Erratum: On the thermodynamic stability of rotating black holes in higher dimensions—a comparison of thermodynamic ensembles (2014 Class. Quantum Grav. 31 135012)

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The following errors were not corrected due to an error in the production process:

(1) Page 3, last sentence: ‘Hints of the instability can be seen in the microcanonical ensemble in which the entropy, which is a monotonically decreasing concave function of angular momenta, at constant mass, until the temperature hits a minimum at which point the entropy has an inflection point and becomes concave in at least one direction [9].’ should read ‘Hints of the instability can be seen in the microcanonical ensemble in which the entropy is a monotonically decreasing concave function of angular momenta, at constant mass, until the temperature hits a minimum at which point the entropy has an inflection point and becomes concave in at least one direction [9].’ i.e. remove ‘, which’.

(2) Page 4, section 2, first line: replace ‘Rotating black holes in $D$ space–time dimensions...’ with ‘Rotating black holes in $D > 4$ space–time dimensions...’.

(3) Page 6, first line after equation (13): replace ‘...where $\beta = \frac{1}{T}$; the symmetric matrix ...’ with ‘where the symmetric matrix...’.

(4) Page 6, first line after equation (14): replace ‘...; and $C_J$ is the specific heat...’ with ‘...; $C_J$ is the specific heat...’, i.e. remove ‘and’.

(5) Page 9, section 3, first sentence: replace ‘...dimensions are assumed to have...’ with ‘... dimensions have...’, i.e. remove ‘are assumed to’.
(6) Page 9: all occurrences of small \( x \) with a subscript on this page should have been replaced with a different font, \( \times \). There are six such occurrences, in the second and third sentences of page 9.

(7) Page 11, first sentence after equation (53): replace ‘...where \( j_s = \frac{2\pi l_s}{3} \) are dimensionless angular momentum,...’ with ‘...where \( j_s = \frac{2\pi l_s}{3} \) is the dimensionless angular momentum,...’.

(8) Page 11, first sentence after equation (54): replace ‘...are saddle points with minima along...’ with ‘...are saddle points of the temperature with minima along...’.

(9) Page 12: line between equations (55) and (56): replace \( J \) with \( j_i \).

(10) Page 14, line after equation (68): sentence should end with close open brackets, i.e. ‘...corresponding to \( 2\sqrt{3} - 3 < \frac{l^i}{M^i} < 1 \)’ should read ‘...corresponding to \( 2\sqrt{3} - 3 < \frac{l^i}{M^i} < 1 \)’.

(11) Page 17, last sentence before section 3.4.4: replace ‘While the determinant itself can positive there...’ with ‘While the determinant itself can be positive there...’.

(12) Page 22, numbered points 1 and 2: all 11 occurrences of capital \( J \) in the numbered points containing equations (96) and (97) should have been replaced with small \( j \).

(13) Page 22, line 13 from bottom: replace ‘...can stabilize the black, provided...’ with ‘...can stabilize the black hole, provided...’.

(14) Page 23, second sentence: replace ‘...but which cancel in the determinant so they are not visible.’ with ‘...but which cancel in the determinant so they are not visible there.’.

(15) Page 25, equation (A.4): \( j_2 \) in the numerator of \( \frac{j^2}{1+j^2} \) in the summation of the first equation should have been replaced with \( j_i \).

(16) Page 26: capital \( J \) in equations (A.12), (A.13), (A.19) and (A.20) should have been replaced with small \( j \).

(17) Page 27: four occurrences of capital \( J \) on the right-hand side of equation (A.21) should have been replaced with small \( j \).

(18) Page 32, equation (D.6): \( (1 - j_j) \) in the numerator in front of \( \delta_{jk} \) should read \( (1 - j_j^2) \), i.e. \( j_j \) should have been replaced with \( j_j^2 \). \( (1 + j_j^2) \) in the denominator in front of \( \delta_{jk} \) should read \( (1 + j_j^2)^2 \). \( (1 + j_j) \) in the denominator of the second term on the right-hand side should read \( (1 + j_j^2) \), i.e. \( j_j \) in the denominator should have been replaced with \( j_j^2 \).

(19) Page 32, equation (D.7): \( (1 - j_j^2) \) in the denominator of the second term on the right-hand side should have been replaced with \( (1 - j_j^2)(1 - j_j^2)^2 \).

(20) Page 24, above appendix A: the following acknowledge should have been added at the end of the conclusions and before the appendices: ‘This research was supported in part by...’
Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

(21) Page 33, reference [33]: the second reference should have been replaced with ‘Phys. Rev. D 47 2234’.
On the thermodynamic stability of rotating black holes in higher dimensions—a comparison of thermodynamic ensembles

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Abstract
Thermodynamic potentials relevant to the microcanonical, the canonical and the grand canonical ensembles, associated with rotating black holes in D-dimensions, are analysed and compared. Such black holes are known to be thermodynamically unstable, but the instability is a consequence of a subtle interplay between specific heats and the moments of inertia and it manifests itself differently in the different ensembles. A simple relation between the product of the specific heat and the determinant of the moment of inertia in both the canonical and the grand canonical ensembles is derived. Myers–Perry black holes in arbitrary dimension are studied in detail. All temperature extrema in the microcanonical ensemble are determined and classified. The specific heat and the moment of inertia tensor are evaluated in both the canonical and the grand canonical ensembles in any dimension. All zeros and poles of the specific heats, as a function of the angular momenta, are determined and the eigenvalues of the isentropic moment of inertia tensor are studied and classified. It is further shown that many of the thermodynamic properties of a Myers–Perry black hole in $D - 2$ dimensions can be obtained from those of a black hole in $D$ dimensions by sending one of the angular momenta to infinity.

Keywords: black holes, higher dimensions, rotation, thermodynamics
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(Some figures may appear in colour only in the online journal)
1. Introduction

An immediate consequence of Hawking’s famous result that Schwarzschild black holes in four dimensions have a temperature that is inversely proportional to their mass [1] is that any such black hole is thermodynamically unstable, due to a negative specific heat. They can however be stabilized by putting them in a box with thermal walls [2] or by introducing a negative cosmological constant of sufficient magnitude [3]. The specific heat of a Schwarzschild black hole can also be rendered positive by making it rotate sufficiently fast, but this does not completely stabilize the black hole as the moment of inertia then becomes negative.

Such an instability is not a disaster: it is often possible, and even useful, to describe thermodynamically unstable systems using the framework of thermodynamics, provided the time-scale of the instability is long enough. For example an ice-cube in a glass of water at room temperature is thermodynamically unstable, nevertheless the melting process can be understood using thermodynamic concepts. Similarly a Schwarzschild black hole is unstable against Hawking radiation, nevertheless the evaporation process, at least the initial stages for a black hole with a mass that is not too small, can be understood from a thermodynamic point of view.

It was shown in [4], utilizing only the Smarr relation and the first law, that all asymptotically flat electrically neutral solutions of the vacuum Einstein equations in $D$-dimensions are thermodynamically unstable. However, as we shall see, this thermodynamic instability manifests itself differently in different thermodynamic ensembles.

The generalization of the Kerr metric to a class of rotating black holes in $D$-dimensions, found by Myers and Perry [5], provides an arena for testing these ideas in a more general context. Higher dimensional black holes are also of central importance in string theory and Kaluza–Klein compactifications as well as in the gauge–gravity correspondence and the AdS-CFT approach to conformal field theories in general.

In four dimensions there is a maximum angular momentum that a rotating black hole can sustain, corresponding to an extremal black hole with vanishing Hawking temperature, but in higher dimensions the situation is more subtle. There is more than one angular momentum in $D > 4$, corresponding to the fact that the rank of $SO(D - 1)$ is greater than one for $D > 4$, and some, but not all, of the angular momenta can become arbitrarily large—the phenomenon of ultra-spinning black holes [5]. Infinite momentum does not however imply infinite angular velocity, rather the corresponding angular velocity vanishes as an angular momentum diverges—the infinite angular momentum is due to a singularity in the moment of inertia of the black hole and is not due to infinite angular velocity.

The thermodynamic quantities of interest are the mass (internal energy),\(^{1}\) the entropy, the temperature, the angular momenta and the angular velocities. In units with Newton’s constant, the speed of light, Planck’s constant and Boltzmann’s constant set to unity (as will be assumed throughout) all thermodynamic quantities have dimensions of length to some power:

| Quantity              | Dimension  |
|-----------------------|------------|
| Mass, $M$             | $D - 3$    |
| Entropy, $S$ (area)   | $D - 2$    |
| Angular momenta, $J^i$| $D - 2$    |
| Temperature, $T$      | $-1$       |

\(^{1}\) When there is a non-zero cosmological constant it is argued in [6] that the mass is more correctly thought of as the enthalpy rather than the internal energy. These are of course the same for zero pressure and in this work we make no distinction between enthalpy and internal energy.
Angular velocity, $\Omega_i$

It is thus natural to consider $M$, $S$, and $J_i$ to be extensive while $T$ and $\Omega_i$ are intensive, and this classification will be adopted here.

In section 2.4 a general relation between the canonical and the grand canonical ensembles, for electrically neutral rotating black holes, is derived. For the canonical ensemble the Hessian, $\partial^2 U$, of the internal energy $U (J_i, S)$ is shown to have determinant

$$\det(\partial^2 U) = \frac{1}{\beta C_J \det I_T},$$

where $C_J$ is the specific heat at constant angular momentum and $I_T$ is the isothermal moment of inertia tensor. For the grand canonical ensemble, on the other hand, the Hessian $-\partial^2 G$ associated with the grand canonical potential, $G (\Omega_i, T)$, is shown to satisfy

$$\det(-\partial^2 G) = \beta C_{\Omega} \det I_S,$$

where $C_{\Omega}$ is the specific heat at constant angular velocity and $I_S$ is the isentropic moment of inertia tensor. Standard thermodynamics arguments then imply that

$$C_J \det I_T = C_{\Omega} \det I_S,$$

which is one of our main results.

We then compare the thermodynamics of Myers–Perry black holes in the different ensembles. In one of the first papers on the stability of Myers–Perry black holes [7] it was observed that, as one of the angular momenta is increased keeping the others zero and the mass fixed (the microcanonical ensemble), there is a minimum in the temperature. The authors suggested that this was a signal of an instability—that there should be dynamical negative modes leading to a more stable solution of Einstein’s equations, but with less symmetry. This gave further support, beyond the non-rotating case studied in [8], to the idea that thermodynamic and dynamical instability of black holes in higher dimensions are intimately related. Hints of the instability can be seen in the microcanonical ensemble in which the entropy, which is a monotonically decreasing concave function of angular momenta, at constant mass, until the temperature hits a minimum at which point the entropy has an inflection point and becomes concave in at least one direction [9].

Thermodynamic instability manifests itself in different ways in the various ensembles. In the grand canonical ensemble the grand canonical potential $G (\Omega_i, T)$ is considered as a function of intensive variables and thermodynamic stability requires that $G$ be a totally concave function of its arguments [10]. The particular cases of asymptotically flat Kerr and Myers–Perry black holes were investigated in [11] for $4 \leq D \leq 6$ and it was shown that the specific heat $C_J$ is negative when all angular momenta vanish, but can become positive when some of the angular momenta become large enough. However when the angular momenta are large enough for the specific heat to be positive the isothermal moment of inertia, all of whose eigenvalues are positive for zero rotation, has at least one negative eigenvalue—there is thus always an instability unless the angular momenta are constrained. One of the results of the present work is to extend the explicit analysis of [11] to all $D$ and show that the same phenomenon persists.

The relationship between the microcanonical and the grand canonical ensembles was studied in [12] and in section 3.4.4 we extend this analysis further and derive a number of
relations between the temperature, the specific heat at constant angular velocity $C_\beta$, and the isentropic moment of inertia tensor $I_\beta$ for Myers–Perry black holes in any dimension. We show that there is a branched hypersurface in angular momentum space where $\beta C_\beta$ (with $\beta = \frac{1}{T}$) develops a pole and that this is the same hypersurface as the one on which the temperature is minimized in the microcanonical ensemble. This hypersurface can be obtained from the extremal $T = 0$ hypersurface by analytically continuing $J_i^2$ to $-(J_i^2)$, keeping the entropy constant. There is yet another significant hypersurface, one with a number of branches on which $\beta C_\beta$ vanishes, and on this hypersurface the isentropic moment of inertia tensor develops an infinite eigenvalue, in the form of a pole. This pole exactly cancels the zero in $\beta C_\beta$ in the Hessian of the grand canonical potential. The branches of this hypersurface divide the space of angular momenta into separate regions determined by the signature of $I_\beta$.

A by product of our analysis is that the thermodynamic properties of a Myers–Perry black hole in $D - 2m$ dimensions, in the micro or the grand canonical ensemble, can be obtained from those of a Myers–Perry black hole in $D$ dimensions by sending $m$ of the angular momenta to infinity in the latter.

There are of course intermediate ensembles between the canonical and the grand canonical cases, the former uses purely extensive thermodynamic parameters and the latter purely intensive, but one can have a mixed set of parameters some being extensive and some intensive. Indeed by imposing constraints and fixing some of the parameters a configuration that is unstable without the constraints can be rendered stable. For example fixing the angular momentum as a constraint means that one need not worry about negative eigenvalues in the moment of inertia tensor and positivity of $C_J$ is sufficient for local thermodynamic stability, as in the analysis of [13] for example.

In section 2 the thermodynamics of rotating black holes in the different ensembles are analysed and equation (3) derived along with other relations between the various thermodynamic quantities. In section 3 Myers–Perry black holes are studied and it is shown explicitly how the specific heats and moments of inertia conspire to satisfy the general relations of section 2. The results are summarized in section 4 and some technical results required in the analysis are relegated to five appendices.

2. Thermodynamics of rotating black holes

Rotating black holes in $D$ space–time dimensions must be treated slightly differently for even and odd $D$ because the rotation group $SO(D - 1)$ has different characterizations of angular momenta in the even and odd dimensional cases. The Cartan sub-algebra has dimension $\frac{D-2}{2}$ for even $D$ and $\frac{D-1}{2}$ for odd $D$ so a general state of rotation is specified by $\frac{D-2}{2}$ independent angular momenta in even $D$ and $\frac{D-1}{2}$ in odd $D$. Let $N = \left\lfloor \frac{D-1}{2} \right\rfloor$, the integral part of $\frac{D-1}{2}$, be the dimension of the Cartan sub-algebra of $SO(D - 1)$, then there are $N$ independent angular momenta $J_i$, $i = 1, \ldots, N$. It is notationally convenient to introduce a parameter $\epsilon = (1 + (-1)^D)/2$ in terms of which

$$N = \frac{D - 1 - \epsilon}{2}.$$ (4)
In the microcanonical ensemble the energy is fixed and we chose as thermodynamic control parameters the extensive quantities, $J^i$ and $M$, with the entropy $S(J^i, M)$ being the thermodynamic potential, which is convenient for differentiation keeping $M$ fixed. In the canonical ensemble the energy is allowed to fluctuate and the internal energy

$$U(J, S) = M,$$

is used as the thermodynamic potential. In the grand canonical ensemble all extensive variables are allowed to fluctuate and the intensive variables are used as control parameters, the relevant thermodynamic potential is then the grand canonical potential

$$G(\Omega, T) = U - TS - \Omega J^i.$$

The grand canonical potential is related to the Euclidean formulation, since the Euclidean action $I_E$ is related to the mass by a Legendre transform

$$I_E = M - TS - \Omega J^i \Rightarrow I_E = G(\Omega, T).$$

### 2.1. Microcanonical ensemble

For completeness we summarize in this sub-section some of the results pertaining to the microcanonical analysis in [14].

With the entropy expressed as a function of $J^i$ and $M, S(J, M)$, we have

$$\frac{\partial S}{\partial J^i} \bigg|_M = -\beta \Omega_i, \quad \frac{\partial S}{\partial M} \bigg|_{J^i} = \beta,$$

where $\beta = \frac{1}{T}$. These thermodynamic quantities have a geometrical interpretation in the Euclidean formulation of the black hole, where demanding absence of a conical singularity requires periodicity in imaginary time, $\tau = it$, with $\tau$ identified with $\tau + \beta$, and periodicity in imaginary angle [14], $\phi = i\phi$, with $\phi$ identified with $\phi - \beta \Omega_i$. Thus the 1-form $dS$ determines the size of the $(\tau, \phi)$ torus,

$$\tau dM + \phi dJ^i \sim \tau dM + \phi dJ^i + dS.$$

For thermodynamic stability the entropy should be purely concave [10], and a necessary (but not sufficient) condition for this is that the Hessian $H_{ab} = -\frac{\partial^2 S}{\partial a \partial b}$, where $S^A = (J^i, M)$, must be a positive definite matrix. The identity

$$\det(H_{ab}) = -\frac{1}{(D-3)MT} \det(H_y),$$

for asymptotically flat black holes, is derived in [4], where $H_y$ is the $N \times N$ matrix

$$H_y = -\left(\frac{\partial^2 S}{\partial J^i \partial J^j}\right)_M.$$

One can immediately conclude that such black holes can never be thermodynamically stable in $D \geq 4$, since $\det(H_{ab}) > 0$ requires $\det(H_y) < 0$, hence at least one eigenvalue of $H_y$ would have to be negative and $S$ cannot be a concave function.
2.2. Canonical ensemble

The canonical ensemble uses the internal energy $U(J, S)$ as thermodynamic potential, depending on extensive arguments that are the Legendre transforms of $\Omega$ and $T$, and for black holes $U$ is identified with the ADM mass, at least in the asymptotically flat case. Stability requires that $U$ be a totally convex function of its arguments. Let $x^A = (J, S)$, with $A = 1, \ldots, N + 1$, $x^I = J$ and $x^{N+1} = S$, then

$$T = \frac{\partial U}{\partial S} \bigg|_J, \quad \Omega = \frac{\partial U}{\partial J} \bigg|_S$$

(12)

and $U_{AB} = \frac{\partial^2 U}{\partial x_A \partial x_B}$ must be a positive matrix. Explicitly

$$U_{AB} = \begin{pmatrix} \frac{\partial^2 U}{\partial J^I \partial J^J} & \frac{\partial^2 U}{\partial J^I \partial S} \\ \frac{\partial^2 U}{\partial S \partial J^I} & \frac{\partial^2 U}{\partial S^2} \end{pmatrix} = \begin{pmatrix} (T^{-1})^{JI} & \zeta_i \\ \zeta_j & (\beta C_j)^{-1} \end{pmatrix},$$

(13)

where $\beta = \frac{1}{T}$; the symmetric matrix

$$(T^{-1})^{JI} = \frac{\partial^2 U}{\partial J^I \partial J^J} = \frac{\partial \Omega_i}{\partial J^J}$$

(14)

is the inverse of the isentropic moment of inertia tensor $I^{II} = \frac{\partial^2 U}{\partial x_A \partial x_B}$; and $C_j$ is the specific heat at constant $J$,

$$C_j = \left( \frac{\partial U}{\partial T} \right)_J = T \left( \frac{\partial S}{\partial T} \right)_J$$

(15)

and

$$\zeta_i = \frac{\partial \Omega_i}{\partial S} \bigg|_J = \frac{\partial T}{\partial J^I} \bigg|_S$$

(16)

(this last equation is a Maxwell relation).

A necessary, but not sufficient, condition for stability is thus

$$\det(\partial^2 U) = \frac{\det\left( I^{-1} - \beta C_j \zeta \zeta^T \right)}{\beta C_j} > 0.$$ 

(17)

2.3. Grand canonical ensemble

In the grand canonical ensemble stability requires that $G(\Omega, T)$ be a concave function. Let $y^A = (\Omega, T)$, with $y^i = \Omega$, and $y^{N+1} = T$, then

$$S = -\frac{\partial G}{\partial T} \bigg|_{y^i}, \quad J^i = -\frac{\partial G}{\partial \Omega} \bigg|_{y^i}$$

(18)
and $G^{AB} = \frac{\delta^2 G}{\delta_{x} \delta_{y}}$ must be a negative matrix. Explicitly

$$
-G^{AB} = \begin{pmatrix}
- \frac{\partial^2 G}{\partial \Omega \partial \Omega} & - \frac{\partial^2 G}{\partial \Omega \partial T} \\
- \frac{\partial^2 G}{\partial T \partial \Omega} & - \frac{\partial^2 G}{\partial T \partial T}
\end{pmatrix} = \begin{pmatrix} I^y_T \eta' \\ \eta' \beta C_\Omega \end{pmatrix}.
$$

(19)

where the symmetric matrix

$$
I^y_T = - \frac{\partial^2 G}{\partial \Omega \partial \Omega} = \frac{\partial J^I}{\partial \Omega} |_T
$$

(20)

is the isothermal moment of inertia tensor; $C_\Omega$ is the specific heat at constant $\Omega$,

$$
C_\Omega = T \left( \frac{\partial S}{\partial T} \right)_\Omega = - T \left( \frac{\partial^2 G}{\partial T^2} \right)_\Omega
$$

(21)

and

$$
\eta' = \frac{\partial J^I}{\partial T} |_\Omega = \frac{\partial S}{\partial \Omega} |_T.
$$

(22)

A necessary, but not sufficient, condition for stability is thus

$$
det \left( - \frac{\partial^2 G}{\partial \Omega} \right) = \beta C_\Omega \det \left( I^y_T - \left( \beta C_\Omega \right)^{-1} \eta' \eta \right) > 0.
$$

(23)

2.4. Relation between the canonical and the grand canonical ensembles

The canonical and the grand canonical ensembles are of course related. An immediate consequence of the Legendre transform (6) is that

$$
-G^{AB} = \left( U^{-1} \right)^{AB}
$$

(24)

and this has important consequences for the individual components.

A relation between the specific heats was derived in [11],

$$
\begin{align*}
\frac{\partial S}{\partial T} |_\Omega & = \frac{\partial S}{\partial T} |_0 + \frac{\partial S}{\partial T} |_T \frac{\partial J^T}{\partial T} |_\Omega = \frac{\partial S}{\partial T} |_0 + \frac{\partial S}{\partial \Omega} |_T \frac{\partial \Omega}{\partial T} |_T \\
& = \frac{\partial S}{\partial T} |_T + \left( I^y_T \right)_\eta' \eta
\end{align*}
$$

(25)

$$
\Rightarrow \beta C_\Omega = \beta C_i + \left( I^y_T \right)_\eta' \eta' \eta
$$

(26)

$$
where the Maxwell relation (22) has been used.

Similar manipulations can be used to relate the isentropic and isothermal moment of inertia tensors,

$$
I^y_T = I^y_2 + \left( \beta C_\Omega \right)^{-1} \eta' \eta'.
$$

(28)
Or equivalently,
\[ (I^{-1}_S)_{ij} = (I^{-1}_T)_{ij} + \beta C_j \zeta_i \zeta_j \]  
(29)

The stability conditions (17) and (23) can thus be expressed as
\[ \det(\partial^2 U) = \frac{1}{\beta C_j \det(I_T)} > 0 \]  
(30)

and
\[ \det(-\partial^2 G) = \beta C_j \det(I_S) > 0. \]  
(31)

Equation (24) now gives the identity
\[ \beta C_j \det(I_S) = \beta C_j \det(I_T). \]  
(32)

A new instability would be expected to develop every time one of the eigenvalues of \(-\partial^2 \beta G\) changes from a positive to a negative value, either by going through zero or infinity. In general this might be expected to happen on a hypersurface on which \(\det(-\partial^2 G)\) is either zero or infinity, but we shall see that, at least in the case of Myers–Perry black holes, there are some subtle cancellations so that the change is not reflected in the determinant.

The form of the Hessians (13) and (19) can be simplified by using a Legendre transform on the scalar variable, respectively \(S\) and \(T\). In the canonical ensemble let \(x^A = (J', T)\) and \(F(J', T) = U - TS. \) In the canonical ensemble let \(x^A = (J', T)\) and \(F(J', T) = U - TS.\) (33)

Then the co-ordinate transformation matrix is
\[ \frac{\partial x^A}{\partial x^B} = \begin{pmatrix} \delta^i_j & 0 \\ \zeta^i_j & (\beta C_j)^{-1} \end{pmatrix} \]  
(34)

with inverse
\[ \frac{\partial x^A}{\partial x^B} = \begin{pmatrix} \delta^i_j & 0 \\ -\beta C_j \zeta_i & \beta C_j \end{pmatrix} \]  
(35)

So, in \((J', T)\) co-ordinates,
\[ U_{AB} = U_{CD} \frac{\partial x^C}{\partial x^A} \frac{\partial x^D}{\partial x^B} = \begin{pmatrix} (I^T_T)^{-1} & 0 \\ 0 & \beta C_j \end{pmatrix} \]  
(36)

is partially diagonalized\(^2\).

Similarly, in the grand canonical ensemble, we can transform from \(y^A = (\Omega, T)\) to \(y^A = (\Omega, S)\) to get
\[ -G^{AB} = -G^{CD} \frac{\partial x^C}{\partial x^A} \frac{\partial x^D}{\partial x^B} = \begin{pmatrix} (I^S_S)^{-1} & 0 \\ 0 & (\beta C_j)^{-1} \end{pmatrix} \]  
(37)

\(^2\) Of course \(U_{AB} \neq \partial^2 F / \partial x^A \partial x^B.\) The stability properties of the canonical ensemble are determined by the signature of the Hessian \(U_{AB},\) which is the same as that of \(U_{AB}.\) The matrix \(\partial^2 F / \partial x^A \partial x^B\) has a different signature.
Note that, although the canonical ensemble implicitly involves $I^{-1}$, its stability properties are most easily seen using $I^{-1}$ in (36) and, while the grand canonical ensemble implicitly involves $I^{-1}$, its stability properties are most easily studied using $I^{-1}$ in (37).

3. Myers–Perry black holes

Myers–Perry black holes in $D$ space-time dimensions are assumed to have an event horizon which has the topology of a $(D-2)$ dimensional sphere. This can be described in terms of Cartesian co-ordinates $x_a$ in $D^{D-1}$ by

$$\sum_{a=1}^{D-1} x_a^2 = 1,$$

and we can write this as

$$\sum_{i=1}^{N} \rho_i^2 + \epsilon y_i^2 = 1,$$

where $x_{2i-1} + ix_{2i} = \rho_i e^{i\phi_i}, i = 1, \ldots, N$, are complex co-ordinates for both the even and odd cases while $y = x_{D-1}$ is only present for even $D$.

Then $\rho$, $\phi$ and $y$ are co-ordinates that can be used to parameterize the sphere and, for the black hole, $J_i$ are angular momenta in the $(x_{2i-1}, x_{2i})$ plane.

The Myers–Perry line element can be expressed as

$$ds^2 = -dr^2 + \frac{2\mu}{U} \left( dt - \sum_{i=1}^{N} \rho_i^2 d\phi_i \right)^2 + \left( \frac{U}{Z - 2\mu} \right) dr^2 + \epsilon r^2 dy^2 + \sum_{i=1}^{N} (r^2 + a_i^2) \left( d\rho_i^2 + \rho_i^2 d\phi_i^2 \right),$$

where the functions $Z$ and $U$ are

$$Z = \frac{1}{r^{2\epsilon}} \prod_{i=1}^{N} \left( r^2 + a_i^2 \right),$$

$$U = Z \left( 1 - \sum_{i=1}^{N} \frac{a_i^2 \rho_i^2}{r^2 + a_i^2} \right).$$

The $a_i$ are rotation parameters in the $(x_{2i-1}, x_{2i})$-plane and $\mu$ is a mass parameter. We use units in which the $D$-dimensional Newton’s constant and the speed of light are set to one.

There is an event horizon at $r_h$, the largest root of $Z = 2\mu = 0$, so

$$\mu = \frac{1}{2r_h^{2\epsilon}} \prod_{i=1}^{N} \left( r_i^2 + a_i^2 \right).$$

Hereinafter we shall not distinguish between upper and lower indices, $I = I'$.  

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3 Hereinafter we shall not distinguish between upper and lower indices, $I = I'$.  

and the area of the event horizon is
\[ \mathcal{A}_h = \frac{\sigma}{r_h^{D-1}} \prod_{i=1}^{N} \left( r_i^2 + a_i^2 \right), \] (42)
where \( \sigma \) is the volume of the round unit \( (D - 2) \)-sphere,
\[ \sigma = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \]. (43)

The Bekenstein–Hawking entropy is
\[ S = \frac{\sigma}{4r_h^{D-1}} \prod_{i=1}^{N} \left( r_i^2 + a_i^2 \right) \] (44)
and the Hawking temperature is
\[ T = \frac{1}{4\pi r_h} \left( D - 3 - 2 \sum_{i=1}^{N} \frac{a_i^2}{r_i^2 + a_i^2} \right) \] (45)

The angular momenta, the entropy and the ADM mass, \( M \), of the black hole are related to each other, and to the metric parameters, via
\[ J_i = \frac{2M a_i}{D - 2} = \frac{\mu}{4\pi} \sigma a_i, \quad M = \frac{(D - 2)\sigma \mu}{8\pi} = \frac{(D - 2)S}{4\pi r_h}, \] (46)
while the angular velocities are
\[ \Omega_i = \frac{a_i}{(r_i^2 + a_i^2)}. \] (47)

3.1. Microcanonical ensemble

The microcanonical ensemble was developed for Myers–Perry black holes in [4, 14, 15]. In particular the Hessian (11) was evaluated explicitly in [14] and is reproduced in appendix A,
\[ H_{ij} = \frac{(D - 2)}{2r_h^2 T M} \left\{ \frac{1}{1 + j_i^2} \right\} \delta_{ij} + \frac{\omega_i \omega_j}{\pi r_h T} \left( \frac{1}{1 + j_i^2} \right) \]
\[ + \frac{1}{1 + j_i^2} - \frac{1}{2} + \frac{2}{t} \Omega^2 \left( \Omega^2 \right) \] (48)
where \( j = \frac{2\pi}{3} \), \( \omega_i = \frac{\lambda}{1 + j_i} \) are dimensionless angular velocities, and \( \Omega^2 = \sum_{i=1}^{N} \omega_i^2 \).

Black hole thermodynamics in \( D > 4 \) has a subtle relation with dynamical instability. It was noted in [7] that, for \( D \geq 6 \), the temperature of a Myers–Perry black hole, with only one \( J' = 0 \), has a minimum as the spin increases at fixed mass. Taking \( J'' \neq 0 \) and \( J' = 0 \) for \( i = 2, \ldots, N \) the minimum is at \( \frac{\rho^2}{\omega_i^2} = \frac{D - 3}{D - 2} \) and in [7] it was suggested that this minimum signals the onset of a dynamical instability for a rotating black hole. Thermodynamic functions are thus giving hints of possible dynamical instability and this was studied in [4] and
where some special cases of non-zero spin were analysed. These authors studied the matrix (48) in the symmetric cases where the non-zero \( a_i \) are all equal,

\[
a_1 = \ldots = a_n = a \neq 0, \quad a_i = 0 \quad \text{for} \quad i = n, \ldots, N. \tag{49}
\]

The entropy and the temperature decrease as the angular momenta are increased, at constant \( M \), until the temperature reaches a minimum, and at precisely that point the matrix \( H_{ij} \) develops a zero eigenvalue, signalling the fact that the entropy ceases to be concave in that direction. The temperature has a minimum for configurations of the form (49) with

\[
a^2 = \frac{D - 3}{D - 3 - 2n}. \tag{50}
\]

In particular \( n = 1 \) gives the original expression in [7], and the two special cases \( n = 1 \) and \( n = N \) were considered in [14], while (50) for general \( n \) appeared in [12].

In appendix A all extrema of the temperature, in the microcanonical ensemble with fixed mass,

\[
\left. \frac{\partial T}{\partial J^i} \right|_{J_i, J_s} = 0, \tag{51}
\]

are found and classified. For finite \( J^i \) they are all of the same from as (49),

\[
J^1 = \ldots = J^n = J_s, \quad J^{n+1} = \ldots = J^N = 0, \tag{52}
\]

(up to permutations of the \( J^i \)) with

\[
J_s^2 = \frac{D - 3}{D - 3 - 2n}, \tag{53}
\]

where \( J_s = \frac{2\nu}{\pi} \) are dimensionless angular momentum, in units of entropy.

The value of the temperature at the extrema (53) is

\[
T_s = \frac{1}{4\pi s} \frac{(D - 3)(D - 3 - 2n)}{(D - 3 - n)}. \tag{54}
\]

The temperature is a maximum, \( T_{\max} = \frac{a^2}{4\pi s} \), for non-rotating Schwarzschild–Tangherlini black holes \( n = 0 \). For finite \( J^i \) the stationary points \( J_s \) are saddle points with minima along the directions \( J_s = (J_s, \ldots, J_s, 0, \ldots, 0) \) satisfying (53) and maxima in the directions orthogonal to these. At the same time \( H_{ij} \) in equation (48) develops a single zero eigenvalue at \( J_s \) in the direction \( J_s \), indicating an inflection point in that direction\(^4\). There are also \( n - 1 \) negative eigenvalues of \( H_{ij} \) at \( J_s \), (A.38), indicating convexity of the entropy in these directions with associated thermodynamic instabilities, while the entropy is concave in all other finite directions. At all stationary points of \( T \), \( \det \left( H_{ij} \right) \) vanishes.

When some number \( m \) of the \( J^i \) are allowed to become infinite equations (53) and (54) are modified to

\(^4\) It is argued in [15] that this point of inflection is not in itself necessarily a sign of dynamical instability: it can indicate a zero mode, taking one Myers–Perry solution into another, rather than a negative mode dynamical instability.
respectively. Indeed in many of the following formulae the thermodynamic properties of a Myers–Perry black hole in $D$ dimensions, with $m$ angular momenta sent to infinity, are seen to be the same as those of a $D - 2m$ dimensional black hole with all angular momenta finite and the moment of inertia tensor, which has zero eigenvalues in the infinite directions, suitably truncated (a caveat to this statement is that we must restrict to $m < \frac{D-3}{4}$, equation (A.11)). This can be seen in the formulae in the appendix, though $C_j$ and $\det(I_j)$ are exceptions and so thermodynamic dimensional reduction using this limit does not work in the canonical ensemble. In this sense lower dimensional black holes can be obtained by starting from large $D$ and sending more and more of the $J_i$ to infinity.

### 3.2. Heat capacities

The heat capacity at constant $J$ is derived in appendix B. It can be expressed fairly concisely by using the functions

$$
\Sigma_{a}^{\pm} = \sum_{j=1}^{N} \frac{j^{2}}{1 \pm j^{2}} \pi^{n}.
$$

The specific heat at constant $J$ is then

$$
C_{j} = \frac{4\pi r_{c} M t}{\left[ t^{2} - (D - 2)(t - 4\Sigma_{a}^{\pm}) \right]},
$$

where

$$
t = D - 3 - 2\Sigma_{a}^{\pm} = 4\pi r_{c} T.
$$

Equation (56) generalizes the formulae for the specific cases $D = 4, 5$ and $6$, given in [11], to arbitrary $D$.

As is well known the specific is negative for $J = 0$,

$$
C_{j=0} = -4\pi r_{c} M,
$$

but can be positive for non-zero $J$.

The specific heat at constant $J$ (the canonical ensemble) is related to the specific heat at constant $\Omega$ (the grand canonical ensemble) by equation (27). Alternatively $C_{\Omega}$ can be evaluated directly for a general $D$ without knowing the moment of inertia explicitly. The details are left to appendix C and here we just quote the result,

$$
C_{\Omega} = \frac{4\pi r_{c} M t (D - 2 + 2\Sigma_{a}^{\pm})}{(D - 2)(D - 3 + 2\Sigma_{a}^{\pm})},
$$

which generalizes the $D = 4$ result of [16] to $D \geq 4$. 

To simplify some later formulae it will be convenient to define, in analogy with (57),

\[ \mathcal{I} = D - 3 + 2 \sum_{i=1}^{N} \frac{j_i^2}{1 - j_i^2} = D - 3 + 2\Sigma_i^-, \]

in terms of which

\[ C_o = -\frac{4\pi r_o M (D - 2 + 2\Sigma_i^-) t}{(D - 2)\mathcal{I}}. \]

Note the signs: in this notation

\[ \Sigma_i^- (j_i^2) = -\Sigma_i^+ (-j_i^2), \quad \mathcal{I} (j_i^2) = \mathcal{I} (-j_i^2). \] (60)

There is a curious parallel between the singularities of \( \beta \mathcal{C}_{\mu\nu} \), where \( \mathcal{I} = 0 \), and extremal black holes for which \( \mathcal{I} \), and hence \( T \), vanishes. Since \( \mathcal{I} (j_i^2) = \mathcal{I} (-j_i^2) \) these are related by mapping \( (J_i^2) \to - (J_i^2) \), keeping the entropy constant.

### 3.3. Moment of inertia tensor

The isothermal moment of inertia tensor,

\[ I_T^{ij} = \left( \frac{\partial \mathcal{I}}{\partial \Omega_j} \right)_T, \] (61)

is derived in appendix D. It is

\[ I_T^{ij} = \frac{2Mr_o^2}{D - 2} \left\{ \frac{(1 + j_i^2)^2}{(1 - j_i^2)} \delta_{ij} - 2jj_j \left( 1 + \frac{4}{t(1 - j_i^2)(1 - j_j^2)} \right) \right\}. \] (62)

Equation (62) generalizes the formulae for the particular cases \( D = 4, 5 \) and 6 derived in [11].

The isentropic moment of inertia tensor

\[ I_s^{ij} = \left( \frac{\partial \mathcal{I}}{\partial \Omega_j} \right)_s \] (63)

was given for general \( D \) in [17]. A derivation is outlined in appendix E and it has the form

\[ I_s^{ij} = \frac{2Mr_o^2}{(D - 2)} \left\{ \frac{(1 + j_i^2)^2}{(1 - j_i^2)} \delta_{ij} - \frac{2jj_j}{(D - 2 + 2\Sigma_i^-)} \left( 1 + j_i^2 \right) \left( 1 + j_j^2 \right) \right\}. \] (64)

The determinant of the isentropic moment of inertia tensor is

\[ \det I_s = \left( \frac{2Mr_o^2}{D - 2} \right)^N \frac{(D - 2)}{(D - 2 + 2\Sigma_i^-)^N} \prod_{i=1}^{N} \frac{(1 + j_i^2)^2}{(1 - j_i^2)}. \] (65)

Note that, in the determinant of the Hessian for the grand canonical ensemble (2), the factor \( D - 2 + 2\Sigma_i^- \) in the denominator of (65) exactly cancels the same factor in the numerator of (59).
3.4. Stability analysis in the canonical and grand canonical ensembles

In this section we examine the thermodynamic stability of Myers–Perry black holes in the canonical and the grand canonical ensembles, using the formulae of sections 3.2 and 3.3. We first summarize the well known case of $D = 4$ and the results of [11] for $D = 5$ and 6, before going on to describe the situation for general $D$.

3.4.1. $D = 4$. The case $D = 4$ is well known, but is included here for completeness. In four dimensions $N = 1$ and there is only one $J$. The temperature is

$$T = \frac{1}{4\pi r_h} \left( \frac{1 - j^2}{1 + j^2} \right)$$

so we must restrict to $0 \leq j \leq 1$, with $j = 1$ being extremal. $C_J$ evaluates to

$$C_J = -2\pi r_h^4 \left( \frac{1 - j^2}{1 + j^2}. \right) \left( \frac{1 + j^4}{1 - 6j^2 - 3j^4} \right)$$

which is positive for $\frac{2}{\sqrt{3}} - 1 < j^2 < 1$, (in terms of $J$ and $M$,

$$\frac{J^2}{M^4} = \frac{4j^2}{(1 + j^2)^2}. \quad (68)$$

and these limits corresponding to $2\sqrt{3} - 3 < j^2 < 1$.

The isothermal moment of inertia is

$$I_r := \frac{r_h^3}{2} \left( 1 - 6j^2 - 3j^4 \right), \quad (69)$$

which is only positive when $C_J$ is negative. Indeed

$$\beta C_J I_r = -4\pi^2 r_h^6 \left( 1 + j^2 \right)^3,$n

(70)

clearly illustrating that Kerr metrics are thermodynamically unstable in the grand canonical ensemble for all values of the angular momentum: when the specific heat is positive the moment of inertia is negative and vice versa. Note that the pole in $C_J$ exactly cancels a zero in $\Omega_r$—a phenomenon that we shall see persists for all $D$.

Equation (32) immediately shows that an instability must be present in the canonical ensemble, though the full story is a little simpler there. Explicitly

$$\beta C_{\mu} I_r = -8\pi^2 r_h^6 \left( 1 + j^2 \right)^3,$n

(71)

and indeed $\beta C_J I_r = \beta C_{\mu} I_r$ as it should be, even though the instability can shift between the specific heat and the moment of inertia in the former case while it always resides in the specific heat in the latter, the isentropic moment of inertia always being positive.

3.4.2. $D = 5$. In five dimensions the Hawking temperature is

$$T = \frac{1}{2\pi r_h} \left( \frac{1 - j_1^2 j_2^2}{(1 + j_1^2)(1 + j_2^2)} \right),$$

so $j_1^2 j_2^2 \leq 1$, with the locus of extremal black holes being the hyperbolae $j_1^2 j_2^2 = 1$. 

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The specific heat at constant $J$ and the isothermal moment of inertia tensor are easily determined from the general formulae in section 2, with $N = 2$, but the explicit forms are not illuminating and we shall resort to a graphical representation. The specific heat is positive in the region of the $j_1 - j_2$ plane indicated in figure 1: it diverges on the boundary of the red inner region and vanishes on the outer hyperbolae (the latter being the $T = 0$ curve); and is positive in the yellow region enclosed by the curves.

The eigenvalues of the isothermal moment of inertia tensor are plotted in figure 2. Both eigenvalues are positive for small $j_1$, and one is always positive, but the other vanishes on the same curve that bounds the red region in figure 1 and is negative outside this region. The innermost surface on which the moment of inertia tensor develops a negative eigenvalue is termed the ultra-spinning surface in reference [14] and it was shown there that there is no ultra-spinning surface in the microcanonical ensemble for a singly spinning Myers–Perry black hole $D = 5$. In contrast we see here that there is an ultra-spinning surface for $I_T$ in the grand canonical ensemble—the concept of an ultra-spinning surface depends on the ensemble used.

Thermodynamic instability can nevertheless be seen directly from

$$C_j \det I_T = -\frac{3\pi^4}{32} \left( 1 + j_1^2 \right)^4 \left( 1 + j_2^2 \right)^4 < 0.$$  \hspace{3cm} (73)

$I_T$ has a negative eigenvalue when $C_j$ is positive and when $I_T$ has two positive eigenvalues, $C_j < 0$. Hence $C_j \det I_T$ is always negative for any black-hole, and so these black-holes are thermodynamically unstable for any choice of angular-momenta with positive temperature.
3.4.3. $D = 6$. In six dimensions the temperature

$$T = \frac{1}{4\pi r_0} \frac{\left( 3 + j_1^2 + j_2^2 - j_1^2 j_2^2 \right)}{\left( 1 + j_1^2 \right) \left( 1 + j_2^2 \right)}$$

(74)

again vanishes on hyperbolae in the $j_1 - j_2$ plane.

The specific heat at constant $J$ looks a little more complicated than in $5D$, but only because some of the hyperbolae overlap. Figure 3 displays similar information to figure 1, but

Figure 2. Eigenvalues of the isothermal moment of inertia tensor, $\mathbf{I}_T^{ij}$, for $D = 5$. The boundary of the central peak in the right-hand figure, the zero locus of this eigenvalue, coincides with the boundary between the red and yellow regions of $C_J$ in figure 1.

Figure 3. $C_J$ in 6-dimensions, with the same colour coding as in figure 1. (This figure is essentially the same as one in [11], using slightly different variables.)
for \( D = 6 \)—this figure is essentially the same as one in [11] and is reproduced here for comparison with figure 4.

The regions in the \( j_1-j_2 \) plane where the eigenvalues of the isothermal moment of inertia tensor take positive values are plotted in figure 4. The yellow cross-shaped shaded region in the left-hand graph of figure, where one of the eigenvalues of \( I_T \) is positive, exactly co-incides with the inner red region of \( C_J \) in figure 3, thus the product of these two quantities is always negative. The determinant

\[
\beta C_J \det I_T = -\frac{32}{27} \pi^5 r_0^{15} \left( \frac{1 + j_1^2}{3 - j_1^2 - j_2^2 - j_2^2} \right) \left( \frac{1 + j_2^2}{3 - j_1^2 - j_2^2 - j_2^2} \right),
\]

is thus negative in the central yellow region of the other eigenvalue of \( I_T \), indicated in the right-hand picture in figure 4, but it is positive in the red region of the right-hand figure, where both eigenvalues of \( I_T \) are negative and \( C_J > 0 \). While the determinant itself can positive there is no regime in which \( C_J \) and both eigenvalues of \( I_T \) are simultaneously positive. The grand canonical ensemble for six-dimensional Myers–Perry black holes is thus everywhere unstable.

### 3.4.4. General \( D \geq 7 \)

In sections 3.4.1–3.4.3 the stability properties of Myers–Perry black holes in the canonical ensemble were analysed in terms of \( C_J \) and isothermal moment of inertia, associated to the canonical ensemble through (30). We focus in this section on the grand canonical ensemble, partly because the canonical ensemble has already been analysed (albeit only for \( D = 4, 5, 6 \)) but primarily because it is algebraically somewhat simpler than the canonical ensemble. The general principles of section 2.4 ensure that the stability properties are the same: since \( \partial^\alpha \partial_\beta U \) and \( -\partial^\alpha \partial^\beta G \) are inverses of each other their signature is the same.
One necessary condition for thermodynamic stability is

\[ \beta C_\Omega = -\frac{16\pi^2 M r_+^2}{(D-2)} \left[ \frac{D - 2 + 2\Sigma_i^-}{D - 3 + 2\Sigma_i^-} \right] > 0, \tag{76} \]

in particular \( \beta C_\Omega \) is negative for non-rotating black holes with all \( j_i = 0 \). More generally we must examine the condition

\[ \frac{D - 2 + 2\Sigma_i^-}{D - 3 + 2\Sigma_i^-} < 0. \tag{77} \]

In terms of the variables

\[ x_i = \frac{j_i^2}{1 - j_i^2}, \tag{78} \]

(77) is a simple ratio of linear functions

\[ \frac{D - 2 + 2\sum_j^n x_j}{D - 3 + 2\sum_j^n x_j} < 0, \tag{79} \]

and positivity of \( \beta C_\Omega \) requires that

\[ -\frac{D - 2}{2} < \sum_{i=1}^N x_i < -\frac{D - 3}{2}, \tag{80} \]

i.e. the \( x \)'s are constrained to lie between two hyperplanes in \( x \)-space, which never intersect for finite \( x_i \). However \( x_i \) diverges when \( j_i^2 \) passes through 1, and this description pushes some subtleties around \( j_i^2 = 1 \) out to infinity.

So we consider instead the condition

\[ \frac{(D - 2 + 2\Sigma_i^-)}{(D - 3 + 2\Sigma_i^-)} \prod_{j=1}^2 \left( 1 - j_i^2 \right) < 0. \tag{81} \]

This ratio can only change sign either across the hypersurface

\[ (D - 2 + 2\Sigma_i^-) \prod_{i}^N \left( 1 - j_i^2 \right) = 0, \tag{82} \]

where it has a zero, or across the hypersurface

\[ (D - 3 + 2\Sigma_i^-) \prod_{i}^N \left( 1 - j_i^2 \right) = 0, \tag{83} \]

where it has a pole. Both these hypersurfaces are of the form

\[ C_{d,i} := (D - s) \prod_{i}^N \left( 1 - j_i^2 \right) + 2 \sum_{k=1}^N \left( j_k^2 \prod_{i \neq k}^N \left( 1 - j_i^2 \right) \right) = 0, \tag{84} \]

with \( s = 2 \) or 3. If any \( j_i^2 = 1 \), for example if \( j_1^2 = 1 \), then

\[ C_{d,i} = 2 \prod_{i=2}^N \left( 1 - j_i^2 \right) = 0, \tag{85} \]
and at least one other $j^i$ must be one, the remaining $j^i$'s, $N-2$ of them, are arbitrary. Indeed the hypersurfaces $C_{D,2}$ and $C_{D,3}$ intersect on a manifold of co-dimension 2, which is actually a flat $\mathbb{R}^{N-2}$ in $j$-space. In $D=7$ for example, the relevant hypersurfaces are $C_{7,2}$ and $C_{7,3}$, while in $D=8$ they are $C_{8,3}$ and $C_{8,2}$. These various hypersurfaces are shown in figure 5.

We can determine whether or not $\beta C_{D}$ changes sign across these hypersurfaces by following it out along rays from the origin in specific directions. For example in the direction $j_1 = j_2 = \cdots = j_N = 0$,
\[
C_{D,2} \frac{D - 2 - (D - 4)j^2}{D - 3 - (D - 5)j^2},
\]
which is negative between \(C_{D,2} = 0\) and \(C_{D,3} = 0\) where
\[
\frac{D - 2}{D - 4} < j^2 < \frac{D - 3}{D - 5}.
\]

Thus \(\beta C_{D}\) does indeed change sign when it crosses either of the hypersurface \(C_{D,2} = 0\) or \(C_{D,3} = 0\) in this direction (in this specific direction each hypersurface has only one branch, and so is only crossed once). We note in passing that the hypersurface \(C_{D,3}\), on which \(j^2 = \frac{D - 1}{D - 3} - 2\), coincides with the surface on which \(T\) is minimized in the microcanonical ensemble, equation (53) with \(n = 1\).

The determinant of the Hessian for Myers–Perry black holes, in the grand canonical ensemble, (31) is derived in appendix E, equation (E.8),
\[
\beta C_{D} \det I_s = - \frac{8\pi^2 (D - 2)}{(D - 3 + 2\Sigma_i^-)} \left( \frac{2M r_s^2}{D - 2} \right)^{N+1} \prod_{n=1}^{N} \left( \frac{1 + j_i^2}{1 - j_i^2} \right)^2.
\]

Thus the factor \(D - 2 + 2\Sigma_i^-\) in the numerator of \(C_{D}\), giving rise to a zero in the specific heat, is cancelled by a similar factor in the denominator of \(\det \{I_s\}\). It is actually more convenient to examine \(I_s^{-1}\) rather than \(I_s\), as it has the slightly simpler form (E.4),
\[
(I_s^{-1})_{ij} = \frac{\partial \Omega}{\partial j_i} \left( \begin{array}{c} (D - 2) \\ 2M r_s^2 \\ 0 \\ \vdots \\ 0 \\ \delta_{ij} \\ (D - 2)(1 + j_i^2)^2 \delta_{ij} \\ 2j_i j_j \\ (D - 2)(1 + j_i^2)(1 + j_j^2) \end{array} \right).
\]

Focusing first on the determinant, stability requires
\[
\det I_s^{-1} = \left( \frac{D - 2}{2M r_s^2} \right)^N \left( \frac{D - 2 + 2\Sigma_i^-}{D - 2} \right) \prod_{n=1}^{N} \left( \frac{1 - j_i^2}{1 + j_i^2} \right) > 0.
\]

Of course positivity, while necessary for stability, is not sufficient, (90) is satisfied when there are an even number of negative eigenvalues, but we do know that \(\det I_s^{-1}\) can only change sign when \(C_{D,2} = 0\).

To understand the eigenvalue structure in more detail consider first the two cases \(D = 7\) and \(D = 8\). The relevant surfaces are \(C_{7,2}\) and \(C_{8,2}\) shown in figure 5. Each surface \(C_{D,2}\) consists of two branches, on which at least one eigenvalue of (89) must vanish, touching at the symmetric point \(j_i^2 = j_j^2 = j_k^2 = 1\) where two eigenvalues vanish and the third is positive. These two surfaces divide the parameter space into three regions. All three eigenvalues are positive in the interior region, inside the inner surface that is visible through the holes in the outer surface, because they are positive at the origin where \(I_s^{-1}\) is a positive multiple of the identity matrix.

We can determine explicitly how many negative eigenvalues there are in the intermediate region between the two surfaces simply by checking the number at any one point in the region, there must be the same number at any other point in the region as none can change...
sign unless we cross one of the surfaces. Similarly we can find the number in the exterior region outside both surfaces.

For the region between the two surfaces we need merely set \( j_i = j_{i+1} = 0 \) and choose \( j^2_i = j^2 \) large enough to ensure that we are outside the interior region. Then \( \left( \frac{2M \Omega^2}{D-2} \right) \mathcal{I}_S^{-1} \) in (89) is diagonal and the eigenvalues are easily read off as

\[
1, 1 \quad \text{and} \quad \frac{(D - 2) - (D - 4)j^2}{(D - 2)(1 + j^2)}, \tag{91}
\]

Hence there is one negative eigenvalue if \( j^2 > \frac{D-2}{D-4} \), with \( j^2 = \frac{D-2}{D-4} \) marking the boundary of the interior region in the \( j_i \)-direction.

For the region exterior to both the surfaces we can set \( j_i = j^2_i = j^2 = j^2 \), with \( j \) large enough to ensure that we are in the exterior region. Now (89) shows that that the eigenvalues of \( \left( \frac{2M \Omega^2}{D-2} \right) \mathcal{I}_S^{-1} \) are

\[
\frac{(1 - j^2)}{(1 + j^2)}, \quad \frac{(1 - j^2)}{(1 + j^2)} \quad \text{and} \quad \frac{(D - 2) - (D - 8)j^2}{(D - 2)(1 + j^2)}. \tag{92}
\]

Hence there are two degenerate negative eigenvalue if \( j^2 > 1 \) and always one other positive eigenvalue for \( D = 7 \) or \( 8 \). We have thus shown that, for every point in the interior region, \( \mathcal{I}_S^{-1} \) has three positive eigenvalues, every point in the intermediate region has two positive eigenvalues and one negative one while every point in the exterior regions has two negative eigenvalues and one positive one. Since \( \beta \mathcal{C}_D \) vanishes on the same surfaces, and is negative in the interior region since it is negative at the origin, we see that the canonical ensemble is never stable in seven or eight dimensions.

The above analysis is easily extended to \( D > 8 \). We only need determine the signs of the eigenvalues of (89) in special directions \( j_i = \cdots = j_{n-1} = j = 0 \), \( j_{n+1} = \cdots = j_N = 0 \), and this gives the signs in each of regions separated by the roots of

\[
C_{D,2} = \left[ D - 2 - (D - 2 - 2n)j^2 \right] \left( 1 - j^2 \right)^{n-1} = 0. \tag{93}
\]

The number of regions in any specific direction is determined by the number of roots, with \( j^2 > 0 \), and the greatest number is when \( n = N \): there are then \( N - 1 \) such roots and the different branches of \( C_{D,2} = 0 \) divide \( j \)-space into \( N \) regions.

The form of (89) in these directions is

\[
\left( \frac{2M \Omega^2}{D-2} \right) \mathcal{I}_S^{-1} = \begin{pmatrix}
\frac{1 - j^2}{(1 + j^2)^2} & \mathbf{I}_{n \times n} & \frac{2}{D - 2} & \frac{j^2}{(j^2 + 1)^3} & \mathbf{Q}_{n \times n} & 0 \\
0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}_{(N-n) \times (N-n)}, \tag{94}
\]

where \( \mathbf{I}_{d \times d} \) are \( d \times d \) identity matrices and \( \mathbf{Q}_{n \times n} \) is the \( n \times n \) matrix whose entries are all one\(^5\). There are \( N - n \) eigenvalues \( +1 \) and the remaining eigenvectors \( \mathbf{V} = (V_1, \ldots, V_n, 0, \ldots, 0) \) and eigenvalues \( \lambda \) are determined by

\(^5\) We use the same notation as [14].
\[
\frac{1 - j^2}{(1 + j^2)^2} V + \frac{2}{(D - 2)(1 + j^2)} j^2 \sum_{i=1}^n V_i = \lambda V_i.
\] (95)

There are two possibilities:

1. \(\sum_{i=1}^n V_i \neq 0\): this requires \(V_i = \cdots = V_n\) which implies that, for \(i = 1, \ldots, n\),

\[
\lambda = \frac{1 - j^2}{(1 + j^2)^2} + \frac{2n}{(D - 2)(1 + j^2)} = \frac{D - 2 - (D - 2 - 2n)j^2}{(D - 2)(1 + j^2)}.
\] (96)

For \(1 \leq n < N\), this configuration returns a negative eigenvalue for \(j^2 > \frac{D - 2}{D - 2 - 2n}\), while for \(n = N = \frac{D - 1 - \epsilon}{2}\) this eigenvalue is positive for all values of \(j^2\).

2. \(\sum_{i=1}^n V_i = 0\): giving \(\lambda = \frac{1 - j^2}{(1 + j^2)}\), and

\[
V = \begin{pmatrix}
1 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 1 & -2 & 0 & \cdots & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}.
\] (97)

This requires \(n \geq 2\) and has degeneracy \(n - 1\). This gives \(n\) negative eigenvalues when \(j^2 > 1\).

The overall picture is then that there are \(N - 1\) branches to the hypersurface \(C_{\Omega, j}\) which divide \(j\)-space into \(N\) regions. All eigenvalues of \(T_\Omega^j\) are positive at the origin and at every point inside the first branch. Every time a branch is crossed by a ray emanating from the origin, one of the positive eigenvalues of \(T_\Omega^{-1}\) changes sign and becomes negative until, in the outer region after all \(N - 1\) branches have been crossed, there are \(N - 1\) negative eigenvalues and one remaining positive one. The only region in which \(T_\Omega^{-1}\), and hence \(I_\Omega\) is a positive matrix is the innermost one. But we have already seen that \(\beta C_{\Omega, j}\) is negative in the innermost region, hence the canonical ensemble is always unstable for any choice of metric parameters in any \(D\).

Another thing that we learn from the analysis here is that there can be singularities in \(C_j\) in the positive \(T\) regime which are not present in \(C_\Omega\). For example in four dimensions, (67), there is a pole in \(C_j\) for \(T > 0\) and while there is a pole in \(C_\Omega\), it is in the negative \(T\) regime and \(C_{\Omega, j}\) is negative for \(T > 0\). When \(C_j\) is positive the instability lies in the moment of inertia, thus fixing \(J\) can stabilize the black, provided \(J\) is large enough, but fixing \(\Omega\) cannot. In higher dimensions the structure is even richer, there are poles in \(C_{\Omega, j}\) for both \(T > 0\) and \(T < 0\), depending on the angular momenta.

In addition to the positive mass Myers–Perry black holes in odd dimensions there are also negative mass Myers–Perry black holes [5]. However, as pointed out in [18], there is a subtlety with these space-times: geodesics are repelled from the would-be event horizon and do not pass through it, so in a sense there is no event horizon. Nevertheless one expects a non-zero Hawking temperature, determined by demanding regularity of the Euclidean time metric, so the entropy cannot be zero. The thermodynamics of these space-times is not analysed here, but would be an interesting future project.

One of the lessons that we learn from this analysis of Myers–Perry black holes in this section is that the general thermodynamic instability of rotating black holes in asymptotically flat space-times, exhibited in equation (10), can manifest itself in rather subtle ways in explicit
cases. Although the overall determinant is negative there can be zeros and poles that cause eigenvalues to change sign but which cancel in the determinant so they are not visible.

4. Conclusions

We have compared the microcanonical, the canonical and the grand canonical ensembles in the thermodynamic description of asymptotically flat rotating black holes in arbitrary dimensions. These black holes are always thermodynamically unstable but the thermodynamic instability manifests itself differently in the different ensembles. There is however an elegant and simple relation between the specific heats and moment of inertia tensors in the canonical and the grand canonical ensembles, given by equation (3),

\[ C_i \det I_r = C_\alpha \det I_\alpha. \]  

(98)

The case of Myers–Perry black holes has been analysed in detail and all extrema of the temperature in the microcanonical ensemble have been found and classified and shown to correspond to inflection points of the entropy.

In the canonical ensemble it has been shown that, in \( D \) dimensions, the specific heat \( C_j \) in equation (56) vanishes when \( T = 0 \) and changes sign on a hypersurface in angular momentum space given by

\[ (4\pi r_\alpha T)^2 = (D - 2) \left( 4\pi r_\alpha T - 4 \sum_{i=1}^{N} \frac{j_i^2}{1 + j_i^2} \right). \]  

(99)

(where \( j_i = \frac{2\pi r_i}{r} \)), on which it diverges. In the determinant of the Hessian this singularity in \( C_j \) is exactly cancelled by an equivalent zero in \( \det (I_r) \). There are also singularities in \( \det (I_r) \) when \( C_{\alpha,3} \) in equation (83) vanishes.

In the grand canonical ensemble \( C_\rho \) in equation (59) also vanishes when \( T = 0 \) and has divergences, this time on the hypersurface defined by \( C_{\alpha,3} = 0 \) rather than that given by (99). In addition \( C_\rho \) also has zeros on the hypersurface \( C_{\alpha,2} = 0 \) in equation (82). In the determinant of the Hessian for the grand canonical ensemble (31) the zeros of \( C_\rho \) are cancelled by corresponding poles in \( \det (I_\rho) \) on \( C_{\alpha,2} = 0 \). The locus of these singular points of \( \det (I_\rho) \) corresponds to a branched hypersurface in angular momentum space which divides the space into \( N \) separate regions. Every time a branch of this hypersurface is crossed an eigenvalue of \( I_\rho \) changes sign and the moment of inertia tensor has different signature in the \( N \) separate regions. Only the region surrounding the origin in angular momentum space gives a positive definite moment of inertia tensor and this region corresponds precisely to the region where \( C_\rho \) is negative.

There is a curious relation between the hypersurface \( C_{\alpha,3} = 0 \) on which both \( C_\rho \) and \( \det (I_\rho) \) diverge on the one hand and extremal \( T = 0 \) Myers–Perry black holes on the other: the algebraic equations defining these two hypersurfaces are related by analytic continuation \( (j^2)^\alpha \to - (j^2)_\beta \), with the entropy held constant.

Our analysis has also shown that, in the microcanonical and the grand canonical ensembles, many of the thermodynamic properties of Myers–Perry black holes in \( D - 2 \) dimensions can be obtained from those of a black hole in \( D \) dimensions by letting one of the angular momenta in \( D \) dimensions tend to infinity, keeping the entropy constant.
The thermodynamic instabilities of Myers–Perry black holes thus have a very rich structure, beyond that of the ultra-spinning surface upon which the moment of inertia tensor develops its first negative eigenvalue.

An obvious direction for future work on this topic is to include a charge on the black hole and/or to introduce a cosmological constant to encompass the case of asymptotically anti-de Sitter rotating black holes. The latter should prove particularly interesting as the black holes will become thermodynamically stable when the magnitude of the cosmological constant is large enough and much could be learned by mapping out the boundary of the stability region.

In four dimensions the shape of the thermodynamic potential surfaces for black holes has been related to catastrophe theory [19–23], for example the Gibbs free energy for a rotating black hole is in the universality class of the swallowtail catastrophe (A in the A–D–E classification of critical points of functions [24]). The higher dimensional case promises to be much richer as there are more control parameters. It would be interesting to determine which are the relevant A – D – E series in even and odd dimensions.

Since the work of Gregory and Laflamme on black brane instabilities [8] it has been known that there is a close relation between thermodynamic stability and dynamical instability and this relation was elucidated for Myers and Perry black holes in [7]. An extensive literature on the subject of the thermodynamic and dynamical instability of rotating black holes in higher dimensions has since emerged [4, 9, 11, 12, 15, 16, 25–33]. Most of this work is concerned with specific solutions and perturbations thereof (in particular black branes have received a lot of attention). Equivalence of these two types of instability was proven for a special type of black brane solutions in [25] and more generally for black branes in [26], building on earlier work of [33]. However they are certainly not equivalent in general: a Schwarzschild black hole is dynamically stable but thermodynamically unstable. This link is an active direction of research and appears to be closest for black branes. The explicit examples considered in this paper are restricted to Myers–Perry black holes for which the relation between thermodynamic and dynamical stability is less clear. Another direction for future research would be a further investigation of the relation between dynamical and thermodynamic instability for black branes in the light of the general results on thermodynamical instability presented here, though this would probably involve numerical analysis in specific cases.

Appendix A. Temperature extrema and inflection points of the entropy

In this appendix we extend the study in [4] and [14] to find all isenthalpic (i.e. constant mass) extrema of $T$ for Myers–Perry black holes, as the $J^i$ are varied in asymptotically flat space-times.

At constant $M$ equation (46) implies that

$$\frac{\partial \ell_i}{\partial J^j} \bigg|_M = \frac{(D - 2)}{2M} \delta_{ij},$$

(A.1)

with, which the expression

$$M = \frac{(D - 2) \sigma r_0^{D-3}}{16\pi} \prod_{i=1}^{N} \left(1 + j_i^2\right)$$

(A.2)
for the mass, gives
\[ dM = 0 \quad \Rightarrow \quad \frac{\partial r_h}{\partial J} \bigg|_\omega = -\frac{(D-2)}{4\pi TM} \Omega_i = -\frac{(D-2)}{tM} \omega_i, \quad (A.3) \]
where we have defined
\[ t := 4\pi r_h T = \left( D - 3 - 2\sum_{i=1}^{\infty} \frac{j_i^2}{1 + j_i^2} \right) \quad \text{and} \quad \omega_i = r_h \Omega_i = \frac{j_i}{1 + j_i^2} \quad (A.4) \]
(both \( t \) and \( \omega_i \) are invariant under \( j_i \to \frac{1}{j_i} \)).

Equations (A.1) and (A.3) together imply
\[ \frac{\partial j_i}{\partial J} \bigg|_\omega = \left( D - 2 \right) \frac{2r_h}{2r_h M} \left( \delta_{ij} + \frac{2j_ij_j}{t \left( 1 + j_i^2 \right)} \right). \quad (A.5) \]
We now have all the information we need to calculate \( \frac{\partial T}{\partial J} \bigg|_M \) from
\[ T = \frac{1}{4\pi r_h} \left( D - 3 - 2\sum_{i=1}^{\infty} \frac{j_i^2}{1 + j_i^2} \right), \quad (A.6) \]
we find
\[ \frac{\partial T}{\partial J} \bigg|_M = \frac{(D-2)}{4\pi r_h M} \left( 1 - \frac{2}{1 + j_i^2} - \frac{4 \Omega^2}{t} \right) \omega_i, \quad (A.7) \]
where \( \Omega^2 := \sum \omega_i^2 \).

For fixed \( M \) extrema of \( T \) occur for
\[ \omega_i = 0 \Leftrightarrow j_i = 0 \quad \text{or} \quad 1 - \frac{2}{1 + j_i^2} - \frac{4 \Omega^2}{t} = 0. \quad (A.8) \]
In particular any finite non-zero \( j_i \) are all equal at an extremum.

It is also possible that some of the \( j_i \) might tend to infinity. Suppose \( m \) of the \( j_i \) diverge as \( j_i \to \Lambda \to \infty \). Then, at fixed finite mass,
\[ r_h \approx \Lambda^{-\frac{2m}{D-3}}, \quad (A.9) \]
and hence
\[ \frac{\partial T}{\partial J} \bigg|_M \approx \Lambda^{-\frac{m}{D-3}} \left( 1 - \frac{2}{1 + j_i^2} - \frac{4 \Omega^2}{t} \right) \omega_i. \quad (A.10) \]
For \( j_i \approx \Lambda, \omega_i \approx \Lambda^{-1} \) so
\[ \frac{\partial T}{\partial J} \bigg|_M \approx \Lambda^{-\frac{m}{D-3} - 1}, \quad (A.11) \]
which tends to zero for \( \Lambda \to \infty \) provided \( m < \frac{D-3}{4} \), which is possible for \( D \geq 8 \).

To keep the discussion general we shall suppose \( n \) of the \( j_i \) are finite and equal, \( m \) are infinite and \( N - n - m \) are zero. Up to permutations of the \( j_i \), extrema of \( T \) can only occur for configurations with
\[ J_i = \cdots = j_n = j > 0, \quad j_{n+1} = \cdots = j_{n-m} = 0, \]
\[ j_{n-m+1} = \cdots = j_m = \Lambda \to \infty \]  
(A.12)

so we focus on the symmetric angular momentum configurations

\[
\tilde{f} = \lim_{\Lambda \to \infty} \left[ \frac{J_n, \ldots, J_1, J, 0, \ldots, 0, \Lambda, \ldots, \Lambda}{J_n, \ldots, J_1} \right].
\]  
(A.13)

The extrema require \( j \) to satisfy the second equation in (A.8), \( j = j_n \) with

\[ j_n^2 = \frac{t + 4\Omega^2}{t - 4\Omega^2}, \]  
(A.14)

At \( \tilde{J}_n \)

\[ \Omega^2 = \frac{n j_n^2}{(1 + j_n^2)^2} \quad \text{and} \quad t = D - 3 - 2m - \frac{2nj_n^2}{(1 + j_n^2)}, \]  
(A.15)

which gives the solution of (A.14) to be

\[ j_n^2 = \frac{D - 3 - 2m}{D - 3 - 2m - 2n}. \]  
(A.16)

The temperature at these extrema is, from (A.6),

\[ T_n = \frac{1}{4\pi r_n} \frac{(D - 3 - 2m)(D - 3 - 2m - 2n)}{(D - 3 - 2m - n)}. \]  
(A.17)

Demanding \( T_n \geq 0 \) imposes the restriction

\[ m + n \leq \frac{D - 3}{2}. \]  
(A.18)

More generally when the angular momenta are of the form (A.13), but not necessarily at \( \tilde{J}_n \), the temperature is

\[ T = \frac{1}{4\pi r_n} \frac{(D - 3 - 2m) + (D - 3 - 2m - 2n) J^2}{(1 + J^2)} \]  
(A.19)

and vanishes for

\[ J^2 = J_0^2 = \frac{D - 3 - 2m}{2m + 2n - (D - 3)}, \]  
(A.20)

which is only possible for \( m + n \geq \frac{D - 3}{2} \).

To analyse the nature of the extrema we need the second derivative of \( T \). A straightforward but tedious calculation gives
$$\frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{\mu} = \frac{(D - 2)^2}{4\pi M^2 \ell_s^4} \left\{ \left[ \frac{4J_i^2}{(1 + J_i^2)^3} - \frac{2Q_i^2}{\ell} \left( \frac{1 - J_i^2}{1 + J_i^2} \right) - \frac{1}{2} \right] \delta_{ij} ight. \\
- \left[ \frac{4}{\ell} \left( \frac{1 - j^2}{(1 + j^2)^3} + \frac{1 - j_i}{(1 + j_i)^2} - \frac{1}{4} \right) + \frac{8Q^2}{\ell^2} \left( \frac{1}{1 + j_i^2} + \frac{1}{1 + j_i^2} + \frac{1}{2} \right) \\
+ \frac{8}{\ell^2} \left( \frac{2Q_i^2}{\ell} + \sum \right) \right\} \delta_{ij}, \quad \text{(A.21)}$$

where

$$\sum := \sum_i \left( \frac{1 - j_i^2}{(1 + j_i^2)^2} \right). \quad \text{(A.22)}$$

Sums like $\sum$ crop up frequently in this analysis and it will prove convenient to define

$$\Sigma_e := \sum_{\ell=1}^{N} \left( \frac{j_i^2}{(1 \pm j_i^2)^2} \right)^{\ell}. \quad \text{(A.23)}$$

in terms of which

$$\sum = 2\Sigma_e - \Sigma_e^+ \quad \text{(A.24)}$$

We wish to determine the signs of the eigenvalues of (A.21) at $\vec{j}_e$. There are three cases to consider:

- At an extremum of the form (A.12), if either of the indices $i$ and $j$ is in the range $[N - n - m, \cdots, N - m]$ then $\frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{\mu} = 0$ unless the other index is in the same range.

When both $i$ and $j$ are in the range $[N - n - m, \cdots, N - m]$.

$$\frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{\mu} = -\frac{(D - 2)^2}{4\pi M^2 \ell_s^4} \left( \frac{2Q^2}{\ell} + \frac{1}{2} \right) \delta_{ij} \quad \text{(A.25)}$$

The eigenvalues are all negative in these directions, corresponding to a maximum of $T$ around $J_e$.

- If one of the indices $i$ or $j$ is in the range $[N - m + 1, \cdots, N]$ and the other is in the range $[1, \cdots, N - m]$ then $\frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{\mu} = 0$. If both are in the range $[N - m + 1, \cdots, N]$ then

$$\frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{\mu} \approx -\frac{1}{\ell_s^4} \sum \approx \frac{1}{\ell_s^4} \sum A^{m - 1} \sim 0 \quad \text{as} \quad A \to \infty \quad \text{(A.26)}$$

since $m < \frac{D - 3}{4}$.

- If both indices $i$ and $j$ are in the range $[1, n]$, then the nature of the extremum is determined by
\[ \frac{\partial^2 T}{\partial J^i \partial J^j} \bigg|_{J^i} = \frac{(D - 2)^2}{4\pi M^2} \left( A_n \delta_{ij} + B_n Q_{ij} \right), \]  
(A.27)

where \( Q_{ij} \) is the \( n \times n \) matrix whose entries are all unity and \( A_n \) and \( B_n \) are ratios of polynomials in \( j \). Since \( Q_{ij} \) has \( n - 1 \) zero eigenvalues and one eigenvalue equal to \( n \), (A.27) has \( n - 1 \) degenerate eigenvalues \( \lambda_i = A_n \) and one eigenvalue \( \lambda_2 = A_n + nB_n \).

Evaluating \( A_n \) and \( B_n \) gives

\[ \lambda_i = \frac{(D - 3 - 2m)(D - 3 - 2m - 2n)^2}{4(D - 3 - 2m - n)} > 0, \]  
(A.28)

\[ \lambda_2 = \frac{1}{4}. \]  
(A.29)

Thus \( \bar{J}_n \) is in general a saddle point, with \( T \) minimized in the directions \( i = 1, \ldots, n \) and maximized in the directions \( i = N - n - m, \ldots, N - m \).

A necessary condition for stability is that the eigenvalues of

\[ H_j = -\frac{\partial^2 S}{\partial J^i \partial J^j} \bigg|_{J^i} \]  
(A.30)

be positive, [4]. Using (A.3) and

\[ S = \frac{m}{4} e^{D/2} \prod_k \left( 1 + j_k^2 \right), \]  
(A.31)

gives

\[ \frac{\partial S}{\partial J^i} \bigg|_{J^i} = -\frac{2}{t} \frac{j_i}{(1 + j_i^2)}. \]  
(A.32)

The entropy is thus a monotonically decreasing function of each of the \( j \) as \( |j| \) increases. It was shown in [14] that the extrema of the temperature in the two special cases \( n = 1 \) and \( n = N \) (with \( m = 0 \)) are inflection points of the entropy. We now show that this is a general property of all the extrema of \( T \).

An inflection point corresponds to a zero eigenvalue of the matrix (A.30), so we first determine

\[ H_j = -\frac{\partial^2 S}{\partial J^i \partial J^j} \bigg|_{J^i} = \frac{(D - 2)}{2\pi M^2} \left\{ \frac{1 - j_i^2}{(1 + j_i^2)^2} \delta_{ij} + \frac{4}{t} \left( \frac{1}{1 + j_i^2} \right)^2 \frac{1}{1 + j_i^2} \right. \]

\[ \left. - \frac{1}{2} \omega^2 \left( a_i a_j \right) \right\}, \]  
(A.33)

which agrees with equation (2.13) in [14], apart from some minor typos.

The eigenvalues at the symmetric points (A.13) are easily determined:
If both indices $i$ and $j$ are in the range $[1, \ldots, n]$ we write

$$H_y = \frac{(D - 2)}{2 r_y^2 MT} \left( \mathcal{A} \delta_{ij} + B Q_y \right)$$

(A.34)

and the eigenvalues of $\mathcal{A} \delta_{ij} + B Q_y$, with $i, j = 1, \ldots, n$, are

$$\lambda_1 = \mathcal{A} \quad \text{and} \quad \lambda_2 = \mathcal{A} + n B,$$

(A.35)

where $\lambda_1$ has degeneracy $n - 1$. These are

$$\lambda_1 = \frac{(1 - j^2)}{(1 + j^2)}$$

$$\lambda_2 = \frac{(D - 3 - 2m) \left[ D - 3 - 2m - (D - 3 - 2m - 2n) j^2 \right]}{\left[ D - 3 - 2m + (D - 3 - 2m - 2n) j^2 \right]^2} .$$

(A.36)

If either of the indices $i$ and $j$ is in the range $[n + 1, \ldots, N - m]$, then $H_y$ vanishes unless the other index is in the same range in which case $\lambda_1 = j = 0$ and

$$H_y = \frac{(D - 2)}{2 r_y^2 TM} \delta_{ij} .$$

(A.37)

If either of the indices $i$ and $j$ is in the range $(N - m + 1, \ldots, N)$ then $\lambda = 0$.

As observed in [14] the eigenvalues are all positive near $J^i = 0$ and the first negative eigenvalue is encountered for $n = 1$ when $j = \frac{D - 3}{D - 5}$, which is precisely the inflection point of [7] at the first temperature minimum. This hypersurface $j = \frac{D - 3}{D - 5}$ on which $H_y$ first develops a zero eigenvalue is the ultra-spinning surface of [14]. For $D = 5$ the ultra-spinning surface in the microcanonical ensemble is not closed and $j_1$ can reach infinity when $j_2 = 0$. It is shown in section 3.4.2 that the $D = 5$ ultra-spinning surface in the canonical ensemble is closed, see figure 2.

At temperature minima, where $j = j_n$, the eigenvalues (A.36) above evaluate to

$$\lambda_1 \bigg|_n = -\frac{n (D - 3 - 2m - 2n)}{2 \left( D - 3 - 2m - n \right)^2} \leq 0, \quad (n - 1) \text{ times};$$

$$\lambda_2 \bigg|_n = 0.$$  (A.38)

In particular there is always one zero eigenvalue, corresponding to an inflection point in the entropy in the direction of the associated eigenvector.

**Appendix B. Specific heat at constant angular momentum**

To calculate the heat capacity at constant $J$

$$C_J = \frac{\partial M}{\partial T} \bigg|_J .$$

(B.1)
we first observe that (46) gives

\[ J' = \frac{2M}{D - 2} a_j, \]  

so \( J' = \text{const} \) implies

\[ \frac{d a_j}{a_j} = - \frac{d M}{M}. \]  

Next, combining (41), (44) and (46), \( j = \frac{a_j}{r_h} = \frac{2a_j}{r_h} \) vary as

\[ \frac{d j_i}{j_i} = - \frac{d S}{S} \bigg|_{j_i} \left( \frac{d a_i}{a_i} - \frac{d r_i}{r_i} \right) \bigg|_{j_i} = - \left( \frac{d M}{M} + \frac{d r_i}{r_i} \right) \bigg|_{j_i}. \]  

But explicitly from (44)

\[ \frac{d S}{S} \bigg|_{j_i} = (D - 2) \frac{d r_i}{r_i} \bigg|_{j_i} + 2 \sum_{k=1}^{\infty} \frac{j_k}{1 + j_k^2} \bigg|_{j_i} \]

\[ = (D - 2) \frac{d r_i}{r_i} \bigg|_{j_i} - 2 \left( \frac{d M}{M} + \frac{d r_i}{r_i} \right) \bigg|_{j_i} \sum_{k=1}^{\infty} \frac{j_k^2}{1 + j_k^2}. \]

Equations (B.4) and (B.5) together now give

\[ \frac{d M}{d r_i} \bigg|_{j_i} = \left( \frac{t}{D - 2 - t} \right) \frac{M}{r_h}. \]  

Similar manipulations on \( T = \frac{1}{4\pi r_h} \) yield

\[ \frac{d T}{d r_i} \bigg|_{j_i} = \frac{1}{4\pi r_h} \left\{ \frac{t^2 - (D - 2)(t - 4\Sigma_{d}^+)}{D - 2 - t} \right\}. \]  

and then (B.6) and (B.7) can be combined to give the specific heat in the canonical ensemble with fixed \( J \),

\[ C_{j} = \frac{16\pi^2 r_h^2 M T}{\left[ t^2 - (D - 2)(t - 4\Sigma_{d}^+) \right]^2}. \]  

This can be expressed in terms of \( M \) and \( j_i \) by noting that

\[ M = \left( \frac{(D - 2)\pi}{16\pi} \right) r_h^{D-3} \prod_{i=1}^{N} \left( 1 + j_i^2 \right). \]  

**Appendix C. Specific heat at constant angular velocity**

The specific heat at constant \( \Omega \) is straightforward to determine, using similar techniques to those of section 3.2. In terms of \( j_i \) the entropy (44) is
\[ S = \frac{m r_n^{D-2}}{4} \prod_{n=1}^{N} \left( 1 + j_0^2 \right), \quad (C.1) \]
\[ T = \frac{1}{4 \pi r_n} = \left( \frac{D - 3 - 2 \Sigma^+}{4 \pi r_n} \right) \quad (C.2) \]

and
\[ \Omega = \frac{j}{r_n \left( 1 + j_0^2 \right)} \quad (C.3) \]

The specific heat at constant angular velocity is defined as
\[ C_\omega = T \left( \frac{\partial S}{\partial T} \right) \quad (C.4) \]

From (C.3)
\[ \Omega = \text{const} \quad \Rightarrow \quad \left. \frac{d j}{d r} \right|_{\Omega} = \left( \frac{1 + j_0^2}{1 - j_0^2} \right) \left( \frac{d r}{r_n} \right) \quad (C.5) \]

Using these it is straightforward to show that
\[ \frac{\partial T}{\partial r} \bigg|_{\Omega} = - \frac{(D - 3 + 2 \Sigma^-)}{4 \pi r_n^2} \quad (C.6) \]

and
\[ \frac{\partial S}{\partial r} \bigg|_{\Omega} = \frac{S}{r_n} \left( D - 2 + 2 \Sigma^- \right) \quad (C.7) \]

Combining these we immediately arrive at equation (59) in the text,
\[ C_\omega = - \frac{4 \pi r_n TS \left( D - 2 + 2 \Sigma^- \right)}{(D - 3 + 2 \Sigma^-)} \quad (C.8) \]

This generalizes the \( D = 4 \) case derived in [16] to arbitrary \( D \).

**Appendix D. Isothermal moment of inertia**

To calculate the isothermal moment of inertia tensor
\[ I^y_t = \left( \frac{\partial T}{\partial \Omega} \right) \quad (D.1) \]

in asymptotically flat Myers–Perry space-times our starting point is again
\[ T = \frac{D - 3 - 2 \Sigma^+}{4 \pi r_n}, \quad (D.2) \]
from which we find
\[ dr_i |_r = - \frac{1}{\pi r} \frac{j d_j}{(1 + j_i^2)^2} \] (D.3)

Now use this in (46), re-written using (41) in the form
\[ J' = \frac{\sigma r^{D-2}}{8\pi} \prod_{i=1}^{N} (1 + j_i^2) j_i, \] (D.4)
to deduce that
\[ \frac{\partial J'}{\partial j_i} |_r = \frac{2Mr_i}{D-2} \left( \delta_{i,2} + 2 \left[ \frac{(D - 3 - 2 \Sigma^+)(1 + j_i^2) - 2(D - 2)}{(D - 3 - 2 \Sigma^+)(1 + j_i^2)} \right] j_k \right), \] (D.5)

Similar manipulations applied to (47) produce
\[ \frac{\partial \Omega}{\partial j_k} |_r = \frac{1}{r_h} \left\{ \left( 1 - j_i \right) \delta_{ij} + \frac{4j_i j_j}{(D - 3 - 2 \Sigma^+)(1 + j_i)(1 + j_i^2)} \right\}, \] (D.6)

the inverse of which is
\[ \frac{\partial j_i}{\partial \Omega} |_r = r_h \left\{ \frac{\left( 1 + j_i^2 \right)^2}{(1 - j_i^2) \delta_{ij} - \frac{4(1 + j_i^2)}{(t + 4 \Sigma)(1 - j_i^2)} j_j} \right\}, \] (D.7)

where
\[ \Sigma := \sum_{i=1}^{N} \frac{j_i^2}{1 - j_i^2} = \frac{1}{2}(\Sigma_i^+ + \Sigma_i^-). \] (D.8)

Equations (D.5)–(D.7) are now easily combined to give the symmetric isothermal moment of inertia tensor
\[ \mathcal{I}^\gamma = \frac{2Mr_i}{D-2} \left\{ \frac{(1 + j_i^2)^2}{(1 - j_i^2) \delta_{ij} - \frac{4(1 + j_i^2)}{(t + 4 \Sigma)(1 - j_i^2)} j_j} \right\}, \] (D.9)

where
\[ \mathcal{I} := D - 3 + 2 \Sigma_i^- . \] (D.10)

This is equation (62) in the text.

The determinant of \( \mathcal{I}_r \) can be evaluated by observing that the components of matrix are of the form
\[ \mathcal{I}_r = \frac{2Mr_i^2}{D-2} \left( A_i \delta_{ij} - B_i B_j - C_i C_j \right) \] (D.11)
with
\[ A_i = \left( \frac{1 + j_i^2}{1 - j_i^2} \right) \]
\[ B_i = \sqrt{2} j_i \]
and
\[ C_i = \frac{8}{\sqrt{q_i}} \left( \frac{j_i}{1 - j_i^2} \right). \]  \tag{D.12}

The determinant of (D.11) has a compact expression, because the off-diagonal entries factorize,
\[ \text{det}(I_r) = \left( \frac{2M_{r}^2}{D - 2} \right)^N \left( \frac{1}{\prod_{i=1}^N A_i} \right)^{N+1} \left\{ 1 - \sum_{i=1}^N \left( \frac{B_i^2 + C_i^2}{A_i} \right) \right. \\
+ \left\{ \sum_{i=1}^N \left( \frac{B_i^2}{A_i} \right) \left( \frac{C_i^2}{A_i} \right) - \left( \sum_{i=1}^N \frac{B_i C_i}{A_i} \right)^2 \right\}. \]  \tag{D.13}

A little manipulation, using (D.12), then yields
\[ \text{det}I_r = -\left( \frac{2M_{r}^2}{D - 2} \right)^N \left[ r^2 - \left( \frac{D - 2}{t} \right) \left( t - 4\sum_{i}^N \right) \right] \prod_{i=1}^N \left( \frac{1 + j_i^2}{1 - j_i^2} \right)^2, \]  \tag{D.14}
with \(\tilde{r}\) defined in (D.10). This, together with the expression for \(C_j\) in (56), leads to the compact expression
\[ \beta C_j \text{det}I_r = -\frac{8\pi^2 (D - 2)}{(D - 3 + 2\Sigma_i)} \left( \frac{2M_{r}^2}{D - 2} \right)^{N+1} \prod_{i=1}^N \left( \frac{1 + j_i^2}{1 - j_i^2} \right)^2. \]  \tag{D.15}

**Appendix E. Isentropic moment of inertia**

To calculate the isentropic moment of inertia in asymptotically flat Myers–Perry space-times, again re-write (44) as
\[ S = \frac{\pi r_b^{D-2}}{4} \prod_{i=1}^N \left( 1 + j_i^2 \right) \]  \tag{E.1}
from which
\[ \frac{dr}{\sqrt{\pi}} = -\frac{2r_b}{D - 2} \sum_{i=1}^N \frac{j_i^2}{1 + j_i^2} \left| \text{d}j_i \right|, \]  \tag{E.2}
with \(\text{d}J_i = \frac{\pi}{2\pi} \text{d}j_i \). Then
\[ \Omega_i = \frac{j_i}{r_b \left( 1 + j_i^2 \right)} \]  \tag{E.3}
yields
\[ (I_{s}^{-1})_{ij} = \left( \frac{\partial \Omega}{\partial j_i} \right)_{s} = \frac{2\pi}{nS} \frac{1 - j_i^2}{(1 + j_i^2)^{\delta_j}} \left( \frac{1 - j_i^2}{(1 + j_i^2)^{\delta_j}} \delta_{ij} + \frac{2jj_i}{(D - 2)(1 + j_i^2)(1 + j_i^2)} \right), \] (E.4)

which was reported in [17]. Equation (E.4) is easily inverted to give
\[ I_{s}^{ij} = \frac{2Mr_{s}^2}{D - 2} \frac{(1 + j_i^2)^2}{(1 - j_i^2)^{\delta_j}} - \frac{2jj_i}{(D - 2 + 2\Sigma_{i}^{1}) (1 + j_i^2)(1 - j_i^2)}. \] (E.5)

Similar manipulations to those of appendix D, with
\[ A_{i} = \frac{(1 + j_i^2)^{1/2}}{(1 - j_i^2)}, \quad B = \sqrt{\frac{2}{\Gamma}} \frac{(1 + j_i^2)^{1/2}}{(1 - j_i^2)}, \quad \text{and} \quad C_{j} = 0, \] (E.6)

reveal that
\[ \det I_{s} = \left( \frac{2Mr_{s}^2}{D - 2} \right)^{N} \frac{(D - 2)}{(D - 2 + 2\Sigma_{i}^{1})} \prod_{i=1}^{N} \frac{(1 + j_i^2)^2}{(1 - j_i^2)}. \] (E.7)

Combining this with $C_{i}$ in (59) leads to the same expression as (D.15),
\[ \beta C_{i} \det I_{s} = - \frac{8\pi^2 (D - 2)}{(D - 3 + 2\Sigma_{i}^{1})} \left( \frac{2Mr_{s}^2}{D - 2} \right)^{N+1} \prod_{i=1}^{N} \frac{(1 + j_i^2)^2}{(1 - j_i^2)}. \] (E.8)

so (3) is indeed satisfied.

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