Multiloop Amplitudes of Light-cone Gauge String Field Theory for
Type II Superstrings

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Abstract

Feynman amplitudes of light-cone gauge string field theory for Type II superstrings are shown to be
equivalent to those of the covariant first quantized formulation. In order to regularize the contact term
divergences, we consider the theory in a linear dilaton background $\Phi_{\text{dilaton}} = -iQX^1$. We show that the
scattering amplitudes are correctly reproduced in the limit $Q \to 0$, even with Ramond sector external
lines.

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1 Introduction

It has been a long standing problem to construct a string field theory for closed superstrings, because of the problems with the method to calculate multiloop amplitudes using the picture changing operators. Recently, Sen has constructed a manifestly Lorentz invariant and BRST invariant string field theory for closed superstrings [1, 2, 3, 4, 5, 6] based on the recently established method [7, 8] to calculate multiloop amplitudes using the picture changing operators, generalizing the string field theory [9, 10] for closed bosonic strings with a nonpolynomial action.

Light-cone gauge closed string field theory is a string field theory which involves only cubic interaction terms [11, 12, 13, 14, 15]. The equivalence of the scattering amplitudes of the light-cone approach and the covariant approach has been established for closed bosonic strings [16, 17]. The light-cone gauge amplitudes for superstrings suffer from the so-called contact term divergences [18, 19, 20]. Up to these problems, the equivalence of the amplitudes for the two approaches are shown in [21] when all the external lines are in the (NS,NS) sector for Type II superstrings or in the NS sector for heterotic strings.

In [25, 26, 27], we show that the contact term divergences are regularized by dimensional regularization. Light-cone gauge string field theory can be formulated in noncritical dimensions or taking the worldsheet theory for the transverse variables to be the one with central charge $c \neq 12$. One convenient choice of the worldsheet theory is that in a linear dilaton background $\Phi_{\text{dilaton}} = -iQ X^1$, with a space-like direction $X^1$ and a real constant $Q$. Although the theory becomes Lorentz noninvariant, it is equivalent to a BRST invariant conformal gauge formulation with an unusual longitudinal part. As in the conventional field theory, one can define the amplitudes as analytic functions of $Q$ and take the limit $Q \to 0$ to obtain those in the critical dimensions. In [28, 29, 30, 31], we have found that the correct tree level amplitudes can be obtained by taking the limit $Q \to 0$ in the dimensionally regularized amplitudes without adding any counterterms. In [32, 33], we have shown the same thing for multiloop amplitudes when all the external lines are in the (NS,NS) or the NS sector.

What we would like to do in this paper is to generalize the results obtained so far to the case where some of the external lines are in the (NS,R), (R,NS) or (R,R) sector, in the case of Type II superstrings. Since our approach does not rely on the superspace formalism on the worldsheet, it is straightforward to deal with the spin fields, which are necessary for constructing amplitudes with Ramond sector external lines. We show that the light-cone gauge amplitudes can be recast into the conformal gauge ones in the same way as in [32, 33]. We find that the regularization works even in the presence of spin fields and the correct amplitudes are obtained by taking the limit $Q \to 0$. The heterotic strings are discussed in a separate paper, for the reasons explained in section 7.

The organization of this paper is as follows. In section 2, we present the form of the light-cone gauge amplitude as an integration of a correlation function of vertex operators made from the transverse variables, over the moduli parameters. In sections 3, 4 and 5 we explain how we can transform the integrands of the light-cone gauge amplitudes into correlation functions of conformal gauge worldsheet theory and obtain the expression of the scattering amplitudes of the covariant approach. In section 6 we show that the dimensional regularization works as in the previous papers and the desired results are obtained in the limit $Q \to 0$. Section 7 is devoted to discussions. Several technical points are treated in the appendices.

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Footnote: In this paper, we deal with the light-cone gauge string field theory in the NSR formalism. The light-cone gauge string field theory in the Green-Schwarz formalism [22, 23] can be shown to be equivalent to that in the NSR formalism [24].
2 Light-cone gauge amplitudes

A Feynman diagram in the light-cone gauge string field theory, as schematically shown in figure [1] is made from the propagator and the vertex depicted in the same figure. A g-loop N-point amplitude is given in the form

\[ A_{N}^{(g)} = (ig_e)^{2g-2+N} e^{\sum_{\text{spin structure}} F_{N}^{(g)}} \prod_{K} dT_{K} \int dX_{g}^{(2g-2+N)C} \]

(2.1)

where \( T_{K} \) \((K = 1, \cdots, 6g - 6 + 2N)\) denote the moduli parameters of the light-cone diagram. In the case of Type II superstrings, the integrand \( F_{N}^{(g)} \) is given as a path integral over the transverse variables on the light-cone diagram. In this paper, we consider the flat background for simplicity and the transverse variables are \( X^{i}, \psi^{i}, \bar{\psi}^{i} \) \((i = 1, \ldots, 8)\). The light-cone diagram can be regarded as a punctured Riemann surface \( \Sigma \). On \( \Sigma \), there exists a unique holomorphic coordinate \( \rho \) whose real part is proportional to the light-cone time. \( \rho \) can be expressed in terms of a local coordinate \( z \) as [34]

\[ \rho(z) = \sum_{r=1}^{N} \alpha_{r} \left[ \ln E(z, Z_{r}) - 2\pi i \int_{p_{0}}^{z} \frac{1}{\Im \omega} \Im \int_{p_{0}}^{Z_{r}} \omega \right] . \]

(2.2)

Here \( E(z, w) \) is the prime form of the surface, \( \omega \) is the canonical basis of the holomorphic abelian differentials, \( \Omega \) is the period matrix [3] and \( z = Z_{r} \) \((r = 1, \cdots, N)\) are the punctures. The path integral on the light-cone diagram is defined by using the metric

\[ ds^{2} = d\rho d\bar{\rho} . \]

(2.3)

This metric is not well-defined at the punctures, and the interaction points of the light-cone diagram \( z = z_{I} \) \((I = 2g - 2 + N)\), which satisfy

\[ \partial \rho (z_{I}) = 0 . \]

\( F_{N}^{(g)} \) can be expressed in terms of correlation functions defined with a metric \( ds^{2} = 2\hat{g}_{z\bar{z}} dz d\bar{z} \) which is regular everywhere on the worldsheet, as

\[ F_{N}^{(g)} \propto (2\pi)^{2} \delta^{2} \left( \sum_{p_{r}^{+}} \right) e^{-4\Gamma_{[\varphi; \hat{g}_{z\bar{z}}]} } \times \left[ dB^{(2g-2+N)/2} \prod_{I=1}^{2g-2+N} \left[ |\partial^{2} \rho(z_{I})|^{-\frac{1}{2}} T_{F}^{I} (z_{I}) T_{\bar{F}}^{I} (\bar{z}_{I}) \right] \prod_{r=1}^{N} V_{r}^{LC} (Z_{r}, \bar{Z}_{r}) \right] . \]

(2.4)

Here \( S^{LC} \) denotes the worldsheet action of the transverse variables, \( T_{F}^{I} \), \( T_{\bar{F}}^{I} \) are the supercurrents for these variables and \( V_{r}^{LC} \) denotes the vertex operator for the \( r \)-th external line inserted at the puncture \( z = Z_{r} \). The explicit form of the vertex operator can be found in appendix [A.1]. The path integral measure \( [dX^{i} d\psi^{i} d\bar{\psi}^{i}]_{\hat{g}_{z\bar{z}}} \) is defined with the metric \( ds^{2} = 2\hat{g}_{z\bar{z}} dz d\bar{z} \). Since the integrand is defined by using the metric (2.3), we need the Weyl anomaly factor \( e^{-4\Gamma_{[\varphi; \hat{g}_{z\bar{z}}]} } \), where

\[ \varphi = \ln \partial \rho \partial \bar{\rho} - \ln \hat{g}_{z\bar{z}} , \]

\[ \Gamma_{[\varphi; \hat{g}_{z\bar{z}}]} = -\frac{1}{4\pi} \int dz \wedge d\bar{z} \sqrt{\hat{g}} \left( \hat{g}^{a\bar{b}} \partial_{a} \varphi \partial_{\bar{b}} \varphi + 2\hat{R} \varphi \right) . \]

(2.5)

\(^{3}\)For the mathematical background relevant for string perturbation theory, we refer the reader to [35].
The right hand side of (2.4) does not depend on the choice of $\hat{g}_{z \bar{z}}$. The most convenient choice for $\hat{g}_{z \bar{z}}$ is the Arakelov metric $g^A_{z \bar{z}}$ [36 37 38]. The explicit form of $e^{-\Gamma[\varphi; g^A_{z \bar{z}}]}$ is given as [34]

$$e^{-\Gamma[\varphi; g^A_{z \bar{z}}]} = \prod_r e^{-2 \Re {\hat{N}_0}} \left( 2g^A_{Z_r \bar{Z}_r} \right)^{-1} \prod_I \left| \partial^2 \rho (z_I) \right|^{-3} \left( 2g^A_{z_I \bar{z}_I} \right)^3$$

$$\times \exp \left[ -2 \sum_{I < J} G^A (z_I; z_J) - 2 \sum_{r < s} G^A (Z_r; Z_s) + 2 \sum_{I, r} G^A (z_I; Z_r) \right]. \quad (2.6)$$

Here $G^A (z; w)$ is the Arakelov Green’s function expressed as

$$G^A (z; w) = -\ln |E(z, w)|^2 + 2\pi \Im \int_w^z \omega \frac{1}{\Im \omega} \Im \int_w^z \omega - \frac{1}{2} \ln (2g^A_{z \bar{z}}) - \frac{1}{2} \ln (2g^A_{w \bar{w}}), \quad (2.7)$$

and

$$\hat{N}_0 = \frac{\rho(z_I^R)}{\alpha_r} - \sum_{s \neq r} \frac{\alpha_s}{\alpha_r} \ln E(Z_r, Z_s) + \frac{2\pi i}{\alpha_r} \int_{p_0}^{Z_r} \omega \frac{1}{\Im \Omega} \sum_{s=1}^{N} \alpha_s \Im \int_{p_0}^{Z_s} \omega,$$

where $z_I^R$ is defined to be the coordinate of the interaction point at which the $r$-th external line interacts.

With the $e^{-\Gamma[\varphi; g^A_{z \bar{z}}]}$ given by (2.6), it is possible to write down the amplitude (2.4) explicitly using the formulas for correlation functions of the transverse variables on higher genus Riemann surfaces given in sections 4.1 and 4.2.

### 3 Amplitudes in the conformal gauge

The amplitude (2.4) can be expressed by conformal gauge worldsheet theory. Namely, as we will show in section 2 [2.4], it is proportional to

$$\int D[XBC] e^{-S_{\text{tot}}} \prod_{K=1}^{6g-6+2N} \left[ \int_{C_K} \frac{dz}{\partial \omega} b_{z \bar{z}} + \varepsilon_K \int_{C_K} \frac{d\bar{z}}{\partial \bar{\omega}} b_{z \bar{z}} \right]^{2g-2+N} \prod_{I=1}^{N} [X(z_I) X(\bar{z}_I)] \prod_{r=1}^{N} V_r (Z_r, \bar{Z}_r), \quad (3.1)$$

3
with vertex operators \( V_r(Z_r, \bar{Z}_r) \) whose explicit forms will be given shortly. Here \( S^{\text{tot}} \) denotes the worldsheet action for the conformal gauge variables, \( D[XBC] \) denotes the path integral measure for them,

\[
X(z) = \left[ c \partial \xi - e^{\phi} T_F + \frac{1}{4} \partial b \eta e^{2\phi} + \frac{1}{4} b \left( 2 \partial \eta e^{2\phi} + \eta \partial e^{2\phi} \right) \right](z)
\]  

(3.2)

is the picture changing operator (PCO), \( \bar{X}(\bar{z}) \) is its antiholomorphic counterpart and \( T_F \) denotes the supercurrent for the matter part of the worldsheet theory. The contours \( C_K \) and \( \varepsilon_K = \pm 1 \) are chosen so that the antighost insertions correspond to the moduli parameters for the light-cone diagram \(^{[22]}\). The expression (3.1) coincides with the integrand of the amplitude obtained by the covariant approach \(^{[39, 40]}\), in which the locations of the PCO’s are taken to be the interaction points of the light-cone diagram \(^{[4]}\).

The vertex operators \( V_r(Z_r, \bar{Z}_r) \) are chosen from those constructed in appendix \( [A.2] \). Suppose that the left moving part of \( V_r^{LC} \) for \( r = 1, \ldots, N - 2M \) are in the NS sector and those for \( r = N - 2M + 1, \ldots, N \) are in the R sector. We use indices \( s, s', s'' \) to denote \( r \) satisfying \( 1 \leq r \leq N - 2M, N - 2M + 1 \leq r \leq N - M, N - M + 1 \leq r \leq N \) respectively. In constructing the conformal gauge amplitudes, we need to treat the following two cases separately:

1. \( M \neq 0 \) or the spin structure is even.
2. \( M = 0 \) and the spin structure is odd.

Although the case \( M \neq 0 \) is our main concern in this paper, here and in the following, we also discuss other cases for completeness. If \( M \neq 0 \) or the spin structure is even, the left moving part of \( V_r(Z_r, \bar{Z}_r) \) \( (r = 1, \ldots, N) \) denoted by \( V^{(pl)}_{rL}(Z_r) \) are taken to be \(^{[4]}\)

\[
\begin{align*}
V^{(pl)}_{sL}(Z_s) &= V^{(-1)}_{sL}(Z_s), \\
V^{(pl)}_{s'L}(Z_{s'}) &= V^{(-\frac{1}{2})}_{s'L}(Z_{s'}), \\
V^{(pl)}_{s''L}(Z_{s''}) &= V^{(-\frac{3}{2})}_{s''L}(Z_{s''}).
\end{align*}
\]

(3.3)

When all the external lines are in the NS sector and the spin structure is odd, two of the vertex operators should be taken to be in the pictures 0 and \(-2 \) \(^{[33]}\). For example, we take

\[
\begin{align*}
V^{(pl)}_{1L}(Z_1) &= V^{(-2)}_{1L}(Z_1), \\
V^{(pl)}_{2L}(Z_2) &= V^{(0)}_{2L}(Z_2), \\
V^{(pl)}_{rL}(Z_r) &= V^{(-1)}_{rL}(Z_r) \quad (r \geq 3).
\end{align*}
\]

(3.4)

The right moving part of the vertex operator \( V^{(pr)}_{rR}(Z_r) \) \( (r = 1, \ldots, N) \) are defined following the same rule depending on \( V^{LC}_{rR} \) and the spin structure of the right moving sector. Combining these, we get the vertex operator \( V_r(Z_r, \bar{Z}_r) \) expressed as

\[
V_r(Z_r, \bar{Z}_r) = V^{(pl)}_{rL}(Z_r)V^{(pr)}_{rR}(\bar{Z}_r).
\]

\(^{4}\)Notice that in the light-cone setup, the positions of the PCO’s have a fixed coordinate in the coordinate patch on the surface and we do not need the \( \partial \xi \) terms.

\(^{5}\)The amplitude does not depend on which of the R sector vertex operators are chosen to be in the \(-\frac{1}{2}\) picture.
4 Correlation functions on higher genus Riemann surfaces

In order to show that (4.1) is proportional to (2.4), we need to know the correlation functions of the longitudinal variables $X^\pm, \psi^\pm$ and the ghosts. These correlation functions are derived from the so-called bosonization formula [41, 42, 43].

The correlation functions for a free field theory with central charge $c$ are recast into the form [44, 45]

\[(\text{left moving part}) \times (\text{right moving part}) \times e^{-cS}, \tag{4.1}\]

using [32]

\[ (g_{zz}^A)^2 \exp \left\{ -\frac{2\pi}{g-1} \frac{\text{Im} \int_{(g-1)z}^{\Delta} \omega (\text{Im} \Omega)^{-1} \text{Im} \int_{(g-1)z}^{\Delta} \omega} \right\} = |\sigma(z)|^2 e^{\frac{3}{2}cS}, \tag{4.2} \]

for $g \neq 1$ and

\[ (g_{zz}^A)^2 \exp \left\{ 4\pi \text{Im} \int_{P_0}^{z} \omega (\text{Im} \Omega)^{-1} \text{Im} \int_{P_0}^{\Delta} \omega \right\} \equiv |\sigma(z)|^2 e^A, \]

\[ \exp \left\{ -2\pi \text{Im} \int_{P_0}^{\Delta} \omega (\text{Im} \Omega)^{-1} \text{Im} \int_{P_0}^{\Delta} \omega \right\} \equiv e^{3S}, \tag{4.3} \]

for $g = 1$. Here $\Delta$ is the Riemann class and $\sigma(z)$ is the holomorphic $\frac{\Delta}{2}$ form which transforms as

\[ \sigma(z) \rightarrow e^{-2\pi i \int_{(g-1)z}^{\Delta} \omega J + \pi i (g-1) \Omega_{J,J}} \sigma(z), \tag{4.4} \]

when $z$ is moved around the $B_J$ cycle once, and invariant when $z$ is moved around the $A_J$ cycles. $S, A$ are quantities which do not depend on $z$. Since the total central charge of the worldsheet theory vanishes, we need only the left moving or right moving part of the correlation function to calculate $F_N^{(g)}$. In this section, we collect the necessary formulas to evaluate (37).

4.1 Free bosons

Let us consider a free scalar boson $X$. The correlation functions of local operators $e^{ipX}$ can be expressed as an integration of the holomorphically factorized correlation function at the fixed internal momenta [35]:

\[
\int [dX]_{\phi^2} e^{-S^X} \prod_r e^{ip_r X} (Z_r, \bar{Z}_r)
\]

\[
\propto 2\pi \delta \left( \sum_p \right) e^{-S} \int dP_j \left\langle \prod_{r,s} e^{ip_r X_r} (Z_r) \right\rangle_{X_r, P_j} \left( \left\langle \prod_{r,s} e^{ip_r X_r} (Z_r) \right\rangle_{X_r, P_j} \right)^*, \tag{4.5}
\]

where $S^X$ denotes the action for $X$ and

\[
\left\langle \prod_{r,s} e^{ip_r X_r} (Z_r) \right\rangle_{X_r, P_j} = \langle 1 \rangle_{X_r, P_j} \prod_{r<s} E(Z_r, Z_s)^{p_r p_s} \exp \left\{ \frac{\pi i}{2} \sum_{J,J'} P_j \Omega_{J,J'} P_{J'} + 2\pi i \sum_{J,J'} \sum_r p_r \int_{P_0}^{\Delta_{J,J'}} \omega \right\},
\]

\[
\langle 1 \rangle_{X_r} = \left( \frac{\prod_r E(z_i, R) \sigma(R) \det \omega_J(z_i)}{\theta[0] \left( \sum_r \int_{P_0}^{\Delta_{J,J'}} \omega - \int_{P_0}^{\Delta_{J,J'}} \omega \right) \prod_{J,J'} E(z_i, z_j) \prod_{r} \sigma(z_i) \right)^{\frac{1}{2}}.
\]

5
$X_L$ here denotes the left moving part of $X$ and $\theta[\alpha](\zeta|\Omega)$ denotes the theta function with characteristics $[\alpha] = [\alpha']$. Roughly speaking, $(1)_{X_L}$ is the left moving part of the partition function of $X$ and it satisfies

$$
\left( \frac{\det' (-g^{\alpha\beta} \partial_\alpha \partial_\beta)}{\int d^2 z \sqrt{g^L}} \right)^{-\frac{1}{2}} \propto \int \prod_j dP_j \left| \langle 1 \rangle_{X_L} \exp \left[ \pi i \sum P_j \Omega_{j,j'} P_{j'} \right] \right|^2 e^{-S} \nonumber \\
\propto \left( \det \text{Im} \Omega \right)^{-\frac{1}{2}} \left| \langle 1 \rangle_{X_L} \right|^2 e^{-S} .
$$

Correlation functions involving derivatives of $X$ can be derived from (4.6) in a holomorphically factorized form.

### 4.2 Free fermions

Let us bosonize the free fermions $\psi^\pm$ so that

$$
\psi^+ = e^{iH}, \psi^- = -e^{-iH}.
$$

For an even spin structure, the left moving part of the correlation function of $\psi^\pm$ is given by

$$
\left\langle \prod_{i=1}^n \psi^-(x_i) \prod_{j=1}^n \psi^+(y_j) \right\rangle_{\psi^\pm} = \langle 1 \rangle_{X_L} \theta[\alpha_L](0) \det \left[ -S_{\alpha_L}(x_i,y_j) \right] ,
$$

where

$$
S_{\alpha}(z,w) = \frac{1}{E(z,w)} \theta[\alpha] \left( \int_0^\pi \omega \right)
$$

(4.7)
is the Szegö kernel and $\alpha_L$ corresponds to the spin structure of the fermion. When there are Ramond sector external lines, the correlation function we need for calculating the amplitude (8.1) is

$$
\left\langle \prod_{i=1}^n \psi^-(x_i) \prod_{j=1}^n \psi^+(y_j) \prod_{s'} e^{-\frac{i}{2} H} (Z_{s'}) \prod_{s''} e^{\frac{i}{2} H} (Z_{s''}) \right\rangle_{\psi^\pm}
$$

$$
= \langle 1 \rangle_{X_L} \theta[\alpha_L] \left( \frac{1}{2} \sum_{s'} \int_{P_0} ^{Z_{s'}} \omega - \frac{1}{2} \sum_{s''} \int_{P_0} ^{Z_{s''}} \omega \right) \prod_{s'<s''} E(z_{s'},z_{s''})^{\frac{1}{2}} \prod_{s''} E(z_{s''})^{\frac{1}{2}} \prod_{s'} E(z_{s'})^{\frac{1}{2}}
$$

$$
\times \det \left( -S_{\alpha_L}(x_i,y_j) \right) ,
$$

(4.8)

where

$$
S_{\alpha_L}(x,y) = \frac{\theta[\alpha_L] \left( \int_{P_0} ^{x} \omega - \int_{P_0} ^{y} \omega + \frac{1}{2} \sum_{s'} \int_{P_0} ^{Z_{s'}} \omega - \frac{1}{2} \sum_{s''} \int_{P_0} ^{Z_{s''}} \omega \right)}{\theta[\alpha_L] \left( \frac{1}{2} \sum_{s'} \int_{P_0} ^{Z_{s'}} \omega - \frac{1}{2} \sum_{s''} \int_{P_0} ^{Z_{s''}} \omega \right)}
$$

$$
\times \frac{1}{E(x,y)} \prod_{s'} E(x,Z_{s'})^{\frac{1}{2}} \prod_{s''} E(y,Z_{s''})^{\frac{1}{2}} \prod_{s'} E(y,Z_{s'})^{\frac{1}{2}}.
$$

(4.9)

$S_{\alpha_L}(x,y)$ can be considered as the propagator of the fermions in the presence of the spin fields. Notice that $\theta[\alpha_L] \left( \frac{1}{2} \sum_{s'} \int_{P_0} ^{Z_{s'}} \omega - \frac{1}{2} \sum_{s''} \int_{P_0} ^{Z_{s''}} \omega \right) \neq 0$ for generic $Z_{s'}, Z_{s''}$ and there are no problems about (4.8) even when $\alpha_L$ corresponds to an odd spin structure.
When all the external lines are in the (NS,NS) sector and the spin structure is odd, the formula we need is
\[
\left\langle \prod_{i=1}^{n} \psi^{-}(x_{i}) \prod_{j=1}^{n} \psi^{+}(y_{j}) \right\rangle_{\psi^{\pm}} = (1)_{X_{L}} \int d\psi^{+}_{0} d\psi^{-}_{0} \text{det} \left[-S_{\alpha L}(x_{i}, y_{j})\right],
\]
where
\[
S_{\alpha L}(x, y) = \psi^{-}_{0} h_{\alpha L}(x) \psi^{+}_{0} h_{\alpha L}(y) + \frac{1}{E(x, y)} \sum_{\nu} \frac{\partial_{\nu} \vartheta [\alpha_{L}]}{\sum_{\nu} \partial_{\nu} [\alpha_{L}]}(0, \Omega) \omega_{\nu}(p),
\]
and
\[
h_{\alpha L}(z) = \sqrt{\sum_{j} \partial_{j} \vartheta [\alpha_{L}](0) \omega_{j}(z)}.
\]

4.3 The reparametrization ghost

The correlation function of the \(bc\) system which appear in \([31]\) is evaluated as
\[
\int [dbdbdc\bar{c}]_{g_{2}z} e^{-S_{bc}} \prod_{r=1}^{N} \left[c(Z_{r})\bar{c}(\bar{Z}_{r})\right] \prod_{K=1}^{6g-6+2N} \left[\int_{C_{K}} dz \frac{b_{\nu}}{2\pi i} \partial_{\rho}(z) + \varepsilon_{K} \int_{C_{K}} \frac{dz}{2\pi i} \bar{b} \right](\bar{z}) \propto \text{det} \left(-g^{A}z_{K}z_{L} \right)^{N} \prod_{r=1}^{N} \left(\alpha_{r} e^{2\text{Re} \vartheta[\alpha_{r}]}\right) e^{-\Gamma[\psi^{A}z_{K}]}.
\]
A proof of this formula was given in \([34]\).

4.4 The superghost

The superghost system is bosonized so that
\[
\beta = e^{-\phi} \partial_{\xi}, \quad \gamma = \eta e^{\phi}.
\]
The left moving part of the correlation function relevant to superstring amplitudes involving Ramond sector external lines is
\[
\left\langle \prod_{l} e^{\phi}(z_{l}) \prod_{s} e^{-\phi}(z_{s}) \prod_{s'} e^{-\frac{1}{2} \phi}(z_{s'}) \prod_{s''} e^{-\frac{1}{2} \phi}(z_{s''}) \right\rangle_{\beta \gamma} \sim \left[ (1)_{X_{L}} \vartheta[\alpha_{L}] \left( \sum_{l} \int_{P_{0}}^{z_{l}} \omega - \sum_{r} \int_{P_{0}}^{Z_{r}} \omega - \frac{1}{2} \sum_{s} \int_{P_{0}}^{Z_{s}} \omega - \frac{3}{2} \sum_{s'} \int_{P_{0}}^{Z_{s'}} \omega - 2 \sum_{P_{0}}^{Z_{\Delta}} \omega \right) \right]^{-1} \times \frac{\prod_{l>l'} E(z_{l}, z_{s}) \prod_{l>s} E(z_{l}, z_{s})^{\frac{1}{2}} \prod_{l>s'} E(z_{l}, z_{s'})^{\frac{1}{2}} \prod_{s>s'} E(z_{s}, z_{s'})^{\frac{1}{2}}}{\prod_{s>s} E(z_{s}, z_{s'})^{\frac{1}{2}} \prod_{s>s'} E(z_{s}, z_{s'})^{\frac{1}{2}} \prod_{s>s''} E(z_{s}, z_{s''})^{\frac{1}{2}} \prod_{s'>s''} E(z_{s'}, z_{s''})^{\frac{1}{2}} \prod_{s<s'} E(z_{s'}, z_{s''})^{\frac{1}{2}} \prod_{s<s''} E(z_{s}, z_{s''})^{\frac{1}{2}} \prod_{s'<s''} E(z_{s'}, z_{s''})^{\frac{1}{2}} \prod_{s<s'} E(z_{s'}, z_{s''})^{\frac{1}{2}} \prod_{s<s''} E(z_{s}, z_{s''})^{\frac{1}{2}} \prod_{s'<s''} E(z_{s'}, z_{s''})^{\frac{1}{2}}}
\times \frac{1}{\prod_{s} \sigma(z_{s})^{2} \prod_{s} \sigma(z_{s}) \prod_{s} \sigma(z_{s})^{3}}.
\]

(4.13)
The formula relevant for odd spin structure amplitudes is
\[
\left\langle \prod_{I} e^{\phi (z_I)} e^{-2\phi (Z_I)} \prod_{r \geq 3} e^{-\phi (Z_r)} \right\rangle_{\beta \gamma}
\]
\[
\sim \left\{ (1)_{\text{Fl}} \frac{\rho}{\alpha} \left[ \sum_{I} \int_{P_0}^{Z_1} \omega - 2 \sum_{r \geq 3} \int_{P_0}^{Z_r} \omega - 2 \int_{P_0}^{\Delta} \omega \right]^{-1}
\times \frac{\prod_{I} E(z_I, Z_I) \prod_{I, r \geq 3} E(z_I, Z_r)}{\prod_{I, r \geq 3} E(z_r, Z_r) \prod_{r \geq 3} E(Z_r, Z_s) \prod_{r \geq 3} \sigma(Z_r) \prod_{r \geq 3} \sigma(Z_r)} \right\}.
\]

### 4.5 Useful formulas

Since \( z_I (I = 1, \cdots, 2g - 2 + N) \) and \( Z_r (r = 1, \cdots, N) \) are the zeros and poles of the one form \( \partial \rho (z) dz \) respectively,
\[
\sum_{I=1}^{2g-2+N} \int_{P_0}^{Z_I} \bar{\omega} - \sum_{r=1}^{N} \int_{P_0}^{Z_r} \bar{\omega} - 2 \int_{P_0}^{\Delta} \bar{\omega} = \bar{m} + \Omega \bar{n},
\]
with \( \bar{m}, \bar{n} \in \mathbb{Z}^2 \). Using this, we get the following expression of \( e^{-\Gamma_{[\bar{s};\bar{g},z_1]} \cdot} \) in \( (4.6) \)
\[
e^{-\Gamma_{[\bar{s};\bar{g},z_1]} \cdot} = \vert \langle 1 \rangle_{\text{LC}} \vert^2 e^{24S},
\]
where
\[
\langle 1 \rangle_{\text{LC}} = \frac{\prod_{I,J} E(z_I, z_J) \prod_{I,r \geq 3} E(Z_r, Z_s)}{\prod_{I,r \geq 3} E(z_r, Z_s) \prod_{r \geq 3} E(Z_r, Z_s) \prod_{r \geq 3} \sigma(Z_r) \prod_{r \geq 3} \sigma(Z_r)} \times e^{-2\pi i\bar{m}\bar{n}} \prod_{r} e^{-\frac{N_{\bar{r}}}{2}} \prod_{I} \partial^2 \rho (z_I) \frac{1}{z_I^2}.
\]

Other useful formulas can be derived from
\[
\vert \partial \rho (z) \vert^2 = C g_{zz}^A \exp \left[ \sum_{r} G^A (z; Z_r) - \sum_{I} G^A (z; z_I) \right],
\]
where \( C \) is a quantity which does not depend on \( z \). Substituting \( (2.7), (4.2), (4.3) \) and \( (4.15) \) into this, we get
\[
\partial \rho (z) = C' \sigma (z^2) \prod_{I} \frac{E(z, z_I)}{E(z, Z)} \exp \left[ -2\pi i\bar{n} \cdot \int_{z}^{z_I} \bar{\omega} \right],
\]
where \( C' \) is a quantity which does not depend on \( z \). From this formula, it is easy to get
\[
\alpha_s = \lim_{z \to Z_s} (z - Z_s) \partial \rho (z) = C' \sigma (Z_s^2) \prod_{I} \frac{E(z, z_I)}{E(Z_s, Z) \prod_{r \neq s} E(Z_s, Z_r)} \exp \left[ -2\pi i\bar{n} \cdot \int_{z}^{Z_s} \bar{\omega} \right],
\]
\[
\partial^2 \rho (z_I) = \lim_{z \to z_I} (z - z_I) \partial \rho (z) = C' \sigma (z_I^2) \prod_{I} \frac{E(z_I, z_J)}{E(z_I, Z) \prod_{r \neq I} E(z_I, Z_r)} \exp \left[ -2\pi i\bar{n} \cdot \int_{z}^{z_I} \bar{\omega} \right].
\]
\( ^6 \)Since \( \rho \) and \( z_I \) depend on the antiholomorphic moduli, it may not be appropriate to identify \( \langle 1 \rangle_{\text{LC}} \) with the left moving part of \( e^{-\Gamma_{[\bar{s};\bar{g},z_1]} \cdot} \).
\section{Equivalence of the light-cone gauge amplitude and the conformal gauge amplitude}

With the formulas collected in the previous section, let us prove that (3.1) is proportional to (2.4). It is possible to show that (3.1) is equal to

\[ \int D [X^{\pm} BC] e^{-S_{\text{tot}}^{\text{light-cone}}} \prod_{K}^{2g-2+N} \left( \oint \frac{dz}{\partial \rho} b_{zz} + \varepsilon_{K} \oint \frac{dz}{\partial \rho} b_{zz} \right) \] \[ \times \prod_{j=1}^{N} \left[ e^{\phi} (z_{j}) e^{\phi} (\bar{z}_{j}) \right] \prod_{r} \left[ V_{r}(Z_{r}, \bar{Z}_{r}) \right]. \] \tag{5.1}

A proof is given in appendix B. Eq. (3.1) can be recast into the form

\[ \int [dX^{\pm} d\psi^{\pm} d\bar{\psi}]_{g_{A}^{\pm}} e^{-S_{\text{conf}}^{\text{light-cone}}} \prod_{I}^{[T_{F}^{\text{light-cone}} (z_{I}) T_{F}^{\text{light-cone}} (\bar{z}_{I})]} \] \[ \times \int D [X^{\pm} BC]_{g_{A}^{\pm}} e^{-S^{X^{\pm} BC}} \prod_{I}^{[e^{\phi} (z_{I}) e^{\phi} (\bar{z}_{I})]} \] \[ \times \prod_{K}^{\varepsilon_{K}} \left[ \oint \frac{dz}{\partial \rho} b_{zz} + \varepsilon_{K} \oint \frac{dz}{\partial \rho} b_{zz} \right] \prod_{r}^{V_{r}(Z_{r}, \bar{Z}_{r})}. \] \tag{5.2}

Here $S^{X^{\pm} BC}$ and $D [X^{\pm} BC]_{g_{A}^{\pm}}$ denote the action and the path integral measure of the longitudinal variables and the ghosts.

From (1.8), (1.10), (4.13), (4.14), (4.17), (4.19) and (4.20), we obtain formulas useful in performing the path integral over the longitudinal variables and ghosts in (5.2). When there are vertex operators in the R sector or the spin structure is even, we get

\[ \langle \prod_{s} e^{-\frac{1}{2} H} (Z_{s}) \prod_{s^\prime} e^{\frac{1}{2} H} (Z_{s'}) \rangle_{\psi^{\pm}} \] \[ = \pm \prod_{\alpha_{\beta}} \frac{\alpha_{\beta}}{\alpha_{\beta}} \langle (1)_{\text{LC}} \rangle^{\frac{1}{2}} \prod_{r}^{e^{-\frac{1}{2} \text{Re} N_{0}^{\text{rc}}} \prod_{l} (\partial^{2} \rho (z_{l}))^{-\frac{1}{2}}, \] \tag{5.3}

and for odd spin structures we have

\[ \langle \psi^{+} (Z_{1}) \psi^{-} (Z_{2}) \rangle_{\psi^{\pm}} \] \[ = \pm \frac{\alpha_{1}}{\alpha_{2}} \langle (1)_{\text{LC}} \rangle^{\frac{1}{2}} \prod_{r}^{e^{-\frac{1}{2} \text{Re} N_{0}^{\text{rc}}} \prod_{l} (\partial^{2} \rho (z_{l}))^{-\frac{1}{2}}. \] \tag{5.4}

(4.12), (5.3) and (5.4) imply

\[ \int D [X^{\pm} BC]_{g_{A}^{\pm}} e^{-S^{X^{\pm} BC}} \prod_{I}^{[e^{\phi} (z_{I}) e^{\phi} (\bar{z}_{I})]} \prod_{K}^{\varepsilon_{K}} \left[ \oint \frac{dz}{\partial \rho} b_{zz} + \varepsilon_{K} \oint \frac{dz}{\partial \rho} b_{zz} \right] \prod_{r}^{V_{r}(Z_{r}, \bar{Z}_{r})} \] \[ \propto (2\pi)^{2} \delta^{2} \left( \sum_{r}^{P_{r}} \right) \prod_{l}^{e^{-\frac{1}{2} \Gamma (\partial^{2} \rho (z_{l}))^{-\frac{1}{2}}} \prod_{r}^{V_{r}^{\text{LC}} (Z_{r}, \bar{Z}_{r})}. \] \tag{5.2}

Substituting this into (5.2), one can show that the conformal gauge expression (3.1) is proportional to the light-cone gauge expression (2.4). Comparing the factorization properties of these two expressions, one can fix the proportionality constant.
6 Dimensional regularization

Although the integrand $F_N^{(g)}$ in (2.1) is proportional to the conformal gauge expression (6.1), the amplitude (2.1) itself is divergent because of the contact term divergences [18, 19, 20]. Fortunately, these are the only spurious singularities to worry about in the light-cone gauge formulation. In the previous papers [32, 33], we have shown that it is possible to regularize the divergences by dimensional regularization, if all the external lines are in the (NS,NS) sector. In this section, we would like to show that the same results hold for the other cases.

The contact term divergences are regularized by taking the worldsheet superconformal field theory to be the one with central charge $c \neq 12$. We take the worldsheet theory to be the one in a linear dilaton background $\Phi_{\text{dilaton}} = -iQX^1$, with a real constant $Q$ and a space-like direction $X^1 \ [46]$. The worldsheet action of $X^1$ and its fermionic partners $\psi^1, \bar{\psi}^1$ on a worldsheet with metric $ds^2 = 2\hat{g}_{zz}dzd\bar{z}$ becomes

$$S[X^1, \psi^1, \bar{\psi}^1; \hat{g}_{zz}] = \frac{1}{8\pi} \int dzd\bar{z} \sqrt{g} \left( \hat{g}^{ab} \partial_aX^1 \partial_bX^1 - 2iQ\hat{R}X^1 \right) + \frac{1}{4\pi} \int dzd\bar{z} \left( \bar{\psi}^1 \bar{\partial} \psi^1 + \psi^1 \partial \bar{\psi}^1 \right), \quad (6.1)$$

Since the number of fermionic variables $\psi^i$ and $\bar{\psi}^i$ does not depend on $Q$, we do not have difficulties in dealing with chiral fermions in the regularization.

It is straightforward to formulate the light-cone gauge string field theory with such a worldsheet theory. The light-cone gauge amplitude becomes (2.1) with

$$F_N^{(g)} \propto (2\pi)^2 \delta^2 \left( \sum p_r^+ \right) e^{-\frac{i}{2}(1-Q^2)\Gamma[\varphi; \hat{g}_{zz}]} \times \int [dX^i d\psi^i d\bar{\psi}^i]_{\hat{g}_{zz}} e^{-S_{\text{LC}}} \prod_{I=1}^{2g-2+N} \left( \partial^2 \rho (z_I) \right)^{-\frac{1}{2}} T_L^{\psi^i C} (z_I) \bar{T}_F^{\psi^i C} (\bar{z}_I) \prod_{r=1}^{N} V_r^{LC} (Z_r, \bar{Z}_r). \quad (6.2)$$

In [46], it was shown that the amplitude given by (2.1) with the integrand (6.2) is finite\footnote{In [46], it was shown that integrand becomes regular at the possible singularities for $Q^2 > 10$. These singularities may be harmless and the integral (2.1) may be well-defined for $Q^2$ slightly less than 10.} for $Q^2 > 10$. We use this light-cone gauge expression of the amplitude for $Q^2 > 10$ to define it as an analytic function of $Q^2$, which is denoted by $A^{LC} (Q^2)$. The amplitude in the critical dimensions may be given by the limit $\lim_{Q \to 0} A^{LC} (Q^2)$. In order to study what happens in the limit $Q \to 0$, we recast the expression of the integrand into the one given by correlation functions in the conformal gauge worldsheet theory.

6.1 Supersymmetric $X^\pm$ CFT

As is explained in [27], the light-cone gauge worldsheet theory in the linear dilaton background corresponds to the conformal gauge worldsheet theory with an unusual longitudinal part which is called the supersymmetric $X^\pm$ CFT. The action of the supersymmetric $X^\pm$ CFT for Type II superstrings is given in the form

$$S^{X^\pm} = -\frac{1}{2\pi} \int d^2z \left( \bar{D}X^+ D\bar{X}^- + D\bar{X}^- D\bar{X}^+ \right) - Q^2 \Gamma_{\text{super}} [X^+, \hat{g}_{zz}], \quad (6.3)$$

where the supercoordinate $z$ is given by

$$z = (z, \theta), \quad (6.4)$$
the superfield $X^\pm$ is defined as

$$X^\pm (z, \bar{z}) = X^\pm (z) + i\theta \psi^\pm (z) + i\bar{\theta} \bar{\psi}^\pm (\bar{z}) + \theta \bar{\theta} F^\pm ,$$  \hspace{1cm} (6.5)

and

$$D = \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{z}},$$

$$\bar{D} = \frac{\partial}{\partial \bar{\theta}} + \theta \frac{\partial}{\partial z},$$

$$d^2 z = d(Re z) d(Im z) d\theta d\bar{\theta} .$$  \hspace{1cm} (6.6)

The interaction term $\Gamma_{\text{super}}$ is given by

$$\Gamma_{\text{super}} [X^+, g_{zz}] = - \frac{1}{2\pi} \int d^2 z \left( \bar{D} \Phi D \Phi + \theta \bar{\theta} g_{zz} \bar{R} \Phi \right) ,$$

$$\Phi (z, \bar{z}) = \ln \left( (D \Theta^+)^2 (z) (\bar{D} \bar{\Theta}^+)^2 (\bar{z}) \right) - \ln g_{zz} ,$$

$$\Theta^+ (z) = \frac{D X^+}{(\partial X^+)^2} (z) ,$$

which is the super Liouville action defined for $\Phi$ with the background metric $ds^2 = 2g_{zz} dz d\bar{z}$.

When the spin structures are both even, the correlation functions of the supersymmetric $X^\pm$ CFT are evaluated as \[32\]

$$\int [dX^+ dX^-] g_{zz} e^{-S_{X^\pm}} \prod_r e^{-ip_r^+ X^- (Z_r, \bar{Z}_r)} \prod_t e^{-ip_r^- X^+ (w_t, \bar{w}_t)}$$

$$= \int [dX^+ dX^-] g_{zz} e^{-S_{X^\pm}^{\text{free}}} \prod_r e^{-ip_r^+ X^- (Z_r, \bar{Z}_r)} \times e^{Q^2 \Gamma_{\text{super}} [X^+, g_{zz}]} \prod_t e^{-ip_r^- X^+ (w_t, \bar{w}_t)}$$

$$= (2\pi)^2 \delta \left( \sum p_t^+ \right) \delta \left( \sum p_r^+ \right) \left( \frac{\det (-g^{zz} \partial_z \partial_{\bar{z}})}{\int d^2 z \sqrt{g}} \right)^{-\frac{1}{2}} \left( \frac{\det (-g^{z\bar{z}} \partial_z \partial_{\bar{z}})}{\int d^2 z \sqrt{g}} \right)^{-\frac{1}{2}} \sigma [\alpha] (0) \sigma [\bar{\alpha}] (0)^*$$

$$\times \prod_t e^{-p_t^+ \frac{2i}{g} (w_t, \bar{w}_t)} e^{Q^2 \Gamma_{\text{super}} [-\frac{1}{2}(p_t^+ + \bar{p}_t^\alpha) g_{zz}]} ,$$  \hspace{1cm} (6.8)

where $S_{X^\pm}^{\text{free}}$ is the free action for the superfield $X^\pm$. Regarding the second line as a correlation function of

$$e^{Q^2 \Gamma_{\text{super}} [X^+, g_{zz}]} \prod_t e^{-ip_r^- X^+ (w_t, \bar{w}_t)}$$

for the free theory with the source term

$$\prod_r e^{-ip_r^+ X^- (Z_r, \bar{Z}_r)} ,$$

we can calculate it by replacing the $X^+ (z, \bar{z})$ by its expectation value $-\frac{1}{2} (\rho_s (z) + \bar{\rho}_s (\bar{z}))$ and derive the third line. Here $\rho_s, \bar{\rho}_s$ are the supersymmetric version of $\rho, \bar{\rho}$ and expressed as

$$\rho_s (z) = \rho (z) + \theta f (z) ,$$

$$\bar{\rho}_s (\bar{z}) = \bar{\rho} (\bar{z}) + \bar{\theta} \bar{f} (\bar{z}) ,$$
with

\[ f(z) = -\sum_r \alpha_r \Theta_r S_{\alpha L}(z, Z_r), \]
\[ \tilde{f}(\bar{z}) = -\sum_r \alpha_r \Theta_r S_{\alpha R}(\bar{z}, \bar{Z}_r). \]  

(6.9)

\( S_{\alpha L} \) and \( S_{\alpha R} \) are taken to be the Szegö kernel (4.7). The explicit form of \( e^{-\Gamma_{\text{super}}[-\frac{1}{8}(\rho_r + \bar{\rho}_r), \tilde{g}_{\alpha}^A]} \) is given by

\[ e^{-\Gamma_{\text{super}}[-\frac{1}{8}(\rho_r + \bar{\rho}_r), \tilde{g}_{\alpha}^A]} = \exp \left[ -\frac{1}{2} \Gamma [\varphi; \tilde{g}_{\alpha}^A] - \sum_r (\Delta \Gamma_{rL} + \Delta \Gamma_{rR}) - \sum_I (\Delta \Gamma_{IL} + \Delta \Gamma_{IR}) \right], \]  

(6.10)

with

\[ -\Delta \Gamma_{rL} = \frac{1}{2\alpha_r} \frac{\partial f f}{\partial^2 \rho}(z_{I(r)}), \]
\[ -\Delta \Gamma_{rR} = -(\Delta \Gamma_{rL})^*, \]
\[ -\Delta \Gamma_{IL} = \left\{ -\left( \frac{5}{12} \frac{\partial^3 \rho}{(\partial^2 \rho)^3} - \frac{3}{4} \frac{(\partial^3 \rho)^2}{(\partial^2 \rho)^4} \right) \frac{\partial f f}{\partial^2 \rho} + \frac{2}{3} \frac{\partial^3 f f}{(\partial^2 \rho)^4} - \frac{\partial^3 \rho}{(\partial^2 \rho)^3} \frac{\partial^2 f f}{\partial^2 \rho} \right\} (z_I), \]
\[ -\Delta \Gamma_{IR} = -(\Delta \Gamma_{IL})^*, \]

and \( \Gamma [\varphi; \tilde{g}_{\alpha}^A] \) given in (2.6).

The correlation function \( (6.8) \) can be expressed as an integration of the holomorphically factorized correlation function at the fixed internal momenta with respect to \( X^+ + \frac{1}{8}(\rho + \bar{\rho}), X^- \):

\[ (2\pi)^2 \delta \left( \sum_{r} p^-_r \right) \delta \left( \sum_{r} p^+_r \right) e^{-3 + 12Q^2} S \times \int \prod_j d^2 P_j^+ \langle \prod e^{-ip^-_r X^+_r}(Z_r) \prod e^{-ip^+_r X^-_r}(w_t) \rangle_{\chi^+_r, p^+_j} \times \langle \prod e^{-ip^-_r X^-_r}(Z_r) \prod e^{-ip^+_r X^+_r}(w_t) \rangle_{\chi^-_r, p^-_j}, \]  

(6.11)

where

\[ \langle \prod e^{-ip^-_r X^+_r}(Z_r) \prod e^{-ip^+_r X^-_r}(w_t) \rangle_{\chi^+_r, p^+_j} \]
\[ = \langle (1)_{LC} \rangle^{-\frac{Q^2}{2}} \langle (1)_{X^+_r} \rangle^2 \exp \left[ -2\pi i \sum_{j,j'} P^+_j \Omega_{I_j I'_j} P^-_{j'} \right] \langle 1 \rangle_{\psi z} \times \exp \left[ Q^2 \left( \sum_r \Delta \Gamma_{rL} + \sum_I \Delta \Gamma_{IL} \right) - \frac{1}{2} \sum_r p^-_r \rho_r(w_t) \right], \]
\[ \langle \prod e^{-ip^-_r X^-_r}(Z_r) \prod e^{-ip^+_r X^+_r}(w_t) \rangle_{\chi^-_r, p^-_j} \]
\[ = \langle \prod e^{-ip^-_r X^-_r}(Z_r) \prod e^{-ip^+_r X^+_r}(w_t) \rangle_{\chi^+_r, p^+_j} \times \]  

(6.12)
The correlation functions involving spin fields or those for odd spin structure can be dealt with in the same way. With the spin fields, we get

\[
\int [d\chi^+ d\chi^-]_{\hat{g}_{zz}} e^{-S_{\text{super}}[\chi^+,\chi^-]} \prod_r e^{-ip_r^\pm \chi^-(\mathbf{z}_r, \bar{\mathbf{z}}_r)} \prod_t e^{-ip_t^\pm \chi^+(\mathbf{w}_t, \bar{\mathbf{w}}_t)}
\times \prod_{s''} e^{-\hat{\tau} H(Z_{s''})} \prod_{s'} e^{\hat{\tau} H(Z_{s'})}
= \int [d\chi^+ d\chi^-]_{\hat{g}_{zz}} e^{-S_{\text{free}}[\chi^+,\chi^-]} \prod_r e^{-ip_r^\pm \chi^-(\mathbf{z}_r, \bar{\mathbf{z}}_r)} \prod_t e^{-ip_t^\pm \chi^+(\mathbf{w}_t, \bar{\mathbf{w}}_t)}
\times \prod_{s''} e^{-\hat{\tau} H(Z_{s''})} \prod_{s'} e^{\hat{\tau} H(Z_{s'})} \times e^{Q^2 \Gamma_{\text{super}}[-\hat{\tau}(\rho_+ + \bar{\rho}_-), \hat{g}_{zz}^\Lambda]},
\]

(6.13)

where \(S_{\alpha_L} \) and \(S_{\alpha_R} \) in (5.9) should be chosen from those given in 4.2 depending on the situation. It is straightforward to obtain the holomorphically factorized correlation function at the fixed internal momenta

\[
\left\langle \prod_r e^{-ip_r^\pm X_L^-(\mathbf{z}_r)} \prod_r e^{-ip_r^\pm X_L^+(\mathbf{w}_t)} \right\rangle_{X_L^\pm, p_j^\pm}
= \left(\langle 1 \rangle_{\text{LC}}\right)^{-\hat{\Delta}^2} \left(\langle 1 \rangle_{X_L}\right)^2 \exp \left[-2\pi i \sum_{J, J'} P^J_J \Omega_J J' P^{J'}_{J'} \right]
\times \prod_{s'} e^{-\hat{\tau} H(Z_{s'})} \prod_{s''} e^{\hat{\tau} H(Z_{s''})} \times \exp \left[Q^2 \left(\sum_r \Delta \Gamma_{r L} + \sum_t \Delta \Gamma_{r L} \right) - \frac{1}{2} \sum_t \bar{p}_t \rho_+(\mathbf{w}_t) \right].
\]

(6.14)

When \(\alpha_L \) corresponds to an odd spin structure, we get

\[
\left\langle \prod_r e^{-ip_r^\pm X_L^-(\mathbf{z}_r)} \prod_r e^{-ip_r^\pm X_L^+(\mathbf{w}_t)} \right\rangle_{X_L^\pm, p_j^\pm}
= \left(\langle 1 \rangle_{\text{LC}}\right)^{-\hat{\Delta}^2} \left(\langle 1 \rangle_{X_L}\right)^2 \exp \left[-2\pi i \sum_{J, J'} P^J_J \Omega_J J' P^{J'}_{J'} \right]
\times \int d\psi_0^+ d\psi_0^- \exp \left[Q^2 \left(\sum_r \Delta \Gamma_{r L} + \sum_t \Delta \Gamma_{r L} \right) - \frac{1}{2} \sum_t \bar{p}_t \rho_+(\mathbf{w}_t) \right].
\]

(6.15)

in which \(S_{\alpha_L} \) is taken to be the one given in 4.11. The right moving part can be defined in the same way.

From these correlation functions, it is straightforward to check that the theory for the longitudinal variables \(X^\pm, \psi^\pm, \bar{\psi}^\pm \) is a superconformal field theory, in the same way as was done in [47]. The super stress tensor \(T^{X^\pm}(\mathbf{z}) \) becomes

\[
T^{X^\pm}(\mathbf{z}) = \frac{1}{2} \partial \chi^+ D\chi^- + \frac{1}{2} \partial \chi^- D\chi^+ + 2Q^2 S(\mathbf{z}, X_L^+) \]

(6.16)

where \(S(\mathbf{z}, X_L^+) \) denotes the super Schwarzian derivative

\[
S(\mathbf{z}, X_L^+) = \frac{\partial^2 \Theta^+}{D\Theta} - 2 \frac{\partial D\Theta^+ \partial \Theta^+}{(D\Theta^+)^2}.
\]

(6.17)

\(T^{X^\pm}(\mathbf{z}) \) satisfies the super Virasoro algebra with the central charge

\[
c = 3 + 12Q^2.
\]

(6.18)
With the ghost system and the stress tensor of the transverse variables, one can construct the BRST charge which turns out to be nilpotent. Therefore the regularization can be considered to be a gauge invariant one.

6.2 \( Q \to 0 \)

The conformal gauge worldsheet theory corresponding to the light-cone theory in noncritical dimensions consists of the supersymmetric \( X^\pm \) CFT, the supersymmetric ghost system and the worldsheet theory of the transverse variables. The integrand \( (6.2) \) can be expressed in terms of correlation functions of the conformal gauge worldsheet theory. It is possible to show that \( F_N^{(g)} \) in \( (6.2) \) is proportional to

\[
\int D[X^{BC}] e^{-S^{\text{tot}}_{\mu \nu} \sum_{K=1}^{6g-6+2N} \left( \oint b_{zz} + \varepsilon_K \oint b_{zz} \right) \prod_{I=1}^{2g-2+N} \left[ X(z_I) \tilde{X}(\tilde{z}_I) \right]}
\]

\[
\times \prod_{r=1}^{N} \left[ e^{-\frac{\alpha^2}{4}} X^+ \left( \hat{z}_{I(r)} \right) V_r(Z_r, \tilde{Z}_r) \right],
\]

where \( \hat{z}_{I(r)} \), \( \hat{\tilde{z}}_{I(r)} \) are the operator valued coordinates defined in [33] and the vertex operators are those constructed in appendix A.2. The proof goes in a way similar to the critical case. As is proved in appendix B the conformal gauge expression \( (6.19) \) is equal to

\[
\int D[X^{BC}] e^{-S_{\mu \nu}^{\text{tot}} \sum_{K=1}^{6g-6+2N} \left( \oint b_{zz} + \varepsilon_K \oint b_{zz} \right) \prod_{I=1}^{2g-2+N} \left[ e^\phi T^{LC}_F(z_I) e^{\tilde{\phi} T^{LC}_{\tilde{F}}(\tilde{z}_I)} \right]}
\]

\[
\times \prod_{r=1}^{N} \left[ e^{-\frac{\alpha^2}{4}} X^+ \left( \hat{z}_{I(r)} \right) V_r(Z_r, \tilde{Z}_r) \right] \cdot
\]

\[
\propto \int [dX^i d\psi^i d\tilde{\psi}^i] e^{-S^{\text{tot}}_{\mu \nu} \sum_{I=1}^{2g-2+N} \left[ \oint \partial^\rho b_{zz} \right] \prod_{r=1}^{N} \left[ e^{-\frac{\alpha^2}{4}} X^+ \left( \hat{z}_{I(r)} \right) V_r(Z_r, \tilde{Z}_r) \right].
\]

Using \( (4.12), (4.13), (4.14), (6.15) \) and \( (6.14) \), we obtain

\[
\int D[X^{BC}] e^{-S^{\pm BC}_{\mu \nu} \sum_{K=1}^{6g-6+2N} \left( \oint b_{zz} + \varepsilon_K \oint b_{zz} \right) \prod_{I=1}^{2g-2+N} \left[ e^\phi (z_I) e^{\tilde{\phi} (\tilde{z}_I)} \right]}
\]

\[
\times \prod_{r=1}^{N} \left[ e^{-\frac{\alpha^2}{4}} X^+ \left( \hat{z}_{I(r)} \right) V_r(Z_r, \tilde{Z}_r) \right] \cdot
\]

\[
\propto (2\pi)^2 \delta^3 \left( \sum_{r=1}^{N} p^+_r \right) \times \sum_{\text{spin structure}} e^{-\frac{1}{2}(1-Q^2)F[\hat{z}_{I(r)}] \prod_{I=1}^{N} \left[ \partial^\rho (z_I) \right]^{-\frac{4}{3}} \prod_{r=1}^{N} V^{LC}_r (Z_r, \tilde{Z}_r) \cdot
\]

Substituting this into \( (6.20) \), one can see that \( (6.19) \) is proportional to \( (6.2) \).

With the conformal gauge expression \( (6.19) \), we can show that the limit \( \lim_{Q \to 0} A^{LC}_r (Q^2) \) coincides with the result of the covariant approach in the same way as was done in previous papers [32, 33]. In the covariant
formulation, the amplitudes can be obtained by the method given by Sen and Witten in \cite{7,8} using the PCO’s. Even with $Q \neq 0$, since the vertex operators, PCO’s and the insertions $e^{-\frac{Q^2}{2}X^+} \left( \hat{z}_{(r)} , \hat{\bar{z}}_{(r)} \right)$ commute with the BRST charge, the conformal gauge expression (6.19) can be deformed to define the amplitudes following the Sen-Witten prescription. We can divide the moduli space into patches and put the PCO’s avoiding the spurious singularities as was explained in \cite{7} and define the amplitude $A_{SW}(Q^2)$. Moving the locations of the PCO’s, the amplitudes change by total derivative terms in moduli space. For $Q^2$ large enough, these total derivative terms do not contribute to the amplitudes, because the infrared divergences are regularized. Therefore

$$A_{SW}(Q^2) = A^{LC}(Q^2),$$

as an analytic function of $Q^2$. Since $A_{SW}(Q^2)$ is free from the spurious singularities, it can be well-defined for $Q^2 < 10$ and

$$\lim_{Q \to 0} A^{LC}(Q^2) = A^{SW}(0),$$

if the right hand side is well-defined. $A^{SW}(0)$ is exactly the amplitude in the critical dimensions obtained by the covariant approach.

7 Conclusions

In this paper, we have shown that the Feynman amplitudes of the light-cone gauge string field theory for Type II superstrings coincide with those of the covariant first quantized approach, even with Ramond sector external lines. The divergences due to the collisions of the supercurrents inserted at the interaction points are regularized by taking the worldsheet theory of the transverse variables to be the one with a linear dilaton background $\Phi_{dilaton} = -iQX^1$. The amplitudes are defined as analytic functions of $Q^2$ and in the limit $Q \to 0$ they coincide with those of the critical string calculated by using the Sen-Witten prescription.

With the formulas given in this paper, it should be possible to do the same thing for heterotic strings and Type I strings. A problem with heterotic strings is the holomorphic factorization. The integrand $F^{(j)}_{N^2}$ for heterotic strings should be given as a product of the holomorphic part of that for Type II superstrings and the antiholomorphic part of that for bosonic strings. Such a holomorphic factorization is subtle in the formulation of light-cone gauge string field theory, because the $\rho$ coordinate depends on the antiholomorphic moduli parameters. We will discuss this issue in a separate publication.

We have shown that the scattering amplitudes of Type II strings can be reproduced by the light-cone gauge string field theory with only cubic interaction terms. The regularization we propose regularizes the infrared divergences of superstring theory in a gauge invariant way. With such a formulation of superstring theory, we should be able to study nonperturbative dynamics of the theory. We will leave this subject for future investigation.

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A Vertex operators

In this appendix, we present the forms of the vertex operators which are used in the main text. Here we consider the vertex operators in the noncritical case $Q \neq 0$. Those in the critical case can be obtained by putting $Q = 0$.

A.1 Light-cone gauge vertex operators

The light-cone vertex operators are local fields corresponding to the external states. The vertex operator $V_{r}^{LC}(Z_r, \bar{Z}_r)$ corresponding to the $r$-th external state $|r\rangle$ given by

$$|r\rangle = |r\rangle_L \otimes |r\rangle_R,$$

as a tensor product of left and right moving states, can be given in the factorized form

$$V_{r}^{LC}(Z_r, \bar{Z}_r) = V_{r}^{LC}(Z_r) V_{r}^{LC}(\bar{Z}_r),$$

accordingly.

For a left moving state

$$\alpha_{-n_1} \cdots \psi_{-s_1} \cdots |p_r\rangle,$$

in the NS sector, the vertex operator is given by

$$V_{r}^{LC}(Z_r) = (\alpha_r)^{\frac{1}{2}} \int_{Z_r} \frac{dz}{2\pi i} i\partial \bar{X}_i(z) w_r^{-n_1} \cdots \int_{Z_r} \frac{dz}{2\pi i} \left( \frac{\partial w_r}{\partial z} \right)^{\frac{1}{2}} \psi_{j_1} (z) w_r^{-s_1 - \frac{1}{2}} \cdots \times e^{i\bar{r}_- \cdot \bar{X}_L(Z_r)} \left( \frac{\partial w_r}{\partial z} \right)^{-\frac{1}{2}} |p_r|^2 e^{-\frac{1}{2}p_r \cdot \rho(z_{I(r)})},$$

where

$$\bar{X}_L(z) = X_i + iQ \delta^{i1} \ln(2g_{z\bar{z}}),$$

$$w_r = \exp \left[ \frac{1}{\bar{r}_-} (\rho(z) - \rho(z_{I(r)})) \right].$$

$\bar{X}_L(z)$ denotes the left moving part of the variable $\bar{X}_i(z, \bar{z})$. The momentum $p_r$ satisfies the on-shell condition

$$\frac{1}{2} \left( -2p_+^- p_- + p_+^- p_+^- \right) + Qp_+^- + N_r = \frac{1}{2} (1 - Q^2), \quad N_r = \sum_k n_k + \sum_s s_l,$$

with $n_k \in \mathbb{Z}$, $s_l \in \mathbb{Z} + \frac{1}{2}$. The factor $(\alpha_r)^{\frac{1}{2}}$ on the right hand side of (A.1) comes from the normalization of the three string vertex and is essential for the Lorentz invariance in the critical case.

For a left moving state

$$\alpha_{-n_1} \cdots \psi_{-s_1} \cdots |p_r\rangle,$$

in the R sector,

$$V_{r}^{LC}(Z_r) = \alpha_r \int_{Z_r} \frac{dz}{2\pi i} i\partial \bar{X}_i(z) w_r^{-n_1} \cdots \int_{Z_r} \frac{dz}{2\pi i} \left( \frac{\partial w_r}{\partial z} \right)^{\frac{1}{2}} \psi_{j_1} (z) w_r^{-s_1 - \frac{1}{2}} \cdots \times S_\alpha e^{i\bar{r}_- \cdot \bar{X}_L(Z_r)} \left( \frac{\partial w_r}{\partial z} \right)^{-\frac{1}{2}} |p_r|^2 e^{-\frac{1}{2}p_r \cdot \rho(z_{I(r)})},$$

(A.3)
[The page text contains mathematical expressions and equations. The content is too complex to be transcribed accurately without the context of the surrounding text.]

where \( S_\alpha \) is the spin field and the on-shell condition is

\[
\frac{1}{2}(-2p_\alpha^- p_- + p_\alpha^+ p_+) + Qp_\alpha^+ + \mathcal{N}_r = -\frac{Q^2}{2}, \quad \mathcal{N}_r = \sum_k n_k + \sum_l m_l,
\]

with \( n_k \in \mathbb{Z}, \, s_l \in \mathbb{Z}. \)

The right moving part of the vertex operator \( V_{rR}^{LC} (\bar{Z}_r) \) can be defined in the same way.

### A.2 Conformal gauge vertex operators

We define the conformal gauge vertex operators corresponding to the vertex operator (A.2) in the NS sector by

\[
V_{rL}^{(-2)}(Z_r) = \frac{2}{p_+} e^{-2\phi \psi^+} A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r),
\]

\[
V_{rL}^{(-1)}(Z_r) = e^{-\phi} A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r),
\]

\[
V_{rL}^{(0)}(Z_r) = \left[ cG \left( 1 - \frac{1}{4} \right) A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r),
\]

(A.4)

with the DDF operators \( A_{-1}^{(r)}, B_{-1}^{(r)} \) for the \( r \)-th string defined as

\[
A_{-1}^{(r)} = \oint_{Z_r} \frac{dz}{2\pi i} D \left( \bar{X}^i + iQ\delta^{i1} \Phi \right) e^{-i\frac{\alpha}{\alpha_r} X_r^L} (z),
\]

\[
B_{-1}^{(r)} = \oint_{Z_r} \frac{dz}{2\pi i} D^{-1} \left( 1 - \frac{1}{4} \right) D \left( \bar{X}^i + iQ\delta^{i1} \Phi \right) e^{-i\frac{\alpha}{\alpha_r} X_r^L} (z).
\]

(A.5)

Here \( \Phi \) is defined in (6.7) and \( \mathcal{X}_L^+ \) denotes the left moving part of \( \mathcal{X}^+ \). The vertex operators in (A.4) satisfy

\[
XV_{rL}^{(-2)} (Z_r) = V_{rL}^{(-1)} (Z_r),
\]

\[
XV_{rL}^{(-1)} (Z_r) = V_{rL}^{(0)} (Z_r),
\]

\[
Q_B V_{rL}^{(-2)} (Z_r) = Q_B V_{rL}^{(-1)} (Z_r) = Q_B V_{rL}^{(0)} (Z_r) = 0,
\]

where \( X \) is the picture changing operator (3.2) and \( Q_B \) denotes the BRST charge. The matter supercurrent \( T_F (z) \) satisfies the OPE

\[
T_F (z) \leftrightarrow A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r)
\]

\[
\sim \frac{1}{z - Z_r} G_{-\frac{1}{4}} A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r),
\]

with

\[
G_{-\frac{1}{4}} A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r)
\]

\[
= \frac{p_+^+}{2} \psi^+ A_{-1}^{(r)} \cdots B_{-1}^{(r)} \cdots e^{-ip_-^+ X_r^+ - i\left(p_- - \frac{2N_r}{\alpha} - \frac{Q^2}{\alpha_r}\right) X_L^+ + ip_+^+ X_r^L} (Z_r) + \ldots .
\]

The ellipses on the right hand side denote the terms which do not involve \( \psi^+ \). \( V_{rL}^{(-2)} (Z_r), V_{rL}^{(-1)} (Z_r) \) and \( V_{rL}^{(0)} (Z_r) \) are the BRST invariant vertex operators in the \(-2, -1, 0\) pictures respectively.
The conformal gauge vertex operators corresponding to (A.3) in the R sector are defined to be

\[
V_{rL}^{(-\frac{1}{2})} (Z_r) = ce^{-\frac{1}{2} S} e^{H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)} ,
\]

\[
V_{rL}^{(-\frac{1}{2})} (Z_r) = ce^{-\frac{1}{2} S} G_0 e^{H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)} .
\]

(A.6)

These are the vertex operators in the $-\frac{1}{2}, -\frac{1}{2}$ pictures respectively and satisfy

\[
XV_{rL}^{(-\frac{1}{2})} (Z_r) = V_{rL}^{(-\frac{1}{2})} (Z_r) ,
\]

\[
Q_B V_{rL}^{(-\frac{1}{2})} (Z_r) = Q_B V_{rL}^{(-\frac{1}{2})} (Z_r) = 0 .
\]

The matter supercurrent \(T_F (z)\) satisfies the OPE

\[
T_F (z) e^{\frac{1}{2} H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)}
\]

\[
\sim (z - Z_r)^{-\frac{1}{2}} G_0 e^{\frac{1}{2} H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)} ,
\]

with

\[
G_0 e^{\frac{1}{2} H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)}
\]

\[
= - \frac{p^+_r}{2} e^{\frac{1}{2} H} A_{-n_1}^{i_1(r)} \cdots B_{-n_1}^{j_1(r)} \cdots S_{n_1} e^{-ip^+_r X^-_r - i(p^-_r - \frac{2N}{c^2} - \frac{2q^2}{c^4}) X^+_r + ip^+_r X^+_r (Z_r)} + \cdots .
\]

The ellipses on the right hand side denote the terms proportional to \(e^{\frac{1}{2} H}\).

The right moving version of (A.4) and (A.6) can be defined in the same way and the vertex operators in the \((p_L, p_R)\) picture for Type II superstring theory are given in the form

\[
V_{rL}^{(p_L, p_R)} (Z_r, \bar{Z}_r) = V_{rL}^{(p_L)} (Z_r) V_{rR}^{(p_R)} (\bar{Z}_r) .
\]

B A proof of the equality of (6.19) and (6.20)

In this appendix, we show that (6.19) is equal to (6.20). This can be done by using fermionic charges

\[
\hat{Q} = \oint \frac{dz}{2\pi i} \left[ - \frac{b}{4\partial \bar{\rho}} \left( iX^+_L - \frac{1}{2} \rho \right) (z) + \frac{\beta}{2\partial \bar{\rho}} \bar{\psi}^+ (z) \right] ,
\]

\[
\hat{Q} = \oint \frac{d\bar{z}}{2\pi i} \left[ - \frac{\bar{b}}{4\partial \rho} \left( iX^+_R - \frac{1}{2} \bar{\rho} \right) (\bar{z}) + \frac{\bar{\beta}}{2\partial \rho} \bar{\psi}^+ (\bar{z}) \right] ,
\]

(B.1)

where

\[
\left( iX^+_L - \frac{1}{2} \rho \right) (z) = \int_{w_0}^{z} dz' \left( i\partial X^+ - \frac{1}{2} \partial \rho \right) (z') ,
\]

\[
\left( iX^+_R - \frac{1}{2} \bar{\rho} \right) (\bar{z}) = \int_{\bar{w}_0}^{\bar{z}} d\bar{z'} \left( i\partial X^+ - \frac{1}{2} \partial \bar{\rho} \right) (\bar{z'}) ,
\]

(B.2)

with a generic point \(w_0\) on the surface and the integration contours on the right hand sides are taken to be the same. Since \(\left( iX^+_L - \frac{1}{2} \rho \right) (z)\), \(\left( iX^+_R - \frac{1}{2} \bar{\rho} \right) (\bar{z})\) thus defined depend on the contours and not single
valued with respect to \( z, \bar{z} \), the right hand sides of \((\text{B.1})\) are actually ambiguous. Here let us consider the combination \( \hat{Q} - \hat{\bar{Q}} \). One can see that the ambiguity of this combination is proportional to

\[
\oint dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) \partial \rho c (w_0) + \oint d\bar{z} \frac{\bar{b}}{2\pi i} \frac{\partial}{\partial \rho} (\bar{z}) \partial \bar{\rho} c (\bar{w}_0),
\]

which coincides with the \( b_0 - \bar{b}_0 \) in light-cone gauge string field theory. All the external states are annihilated by this combination of antighosts and it is inserted in all the nontrivial cycles. Therefore the combination \( \hat{Q} - \hat{\bar{Q}} \) is well-defined in the correlation functions discussed in this paper.

In order to use \( \hat{Q} - \hat{\bar{Q}} \), we need to rewrite the ghost part of the correlation function. Inserting

\[
1 = \left| \oint_{w_0} dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) \partial \rho c (w_0) \right|^2,
\]

into \((\text{B.2})\) and deforming the contours of the antighost insertions, \((\text{B.2})\) is transformed into

\[
\int D[XBC] e^{-S^{\text{tot}}} \partial \rho c (w_0) \bar{\partial} \bar{\rho} c (\bar{w}_0)
\]

\[
\times \prod_{j=1}^{\bar{g}} \left[ \left( \oint A_j \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) + \oint A_j \frac{d\bar{z}}{2\pi i} \frac{\bar{b}}{\partial \bar{\rho}} (\bar{z}) \right) \left( \oint B_j \frac{dz}{2\pi i} \frac{b}{\partial \rho} (z) + \oint B_j \frac{d\bar{z}}{2\pi i} \frac{\bar{b}}{\partial \bar{\rho}} (\bar{z}) \right) \right]
\]

\[
\times \prod_{j} \left\{ \oint (z) X (z_l) \oint \frac{d\bar{z}}{2\pi i} \frac{\bar{b}}{\partial \bar{\rho}} (\bar{z}) \bar{Z} (\bar{z}_l) \right\}
\]

\[
\times \prod_{r} e^{-\frac{\omega^2}{4} X^+ (\hat{\bar{x}}_{(r)}, \hat{x}_{(r)})} \prod_{r=1}^{N} V_r (Z_r, \bar{Z}_r).
\]

(B.3)

The operators inserted at \( z = z_l \) can be expressed as

\[
\oint_{z_l} dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) X (z_l)
\]

\[
= -\oint_{z_l} dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) e^{\phi^{\text{LC}}_2} (z_l)
\]

\[
- \left\{ \hat{Q} - \hat{\bar{Q}}, \oint dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) \oint_{z_l} dw A (w) e^{\phi} (z_l) \right\}
\]

\[
+ \frac{1}{4} \oint_{z_l} dz \frac{b}{2\pi i} \frac{\partial}{\partial \rho} (z) \oint_{z_l} dw \partial \rho \psi^- (w) e^{\phi} (z_l),
\]

(B.4)

where

\[
A (w) = -i \partial X^+ \partial \rho \gamma (w) - 2 \partial (\partial \rho c) \psi^- (w)
\]

\[
= \frac{5}{4} \left( \frac{\partial^2 X^+}{\partial \rho \gamma (w)} - \frac{\partial^3 X^+}{\partial \rho X^+} \frac{\partial^2 X^+}{\partial \rho \gamma (w)} \right) (w).
\]

The statement that \( iX_0^+ - \frac{1}{2} \rho \) is single valued made in appendix C.2 of \([33]\) is wrong. However, by replacing \( \hat{Q}' \) there by \( Q' - \hat{\bar{Q}}' \) which is well-defined, all the results there still hold as we show in the following.
Substituting (B.4) into (B.3) and using the fact that $\hat{Q} - \bar{\hat{Q}}$ commutes with other insertions, we can show that (B.3) is equal to

$$\int D[XBC] e^{-S_{tot}} \partial \rho (w_0) \bar{\partial} \bar{\rho} (\bar{w}_0)$$

$$\times \prod_{J=1}^{q} \left( \int_{\Lambda_J} \frac{d\bar{z}}{\partial \bar{\rho}} b_{zz} + \int_{B_J} \frac{d\bar{z}}{\partial \bar{\rho}} b_{\bar{z}\bar{z}} \right) \left( \int_{B_J} \frac{d\bar{z}}{\partial \bar{\rho}} b_{zz} + \int_{\Lambda_J} \frac{d\bar{z}}{\partial \bar{\rho}} b_{\bar{z}\bar{z}} \right)$$

$$\times \prod_{I} \int_{z_I} \frac{d\bar{z}}{2\pi i} \frac{\delta (w - z_I)}{\partial \bar{\rho}} \bar{\rho}$$

$$\times \prod_{I} \int_{\bar{z}_I} \frac{d\bar{z}}{2\pi i} \frac{\delta (\bar{w} - \bar{z}_I)}{\partial \bar{\rho}} \bar{\rho}$$

$$\times \prod_{r} e^{- \frac{w^2}{\alpha_r}} \chi^+(\hat{z}_I(r), \hat{\bar{z}}_I(r)) \prod_{r=1}^{N} V_r(Z_r, \bar{Z}_r). \quad \text{(B.5)}$$

It is easy to show that the term $\int_{z_I} \frac{d\bar{z}}{2\pi i} \frac{\delta (w - z_I)}{\partial \bar{\rho}} \bar{\rho}$ does not contribute to the correlation function $\langle \rangle$. We can do the same thing for the antiholomorphic part and eventually prove that (6.19) is equal to (6.20).

Putting $Q = 0$, we also find that (3.1) is equal to (5.1).

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