Learning low-dimensional dynamical-system models from noisy frequency-response data with Loewner rational interpolation

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Loewner rational interpolation provides a versatile tool to learn low-dimensional dynamical-system models from frequency-response measurements. This work investigates the robustness of the Loewner approach to noise. The key finding is that if the measurements are polluted with Gaussian noise, then the error due to the noise grows at most linearly with the standard deviation with high probability under certain conditions. The analysis gives insights on how to make the Loewner approach robust against noise via linear transformations and judicious selections of measurements. Numerical results demonstrate the linear growth of the error on benchmark examples.

Keywords: data-driven modeling; nonintrusive model reduction; interpolatory model reduction; Loewner

1 Introduction

Learning dynamical-system models from measurements is a widely studied task in science, engineering, and machine learning; see, e.g., system identification originating from the systems & control community [1, 38, 28, 26, 50, 21, 37, 39, 14, 10, 11], sparsity-promoting methods [42, 43, 8, 41], dynamic mode decomposition [45, 44, 40, 49, 23, 9], and operator inference [32, 31, 36, 22, 33, 48, 29]. Antoulas and collaborators introduced the Loewner approach [2, 24, 27, 5] that constructs models directly from frequency-response measurements, without requiring computationally expensive training phases and without solving potentially non-convex optimization problems. This work investigates the robustness of the Loewner approach to noise in the frequency-response measurements. The main finding of this work is that under certain conditions and with high probability the error introduced by noise into Loewner models grows at most linearly with the standard deviation of the noise.

The Loewner approach [2, 24, 27, 5] derives dynamical-system models from frequency-response data, i.e., from values of the transfer function of the high-dimensional dynamical system of interest. A series of works have extended the Loewner approach from linear time-invariant systems to bilinear systems [4], quadratic-bilinear systems [12], parametrized systems [17], time-delay systems [46], and structured systems [47]. Learning Loewner models from time-domain data, instead of frequency-response measurements, is discussed in [19, 30]. Learning Loewner models from noisy data has received relatively little attention. The work [25] provides a numerical investigation of the effect of noise on the accuracy of Loewner models. The work [16] discusses the rank of Hankel matrices if measurements are polluted by additive Gaussian noise.

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noise. In the thesis [18, Section 2.1], experiments demonstrate that the selection of frequencies at which to obtain measurements and how to partition the measurements has a significant influence on the robustness against perturbations in the data such as noise. In [20], the Loewner approach is applied to control tasks where only noisy measurements are available. In [6], the robustness of interpolatory model reduction against perturbations in evaluations of the transfer function of the high-dimensional system is discussed; however, the perturbations are considered deterministic and stem from, e.g., numerical approximations via iterative methods.

In this work, the robustness of the Loewner approach to Gaussian noise is considered. The contribution is an analysis that bounds the error introduced by noise. In particular, the analysis shows that the error grows at most linearly in the standard deviation of the noise under certain conditions and with high probability. The conditions under which our analysis holds gives insights on how to select the frequencies at which to measure and how to partition the data to reduce the effect of noise. The linear growth of the error with respect to the standard deviation of the noise is observed in numerical examples.

2 Learning low-dimensional dynamical-system models with Loewner rational interpolation

This section recapitulates model reduction with the Loewner approach; see, e.g., [3, 7, 34] for general introductions to model reduction and related concepts.

2.1 Linear time-invariant dynamical systems

Consider the linear time-invariant system

\[
\dot{E} \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)
\]

of order \(N \in \mathbb{N}\) with system matrices \(E, A \in \mathbb{R}^{N \times N}, B \in \mathbb{R}^{N \times 1}\), and \(C \in \mathbb{R}^{1 \times N}\). The state at time \(t\) is \(x(t) \in \mathbb{R}^N\) and the output at time \(t\) is \(y(t) \in \mathbb{R}\). The transfer function is

\[
H(s) = C(sE - A)^{-1}B, \quad s \in \mathbb{C}.
\]

In the following, we only consider systems with full-rank matrix \(E\).

2.2 Loewner rational interpolation

To derive a reduced model of dimension \(n \in \mathbb{N}\) of system (1), consider \(2n\) interpolation points \(s_1, \ldots, s_{2n} \in \mathbb{C}\). The interpolation points are partitioned into two sets \(\{\mu_1, \ldots, \mu_n\}\) and \(\{\gamma_1, \ldots, \gamma_n\}\) of equal size. Define the \(n \times n\) Loewner matrix \(L\) and \(n \times n\) shifted Loewner matrix \(L(s)\) as

\[
L_{ij} = \frac{H(\mu_i) - H(\gamma_j)}{\mu_i - \gamma_j}, \quad L_{ij}^{(s)} = \frac{\mu_i H(\mu_i) - \gamma_j H(\gamma_j)}{\mu_i - \gamma_j}, \quad i, j = 1, \ldots, n,
\]

together with the \(n \times 1\) input matrix \(\hat{B}\) and the \(1 \times n\) output matrix \(\hat{C}\) with components

\[
\hat{B}_i = H(\mu_i), \quad \hat{C}_i = H(\gamma_i), \quad i = 1, \ldots, n.
\]

The Loewner model is

\[
\hat{E} \dot{\hat{x}}(t) = \hat{A} \dot{x}(t) + \hat{B} u(t), \quad \hat{y}(t) = \hat{C} \dot{x}(t)
\]

with \(\hat{E} = -L\) and \(\hat{A} = -L^{(s)}\). The \(n\)-dimensional state at time \(t\) is \(\dot{x}(t)\) and the output at time \(t\) is \(\hat{y}(t)\). The transfer function of the Loewner model is

\[
\hat{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}, \quad s \in \mathbb{C}.
\]
The Loewner approach guarantees that the transfer function $\hat{H}$ of the Loewner model interpolates the transfer function $H$ of system (1) at the interpolation points $s_1, \ldots, s_{2n}$, which means
\[ H(s_i) = \hat{H}(s_i), \quad i = 1, \ldots, 2n. \]

3 Learning Loewner models from noisy frequency-response measurements

We now study the robustness of the Loewner approach to noise in the transfer-function values of system (1). The key contribution is Theorem 1 that bounds the error that is introduced by noise under certain conditions.

3.1 Noisy transfer-function values

Let $\mu \in \mathbb{C}$ with real part $\Re(\mu)$ and imaginary part $\Im(\mu)$ and let $0 < \sigma \in \mathbb{R}$. We denote with $\epsilon \sim \mathcal{CN}(\mu, \sigma)$ a complex random variable, where the real $\Re(\epsilon)$ and imaginary part $\Im(\epsilon)$ are independently normally distributed. The real part $\Re(\epsilon)$ has mean $\Re(\mu)$, the imaginary part $\Im(\epsilon)$ has mean $\Im(\mu)$. The real and the imaginary part of $\epsilon$ have standard deviation $\sigma$.

Let $\epsilon_1, \ldots, \epsilon_n \sim \mathcal{CN}(0,1)$ and $\eta_1, \ldots, \eta_n \sim \mathcal{CN}(0,1)$ be independent random variables. Define the noisy transfer-function values as
\[
H_{\sigma}(\mu_i) = H(\mu_i)(1 + \sigma \epsilon_i), \quad H_{\sigma}(\gamma_i) = H(\gamma_i)(1 + \sigma \eta_i), \quad i = 1, \ldots, n
\]
so that
\[
H_{\sigma}(\mu_i) \sim \mathcal{CN}(H(\mu_i), H(\mu_i)\sigma), \quad H_{\sigma}(\gamma_i) \sim \mathcal{CN}(H(\gamma_i), H(\gamma_i)\sigma), \quad i = 1, \ldots, n
\]
The noise pollutes the transfer-function values in a relative sense, i.e., the standard deviation of $H_{\sigma}(\mu_i)$ is scaled by $H(\mu_i)$. We consider such a relative noise model to be realistic in our situation because measurement errors typically are relative to the value of the quantity that is measured.

3.2 Loewner and noisy transfer-function values

From the noisy transfer-function values, we derive the noisy Loewner matrices
\[
\bar{L}_{ij} = \frac{H_{\sigma}(\mu_i) - H_{\sigma}(\gamma_j)}{\mu_i - \gamma_j}, \quad \bar{L}_{ij}^{(s)} = \frac{\mu_i H_{\sigma}(\mu_i) - \gamma_j H_{\sigma}(\gamma_j)}{\mu_i - \gamma_j}, \quad i, j = 1, \ldots, n
\]
which have the same structure as in Section 2.2 except that the noisy transfer-function values
\[
H_{\sigma}(\mu_1), \ldots, H_{\sigma}(\mu_n), H_{\sigma}(\gamma_1), \ldots, H_{\sigma}(\gamma_n)
\]
are used rather than the noiseless values $H(\mu_1), \ldots, H(\mu_n), H(\gamma_1), \ldots, H(\gamma_n)$. We decompose the noisy Loewner and the noisy shifted Loewner matrix into deterministic and random parts as
\[
\tilde{L} = L + \sigma \delta L, \quad \tilde{L}^{(s)} = L^{(s)} + \sigma \delta L^{(s)},
\]
with
\[
\delta L_{ij} = \frac{H(\mu_i)\epsilon_i - H(\gamma_j)\eta_j}{\mu_i - \gamma_j}, \quad \delta L_{ij}^{(s)} = \frac{\mu_i H(\mu_i)\epsilon_i - \gamma_j H(\gamma_j)\eta_j}{\mu_i - \gamma_j}, \quad i, j = 1, \ldots, n.
\]
We obtain the system matrices
\[
\tilde{E} = \hat{E} + \sigma \delta E, \quad \tilde{A} = \hat{A} + \sigma \delta A,
\]
where \( \delta E = -\delta L_{ij} \) and \( \delta A = -\delta L_{ij}^{(s)} \). Similarly, we define \( \tilde{B} = \hat{B} + \sigma \delta B \) and \( \tilde{C} = \hat{C} + \sigma \delta C \) with
\[
\delta B_i = H(\mu_i)\epsilon_i, \quad \delta C_i = H(\gamma_i)\eta_i, \quad i = 1, \ldots, n. \tag{3}
\]
The Loewner model learned from the noisy transfer-function values is then given by
\[
\tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \quad \tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{4}
\]
with the \( n \)-dimensional state \( \tilde{x}(t) \) at time \( t \) and the output \( \tilde{y}(t) \) at time \( t \). The transfer function of the model \( \tilde{H} \) is
\[
\tilde{H}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}, \quad s \in \mathbb{C}.
\]
We call \( \tilde{H} \) the noisy Loewner model.

### 3.3 Noise structure

The Loewner and the shifted Loewner matrices introduce structure in the noise that is added to the system matrices of the Loewner model learned from noisy transfer-function values. Consider the matrix \( s\delta E - \delta A \) and obtain
\[
s\delta E_{ij} - \delta A_{ij} = -\frac{1}{\mu_i - \gamma_j} (sH(\mu_i)\epsilon_i - sH(\gamma_j)\eta_j - \mu_iH(\mu_i)\epsilon_i + \gamma_jH(\gamma_j)\eta_j)
\tag{5}
\]
\[
= \frac{1}{\gamma_j - \mu_i} ((s - \mu_i)H(\mu_i)\epsilon_i + (-s + \gamma_j)H(\gamma_j)\eta_j)
\]
to write in matrix form as
\[
s\delta E - \delta A = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \cdots & \epsilon_n \end{bmatrix} \begin{bmatrix} H(\mu_1)_{\gamma_1 - \mu_1} & \cdots & H(\mu_1)_{\gamma_n - \mu_n} \\ \vdots & \ddots & \vdots \\ H(\mu_n)_{\gamma_1 - \mu_1} & \cdots & H(\mu_n)_{\gamma_n - \mu_n} \end{bmatrix} F_E + \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{bmatrix}.
\tag{6}
\]

Equation (6) reveals that the random parts of \( s\delta E - \delta A \) can be singled out into the two diagonal random matrices \( \epsilon \) and \( \eta \) of dimension \( n \times n \). The diagonal entries of \( \epsilon \) and \( \eta \) are independent and have distribution \( \mathcal{CN}(0,1) \).

### 3.4 Bounding the error due to noisy transfer-function values

The task is to bound
\[
\tilde{H}(s) - \hat{H}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B} - \tilde{C}(s\hat{E} - \hat{A})^{-1}\hat{B},
\tag{7}
\]
where \( s \) typically takes the values in some specified \( \Omega \subset \mathbb{C} \); for instance on the imaginary axis, possibly within a specified frequency range. In any case, we assume in the following that \( \Omega \) is free of the poles of \( \hat{H} \).

If \( \lambda_i, \phi_i (\hat{\lambda}_i, \phi_i) \) are the poles with the corresponding residues of \( \hat{H} (\tilde{H}) \), then \[13\) Proposition 3.3] holds
\[
\| \tilde{H} - \hat{H} \|_{\mathcal{H}_2}^2 = \sum_{i=1}^{n} \left| \phi_i(\tilde{H}(\lambda_i) - \hat{H}(\lambda_i)) + \sum_{j=1}^{n} \phi_j(\tilde{H}(\lambda_j) - \hat{H}(\lambda_j)) \right|.
\tag{7}
\]
which shows that the error \( \varepsilon \) is of particular interest at the reflected poles of both systems (for \( \eta \) to hold assumed stable, of the same order \( n \), and all poles assumed simple). We do not tackle the issue of a probabilistic error bound for the \( H_2 \) norm; this topic is left for our future work.

For an estimate of the error \( \varepsilon \), we need to understand the effect of the random noise in the matrices \( \hat{E} = \hat{E} + \sigma \delta E \), \( \hat{A} = \hat{A} + \sigma \delta A \), \( \hat{B} = \hat{B} + \sigma \delta B \) to the solution of the linear system \( s(\hat{E} - \hat{A})^{-1}\hat{B} \). To that end, we first briefly review the (deterministic) error bound for a solution of the perturbed system. The key is the condition number \( \kappa_2(s(\hat{E} - \hat{A})) \equiv \|((s\hat{E} - \hat{A})^{-1}\|_{2}\|s\hat{E} - \hat{A}\|_2 \), see, e.g., [15, Theorem 7.2].

**Proposition 1.** Let \( s \) be different from the poles of \( \hat{H} \), and consider the perturbations \( \delta \) and \( \varepsilon \) deterministic and bounded as \( \|\sigma(\delta E - \Delta A)\|_2 \leq \|\varepsilon\| \|s\hat{E} - \hat{A}\|_2, \|\sigma\delta B\|_2 \leq \|\varepsilon\| \|\hat{B}\|_2, \) where \( z > 0 \) is such that \( \zeta \kappa_2(s\hat{E} - \hat{A}) < 1 \). Then \( s\hat{E} - \hat{A} \) is nonsingular and

\[
\|((s\hat{E} - \hat{A})^{-1}\hat{B} - (s\hat{E} - \hat{A})^{-1}\hat{B})_2 \leq \frac{2\zeta}{1 - \zeta \kappa_2(s\hat{E} - \hat{A})}\kappa_2(s\hat{E} - \hat{A}).
\]

This is the standard perturbation bound for the linear system \( \hat{G}\hat{x} = \hat{B}, \hat{G} = s\hat{E} - \hat{A} \), under the deterministic perturbations \( \delta G = \sigma(\delta E - \Delta A) \) and \( \delta \hat{B} = \sigma \delta B \).

Recall that by \( \delta \), \( s\hat{E} - \Delta A = \varepsilon F_E + F_A \eta, \) where \( \varepsilon \) and \( \eta \) are diagonal matrices whose diagonals are random vectors. These can be bounded in probabilistic sense, using concentration inequalities which we briefly review next.

**Proposition 2.** Let \( Z = [z_1, \ldots, z_n]^T \) be a random vector with independent standard normal components \( z_i \sim \mathcal{N}(0,1), i = 1, \ldots, n. \) Then with probability at least \( 1 - \exp(-n/2) \)

\[
\|Z\|_2 \leq 2\sqrt{n}.
\]

If \( Z = [z_1, \ldots, z_n]^T \) is a vector of independent complex random variables \( z_i \sim \mathcal{CN}(0,1) \), with \( \mathcal{CN} \) defined in Section 3.1 then

\[
\|Z\|_2 \leq 4\sqrt{n}
\]

holds with probability at least \( 1 - 2\exp(-n/2) \). The estimate \( \|Z\|_2 \leq 2\sqrt{n} \) follows from Gaussian concentration and because the \( \chi^2 \) distribution is sub-Gaussian; see, e.g., [51, Example 2.28]. The statement \( \|Z\|_2 \leq 4\sqrt{n} \) follows from \( \|Z\|_2 \leq \|\mathfrak{R}(Z)\|_2 + \|\mathfrak{I}(Z)\|_2 \), where \( \mathfrak{R}(Z) \) is the vector that has as components the real parts of the components of \( Z \) and \( \mathfrak{I}(Z) \) is the vector that has as components the imaginary parts of \( Z \).

**Lemma 1.** Let \( s \in \Omega, i.e, s \) is different from the poles of \( \hat{H} \), be such that

\[
0 < \sigma < \frac{1}{\kappa_2(s\hat{E} - \hat{A})} \min\left\{\frac{\|s\hat{E} - \hat{A}\|_2}{4\sqrt{n}}\frac{\|\hat{B}\|_2}{\|F_E\|_2 + \|F_A\|_2}, \frac{\|\hat{B}\|_2}{4\sqrt{n}}\right\}.
\]

Then, with probability at least \( 1 - 4\exp(-n/2) \), \( s \) is not a pole of \( \hat{H} \) and the error bound \( \|\varepsilon\| \) in Proposition 1 holds.

**Proof.** Since, by \( \|s\delta E - \delta A = \varepsilon F_E + F_A \eta \), we have

\[
\|s\delta E - \delta A\|_2 \leq \|\varepsilon\|_2\|F_E\|_2 + \|F_A\|_2\|\eta\|_2.
\]

Because \( \varepsilon \) is diagonal with the elements \( \epsilon_1, \ldots, \epsilon_n \) on the diagonal, we obtain, using Proposition 2 that

\[
\|\varepsilon\|_2 = \max_{i=1, \ldots, n} |\epsilon_i| = \|\epsilon_1, \ldots, \epsilon_n\|_\infty \leq \|\epsilon_1, \ldots, \epsilon_n\|_2 \leq 4\sqrt{n},
\]
with probability at least \(1 - 2\exp(-n/2)\). Equation (13) also means that
\[
\|\delta B\|_2 \leq \|\delta \hat{B}\|_\infty \|\epsilon_1, \ldots, \epsilon_n\|_2 \leq 4\sqrt{n}\|\hat{B}\|_\infty
\]
\[
\|\hat{B}\|_2 \leq (1 + \sigma 4\sqrt{n})\|\hat{B}\|_2
\]
holds with probability at least \(1 - 2\exp(-n/2)\) because of the definition of \(\hat{B}\) given in (2). Similar arguments show that
\[
\|\eta\|_2 \leq 4\sqrt{n}
\]
and
\[
\|\delta C\|_2 \leq 4\sqrt{n}\|\hat{C}\|_\infty
\]
hold with probability at least \(1 - 2\exp(-n/2)\). Set now
\[
\zeta = \sigma\hat{\zeta}, \quad \hat{\zeta} = \max \left\{ \frac{4\sqrt{n}(\|F_E\|_2 + \|F_A\|_2)}{\|s\hat{E} - \hat{A}\|_2}, \frac{4\sqrt{n}\|\hat{B}\|_\infty}{\|\hat{B}\|_2} \right\}
\]
and observe that (11) guarantees \(\zeta \kappa(s\hat{E} - \hat{A}) < 1\). Thus, together with (12), it follows that
\[
\|\sigma(s\hat{E} - \delta A)\|_2 \leq \sigma 4\sqrt{n}(\|F_E\|_2 + \|F_A\|_2)
\]
\[
= \sigma 4\sqrt{n}(\|F_E\|_2 + \|F_A\|_2)\|s\hat{E} - \hat{A}\|_2 \leq \zeta\|s\hat{E} - \hat{A}\|_2
\]
with probability at least \((1 - 2\exp(-n/2))^2 \geq 1 - 4\exp(-n/2)\), where we used (18).

With (14) and the definition of \(\zeta\) in (18), we also obtain that
\[
\|\sigma \delta B\|_2 \leq \sigma 4\sqrt{n}\|\hat{B}\|_\infty = \sigma \frac{4\sqrt{n}\|\hat{B}\|_\infty}{\|\hat{B}\|_2}\|\hat{B}\|_2 \leq \zeta\|\hat{B}\|_2
\]
holds with probability at least \(1 - 4\exp(-n/2)\). Thus, with (19), (20), and because (11) implies \(\zeta \kappa(s\hat{E} - \hat{A}) < 1\), the error bound (8) is applicable with probability at least \(1 - 4\exp(-n/2)\), which also means that \(s\hat{E} - \hat{A}\) is nonsingular. \(\square\)

The following theorem bounds the error due to noise in the transfer-function values.

**Theorem 1.** Under the same assumptions as Lemma 1 for each \(s \in \Omega\)
\[
|\tilde{H}(s) - \hat{H}(s)| \in \mathcal{O}(\sigma)
\]
holds with probability at least \(1 - 4\exp(-n/2)\).

**Proof.** Consider now
\[
\hat{H}(s) - \tilde{H}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} - \hat{C}(s\hat{E} - \hat{A})^{-1}\delta B
\]
\[
= \hat{C} \left( (s\hat{E} - \hat{A})^{-1}\hat{B} - (s\hat{E} - \hat{A})^{-1}\delta B \right) - \sigma \delta C(s\hat{E} - \hat{A})^{-1}\delta B
\]
and take the absolute value to obtain
\[
|\hat{H}(s) - \tilde{H}(s)| \leq c_1 \frac{\zeta}{1 - \zeta \kappa(s\hat{E} - \hat{A})} + |\sigma \delta C(s\hat{E} - \hat{A})^{-1}\delta B|
\]
where we invoked (8) with \(c_1 = 2\|\hat{C}\|_2\|(s\hat{E} - \hat{A})^{-1}\delta B\|_2\kappa(s\hat{E} - \hat{A})\) and \(\zeta\) set as in (18). Note that (8) holds with probability at least \(1 - 4\exp(-n/2)\), and thus (22) holds with the same probability.
We now bound \( \| (s\hat{E} - \hat{A})^{-1} \|_2 \) in probability. Consider the Neumann expansion

\[
(s\hat{E} - \hat{A})^{-1} = (s\hat{E} - \hat{A} + \sigma(s\delta E - \delta A))^{-1} = (s\hat{E} - \hat{A})^{-1} \sum_{i=0}^{\infty} (-1)^i \sigma^i \left( (s\delta E - \delta A)(s\hat{E} - \hat{A})^{-1} \right)^i,
\]

(23)

where the series converges to the inverse of \( I + \sigma(s\delta E - \delta A)(s\hat{E} - \hat{A})^{-1} \) provided that \( \| \sigma(s\delta E - \delta A)(s\hat{E} - \hat{A})^{-1} \|_2 < 1 \). Because (19) holds with probability at least \( 1 - 4\exp(-n/2) \), we obtain that with the same probability of at least \( 1 - 4\exp(-n/2) \) holds

\[
\| \sigma(s\delta E - \delta A)(s\hat{E} - \hat{A})^{-1} \|_2 \leq \zeta \| s\hat{E} - \hat{A} \|_2 \| (s\hat{E} - \hat{A})^{-1} \|_2 < 1,
\]

where we used assumption (11) in the second inequality. Note that the second inequality is strict. Set

\[
\nu = 4\sqrt{n}(\| F_E \|_2 + \| F_A \|_2)(\| s\hat{E} - \hat{A} \|^{-1}_2),
\]

and obtain with (19), (23), and \( \sigma \nu < 1 \) because of (11) that

\[
\| (s\hat{E} - \hat{A})^{-1} \|_2 \leq \| (s\hat{E} - \hat{A})^{-1} \|_2 \sum_{i=0}^{\infty} (\nu \sigma)^i = \| (s\hat{E} - \hat{A})^{-1} \|_2 \frac{1}{1 - \nu \sigma}
\]

(24)

holds with probability at least \( 1 - 4\exp(-n/2) \).

Then, we obtain the bound

\[
| \sigma \delta C(s\hat{E} - \hat{A})^{-1} \hat{B} | \leq \sigma \|\delta C\|_2 \| (s\hat{E} - \hat{A})^{-1} \|_2 \| \hat{B} \|_2 \leq 4\sqrt{n} \|\hat{C}\|_\infty \| \hat{B} \|_2 \| (s\hat{E} - \hat{A})^{-1} \|_2 \frac{\sigma (1 + \sigma 4\sqrt{n})}{1 - \nu \sigma} \leq c_2 \frac{\sigma + \sigma^2 4\sqrt{n}}{1 - \nu \sigma}
\]

(25)

where we used (15), (16), and (21), which together hold with probability at least \( 1 - 4\exp(-n/2) \), and we set \( c_2 = 4\sqrt{n} \|\hat{C}\|_\infty \| \hat{B} \|_2 \| (s\hat{E} - \hat{A})^{-1} \|_2 \). We obtain with (22), (25) the following bound

\[
| \hat{H}(s) - \tilde{H}(s) | \leq \sigma \left[ \frac{c_1 \zeta}{1 - \sigma^2 \kappa_2 (s\hat{E} - \hat{A})} + \frac{c_2 (1 + \sigma 4\sqrt{n})}{1 - \nu \sigma} \right],
\]

(26)

which grows as at most linearly in \( \sigma \) and thus shows (21).

\[\square\]

**Corollary 1.** Under the same conditions as Theorem 11 and for \( s \in \Omega \) that are not zeros of \( \hat{H} \),

\[
\frac{| \tilde{H}(s) - \hat{H}(s) |}{| \hat{H}(s) |} \in \mathcal{O}(\sigma)
\]

(27)

holds with probability at least \( 1 - 4\exp(-n/2) \).

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2 The necessary and sufficient condition for the convergence is that the spectral radius of \( \sigma(s\delta E - \delta A)(s\hat{E} - \hat{A})^{-1} \) is strictly less than one.
Proof. Note that $\tilde{H}(s)$ is independent of $\sigma$ and thus dividing $|\tilde{H}(s)|$ by $|\tilde{H}(s)|$ is sufficient to show if $s$ is no zero of $\tilde{H}$. To highlight a geometric interpretation of (27), we show (27) via a different approach. Since 

$$\tilde{H}(s) = \|C\|_2 \|(sE - \tilde{A})^{-1}B\|_2 \cos \angle(C, (sE - \tilde{A})^{-1}B),$$

the factor $c_1$ can be interpreted as

$$c_1 = \frac{2\kappa_2(s\tilde{E} - \tilde{A})}{\cos \angle(C, (sE - \tilde{A})^{-1}B)} = \frac{2\kappa_2(s\tilde{E} - \tilde{A})}{|\cos \angle(C, (sE - \tilde{A})^{-1}B)|}$$

so that the first term on the right-hand side in (26) contains the bound on the relative error $|\tilde{H}(s) - H(s)|/|\tilde{H}(s)|$ with the two natural condition numbers $\kappa_n(s\tilde{E} - \tilde{A})$ and $|\cos \angle(C, (sE - \tilde{A})^{-1}B)|$. The interpretation is as follows: Evaluating $\tilde{H}$ essentially means solving a perturbed linear system $(s\tilde{E} - \tilde{A})^{-1}B$ and then computing an inner product $(C, (sE - \tilde{A})^{-1}B)$ with a perturbed vector $\tilde{C}$. The sensitivity of the solution of a system of equations to perturbations is quantified by the condition number $\kappa_2(s\tilde{E} - \tilde{A})$ and the sensitivity of the inner product is quantified by $|\cos \angle(C, (sE - \tilde{A})^{-1}B)|$. Similarly, the second term on the right-hand side in (26) can be modified as follows: Instead of (26), estimate the second term in (22) as

$$|\sigma \delta C(s\tilde{E} - \tilde{A})^{-1}B| \leq \sigma \sqrt{n} \|C\|_\infty \|(sE - \tilde{A})^{-1}B\|_2 \left(1 + \frac{2\zeta \kappa_2(s\tilde{E} - \tilde{A})}{1 - \zeta \kappa_2(s\tilde{E} - \tilde{A})}\right)$$

$$= \sigma |\tilde{H}(s)| \frac{4\sqrt{n}}{|\cos \angle(C, (sE - \tilde{A})^{-1}B)|} \frac{1 + \zeta \kappa_2(s\tilde{E} - \tilde{A})}{1 - \zeta \kappa_2(s\tilde{E} - \tilde{A})} \|C\|_\infty .$$

Hence, because $\tilde{H}(s) \neq 0$ per assumption, we can write

$$\frac{|\tilde{H}(s) - H(s)|}{|H(s)|} \leq \sigma \left(\frac{1}{\cos \theta} \left(\frac{2\zeta \kappa_2^{(s)}}{1 - \sigma \zeta \kappa_2^{(s)}} + 4\sqrt{n} \frac{1 + \zeta \kappa_2^{(s)}}{1 - \zeta \kappa_2^{(s)}} \|C\|_\infty \right)\right)$$

(28)

where $\theta = \angle(C, (sE - \tilde{A})^{-1}B)$ and $\kappa_2^{(s)} = \kappa_2(s\tilde{E} - \tilde{A})$.

3.5 Remarks on Theorem 1

The following remarks are in order. First, note that condition 111 implies that $\tilde{H}(s)$ exists with probability at least $1 - 4 \exp(-n/2)$. Second, a similar result as in Theorem 1 can be shown for an absolute noise model, i.e., where the noisy transfer-function values $\tilde{H}(s)$ have a standard deviation that is independent of the transfer-function value $H(s)$. Third, condition 111 in Theorem 1 depends on the scaling of the entries of the system matrices $\tilde{E}, \tilde{A}, \tilde{B},$ and $\tilde{C}$. Consider two regular matrices $D_1$ and $D_2$ of size $n \times n$. Then, the Loewner model given by $\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C},$ derived from noiseless transfer-function values, can be transformed as

$$\tilde{E} = D_1\tilde{E}D_2, \quad \tilde{A} = D_1\tilde{A}D_2, \quad \tilde{B} = D_1\tilde{B}, \quad \tilde{C} = D_1\tilde{C}D_2,$$

with transfer function $\tilde{H}(s) = \tilde{C}(s\tilde{E} - \tilde{A})^{-1}\tilde{B}$. It holds $\tilde{H}(s) = \tilde{H}(s)$ for $s \in \mathbb{C}$; however, the condition number $\kappa_2(s\tilde{E} - \tilde{A})$ is not invariant under the transformation given by $D_1$ and $D_2$, which means that condition 111 in Theorem 1 is not invariant if the system is transformed. A component-wise analysis 113 Section 7.2 could be an option to derive a version of Theorem 1 that is invariant under diagonal linear transformations $D_1$ and $D_2$. Fourth, condition 111 in Theorem 1 depends on the interpolation points $s_1, \ldots, s_{2n}$ and on the partition $\{\mu_1, \ldots, \mu_n\}, \{\gamma_1, \ldots, \gamma_n\}$, which shows that the interpolation points and their partition can influence the robustness of the Loewner approach to noise; cf., e.g., 113 Section 2.1. Our numerical results demonstrate that different choices of interpolation points indeed influence condition 111 and thus when the linear growth of the error 24 with the standard deviation $\sigma$ of the noise holds.
4 Numerical results

We demonstrate Theorem 1 on numerical experiments with benchmark examples in this section.

4.1 CD player

The system of a CD player is a common benchmark problem for model reduction and can be downloaded from the SLICOT website.

4.1.1 Problem setup

We consider single-input-single-output (SISO) systems and therefore we use only the first input and the second output of the CD-player system. The frequency range is $[2\pi, 200\pi]$, which contains some of the major dynamics of the CD-player system. The order of the system is $N = 120$. To derive a Loewner model of order $n$, we select $2n$ interpolation points $s_1, \ldots, s_{2n}$ as follows. First, $n$ points $s_1, \ldots, s_n$ are selected logarithmically equidistant in the range $[2\pi, 200\pi]$ on the imaginary axis. The points $s_{n+1}, \ldots, s_{2n}$ are the complex conjugates of the points $s_1, \ldots, s_n$. The points $s_1, \ldots, s_{2n}$ are sorted descending with respect to their absolute value and then partitioned into two sets $\{\mu_1, \ldots, \mu_n\}$ and $\{\gamma_1, \ldots, \gamma_n\}$ such that complex pairs are in the same set. To test the Loewner models, we select 200 test points $s_{1\text{test}}, \ldots, s_{200\text{test}}$ on the imaginary axis in the range $[2\pi, 200\pi]$.

From the noiseless transfer-function values $H(\mu_1), \ldots, H(\mu_n)$ and $H(\gamma_1), \ldots, H(\gamma_n)$ a Loewner model is derived with transfer function $\tilde{H}$. Then, the transfer-function values are polluted with noise with standard deviation $\sigma$ as described in Section 3.1 and a noisy Loewner model is derived with transfer function $\tilde{H}_\sigma$. Note that we now explicitly denote standard deviation as subscript in the transfer functions of noisy Loewner models.

4.1.2 Results

We consider the error
\[
e(\sigma) = \frac{1}{200} \sum_{i=1}^{200} |\tilde{H}(s_{i\text{test}}) - \tilde{H}_\sigma(s_{i\text{test}})|,\]
which is an average of the error (21) over all 200 test points. Figure 1a shows $e(\sigma)$ for standard deviations $\sigma$ in the range $[10^{-15}, 10^5]$ and dimension $n = 20$. A linear growth of the error (29) with the standard deviation $\sigma$ is observed for $\sigma < 10^{-5}$. Figure 1b shows the number of test points that violate condition (11) of Theorem 1. The results indicate that for $\sigma \geq 10^{-7}$ condition (11) is violated for all 200 test points, which seems to align with Figure 1a that shows a linear growth for $\sigma < 10^{-5}$. Thus, the results in Figure 1a are in agreement with Theorem 1. Similar observations can be made for $n = 28$ in Figure 1c and Figure 1d.

4.2 Penzl

The Penzl system is a benchmark problem that has been introduced in [35, Example 3] and is used in, e.g., [18, 30] to demonstrate the Loewner approach.

4.2.1 Problem setup

The Penzl system is of order $N = 1006$ and we consider the frequency range $[10, 1000]$. We consider two different sets of interpolation points in this example. First, we select $n$ logarithmically equidistant points on the imaginary axis in the range $[10, 1000]$ and include their complex conjugates as described in Section 4.1.1. We denote the corresponding set of $2n$ interpolation points as $E_n$. Second, we select points randomly in the

3http://slicot.org/20-site/126-benchmark-examples-for-model-reduction
imaginary plane in the range \([10, 1000] \times \iota[10, 1000]\), where \(\iota\) is the complex unit \(\iota = \sqrt{-1}\). To select the points randomly, we first select \(10^6\) logarithmically equidistant points \(w_1, \ldots, w_{10^6}\) in \([10, 1000]\) and then draw uniformly \(n\) points from \(w_1, \ldots, w_{10^6}\) for the real and imaginary parts of the \(n\) interpolation points \(s_1, \ldots, s_n\). Then, their complex conjugates are \(s_{n+i} = \bar{s}_i\) for \(i = 1, \ldots, n\). The points are sorted descending with respect to their absolute value and partitioned into two sets of equal size such that complex pairs are in the same set. The set of interpolation points is denoted as \(R_n\). The test points are 200 points selected on the imaginary axis in the range \([10, 1000]\).

4.2.2 Results

Figure 2a shows the error (29) corresponding to the Loewner model with interpolation points \(R_n\) (random) and of dimension \(n = 16\). Figure 2c shows that condition (11) is violated for all test points and all standard deviations \(\sigma\) in the range \([10^{-15}, 10^5]\), which means that Theorem 1 is not applicable. In contrast, Figure 2b shows a linear growth for the error (29) corresponding to the models learned from logarithmically equidistant points \(E_n\). Figure 2d indicates that up to \(\sigma = 10^{-10}\) the condition (11) for Theorem 1 is satisfied, which explains the linear growth for \(\sigma \leq 10^{-10}\). For \(\sigma > 10^{-10}\), Theorem 1 is not applicable, even though a linear growth is observed, which demonstrates that Theorem 1 is rather pessimistic in this example. Figure 3 shows
the magnitude of the transfer functions for $\sigma = 10^{-6}$ and demonstrates that the logarithmically equidistant points $E_n$ seem to provide more robustness against noise than the random points $R_n$ in this example.

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Figure 3: Penzl: Plot (a) and (b) show the magnitude of the transfer function of Loewner models learned from noiseless transfer-function values and from transfer-function values polluted with noise with standard deviation $\sigma = 10^{-6}$.
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