Observability inequalities from measurable sets for some evolution equations

Gengsheng Wang ∗ Can Zhang†

Abstract

In this paper, we build up two observability inequalities from measurable sets in time for some evolution equations in Hilbert spaces from two different settings. The equation reads: \( u' = Au, \ t > 0, \) and the observation operator is denoted by \( B. \) In the first setting, we assume that \( A \) generates an analytic semigroup, \( B \) is an admissible observation operator for this semigroup (cf. [35]), and the pair \((A, B)\) verifies some observability inequality from time intervals. With the help of the propagation estimate of analytic functions (cf. [35]) and a telescoping series method provided in the current paper, we establish an observability inequality from measurable sets in time. In the second setting, we suppose that \( A \) generates a \( C_0 \) semigroup, \( B \) is a linear and bounded operator, and the pair \((A, B)\) verifies some spectral-like condition. With the aid of methods developed in [2] and [29] respectively, we first obtain an interpolation inequality at one time, and then derive an observability inequality from measurable sets in time. These two observability inequalities are applied to get the bang-bang property for some time optimal control problems.

Key words. Evolution equations in Hilbert spaces, observability inequality in measurable sets, telescoping series method, propagation estimate of analytic functions, bang-bang property of time optimal controls

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∗School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China (wanggs62@yeah.net). The authors are partially supported by the National Natural Science Foundation of China under grants 11161130003 and 11171264 and by the National Basis Research Program of China (973 Program) under grant 2011CB808002.
†School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China (zhang-cansx@163.com).
1 Introduction and main results

The aim of this study is to present an observability inequality from measurable sets in time for some parabolic-like evolution equations. Such an estimate was built up for the heat equation in [37] and was established for heat equations with lower order terms depending on both space and time variables \( x \) and \( t \) in [29]. To the best of our knowledge, it has not been touched upon for abstract evolution equations so far.

We start with introducing the evolution equation under study:

\[
\frac{du}{dt} = Au, \quad t > 0, \quad u(0) = u_0 \in X, \tag{1.1}
\]

where \( X \) is a Hilbert space and \( A : D(A) \subset X \to X \) is the infinitesimal generator of a \( C_0 \) semigroup \( \{S(t); t \geq 0\} \) in \( X \). Denote by \( \langle \cdot, \cdot \rangle_X \) and \( \| \cdot \|_X \) the inner product and the norm of \( X \) respectively, and endow the space \( D(A) \) with the graph norm.

We next introduce an observation operator \( B : X \to U \) from two cases. Here \( U \) is another Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_U \) and the norm \( \| \cdot \|_U \). For each Banach space \( Z, \mathcal{L}(Z, U) \) stands for the space of all linear bounded operators from \( Z \) to \( U \), with the usual norm \( \| \cdot \|_{\mathcal{L}(Z,U)} \). In the first case, we let \( B \in \mathcal{L}(D(A), U) \) hold the following two properties:

(a) \( B \) is an admissible observation operator for \( \{S(t); t \geq 0\} \), i.e., for each \( \tau > 0 \), there exists a positive constant \( C(\tau) \) such that

\[
\int_0^\tau \|BS(t)u_0\|_U^2 \, dt \leq C(\tau)\|u_0\|_X^2 \quad \text{for all} \quad u_0 \in D(A). \tag{1.2}
\]

(b) The pair \( (A, B) \) verifies the observability inequality from time intervals: There are two positive constants \( d \) and \( k \) such that for any \( L \in (0, 1] \),

\[
\|S(L)u_0\|_X^2 \leq e^{dL} \int_0^L \|BS(t)u_0\|_U^2 \, dt \quad \text{for all} \quad u_0 \in D(A). \tag{1.3}
\]

Here and throughout this paper, \( C(\cdots) \) denotes a positive constant depending on what are inclosed in the brackets, and may vary in different contexts. Our definition of admissible observation operators is quoted from [36, Chapter 4]. For more details on the above-mentioned inequality (1.3), we refer the readers to [5, Chapter 2] or [36, Chapter 6]. In the second case, we let \( B \in \mathcal{L}(X, U) \) be such that the pair \( (A, B) \) verifies the Hypothesis \( (H) \): There is a family of increasing subspaces \( \{E_{\lambda_m}\}_{m \geq 1} \) of \( X \), with

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \to +\infty,
\]

verifying...
(i) for each $m \in \mathbb{N}$, $S(t)E_{\lambda_m} \subset E_{\lambda_m}$ for all $t \geq 0$;

(ii) there is a constant $\mu > 0$ such that for each $m \in \mathbb{N}$,

$$\|S(t)g\|_X \leq e^{-\mu \lambda_m} \|g\|_X \text{ for all } g \in E_{\lambda_m}^\perp \text{ and } t > 0;$$

(iii) there are constants $\gamma \in (0,1)$ and $N \geq 1$ such that for each $m \in \mathbb{N}$,

$$\|f\|_X \leq Ne^{N\lambda_1} \|Bf\|_U \text{ for all } f \in E_{\lambda_m}.$$

Here, $E_{\lambda_m}^\perp$ is the orthogonal complementary subspace to $E_{\lambda_m}$ in $X$. We refer to [32] or [34] for a similar hypothesis condition to (H).

The main results of this paper are included in the following two theorems.

**Theorem 1.1.** Let $A$ generate an analytic semigroup $\{S(t); t \geq 0\}$ in $X$ and $B \in \mathcal{L}(D(A), U)$ verify the admissible observation condition (1.2). Assume that $(A, B)$ holds the observability inequality (1.3). Then, given $T > 0$ and a subset $E \subset (0, T)$ of positive measure, there exists a positive constant $C = C(E, T, d, k, \|B\|_{\mathcal{L}(D(A), U)})$ such that

$$\|S(T)u_0\|_X \leq C \int_E \|BS(t)u_0\|_U \, dt \text{ for all } u_0 \in D(A). \quad (1.4)$$

**Theorem 1.2.** Let $A$ generate a $C_0$ semigroup $\{S(t); t \geq 0\}$ in $X$ and $B \in \mathcal{L}(X, U)$. Assume that $(A, B)$ satisfies the Hypothesis (H). Then the following estimates hold:

(I) There exists a constant $C = C(N, \mu, \|B\|_{\mathcal{L}(X, U)}, \lambda_1) \geq 1$ such that when $t \in (0, 1]$,

$$\|S(t)u_0\|_X \leq \left( C \exp \left( Ct^{-\gamma} \right) \|BS(t)u_0\|_U \right)^{\frac{1}{2}} \|u_0\|_X^{\frac{3}{2}} \text{ for all } u_0 \in X. \quad (1.5)$$

(II) Given $T > 0$ and a subset $E \subset (0, T)$ of positive measure, there is a constant $C = C(E, T, N, \mu, \|B\|_{\mathcal{L}(X, U)}, \lambda_1, \gamma)$ such that

$$\|S(T)u_0\|_X \leq C \int_E \|BS(t)u_0\|_U \, dt \text{ for all } u_0 \in X. \quad (1.6)$$

Several remarks are given in order:

1. Theorem 1.1 can be applied to get the null controllability from measurable sets in time for several important equations: the internally controlled Stokes equations; the internally controlled degenerate parabolic equations associated with the Grushin operator in dimension 2; the boundary controlled heat equations, and so on. More importantly, with the aid of Theorem 1.1 we can build up the bang-bang property of time optimal control problems for the above-mentioned controlled equations. This property is extremely...
important in the studies of time optimal control problems (cf., e.g., [20], [21], [27], [30], [39], [40], [41]). These applications will be presented in Section 3 of this paper. It is worth mentioning that for the first two equations above-mentioned, the corresponding observability inequality (1.3) was built up in [6] and [4] respectively; while for the last equation, it was provided in [36].

(2) The inequality (1.5) is a quantitative unique continuation estimate at one time, while the inequality (1.6) is an observability estimate from measurable sets in time. They have been studied for heat equations with lower order terms depending on both space and time variables $x$ and $t$ in [28], [29] and [30]. We derive the estimate (1.6) from the inequality (1.5), via the method provided in [29]. In the case where $U = X$ and $B = I$ (the identity operator on $X$), one can directly check that

$$\|S(t)u_0\|_X \leq (C\|S(t)u_0\|_X)^{\frac{1}{2}}\|u_0\|_X^{\frac{1}{2}}$$

for all $u_0 \in X$, $t \in (0,1]$, which leads to (1.5). Consequently, (1.6) holds. Hence, the assumption (H) is not necessary in this case. From (1.6), the bang-bang property for the corresponding time optimal control problem follows. Such property for this special case was first established in [9] by a different way.

(3) Consider the more general evolution equation:

$$\frac{du}{dt} = A(t)u, \ t > 0, \ u(0) = u_0,$$

where $A(\cdot)$ verifies certain conditions such that the above equation is well-posed and the solution is analytic in time (cf. [11] Part 3, Theorem 2.2], [25, Chapter 5]). It seems for us that one can get a similar estimate to (1.4) for the aforementioned time-varying equation, through utilizing a similar method to that used in the proof of Theorem 1.1.

(4) We call the inequality (1.3) an $L^2$-observability inequality on time intervals, since the integral on its right hand side is the $L^2(0,T;U)$-norm of $BS(\cdot)u_0$. Sometimes, we prefer such estimate with the $L^2$-norm replaced by the $L^1$-norm. The latter is called the $L^1$-observability inequality on time intervals. In Section 2, we provide a telescoping series method, by which one can derive the $L^1$-observability inequality on time intervals from the $L^2$-observability inequality on time intervals.

(5) Observability inequalities from time intervals for linear parabolic equations, which grows like (1.3), have been studied in many publications (cf., e.g., [2], [7], [12], [13], [18], [22], [31] and the references therein). Recently, the observability inequality from measurable sets of positive measure for the heat equation has been established in [1], [2] and [12] (with the help of a propagation estimate of smallness for analytic functions). For some general parabolic equations (or systems) with time-independent and analytic coefficients, we refer the reader to [8].
The rest of this paper is organized as follows. Section 2 is devoted to the proofs of
Theorems 1.1 and 1.2. Section 3 presents some applications of Theorems 1.1 and 1.2 to
time optimal control problems.

Notation. For each measurable set $E \subset \mathbb{R}^n$, $\chi_E$ and $|E|$ stand for the characteristic
function and the Lebesgue measure of the set, respectively. For a smooth function $g : \mathbb{R} \to \mathbb{R}$, we write $g^{(\beta)}$, $\beta \in \mathbb{N}$, for the $\beta$-th order derivative. Sometimes we also write $e^{tA}$
for the semigroup generated by $A$, instead of $\{S(t); \ t \geq 0\}$. Write $\mathbb{R}^+$ for the interval $[0, \infty)$. Denote by $A^*$ and $B^*$ the adjoint operators of $A$ and $B$ respectively. Write $D(A)$
and $D(A^*)$ for the domains of $A$ and $A^*$ respectively.

2 Proofs of Theorems 1.1 and 1.2

In this section, we first prove Theorem 1.1 and Theorem 1.2 respectively, and then intro-
duce a telescoping series method, by which one can derive the $L^1$-observability inequality
on time intervals from the $L^2$-observability inequality on time intervals.

2.1 The proof of Theorem 1.1

The proof of Theorem 1.1 is based on several lemmas. The first one concerns with an
analyticity property of the function:

\[ g(t; u_0) \triangleq \| BS(t)u_0 \|_U^2, \quad t > 0, \]  

(2.1)

where \( \{S(t); t \geq 0\} \) is an analytic semigroup with the generator $A$, $u_0 \in D(A)$ and
$B \in \mathcal{L}(D(A), U)$.

Lemma 2.1. For each $u_0 \in D(A)$, the function $g(\cdot) \triangleq g(\cdot; u_0)$ is analytic in $(0, +\infty)$. Furthermore, there are constants $K \geq 1$ and $\rho \in (0, 1)$ independent of $u_0$
such that

\[ |g^{(\beta)}(t)| \leq K \frac{(t-s)^{-2\beta!}}{\rho(t-s)^2} \| u(s) \|_X^2 \quad \text{for all } \beta \in \mathbb{N}, \]  

when $0 < t - s \leq 1$, where $u(\cdot) \triangleq S(\cdot)u_0$.

Proof. By the translation, it suffices to prove the desired estimate for the case that $s = 0$
and $0 < t \leq 1$. We first assume that $\{S(t); t \geq 0\}$ is an uniformly bounded analytic
semigroup with

\[ \| S(t) \|_{\mathcal{L}(X,X)} \leq M \quad \text{for all } t > 0, \]  

when $0 < t - s \leq 1$. The proof of Lemma 2.1 is analogous to that of Lemma 2.1 in
[17].
for some positive constant $M$. By (2.1) and the binomial formula, we have

$$g^{(\beta)}(t) = \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1! \beta_2!} (Bu^{(\beta_1)}(t), Bu^{(\beta_2)}(t))_U$$

for all $\beta \in \mathbb{N}$ and $t \in (0, 1]$.

It follows from the Cauchy-Schwartz inequality that for any $t \in (0, 1]$,\textcolor{red}{\text{missing content}}

Meanwhile, since

$$\|AS(t)\|_{\mathcal{L}(X, X)} = \|S'(t)\|_{\mathcal{L}(X, X)} \leq \frac{C}{t}, \quad t > 0,$$

for some constant $C > 0$ (cf., e.g., [23, Chapter 2, Theorem 5.2]), and because

$$S^{(m)}(t) = (AS\left(\frac{t}{m}\right))^m = (S'\left(\frac{t}{m}\right))^m, \quad t > 0, m \in \mathbb{N},$$

there is a constant $\rho \in (0, 1)$ independent of $u_0$ such that

$$\|u^{(m)}(t)\|_X \leq \left(\frac{C}{(t/m)^m}\right)^m \|u_0\|_X \leq \frac{m!}{(pt)^m} \|u_0\|_X \quad \text{for all} \quad t \in (0, 1] \quad \text{and} \quad m \in \mathbb{N}.$$\textcolor{red}{\text{missing content}}

In the last inequality above, we used the Stirling formula: $m^m \lesssim e^m m!$, $m \in \mathbb{N}$. Consequently,

$$\|Au^{(m)}(t)\|_X = \|u^{(m+1)}(t)\|_X \leq \frac{(m+1)!}{(pt)^{m+1}} \|u_0\|_X \quad \text{for all} \quad t \in (0, 1] \quad \text{and} \quad m \in \mathbb{N}.$$\textcolor{red}{\text{missing content}}

Along with the above two estimates, (2.2) leads to

$$|g^{(\beta)}(t)| \leq 4\|B\|_{\mathcal{L}(D(A), U)}^2 \sum_{\beta_1 + \beta_2 = \beta} \frac{\beta!}{\beta_1! \beta_2!} \frac{(\beta_1 + 1)! (\beta_2 + 1)!}{(pt)^{\beta_1+1} (pt)^{\beta_2+1}} \|u_0\|^2_X$$

$$\leq 4\|B\|_{\mathcal{L}(D(A), U)}^2 \beta! (pt)^{-\beta-2} \|u_0\|^2_X \sum_{\beta_1 + \beta_2 = \beta} (\beta_1 + 1)(\beta_2 + 1)$$

$$\leq 4\|B\|_{\mathcal{L}(D(A), U)}^2 \beta! (pt/8)^{-\beta-2} \|u_0\|^2_X \quad \text{for all} \quad t \in (0, 1].$$
Thus,
\[ |g^{(\beta)}(t)| \leq N\beta!(\rho t)^{-\beta} \] for all $\beta \in \mathbb{N}$, $t > 0$, with $N = 4\|B\|_{L(C(A),U)}^2(\rho t)^{-2}\|u_0\|_X^2$,
for some new constant $\rho \in (0,1)$ independent of $u_0$. This implies the desired estimate for the case where the analytic semigroup $\{S(t); t \geq 0\}$ is uniformly bounded.

Next, we remove the assumption of the uniform boundedness from the analytic semigroup $\{S(t); t \geq 0\}$. Since $\|S(t)\|_{L(X,X)} \leq Me^{\alpha t}$, $t > 0$, for some constants $M > 0$ and $\alpha > 0$, the semigroup $\{\tilde{S}(t); t \geq 0\}$ with $\tilde{S}(t) \triangleq e^{-\alpha t}S(t)$ for $t > 0$, is uniformly bounded and analytic. Given $u_0 \in D(A)$, define
\[ \tilde{g}(t) \triangleq \|B\tilde{S}(t)u_0\|_{U}^2, \quad t > 0. \]
It is clear that
\[ g(t) = e^{2\alpha t}\tilde{g}(t), \quad t > 0, \]
where $g$ is the function given by (2.1) corresponding to the same $u_0$ as above. We have already verified that there is a $\tilde{\rho} \in (0,1)$ independent of $u_0$ such that
\[ |g^{(\beta)}(t)| \leq N\beta!(\tilde{\rho} t)^{-\beta}, \quad \text{with} \quad N = 4\|B\|_{L(C(A),U)}^2(\tilde{\rho} t)^{-2}\|u_0\|_X^2, \quad \text{for all} \quad \beta \in \mathbb{N}, \quad t > 0. \] (2.3)
Notice that
\[ g^{(\beta)}(t) = \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} \frac{\beta!}{\beta_1!\beta_2!\beta_3!} (2\alpha)^{-\beta} e^{2\alpha t} \tilde{g}^{(\beta_2)}(t), \quad t > 0. \]
This, along with (2.3), implies the desired inequality for the case when $s = 0$ and $0 < t \leq 1$, and completes the proof. \(\square\)

Next, we recall the following lemma, which is a propagation of smallness estimate from measurable sets for analytic functions in $\mathbb{R}$ (cf., e.g., [35, 1, Lemma 2] or [2, Lemma 13]).

**Lemma 2.2.** Let $f: [a, a + s] \to \mathbb{R}$, where $a \in \mathbb{R}$ and $s > 0$, be an analytic function satisfying
\[ |f^{(\beta)}(x)| \leq M\beta!(s\rho)^{-\beta} \] for all $x \in [a, a + s]$ and $\beta \in \mathbb{N},$
with some constants $M > 0$ and $\rho \in (0,1]$. Assume that $\tilde{E} \subset [a, a + s]$ is a subset of positive measure. Then there are two constants $C = C(\rho, |\tilde{E}|/s) \geq 1$ and $\vartheta = \vartheta(\rho, |\tilde{E}|/s)$ with $\vartheta \in (0,1)$ such that
\[ \|f\|_{L^\infty(a,a+s)} \leq CM^{1-\vartheta} \left( \frac{1}{|\tilde{E}|} \int_{\tilde{E}} |f(x)| \, dx \right)^{\vartheta}. \]
When \((A, B)\) verifies the observability inequality (1.3), we can make use of Lemma 2.1 and Lemma 2.2 to prove the interpolation inequality presented in the following lemma.

**Lemma 2.3.** Suppose that the conditions in Theorem 1.1 hold. Let \(0 \leq t_1 < t_2\) with \(0 < t_2 - t_1 \leq 1\). Assume that \(E \subset [t_1, t_2]\) is a subset of positive measure and verifies \(|E \cap (t_1, t_2)| \geq \eta(t_2 - t_1)\) with \(\eta \in (0, 1)\). Then there are two positive constants \(C = C(d, k, \rho, \eta, \|B\|_{L(D(A), U)})\) (where \(d, k\) are given by (1.3) and \(\rho\) is given by Lemma 2.1) and \(\theta = \theta(\rho, \eta) \in (0, 1)\) such that for any \(u_0 \in D(A)\), the corresponding solution \(u\) to Equation (1.1) satisfies

\[
\|u(t_2)\|_X \leq \left( Ce^{\frac{C}{(t_2-t_1)^2}} \int_{t_1}^{t_2} \chi_E(t) \|Bu(t)\|_U dt \right)^{\theta} \|u(t_1)\|_X^{1-\theta}. \tag{2.4}
\]

**Proof.** Set

\[
\tau = t_1 + \frac{\eta}{10}(t_2 - t_1) \quad \text{and} \quad \hat{E} = E \cap [\tau, t_2].
\]

Clearly,

\[
|\hat{E}| \geq \frac{\eta(t_2 - t_1)}{2}. \tag{2.5}
\]

By Lemma 2.1, we get that for any \(t \in [\tau, t_2]\),

\[
|g(\beta)(t)| \leq K \left( \frac{(t-t_1)^{-2}}{(\rho(t-t_1))^\beta} \right) \|u(t_1)\|_X^2 \leq K \left( \frac{(\tau-t_1)^{-2}}{(\rho(\tau-t_1))^\beta} \right) \|u(t_1)\|_X^2
\]

\[
\leq K \left( \frac{(\eta(t_2-t_1)/10)^{-2}}{(\rho\eta(t_2-\tau)/10)^\beta} \right) \|u(t_1)\|_X^2
\]

\[
\triangleq M \beta! (s\rho_1)^{-\beta} \quad \text{for all} \quad \beta \in \mathbb{N},
\]

with

\[
M = 100\eta^{-2}(t_2 - t_1)^{-2}K\|u(t_1)\|_X^2, \quad \rho_1 = \frac{\rho\eta}{10} \quad \text{and} \quad s = t_2 - \tau.
\]

According to Lemma 2.2, there are positive constants \(C = C(K, \rho, \eta)\) and \(\vartheta = \vartheta(\rho, \eta) \in (0, 1)\) such that

\[
\|g\|_{L^\infty(\tau, t_2)} \leq (t_2 - t_1)^{-2}C\|u(t_1)\|_X^{2(1-\vartheta)} \left( \frac{1}{|\hat{E}|} \int_{\hat{E}} |g(s)| ds \right)^\vartheta,
\]

which is equivalent to

\[
\|Bu(t)\|_{L^2}^2 \leq (t_2 - t_1)^{-2}C\|u(t_1)\|_X^{2(1-\vartheta)} \left( \frac{1}{|\hat{E}|} \int_{\hat{E}} \|Bu(s)\|_{L^2}^2 ds \right)^\vartheta \quad \text{for all} \quad t \in [\tau, t_2]. \tag{2.6}
\]
By the translation and the observability inequality (1.3), we have
\[ \|u(t_2)\|_X^2 \leq e^{\frac{d}{(t_2-\tau)^r}} \int_{\tau}^{t_2} \|Bu(t)\|_{U}^2 \, dt \leq e^{\frac{C(d,k,r)}{(t_2-\tau)^r}} \int_{\tau}^{t_2} \|Bu(t)\|_{U}^2 \, dt. \]

This, along with (2.5) and (2.6), leads to
\[ \|u(t_2)\|_X^2 \leq e^{\frac{C(n,d,k)}{(t_2-\tau)^r}(t_2-\tau)(t_2-t_1)^{-2}C\|u(t_1)\|_X^{2(1-\theta)}} \left( \frac{1}{|E|} \int_E \|Bu(s)\|_{U}^2 \, ds \right)^{\theta}, \]
\[ \leq e^{\frac{C(n,d,k)}{(t_2-\tau)^r}(t_2-t_1)^{-2}C\|u(t_1)\|_X^{2(1-\theta)}} \max_{t \in [\tau,t_2]} \|Bu(t)\|_{U}^\theta \left( \int_E \|Bu(s)\|_{U} \, ds \right)^{\theta}. \]

By the properties of analytic semigroups, we see that for any \( t \in [\tau,t_2], \)
\[ \|Bu(t)\|_U \leq B\|L(D(A),U)(\|Au(t)\|_X + \|u(t)\|_X) \leq C(t_2-t_1)^{-1}\|u(t_1)\|_X, \]
with some constant \( C = C(B\|L(D(A),U)) > 0. \) This, together with (2.7), indicates that
\[ \|u(t_2)\|_X^2 \leq e^{\frac{C(n,d,k)}{(t_2-\tau)^r}(t_2-t_1)^{-2}C\|u(t_1)\|_X^{2(1-\theta)}} \left( \frac{1}{|E|} \int_E \|Bu(s)\|_{U}^2 \, ds \right)^{\theta} \]
\[ \leq (t_2-t_1)^{-3} e^{\frac{C(n,d,k)}{(t_2-\tau)^r}} \|u(t_1)\|_X^{2-\theta} \left( \int_E \|Bu(s)\|_{U} \, ds \right)^{\theta}. \]

This, along with the estimate \((t_2-t_1)^{-3} \leq e^{3/[k(t_2-t_1)^k]}, \) leads to (2.4), and completes the proof. \( \square \)

We end this subsection with presenting the proof of Theorem 1.1. The proof is based on Lemma 2.3 and the telescoping series method (provided in [2]), which is a modified version of that in [29].

**The proof of Theorem 1.1.** Let \( \ell \in (0,T) \) be a Lebesgue density point of \( E. \) Then for each constant \( q \in (0,1) \) which is to be fixed later, there exists a monotone decreasing sequence \( \{\ell_m\}_{m \geq 1} \subset (0,T), \) with \( 0 < \ell_1 - \ell_2 \leq 1, \) such that (cf. [29, Proposition 2.1])
\[ \ell_{m+1} - \ell_{m+2} = q(\ell_m - \ell_{m+1}), \quad |E \cap (\ell_{m+1}, \ell_m)| \geq \frac{\ell_m - \ell_{m+1}}{3} \quad \text{for all } m \geq 1, \quad (2.8) \]
and such that
\[ \lim_{m \to +\infty} \ell_m = \ell. \quad (2.9) \]

Given \( u_0 \in D(A), \) write \( u(\cdot) = S(\cdot)u_0. \) According to Lemma 2.3, there are constants \( C = C(d,k,\rho,\|B\|_{L(D(A),U)}) \geq 1 \) and \( \theta = \theta(\rho) \in (0,1) \) such that when \( m \geq 1, \)
\[ \|u(\ell_m)\|_X \leq \left( C e^{\frac{C}{(\ell_m - \ell_{m+1})^r}} \int_{\ell_{m+1}}^{\ell_m} \chi_E \|Bu(t)\|_{U} \, dt \right)^{\theta} \|u(\ell_{m+1})\|_X^{1-\theta}. \]
This, together with Young’s inequality:

\[ ab \leq \varepsilon a^p + \varepsilon^{-\frac{p}{r}} b^r, \quad \text{when} \quad a > 0, b > 0, \varepsilon > 0, \]

with \( \frac{1}{p} + \frac{1}{r} = 1, \ p > 1, \ r > 1, \)

indicates that when \( m \geq 1, \)

\[
\| u(\ell_m) \|_X \leq \varepsilon \| u(\ell_{m+1}) \|_X + \varepsilon^{-\theta} C e^{(\ell_m - \ell_{m+1})k} \int_{\ell_{m+1}}^{\ell_m} \chi_E \| Bu(t) \|_U dt \quad \text{for all} \quad \varepsilon > 0,
\]

which is equivalent to

\[
\varepsilon^{1-\theta} e^{\frac{C}{(\ell_m - \ell_{m+1})k}} \| u(\ell_m) \|_X - \varepsilon e^{-\frac{C}{(\ell_{m+1} - \ell_m)k}} \| u(\ell_{m+1}) \|_X \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E \| Bu(t) \|_U dt \quad \text{for all} \quad \varepsilon > 0. \tag{2.10}
\]

By letting \( \varepsilon = e^{-1/[(\ell_m - \ell_{m+1})k]} \) in (2.10), we have

\[
\varepsilon^{1-\theta} e^{\frac{C+1-\theta}{(\ell_m - \ell_{m+1})k}} \| u(\ell_m) \|_X - \varepsilon e^{-\frac{C+1-\theta}{(\ell_{m+1} - \ell_m)k}} \| u(\ell_{m+1}) \|_X \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E \| Bu(t) \|_U dt \quad \text{for all} \quad m \geq 1. \tag{2.11}
\]

We now take

\[
q = \left( \frac{C + 1 - \theta}{C + 1} \right)^{\frac{1}{k}} \in (0, 1), \quad \text{where} \ C \text{ and } \theta \text{ are given in (2.11)}.
\]

It follows from (2.11) and the first formula of (2.8) that

\[
\varepsilon^{1-\theta} e^{-\frac{C+1-\theta}{(\ell_m - \ell_{m+1})k}} \| u(\ell_m) \|_X - \varepsilon e^{-\frac{C+1-\theta}{(\ell_{m+1} - \ell_m)k}} \| u(\ell_{m+1}) \|_X \leq C \int_{\ell_{m+1}}^{\ell_m} \chi_E \| Bu(t) \|_U dt \quad \text{for all} \quad m \geq 1.
\]

Summing the above inequality from \( m = 1 \) to \(+\infty\), and noticing the convergence (2.9), as well as

\[
\sup_{t \in (0, T)} \| u(t) \|_X < +\infty,
\]

we see that

\[
\| u(\ell_1) \|_X \leq C e^{(\ell_1 - t_2)k} \int_{\ell}^{\ell_1} \chi_E \| Bu(t) \|_U dt.
\]

Because \( \| u(T) \|_X \leq C \| u(\ell_1) \|_X \), the above leads to (1.4). This completes the proof. \( \square \)
2.2 The proof of Theorem 1.2

The main idea of the proof is borrowed from [2, Theorem 6].

The proof of Theorem 1.2. We begin with proving the interpolation inequality (1.3). For each \( \lambda \geq \lambda_1 \), we define

\[
E_\lambda = \bigcup_{\lambda_k \leq \lambda} E_{\lambda_k},
\]

which is a subspace of \( X \). Denote by \( E_\lambda \) the orthogonal projection operator from \( X \) to \( E_\lambda \). Given \( u_0 \in X \), write \( E_\lambda^\perp u_0 = u_0 - E_\lambda u_0 \). Because

\[
\|S(t)u_0\|_X \leq \|S(t)E_\lambda u_0\|_X + \|S(t)E_\lambda^\perp u_0\|_X,
\]

we conclude from the properties (i) and (iii) of Hypothesis (H) that

\[
\|S(t)E_\lambda u_0\|_X \leq Ne^{N\lambda \gamma} \|BS(t)E_\lambda u_0\|_U \\
\leq Ne^{N\lambda \gamma} (\|BS(t)u_0\|_U + \|BS(t)E_\lambda^\perp u_0\|_U) \\
\leq Ne^{N\lambda \gamma} (\|BS(t)u_0\|_U + \|B\|_{L(X,U)} \|S(t)E_\lambda^\perp u_0\|_X).
\]

This, together with (2.12), implies that

\[
\|S(t)u_0\|_X \leq Ne^{N\lambda \gamma} (1 + \|B\|_{L(X,U)}) (\|BS(t)u_0\|_U + \|S(t)E_\lambda^\perp u_0\|_X).
\]

By the property (ii) of Hypothesis (H), we have

\[
\|S(t)E_\lambda^\perp u_0\|_X \leq e^{-\mu \lambda t} \|E_\lambda^\perp u_0\|_X \leq e^{-\mu \lambda t} \|u_0\|_X.
\]

Along with (2.13), this yields that for any \( \lambda \geq \lambda_1 \),

\[
\|S(t)u_0\|_X \leq N(1 + \|B\|_{L(X,U)}) \left[ \exp \left( N\lambda \gamma - \frac{\mu \lambda t}{2} \right) \right] (e^{\mu \lambda t/2} \|BS(t)u_0\|_U + e^{-\mu \lambda t/2} \|u_0\|_X).
\]

Because

\[
\max_{\lambda > 0} \left\{ N\lambda \gamma - \frac{\mu \lambda t}{2} \right\} \leq N\left( \frac{2\gamma N}{\mu t} \right)^{\frac{1}{1-\gamma}}, \text{ when } \gamma \in (0,1),
\]

there is a constant \( K = K(N, \mu, \gamma, \|B\|_{L(X,U)}) \) such that

\[
\|S(t)u_0\|_X \leq Ke^{Kt} \left( e^{\mu \lambda t/2} \|BS(t)u_0\|_U + e^{-\mu \lambda t/2} \|u_0\|_X \right) \text{ for all } \lambda \geq \lambda_1,
\]

which is equivalent to

\[
\|S(t)u_0\|_X \leq Ke^{Kt} \left( e^{-1} \|BS(t)u_0\|_U + \varepsilon \|u_0\|_X \right) \text{ for all } \varepsilon \in (0,e^{-\mu \lambda_1 t/2}].
\]
Since
\[ \|S(t)u_0\|_X \leq M\|u_0\|_X, \quad \text{when } t \in (0, 1), \quad \text{for some } M > 0, \]
it holds that for each \( t \in (0, 1), \)
\[ \|S(t)u_0\|_X \leq Me^{\mu \lambda t/2} \varepsilon \|u_0\|_X \quad \text{for all } \varepsilon \geq e^{-\mu \lambda t/2}. \]
This, combined with (2.14), leads to
\[ \|S(t)u_0\|_X \leq Me^{\mu \lambda t} Ke^{K't} \left( \varepsilon^{-1} \|BS(t)u_0\|_U + \varepsilon \|u_0\|_X \right) \quad \text{for all } \varepsilon \in (0, +\infty). \]
Minimizing the above inequality with respect to \( \varepsilon \) gives the desired estimate (1.5).

We next show the observability inequality (1.6) through utilizing a telescoping series method. Let \( \ell \in (0, T) \) be a Lebesgue point of \( E \). For each constant \( q \in (0, 1) \) which will be precised later, there exists a monotone decreasing sequence \( \{\ell_m\}_{m \geq 1} \) satisfying (2.8), (2.9) and \( 0 < \ell_1 - \ell_2 \leq 1 \) (cf. [29, Proposition 2.1]). Let us set
\[ \tau_m = \ell_{m+1} + \frac{\ell_m - \ell_{m+1}}{6} \quad \text{for all } m \geq 1. \]
By the inequality (1.5), we deduce that for any \( t \in [\tau_m, \ell_m] \) and any \( u_0 \in X, \)
\[ \|S(t)u_0\|_X \leq \left( C \exp \left( C(t - \ell_{m+1})^{-\gamma} \right) \|BS(t)u_0\|_U \right)^{1/2} \|S(\ell_{m+1})u_0\|_X^{1/2}, \]
with some constant \( N \geq 1. \) Because
\[ \|S(\ell_m)u_0\|_X \leq M\|S(t)u_0\|_X \quad \text{for some } M > 0 \quad \text{and for all } t \in [\tau_m, \ell_m], \]
the estimate (2.15) implies that
\[ \|S(\ell_m)u_0\|_X \leq \left( N \exp \left( N(\ell_m - \ell_{m+1})^{-\gamma} \right) \|BS(t)u_0\|_U \right)^{1/2} \|S(\ell_{m+1})u_0\|_X^{1/2} \quad \text{for all } t \in [\tau_m, \ell_m]. \]
Then by Young’s inequality, we have
\[ \|S(\ell_m)u_0\|_X \leq \varepsilon \|S(\ell_{m+1})u_0\|_X + \varepsilon^{-1} N \exp \left( N \right) \|BS(t)u_0\|_U \quad \text{for all } t \in [\tau_m, \ell_m]. \]
Integrating the above inequality over \([\tau_m, \ell_m] \cap E \) and noting that
\[ |(\tau_m, \ell_m] \cap E | \geq \left( \ell_m - \ell_{m+1} \right)/6, \]

The text continues with further mathematical analysis and derivations.
we obtain that for any $\varepsilon > 0$,
\[
\|S(\ell_m)u_0\|_X \leq \varepsilon \|S(\ell_{m+1})u_0\|_X + \varepsilon^{-1}N \exp \left( - \frac{N}{(\ell_m - \ell_{m+1})^{\frac{1}{\gamma}}} \right) \int_{\ell_{m+1}}^{\ell_m} \chi_E(t)\|BS(t)u_0\|_U \, dt.
\]
By taking
\[
\varepsilon = \exp \left( - \frac{1}{2(\ell_m - \ell_{m+1})^{\frac{1}{\gamma}}} \right)
\]
in the above inequality, we see that
\[
\exp \left( - \frac{N + \frac{1}{2}}{(\ell_m - \ell_{m+1})^{\frac{1}{\gamma}}} \right) \|S(\ell_m)u_0\|_X - \exp \left( - \frac{N + 1}{(\ell_m - \ell_{m+1})^{\frac{1}{\gamma}}} \right) \|S(\ell_{m+1})u_0\|_X \\
\leq N \int_{\ell_{m+1}}^{\ell_m} \chi_E(t)\|BS(t)u_0\|_U \, dt.
\] (2.16)

We now take
\[
q = \left( \frac{N + \frac{1}{2}}{N + 1} \right)^{\frac{1}{\gamma}} \in (0, 1).
\]
It follows from (2.16) that
\[
\exp \left( - \frac{N + \frac{1}{2}}{(\ell_m - \ell_{m+1})^{\frac{1}{\gamma}}} \right) \|S(\ell_m)u_0\|_X - \exp \left( - \frac{N + \frac{1}{2}}{[q(\ell_m - \ell_{m+1})]^{\frac{1}{\gamma}}} \right) \|S(\ell_{m+1})u_0\|_X \\
\leq N \int_{\ell_{m+1}}^{\ell_m} \chi_E(t)\|BS(t)u_0\|_U \, dt.
\]

Summing the above inequality with respect to $m$ from 1 to $+\infty$, using (2.8) and (2.9), we deduce the desired estimate (1.6) immediately. This completes the proof. \hfill \square

2.3 A telescoping series method

In this subsection, we introduce a telescoping series method, by which one can derive the $L^1$-observability inequality from time intervals through the $L^2$-observability inequality from time intervals for the equation (1.1). The main result of this subsection is as follows.

**Proposition 2.4.** Let $A : D(A) \subset X \to X$ generate a $C_0$ semigroup $\{S(t); t \geq 0\}$ in $X$, such that
\[
\|S(t)\|_{\mathcal{L}(X,X)} \leq Me^{\alpha t} \text{ for all } t \geq 0,
\]
where $M > 0$ and $\alpha \in \mathbb{R}^+$ are independent of $t$. Let $B \in \mathcal{L}(X,U)$. Suppose that there are two positive constants $d$, $k$ and a nondecreasing function $\theta(\cdot)$ from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that
\[
\|S(L)u_0\|_X \leq \theta(L)e^{\frac{d}{2}}\left(\int_0^L \|BS(t)u_0\|_U^2 \, dt \right)^{1/2} \quad \text{for all } L > 0 \text{ and } u_0 \in X. \tag{2.17}
\]
Then there exists a positive constant $N = N(d,k)$ such that
\[
\|S(T)u_0\|_X \leq F(T)e^{\frac{N}{2}}\int_0^T \|BS(t)u_0\|_U \, dt \quad \text{for all } T > 0 \text{ and } u_0 \in X, \tag{2.18}
\]
where $F(\cdot)$ is a function defined by
\[
F(T) = \theta(T)^2\|B\|_{\mathcal{L}(X,U)}Me^{\alpha T}, \quad T > 0. \tag{2.19}
\]
\textbf{Proof.} Let $T > 0$ and $u_0 \in X$. For each $q \in (0,1)$, we define a sequence of real numbers $\{\ell_m\}_{m \geq 0}$ by
\[
\ell_m = q^mT \quad \text{for all } m \geq 0.
\]
Clearly,
\[
\ell_{m+1} - \ell_{m+2} = q(\ell_m - \ell_{m+1}) \quad \text{and} \quad \lim_{m \to +\infty} (\ell_m - \ell_{m+1}) = 0. \tag{2.20}
\]
By the translation, we see from (2.17) that for any $m \geq 0$,
\[
\|S(\ell_m)u_0\|_X \leq \theta(\ell_m - \ell_{m+1})e^{\frac{d}{2}(\ell_m - \ell_{m+1})^2\left(\int_{\ell_{m+1}}^{\ell_m} \|BS(t)u_0\|_U^2 \, dt \right)^{1/2}}. \tag{2.21}
\]
Since
\[
\max_{t \in (\ell_{m+1}, \ell_m)} \|BS(t)u_0\|_X \leq \|B\|_{\mathcal{L}(X,U)}Me^{\alpha(\ell_m - \ell_{m+1})}\|S(\ell_{m+1})u_0\|_X,
\]
the estimate (2.21), together with (2.19), leads to
\[
\|S(\ell_m)u_0\|_X \leq \left(F(\ell_m - \ell_{m+1})e^{\frac{2d}{(\ell_m - \ell_{m+1})^2}}\int_{\ell_{m+1}}^{\ell_m} \|BS(t)u_0\|_U \, dt \right)^{1/2} \|S(\ell_{m+1})u_0\|_X^{1/2}. \tag{2.22}
\]
Because
\[
F(\ell_m - \ell_{m+1}) \leq F(T) \quad \text{for all } m \geq 0,
\]
by applying the Young inequality to (2.22), we see that
\[
\|S(\ell_m)u_0\|_X \leq \varepsilon\|S(\ell_{m+1})u_0\|_X + \varepsilon^{-1}F(T)e^{\frac{2d}{(\ell_m - \ell_{m+1})^2}}\int_{\ell_{m+1}}^{\ell_m} \|BS(t)u_0\|_U \, dt \quad \text{for each } \varepsilon > 0.
\]
Multiplying the above inequality by $\varepsilon e^{-\frac{2d}{(\ell_m-\ell_{m+1})^k}}$ and then taking $\varepsilon = e^{-\frac{1}{(\ell_m-\ell_{m+1})^k}}$ in the resulting inequality, we obtain that

$$e^{-\frac{2d}{(\ell_{m+1})^k}}\|S(\ell_m)u_0\|_X - e^{-\frac{2d+2}{(\ell_{m+1})^k}}\|S(\ell_{m+1})u_0\|_X \leq F(T) \int_{\ell_{m+1}}^{\ell_m} \|BS(t)u_0\|_U dt.$$  \hspace{1cm} (2.23)

Now, we choose

$$q = \left(\frac{2d+1}{2d+2}\right)^{\frac{1}{k}}.$$  

It is obvious that $q \in (0, 1)$. Therefore, it follows from (2.23) and (2.20) that

$$e^{-\frac{2d+1}{(\ell_{m+1})^k}}\|S(\ell_m)u_0\|_X - e^{-\frac{2d+2}{(\ell_{m+1})^k}}\|S(\ell_{m+1})u_0\|_X \leq F(T) \int_{\ell_{m+1}}^{\ell_m} \|BS(t)u_0\|_U dt.$$  

Summing the above inequality with respect to $m$ from 0 to $+\infty$ (the telescoping series) and noting that

$$\lim_{m \to +\infty} e^{-\frac{2d+1}{(\ell_{m+1}-\ell_m)^k}} = 0 \quad \text{and} \quad \max_{t \in [0,T]} \|S(t)u_0\|_X < +\infty,$$

we derive that

$$\|S(T)u_0\|_X \leq F(T)e^{\frac{2d+1}{(1-q)^kT}} \int_0^T \|BS(t)u_0\|_U dt.$$  

This leads to (2.18) and completes the proof. \hfill \Box

**Remark 2.5.** It is worth mentioning that in Proposition 2.4, the pair $(A, B)$ does not hold conditions in either Theorem 1.1 or Theorem 1.2.

We next give two applications of Proposition 2.4 as well as the telescoping series method presented in the proof of this proposition.

**Example 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$, and let $\omega$ be a nonempty open subset of $\Omega$. Consider the following Schrödinger equation

$$\begin{cases}
  iu_t + \Delta u = 0 & \text{in } \Omega \times \mathbb{R}, \\
  u = 0 & \text{on } \partial \Omega \times \mathbb{R}, \\
  u(\cdot, 0) = u_0 & \text{in } \Omega.
\end{cases} \hspace{1cm} (2.24)$$

Under the geometric optic condition on $\Omega$ and $\omega$, it follows from [22, Theorem 1.3] that there exists a positive constant $C = C(\Omega, \omega)$ such that for any $u_0 \in L^2(\Omega)$, the corresponding solution $u$ to Equation (2.24) verifies

$$\|u(\cdot, L)\|_{L^2(\Omega)} \leq Ce^\frac{C}{T} \left( \int_0^L \int_\omega |u(x, t)|^2 \, dx \, dt \right)^{1/2} \text{ for all } L \in (0, 1].$$
According to Proposition 2.4 (with $X = L^2(\Omega)$, $U = L^2(\omega)$, $A = i\Delta$ and $B = \chi_\omega I$, here $I$ is the identity on $X$ and $\chi_\omega$ is the characteristic function of $\omega$), it holds that for each $u_0 \in L^2(\Omega)$,

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq C e^{\frac{C}{T}} \int_0^T \|u(\cdot, t)\|_{L^2(\omega)} dt \text{ for all } T \in (0, 1].$$

Because of the property of isometry:

$$\|u(\cdot, t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)} \text{ for all } t > 0,$$

we find that for each $u_0 \in L^2(\Omega)$,

$$\|u_0\|_{L^2(\Omega)} \leq C e^{\frac{C}{T}} \int_0^T \|u(\cdot, t)\|_{L^2(\omega)} dt \text{ for all } T \in (0, 1].$$

With regard to the observability for the Schrödinger equation, we also would like to mention [26] and [19, Proposition 2.2].

Example 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^2$-boundary $\partial \Omega$. Consider the parabolic equation:

$$\begin{cases}
u_t - \text{div}(\bar{a}(x)\nabla u) = 0 & \text{in } \Omega \times \mathbb{R}^+,
\end{cases}$$

$$\begin{cases}
\quad u = 0 & \text{on } \partial \Omega \times \mathbb{R}^+,
\end{cases}$$

$$\begin{cases}
\quad u(\cdot, 0) = u_0, & \text{in } \Omega,
\end{cases}$$

(2.25)

where $\bar{a}(\cdot) \triangleq (a_{ij}(\cdot)) \in C^1(\overline{\Omega}; \mathbb{R}^{n \times n})$ are such that $a_{ij} = a_{ji}$ over $\overline{\Omega}$ for all $i, j$ and such that for some $0 < \mu_1 < \mu_2$,

$$\mu_1 \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i \xi_j \leq \mu_2 \sum_{i=1}^n \xi_i^2 \text{ for all } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \; x \in \overline{\Omega}.$$

Let $\omega$ be a nonempty and open subset of $\Omega$. The following observability inequality from time intervals has been proved (cf. [7, Theorem 2.1]): There is a constant $C = C(\Omega, \omega, \mu_1, \mu_2) \geq 1$ such that for each $u_0 \in L^2(\Omega)$, the corresponding solution $u$ to Equation (2.25) verifies

$$\|u(\cdot, L)\|_{L^2(\Omega)} \leq C e^{\frac{C}{L}} \left( \int_0^L \int_\omega |u(x, t)|^2 dx dt \right)^{1/2} \text{ for all } L > 0.$$

From this, we can apply Proposition 2.4 to get that for each $u_0 \in L^2(\Omega)$, the corresponding solution $u$ to Equation (2.25) verifies

$$\|u(\cdot, T)\|_{L^2(\Omega)} \leq C e^{\frac{C}{T}} \int_0^T \|u(\cdot, t)\|_{L^2(\omega)} dt \text{ for all } T > 0.$$
Then from Nash's inequality:
\[ \|g\|_{L^2(\omega)} \leq C(\Omega, \omega, \tilde{\omega}) \|g\|_{L^1(\tilde{\omega})}^{\theta} \|\nabla g\|_{L^2(\tilde{\omega})}^{1-\theta}, \]  
with \( \theta = \frac{2}{n+2} \), for all \( g \in H^1_0(\Omega) \),

(where \( \tilde{\omega} \) is a nonempty open subset satisfying \( \omega \subset \subset \tilde{\omega} \subset \Omega \), Hölder’s inequality and the standard energy estimate for solutions to Equation (2.25):

\[ \|u\|_{L^2(0,T;H^1_0(\Omega))} \leq C \|u_0\|_{L^2(\Omega)}, \]

it follows that

\[ \|u(\cdot, T)\|_{L^2(\Omega)} \leq \left( C e^{C_T} \int_0^T \int_\omega |u(x, t)| \, dx \, dt \right)^{\theta} \|u_0\|_{L^2(\Omega)}^{1-\theta} \quad \text{for all } T > 0 \text{ and } u_0 \in L^2(\Omega). \]

Finally, making use of the telescoping series method provided in the proof of Proposition 2.4, we obtain the refined observability inequality:

\[ \|u(\cdot, T)\|_{L^2(\Omega)} \leq C e^{C_T} \int_0^T \int_\omega |u(x, t)| \, dx \, dt \quad \text{for all } T > 0 \text{ and } u_0 \in L^2(\Omega). \]

This inequality has been built up respectively in [3] and [12] by different methods from ours.

### 3 Applications of Theorems 1.1 and 1.2

#### 3.1 Time optimal control problems in Hilbert spaces

We first set up a time optimal control problem for a controlled evolution equation. Let \( X \) and \( U \) be two Hilbert spaces (which are identified with their dual spaces) and \( A \) generate a \( C_0 \) semigroup \( \{S(t); t \geq 0\} \) on \( X \). Denote by \( X_{-1} \) the dual of \( D(A^*) \) with respect to the pivot space \( X \). Then \( \{S(t); t \geq 0\} \) can be extended into a \( C_0 \) semigroup on \( X_{-1} \) (cf. [36, Proposition 2.10.4]). We still use \( \{S(t); t \geq 0\} \) to denote the extended semigroup. Let \( B \in \mathcal{L}(U, X_{-1}) \) be an admissible control operator for \( \{S(t); t \geq 0\} \) (cf., e.g., [36, Definition 4.2.1]), i.e., there is a \( \tau > 0 \) such that \( \text{Ran} \Psi_{\tau} \subset X \), where

\[ \Psi_{\tau} f = \int_0^\tau S(\tau - t) B f(t) \, dt, \quad f \in L^2(0, \tau; U). \]

The controlled equation reads:

\[ \frac{dz}{dt} = Az + B f, \quad t > 0, \quad z(0) = z_0. \]  

(3.1)
Here, $z_0 \in X$ and $f \in L^2_{\text{loc}}(\mathbb{R}^+; U)$. Write $z(\cdot; f, 0, z_0) \in C(\mathbb{R}^+; X)$ for the unique solution of the equation (3.1) corresponding to $f$ and $z_0$ (cf. [5, Theorem 2.37] or [36, Proposition 4.2.5]). The time optimal control problem is as

$$(TP)^M : \quad T(M) \triangleq \inf_{f \in U_M} \{ t > 0 : z(t; f, 0, z_0) = z_1 \},$$

where $z_1 \in X$ is the target which differs from $z_0$ and

$$U_M = \{ f : \mathbb{R}^+ \to U \text{ measurable} : \| f(t) \|_U \leq M, \text{ a.e. } t > 0 \}, \quad \text{with } M > 0.$$  

In this problem, $T(M)$ is called the optimal time, $f^* \in U_M$ is called an optimal control if $z(T(M); f^*, 0, z_0) = z_1$. We say that the problem $(TP)^M$ holds the bang-bang property if any optimal control $f^*$ to this problem verifies $\| f^*(t) \|_U = M$ for a.e. $t \in (0, T(M))$.

The bang-bang property is very important in studies of time optimal control problems. For instance, the uniqueness of the optimal control follows immediately from this property; some equivalence of several different kinds of optimal control problems can be derived with the aid of this property (cf. [39, 40, 43, 44]). The bang-bang property for the problem $(TP)^M$ (where $X = U$ is a Banach space and $B$ is the identity on $X$) was first established in [9] via a very special and smart way. It was first realized in [24] that the bang-bang property can be derived from the observability inequality from measurable sets in time. When the target $z_1$ is replaced by a ball in $X$, the bang-bang property follows from Pontryagin’s maximum principle and the unique continuation property of adjoint equations. With respect to studies on the bang-bang property, we refer the readers to [2, 8, 10, 17, 19, 23, 29, 30, 37] (where the target is allowed to be a single point in the state space) and [15, 16, 38] (where the target is a ball in the state space).

Our main results about the problem $(TP)^M$ are as follows.

**Theorem 3.1.** Let $A$ generate an analytic semigroup $\{ S(t); t \geq 0 \}$ in $X$. Let $B \in \mathcal{L}(U, X)$ be an admissible control operator for $\{ S(t); t \geq 0 \}$. Assume that $(A^*, B^*)$ satisfies the observability inequality from time intervals:

$$\| S(L)^* \varphi_0 \|_X^2 \leq e^{\frac{dL}{k}} \int_0^L \| B^* S(t)^* \varphi_0 \|_U^2 \, dt \text{ for all } \varphi_0 \in D(A^*) \text{ and } L \in (0, 1],$$  

where positive constants $d$ and $k$ are independent of $L$ and $\varphi_0$. Then the problem $(TP)^M$ holds the bang-bang property.

**Theorem 3.2.** Let $A$ generate a $C_0$ semigroup $\{ S(t); t \geq 0 \}$ in $X$ and $B \in \mathcal{L}(U, X)$. Assume the pair $(A^*, B^*)$ verifies the Hypothesis $(H)$. Then the problem $(TP)^M$ holds the bang-bang property.
The proofs of the above theorems are based on the null controllability of the equation \((3.1)\) from measurable sets in time, which is equivalent to the observability inequality from measurable sets in time for the dual equation of \((3.1)\) (cf., e.g., [5, Theorem 2.44] or [36, Theorem 11.2.1]). The latter has been built up in Theorem 1.1 and Theorem 1.2 from different cases. Though the above theorems can be proved by the standard way (cf. [24, 37]), we provide the proof of Theorem 3.1 for the completeness of the current paper.

**The proof of Theorem 3.1.** Since \(A\) generates an analytic semigroup \(\{S(t); t \geq 0\}\) in \(X\), it follows from Theorem 5.2 of Chapter 2 and Lemma 10.2 of Chapter 1 in [25] that the semigroup \(\{S(t)^*; t \geq 0\}\) generated by \(A^*\) is also analytic. Because \(B \in \mathcal{L}(U, X_\delta)\) is an admissible control operator for \(\{S(t); t \geq 0\}\), it follows from Theorem 4.4.3 in [36] that \(B^* \in \mathcal{L}(D(A^*), U)\) is an admissible observation operator for \(\{S(t); t \geq 0\}\). From these, as well as \((3.2)\), we can apply Theorem 1.4 to get the observability inequality from measurable sets in time for the pair \((A^*, B^*)\) (i.e., the inequality \((1.4)\) with \((A, B)\) replaced by \((A^*, B^*)\)):

Given \(T > 0\) and \(E \subset (0, T)\) of positive measure, there exists a constant \(C = C(T, E, k, d, \|B^*\|_{\mathcal{L}(D(A^*), U)})\) such that

\[
\|S(T)^*\varphi_0\|_X \leq C \int_E \|B^*S(t)^*\varphi_0\|_U dt \quad \text{for all } \varphi_0 \in D(A^*). \tag{3.3}
\]

Let \(f^*\) be an optimal control for \((TP)^M\). We aim to show that \(\|f^*(t)\|_U = M\) for a.e. \(t \in (0, T(M))\). Seeking for a contradiction, we suppose that this did not hold. Then there would exist an \(\varepsilon > 0\) and a subset \(E \subset (0, T(M))\) of positive measure such that

\[
\|f^*(t)\|_U \leq M - \varepsilon \quad \text{for each } t \in E.
\]

Set \(\delta_0 = |E|/2\) and \(\hat{E} = E \cap (\delta_0, T(M))\). Clearly, \(|\hat{E}| > 0\). Write \(z^*(\cdot) \triangleq z(\cdot; f^*, 0, z_0)\). Then \(z^*(T(M)) = z_1\). By \((3.3)\) and by the equivalence of the null controllability and the observability inequality (cf., e.g., [5, Theorem 2.44] or [36, Theorem 11.2.1]), we obtain the null controllability from measurable sets for the pair \((A, B)\), i.e., for each constant \(\delta \in (0, \delta_0)\), there is a control \(f\), with

\[
\|f\|_{L^\infty(\mathbb{R}^+, U)} \leq C\|z_0 - z^*(\delta)\|_X \quad \text{for some } C > 0 \text{ independent of } \delta,
\]

such that \(z(\cdot) \triangleq z(\cdot; f, \hat{E}, \delta, z_0 - z^*(\delta))\) verifies \(z(T(M)) = 0\). Let \(\hat{f} = f^* + f\chi_{\hat{E}}\) and \(w = z^* + z\). Then

\[
\frac{dw}{dt} = Aw + B\hat{f} \quad \text{over } (\delta, T(M)), \quad w(\delta) = z_0, \quad w(T(M)) = z_1.
\]

It is easy to verify that \(\|\hat{f}\|_{L^\infty(\delta, T(M); U)} \leq M\) when \(\delta > 0\) is small enough. Finally, by setting \(\hat{f}(t) = \hat{f}(t + \delta)\) and \(\hat{z}(t) = w(t + \delta), \ t \in (0, T(M) - \delta)\), we have

\[
\frac{d\hat{z}}{dt} = A\hat{z} + B\hat{f} \quad \text{over } (0, T(M) - \delta), \quad \hat{z}(0) = z_0, \quad \hat{z}(T(M) - \delta) = z_1.
\]
This leads to a contradiction with the optimality of $T(M)$ for $(TP)^M$, and completes the proof.

3.2 Examples

This subsection presents some examples which are under the framework of Theorem 3.1 or Theorem 3.2.

3.2.1 Time optimal boundary control problem for the heat equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$. Let $\Gamma \subset \partial \Omega$ be a nonempty open subset. For each $M > 0$, we define

$$U_M = \{ f : \mathbb{R}^+ \to L^2(\Gamma) \text{ measurable} : \| f(t) \|_{L^2(\Gamma)} \leq M \text{ for a.e. } t > 0 \}.$$  

The time optimal boundary control problem reads:

$$(TP)_1^M : \quad T(M) \triangleq \inf_{f \in U_M} \{ t > 0 : y(t; f) = 0 \},$$

where $y(\cdot; f)$ solves the equation

$$\begin{cases}
y_t - \Delta y = 0, & \text{in } \Omega \times \mathbb{R}^+,
y = f, & \text{on } \Gamma \times \mathbb{R}^+,
y = 0, & \text{on } (\partial \Omega \setminus \Gamma) \times \mathbb{R}^+,
y(0) = y_0, & \text{in } \Omega,
\end{cases} \quad (3.4)$$

where $y_0 \in L^2(\Omega) \setminus \{0\}$ is arbitrarily fixed.

Let $X = H^{-1}(\Omega)$, $U = L^2(\Gamma)$, $A = \Delta$, with $D(A) = H^1_0(\Omega)$, $B = -\Delta D$, with $D$ the Dirichlet map. The space $L^2(\Gamma)$ is regarded as a subspace of $L^2(\partial \Omega)$ by extending any element $f \in L^2(\Gamma)$ to be zero outside $\Gamma$. Then, from [36, Proposition 10.7.1], $A$ is positive, and consequently generates an analytic semigroup $\{S(t); t \geq 0\}$ in $X$; $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\{S(t); t \geq 0\}$; and the equation (3.4) can be rewritten as

$$\frac{dy}{dt} = Ay + Bf, \quad t > 0; \quad y(0) = y_0. \quad (3.5)$$

Using [32, Theorem 3.2] (see also [2, Remark 2]) and then modifying slightly the proof of [36, Proposition 11.5.4], we can easily verify that the pair $(A, B)$ is null controllable in any time interval $(0, L)$, and the cost of the fast control is $e^{C/L}$, where $C = C(\Omega, \Gamma) > 0$.  

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By the equivalence of the null controllability and the observability inequality (cf., e.g., [5, Theorem 2.44] or [36, Theorem 11.2.1]), \((A^*, B^*)\) verifies the observability inequality
\[
\|e^{LA^*}\varphi_0\|_X^2 \leq e^C \int_0^L \|B^* e^{tA^*}\varphi_0\|_U^2 \, dt \quad \text{for all} \quad \varphi_0 \in D(A^*) \text{ and } L \in (0, 1].
\]

Then, one can utilize Theorem 3.1 to derive the following result:

**Corollary 3.3.** The problem \((TP)^M_1\) holds the bang-bang property.

### 3.2.2 The 3-dimensional Stokes system with 2 scalar controls

Assume that \(\Omega \subset \mathbb{R}^3\) is a bounded domain with a smooth boundary \(\partial\Omega\). Let \(\omega \subset \Omega\) be a nonempty open subset with its characteristic function \(\chi_\omega\). Treat \(L^2(\omega)\) as a subspace of \(L^2(\Omega)\) by extending functions in \(L^2(\omega)\) to be zero outside \(\omega\). Consider the controlled Stokes system

\[
\begin{aligned}
  y_t - \Delta y + \nabla p &= f, \quad \text{in } \Omega \times \mathbb{R}^+, \\
  \text{div } y &= 0, \quad \text{on } \Omega \times \mathbb{R}^+, \\
  y &= 0, \quad \text{in } \partial\Omega \times \mathbb{R}^+, \\
  y(\cdot, 0) &= y_0, \quad \text{in } \Omega,
\end{aligned}
\]

where \(y_0\) is arbitrarily fixed in the space:

\[
L^2_0(\omega) \triangleq \{ y \in (L^2(\Omega))^3 : \text{div } y = 0, y \cdot \nu = 0 \text{ on } \partial\Omega \},
\]

and \(f\) is taken from the control constraint set:

\[
U_M \triangleq \left\{ f = (0, f_2, f_3) \in L^\infty(\mathbb{R}^+; (L^2(\omega))^3) : \|f(t)\|_{(L^2(\omega))^3} \leq M \text{ for a.e. } t > 0 \right\},
\]

with \(M > 0\). The time optimal control problem reads:

\[
(TP)^M_2 : \quad T(M) \triangleq \inf_{f \in U_M} \left\{ t > 0 : y(t; f) = 0 \right\},
\]

where \(y(\cdot; f)\) is the solution to Equation (3.6) corresponding to the control \(f\).

Write \(X = L^2_0(\omega)\) and \(U = \{0\} \times L^2(\omega) \times L^2(\omega)\). Define the operator \(A\) on \(X\) by

\[
\begin{aligned}
  D(A) &= (H^2(\Omega) \cap H_0^1(\Omega))^3 \cap L^2_0(\omega), \\
  Ay &= P(\Delta y) \quad \text{for all } y \in D(A),
\end{aligned}
\]
where $P$ is the Helmholtz projection operator from $(L^2(\Omega))^3$ into $X$ (cf., e.g., [33, Chapter 3]). Let $B \in L(U, X)$ be defined by $Bf = Pf$ for all $f \in U$ (i.e., $B$ is the composition of the Helmholtz projection operator and the imbedding of $U$ into $(L^2(\Omega))^3$). Clearly, $A$ is self-adjoint and generates an analytic semigroup in $X$ (cf., e.g., [14]); $B$ is an admissible control operator for $\{e^{tA}; t \geq 0\}$ and $B^*: X \to U$ is given by

$$B^* \varphi = (0, \chi_\omega \varphi_2, \chi_\omega \varphi_3) \quad \text{for all} \quad \varphi = (\varphi_1, \varphi_2, \varphi_3) \in X;$$

and the equation (3.6) can be rewritten as (cf., e.g., [33, Chapter 4, Section 1.5])

$$\frac{dy}{dt} = Ay + Bf, \quad t > 0. \quad y(0) = y_0.$$

Meanwhile, it follows from [3, Theorem 1] that there exists a positive constant $C = C(\Omega, \omega)$ such that for each $L \in (0, 1]$,

$$\sum_{j=1}^{3} \int_\Omega |\varphi_j(x, L)|^2 \, dx \leq e^{CL} \int_0^L \int_\omega |\varphi_2(x, t)|^2 + |\varphi_3(x, t)|^2 \, dx \, dt \quad \text{for all} \quad \varphi_0 \in L^2_\sigma(\Omega),$$

where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ solves the equation

$$\begin{cases}
\varphi_t - \Delta \varphi + \nabla p = 0, & \text{in } \Omega \times (0, L), \\
\text{div } \varphi = 0, & \text{in } \Omega \times (0, L), \\
\varphi = 0, & \text{in } \partial \Omega \times (0, L), \\
\varphi(\cdot, 0) = \varphi_0.
\end{cases}$$

In other words, the pair $(A^*, B^*)$ satisfies observability inequality:

$$\|e^{LA^*} \varphi_0\|_X^2 \leq e^{CL} \int_0^L \|B^* e^{LA^*} \varphi_0\|_{U'}^2 \, dt \quad \text{for all} \quad \varphi_0 \in X \quad \text{and} \quad L \in (0, 1].$$

Therefore, we can apply Theorem 3.1 to get

**Corollary 3.4.** Problem $(TP)^M_2$ has the bang-bang property.

### 3.2.3 Parabolic equations associated with second order elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial \Omega$, and $\omega$ be a nonempty open subset of $\Omega$. Regard $L^2(\omega)$ as a subspace of $L^2(\Omega)$ by extending functions in $L^2(\omega)$ to be zero outside $\omega$. Consider the second order elliptic differential operator

$$\mathbb{L} y = \sum_{i,j=1}^{n} \text{div} (a_{ij}(x) \nabla y) + \sum_{i=1}^{n} b_i(x) \partial_i y + c(x)y.$$
Here, all the coefficients belong to $C^2(\Omega)$; $a_{ij}(x) = a_{ji}(x)$, when $1 \leq i, j \leq n$ and $x \in \Omega$; and
\[
\sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2 \text{ for all } x \in \Omega, \xi \in \mathbb{R}^n, \text{ with } \theta > 0.
\]
The controlled parabolic equation is as
\[
\begin{cases}
y_t - \mathbb{L}y = f, & \text{in } \Omega \times \mathbb{R}^+, \\
y = 0, & \text{on } \partial \Omega \times \mathbb{R}^+, \\
y(\cdot, 0) = y_0, & \text{in } \Omega,
\end{cases}
\tag{3.7}
\]
where $y_0 \in L^2(\Omega) \setminus \{0\}$ and $f$ is a control function taken from
\[
\mathcal{U}_M \triangleq \{ f \in L^\infty(\mathbb{R}^+; L^2(\omega)) : \|f(t)\|_{L^2(\omega)} \leq M, \text{ a.e. } t > 0 \}, \text{ with } M > 0.
\]
We are concerned with the time optimal control problem
\[
(\text{TP})_M^3 : \quad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \{ t > 0 : y(t; f) = 0 \},
\]
where $y(\cdot, f)$ is the solution to Equation (3.7) corresponding to the control $f$.
Let $X = L^2(\Omega)$ and $U = L^2(\omega)$. Define the operator $A$ on $X$ by setting
\[
\begin{cases}
D(A) = H^2(\Omega) \cap H^1_0(\Omega), \\
Ay = \mathbb{L}y \text{ for all } y \in D(A).
\end{cases}
\]
Let $B \in \mathcal{L}(U, X)$ be defined by $Bf = f$ for all $f \in U$ (i.e., $B$ is the imbedding of $U$ into $X$). Clearly, $A$ generates an analytic semigroup in $L^2(\Omega)$ (cf., e.g., [25, Chapter 7, Theorem 3.5]); $B$ is an admissible control operator for $\{e^{tA}; t \geq 0\}$ and $B^* : X \to U$ is given by $B^* \varphi = \chi_\omega \varphi$ for all $\varphi \in X$ (i.e., $B^*$ is the restriction from $X$ to $U$); and Equation (3.7) can be rewritten as
\[
\frac{dy}{dt} = Ay + Bf, \quad t > 0, \quad y(0) = y_0.
\]
Meanwhile, according to [7, Theorem 2.1], there exists a constant $C = C(\Omega, \omega) > 0$ such that for each $L \in (0, 1]$,
\[
\|e^{LA^*} \varphi_0\|_X^2 \leq e^C \int_0^L \|B^* e^{tA^*} \varphi_0\|_U^2 dt, \quad \text{when } \varphi_0 \in X.
\]
Hence, we have the following consequence of Theorem 3.1.

**Corollary 3.5.** Any time optimal control $f^*$ of Problem $(\text{TP})_M^3$ verifies the bang-bang property: $\|f^*(t)\|_{L^2(\omega)} = M$, a.e. $t \in (0, T(M))$. 

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3.2.4 Degenerate parabolic equations associated with the Grushin operator

Let \( \gamma \in (0, 1) \), \( \Omega = (-1, 1) \times (0, 1) \) and \( \omega = (a, b) \times (0, 1) \), \( 0 < a < b < 1 \). Treat \( L^2(\omega) \) as a subspace of \( L^2(\Omega) \) by extending functions in \( L^2(\omega) \) to be zero outside \( \omega \). Consider the controlled system

\[
\begin{aligned}
\frac{dz}{dt} - \partial^2_x z - |x|^{2\gamma} \partial^2_y z &= f, & \text{in } \Omega \times \mathbb{R}^+, \\
z &= 0, & \text{on } \partial \Omega \times \mathbb{R}^+, \\
z(x, y, 0) &= z_0, & \text{in } \Omega,
\end{aligned}
\tag{3.8}
\]

where \( z_0 \in L^2(\Omega) \setminus \{0\} \) and the control function \( f \) is taken from

\[
U_M \triangleq \{ f \in L^\infty(\mathbb{R}^+; L^2(\omega)) : \| f(t) \|_{L^2(\omega)} \leq M \text{ for a.e. } t > 0 \}, \quad \text{with } \ M > 0.
\]

We are interested in the time optimal control problem

\[
(TP)_M^M : \quad T(M) \triangleq \inf_{f \in U_M} \{ t > 0 : z(t; f) = 0 \},
\]

where \( z(\cdot; f) \) is the solution to Equation (3.8) corresponding to the control \( f \).

We next recall the well-posedness of Equation (3.8) (see [4, Section 2.1]). Let

\[
(g, h) \triangleq \int_{\Omega} (\partial_x g \partial_x h + |x|^{2\gamma} \partial_y g \partial_y h) \, dx \, dy \quad \text{and} \quad |g|_V \triangleq (g, g)^{1/2}, \quad g, h \in C_0^\infty(\Omega).
\]

Set \( V = \overline{C_0^\infty(\Omega)}^{| \cdot |_V} \). Define a bilinear form \( a(\cdot, \cdot) \) over \( V \) by

\[
a(g, h) = -(g, h) \quad \text{for all } \ g, h \in V,
\]

and an operator \( A \) on \( X \triangleq L^2(\Omega) \) by

\[
D(A) = \{ g \in V : \text{there is a constant } C \text{ such that } |a(g, h)| \leq C \| h \|_{L^2(\Omega)} \text{ for all } h \in V \},
\]

\[
\langle Ag, h \rangle_{L^2(\Omega)} = a(g, h) \quad \text{for all } \ g \in D(A) \text{ and } \ h \in V.
\]

Let \( U = L^2(\omega) \). Define \( B \in \mathcal{L}(U, X) \) by \( Bf = f \) for all \( f \in U \). Then, \( A \) is a self-adjoint operator and generates an analytic semigroup in \( X \); \( B \) is an admissible control operator for \( \{ e^{tA} ; t \geq 0 \} \) and \( B^* : X \to U \) is given by \( B^* \varphi = \chi_\omega \varphi \) for all \( \varphi \in X \); and Equation (3.8) can be rewritten as

\[
\frac{dz}{dt} = Az + Bf, \quad t > 0, \quad z(0) = z_0.
\]

Meanwhile, from Proposition 6 and from the proof of Proposition 8 in [4], there is a constant \( C = C(\Omega, \omega) > 0 \) such that for any \( L \in (0, 1] \),

\[
\int_{\Omega} |\varphi(x, y, L)|^2 \, dx \, dy \leq e^{CL} \int_0^L \int_\omega |\varphi(x, y, t)|^2 \, dx \, dy \, dt \quad \text{for all } \ \varphi_0 \in L^2(\Omega),
\]

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where \( \varphi \) solves
\[
\begin{cases}
\varphi_t - \partial_x^2 \varphi - |x|^{2\gamma} \partial_y^2 \varphi = 0, & \text{in } \Omega \times (0, L), \\
\varphi = 0, & \text{on } \partial \Omega \times (0, L), \\
\varphi(x, y, 0) = \varphi_0, & \text{in } \Omega.
\end{cases}
\]
In other words, \((A^*, B^*)\) satisfies observability inequality
\[
\|e^{LA^*} \varphi_0\|_X^2 \leq e^{CL^{-1+\epsilon}} \int_0^L \|B^* e^{tA^*} \varphi_0\|_U^2 dt \text{ for all } \varphi_0 \in X \text{ and } L \in (0, 1].
\]
Therefore, one can apply Theorem 3.1 to deduce the following corollary:

**Corollary 3.6.** Problem \((TP)_4^M\) holds the bang-bang property.

### 3.2.5 Parabolic equations with coefficients jumping at an interface

Let \( \Omega \) be a smooth and bounded domain in \( \mathbb{R}^n \) \( (n \geq 2) \) and \( \omega \subset \Omega \) be a nonempty open subset. Regard \( L^2(\omega) \) as a subspace of \( L^2(\Omega) \) to be zero in \( \Omega \setminus \omega \). Define an operator \( \mathbb{L} \) in \( L^2(\Omega) \) by
\[
\mathbb{L} = \text{div}(a(x) \nabla),
\]
with
\[
D(\mathbb{L}) = \{ u \in H^1_0(\Omega) : \text{div}(a(x) \nabla u) \in L^2(\Omega) \},
\]
where \( a \) verifies
\[
0 < a_1 \leq a(x) \leq a_2 < +\infty \text{ over } \Omega.
\]
The coefficient \( a(\cdot) \) is further assumed smooth apart from across an interface \( \Gamma \), where it may jump. The interface \( \Gamma \) is the boundary of a smooth open subset of \( \Omega \).

Consider the following time optimal control problem:
\[
(TP)_5^M : \quad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \left\{ t > 0 : y(t; f) = 0 \right\},
\]
where
\[
\mathcal{U}_M \triangleq \left\{ f \in L^\infty(\mathbb{R}^+; L^2(\omega)) : \|f(t)\|_{L^2(\omega)} \leq M, \text{ a.e. } t > 0 \right\}, \quad \text{with } M > 0,
\]
and \( y(\cdot, f) \) is the solution to
\[
\begin{cases}
y_t - \mathbb{L}y = f, & \text{in } \Omega \times \mathbb{R}^+, \\
y = 0, & \text{on } \partial \Omega \times \mathbb{R}^+, \\
y(\cdot, 0) = y_0, & \text{in } \Omega,
\end{cases}
\]
with \( y_0 \in L^2(\Omega) \setminus \{0\} \).

Let \( \{\lambda_m\}_{m \geq 1} \), sorted in an increasing sequence, and \( \{e_m\}_{m \geq 1} \) be the sets of the eigenvalues and of the associated \( L^2(\Omega) \)-normalized eigenfunctions of the operator \(-L\), respectively. According to [31, Theorem 1.2], there exists a constant \( N = N(\Omega, \omega) \geq 1 \) such that the spectral inequality

\[
\|g\|_{L^2(\Omega)} \leq Ne^{N\sqrt{\lambda_m}}\|\chi_\omega g\|_{L^2(\omega)},
\]

holds for all \( m \in \mathbb{N} \) and every function \( g \in E_{\lambda_m} \triangleq \text{span}\{e_j : j \leq m\} \).

Let \( X = L^2(\Omega), U = L^2(\omega) \) and \( A = L \). Define \( B \in \mathcal{L}(U, X) \) by \( Bf = f \) for all \( f \in U \). From (3.9), it is easy to see that Hypothesis (H) (in Theorem 3.2) holds in this case. Hence, we have the following consequence of Theorem 3.2.

**Corollary 3.7.** Problem \((TP)_s^M\) holds the bang-bang property.

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