Notes on supersymmetric $Sp(N)$ theories with an antisymmetric tensor

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We study supersymmetric $Sp(N)$ gauge theories with an antisymmetric tensor and degree $n+1$ tree level superpotential. The generalized Konishi anomaly equations derived in hep-th/0304119 and hep-th/0304138 are used to compute the low energy superpotential of the theory. This is done by imposing a certain integrality condition on the periods of a meromorphic one form. Explicit computations for $Sp(2)$, $Sp(4)$, $Sp(6)$ and $Sp(8)$ with cubic superpotential are done and full agreement with the results of the dynamically generated superpotential approach is found. As a byproduct, we find a very precise map from $Sp(N)$ to a $U(N+2n)$ theory with one adjoint and a degree $n+1$ tree level superpotential.

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1. Introduction

$\mathcal{N} = 1$ supersymmetric gauge theories have provided a very rich arena to explore nonperturbative quantum effects. Holomorphy has been the key tool in controlling the strong coupling dynamics. In the 90’s, these techniques were extensively used to determine the structure of holomorphic quantities in the infrared of asymptotically free theories (for a review, see e.g. [1,2]). For pure $\mathcal{N} = 1$ $SU(N)$ a massive vacuum is generated with low energy superpotential

$$W_{\text{low}} = N(\Lambda^{3N})^{1/N}. \quad (1.1)$$

This has the same low energy physics as the proposal of Veneziano and Yankielowicz [3]. They introduced a single scalar superfield $S$ which is a fundamental field in the infrared but it is composite in the microscopic theory. In the latter, $S$ is given by the glueball superfield $-\frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$. In the infrared, the theory develops an effective superpotential,

$$W_{\text{VY}} = S \left[ \ln \left( \frac{\Lambda^{3N}}{S^N} \right) + N \right]. \quad (1.2)$$

At the extremum of (1.2), the fundamental field $S$ acquires an expectation value $S^N = \Lambda^{3N}$ and (1.2) reduces to (1.1). Clearly, the only contribution to this vev is nonperturbative and in perturbation theory we can write $S^N_{\text{p.t.}} = 0$. It was proven in [4] that in the microscopic theory, $S$, defined as the glueball superfield, satisfies the identity $S^N \simeq 0$, where $\simeq$ means valid in the chiral ring of the theory. In [4], it was also suggested that the exchange of the identity $S^N \simeq 0$ in the microscopic theory by the field equation $S^N_{\text{p.t.}} = 0$ in the effective theory is a sign of dual descriptions. If this is the case, no information of the constraint $S^N_{\text{p.t.}} = 0$ should be visible in the off-shell low energy effective superpotential for $S$.

In general, for any pure $\mathcal{N} = 1$ gauge theory, with a simple Lie group $G$, the Veneziano-Yankielowicz superpotential is given by

$$W_{\text{VY}} = S \left[ \ln \left( \frac{\Lambda^{3h}}{S^h} \right) + h \right]. \quad (1.3)$$

where $h$ is the dual Coxeter number of the Lie algebra of $G$.

In [4], it was conjectured that the effective equation of motion $S^h = \Lambda^{3h}$ or $S^h_{\text{p.t.}} = 0$ was reproduced as an identity in the chiral ring of the microscopic theory, i.e. $S^h \simeq 0$. This was proven for $SO(N)$ and $Sp(N)$ gauge groups in [3].
Recently, a new way of computing low energy holomorphic physics of a large class of \( \mathcal{N} = 1 \) gauge theories using bosonic matrix models was conjectured \cite{6,7,8}. This conjecture motivated the development of superfield techniques \cite{9} and of generalized Konishi anomalies \cite{4} to prove it.

Using the superfield techniques \cite{10}, and using the generalized Konishi anomalies \cite{11,12}, the low energy superpotential for \( Sp(N) \) gauge theories with one matter field in the antisymmetric representation and tree level superpotential was computed. These theories were also studied in the 90’s using holomorphy to determine the dynamically generated superpotentials \cite{13,14}. In the far IR, after all fields have been integrated out, the two approaches should give the same answer. By this we mean a single function \( W_{\text{low}} \), the low energy superpotential, that depends on the dynamically generated scale \( \Lambda \) and tree level superpotential parameters.

However, a discrepancy was found in \cite{10} for several examples, namely, \( Sp(4) \), \( Sp(6) \) and \( Sp(8) \) in the unbroken classical vacuum with a cubic superpotential. The difference always sets in at order \( \Lambda^{3h} \) where \( h = N/2 + 1 \) is the dual Coxeter number of \( Sp(N) \). It was then suggested that perhaps \( S \) satisfies relations coming from its glueball origin. This explanation, however, would contradict the dual picture mentioned above.

In \cite{15}, a possible explanation to the discrepancy was suggested. An ambiguity in the UV behavior of a theory was discovered if one allows for supergroups. A prescription to “F-term complete” a theory was given by thinking about the gauge group \( G(N) \) as embedded in a supergroup \( G(N+k|k) \) with \( k \to \infty \). Moreover, the matrix model, glueball superpotential, or Konishi anomaly computations, were claimed to be computing the superpotential of the F-completion \( G(N+k|k) \). Therefore, the analysis of \cite{10,11} and \cite{12}, which was compared to a standard \( Sp(N) \) field theory calculation \cite{13,14}, had actually being done for \( Sp(N) \times Sp(0) \). Here \( Sp(0) \) is a remnant of the F-completion \( Sp(N+k_1+k_2|k_1+k_2) \) broken down to \( Sp(N+k_1|k_1) \times Sp(k_2|k_2) \). This completion produces residual instantons that are not present in the standard UV completion of \( Sp(N) \).

In these notes we show that by using the generalized Konishi anomaly equations derived in \cite{11} and \cite{12} and imposing that the periods of the generating function of chiral operators, \( T(z) = \langle \text{Tr} \frac{1}{z-\Phi} \rangle \), satisfy

\[
\frac{1}{2\pi i} \oint_{A_1} T(z)dz = N, \quad \frac{1}{2\pi i} \oint_{A_2} T(z)dz = 0, \tag{1.4}
\]
we get full agreement with the dynamically generated superpotential approach [13,14], i.e.,
no discrepancy was observed at any order computed in the examples. In particular, these
contain the examples considered in [10].

The $A_k$’s in (1.4) refer to the $A$-cycles of a genus $g = 1$ Riemann surface

$$y^2 = W'(z)^2 + f(z)$$

(1.5)

where $W'(z) = 0$ is the classical F-term field equation.

The IR dynamics is taken into account by imposing that the period of $T(z)dz$ over the
$B$-cycle of (1.5) be an integer [16,17].

We find that the cut $A_2$ does not close up on-shell. This is very surprising at first
due to the second equation in (1.4). The fact that the vanishing of the period of $T(z)dz$
through a given cut implies that the cut disappears on-shell has been usually assumed
for $U(N)$ theories with excellent results so far (see e.g. [18,4]). Here we show that this
assumption is not valid in general and it is the reason for the discrepancy observed in
[10,11,12].

The main difference between $Sp(N)$ with antisymmetric tensor and $U(N)$ with adjoint
is that in the former the branch points of (1.5) are simple poles of $T(z)dz$ while in the
latter they are regular points.

We also consider general degree $n + 1$ superpotentials and generic classical vacua. In
this case, the IR dynamics is also taken into account by imposing that the periods over
the $B$-cycles of (1.3), which in general has genus $n - 1$, be integers [16,17].

The actual computation is carried out by mapping the problem to a $U(N + 2n)$ theory.
This is a very precise map which might suggest a new kind of duality.

To give an example, $Sp(N)$ with one antisymmetric tensor corresponds to $U(N + 2n)$
with one adjoint. If $Sp(N)$ is classically unbroken, then $U(N + 2n)$ is classically broken to
$U(N + 2) \times U(2)^{n-1}$. Quantum mechanically, the vacua of $Sp(N)$ correspond to $U(N + 2n)$
vacua with confinement index $t = 2$ (defined in [19]). Given that $t = 2$, the computation
is effectively carried out in $U(N/2 + n)$ classically broken to $U(N/2 + 1) \times U(1)^{n-1}$. In
particular, for a cubic superpotential $n = 2$ and $U(N/2 + 2)$ is broken to $U(N/2 + 1) \times U(1)$.
The $U(1)$ factor enters in the low energy superpotential starting at order $\Lambda^{3(N/2+1)}$. This
explains why the discrepancy found in [10] started at different orders for different values
of $N$.

1 We follow the convention used in [10] where $N$ is an even integer, and $Sp(2) \simeq SU(2)$.
The presence of the $U(1)$ factors is reminiscent of the $Sp(0)$ factors in [15]. However, it is important to note that these $U(1)$ factors are forced upon us and are crucial to the agreement with the standard $Sp(N)$ dynamically generated superpotential results of [13,14].

This paper is organized as follows: In section 2 we consider $Sp(N)$ theories with one antisymmetric tensor in detail. The anomaly equations are solved for a general degree $n + 1$ superpotential. The mapping to $U(N + 2n)$ is shown and used to compute $W_{\text{low}}$. In section 3, the low energy superpotential for $Sp(4)$, $Sp(6)$ and $Sp(8)$ is computed. Also, as a consistency check, $Sp(0)$ and $Sp(2)$ are considered. In section 4, we end with some conclusions and open questions. In appendix A, we review and compute relevant $U(N + 2n)$ results. Finally, in appendix B, we review the dynamically generated superpotential computation of $W_{\text{low}}$ for $Sp(4)$, $Sp(6)$ and $Sp(8).

2. $Sp(N)$ with antisymmetric matter

We consider $\mathcal{N} = 1$ supersymmetric $Sp(N)$ gauge theories with a chiral superfield in the antisymmetric representation $\Phi \equiv AJ$. $A$ is a $N \times N$ antisymmetric matrix and $J$ is the invariant antisymmetric tensor of $Sp(N)$, i.e., $1_{N/2 \times N/2} \times i\sigma_2$.

The theory is deformed by a degree $n + 1$ tree level superpotential

$$W_{\text{tree}} = \sum_{k=0}^{n} \frac{g_k}{k + 1} \text{Tr} \, \Phi^{k+1}. \quad (2.1)$$

Note that we have explicitly included the term $g_0 \text{Tr} \, \Phi$. In the traceless case, $g_0$ can be used as a Lagrange multiplier imposing the tracelessness constraint, $\langle \text{Tr} \, \Phi \rangle = 0$.

Classically, the gauge group is generically broken to $Sp(N_1) \times \ldots \times Sp(N_n)$ with $N = \sum_{i=1}^{n} N_i$ and $N_i$ even.

We are interested in the generating functions of chiral operators [4,11,12]:

$$T(z) = \text{Tr} \frac{1}{z - \Phi}, \quad R(z) = -\frac{1}{32\pi^2} \text{Tr} \frac{W_\alpha W_\alpha}{z - \Phi}. \quad (2.2)$$

Following [4], a set of equations which constraint the generating functions (2.2) was derived in [11,12]. These equations are obtained by studying a generalization of the Konishi anomaly [20,21] and they can be used to determine $T(z)$ and $R(z)$ up to $2n$ complex parameters.
2.1. Anomaly equations and solution

The equations are given by \([11,12]\),

\[
[W'(z)R(z)]_+ = \frac{1}{2} R^2(z), \tag{2.3}
\]

\[
[W'(z)T(z)]_+ = T(z)R(z) + 2 \frac{d}{dz} R(z).
\]

The first equation gives \(R(z)\) in terms of \(W'(z) = \sum_{k=0}^{n} g_k z^k\) and a degree \(n - 1\) polynomial \(f(z) \equiv -2[W'(z)R(z)]_+\) as follows,

\[
R(z) = W'(z) - \sqrt{W'(z)^2 + f(z)}. \tag{2.4}
\]

The branch of the square root is chosen so that \(R(z) \sim 1/z\) for \(z \to \infty\). This is the correct classical behavior since only the first sheet is visible classically and the appearance of a second sheet is a quantum effect.

Classically, \(R(z)\) is a meromorphic function on a sphere. Quantum mechanically, it is a meromorphic function on a Riemann surface of genus \(n - 1\) defined by

\[
y^2 = W'(z)^2 + f(z). \tag{2.5}
\]

Let us denote by \(A_i\) and \(B_i\), with \(i = 1, \ldots, n - 1\), canonical basis of 1-cycles in \((2.5)\). \(A_i\) is defined to circle the \(i^{\text{th}}\) branch cut. The \(A_n\) cycle can also be defined as the cycle around the \(n^{\text{th}}\) cut. This cycle is not relevant when only holomorphic differentials are considered but here we will be dealing with meromorphic differentials.

The equation for \(T(z)\) can be solved in terms of \(W'(z)\), \(f(z)\), and a degree \(n - 1\) polynomial \(c(z) \equiv [W'(z)T(z)]_+\),

\[
T(z) = \frac{c(z)}{y(z)} + 2 \frac{W''(z)}{y(z)} - 2 \frac{y'(z)}{y(z)}. \tag{2.6}
\]

Note that the combination \(\tilde{c}(z) = c(z) + 2W''(z)\) is again a polynomial of degree \(n - 1\). In addition, note that the last term in \((2.6)\) can be written as a logarithmic derivative. Combining these observations \((2.6)\) becomes

\[
T(z) = \frac{\tilde{c}(z)}{y(z)} - \frac{d}{dz} \log \left(W'(z)^2 + f(z)\right). \tag{2.7}
\]

From \((2.7)\) we learn that \(T(z)dz\) is a meromorphic differential on the Riemann surface \((2.3)\). It has simple poles at \(z = \infty\) on the first and second sheets, and in all branch points. \(T(z)dz\) depends on \(2n\) complex parameters given by the coefficients of \(f(z)\) and \(\tilde{c}(z)\).
The $n$ coefficients in $\tilde{c}(z)$ can be fixed by imposing
\[
\frac{1}{2\pi i} \oint_{A_i} T(z) dz = N_i \quad \text{for} \quad i = 1, \ldots, n. \tag{2.8}
\]
The $n^{th}$ $A$-cycle has to be included since $T(z)dz$ has a pole at infinity. In the unbroken classical vacuum $N_1 = N$ and $N_i = 0$ for $i = 2, \ldots, n$.

On the other hand, $n - 1$ of the coefficients in $f(z)$ can be expressed in terms of
\[
\frac{1}{2\pi i} \oint_{B_i} T(z) dz = b_i \quad \text{for} \quad i = 1, \ldots, n - 1. \tag{2.9}
\]
The $n^{th}$ parameter in $f(z)$ is related to the scale of the theory, $\Lambda$, and depends on the regularization scheme.

The parameters $b_i$’s are determined by the IR dynamics. In [17], it was shown that for a $U(\tilde{N})$ theory with adjoint and fundamental matter, solving the field equations of the low energy effective superpotential is equivalent to imposing the integrality of the periods $b_i$’s in (2.9).

Here we assume that the IR dynamics of the $Sp(N)$ theory is also determined by imposing the integrality of $b_i$. This surprisingly simple condition seems to be a generic feature of theories where a hyperelliptic Riemann surface emerges. It would be interesting to explore this in more detail. We comment more on this issue in section 4.

At this point we have all the ingredients to compute $T(z)$ and $W_{\text{low}}$. However, it proves more convenient to map this problem to a known one. As a byproduct we find a surprisingly precise relation to a $U(N + 2n)$ theory.

2.2. Mapping of $Sp(N)$ to $U(N + 2n)$

Consider $U(\tilde{N})$ with a single adjoint chiral superfield $\Phi_U$ and tree level superpotential
\[
W_{\text{tree}}(U) = \sum_{k=0}^{n} \frac{h_k}{k+1} \text{Tr} \Phi_U^{k+1}. \tag{2.10}
\]
The generalized Konishi anomaly equations for this theory are given by [4],
\[
[W'_U(z)R_U(z)]_- = R_U^2(z), \quad [W'_U(z)T_U(z)]_- = 2R_U(z)T_U(z). \tag{2.11}
\]
The generating functions of chiral operators
\[
R_U(z) = -\frac{1}{32\pi^2} \text{Tr} \frac{W_{U\alpha}W_U^\alpha}{z - \Phi_U}, \quad T_U(z) = \text{Tr} \frac{1}{z - \Phi_U} \tag{2.12}
\]
are given in terms of two degree \( n - 1 \) polynomials \( f_U(z) = -4[W'_U(z)R_U(z)]_+ \) and \( c_U(z) = [W'_U(z)T(z)]_+ \) as follows

\[
T_U(z) = \frac{c_U(z)}{y_U(z)}, \quad R_U(z) = \frac{1}{2} \left( W'_U(z) - y_U(z) \right).
\] (2.13)

These are meromorphic functions on the Riemann surface \( y_U^2 = W'_U(z)^2 + f_U(z) \). As mentioned before, the IR dynamics of this theory is determined by simply imposing the integrality of all the periods of \( T_U(z)dz \).

Motivated by the form of the equations (2.3) and (2.11) we propose the following map,

\[
W'_U(z) = 2W'(z), \quad R_U(z) = R(z).
\] (2.14)

From these equations it is easy to get that

\[
f_U(z) = 4f(z), \quad y_U(z) = 2y(z).
\] (2.15)

The key observation is that \( T(z) \) for \( Sp(N) \) given by (2.7) has the same form as \( T_U(z) \) in (2.13) except for a logarithmic derivative of a meromorphic function on (2.5). Therefore, the extra term does not affect the dynamics since it has integer periods automatically.

This leads us to identify

\[
T_U(z) = T(z) + \frac{d}{dz} \log \left( W'(z)^2 + f(z) \right)
\] (2.16)

or equivalently \( c_U(z) = 2\tilde{c}(z) \). It is not difficult to check that these identifications are consistent as we will see next.

**Chiral operators**

It is important to get the relation between the \( Sp(N) \) invariants \( \text{Tr } \Phi^k \) with \( k \geq 0 \) and the \( U(\tilde{N}) \) invariants \( \text{Tr } \Phi_U^k \) with \( k \geq 0 \). This is done by solving (2.16) order by order in a \( 1/z \) expansion. Let us rewrite (2.16) more explicitly, solving for \( T(z) \),

\[
\left\langle \text{Tr } \frac{1}{z - \Phi} \right\rangle = \left\langle \text{Tr } \frac{1}{z - \Phi_U} \right\rangle - \frac{d}{dz} \log \left( W'(z)^2 + f(z) \right). \] (2.17)

Note that the first \( n \) orders in (2.17), i.e., up to \( 1/z^n \) are equivalent to

\[
[W'(z)T(z)]_+ + 2W''(z) = [W'(z)T_U(z)]_+.
\] (2.18)
Therefore, we do not get either extra information or an inconsistency by studying \( c_U(z) = 2\tilde{c}(z) \).

Of particular interest is the leading order in (2.17),

\[ \text{Tr } 1 = \text{Tr } 1_U - 2n. \]  

(2.19)

This gives \( \tilde{N} = N + 2n \) as claimed in the introduction.

Let us also mention a relation which follows from (2.14) and will play a key role in the sequel

\[ S_U = S. \]  

(2.20)

**Classical breaking pattern**

In order to complete the map we have to identify the correct classical breaking pattern. A \( U(\tilde{N}) \) theory with adjoint \( \Phi_U \) and tree level superpotential (2.10) generically has classical vacua with unbroken \( U(\tilde{N}_1) \times \ldots \times U(\tilde{N}_n) \). The relation between \( \tilde{N}_i \) and \( N_i \) can be found by studying the periods of \( T_U(z) \) through the \( A \)-cycles.

In the classical vacuum where \( \text{Sp}(N) \) is broken to \( \text{Sp}(N_1) \times \ldots \times \text{Sp}(N_n) \) we have

\[ \frac{1}{2\pi i} \oint_{A_i} T(z)dz = N_i \quad \text{for} \quad i = 1, \ldots, n. \]  

(2.21)

On the other hand, the logarithmic derivative \( \psi(z) = \frac{d}{dz} \log (W'(z)^2 + f(z)) \) satisfies

\[ \frac{1}{2\pi i} \oint_{A_i} \psi(z)dz = 2 \quad \text{for} \quad i = 1, \ldots, n. \]  

(2.22)

Therefore,

\[ \tilde{N}_i = \frac{1}{2\pi i} \oint_{A_i} T_U(z)dz = \frac{1}{2\pi i} \oint_{A_i} T(z)dz + \frac{1}{2\pi i} \oint_{A_i} \psi(z)dz = N_i + 2 \quad \text{for} \quad i = 1, \ldots, n. \]  

(2.23)

Note that this is consistent with \( \tilde{N} = \sum_{i=1}^{n} \tilde{N}_i \).

In the \( \text{Sp}(N) \) unbroken vacuum, \( U(N + 2n) \) is broken to \( U(N + 2) \times U(2)^{n-1} \).

**Vacua identification**

The last step in defining the associated \( U(\tilde{N}) \) problem is the identification of vacua in the quantum theory. Recall that in our normalization of \( \text{Sp}(N) \), \( N \) is an even number. Therefore, the set of numbers \( \tilde{N}_i = N_i + 2 \) with \( i = 1, \ldots, n \) has 2 as common divisor.
Equivalently, if \( m \) is the maximum common divisor of the set \( \tilde{N}_i, i = 1, \ldots, n \), then \( m \) is even.

The number of vacua for \( U(N + 2n) \) broken to \( \prod_{i=1}^{n} U(N_i + 2) \) is \( \prod_{i=1}^{n} (N_i + 2) \). As shown in [19], these vacua fall into separate phases distinguished by the “confinement index” \( t \) (defined in [19]). \( t \) takes values in the set of integer factors of \( m \). As mentioned above, \( m \) is always even. Therefore, at least two possibilities are present in this theory: \( t = 1 \) and \( t = 2 \).

A very useful fact about vacua with confinement index \( t > 1 \) is the following. \( T_U(z) \) evaluated at vacua of \( U(N + 2n) \) with classical breaking \( U(N_1 + 2) \times \ldots \times U(N_n + 2) \) and confinement index \( t \) can be effectively computed in a \( U((N + 2n)/t) \) theory with classical breaking \( U((N_1 + 2)/t) \times \ldots \times U((N_n + 2)/t) \) and same tree level superpotential. Let us denote by \( T_u(z) \) the generating function of chiral operators of such a \( U((N + 2n)/t) \) theory. The relation between the two is very simple [19],

\[
T_U(z) = t T_u(z). \tag{2.24}
\]

Consider for example the unbroken \( Sp(N) \) vacuum. This theory is maximally confining. Clearly \( m = 2 \) since \( N_i = 0 \) for \( i = 2, \ldots, n \). Note also that \( U(N + 2n) \) vacua with \( t = 1 \) are in a non-confining phase. Therefore \( t \) must be equal to 2. Motivated by this result, we propose that in general \( t = 2 \). This is consistent with the fact that the center of \( Sp(N) \) is \( \mathbb{Z}_2 \).

Let us check this proposal by counting the number of vacua in the general case. Pure supersymmetric \( Sp(N_i) \) is expected to have \( h_i = N_i/2 + 1 \) vacua distinguished by the solutions to the effective superpotential (1.3), i.e., \( S_i^{h_i} = \Lambda_i^{3h_i} \). Therefore, when \( Sp(N) \) is classically broken to \( \prod_{i=1}^{n} Sp(N_i) \) we find \( \prod_{i=1}^{n} h_i \) vacua. In the \( U(N + 2n) \) theory we have \( \prod_{i=1}^{n} (N_i + 2) \) which is \( 2^n \) times larger. However, the number of vacua with \( t = 2 \) is less and it is computed in \( U(N/2 + n) \) broken to \( \prod_{i=1}^{n} U(N_i/2 + 1) \). Therefore, the number of vacua with \( t = 2 \) is \( \prod_{i=1}^{n} (N_i/2 + 1) \) which is the desired answer.

However, this counting is not completely correct. The reason is that for each vacuum of \( U(N/2 + n) \) there are \( t = 2 \) vacua in \( U(N + 2n) \) [19]. This would lead to twice the expected number. The way this happens is through the relation between the scales of the theories [22,19]. If we denote by \( \Lambda_u \) and \( \Lambda_U \) the scales of \( U(N/2 + n) \) and \( U(N + 2n) \) respectively, then

\[
\Lambda_u^{N+2n} = \eta \Lambda_U^{N+2n} \quad \text{with} \quad \eta^t = 1. \tag{2.25}
\]
As we will see next, in the case of a cubic superpotential and unbroken $Sp(N)$ classical vacuum this puzzle is resolved because the $Sp(N)$ low energy superpotential only depends on $\Lambda_u^{2(N+2n)}$ and therefore it is invariant under the $Z_2$ action $\eta \rightarrow -\eta$. We believe that this is also true in the general case. We leave the proof of this for future work.

2.3. Low energy superpotential

The low energy superpotential, $W_{\text{low}}$, is a single function of the dynamically generated scale of the theory $\Lambda$ and the tree level superpotential parameters. $W_{\text{low}}$ is defined so that the expectation value of the chiral operators $\text{Tr} \Phi^k$ and $S = -\frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$ are given by

$$\frac{\partial W_{\text{low}}}{\partial g_k} = \frac{1}{k+1} \langle \text{Tr} \Phi^{k+1} \rangle,$$

$$\frac{\partial W_{\text{low}}}{\partial \log \Lambda_H} = (N+4) S. \tag{2.26}$$

The factor $(N+4)$ in the last equation is the coefficient of the holomorphic beta function of $Sp(N)$ with $\Phi$ an antisymmetric tensor. We denote by $\Lambda_H$ the scale of the high energy theory before $\Phi$ is integrated out.

The equations (2.26) can be used to find $W_{\text{low}}$ up to an irrelevant constant independent of the couplings in the superpotential and $\Lambda_H$.

For simplicity, and also because it is the case used in the examples, let us consider a cubic superpotential

$$W_{\text{tree}} = \frac{g}{3} \text{Tr} \Phi^3 + \frac{m}{2} \text{Tr} \Phi^2 + \lambda \text{Tr} \Phi. \tag{2.27}$$

Using (2.17) we get

$$\langle \text{Tr} 1 \rangle = \langle \text{Tr} 1_U \rangle - 4, \quad \langle \text{Tr} \Phi \rangle = \langle \text{Tr} \Phi_U \rangle + 2 \frac{m}{g},$$

$$\langle \text{Tr} \Phi^2 \rangle = \langle \text{Tr} \Phi_U^2 \rangle - 2 \left( \frac{m^2}{g^2} - 2 \frac{\lambda}{g} \right), \quad \langle \text{Tr} \Phi^3 \rangle = \langle \text{Tr} \Phi_U^3 \rangle - 2 \left( \frac{3S}{g} - \frac{m^3}{g^3} + 3 \frac{m\lambda}{g^2} \right). \tag{2.28}$$

These equations, together with (2.26), determine $W_{\text{low}}$ once the corresponding vev’s of $U(N+4)$ are known.

An important consistency check is the integrability of (2.26). We assume that the $U(N+4)$ problem has been solved, i.e., $W_{\text{low}(U)}$ is known. More explicitly, we assume that the system

$$\frac{\partial W_{\text{low}(U)}}{\partial h_k} = \frac{1}{k+1} \langle \text{Tr} \Phi_U^{k+1} \rangle,$$

$$\frac{\partial W_{\text{low}(U)}}{\partial \log \Lambda_U} = 2(N+4) S_U \tag{2.29}$$

$$10$$
has been integrated. Recall that \( h_2 = 2g, h_1 = 2m, \) and \( h_0 = 2\lambda \) are the couplings of \( W_{\text{tree}(U)} \) (2.10).

Let us propose an ansatz for \( W_{\text{low}} \),

\[
W_{\text{low}}(\Lambda_H, g, m, \lambda) = \frac{1}{2} W_{\text{low}(U)}(\Lambda_U(\Lambda_H, g), 2g, 2m, 2\lambda) - \frac{1}{3} \frac{m^3}{g^2} + 2 \frac{\lambda m}{g}. \tag{2.30}
\]

The ansatz for \( \Lambda_U = \Lambda_U(g, \Lambda_H) \) implies, by dimensional analysis, that \( \Lambda_U \) is proportional to \( \Lambda_H \) up to a \( g \) dependent factor. This can be shown by taking a derivative with respect to \( \log \Lambda_H \) of (2.30). This leads to

\[
\frac{\partial W_{\text{low}}}{\partial \log \Lambda_H} = (N + 4) S_U \frac{\Lambda_H}{\Lambda_U} \frac{\partial \Lambda_U}{\partial \Lambda_H}. \tag{2.31}
\]

Using the last equation in (2.26) together with (2.20) i.e., \( S = S_U \), we get that the equation is satisfied by the ansatz.

It is straightforward to check that the equations for \( \langle \text{Tr} \Phi \rangle \) and \( \langle \text{Tr} \Phi^2 \rangle \) in (2.28) are satisfied. The equation for \( \langle \text{Tr} \Phi^3 \rangle \) is more interesting and leads to the following equation,

\[
\frac{\partial W_{\text{low}(U)}}{\partial \Lambda_U} \frac{\partial \Lambda_U}{\partial g} = -4 \frac{S}{g}. \tag{2.32}
\]

Using (2.20) and the second equation in (2.23) we get,

\[
\Lambda_U(\Lambda_H, g) = g^{-\frac{N+2}{N+4}} \Lambda_H. \tag{2.33}
\]

up to a numerical constant. This \( g \) dependence will play a crucial role in the next section. In the unbroken \( Sp(N) \) vacuum the leading order of \( W_{\text{low}} \) does not depend on \( g \). The reason is that no \( W \)-boson is massive and has to be integrated out in this vacuum. On the other hand, in \( U(N+4) \) broken to \( U(N+2) \times U(2) \), massive \( W \)-bosons are integrated out. It is very satisfying that the power of \( g \) in (2.33) is precisely correct to cancel the \( g \) dependence in the \( U(N+4) \) answer as shown in the next section.

For future reference let us write the threshold matching relation between the scales \( \Lambda_H \) and \( \Lambda \), the pure \( Sp(N) \) scale after integrating out the antisymmetric tensor,

\[
\Lambda^{3(N+2)} = \Lambda_H^{2(N+4)} m^{N-2}. \tag{2.34}
\]

Recall that \( W_{\text{low}(U)} \) in (2.30) has to be computed around the classical vacuum where \( U(N+4) \) is broken to \( U(N+2) \times U(2) \) and evaluated in the quantum vacua with confinement index \( t = 2 \). This means that \( W_{\text{low}(U)} = 2W_{\text{low}(u)} \) with \( W_{\text{low}(u)} \) the low energy
superpotential of $U(N/2 + 2)$ classically broken to $U(N/2 + 1) \times U(1)$. The scales of the theories are related by \[ \Lambda_u^{N+4} = \eta \Lambda_u^{N+4}. \] (2.35) with $\eta^2 = 1$. This is the source of the doubling of vacua discussed at the end of section 2.2. However, note that (2.33) and (2.34) imply that $\Lambda^3$, which is the expansion parameter in the $Sp(N)$ theory, is invariant under the $\mathbb{Z}_2$ action $\eta \rightarrow -\eta$.

**Effective superpotential for $S$**

Finally, in the case of the unbroken $Sp(N)$ vacuum, an effective superpotential for $S$ can be easily computed by integrating it in. This is done by performing a Legendre transform in two steps. First, we introduce the intermediate superpotential

\[ W_{\text{int}}(S, \Lambda_H, C, g, m, \lambda) = W_{\text{low}}(C, g, m, \lambda) + (N + 4)S \log \frac{\Lambda_H}{C}. \] (2.36)

Second, we integrate out $C$ and use the matching relation (2.34) to get

\[ W_{\text{eff}}(S, \Lambda) = S \left[ \log \left( \frac{\Lambda^3 N/2 + 1}{S(N/2 + 1)} \right) + \frac{N}{2} + 1 \right] + \sum_{k=2}^{\infty} a_k S^k. \] (2.37)

**3. Examples**

In this section we consider the six cases studied in [10]. Namely, $Sp(4)$, $Sp(6)$ and $Sp(8)$ with $\Phi$ in the antisymmetric traceful representation and with $\Phi$ in the antisymmetric traceless representation. The theory is deformed by a cubic tree level superpotential and studied around the unbroken classical vacuum.

As a consistency check of our approach we also include at the end of this section the analysis of $Sp(0)$ and $Sp(2)$.

The tree level superpotential is given by (2.27),

\[ W_{\text{tree}} = \frac{g}{3} \text{Tr } \Phi^3 + \frac{m}{2} \text{Tr } \Phi^2 + \lambda \text{Tr } \Phi. \] (3.1)

In the traceful case we set $\lambda = 0$ and for the traceless case we use it as a Lagrange multiplier imposing the vanishing of $\langle \text{Tr } \Phi \rangle$.

The strategy to compute the low energy superpotential for $Sp(4)$, $Sp(6)$ and $Sp(8)$ is based on the integrability of (2.26), which was proven in the section 2.3. $W_{\text{low}}$ is obtained by first computing $\frac{1}{2} \langle \text{Tr } \Phi_u^2 \rangle$ in the $U(N/2 + 2)$ theory classically broken to $U(N/2 + 1) \times U(1)$. 
Then (2.24) is used to get $\frac{1}{2}\langle \text{Tr} \Phi^2 \rangle = \langle \Phi^2 \rangle$. The $Sp(N)$ vev $\frac{1}{2}\langle \text{Tr} \Phi^2 \rangle$ is obtained from the third equation in (2.28). Finally, $\frac{1}{2}\langle \text{Tr} \Phi^2 \rangle$ is integrated with respect to $m$ to get $W_{\text{low}}$.

The explicit computation is carried out in appendix A. The results come out in terms of $\Lambda_U$. Therefore we have to find the exact relation between the scales of the theories. Using (2.33) and (2.34), we find,

$$
\Lambda_U^{N+4} = g^{-2}m^{1-N/2}\Lambda^{3(N/2+1)}
$$

(3.2)

up to a numerical constant.

Comparing the leading order term of the superpotentials in appendix A

$$
W_{\text{low}} = (N+2)\frac{m^3}{g^2} \left( \frac{g\Lambda_U}{m} \right)^{2(N+4)/(N+2)} + O(\Lambda^{4(N+4)/(N+2)})
$$

(3.3)

with the standard definition of $\Lambda$, i.e.,

$$
W_{\text{low}} = (N/2 + 1)\Lambda^3 + O(\Lambda^6),
$$

(3.4)

the exact relation turns out to be

$$
\Lambda_U^{N+4} = g^{-2}m^{1-N/2} \left( \frac{\Lambda^3}{2} \right)^{N/2+1}.
$$

(3.5)

The final result of using (3.3) in the superpotentials of appendix A are listed below. Note that $W_{\text{eff}}(S)$ can easily be computed using (2.36) but we will not do it here.

**Sp(4) low energy superpotential:**

Traceful case:

$$
W_{\text{low}} = 3\Lambda^3 - \frac{1}{2}\Lambda^6 \frac{g^2}{m^3} - \frac{1}{2}\Lambda^9 \frac{g^4}{m^6} - \frac{187}{216}\Lambda^{12} \frac{g^6}{m^9} - \frac{1235}{648}\Lambda^{15} \frac{g^8}{m^{12}} + O(\Lambda^{18}).
$$

(3.6)

Traceless case:

$$
W_{\text{low}} = 3\Lambda^3 + O(\Lambda^{18}).
$$

(3.7)

**Sp(6) low energy superpotential:**

Traceful case:

$$
W_{\text{low}} = 4\Lambda^3 - \frac{3}{2}\Lambda^6 \frac{g^2}{m^3} - \frac{47}{24}\Lambda^9 \frac{g^4}{m^6} - \frac{75}{16}\Lambda^{12} \frac{g^6}{m^9} - \frac{7437}{512}\Lambda^{15} \frac{g^8}{m^{12}} + O(\Lambda^{18}).
$$

(3.8)
Traceless case:

\[ W_{\text{low}} = 4\Lambda^3 - \frac{1}{6} \Lambda^6 \frac{g^2}{m^3} - \frac{7}{216} \Lambda^9 \frac{g^4}{m^6} - \frac{5}{432} \Lambda^{12} \frac{g^6}{m^9} - \frac{221}{41472} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}). \]  

(3.9)

\( Sp(8) \) low energy superpotential:

Traceful case:

\[ W_{\text{low}} = 5\Lambda^3 - \frac{5}{2} \Lambda^6 \frac{g^2}{m^3} - \frac{13}{4} \Lambda^9 \frac{g^4}{m^6} - \frac{65}{8} \Lambda^{12} \frac{g^6}{m^9} - \frac{2147}{80} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}). \]  

(3.10)

Traceless case:

\[ W_{\text{low}} = 5\Lambda^3 - \frac{1}{4} \Lambda^6 \frac{g^2}{m^3} - \frac{1}{10} \Lambda^9 \frac{g^4}{m^6} - \frac{7}{100} \Lambda^{12} \frac{g^6}{m^9} - \frac{1}{16} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}). \]  

(3.11)

Comparing these results with the ones obtained by using the dynamically generated superpotentials reviewed in appendix B, we find complete agreement to all orders computed.

\( Sp(0) \) and \( Sp(2) \) low energy superpotentials. A consistency check:

\( Sp(0) \) and \( Sp(2) \) are special. The former should clearly give a trivial result. The latter is interesting since in our convention \( Sp(2) \simeq SU(2) \) and the antisymmetric matter is actually a singlet of the gauge group. This means that it does not participate in the strong coupling dynamics of the theory. This is why \( Sp(2) \) is a non trivial consistency check of our formalism.

According to the general analysis of section 2, we are instructed to consider \( U(4) \rightarrow U(2) \times U(2) \) and \( U(6) \rightarrow U(4) \times U(2) \) respectively. Compute \( W_{\text{low(U)}} \) in the vacua with \( t = 2 \) and use it in (2.30). Since \( t = 2 \) we only have to consider \( U(2) \rightarrow U(1) \times U(1) \) and \( U(3) \rightarrow U(2) \times U(1) \) respectively.

In the case of \( U(2) \rightarrow U(1) \times U(1) \) it is known that \( S_u = 0 \) \[16\] and therefore the low energy superpotential does not depend on \( \Lambda_u \). This implies that \( W_{\text{low(u)}} \) can be evaluated at any \( \Lambda_u \) in particular, at \( \Lambda_u = 0 \) to get the classical answer,

\[ W_{\text{low(u)}}(\Lambda_u, g_u, m_u, \lambda_u) = \frac{1}{6} \frac{m_u^3}{g_u^2} - \frac{m_u \lambda_u}{g_u} \]  

(3.12)

where \( g_u = h_2, m_u = h_1 \) and \( \lambda_u = h_0 \) in (2.10).
Using this in (2.30) we find

\[ W_{\text{low}} = 0. \]  

(3.13)

This is indeed the correct result.

In the case \( U(3) \rightarrow U(2) \times U(1) \), the low energy superpotential is given by \[24,22\],

\[ W_{\text{low}(u)}(\Lambda_u, g_u, m_u, \lambda_u = 0) = \frac{1}{6} \frac{m_u^3}{g_u^2} + 2 g_u \Lambda_u^3. \]  

(3.14)

Using this in (2.30) together with the matching relation (3.5) we get,

\[ W_{\text{low}} = 2 \Lambda^3. \]  

(3.15)

This is the correct answer for pure \( \mathcal{N} = 1 \) \( SU(2) \) gauge theory as can be seen from (1.1).

4. Conclusions and open questions

The low energy superpotential of \( \mathcal{N} = 1 \) \( Sp(N) \) theories with matter in the antisymmetric tensor representation and tree level superpotential was computed using the generalized Konishi anomaly equations found in \[11\] and \[12\]. A key role was played by the generating function of chiral operators \( T(z) \). Quantum mechanically, \( T(z)dz \) becomes a meromorphic differential on a Riemann surface.

A remarkably simple condition on \( T(z)dz \) was imposed which accounts for the full IR dynamics. The condition is the integrality of its \( A \)- and \( B \)-periods. Based on all the examples in the literature and the ones presented here, it is reasonable to propose that this is always the case in theories where a hyperelliptic Riemann surface emerges. One possible explanation is the following: In such theories a string theory realization with a dual involving fluxes might be available. On the dual side, the fluxes through compact cycles are quantized and are given as the periods of a closed real form. In the projection to a Riemann surface\footnote{Recall that \( W_{\text{low}(U)} = 2W_{\text{low}(u)} \).} this form gives rise to a one form with integer periods. However, the one form is real instead of meromorphic. It is only on-shell that this real one form becomes meromorphic and agrees with \( T(z)dz \). On the other hand, \( T(z)dz \) is meromorphic by definition but the integrality of its periods is valid only on-shell. It would be interesting to explore the full range of validity of this dual description.

\footnote{The compact cycles are in general embedded in manifolds of complex dimension three. If two of them are trivial, then a projection to a Riemann surface is possible.}
In the process of imposing the integrality condition of the periods of $T(z)dz$ in the $Sp(N)$ theory, we found a very precise map to a $U(N + 2n)$ theory. The map was given for a general degree $n + 1$ tree level superpotential. In the vacuum where $Sp(N)$ is classically broken to $\prod_{i=1}^n Sp(N_i)$, $U(N + 2n)$ is broken to $\prod_{i=1}^n U(N_i + 2)$.

However, one point we did not discuss is the following: In the IR, $U(N + 2n)$ has low energy group $U(1)^n$. Their couplings are frozen and depend on the parameters in the superpotential and $\Lambda_U$. What is the interpretation of this on the $Sp(N)$ side?

In section 2 we found a very simple expression (2.30) for the low energy superpotential $W_{low}$ of $Sp(N)$ with a cubic tree level superpotential. It would be interesting to generalize it to tree level superpotentials of arbitrary degree.

Generalizations of the approach presented in this work to $Sp(N)$ and $SO(N)$ with symmetric/antisymmetric tensors and fundamentals is surely possible and interesting. We leave it for future work.

As mentioned in the introduction we have shown that the vanishing of the period of $T(z)dz$ through a given cut does not imply that the cut closes up on-shell. Also surprising is the opposite case, namely, the period of $T(z)dz$ is non zero but the cut closes up on-shell. This does not arise in the cases studied in this work but we believe it will show up for $SO(N)$ with a symmetric tensor. This is currently under investigation.

We also would like to comment on the role of the term $\frac{d}{dz}R(z)$ in the anomaly equation for $T(z)$. Its presence prevents a given cut from closing up on-shell when the period of $T(z)dz$ is zero through it as one would naively have expected. It would be interesting to get a more geometrical explanation of this phenomenon. In terms of the $U(N/2 + n)$ theory, the role of such a term is to produce a classical breaking pattern of the form $\prod_{i=1}^n U(h_i)$ where $h_i$ is the dual Coxeter number of the original $Sp(N_i)$ factor supported at the $i^{th}$ cut. It is reasonable to think that this is generic and it would be interesting to explore it in other examples which might include exceptional groups.

Finally, we believe that the techniques used here are a reliable and simple way to study the IR dynamics of supersymmetric field theories and should be explored in more detail.

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Appendix A. $U(N+4)$ results

In this appendix we review and derive the relevant $U(N+4)$ field theory results needed to compute the low energy superpotential $W_{\text{low}}$ of the $Sp(N)$ theory discussed in section 3. The theory has a cubic tree level superpotential

$$W_{\text{tree}}(U) = \frac{g_u}{3} \Tr \Phi_U^3 + \frac{m_u}{2} \Tr \Phi_U^2 + \lambda_u \Tr \Phi_U,$$

and it is classically broken to $U(N+2) \times U(2)$. Here we use the notation $g_u$, $m_u$ and $\lambda_u$ for both the $U(N+4)$ and the $U(N/2 + 2)$ theories as they share the same tree level superpotential.

There are two possible ways to proceed. We will briefly discuss the first one, which is very powerful and can be applied to arbitrary high values of $N$. However, this is not what we use since the three cases considered in section 3 map to three cases already considered in [19] using a strong coupling analysis. The reader can skip the first one unless interested in larger values of $N$ for which the strong coupling analysis is complicated.

Matrix model effective superpotential

The first method is to use the matrix model [6,7,8] to compute an effective superpotential for $U(N+4)$ around the classical vacuum $U(N+2) \times U(2)$. This superpotential is a function of $S_1 = -\frac{1}{4\pi i} \oint_{A_1} y_U(z)dz$ and $S_2 = -\frac{1}{4\pi i} \oint_{A_2} y_U(z)dz$ given by

$$W_{\text{eff}}(S_1, S_2) = -\frac{1}{2} (N+2) \int_{\tilde{B}_1} y_U(z)dz - \int_{\tilde{B}_2} y_U(z)dz + (N+4)W(\Lambda_0)$$

$$- 2(N+4) S \log \left(-\frac{\Lambda_0}{\Lambda_U}\right) + 2\pi i (b_1 S_1 + b_2 S_2),$$

where $\tilde{B}_i$ are regularized non-compact cycles, $\Lambda_0$ is a cut off, which should be taken to infinity at the end of the computation and $b_i$’s are integers (for more details see e.g. [17]).

At the extremum of (A.2) we get $W_{\text{low}(U)} = W_{\text{eff}}(<S_1>, <S_2>)$. This result can be used in (2.30) to get the $Sp(N)$ low energy superpotential $W_{\text{low}}$.

Strong coupling approach

The second method is to use a strong coupling analysis where the tree level superpotential is thought of as a deformation of a $\mathcal{N} = 2 \ U(N+4)$ theory. Since we are only
interested in vacua with confinement index \( t = 2 \) we can consider a \( \mathcal{N} = 2 \) \( U(N/2 + 2) \) theory. The only \( \mathcal{N} = 1 \) supersymmetric vacua are located at points in the \( \mathcal{N} = 2 \) Coulomb moduli space that satisfy

\[
P^2_{\tilde{N}}(z) - 4\Lambda_u^{2\tilde{N}} = \frac{1}{g_u^2}(W_u''(z) + f_u(z))H_{\tilde{N}-n}^2(z), \tag{A.3}
\]

where \( \tilde{N} = N/2 + 2 \) and

\[
P_{\tilde{N}}(z) = \det(z1 - \Phi_u) = z^{\tilde{N}} - \langle \text{Tr} \Phi_u \rangle z^{\tilde{N}-1} + \left( \frac{1}{2} (\text{Tr} \Phi_u)^2 - \frac{1}{2} \langle \text{Tr} \Phi_u^2 \rangle \right) z^{\tilde{N}-2} + ... \tag{A.4}
\]

As shown in section 2, the cases \( Sp(4), Sp(6) \) and \( Sp(8) \) map to \( U(8) \rightarrow U(6) \times U(2), U(10) \rightarrow U(8) \times U(2), \) and \( U(12) \rightarrow U(10) \times U(2), \) respectively. Fortunately, as mentioned above we are interested in the vacua with confinement index \( t = 2 \). Therefore, all we need to consider is \( U(4) \rightarrow U(3) \times U(1), U(5) \rightarrow U(4) \times U(1), \) and \( U(6) \rightarrow U(5) \times U(1), \) respectively. The problem of finding \( P_{\tilde{N}}(z), f_u(z) \) and \( H_{\tilde{N}-n}(z) \) satisfying (A.3) was solved for these three cases in [19], from where we borrow the results.

In this appendix we introduce a subscript Sp to all \( Sp(N) \) quantities in order to avoid possible confusions.

\( U(4) \) case:

The solution to (A.3) is:

\[
P_4(z) = (z - a)^2((z + a)^2 + v(z + 2a)) - 2\Lambda_u^4 \quad \text{with} \quad a^3 = \frac{\Lambda_u^4}{v}. \tag{A.5}
\]

Clearly, in the semiclassical limit \( \Lambda_u \rightarrow 0, a \rightarrow 0 \) and \( P_4(z) \rightarrow z^3(z + v) \), showing that here \( U(4) \) is broken to \( U(3) \times U(1) \).

From (A.3) and (A.5), we find that

\[
\frac{1}{g_u}W'(z) = z^2 + vz - a^2. \tag{A.6}
\]

In this computation the freedom to shift \( z \) was used. In order to recover this degree of freedom we shift \( z \rightarrow z + \delta \). Note that \( W'(z) \) in (A.6) depends on \( \delta, v \) and \( a \). Comparing it to \( W'(z) = g_u z^2 + m_u z + \lambda_u \) we get two equations which together with the constraint \( a^3 = \frac{\Lambda_u^4}{v} \), determine \( \delta, v \) and \( a \) as functions of \( m_u/g_u, \lambda_u/g_u \) and \( \Lambda_u \).

\footnote{\( U(5) \rightarrow U(4) \times U(1) \) was first studied in [22]. \( U(4) \rightarrow U(3) \times U(1) \) was also studied in [25].}
These equations can be solved in a power expansion in $\Lambda_u$ around $\Lambda_u = 0$. We also choose to expand around the classical solution

$$v_{\text{cl}} = \sqrt{\left(\frac{m_u}{g_u}\right)^2 - 4\frac{\lambda_u}{g_u}}. \quad \text{(A.7)}$$

Consider first the traceful case. We set $\lambda_u = 0$ and use the values of $\delta$, $a$ and $v$ to compute $P_4(z)$. It is important to remember the shift $z \to z + \delta$. From $P_4(z)$ and (A.4) it is easy to compute $\langle \text{Tr} \Phi_u^2 \rangle$ which is equal to $\frac{1}{2}\langle \text{Tr} \Phi_U^2 \rangle$. Using this last result in (2.28) we get

$$\frac{1}{2}\langle \text{Tr} \Phi^2 \rangle_{\text{Sp}} = \frac{m_u^2}{g_u^2} \left(2T + \frac{14}{3}T^2 + 20T^3 + \frac{8602}{81}T^4 + \frac{153140}{243}T^5 + O(T^6) \right) \quad \text{(A.8)}$$

with

$$T = \left(\frac{g_u\Lambda_u}{m_u}\right)^{8/3}. \quad \text{(A.9)}$$

Finally, note that only the ratio $m_u/g_u$ appears and can be replaced by $m/g$. Recall that the $Sp(N)$ tree level superpotential is given by (3.1) and it is proportional to (A.4). $W_{\text{low}}$ can then be obtained by integrating (A.8) with respect to $m$,

$$W_{\text{low(Sp)}}^\text{Traceful} = \frac{m^3}{g^2} \left(6T - 2T^2 - 4T^3 - \frac{374}{27}T^4 - \frac{4940}{81}T^5 + O(T^6) \right) \quad \text{(A.10)}$$

with

$$T = \left(\frac{g\Lambda_u}{m}\right)^{8/3}. \quad \text{(A.11)}$$

For the traceless case we have to use $\lambda$ to set the second equation in (2.28) to zero, i.e.,

$$\langle \text{Tr} \Phi_U \rangle + 2\frac{m_u}{g_u} = 0 \quad \text{(A.12)}$$

or equivalently,

$$\langle \text{Tr} \Phi_u \rangle + \frac{m_u}{g_u} = 0. \quad \text{(A.13)}$$

The solution to (A.13) is

$$\frac{\lambda_u}{g_u} = -\frac{m_u^2}{g_u^2}T + O(T^6). \quad \text{(A.14)}$$

Using this and following the same procedure as before we compute

$$\frac{1}{2}\langle \text{Tr} \Phi^2 \rangle_{\text{Sp}} = \frac{m_u^2}{g_u^2} \left(T + O(T^6) \right). \quad \text{(A.15)}$$
Integrating with respect to \( m \) we find

\[
W_{\text{low(Sp)}}^{\text{Traceless}} = \frac{m^3}{g^2} \left( 6T + \mathcal{O}(T^6) \right) \tag{A.16}
\]

with \( T \) given by (A.11).

**U(5) case:**

The solution to (A.3) is:

\[
P_5(z) = (z^2 + az - 2ac)^2(z + c) - 2\Lambda^5_u \quad \text{with} \quad a^2 = \frac{\Lambda^5_u}{c^3}. \tag{A.17}
\]

In the semiclassical limit \( \Lambda_u \to 0, \ a \to 0 \) and \( P_5(z) \to z^4(z + c) \), showing that here \( U(5) \) is broken to \( U(4) \times U(1) \).

From (A.3) and (A.17), we find that

\[
\frac{1}{g_u} W'(z) = z^2 + (a + c)z - ac. \tag{A.18}
\]

Following the same steps as in the \( U(4) \) case, we expand around the classical solution

\[
c_{\text{cl}} = \sqrt{\left( \frac{m_u}{g_u} \right)^2 - \frac{4\lambda_u}{g_u}}. \tag{A.19}
\]

In the traceful case with \( \lambda_u = 0 \) we get

\[
\frac{1}{2} \langle \text{Tr} \Phi^2 \rangle_{Sp} = \frac{m_u^2}{g_u^2} \left( 4T + 12T^2 + \frac{141}{2} T^3 + 525T^4 + \frac{141303}{32} T^5 + \mathcal{O}(T^6) \right) \tag{A.20}
\]

with \( T = (g_u\Lambda_u/m_u)^{5/2} \).

Integrating with respect to \( m \) after replacing \( m_u/g_u \) by \( m/g \) we get,

\[
W_{\text{low(Sp)}}^{\text{Traceful}} = \frac{m^3}{g^2} \left( 8T - 6T^2 - \frac{47}{3} T^3 - 75T^4 - \frac{7437}{16} T^5 + \mathcal{O}(T^6) \right) \tag{A.21}
\]

with \( T = (g\Lambda_u/m)^{5/2} \).

For the traceless case we have to solve (A.13) to get,

\[
\frac{\lambda_u}{g_u} = \frac{m_u^2}{g_u} \left( -\frac{4}{3} T - \frac{4}{9} T^2 - \frac{7}{18} T^3 - \frac{35}{81} T^4 - \frac{4199}{7776} T^5 + \mathcal{O}(T^6) \right). \tag{A.22}
\]

Using this we compute

\[
\frac{1}{2} \langle \text{Tr} \Phi^2 \rangle_{Sp} = \frac{m_u^2}{g_u^2} \left( 4T + \frac{4}{3} T^2 + \frac{7}{6} T^3 + \frac{35}{27} T^4 + \frac{4199}{2592} T^5 + \mathcal{O}(T^6) \right). \tag{A.23}
\]
Integrating with respect to $m$ we get,

\[ W_{\text{Traceless}}^{\text{low(Sp)}} = \frac{m^3}{g^2} \left( 8T - \frac{2}{3}T^2 - \frac{7}{27}T^3 - \frac{5}{27}T^4 - \frac{221}{1296}T^5 + O(T^6) \right). \]  

(A.24)

**$U(6)$ case:**

The solution to (A.3) is:

\[ P_6(z) = \left[ z^2 + (h + g)z + \frac{(3h + g)(9h^3 + 15h^2g - hg^2 + g^3)}{108h^2} \right]^2 \left[ z^2 - \frac{(h - g)(3h - g)^2(3h + g)}{108h^2} \right] - 2\Lambda_u^6 \]

with $g$ and $h$ satisfying the constraint

\[ g^5(g^2 - 9h^2)^2 = 27^3h^3\Lambda_u^6. \]

(A.26)

The classical limit $\Lambda_u \to 0$ with $g \to 0$ gives $P_6(z) \to (z + \frac{h}{2})^5(z - \frac{h}{2})$; i.e. $U(6) \to U(5) \times U(1)$.

\[ \frac{1}{g_u} W'(z) = z^2 + \frac{2g}{3}z + \frac{g^4 - 6g^2h^2 - 27h^4}{108h^2}. \]

(A.27)

Following again the same steps as in the $U(4)$ case, we expand around the classical solution

\[ h_{\text{cl}} = -\sqrt{\left( \frac{m_u}{g_u} \right)^2 - 4\frac{\lambda_u}{g_u}}. \]

(A.28)

In the traceful case with $\lambda_u = 0$ we get

\[ \frac{1}{2} \langle \text{Tr} \Phi^2 \rangle_{Sp} = \frac{m_u^2}{g_u^2} \left( 6T + 18T^2 + \frac{546}{5}T^3 + 858T^4 + \frac{38646}{5}T^5 + O(T^6) \right) \]

(A.29)

with $T = (g_u\Lambda_u/m_u)^{12/5}$.

Integrating with respect to $m$,

\[ W_{\text{Traceful}}^{\text{low(Sp)}} = \frac{m^3}{g^2} \left( 10T - 10T^2 - 26T^3 - 130T^4 - \frac{4294}{5}T^5 + O(T^6) \right). \]

(A.30)

For the traceless case we have to solve (A.13) to get,

\[ \frac{\lambda_u}{g_u} = \frac{m_u^2}{g_u^2} \left( -\frac{3}{2}T - \frac{9}{20}T^2 - \frac{21}{25}T^3 - \frac{231}{125}T^4 - \frac{9}{2}T^5 + O(T^6) \right). \]

(A.31)
Using this we compute
\[
\frac{1}{2} \langle \text{Tr} \Phi^2 \rangle_{Sp} = \frac{m_u^2}{g_u^2} \left( 6T + \frac{9}{5}T^2 + \frac{84}{25}T^3 + \frac{924}{125}T^4 + 18T^5 + O(T^6) \right). \tag{A.32}
\]

Integrating with respect to \( m \),
\[
W_{\text{low}(Sp)}^{\text{Traceless}} = \frac{m^3}{g^2} \left( 10T - T^2 - \frac{4}{5}T^3 - \frac{28}{25}T^4 - 2T^5 + O(T^6) \right). \tag{A.33}
\]

Appendix B. \( Sp(N) \) results from \( W_{\text{dyn}} \)

Exact results for the dynamically generated superpotential of \( \mathcal{N} = 1 \) \( Sp(4) \) and \( Sp(6) \) gauge theories with one traceless antisymmetric tensor and some number of flavors were computed in \cite{13} and \cite{14}. Very recently, results for \( Sp(8) \) and for a traceful antisymmetric tensor were obtained in \cite{10} by applying the general strategy of \cite{14}.

Here we will only need the results for the case without flavors. In \cite{10}, a cubic tree level superpotential,
\[
W_{\text{tree}} = \frac{g}{3} \tilde{O}_3 + \frac{m}{2} \tilde{O}_2 \tag{B.1}
\]
where \( \tilde{O}_i = \text{Tr} \Phi^i \), was added to \( W_{\text{dyn}} \) and the F-term equations were solved. Those are the results we will write here.

\( Sp(4) \):

The dynamically generated superpotential is
\[
W_{\text{dyn}} = \frac{2\sqrt{2}}{\sqrt{m}} \Lambda^{9/2}. \tag{B.2}
\]

The solution to the F-term equations leads to
\[
W_{\text{low}}^{\text{Traceless}} = 3\Lambda^3,
\]
\[
W_{\text{low}}^{\text{Traceful}} = 3\Lambda^3 - \frac{1}{2} \Lambda^6 \frac{g^2}{m^3} - \frac{1}{2} \Lambda^9 \frac{g^4}{m^6} - \frac{187}{216} \Lambda^{12} \frac{g^6}{m^9} - \frac{1235}{648} \Lambda^{15} \frac{g^8}{m^{12}} + O(\Lambda^{18}). \tag{B.3}
\]

Here we fix a misprint in \cite{10} in the coefficient of the term \( \Lambda^{15} \) of \( W_{\text{low}}^{\text{Traceful}} \).

\( Sp(6) \):

The dynamically generated superpotential is
\[
W_{\text{dyn}} = \frac{8\Lambda^6}{m\tilde{O}_2 \left( (\sqrt{R} + \sqrt{R + 1})^{2/3} + (\sqrt{R} + \sqrt{R + 1})^{-2/3} - 1 \right)} \tag{B.4}
\]
where $R = -12 \tilde{O}_3^2/\tilde{O}_2^3$.

The solution to the F-term equations leads to

\begin{align}
W_{\text{Traceless}}^{\text{low}} &= 4\Lambda^3 - \frac{1}{6} \Lambda^6 \frac{g^2}{m^3} - \frac{7}{216} \Lambda^9 \frac{g^4}{m^6} - \frac{5}{432} \Lambda^{12} \frac{g^6}{m^9} - \frac{221}{41472} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}), \\
W_{\text{Traceful}}^{\text{low}} &= 4\Lambda^3 - \frac{3}{2} \Lambda^6 \frac{g^2}{m^3} - \frac{47}{24} \Lambda^9 \frac{g^4}{m^6} - \frac{75}{16} \Lambda^{12} \frac{g^6}{m^9} - \frac{7437}{512} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}).
\end{align}

\textbf{(B.5)}

\textbf{Sp(8) :}

The dynamically generated superpotential is

\begin{align}
W_{\text{dyn}} = K \left( -36 R_4 + 144 b^2 R_4 + 288 c R_4 + 8 R_3^2 + 192 b c R_3 + 1152 b^2 c^2 - 36 b^2 - 72 c + 9 \right)^{-1},
\end{align}

where $K = 24 \Lambda^{15/2}/(m\tilde{O}_2)^{3/2}$, $R_3 = \tilde{O}_3/\tilde{O}_2^{3/2}$, $R_4 = \tilde{O}_4/\tilde{O}_2^2$, and with $b$ and $c$ solutions of the following set of polynomial equations

\begin{align}
12 R_4 + 16 b R_3 - 192 b^2 c + 24 b^2 + 96 c^2 - 3 = 0, \\
12 b R_4 + 8 b^2 R_3 + 8 R_3 c - 96 b c^2 + 24 b c - 3 b = 0.
\end{align}

\textbf{(B.7)}

The solution to the F-term equations leads to

\begin{align}
W_{\text{Traceless}}^{\text{low}} &= 5\Lambda^3 - \frac{1}{4} \Lambda^6 \frac{g^2}{m^3} - \frac{1}{16} \Lambda^9 \frac{g^4}{m^6} - \frac{14}{200} \Lambda^{12} \frac{g^6}{m^9} - \frac{1}{16} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}), \\
W_{\text{Traceful}}^{\text{low}} &= 4\Lambda^3 - \frac{5}{2} \Lambda^6 \frac{g^2}{m^3} - \frac{13}{4} \Lambda^9 \frac{g^4}{m^6} - \frac{65}{8} \Lambda^{12} \frac{g^6}{m^9} - \frac{2147}{80} \Lambda^{15} \frac{g^8}{m^{12}} + \mathcal{O}(\Lambda^{18}).
\end{align}

\textbf{(B.8)}
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