FINITE \( p \)-GROUPS OF CONJUGATE TYPE \( \{1, p^3\} \)

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ABSTRACT. We classify finite \( p \)-groups, upto isoclinism, which have only two conjugacy class sizes 1 and \( p^3 \). It turns out that the nilpotency class of such groups is 2.

1. Introduction

A finite group \( G \) is said to be of conjugate type \( \{m_1, m_2, \ldots, m_n\} \), if the set of conjugacy class sizes of \( G \) is \( \{m_1, m_2, \ldots, m_n\} \). Finite groups of conjugate type \( \{1, n\} \) were first investigated by Ito [9] in 1953. He proved that if \( G \) is of conjugate type \( \{1, n\} \), then \( n \) is a power of some prime \( p \) and \( G \) is a direct product of a non-abelian Sylow \( p \)-subgroup and an abelian \( p' \)-subgroup; in particular \( G \) is nilpotent. Hence, to understand such groups, it is sufficient to study finite \( p \)-groups of conjugate type \( \{1, p^n\} \) for \( n \geq 1 \). Half a century later, Ishikawa [7] proved that finite \( p \)-groups of nilpotency class at most 3. In a different paper [6], he classified \( p \)-groups of conjugate type \( \{1, p\} \) and \( \{1, p^2\} \) (definition is recalled in Section 2).

In this paper, we investigate finite \( p \)-groups of conjugate type \( \{1, p^3\} \) and present a classification up to isoclinism. Surprisingly we found that such groups can not be of nilpotency class 3. Before stating our results, we exhibit some examples.

A finite group \( G \) is said to be a Camina group if \( x^G = xG' \) for all \( x \in G − G' \), where \( x^G \) denotes the conjugacy class of \( x \) in \( G \). Let \( H \) be a finite Camina \( p \)-group of nilpotency class 2 with \( |H'| = p^m \geq p^3 \). Let \( A \) be any subgroup of \( H' \) of order \( p^{m−3} \). Then it is easy to see that \( H/A \) is a Camina group of conjugate type \( \{1, p^3\} \). That such groups exist, follows from the following examples. For an integer \( m \geq 3 \), let

\[
H = \left\{ \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 1 & \alpha_3 \\ 0 & 0 & 1 \end{pmatrix} : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}_{p^m} \right\},
\]

where \( \mathbb{F}_{p^m} \) is a finite field of \( p^m \) elements. This is a Camina \( p \)-group of nilpotency class 2 with \( |H'| = p^m \).

For any positive integer \( r \geq 1 \) and prime \( p > 2 \), consider the following group constructed by Ito [9].

\[
G_r = \langle a_1, \ldots, a_{r+1} \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \quad a_i^p = a_{i+1}^p = b_{ij}^p = 1, 1 \leq i < j \leq r + 1, 1 \leq k \leq r + 1 \rangle.
\]  

(1.1)

It follows from [9] Example 1) that the group \( G_r \) defined in (1.1) is a special \( p \)-group of order \( p^{(r+1)(r+2)/2} \) and exponent \( p \), and \( |G_r'| = p^{(r+1)/2} \). This group has only two different conjugacy class sizes, namely 1 and \( p' \). Thus \( G_3 \) is of conjugate type \( \{1, p^3\} \). For simplicity of notation, we assume that \( G_3 \) is generated by \( a, b, c \) and \( d \).

In the following theorem we provide a classification of all finite \( p \)-groups of conjugate type \( \{1, p^3\}, \ p > 2 \), upto isoclinism.

**Theorem 1.2.** Let \( G \) be a finite \( p \)-group of conjugate type \( \{1, p^3\}, \ p > 2 \). Then the nilpotency class of \( G \) is 2 and \( G \) is isoclinic to one of the following groups:
A finite Camina $p$-group of nilpotency class 2 with commutator subgroup of order $p^2$;

(ii) The group $G_3$, defined in (1) for $r = 3$;

(iii) The quotient group $G_3/M$, where $M$ is a normal subgroup of $G_3$ given by $M = \langle [a, b][c, d] \rangle$;

(iv) The quotient group $G_3/N$, where $N$ is a normal subgroup of $G_3$ given by $N = \langle [a, b][c, d], [a, c][b, d] \rangle$ with $t$ any fixed integer non-square modulo $p$.

We remark that the number of generators of Camina groups occurring in (i) can not be bounded as shown by the group $H/A$, where $H$ is defined above and $A$ is a subgroup of $H'$ of order $p^{n-3}$. This group is minimally generated by $2n$ elements.

Since the nilpotency class of a finite 2-group of conjugate type $\{1, 2^n\}$ for all $n \geq 1$ is 2 (see Corollary 2.5), classification problem reduces to finite 2-groups of class 2. To include the case $p = 2$, we consider a more general class of finite $p$-groups of class 2 and conjugate type $\{1, p^3\}$.

Let $G_n$ denote the family consisting of $(n + 1)$-generator non-abelian special $p$-groups $G$ of order $p^{(n+1)(n+2)/2}$. Then it follows that all groups of this family are of conjugate type $\{1, p^n\}$. It also turns out that any two groups in $G_n$ are isoclinic (see the remark following Lemma 3.10). So, all groups in the family $G_3$ are of conjugate type $\{1, p^3\}$, where $p$ is any prime including 2.

Let $\hat{G}_3$ denote the subfamily of $G_3$ consisting of 2-groups. For simplicity of notation, we assume that a group $G$ from $\hat{G}_3$ is minimally generated by the set $\{a, b, c, d\}$. A magma check shows that this family has exactly 989 non-isomorphic groups $[1]$.

We are now well prepared to state our next result which provides a classification of 2-groups of conjugate type $\{1, 8\}$ up to isoclinism.

**Theorem 1.3.** Let $G$ be a finite 2-group of conjugate type $\{1, 8\}$ and nilpotency class 2. Then $G$ is isoclinic to one of the following groups:

(i) A finite Camina 2-group with commutator subgroup of order 8;

(ii) A fixed group $G$ in the family $\hat{G}_3$, defined above;

(iii) The quotient group $G/M$, where $M$ is a normal subgroup of $G$ such that $M = \langle [a, b][c, d] \rangle$;

(iv) The quotient group $G/N$, where $N$ is a normal subgroup of $G$ such that $N = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle$.

Unlike 2-groups, there do exist $p$-groups of nilpotency class 3 and conjugate type $\{1, p^2\}$. These groups are all isoclinic to the group $W$ given in [3] and presented as

$$W = \langle a_1, a_2, b, c_1, c_2 | [a_1, a_2] = b, [a_1, b] = c \rangle$$

$$a_i^p = b^p = c_i^p = [a_1, c_i] = [a_1, c_2] = [a_2, c_1] = 1 \ (i = 1, 2) \rangle.$$

A natural question which arises here is

**Question.** Does there exist a finite $p$-group of nilpotency class 3, for an odd prime $p$, and conjugate type $\{1, p^n\}$, $n \geq 1$, which is not isoclinic to $W$? If yes, for which values of $n \geq 4$, there exists a finite $p$-group of nilpotency class 3 and conjugate type $\{1, p^n\}$?

Before concluding this section, we set some notations for a multiplicatively written finite group $G$ which are mostly standard. We denote by $G'$ the commutator subgroup of $G$. For a subgroup $H$ of $G$, by $H^\#$ we denote the set of non-trivial elements of $H$. For the elements $x, y, z \in G$, the commutator $[x, y]$ of $x$ and $y$ is defined by $x^{-1}y^{-1}xy$, and $[x, y, z] = [[x, y], z]$. Frattini subgroup of $G$ is denoted by $\Phi(G)$. For an element $x \in G$, $x^G$ denotes the conjugacy class of $x$ in $G$. To say that some $H$ is a subgroup or a normal subgroup of $G$ we write $H \leq G$ or $H \trianglelefteq G$, respectively. To indicate, in addition, that $H$ is properly contained in $G$, we write $H < G$ or $H < G$, respectively.
2. Preliminaries

The following concept of isoclinism of groups was introduced by P. Hall \cite{Hall}.

Let $X$ be a finite group and $\bar{X} = X / Z(X)$. Then commutation in $X$ gives a well-defined map $a_X : \bar{X} \times \bar{X} \rightarrow \gamma_2(X)$ such that $a_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups $G$ and $H$ are called isoclinic if there exists an isomorphism $\phi$ of the factor group $\bar{G} = G / Z(G)$ onto $\bar{H} = H / Z(H)$, and an isomorphism $\theta$ of the subgroup $G'$ onto $H'$ such that the following diagram is commutative

$$
\begin{array}{ccc}
\bar{G} \times \bar{G} & \xrightarrow{a_{\bar{G}}} & G' \\
\phi \times \phi \downarrow & & \downarrow \theta \\
\bar{H} \times \bar{H} & \xrightarrow{a_{\bar{H}}} & H'.
\end{array}
$$

The resulting pair $(\phi, \theta)$ is called an isoclinism of $G$ onto $H$. Notice that isoclinism is an equivalence relation among finite groups.

The following two results follow from \cite{Hall}.

**Proposition 2.1.** Let $G$ and $H$ be two isoclinic finite $p$-groups. Then $G$ and $H$ are of the same conjugate type.

**Proposition 2.2.** Let $G$ be a finite $p$-group. Then there exists a group $H$ in the isoclinism family of $G$ such that $Z(H) \leq H'$.

Group $H$ which occurred in Proposition 2.2 is called a stem group in its isoclinism class. In the light of the preceding two results, for the classification of finite $p$-groups of conjugate type $\{1, p^n\}$ up to isoclinism, we only need to consider a stem group from the respective isoclinism family.

The following result is due to Vaughan-Lee \cite{Vaughan-Lee} p. 270, Theorem].

**Proposition 2.3.** Let $G$ be a finite $p$-group. Suppose that every conjugacy class of $G$ contains at most $p^n$ elements. Then $|G'| \leq p^{n(n+1)/2}$.

The following result is due to Ito \cite{Ito} Proposition 3.2].

**Proposition 2.4.** Let $G$ be a finite $p$-group of conjugate type $\{1, p^n\}$. Then the number of elements in any minimal generating set is at least $n$, and order of subgroup of all the elements of order $p$ of $Z(G)$ is at least $p^n$.

Let $G$ be a finite $p$-group and $x \in G$ be such that $|x^G| = p^{b(x)}$. Then $b(x)$ is called the breadth of $x$. The breadth of $G$, denoted by $b(G)$, is defined as $\max\{b(x) \mid x \in G\}$.

The following result is due to Parmeggiani and Stellmacher \cite{Parmeggiani-Stellmacher} p. 59, Corollary]

**Proposition 2.5.** Let $G$ be a $p$-group, $p > 2$. Then $b(G) = 3$ if and only if one of the following holds:

1. $|G'| = p^3$ and $[G : Z(G)] \geq p^4$.
2. $|G'| = p^3$ and there exists $H \triangleleft G$ with $|H| = p$ and $[G/H : Z(G/H)] = p^3$.
3. $|G'| \geq p^4$ and $[G : Z(G)] = p^4$.

A similar result for $p = 2$ is proved by Wilkens \cite{Wilkens}, a consequence of which is stated in the last section for the groups having conjugate type $\{1, 8\}$.

The following result is a part of \cite{Hall} p. 501, Theorem].

**Proposition 2.6.** Let $G$ be a finite group which contains a proper normal subgroup $N$ such that all of the conjugacy classes of $G$ which lie outside of $N$ have the same lengths. Then either $G/N$ is cyclic or every non-identity element of $G/N$ has prime order.

Direct consequences of this result are the following corollaries.

**Corollary 2.7.** Let $G$ be a finite $p$-group of conjugate type $\{1, p^n\}$ such that $Z(G) = G'$. Then $G/Z(G)$ and $G'$ are elementary abelian $p$-groups.
Corollary 2.8. Let $G$ be a finite 2-group of conjugate type $\{1, 2^n\}$, $n \geq 1$. Then the nilpotency class of $G$ is 2.

For a given group $G$, an isoclinism $(\phi, \theta)$ from $G$ onto itself is called an autoclinism of $G$. It is not difficult to prove the following result.

Lemma 2.9. Let $G$ be a group from the family $\hat{G}_n$. Then a bijection between any two minimal generating sets for $G$ extends to an autoclinism of $G$.

For the groups $G := G_r$ defined in (1.1), the following more general result holds true.

Lemma 2.10. A bijection between any two minimal generating sets for $G := G_r$, extends to an automorphism of $G$.

3. Key Lemmas

We start with the following elementary fact, proof of which is immediate from the Hall-Witt identity.

Lemma 3.1. Let $G$ be a group of class 3 and $x, y, z \in G$ such that $[x, z], [y, z] \in Z(G)$. Then $[x, y, z] = 1$.

Lemma 3.2. Let $G$ be a finite $p$-group of conjugate type $\{1, p^3\}$, $p > 2$. Then one of following holds: (i) $|G| = p^3$ and $|G : Z(G)| \geq p^3$; (ii) $|G| \geq p^4$ and $|G : Z(G)| = p^4$.

Proof. Suppose that there exists a normal subgroup $H$ of $G$ such that $|H| = p$ and $|G/H : Z(G/H)| = p^3$. Then, since $G$ is of conjugate type $\{1, p^3\}$, it follows that $Z(G/H) = Z(G)/H$. Thus $p^3 = [G/H : Z(G/H)] = [G/H : Z(G)/H] = [G : Z(G)]$. But, for all $x \in G - Z(G)$ we get $[G : CG(x)] = p^3$ (as $G$ is of conjugate type $\{1, p^3\}$). Since $x \in CG(x) - Z(G)$, we have $Z(G) < CG(x)$; contradicting the equality $[G : Z(G)] = [G : CG(x)] = p^3$. Hence there can not exist any $H < G$ with $|H| = p$ and $|G/H : Z(G/H)| = p^3$. The proof is now complete from Proposition 2.5.

We have noticed above that any group from the family $\hat{G}_{n-1}$ is of conjugate type $\{1, p^{n-1}\}$. The following two results characterize all finite $n$-generator special $p$-groups of order $p^{n(n+1)/2 - 1}$ and conjugate type $\{1, p^{n-1}\}$.

Lemma 3.3. Let $G \in \hat{G}_{n-1}$ be a group generated by $n \geq 4$ elements $a_1, a_2, \ldots, a_n$. Suppose that $M < Z(G) = G' = [G, G]$ with $|M| = p$. Then $G/M$ is of conjugate type $\{1, p^{n-1}\}$ if and only if $M$ can be reduced to the form

$$M = \langle [a_1, a_2][a_3, a_4][a_5, a_6] \ldots [a_{2m-1}, a_{2m}] \rangle, \text{ where } 2 \leq m \leq \lfloor n/2 \rfloor.$$ 

Proof. Notice that $|G| = p^{n(n+1)/2}$. Also notice that any bijection between two minimal generating sets for $G$ extends to an autoclinism of $G$ by Lemma 2.9. Set $\overline{G} = G/M$; then $|\overline{G}| = p^{n(n+1)/2 - 1}$. Since $G$ is of conjugate type $\{1, p^{n-1}\}$ and $|M| = p$, we have that $Z(G)/M = Z(\overline{G}) = \overline{G}' = \Phi(\overline{G})$.

is an elementary abelian $p$-group of order $p^{n(n-1)/2 - 1}$. Thus $[\overline{G} : Z(\overline{G})] = p^n$. Hence $\overline{G}$ is of conjugate type $\{1, p^{n-1}\}$ if and only if each non-central element of $\overline{G}$ commutes only with its own powers up to the central elements.

Let $\bar{x}, \bar{y} \in \overline{G} - Z(\overline{G})$ be such that no one is a power of the other (reading modulo $Z(\overline{G})$). Then it is not difficult to see that $[\bar{x}, \bar{y}] \neq 1$ in $G$. Hence, if $[\bar{x}, \bar{y}] = 1$ in $\overline{G}$, then $[\bar{x}, \bar{y}] \in M^\#$. Any given central subgroup $M_1$ of order $p$, without loss of generality, can be written as

$$M_1 = \langle [a_1, a_2][a_1, a_3][a_1, a_4] \ldots [a_1, a_j]^{a_{i-j}} \ldots [a_{n-1}, a_n]^{a_{n-1,n}} \rangle,$$
where \(1 \leq i < j \leq n\). Now applying the autoclinism induced by the map \(a_2 \mapsto a_2 a_3^{\alpha_{i,j}} \cdots a_n^{\alpha_{1,n}}\), \(a_i \mapsto a_i\) for \(i \neq 2\), \(M_1\) gets mapped to

\[
M_2 = \langle [a_1, a_2][a_3, a_4]^{\alpha_{2,3}} \cdots [a_i, a_j]^{\alpha_{i,j}} \cdots [a_{n-1}, a_n]^{\alpha_{n-1,n}} \rangle
\]

with \(2 \leq i < j \leq n\) and modified \(\alpha_{i,j}\). Notice that \(G/M_1\) is isoclinic to \(G/M_2\), and therefore both \(G/M_1\) and \(G/M_2\) are of the same conjugate type. We now apply another autoclinism induced by the map \(a_1 \mapsto a_1 a_3^{\alpha_{2,3}} \cdots a_n^{\alpha_{2,n}}, a_i \mapsto a_i\) for \(i \neq 1\), and see that \(M_2\) gets mapped to

\[
M_3 = \langle [a_1, a_2][a_3, a_4]^{\alpha_{3,4}} \cdots [a_i, a_j]^{\alpha_{i,j}} \cdots [a_{n-1}, a_n]^{\alpha_{n-1,n}} \rangle
\]

with \(3 \leq i < j \leq n\) and modified \(\alpha_{i,j}\).

Take \(x = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}\) and \(y = a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n}\) be such that none is power of the other (reading modulo \(\mathbb{Z}(G)_p\)). Then \([x, y] \in M_3^p\) only when the following two conditions hold true:

\[
\begin{align*}
(i) & \quad j_3 = j_4 = \cdots = j_n = k_3 = k_4 = \cdots = k_n = 0; \\
(ii) & \quad \alpha_{i,j} = 0, \quad 3 \leq i < j \leq n.
\end{align*}
\]

Hence \(G/M_3\) is of conjugate type \(\{1, p^{n-1}\}\) if and only if at least one \(\alpha_{i,j}\) is non-zero modulo \(p\). We now conclude that \(G/M_3\) is of conjugate type \(\{1, p^{n-1}\}\) if and only if

\[
M_3 = \langle [a_1, a_2][a_3, a_4]^{\alpha_{3,4}} \cdots [a_i, a_j]^{\alpha_{i,j}} \cdots [a_{n-1}, a_n]^{\alpha_{n-1,n}} \rangle
\]

with \(3 \leq i < j \leq n\) and at least one \(\alpha_{i,j}\) is non-zero modulo \(p\). We can assume that \(\alpha_{3,4} \neq 0\).

Make further reductions by applying an autoclinism of \(G\) which is a composition of the autoclinisms induced by the maps (1) \(a_4 \mapsto a_4^{\alpha_{3,4}}, a_1 \mapsto a_1\) for \(i \neq 4\), (2) \(a_4 \mapsto a_4 a_2^{\alpha_{0,3},6} a_6^{\alpha_{3,6}}, a_1 \mapsto a_1\) for \(i \neq 4\) and (3) \(a_3 \mapsto a_3 a_5^{\alpha_{4,5},6} a_6^{\alpha_{4,6}}, a_1 \mapsto a_1\) for \(i \neq 3\). This action reduces \(M_3\) to \(M_4\), which is of the following form

\[
M_4 = \langle [a_1, a_2][a_3, a_4][a_5, a_6]^{\alpha_{5,6}} \cdots [a_i, a_j]^{\alpha_{i,j}} \cdots [a_{n-1}, a_n]^{\alpha_{n-1,n}} \rangle
\]

with \(5 \leq i < j \leq n\) and modified \(\alpha_{i,j}\) again.

Finally, if all \(\alpha_{i,j}\) are 0 modulo \(p\), we are done by taking \(M = M_4\). If not, then finite repetitions of the above process reduce \(M_4\) to the desired form \(M\), completing the proof. 

\[
\square
\]

**Corollary 3.4.** Let \(K\) be an \(n\)-generator special \(p\)-group of order \(p^{n(n+1)/2-1}\) and conjugate type \(\{1, p^{n-1}\}\). Then \(K\) is isoclinic to \(G/M\), where \(G = \langle a_1, a_2, \ldots, a_n \rangle \in \hat{G}_{n-1}\) and \(M < Z(G) = G'\) with \(|M| = p\) is of the form

\[
M = \langle [a_1, a_2][a_3, a_4][a_5, a_6] \cdots [a_{2m-1}, a_{2m}] \rangle, \quad 2 \leq m \leq \lfloor n/2 \rfloor.
\]

**Proof.** Notice that the group \(\hat{G}_{n-1}\), given in the statement, is isomorphic to a quotient of some group \(G\) from the family \(\hat{G}_{n-1}\) by a subgroup of order \(p\) contained in \(G'\). Now the proof follows from the preceding lemma. 

\[
\square
\]

In particular, if, for an odd prime \(p\), we take a \(p\)-group \(G\) from the class \(\hat{G}_{n-1}\) such that the exponent of \(G\) is \(p\), then \(G\) is isomorphic to the group \(G_{n-1}\) defined in (1.1). Then by Lemma 2.10 a bijection between any two minimal generating sets for \(G := G_{n-1}\), extends to an automorphism of \(G\). Therefore, on the lines of the proofs of Lemma 3.3 and Corollary 3.4 (replacing autoclinism by automorphism and isoclinic by isomorphic), we can prove the following.

**Lemma 3.5.** Let \(K\) be an \(n\)-generator special \(p\)-group of order \(p^{n(n+1)/2-1}\), exponent \(p\) and conjugate type \(\{1, p^{n-1}\}\), where \(p\) is an odd prime. Then \(K\) is isomorphic to
\( G_{n-1}/M, \) where \( G_{n-1} \) is the group as defined in \( \text{(1.1)} \) generated by \( a_1, a_2, \ldots, a_n \) and \( M < Z(G) = G' \) with \( |M| = p \) is of the form
\[
M = \langle [a_1, a_2][a_3, a_4][a_5, a_6] \ldots [a_{2m-1}, a_{2m}] \rangle, \quad 2 \leq m \leq \lfloor n/2 \rfloor.
\]

For the case \( n = 3 \), the preceding lemma was proved by Brahana [2]. For the application point of view, we state it explicitly as a corollary.

**Corollary 3.6.** Let \( K \) be a 4-generator special \( p \)-group of order \( p^9 \), exponent \( p \) and conjugate type \( \{1, p^3\} \), where \( p \) is an odd prime. Then \( K \) is isomorphic to \( G_3/M, \) where \( G_3 \) is generated by \( a, b, c, d \) and \( M < Z(G) = G' \) with \( |M| = p \) is of the form \( H = \langle [a, b][c, d] \rangle \).

Now onward we concentrate only on the groups \( G \) from the family \( \hat{G}_3 \).

**Lemma 3.7.** Let \( G \) be a group from the family \( \hat{G}_3 \) which is generated by \( a, b, c, d \).

Suppose that \( N < Z(G) = G' \) with \( |N| = p^2 \). Then \( G/N \) is of conjugate type \( \{1, p^3\} \) if and only if \( N \) can be reduced to the following form
\[
N = \langle [a, b][c, d], [a, c][b, d] \rangle, \quad \text{where} \ r \ \text{is any fixed non-square integer modulo} \ p.
\]

**Proof.** Notice that \( |G| = p^{10} \). Set \( \overline{G} = G/N; \) then \( |\overline{G}| = p^8 \). Since \( G \) is of conjugate type \( \{1, p^3\} \) and \( |N| = p^2 \), we have that
\[
Z(G)/N = Z(\overline{G}) = \Phi(\overline{G})
\]
is an elementary abelian \( p \)-group of order \( p^4 \). Thus \( |\overline{G} : Z(\overline{G})| = p^4 \). Hence \( \overline{G} \) is of conjugate type \( \{1, p^3\} \) if and only if each non-central element of \( \overline{G} \) commutes only with its own powers up to the central elements.

Let \( \bar{x}, \bar{y} \in \overline{G} - Z(\overline{G}) \) be such that no one is a power of the other (reading modulo \( Z(\overline{G}) \)). Then it is not difficult to see that \( [x, y] \neq 1 \) in \( G \). Hence, if \( [\bar{x}, \bar{y}] = 1 \) in \( \overline{G} \), then \( [x, y] \in N^\# \). Any given central subgroup \( N_1 \) of order \( p^2 \), without loss of generality, can be written as one of the following two types:

(i) \( N_1 = \langle [a, b][a, d]^i[a, d]^i b, c, d]^{i^4}, [c, d] \rangle. \)

(ii) \( N_1 = \langle [a, b][a, d]^i[a, d]^i b, [c, d]^{i^4}, [a, c][a, d]^i b, c, d]^{i^4} \rangle. \)

If \( N_1 \) is of type (i), then \( \bar{c} \) commutes with \( \bar{d} \), although \( \bar{c} \notin \langle Z(\overline{G}), \bar{d} \rangle \). Hence \( \overline{G} \) can not be of conjugate type \( \{1, p^3\} \). Therefore we only need to consider \( N_1 \) as in type (ii). Now applying the autoclinism induced by the map \( a \mapsto a, b \mapsto bd^{-i}, c \mapsto cd^{-j}, d \mapsto d \), \( N_1 \) gets mapped to \( N_2 \), where
\[
N_2 = \langle [a, b][b, c]^j a[b, c]^j, [c, d]^{i^4}, [a, c][b, c]^j b, j^2, [c, d]^{j^4} \rangle
\]
with modified powers of the basic commutators. Notice that \( G/N_1 \) and \( G/N_2 \) are isoclinic. We now apply another autoclinism induced by the map \( a \mapsto ac^{i^4}d^{j^2}, b \mapsto b, c \mapsto c, d \mapsto d \), and see that \( N_2 \) gets mapped to
\[
N_3 = \langle [a, b][c, d]^j, [a, c][b, c]^j a[b, c]^j b, j^2 [c, d]^{j^4} \rangle
\]
again with modified powers of commutators. Notice that \( i \) is non-zero modulo \( p \), otherwise \( G/N_3 \) can not be of conjugate type \( \{1, p^3\} \). Thus the map \( a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto d^{i^{-1}} \) extends to an autoclinism of \( G \) and maps \( N_3 \) to
\[
N_4 = \langle [a, b][c, d], [a, c][b, c]^j a[b, c]^j b, d^{j^2} \rangle
\]
with modified \( j_2 \) and \( j_3 \). Again note that \( j_2 \) can not be zero modulo \( p \), otherwise \( [ab^j d^{-j}, c] = 1 \) and so \( G/N_3 \) can not be of conjugate type \( \{1, p^3\} \). Therefore the map \( a \mapsto a, b \mapsto b, c \mapsto c, d \mapsto c^{-j_1}d^{j_2}d \) is well defined. The autoclinism of \( G \) induced by this map takes \( N_4 \) to \( N_5 \), where, after modifying powers,
\[
N_5 = \langle [a, b][c, d], [a, c][b, d]^{i^4}[c, d]^{i^2} \rangle.
\]
Now let \( x = a^{j_1}b^{j_2}c^{j_3}d^{j_4} \) and \( y = a^{k_1}b^{k_2}c^{k_3}d^{k_4} \) be such that none is power of the other (reading modulo \( Z(G) \)). If \( [x, y] \in N_5^# \), then at least one of \( j_1 \) and \( k_1 \) has to be non-zero modulo \( p \). Without loss of generality we take \( j_1 \) to be non-zero. Now we can write \( y \) as \( y = a^{k_1}b^{k_2}c^{k_3}d^{k_4} = (a^{j_1}b^{j_2}c^{j_3}d^{j_4})^{k_1j_1^{-1}}b^{j_2}c^{j_3}d^{j_4}z_1 \), where \( z_1 \in Z(G) \) and \( l_2 \), \( l_3 \), \( l_4 \) are some suitable integers. So we can modify \( x \) and \( y \) as \( x = a^{j_1}b^{j_2}c^{j_3}d^{j_4} \) with \( j_1 \) non-zero and \( y = b^{j_2}c^{j_3}d^{j_4} \). Now \( l_2 \) has to be non-zero modulo \( p \) and \( l_4 \) has to be 0. Using similar argument, we can remove power of \( b \) in \( x \). So we can modify \( x \) and \( y \) as \( x = a^{j_1}b^{j_2}c^{j_3}d^{j_4} \) and \( y = b^{j_2}c^{j_3}d^{j_4} \). Now \( j_3 \) has to be 0. So, finally we have reduced \( x \) and \( y \) to \( x = a^{j_1}d^{j_4} \) and \( y = b^{j_2}c^{j_3}d^{j_4} \). If \( [x, y] \in N_5^# \), then \([x^{j_1^{-1}}, y^{j_1^{-1}}] \) also belongs to \( N_5^# \). Also \( x^{j_1^{-1}} = adl_z^{j_2} \) and \( y^{j_1^{-1}} = bek_z^{j_3} \), where \( z_2 \) and \( z_3 \) are some central elements. Therefore \([x^{j_1^{-1}}, y^{j_1^{-1}}] = [a, b][a, c]^{j_1}[b, d]^{j_1}[c, d]^{j_1} \). So if \( [x, y] \in N_5^# \), then \([a, b][a, c]^{j_1}[b, d]^{j_1}[c, d]^{j_1} \in N_5^# \), and therefore can be written as a product of powers of generators of \( N_5^# \). Now comparing power of the basic commutators, we get

\[
j \equiv ki_1 \pmod{p} \text{ and } jk \equiv ki_2 + 1 \pmod{p}.
\]

Solving these we have

\[
k^2i_1 - ki_1 - 1 \equiv 0 \pmod{p}.
\]

This is possible only when \( i_2 + 4i_1 \) is a square modulo \( p \). From this, we conclude that \( G/N_5 \) is of conjugate type \( \{1, p^\ell \} \) if and only if \( N_5 \) is of the following form

\[
N_5 = \langle [a, b][c, d], [a, c][b, d]^{j_1}[c, d]^{j_1} \rangle; \text{ where } i_2 + 4i_1 \text{ is a non-square modulo } p.
\]

Now we consider two cases, namely: Case 1. \( i_2 \neq 0 \); Case 2. \( i_2 = 0 \), and take these one by one.

**Case 1:** Let \( r \) be a fixed integer non-square modulo \( p \). Then \( r \) must be non-zero. Being non-square, \( i_2 + 4i_1 \) is also non-zero. Thus \( i_2 + 4i_1 \) is a non-zero square modulo \( p \). Thus there exists a non-zero \( l \) such that \( l^2 = \frac{i_2 + 4i_1}{4i_1} \). Set \( t = \frac{i_2}{2} \).

Now applying the autoclinism of \( G \) induced by the map \( a \mapsto a^d \), \( b \mapsto b \), \( c \mapsto b^c \), \( d \mapsto d \), \( N_5 \) gets mapped to

\[
N_6 = \langle [a, b][c, d]^l, [a, b][c, d]^{i_2 + i_1 - l^2}[c, d]^{i_2 - l^2} \rangle
\]

\[
= \langle ([a, b][c, d])^l, ([a, b][c, d])^{i_2 + i_1 - l^2} \rangle
\]

\[
= \langle [a, b][c, d], [a, c]^{l_1}[b, d]^{i_2 + i_1 - l^2} \rangle
\]

\[
= \langle [a, b][c, d], [a, c]^{l_1}[b, d]^{i_2 + i_1 - l^2} \rangle
\]

\[
= \langle [a, b][c, d], [a, c][b, d]^{i_2 + i_1 - l^2} \rangle
\]

\[
= \langle [a, b][c, d], [a, c][b, d]^{i_2 + i_1 - l^2} \rangle
\]

\[
= N.
\]

Hence we are done in this case.

**Case 2:** In this case \( i_1 \) must be a non-square. If \( i_1 = r \), then we are done. If not, then \( i_1^{-1}r \) must be a non-square and therefore there exists a non-zero integer \( l \) such that \( i_1^{-1}r = l^2 \).
Now the autoclinism of $G$ induced by the map $a \mapsto a$, $b \mapsto b$, $c \mapsto c^{-1}$, $d \mapsto d'$ maps $N_7$ to
\[
N_7 = \langle [a, b][c, d], [a, c]^{t^{-1}}[b, d]^{t} \rangle
\]
\[
= \langle [a, b][c, d], ([a, c][b, d])^{t} \rangle^{t^{-1}}
\]
\[
= \langle [a, b][c, d], [a, c][b, d]^{t} \rangle
\]
\[
= N.
\]
The proof is now complete. \[\square\]

The following result characterizes all 4-generator special groups of order $p^8$ and conjugate type $\{1, p^3\}$.

**Corollary 3.8.** Let $K$ be a 4-generator special $p$-group of order $p^8$ and conjugate type $\{1, p^3\}$. Then $K$ is isoclinic to $G/N$, where $G = \langle a, b, c, d \rangle \in \hat{G}_3$ and $N < Z(G) = G'$ with $|N| = p^2$ is of the form
\[
N = \langle [a, b][c, d], [a, c][b, d]^{r} \rangle, \text{ where } r \text{ is any fixed non-square integer modulo } p.
\]

Proof. Notice that the group $K$, given in the statement, is isomorphic to a quotient of some group $G$ from the family $\hat{G}_3$ by a subgroup of order $p^2$ contained in $G'$. Now the proof follows from the preceding lemma. \[\square\]

In particular, if, for an odd prime $p$, we take a $p$-group $G$ from the class $\hat{G}_3$ such that the exponent of $G$ is $p$, then $G$ is isomorphic to the group $G_3$ defined in [1.1] for $r = 3$. Then by Lemma 2.10 a bijection between any two minimal generating sets for $G_3$ extends to an automorphism of $G$. Therefore, on the lines of the proofs of Lemma 3.7 and Corollary 3.8 (replacing autoclinism by automorphism and isoclinic by isomorphic), we can prove the following result, which has also been proved by Brahana [2, Section 2]. But the proof in the present text is in modern terminology.

**Lemma 3.9.** Let $K$ be a 4-generator special $p$-group of order $p^8$, exponent $p$ and conjugate type $\{1, p^3\}$, where $p$ is an odd prime. Then $K$ is isomorphic to $G_3/N$, where $G_3$ is the group defined in [1.1] generated by $a, b, c$ and $d$ and $N < Z(G) = G'$ with $|N| = p^2$ is of the form
\[
N = \langle [a, b][c, d], [a, c][b, d]^{r} \rangle, \text{ where } r \text{ is any fixed non-square integer modulo } p.
\]

Now we consider the family $\hat{G}_3$ of 2-groups defined in the introduction. We start with the following result which tells that certain type of quotient groups of any two groups in $\hat{G}_3$ are isoclinic.

**Lemma 3.10.** Let $G = \langle a, b, c, d \rangle$ and $G^* = \langle s, u, v, w \rangle$ be two groups from the family $\hat{G}_3$. Then the following hold true.

(i) $G$ and $G^*$ are isoclinic.

(ii) If $M = \langle [a, b][c, d] \rangle \leq G'$ and $M^* = \langle [s, u][v, w] \rangle \leq (G^*)'$, then $G/M$ and $G^*/M^*$ are isoclinic.

(iii) If
\[
N = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle \leq G'
\]
and
\[
N^* = \langle [s, u][v, w], [s, v][u, w][v, w] \rangle \leq (G^*)',
\]
then $G/N$ are $G^*/N^*$ are isoclinic.
Proof. We sketch proof only for (i). Note that both \( G/Z(G) \) and \( G'/Z(G') \) are elementary abelian 2-groups of order 2^4, generated by \( \{aZ(G), bZ(G), cZ(G), dZ(G)\} \) and \( \{sZ(G'), uZ(G'), vZ(G'), wZ(G')\} \) respectively. Similarly, both \( G' \) and \( (G')' \) are elementary abelian 2-groups generated by the sets consisting of all 6 basic commutators
\[
\{(a, b), (a, c), (a, d), [b, c], [b, d], [c, d]\}
\]
and
\[
\{(s, u), [s, v], [s, w], [u, v], [u, w], [v, w]\}
\]
respectively. Now the map \( a \mapsto s, b \mapsto u, c \mapsto v \) and \( d \mapsto w \) extends to an isomorphism from \( G/Z(G) \) onto \( G'/Z(G') \), which induces an isomorphism from \( G' \) onto \( (G')' \), making \( G \) and \( G' \) isoclinic.

We remark that the second and third assertions of the preceding lemma hold true in the bigger family \( \hat{G}_3 \). And by the same argument as given in the proof, one can easily prove that any two groups in \( \hat{G}_n \) are isoclinic. We have stated this result for the family \( \hat{G}_3 \) because we here need it only for 2-groups.

The following lemma is immediate from Corollary \( \ref{cor:3.11} \) using Lemma \( \ref{lem:3.10} \) when restricted to the family \( \hat{G}_3 \).

Lemma 3.11. Let \( K \) be a 4-generator special 2-group of order 2^9 and conjugate type \( \{1, 8\} \). Then \( K \) is isoclinic to \( G/M \), where \( G = \langle a, b, c, d \rangle \in \hat{G}_3 \) is any fixed group and \( M < Z(G) = G' \) with \( |M| = 2 \) is of the form \( M = \langle [a, b][c, d] \rangle \).

The following lemma is analogous to Lemma \( \ref{lem:3.7} \) for \( p = 2 \), and therefore the proof is mostly a duplication of the the proof of Lemma \( \ref{lem:3.7} \) with necessary modifications.

Lemma 3.12. Let \( G = \langle a, b, c, d \rangle \in \hat{G}_3 \). Then \( G/N \) with \( |N| = 4 \) is of conjugate type \( \{1, 8\} \) if and only \( N \) can be reduced to the form \( N = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle \).

Proof. Notice that \( |G| = 2^{10} \) and \( |G'| = 2^6 \). Both \( G/Z(G) \) and \( G' = Z(G) \) are elementary abelian. Set \( \overline{G} = G/N \); then \( |\overline{G}| = 2^8 \). Since \( G \) is of conjugate type \( \{1, 8\} \) and \( |N| = 4 \), it follows that
\[
Z(G)/N = Z(\overline{G}) = \overline{G} = \Phi(\overline{G})
\]
is an elementary abelian 2-group of order 2^4. Thus \( [\overline{G} : Z(\overline{G})] = 2^4 \). Hence \( \overline{G} \) is of conjugate type \( \{1, 8\} \) if and only if each non-central element of \( \overline{G} \) commutes only with its own powers up to the central elements.

Let \( \overline{x}, \overline{y} \in \overline{G} \) be such that no one is a power of the other (reading modulo \( Z(\overline{G}) \)). Then it is not difficult to see that \( [\overline{x}, \overline{y}] \neq 1 \) in \( G \). Hence, if \( \overline{[x, y]} = 1 \) in \( \overline{G} \), then \( [x, y] \in N^\# \). Any given central subgroup \( N_1 \) of order 4, without loss of generality, can be written as one of the following two types:

\[
\begin{align*}
(i) \quad N_1 &= \langle [a, b][a, d]^{i_1}[b, c]^{i_2}[b, d]^{i_3}[c, d]^{i_4}, [c, d] \rangle. \\
(ii) \quad N_1 &= \langle [a, b][a, d]^{i_1}[b, c]^{i_2}[b, d]^{i_3}[c, d]^{i_4}, [a, c][a, d]^{j_1}[b, c]^{j_2}[b, d]^{j_3}[c, d]^{j_4} \rangle.
\end{align*}
\]

If \( N_1 \) is of type (i), then \( \overline{c} \) commutes with \( \overline{d} \), although \( \overline{c} \notin \langle Z(\overline{G}), \overline{d} \rangle \). Hence \( \overline{G} \) cannot be of conjugate type \( \{1, p^3\} \). Therefore we only need to consider \( N_1 \) as in type (ii). Now, as done in the proof of Lemma \( \ref{lem:3.4} \), we can reduce \( N_1 \) to the form
\[
N_2 = \langle [a, b][c, d], [a, c][b, d]^{i_1}[c, d]^{i_2} \rangle.
\]
Here \( i_1 \) can not be 0, else \( c \) will commute with \( ad^{-i_2} \); so \( i_1 = 1 \). Again \( i_2 \) can not be 0, else \( ad^{-1} \) will commute with \( bc \); so \( i_2 = 1 \), and hence \( N_2 = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle \).

Now consider \( x = a^{j_1}b^{j_2}c^{j_3}d^{j_4} \) and \( y = a^{k_1}b^{k_2}c^{k_3}d^{k_4} \) be such that none is power of the other (reading modulo \( Z(G) \)). If \( [x, y] \in N_2^\# \), then, on the lines of the proof of Lemma \( \ref{lem:3.7} \) it follows that \( [a, b][a, c][b, d][c, d]^{j_2}[c, d]^{j_3} \in N_2^\# \), and therefore can be written as a product of powers of generators of \( N_2^\# \).
Now comparing powers of the basic commutators, we get
\[ j \equiv k \pmod{2} \] and \[ jk \equiv k + 1 \pmod{2} \]
Solving these we have
\[ k^2 - k - 1 \equiv 0 \pmod{2} \]
This is not possible. Hence no non-central element commutes with other elements except its power (modulo center) in \( G/N \) if and only if \( N_2 = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle \).

Now using Lemma 3.14, the preceding lemma gives

**Corollary 3.13.** Let \( K \) be a 4-generator special 2-group of order \( 2^k \) and conjugate type \( \{1, 8\} \). Then \( K \) is isoclinic to \( G/N \), where \( G = \langle a, b, c, d \rangle \in \mathcal{H}_3 \) be any fixed group and \( N < Z(G) = G' \) with \( |N| = 4 \) is of the form
\[ N = \langle [a, b][c, d], [a, c][b, d][c, d] \rangle . \]

We conclude this section with the following result which is valid only for odd primes.

**Lemma 3.14.** Every isoclinism family of finite \( p \)-groups of nilpotency class 2 and conjugate type \( \{1, p^n\} \) contains a group of exponent \( p \), where \( p \) is an odd prime.

**Proof.** Notice that the isoclinism family of a finite \( p \)-group \( G \) of nilpotency class 2 and conjugate type \( \{1, p^n\} \) contains a special \( p \)-group \( H \) (say). Then \( H \) has the following presentation.
\[ H = \langle x_1, x_2, \ldots, x_d : [x_i, x_j, x_k] = 1, [x_i, x_j]^p = 1, x_i^p = \prod_{j<k} [x_j, x_k]^{c_{ijk}}, \prod_{j<k} [x_j, x_k]^{d_{ijk}} = 1 \rangle , \]
where \( c_{ijk}, d_{ijk} \in \mathbb{Z} \). Let \( F/R \) be a free presentation of \( H \), and \( R_1 \) denote the subgroup of \( R \) which is the normal closure of \( \langle [x_i, x_j, x_k], [x_i, x_j]^p, \prod_{j<k} [x_j, x_k]^{d_{ijk}} \rangle \) in \( F \). Let \( F : = F/R_1 \). Then the group \( F/F^p \) lies in the isoclinism class of \( G \) and is of exponent \( p \). \( \square \)

4. **Proof of Theorems 1.2 and 1.3**

We are now ready to prove our main results.

**Proof of Theorem 1.2.** Let \( G \) be a finite \( p \)-group of conjugate type \( \{1, p^3\} \), \( p > 2 \). Then by [10, Main Theorem], \( G \) can be of nilpotency class 2 or 3. Without loss of any generality, we can always assume that \( Z(G) \leq G' \). First assume that \( G \) is of class 3. We are going to show that this case can not occur, and therefore \( G \) must have nilpotency class 2.

By Proposition 2.4 \( |Z(G)| \geq p^3 \) and \( Z(G) < G' \); so \( |G'| \geq p^4 \). Then it follows from Lemma 3.2 that \( [G : Z(G)] = p^4 \). Since \( Z(G) < G' \), we have \( |G : G'| \leq p^3 \). But, if \( |G : G'| \leq p^2 \), then \( G \) can be minimally generated by at most 2 elements, which contradicts Proposition 2.4. Thus \( G : G' = p^3 \) and minimal generating set for \( G \) has exactly 3 elements. Assume that \( G = \langle a, b, c \rangle \).

Now we have \( |G : Z(G)| = p^4 \) and \( |G : G'| = p^3 \). So, at least one of the three commutators \( [a, b], [a, c] \) and \( [b, c] \) lies outside center. By the symmetry, we can assume that \( [a, b] \in G' - Z(G) \). Set \( [a, b] = \alpha \). Then clearly \( G' = \langle \alpha, Z(G) \rangle \). So there exist integers \( i_1 \) and \( i_2 \) such that
\[ [a, c] = [a, b]^{i_1} \beta_1, \text{ where } \beta_1 \in Z(G) \]
and
\[ [b, c] = [a, b]^{i_2} \beta_2, \text{ where } \beta_2 \in Z(G) . \]
Replacing $c$ by $a^{i_2}b^{-i_2}c$, we get $[a, c], [b, c] ∈ Z(G)$. Then $[a, c] = 1$ (by Lemma 3.1). 

An arbitrary element of $G$ can be written as $g = a^{i_3}b^{j_3}c^{j_3}d^{j_3}z$, where $z ∈ G' = Z(G)$ and $0 ≤ j_k ≤ p - 1$ for $k = 1, 2, 3, 4$. Then 

$$α^G = \{(a^{i_3}b^{j_3})^{-1}α a^{j_3}b^{j_3} | 0 ≤ j_k ≤ p - 1, k = 1, 2, 3\}.$$ 

Thus $|α^G| ≤ p^2$, which contradicts the fact that $G$ is of conjugate type $\{1, p^3\}$. Hence the nilpotency class of $G$ must be 2.

Now onward we assume that the nilpotency class of $G$ is 2. Recall that $G/Z(G)$ and $G'$ are elementary abelian $p$-groups (by Lemma 2.7). By our assumption that $Z(G) ≤ G'$, we have $Z(G) = G' = Φ(G)$. By Proposition 2.1 and Proposition 2.2, we have $p^3 ≤ |Z(G)| = |G'| ≤ p^6$. Thus, by Lemma 3.2, there can be two possibilities, namely

(i) $|G'| = p^3$ and $|G : Z(G)| ≥ p^4$ or
(ii) $|G'| ≥ p^4$ and $|G : Z(G)| = p^4$.

In case (i), $G$ is a Camina group with $|G'| = p^3$.

So it remains to consider case (ii) only. In this case, we have $p^4 ≤ |G'| = |Z(G)| ≤ p^6$ and $|G : Z(G)| = p^4 = |G : G'| = |G : Φ(G)|$. Thus $G/Φ(G)$ is an elementary abelian $p$-group of order $p^4$. Hence $G$ is minimally generated by 4 elements. By Lemma 3.11, we can assume $G$ to be of exponent $p$ upto isoclinism. Thus $G$ is isoclinic to $G_3$ or to a central quotient $G/H$, where $H$ is a non-trivial central subgroup of $G_3$ with $|H| ≤ p^2$. Hence the order of $H$ is either $p$ or $p^2$. Proof of the theorem is now complete by Corollary 3.6 and Lemma 3.9.

Before proceeding to the proof of Theorem 4.2, we state the following result which is a consequence of the main result of Wilkens [12] stated on pages 203 – 204.

**Theorem 4.1.** Let $G$ be a finite 2-group of nilpotency class 2 and conjugate type $\{1, 8\}$. Then one of the following holds:

(i) $|G'| = 2^4$.
(ii) $|G : Z(G)| = 2^4$.
(iii) $|G'| = 2^4$ and there exists $R$ with $R ≤ Ω_1(Z(G))$ and $|R| = 2$ such that $|G/R : Z(G/R)| = 2^3$.
(iv) $|G'| = 2^4$ and $G$ is central product $HC_G(H)$, where $C_G(H)$ is abelian and $H$ is the group given as follows:

There are $i, j, k, l$ and $m ∈ \mathbb{N}$ such that $H ≅ \tilde{H}/\langle x^{2i}, y^{2j}, v_1, v_2, v_3, v_4 \rangle$, where $\tilde{H} = \langle x, y, v_1, v_2, v_3 \rangle$ is of class 2 with $Φ(\tilde{H}) ≤ Z(\tilde{H})$ and otherwise defined by $[v_2, x] = [v_1, v] = [v_3, x] = [v_3, v] = 1$, $[v_1, v_j] ∈ \langle [v_3, x] \rangle$.

We are now ready for the final proof.

**Proof of Theorem 4.1.** Let $G$ be a finite 2-group of nilpotency class 2 and conjugate type $\{1, 8\}$. Then $G$ is isomorphic to one of the groups $G$ in (i), (ii), (iii) and (iv) of the preceding theorem. We are going to show that third and fourth possibilities can not occur. Suppose that (iii) occurs. Then $|G'| = 2^4$ and there exists $R$ with $R ≤ Ω_1(Z(G))$ and $|R| = 2$ such that $|G/R : Z(G/R)| = 2^3$. Since $G$ is of conjugate type $\{1, 8\}$ and $|R| = 2$, we have $Z(G/R) = Z(G)/R$. Then from the fact that $|G/R : Z(G/R)| = 2^3$, we get $|G : Z(G)| = 2^3$, which contradicts our hypothesis that $G$ is of conjugate type $\{1, 8\}$.

Next consider the case (iv)(5). So $G ≅ HC_G(H)$, where $\tilde{H}$ is a 2-group of class 2. It is easy to see that the conjugacy class of the image of $v_3$ in $H$ is of length at most 2. Hence $H$ is not of conjugate type $\{1, 8\}$ and so is for $G$. So we are left with only two cases (i) and (ii).

In case (i), $|G'| = 2^3$, which forces $G$ to be isoclinic to a Camina 2-group with commutator subgroup of order 8.

Finally we consider the case (ii). For any group $G$, in this case, we have $|G : Z(G)| = 2^4$. Since $G$ is of conjugate type $\{1, 8\}$, we have $|G : C_G(x)| = 2^4$, and consequently $|C_G(x) : Z(G)| = 2$ for all $x ∈ G − Z(G)$. Hence for all $x ∈ G − Z(G)$, $x^2 ∈ Z(G)$, i.e.,
\[ G/Z(G) \text{ is an elementary abelian 2-group. Thus } G' \leq Z(G), \text{ and therefore } G \text{ is of class 2.} \]

By Proposition 2.2, we can assume \( Z(G) = G' \). By Proposition 2.3, \( |G'| = |Z(G)| \leq 2^6 \).

Since, \( G \) being conjugate type \( \{1, 8\} \), \( |G'| \geq 2^3 \), it follows that \( 2^7 \leq |G| = 2^{10} \). Since \( G \) is of class 2, obviously \( G' = Z(G) \) is elementary abelian. Therefore the exponent of \( G \) is 4. If \( |G| = 2^7 \), then \( G \) is a Camina group, which is not possible by [8, Theorem 3.2]. Hence \( 2^8 \leq |G| \leq 2^{10} \), and therefore \( G \) must be isomorphic to some group \( T \) in the family \( \hat{G} \) or its central quotient \( T/K \) with \( |K| \leq 4 \). Now the proof is complete by Lemmas 3.10, 3.11 and Corollary 3.13.

\[ \square \]

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