Stationary state solutions for a gently stochastic nonlinear wave equation with ultraviolet cutoffs

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Abstract

We consider a non-linear, one-dimensional wave equation system with finite-dimensional stochastic driving terms and with weak dissipation. A stationary process that solves the system is used to model steady-state non-equilibrium heat flow through a non-linear medium. We show existence and uniqueness of invariant measures for the system modified with ultraviolet cutoffs, and we obtain estimates for the field covariances with respect to these measures, estimates that are uniform in the cutoffs. Finally, we discuss the limit of these measures as the ultraviolet cutoffs are removed.

PACS numbers: 44.10.+i, 05.70.Ln, 05.10.Gg

Key words: Non-equilibrium statistical mechanics, stationary states.

1 Introduction

In this article, we consider a dynamical system of equations for a scalar field in one space dimension with ultraviolet cutoff, where the equations have a non-linear term, dissipative terms, and stochastic driving terms. We will be concerned with the problems of constructing measures on the space of field configurations invariant under the time evolution of the system, and of estimating the covariance of the field with respect to these measures.

This system of equations provides a model for heat flow; the system is a variant of models introduced and examined in detail by Eckmann, Pillet, and Rey-Bellet ([10, 11], see also [24, 25, 26, 7, 22]). These authors considered a finite system of non-linear oscillators coupled to two (or more) free fields that are governed by linear wave equations. The free fields, which model heat reservoirs, are given Gaussian-distributed random initial conditions (Gibbs states), but in general at different temperatures $T_i$ for each of the free fields. The equations of motion for the free fields are readily integrated, leaving a system of stochastic equations for the oscillators. For suitable couplings, these authors found that the equations were Markovian stochastic differential equations. The equations given below, then, are simply the analogue of these stochastic differential equations, but with the oscillators replaced by a non-linear scalar field in one space dimension with bounded support. We examine these equations with an ultraviolet cutoff, i.e. where Fourier modes of the field are set to zero for mode numbers $\{n\}; n > M$, and we consider the limiting situation where the cutoff $M$ is removed as $M \to \infty$.

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The equations of motion for the system are given by
\[
\begin{align*}
\partial_t \phi(x,t) &= \pi(x,t) \\
\partial_t \pi(x,t) &= (\partial^2_x - 1)\phi(x,t) - g(\phi(x,t)) - r(t)\alpha(x) \\
\mathrm{d}r(t) &= -(r(t) - (\alpha, \pi(t)))\,dt + \sqrt{T}\,d\omega(t).
\end{align*}
\] (1.1)

Here \((\phi, \pi)(x,t)\) is a pair of scalar fields; we assume, for convenience, that they satisfy periodic boundary conditions for \(x \in [0,2\pi]\). The fixed vector-valued function (or distribution) \(\alpha = (\alpha_1(x), \alpha_2(x))\) has two components, each in a Sobolev space \(H_s\), that will be further specified below. The (dependent) random vector \(r(t) = (r_1(t), r_2(t))\) takes values in \(\mathbb{R}^2\); its components are artifacts of the reservoirs. The \(\alpha_i\) couple the oscillators to the reservoirs, and here \(r(t)\alpha = \sum_{i=1}^2 r_i(t)\alpha_i(x)\). The quantity \((\alpha_i, \pi)\) is simply the \(L^2([0,2\pi], dx)\)-inner product of \(\alpha_i(x)\) and \(\pi(x,t)\). The driving term \(\omega(t) = (\omega_1, \omega_2)(t)\) is standard two-dimensional Brownian motion and the \(T_i\), \(i = 1,2\), are the reservoir temperatures. The non-linearity \(g\) is assumed to be an odd, bounded, Lipschitz-continuous function.

The overarching goal of the study of these equations is to construct a stationary (non-equilibrium) process \(\Phi(t) = (\phi(t), \pi(t), r(t))\) governed by the above equations and having physically reasonable sample path properties. In particular, the field component \(\phi = \phi(x,t)\) should be at least square integrable in \(x\) a.s., or actually continuous in \(x\) a.s. as in the equilibrium case \(T_1 = T_2\) and in the non-equilibrium but linear \(g = 0\) case, all of which we review below.

In this article, we establish the existence and uniqueness of stationary states for these equations with ultraviolet cutoff \(M\), i.e. such that Fourier modes of the field \(\phi\) and its momentum \(\pi\) are set to zero for mode numbers \(|n| > M\). We also obtain estimates on the field covariance uniform in the cutoff with respect to the stationary cutoff measures. For certain choices of the coupling functions \(\alpha\), we show that \(\phi\) is indeed \(L^2\)-integrable a.s., and that \(\|\phi\|_2\) has variance uniformly bounded in \(M\). We believe that these estimates are an important step in establishing bounds on the mode variances uniform in both the mode number \(n\) and the cutoff \(M\), as one might expect on physical grounds, an issue that we will address in subsequent work. We conclude with a discussion of the limit of the cutoff measures, \(M \to \infty\).

It is known that in equilibrium, temperatures \(T_1 = T_2\), and even for a non-focusing unbounded nonlinearity \(g(x) \propto x^3\), there is an invariant measure for the system (essentially a Gibbs state) which is supported on field configurations \(\{\phi\}\) that are almost surely continuous in \(x\) (they are Brownian motion-like, in fact, Hölder continuous with any index \(< 1/2\)). The measure is absolutely continuous with respect to the \(g = 0\) Gaussian measure (see [27]). For the non-equilibrium case \(T_1 \neq T_2\) but with \(g = 0\), there is steady-state energy flow, i.e., there is a random variable measuring heat flow through any fixed point \(x \in [0,2\pi]\), and with respect to this measure, the expectation of this heat flow is non-zero. Moreover, the measure has support properties similar to those of the equilibrium case (see [30]). But for the non-linear \(g \neq 0\) non-equilibrium case, the regularity of the field is at present not known. If the above equations are to serve as a physical model for a non-linear vibrating string in a non-equilibrium stationary state, we expect the measure to have support properties on field configurations similar to those in the equilibrium case. In particular, we expect the field configurations of \(\phi\) to be a.s. continuous, not shattered. It is really this latter problem–
establishing the regularity of the field with respect to the invariant measure— that is our primary concern in this article and its sequel. The first step in this program is to obtain estimates on the field covariance uniform in the ultraviolet cutoff.

We thus consider ultraviolet cutoff versions of the system (1.1) where we retain the Fourier modes \( \{ \hat{\phi}(n) \} \) for the field \( \phi \) and \( \{ \hat{\pi}(n) \} \) for momentum \( \pi \) for \( |n| \leq M \) with \( M \) a positive integer. Let \( P_{\leq M} \) be projection onto the Fourier modes \( \{ n \} \), with \( |n| \leq M \). Then setting \( \Phi_M = (\phi_M, \pi_M, r_M)^T \) as the cutoff field having non-zero modes only for \( |n| \leq M \), \( \alpha_{M,i} = P_M \alpha_i \), we have that our equations can be written in matrix form as

\[
d\Phi_M = (A_M \Phi_M - G_M(\Phi_M)) dt + \sqrt{T} d\omega, \tag{1.2}
\]

where

\[
A_M \equiv \begin{pmatrix}
0 & 1 & 0 & 0 \\
\partial_x^2 - 1 & 0 & -\alpha_{M,1}(x) & -\alpha_{M,2}(x) \\
0 & \langle \alpha_{M,1} \rangle & -1 & 0 \\
0 & \langle \alpha_{M,2} \rangle & 0 & -1
\end{pmatrix}, \tag{1.3}
\]

with \( \langle \alpha_{M,i} \rangle \) the linear functional of integration against \( \alpha_{M,i} \), and with

\[- G_M(\Phi_M) \equiv \begin{pmatrix}
0 \\
-P_{\leq M} g(\phi_M(x,t)) \\
0 \\
0
\end{pmatrix} \quad \text{and} \quad \sqrt{T} d\omega \equiv \begin{pmatrix}
0 \\
\sqrt{T_1} d\omega_1(t) \\
0 \\
\sqrt{T_2} d\omega_2(t)
\end{pmatrix}. \tag{1.4}\]

We make the following assumptions about the coupling functions and the nonlinearity \( g \) for the system of equations (1.1) and (1.2): 

**Assumption 1.1.** The coupling functions \( \alpha_i \), \( i = 1, 2 \) are real, with all of their Fourier coefficients \( \{ \hat{\alpha}_i(n) \} \) non-zero. There exists a positive constant \( c_1 \) (independent of \( n \)) such that

\[
c_1(|\hat{\alpha}_1(n)|^2 + |\hat{\alpha}_2(n)|^2) \leq |\hat{\alpha}_1(n)|^2 + \hat{\alpha}_2(n)^2|, \tag{1.5}
\]

and there exist both a positive constant \( c_2 \) and \( \theta \), \( -1/2 < \theta < 1/2 \), such that, for all \( n \),

\[
c_2 (n^2 + 1)^{\theta/2} \leq |\hat{\alpha}_i(n)| \leq \frac{1}{c_2} (n^2 + 1)^{\theta/2}. \tag{1.6}
\]

The non-linearity \( g \) is assumed to be bounded and uniformly Lipschitz.

The assumptions on the coupling functions are made in order to assure that the perturbed eigenvalues of \( A \) and \( A_M \) are non-degenerate and have negative real parts. In our model, the dissipation of large \( n \) modes is very weak, with rate \( \sim -n^{2\theta-2} \). It is thus important that the coupling functions themselves have Fourier coefficients large enough so that the dissipation is not overwhelmed by the noise driving these modes, and at the same time not so large that the equations of motion make no sense as the ultraviolet cutoff is removed. The assumptions on the \( \alpha \)'s above appear to capture the right growth/decay rates in the \( \hat{\alpha}(n) \)'s.
The problem of constructing an invariant measure for this gently stochastic wave equation may be contrasted with the problem of constructing stationary states for stochastic parabolic and even 2-dimensional stochastic Navier-Stokes equations with viscosity, where the dissipation of large $n$ Fourier modes of the field is strong (for Navier-Stokes, see [4, 20, 12, 14, 17]). For a heat equation the $n^{th}$ mode decays at a rate $\sim -n^2$, compared with our rate $\sim -n^{2\theta-2}$ mentioned above. In [11], an invariant measure is obtained for a non-linear wave equation with cylindrical Brownian motion driving terms but with strong dissipation for all Fourier modes. In [27], an invariant measure is constructed for a one-dimensional stochastic non-linear Klein-Gordon field but at thermal equilibrium, with weak dissipation at high Fourier modes; see also [30] where an invariant measure for the non-equilibrium but linear case is given as an invariant measure for an Ornstein-Uhlenbeck process. See also [18], for an invariant measure for a harmonic crystal in non-equilibrium.

For invariant measures for wave equations in equilibrium but with no stochastic driving terms, i.e., equilibrium statistical mechanics for wave equations, see for example [21, 31], and for the non-linear Schrödinger equation, see [2, 5, 19].

In the following section, we show existence and uniqueness of invariant measures for the ultraviolet cutoff systems, and in section 3 we obtain bounds on the field covariance modes with respect to these invariant measures. In section 4, we give a general discussion of tightness for these cutoff measures. Much of our analysis will use perturbation theory for linear operators, with estimates on the tails of the relevant perturbation series. The appendix summarizes these perturbation calculations.

2 Existence and Uniqueness of the Stationary Measures for Systems with Ultraviolet Cutoff

In the following, let $\mathcal{H}$ be the Hilbert space of complex-valued functions $\{\Phi = (\phi, \pi, r)^T\}$ equipped with inner product

$$\langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = \int_0^{2\pi} \left( \partial_x \phi_1 \partial_x \phi_2^* + \phi_1 \phi_2^* + \pi_1 \pi_2^* \right) dx + r_1 \cdot r_2^*, \quad (2.1)$$

and let $\| \cdot \|_{\mathcal{H}}$ denote the corresponding norm for $\mathcal{H}$.

The solution $\Phi_M(t) = \Phi_M(t; \Psi)$ to Equation (1.2) with initial data $\Phi_M(0) = \Psi$ satisfies the Duhamel integral equation

$$\Phi_M(t) = -\int_0^t e^{(t-s)A_M} G_M(\Phi_M(s)) ds + \int_0^t e^{(t-s)A_M} \sqrt{T} d\omega(s) + e^{tA_M} \Psi, \quad (2.2)$$

(see [29], chapter 5). The field $\Phi_M(t)$ is a vector in the $(4M + 4)$-dimensional subspace $\mathcal{H}_M$ of $\mathcal{H}$ spanned by $(2M + 1)$ Fourier modes each for $\phi$ and $\pi$, and by the two $r$ modes. Using $\Phi_M(t)$, one defines a probability semigroup $P^t_M$ acting on bounded Borel functions on $\mathbb{R}^{4M+4}$ via

$$P^t_M f(\Psi) = E[f(\Phi_M(t; \Psi))] \quad (2.3)$$
with $E[\cdot]$ the Brownian motion expectation. The operator $P^t_M$ is a Feller semigroup, i.e., $P^t_M f(\Psi)$ is continuous in $\Psi$ if $f$ is continuous which follows from the integral equation above for $\Phi_M(t)$ and the assumption that $g$ is Lipschitz.

**Proposition 2.1.** There exists an invariant measure for the semigroup $P^t_M$.

**Proof:** To show the existence of an invariant measure it is sufficient to prove that the family of measures $\{\mu_{M,T}\}$ defined by

$$\left\{ \int f d\mu_{M,T} = \frac{1}{T} \int_0^T P^t_M f(\Psi) dt \right\}_{T > 0}$$

is tight on the space $\mathcal{H}_M$, for $f$'s bounded and continuous. Since our system is finite dimensional, one need only show that for each $\Psi \in \mathcal{H}_M$, and $\epsilon > 0$, there exist an $R > 0$ such that for time $T$ sufficiently large,

$$\frac{1}{T} \int_0^T P(\|\Phi_M(t; \Psi)\|_{\mathcal{H}} \geq R) dt < \epsilon. \quad (2.5)$$

By the Markov inequality, we have

$$P(\|\Phi_M(t; \Psi)\|_{\mathcal{H}} \geq R) \leq \frac{1}{R} E \left[ \|\Phi_M(t; \Psi)\|_{\mathcal{H}} \right], \quad (2.6)$$

so it suffices to show that $E[\|\Phi_M(t, \Psi)\|_{\mathcal{H}}]$ is bounded uniformly in $t$. But from (2.2),

$$E \left[ \|\Phi_M(t; \Psi)\|_{\mathcal{H}} \right] \leq E \left[ \left\| \int_0^t e^{(t-s)A_M} (G_M(\Phi_M(s))) ds \right\|_{\mathcal{H}} \right] + E \left[ \left\| \int_0^t e^{(t-s)A_M} \sqrt{T} d\omega(s) \right\|_{\mathcal{H}} \right] + \|e^{tA_M} \Psi\|_{\mathcal{H}}. \quad (2.7)$$

Now the unperturbed matrix operator $B_M$ obtained from $A_M$ by setting the coupling $\alpha$'s to zero has spectrum consisting of a doubly degenerate eigenvalue $\lambda_{-1}(0) = -1$, simple eigenvalues $\lambda_{\pm 0}(0) = \pm i$, and doubly degenerate eigenvalues $\lambda_{\pm n}(0) = \pm i\sqrt{n^2 + 1}$, $1 \leq n \leq M$. As seen in perturbation theory, the matrix $A_M$ has eigenvalues near those of $B_M$, but with small negative real parts (see the Appendix, Lemma (A.1), Eq.(A.7)).

Given these eigenvalue estimates, it is then easy to see that the first and third terms on the right side of (2.7) are bounded ($G_M$ is bounded) uniformly in $t$. The second term involving the Itô integral is estimated via Itô calculus,

$$E \left[ \left\| \int_0^t e^{(t-s)A_M} \sqrt{T} d\omega(s) \right\|_{\mathcal{H}}^2 \right] \leq E \left[ \left\| \int_0^t e^{(t-s)A_M} \sqrt{T} d\omega(s) \right\|_{\mathcal{H}}^2 \right] = \text{Tr}_{\mathcal{H}} \int_0^t e^{sA_M} T e^{sA_M^*} ds, \quad (2.8)$$

which is uniformly bounded in $t$, again by the eigenvalue estimates for $A_M$. $\blacksquare$
Remark: The above argument can also be used to show boundedness of higher moments of the field $\Phi_M$ with respect to the invariant measure.

We next proceed to showing uniqueness of the invariant measure $\mu_M$ satisfying $\mu_M P_M^t = \mu_M$ for the semigroup $P_M^t$.

**Proposition 2.2.** Under the assumptions (1.1) on the coupling functions $\alpha$ and nonlinearity $g$, the invariant measure $\mu_M$ for $P_M^t$ is unique.

**Proof:** In order to show uniqueness of the invariant measure, it suffices to show that the system in which the random field $\Phi_M(t)$ is replaced by a deterministic function $X_M(t) \in \mathcal{H}_M$ satisfying
\[
dX_M(t) = (A_MX_M(t) - G_M(X_M(t)))dt + \sqrt{T}u(t)dt
\]
is controllable. This means that, given $X_0$, $X_1$, and a time $t_1$, one can find a smooth control $u(t) = (u_1(t), u_2(t))$ such that $X_M(t)$ is a solution to this differential equation, and $X_M(0) = X_0$, $X_M(t_1) = X_1$ (see [6], page 144). But controllability of this system is equivalent to controllability of the simpler system
\[
dX_M(t) = (\tilde{A}_M X_M(t) - G_M(X_M(t)))dt + \sqrt{T}u(t)dt, \tag{2.10}
\]
where the $dr_M(t)$-equations are replaced by $dr_M = \sqrt{T}u(t)dt$, i.e., the new and the old controls are related by
\[
\sqrt{T}u = -(r_M(t) - \beta\langle \alpha_M, \pi_M(t) \rangle) + \sqrt{T}u(t), \tag{2.11}
\]
and the modified matrix $\tilde{A}_M$ is obtained from $A_M$ by replacing its bottom two rows by zero rows. The solution of equation (2.10) satisfies
\[
X_M(t) = - \int_0^t e^{(t-s)\tilde{A}_M}G_M(X_M(s))ds + \int_0^t e^{(t-s)\tilde{A}_M}\sqrt{T}u(s)ds + e^{t\tilde{A}_M}X_0. \tag{2.12}
\]

We will show the controllability by first replacing $X_M(s)$ in the first integral on the right side of this last equation (2.12) by an arbitrary continuous function $Z_M(s)$, thereby obtaining an expression for $u(t)$ explicitly in terms of $Z_M(s)$, and then by using a fixed point theorem. We thus consider
\[
X_M(t) = - \int_0^t e^{(t-s)\tilde{A}_M}G_M(Z_M(s))ds + \int_0^t e^{(t-s)\tilde{A}_M}\sqrt{T}u(s)ds + e^{t\tilde{A}_M}X_0, \tag{2.13}
\]
for which we seek a control $u(t)$ such that for fixed $Z_M(s)$, $X_M(t_1) = X_1$.

Let
\[
D_t = \int_0^t e^{-s\tilde{A}_M}Te^{-s\tilde{A}_M}ds, \tag{2.14}
\]

**Lemma 2.3.** The matrix $D_t$ is a non-negative self-adjoint invertible $(4M+4) \times (4M+4)$ matrix for $t > 0$, provided the Fourier coefficients of $\alpha_1$ and $\alpha_2$ satisfy the assumption Eq. (1.3).
Proof: Clearly, the integrand of the integral in (2.14) is non-negative and continuous in $s$. If $D_t$ were not invertible, then for some $X \neq 0$, $\langle X, D_t X \rangle_{\mathcal{H}} = 0$, and in particular $\sqrt{\mathbf{1}} e^{-s\hat{A}_M^\dagger} X = 0$ for all $0 \leq s \leq t$. Since $\sqrt{\mathbf{1}} e^{-s\hat{A}_M^\dagger} X$ is smooth in $s$, we have

$$\frac{d^n}{ds^n} \left( \sqrt{\mathbf{1}} e^{-s\hat{A}_M^\dagger} X \right) \bigg|_{s=0} = 0 \text{ for all } n, \text{ so } \sqrt{\mathbf{1}} \hat{A}_M^\dagger X = 0, \ n = 0, 1, 2, \ldots.$$ However we know that for any $Y \in \mathcal{H}_M$, $\langle Y, \sqrt{\mathbf{1}} \hat{A}_M^\dagger X \rangle_{\mathcal{H}} = 0$, which implies that

$$\langle (\hat{A}_M^n)^\dagger \sqrt{\mathbf{1}} Y, X \rangle_{\mathcal{H}} = 0$$

for all $Y$, and thus the range of $\{(\hat{A}_M^n)^\dagger \sqrt{\mathbf{1}}\}^\infty_{n=0}$ is not the whole space $\mathcal{H}_M$. But this contradicts the following claim.

Claim 2.4. Vectors consisting of the columns of $\{(\hat{A}_M^n)^\dagger \sqrt{\mathbf{1}}\}^\infty_{n=0} n = 0, 1, 2, \ldots$ span $\mathcal{H}_M$.

Remark: The generator of the Ornstein-Uhlenbeck semigroup defined by the linear equation $d\Phi_M(t) = \hat{A}_M\Phi_M(t)dt + \sqrt{T}dw$ is

$$\mathcal{G} = T_1 \frac{\partial^2}{\partial r_1^2} + T_2 \frac{\partial^2}{\partial r_2^2} + \int dx \pi_M \frac{\delta}{\delta \phi_M}$$

$$+ \int dx ((\partial_r^2 - 1)\phi_M + \beta r\phi_M) \frac{\delta}{\delta \pi_M}. \quad (2.16)$$

If we denote

$$X_1 = \frac{\partial}{\partial r_1}, \ X_2 = \frac{\partial}{\partial r_2}, \text{ and }$$

$$Y = \int dx \left( \pi_M \frac{\delta}{\delta \phi_M} + ((\partial_r^2 - 1)\phi_M + \beta r \phi_M) \frac{\delta}{\delta \pi_M} \right), \quad (2.17)$$

then the conclusion in claim (2.4) is equivalent to the condition that the Lie algebra generated by the vector fields $X_1, X_2, \text{ and } Y$ has full rank at each point of $\mathbb{R}^{4M+4}$, and this implies that $\mathcal{G}$ is hypoelliptic (See [15]).

Proof of Claim: Proving that the columns of $\{(\hat{A}_M^n)^\dagger \sqrt{\mathbf{1}}\}^\infty_{n=0}$ span $\mathcal{H}_M$ amounts to showing, in the end, that $\{P_{\leq M}(\partial_x^2 - 1)^k \alpha_1 \}_{k=0}^\infty$ and $\{P_{\leq M}(\partial_x^2 - 1)^k \alpha_2 \}_{k=0}^\infty$ span $P_{\leq M}L^2[0,2\pi]$. They do so provided that all Fourier coefficients $\hat{\alpha}_1(m)$ and $\hat{\alpha}_2(m)$ of the $\alpha$’s are nonzero and that $(\hat{\alpha}_1(m), \hat{\alpha}_1(-m))$ and $(\hat{\alpha}_2(m), \hat{\alpha}_2(-m))$ are linearly independent, $m \neq 0$. But this is implied by the assumption Eq. (1.5). This concludes the proof of the claim and hence the Lemma (2.3).

Returning now to the proof of Proposition (2.2), we identify $u(t)$ with the column vector having the four components $(0, 0, u_1(t), u_2(t))$, and we set

$$u(t) = -\sqrt{\mathbf{1}} e^{-t\hat{A}_M^\dagger} D_{t_1}^{-1} \times$$

$$\left( X_0 - \int_0^{t_1} e^{-s\hat{A}_M} G_M(Z_M(s)) ds - e^{-t_1\hat{A}_M} X_1 \right). \quad (2.18)$$
Putting \( u(t) \) into equation (2.13), we have

\[
X_M(t) = -\int_0^t e^{(t-s)\tilde{A}_M} G_M(Z_M(s))ds \\
- e^{t\tilde{A}_M} D_1 D_1^{-1} \left( X_0 - \mu \int_0^{t_1} e^{-s'\tilde{A}_M} G_M(Z_M(s'))ds' - e^{-t_1\tilde{A}_M} X_1 \right) \\
+ e^{t\tilde{A}_M} X_0.
\]

(2.19)

It is then easy to see from this equation that \( X_M(t_1) = X_1 \).

We consider the right side of this last equation (2.19) as a nonlinear operator \( K(Z_M(\cdot)) \) which maps the Banach space \( C_{[0,t_1]}(H_M) \) (continuous functions mapping \([0,t_1]\) to \(H_M)\) into itself, with \( X_M(t) = K(Z_M(t)) \). Let \( \Theta = \{ Z \in C_{[0,t_1]}(H_M) : \|Z(t)\|_{\infty} \leq C \} \) with \( C \) a uniform bound of the right hand side of (2.19). Then \( \Theta \) is closed and convex. Denote the range of the operator \( K \) by \( \Omega \); then \( \Omega \subset \Theta \), and functions in \( \Omega \) are equicontinuous in \( t \) (\( G_M \) is bounded), so \( \Omega \) is compact by the Arzela-Ascoli theorem. By Schauder’s fixed point theorem, (23, page 151), \( K \) has a fixed point \( X_M^*(t) \) such that \( X_M^*(t) = K(X_M^*(t)) \), which satisfies the conditions \( X_M^*(0) = X_0 \) and \( X_M(t_1) = X_1 \). Here \( X_M^* \) is a solution of Eq. (2.13) with control \( u(t) \) given by Eq. (2.18). This concludes the proof of the controllability and hence the uniqueness of the invariant measure, Proposition (2.2). \( \blacksquare \)

3 Estimates on the covariance of the field in the stationary state

Let \( \Phi_M(x,t) = (\phi_M(x,t), \pi_M(x,t), r_{1,M}(t), r_{2,M}(t))^T \) be the stationary field corresponding to the M-cutoff stochastic differential equation (1.2) with invariant measure \( \mu_M \). We write \( f_{M,n,\sigma}^\pm = (f_{M,n,\sigma,\phi}^\pm, f_{M,n,\sigma,\pi}^\pm, f_{M,n,\sigma,r}^\pm) \) as a left eigenvector for \( A_M \), \( f_{M,n,\sigma}^\pm A_M = \lambda_{M,n,\sigma}^\pm \), expressing \( \phi_{M,n,\sigma}^\pm \) in its components, and similarly we write \( e_{M,n,\sigma}^\pm = (e_{M,n,\sigma,\phi}^\pm, e_{M,n,\sigma,\pi}^\pm, e_{M,n,\sigma,r}^\pm)^T \) for the column components of the right eigenvector corresponding to the same eigenvalue \( \lambda_{M,n,\sigma}^\pm \). The eigenvectors are normalized so that

\[
\langle f_{M,n,\sigma}^\pm, e_{M,n,\sigma}^\pm \rangle = \int dx \left( f_{M,n,\sigma,\phi}^\pm(x) e_{M,n,\sigma,\phi}^\pm(x) + f_{M,n,\sigma,\pi}^\pm(x) e_{M,n,\sigma,\pi}^\pm(x) + f_{M,n,\sigma,r}^\pm e_{M,n,\sigma,r}^\pm \right) = 1
\]

(3.1)

(making \( e_{M,n,\sigma,\phi}^\pm \otimes f_{M,n,\sigma,\phi}^\pm \) the kernel of a projection), and, for the sake of definiteness, the \( \pi \)-component \( f_{M,n,\sigma,\pi}^\pm \) has \( L^2[0,2\pi] \)-norm \( 1/\sqrt{2} \). For this normalization, asymptotically for \( n \) and \( M \) large, \( e_{M,n,\sigma,\pi}^\pm \) has the same \( 1/\sqrt{2} \) normalization.

In this section, we obtain estimates on the covariance of \( \Phi_M(t) \) at equal times with respect to the invariant measure \( \mu_M \), \( E_{\mu_M} [\Phi_M(f_{M,n,\sigma}^\pm, t) \Phi_M(f_{M,n,\sigma}^\pm, t)^*] \), where

\[
\Phi_M(f,t) = \langle f, \Phi_M(t) \rangle = \int dx \left( f_{M,n,\sigma,\phi}^\pm(x) \phi(x,t) + f_{M,n,\sigma,\pi}^\pm(x) \pi(x,t) \right) + f_{M,n,\sigma,r}^\pm r(t).
\]
These covariances are independent of time, and so we henceforth suppress the time $t$ in their expressions.

The variances of the $r_M$ variables are bounded by the average of the temperatures, as implied by the following identity.

**Lemma 3.1.** We have

$$E_{\mu_M}[r_M^2] \equiv E_{\mu_M}[r_{1,M}^2 + r_{2,M}^2] = \frac{1}{2}(T_1 + T_2). \quad (3.3)$$

**Remark:** The lemma says that the average expected energy of each of the $r$ variables is given by one-half the average of the temperatures, as one might anticipate from equipartition of energy in equilibrium.

**Proof:** Let $H_M(t)$ be the (degenerate) Lyapunov function defined

$$H_M(t) = \frac{1}{2}\|\Phi_M(t)\|^2 + \int_0^{2\pi} dx h(\phi_M(x,t)) \quad (3.4)$$

with $h(x)$ an antiderivative of $g(x)$. The Ito differential of this quantity is given by

$$dH_M(t) = \left(\frac{1}{2}(T_1 + T_2) - r_M^2\right) dt + r_M\sqrt{T}d\omega(t). \quad (3.5)$$

By stationarity of $\Phi_M$, we have that $dE_{\mu_M}[H(t)] = 0$, and so by the non-anticipating property of $r_M(t)$ and thus independence of $r_M(t)$ and $d\omega$,

$$0 = E_{\mu_M}[dH_M(t)] = E_{\mu_M}\left[\frac{1}{2}(T_1 + T_2) - r_M^2\right] dt = 0. \quad (3.6)$$

The next task is to obtain an estimate on the covariance of different modes of the field. In the following, $\|f_{M,n,\sigma,\pi}^\pm\|$ is the $L^2[0,2\pi]$-norm of $f_{M,n,\sigma,\pi}^\pm$, and $\langle f_{M,n,\sigma,\pi}^\pm, e \rangle$ is the $L^2[0,2\pi]$-inner product of $f_{M,n,\sigma,\pi}^\pm$ with any other $L^2[0,2\pi]$-function $e(x)$.

**Lemma 3.2.** (Equal time covariance). We have, for $n \neq m$, or for $n = m$ and the $\pm$’s differing,

$$\left|E_{\mu_M}[\Phi_M(f_{M,n,\sigma}^\pm)\Phi_M(f_{M,m,\sigma'}^\mp)]\right| \leq \frac{1}{|\lambda_{M,n,\sigma}^\pm + \lambda_{M,m,\sigma'}^\pm|} \left(E[|\Phi_M(f_{M,n,\sigma}^\pm)|^2]^{1/2}|f_{M,m,\sigma',\pi}^\pm|2\|g\|_\infty \right.$$

$$+ E[|\Phi_M(f_{M,m,\sigma}^\pm)|^2]^{1/2}|f_{M,n,\sigma,\pi}^\pm|2\|g\|_\infty + \left|\langle f_{M,n,\sigma}^\pm, T f_{M,m,\sigma'}^\mp \rangle_H \right|. \quad (3.7)$$

**Proof:** Again by stationarity and by using Eq. (12), we have that

$$dE_{\mu_M}[\Phi(f_{M,n,\sigma}^\pm)\Phi(f_{M,m,\sigma'}^\mp)] = 0$$

$$= (\lambda_{M,n,\sigma}^\pm + \lambda_{M,m,\sigma'}^\pm)E_{\mu_M}[\Phi_M(f_{M,n,\sigma}^\pm)\Phi_M(f_{M,m,\sigma'}^\mp)] dt$$

$$- \left(E_{\mu_M}[\Phi_M(f_{M,n,\sigma}^\pm)(f_{M,m,\sigma',\pi}^\pm, g)^*] + E_{\mu_M}[\langle f_{M,n,\sigma,\pi}^\pm, \Phi_M(f_{M,m,\sigma'}^\mp) \rangle_H dt \right). \quad (3.8)$$
Ineq. (3.7) follows immediately.

Finally, we estimate the variances of individual modes (actually their sum). We use the notation $|e_{M,n,\sigma,r}^\pm| = \left( e_{M,n,\sigma,r}^\pm \cdot e_{M,n,\sigma,r}^{\pm *} \right)^{1/2}$, where we have made explicit the dot product of the two $r_M$ components for $e_{M,n,\sigma,r}^\pm$.

**Proposition 3.3.** There exists a constant $C > 0$ independent of the ultraviolet cutoff $M$ such that

$$
\sum_{n,\sigma,\pm} |e_{M,n,\sigma,r}^\pm|^2 E \left[ |\Phi_M(f_{M,n,\sigma}^\pm)|^2 \right] \leq C. \tag{3.9}
$$

The weight $|e_{M,n,\sigma,r}^\pm|^2$ is comparable to $(n^2 + 1)^{\theta - 1}$; there exists a constant $c_1$ with

$$
c_1(n^2 + 1)^{\theta - 1} \leq |e_{M,n,\sigma,r}^\pm|^2 \leq c_1^{-1}(n^2 + 1)^{\theta - 1}. \tag{3.10}
$$

**Proof:** We begin by writing $r_M = (r_{M,1}, r_{M,2})^T$ in an eigenfunction expansion,

$$
r_M = \sum_{n,\sigma,\pm} e_{M,n,\sigma,r}^\pm \Phi_M(f_{M,n,\sigma}^\pm), \tag{3.11}
$$

(the expansion is complete!) and denoting the sum on the left side of the proposition inequality (3.9) by $\|\hat{r}_M\|_2^2$. By Lemma (3.1) above and the expansion for $r_M$, we have that

$$
\frac{1}{2}(T_1 + T_2) = E[r_M^2]
= \|\hat{r}_M\|_2^2 + \sum_{(n,\sigma,\pm) \neq (m,\sigma',\pm')} \sum_{(n,\sigma,\pm)} \left( e_{M,n,\sigma,r}^\pm \cdot e_{M,m,\sigma',r}^{\pm *} \right) E \left[ \Phi_M(f_{M,n,\sigma}^\pm) \Phi_M(f_{M,m,\sigma'}^{\pm *}) \right],
$$

where the double sum is over off-diagonal terms, $n \neq m$, or the $\pm$‘s different, or $\sigma \neq \sigma'$.

For this equation, we use Lemma (3.2) to estimate the double sum over the non-resonant terms with $n \neq m$ or the $\pm$‘s differing. For these terms, the denominators $|\lambda_{M,n,\sigma}^\pm + \lambda_{M,m,\sigma'}^{\pm *}|^{-1}$ behave like $|n - m|$ if $n \neq m$ and $\pm = \pm'$, or like $|n + m|$ if $\pm = \mp'$, and so in either case these denominators are not dangerous (see Lemma A.1 of the appendix). Thus, the “kernel” $|\lambda_{M,n,\sigma}^\pm + \lambda_{M,m,\sigma'}^{\pm *}|^{-1}$ is $\ell^p$-summable for any $p > 1$. We also use the fact that $e_{M,n,\sigma,r}^\pm = (\alpha, e_{M,n,\sigma,r}^\pm)/(1 + \lambda_{M,n,\sigma}^\pm) = O(n^{\theta - 1})$, by Eqs. (A.2 A.6) of the appendix, which happens to establish Eq. (3.10) of the Proposition as well. It follows that $|e_{M,n,\sigma,r}^\pm|$ is $\ell^p$-summable for $p > 1/(1 - \theta)$. These estimates, together with Young’s inequality, then show that the non-resonant part of the double sum in Eq. (3.12) is bounded below by

$$
- c_1 \|\hat{r}_M\|_2^2 \left( \sum_{n,\sigma,\pm} |e_{M,n,\sigma,r}^\pm|^p \right)^{1/p} \geq -c \|\hat{r}_M\|_2^2 \tag{3.13}
$$

for suitable positive constants $c, c_1$ and suitably large $p < 2$. 

The near-resonant terms in the double sum of Eq. (3.12), terms with \( n = m \) and \( \pm \)'s the same but \( \sigma \neq \sigma' \), are more delicate. Fortuitously \( e_{M, n, \sigma, r}^{\pm} \) and \( e_{M, m, \sigma', r}^{\pm} \) are nearly orthogonal for \( n \to \infty \), i.e.,

\[
|e_{M, n, \sigma, r}^{\pm, *} \cdot e_{M, n, \sigma', r}^{\pm}| = |e_{M, n, \sigma, r}^{\pm, *}| |e_{M, n, \sigma', r}^{\pm}| \times O \left( n^{2\theta - 1} \ln n \right),
\]

(3.14)

by Lemma (A.1), Eq. (A.8) of the appendix. This inequality implies that for some fixed \( N \) (independent of the ultraviolet cutoff and chosen so that the \( O \left( n^{2\theta - 1} \ln n \right) \) factor is \( < 1/2 \) for \( n \geq N \)), the tail series satisfies

\[
\sum_{\{(n, \sigma, \pm): n \geq N\}} \left( e_{M, n, \sigma, \sigma}^{\pm, *} \cdot e_{M, n, \sigma', r}^{\pm}\right) E \left[ \Phi(f_{M, n, \sigma}) \Phi(f_{M, n, \sigma', r}) \right] \geq -\frac{1}{2} \| \hat{r}_M \|_{L^2}^2.
\]

(3.15)

Now the sum of the variances of the low \( n \) modes \( n \leq N \) is bounded by a constant \( c_2 \), since the variance of an individual mode \( \Phi_M(f_{M, n, \sigma}) \) is certainly bounded by \( const/(\text{Im} \lambda_{M, n, \sigma})^2 \), as seen from the Duhamel integral representation, Eq. (2.2), for \( \Phi_M \) in the \( t \to \infty \) limit. Thus, combining Eq. (3.12) and Ineqs. (3.13, 3.15), we obtain a quadratic inequality for \( \| \hat{r}_M \|_{L^2} \)

\[
\frac{1}{2} (T_1 + T_2) \geq \| \hat{r}_M \|_{L^2} - c_1 |\hat{r}_M|_{L^2} - \frac{1}{2} \| \hat{r}_M \|_{L^2} - c_2,
\]

(3.16)

which gives the bound of the proposition. 

In the inequality (3.9) of Proposition (3.3) we can actually replace function \( f_{M, n, \sigma}^{\pm} \) in the expectation \( E \left[ \Phi(f_{M, n, \sigma})^2 \right] \) with the free eigenfunctions \( f_{M, n, \sigma}(0) \), that is, left eigenfunctions of \( B_M \), i.e. the matrix operator obtained from \( A_M \) by setting the coupling functions \( \alpha \) to zero. For \( n \geq 0 \), we take \( f_{M, n, \sigma}(0) = \frac{1}{\sqrt{4\pi}} (\pm i (n+1)^{-1/2} e^{i \alpha_n} x, e^{i \alpha_n} x, 0) \), with \( \sigma = 1 \) or \(-1\) (except when \( n = 0 \), where there is no \( \sigma \)-dependence), and \( f_{M, -1, \sigma}(0) = (0, 0, 1, 0) \) or \((0, 0, 0, 1)\).

**Corollary 3.4.** There exists a finite constant \( C > 0 \) independent of the cutoff \( M \) such that

\[
\sum_{n, \sigma, \pm} (n^2 + 1)^{\theta - 1} E \left[ |\Phi_M(f_{M, n, \sigma}(0))|^2 \right] \leq C.
\]

(3.17)

**Remark:** The corollary implies that

\[
\sum_n (n^2 + 1)^{\theta} E[|\hat{\phi}_M(n)|^2] \leq C
\]

(3.18)

uniformly in \( M \), with \( \hat{\phi}_M(n) = \int e^{-i \alpha x} \phi_M(x) \) the usual Fourier coefficient of the field component \( \phi_M \) of \( \Phi_M \). If \( \theta > 0 \), then the field \( \phi_M \) is in \( L^2 \) a.s., with the variance of \( \| \hat{\phi}_M \|_{L^2} \) uniformly bounded in \( M \); obviously, each mode \( \hat{\phi}_M(n) \) has variance uniformly bounded in \( n \). However, we believe that this bound can be improved, that in fact \( \text{var}(\hat{\phi}_M(n)) = O(n^{-2}) \) and that \( \text{var}(\hat{\pi}_M(n)) = O(1) \) uniformly in \( M \) as in the linear and non-linear equilibrium cases and the linear \( g = 0 \) non-equilibrium case. This improvement will be the subject of a
subsequent investigation.

**Proof:** We expand

$$\Phi_M(f^\pm_{M,n,\sigma}(0)) = \sum_{m,\sigma',\pm'} \langle f^\pm_{M,n,\sigma}(0), e^\pm_{M,m,\sigma'} \rangle \Phi_M(f^\pm_{M,m,\sigma'}).$$  \hspace{1cm} (3.19)

The coefficients of this expansion (with inner products as in Eq. (3.11)) satisfy

$$\langle f^\pm_{M,n,\sigma}(0), e^\pm_{M,m,\sigma'} \rangle = \begin{cases} \mathcal{O}(m^{\theta-1}), & n = -1, m \to \infty \\ \mathcal{O}(n^{\theta-1}), & m = -1, n \to \infty \\ \mathcal{O}\left(\frac{n^\theta m^{\theta-1}}{(n-m)^2}\right), & m \neq n, m, n \to \infty \\ \mathcal{O}(1), & \text{otherwise,} \end{cases} \hspace{1cm} (3.20)$$

all uniform in the cutoff \(M\) (see Eq. (A.21) where these estimates are shown). Substituting the expansion Eq. (3.19) into \(E[\Phi_M(f^\pm_{M,n,\sigma}(0))]^2\), we obtain the double sum of terms

$$\langle f^\pm_{M,n,\sigma}(0), e^\pm_{M,m,\sigma'} \rangle \langle f^\pm_{M,n,\sigma}(0), e^{\pm''}_{M,m',\sigma''} \rangle E[\Phi_M(f^\pm_{M,m,\sigma'}) \Phi_M(f^{\pm''}_{M,m',\sigma''})^*].$$ \hspace{1cm} (3.21)

Off-diagonal terms \(m \neq m'\) must be estimated using the non-resonant Ineq. (3.2) of Lemma 3.2 and the above inequalities (3.20). For \(m, m' \to \infty\) these terms are

$$\mathcal{O}\left(\frac{n^\theta m^{\theta-1} m'^{\theta-1}}{(n-m)(n-m')(m-m')}\right)^{1/2} \times \left(E[\Phi_M(e^{\pm}_{M,m,\sigma'})]^2\right)^{1/2} + \mathcal{O}(m^{\theta-1} m'^{\theta-1}) \hspace{1cm} (3.22)$$

It remains to sum the on- and off-diagonal terms (3.21) over \(m, m'\), and over \(n\) with the prefactor \(n^{2\vartheta-2}\). One uses the fact that \(m^{\theta-1} E[\Phi_M(f^{\pm}_{M,m,\sigma'})^2]^{1/2} \) is \(\ell^2\)-summable in \(m\), the content of Proposition (3.3). The near-resonant \(m = m'\) terms are estimated using the inequalities (3.20). By extensive use of Young’s inequality and Hölder’s inequality, the triple sum is shown to be finite. \[\]

## 4 On the tightness of the ultraviolet cutoff stationary measures

Let \(\{\mu_M\}\) denote the unique stationary measures for the ultraviolet cutoff systems of the previous sections, \(M\) labeling the cutoff. Let \(\Phi_M(t)\) be the canonical stationary process associated with \(\mu_M\) so that in particular \(\mu_M\) is the law for \(\Phi_M(t)\) for any time \(t\). It will be convenient to regard the field \(\Phi_M(t)\) as taking values in a space of distributions dual to the Schwartz space \(\mathcal{S} \equiv C^\infty_{\text{per}}[0,2\pi] \times C^\infty_{\text{per}}[0,2\pi] \times \mathbb{R} \times \mathbb{R}\), but where the Fourier modes \(\{\hat{\phi}_M(n), \hat{\pi}_M(n)\}\) of \(\Phi_M\) are all zero for \(|n| > M\).

Corollary (3.4) provides a bound on the variance of the Fourier mode of the field, \(\text{var}_{\mu_M}(\Phi_M(f^\pm_{n,\sigma}(0))) \sim n^{2-2\vartheta}\) uniform in \(M\) at a fixed time, say \(t = 0\). As before, we suppress the explicit time dependence. Let \(\mathcal{H}_s = H_s \oplus H_{s-1} \oplus C \oplus C\). Then generally, we note the following:
Proposition 4.1. Assume that the fields \( \Phi_M \) are of mean zero with respect to \( \mu_M \) for all ultraviolet cutoffs \( M \), and that the field modes are of variance

\[
\var_{\mu_M} [\Phi_M(f_{n,\sigma}^+(0))] \equiv E_{\mu_M} [\| \Phi_M(f_{n,\sigma}^+(0)) \|^2] \leq C(1 + n^2)^p \tag{4.1}
\]

for some constants \( C \) and power \( p \) independent of \( M \). Then the measures \( \{\mu_M\} \) are tight in the weak-* sense that there exist a subsequence \( \{\mu_{M_j}\}, \ j = 1, 2, \ldots \), a limiting measure \( \mu \), and a limiting field \( \Phi \) such that

\[
\lim_{j \to \infty} E_{\mu_{M_j}} [F(\Phi_{M_j}(f_1), \Phi_{M_j}(f_2), \ldots, \Phi_{M_j}(f_k))] = E_{\mu} [F(\Phi(f_1), \Phi(f_2), \ldots, \Phi(f_k))] \tag{4.2}
\]

for all bounded continuous functions \( F(x_1, x_2, \ldots, x_k) \) on \( \mathbb{R}^k \), and with \( f_1, f_2, \ldots, f_k \in S \) for all \( k \). The limiting measure has support in the space of distributions \( S' \) and is \( \sigma \)-additive on the Borel \( \sigma \)-algebra generated by cylinder sets of the form

\[
\{ \Phi : (\Phi_M(f_1), \Phi_M(f_2), \ldots, \Phi_M(f_k)) \in B \} \tag{4.3}
\]

with base \( f_1, f_2, \ldots, f_k \in S \) and \( B \) a Borel set in \( \mathbb{R}^k \).

We have that for \( s < -p - 1/2 \)

\[
E_{\mu} [\| \hat{\Phi} \|^2_{H_s}] < C_1 \tag{4.4}
\]

for some finite constant \( C_1 \); the same bound holds for the \( \mu_M \)'s. In particular, \( \Phi \) is in \( H_s \) a.s.

Remark: Again, if the Fourier coefficients of the \( \alpha \)'s behave as a power, \( |\hat{\alpha}(n)| \sim n^\theta \), then the variances of the Fourier modes indeed satisfy Ineq.\,(4.1), and so we obtain a limiting measure for which \( \Phi \in H_s, \ s < \theta - 3/2 \). However, we do not claim that this measure is an invariant measure for the nonlinear stochastic wave equation; this remains an open problem.

Proof: The marginals of \( \{\mu_M\} \) restricted to functions \( F \) just depending on a fixed and finite number of Fourier modes \( \{\Phi_M(f_{n,\sigma}^+(0))\}_{|n| \leq n_0} \) are tight. This is the case since a closed ball in a finite-dimensional Euclidean space is compact, and given \( \epsilon > 0 \),

\[
P_{\mu_M} \{ \| P_{|n| \leq n_0} \Phi \|_2 > R \} \leq \frac{1}{R^2} E_{\mu_M} \left[ \sum_{|n| \leq n_0, \sigma, \pm} |\Phi_M(f_{n,\sigma}^+(0))|^2 \right] \leq \frac{C}{R^2} \sum_{|n| \leq n_0, \sigma, \pm} (1 + n^2)^p < \epsilon \tag{4.5}
\]

for another constant \( C \) for \( R \) sufficiently large, by our bounds on the Fourier coefficient variances (see [25]). Here, \( P_{|n| \leq n_0} \Phi \) is projection of \( \Phi \) onto its Fourier modes with \( |n| \leq n_0 \). By this tightness, one can construct a subsequence \( \{\mu_{M_j, n_0}\} \) with convergent marginals based on the random variables \( \{\Phi_M(f_{n,\sigma}^+(0))\}_{|n| \leq n_0} \).

One then passes to a sub-subsequence to get convergence for marginals based on a larger collection \( \{\Phi_M(f_{n,\sigma}^+(0))\}, |n| \leq n_1 \) for \( n_1 > n_0 \) and diagonalization, one obtains a
subsequence \( \{ \mu_{M_j} \} \) which, integrated against any continuous function \( F \) of a finite number of Fourier modes, converges, i.e., \( \lim_{j \to \infty} E_{\mu_{M_j}}(F) \) exists.

Again by Chebyshev, given \( \epsilon \)

\[
P_{\mu_M} \{ \Phi : \| \Phi \|_{H^{p-1}} > R \}
\leq \frac{1}{R^2} E_{\mu_M} \left[ \sum_{n,\sigma,\pm} \frac{1}{1 + n^2} \| \Phi_{M}(f_{n,\sigma}(0)) \|^2 \right]
\leq \frac{C}{R^2} \sum_{n,\sigma,\pm} \frac{(1 + n^2)^p}{(1 + n^2)^{p+1}} < \epsilon
\]

for \( R \) sufficiently large, uniformly in \( M \). This implies the last inequality of the lemma, Ineq. (4.4).

It is then easy to see that the domain of definition of \( \mu \) extends uniquely to functions of the form \( F(\Phi(f_1), \Phi(f_2), \ldots, \Phi(f_k)) \), with \( f_1, f_2, \ldots, f_k \in \mathcal{S} \), \( F(x_1, x_2, \ldots, x_k) \) bounded continuous. Ineq. (4.6) reduces the problem of showing the convergence of \( \{ E_{\mu_{M_j}}[F] \} \) to showing that of \( \{ E_{\mu_{M_j}}[F; B_R] \} \), where \( B_R \equiv \{ \Phi : \| \Phi \|_{H^s} \leq R \} \). We write \( f_{\leq N} \) as the projection of \( f \) onto the subspace of Fourier modes with \( |n| \leq N \) and \( f_{> N} \) for the tail of its series. Now for \( \Phi \in B_R \) and for any \( \delta > 0 \), \( \| f_{N} \Phi - f_{\leq N}, \Phi \| \leq R\| f_{> N} \|_{H^{-s}} < \delta \) for \( N \) sufficiently large. The uniform continuity of \( F \) on the bounded set \( \{ x : |x_j| \leq R \} \) then gives the convergence.

That \( \mu \) is countably additive on the Borel sets generated by the cylinder sets is also a consequence of Ineq. (4.6) (13). This concludes the proof of the proposition.

A Perturbation theory for the eigenfunctions of the operator \( A \)

This appendix summarizes properties of the left eigenfunctions \( \{ f_{n,\sigma}^\pm \} \) and right eigenfunctions \( \{ e_{n,\sigma}^\pm \} \) and their corresponding eigenvalues \( \{ \lambda_{n,\sigma}^\pm \} \) for the matrix operator \( A \). All estimates are uniform with respect to the ultraviolet cutoff \( M \), and so the index \( M \) is suppressed. We assume throughout this appendix that the coupling functions \( \alpha_1 \) and \( \alpha_2 \) satisfy the assumptions (1.1). See [16] for the perturbation theory methods utilized.

Fixing the mode number \( n \geq 0 \) and the \( \pm \)'s and then suppressing these and other indices except as needed, we write \( f_{n,\sigma}^\pm = (f_{\sigma,\phi}, f_{\sigma,\pi}, f_{\sigma,r_1}, f_{\sigma,r_2}) \) for the components of \( f_{n,\sigma}^\pm \), and we write \( e_{n,\sigma}^\pm = (e_{\sigma,\phi}, e_{\sigma,\pi}, e_{\sigma,r_1}, e_{\sigma,r_2})^T \) for the column components of the right eigenfunction. The corresponding eigenvalue \( \lambda_{n,\sigma}^\pm \) is near \( \pm i \sqrt{n^2 + 1} \); for \( n > 0 \) and for a given choice of \( \pm \), the eigenvalue is nearly doubly degenerate, whence the index \( \sigma = 1 \) or \( -1 \).

From the eigenvalue equation \( \langle f_{n,\sigma}^\pm, A \cdot \rangle_\mathcal{H} = \lambda_{n,\sigma}^\pm \langle f_{n,\sigma}^\pm, \cdot \rangle_\mathcal{H} \), one finds the relations

\[
f_{\sigma,\phi} = \frac{1}{\lambda_{\sigma}} f_{\sigma,\pi} (\partial_x^2 - 1), \quad f_{\sigma,r} = \frac{-\langle f_{\sigma,\pi}, \alpha \rangle}{1 + \lambda_{\sigma}}, \quad (A.1)
\]

and similarly, from \( A e_{n,\sigma}^\pm = \lambda_{n,\sigma}^\pm e_{n,\sigma}^\pm \),

\[
e_{\sigma,\phi} = \frac{1}{\lambda_{\sigma}} e_{\sigma,\pi}, \quad e_{\sigma,r} = \frac{\langle \alpha, e_{\sigma,\pi} \rangle}{1 + \lambda_{\sigma}}. \quad (A.2)
\]
Note that $f_{\sigma,r}$ and $e_{\sigma,r}$ each have two components, e.g., $f_{\sigma,r} = (f_{\sigma,r_1}, f_{\sigma,r_2})$, corresponding to the two components of $\alpha$ in the above relations. One can then use these relations to write the eigenvalue equation as a non-linear eigenvalue equation just involving the $\pi$-components,

$$
\lambda_\sigma^2 f_{\sigma,\pi} = f_{\sigma,\pi}(\partial_x^2 - 1) - \frac{\lambda_\sigma}{1 + \lambda_\sigma} \langle f_{\sigma,\pi}, \alpha \rangle \langle \alpha \rangle
$$

(A.3)

or

$$
\lambda_\sigma^2 e_{\sigma,\pi} = (\partial_x^2 - 1)e_{\sigma,\pi} - \frac{\lambda_\sigma}{1 + \lambda_\sigma} \langle \alpha \rangle \langle \alpha, e_{\sigma,\pi} \rangle.
$$

(A.4)

In these equations, $|\alpha\rangle\langle\alpha| = \sum_{i=1}^{2} |\alpha_i\rangle\langle\alpha_i|$. For large $n$, $\lambda_\sigma/(1 + \lambda_\sigma) = 1 + i/n + O(n^{-2})$, so that these latter two equations are nearly self-adjoint eigenvalue equations for

$$
\lambda_\sigma^2 e_{\sigma,\pi} = (\partial_x^2 - 1)e_{\sigma,\pi} - \beta \alpha \langle \alpha, e_{\sigma,\pi} \rangle,
$$

(A.5)

with $\lambda_\sigma$ determined implicitly by substituting in $\beta = \lambda_\sigma/(1 + \lambda_\sigma)$. Evidently, $e_{\sigma,\pi} = -\beta(\lambda_\sigma^2 - \partial_x^2 + 1)^{-1}\alpha(e_{\sigma,\pi})$, and from this representation, one sees that the $m^{th}$ Fourier coefficient $\hat{e}_{\sigma,\pi}(m)$ of $e_{\sigma,\pi}$, is $O(n^{\theta})$, for $m^2 \neq m^2$. This implies in particular that

$$
\langle \alpha, e_{\sigma,\pi} \rangle = O(n^\theta).
$$

(A.6)

In the following, let $P_0 = P_0(n)$ be the projection onto the eigenspace spanned by $e^{inx}$ and $e^{-inx}$ in $L^2[0,2\pi]$.

**Lemma A.1.** We have that for $n$ large, the eigenvalues $\{\lambda_{\alpha_n}^\pm\}$ of $A$ are given by

$$
\lambda_{\alpha_n}^\pm = \pm \left( i\sqrt{n^2 + 1} + \frac{(1 \mp i/n)\mu_{n,\sigma}}{2in} \right)
\qquad + iO(n^{4\theta - 2}\ln n) + O(n^{4\theta - 3}\ln n)
$$

(A.7)

where the $\mu_{n,\sigma}$ are the two eigenvalues of the operator $-P_0(n)\alpha\langle\alpha P_0(n)$, and where the error terms are, respectively, imaginary and real.

Let $e_{\alpha,n,\sigma,r}^\pm$ be the $r$-components of the right eigenvector $e_{\alpha,n,\sigma}^\pm$ of $A$. Then for $\sigma \neq \sigma'$, we have

$$
e_{\alpha,n,\sigma,r}^\pm : e_{\alpha,n,\sigma',r}^\pm = O(n^{2\theta - 1}\ln n),
$$

(A.8)

where the dot indicates the dot product of the components, $| \cdot |$ being the Euclidean length.

**Sketch of proof:** For $n > 0$, let $P = P(n)$ be the projection onto the subspace spanned by the two eigenvectors corresponding to the two eigenvalues $\lambda_{\sigma}^2$ near $-(n^2 + 1)$ for the operator $\partial_x^2 - 1 - \beta |\alpha| \langle |\alpha| - z \rangle^{-1}$ expanded in a Neumann series. Then the shift in the eigenvalues is determined from the $2 \times 2$ matrix equation

$$
(\lambda_{\sigma}^2 + (n^2 + 1))P_0P\xi_{\sigma} = -\beta P_0\alpha \langle \alpha P_0\xi_{\sigma}
$$

(A.9)

for $\xi_{\sigma}$ in the span of $P_0$. The projection $P$ itself can be estimated from its representation as a contour integral of the resolvent $(\partial_x^2 - 1 - \beta |\alpha| \langle |\alpha| - z \rangle^{-1}$ expanded in a Neumann series. One finds that

$$
\lambda_{\sigma}^2 = -(n^2 + 1) + \beta \mu_{\sigma} + \beta^2 O(n^{4\theta - 1}\ln n),
$$

(A.10)
where \( \mu_\sigma \) is one of the two eigenvalues of the rank 2 matrix \(-P_0 \alpha \rangle \langle \alpha P_0\) and is of the order of \(|\hat{\alpha}(n)|^2 \equiv \sup |\hat{\alpha}_i(n)|^2 = O(n^{2\theta})\). The Neumann series for the resolvent above is in powers of \(|\alpha, (\partial^2_z - 1 - z)^{-1} \alpha\rangle\) with \(z\) traversing a circle \(|z + n^2 + 1| = n/2\). These powers are estimated using

\[
|\langle \alpha, (\partial^2_z - 1 - z)^{-1} \alpha\rangle| \\
\leq c \sum_{\{m: m \leq n/2\}} \frac{|\hat{\alpha}(m)|^2}{n^2} + c \sum_{\{m: n/2 \leq m \leq 3n/2, m \neq n\}} \frac{|\hat{\alpha}(n)|^2}{|n^2 - m^2|} \\
+ c \frac{|\hat{\alpha}(n)|^2}{n} + c \sum_{\{m: m \geq 3n/2\}} \frac{|\hat{\alpha}(m)|^2}{m^2} \\
\leq c |\hat{\alpha}(n)|^2 \ln n \frac{n}{n}. \tag{A.11}
\]

(The second sum in the second line accounts for the \(\ln n/n\) factor, by an integral test.) One uses this estimate to obtain the correction to the eigenvalue shift in Eq. (A.10). See [30] for additional details. We then set \( \beta = \lambda_\sigma/(1 + \lambda_\sigma) \) into Eq. (A.10) to determine \( \lambda_\sigma \) implicitly; this gives the first assertion of the lemma, Eq. (A.7).

We also need an approximate orthogonality relation for \( e_{\sigma, \pi} \) and \( e_{\sigma', \pi}, \sigma \neq \sigma' \). In Eq. (A.4), we decompose \( e_{\sigma, \pi} \) as

\[
e_{\sigma, \pi} = \xi_\sigma^0 + \tilde{e}_{\sigma, \pi}, \tag{A.12}
\]

where \( \xi_\sigma^0 \) is an \( L^2 \)-normalized eigenfunction of \( P_0 \alpha \rangle \langle \alpha P_0 \), and hence in the subspace spanned by \( P_0 \). Then

\[
P_0(\lambda_\sigma^2 + n^2 + 1 - \beta \alpha) \langle \alpha|)(\xi_\sigma^0 + P_0 \tilde{e}_{\sigma, \pi} + Q_0 \tilde{e}_{\sigma, \pi}) = 0, \tag{A.13}
\]

where \( Q_0 = 1 - P_0 \). We have that

\[
\|P_0(\lambda_\sigma^2 + n^2 + 1 - \beta \alpha) \langle \alpha|\xi_\sigma^0\rangle\| = O(n^{4\theta - 1} \ln n) \tag{A.14}
\]

by the eigenvalue shift estimate above, Ineq. (A.10). Also, we have that

\[
Q_0 \tilde{e}_{\sigma, \pi} = -\beta Q_0(\lambda_\sigma^2 - \partial^2_z + 1)^{-1} \alpha \langle \alpha, e_{\sigma, \pi}, \tag{A.15}
\]

so

\[
|\langle \alpha Q_0 \tilde{e}_{\sigma, \pi}\rangle| = O(n^{3\theta - 1} \ln n), \tag{A.16}
\]

and thus

\[
\|P_0(\lambda_\sigma^2 + n^2 + 1 - \beta \alpha) \langle \alpha|Q_0 \tilde{e}_{\sigma, \pi}\rangle\| = O(n^{4\theta - 1} \ln n). \tag{A.17}
\]

From the identity Eq. (A.13), this last equation, and Eq. (A.14) above, it follows that

\[
(\mu_\sigma - \mu_{\sigma'}) + O(n^{4\theta - 1} \ln n) \|P_0 \tilde{e}_{\sigma, \pi}\| = O(n^{4\theta - 1} \ln n), \tag{A.18}
\]

where \( \sigma' \) refers to the complementary value of \( \sigma \). The assumption on the coupling functions \( \alpha \), Eq. (1.1), assures that \(|\mu_\sigma - \mu_{\sigma'}| \geq c |\hat{\alpha}(n)|^2 \), so that

\[
\|P_0 \tilde{e}_{\sigma, \pi}\| = O(n^{2\theta - 1} \ln n). \tag{A.19}
\]
The last equation of the lemma, Eq. (A.8), follows from the equation for $e_{\sigma,r}$, Eq. (A.2); the decomposition for $e_{\sigma,\pi}$ in Eq. (A.12); and the orthogonality of $\xi_0^\sigma$ and $\xi_0^{\sigma'}$ under $P_0\alpha\langle \alpha P_0$. We have that

$$
\langle e_{\sigma,\pi}, \alpha \rangle \langle \alpha, e_{\sigma',\pi} \rangle = \langle \xi_0^\sigma, \alpha \rangle \langle \alpha, \xi_0^{\sigma'} \rangle + \langle Q_0 \tilde{e}_{\sigma,\pi}, \alpha \rangle \langle \alpha, Q_0 \tilde{e}_{\sigma',\pi} \rangle + \langle Q_0 \tilde{e}_{\sigma,\pi}, \alpha \rangle \langle \alpha, Q_0 \tilde{e}_{\sigma',\pi} \rangle
$$

by Eqs. (A.16, A.19). Combining this equation with Eq. (A.2), we obtain Eq. (A.8) of the lemma

$$
\text{Eq. (3.20) of the text provides estimates on inner products } \langle f_{n,\sigma}^\pm(0), e_{m,\sigma'}^\pm \rangle_H. \text{ To illustrate how these estimates are obtained, consider a case } m \neq n, m, n \text{ large. By familiar resolvent identities, and with } B \text{ the unperturbed matrix operator with the } \alpha \text{'s turned off and with the left eigenfunction } f_{n,\sigma}^\pm(0),
$$

$$
\langle f_{n,\sigma}^\pm(0), e_{m,\sigma'}^\pm \rangle_H = \frac{1}{2\pi i} \int_{\gamma_m} dz (f_{n,\sigma}^\pm(0), \frac{1}{B - z} C \frac{1}{A - z} e_{m,\sigma'}^\pm)_H
$$

$$
= \frac{1}{2\pi i} \int_{\gamma_m} dz \frac{(f_{n,\sigma,\pi}^\pm(0), \alpha) \epsilon_{m,\sigma',\pi}^\pm}{(\lambda_{n}^\pm(0) - z)(\lambda_{m,\sigma'}^\pm - z)} \sim \frac{\hat{\alpha}(n)}{(\lambda_{n}^\pm(0) - \lambda_{M,m,\sigma'}^\pm)(\lambda_{M,m,\sigma'}^\pm + 1)} = O \left(\frac{n^{\theta - 1} \ln n}{n - m}\right).
$$

Here, $\gamma_m$ is the circle $\{ z : |z - \lambda_{M,m,\sigma'}^\pm| = 1/2 \}$ and, in the second line, $C = A - B$ is the $4 \times 4$ perturbation matrix of just the $\alpha$ entries, all other entries being zero. In the last line we have used Eq. (A.4) for $e_{m,\sigma',\pi}^\pm$ and the fact that $\langle \alpha, e_{m,\sigma',\pi}^\pm \rangle \sim \hat{\alpha}(n) = O(n^\theta)$, Eq. (A.6). The other relations of Eq. (3.20) are analyzed similarly.

**Acknowledgements** This article is based on the Ph.D. thesis of YW at the University of Virginia.

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