Unified hydrodynamics theory of the lowest Landau level

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(Dated: August 24, 2018)

We propose a hydrodynamics theory of collective quantum Hall states, which describes incompressible liquids, hexatic liquid crystals, a bubble solid and a Wigner crystal states within a unified framework. The structure of the theory is uniquely determined by the space-time symmetry, and a symmetry with respect to static shear deformations. In agreement with recent experiments the theory predicts two gapped collective modes for incompressible liquids. We argue that the presence of the above two modes is a universal property of a magnetized two-dimensional collective liquid.

PACS numbers: 73.43.Cd, 73.43.Lp, 46.05.+b, 47.10.-g

I. INTRODUCTION

Hydrodynamics or, more generally, continuum mechanics is a common tool to describe collective degrees of freedom in condensed matter systems. This approach becomes especially powerful in the linear regime as it provides a unified view on collective dynamics in solids, liquid crystals, normal liquids, superfluids, etc. (see, for example, Ref 1). The hydrodynamics equations of motion follow the universal conservation laws, while a particular state of matter enters the theory via a linearized “equation of state” that is parametrized by a set of visco-elastic constants. A general form of the “equation of state” (normally this is a dependence of the stress tensor on basic variables), and a number of independent parameters are determined by the macroscopic symmetry.

Hydrodynamics represents the response of a many-particle system to external fields in terms of a few collective variables. This assumes that for whatever reason the underlying particles loose their individual properties and get collectivized. Apparently the above assumption is perfectly valid for quantum collective many-body states, such as superfluids or strongly correlated states of an electron gas in the fractional quantum Hall (FQH) regime²,³. The hydrodynamics of superfluids has indeed a long and reach history (a general review as well as recent applications to trapped atomic Bose condensates can be found in Refs. 4, 5). Yet the power of hydrodynamics approach in application to FQH fluids has been not fully explored. A notable exception is a low energy (in fact, adiabatic) extreme, where the structure of hydrodynamic theory is well understood. In this regime the hydrodynamics of FQH liquids universally reduces to a topological Chern-Simons theory⁶ (see also Ref. 7), which reflects a topological order of incompressible FQH states. An effective Chern-Simons theory successfully describes gapless edge excitations, but, by its low-energy nature, fails to reproduce fundamental bulk modes with a gapped spectrum⁹.

A principal possibility to describe gapped magneto-roton modes within a continuum mechanics formalism has been demonstrated recently⁹. The most important observation of this work is that the gap ∆ in the spectrum of collective excitations requires a highly unusual, resonant frequency dependence of the shear stress. An effective dynamic shear modulus should diverge at ω = ∆, otherwise collective modes will be gapless. It has been shown that the required resonant structure of the shear modulus naturally appears in hydrodynamics with an additional tensor collective variable that describes nontrivial precession dynamics of a shear stress. Using heuristic physical arguments in Ref. 9 we constructed the simplest equation of motion for the new tensor variable, and thus derived a “minimal” hydrodynamic theory that reproduces a qualitatively correct dispersion of a gapped collective mode.

The present paper reports a further progress in that direction. We show that there is a fundamental local symmetry behind the presence of a tensor field in the hydrodynamics of incompressible FQH liquids. That is an invariance with respect to arbitrary static shear deformations, which is a universal property of a liquid state of matter. The tensor dynamic variable is, in fact, a gauge field that supports the above local symmetry.

From experimental side an interest to bulk collective excitations of FQH liquids has been recently renewed by the first observation of the long wavelength dispersion of gapped modes at the filling factor ν = 1/3.10 The most intriguing result of these experiments is a two-branch structure of the gapped excitation spectrum at small wave vectors.

In the present paper we propose a universal hydrodynamical interpretation of these experimental findings. We derive a hydrodynamics of FQH liquids, which relies solely on general symmetry arguments. The only assumption, which underlies our theory, is that the system is a two-dimensional (2D) collective liquid with a proper space-time symmetry (i. e. rotationally and time-reversal invariant). Incompressibility of the liquid, nonvanishing dissipationless Hall conductivity, a gapped spectrum of collective modes, etc., follow automatically from that assumption. The theory predicts two gapped modes (one with upward, and another one with downward dispersions), which is in excellent agreement with recent experiments¹⁰. The key idea in constructing the hydro-
dynamics of incompressible liquids is to preserve the fundamental property of any liquid state—a local symmetry under static volume-preserving deformations. By a successive reduction of the above local invariance we deduce theories of a hexatic liquid crystal, and two different crystalline phases.

The paper is organized as follows. In Sec. II we identify basic symmetries, and present a phenomenological derivation of the linearized hydrodynamics of incompressible QH liquids. We construct a Lagrangian of the theory, derive equations of motion for basic collective variables, and analyze the spectrum of collective excitations. In Sec. III the theory is generalized to describe liquid crystal, and crystal states with hexagonal symmetries. In Sec. IV we summarize our main results.

II. HYDRODYNAMICS OF INCOMPRESSIBLE QUANTUM HALL LIQUIDS

A. Lagrangian of a magnetized collective liquid

Our goal is to describe linear dynamics of an interacting 2D many-particle system confined to the x,y-plane, and subjected to a strong transverse magnetic field \( \mathbf{B} = e_2 B \). The hydrodynamic theory, which we derive in this section, is based on the following two, quite general assumptions about the ground/equilibrium state of the system: (i) The system is in a collective state that is rotationally and Galilee invariant; (ii) The system is in a liquid state.

A few comments on the above assumptions are in order. The term collective state assumes the absence of a continuum of single particle excitations. Hence the dynamics can be represented by a finite number of collective variables. One of those variables, namely the displacement vector \( \mathbf{u}(r,t) \), is fixed as it should appear in any linearized continuum mechanics. The velocity field \( \mathbf{v}(r,t) \), and the variation of the density \( \delta n(r,t) \) are related to the displacement \( \mathbf{u} \) as follows,

\[
\mathbf{v} = \partial_t \mathbf{u}, \quad \delta n = -n \nabla \mathbf{u},
\]

where \( n = \nu/2\pi l^2 \) is the equilibrium density (\( \nu \) is the filling factor, and \( l \) is the magnetic length). In general the displacement does not exhaust all relevant dynamic variables. It is well known that an adequate hydrodynamic description of ordered states of matter requires an introduction of additional fields that are specific for every system (a magnetization in ferromagnets, a director vector in nematic liquid crystals, a superfluid velocity in superfluids, etc.). We will see that a somewhat similar (though not precisely the same) situation takes place for incompressible QFH liquids.

The assumption of a liquid state, (ii), can be also formalized in terms of the displacement vector. By definition, a liquid is a state of matter that does not respond to an arbitrary static shear (i.e., volume-preserving) deformation. Mathematically shear deformations are described by the divergenceless displacement vector. Hence by displacing a liquid system with a static, purely transverse vector \( \mathbf{f}(r) \) we should not produce any physical effect. Formally this means that the equations of motion for any 2D liquid should be invariant under the following transformation \(^{12}\)

\[
u_i(t) = u_i(r,t) + \epsilon_{ij} \partial_j \psi(r),
\]

where \( \psi(r) \) is an arbitrary function of only spatial variables. The invariance under a volume preserving diffeomorphism of Eq. (1) has a transparent interpretation within a Lagrangian formulation of fluid mechanics. It reflects the symmetry with respect to relabeling of different infinitesimal fluid elements\(^{13}\). It is worth noting that the above diffeomorphism symmetry can serve as a most general mathematical definition of a liquid state. In terms of visco-elastic coefficients the symmetry under the transformation of Eq. (1) implies a vanishing static shear modulus.

An important requirement, which we impose on the theory, follows from the universality of the high-frequency response. In the high-frequency, long wavelength limit any interacting many-body system behaves as an elastic continuum (see, for example, Ref. 14, and a recent paper of Ref. 15 for a most general demonstration of this fact). Formally this means that in the high-frequency regime the exact stress tensor takes the standard elastic form\(^{16}\)

\[
\Pi_{ij} = -\delta_{ij} K \nabla \mathbf{u} - \mu_\infty \left( \partial_i u_j + \partial_j u_i - \delta_{ij} \nabla \mathbf{u} \right),
\]

where the high-frequency shear modulus \( \mu_\infty \), and the bulk modulus \( K \) are universal constants that are related to the ground state properties\(^{15}\). Therefore in the limit \( \omega \to \infty \) the theory should reduce to the following equation for displacement vector \( \mathbf{u} \), which describes linear dynamics of the classical elastic medium\(^{16}\)

\[
m \partial_t^2 \mathbf{u} + B \partial_t \mathbf{u} \times \mathbf{e}_z - \frac{K}{n} \nabla (\nabla \mathbf{u}) - \frac{\mu_\infty n}{\nu} \nabla^2 \mathbf{u} + \nabla U_H = \mathbf{F},
\]

where \( U_H \), and \( \mathbf{F} \) are the Hartree potential, and the external force respectively. The high-frequency equation of motion, Eq. (2), is valid for all rotationally and Galilee invariant systems in the presence of an external magnetic field. The above requirement uniquely determines the high-frequency form of the Lagrangian that generates the correct left hand side of the equation of motion, Eq. (2)

\[
L_\infty = L_0 + L_S^\infty,
\]

where \( L_0 \) is a linearized Lagrangian of an ideal liquid\(^{17}\)

\[
L_0 = \frac{1}{2} \mu_0 (\partial_t \mathbf{u})^2 - \frac{1}{2} n B \mathbf{u} (\partial_t \mathbf{u} \times \mathbf{e}_z) - \frac{1}{2} K (\nabla \mathbf{u})^2,
\]

and \( L_S^\infty \) is the standard elastic shear Lagrangian

\[
L_S^\infty = -\frac{1}{2} \mu_\infty (\partial_t u_j)^2.
\]
For the further discussion it convenient to introduce a traceless shear strain tensor

$$S_{ij} = \partial_i u_j + \partial_j u_i - \delta_{ij} \nabla u,$$  \hspace{1cm} (6)

and to represent the shear Lagrangian of Eq. (5) in the following “natural” form

$$L^\infty_\phi = -\frac{1}{4}\mu_\infty \text{Tr}\hat{S}^2 = -\frac{1}{4}\mu_\infty S_{ij} S_{ji},$$  \hspace{1cm} (7)

One can straightforwardly check that Eqs. (5) and (7) are identical up to an irrelevant total derivative. It is important to outline that the high-frequency form of the Lagrangian, Eq. (3), is not an additional assumption, but the exact property of any rotationally and Galilean invariant interacting many-body system.

Let us analyze the symmetries of the Lagrangian $L^\infty$ defined by Eqs. (3), (4), and (7). Obviously it is rotationally, time-reversal, and Galilean invariant. [It is worth mentioning that the requirement of Galilean invariance fixes the coefficients in the first two terms in Eq. (4) to $nm$ and $nB$ respectively. Hence the our first assumption, (i), is fulfilled at the high frequency level. However, the assumption (ii), which corresponds to the local invariance of Eq. (1), is violated by the presence of the shear term in $L^\infty$. Apparently the first term, $L_0$, in Eq. (3) is invariant under the transformation Eq. (1). The second, shear term, $L^\infty_\phi$, fails to satisfy this property since the shear strain tensor, Eq. (6), acquires a correction

$$\hat{S}'(r, t) = \hat{S}(r, t) + \delta\hat{S}(r) \hspace{1cm} (8)$$

$$\delta S_{ij}(r) = \varepsilon_{ik} \partial_k \psi(r) + \varepsilon_{jk} \partial_k \psi(r),$$  \hspace{1cm} (9)

which changes $L^\infty_\phi$. A common prescription to restore any local symmetry is offered by the gauge theory – one has to introduce a gauge field $\tilde{\eta}(r, t)$ that compensates unwanted changes in the Lagrangian. Namely, we replace $\hat{S}$ in Eq. (7) by the form $(\hat{S} - \tilde{\eta})^2$, and add a Lagrangian $L_\eta$ that describes dynamics the gauge field $\tilde{\eta}$, i.e.

$$L^\infty \rightarrow L^\eta_\phi = -\frac{1}{4}\mu_1 \text{Tr}(\hat{S} - \tilde{\eta})^2 + L_\eta.$$  \hspace{1cm} (10)

The local symmetry of Eq. (1) is guaranteed if the term $L_\eta$ in the Lagrangian is invariant under the transformation

$$\hat{\eta}'(r, t) = \hat{\eta}(r, t) + \delta\hat{\eta}(r),$$  \hspace{1cm} (11)

where $\delta\hat{\eta}(r)$ is defined after Eq. (9). Additional restrictions to the form of $L^\eta_\phi$ are imposed by the space-time symmetry, and a fixed high-frequency asymptotic form of Eq. (7). Apparently, since the shear strain tensor $\hat{S}$, Eq. (6), is symmetric and traceless, the gauge field $\tilde{\eta}$ also has to be a symmetric, traceless tensor. Following the outlined route we construct the following most general (quadratic) shear Lagrangian that is rotationally and time-reversal invariant, and enjoys the local symmetry of Eq. (1):

$$L^\eta_\phi = -\frac{1}{4}\mu_1 \text{Tr}(\hat{S} - \tilde{\eta})^2 + \frac{1}{4}\mu_1 \left[\text{Tr}(\partial_t \tilde{\eta})^2 + \Delta_2 \text{Tr}\hat{\eta}(\partial_t \tilde{\eta} \times e_z) + 2\Delta_2' \text{Tr}\partial_t \hat{S}(\partial_t \tilde{\eta} \times e_z)\right],$$  \hspace{1cm} (12)

where the cross product of $e_z$ and a symmetric tensor is defined as follows

$$(\tilde{\eta} \times e_z)_{ij} = (\varepsilon_{ik} \eta_{kj} + \varepsilon_{jk} \eta_{ki})/2.$$  \hspace{1cm} (13)

The term in the square brackets in Eq. (11) contains all allowed scalar invariants up to the second order in space-time derivatives. It should be noted that the rotational and the time-reversal symmetries allow two more quadratic invariants:

$$\text{Tr}\tilde{\eta}^2, \hspace{0.5cm} \text{Tr}(\partial_t \tilde{\eta})(\partial_t \tilde{\eta}).$$  \hspace{1cm} (14)

However these terms are forbidden by the invariance under the local transformation Eq. (10). As a result only the terms with at least one time derivative are left in $L^\eta_\phi$.

The final Lagrangian of a collective quantum Hall liquid takes the form

$$L^\eta_\phi = L_0 + L^\eta_\phi + n u \partial_t a - n u \nabla \varphi,$$  \hspace{1cm} (15)

where $L_0$ and $L^\eta_\phi$ are given by Eqs. (4) and (11) respectively, and the last two terms describe the interaction with external potentials, $a(r, t)$ and $\varphi(r, t)$.

The Lagrangian Eq. (13) defines a theory of two coupled fields – the vector field $u(r, t)$ and a symmetric traceless tensor field $\tilde{\eta}(r, t)$. Both our basic assumptions, (i) and (ii), are satisfied by Eq. (13). This is the most general harmonic Lagrangian for a 2D collective liquid in the presence of a transverse magnetic field. The theory contains four still undefined phenomenological parameters, $\mu_1$, $\Delta_1^2$, $\Delta_2$ and $\Delta'$. The requirement of the high-frequency elastic response provides one constraint, which allows us to express $\mu_1$ in terms of $\mu_\infty$. Three remaining parameters can be related to observable quantities – the excitation gaps, and the corresponding oscillator strength (see Sec. IIIB below).

### B. Equations of motion and response functions

The Lagrangian Eq. (13) generates the following equation of motion for the displacement vector

$$m n \partial_t^2 u + n B \partial_t u \times e_z - K \nabla(\nabla u) + \nabla \tilde{\pi} = n F,$$  \hspace{1cm} (16)

where $F = \partial_t a - \nabla \varphi$, and $\tilde{\pi}$ is the traceless shear stress tensor that is defined as follows

$$\tilde{\pi} = -\mu_1 \left[\hat{S} - \tilde{\eta} - (\Delta' / \Delta_1^2) \partial_t \tilde{\eta} \times e_z\right].$$  \hspace{1cm} (17)

Similarly, variation of the Lagrangian with respect to $\tilde{\eta}$ yields the equation of motion for the gauge field

$$\partial_t^2 \tilde{\eta} + \Delta_1^2 \tilde{\eta} - \Delta_2 \partial_t \tilde{\eta} \times e_z = \Delta_1^3 \hat{S} + \Delta' \partial_t \hat{S} \times e_z.$$  \hspace{1cm} (18)
The tensor field $\mathbf{\hat{\eta}}$ enters the equation of motion for the displacement vector $\mathbf{u}$ only via the stress tensor $\mathbf{\hat{\pi}}$, Eq. (15). Therefore it is more convenient technically to consider $\mathbf{\hat{\pi}}$ as an independent dynamic variable. Making use of Eqs. (15) and (16) we can eliminate the gauge field $\mathbf{\hat{\eta}}$ in favor of $\mathbf{\hat{\pi}}$, and thus derive the following equation of motion for the shear stress tensor

$$\partial_t^2 \mathbf{\hat{\pi}} + \Delta_1^2 \mathbf{\hat{\pi}} - \Delta_2 \partial_t \mathbf{\hat{\pi}} \times \mathbf{e}_z = -\mu_1 (1 + \Delta'^2/\Delta_1^2) \partial_t^2 \mathbf{\hat{S}}$$

$$- \mu_1 (\Delta_2 + 2\Delta') \partial_t \mathbf{\hat{S}} \times \mathbf{e}_z. \quad (17)$$

Using Eq. (17) we can identify one of the phenomenological parameters entering the theory, namely the parameter $\mu_1$. In the high-frequency regime the equation of motion for the displacement, Eq. (14), should reduce to its asymptotic form, Eq. (2). This means that as $\omega \to \infty$ the solution to Eq. (17) should take the form $\mathbf{\hat{\pi}} = -\mu_\infty \mathbf{\hat{S}}$. In the $\omega \to \infty$ regime the behavior of Eq. (17) is dominated by terms with the highest order of time derivatives, i.e. by the first terms in the right and left hand sides of Eq. (17). Thus the requirement of the elastic high-frequency response fixes parameter $\mu_1$ in the Lagrangian to the value

$$\mu_1 = \mu_\infty (1 + \Delta^2/\Delta_1^2)^{-1}. \quad (18)$$

The system of Eqs. (14), (17) completely determine the dynamics of the system. Using Eqs. (14) and (17) we can calculate the response to any configuration of external fields, and find all collective modes of the system. Let us first analyze the equation of motion for $\mathbf{\hat{\pi}}$, Eq. (17). This equation determines a dynamic “equation of state” that relates the stress tensor $\mathbf{\hat{\pi}}$ to the strain tensor $\mathbf{\hat{S}}$. By symmetry the Fourier component of the shear stress tensor $\mathbf{\hat{\pi}}$ should be representable in the form

$$\mathbf{\hat{\pi}} = -\mu(\omega)\mathbf{\hat{S}} + \imath \omega \Lambda(\omega) \mathbf{\hat{S}} \times \mathbf{e}_z, \quad (19)$$

where $\mu(\omega)$ is the dynamic shear modulus, and $\Lambda(\omega)$ is a “magnetic” modulus that is responsible for a Lorentz shear stress. Substituting the representation of Eq. (19) into Eq. (17) we arrive at the following system of two equations for functions $\mu(\omega)$ and $\Lambda(\omega)$

$$(\omega^2 - \Delta_1^2) \mu + \omega^2 \Delta_2 \Lambda = \omega^2 \mu_\infty, \quad (20)$$

$$(\omega^2 - \Delta_1^2) \Lambda + \Delta_2 \mu = -\mu_\infty \frac{\Delta_2 + 2\Delta'}{1 + \Delta'^2/\Delta_1^2}. \quad (21)$$

where we have used the relation of $\mu_1$ to $\mu_\infty$, Eq. (18). Straightforward solution of Eqs. (20), (21) yields the dynamic moduli $\mu(\omega)$ and $\Lambda(\omega)$ for the theory defined by the Lagrangian Eq. (13)

$$\mu(\omega) = \mu_\infty \omega^2 \left[ \frac{f_-}{\omega^2 - \Delta_2^2} + \frac{f_+}{\omega^2 - \Delta_2^2} \right], \quad (22)$$

$$\Lambda(\omega) = \mu_\infty \left[ \frac{f_- - \Delta_-}{\omega^2 - \Delta_-^2} - \frac{f_+ + \Delta_+}{\omega^2 - \Delta_+^2} \right]. \quad (23)$$

The quantities $\Delta_+$ and $\Delta_-$ are related to parameters $\Delta_1^2$ and $\Delta_2$ in the shear Lagrangian of Eq. (11) as follows

$$\Delta_+ + \Delta_- = \Delta_1^2, \quad \Delta_+ - \Delta_- = \Delta_2 \quad (24)$$

The oscillator strengths $f_{\pm}$ are given by the expressions

$$f_\pm = \frac{\Delta_+ (\Delta_+ + \Delta_-)^2}{(\Delta_+ + \Delta_-)(\Delta_+ - \Delta_- + \Delta_2^2)}, \quad (25)$$

and satisfy the sum rule $f_+ + f_- = 1$.

The frequency dependence of shear modulus $\mu(\omega)$, Eq. (22), agrees with our general expectations. It vanishes at $\omega \to 0$, and approaches $\mu_\infty$ at $\omega \to \infty$. A resonant structure of Eqs. (22) and (23) is responsible for a gapped spectrum of collective excitations (we will see that $\Delta_\pm$ are, in fact, the excitation gaps).

To calculate the response functions we need to substitute the stress tensor, Eq. (19), into the equation for $\mathbf{u}$, Eq. (14), and solve it for given external potentials $\mathbf{a}$ and $\phi$. Below we consider the most interesting limit of a strong magnetic field when the first (acceleration) term in Eqs. (4) and (14) becomes irrelevant. Physically this corresponds the situation with all particles occupying the lowest Landau level (LLL). Formally the intra-LLL dynamics is described by the Lagrangian Eq. (13) [or, equivalently, by the equations of motions Eqs. (14), (17)] in the limit of vanishing bare mass, $m \to 0$. Substituting Eq. (19) into Eq. (14), and neglecting the acceleration term we get the equation that describes the LLL-projected dynamics of the displacement vector $\mathbf{u}(\mathbf{q}, \omega)$

$$- \imath \omega (nB + \Lambda q^2) \mathbf{u} \times \mathbf{e}_z + \mu q^2 \mathbf{u} + K q(\mathbf{q} u) = n \mathbf{F}, \quad (26)$$

where $\mu$ and $\Lambda$ are given by Eqs. (22) and (23), and the force in the right hand side is related to the external potentials as follows, $\mathbf{F} = -\imath \omega \mathbf{a} - \imath q \phi$.

Let us first calculate the Hall conductivity $\sigma_H(\omega, q)$. By definition, $\sigma_H(\omega, q)$ relates the $x$-component of the current to the $y$-component of an electric field,

$$j_x = \sigma_H(\omega, q) E_y$$

Noting that $j = n v = -i\omega n \mathbf{u}$ and solving Eq. (26) with $a = 0$, and $-i q \phi = E$ we find

$$\sigma_H(\omega, q) = \frac{n^2}{\omega^2(nB + \Lambda q^2)^2} \frac{\omega^2(nB + \Lambda q^2)}{(nB + \Lambda q^2)^2 - \mu(\mu + K) q^4}. \quad (27)$$

The dissipationless (equilibrium) Hall conductivity is calculated from Eq. (27) by taking first the limit $\omega \to 0$, and then the limit $q \to 0$. Since $\mu(\omega)$, Eq. (22), vanishes as $\omega^2$, this order of limits gives the famous nonzero result

$$\lim_{q \to 0} \lim_{\omega \to 0} \sigma_H(\omega, q) = \frac{n}{B} = \frac{\nu}{2\pi}. \quad (28)$$

It is worth mentioning that in a normal viscous liquid as well as in a Fermi liquid at $T = 0$ we have $\mu(\omega) \sim i\omega,$
which leads to a vanishing dissipationless Hall conductivity, \( \sigma_H(0, q) = 0 \).

Using Eq. (26) one can easily show that the Hall conductivity also determines the response of the density, \( \delta n = -n \nabla u \), to an external magnetic field \( \mathbf{b} = \nabla \times \mathbf{a} \). Indeed, setting \( a_i(\omega, q) = i\varepsilon_{ij} q_j b_q / q^2 \), and solving Eq. (26) we find

\[
\delta n_q = \sigma_H(\omega, q) b_q
\]

Hence a nonvanishing limit in Eq. (28) implies a creation of a charge \( Q = \nu \Phi / 2\pi \) by an adiabatic insertion of a magnetic flux \( \Phi = b_q = 0 \). By inserting precisely one flux quantum, \( \Phi = 2\pi \), (the Laughlin’s gedanken experiment) we create a fractional charge \( \nu \).

Another important quantity is the density response function \( \chi(\omega, q) \), which relates \( \delta n = -n \nabla u \) to the external scalar potential: \( \delta n = \chi(\omega, q) \varphi \). The solution of the equation of motion, Eq. (26), with \( \mathbf{F} = -i q \varphi \) yields

\[
\chi(\omega, q) = n^2 \frac{\mu q^4}{\omega^2 (nB + \Lambda q^2)^2 - \mu(\mu + K) q^4}.
\]

Inserting Eqs. (22) and (23) into Eq. (29) we find that at \( \omega \to 0 \) and small wave vectors \( \chi \sim q^4 \), which signifies the proper incompressibility of the FQH liquid\(^8\).

It should be stressed out that all abovementioned fundamental low energy properties of FQH liquids are guaranteed by the low frequency behavior of the shear modulus, \( \mu \sim \omega^2 \). In fact, this behavior guarantees that the low energy physics is dominated by the Lorentz force term in the Lagrangian/equations of motion. In other words, in the limit \( \omega \to 0 \) the Lagrangian Eq. (13) reduces to the form

\[
\mathcal{L} = \frac{1}{2} n B (\partial_i u \times \mathbf{e}_z) + n u \partial_j a - n u \nabla \varphi
\]

\[
= -\frac{1}{2} n B \varepsilon_{ij} u_i \partial_j u_j + n u_i \partial_i a_i - n u_i \partial_i \varphi,
\]

which is precisely a Chern-Simons Lagrangian in a temporal gauge. Indeed, considering the standard Chern-Simons theory

\[
L^{CS} = \frac{\nu^{-1}}{4\pi} \varepsilon_{\alpha \beta \gamma} A_\alpha \partial_\beta A_\gamma - \frac{1}{2\pi} \varepsilon_{\alpha \beta \gamma} A_\alpha \partial_\beta a_\gamma,
\]

and setting \( A_\alpha = 2\pi n \varepsilon_{ij} u_j \), \( A_0 = 0 \), and \( a_0 = \varphi \) we recover the low frequency Lagrangian Eq. (30). Thus our theory, defined by the Lagrangian of Eq. (13), smoothly interpolates between two exactly known limiting forms – the low energy Chern-Simons theory, Eq. (31), and the high-frequency elasticity theory, Eq. (3). Below we show that this interpolation predicts a spectrum of bulk collective excitations, which is in excellent agreement with experimental observations. It is worth noting that since in the low frequency limit our theory reduces to the Chern-Simons theory, it should reproduce the gapless low energy chiral edge excitations\(^8\). However, the full theory defined by Eq. (13) is also valid beyond the limit of asymptotically small frequencies. Therefore it should be able to predict a modification of the edge spectrum at higher energies.

C. Spectrum of collective excitations

To find collective modes of the system one has to set \( \mathbf{F} = 0 \) in Eq. (26), and to solve the resulting eigenvalue problem. Alternatively, the frequencies of eigenmodes can be determined from the poles of the density response function \( \chi(\omega, q) \), Eq. (29). The corresponding dispersion equation takes the form

\[
\omega^2 [nB + \Lambda(\omega) q^2] - \mu(\mu + K) q^4 = 0.
\]

Using the dynamic moduli \( \mu(\omega) \), Eq. (22), and \( \Lambda(\omega) \), Eq. (23), and solving Eq. (32) we find that the density response function has only two poles. These poles correspond to two collective modes with frequencies \( \Omega_+(q) \) and \( \Omega_-(q) \). Both modes are gapped and have the following small-\( q \) dispersion

\[
\Omega_{\pm}(q) = \Delta_{\pm} \pm \mu_\infty f_{\pm} q^2 t^2,
\]

where \( \mu_\infty = \mu_{\infty} / n \) is the high frequency shear modulus per particle, and \( t = 1/\sqrt{B} \) is the magnetic length. Since \( f_{\pm} \), Eq. (25), are positive, the dispersions of two collective modes always have opposite curvatures.

The origin of gapped collective modes can be traced back to the dynamics of the tensor gauge field \( \mathbf{\eta} \), which is governed by the shear Lagrangian Eq. (11). In fact, the gaps, \( \Delta_{\pm} \), correspond to eigenvalues of the operator in the left hand side of Eq. (16). The presence of two modes with opposite curvatures of small-\( q \) dispersion relations agrees very well with recent experiments\(^10\). At \( q = 0 \) both gapped modes are purely quadrupole, while at any finite \( q \) they have a mixed dipole-quadrupole character with an increasing dipole component at larger \( q \). Microscopically the two gapped modes are most likely related to a mixture of small \( q \) magneto-roton excitations\(^8\) and two-roton bound states\(^18,19\).

In the limit of small wave vectors the density response function, Eq. (29), takes the form

\[
\chi(\omega, q) = \mu_\infty \left[ \frac{f_-}{\omega^2 - \Omega_-^2(q)} + \frac{f_+}{\omega^2 - \Omega_+^2(q)} \right] q^4 t^4.
\]

Hence the quantities \( f_{\pm} \geq 0 \) determine the physical oscillator strengths for two collective modes. Thus Eqs. (24) and (25) relate all phenomenological parameters of the theory to experimentally measurable quantities.

Strictly speaking, the Lagrangian of Eq. (13) defines only a long wavelength theory. However, the results for collective modes look quite reasonable for any value of \( q \). Indeed, a straightforward solution of Eq. (32) shows that with increase of \( q \) the function \( \Omega_-(q) \) monotonically increases, while \( \Omega_+(q) \) passes through a roton-like minimum, and approaches a constant value at \( q l \to \infty \). Thus an overall behavior of the lower branch, \( \Omega_-(q) \), surprisingly well reproduces a physically expected structure of the magneto-exciton spectrum, at least for the Laughlin fractions, \( \nu = 1/(2k + 1) \).
The most important result of the present theory is a prediction of a double-mode structure of gapped collective excitations. The existence of two gapped modes is a generic fact for almost any set of parameters, $\Delta^2$, $\Delta_2$, and $\Delta'$ (it is somewhat more convenient to fix $\Delta_+$, $\Delta_-$, and $\Delta'$). There is, however, a special, very narrow region in the parameter space where the upper branch of collective excitations disappears, and only one gapped mode with a roton minimum is left. In this region our general construction reduces to a simplified single-mode magnetoelasticity theory of Ref. 9. In the single-mode region the dynamic moduli, $\mu$ and $\Lambda$ take the form\(^9\)

$$\mu(\omega) = \mu_\infty \frac{\omega^2}{\omega^2 - \Delta^2}, \quad \Lambda(\omega) = \mu_\infty \frac{\Delta}{\omega^2 - \Delta^2}. \quad (35)$$

Using Eqs. (22)-(25) we find that expressions of Eq. (35) are recovered in two limiting cases: (1) $\Delta_+ \to \infty$, $\Delta_- = \Delta$, and $\Delta'$ is an arbitrary finite constant; (2) $\Delta_+ = \Delta_- = \Delta' = \Delta$. In this single-mode region of the parameter space an effective shear Lagrangian, Eq. (11), reduces to the following simple form

$$L^\text{liq}_S = -\frac{1}{4\mu_\infty} \text{Tr}(\hat{\Delta} - \hat{\eta})^2 + \frac{\mu_\infty}{4\Delta} \text{Tr}(\partial_t \hat{\eta} \times \mathbf{e}_z).$$

All the rest of the three-dimensional parameter space, $(\Delta_+, \Delta_-, \Delta')$, corresponds to a theory with two gapped collective modes. It has been shown in Ref. 9 that the single-mode magnetoelasticity can be derived from the fermionic Chern-Simons theory\(^{20,21,22,23}\) within the random phase approximation (RPA). In the present section we have demonstrated that the symmetry allows for a more general construction, which leads to a double-mode excitation spectrum. Microscopically this general FQH hydrodynamics should most likely emerge from post-RPA vertex corrections.

III. HYDRODYNAMICS OF QUANTUM HALL CRYSTAL AND LIQUID CRYSTAL PHASES

In Sec. II we derived a hydrodynamic theory of incompressible quantum Hall liquids. In the present section this approach is generalized to non-liquid collective states. The hydrodynamics of FQH liquids was based on two principal assumptions, (i) and (ii). Note that at the long wavelength, i.e., for small $q$, in 2D systems the rotational symmetry is indistinguishable from the hexagonal one (with an accuracy up to $q^2$). Hence in the long wavelength limit the assumption (i) also covers all collective states of hexagonal symmetry. Let us keep the assumption (i), but relax (ii) by a successive reduction of the diffeomorphism symmetry defined by Eq. (1).

A. Hexatic liquid crystal

On the first stage we reduce the general local symmetry of Eq. (1) to the invariance under global shear deformations. The later correspond to transformations Eq. (1) with $\varphi(r)$ being a second order polynomial of $r_i$. Such transformations produce a constant in space correction $\delta \hat{S}$, Eq. (9), to the shear strain tensor. The rotational symmetry combined with the global invariance under the transformations Eq. (1) allow only one term to be added to the shear Lagrangian Eq. (11),

$$\delta L^\text{hex}_S = -\frac{\kappa}{4} \text{Tr}(\partial_t \hat{\eta})(\partial_t \hat{\eta}), \quad (36)$$

where $\kappa$ is a positive constant. The full Lagrangian of a collective state that is invariant with respect to global shear deformations takes the form

$$L^\text{hex} = L_0 + L^\text{liq}_S + \delta L^\text{hex}_S + \mathbf{u}\partial_t \mathbf{a} - \mathbf{u} \nabla \varphi. \quad (37)$$

Apparently an additional term, $\delta L^\text{hex}_S$, in the Lagrangian influences only the equation of motion for the gauge field $\hat{\eta}$. Hence the equation for the displacement vector, Eq. (14), as well as the relation of Eq.(15) remain unchanged. Only the equation of motion for the shear stress tensor $\hat{\pi}$ gets modified and acquires gradient corrections

$$\partial_t \hat{\pi} + \hat{\Delta} \left[ 1 - \frac{\kappa}{\mu_1} \nabla^2 \right] \hat{\pi} - \Delta_2 \partial_t \hat{\pi} \times \mathbf{e}_z = -\mu_\infty \partial^2_t \hat{S}$$

$$\quad + \Delta_2^2 \kappa \nabla^2 \hat{S} - \mu_1 (\Delta_2 + 2 \Delta') \partial_t \hat{S} \times \mathbf{e}_z \quad (38)$$

[we note that Eq. (18) still holds, which follows from the universality of the $\omega \to \infty$ asymptotics].

The rotational, and the time-reversal symmetries require\(^9\) the solution of Eq. (38) to be of the general form given by Eq. (19). Hence the strong field ($m \to 0$) expressions for the Hall conductivity and for the density response function, Eqs. (27) and (29), as well as the dispersion equation, Eq. (32), do not change. The analytic structure of the elastic moduli $\mu$ and $\Lambda$ is, however, different. Due to the gradient terms in Eq. (38) the dynamics moduli become functions of both the frequency $\omega$ and the wave vector $q$.

The most important effect of the gradient term, $\delta L^\text{hex}_S$, in the Lagrangian of Eq. (37) is a nonzero value of the shear modulus $\mu(\omega, q)$ at $q = 0$ and finite $q$. Indeed, neglecting all time derivatives in Eq. (38) we get the following static equation of state

$$\hat{\pi} = -\frac{\kappa q^2}{1 + \kappa q^2/\mu_1} \hat{S} = -\mu(0, q) \hat{S}.$$ 

It the long wavelength limit the static shear modulus $\mu_0(q)$ takes the form

$$\mu_0(q) = \mu(0, q) \approx \kappa q^2 \quad (39)$$

(by construction the high-frequency limit of $\mu(\omega, q)$ is fixed to the value $\mu_\infty$). The static shear modulus Eq. (39) vanishes in the limit $q \to 0$. Therefore the state described by the Lagrangian Eq. (37) can not be a solid. To identify the nature of this state we calculate the equal-time correlation function of the orientational order parameter
\[ \Psi(r) = e^{i\nabla \times u(r)} \] At finite temperature \( T \) the behavior of the correlation function \( \langle \Psi^*(r) \Psi(0) \rangle \) at \( r \to \infty \) is determined by the static part of the Lagrangian \( L^{\text{hex}} \). Using an explicit form of Eq. (37) (without external fields), and performing the standard calculations we arrive at the following result

\[ \langle \Psi^*(r) \Psi(0) \rangle \sim \exp \left( -\frac{q^2}{\pi \sigma_{\text{g}}} \right). \] (40)

An algebraic decay of the correlator Eq. (40) is a clear signature of a hexatic liquid crystal.24

A nonzero static shear modulus \( \mu_0 \) of hexatic state dramatically changes the low energy physics. Using the general formulas for the response functions, Eqs. (27) and (29), and the limiting form of the shear modulus, Eq. (39), we get

\[ \lim_{q \to 0} \lim_{\omega \to 0} \sigma_H(\omega, q) = 0, \quad \lim_{q \to 0} \lim_{\omega \to 0} \chi(\omega, q) = -\frac{n^2}{K}. \] (41)

Thus violation of the local symmetry of Eq. (1) destroys the dissipationless quantum Hall effect, and recovers the standard form of the compressibility sum rule.25

The low energy behavior of the response functions, Eq. (41), is closely related to a dramatic modification of the structure of collective modes. In addition to two gapped modes, which remain practically unchanged in comparison with Eq. (33), we find one more nontrivial gapless solution to the dispersion equation, Eq. (32). The frequency of the gapless mode at small \( q \) is readily obtained by inserting the static shear modulus \( \mu_0 \), Eq. (39), into Eq. (32):

\[ \Omega_0 = n^{-1} \sqrt{\mu_0 (\mu_0 + K)} q^2 \]

In a system with Coulomb interaction the small-\( q \) behavior of the bulk modulus is dominated by the Hartree contribution, \( K \approx 2\pi n^2 / q \). In this case the dispersion of the gapless mode in the hexatic phase takes the form

\[ \Omega_0^{\text{hex}}(q) = B^{-1} \sqrt{2\pi \epsilon a} q^5 / 2 \]

(for a short range interaction we get \( \Omega_0^{\text{hex}} \sim q^3 \)). Interestingly, the above dispersion law is similar to that predicted for a Goldstone mode in a quantum Hall nematic phase.26

Recently a number of microscopic trial wave functions for different quantum Hall liquid crystal states have been proposed.27 The single mode approximation (SMA) applied to the hexatic wave function yields a gapped collective mode with a roton minimum. This behavior excellently correlates with our results for the lower gapped mode, \( \Omega_{-}(q) \) – the dispersion of this mode has practically the same form both for hexatic and for a liquid states. The microscopic SMA fails to produce a gapless mode that should be present in liquid crystal phases. In contrast, our approach simultaneously predicts both the gapped modes and the dispersion of the Goldstone mode for the quantum Hall hexatic liquid crystal.

### B. Crystalline states

Finally we completely destroy any kind of invariance under shear deformations. The most radical way to do that is to drop out the gauge field \( \hat{\eta} \) by setting \( \Delta_1, \Delta_2 \to 0 \) in Eq. (11). The resulting Lagrangian is simply \( L^\infty \), Eq. (3). It describes the long wavelength dynamics of a hexagonal 2D solid with a frequency independent shear modulus \( \mu = \mu_\infty \). Apparently this state corresponds to a Wigner crystal. In the limit of a strong magnetic field \( (m \to 0) \) we get only one collective mode that is a well known magneto-phonon with the dispersion

\[ \Omega_0^W = B^{-1} \sqrt{2\pi \mu_\infty q^3 / 2}. \]

A more delicate way to violate the symmetry of Eq. (1) is to add to Eq. (37) the last invariant, \( \text{Tr}(\hat{\eta})^2 \), that is allowed by the rotational and the time-reversal symmetries. The resulting total Lagrangian takes the form

\[ L^{\text{bub}} = L^{\text{hex}} - \frac{1}{4} \mu_2 \text{Tr}(\hat{\eta})^2, \] (42)

where \( L^{\text{hex}} \) is defined after Eq. (37), and \( \mu_2 \) is an additional phenomenological constant. The second term in Eq. (42) leads to the following simple modification of the equation of motion for the shear stress tensor \( \hat{\sigma} \): in Eq. (38) the term \( -K \nabla^2 \) is replaced by the combination \( -K \nabla^2 + \mu_2 \). Noting this fact we easily realize that the Lagrangian of Eq. (42) describes a crystal state with a constant static shear modulus of the form

\[ \mu_0 = \mu(0, 0) = \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}, \]

where \( \mu_1 \) is related to \( \mu_\infty \) by Eq. (18).

A nonzero value of \( \mu_0 \) leads to a gapless magneto-phonon mode with a small-\( q \) dispersion similar to that for a classical Wigner crystal

\[ \Omega_0^{\text{bub}} = B^{-1} \sqrt{2\pi \mu_0 q^3 / 2}. \]

In addition, the dynamics of the gauge field \( \hat{\eta} \), which is encoded in the equation of motion for the stress tensor \( \hat{\sigma} \), still produce two FQH liquid-like gapped modes. The corresponding \( q = 0 \) gaps, \( \Delta_{+}^{\text{bub}} \) and \( \Delta_{-}^{\text{bub}} \), are related to parameters of the Lagrangian as follows

\[ \Delta_+^{\text{bub}} = \Delta_-^{\text{bub}} = \Delta_1^2 \left( 1 + \frac{\mu_2}{\mu_1} \right), \quad \Delta_+^{\text{bub}} - \Delta_-^{\text{bub}} = \Delta_2 \]

It is natural to interpret this intermediate state as a triangular lattice of liquid “bubbles”. The described structure of collective modes is indeed in a qualitative agreement with the Hartree-Fock excitation spectrum of a quantum Hall bubble phase.28

### IV. Conclusion

In conclusion we proposed a unified hydrodynamical framework for the description of collective quantum Hall
states. In particular, the present theory covers incompressible FQH liquids, hexatic liquid crystals, and two different hexagonal crystal phases that can be identified as a Wigner crystal, and a bubble solid. We predicted the dispersion of the Goldstone mode for quantum Hall hexatics, $\Omega_{\text{hex}}^0 \sim q^{5/2}$, and the presence of two gapped modes for incompressible quantum Hall liquids. The later result naturally explains recent experimental observations of the long wavelength dispersion of collective modes in the $\nu = 1/3$ FQH state\textsuperscript{10}.

In short, the hydrodynamics of FQH liquids is based on the following observations. On the one hand, the high-frequency response of any system is universally elastic. On the other hand, a liquid state is insensitive to static volume preserving deformations. The former requirement fixes the high-frequency form of the theory, while the later one implies a particular local symmetry which can be implemented using a proper gauge field. Since our approach relies only on symmetry, we believe that the predicted two gapped modes should be a universal property of a 2D collective magnetized liquid.

Acknowledgment

I am grateful to G. Vignale for the encouragement and for numerous illuminating discussions.