Quantum Hall Effect on Higher Dimensional Spaces

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Abstract

We analyze the quantum Hall effect exhibited by a system of particles moving in a higher dimensional space. This can be done by considering particles on the Bergman ball $\mathbb{B}_\rho^d$ of radius $\rho$ in the presence of an external magnetic field $B$ and investigate its basic features. Solving the corresponding Hamiltonian to get the energy levels as well as the eigenfunctions. This can be used to study quantum Hall effect of confined particles in the lowest Landau level where density of particles and two point functions are calculated. We take advantage of the symmetry group of the Hamiltonian on $\mathbb{B}_\rho^d$ to make link to the Landau problem analysis on the complex projective spaces $\mathbb{C}P^d$. In the limit $\rho \to \infty$, our analysis coincides with that corresponding to particles on the flat geometry $\mathbb{C}^d$. This task has been done for $d = 1, 2$ and finally for the generic case, i.e. $d \geq 3$. 

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1 Introduction

Quantum Hall effect (QHE) [1] is a fascinating subject because not only of the precise quantized Hall conductivity but also of its relationships to other subjects of theoretical physics and mathematics. Since its appearance, the QH committee was and is still considering particles constrained to move in two-dimensional space. This is true because of the experiment evidences and obligations.

More recently, people are talking about higher dimensional QHE on different manifolds. One may ask how QHE can be realized on these spaces? The answer to this question was done in 2001 from a theoretical point of view where an embedding of QHE on 4-dimensional space was achieved by Hu and Zhang [2]. The key point was to generalize the Hall current from $SO(3)$ two-sphere $S^2$ to four-sphere $S^4$ of the invariant group $SO(5)$. This is not the end, but it appeared very exciting and interesting works on the subject. Indeed, QH droplets were considered on complex projective spaces $\mathbb{C}P^d$ [3] where the wavefunctions were obtained and the incompressibility of the Hall liquid was shown. Also based on $\mathbb{C}P^d$, a relation between QHE and the fuzzy spaces was discussed [4]. Other important developments on the subject can be found in [5]. The common feature in these works is to generalize the Landau problem on different higher dimensional manifolds. This is because the Landau problem is the cornerstone of QHE.

On the other hand, investigating the Laplacian operator properties on different manifolds have attracted several authors. Many interesting results on this subject have been reported by mathematicians. For instance, by considering Laplacian on the ball $B$ with unit radius in different dimensions the spectral theories were investigated. For early work with $d = 1$, for example one may see [6, 7]. In the case where $d \geq 2$, we refer to old work [8] and the recent papers can be found in [9]. Very recently, by taking into account of radius as an additional degree of freedom Ghanmi and Intissar [10] analyzed the same problem on the ball $B^d_\rho$ with $\rho$ positive real value. Based on these works, we develop our basic idea.

More specifically, we consider a system of particles on the Bergman Ball $B^d_\rho$ and investigate its basic features. This can be done by generalizing the Landau problem on the plane to higher dimensional space $\mathbb{B}^d_\rho$ and getting its spectrum. The generalized Hamiltonian is invariant under the group $SU(d,1)$ as well as $U(d)$. This will be used to discuss a link to the Karabali and Nair work [3] on $\mathbb{C}P^d$. We analytically start by analyzing the problem on the disc that is $\mathbb{B}^1_\rho$. This is isomorph to the Poincaré half plane $\mathbb{H}$ and therefore one may use the Cayley transform to get the $\mathbb{B}^1_\rho$ spectrum from that of particles living on $\mathbb{H}$. As we will see, the Landau problem on $\mathbb{B}^1_\rho$ can be solved straightforwardly. This allows us to study particles confined in the lowest Landau level (LLL) and discuss QHE in terms of the density of particles and the incompressibility condition. This will be done after building the wavefunction for the fully occupied state $\nu = 1$. Algebraically we make contact with particles moving in two-sphere $S^2$, which is $\mathbb{C}P^1$. This will be the first overlapping to [3]. Subsequently, we treat the same problem but on the ball $\mathbb{B}^2_\rho$ by doing the same job as for the disc and looking for link to $\mathbb{C}P^2$. Finally, we consider the generic case and this will be done by generalizing $\mathbb{B}^1_\rho$ to higher dimensions. Also
QHE of particles in LLL will be investigated and link to CP\(^d\) will be emphasized. In all case we will refer to the flat geometry limit, which corresponds to \(\rho\) goes to infinity.

In section 2, for the necessity we review the Landau problem on the plane and give its connection to QHE of particles in LLL. In section 3, we consider particles living on \(\mathbb{H}\) in magnetic field \(B\) and present the group theory approach to get the spectrum. This can be done by realizing the appropriate group and exploiting its representation. This will be used in section 4 to discuss the link to CP\(^1\) after solving the Landau problem on the disc and discussing QHE in LLL. This task will be done by analyzing the Landau problem on \(\mathbb{B}_2^d\) and compared to CP\(^2\) in section 5. We generalize our results to higher dimensional space \(\mathbb{H}_\rho^d\) in section 6. We conclude and give some perspectives in the final section.

## 2 Flat geometry

Particles of mass \(m\) living on the plane in an uniform magnetic field \(B\) is very interesting problem. Since it has much to do with many areas of theoretical physics and in particular the QHE subject. This is because of its basic properties and the most important is its projection to LLL where particles try to be near the minimal potential. This can be caused by applying a strong \(B\) that generates a gap, which pushes particles to be in LLL. This gives a natural example of the non-commutative geometry and therefore a connection to beautiful theories like Laughlin approach \[11\].

### 2.1 Particle in the plane

For the necessity, we review the results of the Landau problem on the plane. One particle Hamiltonian can be written as

\[
H^F = \frac{1}{4} \left\{ -4\partial_z \partial_{\bar{z}} + B (z \partial_z - \bar{z} \partial_{\bar{z}}) - \left( \frac{B}{2} \right)^2 |z|^2 \right\}
\]

in the complex plane \((z, \bar{z})\) and the symmetric gauge

\[
A^F = \frac{B}{2} (y, -x).
\]

\(F\) refers to Landau problem on the flat surface. The unit system \((c, e, \hbar)\) is chosen to be equal to 1 and \(m = 2\). Without loss of generality, through this paper we assume that \(B\) is strictly positive.

The spectrum can be obtained by diagonalizing \(H^F\) in terms of creation and annihilation operators. These are

\[
a^\dag = 2\partial_z + \frac{B}{2} z, \quad a = -2\partial_{\bar{z}} + \frac{B}{2} \bar{z}
\]

which satisfy the commutation relation

\[
[a^\dag, a] = 2B.
\]
These can be used to write $H^F$ as

$$H^F = \frac{1}{8} \left( a^\dagger a + a a^\dagger \right).$$

It is easily to see that the energy levels are

$$E_l^F = \frac{B}{4} (2l + 1), \quad l = 0, 1, 2 \ldots$$

and the eigenstates as well

$$|l\rangle^F = \frac{1}{\sqrt{(2B)^l l!}} (a^\dagger)^l |0\rangle$$

where $|0\rangle$ is the vacuum such as $a|0\rangle = 0$.

The previous analysis can be generalized to many-particles system described by the total Hamiltonian

$$H_{\text{tot}}^F = \frac{1}{4} \sum_{i=1}^{N} \left\{ -4\partial z_i \partial \bar{z}_i + B (z_i \partial z_i - \bar{z}_i \partial \bar{z}_i) - \left( \frac{B}{2} \right)^2 |z_i|^2 \right\}$$

where the total energy is $N$ copies of (5) and the eigenvalues is basically the tensorial product of $N$ those given in (7).

### 2.2 LLL and QHE

The connection to QHE can be investigated by considering many particles confined in LLL. We start by analyzing the spectrum of one particle, which corresponds to $l = 0$ in the above study. In this case, the complex space becomes non-commutative and we have

$$[z, \bar{z}] = \frac{2}{B}.$$ 

This is similar to the relation (4). Therefore, by analogy to the previous analysis for one particle, we can define a Hamiltonian in LLL as

$$H_{\text{LLL}}^F = \frac{1}{8} \left( b^\dagger b + b b^\dagger \right)$$

where the creation and annihilation operators can be defined as

$$b^\dagger = z, \quad b = \bar{z}.$$ 

It is clear that the spectrum can be read as

$$E_{n,\text{LLL}}^F = \frac{B}{4} (2n + 1), \quad |n\rangle^F_{\text{LLL}} = \frac{(2n)^n}{2^n n!} (a^\dagger)^n |0\rangle.$$ 

It is convenient to project the states on the complex plane $(z, \bar{z})$. Doing this to get the wavefunctions

$$\psi(z, \bar{z}) = \text{const} \, z^n \exp \left( -\frac{B}{4} |z|^2 \right)$$

3
Let us consider \( N \)-particles in LLL, which of course means that all \( l_i = 0 \) with \( i = 1, \ldots, N \) and each \( l_i \) corresponds to the spectrum (6–7). The total wavefunction can be written in terms of the Slater determinant. This is

\[
\psi(z, \bar{z}) = \varepsilon_{i_1 \cdots i_N} z_{i_1}^{n_1} \cdots z_{i_N}^{n_N} \exp \left( -\frac{B}{4} \sum_i |z_i|^2 \right) \tag{14}
\]

where \( \varepsilon_{i_1 \cdots i_N} \) is the fully antisymmetric tensor and \( n_i \) are integers. It is relevant to write this wavefunction as Vandermonde determinant. We have

\[
\psi(z, \bar{z}) = \text{const} \prod_{i,j} (z_i - z_j) \exp \left( -\frac{B}{4} \sum_i |z_i|^2 \right). \tag{15}
\]

This can be interpreted by remembering the Laughlin wavefunction

\[
\psi_{\text{Laugh}}^m(z, \bar{z}) = \prod_{i,j} (z_i - z_j)^m \exp \left( -\frac{B}{4} \sum_i |z_i|^2 \right). \tag{16}
\]

It is well known that it has many interesting features and good ansatz to describe the fractional QHE at the filling factor \( \nu = \frac{1}{m} \). It is clear that (15) is nothing but the first Laughlin state that corresponds to \( \nu = 1 \). Actually, (15) is describing the first quantized Hall plateau of the integer QHE. Note that (16) can also be written as

\[
|m\rangle = \{ \varepsilon_{i_1 \cdots i_N} z_{i_1}^{n_1} \cdots z_{i_N}^{n_N} \}^m |0\rangle. \tag{17}
\]

Before closing this section, we say some words about the filling factor because of its relevance in the QHE world. In the unit system, this is

\[
\nu = \frac{2\pi N}{B} \tag{18}
\]

where \( N \) is the density of particles

\[
N = \frac{N}{S} \tag{19}
\]

and \( S \) is the plane surface. It is obvious that to obtain \( \nu = 1 \), \( N \) should be equal to the finite quantity \( \frac{B}{2\pi} \). The QHE world tells us that \( \nu \) (18) must be quantized and reads as

\[
\nu = \frac{N}{N_\phi}. \tag{20}
\]

This can be either fractional or integer, it depends to what the QHE kind is involved. \( N_\phi \) is the number of the quantum flux

\[
\Phi^F = \int_{\text{plane}} B dx dy \tag{21}
\]

per unit of the flux \( \Phi^F_0 = \frac{\hbar e}{c} \), which equal to 1 in our choice of unit. From this, one can learn that in getting the QHE the magnetic field should be quantized. Note that \( N_\phi \) plays a crucial role since it determines also the degree of degeneracy of Landau levels.
3 Poincaré half plane $\mathbb{H}$

It will be clear in the next that the Poincaré half plane $\mathbb{H}$ is isomorph to the ball $\mathbb{B}^1_\rho$. For this reason, we start by treating the quantum mechanics of one particle living on $\mathbb{H}$ in the presence of a magnetic field. This isomorphism can be used to analysis the Landau problem on $\mathbb{B}^1_\rho$. Our study will be done analytically but we focus on the group theory approach since it will involved in the next. $\mathbb{H}$ is defined by

$$\mathbb{H} = \{ z = x + iy \in \mathbb{C}, y > 0 \}$$

and endowed with the metric

$$ds^2 = \frac{\rho^2}{y^2} (dx^2 + dy^2).$$

The corresponding measure is

$$d\mu = \frac{\rho^2}{y^2} dxdy.$$

3.1 Hamiltonian formalism

One particle Hamiltonian $H^P$ on the Poincaré half plane $\mathbb{H}$ can be derived from the Laplace-Beltrami operator $H^{LB}$ of a particle of mass $m$ on a Riemann surface of metric $g_{ab}$ with a monopole field \[13\]. This operator is given by

$$H^{LB} = \frac{1}{2m} \frac{1}{\sqrt{g}} p^a (\sqrt{gg^{ab}}) p^b$$

where $p_a$ is a covariant derivative and $g$ is the metric determinant. To obtain $H^P$, first one can fix the gauge

$$A^P = B \rho^2 \left( -\frac{1}{y}, 0 \right)$$

and second take into account of the associate metric \[23\]. Doing these and setting $m = 2$, \[25\] becomes

$$H^P = \frac{y^2}{\rho^2} \partial_z \partial_{\bar{z}} + \frac{i}{2} By (\partial_z + \partial_{\bar{z}}) + \frac{B^2 \rho^2}{4}.$$  

$P$ reflects the Landau problem on $\mathbb{H}$. \[27\] generalizes the Hamiltonian \[1\] and can be recovered by taking the limit $\rho \rightarrow \infty$.

There are tow possibilities to obtain the spectrum of $H^P$. This can be done either algebraically or analytically. As it will be clear soon, the first is based on the unitary irreducible representation of $SL(2, \mathbb{R})$ that is the invariant group of $H^P$ but the second is related to the eigenvalue equation. This is

$$H^P \Psi^P = E^P \Psi^P.$$ 

It can be solved on the compact manifold to get the eigenvalues

$$E_i^P = \frac{1}{\rho^2} \left[ \left( l + \frac{1}{2} - b \right)^2 + b^2 + \frac{1}{4} \right].$$
where \( b = \rho^2 B \) and the condition \( 0 \leq l < b - \frac{1}{2} \) must be fulfilled. The corresponding eigenfunctions are given by

\[
\Psi_{l,k}^P(x,y) = \sqrt{\frac{l! (2b - 2l - 1)}{4\pi \rho^2 |k| \Gamma (2b - l)}} \exp (-ikx - |k|y) (2|k|y)^{b-l} L_l^{2b-2l-1} (2|k|y)
\]

(30)

where \( L_\alpha^\beta(x) \) is the generalized Laguerre functions and \( k \) is such as

\[
L_2 \Psi_{l,k}^P(x,y) = -k \Psi_{l,k}^P(x,y).
\]

(31)

\( L_2 = -\partial_x \) is a generator of the group \( SL(2, \mathbb{R}) \), see the next subsection. This shows that (30) are degenerate wavefunctions. Note in passing that the present system has also a continue part, more detail can be found in [12]. In the flat limit, the above spectrum coincides with that obtained for one particle in section 2.

### 3.2 Invariant group

The system described by the Hamiltonian (27) on the space \( \mathbb{H} \) can be analyzed by making use of the group theoretical technology. This can be done by noting that \( H^P \) is invariant under the group \( SL(2, \mathbb{R}) \). Its generators can be mapped in terms of the phase space variables of \( H^P \) such as

\[
L_1 = -i (z \partial_z + \bar{z} \partial_{\bar{z}}), \\
L_2 = - (\partial_z + \partial_{\bar{z}}), \\
L_3 = i (z^2 + \bar{z}^2) \partial_{\bar{z}} - ib (z - \bar{z}).
\]

(32)

They satisfy the \( SL(2, \mathbb{R}) \) commutation relations

\[
[L_1, L_2] = -iL_2, \quad [L_1, L_3] = -iL_3, \quad [L_2, L_3] = 2iL_1.
\]

(33)

Another group can be realized that is isomorph to \( SL(2, \mathbb{R}) \). This can be achieved by defining the generators

\[
J_0 = \frac{1}{2} (L_2 - L_3), \quad J_1 = \frac{1}{2} (L_2 + L_3), \quad J_2 = L_1.
\]

(34)

One can check that they generate the unitary group \( SU(1, 1) \)

\[
[J_0, J_1] = iJ_2, \quad [J_0, J_2] = -iJ_1, \quad [J_1, J_2] = -iJ_0.
\]

(35)

An explicit relation between these generators and \( H^P \) can be fixed by introducing the Casimir operator of \( SU(1, 1) \). This is

\[
C = J_0^2 - J_1^2 - J_2^2.
\]

(36)

Using (32) and (34) to find

\[
H^P = -\frac{1}{4\rho^2} (C - b^2).
\]

(37)
From the last equation, one can learn that the spectrum of $H^P$ can be obtained by handling the representation theory that corresponds to the operator $C$. To clarify this point, we consider an unitary irreducible representation of $SL(2, \mathbb{R})$ as eigenstates of $C$ as well as the compact generator $J_0$ \[14\]. Otherwise, let us choose a basis $\{|j, m\rangle\}$ such as

$$C|j, m\rangle = j(j + 1)|j, m\rangle$$ \hspace{1cm} (38)

where $m$ is an eigenvalue of $J_0$

$$J_0|j, m\rangle = m|j, m\rangle.$$ \hspace{1cm} (39)

These results can be connected to those obtained by applying the analytical approach. This can be done by setting

$$j = l - b, \quad m = -l - b$$ \hspace{1cm} (40)

to end up with the derived energy levels as well as the eigenfunctions of $H^P$. More discussion about this issue can be found in \[12\].

We have some remarks in order. The generalization to $N$-particles without interaction is immediate. The quantum mechanics of particles on the plane can be found by taking the limit $\rho \rightarrow \infty$ \[12\]. The QHE exhibited by the present system on $\mathbb{H}$ was extremely studied in reference \[13\]. In conclusion let us emphasis that the above realization of $SU(1, 1)$ will be helpful in the forthcoming analysis. With this, we discuss the connection to $\mathbb{C}P^1$, i.e. two-sphere. Also it can be generalized to the generic case, i.e. $d \geq 2$, and used to go deeply in investigating other links to different spaces.

4 Hyperbolic disc $\mathbb{B}^1_\rho$

Before considering particle living on higher dimensional manifold in an external magnetic field and discuss the possibility to have QHE, we start from the ball $\mathbb{B}^1_\rho$ of radius $\rho$ that is a particular case and corresponds to the disc. This is interesting task because $\mathbb{B}^1_\rho$ has much to do with two-sphere $S^2$ that is the complex projective space $\mathbb{C}P^1$. This point will be clarified in the next study. $\mathbb{B}^1_\rho$ is

$$\mathbb{B}^1_\rho = \{ w \in \mathbb{C}, |w|^2 < \rho^2 \}$$ \hspace{1cm} (41)

and equipped with the Bergman-Kähler metric

$$(ds^1_\rho)^2 = \frac{1}{(1 - \frac{|w|^2}{\rho^2})^2} dw \otimes d\bar{w}.$$ \hspace{1cm} (42)

In this space the integration can be done with respect to the volume measure

$$d\mu^1_\rho(w) = \frac{1}{(1 - \frac{|w|^2}{\rho^2})^2} dm^1(w)$$ \hspace{1cm} (43)
where \( dm^1(w) = dx dy \) is the Lebesgue form. Therefore the inner product on \( B^1_{\rho} \) of two functions \( \Psi_1 \) and \( \Psi_2 \) in the Hilbert space reads as

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_{B^1_{\rho}} d\mu^1_{\rho}(w) \Psi_1^* \times \Psi_2.
\]

(44)

In this section, we refer to 1 as \( d = 1 \) and hereafter \( \rho \in ]0, \infty[ \).

As we claimed before, the Landau problem on the flat geometry is the cornerstone of QHE. It may be a good task to treat the same problem but at this time on the disc. This will allow us to investigate the basic features of QHE and make contact with some relevant works. For instance, we propose to discuss a link to the results obtained by considering particles on \( \mathbb{C}P^1 \).

4.1 Spectrum

To clarify the above ideas, we consider the Landau problem on the real hyperbolic disc and investigate its properties. One particle Hamiltonian on \( B^1_{\rho} \) can be obtained from the operator (25) associated to the metric (42). This is

\[
H^1 = \left( 1 - \frac{|w|^2}{\rho^2} \right) \left\{ - \left( 1 - \frac{|w|^2}{\rho^2} \right) \partial_w \partial_{\bar{w}} - B (w \partial_w - \bar{w} \partial_{\bar{w}}) + \frac{B^2}{4} |w|^2 \right\}.
\]

(45)

The constant magnetic field is

\[
F^1 = B d\mu^1_{\rho}(w) = (\partial_w A^1_w - \partial_{\bar{w}} A^1_{\bar{w}}) \, dw \wedge d\bar{w}
\]

(46)

where \( A^1_w \) and \( A^1_{\bar{w}} \) are the complex compounds of the gauge \( A^1 \). These can be adjusted to get the corresponding magnetic flux as

\[
\Phi^1 = \int_{B^1_{\rho}} F^1 = B S^1
\]

(47)

where \( S^1 = \pi \rho^2 \) is the area of \( B^1_{\rho} \) measured by the flat metric. Because of the Dirac quantization \( \Phi^1 \) must be integral quantized. Therefore, we have

\[
2k = B \rho^2
\]

(48)

where \( k \) is integer value. This relation will be helpful in discussing QHE generated from particles confined in LLL.

We make some comments about (45). This generalizes the Landau Hamiltonian (1) on the plane and can be recovered in the limit \( \rho \longrightarrow \infty \). Since \( \mathbb{H} \) is isomorphic to \( B^1_{\rho} \), \( H^1 \) has something to do with \( H^P \). Effectively there is a relation between them and can be fixed by making use of the Cayley transformation to get

\[
H^1 = \left( \frac{z + i}{i - \bar{z}} \right)^{-k} H^P \left( \frac{z + i}{i - \bar{z}} \right)^k
\]

(49)
such that \( w \) and \( z \) are governed by

\[
w = \frac{z - i}{z + i}.
\]

(50)

Recalling that \( w \in B_\rho^1 \) and \( z \in \mathbb{H} \). Equation (49) tells us that the eigenvalues and eigenstates of (45) can be derived from those corresponding to \( H^P \). With this transformation the eigenvalue equation reads as

\[
H^1 \Psi^1 (w, \bar{w}) = \left( \frac{z + i}{i - \bar{z}} \right)^k \left[ \left( \frac{z + i}{i - \bar{z}} \right)^k \Psi^1 (w, \bar{w}) \right].
\]

(51)

We present another way to get the analytical solution of the present problem instead of using (49). This first can be done by noting that \( H^1 \) is an elliptical operator on \( B_\rho^1 \), the corresponding eigenfunctions are \( C^\infty \)–functions. They form a Hilbert space as

\[
H (B_\rho^1) = \{ \Psi^1 \in C^\infty, H^1 \Psi^1 (w, \bar{w}) = E^1 \Psi^1 (w, \bar{w}) \}.
\]

(52)

This property is important, because now the eigenfunctions \( \Psi^1 (w, \bar{w}) \) can be expanded into a spherical series. This is

\[
\Psi^1 (w, \bar{w}) = \sum_{p,q \geq 0} g_{p,q} (r^2) h^{p,q} (w, \bar{w})
\]

(53)

where the Fourier coefficient \( g_{p,q} (r^2) \) are \( C^\infty \)–functions in \( [0, \rho[ \) with \( |w| = r \). \( h^{p,q} (w, \bar{w}) \) are homogeneous harmonic polynomials on \( \mathbb{C}^n \) and of degree \( p \) in \( w \) and \( q \) in \( \bar{w} \). They form a space of restriction to the boundary of the disk that is \( U(1) \)-irreducible representation. Note that \( h^{p,q} (w, \bar{w}) \) are similar to those obtained by Karabali and Nair in analyzing the algebraic structure of particles on \( \mathbb{C}P^1 \). These are

\[
h^{p,q} (w, \bar{w}) = w^p \bar{w}^q.
\]

(54)

On the other hand, one can easily check that (53) are also eigenfunctions of the angular momenta operator

\[
L_w = w \partial_w - \bar{w} \partial_{\bar{w}}
\]

(55)

with the eigenvalues

\[
L_w \sum_{p,q \geq 0} g_{p,q} (r^2) h^{p,q} (w, \bar{w}) = \sum_{p,q \geq 0} (p - q) g_{p,q} (r^2) h^{p,q} (w, \bar{w}).
\]

(56)

With the above equations and starting from

\[
H^1 \Psi^1 (w, \bar{w}) = E^1 \Psi^1 (w, \bar{w})
\]

(57)

we end up with a differential equation for \( g_{p,q} (r^2) \). This is

\[
\left[ \left( 1 - \frac{r^2}{\rho^2} \right) \left( 1 - \frac{r^2}{\rho^2} \right) \right] \left[ r^2 \partial^2_{r^2} + (1 + p + q) \partial_{r^2} \right] - B (p - q) + \frac{B^2}{4} |w|^2 \right] = E^1 \right] g = 0.
\]

(58)
The solution of (58) can be worked out by setting
\[ g_{p,q}(r^2) = (1 - u)^{\frac{l}{2} - l} F_{p,q}(u) \] (59)
and making the variable change \( \frac{r^2}{\rho^2} = u \). Injecting this in (58), we find a hypergeometric differential equation for \( F(u) \) such as
\[
\left\{ u(1-u) \frac{d^2}{du^2} + [p + q + 1 - (p + q + k - 2l + 1) u] - (q - l) (k + p - l) \right\} F_{p,q}(u) = 0. \] (60)
It has a solution of the hypergeometric function type \( _2F_1 \)
\[ 2F_1 \left( q - l, k + p - l; p + q + 1; \frac{r^2}{\rho^2} \right). \] (61)

Therefore combining all, we obtain the eigenfunctions
\[ \Psi_1^l(w, \bar{w}) = \left(1 - \frac{r^2}{\rho^2}\right)^{\frac{l}{2} - l} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2F_1 \left( q - l, k + p - l; p + q + 1; \frac{r^2}{\rho^2} \right) h_{p,q}^{k,l}(w, \bar{w}) \] (62)
associate to the eigenvalues
\[ E_1^l = \frac{B}{4} (2l + 1) - \frac{1}{\rho^2} l (l + 1) \] (63)
where \( l \) labels different Landau levels of particle on the disc. It must satisfy the condition \( 0 \leq l < \frac{k - 1}{2} \), recalling that \( 2k = B\rho^2 \).

We introduce a physical quantity since it has much to do with QHE. In particular, it can be used to check the incompressibility condition of the present system in LLL. This is the probability density, which can be calculated to get
\[ |\Psi_1^l|^2 = \sum_{p=0}^{\infty} \sum_{q=0}^{l} Y_{k,l}^{1,p,q} |h_{p,q}^{k,l}|^2 \] (64)
where \( Y_{k,l}^{1,p,q} \) is a function of the radius \( \rho \)
\[ Y_{k,l}^{1,p,q} = \frac{(l - q)! \rho^{2(1+p+q)} \Gamma^2(1+p+q)\Gamma(k-q-l)}{2(k-1-2l)\Gamma(1+p+l)\Gamma(k+p-l)}. \] (65)
We remark that only \( |\Psi_1^l|^2 \) and \( |h_{p,q}^{k,l}|^2 \) are complex coordinate dependent.

Since the above results generalize those of the Landau problem on the flat surface, it is relevant to check the asymptotic behavior. This can be achieved by sending \( \rho \) to infinity. Doing this to obtain the energy levels
\[ E_i^F = \frac{B}{4} (2l + 1). \] (66)
This coincides exactly with (6). The corresponding wavefunctions are

\[ \Psi^F_l(w, \bar{w}) = e^{-\frac{B}{2} r^2} \sum_{p=0}^{\infty} \sum_{q=0}^{l} \mathcal{F}_1 \left( q - l, p + q + 1; \frac{B}{2} r^2 \right) h^p,q_l(w, \bar{w}). \] (67)

They are analogue to the common eigenfunctions of the Hamiltonian (11) and the angular momenta

\[ L_z = z \partial_z - \bar{z} \partial_{\bar{z}}. \] (68)

One may recall the formula [15]

\[ 1 \mathcal{F}_1 (-\mu, 1 + \beta; x) = \frac{\Gamma(1 + \beta)\Gamma(1 + \mu)}{\Gamma(\beta + \mu + 1)} L^\beta_\mu(x) \] (69)

where \( L^\beta_\mu(x) \) is the generalized Laguerre functions.

4.2 QHE on \( \mathbb{R}^1_\rho \)

Particles in LLL are confined in a potential that is grand enough to neglect the kinetic energy. This produces a gap such that particles are not allowed to jump to the next level. LLL is rich and contains many interesting features that are relevant in discussing QHE. With this, it is interesting to consider our system on LLL.

We keep particles in LLL and investigate their basic features. We start by giving the spectrum for one particle in LLL, which can be obtained just by fixing \( l = 0 \) in the previous analysis. This gives the ground state

\[ \Psi^1_0(w) = \left( 1 - \frac{r^2}{\rho^2} \right) \sum_{p=0}^{\infty} \mathcal{F}_1 \left( k + p, p + 1; \frac{r^2}{\rho^2} \right) h^p_k(w) \] (70)

that has the energy

\[ E^1_0 = \frac{B}{4}. \] (71)

This value coincides with that corresponds to the Landau problem on the plane. Actually \( h^p_k(w) \) is a polynomial of degree \( p \) in \( w \). The density in LLL reads as

\[ |\Psi^1_0|^2 = \sum_{p=0}^{\infty} \rho^{2(1+p)} \frac{\Gamma(1+p)\Gamma(k)}{2(k-1)\Gamma(k+p)} |h^p_k|^2. \] (72)

One particle spectrum in LLL can be generalized to that for \( N \)-particles. It is obvious that the total energy is given by

\[ E^1_N = \frac{NB}{4}. \] (73)

The corresponding wavefunction can be constructed as the Slater determinant. This is

\[ \Psi^1_N(w) = e^{i \sum_{i=1}^{N} w_{i}} \psi^1_{i_1}(w_{i_1}) \psi^1_{i_2}(w_{i_2}) \cdots \psi^1_{i_N}(w_{i_N}) \] (74)
where each $\Psi_{i_j}(w_{i_j})$ has the form given in (70). This is similar to the wavefunction (14) on the plane and corresponds to the filling factor $\nu = 1$. Other similar Laughlin states can be obtained as we have in (17).

The definition (18) tells us that the density of particle is an important ingredient. To get QHE, this parameter should be kept constant by varying the magnetic field. For its relevance, we evaluate the density by defining the number of particles as

$$N = \int_{B^1} N^1(w) \, d\mu_\rho = \pi \rho^2 N_0^1. \quad (75)$$

This gives

$$N_0^1 = \frac{N}{\pi \rho^2} = \frac{BN}{2\pi k}. \quad (76)$$

In the thermodynamic limit $N, \rho \rightarrow \infty$, it goes to the finite quantity

$$N_0^1 \sim \frac{B}{2\pi}. \quad (77)$$

This is exactly the density of particles on flat geometry and therefore corresponds to the fully occupied state $\nu = 1$.

In QHE the quantized plateaus come from the realization of an incompressible liquid. This property is important since it is related to the energy. It means that by applying an infinitesimal pressure to an incompressible system the volume remains unchanged [17]. This condition can be checked for our system by considering two-point function and integrating over all particles except two. This is

$$I^1(w_{i_1}, w_{i_2}) = \int_{B^1} d\mu_\rho^1(w_3, w_4, \cdots, w_N) \left[ \Psi_N^1(w) \right]^* \Psi_N^1(w). \quad (78)$$

It is easy to see that $I^1(w_{i_1}, w_{i_2})$ is

$$I^1(w_{i_1}, w_{i_2}) \sim |\Psi_{0,i_1}^1|^2 |\Psi_{0,i_2}^1|^2 - |(\Psi_{0,i_1}^1)^* \Psi_{0,i_2}^1|^2. \quad (79)$$

This can also be evaluated in the plane limit. Taking $\rho \rightarrow \infty$, we obtain

$$I^1(w_{i_1}, w_{i_2}) \sim e^{-\frac{B}{2}(r_{i_1}^2 - r_{i_2}^2)} \sum_{p_1, p_2 \geq 0} \left( |w_{p_1}^1|^2 |w_{p_2}^2|^2 - |w_{p_1}^1 w_{p_2}^2|^2 \right). \quad (80)$$

This equation tells us that the probability of finding two particles at the same position is zero, as should be.

### 4.3 Two-sphere

In 1983, Haldane [16] proposed an approach to overcome the symmetry problem that brought by the Laughlin theory for the fractional QHE describing by the wavefunction (16) at the filling factor $\nu = \frac{1}{m}$. More precisely, (16) is rotationally invariance due to the angular momenta
but not transitionally. By considering particles living on two-sphere in a magnetic monopole, Haldane formulated a theory that possess all symmetries and generalizes the Laughlin proposal. Very recently, Karabali and Nair \[3\] elaborated an algebraic analysis that supports the Haldane statement and gives a more general results. These developments on $S^2$ will be connect to our study on the disc, i.e. $B^1_{\rho}$.

The link to two-sphere $S^2$ can be done via the following considerations. First, $S^2$ can be realized on the disc as

$$\partial B^1_{\rho} = S^2 = \{ \omega \in \mathbb{C}, |w| = \rho \}.$$  \(81\)

It is basically the ball $B^1_{\rho}$ boundary. This tells that from the basic features of $B^1_{\rho}$ one can derive those of $S^2$. Second, $H^1$ is invariant on the symmetric space

$$\frac{SU(1,1)}{U(1)}.$$  \(82\)

To get the complex projective space $\mathbb{CP}^1$

$$\mathbb{CP}^1 \equiv \frac{SU(2)}{U(1)}$$  \(83\)

that is $S^2$, one can use an analytic continuation of $SU(1,1)$ to $SU(2)$ (as in the Weyl unitary trick for groups). It suggests that the obtained spectrum on $B^1_{\rho}$ is similar to that of the Landau problem on the sphere except that our eigenfunctions should be invariant under $U(1)$.

Algebraically to make contact with our analysis on $B^1_{\rho}$, we refer to the symmetry group \(82\) and report what Karabali and Nair \[3\] have done in terms of our language. In this case, one can factorize the Hamiltonian $H^1$ as

$$H^1 = - (D_+D_- + D_-D_+).$$  \(84\)

At this stage, one may do recall to the generators of $SL(2,\mathbb{R})$. This can be done by mapping the covariant derivatives $D_\pm$ as follows

$$D_+ = \frac{1}{\rho}L_2, \quad D_- = \frac{1}{\rho}L_3$$  \(85\)

and the relation must be fulfilled

$$[D_+, D_-] = -\frac{B}{2}.$$  \(86\)

This and \(85\) fix the eigenvalue of the generator $L_1$ as $i\frac{k}{2}$. The Hamiltonian can be written in terms of the Casimir and $J^2_2$ of the group $SU(1,1)$ as

$$H^1 = -\frac{1}{\rho^2} \left( C + J^2_2 \right).$$  \(87\)

It is clear that to get the spectrum of $H^1$, one can use the representation theory where the eigenvalues of $C$ are $j(j+1)$. This gives the energy levels for

$$E^1_l = -\frac{1}{\rho^2} \left[ \left( l - \frac{k}{2} \right) \left( l - \frac{k}{2} + 1 \right) - \frac{k^2}{4} \right]$$  \(88\)
by fixing
\[ j = l - \frac{k}{2} \quad (89) \]
and taking into account the Dirac quantization \( 2k = B\rho^2 \). This leads to the same result as we have derived analytically for one particle on the disc. Therefore, one may use the Karabali and Nair technology to make contact with our analysis on QHE.

In conclusion, the above study suggests to go further and get other link to the complex projective spaces in higher dimensions. Next, we consider the ball \( \mathbb{B}^2_\rho \) and see what has common with \( \mathbb{CP}^2 \).

5 Ball \( \mathbb{B}^2_\rho \)

We deal with another interesting case that is the Landau problem on the ball \( \mathbb{B}^2_\rho \) and compare our results to those derived by analyzing the same problem on the complex projective space \( \mathbb{CP}^2 \). We start by defining \( \mathbb{B}^2_\rho \) as
\[
\mathbb{B}^2_\rho = \{ w = (w_1, w_2) \in \mathbb{C}^2, |w|^2 = |w_1|^2 + |w_2|^2 < \rho \}. \quad (90)
\]
Its Kähler-Bergman metric is given by
\[
(ds^1_\rho)^2 = \frac{\rho^2}{(\rho^2 - |w|^2)^2} \sum_{i,j=1}^{2} \left[(\rho^2 - w_i\bar{w}_j) \delta_{ij} + w_i\bar{w}_j \right] dw_i \otimes d\bar{w}_j. \quad (91)
\]
and the measure reads as
\[
d\mu^2_\rho(w) = \frac{1}{\left(1 - \frac{|w|^2}{\rho^2}\right)^3} dm^2(w) \quad (92)
\]
The volume of the ball \( \mathbb{B}^2_\rho \) measured by the Euclidean metric is
\[
S^2 = \frac{1}{2} \pi^2 \rho^4. \quad (93)
\]
This is interesting because it will be used in calculating the density of particles living on \( \mathbb{B}^2_\rho \). Of course the indices 2 refers to \( d = 2 \).

5.1 Landau problem on \( \mathbb{B}^2_\rho \)

We study the Landau problem on the ball \( \mathbb{B}^2_\rho \). This will be a generalization to \( d = 2 \) of our previous results obtained by treating the same problem on the disc. Let us consider one particle living on \( \mathbb{B}^2_\rho \) in an external magnetic field \( B \). The Hamiltonian can be written as
\[
H^2 = \left(1 - \frac{|w|^2}{\rho^2}\right) \sum_{i=1}^{2} \left\{ - \sum_{j=1}^{2} \left( \delta_{ij} + \frac{w_i\bar{w}_j}{\rho^2} \right) \partial_{w_i} \partial_{\bar{w}_j} - B (w_i\partial_{\bar{w}_i} - \bar{w}_i\partial_{w_i}) + \frac{B^2}{4} |w_i|^2 \right\}. \quad (94)
\]
In this case, the limit $\rho \to \infty$ corresponds to the Landau Hamiltonian on the complex space of dimension $d = 2$, i.e. $\mathbb{C}^2$.

The spectrum can be obtained by solving the eigenvalue equation

$$H^2 \Psi^2 = E^2 \Psi^2.$$  \hfill (95)

It can be worked out by applying the same method as we have done for the disc. This gives the energy levels

$$E^2_l = \frac{B}{4} (2l + 2) - \frac{1}{\rho^2} l (l + 2)$$  \hfill (96)

and the eigenfunctions

$$\Psi^2_l (w, \bar{w}) = \left(1 - \frac{|w|^2}{\rho^2}\right)^{\frac{k-1}{2}} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2F1 \left( q - l, k + p - l; p + q + 2; \frac{|w|^2}{\rho^2} \right) h_{k,l}^{p,q} (w, \bar{w})$$  \hfill (97)

where $0 \leq l < \frac{k}{2} - 1$ and (98) holds. Note that $h_{k,l}^{p,q} (w, \bar{w})$ form a space of restriction to $\partial \mathbb{B}_\rho^2$ and have the same form as given in (54) except that $w$ has two-components. We remark that the spectrum is dimension $d = 2$ dependent. This effect will manifest in discussing QHE on $\mathbb{B}_\rho^2$ and of course make difference with respect to the case $d = 1$.

The probability density is

$$|\Psi^2_l|^2 = \sum_{p=0}^{\infty} \sum_{q=0}^{l} Y^2_{k,l} |h_{k,l}^{p,q}|^2$$  \hfill (98)

where $Y^2_{k,l}$ reads as

$$Y^2_{k,2} = (l - q)! \rho^{2(p+q)} \frac{\Gamma^2(2+p+q) \Gamma(k-q-l-1)}{2(k-2l-2) \Gamma(2+p+l) \Gamma(k+p-l)}.$$  \hfill (99)

The spectrum on the complex plane $\mathbb{C}^2$ can be obtained in the limit $\rho \to \infty$. This leads to the energy

$$E^2_l = \frac{B}{4} (2l + 2)$$  \hfill (100)

and the wavefunctions

$$\Psi^2_l (w, \bar{w}) = e^{-\frac{B}{2}r^2} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2F1 \left( q - l, p + q + 2; \frac{B}{2} r^2 \right) h_{l}^{p,q} (w, \bar{w}).$$  \hfill (101)

5.2 LLL analysis

By restricting to LLL we discuss the QHE on $\mathbb{B}_\rho^2$ in terms of the density of particles and the incompressibility of our system. As usual to be in LLL, we consider the ground state energy that is corresponds to $l = 0$ in (98). We have

$$E^2_0 = \frac{B}{2}.$$  \hfill (102)
This value is similar to that obtained by Karabali and Nair [3]. Its wavefunction can be obtained in the same way from (97)

\[ \Psi_0^2(w) = \left( 1 - \frac{|\omega|^2}{\rho^2} \right) \sum_{p=0}^{\infty} \frac{\Gamma(2 + p) \Gamma(k - 1)}{4(k + p) \Gamma(k + p)} h_k^p(w). \]  

(103)

It has the density

\[ |\Psi_0^2|^2 = \sum_{p=0}^{\infty} \frac{\rho^{2(d+p)} \Gamma(2 + p) \Gamma(k - 1)}{4(k + p) \Gamma(k + p)} |h_k^p|^2. \]  

(104)

It is obvious that for \( N \) particles in LLL, the energy is \( N \) copies of \( E_0^2 \). This is

\[ E_N^2 = \frac{NB}{2} \]  

(105)

and its wavefunction can be built in similar way as we have done in (74)

\[ \Psi_N^2(w) = e^{i\sum_{i=1}^N} \Psi_{i_1}^2(w_{i_1}) \Psi_{i_2}^2(w_{i_2}) \cdots \Psi_{i_N}^2(w_{i_N}) \]  

(106)

where \( \Psi_{i_1}^2(w_{i_j}) \) is given by (103).

The density of \( N \)-particles can be calculated by using the area of \( B^2_\rho \) and adopting a definition of \( N \) analogue to (75). We have

\[ N_0^2 = \frac{2N}{\pi^2 \rho^4} = \frac{NB^2}{2\pi^2 k^2}. \]  

(107)

In the thermodynamics limit, we obtain

\[ N_0^2 \sim \frac{1}{2} \left( \frac{B}{\pi} \right)^2. \]  

(108)

This agrees with Karabali and Nair [3] result and conclusion. More precisely, it means that there is no need to introduce an infinite internal degrees of freedom as it has been done in [2].

In similar fashion to the disc, we can check the incompressibility of the system by evaluating two-point function. This is

\[ I^2(w_{i_1}^2, w_{i_2}^2) = \int_{B^2_\rho} d\mu_\rho(w_3, w_4, \cdots, w_N) \left[ \Psi_N^2(w) \right]^* \Psi_N^2(w). \]  

(109)

This can be integrated to obtain

\[ I^2(w_{i_1}^2, w_{i_2}^2) \sim |\Psi_{0,i_1}^2|^2 |\Psi_{0,i_2}^2|^2 - |(\Psi_{0,i_1}^2)^* \Psi_{0,i_2}^2|^2. \]  

(110)

If we consider the limit \( \rho \rightarrow \infty \), we find

\[ I^2(w_{i_1}^2, w_{i_2}^2) \sim e^{-\frac{B}{2} (r_1^2 - r_2^2)} \sum_{p_1, p_2 \geq 0} (|w_{1}^{p_1}|^2 |w_{2}^{p_2}|^2 - |w_{1}^{p_1} w_{2}^{p_2}|^2). \]  

(111)

It leads to the same conclusion as for the disc except that each \( w_{ij} \) has two components.
To close this section, we note that to make contact with the Landau problem on the complex projective space \( \mathbb{CP}^2 \) one can use the theory group approach. This can be done by considering the invariant symmetric space

\[
\frac{SU(2,1)}{U(2)}
\]

of the Hamiltonian \( H^2 \) and using an analytic continuation of \( SU(2,1) \) to \( SU(3) \), in similar way as for the disc and \( \mathbb{CP}^1 \), to have a link to

\[
\mathbb{CP}^2 \equiv \frac{SU(3)}{U(2)}.
\]

6 Spaces \( \mathbb{B}_\rho^d \)

We generalize the task done in two last sections by considering particles on the Bergman ball \( \mathbb{B}_\rho^d \) of dimension \( d \). This is a generalization to higher dimensional complex spaces, which can be achieved by replacing the plane \( \mathbb{C} \) by the Hermitian complex space \( \mathbb{C}^d (d \geq 1) \) and the real hyperbolic disc by the Bergman complex ball of radius \( \rho > 0 \). We have

\[
\mathbb{B}_\rho^d = \{ w = (w_1, w_2, \cdots, w_d) \in \mathbb{C}^d, |w|^2 = |w_1|^2 + |w_2|^2 + \cdots + |w_d|^2 < \rho \}.
\]

This is endowed with the Kähler-Bergman metric

\[
(ds^d)_{\rho}^2 = \frac{\rho^2}{(\rho^2 - |w|^2)^2} \sum_{i,j=1}^{d} \left[ (\rho^2 - w_i \bar{w}_j) \delta_{ij} + w_i \bar{w}_j \right] dw_i \otimes d\bar{w}_j.
\]

The measure is

\[
d\mu_{\rho}^d(w) = \frac{1}{(1 - |w|^2)^{d+1}} dm^d(w)
\]

where \( dm^d(w) = dw_1 dw_2 \cdots dw_d \) is the Lebesgue measure in the flat geometry \( \mathbb{C}^d \). In general the surface of \( \mathbb{B}_\rho^d \) measured by the flat metric is given by

\[
S^d = \frac{\pi^{\frac{d}{2}} \rho^d}{\Gamma (\frac{d}{2} + 1)}
\]

where \( \Gamma \left( \frac{d}{2} + 1 \right) \) is the gamma function.

6.1 Generalized Hamiltonian

We start by considering the Hamiltonian \( H^d \) of one particle living on \( \mathbb{B}_\rho^d \) in an external magnetic field \( B \). This is the sum over \( d \) of \( H^1 \) such as

\[
H^d = \left( 1 - \frac{|w|^2}{\rho^2} \right) \sum_{i=1}^{d} \left\{ -\sum_{j=1}^{d} \left( \delta_{ij} + \frac{w_i \bar{w}_j}{\rho^2} \right) \partial_{w_i} \partial_{\bar{w}_j} - B (w_i \partial_{w_i} - \bar{w}_i \partial_{\bar{w}_i}) + \frac{B^2}{4} |w|^2 \right\}.
\]
In the limit $\rho \to \infty$, $H^d$ goes to the Landau Hamiltonian on the complex space $\mathbb{C}^d$.

The spectrum of (118) can be obtained via two methods. The first one based on the weighted Plancherel formula [10]. The second is related to the standard method that is the eigenvalue equation

$$H^d \Psi^d = E^d \Psi^d$$

This can solved in similar fashion as we have done for the disk. The obtained results can be summarized as follows. With the condition $0 \leq l < \frac{k-d}{2}$ and (48), the energy levels read as

$$E^d_l = \frac{B}{4} (2l + d) - \frac{1}{\rho^2} l (l + d)$$

and the eigenfunctions are

$$\Psi^d_l(w, \bar{w}) = \left(1 - \frac{|w|^2}{\rho^2}\right)^{\frac{k-d}{2}} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2 F_1 \left( q - l, k + p - l; p + q + d; \frac{|w|^2}{\rho^2} \right) h_{k,l}^{p,q}(w, \bar{w}).$$

Recalling that $w$ has $d$-components. Again $h_{k,l}^{p,q}(w, \bar{w})$ form a space of restriction to the boundary of $B^d_\rho$. We note that both of energy and wavefunctions are dimensions $d$-dependent.

The density of (121) can be calculated to obtain

$$|\Psi^d_l|^2 = \sum_{p=0}^{\infty} \sum_{q=0}^{l} Y_{k,l}^{d,p,q} |h_{k,l}^{p,q}|^2$$

where $Y_{k,l}^{d,p,q}$ is

$$Y_{k,l}^{d,p,q} = \frac{(l-q)! \rho^{2(d+p+q)} \Gamma^2(d+p+q) \Gamma(k-d-q-l)}{2(k-d-2l) \Gamma(d+p+q) \Gamma(k-p-l)}$$

It is clear that for $d = 1, 2$ we recover the spectrum of the disc and $B^2_\rho$, respectively.

We make contact with the flat geometry $\mathbb{C}^d$ by sending $\rho$ to infinity. We have the energy

$$E^*_l = \frac{B}{4} (2l + d)$$

and the corresponding wavefunctions

$$\Psi^*_l(w, \bar{w}) = e^{-\frac{B}{2} r^2} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2 F_1 \left( q - l, p + q + d; \frac{B}{2} r^2 \right) h_l^{p,q}(w, \bar{w}).$$

6.2 Particles in LLL on $B^d_\rho$

We generalize our former study on LLL to higher dimensions and investigate the QHE basic features. We start with one particle ground state

$$\Psi^B_0(w) = \left(1 - \frac{|w|^2}{\rho^2}\right)^{\frac{k-d}{2}} \sum_{p=0}^{\infty} 1 F_1 \left( k + p, p + d; \frac{|w|^2}{\rho^2} \right) h_k^p(w).$$
It corresponds to the eigenvalue
\[ E_0^d = \frac{Bd}{4}, \] (127)

It is also similar to the value obtained by Karabali and Nair \[3\] and shows the first overlapping to \( \mathbb{C}P^d \) analysis.

For \( N \) particles in LLL, the above results generalize to
\[ E_N^d = \frac{dNB}{4} \] (128)

and the wavefunction is
\[ \Psi_N^d(w) = e^{i\sum_{i=1}^{N} \Psi_{i_1}^d(w_{i_1}) \Psi_{i_2}^d(w_{i_2}) \cdots \Psi_{i_N}^d(w_{i_N})} \] (129)

where \( \Psi_{i_j}^d(w_{i_j}) \) read as \[126\].

The generalized density of particles can be calculated by using our definition for \( N \). We have
\[ \mathcal{N}_0^d = \Gamma \left( \frac{d}{2} + 1 \right) \frac{B^d N}{2^d \pi^d \hbar^d}. \] (130)

In the thermodynamics limit \( N, \rho \rightarrow \infty \) and for an even \( d = 2p \) we obtain
\[ \mathcal{N}_0^d \sim p! \left( \frac{B}{2\pi} \right)^{2p}. \] (131)

This relation generalizes those derived by considering particles on the disc and \( \mathbb{B}^2_\rho \). On the other hand, \( \mathcal{N}_0^d \) is proportional to \( B^d \) that is Karabali and Nair \[3\] obtained for the Landau problem on \( \mathbb{C}P^d \).

As usual to check the incompressibility condition for the present system, we evaluate two-point function. In the limit \( \rho \rightarrow \infty \), we have
\[ I^d(w_{i_1}^d, w_{i_2}^d) \sim e^{-\frac{B}{2}(r_1^2 - r_2^2)} \sum_{p_1, p_2 \geq 0} \left( |w_{i_1}^{p_1}|^2 |w_{i_2}^{p_2}|^2 - |w_{i_1}^{p_1} w_{i_2}^{p_2}|^2 \right). \] (132)

This is the same as we have derived before. There is only difference of dimensions.

### 6.3 \( \mathbb{C}P^d \)

As before the connection to the complex projective space \( \mathbb{C}P^d \) can be done via the invariant symmetric space of the generalized Hamiltonian. This is
\[ \frac{SU(d, 1)}{U(d)}. \] (133)

More specifically, one can use an analytic continuation of \( SU(d, 1) \) to \( SU(d + 1) \) to get
\[ \mathbb{C}P^d \equiv \frac{SU(d + 1)}{U(d)} \] (134)
An element of $SU(d, 1)$ is of the form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$  \hspace{1cm} (135)$$

where $A, B, C, D$ are $d \times d$, $d \times 1$, $1 \times d$, $1 \times 1$ matrices, respectively. The element $g$ should be invariant under $U(d)$, i.e. $g \rightarrow gh$, $h \in U(d)$. It acts on $\mathbb{B}_\rho^d$ by fractional linear mappings

$$w \rightarrow g \cdot w = \frac{Aw + B}{Cw + D}$$ \hspace{1cm} (136)$$

As usual the sphere of radius $\rho$ can be seen as the boundary of $\mathbb{B}_\rho^d$

$$\partial \mathbb{B}_\rho^d = \{ \omega \in \mathbb{C}^d, |w|^2 = |w_1|^2 + |w_2|^2 + \cdots |w_d|^2 = \rho^2 \}.$$ \hspace{1cm} (137)$$

The algebraic analysis of the Landau problem on $\mathbb{C}P^d$ leads to the same energy levels as we have reported here.

In conclusion, it may be interesting to look at $\mathbb{B}_\rho^3$ that corresponds to $\mathbb{C}P^3$. If we use the same analysis as before, we get the energy

$$E_l^3 = \frac{B}{4} (2l + 3) - \frac{1}{\rho^2} l (l + 3)$$ \hspace{1cm} (138)$$

and the eigenfunctions

$$\Psi_{l}^3(w, \bar{w}) = \left(1 - \frac{|w|^2}{\rho^2}\right)^{\frac{\lambda}{2}-l} \sum_{p=0}^{\infty} \sum_{q=0}^{l} 2F1 \left(q - l, k + p - l; p + q + 3; \frac{|w|^2}{\rho^2}\right) h_{k,l}^p(w, \bar{w}).$$ \hspace{1cm} (139)$$

The energy levels have been obtained by Karabali and Nair \cite{3} for $\mathbb{C}P^3$. Also it confirm those derived by Hu and Zhang \cite{2} on four-sphere $S^4$.

7 Conclusion

We have analyzed the Landau problem on higher dimensional spaces. This has been done by considering a system of particles living on the Bergman ball $\mathbb{B}_\rho^d$ of radius $\rho$ in presence of an external magnetic field $B$ and investigating its basic features. It can be seen as a generalization of the present system on the flat geometry $\mathbb{C}^d$ with $d \geq 1$. Our task allowed us to make a link to the same problem on the complex projective spaces $\mathbb{C}P^d$. This link was showing that two spaces are sharing some common features. In fact, they possess the same energy levels as well as other overlapping.

More precisely, after writing down the corresponding Hamiltonian and getting its spectrum we have analyzed the QHE of particles confined in LLL. This task has been done first for the elementary case that is the disc where its connection to two-sphere has been discussed. Second, we have studied another interesting case which is $\mathbb{B}_\rho^2$ and its relation to $\mathbb{C}P^2$. Finally, these results have been generalized to $\mathbb{B}_\rho^d$. 

20
In each case above we have evaluated the density of particles and two-point function that is the condition to have a realized incompressible liquid. In particular, in the thermodynamic limit $N, \rho \to \infty$, we have found that the density goes to a finite quantity. This agreed with that derived by Karabali and Nair [3]. Subsequently we have gave the spectrum of $\mathbb{B}_\rho^3$ where the energy levels are similar to those obtained on $\mathbb{C}P^3$ [3] as well as four-sphere $S^4$ [2].

Still some important questions remain to be answered. One could consider the particles with spin as additional degree of freedom and do the same job as we have presented here. On the other hand, a supersymmetric extension of the present work may be an interesting task. A link to QHE on the fuzzy spaces is also interesting.

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References

[1] For instance see R.E. Prange and S.M. Girvin (editors), ”The Quantum Hall Effect” (Springer, New York 1990).

[2] S.C. Zhang and J.P Hu, Science 294 (2001) 823, cond-mat/0110572. J.P. Hu and S.C. Zhang, cond-mat/0112432 S.C. Zhang, Quantum Hall effect in higher dimensions, (Talk given at the Conference on Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and Non-Commutative Geometry in Condensed Matter Physics and Field Theory, 1-4/03/2005 AS-ICTP Trieste).

[3] D. Karabali and V.P. Nair, Nucl. Phys. B641 (2002) 533-546, hep-th/0203264. D. Karabali, Quantum Hall droplets on $\mathbb{C}P^k$ and edge effective actions, (Talk given at the Conference on Higher Dimensional Quantum Hall Effect, Chern-Simons Theory and Non-Commutative Geometry in Condensed Matter Physics and Field Theory, 1-4/03/2005 AS-ICTP Trieste).

[4] D. Karabali, V.P. Nair and S. Randjbar-Daemi, hep-th/0407007

[5] J.P Hu and S.C. Zhang, Phys. Rev. B66 (2002) 125301; B.A. Bernevig, J.P Hu, N. Thombas and S.C. Zhang, Phys. Rev. Lett. 91 (2003) 236803; M. Fabinger, JHEP 0205 (2002) 037; G. Sparling, cond-mat/0211679 Y.X. Chen, hep-th/0210059 hep-th/0201182 Y.X. Chen, B.Y. Hou, Nucl. Phys. B638 (2002) 220; H. Elvang and J. Polchinski, hep-th/0209104 Y.D. Chong and R.B. Laughlin, Ann. Phys. 308 (2003) 237; S. Bellucci, P.Y. Casteill and A. Nersessian, Phys. Lett. B574 (2003) 21; B.P. Dolan, JHEP 0305 (2003) 018; G. Meng, J.Phys. A36 (2003) 9415; V.P. Nair and S. Randjbar-Daemi, hep-th/0309212 D. Karabali
and V.P. Nair, *Nucl. Phys.* **B697** (2004) 513-540, hep-th/0403111; A.P. Polychronakos, *Nucl. Phys.* **B705** (2005) 457, hep-th/0408194; K. Hasebe and Y. Kimura, *Phys. Lett.* **B602** (2004) 255; A.P. Polychronakos, *Nucl. Phys.* **B711** (2005) 505, hep-th/0411065; G. Landi, hep-th/0504092.

[6] J. Elstrodt, *Math. Ann.* **203** (1973) 295 (in German).

[7] J. Patterson, *it Compositio Math.* **31** (1975) 83.

[8] G.B. Folland, *Proc. Am. Math. Soc.* **47** (1975) 401.

[9] G. Zhang, *Stud. Math.* **102** (1992) 103; P. Ahern, J. Bruna and C. Cascante, *Indiana Univ. Math. J.* **45** (1996) 103; A. Boussejra and A. Intissar, *J. Funct. Anal.* **160** (1998) 115; K. Ayaz and A. Intissar, *Diff. Geom. Applic.* **15** (2001) 1.

[10] A. Ghanmi and A. Intissar, *J. Math. Phys.* **46** (2005) 032107.

[11] R.B. Laughlin, *Phys. Rev. Lett.* **50** (1989) 1559.

[12] A. Comtet, *Ann. Phys.* **173** (1987) 185.

[13] R. Iengo and D. Li, *Nucl. Phys.* **B413** (1994) 735.

[14] V. Bargmann, *Ann. Math.* **48** (1947) 568.

[15] W. Magnus, F. Oberhettinger and R.P. Soni, ”Formulas and Theorems for the Special Functions of Mathematical Physics” (Springer, Berlin 1966).

[16] F.D Haldane, *Phys. Rev. Lett.* **51** (1983) 605.

[17] Z.F. Ezawa, ”Quantum Hall Effects: Field Theoretical Approach and Related Topics” (World Scientific, Singapore 2000).