Full Counting Statistics and Field Theory

Yuli V. Nazarov
Kavli Institute of NanoScience, Delft University of Technology, 2628 CJ Delft, The Netherlands

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We review the relations between the full counting statistics and the field theory of electric circuits. We demonstrate that for large conductances the counting statistics is determined by non-trivial saddle-point of the field. Coulomb effects in this limit are presented as quantum corrections that can strongly renormalize the action at low energies.

1 Introduction

The concept of Full Counting Statistics has been introduced in very early days of quantum transport. That time, nobody ever thought of such an abstract and complicated problem as the evaluating and measuring the higher cumulants of electronic noise. The research has been driven by pure curiosity and has resulted in compact and deep Levitov formula.[1] The significance of this contribution has been underappreciated for a number of years. The direct experimental verification seemed to be out of question, while the theorists opted to address more primitive and standard problems.

Nowadays Full Counting Statistics is a reasonably established field attracting attention of many, although the peak of interest is probably in the past.[2] There are beautiful experiments where the higher cumulants have been measured[3] and even single electron transfers have been actually counted.[4] Theoretically, the Full Counting Statistics has been evaluated for virtually any important electron transport system including even so generic as Anderson impurity model.[5]

Still it remains underappreciated that the Full Counting Statistics is eventually not about the marginal deviations of electric currents. Let as draw a parallel with general relativity. The general relativity is not an art to calculate ridiculously small corrections to Newton’s law, although only those can be verified experimentally. The general relativity brings us true knowledge about the Universe. Similar to that, Full Counting Statistics lies near the heart of quantum theory of electricity and is in fact an indispensible element for this.

The present contribution aims to explicate the link between Full Counting Statistics and quantum field theory of electric circuits. The most established example of such theory is the quantum theory of a superconducting Josephson junction in a dissipative electromagnetic environment[6] that is readily reduced to a single-variable field theory for the superconducting phase across the junction[7]. The action of the same type governs Coulomb blockade phenomena in non-superconducting systems.[8, 9, 10] The physics of one-dimensional interacting electrons in the framework of Luttinger model is often reduced to similar schemes[11, 12], where the variable is a drop of a phase over the barrier present in the one-dimensional setup and can be associated with the voltage drop at the barrier.

The structure of the article is as follows. We start (Sections 2,3) by formulating a general quantum theory of a simplest electric circuit and see the need and advantage of the FCS in this respect. We show that the classical limit of the field theory is not trivial as far as FCS is concerned and obtain the FCS at
current bias (Section 4). Quantum effects at big conductances (as compared with the conductance quantum $G_Q \equiv e^2/2\pi\hbar$) can be incorporated by the renormalization of the action parameters. This is frequently the case in field theories [13] We perform the renormalization procedure explicitly in Section 5 for a quantum contact in series with an Ohmic one and for a set of quantum contacts connected to a single node (Section 6).

The concrete results revisited here were first published in [14], [15], and [16].

2 Field theory

Let us first start with elementary electric circuit theory and reason the quantum extension of it. A circuit is made of three archetypal elements: terminals, connectors, and nodes. The voltage is fixed in terminals. A connector is characterized by its $I - V$ characteristics:

$$I(t) = I(t; \{V(t)\})$$ (1)

where we have assumed general relation between the voltage and current so that current at the time moment $t$ depends on time-dependent voltage at all (previous) time moments, $\{V(t)\}$. A simplest circuit contains two connectors ($A$ and $B$) in series, so that a single node and two terminals. (Fig. 1a) Connecting elements in this way brings about an extra variable: Voltage $V_1(t)$ in the node. In the elementary circuit theory under consideration, this voltage $V_1(t)$ is determined from the current conservation in the node,

$$I_A(t; \{V_1(t)\}) = I_B(t; \{V(t) - V_1(t)\}).$$ (2)

assuming the terminal voltages are fixed to 0 and $V(t)$. Once the voltage is determined, one finds the $I - V$ characteristics of the whole circuit. Thereby, the full description of the system naturally emerges from the two descriptions of the separate connectors.

Let us try to reason a quantum extension of this theory. First of all, it is convenient to change from voltages to phases defined as $\Phi(t) = (e/\hbar)V(t)$. This allows to treat superconducting and non-superconducting systems on equal footing. The phases(voltages) of the terminals can be regarded as time-dependent external parameters while the phase of the node becomes a real quantum variable, that is, an operator $\hat{\Phi}_1(t)$. However, it is hardly an option to formulate the theory in operator formalism since finding the classical correspondence becomes a formidable task. Rather, we shall opt for Feynman-Vermon or Keldysh-action description of the system where all observables can be presented as path integrals over the time-dependent non-operator variable $\Phi(t)$. The price to pay for this convenience is quite known: the variable "doubles".

![Fig. 1](image.png)
The point is that the path integral should performed over two parts of the Keldysh contour that correspond to coherent evolution of "kets" and "bras". The variable in principle takes different values $\Phi_1^+(t), \Phi_1^-(t)$ at these two contours.

We still want the description of the compound system under consideration to emerge from the two descriptions of the separate connectors $A$ and $B$. What descriptions? Since the variable has "doubled", the simple $I - V$ characteristics does not suffice. A connector has to be described by something that depends on both variables $\Phi_1^+(t), \Phi_1^-(t)$. It is also clear that from all possible quantum variables of the connector such description should involve only one: the operator of electric current through the connector, $\bar{I}(t)$, and present the reaction of the current on the variables $\Phi_1^+(t), \Phi_1^-(t)$. The proper description of the connector is thus provided by the Feynman-Vernon influence functional of two variables $\Phi(t) \equiv (\Phi_1^+(t) + \Phi_1^-(t))/2, \chi(t) \equiv \Phi_1^+(t) - \Phi_1^-(t)$,

$$Z[\Phi, \chi] = \left\langle T \exp \left\{ \frac{i}{e} \int dt [\Phi(t) + \frac{1}{2} \chi(t)] \bar{I}(t) \right\} \hat{\rho} T \exp \left\{ \frac{i}{e} \int dt [-\Phi(t) + \frac{1}{2} \chi(t)] \bar{I}(t) \right\} \right\rangle. \quad (3)$$

Here, the trace is over the electronic degrees of freedom specific for the connector: Thereby they are "traced out" and never explicitly enter our field theory. The notation $T \langle T \rangle$ denotes time-ordering of the exponents in ascending (descending) order, these exponents presenting the quantum evolution of density matrix $\hat{\rho}$ subject to the fields $\Phi, \chi$. The easiest way to understand the functional is to expand its log in terms of $\chi$ at $\chi \to 0$. The coefficients of the expansion present the cumulants of the time-ordered current operators in the connector that is subject to the external classical phase $\Phi(t)$. In particular, $\ln Z(\chi = 0) = 0, \langle I(t) \rangle = \partial \ln Z / \partial \chi(t), \langle (I(t_1) I(t_2)) \rangle = e^2 \partial^2 \ln Z / \partial \chi(t_1) \partial \chi(t_2)$ and so on. Therefore, the functional $Z$ is nothing but the generating function of the current fluctuations. For quantum conductors and slow-varying $\Phi, \chi$ it has been first considered in [1][17] Now we are ready to build up the quantum description of the whole circuit. Since the circuit is nothing but a compound connector, it has to be characterized by the similar generating function $Z_{A+B}$ that depends on the drop of the fields $\Phi, \chi$ over the circuit. Such functional is nothing but a path integral convolution of the functions of the separate connectors $Z_A$ and $Z_B$,

$$Z_{A+B}[\Phi, \chi] = \int D\Phi_1 D\chi_1 Z_A[\Phi_1, \chi_1] Z_B[\Phi - \Phi_1, \chi - \chi_1]. \quad (4)$$

where the path integration measure $D\Phi_1 D\chi_1 \equiv \prod_t d\Phi(t) d\chi(t)$. The overall generating function is the average over fluctuating phases $\Phi_1, \chi_1$ at the node of the circuit shared by both conductors. Such convolution law is nothing but the presentation of the current conservation in the node. One can see it if one substitutes $Z_{A+B}$ in the form [3] and carries out the integration over $\Phi_1, \chi_1$.

This is the field theory of the simplest single-node circuit. The extension to a more complicated circuit is straightforward. The functional is a product of $Z$’s of all conductors integrated over the extra variables $\Phi, \chi$ defined in each node of the circuit. This functional depends on the phases $\Phi$ applied in each terminal and counting fields $\chi$ defined in terminals so that it gives statistics of the currents to/from each terminal of the circuit.

### 3 General Properties and Concrete Connectors

The difference between classical and quantum effects is not readily manifested in the field theory under consideration. Indeed, the current fluctuations in the connectors may be of classical as well as of quantum origin but the presentation of these fluctuations is almost the same. For the theory in hand, it is constructive to define the difference between classical and quantum as the difference between low-frequency and high-frequency regimes. This is very much like the theory of frequency-dependent noise: It is known that at sufficiently low frequencies ($\ll k_B T/h$ for equilibrium systems) any noise can be regarded as classical irrespective of its origin while quantum mechanics becomes important at higher frequencies.
In general the functional dependence of a connector functional $Z$ one the phases may be complicated and non-local in time. However, one expects on physical grounds the non-locality to vanish at sufficiently low frequencies: The current and its statistics at the time moment $t$ would only depend on the voltage at the same moment. Therefore, at sufficiently slow realizations of the fluctuating phases the functional $Z$ can be expressed in terms of a single function $S(\Phi, \chi)$,

$$\ln Z[\Phi(t), \chi(t)] = \int dt S(\dot{\Phi}(t), \chi(t)),$$  \hspace{1cm} (5)

(here we for simplicity specify to non-superconducting systems).

Let us now concatenate two connectors and try to assess the low-frequency limit of the resulting $Z_{A+B}$ taking the path integral in (3). There can be two importantly distinct cases. It may be that the path integral is dominated by low-frequency $\dot{\Phi}_1, \chi_1$ so that the result is determined by low-frequency limits of $Z_A, Z_B$. Moreover, in this case the path integral can be evaluated in the saddle-point approximation with the result

$$S_{A+B}^{(cl)}(\Phi, \chi) = S_A(\Phi_s, \chi_s) + S_B(\Phi - \Phi_s, \chi - \chi_s).$$  \hspace{1cm} (6)

Here $\Phi_s$ and $\chi_s$ stand for the (generally complex) values of $\dot{\Phi}_1$ and $\chi_1$ at the saddle point where the derivatives with respect to these phases vanish. Since in field theory the relevance of saddle-point approximation is normally associated with classical behavior, we call this classical limit. Generally, high-frequency fluctuations of the fields also contribute to the path integral. The effect of this contribution is that the actual $S_{A+B}$ deviates from the $S_{A+B}^{(cl)}$ evaluated with the saddle-point method. Adopting again the common field theory terms, we call the deviation the quantum correction. Depending on the connector parameters, the correction can be vanishingly small or overwhelming.

The next statement might look less obvious: The quantum correction present the effect of electron-electron interaction in the system and in the context of quantum transport is commonly referred to as (dynamical) Coulomb blockade effect. First puzzled question: Where is a capacitor of capacitance $C$ providing the charging energy $E_C = e^2/2C$ that should accompany any passage about Coulomb blockade? Well, it always present (Fig. 1b) as a capacitance between the node and ground. The capacitive connector contributes to $Z_C$ with $^{18}$

$$\ln Z_C = \frac{i\hbar}{E_C} \int dt \Phi(t) \chi(t)$$  \hspace{1cm} (7)

This term suppresses the high-frequency fluctuations of the fields in the node and is usually needed for proper regularization of the theory since it provides the high-frequency cut-off.

Common wisdom of quantum transport suggests that Coulomb interaction is weak provided the typical conductance of the connectors $G$ exceeds by far the conductance quantum and is dominating otherwise. One can see this from the estimation of phase fluctuations around the saddle point, $\langle \Phi, \chi \rangle_\omega \simeq (iG/G_Q)\omega + E_C\omega^2/\hbar)^{-1}$. The fluctuation is $\ll 1$ provided $G \gg G_Q$. We will see that in this regime the quantum corrections are small although logarithmically diverge at small frequency. In opposite case our system develops a strong Coulomb gap and becomes a sort of SET transistor.

We end this Section with concrete examples of connectors. Ohmic connectors are linear conductors and exhibit Gaussian current fluctuations. In terms of Fourier components of the fields,

$$\ln Z_{Ohm} = \int \frac{d\omega}{2\pi G_Q} (\Phi_\omega(-i\omega Z^{-1}(\omega))\chi_{-\omega} + \chi_\omega 2\omega \Re(Z^{-1}(\omega)) \coth(h\omega/2k_BT)\chi_{-\omega})$$  \hspace{1cm} (8)

$Z(\omega)$ being the frequency-dependent impedance (resistance) of the connector. Since the action is of Gaussian type, the field theory is completely trivial. Any circuits made of Ohmic connectors are reduced to Ohmic connectors as well.
The Ref. [6] has provided the connector action for a tunnel connector between the superconducting and/or normal leads, that appeared to be non-Gaussian even for normal case. In the simplest low-energy limit such connector is a Josephson junction characterized by Josephson energy $E_J$ and corresponding action

$$\ln Z_J = -\frac{i E_J}{\hbar} \int dt (\cos(2\Phi^+(t)) - \cos(2\Phi^-(t))). \quad (9)$$

An arbitrary coherent quantum connector is characterized by the set of transmission coefficients $T_n$. The FCS studies [1] have demonstrated that in low-frequency limit the functional corresponds to

$$S_A(\dot{\Phi}, \chi) = \frac{|\dot{\Phi}|}{2\pi} S(i \text{sgn}(\dot{\Phi})) \chi, \quad S(\xi) \equiv \sum_{n=1}^{N} \ln \left[ 1 + (e^\xi - 1)T_n \right]. \quad (10)$$

4 Saddle point and FCS at current bias

Let us investigate the saddle point of our field theory for a quantum conductor (A) of conductance $G$ in series with an Ohmic conductor (B) of low-frequency resistance $Z$. It is instructive to assume that the Ohmic conductor is noiseless, that is, is kept at vanishing temperature. Its contribution to the action thus reads (see (8))

$$S_B(\Phi, \chi) = \frac{i \dot{\Phi} \chi}{2\pi ZG_Q} \quad (11)$$

The subject of interest is how the current fluctuations produced in the quantum conductor are distributed in the whole circuit. If the Ohmic resistor is small, $ZG \ll 1$, we achieve the current bias for this quantum conductor and access the FCS in this limit.

We apply general Eqs. (5) and (6) to our specific circuit that is driven by the voltage source $V_0$. To avoid significant quantum corrections to this field theory (Coulomb blockade effects [9]) we assume that $|Z(\omega)|G_Q \ll 1$ at frequencies $\hbar \omega \simeq \max(eV, k_BT)$ where the quantum corrections start to form. The zero-frequency impedance $Z$ can have any value.

Both the voltage drop $V$ at the quantum conductor and the current $I$ through the conductor fluctuate in time for finite $Z$, with averages $\bar{I} = V_0G(1 + ZG)^{-1}$, $\bar{V} = V_0(1 + ZG)^{-1}$. Voltage bias corresponds to $ZG \ll 1$ and current bias to $ZG \gg 1$, with $I_0 = V_0/Z$ the imposed current. There are three characteristic time scales: $\hbar/\max(eV, kBT)$, $e/I$, and the $RC$-time of the circuit. The low-frequency regime on which we concentrate is reached for current and voltage fluctuations that are slow on any of these time scales.

We seek the cumulant generating function of charge

$$\mathcal{F}(\xi) = \ln \left( \sum_{q=0}^{\infty} e^{q\xi} P(q) \right) = \sum_{p=1}^{\infty} \frac{\langle\langle q^p \rangle\rangle}{p!} \xi^p \quad (12)$$

where $\langle\langle q^p \rangle\rangle$ is the $p$-th cumulant of the charge transferred during the time interval $\tau$. It is directly related to the Keldysh action in the saddle point (6) by

$$\mathcal{F}(\xi) = \tau S_{A+B}(eV_0/\hbar, -i\xi). \quad (13)$$

To characterize the fluctuations of the voltage across the quantum contact, we will also need the cumulant generating function of phase, $\mathcal{G}(\xi)$. We use that in the absence of noise in the Ohmic connector $V = V_0 - ZI$. Therefore, $\mathcal{G}$ is related to $\mathcal{F}(\xi)$ by a change of variables. The relation is

$$\mathcal{G}(\xi) = \sum_{p=1}^{\infty} \frac{\langle\langle \phi^p \rangle\rangle}{p!} = \phi_0 \xi + \mathcal{F}(-ZG_Q\xi/2), \quad (14)$$
\( \phi_0 \) being the phase change induced by external voltage in time interval \( \tau \). \( \phi_0 = eV_0 \tau / 2 \pi \hbar \). In the limit \( Z \to 0 \) of voltage bias the saddle point of the Keldysh action is at \( \Phi_1 = \Phi \), \( \chi_1 = \chi \), and from Eqs. (6), (12), and (14) one recovers the results of Ref. 1: The cumulant generating function
\[ F_0(\xi) = \tau S_A(eV_0/\hbar, -i\xi) = \phi_0 S(\xi) \] and the corresponding probability distribution
\[ P_{\phi_0}(q) = \lim_{z \to q} \frac{1}{q!} \prod_{n=1}^{N} [1 + (1 - 1)/T_n]^{\phi_0}. \]

We note that the parameter \( \phi_0 \) is in fact the number of attempted transmissions per channel. The first few cumulants are \( \langle q \rangle_0 = \phi_0 G/G_0 \), \( \langle q^2 \rangle_0 = \phi_0 \sum_n T_n(1 - T_n) \), \( \langle q^3 \rangle_0 = \phi_0 \sum_n T_n(1 - T_n)(1 - 2T_n) \). In the single-channel case \( (N = 1) \) the distribution (15) has the binomial form (20).

After these preparations we are now ready to generalize all of this to finite \( Z \), and in particular to derive the dual distribution of phase (21) under current bias. Calculating saddle-point values of \( \Phi_1, \chi_1 \) from Eqs. (6) and (13) we observe that \( (z \equiv ZG) \)
\[ F(\xi) = \frac{\phi_0}{z} [\xi - \sigma(\xi)] \], \( \sigma + zS(\sigma) = \xi \).

The implicit function \( \sigma(\xi) \) (which is determined from the saddle point of the action) provides the cumulant generating function of charge \( F \) for arbitrary series resistance \( Z \). One readily checks that \( F(\xi) \to \phi_0 S(\xi) \) in the limit \( z \to 0 \), as it should.

By expanding Eq. (16) in powers of \( \xi \) we obtain a series of relations between the cumulants \( \langle q^p \rangle \) of charge at \( Z \neq 0 \) and the cumulants \( \langle q^p \rangle_0 \) at \( Z = 0 \). For example, to linear order we find \( \langle q \rangle = (1 + ZG)^{-1} \langle q \rangle_0 \), which nothing but the trivial division of voltage: The mean current \( I \) is rescaled by a factor \( 1 + ZG \) coming from the series resistance. Naively, one may assume that the same rescaling applies to the fluctuations. Indeed, to second order one finds \( \langle q^2 \rangle = (1 + ZG)^{-3} \langle q^2 \rangle_0 \), in agreement with elementary circuit theory.

However, if we go to higher cumulants we find that other terms appear, which can not be incorporated by any rescaling. For example, Eq. (16) gives for the third cumulant
\[ \langle q^3 \rangle = \frac{\langle q^3 \rangle_0}{(1 + ZG)^3} - \frac{3ZG}{(1 + ZG)^2} \frac{(\langle q^2 \rangle_0)^2}{\langle q \rangle_0}. \]

While the first term on the the right-hand-side has the expected scaling form, the second term does not. This is generic for \( p \geq 3 \): \( \langle q^p \rangle = (1 + ZG)^{-p-1} \langle q^p \rangle_0 \) plus a non-linear (rational) function of lower cumulants (19). All terms are of the same order of magnitude in \( ZG \).

Turning now to the limit \( ZG \to \infty \) of current bias, we see from Eq. (16) that \( F \to F_\infty \) with
\[ F_\infty(\xi) = q_0\xi - q_0 S^{\text{inv}}(\xi/z) \]
defined in terms of the functional inverse \( S^{\text{inv}} \) of \( S \). The parameter \( q_0 = \phi_0/z = I_0\tau/e \) (which assumed to be an integer \( \gg 1 \)) is the number of charges transferred by the bias current \( I_0 \) during the time interval \( \tau \). Transforming from charge to phase variables by means of Eq. (14), we find that \( G \to G_\infty \) with
\[ G_\infty(\xi) = -q_0 S^{\text{inv}}(-\xi). \]

It is interesting to discuss a single-channel conductor (transmission \( T_1 \)) separately. In this case the functional inverse gives the function of a similar form. Eq. (19) reduces to \( G_\infty(\xi) = -q_0 \ln[1 + T_1^{-1}(e^{\xi} - 1)] \), corresponding to the Pascal distribution (21). The first three cumulants are \( \langle \phi \rangle = q_0/T_1 \), \( \langle \phi^2 \rangle = (q_0/T_1^2)(1 - T_1) \), \( \langle \phi^3 \rangle = (q_0/T_1^3)(1 - T_1)(2 - T_1) \).

While the charge \( Q \equiv qe \) for voltage bias \( V_0 \equiv \hbar \phi_0/e\tau \) is known to have the binomial distribution (11)
\[ P_{\phi_0}(q) = \frac{\phi_0}{q} T_1^q (1 - T_1)^{\phi_0 - q}, \]
we find that the dual distribution of phase $\Phi = 2\pi \phi$ for current bias $I_0 \equiv e q_0 / \tau$ is the Pascal distribution \[^{[20]}\]

$$P_{q_0}(\phi) = \left( \frac{\phi - 1}{q_0 - 1} \right) \Gamma^{q_0}(1 - \Gamma)^{\phi - q_0}. \quad (21)$$

(Both $q$ and $\phi$ are integers for integer $q_0$ and $q_0$.)

In the more general case not depending on the type of the quantum conductor we have found that the distributions of charge and phase are related in a remarkably simple fashion for $q, \phi \to \infty$:

$$\ln P_q(\phi) = \ln P_{\phi}(q) + O(1). \quad (22)$$

(The remainder $O(1)$ equals $\ln(q/\phi)$ in the shot-noise limit.) This formula, which valid with logarithmic accuracy, is a manifestation of charge-phase duality.\[^{[8]}\] and holds for any conductors.

The binomial distribution \[^{[20]}\] for voltage bias has the interpretation \[^{[1]}\] that electrons hit the barrier with frequency $e V_0 / 2 \pi \hbar$ and are transmitted independently with probability $T_1$. For current bias the transmission rate is fixed at $I_0/e$. Deviations due to the probabilistic nature of the transmission process are compensated for by an adjustment of the voltage drop over the barrier. If the transmission rate is too low, the voltage $V(t)$ rises so that electrons hit the barrier with higher frequency. The number of transmission attempts (“trials”) in a time $\tau$ is given by $(e/2 \pi \hbar) \int_0^\tau V(t) dt \equiv \phi$. The statistics of the accumulated phase $\phi$ is therefore given by the statistics of the number of trials needed for $I_0 \tau/e$ successful transmission events. This stochastic process has the Pascal distribution \[^{[21]}\].

For the general multi-channel case a simple expression for $P_{q_0}(\phi)$ can be obtained in the ballistic limit (all $T_n$’s close to 1) and in the tunneling limit (all $T_n$’s close to 0). In the ballistic limit one has $G_{\infty}(\xi) = q_0 \xi / N + g_0 (N - g) / (e^{\xi/N} - 1)$, corresponding to a Poisson distribution in the discrete variable $N \phi - q_0 = 0, 1, 2, \ldots$. In the tunneling limit $G_{\infty}(\xi) = -q_0 \ln(1 - \xi/j)$, corresponding to a chi-square distribution $P_{q_0}(\phi) \propto e^{q_0 - 1} e^{-q \phi}$ in the continuous variable $\phi > 0$. In contrast, the charge distribution $P_{\phi_0}(q)$ is Poissonian both in the tunneling limit (in the variable $q$) and in the ballistic limit (in the variable $N \phi_0 - q$).

For large $q_0$ and $\phi$, when the discreteness of these variables can be ignored, we may calculate $P_{q_0}(\phi)$ from $G_{\infty}(\xi)$ in saddle-point approximation. If we also calculate $P_{\phi_0}(q)$ from $F_{\phi_0}(\xi)$ in the same approximation (valid for large $\phi_0$ and $q$), we find that the two distributions have a remarkably similar form:

$$P_{\phi_0}(q) = N_{\phi_0}(q) \exp[\tau \Sigma(2 \pi q_0 \phi / \tau, q / \tau)], \quad (23)$$

$$P_{\phi_0}(\phi) = N_{\phi_0}(\phi) \exp[\tau \Sigma(2 \pi \phi / \tau, q_0 / \tau)]. \quad (24)$$

The same exponential function

$$\Sigma(x, y) = S_A(x, -i \xi) - y \xi \quad (25)$$

appears in both distributions (with $\xi$ the location of the saddle point). The pre-exponential functions $N_{\phi_0}$ and $N_{q_0}$ are different, determined by the Gaussian integration around the saddle point. Since these two functions vary only algebraically, rather than exponentially, we conclude that Eq. \[^{[22]}\] holds with the remainder $O(1) = \ln(q/\phi)$ obtained by evaluating $\ln[2 \pi (\partial^2 \Sigma / \partial x^2)^{1/2} (\partial^2 \Sigma / \partial y^2)^{-1/2}]$ at $x = 2 \pi \phi / \tau$, $y = q/\tau$.

5 **Renormalization by Ohmic connector**

We consider the same circuit and turn to analysis of quantum corrections assuming $Z(\omega)G_Q \ll 1$ in the relevant frequency region. We demonstrate that the main effect of the corrections can be incorporated into the renormalization of energy dependence of the transmission eigenvalues of the quantum connector. We study this dependence in a non-perturbative limit to obtain an unexpected result: owing to accumulation of
quantum corrections, all quantum conductors behave at low energies like either a single or a double tunnel junction, which divides them into two broad classes.

It has been shown that at low energy scales the relevant part of the electron-electron interaction in mesoscopic conductors comes from their electromagnetic environment [21] [9]. The resulting dynamical Coulomb blockade has been thoroughly investigated for tunnel junctions [10]. The measure of the interaction strength is the external impedance $Z(\omega)$ at the frequency scale $\Omega = \max(eV, k_B T)$ determined by either the voltage $V$ at the conductor or its temperature $T$. If $z \equiv G t Z(\Omega) < 1$ the interaction is weak, otherwise Coulomb effects strongly suppress electron transport.

A tunnel junction is the simplest quantum conductor with all transmission eigenvalues $T_n \ll 1$. Interaction effects for general connectors with $T_n \simeq 1$ are difficult to quantify for arbitrary $z$. For $z \ll 1$, one can employ perturbation theory to first order in $z$ [22]. The contributions [23] [24] associated the resulting interaction correction to the conductance with shot noise properties of the conductor, while the interaction correction to noise has been associated with the third cumulant of charge transfer [25]. To sort this out, one shall proceed in the framework of the field theory outlined where all cumulants are incorporated into functional dependence of the action on the field $\chi$. The recent experiment [26] addresses the correction to the conductance at arbitrary transmission.

A tunnel junction in the presence of an electromagnetic environment exhibits an anomalous power-law $I$-$V$ characteristic, $I(V) \simeq V^{2z+1}$. The same power law behavior is typical for tunnel contacts between one-dimensional interacting electron systems, the so-called Luttinger liquids [11]. It has also been found for contacts with arbitrary transmission between single-channel conductors in the limit of weak interactions [27]. In this case, the interactions have been found to renormalize the transmission.

In our model of a quantum connector, its transmission probabilities $T_n$ are energy independent in the absence of interactions. We first analyze the quantum correction to first order in $z$. We identify an elastic and an inelastic contribution. The elastic contribution comes with a logarithmic factor that diverges at low energies suggesting that even weak interactions can suppress electron transport at sufficiently low energies. To quantify this we sum up quantum corrections to the action in all orders in $z$ by a renormalization group analysis. We show that the result is best understood as a renormalization of the transmission eigenvalues similar to that proposed in [27]. The renormalization brings about an energy dependence of the transmission eigenvalues according to the flow equation

$$\frac{d T_n(E)}{d \ln E} = 2z T_n(E) [1 - T_n(E)].$$  

To calculate transport properties in the presence of interactions, one evaluates $T_n(E)$ at the energy $E \simeq \Omega$.

With relation (26) we explore the effect of quantum corrections on the distributions of transmission probabilities for various types of mesoscopic conductors. In general, their conductance $G$ and their noise properties display a complicated behavior at $z | \ln E | \simeq 1$ that depends on details of the conductor. However, in the limit of very low energies $z | \ln E | \gg 1$ we find only two possible scenarios. The first one is that the conductor behaves like a single tunnel junction with $G(V) \simeq V^{2z}$. In the other scenario, the transmission distribution approaches that of a symmetric double tunnel junction. The conductance scales then as $G(V) \simeq V^2$. Any given conductor follows one of the two scenarios. This divides all mesoscopic conductors into two broad classes.

We still analyze a simple circuit that consists of a mesoscopic conductor in series with an external resistor $Z(\omega)$ biased with a slow-varying voltage source $V_0(t)$ (Fig. 1) but now concentrate on quantum corrections.

As we have already done, we present the generating function $Z([\chi, \Phi])$ of the low-frequency current fluctuations in the circuit is as a path integral over the fields $\Phi_1(t), \chi_1(t)$ (Eq. 4). It is convenient for us to change the order of the connectors so that

$$Z(\Phi, \chi) = \int D\Phi_1 D\chi_1 \exp \{ \ln Z_c [\Phi - \Phi_1, \chi - \chi_1] \ln Z_{Ohm} [\Phi_1, \chi_1] \}$$  

Equation (27)
where $d\Phi(t)/dt \equiv eV(t)$ and $Z_{\text{Ohm}}$ is given by (8). Let us assume that $\Phi, \chi$ are slow fields. In order not to repeat the considerations of the previous Section, we will simply set $Z(0)$ to 0. In this case, the saddle point is trivial: $\Phi_1, \chi_1 = 0$. In physical terms, all the voltage drops on the quantum contact.

We start the renormalization by concentrating on the “fast” part of the fields $\Phi_1, \chi_1$ and expanding the action till quadratic terms in these fields. Doing so, we neglect the time-dependence of slow fields in comparison with that of fast fields, so the corresponding part of the action reads

$$
\delta S_c(\Phi(t), \chi(t)) = \int dt \int \frac{d\omega}{2\pi} \delta^2 S_c(\Phi(t), \chi(t)) \left( \frac{\delta^2 S_c(\Phi(t), \chi(t))}{\delta \phi^\alpha_\omega \delta \phi^\beta_{-\omega}} \right) \phi^\beta_{-\omega}
$$

(28)

where $\alpha, \beta = \pm, \phi^\pm = \Phi_1(\omega) \pm \chi_1(\omega)/2$, and $M_{\text{Ohm}}$ presents (8). We require that $Z(\omega)G \ll 1$ at any frequency. Under these conditions, the fluctuations of the fast fields are determined by the Ohmic term while the part of the action that comes from the fluctuation and does depend on the slow fields $\Phi, \chi$ is determined by the quantum conductor. Indeed, taking the Gaussian integral (28) we obtain the contribution to the action

$$
\delta S_c(\Phi(t), \chi(t)) = \int dt \int \frac{d\omega}{2\pi} \delta^2 S_c(\Phi(t), \chi(t)) \left( \frac{\delta^2 S_c(\Phi(t), \chi(t))}{\delta \phi^\alpha_\omega \delta \phi^\beta_{-\omega}} \right) (M^{-1})^{\alpha\beta}(\omega)
$$

(29)

This is the renormalization sought. The correction to the conductance of the quantum conductor it gives is of the order of $G_z$.

To proceed, we need the action of the quantum conductor at fast fields, not just at slow ones as given by Eq. (10). It is expressed in terms of Keldysh Green functions $\tilde{G}_{R,L}$ (the “check” denotes $2 \times 2$ matrices in Keldysh space) of electrons in the two reservoirs adjacent to the conductor (17). It takes the form of a trace over frequency and Keldysh indices,

$$
S_c = \frac{i}{2} \sum_n \text{Tr} \ln \left[ 1 + \frac{T_n}{4} \{ \tilde{G}_L, \tilde{G}_R \} - 2 \right]
$$

(30)

and depends on the set of transmission eigenvalues $T_n$ that characterizes the conductor. The fields $\phi^\pm(t)$ enter the expression as a gauge transform of $\tilde{G}$ in one of the reservoirs,

$$
\tilde{G}_R = \tilde{G}_L^\text{res} \quad \text{and} \quad \tilde{G}_L(t, t') = \begin{bmatrix} e^{i\phi^+_{L}(t)} & 0 \\ 0 & e^{i\phi^-_{L}(t)} \end{bmatrix} \tilde{G}_L^\text{res}(t - t') \begin{bmatrix} e^{-i\phi^+_{L}(t')} & 0 \\ 0 & e^{-i\phi^-_{L}(t')} \end{bmatrix},
$$

(31)

$\phi^\pm = \Phi \pm \chi/2 - \phi^{\text{res}}$ being the drop of the phase over the quantum conductor, $G^\text{res}$ being the equilibrium Keldysh Green function

$$
\tilde{G}_L^\text{res}(\epsilon) = \begin{bmatrix} 1 - 2f(\epsilon) & 2f(\epsilon) \\ 2[1 - f(\epsilon)] & 2f(\epsilon) - 1 \end{bmatrix},
$$

(32)

at a given equilibrium electron distribution function $f(\epsilon)$.

To zeroth order in $z$ the fields $\phi^\pm(t)$ do not fluctuate and are fixed to $eVt \pm \chi/2$. Substituting this into Eq. (30) we recover the slow-field action (10)

$$
S^{(0)}(V, \chi) = \int \frac{d\epsilon}{2\hbar \pi} \sum_n \ln \left[ 1 + T_n \left( e^{i\chi} - 1 \right) f_L(1 - f_R) + \left( e^{-i\chi} - 1 \right) f_R(1 - f_L) \right]
$$

(33)
(f_R \equiv f \text{ and } f_L(\epsilon) \equiv f(\epsilon - eV)). To assess the renormalization correction, we expand the non-linear \( S_e \) to second order in the fluctuating fields \( \phi^\pm \) and use (29). The expression for the correction can be presented as

\[
S^{(1)}(V, \chi) = \int_0^\infty d\omega \frac{\text{Re} z(\omega)}{\omega} \left\{ [2N(\omega) + 1]S^{(1)}_{el} + N(\omega)S^{(1)}_{in}(\omega) + [N(\omega) + 1]S^{(1)}_{in}(\omega) \right\}.
\]  

(34)

The three terms in square brackets correspond to \textit{elastic} electron transfer, inelastic transfer with absorption of energy \( \hbar \omega \) from the environment, and inelastic electron transfer with emission of this energy respectively. It is crucial to note that inelastic processes can only occur at frequencies \( \omega \leq \Omega \) and that their contribution to the integral is thus restricted to this frequency range. In contrast, elastic contributions come primarily from frequencies exceeding the scale \( \Omega \). If \( z = \text{const}(\omega) \) for \( \omega \leq \Lambda \), the elastic correction diverges logarithmically, its magnitude being \( \approx z \ln \Lambda/\Omega \). This suggests that i. the elastic correction is more important than the inelastic one and ii. a small value of \( z \) can be compensated for by a large logarithm, indicating the breakdown of perturbation theory. The upper cut-off energy \( \Lambda \) is set either by the inverse \( RC \)-time of the environment circuit or the Thouless energy of the electrons in the mesoscopic conductor.

The concrete expression for \( S^{(1)}_{in} \) reads

\[
S^{(1)}_{in}(\omega, \chi) = i \sum_n \int \frac{dz}{2\pi} D_n D_n^+ \left\{ T_n(f_L - f_R^+) + 2T_n(e^{i\chi} - 1)f_L(1 - f_R^+) \right. \\
\left. + 2T_n^2 \cos(\chi - 1)f_L(1 - f_L^+)(f_R - f_R) + T_n D_n + (1 - D_n)(1 - D_n^+) \right\} \\
+ \{ R \leftrightarrow L, \chi \leftrightarrow -\chi \},
\]

(35)

where we have introduced the functions

\[
D_n = \left\{ 1 + T_n [f_L(1 - f_R)(e^{i\chi} - 1) + f_R(1 - f_L)(e^{-i\chi} - 1)] \right\}^{-1},
\]

(36)

and the notation

\[
f^\pm(\epsilon) = f(\epsilon + \omega), \quad D_n^\pm(\epsilon) = D_n(\epsilon + \omega).
\]

(37)

We do not analyze \( S^{(1)}_{in} \) further and instead turn to the analysis of the elastic correction. It is important that the explicit form of this correction can be presented as

\[
S^{(1)}_{el} = \sum_n \delta T_n \frac{\partial S^{(0)}}{\partial T_n} \text{ with } \delta T_n = -2T_n(1 - T_n).
\]

(38)

This suggests that the main effect of renormalization is to change the transmission coefficients \( T_n \). It also suggests that we can go beyond perturbation theory by a renormalization group analysis that involves the \( T_n \) only. In such an analysis one concentrates at each renormalization step on the “fast” components of \( \phi^\pm \) with frequencies in a narrow interval \( \delta \omega \) around the running cut-off frequency \( E \). Integrating out these fields one obtains a new action for the slow fields. Subsequently one reduces \( E \) by \( \delta \omega \) and repeats the procedure until the running cut-off approaches \( \Omega \). We find that at each step of renormalization the action indeed retains the form given by Eq. (30) and only the \( T_n \) change, provided \( z \ll \min\{1, G_Q/G\} \). The resulting energy dependence of the \( T_n \) obeys Eq. (26). The approximations that we make in this renormalization procedure amount to a summation of the leading logarithms in every order of the perturbation series.

In the rest of the Section we analyze the consequences of Eq. (26) for various mesoscopic conductors. Equation (26) can be explicitly integrated to obtain

\[
T_n(E) = \frac{\xi T_n^\Lambda}{1 - T_n^\Lambda (1 - \xi)}, \quad \xi \equiv \left( \frac{E}{\Lambda} \right)^{2z}
\]

(39)
in terms of the “high energy” (non-interacting) transmission eigenvalues $T^\Lambda$. A mesoscopic conductor containing many transport channels is most conveniently characterized by the distribution $\rho_\Lambda(T)$ of its transmission eigenvalues \cite{28}. It follows from Eq. (39) that the effective transmission distribution at the energy scale $E$ reads

$$
\rho_E(T) = \frac{\xi}{[\xi + T(1 - \xi)]^2} \rho_\Lambda \left( \frac{T}{\xi + T(1 - \xi)} \right). \tag{40}
$$

We now analyze its low energy limit $\xi \to 0$. Any given transmission eigenvalue will approach zero in this limit. Seemingly this implies that for any conductor the transmission distribution approaches that of a tunnel junction, so that all $T_n \ll 1$. The overall conductance would be proportional to $\xi$ in accordance with Ref. \cite{22}.

Indeed, this is one of the possible scenarios. A remarkable exception is the case that the non-interacting $\rho_\Lambda$ has an inverse square-root singularity at $T \to 1$. Many mesoscopic conductors display this feature, most importantly diffusive ones \cite{28}. In this case, the low-energy transmission distribution approaches a limiting function

$$
\rho_*(T) \propto \frac{\xi}{T^3(1 - T)}. \tag{41}
$$

The conductance scales like $\xi^{1/2}$. $\rho_*$ is known to be the transmission distribution of a double tunnel junction: two identical tunnel junctions in series \cite{29}. Indeed, one checks that for a double tunnel junction the form of the transmission distribution is unaffected by interactions. This sets an alternative low-energy scenario. We are not aware of transmission distributions that would give rise to other scenarios.

We believe that this is an important general result in the theory of quantum transport and suggest now a qualitative explanation. The statement is that the conductance of a phase-coherent conductor at low voltage and temperature $\Omega \ll \Lambda$ asymptotically obeys a power law with an exponent that generically takes two values,

$$
G \propto \left( \frac{\Omega}{\Lambda} \right)^{2z}, \quad \text{or} \quad G \propto \left( \frac{\Omega}{\Lambda} \right)^z. \tag{42}
$$

For tunneling electrons the exponent is $2z$. An electron traverses the conductor in a single leap. The second possible exponent $z$ has been discussed in the literature as well, in connection with resonant tunneling through a double tunnel barrier in the presence of interactions \cite{11}. This resonant tunneling takes place via intermediate discrete states contained between the two tunnel barriers. The halved exponent $\alpha = z$ occurs in the regime of the so-called successive electron tunneling. In this case, the electron first jumps over one of the barriers ending up in a discrete state. Only in a second jump over the second barrier the charge transfer is completed. Since it takes two jumps to transfer a charge, the electron feels only half the counter voltage due to interactions with electrons in the environmental impedance $Z$ at each hop. Consequently, the exponent at each jump takes half the value for direct tunneling. Our results strongly suggest that this transport mechanism is not restricted to resonant tunneling systems, or, in other words, that resonant tunneling can occur in systems of a more generic nature than generally believed. As far as transport is concerned, a mesoscopic conductor is characterized by its scattering matrix regardless of the details of its inner structure. In this approach it is not even obvious that the conductor can accommodate discrete states. Nevertheless, the transmission distribution of this scattering matrix does depend on the internal structure of the conductor. The inverse square root singularity of this distribution at $T \to 1$ for a double tunnel barrier is due to the formation of Fabry-Perot resonances between the two barriers. Probably similar resonances are at the origin of the same singularity for more complicated mesoscopic conductors with multiple scattering. They are then the intermediate discrete states that give rise to the modified scaling of the conductance in
presence of interactions. One may speculate that in diffusive conductors these resonances are the so-called “nearly localized states” found in \[30\].

From equation \(29\) one concludes that the resonant tunneling scaling holds only if \(G(E) \gg G_Q\) so that many transport channels contribute to the conductance. At sufficiently small energies, \(G(E)\) becomes of the order of \(G_Q\). All transmission eigenvalues are then small and the conductance crosses over to the tunneling scaling.

6 Renormalization by quantum connectors

One could wonder about the generality of the results obtained in the previous Section and expressed by Eq. \(26\). Indeed, it has been proven under rather restrictive assumptions of an Ohmic connector of negligible resistance. Here, we consider a more general model of several quantum connectors coming together in a single node (Fig. 2). Each connector labeled by \(k\) is characterized by the set of the transmission eigenvalues \(T_n^{[k]}\). In traditional Coulomb blockade situation \((G \ll G_Q)\) this setup is called Coulomb island or SET transistor \([9]\) and is seen very different from the junction-in-the-environment setup considered in the previous Section. However, we show that in the limit of large conductance \(G \gg G_Q\) the Coulomb island is governed by very similar renormalization equations:

\[
\frac{d T_n^{[k]}}{d \ln E} = \frac{2 T_n^{[k]}(1 - T_n^{[k]})}{\sum_{n,k} T_n^{[k]}}.
\]

It looks like each quantum channel sees all others as an “environment” characterized by the effective island conductance \(g = \sum_{n,k} T_n^{[k]}\). If we use \(z = g^{-1}\), Eqs. \(43\) and \(26\) are identical. The difference is that \(g\) by itself is subject to renormalization.

This results in very different low-energy behavior. In contrast to the considerations of the previous Section, the renormalization of all transmission eigenvalues may break down at finite energy — effective Coulomb gap — \(E_C \propto g_0 E_C e^{-\alpha |d_0|}\), \(\alpha\) being a numerical factor depending on the details of the initial transmission distribution, \(g_0\) is the island conductance at high energies \(E_C g_0\). Remarkably, \(E_C\) coincides with the effective charging energy evaluated with instanton technique \([31]\). However, the renormalization stops at the effective Thouless energy \(E_{\text{Th}} \sim G(E) d / G_Q\), \(d\) being mean level spacing in the island. This gives rise to two distinct scenarios at low energy. If \(g_0 > \alpha^{-1} \ln(E_C / \delta)\), Coulomb blockade does not occur with zero-bias conductance being saturated at the value \(G(E_{\text{Th}}) \approx G_Q\). Alternatively, \(G(0) \approx 0\) and \(E_C\) defines the Coulomb gap.

Let us give the details of the model in use. The Coulomb island is characterized by two parameters: charging energy \(E_C\) and the mean level spacing \(\delta\), \(E_C \gg \delta\). \([32]\) The island is connected to \(M \geq 2\) external leads by means of \(M\) arbitrary quantum connectors (Fig. 2) characterized by the set of transmission eigenvalues \(T_n^{[i]}\). We assume that the island is strongly coupled to the leads, \(g_0 = \sum_{n,m} T_n^{[i]} \gg 1\). Our goal is to evaluate the functional \(\mathcal{Z}\) for the whole circuit that now depends on voltages and counting fields in each terminal, \(\mathcal{Z}([V, \chi_i])\). To evaluate this for the Coulomb island, we have extended the semiclassical approach for the FCS of the non-interacting electrons \([33]\). The node houses a dynamical phase variable \(\phi(t)\), its time derivative, \(\dot{\phi}(t) / e\), presents the fluctuating electrostatic potential of the island. According to the rules of our field theory, the functional is represented in the form of a real-time path integral over the fields \(\phi^{\pm}(t)\) residing at two branches of the Keldysh contour

\[
\mathcal{Z}([V, \chi_i]) = \int D\phi_{\pm}(t) \exp \left\{ \frac{i}{2} E_C^{-1} \int_{-\infty}^{+\infty} dt (\dot{\phi}^+)^2 - (\dot{\phi}^-)^2 \right\}
- \sum_k S_{\text{con}}^{[k]} \{ \{ \hat{G}, \hat{G}_k^\dagger \} \} - i \pi \delta^{-1} \text{Tr} \{ (i \dot{\phi} - \dot{\Phi}) \hat{G} \}
\]
\(44\)
Here \( \hat{\Phi} = \begin{pmatrix} \phi^+(t) & 0 \\ 0 & \phi^-(t) \end{pmatrix} \) is the matrix in Keldysh space. \( 2 \times 2 \) matrix \( \hat{G}(t_1, t_2) \) presents the electron Green function in the island that implicitly depends on \( \phi^\pm(t) \). The trace operation includes the summation over Keldysh indices and the integration in time. The contribution of each connector \( S^\text{con} \) has a form
\[
S^\text{con} = -\frac{1}{2} \sum_n \text{Tr} \ln \left[ 1 + \frac{1}{4} T_n^\text{[k]} \{ \hat{G}, \hat{G}^\chi \} - 2 \right] 
\] (45)

\( \{ \hat{G}, \hat{G}^\chi \} \) denoting the anticommutator of the Green functions with respect to both Keldysh and time indices. The Green functions in the leads \( \hat{G}_k(\chi) \) are obtained by \( \chi \)-dependent gauge transformation \( [34] \) of the equilibrium Green functions in the reservoir \( k \), \( \hat{G}^{[0]}_k, \hat{G}^\chi_k(\epsilon) = \exp(i\chi k \bar{\tau}_3/2) \hat{G}^{[0]}_k(\epsilon) \exp(-i\chi k \bar{\tau}_3/2) \), where \( \hat{G}^{[0]}_k \) are given by \( \hat{G}^{[0]}_k = \begin{pmatrix} 1 - 2 f_k & -2 f_k \\ -2(1 - f_k) & 2 f_k - 1 \end{pmatrix} \). Here \( f_k(\epsilon) \) presents the electron distribution function in the \( k \)-th reservoir. The expression \( 45 \) is valid under assumption of instantaneous electron transfer via a connector, thus corresponding to energy-independent \( T_n^\text{[k]} \).

In order to find \( \hat{G}(t_1, t_2) \) at given \( \phi^\pm(t) \), we minimize the action with respect to all \( \hat{G}(t_1, t_2) \) subject to the constran \( \hat{G} \circ \hat{G} = \delta(t_1 - t_2) \). This yields the saddle point equation for \( \hat{G}(t_1, t_2) \):
\[
\sum_{n, k} \frac{T_n^\text{[k]} \{ \hat{G}_k^{\chi}, \hat{G} \}}{4 + T_n^\text{[k]} \{ \hat{G}_k^{\chi}, \hat{G} \} - 2} = i\pi \delta^{-1} [ i \partial_t - \hat{\Phi}, \hat{G} ] 
\] (46)

where \([...,] \) denotes the commutator in the Keldysh-time space. This relation expresses \( \hat{G}(t_1, t_2) \equiv G(t_1, t_2; [\phi^\pm(t)]) \) via the reservoir Green functions \( \hat{G}^{[k]} \). This circuit theory relation is similar to obtained in \( [33] \). It disregards the mesoscopic fluctuations, since those lead to corrections of the order of \( \sim 1/g_0 \) at all energies, whereas the interaction corrections are of the order of \( \sim 1/g_0 \ln(E) \) tending to diverge at small energies. If \( \phi^\pm(t) = 0 \), Eq. \( 46 \) separates in energy representation and coincides with that of Ref. \( [33] \).

This sets the model. We start the analysis of the model with perturbation theory in \( \phi^\pm \) around the semiclassical saddle point \( \hat{G}(t_1, t_2) = \hat{G}_0, \phi^\pm(t) = 0 \). The phase fluctuations are small, \( \delta\phi^2 \sim 1/g_0 \), so we keep only quadratic terms to the action \( 44 \). The resulting Gaussian path integral over \( \phi^\pm \) can be readily done. This procedure is equivalent to the summation of all one-loop diagrams of the conventional perturbation theory, i.e. to the "random-phase approximation" (RPA).
We restrict ourselves to the most interesting low voltage/temperature limit, \( \max\{eV, kT\} \ll g_0 E_C \). In this limit, we evaluate the interaction correction to the CGF with the logarithmic accuracy. It reads

\[
\Delta S_\chi = \frac{t_0}{g_0} \ln \left( \frac{g_0 E_C}{\max\{eV, kT\}} \right) \times \int \frac{d\varepsilon}{2\pi} \sum_{n,k} 2T_n^{[k]} (1 - T_n^{[k]}) \left( \{G_n^\chi, \hat{G}_0\} - 2 \right)
\]

provided \( \max\{eV, kT\} > E_{\text{Th}} \), where \( E_{\text{Th}} = g_0 \delta \) is the Thouless energy of the island. In the opposite case, \( \max\{eV, kT\} < E_{\text{Th}} \), the voltage/temperature should be replaced with \( E_{\text{Th}} \). Note, that the correction (47) is contributed by only virtual inelastic processes that change the probabilities of real elastic scatterings.

For simplicity, we consider the shot-noise limit \( eV \gg kT \) only. Then the magnitude of the correction shall be compared with the zero-order CGF \( S_\chi^{[0]} \sim t_0 eV g_0 \). This implies that the perturbative RPA result (47) is applicable only if \( g_0^{-1} \ln(g_0 E_C/eV) \ll 1 \). At lower voltages \( \Delta S_\chi \) logarithmically diverges. This indicates that we should proceed with a renormalization group (RG) analysis.

We perform the RG analysis of the action (44) along the lines of the previous section decomposing \( \phi^\pm(t) \) onto the fast and slow parts. On each step of RG procedure we eliminate the fast degrees of freedom in the energy range \( E - \delta E < \omega < E \) to obtain new action \( S_{E-\delta E}[\phi_\omega] \), \( E \) being the current ultraviolet cutoff. Our key result is that the change in the action at each step of RG procedure can be presented as a change of transmission eigenvalues \( T_n^{[k]} \). Therefore, the RG equations can be written directly for transmission eigenvalues and take a simple form (43). The equations are to be solved with initial conditions at the upper cutoff energy \( E = g_0 E_C \), those are given by "bare" transmission eigenvalues \( T_n^{[k]}(E = g_0 E_C) = T_n^{[k]} \). The RG equations resemble those for the transmission coefficient for a scatterer in the weakly interacting one-dimensional electron gas [27] and for a single multi-channel scatterer in the electromagnetic environment [15]. The effective impedance \( Z \) is just replaced by inverse conductance of the island to all reservoirs, \( G(E) = G_Q \sum_{n,k} T_n^{[k]}(E) \). The important difference is that this conductance is itself subject to renormalization. The difference becomes most evident in the case when all contacts are tunnel junctions, \( T_n^{[k]} \ll 1 \). In this case, one can sum up over \( k, n \) in Eqs. (43) to obtain the RG for the conductance only: \( dG/d\ln E = 2G_Q \). This renormalization law [35] was recently applied to conductance of granular metals. The Eqs. (43) could be also derived in the framework of functional RG approach to \( \sigma \)-model of disordered metal [36].

We solve the RG Eqs. (43) in general case to obtain

\[
T_n^{[k]}(E) = T_n^{[k]}(g_0 E_C) \frac{y}{1 - T_n^{[k]}(1 - y)} \quad \text{(48)}
\]

\[
\ln(g_0 E_C/E) = - \frac{1}{2} \sum_{n,k} \ln(1 - T_n^{[k]}(1 - y)) \quad \text{(49)}
\]

The first equation gives the renormalized transmission eigenvalues at a given value \( E \) of the upper cutoff in terms of variable \( y(E) \), \( 0 \leq y \leq 1 \). The second equation implicitly expresses \( y(E) \).

We note that the energy dependence of transmission coefficients induced by interaction is very weak provided \( G(E) \gg G_Q \): If energy is changed by a factor of two, the conductance is changed by \( \sim G_Q \). To use the equations for evaluation of FCS at given voltages \( V^{[k]} \) of the leads, one takes \( T_n^{[k]}(E) \) at upper cutoff \( E = \max_k(V^{[k]}) \), and further disregards their energy dependence. Then one can follow the lines of Ref. [33]: It is convenient to introduce the function \( S^{[k]}(x) = - \sum_n \ln[1 + \frac{1}{2} T_n^{[k]}(x - 1)] \) to incorporate all required information about transmission eigenvalues. The renormalization of \( S^{[k]} \) in terms of \( y \) is especially simple: \( S^{[k]}(x, y) = S^{[k]}(x + 1) y - 1 - S^{[k]}(2y - 1) \). From this one readily finds the conductance of each scatterer, \( G^{[k]}(y) = 2G_Q \partial S^{[k]}/\partial x(1, y) \), as well as the renormalized transmission distribution \( T^2 \rho^{[k]}(T, y) = (2/\pi) \Im \{\partial S^{[k]}/\partial x (1 - 2/T - i0, y)\} \).
The total conductance of the Coulomb island versus the energy: two scenarios. We assume $\ln\left(E_c/\delta\right) = 10.0$. Arrows show the energy scale $\sim \delta$. Pane (a): tunnel connectors, $g_0$ changes from 42 (upper curve) to 14 (lowermost curve) with the step 4. Pane (b): diffusive connectors, $g_0$ changes from 18 to 6 with the step 2. The conductance either hits 0 manifesting the Coulomb gap or saturates at finite value.

The RG equations (1) have a fixed point at $T_n^{[k]} = 0, y = 0$ that occur at finite energy

$$E = \bar{E}_C = g_0 E_C \prod_{k,n} (1 - T_n^{[k]})^{1/2}$$

(50)

This indicates the breakdown of RG and formation of Coulomb blockade with the exponentially small gap $\bar{E}_C$. The same energy scale was obtained from equilibrium instanton calculation of Ref. [31]. For a field theory, one generally expects different physics and different energy scales for instantons and perturbative RG. The fact that these scales are the same shows a hidden symmetry of the model which is yet to understand.

Alternative low-energy behavior is realized if the current cut-off reaches $E_{Th} = G(E)\delta/G_Q$. (Fig. 3) The log renormalization of the transmission eigenvalues stops at this point and their values saturate. We thus predict a sharp crossover between the two alternative scenarios, that occur at value of $g_0 = g_c$ corresponding to $\bar{E}_C \simeq \delta$. This value equals $g_c = \alpha^{-1} \ln(E_C/\delta)$, where $\alpha = \frac{1}{2} g_0^{-1} \sum_{n,k} \ln(1 - T_n^{[k]})$, and depends on transmission distribution of all connectors. If all connectors are tunnel junctions, $\alpha_T = 2$. For diffusive connectors, $\alpha_D = \pi^2/8$ and the energy dependence of the total conductivity is given by $g_D(V) \sim g_0 \sqrt{\xi} \tanh \sqrt{\xi} \xi = 2 g_0^{-1} \ln(g_0 E_C/eV)$. (Fig. 3)

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References

[1] L. S. Levitov and G. B. Lesovik, JETP Lett. 58, 230 (1993); [cond-mat/9401004], L. S. Levitov, H. Lee, and G. B. Lesovik, J. Math Phys. 37, 4845 (1996).
[2] Yu. V. Nazarov (editor), NATO ASI Series II, 97, Quantum Noise in Mesoscopic Physics, (Dordrecht, Kluwer, 2003).
[3] B. Reulet, J. Senzier, and D. E. Prober, Phys. Rev. Lett. 91, 196601 (2003).
[4] J. Bylander, T. Duty, P. Delsing, Nature 434, 361 (2005); T. Fujisawa, T. Hayashi, R. Tomita, Y. Hirayama, S. Gustavsson, R. Leturcq, T. Ihn, et al. Science 312, 1634 (2006); Phys. Rev. B 75 075314 (2007).
[5] A. O. Gogolin, A. Komnik, Phys. Rev. B 73, 195301 (2006).
[6] V. Ambegaokar, U. Eckern, G. Schön, Phys. Rev. Lett. 48, 1745 (1982).
[7] H. Grabert, P. Schramm, G. L. Ingold, Phys. Rep. 168, 115 (1988).
[8] D. V. Averin and K. K. Likharev, in *Mesoscopic Phenomena in Solids*, edited by B. L. Altshuler, P. A. Lee, and R. A. Webb (Elsevier, Amsterdam, 1991).

[9] G.-L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret, NATO ASI Series B294 (Plenum, New York, 1992).

[10] M. H. Devoret, D. Esteve, H. Grabert, G. L. Ingold, H. Pothier, C. Urbina, Phys. Rev. Lett. 64, 1824 (1990).

[11] G.-L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling*, edited by H. Grabert and M. H. Devoret, NATO ASI Series B294 (Plenum, New York, 1992).

[12] M. Kindermann, Yu. V. Nazarov and C. W. J. Beenakker, Phys. Rev. Lett. 90, 246805 (2003).

[13] M. Kindermann and Yu. V. Nazarov, Phys. Rev. Lett. 91, 136802 (2003).

[14] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1993).

[15] A. D. Zaikin, Phys. Rev. Lett. 87, 046802 (2001).

[16] R. Egger, H. Grabert, Phys. Rev. Lett. 75, 3505 (1995).

[17] Yu. V. Nazarov, Zh. Eksp. Teor. Fiz. 95, 975 (1989);

[18] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[19] Yu. V. Nazarov, in *Quantum Dynamics of Submicron Structures*, edited by H. A. Cerdeira, B. Kramer, G. Schön, Kluwer, 1995, p. 687; [cond-mat/9410011](https://arxiv.org/abs/cond-mat/9410011).

[20] Yu. V. Nazarov, Phys. Rev. Lett. 88, 196801 (2002).

[21] M. Kindermann and Yu. V. Nazarov, Phys. Rev. Lett. 91, 136802 (2003).

[22] Yu. V. Nazarov, Zh. Eksp. Teor. Fiz. 95, 975 (1989);

[23] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[24] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[25] Yu. V. Nazarov, Zh. Eksp. Teor. Fiz. 95, 975 (1989);

[26] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[27] Yu. V. Nazarov, Zh. Eksp. Teor. Fiz. 95, 975 (1989);

[28] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[29] Yu. V. Nazarov, Sol. St. Comm. 75, 669 (1990).

[30] Yu. V. Nazarov, Phys. Rev. B 51, 5482 (1995).

[31] Yu. V. Nazarov, Phys. Rev. Lett. 82, 1245 (1999).

[32] Strictly speaking, a purely electric field theory corresponds to $\delta_S = 0$. A finite $\delta_S$ enables electrons rather than electricity to escape the node and may provide coherent electron transfer through two connectors.

[33] Yu. V. Nazarov, in *Quantum Dynamics of Submicron Structures*, edited by H. A. Cerdeira, B. Kramer, G. Schön, Kluwer, 1995, p. 687; [cond-mat/9410011](https://arxiv.org/abs/cond-mat/9410011).

[34] Yu. V. Nazarov and D. A. Bagrets, Phys. Rev. Lett., 88, 196801 (2002).

[35] W. Belzig and Yu. V. Nazarov, Phys. Rev. Lett. 87.

[36] J. M. Kosterlitz, Phys. Rev. Lett. 37, 1577 (1976);

[37] M. V. Feigelman, A. Kamenev, A. I. Larkin, and M. A. Skvortsov Phys. Rev. B 66, 054502 (2002)