ADAPTED LINEAR-NONLINEAR DECOMPOSITION AND GLOBAL WELL-POSEDNESS FOR SOLUTIONS TO THE DEFOCUSING CUBIC WAVE EQUATION ON $\mathbb{R}^3$

TRISTAN ROY

Abstract. We prove global well-posedness for the defocusing cubic wave equation

$$
\left\{ \begin{array}{ll}
\partial_{tt} u - \Delta u &= -u^3 \\
\partial_t u(0, x) &= u_0(x) \\
\partial_t u(0, x) &= u_1(x)
\end{array} \right.
$$

with data $(u_0, u_1) \in H^s \times H^{s-1}$, $1 > s > \frac{13}{18} \simeq 0.722$. The main task is to estimate the variation of an almost conserved quantity on an arbitrary long time interval. We divide it into subintervals. On each of these subintervals we write the solution as the sum of its linear part adapted to the subinterval and its corresponding nonlinear part. Some terms resulting from this decomposition have a controlled global variation and other terms have a slow local variation.

1. Introduction

We shall study the defocusing cubic wave equation on $\mathbb{R}^3$

$$
\partial_{tt} u - \Delta u = -u^3
$$

with data $u(0) = u_0$, $\partial_t u(0) = u_1$ lying in $H^s$, $H^{s-1}$ respectively. Here $H^s$ is the standard inhomogeneous Sobolev space i.e $H^s$ is the completion of the Schwartz space $S(\mathbb{R}^3)$ with respect to the norm

$$
\| f \|_{H^s} := \| (1 + D)^s f \|_{L^2(\mathbb{R}^3)}
$$

where $D$ is the operator defined by

$$
\hat{D}f(\xi) := |\xi|\hat{f}(\xi)
$$

and $\hat{f}$ denotes the Fourier transform

$$
\hat{f}(\xi) := \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi} dx
$$

We shall focus on the strong solutions of the defocusing cubic wave equation on some interval $[0, T]$ i.e real-valued maps $u$, $\partial_t u$ that lie in $C([0, T], H^s(\mathbb{R}^3))$, $C([0, T], H^{s-1}(\mathbb{R}^3))$ respectively and that satisfy for $t \in [0, T]$ the following integral equation

$$
u(t) = \cos(tD)u_0 + D^{-1}\sin(tD)u_1 - \int_0^t D^{-1}\sin \left((t - t')D\right) u^3(t') \, dt'.
$$
It is known \cite{10} that (1.1) is locally well-posed for \( s > \frac{1}{2} \) in \( H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \) endowed with the standard norm \( \| (f, g) \|_{H^s \times H^{s-1}} := \| f \|_{H^s} + \| g \|_{H^{s-1}} \). Moreover the time of local existence does only depend on the norm of the initial data \( \|(u_0, u_1)\|_{H^s \times H^{s-1}} \).

Now we turn our attention to the global well-posedness theory of (1.1). In view of the above local well-posedness theory and standard limiting arguments it suffices to establish an a priori bound of the form

\begin{equation}
\| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}} \leq C(s, \| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}}, T)
\end{equation}

for all times \( 0 < T < \infty \) and all smooth-in-time Schwartz-in-space solutions \((u, \partial_t u) : [0, T] \times \mathbb{R}^3 \to \mathbb{R} \), where the right-hand side is a finite quantity depending only on \( s, \| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}} \) and \( T \). Therefore in the sequel we shall restrict attention to such smooth solutions.

The defocusing cubic wave equation (1.1) enjoys the following energy conservation law

\begin{equation}
E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u)^2(t, x) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |Du(t, x)|^4 \, dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} u^4(t, x) \, dx
\end{equation}

Combining this conservation law to the local well-posedness theory we immediately have global well-posedness for (1.1) and for \( s = 1 \).

In this paper we are interested in studying global well-posedness of (1.1) for data whose norm is below the energy norm, i.e \( s < 1 \). It is conjectured that (1.1) is globally well-posed in \( H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3) \) for \( s > \frac{1}{2} \). The study of global existence for the defocusing cubic wave equation has attracted the attention of many researchers. Let us some mention some results for data \((u_0, u_1)\) lying in a slightly different space than \( H^s \times H^{s-1} \) i.e \( H^s \cap L^4 \times H^{s-1} \). Here \( H^s \) is the standard homogeneous Sobolev space i.e the completion of Schwartz functions \( \mathcal{S}(\mathbb{R}^3) \) with respect to the norm

\begin{equation}
\| f \|_{\dot{H}^s} := \| D^s f \|_{L^2(\mathbb{R}^3)}
\end{equation}

Kenig, Ponce and Vega \cite{8} were the first to prove that (1.1) is globally well-posed for \( \frac{1}{4} < s < 1 \). They used the Fourier truncation method discovered by Bourgain \cite{2}. I. Gallagher and F. Planchon \cite{10} proposed a different method to prove global well-posedness for \( 1 > s > \frac{3}{4} \). H. Bahouri and Jean-Yves Chemin \cite{11} proved global-wellposedness for \( s = \frac{3}{4} \) by using a non linear interpolation method and logarithmic estimates from S. Klainerman and D. Tataru \cite{9}. Recently it was proved \cite{12} that the defocusing cubic wave equation under spherically symmetric data is globally well-posed in \( H^s \times H^{s-1} \) for \( 1 > s > \frac{7}{16} \). The main result of this paper is the following one

\begin{thm}
The defocusing cubic wave equation is globally well-posed in \( H^s \times H^{s-1} \), \( 1 > s > \frac{13}{28} \simeq 0.722 \). Moreover if \( s > \frac{13}{28} \) is close to \( \frac{13}{28} \) then

\begin{equation}
\| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}} \leq C \left( \| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}} \right) T^{\frac{28s-13}{28}}
\end{equation}

Here \( C \left( \| u_0 \|_{H^s}, \| u_1 \|_{H^{s-1}} \right) \) is a constant depending only on \( \| u_0 \|_{H^s} \) and \( \| u_1 \|_{H^{s-1}} \).
We set some notation that appear throughout the paper. Given $A, B$ positive number $A \lesssim B$ means that there exists a universal constant $K$ such that $A \leq KB$. We say that $K_0$ is the constant determined by the relation $A \lesssim B$ if $K_0$ is the smallest $K$ such that $A \leq KB$ is true. We write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. $A \ll B$ denotes $A \leq KB$ for some universal constant $K < \frac{1}{100}$. We also use the notations $A + = A + \epsilon$, $A + + = A + 2\epsilon$, $A - = A - \epsilon$ and $A -- = A - 2\epsilon$, etc. for some universal constant $0 < \epsilon << 1$. We shall abuse the notation and write $+$, $-$ for $0+, 0-$ respectively. Let $\nabla$ denote the gradient operator. If $J$ is an interval then $|J|$ is its size. Let $I$ be the following multiplier

$$\hat{f}(\xi) := m(\xi)\hat{f}(\xi)$$

where $m(\xi) := \eta\left(\frac{\xi}{N}\right)$, $\eta$ is a smooth, radial, nonincreasing in $|\xi|$ such that

$$\eta(\xi) := \begin{cases} 1, |\xi| \leq 1 \\ \left(\frac{1}{|\xi|}\right)^{1-s}, |\xi| \geq 2 \end{cases}$$

and $N >> 1$ is a dyadic number playing the role of a parameter to be chosen. We shall abuse the notation and write $m(|\xi|)$ for $m(\xi)$, thus for instance $m(N) = 1$. We denote by $E(Iu(t))$ the so-called mollified energy

$$E(Iu(t)) := \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t Iu(t,x))^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |DIu(t,x)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} (Iu(t,x))^4 \, dx$$

The following result establishes the link between $\|(u(T), \partial_t u(T))\|_{H^s \times H^{s-1}}$ and the mollified energy $E(Iu)$ for a function $u$.

**Proposition 1.2.** [12] Let $T > 0$. Then

$$\| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}} \lesssim \| u_0 \|_{H^s}^2 + (T^2 + 1) \sup_{t \in [0, T]} E(Iu(t))$$

We recall some basic results regarding the defocusing cubic wave equation. Let $\lambda \in \mathbb{R}$ and $u_\lambda$ denote the following function

$$u_\lambda(t,x) := \frac{1}{\lambda} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$$

If $u$ satisfies (1.1) with data $(u_0, u_1)$ then $u_\lambda$ also satisfies (1.1) but with data $\left(\frac{1}{\lambda} u_0\left(\frac{t}{\lambda}\right), \frac{1}{\lambda^2} u_1\left(\frac{t}{\lambda}\right)\right)$.

Now we recall the Strichartz estimates with derivative. These estimates are proved in [12] and follow from the standard Strichartz estimates for the wave equation ([7], [10]).

**Proposition 1.3.** "Strichartz estimates with derivative in 3 dimensions" Let $m \in [0, 1]$ and $0 \leq \tau < \infty$. If $u$ is a strong solution to the IVP problem

$$\begin{cases} \partial_\tau u - \Delta u = F \\ u(0, x) = f(x) \\ \partial_\tau u(0, x) = g(x) \end{cases}$$


then we have the m-Strichartz estimate with derivative

\begin{equation}
\|u\|_{L^q_t([0, \tau]) L^r_x} + \|\partial_t D^{-1} u\|_{L^q_t([0, \tau]) L^r_x} + \|u\|_{L^\infty_t([0, \tau]) \dot{H}^m} + \|\partial_t u\|_{L^\infty_t([0, \tau]) \dot{H}^{m-1}} \\
\lesssim \|f\|_{\dot{H}^m} + \|g\|_{\dot{H}^{m-1}} + \|F\|_{L^q_t([0, \tau]) L^r_x}
\end{equation}

under two assumptions

• \((q, r)\) lie in the set \(W\) of wave-admissible points i.e.

\begin{equation}
W := \{(q, r) : (q, r) \in (2, \infty] \times (2, \infty), \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}\}
\end{equation}

• \((\tilde{q}, \tilde{r})\) lie in the dual set \(\tilde{W}\) of \(W\) i.e.

\begin{equation}
\tilde{W} := \{(\tilde{q}, \tilde{r}) : \frac{1}{\tilde{q}} + \frac{1}{q} = 1, \frac{1}{r} + \frac{1}{\tilde{r}} = 1, (q, r) \in W\}
\end{equation}

• \((q, r, \tilde{q}, \tilde{r})\) satisfy the dimensional analysis conditions

\begin{equation}
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m
\end{equation}

and

\begin{equation}
\frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{3}{2} - m
\end{equation}

Some variables frequently appear in this paper. We define them now.

We say that \((q, r)\) is a m-wave admissible pair if \(0 \leq m \leq 1\) and \((q, r)\) satisfy the two following conditions

• \((q, r) \in W\)
• \(\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m\)

Let \(J = [a, b]\) be an interval included in \([0, \infty)\). Given a function \(u\) we define

\begin{equation}
Z_{m,s}(J, u) := \sup_{q, r} \|D^{1-m} I u\|_{L^q_t(J) L^r_x} + \|D^{-m} \partial_t I u\|_{L^q_t(J) L^r_x}
\end{equation}

where the sup is taken over m-wave admissible \((q, r)\) and let

\begin{equation}
Z(J, u) := \sup_{m \in [0, 1]} Z_{m,s}(J, u)
\end{equation}

If \(u\) satisfies (1.15) then for \(t \in J\)

\begin{equation}
u(t) = u^{l,J}(t) + u^{nl,J}(t)
\end{equation}

with \(u^{l,J}\) denoting the linear part of \(u\) adapted to \(J\) i.e.

\begin{equation}
u^{l,J}(t) := \cos((t - a) D) u(a) + \frac{\sin((t - a) D)}{D} \partial_t u(a)
\end{equation}

and \(u^{nl,J}\) denoting its corresponding nonlinear part i.e.

\begin{equation}
u^{nl,J}(t) := -\int_a^t \frac{\sin((t - t') D)}{D} u^3(t') dt'
\end{equation}
Some estimates that we establish throughout the paper require a Paley-Littlewood decomposition. We set it up now. Let $\phi(\xi)$ be a real, radial, nonincreasing function that is equal to 1 on the unit ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ and that is supported on $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$. Let $\psi$ denote the function

\begin{equation}
\psi(\xi) := \phi(\xi) - \phi(2\xi)
\end{equation}

If $M \in 2\mathbb{Z}$ is a dyadic number we define the Paley-Littlewood operators in the Fourier domain by

\begin{align}
P_{\leq M}f(\xi) &:= \phi\left(\frac{\xi}{M}\right) \hat{f}(\xi) \\
P_Mf(\xi) &:= \psi\left(\frac{\xi}{M}\right) \hat{f}(\xi) \\
P_{> M}f(\xi) &:= \hat{f}(\xi) - P_{\leq M}f(\xi)
\end{align}

Since $\sum_{M \in 2\mathbb{Z}} \psi\left(\frac{\xi}{M}\right) = 1$ we have

\begin{equation}
f = \sum_{M \in 2\mathbb{Z}} P_Mf
\end{equation}

We conclude this introduction by giving the main ideas of the proof of Theorem 1.1 and explaining how the paper is organized. We are interested in finding an a priori upper bound of $\| (u(T), \partial_t u(T)) \|_{H^s \times H^{s-1}}$. Proposition 1.2 shows that it suffices to estimate $\sup_{t \in [0,T]} E(Iu(t))$. The variation of the mollified energy is expected to be slow. Therefore our strategy is to estimate the supremum of the mollified energy by applying the fundamental theorem of calculus. This is the $I$-method originally invented by J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao in [5] to prove global well-posedness for semilinear Schrödinger equations and for rough data and designed in [12] for the defocusing cubic wave equation. We divide the whole interval $[0, T]$ into same length intervals. On each of these subintervals we estimate the variation of the mollified energy by performing a Paley-Littlewood decomposition and, roughly speaking, by dividing the pieces of the solution supported on high frequencies into their linear part adapted to the subinterval and their corresponding nonlinear part. We prove in Section 3 that we can locally control some quantities depending on the solution and globally control other quantities depending on its linear part. Kenig, Ponce and Vega [8] observed that the nonlinear part of $u$ is smoother than the linear part on high frequencies. In the same spirit we prove a local inequality in Section 4 that brings out this fact. The variation of the mollified energy comprises three types of terms. Some of them are only made up of the nonlinear part of the solution: they are estimated by using the gain of regularity found in Section 4 and they are locally small but globally large. Some other are composed of the linear part of the solution: they are estimated by using the global estimates found in Section 3 and they are locally larger but globally smaller. The other ones are mixed terms and are estimated by using the results of Sections 3 and 4: we expect a combination of both effects. We estimate in Section 5 the variation of the smoothed energy on each of these subintervals. Then we iterate to cover the whole interval. The upper bound of the total variation depends on the size of the subintervals. This one plays the role of a parameter to be chosen. By minimizing the upper bound we find the optimal value that yields the sharpest estimate. This process is explained in Section 2: Theorem 1.1 follows.
Remark 1.4. If we had used the original I method \[5\] we would have obtained a \(O\left(\frac{1}{N}\right)\) increase of the smoothed energy on time intervals of size one and we would have found global well-posedness for \(s > \frac{3}{4}\) (see \[11\]). In this paper we prove that we have the same increase but on time intervals of size larger than one and this is why we beat \(\frac{3}{4}\).

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2. Proof of Global Well-Posedness in \(H^s \times H^{s-1}, 1 > s > \frac{13}{18}\)

In this section we prove the global existence of \(1.1\) in \(H^s \times H^{s-1}, 1 > s > \frac{13}{18}\). Our proof relies on some intermediate results that we prove in the next sections. More precisely we shall show the following

**Proposition 2.1. "Local and Global Boundedness"

Let \(J = [a, b]\) be an interval included in \([0, \infty)\). Assume that \(u\) satisfies \((1.1)\) and that

\[
\sup_{t \in J} E(Iu(t)) \leq 2
\]

Then

\[
Z(J, u_{l,J}) \lesssim 1
\]

Moreover if \((q, r)\) are \(m\)-wave admissible then

\[
\|D^{1-m}Iu\|_{L^q_t(J)L^r_x} \lesssim (\max(1, |J|))^{\frac{1}{r}}
\]

and

\[
\|D^{1-m}Iu^{nl,J}\|_{L^q_t(J)L^r_x} \lesssim (\max(1, |J|))^{\frac{1}{r}}
\]

**Proposition 2.2. "Local Gain of Regularity of the Nonlinear Term"

Let \(J = [a, b]\) be an interval included in \([0, \infty)\) and \(u\) such that \((1.1)\) and \((2.1)\) hold. Then

\[
\|\partial_t Iu^{nl,J}\|_{L^q_t(J)L^r_x} + \|DIu^{nl,J}\|_{L^q_t(J)L^r_x} \lesssim (\max(1, |J|))^{\frac{1}{r}}
\]

**Proposition 2.3. "Almost Conservation Law"

Let \(J = [a, b]\) be an interval included in \([0, \infty)\) and \(u\) such that \((1.1)\) and \((2.1)\) hold. Then

\[
|\sup_{t \in J} E(Iu(t)) - E(Iu(a))| \lesssim \max\left(\frac{(\max(1, |J|))^{\frac{1}{r}}}{N}, \frac{(\max(1, |J|))^{\frac{1}{r}}}{N}\right)
\]

\[\text{More precisely the size is } \sim N^{\frac{1}{r}}: \text{ see } (2.12).\]
For the remainder of the section we show that Proposition 2.3 implies Theorem 1.1.

Let $T > 0$ and $N = N(T) >> 1$ be a parameter to be chosen later. There are three steps to prove Theorem 1.1.

1. **Scaling.** It was proved in [12] that there exists $C_0 = C_0 (\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})$ such that if $\lambda$ satisfies

$$
\lambda = C_0 N^{\frac{2(1-s)}{2s-1}}
$$

then

$$
E(Iu_\lambda(0)) \leq \frac{1}{2}
$$

2. **Boundedness of the mollified energy.** Let $F_T$ denote the following set

$$
F_T = \left\{ T' \in [0, T] : \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) \leq 1 \right\}
$$

with $\lambda$ defined in (2.7). We claim that $F_T$ is the whole set $[0, T]$ for $N = N(T) >> 1$ to be chosen later. Indeed

- $F_T \neq \emptyset$ since $0 \in F_T$ by (2.8).
- $F_T$ is closed by the dominated convergence theorem.
- $F_T$ is open. Let $\tilde{T} \in F_T$. By continuity there exists $\delta > 0$ such that for every $T' \in (\tilde{T} - \delta, \tilde{T} + \delta) \cap [0, T]$ we have

$$
\sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) \leq 2
$$

Assume that $\lambda T' \leq 1$. Then by (2.8), (2.11) and by Proposition 2.3 and we have

$$
\left| \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) - \frac{1}{2} \right| \lesssim \frac{1}{N^{\frac{1}{2}}}
$$

Now assume that $\lambda T' > 1$. We divide the interval $[0, \lambda T']$ into subintervals $(J_i)_{i \in \mathbb{N}}$ such that $|J_1| = \ldots = |J_{i-1}| = \epsilon$, $\lambda T' \geq \epsilon > 1$ to be determined and $|J_i| \leq \epsilon$. By (2.8), (2.10) and Proposition 2.3 we have

$$
\left| \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) - \frac{1}{2} \right| \lesssim \frac{\lambda T'}{\epsilon} \left( \frac{\lambda T'}{N^{\frac{1}{2}}} + \frac{\lambda T'}{N^{\frac{3}{2}}} \right)
$$

We are seeking to minimize the right-hand side of (2.12) with respect to $\epsilon$. If $\lambda T' >> N^{\frac{1}{2}}$ then choosing $\epsilon \sim N^{\frac{1}{2}}$ we have

$$
\left| \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) - \frac{1}{2} \right| \lesssim \frac{\lambda T'}{N^{\frac{1}{2}}}
$$

Now if $\lambda T' \lesssim N^{\frac{1}{2}}$ then letting $\epsilon = \lambda T'$ we have

$$
\left| \sup_{t \in [0, \lambda T']} E(Iu_\lambda(t)) - \frac{1}{2} \right| \lesssim \frac{1}{N^{\frac{1}{2}}}
$$
Let $C_0$, $C_1$ and $C_2$ be the constants determined by \( \lesssim \) in (2.11), (2.13) and (2.14) respectively and let $C = \max(C_0, C_1, C_2)$. Since $s > \frac{13}{18}$ we can always choose for every $T > 0$ a $N = N(T) \gg 1$ such that

\[
C \max \left( \frac{1}{N}, \frac{\lambda T}{N^{\frac{5}{4}}}, \frac{1}{N^{\frac{3}{4}}} \right) \leq \frac{1}{2}
\]

With this choice of $N = N(T) \gg 1$ we have $\sup_{t \in [0, \lambda T]} E(Iu_\lambda(t)) \leq 1$.

Hence $F_T = [0, T]$ with $N = N(T) \gg 1$ satisfying (2.15).

(3) Conclusion. Following the I- method described in [5] we have global well-posedness in $H^s \times H^{s-1}, 1 > s > \frac{13}{18}$. Now let $T$ be large and let $s > \frac{13}{18}$ be close to $\frac{13}{18}$. Then let $N$ such that

\[
0.9 \leq C \frac{\lambda T}{N^{\frac{5}{4}}} \leq \frac{1}{2}
\]

Notice that (2.16) is satisfied with this choice of $N$. We plug (2.17) into (2.16) and we apply Proposition 1.2 to get (1.9).

3. Proof of "Local and Global Boundedness"

In this section we prove Proposition 2.1. In what follows we also assume that $J = [0, \tau]$; the reader can check after reading the proof that the other cases come down to this one. We slightly modify an argument in [12]. We multiply the $m$-Strichartz estimate with derivative (1.16) by $D^{1-m}I$ and we have

\[
Z_{m,s}(\tau, u^{l,J}) \lesssim \|DIu_0\|_{L^2} + \|Iu_1\|_{L^2}
\]

This proves (2.2).

Now let us prove (2.3) and (2.4). Notice that it suffices by (2.2) and the triangle inequality to prove (2.3). We divide $[0, \tau]$ into subintervals $(J_k)_{k \in \{1, \ldots, l\}}$ such that $|J_1| = \ldots = |J_{l-1}| = \tau_0$ and $|J_l| \leq \tau_0 \tau_0 > 0$ constant to be determined. By concatenation it suffices to establish that $Z(J_k, u) \lesssim 1, k \in \{1, \ldots, l\}$. We will prove the claim for $k = 1$. By iteration it is also true for $k > 1$. There are two steps

- First Step We assume that $m \leq s$. We multiply the $m$-Strichartz estimate with derivative (1.16) by $D^{1-m}I$ and we get from the fractional Leibnitz rule, the Hölder in time and the Hölder in space inequalities
\begin{align}
\tag{3.2}
Z_{m,s}(\tau_0, u) &\lesssim \|DIu_0\|_{L^2} + \|Iu_1\|_{L^2} + \|D^{1-m}I(uuu)\|_{L^1_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} \\
&\lesssim 1 + \|D^{1-m}Iu\|_{L^\infty_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} \|u\|_{L^2_t([0, \tau_0])L^6_x}^2 \\
&\lesssim 1 + Z_{m,s}(\tau_0, u) \left( \tau_0^{\frac{1}{2}} \|P_{\leq N} u\|_{L^6_t([0, \tau_0])L^6_x} + \tau_0^{\frac{s}{2}} \|P_{> N} u\|_{L^\infty_t([0, \tau_0])L^6_x} \right)^2 \\
&\lesssim 1 + Z_{m,s}(\tau_0, u) \left( \tau_0^{\frac{1}{2}} \|Iu\|_{L^6_t([0, \tau_0])L^6_x} + \tau_0^{\frac{s}{2}} Z_{s,s}(\tau_0, u) \right)^2 \\
&\lesssim 1 + Z_{m,s}(\tau_0, u) \left( \tau_0^{\frac{1}{2}} \left( \sup_{t \in [0, \tau_0]} E(Iu(t)) \right)^{\frac{1}{2}} + \tau_0^{\frac{s}{2}} Z_{s,s}(\tau_0, u) \right)^2 \\
\end{align}

Assume that \( m = s \). If \( \tau_0 > 0 \) is small enough then after applying a continuity argument to (3.2) we get from (2.1)

\begin{align}
\tag{3.3}
Z_{s,s}(\tau_0, u) &\lesssim 1 \\
\end{align}

Assume that \( m < s \). Then by (3.2) and (3.3)

\begin{align}
\tag{3.4}
Z_{m,s}(\tau_0, u) &\lesssim 1 \\
\end{align}

- **Second step** We assume that \( m > s \). By (3.2), (3.3) and (3.4) we have

\begin{align}
\tag{3.5}
\|D^{1-r}I(uuu)\|_{L^1_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} &\lesssim Z_{r,s}(\tau_0, u) \left( \tau_0^{\frac{1}{2}} \left( \sup_{t \in [0, \tau_0]} E(Iu(t)) \right)^{\frac{1}{2}} + \tau_0^{\frac{s}{2}} Z_{s,s}(\tau_0, u) \right)^2 \\
&\lesssim 1 \\
\end{align}

for \( r \leq s \). The inequality

\begin{align}
\tag{3.6}
\|D^{1-m}I(uuu)\|_{L^1_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} &\lesssim \|D^{1-r}I(uuu)\|_{L^1_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} \\
\end{align}

follows from the application of Sobolev homogeneous embedding. We multiply the \( m \)-Strichartz estimate with derivative (1.10) by \( D^{1-m}I \) and get from (3.3) and (3.6)

\begin{align}
\tag{3.7}
Z_{m,s}(\tau_0, u) &\lesssim \|DIu_0\|_{L^2} + \|Iu_1\|_{L^2} + \|D^{1-m}I(uuu)\|_{L^1_t([0, \tau_0])L^6_x}^{\frac{6}{2m}} \\
&\lesssim 1 \\
\end{align}

4. Proof of \"Local Gain of regularity of the nonlinear part\"

In this section we prove (2.3). In what follows we also assume that \( J = [0, \tau] \); the reader can check after reading the proof that the other cases come down to that one. We get from Proposition (1.3) and Proposition (2.1)

\begin{align}
\tag{4.1}
\|\partial_t Iu^{nl,J}\|_{L^6_t([0, \tau])L^6_x} + \|D^{1-m}Iu^{nl,J}\|_{L^1_t([0, \tau])L^6_x} &\lesssim \|DI(uuu)\|_{L^\infty_t([0, \tau])L^6_x}^{\frac{2}{5}} \\
\end{align}

But
We perform a Paley-Littlewood decomposition to estimate the right hand side of (5.4)
\[ \| DI(uu) \|_{L^p_t([0,\tau])L^q_x} \leq \| DIu \|_{L^p_t([0,\tau])L^q_x} \| u \|_{L^p_t([0,\tau])L^q_x}^2 \]

\[ \leq \| DIu \|_{L^p_t([0,\tau])L^q_x} \left( \| P_{<\mathcal{N}}u \|_{L^2_tL^q_x} + \| P_{\geq\mathcal{N}}u \|_{L^2_tL^q_x} \right)^2 \]

\[ \leq \| DIu \|_{L^p_t([0,\tau])L^q_x} \left( \tau^{\frac{3}{2}} \| D^{-1}Iu \|_{L^p_tL^q_x} + \| D^{-1}Iu \|_{L^2_tL^q_x} \right)^2 \]

Combining (4.1) and (4.2) we get

\[ \| \partial_tIu^{nl,j} \|_{L^q_t([0,\tau])L^2_x} + \| DIu^{nl,j} \|_{L^2_t([0,\tau])L^2_x} \leq \tau^{\frac{3}{2}} \]

5. Proof of "Almost conservation law"

Let \( J = [a, b] \) be an interval included in \([0, \infty)\) and \( u \) such that (1.1) and (2.1) hold. Let \( \tau \in J \). Then the Plancherel formula and the fundamental theorem of calculus yield

\[ |E(Iu(\tau)) - E(Iu(a))| = \left| \int_a^\tau \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \partial_t Iu(t, \xi_1) \tilde{I}u(t, \xi_2) \tilde{I}u(t, \xi_3) \tilde{I}u(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

with

\[ \mu(\xi_2, \xi_3, \xi_4) := 1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2) m(\xi_3) m(\xi_4)} \]

We perform a Paley-Littlewood decomposition to estimate the right hand side of (5.1). Let \( u_i := P_{N_i} u \) and let \( X \) denote the following number

\[ X := \left| \int_a^\tau \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \partial_t Iu_1(t, \xi_1) \tilde{I}u_2(t, \xi_2) \tilde{I}u_3(t, \xi_3) \tilde{I}u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

The strategy to estimate \( X \) is explained in [4, 12]. We recall the main steps.

Overview of the strategy.

(1) First step We seek a pointwise bound of the symbol

\[ |\mu(\xi_2, \xi_3, \xi_4)| \leq B(N_2, N_3, N_4) \]

Then for some \( A \subset \{1, \ldots, 4\} \) to be chosen we decompose for every \( i \in A \) \( u_i \) into its linear part \( u_i^l := P_{N_i} u_i^l \) and its nonlinear part \( u_i^{nl,i} := P_{N_i} u_i^{nl,i} \) and after expansion we need to evaluate expressions of the form

\[ Y := \left| \int_a^\tau \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \xi_3, \xi_4) \partial_t Iu_1(t, \xi_1) \tilde{I}v_2(t, \xi_2) \ldots \tilde{I}v_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

with \( v_j, j \in \{1, \ldots, 4\} \) denoting \( u_j^{nl} \) or \( u_j^l \) or \( u_j^3 \). We get from the Coifman-Meyer theorem (3, p179)

\[ \text{the value of } v_j \text{ depends on the choice of } A \]
This ends the overview of the strategy.

Let us get back to the proof. By symmetry we may assume that $p_j \geq q_j \geq q_j^*$ for some $j$ by considering slight variations $(p_j \pm, q_j \pm) \in [1, \infty] \times (1, \infty)$ of $(p_1, q_1), (p_2, q_2)$ that are $m_1 \pm, m_2 \pm$ wave admissible and such that $\frac{1}{p_j \pm} + \frac{1}{q_j \pm} = \frac{1}{2} \cdot \frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2}$ respectively. For instance if we create slight variations $(p_1+q_1^+, (p_2+q_2^+)$ of $(p_1, q_1), (p_2, q_2)$ respectively then we get from Bernstein and Hölder in time inequalities

\begin{align}
\|I v_j\|_{L_t^{p_j^*}(J)L_x^q} &\lesssim N_j^{-1}\|D^{1-(m_j-1)} v_j\|_{L_t^{p_j^*}(J)L_x^{q}} \\
\|\partial_t I v_j\|_{L_t^{p_j^*}(J)L_x^q} &\lesssim N_j^{-1}\|D^{-(m_j)} \partial_t I v_j\|_{L_t^{p_j^*}(J)L_x^{q}} \\
\|I v_j\|_{L_t^{p_j^*}(J)L_x^{q}} &\lesssim N_j^{2}\|\partial_t I v_j\|_{L_t^{q}(J)L_x^{q}} \\
\|\partial_t I v_j\|_{L_t^{q}(J)L_x^{q}} &\lesssim N_j^{2}\|\partial_t I v_j\|_{L_t^{q}(J)L_x^{q}}
\end{align}

It was proved \cite{12} that the following inequality holds

\begin{align}
\|I v_j\|_{L_t^{p_j^*}(J)L_x^{q}} &\lesssim N_j^{-1}\|D^{1-(1-\varepsilon)} v_j\|_{L_t^{p_j^*}(J)L_x^{q}} \\
\|\partial_t I v_j\|_{L_t^{p_j^*}(J)L_x^{q}} &\lesssim N_j^{-1}\|D^{-(1-\varepsilon)} \partial_t I v_j\|_{L_t^{q}(J)L_x^{q}}
\end{align}

by using the localization in time to our advantage. The creation of $N_j^*$ allows to make the summation with respect to $N_j$ whenever $N_j < 1$.

This ends the overview of the strategy.

Let us get back to the proof. By symmetry we may assume that $N_2 \geq N_3 \geq N_4$. Let $N_1^* ..., N_4^*$ be the four numbers $N_1, ..., N_4$ in order so that $N_1^* \geq N_2^* \geq N_3^* \geq N_4^*$. We can assume that $N_1^* > N$ since if not the multiplier $\mu$ of $X$ vanishes and $X = 0$. We can also assume that $N_1^* \sim N_1$ since if not the convolution constraint $\xi_1 + ... + \xi_4 = 0$ imposes $X = 0$. There are three cases

- **Case 1**: $N_1^* = N_2$ and $N_2^* = N_1$

  We write $u_i = u^{l,j}_i + u^{n,j}_i$, $i \in \{1, 2\}$. We need to estimate

\footnote{In other words $(p_j, q_j) = \left(\frac{2}{m_j}, \frac{2}{1-m_j}\right)$ \footnote{More precisely $\|I v_j\|_{L_t^{p_j^*}(J)L_x^q} \lesssim N_j^{-1}\|D^{1-(1-\varepsilon)} v_j\|_{L_t^{p_j^*}(J)L_x^{q}}$ with $\varepsilon = 5\varepsilon$.}
\[ X_1 = \left| \int_0^T \int_{\xi_1+\ldots+\xi_4=0} \mu(\xi_2, \ldots, \xi_4) \frac{\partial}{\partial t} Iu_1^{l,j}(t, \xi_1) Iu_2^{n,l,j}(t, \xi_2) \Gamma u_3(t, \xi_3) \Gamma u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

\[ X_2 = \left| \int_0^T \int_{\xi_1+\ldots+\xi_4=0} \mu(\xi_2, \ldots, \xi_4) \frac{\partial}{\partial t} Iu_1^{l,j}(t, \xi_1) Iu_2^{n,l,j}(t, \xi_2) \Gamma u_3(t, \xi_3) \Gamma u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

\[ X_3 = \left| \int_0^T \int_{\xi_1+\ldots+\xi_4=0} \mu(\xi_2, \ldots, \xi_4) \frac{\partial}{\partial t} Iu_1^{n,l,j}(t, \xi_1) Iu_2^{l,j}(t, \xi_2) \Gamma u_3(t, \xi_3) \Gamma u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

and

\[ X_4 = \left| \int_0^T \int_{\xi_1+\ldots+\xi_4=0} \mu(\xi_2, \ldots, \xi_4) \frac{\partial}{\partial t} Iu_1^{n,l,j}(t, \xi_1) Iu_2^{l,j}(t, \xi_2) \Gamma u_3(t, \xi_3) \Gamma u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right| \]

There are two subcases

- Case 1.a: \( N_3 \gtrsim N \)

We have

\[ |\mu| \lesssim \frac{m(N_1)}{m(N_2)m(N_3)m(N_4)} \]

By (2.22), (2.31), (2.34), (5.3) and (5.14) we have

\[ X_1 \lesssim \frac{1}{m(N_3)m(N_2)} \left\| \frac{\partial}{\partial t} Iu_1^{l,j} \right\|_{L^2_\omega(J)} \left\| Iu_2^{n,l,j} \right\|_{L^\infty_\omega(J)} \left\| Iu_3 \right\|_{L^\infty_\omega(J)} \left\| Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_3 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \lesssim (\max(1, |J|))^\frac{1}{2} \frac{N_3^{-N_2^{-1}}}{N_1^{-N_4^{-1}}} \]

Similarly

\[ X_2 \lesssim \frac{1}{m(N_3)m(N_2)} \left\| \frac{\partial}{\partial t} Iu_1^{l,j} \right\|_{L^2_\omega(J)} \left\| Iu_2^{n,l,j} \right\|_{L^\infty_\omega(J)} \left\| Iu_3 \right\|_{L^\infty_\omega(J)} \left\| Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_3 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \lesssim (\max(1, |J|))^\frac{1}{2} \frac{N_3^{-N_2^{-1}}}{N_1^{-N_4^{-1}}} \]

We have

\[ X_3 \lesssim \frac{1}{m(N_3)m(N_2)} \left\| \frac{\partial}{\partial t} Iu_1^{n,l,j} \right\|_{L^\infty_\omega(J)} \left\| Iu_2^{l,j} \right\|_{L^2_\omega(J)} \left\| Iu_3 \right\|_{L^\infty_\omega(J)} \left\| Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_3 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^2_\omega(J)} \left\| D^{(1-0)} Iu_4 \right\|_{L^\infty_\omega(J)} \lesssim (\max(1, |J|))^\frac{1}{2} \frac{N_3^{-N_2^{-1}}}{N_1^{-N_4^{-1}}} \]
As for $X_4$ we make further decompositions. We write $u_3 = u_{3,1}^{l,j} + u_{3,2}^{n,l,j}$ and we need to estimate

\begin{equation}
X_{4,1} = \left| \int_a^x \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \overline{u_1^{1,l,j}}(t, \xi_1) \overline{u_2^{n,l,j}}(t, \xi_2) \overline{u_3^{1,l,j}}(t, \xi_3) \overline{u_4}(t, \xi_4) \ d\xi_2 \ldots d\xi_4 dt \right|
\end{equation}

and

\begin{equation}
X_{4,2} = \left| \int_a^x \int_{\xi_1 + \ldots + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \overline{u_1^{1,l,j}}(t, \xi_1) \overline{u_2^{n,l,j}}(t, \xi_2) \overline{u_3^{1,l,j}}(t, \xi_3) \overline{u_4}(t, \xi_4) \ d\xi_2 \ldots d\xi_4 dt \right|
\end{equation}

We have

\begin{equation}
X_{4,1} \lesssim \frac{1}{m(N) m(N)} \| \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_2^{n,l,j} \|_{L^\infty_t \cap L^2_x} \| u_3^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_4 \|_{L^\infty_t \cap L^2_x} - \| D^{-1-\alpha} \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| D u_2^{n,l,j} \|_{L^\infty_t \cap L^2_x}
\end{equation}

\begin{equation}
\lesssim (\max(1, |J|)) \left( \frac{N^+}{N^-} \right).
\end{equation}

and by (2.3) we have

\begin{equation}
X_{4,2} \lesssim \frac{1}{m(N) m(N)} \| \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_2^{n,l,j} \|_{L^\infty_t \cap L^2_x} \| u_3^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_4 \|_{L^\infty_t \cap L^2_x} - \| D^{-1-\alpha} \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| D u_2^{n,l,j} \|_{L^\infty_t \cap L^2_x}
\end{equation}

\begin{equation}
\lesssim (\max(1, |J|)) \left( \frac{N^+}{N^-} \right).
\end{equation}

– **Case 1.b:** $N_3 \ll N$ By the Mean Value Inequality

\begin{equation}
|\mu| \lesssim \frac{|\nabla m(\xi)|}{m(\xi)} \lesssim \frac{N^+}{N^-}
\end{equation}

Now if for $X_1$, $X_2$, $X_3$, $X_{4,1}$ and $X_{4,2}$ we apply the same procedure to that of Case 1.a and if we use (5.15, 22) we see that the factor $N^+_2$ that appears always satisfies $\alpha \geq 0$ and consequently is comparable to $N^\alpha$. Therefore the results are the same. For instance

\begin{equation}
X_1 \lesssim \frac{N_2}{N^-} \| \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_2^{l,j} \|_{L^\infty_t \cap L^2_x} \| u_3^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| u_4 \|_{L^\infty_t \cap L^2_x} - \| D^{-1-\alpha} \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| D u_2^{l,j} \|_{L^\infty_t \cap L^2_x}
\end{equation}

\begin{equation}
\lesssim \frac{N_2}{N^-} N^+ N_1 \frac{N^+}{N^-} N^+ \| D^{-1-\alpha} \partial_t u_1^{1,l,j} \|_{L^\infty_t \cap L^2_x} \| D u_2^{l,j} \|_{L^\infty_t \cap L^2_x}
\end{equation}

\begin{equation}
\lesssim (\max(1, |J|)) \left( \frac{N^+}{N^-} \right)
\end{equation}

and here the factor $N^+_2 \frac{N_2}{N^-} = N^+_3$ appears.

**Case 2:** $N^+_1 = N_1$ and $N^+_2 = N_2$

Since $N_1 \sim N_2$ then this case boils down to the previous one.
• Case 3: $N_1 = N_2$ and $N_2 = N_3$

We write $u_i = \tilde{u}_i^{l,j} + \tilde{u}_i^{nl,j}$, $i \in \{2, 3\}$. We need to estimate

\begin{equation}
X_1 = \left| \int_0^\tau \int_{\xi_1 + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \tilde{I} u_1(t, \xi_1) \tilde{I} u_2(t, \xi_2) \tilde{I} u_3(t, \xi_3) \tilde{I} u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right|
\end{equation}

\begin{equation}
X_2 = \left| \int_0^\tau \int_{\xi_1 + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \tilde{I} u_1(t, \xi_1) \tilde{I} u_2(t, \xi_2) \tilde{I} u_3^{nl,j}(t, \xi_3) \tilde{I} u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right|
\end{equation}

\begin{equation}
X_3 = \left| \int_0^\tau \int_{\xi_1 + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \tilde{I} u_1(t, \xi_1) \tilde{I} u_2^{nl,j}(t, \xi_2) \tilde{I} u_3^{l,j}(t, \xi_3) \tilde{I} u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right|
\end{equation}

and

\begin{equation}
X_4 = \left| \int_0^\tau \int_{\xi_1 + \xi_4 = 0} \mu(\xi_2, \ldots, \xi_4) \partial_t \tilde{I} u_1(t, \xi_1) \tilde{I} u_2^{nl,j}(t, \xi_2) \tilde{I} u_3^{nl,j}(t, \xi_3) \tilde{I} u_4(t, \xi_4) \, d\xi_2 \ldots d\xi_4 \, dt \right|
\end{equation}

We have

\begin{equation}
|\mu| \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)}
\end{equation}

By \ refers to \ref{e2.23}, \ref{e2.30}, \ref{e2.41}, \ref{e5.9} and \ref{e5.28} we have

\begin{equation}
X_1 \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \left[ \partial_t I u_1 \right]_{L^\infty_t(0,T) L^2_x} \left( \left| I u_2^{l,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_3^{l,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_4 \right|_{L^\infty_t(0,T) L^2_x} \right]
\end{equation}

\begin{equation}
X_2 \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \left[ \partial_t I u_1 \right]_{L^\infty_t(0,T) L^2_x} \left( \left| I u_2^{l,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_3^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_4 \right|_{L^\infty_t(0,T) L^2_x} \right]
\end{equation}

\begin{equation}
X_3 \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \left[ \partial_t I u_1 \right]_{L^\infty_t(0,T) L^2_x} \left( \left| I u_2^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_3^{l,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_4 \right|_{L^\infty_t(0,T) L^2_x} \right]
\end{equation}

\begin{equation}
X_4 \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \left[ \partial_t I u_1 \right]_{L^\infty_t(0,T) L^2_x} \left( \left| I u_2^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_3^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_4 \right|_{L^\infty_t(0,T) L^2_x} \right]
\end{equation}

Similarly since $N_2 \sim N_3$ we have

\begin{equation}
X_3 \lesssim \frac{m(N_1)}{m(N_2) m(N_3) m(N_4)} \left[ \partial_t I u_1 \right]_{L^\infty_t(0,T) L^2_x} \left( \left| I u_2^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_3^{nl,j} \right|_{L^\infty_t(0,T) L^2_x} \left| I u_4 \right|_{L^\infty_t(0,T) L^2_x} \right]
\end{equation}

As for $X_4$ we make further decompositions. We write $u_1 = u_1^{l,j} + u_1^{nl,j}$ and we need to estimate
(5.32)
\[X_{4,1} = \left| \int_a^b \int_{\xi_1+...+\xi_4=0} \rho(t,\xi_1,...,\xi_4) \partial_t Iu_1^{n,J} (t,\xi_1) Iu_2^{n,J} (t,\xi_2) Iu_3^{n,J} (t,\xi_3) Iu_4(t,\xi_4) \, d\xi_2...d\xi_4 \, dt \right| \]

and

(5.33)
\[X_{4,2} = \left| \int_a^b \int_{\xi_1+...+\xi_4=0} \rho(t,\xi_1,...,\xi_4) \partial_t Iu_1^{n,J} (t,\xi_1) Iu_2^{n,J} (t,\xi_2) Iu_3^{n,J} (t,\xi_3) Iu_4(t,\xi_4) \, d\xi_2...d\xi_4 \, dt \right| \]

We have

(5.34)
\[X_{4,1} \lesssim \frac{m(N)}{m(N_1)m(N_2)m(N_4)} \| \partial_t Iu_1^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_2^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_3^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_4 \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \]
\[\lesssim \frac{m(N)}{m(N_1)m(N_2)m(N_4)} N_1 N_2 N_4 \| \partial_t Iu_1^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_2^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_3^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \]
\[\| D^{1-(1/2)} Iu_4 \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \]
\[\lesssim (\max(1,|J|))^{\frac{N_1}{N_2} - \frac{N_1}{N_4}} \]

and by (2.5) we have

(5.35)
\[X_{4,2} \lesssim \frac{m(N)}{m(N_1)m(N_2)m(N_4)} \| \partial_t Iu_1^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_2^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \| Iu_3^{n,J} \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \]
\[\| D^{1-(1/2)} Iu_4 \|_{L^2(J)} \| L^{\infty}(J) \|_{L^2} \]
\[\lesssim (\max(1,|J|))^{\frac{N_1}{N_2} - \frac{N_1}{N_4}} \]

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University of California, Los Angeles
E-mail address: triroy@math.ucla.edu