Class of solutions of the Wheeler-DeWitt equation with ordering parameter

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Abstract

In this letter, we discuss the Wheeler-DeWitt equation with an ordering parameter in the Friedmann-Robertson-Walker universe. The solutions when the universe was very small and at the end of the expansion are obtained in terms of Bessel and Heun functions, respectively. We also obtain a boundary condition which should be satisfied by the ordering parameter, namely, \( p \leq 1 \). We investigate the minimum value of the scale factor with respect to the maximum value of the probability density.

Keywords: quantum relativistic cosmology, ordering parameter, wave function, energy density, boundary condition

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1. Introduction

In order to write a quantum cosmology theory for the evolution of the universe, DeWitt \textsuperscript{[1]} and Wheeler \textsuperscript{[2]} proposed an equation inspired on the Hamilton-Jacobi formulation of General Relativity \textsuperscript{[3]}. In this approach, the wave function of the universe can be obtained and as a consequence, it is possible, in principle, to discuss the quantum properties of the spacetime.

This line of research, called quantum relativistic cosmology, has inspired a lot of investigations such as the tunneling wave function of the universe \textsuperscript{[4]}, the interaction with another universe and effects on the cosmic microwave background \textsuperscript{[5]}, as well as some aspects concerning the consistence of the Wheeler-DeWitt equation in a modified theory of gravity \textsuperscript{[6]}, among others.

In the middle 1980’s, Vilenkin \textsuperscript{[7]} analyzed the problem of boundary conditions in quantum cosmology. In his work, the Wheeler-DeWitt equation plays the role of the Schrödinger equation for the wave function of the universe. The evolution from a universe with “nothing” to a large scale one was his great achievement. Based in this approach, He et al. \textsuperscript{[8]} studied the dynamical interpretation of the wave function of the universe by considering the Wheeler-DeWitt equation with an ordering parameter in the Friedmann-Robertson-Walker universe with open geometry.

Recently, we found a class of solutions of the Wheeler-DeWitt equation in a homogeneous and isotropic universe for the spatially closed, flat, and open geometries of the Friedmann-Robertson-Walker universe filled with different forms of energy, namely, matter, radiation, and vacuum \textsuperscript{[9]}. Soon after, we have generalized our analysis and then studied the dynamical interpretation of the Wheeler-DeWitt equation by taking into account all different kinds of energy \textsuperscript{[10]}.

In this work we examine the Wheeler-DeWitt equation with an ordering parameter in a universe filled with some kinds of energy, namely, vacuum, matter, radiation, string network, and quintessence. Furthermore, we do not fix a geometry for the spacetime, that is, the universe can be spatially closed, flat, or open.

This Letter is organized as follows. In Section 2 we present the Wheeler-DeWitt equation with an ordering parameter in the Friedmann-Robertson-Walker universe. In Section 3 we discuss the asymptotic behaviors of the wave function of the universe and obtain a boundary condition which should be satisfied by the ordering parameter. Finally, in Section 4 we close the paper with the final remarks.

2. Wheeler-DeWitt equation with ordering term

We want to discuss some features of the quantum relativistic cosmology. In order to do this, let us establish the Wheeler-DeWitt equation in the minisuperspace approximation as follows.
The classical evolution of the universe at different stages, valid for all forms of energy, can be obtained from the following Hamiltonian

$$H(P_a, a) = \frac{3\pi c^2}{4G} a^3 \left( \frac{4G P_a^2}{9\pi^2 a^4} + \frac{k c^2}{a^2} - \frac{8\pi G}{3c^2} \rho \right),$$

(1)

where $P_a$ is the canonically conjugate momentum, $k = -1, 0, +1$ is the curvature parameter, and $\rho$ is the energy density. The scale factor, $a$, is such that $0 \leq a < \infty$.

Now, in order to write down a most general form of the Wheeler-DeWitt equation with an ordering parameter, let us substitute Eqs. (5) and (6) into

$$P_a^2 \rightarrow -\hbar^2 \frac{\partial}{\partial a} \left( a^p \frac{\partial}{\partial a} \right),$$

(2)

and then impose the constraint $H\Psi(a) = 0$, where $\Psi(a)$ is the wave function of the universe which depends only on the scale factor. The parameter $p$ represents the uncertainty in the ordering of the operator factors $a$ and $a^{-2}$.

Therefore, the Wheeler-DeWitt equation with an ordering parameter, which corresponds to the term with the ordering factor $p$, in the Friedmann-Robertson-Walker universe can be written as

$$\left[ \frac{d^2}{da^2} + \frac{p}{a} \frac{d}{da} - \frac{9\pi^2 \omega^2 a^2}{4\hbar^2 G^2} \left( k c^2 - \frac{8\pi G a^2}{3c^2} \rho \right) \right] \Psi(a) = 0.$$  

(3)

The total energy density $\rho$ can be written as the sum of all kinds of energy, namely, vacuum, matter, radiation, string network and quintessence. It is given by

$$\rho = \sum_{\omega} \rho_\omega = \rho_v + \rho_m + \rho_r + \rho_s + \rho_q.$$  

(4)

The energy density of the vacuum, $\rho_v$, is expressed in terms of the cosmological constant as

$$\rho_v = \frac{\Lambda c^4}{8\pi G}.$$  

(5)

For the other kinds of energy, the energy density $\rho_\omega$ is given by

$$\rho_\omega = \frac{A_\omega}{a^{3(\omega + 1)}},$$  

(6)

where the energy density parameter $A_\omega$ is summarized in Table 1.

In what follows, we will obtain the class of solutions of the Wheeler-DeWitt equation with an ordering parameter, given by Eq. (3).

### 3. Class of solutions

In order to solve the Wheeler-DeWitt equation with an ordering parameter, let us substitute Eqs. (5) and (6) into Eq. (3) and define the wave function as $\Psi(a) = a^{-\frac{p}{2}} y(a)$. Thus, we obtain

$$-\hbar^2 \frac{d^2 y(a)}{da^2} + V_{eff}(a,p)y(a) = 0,$$

(7)

where $V_{eff}(a, p)$ is the effective potential given by

$$V_{eff}(a, p) = \frac{9\pi^2 \omega^2 k^2}{4G^2} a^2 - \frac{6\pi^2 G a^2}{G} \left( A_v + A_m a \right) + A_s a^2 + A_q a^3 + \left( \frac{A c^4}{8\pi G} \right) \frac{3(k - 1)}{8} + \frac{(p^2 - 2)p\hbar^2}{4a^2}.$$  

(8)

At this point we can compare this effective potential with the one that we obtained in Ref. [10]. In fact, the unique difference is the last term proportional to $\frac{1}{a^2}$, which depends on the ordering parameter $p$. This means that the ordering term will contribute only in the case when the scale factor goes to zero, that is, in the beginning of the universe.

For the simple minisuperspace model, Hawking et al. [11] proposed that the differential operator in the Wheeler-DeWitt equation should be the Laplace operator and hence $p = 1$. On the other hand, He et al. [8] analyzed the dynamical interpretation of the wave function of the universe and obtained two boundary conditions which determine the value of the ordering parameter as $p = -2$.

Thus, we will discuss the Wheeler-DeWitt equation for these two values of the ordering parameter and compare with the case when $p = 0$ (or $p = 2$), in which we recover the results known in the literature.

The behaviors of $V_{eff}(a, p)$ with $p = -2, 1$ are shown in Figures 1-8 and 9-12 for positive and negative values of the cosmological constant, respectively.

From Figures 1-8 we can conclude that the term which involves the ordering parameter $p$ will contribute only in the asymptotic region near the origin.

Now, let us solve Eq. (7) in a general way, in a sense that the solution will be valid for all forms of energy density. In order to do this, we can write the Wheeler-DeWitt equation with ordering term in the Friedmann-Robertson-Walker universe as

$$\frac{d^2 y(a)}{da^2} + \left( B_0 + B_0 a + B_2 a^2 + B_3 a^3 \right) y(a) = 0,$$

where $B_0 = \frac{3(k - 1)}{8}$, $B_0 = \frac{9\pi^2 \omega^2 k^2}{4G^2} a^2 - \frac{6\pi^2 G a^2}{G} \left( A_v + A_m a \right) + A_s a^2 + A_q a^3 + \left( \frac{A c^4}{8\pi G} \right) \frac{3(k - 1)}{8} + \frac{(p^2 - 2)p\hbar^2}{4a^2}$.

### Table 1: The energy density parameter $A_\omega = \rho_\omega a^{3(\omega + 1)}$ related to the state parameter $\omega$. Here $\rho_\omega a_0^3$ stands for the value of $\rho_\omega$ at present time, as well as $a_0$ does for $a$.

| Nature of energy | $\omega$ | $A_\omega$ |
|------------------|---------|-----------|
| matter           | 0       | $A_m = \rho_m a_0^3$ |
| radiation        | $\frac{1}{3}$ | $A_r = \rho_r a_0^3$ |
| string network   | $-\frac{1}{4}$ | $A_s = \rho_s a_0^3$ |
| quintessence     | $-\frac{1}{4}$ | $A_q = \rho_q a_0^3$ |
Figure 1: The effective potential $V_{eff}(a,p)$ for $\Lambda > 0$. We focus on the $p = -2,1$ cases and compare with the case where $p = 0$.

Figure 2: The effective potential $V_{eff}(a,p)$ for $\Lambda > 0$. We focus on the $k = -1$ case and compare with each kind of energy density $\omega$.

Figure 3: The effective potential $V_{eff}(a,p)$ for $\Lambda > 0$. We focus on the $k = 0$ case and compare with each kind of energy density $\omega$.

Figure 4: The effective potential $V_{eff}(a,p)$ for $\Lambda > 0$. We focus on the $k = +1$ case and compare with each kind of energy density $\omega$.

\[ + B_4 a^4 + \frac{B_7}{a} \right) y(a) = 0, \]

where the coefficients $B_0$, $B_1$, $B_2$, $B_3$, $B_4$, and $B_5$ are given by

\[ B_0 = \frac{6\pi^3c^2}{h^2G} A_r, \]

\[ B_1 = \frac{6\pi^3c^2}{h^2G} A_m, \]

\[ B_2 = \frac{6\pi^3c^2}{h^2G} A_4 - \frac{9\pi^2c^6}{4h^2G^2} k, \]

\[ B_3 = \frac{6\pi^3c^2}{h^2G} A_q. \]
Figure 5: The effective potential $V_{\text{eff}}(a, p)$ for $\Lambda = -|\Lambda|$. We focus on the $p = -2, 1$ cases and compare with the case where $p = 0$.

Figure 6: The effective potential $V_{\text{eff}}(a, p)$ for $\Lambda = -|\Lambda|$. We focus on the $k = -1$ case and compare with each kind of energy density $\omega$.

Figure 7: The effective potential $V_{\text{eff}}(a, p)$ for $\Lambda = -|\Lambda|$. We focus on the $k = 0$ case and compare with each kind of energy density $\omega$.

Figure 8: The effective potential $V_{\text{eff}}(a, p)$ for $\Lambda = -|\Lambda|$. We focus on the $k = +1$ case and compare with each kind of energy density $\omega$.

No general solution in terms of standard functions is known for this equation over the entire range $0 \leq a < \infty$. However, we may obtain solutions near $a = 0$ and at infinity, as follows.

3.1. Case 1: scale factor very small ($a \to 0$)

By expanding all terms in Eq. (9) we have for small $a$, which means when the universe was very small, the

$$B_4 = \frac{6\pi^3\hbar^3}{8\pi^2 G^2 \Lambda},$$  \hspace{1cm} (14)$$

$$B_5 = \frac{(2p - p^2)}{4}. \hspace{1cm} (15)$$

$$k = -1$$

$$k = +1$$
following equation
\[
\frac{d^2 y(a)}{da^2} + \left( B_0 + \frac{B_3}{a^2} \right) y(a) = 0.
\]
(16)

Therefore, we can write the analytical solutions of Eq. (16) in terms of the Bessel functions as
\[
y(a) = a^\frac{\nu}{2} [i C_1 J_\nu(\sqrt{B_0} a) + C_2 N_\nu(\sqrt{B_0} a)],
\]
(17)

where \( J_\nu \) is the Bessel functions of the first kind, \( N_\nu \) is the Bessel functions of the second kind (Neumann functions), with \( C_1 \) and \( C_2 \) being constants. The parameter \( \nu \) is defined by
\[

\nu = \frac{\sqrt{1 - 4B_3}}{2} = p - \frac{1}{2}.
\]
(18)

On the other hand, the series expansion of the Bessel functions, for small \( x \), are given by
\[

J_\nu(x) \sim \frac{1}{\nu!} \left( \frac{x}{2} \right)^\nu + \ldots,
\]
(19)

\[
N_\nu(x) \sim -\frac{(\nu - 1)!}{\pi} \left( \frac{x}{2} \right)^{-\nu} + \ldots,
\]
(20)

so that as \( a \to 0 \) we have the following asymptotic solution
\[

\Psi(a) \sim C \sqrt{\frac{\pi}{2}} \left[ \frac{1}{\nu!} \left( \frac{\sqrt{B_0}}{2} \right)^\nu - \frac{(\nu - 1)!}{\pi} \left( \frac{\sqrt{B_0}}{2} \right)^{-\nu} a^{1-p} \right].
\]
(21)

Here we have chosen \( C_1 = C_2 = C \sqrt{\frac{\pi}{2}} \), where \( C \) is a constant to be determined.

From this solution we can conclude that the wave function approaches to a constant when \( a \to 0 \), for \( p \leq 1 \). Furthermore, in this case the wave function is divergent for \( p > 1 \). Therefore, from a dynamical interpretation of the wave function of the universe, we have obtained a boundary condition for the ordering parameter, namely, \( p \leq 1 \).

It is worth calling attention to the fact that this asymptotic solution is valid for any value of the curvature parameter \( k = -1, 0, +1 \) and depends on the energy density of the radiation (coefficient \( B_0 \)), which is the predominant form of energy at early stages of the universe. From this point of view, we are extending the solution obtained by He et al. [8], which is valid only for the case of the flat universe \( k = 0 \).

We can also obtain a relationship between the quantity associated to the radiation energy density of the universe in a specific instant, \( B_0 \), and a postulated initial value for the scale factor, \( a_0 \), by considering the maximum probability density at such a value, calculated from \( \partial_a |\Psi(a)|^2 = 0 \) [13]. Taking into account Eq. (17), the value of \( a_0 \) arises thus from the leading terms of the derivative expansion,
\[
a_0 \approx \frac{\pi^{-2/\nu}}{\sqrt{B_0}} \left[ \frac{2^{\nu+2} \cos(\nu \pi \Gamma(\nu + 1)^2)}{C^2[\Gamma(\nu) + 4\Gamma(\nu + 1)]} \right]^{1/\nu} \times [2\Gamma(-\nu)\Gamma(\nu + 1) + \Gamma(-\nu)\Gamma(\nu) - \Gamma(1 - \nu)\Gamma(\nu)]^{1/\nu},
\]
(22)

in the limit for which \( a \ll 1 \). In Figure (9) we depict the minimum scale factor, which corresponds to the maximum probability density near the origin, as a function of the parameter \( \nu \). Notice the local minimum around \( \nu \approx -0.35 \).

3.2. Case 2: large scale factor \( a \to \infty \)

We now expand all terms in Eq. (9) for large \( a \), which means the scale factor at the end of the expansion, and write the equation in the form
\[
\frac{d^2 y(a)}{da^2} + \left( B_0 + B_1 a + B_2 a^2 + B_3 a^3 \right) \frac{B_4 a^5}{10} y(a) = 0.
\]
(23)

Therefore, we can write the analytical solution of Eq. (23) in terms of the Triconfluent Heun function. It is given by
\[
y(x) = e^{-\frac{1}{2}(x^2 + \gamma x)} \times \{ C_1 \ HeunT(\alpha, \beta; \gamma; x) + C_2 \ e^{\gamma x} \ HeunT\left( \lambda^2 \alpha, -\beta, -\lambda \gamma; \frac{x}{\lambda} \right) \}
\]
(24)

where \( \lambda^3 = -1 \). The new variable \( x \) is defined by
\[
x = \tau (a + \xi),
\]
(25)

where the parameters \( \tau \) and \( \xi \) are expressed as
\[
\tau = \left( \frac{4B_4}{9} \right)^{1/6},
\]
(26)

\[
\xi = \frac{B_3}{4B_4}.
\]
(27)

The parameters \( \alpha, \beta, \) and \( \gamma \) are defined by
\[
\alpha = b_0 + \frac{1}{9} b_2,
\]
(28)

\[
\beta = b_1,
\]
(29)
Here we have chosen $C = \frac{C_1}{C_2}$, where $C$ is a constant to be determined.

In this case, we conclude that the wave function of the universe goes to zero when $a \to \infty$, independent of the value of the order parameter. Therefore, as expected from a dynamical point of view, the ordering term does not play a role in the behavior of the universe at the end of the expansion.

4. Summary

This work has illustrated the difficulty of solving the Wheeler-DeWitt equation with an ordering parameter in the Friedmann-Robertson-Walker universe when it is taken into account all different kinds of energy and considering any value of curvature. In the absence of analytical solutions, we examined the solutions only for very small $(a \to 0)$ and very large values of the scale factor $(a \to \infty)$.

We have found the asymptotic behavior of the wave function of the universe in a quantum cosmological scenario. In the cases 1 and 2 therefore, the asymptotic solutions of Eq. (9) near the origin and at infinity are expressed in terms of Bessel and triconfluent Heun functions, respectively. From these results, we obtained a boundary condition for the ordering parameter, namely, $p \leq 1$.

Supposing that the universe has started its existence with a minimum size, we have obtained the correspondent minimum scale factor, $a_0$, which is related to the maximum probability density near the origin $a = 0$, having been expressed as a function of the radiation density energy and of the parameter $\nu = (p - 1)/2 \leq 0$. Though the probabilistic interpretation of the Universe wave function is still controversial, it is admitted by some authors [8]. Furthermore, we have shown that the ordering term does not contribute at the end of the expansion of the universe.

Therefore, the ordering parameter has only a quantum dynamical interpretation at the early stages of the universe.

Finally, the approach developed here leads to a full representation of the solutions at the asymptotic regimes in terms of standard functions in the most general case. It is worth calling attention to the fact that for many quantum calculations such solutions are all that should be required.

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