Darboux transformations for quasi-exactly solvable Hamiltonians

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Abstract

We construct new quasi-exactly solvable one-dimensional potentials through Darboux transformations. Three directions are investigated: Reducible and two types of irreducible second-order transformations. The irreducible transformations of the first type give singular intermediate potentials and the ones of the second type give complex-valued intermediate potentials while final potentials are meaningful in all cases. These developments are illustrated on the so-called radial sextic oscillator.

1 Introduction

The resolution of the one-dimensional Schrödinger equation has attracted much attention since the first statements of quantum mechanics. It is now well known that this equation can be analytically solved for a small number of interactions only. In that case, it is referred to as an exactly solvable (ES) equation. Because of their major interest (analytic solutions describe more finely the physical reality), several methods have been proposed in order to increase the number of these ES Schrödinger equations. One can cite the factorization method introduced by Schrödinger \[1\] and actualized by Witten in the form of supersymmetric quantum mechanics \[2\]. It is based on the fact that a one-dimensional Schrödinger Hamiltonian

\[ H_0 \equiv -\frac{d^2}{dx^2} + V_0(x) \] (1)


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can be factorized as
\[ H_0 = L_0^\dagger L_0 + \alpha , \]  
with
\[ L_0 = \frac{d}{dx} + W_0(x) , \] 
the constant \( \alpha \) being called the factorization constant. One can then obtain in a straightforward way the eigenfunctions of
\[ H_1 \equiv L_0L_0^\dagger + \alpha = -\frac{d^2}{dx^2} + V_1(x) , \] 
where the prime denotes the derivative with respect to \( x \). They are precisely given by
\[ \phi_N(x) = L_0 \psi_N(x) , \] 
where \( \psi_N(x) \) stands for the eigenfunction (of eigenvalue \( E_N \)) related to \( H_0 \). The relation (4) means that the operator \( L_0^\dagger \) realizes the transformation in the opposite direction. The function
\[ \tilde{\psi}_N(x) = L_0^\dagger \phi_N(x) , \] 
is an eigenfunction of \( H_0 \) differing from \( \psi_N \) by a normalization constant only. Note that neither \( L_0 \) nor \( L_0^\dagger \) are unitary and they do not preserve the norms of the functions. The relations (3) and (4) imply that \( H_0 \) and \( H_1 \) are isospectral up to the fact that either the ground state of \( H_0 \) could belong to the kernel of the operator \( L \) or the ground state of \( H_1 \) could belong to the kernel of the operator \( L_0^\dagger \). In this case the spectra of \( H_0 \) and \( H_1 \) differ by the ground state level only and they are strictly isospectral otherwise.

Another method for giving rise to ES Hamiltonians is the so-called Darboux transformation [3]. This method is based on a general notion of transformation operators [4]. The main element in definition of such an operator is the so-called intertwining relation
\[ L_0 H_0 = H_1 L_0^\dagger . \] 
The factorization method (a fortiori the supersymmetric quantum mechanics) appeared long after the publication of the Darboux paper which was unfamiliar to physicists. But since the paper by Andrianov et. al. [5] it was realized that the methods are equivalent. The relation (8) gives the same expression for the transformation operator as in the original paper by Darboux [3] which coincides with (3)-(6) and precises the function \( W_0(x) \) to be the logarithmic derivative
\[ W_0(x) = -\frac{d}{dx} \frac{\ln \psi(x)}{} . \] 
of a solution to the initial Schrödinger equation
\[ (H_0 - \alpha) \psi(x) = 0 . \] 
Note that \( \alpha \) here is exactly the same factorization constant as in Eqs. (2) and (4). The function \( \psi(x) \) is called the transformation function (sometimes the factorization function).
The only condition imposed on $\psi(x)$ is its absence of zeros inside the interval where the equation (11) is solved. This condition may be satisfied only if $\alpha \leq E_0$ (see e.g. [6]) where $E_0$ is the ground state energy of $H_0$ if it has a discrete spectrum or the lower bound of the continuous spectrum otherwise. For $\alpha = E_0$ the function $\psi(x)$ is evidently the ground state function of $H_0$ and for $\alpha < E_0$ it is an unphysical solution of (11).

It is evident that the Hamiltonian $H_1$ can be taken as the initial Hamiltonian for the next transformation step etc. In such a way one gets a chain of so called reducible transformations. The main feature of such a chain is that every element of the chain has a well-defined (this means essentially self-adjoint) Hamiltonian acting in the same Hilbert space and the resulting action of the whole chain may be obtained with the help of a higher order transformation operator only acting on solutions of the initial equation. A natural question then arises. Is it possible to get something new with respect to a chain of transformations by defining an $n$th order transformation operator as an $n$th order differential operator satisfying the intertwining relation (8)? The answer is positive: there are the irreducible chains of transformations. The main feature of such chains is that despite of the fact that they can also decompose into a superposition of first order operators, some of the corresponding intermediate Hamiltonians can not be defined properly: They are either singular or non Hermitian. More precisely, it was first established that some intermediate potentials may have poles inside the interval where the initial Schrödinger equation is solved but the final potential is regular [5] (the supersymmetric interpretation can be found in [7] and a generalization of Krein’s statement is in [10]). In this paper we refer to this case as irreducible transformations of the first type.

Another possibility to get an irreducible chain consists in choosing complex factorization energies $\alpha_1 = \alpha$ and $\alpha_2 = \overline{\alpha}$ (the bar over a symbol means the complex conjugation). In this case some intermediate potentials are complex but the final potential is real [11]. We refer to this case as irreducible transformations of the second type.

Irreducible Darboux transformations have already been exploited in order to give rise to a large amount of new ES Schrödinger equations [7], [9]-[12]. However a new kind of Schrödinger equations appeared in the literature some twenty years ago [13]. These are the so-called quasi-exactly solvable (QES) Schrödinger equations, i.e., those for which a finite number of solutions can be analytically determined. These equations are rather exceptional. A previous attempt to list them in an exhaustive way has been performed by Turbiner [14] on the basis of the link between these equations and the $sl(2, R)$ vector fields preserving a finite-dimensional space. However it is now understood that some of these equations [15] do not have any connection with the Lie algebra $sl(2, R)$. Therefore it is interesting to find other approaches giving rise to new QES equations. Supersymmetric quantum mechanics is one of these approaches but it has been mainly used in the context of ES equations (for a recent review, see e.g. [16]). In what concerns QES equations there are only several papers on this subject (see e.g. [17]). The purpose of this paper is to fill in this gap. Thus, in Sections 2 and 3, we focus on the reducible and irreducible second-order Darboux transformations, respectively. In each of these Sections we produce new QES potentials starting from the same QES potential. For simplicity we choose it as the one corresponding to the radial sextic oscillator [18] even if our developments can evidently be applied to any of the already known QES interactions.
2 Reducible second-order Darboux transformations

Let us start with the one-dimensional Schrödinger Hamiltonian $H$ as given in Eq. (1) with

$$V_0(x) = a^2 x^6 - 2a(2M + 2s + 1)x^2 + \frac{4(s - 1/4)(s - 3/4)}{x^2}, \quad x \in \mathbb{R}^+$$

which describes the radial sextic oscillator, a prototype of QES systems. In Eq. (11), $a$ and $s$ are positive constants while the positive integer $M$ is related to the number (equal to $M + 1$) of analytic eigenfunctions. These functions are expressed as

$$\psi_N(x) = \exp \left( -\frac{a^4}{4}x^4 \right) x^{2s-\frac{1}{2}} P_M^{(N)}(x^2), \quad N = 0, 1, \ldots, M$$

where

$$P_M^{(N)}(x^2) = \sum_{n=0}^{M} c_n^{(N)} x^{2n}.$$ (13)

The coefficients $c_n^{(N)}$ have to be determined through the resolution of a system of algebraic equations. Precisely these coefficients are

$$c_n^{(N)} = \frac{(-1)^{M-n}}{(M-n)!} Y_m^{(N)}(\alpha)$$

$$\equiv 1 \text{ if } n = M$$

$$\equiv \frac{(-1)^{M-n}}{(M-n)!} \sum_{i_1 \neq \ldots \neq i_{M-n}} \alpha_i^{(N)} \ldots \alpha_{i_{M-n}} \text{ if } n \neq M,$$ (14)

with the $\alpha_j^{(N)}$’s being known through [18].

$$\sum_{k=1}^{M} \frac{(4ax^4 - 8s)}{x^2 - \alpha_k^{(N)}} - \sum_{k \neq l=1}^{M} \frac{4x^2}{(x^2 - \alpha_k^{(N)})(x^2 - \alpha_l^{(N)})} - 4aMx^2 - E_N = 0.$$ (15)

Let us now turn to the Darboux transformation. In the reducible case we are concerned with, it is a product of two Darboux transformations as given in Eq. (3) with $W_0(x)$ being fixed according to Eq. (9). We first choose $\psi(x)$ to be the ground state of $H_0$ in order to prove that even in this simplified context we can generate new QES potentials. Indeed with such a choice we obtain

$$W_0(x) = ax^3 - \frac{2s - 1/2}{x} - \sum_{k=1}^{M} \frac{2x}{x^2 - \alpha_k^{(0)}}.$$ (16)

Therefore one can immediately obtain $M$ eigenfunctions $\phi_N(x)$ (as expressed in Eq. (6)) of the Hamiltonian $H_1$ defined in Eqs. (4) and (5)

$$\phi_N(x) = \exp \left( -\frac{a^4}{4}x^4 \right) x^{2s-\frac{1}{2}} W(P_M^{(0)}(x^2), P_M^{(N)}(x^2)), \quad N = 1, 2, \ldots, M.$$ (17)
The symbol $W$ stands for the usual Wronskian i.e.

$$W(f_1(x), f_2(x)) = f_1(x) \frac{df_2(x)}{dx} - f_2(x) \frac{df_1(x)}{dx}. \quad (18)$$

Taking $f_1(x)$ and $f_2(x)$ being respectively the polynomials $P^R_M(x^2)$ and $P^N_M(x^2)$ this Wronskian will be denoted by $W_{RN}(x)$. It is also clear from (17) that $\phi_0(x) = 0$ in accordance with Eq. (16).

The potential $V_1(x)$ related to these $M$ nontrivial solutions $\phi_N(x)$ is expressed through Eq. (5) as

$$V_1(x) = a^2 x^6 - 4a(M + s - 1)x^2 + \frac{4s^2 - 1/4}{x^2} + 4 \sum_{k=1}^{M} \frac{(2s + M - a(\alpha_k^{(0)})^2)}{x^2 - \alpha_k^{(0)}}$$

$$+ 8 \sum_{k=1}^{M} \frac{\alpha_k^{(0)}}{(x^2 - \alpha_k^{(0)})^2} + 2 \sum_{k \neq l=1}^{M} \frac{\alpha_k^{(0)} + \alpha_l^{(0)}}{(x^2 - \alpha_k^{(0)})(x^2 - \alpha_l^{(0)})}. \quad (19)$$

It is thus a QES potential and being not subtended by $sl(2, R)$, it has not been listed in [14]. Yet at this stage we have thus produced a new QES potential. This is a generalization of a previously obtained potential [17] which corresponds to the choice $a = 1/2$ and $s = 1/4$.

We now proceed further by going to the second first-order Darboux operator $L_1$ such that

$$H_1 = -\frac{d^2}{dx^2} + V_1(x) = L^\dagger_1 L_1 + E_1, \quad (20)$$

with

$$L_1 = \frac{d}{dx} + W_1(x) \equiv \frac{d}{dx} - \frac{d(\ln \phi_1(x))}{dx}. \quad (21)$$

We will then obtain once again a new QES potential $V_2(x)$ defined according to Eq. (5)

$$V_2(x) = V_1(x) + 2W_1'(x) \quad (22)$$

and the corresponding $(M - 1)$ eigenfunctions

$$\chi_N(x) \equiv L_1 \phi_N(x), \quad N = 2, ..., M. \quad (23)$$

The function $W_1(x)$ characterizing this second Darboux operator is determined according to Eqs. (21) and (17) and it is given by

$$W_1(x) = -W_0(x) + (E_1 - E_0) \frac{P_M^{(0)}(x^2) P_M^{(1)}(x^2)}{W_{01}(x)}. \quad (24)$$

It is then straightforward to obtain $V_2(x)$ and $\chi_N(x)$ through Eqs. (22) and (23), respectively. They are

$$V_2(x) = V_0(x) + 2(E_1 - E_0) \frac{d}{dx} \left( \frac{P_M^{(0)}(x^2) P_M^{(1)}(x^2)}{W_{01}(x)} \right) \quad (25)$$
and
\[ \chi_N(x) = \exp \left( -\frac{a}{4} x^4 \frac{x^{2s-1/2}}{W_{01}(x)} \right) \times \left[ -E_0 P_M^{(0)}(x^2) W_{1N}(x) + E_1 P_M^{(1)}(x^2) W_{0N}(x) - E_N P_M^{(N)}(x^2) W_{01}(x) \right], \]
\[ N = 2, \ldots, M. \]

Once again it is immediate to observe from Eq. (27) that \(\chi_0(x) = \chi_1(x) = 0\). We thus obtain in Eq. (27) the \((M - 1)\) eigenfunctions of the new QES potential \(V_2(x)\) given in Eq. (25).

As stated in the Introduction, this succession of two first-order Darboux operators is equivalent to a unique second-order Darboux operator defined by
\[ L \equiv L_1 L_0. \quad (27) \]
Following the general theory of Darboux transformations [3], and this can be directly checked through Eqs. (3), (16), (21) and (24), this operator is such that
\[ L \psi_N(x) = W^{-1}(\psi_0(x), \psi_1(x)) W(\psi_0(x), \psi_1(x), \psi_N(x)) \quad (28) \]
giving again the eigenfunctions \(\chi_N(x)\) defined in Eq. (27), while
\[ V_2(x) = V_0(x) - 2 \frac{d^2}{dx^2} \left[ \ln W(\psi_0(x), \psi_1(x)) \right]. \quad (29) \]
This Darboux operator \(L\) is thus the one connecting the two QES potentials \(V_0(x)\) and \(V_2(x)\) given in (24). The intermediate step \(V_1(x)\) (see formula (13)) is also physically meaningful. It is precisely the statement of reducible Darboux transformations but adapted here to QES equations.

We observe here the usual situation inherent for the potentials with the centrifugal term of the type \(l(l+1)/x^2\), \((l = 2s - 3/2)\). The Darboux transformation results in the change \(l \rightarrow l+1\) for this term even if it is not present in the initial potential (i.e. \(l = 0\)). This is due to zero boundary condition for the functions (12) and their specific asymptotic behaviour when \(x \rightarrow 0\), \(\psi_N(x) \sim x^{l+1}\).

We end this Section by a specific example. This will be in particular useful in order to compare the results we have obtained here with the ones of the irreducible context in the next Section. We thus take
\[ a = \frac{1}{2}, \quad M = 2 \quad (30) \]
so that the starting potential \(V_0(x)\) is
\[ V_0(x) = \frac{1}{4} x^6 - (5 + 2s)x^2 + \frac{4(s - 1/4)(s - 3/4)}{x^2}. \quad (31) \]
The corresponding eigenfunctions and eigenvalues are given by
\[ \psi_0(x) = \exp \left( -\frac{1}{8} x^4 x^{2s-1/2}(x^4 + 2\sqrt{4s + 1}x^2 + 4s) \right), \quad E_0 = -4\sqrt{4s + 1}, \quad (32) \]
\[ \psi_1(x) = \exp \left( -\frac{1}{8} x^4 x^{2s-1/2}(x^4 - 4s - 2) \right), \quad E_1 = 0, \quad (33) \]
\[ \psi_2(x) = \exp \left( -\frac{1}{8} x^4 x^{2s-1/2}(x^4 - 2\sqrt{4s + 1}x^2 + 4s) \right), \quad E_2 = 4\sqrt{4s + 1} \quad (34) \]
as seen from Eq. (12). The new QES potential $V_2(x)$ resulting from the second-order reducible operator $L$ given by (27) is (see Eq. (25))

$$V_2(x) = \frac{1}{4}x^6 - (2s - 1)x^2 + \frac{8s}{x^2} + \frac{8(x^2 - \sqrt{4s + 1})}{(x^4 + 2\sqrt{4s + 1}x^2 + 4s + 2)} - \frac{32x^2}{(x^4 + 2\sqrt{4s + 1}x^2 + 4s + 2)^2},$$

while its unique analytic eigenfunction $\chi_2(x)$ as well as the corresponding energy $E_2$ are

$$\chi_2(x) = \exp\left(-\frac{1}{8}x^4\right)\frac{x^{2s+3/2}}{(x^4 + 2\sqrt{4s + 1}x^2 + 4s + 2)}, \quad E_2 = 4\sqrt{4s + 1}. \quad (36)$$

Moreover there exists in this case a meaningful intermediate step characterized by the potential $V_1(x)$, i.e.,

$$V_1(x) = \frac{1}{4}x^6 - 2(s + 1)x^2 + \frac{(4s^2 - 1/4)}{x^2} + \frac{4(1 + \sqrt{4s + 1})}{x^2 + \sqrt{4s + 1} - 1} + \frac{4(1 - \sqrt{4s + 1})}{x^2 + \sqrt{4s + 1} + 1} + \frac{8(1 - \sqrt{4s + 1})}{(x^2 + \sqrt{4s + 1} + 1)^2} - \frac{8\sqrt{4s + 1}}{(x^2 + \sqrt{4s + 1} + 1)^2} - \frac{8\sqrt{4s + 1}}{x^2 + 2\sqrt{4s + 1}x^2 + 4s}, \quad (37)$$

Two eigenfunctions and eigenvalue related to this potential are also known. They are given by

$$\phi_1(x) = \exp\left(-\frac{1}{8}x^4\right)x^{2s+1/2}\frac{x^4 + 2\sqrt{4s + 1}x^2 + 4s + 2}{x^4 + 2\sqrt{4s + 1}x^2 + 4s}, \quad E_1 = 0, \quad (38)$$

$$\phi_2(x) = \exp\left(-\frac{1}{8}x^4\right)x^{2s+1/2}\frac{x^4 - 4s}{x^4 + 2\sqrt{4s + 1}x^2 + 4s}, \quad E_2 = 4\sqrt{4s + 1}. \quad (39)$$

### 3 Irreducible second-order Darboux transformations

#### 3.1 Transformations of the first type

We now turn to the irreducible context. We still consider at the start the potential $V_0(x)$ given in Eq. (11) as well as the corresponding eigenfunctions $\psi_N(x)$, $N = 0, 1, \ldots, M$ specified in Eq. (12). The main purpose here is to find an operator similar to the one given in Eq. (28) but being such that the factorization (27) cannot give rise to a physically meaningful $V_1(x)$. We thus propose to consider an operator $L$ being built on $\psi_1(x)$ and $\psi_2(x)$ i.e.

$$L\psi_N(x) \equiv W^{-1}(\psi_1(x), \psi_2(x))W(\psi_1(x), \psi_2(x), \psi_N(x)) \quad (40)$$

such that $L\psi_N(x)$ will stand for the $(M - 1)$ $(N = 0, 3, 4, \ldots, M)$ eigenfunctions of the new QES potential replacing $V_2(x)$ given in (29).
One can see, comparing Eq. (40) with Eq. (28), that the irreducible case can be deduced from the reducible one by the simple changes of indices: 0 → 1 and 1 → 2, in Eqs. (28) and (29) respectively. Through example (35) we thus get

\[ V_2(x) = \frac{1}{4} x^6 - (2s - 1)x^2 + \frac{8s}{x^2} + \frac{8(x^2 + \sqrt{4s + 1})}{(x^4 - 2\sqrt{4s + 1}x^2 + 4s + 2)} - \frac{32x^2}{(x^4 - 2\sqrt{4s + 1}x^2 + 4s + 2)^2} \]  

(41)

and

\[ \chi_0(x) = \exp\left(-\frac{1}{8} x^4\right) \frac{x^{2s+3/2}}{(x^4 - 2\sqrt{4s + 1}x^2 + 4s + 2)} , \quad E_0 = -4\sqrt{4s + 1} . \]  

(42)

One immediately notice that in this irreducible case, the potential \( V_1(x) \) as well as its eigenfunction \( \chi_0(x) \) can be deduced from their reducible analogues \( V(x) \) and \( \chi(x) \) by a simple change of sign in front of the square roots. However it is a new QES potential with a unique eigenfunction \( \chi_0(x) \) given in Eq. (42) and characterized this time by the lowest energy of the potential (11) when restricted to the choices (30).

As stated before, the Darboux operator \( L \) cannot give rise to a physically relevant intermediate potential \( V_1(x) \). Indeed, the operator \( L \) defined in Eq. (40) can be written as

\[ L_j = \frac{d}{dx} + w_j(x), \quad j = 0, 1 . \]  

(43)

More precisely, the function \( w_0(x) \), the irreducible analogue of \( W_0(x) \) given in Eq. (16), is determined as

\[ w_0(x) = \frac{1}{2} x^3 - \frac{(2s - 1/2)}{x} - 2x \left( \frac{1}{x^2 - \sqrt{4s + 1} - 1} + \frac{1}{x^2 - \sqrt{4s + 1} + 1} \right) , \]  

(44)

which effectively coincides with the one of Eq. (14) up to the choices (30) and the replacement of \( a_k^{(0)} \) \((k = 1, \ldots, M)\) by \( a_k^{(2)} \). As clear from Eq. (44), this leads to singularities which were not present in the reducible case (19). These singularities are also recovered at the step of the potential \( V_1(x) \) obtained through Eq. (4). Indeed we have (compare with Eq. (43))

\[ V_1(x) = \frac{1}{4} x^6 - 2(s + 1)x^2 + \frac{(4s^2 - 1/4)}{x^2} + \frac{4(1 + \sqrt{4s + 1})}{x^2 - \sqrt{4s + 1} - 1} + \frac{4(1 - \sqrt{4s + 1})}{x^2 - \sqrt{4s + 1} + 1} - \frac{8(1 + \sqrt{4s + 1})}{(x^2 - \sqrt{4s + 1} + 1)^2} + \frac{8(1 - \sqrt{4s + 1})}{(x^2 - \sqrt{4s + 1} - 1)^2} - \frac{8\sqrt{4s + 1}}{x^4 - 2\sqrt{4s + 1}x^2 + 4s} . \]  

(45)

This potential is ill-defined in the space \( \mathbb{R}_0^+ \) we are working in and cannot be considered as a convenient intermediate step proving then the irreducible character of \( L \).
3.2 Transformations of the second type

Let us now conclude with the third specific feature of the Darboux transformation namely the fact that one can construct physically significant new potentials from complex solutions of the initial equation. Once again we concentrate on the radial sextic oscillator characterized by the potential $V_0(x)$ given in Eq. (11). In order to illustrate our approach, we only need to find non-physical eigenfunctions corresponding to complex eigenvalues.

We shall now show that the potential (11) with the integer and half-integer values of $s$ has analytic solutions for some complex values of the "energy" (for simplicity we shall continue to use this terminology). For this purpose we note that this equation is covariant under the transformation $s \rightarrow 1 - s$, $M \rightarrow M + 2s - 1$. This means that for a given value of the positive integer $M$ and some values of the integer or half-integer $s$, the eigenfunction of $H_0$ with $s$ replaced by $\tilde{s} = 1 - s$ and $M$ by $\tilde{M} = M + 2s - 1$ is also the eigenfunction of $H_0$ with the given $M$ and $s$. Note that $\tilde{s}$ is negative for $s > 1$. The values of the energy calculated according to [18] with negative $s$ might be either complex or real. When they are real they lead to known solutions but for complex values of the energy we obtain new solutions of the Schrödinger equation. These new solutions when used to construct second order transformations lead to new real QES potentials. The second-order Darboux operator $\Lambda$ is defined in complete analogy with Eq. (28) by

$$\Lambda \psi_0(x) = W^{-1}(\psi(x), \bar{\psi}(x))W(\psi(x), \bar{\psi}(x), \psi_0(x)).$$  (46)

The state (46) will then be the eigenfunction (of the same eigenvalue as the one of $\psi_0(x)$) of the QES potential

$$\nu_2(x) = V_0(x) - 2 \frac{d^2}{dx^2} (\ln W(\psi(x), \bar{\psi}(x))).$$  (47)

Let us illustrate now this possibility by a specific example. We take $a = 1/2$, $M = 0$ and $s = 2$. In this case $\tilde{M} = 3$ and $\tilde{s} = -1$. The complex eigenfunction of the potential

$$V_0(x) = \frac{1}{4} x^6 - 5x^2 + \frac{35}{4x^2}$$  (48)

is

$$\psi(x) = x^{-5/2} \exp(-x^4/8)[\frac{1}{2} x^2(\frac{1}{2} x^4 - 5) + i\sqrt{5}(\frac{1}{2} x^4 - 1)].$$

It corresponds to the energy $E = -i4\sqrt{5}$. The single analytic eigenfunction (corresponding to $E_0 = 0$) of this potential (48) is

$$\psi_0(x) = \exp(-\frac{1}{8} x^4) x^{\frac{3}{2}}.$$  (49)

The new QES potential reads

$$\nu_2(x) = \frac{1}{4} x^6 + x^2 + \frac{3}{4x^2} + 16 \frac{-6x^2 + x^6}{20 + 4x^4 + x^8} - 2048 \frac{x^6}{(20 + 4x^4 + x^8)^2}$$  (50)

and its solution is

$$\Lambda \psi_0(x) = \frac{\exp(-\frac{1}{8} x^4) x^{\frac{3}{2}}(x^4 + 6)}{x^8 + 4x^4 + 20}, \quad E_0 = 0.$$  (51)
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References

[1] E. Schrödinger, Proc. Irish Acad. A46 (1940) 9; L. Infeld and T.E. Hull, Rev. Mod. Phys. 23 (1951) 21.

[2] E. Witten, Nucl. Phys. B185 (1981) 513.

[3] G. Darboux, Compt. Rend. Acad. Sci. Paris 94 (1882) 1343; V.B. Matveev and M.A. Salle, Darboux transformations and solitons. Springer, Berlin (1991).

[4] B.M. Levitan, Inverse Sturm-Liouville problems Nauka, Moscow (1984) (in Russian).

[5] A.A. Andrianov, Borisov N.V., M.V. Ioffe and M.I. Eides, Theor. Math. Phys. 61 (1984) 17 (in Russian).

[6] C.V. Sukumar, J. Phys. A 19 (1986) 2297

[7] V.G. Bagrov and B.F. Samsonov, Theor. Math. Phys. 104 (1995) 1051; V.G. Bagrov and B.F. Samsonov, Phys. Part. Nucl. 28 (1997) 951;

[8] M.G. Krein, Dokl. Akad. Nauk SSSR, 113 (1957) 970 (in Russian).

[9] B.F. Samsonov, Mod. Phys. Lett. A 11 (1996) 1563.

[10] B.F. Samsonov, Phys. Lett. A 263 (1999) 274.

[11] V.G. Bagrov and B.F. Samsonov, J. Moscow Phys. Soc. 5 (1995) 191; A.A. Andrianov, M.V. Ioffe, F. Cannata and J.P. Dedonder, Int. J. Mod. Phys. A10 (1995) 2683.

[12] J.-M. Sparenberg and D. Baye, Phys. Rev. C 55 (1997) 2175.

[13] M. Razavy, Phys. Lett. A82 (1981) 7; A.V. Turbiner and A.G. Ushveridze, Phys. Lett. A126 (1987) 181.

[14] A.V. Turbiner, Commun. Math. Phys. 118 (1988) 467.

[15] V.M. Tkachuk, Phys. Lett. A245 (1998) 177; I.V. Kuliy and V.M. Tkachuk, J. Phys. A32 (1999) 2157; S.N. Dolya and O.B. Zaslavskii, J. Phys. A34 (2001) 1981.

[16] F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics, World Scientific, Singapore (2001).

[17] M.A. Shifman, Int. J. Mod. Phys. A4 (1989) 3305.
[18] A.G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics*, IOP Publishing Ltd (1994).