta_{S_1 S_2} = (\Omega_1 - \Omega_2)t + \phi_0.$$ \hspace{1cm} (15)

Note that $S$, $\lambda_{LS}$, $S_1^V$ ($S_1^V = S_1 \cos \eta_{SS_1}$), and $\lambda_{LS}^\parallel$ in Eq(10) are function of time, which leads to the change of the precession rate of orbit, $\Omega_0$.

$$\dot{\Omega}_0 = \rho \dot{\Omega} + \rho \Omega,$$ \hspace{1cm} (16)

where

$$\dot{\Omega} = \Omega_2 \Omega_{12} X_3 X_4 - \Omega_{12} X_1 (\Omega_0 X_2 + \Omega_{12} X_3) + \Omega_{12} X_1 X_5,$$ \hspace{1cm} (17)

where $\Omega_{12} = \Omega_1 - \Omega_2$, $\Omega_0 = \Omega_1 - \Omega_0$,

$$X_1 = \frac{S_1^V}{S} \sin \lambda_{LS_1}, X_2 = \tan \eta_{SS_1}, X_3 = \frac{S_1^V S_2^V}{S} \sin \eta_{SS_1}, X_4 = \frac{\cos^2 \lambda_{LS}}{\sin \lambda_{LS}}, X_5 = \cot \lambda_{LS_1} \lambda_{LS_1},$$

$$\dot{\rho} = -\frac{\Omega_{12}}{\sin^2 \lambda_{LS}} [\dot{X}_3 X_4 + \dot{X}_4 X_3 - \frac{2 \Omega_{12} X_3^2 X_4^2}{\sin \lambda_{LS}}].$$

$$\dot{X}_4 = -\Omega_{12} X_3 X_4 (2 + \frac{X_4}{\sin \lambda_{LS}}),$$

with $\dot{\Omega}_0 = \dot{\rho} \dot{\Omega} + 2 \dot{\rho} \dot{\Omega} + \dot{\rho}$.

where

$$\dot{\Omega} = \Omega_2 \Omega_{12} (\dot{X}_3 X_4 + X_3 \dot{X}_4) - \Omega_{12} \dot{X}_1 (\Omega_0 X_2 + \Omega_{12} X_3) -$$

$$\Omega_{12} X_1 (\Omega_0 \dot{X}_2 + \Omega_{12} \dot{X}_3) + \Omega_{12} (X_1 \dot{X}_5 + \dot{X}_1 X_5),$$ \hspace{1cm} (19)

where

$$X_1 = -\Omega_0 X_1 \tan \eta_{SS_1} - (\Omega_{12} X_2 X_3)$$

$$X_2 = -\Omega_0 \sec^2 \eta_{SS_1},$$

$$X_3 = \dot{Y} \sigma + Y \dot{\sigma},$$

where $Y = \frac{S_{1V} S_{2V} \sin \eta_{S_1 S_2}}{S^V}, \sigma = \frac{1}{\sin \lambda_{LS}},$

$$\dot{Y} = \Omega_{12} (Y \cot \eta_{S_1 S_2} - Y^2 \sigma)$$

$$\dot{\sigma} = -\frac{1}{(\sin \lambda_{LS})^2} \sin \lambda_{JS} \dot{X}_4$$

$$+ (2 \sin \lambda_{JS} + X_4) \frac{\cos^2 \lambda_{JS} \sin X_3 \Omega_{12}}{\sin \lambda_{JS}},$$

$$\ddot{\rho} = -\frac{\Omega_{12}}{\sin^2 \lambda_{JS}} [\ddot{X}_3 X_4 + \ddot{X}_4 X_3 - \frac{2 \Omega_{12} X_3^2 X_4^2}{\sin \lambda_{LS}}].$$

$$\dot{X}_4 = -\Omega_{12} X_3 X_4 (2 + \frac{X_4}{\sin \lambda_{LS}}),$$

$$\dot{X}_5 = \dot{\lambda}_{LS_1} \cot \lambda_{LS_1} - (\lambda_{LS_1})^2 \csc^2 \lambda_{LS_1},$$

where $\lambda_{LS_1} = \tan^{-1} (\tan \lambda_{JS_1} \cos \eta_{SS_1})$, $\dot{\lambda}_{LS_1} = \frac{z}{1 + z^2}$, $\dot{\lambda}_{LS_1} = \frac{z (1 + z^2) - 2 z^2}{(1 + z^2)^2}$. In which $z = \tan \lambda_{JS_1} \cos \eta_{SS_1}$, $\ddot{z} = -\Omega_0 \tan \lambda_{JS_1} \sin \eta_{SS_1}$, $\ddot{z} = -\Omega_0 \tan \lambda_{JS_1} \cos \eta_{SS_1}$. By Eq(12), the third order derivative of $\Omega_0$ are given,

$$\frac{d^3 \Omega_0}{dt^3} = \Omega \frac{d^2 \rho}{dt^2} + \rho \frac{d^2 \Omega}{dt^2} + 3 \dot{\rho} \dot{\Omega} + 3 \rho \dot{\Omega},$$ \hspace{1cm} (20)

where

$$\frac{d^3 \Omega}{dt^3} = \Omega_2 \Omega_{12} (2 \dot{X}_3 X_4 + \dot{X}_3 X_4 + X_3 \dot{X}_4) -$$

$$2 \Omega_{12} X_1 (\Omega_0 \dot{X}_2 + \Omega_{12} \dot{X}_3) - \Omega_{12} \dot{X}_1 (\Omega_0 X_2 + \Omega_{12} X_3) -$$

$$\Omega_{12} X_1 (\Omega_0 \dot{X}_2 + \Omega_{12} \dot{X}_3) + \Omega_{12} (X_1 \dot{X}_5 + \dot{X}_1 X_5 + 2 \dot{X}_1 X_5),$$ \hspace{1cm} (21)

$$\ddot{X}_1 = -\Omega_0 \dot{X}_1 \tan \eta_{SS_1} - \Omega_0^2 X_1 \sec^2 \eta_{SS_1},$$

$$-\Omega_{12} \dot{X}_1 X_3 - \Omega_{12} \dot{X}_3 X_1$$

$$\dot{X}_2 = 2 \Omega_0^2 \tan \eta_{SS_1} \sec^2 \eta_{SS_1}$$
\[
\ddot{X}_3 = \dot{Y}\sigma + \dot{\sigma}Y + 2\dot{Y}\dot{\sigma}
\]
\[
\ddot{X}_4 = -\Omega_{12}(2 + \cot \lambda_{LS})(\dot{X}_3 X_4 + \dot{X}_4 X_3) - \Omega_{12}X_3 X_4 (\dot{X}_4 \csc \lambda_{LS} - \cos^2 \lambda_{LS} \csc^3 \lambda_{LS} \Omega_{12}X_3 X_4)
\]
where
\[
\ddot{Y} = \Omega_{12} \dot{Y} \cot \eta_{S1,S2} - \Omega_{12}^2 Y \csc^2 \eta_{S1,S2} - 4\Omega_{12} Y \dot{Y} \sigma - 2\Omega_{12} Y^2 \dot{\sigma}
\]
\[
\dot{\sigma} = 2\sigma^2 (\alpha \sin \lambda_{JS})^2 + \sigma^2 [\sin \lambda_{JS} \ddot{X}_4 + \dot{X}_4 \dot{\lambda}_{JS} \cos \lambda_{JS}]
\]
\[
+ \Omega_{12} (2\dot{\lambda}_{JS} \cos \lambda_{JS} + \dot{X}_4) \cos^2 \lambda_{JS} \csc \lambda_{JS} X_3
\]
\[
+ \Omega_{12} (2\sin \lambda_{JS} + X_4) (\ddot{\xi} X_3 + \dot{\xi} \dot{X}_3)
\]
\[
\ddot{\xi} = \cos^2 \lambda_{JS} \csc \lambda_{JS}
\]
\[
\dot{\lambda}_{JS} = \Omega_{12} X_3 \cos \lambda_{JS} \csc \lambda_{JS}
\]
\[
\dot{X}_5 = 2 (\dot{\lambda}_{LS1})^2 \cos \lambda_{LS1} \csc \lambda_{LS1} - 3 \dot{\lambda}_{LS1} \dot{\lambda}_{LS1} \csc^2 \lambda_{LS1} + \frac{d^3 \lambda_{LS1}}{dt^3} \cot \lambda_{LS1}
\]
where
\[
\frac{d^3 \lambda_{LS1}}{dt^3} = \Omega_{61}^3 \tan \lambda_{JS} \sin \eta_{SS1}.
\]

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FIG. 1: Angles and orientation conventions relating vectors, $S$, $L$ and $J$ to the coordinate system. $S$, $L$ and $J$ form a triangle, and determine a plane $LJS$. At any instant $S$ and $L$ must be at the opposite side of the fixed vector, $J$ (invariable both in direction and magnitude). $S_{1\parallel}$ and $S_{1\perp}$ are components of $S_1$ projected in the plane $LJS$ and perpendicular to it respectively. $(\tau_1)_R$ and $(\tau_2)_R$ are torques corresponding to the precession and nutation of $L$ (or $S$) in a binary system respectively.