Stochastic Dynamics of Vortex Loop.
Large Scale Stirring Force

S.K. Nemirovskii and A. Ja. Baltsevich

Institute of Thermophysics, 630090, Novosibirsk, Lavrent’eva, 1.

Abstract. Stochastic dynamics of a vortex filament obeying local induced approxima-
tion equation plus random agitation is investigated by analytical and numerical
methods. The character of a stirring force is supposed to be a white noise with spatial
correlator concentrated at large distances comparable with size of the loop. Dependence
of the spectral function \( \langle s_\alpha^* s_\beta \rangle \) of the vortex line on both the one-di-

dimensional wave vector \( \kappa \) and intensity of the external force correlator \( \langle \zeta_\alpha^* \zeta_\alpha \rangle \) was studied. Here \( s_\alpha \) is

the Fourier transform of the line element position \( s^\alpha(\xi, t) \). It is shown that under the

influence of an external random force a vortex ring becomes a small tangle whose mean

size depends on external force intensity. The theoretical predictions and the numerical

results are in reasonable agreement.

1 Introduction

In the previous paper [1] we discuss how the large-scale perturbations can de-

stroy the thermal equilibrium state in the space of vortex loop configurations.

In this paper we elaborate that idea and present results of the both analytical

and numerical investigations on stochastic dynamics of a vortex filament in HeII

undergoing an action of the large scale random displacements. Moreover we con-

sider the case when the smooth dissipation connected with normal component

is small (that correspond to the case of very small temperature ) and the only

strong dissipative mechanisms appear at very small scales comparable with the

core radius of vortex. Thus detailed (at which scale) balance between the pump-

ing and dissipation required for themal equilibrium is violated and, as it has

been discussed in [1], essentially nonequilibrium picture state must develop.

From a formal point of view that problem is significantly more involved,

therefore we restrict ourselves to a consideration of the local induction approx-

imation [2],[3] in the equation of motion and omit processes of reconnection.

This statement of problem is, of course, far from real superfluid turbulence in

He II. A value of that work is that it enables us to understand mechanisms of

entanglement of vortex filament and of appearing the strongly non equilib-

rium state. We would remind that idea of the vortex tangle had been launched by

Feynman more than 40 years ago [2] and only about 10-15 years ago Schwarz

demonstrated and confirmed that idea in his famous numerical simulations [3].

To our knowledge a similar success in analytic study is absent.

A structure of the paper is following. In the first part of this paper we de-

develop nonequilibrium. diagram technique analogous to the one elaborated by
Wyld \cite{Wyld} for classical turbulence. Using further method of direct interaction approximation we derive a set of Dyson equation for the pair correlators and for the Green functions. Assuming that region of stirring force and dissipation are widely separated in space of scales we seek for a scale invariant solution in the so called inertial interval. We also present results of direct numerical simulation of the vortex tangle dynamics. Numerical results confirm the ones obtained in the analytical investigations, however some discrepancies remained.

2 Analytical Investigation

In the local induction approximation (LIA) the equation of motion of quantized vortex filament in HeII reads

\[
\frac{d s(\xi, t)}{d t} = \tilde{\beta} s' \times s'' + \delta + \zeta(\xi, t).
\]  

(1)

Here \(s(\xi, t)\) is a point of the filament labeled by the Lagrangian variable \(\xi\), \(0 \leq \xi \leq 2\pi\), which coincides here with the arclength. The quantity \(\delta\) stands for dissipation, which is small for usual scales and large for marginally small scales comparable with the core size \(r_0\). External Langevin force \(\zeta(\xi, t)\) is supposed to be Gaussian with correlator

\[
\langle \zeta^\alpha(\xi_1, t_1) \zeta^\beta(\xi_2, t_2) \rangle = F^\alpha(\xi_1 - \xi_2) \delta(t_1 - t_2) \delta^{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3,
\]  

(2)

where \(F^\alpha(\xi_1 - \xi_2)\) is changing on the large scale of order of the line length (\(\sim 2\pi\)).

The quantity \(\tilde{\beta}\) is \(\tilde{\beta} = \frac{\kappa}{4\pi} \log \frac{R}{r_0}\), with circulation \(\kappa\) and cutting parameters \(R\) (external size, e.g. averaged radius of curvature) and \(r_0\). For our numerical calculations we have chosen \(\kappa = 10^{-3} cm^2/s\). This value corresponds to the case of superfluid helium.

In Fourier space equation (2) has the form

\[
- i\omega s^\alpha_q = \int \Gamma^{\alpha\beta\gamma}_{K_1 K_2} s^\beta_{q_1} s^\gamma_{q_2} \delta(q - q_1 - q_2) dq_1 dq_2 + \delta_q + \zeta^\alpha_q.
\]  

(3)

Here \(s^\alpha_q\) is the spacial and temporal Fourier component of \(s^\alpha(\xi, t)\), defined as follows:

\[
s^\alpha_q = \int \int s^\alpha(\xi, t) e^{i(\omega t - \kappa \xi)} dt d\xi.
\]  

(4)

The vertex \(\Gamma^{\alpha\beta\gamma}_{K_1 K_2}\) responsible for nonlinear interaction has the form

\[
\Gamma^{\alpha\beta\gamma}_{K_1 K_2} = \frac{i\tilde{\beta}}{2\sqrt{2\pi}} \epsilon^{\alpha\beta\gamma} K_1 K_2 (K_2 - K_1),
\]  

(5)

where \(\epsilon^{\alpha\beta\gamma}\) it the antisymmetric unit tensor. One can show the vertex \(\Gamma^{\alpha\beta\gamma}_{K_1 K_2}\) to satisfy the so called Jacoby identities.
\[ \kappa_n^2 \Gamma_\kappa + \kappa_2^2 \Gamma_\kappa \alpha \beta + \kappa_1^2 \Gamma_\kappa \alpha \beta \] 
\( \delta (\kappa + \kappa_1 + \kappa_2) = 0, \quad n = 2, 4. \) (6)

This relation expresses are tightly connected with the laws of conservation of total length \( L \) and curvature \( K \):

\[ L = \int_0^{2\pi} s' s' \, d\xi = \text{const}, \quad K = \int_0^{2\pi} s'' s'' \, d\xi = \text{const}. \] (7)

Conservation of these quantities is readily derived from either of relations (1), (3). It is understood that conservation law is held in absence of both dissipation and stirring force.

One of the regular approaches to describe random fields is based on the Wyld diagram technique \[4\], originally developed to study hydrodynamic turbulence. Following this technique we introduce for the description of random processes the following averages: the spectral density tensor (or correlator, or simply spectrum) \( S_{\alpha \beta}^q \) and the Green tensor \( G_{\alpha \beta}^q \) (or simply Green function) which are defined by

\[ S_{\alpha \beta}^q \delta(q + q_1) = \langle s_{\alpha}^q s_{\beta}^{q_1} \rangle, \] (8)

\[ G_{\alpha \beta}^q \delta(q + q_1) = \langle \delta s_{\alpha}^q \delta s_{\beta}^{q_1} \rangle. \] (9)

Analysis of diagrams shows that due to the antisymmetry of tensor \( \epsilon^{\alpha \beta \gamma} \) contained in the expression for the vertex \( \Gamma_{\kappa \kappa_1 \kappa_2}^{\alpha \beta \gamma} \), both \( S_{\alpha \beta}^q \) and \( G_{\alpha \beta}^q \) are proportional to \( \delta_{\alpha \beta} \), i.e. \( S_{\alpha \beta}^q \equiv S_{\alpha}^q \) and \( G_{\alpha \beta}^q \equiv G_{\alpha}^q \). Details of that technique are described in \[5\].

The renormalized quantities \( S_{\alpha}^q \) and \( G_{\alpha}^q \) (taking into account interactions) satisfy a Dyson set of diagram equations:

\[ G_{\alpha}^q = G_{\alpha}^{\circ} + G_{\alpha}^{\circ} \Sigma_{\alpha}^q G_{\alpha}^{\circ}, \] (10)

\[ S_{\alpha}^q = G_{\alpha}^{\circ} (F_{\alpha}^q + \Phi_{\alpha}^q) G_{\alpha}^{\circ}. \] (11)

Here \( G_{\alpha}^{\circ} \) is the “bare” Green function which is equal to \( (\omega - \delta_\kappa)^{-1} \). The mass operators \( \Phi_{\alpha}^q \) and \( \Sigma_{\alpha}^q \) can be written in form of diagram series: These series frequently used in nonequilibrium processes have a standard form, explicit form of them is given in \[4\], \[5\].

### 3 Conservation Laws and Pair Correlators

Dyson equations have shapes indicating a cumbersome handling, therefore they can be studied for some special cases. One of them is considered in the present paper. It is connected with conservation laws for the total length and the curvature expressed by (6). Let us consider conservation of total curvature (for total...
length there is the same consideration). In Fourier space the conservation laws for total curvature $\kappa$ can be expressed in the following form

$$\frac{\partial K_\kappa}{\partial t} + \frac{\partial P^K_\kappa}{\partial \kappa} = I^K_+ - I^K_-$$

(12)

where $K_\kappa = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} s'' s'' e^{-i\kappa \xi} d\xi$ is the curvature density and $P^K_\kappa$ is the flux of this quantity in Fourier space (or, equally, in space of scales). The right-hand side of equation (12) describes creation of additional curvature (with rate $I^K_+$) due to external force and annihilation of it due to dissipative mechanism (with rate $-I^K_-$). In the equilibrium case the flux $P^K_\kappa$ is absent and source and sink terms must compensate each other locally for each $\kappa$, i.e. $I^K_+ = I^K_-$. In the case under consideration when source and sink terms are widely separated in $\kappa$-space that condition is obviously violated. Therefore a flux of curvature $P^K_\kappa$ in Fourier space appears. In the case of large numbers $\kappa$ far from both region of the pumping $\kappa_+$ and of the sink $\kappa_-$, $\kappa_+ \ll \kappa \ll \kappa_+$, the so called inertial interval, derivative $\partial P^K_\kappa / \partial \kappa = 0$, so $P^K_\kappa$ is constant equal, say, $P^K_\kappa$. Resuming we conclude that the problem reduces to study the set of Dyson equations (10)-(11) in inertial interval under condition of constant flux of the curvature. In this case $S^\alpha_q$ and $G^\alpha_q$ are expected to be independent on the concrete type of both source and sink but to be dependent on value of $P^K$. Furthermore, the vertices $\Gamma^{\alpha\beta\gamma}_{\kappa_1\kappa_2}$ are homogeneous functions of its arguments. This property, as well as the condition $\kappa_+ \ll \kappa_- \ll \kappa_+$ by virtue of which one can put $\kappa_+ = 0$ and $\kappa_- = \infty$, leads to the assumption that the problem is the scale invariant, i.e. it has no characteristic scale for $\kappa$. This suggests a power-law form of $S^\alpha_q$ and $G^\alpha_q$

$$S^\alpha_q = \frac{1}{\kappa^r + p} \left[ \frac{\omega}{\kappa^r} \right], \quad G^\alpha_q = \frac{1}{\kappa^g} \left[ \frac{\omega}{\kappa^g} \right].$$

(13)

Here both $f$ and $g$ are dimensionless functions of their arguments. We aim now to find the scaling indices $r$ and $p$.

The first relation between indices $r$ and $p$ can be found from an analysis of diagram series, claiming all terms to have the same powers of argument $\kappa$. This leads to the first scaling condition

$$2r + p = 7.$$

(14)

Another relation between $r$ and $p$ can be obtained from the Dyson equations (10), (11) which can be rewritten in the form (see e.g. [3])

$$\int d\omega \ Im \ \left\{ S^\alpha_q \Sigma^\alpha_q - \Phi^\alpha_q G^{\alpha \ast}_q \right\} = 0.$$  

(15)

This relation plays the role of kinetic equations for systems with a weak interaction. It has been obtained with help of the expression for the Green function $G^\alpha_q = (\omega - \Sigma^\alpha_q)^{-1}$; the external force correlator $F^\alpha_q$ disappears in the inertial interval. To find a relation of interest between $r$ and $p$ we rewrite relation (15) disclosing expressions for mass operators $\Phi^\alpha_q$, $\Sigma^\alpha_q$ and restricting ourselves
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to first order terms in diagram series. That procedure called direct interaction approximation is frequently used in classical turbulence (see e.g. [6]). After some calculation we arrive at the following relation (see also [8]):

\[
\text{Im} \int d\omega d\omega_1 d\kappa_1 d\kappa_2 \delta(q + q_1 + q_2) \times \Gamma_{\kappa_1 \kappa_2}^{\alpha \beta \gamma} \\
\times \left\{ \Gamma_{\kappa_1 \kappa_2}^{\alpha \beta \gamma} G_\alpha^{q_1} S_{\beta q_1} S_{\gamma q_2}^\gamma + \Gamma_{\kappa_2 \kappa_1}^{\gamma \alpha \beta} G_\gamma^{q_2} S_{\alpha q_2} S_{\beta q_1}^\beta + \Gamma_{\kappa_1 \kappa_2}^{\beta \gamma \alpha} G_\beta^{q_1} S_{\gamma q_1} S_{\alpha q_2}^\alpha \right\} = 0 .
\]

(16)

To move further we perform conformal transformations in the second and third term within the braces in integrand, known as Zakharov transformations (see e.g. Zakharov [7] and Kuznetsov and L’vov [8]). For example for the second term these transformations have the form

\[
\kappa = \kappa ''(\kappa/\kappa '') , \quad \kappa_1 = \kappa ' (\kappa/\kappa '') , \quad \kappa_2 = \kappa (\kappa/\kappa '') ,
\]

(17)

\[
\omega = \omega ''(\kappa/\kappa '') r , \quad \omega_1 = \omega ' (\kappa/\kappa '') r , \quad \omega_2 = \omega (\kappa/\kappa '') r .
\]

(18)

The third term is transformed in similar manner. As a result the integrand in (16) becomes

\[
\Gamma_{\kappa_1 \kappa_2}^{\alpha \beta \gamma} G_\alpha^{q_1} S_{\beta q_1} S_{\gamma q_2}^\gamma \left\{ \Gamma_{\kappa_1 \kappa_2}^{\alpha \beta \gamma} + \left[ \frac{\kappa}{\kappa_2} \right]^x \Gamma_{\kappa_2 \kappa_1}^{\gamma \alpha \beta} + \left[ \frac{\kappa}{\kappa_1} \right]^x \Gamma_{\kappa_1 \kappa_2}^{\beta \gamma \alpha} \right\}
\]

(19)

where

\[
x = 7 - r - 2p .
\]

(20)

Due to Jacoby identities (6) the integrand vanishes when \( x = -2 \) for conservation of total length and \( x = -4 \) for conservation of total curvature. Substituting these values into (20) and solving equations (14), (20) we obtain a set of couples of indices \( r, p \) corresponding to nonequilibrium states with fluxes

Fig. 1.
of the length \((r = 5/3, \ p = 11/3)\) and of the curvature \((r = 1, \ p = 5)\). One
time correlators can be found then integrating over frequecnes \(\omega\)

\[
S_\kappa^\alpha = \int d\omega \frac{1}{k^{r+p}} f\left(\frac{\omega}{k^s}\right) \propto \begin{cases} 
\kappa^{-\frac{11}{3}} & \text{for length} \\
\kappa^{-5} & \text{for curvature} 
\end{cases} \tag{21}
\]

So we have got solutions for the correlators \(S_\kappa^\alpha\) which correspond to different
conservation laws in \([8]\). Since there are no sources and sinks acting in the intermediate range these solutions guarantee that the according fluxes are constant. Depending on the way of agitation of the system one can get the real spectrum as some mixture of the obtained solutions in which the fluxes of length and curvature are present simultaneously. A similar situation for wave systems has been discussed earlier (see e.g. \([6]\) and bibliography therein) and it is called multi-flux solution.

Having in mind to compare our result with the both numerical and experimental investigation we have to take into account a presence of \(\delta\)-correlated (in \(\xi\)-space) random force considered in our previous paper \([1]\). In the local induction approximation the energy \(H\{s\}\) of line is proportional of its length and in parametrization when \(\xi\) is arclength can be expressed as

\[
H\{s\} = \frac{\rho_0 k}{4\pi} \ln \frac{R}{r_0} \int s'(\xi)s'(\xi) d\xi \tag{22}
\]

It is easy to see that the equilibrium distribution described in \([1]\) leads in that case to correlator \(S_\kappa^\alpha \propto 1/\kappa^2\). The final solution is a mix of equilibrium solution and of the ones expressed by relation \([21]\). Because of nonlinearity it, in general, is not a simple superposition except of the cases when one of stirring action prevails and the other can be considered as small deviations. For instance if a

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Fig. 2.
large-scale random stirring is small in comparison with $\delta$-correlated (in $\xi$-space) action we have

$$S^\alpha_\kappa = \frac{A}{\kappa^2} + \frac{B}{\kappa^{11/3}} + \frac{C}{\kappa^5}. \quad (23)$$

The second and third terms in the right-hand side of (23) are small. The constants $A, B, C$ entering are connected with both intensity of random stirring and its structure. The further specification requires some additional analysis.

\section*{4 Some Numerical Results}

In this section we present some preliminary results on a direct numerical simulations of a vortex ring evolution under action of a random stirring displacements. The large scale character of noise was guaranteed by calculating it from a Fourier series taking into account only the first few harmonics. Besides some (uncontrolled) white noise due to numerical procedure has been excited. Fig. 1 shows the projection of the line in the $x, y$ - plane (where the ring was placed initially) for several times. As predicted, an consequent arising of higher harmonics takes place leading eventually to an entanglement of the initially smooth vortex loop.

Another numerical results is shown in Fig. 2 where logarithm of quantity $S_{xx}^\kappa$ averaged over several realizations is depicted as a functions of log $\kappa$. The average slopes the graphs depend on intensity large-scale stirring force. In several realizations the slope lies between $-2.5$ and $-3.5$, which agrees with theoretical prediction (23).

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