Almost contact metric structures defined by an 
$N$-prolonged connection

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Abstract

On a manifold with an almost contact metric structure $(\varphi, \vec{\xi}, \eta, g, X, D)$ the notions of the interior and the $N$-prolonged connections are introduced. Using the $N$-prolonged connection, a new almost contact metric structure is defined on the distribution $D$. The properties of this structure are studied.

Key words: almost contact metric structure, interior connection, $N$-prolonged connection, prolonged almost contact metric structure.

1 Introduction

The study of the geometry of the tangent bundles was initiated in the fundamental paper [1] by Sasaki published in 1958. Using a Riemannian metric $g$ on a manifold $X$ Sasaki defines a Riemannian metric $G$ on the tangent bundle $TX$ of the manifold $X$. This construction is grounded on the natural splitting (that takes a place due to the existence of the Levi-Civita connection) of the tangent bundle $TTX$ of the manifold $TX$ into the direct sum of the vertical and horizontal distributions, the fibers of these distributions are isomorphic to the fibers of the distribution $TX$. The odd analogue of the tangent bundle is a distribution $D$ of an almost contact metric structure $(\varphi, \vec{\xi}, \eta, g)$. Similarly to the bundle $TTX$, the bundle $TD$ due to a connection over the distribution [2] (and later an $N$-prolonged connection, i.e. connection in the vector bundle $(X, D)$) splits into the direct sum of the vertical and horizontal distributions.

In [2,3] it is shown that on the manifold $D$ can be defined in a natural way an almost contact metric structure allowing e.g. to give an invariant character to the analytical description of the mechanics with constraints. In [3], on the manifold $D$ the geodesic pulverization of the connection over the distribution is defined; this is an analogue of the geodesic pulverization, defined on the space of the tangent bundle $TX$ and having the following physical interpretation: the projections of the integral curves of the geodesic pulverization of the connection over the distribution coincide with the admissible geodesics (the trajectories of the mechanical system with constrains).

The present work is an introduction to the geometry of prolonged almost contact metric structures and it is dedicated to the development of the following two ideas: the idea of the generalization of the Sasaki construction [1] for the case of odd dimension, and the idea of the extension of the interior connection.

The paper has the following structure. The second section consists of three subsections; the first of them contains short information about the interior geometry of almost contact metric spaces. This stuff can be found in [4].

In section 2.2, the notion of the $N$-prolonged metric connection is introduced. The interior connection defines the parallel transport of the admissible vectors (i.e. the vectors belonging to...
the distribution $D$) along admissible curves. Each corresponding $N$-prolonged connection is a
connection in the vector bundle $(D, \pi, X)$ defined by the interior connection and an endomorph-
ism $N : D \to D$. The choice of the endomorphism $N : D \to D$ effects the properties of the
prolonged connection and also the properties of the (prolonged) almost contact metric structure
appearing on the total space $D$ of the vector bundle $(D, \pi, X)$. The central of this subsection
is the theorem about the existence and uniqueness of the $N$-prolonged metric connection with
zero torsion. In section 2.3, the relation of the interior and prolonged connections with the
known connections appearing on almost contact metric spaces is shown.

In the third section, on the manifold $D$ with an prolonged metric connection, the prolonged
almost contact metric structure is defined. In subsection 3.1, the properties of the prolonged
almost contact metric structure are investigated. In subsection 3.2, a negative answer to the
question about the possibility of the isometrical imbedding of the manifold $D$ to the manifold
$TX$ with the Sasaki metric is given.

2 Interior and $N$-prolonged connections

2.1 Preliminaries on the interior geometry of almost contact metric
spaces

Let $X$ be a smooth manifold of an odd dimension $n$. Denote by $\Gamma TX$ the $C^\infty(X)$-module of
smooth vector fields on $X$. All manifolds, tensors and other geometric objects will be assumed
to be smooth of the class $C^\infty$. An almost contact metric structure on $X$ is an aggregate
$(\varphi, \xi, \eta, g)$ of tensor fields on $X$, where $\varphi$ is a tensor field of type $(1, 1)$, which is called
the structure endomorphism, $\xi$ and $\eta$ are a vector and a covector, which are called the structure
vector and the contact form, respectively, and $g$ is a (pseudo-)Riemannian metric. Moreover,

\begin{align*}
\eta(\xi) &= 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\
\varphi^2 \tilde{X} &= -\tilde{X} + \eta(\tilde{X}) \xi, \quad g(\varphi \tilde{X}, \varphi \tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \eta(\tilde{X}) \eta(\tilde{Y}), \\
d\eta(\xi, \tilde{X}) &= 0
\end{align*}

for all $\tilde{X}, \tilde{Y} \in \Gamma TM$. The skew-symmetric tensor $\Omega(\tilde{X}, \tilde{Y}) = g(\tilde{X}, \varphi \tilde{Y})$ is called the funda-
mental form of the structure. A manifold with a fixed almost contact metric structure is called
an almost contact metric manifold. If $\Omega = d\eta$ holds, then the almost contact metric structure
is called a contact metric structure. An almost contact metric structure is called normal if

\[ N_\varphi + 2d\eta \otimes \xi = 0, \]

where $N_\varphi$ is the Nijenhuis torsion defined for the tensor $\varphi$. A normal contact metric structure
is called a Sasakian structure. A manifold with a given Sasakian structure is called a Sasakian
manifold. Let $D$ be the smooth distribution of codimension 1 defined by the form $\eta$, and
$D^\perp = \text{span}(\xi)$ be the closing of $D$. If the restriction of the 2-form $\omega = d\eta$ to the distribution
$D$ is non-degenerate, then the vector $\xi$ is uniquely defined by the condition

\[ \eta(\xi) = 1, \quad \ker \omega = \text{span}(\xi), \]

and it is called the Reeb vector field. In this subsection we will pay more attention to the
so called almost contact Kählerian spaces \[5\], other basic classes of of almost contact metric
spaces will be considered in the next section.

An almost contact metric structure is called almost normal, if it holds

\[ N_\varphi + 2(d\eta \circ \varphi) \otimes \xi = 0. \tag{1} \]
In what follows, an almost normal almost contact metric space will be called an almost contact Kählerian space if its fundamental form is closed. An almost contact metric space is called almost K-contact metric space if $L\xi g = 0$. The last equality is usually used in the case, when the form $\omega$ has the maximal rank, then the corresponding space is called K-contact.

An almost normal contact metric structure is obviously a Sasakian structure. Sasakian manifolds are popular among the researchers of almost contact metric spaces by the following two reasons. On one hand, there exist a big number of interesting and deep examples of Sasakian structures, on the other hand, the Sasakian manifolds have very important and natural properties. In the same time, the almost contact Kählerian spaces inherit many important properties of the Sasakian spaces, this turns out to be very essential in the cases when an almost contact metric space can not in principle be a Sasakian space [6].

We say that a coordinate chart $K(x^\alpha)$ ($\alpha, \beta, \gamma = 1, \ldots, n, a, b, c, e = 1, \ldots, n-1$) on a manifold $X$ is adapted to the non-holonomic manifold $D$ if

$$D^\perp = \text{span} \left( \frac{\partial}{\partial x^n} \right)$$

holds [4]. Let

$$P : TX \rightarrow D$$

be the projection map defined by the decomposition

$$TX = D \oplus D^\perp$$

and let $K(x^\alpha)$ be an adapted coordinate chart. Vector fields

$$P(\partial_a) = \vec{e}_a = \partial_a - \Gamma^a_{\alpha n} \partial_n$$

are linearly independent, and linearly generate the system $D$ over the domain of the definition of the coordinate map:

$$D = \text{span}(\vec{e}_a).$$

Thus we have on $X$ the non-holonomic field of bases

$$(\vec{e}_a, \partial_n)$$

and the corresponding field of cobases

$$(dx^a, \theta^n = dx^n + \Gamma^n_{\alpha} dx^\alpha).$$

It can be checked directly that

$$[\vec{e}_a, \vec{e}_b] = M^a_{\alpha b} \partial_n,$$

where the components $M^a_{\alpha b}$ form the so-called tensor of non-holonomicity [7]. Under assumption that for all adapted coordinate systems it holds $\xi = \partial_n$, the following equality takes the place

$$[\vec{e}_a, \vec{e}_b] = 2\omega_{ba} \partial_n,$$

where $\omega = d\eta$. We say also that the basis

$$\vec{e}_a = \partial_a - \Gamma^a_{\alpha n} \partial_n$$

is adapted, as the basis defined by an adapted coordinate map. Note that

$$\partial_n \Gamma^a_{\alpha} = 0$$
Let $K(x^a)$ and $K(x^{a'})$ be adapted charts, then under the condition
\[
\bar{\xi} = \partial_n
\]
we get the next formulas for the coordinate transformation:
\[
x^a = x^a(x^{a'}), \quad x^n = x^n + x^n(x^{a'}).
\]

A tensor field of type $(p,q)$ defined on an almost contact metric manifold is called admissible (to the distribution $D$) if in adapted coordinate map it looks like
\[
t = t^{a_1}_{b_1}...a_p^{b_p} \, \bar{e}_{a_1} \otimes ... \otimes \bar{e}_{a_p} \otimes dx^{b_1} \otimes ... \otimes dx^{b_q}.
\]
From the definition of an almost contact structure it follows that the field of endomorphisms $\varphi$ is an admissible tensor field of type $(1,1)$. The field of endomorphisms $\varphi$ we call an admissible almost complex structure, taking into the account its properties. The 2-form $\omega = d\eta$ is also an admissible tensor field and it is natural to call it an admissible symplectic form.

The transformation of the components of an admissible tensor field in adapted coordinates satisfies the following low:
\[
t^a_b = A^a_{a'}A^b_{b'}t^{a'}_{b'},
\]
where $A^a_{a'} = \frac{\partial x^a}{\partial x^{a'}}$.

**Remark 1.** From the formulas for the transformation of the components of an admissible tensor field it follows that the derivatives $\partial_n t^a_b$ are again components of an admissible field. Moreover, the vanishing of $\partial_n t^a_b$ does not depend on the choice of adapted coordinates. The last statement also follows from the equality $(L_{\bar{\xi}})^a_b = \partial_n t^a_b$.

**Remark 2.** An admissible tensor structure satisfying $\partial_n t^a_b = 0$ we will call projectible (in the literature there are other names for the structures with similar properties: basic, semi basic, etc.). In what follows we will see that admissible projectible structures can be naturally considered as structures defined on a submanifold of a smaller dimension.

Using adapted coordinates we introduce the following admissible tensor fields:
\[
h^a_b = \frac{1}{2}\partial_n \bar{\chi}^a_b, \quad C_{ab} = \frac{1}{2}\partial_n g_{ab}, \quad C^a_b = g^{da}C_{db}, \quad \psi^b_a = g^{db}\bar{\omega}_{da}.
\]
We denote by $\bar{\nabla}$ and $\bar{\Gamma}^a_{\beta\gamma}$ the Levi-Civita connection and the Christoffel symbols of the metric $g$. The proof of the following theorem follows from direct computations.

**Theorem 1** The Christoffel symbols of the Levi-Civita connection of an almost contact metric space with respect to adapted coordinates are the following:
\[
\bar{\Gamma}^{c}_{ab} = \Gamma^{c}_{ab}, \quad \bar{\Gamma}^{n}_{ab} = \omega_{ba} - C_{ab}, \quad \bar{\Gamma}^{b}_{an} = \bar{\Gamma}^{b}_{na} = C_{ab}^b - \psi_{a}^b, \quad \bar{\Gamma}^{n}_{na} = \bar{\Gamma}^{n}_{nn} = 0,
\]
where
\[
\Gamma^{a}_{bc} = \frac{1}{2}g^{ad}(\bar{e}_{b}g_{cd} - \bar{e}_{c}g_{bd} - \bar{e}_{d}g_{bc}).
\]

### 2.2 $N$-prolonged metric connection

An intrinsic linear connection on a manifold with an almost contact metric structure [4] is defined as a map
\[
\nabla : \Gamma D \times \Gamma D \to \Gamma D
\]
that satisfies the following conditions:

1) $\nabla_{f_1\bar{a}_1 + f_2\bar{a}_2} = f_1\nabla_{\bar{a}_1} + f_2\nabla_{\bar{a}_2};$

2) $\nabla_{\bar{a}}f \bar{v} = f\nabla_{\bar{a}}\bar{v} + (\bar{u}f)\bar{v},$


where $\Gamma D$ is the module of admissible vector fields. The Christoffel symbols are defined by the relation
\[ \nabla_{\vec{e}_a} \vec{e}_b = \Gamma^{c}_{ab} \vec{e}_c. \]

The torsion $S$ of the intrinsic linear connection is defined by the formula
\[ S(\vec{X},\vec{Y}) = \nabla_{\vec{X}} \vec{Y} - \nabla_{\vec{Y}} \vec{X} - p[\vec{X}, \vec{Y}]. \]

Thus with respect to an adapted coordinate system it holds
\[ S_{ab}^c = \Gamma^{c}_{ab} - \Gamma^{c}_{ba}. \]

The action of an intrinsic linear connection can be extended in a natural way to arbitrary admissible tensor fields. An important example of an intrinsic linear connection is the intrinsic metric connection that is uniquely defined by the conditions $\nabla g = 0$ and $S = 0$ [7]. With respect to the adapted coordinates it holds
\[ \Gamma^{a}_{bc} = \frac{1}{2}g^{ad}(\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc}). \] (2)

Note that $\Gamma^{a}_{bc} = \tilde{\Gamma}^{a}_{bc}$ (see Theorem 1).

In the same way as a linear connection on a smooth manifold, an intrinsic connection can be defined by giving a horizontal distribution over the total space of some vector bundle. In the case of the interior connection, the role of such bundle plays the distribution $D$. One says that over a distribution $D$ a connection is given if the distribution
\[ \tilde{D} = \pi^{-1}_*(D), \]
where $\pi : D \to X$ is the natural projection, can be decomposed into a direct some of the form
\[ \tilde{D} = HD \oplus VD, \]
where $VD$ is the vertical distribution on the total space $D$.

Let us introduce a structure of a smooth manifold on $D$. This structure is defined in the following way. To each adapted coordinate chart $K(x^n)$ on the manifold $X$ we put in correspondence the coordinate chart $\tilde{K}(x^n, x^{n+a})$ on the manifold $D$, where $x^{n+a}$ are the coordinates of an admissible vector with respect to the basis
\[ \tilde{e}_a = \partial_a - \Gamma^n_a \partial_n. \]

The constructed over-coordinate chart will be called adapted. Thus the assignment of a connection over a distribution is equivalent to the assignment of the object
\[ G^{a}_{b}(X^a, X^{n+a}) \]
such that
\[ HD = \text{span}(\tilde{e}_a), \]
where
\[ \tilde{e}_a = \partial_a - \Gamma^n_a \partial_n - G^b_a \partial_{n+b}. \]

If it holds
\[ G^{a}_{b}(x^a, x^{n+a}) = \Gamma^{a}_{bc}(x^a)x^{n+c}, \]
then the connection over the distribution $D$ is defined by the intrinsic linear connection. In [2], the notion of the prolonged connection was introduced. The prolonged connection is always considered as a connection over a distribution and it is defined by the decomposition
\[ TD = HD \oplus VD, \]
where $HD \subset \hat{HD}$. The prolonged connection is a connection in a vector bundle. As it follows from the definition of the prolonged connection, for its assignment (under the condition that a connection on the distribution is already defined) it is enough to define a vector field $\tilde{u}$ on the manifold $D$ that has the following coordinate form: $\tilde{u} = \partial_n - N^a_b x^{n+b} \partial_{n+a}$, where the endomorphism $N : D \to D$ can be chosen in an arbitrary way. We call the torsion of the of the prolonged connection the torsion of the initial connection. In what follows we call a prolonged connection an $N$-prolonged connection.

In [7] the admissible tensor field

$$R(\tilde{u}, \tilde{v}) \tilde{w} = \nabla_{\tilde{u}} \nabla_{\tilde{v}} \tilde{w} - \nabla_{\tilde{v}} \nabla_{\tilde{u}} \tilde{w} - \nabla_{[\tilde{u}, \tilde{v}]} \tilde{w}$$

is called by Wagner the first Schouten curvature tensor. With respect to the adapted coordinates it holds

$$R^a_{bcd} = 2\tilde{e}^a_i \Gamma^d_{bj} + 2\Gamma^d_{[a|e]} \Gamma^e_{bj}.\tag{7}$$

If the distribution $D$ does not contain any integrable subdistribution of dimension $n - 2$, then the Schouten curvature tensor is zero if and only if the parallel transport of admissible vectors does not depend on the curve [7]. We say that the Schouten tensor is the curvature tensor of the distribution $D$. If this tensor is zero, we say that the distribution $D$ is a zero-curvature distribution. Note that the partial derivatives $\partial_n \Gamma^a_{bc} = P^a_{bc}$ are components of an admissible tensor field [7].

**Remark 3.** In the case of (almost) K-contact spaces the Schouten tensor has the same properties as the Riemannian tensor of a manifold. In general this is not true. The vector fields

$$\tilde{e}_a = \partial_n - \Gamma^a_n \partial_n - \Gamma^a_{ac} x^{n+c} \partial_{n+a}, \quad \tilde{u} = \partial_n - N^a_b x^{n+b} \partial_{n+a}, \quad \partial_{n+a}$$

define on $D$ a non-holonomic (adapted) field of bases, and the forms

$$dx^a, \quad \Theta^n = dx^n + \Gamma^a_a dx^a, \quad \Theta^{n+a} = dx^{n+a} + \Gamma^a_{bc} x^{n+c} dx^b + N^a_b x^{n+b} dx^n$$

define the corresponding field of cobases. The following structure equations can be obtained:

$$[\tilde{e}_a, \tilde{e}_b] = 2\omega_{ba} \tilde{u} + x^{n+d}(2\omega_{ba} N^c_d + R^c_{bad}) \partial_{n+c}, \tag{3}$$

$$[\tilde{e}_a, \tilde{u}] = x^{n+d}(\partial_n \Gamma^c_{ad} - \nabla_a N^c_d) \partial_{n+c}, \tag{4}$$

$$[\tilde{e}_a, \partial_{n+b}] = \Gamma^c_{ab} \partial_{n+c}. \tag{5}$$

From (3) and (4) we obtain the the expression for the curvature tensor of the prolonged connection

$$K(\tilde{u}, \tilde{v}) \tilde{w} = 2\omega(\tilde{u}, \tilde{v}) N \tilde{w} + R(\tilde{u}, \tilde{v}) \tilde{w}, \tag{6}$$

$$K(\tilde{\xi}, \tilde{u}) \tilde{v} = P(\tilde{u}, \tilde{v}) - (\nabla_{\tilde{u}} N) \tilde{v}, \tag{7}$$

where $\tilde{u}, \tilde{v}, \tilde{w} \in \Gamma D$.

**Theorem 2** There exists an $N$-prolonged metric connection uniquely determined by the following conditions:

1. $\tilde{Z}g(\tilde{X}, \tilde{Y}) = g(\nabla_{\tilde{Z}} \tilde{X}, \tilde{Y}) + g(\tilde{X}, \nabla_{\tilde{Z}} \tilde{Y})$ (metricity property),
2. $\nabla_{\tilde{X}} \tilde{Y} - \nabla_{\tilde{Y}} \tilde{X} - P[\tilde{X}, \tilde{Y}] = 0$ (connection is torsion-free),
3. $N$ is a symmetric endomorphism such that

$$g(N \tilde{X}, \tilde{Y}) = \frac{1}{2} L_{\tilde{\xi}} g(\tilde{X}, \tilde{Y}), \tag{8}$$

where $\tilde{X}, \tilde{Y}, \tilde{Z} \in \Gamma D$ are sections of the distribution $D$, and $p : TX \to D$ is the projection.
Proof. The first two conditions of the theorem uniquely define the interior metric connection [7]. Alternating the second covariant derivative we get
\[ \nabla[v\nabla_a]g_{bc} = 2\omega_{ea}
abla_n g_{bc} - g_{dc} R^d_{eab} - g_{bd} R^d_{eac}. \]
Comparing the obtained equality with (8) we find the implicit expression for the endomorphism \( N \):
\[ N^f_b = \frac{1}{4(n-1)}\omega^e_a (R^f_{eab} + g_{bd} g^{cf} R^d_{eac}). \]
If \( \partial_n g_{ab} - 0 \), then \( N = 0 \). This proves the theorem. □

We call the prolonged connection with the properties form theorem 2 the \( N \)-prolonged metric connection. We will use the notation \( \nabla^N = (\nabla, N) \) for the prolonged connection. In particular, \( \nabla^1 = (\nabla, 0) \).

2.3 Special connections on manifolds with almost compact metric structure
E.Cartan [8–10] was the first who considered linear metric connection with a torsion instead of the Levi-Civita connection. The most interesting among the metric connections with torsion is the semi-symmetric connection investigated by K.Yano in [11]. The quarter-symmetric connection defined in 1975 Golab [12]. There is a big number of works dedicated to metric and non-metric connections with torsion defined on manifolds with almost contact structures. Here we fix attention to the paper by Bejancu [13]. Bejancu defines the connection \( \nabla^B \) on a Sasaki manifold by the formula
\[ (\nabla^B \vec{X}, \vec{Y}) = \eta(\vec{X}) \nabla^B \vec{Y} - \eta(\vec{Y}) \nabla^B \vec{X} + (\omega + c)(\vec{X}, \vec{Y}) \vec{\xi}. \]
With respect to an adapted coordinates, the non-zero components of the connection \( \nabla^B \) are
\[ \Gamma^{Ba}_{bc} = \Gamma^a_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc}). \]
The constructed by Bejancu connection is in general not metric in the common case of almost contact structure more general then the Sasaki structure. Indeed, by the equality
\[ \nabla^B_n g_{ab} = \partial g_{ab}, \]
the metric Bejancu connection is equivalent to an almost K-contact almost contact metric structure. Define on a manifold with an almost contact metric structure the connection \( \nabla^N \) by the equality
\[ \nabla^N_X Y = \nabla^B_X Y - \eta(X)NY, \]
where \( N \) is the endomorphism from theorem 2. Let us call the introduced connection the \( N \)-connection. The non-zero components of this connection are
\[ \Gamma^{Na}_{bc} = \Gamma^n_{bc} = \frac{1}{2} g^{ad}(\vec{e}_b g_{cd} - \vec{e}_c g_{bd} - \vec{e}_d g_{bc}), quad \Gamma^{Na}_{nc} = N^a_c. \]
The torsion of the \( N \)-connection is defined by the equality
\[ S^N(X, Y) = 2\omega(X, Y)\xi + \eta(X)NY - \eta(Y)NX. \]
The following theorem can be proved by direct computations.

**Theorem 3** An \( N \)-connection is a metric connection.
3 $N$-prolonged connection as an almost contact metric structure

Consider on a manifold $X$ a contact metric structure $(D, \varphi, \xi, \eta, g, X)$. We define on the distribution $D$ as on a smooth manifold the almost contact metric structure $(\tilde{D}, \tilde{J}, \tilde{\eta} = \eta \circ \pi_\ast, \tilde{g}, \tilde{D})$ by setting

\[
\tilde{g}(\tilde{e}_a, \tilde{e}_b) = \tilde{g}(\partial_{n+a}, \partial_{n+b}) = \tilde{g}(\tilde{e}_a, \tilde{e}_b), \quad \tilde{g}(\tilde{e}_a, \partial_{n+b}) = \tilde{g}(\tilde{e}_a, \tilde{u}) = \tilde{g}(\tilde{u}, \partial_{n+b}) = 0,
\]

\[
\tilde{J}(\tilde{e}_a) = \partial_{n+a}, \quad \tilde{J}(\partial_{n+a}) = -\tilde{e}_a, \quad \tilde{J}(\tilde{u}) = 0.
\]

The vector fields

\[
\tilde{e}_a = \partial_a - \Gamma^a_n \partial_n - \Gamma^b_{ac} x^{n+c} \partial_{n+b}, \quad \tilde{u} = \partial_n - N^a_b x^{n+b} \partial_{n+a}
\]

are defined here by the prolonged connection. Let $\tilde{\omega} = d\lambda$. It can be directly checked that the non-zero components of the form $\tilde{\omega}$ are given by the equality $\tilde{\omega}_{ab} = \omega_{ab}$. Consequently, $rk\tilde{\omega} = \frac{n-1}{2}$. This implies that the constructed structure is not contact and, in particular, it is not a Sasaki structure.

**Theorem 4** The prolonged almost contact metric structure is almost K-contact if and only if the initial structure is K-contact.

**Proof.** Possibly non-zero components of the curvature tensor with respect to adapted coordinates are of the form

\[
(L_{\tilde{a}} \tilde{g})_{ab} = \partial_n g_{ab}, \quad (L_{\tilde{a}} \tilde{g})_{n+a,n+b} = \partial_n g_{ab} - g_{ac} N^c_b - g_{cb} N^c_a, \quad (L_{\tilde{a}} \tilde{g})_{n+a,b} = g_{ac} (P^c_{bd} - \nabla_b N^c_d) x^{n+d}.
\]

In fact, the components from (11) are also zero, since these are the components of the covariant derivative of the metric tensor. The equality $\partial_n g_{ab}$ implies $N^c_{ab} = 0$ and $P^c_{ab} = 0$ (see (9) and (10)). This proofs the theorem. □

Suppose now that at the initial structure is K-contact ($N = 0$), then the following theorem takes a place.

**Theorem 5** The almost contact metric structure $(\tilde{D}, \tilde{J}, \tilde{\eta}, \lambda = \eta \circ \pi_\ast, \tilde{g}, \tilde{D})$ is is almost normal if and only if the distribution $D$ is a distribution of zero curvature.

**Proof.** Let us rewrite the equality (11) in new notation,

\[
N_J + 2(d\tilde{\eta} \circ J) \otimes \tilde{u} = 0.
\]

In [4] it was shown that an almost contact structure is almost normal if and only if $\tilde{P} \circ N_J = 0$, where $\tilde{P} : TD \to \tilde{D}$ is the projection.

Using the equalities (12)–(15), for the case of the connection $\nabla^1$, we get the following expressions for the Nijenhuis torsion of the operator $J$:

\[
N_J (\tilde{e}_a, \tilde{e}_b) = -R^c_{abc} x^{n+c} \partial_{n+e},
\]

\[
N_J (\partial_{n+a}, \partial_{n+b}) = 2\omega_{ba} \partial_n + R^c_{abc} x^{n+c} \partial_{n+e},
\]

\[
N_J (\tilde{e}_a, \partial_{n+b}) = 0,
\]

\[
N_J (\tilde{e}_a, \partial_n) = N_J (\partial_{n+a}, \partial_n) = -x^{n+c} P^b_{ac} \partial_{n+b}.
\]
Thus the prolonged almost contact metric structure is almost normal if and only if the Schouten curvature tensor is zero. □

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