Representations and fusion rules for the orbifold vertex operator algebras $L_{\hat{\mathfrak{sl}}_2}(k,0)^{\mathbb{Z}_p}$

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ABSTRACT
For the cyclic group $\mathbb{Z}_p$ ($p$ is a prime) and a positive integer $k$, we study the representations of the orbifold vertex operator algebra $L_{\hat{\mathfrak{sl}}_2}(k,0)^{\mathbb{Z}_p}$. All the irreducible modules for $L_{\hat{\mathfrak{sl}}_2}(k,0)^{\mathbb{Z}_p}$ are classified and constructed explicitly. Quantum dimensions and fusion rules for $L_{\hat{\mathfrak{sl}}_2}(k,0)^{\mathbb{Z}_p}$ are completely determined.

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1. Introduction
The orbifold construction is a powerful tool for constructing new vertex algebras from given ones. Let $V$ be a vertex operator algebra and $G$ a finite automorphism group of $V$, the fixed point subalgebra $V^G = \{ v \in V \mid gv = v, g \in G \}$ is called an orbifold vertex operator subalgebra of $V$. Many interesting examples, especially orbifold vertex operator algebras related to affine vertex operator algebras and lattice vertex operator algebras, have been extensively studied in both physics and mathematics literature ([3, 4, 13–15, 17, 26, 24, 30, 31], etc.).

The orbifold theory is to study the properties and representation theory of the fixed point vertex operator subalgebra $V^G$. It is natural to ask whether $V^G$ inherits some properties from $V$, such as simplicity, rationality, $C_2$-cofiniteness, and regularity. It has been established that if $V$ is a regular and selfdual vertex operator algebra of CFT type and $G$ is a finite solvable group, then $V^G$ is again a regular and selfdual vertex operator algebra of CFT type [3, 30]. The decomposition of $V$ into a direct sum of irreducible $V^G$-modules was initiated in [8] and [13]. The decomposition of an arbitrary irreducible $g$-twisted $V$-module into a direct sum of $V^G$-modules was achieved in [17] and [31]. It follows from [15] that if $V^G$ is a regular and selfdual vertex operator algebra of CFT type, then any irreducible $V^G$-module occurs in an irreducible $g$-twisted $V$-module for some $g \in G$. In other words, the irreducible $V^G$-modules were completely classified if $V^G$ is a regular and selfdual vertex operator algebra of CFT type.

This article is a continuation of our investigation on $\mathbb{Z}_3$-orbifold vertex operator algebras $L_{\hat{\mathfrak{sl}}_2}(k,0)^{\mathbb{Z}_3}$ [33]. It is well known that $L_{\hat{\mathfrak{sl}}_2}(k,0)$ is a regular and selfdual vertex operator algebra of

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CFT type for $k \in \mathbb{Z}_{\geq 1}$ [20, 27]. It is natural to consider the orbifold vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)^G$ for $k \in \mathbb{Z}_{\geq 1}$ and some finite subgroup $G$ of $\text{Aut}(L_{\mathfrak{sl}_2}(k,0))$. Representations and fusion rules of the $\mathbb{Z}_2$-orbifold and $\mathbb{Z}_3$-orbifold of the vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)(k \in \mathbb{Z}_{\geq 1})$ were given in [24] and [33], respectively. For the Klein group $K$, $k \in \mathbb{Z}_{\geq 1}$, representations of the orbifold vertex operator algebras $L_{\mathfrak{sl}_2}(k,0)^K$ were constructed in [25]. For a prime $p$, let $\mathbb{Z}_p$ be the cyclic subgroup of $\text{Aut}(L_{\mathfrak{sl}_2}(k,0))$ generated by $\sigma_p$ which is defined by $\sigma_p(h) = h, \sigma_p(e) = \xi_p e, \sigma_p(f) = \xi_p^{-1}f$, where $\xi_p = e^\frac{2\pi i}{p} - 1$, $\{h, e, f\}$ is a standard Chevalley basis of $\mathfrak{sl}_2$ with Lie brackets $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Then any irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-module occurs in an irreducible $\tau$-twisted $L_{\mathfrak{sl}_2}(k,0)$-module for some $\tau \in \mathbb{Z}_p$ [15]. In this article, we classify and construct all the irreducible modules for the orbifold vertex operator algebras $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$ for $k \geq 1$. We construct $\tau$-twisted modules of $L_{\mathfrak{sl}_2}(k,0)$ for each $\tau \in \mathbb{Z}_p$, and give the decomposition of each irreducible $\tau$-twisted $L_{\mathfrak{sl}_2}(k,0)$-module into a direct sum of irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-modules. It turns out that there are exactly $p^2(k+1)$ inequivalent irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-modules. We call the irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-module coming from the irreducible $L_{\mathfrak{sl}_2}(k,0)$-module the untwisted type $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-module. We call the irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-module coming from the twisted $L_{\mathfrak{sl}_2}(k,0)$-module the twisted type $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-module.

The quantum dimensions of the irreducible modules introduced in [6] are the important invariants of $V$ and the product formula \( q\text{dim}_V(M \boxtimes_V N) = q\text{dim}_V M \cdot q\text{dim}_V N \) ([6]) for any $V$-modules $M, N$ plays an essential role in computing the fusion rules. An explicit relation between the quantum dimension of an irreducible $g$-twisted $V$-module $M$ and the quantum dimension of an irreducible $V^G$-submodule of $M$ was given in [15]. We use this powerful relation to compute the quantum dimension of any irreducible module of the orbifold vertex operator algebras $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$.

The fusion rules for the orbifold vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$ are completely determined in Quantum dimensions and fusion rules for the orbifold vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$ section. The initial inspiration for the main idea is the fusion rules of the $\mathbb{Z}_3$-orbifold of the vertex operator algebra $L_{\mathfrak{sl}_2}(k,0)$ [33]. The determination of the fusion products between twisted type $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-modules and twisted type $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-modules is the hardest step. The main strategy is to employ the Proposition 2.8. in [9] which describes that if $W = M_1 \boxtimes_V M_2$ for any $g_r$-twisted $V$-module $M_i (i = 1, 2)$ together with some other conditions then $\tilde{W} = M_1 \boxtimes_V \tilde{M_2}$ (the notation of $\tilde{W}$ is defined in [9] Lemma 2.6.). Furthermore, we determine the contragredient modules of all the irreducible $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$-modules, thus the fusion rules for $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$ are completely determined. Finally, we use the lattice realization of $L_{\mathfrak{sl}_2}(1,0)^{\mathbb{Z}_p}$ to determine the fusion rules for $L_{\mathfrak{sl}_2}(1,0)^{\mathbb{Z}_p}$ which is compatible with our main results.

This article is organized as follows. In Preliminary section, we briefly review some basic notations and facts on vertex operator algebras. In Classification and construction of irreducible modules of $L_{\mathfrak{sl}_2}(k,0)^{\mathbb{Z}_p}$ section, we first give the action of the cyclic group $\mathbb{Z}_p$ on $L_{\mathfrak{sl}_2}(k,0)$ and realize each element of $\mathbb{Z}_p$ as an inner automorphism of $\mathfrak{sl}_2$. Then we classify and construct all the
irreducible modules of the orbifold vertex operator algebras $L_{\gamma}(k,0)^{Z_p}$ for $k \geq 1$. In Quantum dimensions and fusion rules for the orbifold vertex operator algebra $L(k,0)^{Z_p}$ section, we compute the quantum dimension of any irreducible module of $L_{\gamma}(k,0)^{Z_p}$ for $k \geq 1$. Finally, the fusion rules for the orbifold vertex operator algebras $L_{\gamma}(k,0)^{Z_p}$ are completely determined.

We use the usual symbols $\mathbb{C}$ for the complex numbers, $\mathbb{Z}$ for the integers, $\mathbb{Z}_{\geq 0}$ for the non-negative integers, and $\mathbb{Z}_{\geq 1}$ for the positive integers. In this article, $j$ means the residue of the integer $j$ modulo $p$.

**Preliminary**

Let $(V,Y,1,\omega)$ be a vertex operator algebra [2, 19]. We first review basics from [11, 12, 18] and [27]. Let $g$ be an automorphism of the vertex operator algebra $V$ of finite order $T$. Denote the decomposition of $V$ into eigenspaces of $g$ as:

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r,$$

where $V^r = \{v \in V | gv = e^{-2\pi \sqrt{-1}r}v\}$, $0 \leq r \leq T - 1$. We use $r$ to denote both an integer between 0 and $T - 1$ and its residue class modulo $T$ in this situation.

**Definition 2.1.** Let $V$ be a vertex operator algebra. A weak $g$-twisted $V$-module is a vector space $M$ equipped with a linear map

$$Y_M(\cdot, x) : V \rightarrow (\text{End}M)[[x^r, x^{-r}]]$$

$$v \rightarrow Y_M(v, x) = \sum_{n \in \mathbb{Z}/T\mathbb{Z}} v_n x^{-n-1},$$

where $v_n \in \text{End} M$, satisfying the following conditions for $0 \leq r \leq T - 1, u \in V^r, v \in V, w \in M$:

$$Y_M(u, x) = \sum_{n \in \mathbb{Z}/T\mathbb{Z}} u_n x^{-n-1},$$

$$u_s w = 0 \quad \text{for} \quad s \gg 0,$$

$$Y_M(1, x) = \text{id}_M,$$

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) Y_M(u, x_1) Y_M(v, x_2) - x_0^{-1} \delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_M(v, x_2) Y_M(u, x_1) = x_2^{-1} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_M(1, x_0)v_n x_n,$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ and all binomial expressions are to be expanded in non-negative integral powers of the second variable.

The following Borcherds identities can be derived from the twisted-Jacobi identity [11, 35].

$$[u_{m+\frac{r}{i}}, v_{n+\frac{r}{i}}] = \sum_{i=0}^{\infty} \binom{m}{i} \binom{r}{i} (u_i v)_{m+n+\frac{r}{i}-i},$$

$$\sum_{i=0}^{\infty} \binom{r}{i} (u_{m+i} v)_{n+\frac{r}{i}-i} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} (u_{m+\frac{r}{i}-i} v_{n+\frac{r}{i}} - (-1)^m v_{m+n+\frac{r}{i}-i} u_{\frac{r}{i}}),$$

where $u \in V^r, v \in V^s, m, n \in \mathbb{Z}$.

**Definition 2.2.** An admissible $g$-twisted $V$-module is a weak $g$-twisted $V$-module which carries a $\frac{r}{i} \mathbb{Z}_{\geq 0}$-grading $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ satisfying $v_n M(n) \subseteq M(n + r - m - 1)$ for homogeneous $v \in V_r, m, n \in \frac{1}{r} \mathbb{Z}$. 
Definition 2.3. A g-twisted V-module is a weak g-twisted V-module which carries a $C$-grading:

$$M = \bigoplus_{i \in \mathbb{Z}} M_i,$$

such that $\dim M_i < \infty, M_{i+\frac{d}{2}} = 0$ for fixed $\lambda$ and $n \ll 0, L(0)w = \lambda w = (\lambda w)w$ for $w \in M$, where $L(0)$ is the component operator of $Y_M(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$.

Remark 2.4. If $g = id_V$, we have the notations of weak, admissible, and ordinary V-modules [10].

Definition 2.5. A vertex operator algebra V is called g-rational if the admissible g-twisted V-module category is semisimple. V is called rational if V is id$_V$-rational.

If $M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z} \geq 0} M(n)$ is an admissible g-twisted V-module, the contragredient module $M'$ is defined as follows:

$$M' = \bigoplus_{n \in \frac{1}{2}\mathbb{Z} \geq 0} M(n)^*,$$

where $M(n)^* = \text{Hom}_C(M(n), C)$. The vertex operator $Y_{M'}(a, z)$ is defined for $a \in V$ via

$$\langle Y_{M'}(a, z)f, u \rangle = (f, Y_M(e^{L(1)}(-z^{-2})L(0)a, z^{-1})u),$$

where $\langle f, u \rangle = f(u)$ is the natural pairing $M' \times M \rightarrow \mathbb{C}$. It follows from [18] and [34] that $(M', Y_{M'})$ is an admissible $g^{-1}$-twisted V-module. We can also define the contragredient module $M'$ for a g-twisted V-module M. In this case, $M'$ is a $g^{-1}$-twisted V-module. Moreover, M is irreducible if and only if $M'$ is irreducible. M is said to be selfdual if M is V-isomorphic to $M'$. In particular, V is said to be a selfdual vertex operator algebra if V is isomorphic to $V'$. We recall the following concept from [36].

Definition 2.6. A vertex operator algebra is called $C_2$-cofinite if $C_2(V)$ has a finite codimension (i.e., $\dim V/C_2(V) < \infty$), where $C_2(V)$ is a subspace of V spanned by $u_{-2}v$ for $u, v \in V$.

We have the following results from [11] and [12].

Theorem 2.7. If V is g-rational vertex operator algebra, then

1. Any irreducible admissible g-twisted V-module M is a g-twisted V-module. Moreover, there exists a number $\lambda \in \mathbb{C}$ such that $M = \bigoplus_{n \in \frac{1}{2}\mathbb{Z} \geq 0} M_{\lambda+n}$, where $M_{\lambda} \neq 0$. The number $\lambda$ is called the conformal weight of M;

2. There are only finitely many irreducible admissible g-twisted V-modules up to isomorphism.

Definition 2.8. A vertex operator algebra V is called regular if every weak V-module is a direct sum of irreducible V-modules, i.e., the weak module category is semisimple.

Definition 2.9. A vertex operator algebra $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is said to be of CFT type if $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}1$.

Theorem 2.10 ([3, 30]). If V is a regular and selfdual vertex operator algebra of CFT type, and G is a solvable automorphism group of V, then $V^G$ is a regular and selfdual vertex operator algebra of CFT type.
Classification and construction of irreducible modules of $L_{\mathfrak{sl}_2}(k,0)^{Z_p}$

In this section, for a prime $p$, we will introduce the cyclic automorphism group $Z_p$ of $L_{\mathfrak{sl}_2}(k,0)$, and realize each element of $Z_p$ as an inner automorphism of $\mathfrak{sl}_2(\mathbb{C})$. We will classify and construct explicitly the irreducible modules of the orbifold vertex operator algebras $L_{\mathfrak{sl}_2}(k,0)^{Z_p}$ for $k \geq 1$. In the following statement, we denote $L_{\mathfrak{sl}_2}(k,0)$ by $L(k,0)$ for simplicity, $k$ is a fixed positive integer and $p$ is a fixed prime unless stated otherwise.

We first review some notations and facts about the action of the automorphism group on twisted modules of vertex operator algebra $V$ from [12, 15, 17, 31].

Let $g, h$ be two automorphisms of $V$. If $(M, Y_M)$ is a weak $g$-twisted $V$-module, there is a weak $h^{-1}gh$-twisted $V$-module $(M \circ h, Y_{Mh})$ where $M \circ h = M$ as vector spaces and $Y_{Mh}(v, z) = Y_M(hv, z)$ for $v \in V$. This gives a right action of $\text{Aut}(V)$ on weak twisted $V$-modules. The $V$-module $M$ is called $h$-stable if $M \circ h$ and $M$ are isomorphic $V$-modules.

Let $G$ be a finite group of automorphisms of $V$, $g \in G$ of finite order $T$, and $M = (M, Y_M)$ an irreducible $g$-twisted $V$-module. Define a subgroup $G_M$ of $G$ consisting of all $h \in G$ such that $M$ is $h$-stable. For $h \in G_M$, there is a $g$-twisted $V$-module isomorphism $\phi(h) : M \rightarrow M \circ h$ satisfying

$$\phi(h) Y_M(v, z) \phi(h)^{-1} = Y_{Mh}(v, z) = Y_M(hv, z)$$

for $v \in V$. The simplicity of $M$ together with Schur’s lemma shows that $h \mapsto \phi(h)$ is a projective representation of $G_M$ on $M$. Let $\alpha_M$ be the corresponding 2-cocycle in $C^2(G, C^*)$. Then $M$ is a module for the twisted group algebra $C^G_{\alpha_M}[G_M]$ which is a semisimple associative algebra. A basic fact is that $g$ belongs to $G_M$. In fact, from the definition, $M$ has a decomposition $M = \bigoplus_{h \in \mathbb{Z}_p} M(n)$.

We define $\phi(g)$ acting on $M(n)$ as $e^{2\pi \sqrt{-1}n}$ for all $n$. It is easy to check that

$$\phi(g) Y_M(v, z) \phi(g)^{-1} = Y_M(gv, z)$$

for $v \in V$. This implies $g \in G_M$. For $r = 0, \ldots, T - 1$, let $M^r = \bigoplus_{n \in \mathbb{Z}} M(n)$. Then $M = \bigoplus_{r=0}^{T-1} M^r$ and each $M^r$ is an irreducible $V^{(r)}$-module on which $\phi(g)$ acts as scalar $e^{2\pi \sqrt{-1}r}$ [15].

Let $\Lambda_{G_M, \alpha_M}$ be the set of all irreducible characters $\lambda$ of $C^G_{\alpha_M}[G_M]$. Then

$$M = \bigoplus_{\lambda \in \Lambda_{G_M, \alpha_M}} W_\lambda \otimes M_\lambda,$$

(3.1)

where $W_\lambda$ is the simple $C^G_{\alpha_M}[G_M]$-module affording $\lambda$ and $M_\lambda = \text{Hom}_{C^G_{\alpha_M}[G_M]}(W_\lambda, M)$ is the multiplicity of $W_\lambda$ in $M$. Each $M_\lambda$ is a module for the vertex operator subalgebra $V_{G_M}$.

We now recall the following results from [15, 17], and [31].

**Theorem 3.1** ([15, 17]). *With the same notations as above we have*

1. $W_\lambda \otimes M_\lambda$ is nonzero for any $\lambda \in \Lambda_{G_M, \alpha_M}$.
2. Each $M_\lambda$ is an irreducible $V_{G_M}$-module.
3. $M_\lambda$ and $M_\mu$ are equivalent $V_{G_M}$-module if and only if $\lambda = \mu$.

**Theorem 3.2** ([15, 31]). *Let $g, h \in G$, $M$ be an irreducible $g$-twisted $V$-module, and $N$ an irreducible $h$-twisted $V$-module. If $M, N$ are not in the same orbit under the action of $G$, then the irreducible $V^G$-modules $M_\lambda$ and $N_\mu$ are inequivalent for any $\lambda \in \Lambda_{G_M, \alpha_M}$ and $\mu \in \Lambda_{G_N, \alpha_N}$.*

**Theorem 3.3** ([15]). *Let $V^G$ be a regular and selfdual vertex operator algebra of CFT type. Then any irreducible $V^G$-module is isomorphic to $M_\lambda$ for some irreducible $g$-twisted $V$-module $M$ and some $\lambda \in \Lambda_{G_M, \alpha_M}$. In particular, if $V$ is a regular and selfdual vertex operator algebra of CFT type and $G$ is solvable, then any irreducible $V^G$-module is isomorphic to some $M_\lambda$.***
Let \{h, e, f\} be a standard Chevalley basis of \(\mathfrak{sl}_2(\mathbb{C})\), \(\xi_p = e^{\frac{2\pi i}{p}}\), define automorphism \(\sigma_p\) of \(\mathfrak{sl}_2(\mathbb{C})\) as follows:

\[
\sigma_p(h) = h, \sigma_p(e) = \xi_p e, \sigma_p(f) = \xi_p^{-1} f.
\]

It is obvious that the automorphism subgroup generated by \(\sigma_p\) is isomorphic to the cyclic group \(\mathbb{Z}_p\), and \(\mathbb{Z}_p\) can be lifted to an automorphism subgroup of the vertex operator algebra \(L_{\mathfrak{sl}_2}(k, 0)\).

Let \(\alpha\) be the simple root of \(\mathfrak{sl}_2(\mathbb{C})\) with \(\langle \alpha, \alpha \rangle = 2\). From [20], the integrable highest weight \(L(k, 0)\)-modules \(L(k, i)\) for \(0 \leq i \leq k\) provide a complete list of irreducible \(L(k, 0)\)-modules with the lowest weight spaces being \((i + 1)\)-dimensional irreducible \(\mathfrak{sl}_2(\mathbb{C})\)-modules \(L(\frac{i}{2})\), respectively. For \(0 \leq i \leq k\), let \(v^{i,0}\) be a highest weight vector of the \(\mathfrak{sl}_2(\mathbb{C})\)-module \(L(\frac{i}{2})\) according to the \(\mathfrak{sl}_2\)-triple \(\{h, e, f\}\) with the weight \(i\) and set

\[
v^{i,j} = \frac{1}{j!}f(0)^j v^{i,0}, 0 \leq j \leq i.
\]

Then \(\{v^{i,j} \mid 0 \leq j \leq i\}\) is naturally a basis of \(L(\frac{i}{2})\) with the following relations

\[
\begin{align*}
&h(0)v^{i,j} = (i - 2j)v^{i,j} \quad \text{for} \quad 0 \leq j \leq i, \\
e(0)v^{i,0} = 0, \quad &e(0)v^{i,j} = (i - j + 1)v^{i,j-1} \quad \text{for} \quad 1 \leq j \leq i, \\
f(0)v^{i,0} = 0, \quad &f(0)v^{i,j} = (j + 1)v^{i,j+1} \quad \text{for} \quad 0 \leq j \leq i - 1, \\
&a(n)v^{i,j} = 0 \quad \text{for} \quad a \in \{h, e, f\}, \quad n \geq 1.
\end{align*}
\]

It is well known that \(L(k, 0)\) is a regular and selfdual vertex operator algebra of CFT type for \(k \in \mathbb{Z}_{\geq 1}\) [20, 27]. From Theorem 2.10, \(L(k, 0)^{\mathbb{Z}_p}\) is again a regular and selfdual vertex operator algebra of CFT type. Thus, from Theorem 3.3, any irreducible \(L(k, 0)^{\mathbb{Z}_p}\)-module occurs in an irreducible \(\sigma_p^r\)-twisted \(L(k, 0)\)-module for some \(0 \leq r \leq p - 1\).

To classify and construct all the irreducible \(L(k, 0)^{\mathbb{Z}_p}\)-modules coming from the irreducible untwisted (i.e., \(id\)-twisted) \(L(k, 0)\)-modules \(L(k, i)\) \((0 \leq i \leq k)\), we first determine the subgroup \((\mathbb{Z}_p)^{L(k,i)}\) of \(\mathbb{Z}_p\) which contains \(\tau \in \mathbb{Z}_p\) such that \(L(k, i)\) is \(\tau\)-stable. The proof of the following Lemma is similar to that of Lemma 4.9. of [25].

**Lemma 3.4.** \((\mathbb{Z}_p)^{L(k,i)} = \mathbb{Z}_p\) for any \(0 \leq i \leq k\).

For \(0 \leq i \leq k\), we define linear operators \(\phi(\sigma_p^r) \quad (r = 0, 1, \cdots, p - 1)\) on the top level of \(L(k, i)\) as follows:

\[
\phi(\sigma_p^r) : v^{i,j} \mapsto \sigma_p^r(v^{i,j}). \quad (3.2)
\]

Then \(\phi(\sigma_p^r)\) can be extended to irreducible \(L(k, 0)\)-module isomorphism from \(L(k, i)\) to \(L(k, i) \circ \sigma_p^r\) as follows:

\[
\phi(\sigma_p^r)(v_n v^{i,j}) = (\sigma_p^r(v))_n \phi(\sigma_p^r)(v^{i,j}), \quad v \in L(k, 0), n \in \mathbb{Z}. \quad (3.3)
\]

Using Theorem 3.1. and Theorem 3.2., we have the following result.

**Theorem 3.5.** For any \(0 \leq i \leq k\), we have the following irreducible \(L(k, 0)^{\mathbb{Z}_p}\)-module decomposition

\[
L(k, i) = \bigoplus_{j=0}^{p-1} L(k, i)^j,
\]

where \(L(k, 0)^0(= L(k, 0)^{\mathbb{Z}_p})\) is a simple vertex operator algebra and \(L(k, i)^j\) is the eigenspace for \(\phi(\sigma_p)\) on \(L(k, i)\) with eigenvalue \(e^{\frac{2\pi i}{p}}\).
Proof. The simplicity of $L(k, i)$ shows that $\sigma^r_p \mapsto \phi(\sigma^r_p)$ gives a projective representation of $\mathbb{Z}_p$ on $L(k, i)$. By Lemma 3.4., the $\mathbb{Z}_p$-orbit $L(k, i) \circ \mathbb{Z}_p$ of $L(k, i)$ only contains itself. Let $\mathfrak{z}_{L(k, i)}$ be the corresponding 2-cocycle in $C^2(\mathbb{Z}_p, C^r)$. Then $L(k, i)$ is a module for the twisted group algebra $C^{2Z_k, 0}[\mathbb{Z}_p]$ with relation $\phi(\sigma^r_p)\phi(\sigma^r_p) = \phi(\sigma^{r+1}_p)$. The twisted group algebra $C^{2Z_k, 0}[\mathbb{Z}_p]$ is a commutative semisimple associative algebra which has $p$ irreducible modules of dimension one. From Theorem 3.1., we know that $L(k, i)^j, j = 0, 1, \ldots, p - 1$ are inequivalent irreducible $L(k, 0)^{\mathbb{Z}_p}$-modules for fixed $0 \leq i \leq k$. Therefore, the decomposition of $L(k, i)$ into inequivalent irreducible $L(k, 0)^{\mathbb{Z}_p}$-modules is $L(k, i) = \bigoplus_{j=0}^{p-1} L(k, i)^j$.

Let $h^{(p, r)} = \frac{r}{2p} h, r \in \mathbb{Z}_{\geq 0}$. From [28], we know that $e^{2\pi \sqrt{-1} h^{(p, r)}(0)}$ is an automorphism of $L(k, 0)$. It is straightforward to see that $e^{2\pi \sqrt{-1} h^{(p, r)}(0)} = \sigma^r_p$. For $r \in \mathbb{Z}_{\geq 0}$, let

\[ \Delta(h^{(p, r)}, z) = z^{h^{(p, r)}(0)} \exp \left( \sum_{n=1}^{\infty} \frac{h^{(p, r)}(n)}{n} (-z)^{-n} \right). \]

It is easy to verify that $\Delta(h^{(p, r)}, z) = \Delta(h^{(p, 1)}, z)^r$. From [29], we have the following result.

Lemma 3.6. For each $r \in \mathbb{Z}_{\geq 0}, (L(k, i)^{\mathbb{Z}_p}, Y_{\sigma^r_p}(\cdot, z)) = (L(k, i), Y(\Delta(h^{(p, r)}, z) \cdot, z))(0 \leq i \leq k)$ provide a complete list of irreducible $\sigma^r_p$-twisted $L(k, 0)$-modules. In particular, $(L(k, i)^{\mathbb{Z}_p}, Y_{\sigma^r_p}(\cdot, z)) = (L(k, i), Y(\cdot, z))(0 \leq i \leq k)$ are all the irreducible untwisted $L(k, 0)$-modules.

Direct calculations yield that

\[ h^{(p, r)}(0) \omega = 0, \quad h^{(p, r)}(1) \omega = h^{(p, r)}, \quad h^{(p, r)}(1)^2 \omega = \frac{r^2 k}{2p^2} \mathbb{1}, \]

\[ h^{(p, r)}(n) \omega = 0 \text{ for } n \in \mathbb{Z}_{>1}, \]

\[ \Delta(h^{(p, r)}, z) \omega = \omega + z^{-1} h^{(p, r)} + z^{-2} \frac{r^2 k}{4p^2} \mathbb{1}, \]

\[ Y_{\sigma^r_p}(h^{(p, r)}, z) = Y \left( h^{(p, r)} + \frac{r^2 k}{2p^2} z^{-1}, z \right), \]

\[ Y_{\sigma^r_p}(h, z) = Y \left( h + \frac{r k}{p} z^{-1}, z \right), \]

\[ Y_{\sigma^r_p}(e, z) = z^r Y(e, z), \]

\[ Y_{\sigma^r_p}(f, z) = z^{-r} Y(f, z). \]

To distinguish the components of $Y(v, z)$ from those of $Y_{\sigma^r_p}(v, z)$, for fixed $r$, we denote the following expansions

\[ Y_{\sigma^r_p}(v, z) = \sum_{n \in \mathbb{Z}^{\mathbb{Z}_p}} v_n z^{-n-1}, \quad Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}, \]

where $v \in L(k, 0), t \in \{0, 1, \ldots, p - 1\}$ such that $\sigma^r_p(v) = e^{2\pi \sqrt{-1} r t} v$. Let $L_n^{(p, r)}$ be the component operator of $Y_{\sigma^r_p}(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^{(p, r)} z^{-n-2}$. Note that $L_n^{(p, 0)} = L(n)$. By (3.6)–(3.10) and direct calculations, we have the following lemmas.
Lemma 3.7. \( L_0^{(p, r)} v_i^r = a_{i, k, i}^{p, r} v_i^r \), where \( a_{i, k, i}^{p, r} = \frac{i(i+2)}{4(k+2)} + \frac{r^2k-2rp_i}{4p^2} \) is the eigenvalue of the operator \( L_0^{(p, r)} \) on \( v_i^r \). Thus, for \( r = 0, 1, ..., p - 1 \), \( a_{i, k, i}^{p, r} \) is the conformal weight of the irreducible \( \sigma_p^r \)-twisted \( L(k, 0) \)-module \( L(k, i)^T_r \).

Lemma 3.8. For \( 0 \leq r \leq p - 1 \), \( 0 \leq i \leq k \), we have

1. \( a_{i, k, i}^{p, r} = a_{i, k, k-i}^{p, r} \),
2. \( a_{i, k, i}^{p, r} + \frac{r}{p} = a_{i, k, k-i}^{p, r} \).

For \( 0 \leq r \leq p - 1 \), \( 0 \leq i \leq k \), we can write \( L(k, i)^T_r = \bigoplus_{n \in \frac{1}{p} \mathbb{Z}} L(k, i)^T_r(n) \) as an admissible \( \sigma_p^r \)-twisted \( L(k, 0) \)-module, where

\[
L(k, i)^T_r(n) = \{ v \in L(k, i)^T_r | L_0^{(p, r)} v = (a_{i, k, i}^{p, r} + n)v \}.
\]

In particular, \( L(k, i)^T_r(0) = L(\frac{ip}{2}) \) is the top level of the irreducible untwisted \( L(k, 0) \)-module \( L(k, i) \). Let

\[
L(k, i)^T_r = \bigoplus_{n \in \frac{1}{p} \mathbb{Z}} L(k, i)^T_r(n),
\]

then \( L(k, i)^T_r \) is an irreducible \( L(k, 0)^\mathbb{Z}_p \)-module for \( j = 0, 1, ..., p - 1 \) [13]. Actually, from the definition of \( \phi(\sigma_p^r) \), we can see that \( L(k, i)^T_r \) is the eigenspace for \( \phi(\sigma_p^r) \) on \( L(k, i)^T_r \) with eigenvalue \( e^{\frac{2\pi i r}{p}} \). In particular, if \( p - 1 \leq i \leq k \), then

\[
L_0^{(p, r)} v_i^{r-j} = \left( a_{i, k, i}^{p, r} + \frac{r}{p} \right) v_i^{r-j}, \text{ for } 0 \leq r \leq p - 1, 0 \leq j \leq i,
\]

that is \( v_i^{r-j} \in L(k, i)^T_r \) and \( v_i^{r-j} \in L(k, i)^T_r \) for \( 1 \leq r \leq p - 1 \).

Remark 3.9. For the cases \( p = 2, 3 \), the lowest weight vectors of the irreducible \( L(k, 0)^\mathbb{Z}_r \)-modules \( L(k, i)^T_r(0 \leq i \leq k, j = 0, 1, ..., p - 1) \) with their lowest weights have been listed in [24] and [33], respectively.

Now we are poised to give the classification of the irreducible \( L(k, 0)^\mathbb{Z}_p \)-modules coming from \( \sigma_p^r \)-twisted \( L(k, 0) \)-modules \( L(k, i)^T_r(0 \leq i \leq k, 0 \leq r \leq p - 1) \).

Theorem 3.10. For any \( 0 \leq i \leq k, 0 \leq r \leq p - 1 \), we have the following inequivalent irreducible \( L(k, 0)^\mathbb{Z}_p \)-module decomposition:

\[
L(k, i)^T_r = \bigoplus_{j=0}^{p-1} L(k, i)^T_r(j).
\]

Proof. For any \( 0 \leq r \leq p - 1 \), a basic fact is that \( \sigma_p^r \) belongs to \( (\mathbb{Z}_p)^{L(k, i)^T_r} \), thus \( (\mathbb{Z}_p)^{L(k, i)^T_r} = \mathbb{Z}_p \) for any \( 0 \leq i \leq k \). Then the theorem follows from (3.1) and Theorem 3.1. \( \square \)

We are now in a position to state the main result of this section.

Theorem 3.11. There are exactly \( p^2(k + 1) \) irreducible \( L(k, 0)^\mathbb{Z}_p \)-modules up to isomorphism as follows:

\[
L(k, i)^T_r, L(k, i)^T_r(j), \quad 0 \leq i \leq k, \quad 1 \leq r \leq p - 1, \quad 0 \leq j \leq p - 1.
\]
Proof. It follows from the Theorem 3.3. that all the irreducible modules of the orbifold vertex operator algebra \( L(k,0)^{Z_p} \) come from \( \{ L(k,i), L(k,i)^{T_r} \mid 1 \leq r \leq p - 1, 0 \leq i \leq k \} \). Then the theorem follows from Theorem 3.2., Theorem 3.5., and Theorem 3.10.

Remark 3.12. For \( k = 1 \), the orbifold vertex operator algebra \( L(1,0)^{Z_p} \) was realized as the lattice vertex operator algebra \( V_{Z\beta} \) associated to the positive definite even lattice \( Z\beta = Zp\alpha \) with \( (\beta, \beta) = 2p^2 \) [5]. Moreover, it is well known that there are \( 2p^2 \) inequivalent irreducible \( V_{Z\beta} \)-modules: \( \{ V_{Z\beta + s\beta} \mid 0 \leq s < 2p^2 \} \) [2, 19]. Therefore, from [5] together with the Proposition 2.15. in [9], we have the following \( L(1,0)^{Z_p} \)-module isomorphisms:

\[
L(1,i) \cong \bigoplus_{s=0}^{p-1} V_{Zs + \frac{2s+1}{2p}\beta}, \quad \text{for } i = 0, 1,
\]

\[
L(1,i)^{T_r} \cong \bigoplus_{s=0}^{p-1} V_{Zs + \frac{(2s+1)p + r}{2p}\beta}, \quad \text{for } 1 \leq r \leq p - 1, i = 0, 1.
\]

\[
L(1,0)^i \cong V_{Zs + \frac{i}{p}\beta}, \quad \text{for } 1 \leq j \leq p - 1,
\]

\[
L(1,1)^j \cong V_{Zs + \frac{j-1}{p}\beta + \frac{1}{2p}\beta}, \quad \text{for } 1 \leq j \leq p - 1,
\]

\[
L(1,0)^{T_r,\overline{\eta}} \cong V_{Zs + \frac{j-1}{p}\beta + \frac{1}{2p}\beta}, \quad \text{for } 1 \leq r, j \leq p - 1,
\]

\[
L(1,1)^{T_r,\overline{\eta}} \cong V_{Zs + \frac{j-1}{p}\beta + \frac{1}{2p}\beta}, \quad \text{for } 1 \leq r, j \leq p - 1.
\]

Quantum dimensions and fusion rules for the orbifold vertex operator algebra \( L(k,0)^{Z_p} \)

In this section, we first recall from [15] some results on the quantum dimensions of irreducible \( g \)-twisted \( V \)-modules and irreducible \( V^G \)-modules for \( G \) being a finite automorphism group of the vertex operator algebra \( V \). Then we compute the quantum dimensions for irreducible modules of the orbifold vertex operator algebra \( L(k,0)^{Z_p} \). Finally, we determine the fusion rules for the orbifold vertex operator algebras \( L(k,0)^{Z_p} \).

Let \( V \) be a vertex operator algebra, \( g \) an automorphism of \( V \) with order \( T \) and \( M = \bigoplus_{n \in \mathbb{Z} \geq s} M_{k+n} \) a \( g \)-twisted \( V \)-module. For any homogeneous element \( v \in V \), we define a trace function associated to \( v \) as follows:

\[
Z_M(v,q) = tr_M o(v) q^{L(0)-\frac{c}{2}} = q^{\frac{c}{2}} \sum_{n \in \mathbb{Z} \geq 0} tr_{M_{k+n}} o(v) q^n
\]  \hspace{1cm} (4.1)

where \( o(v) = v_{wT-1} \) is the degree zero operator of \( v \), \( c \) is the central charge of the vertex operator algebra \( V \), and \( \lambda \) is the conformal weight of \( M \). This is a formal power series in variable \( q \). It is proved in [11, 36] that \( Z_M(v,q) \) converges to a holomorphic function, denoted by \( Z_M(v,\tau) \), in the domain \( |q| < 1 \) if \( V \) is \( C_\tau \)-cofinite. Here and below, \( \tau \) is in the upper half plane \( \mathbb{H} \) and \( q = e^{2\pi \sqrt{-1}\tau} \). Note that if \( v = 1 \) is the vacuum vector, then \( Z_M(1,q) \) is the formal character of \( M \). We simply denote \( Z_M(1,q) \) and \( Z_M(1,\tau) \) by \( \chi_M(q) \) and \( \chi_M(\tau) \), respectively. \( \chi_M(q) \) is called the character of \( M \).
Let $V$ be a regular and selfdual vertex operator algebra of CFT type and $G$ a finite automorphism group of $V$. Let $g \in G$ and $M$ a $g$-twisted $V$-module. Then $M$ is a finite sum of irreducible $g$-twisted $V$-modules. In particular, each homogeneous subspace of $M$ is finite dimensional. From the above discussion, we know that $\chi_V(\tau)$ and $\zeta_M(\tau)$ are holomorphic functions on $\mathbb{H}$. In [6], the quantum dimension of $M$ over $V$ is defined to be

$$q\text{dim}_V M = \lim_{y \to 0^+} \frac{\chi_M(\sqrt{-1}y)}{\chi_V(\sqrt{-1}y)} = \lim_{q \to 1^+} \frac{\zeta_M(q)}{\zeta_V(q)}$$

where $y$ is real and positive, $q = e^{2\pi \sqrt{-1}\tau}$, $\tau = \sqrt{-1}y$. From [6], we know that for any $V$-module $M$, $q\text{dim}_V M$ always exists and is greater than or equal to 1 if the conformal weight of each irreducible $V$-module is positive except $V$ itself. It was proved in [15] that for any $g \in G$ and any $g$-twisted $V$-module $M$, $q\text{dim}_V M$ always exists and is non-negative. Also, $q\text{dim}_V (M \circ h) = q\text{dim}_V M$ for any $h \in G$ and $q\text{dim}_V M = q\text{dim}_V M'$.

We now recall from [18] the notions of intertwining operators and fusion rules.

**Definition 4.1.** Let $(V, Y)$ be a vertex operator algebra and let $(W^1, Y^1), (W^2, Y^2)$, and $(W^3, Y^3)$ be $V$-modules. An intertwining operator of type $$\begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$$ is a linear map

$$I(\cdot, z) : W^1 \to \text{Hom}(W^2, W^3)(z)$$

$$u \mapsto I(u, z) = \sum_{n \in \mathbb{C}} u_n z^{-n-1}$$

satisfying:

1. for any $u \in W^1$ and $v \in W^2$, $u_n v = 0$ for $n$ sufficiently large;
2. $I(L(-1)v, z) = \frac{d}{dz} I(v, z)$;
3. (Jacobi identity) for any $u \in V, v \in W^1$,

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^3(u, z_1)I(v, z_2) - z_0^{-1} \delta \left( \frac{-z_2 + z_1}{z_0} \right) I(v, z_2) Y^2(u, z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y^1(u, z_0)v, z_2).$$

The space of all intertwining operators of type $$\begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix}$$ is denoted by $I_V \left( \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix} \right)$. Let $N_{W^1, W^2}^{W^3} = \text{dim} I_V \left( \begin{pmatrix} W^3 \\ W^1 W^2 \end{pmatrix} \right)$. These integers $N_{W^1, W^2}^{W^3}$ are usually called the fusion rules.

**Definition 4.2.** Let $V$ be a vertex operator algebra, and $W^1, W^2$ be two $V$-modules. A pair $(W, F(\cdot, z))$, which consists of a $V$-module $W$ and an intertwining operator $F(\cdot, z)$ of type $$\begin{pmatrix} W \\ W^1 W^2 \end{pmatrix},$$ is called a tensor product (or fusion product) of the ordered pair $W^1$ and $W^2$ if for any $V$-module $M$ and any intertwining operator $I(\cdot, z)$ of type $$\begin{pmatrix} M \\ W^1 W^2 \end{pmatrix},$$ there exists a unique $V$-module homomorphism $f$ from $W$ to $M$ such that $I(\cdot, z) = f \circ F(\cdot, z)$. We use $W^1 \boxtimes_V W^2$ for a tensor product module of $W^1$ with $W^2$.

The following result is obtained in [21–23].
Theorem 4.3. Let $V$ be a regular and selfdual vertex operator algebra of CFT type. Assume that $M^0 \cong V, M^1, \ldots, M^d$ are all inequivalent irreducible $V$-modules and the conformal weights $\lambda_i$ of $M^i$ are positive for all $i > 0$. Then the tensor product of any two $V$-modules $M \otimes V N$ exists. In particular,

$$M^i \otimes_V M^j = \sum_{k=0}^{d} N_{M^i, M^j}^{M^k},$$

for any $i, j \in \{0, 1, \ldots, d\}$.

Fusion rules have the following symmetric property [18].

Proposition 4.4. Let $W^i(i = 1, 2, 3)$ be $V$-modules. Then

$$N_{W^1, W^2}^{W^3} = N_{W^2, W^1}^{W^3}, N_{W^1, W^2}^{W^3} = N_{W^1, (W^2)}^{(W^3)}.$$

Definition 4.5. Let $V$ be a simple vertex operator algebra. A simple $V$-module $M$ is called a simple current if for any irreducible $V$-module $W$, $M \otimes V W$ exists, and is also an irreducible $V$-module.

Lemma 4.6. ([6]) Let $V$ be a regular and selfdual vertex operator algebra of CFT type. Assume that $M^0 \cong V, M^1, \ldots, M^d$ are all inequivalent irreducible $V$-modules. We also assume that the conformal weights $\lambda_i$ of $M^i$ are positive for all $i > 0$. Then

$$qdim_V(M^i \otimes_V M^j) = qdim_V M^i \cdot qdim_V M^j$$

for $0 \leq i, j \leq d$.

Lemma 4.7. ([15]). Let $V$ be a regular and selfdual vertex operator algebra of CFT type, $G$ a finite automorphism group of $V$, $g \in G$ and $M$ a $g$-twisted $V$-module, $\lambda \in \Lambda_{G_M, \varpi_M}$. If the conformal weight of any irreducible $g$-twisted $V$-module is positive except $V$ itself, then

$$qdim_{V \otimes G} M = |G| \cdot qdim_V M,$$

$$qdim_{V \otimes G} M_{\lambda} = [G: G_M] \cdot dim W_{\lambda} \cdot qdim_V M.$$

Moreover, $qdim_V M$ takes values in $\{2 \cos \frac{n \pi}{n} | n \geq 3\} \cup \{2, \infty\}$.

Using Theorem 4.5. of [1] and Theorem 3.15 of [16], we have the quantum dimensions of irreducible $L(k, 0)$-modules

$$qdim_{L(k, 0)} L(k, i) = \frac{\sin \frac{\pi (i+1)}{k+2}}{\sin \frac{\pi}{k+2}},$$

for $0 \leq i \leq k$. As a consequence, we have the following result from [5].

Lemma 4.8. For $0 \leq r \leq p - 1$, the quantum dimensions of $\sigma_p^r$-twisted $L(k, 0)$-modules are

$$qdim_{L(k, 0)} L(k, i)^{\sigma_p^r} = \frac{\sin \frac{\pi (i+1)}{k+2}}{\sin \frac{\pi}{k+2}}, 0 \leq i \leq k.$$

Note that $L(k, 0)$ satisfy all the conditions in Lemma 4.7., $(\mathbb{Z}_p)_{L(k, i)} = \mathbb{Z}_p$ and $(\mathbb{Z}_p)_{L(k, i)^{\sigma_p^r}} = \mathbb{Z}_p$ for $0 \leq i \leq k, r = 1, 2$. Using Lemmas 4.7. and 4.8., we can compute the quantum dimensions of $L(k, 0)^{\mathbb{Z}_p}$-modules:
Lemma 4.10. The quantum dimensions of irreducible \( L(k,0) \) modules are
\[
qdim_{L(k,0)^{\mathbb{Z}_{p}}\gamma} L(k,i) = qdim_{L(k,0)^{\mathbb{Z}_{p}}\gamma} L(k,i)^{T_{r}} = p \frac{\sin \frac{n(i+1)}{k+2}}{\sin \frac{\pi}{k+2}},
\]
for \( 0 \leq i \leq k, 1 \leq r \leq p - 1 \). Therefore, we can easily obtain the quantum dimensions of irreducible \( L(k,0)^{\mathbb{Z}_{p}} \)-modules.

Theorem 4.9. The quantum dimensions of irreducible \( L(k,0)^{\mathbb{Z}_{p}} \)-modules are
\[
qdim_{L(k,0)^{\mathbb{Z}_{p}}\gamma} L(k,i)^{j} = qdim_{L(k,0)^{\mathbb{Z}_{p}}\gamma} L(k,i)^{T_{r},j} = \frac{\sin \frac{n(i+1)}{k+2}}{\sin \frac{\pi}{k+2}},
\]
for \( 0 \leq i \leq k, 1 \leq r \leq p - 1, 0 \leq j \leq p - 1 \).

By observation, we have \( qdim_{L(1,0)^{\mathbb{Z}_{p}}\gamma} M = 1 \) for any irreducible \( L(1,0)^{\mathbb{Z}_{p}} \)-module \( M \). As a consequence, all the irreducible \( L(1,0)^{\mathbb{Z}_{p}} \)-modules are simple currents [6].

Let \( V \) be a vertex operator algebra with only finitely many irreducible modules. The global dimension of \( V \) is defined as \( \text{glob}(V) = \sum_{M \in \text{irr}(V)} qdim(M)^{2} \) [6]. Assume \( G \) is a finite subgroup of \( \text{Aut}(G) \), it is proved that \( |G|^{2} \text{glob}(V) = \text{glob}(V_{G}) \) [1, 15]. One immediately gets that
\[
\text{glob}(L(k,0)^{\mathbb{Z}_{p}}) = p^{2} \sum_{i=0}^{k} \left( \frac{\sin \frac{n(i+1)}{k+2}}{\sin \frac{\pi}{k+2}} \right)^{2}.
\]

Now we recall from [32], the fusion rules for the simple affine vertex operator algebra \( L(k,0) \).

Lemma 4.10.
\[
L(k, i) \boxtimes_{L(k,0)} L(k, j) = \sum_{\ |i-j| \leq i+j \leq 2k \atop i+j+1 \leq 2k} L(k, l).
\]

Lemma 4.11 ([9]). For \( 0 \leq i, j, l \leq k, i + j + l \leq 2k \), let \( \mathcal{U}(\cdot, z) \) be an intertwining operator of type \( \left( \begin{array}{c}
L(k, l) \\
L(k, i) \end{array} \right) \left( \begin{array}{c}
L(k, j) \\
L(k, i) \end{array} \right) \). Define \( \mathcal{U}_{\gamma}(\cdot, z) = \mathcal{U}(\h_{[p-r]}, z) \cdot z \). Then \( \mathcal{U}_{\gamma}(\cdot, z) \) is an intertwining operator of type \( \left( \begin{array}{c}
L(k, l)^{T_{r}} \\
L(k, i)^{T_{r}} \end{array} \right) \).

To determine the contragredient modules of irreducible \( L(k,0)^{\mathbb{Z}_{p}} \)-modules, we recall from [7] that the irreducible \( L(k,0) \)-modules \( L(k, i)(0 \leq i \leq k) \) can be realized in the module \( V_{L^{\perp}} \) of the lattice vertex operator algebra \( V_{L} \), where \( L = \mathbb{Z}a_{1} + \cdots + \mathbb{Z}a_{k} \) with \( \langle a_{i}, a_{j} \rangle = 2\delta_{i,j} \), and \( L^{\perp} \) is the dual lattice of \( L \). Let
\[
\gamma = a_{1} + a_{2} + \cdots + a_{k}, \quad H = \gamma(-1) \mathbb{1},
\]
\[
E = e^{a_{1}} + \cdots + e^{a_{k}}, \quad F = e^{-a_{1}} + \cdots + e^{-a_{k}},
\]
then the component operators \( H_{n}, E_{n}, F_{n} \) \( (n \in \mathbb{Z}) \) of the vertex operators \( Y(u, z), Y(E, z), Y(F, z) \) give a representation of \( \mathfrak{sl}_{2} \) with level \( k \) under the correspondence
\[
H_{n} \leftrightarrow h(n), \quad E_{n} \leftrightarrow e(n), \quad F_{n} \leftrightarrow f(n).
\]
Furthermore, the top level of \( L(k, i) \) is an \( i + 1 \) dimensional vector space which is spanned by \( \{ v^{i,j} \ | \ 0 \leq j \leq i \} \) and \( v^{i,j} \) has the explicit form in \( V_{L^{\perp}} \).
\begin{equation}
\nu^{0,0} = 1, \nu^{i,0} = \sum_{I \subseteq \{1, 2, \ldots, k\}} e^{\frac{z}{I}} \quad \nu^{i,i} = \sum_{|I| = i} e^{\frac{z}{I}}, \quad (4.13)
\end{equation}

\begin{equation}
\nu^{i,j} = \sum_{I \subseteq \{1, 2, \ldots, k\}} \sum_{|J| = j} e^{\frac{z_{I,j}}{z}} \quad (4.14)
\end{equation}

where \(x_I = \sum_{r \in I} x_r\) for a subset \(I\) of \(\{1, 2, \ldots, k\}\), and the vertex operator associated with \(e^z, x \in L^+\) is defined on \(V_L\) by

\[
\mathcal{Y}(e^z, z) = \exp \left( \sum_{n=1}^{\infty} \frac{z(-n)}{n} z^n \right) \exp \left( \sum_{n=1}^{\infty} \frac{z(n)}{-n} z^{-n} \right) e^z z^{(0)}. \quad (4.15)
\]

Moreover, the operator \(\mathcal{Y}\) by restriction gives rise to an intertwining operator for \(V_L\) of type

\[
\left(\begin{array}{c}
V_{\lambda_1 + \lambda_2 + L} \\
V_{\lambda_1 + L V_{\lambda_2 + L}}
\end{array}\right)
\]

for \(\lambda_1, \lambda_2 \in L^+\).

**Theorem 4.12.** For \(0 \leq i \leq k, 0 \leq r, j \leq p - 1, k \in \mathbb{Z} \geq 1\). \(\hspace{1cm}\)

1. \((L(k,i))^{(i)} \cong L(k,0)^{Z_p}\) as irreducible \(L(k,0)^{Z_p}\)-modules.
2. \((L(k,i))^{T_{p^{-1}}(i)} \cong L(k,0)^{Z_p}\) as irreducible \(L(k,0)^{Z_p}\)-modules.

**Proof.** The contragredient module of \(L(1, i)\) and \(L(1, i)^{T_{p^{-1}}(i)}\) \((i = 0, 1, 0 \leq r, j \leq p - 1)\) can be easily determined by using the lattice vertex operator algebra \(V_{Z_p}\) in Remark 3.12. Indeed, the contragredient module of \(V_{Z_p}^2\) is \(V_{Z_p}^{Z_p}\) for any \(0 \leq s < 2p^2\).

A basic fact is that if \(V\) is a selfdual vertex operator algebra, \((M, Y_M)\) is a \(V\)-module and \((M', Y_{M'})\) is the contragredient module of \(M\), then \(V \subset M \otimes V M'\). From Theorem 2.10, we know that \(L(k,0)^{Z_p}\) is a selfdual vertex operator algebra. Note that \(v^{i,i} \in L(k,i)^0\). By induction, for any \(0 \leq i \leq k, 0 \leq j \leq p - 1\), there exists a nonzero vector \(v := e(-n_1) e(-n_2) \cdots e(-n_s) v^{i,i} \) \((n_1 \geq n_2 \geq \cdots \geq n_s, s \geq 0)\) in \(L(k,i)^{\lambda}\). Then, \(s = j\) and

\[
e^{0} v = \frac{1}{e} e(-n_1) e(-n_2) \cdots e(-n_s) v^{i,0} e^{(j)}, \quad u : = f(-n_1) f(-n_2) \cdots f(-n_s) v^{i,i} \in L(k,i)^{\lambda}.
\]

Since

\[
\nu^{0,0} = 1 \in L(k,0)^0 = L(k,0)^{Z_p} \subseteq L(k,i)^{\lambda} \otimes_{L(k,0)^{Z_p}} (L(k,i)^{\lambda})',
\]

by using (4.12)-(4.15), we can deduce that 1 can be obtained from \(\mathcal{Y}(e(0)^{j} v, z) u\), where \(\mathcal{Y}\) is the nonzero intertwining operator for \(V_L\) of type \(\left(\begin{array}{c}
V_{\lambda_1 + \lambda_2 + L} \\
V_{\lambda_1 + L V_{\lambda_2 + L}}
\end{array}\right)\) for \(\lambda_1, \lambda_2 \in L^+\). This implies that \((L(k,i)^{\lambda})' \cong L(k,i)^{-\lambda}. \) This proves the assertion 1.

Next, we prove the assertion 2. From the definition of a contragredient module, we know that any \(g\)-twisted \(V\)-module \(M\) and its contragredient module \(M' (g^{-1}\text{-twisted } V\text{-module})\) have the same lowest weight. Note that \(a_{k,i}^r = a_{k,i}^{p-r}\), where \(a_{k,i}^r (0 \leq r \leq p - 1)\) is the conformal weight of \(L(k,i)^{T_{p^{-1}}(i)}\) defined in Lemma 3.7. Therefore, \((L(k,i)^{T_{p^{-1}}(i)})' \cong L(k,k - i)^{T_{p^{-1}}(i)}\) holds for any \(0 \leq i \leq k, 0 \leq j \leq p - 1\). \(\square\)

**Lemma 4.13.** For \(0 \leq i \leq k, 0 \leq r_1, r_2 \leq p - 1\), we have the following \(L(k,0)\)-isomorphisms.
(1) \((L(k, i)^{T_{n_1}}, Y_{\phi^p}(\Delta(h^{(p, r_2)}, z) \cdot, z)) \cong (L(k, i)^{T_{n_1}+r_2}, Y_{\phi^{p+r_2}}(\cdot, z))\) if \(0 \leq r_1 + r_2 < p\);
(2) \((L(k, i)^{T_{n_1}}, Y_{\phi^p}(\Delta(h^{(p, r_2)}, z) \cdot, z)) \cong (L(k, k - i), Y(\cdot, z))\) if \(r_1 + r_2 = p\);
(3) \((L(k, i)^{T_{n_1}}, Y_{\phi^p}(\Delta(h^{(p, r_2)}, z) \cdot, z)) \cong (L(k, k - i)^{T_{n_1}+r_2}, Y_{\phi^{p+r_2}}(\cdot, z))\) if \(r_1 + r_2 > p\).

Proof. Note that \(\Delta(h^{(p, r)}, z) = \Delta(h^{(p, 1)}, z)^r\) for \(r \in \mathbb{Z}_{\geq 0}\), and therefore the assertion 1 is obvious. From Lemma 2.6 in [9], we know that \((L(k, i), Y(\Delta(h^{(p, o)}, z) \cdot, z))\) is an irreducible \(L(k, 0)\)-module with the eigenvalue \(a_{k, i}^{p, 0}\) of the operator \(L_0^{(p, o)}\) on \(v^{i, i}\) defined in Lemma 3.7. Since \(a_{k, i}^{p, 0} = a_{k, j}^{p, 0}\) if and only if \(j = k - i\), we can deduce that the assertion 2 holds. Indeed, we can define the map \(\psi_{p, 1}\) as follows:

\[
\psi_{p, 1}: (L(k, i), Y(\Delta(h^{(p, o)}, z) \cdot, z)) \rightarrow (L(k, k - i), Y(\cdot, z)) \quad (4.16)
\]

\[
v^{i, i} \mapsto v^{k-i, 0}. \quad (4.17)
\]

It is easy to verify that \(\psi_{p, 1}\) extended by (3.3) is an \(L(k, 0)\)-isomorphism.

Finally, the assertion 3 is immediate by using the assertion 2:

\[
(L(k, i)^{T_{n_1}}, Y_{\phi^p}(\Delta(h^{(p, r_2)}, z) \cdot, z)) \cong (L(k, i), Y(\Delta(h^{(p, r_1+r_2)}, z) \cdot, z))
\]

\[
\cong (L(k, i), Y(\Delta(h^{(p, r_2)}, z)) \Delta(h^{(p, r_1+r_2-p)}), z) \cdot, z))
\]

\[
\cong (L(k, k - i), Y(\Delta(h^{(p, r_1+r_2-p)}, z) \cdot, z))
\]

\[
\cong (L(k, k - i)^{T_{n_1+r_2}}, Y_{\phi^{p+r_2}}(\cdot, z)).
\]

Moreover, for \(p < r_1 + r_2 < 2p - 1\), we can also construct a \(\sigma_p^{r_1+r_2-p}\)-twisted \(L(k, 0)\)-isomorphism

\[
\psi_{p, 2}: (L(k, i), Y(\Delta(h^{(p, r_1+r_2)}, z) \cdot, z)) \rightarrow (L(k, k - i)^{T_{n_1+r_2}}, Y_{\phi^{p+r_2}})(\cdot, z)) \quad (4.18)
\]

\[
v^{i, i} \mapsto v^{k-i, 0}. \quad (4.19)
\]

which is extended by (3.3).

Lemma 4.14. For \(0 \leq i \leq k, 0 \leq r, r_1, r_2, j \leq p - 1\), we have the following \(L(k, 0)^{T_{n_1}}\)-isomorphisms.

1. \((L(k, i)^{T_{n_1}}, Y(\Delta(h^{(p, r)}, z) \cdot, z)) \cong (L(k, i)^{T_{n_1}}, Y_{\phi^{p+r_2}}(\cdot, z))\);
2. \((L(k, i)^{T_{n_1}}, Y_{\phi^{p+r_2}}(\cdot, z)) \cong (L(k, i)^{T_{n_1}+r_2}, Y_{\phi^{p+r_2}}(\cdot, z))\) if \(0 \leq r_1 + r_2 < p\);
3. \((L(k, i)^{T_{n_1}}, Y_{\phi^{p+r_2}}(\cdot, z)) \cong (L(k, k - i)^{T_{n_1+r_2}}, Y_{\phi^{p+r_2}}(\cdot, z))\) if \(r_1 + r_2 = p\);
4. \((L(k, i)^{T_{n_1}}, Y_{\phi^{p+r_2}}(\cdot, z)) \cong (L(k, k - i)^{T_{n_1+r_2-p}}, Y_{\phi^{p+r_2}}(\cdot, z))\) if \(r_1 + r_2 > p\).

Proof. We first show the assertion 1. Since \((L(k, i), L(k, i)^{T_{n_1}})\) as vector spaces and \(Y(\Delta(h^{(p, r)}, z) \cdot, z) = Y_{\phi^{p+r_2}}(\cdot, z)\) as vertex operators on \((L(k, i), i)\), it follows from the nonzero vector \(e(-n_1)e(-n_2)\) if \(n_1 \geq n_2 \geq \cdots \geq n_s \geq 0, s = j\) belongs to \((L(k, i)^{T_{n_1}}, L(k, i)^{T_{n_1}})\) that \((L(k, i)^{T_{n_1}}, L(k, i)^{T_{n_1}})\) as vector spaces. Then we obtain the assertion 1.

The assertion 2 can be proved by using similar arguments. Just note that the nonzero vector \(e(-n_1)e(-n_2)\) if \(n_1 \geq n_2 \geq \cdots \geq n_s \geq 0, s = j\) belongs to \((L(k, i)^{T_{n_1}}, L(k, i)^{T_{n_1}})\) if \(r_1 + r_2 = p\).

Next, we show the assertion 3. Recall the \(L(k, 0)^{T_{n_1}}\)-isomorphism \(\psi_{p, 1}\) defined in (4.16) and the fact that \(a_{k, i}^{p, 0} = a_{k, j}^{p, 0}\) if \(k - i \in p\mathbb{Z} + r\). Then, using the assertion 1 and
Lemma 4.13, we can deduce that, if $r_1 + r_2 = p$ and $k - i \in p\mathbb{Z} + r$,
\[
(L(k,i)^{i_1}, r_j) Y_{n_0}(\Delta(h^{(p,r_2)}, z) \cdot z)) \cong (L(k,i)^{i_1}, Y(\Delta(h^{(p,r_1)}, z) \cdot z)).
\]
\[
\cong (L(k,k - i)^{i_1}, Y(\cdot, z)).
\]
This proves the assertion 3.
Finally, the assertion 4 follows from the assertions 1, 3, and Lemma 4.13. \(\square\)

For $j_1, j_2 \in \mathbb{Z}$, $0 \leq i_1, i_2, i_3 \leq k$, such that $i_1 + i_2 + i_3 \in 2\mathbb{Z}$, $i_1 + i_2 + i_3 \leq 2k$, we define
\[
\text{sgn}(i_1, i_2, i_3, j_1, j_2) := j_1 + j_2 - \frac{1}{2}(i_1 + i_2 - i_3).
\] (4.20)

Now we are in a position to determine the fusion rules for all the irreducible $L(k,0)^{\mathbb{Z}_p}$-modules. For the irreducible $L(k,0)^{\mathbb{Z}_p}$-modules $W$ and $W'$, we drop the subscript $L(k,0)^{\mathbb{Z}_p}$ in the fusion product $W \boxtimes L(k,0)^{\mathbb{Z}_p} W'$ and simply denote $W \boxtimes W'$ without causing confusion. The following theorem together with Proposition 4.4. and Theorem 4.12. give all the fusion rules for the $\mathbb{Z}_p$-orbifold vertex operator algebra $L(k,0)^{\mathbb{Z}_p}$.

**Theorem 4.15.** The fusion rules for the $\mathbb{Z}_p$-orbifold affine vertex operator algebra $L(k,0)^{\mathbb{Z}_p}$ are given as follows:

1. For $0 \leq i_1, i_2, i_3 \leq k$, $0 \leq j_1, j_2 \leq p - 1$,
\[
(L(k,i_1)^{i_1} \boxtimes L(k,i_2)^{i_2} \cong \sum_{\substack{|i_1 - i_2| \\ i_1 + i_2 + i_3 \in 2\mathbb{Z}}} L(k,i_3)^{\text{sgn}(i_1,i_2,i_3,j_1,j_2)}; \quad (4.21)
\]

2. For $0 \leq i_1, i_2, i_3 \leq k$, $0 \leq j_1, j_2 \leq p - 1$, $0 < r < p$,
\[
(L(k,i_1)^{i_1} \boxtimes L(k,i_2)^{i_2} \cong \sum_{\substack{|i_1 - i_2| \\ i_1 + i_2 + i_3 \in 2\mathbb{Z}}} L(k,i_3)^{r \text{sgn}(i_1,i_2,i_3,j_1,j_2)}; \quad (4.22)
\]

3. For $0 \leq i_1, i_2, i_3 \leq k$, $0 \leq j_1, j_2 \leq p - 1$, $0 < r_1 + r_2 < p$,
\[
(L(k,i_1)^{r_1} \boxtimes L(k,i_2)^{r_2} \cong \sum_{\substack{|i_1 - i_2| \\ i_1 + i_2 + i_3 \in 2\mathbb{Z}}} L(k,i_3)^{r_1 + r_2 \text{sgn}(i_1,i_2,i_3,j_1,j_2)}; \quad (4.23)
\]

4. For $0 \leq i_1, i_2, i_3 \leq k$, $0 \leq j_1, j_2 \leq p - 1$, $r_1 + r_2 = p$,
\[
(L(k,i_1)^{r_1} \boxtimes L(k,i_2)^{r_2} \cong \sum_{\substack{|i_1 - i_2| \\ i_1 + i_2 + i_3 \in 2\mathbb{Z}}} L(k,k - i_3)^{r_1 + r_2 - p \text{sgn}(i_1,i_2,i_3,j_1,j_2)}; \quad (4.24)
\]

5. For $0 \leq i_1, i_2, i_3 \leq k$, $0 \leq j_1, j_2 \leq p - 1$, $r_1 + r_2 > p$,
\[
(L(k,i_1)^{r_1} \boxtimes L(k,i_2)^{r_2} \cong \sum_{\substack{|i_1 - i_2| \\ i_1 + i_2 + i_3 \in 2\mathbb{Z}}} L(k,k - i_3)^{r_1 + r_2 - p \text{sgn}(i_1,i_2,i_3,j_1,j_2)}; \quad (4.25)
\]

**Proof.** Proof of (4.22): From Lemma 4.11, we know that $\gamma^p(\cdot, z)$ is an intertwining operator of type $\left(\begin{array}{c}
L(k,i_3)^{i_1} \\
L(k,i_1)L(k,i_2)^{i_2}
\end{array}\right)$ where $0 \leq i_1, i_2, i_3 \leq k$, $|i_1 - i_2| \leq i_3 \leq i_1 + i_2 + i_3 \in 2\mathbb{Z}$ and $i_1 + i_2 + i_3 \leq 2k$. For $0 \leq j_1, j_2, j_3 \leq p - 1$, there exist two nonzero vectors $u^{(j_1)}$ and $u^{(j_2)}$:
\[ u^{(j)} := e(-n_1)e(-n_2) \cdots e(-n_s)e^{i \cdot i}(n_1 \geq \cdots \geq n_s \geq 1, s \geq 0, s = j) \in L(k, i)^{h}, \]
\[ u^{(j)} := e(-m_1)e(-m_2) \cdots e(-m_t)e^{i \cdot i}(m_1 \geq \cdots \geq m_t \geq 1, t \geq 0, t = j) \in L(k, i)^{r, \overline{r}}. \]

Note that \( h^{(p, r)}(0)u^{(j)} = \frac{r(2s-i)}{2p} u^{(j)} \). Thus, we have
\[ \mathcal{Y}_{\sigma_p}(u^{(j)}, z) u^{(j)} = \frac{r(2s-i)}{2p} \mathcal{Y}(u^{(j)}, z) u^{(j)}. \]

Recall that \( d_{k, i}^{p, r} = \frac{\mathbb{R}^{k-2r+1}}{2p} \) is the conformal weight of the irreducible \( \sigma_p \)-twisted \( L(k, 0) \)-module \( L(k, i)^r \) for \( 0 \leq r \leq p - 1 \). Then we can deduce that a restriction \( \mathcal{Y}_{\sigma_p}(\cdot, z) \) gives an intertwining operator of type \( \left( L(k, i)^{r, \overline{r}}, L(k, i)^{r, \overline{r}} \right) \) if and only if
\[ d_{k, i}^{p, 0} + d_{k, i}^{p, 0} - d_{k, i}^{p, 0} - d_{k, i}^{p, r} + d_{k, i}^{p, r} + \frac{r_1 - rs - rt + r_j}{2p} \in \mathbb{Z} \]
which is equivalent to \( \frac{1 + i + i}{2p} - \frac{h + j + i}{p} \in \mathbb{Z} \). Hence, \( j_3 = \text{sgn}(i_1, i_2, i_3, j_1, j_2) \). Since \( (r, p) = 1 \), we know that, for fixed \( r \), \( j_3 \in [0, j_3 \leq p - 1) \) run over all the numbers of the set \( \{0, 1, \ldots, p - 1\} \). Recall the quantum dimensions of irreducible \( L(k, 0)^{r, r} \)-modules along with the fact that
\[ \frac{\sin \frac{\pi (i+1)}{k+2}}{\sin \frac{\pi}{k+2}} \cdot \frac{\sin \frac{\pi (i+1)}{k+2}}{\sin \frac{\pi}{k+2}} = \sum_{|u-v| \leq h \leq n \leq v+i} \frac{\sin \frac{\pi (i+1)}{k+2}}{\sin \frac{\pi}{k+2}}. \]

Then we can infer that (4.22) holds.

**Proof of (4.21):** Note that the nonzero vector \( e(-n_1)e(-n_2) \cdots e(-n_s)e^{i \cdot i}(n_1 \geq \cdots \geq n_s \geq 1, s \geq 0, s = j) \in L(k, i)^r \cap L(k, i)^{r, \overline{r}} \), then (4.21) follows from (4.22).

**Proof of (4.23):** From Lemma 4.14, we know that
\[ (L(k, i)^{r, \overline{r}}^{\Delta}, Y_{\sigma_p}^{r, \overline{r}}(\Delta(h^{(p, r)}, z), \cdot, z)) \cong (L(k, i)_{C_22}^{r, \overline{r}}^{\Delta}, Y_{\sigma_p}^{r, \overline{r}}(\cdot, \cdot, z)). \]

Using the symmetry property along with (4.22), we have
\[ L(k, i)^{r, \overline{r}} \cong L(k, i)^{h, r} \cong L(k, i)^{r, \overline{r}} \]
\[ = \sum_{|u-v| \leq h \leq n \leq v+i} \frac{L(k, i)_{C_22}^{r, \overline{r}}^{\Delta(h^{(p, r)}, z), \cdot, z)}{\sum_{|u-v| \leq h \leq n \leq v+i} \frac{L(k, i)_{C_22}^{r, \overline{r}}^{\Delta(h^{(p, r)}, z), \cdot, z)}}. \]

Therefore, using the Proposition 2.8. in [9], we have
\[ L(k, i)^{r, \overline{r}} \cong L(k, i)^{r, \overline{r}} \cong L(k, i)^{r, \overline{r}} \]
\[ \cong \sum_{|u-v| \leq h \leq n \leq v+i} \frac{L(k, i)^{r, \overline{r}}^{\Delta(h^{(p, r)}, z), \cdot, z)}{\sum_{|u-v| \leq h \leq n \leq v+i} \frac{L(k, i)^{r, \overline{r}}^{\Delta(h^{(p, r)}, z), \cdot, z)}}. \]

Then (4.23) is clear.
In almost exactly the same way, we can prove (4.24) and (4.25) by using the Proposition 2.8 in [9] and Lemma 4.14 again.

Remark 4.16. For the case of $k = 1$, recall from Remark 3.12 that $L(1,0)^{\mathbb{Z}_p}$ can be realized as the lattice vertex operator algebra $V_{\mathbb{Z}_p} \beta$ with $(\beta, \beta) = 2p^2$ and the correspondence between irreducible $L(1,0)^{\mathbb{Z}_p}$-modules and $\{ V_{\mathbb{Z}_p + \frac{s}{2p} \beta} | 0 \leq s < 2p^2 \}$ has been listed explicitly. It is well known that

$$V_{\mathbb{Z}_p + \frac{s}{2p} \beta} \otimes V_{\mathbb{Z}_p + \frac{t}{2p} \beta} = V_{\mathbb{Z}_p + \frac{s+t}{2p} \beta},$$

where we use $s, t$ to denote both integers between 0 and $2p^2 - 1$ and its residue class modulo $2p^2$ in this situation. This formula also gives the fusion rules for all the irreducible $L(1,0)^{\mathbb{Z}_p}$-modules. It is not difficult to verify that the fusion rules given in this manner are consistent with the results in Theorem 4.15.

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