Solvable Lie algebras of derivations of rank one

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Let $\mathbb{K}$ be a field of characteristic zero and $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation $D$ of $A$ is a $\mathbb{K}$-linear mapping $D: A \to A$ that satisfies the rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. The set $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the polynomial ring $A$ forms a Lie algebra over $\mathbb{K}$. This Lie algebra is simultaneously a free module over $A$ with the standard basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}$. Therefore, for each subalgebra $L$ of $W_n(\mathbb{K})$ one can define the rank $\text{rank}_AL$ of $L$ over the ring $A$. Note that for any $f \in A$ and $D \in W_n(\mathbb{K})$ a derivation $fD$ is defined by the rule: $fD(a) = f \cdot D(a)$ for all $a \in A$.

Finite dimensional subalgebras $L$ of $W_n(\mathbb{K})$ such that $\text{rank}_AL = 1$ were described in [1]. We study solvable subalgebras $L \subseteq W_n(\mathbb{K})$ of rank 1 over $A$ without restrictions on the dimension over the field $\mathbb{K}$.

Recall that a polynomial $f \in A$ is said to be a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $f \neq 0$ and $D(f) = \lambda f$ for some polynomial $\lambda \in A$. The polynomial $\lambda$ is called the polynomial eigenvalue of $f$ for the derivation $D$. Some properties of Darboux polynomials and their applications in the theory of differential equations can be found in [3]. Denote by $A^1_D$ the set of all Darboux polynomials for $D \in W_n(\mathbb{K})$ with the same polynomial eigenvalue $\lambda$ and of the zero polynomial. Obviously, the set $A^1_D$ is a vector space over $\mathbb{K}$. If $V$ is a subspace of $A^1_D$ for any derivation $D \in W_n(\mathbb{K})$, then we denote by $VD$ the set of all derivations $fD$, $f \in V$.

THEOREM 1. Let $L$ be a subalgebra of the Lie algebra $W_n(\mathbb{K})$ of rank 1 over $A$ and $\dim_\mathbb{K} L \geq 2$. The Lie algebra $L$ is abelian if and only if there exist a derivation $D \in W_n(\mathbb{K})$ and a Darboux polynomial $f$ for $D$ with the polynomial eigenvalue $\lambda$ such that $L = VD$ for some $\mathbb{K}$-subspace $V \subseteq A^1_D$.

Using this result one can characterize nonabelian subalgebras of rank 1 over $A$ of the Lie algebra $W_n(\mathbb{K})$. For the Lie algebra $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the field $\mathbb{K}(x_1, x_2, \ldots, x_n)$ this problem is simpler and was considered in [2].

References
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### Classification of quasigroups according to their parastrophic symmetry groups

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Let $Q$ be a set, a mapping $f : Q^3 \to Q$ is called an invertible ternary operation (=function), if it is invertible element in all semigroups $(O_3; \oplus)$, $(O_3; \circ)$ and $(O_3; \odot)$, where $O_3$ is the set of all ternary operations defined on $Q$ and

\[
(f \oplus f_1)(x_1, x_2, x_3) := f(f_1(x_1, x_2, x_3), x_2, x_3), \quad (f \odot f_1)(x_1, x_2, x_3) := f(x_1, f_1(x_1, x_2, x_3), x_3),
\]

\[
(f \odot f_1)(x_1, x_2, x_3) := f(x_1, x_2, f_1(x_1, x_2, x_3)).
\]

The set of all ternary invertible functions is denoted by $\Delta_3$. If an operation $f$ is invertible and $(14)f$, $(24)f$, $(34)f$ are its inverses in those semigroups, then the algebra $(Q; f, (14)f, (24)f, (34)f)$ (in brief, $(Q; f)$) is called a ternary quasigroup [1]. The inverses are also invertible. All inverses to inverses are called $\sigma$-parastrophes of the operation $f$ and can be defined by

\[
\sigma f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \iff f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,
\]

where $S_4$ denotes the group of all bijections of the set $\{0, 1, 2, 3\}$. Therefore in general, every invertible operation has 24 parastrophes. Since parastrophes of a quasigroup satisfy the equalities $\eta(\tau f) = \tau f$, then the symmetric group $S_4$ defines an action on the set $\Delta_3$. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|Ps(f)|$, where $Ps(f)$ denotes a stabilizer group of $f$ under this action which is called *parastrophic symmetry group* of the operation $f$.

Let $\mathfrak{P}(H)$ denote the class of all quasigroups whose parastrophic symmetry group contains the group $H \in S_4$. A ternary quasigroup $(Q; f)$ belongs to $\mathfrak{P}(H)$ if and only if $\tau f = f$ for all $\tau$ from a set $G$ of generators of the group $H$, therefore, the class of quasigroup $\mathfrak{P}(H)$ is a variety.

For every subgroup $H$ of the group $S_4$ the variety $\mathfrak{P}(H)$ are described and its subvariety of ternary group isotopes are found. For example, let

\[D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\} \leq S_4.\]

**Theorem 1.** A ternary quasigroup $(Q; f)$ belong to the variety $\mathfrak{P}(D_8)$ if and only if

\[
f(x, y, z) = f(y, x, z), \quad f(x, y, f(x, y, z)) = z, \quad f(z, f(x, y, z), x) = y.
\]