Finiteness of cohomology of local systems on rigid analytic spaces

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Abstract

We prove that the cohomology groups of an étale $\mathbb{Q}_p$-local system on a smooth proper rigid analytic space are finite-dimensional $\mathbb{Q}_p$-vector spaces, provided that the base field is either a finite extension of $\mathbb{Q}_p$ or an algebraically closed nonarchimedean field containing $\mathbb{Q}_p$. This result manifests as a special case of a more general finiteness result for the higher direct images of a relative $(\varphi, \Gamma)$-module along a smooth proper morphism of rigid analytic spaces over a mixed-characteristic nonarchimedean field.

Throughout this paper, fix a prime number $p$, and let $K$ be a field of characteristic 0 which is complete with respect to a multiplicative absolute value extending the $p$-adic absolute value on $\mathbb{Q}$. We prove the following theorem.

**Theorem 0.1.** Let $X$ be a smooth proper rigid analytic variety over $K$. Let $V$ be an étale $\mathbb{Q}_p$-local system on $X$.

(a) Suppose that $K$ is algebraically closed. Then the (continuous) cohomology groups $H^i(X, V)$ are finite-dimensional $\mathbb{Q}_p$-vector spaces for all $i \geq 0$ and vanish for $i > 2 \text{dim}(X)$; moreover, they remain invariant under base extension from $K$ to a larger algebraically closed field.

(b) Suppose that $K$ is a finite extension of $\mathbb{Q}_p$. Then the cohomology groups $H^i(X, V)$ are finite-dimensional $\mathbb{Q}_p$-vector spaces for all $i \geq 0$ and vanish for $i > 2 \text{dim}(X) + 2$.

In Theorem 0.1 part (b) may be deduced from (a) plus Tate’s local duality theorem, which implies the special case of (b) where $X$ is a point; however, our proof of (b) will not explicitly invoke Tate’s theorem. (See Theorem 8.4 for the proof of (b) and Theorem 10.6 for the proof of (a).)

The case of Theorem 0.1 where $V$ is the trivial local system is a consequence of Scholze’s results on $p$-adic comparison isomorphisms, e.g., see [19, Corollary 1.8]. To a first approximation, the proof of Theorem 0.1 uses the same basic ingredients as in [19], namely the pro-étale topology on $X$, the presence of perfectoid spaces in the pro-étale site, the perfectoid (tilting) correspondence between these spaces and certain spaces of characteristic $p$, and
the Artin-Schreier exact sequence in the geometric setting. However, the approach of [19] is ultimately limited to $\mathbb{Q}_p$-local systems which arise by isogeny from $\mathbb{Z}_p$-local systems; this is sufficient for local systems occurring in algebraic geometry, but in the analytic category one acquires additional local systems that do not admit a global reduction of structure from $\text{GL}_n(\mathbb{Q}_p)$ to $\text{GL}_n(\mathbb{Z}_p)$. The first example of this is the case where $X$ is an elliptic curve with multiplicative reduction, in which case its étale fundamental group admits a representation into $\mathbb{Q}_p^* \times \mathbb{Z}_p$ coming from the Tate uniformization.

To deal with arbitrary $\mathbb{Q}_p$-local systems, we translate the theorem into a statement about relative $(\varphi, \Gamma)$-modules on rigid analytic spaces, as developed in [12, 13]. Briefly put, this amounts to viewing $V$ as a module over the constant sheaf $\mathbb{Q}_p$ on the pro-étale site, then tensoring with a certain $p$-adic period sheaf $\tilde{C}$ equipped with a Frobenius endomorphism $\varphi$, to obtain a locally free sheaf $\mathcal{F}$ of $\tilde{C}$-modules equipped with a semilinear $\varphi$-action (i.e., a relative $(\varphi, \Gamma)$-module). The resulting sheaf $\mathcal{F}$ then sits in an Artin-Schreier sequence

$$0 \to V \to \mathcal{F} \xrightarrow{\varphi^{-1}} \mathcal{F} \to 0,$$

so we may compute the cohomology of $V$ as the hypercohomology of the complex $\mathcal{F} \xrightarrow{\varphi^{-1}} \mathcal{F}$. (This is analogous to Herr’s computation of Galois cohomology in terms of $(\varphi, \Gamma)$-modules [5, 6], except that the role of $\Gamma$ has been absorbed by the pro-étale site.)

In that context, it is natural to state and prove some more general results. For example, in the context of Theorem 0.1 $\mathbb{Q}_p$-local systems give rise to relative $(\varphi, \Gamma)$-modules satisfying a certain local condition (that of being étale), but the proof of finite-dimensionality applies uniformly to arbitrary relative $(\varphi, \Gamma)$-modules; it moreover can be generalized to a relative statement about higher direct images along a smooth proper morphism of rigid spaces (Theorem 8.1). The more general relative $(\varphi, \Gamma)$-modules are of some interest in their own right, in part because they may be interpreted as vector bundles on certain schemes or adic spaces related to the curves in $p$-adic Hodge theory introduced by Fargues and Fontaine [4] and further studied in [12]. In particular, moduli spaces of such bundles are thought to give rise to the local Langlands correspondence in a fashion analogous to that seen in the case of function fields of positive characteristic; this imparts some urgency to the study of cohomology of such spaces.

In addition to constructing higher direct images, we establish their compatibility with base change (Theorem 8.4) and give a relative version of the étale-de Rham comparison isomorphism for rigid analytic spaces constructed by Scholze [19] (Theorem 10.12). Note that the latter applies to $\mathbb{Q}_p$-local systems, whereas the methods of [19] are only capable of handling $\mathbb{Z}_p$-local systems; the latter tend to arise from algebraic-geometric constructions, whereas the former typically arise from genuinely analytic constructions (e.g., the Tate uniformization of an elliptic curve, or the Gross-Hopkins period morphism).

Even with the present work, some properties of relative cohomology of $(\varphi, \Gamma)$-modules remain to be developed, notably cohomology with compact supports and sheaf duality. We defer these topics to a later occasion.
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1 Completely continuous homomorphisms

We begin with a key ingredient for finiteness results: the method of Cartan-Serre for proving finiteness of cohomology, using the Schwartz lemma on completely continuous homomorphisms of Banach spaces. We follow the treatment given by Kiehl in his proof of preservation of coherence along proper morphisms of rigid analytic spaces [15], except that we make some crucial technical modifications to avoid unwanted dependence on noetherian hypotheses (Remark 1.12).

Hypothesis 1.1. Throughout §1 fix a Banach ring $A$ according to the convention of [12, 13], i.e., a commutative ring complete with respect to a submultiplicative nonarchimedean absolute value and containing a topologically nilpotent unit. We work in the category $\text{BMod}_A$ whose objects are Banach modules over $A$ and whose morphisms are bounded $A$-linear homomorphisms; note that these homomorphisms satisfy the open mapping theorem [12, Theorem 2.2.8], so in particular every surjective morphism is strict.

Remark 1.2. If $A$ is noetherian, then any finite $A$-module is a Banach module [12, Remark 2.2.11]; in particular, any morphism in $\text{BMod}_A$ with finitely generated image is strict (by the open mapping theorem). By contrast, if $A$ is not noetherian, every finite $A$-module $M$ has a unique natural topology (induced by any $A$-linear surjection from a finite free module), but need not be a Banach module for this topology; this occurs if and only if $M$ is Hausdorff (by the open mapping theorem; see [13, Remark 1.2.6]). In particular, any finite $A$-submodule of a Banach module over $A$ is itself complete for its natural topology, although not necessarily for the subspace topology.

One important class of finite $A$-modules which are complete for the natural topology is the class of pseudocoherent $A$-modules; see Definition 1.3.

Definition 1.3. Let $f : M \to N$ be a morphism in $\text{BMod}_A$. We say that $f$ is completely continuous if there exists a sequence of finite $A$-submodules $N_i$ of $N$ such that the operator norms of the compositions $M \to N \to N/N_i$, for some fixed norms on $M$ and $N$ and the quotient seminorm on $N/N_i$, converge to 0. (This is slightly weaker than the usual definition; see Remark 1.12.)

A basic fact about completely continuous morphisms is the Schwartz lemma (compare [15, Satz 1.5]). We will need a slightly different result to control cohomology (see Lemma 1.9 below), but giving the proof of this statement provides a useful illustration of technical details to appear later.
Lemma 1.4. Let \( f, g : M \to N \) be morphisms in \( \text{BMod}_A \) such that \( f \) is surjective and \( g \) is completely continuous. Then \( \text{coker}(f + g) \) is a finite \( A \)-module. (Note that this does not immediately imply that \( f + g \) is strict unless \( A \) is noetherian; see Remark 1.5.)

Proof. Choose a finite free \( A \)-module \( U \) admitting an \( A \)-linear morphism \( \pi' : U \to N \) such that \( U \xrightarrow{\pi'} N \to \text{coker}(f) \) is surjective. Fix norms on \( M, N, U \) and a real number \( t > 1 \). By the open mapping theorem, \( f \oplus \pi' : M \oplus U \to N \) admits a bounded set-theoretic section \( s \oplus s' : N \to M \oplus U \); let \( c \) be the operator norm of \( s \). By Remark 1.2, Since \( g \) is completely continuous, we can find a finite \( A \)-submodule \( V \) of \( N \) such that for the quotient seminorm on \( N/V \), the composition of \( g \) with the projection \( \pi'' : N \to N/V \) has operator norm at most \( t^{-2}c^{-1} \). The map \( \pi \) admits a set-theoretic section \( s'' : N/V \to N \) which is bounded of norm at most \( t \).

For \( n \in N \), define \( m_i \in M, u_i \in U, v_i \in V \) for \( i = 0, 1, \ldots \) recursively as follows. First set \( n_0 := n \). Given \( n_i \), put
\[
(m_i, u_i) := (s \oplus s')(n_i), \quad v_i := ((1 - s'' \circ \pi'') \circ g)(m_i), \quad n_{i+1} := (s'' \circ \pi'' \circ g)(m_i).
\]
We then have \( |n_{i+1}| \leq t^{-1}|n_i| \), so the \( n_i \) converge to 0; since all maps in the construction are bounded, the \( m_i, u_i, v_i \) also converge to 0. By construction,
\[
(f + g)(m_i) + \pi(u_i) = n_i + g(m_i) = n_i + v_i - n_{i+1}.
\]
if we sum this relation over \( i \) and set \( m := \sum_{i=0}^{\infty} m_i, u = \sum_{i=0}^{\infty} u_i, v = \sum_{v=0}^{\infty} v_i, \) we have
\[
(f + g)(m) + \pi(u) = n + v.
\]
Moreover, \( v \in N \) belongs to \( V \) because the latter is complete by Remark 1.2. It follows that \( V \to \text{coker}(f + g) \) is surjective, so \( \text{coker}(f + g) \) is a finite \( A \)-module. \( \square \)

Remark 1.5. The usual statement of Lemma 1.4 requires \( f \) to be surjective, rather than allowing the cokernel to be nonzero but finitely generated; this is the level of generality needed in most applications, including ours. In fact, the formally stronger assertion is itself an immediate consequence of the surjective case, as one may replace the original maps \( f, g : M \to N \) by maps \( f \oplus \pi, g \oplus 0 : M \oplus U \to N \) where \( \pi : U \to N \) is an \( A \)-linear morphism from a finite free \( A \)-module \( U \) such that the composition \( U \xrightarrow{\pi} N \to \text{coker}(f) \) is surjective.

It is therefore not an obvious design choice to proceed as we have, by bundling the finite cokernel into the proof of Lemma 1.4. The justification for this choice is to illustrate the fact that the constant \( c \) used to choose the subspace \( V \) depends only on \( s \), not on \( s' \). In the context of Lemma 1.4 this distinction is immaterial, but we use a similar argument to crucial effect in the proof of Lemma 1.9 and it is the latter statement that is needed for cohomological applications via Lemma 1.10.

Remark 1.6. Let \( f : M \to N \) be a morphism in \( \text{BMod}_A \). If \( f \) is completely continuous, then it is obvious that the precomposition of \( f \) with any morphism \( g : M' \to M \) in \( \text{BMod}_A \) is again completely continuous. Conversely, if \( g \) is a (necessarily strict) surjection and \( f \circ g \) is completely continuous, then so is \( f \).
In the other direction, if \( f \) is completely continuous, then the postcomposition of \( f \) with a morphism \( g : N \to N' \) in \( \text{BMod}_A \) is again completely continuous. However, if \( g \) is a strict inclusion and \( g \circ f \) is completely continuous, then it is not obvious that \( f \) is completely continuous unless \( g \) splits in \( \text{BMod}_A \), as otherwise it is not clear how to “project” finite \( A \)-submodules of \( N' \) to finite \( A \)-submodules of \( N \) in a sufficiently uniform way.

**Remark 1.7.** Let \( R \to S \) be a bounded morphism of Banach algebras over \( A \) which is completely continuous as a morphism in \( \text{BMod}_A \). Let \( M \) be a finite Banach module over \( R \) such that \( M \otimes_R S \) is a Banach module over \( S \). For \( F \to M \) an \( R \)-linear surjection from a finite free \( R \)-module, it is obvious that \( F \to F \otimes_R S \) is completely continuous, as then is \( F \to F \otimes_R S \to M \otimes_R S \). Refactoring this composition as \( F \to M \to M \otimes_R S \), we may remove the surjection \( F \to M \) as per Remark 1.6 to deduce that \( M \to M \otimes_R S \) is again completely continuous.

The crucial example of Remark 1.7 in [15] involves the following class of morphisms.

**Example 1.8.** For \( i = 1, \ldots, n \), choose real numbers \( r_i, r'_i, s_i, s'_i \) such that \( 0 < s_i < s'_i \leq r'_i < r_i \). Then the natural inclusions

\[
A\{T_1/r_1, \ldots, T_n/r_n\} \to A\{T_1/r'_1, \ldots, T_n/r'_n\}
\]

\[
A\{s_1/T_1, T_1/r_1, \ldots, s_n/T_n, T_n/r_n\} \to A\{s'_1/T_1, T_1/r'_1, \ldots, s'_n/T_n, T_n/r'_n\}
\]

are strictly completely continuous, taking the submodules generated by the initial segments of an enumeration of monomials.

In light of the previous remark, we formulate the following variant of Lemma 1.4.

**Lemma 1.9.** Let \( f, g : M \to N, h : N \to N' \) be morphisms in \( \text{BMod}_A \) such that \( \text{coker}(f) \) is a finite \( A \)-module, \( h \circ g \) is completely continuous, and \( h \) is a strict inclusion. Then \( \text{coker}(f + g) \) is contained in a finite \( A \)-module.

**Proof.** To simplify notation slightly, we proceed as in Remark 1.6 to formally reduce to the case where \( f \) is surjective. Fix norms on \( M, N, N' \) and a real number \( t > 1 \). By the open mapping theorem, \( f \) admits a bounded set-theoretic section \( s : N \to M \); let \( c \) be the operator norm of \( s \). By Remark 1.2 Since \( h \circ g \) is completely continuous, we can find a finite \( A \)-submodule \( V \) of \( N' \) such that for the quotient seminorm on \( N'/V \), the composition of \( h \circ g \) with the projection \( \pi'' : N' \to N'/V \) has operator norm at most \( t^{-3}c^{-1} \). The map \( \pi'' \) admits a set-theoretic section \( s'' : N'/V \to N' \) which is bounded of norm at most \( t \).

Since \( V \) is a finite \( A \)-module, its image in \( N'/N \) is again a finite \( A \)-module, and hence a Banach module (Remark 1.2); let \( N'' \) be the inverse image of this module in \( N' \), which is then also a Banach module. Choose a finite free \( A \)-module \( U \) and an \( A \)-linear morphism \( \pi : U \to N'' \) such that \( U \xrightarrow{\pi} N'' \to N'/N \) is surjective. Choose a set-theoretic section \( s'' \) of the projection \( \pi'' : N'' \to N''/N \) of operator norm at most \( t \) taking 0 to 0. By composing \( 1-s'' \circ \pi'' : N'' \to N \) with the previously chosen section \( s : N \to M \), we obtain a set-theoretic map \( s : N'' \to M \) of operator norm at most \( ct \) extending \( s \), which can be complemented to a
bounded set-theoretic section \( s \oplus s' : N'' \to M \oplus U \) of the surjective map \( f \oplus \pi \). (We cannot control the norm of \( s' \), but as per Remark 1.13 this will have no effect on the sequel.)

We now emulate the proof of Lemma 1.4. For \( n \in N'' \), define \( m_i \in M, u_i \in U, v_i \in V \) for \( i = 0, 1, \ldots \) recursively as follows. First set \( n_0 := n \). Given \( n_i \), put

\[
(m_i, u_i) := (s \oplus s')(n_i), \quad v_i := ((1 - s'' \circ \pi'') \circ g)(m_i), \quad n_{i+1} := (s'' \circ \pi'' \circ g)(m_i).
\]

We then have \( |n_{i+1}| \leq t^{-1} |n_i| \), so the \( n_i \) converge to 0; since all maps in the construction are bounded, the \( m_i, u_i, v_i \) also converge to 0. By construction,

\[
(f + g)(m_i) + \pi(u_i) = n_i + g(m_i) = n_i + v_i - n_{i+1};
\]

if we sum this relation over \( i \) and set \( m := \sum_{i=0}^{\infty} m_i, u = \sum_{i=0}^{\infty} u_i, \) we have

\[
(f + g)(m) + \pi(u) = n + v.
\]

Moreover, \( v \in N' \) belongs to \( V \) because the latter is complete by Remark 1.2. It follows that the map \( U \oplus V \to \pi(U) \oplus V \to \text{coker}(f + g : M \to N'') \) is surjective, and hence \( \text{coker}(f + g : M \to N'') \) is a finite \( A \)-module. From the exact sequence

\[
0 \to \text{coker}(f + g : M \to N) \to \text{coker}(f + g : M \to N'') \to N''/N \to 0,
\]

we deduce the claim. \( \square \)

The following consequence of Lemma 1.9 is essentially 13, Satz 2.5, except with weaker hypotheses (see Remark 1.12).

**Lemma 1.10** (after Cartan-Serre). Let \( f^\bullet : C_1^\bullet \to C_2^\bullet \) be a morphism of complexes in \( \text{BMod}_A \). Suppose that for some \( i \), the following conditions hold.

(a) The morphism \( f^i \) is completely continuous.

(b) The map \( H^i(C_1) \to H^i(C_2) \) induced by \( f^i \) has cokernel which is a finite \( A \)-module.

Then the group \( H^i(C_2) \) is contained in a finite \( A \)-module. In particular, if \( f^\bullet \) is a quasi-isomorphism, then \( H^i(C_1) \) is contained in a finite \( A \)-module.

**Proof.** Put \( Z_j^i := \ker(d_j^i : C_j^i \to C_j^{i+1}) \). By (a) and (b) and Remark 1.6 the map

\[
Z_j^i \oplus C_j^{i-1} \to Z_j^i, \quad (a, b) \mapsto -f^i(a)
\]

is the composition of a completely continuous morphism with a strict inclusion, while the map

\[
Z_j^i \oplus C_j^{i-1} \to Z_j^i, \quad (a, b) \mapsto f^i(a) + d_j^{i-1}(b)
\]

has cokernel which is a finite \( A \)-module. We may identify \( H^i(C_2) \) with the cokernel of the map

\[
Z_j^i \oplus C_j^{i-1} \to Z_j^i, \quad (a, b) \mapsto d_j^{i-1}(b),
\]

and by Lemma 1.9 this cokernel is contained in a finite \( A \)-module. This proves the claim. \( \square \)
Remark 1.11. Our initial applications of Lemma 1.10 will occur in the following context: we have a bounded homomorphism $R \to S$ of Banach algebras over $A$ which is completely continuous in $\text{BMod}_A$ and a complex $C^\ast_1$ of finite Banach modules over $R$ such that each module $C^\ast_2 = C^\ast_1 \otimes_R S$ is complete and the induced map $C^\ast_1 \to C^\ast_2$ is a quasi-isomorphism. In this setting, condition (a) of Lemma 1.10 will be satisfied thanks to Remark 1.7.

Our core application of Lemma 1.10 will be in a somewhat more complicated setting. See Remark 2.8.

Remark 1.12. We say that the morphism $f : M \to N$ in $\text{BMod}_A$ is completely continuous in the sense of Kiehl (i.e., vollständig stetig in the sense of [15, Definition 1.1]) if it can be uniformly approximated by a sequence of morphisms $f_i : M \to N$ whose images are finite $A$-modules. This implies that $f$ is completely continuous in the sense of Definition 1.3 by taking the submodules of $N$ to be the images of the $f_i$. The converse implication is not clear in general, but it does hold when $A$ is a nonarchimedean field: the projection $\pi_i : N \to N/N_i$ admits an $A$-linear section $s$ whose operator norm is bounded uniformly in $i$ (e.g., because [9, Lemma 1.3.7] allows the operator norm to be made arbitrarily close to 1), and the sequence of maps $f_i = (1 - s \circ \pi_i) \circ f$ then has the desired effect. It also holds if $M$ is topologically projective, i.e., it is a direct summand of the completion of a free $A$-module with respect to the supremum norm; in this case, one may construct the maps $f_i$ by assigning images to topological generators. The latter observation has the net effect of minimizing any substantive difference between our definition and Kiehl’s definition, and also explains why various intermediate results in [15] require precomposition with a surjection from a topologically free module.

However, there is no exact analogue in [15] of Lemma 1.9. Instead, Kiehl formulates in [15, Definition 1.1] a stronger version of the complete continuity condition, defining what it means for a morphism $f$ to be strictly completely continuous (streng vollständig stetig) by imposing a certain uniformity condition on the maps $f_i$. The purpose of this stronger condition is to make it possible to deduce strict complete continuity after postcomposition with a strict inclusion. Unfortunately, Kiehl is only able to achieve this implication in case $A$ is an affinoid algebra over a nonarchimedean field [15, Satz 1.4], because the argument requires a very strong noetherian property which is only verified in this case [15, Satz 5.1]: for $A_0$ a ring of definition of $A$, every $A_0$-submodule of every finite $A_0$-module is almost finitely generated in the sense of almost ring theory. While it is reasonable to expect this for other strongly noetherian Banach rings, such as the coordinate rings of affinoid subsets of Fargues-Fontaine curves [10], it is immaterial for our purposes: if $A$ is noetherian, then the conclusion of Lemma 1.10 promotes to the statement that $H^i(C_2)$ is a finite $A$-module. Moreover, in our application of Lemma 1.10 to the proof of Proposition 6.1 the base ring $A$ will not be noetherian; this means that we will have to use extra structure on cohomology, plus some deep finiteness results from [13], in order to promote the conclusion (that cohomology groups are contained in finite $A$-modules) to stronger finite generation assertions.

In any case, we leave it as an easy exercise to recover the main theorem of [15], with an arguably simpler proof, using the lemmas stated above.
2 Totalizations

As much of this paper is concerned with matters of homological algebra, it is best to review some basic formalism in order to fix notation and conventions, especially with regard to converting multidimensional complexes into single complexes; this allows us to state a crucial formal promotion of Lemma 1.10. Throughout §2 fix an exact additive category $\mathcal{A}$.

**Definition 2.1.** Let $C^+(\mathcal{A})$ be the category of bounded below complexes in $\mathcal{A}$ with cohomological indexing, so that for $C \in C^+(\mathcal{A})$, the differential $d^i_C$ maps from $C^i$ to $C^{i+1}$; recall that a morphism $f^\bullet : C_1 \to C_2$ in $C^+(\mathcal{A})$ consists of the vertical arrows in a commutative diagram of the form

$$
\cdots \longrightarrow C_{i-1}^i \xrightarrow{d_{C_1}^{i-1}} C_i^i \xrightarrow{d_{C_1}^i} C_{i+1}^i \longrightarrow \cdots \\
\cdots \longrightarrow C_{i-1}^{i-1} \xrightarrow{d_{C_2}^{i-1}} C_i^{i-1} \xrightarrow{d_{C_2}^i} C_{i+1}^{i-1} \longrightarrow \cdots
$$

Such a morphism is null-homotopic if there exist maps $h^i : C_i^i \to C_i^{i-1}$ such that

$$f^i = d_{C_2}^{i-1} \circ h^i + h^{i+1} \circ d_{C_1}^i; \quad (2.1.1)$$

this implies that $f$ induces zero maps $h^\bullet(C_1) \to h^\bullet(C_2)$ of cohomology groups. (Note that (2.1.1) alone implies that $f$ is a morphism in $C^+(\mathcal{A})$, i.e., that the diagram commutes.) One checks that the morphisms which are null-homotopic form an ideal under composition; quotienting the morphism spaces by this ideal yields the homotopy category $K^+(\mathcal{A})$.

**Definition 2.2.** For $f : C_1 \to C_2$ a morphism in $C^+(\mathcal{A})$, the mapping cone is the object $\text{Cone}(f) \in C^+(\mathcal{A})$ such that $\text{Cone}(f)^i = C_i^1 \oplus C_i^{i-1}$ and

$$d_{\text{Cone}(f)}^i(x, y) = (d_{C_1}^i(x), d_{C_2}^{i-1}(y) + (-1)^i f^i(x));$$

there are obvious morphisms $C_2[-1] \to \text{Cone}(f) \to C_1$ which compose to zero (and close up into a distinguished triangle). The class of $\text{Cone}(f)$ in $K^+(\mathcal{A})$ is functorially determined by the class of $f$ in $K^+(\mathcal{A})$.

Suppose that $g : C_0 \to C_1$ is a second morphism in $C^+(\mathcal{A})$ such that the composition $C_0 \to C_1 \to C_2$ is null-homotopic as witnessed by $h^i : C_0^i \to C_2^{i-1}$. We may then construct a morphism $C_0 \to \text{Cone}(f)$ in $C^+(\mathcal{A})$ mapping $C_0^i$ to $C_1^i \oplus C_2^{i-1}$ via $g^i \oplus -h^i$. The class of $C_0 \to \text{Cone}(f)$ in $K^+(\mathcal{A})$ is functorially determined by the class of $g$ in $K^+(\mathcal{A})$.

**Remark 2.3.** The second observation in Definition 2.2 may be restated as follows. Let

$$
\begin{array}{ccc}
C_1 & \xrightarrow{f} & C_2 \\
| e_1 \downarrow & & \downarrow e_2 \\
D_1 & \xrightarrow{g} & D_2
\end{array}
$$
be a diagram in $C^+(\mathcal{A})$ which commutes in $K^+(\mathcal{A})$, and choose a homotopy $h^\bullet : C^i_1 \to D^i_2$ witnessing the vanishing of $e_2 \circ f - g \circ e_1$ in $K^+(\mathcal{A})$. Then the induced morphism $\text{Cone}(f) \to \text{Cone}(g)$ can be represented by a morphism of complexes whose component $C^i_1 \oplus C^i_2 \to D^i_1 \oplus D^i_2$ acts as

$$(a, b) \mapsto (e^i_1(a), e^i_2(b) + (-1)^{i+1}h^i(a)).$$

In particular, this map depends on $h$.

This observation has the following consequence in our setting. Take $\mathcal{A} = \text{BMod}_A$ for $A$ a Banach ring. If $e_1, e_2$ consist of completely continuous morphism, it does not follow that the resulting map $\text{Cone}(f) \to \text{Cone}(g)$ has the same property unless we can ensure that the homotopy $h$ is itself completely continuous. The easiest way to achieve this is to ensure that in fact $h = 0$, which is to say the original diagram commutes already in $C^+(\mathcal{A})$; fortunately, this is what will ultimately happen in the case of interest.

**Definition 2.4.** Let $C_\bullet := C_0 \to \cdots \to C_n$ be a finite sequence in $C^+(\mathcal{A})$ that represents a complex in $K^+(\mathcal{A})$; that is, the compositions $C_{i-1} \to C_i \to C_{i+1}$ are null-homotopic for all $i$. We now repeat the following operation: take the last two terms $C_{n-1} \to C_n$ and replace them by the single term $\text{Cone}(C_{n-1} \to C_n)$. After $n$ steps, we end up with an element of $C^+(\mathcal{A})$ whose class in $K^+(\mathcal{A})$ is uniquely determined by the image of $C_\bullet$ in the category of bounded complexes in $K^+(\mathcal{A})$. We call this element the **totalization** of $C_\bullet$, and denote it by $\text{Tot}(C_\bullet)$.

**Remark 2.5.** In the case that the sequence $C_\bullet$ is indeed a complex in $C^+(\mathcal{A})$ (i.e., the compositions $C_{i-1} \to C_i \to C_{i+1}$ are actually zero, not merely null-homotopic), the totalization of $C_\bullet$ is represented by the total complex of the double complex $C^n,j := C^n_j$.

**Remark 2.6.** For $i = 1, 2$, let $C_{i, \bullet} := C_{i,0} \to \cdots \to C_{i,n}$ be a sequence in $C^+(\mathcal{A})$ representing a complex in $K^+(\mathcal{A})$. Let $f_\bullet : C_{1, \bullet} \to C_{2, \bullet}$ be a family of morphisms in $C^+(\mathcal{A})$ such that for each $j$, the diagram

$$
\begin{array}{c}
C_{1,j-1} \longrightarrow \rightarrow C_{1,j} \longrightarrow \rightarrow C_{1,j+1} \\
\downarrow f_{j-1} \quad \downarrow f_j \quad \downarrow f_{j+1} \\
C_{2,j-1} \longrightarrow \rightarrow C_{2,j} \longrightarrow \rightarrow C_{2,j+1}
\end{array}
$$

commutes in $C^+(\mathcal{A})$, not only in $K^+(\mathcal{A})$. Moreover, suppose that one can choose homotopies witnessing that each row is a complex in $K^+(\mathcal{A})$ which themselves form commutative diagrams with the $f_j$. It is straightforward to see that the family $f_\bullet$ naturally induces a morphism $\text{Tot}(f_\bullet) : \text{Tot}(C_{1, \bullet}) \to \text{Tot}(C_{2, \bullet})$ in $C^+(\mathcal{A})$.

Unfortunately, it is unclear how to formulate an analogue of Lemma 1.10 which up to full homotopy equivalence. Instead, we give a more rigid statement which provides just enough flexibility for our purposes.

**Lemma 2.7.** Let $A$ be a Banach ring and let $n$ be a nonnegative integer. For $i = 1, 2$, let $C_{i, \bullet} := C_{i,0} \to \cdots \to C_{i,n}$ be a sequence in $C^+(\text{BMod}_A)$ representing a complex in
$K^+(\text{BMod}_A)$. Let $f_\bullet : C_1 \rightarrow C_2$ be a family of morphisms in $C^+(\text{BMod}_A)$. For each $0 \leq j \leq n$, let $\alpha_{i,j} : D_{i,j} \rightarrow C_{i,j}$, $\beta_{i,j} : C_{i,j} \rightarrow D_{i,j}$ be morphisms in $C^+(\text{BMod}_A)$ which are inverses to each other in $K^+(\text{BMod}_A)$. Suppose that the following conditions are satisfied.

(a) For each $j$, the diagram

$$
\begin{array}{ccc}
C_{1,j-1} & \rightarrow & C_{1,j} & \rightarrow & C_{1,j+1} \\
\downarrow f_{j-1} & & \downarrow f_j & & \downarrow f_{j+1} \\
C_{2,j-1} & \rightarrow & C_{2,j} & \rightarrow & C_{2,j+1}
\end{array}
$$

commutes in $C^+(\text{BMod}_A)$. Moreover, one can choose homotopies witnessing that each row is a complex in $K^+(\text{BMod}_A)$ which themselves form commutative diagrams with the $f_j$.

(b) For each $j$, there exists a morphism $g_j : D_{1,j} \rightarrow D_{2,j}$ in $C^+(\text{BMod}_A)$ consisting of completely continuous morphisms in $\text{BMod}_A$ such that the diagram

$$
\begin{array}{ccc}
D_{1,j} & \rightarrow & C_{1,j} & \rightarrow & C_{1,j} \\
\downarrow g_j & & \downarrow \beta_{1,j} & & \downarrow \alpha_{1,j} \\
D_{2,j} & \rightarrow & C_{2,j} & \rightarrow & C_{2,j}
\end{array}
$$

commutes in $C^+(\text{BMod}_A)$. Moreover, one can choose homotopies witnessing that each row is a complex in $K^+(\text{BMod}_A)$ which themselves form commutative diagrams with the $f_j$ and the $g_j$.

(c) The morphism $\text{Tot}(f_\bullet) : \text{Tot}(C_1) \rightarrow \text{Tot}(C_2)$ is a quasi-isomorphism.

Then each cohomology group $h^i(\text{Tot}(C_1)) \cong h^i(\text{Tot}(C_2))$ is contained in some finite $A$-module.

Proof. For $0 \leq j \leq n$, we will construct by descending induction a quasi-isomorphism

$$
T_{i,j} : \text{Tot}(C_{i,j} \rightarrow \cdots \rightarrow C_{i,n}) \rightarrow T(D)_{i,j}
$$

in $C^+(\text{BMod}_A)$ as follows. For $j = n$, put $T(D)_{i,n} = D_{i,n}$ and $T_{i,n} = \beta_{i,n}$. Suppose we have constructed $T_{i,j}$. Consider the diagram

$$
\begin{array}{ccc}
C_{i,j-1} & \rightarrow & \text{Tot}(C_{i,j} \rightarrow \cdots \rightarrow C_{i,n}) \\
\downarrow \beta_{i,j-1} & & \downarrow T_{i,j} \\
D_{i,j-1} & \rightarrow & T(D)_{i,j}
\end{array}
$$

where $D_{i,j-1} \rightarrow T(D)_{i,j}$ is given by the composition

$$
D_{i,j-1} \overset{\alpha_{i,j-1}}{\rightarrow} C_{i,j-1} \overset{T_{i,j}}{\rightarrow} \text{Tot}(C_{i,j} \rightarrow \cdots \rightarrow C_{i,n}) \rightarrow T(D)_{i,j}.
$$
Since \( \alpha_{i,j-1} \) and \( \beta_{i,j-1} \) are inverse to each other in \( K^+(\text{BMod}_A) \), this diagram is commutative in \( K^+(\text{BMod}_A) \). We set

\[
T(D)_{i,j-1} = \text{Cone}(D_{i,j-1} \to T(D)_{i,j})
\]

and \( T_{i,j-1} \) to be the morphism

\[
\text{Tot}(C_{i,j-1} \to \cdots \to C_{i,n}) = \text{Cone}(C_{i,j-1} \to \text{Tot}(C_{i,j} \to \cdots \to C_{i,n})) \to T(D)_{i,j-1}.
\]

Since \( \beta_{i,j-1} \) and \( T_{i,j} \) are quasi-isomorphisms, \( T_{i,j-1} \) is a quasi-isomorphism as well. Note that by (b), the family \( g_\bullet \) induces a morphism \( T(D)_{1,j} \to T(D)_{2,j} \) making the diagram

\[
\begin{array}{ccc}
T(D)_{1,j} & \longrightarrow & \text{Tot}(C_{1,j} \to \cdots \to C_{1,n}) \\
\downarrow & & \downarrow \\
T(D)_{2,j} & \longrightarrow & \text{Tot}(C_{2,j} \to \cdots \to C_{2,n})
\end{array}
\]

commutative in \( C^+(\text{BMod}_A) \). By (c), we thus get a quasi-isomorphism \( T(D)_{1,0} \to T(D)_{2,0} \), to which we may directly apply Lemma 1.10. \( \square \)

Remark 2.8. Lemma 2.7 is required because the Proposition 6.1 involves a more complicated arrangement than the one described in Remark 1.11: we cannot represent the desired comparison of cohomology using a single completely continuous morphism in \( C^+(\text{BMod}_A) \). Rather, what arises most naturally is a sequence of (pairs of) complexes, each derived as Remark 1.11 for a different ring homomorphism, which fit together in \( K^+(\text{BMod}_A) \) but not in \( C^+(\text{BMod}_A) \). Moreover, the comparison maps between the individual complexes are not quasi-isomorphisms; this only occurs for the total complexes. The commutativity conditions will arise from the construction of the comparison maps as certain base extensions.

3 Cohomology of \((\varphi, \Gamma)\)-modules

As an example of the preceding discussion, we rederive some existing finiteness results for the cohomology of \((\varphi, \Gamma)\)-modules in classical (nonrelative) \( p \)-adic Hodge theory.

Definition 3.1. Let \( A_{\mathbb{Q}_p} \) be the ring of formal sums \( \sum_{n \in \mathbb{Z}} c_n \pi^n \) with \( c_n \in \mathbb{Z}_p \) such that \( c_n \to 0 \) as \( n \to -\infty \). For \( \gamma \in \mathbb{Z}_p^\times \), this ring admits commuting endomorphisms \( \varphi, \gamma \) given by the formulas

\[
\varphi\left(\sum_n c_n \pi^n\right) = \sum_n c_n((1 + \pi)^p - 1)^n, \quad \gamma\left(\sum_n c_n \pi^n\right) = \sum_n c_n((1 + \pi)^\gamma - 1)^n. \quad (3.1.1)
\]

A \((\varphi, \Gamma)\)-module over \( A_{\mathbb{Q}_p} \) is a finite projective module over \( A_{\mathbb{Q}_p} \) equipped with commuting semilinear actions of \( \varphi \) and \( \Gamma \); if we allow the module to be finitely generated but not necessarily projective, we obtain a generalized \((\varphi, \Gamma)\)-module over \( A_{\mathbb{Q}_p} \). We define the cohomology
groups $H^i_{\varphi, \Gamma}(M)$ of a generalized $(\varphi, \Gamma)$-module $M$ to be the total cohomology of the double complex
\[ C_{\varphi, \Gamma}(M) : 0 \to C^i_{\text{cont}}(\Gamma, M) \xrightarrow{\varphi-1} C^i_{\text{cont}}(\Gamma, M) \to 0, \]
where $C^i_{\text{cont}}(\Gamma, M)$ is the complex of continuous group cochains.

**Lemma 3.2.** Let $R$ be a $p$-adically separated and complete ring. Let $A$ be an abelian subcategory of the category of $p$-adically separated and complete $R$-modules such that for any $M \in A$ which is finitely generated as an $R$-module, $\text{Tor}_1^R(M, R/pR)$ is a finitely generated $R/pR$-module. Now suppose that $C : 0 \to C^0 \to \cdots \to C^n \to 0$ is a bounded complex in $A$ such that the $R/pR$-modules $H^n(C \otimes_R R/pR)$ are finitely generated. Then the $R$-modules $H^n(C^\bullet)$ are themselves finitely generated.

**Proof.** We proceed by induction on $n$. Since base extension of modules is right exact, we have $H^n(C^\bullet \otimes_R R/p^mR) = H^n(C^\bullet) \otimes_R R/p^mR$; in particular, for $m_1 \leq m_2$, we have
\[ H^n(C^\bullet \otimes_R R/p^{m_2}R) \otimes_{R/p^{m_2}R} R/p^{m_1}R \cong H^n(C^\bullet \otimes_R R/p^{m_1}R). \]

By hypothesis, $H^n(C^\bullet \otimes_R R/pR)$ is a finitely generated $R/pR$-module; we may thus choose a finitely generated $R$-submodule $S$ of $C^n$ which surjects onto $H^n(C^\bullet \otimes_R R/pR)$. Now
\[ C^{n-1} \oplus S \to C^n \]
is a map between $p$-adically separated and complete $R$-modules which is surjective modulo $p$, hence it is surjective; that is, $S \to H^n(C^\bullet)$ is surjective, so $H^n(C^\bullet)$ is finitely generated.

Now put $C^{n-1} = \ker(C^n \to C^n)$; the truncated complex
\[ C' : 0 \to C^0 \to \cdots \to C^{n-2} \to C^{n-1} \to 0 \]
is again a complex in $A$. The $R/pR$-modules $H^n(C^\bullet \otimes_R R/pR)$ are again finitely generated: namely, this is obvious except for the final group
\[ H^{n-1}(C^\bullet \otimes_R R/pR) = \text{coker}(C^{n-2}/pC^{n-2} \to C^{n-1}/pC^{n-1}), \]
which fits into a right exact sequence
\[ \text{Tor}_1^R(H^n(C^\bullet), R/pR) \to H^{n-1}(C^\bullet \otimes_R R/pR) \to H^{n-1}(C^\bullet \otimes_R R/pR) \to 0. \]

By assumption we have that $\text{Tor}_1^R(H^n(C^\bullet), R/pR)$ is a finitely generated $R/pR$-module. This yields that $H^{n-1}(C^\bullet \otimes_R R/pR)$ is a finitely generated $R/pR$-module as well. We may thus apply the induction hypothesis to $C'$ to complete the proof.

The following result was proved for $p > 2$ by Herr [5, Théorème B] and in full by the second author [16, Theorem 3.3]; here, we derive it as an application of the Cartan-Serre method.

**Theorem 3.3 (Herr, Liu).** Let $M$ be a generalized $(\varphi, \Gamma)$-module over $A_{\mathbb{Q}_p}$. Then the groups $H^i_{\varphi, \Gamma}(M)$ are finite $\mathbb{Z}_p$-modules for $i = 0, 1, 2$ and zero for $i > 2$. 

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Proof. Write $\Gamma$ as the product of a finite subgroup $\Delta$ and a subgroup $\Gamma'$ admitting a topological generator $\gamma$. Then $C^\bullet_{\text{cont}}(\Gamma, M)$ is quasi-isomorphic to the complex

$$0 \rightarrow M^\Delta \xrightarrow{-1} M^\Delta \rightarrow 0$$

so its cohomology vanishes outside of degrees 0 and 1; this implies that $H^i_{\varphi, \Gamma}(M) = 0$ for $i > 2$.

To prove that the groups $H^i_{\varphi, \Gamma}(M)$ are finite $\mathbb{Z}_p$-modules for $i = 0, 1, 2$, by Lemma 3.2 we may reduce to the case where $M$ is killed by $p$, and hence finite free over $\mathcal{A}_{\mathbb{Q}_p}/(p) \cong \mathbb{Q}_p((\pi))$. Fix a basis of $M$. For $0 \leq s \leq r < 1$, note that $\mathbb{Q}_p((\pi))$ is complete with respect to the norm $|\cdot|_{s, r}$ given by

$$|c_n \pi^n|_{s, r} = \max\{\max\{r^n, s^n\} : n \in \mathbb{Z}, c_n \neq 0\},$$

so $M$ is complete with respect to the supremum norm $|\cdot|_{s, r, M}$. For any $s, r$, the morphisms in $C^\bullet_{\text{cont}}(\Gamma, M)$ are bounded with respect to $|\cdot|_{s, r, M}$; if $s \leq r^p$, the morphism $\varphi - 1$ is bounded with respect to $|\cdot|_{s, r^p, M}$ on the source and $|\cdot|_{s, r, M}$ on the target. Let $C_{\varphi, \Gamma}(M)_{s, r}$ be the double complex $C_{\varphi, \Gamma}(M)$ equipped with norms in this manner.

Now choose $r, s, r', s' \in (0, 1)$ with $s \leq r^p$, $s' < s$, and $r < r'$. Then

$$C_{\varphi, \Gamma}(M)_{s', r'} \rightarrow C_{\varphi, \Gamma}(M)_{s, r}$$

is a morphism of double complexes which is a quasi-isomorphism (since it is the identity map on underlying modules) in which each individual map is completely continuous. We would then be in the situation where the desired finiteness would follow from Lemma 3.10, except that the base ring $\mathbb{F}_p$ does not contain a topologically nilpotent unit. However, we may get around this issue using the method of [11, Lemma 8.9]: extend base from $\mathbb{F}_p$ to $\mathbb{F}_p((z))$, apply Lemma 3.10 (keeping in mind Remark 1.11), then project onto the coefficient of $z^0$. We thus recover the desired finiteness.

Definition 3.4. Let $C_{\mathbb{Q}_p}$ be the union of the rings of rigid analytic functions on the discs $c < |\pi| < 1$ over all $c \in (0, 1)$ (commonly known as the Robba ring over $\mathbb{Q}_p$); this ring carries a natural locally convex topology as the inductive limit of Fréchet spaces. The equations (3.1.1) again define commuting operators $\varphi, \gamma$ on $C_{\mathbb{Q}_p}$ for $\gamma \in \Gamma$. (The notation is in the style of [12, 13]; another common label is $\mathcal{B}_{\text{rig}}^{\psi}(\mathbb{Q}_p)$.)

A $(\varphi, \Gamma)$-module over $C_{\mathbb{Q}_p}$ is a finite projective module over $C_{\mathbb{Q}_p}$ equipped with commuting semilinear actions of $\varphi$ and $\Gamma$, with the action of $\Gamma$ being continuous; if we allow the module to be finitely presented but not necessarily projective, we obtain a generalized $(\varphi, \Gamma)$-module over $C_{\mathbb{Q}_p}$. We define cohomology groups as before.

The following statement is due to the second author [16], but again we prefer to prove it as an illustration of the Cartan-Serre method.

Theorem 3.5 (Liu). Let $M$ be a generalized $(\varphi, \Gamma)$-module over $C_{\mathbb{Q}_p}$. Then the groups $H^i_{\varphi, \Gamma}(M)$ are finite $\mathbb{Q}_p$-modules for $i = 0, 1, 2$ and zero for $i > 2$. 

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Proof. As in the proof of Theorem 3.3, we see that $H^i_{\varphi,\Gamma}(M) = 0$ for $i > 2$. To prove that the groups $H^i_{\varphi,\Gamma}(M)$ are finite $\mathbb{Q}_p$-modules for $i = 0, 1, 2$, for any interval $I$, let $A(I)$ denote the rigid analytic annulus $|\pi| \in I$ over $\mathbb{Q}_p$, omitting the parentheses when $I$ is written as an explicit interval with endpoints. For some $c_1 > 0$, we may then realize $M$ as a module $M_{[c_1,1]}$ over $\mathcal{O}(A[c_1,1])$ equipped with an action of $\Gamma$ and a compatible isomorphism

$$\varphi^* M_{[c_1,1]} \cong M_{[c_1,1]} \otimes \mathcal{O}(A[c_1,1]) \mathcal{O}(A[c_1^{1/p},1]).$$

For $d_1 \in [c_1^{1/p}, 1)$, put

$$M_{[c_1,d_1]} = M_{[c_1,1]} \otimes \mathcal{O}(A[c_1,1]) \mathcal{O}(A[c_1,d_1]);$$

by [13, Theorem 5.7.11], the cohomology of $C_{\varphi,\Gamma}(M)$ is also computed by the double complex

$$0 \to C^\bullet_{\text{cont}}(\Gamma, M_{[c_1,d_1]}) \xrightarrow{\varphi^{-1}} C^\bullet_{\text{cont}}(\Gamma, M_{[c_1^{1/p},d_1]}) \to 0.$$

We may thus deduce the claim by computing the cohomology using two intervals $[c_1, d_1], [c_2, d_2]$ with one contained in the interior of the other, then applying Lemma 1.10 (keeping in mind Remark 1.11).

Remark 3.6. Both Theorem 3.3 and Theorem 3.5 promote immediately to $(\varphi, \Gamma)$-modules for any positive integer $a$: if $M$ is such an object, then $M \oplus \varphi^* M \oplus \cdots \oplus (\varphi^{a-1})^* M$ carries an action of $\varphi$ and its $(\varphi, \Gamma)$-cohomology coincides with the $(\varphi^a, \Gamma)$-cohomology of $M$. Similar considerations will hold later in the paper; to simplify the exposition, we make all statements exclusively for $(\varphi, \Gamma)$-modules.

Remark 3.7. Theorem 3.3 and Theorem 3.5 also promote immediately to theorems concerning $(\varphi, \Gamma)$-modules for which the base field $\mathbb{Q}_p$ has been replaced by a finite extension (this being the level of generality considered in [5], [16], and elsewhere).

Remark 3.8. The argument of Theorem 3.5 can also be used to obtain the finiteness theorem for cohomology of arithmetic families of $(\varphi, \Gamma)$-modules given by the first author with Pottharst and Xiao in [14]. The proof therein is more subtle; it involves reducing to the result of [16] via an intricate dévissage argument and a duality computation.

4 Relative $(\varphi, \Gamma)$-modules

We continue by summarizing the main points of the theory of relative $(\varphi, \Gamma)$-modules on rigid analytic spaces, as developed in [12, 13].

Definition 4.1. For $X$ an adic space, let $X_{\text{pro\-ét}}$ denote the pro-étale site in the sense of [19, §3], [12 §9.1]; that is, a basic pro-étale open in $X$ is a tower of finite étale surjective covers over an étale open. (We do not consider the finer pro-étale topology used in [20].) If $p$ is topologically nilpotent on $X$, then the pro-étale topology is generated by towers of finite étale covers of affinoids for which the completed direct limit of the associated rings is a perfectoid ring [13 Lemma 3.3.26]; such pro-étale opens are called perfectoid subdomains of $X$ (or better, of $X_{\text{pro\-ét}}$).
We recall some basic facts about pseudocoherent modules from [13].

**Definition 4.2.** A module $M$ over a ring $R$ is **pseudocoherent** if it admits a projective resolution (possibly of infinite length) by finite projective modules. For example, any finitely generated module over a noetherian ring is pseudocoherent, as is any finitely presented module over a coherent ring. A pseudocoherent module over a Banach ring is complete for its natural topology [13, Corollary 3.4.9] and over perfectoid adic Banach rings for the analytic topology and the étale topology [13, Theorem 2.5.7] and over perfectoid adic Banach rings for the pro-étale topology [13, Corollary 4.3.9].

A module $M$ over $R$ is **pseudoflat** if for every $R$-module $N$ admitting a partial projective resolution $P_2 \to P_1 \to P_0 \to M \to 0$ with $P_0, P_1, P_2$ finite projective, we have $\text{Tor}_1^R(M, N) = 0$; this implies that tensoring with $M$ is an exact functor from pseudocoherent $R$-modules to $R$-modules (or to pseudocoherent $M$-modules if $M$ is itself an $R$-algebra). For example, every flat module is pseudoflat, and conversely if $R$ is coherent (because omitting the condition of finiteness of $P_2$ leads back to the definition of a flat module). While rational localizations of affinoid algebras are flat, rational localizations of arbitrary adic Banach rings are only known to be pseudoflat [13, Theorem 2.4.8].

**Definition 4.3.** For any adic space $X$ on which $p$ is topologically nilpotent, define the $p$-adic period sheaves $\tilde{A}_X$ and $\tilde{C}_X$ on the pro-étale site $X_{\text{pro-ét}}$ as in [12, Definition 9.3.3]; by construction, both admit a *Frobenius endomorphism* $\varphi$. For real numbers $r, s$ such that $0 < s \leq r$, we also have a sheaf $\tilde{C}_X^{[s, r)}$ on $X_{\text{pro-ét}}$ defined as in [12, Definition 9.3.3]. We have an inclusion $\tilde{C}_X^{[s, r)} \to \tilde{C}_X^{[s', r')}$ whenever $[s', r') \subseteq [s, r]$, and a Frobenius endomorphism $\varphi : \tilde{C}_X^{[s, r)} \to \tilde{C}_X^{[s/p, r/p)}$. Pseudocoherent $\tilde{A}_X$-modules and pseudocoherent $\tilde{C}_X^{[s, r)}$-modules form stacks over perfectoid adic Banach rings for the pro-étale topology [13, Theorem 4.3.3].

**Remark 4.4.** As a quick summary, we recall that for $Y$ a perfectoid subdomain of $X$, $\tilde{A}_X$ evaluates to the ring of Witt vectors over the ring of sections of the tilted completed structure sheaf on $Y$, while $\tilde{C}_X^{[s, r)}$ evaluates to an “extended Robba ring” derived from the Witt vectors by restricting to elements satisfying a growth condition, inverting $p$, then completing for a suitable Banach norm. The evaluation of $\tilde{C}_X$ is obtained from the evaluations of the $\tilde{C}_X^{[s, r)}$ by first taking an inverse limit as $s \to 0^+$, then a direct limit as $r \to 0^+$.

**Remark 4.5.** For $Y$ a perfectoid subdomain of a perfectoid space $X$, the morphism $\tilde{A}_X \to \tilde{A}_Y$, (resp. $\tilde{C}_X^{[s, r]} \to \tilde{C}_Y^{[s, r]}$) is pseudoflat [13, Proposition 4.2.3].

**Lemma 4.6.** Let $(A, A^+)$ be a perfectoid adic Banach ring. Let $\text{Spa}(B, B^+)$ be a perfectoid subdomain of $\text{Spa}(A, A^+)$. Let $C^\bullet$ be a complex of Banach modules over $\tilde{C}_A^{[s, r]}$. Suppose that $C$ is a perfect complex of $\tilde{C}_A^{[s, r]}$-modules with pseudocoherent cohomology and that $C \otimes_{\tilde{C}_A^{[s, r]}} \tilde{C}_B^{[s, r]}$ is a perfect complex of $\tilde{C}_B^{[s, r]}$-modules with pseudocoherent cohomology. Then the natural morphisms

$$h^\bullet(C) \otimes_{\tilde{C}_A^{[s, r]}} \tilde{C}_B^{[s, r]} \to h^\bullet(C \otimes_{\tilde{C}_A^{[s, r]}} \tilde{C}_B^{[s, r]})$$

are isomorphisms.
Upon taking the completed tensor product over \( \mathbb{Q}_p \) with the \( p \)-cyclotomic extension \( L \), the rings \( \tilde{C}_A^{[s,r]} \), \( \tilde{C}_B^{[s,r]} \) become perfectoid [13, Proposition 4.1.13] and the map between them becomes a perfectoid subdomain [13, Lemma 4.2.2]. We may then apply Remark 4.5, then use a splitting \( L \to \mathbb{Q}_p \) of the inclusion to recover the desired isomorphisms.

**Definition 4.7.** A \((\varphi, \Gamma)\)-module (resp. a pseudocoherent \((\varphi, \Gamma)\)-module) over \( \tilde{A}_X \) or \( \tilde{C}_X \) on \( X \) is a sheaf \( F \) of modules which is locally represented by a finite projective (resp. pseudocoherent) module over the corresponding ring, together with an isomorphism \( \varphi^*F \to F \). We sometimes refer to a \((\varphi, \Gamma)\)-module also as a projective \((\varphi, \Gamma)\)-module for emphasis.

Denote by \( \mathcal{C}_X^{\Phi} \) the category of pseudocoherent \((\varphi, \Gamma)\)-modules over \( \tilde{C}_X \).

**Remark 4.8.** In this context, the symbol \( \Gamma \) in the term \((\varphi, \Gamma)\)-module is but a skeuomorph: the former role of \( \Gamma \), as a provider of descent data, is now assumed by the pro-étale topology.

**Remark 4.9.** The module structure of pseudocoherent \((\varphi, \Gamma)\)-modules over \( \tilde{A}_X \) is quite simple: the only possible torsion is \( p \)-power torsion, and a \( p \)-torsion-free object is projective [13, Lemma 4.5.4] (see Theorem 4.14 for an even stronger statement). In particular, without risk of confusion, we may refer to these objects also as coherent \((\varphi, \Gamma)\)-modules over \( \tilde{A}_X \).

The situation in type \( \tilde{C} \) is a bit subtler; see Remark 4.17.

**Remark 4.10.** In case \( s \leq r/p \), the category \( \mathcal{C}_X^{\Phi} \) is equivalent to the category of pseudocoherent \( \tilde{C}_X^{[s,r]} \)-modules \( F^{[s,r]} \) equipped with isomorphisms

\[
\varphi^*F^{[s,r]} \otimes \tilde{C}_X^{[s,r/p]} \cong F^{[s,r]} \otimes \tilde{C}_X^{[s,r/p]},
\]

see [13, Theorem 4.6.10]. (We sometimes refer to these objects as pseudocoherent \((\varphi, \Gamma)\)-modules over \( \tilde{C}_X^{[s,r]} \).) In particular, for any perfectoid subdomain \( Y \) of \( X \), any pseudocoherent \((\varphi, \Gamma)\)-module over \( X \) is represented by a pseudocoherent \( \tilde{C}_X(Y) \)-module. Moreover, if \( F \in \mathcal{C}_X^{\Phi} \) corresponds to \( F^{[s,r]} \) in this fashion, then the complexes

\[
0 \to F \xrightarrow{\varphi^{-1}} F \to 0, \quad 0 \to F^{[s,r]} \xrightarrow{\varphi^{-1}} F^{[s,r/p]} \to 0
\]

have the same cohomology sheaves [13, Theorem 4.6.9].

**Definition 4.11.** By a \( \mathbb{Z}_p \)-local system or \( \mathbb{Q}_p \)-local system over an adic space \( X \), we will mean a finite free module over the corresponding locally constant sheaf \( \mathcal{Z}_p \) or \( \mathcal{Q}_p \) (defining locally constant sheaves in the sense of [12, Definition 1.4.10]).

**Theorem 4.12.** The functor \( T \mapsto F = T \otimes_{\mathbb{Z}_p} \tilde{A}_X \) defines an equivalence of categories between locally finite \( \mathbb{Z}_p \)-modules on \( X_{\text{proét}} \) and pseudocoherent \((\varphi, \Gamma)\)-modules over \( \tilde{A}_X \), with the one-sided inverse being \( F \mapsto F^\varphi \); in particular, \( \mathbb{Z}_p \)-local systems correspond to projective \((\varphi, \Gamma)\)-modules. Moreover, the pro-étale cohomology of \( T \) is computed by the \((\varphi, \Gamma)\)-hypercohomology of \( F \).

**Proof.** See [13, Corollary 4.5.8].
Theorem 4.13. The functor $V \mapsto F = V \otimes_{\mathbb{Q}_p} \tilde{C}_X$ defines a fully faithful functor from $\mathbb{Q}_p$-local systems on $X$ to $(\varphi, \Gamma)$-modules over $\tilde{C}_X$, with the one-sided inverse being $F \mapsto F^\varphi$. Moreover, the pro-étale cohomology of $V$ is computed by the $(\varphi, \Gamma)$-hypercohomology of $F$.

Proof. See [13, Theorem 4.5.11]. □

Theorem 4.14. Let $X$ be an affinoid space over $K$. Let $\psi$ be a finite étale tower over $X$ whose total space $Y$ is perfectoid. Let $\mathcal{F}$ be a sheaf of $\tilde{A}_X$-modules. Then $\mathcal{F}$ is pseudocoherent if and only if its restriction to $Y$ is represented by a finitely generated $\tilde{A}(Y)$-module.

Proof. See [13, Proposition 8.3.5]. □

Theorem 4.15. Let $X$ be an affinoid space over $K$. Let $\psi$ be a finite étale tower over $X$ whose total space $Y$ is perfectoid. Let $\mathcal{F}$ be a sheaf of $\tilde{C}_X^{[s,r]}$-modules. Then $\mathcal{F}$ is pseudocoherent if and only if its restriction to $Y$ is represented by a finitely generated $\tilde{C}^{[s,r]}(Y)$-module.

Proof. See [13, Proposition 8.9.2]. □

Theorem 4.16. Let $X$ be an affinoid space over $K$.

(a) The category $\mathcal{C}_X^\Phi$ of pseudocoherent $(\varphi, \Gamma)$-modules over $\tilde{C}_X$ is an abelian subcategory of the category of sheaves of $\tilde{C}_X$-modules on $X_{pro\acute{e}t}$ (that is, the formation of kernels and cokernels commutes with the embedding).

(b) If $X$ is quasicompact, then $\mathcal{C}_X^\Phi$ satisfies the ascending chain condition: given any sequence $\mathcal{F}_0 \to \mathcal{F}_1 \to \cdots$ of epimorphisms in $\mathcal{C}_X^\Phi$, there exists $i_0 \geq 0$ such that for all $i \geq i_0$, the map $\mathcal{F}_i \to \mathcal{F}_{i+1}$ is an isomorphism.

(c) For $i \geq 0$, the bifunctors $\operatorname{Ext}^i$ and $\operatorname{Tor}_i$ take $\mathcal{C}_X^\Phi \times \mathcal{C}_X^\Phi$ into $\mathcal{C}_X^\Phi$.

(d) For any morphism $f : Y \to X$ of rigid spaces over $K$, for $i \geq 0$, the functor $L_i f^*_{pro\acute{e}t}$ takes $\mathcal{C}_X^\Phi$ into $\mathcal{C}_Y^\Phi$.

Proof. See [13, Theorem 8.10.6, Theorem 8.10.7]. □

Remark 4.17. In light of Theorem 4.16 when $X$ is a rigid space over a field rather than a more general adic space, it is natural to refer to objects of $\mathcal{C}_X^\Phi$ also as coherent $(\varphi, \Gamma)$-modules over $\tilde{C}_X$. The proof of Theorem 4.16 yields some additional information about the module structure of these objects. This is more easily described in terms of the corresponding modules over $\tilde{C}_X^{[s,r]}$: any such module admitting a $\Gamma$-action (i.e., which is a sheaf for the pro-étale topology) becomes finite projective after inverting some element $t$ of $\tilde{C}_L^{[s,r]}$ for $L$ a perfectoid extension of $K$.

In case $1 \in [s, r]$, one candidate for such an element is an element $t_0$ generating the kernel of the canonical map $\theta : \tilde{C}_L^{[s,r]} \to L$. In case $K$ is discretely valued, the element $t$ can always be taken to be a product of images of $t_0$ under various powers of $\varphi$; by contrast, when $K$ is perfectoid, we have many more options for $t$. However, most of the module-theoretic
complexity arises from $t_\theta$, in the following sense: if $t$ is irreducible and coprime to $t_\theta$, then any $t$-torsion coherent $\Gamma$-module over $\hat{\mathcal{C}}_{\mathbf{X}}^{[s,r]}$ is finite projective over $\hat{\mathcal{C}}_{\mathbf{X}}^{[s,r]} / (t)$ [13 Corollary 8.8.10].

By contrast, if $t = t_\theta$, then $t$-torsion coherent $\Gamma$-modules over $\hat{\mathcal{C}}_{\mathbf{X}}^{[s,r]}$ need not be finite projective over $\hat{\mathcal{C}}_{\mathbf{X}}^{[s,r]} / (t) \cong \hat{\mathcal{O}}_{\mathbf{X}}$; for instance, the pullback of any coherent $\mathcal{O}_{\mathbf{X}}$-module occurs in this category. However, one can at least say that these modules have Fitting ideals which descend to $\mathcal{O}_{\mathbf{X}}$ (see [13 §8.3]), so one can establish the desired module-theoretic properties using noetherian induction and resolution of singularities.

**Definition 4.18.** We say that a pseudocoherent $(\varphi, \Gamma)$-module $\mathcal{F}$ over $\hat{\mathcal{C}}_{\mathbf{X}}^{[s,r]}$ is $\theta$-local (resp. $\co$-$\theta$-projective) if it is annihilated by (resp. becomes projective after inverting) some product of images of $t_\theta$ under various powers of $\varphi$, and similarly for an object of $\mathcal{C}\Phi_{\mathbf{X}}$.

By Remark 4.17 if $k$ is discretely valued, then every object of $\mathcal{C}\Phi_{\mathbf{X}}$ is $\co$-$\theta$-projective; moreover, for general $k$, most module-theoretic difficulties are concentrated in the subcategory of $\theta$-local objects. For example, by [13 Corollary 8.8.10] (as applied in Remark 4.17), in the quotient of $\mathcal{C}\Phi_{\mathbf{X}}$ by the Serre subcategory of $\theta$-local objects, every object has projective dimension at most 1; in other words, any torsion-free object of $\mathcal{C}\Phi_{\mathbf{X}}$ is $\co$-$\theta$-projective. By similar arguments, one sees that the essential images of $\operatorname{Ext}^i$ for $i > 1$, $\operatorname{Tor}$, for $i > 1$, and $L_i \mathcal{F}_{\proet}^*$ for $i > 0$ all consist of $\theta$-local objects [13 Theorem 8.10.6, Theorem 8.10.7].

**Remark 4.19.** Although we will not use this result here, we note that the ascending chain condition also holds for the localization of $\mathcal{C}\Phi_{\mathbf{X}}$ at a point of $X$. See [13 Theorem 8.10.9].

The following example shows that one cannot hope to extend Theorem 4.16 to arbitrary higher direct images along a general morphism of rigid spaces.

**Example 4.20.** Put $X = \operatorname{Spa}(K, K^\circ), Y = \operatorname{Spa}(K\{T, T^{-1}\}, K\{T, T^{-1}\}^\circ)$, and $\mathcal{F} = \hat{\mathcal{C}}_Y$. Then the higher direct images of $\mathcal{F}$ along $Y_{\proet} \to X_{\proet}$ are not coherent $(\varphi, \Gamma)$-modules over $\hat{\mathcal{C}}_X$. This amounts to the fact that for $R$ a Banach algebra over $\mathbb{Q}_p$ containing a topologically nilpotent unit $\pi$, if we equip $S = R\{T, T^{-1}\}$ with the action of $\mathbb{Z}_p^\times$ sending $T$ to $(1 + \pi)T$, then for any $\gamma \in \mathbb{Z}_p^\times$ of infinite order, the action of $\gamma - 1$ on $S$ is not strict. A closely related fact is that the de Rham cohomology of the closed unit disc over a $p$-adic field is not finite-dimensional: integration of a power series preserves the radius of convergence but not the behavior at the boundary.

A related example is the following.

**Example 4.21.** Let $Y$ be an open annulus over $X$. Then the failure of finiteness described in Example 4.20 does not arise, but there are additional subtleties. For instance, for each character of $\mathbb{Z}_p^\times$, we may define a $\Gamma$-module over $\hat{\mathcal{C}}_Y$ which is free of rank 1 with the action of $\mathbb{Z}_p^\times$ on the generator being given by the chosen character. When this character is exponentiation by a $p$-adic Liouville number, we expect the $\Gamma$-cohomology not to be finite-dimensional, again by analogy with a corresponding pathology in the theory of $p$-adic differential equations (see [9] Chapter 13 for further discussion).

However, note that this example does not admit an action of $\varphi$. One may thus still hope to prove coherence of higher direct images of coherent $(\varphi, \Gamma)$-modules along $Y_{\proet} \to$
This might be likened to the fact that finiteness results for the rigid cohomology of overconvergent isocrystals on varieties of characteristic $p$ are generally only known in the presence of Frobenius structures (e.g., see [3]). While this makes such a result plausible, we will only prove such a theorem for smooth proper morphisms.

5 Properness for rigid analytic varieties

We next recall Kiehl’s definition of properness for rigid analytic varieties, and its interaction with completely continuous morphisms.

Definition 5.1. Let $f : (B, B^+) \to (B', B'^+)$ be an affinoid localization of adic Banach algebras over an adic Banach ring $(A, A^+)$. We say that $f$ is inner (relative to $(A, A^+)$) if there exists a strict surjection $A\{S\} \to B$ for some (possibly infinite set) $S$ such that each element of $S$ maps to a topologically nilpotent element of $B'$. Since this condition does not depend on the plus subrings, we will also say that $B \to B'$ is inner relative to $A$.

Lemma 5.2. Let $(A, A^+) \to (A', A'^+) \to (A'', A''^+)$ be morphisms of perfectoid adic Banach algebras, and apply the perfectoid correspondence [13, Theorem 3.3.8] to obtain morphisms $(R, R^+) \to (R', R'^+) \to (R'', R''^+)$ of perfect uniform adic Banach algebras. If $(R', R'^+) \to (R'', R''^+)$ is an inner rational localization relative to $(R, R^+)$, then $(A', A'^+) \to (A'', A''^+)$ is an inner rational localization relative to $(A, A^+)$. 

Proof. By [13, Theorem 3.3.18], $(A', A'^+) \to (A'', A''^+)$ is also a rational localization. Choose a strict surjection $f : R\{S\} \to R'$ such that each element of $S$ maps to a topologically nilpotent element of $R''$. We then obtain a strict surjection $A\{S\} \to A'$ sending $s$ to $\theta([f(s)])$; hence $(A', A'^+) \to (A'', A''^+)$ is inner. 

Remark 5.3. We do not know if the converse to Lemma 5.2 holds. An argument to prove this would likely encounter difficulties similar to those described in [12, Remark 3.6.12].

Hypothesis 5.4. For the remainder of §5 let $(A, A^+)$ be an adic affinoid algebra over $K$ (a nonarchimedean field of mixed characteristics).

Remark 5.5. Let $f : (B, B^+) \to (B', B'^+)$ be an affinoid localization of adic affinoid algebras over $(A, A^+)$. In light of Hypothesis 5.4, the condition that $f$ is inner can be reformulated in the following ways.

(a) The Banach ring $B$ is topologically generated over $A$ by power-bounded elements which become topologically nilpotent in $B'$.

(b) For some $n \geq 0$, there exists a strict $A$-linear surjection $A\{T_1, \ldots, T_n\} \to B$ which extends to a morphism $A\{T_1/r, \ldots, T_n/r\} \to B'$ for some $r \in (0, 1)$.

(c) For the spectral seminorms on $A, B, B'$, the image of $\kappa_B$ in $\kappa_{B'}$ is a finite $\kappa_A$-algebra.
Example 5.6. For \( r_i, s_i \in p^\mathbb{Q} \) with \( s_i < r_i \), the morphism
\[
A\{T_1/r_1, \ldots, T_n/r_n\} \to A\{T_1/s_1, \ldots, T_n/s_n\}
\]
is inner relative to \( A \).

Lemma 5.7. Let \( f : (B, B^+) \to (B', B'^+) \) be an inner affinoid localization of adic affinoid algebras over \((A, A^+)\). Then the underlying morphism \( B \to B' \) of Banach spaces over \( A \) is completely continuous.

Proof. See the proof of [15, Satz 2.5].

Lemma 5.8. Let \( f : (B, B^+) \to (B', B'^+) \) be an inner affinoid localization of affinoid algebras over \((A, A^+)\). Let \( g : (C, C^+) \to (C', C'^+) \) be the base extension of \( f \) along an integral morphism \( B \to C \) of affinoid algebras over \( A \). Then \( g \) is also inner.

Proof. This is immediate from Remark 5.5(c).

Definition 5.9. Let \( f : Y \to X \) be a morphism of rigid analytic spaces over \( K \). We say that \( f \) is separated if the diagonal morphism \( f : Y \to Y \times_X Y \) is a closed immersion. We say that \( f \) is proper if \( f \) is separated and additionally, for every affinoid subspace \( U \) of \( X \), there exist two finite coverings \( \{V_i\}_{i=1}^n, \{W_i\}_{i=1}^n \) of \( f^{-1}(U) \) by affinoid subspaces with the property that for \( i = 1, \ldots, n \), \( W_i \) is an affinoid subdomain of \( V_i \) which is inner relative to \( U \). (That is, for \( A, B', B'' \) the rings of global sections of \( U, V_i, W_i \), the morphism \( B' \to B'' \) is an affinoid localization which is inner relative to \( A \).)

Remark 5.10. The definition of properness given in Definition 5.9 is the original definition of Kiehl [15]. Alternate characterizations are described in [21, 22]; such characterizations can be used to check for instance that properness is stable under composition, and that the analytification of a proper algebraic variety is proper.

Remark 5.11. It will be useful in what follows to recall that for any adic Banach algebra \((A, A^+)\), the disc \( \text{Spa}(A\{T\}, A^+(\{T\})) \) is covered by the annuli \( \text{Spa}(A\{T^\pm\}, A^+(\{T^\pm\})) \) and \( \text{Spa}(A^+\{(T - 1)^\pm\}, A^+\{(T - 1)^\pm\}) \).

Lemma 5.12. Let \((A, A^+)\) be an adic affinoid algebra over \( K \), and let \( f : X \to \text{Spa}(A, A^+) \) be a smooth proper morphism of relative dimension \( n \). Then for each \( x \in X \) and each affinoid subdomain \( x \in V_0 \) of \( X \), there exist a rational localization \((A, A^+) \to (B, B^+) \) with \( f(x) \in \text{Spa}(B, B^+) \), some affinoid subdomains \( x \in V_2 = \text{Spa}(C_2, C_2^+) \subseteq V_1 = \text{Spa}(C_1, C_1^+) \subseteq V_0 \) of \( X \), and a commutative diagram of the form
\[
\begin{array}{c}
B\{\frac{s_1}{T_1}, \ldots, \frac{s_n}{T_n}; T_1, \ldots, T_n\} \\
\downarrow C_1 \\
C_2
\end{array}
\]
\[
\to \begin{array}{c}
B\{\frac{s_2}{T_1}, \ldots, \frac{s_n}{T_n}; T_1, \ldots, T_n\} \\
\downarrow C_2
\end{array}
\]
for some \( 0 < s_{1,i} < s_{2,i} \leq r_{2,i} < r_{1,i} \) in which the vertical arrows are finite étale morphisms of \( A \)-algebras. In particular, \((C_1, C_1^+) \to (C_2, C_2^+) \) is inner relative to \((B, B^+)\).
Proof. Since $X$ is proper, we can find affinoid subdomains $x \in V'_2 \subseteq V'_1$ of $X$ such that $V'_2$ is inner in $V'_1$ relative to $A$. Since $X$ is also smooth, we can find an affinoid subdomain $x \in V''_2 \subseteq V'_2 \cap V_0$ such that $V_2 = \mathrm{Spa}(C'', C''_2)$ and $C''_2$ is finite étale over $B\{T_1, \ldots, T_n\}$ for some rational localization $(A, A^+) \to (B, B^+)$. As in Remark 5.11 we may ensure that $T_1, \ldots, T_n$ do not vanish at $x$.

Let $f_1, \ldots, f_n \in C''_2$ be the pullbacks of $T_1, \ldots, T_n$. Then any sufficiently close approximations of $f_1, \ldots, f_n$ also define a finite étale morphism $B\{T_1, \ldots, T_n\} \to C''_2$. In particular, these approximations may be chosen so that this morphism extends to a commutative diagram

$$
\begin{array}{ccc}
B\{\frac{T_1}{r_{1,1}}, \ldots, \frac{T_n}{r_{1,n}}\} & \longrightarrow & B\{T_1, \ldots, T_n\} \\
\downarrow & & \downarrow \\
C''_1 & \longrightarrow & C''_2
\end{array}
$$

for some $r_{1,1}, \ldots, r_{1,n} > 1$ in such a way that $\mathrm{Spa}(C''_2, C''_2) \subseteq \mathrm{Spa}(C'', C''_2)$ is an inner rational subdomain inclusion of affinoid subdomains of $X$ and $B\{T_1/r_{1,1}, \ldots, T_n/r_{1,n}\} \to C''_1$ is finite étale. We thus obtain a diagram of the desired form by taking $r_{2,i} = 1$ and choosing $s_{1,i}, s_{2,i}$ with $|T_i(x)|^{-1} \leq s_{1,i} < s_{2,i}$.

6 Finiteness of relative $\Gamma$-cohomology

We now begin our treatment of higher direct images by applying the method of Cartan-Serre to get a finiteness result. Due to the unavoidably local nature of the construction of toric towers, some care is required with the homological algebra setup (which we already took in §2). Note that this approach is not enough by itself to establish coherence of higher direct images, as it does not immediately establish compatibility with base change; see §7.

Proposition 6.1. Suppose that $K$ contains all $p$-power roots of unity. Let $f : Y \to X$ be a smooth proper morphism of relative dimension $n$ of rigid analytic spaces over $K$. Let $\mathcal{F}$ be a coherent $(\varphi, \Gamma)$-module over $\tilde{\mathbf{A}}_Y$ (resp. $\mathcal{C}_Y$). Then there exists an open covering $\{U_j\}_j$ of $X$ such that for any index $j$ and any perfectoid subdomain $\tilde{X}$ of $U_j$, the groups $H^i((Y \times_X \tilde{X})_{\text{proét}}, \mathcal{F}|_{Y \times_X \tilde{X}})$ are pseudocoherent modules over $\tilde{\mathbf{A}}_{\tilde{X}}$ (resp. $\mathcal{C}_{\tilde{X}}$). Moreover, these groups vanish for $i > 2n$, and their formation is compatible with passage from $\tilde{X}$ to a pro-finite étale cover.

Since the proof of Proposition 6.1 is rather involved, we break it up into a series of steps. We begin with some initial reductions. (We distinguish the two phrasings in this statement, and all subsequent ones exhibiting a similar duality, as the type $\mathbf{A}$ case and the type $\mathbf{C}$ case.)

Remark 6.2. In the type $\mathbf{C}$ case of Proposition 6.1 we may invoke Remark 4.10 to replace $\mathcal{F}$ with a pseudocoherent module over $\mathcal{C}_{\tilde{X}}^{[s,r]}$.

We continue with a geometric reduction setting up a suitable framework for the Cartan-Serre method.
Definition 6.4. Using Lemma 5.12 and Remark 6.3 we may reduce to the case where $X = \text{Spa}(A, A^\pm)$ is affinoid; $Y$ admits two coverings $\{V_{i, j}\}_{i, j = 1}^m$, $\{V_{2, i}\}_{i = 1}^m$ with $V_{2, i} = \text{Spa}(C_{2, i}, C_{2, i}^+) \subseteq V_{1, i} = \text{Spa}(C_{1, i}, C_{1, i}^+)$; and for each $i$ there exists a commutative diagram

$$A\{\frac{s_{1, i, 1}}{T_1}, \ldots, \frac{s_{1, i, n}}{T_n}, \frac{T_1}{r_{1, i, 1}}, \ldots, \frac{T_n}{r_{1, i, n}}\} \longrightarrow A\{\frac{s_{2, i, 1}}{T_1}, \ldots, \frac{s_{2, i, n}}{T_n}, \frac{T_1}{r_{2, i, 1}}, \ldots, \frac{T_n}{r_{2, i, n}}\}$$

(6.4.1)

for some $0 < s_{1, i, j} < s_{2, i, j} \leq r_{2, i, j} < r_{1, i, j}$ in which the vertical arrows are finite étale morphisms of $A$-algebras. We may further assume that for any nonempty subset $I \subseteq \{1, \ldots, m\}$, the spaces $V_{1, I} = \cap_{i \in I} V_{i, i}$, $V_{2, I} = \cap_{i \in I} V_{2, i}$ have the property that for each $i \in I$, $V_{1, i}$ is a rational subspace of $V_{1, i}$ and $V_{2, i}$ is a rational subspace of $V_{2, i}$.

Remark 6.5. In the setting of Definition 6.4, we will prove that the claim holds on all of $X$. To this end, we may assume without loss of generality that $\tilde{X} = \text{Spa}(\tilde{A}, \tilde{A}^\pm)$ is the total space of a finite étale perfectoid tower $\psi_0$ over $X$ (otherwise, there is a nonfinite étale morphism at the bottom of the tower, but we can pull back the data of Definition 6.4 along that étale morphism). Note that the claimed compatibility with passage from $\tilde{Y}$ to a pro-finite étale cover $\tilde{Y}'$ follows from the fact that $\tilde{A}_{\tilde{X}}/(p^n)$ (resp. $\tilde{C}_{\tilde{X}}^{[s, r]}$) is a topological direct sum of finite projective $\tilde{A}_{\tilde{X}}/(p^n)$-modules (resp. $\tilde{C}_{\tilde{X}}^{[s, r]}$-modules) [13, Remark 4.2.4].

We now build a particularly useful set of finite étale towers over subspaces of $Y$.

Definition 6.6. For $* = 1, 2$, put

$$(\tilde{C}_{1, i}, \tilde{C}_{1, i}^+) = (C_{1, i}, C_{1, i}^+) \hat{\otimes}_{(A, A^\pm)}(\tilde{A}, \tilde{A}^+) .$$

Let $\psi_{*, i}$ denote the tower of finite étale coverings $\{\text{Spa}(\tilde{C}_{*, i, k}, \tilde{C}_{*, i, k}^+) \rightarrow \text{Spa}(\tilde{C}_{*, i}, \tilde{C}_{*, i}^+)\}_{k = 0}^\infty$ in which

$$(\tilde{C}_{*, i, k}, \tilde{C}_{*, i, k}^+) = \tilde{C}_{*, i}[T_1/p^k, \ldots, T_n/p^k].$$

Note that $\psi_{*, i}$ is a restricted toric tower in the sense of [13, Definition 7.1.4]; according to [13, Theorem 7.1.9], all of the results of [13, §5] apply to such towers.

We next describe the desired cohomology groups explicitly in terms of the specified towers.

Definition 6.7. For $* = 1, 2$ and $I \subseteq \{1, \ldots, m\}$ nonempty, let $\psi_{*, I}'$ be the tower whose $k$-th term is the fiber product over $Y$ of the spaces $\text{Spa}(\tilde{C}_{*, i, k}, \tilde{C}_{*, i, k}^+)\}_{i \in I}$. Let $\tilde{M}_{*, I}'$ be the Čech complex for the sheaf $\mathcal{F}$ and the cover $\psi_{*, I}'$; that is, $\tilde{M}_{*, I}'$ is the set of sections of $\mathcal{F}$ on the $(i + 1)$-fold fiber product of the total space of $\psi_{*, I}'$. 22
For $s = 1, \ldots, m$, we may interpolate the complexes $M_{s,I}^*$ for $I \subseteq \{1, \ldots, s\}$ into an $(s + 1)$-dimensional complex $M_{s,I}^*$, taking $M_{s,I}^*$ to be the zero complex for $I = \emptyset$. We then have a natural morphism $M_{1,s}^* \to M_{2,s}^*$. By \cite{13} Theorem 5.7.11, the total complex $M_{1,s}^*$ (resp. $M_{2,s}^*$) computes the cohomology of $F$ on the inverse image of $U_1 \cup \cdots \cup U_s$ (resp. $V_1 \cup \cdots \cup V_s$) in $X$. In particular, for $s = m$, both total complexes compute the cohomology groups $H^i(\hat{X}_{\proet}, F)$, so the morphism between them is a quasi-isomorphism of total complexes.

We next produce two alternate models for the single complex $\tilde{M}_{s,I}^*$.

**Definition 6.8.** For $I \subseteq \{1, \ldots, m\}$ nonempty, let $i = i(I)$ be the largest element of $M$. Let $\psi_{1,I}$ (resp. $\psi_{2,I}$) be the tower obtained from $\psi_{1,I}$ (resp. $\psi_{2,I}$) by pullback from $U_i$ (resp. $V_i$) to $U_I$ (resp. $V_I$). Let $\tilde{M}_{s,I}^*$ be the Cech complex for the sheaf $F$ and the cover $\psi_{s,I}$. There is a natural morphism $\tilde{M}_{s,I}^* \to \tilde{M}_{s,I}^*$ of complexes which is a quasi-isomorphism \cite{13} Theorem 4.6.9.

Let $A_{\psi_{s,I}}$, $C_{\psi_{s,I}}^{[s,r]}$ be the imperfect period rings associated to the tower $\psi_{s,I}$ as per \cite{13} Definition 5.2.1. Note that the morphism $A_{\psi_{1,I}}/(p) \to A_{\psi_{2,I}}/(p)$ (resp. $C_{\psi_{1,I}}^{[s,r]} \to C_{\psi_{2,I}}^{[s,r]}$) of Banach algebras over $\tilde{A}_X/(p)$ (resp. $\tilde{C}_{X}^{[s,r]}$) are completely continuous as morphisms of Banach modules.

In case (a) (resp. case (b)) of Proposition 6.1, we may apply \cite{13} Theorem 5.8.16 (resp. \cite{13} Theorem 5.9.4) to realize $F|_{U_I}, F|_{V_I}$ as pseudocoherent $(\varphi, \Gamma)$-modules over $A_{\psi_{1,I}}, A_{\psi_{2,I}}$ (resp. over $C_{\psi_{1,I}}^{[s,r]} C_{\psi_{2,I}}^{[s,r]}$). Let $M_{s,I}^*$ be the resulting complex of continuous $\Gamma$-cochains for $\Gamma \cong \mathbb{Z}_p$ the automorphism group of the tower $\psi_{s,I}$ (which is Galois because we forced $\tilde{A}$ to contain all $p$-power roots of unity). There is a natural morphism $M_{s,I}^* \to \tilde{M}_{s,I}^*$ of complexes which is a quasi-isomorphism \cite{13} Theorem 5.8.15, and the diagram

$$
\begin{array}{ccc}
M_{1,I}^* & \longrightarrow & \tilde{M}_{1,I}^* \\
\downarrow & & \downarrow \\
M_{2,I}^* & \longrightarrow & \tilde{M}_{2,I}^*
\end{array}
$$

(6.8.1)

commutes.

We next define splittings associated to the quasi-isomorphisms we have just introduced.

**Definition 6.9.** As in Definition 6.4, we may use \cite{13} Remark 4.2.4 to construct a morphism $\sigma_{s,I}^* : M_{s,I}^* \to \tilde{M}_{s,I}^*$ splitting the given quasi-isomorphism in the other direction; more precisely, it is an inverse in the homotopy category to the given quasi-isomorphism.

Since $\psi_{s,I}$ is a toric tower, the map $A_{\psi_{s,I}}/(p) \to A_{\psi_{s,I}}/(p)$ (resp. $C_{\psi_{s,I}}^{[s,r]} \to C_{\psi_{s,I}}^{[s,r]}$) admits a unique $\Gamma$-equivariant splitting induced by projection onto monomials in the $T_i$ with integer exponents, which is again an inverse in the homotopy category (compare \cite{13} Remark 7.1.11). Using such splittings, we obtain a morphism $\sigma_{s,I}^* : M_{s,I}^* \to M_{s,I}^*$. 

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In both cases, the splittings may (and will) be chosen compatibly with enlarging $I$ without changing $i(I)$, by localizing the splitting constructed for $I = \{i(I)\}$. Also, the splittings for $* = 2$ may (and will) be taken to be base extensions of the splittings for $* = 1$, so that the diagram

\[
\begin{array}{ccc}
\tilde{M}'_{1,I} & \xrightarrow{\sigma'_{1,I}} & \tilde{M}'_{2,I} \\
\downarrow & & \downarrow \\
\tilde{M}'_{2,I} & \xrightarrow{\sigma'_{2,I}} & \tilde{M}'_{2,I}
\end{array}
\]

(commutes. However, we cannot ensure any compatibility with enlarging $I$ when this changes the value of $i(I)$; this gives rise to some of the complexity in the ensuing arguments.

We next use the splittings to interpolate the model complexes $M'_{s,I}$.

**Definition 6.10.** For $s \in \{1, \ldots, m\}$, let $\tilde{N}'_{s,m}$ be the subcomplex of $\tilde{M}'_{s,m}$ composed of those terms for which the index $I$ satisfies $i(I) = s$. Let $N_{s,m}$ be the subcomplex of $\tilde{N}'_{s,m}$ defined by the inclusions $\tilde{M}'_{s,m} \to \tilde{M}'_{s,m}$ for all $I$ with $i(I) = s$; the fact that these inclusions define a subcomplex is due to the uniform derivation of the complexes $M'_{s,I}$ from the single tower $\psi_{*s}$. For the same reason, the composition $\sigma_{*s} \circ \sigma'_{*s}$ defines a splitting $\tilde{N}'_{s,*s} \to N_{s,*s}$ of the inclusion map, which provides an inverse in the homotopy category.

We may now view $\tilde{M}'_{s,m}$ as the total complex associated to a sequence of morphisms

$\tilde{N}'_{s,1} \to \cdots \to \tilde{N}'_{s,m}$,

where each term $\tilde{N}'_{s,i}$ is isomorphic in the homotopy category to $N_{s,i}$. Moreover, the comparison map $\tilde{N}'_{1,*s} \to \tilde{N}'_{2,*s}$ is represented in the homotopy category by the base extension morphism $\tilde{N}'_{1,*s} \to N_{2,*s}$, which by Remark 6.11 is completely continuous as long as this is true of the underlying ring homomorphism.

**Remark 6.11.** We will prove below that the cohomology groups in Proposition 6.1 are pseudo-coherent. Concerning other assertions of the proposition, we have already addressed compatibility with pro-finite étale covers (see Definition 6.4). To obtain vanishing in degrees above $2n$, it suffices to combine the following two observations: each individual complex $M'_{s,I}$ has no cohomology above degree $n$; and (if $X$ is quasicompact) the space $Y$ has cohomological dimension $\leq n$ [3 Proposition 2.5.8].

We are now ready to prove Proposition 6.1, starting with the type A case.

**Proof of Proposition 6.1, type A case.** As per Remark 6.11, it suffices to check that the cohomology groups are pseudo-coherent, or even just finitely generated in light of Theorem 4.14.

In the type A case, the setup culminating in Definition 6.10 is immediately susceptible to Lemma 2.7 only when $F$ is killed by $p$, because $\tilde{A}_X/(p^n)$ only admits the structure of a Banach ring for $n = 1$. The commutativity conditions are all satisfied: the conditions in (a) hold because $\tilde{N}_{1,*s} \to \tilde{N}^s_{2,*s}$ is a genuine morphism of genuine complexes (so in particular...
we may take all homotopies to be zero), whereas the conditions in (b) hold thanks to the commutativity of the diagrams \( \text{(6.8.1), (6.9.1)} \).

In light of the previous discussion, when \( F \) is killed by \( p \), we may apply Lemma \( \text{(2.7)} \) to deduce that the cohomology groups are contained in finitely generated modules, then \( \text{(13, Corollary 8.2.15)} \) to deduce that they are finitely generated. For general \( F \), we use the \( p \)-torsion case to deduce finite generation of the cohomology groups using Lemma \( \text{(3.2)} \), taking \( R = \check{A}_X \) and \( A \) to be the category of \( \check{A}_X \)-modules equipped with commuting \( \varphi^{\pm 1}, \Gamma \)-actions. Note that the assumption on \( A \) is ensured by \( \text{(13, Proposition 8.3.9)} \) and Theorem \( \text{(4.14)} \).

We next treat the type \( C \) case.

Proof of Proposition \( \text{(6.1)} \) type \( C \) case. As per Remark \( \text{(6.11)} \) it suffices to check that the cohomology groups are pseudocoherent, or even just finitely generated in light of Theorem \( \text{(4.15)} \). In the type \( C \) case, the setup culminating in Definition \( \text{(6.10)} \) is immediately susceptible to Lemma \( \text{(2.7)} \) with the same comments about the commutativity conditions; it thus follows that the cohomology groups are complete and contained in finitely generated modules. To promote this containment, we work through several layers of generality.

(i) For \( F \) killed by \( t_\theta \), we proceed primarily by noetherian induction on the support of \( F \), and secondarily by descending induction on cohomological degree. Suppose that \( F \) is killed by some ideal \( J \) of \( A \). To establish finiteness of \( h^i(\check{M}_s, \phi_m) \) for some \( i \) (given the same for all larger \( i \)), we work in the context of \( \text{(13, Remark 8.3.10)} \) (except that the letter \( f \) is presently in use as a morphism, so we use \( g \) for the ring element denoted \( f \) in loc cit.).

Let \( Z_i \) and \( Y_i \) be the modules of \( i \)-cocycles and \( i \)-coboundaries in \( \check{M}_s, \phi_m \), so that the desired cohomology group is \( M_i = Z_i / Y_i \). As described in \( \text{(13, Remark 8.3.10)} \), we can find \( g \in A \) which is not a zero-divisor in \( A/J \) or \( \check{A}/JA \check{A} \), such that \( Z_i / Y_i \) admits a homomorphism to a finitely generated \( \check{A}/JA \check{A} \)-module which becomes a split inclusion after inverting \( g \); this then implies that the quotient of \( M_i \) by its \( g \)-power-torsion submodule is pseudocoherent (using the induction hypotheses to verify conditions (a) and (b) of \( \text{(13, Remark 8.3.10)} \)).

Now suppose \( M_i \) is not pseudocoherent, then the sequence of \( g^n \)-torsion submodules of \( M_i \) does not stabilize as every \( M_i[g^n] \) is pseudocoherent, and the union of this sequence contains the closure of 0 in \( M_i \). Moreover, the closure of 0 in \( M_i \) is not contained in any \( M_i[g^n] \). Otherwise, let \( N_i \) be the quotient of \( M_i \) by the closure of 0. Since \( N_i \) is a Banach module, we may apply \( \text{(13, Remark 8.3.10)} \) to deduce that \( N_i[g^\infty] = N_i[g^n] \) for some \( n \) because an infinite ascending union of Banach modules cannot be a Banach module unless the union stabilizes at some finite stage. Therefore, we may conclude that the \( g \)-power-torsion submodules of \( M_i \) stabilize at some finite stage, yielding a contradiction.

Now repeat the construction after performing a base extension from \( A \) to \( A\{T\} \); let \( Z'_i \) and \( Y'_i \) be the resulting modules of \( i \)-cocycles and \( i \)-coboundaries, and let \( M'_i = Z'_i / Y'_i \) be the quotient. Again, the closure of 0 in \( M'_i \) consists entirely of \( g \)-power-torsion
elements; however, if $M_i$ is not pseudocoherent, then for each sufficiently large $n$ there
must exist some $z_n \in Z_i$ in the closure of $Y_i$ whose image in $M_i$ is killed by $g^n$ but not
by $g^{n-1}$. By scaling each $z_n$ suitably, we may ensure that the sum $\sum_{n=0}^{\infty} z_n T^n$
converges to an element $z \in Z_i'$ in the closure of $Y_i'$ whose image in $M_i'$ is not killed by
any power of $g$, a contradiction. This proves the desired finiteness result.

(ii) For $F$ killed by some $t \in \tilde{C}_{\bar{K}}^{[s,r]}$ coprime to $t_\theta$, [13, Corollary 8.8.10] (as
applied in Remark 4.17) implies the desired finite generation.

(iii) Using (i) and (ii), we deduce the claim whenever $F$ is killed by any nonzero $t \in \tilde{C}_{\bar{K}}^{[s,r]}$;
we may thus argue as in (i), using some suitable $t \in \tilde{C}_{\bar{K}}^{[s,r]}$ in place of $g$, to conclude.

The proof is complete.

\section{Base change}

As noted previously, to establish coherence of higher direct images, one must combine Proposition 6.1 with a certain compatibility with base change. We address this point next.

\textbf{Proposition 7.1.} With hypotheses and notation as in Proposition 6.1, for any index $j$, any
perfectoid subdomain $\tilde{X}$ of $U_{j, \text{pro\text{ê}t}}$, and any perfectoid subdomain $\tilde{X}'$ of $\tilde{X}_{\text{pro\text{ê}t}}$, the morphisms

\[ H^i(\tilde{X}_{\text{pro\text{ê}t}}, F) \otimes \tilde{A}_{\tilde{X}} \to H^i(\tilde{X}'_{\text{pro\text{ê}t}}, F) \quad \text{(resp. } H^i(\tilde{X}_{\text{pro\text{ê}t}}, F) \otimes \tilde{C}_{\bar{X}}^{[s,r]} \to H^i(\tilde{X}'_{\text{pro\text{ê}t}}, F)) \]

(7.1.1)

are isomorphisms for all $i \geq 0$.

Again, we devote the entire remainder of this section to the proof of Proposition 7.1 and thus retain notation as therein.

\textbf{Remark 7.2.} To begin, note that if $\tilde{X}'$ is a pro-finite étale cover of $\tilde{X}$, then the claimed
compatibility (i.e., the fact that the maps in (7.1.1) are isomorphisms) is already included
in Proposition 6.1.

\textbf{Remark 7.3.} From the proof of Proposition 6.1, we obtain a bounded complex $\tilde{M}^\bullet$ such
that $H^i(\tilde{X}_{\text{pro\text{ê}t}}, F) \cong h^i(\tilde{M}^\bullet)$ (e.g., the totalization of the complex denoted $\tilde{M}_{1.\text{sm}}^\bullet$ in Definition 6.7). Define the second complex

\[ \tilde{M}'^\bullet = \tilde{M}^\bullet \otimes \tilde{A}_{\tilde{X}} \quad \text{(resp. } \tilde{M}'^\bullet = \tilde{M}^\bullet \otimes \tilde{C}_{\bar{X}}^{[s,r]} \tilde{C}_{\bar{X}}^{[s,r]}). \]

Define the third complex

\[ \tilde{M}''^\bullet = \tilde{M}'' \otimes \tilde{A}_{\tilde{X}} \quad \text{(resp. } \tilde{M}'' \otimes \tilde{C}_{\bar{X}}^{[s,r]} \tilde{C}_{\bar{X}}^{[s,r]}); \]

then $H^i(\tilde{X}_{\text{pro\text{ê}t}}, F) \cong h^i(\tilde{M}'^\bullet)$.
By hypothesis, we know that the $h^i(\tilde{M}^\bullet)$ are pseudocoherent modules which vanish in large enough degrees. By Remark 4.3, we have isomorphisms
\[
h^i(\tilde{M}^\bullet) \otimes_{\tilde{A}_X} \tilde{A}_{\tilde{X}} \cong h^i(\tilde{M}^\bullet) \quad \text{(resp. } h^i(\tilde{M}^\bullet) \otimes_{\tilde{C}^{[s,r]}_X} \tilde{C}^{[s,r]}_{\tilde{X}} \cong h^i(\tilde{M}^\bullet)); \quad (7.3.1)\]
the content of Proposition 7.4 is thus that the morphism $\tilde{M}^\bullet \to \tilde{M}^\bullet$ is a quasi-isomorphism.

**Lemma 7.4.** Suppose that there exists $f \in \mathcal{O}(X)$ such that $\tilde{X}' = \{v \in \tilde{X} : v(f) \geq 1\}$. Then the morphisms in (7.1.1) are isomorphisms.

**Proof.** Retain notation as in Remark 7.3. By approximating $f$ suitably as in [12, Corollary 3.6.7], we can find $\tilde{f} \in \mathcal{O}(\tilde{X})$ such that $\tilde{X}' = \{v \in \tilde{X} : v(\theta(\tilde{f})) \geq 1\}$. Put $\tilde{A} = \tilde{A}_X, \tilde{A}' = \tilde{A}_{\tilde{X}}$ (resp. $\tilde{A} = \tilde{C}^{[s,r]}_X, \tilde{A}' = \tilde{C}^{[s,r]}_{\tilde{X}}$); we then have the exact sequence
\[
0 \to \tilde{A}\{T\} \xrightarrow{\tilde{T}_T} \tilde{A}\{T\} \to \tilde{A}' \to 0. \quad (7.4.1)\]
Similarly, for any perfectoid subdomain $\tilde{U}$ of $Y$ lying over $\tilde{X}$, if we put $\tilde{U}' = \tilde{U} \times_{\tilde{X}} \tilde{X}'$ and $\tilde{B} = \tilde{A}_Y, \tilde{B}' = \tilde{A}_Y$ (resp. $\tilde{B} = \tilde{C}^{[s,r]}_Y, \tilde{B}' = \tilde{C}^{[s,r]}_Y$); we have an exact sequence
\[
0 \to \tilde{B}\{T\} \xrightarrow{\tilde{T}_T} \tilde{B}\{T\} \to \tilde{B}' \to 0. \quad (7.4.2)\]
Put
\[
\tilde{Z}^i = \ker(\tilde{M}^i \to \tilde{M}^{i+1}), \quad \tilde{Y}^i = \text{image}(\tilde{M}^{i-1} \to \tilde{M}^i),
\tilde{Z}^{ni} = \ker(\tilde{M}^{ni} \to \tilde{M}^{ni+1}), \quad \tilde{Y}^{ni} = \text{image}(\tilde{M}^{ni-1} \to \tilde{M}^{ni}).
\]
For $* = \tilde{M}^i, \tilde{Y}^i, \tilde{Z}^i, h^i(\tilde{M}^\bullet)$, let $*\{T\}$ be the completion of $*\{T\}$ (i.e., the module $* \otimes_A A[T]$ viewed as polynomials with coefficients in $*$) for the Gauss norm. For each $i$, the sequence
\[
0 \to \tilde{M}^i\{T\} \xrightarrow{\tilde{T}_T} \tilde{M}^i\{T\} \to \tilde{M}^{ni} \to 0
\]
is exact: exactness at the middle and right follow by (uncompleted) base extension from (7.4.2), and exactness at the left follows by considering formal power series in $T$ (compare [13, Remark 1.2.5]).

We prove by descending induction on $i$ that the sequences
\[
0 \to \tilde{Y}^i\{T\} \xrightarrow{\tilde{T}_T} \tilde{Y}^i\{T\} \to \tilde{Y}^{ni} \to 0 \quad (7.4.3)
0 \to \tilde{Z}^i\{T\} \xrightarrow{\tilde{T}_T} \tilde{Z}^i\{T\} \to \tilde{Z}^{ni} \to 0 \quad (7.4.4)
0 \to h^i(\tilde{M}^\bullet)\{T\} \xrightarrow{\tilde{T}_T} h^i(\tilde{M}^\bullet)\{T\} \to h^i(\tilde{M}^\bullet) \to 0 \quad (7.4.5)
\]
are exact. Given (7.4.3) with \( i \) replaced by \( i + 1 \), in the diagram

\[
\begin{array}{c}
0 \quad 0 \quad 0 \\
\downarrow & & \downarrow \\
\tilde{Z}^i \{ T \} \rightarrow \tilde{M}^i \{ T \} \rightarrow \tilde{Y}^{i+1} \{ T \} \rightarrow 0 \\
\downarrow & & & \downarrow \\
0 \quad 0 \quad 0 \\
\downarrow & & \downarrow \\
\tilde{Z}^i \{ T \} \rightarrow \tilde{M}^i \{ T \} \rightarrow \tilde{Y}^{i+1} \{ T \} \rightarrow 0 \\
\downarrow & & \downarrow \\
\tilde{Y}^i \rightarrow \tilde{Y}^{i+1} \rightarrow 0 \\
0 \quad 0 \quad 0
\end{array}
\]  

all three rows and the middle and right columns are exact; by the snake lemma, we infer that (7.4.4) is exact. In the diagram

\[
\begin{array}{c}
0 \quad 0 \quad 0 \\
\downarrow & & \downarrow \\
\tilde{Y}^i \{ T \} \rightarrow \tilde{Z}^i \{ T \} \rightarrow h^i(\tilde{M}^*) \{ T \} \rightarrow 0 \\
\downarrow & & \downarrow \\
0 \quad 0 \quad 0 \\
\downarrow & & \downarrow \\
\tilde{Y}^i \{ T \} \rightarrow \tilde{Z}^i \{ T \} \rightarrow h^i(\tilde{M}^*) \{ T \} \rightarrow 0 \\
\downarrow & & \downarrow \\
\tilde{Y}^i \rightarrow \tilde{Z}^i \rightarrow h^i(\tilde{M}^*) \rightarrow 0 \\
0 \quad 0 \quad 0
\end{array}
\]  

all three rows and the middle column are exact. In the right column, exactness at the top holds by [13, Remark 1.2.5] and exactness at the bottom holds because the bottom row and the middle column are exact. By the snake lemma again, we deduce that the left column is exact at the top and middle.

We now peek ahead to the diagram (7.4.6) with \( i \) replaced by \( i - 1 \); again, all three rows and the middle column are exact, which forces the right column to be exact at the bottom as well as in the other two positions. Returning to (7.4.7), we now have exactness in all rows and the left and middle columns; by the snake lemma once more, we deduce that the right column is exact. To summarize, we now have exactness of all three equations (7.4.3), (7.4.4), (7.4.5), thus completing the induction.

From (7.4.1) and (7.4.5), we see that \( h^i(\tilde{M}^{**}) \rightarrow h^i(\tilde{M}^*) \) is an isomorphism for all \( i \). This is the desired result.
Corollary 7.5. There exists a neighborhood basis \( B \) of \( \tilde{X}_{\text{pro}} \) such that for each \( \tilde{X}' \in B \), the morphisms (7.1.1) are isomorphisms.

Proof. Combine Lemma 7.4 with [13, Lemma 2.4.6] (to get a neighborhood basis of \( X_\text{ ét} \) and hence of \( \tilde{X}_\text{ ét} \)) and Remark 7.2 (to enlarge this to a neighborhood basis of \( \tilde{X}_{\text{pro}} \)).

Proof of Proposition 7.1 By Corollary 7.5, there exists a covering \( \{ \tilde{U}_j \} \) of \( \tilde{X}' \) such that in the composition

\[
\bigoplus_j H^i(\tilde{X}_{\text{pro}}, F) \otimes_{\tilde{A}_{\tilde{X}}} \tilde{A}_{\tilde{U}_j} \to \bigoplus_j H^i(\tilde{X}'_{\text{pro}}, F) \otimes_{\tilde{A}_{\tilde{X}'} \tilde{A}_{\tilde{U}_j}} \tilde{A}_{\tilde{U}_j} \to \bigoplus_j H^i(\tilde{U}_{j, \text{pro}}, F)
\]

(resp. \( \bigoplus_k H^i(\tilde{X}_{\text{pro}}, F) \otimes_{\tilde{C}[s,r]} \tilde{C}[s,r]_{\tilde{U}_j} \to \bigoplus_k H^i(\tilde{X}'_{\text{pro}}, F) \otimes_{\tilde{C}[s,r]} \tilde{C}[s,r]_{\tilde{U}_j} \to \bigoplus_k H^i(\tilde{U}_{j, \text{pro}}, F) \)), both the second arrow and the composition are isomorphisms; consequently, the first arrow is also an isomorphism. That is, the morphisms in (7.1.1) become isomorphisms after tensoring over \( \tilde{A}_{\tilde{X}} \) (resp. over \( \tilde{C}[s,r]_{\tilde{X}} \)) with \( \bigoplus_j \tilde{A}_{\tilde{U}_j} \) (resp. over \( \bigoplus_j \tilde{C}[s,r]_{\tilde{X}_{\tilde{U}_j}} \)). Since this tensoring is exact on pseudocoherent modules (by Remark 4.5) and faithful (by [12, Lemma 2.3.12]), this implies the desired isomorphism.

8 Higher direct images and applications

With the preceding calculations in hand, we derive global conclusions about the relative cohomology of coherent \((\varphi, \Gamma)\)-modules, and consequences for local systems.

Theorem 8.1. Let \( f : Y \to X \) be a smooth proper morphism of relative dimension \( n \) of rigid analytic spaces over \( K \). Let \( F \) be a coherent \((\varphi, \Gamma)\)-module over \( \tilde{A}_Y \) (resp. \( \tilde{C}[s,r]_Y \)). Then the higher direct images \( R^i f_{\text{pro}}^* F \) are coherent \((\varphi, \Gamma)\)-modules over \( \tilde{A}_X \) (resp. \( \tilde{C}[s,r]_X \)); moreover, these sheaves vanish for \( i > 2n \).

Proof. By combining Proposition 6.1 and Proposition 7.1, we immediately obtain the desired conclusion.

Remark 8.2. In light of the acyclicity of pseudocoherent sheaves on affinoid perfectoid spaces [13, Corollary 3.5.6], Theorem 8.1 implies a posteriori that the assertions of Proposition 6.1 and Proposition 7.1 both hold for the trivial covering of \( X \), without any restriction on \( K \). (Namely, compute the cohomology groups from the higher direct images using the Leray spectral sequence.)

By specializing \( X \) to a point, we obtain the following corollary.

Theorem 8.3. Assume that \( K \) is a finite extension of \( \mathbb{Q}_p \). Let \( X \) be a smooth proper rigid analytic variety of dimension \( n \) over \( K \). Let \( F \) be a coherent \((\varphi, \Gamma)\)-module over \( \tilde{A}_X \) (resp. \( \tilde{C}_X \)). Then the cohomology groups \( H^i_{\varphi, \Gamma}(F) \) are finite \( \mathbb{Z}_p \)-modules (resp. finite-dimensional \( \mathbb{Q}_p \)-vector spaces); moreover, these groups vanish for \( i > 2n + 2 \).
Proof. This follows by combining Theorem 8.3 (resp. Theorem 8.5), [13, Theorem 5.7.10] (resp. [13, Theorem 5.7.11]), and Theorem 8.1.

By further specializing to étale \((\varphi, \Gamma)\)-modules, we obtain the following corollary, which includes Theorem 0.1(b).

**Theorem 8.4.** Assume that \( K \) is a finite extension of \( \mathbb{Q}_p \). Let \( X \) be a smooth proper rigid analytic variety over \( K \).

(a) Let \( T \) be an étale \( \mathbb{Z}_p \)-local system on \( X \), or more generally a locally finite \( \mathbb{Z}_p \)-module on \( X_{\pro\et} \). Then the pro-étale cohomology groups \( H^i(X_{\pro\et}, T) \) are finite \( \mathbb{Z}_p \)-modules.

(b) Let \( V \) be an étale \( \mathbb{Q}_p \)-local system on \( X \). Then the pro-étale cohomology groups \( H^i(X_{\pro\et}, V) \) are finite-dimensional \( \mathbb{Q}_p \)-vector spaces.

Moreover, in both cases, these groups vanish for \( i > 2n + 2 \).

**Proof.** Both parts follow from combining Theorem 8.3 with [12, Theorem 9.4.5].

**Remark 8.5.** Theorem 8.4(a) may also be deduced by reducing to the corresponding statement for an \( \mathbb{F}_p \)-local system, which is a finiteness theorem of Scholze [19, Theorem 5.1]. One can continue in this fashion to establish Theorem 8.4(b) for isogeny \( \mathbb{Z}_p \)-local systems, i.e., étale \( \mathbb{Z}_p \)-local systems admitting a stable lattice. However, it is unclear how to extend the methods used to prove [19, Theorem 5.1] to establish Theorem 8.4(b) for general \( \mathbb{Q}_p \)-local systems, or for more general \((\varphi, \Gamma)\)-modules.

### 9 Base change revisited

In this section, we establish the following general base change theorem. The statement can be formally promoted to a derived version; see Remark 9.3.

**Theorem 9.1.** With notation as in Theorem 8.7, let \( g : X' \to X \) be any morphism of rigid spaces over \( K \). Put \( Y' = Y \times_X X' \) and let \( f' : Y' \to X' \), \( g' : Y' \to Y \) be the base extensions of \( f, g \). Then the natural map

\[
\mathbb{L}g^*_{\pro\et} \mathbb{R}f_{\pro\et} \mathcal{F} \to \mathbb{R}f'_{\pro\et} \mathbb{L}g'^*_{\pro\et} \mathcal{F}
\]

is an isomorphism.

**Remark 9.2.** In the setting of Theorem 9.1, let \( \tilde{X} \) be a perfectoid subdomain of \( X \), and let \( \tilde{M}^\bullet \) be a bounded complex computing \( H^i(X_{\pro\et}, \mathcal{F}) \). Let \( \tilde{X}' \) be a perfectoid subdomain of \( X' \) mapping to \( \tilde{X} \). As in Remark 7.3, define the second complex

\[
\tilde{M}^\bullet = M^\bullet \otimes_{\tilde{A}_{\tilde{X}}} A_{\tilde{X}'} \quad (\text{resp. } \tilde{M}^{\bullet\bullet} = M^{\bullet\bullet} \otimes_{\tilde{C}^{[s,r]}_{\tilde{X}}} \tilde{C}^{[s,r]}_{\tilde{X}'})
\]
and the third complex
\[ \tilde{M}'^\bullet := \tilde{M} \otimes_{\tilde{A}_{\tilde{X}}} \tilde{A}_{\tilde{X}'} \] (resp. \( \tilde{M}'^\bullet = \tilde{M} \otimes_{\tilde{C}_{\tilde{X}'}^{[s,r]}} \tilde{C}_{\tilde{X}'}^{[s,r]} \));
then \( H^i(\tilde{X}^\prime_{\pro\et}, \mathbb{L}\mathcal{g}^\ast_{\pro\et} \mathcal{F}) \cong h^i(\tilde{M}'^\bullet) \), and the claim is that the map \( \tilde{M}'^\bullet \to \tilde{M}^\bullet \) is a quasi-isomorphism.

The following argument is inspired by [19, Corollary 5.12], specifically in its use of [7, Proposition 2.6.1]; however, we need to recast the latter in terms of a suitable topology.

**Lemma 9.3.** In Theorem 9.1, suppose that \( \mathcal{F} \) is killed by \( p \) (resp. by some \( t \in \tilde{C}_{\tilde{X}'}^{[s,r]} \) coprime to \( t_\theta \)). Then the map (9.1.1) is an isomorphism.

**Proof.** As in [13, §3.5], define the \( v \)-topology on the category of perfectoid spaces in which a covering is simply a surjective morphism for which the inverse image of each quasicompact open subset is contained in a quasicompact open subsets (by analogy with the \( h \)-topology for schemes). This topology has the following properties (for \( \tilde{X} \) a perfectoid space).

- Any finite projective \( \mathcal{O}_{\tilde{X}} \)-module is acyclic for the \( v \)-topology [13, Theorem 3.5.5].

- Let \( \nu \) be the morphism from the \( v \)-topology of \( \tilde{X} \) to the analytic site. Then pullback via \( \nu \) defines an equivalence of categories between the categories of vector bundles for the analytic topology and the \( v \)-topology [13, Theorem 3.5.8].

- For \( \mathcal{F} \) a finite projective \( \mathcal{O}_{\tilde{X}} \)-module for the \( v \)-topology, we have \( R^i\nu_\ast \mathcal{F} = 0 \) for \( i > 0 \) [13, Corollary 3.5.9]. In particular, for \( \mathcal{F} \) a finite projective \( \mathcal{O}_{\tilde{X}} \)-module for the analytic topology, the map \( \mathcal{F} \to R^\nu_\ast (\nu^\ast \mathcal{F}) \) is an isomorphism.

Using the perfectoid correspondence, we deduce the corresponding assertions with \( \mathcal{O} \) replaced by \( \tilde{A}/(p) \) or \( \tilde{C}_{\tilde{X}'}^{[s,r]}/(t) \).

By Remark 4.9 (resp. Remark 4.17), \( \mathcal{F} \) is finite projective over \( \tilde{A}_Y/(p) \) (resp. over \( \tilde{C}_{\tilde{Y}'}^{[s,r]}/(t) \)). Similarly, by Proposition 6.1 the sheaves \( R^if_{\pro\et} \ast \mathcal{F} \) are finite projective over \( \tilde{A}_X/(p) \) (resp. over \( \tilde{C}_{\tilde{X}}^{[s,r]}/(t) \)). In particular, the ordinary and derived pullbacks of these sheaves along any morphism of perfectoid spaces coincide.

In light of the previous discussion, we may argue as follows. Let \( \tilde{X} \) be a perfectoid subdomain of \( X \), let \( \tilde{Y} \) be a perfectoid subdomain of \( Y \) lying over \( \tilde{X} \), let \( \tilde{X}' \) be a perfectoid subdomain of \( X' \) lying over \( \tilde{X} \times_X X' \) (which may not itself be perfectoid), and let \( \tilde{Y}' \) be a perfectoid subdomain of \( Y' \) lying over \( \tilde{X}' \times_X Y \). If we restrict the morphisms \( f, g, f', g' \) to these perfectoid subdomains, then the morphism in (9.1.1) is naturally isomorphic to the same morphism computed using the \( v \)-topology instead of the pro-étale topology, and the latter is an isomorphism because every morphism of perfectoid spaces is part of a \( v \)-covering. This proves the claim.

**Lemma 9.4.** In Theorem 9.1, suppose that \( X' \) is a point and that \( \mathcal{F} \) is killed by \( t_\theta \). Then the map (9.1.1) is an isomorphism.
Proof. Since the formation of both sides commutes with extension of $K$, we may reduce to the case where $X'$ is a $K$-rational point. In the notation of Remark 9.2, we must check that the morphism from the uncompleted base extension $\tilde{M}^\bullet$ to the completed base extension $\tilde{M}'^\bullet$ is a quasi-isomorphism. However, in this case $\mathcal{F}$ is a coherent $\hat{O}_Y$-module and the morphisms

$$\hat{O}_X \to g_{\text{pro}\acute{e}t}^* \hat{O}_{X'}, \quad \hat{O}_Y \to g_{\text{pro}\acute{e}t}^* \hat{O}_{Y'},$$

are surjective, so in fact each individual morphism $\tilde{M}^\bullet_i \to \tilde{M}'^\bullet_i$ is an isomorphism. \hfill \square

The following argument emulates [14, Lemma 4.1.5]

**Proof of Theorem 9.1.** We wish to check that that (9.1.1) induces isomorphisms of cohomology groups, or equivalently that the associated mapping cone $C = \text{Cone}(\tilde{M}^\bullet \to \tilde{M}'^\bullet)$ is acyclic. We know that the cohomology groups of $\tilde{M}'^\bullet$ are pseudocoherent, and that the cohomology groups of $\tilde{M}^\bullet$ are the base extensions of pseudocoherent modules. Using Remark 4.9 in the type $A$ case, or Theorem 4.16 in the type $C$ case, it follows that the cohomology groups of $C$ are coherent $(\varphi, \Gamma)$-modules. If one of them were nonzero, then this could be detected from its reduction modulo $p$ (in the type $A$ case) or some irreducible $t \in \tilde{C}^{[s,r]}_K$ (in the type $C$ case) evaluated at a suitable point; however, this is ruled out by Lemma 9.3 and Lemma 9.4.

We conclude this discussion with a formal observation.

**Remark 9.5.** Since coherent $(\varphi, \Gamma)$-modules form an abelian category and are preserved under derived pullback along an arbitrary morphism (by Remark 4.9 in the type $A$ case and Theorem 4.16 in the type $C$ case) and derived pushforward along a smooth proper morphism (by Theorem 8.1), Theorem 9.1 as stated immediately implies the corresponding statement with the sheaf $\mathcal{F}$ replaced by a bounded complex of coherent $(\varphi, \Gamma)$-modules, or more generally a perfect complex of sheaves of $\hat{A}$-modules (resp. $\hat{C}^{[s,r]}$-modules) whose cohomology groups are coherent $(\varphi, \Gamma)$-modules. Note that in the latter, it is not necessary to equip the complex itself with an action of $\varphi$. Similar observations apply to our other results.

10 Projectivity of higher direct images

In Theorem 8.1, if $\mathcal{F}$ is a projective $(\varphi, \Gamma)$-module, it is not apparent under what conditions its higher direct images will themselves be projective, rather than merely coherent. We establish some sufficient conditions for this, starting with the type $A$ case.

**Theorem 10.1.** With notation as in Theorem 8.1, suppose that $\mathcal{F}$ is the coherent $(\varphi, \Gamma)$-module over $\hat{A}_Y$ associated to the locally finite $\mathbb{Z}_p$-module $T$ on $Y$. Then the higher direct images of $\mathcal{F}$ along $Y_{\text{pro}\acute{e}t} \to X_{\text{pro}\acute{e}t}$ are the coherent $(\varphi, \Gamma)$-modules over $\hat{A}_X$ associated to the higher direct images of $T$; in particular, the higher direct images of $\mathcal{F}$ are projective if and only if the higher direct images of $T$ are $p$-torsion-free (i.e., are $\mathbb{Z}_p$-local systems).
Proof. By the second part of Theorem 4.12, we may recover the higher direct images of $T$ as the cohomology groups of the mapping cone of $\varphi - 1$ on the derived pushforward of $\mathcal{F}$. Now note that for any $\varphi$-module over $\tilde{A}_X$, the map $\varphi - 1$ is surjective as a morphism of pro-étale sheaves; we may thus recover the higher direct images of $T$ from the individual higher direct images of $\mathcal{F}$ by taking $\varphi$-invariants. By the first part of Theorem 4.12, we may turn around and recover the higher direct images of $\mathcal{F}$ from the higher direct images of $T$ by tensoring over $\mathbb{Z}_p$ with $\tilde{A}_X$. This proves the claim. \hfill \Box

Remark 10.2. It is possible for the higher direct images of $T$ to have nontrivial $p$-torsion even if $T$ is $p$-torsion-free. See for example [1, §2.2].

Theorem 10.3. Assume that $K$ is algebraically closed. Let $X$ be a smooth proper rigid analytic variety over $K$. Let $T$ be an étale $\mathbb{Z}_p$-local system on $X$. Then the pro-étale cohomology groups $H^i(X_{\text{pro-\acute{e}t}}, T)$ are finite $\mathbb{Z}_p$-modules; moreover, these groups vanish for $i > 2n$.

Proof. This is immediate from Theorem 8.1 plus Theorem 10.1. \hfill \Box

Remark 10.4. The analogue of Theorem 10.1 in type C is subtler because the functor from $\mathbb{Q}_p$-local systems to $(\varphi, \Gamma)$-modules is only fully faithful, not essentially surjective; moreover, the surjectivity of $\varphi - 1$ is true on the essential image of this functor, but not on the whole category. We thus can only say a priori that there is a natural quasi-isomorphism

$$Rf_*V \cong \text{Cone}(\varphi - 1, Rf_*\mathcal{F}).$$

(10.4.1)

In particular, there is a natural morphism

$$(R^if_*V) \otimes_{\mathbb{Q}_p} \tilde{C}_X^{[s,r]} \to R^if_*\mathcal{F}$$

(10.4.2)

but it is not guaranteed to be either injective or surjective.

To address the previous discussion, we first treat the absolute case (i.e., where the base space is a point) in detail.

Lemma 10.5. Suppose that $K$ is algebraically closed. Let $X = \text{Spa}(A, A^+)$ be a smooth affinoid space over $K$. Let $T$ be a locally finite $\mathbb{Z}_p$-module on $X$. Then for all $i \geq 0$, the following statements hold.

(a) The group $H^i(X_{\text{pro-\acute{e}t}}, T)$ is a finite $\mathbb{Z}_p$-module.

(b) Let $K'$ be an algebraically closed nonarchimedean field containing $K$, put $X' = X \times_K K'$, and let $T'$ be the pullback of $T$ to $X'$. Then the morphism $H^i(X_{\text{pro-\acute{e}t}}, T) \to H^i(X'_{\text{pro-\acute{e}t}}, T')$ is an isomorphism.

Proof. Since both claims are local, we may assume that $X$ is the base of a restricted toric tower $\psi$. Let $\psi'$ be the base extension of this tower to $X'$. Let $A_\psi$ (resp. $A_{\psi'}$) be the imperfect
period ring associated to the tower \( \psi \) (resp. \( \psi' \)); by [13, Theorem 7.1.5], the vertical arrows in the commutative diagram

\[
\begin{array}{ccc}
H^i(X_{\pro\et}, T) & \longrightarrow & H^i(X_{\pro\et}, T') \\
\downarrow & & \downarrow \\
H^i_{\varphi, \Gamma}(\mathcal{F}) & \longrightarrow & H^i_{\varphi, \Gamma}(\mathcal{F}')
\end{array}
\]

are isomorphisms. To check that the horizontal arrows are isomorphisms, by Lemma 3.2 it suffices to check the case where \( T \) is killed by \( p \); using the adjunction morphism for a suitable finite étale cover of \( X \), we may further assume that \( T \) is trivial and \( X \) is connected. In this case, let \( R_\psi \) (resp. \( R_{\psi'} \)) denote the reduction of \( A_\psi \) (resp. \( A_{\psi'} \)) modulo \( p \). Let \( K^\flat \) (resp. \( K'^\flat \)) denote the tilt of \( K \) (resp. \( K' \)). The residue map

\[
f \mapsto \text{Res} f \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_n}{T_n}
\]

defines a projection of \( R_\psi \) onto \( K^\flat \) in the category of Banach modules over \( K^\flat \); the kernel of this morphism can be expressed as the completed direct sum of \( \varphi^s(S) \) over all \( s \geq 0 \), where

\[
S = \bigoplus (e_1, \ldots, e_n) \in (\mathbb{Z}/p\mathbb{Z})^n \setminus (0, \ldots, 0)T_1^{e_1} \cdots T_n^{e_n} R_\psi^p.
\]

In particular, on this kernel, \( \varphi - 1 \) vanishes and its cokernel is isomorphic to \( S \).

Using the construction from [13, Lemma 7.1.7], we may exhibit a chain homotopy witnessing the vanishing of \( H^i_{\varphi, \Gamma}(S) \) for all \( i \geq 0 \); it follows that the morphisms

\[
H^i_{\varphi, \Gamma}(K^\flat) \to H^i_{\varphi, \Gamma}(R_\psi), \quad H^i_{\varphi, \Gamma}(K'^\flat) \to H^i_{\varphi, \Gamma}(R_{\psi'})
\]

are isomorphisms. Since

\[
\ker(\varphi - 1, K^\flat) = \ker(\varphi - 1, K'^\flat) = \mathbb{F}_p, \quad \text{coker}(\varphi - 1, K^\flat) = \text{coker}(\varphi - 1, K'^\flat) = 0,
\]

this yields the desired result.

We now recover Theorem 0.1(a) and a bit more.

**Theorem 10.6.** Assume that \( K \) is algebraically closed. Let \( X \) be a smooth proper rigid analytic variety over \( K \). Let \( V \) be an étale \( \mathbb{Q}_p \)-local system on \( X \).

(a) The pro-étale cohomology groups \( H^i(X_{\pro\et}, V) \) are finite-dimensional \( \mathbb{Q}_p \)-vector spaces; moreover, these groups vanish for \( i > 2n \).

(b) The formation of the groups in (a) commutes with base change from \( K \) to a larger algebraically closed nonarchimedean field.

(c) Let \( f : X \to \text{Spa}(K, K^+) \) be the structure morphism, and let \( \mathcal{F} \) be the étale \((\varphi, \Gamma)\)-module over \( \mathcal{C}[^s,r]_X \) associated to \( V \). Then for each \( i \), \( R^if_*\mathcal{F} \) is an étale projective \((\varphi, \Gamma)\)-module.
Proof. We consider the category of Banach-Colmez spaces over $K$ (i.e., the Espaces de Banach de dimension finie of [2]); roughly speaking, these are objects in the category of topological $\mathbb{Q}_p$-vector spaces which, in the quotient by the Serre subcategory of finite-dimensional $\mathbb{Q}_p$-vector space, becomes isomorphic to the restriction of scalars of a finite-dimensional $K$-vector space. Each such space has a $K$-dimension, which is a nonnegative integer, and a $\mathbb{Q}_p$-dimension, which can be any integer in general but must be nonnegative if the $K$-dimension vanishes; these are both additive in short exact sequences.

Let $\mathcal{C}$ denote the category of $\varphi$-modules over the Robba ring associated to $K$ (or equivalently, vector bundles on the Fargues-Fontaine curve associated to $K$). The basic properties of $\mathcal{C}$ that we need (e.g., see [4]) are that every object of $\mathcal{C}$ admits a direct sum into copies of certain standard objects $\mathcal{O}(s)$ parametrized by $s \in \mathbb{Q}$, and the cohomology groups of these objects (which are only supported in degrees 0 and 1) are Banach-Colmez spaces over $K$ with the following additional constraints.

- For $s < 0$, $H^0$ vanishes and $H^1$ has positive $K$-dimension (determined by $s$).
- For $s = 0$, $H^0$ has $K$-dimension 0 (but nonzero $\mathbb{Q}_p$-dimension) and $H^1$ vanishes.
- For $s > 0$, $H^0$ has positive $K$-dimension (determined by $s$) and $H^1$ vanishes.

From [10.4.1] and the preceding discussion, it follows that each group $H^i(Y_{\text{pro}^\text{et}}, V)$ is a Banach-Colmez space; moreover, base extension from $K$ to $K'$ preserves the $K$-dimension. However, by applying Lemma [10.5] locally on $Y$, we see that $R^if_*V$ is invariant under an algebraically closed base field extension, so the $K$-dimension of $H^i(Y_{\text{pro}^\text{et}}, V)$ must vanish. It follows that $H^i(Y_{\text{pro}^\text{et}}, V)$ is a finite-dimensional $\mathbb{Q}_p$-vector space; this proves (a) and (b). Again from [10.4.1] and the classification, it follows that $R^if_*\mathcal{F}$ splits as a direct sum of copies of $\mathcal{O}(s)$ where only $s = 0$ is allowed. This proves (c).

We now promote the previous statement to the relative case.

**Theorem 10.7.** With notation as in Theorem 8.1, suppose that $\mathcal{F}$ is the étale ($\varphi, \Gamma$)-module over $\widehat{\mathcal{C}}^\eta$ associated to the $\mathbb{Q}_p$-local system $V$ on $Y$. For each $i$, the map [10.4.2] is an isomorphism identifying $R^if_*V$ with $(R^if_*\mathcal{F})^{\varphi = 1}$. In particular, $R^if_*V$ is a $\mathbb{Q}_p$-local system and $R^if_*\mathcal{F}$ is an étale projective ($\varphi, \Gamma$)-module.

**Proof.** We proceed by descending induction on $i$. Given the claim for degree $\geq i + 1$, Theorem 9.1 implies that the formation of $R^if_*\mathcal{F}$ commutes with arbitrary base change.

Suppose that $X$ is a curve and $K$ is algebraically closed. By Remark 4.17, the torsion submodule of $R^if_*\mathcal{F}$ is annihilated by some $t \in \widehat{\mathcal{C}}_K^{\text{fil}}$; in particular, any torsion persists upon base extension to a geometric point. By Theorem 10.6 this means that $R^if_*\mathcal{F}$ is torsion-free, and in particular co-$\theta$-projective (see Definition 4.18). Moreover, there exists a Zariski-closed nowhere dense subspace $Z$ of $X$ such $R^if_*\mathcal{F}|_{X \setminus Z}$ is projective (namely the support of the Fitting ideal of $R^if_*\mathcal{F}$; see [13, Remark 8.7.6]).

By Theorem 4.16 we may form the double dual $\mathcal{G}$ of $R^if_*\mathcal{F}$ in the category of coherent ($\varphi, \Gamma$)-modules; by the previous paragraph, the natural morphism $R^if_*\mathcal{F} \to \mathcal{G}$ is injective. By
Theorem 5.9.4, Theorem 7.1.9, \( G \) descends to a reflexive module over a two-dimensional regular noetherian ring, which is therefore projective.

If we pull back to a geometric point of \( X \setminus Z \), then \( R^i f_* F \to G \) becomes an isomorphism and both sides are étale by Theorem 10.6. Since \( Z \) is nowhere dense, this implies that the degree of \( G \) is everywhere 0; moreover, the cokernel \( \mathcal{H} \) of \( R^i f_* F \to G \) has generic rank 0, so it must be \( \theta \)-local.

Let \( g : X' \to X \) be a geometric point supported in \( Z \). Then \( L_1 g^* \mathcal{H} \) is again killed by some power of \( t_\theta \); since

\[
0 \to L_1 g^* \mathcal{H} \to g^* R^i f_* F \to g^* G \to g^* \mathcal{H} \to 0
\]

is exact and \( g^* R^i f_* F \) is torsion-free (again by Theorem 10.6), we must have \( L_1 g^* \mathcal{H} = 0 \). Now \( g^* R^i f_* F \to g^* G \) is an injection with \( \theta \)-local cokernel, so \( \deg(g^* R^i f_* F) \leq \deg(g^* G) \) with equality if and only if \( \mathcal{H} = 0 \). However, \( g^* R^i f_* F \) is étale, so its degree is 0, whereas \( g^* G \) has degree 0 as calculated above. We conclude that the support of \( \mathcal{H} \) is empty, and so \( R^i f_* F \cong G \).

We now drop the restriction on \( X \). By the curve case, the fibers of \( R^i f_* F \) over geometric points of \( X \) are projective of locally constant rank; by [12, Proposition 2.8.4], \( R^i f_* F \) is projective. Since \( R^i f_* F \) is fiberwise étale, [12, Corollary 7.3.9] implies that \( R^i f_* F \) is étale, that is, the map

\[
(R^i f_* F)^\varphi = 1 \otimes_{\mathbb{Q}_p} \tilde{C}^{[s,r]}_X \to R^i f_* F
\]

is an isomorphism. It follows that \( \varphi - 1 \) is surjective on \( R^i f_* F \), so the fact that [10.4.1] is a quasi-isomorphism now implies that [10.4.2] is an isomorphism, as desired.

Remark 10.8. Recall that by combining Theorem 4.13 with the properties of the \( v \)-topology described in the proof of Lemma 9.3, we may immediately deduce that the categories of \( \mathbb{Q}_p \)-local systems on \( X \) for the pro-étale topology and the \( v \)-topology coincide, and the pullback equivalence is moreover exact (compare [13, Remark 4.5.2].) In light of Theorem 9.1 and Theorem 10.7, we may further deduce that higher direct images of \( \mathbb{Q}_p \)-local systems along smooth proper morphisms can be computed using the \( v \)-topology in place of the pro-étale topology. In particular, formation of these higher direct images commutes with arbitrary base change on \( X \), as expected by analogy with the proper base theorem for étale cohomology of schemes.

The question of when the higher direct images of a projective but not étale \((\varphi, \Gamma)\)-module are again projective is somewhat subtler. One easy observation is the following.

Proposition 10.9. With notation as in Theorem 8.7, suppose that \( F \) is a coherent \((\varphi, \Gamma)\)-module over \( \tilde{C}^{[s,r]}_X \). Suppose also that each connected component of \( X \) contains a point \( x \) such that:

- \( F \) is projective on some neighborhood of \( f^{-1}(x) \); and
- \( f^{-1}(x) \) is contained in the étale locus of \( F \).
Then the higher direct images of $\mathcal{F}$ are co-$\theta$-projective.

**Proof.** By Remark 4.17, the prime-to-$t_\theta$ torsion of each higher direct image has locally constant rank on $X$. By Theorem 9.1, it thus suffices to rule out the existence of such torsion on one fiber per connected component of $X$; this is immediate from Theorem 10.7.

We next consider the case of a de Rham $(\varphi, \Gamma)$-module.

**Definition 10.10.** Suppose that $K$ is discretely valued with perfect residue field. Define the sheaves $\mathcal{O}_{B_{dR,X}}^+, \mathcal{O}_{B_{dR,X}}$ on $X_{pro\acute{e}t}$ as in [13, Definition 8.6.5]. For $\mathcal{F}$ a $(\varphi, \Gamma)$-module over $\tilde{\mathbb{C}}^{[s,r]}_X$ with $1 \in [s,r]$, let $\nu_{pro\acute{e}t} : X_{pro\acute{e}t} \to X$ be the canonical morphism, and define

$$D_{dR}(\mathcal{F}) := \nu_{pro\acute{e}t}^*(\mathcal{F} \otimes_{\tilde{\mathbb{C}}^{[s,r]}_X} \mathcal{O}_{B_{dR,X}}).$$

This is a coherent $\mathcal{O}_X$-module: namely, by [13, Theorem 8.6.2] this holds if $X$ is smooth, and the general case then follows by resolution of singularities. If $X$ is smooth, we moreover have a canonical $\mathcal{O}_X$-linear connection on $D_{dR}(\mathcal{F})$ (see [19, Theorem 7.2]), forcing $D_{dR}(\mathcal{F})$ to be projective.

There is a canonical injective morphism

$$D_{dR}(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{O}_{B_{dR,X}} \to \mathcal{F} \otimes_{\tilde{\mathbb{C}}^{[s,r]}_X} \mathcal{O}_{B_{dR,X}}.$$

We say that $\mathcal{F}$ is *de Rham* if this morphism is an isomorphism. In case $\mathcal{F}$ is the étale $(\varphi, \Gamma)$-module associated to a $\mathbb{Q}_p$-local system $V$, we say that $V$ is de Rham if $\mathcal{F}$ is; by a theorem of the second author and X. Zhu [17], this holds if and only if each connected component of $X$ contains a classical point at which $V$ is de Rham in the usual sense of Fontaine.

**Theorem 10.11.** With notation as in Theorem 8.1, suppose that $K$ is discretely valued with perfect residue field, $X$ is smooth, and $\mathcal{F}$ is a de Rham $(\varphi, \Gamma)$-module over $\tilde{\mathbb{C}}^{[s,r]}_Y$ along $Y_{pro\acute{e}t} \to X_{pro\acute{e}t}$. Then the higher direct images of $\mathcal{F}$ along $Y_{pro\acute{e}t} \to X_{pro\acute{e}t}$ are de Rham $(\varphi, \Gamma)$-modules over $\tilde{\mathbb{C}}^{[s,r]}_X$, and there are natural isomorphisms

$$R^i f_* D_{dR}(\mathcal{F}) \cong D_{dR}(R^i f_{pro\acute{e}t,*} \mathcal{F}). \quad (10.11.1)$$

**Proof.** In light of Theorem 8.1 and the assumption that $K$ is discretely valued, it suffices to check projectivity on the level of $t_\theta$-adic completions, i.e., to check that (10.11.1) is an isomorphism. This follows as in [19, Theorem 7.11].

Combining Theorem 10.7 and Theorem 10.11 gives a relative version of the étale-de Rham comparison isomorphism, generalizing [19, Theorem 1.10] which treats the case of $\mathbb{Z}_p$-local systems.

**Theorem 10.12.** Let $K$ be a complete discretely valued field of characteristic 0 whose residue field is perfect of characteristic $p$. Let $f : Y \to X$ be a smooth proper morphism of rigid analytic spaces over $K$. Let $V$ be a de Rham $\mathbb{Q}_p$-local system on $Y$, and let $\mathcal{F}$ be its associated étale $(\varphi, \Gamma)$-module.
(a) The higher direct images $R^i f_{\text{pro\-ét}} \ast \mathcal{F}$ are étale de Rham $(\varphi, \Gamma)$-modules.

(b) The canonical morphisms $(R^i f_{\text{pro\-ét}} \ast V) \otimes_{\mathbb{Q}_p} \bar{\mathbb{C}}_X \to R^i f_{\text{pro\-ét}} \ast \mathcal{F}$ are isomorphisms. Equivalently, the canonical morphisms $R^i f_{\text{pro\-ét}} \ast V \to (R^i f_{\text{pro\-ét}} \ast \mathcal{F})^{\varphi=1}$ are isomorphisms.

(c) The canonical morphisms $R^i f_\ast D_{\text{dR}}(\mathcal{F}) \to D_{\text{dR}}(R^i f_{\text{pro\-ét}} \ast \mathcal{F})$ are isomorphisms.

We point out a corollary suggested by David Hansen.

**Corollary 10.13.** For $K, f$ as in Theorem 10.12, suppose that $X$ is itself smooth over $K$ (as then is $Y$). Then the sheaves $R^i f_{\ast} \Omega^j_{Y/X}$ on $X$ are vector bundles for all $i, j \geq 0$.

**Proof.** Take $V = \mathbb{Q}_p$ in Theorem 10.12 and identify the sheaves in question with the graded quotients of the filtered sheaf $D_{\text{dR}}(\mathcal{F})$. By [17, Theorem 3.8(ii)], these graded quotients are locally free. \qed

**Remark 10.14.** The point of view implicit in Theorem 10.12 is that it is the $(\varphi, \Gamma)$-module $\mathcal{F}$, rather than the $\mathbb{Q}_p$-local system $V$, that lies at the heart of the comparison isomorphism. This point of view is shared by the work of Bhatt, Morrow, and Scholze [1], in which the crystalline comparison isomorphism is constructed via a certain $A_{\text{inf}}$-valued cohomology theory (for more on which see [18]).

**Remark 10.15.** The restriction to discretely valued fields in the last few results is a side effect of the fact that the de Rham condition on $(\varphi, \Gamma)$-modules requires discreteness in order to make the definition. If one could extend this definition in a sensible way to more general $K$, one could hope to generalize these results also.

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