The Maximum Negative Hypergeometric Distribution

Daniel Zelterman
Division of Biostatistics
School of Medicine
Yale University
New Haven, CT 06520

November 9, 2018

Email address for author: daniel.zelterman@yale.edu. This research was supported by grant P30-CA16359 awarded by the U.S. National Institutes of Health to the Yale Comprehensive Cancer Center.
Abstract

An urn contains a known number of balls of two different colors. We describe the random variable counting the smallest number of draws needed in order to observe at least $c$ of both colors when sampling without replacement for a prespecified value of $c = 1, 2, \ldots$. This distribution is the finite sample analogy to the maximum negative binomial distribution described by Zhang, Burtness, and Zelterman (2000). We describe the modes, approximating distributions, and estimation of the contents of the urn.

Keywords: discrete distributions; negative binomial distribution; riff-shuffle distribution; hypergeometric distribution; negative hypergeometric distribution; maximum negative binomial distribution

1 Introduction

*And the LORD said unto Noah, Come thou and all thy house into the ark; for thee have I seen righteous before me in this generation.*

*Of every clean beast thou shalt take to thee by sevens, the male and his female: and of beasts that are not clean by two, the male and his female.*

Genesis 7:1–2. King James translation

This charge to Noah required seven pairs of clean animals. How many animals did Noah plan on catching in order to be reasonably sure of achieving $c = 7$ male and female pairs? He didn’t want to handle more dangerous, wild creatures than necessary. In the case of a rare or endangered species, the finite population size $N$ could be small. A "clean" animal
meant it was suitable for consumption or sacrifice.

In a sequence of independent and identically distributed Bernoulli \((p)\) random variables, the *negative binomial distribution* describes the behavior of the number of failures \(Y\) observed before observing \(c\) successes, for integer valued parameter \(c \geq 1\). This well-known distribution has probability mass function

\[
\Pr[Y = y] = \binom{c + y - 1}{c - 1} p^c (1 - p)^y
\]

(1)
defined for \(y = 0, 1, \ldots\).

The negative binomial distribution (1) is discussed in detail by Johnson, Kotz, and Kemp (1992, Ch. 5). In this introductory section we will describe several sampling schemes closely related to the negative binomial. Table 1 may be useful in illustrating the various relations between these distributions.

The *maximum negative binomial distribution* is the distribution of the smallest number of trials needed in order to observe at least \(c\) successes and \(c\) failures for integer valued parameter \(c \geq 1\). This distribution is motivated by the design of a medical trial in which we want to draw inference on the Bernoulli parameter \(p\) in an infinitely large population. If the prevalence \(p\) of a binary valued genetic trait in cancer patients is very close to either zero or one then there is little to be gained in screening them for it. The statistical test of interest then, is whether \(p\) is moderate or whether it is extremely close to either 0 or 1.

In order to test this hypothesis we have decided to sequentially test patients until we have observed at least \(c\) of both the wildtype (normal) and abnormal genotypes. A small number of observations necessary to obtain at least \(c\) of both genotypes is statistical evidence that \(p\) is not far from 1/2. Similarly, a large number of samples needed to observe at least \(c\) of both genotypes is statistical evidence that the Bernoulli parameter \(p\) is extreme.

Let \(Y\) denote the ‘excess’ number of trials needed beyond the minimum of \(2c\). The probability mass function of the maximum negative binomial distribution is

\[
\Pr[Y = y] = \binom{2c + y - 1}{c - 1} (p^y + q^y)(pq)^c
\]

(2)
for \(y = 0, 1, \ldots\) and \(q = 1 - p\).
The maximum negative binomial distribution is so-named because it represents the larger of two negative binomial distributions: the number of failures before the \( c \)-th success is observed and the number of successes until the \( c \)-th failure is observed. This distribution is also a mixture of two negative binomial distributions \([1]\) that are left-truncated at \( Y = c \).

An intuitive description of the terms in \([2]\) are as follows. There are \( c \) successes and \( c \) failures that occur with probability \((pq)^c\). All of the \( y \) extra trials beyond \( 2c \) must all be either successes or failures, hence the \( p^y + q^y \) term. Finally, the last \( 2c + y \) Bernoulli trial must be the one that completes the experiment ending with either the \( c \)-th success or the \( c \)-th failure.

In Zhang, et al. (2000) we describe properties of the distribution \([2]\). The maximum negative hypergeometric distribution given at \([7]\) below and developed in the following sections is the finite sample analogue to the maximum negative binomial distribution \([2]\).

The parameters \( p \) and \( q = 1 - p \) are not identifiable in \([2]\). Specifically, the same distribution in \([2]\) results when \( p \) and \( q \) are interchanged. Similarly, it is impossible to distinguish between inference on \( p \) and on \( 1 - p \) without additional information. In words, we can’t tell if we are estimating \( p \) or \( q \) unless we also know how many successes and failures were observed at the point at which we obtained at least \( c \) of each. A similar identifiability problem is presented for the maximum negative hypergeometric distribution described in Section 4.

The minimum negative binomial or riff-shuffle distribution is the distribution of the smallest number of Bernoulli trials needed in order to observe either \( c \) successes or \( c \) failures. Clearly, at least \( c \) and fewer than \( 2c \) Bernoulli trials are necessary. The random variable \( Y + c \) counts the total number of trials needed until either \( c \) successes or \( c \) failures are observed for \( Y = 0, 1, \ldots, c - 1 \). The experiment ends with sample numbered \( Y + c \) from the Bernoulli population.

The mass function of the minimum negative binomial distribution is

\[
\Pr[Y = y] = \binom{c + y - 1}{c - 1} (p^c q^y + p^y q^c) \quad (3)
\]
Table 1: Comparison of related sampling distributions

| Sampling scheme               | Infinite population or with replacement | Finite population without replacement |
|-------------------------------|----------------------------------------|--------------------------------------|
| Predetermined number of items | Binomial distribution                  | Hypergeometric distribution          |
| Until $c$ successes           | Negative binomial distribution (1)     | Negative hypergeometric distribution (4) |
| Until either $c$ successes or | Riff shuffle or Minimum negative        | Minimum negative hypergeometric      |
| $c$ failures                  | Minimum negative binomial distribution (3) | distribution (5)                     |
| Until $c$ successes and       | Maximum negative binomial distribution (2) | Maximum negative hypergeometric      |
| $c$ failures                  |                                         | distribution (6), (7)                |

for $y = 0, 1, \ldots, c - 1$.

The naming of (3) as the minimum negative binomial refers to the smaller of two dependent negative binomial distributions: the number of failures before the $c$—th success, and the number of successes before the $c$—th failure. In words, distribution (3) says that there will be either $c$ Bernoulli successes and $y$ failures or else $c$ failures and $y$ Bernoulli successes. This distribution is introduced by Uppuluri and Blot (1970) and described in Johnson, Kotz, and Kemp (1992, pp 234–5). Lingappaiah (1987) discusses parameter estimation for distribution (3).

The three discrete distributions described up to this point are based on sampling from an infinitely large Bernoulli ($p$) parent population. Each of these distributions also has a finite sample analogy. These will be described next.

The negative hypergeometric distribution (Johnson, Kotz, and Kemp, 1992, pp 239–42) is the distribution of the number of unsuccessful draws from an urn with two different colored balls until a specified number of successful draws have been obtained. If $m$ out of $N$ balls are of the ‘successful’ type then the number of unsuccessful draws $Y$ observed before $c$ of
the successful types are obtained is
\[
\Pr[Y = y] = \binom{c + y - 1}{c - 1} \binom{N - c - y}{m - c} / \binom{N}{m}
\]
with parameters satisfying \(1 \leq c \leq m < N\) and range \(y = 0, 1, \ldots, N - m\). The expected value of \(Y\) in (4) is \(mc/(N - m - 1)\).

The negative hypergeometric distribution (4) is the finite sample analogy to the negative binomial distribution (1). Unlike the negative binomial distribution, the negative hypergeometric distribution has a finite range. The maximum negative hypergeometric distribution described in the following sections is the larger of two, dependent negative hypergeometric distributions.

The minimum negative hypergeometric distribution describes the smallest number of urn draws needed in order to observe either \(c\) successes or \(c\) failures. This distribution is the finite sample analogy to the riff-shuffle distribution (3). The probability mass function of the minimum negative hypergeometric distribution is
\[
\Pr[Y = y] = \binom{c + y - 1}{c - 1} \left[ \binom{m}{c} \binom{N - m}{y} + \binom{m}{y} \binom{N - m}{c} \right] / \left[ \binom{c + y}{c} \binom{N}{c + y} \right]
\]
for \(y = 0, 1, \ldots, c - 1\).

In the example of the charge to Noah, we have \(c = 7\) male/female pairs of animals captured from a finite population of \(m\) males and \(N - m\) females.

In Section 2 we give the probability mass function of the maximum negative hypergeometric distribution. Section 3 details some approximations to this distribution. In Section 4 we discuss estimation of the parameter that describes the contents of the urn.

\section{The distribution}

An urn contains \(N\) balls: \(m\) of one color; and the remaining \(N - m\) of another color. We continue sampling from the urn without replacement until we have observed \(c\) balls of both colors, for integer parameter \(c \geq 1\). Sampling with replacement is the same as sampling from the maximum negative binomial distribution (2) with parameter \(p = m/N\).
Let $Y$ denote the random variable counting the number of extra draws needed beyond the minimum $2c$. That is, on draw numbered $Y+2c$ we will have first observed at least $c$ of both colors. All of the $Y$ extra draws from the urn must be of the same color so there will be $c$ of one color and $Y+c$ of the other color at the end of the experiment. We will describe the distribution and properties of this random variable.

For $k = 1, 2, \ldots$ define the factorial polynomial

$$z^{(k)} = z(z - 1) \cdots (z - k + 1).$$

We also define $z^{(0)} = 1$.

The maximum negative hypergeometric distribution probability mass function can be written as

$$\Pr[Y = y] = \binom{2c + y - 1}{c - 1} \left\{ m^{(c+y)}(N - m)^{(c)} + m^{(c)}(N - m)^{(c+y)} \right\} / N^{(2c+y)} \quad (6)$$

defined for the range of $Y$:

$$0 \leq y \leq \max\{m - c, N - m - c\}.$$

The integer valued parameters $(N, m, c)$ are constrained to

$$1 \leq c \leq m < N \quad \text{and} \quad c \leq N - m.$$

Similarly,

$$\Pr[Y = y] = \left\{ c/(2c + y) \right\} \left\{ \binom{m}{c+y} \binom{N - m}{c} + \binom{m}{c} \binom{N - m}{c+y} \right\} / \binom{N}{2c+y} \quad (7)$$

expresses the maximum negative hypergeometric distribution $(6)$ in terms of binomial coefficients.

The same distribution in $(6)$ and $(7)$ result when the parameter $m$ is interchanged with $N - m$. This remark illustrates the identifiability problem with the parameters in the maximum negative hypergeometric distribution. A similar identifiability problem occurs in the maximum negative binomial distribution given at $(2)$. We will describe the estimation of the $m$ parameter in Section 4.
Special cases of this distribution are as follows. For general parameter values,

\[ \Pr[Y = 0] = \binom{N - 2c}{m - c} \binom{2c}{c} / \binom{N}{m}. \]

If \( c = m = N/2 \) then the maximum negative hypergeometric distribution is degenerate and all of its probability is a point mass at \( Y = 0 \). In words, if \( c = m = N - m \) then there can be only one possible outcome. In this case, all of the balls in the urn must be drawn before we can observe \( c \) balls of both colors.

The special case of \( c = m = 1 \) with \( N > 2 \) has the form

\[ \Pr[Y = y | m = c = 1] = \begin{cases} 
2/N & \text{for } y = 0 \\
1/N & \text{for } y = 1, \ldots, N - 2 
\end{cases} \]

and zero otherwise. This is also the form of the distribution for \( c = 1 \) and \( m = N - 1 \).

The special case for \( c = m \) and \( N = 2m + 1 \) has mass function

\[ \Pr[Y = y | m = c, N = 2c + 1] = \begin{cases} 
(m + 1)/(2m + 1) & \text{for } y = 0 \\
m/(2m + 1) & \text{for } y = 1 
\end{cases} \]

and zero otherwise. This is also the distribution of \( Y \) for \( m = c + 1 \) and \( N = 2m + 1 \).

In words, this represents the distribution of the color of the last ball remaining after all but one have been drawn from the urn.

3 Properties and Approximations

There are five basic shapes that the maximum negative hypergeometric distribution will assume. These are illustrated in Figs. 1 through 5. In each figure, the limiting maximum negative binomial distribution [2] is also presented. This limit can be expressed, more formally, as follows.

**Lemma 1.** For fixed values of \( c \geq 1 \), let \( N \) and \( m \) both grow large such that \( m/N = p \) for \( p \) bounded between zero and one. Then the behavior of the maximum negative hypergeometric random variable [3] approaches the maximum negative binomial distribution [2] with parameters \( c \) and \( p \).
Table 2: Ranges of $m$ parameter that result in unimodal maximum negative hypergeometric distributions for specified values of $c$ and $N$. Omitted distributions are either degenerate or the parameter values are invalid.

| $c$ | $N = 10$ | $N = 50$ | $N = 250$ |
|-----|---------|---------|---------|
| 1   | Unimodal for all $m = 1, \ldots, N - 1$ |         |         |
| 2   | $3 \leq m \leq 7$ | $9 \leq m \leq 41$ | $38 \leq m \leq 212$ |
| 3   | $4 \leq m \leq 6$ | $13 \leq m \leq 37$ | $55 \leq m \leq 195$ |
| 4   | $m = 5$ | $15 \leq m \leq 35$ | $65 \leq m \leq 185$ |
| 5   | $- - -$ | $16 \leq m \leq 34$ | $73 \leq m \leq 177$ |
| 10  | $- - -$ | $20 \leq m \leq 30$ | $90 \leq m \leq 160$ |
| 15  | $- - -$ | $22 \leq m \leq 28$ | $98 \leq m \leq 152$ |
| 20  | $- - -$ | $24 \leq m \leq 26$ | $103 \leq m \leq 147$ |
| 25  | $- - -$ | $- - -$ | $106 \leq m \leq 144$ |

**Proof.** Values of $Y$ remain bounded with high probability under these conditions. In (6) we write

$$m^{(c+y)} (N-m)^{(c)} / N^{(2c+y)} \geq (m-c-y)^{c+y} (N-m-c)^{c} / N^{2c+y}$$

$$= (m/N)^{c+y} \{ (N-m)/N \}^{c} \{ 1 - (c+y)/m \}^{c+y} \{ 1 - c/(N-m) \}^{c}$$

$$= p^{c+y} q^{c} \{ 1 + O_p(N^{-1}) \}$$

where $p = m/N$ and $q = 1 - p = (N-m)/N$.

We can also write

$$m^{(c+y)} (N-m)^{(c)} / N^{(2c+y)} \leq m^{c+y} (N-m)^{c} / (N-2c-y)^{2c+y}$$

$$= (m/N)^{c+y} \{ (N-m)/N \}^{c} / \{ 1 - (2c+y)/N \}^{2c+y}$$

$$= p^{c+y} q^{c} \{ 1 + O_p(N^{-1}) \} .$$
A similar argument shows
\[
m^{(c)}(N - m)^{c+y}/N^{(2c+y)} = p^c q^{c+y} \{1 + O_p(N^{-1})\}
\]
completing the proof. ■

In words, if \( m \) and \( N \) are both large then sampling from the urn without replacement is almost the same as sampling with replacement. Sampling with replacement is the same as sampling from a Bernoulli parent population yielding the maximum negative binomial distribution \((2)\).

We next describe the modes for this distribution. The maximum negative hypergeometric distribution can have either one or two modes. Write
\[
\Pr[Y = 0]/\Pr[Y = 1] = (c + 1)/c
\]
to show that this distribution always has at least one local mode at \( Y = 0 \).

The maximum negative binomial distribution \((2)\) also has at least one local mode at \( Y = 0 \) for all values of the parameter \( p \). The local mode of the maximum negative hypergeometric distribution at \( Y = 0 \) is clearly visible in Figs. 1, 2, 4, and 5. The local mode at \( Y = 0 \) in Fig. 3 is also present but it is very small.

Table \( 2 \) presents examples of parameter values corresponding to unimodal distributions in \((7)\). In general, there will be only one mode at \( Y = 0 \) when \( m/N \) is not too far from 1/2. The range of \( m \) with unimodal distributions becomes narrower as \( c \) becomes larger when \( N \) is fixed. If \( m = N/2 \) then the distribution is always unimodal.

### 3.1 A gamma approximation

An approximate gamma distribution is illustrated in Fig. 4. Under the conditions of the following lemma, the local mode at \( Y = 0 \) becomes negligible.

**Lemma 2.** For fixed \( c \geq 1 \), if \( m \) grows as \( \theta N^{1/2} \) for large \( N \) and some \( \theta > 0 \) then \( \theta Y/N^{1/2} \) behaves approximately as the sum of \( c \) independent standard exponential random variables.
Proof. Begin at (6) and write

\[
\left(\frac{2c + y - 1}{c - 1}\right) = \prod_{i=1}^{c-1} \frac{(y + 2c - i)}{i} = y^{c-1}(1 + O_p(N^{-1/2}))/\Gamma(c)
\]

Define \( \Delta \) as

\[
\Delta = m^c(N - m)^{y + c}/N^{2c+y}.
\]

Under the conditions of his lemma, the term

\[
m^{c+y}(N - m)^c/N^{2c+y}
\]

will be much smaller than \( \Delta \) and can be ignored.

We have

\[
\log \Delta = \sum_{i=0}^{c-1} \log\{(m - i)/(N - i)\} + \sum_{j=0}^{y+c-1} \log\{(N - m - j)/(N - c - j)\}
\]

\[
= c \log\{\theta N^{-1/2} + O(N^{-1})\} + \sum_{j=0}^{y+c-1} \log\{1 - (m - c)/(N - c - j)\}
\]

For \( \epsilon \) near zero, write

\[
\epsilon - \epsilon^2/2 \leq \log(1 + \epsilon) \leq \epsilon
\]

so that

\[
\log \Delta = c \log\{\theta N^{-1/2} + O(N^{-1})\} - \theta y/N^{1/2} + O_p(y/N)
\]

The transformation \( X = \theta Y/N^{1/2} \) has Jacobian \( N^{1/2}/\theta \) so

\[
(N^{1/2}/\theta) \Pr[Y] = x^{c-1}e^{-x}/\Gamma(c)
\]

ignoring terms that tend to zero for large values of \( N \). This is the density function of the sum of \( c \) independent, standard exponential random variables.
3.2 A half-normal approximation

If $Z$ has a standard normal distribution then the distribution of $|Z|$ is said to be standard half-normal or folded normal. The density function of the random variable $X = |Z|$ is

$$(2/\pi)^{1/2} \exp(-x^2/2)$$

for $x \geq 0$ (Stuart and Ord, 1987, p 117). The approximate half-normal behavior of the maximum negative hypergeometric distribution is illustrated in Fig. 5.

**Lemma 3.** When $N$ becomes large, if $m = N/2$ and $c$ grows as $N^{1/2}$ then $Y/(2c)^{1/2}$ behaves approximately as a standard half-normal random variable.

The proof involves expanding all factorials in (7) using Stirling’s approximation. The details are provided in Appendix A.

3.3 A normal approximation

The normal approximation to the maximum negative hypergeometric distribution can be seen in Fig. 3. This is proved more formally in Lemma 4, below. No generality is lost by requiring $m > N/2$ because $m$ and $N - m$ can be interchanged to yield the same distribution.

**Lemma 4.** For large values of $N$, suppose $c$ grows as $N^{1/2}$ and $m = Np$ for $1/2 < p < 1$. Then $(Y - \mu)/\sigma$ behaves approximately as standard normal where

$$\mu = c(p - q)/q$$

for $q = 1 - p$ and

$$\sigma = (cp)^{1/2}/q.$$

The proof of this lemma is given in Appendix B. The details involve using Stirling’s approximation to all of the factorials in (7) and expanding these in a two-term Taylor series.
4 Estimation

The most practical situation concerning parameter estimation involves estimating the $m$ parameter when $c$ and $N$ are both known. In terms of the original, motivating example drawing inference on the genetic markers in cancer patients, the finite population size $N$ will be known, and the parameter $c$ is chosen by the investigators in order to achieve specified power and significance levels. The $m$ parameter describes the composition of the $N$ individuals in the finite-sized population. The value of $m$ is known without error if all $N$ subjects are observed.

The estimation of $m$ in this section is made on the basis of a single observation of the random variable $Y$. We will treat the unknown $m$ parameter as continuous valued rather than as a discrete integer as it has been used in previous sections.

The log-likelihood kernel function of $m$ in (6) is

$$
\Lambda(m) = \log \left\{ \left(m^c(N - m)^{(c+y)} + m^{(c+y)}(N - m)^c \right) / N^{(2c+y)} \right\}.
$$

As a numerical illustration, the function $\Lambda(m)$ is plotted in Fig. 6 for $N = 20$ and $c = 3$. Observed values of $y$ are given as $0, 1, \ldots, 7$ in this figure. The range of valid values of the $m$ parameter are $3 \leq m \leq 17$ for the values of $c$ and $N$ in this example. Smaller observed values of $y = 0, 1, 2$ in this example exhibit log-likelihood functions with a single mode corresponding to maximum likelihood estimates of $\hat{m} = N/2$. For values of $y \geq 3$ the likelihood $\Lambda$ has two modes, symmetric about $N/2$.

Intuitively, if the observed value $Y$ is small then we are inclined to believe that the urn is composed of an equal number of balls of both colors. That is, if we quickly observe $c$ of both colored balls then this is good statistical evidence of an even balance of the two colors in the urn. Conversely, if the observed $Y$ is relatively large then we will estimate an imbalance in the composition of the urn. Without the additional knowledge of the number of successes and failures observed then we are unable to tell if we are estimating $m$ or $N - m$.

More generally, there will be either one mode of $\Lambda(m)$ at $\hat{m} = N/2$ or else two modes,
symmetric about $N/2$ depending on the sign of

$$\Lambda''(m) = (\partial/\partial m)^2 \Lambda(m)$$

evaluated at $m = N/2$. If $\Lambda''(N/2)$ is negative then there will be one mode of $\Lambda$ at $N/2$.

Useful rules for differentiating factorial polynomials are as follows. For $c \geq 1$,

$$(\partial/\partial m) m^{(c)} = m^{(c)} \sum_{i=0}^{c-1} (m-i)^{-1}$$

and for $c \geq 2$,

$$(\partial/\partial m)^2 m^{(c)} = 2m^{(c)} \sum_{i=0}^{c-1} (m-i)^{-1}(m-i')^{-1}.$$  

Use these rules to write

$$(\partial/\partial m) m^{(c)}(N-m)^{(y+c)} = m^{(c)}(N-m)^{(y+c)} \left\{ \sum_{i=0}^{c-1} (m-i)^{-1} - \sum_{j=0}^{y+c-1} (N-m-j)^{-1} \right\}$$

and show that the likelihood $\Lambda(m)$ always has a critical value at $m = N/2$:

$$\left. (\partial/\partial m)\Lambda(m) \right|_{m=N/2} = 0.$$  

The critical point of $\Lambda$ at $m = N/2$ may either be a global maximum or else a local minimum as seen in the example of Fig. 6. This distinction depends on the sign of the second derivative $\Lambda''$ of $\Lambda$.

The second derivative of $\Lambda$ can be found from

$$(\partial/\partial m)^2 m^{(c)}(N-m)^{(y+c)} = m^{(c)}(N-m)^{(y+c)} \times \left\{ \sum_{i=0}^{c-1} (m-i)^{-1} - \sum_{j=0}^{y+c-1} (N-m-j)^{-1} \right\}^2 - \sum_{i=0}^{c-1} (m-i)^{-2} - \sum_{j=0}^{y+c-1} (N-m-j)^{-2}.$$  

The sign of $\Lambda''(N/2)$ is the same as that of

$$\phi(N, c, y) = \sum_{k=0}^{y-1} (N/2 - c - k)^{-1} (N/2 - c - k')^{-1} - \sum_{i=0}^{c-1} (N/2 - i)^{-2}.$$  

13
The first summation in $\phi$ is zero when $y = 0$ or 1. The function $\phi$ is then negative for small values of $y$ demonstrating that the maximum likelihood estimate of $m$ is $\hat{m} = N/2$ in these cases. Similarly, $\phi$ is an increasing function of $y$ and may eventually become positive for larger values of $y$, so that $\Lambda$ will have two modes. These modes are symmetric about $N/2$ because

$$\Lambda(m) = \Lambda(N - m)$$

for all $0 < m < N$.

In other words, a small observed value of $y$ leads us to believe that there are an equal number of balls of both colors in the urn and estimate $m$ by $N/2$. Similarly, a large observed value of $y$ relative to $c$ leads us to estimate an imbalance in the composition of the urn.

References

Johnson, N.L., S. Kotz, and A.W. Kemp (1992). *Univariate Discrete Distributions*. New York: John Wiley & Sons.

Lingappiah, G.S. (1987). Some variants of the binomial distribution. *Bulletin of the Malaysian Mathematical Society* 10: 82–94.

Stuart, A. and J.K. Ord (1987). *Kendall’s Advanced Theory of Statistics* Vol 1, 5th Edition: *Distribution Theory*, New York: Oxford University Press.

Uppuluri, V.R.R., and W.J. Blot (1970). “A probability distribution arising in a riff-shuffle.” *Random Counts in Scientific Work, 1: Random Counts in Models and Structures*, G.P. Patil (editor), University Park: Pennsylvania State University Press, pp 23–46.

Zhang, Z., B.A. Burtness, and D. Zelterman (2000). The maximum negative binomial distribution. *Journal of Statistical Planning and Inference* 87: 1–19.
Appendix A: Proof of the half-normal approximation

The proof of Lemma 3 is provided here. We assume that \( m = N/2 \) and \( c = N^{1/2} \). The random variable \( Y \) is \( O_p(N^{1/4}) \).

Expand all of the factorials in (7) using Stirling’s approximation giving

\[
\log \Pr[Y = y] = -1/2 \log(2\pi) + T_1(c) + T_2(N) + O_p(N^{-1/2})
\]

where

\[
T_1 = \log\{2c/(2c + y)\} - (c + y + 1/2) \log(c + y)
\]
\[
- (c + 1/2) \log(c) + (2c + y + 1/2) \log(2c + y)
\]

contains terms in \( O_p(c \log c) \) and

\[
T_2 = (N + 1) \log(N/2) - (N + 1/2) \log(N) - (N/2 - c + 1/2) \log(N/2 - c)
\]
\[
- (N/2 - c - y + 1/2) \log(N/2 - c - y) + (N - 2c - y + 1/2) \log(N - 2c - y)
\]

contains terms that are \( O_p(N \log N) \).

In all of the following expansions it is useful to keep in mind that \( c = c_N \) is approximately equal to \( N^{1/2} \) and \( Y = O_p(N^{1/4}) \). Write \( T_1 \) as

\[
T_1 = -\log\{(2c + y)/2c\} + (2c + y + 1/2) \log 2 - 1/2 \log(c)
\]
\[
+ c \log\{(c + y/2)^2/c(c + y)\} + (y + 1/2) \log\{(c + y/2)/(c + y)\}
\]
\[
= -\log(1 + y/2c) + (2c + y + 1/2) \log 2 - 1/2 \log(c)
\]
\[
+ c \log\{1 + y^2/4c(c + y)\} + (y + 1/2) \log\{1 - y/2(c + y)\}.
\]

Expand every appearance of \( \log(1 + \epsilon) = \epsilon + O(\epsilon^2) \) for \( \epsilon \) near zero to show

\[
T_1 = (2c + y + 1/2) \log 2 - 1/2 \log(c) - y^2/4c + O_p(N^{-1/4}).
\]
Similarly, we can write

\[ T_2 = \frac{N}{2} - c \log \left\{ \frac{N - 2c - y}{N - 2c} \right\} \]

\[ + y \log \left\{ \frac{N - 2c - 2y}{N - 2c - y} \right\} - (2c + y - 1) \log 2 \]

\[ + \frac{1}{2} \log \left\{ \frac{N}{(N - 2c)} \right\} + \frac{1}{2} \log \left\{ \frac{N - 2c - y}{N - 2c - 2y} \right\} \]

\[ = \left( \frac{N}{2} - c \right) \log \left\{ 1 + \frac{y^2}{O_p(N^2)} \right\} + y \log \left\{ 1 - y/O_p(N) \right\} \]

\[ - (2c + y - 1) \log 2 + \frac{1}{2} \log \left\{ 1 + O(c/N) \right\} + 1 \log \left\{ 1 + y/O_p(N) \right\} . \]

Then write \( \log(1 + \epsilon) = O(\epsilon) \) for \( \epsilon \) near zero, giving

\[ T_2 = -(2c + y - 1) \log 2 + O_p(N^{-1/2}) . \]

These expressions for \( T_1 \) and \( T_2 \) in \( \Pr[Y] \) give

\[ \log \Pr[Y = y] = \log 2 - 1/2 \log(\pi c) - y^2/4c + O_p(N^{-1/4}) . \]

Finally, we note that \((2c)^{1/2}\) is the Jacobian of the transformation \( X = Y/(2c)^{1/2} \). Then

\[ \log \left\{ (2c)^{1/2} \Pr[Y = y] \right\} = \frac{1}{2} \log(2/\pi) - x^2/2 + O_p(N^{-1/4}) \]

is the density of the folded normal distribution, except for terms that tend to zero with high probability. ■

**Appendix B: Standard normal approximate distribution**

The details of the proof of Lemma 4 are given here. Define \( \Omega \) as

\[ \Omega = \left\{ \frac{c}{2c+y} \right\} \left( \frac{m}{c+y} \right) \left( \frac{N-m}{c} \right) \left/ \left( \frac{N}{2c+y} \right) \right. \]

The term

\[ \left\{ \frac{c}{2c+y} \right\} \left( \frac{m}{c+y} \right) \left( \frac{N-m}{c} \right) \left/ \left( \frac{N}{2c+y} \right) \right. \]
in (7) is much smaller than $\Omega$ and can be ignored under the conditions of this lemma.

Expand all of the factorials in $\Omega$ using Stirling’s formula giving

$$\log \Omega = -1/2 \log(2\pi) + S_1(N) + S_2(N - c) + S_3(y) + O_p(N^{-1/2})$$

where

$$S_1 = (Np + 1/2) \log(Np) + (Nq + 1/2) \log(Nq) - (N + 1/2) \log N$$

corresponds to $m!$, $(N - m)!$, and $N!$;

$$S_2 = (N - 2c - y + 1/2) \log(N - 2c - y) - (Np - c - y + 1/2) \log(Np - c - y)$$
$$- (Nq - c + 1/2) \log(Nq - c)$$

corresponds to $(N - 2c - y)!$, $(m - c - y)!$, and $(N - m - c)!$; and

$$S_3 = \log\left\{c/(2c + y)\right\} + (2c + y + 1/2) \log(2c + y)$$
$$- (c + y + 1/2) \log(c + y) - (c + 1/2) \log(c)$$

corresponds to $c/(2c + y)$, $(2c + y)!$, $(c + y)!$ and $c!$.

Write out all of the terms in $S_1$ to show

$$S_1 = Np \log p + Nq \log q + 1/2 \log(Npq).$$

We can write $S_2$ as

$$S_2 = (N - 2c - y + 1/2) \log N + (N - 2c - y + 1/2) \log\{1 - (2c + y)/N\}$$
$$- (Np - c - y + 1/2) \log(Np) - (N - c - y + 1/2) \log\{1 - (c + y)/Np\}$$
$$- (Nq - c + 1/2) \log(Nq) - (Nq - c + 1/2) \log(1 - c/N).$$

Then

$$S_1 + S_2 = c \log(pq) + y \log p + (N - 2c - y + 1/2) \log\{1 - (2c + y)/N\}$$
$$- (Np - c - y + 1/2) \log\{1 - (c + y)/Np\} - (Nq - c + 1/2) \log(1 - c/Nq).$$
Since
\[(c + y)/N = O_p(N^{-1/2})\]
we can expand
\[\log(1 + \epsilon) = \epsilon - \epsilon^2/2 + O(\epsilon^3)\]
for \(\epsilon\) near zero to show
\[S_1 + S_2 = c\log(pq) + y\log p + (2c + y)^2/2N\]
\[-(c + y)^2/2Np - c^2/2Nq + O_p(N^{-1/2}) .\]

Then write
\[y = \mu + Z\sigma\]
where \(Z\) is a \(O_p(1)\) random variable, giving
\[S_1 + S_2 = c\log(pq) + y\log p + \{(2c + \mu)^2pq - (c + \mu)^2q - c^2p\}/2Npq\]
\[+ Z\sigma\{(2c + \mu)p - c - \mu\}/Np + O_p(N^{-1/2}) .\]

Substitute \(\mu = c(p - q)/q\) to show
\[S_1 + S_2 = c\log(pq) + y\log p + O_p(N^{-1/2}) .\] (8)

Next write \(S_3\) as
\[S_3 = (c + y)\log\{(2c + y)/(c + y)\} + (c - 1/2)\log\{(2c + y)/c\} - 1/2\log(c + y) .\]

Expand the argument of the first logarithm in \(S_3\) here in a two-term Taylor series showing
\[(2c + y)/(c + y) = (2c + \mu + Z\sigma)/(c + \mu + Z\sigma)\]
\[= \{(2c + \mu)/(c + \mu)\} \{1 + Z\sigma/(2c + \mu)\}/\{1 + Z\sigma/(c + \mu)\}\]
\[= \{(2c + \mu)/(c + \mu)\}\]
\[\times \left[ 1 - Z\sigma c/\{(c + \mu)(2c + \mu)\} + Z^2\sigma^2 c/\{(c + \mu)^2(2c + \mu)\} + O(N^{-3/4}) \right] .\]
Then expand
\[ \log(1 + \epsilon) = \epsilon - \epsilon^2/2 + O(\epsilon^3) \]
to show
\[
(c + y) \log\{(2c + y)/(c + y)\} = (c + \mu + Z\sigma) \log\{(2c + \mu)/(c + \mu)\} \\
+ \left( c + \mu + Z\sigma \right) Z\sigma c / \{(c + \mu)(2c + \mu)\} \\
- Z^2\sigma^2 c / \{(c + \mu)(2c + \mu)\} \\
- 1/2 (Z\sigma c)^2 / \{2(c + \mu)(2c + \mu)^2\} + O_p(N^{-1/4}).
\]

Similarly,
\[
(c - 1/2) \log\{(2c + y)/c\} = (c - 1/2) \log(2 + \mu/c) + (c - 1/2) \log\{1 + Z\sigma/(2c + \mu)\} \\
= (c - 1/2) \log(2 + \mu/c) + Z\sigma c/(2c + \mu) \\
- Z^2\sigma^2 c / \{2(2c + \mu)^2\} + O_p(N^{-1/4})
\]
and
\[
1/2 \log(c + y) = 1/2 \log\{(c + \mu)(1 + Z\sigma/(c + \mu))\} \\
= 1/2 \log(c + \mu) + O_p(N^{-1/4})
\]
so that
\[
S_3 = (c + y) \log\{(2c + \mu)/(c + \mu)\} - 1/2 \log(c + \mu) \\
+ (c - 1/2) \log\{(2c + \mu)/c\} - Z^2\sigma^2 c / \{2(c + \mu)(2c + \mu)\} + O_p(N^{-1/4}).
\]

Substitute values of \( \mu = c(p - q)/q \) and \( \sigma^2 = cp/q^2 \) giving
\[
S_3 = (c + y) \log p - \log \sigma + (c - 1/2) \log q - Z^2/2 + O_p(N^{-1/4}).
\]

This expression, together with the form of \( S_1 + S_2 \) given at \( \text{[N]} \) shows that
\[
\log \Omega = -1/2 \log(2\pi) - \log \sigma - Z^2/2 + O_p(N^{-1/4})
\]
demonstrating the approximately standard normal behavior of the random variable $Z = (Y - \mu) / \sigma$.

Figure 1: The maximum negative hypergeometric distribution for $c = 3$ and $m = 0.4N$ with values of $N = 15, 20, 30, 60,$ and $120$. The distribution corresponding to the maximum negative binomial (MxNB) given at (2) with parameters $c = 3$ and $p = 0.4$ is the limit when $N$ is large. Mass points of these discrete distributions are joined with lines for clarity in this and other figures.
Figure 2: The maximum negative hypergeometric distribution for $c = 6$ and $m = N/3$ with $N = 24, 27, 30, 36, 48,$ and $96$. Bimodal distributions generally appear whenever $c$ is moderately large and $N$ is small. Examples of these conditions are given in Table 2. The limiting maximum negative binomial distribution (MxNB) has parameters $c = 6$ and $p = 1/3$. 

![Graph showing the maximum negative hypergeometric distribution for different values of N]
Figure 3: The maximum negative hypergeometric distribution for $c = 20$ and $m = N/4$ with $N = 100, 120, 200, \text{ and } 400$. A normal approximate distribution appears when the $(c, m, N)$ parameters are all large and $m/N$ is not close to zero, one, or $1/2$. This is proved in Lemma 4.
Figure 4: The maximum negative hypergeometric distribution for $c = 2$ and $m = N/10$ with $N = 40$, 50, and 100. A gamma approximate distribution is proved in Lemma 2 when $m$ is much smaller than $N$. There is always a local mode at $Y = 0$. The asymptotic maximum negative binomial (MxNB) distribution has parameters $c = 2$ and $p = 1/10$. 

![Graph showing the maximum negative hypergeometric distribution for different values of N.](image)

$0.05$

$N = 40$

MxNB

$Pr[Y]$

0 20 40 60 80

$Pr[Y]$
Figure 5: The maximum negative hypergeometric distribution for $c = 20$ and $m = N/2$ for $N = 50, 54, 60, 70, 100, \text{ and } 300$. A half normal approximate distribution appears when $c$ and $N$ are large and $m = N/2$. The limiting distribution is proved in Lemma 3.
Figure 6: The maximum negative hypergeometric log-likelihood kernel function $\Lambda(m)$ for parameter values $N = 20$ and $c = 3$ and observed values $y = 0, \ldots, 7$, from top to bottom, respectively.