An empirically equivalent random field for the quantized electromagnetic field

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A random field is constructed that is equivalent to the quantized electromagnetic field in the sense that both may be generated from the same algebra of creation and annihilation operators, but with different relationships between the algebra and space-time. As for the comparable construction for the complex Klein-Gordon quantum field, the roles played by positive and negative frequency modes of the field are modified, with additional use made here of helicity projections.

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INTRODUCTION

A presentation of the Klein-Gordon random field in terms of creation and annihilation operators was introduced in [1], and shown to be empirically equivalent to the complex Klein-Gordon quantum field. Here, we will construct two random fields that are equivalent to the quantized electromagnetic field in the same sense. This has greater empirical significance than the comparable construction for the Klein-Gordon quantum field because of the wide use of the quantized electromagnetic field in quantum optics and other Physics. Although we will construct two random fields that are equivalent to the quantized electromagnetic field, one is relegated to an appendix because it is not complex linear, which makes it much less mathematically attractive.

These constructions show that quantum fields and random fields are significantly closer to one another than has generally been understood, with the reservation that at present there is no similar construction for fermion fields. It is proposed that it is possible to understand quantum fields as a classical mathematics of stochastic signal analysis in the presence of a Lorentz invariant quantum noise that is distinguishable from thermal noise because of their different symmetry properties. Although the random fields constructed here and in [1] are commutative algebras when taken in isolation, the associated algebra of creation and annihilation operators is a non-commutative algebra, with the same structure as the creation and annihilation algebra associated with the equivalent quantum field.

Random fields are of constructive and of comparative interest, as well as of interpretational interest, because they admit Lie field deformations that preserve commutativity [2], which may be empirically effective models in some contexts or which may by comparison illuminate interacting quantum fields, whereas Lie field deformations of Wightman fields that preserve non-trivial micro-causality are not possible [3]. Citations were given in [1] to a few other approaches that use mathematics that is more-or-less similar to the mathematics of random fields. Random fields are discussed in a quantum field context in [4], for example. I have previously shown that Bell in- equalities are generally not satisfied by random fields [3], so empirically effective random field models for nature are not ruled out by the violation of Bell inequalities by experiment.

THE QUANTIZED ELECTROMAGNETIC FIELD

The quantized electromagnetic field, which is properly speaking an operator-valued distribution, can be constructed much more effectively using test functions than is possible using standard textbook methods. We use a Schwartz space of test functions $S$ to smear the quantized electromagnetic field $\phi^{\mu\nu}(x)$ to give a complex linear map of test functions into an algebra of operators $A$,

$$\hat{\phi} : S \rightarrow A; f \mapsto \hat{\phi}_f = \int \phi^{\mu\nu}(x)f_{\mu\nu}(x)d^4x.$$ (1)

To determine the expectation values of operators constructed using $\hat{\phi}_f$ in the vacuum state, we write it as the sum of abstract creation and annihilation operators, $\hat{\phi}_f = a_f^\dagger + a_f$, and write down the commutation bracket

$$[a_f, a_g^\dagger] = \langle f, g \rangle$$

$$= -\hbar \int \frac{d^4k}{(2\pi)^4}2\pi\delta(k_\mu k^\nu)\theta(k_0)\tilde{f}_\alpha^\dagger (k)k^{\beta\gamma}\tilde{g}_\beta^\dagger (k),$$ (2)

then we introduce a vacuum vector $|0\rangle$, on which annihilation operators have a trivial action, $a_f |0\rangle = 0$, which allows us to define a vacuum state, $\omega_0(\hat{A}) = \langle 0|\hat{A}|0\rangle$, and hence to use the GNS-construction of a Hilbert space [4 §III.2.2]. The GNS construction introduces almost no extraneous mathematical structure, so it is significantly more natural than the construction of a Fock space as a direct sum of tensor products of a single-particle Hilbert space. [Notation: $f^*$ is a real-space conjugation, so that $f^*(k) = \tilde{f}(-k).$]

The algebraic structure and the trivial action of the annihilation operators on the vacuum state determines the $c$-number expectation value of operators that are constructed as sums and products of creation and annihilation operators. There are of course subtleties of analysis...
(particularly if we insist that the Hilbert space must support a continuous representation of the Poincaré group), but we do not have to be concerned with them here. The same algebraic structure, $[a_f, a^*_g] = (f, g)$, works for any free quantum field, with different space-time properties encoded in different forms of the inner product $(f, g)$. The form given above for the quantized electromagnetic field, which has the advantage of being expressed without using the electromagnetic potential, is derived in [7, Eq. (3.27)]. Note that this manifestly covariant algebraic approach fixes all empirical consequences without introducing a non-measurable Hamiltonian operator as a fundamental object.

A test function approach to quantum fields in principle fixes attention on how we construct measurement operators and apparatus and on how we prepare the quantum state, in contrast to a focus on the quantum field at a point, which is unmeasurable, although quantum field states are undoubtedly remote from a detailed description of a whole experimental apparatus. In practice, quantum optics often specializes, very effectively, to a handful of wave numbers and uses an appropriate handful of quantized simple harmonic oscillators as a model, even though this is a remarkably nonlocal practice. Test functions are known as “window functions” in signal analysis, which represent the signal response of a measurement process.

A RANDOM FIELD EQUIVALENT

The construction of a Klein-Gordon random field that is empirically equivalent to the complex Klein-Gordon field [1] depended on two types of Lorentz invariant projections, to positive/negative frequency and to real/imaginary components; for the quantized electromagnetic field, there is an additional Lorentz invariant projection, to positive/negative helicity. Using positive/negative frequency and helicity projections, we can project a test function into four parts,

$$
\hat{f}(k) = \theta(k_0)\frac{1}{2}(1+i\star)\hat{f}(k) + \theta(-k_0)\frac{1}{2}(1-i\star)\hat{f}(k) + \theta(k_0)\frac{1}{2}(1-i\star)\hat{f}(k) + \theta(-k_0)\frac{1}{2}(1+i\star)\hat{f}(k),
$$

where $\star$ denotes the Hodge dual, $(\star f)_{\mu\nu} = \epsilon_{\mu\nu}^{\alpha\beta}f_{\alpha\beta}$, and we have omitted indices from the above equation. In terms of these components, we may construct an invertible map $S \rightarrow \mathcal{S}; f \mapsto f^\star$,

$$
\tilde{f}^\star(k) = \theta(k_0)\frac{1}{2}(1+i\star)\hat{f}(k) + \theta(k_0)\frac{1}{2}(1-i\star)\hat{f}(k) + \theta(-k_0)\frac{1}{2}(1+i\star)\hat{f}(k) + \theta(-k_0)\frac{1}{2}(1-i\star)\hat{f}(k),
$$

in terms of which we can compute the commutation relations

$$
[a_{f,\star}, a^*_{g,\star}] = (f^\star, g^\star)
$$

$$
= -\hbar \int \frac{dk}{(2\pi)^4} 2\pi \delta(k_\mu k^\nu)\theta(k_0)k^\alpha \left[ \frac{1}{2}(1+i\star)\hat{f}(k) + \frac{1}{2}(1-i\star)\hat{f}(k) \right]^\star_{\alpha\mu} k^\beta \left[ \frac{1}{2}(1+i\star)\hat{g}(k) + \frac{1}{2}(1-i\star)\hat{g}(k) \right]_{\beta},
$$

$$
[a_{f,\star}, a^*_{g,\star}] = (f^\star, g^\star)
$$

$$
= -\hbar \int \frac{dk}{(2\pi)^4} 2\pi \delta(k_\mu k^\nu)\theta(k_0)k^\alpha \left[ \frac{1}{2}(1+i\star)\hat{f}(k) + \frac{1}{2}(1-i\star)\hat{f}(k) \right]_{\alpha\mu} k^\beta \left[ \frac{1}{2}(1+i\star)\hat{g}(k) + \frac{1}{2}(1-i\star)\hat{g}(k) \right]_{\beta}.
$$

Both integrands are invariant under $k \mapsto -k$. The latter expression is symmetric in $f$ and $g$, so for the complex linear operator-valued distribution $\hat{\chi}_f = a_{f,\star} + a^*_{f,\star}$,

$$
[\hat{\chi}_f, \hat{\chi}_g] = [a_{f,\star}, a^*_{g,\star}] - [a^*_{f,\star}, a_{f,\star}] = 0,
$$

hence $\hat{\chi}_f$ is a Lorentz covariant random field for an arbitrary complex-valued test function $f_{\mu\nu}(x)$. Furthermore, we find that $f^{\star\star} = f^{\star\star}$, so we can introduce an operator $b_f = a_{f,\star}$, satisfying the commutation relation $[b_f, b^*_g] = [a_{f,\star}, a^*_{g,\star}] = (f^\star, g^\star)$, in terms of which we can write $\hat{\chi}_f = b_{f,\star} + b^{\dagger}_{f,\star}$, which is a self-adjoint observable if $f^\star = f$. Note that if $f = f^\star$ then $f^{\star\star} = f^{\star\star}$, so $f \mapsto f^\star$ maps real-valued test functions to real-valued test func-
Discussion

The algebra of creation and annihilation operators that is directly associated with the random field $\chi_f$ is the same as that for the quantized electromagnetic field, however the relationship to space-time is different. The different relationship to space-time has an extreme consequence for our intuition, since it requires us to use different functions to construct a given algebraic relationship between operators. To use a random field approach, we have to suppose that there is no a priori set of test functions associated with a given experimental apparatus, even though the usual associations have become very familiar. Instead, if we can use a set of operators $S_a = \{ a_f \}$ to construct a prepared state and measurement operators as an empirically effective model for an experiment, we can equally well use $S_b = \{ b_f \}$, in the same expressions, which gives the same expected values for the same experimental data. In the experimental description given by $S_b$, we can ask what we would observe if we had an ideal apparatus that could measure $\chi_f$, even though we do not have such a thing, allowing a semblance of classical thinking. Although a sufficient degree of classical thinking is possible to encourage a reassessment of quantum field theory, the differences between random fields and classical differentiable fields are already substantial, so we should have limited expectations, particularly if we are to consider ourselves to the non-commutative algebra of creation and annihilation operators, not just to the commutative algebra of the random field observables. When we return to considering the results of real experimental apparatuses, we will of course have to ensure that the effects of quantum noise are properly modeled, since indeed we do not have any measurement apparatuses that are free of and unaffected by quantum noise.

I am grateful to Iwo Bialynicki-Birula for encouraging me to construct a random field equivalent for the quantized electromagnetic field and for comments that focused my attention on helicity \[8\]. The relationship between $f$ and $f^*$ is comparable to the relationship between the electric and magnetic test functions in \[8, \S V\].

An alternative random field equivalent

We may also construct an invertible map $S \rightarrow S; f \mapsto f^\square$, \[\begin{align*}
\tilde{f}^\square(k) &= \theta(k_0) \frac{1}{2} (1 + i\star) \tilde{f}(k) \\
&+ \theta(k_0) \frac{1}{2} (1 - i\star) \tilde{f^*}(k) \\
&+ \theta(-k_0) \frac{1}{2} (1 + i\star) \tilde{f^*}(k) \\
&+ \theta(-k_0) \frac{1}{2} (1 - i\star) \tilde{f}(k),
\end{align*}\] (A.7) in terms of which we can compute the commutation relation

$$[a_f, a^\dagger_g] = (f^\square, g^\square)$$

$$= -\hbar \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k_\mu k^\mu) \theta(k_0) k^\alpha \left[ \frac{1}{2} (1 + i\star) \tilde{f}(k) + \frac{1}{2} (1 - i\star) \tilde{f^*}(k) \right]_{\alpha\mu}^\ast k^\beta \left[ \frac{1}{2} (1 + i\star) \tilde{g}(k) + \frac{1}{2} (1 - i\star) \tilde{g^*}(k) \right]_{\beta}^\mu$$

$$= -\hbar \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k_\mu k^\mu) \theta(k_0) k^\alpha \left[ \frac{1}{2} (1 - i\star) \tilde{f^*}(k) + \frac{1}{2} (1 + i\star) \tilde{f}(k) \right]_{\alpha\mu}^\ast k^\beta \left[ \frac{1}{2} (1 + i\star) \tilde{g}(k) + \frac{1}{2} (1 - i\star) \tilde{g^*}(k) \right]_{\beta}^\mu, \quad (A.8)$$

which is symmetric in $f$ and $g$, so that for the self-adjoint operator-valued distribution $\xi_f = a_f^\square + a^\dagger_f^\square$, \[\begin{align*}
[\xi_f, \xi_g] &= [a_f^\square, a^\dagger_g] - [a_g^\square, a^\dagger_f] = 0. \quad (A.9)
\end{align*}\]

$\hat{\xi}_f$ is therefore an observable Lorentz covariant random field for an arbitrary complex-valued test function $f_\mu(x)$. In the context of this alternative random field, we can introduce an operator $c_f = a_f^\square$, satisfying the commutation relation $[c_f, c_g^\dagger] = [a_f^\square, a^\dagger_g^\square] = (f^\square, g^\square)$, in terms of which we can write $\hat{\xi}_f = c_f + c_f^\dagger$. This construction is not complex linear, which makes it much less attractive than the construction in the main text, but it is as well to record this alternative partly as a reminder not

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