Abstract

The $k$-cosymplectic Lagrangian and Hamiltonian formalisms of first-order field theories are reviewed and completed. In particular, they are stated for singular and almost-regular systems. Subsequently, several alternative formulations for $k$-cosymplectic first-order field theories are developed: First, generalizing the construction of Tulczyjew for mechanics, we give a new interpretation of the classical field equations in terms of certain submanifolds of the tangent bundle of the $k^1$-velocities of a manifold. Second, the Lagrangian and Hamiltonian formalisms are unified by giving an extension of the Skinner-Rusk formulation on classical mechanics. Finally, both formalisms are formulated in terms of Lie algebroids.

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Key words: $k$-cosymplectic manifolds, classical field theory, Lagrangian formalism, Hamiltonian formalism, Lie algebroids.

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1 Introduction

Günther’s (k-symplectic) formalism [11 2] [16 38] is the generalization to first order classical field theories of the standard symplectic formalism in mechanics, which is the geometric framework for describing autonomous dynamical systems. In this sense, the k-symplectic formalism is used to give a geometric description of certain kinds of field theories: in a local description, those whose Lagrangian does not depend on the coordinates in the basis (in many of them, the space-time coordinates): that is, it is only valid for Lagrangians $L(q^i, v^i_A)$ and Hamiltonians $H(q^i, p^i_A)$ that depend on the field coordinates $q^i$ and on the partial derivatives of the field $v^i_A$.

The k-cosymplectic formalism is the generalization to field theories of the standard cosymplectic formalism in mechanics, which is the geometric framework for describing non-autonomous dynamical systems [28 29]. This formalism describes field theories involving the coordinates in the basis $(t^1, ..., t^k)$ on the Lagrangian $L(t^A, q^i, v^i_A)$ and on the Hamiltonian $H(t^A, q^i, p^i_A)$. The k-cosymplectic formalism was introduced in [28 29]. One of the advantages of this formalism, and of Günther’s (k-symplectic) formalism, is that only the tangent and cotangent bundle of a manifold are required to develop it. In addition, there are also other polysymplectic formalisms for describing field theories, such as those developed by G. Sardanashvily et al [12, 13, 43], and by I. Kanatchikov [18], as well as the n-symplectic formalism of L. K. Norris [36 40]. Let us remark that
the multisymplectic formalism is the most ambitious program to develop the Classical Field Theory (see for example [5, 9, 10, 14, 15, 19], and references quoted therein). In [24, 25] the equivalence between the multisymplectic and $k$-cosymplectic description is shown, when the theories with trivial configuration bundles are considered.

In this paper, we first review the $k$-cosymplectic formalism for singular field theories, improving previous developments on this topic [29]. After this, the main aims are:

1. **To introduce certain submanifolds of $TM \oplus^k$. $\oplus TM$, the Whitney sum of $k$ copies of $TM$, which allows us to describe the Euler-Lagrange and Hamilton equations.**

This part of the paper is inspired in [35, 46], where Tulczyjew formulates Hamiltonian dynamics in terms of Lagrangian submanifolds of a symplectic manifold. M. de León et al. generalize Tulczyjew’s paper to higher-order Lagrangian systems in [20] and to classical field equations in terms of submanifolds of multisymplectic manifolds in [21, 24]. In [23], Tulczjew’s construction is extended to Lagrangian Mechanics on Lie algebroids in terms of Lagrangian Lie subalgebroids, which are submanifolds of a Lie algebroid $\tau : E \rightarrow Q$.

2. **To extend the Skinner-Rusk formalism to $k$-cosymplectic classical field theories.**

In [44], the authors developed the Skinner-Rusk formalism in order to give a geometrical unified formalism for describing mechanical systems. It incorporates all the characteristics of Lagrangian and Hamiltonian descriptions of these systems (including dynamical equations and solutions, constraints, Legendre map, evolution operators, equivalence, etc.). This formalism has been generalized to time-dependent mechanical systems [7], to the multisymplectic description of first-order field theories [11, 22], and also to the $k$-symplectic formulation of field theories [41].

We extend this unified framework to the $k$-cosymplectic description of first-order classical field theories [28, 29], and to show how this description comprises the main features of the Lagrangian and Hamiltonian formalisms, both for the regular and singular cases.

3. **To develop the $k$-cosymplectic formalism in terms of Lie algebroids.**

In [32], a theory of Lagrangian Mechanics is developed in a similar way to the formulation of the standard Lagrangian Mechanics. This approach differs from that followed by A. Weinstein [48]. A good survey on this subject is [23]. The multisymplectic formalism for classical field theories is extended to the setting of Lie algebroids in [34, 35], and in [47] a geometric framework for discrete Classical Field theories on Lie groupoids is presented.

The organization of the paper is as follows:

Section 2 is devoted to reviewing the main features of the $k$-cosymplectic formalism of Lagrangian and Hamiltonian field theories, and to stating these formalisms for singular systems. First, the field theoretic phase space for the Hamiltonian approach space is $\mathbb{R}^k \times (T^*_k)^* Q$, where $(T^*_k)^* Q = T^* Q \oplus^k$. $\oplus T^* Q$ is the Whitney sum of $k$-copies of the cotangent bundle $T^* Q$ of a manifold $Q$. This space is the canonical example of a $k$-cosymplectic manifold. The field phase space for the Lagrangian description is $\mathbb{R}^k \times T^*_k Q$, where $T^*_k Q = TQ \oplus^k$. $\oplus TQ$ is the Whitney sum of $k$-copies of the tangent bundle $TQ$. $T^*_k Q$ has the canonical $k$-tangent structure, given by $k$ canonical tensor fields of type $(1, 1)$ satisfying certain properties. This structure on $T^*_k Q$ can be lifted to $\mathbb{R}^k \times T^*_k Q$. Using the extended tensor fields or the Legendre map and a Lagrangian function, we can construct a $k$-cosymplectic (or $k$-precosymplectic) structure on $\mathbb{R}^k \times T^*_k Q$ whose 1-forms and 2-forms enable us to develop the Lagrangian formalism.

In Section 3 we develop the first aim of the paper. We give a new interpretation of the classical field equations in terms of certain submanifolds of $T^*_k (\mathbb{R}^k \times (T^*_k)^* Q)$. In order to do this, we introduce $2k$ derivations $\tau_{TA}$ and $dT_A$, $1 \leq A \leq k$ from $\bigwedge M$ to $\bigwedge T^*_k M$, for a differentiable manifold $M$. These derivations are the main tools for developing the rest of the section and there is a generalization of the derivations introduced by Tulczyjew in [45, 46].

In Section 4 we develop the unified formalism for field theories (second aim), which is based
on the use of the Whitney sum $\mathcal{M} = (\mathbb{R}^k \times T^1_1 Q) \oplus \mathbb{R}^k \times (T^1_1)^* Q$. There are canonical “precosymplectic” forms on $\mathcal{M}$ (the pull-back of the canonical cosymplectic forms on each $\mathbb{R} \times T^* Q$) and a natural coupling function, which is defined by the contraction between vectors and covectors. Then, given a Lagrangian $L \in C^\infty(\mathbb{R}^k \times T^1_1 Q)$, we can state a field equation on $\mathcal{M}$. This equation has solution only on a submanifold $M_L$, which is the graph of the Legendre map. Then we prove that if $Z = (Z_1, \ldots, Z_k)$ is an integrable $k$-vector field, which is a solution to this equation and tangent to $M_L$, then the projection onto the first factor $\mathbb{R}^k \times T^1_1 Q$ of the integral sections of $Z$ are solutions to the Euler-Lagrange field equations. If $L$ is regular, the converse also holds. Furthermore, we establish the relationship between $Z$ and the Hamiltonian and the Lagrangian $k$-vector fields of the $k$-cosymplectic formalism, $X_H$ and $X_L$.

In Section 5 we present some basic facts on Lie algebroids, including results form differential calculus, morphisms and prolongations of Lie algebroids. In this section we also introduce a bundle $\mathcal{T}_k^E P = (\mathbb{R}^k \times \oplus k \oplus E) \times_{\mathbb{R}^k \times T^1_1 Q} T^1_1(\mathbb{R}^k \times P) \rightarrow P$ which is said to be the $k$-prolongation of a Lie algebroid $\tau : E \rightarrow Q$ over a fibration $\pi : P \rightarrow Q$. This space is a generalization of the prolongation of Lie algebroids and it is the fundamental geometric element to develop the Lagrangian and Hamiltonian $k$-cosymplectic field theory on Lie algebroids.

Section 6 is devoted to developing a Lagrangian and Hamiltonian $k$-cosymplectic description of field theories on Lie algebroids. In particular, in section 6.1 we develop the $k$-cosymplectic Lagrangian formalism on Lie algebroids. The fundamental point of this development is to consider the manifold $\mathcal{T}_k^E P$ with $P = \oplus E$ and the geometric objects defined on $\mathcal{T}_k^E (\oplus E)$. Given a Lagrangian function $L : \mathbb{R}^k \times \oplus k \oplus E \rightarrow \mathbb{R}$, solving certains equations we obtain a section $\xi_L$ of $\mathcal{T}_k^E (\oplus E)$ such that its integral sections are solutions to the Euler-Lagrange equations for $L$. Finally in section 6.1.3 we recover the standard Lagrangian $k$-cosymplectic formalism described in section 2.2 as a particular case of the section 6 when $E = T^* Q$. Section 6.2 is devoted to developing a Hamiltonian $k$-cosymplectic description of field theory on Lie algebroids. This description is similar to the Lagrangian case; now we consider the vector bundle $\mathcal{T}_k^E (\oplus E^*)$, where $E^* \rightarrow Q$ is the dual bundle of $E$. Given a Hamiltonian function $H : \mathbb{R}^k \times \oplus k \oplus E^* \rightarrow \mathbb{R}$, solving certains equations we obtain a section $\xi_H$ of $\mathcal{T}_k^E (\oplus E^*)$ such that its integral sections are solutions to the Hamilton equations for $H$. Taking $E = T^* Q$, the results in sections 5 and 5 coincide with the results of the standard $k$-cosymplectic formalism described in section 2. Thus the standard $k$-cosymplectic formalism can be recovered as a particular case of the description on Lie algebroids.

Manifolds are real, paracompact, connected and $C^\infty$. Maps are $C^\infty$. Sum over crossed repeated indices is understood.

2 The standard $k$-cosymplectic formalism in field theory

2.1 Hamiltonian formalism [28]

2.1.1 Geometric elements

Let $Q$ be a differentiable manifold, dim $Q = n$, and $\tau^*_Q : T^* Q \rightarrow Q$ its cotangent bundle. Denote by $(T^1_1)^* Q = T^* Q \oplus \ldots \oplus T^* Q$, the Whitney sum of $k$ copies of $T^* Q$. The manifold $(T^1_1)^* Q$ can be identified with the manifold $J^1(Q, \mathbb{R}^k)_0$ of 1-jets of mappings from $Q$ to $\mathbb{R}^k$ with target at $0 \in \mathbb{R}^k$,

$$J^1(Q, \mathbb{R}^k)_0 = T^* Q \oplus \ldots \oplus T^* Q \equiv (\sigma^1(q), \ldots, \sigma^k(q)),$$
where \( \sigma^A = \pi^A \circ \sigma : Q \to \mathbb{R} \) is the \( A \)th component of \( \sigma \), and \( \pi^A : \mathbb{R}^k \to \mathbb{R} \) is the canonical projection onto the \( A \)th component, for \( 1 \leq A \leq k \). \( (T^*_k)^{\pi}Q \) is called the cotangent bundle of \( k \)-covelocities of the manifold \( Q \).

The manifold \( J^1\pi_Q \) of 1-jets of sections of the trivial bundle \( \pi_Q : \mathbb{R}^k \times Q \to Q \) is diffeomorphic to \( \mathbb{R}^k \times (T^*_k)^{\pi}Q \), via the diffeomorphism given by

\[
J^1\pi_Q \quad \mathbb{R}^k \times (T^*_k)^{\pi}Q
\]

\[
j_q^1 \phi = j_q^1(\phi_Q, Id_Q) \quad (\phi_Q(q), \alpha^1_q, \ldots, \alpha^k_q),
\]

where \( \phi_Q : Q \xrightarrow{\phi} \mathbb{R}^k \times Q \xrightarrow{\pi^k} \mathbb{R}^k \), \( \alpha^A_q = d(\phi_Q)^A(q), 1 \leq A \leq k \), and \( (\phi_Q)^A : Q \xrightarrow{d\phi_Q} \mathbb{R}^k \xrightarrow{\pi^A} \mathbb{R} \).

Throughout the paper we use the following notation for the canonical projections

\[
\mathbb{R}^k \times (T^*_k)^{\pi}Q \xrightarrow{\pi_Q} \mathbb{R}^k \xrightarrow{\pi_Q} Q,
\]

and \( (\pi_Q)_1 = \pi_Q \circ (\pi_Q)_{1,0} \), where

\[
\pi_Q(t, q) = q, \quad (\pi_Q)_{1,0}(t, (\alpha^1_q, \ldots, \alpha^k_q)) = (t, q), \quad (\pi_Q)_1(t, (\alpha^1_q, \ldots, \alpha^k_q)) = q,
\]

with \( t \in \mathbb{R}^k \), \( q \in Q \) and \( (\alpha^1_q, \ldots, \alpha^k_q) \in (T^*_k)^{\pi}Q \).

If \( (q^i) \) are local coordinates on \( U \subseteq Q \), then the induced local coordinates \( (q^i, p_i) \), \( 1 \leq i \leq n \), on \( (\pi_Q)^{-1}(U) = T^*_U \subseteq T^*Q \), are given by

\[
q^i(\alpha_q) = q^i(q), \quad p_i(\alpha_q) = \alpha_q \left( \frac{\partial}{\partial q^i} \big| q \right),
\]

and the induced local coordinates \((t^A, q^i, p_i^A)\) on \((\pi_Q)_1^{-1}(U) = \mathbb{R}^k \times (T^*_k)^{\pi}U\) are given by

\[
t^A(j_q^1 \phi) = (\phi_Q(q))^A, \quad q^i(j_q^1 \phi) = q^i(q), \quad p_i^A(j_q^1 \phi) = d(\phi_Q)^A(q) \left( \frac{\partial}{\partial q^i} \big| q \right),
\]

or equivalently, for \( 1 \leq i \leq n \) and \( 1 \leq A \leq k \),

\[
t^A(t, (\alpha^1_q, \ldots, \alpha^k_q)) = t^A, \quad q^i(t, (\alpha^1_q, \ldots, \alpha^k_q)) = q^i(q), \quad p_i^A(t, (\alpha^1_q, \ldots, \alpha^k_q)) = \alpha^A_q \left( \frac{\partial}{\partial q^i} \big| q \right).
\]

On \( \mathbb{R}^k \times (T^*_k)^{\pi}Q \), we define the differential forms

\[
\eta^i = (\pi^A_1)^* dt^A, \quad \theta_0^A = (\pi^A_2)^* \theta_0, \quad \omega_0^A = (\pi^A_2)^* \omega_0,
\]

where \( \pi^A_1 : \mathbb{R}^k \times (T^*_k)^{\pi}Q \to \mathbb{R} \) and \( \pi^A_2 : \mathbb{R}^k \times (T^*_k)^{\pi}Q \to T^*Q \) are the projections defined by

\[
\pi_1^A(t, (\alpha^1_q, \ldots, \alpha^k_q)) = t^A, \quad \pi_2^A(t, (\alpha^1_q, \ldots, \alpha^k_q)) = \alpha^A_q,
\]

\( \omega = -d\theta = dq^i \wedge dp_i \) is the canonical symplectic form on \( T^*Q \) and \( \theta = p_i dq^i \) is the Liouville 1-form on \( T^*Q \). Obviously \( \omega^A = -d\theta^A \).

In local coordinates we have

\[
\eta^i = dt^A, \quad \theta^A = p_i^A dq^i, \quad \omega^A = dq^i \wedge dp_i^A.
\]

(1)

Finally, consider the vertical distribution of the bundle \((\pi_Q)_1,0 : \mathbb{R}^k \times (T^*_k)^{\pi}Q \to \mathbb{R}^k \times Q, \)

\[
V^* = ker ((\pi_Q)_1,0)_* = \left( \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right)_{i=1,\ldots,n}
\]

A simple inspection of the expressions in local coordinates \( [1] \) shows that the forms \( \eta^A \) and \( \omega^A \) are closed, and the following relations hold

A simple inspection of the expressions in local coordinates \( [1] \) shows that the forms \( \eta^A \) and \( \omega^A \) are closed, and the following relations hold
1. $\eta^1 \wedge \cdots \wedge \eta^k \neq 0$, \quad $(\eta^A)|_V \cdot = 0$, \quad $(\omega^A)|_V \cdot \cdot V \cdot = 0$,
2. $(\bigcap_{A=1}^k \ker \eta^A) \cap (\bigcap_{A=1}^k \ker \omega^A) = \{0\}$, \quad $\dim(\bigcap_{A=1}^k \ker \omega^A) = k$.

Then, from the above geometrical model, we have the following definition [28]:

**Definition 2.1** Let $M$ be a differentiable manifold of dimension $k(n+1)+n$. A $k$-cosymplectic structure on $M$ is a family $(\eta^A, \omega^A, V)$, where each $\eta^A$ is a closed 1-form, each $\omega^A$ is a closed 2-form and $V$ is an integrable $nk$-dimensional distribution on $M$, satisfying 1 and 2. In this case, $M$ is said to be an $k$-cosymplectic manifold.

**Theorem 2.1** [28] (Darboux Theorem) If $M$ is a $k$-cosymplectic manifold, then around each point of $M$ there exist local coordinates $(x^A, y^i, z_i^A)$; $1 \leq A \leq k, 1 \leq i \leq n$, such that

$$\eta^A = dx^A, \quad \omega^A = dy^i \wedge dz_i^A, \quad V = \left\langle \frac{\partial}{\partial z_i^1}, \ldots, \frac{\partial}{\partial z_i^k} \right\rangle_{i=1,\ldots,n}.$$  

The canonical model for these geometrical structures is $(\mathbb{R}^k \times (T^1_k)*Q, \eta^A, \omega^A, V^*)$.

For every $k$-cosymplectic structure $(\eta_A, \omega_A, V)$ on $M$, there exists a family of $k$ vector fields $\{R_A\}$ characterized by the following conditions

$$\iota_{R_A} \eta^B = \delta^B_A, \quad \iota_{R_A} \omega^B = 0, \quad 1 \leq A, B \leq k$$

They are called the *Reeb vector fields* associated to the $k$-cosymplectic structure. In the canonical model $R_A = \partial / \partial t^A$. Observe that the vector fields $\{\partial / \partial t^A\}$ are defined intrinsically in $\mathbb{R}^k \times (T^1_k)*Q$, and span locally the vertical distribution with respect to the projection $\mathbb{R}^k \times (T^1_k)*Q \to (T^1_k)*Q$.

### 2.1.2 $k$-vector fields and integral sections

Let $M$ be an arbitrary manifold, $T^1_k M$ the Whitney sum $TM \oplus k$. $\oplus TM$ of $k$ copies of $TM$, and $\tau_M : T^1_k M \to M$ its canonical projection. $\tau_M : T^1_k M \to M$ is usually called the tangent bundle of $k^1$-velocities of $M$, the reason for this name will be explained later in Section 2.2.1.

**Definition 2.2** A $k$-vector field on $M$ is a section $X : M \to T^1_k M$ of the projection $\tau_M$.

Since $T^1_k M$ is the Whitney sum $TM \oplus k$. $\oplus TM$ of $k$ copies of $TM$, we deduce that giving a $k$-vector field $X$ is equivalent to giving a family of $k$ vector fields $X_1, \ldots, X_k$ on $M$ by projecting $X$ onto every factor. For this reason we will denote a $k$-vector field by $(X_1, \ldots, X_k)$.

**Definition 2.3** An integral section of the $k$-vector field $(X_1, \ldots, X_k)$ passing through a point $x \in M$ is a map $\phi : U_0 \subset \mathbb{R}^k \to M$, defined on some neighborhood $U_0$ of $0 \in \mathbb{R}^k$, such that

$$\phi(0) = x, \quad \phi_*(t) \left( \left. \frac{\partial}{\partial t^A} \right|_{t=1} \right) = X_A(\phi(t)),$$  

for $t \in U_0$.

We say that a $k$-vector field $(X_1, \ldots, X_k)$ on $M$ is integrable if there is an integral section passing through each point of $M$.

Observe that, if $k = 1$, this definition coincides with the definition of integral curve of a vector field. In the $k$-cosymplectic formalism, the solutions to the field equations are described as the integral sections of some $k$-vector fields.
2.1.3 $k$-cosymplectic Hamiltonian formalism

Let $H: \mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R}$ be a Hamiltonian function. Let $X = (X_1, \ldots, X_k)$ be a $k$-vector field on $\mathbb{R}^k \times (T^1_k)^*Q$, which is a solution to the following equations

$$
dt^A(X_B) = \delta_B^A, \quad \sum_{i=1}^k \iota_{X_A} \omega^A = dH - \sum_{A=1}^k \frac{\partial H}{\partial q^A} \dt^A. \tag{2}
$$

If $X = (X_1, \ldots, X_k)$ is an integrable $k$-vector field, locally given by

$$
X_A = (X_A)^B \frac{\partial}{\partial t^B} + (X_A)^i \frac{\partial}{\partial q^i} + (X_A)_B \frac{\partial}{\partial p^B_i}
$$

then

$$
(X_A)^B = \delta_B^A, \quad \frac{\partial H}{\partial p^A_i} = (X_A)^i, \quad \frac{\partial H}{\partial q^i} = - \sum_{A=1}^k (X_A)^A_i, \tag{3}
$$

and if $\psi: \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^*Q$, locally given by $\psi(t) = (\psi^A(t), \psi^i(t), \psi^A_i(t))$, is an integral section of $X$, then

$$
\frac{\partial \psi^A}{\partial t^B} = \delta_{AB}, \quad \frac{\partial \psi^i}{\partial t^B} = (X_B)^i, \quad \frac{\partial \psi^A_i}{\partial t^B} = (X_B)_i^A.
$$

Therefore, from (3) we obtain that $\phi(t)$ is a solution to the Hamiltonian field equations

$$
\frac{\partial H}{\partial q^i} = - \sum_{A=1}^k \frac{\partial \psi^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \psi^i}{\partial t^A}. \tag{4}
$$

So, equations (2) can be considered as a geometric version of the Hamiltonian field equations.

**Remark 2.1** The above Darboux theorem allows us to write the Hamiltonian formalism in an arbitrary $k$-cosymplectic manifold $(M, \eta^A, \omega^A, V)$, substituting the equations (2) by

$$
\eta^A(X_B) = \delta_B^A, \quad \sum_{A=1}^k \iota_{X_A} \omega^A = dH + \sum_{A=1}^k (1 - R_A(H)) \eta^A.
$$

If $(M, \eta^A, \omega^A, V)$ is a $k$-cosymplectic manifold, we can define the vector bundle morphism

$$
\Omega^\sharp: \quad T^1_k M \quad \to \quad T^* M
$$

$$(X_1, \ldots, X_k) \quad \mapsto \quad \Omega^\sharp(X_1, \ldots, X_k) = \sum_{A=1}^k \iota_{X_A} \omega^A + \eta^A(X_A) \eta^A
$$

and denoting by $\mathcal{M}_k(C^\infty(M))$ the space of matrices of order $k$ whose entries are functions on $M$, we can also define the vector bundle morphism

$$
\eta^\sharp: \quad T^1_k M \quad \to \quad \mathcal{M}_k(C^\infty(M))
$$

$$(X_1, \ldots, X_k) \quad \mapsto \quad \eta^\sharp(X_1, \ldots, X_k) = (\eta^A(X_B)).
$$

Then, the solutions to (2) are given by $(X_1, \ldots, X_k) + (\ker \Omega^\sharp \cap \ker \eta^\sharp)$, where $(X_1, \ldots, X_k)$ is a particular solution.
2.2 Lagrangian formalism

2.2.1 Geometric elements

The tangent bundle of \(k^1\)-velocities of a manifold

Let \(\tau_Q : TQ \rightarrow Q\) be the tangent bundle of \(Q\). Let us denote by \(T^1_kQ\) the Whitney sum \(TQ \oplus \cdots \oplus TQ\) of \(k\) copies of \(TQ\), with projection \(\tau^k_Q : T^1_kQ \rightarrow Q\), \(\tau^k_Q(v_{1q}, \ldots, v_{kq}) = q\). \(T^1_kQ\) can be identified with the manifold \(J_0^1(\mathbb{R}^k, Q)\) of the \(k^1\)-velocities of \(Q\), that is, 1-jets of maps \(\sigma : \mathbb{R}^k \rightarrow Q\) with source at \(0 \in \mathbb{R}^k\), say

\[
J_0^1(\mathbb{R}^k, Q) = TQ \oplus \cdots \oplus TQ \equiv (v_{1q}, \ldots, v_{kq}),
\]

where \(q = \sigma(0)\), and \(v_{Aq} = \sigma_*(0)(\frac{\partial}{\partial t^A}|_0)\). \(T^1_kQ\) is the tangent bundle of \(k^1\)-velocities of \(Q\) [37].

If \((q^i)\) are local coordinates on \(U \subseteq Q\), the induced coordinates \((q^i, v^i)\) on \(TU = \tau_Q^{-1}(U)\) are

\[
q^i(v_q) = q^i(q), \quad v^i(v_q) = v_q(q^i),
\]

and the induced coordinates \((q^i, v^i_A)\), \(1 \leq i \leq n\), \(1 \leq A \leq k\), on \(T^1_kU = (\tau^k_Q)^{-1}(U)\) are given by

\[
q^i(v_{1q}, \ldots, v_{kq}) = q^i(q).
\]

In general, if \(F : M \rightarrow N\) is a differentiable map, then the induced map \(T^1_kF : T^1_kM \rightarrow T^1_kN\) defined by \(T^1_k(F)(j^0_0g) = j_0^0(F \circ g)\) is given by

\[
T^1_k(F)(v_{1q}, \ldots, v_{kq}) = (F_*(q)v_{1q}, \ldots, F_*(q)v_{kq}),
\]

where \(v_{1q}, \ldots, v_{kq} \in T_Q\mathbb{R}^k\), \(q \in \mathbb{R}^k\), and \(F_*(q) : T^1_q\mathbb{R}^k \rightarrow T^1_{F(q)}\mathbb{R}^k\) is the induced map.

The manifold \(T^1_k(\mathbb{R}^k \times P)\)

Let \(\pi : P \rightarrow Q\) be an arbitrary fibration. We use the tangent bundle of \(k^1\)-velocities of the manifold \(\mathbb{R}^k \times P\), then we consider the vector bundle \(\tau^k_{\mathbb{R}^k \times P} : T^1_k(\mathbb{R}^k \times P) \rightarrow \mathbb{R}^k \times P\).

An element \(W_{(t,p)} \in (\tau^k_{\mathbb{R}^k \times P})^{-1}(t,p)\) is given by \(W_{(t,p)} = (v_{1(t,p)}, \ldots, v_{k(t,p)})\).

Since \((t^A, q^i, u^\vartheta)\) are the local coordinates on \(\mathbb{R}^k \times P\), we write each vector \((v_A)(t,p)\) as follows

\[
(v_A)(t,p) = (v_A)_B \frac{\partial}{\partial t^B}|_{(t,p)} + (v_A)^i \frac{\partial}{\partial q^i}|_{(t,p)} + (v_A)^\vartheta \frac{\partial}{\partial u^\vartheta}|_{(t,p)}.
\]

Thus the local coordinates \((t^A, q^i, u^\vartheta)\) induce the local coordinates \((t^A, q^i, u^\vartheta, (v_A)_B, (v_A)^i, (v_A)^\vartheta)\) in \(T^1_k(\mathbb{R}^k \times P)\).

Taking into account the identification \(T^1_k(\mathbb{R}^k \times P) \equiv T^1_k\mathbb{R}^k \times T^1_k(P)\) given by

\[
(v_{1(t,p)}, \ldots, v_{k(t,p)}) = \left((v_1)_B \frac{\partial}{\partial t^B}|_t, \ldots, (v_k)_B \frac{\partial}{\partial t^B}|_t; (v_1)^i \frac{\partial}{\partial q^i}|_p, (v_1)^\vartheta \frac{\partial}{\partial u^\vartheta}|_p, \ldots, (v_k)^i \frac{\partial}{\partial q^i}|_p, (v_k)^\vartheta \frac{\partial}{\partial u^\vartheta}|_p\right),
\]

we consider the map

\[
F = \tau^k_{\mathbb{R}^k} \times T^1_k(\pi) : T^1_k(\mathbb{R}^k \times P) \equiv T^1_k\mathbb{R}^k \times T^1_k(P) \rightarrow \mathbb{R}^k \times T^1_kQ.
\]
which is given by

\[ F(W(t,p)) = F(v_1(t,p), \ldots, v_k(t,p)) = (t, (v_1)^i \frac{\partial}{\partial q^i}|_q, \ldots, (v_k)^i \frac{\partial}{\partial q^i}|_q), \quad (6) \]

since \( T^1_k(\pi) = T\pi \times \ldots \times T\pi \), and

\[ T^k_p\pi \left( (v_A)^i \frac{\partial}{\partial q^i}|_p + (v_A)^\theta \frac{\partial}{\partial q^i}|_p \right) = (v_A)^i \frac{\partial}{\partial q^i}|_q \quad 1 \leq A \leq k. \]

Thus

\[ F = \tau^{k}_{\mathbb{R}^k} \times T^1_k : (t^A, q^i, u^\theta, (v_A)^B, (v_A)^i, (v_A)^\theta) \rightarrow (t^A, q^i, (v_A)^i) \]

This map \( F \) will be used in the description of the Tulczyjew’s Lagrangian formalism and in the definition of the fiber bundle \( T^E_k P \) (see section \ref{sec:6}), which is the fundamental geometric element to develop the \( k \)-cosymplectic formalism on Lie algebroids (see section \ref{sec:6}).

The manifold \( \mathbb{R}^k \times T^1_k Q \)

Next we see that the manifold \( \mathbb{R}^k \times T^1_k Q \) is a cosymplectic manifold when a regular Lagrangian \( L : \mathbb{R}^k \times T^1_k Q \rightarrow \mathbb{R} \) is given.

The manifold \( J^1\pi_{\mathbb{R}^k} \) of 1-jets of sections of the trivial bundle \( \pi_{\mathbb{R}^k} : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \) is diffeomorphic to \( \mathbb{R}^k \times T^1_k Q \), via the diffeomorphism given by

\[ J^1\pi_{\mathbb{R}^k} \rightarrow \mathbb{R}^k \times T^1_k Q \]

\[ j^1_t \phi = j^1_t(Id_{\mathbb{R}^k}, \phi_Q) \rightarrow (t, v_1, \ldots, v_k), \]

where \( \phi_Q : \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q \) and \( v_A = (\phi_Q)_* (t) \left( \frac{\partial}{\partial A} \right)_t \).

Denote by \( p_Q : \mathbb{R}^k \times T^1_k Q \rightarrow Q \) the canonical projection, that is \( p_Q(t, v_1, \ldots, v_k) = q \). If \( (q^i) \) are local coordinates on \( U \subseteq Q \), then the induced local coordinates \( (q^i, v_i), 1 \leq i \leq n, \) on \( \tau^{-1}_Q(U) = TU \subseteq TQ \), are given by

\[ q^i(v_q) = q^i(q), \quad v_i(v_q) = v_q(q^i), \]

and then the induced local coordinates \( (t^A, q^i, v^i_A) \) on \( p^{-1}_Q(U) = \mathbb{R}^k \times T^1_k U \) are given by

\[ t^A(j^1_t \phi) = t^A; \quad q^i(j^1_t \phi) = q^i(\phi_Q(t)); \quad v^i_A(j^1_t \phi) = \frac{\partial(q^i \circ \phi_Q)}{\partial t^A}(t) \]

or equivalently,

\[ t^A(t, v_1, \ldots, v_k) = t^A; \quad q^i(t, v_1, \ldots, v_k) = q^i(q); \quad v^i_A(t, v_1, \ldots, v_k) = v^i_A(q^i), \]

Throughout the paper we use the following notation for the canonical projections

\[ \mathbb{R}^k \times (T^1_k Q) \xrightarrow{(\pi_{\mathbb{R}^k})^1,0} \mathbb{R}^k \times Q \xrightarrow{\pi_{\mathbb{R}^k}} \mathbb{R}^k \]

and \( (\pi_{\mathbb{R}^k})_1 = \pi_{\mathbb{R}^k} \circ (\pi_{\mathbb{R}^k})_1,0 \), where, for \( t \in \mathbb{R}^k \), \( q \in Q \) and \( (v_1, \ldots, v_k) \in T^1_k Q \),

\[ \pi_{\mathbb{R}^k}(t, q) = t, \quad (\pi_{\mathbb{R}^k})_1(t, v_1, \ldots, v_k) = (t, q), \quad (\pi_{\mathbb{R}^k})_1(t, v_1, \ldots, v_k) = t. \]
Canonical vector fields and tensor fields on $\mathbb{R}^k \times T^1_k Q$. Poincaré-Cartan forms

Denote by $\Delta$ the canonical vector field (Liouville vector field) of the vector bundle $(\pi_{\mathbb{R}^k})_{1,0} : \mathbb{R}^k \times T^1_k Q \rightarrow \mathbb{R}^k \times Q$. This vector field $\Delta$ is the infinitesimal generator of the following flow

$$\mathbb{R} \times (\mathbb{R}^k \times T^1_k Q) \rightarrow \mathbb{R}^k \times T^1_k Q$$

$$(s, (t, v_{1q}, \ldots, v_{kq})) \mapsto (t, e^sv_{1q}, \ldots, e^sv_{kq})$$

and in local coordinates it has the form

$$\Delta = \sum_{i,A} v_A^i \frac{\partial}{\partial v_A^i}.$$

$\Delta$ can be written as the sum $\Delta = \sum_{A=1}^k \Delta_A$, where each vector field $\Delta_A$ is the infinitesimal generator of the following flow

$$\mathbb{R} \times (\mathbb{R}^k \times T^1_k Q) \rightarrow \mathbb{R}^k \times T^1_k Q$$

$$(s, (t, v_{1q}, \ldots, v_{kq})) \mapsto (t, v_{1q}, \ldots, v_{A-1q}, e^sv_{Aq}, v_{A+1q}, \ldots, v_{kq}).$$

**Definition 2.4** For a vector $X_q$ at $Q$, and for $A = 1, \ldots, k$, we define its vertical $A$-lift $(X_q)^A$ as the local vector field on $\tau_Q^{-1}(q) \subset T^1_k Q$ given by

$$(X_q)^A(w_q) = \frac{d}{ds}\bigg|_{s=0} \left(w_q + (0, \ldots, 0, s \ A \ X_q, 0, \ldots, 0) \right), \text{ for } w_q = (v_{1q}, \ldots, v_{kq}) \in T^1_k Q.$$  

In local coordinates, for a vector $X_q = a^i \frac{\partial}{\partial q^i}$ we have

$$(X_q)^A = a^i \frac{\partial}{\partial v_A^i}.$$  

(8)

The canonical $k$-tangent structure on $T^1_k Q$ is the set $(S^1, \ldots, S^k)$ of tensor fields of type $(1, 1)$ defined by

$$S^A(w_q)(Z_{w_q}) = (\tau_q(w_q)(Z_{w_q}))^A, \text{ for all } Z_{w_q} \in T_{w_q}(T^1_k Q), w_q \in T^1_k Q.$$  

From (3), in local coordinates we have

$$S^A = \frac{\partial}{\partial v_A^i} \otimes dq^i.$$  

(9)

The tensors $S^A$ can be regarded as the $(0, \ldots, 0, 1, 0, \ldots, 0)$-lift of the identity tensor on $Q$ to $T^1_k Q$ defined in [37].

In an obvious way, we consider the extension of $S^A$ to $\mathbb{R}^k \times T^1_k Q$, which we also denote by $S^A$, and they have the same local expressions [29].

The $k$-tangent manifolds were introduced as a generalization of the tangent manifolds in [26, 27]. The canonical model of these manifolds is $T^1_k Q$ with the structure given by $(S^1, \ldots, S^k)$.

As in the case of mechanical systems, these tensor fields $S^A$ allow us to introduce the forms $\theta_L^A$ and $\omega_L^A$ on $\mathbb{R}^k \times T^1_k Q$ as follows

$$\theta_L^A = dL \circ S^A, \quad \omega_L^A = -d\theta_L^A,$$

with local expressions

$$\theta_L^A = \frac{\partial L}{\partial v_A^i} dq^i, \quad \omega_L^A = dq^i \wedge d \left( \frac{\partial L}{\partial v_A^i} \right).$$  

(10)
Second order partial differential equations on $\mathbb{R}^k \times T^1_k Q$

Now we characterize the integrable $k$-vector fields on $\mathbb{R}^k \times T^1_k Q$ whose integral sections are canonical prolongations of maps from $\mathbb{R}^k$ to $Q$.

**Definition 2.5** A $k$-vector field $X = (X_1, \ldots, X_k)$ on $\mathbb{R}^k \times T^1_k Q$ is a second order partial differential equation (SOPDE for short) if:

$$dt^A(X_B) = \delta^A_B, \quad S^A(X_A) = \Delta_A.$$ 

A direct computation in local coordinates shows that the local expression of a SOPDE $X = (X_1, \ldots, X_k)$ is

$$X_A(t, q^i, v^i_A) = \frac{\partial}{\partial t^A} + v^i_A \frac{\partial}{\partial q^i} + (X_A)_B^{i} \frac{\partial}{\partial v^B_i}, \quad (11)$$

where $(X_A)_B^{i}$ are functions on $\mathbb{R}^k \times T^1_k Q$. As a direct consequence of the above local expressions, we deduce that the family of vector fields $\{X_1, \ldots, X_k\}$ are linearly independent.

**Definition 2.6** Let $\phi : \mathbb{R}^k \to Q$ be a map, the first prolongation $\phi^{[1]}$ of $\phi$ is the map

$$\phi^{[1]} : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q, \quad t \mapsto (t, j_0^1 \phi_t) \equiv \left( t, \phi_t(t) \left( \frac{\partial}{\partial t^1} \right), \ldots, \phi_t(t) \left( \frac{\partial}{\partial t^k} \right) \right),$$

where $\phi_t(s) = \phi(t + s)$. In local coordinates

$$\phi^{[1]}(t^1, \ldots, t^k) = \left( t^1, \ldots, t^k, \phi_t(t^1, \ldots, t^k), \frac{\partial \phi}{\partial t^A}(t^1, \ldots, t^k) \right),$$

**Lemma 2.1** Let $(X_1, \ldots, X_k)$ be a SOPDE. A map $\psi : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q$, given by $\psi(t) = (\psi^A(t), \psi^i(t), \psi_A^i(t))$, is an integral section of $(X_1, \ldots, X_k)$ if, and only if,

$$\psi^A(t) = t^A + c^A, \quad \psi_A^i(t) = \frac{\partial \psi^i}{\partial t^A}(t), \quad \frac{\partial^2 \psi^i}{\partial t^A \partial t^B}(t) = (X_A)_B^i(\psi(t)). \quad (12)$$

(Proof) Equations (12) follow from Definition 2.3 and (11).

**Remark 2.2** The integral sections of a SOPDE are given by $\psi(t) = \left( t^A + c^A, \psi^i(t), \frac{\partial \psi^i}{\partial t^A}(t) \right)$, where the functions $\psi^i(t)$ satisfy the third equation in (12) and $c^A$ are constants. In the particular case $c = 0$, we have that $\psi = \phi^{[1]}$ where $\phi = p_Q \circ \psi : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q \overset{p_Q}{\to} Q$; that is, $\phi(t) = (\psi^i(t))$.

Conversely if $\phi : \mathbb{R}^k \to Q$ is any map such that

$$\frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (X_A)_B^i(\phi^{[1]}(t)),$$

then $\phi^{[1]}$ is an integral section of $(X_1, \ldots, X_k)$.

Observe that if $(X_1, \ldots, X_k)$ is integrable, from (12) we deduce that $(X_A)_B^i = (X_B)_A^i$. 

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Lemma 2.2 Let $X = (X_1, \ldots, X_k)$ be an integrable $k$-vector field on $\mathbb{R}^k \times T^1_kQ$. If every integral section of $X$ is the first prolongation $\phi^{[1]}$ of map $\phi : \mathbb{R}^k \rightarrow Q$, then $X$ is a sopde.

(Proof) Let us suppose that each $X_A$ is locally given by

$$X_A(t, q^i, v^j_B) = (X_A)^B \frac{\partial}{\partial t^B} + (X_A)^i_q \frac{\partial}{\partial q^i} + (X_A)^i_B \frac{\partial}{\partial v^j_B}. \quad (13)$$

Let $\psi = \phi^{[1]} : \mathbb{R}^k \rightarrow \mathbb{R}^k \times T^1_kQ$ be an integral section of $X$, then from (11), (13) and Definitions 2.3 and 2.6 we obtain

$$(X_A)^B(\phi^{[1]}(t)) = \delta^B_A, \quad (X_A)^i(\phi^{[1]}(t)) = \frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{[1]}(t)), \quad (X_A)^i_B(\phi^{[1]}(t)) = \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t)$$

thus $X_A$ is locally given as in (11). \qed

2.2.2 The Legendre map and the Lagrangian forms

Given a Lagrangian $L : \mathbb{R}^k \times T^1_kQ \rightarrow \mathbb{R}$ the Legendre map $FL : \mathbb{R}^k \times T^1_kQ \rightarrow \mathbb{R}^k \times (T^1_kQ)^*$ is defined as follows:

$$FL(t, v_{1q}, \ldots, v_{kq}) = (t, \ldots, [FL(t, v_{1q}, \ldots, v_{kq})]^A, \ldots),$$

where

$$[FL(t, v_{1q}, \ldots, v_{kq})]^A_q(u_q) = \frac{d}{ds}\big|_{s=0} L(t, v_{1q}, \ldots, v_{Aq} + su_q, \ldots, v_{kq}).$$

It is locally given by

$$FL : (t^A, q^i, v^j_A) \longrightarrow \left(t^A, q^i, \frac{\partial L}{\partial v^j_A}\right). \quad (14)$$

From (10) and (11) the following identities hold

$$\theta^A_L = FL^* \theta^A, \quad \omega^A_L = FL^* \omega^A. \quad (15)$$

Definition 2.7 A Lagrangian function $L : \mathbb{R}^k \times T^1_kQ \rightarrow \mathbb{R}$ is said to be regular (resp. hyperregular) if the corresponding Legendre map $FL$ is a local (resp. global) diffeomorphism. Elsewhere $L$ is called a singular Lagrangian.

From (14) we obtain that $L$ is regular if, and only if, $\det \left( \frac{\partial^2 L}{\partial v^i_A \partial v^j_B} \right) \neq 0, 1 \leq i, j \leq n, 1 \leq A, B \leq k$. Therefore (see [29]):

Proposition 2.1 The following conditions are equivalent: (i) $L$ is regular, (ii) $FL$ is a local diffeomorphism. (iii) $(dt^A, \omega^A_L, V_*)$ is a $k$-cosymplectic structure on $\mathbb{R}^k \times T^1_kQ$ where $V_* = \ker ((\pi_{\mathbb{R}^k})_{1,0})_* = \langle \frac{\partial}{\partial v^1_1}, \ldots, \frac{\partial}{\partial v^1_n} \rangle_{i=1,\ldots,n}$ is the vertical distribution of the bundle $(\pi_{\mathbb{R}^k})_{1,0} : \mathbb{R}^k \times T^1_kQ \rightarrow \mathbb{R}^k \times Q$.  

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Suppose that a physical system is described by a singular Lagrangian function $L : \mathbb{R}^k \times T^1_k Q \to \mathbb{R}$ is called almost-regular if $P := FL(T^1_k Q)$ is a closed submanifold of $\mathbb{R}^k \times (T^1_k)^* Q$ (we will denote the natural imbedding by $j_0 : P \hookrightarrow \mathbb{R}^k \times (T^1_k)^* Q$, FL is a submersion onto its image, and the fibres $FL^{-1}(FL(t, w_q))$, for every $(t, w_q) \in \mathbb{R}^k \times T^1_k Q$, are connected submanifolds of $\mathbb{R}^k \times T^1_k Q$.

Observe that the vector fields $\frac{\partial}{\partial t^i}$ are tangent to $P$.

2.2.3 $k$-cosymplectic Lagrangian formalism

Suppose that a physical system is described by $n$ functions $\psi^i(t^1, \ldots, t^k)$. Associated with this system there is a Lagrangian $L(t^A, \psi^i, \psi^j_A)$, with $\psi^i_A = \frac{\partial \psi^i}{\partial t^A}$, then the Euler-Lagrange equations are

$$ \sum_{A=1}^k \left( \frac{\partial^2 L}{\partial A^A \partial \psi^i_A} + \frac{\partial^2 L}{\partial q^j \partial \psi^i_A} \frac{\partial \psi^j_A}{\partial A^A} + \frac{\partial^2 L}{\partial \psi^j_B^B \partial \psi^i_A^A} \frac{\partial^2 \psi^j_B}{\partial A^A \partial B} \right) = \frac{\partial L}{\partial q^i}, $$

and we can consider that the Lagrangian $L$ is defined on $\mathbb{R}^k \times T^1_k Q$, that is $L = L(t^A, q^i, \psi^j_A)$, and we can write the Euler-Lagrange equations as

$$ \sum_{A=1}^k \frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial \psi^i_A}(\psi(t)) \right) = \frac{\partial L}{\partial q^i}(\psi(t)), \quad \psi^i_A(\psi(t)) = \frac{\partial (q^j \circ \psi)}{\partial t^A}(t), \quad (16) $$

where each solution $\psi : U_0 \subset \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q$ is given by $\psi(t) = \left( t, \psi^i(t), \frac{\partial \psi^i}{\partial t^A}(t) \right)$.

Thus each solution $\psi(t)$ of the Euler-Lagrange equations is a first prolongation of a map $\phi : U_0 \subset \mathbb{R}^k \to Q$ given by $\phi(t) = (\psi^i(t))$.

Next we give a geometrical description of these equations. Let us consider the equations

$$ dt^A((X_L)_B) = \delta^A_B, \quad \sum_{A=1}^k i_{(X_L)_A} \omega^A_L = dE_L + \sum_{A=1}^k \frac{\partial L}{\partial A^A} dt^A, \quad (17) $$

where $E_L = \Delta(L) - L$. If $(X_L)_A$ is locally given by

$$ (X_L)_A = ((X_L)_A)_B^B \frac{\partial}{\partial t^B} + ((X_L)_A)^i_q \frac{\partial}{\partial q^i} + ((X_L)_A)_A^B \frac{\partial}{\partial v_B^i} , $$

we obtain that (17) is equivalent to the equations

$$ ((X_L)_A)_B^B = \delta_A^B, \quad ((X_L)_B)^i_q \frac{\partial^2 L}{\partial t^A \partial v_B^i} = v_B^i \frac{\partial^2 L}{\partial t^A \partial v_B^i}, $$

$$ ( (X_L)_C^j \frac{\partial^2 L}{\partial v_B^i \partial v_C^j} = v_B^i \frac{\partial^2 L}{\partial v_B^i \partial v_C^j} \quad (18) $$

and

$$ \frac{\partial^2 L}{\partial q^i \partial v_B^i} (v_B^i - ((X_L)_B)^i_q) + \frac{\partial^2 L}{\partial t^B \partial v_B^i} + v_B^k \frac{\partial^2 L}{\partial q^k \partial v_B^i} + ( (X_L)_B)_A^B \frac{\partial^2 L}{\partial v_B^i \partial v_C^j} = \frac{\partial L}{\partial q^i} \quad (19) $$

When $L$ is regular, from (18) we obtain that this last equation can be written as follows

$$ \frac{\partial^2 L}{\partial t^B \partial v_B^i} + v_B^k \frac{\partial^2 L}{\partial q^k \partial v_B^i} + ( (X_L)_B)_A^B \frac{\partial^2 L}{\partial v_B^i \partial v_C^j} = \frac{\partial L}{\partial q^i}, \quad (20) $$
and then \((X_L)_A\) is locally given by
\[
(X_L)_A = \frac{\partial}{\partial t^A} + v^{i}_B \frac{\partial}{\partial q^i} + ((X_L)_A)^{i}_B \frac{\partial}{\partial v^i_B};
\]
that is, \(((X_L)_1, \ldots, (X_L)_k)\) is a sopde.

**Theorem 2.2** Let \(L\) be a Lagrangian and \(X_L = ((X_L)_1, \ldots, (X_L)_k)\) a \(k\)-vector field such that
\[
dt^A((X_L)_B) = \delta^A_B, \quad \sum_{A=1}^k i_{(X_L)_A} \omega^A_L = dE_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} \dt^A. \tag{21}
\]

1. If \(L\) is regular, then \(X_L = ((X_L)_1, \ldots, (X_L)_k)\) is a sopde. If \(\psi : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q\) is an integral section of \(X_L\), then \(\phi : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q\) \(\psi\) is a solution to the Euler-Lagrange equations \([17]\).

2. If \(((X_L)_1, \ldots, (X_L)_k)\) is integrable, and \(\phi^{[1]} : \mathbb{R}^k \to \mathbb{R}^k \times T^1_k Q\) is an integral section, then \(\phi : \mathbb{R}^k \to Q\) is a solution to the Euler-Lagrange equations \([16]\).

(Proof) 1 is a consequence of \([18]\) and \([20]\). If \(\phi^{[1]}\) is an integral section, from \([19]\) and the local expression of \(\phi^{[1]}\) we have that \(\phi\) is a solution to the Euler-Lagrange equations \([16]\).

**Remark 2.3** If \(L : \mathbb{R}^k \times T^1_k Q \to \mathbb{R}\) is a regular Lagrangian, then \((dt^A, \omega^A_L, V_\phi)\) is a \(k\)-cosymplectic structure on \(\mathbb{R}^k \times T^1_k Q\). The Reeb vector fields \((R_L)_A\) corresponding to this \(k\)-cosymplectic structure are characterized by
\[
i_{(R_L)_A} dt^B = \delta^B_A, \quad i_{(R_L)_A} \omega^B_L = 0,
\]
and satisfy \((R_L)_A (E_L) = -\partial L / \partial t^A\).

If the Lagrangian \(L\) is hyper-regular, that is, \(FL\) is a diffeomorphism, then we can define a Hamiltonian function \(H : \mathbb{R}^k \times (T^1_k)^* Q \to \mathbb{R}\) by \(H = E_L \circ FL^{-1}\) where \(FL^{-1}\) is the inverse map of \(FL\). Then we have the following:

**Theorem 2.3**

1. If \(X_L = ((X_L)_1, \ldots, (X_L)_k)\) is a solution to \([17]\), then \(X_H = ((X_H)_1, \ldots, (X_H)_k)\), where \((X_H)_A = FL_\ast ((X_L)_A)\) is a solution to the equations \([3]\) in \(\mathbb{R}^k \times (T^1_k)^* Q\), with \(\eta^A = \eta^A_0\), \(\omega^A = \omega^A_0\), and \(H = E_L \circ FL^{-1}\).

2. If \(X_L = ((X_L)_1, \ldots, (X_L)_k)\) is integrable and \(\phi^{[1]}\) is an integral section, then \(\varphi = FL \circ \phi^{[1]}\) is an integral section of \(X_H = ((X_H)_1, \ldots, (X_H)_k)\) and thus it is a solution to the Hamilton field equations \([4]\) for \(H = E_L \circ FL^{-1}\).

(Proof)

1. It is an immediate consequence of \([2]\) and \([17]\) using that \(FL_\ast \eta^A_0 = dt^A\), \(FL_\ast \omega^A_0 = \omega^A_L\), and \(E_L = H \circ FL^{-1}\).
Remark 2.4 If we rewrite the equations (21) for the case $k = 1$, we have
\[
dt(X_L) = 1 \quad , \quad i_{X_L} \omega_L = dE_L + \frac{\partial L}{\partial t} dt ,
\]
which are equivalent to the dynamical equations
\[
dt(X_L) = 1 \quad , \quad i_{X_L} \Omega_L = 0
\]
where $\Omega_L = \omega_L + dE_L \wedge dt$ is the Poincaré-Cartan 2-form associated to the Lagrangian $L$ (see [8]). This describes the non-autonomous Lagrangian mechanics. Then, applying theorem 2.3 the non-autonomous Hamiltonian mechanics is obtained.

If the Lagrangian $L$ is singular, then the existence of solutions to the equations (17) is not assured except, perhaps, in a submanifold of $\mathbb{R}^k \times T^1_k Q$ (see [29]). Furthermore, when these solutions exist, they are not SOPDE, in general. Thus, in order to recover the Euler-Lagrange equations (16), the following condition must be added to the equations (17) (see definition 2.5):
\[
S^A(X_A) = \Delta_A .
\]

If the Lagrangian is almost-regular, then there exists $H_0 \in C^\infty(P)$ such that $(FL_0)^* H_0 = E_L$, where $FL_0: \mathbb{R}^k \times T^1_k Q \rightarrow P$ is defined by $j_0 \circ FL_0 = FL$. The Hamiltonian field equation analogous to (2) should be
\[
\eta^A((X_0)_B) = \delta^A_B, \quad \sum_{i=1}^k u_{(X_0)_A}(j_0^*(\omega^A_0)) = dH_0 - \sum_{A=1}^k \frac{\partial H_0}{\partial A} j_0^*(\eta^A).
\]
where $X_0 = ((X_0)_1, \ldots, (X_0)_k)$ (if it exists) is a $k$-vector field on $P$. The existence of $k$-vector fields in $P$ solution to the above equations is not assured except, perhaps, in a submanifold of $P$.

3 Tulczyjew’s submanifolds for $k$-cosymplectic field theories

This approach is inspired in papers [45, 46] and [21, 24].

3.1 Derivations on $T^1_k M$

Let us denote with $\Lambda N$ the algebra of the exterior differential forms on an arbitrary manifold $N$. In [45, 46], a derivation $i_T$ of degree $-1$ from $\Lambda M$ on $\Lambda TM$ over $\tau_M : TM \rightarrow M$ was defined in an arbitrary manifold $M$ by $i_T \mu = 0$ if $\mu$ is a function on $M$, and by
\[
i_T \mu(v_x)(Z^1_{v_x}, \ldots, Z^l_{v_x}) = \mu(p)(v_x, (\tau_M)_*(v_x)(Z^1_{v_x}), \ldots, (\tau_M)_*(v_x)(Z^l_{v_x})),
\]
if $\mu$ is a $l + 1$-form, where $x \in M$, $Z^i_{v_x} \in T_{v_x}(TM)$.

A derivation $d_T$ of degree 0 from $\Lambda (M)$ on $\Lambda TM$ over $\tau$ is defined by $d_T \mu = i_T d\mu + di_T \mu$, where $d$ is the exterior derivative. We have $dd_T = d_T d$.  

2. It is an immediate consequence of Definition [23] of integral section of a $k$-vector field.  

\[\square\]
We extend the above definition as follows: for every $A = 1, \ldots, k$ we define a derivation $i_T A$ of degree $-1$ from $\bigwedge(M)$ on $\bigwedge T^1_k M$ over $\tau : T^1_k M \to M$ by $i_T A \mu = 0$ if $\mu$ is a function on $M$, and by

$$i_T A \mu(w_x)(Z^1_{w_x}, \ldots, Z^l_{w_x}) = \mu(p)((\tau_A w_x), \tau_k (w_x) (Z^1_{w_x}), \ldots, \tau_k (w_x) (Z^l_{w_x})), \quad (22)$$

if $\mu$ is an $l + 1$-form, where $\tau_A : T^1_k M \to M$ is the projection on the $A^{th}$-component of $T^1_k M$, $w_x \in T^1_k M$ and $Z^i_{w_x} \in T_{w_x} (T^1_k M), \quad 1 \leq i \leq l$.

A derivation $d_T A$ of degree $0$ from $\bigwedge(M)$ on $\bigwedge T^1_k M$ over $\tau$ is defined by $d_T A \mu = i_T A d\mu + d i_T A \mu$, where $d$ is the exterior derivative. We have $dd_T A = d_T A d$.

### 3.2 Tulczyjew’s Hamiltonian formulation

In this subsection it is important to consider the paragraph “The manifold $T^1_k (\mathbb{R}^k \times P)$” of the Section 2.2.1. Here we consider the particular case $P = (T^1_k)^* Q$.

Let $W_{(t, \alpha)} = (v_1(t, \alpha), \ldots, v_k(t, \alpha))$ be a point in $T^1_k (\mathbb{R}^k \times (T^1_k)^* Q)$ where $(t, \alpha) \in \mathbb{R}^k \times (T^1_k)^* Q$, that is $v_{A(t, \alpha)} \in T_{(t, \alpha)} (\mathbb{R}^k \times (T^1_k)^* Q), \quad 1 \leq A \leq k$. We write

$$v_A (t, \alpha) = (v_A)_B \frac{\partial}{\partial t^B} |_{(t, \alpha)} + (v_A)_i \frac{\partial}{\partial q^i} |_{(t, \alpha)} + (v_A)_B^i \frac{\partial}{\partial p_B^i} |_{(t, \alpha)}$$

and we introduce the canonical coordinates $(t^A, q^i, p_A^i, (v_A)_B, (v_A)_i, (v_A)_B^i)$ on $T^1_k (\mathbb{R}^k \times (T^1_k)^* Q)$.

Let $\omega^A$ be the canonical 2-forms on $\mathbb{R}^k \times (T^1_k)^* Q$ introduced in the subsection 2.1.1. From (22) we introduce the 1-forms $i_T A \omega^A$ on $T^1_k (\mathbb{R}^k \times (T^1_k)^* Q)$, which are locally given by

$$i_T A \omega^A = \sum_{i=1}^{k} ((v_A)_i^i dP_i^i - (v_A)_i^A dq^i). \quad (23)$$

Let $H : \mathbb{R}^k \times (T^1_k)^* Q$ be a Hamiltonian function. Associated with $H$ we define the submanifold

$$D_H = \{ W_{(t, \alpha)} \in T^1_k (\mathbb{R}^k \times (T^1_k)^* Q) : dH (v_{A(t, \alpha)}) = \delta^B_A \},$$

$$\left( \sum_{A=1}^{k} i_T A \omega^A \right) (W_{(t, \alpha)}) = \left( \tau_{\mathbb{R}^k \times (T^1_k)^* Q} (dH - \frac{\partial H}{\partial t^A} dt^A) \right) (W_{(t, \alpha)}) \} ;$$

where $\tau_{\mathbb{R}^k \times (T^1_k)^* Q} : T^1_k (\mathbb{R}^k \times (T^1_k)^* Q) \to \mathbb{R}^k \times (T^1_k)^* Q$ is the canonical projection of the tangent bundle of $k^1$-velocities of $\mathbb{R}^k \times (T^1_k)^* Q$.

From (23) we deduce that $D_H$ is locally defined by the constraints

$$\frac{(v_A)_B (W_{(t, \alpha)})}{(v_A)_i (W_{(t, \alpha)})} = \delta^B_A, \quad \frac{(v_A)_i (W_{(t, \alpha)})}{(v_A)_i^A (W_{(t, \alpha)})} = \frac{\partial H}{\partial p_B^i} |_{\tau_{\mathbb{R}^k \times (T^1_k)^* Q} (W_{(t, \alpha)})}, \quad (24)$$

$$- \sum_{A=1}^{k} \frac{(v_A)_i^A (W_{(t, \alpha)})}{(v_A)_i^A (W_{(t, \alpha)})} = \frac{\partial H}{\partial q^i} |_{\tau_{\mathbb{R}^k \times (T^1_k)^* Q} (W_{(t, \alpha)})},$$

and thus $\text{dim} \, D_H = k + (nk) + k(nk)$.

**Proposition 3.1** Let $X = (X_1, \ldots, X_k)$ be an integrable $k$-vector field on $\mathbb{R}^k \times (T^1_k)^* Q$ such that $\text{Im} \, X \subset D_H$. Then its integral sections are solutions of the HDW-equations.
Let $X = (X_A)^B \frac{\partial}{\partial t^B} + (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^i \frac{\partial}{\partial p^i}$, then from (24) and definition of $D$ we have that

$$
(X_A)^B(W_{(t,\alpha)}) = \delta_A^B, \quad (X_A)^i(W_{(t,\alpha)}) = \frac{\partial H}{\partial p^i} \bigg|_{\tau_{R^k \times (T^1_k)^*Q}(W_{(t,\alpha)})},
$$

and thus

$$
- \sum_{A=1}^k (X_A)^A(W_{(t,\alpha)}) = \frac{\partial H}{\partial q^i} \bigg|_{\tau_{R^k \times (T^1_k)^*Q}(W_{(t,\alpha)})}.
$$

If $X$ is integrable and if $\psi : \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^*Q$, given by $\psi(t) = (\psi^A(t), \psi^i(t), \psi^A(t))$, is an integral section of $X$, then we obtain

$$
\frac{\partial \psi^A}{\partial t} \bigg|_t = (X_A)^B(\psi(t)), \quad \frac{\partial \psi^i}{\partial t} \bigg|_t = (X_A)^i(\psi(t)), \quad \frac{\partial \psi^A}{\partial p^i} \bigg|_t = (X_B)^A(\psi(t)).
$$

Therefore, from (25) and (26) we obtain that $\psi(t)$ is a solution to the Hamilton field equations (4) where $\psi(t) = (t + cte, \psi^i(t), \psi^A(t))$.

3.3 Tulczyjew's Lagrangian formulation

Let $\theta^A$ be the canonical 1-forms on $\mathbb{R}^k \times (T^1_k)^*Q$ introduced in subsection 2.1.1. Then using that $d_{T^A} \theta^A = -i_{T^A} \omega^A + di_{T^A} \theta^A_0$, where $\omega^A = -d\theta^A$, we obtain from (22) that the 1-forms $d_{T^A} \theta^A$ on $T^1_k(\mathbb{R}^k \times (T^1_k)^*Q)$ are locally given by

$$
d_{T^A} \theta^A = (v_A)^A_i \, dq^i + p^A_i \, d(v_A)^i \in \bigcup_{A=1}^k T^1_k(\mathbb{R}^k \times (T^1_k)^*Q). \tag{27}
$$

Let $L(t^A, q^i, v^i_A)$ be a function on $\mathbb{R}^k \times T^1_k Q$ and let $D_L$ be the submanifold

$$
D_L = \{ W_{(t,\alpha)} \in T^1_k(\mathbb{R}^k \times (T^1_k)^*Q) : dt^B(v_{A(t,\alpha)}) = \delta_A^B, \sum_{A=1}^k d_{T^A} \theta^A(W_{(t,\alpha)}) = F^* (dL - \frac{\partial L}{\partial t^A} dt^A)(W_{(t,\alpha)}) \},
$$

$F = (\tau_{R^k \times (T^1_k)^*Q})_t : T^1_k(\mathbb{R}^k \times (T^1_k)^*Q) \to \mathbb{R}^k \times T^1_k Q$ being the map defined by (7), which, in this particular case (i.e., with $P = (T^1_k)^*Q$ and $\pi = \pi_Q$) is locally given by

$$
F(t^A, q^i, p^A_i, (v_A)^i_B, (v_A)^A_i, (v_A)^B_i) = (t^A, q^i, (v_A)^i_A). \tag{27}
$$

From (27) we deduce that $D_L$ is characterized by the equations

$$
(v_A)_B(W_{(t,\rho)}) = \delta_A^B, \quad p^A_i(W_{(t,\rho)}) = \frac{\partial L}{\partial v^A_i}|_{F(W_{(t,\rho)})}, \quad \sum_{A=1}^k (v_A)^A_i(W_{(t,\rho)}) = \frac{\partial L}{\partial q^i}|_{F(W_{(t,\rho)})}. \tag{28}
$$

and thus $D_L$ has dimension $k + nk + nk^2$.

**Proposition 3.2** Let $\Psi : \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^*Q$ be a section of $\mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R}^k$. If

$$
\Psi(1)(t) = (\Psi_*(t)(\frac{\partial}{\partial t}(t)), \ldots, \Psi_*(t)(\frac{\partial}{\partial k}(t))) \in D_L
$$

then $\psi = (\pi_Q)_* \circ \Psi : \mathbb{R}^k \to Q$ is a solution to the Euler-Lagrange equations.
(Proof) If the local expression of \( \Psi \) is given by \( \Psi(t) = (t^A, \Psi^i(t), \Psi^A_i(t)) \), then

\[
\Psi^i_s(t) \left( \frac{\partial}{\partial t^A} (t) \right) = \frac{\partial}{\partial t^A} \Psi(t) + \frac{\partial \Psi^i}{\partial t^A}(t) \frac{\partial}{\partial q^i} \Psi(t) + \frac{\partial \Psi^B_i}{\partial t^A}(t) \frac{\partial}{\partial p^B_i} \Psi(t)
\]

and thus, we deduce from \( \text{[23]} \) that

1) \( (v_A)_B(\Psi^1(t)) = \delta^B_A \),

2) \( \dot{p}^A_i(\Psi^1(t)) = \dot{p}^A_i(\Psi(t)) = \Pi^A_i(t) = \frac{\partial L}{\partial \dot{v}^A_i} \bigg|_{F(\Psi^1(t))} \),

3) \( \sum_{A=1}^{k} (v_A)_i(\Psi^1(t)) = \sum_{A=1}^{k} \frac{\partial \Psi^A_i}{\partial t^A}(t) = \frac{\partial L}{\partial \dot{q}} \bigg|_{F(\Psi^1(t))} \).

From 2) and 3) we obtain

\[
\sum_{A=1}^{k} \frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial v^A} \bigg|_{F(\Psi^1(t))} \right) = \frac{\partial L}{\partial \dot{q}} \bigg|_{F(\Psi^1(t))}.
\]

Now, we consider the map \( \psi = (\pi_Q)_1 \circ \Psi \) locally given by \( \psi(t) = (\Psi^1(t)) \). It is easy to show that \( F \circ \Psi^1(t) = (t, \psi_s(t)(\frac{\partial}{\partial t^1}(t)), \ldots, \psi_s(t)(\frac{\partial}{\partial t^k}(t))) = \psi^1(t) \), where \( \psi^1 \) is defined in Definition \( \text{[26]} \).

So, we obtain

\[
\sum_{A=1}^{k} \frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial v^A} \bigg|_{\psi^1(t)} \right) = \frac{\partial L}{\partial \dot{q}} \bigg|_{\psi^1(t)},
\]

that is, \( \psi \) is a solution to the Euler-Lagrange equations. \( \blacksquare \)

4 Skinner-Rusk formulation

4.1 The Skinner-Rusk formalism for \( k \)-cosymplectic field theories

Let us consider the Whitney sum \( \mathcal{M} = (\mathbb{R}^k \times T^1_k Q) \oplus_{\mathbb{R}^k \times Q} (\mathbb{R}^k \times (T^1_k)^* Q) \), with natural coordinates \((t^A, q^i, v^i_A, p^A_i)\). It has natural bundle structures over \( \mathbb{R}^k \times T^1_k Q \) and \( \mathbb{R}^k \times (T^1_k)^* Q \). Let us denote by \( pr_1 : \mathcal{M} \to \mathbb{R}^k \times T^1_k Q \) the projection into the first factor, \( pr_1(t^A, q^i, v^i_A, p^A_i) = (t^A, q^i, v^i_A) \) and by \( pr_2 : \mathcal{M} \to \mathbb{R}^k \times (T^1_k)^* Q \) the projection into the second factor, \( pr_2(t^A, q^i, v^i_A, p^A_i) = (t^A, q^i, p^A_i) \).

Let \((\eta^1, \ldots, \eta^k, \omega^1, \ldots, \omega^k)\) be the canonical forms of the canonical \( k \)-cosymplectic structure on \( \mathbb{R}^k \times (T^1_k)^* Q \). We denote

\[
\partial^A = (pr_2)^* dt^A = dt^A, \quad \Omega^A = (pr_2)^* \omega^A,
\]

and so we have the family \((\partial^1, \ldots, \partial^k, \Omega^1, \ldots, \Omega^k)\) in \( \mathcal{M} \).

Now, taking the \( k \)-vector field \( \left( \frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^k} \right) \) in \( \mathbb{R}^k \times (T^1_k)^* Q \), we can define a family of \( k \)-vector fields \((\xi_1, \ldots, \xi_k)\) in \( \mathcal{M} \) such that

\[
(pr_2)_s \xi_A = \frac{\partial}{\partial t^A}.
\]

These \( k \)-vector fields \((\xi_1, \ldots, \xi_k)\) satisfy that, for \( 1 \leq A, B \leq k \),

\[
\iota_{\xi_A} \partial^B = \iota_{\xi_A} (pr_2^* dt^B) = pr_2^* (\iota_{\frac{\partial}{\partial t^A}} dt^B) = \delta^B_A
\]

\[
\iota_{\xi_A} \Omega^B = \iota_{\xi_A} (pr_2^* \omega^B) = pr_2^* (\iota_{\frac{\partial}{\partial t^A}} \omega^B) = 0,
\]

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and they are locally given by
\[ \xi_A = \partial_{\partial t^A} + (\xi_A)_B^i \partial_{\partial v^B_i}, \]
where \((\xi_A)_B^i\) are arbitrary local functions in \(\mathcal{M}\). Hence, this \(k\)-vector field is not unique.

Finally, the coupling function in \(\mathcal{M}\), denoted by \(C\), is defined as follows:
\[
C : \mathcal{M} = (\mathbb{R}^k \times T^1_k Q) \oplus_{\mathbb{R}^k \times Q} (\mathbb{R}^k \times (T^1_k)^* Q) \rightarrow \mathbb{R}
\]
\[
(t, v_{1q}, \ldots, v_{kq}, \alpha_q^1, \ldots, \alpha_q^k) \rightarrow \sum_{A=1}^k \alpha_q^A(v_{Aq})
\]

Given a Lagrangian \(L \in C^\infty(\mathbb{R}^k \times T^1_k Q)\), we define the Hamiltonian function \(H \in C^\infty(\mathcal{M})\) as
\[
H = C - pr_1^* L ,
\]
which, in coordinates, is given by
\[
H = p^A_i v^i_A - L(t^A, q^i, v^i_A) .
\]

Then, in this formalism, we have the following problem:

**Statement 4.1** Suppose that an integrable \(k\)-vector field \(Z = (Z_1, \ldots, Z_k)\) exists in \(\mathcal{M}\), such that
\[
\partial^A(Z_B) = \delta^A_B , \quad \sum_{A=1}^k v_{ZA} \Omega^A = dH - \sum_{A=1}^k \xi_A(H) \partial^A .
\]

The problem is to find the integral sections \(\psi : \mathbb{R}^k \rightarrow \mathcal{M}\) of \(Z = (Z_1, \ldots, Z_k)\).

Equations (32) give different kinds of information. In fact, writing locally each \(Z_A\) as
\[
Z_A = (Z_A)^B \partial_{\partial t^B} + (Z_A)^i \partial_{\partial q^i} + (Z_A)_B^i \partial_{\partial v^B_i} + (Z_A)_B^i \partial_{\partial p^i_B} ,
\]
from (1), (31) and (32) we obtain
\[
(Z_A)^i = v^i_A
\]
\[
\sum_{A=1}^k (Z_A)^A_i = \frac{\partial L}{\partial q^i} \circ pr_1 .
\]

Then the vector fields \(Z_A\) are locally given by
\[
Z_A = \partial_{\partial t^A} + v^i_A \partial_{\partial q^i} + (Z_A)^i \partial_{\partial v^i} + (Z_A)_B^i \partial_{\partial p^i_B} ,
\]
where the coefficients \((Z_A)_i^B\) are related by the equations (36). Observe that these equations do not depend on the arbitrary functions \((\xi_A)_B^i\), that is, on the family of vector fields \(\{\xi_A\}\) that we have chosen to extend the vector fields \(\{\partial_{\partial t^A}\}\).

So, in particular, we have obtained information on four different classes:
1. The constraint equations (31), which are algebraic (not differential) equations defining a submanifold $M_L$ of $\mathcal{M}$ where the equation (32) has solution. Observe that this submanifold is just the graph of the Legendre map $FL$ defined by the Lagrangian $L$.

2. Let us observe that, as a consequence of (31), the $k$-vector field $Z = (Z_1, \ldots, Z_k)$, $Z_A \in \mathfrak{X}(\mathcal{M})$, satisfies equation (32) only on $M_L$.

3. Equations (35), called the sopde condition, will be used in the following subsection (see Theorem 4.1), to show that the integral sections of $Z = (Z_1, \ldots, Z_k)$ can be obtained from first prolongations $\phi^{[1]}$ of maps $\phi : \mathbb{R}^k \to Q$.

4. Equations (36), which, taking into account (33), (34) and (35), will give the classical Euler-Lagrange equations for the integral sections of $Z$ (see Theorem 4.1).

5. From (33), (34), (35) and (36), we deduce that the solutions of equations (32) do not depend on the $k$-vector field $(\xi_1, \ldots, \xi_k)$.

If $Z = (Z_1, \ldots, Z_k)$ is a solution to (32), then each $Z_A$ is tangent to the submanifold $M_L$, and only if, the functions $Z_A \left( p^B_j - \frac{\partial L}{\partial v^i_B} \right)$ vanish at the points of $M_L$. Then from (37) we deduce that this is equivalent to the following equations

$$
(Z_A)^B_j = \frac{\partial^2 L}{\partial t^A \partial v^i_B} + v^i_A \frac{\partial^2 L}{\partial q^i \partial v^j_B} + (Z_A)^i_C \frac{\partial^2 L}{\partial v^i_C \partial v^j_B},
$$

which are conditions for the coefficients $(Z_A)^i_C$.

Taking into account that the $k$-vector fields $Z$ must be tangent to the submanifold $M_L$, the above problem can be stated in $M_L$, instead of in $\mathcal{M}$. First observe that the family composed of the $k$ vector fields $(\xi_1, \ldots, \xi_k)$ on $\mathcal{M}$ are tangent to $M_L$ if and only if

$$
\frac{\partial^2 L}{\partial t^A \partial v^i_B} \circ pr_1 + (\xi_A)^j_C \frac{\partial^2 L}{\partial v^j_C \partial v^i_B} \circ pr_1 = 0,
$$

since the constraint function defining $M_L$ is $p^A_i - \frac{\partial L}{\partial v^i_A} \circ pr_1$. Thus taking into account 3, we have

**Statement 4.2** We denote by $j : M_L \to \mathcal{M}$ the natural imbedding. The problem is to find the integral sections $\psi : \mathbb{R}^k \to M_L \subset \mathcal{M}$ of integrable $k$-vector fields $Z_L = ((Z_L)_1, \ldots, (Z_L)_k)$ on $M_L$ solution to the following equations

$$
(j^*\theta^A)((Z_L)_B) = \delta^i_B, \quad \sum_{A=1}^k \iota_{(Z_L)_A}(j^*\mathcal{O}^A) = d(j^*\mathcal{H}) - j^* \left[ \sum_{A=1}^k \xi_A(\mathcal{H}) \right](j^*\theta^A),
$$

(39)

Of course, $j^*(Z_L)_A = Z_A|_{M_L}$, where $Z = (Z_1, \ldots, Z_k)$ is the $k$-vector field on $\mathcal{M}$ solution to (32).

It is interesting to remark that:

1. In general, equations (32) (or, what is equivalent, equations (39)) do not have a unique solution. Solutions to (32) are given by $(Z_1, \ldots, Z_k) + (\ker \Omega^i \cap \ker \theta^j)$, where $(Z_1, \ldots, Z_k)$ is a particular solution, and $\Omega^i$ is the morphism defined by

$$
\Omega^i : T^A_k \mathcal{M} \longrightarrow T^*\mathcal{M},
$$

$$(Y_1, \ldots, Y_k) \rightarrow \Omega^i(Y_1, \ldots, Y_k) = \sum_{A=1}^k \iota_{Y_A} \Omega^A + \theta^A(Y_A) \theta^A,
$$

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and, denoting by $\mathcal{M}_k(C^\infty(\mathcal{M}))$ the space of matrices of order $k$ whose entries are functions on $\mathcal{M}$, the vector bundle morphism $\vartheta^\sharp$ is defined by

$$
\vartheta^\sharp : T_k^1 \mathcal{M} \rightarrow \mathcal{M}_k(C^\infty(\mathcal{M}))
$$

$$(Y_1, \ldots, Y_k) \rightarrow \vartheta^\sharp(Y_1, \ldots, Y_k) = (\vartheta^A(Y_B)).$$

2. If $L$ is regular, then taking into account (33), (35) and (36) we can define a local $k$-vector field $(Z_1, \ldots, Z_k)$ on a neighborhood of each point in $M_L$ which is a solution to (32). Each $Z_A$ is locally given by

$$(Z_A)^B = \delta_A^B, \quad (Z_A)^i = v_A^i, \quad (Z_A)_i^B = \frac{1}{k} \frac{\partial L}{\partial q^i} \delta_A^B,$$

with $(Z_A)^B$ satisfying (38). Now, by using a partition of the unity, one can construct a global $k$-vector field which is a solution to (32).

When the Lagrangian function $L$ is singular, we cannot ensure the existence of solutions to the equations (32) or (39). Thus we must develop a constraint algorithm for obtaining a constraint submanifold (if it exists) where these solutions exist. Next, we outline this procedure (see also [22], where a similar algorithm is sketched in the multisymplectic formulation).

Assuming that the Lagrangian is almost-regular, we start with $P_0 = M_L$. Then, let $P_1$ be the subset of $P_0$ composed of those points where a solution to (39) exists, that is,

$$P_1 = \{z \in P_0 \mid \exists ((Z_L)_1, \ldots, (Z_L)_k) \in (T_k^1)_z P_0 \text{ solution to (39)} \}.$$

If $P_1$ is a submanifold of $P_0$, there exists a section of the canonical projection $\tau_{P_0} : T_k^1 P_0 \rightarrow P_0$ defined on $P_1$ which is a solution to (39), but which does not define a $k$-vector field on $P_1$, in general. In order to find solutions taking values into $T_k^1 P_1$, we define a new subset $P_2$ of $P_1$ as

$$P_2 = \{z \in P_1 \mid \exists ((Z_L)_1, \ldots, (Z_L)_k) \in (T_k^1)_z P_1 \text{ solution to (39)} \}.$$  

If $P_2$ is a submanifold of $P_1$, then there exists a section of the canonical projection $\tau_{P_1} : T_k^1 P_1 \rightarrow P_1$ defined on $P_2$ which is a solution to (39), but which does not define, in general, a $k$-vector field on $P_2$. Proceeding further, we get a family of constraint manifolds

$$\ldots \hookrightarrow P_2 \hookrightarrow P_1 \hookrightarrow P_0 = M_L \hookrightarrow \mathcal{M}.$$  

If there is a natural number $f$ such that $P_{f+1} = P_f$ and $\dim P_f > k$, then we call $P_f$ the final constraint submanifold on which we can find solutions to equation (39). The solutions are not unique (even in the regular case) and, in general, they are not integrable. In order to find integrable solutions to equation (39), a constraint algorithm based on the same idea must be developed.

### 4.2 The field equations for sections

Consider the following restrictions of the projections $pr_1$ and $pr_2$:

$$pr^0_1 : M_L \rightarrow \mathbb{R}^k \times T_k^1 Q, \quad pr^0_2 : M_L \rightarrow \mathbb{R}^k \times (T_k^1)^* Q.$$  

**Remark 4.1** Observe that, as $M_L$ is the graph of $FL$, it is diffeomorphic to $\mathbb{R}^k \times T_k^1 Q$, and this means that $pr^0_1$ is really a diffeomorphism.
Let \( \mathbf{Z} = (Z_1, \ldots, Z_k) \) be an integrable \( k \)-vector field solution to (32). Every integral section \( \psi: \mathbb{R}^k \rightarrow (\psi^A(t), \psi^i(t), \psi^i_A(t), \psi^i_i(t)) \in \mathcal{M} \) of \( \mathbf{Z} \) is of the form \( \psi = (\psi_L, \psi_H) \), with \( \psi_L = pr_1 \circ \psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times T^1_k Q \), and if \( \psi \) takes values in \( M_L \) then \( \psi_H = FL \circ \psi_L \). In fact, from (34),

\[
\psi_H(t) = (pr_2 \circ \psi)(t) = (\psi^A(t), \psi^i(t), \psi^i_A(t)) = \left( \psi^A(t), \psi^i(t), \frac{\partial L}{\partial p^j_A}(\psi_L(t)) \right) = (FL \circ \psi_L)(t).
\]

In this way, every constraint, differential equation, etc. in the unified formalism can be translated into the non autonomous Lagrangian or Hamiltonian formalism by restriction to the first or second factors of the product bundle. In particular, conditions (34) generate, by \( pr_2 \)-projection, the primary constraints of the Hamiltonian formalism for singular Lagrangians (i.e., the image of the Legendre transformation, \( FL(\mathbb{R}^k \times T^1_k Q) \subset \mathbb{R}^k \times (T^1_k)^* Q \)), and they can be called the primary Hamiltonian constraints.

Hence the main result in this subsection is the following:

**Theorem 4.1** Let \( \mathbf{Z} = (Z_1, \ldots, Z_k) \) be an integrable \( k \)-vector field in \( \mathcal{M} \) solution to (32), and let \( \psi: \mathbb{R}^k \rightarrow M_L \subset \mathcal{M} \) be an integral section of \( \mathbf{Z} = (Z_1, \ldots, Z_k) \), with \( \psi = (\psi_L, \psi_H) = (\psi_L, FL \circ \psi_L) \). Then \( \psi_L \) is the canonical lift \( \phi^{[1]} \) of the projected section \( \phi = p_Q \circ pr_0 \circ \psi: \mathbb{R}^k \rightarrow M_L \approx \mathbb{R}^k \times T^1_k Q \rightarrow Q \), and \( \phi \) is a solution to the Euler-Lagrange field equations (17).

\[
\psi_L = \phi^{[1]}
\]

(Proof) If \( \psi(t) = \left( \psi^A(t), \psi^i(t), \psi^i_A(t), \psi^i_i(t) = \frac{\partial L}{\partial v^j_A}(\psi_L(t)) \right) \) is an integral section of \( \mathbf{Z} \), then

\[
Z_A(\psi(t)) = \frac{\partial \psi^B}{\partial t^A}(t) \frac{\partial}{\partial \psi^B}(\psi(t)) + \frac{\partial \psi^i_A}{\partial t^A}(t) \frac{\partial}{\partial \psi^i_A}(\psi(t)) = \frac{\partial \psi^i_B}{\partial t^A}(t) \frac{\partial}{\partial \psi^i_B}(\psi(t)) + \frac{\partial \psi^i_i}{\partial t^A}(t) \frac{\partial}{\partial \psi^i_i}(\psi(t)) \quad (40)
\]

From (33), (34), (35) and (40) we obtain

\[
\frac{\partial \psi^B}{\partial t^A}(t) = (Z_A)^B(\psi(t)) = \delta^B_A \quad (41)
\]

\[
\psi^i_A(t) = p^i_A(\psi(t)) = \left( \frac{\partial L}{\partial v^j_A} \circ pr_1 \right)(\psi(t)) = \frac{\partial L}{\partial v^j_A}(\psi_L(t)) \quad (42)
\]

\[
\psi^i_A(t) = v^i_A(\psi(t)) = (Z_A)^i(\psi(t)) = \frac{\partial \psi^i}{\partial t^A}(t) \quad (43)
\]

\[
\frac{\partial \psi^B}{\partial t^A}(t) = (Z_A)^B(\psi(t)) \quad (44)
\]
Therefore from (36), (32) and (11) we obtain
\[
\frac{\partial L}{\partial q^A}(\psi_L(t)) = \sum_{A=1}^{k} (Z_A)^A_i(\psi(t)) = \sum_{A=1}^{k} \frac{\partial \psi^A}{\partial t^A}(t) = \sum_{A=1}^{k} \frac{\partial}{\partial t^A} \left( \frac{\partial L}{\partial \psi^A}(\psi_L(t)) \right),
\]
and from (41) we obtain \(\psi^A(t) = t^A + c^A\). Taking \(c^A = 0\), from (43) we have
\[
\psi_L(t) = \left( t, \psi^i(t), \frac{\partial \psi^i}{\partial t^A}(t) \right),
\]
and from the last two equations we deduce that \(\psi_L = \phi[1]\) and \(\phi = p_Q \circ pr_0 \circ \psi : \mathbb{R}^k \times T_k^1 Q \rightarrow M \approx \mathbb{R}^k \times T_k^1 Q \rightarrow Q\), is a solution to the Euler-Lagrange field equations (16), where \(\phi(t) = (\psi^i(t))\).

**Proposition 4.1** According to the hypothesis of Theorem 4.1, if \(L\) is regular then \(\psi_H = FL \circ \psi_L\) is a solution to the Hamilton field equations (4), where the Hamiltonian \(H\) is given by \(H \circ FL = E_L\).

(Proof) Since \(L\) is regular, \(FL\) is a local diffeomorphism, and thus we can choose for each point in \(\mathbb{R}^k \times T_k^1 Q\) an open neighborhood \(U \subset \mathbb{R}^k \times T_k^1 Q\) such that \(FL|_U : U \rightarrow FL(U)\) is a diffeomorphism. So we can define \(H_U : FL(U) \rightarrow \mathbb{R}\) as \(H_U = (E_L)|_U \circ (FL|_U)^{-1}\).

Denoting by \(H \equiv H_U\), \(E_L \equiv (E_L)|_U\) and \(FL \equiv FL|_U\), we have \(E_L = H \circ FL\), which provides the identities
\[
\frac{\partial H}{\partial p_i^A} \circ FL = v^A_i, \quad \frac{\partial H}{\partial q^A} \circ FL = -\frac{\partial L}{\partial q^A}.
\]
Now considering the open subset \(V = \psi_L^{-1}(U) \subset \mathbb{R}^k\), we have \(\psi|_V : V \subset \mathbb{R}^k \rightarrow U \oplus FL(U) \subset M\), where \((\psi_L)|_V : V \subset \mathbb{R}^k \rightarrow U \subset \mathbb{R}^k \times T_k^1 Q\) and \((\psi_H)|_V = FL \circ (\psi_L)|_V : V \subset \mathbb{R}^k \rightarrow FL(U) \subset \mathbb{R}^k \times (T_k^1)^* Q\). Therefore from (36), (43), (44) and (45), for every \(t \in V \subset \mathbb{R}^k\) we obtain
\[
\frac{\partial H}{\partial p_i^A}(\psi_H(t)) = \left( \frac{\partial H}{\partial p_i^A} \circ FL \right)(\psi_L(t)) = v^i_A(\psi_L(t)) = \frac{\partial \psi^i}{\partial t^A}(t),
\]
\[
\frac{\partial H}{\partial q^A}(\psi_H(t)) = \left( \frac{\partial L}{\partial q^A} \circ FL \right)(\psi_L(t)) = -\frac{\partial L}{\partial q^A}(\psi_L(t)) = -(Z_A)^A_i(\psi(t)) = -\frac{\partial \psi^i}{\partial t^A}(t),
\]
from which we deduce that \((\psi_H)|_V\) is a solution to the Hamilton field equations (4).

Conversely, we can state:

**Proposition 4.2** If \(L\) is regular and \(X = (X_1, \ldots, X_k)\) is a solution to (17) then:

1. The \(k\)-vector field \(Z = (Z_1, \ldots, Z_k)\) given by \(Z_A = (Id_{\mathbb{R}^k \times T_k^1 Q} \oplus FL)_*(X_A)\) is a solution to (32).

2. If \(\psi_L : \mathbb{R}^k \rightarrow \mathbb{R}^k \times T_k^1 Q\) is an integral section of \(X = (X_1, \ldots, X_k)\) (and thus, from Remark 2.2 and from Theorem 2.2 \(\phi = \rho \circ \psi_L : \mathbb{R}^k \times T_k^1 Q \rightarrow \mathbb{R}^k \times T_k^1 Q \rightarrow Q\) is a solution to the Euler-Lagrange field equations) then \(\psi = (\psi_L, FL \circ \psi_L) : \mathbb{R}^k \rightarrow M_L \subset M\) is an integral section of \(Z = (Z_1, \ldots, Z_k)\).
(Proof) Now from the local expression (14) of $FL$ we deduce that

1. If $L$ is regular and $X = (X_1, \ldots, X_k)$ is a solution to (17), then from Theorem 2.2 we know that $X_A$ is a SOPDE, and thus $X_A$ is locally given by

$$X_A = \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (X_A)_B \frac{\partial}{\partial v_B^i},$$

where $(X_A)_B$ satisfy

$$\frac{\partial^2 L}{\partial t^A \partial v_A^i} + v_A^i \frac{\partial^2 L}{\partial q^i \partial v_A^i} + (X_A)_B \frac{\partial^2 L}{\partial v_B^i \partial v_A^i} = \frac{\partial L}{\partial q^i}. \quad (46)$$

Now from the local expression (14) of $FL$ we deduce that

$$Z_A = (Id_{\mathbb{R}^k \times T^1_k \mathbb{Q}} \oplus FL)_\ast (X_A) = \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (X_A)_B \frac{\partial}{\partial v_B^i} + \left( \frac{\partial^2 L}{\partial t^A \partial v_A^i} + v_A^i \frac{\partial^2 L}{\partial q^i \partial v_A^i} + (X_A)_B \frac{\partial^2 L}{\partial v_B^i \partial v_A^i} \right) \frac{\partial}{\partial p_j^c}.$$

Then from (47) and (48) we deduce that $Z = (Z_1, \ldots, Z_k)$ satisfy equations (33), (35), (36) and each $Z_A$ is tangent to $M_L$.

2. It is an immediate consequence of the definition of $Z$ and the definition of integral section.

**Remark 4.2** The last result really holds for regular and almost-regular Lagrangians. In the almost-regular case, assuming as additional hypothesis that $X_L$ is a SOPDE, the proof is the same, but the sections $\psi$, $\psi_L$ and $\psi_H$ take values not on $M_L$, $\mathbb{R}^k \times T^1_k \mathbb{Q}$ and $\mathbb{R}^k \times (T^1_k)^* \mathbb{Q}$, but in the final constraint submanifold $P_f$ and on the projection submanifolds $pr_1(P_f) \subset \mathbb{R}^k \times T^1_k \mathbb{Q}$ and $pr_2(P_f) \subset \mathbb{R}^k \times (T^1_k)^* \mathbb{Q}$, respectively.

### 4.3 The field equations for $k$-vector fields

Next we establish the relationship between $k$-vector fields that are solutions to (17) and $k$-vector fields that are solutions to (32) or, what is equivalent, solutions to (39). First, observe that:

**Lemma 4.1** We have that

$$j^* \vartheta^A = (pr_1^0)^* dt^A, \quad j^* \Omega^A = (pr_1^0)^* \omega_L^A.$$ 

(Proof) It is immediate from (15), taking into account that $FL \circ pr_1^0 = pr_2 \circ j$. □

**Theorem 4.2** a) Let $L : \mathbb{R}^k \times T^1_k \mathbb{Q} \to \mathbb{R}$ be a Lagrangian and let $Z_L = ((Z_L)_1, \ldots, (Z_L)_k)$ be a $k$-vector field on $M_L$ solution to (39). Then the $k$-vector field $X_L = ((X_L)_1, \ldots, (X_L)_k)$ on $\mathbb{R}^k \times T^1_k \mathbb{Q}$ defined by

$$X_L \circ pr_1^0 = T^1_k (pr_1^0) \circ Z_L \quad (50)$$

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is a $k$-vector field solution to (17), where $T^1_k(pr^0_1): T^1_k(M_L) \rightarrow T^1_k(\mathbb{R}^k \times T^1_kQ)$ is the natural extension of $pr^0_1$, introduced in (3).

Conversely, every $k$-vector field $X_L$ solution to (17) can be recovered in this way from a $k$-vector field $Z_L$ in $M_L$ solution to (39).

b) The $k$-vector field $Z_L$ is integrable if, and only if, the $k$-vector field $X_L$ is an integrable SOPDE.

(Proof) a) Since $pr^0_1: M_L \rightarrow \mathbb{R}^k \times T^1_kQ$ is a diffeomorphism, then the $k$-vector field $X_L$ on $\mathbb{R}^k \times T^1_kQ$ defined by (50) is given by

\[(X_L)_A = ((pr^0_1)^{-1})^* (Z_L)_A. \tag{51}\]

Furthermore, we obtain that

\[j^*H = j^*(C - (pr^1)_*L) = j^*C - j^*(pr^1)_*L = (pr^0_1)^*(C(L)) - (pr^0_1)^*L = (pr^0_1)^*E_L. \tag{52}\]

From (49) and (51) we deduce that

\[j^*\theta^A((Z_L)_B) = ((pr^0_1)^*dt^A)((pr^0_1)^*(X_L)_B) = (pr^0_1)^*(dt^A((X_L)_B)), \tag{53}\]

and from (29), (30), (31) and (34)

\[j^*[\xi_A(H)] = j^* \left[ \left( \frac{\partial}{\partial t^A} + (\xi_A)_B \frac{\partial}{\partial v^i_B} \right) (p^C_j v^j_C - (pr^1)_*L) \right] \]

\[= j^* \left[ (\xi_A)_B (\delta^B_i \circ pr^1) - pr^1_*(\frac{\partial L}{\partial t^A}) \right] = -(pr^0_1)^* \left( \frac{\partial L}{\partial t^A} \right). \tag{54}\]

Therefore from (39), (49), (51), (52) and (54), we obtain

\[\sum_{A=1}^{k} i(z_L)_A j^* \Omega^A - d(j^*H) + j^* \left[ \sum_{A=1}^{k} \xi_A (H) \right] (j^*\theta^A) \]

\[= \sum_{A=1}^{k} i((pr^0_1)^*(X_L)_A (pr^1)_*T^A) - d((pr^0_1)^*E_L) - \sum_{A=1}^{k} (pr^0_1)^* \left( \frac{\partial L}{\partial t^A} \right) (pr^0_1)^*dt^A \tag{55}\]

\[= (pr^0_1)^* \left( \sum_{A=1}^{k} i(X_L)_A \omega^A_L - dE_L - \sum_{A=1}^{k} \frac{\partial L}{\partial t^A} dt^A \right).\]

Since $pr^0_1$ is a diffeomorphism, from (53) and (55) we deduce that the $k$-vector field $Z_L$ is a solution to (39) if, and only if, the $k$-vector field $X_L$ is a solution to (17).

b) Suppose now that the $k$-vector field $Z_L$ is integrable. Let $\varphi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times T^1_kQ$ be an integral section of $X_L$, that is, $(X_L)_A(\varphi(t)) = \varphi_*(t) \left( \frac{\partial}{\partial t^A} \right)_t$. Thus

\[(Z_L)_A((pr^0_1)^{-1} \circ \varphi(t)) = ((pr^0_1)^{-1})*(X_L)_A((pr^0_1)^{-1} \circ \varphi(t)) = ((pr^0_1)^{-1})*_*(\varphi(t))(X_L)_A(\varphi(t)) \]

\[= ((pr^0_1)^{-1})*_*(\varphi(t)) \left( \varphi_*(t) \left( \frac{\partial}{\partial t^A} \right)_t \right) = ((pr^0_1)^{-1} \circ \varphi(t))_* \left( \frac{\partial}{\partial t^A} \right)_t, \]

which means $\psi = (pr^0_1)^{-1} \circ \varphi: \mathbb{R}^k \rightarrow M_L$ is an integral section of $Z_L$. 

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Since $\psi : \mathbb{R}^k \to M_L$, then we know that the integral section $j \circ \psi : \mathbb{R}^k \to \mathcal{M}$ is given by $(j \circ \psi)_L$, $FL \circ (j \circ \psi)_L$, and from Theorem 4.1 we know that $(j \circ \psi)_L = \phi^{[1]}$, where $\phi = pQ \circ \psi : \mathbb{R}^k \to M_L \approx \mathbb{R}^k \times T^1_kQ \overset{pQ}{\to} Q$. Then we have

$$\phi^{[1]} = (j \circ \psi)_L = pr_1 \circ j \circ \psi = pr_1^0 \circ \psi = \varphi.$$  

Since every integral section $\varphi$ of $X_L$ is a first prolongation $\phi^{[1]}$ of a map $\phi : \mathbb{R}^k \to Q$ space, we deduce from Lemma 2.2 that $X_L$ is a SOPDE.

If $m$ is an arbitrary point of $\mathbb{R}^k \times T^1_kQ$, we consider the integral section $\psi$ of $Z_L$ passing through $(pr_0^1)^{-1}(m)$, then $pr_0^0 \circ \psi$ is an integral section of $X_L$ passing through $m$. Thus, $X_L$ is integrable.

Conversely, let $X_L$ be an integrable SOPDE. If $m$ is an arbitrary point of $M_L$, we consider the integral section $\varphi$ of $X_L$ passing through $(pr_0^1)(m)$ then $(pr_0^0)^{-1} \circ \varphi$ is an integral section of $Z_L$ passing through $m$. Thus, $Z_L$ is integrable.

If $L$ is regular, in a neighborhood of each point of $\mathbb{R}^k \times T^1_kQ$ there exists a local solution $X_L = ((X_L)_1, \ldots, (X_L)_k)$ to (17). As $L$ is regular, $FL$ is a local diffeomorphism, so this open neighborhood can be chosen in such a way that $FL$ is a diffeomorphism onto its image. Thus in a neighborhood of each point of $FL(\mathbb{R}^k \times T^1_kQ)$ we can define $(X_H)_A = [(FL)^{-1}]^*(X_L)_A$, or equivalently, in terms of $k$-vector fields $T^1_k(FL) \circ X_L = X_H$.

**Proposition 4.3** 1. The local $k$-vector field $X_H = ((X_H)_1, \ldots, (X_H)_k)$ is a solution to (2), where the Hamiltonian $H$ is locally given by $H \circ FL = E_L$. (In other words, the local $k$-vector fields $X_L$ and $X_H$ solution to (17) and (4), respectively, are $FL$-related).

2. Every local integrable $k$-vector field solution to (4) can be recovered in this way from a local integrable $k$-vector field $Z$ in $\mathcal{M}$ solution to (32).

(Proof)

1. This is the local version of Theorem 2.3

2. Furthermore, if $X_H$ is a local integrable $k$-vector field solution to (4), then we can obtain the $FL$-related local integrable $k$-vector field $X_L$ solution to (17). By Theorem 4.2 we recover $X_L$ by a local integrable $k$-vector field $Z_L$ solution to (32).

It is interesting to point out that the Skinner-Rusk formalism developed in [3] and [7] for the time-dependent mechanics is just a particular case of the Skinner-Rusk formalism which we present here for the $k$-cosymplectic formulation of first-order field theories.

5 Lie algebroids and associated spaces

In this section we present some basic facts on Lie algebroids that are necessary for further developments. We refer to the reader to [4, 17, 30, 31] for details about Lie groupoids, Lie algebroids and their role in differential geometry.
5.1 Lie algebroids

Let $E$ be a vector bundle of rank $m$ over a manifold $Q$ of dimension $n$, and let $\tau : E \to Q$ be the vector bundle projection. Denote by $\text{Sec}(E)$ the $C^\infty(Q)$-module of sections of $\tau : E \to Q$. A Lie algebroid structure $([\cdot,\cdot]_E, \rho)$ on $E$ is a Lie bracket $[\cdot,\cdot]_E : \text{Sec}(E) \times \text{Sec}(E) \to \text{Sec}(E)$ on the space $\text{Sec}(E)$, together with a bundle map $\rho : E \to TQ$, called the anchor map, such that if we denote by $\rho : \text{Sec}(E) \to \mathfrak{X}(Q)$ the homomorphism of the $C^\infty(Q)$-module induced by the anchor map, then they satisfy the compatibility condition

$$[\sigma_1, f\sigma_2]_E = f[\sigma_1, \sigma_2] + (\rho(\sigma_1)f)\sigma_2.$$ 

Here $f$ is a smooth function on $Q$; $\sigma_1, \sigma_2$ are sections of $E$, and we denote by $\rho(\sigma_1)$ the vector field on $Q$ given by $\rho(\sigma_1)(q) = \rho_\sigma_1(q)$. The triple $(E, [\cdot,\cdot]_E, \rho)$ is called a Lie algebroid over $Q$. From the compatibility condition and the Jacobi identity, it follows that the anchor map $\rho : \text{Sec}(E) \to \mathfrak{X}(Q)$ is a homomorphism between the Lie algebras $(\text{Sec}(E), [\cdot,\cdot]_E)$ and $(\mathfrak{X}(Q), [\cdot,\cdot])$.

In this paper, we consider a Lie algebroid as a substitute of the tangent bundle of $Q$. In this way, one regards an element $a$ of $E$ as a generalized velocity, and the actual velocity $v$ is obtained when applying the anchor map to $a$, i.e. $v = \rho(a)$.

Let $(q^i)_{i=1}^n$ be local coordinates on $Q$ and $(e_a)_{a=1}^m$ be a local basis of sections of $\tau$. Given $a \in E$ such that $\tau(a) = q$, we can write $a = y^\alpha(a)e_\alpha(q) \in E_q$, thus the coordinates of $a$ are $(q^i(a), y^\alpha(a))$. Therefore, each section $\sigma$ is locally given by $\sigma_U = y^\alpha e_\alpha$.

In local form, the Lie algebroid structure is determined by the local functions $\rho^i_\alpha$, $C^\gamma_{\alpha\beta}$ on $Q$. Both are determined by the relations

$$\rho(e_\alpha) = \rho^i_\alpha \frac{\partial}{\partial q^i}, \quad [e_\alpha, e_\beta]_E = C^\gamma_{\alpha\beta} e_\gamma.$$  

(56)

The functions $\rho^i_\alpha$ and $C^\gamma_{\alpha\beta}$ are said to be the structure functions of the Lie algebroid in the above coordinate system. They satisfy the following relations (as a consequence of the compatibility condition and Jacobi’s identity)

$$\sum_{cyclic(\alpha,\beta,\gamma)} \left( \rho^i_\alpha \frac{\partial C^\mu_{\beta\gamma}}{\partial q^i} + C^\mu_{\alpha\rho} C^\rho_{\beta\gamma} \right) = 0, \quad \rho^i_\alpha \frac{\partial \rho^j_\beta}{\partial q^i} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^j_\gamma C^\gamma_{\alpha\beta},$$  

(57)

which are usually called the structure equations of the Lie algebroid.

Exterior differential

The structure of the Lie algebroid on $E$ allows us to define the exterior differential of $E$, $d^E : \text{Sec}(\wedge^i E^*) \to \text{Sec}(\wedge^{i+1} E^*)$, as follows

$$d^E \mu(\sigma_1, \ldots, \sigma_{i+1}) = \sum_{i=1}^{i+1} (-1)^{i+1} \rho(\sigma_i)\mu(\sigma_1, \ldots, \tilde{\sigma}_i, \ldots, \sigma_{i+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([\sigma_i, \sigma_j]_E, \sigma_1, \ldots, \tilde{\sigma}_i, \ldots, \tilde{\sigma}_j, \ldots, \sigma_{i+1}) ,$$

for $\mu \in \text{Sec}(\wedge^i E^*)$ and $\sigma_1, \ldots, \sigma_{i+1} \in \text{Sec}(E)$. It follows that $d$ is a cohomology operator, that is, $d^2 = 0$.

In particular, if $f : Q \to \mathbb{R}$ is a real smooth function then $df(\sigma) = \rho(\sigma)f$, for $\sigma \in \text{Sec}(E)$. Locally, the exterior differential is determined by

$$dq^i = \rho^i_\alpha e_\alpha \quad \text{and} \quad dc^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} e_\alpha \wedge e_\beta,$$  

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where \{e^\alpha\} is the dual basis of \{e_\alpha\}.

The usual Cartan calculus extends to the case of Lie algebroids: for every section \sigma of \cal E we have a derivation \iota_\sigma (contraction) of degree \(-1\) and a derivation \cal L_\sigma = \iota_\sigma \circ d + d \circ \iota_\sigma (Lie derivative) of degree \(0\), (for more details, see [30, 31]).

Morphisms

Let \((E,\cdot,E,\rho)\) and \((E',\cdot,E',\rho')\) be two Lie algebroids over \(Q\) and \(Q'\) respectively. Suppose that \(\Phi = (\overline{\Phi}, \overline{\Phi})\) is a vector bundle map, that is \(\Phi : E \to E'\) is a fiberwise linear map over \(\overline{\Phi} : Q \to Q'\). The pair \((\overline{\Phi}, \overline{\Phi})\) is said to be a Lie algebroid morphism if

\[
d^E(\Phi^*\sigma') = \Phi^*(d^{E'}\sigma'), \quad \text{for all } \sigma' \in Sec((E')^*) \text{ and for all } l.
\]

Here \(\Phi^*\sigma'\) is the section of the vector bundle \(\wedge^k E^* \to Q\) defined for \(l > 0\) by

\[
(\Phi^*\sigma')_q(a_1, \ldots, a_l) = \sigma'_{{\overline{\Phi}(a_1)}, \ldots, {\overline{\Phi}(a_l)}},
\]

for \(q \in Q\) and \(a_1, \ldots, a_l \in E_q\). In the particular case when \(Q = Q'\) and \(\overline{\Phi} = id_Q\) then (58) holds if, and only if,

\[
(\overline{\Phi} \circ \sigma_1, \overline{\Phi} \circ \sigma_2)_{E'} = (\overline{\Phi}[\sigma_1, \sigma_2])_E, \quad \rho'(\overline{\Phi} \circ \sigma) = \rho(\sigma), \quad \text{for } \sigma, \sigma_1, \sigma_2 \in Sec(E).
\]

The prolongation of a Lie algebroid over a fibration

(See [6, 17, 23, 33]). Let \((E,\cdot,E,\rho)\) be a Lie algebroid over a manifold \(Q\) and \(\pi : P \to Q\) a fibration. Consider the subset of \(E \times TP\)

\[
\cal T^E P = \{(b,v) \in E_q \times T_p P \mid \rho(b) = T_p \pi(v)\}
\]

where \(T \pi : TP \to TQ\) is the tangent map to \(\pi, p \in P_q\), and \(\pi(p) = q\). \(\cal T^E P = \cup_{p \in P} \cal T^E P\) is a vector bundle over \(P\) and the vector bundle projection is \(\pi^E : \cal T^E P \to P\). We can consider a structure of Lie algebroid on \(\cal T^E P\) given by the anchor \(\rho^\pi : \cal T^E P \to TP\), \(\rho^\pi(b,v_p) = v_p\).

This Lie algebroid will be used in Section 6.2.3 when we introduce the solutions of the Hamilton field equations on Lie algebroids.

5.2 The manifold \(\oplus E\)

The standard \(k\)-cosymplectic Lagrangian formalism is developed in the bundle \(\bb R^k \times T^1_1 Q\). When we consider a Lie algebroid \(E\) as a substitute of the tangent bundle, it is natural, in this situation,
to consider that the analog of the bundle of $k^1$-velocities $T_k^1Q$ is the Whitney sum of $k$ copies of the algebroid $E$, and thus the analog of the manifold $\mathbb{R}^k \times T_k^1Q$ is $\mathbb{R}^k \times \oplus E$.

We denote by $\bigoplus^k E = E \oplus \cdots \oplus E$, the Whitney sum of $k$ copies of the vector bundle $E$, with projection map $\bar{\tau} : \bigoplus^k E \to Q$, given by $\bar{\tau}(a_{1q}, \ldots, a_{kq}) = q$. The Lie structure of $E$ allows us to introduce the following maps:

$$\bar{\tau} : \bigoplus^k E \to Q$$

where $\bar{\rho}(a_{1q}, \ldots, a_{kq}) = (\rho(a_{1q}), \ldots, \rho(a_{kq}))$, and $\rho : E \to TQ$ is the anchor map of $E$.

Local basis of sections of $\bar{\tau} : \bigoplus^k E \to Q$

A local basis $\{e_a\}_{a=1}^m$ of $\text{Sec}(E)$ induces a local basis of sections of the bundle $\bar{\tau} : \bigoplus^k E \to Q$. In fact, let $a_q = (a_{1q}, \ldots, a_{kq})$ be an arbitrary point of $\bigoplus E$, then for each $A (A = 1, \ldots, k)$, $a_{Aq} \in E$ and since $\{e_a\}$ is a local basis of sections of $E$ we have $a_{Aq} = y^\alpha(a_{Aq})e_a(q)$. Therefore

$$a_q = (y^\alpha(a_{1q})e_a(q), \ldots, y^\alpha(a_{kq})e_a(q)) = y^\alpha(a_{aq})\bar{e}_A^a(q),$$

where $\bar{e}_A^a(q) = (\ldots, e_a(q), \ldots, 0)$. Thus a local basis $\{e_a\}$ of $\text{Sec}(E)$ induces a local basis $\{\bar{e}_A^a\}$ of $\text{Sec}(\bigoplus E)$ defined by $\bar{e}_A^a(q) := (0, \ldots, e_A(q), \ldots, 0)$, where $\hat{A}$ indicates the $A^{th}$ position of $\bar{e}_A^a(q)$.

If $(q^i, y^\alpha)$ are local coordinates on $\bar{\tau}^{-1}(U) \subseteq E$, then the induced local coordinates $(\hat{q}^i, y_A^\alpha)$ on $\bar{\tau}^{-1}(U) \subseteq \bigoplus E$ are given by

$$q^i(a_{1q}, \ldots, a_{kq}) = \hat{q}^i(q), \quad y_A^\alpha(a_{1q}, \ldots, a_{kq}) = y^\alpha(a_{Aq}).$$

5.3 The $k$-prolongation of a Lie algebroid over a fibration

Let $\pi : P \to Q$ be a bundle. Now we define a vector bundle which generalizes the concept of prolongation of a Lie algebroid over a fibration $\pi : P \to Q$. We denote this bundle by $T_k^E P$ and it is called the $k$-prolongation of $P$ with respect to a Lie algebroid $E$.

Throughout this paper we consider two particular cases of $k$-prolongations. The first corresponds to the case $P = \bigoplus E$, and the bundle $T_k^E(\bigoplus E)$ will allow us to develop the Lagrangian formalism on Lie algebroids, (see section 6.1). This bundle plays the role of the bundle $T_k^1(\mathbb{R}^k \times T_k^1Q) \to \mathbb{R}^k \times T_k^1Q$ in the Lagrangian $k$-cosymplectic formalism. The second particular case is $P = \bigoplus E^\ast$, where we will develop the Hamiltonian formalism (see section 6.2).

The total space of the $k$-prolongation of $P$ with respect to $E$

$$T_k^E P = \left(\mathbb{R}^k \times \bigoplus^k E\right) \times_{\mathbb{R}^k \times T_k^1Q} T_k^1(\mathbb{R}^k \times P)$$
is the total space of the pull-back of the map $F = \tau^k_{\mathbb{R}^k} \times T^1 \pi : T^1_k(\mathbb{R}^k \times P) \to \mathbb{R}^k \times T^1_k Q$, (locally defined by \((7)\)), by the map $Id_{\mathbb{R}^k} \times \tilde{\rho} \equiv Id_{\mathbb{R}^k} \times \rho \circ \cdot k. \circ \cdot : \mathbb{R}^k \times \oplus E \to \mathbb{R}^k \times T^1_k Q$,

$$\mathcal{T}^E_k P = \{(s, a_q), W_{(t,p)}\} \in (\mathbb{R}^k \times \oplus E) \times T^1_k(\mathbb{R}^k \times P) : Id_{\mathbb{R}^k} \times \tilde{\rho}(s, a_q) = F(W_{(t,p)})\},$$

where $W_{(t,p)} = ((v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)}) \in T^1_k(\mathbb{R}^k \times P)$.

Let us observe that $(s, a_q, (v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)}) \in T^E_k P$ means that $s = t$ and $q = \pi(p)$. Therefore, an element $(s, a_q, (v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)})$ of $T^E_k P$ can be identified with a family $(a_{\pi(p)}, (v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)})$.

**Remark 5.1** Throughout the paper we shall denote the elements of $T^E_k P$ as $(a_q, W_{(t,p)})$ where $a_q \in \oplus E$, $W_{(t,p)} \in T^1_k(\mathbb{R}^k \times P)$ and $\pi(p) = q$.

The $k$-prolongacion of a Lie algebroid $E$ over a fibration $\pi : P \to Q$ is the space $\mathcal{T}^E_k P$ fibered over $\mathbb{R}^k \times P$ with the projection

$$\tau^k_{\mathbb{R}^k \times P} : \mathcal{T}^E_k P \to \mathbb{R}^k \times P$$

$$(a_q, W_{(t,p)}) \to \tau^k_{\mathbb{R}^k \times P}(a_q, W_{(t,p)}) = \tau^k_{\mathbb{R}^k \times F}(W_{(t,p)}) = (t, p)$$

where $\tau^k_{\mathbb{R}^k \times P} : T^1_k(\mathbb{R}^k \times P) \to \mathbb{R}^k \times P$ is the projection of the bundle of $k^1$-velocities of $P$.

If $q \in Q$, $a_q \in \oplus E$, $W_{(t,p)} = (v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)} \in T^1_k(\mathbb{R}^k \times P)$ and $(a_q, W_{(t,p)}) \in T^E_k P$, we have the following natural projections

$$\tau^k_{1, \oplus E}(a_q, W_{(t,p)}) = a_q \ , \ \tau^A_k(a_q, W_{(t,p)}) = v_{A(t,p)}$$

and we have the following diagram

**Local coordinates on $T^E_k P$**

Let $(a_q, W_{(t,p)})$ be an arbitrary point of $T^E_k P$, then $a_q = (a_{1_q}, \ldots, a_{k_q}) \in \oplus E$ and $W_{(t,p)} = ((v_1)_{(t,p)}, \ldots, (v_k)_{(t,p)}) \in T^1_k(\mathbb{R}^k \times P)$, satisfy

$$Id_{\mathbb{R}^k} \times \tilde{\rho}(t, a_q) = \tau^k_{\mathbb{R}^k} \times T^1_k \pi(v_1)_{(t,p)}, \ldots, v_k)_{(t,p)}$$

(60)
Given local coordinates \((q^i, u^\vartheta)\), \(1 \leq i \leq \dim Q\), \(1 \leq \vartheta \leq s\) on \(P\), and a local basis \(\{e_\alpha\}_{\alpha=1}^m\) of sections of \(E\), we have that

\[
a_q = y_A^a(a_q)e^A_\alpha(q), \quad (v_A)(t,p) = (v_A)^B \frac{\partial}{\partial q^A}(t,p) + (v_A)^i \frac{\partial}{\partial q^i}(t,p) + (v_A)^\vartheta \frac{\partial}{\partial u^\vartheta}(t,p). \tag{61}
\]

As

\[
\rho(a_Aq) = \rho(y_A^a(a_q)e_\alpha(q)) = y_A^a(a_q)\rho(e_\alpha(q)) = y_A^a(a_q)\rho^i(a_q)\frac{\partial}{\partial q^i}(t,p),
\]

and from (61) we have that the condition (60) is equivalent to

\[
(v_A)^i = y_A^a(a_q)\rho^i(a_q). \tag{62}
\]

Taking into account (62) we introduce the local coordinates \((t^A, q^i, u^\vartheta, z^A_A, v^B_A, (v_A)^\vartheta)\) on \(T_k^E P\) given by

\[
t^A(a_q, W(t,p)) = t^A(t), \quad q^i(a_q, W(t,p)) = q^i(q), \quad u^\vartheta(a_q, W(t,p)) = u^\vartheta(p),
\]

\[
z^A_A(a_q, W(t,p)) = y_A^a(a_q), \quad v^B_A(a_q, W(t,p)) = v_{A(t,p)}(t^B), \quad (v_A)^\vartheta(a_q, W(t,p)) = v_{A(t,p)}(u^\vartheta). \tag{63}
\]

**Local basis of sections of the bundle** \(\overline{\tau}_{\mathbb{R}^k \times P} : T_k^E P \to \mathbb{R}^k \times P\)

Given local coordinates \((t^A, q^i, u^\vartheta)\) on \(\mathbb{R}^k \times P\) and a local basis \(\{e_\alpha\}\) of sections of \(E\), we can define a local basis \(\{X^A_A, Y^B_B, V^A_\vartheta\}\) of sections of \(\overline{\tau}_{\mathbb{R}^k \times P} : T_k^E P \to \mathbb{R}^k \times P\). In fact, from (61) and (62) we deduce that for a point \((a_q, W(t,p))\) one has

\[
(a_q, W(t,p)) = y_A^a(a_q)(e^A_\alpha(q), (0, \ldots, \rho^i_A(x)\frac{\partial}{\partial q^i}(t,p), \ldots, 0))
\]

\[
+ v^B_A(0_q, (0, \ldots, \frac{\partial}{\partial H^B}(t,p), \ldots, 0)) + (v_A)^\vartheta(0_q, (0, \ldots, \frac{\partial}{\partial u^\vartheta}(t,p), \ldots, 0)). \tag{64}
\]

Thus the set \(\{X^A_A, Y^B_B, V^A_\vartheta\}\) with \(1 \leq A, B \leq k; 1 \leq \alpha \leq m; 1 \leq \vartheta \leq s\), defined by

\[
X^A_A(t,p) = (e^A_\alpha(q), (0, \ldots, \rho^i_A(q)\frac{\partial}{\partial q^i}(t,p), \ldots, 0))
\]

\[
Y^B_B(t,p) = (0_q, (0, \ldots, \frac{\partial}{\partial H^B}(t,p), \ldots, 0)) \tag{65}
\]

\[
V^A_\vartheta(t,p) = (0_q, (0, \ldots, \frac{\partial}{\partial u^\vartheta}(t,p), \ldots, 0)
\]

is a local basis of sections of the vector bundle \(\overline{\tau}_{\mathbb{R}^k \times P} : T_k^E P \to \mathbb{R}^k \times P\).

**Remark 5.2** Throughout this paper, a section of \(T_k^E P\) means a section of \(\overline{\tau}_{\mathbb{R}^k \times P} : T_k^E P \to \mathbb{R}^k \times P\).
$k$-vector field on $\mathbb{R}^k \times P$ associated to a section of $T_k^E P$

Every section $\sigma$ of $T_k^E P$ has associated a $k$-vector field on $\mathbb{R}^k \times P$ given by

$$\tilde{\tau}_2(\sigma) = (\tilde{\tau}_2^1(\sigma), \ldots, \tilde{\tau}_2^k(\sigma)).$$

Let $\sigma: \mathbb{R}^k \times P \to T_k^E P$ be an arbitrary section of $\tau_{\mathbb{R}^k \times P}$, such that $\sigma(t, p) = (t, p, \sigma_A^B(t, p), \sigma_A^B(t, p)), \sigma_A^B(t, p), \sigma_A^B(t, p))$, then from (64) and (65) we have that the expression of $\sigma$ in terms of the basis $\{X^A, B^A, Y^A\}$ is

$$\sigma = \sigma_A^B X^A + \sigma_A^B Y^A + \sigma_A^2 Y^A,$$

and the associated $k$-vector field $\tilde{\tau}_2(\sigma) = (\tilde{\tau}_2^1(\sigma), \ldots, \tilde{\tau}_2^k(\sigma))$ is locally given by

$$\tilde{\tau}_2^k(\sigma) = \sigma_A^B \frac{\partial}{\partial t^B} + \rho_\alpha^k \sigma_A^\alpha \frac{\partial}{\partial q^\alpha} + \sigma_A^2 \frac{\partial}{\partial \theta^\gamma} \in \mathfrak{X}(\mathbb{R}^k \times P). \tag{66}$$

These $k$-vector fields play an important role in the development of the Lagrangian and Hamiltonian formalism on Lie algebroids. (See section 6.1.2 and 6.2.2)

**Lie bracket of section of $T_k^E P$**

A Lie bracket associated to the Lie bracket on $\text{Sec}(E)$ can be easily defined in terms of projectable sections as follows: a section $Z$ of $T_k^E P$ is said to be projectable if there exists a section $\sigma$ of $\oplus^k E$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathbb{R}^k \times P & \xrightarrow{Z} & T_k^E P \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\sigma} & \oplus^k E
\end{array}$$

Equivalently, a section $Z$ is projectable if, and only if, it is of the form $Z(t, p) = (\sigma(\pi(p)), X(t, p))$, for some section $\sigma = (\sigma_1, \ldots, \sigma_k)$ of $E$ and some $k$-vector field $X = (X_1, \ldots, X_k)$ on $\mathbb{R}^k \times P$. The Lie bracket of two projectable sections $Z$ and $Z'$ is then given by

$$[Z, Z']^\pi(t, p) = ([\sigma_1, \sigma_1]^E(q), \ldots, [\sigma_k, \sigma_k]^E(q), [X_1, X_1](t, p), \ldots, [X_k, X_k](t, p)),$$

where $(t, p) \in \mathbb{R}^k \times P, q = \pi(p)$. It is easy to see that $[Z, Z']^\pi(t, p)$ is an element of $T_k^E P$. Since any section of $T_k^E P$ can be locally written as a linear combination of the projectable sections $\{X^A, B^A, Y^A\}$, the definition of a Lie bracket for arbitrary sections of $T_k^E P$ follows.

The Lie brackets of the elements of the local basis $\{X^A, B^A, Y^A\}$ are

$$\begin{align*}
[X^A, B^B]^\pi &= \delta^A_B C^C_{\alpha\beta} X^A \\
[X^A, Y^B]^\pi &= 0 \\
[Y^A, B^B]^\pi &= \theta_A^B X^A = 0 \\
[Y^A, Y^B]^\pi &= \theta_A^B X^A = 0 .
\end{align*} \tag{67}$$

**The derivation $d^A$**

Now, for each $A$, we define a derivation of degree 1 on the set $\text{Sec}(\Lambda^l(T_k^E P)^*)$, that is, $d^A: \text{Sec}(\Lambda^l(T_k^E P)^*) \to \text{Sec}(\Lambda^{l+1}(T_k^E P)^*)$ given by

$$d^A = \sum_{i=1}^l (-1)^{i+1} \tilde{\rho}_A(Z_i)\mu(Z_1, \ldots, Z_{i-1}, \ldots, Z_{i+l})$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([Z_i, Z_j]^\pi, Z_1, \ldots, \tilde{Z}_i, \ldots, \tilde{Z}_j, \ldots, Z_{l+1}), \tag{68}$$
for \( \mu \in \text{Sec}(\wedge^1(T_k^E P)^*) \) and \( Z_1, \ldots, Z_k \in \text{Sec}(T_k^E P) \). In particular, if \( f : \mathbb{R}^k \times P \to \mathbb{R} \) is a real smooth function, then \( d^A f : \mathbb{R}^k \times P \to (T_k^E P)^* \) is defined by \( d^A f(Z) = \tilde{\tau}_2^A(Z) f \), for \( Z \in \text{Sec}(T_k^E P) \).

From (65) and (66) we deduce that \( d^A f \) is locally given by

\[
d^A f = \frac{\partial f}{\partial t} \gamma^C_A + \rho_0^i \frac{\partial f}{\partial y^i_A} A^i_A + \frac{\partial f}{\partial y^j} Y^j_A ,
\]

(69)

where \( \{ X^A_A, Y^j_A, V^j_A \} \) is the local basis of sections of \( (T_k^E P)^* \) dual to the local basis \( \{ X^A_A, Y^B_B, V^j_A \} \) of sections of \( T_k^E P \).

The derivation \( d^A \) allows us to introduce the Poincaré-Cartan 2-sections of the Lagrangian formalism on Lie algebroids (see section 6.1.2), and the Liouville 2-section of the Hamiltonian formalism (see section 6.2.2).

6 \( k \)-cosymplectic classical field theory on Lie algebroids

6.1 Lagrangian formalism

In this subsection we give a description of Lagrangian \( k \)-cosymplectic first-order classical field theory on Lie algebroids. The Lagrangian field theory on Lie algebroids is developed in the \( k \)-prolongation \( T_k^E(\oplus E) \) of a Lie algebroid \( E \) over the vector bundle projection \( k : \oplus E \to Q \). This vector bundle \( T_k^E(\oplus E) \) plays the role of \( \tau^k_{\mathbb{R}^k \times T^1_k Q} : T^1_k(\mathbb{R}^k \times T^1_k Q) \to \mathbb{R}^k \times T^1_k Q \), the tangent bundle of \( k \)-velocities of \( \mathbb{R}^k \times T^1_k Q \), in the standard Lagrangian \( k \)-cosymplectic formalism.

6.1.1 Geometric elements

The vector bundle \( T_k^E(\oplus E) \)

Consider the \( k \)-prolongation \( T_k^E(\oplus E) \) of a Lie algebroid \( E \) over the fibration \( k : \oplus E \to Q \) (observe that, in this case, the fiber \( \pi : P \to Q \) is \( k : \oplus E \to Q \)). Taking into account the general description of the \( k \)-prolongation (see the previous section), if \( (q^i, y^A_A) \) are local coordinates on \( \oplus E \), then we have the local coordinates \( (t^A, q^i, y^A_A, z^A_B, v^A_B) \) on \( T_k^E(\oplus E) \) given by (see (63))

\[
\begin{align*}
t^A(a_q, W(t, b)) &= t^A(t) , \\
q^i(a_q, W(t, b)) &= q^i(q) , \\
y^A_A(a_q, W(t, b)) &= y^A_A(p) , \\
z^A_B(a_q, W(t, b)) &= y^A_B(a_q) , \\
v^A_B(a_q, W(t, b)) &= v^A_B(a_q, t^B) ,
\end{align*}
\]

(70)

and the local basis \( \{ X^A_A, Y^A_A, V^A_B \} \) of sections of \( \tau^k_{\mathbb{R}^k \times \oplus E} : T_k^E(\oplus E) \to \mathbb{R}^k \times \oplus E \), defined in (65), is written here as

\[
\begin{align*}
X^A_A(t, b_q) &= (\tilde{\tau}^A_A(q), (0, \ldots, \rho^i_A(q) \frac{\partial}{\partial q^i}|_{(t, b_q)}, \ldots, 0)) \\
Y^A_A(t, b_q) &= (0, (0, \ldots, \frac{\partial}{\partial B}|_{(t, b_q)}, \ldots, 0)) \\
(V^A_B(t, b_q)) &= (0, (0, \ldots, \frac{\partial}{\partial y^B_B}|_{(t, b_q)}, \ldots, 0))
\end{align*}
\]

(71)
From (65), we know that the $k$-vector fields associated to the basis $\{\mathcal{X}_\alpha^A, \mathcal{Y}_a^B, (\mathcal{V}^A)^B\}$ are

$$
\hat{\tau}_1^A(\mathcal{X}_\alpha^B) = \delta_B^A \rho_0^\alpha \frac{\partial}{\partial q^i}, \quad \hat{\tau}_1^A(\mathcal{Y}_a^B) = \delta_B^a \frac{\partial}{\partial q^i}, \quad \hat{\tau}_1^A((\mathcal{V}^C)^B) = \delta_A^C \frac{\partial}{\partial y_A^B}.
$$

(72)

The Lie brackets (67) of the elements of the local basis of sections are now

$$
\begin{align*}
[X^\alpha, X^\beta]_\tau^\gamma &= \delta^\alpha_\beta \gamma^\gamma_\alpha, \quad [X^\alpha, Y^B]_\tau^\gamma = 0, \quad [X^\alpha, (\mathcal{V}^C)^B]_\tau^\gamma = 0 \\
[Y^B, Y^C]_\tau^\gamma &= 0, \quad [Y^B, (\mathcal{V}^C)^D]_\tau^\gamma = 0, \quad [(\mathcal{V}^A)^B, (\mathcal{V}^C)^D]_\tau^\gamma = 0.
\end{align*}
$$

(73)

In $T_k^E(\oplus E)$ there are two families of canonical objects: the Liouville sections and the vertical endomorphism whose definitions and properties mimic those of the corresponding canonical objects in $\mathbb{R}^k \times T_k^1 Q$ (see [25 28 29 39]). First, we need to introduce:

**Vertical $A$-lifts**

An element $(a_q, W(t,b_q))$ of $T_k^E(\oplus E)$ is said to be vertical if

$$
\hat{\tau}_{1, \oplus E}^k (a_q, W(t,b_q)) = 0_q \equiv (0_q, \ldots, 0_q) \in \oplus E
$$

where $\hat{\tau}_{1, \oplus E}^k : T_k^E(\oplus E) \to \oplus E$ is the projection on the first factor. This condition means that

1) $a_{Aq} = 0_q$.

2) If $Z(t,b_q) = (0_q, W(t,b_q))$, with $W(t,b_q) = (v_1(t,b_q), \ldots, v_k(t,b_q)) \in T_k^l(\mathbb{R}^k \times \oplus E)$, since $Z \in T_k^E(\oplus E)$ from (62) we have $(v_A)^j = y_A^\alpha(0_q) \rho_\alpha = 0$

Therefore each $v_A(t,b_q)$ is locally given by

$$
v_A(t,b_q) = v_A^B \frac{\partial}{\partial t^B}(t,b_q) + (v_A)^\alpha_B \frac{\partial}{\partial y_A^\alpha}(t,b_q) \in T(t,b_q)(\mathbb{R}^k \times \oplus E)
$$

and it is vertical with respect to the canonical projection $\mathbb{R}^k \times \oplus E \to Q$.

For each $A = 1, \ldots, k$, we call the following map the vertical $A$th-lifting map

$$
\xi^{VA}: \mathbb{R}^k \times \oplus E \times Q \mathbb{R}^k \times \oplus E \to T_k^E(\oplus E),
$$

(74)

$$
(t, a_q, b_q) \mapsto \xi^{VA}(t, a_q, b_q) = (0_q, (0, \ldots, (a_q(V_A), \ldots, 0))
$$

where for an arbitrary function $f$ defined on $\mathbb{R}^k \times \oplus E$, $(a_q)^V_A$ is given by

$$
(a_q(V_A)) f = \frac{d}{ds}|_{s=0} f(t, b_1, \ldots, b_n, s a_q, \ldots, b_k).
$$

(75)

From (75) we deduce that the local expression of $(a_q)^V_A$ is

$$
(a_q)^V_A(t,b_q) = y_A^\alpha(a_q) \frac{\partial}{\partial y_A^\alpha}(t,b_q) \in T(t,b_q)(\mathbb{R}^k \times \oplus E).
$$

(76)
From (71), (76) and (74) we obtain
\[
\xi_{\alpha}(t, a_\alpha, b_\alpha) = (0_\alpha, (0, \ldots, y_{A}^{\alpha}(a_\alpha) \partial A_{A}^{\alpha}(t, b_\alpha), \ldots, 0)) = y_{A}^{\alpha}(a_\alpha)(V_{\alpha}^{A})(t, b_\alpha).
\] (77)

The vertical $A^h$-lifting map allows us to define vertical lifts of section of $\tau : \mathbb{R}^{k x \oplus} E \to Q$ to sections of $\tau_{\mathbb{R}^{k x \oplus}} : \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E) \to \mathbb{R}^{k x \oplus E}$. If $\sigma$ is a section of $\oplus E \to Q$, then the section $\sigma_{\mathbb{R}^{k x \oplus}} : \mathbb{R}^{k x \oplus E} \to \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E)$ of $\tau_{\mathbb{R}^{k x \oplus}}$ is defined by $\sigma_{\alpha}(t, b_\alpha) = \xi_{\alpha}(t, \sigma_\alpha, b_\alpha)$, and it will be called the vertical $A^h$-lift of $\sigma$. In particular, from (77) we obtain
\[
(\tau_{\alpha})_{\mathbb{R}^{k x \oplus}}(t, b_\alpha) = \delta_{\alpha}^{A}(V_{\alpha}^{A})(t, b_\alpha),
\]
(78)

The Liouville sections

The Liouville $A^h$-section $\Delta_{\alpha}$ is the section of $\tau_{\mathbb{R}^{k x \oplus}} : \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E) \to \mathbb{R}^{k x \oplus E}$ given by
\[
\Delta_{\alpha} : \mathbb{R}^{k x \oplus E} \to \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E)
\]
\[
(t, b_\alpha) \mapsto \Delta_{\alpha}(t, b_\alpha) = \xi_{\alpha}(t, b_\alpha),
\]
From (77) we obtain that $\Delta_{\alpha}$ is locally given by
\[
\Delta_{\alpha} = \sum_{\alpha} y_{\alpha}(V_{\alpha}^{A})_{\alpha}.
\] (79)

The set $\{\tau_{\mathbb{R}^{k x \oplus}}(\Delta_{\alpha}) = (\tau_{1}^{\alpha}(\Delta_{\alpha}), \ldots, \tau_{k}^{\alpha}(\Delta_{\alpha}))\}$, where each $\tau_{\mathbb{R}^{k x \oplus}}(\Delta_{\alpha})$ is the $k$-vector field associated to each $\Delta_{\alpha}$ given by (69), enables us to introduce the Lagrangian energy function $E_{L}$

**Remark 6.1** In the standard case, every section $\Delta_{\alpha}$ translates into the $k$-vector field on $\mathbb{R}^{k x T_{k}^{1}Q}$ given by $(0, \ldots, v_{\alpha} \partial A_{A}^{\alpha}, \ldots, 0)$, and since $\Delta$ is fixed, $\Delta_{\alpha}$ is identified with the canonical vector field, $\Delta_{\alpha} = v_{\alpha} \partial A_{A}^{\alpha}$. (See [23, 28, 29, 38]).

The vertical endomorphism

**Definition 6.1** The $A^h$-vertical endomorphisms of $\mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E)$, for $1 \leq A \leq k$, are defined as
\[
\tilde{S}_{A} : \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E) \to \mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E)
\]
\[
(a_\alpha, W_{(t, b_\alpha)}) \mapsto \tilde{S}_{A}(a_\alpha, W_{(t, b_\alpha)}) = \xi_{\alpha}(t, a_\alpha, b_\alpha).
\]

Next, we express $\tilde{S}_{A}$ using the basis of local sections $\{A_{\alpha}^{A}, B_{\beta}^{A}, (V_{A}^{A})_{\beta}^{A}\}$ of $\mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E)$ and its dual basis $\{A_{\alpha}^{\alpha}, B_{\beta}^{A}, (V_{A}^{A})_{\alpha}^{B}\}$ of $(\mathcal{T}_{\mathbb{R}^{k x \oplus}}^E(k \oplus E))^{*}$. In fact, from (71) and (78) we obtain
\[
\tilde{S}_{A} = \sum_{\alpha} (V_{A}^{A})_{\alpha} \otimes A_{\alpha}^{\alpha}.
\] (80)

**Remark 6.2** In the standard case, the endomorphism $\tilde{S}_{A}$ coincides with the $A^h$ element of the family of the canonical $k$-tangent structure $(S_{1}, \ldots, S_{A})$.  

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Second order partial differential equations

In the standard $k$-cosymplectic Lagrangian formalism, the solutions to the Euler-Lagrange equations are obtained as integral sections of second order partial differential equations (SOPDE). Taking into account the Definition 2.5 and the remarks 6.1 and 6.2, we introduce the following

**Definition 6.2** A section $\xi : \mathbb{R}^k \times \oplus E \to T^E_k \oplus E$ of $T^E_k \oplus E$ is called a second order partial differential equation (SOPDE) if

$$\tilde{S}^A(\xi) = \Delta_A \quad \text{and} \quad \tilde{V}^B_A(\xi) = \delta^A_B.$$

It is easy to deduce that the local expression of a SOPDE $\xi$ is

$$\xi = \gamma^A_A + y^\alpha_A \chi^A_A + (\xi_A)^B_B (\gamma^A_A),$$

where $(\xi_A)^B_A$ are functions on $\mathbb{R}^k \times \oplus E$.

From (80), we note that the $k$-vector field $\tilde{\tau}_2(\xi) = (\tilde{\tau}_2^1(\xi), \ldots, \tilde{\tau}_2^k(\xi))$ on $\mathbb{R}^k \times \oplus E$ associated to $\xi$ is locally given by

$$\tilde{\tau}_2^A(\xi) = \frac{\partial}{\partial t^A} + \rho^i_A \frac{\partial}{\partial q^i} + (\xi_A)^B_B \frac{\partial}{\partial y^B_B}.$$  \hspace{1cm} (81)

**Definition 6.3** A map $\eta : \mathbb{R}^k \to \mathbb{R}^k \times \oplus E$ is an integral section of the SOPDE $\xi$, if $\eta$ is an integral section of the associated $k$-vector field $\tilde{\tau}_2(\xi)$, that is,

$$\tilde{\tau}_2^A(\xi)(\eta(t)) = \eta_a(t) \left( \frac{\partial}{\partial t^A} \bigg|_t \right).$$  \hspace{1cm} (82)

Locally, if $\eta(t) = (\eta^A(t), \eta^i(t), \eta^B(t))$, from (81) we deduce that (82) is equivalent to

$$\frac{\partial \eta^B}{\partial t^A} \bigg|_t = \delta^A_B, \quad \frac{\partial \eta^i}{\partial t^A} \bigg|_t = \eta^i_A(t) \rho^i_A, \quad \frac{\partial \eta^B}{\partial t^A} \bigg|_t = (\xi_A)^B_B(\eta(t)).$$  \hspace{1cm} (83)

### 6.1.2 Lagrangian formalism

In this section, we develop a geometric framework, enabling us to write the Euler-Lagrange equations associated with the Lagrangian function $L$ in an intrinsic way.

Let $L : \mathbb{R}^k \times \oplus E \to \mathbb{R}$ be a function which we call a Lagrangian function.

**Poincaré-Cartan sections and the Lagrangian energy function**

We introduce the Poincaré-Cartan 1-sections associated with $L$:

$$\Theta^A_L : \mathbb{R}^k \times \oplus E \to (T^E_k \oplus E)^*, \hspace{1cm} (t, b_q) \mapsto \Theta^A_L(t, b_q),$$

where $\Theta^A_L(t, b_q)$ is defined by

$$\Theta^A_L(t, b_q) : (T^E_k \oplus E)(t, b_q) \to \mathbb{R}, \hspace{1cm} Z(t, b_q) \mapsto (\Theta^A_L(t, b_q)Z(t, b_q)) = (d^A_L)(t, b_q)(Z(t, b_q)).$$
From (68) we obtain that
\[(\Theta^A_L)(t, b_q)(Z(t, b_q)) = (d^A L)(t, b_q)(\tilde{S}^A(t, b_q)(Z(t, b_q))) = \tau^A_2((\tilde{S}^A)(t, b_q)(Z(t, b_q)))L , \]

where \((t, b_q) \in \mathbb{R}^k \times \oplus E\), and \(Z(t, b_q) \in (\mathcal{T}^E_k(\oplus E))(t, b_q)\).

From (72) and (80) we obtain the local expression of \(\Theta^A_L\),
\[
\Theta^A_L = \frac{\partial L}{\partial y^\alpha_A} X^\alpha_A . \tag{84}
\]

The Poincaré-Cartan 2-sections \(\Omega^A_L: \mathbb{R}^k \times \oplus E \rightarrow (\mathcal{T}^E_k(\oplus E))^* \wedge (\mathcal{T}^E_k(\oplus E))^*\) associated with \(L\) are given by
\[
\Omega^A_L = -d^A \Theta^A_L . \tag{85}
\]

From (57), (68), (72), (73) and (84) we obtain
\[
\omega^A_L = \frac{\partial^2 L}{\partial t^A \partial y^\alpha_A} X^\alpha_A \wedge \tilde{Y}^B_B + \frac{1}{2} \left( \rho^\beta A \frac{\partial^2 L}{\partial q^\beta \partial y^\alpha_A} - \rho^\beta \frac{\partial^2 L}{\partial q^\alpha \partial y^\beta_A} + C^\gamma_{\alpha \beta} \frac{\partial L}{\partial y^\gamma_A} \right) X^\alpha_A \wedge X^\beta_A + \frac{\partial^2 L}{\partial y^\beta_A \partial y^\alpha_A} X^\alpha_A \wedge (V^A_B) \tag{85}
\]

The energy function \(E_L: \mathbb{R}^k \times \oplus E \rightarrow \mathbb{R}\) defined by \(L\) is
\[
E_L = \sum_{A=1}^k \tau^A_2(\Delta_A)L - L , \tag{86}
\]

and from (72) and (79) one deduces that \(E_L\) is locally given by
\[
E_L = \sum_{A=1}^k y^\alpha_A \frac{\partial L}{\partial y^\alpha_A} - L . \tag{86}
\]

**Morphisms**

Before addressing the Euler-Lagrange equations on Lie algebroids, we show a new point of view for the solutions to the standard Euler-Lagrange equations, which allows us to consider a solution as a morphism of Lie algebroids.

In the standard Lagrangian \(k\)-cosymplectic description of first order classical field theories, a solution to the Euler-Lagrange equation is a field \(\phi: \mathbb{R}^k \rightarrow Q\) such that its first prolongation \(\phi^{[1]}: \mathbb{R}^k \rightarrow \mathbb{R}^k \times T^1_k Q\) (see Definition 2.6) satisfies the Euler-Lagrange field equations, that is,
\[
\sum_{A=1}^k \frac{\partial}{\partial t^A} \left| \frac{\partial L}{\partial v^A_{[1]}(t)} \right| = \frac{\partial L}{\partial q^\alpha} \left| \phi^{[1]}(t) \right| .
\]

The map \(\phi\) induces the following morphism of Lie algebroids
\[
\begin{array}{ccc}
T\mathbb{R}^k & \xrightarrow{T\phi} & TQ \\
\tau_{\phi} & \downarrow & \tau_Q \\
\mathbb{R}^k & \xrightarrow{\phi} & Q
\end{array}
\]
Remark 6.3

In the standard case where the Euler-Lagrange equations to be considered here is a morphism of Lie algebroid $\Phi = (\mathcal{F}, \Phi)$:

$$\Phi : \mathcal{T}_k^E \to \mathcal{F}$$

Taking a local basis $\{e_A\}_{A=1}^k$ of local sections of $\mathcal{T}_k^E$, one can define a section $\tilde{\Phi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus E$ associated to $\Phi$ and given by

$$\tilde{\Phi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus E \equiv E \oplus \mathbb{R}^k \oplus E$$

Let $(t^A)$ and $(q^i)$ be a local coordinate system on $\mathbb{R}^k$ and $Q$, respectively. Let $\{e_A\}$ be a local basis of sections of $\tau_{\mathbb{R}^k}$ and $\{e_\alpha\}$ be a local basis of sections of $E$; we denote by $\{e^A\}$ and $\{e_\alpha\}$ the dual basis. Then $\Phi$ is determined by the relations $\Phi(t^A) = (\phi^A(t))$ and $\Phi^\star e_\alpha = \phi_\alpha^A e^A$ for certain local functions $\phi^A$ and $\phi_\alpha^A$ on $\mathbb{R}^k$. Thus, the associated map $\tilde{\Phi}$ is locally given by $\tilde{\Phi}(t) = (t^A, \phi(t), \phi_\alpha^A(t))$.

In this case, the condition of Lie morphism is written

$$\rho_\alpha^A \phi_\alpha^A = \frac{\partial \phi^A}{\partial t^A}, \quad o = \frac{\partial \phi_\alpha^A}{\partial t^A} \quad \text{and} \quad \frac{\partial \phi^A}{\partial t^A} = \frac{\partial \phi_\alpha^A}{\partial t^A}. \quad (87)$$

**Remark 6.3** In the standard case where $E = TQ$, the above morphism conditions reduce to

$$\phi^A = \frac{\partial \phi^A}{\partial t^A} \quad \text{and} \quad \frac{\partial \phi^A}{\partial t^A} = \frac{\partial \phi_\alpha^A}{\partial t^A}. \quad (88)$$

Then, by considering morphisms we are just considering the first prolongation of fields $\phi : \mathbb{R}^k \to Q$.

**The Euler-Lagrange equations**

For an arbitrary section $\xi : \mathbb{R}^k \times \oplus E \to \mathcal{T}_k^E$ of $\tau_{\mathbb{R}^k} : \mathcal{T}_k^E \to \mathbb{R}^k \times \oplus E$, consider the equations

$$\nabla_A C^C \xi_A = \delta_A^C, \quad L^C A + \frac{1}{k} \left( d^A E^B + \sum_{C=1}^k \frac{\partial L}{\partial C} C^C A \right), \quad (89)$$

Writing $\xi = \xi^C B C^C B + \xi_\alpha^A \xi^B_\alpha + (\xi_B)^C_\alpha (V^B)^C_\alpha$, from (69), (85), (86) we have that (88) is equivalent to

$$\nabla_A C^C \xi_A = \delta_A^C, \quad \frac{\partial^2 L}{\partial C \partial Y^C A} \xi_A = \frac{1}{k} \frac{\partial^2 L}{\partial C \partial Y^C B} y^B \xi_A, \quad \frac{\partial^2 L}{\partial C \partial Y^C A} \xi_A = \frac{1}{k} \frac{\partial^2 L}{\partial C \partial Y^C B} y^B \xi_A \quad \text{and} \quad (89)$$

$$\frac{\partial^2 L}{\partial C \partial Y^C A} \xi_A + \left( \rho ^2 B \frac{\partial^2 L}{\partial C \partial Y^C A} - \rho ^2 A \frac{\partial^2 L}{\partial C \partial Y^C A} + C^C_\alpha \frac{\partial L}{\partial C \partial Y^C A} \right) \xi_\alpha^A + \frac{\partial^2 L}{\partial C \partial Y^C B} (\xi_\alpha^B) \xi_\alpha^A = \frac{1}{k} \left( \frac{\partial L}{\partial C \partial Y^C B} - \frac{\partial^2 L}{\partial C \partial Y^C B} y^B \right) \rho ^2 A \xi_A$$

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From (89) we obtain
\[ \xi_C^B = \delta_C^B , \quad \sum_{A=1}^k \frac{\partial^2 L}{\partial t \partial y_C^A} \xi_A^\alpha = \frac{\partial^2 L}{\partial t \partial y_C^B} \eta^\beta, \quad \sum_{A=1}^k \frac{\partial^2 L}{\partial y_C^A \partial y_C^B} \xi_A^\alpha = \frac{\partial^2 L}{\partial y_C^B \partial y_C^B} \eta^\beta, \]
\[ \sum_{A=1}^k \frac{\partial^2 L}{\partial t \partial y_C^A} + \sum_{A=1}^k \left( \rho_\beta^\gamma \frac{\partial^2 L}{\partial q^\gamma \partial y_C^A} + \rho_\alpha^\gamma \frac{\partial^2 L}{\partial t \partial y_C^A} \right) \xi_A^\alpha = \sum_{A=1}^k \frac{\partial^2 L}{\partial y_C^B \partial y_C^A} (\xi_A^B) \right) \right). \quad (90) \]
\[ = \left( \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial q^i \partial y_C^B} \eta^\beta \right) \rho_A^\alpha. \]

**Definition 6.4** L is said to be a regular Lagrangian if the matrix \( \left( \frac{\partial^2 L}{\partial y_C^\alpha \partial y_C^\beta} \right) \) is regular.

When L is regular, from the three identity of (90) we obtain
\[ \xi_A^\alpha = y_A^\alpha. \quad (91) \]

In this case, the solution \( \xi \) to the equations (83) is a SOPDE and the functions \((\xi_A)^\beta_B\) are the solutions to the equation
\[ \sum_{A=1}^k \left( \frac{\partial^2 L}{\partial t \partial y_C^A} + \rho_\beta^\gamma \frac{\partial^2 L}{\partial q^\gamma \partial y_C^A} y_A^\beta + C_{\alpha \beta}^\gamma \frac{\partial L}{\partial y_C^A} y_A^\beta + \frac{\partial^2 L}{\partial y_C^B \partial y_C^A} (\xi_A)^B \right) = \frac{\partial L}{\partial q^i} \rho_A^\alpha. \quad (92) \]

Let \( \Phi = (\overline{\Phi}, \Phi) \) be a morphism on Lie algebroids between \( \tau_{k}E : \mathbb{T} \mathbb{R}^k \to \mathbb{R}^k \) and \( \tau : E \to Q \), and \( \Phi : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \) is the associated map. If \( \Phi : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \) is an integral section of the SOPDE \( \xi \) locally given by \( \Phi(t) = (\phi^B(t), \phi^i(t), \phi^\alpha(t)) \). From (83), (87), (90), (91) and (92) we obtain
\[ \frac{\partial \phi^B}{\partial t} |_{\phi(t)} = \delta^B_A, \quad \frac{\partial \phi^i}{\partial t} |_{\phi(t)} = \phi^i_A(\rho_A^\alpha(\phi^j(t)) , \quad 0 = \frac{\partial \phi^\alpha}{\partial t} |_{\phi(t)} = \phi^\alpha_B(\rho_B^\alpha(\phi^i(t)) \frac{\partial L}{\partial y_C^B} - \frac{\partial \phi^\alpha}{\partial t} |_{\phi(t)} + C_{\alpha \beta}^\gamma \frac{\partial L}{\partial y_C^A} \frac{\partial L}{\partial y_C^B} \frac{\partial \phi}{\partial t} |_{\phi(t)} \right) \right), \]
where the last equation is a consequence of \( \Phi \) being a Lie morphism. The above equations can be written as follows
\[ \frac{\partial \phi^B}{\partial t} |_{t} = \delta^B_A , \quad \frac{\partial \phi^i}{\partial t} |_{t} = \phi^i_A(\rho_A^\alpha(\phi^j(t)) , \quad 0 = \frac{\partial \phi^\alpha}{\partial t} |_{t} - \frac{\partial \phi^\alpha}{\partial t} |_{t} + C_{\alpha \beta}^\gamma \phi^\beta(\phi^i(t)) \frac{\partial L}{\partial y_C^B} \frac{\partial \phi}{\partial t} |_{t} \right) \right), \quad (93) \]
which are called the Euler-Lagrange field equations written in terms of a Lie algebroid \( E \) in the \( k \)-cosymplectic setting.

The results of this section can be summarized in the following

**Theorem 6.1** Let \( L : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \) be a regular Lagrangian and let \( \xi : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \) be a section of \( \tau_{\mathbb{R}^k \times \mathbb{R}^k} \) such that
\[ \mathcal{Y}^C_B(\xi) = \delta^C_B , \quad \tau_\xi \Omega_L^A = \frac{1}{k} \left( d^A E_L + \sum_{C=1}^k \frac{\partial L}{\partial y_C^A} \mathcal{Y}^C_A \right). \quad (94) \]
1. Then $\xi$ is a SOPDE.

2. Let $\Phi = (\overline{T}, \Phi)$ be a morphism on Lie algebroids between $\tau_{\mathbb{R}^k} : T\mathbb{R}^k \to \mathbb{R}^k$, and $\tau : E \to Q$, and let $\overline{\Phi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus E$ be the associated map. If $\overline{\Phi} : \mathbb{R}^k \to \mathbb{R}^k \times \oplus E$ is an integral section of the SOPDE $\xi$, then it is a solution of the the Euler-Lagrange field equations (93) written in terms of a Lie algebroid $E$.

6.1.3 Relation with the standard Lagrangian $k$-cosymplectic formalism

As a final remark in this Subsection, it is interesting to point out that the standard Lagrangian $k$-cosymplectic formalism is a particular case of the Lagrangian formalism on Lie algebroids, when $E = TQ$, the anchor map $\rho$ is the identity on $TQ$, and the structure constants $C^i_{\alpha\beta} = 0$.

In this case we have:

- The manifold $\mathbb{R}^k \times \oplus E$ is identified with $\mathbb{R}^k \times T^1_kQ$ and $\tau_{T^1_kQ}(T^1_kQ)$ with $T^1_k(\mathbb{R}^k \times T^1_kQ)$.
- The energy function $E_L : \mathbb{R}^k \times T^1_kQ \to \mathbb{R}$ is given by $E_L = \sum_{A=1}^{\Delta^1} \Delta_A(L) - L$, where the vector fields $\Delta_A$ on $\mathbb{R}^k \times T^1_kQ$ have been explained in Remark 6.1.

- A section $\xi : \mathbb{R}^k \times \oplus E \to T^E_{\mathbb{R}^k}(\oplus E)$ corresponds to a $k$-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $\mathbb{R}^k \times T^1_kQ$, that is, $\xi$ is a section of $\tau_{T^1_kQ} : T^1_k(\mathbb{R}^k \times T^1_kQ) \to \mathbb{R}^k \times T^1_kQ$.

- Let $f$ be a function on $\mathbb{R}^k \times T^1_kQ$, then
  
  $$d^A f(Y_1, \ldots, Y_k) = df(Y_A),$$

  where $df$ denotes the standard differential and $(Y_1, \ldots, Y_k)$ is a $k$-vector field on $\mathbb{R}^k \times T^1_kQ$.

- We have that

  $$\mathcal{T}_B^A((X_1, \ldots, X_k)) = dt^B(X_C),$$
  $$\Omega^A_L((X_1, \ldots, X_k), (Y_1, \ldots, Y_k)) = \omega^A_L(X_A, Y_A),$$

  where $\omega^A_L$ are the Poincaré-Cartan 2-form of the standard $k$-cosymplectic formalism defined in section 2.1.2.

- Thus, in the standard $k$-cosymplectic formalism, the equation (94) can be written as follows:

  $$dt^A(\xi_B) = \delta^A_B, \quad i_{\xi_A} \omega^A_L = \frac{1}{k} dE_L + \sum_{C=1}^{k} \frac{\partial L}{\partial t^C} dt^A \delta^A_C,$$

  which implies

  $$dt^A(\xi_B) = \delta^A_B, \quad \sum_{A=1}^{k} i_{\xi_A} \omega^A_L = dE_L + \sum_{A=1}^{k} \frac{\partial L}{\partial t^A} dt^A.$$  

- In the standard case, a map $\phi : \mathbb{R}^k \to Q$ induces a morphism on Lie algebroids $(T\phi, \phi)$ between $T\mathbb{R}^k$ and $TQ$. In this case, the associated map $\overline{\Phi}$ of this morphism is defined as the first prolongation $\phi^{[1]}$ of $\phi$ given by

  $$\overline{\Phi}(t) = (t, T\phi(\frac{\partial}{\partial t^1}|_t), \ldots, T\phi(\frac{\partial}{\partial t^k}|_t)).$$

Let us observe that $\overline{\Phi} = \phi^{[1]}$ (see 2.6).
Thus, from the Theorem 6.1 and the above remarks, we deduce the following corollary, which is a summary of the Lagrangian \( k \)-cosymplectic formalism.

**Corollary 6.1** Let \( L : \mathbb{R}^k \rightarrow \mathbb{R}^k \times T_1^kQ \) be a regular Lagrangian and \( \xi = (\xi_1, \ldots, \xi_k) \) a \( k \)-vector field on \( \mathbb{R}^k \times T_1^kQ \) such that

\[
\begin{align*}
    dt^A(\xi_B) &= \delta^A_B, \\
    \sum_{A=1}^k i_{\xi_A} \omega^A_L &= dE_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} dt^A.
\end{align*}
\]

1. Then \( \xi \) is a SOPDE.

2. If \( \tilde{\Phi} \) is an integral section of the \( k \)-vector field \( \xi \), then it is a solution to the Euler-Lagrange field equations (16) in the standard Lagrangian \( k \)-cosymplectic field theories. Let us observe that \( \tilde{\Phi} = \phi^{[1]} \).

Finally, we introduce a comparative table between the standard \( k \)-cosymplectic Lagrangian formalism and the case on Lie algebroids:

| Phase space                  | \( \mathbb{R}^k \times T_1^kQ \) | \( \mathbb{R}^k \times \oplus^k E \) |
|-----------------------------|-----------------------------------|--------------------------------------|
| Canonical forms             | \( \omega^A \in \Lambda^2(\mathbb{R}^k \times T_1^kQ) \) | \( \Omega^A \in \text{Sec}(T^E_k(\oplus^k E)) \wedge \text{Sec}(T^E_k(\oplus^k E))^* \) |
| Lagrangians                 | \( L : \mathbb{R}^k \times T_1^kQ \rightarrow \mathbb{R} \) | \( L : \mathbb{R}^k \times \oplus^k E \rightarrow \mathbb{R} \) |
| Geometric equations         | \[
\begin{align*}
    dt^A((X_L)_B) &= \delta^A_B, \\
    \sum_{A=1}^k i_{(X_L)_A} \omega^A &= dE_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} dt^A, \\
    ((X_L)_1, \ldots, (X_L)_k)
\end{align*}
\] | \[
\begin{align*}
    \mathcal{Y}^C_{\xi}(\xi) &= \delta^C_B, \\
    i_{\xi} \Omega^A &= \frac{1}{k} \left( d^A L + \sum_{C=1}^k \frac{\partial L}{\partial y^C} \mathcal{Y}^C_A \right), \\
    \xi &\in \text{Sec}(T^E_k(\oplus^k E)) \\
    \mathcal{T}^E_k(\oplus^k E) &= (\mathbb{R}^k \times \oplus^k E) \times_{\mathbb{R}^k \times T_1^kQ} T_1^k(\mathbb{R}^k \times \oplus^k E)
\end{align*}
\] |

### 6.2 Hamiltonian formalism

In this subsection we develop on Lie algebroids the equivalent to section 2.1 in the standard \( k \)-cosymplectic formalism.

We begin this section by introducing the manifold \( \oplus^k E^* \), which plays the role of \( T^*_k Q \) in the classical \( k \)-cosymplectic Hamiltonian setting.

Let \( (E, [\cdot, \cdot], \rho) \) be a Lie algebroid over a manifold \( Q \). For the Hamiltonian approach we consider the dual bundle, \( \tau^* : E^* \rightarrow Q \) of \( E \).

**6.2.1 Geometric elements**

The manifold \( \oplus^k E^* \)
The standard \( k \)-cosymplectic Hamiltonian formalism is developed on the manifold \( \mathbb{R}^k \times (T_k^1)^*Q \). Considering a Lie algebroid \( E \) as a substitute for the tangent bundle, it is natural to consider that the analog of \( (T_k^1)^*Q \) is the Whitney sum over \( Q \) of \( k \) copies of the dual space \( E^* \).

We denote by \( \bigoplus^k E^* = E^* \oplus \ldots \oplus E^* \) the Whitney sum of \( k \) copies of the vector bundle \( E^* \), and the projection map \( \tilde{\tau}^* : \bigoplus^k E^* \to Q \), which is \( \tilde{\tau}^*(a_{1q}^*, \ldots, a_{kq}^*) = q \).

**Local basis of sections of** \( \tilde{\tau}^* : \bigoplus^k E^* \to Q \)

Let \( a_q^* = (a_{1q}^*, \ldots, a_{kq}^*) \) be an arbitrary point of \( \bigoplus^k E^* \), since \( a_{Aq}^* \subset E^* \), and \( \{e^\alpha\} \) is a local basis of sections of \( E^* \) \( \{e^\alpha\} \) is the dual basis of the basis of sections of \( E \), \( \{e^\alpha\} \), we have

\[
a_{Aq}^* = y_\alpha(a_{Aq}^*)e^\alpha(q),
\]

where \( \tilde{e}_\alpha^A(q) = (0, \ldots, e^\alpha(q), \ldots, 0) \), and where \( \tilde{A} \) indicates the \( A^{th} \) position of \( \tilde{e}_\alpha^A(q) \).

Thus, \( \{\tilde{e}_\alpha^A\} \) is a local basis of sections of \( \bigoplus^k E^* \), and if \( (q^i, y^A) \) are local coordinates on \( (\tilde{\tau}^*)^{-1}(U) \subseteq E^* \), the induced local coordinates \( (q^i, y^A) \) on \( (\tilde{\tau}^*)^{-1}(U) \subseteq \bigoplus^k E^* \) are given by

\[
q^i(a_{1q}^*, \ldots, a_{kq}^*) = q^i(q), \quad y^A(a_{1q}^*, \ldots, a_{kq}^*) = y_\alpha(a_{Aq}^*).
\]

**The vector bundle** \( \mathcal{T}_k^E(\bigoplus^k E^*) \)

We now consider the \( k \)-prolongation \( \mathcal{T}_k^E(\bigoplus^k E^*) \subset \bigoplus^k E \times T^1_k(\mathbb{R}^k \times \bigoplus^k E^*) \) of a Lie algebroid \( E \) over the fibration \( \tilde{\tau}^* : \bigoplus^k E^* \to Q \) (let us observe that in this case the fiber \( \pi : P \to Q \) is \( \tilde{\tau}^* : \bigoplus^k E^* \to Q \)).

The vector bundle \( \mathcal{T}_k^E(\bigoplus^k E^*) \) plays the role of \( T^1_k(\mathbb{R}^k \times (T_k^1)^*Q) \to \mathbb{R}^k \times (T_k^1)^*Q \), and its sections corresponds to the \( k \)-vector fields on \( \mathbb{R}^k \times (T_k^1)^*Q \).

Recalling section [5.3], for this particular case we obtain that if \( (q^i, y^A) \) are local coordinates on \( \bigoplus^k E^* \), we have the local coordinates \( (t^A, q^i, y^A, z^A, y^B, (v^A)_\beta) \) on \( \mathcal{T}_k^E(\bigoplus^k E^*) \) given by (see [63])

\[
t^A(a_q^*, W_{(t,b_q^*)}) = t^A(t), \quad q^i(a_q^*, W_{(t,b_q^*)}) = q^i(q), \quad (v^A)_\beta(a_q^*, W_{(t,b_q^*)}) = (v^A)_\beta(b_q^*),
\]

and the local basis \( \{X^A, Y^A, (Y^A)_\beta\} \) of sections of \( \tau \) \( \mathcal{T}_k^E(\bigoplus^k E^*) \to \mathbb{R}^k \times \bigoplus^k E^* \), defined in [5.5], is written here as follows

\[
X^A(t, b_q^*) = (\tilde{e}_\alpha^A(q), 0, \ldots, \rho^A(q) \frac{\partial}{\partial q^i}|_{(t,b_q^*)}, \ldots, 0))
\]

\[
Y^A(t, b_q^*) = (0, 0, \ldots, \frac{\partial}{\partial t^A}|_{(t,b_q^*)}, \ldots, 0)) \quad \text{(95)}
\]

\[
(Y^A)_\beta(t, b_q^*) = (0, 0, \ldots, \frac{\partial}{\partial y^B_\beta}|_{(t,b_q^*)}, \ldots, 0))
\]
6.2.2 Hamiltonian formalism

Let \((E, [\cdot, \cdot]_E, \rho)\) be a Lie algebroid on a manifold \(Q\) and \(H : \mathbb{R}^k \times \oplus E^* \to \mathbb{R}\) a Hamiltonian function.

The Liouville sections

We may introduce \(k\) sections of the vector bundle \(\mathcal{T}_k^E(\oplus E^*)^* \to \mathbb{R}^k \times \oplus E^*\) as follows.

\[
\Theta^A : \mathbb{R}^k \times \oplus E \longrightarrow (\mathcal{T}_k^E(\oplus E^*))^*
\]

\[
(t, b^*_q) \mapsto \Theta^A_{(t, b^*_q)} : (\mathcal{T}_k^E(\oplus E^*))_{(t, b^*_q)} \longrightarrow \mathbb{R}
\]

\[
(a_q, W(t, b^*_q)) \mapsto \Theta^A_{(t, b^*_q)}(a_q, W(t, b^*_q)) = b^*_q(a_A).
\]

In local coordinates we have

\[
\Theta^A = \sum_{\beta} y^A_{\beta} \iota^A_{\beta} ,
\]

where \(\{\iota^A_{\alpha}, \gamma^B_{\alpha}, (\gamma^A_{\beta})_B\}\) is the local basis of sections of \(\mathcal{T}_k^E(\oplus E^*)^*\), which is the dual basis of the local basis \(\{\iota^A_{\alpha}, \gamma^A_{\beta}, (\gamma^A_{\beta})_B\}\) of sections of \(\mathcal{T}_k^E(\oplus E)\).

Now for each \(A\) we define the 2-section \(\Omega^A : \mathbb{R}^k \times \oplus E^* \to (\mathcal{T}_k^E(\oplus E^*))^* \land (\mathcal{T}_k^E(\oplus E^*))^*\) as

\[
\Omega^A = -d^A \Theta^A ,
\]

where \(d^A\) denotes the derivation introduced in \([68]\) with \(P = \oplus E^*\).

By a straightforward computation from \([66], [67], [68], [95]\) and \([96]\), we obtain

\[
\Omega^A = \sum_{\beta} \iota^A_{\beta} \land (\gamma^A_{\beta}) + \frac{1}{2} \sum_{\beta, \gamma, \delta} C^A_{\beta \gamma} y^A_{\beta} \iota^A_{\gamma} \land \iota^A_{\delta} .
\]

Hamilton’s equations

Let \(H : \mathbb{R}^k \times \oplus E^* \to \mathbb{R}\) be a Hamiltonian function. For an arbitrary section \(\xi : \mathbb{R}^k \times \oplus E^* \rightarrow \mathcal{T}_k^E(\oplus E^*)\) of \(\tilde{T}_k^E : \mathcal{T}_k^E(\oplus E^*) \rightarrow \mathbb{R}^k \times \oplus E^*\), we consider the system of equations

\[
\mathcal{T}_A^C(\xi) = \delta^C_A , \quad \iota_\xi \Omega^A = \frac{1}{k} \left( d^A H - \sum_{B=1}^k \frac{\partial H}{\partial y^B} \gamma^B_{\alpha} \right) .
\]

Writing \(\xi = \xi^B_C \gamma^B_{\alpha} + \xi^\alpha_A \gamma^A_{\beta} + (\xi^B_C)^C(\gamma^B_{\alpha})_C\), from \([69], [95]\) and \([97]\) we obtain that \([98]\) is equivalent to the equations

\[
\xi^C_B = \delta^C_B , \quad \delta^A_B \xi^\alpha_A = \frac{1}{k} \frac{\partial H}{\partial y^B} , \quad (\xi^\alpha_A)_C - C^A_{\beta \gamma} y^A_{\beta} \xi^\alpha_A = \frac{1}{k} \rho^A_{\beta} \frac{\partial H}{\partial q^\beta} .
\]

From \([99]\) we obtain

\[
\xi^C_B = \delta^C_B , \quad \sum_{A=1}^k \delta^A_{\alpha} \xi^\alpha_A = \xi^\alpha_B = \frac{\partial H}{\partial y^B} , \quad \sum_{A=1}^k (\xi^\alpha_A)_C - \sum_{A=1}^k C^A_{\beta \gamma} y^A_{\beta} \xi^\alpha_A = -\rho^A_{\beta} \frac{\partial H}{\partial q^\beta} .
\]

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Substituting the second identities of (100) in the three equations, we have

\[ \xi_B^C = \delta_C^B, \quad \xi_B^i = \frac{\partial H}{\partial y_B^i}, \quad \sum_{A=1}^{k} (\xi_A)^{\beta}_{\alpha} = -\left( \rho^\beta_{\alpha} \frac{\partial H}{\partial q^i} + \sum_{A=1}^{k} C^\gamma_{\alpha\beta} y^A_{\gamma} \frac{\partial H}{\partial y^A_{\beta}} \right). \quad (101) \]

From (101) and (102) we obtain

\[ \xi_B^A = \frac{\partial \psi_B}{\partial H^A}, \quad \xi_B^i \rho^i_{\beta} = \frac{\partial \psi_i}{\partial H^A} \quad (\xi_A)^{\beta}_{\alpha} = \frac{\partial \psi_B}{\partial H^A}. \quad (102) \]

From (101) and (102) we obtain

\[ \frac{\partial \psi^C}{\partial H^B} = \delta^C_B, \quad \frac{\partial \psi^i}{\partial H^A} \bigg|_t = \rho^i_{\alpha} \frac{\partial H}{\partial y_B^i} \bigg|_{\psi(t)} + \sum_{A=1}^{k} \frac{\partial \psi^A}{\partial H^A} = -\left( \rho^\beta_{\alpha} \frac{\partial H}{\partial q^i} \bigg|_{\psi(t)} + \sum_{A=1}^{k} C^\gamma_{\alpha\beta} \psi^A \frac{\partial H}{\partial y_B^i} \bigg|_{\psi(t)} \right). \quad (103) \]

In our case, a solution to the Hamilton equations must be a morphism \( \psi = (\overline{\psi}, \psi) \) of Lie algebroids between \( \tau_{\mathbb{R}^k} : T\mathbb{R}^k \rightarrow \mathbb{R}^k \) and \( \tau^{E}_{\mathbb{R}^k \times \oplus E^*} : T^{E}(\mathbb{R}^k \times \oplus E^*) \subset E \times T(\mathbb{R}^k \times \oplus E^*) \rightarrow \mathbb{R}^k \times \oplus E^* \), where \( T^{E}(\mathbb{R}^k \times \oplus E^*) \) is the prolongation of the Lie algebroid \( E \) over the fibration \( \mathbb{R}^k \times \oplus E^* \rightarrow Q \).

\[
\begin{array}{c}
T\mathbb{R}^k \xrightarrow{T}\mathbb{R} \times (\mathbb{T}_k)^*Q \\
\tau_{\mathbb{R}^k} \downarrow \quad \downarrow \tau^{E}_{\mathbb{R}^k \times \oplus \mathbb{E}^*} \quad \tau^{E}_{\mathbb{R}^k \times \oplus \mathbb{E}^*} \quad \mathbb{R}^k \times (\mathbb{T}_k)^*Q \\
\mathbb{R} \xrightarrow{\psi} \mathbb{R}^k \times (\mathbb{T}_k)^*Q
\end{array}
\]

Let \( \{e_{\alpha}\} \) and \( \{X_{\alpha}, V_{A}, V^A_{\alpha}\} \) be a local basis of \( \text{Sec}(\tau_{\mathbb{R}^k}) \) and \( \text{Sec}(\tau^{E}_{\mathbb{R}^k \times \oplus \mathbb{E}^*}) \), respectively, and \( \{e^\alpha\} \) and \( \{X^\alpha, V^A, V^A_{\alpha}\} \) their dual basis. (Here \( X_{\alpha}(t, b^q_A) = (e_{\alpha}(\mathbf{q}), \rho^\alpha_{\beta}(\mathbf{q}) \frac{\partial}{\partial q^i}|_{(t, b^q_A)}), \quad V_{A}(t, b^q_A) = (0, \frac{\partial}{\partial y_B^i}|_{(t, b^q_A)}). \) If \( \psi = (\overline{\psi}, \overline{\psi}) \) is locally given by the relations

\[ \overline{\psi}(t) = (t^A, \psi^i(t), \psi^A_{\alpha}(t)) \quad , \quad \psi = (\psi^A_{\alpha}X_{\alpha} + V_{A} + \psi^B_{\beta\alpha}V^B_{\beta}) \otimes e^A \]

then the morphism condition (59) can be written

\[ \rho^i_{\alpha}\psi^A_{\alpha} = \frac{\partial \psi^i}{\partial H^A}, \quad \psi^B_{\alpha} = \frac{\partial \psi^B}{\partial H^A}, \quad 0 = \frac{\partial \psi^A}{\partial H^B} \frac{\partial H}{\partial H^A} + C^\gamma_{\alpha\beta} \psi^B \psi^B_{\gamma}. \quad (104) \]

From (103) and (104) we obtain the following:
Let $H : \mathbb{R}^k \times \bigoplus E^* \to \mathbb{R}$ be a Hamiltonian and $\xi : \mathbb{R}^k \times \bigoplus E \to T_{\mathbb{R}^k}^E(k \bigoplus E^*)$ be a section of $\tilde{\tau}_{k \bigoplus E^*}$ such that

$$\mathcal{Y}^E_A(\xi) = \delta_A^C, \quad \iota_\xi \Omega^A = \frac{1}{k} (d^A H - \sum_{B=1}^k \frac{\partial H}{\partial y^B} \mathcal{Y}^B_A).$$

Let $\psi = (\mathbf{\tilde{v}}, \mathbf{\tilde{w}})$ be a morphism of Lie algebroids between $\tau_{\mathbb{R}^k} : T\mathbb{R}^k \to \mathbb{R}^k$ and $\tau^E_{\mathbb{R}^k \bigoplus E^*} : T^E_{\mathbb{R}^k \bigoplus E^*}$ $E^* \subset E \times T(\mathbb{R}^k \times \bigoplus E^*) \to \mathbb{R}^k \times \bigoplus E^*$. If $\psi : \mathbb{R}^k \to \mathbb{R}^k \times \bigoplus E^*$ is an integral section of $\xi$, then $\psi$ is a solution of the system of partial differential equations

$$\rho^{i_j}_{A_k} = \frac{\partial \psi^i}{\partial t_A} |_{\psi(t)}, \quad \rho^{i_j}_{A_k} = \frac{\partial \psi^i}{\partial H |_{\psi(t)}} \sum_{A=1}^k \frac{\partial \psi^A}{\partial y^B |_{\psi(t)}} = -\left( \rho^{i_j}_{A_k} \frac{\partial H}{\partial y^B |_{\psi(t)}} + \sum_{A=1}^k C^i_{\alpha \beta} \psi^A(t) \frac{\partial H}{\partial y^B |_{\psi(t)}} \right).$$

which are called the Hamilton field equations on Lie algebroids.

6.2.3 Relation with the standard Hamiltonian $k$-cosymplectic formalism

As a final remark, it is interesting to point out that the standard Hamiltonian $k$-symplectic formalism is a particular case of the Hamiltonian formalism on Lie algebroids.

In this case we have:

- The manifold $\mathbb{R}^k \times \bigoplus E^*$ is identified with $\mathbb{R}^k \times (T_{\mathbb{R}^k}^k)^* Q$ and $T_{\mathbb{R}^k}^TQ((T_{\mathbb{R}^k}^k)^* Q)$ with $T_{\mathbb{R}^k}^TQ((T_{\mathbb{R}^k}^k)^* Q)$.

- A section $\xi : \mathbb{R}^k \times \bigoplus E^* \to T_{\mathbb{R}^k}^E(k \bigoplus E^*)$ corresponds to a $k$-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on $\mathbb{R}^k \times (T_{\mathbb{R}^k}^k)^* Q$, that is, $\xi$ is a section of $\tau^E_{\mathbb{R}^k \bigoplus E^*} : T_{\mathbb{R}^k}^E(k \bigoplus E^*) \to \mathbb{R}^k \times (T_{\mathbb{R}^k}^k)^* Q$.

- Let $f$ be a function on $\mathbb{R}^k \times (T_{\mathbb{R}^k}^k)^* Q$, then

$$d^A f(Y_1, \ldots, Y_k) = df(Y_A),$$

where $df$ denotes the standard differential and $(Y_1, \ldots, Y_k)$ is a $k$-vector field on $\mathbb{R}^k \times (T_{\mathbb{R}^k}^k)^* Q$.

- We have that

$$\mathcal{Y}^E_B((X_1, \ldots, X_k)) = dt^C(X_C) \quad \Omega^A((X_1, \ldots, X_k), (Y_1, \ldots, Y_k)) = \omega^A(X_A, Y_A).$$

Thus, in the standard $k$-cosymplectic formalism the equation (98) can be written as follows

$$dt^A(\xi_B) = \delta^A_B, \quad i_{\xi_A} \omega^A = \frac{1}{k} dH - \sum_{C=1}^k \frac{\partial H}{\partial t_C} dt^A \delta^A_C,$$

which implies

$$dt^A(\xi_B) = \delta^A_B, \quad \sum_{A=1}^k i_{\xi_A} \omega^A = dH - \sum_{A=1}^k \frac{\partial H}{\partial t_A} dt^A.$$
Thus, from the Theorem 6.2 and the above remarks, we deduce the following corollary, which summarizes the Hamiltonian $k$-cosymplectic formalism.

**Corollary 6.2** Let $H: \mathbb{R}^k \times \bigoplus E^* \to \mathbb{R}$ be a Hamiltonian function and $\xi = (\xi_1, \ldots, \xi_k)$ a $k$-vector field on $\mathbb{R}^k \times (T^1_k)^*Q$ such that

$$dt^A(\xi_B) = \delta^A_B, \quad \sum_{A=1}^k i_{\xi_A} \omega_A = dH - \sum_{A=1}^k \frac{\partial H}{\partial t^A} dt^A.$$

Thus, if $\psi: \mathbb{R}^k \to \mathbb{R}^k \times (T^1_k)^*Q$ is an integral section of the $k$-vector field $\xi$, then it is a solution of the Hamilton equations (4).

In the following table we compare the $k$-cosymplectic Hamiltonian formalism in the standard case and on Lie algebroids:

| $k$-cosymplectic | Lie Algebroids |
|------------------|----------------|
| Phase space      | $\mathbb{R}^k \times \bigoplus E^*$ |
| Canonical forms  | $\Omega^A \in Sec((T^E_k(\bigoplus E^*))^*) \cap Sec((T^E_k(\bigoplus E^*))^*)$ |
| Hamiltonians     | $H: \mathbb{R}^k \times \bigoplus E^* \to \mathbb{R}$ |
| Geometric equations | $\sum_{A=1}^k i_{(X_H)_A} \omega_A = dH - \sum_{A=1}^k \frac{\partial H}{\partial t^A} dt^A$
|                  | $((X_H)_1, \ldots, (X_H)_k)$ |

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