Un-equivalency Theorem between Deformed and undeformed Heisenberg-Weyl’s Algebras

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Abstract

Two fundamental issues about the relation between the deformed Heisenberg-Weyl algebra in noncommutative space and the undeformed one in commutative space are elucidated. First the un-equivalency theorem between two algebras is proved: the deformed algebra related to the undeformed one by a non-orthogonal similarity transformation is explored; furthermore, non-existence of a unitary similarity transformation which transforms the deformed algebra to the undeformed one is demonstrated. Secondly the uniqueness of realizing the deformed phase space variables via the undeformed ones is elucidated: both the deformed Heisenberg-Weyl algebra and the deformed bosonic algebra should be maintained under a linear transformation between two sets of phase space variables which fixes that such a linear transformation is unique. Elucidation of this un-equivalency theorem has basic meaning both in theory and experiment.
Spatial noncommutativity is an attractive basic idea for a long time. Recent interest on this subject is motivated by studies of the low energy effective theory of D-brane with a nonzero NS - NS $B$ field background [1–3]. It shows that such low energy effective theory lives on noncommutative space. For understanding low energy phenomenological events quantum mechanics in noncommutative space (NCQM) is an appropriate framework. NCQM have been extensively studied and applied to broad fields [4–16]. But up to now it is not fully understood.

In literature there is an extensively tacit understanding about equivalency between the deformed Heisenberg-Weyl algebra in noncommutative space and the undeformed one in commutative space. As is well known, the deformed phase space variables are related to the undeformed ones by a linear transformation, thus one concludes that the algebra of noncommutative quantum mechanical observables is the standard one. This leads to the tacit understanding of fully equivalency between two algebras. A related tacit understanding is that there are many equivalent linear transformations between two sets of phase space variables.

In this paper we elucidate these two subtle points. First we clarify equivalency conditions between two algebras. We demonstrate that the deformed algebra is related to the undeformed one by a similarity transformation with a non-orthogonal real matrix. Furthermore, we prove that a unitary similarity transformation which transforms two algebras to each other does not exist. The results are summarized in the un-equivalency theorem between two algebras. Secondly we clarify that among deferent types of linear transformations of realizing deformed phase space variables via undeformed ones only a unique one maintains both the deformed Heisenberg-Weyl algebra and the deformed bosonic algebra.

In order to develop the NCQM formulation we need to specify the phase space and the Hilbert space on which operators act. The Hilbert space is consistently taken to be exactly the same as the Hilbert space of the corresponding commutative system [4]. As for the phase space we consider both position-position noncommutativity (position-time noncommutativity is not considered) and momentum-momentum noncommutativity [3, 15]. In this case the consistent deformed Heisenberg-Weyl algebra is as follows:

\[ [\hat{x}_i, \hat{x}_j] = i\xi^2 \theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\xi^2 \eta_{ij}, \quad (i, j = 1, 2), \]

(1)
where $\theta$ and $\eta$ are the constant, frame-independent parameters. Here we consider the intrinsic \footnote{A note is placed here to provide additional context or information about the parameter.} momentum-momentum noncommutativity. It means that the parameter $\eta$, like the parameter $\theta$, should be extremely small. $\epsilon_{ij}$ is an antisymmetric unit tensor, $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$; $\xi = (1 + \theta\eta/4\hbar^2)^{-1/2}$ is the scaling factor. For the case of both position - position and momentum - momentum noncommuting the scaling factor $\xi$ in Eq. (1) guarantees consistency of the framework, and plays an essential role in dynamics as well. For example, in the discussion of deformed two - mode quadrature operators it revealed that effects of \footnote{Another note is placed here to provide additional context or information about the parameter.} spatial noncommutativity are included in the scaling factor $\xi$ [16]. When $\eta = 0$, we have $\xi = 1$. The deformed Heisenberg-Weyl algebra (1) reduces to the one of only position-position noncommuting.

The deformed phase space variables $\hat{x}_i$ and $\hat{p}_i$ are related to the undeformed ones $x_i$ and $p_i$ by the following linear transformation [15]

$$
\hat{x}_i = \xi(x_i - \frac{1}{2\hbar}\theta\epsilon_{ij}p_j), \quad \hat{p}_i = \xi(p_i + \frac{1}{2\hbar}\eta\epsilon_{ij}x_j).
$$

(2)

where $x_i$ and $p_i$ satisfy the undeformed Heisenberg-Weyl algebra $[x_i, x_j] = [p_i, p_j] = 0, [x_i, p_j] = i\hbar\delta_{ij}$.

In literature the point of the tacit understanding of equivalency between the deformed Heisenberg-Weyl algebra and the undeformed one is as follows: any Lie algebra generated by relations $[X_a, X_b] = iT_{ab}$ with central $T_{ab}$ satisfying $det(T_{ab}) \neq 0$ can be put into a usual canonical form, like Eqs. (2). Therefore the deformed Heisenberg-Weyl algebra (1) and the undeformed one are the same, and the spectrum of an observable is the same regardless we star with deformed variables $(\hat{x}_i, \hat{p}_i)$ or undeformed ones ($x_i, p_i$).

Now we elucidate this subtle point. Equivalency between the deformed Heisenberg-Weyl algebra and the undeformed one must satisfy two conditions: (i) Two sets of phase space variables $(\hat{x}_i, \hat{p}_i)$ and $(x_i, p_i)$ can be related to each other by a singular-free linear transformation (The inverse transformation should exit for all values of $(\hat{x}_i, \hat{p}_i)$ and $(x_i, p_i)$); (ii) Two algebras can be transformed to each other by a unitary similarity transformation.

First we consider the second condition. We prove the following theorem.

**The Un-equivalency Theorem** The deformed Heisenberg-Weyl algebra in noncommutative space is transformed to the undeformed one in commutative space by a similarity
transformation with a non-orthogonal real matrix. A unitary similarity transformation which relates two algebras to each other does not exist.

The demonstration of the first part of the theorem is trivial. We define a $1 \times 4$ column matrix $\hat{U} = (\hat{U}_1, \hat{U}_2, \hat{U}_3, \hat{U}_4)$ with elements $\hat{U}_1 = \hat{x}_1, \hat{U}_2 = \hat{x}_2, \hat{U}_3 = \hat{p}_1$ and $\hat{U}_4 = \hat{p}_2$, a $4 \times 1$ row matrix $\hat{U}^T$ with elements $\hat{U}_i^T = \hat{U}_i, (i = 1, 2, 3, 4)$, and a $4 \times 4$ matrix $\hat{M}$ with elements $i\hat{M}_{ij} = [\hat{U}_i, \hat{U}_j^T], (i, j = 1, 2, 3, 4)$. The matrix $\hat{M}$ represents the deformed Heisenberg-Weyl algebra. From Eqs. (1) it follows that $\hat{M}$ reads

$$\hat{M} = \begin{pmatrix}
0 & \xi^2 \theta & \hbar & 0 \\
-\xi^2 \theta & 0 & 0 & \hbar \\
-\hbar & 0 & 0 & \xi^2 \eta \\
0 & -\hbar & -\xi^2 \eta & 0
\end{pmatrix}. \quad (3)$$

The corresponding matrixes in commutative space are a $1 \times 4$ column matrix $U$ with elements $U_1 = x_1, U_2 = x_2, U_3 = p_1$ and $U_4 = p_2$, a $4 \times 1$ row matrix $U^T$ with elements $U_i^T = U_i, (i = 1, 2, 3, 4)$, and a $4 \times 4$ matrix $M$ with elements $iM_{ij} = [U_i, U_j^T], (i, j = 1, 2, 3, 4)$. The matrix $M$ represents the undeformed Heisenberg-Weyl algebra, which can be obtained by putting $\theta = \eta = 0$ in the matrix $\hat{M}$ (3),

$$M = \begin{pmatrix}
0 & 0 & \hbar & 0 \\
0 & 0 & 0 & \hbar \\
-\hbar & 0 & 0 & 0 \\
0 & -\hbar & 0 & 0
\end{pmatrix}. \quad (4)$$

From Eq. (2) it follows that $\hat{U}_i = R_{ik} U_k, \hat{U}_i^T = \hat{U}_j = R_{ij} U_i = U_i^T R_{ij}^T$, and the deformed Heisenberg-Weyl algebra is related to the undeformed Heisenberg-Weyl algebra by a similarity transformation $\hat{M}_{ij} = R_{ik} M_{kl} R_{lj}^T$ with a real matrix $R$

$$R = \begin{pmatrix}
\xi & 0 & 0 & -\frac{1}{2\hbar} \xi \theta \\
0 & \xi & \frac{1}{2\hbar} \xi \theta & 0 \\
0 & \frac{1}{2\hbar} \xi \eta & \xi & 0 \\
-\frac{1}{2\hbar} \xi \eta & 0 & 0 & \xi
\end{pmatrix}. \quad (4)$$

It is obvious that $R$ is not orthogonal matrix $RR^T \neq I$.

Now we prove the second part of the un-equivalency theorem. Eq. (2) shows that if there is such a unitary transformation, its elements should be real. That is, it should be an orthogonal matrix $S$ with real elements $S_{ij}, SS^T = S^T S = I$, and satisfies $S_{ik} \hat{M}_{kl} S_{lj}^T =$
\[ M_{ij}, \text{ or } S_{ik} \hat{M}_{kj} = M_{ik} S_{kj}. \] This is a system of 16 homogeneous linear equations for \( S_{ij}, (i, j = 1, 2, 3, 4). \) It is divided into 4 closed sub-systems of 4 homogeneous linear equations. Among them we consider a closed sub-system including \( S_{12}, S_{13}, S_{31} \) and \( S_{34}, \) which reads

\[
\begin{align*}
\xi^2 \theta S_{12} + \hbar S_{13} &= -\hbar S_{31}, \\
\hbar S_{12} + \xi^2 \eta S_{13} &= \hbar S_{34}, \\
\xi^2 \theta S_{31} - \hbar S_{34} &= -\hbar S_{12}, \\
\hbar S_{31} - \xi^2 \eta S_{34} &= -\hbar S_{13}.
\end{align*}
\]

The condition of non-zero solutions of \( S_{12}, S_{13}, S_{31} \) and \( S_{34} \) is

\[
\xi^2 \theta \eta = \pm \hbar (\theta + \eta). \tag{6}
\]

In order to elucidate the physical meaning of Eq. (6), we consider conditions of guaranteeing Bose-Einstein statistics in the case of both position-position and momentum-momentum noncommuting in the context of non-relativistic quantum mechanics. We start from the general construction of deformed annihilation and creation operators \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) \( (i = 1, 2) \) at the deformed level, which are related to the deformed phase space variables \( \hat{x}_i \) and \( \hat{p}_i. \) The general form of \( \hat{a}_i \) can be represented as \( \hat{a}_i = c_1 (\hat{x}_i + ic_2 \hat{p}_i), \) where constants \( c_1 \) and \( c_2 \) can be fixed as follows. The deformed annihilation and creation operators \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) should satisfy \( [\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1 \) (to keep the physical meaning of \( \hat{a}_i \) and \( \hat{a}_i^\dagger \)). From this requirement and the deformed Heisenberg-Weyl algebra \( \{\} \) it follows that \( c_1 = \sqrt{1/2c_2 \hbar}. \)

When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, Bose-Einstein statistics should be maintained at the deformed level described by \( \hat{a}_i, \) thus operators \( \hat{a}_1 \) and \( \hat{a}_2 \) should be commuting. From \( [\hat{a}_i, \hat{a}_j] = 0 \) and the deformed Heisenberg-Weyl algebra \( \{\} \) it follows that \( ic_1^2 \xi^2 \epsilon_{ij} (\theta - c_2^2 \eta) = 0, \) i.e. \( c_2 = \sqrt{\theta/\eta}. \) (The phases of \( \theta \) and \( \eta \) are chosen so that \( \theta/\eta > 0. \) ) The general representations of the

\[ ^1 \text{In Eq. (6) dimensions of different terms are different. If we define a } 1 \times 4 \text{ column matrix } \hat{V} \text{ with elements } \hat{V}_1 = \hat{x}_1, \hat{V}_2 = \hat{x}_2, \hat{V}_3 = \alpha \hat{p}_1 \text{ and } \hat{V}_4 = \alpha \hat{p}_2, \text{ where } \alpha \text{ is an auxiliary arbitrary non-zero constant with the dimension } [\text{mass}]^{-1}[\text{time}]^1. \text{ Thus } \hat{V}_i \text{ (} i = 1, 2, 3, 4 \text{) have the same dimension } [\text{length}]^2. \text{ Then in Eq. (6) dimensions of different terms are same. The introduction of the arbitrary constant } \alpha \text{ does not change the following conclusion.} \]
deformed annihilation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$ are

$$\hat{a}_i = \sqrt{\frac{1}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \hat{x}_i + i \sqrt{\frac{\theta}{\eta}} \hat{p}_i \right), \quad \hat{a}_i^\dagger = \sqrt{\frac{1}{2\hbar}} \sqrt{\frac{\eta}{\theta}} \left( \hat{x}_i - i \sqrt{\frac{\theta}{\eta}} \hat{p}_i \right).$$

(7)

The structure of the deformed annihilation operator $\hat{a}_i$ in Eq. (7) is determined by the deformed Heisenberg-Weyl algebra (1) itself, independent of dynamics. The special feature of a dynamical system is encoded in the dependence of the factor $\eta/\theta$ on characteristic parameters of the system under study.

In the limits $\theta, \eta \to 0$ and $\eta/\theta$ keeping finite, the deformed annihilation operator $\hat{a}_i$ should reduce to the undeformed annihilation operator $a_i$. In commutative space in the context of non-relativistic quantum mechanics the general form of the undeformed annihilation operator $a_i$ can be represented as $a_i = d_1 (x_i + id_2 p_i)$. From $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$ and the undeformed Heisenberg-Weyl algebra it also follows that $d_1 = \sqrt{1/2d_2 \hbar}$ with $d_2 > 0$. But from the undeformed Heisenberg-Weyl algebra the equation $[a_i, a_j] = 0$ is automatically satisfied, thus there is not constraint on the coefficient $d_2$. The general form of the undeformed annihilation operator reads

$$a_i = \sqrt{\frac{1}{2d_2 \hbar}} (x_i + id_2 p_i).$$

(8)

Like the situation of the deformed annihilation operator $\hat{a}_i$, here the structure of $a_i$ is determined by the undeformed Heisenberg-Weyl algebra itself, independent of dynamics. The special feature of a dynamical system is encoded in the dependence of the factor $d_2$ on characteristic parameters of the system under study. If noncommutative quantum theory is a realistic physics, all quantum phenomena should be reformulated at the deformed level. This means that in the limits $\theta, \eta \to 0$ and $\eta/\theta$ keeping finite the deformed annihilation operator $\hat{a}_i$ should reduce to the undeformed one $a_i$. Comparing Eq. (7) and (8), it follows that in the limits $\theta, \eta \to 0$ and $\eta/\theta$ keeping finite the factor $\eta/\theta$ reduces to a positive quantity:

$$\frac{\eta}{\theta} \to \frac{1}{d_2^2} > 0.$$  

(9)

But from Eq. (6), we obtain $\eta/\theta = \pm \hbar/(\xi^2 \mp \hbar)$. This equation shows that in the limits $\theta, \eta \to 0$ and $\eta/\theta$ keeping finite, we have $\eta/\theta \to -1$, which contradicts Eq. (9). We conclude
that Eq. (6) is un-physical. The situation for the rest elements of $S_{ij}$ is the same. Thus the supposed orthogonal real matrix $S$ consistent with physical requirements does not exist. The second part of the un-equivalency theorem is proved.

Now we consider the first condition about equivalency of the two algebras. Eq. (2) shows that the determinant $R$ of the transformation matrix $R$ between $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$ and $(x_1, x_2, p_1, p_2)$ is $R = \xi^4(1 - \theta \eta / 4\hbar^2)^2$. When $\theta \eta = 4\hbar^2$, the matrix $R$ is singular. In this case the inverse of $R$ does not exist. It means that the first condition about equivalency of two algebras is not satisfied.

The above results show that for the case of both position-position and momentum-momentum noncommuting the deformed Heisenberg-Weyl algebra and the undeformed one are not equivalent.

For the case of only position-position noncommuting, $\eta = 0$, the transformation matrix $R$ between $(\hat{x}_1, \hat{x}_2, \hat{p}_1, \hat{p}_2)$ and $(x_1, x_2, p_1, p_2)$ reduces to the matrix

$$
R_0 = \begin{pmatrix}
1 & 0 & 0 & -\frac{1}{2\hbar} \theta \\
0 & 1 & \frac{\hbar}{2\hbar} \theta & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Its determinant $R_0 \equiv 1$, which is singular-free. But in this case $R_0$ is not an orthogonal matrix either, $R_0R_0^T \neq I$. Furthermore, in this case the supposed orthogonal real matrix $S$ reduces to $S_0$, which is obtained from $S$ by setting $\eta = 0$. The closed sub-system of 4 homogeneous linear equations including $S_{0,12}$, $S_{0,13}$, $S_{0,31}$ and $S_{0,34}$ has only zero solutions. The supposed orthogonal real matrix $S_0$ does not exist, either. We conclude that for the case of only position-position noncommuting the deformed and the undeformed Heisenberg-Weyl algebras are also not equivalent.

Now we elucidate the uniqueness of the linear realization of the deformed phase space variables via the undeformed ones. A physical realization should maintain both the deformed Heisenberg-Weyl algebra and the deformed bosonic algebra.

It worth noting that among deferent types of linear transformations between two sets of phase space variables only Eq. (2) maintains both the deformed Heisenberg-Weyl algebra and the deformed bosonic algebra. It is trivial to check that Eq. (2) maintains the deformed Heisenberg-Weyl algebra.
Inserting Eqs. (2) into Eqs. (7), and using Eq. (8), we obtain the linear representation of the deformed annihilation operator by the undeformed one

\[ \hat{a}_i = \xi \left( a_i + \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a_j \right), \quad \hat{a}_i^\dagger = \xi \left( a_i^\dagger - \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a_j^\dagger \right). \]  

(10)

Eq. (10) maintains the deformed bosonic algebra, including the bosonic commutation relations \([\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1\).

In literature there are another types of linear transformations between two sets of phase space variables. One example is to set \(\xi = 1\). In this case the deformed Heisenberg-Weyl algebra reduces to:

\[ [\hat{x}_i, \hat{x}_j] = i\theta \epsilon_{ij}, \quad [\hat{p}_i, \hat{p}_j] = i\eta \epsilon_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad (i, j = 1, 2) \]  

(11)

and representations of deformed variables \(\hat{x}_i\) and \(\hat{p}_i\) by undeformed variables \(x_i\) and \(p_i\) reads:

\[ \hat{x}_i = x_i - \frac{1}{2\hbar} \theta \epsilon_{ij} p_j, \quad \hat{p}_i = p_i + \frac{1}{2\hbar} \eta \epsilon_{ij} x_j. \]  

(12)

Inserting Eqs. (12) into Eqs. (11), the Heisenberg commutation relation in Eqs. (11) is changed to

\[ [\hat{x}_i, \hat{p}_j] = i\hbar \left( 1 + \frac{\theta \eta}{4\hbar^2} \right) \delta_{ij}. \]  

(13)

In order to maintain the Heisenberg commutation relation, one may introduces an effective Planck constant \(\hbar_{\text{eff}} = \hbar \left( 1 + \theta \eta / 4\hbar^2 \right)\) and explains \(\hbar_{\text{eff}}\) as a modification of the Planck constant by spatial noncommutativity. In order to clarify the real physical meaning of Eq. (13) we consider the linear representation of the deformed annihilation operator by the undeformed one again. By the similar procedure of leading to Eq. (7), for the case \(\xi = 1\) we obtain

\[ \hat{a}_i = a_i + \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a_j, \quad \hat{a}_i^\dagger = a_i^\dagger - \frac{i}{2\hbar} \sqrt{\theta \eta \epsilon_{ij}} a_j^\dagger. \]  

(14)

Eq. (14) leads to the following bosonic commutation relations

\[ [\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = (1 + \frac{\theta \eta}{4\hbar^2}). \]  

(15)

Eq. (14) does not maintain the bosonic commutation relations \([\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1\). The physical meaning of Eq. (13) is similar to Eq. (15). The correct physical explanation of
Eq. (13) is that Eq. (12) does not maintain the Heisenberg commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$.

We can demonstrate that except Eq. (2) any other type of linear transformations between two sets of phase space variables can’t maintain both the deformed Heisenberg-Weyl algebra and the deformed bosonic algebra.

Because the deformed Heisenberg-Weyl algebra and the undeformed one are, respectively, the foundations of noncommutative quantum theories and commutative ones, elucidation of the un-equivalency between two algebras has significant meaning both in theories and experiments. Based on such a un-equivalency one can expect essentially new effects of spatial noncommutativity emerged from noncommutative quantum theories.

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