Abstract  The true variables in QED are the transverse photon components and Dirac’s physical electron, constructed out of the fermionic field and the longitudinal components of the photon. We calculate the propagators in terms of these variables to one loop and demonstrate their gauge invariance. The physical electron propagator is shown not to suffer from infrared divergences in any gauge. In general, all physical Green’s functions are gauge invariant and infrared-finite.
The usual covariant formulation of QED exhibits a gauge dependence which implies that the fermion in the Lagrangian and the four components of $A_\mu$ cannot be physical quantities. In this letter we discuss the physical fields and calculate their propagators in lowest order perturbation theory.

In a general Lorentz gauge the QED Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(iD - m)\psi + i\bar{c}\gamma^\mu c,$$  \hspace{1cm} (1)

where $D_\mu = \partial_\mu + igA_\mu$ and the ghosts are Hermitian. This Lagrangian clearly does not only depend on the physical degrees of freedom, thus some mechanism is needed to isolate the true physical states. The Lagrangian (1) has, however, the following BRST invariance

$$\begin{align*}
\delta A_\mu &= \partial_\mu c, \\
\delta c &= 0, \\
\delta \bar{c} &= -\frac{i}{\xi}\partial_\mu A^\mu, \\
\delta \psi &= -igc\psi, \\
\delta \bar{\psi} &= -ig\bar{\psi}c,
\end{align*}$$  \hspace{1cm} (2)

which is generated by the conserved, nilpotent BRST charge

$$Q = \int d^3x \left((\partial_i F^{0i} + gj_0)c - \frac{1}{\xi}\partial_\mu A^\mu \bar{c}\right).$$  \hspace{1cm} (3)

Those states $|\psi\rangle$ for which $Q|\psi\rangle = 0$, which are not of the zero norm type $|\psi\rangle = Q|\chi\rangle$ (recall $Q^2 = 0$), can be identified with the physical states of QED. Some of the complications encountered in excluding these zero norm states can be avoided by the use of a further symmetry of QED which we have recently observed and which may be directly employed to isolate the physical states[2]. The result of this analysis is that the physical degrees of freedom are just the two transverse photon components and Dirac’s physical electrons[3]

$$\begin{align*}
\psi_{\text{phys}}(x) &= \psi(x) \exp(ig\frac{\partial_i A_i(x)}{\nabla^2}) \quad \text{and} \quad \bar{\psi}_{\text{phys}}(x) = \bar{\psi}(x) \exp(-ig\frac{\partial_i A_i(x)}{\nabla^2}).
\end{align*}$$  \hspace{1cm} (4)

The non-local exponential function creates the Coulomb change in the electric field around the electron[3]. Since $\psi_{\text{phys}}$ represents an electron with spin, mass and charge, we will call this the electron and $\psi$ the fermion in this letter. These fields are physical and hence their Greens functions must also be gauge invariant. The purpose of this letter is to start to
develop a perturbation theory for these physical variables: we will do this by calculating their propagators to one loop. These will be seen to yield a gauge invariant, infrared-finite description of QED.

The physical photon, $A^\text{phys}_\mu$, is given by

$$A^\text{phys}_\mu(k) = P_{\mu\nu}(k)A^\nu(k),$$

where

$$P_{\mu\nu}(k) = \left[ g_{\mu\nu} + \frac{k_\mu k_\nu + k^2 \eta_\mu \eta_\nu - (k_\mu \eta_\nu + k_\nu \eta_\mu) k \cdot \eta}{k^2 - (k \cdot \eta)^2} \right],$$

and $\eta$ is the temporal vector, $(1, 0, 0, 0)$. This projector satisfies $P_{0\nu} = k^\mu P_{\mu\nu} = 0$, and so $A^\text{phys}_\mu$ has just two components, the transverse components of the photon. The physical photon propagator is then

$$D^\text{phys}_{\mu\nu}(k) = \int \frac{d^4k}{(2\pi)^4}\langle T(A^\text{phys}_\mu(x)A^\text{phys}_\nu(y)) \rangle e^{ik \cdot (x-y)}.$$  (7)

It must be gauge invariant. We can easily see that this is the case for the free propagator which is

$$D^{(0)}_{\mu\nu}(k) = P_{\mu\lambda}(k)D^{\lambda\sigma(0)}(k)P_{\sigma\nu}(k)$$

$$= \frac{-1}{k^2} \left[ g_{\mu\nu} + \frac{k_\mu k_\nu + k^2 \eta_\mu \eta_\nu - (k_\mu \eta_\nu + k_\nu \eta_\mu) k \cdot \eta}{k^2 - (k \cdot \eta)^2} \right].$$

(8)

where the bracketed superscript refers to the power of the coupling and $D^{\lambda\sigma(0)}(k)$ is the full free propagator from (1). This gauge invariant result is just the free propagator in radiation gauge$^4$.

To one loop, Fig. 1, the propagator receives the following contribution

$$D^{(2)}_{\mu\nu}(k) = P_{\mu\rho}(k)D^{\rho\lambda(0)}(k)\Pi^{(2)}_{\lambda\sigma}(k)D^{\sigma\tau(0)}(k)P_{\tau\nu}(k)$$

$$= \frac{1}{k^2} \left[ g_{\mu\nu} + \frac{k_\mu k_\nu + k^2 \eta_\mu \eta_\nu - (k_\mu \eta_\nu + k_\nu \eta_\mu) k \cdot \eta}{k^2 - (k \cdot \eta)^2} \right] \Pi(k),$$

(9)

where $\Pi^{(2)}_{\lambda\sigma}(k)$ is the one loop photon polarisation tensor and $\Pi(k)$ is the standard scalar polarisation to one loop. This result is gauge invariant and is again just what one would obtain in radiation gauge. We stress that, contrary to some suggestions in the literature, it is not necessary to go to radiation gauge to obtain this result since this projection
on the propagator is gauge invariant. Indeed from the Ward identity we know that the photon polarisation must have the tensor structure $g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}$ to all orders in the coupling. However, from the transversality to the momentum $k$ of both the polarisation and of the projector $P$ on the physical photons the gauge dependence in the photon propagators vanishes in the physical propagator. Any gauge dependence in the scalar polarisation would now directly enter the physical propagator as a multiplicative factor. We thus see that the photon polarisation must be independent of the gauge parameter to all orders in the coupling\cite{5}. This may be contrasted with the full propagator, where the gauge dependence resides in the propagator for longitudinal photons.

The electron propagator is

$$S^{\text{phys}}(k) = \int \frac{d^4k}{(2\pi)^4} T\langle \psi^{\text{phys}}(x) \bar{\psi}^{\text{phys}}(y) \rangle e^{ik \cdot (x-y)},$$ \hspace{1cm} (10)

which implies the free propagator

$$S^{\text{phys}(0)}(k) = \frac{i}{\not{p} + m}. \hspace{1cm} (11)$$

This is, of course, the usual, gauge invariant, free fermion propagator — reflecting the fact that the fermion is identical to the true electron at $O(g^0)$, or, to put it another way, that the fermion is BRST invariant in lowest order in the coupling.

To one loop (see Fig. 2) there are four separate contributions to the electron propagator: the usual self-energy contribution, i.e., the exponentials are treated to lowest order in the coupling, cross terms where one exponential factor is expanded to order $g$ and a term where both exponentials are so expanded. Note that if either of the exponentials is expanded to second order in the coupling we obtain a tadpole that vanishes in dimensional regularisation. The terms coming from the exponential factors describe interactions with longitudinal photons.

As the reader can see from Fig. 2, we now have to directly calculate the propagator itself; the final three diagrams do not let themselves be interpreted as self energy terms. There are two ways of calculating this propagator. In the Lorentz class all four diagrams contribute. In Coulomb gauge the exponentials are unity and only the self energy diagram survives. (Equivalently one could calculate in radiation gauge where the exponentials are also trivial; there an extra diagram from the four fermion interaction appears, which in
the not completely physical Coulomb gauge is duplicated by the contribution to the self energy from the exchange of temporal photons.) Taking for simplicity massless QED it can be seen by direct calculation of the diagrams in Fig. 2, in the full Lorentz class, that the one loop electron propagator becomes

\[ S_{\text{phys}}^{(2)}(k) = -\frac{g^2}{k^2} \int \frac{d^Dp}{(2\pi)^D} \gamma_\mu \frac{p - k}{(p - k)^2} \gamma_\nu \frac{1}{p^2} \left[ -\gamma^{\mu\nu} + \frac{p^\mu p^\nu - \eta \cdot p (p^\mu \eta^\nu + p^\nu \eta^\mu)}{p^2 - (\eta \cdot p)^2} \right] \frac{1}{k}. \]

(12)

All gauge parameter dependent terms from the various diagrams have, as expected, cancelled. In (12) we recognise

\[ \frac{1}{p^2} \left[ -g^{\mu\nu} + \frac{p^\mu p^\nu - \eta \cdot p (p^\mu \eta^\nu + p^\nu \eta^\mu)}{p^2 - (\eta \cdot p)^2} \right] = D_{\mu\nu}^{\text{Coul}(0)}(p), \]

(13)

the free propagator in the Coulomb gauge. This further demonstrates the gauge invariance of this result, since in Coulomb gauge the electron and fermion propagators are equal. The fermion propagator to one loop in Coulomb gauge may be found in Ref. 6.

This analysis may be straightforwardly extended to higher Green’s functions. For the vertices this means explicitly that the physical electron transverse photon vertex, \( \langle \psi_{\text{phys}}(x) \bar{\psi}_{\text{phys}}(y) A_\mu^{\text{phys}}(z) \rangle \), is invariant and that unphysical vertices, like that with the temporal photon component \( \langle \psi_{\text{phys}}(x) \bar{\psi}_{\text{phys}}(y) A_0(z) \rangle \), are not. From the gauge invariance of the physical vertex it must be equivalent to the one calculated in Coulomb gauge, where electrons may be replaced by fermions: i.e., the physical vertex calculated in any gauge will yield the Coulomb gauge result\(^{[6]}\) for \( \langle \psi(x) \bar{\psi}(y) A_\mu^{\text{phys}}(z) \rangle \).

We have in other words the following equivalent recipes for physical Green’s functions in QED, i.e., those defined in terms of \( \psi_{\text{phys}}, \bar{\psi}_{\text{phys}} \) and the transverse photons, \( A_\mu^{\text{phys}} \). One may either calculate them in any gauge, say a Lorentz gauge, directly, or one may work in Coulomb or radiation gauge where one may use the usual fermions, \( \psi \). The end results will be identical. (Note that in Coulomb gauge one must still project out the external photons upon the physical sector, with that gauge condition this is simply a matter of dropping external temporal photons.)

One of the attractions of Coulomb gauge is its infrared finiteness, and we see that all gauges will now share this important property if physical fields are used.

We have seen above that it was possible to construct a sensible perturbation theory for the physical fields of QED despite their non-local nature. The extension of this approach
to vertex corrections and higher loops should not prove significantly more difficult than the usual version of QED even if covariant gauges are employed. Indeed this description, although at first sight appearing more cumbersome than the standard covariant formulation, has many attractive features: the physical Green’s functions are explicitly gauge invariant and the S-matrix is already infrared-finite.

Acting on the vacuum the operator corresponding to Dirac’s electron (4) will create a state outside of the Fock space. This is simply a consequence of the exponential factor. Coherent states provide a natural arena to discuss this state. Thus we see that basing QED on these physical fields produces an infrared-finite theory, at the expense of extending the traditional Fock space to include coherent states upon which these operators are defined. This should be contrasted with the traditional approach to QED, where the infrared divergences of the Green’s functions motivate the introduction of coherent states\cite{7}. A fuller account of the connection between the coherent states of the physical fields and those generally used will be provided elsewhere\cite{8}.

Although, as we have recently shown\cite{9}, the Gribov ambiguity prevents the construction of a physical quark, it is locally, i.e., perturbatively, possible. (This pseudophysical quark will not, however, reduce in Coulomb gauge to the Lagrangian fermion.) Such perturbative quarks are ‘seen’ in deep inelastic scattering. The non abelian extension of this letter would thus provide a gauge invariant description of the propagation and interaction of perturbatively physical quarks in deep inelastic scattering.

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Figure Captions

**Fig. 1** The one loop contribution to the photon propagator.

**Fig. 2** One loop contributions to the electron propagator. The dashed lines represent contributions from the longitudinal components of $A_i$ coming from the expansion of the exponential factors.

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