Vortex membranes in ideal fluids, coadjoint orbits, and characters

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Abstract

In this paper, we describe the coadjoint orbits of the group of volume preserving diffeomorphisms associated to the motion of codimension one singular membranes in ideal fluids, i.e. filaments in 2D, membranes in 3D. We show that they can be identified with a certain class of nonlinear Grassmannian of submanifolds endowed with closed 1-form of a given type and enclosing a given volume, called a decorated nonlinear Grassmannian. Here the orbit symplectic form takes a particularly simple expression. We show that these coadjoint orbits are prequantizable if the period group of the 1-form and the enclosed volume of the submanifold satisfy an integer relation, and we construct explicitly the prequantum bundle in this case, which is also given by a decorated nonlinear Grassmannian. We then find a character for the prequantizable coadjoint orbits, as well as a polarization group on which the character extends, which is a first step beyond prequantization. Finally we interpret geometrically the expression of this character and justify rigorously a construction of this character obtained earlier for membranes in 3D.

1 Introduction

The motion of codimensional one singular membrane solutions of the Euler equations of an ideal fluid is formally given by a Hamiltonian system on particular classes of coadjoint orbits of the group of volume preserving diffeomorphism of the fluid domain. This follows from the geometric interpretation of the solution of the Euler equation of an ideal fluid as geodesics on the group of volume preserving diffeomorphism endowed with the invariant $L^2$ metric \([1]\), together with the understanding of the noncanonical Hamiltonian structure of the Euler equations as a Lie-Poisson bracket arising via reduction by symmetry of the canonical Hamiltonian structure written in terms of the Lagrangian fluid configuration map and its conjugate momentum \([22]\).

In this paper we describe explicitly these coadjoint orbits and their symplectic form in terms of a specific class of nonlinear Grassmannians (manifolds of submanifolds), that we call decorated Grassmannians, and show that under an integral condition, these coadjoint orbits are prequantizable, admit a polarization group, as well as a character that extends to that group. In particular we recover and rigorously justify several steps towards the quantization process of vortex membrane presented in \([9]\).

Descriptions of coadjoint orbits of diffeomorphism groups in terms of nonlinear Grassmannians has been carried out for several situations. For the group of Hamiltonian diffeomorphisms, two classes of coadjoints orbits were described in \([29]\), \([19]\) and \([11]\). These classes were
completed in [7] and obtained via symplectic reduction for the dual pair of momentum maps associated to the Euler equations [5]. For the group of volume preserving diffeomorphisms, coadjoint orbits associated to codimension two singular solutions of the Euler equations were described as nonlinear Grassmannians in [11]. An attempt to describe singular vorticities of codimension one has been done in [15].

The decorated Grassmannians that we obtain can be roughly described as follows. They consists of all codimension one submanifolds of a given type, enclosing a given volume, each of them endowed with a closed 1-form of a given type. While these decorated Grassmannians cannot be identified with a canonical cotangent bundle, the orbit symplectic form takes a particularly simple expression reminiscent of the canonical symplectic form on the cotangent bundle of a nonlinear Grassmannian. This is shown in Section 2.

When the period group of the given closed 1-form is discrete, the 1-form can be written as the logarithmic derivative of some circle valued function, which allows to construct a natural circle bundle over these coadjoint orbits. Under an integral condition relating the enclosed volume and the period group, this circle bundle gives rise to a prequantum bundle, whose connection 1-form can be concretely described, as shown in §3.

In §4, we construct a character that integrates the momenta associated to the prequantizable coadjoint orbits, and we further show the existence of a polarization group on which the character extends, which is a first step beyond prequantization. Finally we interpret geometrically the expression of this character and justify rigorously the steps towards the quantization construction described earlier in [9] for membranes in 3D.

2 Coadjoint orbits of codimension one vortex membranes

In this section, we describe coadjoint orbits associated to the motion of codimension one singular membranes in ideal fluids (filaments in 2D, membranes in 3D). Coadjoint orbits for codimension two membranes (filaments in 3D) were studied in [22], [11].

2.1 The group of exact volume preserving diffeomorphisms

Let \((M, \mu)\) be an \(n\)-dimensional manifold endowed with a volume form. Consider the group

\[
\text{Diff}(M, \mu) = \{ \varphi \in \text{Diff}(M) : \varphi^* \mu = \mu \}
\]

of volume preserving diffeomorphisms and its corresponding Lie algebra of divergence free vector fields \(\mathfrak{X}(M, \mu) = \{ u \in \mathfrak{X}(M) : \mathcal{L}_u \mu = 0 \}\).

If \(M\) is compact, the subgroup \(\text{Diff}_{\text{ex}}(M, \mu)\) of exact volume preserving diffeomorphisms is defined as the kernel of Thurston’s flux homomorphism that integrates \(u \in \mathfrak{X}(M, \mu) \mapsto \int u \mu \in H^{n-1}(M)\), see, e.g., [2]. It is a connected Lie group and its Lie algebra consists of exact divergence free vector fields

\[
\mathfrak{X}_{\text{ex}}(M, \mu) = \{ u \in \mathfrak{X}(M) : \mathbf{i}_u \mu \in B^{n-1}(M) \},
\]

where \(B^{n-1}(M)\) denotes the space of exact \((n - 1)\)-forms on \(M\). An \((n - 2)\)-form \(\alpha\) such that \(\mathbf{i}_u \mu = d\alpha\) is called a potential for the vector field \(u\), in which case we denote \(u = X_\alpha\). If the de Rham cohomology group \(H^{n-1}(M)\) vanishes, then the group of exact volume preserving diffeomorphisms is the identity component of \(\text{Diff}(M, \mu)\).

If \(M\) is noncompact, then the volume form is exact \(\mu = d\nu\). In this case we will always restrict to compactly supported diffeomorphisms (i.e. diffeomorphisms equal to the identity.
map outside a compact set) and compactly supported vector fields on $M$. Thurston’s flux homomorphism becomes $\varphi \mapsto [\varphi^* \nu - \nu]$, so there is a concrete description of the group of exact volume preserving diffeomorphisms, namely the following identity component of the group of compactly supported diffeomorphisms that preserve the $(n-1)$-form $\nu$ up to an exact form:

$$\text{Diff}_{\text{ext}}(M, \mu) = \{ \varphi \in \text{Diff}_c(M) : \varphi^* \nu - \nu \in B^{n-1}(M) \}^0, \quad (2)$$

with corresponding Lie algebra of compactly supported vector fields:

$$\mathfrak{x}_{\text{ext}}(M, \mu) = \{ u \in \mathfrak{x}_c(M) : \mathcal{L}_u \nu \in B^{n-1}(M) \}.$$

Remark that both descriptions don’t depend on the choice of $\nu$ with $d\nu = -\mu$.

**Central extensions.** If $(M, \mu)$ is compact, the Lichnerowicz central extension of the Lie algebra of exact divergence free vector fields [21][25]

$$H^{n-2}(M) \to \mathfrak{x}_{\text{ext}}(M, \mu) := \Omega^{n-2}(M)/B^{n-2}(M) \xrightarrow{d} \mathfrak{x}_{\text{ext}}(M, \mu), \quad (3)$$

using the identification of $\mathfrak{x}_{\text{ext}}(M, \mu)$ with $B^{n-1}(M)$, is endowed with the following Lie bracket on $\Omega^{n-2}(M)/B^{n-2}(M)$:

$$\{[\alpha_1], [\alpha_2]\} = [i_{x_{\alpha_1}} i_{x_{\alpha_2}} \mu].$$

It integrates to the Ismagilov central extension, denoted $\tilde{\text{Diff}}_{\text{ext}}(M, \mu)$, of the group of exact volume preserving diffeomorphisms by a quotient group of the de Rham cohomology group $H^{n-2}(M)$, [13].

When the manifold $M$ is noncompact, in the Lichnerowicz central extension (3) the space $\Omega^{n-2}(M)$ has to be replaced by the space of $(n-2)$-forms that have compactly supported differentials, denoted by $\Omega^{n-2}_c(M)$ (which contains both $\Omega^{n-2}_c(M)$ and the space of closed forms $Z^{n-2}(M)$). Since $\mu = d\nu$, the central extended group $\tilde{\text{Diff}}_{\text{ext}}(M, \mu)$ can be described with a group 2-cocycle,

$$C : \text{Diff}_{\text{ext}}(M, \mu) \times \text{Diff}_{\text{ext}}(M, \mu) \to H^{n-2}(M), \quad (4)$$

due to Ismagilov [13]. Let $b : B^n_c(M) \to \Omega^{n-2}_d(M)$ be a continuous right inverse to $d : \Omega^{n-2}_d(M) \to B^{n-1}_c(M)$, i.e. $db = 1$. Then $P = 1 - bd : \Omega^{n-2}_d(M) \to Z^{n-2}(M)$ is a continuous projection with kernel $\text{Im} b$. Now $\varphi^* \nu - \nu \in B^{n-1}_c(M)$ and the Ismagilov cocycle is given by

$$C(\varphi, \psi) = [b^* b - b^* b^* \varphi^* \psi^*(\varphi^* \nu - \nu)]. \quad (5)$$

The cocycle $C$ depends on the choice of the right inverse $b$, but its cohomology class doesn’t.

**Remark 2.1** For later reference we give here two splittings of the Lichnerowicz extension (3). The Lie algebra 2-cocycle obtained by twice differentiating the group cocycle $C$,

$$\omega_C(u, v) = [P(i_u \mathcal{L}_v \nu - i_v \mathcal{L}_u \nu)] \in H^{n-2}(M), \quad (6)$$

corresponds to the splitting by the section

$$s_C : \mathfrak{x}_{\text{ext}}(M, \mu) \to \Omega^{n-2}_d(M)/B^{n-2}(M), \quad s_C(u) = bi_u \mu + Pi_u \nu. \quad (7)$$

The more familiar cocycle $\omega_L(u, v) = [Pi_u i_v \mu]$, cohomologous to $\omega_C$, corresponds to the section $s_L(u) = bi_u \mu$ [26].
2.2 Principal bundle of embeddings

Let $S$ be a compact oriented manifold with $\dim S < \dim M$. We denote by $\text{Diff}_+(S)$ the group of orientation preserving diffeomorphisms of $S$. Consider the Fréchet manifold $\text{Emb}(S, M)$ of embeddings of $S$ into $M$. We have the principal $\text{Diff}_+(S)$ bundle

$$\pi : \text{Emb}(S, M) \to \text{Gr}^S(M), \quad f \mapsto N = f(S),$$

where $\text{Gr}^S(M)$ is the nonlinear Grassmannian of all oriented submanifolds of $M$ of type $S$. By the infinitesimal transitivity of the $\text{Diff}(M)$ action on $\text{Emb}(S, M)$, see [12], the tangent space at $f$ can be written as $T_f \text{Emb}(S, M) = \{u \circ f : S \to TM : u \in \mathcal{X}(M)\}$. The tangent space to $\text{Gr}^S(M)$ at $N$ is $T_N \text{Gr}^S(M) = \Gamma(TN^\perp)$, the space of sections of the normal bundle $TN^\perp := TM|_N/TN$.

From now on we assume that $M$ is noncompact $n$-dimensional endowed with a volume form $\mu = d\nu$, and $S$ is compact oriented $(n - 1)$-dimensional. We fix $a > 0$ and we define the manifold

$$\text{Emb}_a(S, M) := \{f \in \text{Emb}(S, M) : \int_S f^* \nu = a\},$$

with tangent space at $f$

$$T_f \text{Emb}_a(S, M) = \{u \circ f : S \to TM : u \in \mathcal{X}(M), \int_S f^* i_u \mu = 0\}.$$

By restriction of (8) we get another principal $\text{Diff}_+(S)$ bundle $\pi : \text{Emb}_a(S, M) \to \text{Gr}^S_a(M)$ over the nonlinear Grassmannian

$$\text{Gr}^S_a(M) = \{N \in \text{Gr}^S(M) : \int_N \nu = a\},$$

with tangent space $T_N \text{Gr}^S_a(M) = \{u_N \in \Gamma(TN^\perp) : \int_N i_u \mu = 0\}$.

**Remark 2.2** Independence of the choice of $\nu$ with $d\nu = \mu$ in (9) is achieved by restricting only to those connected components of $\text{Emb}_a(S, M)$ that consist of embeddings $f$ whose image $N := f(S) \subset M$ is the boundary of a compact domain $D \subset M$. Thus the volume enclosed by $N$ is constant:

$$\int_D \mu = \int_N \nu = a.$$

This condition singles out a union of connected components of $\text{Emb}_a(S, M)$ and of $\text{Gr}^S_a(M)$.

**Remark 2.3** With the choice of a Riemannian metric $g$ on $M$, the tangent space to the nonlinear Grassmannian can be identified with

$$T_N \text{Gr}^S_a(M) = \Gamma(TN^\perp_g) \cong C^\infty(N),$$

since any normal vector field on $N$ can be written as $u_N = f_N n_N$, where $n_N$ denotes the unique unit normal vector field to $N$ in $M$ compatible with the orientations. Moreover, $\int_N \int_N i_u \mu = \int_N i_{u_N} \mu = 0$ for the volume form $\mu_N = i_{n_N} \mu$ induced on $N$ by the metric $g$, hence

$$T_N \text{Gr}^S_a(M) \cong C_0^\infty(N) := \{f_N \in C^\infty(N) : \int_N f_N \mu = 0\}. $$

(11)
2.3 Coadjoint orbits as nonlinear Grassmannians

In this section we define the decorated nonlinear Grassmannian \( \text{Gr}^{S,\beta}_{a}(M) \) and identify its connected components with coadjoint orbits of the Ismagilov extension \((4)\). We also obtain an explicit expression of the orbit symplectic form, when a Riemannian metric on \( M \) is chosen.

We fix a closed 1-form \( \beta \) on the compact oriented \((n - 1)\)-dimensional manifold \( S \). We assume that \( \beta \) has only isolated zeros (if any), so, by the compactness of \( S \), it has a finite number of zeros. The group of \( \beta \)-preserving diffeomorphisms \( \text{Diff}(S, \beta) \) is not a Lie group in general, still it has an associated Lie algebra of vector fields

\[
\mathfrak{X}(S, \beta) = \{ v \in \mathfrak{X}(S) : \mathcal{L}_v \beta = 0 \}.
\]

Again we assume that \( M \) is noncompact with \( \mu = d\nu \). We define the decorated nonlinear Grassmannian

\[
\text{Gr}^{S,\beta}_{a}(M) := \{(N, \beta_N) : N \in \text{Gr}^{S}(M), \beta_N \in \Omega^1(N) \text{ s.t. } \exists \Psi \in \text{Diff}(S, N), \Psi^* \beta_N = \beta\}
\]

that consists of all submanifolds \( N \subset M \) of type \( S \), each of them endowed with a 1-form \( \beta_N \) of type \( \beta \), as well as its subset

\[
\text{Gr}^{S,\beta}_{a}(M) := \{(N, \beta_N) \in \text{Gr}^{S,\beta}_{a}(M) : N \in \text{Gr}^{S}_{a}(M)\}.
\]

Note that \( \beta_N \) is closed and has a finite number of zeros (if any), just like \( \beta \). Formally,

\[
\pi^{\beta} : \text{Emb}(S, M) \rightarrow \text{Gr}^{S,\beta}_{a}(M), \quad f \mapsto (N, \beta_N) = (f(S), f_* \beta)
\]

is a principal \( \text{Diff}_+(S, \beta) \) bundle, as well as its restriction

\[
\pi^{\beta} : \text{Emb}_{a}(S, M) \rightarrow \text{Gr}^{S,\beta}_{a}(M). \tag{12}
\]

The forgetful map \( (N, \beta_N) \mapsto N \) provides canonical projections from decorated nonlinear Grassmannians to nonlinear Grassmannians: \( \text{Gr}^{S,\beta}_{a}(M) \rightarrow \text{Gr}^{S}_{a}(M) \) and \( \text{Gr}^{S,\beta}_{a}(M) \rightarrow \text{Gr}^{S}_{a}(M) \). We refer to \([6]\) for the description of principal bundles over several classes of nonlinear Grassmannians.

**Lemma 2.4** Let \( N \) be an orientable codimension one submanifold of the \( n \)-dimensional manifold \( M \) with inclusion \( i_N : N \rightarrow M \). Given \( Y \in \mathfrak{X}(N) \), there exists \( \rho \in \Omega^{n-2}(M) \) such that \( i_N^* \rho = 0 \) and \( X_{\rho}|_{N} = Y \).

**Proof.** We endow \( M \) with a Riemannian metric \( g \). Since \( N \) is orientable, \( TN^{1,g} \) is a trivial line bundle, the trivialization being given by \( TN^{1,g} \rightarrow N \times \mathbb{R} \), \( v_x \mapsto (x, g(v_x, n)) \), for \( n \) the unit normal vector field along \( N \). Thus, we can identify a tubular neighborhood \( U \) of \( N \) in \( M \) with \( N \times (-1, 1) \ni (x, t) \). With this identification, the volume form \( \mu \) reads \( dt \wedge \omega \), where \( \omega \) is a \( t \)-dependent \((n - 1)\)-form on \( N \).

We define \( \widetilde{Y} \in \mathfrak{X}(U) \) by \( \widetilde{Y}(x, t) := Y(x) + 0 \partial_t \) on \( U \). We consider the form \( ti_{\widetilde{Y}} \omega \in \Omega^{n-2}(U) \) and extend it to \( \rho \in \Omega^{n-2}(M) \). Let us check that \( \rho \) satisfies the two required conditions. On one hand, we have \( i_N^* \rho = i_N^* (i_{\widetilde{Y}} \omega) = 0 \), since \( N \subset U \) is characterized by \( t = 0 \). On the other hand, on \( U \) we have \( i_{X_{\rho} - Y} \mu = d\rho - i_{\widetilde{Y}} \mu = d\left(ti_{\widetilde{Y}} \omega\right) - i_{\widetilde{Y}}(dt \wedge \omega) = tdi_{\widetilde{Y}} \omega \). We get \( (X_{\rho} - Y)|_N = 0 \), so \( X_{\rho}|_N = Y \). \( \blacksquare \)

**Proposition 2.5** The Lie algebra \( \mathfrak{X}_{\text{ex}}(M, \mu) \) acts transitively on the manifold \( \text{Emb}_{a}(S, M) \).
Proof. We consider \( u \circ f \in T_f \text{Emb}_a(S, M) \) with \( u \in \mathfrak{X}(M) \), so that \( \int_N i_u \mu = 0 \), where \( N = f(S) \). Therefore, there exists \( \alpha_N \in \Omega^{n-2}(N) \) with \( i_N^* \mu = d\alpha_N \). We choose \( \alpha \in \Omega^{n-2}(M) \) such that \( i_N^* \alpha = \alpha_N \). We have \( i_N^*(i_u - X_\alpha) = d\alpha_N - i_N^* d\alpha = 0 \), hence \( (u - X_\alpha)|_N \in \mathfrak{X}(N) \). By the Lemma 2.4 one can find \( \rho \in \Omega^{n-2}(M) \) with \( i_N^* \rho = 0 \) and such that \( X_\rho|_N = (u - X_\alpha)|_N \). Hence the vector field \( X_{\alpha + \rho} \in \mathfrak{X}_\text{ex}(M, \mu) \) has the property \( u|_N = X_{\alpha + \rho}|_N \), so \( u \circ f = X_{\alpha + \rho} \circ f \), which shows the infinitesimal transitivity of \( \mathfrak{X}_\text{ex}(M, \mu) \) on \( \text{Emb}_a(S, M) \). 

The transitivity of the \( \mathfrak{X}_\text{ex}(M, \mu) \) action on \( \text{Emb}_a(S, M) \), proven above, improves (10) to

\[
T_f \text{Emb}_a(S, M) = \{ u \circ f : S \to TM : u \in \mathfrak{X}_\text{ex}(M, \mu) \}. \tag{13}
\]

We consider the 2-form \( \omega \) on \( \text{Emb}_a(S, M) \) defined at \( f \) by

\[
\omega(u \circ f, v \circ f) = \int_S \beta \wedge f^* i_u \mu, \quad u, v \in \mathfrak{X}_\text{ex}(M, \mu). \tag{14}
\]

In the hat calculus notation [27], \( \omega \) is written as

\[
\omega = \hat{\beta} \cdot \mu = -d(\hat{\beta} \cdot \nu). \tag{15}
\]

Thus \( \omega \) is a closed 2-form. In order to show that it is a \( \pi^\beta \)-basic 2-form, we need to verify \( i_{\zeta_v} \omega = 0 \), where \( \zeta_v(f) = T f \circ v \) is the infinitesimal generator of the \( \text{Diff}(S, \beta) \) action on \( \text{Emb}_a(S, M) \). For all \( v \in \mathfrak{X}(S, \beta) \) and \( u \in \mathfrak{X}_\text{ex}(M, \mu) \),

\[
\omega(Tf \circ v, u \circ f) = \int_S \beta(v) f^* i_u \mu = \beta(v) \int_S f^* i_u \mu = 0. \tag{16}
\]

Since \( \omega \) is a \( \pi^\beta \) basic 2-form, it descends to a closed 2-form on the nonlinear Grassmannian \( \text{Gr}^S,\beta_a(M) \), i.e.

\[
(\pi^\beta)^* \Omega = \omega. \tag{17}
\]

Moreover, the theorem below ensures that \( \Omega \) is a symplectic form.

**Theorem 2.6** Let us denote by \([\alpha]\) with \( \alpha \in \Omega^{n-2}(M) \), the elements of the Lichnerowicz extension \( \mathfrak{X}_\text{ex}(M, \mu) := \Omega^{n-2}(M) / B^{n-2}(M) \). Then the map

\[
J : \text{Gr}^S,\beta_a(M) \to \mathfrak{X}_\text{ex}(M, \mu)^*, \quad (J(N, \beta_N), \alpha) = \int_N \beta_N \wedge i_N^* \alpha, \tag{18}
\]

is a bijection from each connected component of \( \text{Gr}^S,\beta_a(M) \) onto a coadjoint orbit of \( \text{Diff}_\text{ex}(M, \mu) \).

The symplectic form induced on \( \text{Gr}^S,\beta_a(M) \) by the orbit symplectic form \( \omega_{\text{KKS}} \) via \( J \) is the form \( \Omega \) in (17).

**Remark 2.7** The group \( \text{Diff}_\text{ex}(M, \mu) \) acts on the symplectic manifold \( \text{(Gr}^S,\beta_a(M), \Omega) \) in a Hamiltonian way. When passing to the Ismagilov central extension, it admits the equivariant momentum map \( J \) in (18).

**Proof.** We will prove that \( J \) is injective and \( \text{Diff}_\text{ex}(M, \mu) \)-equivariant, and that \( \text{Diff}_\text{ex}(M, \mu) \) acts transitively on connected components of \( \text{Gr}^S,\beta_a(M) \). These three facts will ensure the bijectivity between connected components of \( \text{Gr}^S,\beta_a(M) \) and coadjoint orbits of \( \text{Diff}_\text{ex}(M, \mu) \).

If \( J(N_1, \beta_{N_1}) = J(N_2, \beta_{N_2}) \), then necessarily \( N_1 = N_2 \), since \( \beta_{N_1} \) and \( \beta_{N_2} \) have a finite number of zeros. We thus have

\[
\int_{N_1} (\beta_{N_1} - \beta_{N_2}) \wedge i_{N_1}^* \alpha = 0, \quad \text{for all } \alpha \in \Omega^{n-2}(M) \text{ hence } \beta_{N_1} = \beta_{N_2}. \]

This shows that \( J \) is injective.
To show the equivariance of $J$, we note that for all $\widehat{\varphi} \in \text{Diff}^*_\text{ex}(M, \mu)$ we have $\text{Ad}_{\widehat{\varphi}}[\alpha] = [\varphi_* \alpha]$, where $\varphi \in \text{Diff}^*_\text{ex}(M, \mu)$ is the diffeomorphism induced from $\widehat{\varphi}$. Hence, the equivariance follows from the equality $\langle J(\varphi(N), \varphi_* \beta_N), [\varphi_* \alpha] \rangle = \langle J(N, \beta_N), [\alpha] \rangle$.

By Proposition 2.5, the Lie algebra $\mathfrak{X}_{\text{ex}}(M, \mu)$ acts transitively on $\text{Emb}_a(S, M)$, hence on $\text{Gr}^a_{S, \beta}(M)$. The fact that the constructions in Proposition 2.5 can be done with a smooth dependence on a parameter, ensures the transitivity of the $\text{Diff}^*_\text{ex}(M, \mu)$-action on connected components of $\text{Gr}^a_{S, \beta}(M)$.

Let us fix a coadjoint orbit $\mathcal{O}$ in the image of $J$. For $[\alpha] \in \mathfrak{X}_{\text{ex}}(M, \mu)$, we denote by $\text{ad}^*_{[\alpha]}$ the corresponding infinitesimal generators on $\mathcal{O}$. Using the equivariance of $J$, we have for $f \in \text{Emb}_a(S, M)$ and $\pi^\beta(f) = (N, \beta_N)$ that

$$
\left( (J \circ \pi^\beta)^* \omega_{\text{KKS}} \right)(X_{\alpha_1} \circ f, X_{\alpha_2} \circ f) = \omega_{\text{KKS}} \left( \text{ad}^*_{[\alpha_1]} J(N, \beta_N), \text{ad}^*_{[\alpha_2]} J(N, \beta_N) \right) = (J(N, \beta_N), \{[\alpha_1], [\alpha_2]\}) = \int_N \beta_N \wedge i_N^*(i_{X_{\alpha_1}} i_{X_{\alpha_2}} \theta)
$$

$$
= \omega(X_{\alpha_1}, X_{\alpha_2}) = \left( (\pi^\beta)^* \Omega \right)(X_{\alpha_1} \circ f, X_{\alpha_2} \circ f).
$$

It follows that $\Omega$ is the pull-back by $J$ of the orbit symplectic form $\omega_{\text{KKS}}$. □

**Remark 2.8** In the special cases when either the 1-form $\beta$ is exact or the de Rham cohomology group $H^{n-2}(M)$ vanishes, the coadjoint orbits from Theorem 2.6 are coadjoint orbits of the group $\text{Diff}^*_\text{ex}(M, \mu)$ itself. In each of these two cases, a $\text{Diff}^*_\text{ex}(M, \mu)$-equivariant map $J$ is well-defined as

$$
J : \text{Gr}^a_{S, \beta}(M) \to \mathfrak{X}_{\text{ex}}(M, \mu)^*, \quad \langle J(N, \beta_N), X_\alpha \rangle = \int_N \beta_N \wedge i_N^* \alpha.
$$

Moreover, focusing on the connected component of $(N, \beta_N)$ in $\text{Gr}^a_{S, \beta}(M)$, we notice it is also a coadjoint orbit of $\text{Diff}^*_\text{ex}(M, \mu)$ if the following weaker condition is satisfied: the subspace $i_N^* H^{n-2}(M) \subset H^{n-2}(N)$ lies in the annihilator of $[\beta_N] \in H^1(N)$ via the cup product pairing $H^1(N) \times H^{n-2}(N) \to \mathbb{R}$ in the compact manifold $N$.

**Remark 2.9** The choice of a Riemannian metric $g$ on $M$ splits in a natural way the tangent space to $\text{Gr}^a_{S, \beta}(M)$ (similarly to the tangent space to the weighted nonlinear Grassmannian [7]). The tangent map to $\pi^\beta$ can be written as

$$
T_f \pi^\beta(u \circ f) = \left( u|_{\nabla N}, -L_{u|_{\nabla N}} \beta_N \right),
$$

where $u|_N = u|_{\nabla N} + u|_{\nabla N}$ is the orthogonal decomposition relative to the Riemannian metric. Since $L_{u|_{\nabla N}} \beta_N = d(i_{u|_{\nabla N}} \beta_N)$, this provides the following splitting of the tangent space:

$$
T_{(N, \beta_N)} \text{Gr}^a_{S, \beta}(M) = T_N \text{Gr}^a_{S}(M) \times d(\beta_N(\mathfrak{X}(N))) \quad \text{(11)} \equiv C^\infty_0(N) \times d(\beta_N(\mathfrak{X}(N))).
$$

The tangent space consists of the infinitesimal generators $\zeta_u$ at $(N, \beta_N)$ for $u \in \mathfrak{X}(M, \mu)$, which decompose under the splitting above as

$$
\zeta_u(N, \beta_N) = (f_N, d\lambda_N),
$$

where we write $u|_{\nabla N} = f_N n_N$, i.e. $f_N = g(u, n_N)$, and $-i_{u|_{\nabla N}} \beta_N = \lambda_N$.  

7
Remark 2.11 The expression of $\Omega$ does not depend on the choice of the functions $\lambda_N$ and $\rho_N$ in $\beta_N(\mathcal{X}(N))$, since $f_N\mu_N$ and $g_N\mu_N$ are exact forms.

We check the identity (21) on infinitesimal generators $\zeta_u(N, \beta_N) = (f_N, d\lambda_N)$ and $\zeta_v(N, \beta_N) = (g_N, d\rho_N)$ for $u, v \in \mathcal{X}_{ex}(M, \mu)$, i.e. $u^\top|_N = f_N n_N$ and $-i_{u^\top|_N} \beta_N = \lambda_N$, resp. $v^\top|_N = g_N n_N$ and $-i_{v^\top|_N} \beta_N = \rho_N$. Hence

\[
\Omega_{(N, \beta_N)}((f_N, d\lambda_N), (g_N, d\rho_N)) = \int_N \beta_N \wedge i^*_N (i_{u^\top|_N} \mu)
\]

Only the mixed terms survive at step two: $i_{u^\top|_N} i_{v^\top|_N} \mu = 0$ since all normal vectors to $N$ are colinear and $i^*_N (i_{u^\top|_N} i_{v^\top|_N} \mu) = 0$ since $i^*_N \mu = 0$. This gives the desired expression of the symplectic form.

\[
\Omega_{(N, \beta_N)}((f_N, d\lambda_N), (g_N, d\rho_N)) = \int_N (\rho_N f_N - \lambda_N g_N) \mu_N.
\]

Remark 2.11 The space $\beta_N(\mathcal{X}(N))$ consists of all smooth functions on $N$ that vanish at each zero of $\beta_N$, since the 1-form $\beta_N$ has only a finite number of zeros, and we get a nondegenerate pairing

\[
\mathcal{C}^\infty_0(N) \times d(\beta_N(\mathcal{X}(N))) \to \mathbb{R}, \quad (f_N, d\lambda_N) = \int_N \lambda_N f_N \mu_N.
\]

Thus, using also (11), the tangent space (20) can be written as

\[
T_{(N, \beta_N)} \text{Gr}^{S, \beta}_a(M) = \mathcal{C}^\infty_0(N) \times \mathcal{C}^\infty_0(N)^*.
\]

We note the similarity of the orbit symplectic form (21) to the cotangent bundle symplectic form on $T^* \text{Gr}^S_a(M)$. Still the coadjoint orbit is not a cotangent bundle: the fibers of the projection given by the forgetting map $\text{Gr}^{S, \beta}_a(M) \to \text{Gr}^S_a(M)$, $(N, \beta_N) \mapsto N$, are affine and there is no analogue to the zero section of the cotangent bundle projection $T^* \text{Gr}^S_a(M) \to \text{Gr}^S_a(M)$.

2.4 Low dimensions

In this section we particularize the results of Theorem 2.6 to filaments in 2D ($n = 2$) and membranes in 3D ($n = 3$). In the first case we relate the coadjoint orbits to those obtained by symplectic reduction in the ideal fluid dual pair [7]. The second case is related to the paper [9] of Goldin, Menikoff, and Sharp, where open membranes in 3D are treated (our membranes are closed submanifolds).
Two dimensional case. If $\dim M = 2$ the volume form $\mu = d\nu$ is a symplectic form and $\text{Diff}_{\text{ex}}(M, \mu)$ is the group of compactly supported Hamiltonian diffeomorphisms of the noncompact surface $M$. In this case the Ismagilov group cocycle (5) is cohomologous the ILM cocycle on the Hamiltonian group on $M$ [14], namely

$$(\varphi, \psi) \mapsto \int_{x_0}^{\psi(x_0)} (\varphi^* \nu - \nu) \in H^0(M) = \mathbb{R}, \quad x_0 \in M,$$

that integrates the Kostant-Souriau Lie algebra cocycle $(u, v) \mapsto \mu(u, v)(x_0)$ on the Lie algebra of compactly supported Hamiltonian vector fields on $M$ (with non-zero cohomology class, since $M$ is noncompact). Thus the Lichnerowicz Lie algebra extension is $C^\infty(M)$ with Poisson bracket and $\hat{\text{Diff}}_{\text{ex}}(M, \mu)$ is the quantomorphism group.

Let $S$ be the circle $S^1$ endowed with a 1-form $\beta$ with inly isolated zeros. Theorem 2.6 provides coadjoint orbits of the quantomorphism group, namely connected components of the space of closed curves $N$ in $M$ that enclose a fixed area equal to $a$, each curve $N$ being endowed with a 1-form $\beta_N$.

Ideal fluid dual pair. When $\beta$ has no zeros, these coadjoint orbits of the quantomorphism group are also obtained by symplectic reduction at non-zero momentum in the ideal fluid dual pair, similarly to the reduction at zero performed in [7, Section 6], as we explain below.

Since $\beta$ is a volume form on the circle, $\text{Diff}(S^1, \beta) \cong \text{Rot}(S^1)$, the rotation group. On the manifold of circle embeddings $\text{Emb}(S^1, M)$, the closed 2-form $\omega$ from (15) is weakly nondegenerate, hence symplectic. It comes with two natural Hamiltonian actions: the Hamiltonian group $\text{Diff}_{\text{ex}}(M, \mu)$ acts by composition on the left and the rotation group $\text{Diff}(S^1, \beta) = \text{Rot}(S^1)$ acts by composition on the right. The action of $\text{Diff}(S^1, \beta)$ admits an equivariant momentum map $J_R : \text{Emb}(S^1, M) \to \mathbb{R}$,

$$J_R(f) = \int_{S^1} f^* \nu.$$

In order to get an equivariant momentum map for the action of $\text{Diff}_{\text{ex}}(M, \mu)$ we have to pass to a central extension: the quantomorphism group (with Lie algebra $C^\infty(M)$), so

$$J_L : \text{Emb}(S^1, M) \to C^\infty(M)^*, \quad J_L(f) = f_* \beta.$$

The momentum maps $(J_L, J_R)$ form a dual pair and we can apply the following result: symplectic reduction for one group provides coadjoint orbits for the other group. The symplectic reduced space $J_R^{-1}(a)/\text{Rot}(S^1)$ is the decorated nonlinear Grassmannian $\text{Gr}^{S^1, \beta}(M)$ of all closed curves $N \subset M$ enclosing a given volume $a$, each of them endowed with a volume form $\beta_N$ of type $\beta$. The momentum map induced by $J_L$ on $\text{Gr}^{S^1, \beta}(M)$ recovers the momentum map (18).

Three dimensional case. If $M = \mathbb{R}^3$ with standard volume form, we are in the setting considered in [9]. We fix a Riemannian metric $g$ on $\mathbb{R}^3$, we denote by $n_N$ the unit normal vector field along the oriented surface $N \subset \mathbb{R}^3$, and by $\mu_N = i_{n_N}^* n_N \mu$ the induced volume form on $N$. The momentum map, see Remark 2.8,

$$\langle J(N, \beta_N), X_\alpha \rangle = \int_N \beta_N \wedge i_N^* \alpha$$

can be written in terms of vector fields as in [9]. Let $\alpha^2 \in \mathfrak{X}(\mathbb{R}^3)$ be obtained from the potential $\alpha \in \Omega^1(\mathbb{R}^3)$ by rising indices with the Riemannian metric, and let $v_N \in \mathfrak{X}(N)$ be
uniquely defined by $i_{v_N} \mu_N = -\beta_N$. Then the momentum reads
\[
(J(N, \beta_N), X_\alpha) = - \int_N i_{v_N} \mu_N \wedge i_N^* \alpha = \int_N \alpha(v_N) \mu_N = \int_N g(v_N, \alpha^\sharp|N) \mu_N.
\]

3 Prequantization of coadjoint orbits

3.1 Flux homomorphism for closed 1-forms

In this paragraph we shall define a normal subgroup of $\text{Diff}(S, \beta)$ that is needed for the construction of the prequantum bundle. Let $S$ be a compact connected manifold. Given a closed 1-form $\beta \in \Omega^1(S)$, recall that the period group of $\beta$ is the subgroup of $\mathbb{R}$ defined by
\[
\text{Per}_\beta := \left\{ \int_\gamma \beta : [\gamma] \in H_1(S, \mathbb{Z}) \right\}.
\]
From now on we will assume that the period group of $\beta$ is discrete.

**Lemma 3.1** If the discrete period group of $\beta$ is $\text{Per}_\beta = r \mathbb{Z}$, for $r > 0$, then there exists $F : S \to \mathbb{R}/r \mathbb{Z} =: S^1_r$ such that $\beta = F^* \theta$, where $\theta \in \Omega^1(S^1_r)$ denotes the angle element (i.e. $\beta$ is the logarithmic derivative of $F$, and $F$ is a Cartan developing of $\beta$).

Moreover, if $\beta$ has no zeros, then $F$ is a submersion.

**Proof.** Denoting by $\tilde{\pi} : \tilde{S} \to S$ the universal covering space of $S$, there exists $f \in C^\infty(\tilde{S})$ such that $df = \pi^* \beta$. Since the period group $\text{Per}_\beta$ is $r \mathbb{Z}$, the map $f$ descends to a smooth map $F : S \to \mathbb{R}/r \mathbb{Z} = S^1_r$. Then the pull-back of the angle element by $F$ is $\beta$, because $\pi^* F^* \theta = df = \pi^* \beta$.

The map
\[
c_\beta : \text{Diff}(S, \beta) \to S^1_r, \quad c_\beta(\varphi) := (F \circ \varphi^{-1})/F
\] (22)
is a well-defined group homomorphism, called the flux homomorphism of $\beta$. Indeed, because $\beta = F^* \theta$ is preserved by $\varphi$, the map $(F \circ \varphi^{-1})/F : S \to S^1_r$ is constant. Thus
\[
c_\beta(\varphi_1 \circ \varphi_2) = ((F \circ \varphi_1^{-1})/F) \circ \varphi_1^{-1} \circ (F \circ \varphi_2^{-1})/F) \circ \varphi_2^{-1} = c_\beta(\varphi_1)c_\beta(\varphi_2).
\]
The flux homomorphism $c_\beta$ does not depend on the choice of $F : S \to S^1_r$ with $F^* \theta = \beta$ since, for a given $\beta$, the map $F$ is defined up to multiplication by a constant in $S^1_r$.

The derivative of the flux homomorphism $c_\beta$ at the identity is the Lie algebra homomorphism
\[
d_c c_\beta : \mathfrak{X}(S, \beta) \to \mathbb{R}, \quad d_c c_\beta(v) = -\beta(v).
\] (23)
The kernel of the flux homomorphism is the normal subgroup
\[
\text{Diff}_\text{ex}(S, \beta) = \{ \varphi \in \text{Diff}(S) : F \circ \varphi = F \} \subseteq \text{Diff}(S, \beta),
\]
which doesn’t depend on the choice of $F$. It formally integrates the Lie subalgebra
\[
\mathfrak{X}_\text{ex}(S, \beta) = \{ v \in \mathfrak{X}(S, \beta) : \beta(v) = 0 \} \subseteq \mathfrak{X}(S, \beta).
\]
Remark 3.2 If \( \beta \) has no zeros, then \( \mathcal{X}_{\text{ex}}(S, \beta) \) is a codimension one Lie subalgebra of \( \mathcal{X}(S, \beta) \) and \( \text{Diff}_{\text{ex}}(S, \beta) \) is a codimension one Lie subgroup. The homomorphism \( c_{\beta} \) is surjective, so the quotient group \( \text{Diff}(S, \beta)/\text{Diff}_{\text{ex}}(S, \beta) \) is isomorphic to the circle \( S^1_r \).

If \( \beta \) has isolated zeros, then \( \mathcal{X}_{\text{ex}}(S, \beta) = \mathcal{X}(S, \beta) \), because (23) is the trivial Lie algebra homomorphism. The image of \( c_{\beta} \) is a finite subgroup of \( S^1_r \) and \( \text{Diff}_{\text{ex}}(S, \beta) \) is a subgroup of finite order of \( \text{Diff}(S, \beta) \).

We now define several objects that are needed for the construction of the characters later in §4. This needs the prequantization condition \( ra \in \mathbb{Z} \), discussed in §3.2.

Lemma 3.3 Let \( F : S \to S^1_r \) be as in Lemma 3.1. Then for all \( \varphi \in \text{Diff}(S)_0 \), the map \((F \circ \varphi^{-1})/F : S \to S^1_r \) is liftable to a smooth map \( f_{\varphi} \in \mathcal{C}^\infty(S) \), i.e.

\[
(F \circ \varphi^{-1})/F = f_{\varphi} \mod \mathbb{Z}.
\]

Moreover, the lifts satisfy a 1-cocycle identity modulo \( r\mathbb{Z} \):

\[
f_{\varphi \psi} - \varphi_* f_{\psi} - f_{\varphi} \in r\mathbb{Z}.
\]

Proof. Let \( \varphi \in \text{Diff}(S)_0 \) and \( F_{\varphi} := (F \circ \varphi^{-1})/F : S \to S^1_r \). The pullback of the angle element \( F_{\varphi}^* \theta = (\varphi^{-1})^* \beta - \beta \) is exact, since \( \varphi \) preserves the de Rham cohomology class \([\beta] \in H^1(S)\).

This ensures that \( F_{\varphi} \) is liftable to a map \( f_{\varphi} : S \to \mathbb{R} \), namely \( f_{\varphi}(x) = \int_{x_0}^x F_{\varphi}^* \theta \) with \( x_0 \in S \).

Let us assume that \( ra \in \mathbb{Z} \). In this case, the multiplication by \( a \) is a well defined group homomorphism

\[
m_a : S^1_r = \mathbb{R}/r\mathbb{Z} \to S^1 = \mathbb{R}/\mathbb{Z}, \quad x \mod r\mathbb{Z} \mapsto ax \mod \mathbb{Z}.
\]

By composing with the flux homomorphism \( c_{\beta} \) defined in (22), we get a circle valued group homomorphism:

\[
c_{\beta}^a : \text{Diff}(S, \beta) \to S^1, \quad c_{\beta}^a = m_a \circ c_{\beta}.
\]

Next we endow \( S \) with a differential form \( \nu_S \) of maximal degree (not necessarily a volume form) such that \( \int_S \nu_S = a \). By Lemma 3.3, the map

\[
\kappa_{\beta} : \text{Diff}(S)_0 \to S^1 = \mathbb{R}/\mathbb{Z}, \quad \kappa_{\beta}(\varphi) = \int_S f_{\varphi} \nu_S \mod \mathbb{Z}
\]

is well defined since any two lifts of \( F_{\varphi} \) differ by an integral multiple of \( r \) and \( ra \in \mathbb{Z} \). Moreover, the restriction of \( \kappa_{\beta} \) to the subgroup \( \text{Diff}(S, \beta) \cap \text{Diff}(S)_0 \) is a group homomorphism which coincides with \( c_{\beta}^a \) restricted to this subgroup. Indeed, the map \( f_{\varphi} \) is constant for \( \varphi \in \text{Diff}(S, \beta) \cap \text{Diff}(S)_0 \), so

\[
\kappa_{\beta}(\varphi) \equiv a f_{\varphi} \mod \mathbb{Z} = m_a(f_{\varphi} \mod r\mathbb{Z}) \equiv m_a(c_{\beta}(\varphi)) \equiv c_{\beta}^a(\varphi).
\]

3.2 The prequantum bundle over \( \text{Gr}^{S, \beta}_a(M) \)

In this section we assume that the closed 1-form \( \beta \) with discrete period group \( \text{Per}_{\beta} = r\mathbb{Z} \) has no zeros, see Remark 3.2. We fix a function \( F : S \to S^1_r \) with \( F^* \theta = \beta \) as in Lemma 3.1, uniquely defined up to a constant factor.
We formally have a principal $\text{Diff}(S, \beta)$ endowed with volume form $\mu = d\nu$ for $\nu \in \Omega^{n-1}(M)$:

$$\text{Gr}_{\text{ex}}^{S, \beta}(M) = \text{Emb}(S, M)/\text{Diff}_{\text{ex}}(S, \beta)$$

$$= \{ (N, F_N) \in \text{Gr}^S(M) \times C^\infty(N, S^1_r) : \exists \Psi \in \text{Diff}(S, N), \Psi^* F_N = F \}. \quad (28)$$

We formally have a principal $\text{Diff}_{\text{ex}}(S, \beta)$ bundle

$$\pi^\beta_{\text{ex}} : \text{Emb}(S, M) \to \text{Gr}_{\text{ex}}^{S, \beta}(M), \quad f \mapsto (N, F_N) = (f(S), f_* F). \quad (29)$$

The quotient group $\text{Diff}(S, \beta)/\text{Diff}_{\text{ex}}(S, \beta)$ is isomorphic to $S^1_r = \mathbb{R}/r\mathbb{Z}$, the fiber of the bundle

$$\Pi := \text{Gr}_{\text{ex}}^{S, \beta}(M) \to \text{Gr}^{S, \beta}(M), \quad (N, F_N) \mapsto (N, F_N\theta). \quad (30)$$

From $F^* \theta = \beta$ we obtain that $\Pi \circ \pi^\beta_{\text{ex}} = \pi^\beta$.

**The circle action.** The $S^1_r$ action on $\text{Gr}_{\text{ex}}^{S, \beta}(M)$, reminiscent from the principal $\text{Diff}(S, \beta)$ action on the manifold of embeddings, i.e.

$$c_\beta(\varphi) \cdot \pi^\beta_{\text{ex}}(f) = \pi^\beta_{\text{ex}}(f \circ \varphi), \quad (31)$$

can be written simply as

$$z \cdot (N, F_N) = (N, z F_N), \quad z \in S^1_r. \quad (32)$$

Indeed, since the flux homomorphism $c_\beta$ is surjective, it is sufficient to check the formula for $z = c_\beta(\varphi)$ with $\varphi \in \text{Diff}(S, \beta)$ and $(N, F_N) = \pi^\beta_{\text{ex}}(f)$:

$$z \cdot (N, F_N) = c_\beta(\varphi) \cdot \pi^\beta_{\text{ex}}(f) \overset{(31)}{=} \pi^\beta_{\text{ex}}(f \circ \varphi)$$

$$= (f(\varphi(S)), f_*(F \circ \varphi^{-1})) = (f(S), c_\beta(\varphi)f_* F) \overset{(29)}{=} (N, z F_N),$$

where we use $c_\beta(\varphi) F = F \circ \varphi^{-1}$ at step four.

The expression (32) of the action also shows that the definition (28) of the nonlinear Grassmannian $\text{Gr}_{\text{ex}}^{S, \beta}(M)$ doesn’t depend on the choice of the map $F$ associated to $\beta$.

**Lemma 3.4** The action (32) makes the circle bundle (30) into a principal $S^1_r$-bundle.

Similarly to (28), we define the subset

$$\mathcal{P} := \text{Gr}_{a, \text{ex}}^{S, \beta}(M) = \{(N, F_N) \in \text{Gr}_{\text{ex}}^{S, \beta}(M) : N \in \text{Gr}^S(M)\}$$

the base of the principal $\text{Diff}_{\text{ex}}(S, \beta)$ bundle

$$\pi^\beta_{\text{ex}} : \text{Emb}_a(S, M) \to \text{Gr}_{a, \text{ex}}^{S, \beta}(M). \quad (33)$$

By restriction of the principal bundle from Lemma 3.4 to $\mathcal{P}$, we get a principal $S^1_r$ bundle over the symplectic manifold $(\text{Gr}_{a}^{S, \beta}(M), \Omega)$ from Theorem 2.6:

$$\Pi : \mathcal{P} = \text{Gr}_{a, \text{ex}}^{S, \beta}(M) \to \text{Gr}_{a}^{S, \beta}(M), \quad \Pi(N, F_N) = (N, F_N\theta). \quad (34)$$

Next we show that the pullback $\Pi^* \Omega$ of the symplectic form $\Omega$ is exact. We consider the 1-form on $\text{Emb}_a(S, M)$ given by $\theta := -\beta \cdot \nu$, namely,

$$\theta(u \circ f, v \circ f) = \int_S \beta \wedge f^* i_a \nu, \quad (35)$$
for all \( u \in \mathfrak{X}_{\text{ex}}(M, \mu) \) (see (13)). We first show the identity \( \omega = d\theta \) with \( \omega \) given in (14). Using the infinitesimal generators for the \( \mathfrak{X}_{\text{ex}}(M, \mu) \) action on \( \text{Emb}_a(S, M) \), namely \( \zeta_u(f) = u \circ f \):

\[
d\theta(\zeta_u, \zeta_v)(f) = \mathcal{L}_{\zeta_u}(\theta(\zeta_v)(f)) - \mathcal{L}_{\zeta_v}(\theta(\zeta_u)(f)) - \theta(\zeta_{[u,v]})(f)
= -\int_S \beta \wedge f^* \mathcal{L}_\nu f \nu + \int_S \beta \wedge f^* \mathcal{L}_\nu \nu + \int_S \beta \wedge f^* [u,v] \nu
= \int_S \beta \wedge f^* [u,v] d\nu = \omega(\zeta_u, \zeta_v)(f).
\]

The 1-form \( \theta \) is \( \pi^\beta_{\text{ex}} \) basic: the infinitesimal generator of \( v \in \mathfrak{X}_{\text{ex}}(S, \beta) \), namely \( \zeta_v(f) = Tf \circ v \), annihilates both forms \( d\theta = \omega \) and \( \theta \) by (16) and

\[
\theta(Tf \circ v) = -\int_S \beta \wedge \mathcal{L}_u f \nu = -\beta(v) \int_S f^* \nu = -a\beta(v) = 0.
\]

Thus the \( \text{Diff}(S, \beta) \) invariant 1-form \( \theta \) descends to an \( S^1_\beta \) invariant 1-form \( \Theta \) on \( \text{Gr}^\beta_{a,\text{ex}}(M) \), i.e. \( (\pi^\beta_{\text{ex}})^* \Theta = \theta \). Now the calculation \( (\pi^\beta_{\text{ex}})^* \Pi^* \Omega = (\pi^\beta_{\text{ex}})^* \omega = d\theta = d(\pi^\beta_{\text{ex}})^* \Theta = (\pi^\beta_{\text{ex}})^* d\Theta \) ensures that \( \Pi^* \Omega = d\Theta \). We resume these facts in the next proposition.

**Proposition 3.5** The circle bundle \( \Pi : \mathcal{P} = \text{Gr}^\beta_{a,\text{ex}}(M) \rightarrow \text{Gr}^\beta_a(M) \) is endowed with a principal \( S^1_\beta \) action and an \( S^1_\beta \)-invariant 1-form \( \Theta \) on \( \mathcal{P} \) that satisfies \( \Pi^* \Omega = d\Theta \).

Note that \( \mathcal{P} \) is not yet a prequantum bundle, since the 1-form \( \Theta \) doesn’t reproduce the infinitesimal generators of the principal \( S^1_\beta \) action.

**Remark 3.6** When fixing a Riemannian metric \( g \) on \( M \), the tangent map to \( \pi^\beta_{\text{ex}} \) in (33) can be written similarly to (19) as

\[
T_f \pi^\beta_{\text{ex}}(u \circ f) = \left( g(u, n_N), -i_u N \beta_N \right), \quad \beta_N = f_* \beta.
\]

Since \( \beta \) has no zeros we have \( \beta_N(\mathcal{X}(N)) = C^\infty(N) \), so the tangent space at \( (N, \beta_N) \) reads

\[
T_{(N,F_N)} \mathcal{P} = C^\infty_0(N) \times C^\infty(N).
\]

Under this splitting of the tangent space, the infinitesimal generator for the \( \mathfrak{X}_{\text{ex}}(M, \mu) \) action on \( \mathcal{P} \), namely \( \zeta_u(N, F_N) = (f_N, \lambda_N) \), where \( f_N = g(u, n_N) \) and \( \lambda_N = -i_u N \beta_N \). Thus the 1-form \( \Theta \) can be expressed as

\[
\Theta(N,F_N)(f_N, \lambda_N) = \Theta(\zeta_u)(N, F_N) \overset{(35)}{=} \int_N \beta_N \wedge i_u N \nu = \int_N \lambda_N i_u N \nu - \int f_N \beta_N \wedge i_u N \nu.
\]

**The prequantum bundle.** Let \( x \in \mathbb{R} \), choose \( v \in \mathfrak{X}(S, \beta) \) with \( \beta(v) = -x \), and let \( \phi_t^v \in \text{Diff}(S, \beta) \) denote the flow of the vector field \( v \in \mathfrak{X}(S, \beta) \). Then \( \frac{d}{dt} |_{t=0} c_\beta(\phi_t^v) = -\beta(v) = x \), so the infinitesimal generator \( \zeta_x \) for the \( S^1_\beta \)-action on \( \mathcal{P} \) can be computed with the help of the curve \( c_\beta(\phi_t^v) \in S^1_\beta \). From the identity

\[
\Theta \left( \frac{d}{dt} |_{t=0} (c_\beta(\phi_t^v) \cdot (N, F_N)) \right) = -a\beta(v)
\]

it follows that, for all \( (N, F_N) \in \mathcal{P} \),

\[
\Theta(\zeta_x(N, F_N)) = ax, \quad x \in \mathbb{R},
\]
so Θ doesn’t reproduce the infinitesimal generators of the principal action.

To prove (39), we use \((\pi^β_{ex})^*Θ = θ\) to compute for \((N, F_N) = π^β_{ex}(f)\) with \(f ∈ Emb_q(S, M)\):

\[
Θ\left(\frac{d}{dt}\big|_0 (c_β(φ^v_1) \cdot (N, F_N))\right) = Θ\left(\frac{d}{dt}\big|_0 (c_β(φ^v_1) \cdot π^β_{ex}(f))\right) = Θ\left(\frac{d}{dt}\big|_0 π^β_{ex}(f \circ φ^v_1)\right)
\]

\[
= (π^β_{ex})^*Θ\left(\frac{d}{dt}\big|_0 (f \circ φ^v_1)\right) = θ(Tf \circ φ^v_1) \overset{(36)}{=} -a_β(v).
\]

Now the prequantization condition \(k = ra ∈ N\) comes into play. The kernel of the surjective homomorphism \(m_a : S^1_r → S^1\) in (25) is isomorphic to \(Ζ_k\) and \(S^1_r/Ζ_k ≃ S^1\). We consider the quotient taken relative to the principal \(S^1_r\) action restricted to this kernel \(Ζ_k\):

\[
q : \mathcal{P} → \tilde{\mathcal{P}} := \mathcal{P}/Ζ_k = Gr^a_{S^1_r, ex}(M)/Ζ_k,
\]

We notice that the bundle projection \(Π\) in (30) descends to

\[
\tilde{Π} : \tilde{\mathcal{P}} → Gr^a_{S^1_r}(M)
\]

i.e. \(\tilde{Π} \circ q = Π\), and the 1-form \(Θ\) descends to a 1-form \(\tilde{Θ}\) on \(\tilde{\mathcal{P}}\), i.e. \(q^*Θ = Θ\). Moreover, the principal \(S^1_r\) action on \(\mathcal{P}\) descends to a principal \(S^1\) action on \(\tilde{\mathcal{P}}\),

\[
m_a(z) \cdot q(N, F_N) := q(z \cdot (N, F_N)), \quad z ∈ S^1_r,
\]

(42)

where \(m_a(z)\) runs through the whole \(S^1\). We denote by \(ζ_x \) with \(x ∈ R\) the infinitesimal generators for the circle action on \(\mathcal{P}\).

**Theorem 3.7** If \(k = ra ∈ N\), the symplectic manifold \((Gr^a_{S^1_r}(M), Ω)\) is prequantizable, with prequantum bundle given by the principal \(S^1\) bundle \(Π : \tilde{\mathcal{P}} → Gr^a_{S^1_r}(M)\) with principal connection 1-form \(Θ\).

**Proof.** The 1-form \(Θ\) is \(S^1\)-invariant since \(Θ\) is \(S^1_r\)-invariant. To show that is is a principal connection, it remains to verify that it reproduces the generators of the infinitesimal principal \(S^1\)-action. Note that, by (42), the infinitesimal generators \(ζ_x ∈ Χ(\mathcal{P})\) and \(ζ_{ax} ∈ Χ(\tilde{\mathcal{P}})\) are \(q\)-related. Now, since \(q^*Θ = Θ\), the identity (40) implies that

\[
Θ(ζ_x(q(N, F_N))) = x, \quad x ∈ R.
\]

Thus the 1-form \(Θ\) is indeed a principal connection. That its curvature is the symplectic form \(Ω\), i.e. \(dΘ = Π^*Ω\), follows from Proposition 3.5.

**Corollary 3.8** For \(k = ra ∈ N\), all the coadjoint orbits of the central extension \(\widehat{Diff}_{ex}(M, μ)\), in Theorem 2.6, are prequantizable.

### 4 Characters and polarizations

In this section, assuming that the prequantization condition \(k = ra ∈ N\) holds, we build a character for the momentum \(J(N, β_N) ∈ Χ_{ex}(M, μ)^*\) on the Lichnerowicz extension, where \((N, β_N) ∈ Gr^{S^1_r}_{a, ex}(M)\), i.e. \((J(N, β_N), [α]) = ∫_N β_N ∧ i^*_N α\). Then we present a polarization for the momentum \(J(N, β_N)\) that extends the one defined in [9] to closed membranes, and we show how to extend the character described in section 4 to the polarization group.

A character associated to a momentum \(m ∈ g^*\) is a group homomorphism \(χ : G_m → S^1\) defined on the coadjoint isotropy subgroup associated to \(m\), that integrates the Lie algebra
homomorphism obtained by the restriction of the momentum to its isotropy Lie algebra \( \mathfrak{g}_m \), i.e. \( d_x \chi = m|_{\mathfrak{g}_m} \). It defines a line bundle \( G \times \chi \mathbb{C} \) over the coadjoint orbit \( O_m = G/G_m \).

A polarization group associated to a momentum \( m \in \mathfrak{g}^* \) is a Lie group \( H \) such that \( G_m \subset H \subset G \), whose Lie algebra \( \mathfrak{h} \) satisfies \( m|_{[\mathfrak{h}, \mathfrak{h}]} = 0 \). Moreover, if the character \( \chi \) can be extended to a group homomorphism \( \tilde{\chi} \) on \( H \), then we also get a polarization line bundle \( G \times \tilde{\chi} \mathbb{C} \) over the configuration space \( G/H \). In finite dimensions it is also asked that \( \dim G/H = \frac{1}{2} \dim G/G_m \), so the configuration space is half the dimension of the coadjoint orbit.

Let us first assume that \( H^{n-2}(M) = 0 \). We will consider the general case in \( \S 4.4 \).

### 4.1 Characters on \( \text{Diff}_\text{ex}(M, \mu)_{\beta_N} \)

Since \( H^{n-2}(M) = 0 \), there is no need to consider the Lichnerowicz extension, so \( J(N, \beta_N) \in \mathfrak{x}_{\text{ex}}(M, \mu)^* \). Since the momentum map \( J \) is equivariant, the isotropy subgroups of \( \text{Diff}_\text{ex}(M, \mu) \) at \((N, \beta_N)\) and at \((N, \beta_N)\) coincide:

\[
\text{Diff}_\text{ex}(M, \mu)_{\beta_N} = \{ \psi \in \text{Diff}_\text{ex}(M, \mu) : \psi(N) = N, \psi^* \beta_N = \beta_N \}
\]

with Lie algebra

\[
\mathfrak{x}_{\text{ex}}(M, \mu)_{\beta_N} = \{ v \in \mathfrak{x}_{\text{ex}}(M, \mu) : v|_N \in \mathfrak{x}(N), \beta_N(v) \text{ constant} \}.
\]

**Lemma 4.1** If \( \mu = d\nu \) and \( H^{n-2}(M) = 0 \), then for all \((N, \beta_N) \in \text{Gr}^{S, \beta}(M)\) the map

\[
\sigma_{\beta_N} : \text{Diff}_\text{ex}(M, \mu) \to \mathbb{R}, \quad \sigma_{\beta_N}(\psi) = \int_N \beta_N \wedge \iota_N^*d^{-1}(\psi^*\nu - \nu)
\]

is well-defined and verifies

\[
\sigma_{\beta_N}(\psi_1 \circ \psi_2) = \sigma_{(\psi_2)_*\beta_N}(\psi_1) + \sigma_{\beta_N}(\psi_2).
\]

Viewed as a map \( \sigma : \text{Diff}_\text{ex}(M, \mu) \to C^\infty(\text{Gr}^{S, \beta}(M)) \), it is a group 1-cocycle with derivative at the identity the Lie algebra 1-cocycle

\[
d_x \sigma_{\beta_N}(v) = \int_N \beta_N \wedge \iota_N^*d^{-1} \mathcal{L}_v \nu, \quad v \in \mathfrak{x}_{\text{ex}}(M, \mu).
\]

**Proof.** To \( \psi \in \text{Diff}_\text{ex}(M, \mu) \), we associate the exact form \( \psi^*\nu - \nu = d\tau \) and \( \sigma_{\beta_N}(\psi) = \int_N \beta_N \wedge \iota_N^*\tau \). It is well defined: if \( \tau_1, \tau_2 \) are two such \((n-2)\)-forms with \( d\tau_1 = d\tau_2 = \psi^*\nu - \nu \), then \( \tau_1 - \tau_2 \) is closed, hence exact by \( H^{n-2}(M) = 0 \), and we have \( \int_N \beta_N \wedge \iota_N^*(\tau_1 - \tau_2) = 0 \), since \( \beta_N \) is closed.

Now we verify the cocycle condition. For all \( \psi_1, \psi_2 \in \text{Diff}_\text{ex}(M, \mu) \), we compute

\[
\sigma_{\beta_N}(\psi_1 \circ \psi_2) = \int_N \beta_N \wedge \iota_N^*d^{-1}(\psi_1^*\psi_2^*\nu - \nu) = \int_N \beta_N \wedge \iota_N^*d^{-1}(\psi_2^*\psi_1^*\nu - \psi_2^*\nu)
\]

\[
+ \int_N \beta_N \wedge \iota_N^*d^{-1}(\psi_2^*\nu - \nu) = \int_N \beta_N \wedge (\psi_2|_N)^*\iota_{\psi_2(N)}^*d^{-1}(\psi_1^*\nu - \nu) + \sigma_{\beta_N}(\psi_2)
\]

\[
= \int_{\psi_2(N)} (\psi_2)_*\beta_N \wedge \iota_{\psi_2(N)}^*d^{-1}(\psi_1^*\nu - \nu) + \sigma_{\beta_N}(\psi_2) = (\psi_2)_*\sigma_{\beta_N}(\psi_1) + \sigma_{\beta_N}(\psi_2),
\]

using at step three the obvious identity \( \psi_2 \circ \iota_N = \iota_{\psi_2(N)} \circ \psi_2|_N \).

Note that if \( \psi_2 \) belongs to the isotropy subgroup \( \text{Diff}_\text{ex}(M, \mu)_{\beta_N} \), i.e. \( \psi_2(N) = N \) and \( (\psi_2)_*\beta_N = \beta_N \), then we have that \( \sigma_{\beta_N} \) is a group homomorphism by (43).
The momentum $J(N, \beta_N)$ at $v = X_\alpha$ can be written involving $\sigma_{\beta_N}$ as

$$
\langle J(N, \beta_N), v \rangle = \int_N \beta_N \wedge i_N^o \alpha = \int_N \beta_N \wedge i_N^o d^{-1}(i_\nu \mu) = \int_N \beta_N \wedge i_N^o d^{-1}(L_\nu \nu - d(i_\nu \nu)) = d_e \sigma_{\beta_N}(v) - \int_N \beta_N \wedge i_N^o i_\nu \nu. \tag{45}
$$

If, moreover, $v$ lies in the isotropy Lie algebra $\mathfrak{x}_\text{ex}(M, \mu)_{\beta_N}$ of $(N, \beta_N) \in \text{Gr}_{\alpha}^{S, \beta}(M)$, i.e. $v|_N \in \mathfrak{x}(N)$ and $\beta_N(v)$ is constant, then we get a further description of the momentum:

$$
\langle J(N, \beta_N), v \rangle = d_e \sigma_{\beta_N}(v) - \int_N i|_N \beta_N \wedge i_N^o \nu = d_e \sigma_{\beta_N}(v) - a \beta_N(v),
$$
since $\int_N \nu = a$.

From (23) and (44) we deduce the character for the momentum $J(N, \beta_N)$.

**Theorem 4.2** Assume that $\text{Per}_{\beta} = r\mathbb{Z}$ and the prequantization condition $k = ra \in \mathbb{N}$ holds. If $\mu = d\nu$ and $H^{n-2}(M) = 0$, then a character for the momentum $J(N, \beta_N) \in \mathfrak{x}_\text{ex}(M, \mu)^*$, with $(N, \beta_N) \in \text{Gr}_{\alpha}^{S, \beta}(M)$, is given by

$$
\chi_{\beta_N} : \text{Diff}_{\text{ex}}(M, \mu)_{\beta_N} \to S^1, \quad \chi_{\beta_N}(\psi) = c_{\beta_N}^a(\psi|_N)e^{2\pi i \sigma_{\beta_N}(\psi)},
$$

where $\sigma_{\beta_N}$ is defined in Lemma 4.1 and $c_{\beta_N}^a : \text{Diff}(N, \beta_N) \to S^1$ in (26).

### 4.2 The polarization group

Let $\text{Diff}_{\text{ex}}(M, \mu)_N$ denote the subgroup of $\text{Diff}_{\text{ex}}(M, \mu)_{\beta_N}$ that consists of all exact volume preserving diffeomorphisms of $M$ that leave $N$ invariant as a set. The role of the polarization group is played by its identity component, denoted by $H$. The corresponding Lie algebra $\mathfrak{h} = \mathfrak{x}_\text{ex}(M, \mu)_N$, which consists of those exact divergence free vector fields that are tangent to $N$, satisfies the polarization condition $J(N, \beta_N)|_{[h, h]} = 0$. Indeed, for all $X_{\alpha_1}, X_{\alpha_2} \in \mathfrak{h}$, we have

$$
\langle J(N, \beta_N), [X_{\alpha_1}, X_{\alpha_2}] \rangle = \int_N \beta_N \wedge i_{[X_{\alpha_1}, X_{\alpha_2}]}^o i_{[X_{\alpha_1}, X_{\alpha_2}]}^o \mu = 0,
$$

using that fact that $i_{[X_{\alpha_1}, X_{\alpha_2}]}^o i_{[X_{\alpha_1}, X_{\alpha_2}]}$ is a potential for the divergence free vector field $[X_{\alpha_1}, X_{\alpha_2}]$.

Let $F_N : N \to S^1_\pi$ be such that $F_N^* \theta = \beta_N$ for the angle element $\theta \in \Omega^1(S^1_\pi)$, and let $\varphi \in \text{Diff}_{\text{ex}}(M, \mu)_N$, so that $\varphi|_N \in \text{Diff}(N)$, and, moreover, it is isotopic to the identity. Recall from Lemma 3.3 that we can lift the map $(F|_N \circ \varphi^{-1}|_N)F_N^{-1} : N \to S^1_\pi$ to a map $f_{\varphi} \in C^\infty(N)$, modulo $r\mathbb{Z}$. As in (27) we can defined the map

$$
\kappa_{\beta_N} : \text{Diff}(N)_0 \to S^1, \quad \kappa_{\beta_N}(\varphi|_N) = e^{2\pi i \int_N f_{\varphi} i_N^o \nu}.
$$

**Theorem 4.3** Assume that the prequantization condition $k = ra \in \mathbb{N}$ holds. There exists an extension of the character $\chi_{\beta_N}$ from Theorem 4.2 to the polarization group $H$, given by the homomorphism

$$
\overline{\chi}_{\beta_N} : H \to S^1, \quad \overline{\chi}_{\beta_N}(\varphi) = \kappa_{\beta_N}(\varphi|_N)e^{2\pi i \sigma_{\beta_N}(\varphi)}, \tag{46}
$$

with $\sigma_{\beta_N}$ defined in Lemma 4.1.
Proof. First we check that $\bar{\chi}_{\beta_N}$ is a group homomorphism. On one hand

$$\sigma_{\beta_N}(\phi \circ \psi) - \sigma_{\beta_N}(\phi) - \sigma_{\beta_N}(\psi) = \int_N ((\psi^{-1}|_N)^*\beta_N - \beta_N) \wedge i_N^*d^{-1}(\psi^*\nu - \nu)$$

$$= \int_N df_{\psi} \wedge i_N^*d^{-1}(\psi^*\nu - \nu) = \int_N f_{\psi} \wedge i_N^*(\psi^*\nu - \nu).$$

On the other hand

$$\kappa_{\beta_N}(\phi \circ \psi)\kappa_{\beta_N}(\phi)^{-1}\kappa_{\beta_N}(\psi)^{-1} = e^{2\pi i \int_N f_{\psi} \wedge i_N^*(\psi^*\nu - \nu)}.$$

By derivation we get for all $v \in \mathfrak{x}_{\text{ex}}(M,\mu)_N$

$$d_v\bar{\chi}_{\beta_N}(v) = \int_N \beta_N \wedge i_N^*d^{-1}\mathcal{L}_v\nu - \int_N \beta_N(v)\nu = d_v\sigma_{\beta_N}(v) - \int_N \beta_N \wedge i_N^*1_v\nu. \quad (45)$$

thus $d_v\bar{\chi}_{\beta_N}$ is the restriction of the momentum to the polarization Lie algebra $\mathfrak{x}_{\text{ex}}(M,\mu)_N.$

Codimension two vorticities. We consider the case of a $n$-dimensional volume manifold $(M,\mu)$, compact or not, but with $H^{n-2}(M) = 0$. Let $S$ be a compact manifold of dimension $n - 2$ and let $N \in \text{Gr}^S(M)$, the nonlinear Grassmannian of codimension two submanifolds of type $S$. The momentum $(J(N),v) = \int_N \alpha$ for $v = X_{\alpha} \in \mathfrak{x}_{\text{ex}}(M,\mu)$ realizes each connected component of $\text{Gr}^S(M)$ as a coadjoint orbit of $\text{Diff}_{\text{ex}}(M,\mu)$ [13][11]. We give an argument, similar to the one given in [9], why there exist no polarizations for these coadjoint orbits.

The restriction of $J(N) \in \mathfrak{x}_{\text{ex}}(M,\mu)^*$ to its isotropy Lie algebra $\mathfrak{x}_{\text{ex}}(M,\mu)_N$, of exact divergence free vector fields tangent to $N$, vanishes. Indeed, the vector field $u$ belongs to the isotropy Lie algebra of $J(N)$ if and only if $\langle J(N),[u,v]\rangle = \int_N i_u i_v \mu = 0$ for all exact divergence free vector fields $v$. This happens if and only if $u$ is tangent to $N$. Obviously $J(N)$ vanishes on $[\mathfrak{x}_{\text{ex}}(M,\mu)_N,\mathfrak{x}_{\text{ex}}(M,\mu)_N]$, but the Lie algebra $\mathfrak{x}_{\text{ex}}(M,\mu)_N$ is perfect by [16], so $J(N)$ vanishes on the whole $\mathfrak{x}_{\text{ex}}(M,\mu)_N$.

Let $Q$ be any codimension one submanifold of $M$ that contains $N$ and let $\mathfrak{h} = \mathfrak{x}_{\text{ex}}(M,\mu)_Q$. It is a perfect Lie algebra by [16]. Moreover, for all $u,v \in \mathfrak{h}$ we have

$$\langle J(N),[u,v]\rangle = \int_N i_u i_v \mu = 0$$

since $u,v$ are tangent to $Q$ and $N$ has codimension one in $Q$. We get that $J(N)$ vanishes on $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, hence $J(N)|_{\mathfrak{h}} = 0$. Thus $H = \text{Diff}_{\text{ex}}(M,\mu)_Q$ is not a good candidate for a polarization subgroup associated to $J(N)$.

4.3 A geometric interpretation of the character

We consider as earlier a non-compact $n$-dimensional manifold $M$, endowed with an exact volume form $\mu = d\nu$. Let $(S,\beta)$ be again a compact $(n - 1)$-dimensional manifold endowed with $\beta \in \Omega^1(S)$, closed, without zeroes, and with discrete period group equal to $r\mathbb{Z}$. Moreover, the submersion $F : S \to S^1$ with $F^*\theta = \beta$ from Lemma 3.1 is assumed to be a fibration. Every point $(N,\beta_N)$ of the decorated Grassmannian $\text{Gr}^S(M)$ inherits the same properties: $\beta_N \in \Omega^1(N)$ is closed, without zeroes, with period group $r\mathbb{Z}$, and equal to the logarithmic derivative of the fibration $F_N : N \to S^1$. The existence of a closed 1-form $\beta_N$ without zeroes on the closed codimension one submanifold $N \subset M$, forces $N$ to have zero Euler characteristic (this always holds for odd-dimensional $N$).
We add two more assumptions. First, the smooth singular homology group \( H_{n-1}(M) = 0 \), so that every compact codimension one submanifold \( N \subset M \) is a boundary. We denote by \( W \) the compact smooth singular \( n \)-chain of \( M \) with boundary \( N \) (since \( M \) is non-compact, there exists also a non-compact one with the same property). Thus \( \int_W \mu = \int_N \nu \).

The second assumption concerns the fibers \( \Gamma_z = F_N^{-1}(z) \) with \( z \in S^1 \) of the fibration \( F_N \), which are \((n-2)\)-dimensional submanifolds of \( N \). We ask that the smooth singular homology classes \([\Gamma_z] \in H_{n-2}(W)\) all vanish. This means there exist smooth singular \((n-1)\)-chains \( \Sigma_z \) in \( W \) such that \( \partial \Sigma_z = \Gamma_z \).

A Fubini type theorem permits to write the momentum associated to \( \beta_N \) as:

\[
\langle J(N, \beta_N), X_\alpha \rangle = \int_N \beta_N \wedge i^*_N \alpha = \int_{S^1_+} \left( \int_{F_N^{-1}(z)} \alpha \right) dz = \int_{S^1_+} \left( \int_{\Sigma_z} i_{X_\alpha} \mu \right) dz. \tag{47}
\]

Let \( \{ \varphi_t \}, t \in [0, 1] \), denote a homotopy class of diffeomorphisms in the polarization subgroup \( H \), the identity component of \( \text{Diff}_{\text{ex}}(M, \mu)_N \), starting at the identity. It is the flow of a time dependent vector field \( v_t \in \mathfrak{h} = \mathfrak{X}_{\text{ex}}(M, \mu)_N \), namely \( \frac{d}{dt} \varphi_t = v_t \circ \varphi_t \). Thus the homotopy class \( \{ \varphi_t \} \) belongs to the universal covering group \( \tilde{H} \). The group structure on this universal cover of a diffeomorphism group can be viewed either as composition of diffeomorphisms or as path juxtaposition, just like in [23, Section 10.2] for the Hamiltonian group.

For each \( z \in S^1_+ \), let \( W_z^{\varphi_t} \) denote the smooth singular \( n \)-chain in \( M \) swept out by the \((n-1)\)-chain \( \Sigma_z \) under the path of diffeomorphisms \( \varphi_t \) from time 0 to time 1. The smooth singular \( n \)-chain \( \Sigma_z \) lives on \( W \) and the isotopy path \( \varphi_t \) preserves \( N = \partial W \), hence the chain \( W_z^{\varphi_t} \) stays in \( W \). In particular the following total volume is finite:

\[
\int_{W_z^{\varphi_t}} \mu = \int_0^1 \left( \int_{\Sigma_z} i_{\varphi_t^*} v_t \mu \right) dt = \int_0^1 \left( \int_{\Sigma_z} \varphi_t^*(i_{v_t} \mu) \right) dt = \int_0^1 \left( \int_{\Gamma_z} \varphi_t^* \alpha_t \right) dt, \tag{48}
\]

where \( \alpha_t \in \Omega^{n-2}(M) \) is any potential form for \( v_t \), i.e. \( i_{v_t} \mu = d\alpha_t \).

The map that associates to the homotopy class \( \{ \varphi_t \} \) the average volume of \( W_z^{\varphi_t} \) over the base \( S^1_+ \),

\[
\bar{\chi}_{\beta_N} : \tilde{H} \to \mathbb{R}, \quad \bar{\chi}_{\beta_N}(\{ \varphi_t \}) := \int_{S^1_+} \left( \int_{W_z^{\varphi_t}} \mu \right) dz \overset{\text{(48)}}{=} \int_{S^1_+} \int_0^1 \left( \int_{\Gamma_z} \varphi_t^* \alpha_t \right) dt dz, \tag{49}
\]

is a well defined group homomorphism, since it doesn’t depend on the choice of the path \( \varphi_t \) in the homotopy class and on the potential \( \alpha_t \), as well as on the choices of \( F_N \) and \( \Sigma_z \). More precisely, the independence of the isotopy path in the homotopy class can be proven like it is done for the flux homomorphism in [23, Section 10.2]; the independence of \( \Sigma_z \) such that \( \partial \Sigma_z = \Gamma_z \) follows from the exactness of \( i_{v_t} \mu \) in (48); the independence of \( \alpha_t \) follows from the condition \( H^{n-2}(M) = 0 \); finally, the independence of \( F_N \) follows by the translation invariance of the Haar measure \( dx \) on \( S^1_+ \), see the proof of Lemma 3.1.

The group homomorphism (49) integrates the restriction to \( \mathfrak{h} = \mathfrak{X}_{\text{ex}}(M, \mu)_N \) of the momentum \( J(N, \beta_N) \) associated to \( (N, \beta_N) \in \text{Gr}^\text{Gr}^{\text{ex}}_{2, \mu}(M) \). Indeed, let \( \phi^\varepsilon_t \) denote the flow at time \( t \) of the vector field \( v \in \mathfrak{h} \) and consider the path \( \varepsilon \mapsto \{ \phi^\varepsilon_t \} \) of homotopy classes in \( \tilde{H} \). Its derivative at \( \varepsilon = 0 \) is equal to \( v = X_\alpha \). We compute the Lie algebra homomorphism as

\[
v = X_\alpha \mapsto \frac{d}{d\varepsilon} \big|_{\varepsilon=0} \bar{\chi}_{\beta_N}(\{ \phi^\varepsilon_t \}_{0 \leq t \leq \varepsilon}) \overset{\text{(49)}}{=} \frac{d}{d\varepsilon} \big|_{\varepsilon=0} \int_{S^1_+} \int_0^\varepsilon \int_{\Gamma_z} (\phi^\varepsilon_t)^* \alpha_t \ dt \ dz = \int_{S^1_+} \int_{\Gamma_z} \alpha \ dz \overset{\text{(47)}}{=} \langle J(N, \beta_N), X_\alpha \rangle.
\]

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Figure 1: An illustration of the construction of the character with $M = \mathbb{R}^3$ and $N$ the torus for $\varphi$ in the polarization group. For each $z \in S^1_r$, the volume enclosed by the hatched area represents the volume $W_{\varphi_t}$ swept out by $\Sigma_z$ under the path of diffeomorphisms $\varphi_t$ from time 0 to time 1, with $\varphi_1 = \varphi$.

**Proposition 4.4** Let $(N, \beta_N) \in \text{Gr}^{S^\mu}(M)$ with $a = \int_M \mu$ the total volume enclosed by $N$. Assume that the prequantization condition $ra \in \mathbb{Z}$ holds. The $\mathbb{R}$ valued group homomorphism $\tilde{\chi}_{\beta_N}$ on $H$ in (49) descends to an $S^1$ valued group homomorphism on the polarization group $H$, i.e. the extended character

$$\tilde{\chi}_{\beta_N} : H \to S^1, \quad \tilde{\chi}_{\beta_N}(\varphi) := \tilde{\chi}_{\beta_N}({\varphi}_t) \mod \mathbb{Z} = \int_{S^1_r} \left( \int_{W_{\varphi_t}} \mu \right) dz \mod \mathbb{Z},$$

where $\varphi_t$ is any isotopy in $H$ from the identity to $\varphi$.

**Proof.** We only need to check that every homotopy class of a loop ${\varphi}_t \in \pi_1(H)$ is mapped by $\tilde{\chi}_{\beta_N}({\varphi}_t)$ into $\mathbb{Z}$. Indeed, since $\varphi_1$ is the identity on $M$, the volume of the smooth singular $n$-chain swept out by the $(n-1)$-chain $\Sigma_z$ under this path of diffeomorphisms, $W_{\varphi_t} \subseteq M$, is an integer multiple $m_z \in \mathbb{Z}$ of $a$, the volume of the $n$-chain $W$. But $m_z$ depends continuously on $z \in S^1_r$, hence $m_z = m \in \mathbb{Z}$ constant. Then

$$\tilde{\chi}_{\beta_N}({\varphi}_t) = \int_{S^1_r} \left( \int_{W_{\varphi_t}} \mu \right) dz = \int_{S^1_r} (ma) dz = mar \in \mathbb{Z},$$

by the prequantization condition $ra \in \mathbb{Z}$. ■

**Example 4.5 (Low dimensions: $M = \mathbb{R}^n$ with $n = 3, 4$)** We start with the case $M = \mathbb{R}^3$ endowed with canonical volume form. The existence of a closed 1-form $\beta_N$ without zeros on the oriented surface $N \subset \mathbb{R}^3$, forces $N$ to be a 2-torus. Let $W$ denote the solid torus with boundary the 2-torus, i.e. $N = \partial W$. We impose that the level sets of the fibration $F_N$ on $N$
by circles are closed curves that can be unknotted inside $W$, i.e. the fibers are boundaries of surfaces that lie inside the solid torus $W$.

In the case $M = \mathbb{R}^3$, the codimension one submanifold $N \subset \mathbb{R}^4$ can take the form $N = S^1 \times Q$. Here $Q \subset 0 \times \mathbb{R}^3$ a closed surface that is rotated around the coordinate plane $Ox_3x_4$ to get $N$. If $Q$ is the 2-torus, then $N = S^1 \times S^1 \times S^1$ is a 3-torus and $W = S^1 \times S^1 \times D^2$ a solid 3-torus. If $Q$ is a 2-sphere, then $N = S^1 \times S^2$ and $W = S^1 \times B^3$ is obtained by rotation of a 3-ball around the same coordinate plane.

### 4.4 The general case $H^{n-2}(M) \neq 0$

Let $(M, \mu = d\nu)$ be $n$-dimensional noncompact with $H^{n-2}(M) \neq 0$. We know from Theorem 2.6 that connected components of $\text{Gr} \, S^{\beta} \, (M)$ are coadjoint orbits of the Ismagilov extension $\text{Diff} \,_{ex}(M, \mu) = \text{Diff} \,_{ex}(M, \mu) \times_C H^{n-2}(M)$, where $C$ is the Ismagilov group cocycle (5). Because of the natural expression of the adjoint action in the extended group $\text{Ad} \,\varphi([\alpha]) = [\varphi \dot{\alpha}]$, the coadjoint action restricted to $\text{Gr} \, S^{\beta} \, (M)$ in the central extension $\text{Diff} \,_{ex}(M, \mu)$ is also the natural one:

$$\text{Ad} \,\varphi(N, \beta_N) = (\varphi(N), \varphi \beta_N), \quad [\alpha] \in \check{\mathcal{X}} \,_{ex}(M, \mu) = \Omega^{n-2}/B^{n-2}(M).$$

In particular, the isotropy subgroup of $J(N, \beta_N)$ is again $\text{Diff} \,_{ex}(M, \mu)_{\beta_N}$ with isotropy Lie algebra $\mathcal{X} \,_{ex}(M, \mu)_{\beta_N}$.

Under the decomposition of the Ismagilov extension as $\mathcal{X} \,_{ex}(M, \mu) \oplus H^{n-2}(M)$ by the section (7) that gives the cocycle $\omega_C$ from (6), the momentum $J(N, \beta_N)$ becomes

$$J(N, \beta_N) : (u, [\alpha]) \mapsto \int_N \beta_N \wedge i^*_N(b \mathcal{L}_u \nu - i_u \nu) + \int_N \beta_N \wedge \alpha_0. \quad (50)$$

The analogue of the 1-cocycle $\sigma$ from Lemma 4.1 is

$$\sigma : \text{Diff} \,_{ex}(M, \mu) \times_C H^{n-2}(M) \to C^\infty \left(\text{Gr} \, S^{\beta}(M) \right)$$

defined for all $(N, \beta_N) \in \text{Gr} \, S^{\beta}(M)$ by

$$\sigma_{\beta_N}(\psi, b) = \int_N \beta_N \wedge i^*_N b (\psi^* \nu - \nu) + \int_N [\beta_N] \wedge i^*_N b. \quad (51)$$

Indeed, it verifies the cocycle identity

$$\sigma_{\beta_N}(\psi_1 \circ \psi_2, b_1 + b_2 + C(\psi_1, \psi_2)) = \sigma(\psi_2)_{\beta_N}(\psi_1, b_1) + \sigma_{\beta_N}(\psi_2, b_2). \quad (52)$$

The associated Lie algebra 1-cocycle $\mathcal{X} \,_{ex}(M, \mu) \times_C H^{n-2}(M) \to C^\infty(\text{Gr} \, S^{\beta}(M))$ is given by

$$d_{\varphi} \sigma_{\beta_N}(\psi, b) = \int_N \beta_N \wedge i^*_N b (\mathcal{L}_\nu \nu) + \int_N [\beta_N] \wedge b. \quad (53)$$

We can now write down a generalization of Theorem 4.2 to the case $H^{n-2}(M) \neq 0$.

**Theorem 4.6** Given $(N, \beta_N) \in \text{Gr} \, S^{\beta}(M)$, a character for the restriction of $J(N, \beta_N)$ to the isotropy Lie algebra $\mathcal{X} \,_{ex}(M, \mu)_{\beta_N} = \mathcal{X} \,_{ex}(M, \mu)_{\beta_N} \times_C H^{n-2}(M)$ is

$$\chi_{\beta_N} : \text{Diff} \,_{ex}(M, \mu)_{\beta_N} = \text{Diff} \,_{ex}(M, \mu)_{\beta_N} \times_C H^{n-2}(M) \to S^1, \quad \chi_{\beta_N}(\psi, b) = \epsilon_{\beta_N}^a(\psi |_N)e^{2\pi i \sigma_{\beta_N}(\psi, b)},$$

where $\sigma_{\beta_N}$ is defined in (51) and $\epsilon_{\beta_N}^a$ is given in (26).
Proof. We show that $\mathbf{d}_eX\beta_N = J(N, \beta_N)|_{\mathbf{x}_{\text{ex}}(M, \mu)\beta_N}$. On one hand we notice that

$$\mathbf{d}_e\sigma_{\beta_N}(v, b) - \langle J(N, \beta_N), (v, b) \rangle = \int_N \beta_N \wedge i_N^*v \nu.$$ 

On the other hand we have $\int_N \beta_N \wedge i_N^*v \nu = \beta_N(v) \int_N \nu = a\beta_N(v) = \mathbf{d}_e\sigma_{\beta_N}(v)$ for all $v \in \mathbf{x}_{\text{ex}}(M, \mu)\beta_N$. ■

The polarization group. A polarization Lie algebra for $(N, \beta_N)$ is

$$\mathfrak{h} = \mathbf{x}_{\text{ex}}(M, \mu)_N \times_{\omega_C} H^{n-2}(M) = \{[\alpha] \in \Omega^{n-2}(M)/B^{n-2}(M) : X\alpha \in \mathbf{x}_{\text{ex}}(M, \mu)\}.$$ 

We use again the section $s_C$ in (7) to decompose elements of the Lie algebra central extension.

Theorem 4.7 Assume that the prequantization condition $k = ra \in \mathbb{N}$ holds. If $\mu = dv$, then

$$\overline{\chi}_{\beta_N} : \text{Diff}_{\text{ex}}(M, \mu)_N \times_{\omega_C} H^{n-2}(M) \to S^1, \quad \overline{\chi}_{\beta_N}(\varphi, b) = \kappa_{\beta_N}(\varphi|_N)e^{2\pi i\sigma_{\beta_N}(\varphi, b)}$$

is a character for the momentum $J(N, \beta_N)$ restricted to the polarization Lie algebra $\mathfrak{h} = \mathbf{x}_{\text{ex}}(M, \mu)_N \times_{\omega_C} H^{n-2}(M)$, where $\sigma_{\beta_N}$ is defined in (51) and $\kappa_{\beta_N}$ is given in (27). It extends the character $\chi_{\beta_N}$ from Theorem 4.6.

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