PALEY-WIENER ISOMORPHISM OVER INFINITE-DIMENSIONAL UNITARY GROUPS

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Abstract. An analog of the Paley-Wiener isomorphism for the Hardy space with an invariant measure over infinite-dimensional unitary groups is described. This allows us to investigate on such space the shift and multiplicative groups, as well as, their generators and intertwining operators. We show applications to the Gauss-Weierstrass semigroups and to the Weyl-Schrödinger irreducible representations of complexified infinite-dimensional Heisenberg groups.

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1. Introduction

The work deals with the Hardy space $H^{2}_{\chi}$ of square-integrable complex-valued functions with respect to a probability measure $\chi$ over the infinite-dimensional unitary group $U(\infty) := \bigcup \{U(m): m \in \mathbb{N}\}$, extended by unit $1$, which irreducibly acts on a separable complex Hilbert space $E$. Here, $U(m)$ is the subgroup of unitary $(m \times m)$-matrices endowed with Haar’s measure $\chi_{m}$. In what follows, $U(\infty)$ is densely embedded via a universal mapping $\pi$ into the space of virtual unitary matrices $\mathcal{U} = \lim \leftarrow U(m)$ defined as the projective limit under Livšic’s mappings $\pi_{m+1}^{m}: U(m+1) \to U(m)$. The projective limit $\chi = \lim \leftarrow \chi_{m}$, such that each image-measure $\pi_{m+1}^{m}(\chi_{m+1})$ is equal to $\chi_{m}$, is concentrated on the range $\pi(U(\infty))$ consisting of stabilized sequences (see [18, Neretin 2002], [20, Olshanski 2003]). The measure $\chi$ is invariant under right actions [20, n.4]. We refer to [23, Yamasaki 1974], [5, Borodin and Olshanski 2005] for applications of $\chi$ to stochastic processes. Needed properties of Hardy spaces $H^{2}_{\chi}$ can be found in [15]. Various cases of Hardy spaces in infinite-dimensional settings were considered in [9, Cole and Gamelin 1986], [17, Ørsted and Neeb 1998].

Now, we briefly describe results. Using a unitarily weighted symmetric Fock space $\Gamma_{w}$, defined by $E$ and $\chi$, we find an orthogonal basis in $H^{2}_{\chi}$ of Hilbert-Schmidt polynomials such that the conjugate-linear mapping

$$\Phi: \Gamma_{w} \to H^{2}_{\chi}$$
is a surjective isometry. This allows us to establish in Theorem 4.2 an integral formula for a Fock-symmetric $\mathcal{F}$-transform

$$\mathcal{F}: H^2_{\chi} \ni f \mapsto \hat{f} \in H^2_{\chi}$$

where the Hilbert space $H^2_{\chi}$, uniquely determined by $\gamma$, consists of Hilbert-Schmidt analytic entire functions on $E$. Thus, the $\mathcal{F}$-transform acts as an analog of the Paley-Wiener isomorphism over infinite-dimensional groups.

Furthermore, we investigate two different representations of the additive group from $E$ over the Hardy space $H^2_{\chi}$ by shift and multiplicative groups. Theorem 6.1 states that the $\mathcal{F}$-transform is an intertwining operator between the multiplication group $M_{\alpha}$ on $H^2_{\chi}$ and the shift group $T_{\alpha}$ on $H^2_{\omega}$. On the other hand, Theorem 6.2 shows that $\mathcal{F}$ is the same between the shift group $T_{\alpha}$ on $H^2_{\chi}$ and the multiplication group $M_{\alpha^*}$ on $H^2_{\omega}$. Integral formulas describing interrelations between their generators are established. In Theorem 7.1 suitable commutation relations are stated.

Applications to the Gauss-Weierstrass-type semigroups on $H^2_{\chi}$ are shown in Theorem 8.1. An another application to linear representations of complexified infinite-dimensional Heisenberg groups on $H^2_{\chi}$ in a Weyl-Schrödinger form is given in Theorem 9.1.

Infinite-dimensional Heisenberg groups was considered in [16, Neeb 2000] by using reproducing kernel Hilbert spaces. The Schrödinger representation of infinite-dimensional Heisenberg groups on $L^2_{\chi}$ with respect to a Gaussian measure $\gamma$ over a real Hilbert space is described in [3, I.Belitő, D.Belitő and M. Măntoiu 2016] (see also earlier publications [11, 12]).

2. Hilbert-Schmidt analyticity

Let $E$ stand for a separable complex Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ and a fixed orthonormal basis $\{ e_k : k \in \mathbb{N} \}$. Denote by $E_{\text{alg}}^\otimes_n = E \otimes_n \otimes E \ (n \in \mathbb{N})$ its algebraic tensor power consisted of the linear span of elements $\psi_n = x_1 \otimes \ldots \otimes x_n$ with $x_i \in E$ ($i = 1, \ldots, n$). Set $x_{\otimes n} := x \otimes_n \otimes x$. The symmetric algebraic tensor power $E_{\text{alg}}^\otimes_n = E \otimes \ldots \otimes E$ is defined to be the range of the projector $s_n: E_{\text{alg}}^\otimes_n \ni \psi_n \mapsto x_1 \otimes \ldots \otimes x_n$ with $x_1 \otimes \ldots \otimes x_n := (n!)^{-1} \sum\sigma x_\sigma(1) \otimes \ldots \otimes x_\sigma(n)$ where $\sigma: \{ 1, \ldots, n \} \mapsto \{ \sigma(1), \ldots, \sigma(n) \}$ runs through all permutations. The symmetric algebraic Fock space is defined as the orthogonal sum $\Gamma_{\text{alg}} = \bigoplus_{n \in \mathbb{Z}_+} E_{\text{alg}}^\otimes_n$ with $E_{\otimes 0} = \mathbb{C}$.

Let $E_{\text{h}}^\otimes_n := E \otimes \ldots \otimes E$ be the completion of $E_{\text{alg}}^\otimes_n$ with respect to the Hilbertian norm $\| \psi_n \|_h = \langle \psi_n | \psi_n \rangle^{1/2}$ with $\langle \psi_n | \psi_n \rangle_h = \langle x_1 | x_1 \rangle \ldots \langle x_n | x_n \rangle$. Denote by $E_{\text{h}}^\otimes_n$ the range of continuous extension of $s_n$ on $E_{\text{h}}^\otimes_n$. As usual, the symmetric Fock space is defined to be $\Gamma_{\text{h}} = \bigoplus_{n \in \mathbb{Z}_+} E_{\text{h}}^\otimes_n$.

Denote by $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}_m^m$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ a partition of $n$, that is, $n = |\lambda|$ where $|\lambda| := \lambda_1 + \ldots + \lambda_m$. Any $\lambda$ may be identified with Young’s diagram of length $l(\lambda) = m$. Let $\mathbb{Y}$ denote all diagrams and $\mathbb{Y}_n = \{ \lambda \in \mathbb{Y} : |\lambda| = n \}$. Assume that $\mathbb{Y}_0 = \{ \emptyset \in \mathbb{Y} : |\emptyset| = 0 \}$ and $l(\emptyset) = 1$. Let $\mathbb{N}_n^m := \{ t = (t_1, \ldots, t_m) \in \mathbb{N}_m^m : u \neq t_k, \forall l \neq k \}$. For each $\lambda \in \mathbb{Y}$ we assign the constant

$$C_{|\lambda|, l(\lambda)} := \frac{(l(\lambda) - 1)! |\lambda|!}{(l(\lambda) - 1 + |\lambda|)!} \leq 1. \tag{2.1}$$

The spaces $E_{\text{alg}}^\otimes_n$ and $\Gamma_{\text{alg}}$ may be generated by the basis of symmetric tensors

$$e_{\lambda}^{\otimes Y} = \bigcup \{ e_{\lambda_1}^{\otimes \lambda_1} \otimes \ldots \otimes e_{\lambda_m}^{\otimes \lambda_m} : (\lambda, t) \in \mathbb{Y}_n \times \mathbb{N}_n^m \}$$

$$e_{\otimes Y} = \bigcup \{ e_{\otimes Y}^n : n \in \mathbb{Z}_+ \}$$

with $e_{\otimes 0} = 1$,

respectively. As is known [4, Sec. 2.2.2], norm of basis element in $\Gamma_{\text{h}}$ is equal to

$$\| e_{\lambda}^{\otimes \lambda} \|_h^2 = \frac{\lambda!}{|\lambda|!}, \quad \lambda! := \lambda_1! \cdot \ldots \cdot \lambda_m! \tag{2.2}$$
Let us define a new Hilbertian norm on \( \Gamma_{\text{alg}} \) by the equality \( \| \cdot \|_w = \langle \cdot | \cdot \rangle_w^{1/2} \) where scalar product \( \langle \cdot | \cdot \rangle_w \) is determined via the orthogonal relations

\[
\langle e_i^{\otimes \lambda} | e_i'^{\otimes \lambda'} \rangle_w = \begin{cases} \frac{C_{|\lambda|,|\lambda'|}}{\sqrt{h}} \| e_i^{\otimes \lambda} \|^2 : \lambda = \lambda' \text{ and } i = i', \\ 0 : \lambda \neq \lambda' \text{ or } i \neq i'. \end{cases}
\]

Denote by \( E_w^{\otimes n} \) and \( \Gamma_w \) the appropriate completions of \( E_{\text{alg}}^{\otimes n} \) and \( \Gamma_{\text{alg}} \), respectively. For any \( \lambda \in \mathbb{N}_\ast(\lambda) \) there corresponds in \( E_w^{\otimes n} \) the \( d \)-dimension subspace with \( d = C_{|\lambda|,|\lambda'|}^{-1} \), spanned by elements \( \{ e_i^{\otimes \lambda} : \lambda \in \mathbb{Y}_n \} \). The Hilbertian orthogonal sum

\[
\Gamma_w = \bigoplus_{n \in \mathbb{Z}_+} E_w^{\otimes n}
\]

endowed with \( \langle \cdot | \cdot \rangle_w \) we will call unitarily weighted symmetric Fock space.

Let \( x = \sum e_k x_k \) be the Fourier series of \( x \in E \) with coefficients \( x_k = \langle x | e_k \rangle \). We assign to any \( (\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_\ast(\lambda) \) the \( n \)-homogenous Hilbert-Schmidt polynomial defined via the Fourier coefficients

\[
x_i^\lambda := \langle x^{\otimes n} | e_i^{\otimes \lambda} \rangle_w = x_{i,1}^{\lambda_1} \cdots x_{i,l}^{\lambda_l}, \quad x \in E.
\]

Using the tensor multinomial theorem, we define in \( \Gamma_w \) the Fourier decomposition of exponential vectors (or coherent state vectors)

\[
\varepsilon(x) := \bigoplus_{n \in \mathbb{Z}_+} \frac{x^{\otimes n}}{n!} = \bigoplus_{n \in \mathbb{Z}_+} \frac{1}{n!} \left( \sum_{k \in \mathbb{N}} e_k x_k \right)^{\otimes n}
\]

(2.3)

\[
= \bigoplus_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_\ast(\lambda)} \frac{n!}{\lambda!} e_i^{\otimes \lambda} x_i^\lambda
\]

with respect to the basis \( e^{\otimes Y} \). It is convergent in \( \Gamma_w \) in view of (2.1) and

\[
\| \varepsilon(x) \|^2_w = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!^2} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_\ast(\lambda)} \left( \frac{n!}{\lambda!} \right)^2 \| e_i^{\otimes \lambda} \|^2_w |x_i^\lambda|^2
\]

(2.4)

\[
= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!^2} \sum_{(\lambda, i)} \frac{n!}{\lambda!} C_{|\lambda|,|\lambda'|} |x_i^\lambda|^2 \leq \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i)} \frac{n!}{\lambda!} |x_i^\lambda|^2
\]

\[
= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left( \sum_{k \in \mathbb{N}} |x_k|^2 \right)^n = e \|x\|^2.
\]

Particularly, (2.4) implies that the function \( E \ni x \mapsto \varepsilon(x) \in \Gamma_w \) is entire analytic.

Consider the space of complex-valued functions in the variable \( x \in E \)

\[
H_w^2 := \{ \psi^\ast(x) := \langle \varepsilon(x) | \psi \rangle_w : \psi \in \Gamma_w \} \quad \text{with norm} \quad \| \psi^\ast \| = \| \psi \|_w.
\]

Every function \( \psi^\ast \) is entire analytic as the composition of \( \varepsilon(\cdot) \) with \( \langle \cdot | \psi \rangle_w \). The subspace in \( H_w^2 \) of \( n \)-homogenous Hilbert-Schmidt polynomials is defined to be

\[
H_w^{2,n} := \{ \psi_n^\ast(x) = \langle x^{\otimes n} | \psi_n \rangle_w : \psi_n \in E_w^{\otimes n} \}.
\]

Evidently, \( H_w^2 = \mathbb{C} \oplus H_w^{2,1} \oplus H_w^{2,2} \oplus \ldots \).

It is important that \( H_w^2 \) is uniquely determined by \( \Gamma_w \) since \( \{ \varepsilon(x) : x \in E \} \) is total in \( \Gamma_w \).

Similarly, for the subspace \( H_w^{2,n} \) which is uniquely determined by \( E_w^{\otimes n} \), since \( \{ x^{\otimes n} : x \in E \} \) is total in \( E_w^{\otimes n} \). The last totality follows from the polarization formula for symmetric tensor products

\[
\varepsilon_i^{\otimes \lambda} = \frac{1}{2^n n!} \sum_{\theta_1, \ldots, \theta_n = \pm 1} \theta_1 \cdots \theta_n \theta_i^{\otimes n} \quad \text{with} \quad a = \sum_{i=1}^{l(\lambda)} \theta_i \varepsilon_i^{\otimes \lambda_i}.
\]

(2.5)
which is valid for all $e_i^\lambda \in e^\infty Y_n$ (see e.g., [11, Sec. 1.5]) Thus, the conjugate-portmanteau theorem \[\psi \mapsto \psi^*\] from $\Gamma_w$ onto $H_w^2$ and from $E_w^{\infty n}$ onto $H_w^{\infty n}$ hold.

In conclusion, we can notice that every analytic function $\psi^* \in H_w^2$ determined by $\psi = \sum \psi_n \in \Gamma_w$, $(\psi_n \in E_w^{\infty n})$ has the Taylor expansion at zero

$$\psi^*(x) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, \nu) \in Y_n \times N_n(\lambda)} \frac{\langle e_i^\lambda, \psi_n \rangle_w}{\|e_i^\lambda\|^2_w} x^\lambda, \quad x \in E$$

that follows from (2.3). The function $\psi^*$ is entire Hilbert-Schmidt analytic [15, n.5], [14, n.2].

Note that analytic functions of Hilbert-Schmidt types were considered in [10, 21]. More general classes of analytic functions associated with coherent sequences of polynomial ideals were described in [8].

3. Hardy space over $U(\infty)$

In what follows, we endow each group $U(m)$ with the probability Haar measure $\chi_m$ and assume that $U(m)$ is identified with its range with respect to the embedding $U(m) \ni u_m \mapsto \begin{bmatrix} u_m & 0 \\ 0 & 1 \end{bmatrix} \in U(\infty)$. The Livšč transform from $U(m+1)$ onto $U(m)$ is described in [18, Prop. 0.1] and [20, Lem. 3.1] as the surjective Borel mapping

$$\pi^{-1}_m: u_{m+1} := \begin{bmatrix} z_m & a \\ b & t \end{bmatrix} \mapsto u_m := \begin{cases} z_m - [a(1+t)^{-1}b] & : t \neq -1 \\ z_m & : t = -1. \end{cases}$$

The projective limit $\mathcal{U} := \lim U(m)$ under $\pi^{-1}_m$ has surjective Borel projections $\pi_m: \mathcal{U} \ni u \mapsto u_m \in U(m)$ such that $\pi_m = \pi^{-1}_m \circ \pi_{m+1}$.

Consider a universal dense embedding $\pi: U(\infty) \ni \mathcal{U}$ which to every $u_m \in U(m)$ assigns the stabilized sequence $u = (u_k)$ such that (see [20, n.4])

$$\pi: U(m) \ni u_m \mapsto (u_k) \in \mathcal{U}, \quad u_k = \begin{cases} \pi^{-1}_m(u_m) & : k < m \\ u_m & : k \geq m, \end{cases}$$

where $\pi^{-1}_k := \pi^{-1}_{k-1} \circ \ldots \circ \pi^{-1}_m$ for $k < m$ and $\pi^{-1}_m$ is identity mapping for $k \geq m$. On its range $\pi(U(\infty))$, endowed with the Borel structure from $\mathcal{U}$, we consider the inverse mapping

$$\pi^{-1}: \mathcal{U} \ni (u_k) \mapsto (\pi^{-1}_m(u_m)) \ni U(\infty)$$

The right action $\mathcal{U}_r \ni u \mapsto u.g \in \mathcal{U}_r$ with $g = (v, w) \in U(\infty) \times U(\infty)$ is defined by $\pi_m(u.g) = w^{-1}\pi_m(u)v$ where $m$ is so large that $g = (v, w) \in U(m) \times U(m)$.

Following [18, n.3.1], [20, Lem. 4.8] via the Kolmogorov consistency theorem (see e.g., [19, Thm 1], [25, Cor. 4.2]) we uniquely define on $\mathcal{U} = \lim U(m)$ the probability measure $\chi := \lim \chi_m$ such that each image-measure $\pi^{-1}_m(\chi_m + 1)$ is equal to $\chi_m$. For any Borel subset $A \subset \mathcal{U}_r$ we have $\pi_{m+1}(A) \subseteq [\pi^{-1}_{m+1} - 1][\pi_{m+1}(A)]$, because $\pi_m = \pi^{-1}_m \circ \pi_{m+1}$. It follows that $(\pi_m \circ \pi_{m+1})(A) = \pi^{-1}_{m+1}(\chi_m + 1)[\pi_{m+1}(A)] = \chi_m + 1[\pi^{-1}_{m+1} - 1][\pi_{m+1}(A)] \geq (\chi_m \circ \pi_{m+1})(A)$. Hence, $\chi$ satisfies the condition

$$\chi(A) = \inf(\chi_m \circ \pi_{m+1})(A) = \lim \chi_m(A)$$

and therefore the projective limit $\lim \chi_m$ exists on $\mathcal{U}_r$ via the well known Prohorov theorem [6, Thm IX.52]. Moreover, it is a Radon probability measure concentrated on $\mathcal{U}_r$ [25, Thm 4.1]. By the known portmanteau theorem [13, Thm 13.16] and the Fubini theorem, the invariance of Haar measures $\chi_m$ together with [62] yield the invariance properties under the right action,

$$\int f(u.g) d\chi(u) = \int f(u) d\chi(u), \quad g \in U(\infty) \times U(\infty), \quad f \in L^\infty,$$

$$\int f d\chi = \int d\chi(u) \int_{U(m) \times U(m)} f(u.g) d(\chi_m \otimes \chi_m)(g),$$
where $L^\infty_\chi$ stands for the space of all $\chi$-essentially bounded complex-valued functions defined on $\mathcal{U}_\pi$ and endowed with norm $\|f\|_\infty = \text{ess sup}_{u \in \mathcal{U}_\pi} |f(u)|$.

Let $L^2_\chi$ be the space of square-integrable $\mathbb{C}$-valued functions $f$ on $\mathcal{U}_\pi$ with norm

$$\|f\|_\chi = (\langle f \mid f \rangle_\chi)^{1/2} \quad \text{where} \quad \langle f \mid f \rangle_\chi := \int f\overline{f} \, d\chi.$$ 

The embedding $L^\infty_\chi \ni L^2_\chi$ holds, moreover, $\|f\|_\chi \leq \|f\|_\infty$ for all $f \in L^\infty_\chi$.

To given the $E$-valued mapping $\mathcal{U}_\pi \ni u \mapsto \pi^{-1}(u) e_1$, we can well-define the Borel $\chi$-essentially bounded functions in the variable $u \in \mathcal{U}_\pi$,

$$\phi_k := \phi_{e_k}, \quad \phi_{e_k}(u) = \langle \pi^{-1}(u) e_1 \mid e_k \rangle, \quad k \in \mathbb{N},$$

which do not depend on the choice of $e_1$ in $\bigcup S(m)$ where $S(m)$ is the $m$-dimensional unit sphere in $E$ [13, n.3]. The uniqueness of $\phi_x(u) = \langle \pi^{-1}(u) e_1 \mid x \rangle$ with $x \in E$ results from the total embedding $\pi: U(\infty) \ni \mathcal{U}$. From [3.1] it follows that $\pi^{-1} \circ \pi^{-1}$ coincides with the embedding $U(m) \ni \pi^{-1}(x)$.

Hence, by [3.2] and the portmanteau theorem there exist the limit

$$\int \phi_x \, d\chi = \lim_{m \to \infty} \int U(m) \phi_x \, d(\chi_m \circ \pi_m) = \lim_{m \to \infty} \int U(m) \phi_x \, d(\pi_m^{-1}) \, d\chi_m,$$

where $\phi_x \in L^\infty_\chi$ for any $\phi_x(u) = \langle \pi^{-1}(u) e_1 \mid x \rangle$ with $x \in E$.

By formula (2.5) to every $e^\lambda \in e^\lambda \mathcal{Y}_n$ there uniquely corresponds the Borel function from $L^\infty_\chi$

$$\phi^\lambda(u) := \langle [\pi^{-1}(u) e_1]^\circ m \mid e^\lambda \rangle_w = \phi^\lambda_1(u) \ldots \phi^\lambda_l(u)$$

in the variable $u \in \mathcal{U}_\pi$. It follows that the orthogonal basis $\phi^\lambda = e^\lambda_1 \otimes \ldots \otimes e^\lambda_m$, indexed by $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{Y}$ and $t = (t_1, \ldots, t_m) \in \mathbb{N}_+^m$ with $m = l(\lambda)$, uniquely determines the systems of $\mathcal{B}_\chi$-essentially bounded functions in the variable $u \in \mathcal{U}_\pi$.

$$\phi^\lambda = \bigcup \left\{ \phi^\lambda : (\lambda, t) \in \mathcal{Y} \times \mathbb{N}_+^m, \quad m = l(\lambda) \right\},$$

with $\phi^0 \equiv 1$.

The Hardy space $H^2_\chi$ is defined as the closed complex linear span of $\phi^\lambda$ endowed with $L^2_\chi$-norm. The following assertion is proved in [13, Thm 3.2].

**Theorem 3.1.** The system of Borel functions $\phi^\lambda$ forms an orthogonal basis in $H^2_\chi$ such that

$$\|\phi^\lambda\|_\chi = C_{|\lambda|, l(\lambda)} \|e^\lambda\|_s, \quad \lambda \in \mathcal{Y}, \quad t \in \mathbb{N}_+^m.$$

Define the subspace $H^2_\chi \subset H^2_\chi(n)$ for any $n \in \mathbb{N}$ to be the closed linear span of the subsystem $\phi^\lambda_n$. Theorem 3.1 implies that $H^2_\chi(n) \subset H^2_\chi(n) \subset H^2_\chi(m)$ in $L^2_\chi$ for any $n \neq m$. This provides the orthogonal decomposition

$$H^2_\chi = \mathbb{C} \oplus H^2_\chi \oplus H^2_\chi \oplus \ldots .$$

4. **FOCK-SYMMETRIC $F$-TRANSFORM**

The one-to-one correspondence $e^\lambda \leftrightarrow \phi^\lambda$ allows us to define via the change of orthonormal bases

$$\Phi: \Gamma_w \ni e^\lambda \mapsto \|e^\lambda\|_w^{-1} \mapsto \phi^\lambda \|\phi^\lambda\|_w^{-1} \in H^2_\chi, \quad \lambda \in \mathcal{Y}, \quad t \in \mathbb{N}_+^l(\lambda)$$

the isometric conjugate-linear mapping $\Phi: \Gamma_w \to H^2_\chi$. The adjoint mapping $\Phi^*: H^2_\chi \to \Gamma_w$ is defined by $\langle \Phi^* f \mid g \rangle_\chi = \langle f \mid \Phi^* g \rangle_w$ with $f \in H^2_\chi$. The suitable Fourier decomposition has the form

$$\Phi \psi = \sum_{(\lambda, t) \in \mathcal{Y} \times \mathbb{N}_+^l(\lambda)} \psi(\lambda, t) \phi^\lambda \|\phi^\lambda\|_\chi^{-1}, \quad \psi(\lambda, t) := \langle e^\lambda \mid \psi \rangle_w \|e^\lambda\|_w^{-1}.$$
for any \( \psi \in \Gamma_w \). In particular, the equality \( \Phi x = \sum x_k \phi_k \) is valid for all \( x \in E \). This gives the equalities
\[
\|\Phi x\|^2 = \sum |x_k|^2 = \|x\|^2, \quad x \in E.
\]

Using this, we will examine the composition of \( \Phi \) with the \( \Gamma_w \)-valued function \( \varepsilon : E \ni x \mapsto \varepsilon(x) \). Its correctness justifies the following assertion that substantially uses the \( L^\infty_x \)-valued function
\[
\phi_x : \mathcal{U}_x \ni u \mapsto (\Phi x)(u) = \sum x_k \phi_k(u)
\]
which is linear in the variable \( x \in E \).

Similarly to the known case of Wiener spaces, the function \( \phi_x \) can be seen as a group analog of the Paley-Wiener map (see e.g. [12, n.4.4] or [24]).

**Lemma 4.1.** The composition \( \Phi \varepsilon(x) \), which is understood as the function
\[
[\Phi \varepsilon(x)](u) : \mathcal{U}_x \ni u \mapsto \exp (\phi_x(u)),
\]
takes values in \( L^\infty_x \) for all \( x \in E \).

**Proof.** Applying \( \Phi \) to the Fourier decomposition \( \langle \Phi^* f \rangle_w \), we obtain
\[
\Phi \varepsilon(x) = \sum_{n \in \mathbb{Z}^+} \frac{1}{n!} \sum_{(\lambda,\mu) \in \mathcal{Y}_n \times \mathbb{N}^*_\lambda} \frac{n!}{\lambda!} x^\lambda \phi^\mu = \sum_{n \in \mathbb{Z}^+} \frac{1}{n!} \left( \sum_{k \in \mathbb{N}} x_k \phi_k \right)^n = \exp (\phi_x).
\]

It directly follows that \( \|\Phi \varepsilon(x)\|_\infty \leq \exp \|\phi_x\|_\infty \). \( \square \)

**Theorem 4.2.** For every \( f = \sum f_n \in H^2_{\chi} \), \( (f_n \in H^2_{\chi} \) the entire analytic function \( \hat{f}(x) : = \langle \varepsilon(x) \mid \Phi^* f \rangle_w \) in the variable \( x \in E \) and its Taylor coefficients at origin have the integral representations
\[
\hat{f}(x) = \int \exp (\bar{\phi}_x) f \, d\chi \quad \text{and} \quad d^n_0 \hat{f}(x) = \int \bar{\phi}_x^n f_n \, d\chi,
\]
respectively. The mapping \( \mathcal{F} : H^2_{\chi} \ni f \mapsto \hat{f} \in H^2_{\chi} \) (understanding as a Fock-symmetric \( \mathcal{F} \)-transform) provides the isometries
\[
H^2_{\chi} \cong H^2_{\chi} \quad \text{and} \quad H^2_{\chi} \cong H^2_{\chi}.
\]

**Proof.** First recall that the \( \Gamma_w \)-valued function \( \varepsilon(\cdot) \) is entire analytic on \( E \), therefore \( \hat{f} \) is the same, as the composition of \( \varepsilon(\cdot) \) with \( \langle \cdot \mid \Phi^* f \rangle_w \). Farther on, consider the Fourier decomposition with respect to the basis \( \phi^\nu \),
\[
f = \sum_{n \in \mathbb{Z}^+} f_n = \sum_{(\lambda,\mu) \in \mathcal{Y}_n \times \mathbb{N}^*_\lambda} \frac{1}{\lambda!} \frac{n!}{\lambda!} x^\lambda \phi^\mu = \frac{1}{\lambda!} \frac{n!}{\lambda!} \int f \bar{\phi}_x^\lambda \, d\chi.
\]

Applying \( \Phi^* \) to \( f \) in this decomposition and substituting \( \hat{f}_{\lambda,\mu} \) into \( \hat{f} \), we obtain
\[
\hat{f}(x) = \sum_{n \in \mathbb{Z}^+} \frac{1}{n!} \sum_{(\lambda,\mu) \in \mathcal{Y}_n \times \mathbb{N}^*_\lambda} \frac{n!}{\lambda!} \bar{f}_{\lambda,\mu} \langle \phi^\lambda \mid \phi^\mu \rangle \int f \bar{\phi}_x^\lambda \, d\chi = \int \sum_{n \in \mathbb{Z}^+} \frac{n!}{\lambda!} \bar{f}_{\lambda,\mu} \langle \phi^\lambda \mid \phi^\mu \rangle \int f \bar{\phi}_x^\lambda \, d\chi
\]
where the last equality is valid by Lemma 4.1. It particularly follows that for \( y = \alpha x \),
\[
\hat{f}(y) = \int \exp (\bar{\phi}_x) f \, d\chi = \sum \alpha^n \int \bar{\phi}_x^n f_n \, d\chi, \quad \alpha \in \mathbb{C}.
\]
Differentiating \( \hat{f} \) at \( y = 0 \) and using the \( n \)-homogeneity of derivatives, we obtain
\[
d^n_0 \hat{f}(x) = \frac{d^n}{d\alpha^n} \sum \alpha^n \int \bar{\phi}_x^n f_n \, d\chi \bigg|_{\alpha = 0} = \int \bar{\phi}_x^n f_n \, d\chi.
\]
Finally, we notice that the isometry $H^2_\mathcal{X} \simeq H^2_w$ holds, since the isometry $\Phi^*$ is surjective. In the case of polynomials we similarly get $H^2_\mathcal{X} \simeq H^2_w$.

Note that a different integral formula for analytic functions employing Wiener measures on infinite-dimensional Banach spaces was presented in [22].

5. Exponential creation and annihilation groups

Let us define the linear mapping $j_n: E^{\otimes n}_w \to E^{\otimes n}_h$ to be the continuous extension of identity mapping acting on the dense subspace $E^{\otimes n}_w \subset E^{\otimes n}_w \cap E^{\otimes n}_h$. Such continuous extension $j_n$ is a contractive injection with dense range. In fact, enough to expand elements from $E^{\otimes n}_w$ and $E^{\otimes n}_h$ into the Fourier series with respect to orthogonal basis $e^{\otimes Y_n}$ and apply the inequality

$$\tag{5.1} \|e^{\otimes \lambda}_i\|^2_w = C_{|\lambda|,\ell(\lambda)} \|e^{\otimes \lambda}_i\|^2_h \leq \|e^{\otimes \lambda}_i\|^2_h, \quad \lambda \in \mathbb{Y}_n$$

which follows from Theorem 3.1, taking into account the inequality (2.1). Using subsequently that $E^{\otimes n}_h$ is reflexive, we obtain that its adjoint operator $j^*_n: E^{\otimes n}_h \to E^{\otimes n}_w$ is a contractive injection with dense range. Thus, the mapping $j_n$ is also injective. Moreover, $E^{\otimes n}_h \xrightarrow{i_n} E^{\otimes n}_w \xrightarrow{j_n} E^{\otimes n}_h$ forms a Gelfand triple. Particularly, the operator $s_n$ possesses continuous extension on $E^{\otimes n}_w$.

Using this, we consider the linear operator

$$s_{n/m} := s_n \circ (i_m \otimes j_{n-m}) \quad \text{with} \quad m \leq n$$

defined to be $\phi_m \otimes \psi_{n-m} = s_{n/m}(\phi_m \otimes \psi_{n-m}) \in E^{\otimes n}_w$ for all $\phi_m \in E^{\otimes m}_w$, $\psi_{n-m} \in E^{\otimes (n-m)}_w$.

**Lemma 5.1.** The mapping $s_{n/m}$ from $E^{\otimes m}_w \otimes_h E^{\otimes (n-m)}_w$ to $E^{\otimes (n-m)}_w$ is a contractive injection with dense range.

**Proof.** Expand elements of $E^{\otimes m}_w \otimes_h E^{\otimes (n-m)}_w$ with respect to $e^{\otimes \lambda}_i \otimes e^{\otimes \mu}_j$ for all $\lambda, \mu \in \mathbb{Y}$, $i, j \in \mathbb{N}^{t(\lambda)}$, such that $|\lambda| = m$, $|\mu| = n - m$. Using (5.1), we have

$$\|e^{\otimes \lambda}_i \otimes e^{\otimes \mu}_j\|_{E^{\otimes m}_w \otimes_h E^{\otimes (n-m)}_w} = \|e^{\otimes \lambda}_i\|_w \|e^{\otimes \mu}_j\|_h \leq \|e^{\otimes \lambda}_i\|_h \|e^{\otimes \mu}_j\|_h = \|e^{\otimes \lambda}_i \otimes e^{\otimes \mu}_j\|_h.$$ 

As above, it implies that the mapping $i_m \otimes j_{n-m}: E^{\otimes m}_w \otimes_h E^{\otimes (n-m)}_w \to E^{\otimes (n-m)}_h$, defined to be the continuous extension of identity mapping on $E^{\otimes m}_w \otimes E^{\otimes (n-m)}_w$, is a contractive injection. Using subsequently that $E^{\otimes m}_h \otimes_h E^{\otimes (n-m)}_h$ is reflexive, we get the Gelfand triple

$$E^{\otimes m}_w \otimes_h E^{\otimes (n-m)}_w s_{n/m} \rightarrow E^{\otimes (n-m)}_h i_n \rightarrow E^{\otimes n}_w$$

where injections are contractive and have dense ranges. \(\square\)

**Lemma 5.2.** The exponential creation group, defined on $\{\varepsilon(x): x \in E\}$ by

$$\mathcal{T}_a \varepsilon(x) = \varepsilon(x+a),$$

has a unique linear extension $\mathcal{T}_a: \Gamma_w \ni \psi \mapsto \mathcal{T}_a \psi \in \Gamma_w$ such that

$$\|\mathcal{T}_a \psi\|^2_w \leq \exp(\|a\|^2)|\psi\|^2_w \quad \text{and} \quad \mathcal{T}_{a+b} = \mathcal{T}_a \mathcal{T}_b = \mathcal{T}_b \mathcal{T}_a \quad \text{for all} \quad a, b \in E.$$

**Proof.** Let us define the creation operators $\delta^m_{a,n}: E^{\otimes (n-m)}_w \to E^{\otimes n}_w$ as

$$\delta^m_{a,n} x^{\otimes (n-m)} := s_{n/m}\left[ a^{\otimes m} \otimes x^{\otimes (n-m)} \right] = \frac{(n-m)!}{n!} \frac{d^m(x+ta)^{\otimes n}}{d^m t}|_{t=0}$$

for all $a, x \in E$. Note that the second equality in (5.2) follows from the binomial formula for symmetric tensor elements $(x+ta)^{\otimes n} = \sum_{m=0}^n \binom{n}{m} (ta)^{\otimes m} \otimes x^{\otimes (n-m)}$. Put $\delta^0_{a,n} = 1$. If $a = 0$ then $\delta_{0,n} = 0$. Summing over $n \geq m$ with coefficients $1/(n-m)!$, we get

$$\delta^m_a \varepsilon(x) = \frac{d^m(x+ta)}{d^m t}|_{t=0} = \sum_{n \geq m} \frac{s_{n/m}[a^{\otimes m} \otimes x^{\otimes (n-m)}]}{(n-m)!}, \quad t \in \mathbb{C}. \tag{5.3}$$
This series is convergent, since by Lemma \[5.1\] and \[(5.3)\] the inequality
\[
\|\delta_a^m \varepsilon(x)\|_w \leq \|a\|^m \left( \bigoplus_{n \geq m} x^\otimes(n-m) \right) \|_w = \|a\|^m \|\varepsilon(x)\|_w
\]
holds. From \[(5.3)\] and the tensor binomial formula mentioned above it follows that
\[
\begin{align*}
\sum_{m=0}^{n} \frac{1}{m!} a^m a^\otimes(n-m) & = \sum_{m=0}^{n} \frac{a^m \otimes a^\otimes(n-m)}{m!(n-m)!} = \frac{(x+a)^\otimes n}{n!}.
\end{align*}
\]
Summing over \(n \in \mathbb{Z}_+\) with coefficients \(1/n!\) and using \[(5.3)\], we obtain
\[
\mathcal{T}_a^\varepsilon(x) = \bigoplus_{n \in \mathbb{Z}_+} \sum_{m=0}^{n} \frac{1}{m!} \delta_{a,n}^m x^\otimes(n-m) = \bigoplus_{m \in \mathbb{Z}_+} \frac{m!}{n} \delta_{a,n}^m x^\otimes(n-m) = \exp(\delta_a^\otimes n).
\]
The inequalities \[(2.4)\] and \[(5.3)\] yield \(\|\mathcal{T}_a^\varepsilon(x)\|_w^2 \leq \exp(\|a\|^2 \|\varepsilon(x)\|_w^2)\). Taking into account the totality of \(\{\varepsilon(x): x \in E\}\), this inequality implies the required inequality on \(\Gamma_w\). It also follows that \(\mathcal{T}_{a+b} = \mathcal{T}_a^\varepsilon \mathcal{T}_b = \mathcal{T}_b^\varepsilon \mathcal{T}_a\), since \(\delta_{a+b} = \delta_a + \delta_b\) for all \(a,b \in E\) by linearity of creation operators. This ends the proof.

We define the adjoint operators \(\delta_{a,n}^m \cdot E \supseteq \psi \mapsto \delta_{a,n}^m \psi \in E^\otimes(n-m)\) as
\[
\langle \delta_{a,n}^m x^\otimes(n-m) \mid \psi \rangle = \langle x^\otimes(n-m) \mid \delta_{a,n}^m \psi \rangle_w, \quad a, x \in E
\]
for \(n \geq m\). It immediately follows that for every \(\psi_{n-m} \in E^\otimes(n-m)\) and \(x \in E\),
\[
\langle \delta_{a,n}^m x^\otimes(n-m) \mid \psi_{n-m} \rangle = \langle x^\otimes(n-m) \mid \delta_{a,n}^m \psi_{n-m} \rangle_w
\]
and
\[
\langle \delta_{a,n}^m x^\otimes(n-m) \mid \psi_{n-m} \rangle = \langle x \mid a \rangle^m \langle x^\otimes(n-m) \mid \psi_{n-m} \rangle_w.
\]
Using \(\delta_{a,n}^m\), we can define the exponential annihilation group by the equalities
\[
\mathcal{T}_a^\varepsilon(x) = \exp(\delta_a^\otimes n) \varepsilon(x) = \sum_{m \in \mathbb{Z}_+} \frac{\delta_{a,n}^m \varepsilon(x)}{m!}, \quad \delta_a^m \varepsilon(x) := \sum_{n \geq m} \frac{\delta_{a,n}^m x^\otimes(n-m)}{n!}
\]
for all \(a, x \in E\). Taking into account Lemma \[5.2\] we obtain the following claim.

**Lemma 5.3.** The exponential annihilation group \(\mathcal{T}_a^\varepsilon\) defined by \[(5.6)\] possesses a unique linear extension \(\mathcal{T}_a^\varepsilon : \Gamma_w \ni \psi \mapsto \mathcal{T}_a^\varepsilon \psi \in \Gamma_w\) such that
\[
\|\mathcal{T}_a^\varepsilon \psi\|_w^2 \leq \exp(\|a\|^2 \|\psi\|_w^2) \quad \text{and} \quad \mathcal{T}_{a+b} = \mathcal{T}_{a}^\varepsilon \mathcal{T}_{b}^\varepsilon = \mathcal{T}_{b}^\varepsilon \mathcal{T}_{a}^\varepsilon \quad \text{for all} \quad a, b \in E.
\]

6. **Intertwining properties of \(\mathcal{F}\)-transform**

Let us define on the space \(H_\chi^2\) the multiplicative group \(M_a^\dagger : E \ni a \mapsto M_a^\dagger\) to be
\[
M_a^\dagger f(u) = \exp[\tilde{\phi}_a(u)] f(u), \quad f \in H_\chi^2, \quad u \in \mathcal{U}_\pi.
\]
It can be considered as a linear representation of the additive group from \(E\). By Lemma \[5.1\] the function \(u \mapsto \exp[\tilde{\phi}_a(u)]\) with a fixed \(a\) belongs to \(L_\chi^{\infty}\). Hence, \(M_a^\dagger\) is continuous on \(H_\chi^2\). The generator of the 1-parameter group \(E \ni t \mapsto M_a^\dagger\) coincides with the operator of multiplication by the \(L_\chi^{\infty}\)-valued function
\[
\tilde{\phi}_a : \mathcal{U}_\pi \ni u \mapsto \tilde{\phi}_a(u) \quad \text{where} \quad dM_{a}^\dagger/du|_{t=0} = \tilde{\phi}_a.
\]
The continuity of \(E \ni a \mapsto \exp[\tilde{\phi}_a]\) implies that this 1-parameter group \(M_a^\dagger\) is strongly continuous on \(H_\chi^2\). As a consequence, its generator \(\hat{\phi}_a f)(u) = \tilde{\phi}_a(u) f(u)\) with domain \(\mathcal{D}(\tilde{\phi}_a) = \{f \in H_\chi^2 : \tilde{\phi}_a f \in H_\chi^2\}\) is closed and densely-defined. As well, its power \(\tilde{\phi}_a^m\) defined on \(\mathcal{D}(\tilde{\phi}_a^m) = \{f \in H_\chi^2 : \tilde{\phi}_a^m f \in H_\chi^2\}\) for any \(m \in \mathbb{N}\) is the same.
The additive group contained in $E$ may be also linearly represented on $H^2_w$ as the shift group
\[ T_a \hat{f}(x) = \hat{f}(x + a), \quad f \in H^2_w; \quad x, a \in E. \]
The directional derivative on the space $H^2_w$ along a nonzero $a \in E$ coincides with the generator of the 1-parameter shift subgroup $\mathbb{C} \ni t \mapsto T_{ta}$, that is,
\[ \partial_a \hat{f} = \lim_{t \to 0} t^{-1}(T_{ta} \hat{f} - \hat{f}) \quad \text{with domain } \mathcal{D}(\partial_a) := \{ \hat{f} \in H^2_w : \partial_a \hat{f} \in H^2_w \}. \]
Note that the 1-parameter shift group $T_{ta}$, which is intertwined with $M^\sharp_{ta}$ by the $\mathcal{F}$-transform
\[ (6.1) \quad T_{ta} \hat{f}(x) = \int \exp \left[ \phi_{x+ta} \right] f d\chi = \int \exp(\phi_x)M^\dagger_{ta} f d\chi, \]
is strongly continuous on $H^2_w$. Since $\mathcal{D}(\delta^m_a)$ contains all polynomials from $H^2_w$, each operator $\delta^m_a$ with domain $\mathcal{D}(\delta^m_a) = \{ \hat{f} \in H^2_w : \delta^m_a \hat{f} \in H^2_w \}$ is closed and densely-defined. From (6.1) it directly follows
\[ (6.2) \quad \delta^m_a \hat{f}(x) = \int \exp(\phi_x) \frac{d^m M^\dagger_{ta}}{dt^m} \bigg|_{t=0} f d\chi = \int \exp(\phi_x)T^\dagger_a \hat{f} d\chi \]
for all $f \in \mathcal{D}(\delta^m_a)$ and $x \in E$. On the other hand, by Theorem 4.2 we have
\[ (6.3) \quad T_a \hat{f}(x) = \langle T_a \varepsilon(x) | \Phi^* f \rangle = \langle \varepsilon(x) | T^\dagger_a \Phi^* f \rangle = \int \exp(\phi_x)\Phi T^\dagger_a \Phi^* f d\chi. \]

Theorem 4.2 together with (6.1) and (6.3) imply that $M^\sharp_{ta}$ is connected with the exponential annihilation group $T^\dagger_a$ by the intertwining operator $\Phi$. This can be written as $M^\sharp_{ta} = \Phi T^\dagger_a \Phi^*$. Thus, the $\mathcal{F}$-transform serves as an intertwining operator for the groups $M^\sharp_{ta}$ on $H^2_w$. Moreover, using (6.1), (6.2) and (6.3), we obtain
\[ d^m T_{ta} \hat{f}(x)/dt^m \bigg|_{t=0} = \langle \varepsilon(x) | \delta^m_a \Phi^* f \rangle_w = \delta^m_a \hat{f}(x). \]
As a result, we have proved the following statement.

**Theorem 6.1.** For every $f \in H^2_w$ the following equalities hold,
\[ T_a \mathcal{F}(f) = \mathcal{F}(M^\dagger_a f), \quad M^\dagger_a f = \Phi T^\dagger_a \Phi^* f, \quad a \in E, \]
Moreover, for every $f \in \mathcal{D}(\delta^m_a)$ ($m \in \mathbb{N}$) and a nonzero $a \in E$,
\[ \delta^m_a \hat{f}(x) = \langle \varepsilon(x) | \delta^m_a \Phi^* f \rangle_w = \int \exp(\phi_x)\phi^m_a f d\chi, \quad x \in E. \]

Let us consider on $H^2_w$ the multiplicative group with a nonzero $a \in E$,
\[ M^\ast_a \hat{f}(x) = \hat{f}(x) \exp(x | a), \quad \hat{f} \in H^2_w. \]
The generator on $H^2_w$ of the appropriate 1-parameter subgroup $\mathbb{C} \ni t \mapsto M_{ta^\ast}$ is
\[ d M_{ta^\ast}/dt \bigg|_{t=0} = \langle \cdot | a \rangle := a^\ast, \quad a \in E. \]
Hence, it coincides with the following linear operator of multiplication
\[ (a^\ast \hat{f})(x) = \langle x | a \rangle \hat{f}(x) \quad \text{with domain } \mathcal{D}(a^\ast) = \{ \hat{f} \in H^2_w : a^\ast \hat{f} \in H^2_w \}. \]
Its power $a^{\ast m}$ is densely-defined on $\mathcal{D}(a^{\ast m}) = \{ \hat{f} \in H^2_w : a^{\ast m} \hat{f} \in H^2_w \}$ which contains all polynomials from $H^2_w$.

Using Lemma 5.2 we can represent the additive group from $E$ over the space $H^2_w$ by the shift group
\[ T^\dagger_a = \Phi T_a \Phi^* \quad \text{with the generator } \delta^\dagger_a = \Phi \delta_a \Phi^* \]
defined on $\mathcal{D}(\delta^\dagger_a) = \{ f \in H^2_w \colon \delta^\dagger_a f \in H^2_w \}$. This means that $T^\dagger_a$ is connected via the intertwining operator $\Phi$ with the exponential creation group $T_a$. 

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Theorem 6.2. For every $f \in H^2_\chi$ the following equality holds,

$$M_a \mathcal{F}(f) = \mathcal{F}(T^\dagger_a f), \quad a \in E,$$

that is, the $\mathcal{F}$-transform is an intertwining operator for the groups $M_a*$ on $H^2_w$ and $T^\dagger_a$ on $H^2_\chi$.

Moreover, for every $f \in \mathcal{D}(\delta^{tm}_a) = \{ f \in H^2_\varepsilon : \delta^{tm}_a f \in H^2_\varepsilon \}$ $(m \in \mathbb{N})$ and a nonzero $a \in E$,

$$\langle (a^* m \tilde{f})(x) = \langle \varepsilon(x) | \delta^{tm}_a \Phi^* f \rangle_w = \int \exp(\tilde{\phi}_x) \delta^{tm}_a f \, d\chi, \quad x \in E.$$

Proof. The equality (5.4) yields $\langle x | a^* m \psi_{n-m}(x) = \langle \delta^{tm}_a x^m_n | \psi_{n-m} \rangle_w$ for all $n \geq m$. By Theorem 4.2 for any $f = \sum_n f_n \in H^2_\varepsilon$ there exists a unique $\psi = \oplus_n \psi_n$ in $\Gamma_w$ with $\psi_n \in E^{\otimes n}$ such that $\Phi^* f \psi = \psi$ and $f_n = \psi^n$. Summing over all $m \in \mathbb{Z}_+$ and $n \geq m$ and using (5.6), we obtain that

$$M_a \hat{f}(x) = \exp(\langle x | a \rangle \varepsilon(x) | \Phi^* f \rangle_w = \sum_{m \in \mathbb{Z}_+} \sum_{n \geq m} \frac{(x | a)^m}{m!} \psi^*_{n-m}(x)\varepsilon$$

$$= \langle \mathcal{T}^*_a \varepsilon(x) | \Phi^* f \rangle_w = \langle \varepsilon(x) | \mathcal{T}^*_a \Phi^* f \rangle_w.$$

By Theorem 4.2 and Lemma 5.2 it follows that the equalities hold for all $\hat{f} \in H^2_w$. On the other hand, the equalities (5.6) and (6.5) yield

$$\frac{d^m M_{ta} \hat{f}(x)}{dt^m} \bigg|_{t=0} = \int \exp(\tilde{\phi}_x) \frac{d^m \mathcal{T}^t_a}{dt^m} \bigg|_{t=0} \hat{f} \, d\chi = \int \exp(\tilde{\phi}_x) \delta^{tm}_a f \, d\chi$$

for all $f \in \mathcal{D}(\delta^{tm}_a)$. This in turn yields (6.4). \hfill \square

7. Commutation relations

Describe the commutation relations between $M^\dagger_a$ and $T^\dagger_b$ on the Hardy space $H^2_\varepsilon$.

Theorem 7.1. For any nonzero $a, b \in E$ the commutation relations

$$M^\dagger_a T^\dagger_b = \exp(a \mid b) M^\dagger_b T^\dagger_a, \quad (\tilde{a}_b \delta^\dagger_b - \delta^\dagger_b \tilde{a}_b) f = (a \mid b) f$$

hold, wherein $f$ belongs to the dense subspace $\mathcal{D}(\delta^\dagger_b) \cap \mathcal{D}(\delta^\dagger_a) \subset H^2_\varepsilon$.

Proof. Let us prove that the following equalities hold,

$$(7.1) \quad T^\dagger_a M_b^* = \exp(a \mid b) M_b^* T^\dagger_a, \quad (\tilde{a}_b a^* - b^* \tilde{a}_b) \hat{f} = (a \mid b) \hat{f}$$

where $\hat{f} \in \mathcal{D}(b^{1*}) \cap \mathcal{D}(a^* b)$. First property follows from the direct calculations:

$$M_b^* T^\dagger_a \hat{f}(x) = \exp(\langle x | b \rangle) \hat{f}(x + a),$$

$$T^\dagger_a M_b^* \hat{f}(x) = \hat{f}(x + a) \exp(\langle x | b \rangle) \exp(\langle a | b \rangle) = \exp(\langle a | b \rangle) M_b^* T^\dagger_a \hat{f}(x)$$

for all $\hat{f} \in H^2_w$ and $x \in E$. For any $\hat{f} \in \mathcal{D}(b^{1*}) \cap \mathcal{D}(a^* b)$ and $t \in \mathbb{C}$, we have

$$\frac{d^2}{dt^2} T^\dagger_a M_b^* \hat{f} \bigg|_{t=0} = \left[ \frac{d}{dt} T^\dagger_a M_b^* \hat{f} + 2 \tilde{a}_b T^\dagger_a b^* M_b^* T^\dagger_a \hat{f} + T^\dagger_a b^{1*} M_b^* \hat{f} \right]_{t=0}$$

$$= (a^* b + 2 \tilde{a}_b b^* + b^{1*}) \hat{f}.$$

On the other hand, differentiating again, we have

$$\frac{d}{dt} T^\dagger_a M_b^* \hat{f} \bigg|_{t=0} = \left[ \frac{d}{dt} \exp(\langle ta | \tilde{b} \rangle) M_b^* T^\dagger_a \hat{f} \right]_{t=0} = \left[ \frac{d^2}{dt^2} \exp(\langle ta | \tilde{b} \rangle) M_b^* T^\dagger_a \hat{f} \right]_{t=0}.$$
This yields \(7.1\) where \(\mathcal{D}(b^2) \cap \mathcal{D}(\alpha^2)\) contains the dense subspace in \(H_w^2\) of all polynomials \(\hat{f}\) generating by finite sums \(\Phi^* f = \bigoplus \psi_n \in \Gamma_w\).

Using that \(T_b^\dagger = \mathcal{F}^{-1} M_b^* \mathcal{F}\) and \(M_b^\dagger = \mathcal{F}^{-1} T_b \mathcal{F}\) with \(\mathcal{F}^{-1}: H_w^2 \to H_\chi^2\) and applying \(7.1\), we obtain

\[
M_b^\dagger T_b^\dagger = \mathcal{F}^{-1} T_b M_b^* \mathcal{F} = \exp(x | b) \mathcal{F}^{-1} M_b^* \mathcal{F} = \exp(x | b) T_b^\dagger M_b^\dagger,
\]

\[
(\partial_\alpha^2 \partial_b^\dagger - \partial_b^\dagger \partial_\alpha^2) f = \mathcal{F}^{-1} (\partial_\alpha b^* - b^* \partial_\alpha) \mathcal{F} f = \langle a | b \rangle f
\]

for all \(f \in \mathcal{D}(\partial_\alpha^2) \cap \mathcal{D}(\partial_b^2)\). For any \(f = \sum f_n \in H_\chi^2\) there exists a unique \(\psi = \bigoplus \psi_n \in \Gamma_w\) with \(\psi_n \in E_w^\infty\) such that the equalities \(\Phi^* f = \psi\) and \(f_n = \psi_n^*\) hold. Hence, the following embedding \(\mathcal{D}(\partial_\alpha^2) \cap \mathcal{D}(\partial_b^2) \subset H_\chi^2\) is dense.

\[\square\]

8. Gauss-Weierstrass semigroups

Next we show that the 1-parameter Gauss-Weierstrass semigroups on the Hardy space \(H_w^2\) can be well described by shift and multiplicative groups (a classic case can be found in \([7, n.4.3.2]\)). For this purpose we use the Gaussian kernel

\[
g_r(\tau) = \frac{1}{\sqrt{4\pi r}} \exp \left( -\frac{\tau^2}{4r} \right), \quad \tau \in \mathbb{R}, \quad r > 0.
\]

**Theorem 8.1.** The 1-parameter Gauss-Weierstrass semigroups \(\{W_r^\delta : r > 0\}\) and \(\{W_r^\alpha : r > 0\}\), defined on the Hardy space \(H_\chi^2\) for any nonzero \(a \in E\) as

\[
W_r^\delta f = \int_\mathbb{R} g_r(\tau) T_{r\alpha} f d\tau \quad \text{and} \quad W_r^\alpha f = \int_\mathbb{R} g_r(\tau) M_{r\alpha} f d\tau, \quad f \in H_\chi^2,
\]

are generated by \(\partial_\alpha^2\) and \(\partial_\alpha^2\), respectively.

**Proof.** First it is sufficient to prove that the axillary 1-parameter families of linear operators over \(H_w^2\)

\[
G_r^a \hat{f} = \int_\mathbb{R} g_r(\tau) M_{\tau a} \hat{f} d\tau \quad \text{and} \quad G_r^\alpha \hat{f} = \int_\mathbb{R} g_r(\tau) T_{\tau a} \hat{f} d\tau, \quad \hat{f} \in H_w^2
\]

can be generated by \(a^2\) and \(\partial_\alpha^2\) and satisfy the semigroup property. Properties of Gaussian kernel yield

\[
\int_\mathbb{R} g_r(\tau) \tau^{2k} d\tau = \frac{1}{2^{2k} \pi^{k}} \int_\mathbb{R} e^{-\frac{\tau^2}{4r}} \tau^{2k} d\tau \bigg|_{\tau=2\sqrt{r}u} = \frac{(2\sqrt{r})^{2k}}{\sqrt{\pi}} \int_\mathbb{R} e^{-u^2} u^{2k} dv
\]

\[
= \frac{2^{2k+1}}{(k+1)!} \Gamma \left( \frac{2k+1}{2} \right) = \frac{2(2k-1)!}{(k-1)!} \pi r^k, \quad k \in \mathbb{N}.
\]

We can rewrite \(G_r^a \hat{f}\) on the dense subspace \(\{\hat{f} \in H_w^2 : \exp(\tau a^*) \hat{f} \in H_\chi^2\}\) as

\[
G_r^a \hat{f} = \int_\mathbb{R} g_r(\tau) \exp(\tau a^*) \hat{f} d\tau = \sum_{k \in \mathbb{Z}} \frac{a^k \hat{f}}{k!} \int_\mathbb{R} g_r(\tau) \tau^k d\tau
\]

\[
= \sum_{k \in \mathbb{Z}} \frac{2(2k-1)!}{(k-1)!} \pi r^k a^* \tau^{2k} \frac{\hat{f}}{(2k)!} = \sum_{k \in \mathbb{Z}} \frac{r^k a^* \tau^{2k} \hat{f}}{(k!)^2} = \exp(ra^2) \hat{f}
\]

By first equality in \(8.2\), the family \(G_r^a\) can be extended to the convolution \(g_r \otimes \hat{f} := \int_\mathbb{R} g_r(\tau) M_{\tau a} \hat{f} d\tau, \quad \hat{f} \in H_w^2\)
(dependent on \( a \)) over the whole space \( H^2_w \). Thus, to show that the semigroup property holds, it suffices to show that
\[
\mathfrak{g}_{r+s}\ast \hat{f} = G^\ast_{r+s} \hat{f} = (G^\ast_r \circ G^\ast_s) \hat{f} = \mathfrak{g}_r \ast (\mathfrak{g}_s \ast \hat{f}) = (\mathfrak{g}_r \ast \mathfrak{g}_s) \ast \hat{f}.
\]
But this straightly follows from the known convolution equality \( \mathfrak{g}_{r+s} = \mathfrak{g}_r \ast \mathfrak{g}_s \).

Further, using the equality \( T_a^f = \mathcal{F}^{-1} M_a \mathcal{F} \) we obtain that
\[
W_r^{\delta^1} f = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) \mathcal{F}^{-1} M_{r^a} \mathcal{F} f \, d\tau = \mathcal{F}^{-1} G^r_{a^2} \mathcal{F} f
\]
for all \( f \in H^2_\chi \). By Theorem 6.2 it follows that
\[
\frac{dW_r^{\delta^1} f}{dr} \bigg|_{r=0} = \mathcal{F}^{-1} G^r_{a^2} \frac{\hat{f}}{dr} \bigg|_{r=0} = \mathcal{F}^{-1} a^2 \hat{f} = \delta^1 a^2 f
\]
for all \( f \in \mathcal{D}(\delta^1 f) \), since \( \hat{f} \in \mathcal{D}(a^2) \) and \( \delta^1 a^2 = \mathcal{F}^{-1} a^2 \mathcal{F} \). Hence, the case of semigroup \( W_r^{\delta^1} \) is proven.

Similar reasonings can be applied to the semigroup \( G^{\delta^2} \). As a result, we obtain that the equalities \( W_r^{\delta^2} = \mathcal{F}^{-1} G^{\delta^2} \mathcal{F} \) and \( \delta^2 a^2 = \mathcal{F}^{-1} \delta^2 a^2 \mathcal{F} \) hold.

### 9. Complexified infinite-dimensional Heisenberg group

Let us give yet another application. Consider an infinite-dimensional analog of the Heisenberg group over \( \mathbb{C} \). Namely, let us define the group \( \mathfrak{G} \) of upper triangular matrix-type elements
\[
X(a, b, t) = \begin{bmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \quad t \in \mathbb{C}, \quad a, b \in E
\]
with unit \( X(0, 0, 0) \) and multiplication
\[
\begin{bmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & t' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a + a' & t + t' + \langle a | b' \rangle \\ 0 & 1 & b + b' \\ 0 & 0 & 1 \end{bmatrix}.
\]
Obviously, \( X(a, b, t)^{-1} = X(-a, -b, -t + \langle a | b \rangle) \).

Describe an irreducible linear representation of the group \( \mathfrak{G} \). For this purpose we will use the algebra \( \mathbb{H} \) of quaternions \( \gamma = \alpha_1 + \alpha_2 i + \beta_1 j + \beta_2 k = (\alpha_1 + \alpha_2 i) + (\beta_1 + \beta_2 i) j = \alpha + \beta j \) as pairs of complex numbers \( (\alpha, \beta) \in \mathbb{C}^2 \) with \( \alpha = \alpha_1 + \alpha_2 i, \beta = \beta_1 + \beta_2 i \in \mathbb{C} \) and \( \alpha_1, \beta_1 \in \mathbb{R} \) where basis elements in \( \mathbb{H}^4 \) satisfy the relations \( i^2 = j^2 = k^2 = i j k = -1, k = i j = -j i \). Thus, \( \mathbb{H} = \mathbb{C} \oplus \mathbb{C} j \) is a vector space over \( \mathbb{C} \) [26]. Denote \( \beta := \Im \gamma \) where \( \gamma = \alpha + \beta j \).

Let \( E_\mathbb{H} = E \oplus E j \) be the Hilbert space with \( \mathbb{H} \)-valued scalar product
\[
\langle p | p' \rangle = \langle a + b j | a' + b' j \rangle = \langle a | a' \rangle + \langle b | b' \rangle + \frac{1}{2} \langle (a' | b) - (a | b') \rangle j
\]
where \( p = a + b j \) with \( a, b \in E \) (similarly, for \( p' = a' + b' j \)). Hence,
\[
\Im \langle p | p' \rangle = \langle a' | b \rangle - \langle a | b' \rangle, \quad \Im \langle p | p \rangle = 0.
\]

**Theorem 9.1.** The following linear representation of \( \mathfrak{G} \) over \( H^2_\chi \) (which can be seen as an analog of the Weyl-Schrödinger representation),
\[
\mathcal{W}^{1}: \mathfrak{G} \ni X(a, b, t) \longmapsto \exp \left[ t + \frac{1}{2} \langle a | b \rangle \right] T_a T_b^\dagger M_b^\dagger,
\]
is well defined and irreducible.

**Proof.** First we prove that the following operator representation
\[
\mathcal{W}: \mathfrak{G} \ni X(a, b, t) \longmapsto \exp \left[ t + \frac{1}{2} \langle a | b \rangle \right] M_a T_b
\]
into the operator algebras over $H^2_w$ is well defined and irreducible. Consider the auxiliary group $\mathbb{C} \times E_\mathbb{H}$ with the multiplication

$$(t, p)(t', p') = \left( t + t' - \frac{1}{2}\Im\langle p \mid p' \rangle, p + p' \right)$$

for all $p = a + bj$, $p' = a' + b'j \in E_\mathbb{H}$. It is related to $\mathcal{G}$ via the mapping

$$\mathcal{G} : X(a, b, t) \mapsto \left( t - \frac{1}{2}\langle a \mid b \rangle, a + bj \right).$$

Check that $\mathcal{G}$ is a group isomorphism. In fact,

$$\mathcal{G} (X(a, b, t)X(a', b', t')) = \mathcal{G} (X(a + a', b + b', t + t' + \langle a \mid b' \rangle))$$

$$= \left( t + t' + \langle a \mid b' \rangle - \frac{1}{2}\left[ |a + a'| + |b + b'| \right], a + a' \right)$$

$$= \left( t + t' - \frac{1}{2}\left[ (a \mid b) + (a' \mid b') \right] + \frac{1}{2}\left[ (a \mid b') - (a' \mid b) \right], a + a' \right)$$

$$= \left( t - \frac{1}{2}(a \mid b), a + bj \right) \left( t' - \frac{1}{2}(a' \mid b'), a' + b'j \right)$$

$$= \mathcal{G} (X(a, b, t)) \mathcal{G} (X(a', b', t')).$$

Now let us check that the Weyl-like operator

$$W(p) = \exp\left[ \frac{1}{2}(a \mid b) \right] M_{a^*}T_b,$$  

$$p = a + bj$$

on the space $H^2_w$ satisfies the commutation relation

$$W(p + p') = \exp\left[ -\frac{1}{2}\Im\langle p \mid p' \rangle \right] W(p)W(p').$$

In fact, using \([7,1]\), we obtain

$$\exp\left[ \frac{1}{2}(a \mid b') - \frac{1}{2}(a' \mid b) \right] W(p)W(p')$$

$$= \exp\left[ \frac{1}{2}(a \mid b) + \frac{1}{2}(a' \mid b') \right] \exp\left[ \frac{1}{2}(a \mid b') - \frac{1}{2}(a' \mid b) \right] M_{a^*}T_b M_{a'^*}T_{b'}$$

$$= \exp\left[ \frac{1}{2}(a + a' \mid b + b') \right] M_{a^* + a'^*}T_{b + b'} = W(p + p').$$

As a consequence, the mapping $\mathcal{G} : \mathbb{C} \times E_\mathbb{H} \ni (t, p) \mapsto \exp(t)W(p)$ is a group isomorphism. So, $\mathcal{W}$ is also a group isomorphism as a composition of the group isomorphisms $\mathcal{I}$ and $\mathcal{G}$.

Let us check irreducibility. If there exists an element $x_0 \neq 0$ in $E$ and an integer $n > 0$ such that

$$\exp\left[ t + \frac{1}{2}(a \mid b) \right] e^{a(x)} [x_0^n(x + b)]^n = 0 \quad \text{for all} \quad x, a, b \in E$$

then $x_0 = 0$. This gives a contradiction. Hence the representation $\mathcal{W}$ is irreducible. Finally, using that

$$\exp\left[ t + \frac{1}{2}(a \mid b) \right] T_b M_{a^*} = \mathcal{F}^{-1} \left( \exp\left[ t + \frac{1}{2}(a \mid b) \right] M_{a^*}T_b \right) \mathcal{F},$$

we conclude that the group representation $\mathcal{W}^{-1} = \mathcal{F}^{-1} \mathcal{W} \mathcal{F}$ is irreducible. \(\square\)

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