LINEAR COMBINATIONS OF POLYNOMIALS WITH THREE-TERM RECURRENCE

Khang Tran and Maverick Zhang

Abstract. We study the zero distribution of the sum of the first $n$ polynomials satisfying a three-term recurrence whose coefficients are linear polynomials. We also extend this sum to a linear combination, whose coefficients are powers of $az + b$ for $a, b \in \mathbb{R}$, of Chebyshev polynomials. In particular, we find necessary and sufficient conditions on $a, b$ such that this linear combination is hyperbolic.

1. Introduction

The sequence of Chebyshev polynomials of the first kind $\{T_n(z)\}_{n=0}^{\infty}$ defined by the recurrence

$$T_{n+1}(z) = 2zT_n(z) + T_{n-1}(z)$$

with $T_0(z) = 1$ and $T_1(z) = z$ forms a sequence of orthogonal polynomials whose zeros are real (i.e., hyperbolic polynomials). The location of zeros of polynomials satisfying a more general recurrence

$$R_{n+1}(z) = A(z)R_n(z) + B(z)R_{n-1}(z)$$

(1.1)

where $A(z), B(z) \in \mathbb{C}[z]$ was given in [3]. In [2], the author studied the set of zeros of a linear combination of Chebyshev polynomials $\sum_{k=0}^{m} a_k T_{n-k}(z)$, $m \leq n$, $a_k \in \mathbb{R}$, and provided a connection between this sequence and the theory of Pisot and Salem numbers in number theory. In the special case when $m = n$ and $a_k = 1 \ \forall k$, the sum of the first $n$ Chebyshev polynomials connects to Dirichlet kernel in the Fourier analysis. In Section 2 of this paper, we study the zeros of this sum (cf. Theorem [2.1]) when the sequence of Chebyshev polynomials are replaced by a more general sequence $\{R_n(z)\}$ given in (1.1) where $A(z)$ and $B(z)$ are any linear polynomials with real coefficients.

The sequence of Chebyshev polynomials of the second kind $\{U_n(z)\}$ satisfies the same recurrence as that of the first kind with the initial condition $U_0(z) = 1$ and $U_1(z) = 2z$. This initial condition can be written in the form $U_0(z) = 1$ and $U_1(z) = 2z$.
In Section 3 of this paper, we study the zeros of a linear combination of Chebyshev polynomials of the second kind whose coefficients are powers of \(az + b\). In particular, we consider

\[
Q_n(z) = \sum_{k=0}^{n} (az + b)^k U_{n-k}(z), \quad a, b \in \mathbb{R}.
\]

We find the necessary and sufficient conditions on \(a\) and \(b\) under which the zeros of the resulting polynomials are real (cf. Theorem 3.2).

2. Sum of polynomials with three-term recurrence

For \(a_1, b_1, a_2, b_2 \in \mathbb{R}, a_2 \neq 0\), we let \(R_n(z)\) be the sequence of polynomials satisfying the recurrence

\[
R_{n+1}(z) = (a_1 z + b_1) R_n(z) + (a_2 z + b_2) R_{n-1}(z)
\]

with \(R_0(z) = 1\) and \(R_{-n}(z) = 0, \forall n \in \mathbb{N}\). Equivalently the sequence \(\{R_n(z)\}_{n=0}^\infty\) is generated by

\[
\sum_{n=0}^\infty R_n(z) t^n = \frac{1}{1 - (a_1 z + b_1)t - (a_2 z + b_2)t^2}.
\]

In this section, we study the necessary and sufficient conditions on \(a_1, b_1, a_2, \) and \(b_2\) under which all the zeros of the polynomial \(\sum_{n=0}^n R_{n-k}(z)\) are real. Those polynomials form a sequence whose generating function is

\[
\sum_{n=0}^\infty \sum_{k=0}^n R_k(z) t^n = \sum_{k=0}^\infty t^k \sum_{n=k}^\infty R_{n-k}(z) t^{n-k} = \frac{1}{(1-t)(1 - (a_1 z + b_1)t - (a_2 z + b_2)t^2)}.
\]

With the substitutions \(t\) by \(-t\), \(a_2\) by \(-a_2\), and \(b_2\) by \(-b_2\), and then substitute \(a_2 z + b_2\) by \(z\), we reduce the generating function to the form

\[
\frac{1}{(t+1)((az + b)t^2 + zt + 1)}.
\]

Note that all the substitutions above preserve the reality of the zeros of the generated sequence of polynomials. We state the main theorem of this section.

**Theorem 2.1.** Let \(a, b \in \mathbb{R}\). The zeros of all the polynomials \(P_n(z)\) generated by

\[
\sum_{n=0}^\infty P_n(z) t^n = \frac{1}{(t+1)((az + b)t^2 + zt + 1)}
\]

are real if and only if \(b \geq 1 + 2|a|\). Under this condition the zeros of \(P_n(z)\) lie on

\[
(2a - 2\sqrt{a^2 + b}, 2a + 2\sqrt{a^2 + b})
\]

and are dense there as \(n \to \infty\).
Proof. Sufficiency We assume \( b \geq 1 + 2|a| \). To prove that the zeros of \( P_n(z) \) lie on \((2.2)\), we count the number of real zeros of \( P_n(z) \) on this interval and show that this number is at least the degree of this polynomial which is given by the lemma below.

Lemma 2.1. For each \( n \in \mathbb{N} \), the degree of \( P_n(z) \) is at most \( n \).

Proof. We collect the coefficients in \( t \) of the denominator of the right-hand side of \((2.1)\) and obtain the recurrence

\[
P_n(z) = -(z + 1)P_{n-1}(z) - ((a + 1)z + b)P_{n-2}(z) - (az + b)P_{n-3}(z)
\]

where \( P_0(z) = 1 \) and \( P_{-n}(z) = 0, \forall n \in \mathbb{N} \). The lemma follows from induction. \( \square \)

To count the number of real zeros of \( P(z) \), we construct two auxiliary real-valued functions \( z(\theta) \) and \( \tau(\theta) \) on \( \theta \in (0, \pi) \). The first function is defined as

\[
(2.3) \quad z(\theta) = 2a \cos^2 \theta - 2 \cos \theta \sqrt{a^2 \cos^2 \theta + b}.
\]

By the quadratic formula, \( z(\theta) \) satisfies

\[
(2.4) \quad z(\theta)^2 - 4az(\theta) \cos^2 \theta - 4b \cos^2 \theta = 0.
\]

We will show later that there are \( n \) values of \( \theta \in (0, \pi) \), each of which yields a zero of \( P_n(z) \) on \((2.2)\) via \( z(\theta) \). The lemma below ensures a bijective correspondence between \( \theta \) and \( z(\theta) \).

Lemma 2.2. The function \( z(\theta) \) is increasing on \( (0, \pi) \) and it maps this interval onto the interval

\[
(2a - 2\sqrt{a^2 + b}, 2a + 2\sqrt{a^2 + b}).
\]

Proof. To show \( z(\theta) \) is increasing, we compute its derivative

\[
\frac{dz}{d\theta} = -4a \cos \theta \sin \theta + \frac{4a^2 \cos^2 \theta \sin \theta + 2b \sin \theta}{\sqrt{a^2 \cos^2 \theta + b}}
\]

and see that it suffices to show

\[
2a^2 \cos^2 \theta + b > 2|a \cos \theta| \sqrt{a^2 \cos^2 \theta + b}.
\]

The left-hand side is positive and the squares of both sides reduce the inequality to \( b^2 > 0 \), which shows that \( z(\theta) \) is increasing. We complete the lemma by computing the limits \( \lim_{\theta \to 0} z(\theta) = 2a - 2\sqrt{a^2 + b} \) and \( \lim_{\theta \to \pi} z(\theta) = 2a + 2\sqrt{a^2 + b} \). \( \square \)

To define the second function \( \tau(\theta) \), we need the following lemma.

Lemma 2.3. For any \( \theta \in (0, \pi) \), we have \( az(\theta) + b > 0 \).

Proof. From Lemma 2.2 it suffices to show that \( b + 2a^2 > 2|a| \sqrt{a^2 + b} \). Since we know the left-hand side is positive by \( b \geq 1 + 2|a| \), we obtain the inequality above by squaring both sides. \( \square \)

From Lemma 2.3 we define the functions

\[
\tau(\theta) = \frac{1}{\sqrt{az(\theta) + b}}, \quad t_1(\theta) = \tau(\theta)e^{-i\theta}, \quad t_2(\theta) = \tau(\theta)e^{i\theta}
\]

on \( \theta \in (0, \pi) \).
Lemma 2.4. For any \( \theta \in (0, \pi) \), the two zeros of
\[
(az(\theta) + b)t^2 + z(\theta)t + 1
\]
are \( t_1(\theta) \) and \( t_2(\theta) \).

Proof. We verify that \( \tau(\theta) e^{\pm i\theta} \) satisfy Vieta’s formulas. Indeed, we have
\[
t_1(\theta)t_2(\theta) = \tau(\theta)^2 = \frac{1}{az(\theta) + b},
\]
\[
t_1(\theta) + t_2(\theta) = 2\tau(\theta) \cos \theta = \frac{2 \cos \theta}{\sqrt{az(\theta) + b}}
\]
From (2.3), we note that 
\[
(2.7)
\]
\[
(2.6)
\]
\[
(2.5)
\]
\[
(2.4)
\]

From (2.3), we note that \( z(\theta) \cos \theta < 0 \) since \( b > 0 \). As a consequence, we obtain
\[
(2.7)
\]
by squaring both sides and applying (2.4).

The lemma below shows that for each \( \theta \in (0, \pi) \), the two zeros of (2.5) lie inside the unit ball.

Lemma 2.5. For any \( \theta \in (0, \pi) \), we have \( |\tau(\theta)| < 1 \).

Proof. From (2.6), (2.7), and (2.3), it suffices to show
\[
\sqrt{a^2 \cos^2 \theta + b} > 1 + a \cos \theta.
\]
If the right-hand side is negative, the inequality is trivial. If not, we square both sides and the inequality follows from \( b \geq 1 + 2|a| > 1 + 2a \cos \theta \).

For each \( \theta \in (0, \pi) \), the Cauchy differentiation formula gives
\[
P_n(z(\theta)) = \frac{1}{2\pi i} \int_{|t|=R} \frac{1}{(t+1)((az(\theta) + b)t^2 + z(\theta)t + 1)(t+1)\tau^n}dt.
\]
\[
= \frac{1}{2\pi i} \int_{|t|=R} \frac{1}{(az(\theta) + b)(t+1)(t - t_1(\theta))(t - t_2(\theta))\tau^{n+1}}dt.
\]
We recall that \( az(\theta) + b \neq 0 \) by Lemma 2.3. If we integrate the integrand over the circle \( Re^{it}, 0 \leq t \leq 2\pi \), and let \( R \to \infty \), then the integral approaches 0. Thus the sum of \( P_n(z(\theta)) \) and the residue of the integrand at the three simple poles \(-1, t_1(\theta)\) and \( t_2(\theta) \) is 0. We compute these residues and deduce that \(- (az(\theta) + b)P_n(z(\theta))\) equals to
\[
\frac{(-1)^{n+1}}{(1 + t_1(\theta))(1 + t_2(\theta))} + \frac{1}{1} t_1(\theta)^{n+1}(1 + t_1(\theta))(1 + t_2(\theta))
\]
\[
\frac{1}{1} t_2^{n+1}(\theta)(1 + t_2(\theta))(t_2(\theta) - t_1(\theta)).
\]

We multiply this expression by \((1 + t_1(\theta))(1 + t_2(\theta))\tau(\theta)^n\), which is nonzero \( \forall \theta \in (0, \pi) \), and conclude \( \theta \) is a zero of \( P_n(z(\theta)) \) if and only if it is a zero of
\[
(-1)^n \tau(\theta)^n + \frac{1 + \tau(\theta)e^{i\theta}}{(\tau(\theta)e^{-i\theta} - \tau(\theta)e^{i\theta})e^{-i(n+1)\theta}} + \frac{1 + \tau(\theta)e^{-i\theta}}{(\tau(\theta)e^{i\theta} - \tau(\theta)e^{-i\theta})e^{i(n+1)\theta}}
\]
or equivalently a zero of
\[ (-1)^{n+1} \tau(\theta)^{n+1} - \frac{\sin((n+1)\theta)/\tau(\theta) + \sin((n+2)\theta)}{\sin \theta}. \]

With the trigonometric identity \( \sin(n+2)\theta = \sin((n+1)\theta) \cos \theta + \cos((n+1)\theta) \sin \theta \), we write the expression above as
\[ (-1)^{n+1} \tau(\theta)^{n+1} - \cos((n+1)\theta) - \frac{\sin((n+1)\theta)(\cos \theta + 1/\tau(\theta))}{\sin \theta}. \]

We note that if \( \theta = \frac{k\pi}{n+1}, 1 \leq k \leq n \), then the sign of \((2.8)\) is \((-1)^{k+1}\) since \(\tau(\theta) < 1\) by Lemma 2.3. By the intermediate value theorem, \((2.8)\) has at least \(n-1\) solutions on \((\pi/(n+1), n\pi/(n+1))\). We also note that as \(\theta \to 0\), the sign of \((2.8)\) is negative since \(\sin((n+1)\theta)/\sin \theta\) approaches \(n+1\) and \(\tau(\theta) < 1\). Thus \(2.3\) has another zero on \((0, \pi/(n+1))\). From Lemma 2.2 each zero in \(\theta\) of \((2.8)\) gives exactly one zero in \(z\) of \(P_n(z)\) on \((2a - 2\sqrt{a^2 + b}, 2a + 2\sqrt{a^2 + b})\). Thus all the zeros of \(P_n(z)\) lie on the interval above by the fundamental theorem of algebra and Lemma 2.4. The density of the zeros of \(P_n(z)\) as \(n \to \infty\) on this interval follows directly from the density of the solutions of \((2.8)\) and the continuity of \(z(\theta)\).

**Necessity** In this necessary direction, we will show that if either (1) \(b \leq -1\) or (2) \(-1 < b < 1 + 2|a|\), then not all polynomials \(P_n(z)\) are hyperbolic. By [1] Theorem 1.5, it suffices to find \(z^* \in \mathbb{C} \setminus \mathbb{R}\) such that the zeros of
\[ (t + 1)((az^* + b)t^2 + z^*t + 1) \]
are distinct and the two smallest (in modulus) zeros of this polynomial have the same modulus. Note that every small neighborhood of such \(z^*\) will contain a zero of \(P_n(z)\) for all large \(n\) and consequently \(P_n(z)\) is not hyperbolic for all large \(n\). For more details on this application of the theorem, see [1].

For the first case \(b \leq -1\), we let \(\tau^*\) be any angle with \(a^2 \cos^2 \theta^* < -b\) and let \(\tau^*\) be any zero of \(b\tau^2 - 2a\tau \cos \theta^* - 1\). Note that \(\tau^* \notin \mathbb{R}\) since \(a^2 \cos^2 \theta^* + b < 0\) and consequently \(\tau^* \notin \mathbb{R}\) by the definition of \(\tau^*\). With the note that \(2a\tau^* \cos \theta^* + 1\) is nonreal (and thus nonzero), we choose
\[ z^* = \frac{-2b\tau^* \cos \theta^*}{2a\tau^* \cos \theta^* + 1} \]
which is nonreal since \(1/z^* \notin \mathbb{R}\). From the definitions of \(\tau^*, \theta^*,\) and \(z^*\) above, the two solutions of \((az^* + b)t^2 + z^*t + 1 = 0\) are \(\tau^* e^{\pm i\theta^*}\) since they satisfy Vieta’s formulas
\[ \tau^2 = \frac{1}{az^* + b}, \quad 2\tau^* \cos \theta^* = -\frac{z^*}{az^* + b}. \]
Since \(\tau^*\) and \(\overline{\tau^*}\) are solutions of \(b\tau^2 - 2a\tau \cos \theta^* - 1\), we have \(\tau^* \overline{\tau^*} = |\tau^*|^2 = -1/b \leq 1\). Thus the two smallest (in modulus) zeros of \((2.9)\) equal in modulus and we complete the case \(b \leq -1\).

We now consider the case \(-1 < b < 1 + 2|a|\). We will find \(z^* \notin \mathbb{R}\) so that the smaller (in modulus) zero of \((az^* + b)t^2 + z^*t + 1\) lie on the unit circle. The inequality \(2|a| - b| < 1\) implies that \(1 + 2|a| > |b| > b|2|a| - b|\) and consequently
\[ 1 - b^2 + 2|a| + 2b|a| > 0. \]
We conclude there is $\theta^* \in (0, \pi)$ sufficiently close to 0 when $a \geq 0$ or close to $\pi$ when $a < 0$ such that $b^2 - 2ab \cos \theta^* < 1 + 2a \cos \theta^*$. With this choice of $\theta^*$, we have

$$\frac{|be^{i\theta^*} - a|}{|ae^{i\theta^*} + 1|} = \frac{b^2 + a^2 - 2ab \cos \theta}{a^2 + 1 + 2a \cos \theta} < 1.$$  

We define

$$z^* = \frac{-1 - be^{2i\theta^*}}{ae^{2i\theta^*} + e^{i\theta^*}}$$

and write

$$(az^* + b)t^2 + z^*t + 1$$

as $z^*(at^2 + t) + bt^2 + 1$ to conclude that $e^{i\theta^*}$ is a zero of this polynomial. Since the product of the two zeros of this polynomial is $1/(az^* + b)$, we claim that the other zero of this polynomial is more than 1 in modulus by showing that

$$\frac{1}{|az^* + b|} > 1.$$  

Indeed, from the definition of $z^*$, this inequality is equivalent to (2.10). We note that $z^* \notin \mathbb{R}$ since a solution of (2.11) is $e^{i\theta^*} \notin \mathbb{R}$ and the other solution is more than 1 in modulus.

\[\Box\]

3. Linear combination of Chebyshev polynomials

The goal of this section is to study necessary and sufficient conditions under which the zeros of (1.2) are real. The sequence $\{Q_n(z)\}$ in (1.2) is generated by

$$\sum_{n=0}^{\infty} Q_n(z)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (az + b)^k U_{n-k}(z)t^n$$

$$= \sum_{k=0}^{\infty} (az + b)^k t^k \sum_{n=k}^{\infty} U_{n-k}(z)t^{n-k}$$

$$= \frac{1}{(1 + (az + b)t)(1 - 2zt + t^2)}.$$  

With the substitution $z$ by $-z/2$ and then $-a/2$ by $a$, it suffice to study the hyperbolicity of the sequence generated of polynomials by

$$\frac{1}{(1 + (az + b)t)(1 + zt + t^2)}.$$  

As a small digression of the main goal, we will prove the following theorem which states that the positivity of the $t^2$-coefficient in the factor $1 + zt + t^2$ is important to ensure the hyperbolicity of the generated sequence of polynomials.

**Theorem 3.1.** Suppose $a, b, c \in \mathbb{R}$ where $c \neq 0$. If $c \leq 0$, then not all the polynomials $P_n(z)$ generated by

$$\frac{1}{((az + b)t + 1)(ct^2 + zt + 1)}.$$
are hyperbolic.

We note that if \( c = 0 \), the sequence of generated polynomials satisfies a three-term recurrence and their zeros have been studied in [3]. Under the condition \( c > 0 \), with the substitution \( t \to t/\sqrt{c} \), we can assume \( c = 1 \). The following theorem settles the necessary and sufficient conditions for the hyperbolicity of (1.2).

**Theorem 3.2.** Suppose \( a, b \in \mathbb{R} \). The zeros of all the polynomials \( P_n(z) \) generated by

\[
\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{((az + b)t + 1)(t^2 + zt + 1)}.
\]

are real if and only if \( |b| \leq 1 - 2|a| \). Moreover when \( |b| \leq 1 - 2|a| \), the zeros of \( P_n(z) \) lie on \((-2, 2)\) and are dense there as \( n \to \infty \).

**Proof of Theorem 3.1.** In the case \( c < 0 \), with the substitution \( t \to t/\sqrt{|c|} \), it suffices to show that for any \( a, b \in \mathbb{R} \), not all the polynomials generated by

\[
\frac{1}{((az + b)t + 1)(-t^2 + z^*t + 1)}
\]

are hyperbolic. Recall a consequence of [1, Theorem 1.5] that we will need to find \( z^* \neq \mathbb{R} \) so that the two smallest zeros of

\[
((az^* + b)t + 1)(-t^2 + z^*t + 1)
\]

are equal in modulus.

In the case \( |b| < 1 \), we choose \( z^* = iy^* \) where

\[
0 < y^* < \min \left( \frac{\sqrt{1 - b^2}}{|a|}, 2 \right)
\]

if \( a \neq 0 \) and \( 0 < y^* < 2 \) if \( a = 0 \). The two zeros of \(-t^2 + z^*t + 1\),

\[
iy^* \pm \frac{\sqrt{4 - y^*2}}{2}
\]

lie on the unit circle and thus their modulus is less than

\[
\frac{1}{|az^* + b|} = \frac{1}{\sqrt{a^2y^2 + b^2}}.
\]

For the remainder of Section 3.1, we assume \( |b| \geq 1 \). To make a suitable choice for \( z^* \), we consider the following lemma.

**Lemma 3.1.** With the principal cut, there exists \( \theta^* \neq k\pi, k \in \mathbb{Z} \), such that

\[
|b| + \sqrt{b^2 + 4a^2 - 4ae^{i\theta^*}} \geq |2a - 2e^{i\theta^*}|.
\]

**Proof.** We note that \( b^2 + 4a^2 \geq 4|a| \) since \( 4|a|(1 - |a|) \leq 1 \leq |b| \). Thus with the principle cut, the function

\[
f(z) := \frac{|b| + \sqrt{b^2 + 4a^2 - 4az}}{2a - 2z}
\]
is meromorphic on the open unit ball with the possible pole at \( z = a \) if \( |a| < 1 \). To prove this lemma, we will find \( z \notin \mathbb{R} \) and \( |z| = 1 \) such that \( |f(z)| \geq 1 \).

We note that if \( |a| \geq 1 \), then \( f(z) \) is analytic on the unit ball and
\[
|f(0)| = \frac{|b| + \sqrt{b^2 + 4a^2}}{2|a|} > 1.
\]

Thus by the maximum modulus principle \( |f(z)| > 1 \) for some \( |z| = 1 \). We can choose such \( z \notin \mathbb{R} \) by the continuity of \( f(z) \).

On the other hand if \( |a| < 1 \), then the Cauchy integral formula implies that
\[
\oint_{|z|=1} |f(z)||dz| \geq \oint_{|z|=1} f(z)dz = 2\pi|b| \geq 2\pi.
\]
Consequently \( |f(z)| > 1 \) for some \( |z| = 1 \) or \( |f(z)| = 1 \) for all \( |z| = 1 \) and the lemma follows. \( \square \)

We now define
\[
z^* = \frac{-2ab + be^{i\theta^*} + \text{sign}(b)e^{i\theta^*}\sqrt{b^2 + 4a^2} - 4ae^{i\theta^*}}{2a^2 - 2ae^{i\theta^*}}
\]
where \( \theta^* \) is given in Lemma 3.1. With this definition, \( z^* \) is a solution of
\[
(a^2 - ae^{i\theta^*})z^2 + (2ab - be^{i\theta^*})z + b^2 - e^{2i\theta^*} = 0
\]
from which we deduce that
\[
-\frac{e^{i\theta^*}}{a}z^* + b = -\frac{2a - 2e^{i\theta^*}}{-b + \text{sign}(b)\sqrt{b^2 + 4a^2} - 4ae^{i\theta^*}}
\]
is a zero in \( t \) of
\[-t^2 + z^*t + 1.
\]
The modulus of 3.2 is the same as the modulus of the zero in \( t \) of \((az^* + b)t + 1\) which is at most 1 by the definition of \( \theta^* \). This modulus is larger than the modulus of the other zero of \(-t^2 + z^*t + 1\) since the product of two zeros of this polynomial is \(-1\). We finish the proof of Theorem 3.1 by noting that \( z^* \notin \mathbb{R} \) since the two zeros of \(-t^2 + z^*t + 1\) are neither real nor complex conjugate. \( \square \)

**Proof of Theorem 3.2** Sufficiency Let \( \{P_n(z)\} \) be the sequence of polynomials defined in (3.1) where \( |b| \leq 1 - 2|a| \). The proof of the following lemma is the same as that of Lemma 4 in [4]. For brevity, we omit the proof in this paper.

**Lemma 3.2.** For each \( b \in [-1,1] \), let \( S_b \) be a dense subset of
\[
\left[ \frac{|b| - 1 - |b|}{2}, \frac{1 - |b|}{2} \right]
\]
and \( n \in \mathbb{N} \) be fixed. If for any \( a \in S_b \), the zeros of \( P_n(z) \) lie on \((-2,2)\), then the same conclusion holds for any \( a \in (3.3) \).
Suppose $|b| \leq 1 - 2|a|$. From Lemma 3.2 it suffices to consider $a \neq 0$. We define the monotone function $z(\theta) = -2 \cos \theta$ on $(0, \pi)$ and note that for each $\theta \in (0, \pi)$ the two zeros of $t^2 + z(\theta)t + 1$ are $e^{\pm i\theta}$. We consider the function

$$t_0(\theta) = \frac{-1}{az(\theta) + b}, \quad \theta \in (0, \pi),$$

which has a vertical asymptote at $\theta = \cos^{-1}(b/2a)$ if $|b| < 2|a|$. For any $\theta \in (0, \pi)$ such that $2a \cos \theta \neq b$, the Cauchy differentiation formula gives

$$P_n(z(\theta)) = \frac{1}{az(\theta) + b} \int_{|t|=0} dt \frac{dt}{(t - t_0(\theta))(t - e^{i\theta})(t - e^{-i\theta})t+1}.$$

After computing the residue of the integrand at the three non-zero simple poles $t_0(\theta)$, $e^{\pm i\theta}$, and letting the radius of the integral approach infinity, we apply similar computations in (2.8) to conclude that $\theta \in (0, \pi)$, $2a \cos \theta \neq b$, is a zero of $P_n(z(\theta))$ if and only if it is a zero of

$$(3.4) \quad \frac{-1}{t_0(\theta)^{n+1}} + \cos((n+1)\theta) + \frac{(\cos \theta - t_0(\theta)) \sin((n+1)\theta)}{\sin \theta}.$$ 

From Lemma 3.2 it suffices to consider $|b| \neq 2|a|$. We note that the limits of (3.4) as $\theta \to 0$ and $\theta \to \pi$ are

$$(3.5) \quad n + 2 + \frac{n + 1}{b - 2a} + (-1)^n (b - 2a)^{n+1},$$

$$(3.6) \quad (-1)^{n+1} (n + 2) + (-1)^n \left( \frac{n + 1}{b + 2a} + (b + 2a)^{n+1} \right)$$

respectively.

In the case $|b| > 2|a|$, (3.4) is a continuous function of $\theta$ on $(0, \pi)$ and its sign at $\theta = k\pi/(n+1)$, for $1 \leq k \leq n$, is $(-1)^k$ since

$$|t_0(\theta)| > \frac{1}{2|a| + |b|} \geq 1.$$ 

By the intermediate value theorem, we obtain at least $n - 1$ zeros of (3.4) on $(\pi/(n+1), n\pi/(n+1))$. If $b > 0$, then (3.5) is positive since $0 < b - 2a \leq 1$ and we obtain at least another zero of (3.4) on $(0, \pi/(n+1))$. On the other hand, if $b < 0$, then the inequalities

$$-1 < b + 2a < 0$$

imply that the sign of (3.6) is $(-1)^{n+1}$ and we have at least another zero of (3.4) on $(n\pi/(n+1), \pi)$. We conclude that when $|b| > 2|a|$, (3.4) has at least $n$ zeros on $(0, \pi)$, each of which yields a zero of $P_n(z)$ on the interval $(-2, 2)$ by the map $z(\theta)$. Thus all the zeros of $P_n(z)$ lie on $(-2, 2)$ by the fundamental theorem of algebra.

We now consider the case $|b| < 2|a|$. As a function of $\theta$ on $(0, \pi)$, (3.4) has a vertical asymptote at $\theta = \cos^{-1}(b/2a)$ since $t_0(\theta)$ does. By Lemma 3.2 we can assume

$$\cos^{-1} \frac{b}{2a} \neq \frac{k\pi}{n+1}, \quad 1 \leq k \leq n.$$
Thus for some $0 \leq k_0 \leq n$, the open interval
\begin{equation}
\left( \frac{k_0}{n+1}, \frac{k_0+1}{n+1} \pi \right)
\end{equation}
contains $\cos^{-1}(b/2a)$. We note that this interval may or may not contain a zero of (3.4). In the case $a < 0$, we observe that (3.5) is positive and the sign of (3.6) is $(-1)^{n+1}$. Thus there are at least $n$ zeros of (3.4) on the $n$ intervals
\[(k\pi/(n+1), (k+1)\pi/(n+1)), \text{ for } 0 \leq k \leq n \text{ and } k \neq k_0\]
and we conclude that all the zeros of $P_n(z)$ lie on $(-2, 2)$ by the same argument as in the previous case. On the other hand, if $a > 0$, then the limits (3.4) as $\theta$ approaches the left and right of $\cos^{-1}(b/2a)$ are
\[
\lim_{\theta \to \cos^{-1}(b/2a)-} \frac{-\sin((n+1)\theta)}{b-2a\cos(\theta)} = (-1)^{k_0+1}\infty
\]
and
\[
\lim_{\theta \to \cos^{-1}(b/2a)+} \frac{-\sin((n+1)\theta)}{b-2a\cos(\theta)} = (-1)^{k_0}\infty,
\]
respectively. If $k_0 \neq 0$ and $k_0 \neq n$, then we conclude that (3.7) contains at least two zeros of (3.4). Thus we obtain at least $n$ zeros of this expression on the $n-1$ intervals $(k\pi/(n+1), (k+1)\pi/(n+1))$, for $1 \leq k < n$. In the case $k_0 = 0$ or $k_0 = n$, (3.7) contains at least one zero of (3.4) and thus there are at least $n$ zeros of (3.4) on the $n$ intervals $(k\pi/(n+1), (k+1)\pi/(n+1))$, for $1 \leq k < n$ and $k = k_0$.

**Necessity** Here we assume $|b| + 2|a| > 1$ and show that not all zeros of $P_n(z)$ defined in (3.1) are real when $n$ is large. From [1, Theorem 1.5], it suffices to find $z \notin \mathbb{R}$ so that $|t_0| = |t_1| \leq |t_2|$ where
\begin{equation}
t_0 := -\frac{1}{az + b}
\end{equation}
and $t_1$ and $t_2$ are the two zeros of $1 + zt + t^2$. To motivate the choice of $z$, we provide heuristic arguments by noticing that $t_1t_2 = 1$ and letting
\begin{equation}
t_1 = t_0 e^{i\theta} = -\frac{e^{i\theta}}{az + b}
\end{equation}
\begin{equation}
t_2 = -e^{-i\theta}(az + b).
\end{equation}
The equation $1 + zt_2 + t_2^2 = 0$ yields
\[(az + b)^2 - ze^{i\theta}(az + b) + e^{2i\theta} = 0
\]
or equivalently
\begin{equation}
(a^2 - ae^{i\theta})z^2 + (2ab - be^{i\theta})z + b^2 + e^{2i\theta} = 0.
\end{equation}
With a choice of branch cut which will be specified later, the equation above has two solutions
\[z = \frac{-2ab + be^{i\theta} \pm e^{i\theta} \sqrt{b^2 - 4a^2 + 4ae^{i\theta}}}{2a^2 - 2ae^{i\theta}}
\]
and the corresponding values for $az + b$ are
For a formal proof of the necessary condition, we consider the following cases.

Case 1: $|a| \leq 1$. We have the inequality

$$b^2 - 4a^2 + 4|a| - (|b| + 2|a| - 2)^2 = 4(1 - |a|)(2|a| + |b| - 1) \geq 0.$$  

with equality if and only if $|a| = 1$. This implies

(3.13)  
$$b^2 - 4a^2 + 4|a| \geq 0,$$

(3.14)  
$$\sqrt{b^2 - 4a^2 + 4|a|} + |b| \geq |b| + 2|a| - 2 + |b| \geq 2|a| - 2$$

with equality if and only if $|a| = 1$ and $b = 0$. We define $\theta \in (0, \pi)$ sufficiently close to 0 or $\pi$ such that $e^{i\theta}$ is close to $\text{sign}(a)$ if $a \neq 0$. If $a = 0$, we pick any $\theta \in (0, \pi)$.

With this choice of $\theta$ and the principal cut, we let

(3.15)  
$$z = \begin{cases} 
\frac{-2ab + b e^{i\theta} - \text{sign}(b) e^{i\theta} \sqrt{b^2 - 4a^2 + 4ae^{i\theta}}}{2a^2 - 4ae^{i\theta}} & \text{if } ab \neq 0, \\
\sqrt{a^2 - 4ae^{i\theta}} & \text{if } b = 0, \\
\sqrt{e^{2i\theta}} & \text{if } a = 0.
\end{cases}$$

With this choice of $z$, (3.11) holds and consequently $t_1$ and $t_2$ defined in (3.9) and (3.10) are the zeros of $1 + za + t^2$. If $a = 0$, then

(3.16)  
$$|t_0| = |t_1| < |t_2|$$

since $|b| > 1$. If $b = 0$ then the inequalities $|a| \leq 1$ and (3.13) imply that $|a| = 1$.  

As a consequence, (3.16) follows from (3.8), (3.9), (3.10), and (3.15). Finally, if $ab \neq 0$, then from (3.12) and (3.14), we conclude $|az + b|$ approaches

$$\frac{|b| + \sqrt{b^2 - 4a^2 + 4|a|}}{2 - 2|a|} > 1$$

as $e^{i\theta} \rightarrow \text{sign}(a)$. Thus from (3.9) and (3.10) there is $\theta \in (0, \pi)$ sufficiently close to 0 or $\pi$ such that $|t_0| = |t_1| < |t_2|$. We also note that $z \notin \mathbb{R}$ since if $z \in \mathbb{R}$, then the fact that $t_1,t_2 \notin \mathbb{R}$ by (3.9) and (3.10) implies $t_1 = t_2$ which contradicts to $|t_1| < |t_2|$.

Case 2: $|a| > 1$ and $|b| < 1$. By the intermediate value theorem there is $y \in (0, \infty)$ such that

$$2\sqrt{a^2y^2 + b^2} - \sqrt{y^2 + 4} - y = 0$$

since the left-hand side is $2|b| - 2 < 0$ when $y = 0$ and its limit is $\infty$ when $y \rightarrow \infty$.

With the choice $z = iy$, we have

$$|t_0| = \frac{1}{|az + b|} = \frac{1}{\sqrt{a^2y^2 + b^2}}$$

and the modulus of the smaller zero of $t^2 + iyt + 1$ is

$$\frac{\sqrt{y^2 + 4} - y}{2} = \frac{2}{\sqrt{y^2 + 4} + y} = |t_0|.$$
Case 3: $|b| \geq 1$ and $|a| > 1$. If $2 + |b| > 2|a|$, then with the same choice of $\theta$ and $z$ and the same argument as in the first case, this case follows from

$$|\sqrt{b^2 - 4a^2 + 4ae^{i\theta}} + |b|| > |b| > 2|a| - 2.$$ 

We now consider $2 + |b| \leq 2|a|$. We square both sides of $2|a| - 2 \geq |b|$ to obtain

$$b^2 - 4a^2 \leq 4 - 8|a| < -4|a|$$

which implies that, with the cut $[0, \infty)$, the function

$$f(z) := \frac{-b + \sqrt{b^2 - 4a^2 + 4az}}{2a - 2z}$$

is analytic on a small region containing the closed unit ball. From the maximum modulus principle and the fact that

$$|f(0)| = \frac{|-b + \sqrt{b^2 - 4a^2}|}{|2a|} = 1,$$

we conclude that there is $\theta \in \mathbb{R}$ so that $|f(e^{i\theta})| > 1$. With this $\theta$, we let

$$z = \frac{-2ab + be^{i\theta} + e^{i\theta}\sqrt{b^2 - 4a^2 + 4ae^{i\theta}}}{2a^2 - 2ae^{i\theta}}$$

and apply (3.8), (3.9), (3.10), and (3.12) to conclude $|t_0| = |t_1| < |t_2|$. The fact that $z \not\in \mathbb{R}$ follows from the same argument as in the previous case. \[\Box]\n
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Department of Mathematics
California State University, Fresno
California
U.S.A.
khant@mail.fresnostate.edu

Department of Mathematics
University of California, Berkeley
California
U.S.A.
maverickzhang@berkeley.edu