Hausdorff compactifications in ZF

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Abstract

For a compactification $\alpha X$ of a Tychonoff space $X$, the algebra of all functions $f \in C(X)$ that are continuously extendable over $\alpha X$ is denoted by $C_\alpha(X)$. It is shown that, in a model of $\text{ZF}$, it may happen that a discrete space $X$ can have non-equivalent Hausdorff compactifications $\alpha X$ and $\gamma X$ such that $C_\alpha(X) = C_\gamma(X)$. Amorphous sets are applied to a proof that Glicksberg’s theorem that $\beta X \times \beta Y$ is the Čech-Stone compactification of $X \times Y$ when $X \times Y$ is a Tychonoff pseudocompact space is false in some models of $Z\text{F}$. It is noticed that if all Tychonoff compactifications of locally compact spaces had $C^*$-embedded remainders, then van Douwen’s choice principle would be satisfied. Necessary and sufficient conditions for a set of continuous bounded real functions on a Tychonoff space $X$ to generate a compactification of $X$ are given in $Z\text{F}$. A concept of a functional Čech-Stone compactification is investigated in the absence of the axiom of choice.
1 Introduction

For a topological space $X$, we denote by $C(X)$ the algebra of all continuous real functions on $X$ and by $C^*(X)$ the algebra of all bounded continuous real functions on $X$. We recall that a compactification of $X$ is an ordered pair $\langle \alpha X, \alpha \rangle$ such that $\alpha X$ is a compact (not necessarily Hausdorff) space, while $\alpha : X \to \alpha X$ is a homeomorphic embedding such that $\alpha(X)$ is dense in $\alpha X$. Usually, a compactification $\langle \alpha X, \alpha \rangle$ of $X$ is denoted by $\alpha X$, the space $X$ is identified with $\alpha(X)$ and $\alpha$ is treated as the identity map $\text{id}_X : X \to \alpha X$. Thus, in abbreviation, we can say that a compactification of $X$ is a compact space $\alpha X$ such that $X$ is a dense subspace of $\alpha X$. The remainder of a compactification $\alpha X$ of $X$ is the set $\alpha X \setminus X$. We use the notation $\alpha X \approx \gamma X$ to say that compactifications $\alpha X$ and $\gamma X$ of $X$ are equivalent, i.e. that there exists a homeomorphism $h : \alpha X \to \gamma X$ such that $h \circ \alpha = \gamma$. If there exists a continuous map $f : \alpha X \to \gamma X$ such that $f \circ \alpha = \gamma$, we write $\gamma X \leq \alpha X$. If $\alpha X$ and $\gamma X$ are Hausdorff compactifications of $X$, then $\alpha X \approx \gamma X$ if and only if $\alpha X \leq \gamma X$ and $\gamma X \leq \alpha X$.

Throughout this article, we assume the same system $\text{ZF}$ of set-theoretic axioms as in [22] to develop a theory of Hausdorff compactifications without the axiom of choice $\text{AC}$. Of course, we assume that $\text{ZF}$ is consistent. We clearly denote theorems provable in $\text{ZF}$ or, respectively, in $\text{ZF}$ enriched by an additional axiom $\text{A}$ by putting $[\text{ZF}]$ or $[\text{ZF} + \text{A}]$, respectively, at the beginning of theorems. We use the same notation as in [12] for weaker forms of $\text{AC}$, and we shall refer to a corresponding form from [15]. In particular, $\text{UFT}$ stands for the ultrafilter theorem which states that, for every set $X$, each filter in the power set $\mathcal{P}(X)$ can be enlarged to an ultrafilter (cf. Definition 2.15 in [12] and Form 14 A in [15]).

If $F$ is a set of mappings $f : X \to Y_f$, then the evaluation map $e_F : X \to \prod_{f \in F} Y_f$ is defined by: $e_F(x)(f) = f(x)$ for any $x \in X$ and $f \in F$ (cf. Defini-
tion 1.23 in [2]). For a Tychonoff space $X$, we denote by $\mathcal{E}(X)$ the collection of all $F \subseteq C^*(X)$ such that $e_F : X \to \mathbb{R}^F$ is a homeomorphic embedding. If $F \in \mathcal{E}(X)$, we denote by $e_F X$ the closure of $e_F(X)$ in $\mathbb{R}^F$. In ZFC-theory of Hausdorff compactifications, it is well known that every Hausdorff compactification $\alpha X$ of a non-empty space $X$ is strictly determined by the algebra $C_\alpha(X)$ of all continuous real functions on $X$ that are continuously extendable over $\alpha X$; more precisely, $\alpha X$ is equivalent with $e_F X$ for $F = C_\alpha(X)$. Moreover, if $\alpha X$ and $\gamma X$ are Hausdorff compactifications of $X$, then it holds true in ZFC that $\alpha X$ is equivalent with $\gamma X$ if and only if $C_\alpha(X) = C_\gamma(X)$ (cf. Theorem 2.10 in [2]). However, in ZF, it is equivalent with UFT that, for every non-empty Tychonoff space $X$ and for each $F \in \mathcal{E}(X)$, the space $e_F X$ is compact (see, for instance, Theorem 10.12 of [22] and Theorem 4.37 in [12]). In our article, we pay a special attention to the fact that there is a model of ZF in which there are Tychonoff spaces that have Hausdorff, not completely regular compactifications.

In Section 2, we prove that, in a model of ZF, it may happen that even a discrete space $X$ can have non-equivalent Hausdorff compactifications $\alpha X$ and $\gamma X$ such that $C_\alpha(X) = C_\gamma(X)$ and $\gamma X$ is not completely regular. A not completely regular Hausdorff compactification of a Tychonoff space is called a strange compactification. We show that all Hausdorff compactifications with finite remainders of Tychonoff spaces are completely regular in ZF. We prove that it may happen in a model of ZF that the remainder a Tychonoff compactification of a locally compact space can be not $C^*$-embedded in the compactification. If for every locally compact space $X$ and for every Tychonoff compactification $\alpha X$ of $X$ the remainder $\alpha X \setminus X$ is $C^*$-embedded in $\alpha X$, then van Douwen’s choice principle (Form 119 in [15]) must hold. We notice that an amorphous set exists if and only if there exists an infinite discrete space $X$ whose all Hausdorff compactifications are equivalent with the Alexandroff compactification of $X$. Moreover, we show that Glicksberg’s theorem on when the product of Čech-Stone compactifications of $X$ and $Y$ is the Čech-Stone compactification of $X \times Y$ is false in some models of ZF.

In Section 3, we show a number of necessary and sufficient conditions for $F \in \mathcal{E}(X)$ to have the property that the space $e_F X$ is compact in ZF. We give a definition of a functional Čech-Stone compactification and compare it in ZF with the standard notion of the Čech-Stone compactification of a Tychonoff space. For $F \in \mathcal{E}(X)$, we list a number of new problems on the smallest sequentially closed subalgebra of $C^*(X)$ which contains $F$ and all constant functions from $C^*(X)$. 

3
It is not obvious at all whether every Tychonoff space has a Hausdorff compactification in ZF. In 2016, E. Wajch asked whether there is a model of ZF in which a non-compact metrizable Cantor cube can fail to have a Hausdorff compactification (see Question 3.8 of [27]). The following general problem is unsolved:

**Problem 1.1.** Does there exist a model of ZF in which there is a Tychonoff space which does not have any Hausdorff compactification?

**Proposition 1.2.** In every model of ZF + UFT, every Tychonoff space has a Tychonoff compactification.

**Proof.** Let $X$ be a Tychonoff space in a model $\mathcal{M}$ of ZF + UFT. Since, by Theorem 4.70 of [12], all Tychonoff cubes are compact in $\mathcal{M}$, the space $e_F X$ for $F = C^*(X)$ is compact. Thus, in $\mathcal{M}$, the pair $\langle e_F X, e_F \rangle$ is a Tychonoff compactification of $X$. □

Although we are still unable to solve Problem 1.1, we offer some other results relevant to it. Of course, if $X$ is an infinite $T_1$-space, it is easy to show a compact $T_1$-space $Y$ such that $X$ is a dense subspace of $Y$ and $Y \setminus X$ is a singleton. To do this, for an infinite $T_1$ space $X$, it suffices to take a point $\infty \not\in X$, put $Y = X \cup \{\infty\}$ and define a topology in $Y$ as the collection of all open subsets of $X$ and of all sets of the form $Y \setminus A$ where $A$ is a finite subset of $X$.

We use the same definition of the Čech-Stone compactification in ZF as in [10]:

**Definition 1.3.** A Čech-Stone compactification of a Hausdorff space $X$ is a Hausdorff compactification $\langle \beta X, \beta \rangle$ of $X$ such that, for every compact Hausdorff space $K$ and for each continuous mapping $f : X \to K$, there exists a continuous mapping $\tilde{f} : \beta X \to K$ such that $f = \tilde{f} \circ \beta$.

Any two Čech-Stone compactifications of a space $X$ are equivalent, so, if a Hausdorff space $X$ has a Čech-Stone compactification, we denote any Čech-Stone compactification of $X$ by $\beta X$ and call it the Čech-Stone compactification of $X$.

**Remark 1.4.** In ZF, if a Hausdorff space $X$ has its Čech-Stone compactification, then, for every Hausdorff compactification $\alpha X$ of $X$, we have $\alpha X \leq \beta X$. 

Basic facts about Hausdorff compactifications in ZFC can be found, for instance, in [2], [7], [23] and [5]. An essential role in ZFC-theory of Hausdorff compactifications is played by the following notion of a normal (Wallman) base which can be found, for instance, in [2] and [23]; however, we use its slightly modified form given in Definition 1.7 of [22]:

**Definition 1.5.** A family \( C \) of subsets of a topological space \( X \) is called a normal or Wallman base for \( X \) if \( C \) satisfies the following conditions:

(i) \( C \) is a base for closed sets of \( X \);

(ii) if \( C_1, C_2 \in C \), then \( C_1 \cap C_2 \in C \) and \( C_1 \cup C_2 \in C \);

(iii) for each set \( A \subseteq X \) and for each \( x \in X \setminus A \), if \( A \) is a singleton or \( A \) is closed in \( X \), then there exists \( C \in C \) such that \( x \in C \subseteq X \setminus A \);

(iv) if \( A_1, A_2 \in C \) and \( A_1 \cap A_2 = \emptyset \), then there exist \( C_1, C_2 \in C \) with \( A_1 \cap C_1 = A_2 \cap C_2 = \emptyset \) and \( C_1 \cup C_2 = X \).

For Wallman-type extensions, we use the same notation as in [22]. Namely, suppose that \( C \) is a Wallman base for \( X \). Let \( W(X,C) \) denote the set of all ultrafilters in \( C \). For \( A \in C \), we put

\[
[A]_C = \{ p \in W(X,C) : A \in p \}.
\]

The Wallman space of \( X \) corresponding to \( C \) is denoted by \( W(X,C) \) and it is the set \( W(X,C) \) equipped with the topology having the collection \( \{ [A]_C : A \in C \} \) as a closed base. The canonical embedding of \( X \) into \( W(X,C) \) is the mapping \( h_C : X \to W(X,C) \) defined by the equality \( h_C(x) = \{ A \in C : x \in A \} \) for each \( x \in X \). The pair \( (W(X,C), h_C) \) is called the Wallman extension of \( X \) corresponding to \( C \) and, for simplicity, this extension is denoted also by \( W(X,C) \). In the case when \( X \) is a discrete space, the Wallman space \( W(X, P(X)) \) corresponding to the power set \( P(X) \) is usually called the Stone space of \( X \) and it is denoted by \( S(X) \).

A topological space \( X \) is called semi-normal if \( X \) has a Wallman base (cf. [6] and [22]). In the light of Theorem 2.8 of [22], UFT is an equivalent of the following sentence: For every semi-normal space \( X \) and for every normal base \( C \) of \( X \), the Wallman space \( W(X,C) \) is compact. A Hausdorff compactification \( \alpha X \) of a space \( X \) is called a Wallman-type compactification of \( X \) if there exists a normal base \( C \) of \( X \) such that the space \( W(X,C) \) is
compact and the compactification $W(X,C)$ of $X$ is equivalent with $\alpha X$. It was proved in [25] that not all Hausdorff compactifications in ZFC are of Wallman type; however, a satisfactory solution to the following problem is unknown:

**Problem 1.6.** Can it be proved in ZF that there are discrete spaces that have Hausdorff compactifications which are not of Wallman type?

**Remark 1.7.** It is worth to notice that Šapiro’s result given in [24] that all Hausdorff compactifications of discrete spaces are of Wallman type is provable in ZF. Historical remarks about the results of [24] are given in [3].

For a topological space $X$, let $Z(X)$ stand for the collection of all zero-sets in $X$. Then $Z(X) = \{ f^{-1}(0) ; f \in C^*(X) \}$. Of course, $Z(X)$ is a normal base for $X$ if and only if $X$ is a Tychonoff space. It is well known that, in ZFC, if $X$ is a Tychonoff space, then its Čech-Stone compactification is of Wallman type because it is equivalent with the Wallman extension of $X$ corresponding to the normal base $Z(X)$ (cf., e.g., [2], [6], [7], [23] and [22]). That the Wallman space $W(X,Z(X))$ is compact for every Tychonoff space $X$ is an equivalent of UFT (see, e.g., Theorem 2.8 of [22]). Some relatively new results on Hausdorff compactifications in ZF have been obtained in [14], [10], [11], [18], [19] and, for Delfs-Knebusch generalized topological spaces (applicable to topological spaces), in [22]. In [1], there is a well-written chapter on a history of Hausdorff compactifications in ZFC (see [3]). Unfortunately, not much is known about Hausdorff compactifications in ZF. In this article, to start a systematic study of Hausdorff compactifications in ZF, we put in order basic notions concerning them, as well as we show newly discovered important differences between ZF-theory and ZFC-theory of Hausdorff compactifications.

## 2 Strange Hausdorff compactifications

**Definition 2.1.** Let $\alpha X$ be a Hausdorff compactification of a Tychonoff space $X$. We say that:

(i) $\alpha X$ is *generated by a set of functions* if there exists $F \in \mathcal{F}(X)$ such that the space $e_F X$ is compact and the compactifications $\alpha X$ and $e_F X$ of $X$ are equivalent;
(ii) \(\alpha X\) is generated by a set \(F \in \mathcal{E}(X)\) if the space \(e_F X\) is compact, while the compactifications \(e_F X\) and \(\alpha X\) of \(X\) are equivalent;

(iii) \(\alpha X\) is strange if it is not generated by a set of functions.

**Definition 2.2.** For a topological space \(X\), we say that:

(i) \(\text{UL}(X)\) holds or, equivalently, \(X\) satisfies Urysohn's Lemma or, equivalently, \(X\) is a U space if, for each pair of disjoint closed subsets \(A, B\) of \(X\) there exists \(f \in C^*(X)\) such that \(A \subseteq f^{-1}(0)\) and \(B \subseteq f^{-1}(1)\);

(ii) \(\text{TET}(X)\) holds or, equivalently, \(X\) satisfies Tietze’s Extension Theorem or, equivalently, \(X\) is a T space if, for each closed subspace \(P\) of \(X\), every function \(f \in C(P)\) is extendable to a function from \(C(X)\).

Let us denote Form 78 of [15] (Urysohn’s Lemma) by \(\text{UL}\). Then \(\text{UL}\) is the sentence: \(\text{UL}(X)\) holds for every normal space \(X\). We denote Form 375 of [15] (Tietze-Urysohn Extension Theorem) by \(\text{TET}\). Then \(\text{TET}\) is the sentence: \(\text{TET}(X)\) holds for every normal space \(X\). The principle of dependent choices (Form 43 in [15]) is denoted by \(\text{DC}\).

**Remark 2.3.** In [16], spaces that satisfy Urysohn’s Lemma were called U spaces, while spaces that satisfy Tietze’s Extension Theorem were called T spaces, \(\text{UL}\) was denoted by \(\text{NU}\), while \(\text{TET}\) by \(\text{NT}\). Of course, every T space is a U space and all U spaces are normal. In [16], \(\text{NU (NT, resp.)}\) is an abbreviation to: "Every normal space is a U space." ("Every normal space is a T space.", resp.). However, in this article, we find it more natural to denote Urysohn’s Lemma by \(\text{UL}\) and Tietze’s Extension Theorem by \(\text{TET}\). It is well known that \(\text{UL}\) and \(\text{TET}\) are independent of \(\text{ZF}\). Of course, it is true in \(\text{ZF}\) that if \(X\) is a topological space such that \(\text{TET}(X)\) holds, then \(\text{UL}(X)\) also holds. It is known that \(\text{ZF + DC}\) implies both \(\text{UL}\) and \(\text{TET}\) (cf. entries (43, 78) and (43, 375) on pages 339 and 386 in [15]). Hence, in \(\text{ZF + DC}\), a topological space \(X\) satisfies \(\text{UL}(X)\) if and only if \(\text{TET}(X)\) holds. In [16], it was shown that there is a model \(\mathcal{M}\) of \(\text{ZF}\) in which there is a compact Hausdorff space \(X\) such that \(\text{UL}(X)\) holds and \(\text{TET}(X)\) fails in \(\mathcal{M}\). However, it is an open question, already posed in [16], whether \(\text{UL}\) implies \(\text{TET}\).

We can easily obtain the following results:

**Proposition 2.4.** \([\text{ZF + UL}]\) An arbitrary Hausdorff compactification is not strange.
Proposition 2.5. [ZF] A Hausdorff compactification $\alpha X$ of a non-empty Tychonoff space $X$ is not strange if and only if $\alpha X$ is completely regular.

Proposition 2.6. [ZF] If a Hausdorff compactification $\gamma X$ of a non-empty topological space $X$ is completely regular, then $\gamma X$ is generated by $C_\gamma(X)$.

Definition 2.7. Let $n$ be a positive integer. It is said that a compactification $\alpha X$ of a topological space $X$ is an $n$-point compactification of $X$ if the remainder $\alpha X \setminus X$ consists of exactly $n$ points.

The Alexandroff compactification of a locally compact, non-compact Hausdorff space $X$ is every Hausdorff compactification of $X$ with a one-point remainder. However, the following notion of the Alexandroff compactification of a topological space is also useful:

Definition 2.8. Let $\langle X, \tau_X \rangle$ be a non-compact topological space and let $\infty$ be an element which does not belong to $X$. Denote by $K_X$ the collection of all simultaneously closed and compact sets of $\langle X, \tau_X \rangle$. We put $\alpha_a X = X \cup \{\infty\}$ and $\tau = \tau_X \cup \{U \subseteq \alpha_a X : \alpha_a X \setminus U \in K_X\}$. Then the topological space $\langle \alpha_a X, \tau \rangle$ is called the Alexandroff compactification of $\langle X, \tau_X \rangle$ and we denote it by $\alpha_a X$.

We are going to give partial solutions to the following open problem:

Problem 2.9. Is there a model of $\text{ZF}$ in which there exists a Hausdorff, not completely regular compactification $\gamma X$ of a Tychonoff space $X$ such that $\gamma X \setminus X$ is completely regular?

For a space $X$, let $Coz(X) = \{X \setminus A : A \in Z(X)\}$. Members of $Coz(X)$ are called cozero-sets of $X$. Basic properties of zero-sets and cozero-sets can be found in [7] and [5].

We are going to prove in $\text{ZF}$ that all Hausdorff compactifications with finite remainders of Tychonoff spaces are not strange. To do this well, we need a proof in $\text{ZF}$ of Theorem 3.1.7 of [5]; however, since the axiom of choice is involved in the proof to Theorem 3.1.7 in [5], let us state the following lemma and give its subtle proof in $\text{ZF}$:

Lemma 2.10. [ZF] Let $K$ be a compact subset of a completely regular space $X$ and let $A$ be a closed subset of $X$ such that $K \cap A = \emptyset$. Then there exists a function $f \in C^*(X)$ such that $A \subseteq f^{-1}(0)$ and $K \subseteq f^{-1}(1)$.
Proof. Let \( \mathcal{V} = \{ V \in Coz(X) : \text{cl}_X V \subseteq X \setminus A \} \). Since \( X \) is completely regular, we have \( K \subseteq \bigcup \mathcal{V} \). By the compactness of \( K \), there exists a finite collection \( \mathcal{U} \subseteq \mathcal{V} \) such that \( K \subseteq \bigcup \mathcal{U} \). A finite union of cozero-sets is a cozero-set; thus, the set \( U_0 = \bigcup \mathcal{U} \) is a cozero-set of \( X \). There exists a continuous function \( g : X \to [0,1] \) such that \( U_0 = g^{-1}((0,1]) \). Then \( g^{-1}(0) \cap K = \emptyset \). It follows from the continuity of \( g \) and from the compactness of \( K \) that there exists a positive integer \( n_0 \) such that \( g^{-1}([0,\frac{1}{n_0}]) \cap K = \emptyset \). The sets \( C = g^{-1}(\{0\}) \) and \( D = g^{-1}([\frac{1}{n_0},1]) \) are disjoint zero-sets in \( X \), \( K \subseteq D \) and \( A \subseteq C \). Since disjoint zero-sets are functionally separated (cf. 1.10 in [4] or Theorem 1.5.14 in [3]), there exists a continuous function \( f : X \to [0,1] \) such that \( C \subseteq f^{-1}(0) \) and \( D \subseteq f^{-1}(1) \). Then \( A \subseteq f^{-1}(0) \) and \( K \subseteq f^{-1}(1) \). \( \square 

Corollary 2.11. [ZF] Every compact completely regular space \( X \) satisfies UL(\( X \)).

Proposition 2.12. [ZF] If \( X \) is a Tychonoff, locally compact non-compact space, then the one-point Hausdorff compactification of \( X \) is not strange.

Proof. Let us fix a closed subset \( A \) of \( \alpha_aX \) and suppose that \( x \in \alpha_aX \setminus A \). We consider the following cases:

(i) \( \infty \in A \). In this case \( x \in X \). Since every Hausdorff compact space is normal, there exists a pair \( U, V \) of disjoint open sets in \( \alpha_aX \) such that \( x \in U \) and \( A \subseteq V \). Then \( x \notin \text{cl}_{\alpha_aX} V \). Since \( X \) is Tychonoff, there exists a function \( f \in C^*(X) \) such that \( f(x) = 0 \) and \( X \cap \text{cl}_{\alpha_aX} V \subseteq f^{-1}(1) \). We define a function \( F : \alpha_aX \to \mathbb{R} \) by putting \( F(t) = f(t) \) for each \( t \in X \) and \( F(\infty) = 1 \). To check that \( F \) is continuous, suppose that \( D \) is a closed subset of \( \mathbb{R} \). Then \( F^{-1}(D) = f^{-1}(D) \subseteq \alpha_aX \setminus V \) when \( 1 \notin D \), while \( F^{-1}(D) = [f^{-1}(D) \cap (\alpha_aX \setminus V)] \cup \text{cl}_{\alpha_aX} V \) when \( 1 \in D \). This, together with the continuity of \( f \), implies that \( F^{-1}(D) \) is closed in \( \alpha_aX \), so \( F \in C^*(\alpha_aX) \). Of course, \( A \subseteq F^{-1}(1) \) and \( F(x) = 0 \).

(ii) \( \infty \notin A \). In this case \( A \) is a compact subset of \( X \). Working similarly to case (i), we can find a pair \( U, V \) of disjoint open subsets of \( \alpha_aX \) such that \( A \subseteq U \) and \( \{\infty, x\} \subseteq V \). Since \( X \) is completely regular, it follows from Lemma 2.10 that there exists a function \( f \in C^*(X) \) such that \( A \subseteq f^{-1}(0) \) and \( X \cap \text{cl}_{\alpha_aX} V \subseteq f^{-1}(1) \). We define \( F \in C^*(\alpha_aX) \) by putting \( F(t) = f(t) \) for each \( t \in X \) and \( F(\infty) = 1 \). Then \( A \subseteq F^{-1}(0) \) and \( F(x) = 1 \). \( \square 

Proposition 2.12 can be generalized to the following:
**Proposition 2.13. [ZF]** Every Hausdorff compactification $\alpha X$ a non-compact locally compact Tychonoff $X$ with a finite remainder $\alpha X \setminus X$ is completely regular.

**Proof.** Let $\alpha X$ be a Hausdorff compactification of $X$ such that $\alpha X \setminus X$ is finite. For $n \in \omega$, suppose that $\alpha X \setminus X$ is of cardinality $n$. Let $\alpha X \setminus X = \{y_i : i \in n\}$. Let $A$ be a closed subset of $\alpha X$ and let $y \in \alpha X \setminus A$. If $y \in X$, we know from Proposition 2.12 that there exists a function $h \in C_{\alpha a}(X)$ such that $h(y) = 0$ and $A \cap X \subseteq h^{-1}(1)$. Since $\alpha_aX \leq \alpha X$, the function $h$ is continuously extendable over $\alpha X$. If $h$ is the continuous extension of $h$ over $\alpha X$, then $\tilde{h}(y) = 0$ and $A \subseteq \tilde{h}^{-1}(1)$. 

Now, consider the case when $y \in \alpha X \setminus X$. There is a collection $\{V_i : i \in n\}$ of pairwise disjoint open sets in $\alpha X$ such that $y_i \in V_i$ for each $i \in n$. Let $K = \alpha X \setminus \bigcup_{i \in n} V_i$ and let $A_i = K \cup (A \cap V_i)$ for each $i \in n$. Consider any $i \in n$. Notice that $A_i = K \cup [A \cap (\alpha X \setminus \bigcup\{V_j : j \in n \setminus \{i\}\})]$, so $A_i$ is closed in $K \cup V_i$. Of course, $K \cup V_i = \alpha X \setminus \bigcup\{V_j : j \in n \setminus \{i\}\}$ is a one-point compactification of $K \cup (X \cap V_i)$. Therefore, if $y_i \notin A$, it follows from Proposition 2.12 that there exists a continuous function $f_i : K \cup V_i \to [0, 1]$ such that $f_i(y_i) = 0$ and $A_i \subseteq f_i^{-1}(1)$. If $i \in n$ is such that $y_i \in A$, we put $f_i(z) = 1$ for each $z \in K \cup V_i$. We define a function $f : \alpha X \to [0, 1]$ as follows: if $i \in n$ and $z \in K \cup V_i$, then $f(z) = f_i(z)$. Clearly, $A \subseteq f^{-1}(1)$ and $f(y_i) = 0$ for each $i \in n$ such that $y_i \notin A$. Let us prove that $f$ is continuous. To this aim, consider any closed in $\mathbb{R}$ set $D$. If $i \in n$, the set $f_i^{-1}(D)$ is closed in $K \cup V_i$. Since $K \cup V_i$ is closed in $\alpha X$ for each $i \in n$, we have that $f^{-1}(D)$ is closed in $\alpha X$ because $f^{-1}(D) = \bigcup\{f_i^{-1}(D) : i \in n\}$. Finally, to show that $f(y) = 0$ and $A \subseteq f^{-1}(1)$, it suffices to notice that since $y \in \alpha X \setminus X$, there exists $i \in n$ such that $y = y_i$ and, of course, $y_i \notin A$. 

**Proposition 2.14. [ZF]** Suppose that $\alpha X$ is a Hausdorff compactification of a Tychonoff space $X$ such that $\alpha X \setminus X$ is homeomorphic with the Alexandroff compactification of the discrete space $\omega$. Then $\alpha X$ is completely regular.

**Proof.** We may assume that $\alpha X \setminus X = \omega + 1$ where $\omega + 1$ is equipped with the usual order topology on ordinal numbers. Notice that $X$ is locally compact because $X$ is open in $\alpha X$. Therefore, $\alpha_aX$ is a Hausdorff compactification of $X$. Let $A$ be a closed subset of $\alpha X$ and let $y \in \alpha X \setminus A$. If $y \in X$, we know from Proposition 2.12 that there exists $f \in C(\alpha aX)$ such that $f(y) = 0$ and $(A \cap X) \cup (\alpha_aX \setminus X) \subseteq f^{-1}(1)$. If $f$ is the continuous extension of $f|_X$ over
\( \alpha X \), then \( \tilde{f}(y) = 0 \) and \( A \subseteq \tilde{f}^{-1}(1) \). Now, assume that \( y \in \alpha X \setminus X \) and consider the following cases (i) and (ii):

(i) \( y = \omega \). In this case, there exists an open in \( \alpha X \) set \( V \) such that \( A \cap \text{cl}_{\alpha X}V = \emptyset \), \( y \in V \) and the set \( (\alpha X \setminus X) \setminus V \) is finite. Let \( W \) be an open set in \( \alpha X \) such that \( A \subseteq W \) and \( (\text{cl}_{\alpha X}W) \cap (\text{cl}_{\alpha X}V) = \emptyset \). Put \( Y = X \cap \text{cl}_{\alpha X}W \) and \( \gamma Y = \text{cl}_{\alpha X}Y = \text{cl}_{\alpha X}W \). Then \( \gamma Y \) is a Hausdorff compactification of the Tychonoff space \( Y \) such that \( \gamma Y \setminus Y \) is finite. In view of Proposition 2.13, the space \( \gamma Y \) is completely regular. The sets \( \text{bd}_{\alpha X}W \) and \( A \) are disjoint and both compact in the compact Tychonoff space \( \gamma Y \). Thus, it follows from Lemma 2.10 that there exists a continuous function \( g : \gamma Y \to [0, 1] \) such that \( \text{bd}_{\alpha X}W \subseteq g^{-1}(0) \) and \( A \subseteq g^{-1}(1) \). We define a function \( \tilde{g} : \alpha X \to [0, 1] \) putting \( \tilde{g}(z) = 0 \) for each \( z \in \alpha X \setminus \text{cl}_{\alpha X}W \), while \( \tilde{g}(z) = g(z) \) for each \( z \in \gamma Y \). The function \( \tilde{g} \) is continuous on \( \alpha X \); moreover, \( \tilde{g}(y) = 0 \) and \( A \subseteq \tilde{g}^{-1}(1) \).

(ii) \( y \in \omega \). Then there exists an open neighbourhood \( W(y) \) of \( y \) in \( \alpha X \) such that \( A \cap \text{cl}_{\alpha X}W(y) = \emptyset \) and \( \text{cl}_{\alpha X}W(y) \cap (\alpha X \setminus X) = \{y\} \). Since \( \text{cl}_{\alpha X}W(y) \) is a point Hausdorff compactification of \( X \cap \text{cl}_{\alpha X}W(y) \), it follows from Proposition 2.12 that there exists a continuous function \( h : \text{cl}_{\alpha X}W(y) \to [0, 1] \) such that \( h(y) = 0 \) and \( \text{bd}_{\alpha X}W(y) \subseteq h^{-1}(1) \). We define a function \( \tilde{h} : \alpha X \to [0, 1] \) as follows: if \( z \in \text{cl}_{\alpha X}W(y) \), then \( \tilde{h}(z) = h(z) \); if \( z \in \alpha X \setminus W(y) \), then \( \tilde{h}(z) = 1 \). The function \( \tilde{h} \) is continuous, \( \tilde{h}(y) = 0 \) and \( A \subseteq \tilde{h}^{-1}(1) \).

We recall that a subspace \( P \) of a space \( X \) is called \( C^* \)-embedded in \( X \) if each function from \( C^*(P) \) is continuously extendable over \( X \) (cf. 1.13 in [7] or Definition 1.31 in [2]).

**Theorem 2.15.** [ZF] Let \( \alpha X \) be a Hausdorff compactification of a locally compact Tychonoff space \( X \) such that \( \alpha X \setminus X \) is completely regular and \( C^* \)-embedded in \( \alpha X \). Then \( \alpha X \) is completely regular.

**Proof.** Let \( A \) be a closed subset of \( \alpha X \) and let \( y \in \alpha X \setminus A \). If \( y \in X \), we know from Proposition 2.12 that there exists a function \( \psi \in C(\alpha_\alpha X) \) such that \( \psi(y) = 1 \) and \( A \cap X \cup (\alpha_\alpha X \setminus X) \subseteq \psi^{-1}(0) \). If \( f \) is the restriction of \( \psi \) to \( X \), while \( \tilde{f} \) is the continuous extension of \( f \) over \( \alpha X \), then \( \tilde{f}(y) = 1 \) and \( A \subseteq \tilde{f}^{-1}(0) \).

Now, consider the case when \( y \in \alpha X \setminus X \). Since \( \alpha X \setminus X \) is completely regular, there exists a continuous function \( g : \alpha X \setminus X \to [0, 1] \) such that \( g(y) = 1 \) and \( A \cap (\alpha X \setminus X) \subseteq g^{-1}(0) \). Since \( \alpha X \setminus X \) is \( C^* \)-embedded in \( \alpha X \), the function \( g \) has a continuous extension \( \tilde{g} : \alpha X \to [0, 1] \). Let \( B = A \cap \tilde{g}^{-1}([\frac{1}{2}, 1]) \). Then \( B \) is a compact subset of \( X \). In the light of Proposition
2.12, the Alexandroff compactification \( \alpha X \) is completely regular, so there exists a continuous function \( \phi : \alpha X \to [0, 1] \) such that \( \phi(\alpha X \setminus X) = \{1\} \) and \( B \subseteq \phi^{-1}(0) \). Since \( \alpha_a X \leq \alpha X \), the function \( \phi|_X \) has a continuous extension \( \tilde{h} : \alpha X \to [0, 1] \) defined by \( \kappa = \min\{\tilde{g}, \tilde{h}\} \). It is clear that \( \kappa(y) = 1 \). If \( z \in B \), then \( \kappa(z) = 0 \) because \( \tilde{h}(z) = 0 \). If \( z \in A \setminus B \), then \( \tilde{g}(z) \leq \frac{1}{2} \), so \( \kappa(z) \leq \frac{1}{2} \). This implies that \( A \subseteq \kappa^{-1}([0, \frac{1}{2}]) \). Let \( C = \kappa^{-1}(1) \) and \( D = \kappa^{-1}([0, \frac{1}{2}]) \). Then \( C, D \) are disjoint zero-sets in \( \alpha X \) such that \( y \in C \) and \( A \subseteq D \). This proves that \( \alpha X \) is completely regular because disjoint zero-sets are functionally separated. \( \square \)

Remark 2.16. In view of Remark 2.3 and Lemma 2.10, it holds true in every model of \( ZF + DC \) that if \( X \) is a compact Tychonoff space, then \( TET(X) \) is satisfied. It was shown in Section 3 of [10] that there is a model of \( ZF \) in which a compact Tychonoff space \( X \) need not satisfy \( TET(X) \).

**Proposition 2.17.** [ZF] Suppose that \( \alpha X \) is a Hausdorff compactification of a Tychonoff space \( X \) such that \( \alpha X \setminus X \) is finite. Then \( \alpha X \setminus X \) is \( C^* \)-embedded in \( \alpha X \).

**Proof.** We may assume that \( X \) is non-compact. Let \( n \in \omega \) be equipotent with \( \alpha X \setminus X \). We fix a function \( f : \alpha X \setminus X \to \mathbb{R} \) and put \( D = f(\alpha X \setminus X) \). Assume that \( \alpha X \setminus X = \{y_i : i \in n\} \). There is a collection \( \{V_i : i \in n\} \) of pairwise disjoint open sets in \( \alpha X \) such that \( y_i \in V_i \) for each \( i \in n \). For \( d \in D \), let \( N(d) = \{i \in n : f(y_i) = d\} \) and \( A(d) = \alpha X \setminus \bigcup\{V_i : i \in N(d)\} \).

In the light of Proposition 2.13, the space \( \alpha X \) is Tychonoff. Thus, for each \( d \in D \), there exists a continuous function \( g_d : \alpha X \to [d, 1 + \max D] \) such that \( f^{-1}(d) \subseteq g_d^{-1}(d) \) and \( A(d) \subseteq g_d^{-1}(1 + \max D) \). Let \( g(t) = \min\{g_d(t) : d \in D\} \) for each \( t \in \alpha X \). Then \( g \in C(\alpha X) \) and \( g(t) = f(t) \) for each \( t \in \alpha X \setminus X \). \( \square \)

Of course, one can also deduce from Proposition 2.13 and the \( ZFC \)-proof to Tietze-Urysohn extension theorem that Proposition 2.17 holds in \( ZF \); however, we prefer a simpler, direct \( ZF \)-proof to it. Among other facts, we are going to show that it may happen in a model of \( ZF \) that, for a Hausdorff compactification \( \alpha X \) of a locally compact Tychonoff space such that \( \alpha X \setminus X \) is homeomorphic with \( \omega + 1 \), the remainder \( \alpha X \setminus X \) can fail to be \( C^* \)-embedded in \( \alpha X \). It might be interesting to know the place of the following sentences \( C^*\mathbb{R} \) and \( C^*\mathbb{R}[\omega] \) in the hierarchy of choice principles:

- **\( C^*\mathbb{R} \):** For every locally compact Tychonoff space \( X \) and for every Tychonoff compactification \( \alpha X \) of \( X \), the remainder \( \alpha X \setminus X \) is \( C^* \)-embedded in \( \alpha X \).
$C^*\mathbb{R}[\omega]$: For every locally compact Tychonoff space $X$ and for every Hausdorff compactification $\alpha X$ of $X$ such that $\alpha X \setminus X$ is homeomorphic with $\omega + 1$, the remainder $\alpha X \setminus X$ is $C^*$-embedded in $\alpha X$.

As usual, we denote by CMC the axiom of countable multiple choice which states that for each sequence $(X_n)_{n \in \omega}$ of non-empty sets there exists a sequence $(F_n)_{n \in \omega}$ of non-empty finite subsets $F_n$ of $X_n$ (see Form 126 in [15] and Definition 2.10 in [12]). The axiom of countable choice (Form 8 in [15]), denoted by CC in Definition 2.5 of [12] and by CAC in many articles (see, for instance, [13] and [16]), states that every non-empty countable collection of non-empty sets has a choice function. Let us recall the following van Douwen’s choice principle (Form 119 in [15], as well as CC(Z) on page 79 in [12]) which was introduced in [4] and denoted by $vDCP(\omega)$ in [16]:

$vDCP(\omega)$: For every family $\{\langle A_i, \leq_i \rangle : i \in \omega\}$ such that each $\langle A_i, \leq_i \rangle$ is a linearly ordered set isomorphic with the set $\mathbb{Z}$ of integers equipped with the standard order, the family $\{A_i; i \in \omega\}$ has a choice function.

It was shown in [16] that $vDCP(\omega)$ is strictly weaker than CMC. For significant applications of models in which $vDCP(\omega)$ fails, the following construction was used, for instance, in [4], [16], [27] and in Section 4.7 of [12]:

Let $A = \{\langle A_i, \leq_i \rangle : i \in \omega\}$ be a collection of linearly ordered sets $\langle A_i, \leq_i \rangle$ isomorphic with the set $\mathbb{Z}$ of integers equipped with the standard order. Let $A = \bigcup\{A_i : i \in \omega\}$. Fix sets $\bar{A} = \{a_i : i \in \omega\}$ and $\bar{B} = \{b_i : i \in \omega\}$ of pairwise distinct elements such that $A \cap (\bar{A} \cup \bar{B}) = \emptyset = A \cap \bar{B}$. Put $X_i = A_i \cup \{a_i, b_i\}$ and extend the order $\leq_i$ to a linear order $\leq_i$ on $X_i$ by requiring that $a_i$ is the smallest element of $\langle X_i, \leq_i \rangle$, while $b_i$ is the largest element of $\langle X_i, \leq_i \rangle$. For simplicity, without any loss of generality, we may assume that $X_i \cap X_j = \emptyset$ for each pair $i, j$ of distinct elements of $\omega$. We equip each $X_i$ with the order topology induced by $\leq_i$. We denote by $X[A]$ the disjoint union (the sum) of the linearly ordered topological spaces $X_i$ where $i \in \omega$. The spaces $X_i$ are all metrizable, so Tychonoff. Clearly, the space $X[A]$ is locally compact. It is easy to prove in ZF that sums of completely regular spaces are completely regular. Hence $X[A]$ is also completely regular in ZF.

**Theorem 2.18.** The following implications are true in every model of ZF:

(i) $C^*\mathbb{R}[\omega]$ implies $vDCP(\omega)$;

(ii) the conjunction of UFT and $C^*\mathbb{R}[\omega]$ implies CMC;

(iii) CC implies $C^*\mathbb{R}[\omega]$. 

13
Proof. (i) Let us fix a family \( \mathcal{A} = \{(A_i, \leq_i) : i \in \omega \} \) as in the definition of \( \nu\text{DCP}(\omega) \), as well as sets \( \tilde{A} = \{a_i : i \in \omega \}, \tilde{B} = \{b_i : i \in \omega \} \) as in the construction of \( X[\mathcal{A}] \) described above. We consider the Alexandroff compactification \( \alpha \tilde{X}[\mathcal{A}] = X[\mathcal{A}] \cup \{\infty\} \) of \( X[\mathcal{A}] \), the set \( K = \tilde{A} \cup \{\infty\} \cup \tilde{B} \) and the subspace \( X = (\alpha \tilde{X}[\mathcal{A}]) \setminus K \) of \( \alpha \tilde{X}[\mathcal{A}] \). Of course, the space \( X \) is discrete and dense in \( \alpha \tilde{X}[\mathcal{A}] \). In view of Proposition 2.12, the compactification \( \alpha X = \alpha \tilde{X}[\mathcal{A}] \) of \( X \) is Tychonoff. We define a continuous function \( g : K \to [-1,1] \) by putting \( g(\infty) = 0 \), while \( g(a_i) = \frac{1}{i+1} \) and \( g(b_i) = \frac{1}{i+1} \) for each \( i \in \omega \). Suppose that \( K \) is \( C^* \)-embedded in \( \alpha X \). Then \( g \) has a continuous extension to a function \( \tilde{g} : \alpha X \to [-1,1] \). In much the same way, as in Section 3 of [16], for each \( i \in \omega \), we can define \( t(i) = \max \{x \in A_i : \tilde{g}(x) < 0\} \) to obtain a choice function \( t \in \prod_{i \in \omega} A_i \). This, together with the fact that \( K \) is homeomorphic with the Alexandroff compactification \( \omega + 1 \) of \( \omega \), implies that (i) holds.

(ii) Now, let us suppose that \text{CMC} is false. In this case, notice that it was shown in the proof to Theorem 3 of [16] that there exists a Tychonoff space \( Z \) such that, for a compact subset \( C \) of \( Z \), the set \( Z \setminus C \) is dense in \( Z \), while \( C \) is not \( C^* \)-embedded in \( Z \) and \( C \) is homeomorphic with \( \omega + 1 \). Assume that \text{UFT} is satisfied. It follows from Proposition 1.2 that the space \( Z \) has a Tychonoff compactification \( \gamma Z \). The subspace \( Z_0 = (\gamma Z) \setminus C \) of \( \gamma Z \) is a locally compact space such that the remainder \( C = (\gamma Z) \setminus Z_0 \) is not \( C^* \)-embedded in the Tychonoff compactification \( \gamma Z \) of \( Z_0 \).

(iii) Assume that \text{CC} holds. Let \( \alpha X \) be a Hausdorff compactification of a Tychonoff space \( X \) such that \( \alpha X \setminus X = \omega + 1 \). Denote by \( \mathcal{F} \) the collection of all non-empty finite subsets of \( \omega \). Then \( \mathcal{F} \) is countable. Let \( \mathcal{F} = \{F_n : n \in \omega\} \). For each \( n \in \omega \), let \( \mathcal{G}_n \) be the collection of all functions \( g \in C(\alpha X) \) such that \( (\omega + 1) \setminus F_n \subseteq g^{-1}(0) \) and \( F_n \subseteq g^{-1}(1) \). The collections \( \mathcal{G}_n \) are all non-empty because, by Proposition 2.14, the space \( \alpha X \) is Tychonoff. Since \text{CC} holds, the collection \( \{\mathcal{G}_n : n \in \omega\} \) has a choice function. Let \( G \in \prod_{n \in \omega} \mathcal{G}_n \). Let \( A, B \) be a pair of non-empty disjoint closed subsets of \( \omega + 1 \). Then \( A \in \mathcal{F} \) or \( B \in \mathcal{F} \). Suppose that \( n_0 \in \omega \) is such that \( A = F_{n_0} \). The function \( g = G(n_0) \) is such that \( B \subseteq g^{-1}(0) \) and \( A \subseteq g^{-1}(1) \). With this observation in hand, if \( f \in C(\omega + 1) \), we can slightly modify the well-known standard \text{ZFC}-proof of Tietze-Urysohn Extension Theorem to find in \text{ZF+CC} a continuous extension of \( f \) over \( \alpha X \).

Corollary 2.19. The following sentences are relatively consistent with \text{ZF}:

(i) There exists a Tychonoff compactification \( \gamma X \) of a locally compact space
such that TET(γY) fails but γY \ Y is C*-embedded in γY.

(ii) C*R[ω] is false.

Proof. Let us consider the space X = αX[A] \ K and its compactification αX = αX[A] used in the proof to (i) of Theorem 2.18. Let c ∈ αX be an accumulation point of X, let Y = αX \ {c} and γY = αX. It was shown in the proof to Theorem 2.18 (i) that if A is such that \{A_i : i ∈ ω\} does not have a choice function, then αX is not a T space. To complete the proof, it suffices to use Theorem 2.18 together with the fact that there is a model of ZF in which vDCP fails.

We are going to prove that it is relatively consistent with ZF that there exists a Tychonoff space which has a strange Hausdorff compactification. We shall deduce several surprising consequences of the existence of strange compactifications in some models of ZF.

**Theorem 2.20.** [ZF] Let Y be a given non-empty compact Hausdorff space. Then there exist a discrete space D_Y and a Hausdorff compactification γD_Y of D_Y such that (γD_Y) \ D_Y is homeomorphic with Y.

Proof. Let D_Y = (ω × Y) \ ({0} × Y) be considered with its discrete topology. Let γD_Y = ω × Y be equipped with the following topology:

(i) all points of (γD_Y) \ ({0} × Y) are isolated;

(ii) if y ∈ Y, then a base of neighbourhoods of the point ⟨0, y⟩ in γD_Y consists of all sets of the form: (ω × U) \ K where K is a finite subset of D_Y, while U is open in Y and y ∈ U.

Obviously, the topological space γD_Y is a compact Hausdorff space such that D_Y is a dense subspace of γD_Y, while the space Y is homeomorphic with the remainder (γD_Y) \ D_Y.

**Theorem 2.21.** The following sentences are relatively consistent with ZF:

(i) There exists a discrete space which has a strange compactification.

(ii) There exists a Tychonoff space X which has non-equivalent Hausdorff compactifications αX and γX such that C_α(X) = C_γ(X).
Proof. Let $\mathcal{M}$ be a model of $\text{ZF}$ such there exists in $\mathcal{M}$ an uncountable Hausdorff space $Y$ which is compact and such that all continuous real functions on $Y$ are constant (see [9], as well as Form 78 in models $\mathcal{N}3$ and $\mathcal{N}8$ in [15]). In view of Theorem 2.20, there exist in $\mathcal{M}$ a discrete space $D_Y$ and a Hausdorff compactification $\gamma D_Y$ of $D_Y$ such that $(\gamma D_Y) \setminus D_Y$ is homeomorphic with $Y$. Put $X = D_Y$ and, for simplicity, assume that $\gamma X \setminus X = Y$. Then it holds true in $\mathcal{M}$ that $\gamma X$ is a strange compactification of $X$. Let $\alpha X$ be the one-point compactification in $\mathcal{M}$ of $X$. Of course, $\gamma X$ and $\alpha X$ are non-equivalent, while $C_\gamma(X) = C_\alpha(X)$.

Corollary 2.22. It is not a theorem of $\text{ZF}$ that if $\alpha X$ and $\gamma X$ are Hausdorff compactifications of a Tychonoff space $X$ such that $C_\alpha(X) \subseteq C_\gamma(X)$, then $\alpha X \leq \gamma X$.

One can easily prove the following:

**Theorem 2.23.** [ZF] If Hausdorff compactifications $\alpha X$ and $\gamma X$ of a topological space $X$ are both completely regular, then $\alpha X \leq \gamma X$ if and only if $C_\alpha(X) \subseteq C_\gamma(X)$.

**Problem 2.24.** Find in $\text{ZF}$ reasonable necessary and sufficient internal conditions for a Tychonoff space to have no strange Hausdorff compactification.

We recall that an amorphous set is an infinite set $X$ such that if $A$ is an infinite subset of $X$, then the set $X \setminus A$ is finite (see E.11 in Section 4.1 of [12], Form 64 and Note 57 in [15]). Amorphous sets exist, for instance, in $\text{ZF}$-model $\mathcal{M}37$ of [15] (see also model $\mathcal{N}1$ of [15] together with entries (361,64) and (363, 64) on page 335 in [15]).

**Definition 2.25.** A topological space $\langle X, \tau \rangle$ will be called amorphous if $X$ is an amorphous set.

**Proposition 2.26.** [ZF] Every amorphous Hausdorff space is either discrete or a one-point Hausdorff compactification of an amorphous discrete space.

**Proof.** Let $X$ be an amorphous Hausdorff space. Suppose that $X$ is not discrete. Then $X$ has exactly one accumulation point. Let $x_0$ be the unique accumulation point of $X$ and let $Y = X \setminus \{x_0\}$. Then $Y$ is a discrete amorphous space such that $X$ is a one-point Hausdorff compactification of $Y$. 

The following theorem, together with Theorem 2.21, points out that a satisfactory solution to Problem 2.24 can be complicated even for discrete spaces:
Theorem 2.27. [ZF] Let $X$ be an infinite discrete space. Then every Hausdorff compactification of $X$ is equivalent with the Alexandroff compactification of $X$ if and only if $X$ is amorphous.

Proof. Let $\alpha X$ be a Hausdorff compactification of $X$. If $\alpha X \setminus X$ is not a singleton, then there is a pair $x, y$ of distinct points of $\alpha X \setminus X$, so there exists a pair $U, V$ of disjoint open sets in $\alpha X$ such that $x \in U$ and $y \in V$. Then the sets $U \cap X$ and $V \cap X$ are disjoint infinite subsets of $X$, hence $X$ cannot be amorphous. On the other hand, if $X$ is not amorphous, then there are disjoint infinite subsets $Y, Z$ of $X$ such that $X = Y \cup Z$, which implies that $X$ has a two-point Hausdorff compactification. □

Corollary 2.28. [ZF] If a discrete space $X$ is amorphous, then its Čech-Stone compactification is the Alexandroff compactification of $X$.

Corollary 2.29. [ZF] The following conditions are equivalent:

(i) there do not exist amorphous sets;

(ii) every infinite discrete space has a Hausdorff compactification whose remainder is not a singleton.

Theorem 2.30. [ZF] Let $n \in \omega \setminus \{0\}$ and let $X$ be an infinite discrete space. Then $X$ has an $n$-point Hausdorff compactification and does not have any $(n + 1)$-point Hausdorff compactification if and only if $X$ is a disjoint union of $n$ amorphous sets.

Proof. Necessity. First, assume that $X$ has an $n$-point compactification. Then there exists a collection $\{V_i : i \in n\}$ of infinite pairwise disjoint subsets of $X$ such that the set $X \setminus \bigcup_{i \in n} V_i$ is compact. Suppose that there exists $i_0 \in n$ such that the set $V_{i_0}$ is not amorphous. Then there exists an infinite subset $U$ of $V_{i_0}$ such that the set $V_{i_0} \setminus U$ is also infinite. Since Theorem 6.8 of [2] (cf. Theorem 2.1 of [21]) is provable in ZF, we can apply it to showing that $X$ has an $(n + 1)$-point Hausdorff compactification. Therefore, if $X$ does not have any $(n + 1)$-point Hausdorff compactification, the sets $V_i$ must be amorphous for each $i \in n$.

Sufficiency. Now, assume that $\{X_i : i \in n\}$ is a collection of pairwise disjoint amorphous subsets of $X$ such that $X = \bigcup_{i \in n} X_i$. Then $X$ has an $n$-point Hausdorff compactification $\alpha X$ such that, for each pair $i, j \in n$, the sets $X_i$ and $X_j$ have disjoint closures in $\alpha X$ whenever $i \neq j$. Let $\gamma X$
be an arbitrary Hausdorff compactification of $X$. Put $Y = \alpha X \times \gamma X$ and define a mapping $r : X \to Y$ by: $r(x) = \langle \alpha(x), \gamma(x) \rangle$ for each $x \in X$. Then $\langle rX, r \rangle$, where $rX = \text{cl}_Y r(X)$, is a Hausdorff compactification of $X$ such that $\alpha X \subseteq rX$ and $\gamma X \subseteq rX$. Let $h : rX \to \alpha X$ be such that $h \circ r = \alpha$. Suppose that there exists $z \in \alpha X \setminus X$ such that $h^{-1}(\{z\})$ is not a singleton. Then there exists a collection $\{W_j : j \in n + 1\}$ of pairwise disjoint infinite subsets of $X$ such that, for each $j \in n + 1$, there exists $i \in n$ such that $W_j \subseteq X_i$. There must exist $i_1 \in n$ and a pair $j, k$ of distinct numbers from $n + 1$ such that $W_j \subseteq X_{i_1}$ and $W_k \subseteq X_{i_1}$. This is impossible because $X_{i_1}$ is amorphous. The contradiction obtained shows that the compactifications $\alpha X$ and $rX$ are equivalent. Since $\gamma X \leq rX$ and $rX \setminus r(X)$ consists of exactly $n$ points, we have that $\gamma X \setminus \gamma(X)$ consists of at most $n$ points. This completes the proof.

**Theorem 2.31. [ZF]** Let $n \in \omega \setminus \{0\}$ and let $X$ be a discrete space such that $X$ is a union of $n$ pairwise disjoint amorphous sets. Then the Čech-Stone compactification $\beta X$ of $X$ is the unique (up to the equivalence) $n$-point Hausdorff compactification of $X$.

**Proof.** By Theorem 2.30, $X$ has an $n$-point Hausdorff compactification $\alpha X$. We have shown in the proof to Theorem 2.30 that if $\gamma X$ is a Hausdorff compactification of $X$, then $\gamma X \leq \alpha X$. This implies that $\beta X \approx \alpha X$. Lemma 6.12 of [2] is provable in ZF and we infer from it and from Theorem 2.30 that all $n$-point Hausdorff compactifications of $X$ are equivalent.

One of the most important theorems on products of Čech-Stone compactifications is Glicksberg’s theorem of ZFC which asserts that, for infinite Tychonoff spaces $X$ and $Y$, the Cartesian product $\beta X \times \beta Y$ is the Čech-Stone compactification of $X \times Y$ if and only if $X \times Y$ is pseudocompact (see [8] and Problem 3.12.21 (c) of [5]). We are going to prove that Glicksberg’s theorem fails in every model of ZF in which there is an amorphous set. A well-known fact of ZFC is that if $X$ is a compact Hausdorff space, while $Y$ is a pseudocompact Tychonoff space, then the product $X \times Y$ is pseudocompact (see Corollary 3.10.27 of [5]). Unfortunately, the proof to it in [5] is not a proof in ZF. This is why we show a proof in ZF to the following helpful lemma:

**Lemma 2.32. [ZF]** Suppose that $X, Y$ are non-empty topological spaces such that $X$ is compact and $Y$ is pseudocompact. Then $X \times Y$ is pseudocompact.
Proof. Let $g : X \times Y \to \mathbb{R}$ be a continuous function. Put $f = |g|$ and define a function $F : Y \to \mathbb{R}$ by:

$$F(y) = \sup\{f(x, y) : x \in X\}.$$

To check that $F$ is continuous, consider any point $y_0 \in Y$ and real numbers $a, b$ such that $a < F(y_0) < b$. Let $\mathcal{U}$ be a collection of all non-empty open sets in $X$ such that if $U \in \mathcal{U}$, then there exists an open in $Y$ set $G$ such that $y_0 \in G$ and $f(U \times G) \subseteq (-\infty, b)$. It follows from the continuity of $f$ that $U$ is an open cover of $X$. Since $X$ is compact, there exists a finite subcover $\mathcal{U}_0$ of $\mathcal{U}$. For each $U \in \mathcal{U}_0$, we can choose an open neighbourhood $G(U)$ of $y_0$ such that $f(U \times G(U)) \subseteq (-\infty, b)$. Since $a < F(y_0)$, there exists $x_0 \in X$ such that $a < f(x_0, y_0)$. By the continuity of $f$, there exists an open neighbourhood $V_0$ of $y_0$ such that $f(U \times V_0) \subseteq (a, +\infty)$. Let $V = V_0 \cap \bigcap\{G(U) : U \in \mathcal{U}_0\}$. There exists $U_0 \in \mathcal{U}_0$ such that $x_0 \in U_0$. If $y \in V$, then $a < f(x_0, y) \leq F(y) < b$; hence, $F$ is continuous at $y_0$. Since $Y$ is pseudocompact, the function $F$ is bounded. This implies that $f$ is bounded, so $g$ is also bounded.

Theorem 2.33. $[$ZF$]$ For every amorphous discrete space $X$, the spaces $X$ and $\beta X \times X$ are both pseudocompact, while $\beta X \times \beta X$ is not the Čech-Stone compactification of $\beta X \times X$.

Proof. Let $X$ be an amorphous discrete space. Consider any $f : X \to \mathbb{R}$. Then $f(X)$ is either finite or amorphous. Since there do not exist amorphous linearly ordered sets, the set $f(X)$ is finite. This implies that $X$ is pseudocompact. By Corollary 2.28, $\beta X$ is a one-point Hausdorff compactification of $X$. In view of Lemma 2.32, the space $\beta X \times X$ is pseudocompact. Now, let $A = \{(x, y) \in \beta X \times X : x = y\}$. Then $A$ is clopen in $\beta X \times X$. Let $f : \beta X \times X \to \{0, 1\}$ be defined by $f(z) = 0$ if $z \in A$, while $f(z) = 1$ if $z \in (\beta X \times X) \setminus A$. The function $f$ is continuous but it is not continuously extendable over $\beta X \times \beta X$ since the sets $f^{-1}(0)$ and $f^{-1}(1)$ do not have disjoint closures in $\beta X \times \beta X$. □

Corollary 2.34. The following statement is independent of $\text{ZF}$: there exist a compact Tychonoff space $K$ and a pseudocompact Tychonoff space $X$ such that $X$ has its Čech-Stone compactification, while $K \times \beta X$ is not the Čech-Stone compactification of $K \times X$.  

19
3 Compactifications generated by sets of functions

As we have already informed in section 1, it holds true in every model of \(\text{ZF} + \text{UFT}\) that if \(X\) is a Tychonoff space and \(F \in \mathcal{E}(X)\), then the space \(e_F X\) is compact; however, \(e_F X\) can fail to be compact in a model of \(\text{ZF} + \neg\text{UFT}\). Therefore, it might be useful to find necessary and sufficient conditions for \(F \in \mathcal{E}(X)\) to have the property that the space \(e_F X\) is compact in \(\text{ZF}\).

For a compactification \(\alpha X\) of \(X\) and for a function \(f \in C_{\alpha}(X)\), the unique continuous extension of \(f\) over \(\alpha X\) is usually denoted by \(f_{\alpha}\). For \(F \subseteq C_{\alpha}(X)\), we put \(F_{\alpha} = \{f_{\alpha} : f \in F\}\).

**Theorem 3.1.** [\(\text{ZF}\)] Suppose that \(X\) is a Tychonoff space and that \(F \in \mathcal{E}(X)\). Then the following conditions are equivalent:

(i) \(e_F X\) is compact;

(ii) there exists a (not necessarily Hausdorff) compactification \(\alpha X\) of \(X\) such that \(F \subseteq C_{\alpha}(X)\).

**Proof.** If \(e_F X\) is compact, then \(F \subseteq C_{e_F}(X)\), so (i) implies (ii). Assume that (ii) holds. Let \(\alpha X\) be a compactification of \(X\) such that \(F \subseteq C_{\alpha}(X)\). We define a mapping \(h : \alpha X \to \mathbb{R}^F\) by \(h(t)(f) = f_{\alpha}(t)\) for all \(t \in \alpha X\) and \(f \in F\). Since \(e_F(X) \subseteq h(\alpha X)\), it follows from the compactness of \(\alpha X\) that \(e_F X \subseteq h(\alpha X)\). We shall show that \(e_F X = h(\alpha X)\). To this aim, suppose that \(y \in h(\alpha X) \setminus e_F X\). Let \(t \in \alpha X\) be such that \(h(t) = y\). There exist a non-empty finite set \(K \subseteq F\) and a positive real number \(\varepsilon\), such that if \(V_f = \mathbb{R}\) for \(f \in F \setminus K\), while \(V_f = (y(f) - \varepsilon, y(f) + \varepsilon)\) for \(f \in K\), then \(e_F(X) \cap \prod_{f \in F} V_f = \emptyset\). Then \(t \in \bigcap_{f \in K} (f_{\alpha})^{-1}(V_f)\) and, by the density of \(X\) in \(\alpha X\), there exists \(z \in X \cap \bigcap_{f \in K} (f_{\alpha})^{-1}(V_f)\). Then \(h(z) \in e_F(X) \cap \prod_{f \in F} V_f\) which is impossible. The contradiction obtained implies that \(e_F X = h(\alpha X)\).

In consequence, \(e_F X\) is compact. Hence (ii) implies (i).

**Definition 3.2.** A Hausdorff completely regular compactification \(\gamma X\) of a Tychonoff space \(X\) will be called a functional Čech-Stone compactification of \(X\) if \(C_\gamma(X) = C^*(X)\).

The following proposition is an immediate consequence of Theorem 2.23:
Proposition 3.3. [ZF] If $\gamma_1 X$ and $\gamma_2 X$ are functional Čech-Stone compactifications of $X$, then $\gamma_1 X$ and $\gamma_2 X$ are equivalent.

Remark 3.4. Suppose that a Tychonoff space $X$ has a functional Čech-Stone compactification. Since all functional Čech-Stone compactifications of $X$ are equivalent, let us denote by $\beta^f X$ an arbitrary functional Čech-Stone compactification of $X$.

Proposition 3.5. [ZF] Let $X$ be a Tychonoff space and let $F = C^*(X)$. Then $X$ has its functional Čech-Stone compactification if and only if the space $e_F X$ is compact. Moreover, if $\beta^f X$ exists, then $\beta^f X \approx e_F X$.

Proof. Suppose that $\beta^f X$ exists. It follows from Theorem 3.1 that the space $e_F X$ is compact. If $e_F X$ is compact, then since $C(e_F X) = C^*(X)$, we have $\beta^f X \approx e_F X$ by Theorem 2.23.

It was observed in [22] that Taimanov’s Theorem 3.2.1 of [5] is valid in ZF (see Theorem 5.15 of [22]). By using Taimanov’s theorem and Lemma 2.10, one can easily prove the following proposition:

Proposition 3.6. [ZF] Suppose that a Tychonoff space $X$ has its functional Čech-Stone compactification. If $K$ is a compact Tychonoff space, then every continuous mapping $f : X \to K$ is continuously extendable over $\beta^f X$.

Corollary 3.7. [ZF] Let $X$ be a Tychonoff space such that $\beta^f X$ exists. Then, for every completely regular Hausdorff compactification $\alpha X$ of $X$, we have $\alpha X \leq \beta^f X$.

Theorem 3.8. [ZF] Let $X$ be a non-empty Tychonoff space such that $\beta X$ exists. Then there exists $\beta^f X$ and $\beta^f X \leq \beta X$.

Proof. If $\beta X$ exists, it follows from Theorem 3.1 that $\beta^f X$ exists, too. In view of Remark 1.4, $\beta^f X \leq \beta X$.

From Theorem 3.8 and Corollary 3.7, we deduce the following:

Proposition 3.9. [ZF] Suppose that $X$ is a Tychonoff space such that $\beta X$ exists. If $\beta X$ is completely regular, then $\beta X \approx \beta^f X$.

Theorem 3.10. [ZF] For every non-empty Tychonoff space $X$, the following conditions are equivalent:
(i) the Wallman space \(W(X, \mathcal{Z}(X))\) is compact;

(ii) there exists a compactification \(\alpha X\) of \(X\) such that \(C_{\alpha}(X) = C^*(X)\);

(iii) \(e_F X\) is compact where \(F = C^*(X)\).

Proof. It is obvious that (i) implies (ii) and (iii) implies (ii). That (ii) implies (iii) follows from Theorem 3.1. To show that (ii) implies (i), let us assume (iii) and put \(h = e_F\) where \(F = C^*(X)\). Now, consider any filter \(A\) in \(\mathcal{Z}(X)\).

By the compactness of \(e_F X\), there exists \(p \in \bigcap_{A \in \mathcal{A}} \text{cl}_{e_F X}[h(A)]\). We define

\[
\mathcal{F} = \{f^{-1}(0) : p \in \text{cl}_{e_F X}h(f^{-1}(0)) : f \in F\}.
\]

Then \(A \subseteq \mathcal{F}\). We shall prove that \(\mathcal{F}\) is a filter in \(\mathcal{Z}(X)\).

Let \(Z_1, Z_2 \in \mathcal{F}\). To show that \(Z_1 \cap Z_2 \in \mathcal{F}\), suppose that \(p \notin \text{cl}_{e_F X}[h(Z_1 \cap Z_2)]\). By the complete regularity of \(\mathbb{R}^F\), there exists \(\psi \in C^*(\mathbb{R}^F)\) such that \(\psi(p) = 0\) and \(h(Z_1 \cap Z_2) \subseteq \psi^{-1}(1)\). Let \(A_1 = Z_1 \cap h^{-1}(\psi^{-1}((-\infty, \frac{1}{2}]))\) and \(A_2 = Z_2 \cap h^{-1}(\psi^{-1}((-\infty, \frac{1}{2}]))\). Then \(A_1, A_2 \in \mathcal{Z}(X)\) and \(A_1 \cap A_2 = \emptyset\).

There exists \(g \in C^*(X)\) such that \(A_1 \subseteq g^{-1}(0)\) and \(A_2 \subseteq g^{-1}(1)\). For the projection \(\pi_g : \mathbb{R}^F \to \mathbb{R}\), we have \(h(A_1) \subseteq \pi_g^{-1}(0)\) and \(h(A_2) \subseteq \pi_g^{-1}(1)\), so \(\text{cl}_{e_F X}[h(A_1)] \cap \text{cl}_{e_F X}[h(A_2)] = \emptyset\). This contradicts the fact that \(p \in \text{cl}_{e_F X}[h(A_1)] \cap \text{cl}_{e_F X}[h(A_2)]\). Hence \(p \in \text{cl}_{e_F X}[h(Z_1 \cap Z_2)]\). This implies that \(Z_1 \cap Z_2 \in \mathcal{F}\), so \(\mathcal{F}\) is a filter in \(\mathcal{Z}(X)\). To check that the filter \(\mathcal{F}\) is maximal in \(\mathcal{Z}(X)\), suppose that \(\mathcal{H}\) is a filter in \(\mathcal{Z}(X)\) such that \(\mathcal{F} \subseteq \mathcal{H}\). Suppose that \(Z \in \mathcal{H}\) and \(Z \notin \mathcal{F}\). Then \(p \notin \text{cl}_{e_F X}h(Z)\). By the complete regularity of \(\mathbb{R}^F\), there exists \(A \in \mathcal{Z}(X)\) such that \(A \cap Z = \emptyset\) and \(p \in \text{cl}_{e_F X}(h(A))\). Then \(A \in \mathcal{F}\), so \(A \in \mathcal{H}\). This is impossible because \(Z \in \mathcal{H}\), while \(Z \cap A = \emptyset\).

Therefore, \(\mathcal{F}\) is an ultrafilter in \(\mathcal{Z}(X)\). Since every filter in \(\mathcal{Z}(X)\) is contained in an ultrafilter in \(\mathcal{Z}(X)\), the Wallman space \(W(X, \mathcal{Z}(X))\) is compact. Hence (ii) implies (i).

\[\square\]

**Corollary 3.11. [ZF]** Let \(X\) be a non-empty Tychonoff space which has its functional Čech-Stone compactification. Then \(W(X, \mathcal{Z}(X))\) is a Hausdorff compactification of \(X\) equivalent with \(\beta^f X\).

**Proof.** It follows from Theorem 3.10 that \(W(X, \mathcal{Z}(X))\) is compact. Since, for every pair \(A, B\) of disjoint sets from \(\mathcal{Z}(X)\), the closures of \(A\) and \(B\) in \(\beta^f X\) are also disjoint, it follows from Theorem 5.15 of [22] that the mapping \(h_{\mathcal{Z}(X)} : X \to W(X, \mathcal{Z}(X))\) is continuously extendable over \(\beta^f X\), hence \(W(X, \mathcal{Z}(X)) \leq \beta^f X\). On the other hand, if \(A, B \in \mathcal{Z}(X)\) are disjoint, then
the closures of $A$ and $B$ in $\mathcal{W}(X, Z(X))$ are disjoint; therefore, in view of Theorem 5.15 of [22], the mapping $\text{id}_X : X \to \beta^f X$ has a continuous extension over $\mathcal{W}(X, Z(X))$. This gives that $\beta^f X \leq \mathcal{W}(X, Z(X))$.

**Remark 3.12.** For a compactification $\alpha X$ of $X$, let $Z_{\alpha}(X) = \{f^{-1}(0) : f \in C_{\alpha}(X)\}$. It may happen in a model of ZF that, for a completely regular space $X$, there exists a Hausdorff compactification $\alpha X$ of $X$ such that $Z(X) = Z_{\alpha}(X)$, while the Wallman space $\mathcal{W}(X, Z(X))$ is not compact. To prove this, let us notice that Form 70 of [15] is equivalent to the statement: There are no free ultrafilters in the power set $\mathcal{P}(\omega)$. In the model $\mathcal{M}2$ of [15], Form 70 of [15] is false. This implies that the Wallman space $\mathcal{W}(\omega, \mathcal{P}(\omega))$ is not compact in $\mathcal{M}2$. However, for the Alexandroff compactification $\alpha_{a\omega}$ of the discrete space $\omega$ in $\mathcal{M}2$, we have that $Z(\omega) = Z_{\alpha_{a\omega}}(\omega)$.

For a topological space $X$, let us denote by $\text{Cl}(X)$ the collection of all closed sets of $X$.

**Theorem 3.13.** [ZF] Suppose that $X$ is a $T_1$-space which satisfies $\text{UL}(X)$. Then the following conditions are equivalent:

(i) the Wallman space $\mathcal{W}(X, \text{Cl}(X))$ is compact;

(ii) the Čech-Stone compactification of $X$ exists;

(iii) the functional Čech-Stone compactification of $X$ exists.

**Proof.** It is obvious that if (i) holds, then the compactification $\mathcal{W}(X, \text{Cl}(X))$ is the Čech-Stone compactification of $X$, so (i) implies (ii). In view of Theorem 3.8, (ii) implies (iii).

Assume that $\beta^f X$ exists. Let $K$ be a compact Hausdorff space and let $h : X \to K$ be a continuous mapping. Consider any pair $A, B$ of disjoint closed sets of $K$. Since $\text{UL}(X)$ holds, the sets $h^{-1}(A)$ and $h^{-1}(B)$ are functionally separated in $X$. In consequence, the sets $h^{-1}(A)$ and $h^{-1}(B)$ have disjoint closures in $\beta^f X$. By Theorem 5.15 of [22], the function $h$ is continuously extendable over $\beta^f X$. This proves that $\beta^f X$ is the Čech-Stone compactification of $X$. Therefore, (ii) and (iii) are equivalent.

Assume (ii). Let $A$ be a filter in $\text{Cl}(X)$. There exists $p \in \bigcap_{A \in A} \text{cl}_{\beta^f X} A$. We define

$$\mathcal{F} = \{A \in \text{Cl}(X) : p \in \text{cl}_{\beta^f X} A\}.$$

Let us check that $\mathcal{F}$ is a filter in $\text{Cl}(X)$. Let $C_1, C_2 \in \mathcal{F}$. Suppose that $p \notin \text{cl}_{\beta^f X}(C_1 \cap C_2)$. In much the same way, as in the proof to Theorem 3.10,
we find a set $D \in \mathcal{Z}(\beta X)$ such that $p \in \text{int}_{\beta X}D$ and $D \cap (C_1 \cap C_2) = \emptyset$. We put $A_i = D \cap C_i$ for $i \in \{1, 2\}$. We have that $p \in \text{cl}_{\beta X} A_i$ for $i \in \{1, 2\}$.

On the other hand, since $X$ satisfies UL$(X)$, the sets $A_1, A_2$ have disjoint closures in $\beta X$. This is impossible. In consequence, $C_1 \cap C_2 \in \mathcal{F}$. Therefore, $\mathcal{F}$ is a filter. Of course, $\mathcal{A} \subseteq \mathcal{F}$. We check that $\mathcal{F}$ is an ultrafilter in $\text{Cl}(X)$. To do this, consider any filter $\mathcal{H}$ in $\text{Cl}(X)$ such that $\mathcal{F} \subseteq \mathcal{H}$. Suppose that there exists $A \in \mathcal{H} \setminus \mathcal{F}$.

Then $p \notin \text{cl}_{\beta X}A$, so there exists $Z \in \mathcal{Z}(\beta X)$ such that $p \in \text{int}_{\beta X}Z$ and $Z \cap A = \emptyset$. It is obvious that $Z \cap X \in \mathcal{F}$, so $Z \cap X \in \mathcal{H}$. This is impossible because $A \in \mathcal{H}$ and $(Z \cap X) \cap A = \emptyset$. Therefore, $\mathcal{F}$ is an ultrafilter in $\text{Cl}(X)$. Since every filter in $\text{Cl}(X)$ can be enlarged to an ultrafilter in $\text{Cl}(X)$, the space $\mathcal{W}(X, \text{Cl}(X))$ is compact.

**Corollary 3.14.** [ZF] Let $X$ be a normal $T_1$-space such that $\beta^f X$ exists. If UL$(X)$ holds, then $\beta X \approx \beta^f X \approx \mathcal{W}(X, \text{Cl}(X))$.

**Remark 3.15.** Theorem 18 of [14] follows directly from our Corollary 3.14 and Theorem 3.10.

Since UFT is equivalent to the Boolean Prime Ideal Theorem BPI (Form 14 in [15], denoted by PIT in [12]), all statements equivalent to BPI are also equivalent to conditions (i)-(vii) of the following theorem:

**Theorem 3.16.** [ZF] The following conditions are equivalent:

(i) UFT;

(ii) every Tychonoff space has its functional Čech-Stone compactification;

(iii) every Cantor cube $2^J$ has a Hausdorff compactification $Y$ such that, for each $j \in J$, the projection $\pi_j : 2^J \to \{0, 1\}$ has a continuous extension over $Y$;

(iv) every Cantor cube $2^J$ is compact;

(v) every Tychonoff space $X$ has a Hausdorff compactification $\alpha X$ such that every clopen set of $X$ has a clopen closure in $\alpha X$;

(vi) every Tychonoff space $X$ has a Hausdorff compactification $\alpha X$ such that every continuous function from $X$ into the discrete space $\{0, 1\}$ is continuously extendable to a function from $\alpha X$ into $\{0, 1\}$.
(vii) every discrete space $X$ has a compactification $\alpha X$ such that, for each subset $A$ of $X$, the closure in $\alpha X$ of $A$ is clopen in $\alpha X$.

Proof. That (i) implies (ii) follows Theorem 4.70 of [22] and from our Theorem 3.1. It is obvious that (ii) implies (iii). When we replace the unit interval $[0,1]$ by the two-point discrete space $\{0,1\}$ and use similar argument as in the proof to Theorem 10.12 in [22], we can show that (iii) implies (iv). In view of Theorem 4.70 of [12], (iv) and (i) are equivalent. Of course, (v) and (vi) are equivalent, (ii) implies (vi) and (vi) implies (vii). Suppose that (vii) holds. Let $X$ be a non-empty discrete space and let $f \in C^*(X)$. Consider any pair $C,D$ of disjoint closed subsets of $\mathbb{R}$. Put $A = f^{-1}(C)$ and $B = f^{-1}(D)$. Let $\alpha X$ be a compactification of $X$ such that every subset of $X$ has a clopen closure in $\alpha X$. Then the sets $A,B$ must have disjoint closures in $\alpha X$. It follows from Theorem 5.15 of [22] that $f$ is continuously extendable over $\alpha X$. It follows from Theorem 3.10 that the Stone space $S(X) = W(X, Z(X))$ is compact. Hence (vii) implies (i) (cf. Remark 2.9 of [22]).

By applying our Theorem 3.1 and the proof to Theorem 10.12 of [22], we can immediately deduce the following:

**Proposition 3.17.** [ZF] Let $J$ be an infinite set. Then the Cantor cube $2^J$ is compact if and only if there exists a Hausdorff compactification $Y$ of $2^J$ such that, for each $j \in J$, the projection $\pi_j : 2^J \to \{0,1\}$ is continuously extendable over $Y$. Similarly, the Tychonoff cube $[0,1]^J$ is compact if and only if there exists a Hausdorff compactification $Z$ of $[0,1]^J$ such that, for each $j \in J$, the projection $\pi_j : [0,1]^J \to [0,1]$ is continuously extendable over $Z$.

**Theorem 3.18.** The following sentences are relatively consistent with ZF:

(i) There exists a 0-dimensional $T_1$-space which has no compactification $\alpha X$ such that every clopen subset of $X$ has a clopen closure in $\alpha X$.

(ii) There exists a metrizable 0-dimensional space which has no compactification $\alpha X$ such that every clopen subset of $X$ has a clopen closure in $\alpha X$.

Proof. It was shown in [27] that in some models of ZF (for instance, in the model $\mathcal{M}7$ of [15]) there exists a metrizable non-compact Cantor cube. Let $X$ be a metrizable non-compact Cantor cube. Of course, $X$ is 0-dimensional.
It follows from Theorem 3.1 and Proposition 3.17 that $X$ does not have a compactification $\alpha X$ such that every clopen subset of $X$ has a clopen closure in $\alpha X$. Hence both (i) and (ii) are relatively consistent with ZF.

Remark 3.19. It is known that in the model $\mathcal{M}7$ of [15], the Cantor cube $2^\mathbb{R}$ is not compact (see [17]). Hence, in view of Proposition 3.17 and Theorem 3.1, $2^\mathbb{R}$ is a 0-dimensional $T_1$-space which has no compactifications in $\mathcal{M}7$ in which closures of clopen sets in $2^\mathbb{R}$ are clopen. However, by Theorem 2.2 of [27], the space $2^\mathbb{R}$ cannot be metrizable.

Definition 3.20. Let $X$ be a topological space and $\text{Clop}(X)$ the collection of all clopen subsets of $X$. Every filter in the family $\text{Clop}(X)$ will be called a clopen filter of $X$. Every ultrafilter in $\text{Clop}(X)$ will be called a clopen ultrafilter of $X$.

In the light of Theorem 3.18, it may be useful to have a deeper look at clopen filters.

Theorem 3.21. [ZF] Let $X$ be a topological space. Suppose that $\alpha X$ is a compactification of $X$ such that every set $A \in \text{Clop}(X)$ has a clopen closure in $\alpha X$. Then every clopen filter of $X$ is included in a clopen ultrafilter of $X$.

Proof. Let $\mathcal{H}$ be a clopen filter of $X$. It follows from the compactness of $\alpha X$ that the set $K = \bigcap_{H \in \mathcal{H}} \text{cl}_{\alpha X}(H)$ is non-empty. Let us fix $x \in K$ and put

$$\mathcal{F} = \{ F \in \text{Clop}(X) : x \in \text{cl}_{\alpha X}F \}.$$

We show that $\mathcal{F}$ is a clopen ultrafilter of $X$ such that $\mathcal{H} \subseteq \mathcal{F}$. Clearly, $\emptyset \notin \mathcal{F}$ and $\mathcal{H} \subseteq \mathcal{F}$. For each $A \in \text{Clop}(X)$, we have $\alpha X = \text{cl}_{\alpha X}(A) \cup \text{cl}_{\alpha X}(X \setminus A)$. Hence, since $X$ is dense in $\alpha X$, it follows from our hypothesis that, for each $A \in \text{Clop}(X)$, the point $x$ belongs to exactly one of the sets $\text{cl}_{\alpha X}(A)$ and $\text{cl}_{\alpha X}(X \setminus A)$, so exactly one of $A$ and $X \setminus A$ belongs to $\mathcal{F}$.

Consider any $A, B \in \mathcal{F}$. We show that $A \cap B \notin \mathcal{F}$. Assume the contrary that $A \cap B \notin \mathcal{F}$. Then $(X \setminus A) \cup (X \setminus B) \in \mathcal{F}$ and, consequently, $x \in \text{cl}_{\alpha X}(X \setminus A)$ or $x \in \text{cl}_{\alpha X}(X \setminus B)$. This implies that either $A \notin \mathcal{F}$ or $B \notin \mathcal{F}$ - a contradiction. Hence $\mathcal{F}$ is closed under finite intersections. Of course, for any $A \in \mathcal{F}$ and $B \in \text{Clop}(X)$ such that $A \subseteq B$, we have $B \in \mathcal{F}$. All this taken together implies that $\mathcal{F}$ is a clopen ultrafilter of $X$ such that $\mathcal{H} \subseteq \mathcal{F}$. \hfill \Box
**Remark 3.22.** (a) The requirement from Theorem 3.21 that if \( A \in \text{Clop}(X) \), then \( \text{cl}_{\alpha X}(A) \in \text{Clop}(\alpha X) \), cannot be dropped out even when \( X \) is discrete. Indeed, if \( \mathcal{M} \) is a \( \text{ZF} \)-model such that \( \omega \) has no free ultrafilters in \( \mathcal{M} \), e.g. Feferman’s Model \( \mathcal{M}_2 \) or Solovay’s Model \( \mathcal{M}_5(\aleph) \) in [15], then the clopen filter \( \mathcal{H} \) of all cofinite subsets of the discrete space \( \omega \) does not extend to a clopen ultrafilter of \( \omega \) in \( \mathcal{M} \). The Alexandroff compactification \( \alpha \omega \) of the discrete space \( \omega \) is a Hausdorff compactification of \( \omega \) such that, for each infinite subset \( A \) of \( \omega \), if \( \omega \setminus A \) is infinite, the closure of \( A \) in \( \alpha \omega \) is not clopen.

(b) That conditions (i) and (vii) of Theorem 3.16 are equivalent can be proved by applying Theorem 3.21. Namely, for a discrete space \( X \), the Stone space \( S(X) \) is compact if and only if every filter in the power set \( \mathcal{P}(X) \) is contained in an ultrafilter in \( \mathcal{P}(X) \). By Theorem 3.21, condition (vii) of Theorem 3.16 implies that \( S(X) \) is compact for every discrete space \( X \). Of course, \( S(X) \) is compact for every discrete space \( X \) if and only if \( \text{UFT} \) holds.

(c) It was shown in [18] that the following are equivalent:

(i) every clopen filter of \( 2^\mathbb{R} \) extends to a clopen ultrafilter of \( 2^\mathbb{R} \) (Form 139 in [15]).

(ii) \( \text{BPI}(\omega) \) : every filter on \( \omega \) extends to an ultrafilter (Form 225 in [15]).

(iii) the Cantor cube \( 2^\mathbb{R} \) is compact.

Hence, \( 2^\mathbb{R} \) is not compact if and only if there exists a clopen filter \( \mathcal{H} \) of \( 2^\mathbb{R} \) which does not extend to a clopen ultrafilter of \( 2^\mathbb{R} \). Thus, it follows directly from Theorems 3.1, 3.21 and Proposition 3.17 that the Cantor cube \( 2^\mathbb{R} \) does not have a compactification in which closures of clopen sets of \( 2^\mathbb{R} \) are clopen if and only if there exists a clopen filter of \( 2^\mathbb{R} \) which does not extend to a clopen ultrafilter of \( 2^\mathbb{R} \).

**Remark 3.23.** (a) Let \( X \) be a Tychonoff space and \( \alpha X \) be a Hausdorff compactification of \( X \). It is known that if \( \alpha X \) is a Čech-Stone compactification of \( X \), then the following condition is satisfied:

\[ (\ast) \quad \text{for every clopen set } A \text{ of } X, \text{ cl}_{\alpha X}(A) \text{ is a clopen set of } \alpha X. \]

On the other hand, the interval \([0,1]\) is a non-Čech-Stone compactification of the open interval \((0,1)\) with the usual topology, while \((0,1)\) satisfies trivially \((\ast)\). Thus, \((\ast)\) does not imply \( \alpha X \) is a Čech-Stone compactification of \( X \).

(b) For each \( n \in \omega \), let \( \mathcal{F}_n = \{ A \subseteq \omega : n \in A \} \). Since the set \( W = \{ \mathcal{F}_n : n \in \omega \} \) is dense in \( S(\omega) \), it follows that any compactification of \( S(\omega) \) is also
a compactification of the discrete space $\omega$. In particular, if $S(\omega)$ is compact, then it is the Čech-Stone compactification of $\omega$. Clearly, if in a $\text{ZF}$-model $\mathcal{M}$ there do not exist free ultrafilters in the collection $\mathcal{P}(\omega)$, then $S(\omega) = W$ is discrete in $\mathcal{M}$, and the Alexandroff compactification $\alpha_aW$ is a Hausdorff compactification of $W$ but, in view of Theorem 2.27, $\alpha_aW$ is not a Čech-Stone compactification of $W$ because $W$ is not amorphous. In the model $\mathcal{N}[\Gamma]$ of $[\Pi]$, $S(\omega)$ is a dense-in-itself Tychonoff space which is easily seen not to be locally compact. Hence, the Alexandroff compactification $\alpha_aS(\omega)$ of $S(\omega)$ is a Hausdorff compactification of $S(\omega)$ but, in much the same way, as in the proof to Theorem 3.5.9 of [5], one can check that $eX$ is the Čech-Stone compactification of $X$.

**Proposition 3.24. [ZF]** If a topological space $X$ has a Hausdorff compactification and there exists a set $\mathcal{K}$ of Hausdorff compactifications of $X$ such that every Hausdorff compactification of $X$ is equivalent with a member of $\mathcal{K}$ and, moreover, if the space $\prod_{\gamma X \in \mathcal{K}} \gamma X$ is compact, then $X$ has its Čech-Stone compactification.

**Proof.** Let $\mathcal{K}$ be a set of Hausdorff compactifications of $X$ such that every Hausdorff compactification of $X$ is equivalent with a member of $\mathcal{K}$. Put $Y = \prod_{\gamma X \in \mathcal{K}} \gamma X$ and assume that $Y$ is compact. Let $e: X \to Y$ be defined by: $e(x)(\gamma X) = \gamma(x)$ for all $x \in X$ and $\gamma X \in \mathcal{K}$. Denote by $eX$ the closure in $Y$ of $e(X)$. Then $eX$ is a Hausdorff compactification of $X$. In much the same way, as in the proof to Theorem 3.5.9 of [5], one can check that $eX$ is the Čech-Stone compactification of $X$. $\square$

**Proposition 3.25. [ZF]** Suppose that a topological space $X$ has its Čech-Stone compactification. Then there exists a set $\mathcal{K}$ of Hausdorff compactifications of $X$ such that every Hausdorff compactification of $X$ is equivalent with a member of $\mathcal{K}$.

**Proof.** Let us consider the collection $\mathcal{R}$ of all collections $\mathcal{D}$ of pairwise disjoint closed subsets of $\beta X$ such that each $D \in \mathcal{D}$ is non-empty, $\beta X \setminus X = \bigcup_{D \in \mathcal{D}} D$ and the quotient space $r_{\mathcal{D}}X$ obtained from $\beta X$ by identifying each set $D \in \mathcal{D}$ with a point is a Hausdorff compactification of $X$. It follows from $\text{ZF}$ that the class $\mathcal{K} = \{r_{\mathcal{D}}X : \mathcal{D} \in \mathcal{R}\}$ is a set. Of course, every Hausdorff compactification of $X$ is equivalent with a member of $\mathcal{K}$. $\square$
Remark 3.26. In ZFC, for a non-empty Tychonoff space $X$, the class $K = \{ e_F : F \in \mathcal{E}(X) \}$ is a set of Hausdorff compactifications of $X$ such that every Hausdorff compactification of $X$ is equivalent with a member of $K$. However, perhaps, in a model of ZF, it may happen that every class $K$ of Hausdorff compactifications of a space $X$ such that every Hausdorff compactification of $X$ is equivalent with a member of $K$ is a proper class.

We include a careful proof to the following theorem for completeness:

**Theorem 3.27.** [ZF] The following conditions are equivalent:

(i) UFT;

(ii) for every topological space $X$ it holds true that if $X$ has a Hausdorff compactification, then $X$ has the Čech-Stone compactification;

(iii) every Tychonoff space has its Čech-Stone compactification.

**Proof.** Suppose that $\langle X, \tau \rangle$ is a topological space which has a Hausdorff compactification. Consider the Stone space $S(X)$ of the discrete space $\langle X, \mathcal{P}(X) \rangle$. Assume that (i) holds. Then $S(X)$ is the Čech-Stone compactification of $\langle X, \mathcal{P}(X) \rangle$. Consequently, for every Hausdorff compactification $\gamma \langle X, \tau \rangle$ of $\langle X, \tau \rangle$ the mapping $\text{id}_X : X \to \gamma \langle X, \tau \rangle$ is continuously extendable over $S(X)$ to a mapping $g_{\gamma} : S(X) \to \gamma \langle X, \tau \rangle$. We consider the equivalence relation $\approx_\gamma$ on $S(X)$ defined by: $y \approx_\gamma z$ if and only if $g_{\gamma}(y) = g_{\gamma}(z)$ for $y, z \in S(X)$. Then the space $\gamma \langle X, \tau \rangle$ is homeomorphic with the quotient space $S(X)/\approx_\gamma$ (see Theorem 2.4.3 of [5]). Since the class of all quotient spaces obtained from $S(X)$ is a set, we can deduce from the scheme of replacement (Axiom 6 on page 10 of [20]) that there exists a set $\mathcal{K}$ of Hausdorff compactifications of $\langle X, \tau \rangle$ such that every Hausdorff compactification of $\langle X, \tau \rangle$ is equivalent with a member of $\mathcal{K}$. In the light of Proposition 3.24 and Theorem 4.70 of [12], we infer that (i) implies (ii). We can deduce from Theorem 3.16 that (ii) implies (i) and that (iii) implies (i). In the light of Theorem 4.70 of [12], it follows from Theorem 3.16 and Proposition 3.24 that (i) implies (iii).

**Corollary 3.28.** [ZF + UFT + UL] Every normal $T_1$-space has its Čech-Stone compactification.

**Proof.** We assume ZF + UFT + UL. Let $X$ be a normal $T_1$ space. That $X$ is Tychonoff follows from UL(X). Since UFT holds, it follows from Theorem 3.27 or from Corollary 3.14 and Theorem 3.16 that $\beta X$ exists.
For a non-empty topological space $X$, let us consider $C^* (X)$ with the metric of uniform convergence $\rho_u$ defined by the equality $\rho_u(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$ for $f, g \in C^* (X)$. The topology $\tau(\rho_u)$ induced by $\rho_u$ is called the **topology of uniform convergence** in $C^* (X)$.

**Definition 3.29.** A set $A \subseteq C^* (X)$ will be called:

(i) **sequentially closed** in $C^* (X)$ if, for each uniformly convergent on $X$ sequence $(f_n)$ of functions from $A$, the limit function $f = \lim_{n \to +\infty} f_n$ belongs to $A$;

(ii) **uniformly closed** in $C^* (X)$ if $A$ is closed with respect to the topology of uniform convergence in $C^* (X)$.

Of course, every uniformly closed subset of $C^* (X)$ is sequentially closed. It follows from Theorem 4.54 of [12] that it may not be true in a model of $\text{ZF}$ that every sequentially closed subset of $C^* (X)$ is uniformly closed. The following theorem of $\text{ZFC}$ can be deduced immediately from Theorem 2.12 of [26]:

**Theorem 3.30.** [ZFC] If a compactification $\gamma X$ of a non-empty topological space $X$ is generated by a set $F \in \mathcal{E}(X)$, then $C_{\gamma} (X)$ is the smallest (with respect to inclusion) sequentially closed subalgebra of $C^* (X)$ which contains $F$ and all constant functions from $C^* (X)$.

It is still unknown whether Theorem 3.30 can be proved in $\text{ZF}$. In the original proof to Theorem 2.12 in [26], the axiom of choice was used. Theorem 3.30 is a useful tool for investigations of Hausdorff compactifications in every model of $\text{ZFC}$.

**Conclusions.** In this article, we have proved a considerable number of theorems on Hausdorff compactifications with the absence of the axiom of choice. We have posed non-trivial open problems that are of fundamental importance in $\text{ZF}$-theory of Hausdorff compactifications. More research is needed to solve the problems in a not-too-distant future.

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