Improved bounds for the oriented radius of mixed multigraphs

Jasine Babu | Deepu Benson | Deepak Rajendraprasad

Department of Computer Science and Engineering, Indian Institute of Technology Palakkad, Kerala, India

Correspondence
Deepu Benson, Department of Computer Science and Engineering, Indian Institute of Technology Palakkad, Kerala, India.
Email: bensondeepu@gmail.com

Funding information
Science and Engineering Research Board (SERB-MATRICS), Grant/Award Number: MTR/2019/000505

Abstract
A mixed multigraph \( G \) is an ordered pair \( G = (V, E) \) with \( V \) a set of vertices, \( E \) a multiset of unordered and ordered pairs of vertices, respectively, called undirected and directed edges. An orientation of a mixed multigraph \( G \) is an assignment of exactly one direction to each undirected edge of \( G \). A mixed multigraph \( G \) can be oriented to a strongly connected digraph if and only if \( G \) is bridgeless and strongly connected. For each \( r \in \mathbb{N} \), let \( f(r) \) denote the smallest number such that any strongly connected bridgeless mixed multigraph with radius \( r \) can be oriented to a digraph of radius at most \( f(r) \). We improve the current best upper bound of \( 4r^2 + 4r \) on \( f(r) \) to \( 1.5r^2 + r + 1 \). Our upper bound is tight up to a multiplicative factor of 1.5 since, for every positive integer \( r \), there exists an undirected bridgeless graph of radius \( r \) such that every orientation of it has a radius at least \( r^2 + r \). We prove a marginally better lower bound, \( f(r) \geq r^2 + 3r - 1 \), for mixed multigraphs. While this marginal improvement does not help with asymptotic estimates, it clears a natural suspicion that, like, undirected graphs, \( f(r) \) may be equal to \( r^2 + r \) even for mixed multigraphs. En route, we show that if each edge of \( G \) lies in a cycle of length at most \( \eta \), then the oriented radius of \( G \) is at most \( 1.5r\eta \). All our proofs are constructive and lend themselves to polynomial-time algorithms.

KEYWORDS
mixed multigraph, oriented radius, strong orientation
1 INTRODUCTION

A mixed multigraph (mixed graph for short) $G$ is an ordered pair $G = (V, E)$ with $V$ a set of vertices, $E$ a multiset of unordered and ordered pairs of vertices, respectively, called undirected and directed edges. A walk from a vertex $v_i$ to a vertex $v_k$ in a mixed graph $G$ is a sequence of vertices $v_1, v_2, ..., v_k$ such that for every $i$ belonging to $[k - 1]$, $G$ contains either an undirected edge $\{v_i, v_{i+1}\}$ or an edge directed from $v_i$ to $v_{i+1}$. A path is a walk without repeated vertices, except possibly $v_1 = v_k$. A mixed graph is strongly connected (connected for short) if there is a path from each vertex to every other vertex. Notice that this is a common generalization of connectivity of undirected graphs and strong connectivity of digraphs. A bridge in a connected mixed multigraph $G$ is an undirected edge of $G$ whose removal disconnects the underlying undirected multigraph of $G$. A bridgeless connected mixed graph is called 2-edge connected. An orientation of a mixed graph $G$ is an assignment of exactly one direction to each undirected edge of $G$. A mixed graph $G$ is called strongly orientable if it can be oriented to a strongly connected digraph.

The first major study about strongly orientable graphs was by H. E. Robbins in 1939 [14], where he proved that an undirected graph is strongly orientable if and only if it is 2-edge connected. Robbins motivated the study using the real-world scenario of converting a network of two-way streets to one-way streets without sacrificing reachability from any point to another. In the “real-world” it is very well possible that some, but not all, of the streets in the network are already one-way. Hence the same scenario motivates the study of the strong orientability of mixed graphs. In 1980, Boesch and Tindell [5] extended Robbins’ theorem to mixed graphs. They proved that a mixed graph is strongly orientable if and only if it is 2-edge connected. Neither of these works approached the problem with a view to contain the distance blow-up that may result from the orientation. This was taken up by Chvátal and Thomassen in 1978 for undirected graphs [7], and Chung, Garey, and Tarjan in 1985 for mixed graphs [6]. We need some more terminology before stating their results.

Let $G$ be an undirected, directed, or mixed graph. For a vertex $u$ in $G$, $N[u]$ denotes the set consisting of $u$ and all its in-neighbours, out-neighbours, and undirected neighbours. The length of a path in $G$ is the number of edges (directed and undirected) in the path. The distance $d_G(u, v)$ from a vertex $u$ to a vertex $v$ in $G$ is the length of a shortest path from $u$ to $v$ in $G$. A subset $D$ of the vertex set of $G$ is called an $r$-step dominating set of $G$ if every vertex not in $D$ is at a distance of at most $r$ from at least one vertex of $D$. The out-eccentricity $e_{out}(u)$ of $u$ is $\max\{d_G(u, v) | v \in V(G)\}$ and in-eccentricity $e_{in}(u)$ is $\max\{d_G(v, u) | v \in V(G)\}$. The eccentricity $e(u)$ of $u$ is $\max\{e_{out}(u), e_{in}(u)\}$. The radius of $G$ is $\min\{e(u) | u \in V(G)\}$ and diameter is $\max\{e(u) | u \in V(G)\}$. A vertex $u$ of $G$ is a central vertex of $G$ if $e(u)$ is equal to the radius of $G$. The radius and diameter of a disconnected graph are taken to be infinite. The oriented radius (diameter) of $G$ is the minimum radius (diameter) of an orientation of $G$. For each $r \in \mathbb{N}$, let $f(r)$ (resp., $\tilde{f}(r)$) denote the smallest number such that any 2-edge connected mixed graph (resp., undirected graph) with radius $r$ has oriented radius at most $f(r)$ (resp., $\tilde{f}(r)$). For each $d \in \mathbb{N}$, let $g(d)$ (resp., $\tilde{g}(d)$) denote the smallest number such that any 2-edge connected mixed graph (resp., undirected graph) with diameter $d$ has oriented diameter at most $g(d)$ (resp., $\tilde{g}(d)$). Since every 2-edge connected undirected graph is also a 2-edge connected mixed graph, $\tilde{f} \leq f$ and $\tilde{g} \leq g$.

A seminal work on the oriented radius of undirected graphs was by Chvátal and Thomassen [7]. The major focus of their work was on the parameter $\tilde{f}(r)$. They showed that $\tilde{f}(r) = r^2 + r$. The upper bound was established by exploiting the properties of a carefully chosen ear
decomposition. For the lower bound, they constructed a family of 2-edge connected graphs $G_r$ of radius $r$ and oriented radius $r^2 + r$ (see Figure 1 for $G_3$). In the context of mixed graphs, the major work on oriented radius was by Chung, Garey, and Tarjan [6] in 1985. They extended the work of Chvátal and Thomassen [7] to mixed graphs and showed that $f(r) \leq 4r^2 + 4r$. Hence, $r^2 + r = \bar{f}(r) \leq f(r) \leq 4r^2 + 4r$. Notice that, unlike the case of undirected graphs, there is a gap of 4 between the lower and upper bounds. This is because mixed multigraphs pose a few challenges which are not present in undirected graphs. In this article, we have considerably refined the attack of Chung, Garey, and Tarjan [6] to bring down the upper bound on $f(r)$ from $4r^2 + 4r$ to $1.5r^2 + r + 1$.

1.1 Our results

- We show that $f(r) \leq 1.5r^2 + r + 1$. Moreover, such an orientation can be found in polynomial time.
- We show that $r^2 + 3r - 1 \leq f(r)$.
- We also show that, if every edge of a mixed multigraph $G$ lies in a cycle of length at most $\eta$, then the oriented radius of $G$ is at most $1.5\eta$. Moreover, such an orientation also can be found in polynomial time.

**FIGURE 1** Undirected graph $G_3$ from [7]. While $G_3$ has radius 3, every orientation of $G_3$ has radius at least 12.
We consider our first result, the \((8/3)\)-factor improvement in the upper bound on \(f(r)\) (over that of Chung, Garey, and Tarjan [6]), as the major contribution in this work. Similar to the approach in [6], our proof is constructive and is presented as an algorithm which is easily seen to be polynomial time (Section 2). In comparison, our second result, the improvement to the lower bound (over the optimal lower bound for undirected graphs due to Chvátal and Thomassen) is marginal and does not help improve asymptotic estimates. But it clears a natural suspicion that \(f = \tilde{f}\). The proof is via a construction of an infinite family of mixed multigraphs parameterized by \(r\) (Section 4).

Our third result can be seen as an extension of the work by Huang, Li, Li, and Sun [12] who proved that the oriented radius of an undirected 2-edge connected graph \(G\) is at most \(r(\eta - 1)\). In Section 3, we illustrate that our algorithm provides a template to handle such interesting special cases where there is an upper bound on the unavoidable penalty you pay for orienting an undirected edge, namely, the length of the shortest cycle containing that edge.

The gap between our lower and upper bounds on \(f(r)\) leaves room for a conjecture on the exact behaviour of \(f(r)\). We are inclined to believe that the upper bound has further room for improvement and show that our algorithm does not perform optimally on the family of graphs constructed to establish our lower bound. We conjecture that \(f(r) = r^2 + O(r)\).

Another line of exploration was on improving the bounds on \(g(d)\) over those obtained as a corollary on bounds on \(\tilde{f}(r)\). Since the diameter of directed as well as undirected graphs is sandwiched between radius and twice the radius, it follows that \(g(d) \leq 2\tilde{f}(d)\). From this observation and their theorem \(\tilde{f}(r) \leq r^2 + r\), Chvátal and Thomassen [7] obtained an easy corollary, \(g(d) \leq 2\tilde{f}(d) = 2d^2 + 2d\). The family of graphs constructed by Chvátal and Thomassen [7] to provide the bound \(r^2 + r \leq \tilde{f}(r)\), also showed that \(\frac{1}{2}d^2 + d \leq g(d)\) for every even \(d \geq 2\). This left a 4-factor gap between the lower and upper bounds on \(g(d)\), inviting many investigations to narrow it. Chvátal and Thomassen also showed that \(g(2) = 6\). Kwok, Liu, and West [13] proved that \(9 \leq g(3) \leq 11\). Together with Vaka, we showed that \(g(4) \leq 21\) [2]. The major focus in [2] was to show that \(g(d) \leq 1.373d^2 + 6.971d - 1\). Extensions of this line of attack to mixed graphs to obtain better bounds for \(g(d)\) (bypassing \(f(r)\)) are still open. We believe that closing the gap on \(f(r)\) will generate more interest on \(g(d)\).

1.2 Further literature

Not surprisingly, undirected graphs have overshadowed mixed graphs in this line of research. Moreover, oriented diameter has received more attention than oriented radius. Since \(\tilde{g}\) is quadratic, there have been several studies on special graph classes restricted to which \(\tilde{g}\) is linear. AT-free graphs [10] and chordal graphs [11] are popular such classes.

Bounds (mostly upper bounds) on the oriented diameter of 2-edge connected graphs in terms of other invariants, like, domination number [10], minimum degree [3, 15], and maximum degree [8] are also known. A final result of Chvátal and Thomassen in their 1978 paper was that it is NP-hard to decide whether a given undirected graph has oriented diameter 2. This has prompted a search for polynomial-time algorithms for the problem on special graph classes. Eggemann and Noble [9] studied the oriented diameter of planar graphs. For each constant \(l\), they proposed an algorithm that decides if a planar graph \(G = (V, E)\) has an orientation with diameter at most \(l\) and runs in time \(O(c|V|)\), where the constant \(c\) depends on \(l\).

Orientations of mixed graphs with objectives other than minimizing the radius and diameter have been studied. Aamand, Hjuler, Holm, and Rotenberg [1] studied the necessary
and sufficient conditions for the existence of a strong orientation for an undirected or a mixed graph when the edges of that graph are partitioned into trails. They also provided a polynomial-time algorithm for finding a strong trail orientation, if one exists, for undirected and mixed graphs. Blind [4] studied k-arc-connected orientations in undirected and mixed graphs.

2 | UPPER BOUND

In this section, we show that \( f(r) \leq 1.5r^2 + r + 1 \) by designing and analyzing an orientation algorithm \textsc{StrongOrientation} that takes a mixed graph \( G \) of radius \( r \) as input and outputs an orientation of \( G \) of radius at most \( 1.5r^2 + r + 1 \). This algorithm involves repeated applications of two symmetric orientation algorithms, Algorithm \textsc{OrientOut} and Algorithm \textsc{OrientIn}. To decide the order of execution of these two algorithms, we need to partition the sequence of the first \( r \) positive odd numbers into two nearly equal parts. The following lemma will help us there.

Lemma 1. For every \( n \), the set of first \( n \) odd numbers \( S = \{2i − 1|1 \leq i \leq n\} \) can be partitioned into two sets \( A \) and \( B \) such that

\[
\left| \sum_{a \in A} a - \sum_{b \in B} b \right| \leq \begin{cases} 2, & n = 2, \\ 1, & n \neq 2. \end{cases}
\]

Moreover, this partition can be generated in polynomial time.

The proof of Lemma 1 is simple and thus left to the reader. The following lemma is due to Chung, Garey, and Tarjan.

Lemma 2 (Chung et al. [6, Lemma 5]). Let \( G \) be a strongly connected bridgeless mixed multigraph of radius \( r \) with a central vertex \( u \). Then any edge incident to \( u \) is on a cycle of length at most \( 2r + 1 \).

We generalize this result by extending it to every edge in \( G \) (Lemma 3). The proof given in [6] can be used to show that every edge \( pq \) of \( G \) is on a cycle of length at most \( 2k + 1 \), where \( k \) is the eccentricity of \( p \). In fact further results in [6] make use of this version, even though it is not explicitly stated so. But, we argue in Lemma 3 that any edge \( pq \) of \( G \) is on a cycle of length at most \( 2r + 1 \) itself. The proof of Lemma 3 given below is an extension of the proof given by Chung, Garey, and Tarjan [6].

Lemma 3. Let \( G \) be a strongly connected bridgeless mixed multigraph of radius \( r \). Then any edge \( pq \) of \( G \) is on a cycle of length at most \( 2r + 1 \).

Proof. Let \( G \) be a strongly connected bridgeless mixed multigraph of radius \( r \) with a central vertex \( u \). First we consider the case when \( \overrightarrow{pq} \) is a directed edge oriented from \( p \) to \( q \). Since \( G \) is of radius \( r \), there exists a \( u - p \) path of length at most \( r \) and a \( q - u \) path of length at most \( r \). Hence, there is a walk and therefore a path of length at most \( 2r \) from \( q \) to \( p \). Together with \( \overrightarrow{pq} \), it forms a cycle of length at most \( 2r + 1 \) containing \( \overrightarrow{pq} \).
Now, let $pq$ be an undirected edge. To show that an edge $pq$ lies in a cycle of length at most $2r + 1$, it is enough to show that it lies in a closed walk of length at most $2r + 1$ which passes through $pq$ exactly once. We define two subsets $X$ and $Y$ of $V(G)$ as follows. A vertex $v \in V(G)$ is in $X$ if every shortest path from $u$ to $v$ contains $pq$. A vertex $v \in V(G)$ is in $Y$ if every shortest path from $v$ to $u$ contains $pq$. First, we handle the case when $X \neq Y$. Assume, there exists a vertex $w \in (X \setminus Y)$. Since $w \in X$, all the shortest paths from $u$ to $w$ contain $pq$. Let $P_w$ be one such path. Since $w \notin Y$, at least one of the shortest paths from $w$ to $u$ do not pass through $pq$. Let $P_u$ be one such path. Now, $P_w \cup P_u$ is a closed walk of length at most $2r$ containing the edge $pq$ exactly once. A similar argument could also be made, if there exists a vertex $w \in (Y \setminus X)$. Henceforth, we assume, $X = Y$.

Let $\overline{X} = V(G) \setminus X$ and $\overline{Y} = V(G) \setminus Y$. Notice that, since $X = Y$, $\overline{X} = \overline{Y}$. Further notice that $\overline{X} \neq \emptyset$ as $u \in \overline{X}$. Now, assume vertices $p, q \in \overline{X}$. Since $\overline{X} = \overline{Y}$, there is a shortest path from $u$ to $p$ of length at most $r$ and a shortest path from $q$ to $u$ of length at most $r$, both of which do not pass through $pq$. Hence, it is easy to see that there is a closed walk of length at most $2r + 1$ containing $pq$ exactly once. Next we show that $p$ and $q$ cannot both be in $X$. If $q \in X$, by the definition of $X$, every shortest path from $u$ to $q$ passes through $pq$. Any of those paths on removing the edge $pq$ will give a shortest path from $u$ to $p$ not containing $pq$. Hence, $p \notin X$. Similarly, if $p \in X$ then $q \notin X$.

Henceforth, we can assume that $pq$ crosses the cut $(X, \overline{X})$. Since, $G$ does not contain any bridges, there is an edge $ab$, other than $pq$, crossing the cut $(X, \overline{X})$ with $a \in X$ and $b \in \overline{X}$. First, we consider the case when the edge $ab$ is undirected or directed from $a$ to $b$. Since $a \in X$, all the shortest paths from $u$ to $a$ contain the edge $pq$. Let $P_a$ be one such path. Since $b \in \overline{Y}$, there exists at least one shortest path $P_b$ from $b$ to $u$ which does not contain $pq$. Hence, $P_a \cup P_b \cup \{ab\}$ is a closed walk of length at most $2r + 1$ containing the edge $pq$ exactly once. Finally, we consider the case when $ab$ is oriented from $b$ to $a$. Since $a \in Y$, every shortest path from $a$ to $u$ passes through the edge $pq$. Since $b \notin X$, there exists at least one shortest path from $u$ to $b$ which does not contain the edge $pq$. Hence, in this case also, there exists a closed walk of length at most $2r + 1$ containing the edge $pq$ exactly once.

Let $G$ be a strongly connected bridgeless mixed graph of radius $r$ with a central vertex $u$. We first describe Algorithm OrientOut, in detail, which identifies a subgraph $H$ of $G$ such that $V(H)$ is an $(r - 1)$-step dominating set of $V(G)$ and an orientation $\overrightarrow{H}$ of $H$ such that $\forall v \in V(\overrightarrow{H}), d_{\overrightarrow{H}}(u, v) \leq 2r$ and $d_{\overrightarrow{H}}(v, u) \leq 4r - 1$. Using this algorithm alone, we can show that $f(r) \leq 2r^2 + r$ using an induction on $r$, as discussed below.

When $r = 0$, the graph $G$ is a single isolated vertex and the assumption holds true trivially. Let us assume the assumption holds true for graphs of radius $r - 1$. Let us contract the subgraph $H$ into a single vertex $v_H$ to obtain a graph $G'$. We can see that $G'$ has radius at most $r - 1$. Thus by the induction hypothesis, $G'$ has an orientation with radius at most $2(r - 1)^2 + (r - 1) = 2r^2 - 3r + 1$. Notice that, $G'$ and $H$ do not share any edges. Thus by combining the orientations of $G'$ and $H$, we can get an orientation $\overrightarrow{G}$ of $G$ with radius at most $(2r^2 - 3r + 1) + (4r - 1) = 2r^2 + r$. Hence, by induction, the assumption is true for all values of $r$.

The description of the four-stage algorithm OrientOut is given below.
Algorithm ORIENTOUT

Input: A strongly connected bridgeless mixed multigraph G and a vertex \( u \in V(G) \) with eccentricity at most \( r \).
Output: An orientation \( \vec{H} \) of a subgraph \( H \) of G such that \( N[u] \subseteq V(\vec{H}) \) and for every vertex \( v \) in \( \vec{H} \), \( d_{\vec{H}}(u, v) \leq 2r \) and \( d_{\vec{H}}(v, u) \leq 4r - 1 \).

We create \( \vec{H} \) in four stages. The vertices newly added to \( \vec{H} \) in a stage will be referred to as the vertices captured in that stage.

Stage 0. Let \( v \) be a vertex having multiple edges incident with \( u \). If possible, these edges are oriented in such a way that all the \( uv \) edges are part of a directed 2-cycle. Notice that this does not increase any pairwise distances in \( G \). We can also remove multiple oriented edges in the same direction between any pair of vertices without affecting any distance. Hence the only multiedges left in \( G \) are those which form a directed 2-cycle.

Let \( X \) denote the set of all vertices with at least one edge incident with the vertex \( u \). \( X \) is partitioned into \( X_{\text{in}} \), \( X_{\text{out}} \), and \( X_{\text{un}} \). A vertex \( v \in X \) is said to be in \( X_{\text{in}} \) if it has at least one directed edge towards \( u \). A vertex \( v \in X \) is said to be in \( X_{\text{out}} \) if it has at least one directed edge from \( u \). Finally, \( \bigcup X \subseteq X_{\text{in}} \cup X_{\text{out}} \). Notice that a vertex \( v \in X_{\text{un}} \) has exactly one undirected edge incident with \( u \). We initialize \( X_{\text{conf}} = \emptyset \) (we will later identify this as the set of conflicted vertices in \( X \)). For each \( v \in X \), let \( l(v) \) denote the length of a shortest cycle containing an edge between \( u \) and \( v \). By Lemma 3, \( l(v) \leq 2r + 1 \). Let \( s = \sum_{v \in X} l(v) \).

Stage 1. Let \( G = G_0 \). We will orient some of the unoriented edges of \( G_0 \) incident with \( u \) in this stage to obtain a mixed graph \( G_1 \) as follows. Let \( v_1 , v_2 , \ldots , v_k \) be an arbitrary ordering of \( v \) such that all the vertices in \( X_{\text{in}} \) come before any vertex in \( X_{\text{out}} \) and all the vertices in \( X_{\text{out}} \) come before any vertex in \( X_{\text{un}} \). Let \( c = |X_{\text{in}} \cup X_{\text{out}}| \).

Repeat Steps (i)–(iii) for \( i = c + 1 \) to \( X_1 \):

(i) An edge \( uv_i \) is oriented from \( v_i \) to \( u \) if the parameter \( s \) remains the same even after such an orientation. The vertex \( v_i \) is added to \( X_{\text{in}} \) in this case.

(ii) Otherwise, the edge \( uv_i \) is oriented from \( u \) to \( v_i \) if the parameter \( s \) remains the same even after such an orientation. The vertex \( v_i \) is added to \( X_{\text{out}} \) in this case.

(iii) Otherwise, we leave the edges \( uv_i \) unoriented. Such an edge \( uv_i \) is called a conflicted edge and the vertex \( v_i \) is called a conflicted vertex. The vertex \( v_i \) is added to \( X_{\text{conf}} \) in this case.

Observation 1. If an edge \( uv_i \) is conflicted then there exists an edge \( v_j u \), at the time of processing \( v_i \), where \( v_j \in X_{\text{in}} \) and \( j < i \) such that the edge \( uv_i \) is a part of every shortest cycle containing the edge \( v_j u \). Otherwise the parameter \( s \) would have remained the same even if the edge \( uv_i \) gets oriented from \( u \) to \( v_i \). Hence, for every vertex \( v \in X_{\text{conf}} \), \( uv \) is an undirected edge and there exists a \( w \in X_{\text{in}} \) such that every shortest path from \( u \) to \( w \) starts with the edge \( uv \).

We have the following distance bounds in \( G_1 \): \( \forall x \in X_{\text{in}} \), \( d_{G_1}(u, x) \leq 2r \) and \( \forall y \in X_{\text{out}} \), \( d_{G_1}(y, u) \leq 2r \). These bounds follow from the fact that the edges are oriented in Stage 1 only if the parameter \( s \) remains unchanged.

Stage 2. In this stage, we try to capture all the vertices in \( X_{\text{in}} \). Let \( T_{\text{out}} \) be a minimal tree formed by a breadth-first search in \( G_1 \), starting from \( u \) and including at least one path from \( u \) to
v for each v ∈ X_{in}. Notice that T_{out} has height at most 2r and every leaf of T_{out} is a vertex in X_{in}. Orient the tree T_{out} outward from u in G_1 to obtain G_2. Let S_1 = V(T_{out}). Every vertex in S_1 is part of a directed cycle in G_2 containing u and of length at most 2r + 1. Hence for any vertex x ∈ S_1, we have d_{G_2}(u, x) ≤ 2r and d_{G_2}(x, u) ≤ 2r. Notice that, by Observation 1, any edge e = uv, where v ∈ X_{conf}, will get oriented in Stage 2. Hence (X_{in} ∪ X_{conf}) ⊆ S_1.

Stage 3. Let X' = X_{out} \setminus S_1. In this stage, we try to capture all the vertices in X'. Let G_3 be a graph obtained from G_2 by contracting G[S_1] into a single vertex u*. Every arc u*v, v ∈ X', was part of a cycle of length at most 2r + 1 in G_1. Hence v has a path of length at most 2r – 1 to some vertex in S_1. This path together with the edge u*v ensures that every vertex v in X' lies in a cycle of length at most 2r containing u* in G_3. Let T_{in} be a minimal tree formed by a reverse breadth-first search in G_1 starting from u* such that it contains a path from each v ∈ X' to u*. Notice that, T_{in} has height at most 2r – 1 and every leaf of T_{in} is a vertex in X'. The reduction of 1 in height is because (X_{in} ∪ X_{conf}) ⊆ S_1. Orient the tree T_{in} inward towards u* in G_3 to obtain G_4. Let S_2 = V(T_{in})\{u*\}. Then for any vertex y ∈ S_2, we have the following bounds, d_{G_4}(u*, y) ≤ 2r – 1 and d_{G_4}(y, u*) ≤ 2r – 1. Let H = G[S_1 ∪ S_2]. Notice that, T_{out} and T_{in} do not share any edges in H. The edges of H are oriented consistently with the orientation of T_{out} and T_{in} to obtain H. We have already proved that for any vertex x ∈ S_1, d_{G_4}(x, u) ≤ 2r and d_{G_4}(y, u) ≤ 2r. Hence ∀ x ∈ S_1, d_{H}(u, x) ≤ 2r and d_{H}(x, u) ≤ 2r. Consider a vertex y ∈ S_2. We have proved that d_{G_4}(y, u*) ≤ 2r – 1. This implies d_{H}(y, u) ≤ (2r – 1) + (2r) = 4r – 1. Since height of T_{in} is at most 2r – 1, y has a directed path of length at most 2r – 2 from a vertex v ∈ X' in G_4. This path along with the arc u*v will give us the additional bound d_{H}(u, y) ≤ 2r for all y ∈ S_2. Thus H has radius at most 4r – 1 with d_{H}(u, x) ≤ 2r and d_{H}(x, u) ≤ 4r – 1, ∀x ∈ V(H).

As mentioned earlier, using algorithm ORIENTOUT and an induction on r, we can get the upper bound f(r) ≤ 2r^2 + r. This itself is a 2-factor improvement over the existing bound of 4r^2 + 4r obtained by Chung, Garey, and Tarjan [6]. Notice that, in H, the bound on e_{out}(u) is better than that of e_{in}(u) by a factor of 2. To exploit this gap, we design a partner algorithm ORIENTIN using a symmetric construction.

Algorithm ORIENTIN

\hspace{0.5cm} \textbf{Input:} A strongly connected bridgeless mixed multigraph G and a vertex u ∈ V(G) with eccentricity at most r.
\hspace{0.5cm} \textbf{Output:} An orientation \( \overrightarrow{H} \) of a subgraph H of G such that N[u] ⊆ V(\( \overrightarrow{H} \)) and for every vertex v in \( \overrightarrow{H} \), d_{H}(u, v) ≤ 4r – 1 and d_{H}(v, u) ≤ 2r.

Let G be a strongly connected bridgeless mixed graph of radius r and u be a central vertex of G. To show that f(r) ≤ 1.5r^2 + r + 1, we design an r-phase algorithm in which each phase involves an application of either ORIENTOUT or ORIENTIN. For convenience, we number the phases from r down to 1. A graph G_i and a vertex u_i ∈ V(G_i) with eccentricity at most i are the inputs to the algorithm ORIENTOUT or ORIENTIN executed in Phase i of algorithm STRONGORIENTATION. \( \overrightarrow{H_i} \) is the output of the algorithm executed in Phase i.
**Algorithm STRONGORIENTATION**

**Input:** A strongly connected bridgeless mixed multigraph $G$ and a vertex $u \in V(G)$ of eccentricity $r$.

**Output:** An orientation $\overrightarrow{G}$, of $G$, with radius at most $1.5r^2 + r + 1$.

1: $S \leftarrow \{2i - 1\mid 1 \leq i \leq r\}$.
2: Obtain a partition $(A, B)$ of $S$ such that $|\Sigma_{a \in A} a - \Sigma_{b \in B} b| \leq 2$.
3: $I_A \leftarrow \{i(2i - 1) \in A\}$, $I_B \leftarrow \{i(2i - 1) \in B\}$.
4: $i \leftarrow r$, $G_i \leftarrow G$, $u_i \leftarrow u$.

5: **repeat**
6: if $i \in I_A$ then
7: $\overrightarrow{H_i} \leftarrow \text{ORIENTOUT}(G_i, u_i)$.
8: else
9: $\overrightarrow{H_i} \leftarrow \text{ORIENTIN}(G_i, u_i)$.
10: **end if**
11: Obtain $G_{i-1}$ from $G_i$ by contracting $V(\overrightarrow{H_i})$ to a single vertex $u_{i-1}$.
12: $i \leftarrow i - 1$.
13: **until** $i = 0$.
14: **An edge of $G$ which gets oriented in some $\overrightarrow{H_i}$, $1 \leq i \leq r$, takes that orientation in $\overrightarrow{G}$**.
15: **Remaining edges of $G$ are oriented arbitrarily in $\overrightarrow{G}$**.

The existence of the partition used in Step 2 of the algorithm is guaranteed by Lemma 1 and can be computed in polynomial time. Since $N[u_i] \subseteq V(\overrightarrow{H_i})$, after each phase, the radius of the resultant graph reduces by at least 1. Hence, the vertex $u_{i-1}$ of $G_{i-1}$ has eccentricity at most $i - 1$. Further notice that for $1 \leq i \leq r$, the graphs $\overrightarrow{H_i}$ are pairwise edge disjoint and hence their union gives a consistent orientation $\overrightarrow{G}$ of $G$. For $1 \leq i \leq r$, let $e_{\text{in}}^i = \max\{d_{\overrightarrow{H_i}}(v, u_i)\mid v \in V(\overrightarrow{H_i})\}$ denote the in-eccentricity of $u_i$ in $\overrightarrow{H_i}$ and $e_{\text{out}}^i = \max\{d_{\overrightarrow{H_i}}(u_i, v)\mid v \in V(\overrightarrow{H_i})\}$ denote the out-eccentricity of $u_i$ in $\overrightarrow{H_i}$. By combining the guarantees given by the two algorithms ORIENTOUT and ORIENTIN, for the in-distance and out-distance in $\overrightarrow{H_i}$, we have the following bounds for $e_{\text{out}}(u)$ and $e_{\text{in}}(u)$ of $\overrightarrow{G}$.

- $e_{\text{out}}(u) \leq \Sigma_{i=1}^{r} e_{\text{out}}^i = \Sigma_{i \in I_A}(2i) + \Sigma_{i \in I_B}(4i - 1) = \Sigma_{i=1}^{r}(2i) + \Sigma_{i \in I_B}(2i - 1)$.
- $e_{\text{in}}(u) \leq \Sigma_{i=1}^{r} e_{\text{in}}^i = \Sigma_{i \in I_A}(4i - 1) + \Sigma_{i \in I_B}(2i) = \Sigma_{i=1}^{r}(2i) + \Sigma_{i \in I_B}(2i - 1)$.

The radius of $\overrightarrow{G}$ is at most $\max\{e_{\text{out}}(u), e_{\text{in}}(u)\} \leq \Sigma_{i=1}^{r}(2i) + \max\{\Sigma_{i \in I_A}(2i - 1), \Sigma_{i \in I_B}(2i - 1)\}$. Notice that $\Sigma_{i=1}^{r}(2i - 1) = r^2$ and by Lemma 1, we get $\max\{\Sigma_{i \in I_A}(2i - 1), \Sigma_{i \in I_B}(2i - 1)\}$ is at most $\frac{r^2}{2} + 1$. Thus the radius of $\overrightarrow{G}$ is at most $\Sigma_{i=1}^{r}(2i) + \left(\frac{r^2}{2} + 1\right) \leq (r^2 + r) + \left(\frac{r^2}{2} + 1\right) = 1.5r^2 + r + 1$. Hence, the oriented radius of $G$ is at most $1.5r^2 + r + 1$.

This proves the main theorem of the section.

**Theorem 4.** $f(r) \leq 1.5r^2 + r + 1.$
It is easy to verify that Algorithm StrongOrientation runs in polynomial time. Hence we have a polynomial-time algorithm to find a strong orientation $\tilde{G}_r$ with radius at most $1.5r^2 + r + 1$ for a mixed graph $G_r$ of radius $r$. Since $r \leq d \leq 2r$, we can immediately see that a mixed graph of diameter $d$ has an orientation of diameter at most $3d^2 + 2d + 2$ for all $d$.

3 | Upper Bound for Some Special Classes of Mixed Multigraphs

Let $G$ be a strongly connected mixed graph with radius $r$. Let $\eta(G)$ denote the smallest integer $k$ such that every edge of $G$ belongs to a cycle of length at most $k$. Huang, Li, Li, and Sun [12] proved that the oriented radius of an undirected graph $G$ of radius $r$ is at most $r(\eta(G) - 1)$. We generalize this result to mixed graphs. We start by generalizing Lemma 1 as follows.

**Lemma 5.** Let set $S_i = \{2i - 11 \leq i \leq n\}$. Let $S_2$ be a multiset with $k$ elements of the same value: either $n_2 - 1$ or $n_2$. Then the multiset $S = S_1 \cup S_2$ can be partitioned into two multisets $A$ and $B$ such that $|\Sigma_{a \in A} a - \Sigma_{b \in B} b| \leq 2$. Moreover, this partition can be generated in polynomial time.

**Proof.** When $n = 2$, $S_1 = \{1, 3\}$ which can be partitioned into two sets $X$ and $Y$ such that $|\Sigma_{a \in X} a - \Sigma_{b \in Y} b| = 2$. When $n \neq 2$, by Lemma 1, $S_1$ can be partitioned into two sets $X$ and $Y$ such that $|\Sigma_{a \in X} a - \Sigma_{b \in Y} b| \leq 1$.

- If $k$ is even, $S_2$ can be partitioned into two multisets $P$ and $Q$ with $\frac{k}{2}$ elements each. The sum of elements in both $P$ and $Q$ is the same.
- If $k$ is odd, let $S'_1 = S_1 \setminus \{2n - 1\}$ and $S'_2$ be the multiset $S_2 \setminus \{2n - 1\}$. As done before, $S'_1$ can be partitioned into two sets $X$ and $Y$ such that $\Sigma_{b \in Y} b \leq \Sigma_{a \in X} a \leq (\Sigma_{b \in Y} b) + 2$. Further, since the value of the elements in the multiset $S_2$ is either $2n - 1$ or $2n$, $S'_2$ can be partitioned into two multisets $P$ and $Q$ such that $\Sigma_{b \in Q} b \leq \Sigma_{a \in P} a \leq (\Sigma_{b \in Q} b) + 1$.

Now, $A = X \cup Q$ and $B = Y \cup P$ is the required partition of $S$.

It can be verified that the partition can be generated in polynomial time. □

Let $G$ be a strongly connected bridgeless mixed graph with a vertex $u \in V(G)$ having eccentricity at most $r$ and $\eta(G) = \eta$. By Lemma 3, $\eta \leq 2r + 1$. We use the same terminology of $G_i$, $u_i$, and $H_i$ as used in Algorithm StrongOrientation. Let $\eta_i$ denote the smallest integer $k$ such that every edge incident with $u_i$ belongs to a cycle of length at most $k$ in $G_i$. Since $u_i$ has eccentricity at most $i$, by Lemma 3, it is easy to see that, $\eta_i \leq \min(\eta, 2i + 1)$. Let $k_i = \min(\eta, 2i + 1)$ denote this upper bound. From the details of algorithms OrientIn and OrientOut, one can see that when $G_i$ and $u_i$ are given as input, the distance guarantees of these algorithms only use the value of $\eta_i \leq k_i$. Now, a second look at algorithm OrientIn, with the definition of $\eta_i$ in mind, will show that the oriented subgraph $\tilde{H}_i$ returned by OrientIn($G_i$, $u_i$) has $e_{in}(u_i)$ at most $(k_i - 1)$ and $e_{out}(u_i)$ at most $(2k_i - 3)$. Similarly, $\tilde{H}_i$ returned by OrientOut($G_i$, $u_i$) has $e_{out}(u_i)$ at most $(k_i - 1)$ and $e_{in}(u_i)$ at most $(2k_i - 3)$.
A strong orientation $\overrightarrow{G}$ of $G$ is obtained by an $r$-phase algorithm similar to STRONGORIENTATION, with each phase involving an execution of either ORIENTIN or ORIENTOUT. Let $m = \lceil (\eta - 1)/2 \rceil$. Recall that $2i + 1$ dominates $\eta$ for $i > m$. For $1 \leq i \leq r$, let $e_{in}^i = \max \{d_{\overrightarrow{H}}(vu) | v \in V(\overrightarrow{H})\}$ denote the in-eccentricity of $u_i$ in $\overrightarrow{H}$, and $e_{out}^i = \max \{d_{\overrightarrow{H}}(vu) | v \in V(\overrightarrow{H})\}$ denote the out-eccentricity of $u_i$ in $\overrightarrow{H}$. Let $S$ be the multiset $\{k_i - 21 \leq i \leq r\}$. It can be verified that $S$ satisfies the requirements of Lemma 5. Hence, we can obtain a partition $(A, B)$ of $S$ such that their sums differ at most by 2. Let $\sigma = (\sigma_1, ..., \sigma_r)$ be a nondecreasing ordering of elements in $S$. For $1 \leq i \leq r$, let $e_{in}^i$ denote the in-eccentricity of $u_i$ in $\overrightarrow{H}$, and $e_{out}^i$ denote the out-eccentricity of $u_i$ in $\overrightarrow{H}$. Let $\sigma = (\sigma_1, ..., \sigma_r)$ be a nondecreasing ordering of elements in $S$. For $1 \leq i \leq r$, let $\sigma_i \in I_A$ if and only if $\sigma_i \in A$ and $\sigma_i \in I_B$ otherwise. In Phase $i$, we apply ORIENTOUT if $\sigma_i \in I_A$ and ORIENTIN if $\sigma_i \in I_B$. By combining the guarantees given by ORIENTIN and ORIENTOUT for the in-distance and out-distance in $\overrightarrow{H}$, we can have the following bounds for $e_{in}(u)$ and $e_{out}(u)$ of $\overrightarrow{G}$.

$$e_{in}(u) \leq \sum_{i=1}^{r} e_{in}^i$$

$$\leq \sum_{i \in I_A} (2k_i - 3) + \sum_{i \in I_B} (k_i - 1)$$

$$= \sum_{i=1}^{r} (k_i - 1) + \sum_{i \in I_B} (k_i - 2)$$

$$= k_{in},$$

$$e_{out}(u) \leq \sum_{i=1}^{r} e_{out}^i$$

$$\leq \sum_{i \in I_A} (k_i - 1) + \sum_{i \in I_B} (2k_i - 3)$$

$$= \sum_{i=1}^{r} (k_i - 1) + \sum_{i \in I_B} (k_i - 2)$$

$$= k_{out}. $$

Further, we have the following bound for $|k_{in} - k_{out}|$ and $(k_{in} + k_{out})$.

$$|k_{in} - k_{out}| = \left| \sum_{i \in I_A} (k_i - 2) - \sum_{i \in I_B} (k_i - 2) \right|$$

$$\leq 2 \quad \text{(Lemma 5),}$$

$$(k_{in} + k_{out}) = 2 \sum_{i=1}^{r} (k_i - 1) + \sum_{i=1}^{r} (k_i - 2)$$

$$= 3 \left( \sum_{i=1}^{r} k_i \right) - 4r$$

$$= 3 \left[ m \left( \sum_{i=1}^{m} 2i + 1 \right) + \left( \sum_{i=m+1}^{r} \eta \right) \right] - 4r \quad \text{(By definition of } k_i)$$

$$= 3 \left( \sum_{i=1}^{r} \eta \right) - \sum_{i=1}^{m} (\eta - (2i + 1)) - 4r$$

$$= 3r\eta - 3 \sum_{i=1}^{m} (\eta - (2i + 1)) - 4r.$$
Notice that, \( \eta \in \{2m + 1, 2m + 2\} \). Now consider the term \( \sum_{i=1}^{m}(\eta - (2i + 1)) \) in Equation (2). Notice that, when \( \eta \) is even, this term sums up odd numbers from 1 to \( \eta - 3 \) and when \( \eta \) is odd, the term sums up even numbers from 0 to \( \eta - 3 \). Hence, by taking the worst case for Equation (2), we get

\[
(k_{in} + k_{out}) \leq 3\eta - \frac{3}{4}(\eta - 1)(\eta - 3) - 4r.
\]

(3)

From Equations (1) and (3), we get

\[
\max\{e_{in}(u), e_{out}(u)\} \leq \max\{k_{in}, k_{out}\} \\
\leq \left\lfloor \frac{k_{in} + k_{out}}{2} \right\rfloor + 1 \\
\leq 1.5\eta - 0.375(\eta - 1)(\eta - 3) - 2r + 1.
\]

(4)

Since \( \eta \geq 3, (\eta - 1)(\eta - 3) \) is nonnegative. Hence, by Equation (4), we have the following theorem.

**Theorem 6.** Let \( G \) be a strongly connected bridgeless mixed multigraph of radius \( r \) and let \( \eta(G) = \eta \) denote the smallest integer \( k \) such that every edge of \( G \) belongs to a cycle of length at most \( k \). Then \( G \) has oriented radius at most

\[
1.5\eta - 0.375(\eta - 1)(\eta - 3) - 2r + 1 \leq 1.5r\eta.
\]

A \( k \)-chordal mixed multigraph is a mixed multigraph graph in which all cycles of length more than \( k \) have a chord. In particular when \( k = 3 \), the graph is called a chordal mixed multigraph. Since \( \eta \leq k \), by Theorem 6, we have the following corollary.

**Corollary 7.** Let \( G \) be a strongly connected bridgeless \( k \)-chordal mixed multigraph with radius \( r \). Then \( G \) has oriented radius at most \( 1.5rk - 0.375(k - 1)(k - 3) - 2r + 1 \). In particular, if \( G \) is a strongly connected bridgeless chordal mixed multigraph of radius \( r \) then \( G \) has oriented radius at most \( 2.5r + 1 \).

4 | LOWER BOUND

When we restrict to undirected graphs, the currently known lower bound for \( f(r) \) is \( r^2 + r \). This follows from a construction by Chvátal and Thomassen [7], an instance of which is shown in Figure 1. Using a similar idea, we construct a mixed graph \( G_r \) of radius \( r \) whose oriented radius is \( r^2 + 3r - 1 \). To construct \( G_r \), we first define a ternary tree \( T_r \) of height \( r \), rooted at a vertex \( a \). Each internal vertex \( v \) of \( T_r \) has three children, called the left, middle, and right child of \( v \). The Level of a vertex is its distance from \( a \) in \( T_r \). For a vertex \( v \) in Level \( i \), for \( 0 \leq i \leq r - 2 \), the middle child of \( v \) is a leaf in \( T_r \) and the other two children are internal vertices. Let \( b, c, \) and \( d \) be the left, middle, and right children, respectively, of \( a \). Let \( f \) be the leaf in Level \( r \) of \( T_r \) such that every vertex in the \( a - f \) path other than \( a \) is a left child of its parent. Let \( g \) be the leaf in Level \( r \) of \( T_r \) such that every vertex in the \( a - g \) path other than \( a \) is a right child of its parent. We construct a mixed graph \( H \) from \( T_r \) as follows. For each Level \( i \), for \( i = 0 \) to \( r - 1 \) and for each internal vertex \( p_i \) in Level \( i \), with \( l_{i+1}, m_{i+1}, r_{i+1} \), respectively, being the left, right, and middle
children of $p_i$, we do the following. Add paths of length $2r - 2i - 1$ from $l_{i+1}$ to $m_{i+1}$ and $m_{i+1}$ to $r_{i+1}$. The middle edge of the first path is oriented along the direction from $l_{i+1}$ to $m_{i+1}$ and the middle edge of the second path is oriented along the direction from $m_{i+1}$ to $r_{i+1}$. Notice that, due to the presence of these directed edges, in any strong orientation of $H$, directions of all the edges except the edges between the middle children and their parents are forced. But we leave them unoriented for now so that the eccentricity of the vertex $a$ remains $r$. Now, the mixed graph $G_r$ is constructed by taking two copies of $H, H_1$, and $H_2$ and identifying the root $a$ of $H_1$ and $H_2$. The vertices of $H_1$ retain their labels from $H$. We can easily see that $G_r$ is strongly connected with radius $r$ and $a$ is the unique central vertex. Figure 2 shows the mixed multigraph $G_3$.

It can be verified that in any strong orientation $\overrightarrow{G_r}$ that minimizes the radius, $a$ is the unique central vertex. Hence, there will be an optimum orientation in which the orientation of $H_1$ is symmetric with the orientation of $H_2$. We have already seen that edges from a middle child to its parent are the only edges in $G_r$ whose orientation is not forced in a strong orientation. Hence, in any strong orientation $\overrightarrow{G_r}$ of $G_r$ with an induced orientation $\overrightarrow{H_1}$ of $H_1$, we have the following bounds for every internal vertex $p_i$ of $T_r$ at Level $i$ with $l_{i+1}$ and $r_{i+1}$ as the left and right children, respectively, for $0 \leq i \leq r - 1$.

\[
\begin{align*}
    d_{\overrightarrow{H_1}}(l_{i+1}, p_i) &\in [2(r - i), 4(r - i) - 1], \\
    d_{\overrightarrow{H_1}}(p_i, r_{i+1}) &\in [2(r - i), 4(r - i) - 1], \\
    d_{\overrightarrow{H_1}}(p_i, l_{i+1}) &= 1, \\
    d_{\overrightarrow{H_1}}(r_{i+1}, p_i) &= 1.
\end{align*}
\]

From the first line of Equation (5), it follows that, $d_{\overrightarrow{H_1}}(f, b) \geq \sum_{i=1}^{r-1} 2(r - i) = r^2 - r$. Similarly, from the second line of Equation (5), it follows that $d_{\overrightarrow{H_1}}(d, g) \geq \sum_{i=1}^{r-1} 2(r - i) = r^2 - r$. There are only two possible orientations for the edge $ac$. If the edge $ac$ in $H_1$ is oriented from $a$
to c, then $d_{H_1}(b, a) = 4r - 1$. Hence $d_{H_1}(f, a) = d_{H_1}(f, b) + d_{H_1}(b, a) \geq (r^2 - r) + (4r - 1) = r^2 + 3r - 1$. Similarly, if the edge ac in $H_1$ is oriented from c to a, we can argue that $d_{H_1}(a, g) = d_{H_1}(a, d) + d_{H_1}(d, g) \geq (4r - 1) + (r^2 - r) = r^2 + 3r - 1$. From this, it follows that $G_r$ has oriented radius at least $r^2 + 3r - 1$.

Algorithm \textsc{StrongOrientation} will orient $G_r$ with a radius of $1.5r^2 + O(r)$ only. Here, we show that there is an orientation $\overrightarrow{G_r}$ with radius $r^2 + 3r - 1$ thereby indicating that our algorithm has room improvement. Recall that all the edges except the ones between the middle children and their parents are forced in any strong orientation $\overrightarrow{H_1}$ of $H_1$. First, we orient all the edges whose orientation is forced. Now, the only edges that remain to be oriented are those which go from a parent to its middle child. Among them, the edge ac of $H_1$ is oriented from a to c. Every edge from a vertex y to its middle child is oriented away from y if the edge from y to its parent is oriented towards y and vice-versa. This completes the orientation of $\overrightarrow{G_r}$.

Let $v_r$ be a vertex with maximum in-distance to $a$. It can be verified that $v_r$ is a left or right child in Level r of $T_r$. Let P be a shortest path from $v_r$ to $a$ in $\overrightarrow{H_1}$ and let $v_r, v_{r-1}, ..., v_0 = a$ be a subsequence of P such that for $1 \leq i \leq r, v_{i-1}$ is the parent of $v_i$ in $T_r$. Let $d_i$ denote the distance from $v_i$ to $v_{i-1}$. By Equation (5), $d_i \in \{1, 2(r - i + 1), 4r - i + 1\}$. We can see that $d_{H_1}(v_r, a) = d_r + d_{r-1} + \cdots + d_1$. We have already seen that $d_i \leq 4(r - i + 1) - 1$. Further, if $d_i = 4(r - i + 1) - 1$, then the edge between $v_{i-1}$ and its middle child is oriented away from $v_{i-1}$. Hence, the edge between $v_{i-1}$ and its parent $v_{i-2}$ is oriented away from $v_{i-1}$ and hence $d_{i-1} = 1$, for $2 \leq i \leq r$. Hence, for $2 \leq j \leq r - 1$, it can be verified using induction on ($r - j$) that, if $\sum_{i=j}^r d_i > \sum_{i=j}^r 2(r - i + 1)$, then $\sum_{i=j}^r d_i \leq \sum_{i=j}^r 2(r - i + 1)$. Choosing $j = 2$, we see that, either $d_2 + \cdots + d_r \leq \sum_{i=2}^r 2(r - i + 1) = r^2 - r$ or $d_1 + d_2 + \cdots + d_r \leq \sum_{i=1}^r 2(r - i + 1) = r^2 + r \leq r^2 + 3r - 1$. Since $d_i \leq 4r - 1$, in the former case too, $d_1 + d_2 + \cdots + d_r \leq r^2 + 3r - 1$. Thus, $d_{H_1}(v_r, a) \leq r^2 + 3r - 1$. Hence, $\forall v \in V(H_1), d_{H_1}(v, a) \leq r^2 + 3r - 1$. Since $H_1$ and $H_2$ are isomorphic, there exists a

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Mixed multigraph of radius 3 and oriented radius 17.}
\end{figure}
strong orientation $\overrightarrow{G_r}$ of $G_r$ with radius at most $r^2 + 3r - 1$. Since we have already shown that $G_r$ has oriented radius at least $r^2 + 3r - 1$, the oriented radius of $G_r$ is $r^2 + 3r - 1$.

Now the main theorem of the section follows.

**Theorem 8.** $f(r) \geq r^2 + 3r - 1$.

Notice that, the same bound could be obtained by modifying only the first level in the example of Chvátal and Thomassen (see Figure 3). But still, we chose to analyze $G_r$, because our algorithm will orient $G_r$ with a radius of $1.5r^2 + O(r)$ only. That is, $G_r$ demonstrates the tightness of the analysis of our algorithm. Hence, there is scope for a better algorithm and we conjecture that $f(r) = r^2 + O(r)$.

**Conjecture 9.** $f(r) = r^2 + O(r)$.

**ACKNOWLEDGMENTS**

Supported by SERB-MATRICS Grant MTR/2019/000505. We are grateful to the anonymous referees for their very careful comments which helped us improve the presentation of this paper.

**ORCID**

Deepu Benson <http://orcid.org/0000-0002-5393-1162>

Deepak Rajendraprasad <http://orcid.org/0000-0001-9101-8967>

**REFERENCES**

1. A. Aamand, N. Hjuler, J. Holm, and E. Rotenberg, *One-way trail orientations*, 45th International Colloquium on Automata, Languages, and Programming (ICALP 2018), vol. 107, Leibniz International Proceedings in Informatics (LIPIcs), (I. Chatzigiannakis, C. Kaklamanis, D. Marx, and D. Sannella, eds.), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 2018, pp. 6:1–6:13.

2. J. Babu, D. Benson, D. Rajendraprasad, and S. N. Vaka, *An improvement to Chvátal and Thomassen’s upper bound for oriented diameter*, Discrete Appl. Math. 304 (2021), 432–440.

3. S. Bau and P. Dankelmann, *Diameter of orientations of graphs with given minimum degree*, European J. Combin. 49 (2015), 126–133.

4. S. Blind, *Output-sensitive algorithms for enumeration problems in graphs*, Ph.D. thesis, Université de Lorraine, 2019.

5. F. Boesch and R. Tindell, *Robbins’s theorem for mixed multigraphs*, Amer. Math. Monthly. 87 (1980), no. 9, 716–719.

6. F. R. K. Chung, M. R. Garey, and R. E. Tarjan, *Strongly connected orientations of mixed multigraphs*, Networks. 15 (1985), no. 4, 477–484.

7. V. Chvátal and C. Thomassen, *Distances in orientations of graphs*, J. Combin. Theory Ser B. 24 (1978), no. 1, 61–75.

8. P. Dankelmann, Y. Guo, and M. Surmacs, *Oriented diameter of graphs with given maximum degree*, J. Graph Theory. 88 (2018), no. 1, 5–17.

9. N. Eggemann and S. D. Noble, *Minimizing the oriented diameter of a planar graph*, Electron. Notes Discrete Math. 34 (2009), 267–271.

10. F. V. Fomin, M. Matamala, E. Prisner, and I. Rapaport, *At-free graphs: Linear bounds for the oriented diameter*, Discrete Appl. Math. 141 (2004), no. 1–3, 135–148.

11. F. V. Fomin, M. Matamala, and I. Rapaport, *Complexity of approximating the oriented diameter of chordal graphs*, J. Graph Theory. 45 (2004), no. 4, 255–269.
12. X. Huang, H. Li, X. Li, and Y. Sun, *Oriented diameter and rainbow connection number of a graph*, Discrete Mat. Theoret. Comput. Sci. **16** (2014), no. 3, 51–60.

13. P. K. Kwok, Q. Liu, and D. B. West, *Oriented diameter of graphs with diameter 3*, J. Combin. Theory Ser. B. **100** (2010), no. 3, 265–274.

14. H. E. Robbins, *A theorem on graphs, with an application to a problem of traffic control*, Amer. Math. Monthly. **46** (1939), no. 5, 281–283.

15. M. Surmacs, *Improved bound on the oriented diameter of graphs with given minimum degree*, European J. Combin. **59** (2017), 187–191.

**How to cite this article:** J. Babu, D. Benson, and D. Rajendraprasad, *Improved bounds for the oriented radius of mixed multigraphs*, J. Graph Theory. 2023;103:674–689. https://doi.org/10.1002/jgt.22941