THE FUNDAMENTAL $\infty$-GROUPOID OF A PARAMETRIZED FAMILY

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Abstract. Given an $\infty$-category $\mathcal{C}$, one can naturally construct an $\infty$-category $\text{Fam}(\mathcal{C})$ of families of objects in $\mathcal{C}$ indexed by $\infty$-groupoids. An ordinary categorical version of this construction was used by Borceux and Janelidze in the study of generalized covering maps in categorical Galois theory. In this paper, we develop the homotopy theory of such "parametrized families" as generalization of the classical homotopy theory of spaces. In particular, we study homotopy-theoretical constructions that arise from the fundamental $\infty$-groupoids of families in an $\infty$-category. In the same spirit, we show that $\text{Fam}(\mathcal{C})$ admits a Grothendieck topology which generalizes Carchedi's canonical/epimorphism topology on certain $\infty$-topoi.

1. Introduction

1.1. Motivation. In their generalization of classical Galois theory in [2], Borceux and Janelidze draw connections between the coproduct completion of categories and the theory of locally connected topological spaces. The coproduct completion $\text{Fam}(\mathcal{C})$ of a category $\mathcal{C}$ is the category whose objects are families of objects in $\mathcal{C}$ parameterized by sets and whose morphisms are maps between members of the families in question induced by functions on the indexing sets. As described in [2], every category of the form $\text{Fam}(\mathcal{C})$ is equipped with a "family fibration" functor $\pi_0 : \text{Fam}(\mathcal{C}) \to \text{Set}$, which sends each family to its indexing set. Geometrically, $\pi_0$ is a generalization of the usual connected components functor for topological spaces, since a family of objects in $\mathcal{C}$ can be viewed as a "disjoint union" of its members - the members being the "connected components" of some sort of generalized space. When $\mathcal{C}$ is instead a $(2,1)$-category, its $\text{Fam}(\mathcal{C})$ is the same as the $1$-categorical case except that the families are parametrized by groupoids instead of sets (and is a colimit completion with respect to groupoid-indexed diagrams).

The family fibration functor is $\text{Grpd}$-valued and can be viewed as an analog of the fundamental groupoid of a topological space. This trend continues for $(n,1)$-categories as $n \to \infty$, so it is natural to expect that one can define the "fundamental $\infty$-groupoid" of parametrized families in an $\infty$-category. This would provide an $\infty$-categorical generalization of the analogy between families of objects in categories and locally connected spaces studied by Borceux and Janelidze - and thus lead to a non-trivial homotopy theory of parametrized families in $\infty$-categories.

1.2. Goal and Outline. The goal of this paper is to develop the homotopy theory of families in $\infty$-categories as an extension of the homotopy theory of topological spaces. In §2, we will recall some background information about extensive categories and Grothendieck topologies on an $\infty$-category. In §3, we will introduce the $\infty$-category $\text{Fam}(\mathcal{C})$ of parametrized families in an $\infty$-category and prove general categorical results about $\text{Fam}(\mathcal{C})$ that we need in subsequent sections. In §4, we define and study the fundamental $\infty$-groupoids and fundamental groups of objects in $\text{Fam}(\mathcal{C})$. In §5, we use the construction of §4 to construct a Grothendieck topology on $\text{Fam}(\mathcal{C})$ which generalizes Carchedi's "canonical" topology on the $\infty$-topos of $\infty$-groupoids. In §6, we describe Joyal's notion of a(n) $(\infty,1)$-locus and how our results relate to it. We also state some tangential results of ours there. In the final section, we describe some avenues for future investigation.

1.3. Conventions. We will assume an understanding of basic topos theory and higher category theory. By an $n$-category, we mean an $(n,1)$-category. In particular, an $\infty$-category is a category enriched over $\infty$-groupoids/Kan complexes, i.e an $(\infty,1)$-category. $\text{Cat}_\infty$ (resp. $\text{Grpd}_\infty$) denotes the $(\infty,2)$-category of $\infty$-categories (resp. $\infty$-category of $\infty$-groupoids). A topos always means a Grothendieck/sheaf topos.

2. Background

2.1. Extensive $\infty$-categories. We define extensive $\infty$-categories as a slightly weakened version of Barwick's disjunctive $\infty$-categories defined in Section 4 of [1]. They are identical except that we don't require them to be closed under finite limits.

Definition 2.1. Let $\mathcal{C}$ be an $\infty$-category. $\mathcal{C}$ is extensive if, for an arbitrary collection $\{X_i\}_{i \in I}$ of objects in $\mathcal{C}$, the canonical coproduct functor $\coprod : \coprod_{i \in I} \mathcal{C}/X_i \to \mathcal{C}/\coprod_{i \in I} X_i$ is a categorical equivalence.

Example 2.2. Any $\infty$-topos (e.g $\text{Grpd}_\infty$, $[X^{op}, \text{Grpd}_\infty]$ for a small category $X$, $\text{Shv}_\infty(S)$ for an $\infty$-site $S$, etc.) is extensive.

Definition 2.3. Let $\mathcal{C}$ be an $\infty$-category. An object $X \in \mathcal{C}$ is connected if $\text{Hom}_\mathcal{C}(X, -) : \mathcal{C} \to \text{Grpd}_\infty$ preserves all coproducts.
Remark 2.4. There is a simpler description of connected objects in extensive $\infty$-categories. Namely, if $\mathcal{C}$ is an extensive $\infty$-category, then an object $X$ is connected if and only if for any coproduct decomposition $X = X_1 \coprod X_2$, exactly one of the $X_i$ is not initial.

Example 2.5. A topological space/$\infty$-groupoid is connected (categorically) precisely if it connected in the usual sense. In the 1-truncated case, an object in Set is connected if and only if it is a singleton. Additionally, a scheme is a connected object in the category of schemes if and only if it is a connected scheme, i.e. its underlying topological space is connected.

2.2. Grothendieck Topologies. Let $\mathcal{C}$ be an $\infty$-category. A Grothendieck topology on $\mathcal{C}$ allows us to treat objects of $\mathcal{C}$ like open sets of a topological space. In this subsection, we will briefly review some key ideas relevant to Grothendieck topologies.

Definition 2.6. A covering of an object $X \in \mathcal{C}$ is a set of maps $\{f_i : X_i \rightarrow X\}_{i \in I}$ with a common codomain $X$ that satisfy the following conditions:

- If $X' \rightarrow X$ is an equivalence, then the singleton set $\{X' \rightarrow X\}$ is a covering.
- If $\{f_i : X_i \rightarrow X\}_{i \in I}$ is a covering and $g : Y \rightarrow X$ is a map in $\mathcal{C}$, then the pullbacks $X_i \times_X Y \rightarrow Y$ exist for each $i \in I$ and $\{X_i \times_X Y \rightarrow Y\}_{i \in I}$ is a covering of $Y$.
- If $\{f_i : X_i \rightarrow X\}_{i \in I}$ is a covering and each $X_i$ is equipped with a covering $\{f_{ij} : X_{ij} \rightarrow X_i\}_{j \in I}$, then the composite family $\{f_{ij} \circ f_{ij} : X_{ij} \rightarrow X\}$ covers $X$.

Definition 2.7. A Grothendieck topology $\tau$ on an $\infty$-category $\mathcal{C}$ is an assignment of coverings $\{f_i : X_i \rightarrow X\}_{i \in I}$ to each object $X \in \mathcal{C}$. An $\infty$-category equipped with a Grothendieck topology is a (Grothendieck) $\infty$-site. We will suppress the “$\infty$-” when it is clear that we are in the $\infty$-categorical context. If $\tau$ is a topology on $\mathcal{C}$, then we denote the associated site by $(\mathcal{C}, \tau)$ unless the context is clear.

Remark 2.8. Note that the above definition is often called a Grothendieck pretopology - this acts as a “basis” for a Grothendieck topology on an $\infty$-category.

Example 2.9. Let CartSp denote the category of smooth manifolds of the form $\mathbb{R}^n$ for $n \in \mathbb{N}$ and smooth functions. There is a natural topology on CartSp whose covering families are the usual open covers.

Example 2.10. Let $\mathbb{H}$ be a 1-topos. There is a Grothendieck topology, namely the canonical topology, on $\mathbb{H}$ whose covering families are families $\{f_i : X_i \rightarrow X\}_{i \in I}$ that are jointly epimorphic (the induced map $\coprod_{i \in I} X_i \rightarrow X$ is an epimorphism). There is an $\infty$-toposic refinement of this notion, described in [3, Definition 2.2.5]. Namely, there is a Grothendieck topology on any $\infty$-topos $\mathbb{H}$ whose covering families are (generated by) sets of maps $\{X_i \rightarrow X\}_i$ such that the induced map $\coprod_{i \in I} X_i \rightarrow X$ is an effective epimorphism (see Definitions 2.11 and 2.12 and Example 2.13). This is known as the epimorphism topology.

Definition 2.11. Let $\mathcal{C}$ be an $\infty$-category with pullbacks. The Čech nerve of a map $f : X' \rightarrow X$ is the simplicial object $\check{C}(f)_\bullet : \Delta^{op} \rightarrow \mathcal{C}$ sending $[k]$ to the $k$-fold fiber product $X' \times_X X' \times_X \ldots \times_X X'$ of $X'$ with itself.

Definition 2.12. Let $f : X \rightarrow Y$ be a map in an $\infty$-category $\mathcal{C}$ such that $\check{C}(f)_\bullet$ exists ($\mathcal{C}$ being closed under pullbacks ensures this). Let $\Delta_*$ denote the augmented simplex category. We can construct an augmented simplicial object $\check{C}(f)_\bullet : \Delta^{op} \rightarrow \mathcal{C}$ out of $\check{C}(f)_\bullet$ by attaching $Y$ to $\check{C}(f)_\bullet$ via $f$, i.e. by setting $d^{-1} = f$ and $\check{C}(f)_{[-1]} = Y$. $f$ is an effective epimorphism if $\check{C}(f)_\bullet$ exhibits $Y$ as the colimit of $\check{C}(f)_\bullet$.

Example 2.13 (6, Corollary 7.2.1.15). Let $f : X \rightarrow Y$ be a map of $\infty$-groupoids. $f$ is an effective epimorphism precisely if the induced function $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ is surjective.

The following lemma is immediate from definitions.

Lemma 2.14. Let $\mathcal{C}$ and $\mathcal{D}$ be $\infty$-categories with pullbacks and let $\varphi : X \rightarrow Y$ be an effective epimorphism in $\mathcal{C}$. Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a cocontinuous functor. Then $F(\varphi)$ is an effective epimorphism in $\mathcal{D}$.

3. Parametrized Objects in $\infty$-categories

In this section, we describe the main object of study in this paper: the $\infty$-category of parametrized families of objects in an $\infty$-category. We also develop some general categorical results that we use in subsequent sections.

Definition 3.1. Let $\mathcal{C}$ be an $\infty$-category. Define another $\infty$-category $\text{Fam}(\mathcal{C})$ as follows. The objects of $\text{Fam}(\mathcal{C})$ are pairs $(X, F)$, where $X$ is a small $\infty$-groupoid and $F : X \rightarrow \mathcal{C}$ is a functor. A map $(X_1, F_1) \rightarrow (X_2, F_2)$ is a pair $(\varphi, \varphi_*)$, where $\varphi : X_1 \rightarrow X_2$ is a functor and $\varphi_*$ is a natural transformation $X \xrightarrow{\varphi} F_1 \xrightarrow{\varphi_*} F_2$.

One can think of an object of $\text{Fam}(\mathcal{C})$ as a set of objects in $\mathcal{C}$ “parametrized” by some $\infty$-groupoid/homotopy type $X$.

Example 3.2. Let $\ast$ denote the terminal $\infty$-groupoid. There is an equivalence of categories $\text{Fam}(\ast) \simeq \text{Grpd}_\infty$. This is because we can identify $\text{Fam}(\ast)$ with the slice category $(\text{Grpd}_\infty)_{/\ast}$, which is equivalent to $\text{Grpd}_\infty$. 


Example 3.3. If $X$ happens to be a groupoid of the form $BG$ for a group $G$, then any object $(X, F) \in \text{Fam}(\mathcal{C})$ is just an object of $\mathcal{C}$ equipped with a $G$-action.

Remark 3.4. For any $\infty$-category $\mathcal{C}$, $\text{Fam}(\mathcal{C})$ is an extensive $\infty$-category.

Terminology 3.5. For $(X, F) \in \text{Fam}(\mathcal{C})$, we will refer to $X$ as the shape of $(X, F)$ and $F$ as the arrow of $(X, F)$.

Remark 3.6. There is a fully faithful, left-exact embedding $\sigma : \mathcal{C} \hookrightarrow \text{Fam}(\mathcal{C})$ sending $\mu$ to the family $(\ast, \gamma)$, where $\gamma : \ast \to \mathcal{C}$ is the functor that picks out $X \in \mathcal{C}$.

Remark 3.7. The $\text{Fam}(-)$ construction extends to an $(\infty, 2)$-endofunctor $\text{Fam}(-) : \text{Cat}_\infty \to \text{Cat}_\infty$.

Proposition 3.8. $\text{Fam}(\mathcal{C})$ is the universal colimit completion of $\mathcal{C}$ with respect to diagrams indexed by $\infty$-groupoids. More precisely:

- Any functor $D : K \to \text{Fam}(\mathcal{C})$ where $K$ is an $\infty$-groupoid has a colimit in $\text{Fam}(\mathcal{C})$.
- Let $\mathcal{D}$ be an $\infty$-category with all $\text{Grpd}_\infty$-indexed colimits and denote by $[\text{Fam}(\mathcal{C}), \mathcal{D}]_\ast$, the full subcategory of $[\text{Fam}(\mathcal{C}), \mathcal{D}]$ spanned by functors which preserve $\text{Grpd}_\infty$-indexed colimits. Then there is an equivalence of categories:

$$[\mathcal{C}, \mathcal{D}] \cong [\text{Fam}(\mathcal{C}), \mathcal{D}]_\ast$$

Remark 3.9. Generally, if $\mathcal{C}$ is an $n$-category, then $\text{Fam}(\mathcal{C})$ is the universal colimit completion of $\mathcal{C}$ with respect to diagrams indexed by $(n - 1)$-groupoids. The construction is exactly the same except $\text{Fam}(\mathcal{C})$ has objects $(X, F)$ in which $X$ is an $(n - 1)$-groupoid and $F : X \to \mathcal{C}$ is a functor. For example if $\mathcal{C}$ is an ordinary category, then $\text{Fam}(\mathcal{C})$ is its coproduct completion.

The following proposition reflects a general principle of colimit completions in $(\infty)$-categories: to form the universal completion of a category $\mathcal{C}$ under colimits of a certain shape, one takes something resembling the closure of representable $(\infty)$-presheaves on $\mathcal{C}$ under colimits of that shape.

Proposition 3.10. Let $[\mathcal{C}^{op}, \text{Grpd}_\infty]$ denote the full subcategory of $[\mathcal{C}^{op}, \text{Grpd}_\infty]$ spanned by colimits of representable $\infty$-presheaves indexed by $\infty$-groupoids and let $y : \mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \text{Grpd}_\infty]$ denote the Yoneda embedding. Then the functor

$$\text{Fam}(\mathcal{C}) \to [\mathcal{C}^{op}, \text{Grpd}_\infty]$$

that sends $(X, F) \mapsto \lim(y \circ F)$ induces an equivalence of categories.

We now give an explicit construction of (co)limits in $\text{Fam}(\mathcal{C})$. In this construction and in Lemmas 3.12 and 3.15 we will use some notation introduced in §4, in particular $\Pi_\infty$.

Construction of (co)limits in $\text{Fam}(\mathcal{C})$. Fix an $\infty$-category $K$ and let $D : K \to \text{Fam}(\mathcal{C})$ be a diagram. The shape $\Pi_\infty(\text{lim}(D))$ of $\text{lim}(D)$ (when it exists) is given by:

$$\Pi_\infty(\text{lim}(D)) = \text{lim}(K \xrightarrow{D} \text{Fam}(\mathcal{C}) \xrightarrow{\Pi_\infty} \text{Grpd}_\infty)$$

By definition, each $x \in \Pi_\infty \circ D(K)$ is equipped with a map $\Pi_\infty(D(x)) \to \mathcal{C}$ which commutes with the required triangles. Thus there is induced a functor:

$$\text{lim}(K \xrightarrow{D} \text{Fam}(\mathcal{C}) \xrightarrow{\Pi_\infty} \text{Grpd}_\infty) \to \mathcal{C}$$

This is precisely the arrow of $\text{lim}(D)$. Now we describe limits as before, let $D : K \to \text{Fam}(\mathcal{C})$ be a diagram. the shape $\Pi_\infty(\text{lim}(D))$ of $\text{lim}(D)$ is given by the limit:

$$\Pi_\infty(\text{lim}(D)) = \text{lim}(K \xrightarrow{D} \text{Fam}(\mathcal{C}) \xrightarrow{\Pi_\infty} \text{Grpd}_\infty)$$

For each $x \in K$, there is a canonical projection $p_x : \text{lim}(\Pi_\infty \circ D) \to \Pi_\infty \circ D(x)$. By definition, for such $x$ we have functors $\gamma_x : x \to \mathcal{C}$, so we can define a natural functor $\zeta : K \to [\text{lim}(\Pi_\infty \circ D), \mathcal{C}]$ by $\zeta(x) = \gamma_x \circ p_x : \text{lim}(\Pi_\infty \circ D) \to \mathcal{C}$. The arrow of $\text{lim}(D)$ is given by $\text{lim}(\zeta)$. □

Remark 3.11. Actually, the colimit in (3) must be taken in $\text{Cat}_\infty$ in order for the desired universal property to kick in ($\mathcal{C}$ is not necessarily an $\infty$-groupoid), but this does not affect anything.

Lemma 3.12. Fix an $\infty$-category $\mathcal{C}$ and a small $\infty$-category $\lambda$. Suppose limits indexed by $\lambda$ exist in $\mathcal{C}$. Then limits indexed by $\lambda$ exist in $\text{Fam}(\mathcal{C})$.

Proof. This follows from the fact that the limit of a functor $D : \lambda \to \text{Fam}(\mathcal{C})$ is computed as a combination of $\lambda$-indexed limits of $\infty$-groupoids (which are guaranteed to exist) and $\lambda$-indexed limits in the functor category $[\text{lim}(\Pi_\infty \circ D), \mathcal{C}]$. The claim holds since limits in $[\text{lim}(\Pi_\infty \circ D), \mathcal{C}]$ are computed point-wise as limits in $\mathcal{C}$. □

\footnote{Thanks to MathOverflow user Kyle弗伦多 for describing this to us.}
Definition 3.13. Let $\mathbb{H}$ be an $\infty$-category. Recall that an object $X \in \mathbb{H}$ is $n$-truncated if the mapping $\infty$-groupoids $\text{Hom}_\mathbb{H}(Y, X)$ are $n$-groupoids for all $Y \in \mathbb{H}$. The $n$-truncation functor $\tau_{\leq n} : \mathbb{H} \to \tau_{\leq n} \mathbb{H}$ is left adjoint to the full inclusion $\tau_{\leq n} \mathbb{H} \hookrightarrow \mathbb{H}$ of $n$-truncated objects in $\mathbb{H}$.

Proposition 3.14. Let $\mathcal{C}$ be an $\infty$-category. An object $(X, F) \in \text{Fam}(\mathcal{C})$ is 0-truncated if and only if $X$ is a discrete $\infty$-groupoid/set.

Lemma 3.15. Let $\mathcal{C}$ be an $\infty$-category. An object $(X, F) \in \text{Fam}(\mathcal{C})$ is connected if and only if $X$ is a connected $\infty$-groupoid.

Proof. Since $\text{Fam}(\mathcal{C})$ is extensive, this reduces to showing that in any coproduct decomposition $(X, F) \simeq (X_1, F_1) \coprod (X_2, F_2)$, exactly one of the $(X_i, F_i)$ is not initial if and only if $X$ is connected as an $\infty$-groupoid. But this is immediate since $\Pi_\infty((X_1, F_1) \coprod (X_2, F_2)) = X_1 \coprod X_2$.

This implies that any object in $\text{Fam}(\mathcal{C})$ can be written (essentially uniquely) as a coproduct of connected objects. Thus, we may regard $\text{Fam}(\mathcal{C})$ for $\mathcal{C}$ an $\infty$-category in much the same way as we treat the 1-categorical case (since $\text{Fam}(\mathcal{C})$ for $\mathcal{C}$ a 1-category is a coproduct completion, every object can be written as a coproduct of connected objects). This property of every object admitting coproduct decomposition into connected objects is of course shared by the category $\mathcal{T}_{\text{sp}}$ of topological spaces and continuous maps.

4. The Fundamental $\infty$-Groupoid of a Parametrized Family

In this section, we study phenomena pertaining to the homotopy theory of parametrized families in $\infty$-categories. In particular, we define the fundamental ($\infty$-)groupoid(s) of objects in $\text{Fam}(\mathcal{C})$.

Definition 4.1. Let $\mathcal{C}$ be an $\infty$-category and consider $\text{Fam}(\mathcal{C})$. The fundamental $\infty$-groupoid functor $\Pi_\infty : \text{Fam}(\mathcal{C}) \to \text{Grpd}_\infty$ is the functor sending $(X, F) \mapsto X$. Equivalently, it is the Grothendieck construction $\Pi_\infty \simeq \int[-, \mathcal{C}]$ of the representable prestack $[-, \mathcal{C}] : \text{Grpd}_\infty \to \text{Cat}_\infty$ that sends $X \mapsto [X, \mathcal{C}]$.

Remark 4.2. Let $\tau_{\leq 0} \text{Fam}(\mathcal{C})$ denote the full subcategory of $\text{Fam}(\mathcal{C})$ spanned by 0-truncated objects, i.e., families $(X, F)$ such that $X$ is a set. The inclusion $i : \tau_{\leq 0} \text{Fam}(\mathcal{C}) \subset \text{Fam}(\mathcal{C})$ induces a commutative square of $\infty$-categories:

\[
\begin{array}{ccc}
\text{Fam}(\mathcal{C}) & \xrightarrow{\Pi_\infty} & \text{Grpd}_\infty \\
\tau_{\leq 0} \text{Fam}(\mathcal{C}) & \xrightarrow{i} & \text{Set} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad...
We observe that Proposition 4.3 implies the following statement.

**Corollary 4.4.** Let \( \mathcal{C} \) be an \( \infty \)-category such that \( \text{Fam}(\mathcal{C}) \) is an \( \infty \)-topos (e.g. \( \mathcal{C} = \text{Grpd}^{+/\pi}, \text{Sp} \)). Then the full subcategory \( \tau_{\leq 0} \text{Fam}(\mathcal{C}) \) of \( \text{Fam}(\mathcal{C}) \) on 0-truncated objects is a locally connected 1-topos.

Let \( X \) be a topological space. Recall that its fundamental group \( \pi_1(X, x) \) at basepoint \( x \in X \) can be obtained by the automorphism group \( \text{Aut}_{\Pi_1(X)}(x) \) of \( x \) in its fundamental groupoid \( \Pi_1(X) \), where \( x \) is regarded as an object of \( \Pi_1(X) \). In the rest of this section, we will describe a natural construction of the fundamental groups of parametrized families in an \( \infty \)-category based on this perspective.

**Definition 4.5.** Let \( \mathcal{C} \) be an \( \infty \)-category. Define the fundamental groupoid functor \( \Pi_1 : \text{Fam}(\mathcal{C}) \to \text{Grpd} \) by \( \pi_1 = \tau_{\leq 1} \circ \Pi_\infty \).

**Proposition 4.6.** Let \( \mathcal{C} \) be an \( \infty \)-category. By abuse of notation, denote by \( * \) both the terminal object of \( \mathcal{C} \) and the terminal \( (\infty-) \)groupoid. Then \( \Pi_1 \) induces a functor
\[
\Pi_1^+ : \text{Fam}(\mathcal{C})^{+/\pi} \to \text{Grpd}^{+/\pi}
\]
on categories of pointed objects, i.e \( \Pi_1 \) takes pointed families in \( \mathcal{C} \) to pointed groupoids. Furthermore, \( \Pi_1^+ \) preserves small colimits.

**Proof.** Fix a pointed object \( * \xrightarrow{x} (X, F) \) in \( \text{Fam}(\mathcal{C}) \). That \( \Pi_1 \) induces a functor on categories of pointed objects is immediate from the observation that both \( \Pi_\infty \) and \( \tau_{\leq 1} \) preserve terminal objects, hence \( \Pi_1 = \tau_{\leq 1} \circ \Pi_\infty \) induces a map \( * \xrightarrow{\Pi_1(x)} \Pi_1(X, F) \).

From the construction of colimits in \( \text{Fam}(\mathcal{C}) \), it is clear that \( \Pi_\infty \) preserves them. Since \( \tau_{\leq 1} \) is a left adjoint, it also preserves colimits. By definition, for \( K \) a small \( \infty \)-category and \( D : K \to \text{Fam}(\mathcal{C})^{+/\pi} \) a diagram, we have \( \lim(D) = \lim(U \circ D) \coprod * \xrightarrow{*} * \), where \( U : \text{Fam}(\mathcal{C})^{+/\pi} \to \text{Fam}(\mathcal{C}) \) is the canonical projection. But since the composite \( \Pi_1 = \tau_{\leq 1} \circ \Pi_\infty \) preserves colimits and terminal objects, we have:
\[
\Pi_1(\lim(D)) = \Pi_1(\lim(U \circ D) \coprod * \xrightarrow{*} * )
\]
\[
\simeq \Pi_1(\lim(U \circ D)) \coprod * \xrightarrow{*} *
\]
\[
\simeq \lim(U \circ \Pi_1 \circ D) \coprod * \xrightarrow{*} *
\]
Coupling the fact that \( \Pi_1 \) evidently commutes with \( U \) with the definition of colimits in \( \text{Fam}(\mathcal{C})^{+/\pi} \), we get:
\[
\left( (\lim(U \circ \Pi_1 \circ D) \coprod * \xrightarrow{*} \right) \simeq \lim(\Pi_1 \circ D)
\]
Thus the proposition follows.

**Definition 4.7.** The fundamental group \( \pi_1((X, F), x) \) at \( x \in (X, F) \) of a pointed family \( * \xrightarrow{x} (X, F) \in \text{Fam}(\mathcal{C})^{+/\pi} \) is the automorphism group \( \pi_1((X, F), x) = \text{Aut}_{\tau_{\leq 1}(\Pi_\infty(X, F))}(x) \). This extends to a functor \( \pi_1 : \text{Fam}(\mathcal{C})^{+/\pi} \to \text{Grpd} \). Equivalently, it is the based fundamental group of the pointed \( \infty \)-groupoid \( (\Pi_\infty(X, F), x) \) regarded as a pointed topological space under the homotopy hypothesis.

**Remark 4.8.** \( \pi_1((X, F), x) \) can also be computed as the first based simplicial homotopy group \( \pi_1(N(\Pi_1(X, F)), x) \) of the pointed Kan complex \( N(\Pi_1(X, F)) \) at \( x \in N(\Pi_1(X, F)) \).

**Remark 4.9.** Via the equivalence \( \text{Fam}(*) \simeq \text{Grpd}_\infty \), the \( \pi_1 \) construction of Definition 4.7 recovers the classical fundamental group of a topological space.

**Proposition 4.10.** Let \( \mathcal{C} \) be an \( \infty \)-category and fix a connected object \( (X, F) \in \text{Fam}(\mathcal{C}) \). Let \( (x_0, \phi_0), (x_1, \phi_1) : * \to (X, F) \) be two basepoints in \( (X, F) \). Then there is a canonical isomorphism of groups \( \pi_1((X, F), x_0) \xrightarrow{\simeq} \pi_1((X, F), x_1) \).

**Proof.** By Lemma 3.13, \( X \) (and hence \( \tau_{\leq 1}(X) \)) is a connected \( \infty \)-groupoid. Since \( \tau_{\leq 1}(X) \) is connected, there are canonical equivalences of groupoids
\[
\text{BAut}_{\tau_{\leq 1}(X)}(x_0(*)) \xrightarrow{\simeq} \tau_{\leq 1}(X) \xleftarrow{\simeq} \text{BAut}_{\tau_{\leq 1}(X)}(x_1(*))
\]
Since \( \text{BAut}_{\tau_{\leq 1}(X)}(x_0(*)) \) and \( \text{BAut}_{\tau_{\leq 1}(X)}(x_1(*)) \) are equivalent and have one object each, they must be isomorphic. The proposition follows from the fact that \( B \) is fully faithful and hence reflects isomorphisms.
5. A Grothendieck Topology on Families in an ∞-category

Given an ∞-category C, we can treat objects in Fam(C) as “spaces” inside of C. Thus, it is natural to ask for an appropriate notion of an open covering of a family (X, F) ∈ Fam(C). This can be accomplished by defining a Grothendieck topology on Fam(C). In this section, we endow Fam(C) the structure of a site based on Carchedi’s epimorphism topology on ∞-topoi (Example 2.10). We start with a definition.

**Definition 5.1.** Let I be a set and let ∏ be an ∞-topos. A family of maps of \( \{X_i \xrightarrow{f_i} X\}_{i \in I} \) in ∏ is an effective epimorphic family if the induced map \( \coprod_{i \in I} f_i : \coprod_{i \in I} X_i \to X \) is an effective epimorphism.

The following theorem asserts that a form of the epimorphism topology (See Example 2.10) on Grpd∞ holds in the context of families in an ∞-category.

**Theorem 5.2.** Let C be an ∞-category with pullbacks and let \( (X, F) \) be an object of Fam(C). Define a family of maps \( \{ (X_i, F_i) \xrightarrow{(f_i, F_i)} (X, F) \}_{i \in I} \) with codomain \( (X, F) \) to be a covering family if the induced family

\[
\{ \Pi_\infty(X_i, F_i) \xrightarrow{\Pi_\infty(f_i, F_i)} \Pi_\infty(X, F) \}_{i \in I}
\]

(19)

\[
= \{ X_i \xrightarrow{f_i} X \}_{i \in I}
\]

(20)

is an effective epimorphic family of ∞-groupoids. These covering families define a Grothendieck topology on Fam(C).

**Proof.** Clearly, any equivalence \( \{(\gamma, \gamma_i) : (X', F') \xrightarrow{\sim} (X, F)\} \) is a covering since it must hold that the underlying map of ∞-groupoids \( \{(\gamma) : X' \to X\} \) is also an equivalence. Coupling Lemma 5.12 with the assumption that C has pullbacks implies that for any covering family \( \{(X_i, F_i) \xrightarrow{(f_i, F_i)} (X, F)\}_{i \in I} \) and a map \( (g, \phi) : (X', F') \to (X, F) \), there exists a pullback square:

\[
\begin{array}{ccc}
\coprod_{i \in I}(X_i, F_i) \times_{(X, F)} (X', F') & \xrightarrow{(g, \phi)} & (X', F') \\
\downarrow & & \downarrow \\
\coprod_{i \in I}X_i \xrightarrow{\coprod_{i \in I}f_i} X & \xrightarrow{g} & (X, F)
\end{array}
\]

in Fam(C) for each \( i \in I \). We claim that \( \{(X_i, F_i) \times_{(X, F)} (X', F') \to (X', F')\}_{i \in I} \) is a covering family, i.e that the induced map \( \coprod_{i \in I}(X_i \times X' X') \to X' \) is an effective epimorphism. \( \Pi_\infty \) evidently preserves small (co)limits, so applying \( \Pi_\infty \) induces a pullback square of the underlying ∞-groupoids:

\[
\begin{array}{ccc}
\coprod_{i \in I}X_i \times_{X} X' & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \\
\coprod_{i \in I}X_i & \xrightarrow{\coprod_{i \in I}f_i} & X
\end{array}
\]

for each \( i \in I \). By assumption, the bottom horizontal map is an effective epimorphism, so the top horizontal map is also an effective epimorphism by [6, Proposition 6.2.3.15]. The claim then follows from the fact that coproducts of ∞-groupoids are universal, so that \( \coprod_{i \in I}X_i \times_{X} X' \simeq \coprod_{i \in I}(X_i \times X' X') \) is an effective epimorphism. That covering families are stable under composition is clear from [6, Corollary 7.2.1.12], so we are done. □

We will refer to the Grothendieck topology of Theorem 5.2 as the effective topology and denote the associated ∞-site as (Fam(C), E). Denote the epimorphism topology [3, Definition 2.2.5] on Grpd∞ by (Grpd∞, Epi). By construction, \( \Pi_\infty \) yields a morphism of sites:

\[
\Pi_\infty : (\text{Fam}(C), E) \to (\text{Grpd}_\infty, \text{Epi})
\]

(21)

**Remark 5.3.** The effective topology on Fam(∞) ∼= Grpd∞ is precisely the epimorphism topology.

**Proposition 5.4.** Let C be an ∞-category with pullbacks such that Fam(C) is an ∞-topos (e.g. C = Sp). Then the effective topology on Fam(C) contains the epimorphism topology, i.e every covering family in the epimorphism topology is also an effective covering.

**Proof.** We need to show that if \( \{(f_i, \phi_i) : (X_i, F_i) \to (X, F)\}_{i \in I} \) is an effective epimorphic family, then the induced family of maps of ∞-groupoids \( \{f_i : X_i \to X\}_{i \in I} \) under \( \Pi_\infty \) is also effective epimorphic. This follows from coupling Lemma 2.14 with the fact that \( \Pi_\infty \) preserves small colimits. □
6. Appendix A: (Higher) Loci and Miscellaneous Results

The fundamental group of a parametrized family (definition 6.1) in many ways looks like the geometric homotopy groups of objects an ∞-topos. In fact, there are cases of ∞-categories C such that Fam(C) is a topos. The restriction of our definition of the fundamental ∞-groupoids/groups of parametrized families to families of objects in such categories thus behaves similarly to notions of homotopical invariants of objects in higher topoi. In this section, we discuss such categories in both the ordinary and higher categorical case and state some of our results relevant to the topic.

Definition 6.1 (Joyal[3]). A locus is a locally presentable category C such that Fam(C) is a topos.

In [5], Joyal notes the following:

Observation 6.2. The category Set^{op} of pointed sets is a locus.

Notation 6.3. For convenience, we will write a family in C indexed over a set I as ⟨X_i⟩_{i ∈ I}, where each X_i ∈ C.

Proof of Observation 6.2. It is clear that Set^{op} is locally presentable, since co-slice categories of locally presentable categories are themselves locally presentable. It remains to show that Fam(Set^{op}) is a topos.

Let ∆[1] denote the “walking-arrow-equipped-with-a-section,” i.e the free category on the directed graph 0 → 1 subject to the condition r ◦ s = id[0]. We claim that there is an equivalence of categories Fam(Set^{op}) ≃ [∆[1], Set]. We construct a functor F : [∆[1], Set] → Fam(Set^{op}) as follows. Fix an object X ∈ [∆[1], Set]. For every p ∈ X[0], the fiber X[r]^{-1}(p) is canonically pointed by X[s](p) (this is guaranteed to land in X[r]^{-1}(p) because of the condition X(r) ◦ X(s) = id_X[0]). Now we define:

\[ F : X \mapsto \langle (X[r]^{-1}(p), X[s](p)) \rangle_{p \in X[0]} \]

We now construct a pseudo-inverse G : Fam(Set^{op}) → [∆[1], Set]. Let U : Set^{op} → Set denote the canonical projection and fix ⟨X_i⟩_{i ∈ I} ∈ Fam(Set^{op}). Define the function γ_1 : \prod_{i ∈ I} U(X_i) → I that sends all x ∈ X_i to i ∈ I. There is a canonical function γ_2 : I → \prod_{i ∈ I} U(X_i) that sends i ∈ I to the basepoint of X_i. Clearly, γ_1 ◦ γ_2 = id_I, so we can define the functor

\[ G : ⟨X_i⟩_{i ∈ I} \mapsto \left[ \begin{array}{ccc} ⟨X_i⟩_{i ∈ I}[0] = I & G(⟨X_i⟩_{i ∈ I})[1] = \prod_{i ∈ I} U(X_i) \\ \end{array} \right] \]

It is straightforward to verify that G ◦ F (resp. F ◦ G) is naturally isomorphic to id_{[∆[1], Set]} (resp. id_{Fam(Set^{op})}), so the claim is proven. Since Fam(Set^{op}) is equivalent to a category of diagrams in Set, it is a topos.

Corollary 6.4. [∆[1], Set] is a locally connected topos. Furthermore, for any precosheaf X ∈ [∆[1], Set], there is an isomorphism

\[ π_0(X) ≃ X[0] \]

Remark 6.5. A similar argument can be used to show that the coproduct completion of the category of pointed objects in a topos E is itself a topos. This is clear once we note that the condition r ◦ s = id[0] implies that for every object X ∈ [∆[1], E] and any global element p : * → X[0], we get a commutative diagram in E:

\[ \begin{array}{ccc} X[1] × X[0] & \xrightarrow{id} & X[1] × X[0] \\ \xrightarrow{X[s] ◦ p} & & \xavier \urarrow \swarrow \xrightarrow{p} \\ X[0] & \xrightarrow{X[r]} & X[0] \end{array} \]

in which the square is a pullback and the global element * → X[1] × X[0] * is induced by universal property, i.e the fibered product X[1] ×_{X[0]} * is canonically pointed. By repeating this construction for each p ∈ Hom_E(*, X[0]), we can build a family of pointed objects in E by taking fibers. This assignment is functorial and yields an equivalence of categories

\[ [∆[1], E] \xrightarrow{≃} Fam(E^{op}) \]

\[ ^2 \]Joyal also requires loci to be pointed, i.e to have a zero object.

\[ ^3 \]This part of the equivalence is constructed in [3].
Due to this equivalence, we can slightly generalize Corollary 6.4 to obtain the following:

**Corollary 6.6.** Let \( \mathbb{E} \) be a topos with terminal object \(*\). Then \( [\Delta[1], \mathbb{E}] \) is a locally connected topos. Additionally for any object \( X \in [\Delta[1], \mathbb{E}] \), one has an isomorphism:

\[
\pi_0(X) \cong \text{Hom}_\mathbb{E}(*, X[0])
\]

Corollary 6.6 indicates that the family construction may be of use in computations involving connected components of objects in locally connected topoi.

The next proposition gives a relationship between the coproduct and colimit completions of loci.

**Proposition 6.7.** Let \( \mathcal{C} \) be a locus. Then the Yoneda extension \( (\_ \otimes_\mathcal{C} \sigma : [\mathcal{C}^{op}, \text{Set}] \to \text{Fam}(\mathcal{C}) \) of the singleton embedding \( \sigma : \mathcal{C} \hookrightarrow \text{Fam}(\mathcal{C}) \) is the inverse image component of a canonical geometric morphism \( \text{Fam}(\mathcal{C}) \to [\mathcal{C}^{op}, \text{Set}] \).

**Proof.** By classical results, we can explicitly construct the right adjoint component \( N \) by \( N(\zeta) = \text{Hom}_{\text{Fam}(\mathcal{C})}(\sigma(-), \zeta) : \mathcal{C}^{op} \to \text{Set} \) for \( \zeta \in \text{Fam}(\mathcal{C}) \). By [6, VII.9.1], in order to show that \( (\_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \) is left-exact, it suffices to show that \( \sigma \) is flat in the internal logic of \( \mathcal{C} \). But by our assumptions \( \mathcal{C} \) is locally presentable (and hence finitely complete) and \( \text{Fam}(\mathcal{C}) \) is a topos, so \( \sigma \) is internally flat if it is representably flat. Again by the finite completeness assumption on \( \mathcal{C} \), \( \sigma \) is representably flat precisely if it preserves finite limits, which holds by [2, Corollary 6.2.7]. Thus \( (\_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \) is left-exact, so we obtain the desired geometric morphism \( (\_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \_ \otimes_\mathcal{C} \sigma \) \). \( \square \)

**Definition 6.8 (Joyal).** An infinite-locus is a locally presentable \( \infty \)-category \( \mathcal{C} \) such that \( \text{Fam}(\mathcal{C}) \) is an \( \infty \)-topos.

**Example 6.9.** Important examples include the \( \infty \)-category \( \text{Grpd}_\infty \) of pointed homotopy types and the \( \infty \)-category \( \text{Sp} \) of spectra. In analogy to the 1-categorical case, \( \text{Fam}(\text{Grpd}_\infty) \) is an \( \infty \)-topos due to the equivalence \( [\Delta[1], \text{Grpd}_\infty] \cong \text{Fam}(\text{Grpd}_\infty) \).

The following was conjectured by Joyal in [5] and proven by Hoyois in [4]. The proof in loc cit. relies on the “stable” Giraud Theorem.

**Theorem 6.10.** Any locally presentable stable \( \infty \)-category is an infinite-locus.

### 7. Future Work

In this paper, we have shown that the \( \infty \)-category \( \text{Fam}(\mathcal{C}) \) has a natural homotopy theory which generalizes the homotopy theory of topological spaces. The following are some directions/questions for future work on this topic:

- Generally, if \( \mathcal{C} \) is an \( (\infty, n) \)-category for some \( n \geq 1 \), then the objects of \( \text{Fam}(\mathcal{C}) \) are pairs \((X, F)\), where \( X \) is an \( (\infty, n-1) \)-category and \( F : X \to \mathcal{C} \) is a functor. This implies that an analog of \( \Pi_\infty \) for parametrized families in arbitrary \( (\infty, n) \)-categories would output a directed space instead of a homotopy type (the “fundamental \( (\infty, n-1) \)-category” instead of fundamental \( \infty \)-groupoid). There should be analogs of the contents of this paper in directed homotopy theory. However, there is a general lack of literature on the notions of \( (\infty, n) \)-topoi, etc., so this may be more difficult.

- Can the analogy between the fundamental \( \infty \)-groupoid of a parametrized family and (for instance) the fundamental \( \infty \)-groupoids of objects in a locally \( \infty \)-connected higher topos be made more precise? More specifically, if \( \text{Fam}(\mathcal{C}) \) is an \( \infty \)-topos, then does \( \Pi_\infty \) fit into an adjoint triple resembling an essential geometric morphism?

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