Residual \(\mathbb{Z}_2 \times \mathbb{Z}_2\) symmetries and lepton mixing

L. Lavoura, P.O. Ludl

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ABSTRACT
We consider two novel scenarios of residual symmetries of the lepton mass matrices. Firstly we assume a \(\mathbb{Z}_2 \times \mathbb{Z}_2\) symmetry \(G_L\) for the charged-lepton mass matrix and a \(\mathbb{Z}_2\) symmetry \(G_V\) for the light neutrino mass matrix. With this setting, the moduli of the elements of one column of the lepton mixing matrix are fixed up to a reordering. One may interchange the roles of \(G_L\) and \(G_V\) in this scenario, thereby constraining a row, instead of a column, of the mixing matrix. Secondly we assume a residual symmetry group \(G_L \equiv \mathbb{Z}_m\) \((m > 2)\) which is generated by a matrix with a doubly-degenerate eigenvalue. Then, with \(G_V \equiv \mathbb{Z}_2 \times \mathbb{Z}_2\) the moduli of the elements of a row of the lepton mixing matrix get fixed. Using the library of small groups we have performed a search for groups which may embed \(G_L\) and \(G_V\) in each of these two scenarios. We have found only two phenomenologically viable possibilities, one of them constraining a column and the other one a row of the mixing matrix.

1. Introduction

A group-theoretical philosophy for explaining the phenomenological values of the lepton mixing parameters has emerged during the last few years [1–16]. In that philosophy, those values follow from the distinct Abelian symmetry groups—\(G_L\) and \(G_V\)—under which the lepton mass matrices—\(M_L\) and \(M_V\), respectively—are invariant. Those matrices are defined by the mass terms

\[
L_{\text{mass}} = -\bar{\ell}_L M_L \ell_R + \frac{1}{2} \nu_L^T C^{-1} \nu_L + \text{H.c.},
\]

where \(\ell_L, R\) are the left- and right-handed charged-lepton fields, \(\nu_L\) are the light neutrino fields, and \(C\) is the charge-conjugation matrix in Dirac space. (We assume the neutrinos to be Majorana particles.) Let \(H_L \equiv M_L M_L^T\); if the mass matrices are diagonalized as \(U_L^T H_L U_L = D_L \equiv \text{diag}(m_{12}^2, m_{23}^2, m_{33}^2)\) and \(U_V^T M_V U_V = D_V \equiv \text{diag}(m_1, m_2, m_3)\), then the lepton mixing matrix is given by \(U_{\text{PMNS}} = U_L^T U_V\). (\(m_{1,2,3}\) denote the three neutrino masses.) Let the symmetry group \(G_L\) be generated by a matrix \(L\) such that \(L^{-1} H_L L = H_L\). If we choose a basis in which \(L\) is diagonal and if we assume that the diagonal matrix elements of \(L\) are all distinct, then this invariance forces \(H_L\) to be diagonal. Thus, in that basis \(U_L = I_3\) (up to a permutation of the charged leptons) and \(U_{\text{PMNS}} = U_V\). \(U_{\text{PMNS}}\) is the 3 × 3 unit matrix. In the same basis, let a generator \(N\) of \(G_V\) be a unitary 3 × 3 matrix of order two and with two different eigenvalues, i.e. \(N^2 = 1\) but \(N \neq \pm 1\). Such a matrix can always be written as

\[
N = \gamma (13 - 2u u^T),
\]

where \(\gamma = \pm 1\) and \(u = (u_1, u_2, u_3)^T\) is a normalized column vector, viz. \(u^T u = |u_1|^2 + |u_2|^2 + |u_3|^2 = 1\). Invariance of \(M_V\) under \(N\) means that \(N^T M_V N = M_V\). Then, it follows from \(N u = -\gamma u\) that \(N^3 (M_V u) = N (M_V u) = (N^T M_V N)(N u) = -\gamma (M_V u)\). But, the eigenvalue \(-\gamma\) of \(N^3\) is non-degenerate; therefore, \(M_V u \propto u^\gamma\). Since \(M_V U_V = U_V^* D_V\) and the neutrino masses are non-degenerate, \(u\) must be one of the columns of \(U_V \equiv U_{\text{PMNS}}\). It thence follows that \(|u_{1,2,3}|^2\) are, up to a reordering of the charged leptons, the moduli of the matrix elements of a column (one may still choose which column) of \(U_{\text{PMNS}}\).

The above-mentioned philosophy assumes that there is a finite discrete group \(G\) which has both \(G_L\) and \(G_V\) as subgroups. In our search in Section 2.1, \(G_L\) is generated by two matrices \(L_1\) and \(L_2\) instead of just one.

1 The possibilities for the experimental investigation of the implications of residual symmetries are discussed in Refs. [17–20]. Furthermore, residual symmetries have also been considered in the quark sector [21,22].
to find a suitable $G$ such that the ensuing $|u_{1,2,3}|^2$ agree with the phenomenological values of the moduli of the matrix elements of one of the columns of $U_{PMNS}$. This has been done in Ref. [2] under the assumption that $G$ is a subgroup of SU(3) of order smaller than 512. In Ref. [11] a more complete search has been undertaken, wherein $G$ was assumed to be a subgroup of U(3) of order less than 1536. Both Refs. [2] and [11] assume $G$ to possess a faithful three-dimensional irreducible representation. In Ref. [11] it was moreover assumed that $G$ fully determines $U_{PMNS}$, because its subgroup $G_r$, is generated by two commuting matrices $N$ and $N'$, both of the form in Eq. (2) but with two mutually orthogonal vectors $u$ and $u'$, respectively. (Thus, $G_r \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $G_r \cong \mathbb{Z}_2^2$.) A variant of this philosophy has been employed in Refs. [7,16,21], where the neutrino mass terms have been assumed to be of the Dirac type and, correspondingly, the matrix $\nu$ that commutes with the phenomenological values. Section 4 contains the eigenvalue, in such a way that a $|u_k|^2$ was computed without the need to diagonalize $L_1$ and $L_2$; indeed,

$$|u_k|^2 = \frac{1}{4} \left[ 1 + \frac{\text{tr}(L_k N)}{\text{tr}(L_k)} \right]$$ (4)

Eq. (4) is easily verified in the basis where Eqs. (2) and (3) hold; since it is written in terms of traces, it holds in any other basis—even in one where $D(G)$ is not formed by unitary matrices. One may thus compute the moduli of the matrix elements of one column of $U_{PMNS}$ just from the knowledge of $L_1$, $L_2$, and $N$ in an arbitrary basis.\footnote{The computation of mixing-matrix elements from invariant traces was pioneered in Ref. [23].}

The computer algebra system GAP [24] has access to SmallGroups [25], a library of all the groups (up to isomorphisms) of order smaller than 2000—excluding the 49,487,365,422 groups of order 1024. Since there are 408,641,062 groups of order 1536 = 12 × 128, we have restricted our search to the 1,336,749 groups of order 12n for $n \leq 127$. We have furthermore excluded groups $G$ which are direct products of the form

$$G \cong \mathbb{Z}_m \times G' \quad (m \geq 2),$$ (5)

because such groups do not provide any restrictions beyond those already following from the smaller group $G'$.\footnote{The group $D_{14}$ is of particular interest, especially for quark mixing, because it nicely fits Cabibbo mixing [27], as can be seen in the second line before the last of Table 1.}

Going through these groups, by constructing their character tables, we have sieved out the groups which have a faithful three-dimensional irreducible representation. We have used the GAP package SONATA [26] to find all the subgroups of the groups under investigation. For those groups which have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup and a $\mathbb{Z}_2$ subgroup with trivial intersection, we have explicitly constructed all the non-equivalent three-dimensional irreducible representations $D$ and we have computed all the candidates for pairs ($D(G_1)$, $D(G_2)$). When neither $D(G_1)$ nor $D(G_2)$ contained $-I_3$, we have computed the corresponding $|u_k|^2$ through Eq. (4). The results can be found in Table 1.

In Table 1 (and in the second column of Table 2) one observes that, whenever $G_1$ and $G_2$ together generate a group $D_n$ with even $n$, this leads to $|u_{1}|^2$, $|u_{2}|^2$, $|u_{3}|^2) = (0, \sin^2 \frac{2\pi}{n}, \cos^2 \frac{2\pi}{n})$ with $m = 2n$ and, possibly, smaller (integer) values of $m$.\footnote{The group $D_{14}$ is of particular interest, especially for quark mixing, because it nicely fits Cabibbo mixing [27], as can be seen in the second line before the last of Table 1.} The group $D_n$ may be defined as consisting of the matrices

$$X(p) = \begin{pmatrix} -\cos(p a_{\theta}) & -\sin(p a_{\theta}) \\ -\sin(p a_{\theta}) & \cos(p a_{\theta}) \end{pmatrix}$$ and

$$Y(p) = \begin{pmatrix} \cos(p a_{\theta}) & -\sin(p a_{\theta}) \\ \sin(p a_{\theta}) & \cos(p a_{\theta}) \end{pmatrix},$$ (6)

where $a_{\theta} = 2\pi/n$ and $p = 0, 1, 2, \ldots, n - 1$. For even $n$, this group has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup formed by $I_2$, $Y(n/2), X(n/2), \text{and } X(0)$. The group $D_n$ is a subgroup of SO(3) through its reducible triplet representation.
The eigenvalues of $H$ matrix are non-degenerate. Therefore, the row are of course precisely the same as those obtained in Sec-
ware o w where the matrices diagonalizing $L$ matrix are $\ell = 1, 2, 3$. In this representation of $M$, the eigenvalue $-\gamma$ of $LH$ \(\ell\) is $-\gamma$ of $L$. Since we are in the basis where $E_L$ diagonal $L$ is, however, restricted by the condition $L^2 = 1$, since it generates a group $Z_2$. We now lift this restriction and suppose instead that $L$ generates a group $Z_n$ with $n > 2$, i.e. $L^n = 1$. We thus assume that in the neutrino sector there is a residual symmetry $G_\nu \cong Z_2 \times \mathbb{Z}_2$, generated by a matrix $L$ with a degenerate eigenvalue $\sigma$ and another eigenvalue $\rho \neq \sigma$ (of course $\sigma^n = \rho^n = 1$). Let, in the basis where $E = (10)$ hold, $v = (v_1, v_2, v_3)^T$ denote the normalized eigenvector of $L$ corresponding to the eigenvalue $\rho$. One may then write

$$L = \sigma \mathbb{1}_3 + (\rho - \sigma) vv^T.$$  

In this basis, $M_\nu$ is diagonal and therefore $U_{\text{PMNS}} = U_{\text{L}}^\dagger$ up to a permutation of rows. In the charged-lepton sector the residual symmetry is $Z_2$, generated by a matrix $L$ with a degenerate eigenvalue $\sigma$ and another eigenvalue $\rho \neq \sigma$ (of course $\sigma^n = \rho^n = 1$). Let, in the basis where $E = (10)$ hold, $v = (v_1, v_2, v_3)^T$ denote the normalized eigenvector of $L$ corresponding to the eigenvalue $\rho$. One may then write

$$L = \sigma \mathbb{1}_3 + (\rho - \sigma) vv^T.$$  

The $|v|_k$ are, up to a reordering, the moduli of the matrix elements of one row of $U_{\text{PMNS}}$. They may be computed in a basis-independent way through

$$|v|_k^2 = \frac{1}{2(\rho - \sigma)} \left[ \frac{\text{tr}(N_k L)}{\text{tr}(N_k)} \right].$$  

Thus, we have searched for groups $G$ which fulfill the following conditions:

1. $G$ is finite.
2. $G$ has a faithful three-dimensional irreducible representation $D(G)$.
3. $G$ has two subgroups, $G_e \cong Z_n$ ($n > 2$) and $G_\nu \cong Z_2 \times \mathbb{Z}_2$, which have a trivial intersection, i.e. $G_e \cap G_\nu = \{e\}$.
4. $D(G_\nu)$ does not contain $-\mathbb{1}_3$.

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4. $D(G_\nu)$ does not contain $-\mathbb{1}_3$.
5. $D(G_2)$ is generated by a matrix $I$, which has a twice degenerate eigenvalue $\sigma$ and another eigenvalue $\rho$ which differs from $\sigma$.

6. The group $\langle G_c, G_v \rangle$ generated by $D(G_c)$ and $D(G_v)$ is non-Abelian.

Once again, we have excluded groups of the form $G \cong \mathbb{Z}_m \times G'$ with $m \geq 2$. For each group of order smaller than $9000$ fulfilling the above requirements, we have computed the corresponding $|\psi|^2$ by means of Eq. (12). The results can be found in Table 2.

### Table 2

| $G$ | $\langle |\psi|^2 |, |\psi|^2 | \rangle$ | $G_c$ | $\langle G_c, G_v \rangle$ |
|-----|----------------------------------|-------|-------------------------|
| $\{ 48, 30 \}$ | $\{ \{ 192, 182 \}, \{ 432, 260 \} \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_4, \mathbb{Z}_6$ |
| $\{ 216, 95 \}$ | $\{ \{ 648, 259 \}, \{ 648, 260 \} \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 266 \}$ | $\{ \{ 648, 563 \}, \{ 864, 701 \} \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 96, 64 \}$ | $\{ \{ 384, 568 \}, \{ 864, 701 \} \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 96, 65 \}$ | $\{ \{ 384, 571 \}, \{ 864, 703 \} \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 266 \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 563 \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 864, 701 \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 864, 703 \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 600, 179 \}$ | $\langle 1/2, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 259 \}$ | $\{ \{ 648, 260 \} \}$ | $\langle 1/4, 1/4, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 266 \}$ | $\{ \{ 648, 563 \}, \{ 864, 701 \} \}$ | $\langle 1/4, 1/4, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 648, 563 \}$ | $\langle 1/4, 1/4, 1/2 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 864, 701 \}$ | $\langle 1/4, 3/4 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |
| $\{ 864, 703 \}$ | $\langle 1/4, 3/4 \rangle$ | $\mathbb{Z}_6, \mathbb{Z}_6$ |

### Table 3

| $G$ | $\langle G_c, G_v \rangle$ |
|-----|-------------------------|
| $\{ 24, 12 \}$ | $\mathbb{Z}_4 \times \mathbb{D}(6 \times 2)^2$ |
| $\{ 48, 30 \}$ | $\mathbb{A}_4 \times \mathbb{Z}_4$ |
| $\{ 60, 5 \}$ | $\mathbb{A}_4$ |
| $\{ 96, 64 \}$ | $\mathbb{A}(6 \times 4)^2$ |
| $\{ 96, 65 \}$ | $\mathbb{A}_4 \times \mathbb{Z}_4$ |
| $\{ 168, 42 \}$ | $\mathbb{S}(168) \cong \mathbb{PSL}(2, 7)$ |
| $\{ 192, 182 \}$ | $\mathbb{A}(\mathbb{Z}_4 \times \mathbb{Z}_4) \times \mathbb{Z}_4$ |
| $\{ 192, 186 \}$ | $\mathbb{A}_4 \times \mathbb{Z}_4$ |
| $\{ 216, 95 \}$ | $\mathbb{A}(6 \times 6)^2$ |
| $\{ 384, 568 \}$ | $\mathbb{A}(6 \times 8)^2$ |
| $\{ 384, 571 \}$ | $\mathbb{A}(\mathbb{Z}_4 \times \mathbb{Z}_4) \times \mathbb{Z}_8$ |
| $\{ 384, 581 \}$ | $\mathbb{A}_4 \times \mathbb{Z}_4$ |
| $\{ 432, 260 \}$ | $\mathbb{A}(\mathbb{Z}_4 \times \mathbb{Z}_4) \times \mathbb{Z}_4$ |
| $\{ 500, 179 \}$ | $\mathbb{A}(6 \times 10^3)$ |
| $\{ 648, 259 \}$ | $\mathbb{D}(6) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ |
| $\{ 648, 260 \}$ | $\mathbb{Z}_4$ |
| $\{ 648, 266 \}$ | $\mathbb{Z}_4$ |
| $\{ 648, 563 \}$ | $\mathbb{Z}_4$ |
| $\{ 864, 703 \}$ | $\mathbb{Z}_4$ |
| $\{ 864, 701 \}$ | $\mathbb{Z}_4$ |
| $\{ 1176, 243 \}$ | $\mathbb{A}(6 \times 14^2)$ |

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| $\{ 648, 266 \}$ | $\mathbb{Z}_4$ |
| $\{ 648, 563 \}$ | $\mathbb{Z}_4$ |
| $\{ 864, 703 \}$ | $\mathbb{Z}_4$ |
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6. We have stopped this search at a lower group order because the construction of the irreducible representations becomes, for large groups, extremely expensive in terms of computer time.

7. One might consider the possibility where our predictions only hold as a first approximation and are corrected by other effects—for instance, suppressed terms in the Lagrangian and/or the renormalization-group evolution of the parameters of $U_{\text{PMNS}}$. We shall not entertain such possibilities here.
The sum of Eqs. (13b) and (13c) is equivalent to Eq. (13a). It makes a prediction for $\theta_{23}$ as a function of $\theta_{13}$:

$$s_{23}^2 = \frac{1 - 2s_{13}^2}{2 - 2s_{13}^2}. \quad (14)$$

With $0.0169 \leq s_{23}^2 \leq 0.0315$ at 3σ level [29], this yields $0.4837 \leq s_{23}^2 \leq 0.4914$. This means that the atmospheric mixing angle is maximal for all practical purposes.

The difference between Eqs. (13b) and (13c) yields a prediction for $\cos \delta$:

$$4s_{12}c_{12}s_{23}c_{23}s_{13} \cos \delta = (s_{12}^2 - c_{12}^2)(s_{23}^2 - c_{23}^2) \quad (15)$$

Using Eq. (14), this gives

$$\cos \delta = -\frac{s_{12}^2 - c_{12}^2}{4s_{12}c_{12}} \frac{1 - 3s_{13}^2}{\sqrt{2s_{13}^2 - 2s_{13}^4}}. \quad (16)$$

Since $c_{12} > s_{12}$, $\cos \delta$ is predicted to be negative. Moreover, $|\cos \delta|$ is quite large; the bound $\cos^2 \delta \leq 1$ gives

$$\sin(2\theta_{12}) \geq -\frac{3s_{13}^2}{1 - s_{13}^2} \approx 1 - 2s_{13}^2 - 2s_{13}^4 - 2s_{13}^6 - \cdots. \quad (17)$$

This implies that $\theta_{12}$ and $\theta_{13}$ cannot be both within their 1σ intervals of Ref. [29] and can only marginally be both within their 2σ intervals, see Fig. 1. Anyway, the angle $\delta$ should be close to either 0 or $\pi$, i.e. CP violation in lepton mixing is predicted to be small.

4. Conclusions

In this work, using the software GAP and the SmallGroups Library, we have looked for finite groups $G$ which have a faithful three-dimensional irreducible representation $D(G)$ and have two subgroups, $Z_n$ and $Z_2 \times Z_2$, with a trivial intersection. Moreover, $D(Z_n)$ should have a twice degenerate eigenvalue and neither $D(Z_n)$ (for $n = 2$) nor $D(Z_n \times Z_2)$ should contain the matrix $-I_3$. When $n = 2$ we have taken the search up to order 1536 but for $n = 2$ we only reached group order 1000.

Applying the results of our search to the prediction of lepton mixing, we have noticed that almost all the groups that we have found lead to a zero mixing matrix element, which is phenomenologically disallowed. There are only two exceptions. In one of them, the groups [60, 5] $\cong A_5$ and [1080, 260] $\supset [60, 5]$ may lead to the first column of the lepton mixing matrix having elements with moduli squared (0.6545, 0.25, 0.0955); this is viable and had already been found in a previous paper [28]. In the other exception, many groups—see Tables 1 and 2—may lead to either the second or the third row of the lepton mixing matrix having elements with moduli $(1/2, 1/2, 1/\sqrt{2})$; the consequences of this prediction are a very close to maximal atmospheric mixing angle and $|\cos \delta|$ straddling 1.

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