A First Polynomial Non-Clausal Class in Many-Valued Logic

Gonzalo E. Imaz

Artificial Intelligence Research Institute (IIIA) - CSIC, Barcelona, Spain

email: {gonzalo}@iiia.csic.es

Abstract

The relevance of polynomial formula classes to deductive efficiency motivated their search, and currently, a great number of such classes is known. Nonetheless, they have been exclusively sought in the setting of clausal form and propositional logic, which is of course expressively limiting for real applications. As a consequence, a first polynomial propositional class in non-clausal (NC) form has recently been proposed.

Along these lines and towards making NC tractability applicable beyond propositional logic, firstly, we define the Regular many-valued Horn Non-Clausal class, or RH, obtained by suitably amalgamating both regular classes: Horn and NC.

Secondly, we demonstrate that the relationship between (1) RH and the regular Horn class is that syntactically RH subsumes the Horn class but that both classes are equivalent semantically; and between (2) RH and the regular non-clausal class is that RH contains all NC formulas whose clausal form is Horn.

Thirdly, we define Regular Non-Clausal Unit-Resolution, or RURNC, and prove both that it is complete for RH and that checks its satisfiability in polynomial time. The latter fact shows that our intended goal is reached since RH is many-valued, non-clausal and tractable.

As RH and RURNC are, both, basic in the DPLL scheme, the most efficient in propositional logic, and can be extended to some other non-classical logics, we argue that they pave the way for efficient non-clausal DPLL-based approximate reasoning.

Field: Tractable Approximate Automated Reasoning.
Keywords: Regular Many-Valued Logic; Horn; Non-Clausal; Tractability; Resolution; DPLL; Satisfiability Testing; Logic Programming; Theorem Proving.

1 Introduction

In contrast to the simple clausal form, i.e. a conjunction of clauses, the non-clausal (NC) form, on which focuses this article, allows an arbitrary nesting of the \( \land \) and \( \lor \) connectives. Thus, the NC formulas of a given logic contain an undetermined number of nested \( \land \) and \( \lor \) connectives and their atoms are negated and unnegated literals of the given logic. The expressiveness of NC formulas is exponentially richer than that of clausal formulas and they have found much use in heterogeneous fields and practical settings as discussed below.

Indeed, within classical logic, non-clausal formulas are found in numerous scenarios and reasoning problems such as quantified boolean formulas \[45\], DPLL \[97\], nested logic programming \[90\], knowledge compilation \[35\], description logics \[69\], numeric planning \[92\] and many other fields that are mentioned in \[66\]. In the particular case of first-order logic, one can find approaches on non-clausal theorem proving in the former steps.
of automated reasoning e.g., [25, 6] but such area is still the object of current research activity as the regularly reported novel results show e.g., [48, 99, 88].

And within non-classical logics, non-clausal formulas having different roles and functionalities have been studied in a profusion of languages: signed many-valued logic [86, 18, 96], Lukasiewicz logic [72], Levesque’s three-valued logic [30], Belnap’s four-valued logic [2], M3 logic [2], fuzzy logic [55], fuzzy description logic [54], intuitionistic logic [89], modal logic [59], lattice-valued logic [99] and more.

In many frameworks of non-classical and classical logics, non-clausal formulas are often translated into clausal form e.g. [58, 12, 30] to which clausal reasoning methods are then applied. However, it is well-known that such translations can either blow up exponentially the size of formulas or lose, both, their semantical properties, preventing its application in some settings, and original syntactical structure, proven experimentally to highly decrease practical efficiency. On the other hand, the syntactic form of the formulas involved plays a role [30]: reasoning in Levesque’s three-valued system [74] is polynomial if the formulas are in clausal form, while it is co-NP-complete if no normal form is assumed.

On the other side, Horn clausal formulas can be read naturally as instructions for a computer, and are recognized as central for deductive databases, declarative programming, and more generally, for rule-based systems. In fact, Horn formulas have received a great deal of attention since 1943 [51, 65] and, at present, there is a broad span of areas within artificial intelligence relying on them, and their scope covers a fairly large spectrum of realms spread across many logics and a variety of reasoning settings.

Furthermore, regarding Horn efficiency, the valuable contribution of the conjunction of Horn formulas and Horn-SAT algorithms to clausal efficiency is reflected by the fact that the highly efficient DPLL solvers embed a Horn-SAT-like algorithm, so-called Unit Propagation [37]. These algorithms have been greatly optimized to the point that, the Horn-SAT algorithm devised for propositional logic is even strictly linear [39, 44]. Hence, searching for polynomial (clausal) super-classes of the Horn class in propositional logic has been a key issue for several decades in the quest for improving clausal reasoning, and indeed, currently the existence of a great number of such classes is known; the names of some well-known of these classes are: hidden-Horn, generalized Horn, Q-Horn, extended-Horn, SLUR, Quad, matched, UP-Horn and more (see [91, 66] for short reviews).

In contrast to such remarkable advances in clausal tractability, the non-clausal tractability is enormously delayed as the two following facts clearly reveal: (i) there is only one (recently found) non-trivial \(^2\) polynomial class in propositional logic [66]; and (ii) beyond propositional logic, there is none of such polynomial non-clausal classes.

Thus, since the signed many-valued logic is, one may say, rather close to propositional logic and is employed in a wide range of reasoning scenarios and applications e.g., [21, 79, 73, 8, 47] (discussed in related work), we have selected its sub-class, called regular logic, in order to determine a first tractable non-clausal class within approximate reasoning.

For this purpose, we first introduce the hybrid class of Regular many-valued Horn-NC formulas, or \(\mathcal{R}H\), resulting from suitably merging both regular classes, Horn and NC, or equivalently, by suitably lifting the existing regular Horn pattern [24, 64, 72] to NC form. We then prove that satisfiability testing \(\mathcal{R}H\) is polynomial.

Thus, our first contribution is carried out as follows. By lifting the regular Horn pattern

\(^1\)So the terms Horn-SAT algorithm and Unit Propagation procedure will be used interchangeably.

\(^2\)Trivial classes include, for instance, unsatisfiable formulas whose translation to DNF is polynomial.
(a regular clausal formula is Horn if all its clauses have any number of negative literals and at most one positive literal) to the NC level, we establish the regular Horn-NC pattern as the next recursive non-clausal restriction: a regular NC formula is Horn-NC if all its disjunctions have any number of negative disjuncts and at most one non-negative Horn-NC disjunct. Accordingly, $\mathcal{RH}$ is the class of regular Horn-NC formulas. Note that $\mathcal{RH}$ naturally includes the regular Horn clausal class. Subsequently, we provide a more fine-grained syntactical definition of $\mathcal{RH}$ in a compact and inductive function.

Our second contribution is proving the relationships between $\mathcal{RH}$ and the regular Horn and NC classes which are as follows: (1) $\mathcal{RH}$ is related to the regular Horn class in that every Horn-NC formula is logically equivalent to a Horn formula, and hence, $\mathcal{RH}$ and regular Horn are equivalent semantically but syntactically $\mathcal{RH}$ subsumes regular Horn; and (2) $\mathcal{RH}$ is related to the regular NC class in that $\mathcal{RH}$ contains all regular NCs whose clausal form (to be specified) is Horn. The Venn diagram in Fig. 1 relates the new $\mathcal{RH}$ to the known regular classes Horn ($\mathcal{H}$), Non-Clausal ($\mathcal{N}_C$) and Clausal ($\mathcal{C}$).

As a third contribution, we provide the calculus Regular Non-Clausal Unit-Resolution, or $RUR_{NC}$, prove its completeness for $\mathcal{RH}$ and that it enables checking $\mathcal{RH}$ satisfiability in polynomial time. This claim shows that our initial intended aim is achieved giving that $\mathcal{RH}$ is multi-valued, non-clausal and tractable, and so far as we know, $\mathcal{RH}$ is the first published class with such features. The polynomiality of $\mathcal{RH}$ yields an immediate proof that the computational problem Regular-Horn-NC-SAT is $P$-complete.

Some proven properties [66] of propositional Horn-NC formulas also apply to the (?) regular ones presented here, and among them, we highlight their polynomial recognition, that is, deciding whether any arbitrary regular NC formula is Horn-NC is performed in polynomial time. Altogether, $\mathcal{RH}$ enjoys the advantageous computational properties of being a class both recognized and tested for satisfiability in worst-case polynomial complexity.

We synthesize and illustrate our aforementioned contributions through the specific formula $\varphi$ given below, whose infix notation is explained in detail in Section 3 and wherein $\phi_1, \phi_2$ and $\varphi'$ are regular NCs, and $X^{\geq a}$ and $X^{\leq a}$ denote a literal that is satisfiable if the truth-value assigned to $X$ is, respectively, greater or less than or equal to the threshold $a$:

$$\varphi = \{ \land P^\leq 8 \ (\lor P^\leq 2 \ \{ \land (\lor P^\leq 3 \ Q^\leq 4 \ P^{\geq 1}) \ (\lor \phi_1 \ \{ \land \phi_2 \ P^\leq 6 \}) \ Q^{\geq 7} \}) \ \varphi' \}$$

We will show that $\varphi$ is Horn-NC when $\phi_1, \phi_2$ and $\varphi'$ are Horn-NC and at least one of $\phi_1$ or $\phi_2$ is negative. In that case we will prove that:
1. $\varphi$ can be tested for satisfiability in polynomial time.
2. $\varphi$ can be recognized as Horn-NC in polynomial time [66].
3. $\varphi$ is logically equivalent to a regular Horn formula.
4. $\varphi$ is exponentially smaller than its equivalent regular Horn formula.
5. Applying $\land/\lor$ distributivity to $\varphi$ yields a regular Horn formula.

Section 7 shows that, $\mathcal{RH}$ and $\mathcal{RUR}_{NC}$ in tandem allow logic programming: (i) enriching their syntax from simple regular Horn rules to regular Horn-NC rules in which heads and bodies are NCs with slight syntactical restrictions; and (ii) answering queries with an efficiency comparable to clausal efficiency, that is, in polynomial time. This is possible thanks to the facts that regular Horn-NC formulas are, both, polynomial for satisfiability testing and have only one minimal model (indeed, as above mentioned, they are logically equivalent to a regular Horn clausal formula).

In future work, we outline how the Horn-NC class and NC Unit-Resolution will be defined in other uncertainty logics\(\footnote{For the discussion of their extension to classical logics, the reader may consult \cite{66}.}\), e.g. Lukasiewicz and possibilistic logics. As both entities are basic in DPLL, they can also be a starting point towards developing NC DPLL-based approximate reasoning. Finally, we think that our definition of NC Unit-Resolution is the base to obtain Non-Clausal Resolution for some uncertainty logics, missing so far.

The paper continues as follows. Sections 2 and 3 present background on regular clausal and non-clausal logic, respectively. Section 4 defines $\mathcal{RH}$. Section 5 relates $\mathcal{RH}$ to the Horn and NC classes. Section 6 introduces $\mathcal{RUR}_{NC}$ and proves the tractability of $\mathcal{RH}$. Section 7 applies $\mathcal{RH}$ and $\mathcal{RUR}_{NC}$ to NC logic programming. Sections 8 and 9 focus on related and future work, respectively. Last section summarizes our contributions.

## 2 Regular Many-Valued Clausal

This section presents notation, terminology and background on clausal regular logic, and since regular logic is a sub-class of signed logic, we start by a general presentation of both logics (for a complete presentation, the reader may consult \cite{21, 61, 13, 62}).

Signed logic differs from propositional logic only at the literal level. A signed literal is a pair $S \cdot P$, where $P$ is a proposition and $S$ is a (usually finite) set of truth-values, and it is satisfiable by an interpretation $\mathcal{I}$ only if $\mathcal{I}(P) \in S$. Since $\mathcal{I}(P) \in S$ is true or false, i.e. two-valued, satisfaction of $\land/\lor$-connectives by interpretations is like in classical logic\(\footnote{See \cite{19} for a detailed analyze of the relation between signed and propositional logics.}\).

Signed logic is a generic representation for finite-valued logics since: deciding the satisfiability of formulas of any finite-valued (and some infinite-valued) logic is polynomially reducible to the problem of deciding the satisfiability of signed clausal formulas \cite{58}.

Regular logic is the most studied sub-class of signed logic and derives from it when the truth-value domain is totally ordered and the signs $S$ are intervals of two kinds: $[\alpha, \infty]$ or $[\alpha, \infty]$. The significance of regular logic stems from its close connection with signed logic \cite{59}, that is: every signed formula is logically equivalent to some regular formula.

The regular language is the same for the clausal and the non-clausal frameworks and is defined next.
Definition 2.1. The regular symbol language is formed by an infinite truth-value set $T$ endowed with a total ordering $\succ$, plus the sets: regular signs $\{\geq, \leq\}$, regular propositions $P = \{P, Q, R, \ldots\}$, classical connectives $\{\lor, \land\}$ and auxiliary symbols: $(, )$, $\{$ and $\}$. 

Remark. In the examples throughout the article, the truth-value set $T$ will be the real unit interval $T = [0, 1]$, as is usually considered in the literature.

Next we introduce our notation for atoms. Although to denote regular literals a plethora of notations have been invented in the clausal framework, we use a different notation for our non-clausal setting with a twofold purpose: (i) minimizing the symbols existing in regular NC formulas for improving their readability; and (ii) helping to visually determine whether a regular NC formula is Horn-NC.

Definition 2.2. If $\alpha, \beta \in T$, $\alpha \geq \beta$ is a regular constant. If $X \in P$ and $\alpha \in T$, then $X \geq \alpha$ and $X \leq \alpha$ are, respectively, a regular positive and a regular negative literals. $K$ and $L$ are, respectively, the set of regular constants and regular literals. $K \cup L$ includes the atoms.

Example 2.3. $2 \geq .8$ is a regular constant; $P \geq .7$ and $R \leq .1$ are examples of regular positive and regular negative literals, respectively.

We superscript and subscript positive and negative literals, respectively, in order to be able to recognize, just through visual inspection, whether a regular NC formula is Horn-NC.

Definition 2.4. $C = (\lor L_1 L_2 \ldots L_k)$, the $L_i$ being regular atoms, is a regular clause. $\{\land C_1 C_2 \ldots C_n\}$, the $C_i$ being regular clauses, is a regular clausal formula. $C$ is the set of regular clausal formulas.

Example 2.5. $\{\land (\lor P \geq .7 \ 2 \geq .8 \ R \leq .1) (\lor P \geq .2 \ R \leq .0)\}$ is a regular clausal formula.

Definition 2.6. A regular clause with at most one regular positive literal is regular Horn. If the $h_i$ are regular Horn clauses then $\{\land h_1 h_2 \ldots h_n\}$ is a regular Horn formula. $H$ is the set of regular Horn formulas.

Note. Since this article focuses on regular logic, in most cases we will omit the word regular preceding entities and simply speak of literal, clause, formula, etc.

We will denote $\top$ any satisfiable constant, e.g. $1 \geq .6$, and $\bot$ any unsatisfiable constant, e.g. $.6 \geq 1$, whose formal definitions follow.

Definition 2.8. Let $\alpha, \beta \in T$. We denote $\top$ any constant $\alpha \geq \beta$ such that $\alpha \succ \beta$, and note $\bot$ any constant $\alpha \geq \beta$ such that $\beta \succ \alpha$ and $\alpha \neq \beta$. The empty conjunction $\{\land\}$ is considered equivalent to a $\top$-constant and the empty disjunction $(\lor)$ to a $\bot$-constant.

Definition 2.9. An interpretation $I$ maps the propositions $P$ into the truth-value set $T$ and the clausal formulas $C$ into $\{0, 1\}$ and the mapping is extended from $K \cup P$ to $C$ by means of the rules below, where $X \in P$ and $\alpha \in T$. 

5
• $I(\bot) = I((\lor)) = 0$ and $I(\top) = I(\{\land\}) = 1$.

• $I(X^{\geq \alpha}) = \begin{cases} 1 & \text{if } I(X) \geq \alpha \\ 0 & \text{otherwise} \end{cases}$

• $I((\lor \ell_1 \ldots \ell_i \ldots \ell_k)) = \max\{I(\ell_i) : 1 \leq i \leq k\}$.

• $I(\{\land C_1 \ldots C_i \ldots C_k\}) = \min\{I(C_i) : 1 \leq i \leq k\}$.

Definition 2.10. Some well-known semantical notions follow, $\varphi$ being a formula:

– An interpretation $I$ is a model of $\varphi$ if $I(\varphi) = 1$.
– If $\varphi$ has a model then it is satisfiable and otherwise unsatisfiable.
– $\varphi$ and $\varphi'$ are (logically) equivalent, noted $\varphi \equiv \varphi'$, if $\forall I, I(\varphi) = I(\varphi')$.
– $\varphi'$ is a logical consequence of $\varphi$, noted $\varphi \models \varphi'$, if $\forall I, I(\varphi) = 1 \rightarrow I(\varphi') = 1$.

Example 2.11. For instance, any interpretation $I$ such that $I(R) = .1$ satisfies the formula from Example 2.7, which is hence satisfiable.

Definition 2.12. We identify the next satisfiability problems:

– Reg-SAT is the satisfiability problem of regular clausal formulas.
– Reg-Horn-SAT is Reg-SAT restricted to its Horn subclass.

Clausal Complexity. Reg-SAT is NP-complete [21, 61, 62] and Reg-Horn-SAT has complexity $O(n \log n)$ and $O(n)$ for the infinite- and finite-valued regular logics [46, 59, 61, 62], respectively.

3 Regular Many-Valued Non-Clausal

In this section, we present regular non-clausal syntactical and semantical concepts which are quite straightforwardly obtained by generalizing those from the clausal setting.

For the sake of readability of non-clausal formulas, we next justify our chosen notation of them. Thus, we employ:

1. The prefix notation because it requires only one $\lor/\land$-connective per formula, while infix notation requires $k - 1$, $k$ being the arity of the involved $\lor/\land$-connective.
2. Two symbol formula delimiters (Definition 3.1), $(\lor \ldots)$ for disjunctions and $\{\land \ldots\}$ for conjunctions, to better distinguish them inside nested non-clausal formulas.

Definition 3.1. The set $N_C$ of non-clausal formulas is inductively defined in the usual way exclusively from the following rules:

• $K \cup L \subset N_C$.
• If $\forall i \in \{1, \ldots, k\}, \varphi_i \in N_C$ then $\{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\} \in N_C$.
• If $\forall i \in \{1, \ldots, k\}, \varphi_i \in N_C$ then $(\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in N_C$.

6For background on propositional non-clausal concepts, the reader is referred to [21].
6These formulas are also called "negation normal form formulas" in the literature.
- \{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\} and any \varphi_i are called conjunction and conjunct, respectively.
- \{\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k\} and any \varphi_i are called disjunction and disjunct, respectively.
- \{\odot \varphi_1 \ldots \varphi_i \ldots \varphi_k\} stands for both \{\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k\} and \{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\}.

**Example 3.2.** Three examples of NC formulas are given below. We will show that \varphi_1 is not Horn-NC while \varphi_2 is Horn-NC. As \varphi_3 includes \varphi_1, then \varphi_3 is not Horn-NC either.

- \varphi_1 = \{\land (\lor P_{\leq 4} Q_{\geq 7} .9 \leq .4 ) (\lor Q_{\geq 6} \{\land R_{\leq 8} S_{\geq 9} .4 \geq .2\})\}
- \varphi_2 = (\lor \{\land P_{\leq 4} 1 \geq 0\} \{\land (\lor P_{\leq 3} R_{\geq 8}\} \{\land Q_{\geq 6} (\lor P_{\geq 7} S_{\leq 1})\})\}
- \varphi_3 = (\lor \varphi_1 \{\land Q_{\geq 6} (\lor \varphi_1 Q_{\leq 6} \varphi_2)\} \{\land \varphi_2 .9 \geq .6 \varphi_1\})

**Definition 3.3.** Subformulas are inductively defined as follows. The unique subformula of an atom \((K \cup L)\) is the atom itself, and the sub-formulas of a formula \(\varphi = \{\odot \varphi_1 \ldots \varphi_i \ldots \varphi_k\}\) are \varphi itself plus the sub-formulas of the \varphi_i’s.

**Example 3.4.** The sub-formulas of a clausal formula are the formula itself plus its clauses, literals and constants.

**Definition 3.5.** NC formulas are representable by trees if: (i) the nodes are: each atom is a leaf and each occurrence of a \land/\lor-connective is an internal node; and (ii) the arcs are: each sub-formula \{\odot \varphi_1 \ldots \varphi_i \ldots \varphi_k\} is a k-ary hyper-arc linking the node of \odot with the node of \varphi_i, for every \(i\), if \varphi_i is an atom and with the node of its connective otherwise.

**Example 3.6.** The graphical representation of \varphi from the introduction is given in the illustrative Examples 6.4 in Section 6 Example 6.6 provides further examples of DAGs.

A different, bi-dimensional graphical model of NCs is handled in [S40, 67] and in other works. On the other hand, our approach also applies when non-clausal formulas are represented and implemented as directed acyclic graphs (DAGs), which allow for important savings in both space and time.

**Definition 3.7.** An NC formula \(\varphi\) is modeled by a DAG if each sub-formula \(\phi\) is modeled by a unique DAG \(D_\phi\) and each \(\phi\)-occurrence by a pointer to (the root of) \(D_\phi\).

**Example 3.8.** Let us consider \(\varphi_3\) from Example 3.2 \(\varphi_1\) and \(\varphi_2\) should be represented by unique DAGs, i.e. \(D_{\varphi_1}\) and \(D_{\varphi_2}\), and each of the two occurrences of both \(\varphi_1\) and \(\varphi_2\) within \(\varphi_3\), by a pointer to their corresponding \(D_{\varphi_1}\) or \(D_{\varphi_2}\).

**Remark.** Although our approach is also valid for DAGs, for simplicity, we will use formulas representable by trees in the illustrative examples throughout this article.

- In the remaining of this subsection, we present semantical notions.

**Definition 3.9.** An interpretation \(I\) maps the propositions \(P\) into \(T\) and the non-clausal formulas \(\mathcal{NC}\) into \{0, 1\} and the mapping is extended from \(K \cup P\) to \(\mathcal{NC}\) by mappings atoms as done in Definition 2.9 and non-atomic formulas by the next functions:

- \(I(\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) = \max\{I(\varphi_i) : 1 \leq i \leq k\}\).
- \(I(\land \varphi_1 \ldots \varphi_i \ldots \varphi_k) = \min\{I(\varphi_i) : 1 \leq i \leq k\}\).
Definition 3.10. The concepts of model, un-satisfiable formula, logical equivalence and logical consequence in Definition 2.10 are equally defined for non-clausal formulas.

Example 3.11. Let us take $\varphi_1$ and $\varphi_2$ from Example 3.2 above: (i) any interpretation $I$ s.t. $I(Q) = 1$ is a model of $\varphi_1$; and (ii) one can verify that $\varphi_3 \equiv (\lor \varphi_1 \land Q^\geq 6 \land \varphi_2)$.

Definition 3.12. Akin to the clausal case, we identify the next satisfiability problems:
- Reg-NC-SAT is the satisfiability problem of regular NC formulas.
- Reg-Horn-NC-SAT is Reg-NC-SAT restricted to its Horn-NC subclass.

Non-Clausal Complexity. We can do the next considerations:
- Regarding Reg-NC-SAT, one easily verifies that it is NP-complete: NP-membership follows straightforwardly since checking whether a given interpretation is a model of a regular NC formula is trivially done polynomially. NP-completeness follows from: Reg-NC-SAT includes Reg-SAT which in turn includes classical SAT.
- Regarding Reg-Horn-NC-SAT, among our original results are, both, the definition of the regular Horn-NC class, $RH$, and the proof that its associated satisfiability problem, namely Reg-Horn-NC-SAT, is polynomial. From this polynomiality, we will trivially prove that Reg-Horn-NC-SAT is P-complete.

Next, some simple rules to simplify formulas are supplied.

Definition 3.13. Constant-free, equivalent formulas are straightforwardly obtained by applying to sub-formulas the simplifying rules below:
- Replace $(\lor \top \varphi)$ with $\top$.
- Replace $(\land \bot \varphi)$ with $\bot$.
- Replace $(\land \top \varphi)$ with $\varphi$.
- Replace $(\lor \bot \varphi)$ with $\varphi$.

Example 3.14. The constant-free, equivalent NC of $\varphi_2$ in Example 3.2 is:

$$\varphi = (\lor P_{\leq 4} \land (\lor P_{\leq 3} R_{\geq 8}) \land Q^\geq 6 (\lor P_{\geq 7} S_{\leq 1}) \land P_{\leq 1} R_{\leq 8} \land S_{\leq 1} (\land P_{\leq 3} Q_{\leq 0}))$$

Remark. For simplicity and since free-constant, equivalent formulas are easily obtained, hereafter we will consider only free-constant formulas.

4 Defining the Class $RH$

- HNC is used as a shorthand for Horn-NC.
- First of all, we need to define the negative formulas, which are the generalization of negative literals of the clausal case.

Definition 4.1. Negative formulas are non-clausal formulas having solely negative literals. $N^-$ denotes the set of negative non-clausal formulas.

Example 4.2. Trivially, negative literals are basic negative NC formulas. Another example of negative formula is $(\lor \land P_{\leq 1} R_{\leq 8} \land S_{\leq 1} (\lor P_{\leq 3} Q_{\leq 0}))$. $\Box$
Next we first define RH in a simple way and then, by taking at closer look, proceed to give its fine-grained definition in a compact and inductive function. Below we characterize RH by lifting the Horn-clausal pattern (defined in [59, 61, 62] as ”a regular Horn clause has at most one positive literal”), to the NC level in a straight way that is as follows.

**Definition 4.3.** A regular NC disjunction is HNC if it has at most one disjunct having positive literals. A regular NC formula \( \varphi \) is HNC if all its disjunctions are HNC. We denote \( RH \) the class of regular HNC formulas.

**Important Remark.** As Definition 4.3 is not concerned with how formulas are modeled, our approach also applies when they are represented by DAGs and not just by trees.

Clearly regular Horn formulas are regular HNC, which implies that the published Horn clausal class \( H \) [59, 21, 61, 62] is naturally subsumed by \( RH \), namely \( H \subset RH \).

**Proposition 4.4.** All sub-formulas of any HNC formula are HNC.

Such claim follows trivially from Definition 4.3. The converse does not hold as there are non-HNC formulas whose all sub-formulas are HNC.

**Example 4.5.** One can see that \( \varphi_1 \) below has only one non-negative disjunct and so \( \varphi_1 \) is HNC, while \( \varphi_2 \) is not HNC as it has two non-negative disjuncts.

- \( \varphi_1 = (\lor \{ \land Q_{\leq 6} S_{\leq 7} \} \land R_{\geq 7} P_{\geq 3}) \).
- \( \varphi_2 = (\lor \{ \land Q_{\leq 6} S_{\geq 7} \} \land R_{\geq 7} P_{\geq 3}) \).

Thus superscripting and subscripting positive and negative literals, respectively, enables to check how many disjuncts in a given disjunction contain positive literals, and so to decide, according to Definition 4.3, whether a given regular NC is HNC.

**Example 4.6.** We now consider both formulas \( \varphi \) in Example 3.14 and \( \varphi' \) below, which results from \( \varphi \) by just switching its literal \( P_{\leq 4} \) for \( P_{\geq 4} \): 

\[ \varphi' = (\lor P_{\geq 4} \{ \land (\lor P_{\geq 3} R_{\leq 8}) \land Q_{\geq 6} (\lor P_{\geq 7} S_{\leq 1}) \}) \]

Now we only check whether \( \varphi \) and \( \varphi' \) are HNC, and later, they will be thoroughly analyzed. So, all disjunctions of \( \varphi \), which are \( (\lor P_{\geq 3} R_{\leq 8}), (\lor P_{\geq 7} S_{\leq 1}) \) and \( \varphi \) itself have exactly one non-negative disjunct; thus \( \varphi \) is HNC. Yet, \( \varphi' \) is of the kind \( (\lor P_{\geq 4} \phi), \phi \) being non-negative. As \( \varphi' \) has two non-negative disjuncts, \( \varphi' \) is not HNC.

Towards a fine-grained definition of \( RH \), we individually and inductively specify:

- HNC conjunctions, in Lemma 4.7 and
- HNC disjunctions, in Lemma 4.9

and subsequently, we compactly specify \( RH \) by merging the precedent specifications into an inductive function given in Definition 4.12.

Clearly, conjunctions of Horn clausal formulas are Horn too, and a similar kind of Horn-like compliance also holds in NC, viz. conjunctions of HNC formulas are HNC too, which is straightforwardly formalized next.

---

*Definition 4.3* stipulates that the sub-formulas of \( \varphi \) include \( \varphi \) itself.
Lemma 4.7. Conjunctions of HNC formulas are HNC as well, formally:

\[ \{ \wedge \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \in RH \iff \text{for } 1 \leq i \leq k, \ \varphi_i \in RH. \]

Proof. It is obvious that if all sub-formulas \( \varphi_i \) individually verify Definition 4.3 so does a conjunction thereof, and vice versa. \[ \square \]

Example 4.8. If \( H_1 \) and \( H_2 \) are Horn and \( \varphi \) is from Example 3.14 which by Example 4.6 is HNC, then for instance \( \varphi_1 = \{ \wedge H_1, H_2 \ \varphi \} \) is HNC.

In order to formally define \( RH \), we now take a closer look at Definition 4.3. Thus, it is not hard to check that the definition of HNC disjunction of Definition 4.3 can be equivalently reformulated in the next inductive manner: "a disjunctive NC is HNC if it has any number of negative disjuncts and at most one non-negative HNC disjunct", which leads to the next formalization and statement.

Lemma 4.9. A disjunctive NC \( \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \) with \( k \geq 1 \) disjuncts belongs to \( RH \) iff \( \varphi \) has one HNC disjunct and \( k - 1 \) negative disjuncts, formally

\[ \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in RH \iff \text{there is } i \text{ s.t. } \varphi_i \in RH \text{ and for all } j \neq i, \varphi_j \in N^- \]

Proof. If: Since the sub-formulas \( \forall j, j \neq i, \varphi_j \) have no positive literals, the non-negative disjunctions of \( \varphi = (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k) \) are those of \( \varphi_i \) plus \( \varphi_i \) and \( \varphi \) themselves. Given that by hypothesis \( \varphi_i \in RH \) and that \( \forall j, j \neq i, \varphi_j \) has no positive literals then clearly all of them belong to \( RH \). Furthermore, since \( \varphi \) has only one non-negative disjunct and its sub-formulas verify Definition 3.1 so does \( \varphi \) itself. If: It is proven by contradiction (a similar proof is given in the first theorem in the Appendix): if any of the two conditions of the lemma does not hold, i.e. (i) \( \exists i, \varphi_i \notin RH \) or (ii) \( \exists i, j \neq j, \varphi_i, \varphi_j \notin N^- \), then \( \varphi \notin RH \). \[ \square \]

The next claims follow trivially from Lemma 4.9:

- Horn clauses are non-recursive HNC disjunctions.
- NC disjunctions with all negative disjuncts are HNC.
- NC disjunctions with \( k \geq 2 \) non-negative disjuncts are not HNC.

Next, we first reexamine, bearing Lemma 4.9 in mind, the formulas from Example 4.5 included in Example 4.10 and then those from Example 4.6 included in Example 4.11.

Example 4.10. Below we analyze \( \varphi_1 \) and \( \varphi_2 \) from Example 4.6.

- \( \varphi_1 = (\lor \{ \wedge Q \leq 6 \ S \leq 7 \} \ \{ \wedge R \geq 7 \ P \geq 3 \}) \).
  - Clearly \( \{ \wedge Q \leq 6 \ S \leq 7 \} \in N^- \).
  - By Lemma 4.7 \( \{ \wedge R \geq 7 \ P \geq 3 \} \in RH \).
  - According to Lemma 4.9 \( \varphi_1 \in RH \).

- \( \varphi_2 = (\lor \{ \wedge Q \leq 6 \ S \geq 7 \} \ \{ \wedge R \geq 7 \ P \leq 3 \}) \).
  - Obviously \( \{ \wedge Q \leq 6 \ S \geq 7 \} \notin N^- \) and \( \{ \wedge R \geq 7 \ P \leq 3 \} \notin N^- \).
  - According to Lemma 4.9 \( \varphi_2 \notin RH \). \[ \square \]
Example 4.11. Consider again \( \varphi \) from Example 3.14 and \( \varphi' \) from Example 4.6 and recall that \( \varphi' \) results from \( \varphi \) by just switching its literal \( P \leq 4 \) for \( P \geq 4 \). Below we analyze one-by-one the sub-formulas of both \( \varphi \) and \( \varphi' \).

- By Lemma 4.9, \(( \lor P \leq 3 R \geq 8 ) \in RH\).

- By Lemma 4.9, \(( \lor P \geq 7 S \leq 1 ) \in RH\).

- By Lemma 4.7, \({ \land Q \geq 6 ( \lor P \geq 7 S \leq 1 ) } \in RH\).

- By Lemma 4.7, \( \varphi = { \land ( \lor P \leq 3 R \geq 8 ) ( \land Q \geq 6 ( \lor P \geq 7 S \leq 1 ) ) } \in RH\).

- Using previous formula \( \varphi \), we have \( \varphi = ( \lor P \leq 4 \varphi ) \).

- Since \( P \leq 4 \in N^- \) and \( \varphi \in RH \), by Lemma 4.9, \( \varphi \in RH \).

- The second formula in Example 4.6 is \( \varphi' = ( \lor P \geq 4 \varphi ) \).

- Since \( P \geq 4 \), \( \varphi / \in N^- \), by Lemma 4.9, \( \varphi' / \in RH \).

- By using Lemmas 4.7 and 4.9 the class \( RH \) is syntactically, compactly and inductively defined as follows.

**Definition 4.12.** We inductively define the set of formulas \( RH \) from exclusively the rules below, wherein \( k \geq 1 \) and \( L \) is the set of literals.

1. \( L \subset RH \).
2. If \( \forall i, \varphi_i \in RH \) then \( \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \in RH \).
3. If \( \varphi_i \in RH \) and \( \forall j \neq i, \varphi_j \in N^- \) then \( ( \lor \varphi_1 \ldots \varphi_i \ldots \varphi_k ) \in RH \).

We prove next that the class \( RH \) indeed coincides with the class \( \mathcal{H}R \), namely Definition 4.12 is indeed the detailed, recursive and compact definition of \( \mathcal{H}R \).

**Theorem 4.13.** We have: \( RH = \mathcal{H}R \).

**Proof.** We prove first \( RH \subseteq \mathcal{H}R \) and then \( \mathcal{H}R \supseteq RH \).

- \( RH \subseteq \mathcal{H}R \) is easily proven by structural induction as outlined below:
  1. \( L \subset RH \) trivially holds.
  2. The non-recursive \( RH \) conjunctions are conjunctions of literals, which trivially verify Definition 4.3 and so are in \( RH \). Further, assuming that \( RH \subseteq \mathcal{H}R \) holds until a given inductive step and that \( \varphi_i \in RH \), \( 1 \leq i \leq k \), in the next induction step any formula \( \varphi = \{ \land \varphi_1 \ldots \varphi_i \ldots \varphi_k \} \) may be added to \( RH \); but by Lemma 4.7, \( \varphi \in RH \) and so \( RH \subseteq \mathcal{H}R \) holds.
  3. The non-recursive disjunctions in \( RH \) are obviously Horn clauses, which trivially fulfill Definition 4.3 and so are in \( RH \). Then assuming that for a given recursive level \( RH \subseteq \mathcal{H}R \) holds, in the next recursion, only disjunctions \( \varphi \) in (3) are added to \( RH \). But the condition of (3) and that of Lemma 4.9 are equal; so by Lemma 4.9, \( \varphi \in RH \) also. Therefore \( \mathcal{H}R \subseteq RH \) holds.

- \( \mathcal{H}R \subseteq RH \). Given that the structures to define \( NC \) and \( RH \) in Definition 3.1 and Definition 4.12, respectively, are equal, the potential inclusion of each NC formula \( \varphi \) in \( RH \) is systematically considered. Further, the statement: if \( \varphi \in RH \) then \( \varphi \in RH \), is proven by structural induction on the depth of formulas, by applying a reasoning similar to that of the previous \( RH \subseteq RH \) case and by also using Lemmas 4.7 and 4.9 \( \blacksquare \)
Within the propositional logic setting, the homologue of Definition 4.12 has served in [66] to design a linear algorithm that decides whether a given NC $\varphi$ is HNC. This algorithm is extensible to regular logic albeit its polynomial degree can of course slightly increase.

**Example 4.14.** We analyze $\varphi$ and $\varphi'$ from Example 4.11 w.r.t. Definition 4.12:

- By (3), $(\lor P_{\leq 3} R^{\geq 8}) \in \mathcal{RH}$.
- By (3), $(\lor P^{\geq 7} S_{\leq 1}) \in \mathcal{RH}$.
- By (2), $\{\land Q^{\geq 6} (\lor P^{\geq 7} S_{\leq 1})\} \in \mathcal{RH}$.
- By (2), $\phi = \{\land (\lor P_{\leq 3} R^{\geq 8}) (\land Q^{\geq 6} (\lor P^{\geq 7} S_{\leq 1}))\} \in \mathcal{RH}$.
- By (3), $\varphi = (\lor P_{\leq 4} \phi) \in \mathcal{RH}$.
- By (3), $\varphi' = (\lor P^{\geq 4} \phi) \notin \mathcal{RH}$. \qed

**Example 4.15.** If we assume that $\varphi_1$, $\varphi_2$, and $\varphi_3$ are negative and $\varphi_4$ and $\varphi_5$ are HNC, then according to Definition 4.12 four examples of nested HNC formulas follow.

- By (3), $\varphi_6 = (\lor \varphi_1 \varphi_4) \in \mathcal{RH}$.
- By (2), $\varphi_7 = \{\land \varphi_1 \varphi_5 \varphi_6\} \in \mathcal{RH}$.
- By (3), $\varphi_8 = (\lor \varphi_1 \varphi_2 \varphi_7) \in \mathcal{RH}$.
- By (2), $\varphi_9 = \{\land \varphi_6 \varphi_7 \varphi_8\} \in \mathcal{RH}$. \qed

Next, we analyze a more complete example, concretely the one given in the Introduction.

**Example 4.16.** Let us take $\varphi$ below, wherein $\varphi_1, \varphi_2$ and $\varphi'$ are NC formulas:

$$\varphi = \{\land P_{\leq 8} \quad (\lor P_{\leq 2} \{\land (\lor P_{\leq 3} Q_{\leq 4} P^{\geq 1}) (\lor \varphi_1 \{\land \varphi_2 P_{\leq 6}\}) Q^{\geq 7}\}) \quad \varphi'\}. $$

The disjunctions of $\varphi$ and the proper $\varphi$ can be rewritten as follows:

- $\omega_1 = (\lor P_{\leq 3} Q_{\leq 4} P^{\geq 1})$.
- $\omega_2 = (\lor \varphi_1 \{\land \varphi_2 P_{\leq 6}\})$.
- $\omega_3 = (\lor P_{\leq 2} \{\land \omega_1 \omega_2 Q^{\geq 7}\})$.
- $\varphi = \{\land P_{\leq 8} \omega_3 \varphi'\}.$

We analyze one-by-one such disjunctions and finally the proper $\varphi$:

- $\omega_1$: Trivially, $\omega_1$ is Horn.
- $\omega_2$: $\omega_2 \in \mathcal{RH}$ if $\varphi_1, \varphi_2 \in \mathcal{RH}$ and also if at least one of $\varphi_1$ or $\varphi_2$ is negative.
- $\omega_3$: $\omega_3 \in \mathcal{RH}$ if $\omega_2 \in \mathcal{RH}$ (as $\omega_1 \in \mathcal{RH}$).
- $\varphi$: $\varphi \in \mathcal{RH}$ if $\omega_2, \varphi' \in \mathcal{RH}$ (as $\omega_3 \in \mathcal{RH}$ if $\omega_2 \in \mathcal{RH}$).
Summarizing the conditions on $\varphi$ and on $\omega_2$, we have that:

- $\varphi$ is HNC: if $\varphi'$, $\phi_1$ and $\phi_2$ are HNC and if at least one of $\phi_1$ or $\phi_2$ is negative.

Since by Proposition 4.3, all sub-formulas of an HNC must be HNC, we can conclude that $\varphi$ is HNC if its sub-formulas are HNC and if at least one of $\phi_1$ or $\phi_2$ is negative.

5 Relating $\mathcal{RH}$ to the Classes Horn and NC

In this section, we demonstrate the relationships between $\mathcal{RH}$ and the classes $\mathcal{H}$ and $\mathcal{N}_C$. The formal proofs of the theorems were given in [66] but are provided in an Appendix for the sake of the paper being self-contained.

A new key and simple concept is introduced next, necessary to provide afterwards, the relationship between $\mathcal{RH}$ and the classes $\mathcal{H}$ and $\mathcal{N}_C$.

Definition 5.1. For every $\varphi \in \mathcal{N}_C$, we define $cl(\varphi)$ as the unique clausal formula that results from applying the $\lor$/\$\land$ distributivity laws to $\varphi$ until a clausal formula, viz. $cl(\varphi)$, is obtained. We will call $cl(\varphi)$ the clausal form of $\varphi$.

Example 5.2. Applying $\lor$/\$\land$ distributivity to $\varphi_1$ and $\varphi_2$ in Example 4.5, one obtains the clausal forms below. Note that $cl(\varphi_1)$ is Horn but not $cl(\varphi_2)$.

\[
\begin{align*}
cl(\varphi_1) &= \{\land (\lor Q \leq 6 \ R \geq 7) (\lor Q \leq 6 \ P \geq 3) (\lor S \leq 7 \ R \geq 7) (\lor S \leq 7 \ P \geq 3)\} \\
cl(\varphi_2) &= \{\land (\lor Q \leq 6 \ R \geq 7) (\lor Q \leq 6 \ P \leq 3) (\lor S \geq 7 \ R \geq 7) (\lor S \geq 7 \ P \leq 3)\}
\end{align*}
\]

\[\Box\]

Proposition 5.3. We have that: $\varphi \equiv cl(\varphi)$.

- The proof follows from the fact that $cl(\varphi)$ results by just applying the $\lor$/\$\land$ distributivity laws to $\varphi$ and that such laws of course preserve the logical equivalence.
- $cl(\varphi)$ is key to relate $\mathcal{RH}$ to the classes $\mathcal{H}$ and $\mathcal{N}_C$ as the next statements will show.

Theorem 5.4. The clausal form of all HNC formulas is Horn, formally:

\[\forall \varphi \in \mathcal{RH}, \text{ we have : } cl(\varphi) \in \mathcal{H}.\]

Proof. See Appendix. \[\Box\]

Theorem 5.4 and Proposition 5.3 yield the next characterization of $\mathcal{RH}$.

Corollary 5.5. Every HNC formula is logically equivalent to some Horn formula, formally

\[\forall \varphi \in \mathcal{RH}, \exists H \in \mathcal{H} \text{ such that } \varphi \equiv H.\]

Proof. From Proposition 5.3, and Theorem 5.4 we have that, for every HNC formula: $\forall \varphi \in \mathcal{RH}$, both, $cl(\varphi) \in \mathcal{H}$ and $\varphi \equiv cl(\varphi)$. Hence the corollary holds. \[\Box\]

Taking into account the previous corollary and the fact that $\mathcal{H} \subset \mathcal{RH}$, we verify that:

Corollary 5.6. $\mathcal{RH}$ and $\mathcal{H}$ are logically equivalent, noted $\mathcal{RH} \equiv \mathcal{H}$: every formula in a class is equivalent to another formula in the other class.
Proof. It follows from Corollary 5.5 and the fact that $\mathcal{H} \subset \mathcal{RH}$. ■

Therefore, we previously checked that syntactically $\mathcal{RH}$ subsumes the regular Horn class but now we have proved that semantically both classes are equivalent.

Thus $cl(\varphi)$ is a syntactical and semantical means of characterizing $\mathcal{RH}$; indeed, because $cl(\varphi)$ issues from $\varphi$ by a syntactical operation, and because $cl(\varphi)$ is semantically equivalent to $\varphi$, respectively.

The next theorem specifies which NC formulas are contained in $\mathcal{RH}$, or in other words, which syntactical NC fragment constitutes $\mathcal{RH}$.

Theorem 5.7. All NC formulas $\varphi$ whose clausal form is Horn are HNC, namely

$$\forall \varphi \in \mathcal{N}_C, \text{ if } cl(\varphi) \in \mathcal{H} \text{ then } \varphi \in \mathcal{RH}.$$

Proof. See Appendix. ■

Example 5.8. We apply below Theorem 5.7 to Examples 4.5 and 4.6.

- Example 4.5: by Example 5.2, $cl(\varphi_1) \in \mathcal{H}$ and $cl(\varphi_2) \notin \mathcal{H}$; hence, only $\varphi_1$ is HNC.

- Example 4.6: we do not supply $cl(\varphi)$ nor $cl(\varphi')$ due to their big size, but one has $cl(\varphi) \in \mathcal{H}$ and $cl(\varphi') \notin \mathcal{H}$. Hence only $\varphi$ is HNC. □

Next Theorem 5.9 just puts together previous Theorems 5.4 and 5.7 and can be viewed as an alternative definition of $\mathcal{RH}$ to that given in Definition 4.12.

Theorem 5.9. The next statement holds:

$$\forall \varphi \in \mathcal{N}_C : \, \varphi \in \mathcal{RH} \iff cl(\varphi) \in \mathcal{H}.$$

Proof. It follows immediately from Theorems 5.4 and 5.7. ■

Although the definition of $\mathcal{RH}$ in the previous theorem is concise and simple, trying to recognize HNF formulas via $cl(\varphi)$ is unfeasible, given that obtaining $cl(\varphi)$ takes both exponential time and space. Contrary to this, a polynomial algorithm can be obtained following Definition 4.12 as done in [66] for the propositional logic.

6 Non-Clausal Unit-Resolution and the Tractability of $\mathcal{RH}$

This section defines Regular Non-Clausal Unit-Resolution, or $RUR_{NC}$, which extends the same rule for propositional logic presented in [66]. $RUR_{NC}$ is proven to be complete for $\mathcal{RH}$ and to polynomially test $\mathcal{RH}$ for satisfiability. $RUR_{NC}$ encompasses the main inference rule, called RUR, and several simplification rules. The latter rules are simple but RUR is quite elaborate, and so we present it progressively:

- for almost-clausal HNCs, in Subsection 6.1
- for general HNCs, in Subsection 6.2 and
- for general NCs, in Subsection 6.3.

Remark. If $\varphi$ is a disjunctive HNC with more than one disjunct, then, according to Definition 4.3 it has at least one disjunct containing only negative literals and so assigning 0 to all its propositions satisfies $\varphi$. Therefore, to discard the case in which the input may be trivially satisfiable, we will consider that the input $\varphi$ is a conjunctive HNC formula.
6.1 Almost-Clausal HNC formulas

We start by recalling below regular clausal unit-resolution \cite{59, 21}, wherein \( X \in \mathcal{P} \) is a proposition, \( \alpha, \beta \in \mathcal{T} \) are truth-values and the \( \ell_i \)'s are literals:

\[
\begin{align*}
X^{\geq \alpha} \land (\lor \ell_1 \ldots \ell_j X_{\leq \beta} \ell_{j+1} \ldots \ell_k), \quad \alpha > \beta \\
\quad (\lor \ell_1 \ldots \ell_j \ell_{j+1} \ldots \ell_k)
\end{align*}
\]

- Such rule is refutationally complete for the regular Horn class \cite{59}.
- At first, we introduce \( \text{RUR} \) just for almost-clausal HNC formulas. Assume HNCs with the next almost-clausal pattern, where \( X \in \mathcal{P} \) and \( \alpha, \beta \in \mathcal{T} \):

\[
\{ \land \pi_1 \ X^{\geq \alpha} \pi_2 \ (\lor \phi_1 \ldots \phi_{j-1} X_{\leq \beta} \phi_{j+1} \ldots \phi_k) \pi_3 \}, \quad \alpha > \beta
\]

in which the \( \pi \)'s are lists of HNC formulas, namely \( \pi_1 = \varphi_1 \ldots \varphi_{l-1} ; \pi_2 = \varphi_{l+1} \ldots \varphi_{l-1} ; \) and \( \pi_3 = \varphi_{l+1} \ldots \varphi_n \). These formulas are almost-clausal in the sense that if the \( \varphi \)'s and \( \phi \)'s were clauses and literals, respectively, then such formulas would be clausal. Since \( X^{\geq \alpha} \) and \( X_{\leq \beta} \), for \( \alpha > \beta \), are unsatisfiable, almost-clausal formulas are clearly equivalent to:

\[
\{ \land \pi_1 \ X^{\geq \alpha} \pi_2 \ (\lor \phi_1 \ldots \phi_{j-1} \phi_{j+1} \ldots \phi_k) \pi_3 \}
\]

and thus, one obtains the next simple inference rule:

\[
\begin{align*}
X^{\geq \alpha} \land (\lor \phi_1 \ldots \phi_j X_{\leq \beta} \phi_{j+1} \ldots \phi_k), \quad \alpha > \beta \\
\quad (\lor \phi_1 \ldots \phi_j \phi_{j+1} \ldots \phi_k)
\end{align*}
\]

\( \text{RUR} \)

Note that if almost-clausal formulas are clausal, then the above rule recovers regular clausal unit-resolution. By noting \( \mathcal{D}(X_{\leq \beta}) \) the disjunction \( (\lor \phi_1 \ldots \phi_j \phi_{j+1} \ldots \phi_k) \), the previous rule can be rewritten concisely as:

\[
\begin{align*}
X^{\geq \alpha} \land (\lor X_{\leq \beta} \mathcal{D}(X_{\leq \beta})), \quad \alpha > \beta \\
\quad \mathcal{D}(X_{\leq \beta})
\end{align*}
\]

\( \text{RUR} \) \hspace{1cm} (1)

Example 6.1. Let us consider the next formula:

\[
\{ \land \ P^{\geq 8} \ (\lor \ Q_{\leq 4} \ P_{\leq 5}) \ (\lor \ R^{\geq 9} \ {\land \ Q_{\leq 2} \ P_{\leq 3}}) \}
\]

If we pick up \( P^{\geq 8} \) and \( P_{\leq 5} \), then we have \( \mathcal{D}(P_{\leq 5}) = (\lor Q_{\leq 4}) \). So by applying the previous rule to \( \varphi \) and then by removing the generated redundant \( \lor \)-connective, one deduces:

\[
\{ \land \ P^{\geq 8} \ Q_{\leq 4} \ (\lor \ R^{\geq 9} \ {\land \ Q_{\leq 2} \ P_{\leq 3}}) \} \tag{\Box}
\]

We now extend our analysis from HNCs with pattern \( X^{\geq \alpha} \land (\lor X_{\leq \beta} \mathcal{D}(X_{\leq \beta})) \) to those with pattern \( X^{\geq \alpha} \land (\lor \ C(X_{\leq \beta}) \mathcal{D}(X_{\leq \beta})) \) in which \( C(X_{\leq \beta}) \) is the greatest sub-formula of the input \( \varphi \) that becomes false when \( X_{\leq \beta} \) is false, that is, \( C(X_{\leq \beta}) \) is the greatest sub-formula of \( \varphi \) containing \( X_{\leq \beta} \) and equivalent to a conjunction \( X_{\leq \beta} \land \psi \). For instance, if the input has the sub-formula \( \{ \land \phi_3 \{ \land X_{\leq \beta} \ (\lor \phi_1 \ P_{\geq 3}) \} \phi_2 \} \), we take it as \( C(X_{\leq \beta}) \) because it is equivalent to \( X_{\leq \beta} \land \{ \land \phi_3 \ (\lor \phi_1 \ P_{\geq 3}) \phi_2 \} \). Thus, if \( X_{\leq \beta} \) becomes false so does the formula \( C(X_{\leq \beta}) \).

Remark. \( C(X_{\leq \beta}) \) contains \( X_{\leq \beta} \) while \( \mathcal{D}(X_{\leq \beta}) \) excludes it.
Example 6.2. The next formula is an extension of that from Example 6.1 where \( P_{\leq 5} \) is substituted by the formula \( \{ \land \ P_{\leq 5} \ (\lor \ R^{\geq 7} \ Q_{\leq 1}) \} \):

- \( \varphi = \{ \land \ P_{\geq 8} \ (\lor \ Q_{\leq 4} \ \{ \land \ P_{\leq 5} \ (\lor \ R^{\geq 7} \ Q_{\leq 1}) \} \) \ (\lor \ R^{\geq 9} \ \{ \land \ Q_{\leq 2} \ P_{\leq 3} \} \) \]

If we select \( P_{\geq 8} \) and \( P_{\leq 5} \), then \( \varphi \) has one sub-formula with the \( X^{\geq \alpha} \land (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \) pattern in which \( C(P_{\leq 5}) = \{ \land \ P_{\leq 5} \ (\lor \ R^{\geq 7} \ Q_{\leq 1}) \} \) and \( D(P_{\leq 5}) = (\lor \ Q_{\leq 4}) \).

Thus, the needed RUR for the extended pattern \( X^{\geq \alpha} \land (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \) is:

\[
\frac{X^{\geq \alpha} \land (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta}))}{D(X_{\leq \beta})}, \alpha > \beta
\]

(2)

Example 6.3. By applying RUR with literals \( P_{\geq 8} \) and \( P_{\leq 5} \) to \( \varphi \) in Example 6.2 and then removing the generated redundant \( \lor \)-connective, we obtain:

\( \varphi' = \{ \land \ P_{\geq 8} \ Q_{\leq 4} \ (\lor \ R^{\geq 9} \ \{ \land \ Q_{\leq 2} \ P_{\leq 3} \} \) \}

6.2 General HNC formulas

We now consider arbitrarily nested HNCs to which the rule RUR can indeed be applied, which means HNCs with the next pattern:

\[
\{ \land \ \pi_0 \ \ X^{\geq \alpha} \ \pi_0' \ (\lor_1 \ \pi_1 \ldots \ (\lor_k \ \pi_k \ \ (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \ \pi_k' \ldots \pi_1') \ \pi_0'' \}
\]

where the \( \pi \)'s and \( \pi' \)'s are concatenations of HNC formulas, for instance, for the nesting level \( j \), we have \( \pi_j = \varphi_{j1} \ldots \varphi_{j-1} \) and \( \pi_j' = \varphi_{j+1} \ldots \varphi_{jn} \). It is not hard to check that RUR can be generalized simply as follows:

\[
\frac{X^{\geq \alpha} \land (\lor_1 \ \pi_1 \ldots \ (\lor_k \ \pi_k \ \ (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \ \pi_k' \ldots \pi_1'))}{(\lor_1 \ \pi_1 \ldots \ (\lor_k \ \pi_k \ \ D(X_{\leq \beta}) \ \pi_k' \ldots \pi_1')}, \alpha > \beta
\]

RUR

Remark. RUR should be read as: if the input \( \varphi \) has any sub-formula having the pattern of the right conjunct of the numerator then it can be replaced with the formula in the denominator. In practice, applying RUR amounts to simply remove \( C(X_{\leq \beta}) \).

In order to simplify RUR, we denote \( \Pi \) the right conjunct of the numerator and also denote \( \Pi \cdot (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \) that \( (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta})) \) is a sub-formula of \( \Pi \). Using both notations, the rule above can be compacted giving rise to RUR for arbitrary HNCs:

\[
\frac{X^{\geq \alpha} \land \Pi \cdot (\lor \ C(X_{\leq \beta}) \ D(X_{\leq \beta}))}{\Pi \cdot D(X_{\leq \beta})}, \alpha > \beta
\]

RUR

In the next Examples 6.4 and 6.6 we analyze the formula \( \varphi \) from the Introduction section (and also from Example 4.16).

\[8\] The notation \( \langle \lor \ \varphi_1 \ldots \varphi_k \rangle \) was introduced in Definition 3.1 bottom.
**Example 6.4.** Let us consider $\varphi$ from the Introduction where $\varphi' = P^{\geq,7}$.

\[\varphi = \{\land P_{\leq,8} \quad (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\})\} \]

Its associated DAG (tree, in this case) is depicted in **Fig. 1** below:

![Fig. 1. Tree of Example 6.4.](image)

Selecting the literals $P^{\geq,7}$ and $P_{\leq,6}$, the formula $\Pi \cdot (\lor C(P_{\leq,6}) \ D(P_{\leq,6}))$ is:

\[\land (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\})\]

wherein the sub-formula $(\lor C(P_{\leq,6}) \ D(P_{\leq,6}))$ is $(\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6})$. Applying RUR to $\varphi$ leads to the next simpler formula:

$\varphi' = \{\land P_{\leq,8} \quad (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\}) \}$ $P^{\geq,7}$

whose associated tree is depicted in **Fig. 2** below.

![Fig. 2. Tree of Formula $\varphi'$](image)

- **Simplification Rules.** To complete the calculus $RUR_{NC}$, the simplification rules given below must accompany $RUR$. Recall that, by Definition 2.8 $(\lor)$ is a $\perp$-constant. The first two rules simplify formulas by (upwards) propagating $(\lor)$ from sub-formulas to formulas, in which $\varphi$ is the input HNC, and, as before, $\varphi \cdot \phi$ means that $\phi$ is a sub-formula of $\varphi$:  

\[\land (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\}) \]$ $P^{\geq,7}$

\[\land (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\}) \]$ $P^{\geq,7}$

\[\land (\lor P_{\leq,2} \quad \{\land (\lor P_{\leq,3} \quad Q_{\leq,4} \quad P^{\geq,1} \quad (\lor \phi_1 \quad \{\land \phi_2 \quad P_{\leq,6}) \quad Q^{\geq,7})\}) \]$ $P^{\geq,7}$
\[ \varphi \cdot (\lor \phi_1 \ldots \phi_{i-1} \lor \phi_{i+1} \ldots \phi_k) \lor \bot \] (3)

\[ \varphi \cdot \{ \land \varphi_1 \ldots \varphi_{i-1} \lor \varphi_{i+1} \ldots \varphi_k \} \land \bot \] (4)

The next two rules remove redundant connectives; the first one removes a connective if it is applied to a single sub-formula, i.e. \( \langle \circ_1 \phi_1 \rangle \):

\[ \varphi \cdot \langle \circ_2 \varphi_1 \ldots \varphi_{i-1} \langle \circ_1 \phi_1 \rangle \varphi_{i+1} \ldots \varphi_k \rangle \land_{k+1} \] (5)

and the next rule removes a connective of a sub-formula that is inside another sub-formula with the same connective:

\[ \varphi \cdot \langle \circ_1 \varphi_1 \ldots \varphi_{i-1} \langle \circ_2 \phi_1 \ldots \phi_n \rangle \varphi_{i+1} \ldots \varphi_k \rangle \circ_1 = \circ_2 \land_{k+n} \] (6)

Finally, two further trivial simplification rules are required:

\[ \{ \lor P \geq \alpha \land \land P \leq \beta \} \land_{\max(\alpha, \beta)} \] (7)

\[ \lor P \geq \alpha \land \land P \leq \beta \} \land_{\max(\alpha, \beta)} \] (8)

**Remark.** Rules like the following ones:

\[ \lor P \geq \alpha \land \land P \leq \beta \} \land_{\max(\alpha, \beta)} \] (7)

\[ \lor P \geq \alpha \land \land P \leq \beta \} \land_{\max(\alpha, \beta)} \] (8)

are of course sound and have indeed interest for improving efficiency but are unnecessary for warranting refutational completeness, which is our concern in this paper.

**Definition 6.5.** We define \( RUR_{NC} \) as the calculus formed by the rule RUR and the above described simplification rules, namely,

\[ RUR_{NC} = \{ RUR, \lor, \land, \circ_{k+1}, \circ_{k+n}, \land_{\alpha}, \land_{\beta}, \land_{\max} \} \]

**Example 6.6.** By continuing Example 6.4, we now show how \( RUR_{NC} \) finds that \( \varphi' \) is unsatisfiable. If we select \( P \geq 3 \) and \( P \leq 2 \) (colored blue in Fig. 2), then: \( C(P \leq 2) = P \leq 2; \ D(P \leq 2) = (\lor \{ P \leq 3 \land Q \leq 4 \land P \geq 1 \}) \lor (\land \phi_1) \lor Q \geq 7 \}; \) and

\[ \Pi \cdot (\lor C(P \leq 2) \land D(P \leq 2)) = (\lor C(P \leq 2) \land D(P \leq 2)) \]

By applying RUR to \( \varphi' \), the obtained formula \( \varphi'' \) is depicted in Fig. 3 below:

![Fig. 3. Tree of Formula \( \varphi'' \)]
After two applications of $⊙_{k+1}$ and one of $⊙_{k+n}$, one gets the formula associated with the tree in Fig. 4 below:

$$∧ P_{≤8} \lor \phi_1 Q_{≥7} \land p_{≥7}$$

$$P_{≤3} Q_{≤4} p_{≥1}.$$

Fig. 4. Continuing Example 6.6

In this state, two applications of RUR to the two complementary pairs $Q_{≥7}$ and $Q_{≤4}$, and $p_{≥7}$ and $P_{≤3}$, and then one application of $⊙_{k+1}$ lead the calculus $RUR_{NC}$ to the formula associated with the tree despited in Fig. 5, left:

$$∧ P_{≤8} p_{≥1} \lor \phi_1 Q_{≥7} \land p_{≥7}$$

$$P_{≤8} Q_{≥7} p_{≥1}.$$

Fig. 5. Continuing Example 6.6

Now, the inference rule, called max, is applied and the formula inferred has the right tree in Fig. 5. Finally, the rule $\bot_{β}$ derives $(\lor)$.

Lemma 6.7. An HNC $φ$ is unsatisfiable iff $RUR_{NC}$ applied to $φ$ derives $(\lor)$.

Proof. We analyze below both senses of the lemma.

$⇒$ Let us assume that $φ$ is unsatisfiable. Then $φ$ must have a sub-formula verifying the RUR numerator, otherwise, all unsatisfiable pairs of literals $X_{≥α}$ and $X_{≤β}$ such that $α > β$ are included in disjunctions (if $α = β$, both literals are satisfiable). If this case, since all disjunctions of $φ$, by definition of HNC formula, have at least one negative literal, $φ$ would be satisfied by assigning to all propositions the value 0, which contradicts the initial hypothesis. Therefore, RUR is applied to $φ$ with the literals $X_{≥α}$ and $X_{≤β}$ such that $α > β$ and the resulting formula is simplified. The new formula is equivalent to $φ$ and has at least one literal less than $φ$. Hence, by induction on the number of literals of $φ$, we easily obtain that $RUR_{NC}$ ends only when $(\lor)$ is derived.

$⇐$ Let us assume that $RUR_{NC}$ has been applied without having derived $(\lor)$. Clearly, $RUR_{NC}$ ends when a deduced formula $φ'$ is not a conjunction of a literal $X_{≥α}$ with a disjunction including another literal $X_{≤β}$ such that $α > β$. Firstly, since $RUR_{NC}$ is sound, $φ$ and $φ'$ are equi-satisfiable. Secondly, if $φ'$ has complementary literals, then they are integrated in disjunctions. Thus $φ'$ is satisfied by assigning the value 0 to all its unassigned propositions, since, by definition of HNC formula, all disjunctions have at least one negative disjunct. Therefore, since $φ'$ is satisfiable so is $φ$.

Lemma 6.8. Reg-Horn-NC-SAT is polynomial.

Proof. The number of inferences performed by $RUR_{NC}$ to the input $φ$ is bounded linearly in the size of $φ$. Indeed, the number of RUR rules performed is at most the number of
literals in \( \varphi \), and the number of simplification rules is at most the number of connectives plus the number of literals in \( \varphi \). On the other hand, it is not difficult to find data structures to polynomially execute each inference of the calculus \( RUR_{NC} \). Hence, \( \text{Reg-Horn-NC-SAT} \) is polynomial.

**Proposition 6.9.** \( \text{Reg-Horn-NC-SAT} \) is \( P \)-complete.

**Proof.** It follows straightforwardly from the next two facts: (i) \( \text{Reg-Horn-NC-SAT} \) is polynomial, according to Lemma 6.8, and (ii) \( \text{Reg-Horn-NC-SAT} \) includes \( \text{Reg-Horn-SAT} \) which in turn includes propositional Horn-SAT which is \( P \)-complete [29].

**Remark.** Having established \( RUR_{NC} \), the procedure \( \text{NC Unit-Propagation} \) (viz. the \( \text{Reg-Horn-NC-SAT} \) algorithm) can easily be designed, and on its basis, effective Non-Clausal DPLL-based solvers can be developed.

The best published complexities for \( \text{Reg-Horn-SAT} \) are \( O(n \log n) \) and \( O(n) \) for infinite- and finite-valued regular formulas, respectively [59, 21, 61, 62]. For future work, we will attempt to find data structures and devise algorithms, inspired by \( RUR_{NC} \), able to decide the satisfiability of HNCs with complexity tight to the aforementioned clausal ones.

### 6.3 Further Inferences Rules

In this subsection, we present two further inferences extending \( RUR_{NC} \): Regular General-Unit-Resolution and Regular Hyper-Unit-Resolution.

**General Unit-Resolution.** So far we have defined \( RUR_{NC} \) just for HNCs and thus, its design for general NCs was pending. For such purpose, the application of the RUR rule to sub-formulas \( \phi \) of the input NC \( \varphi \) having the RUR numerator pattern should be authorized. Namely, applying the previous RUR rule to each sub-formula of \( \varphi \) with pattern \( \phi = X^{\geq \alpha} \land \Pi \cdot (\lor C(X^{\leq \beta}) D(X^{\leq \beta})) \) should be permitted and so, \( \phi \) could be replaced with \( X^{\geq \alpha} \land \Pi \cdot D(X^{\leq \beta}) \). Hence, the formal specification of the Regular General-Unit-Resolution rule, RGUR, for any general NC \( \varphi \) is:

\[
\frac{\varphi \cdot (X^{\geq \alpha} \land \Pi \cdot (\lor C(X^{\leq \beta}) D(X^{\leq \beta})))}{\varphi \cdot (X^{\geq \alpha} \land \Pi \cdot D(X^{\leq \beta}))} \quad \text{RGUR}
\]

**Example 6.10.** Consider again Example 6.6. One can check that its sub-formula

\[
(\lor P_{\leq 2} \{\land (\lor P_{\leq 3} Q_{\leq 4} P^{\geq 1}) (\lor \phi_1 \{\land \phi_2 P_{\leq 6}\})
\]

has the pattern of the RGUR numerator, and so RGUR can be applied and replaced it with \( (\lor P_{\leq 2} \{\land (\lor P_{\leq 3} P^{\geq 1}) (\lor \phi_1 \{\land \phi_2 P_{\leq 6}\}) \) in the input formula.

Observe that the introduction of this new RGUR rule applicable to certain sub-formulas habilitates new sequences of inferences, and so, their suitable management can enhance the overall deductive efficiency.

**Proposition 6.11.** Let \( \varphi \) be any NC formula. If \( \varphi' \) results from applying RGUR to \( \varphi \) then \( \varphi' \) and \( \varphi \) are logically equivalent.
Proof. The proof of the soundness of RGUR is straightforward.

Remark. The extension of the simplification rules from HNCs to NCs is similarly obtained and so is the calculus RGURNC.

Hyper Unit-Resolution. The given definition of Regular NC Unit-Resolution can be extended in order to obtain Regular NC Hyper-Unit-Resolution (RHURNC), which is given below. The sub-formula $\lor C(X \leq \beta) \land D(X \leq \beta)$ is denoted $CD(X \leq \beta)$ and since $X \leq \beta_i, 1 \leq i \leq k$, are literal occurrences that are pairwise different, so are $\Pi_i \cdot CD(X \leq \beta_i), 1 \leq i \leq k$:

$$ \frac{X \geq \alpha \land \Pi_1 \cdot CD(X \leq \beta_1) \land \ldots \land \Pi_i \cdot CD(X \leq \beta_i) \land \ldots \land \Pi_k \cdot CD(X \leq \beta_k)}{\Pi_1 \cdot D(X \leq \beta_1) \land \ldots \land \Pi_i \cdot D(X \leq \beta_i) \land \ldots \land \Pi_k \cdot D(X \leq \beta_k)} RHUR_{NC} $$

The soundness and completeness of RHURNC follow from those of RURNC.

7 Non-Clausal Logic-Programming

In spite of both the profusion of many-valued logic programming approaches in clausal form developed since annotated logic programming was conceived [68] and the important advances carried out in propositional logic programming in NC form since nested logic programming was proposed [75], no approach seems to have been developed to deal with many-valued logic programming in NC form. Regarding computational issues, tractability is analyzed, to our knowledge, only in [66] which focuses on propositional logic.

This section shows the usefulness of both RH and RURNC for NC logic-programming in regular-logic. Concretely, we show that the rule syntax can be enriched allowing NCs in heads and bodies with slight restrictions while keeping query-answering efficiency qualitatively comparable to clausal efficiency, specifically answering queries is polynomial.

Definition 7.1. An HNC rule is an expression $B^+ \rightarrow H$ wherein $B^+$ (Body) is an NC formula having only positive literals and $H$ (Head) is any arbitrary HNC formula. An HNC logic program is a set of HNC rules.

Example 7.2. The next rule is an HNC rule, where its body is an NC with only positive literals and its head is a simple HNC formula.

$$ \{ \land R^{g1} (\lor P^{z,7} \{\land S^{z,7} Q^{z,6}\})\} \longrightarrow (\lor \{\land Q^{z,6} S^{z,7}\} \{\land R^{z,7} P^{z,3}\} ) $$

Another HNC rule is given below, where its body is again a positive NC and its head $\varphi$ can be, for instance, any HNC among the ones used in previous illustrative examples:

$$ \{ \land R^{g1} (\lor P^{z,7} \{\land S^{z,7} Q^{z,6}\}) (\lor R^{g,9} \{\land Q^{g,2} P^{g,3}\}) Q^{g,7}\} \longrightarrow \varphi $$

Proposition 7.3. A conjunction of HNC rules, or equivalently, an HNC logic program, is an HNC formula.

Proof. Clearly a rule $(\lor \neg B^+ H)$ verifies Definition [1.3] and hence so does a conjunction thereof, or equivalently, so does an HNC logic program.

Proof. Clearly a rule $(\lor \neg B^+ H)$ verifies Definition [1.3] and hence so does a conjunction thereof, or equivalently, so does an HNC logic program.
The rules from Example 7.2 give an intuitive idea of the potentiality of RH to enrich declarative rules. Next lemma analyzes the dual aspect to expressiveness, i.e., efficiency, stating that the complexity of query-answering is polynomial (as in the clausal framework).

**Lemma 7.4.** Let \( S \) be a positive literal set, \( Lp \) be an HNC logic program and \( \varphi \) be any arbitrary (unrestricted) NC formula. Deciding whether \( S \land Lp \models \varphi \) is polynomial.

**Proof.** By Lemma 6.8 one can polynomially check whether \( S \land Lp \) is satisfiable. If so, then by applying \( RUR_{NC} \), one can polynomially obtain the positive literals that follow logically from \( S \land Lp \), i.e., its minimal model. Finally, one can also polynomially check whether, for such minimal model, \( \varphi \) is evaluated to 1, i.e., whether \( S \land Lp \models \varphi \) holds. ■

Summarizing, the advantages conferred by RH and \( RUR_{NC} \) to NC logic programs are that, clearly NC logic programs are smaller and can be even exponentially smaller than their equivalent clausal logic programs, and that, according to Lemma 7.4, query-answering takes polynomial time.

### 8 Related Work

This research work heavily relies on the areas of regular logic, NC reasoning and Horn formulas. In [66], related work to NC reasoning and Horn formulas is extensively discussed, and in this section, we will only discuss related work to regular logic.

Since regular logic is a (relevant) sub-class of signed logic, we next start by introducing signed logic and its variants, and by discussing general aspects. Subsequently we review the existing complexity sub-classes in signed clausal satisfiability and then the published methods for solving signed non-clausal satisfiability.

#### 8.1 General Presentation

We recall (Section 2) that signed logic differs from propositional logic only at the literal level. A signed literal is a pair \( S \cdot P \) where \( P \in P \) and \( S \) is a (usually finite) subset \( S \subseteq T \) and is satisfied by \( I \) only if \( I(P) \in S \). The remaining concepts given in Sections 2 and 3 are equally applicable to signed logic. \( S \) is called sign of \( S \cdot P \) and the idea of using truth-value sets as signs is due to independently several authors [56, 38, 87].

Regular logic is the most studied and employed sub-class of signed logic and derives from it when \( T \) is totally ordered and the signs are only of two kinds: \([−\infty, α]\), which corresponds to \( P_{≤α} \), or \([α, ∞]\), which corresponds to \( P^{≥α} \). Other main signed sub-classes are issued when \( T \) is partially ordered, specially when \( T \) is a lattice, and when the signs are singletons; the latter is called mono-signed logic. Subsection 8.2 reviews the published polynomial and NP-complete fragments of signed clausal logic.

An outstanding feature [58] of signed clausal forms is that they offer a suitable logical framework for automated reasoning in multiple-valued logics given that the satisfiability problem of any finitely-valued propositional logic, as well as of certain infinitely-valued logics, is polynomially reducible to the satisfiability problem of signed clausal formulas. On the other hand, the significance of regular logic comes from the next property demonstrated by Hahnle [59]: every signed formula is logically equivalent to some regular formula.

Concerning applications, signed logic and annotated logic programs [68] are closely connected to each other [76]. Annotated logic programming is obtained from signed logic
when the formulas are Horn clausal and $T$ is a lattice. Annotated logic programming is a suitable language to manage locally inconsistent, incomplete and uncertain databases, and indeed a large number of systems have been developed during the last decades.

Another main application of signed logic is found interpreting a literal $S \cdot P$ as ”$P$ is constrained to the values in $S$”. This allows signed clausal forms to be used as a powerful knowledge representation language for constraint programming [77] and they have shown to be a practical and competitive approach to solving combinatorial decision problems [50, 23, 9].

As aforementioned, a great deal of research on signed logic has been conducted in satisfiability solving, logic programming and constraint solving, but signed logic and its variants have also been handled, during the last twenty years, in many approximate reasoning scenarios such as model-based diagnosis [49], signed optimization [8], signed randomization [17], combining signed logic and linear integer arithmetic [7], learning in CSP [98], comparing resolution proofs and CDCL-with-restarts [82], regular belief merging [36], in the formalization of a recent real-world multivalued-logic [47] and more.

8.2 Signed Clausal

Outside signed logic, the complexity of fuzzy logics is NP-complete or harder [63] and only few and very restricted classes are polynomial [26, 27]. Next we discuss computational issues in signed logic and its variants and show that a variety of polynomial classes exist.

The central role of signed clausal formulas in automated deduction pointed out above, justified a detailed study of its sub-classes, including algorithms for and complexities of associated satisfiability problems. These results are summarized next, and for more details, the interested reader may consult [21, 61, 60, 62] and the references therein.

Signed Resolution. Efficient decision procedures for signed logic are mostly based on the extension of resolution to signed logic, which appeared in [57, 58] and independently in [87, 85], also see [13]. Regular resolution and regular unit-resolution were given in [59]. Resolution when $T$ is only partially ordered was proposed by several authors [78, 51, 94]. The approach in [51, 94] consists in encoding regular formulas in first-order theories with transitive relations and then applying ordered resolution [15] which includes transitivity axioms. Mono-signed resolution was developed in [12, 13] and in [51, 94]: in the latter approach the authors translate a formula into first-order equations and then use superposition calculus [14].

Complexity Classes. Next we overview the complexity of sub-classes of signed clausal logic. We denote Signed-SAT the satisfiability problem of signed formulas, and Signed-2-SAT denotes Signed-SAT restricted to formulas whose clauses have only two literals. Reg-SAT (Section 2) and Reg-2-SAT are similarly defined for regular logic.

NP-Completeness. Signed-SAT is NP-complete: its NP-membership is verified as for classical SAT and its NP-completeness is obvious because it includes classical SAT. Both Signed-2-SAT and Reg-2-SAT for any $|T| \geq 3$ are NP-complete [19, 80], in contrast with the linearity of classical 2-SAT [11]. Even Reg-2-SAT is NP-complete for general partial orders, which is proved by reducing Signed-2-SAT to Reg-2-SAT [20]. More concretely, in [20] it is proved that Signed-2-SAT is NP-complete (1) if $T$ is a modular lattice and the signs $S$ are complements of regular signs $\leq \alpha$ and $\geq \alpha$ of $T$ (i.e. the literals are $P_{\leq \alpha}$ and $P_{\geq \alpha}$), or (2) if $T$ is a distributive lattice and the signs are regular signs of $T$ and their complements. Further complexities on Signed-2-SAT based on the Helly property
are obtained in \cite{33, 34} and are discussed below.

**Polynomial Signed-2-SAT.** Under certain restrictions, however, Reg-2-SAT is polynomial, e.g. when $\mathcal{T}$ is totally ordered, Reg-2-SAT is $O(n \log n)$, $n$ being the length of $\varphi$, which is proved via a reduction to classical 2-SAT \cite{22}. A polynomial result for the more general case, when $\mathcal{T}$ is a lattice and all occurring signs are of the form $\leq \alpha$ or $\geq \alpha$ ($P_{\leq \alpha}$ or $P_{\geq \alpha}$) is in \cite{20}. Charatonik and Wrona \cite{32} showed that this problem can be solved in quadratic time in the size of the input and in linear time in the size of the formula if the lattice is fixed. For this, they used a reduction of a many-valued satisfiability problem on a lattice to classical SAT. Somewhat different multi-valued 2-SAT problems are studied in \cite{32}. In \cite{12}, the authors proved that mono-signed 2-SAT is polynomial and in \cite{80} a linear-time procedure for such problem is described. In \cite{10}, regular and mono-signed logics are merged and new 2-SAT problems are defined proving some of them are polynomial.

**Polynomial Reg-Horn-SAT.** In the regular Horn-SAT problem, three have been analyzed depending on the structure of $\mathcal{T}$: (1) $\mathcal{T}$ is totally ordered, (2) $\mathcal{T}$ is a lattice, and (3) $\mathcal{T}$ is partially ordered but not a lattice.

(1) When $\mathcal{T}$ is totally ordered, Reg-Horn-SAT can be solved in time linear in size($\varphi$) if $\mathcal{T}$ is finite, and in $O(n \log n)$ otherwise \cite{59, 22}. Many of the results are proven via reduction to classical logic \cite{22}. Some Horn problems are defined in \cite{10} for the aforementioned merged regular and mono-signed logic and their tractability are stated. Complementary results are obtained in \cite{52, 53}, where, given a set of interpretations $M$, the authors obtain a regular Horn formula whose set of models is $M$ (called constraint description problem).

(2) If $\mathcal{T}$ is a finite lattice, then Reg-Horn-SAT is decidable in linear time in the size of the formula and polynomial in the cardinality of $\mathcal{T}$ via a reduction to classical Horn-SAT \cite{19}. For distributive lattices, the bound obtained in \cite{93} is size($\varphi$) $\times$ $n^2$, where $n$ is the cardinality of $\mathcal{T}$. A closer inspection of the proofs in the cited paper yields immediately that all Reg-Horn-SAT problems with fixed size truth values have linear complexity. If $\mathcal{T}$ is infinite, then Reg-Horn-SAT is decidable provided that $\mathcal{T}$ is a locally finite lattice, that is, every sub-lattice generated by a finite subset is finite \cite{19}.

(3) If the partial order of $\mathcal{T}$ is not a lattice, then a natural notion of regular sign can still be obtained by using signs of the form $S^2 = \{i \in \mathcal{T} \mid \exists j \in S \text{ s.t. } i \geq j\}$, where $S \subseteq \mathcal{T}$. This more general Reg-Horn-SAT is still decidable linearly in the length of the formula, but exponential in the cardinality of $\mathcal{T}$ provided that $\mathcal{T}$ possesses a maximal element \cite{19}.

**Helly Property.** We denote $\mathcal{S}$ the set of signs occurring in a given formula $\varphi$. While in the previous works order-theoretic properties of the truth-value domain $\mathcal{T}$ are exploited to make conclusions on the complexity of signed SAT problems, Chepoi et al. \cite{33, 34} completely settle the complexity question in the general case by reverting to combinatorial properties of the set system $\mathcal{S}$. In particular, they prove that: Signed-SAT for $|\mathcal{T}| \geq 3$ is polynomial (even trivial), if $\bigcap_{S \in \mathcal{S}} S \neq \emptyset$ and NP-complete otherwise.

On the other hand, Signed-2-SAT is polynomial if, and only if, $\mathcal{S}$ fulfills the Helly property (every sub-family $\mathcal{V} \subset \mathcal{S}$ satisfying $\bigcap_{S \in \mathcal{V}} S = \emptyset$ contains two sets $S, S' \in \mathcal{V}$ such that $S \bigcap S' = \emptyset$) and NP-complete otherwise. For the case when $\mathcal{S}$ has the Helly property, i.e. the polynomial case, Chepoi et al. show that the satisfiability can in fact be checked in linear time in the spirit of the result for classical 2-SAT \cite{11}. Also, they prove that the Helly property itself can be checked in polynomial time.
8.3 Signed Non-Clausal

Signed non-clausal formulas are defined in [87] as negation-free Boolean formulas with signed literals or signed formulas as atoms. Lehmke [72] observed that every formula of infinite-valued Lukasiewicz logic can be expressed in signed NC, provided that Lukasiewicz sum and product are used instead of classical disjunction and conjunction (this process can blow up a formula exponentially). Next we discuss the three approaches to solve the satisfiability problem of signed non-clausal formulas published so far.

(1) Murray & Rosenthal’s dissolution was first available in classical [84] and then in finite-valued [85] logics. The dissolution rule selects in a signed NC formula an implicitly conjunctively connected pair of literals \( S \cdot P, S' \cdot P \) and restructures it in such a way that at least one conjunct occurrence of \( S \cdot P, S' \cdot P \) is replaced with \( S \cap S' \cdot P \). Producing \( P \cdot P \) leads to obvious simplifications such that any unsatisfiable formula is reduced to the empty formula after a finite number of dissolution steps. The authors in [18] proposed a method to eliminate some redundancies in an input signed NC formula. No complexity issues of path dissolution have been studied.

(2) Non-Clausal Resolution for classical logic, proposed in [83] (see also [16]), was extended to many-valued logic by Z. Stachniak [95, 96]. The basic idea is to derive from formula \( \phi(p) \) and \( \psi(p) \) (where \( p \) is an atom occurring in \( \phi \) and \( \psi \)) a new formula \( \phi(p) \lor \psi(p) \) for certain variable-free formulas and then to perform logic-specific simplifications. Their view of Non-Clausal Resolution is different from ours, and it seems that their approach presents some drawbacks preventing the definition of Non-Clausal Unit-Resolution, which indeed had not been proposed. No complexities issues were discussed.

(3) The method TAS [1] computes a simplified DNF of an NC formula. The input formula is unsatisfiable iff the result is the empty formula. The efficiency of the method comes from the fact that before each application of the distributive laws, unitary models of sub-formulas are computed and used for simplification. The generalization of the TAS method to signed NC formulas has been reported in [3, 4]. Although many experimental running-times were published, no computational complexities were obtained.

9 Future Work

Future work that is likely to receive our attention is divided into four main lines (see (1) to (4) below) and each of them is generalized to several non-classical logics. (1) defining the Horn-NC class \( H_{NC} \) and the Non-Clausal Unit-Resolution calculus \( UR_{NC} \), and proving the completeness of each \( UR_{NC} \) for its \( H_{NC} \); (2) applying \( H_{NC} \) and \( UR_{NC} \) obtained in (1) to NC logic programming; (3) developing NC DPLL-based approximate reasoning using \( H_{NC} \) and \( UR_{NC} \) obtained in (1); and (4) establishing Non-Clausal Resolution.

(1a) Reg-Horn-NC-SAT. Since \( RH \) should play in NC form a rôle similar to that of Horn in clausal form, worthy research efforts remain to devise a highly-efficient Reg-Horn-NC-SAT algorithm. Here we have already shown that Reg-Horn-NC-SAT is polynomial and our next goal will be proving that its complexity is close to \( O(n \log n) \), i.e. the one of Reg-Horn-SAT [10, 59, 61, 62].

(1b) Lattice-Regular Logic. We think that some of the classes published (see related work) for regular Horn formulas when the truth-value set \( T \) is partially ordered as a lattice

\[ T \]
can be lifted to NC. Specifically, we will study different modular and distributive lattices and define classes \( \mathcal{H}_{NC} \) of Horn-NC formulas. Then we will define Non-Clausal Unit-Resolution calculi and prove their completeness for their corresponding \( \mathcal{H}_{NC} \). Then we will verify which clausal classes preserve their polynomiality when lifted to the NC level.

**(1c) Lukasiewicz Logic.** SAT-checking the infinite-valued Lukasiewicz Horn class, \( \mathcal{L}_{\infty}\)-Horn, was proved to be NP-Complete \[28\] and polynomial \[26, 27\] for the 3-valued class, \( \mathcal{L}_3\)-Horn. Thus, our first goal is to NC lift \( \mathcal{L}_3\)-Horn and determine the class \( \mathcal{L}_3\mathcal{H}_{NC} \). Then we will analyze whether tractability is preserved in NC, that is, whether SAT-checking \( \mathcal{L}_3\mathcal{H}_{NC} \) is polynomial. For that purpose, a former step is to define the calculus \( \mathcal{L}_3\mathcal{UR}_{NC} \).

**(1d) Possibilistic Logic.** Surveys of this logic and its numerous applications are in \[40, 41, 42\]. In possibilistic logic, rather than testing satisfiability, the deductive problem comes to determine the inconsistency degree of possibilistic conjunctive formulas. Such problem for the necessity-valued Horn clausal class, \( \mathcal{N}_\alpha\)-Horn, is polynomial \[70\], and its complexity, indirectly discussed in \[5\] via possibilistic logic programming, is \( O(n \log n) \). So our first goal will be defining the necessity-valued Horn-NC conjunctive class, or \( \mathcal{N}_\alpha\mathcal{H}_{NC} \), and attempting to prove that computing its inconsistency degree is indeed polynomial. If so, it would make \( \mathcal{N}_\alpha\mathcal{H}_{NC} \) the first tractable possibilistic class in NC form. Ulterior research is planned to deal with NC formulas when both necessity and possibility measures are available \[71, 64, 70\]. Possibilistic paraconsistent reasoning is highly developed in the clausal setting \[43, 42, 31\], and thus, taking such advances as reference, new issues are open regarding expressiveness and computing aspects of possibilistic NC and HNC formulas.

**(2) Logic Programming.** Once \( \mathcal{H}_{NC} \) and \( UR_{NC} \) have been defined for any of the four above logics, and the next step consists of devising Horn-NC-SAT algorithms for such four logics. The expected efficiency of such algorithms can have a positive impact on NC logic programming based on some non-classical logics. Indeed, along the lines of Section 7 we will define a highly-rich logic programming language, where bodies and heads are NCs with slight syntactical restrictions and which can be efficiently interpreted using the previously designed Horn-NC-SAT algorithms.

**(3) DPLL-Based Reasoning.** A Horn-NC-SAT algorithm is indeed the procedure called NC Unit-Propagation which is essential in the DPLL skeleton, proved so far to be the most efficient in propositional logic. Hence, we think that an important axis for future research is the constructing of NC DPLL-based reasoners to satisfiability solving and theorem proving and for the non-classical logics in (1a) to (1d) above.

**(4) Resolution.** N. Murray in the 1980s \[83\] proposed Non-Clausal Resolution for classical logic (see also the handbook \[16\]) in a combined manner, namely each NC Resolution application must be followed by logical functions to simplify the inferred formulas. NC Resolution has also been extended to some non-classical logics such as multi-valued logic \[95, 90\], fuzzy logic \[55\] and fuzzy description logic \[54\]. Thus a non-functional definition of NC Resolution is missing so far. However, our presented approach has allowed us to establish Regular NC Unit-Resolution in a non-functional classical-like fashion (and similarly for propositional logic in \[66\]). So we think that our approach can be resumed towards defining NC Resolution for some logics including the four mentioned above in (1a) to (1d).
10 Conclusion

Towards characterizing the first polynomial class in multi-valued logic and non-clausal (NC) form, firstly we have defined the class of Regular Horn-NC formulas, $\mathcal{RH}$, by means of both an inductive, compact function and of a convenient merging of the regular Horn and NC classes.

As second contribution, we have analyzed the relationships between $\mathcal{RH}$ and the classes regular Horn and regular NC, and in this respect, we have proved that: (i) $\mathcal{RH}$ syntactically subsumes the Horn class but both classes are semantically equivalent; and (ii) $\mathcal{RH}$ includes all regular NC formulas whose clausal form is Horn.

Our third outcome includes both the definition of Regular Non-Clausal Unit-Resolution, or $\text{RUR}_{NC}$, and the proof that $\text{RUR}_{NC}$ is complete for $\mathcal{RH}$ and tests $\mathcal{RH}$ satisfiability in polynomial time. Hence, the latter fact shows that our initial goal, that is, finding a polynomial NC class beyond propositional logic, is accomplished.

We have also discussed how NC logic programming can benefit from our results, arguing that classical Horn rules can be notably extended by considering HNC rules in which bodies and heads are NCs fulfilling some syntactical constraints and that such syntactical enrichment is accompanied by a polynomial efficiency in query-answering.

Finally, we have discussed several future research lines: (i) defining both the Horn-NC class $\mathcal{H}_{NC}$ and the Non-Clausal Unit-Resolution calculus $\text{UR}_{NC}$ for several logics; (ii) developing NC logic programming based on $\mathcal{H}_{NC}$ and $\text{UR}_{NC}$ for several logics; (iii) enhancing NC DPLL-based approximate reasoning via $\mathcal{H}_{NC}$ and $\text{UR}_{NC}$; and (iv) establishing Non-Clausal Resolution.

**Funding Source.** Spanish project ISINC (PID2019-111544GB-C21).

**References**

[1] G. Aguilera, I. de Guzman, and M. Ojeda. Increasing the efficiency of automated theorem proving. *Journal of Applied Non-classical Logics*, 5(1):9–29, 1995.

[2] G. Aguilera, I. de Guzman, and M. Ojeda. A reduction-based theorem prover for 3-valued logic. *Mathware and Soft Computing*, 4(2):99–127, 1997.

[3] G. Aguilera, I. P. de Guzmán, M. Ojeda-Aciego, and A. Valverde. Reducing signed propositional formulas. *Soft Comput.*, 2(4):157–166, 1998.

[4] G. Aguilera, I. P. de Guzmán, M. Ojeda-Aciego, and A. Valverde. Reductions for non-clausal theorem proving. *Theor. Comput. Sci.*, 266(1-2):81–112, 2001.

[5] T. Alsinet and L. Godo. A complete calculus for possibilistic logic programming with fuzzy propositional variables. In *UAI ’00: Proceedings of the 16th Conference in Uncertainty in Artificial Intelligence, Stanford University, Stanford, California, USA, June 30 - July 3, 2000*, pages 1–10, 2000.

[6] P. Andrews. Theorem proving via general matings. *Journal of Association Computing Machinery*, 28, 1981.

[7] C. Ansótegui, M. Bofill, F. Manyà, and M. Villaret. SAT and SMT technology for many-valued logics. *J. Multiple Valued Log. Soft Comput.*, 24(1-4):151–172, 2015.
[8] C. Ansótegui, M. L. Bonet, J. Levy, and F. Manyà. Resolution procedures for multiple-valued optimization. Inf. Sci., 227:43–59, 2013.

[9] C. Ansótegui, J. Larrubia, C. M. Li, and F. Manyà. Exploiting multivalued knowledge in variable selection heuristics for SAT solvers. Ann. Math. Artif. Intell., 49(1-4):191–205, 2007.

[10] C. Ansótegui and F. Manyà. New logical and complexity results for Signed-SAT. In Proceedings, 33rd International Symposium on Multiple-Valued Logics (ISMVL), Tokyo, Japan, pages 181–187. IEEE CS Press, Los Alamitos, 2003.

[11] B. Aspvall, M. Plass, and R. Tarjan. A linear-time algorithm for testing the truth of certain quantified Boolean formulas. Information Processing Letters, 8(3):121–132, 1979.

[12] M. Baaz and C. G. Fermüller. Resolution-based theorem proving for manyvalued logics. J. Symb. Comput., 19(4):353–391, 1995.

[13] M. Baaz, C. G. Fermüller, and G. Salzer. Automated deduction for many-valued logics. In Handbook of Automated Reasoning (in 2 volumes), pages 1355–1402. 2001.

[14] L. Bachmair and H. Ganzinger. Rewrite-based equational theorem proving with selection and simplification. J. Log. Comput., 4(3):217–247, 1994.

[15] L. Bachmair and H. Ganzinger. Ordered chaining calculi for first-order theories of transitive relations. J. ACM, 45(6):1007–1049, 1998.

[16] L. Bachmair and H. Ganzinger. Resolution theorem proving. In Handbook of Automated Reasoning (in 2 volumes), pages 19–99. 2001.

[17] K. Ballerstein and D. O. Theis. An algorithm for random signed 3-sat with intervals. Theor. Comput. Sci., 524:1–26, 2014.

[18] B. Beckert, R. Hähnle, and G. Escalada-Imaz. Simplification of many-valued logic formulas using anti-links. J. Log. Comput., 8(4):569–587, 1998.

[19] B. Beckert, R. Hähnle, and F. Manyà. Transformations between signed and classical clause logic. In Proc. Int. Symp. on Multiple Valued Logics, ISMVL’99, Freiburg, Germany, 1999.

[20] B. Beckert, R. Hähnle, and F. Manyà. The 2-sat problem of regular signed CNF formulas. In 30th IEEE International Symposium on Multiple-Valued Logic, ISMVL 2000, Portland, Oregon, USA, May 23-25, 2000, Proceedings, pages 331–336, 2000.

[21] B. Beckert, R. Hähnle, and F. Manyà. The SAT problem of signed CNF formulas. In D. Basin, M. D’Agostino, D. Gabbay, S. Matthews, and L. Viganò, editors, Labelled Deduction, pages 59–80. Applied Logic Series, vol 17. Springer, Dordrecht, 2000.

[22] R. Béjar, R. Hähnle, and F. Manyà. A modular reduction of regular logic to classical logic. In 31st IEEE International Symposium on Multiple-Valued Logic, ISMVL 2001, Warsaw, Poland, May 22-24, 2001, Proceedings, pages 221–226, 2001.
[23] R. Béjar, F. Manyà, A. Cabiscol, C. Fernández, and C. P. Gomes. Regular-sat: A many-valued approach to solving combinatorial problems. *Discret. Appl. Math.*, 155(12):1613–1626, 2007.

[24] M. Ben-Ari. *Mathematical Logic for Computer Science, 3rd Edition*. Springer, 2012.

[25] W. W. Bledsoe. Non-resolution theorem proving. *Artif. Intell.*, 9(1):1–35, 1977.

[26] M. Bofill, F. Manyà, A. Vidal, and M. Villaret. The complexity of three-valued Lukasiewicz rules. In *MDAI*, pages 221–229, 2015.

[27] M. Bofill, F. Manyà, A. Vidal, and M. Villaret. New complexity results for Lukasiewicz logic. *Soft Computing*, 23(7):2187–2097, 2019.

[28] S. Borgwardt, M. Cerami, and R. Peñaloza. Many-valued horn logic is hard. In *Proceedings of the First Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness, PRUV 2014*, co-located with 7th International Joint Conference on Automated Reasoning (IJCAR 2014), Vienna, Austria, July 23-24, 2014, pages 52–58, 2014.

[29] F. Bry, N. Eisinger, T. Eiter, T. Furche, G. Gottlob, C. Ley, B. Linse, R. Pichler, and F. Wei. Foundations of rule-based query answering. In *Reasoning Web, Third International Summer School 2007, Dresden, Germany, September 3-7, 2007, Tutorial Lectures*, pages 1–153, 2007.

[30] M. Cadoli and M. Schaerf. On the complexity of entailment in propositional multi-valued logics. *Ann. Math. Artif. Intell.*, 18(1):29–50, 1996.

[31] C. Cayrol, D. Dubois, and F. Touazi. Symbolic possibilistic logic: completeness and inference methods. *J. Log. Comput.*, 28(1):219–244, 2018.

[32] W. Charatonik and M. Wrona. 2-sat problems in some multi-valued logics based on finite lattices. In *37th International Symposium on Multiple-Valued Logic, ISMVL 2007, 13-16 May 2007, Oslo, Norway*, page 21, 2007.

[33] C. Chepoi, N. Creignou, M. Hermann, and G. Salzer. Deciding the Satisfiability of Propositional Formulas in Finitely-Valued Signed Logics. In *38th International Symposium on Multiple Valued Logic*, pages 100–105, 2008.

[34] V. Chepoi, N. Creignou, M. Hermann, and G. Salzer. The helly property and satisfiability of boolean formulas defined on set families. *Eur. J. Comb.*, 31(2):502–516, 2010.

[35] A. Darwiche. Decomposable negation normal form. *J. ACM*, 48(4):608–647, 2001.

[36] P. Dellunde. A characterization of belief merging operators in the regular horn fragment of signed logic. In *Modeling Decisions for Artificial Intelligence - 17th International Conference, MDAI 2020, Sant Cugat, Spain, September 2-4, 2020, Proceedings*, pages 3–15, 2020.

[37] H. E. Dixon, M. L. Ginsberg, and A. J. Parkes. Generalizing boolean satisfiability I: background and survey of existing work. *J. Artif. Intell. Res.*, 21:193–243, 2004.
[38] P. Doherty. A constraint-based approach to proof procedures for multiple-valued logics. In First World Conference on the Fundamentals of Artificial Intelligence WOCFAI-91, Paris, 1991.

[39] W. Dowling and J. Gallier. Linear-time algorithms for testing the satisfiability of propositional Horn formulae. Journal of Logic Programming, (3):267–284, 1984.

[40] D. Dubois, J. Lang, and H. Prade. Possibilistic logic. In Handbook of Logic in Artificial Intelligence and Logic Programming, pages 419–513. New York: Oxford University Press, 1994.

[41] D. Dubois and H. Prade. Possibilistic Logic: a Retrospective and Prospective View. Fuzzy Sets Syst., 144(1):3–23, 2004.

[42] D. Dubois and H. Prade. Possibilistic Logic: An Overview. In J. W. D. M. Gabbay, J. H. Siekmann, editor, Handbook of the History of Logic. Vol 9, Computational Logic, pages 283–342. North-Holland, 2014.

[43] D. Dubois and H. Prade. Inconsistency management from the standpoint of possibilistic logic. Int. J. Uncertain. Fuzziness Knowl. Based Syst., 23(Supplement-1):15–30, 2015.

[44] O. Dubois, P. Andrè, Y. Boufkhad, and Y. Carlie. Chap. SAT vs. UNSAT, volume 26 of Second DIMACS implementation challenge: cliques, coloring and unsatisfiability. DIMACS Series in Discrete Mathematics and Theoretical Computer Sciences, pages 415–436. American Mathematical Society, 1996.

[45] U. Egly, M. Seidl, and S. Woltran. A solver for qbf in negation normal form. Constraints An Int. J., 14(1):38–79, 2009.

[46] G. Escalada-Imaz and F. Manyà. The satisfiability problem for multiple-valued Horn formulae. In Proc. International Symposium on Multiple-Valued Logics, ISMVL’94, pages 250–256, Boston/MA, USA, 1994. IEEE Press, Los Alamitos.

[47] R. Fagin, R. Riegel, and A. G. Gray. Foundations of reasoning with uncertainty via real-valued logics. CoRR, abs/2008.02429, 2020.

[48] M. Färber and C. Kaliszyk. Certification of nonclausal connection tableaux proofs. In Automated Reasoning with Analytic Tableaux and Related Methods - 28th International Conference, TABLEAUX 2019, London, UK, September 3-5, 2019, Proceedings, pages 21–38, 2019.

[49] A. Feldman, J. Pietersma, and A. van Gemund. A multi-valued SAT-based algorithm for faster model-based diagnosis. In C. A. Gonzales, T. Escobert, and B. Pulido, editors, Proceedings of the Seventeenth International Workshop on Principles of Diagnosis (DX-06) Pe naranda de Duero, Burgos, Spain, pages 93–100, June 2006.

[50] A. M. Frisch, T. J. Peugniez, A. J. Doggett, and P. Nightingale. Solving non-boolean satisfiability problems with stochastic local search: A comparison of encodings. J. Autom. Reason., 35(1-3):143–179, 2005.
[51] H. Ganzinger and V. Sofronie-Stokkermans. Chaining techniques for automated theorem proving in many-valued logics. In 30th IEEE International Symposium on Multiple-Valued Logic, ISMVL 2000, Portland, Oregon, USA, May 23-25, 2000, Proceedings, pages 337–344, 2000.

[52] À. J. Gil, M. Hermann, G. Salzer, and B. Zanuttini. Efficient algorithms for constraint description problems over finite totally ordered domains: Extended abstract. In Automated Reasoning - Second International Joint Conference, IJCAR 2004, Cork, Ireland, July 4-8, 2004, Proceedings, pages 244–258, 2004.

[53] À. J. Gil, M. Hermann, G. Salzer, and B. Zanuttini. Efficient algorithms for description problems over finite totally ordered domains. SIAM J. Comput., 38(3):922–945, 2008.

[54] H. Habiballa. Resolution strategies for fuzzy description logic. In New Dimensions in Fuzzy Logic and Related Technologies. Proceedings of the 5th EUSFLAT Conference, Ostrava, Czech Republic, September 11-14, 2007, Volume 2: Regular Sessions, pages 27–36, 2007.

[55] H. Habiballa. Fuzzy Logic: Algorithms, Techniques and Implementations, chapter Resolution Principle and Fuzzy Logic, pages 55–74. InTec, 2012.

[56] R. Hähnle. Towards an efficient Tableau proof procedure for multiple-valued logics. In Proc. of Computer Science Logic CSL’90, Heidelberg, Germany, 1990.

[57] R. Hähnle. Short CNF in finitely-valued logics. In H. J. Komorowski and Z. W. Ras, editors, Methodologies for Intelligent Systems, 7th International Symposium, ISMIS ’93, Trondheim, Norway, June 15-18, 1993, Proceedings, volume 689 of Lecture Notes in Computer Science, pages 49–58. Springer, 1993.

[58] R. Hähnle. Short conjunctive normal forms in finitely-valued logics. Journal of Logic and Computation, 4(6):905–927, 1994.

[59] R. Hähnle. Exploiting data dependencies in many-valued logics. Journal of Applied Non-classical Logics, (6):49–69, 1996.

[60] R. Hähnle. Advanced many-valued logics. In Handbook of Philosophical Logic, volume 2, pages 297–395. Kluwer, Dordrecht, 2nd edition, 2001.

[61] R. Hähnle. Complexity of Many-Valued Logics. In Proceedings ISMVL 2001: 137-148, 2001.

[62] R. Hähnle. Complexity of many-valued logics. In M. Fitting and E. Orlowska, editors, Beyond Two: Theory and Applications of Multi-Valued Logic, volume 114, chapter 9, pages 221–233. Physica, Heidelberg, 2003.

[63] Z. Hanikova. Chapter X: Computational Complexity of Propositional Fuzzy Logics. In Handbook of mathematical fuzzy logic, pages 793–851. College Publications, 2011.

[64] B. Hollunder. An alternative proof method for possibilistic logic and its application to terminological logics. Int. J. Approx. Reason., 12(2):85–109, 1995.
[65] A. Horn. On sentences which are of direct unions of algebras. *J. Symb. Logic*, 16(1):14–21, 1951.

[66] G. E. Imaz. The Horn Non-Clausal Class and its Polynomiality. *CoRR*, cs.AI/2108.13744; http://arxiv.org/abs/2108.13744, 2021.

[67] H. Jain, C. Bartzis, and E. Clarke. Satisfiability Checking of Non-clausal Formulas Using General Matings. In *SAT-2006*, pages 75–89, 2006.

[68] M. Kifer and V. S. Subrahmanian. Theory of generalized annotated logic programming and its applications. *J. Log. Program.*, 12(3&4):335–367, 1992.

[69] S. Klarman, U. Endriss, and S. Schlobach. Abox abduction in the description logic *ALC*. *J. Autom. Reasoning*, 46(1):43–80, 2011.

[70] J. Lang. Possibilistic logic: complexity and algorithms. In e. Gabbay D., Smets Ph, editor, *Handbook of Defeasible Reasoning and Uncertainty Management System*, pages 179–220. Dordrecht, The Netherlands: Kluwer Academic Publishers, 2001.

[71] J. Lang, D. Dubois, and H. Prade. A logic of graded possibility and certainty coping with partial inconsistency. In *UAI ’91: Proceedings of the Seventh Annual Conference on Uncertainty in Artificial Intelligence, University of California at Los Angeles, Los Angeles, CA, USA, July 13-15, 1991*, pages 188–196, 1991.

[72] S. Lehmk. A resolution-based axiomatization of ’bold’ propositional fuzzy logic. In *Linz’96: Fuzzy Sets, Logics, and Artificial Intelligence*, pages 115–119, 1996.

[73] A. Leitsch and C. Fermuller. The resolution principle. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 12. Springer, Dordrecht, 2005.

[74] H. J. Levesque. A knowledge-level account of abduction. In *Proceedings of the 11th International Joint Conference on Artificial Intelligence, Detroit, MI, USA, August 1989*, pages 1061–1067, 1989.

[75] V. Lifschitz, L. R. Tang, and H. Turner. Nested expressions in logic programs. *Ann. Math. Artif. Intell.*, 25(3-4):369–389, 1999.

[76] J. J. Lu. Logic programming with signs and annotations. *J. Log. Comput.*, 6(6):755–778, 1996.

[77] J. J. Lu, J. Henschen, and J. Schu. Computing multi-valued logic programs. *Mathware & Soft Computing*, 4(2):129–153, 1997.

[78] J. J. Lu, N. V. Murray, and E. Rosenthal. A framework for automated reasoning in multiple-valued logics. *J. Automated Reasoning*, 21(1):39–67, 1998.

[79] J. J. Lu, N. V. Murray, and E. Rosenthal. Deduction and search strategies for regular multiple-valued logics. *Multiple-Valued Logic and Soft Computing*, 11(3-4):375–406, 2005.

[80] F. Manyà. The 2-SAT problem in signed CNF formulas. *Journal of Multiple-Valued Logic*, 5(4):307–325, 2000.

32
[81] J. McKinsey. The decision problem for some classes of sentences without quantifiers. *Journal of Symbolic Logic*, (8):61–76, 1943.

[82] D. Mitchell. Resolution and clause-learning with restarts for signed CNF formulas. *FLAP*, 4(7), 2017.

[83] N. Murray. Completely Non-Clausal Theorem Proving. *Artificial Intelligence*, 18(1):67–85, 1982.

[84] N. Murray and E. Rosenthal. Dissolution: making paths vanish. *Journal of the ACM*, 3:504–535, 1993.

[85] N. Murray and E. Rosenthal. Signed formulas: a liftable meta-logic for multiple-valued logics. In *7th ISMIS'93, Trondeim, Norway*, pages 275–284, 1993.

[86] N. Murray and E. Rosenthal. Adapting classical inference techniques to multiple-valued logics using signed formulas. *Fundamenta Informaticae*, 3(21):237–253, 1994.

[87] N. V. Murray and E. Rosenthal. Resolution and path dissolution in multi-valued logics. In Z. W. Ras and M. Zemankova, editors, *Methodologies for Intelligent Systems, 6th International Symposium, ISMIS ’91, Charlotte, N.C., USA, October 16-19, 1991, Proceedings*, volume 542 of *Lecture Notes in Computer Science*, pages 570–579. Springer, 1991.

[88] B. E. Oliver and J. Otten. Equality preprocessing in connection calculi. In *Joint Proceedings of the 7th Workshop on Practical Aspects of Automated Reasoning (PAAR) and the 5th Satisfiability Checking and Symbolic Computation Workshop (SC-Square) Workshop, 2020 co-located with the 10th International Joint Conference on Automated Reasoning (IJCAR 2020), Paris, France, June-July, 2020 (Virtual)*, pages 76–92, 2020.

[89] J. Otten. Non-clausal connection calculi for non-classical logics. In *Automated Reasoning with Analytic Tableaux and Related Methods - 26th International Conference, TABLEAUX 2017, Brasília, Brazil, September 25-28, 2017, Proceedings*, pages 209–227, 2017.

[90] D. Pearce, H. Tompits, and S. Woltran. Characterising equilibrium logic and nested logic programs: Reductions and complexity. *Theory Pract. Log. Program.*, 9(5):565–616, 2009.

[91] S. Prestwich. CNF Encodings. In A. Biere, M. Heule, H. van Maaren, and T. Walsh, editors, *Handbook of Satisfiability: Chapter 1*, pages 75–87. IOS Press, 2009.

[92] E. Scala, P. Haslum, S. Thiébaux, and M. Ramírez. Subgoaling techniques for satisficing and optimal numeric planning. *J. Artif. Intell. Res.*, 68:691–752, 2020.

[93] V. Sofronie-Stokkermans. On translation of finitely-valued logics to classical first-order logic. In *13th European Conference on Artificial Intelligence, Brighton, UK, August 23-28 1998, Proceedings.*, pages 410–411, 1998.

[94] V. Sofronie-Stokkermans. Automated theorem proving by resolution in non-classical logics. *Ann. Math. Artif. Intell.*, 49(1-4):221–252, 2007.
A Proofs

Before Theorems 5.4 and 5.7, a preliminary theorem (below) is required. Thus, we supply successively the next proofs: Preliminary Theorem, Theorem 5.4 and Theorem 5.7.

Preliminary Theorem.

**Theorem.** Let $\varphi$ be an NC disjunction $\bigvee \varphi_1 \ldots \varphi_i \ldots \varphi_k$. $cl(\varphi) \in H$ iff $\varphi$ has $k - 1$ negative disjuncts and one disjunct s.t. $cl(\varphi_i) \in H$, formally

$$cl(\bigvee \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in H$$

if and only if

1. $\exists i$, s.t. $cl(\varphi_i) \in H$ and
2. for all $j \neq i$, $\varphi_j \in N^-$.

**Proof.** If-then. By refutation: let $cl(\bigvee \varphi_1 \ldots \varphi_i \ldots \varphi_k) \in H$ and prove that if (1) or (2) are violated, then $cl(\varphi) \notin H$.

- **Statement (1).**
  - If we take the case $k = 1$, then $\varphi = \varphi_1$.
  - But $cl(\varphi_1) \notin H$ implies $cl(\varphi) \notin H$.

- **Statement (2).**
  - Suppose that, besides $\varphi_i$, one $\varphi_j, j \neq i$, has positive literals too.
  - We take a simple case, concretely $k = 2$, $\varphi_1 = A$ and $\varphi_2 = B$.
  - So, $\varphi = (\varphi_1 \varphi_2) = (A B)$, which implies $cl(\varphi) \notin H$. 
Only-If. For simplicity and without loss of generality, we assume that 
$$(\lor \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1}) = \varphi^- \in N^-$$ and $\varphi_k \in \mathcal{RH}$, and prove that 

$$cl(\varphi) = cl((\lor \varphi_1 \ldots \varphi_i \ldots \varphi_{k}) ) = cl((\lor \varphi^- \varphi_k)) \in \mathcal{H}.$$ 

- To obtain $cl(\varphi)$, one must obtain first $cl(\varphi^-)$ and $cl(\varphi_k)$, and so 
  
  (i) $cl(\varphi) = cl((\lor \varphi^- \varphi_k)) = cl((\lor cl(\varphi^-) cl(\varphi_k))).$

- By definition of $\varphi^- \in N^-$, 
  
  (ii) $cl(\varphi^-) = \{\land D_1^- \ldots D_{m-1}^- D_m^-\}$; the $D_i^-$'s are negative clauses.

- By definition of $\varphi_k \in \mathcal{RH}$, 
  
  (iii) $cl(\varphi_k) = H = \{\land h_1 \ldots h_{n-1} h_n\}$; the $h_i$'s are Horn clauses.

- By (i) to (iii), 
  
  
  $cl(\varphi) = cl((\lor \{\land C_1 \ldots C_i \ldots C_n\} (\lor \{\land D_1^- \ldots D_{m-1}^- D_m^-\} H)) \} ).$

- Since the $C_i = (\lor D_1^- h_i)$'s are Horn clauses, 
  
  $cl(\varphi) = cl((\land H_1 (\lor \{\land D_2^- \ldots D_{m-1}^- D_m^-\} H)) \} ).$

- For $j < m$ we have, 
  
  $cl(\varphi) = cl((\land H_1 \ldots H_{j-1} H_j (\lor \{\land D_{j+1}^- \ldots D_{m-1}^- D_m^-\} H)) \} ).$

- For $j = m$, 
  
  $cl(\varphi) = \{\land H_1 \ldots H_{m-1} H_m H\} = H' \in \mathcal{H}.$

- Hence $cl(\varphi) \in \mathcal{H}$. 

Proof of Theorem 5.4.

Theorem 5.4. We have: $\forall \varphi \in \mathcal{RH}: cl(\varphi) \in \mathcal{H}$.

Proof. We use Definition 4.12 of $\mathcal{RH}$. The proof is done by structural induction on the number of recursions $r(\varphi)$ needed to include $\varphi$ in $\mathcal{RH}$. We define $r(\varphi)$ as:

$$r(\varphi) = \begin{cases} 0 & \varphi = H, \\ 1 + \max \{r(\varphi_1), \ldots, r(\varphi_{k-1}), r(\varphi_k)\} & \varphi = (\lor \varphi_1 \ldots \varphi_{k-1} \varphi_k). \end{cases}$$

- Base Case: $r(\varphi) = 0$.
  
  By definition, $\varphi = H$, and so trivially $cl(\varphi) \in \mathcal{H}$.

- Induction hypothesis:
  
  $\forall \varphi, r(\varphi) \leq n, \varphi \in \mathcal{RH}$ entails $cl(\varphi) \in \mathcal{H}$.

- Induction proof: $r(\varphi) = n + 1$.
  
  According to Definition 4.12 cases (1) and (2) below arise.

35
(1) \( \varphi = \{ \land \varphi_1 \ solnte \varphi_k \} \in \mathcal{RH} \), where \( k \geq 1 \).  
- By definition of \( r(\varphi) \),  
  \[ r(\varphi) = n + 1 \] entails \( 1 \leq i \leq k \), \( r(\varphi_i) \leq n \).  
- By induction hypothesis,  
  \( \varphi_i \in \mathcal{RH} \) and \( r(\varphi_i) \leq n \) entail \( cl(\varphi_i) \in \mathcal{H} \).  
- It is obvious that,  
  \[ cl(\varphi) = \{ \land cl(\varphi_1) \ solnte \lor \varphi_k \} \].  
- Therefore,  
  \[ cl(\varphi) = \{ \land H_1 \ solnte H_i \ solnte H_k \} = H \in \mathcal{H} \].

(2) \( \varphi = (\lor \varphi_1 \ solnte \varphi_k) \in \mathcal{RH} \), where \( k \geq 1 \).  
- By Theorem 4.12, line (3),  
  \[ 0 \leq i \leq k - 1 \], \( \varphi_i \in \mathcal{N}^- \) and \( \varphi_k \in \mathcal{RH} \).  
- By definition of \( r(\varphi) \),  
  \[ r(\varphi) = n + 1 \] entails \( r(\varphi_k) \leq n \).  
- By induction hypothesis,  
  \( d(\varphi_k) \leq n \) and \( \varphi_k \in \mathcal{RH} \) entail \( cl(\varphi_k) \in \mathcal{H} \).  
- By the first theorem, only-if, in the Appendix,  
  \[ 0 \leq i \leq k - 1 \], \( \varphi_i \in \mathcal{N}^- \) and \( cl(\varphi_k) \in \mathcal{H} \) entail:  
  \[ cl((\lor \varphi_1 \ solnte \varphi_k)) \in \mathcal{H} \].

\[ \blacksquare \]

Proof of Theorem 5.7

**Theorem 5.7.** \( \forall \varphi \in \mathcal{N}^- \): if \( cl(\varphi) \in \mathcal{H} \) then \( \varphi \in \mathcal{RH} \).

**Proof.** We use Definition 4.12 of \( \mathcal{RH} \). The next claims are trivial:

- By Definition 5.1 of \( \mathcal{N}^- \), \( \mathcal{C} \subset \mathcal{N}^- \).  
- By Definition 4.12 of \( \mathcal{RH} \), \( \mathcal{H} \subset \mathcal{RH} \).

Now, we define the depth \( d(\varphi) \) of \( \varphi \) as

\[
    d(\varphi) = \begin{cases} 
    0 & \varphi \in \mathcal{C} \\
    1 + \max \{d(\varphi_1), \ldots, d(\varphi_i), \ldots, d(\varphi_k)\} & \varphi = (\lozenge \varphi_1 \ solnte \varphi_i \ solnte \varphi_k) \end{cases}
\]

The proof is by structural induction on \( d(\varphi) \).

- **Base case:** \( d(\varphi) = 0 \) and \( cl(\varphi) \in \mathcal{H} \).  
  - \( d(\varphi) = 0 \) entails \( \varphi \in \mathcal{C} \).
– If $\varphi \not\in \mathcal{H}$, then $\text{cl}(\varphi) \not\in \mathcal{H}$, contradicting the assumption.
– Hence $\varphi \in \mathcal{H}$ and so by Definition 4.12 $\varphi \in \mathcal{R}\mathcal{H}$.

• Inductive hypothesis:
  \[ \forall \varphi \in \mathcal{N}_C, \ d(\varphi) \leq n \text{ and } \text{cl}(\varphi) \in \mathcal{H} \text{ entail } \varphi \in \mathcal{R}\mathcal{H}. \]

• Induction proof: $d(\varphi) = n + 1$.
  By Definition 3.1 of $\mathcal{N}_C$, cases (i) and (ii) below arise.
  
  (i) $\text{cl}(\varphi) = \text{cl}(\{\land \varphi_1 \ldots \varphi_i \ldots \varphi_k\}) \in \mathcal{H}$ and $k \geq 1$.
  – Since $\varphi$ is a conjunction, $1 \leq i \leq k$, $\text{cl}(\varphi_i) \in \mathcal{H}$.
  – By definition of $d(\varphi)$,
    \[ d(\varphi) = n + 1 \text{ entails } 1 \leq i \leq k, \ d(\varphi_i) \leq n. \]
  – By induction hypothesis,
    \[ 1 \leq i \leq k, \ d(\varphi_i) \leq n, \ \text{cl}(\varphi_i) \in \mathcal{H} \text{ entail } \varphi_i \in \mathcal{R}\mathcal{H}. \]
  – By Definition 4.12 line (2),
    \[ 1 \leq i \leq k, \ \varphi_i \in \mathcal{R}\mathcal{H} \text{ entails } \varphi \in \mathcal{R}\mathcal{H}. \]

  (ii) $\text{cl}(\varphi) = \text{cl}(\{\lor \varphi_1 \ldots \varphi_i \ldots \varphi_k\}) \in \mathcal{H}$ and $k \geq 1$.
  – By the first theorem, if-then, in the Appendix,
    \[ 0 \leq i \leq k - 1, \ \varphi_i \in \mathcal{N}^- \text{ and } \text{cl}(\varphi_k) \in \mathcal{H}. \]
  – By definition of $d(\varphi)$,
    \[ d(\varphi) = n + 1 \text{ entails } d(\varphi_k) \leq n. \]
  – By induction hypothesis,
    \[ d(\varphi_k) \leq n \text{ and } \text{cl}(\varphi_k) \in \mathcal{H} \text{ entail } \varphi_k \in \mathcal{R}\mathcal{H}. \]
  – By Definition 4.12 line (3),
    \[ 0 \leq i \leq k - 1, \ \varphi_i \in \mathcal{N}^- \text{ and } \varphi_k \in \mathcal{R}\mathcal{H} \text{ entail: } \]
    \[ (\lor \varphi_1 \ldots \varphi_i \ldots \varphi_{k-1} \varphi_k) = \varphi \in \mathcal{R}\mathcal{H}. \]