Dynamics of entanglement in quantum computers with imperfections

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Any practical implementation of a quantum computer will have to face errors, due to the inevitable coupling to the environment \textsuperscript{1,2} or to device imperfections \textsuperscript{3}. The effect of the environment is to introduce decoherence which sets the time scale over which quantum computation is no longer possible. The presence of static imperfections, although not leading to any decoherence, may be also decremental for the implementation of any quantum computational task. A small inaccuracy in the coupling constants induces errors in gating or an unwanted time evolution in the case in which the Hamiltonian cannot be switched exactly to zero. If the imperfection strength increases, new phenomena occur and above a certain threshold the core of the computer can even “melt” due to the setting in of chaotic behavior \textsuperscript{2}. Previous investigations of the stability of quantum information processing in the presence of such effects mainly studied the fidelity of the quantum evolution as an indicator of the quality of the computation \textsuperscript{2,3,6,7}. In particular, in Ref. \textsuperscript{3} the fidelity was used to measure the stability of the quantum memory, that is of a state loaded on a quantum computer with imperfections. A more complete characterization of the stability of a quantum computation requires a deeper investigation of the state of the system. Fidelity is one of the ways to characterize it. In this Letter we discuss the behavior of entanglement on approaching the transition to quantum chaos. Entanglement is not only one of the most intriguing features predicted by the quantum theory but also a fundamental resource for quantum computation and communication \textsuperscript{4}. Therefore studies of the stability of entanglement under decoherence and imperfection effects are, in our opinion, of interest.

We model the quantum computer as a lattice of interacting spins (qubits) where, due to the unavoidable presence of imperfections, the spacing between the up and down states (external field) and the couplings between the qubits (exchange interactions) are both random. The entanglement properties in spin systems has recently attracted attention and we refer to \textsuperscript{5} for a more detailed introduction. In this work we consider, for the first time, the effect of disorder in the couplings.

We consider \( n \) qubits on a two-dimensional lattice, described by the Hamiltonian

\[
H = \sum_{i=1}^{n} \Delta_i \sigma_i^z + \sum_{i<j=1}^{n} J_{ij} \left( \frac{1+\gamma}{2} \sigma_i^x \sigma_j^x + \frac{1-\gamma}{2} \sigma_i^y \sigma_j^y \right),
\]

where the \( \sigma_i^\alpha \) (\( \alpha = x, y, z \)) are the Pauli matrices for the qubit \( i \) and the second sum runs over nearest-neighbor qubit pairs on the lattice. The energy spacing between the up and down states of a qubit is given by \( \Delta_i = \Delta_0 + \delta_i \), where the \( \delta_i \)'s are uniformly distributed in the interval \([-\delta/2, \delta/2]\). The parameter \( \delta \) gives the width of the \( \Delta_i \) (single qubit energy) distribution around the average value \( \Delta_0 \). The couplings \( J_{ij} \) are also uniformly distributed in the interval \([-J, J]\). We consider \( 0 \leq \gamma \leq 1 \). For \( \gamma = 1 \) Eq. \textsuperscript{(1)} reduces to the disordered Ising model, while for \( \gamma = 0 \) it gives the disordered XY model. For \( \gamma = 1 \) the model defined by Eq. \textsuperscript{(1)} has been proposed by Georgeot and Shepelyansky \textsuperscript{3} to describe the hardware of a quantum computer, in which system imperfections generate unwanted interqubit couplings and energy fluctuations. For \( J, \delta = 0 \) the spectrum of the Hamiltonian is composed of \( n+1 \) degenerate levels, and the interlevel spacing is \( 2\Delta_0 \), which corresponds to the energy required to flip a single qubit. We study the case \( \delta, J \ll \Delta_0 \), in which the degeneracies are resolved and the spectrum is composed by \( n+1 \) bands. Since \( \delta, J \ll \Delta_0 \), the interband couplings in \textsuperscript{(1)} can be neglected and every single band can be studied separately. We concentrate our studies on the central band with zero magnetization \( \left( \sum \sigma_i^z = 0 \right) \). We assume free boundary conditions and set the energy scale \( \Delta_0 = 1 \).

In this Letter, we investigate the evolution in time of the entanglement between two nearest neighbor qubits in the lattice. We confine our interest to the entanglement of formation \textsuperscript{2} between two spins in the lattice. The
propagation of entanglement in disordered-free systems has been analyzed in Ref. [3]. In the same spirit we study the entanglement evolution for an initially maximally entangled pair of qubits:

\[ |\Psi_B(0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes |010\ldots01\rangle. \tag{2} \]

We will also analyze the generation of the entanglement, due to interqubit coupling, of two spins which are initially in a separable state. In this case we start from the wave vector

\[ |\Psi_S(0)\rangle = |01\rangle \otimes |010\ldots01\rangle. \tag{3} \]

In both cases, the initial state, Eqs. (2,3), evolves according to the Hamiltonian defined in Eq. 1. We compute the reduced density matrix \(\rho_{12}(t)\), obtained from the wave function \(|\Psi(t)\rangle\) after all the spins except those at sites 1 and 2 have been traced out: \(\rho_{12}(t) = Tr_{3,\ldots,n}|\Psi(t)\rangle\langle\Psi(t)|\). We choose qubits 1 and 2 to be nearest neighbors on the border of the lattice (their position in the lattice is not crucial for the results here reported). Their state at time \(t\) described by the density matrix \(\rho_{12}(t)\), is mixed since qubits 1 and 2 become entangled with the rest of the lattice. We quantify the entanglement by means of the concurrence \(C\), defined as \(C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}\), where the \(\lambda_i\)'s are the square roots of the eigenvalues of the matrix \(R = \rho_{12}\rho_{12}^\dagger\), in descending order; the spin-flipped density matrix is defined by \(\tilde{\rho}_{12} = (\sigma_1^y \otimes \sigma_2^y)\rho_{12}(\sigma_1^y \otimes \sigma_2^y)\) (in this definition the standard basis \([|00\rangle, |01\rangle, |10\rangle, |11\rangle\) must be used). Another dynamical quantity widely used to characterize the stability of a state under perturbations is the fidelity \(F\). It is defined as \(F(t) = \langle\Psi(0)|\Psi(t)\rangle^2\), where \(\langle\Psi(t)| = \exp(-iHt)|\psi(0)\rangle\) (we set \(\hbar = 1\)). For weak interqubit coupling \(f(t)\) is close to 1 for all times. For strong coupling we enter the so-called quantum chaos regime, and the fidelity essentially drops to zero \(F\). In the following we contrast the behavior of the fidelity with that of the concurrence. In addition to the different regimes found in the analysis of the fidelity, the concurrence gives additional information related to the choice of the initial state.

**Initially entangled state** - We first consider the time evolution of \(C(t)\) starting from the (maximally entangled) initial state \(|\Psi_B(0)\rangle\). If the coupling strength \(J\) is weak, then the time evolution of \(C(t)\) can be obtained by means of perturbation theory in the coupling \(J\). Only the quantum register states directly coupled to the initial state are involved (the quantum register states are the eigenstates of Hamiltonian \(H\) at \(J = 0\)): \(|\alpha\rangle = |\alpha_1, \alpha_2, \ldots, \alpha_n\rangle\), with \(\alpha_i = 0, 1\). Since the exchange interaction has a two-body nature, the matrix elements are zero when the state binary strings differ in more than two digits. Furthermore, only nearest neighbor qubits interact, and therefore to first order in perturbation theory only order of \(n\) quantum register states are involved. It is important to note that the coupling to a state which differs from the initial state \(|\Psi_B(0)\rangle\) in the polarization state of two qubits other that 1 and 2 does not affect the pairwise entanglement between qubits 1 and 2. To first order in the perturbation theory, these two qubits become entangled only with nearest neighbor qubits. Therefore the concurrence can be described in terms of superposition between a small number of quantum register states, independently of the total number \(n\) of qubits. In the perturbative regime, the frequencies of these oscillations are \(\sim \delta\), around an average value \(\sim 1-(J/\delta)^2\). Perturbation theory breaks down when the typical interaction matrix element \(J\) between directly coupled states becomes of the order of their energy separation \(\Delta E \sim \delta/n\). This follows from the fact that two quantum register states are directly coupled when two qubits are flipped. This changes the energy by \(\sim \delta\), and there are the order of \(n\) states directly coupled inside this energy interval. Therefore the perturbative regime is limited to \(J < J_p \sim \delta/n\). Above this threshold, it is known \(\mathbb{R}\) that the fidelity decays exponentially: \(f(t) \approx \exp(-\Gamma_f t)\), with the rate \(\Gamma_f \sim J^2/\rho_f\) determined by the Fermi golden rule, where \(\rho_f = 1/\Delta E \sim n/\delta\) is the density of directly coupled states. The transitions involving qubits other than 1 and 2 do not change the concurrence. Therefore the density of directly coupled states relevant for this quantity is given by \(\rho_c \sim 1/\delta\). Hence we expect the concurrence...
to decay exponentially, \( C(t) \sim \exp(-\Gamma_c t) \), with the rate \( \Gamma_c \sim J^2 \rho_c \sim J^2/\delta \). This expectation is borne out by the numerical data of Fig. 1, which show that the concurrence time scale \( t_c^E \), defined by the condition \( C(t_c^E) = 0.96 \), is inversely proportional to \( J^2 \) (the result does not depend on the value chosen). In the regime of strong coupling, \( J \geq \delta \), the time evolution of the fidelity is given by \( f(t) \approx \prod_{i<j} \cos(J_{ij} t) \), where the product runs over nearest neighbor qubits. Thus the fidelity decay is Gaussian: \( f(t) \approx \exp(-\sum_{i<j} J_{ij}^2 t^2) \sim \exp(-J^2 n t^2) \). In computing the concurrence decay, we must again take into account that the rotations of qubits different from 1 and 2 do not change their pairwise entanglement. Therefore we expect \( C(t) \sim \exp(-J^2 t^2) \). This gives a concurrence time scale \( t_c^E \sim J^{-1} \), in good agreement with the numerical data of Fig. 1. Since in the Fermi golden rule regime the concurrence time scale \( t_c^E \sim \delta/J^2 \), while in the ergodic regime \( t_c^E \sim 1/J \), the transition between these two regimes takes place at \( J = J_E \sim \delta \) (we checked numerically that \( J_E \sim \delta \)). The data of Fig. 1, taken at different number of qubits, seem to collapse on a single curve, in agreement with our theoretical expectations.

It is also interesting to study the saturation value of the concurrence, \( C_\infty = (\lim_{t \to \infty} \int_{\tau}^{t} C(t') dt')/\tau \). In order to smoothen the fluctuations, we average our numerical data over both different random configurations of \( \delta_i \) and \( J_{ij} \) and a large enough time interval \( \tau \) (we take \( \tau \) equal to 1/10 of the total integration time \( t \), with \( t \) sufficiently long to observe the saturation of \( C(t) \)). In Fig. 2, we study \( C_\infty \) as a function of the coupling strength \( J \), starting from the entangled state \( |\Psi_B(0)\rangle \). Again we can distinguish three regimes. In the perturbative regime \( J < J_p \sim \delta/n \) the saturation value is close to one (more precisely, \( 1 - C_\infty \sim (J/\delta)^2 \)). In the ergodic regime at \( J > J_E \sim \delta \) the concurrence \( C_\infty \) is equal to zero. Quantum chaos is characterized by the ergodicity of the eigenfunctions \( |\psi_i\rangle \) \( i = 1, ..., N = 2^n \) of Hamiltonian \( \hat{H} \). This means that, by expanding \( |\psi_i\rangle \) over the quantum register states, \( |\psi_i\rangle = \sum_{\alpha} c_{\alpha} |\alpha\rangle \), the coefficient \( c_{\alpha} \) are randomly fluctuating and have amplitudes \( |c_{\alpha}| \sim 1/\sqrt{N} \).

In this regime, the complexity of the eigenstates, characterized by the entropy \( S_i = -\sum_{\alpha} |c_{\alpha}|^2 \log |c_{\alpha}|^2 \), is maximal, that is \( S_i \sim n \). We note that indeed the maximum entropy criterion \( S_i \sim n \) gives the ergodicity threshold \( J_E \sim \delta \). Due to quantum ergodicity, for times \( t \gg t_c^E \sim 1/J \) the wave function \( |\Psi(t)\rangle \) is a random superposition of the quantum register states: \( |\Psi(t)\rangle = \sum_{\alpha} a_{\alpha}(t) |\alpha\rangle \), where the coefficients \( a_{\alpha}(t) \) have amplitudes \( 1/\sqrt{N} \) and random phases. The diagonal elements of the reduced density matrix \( \rho_2 \) are given by \( [\rho_{11}(t)]_{\alpha_1 \alpha_2} = \sum_{\alpha_3, ..., \alpha_n} |a_{\alpha_3, ..., \alpha_n}(t)|^2 \). The value of these elements is \( \approx 1/4 \), since they are the sum of \( N/4 \) positive terms, and the value of each term is \( \sim 1/N \).

The off diagonal matrix elements of \( \rho_{12} \) are instead given by the sum of \( N/4 \) terms with random signs, and therefore their value is \( \sim 1/\sqrt{N} \). For such a reduced density matrix the concurrence is equal to zero. Physically this means that in the ergodic regime the entanglement is multipartite and shared between all the qubits, and therefore the pairwise entanglement drops to zero. In this regime the coupling to the rest of the system acts as a dephasing channel, and the reduced density matrix becomes essentially diagonal, thus destroying the pairwise entanglement.

**Initially separable state** - Since the Hamiltonian in Eq. \( 4 \) can create entanglement due to the exchange term between neighboring qubits, it is interesting to consider also separable states as initial states. In the perturbative regime \( J < J_p \), the concurrence oscillates...
FIG. 4: Concurrence saturation value $C_{\infty}$ as a function of $J$ for $\gamma = 0$ (black symbols) and $\gamma = 1$ (gray stars), $\delta = 0.2$, initial state $|\Psi_S\rangle$ (filled symbols). Symbols have the same meaning as in Fig. 1. The straight line is proportional to $J^2$. The dashed line is the same as in Fig. 2.

with frequency $|\delta_1 - \delta_2|$ between 0 and a maximum value $2|J_{12}|/|\delta_1 - \delta_2|$. At times $t \ll 1/|\delta_1 - \delta_2|$, the perturbation theory gives $C(t) \approx 2|J_{12}|t$, in agreement with the numerical data of Fig. 3 (after disorder averaging we get $C(t) = Jt$). In the ergodic regime $J > J_E$, a maximum concurrence of the order of one is generated after a time $\sim 1/J$. After this time the entanglement of the pair rapidly drops to zero. In the intermediate regime the concurrence reaches its maximum and then saturates to a value which is nearly independent of $J$. Therefore the generation of pairwise entanglement is an interesting characteristic of this region.

The saturation value $C_{\infty}$ as a function of the coupling $J$ is shown in Fig. 4. In the perturbative regime $C_{\infty} \propto J/\delta$ whereas in the ergodic regime $C_{\infty} = 0$. It is interesting to notice that there is a broad region in which the interactions create a significant amount of entanglement, quantified by a concurrence $C_{\infty} \approx 0.2 - 0.3$. We also note that in the ergodic regime $J > J_E$ the saturation value $C_{\infty} = 0$ does not depend on the initial conditions, as expected. On the contrary, when $J < J_E$ the concurrence saturation value depends on the initial state vector, implying the absence of ergodicity.

As it can be seen from our data shown in Figs. 1-4, we did not find any significant change in the dynamics of entanglement as a function of $\gamma$. Even though a further symmetry is added to the system when $\gamma = 0$ (the invariance with respect to rotations about the $z$-axis), this has little influence on entanglement.

In summary, we have studied the dynamics of the pairwise entanglement in a qubit lattice in the presence of static imperfections, and characterized three different regimes: (i) the perturbative regime, in which the entanglement is stable against imperfections, (ii) the crossover regime, in which the imperfections degrade the concurrence of an initially entangled pair but can also drive a significant entanglement generation, and (iii) the ergodic regime, in which a pair of qubits becomes entangled with the rest of the lattice and therefore the concurrence of the pair drops to zero. We stress two important points of our findings from the point of view of quantum computation. First of all, the pairwise entanglement is destroyed above a coupling strength which is independent of the size of the quantum computer. Moreover, there is a broad crossover region in which the computer imperfections can be used to create a significant amount of pairwise entanglement. Finally we want to emphasize that our analysis leads to the conclusion that the spin system behaves as a dephasing environment only in the ergodic regime. In the integrable limit, even an infinite system is unable to dephase completely the initial state. We believe that this observation may be relevant in understanding the behavior of spin or more complex baths.

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