The nonlinear Schrödinger equations with combined nonlinearities of power-type and Hartree-type

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Abstract

This paper is devoted to a comprehensive study of the nonlinear Schrödinger equations with combined nonlinearities of the power-type and Hartree-type in any dimension \( n \geq 3 \). With some structural conditions, a nearly whole picture of the interactions of these nonlinearities in the energy space is given. The method is based on the Morawetz estimates and perturbation principles.

Keywords: Global well-posedness; scattering; blow up; Morawetz estimates; perturbation principles.

1 Introduction

We are concerned with the Cauchy problem for the following Schrödinger equation

\[
\begin{cases}
    iu_t + \Delta u = \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2)u \\
    u(0, x) = u_0(x),
\end{cases}
\]

where \( u(t, x) \) is a complex-value function in spacetime \( \mathbb{R} \times \mathbb{R}^n (n \geq 3) \), initial datum \( u_0 \) takes value in \( H^1_x(\mathbb{R}^n) \) (or \( \sum \)), \( \lambda_1 \) and \( \lambda_2 \) are nonzero constants, \( 0 < p \leq \frac{4}{n-2} \), and \( 0 < \gamma < 4 \) with \( n > \gamma \). For such a problem, T. Cazenave has given a fundamental discussion in [6]. However, just a few cases he has settled, for example when both nonlinearities are defocusing, the equation must be energy-subcritical; when one nonlinearity is focusing, and the index of the nonlinearity lies between mass-critical and energy-critical, need mass and energy sufficiently small. In very recent years, there are many results on the global well-posedness for the following energy-critical (1.2) and (1.3) or mass-critical (1.4) and (1.5) nonlinear Schrödinger equation have been obtained by T. Tao, J. Colliander and Carlos E. Kenig and so on, respectively. [15, 19, 20, 23, 2, 3, 26, 17, 18]
\[
\begin{aligned}
  iu_t + \Delta u &= \lambda_1 |u|^{\frac{4}{n-2}} u \\
  u(0, x) &= u_0(x)
\end{aligned}
\] (1.3)

\[
\begin{aligned}
  iu_t + \Delta u &= \lambda_2 (|x|^{-2} * |u|^2) u \\
  u(0, x) &= u_0(x)
\end{aligned}
\] (1.4)

\[
\begin{aligned}
  iu_t + \Delta u &= \lambda_1 |u|^4 u \\
  u(0, x) &= u_0(x)
\end{aligned}
\] (1.5)

Therefore, in this paper, we want to give a whole picture of the interactions of these both nonlinearities. First of all, we hope to solve the same problem of (1.1) when one nonlinearity is energy-critical. Then, we discuss the case \(\lambda_1 \cdot \lambda_2 < 0\) that T. Cazenave didn’t take care of but separately. Precisely, we hope that under some structural conditions, that is, under some relations of \(\lambda\) and \(p\), the defocusing term is able to control the focusing term, so that the whole nonlinearities behaviour like the defocusing property, therefore there is a global wellposed behaviour to be appeared because the defocusing nonlinearity will amplify the dispersive effect of the linear equation, but the focusing one usually is to cancel the dispersive effect.

Schrödinger equation (1.1) has two conservation laws: energy conservation and mass conservation, where energy and mass are defined as follow:

\[
E(u(t)) : = \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{\lambda_1}{p+2} \int |u|^{p+2} \, dx + \frac{\lambda_2}{4} \int (|x|^{-\gamma} * |u|^2) |u|^2 \, dx
\]

\[
M(u(t)) : = \int |u|^2 \, dx
\]

As they are conserved, we’ll prefer to write \(E(u)\) for \(E(u(t))\) and \(M(u)\) for \(M(u(t))\).

Our first main theorem is as following:

**Theorem 1.1 (Global well-posedness)** Let \(u_0 \in H^1_x\). Then there exists a unique global solution \(u\) to (1.1) in each of the following cases:

1. when \(\lambda_1, \lambda_2 > 0\), \(0 < p \leq \frac{4}{n-2}\), \(0 < \gamma \leq 4\) with \(\gamma < n\) except for \((p, \gamma) = (\frac{4}{n-2}, 4)\).

2. when \(\lambda_1 > 0\), \(\lambda_2 < 0\),
   
   2.1 \(0 < p \leq \frac{4}{n-2}\), and \(0 < \gamma < \min\{n, \frac{np}{2}\}\).
   
   2.2 \(\frac{np}{2} \leq \gamma < 2\).
   
   2.3 \(\frac{np}{2} \leq \gamma = 2\), and \(\|u_0\|_{L^2}^2 < \frac{1}{|\lambda_2|} \|W\|_{L^2}^2\).
   
   2.4 \(\frac{np}{2} \leq \gamma = 4\) (\(n > 4\)), \(E < \frac{E(W)}{|\lambda_2|}\), \(\|\nabla u_0\|_{L^2}^2 < \frac{1}{|\lambda_2|} \|W\|_{L^2}^2\) and \(u_0\) is radial except for \((p, \gamma) = (\frac{4}{n-2}, 4)\).
   
   2.5 \(\frac{np}{2} \leq \gamma, 2 < \gamma < \min\{4, n\}\), \(EM^{\frac{4-n}{2}} < \left(\frac{1}{2} - \frac{1}{\gamma}\right) \left[\frac{2\gamma E(W)}{|\lambda_2|} \gamma\right]^{\frac{2}{2}}\)
and \(\|\nabla u_0\|_{L^2}^2 M^{\frac{4-n}{2}} < \left(\frac{\|W\|_{L^2}^2}{|\lambda_2|}\right)^{\frac{2}{2}}\).
when the initial datum is radial, there may exist the global solution for Killip and M. Visan have proven the global well-posedness for
For the case 2.4, we need the initial datum to be radial. Because according to [20] Remark 1.1
\[ E(W) := \frac{1}{2} \int |\nabla W|^2 \, dx - \frac{1}{4} \int (|x|^{-\gamma} * |w|^2) |W|^2 \, dx. \]
3. when \( \lambda_1 < 0, \lambda_2 > 0, \)

3.1 \( 0 < p < \min \{ \frac{4}{n}, \frac{4}{2+n-\gamma} \}, \) and \( 0 < \gamma \leq 4 \) with \( \gamma < n. \)

3.2 \( p = \frac{4}{n}, \) \( p \geq \frac{4}{2+n-\gamma}, \) and \( \| u_0 \|_{L^2} < \| \lambda_1 |^{-\frac{4}{n}} \| R \|_{L^2}. \)

3.3 \( \frac{4}{2+n-\gamma} \leq p = \frac{4}{n-2} \) except for \( (p, \gamma) = (\frac{4}{n-2}, 4), \) in addition,
if \( n \geq 5, \) require \( E < \| \lambda_1 |^{\frac{2-n}{2}} \tilde{E}(R), \| \nabla u_0 \|_{L^2}^2 < \| \lambda_1 |^{\frac{2-n}{2}} \| \nabla R \|_{L^2}^2, \)
if \( n = 3, 4, u_0 \) is radial.

3.4 \( \frac{4}{n} < p < \frac{4}{n-2}, \) and \( \frac{4}{2+n-\gamma} \leq p \) with

\[ EM^{\frac{4-(n-2)p}{np-4}} < \| \lambda_1 |^{\frac{4-(n-2)p}{4np}} \left( \frac{2np}{np-4} \right) \tilde{E}(R)^{\frac{2p}{np-4}}, \]

\[ \| \nabla u_0 \|^2_{L^2} M^{\frac{4-(n-2)p}{np-4}} < \| \lambda_1 |^{\frac{4-(n-2)p}{4np}} \| \nabla R \|^2_{L^2}, \]

where \( R \) is the solution of ground state: \( \Delta R + |R|^p R = \frac{4-(n-2)p}{np} R \)
and \( \tilde{E}(R) := \frac{1}{2} \int |\nabla R|^2 \, dx - \frac{1}{p+2} \int |R|^{p+2} \, dx. \)

4. \( \lambda_1 < 0, \lambda_2 < 0, \) \( 0 < p < \frac{4}{n}, \) and \( 0 < \gamma < 2. \)

Moreover, for all compact intervals \( I, \) the global solution satisfies the following spacetime bound:

\[ \| u \|_{S^{1}(I \times \mathbb{R}^n)} \leq C(|I|, E, M). \] (1.6)

**Remark 1.1** For the case 2.4, we need the initial datum to be radial. Because according to [20] when the initial datum is radial, there may exist the global solution for [17,22]. For the case 3.3, R. Killip and M. Visan have proven the global well-posedness for [13] in [17] when the initial datum isn’t radial, but their approach is not suitable for the lower dimension, thus for lower dimension we preserve the radial condition.

We’ll prove this theorem in Section 4. Our chief work is to get a bound of \( \| u \|_{H^1} \) which only depends on energy and mass, and then apply the perturbation principles to get the result. As mentioned above, we hope the defocusing term can control the focusing term, however, this can’t be true usually, but we can prove that under the assumption of 2.1 and 3.1 in Theorem 1.1, it do happen. For other cases, our approach can’t show the defocusing term is able to control the focusing term. So just as what T. Cazenave did, still need some circumstances of the smallness about energy and mass. But the different point from that is the smallness which is characterized by the ground state. Unfortunately, our method isn’t useful for the case that both of the power and Hartree nonlinearities are energy-critical. Because after using Strichartz estimate, we need the dependence in time for the coefficients of nonlinearities, but no such factor for such both cases are energy-critical. The detail is in Section 4.
In Section 5, we consider the asymptotic behavior of these global solutions. It is natural to apply an unconditional scattering theory for (1.4) and (1.5). However, at least till now, we have to demand the initial datum radial and the size of mass is smaller than the one of ground state [26] [18]. Therefore, we need the following assumptions:

**Assumption 1.1** Let \( v_0 \in H^1_x \), \( \lambda_1 > 0 \). Then there exists a unique global solution \( v \) to (1.5) and satisfies
\[
\| v \|_{L^{2(n+2)}_{t,x}(\mathbb{R} \times \mathbb{R}^n)} \leq C(\| v_0 \|_{L^2_x}) (1.7)
\]

**Assumption 1.2** Let \( w_0 \in H^1_x \), \( \lambda_2 > 0 \). Then there exists a unique global solution \( w \) to (1.4) and moreover
\[
\| w \|_{L^{\frac{4n}{p}}_{t}L^{\frac{6n}{n}}_{x}(\mathbb{R} \times \mathbb{R}^n)} \leq C(\| w_0 \|_{L^2_x}) (1.8)
\]

Our second main theorem is:

**Theorem 1.2** *(Energy space scattering)*

Let \( u_0 \in H^1_x \), the conditions in Theorem 1.1 are assumed, and \( u \) be the unique solution to (1.1). In addition, if \( p = \frac{4}{n} \), then we need Assumption 1.1. If \( \gamma = 2 \), then we also need Assumption 1.2. Then in the following case, there exist \( u_+ \), \( u_- \in H^1_x \) such that
\[
\| u - e^{it\Delta}u_\pm \|_{H^1_x} \to 0 \quad \text{as} \ t \to \pm \infty. (1.9)
\]

**case 1:** \( \lambda_1 \), \( \lambda_2 > 0 \), \( \frac{4}{n} \leq p \leq \frac{4}{n-2} \), \( 2 \leq \gamma \leq 4 \) with \( \gamma < n \) except the point \( (p, \gamma) = (\frac{4}{n-2}, 4) \), especially, when \( (p, \gamma) = (\frac{4}{n}, 2) \), we still need the small mass condition;

**case 2:** \( \lambda_1 \cdot \lambda_2 < 0 \), \( \frac{4}{n} \leq p \leq \frac{4}{n-2} \), \( 2 \leq \gamma \leq 4 \) with \( \gamma < n \) and the small mass condition except the point \( (p, \gamma) = (\frac{4}{n-2}, 4) \).

Furthermore,
\[
\| u_+ \|_{L^2_x} = \| u_- \|_{L^2_x} = \| u_0 \|_{L^2_x} \quad \text{and} \quad \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_+|^2 = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla u_-|^2 = E(u_0)
\]

We’ll prove the theorem in Section 5. The main tools are a refined Morawetz estimate and the perturbation principles. Unfortunately, to use such a refined Morawetz estimate, we have to require that \( \lambda_1 > 0 \), \( \lambda_2 > 0 \), \( p > \frac{4}{n} \), \( \gamma > 2 \). So when \( \lambda_1 \cdot \lambda_2 < 0 \) need a kind of smallness, here we demand of mass sufficiently small. The refined Morawetz estimate was firstly used by T. Tao to prove the dispersive property of the cubic Schrödinger equation [4], but the space dimension must be no less than 3. Then, J. Colliander, M. Grillakis and N. Tzirakis get a refined Morawetz estimate for 1-D and 2-D, and obtain the scattering of 2-D power type Schrödinger equation. However, for this case \( \gamma < n = 2 \), in order to apply Morawetz estimate, we need \( \gamma > 2 \). Thus we can’t have scattering for Hartree, as well for (1.1). For \( p = \frac{4}{n} \) and \( \gamma = 2 \), i.e. both nonlinearities are mass-critical, the low frequency of the solution can own an effective control, but there is no such a good luck for the high one. Thus at this time, we view (1.1) as the perturbation of free Schrödinger equation.

At the last section, we describe the blow up phenomena when the initial datum belongs to \( \Sigma \) space. We believe the method also suitable for the initial datum belongs to energy space with radial condition. The detail can be consulted in Chapter 6 of [6].

Our last main theorem is:
Theorem 1.3 (blowup)
Let $u_0 \in \Sigma$. Then blowup occurs in each of the following cases:

1. for $\lambda_1 > 0$, $\lambda_2 < 0$: when $2 \leq \gamma \leq 4$, $0 < p \leq \frac{4}{n-2}$, $\gamma \geq \frac{np}{2}$, and $E < 0$;
2. for $\lambda_1 < 0$, $\lambda_2 > 0$: when $\frac{4}{n-2} \leq p \leq \frac{4}{n-2}$, $0 < \gamma \leq \frac{np}{2}$, and $E < 0$;
3. for $\lambda_1 < 0$, $\lambda_2 < 0$
   - when $\frac{4}{n} < p \leq \frac{4}{n-2}$, $0 < \gamma < 2$, and $4npE + C(M) < 0$.
   - when $0 < p < \frac{4}{n}$, $2 < \gamma \leq 4$, and $8\gamma E + C(M) < 0$.
   - when $\frac{4}{n} \leq p \leq \frac{4}{n-2}$, $2 \leq \gamma \leq 4$, and $E < 0$.

Remark 1.2 The conclusions in Theorem 1.1 and Theorem 1.3 aren’t contrary, since the energy in Theorem 1.1 are nonnegative. One can find that for the case $\lambda_1 < 0$, $\lambda_2 > 0$, we drop a situation:

\[ \frac{np}{2} < \gamma < 2 + n \frac{4-n}{p}, \]  
which was caused by that we could not judge the relationship between $\int (|x|^{-\gamma} |u|^2) |u|^2 dx$ and $\| u \|_{L_{p+2}^s}^{p+2}$. Since the inequality

$$ \| u \|_{L_{p-\gamma}^q} \lesssim \int (|x|^{-\gamma} |u|^2) |u|^2 dx, \| u \|_{L_{p+2}^s}^{p+2} \lesssim \| u \|_{L_p}^s $$

holds true where $q = \frac{2(4+n-\gamma)}{2+n-\gamma}$, $r = \frac{2n+2\gamma}{n}$. If we can get $\int (|x|^{-\gamma} |u|^2) |u|^2 dx \sim \| u \|_{L_p}^s$, for the situation $s > p + 2$, one can apply the method in Subsection 4.2 for case (2), to get the global well-posedness and scattering; for the other $s \leq p + 2$, one can apply the method in Section 6, to say under some condition, it would blow up in finite time.

2 Notation

In this section, we will introduce a few notations and fundamental inequalities which always appear in the following sections.

Definition 2.1: We say a pair $(q, r)$ is Schrödinger-admissible if $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ and $2 \leq q, r \leq \infty$. If $I \times \mathbb{R}^n$ is a spacetime slab, we define:

$$\| u \|_{\dot{S}^0(I \times \mathbb{R}^n)} := \sup \| u \|_{L_q^q L_{p}^p(I \times \mathbb{R}^n)},$$
where the sup is taken over all admissible pairs $(q, r)$,

$$\| u \|_{\dot{S}^1(I \times \mathbb{R}^n)} := \| \nabla u \|_{\dot{S}^0(I \times \mathbb{R}^n)}.$$

Denote $\dot{N}^0(I \times \mathbb{R}^n)$ the dual space of $\dot{S}^0(I \times \mathbb{R}^n)$, and

$$\dot{N}^1(I \times \mathbb{R}^n) := \{ u : \nabla u \in \dot{N}^0(I \times \mathbb{R}^n) \}.$$
We also define the following norms:

\[ \|u\|_{U(I)} := \|u\|_{L_t^1L_x^{\frac{6n}{4n-7}}(I \times \mathbb{R}^n)} \]
\[ \|u\|_{V(I)} := \|u\|_{L_t^{2(n+2)}L_x^{\frac{2n}{n+4}}(I \times \mathbb{R}^n)} \]
\[ \|u\|_{W(I)} := \|u\|_{L_t^{2(n+2)}L_x^{\frac{2n}{n+4}}(I \times \mathbb{R}^n)} \]
\[ \|u\|_{Z(I)} := \|u\|_{L_t^{n+1}L_x^{\frac{2(n+1)}{n}}(I \times \mathbb{R}^n)} \]

By definition and Sobolev embedding, we obtain

**Lemma 2.1** For any \( \dot{S}^1 \) function \( u \) on \( I \times \mathbb{R}^n \), we have

\[
\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^{2(n+2)}L_x^{\frac{2n}{n+4}}} + \|\nabla u\|_{L_t^2L_x^{\frac{2n}{n}}} + \|\nabla u\|_{L_t^{2(n+2)}L_x^{\frac{2n}{n+4}}} \leq \|u\|_{\dot{S}^1}, \tag{2.1}
\]

where all spacetime norms are on \( I \times \mathbb{R}^n \).

**Lemma 2.2** (Strichartz estimates) Let \( I \) be a compact time interval, \( k = 0, 1 \), and \( u : I \times \mathbb{R}^n \rightarrow \mathbb{C} \) be an \( \dot{S}^k \) solution to the forced Schrödinger equation

\[ iu_t + \Delta u = F \]

for a given function \( F \). Then we have

\[
\|u\|_{\dot{S}^k(I \times \mathbb{R}^n)} \lesssim \|u(t_0)\|_{\dot{H}^k(\mathbb{R}^n)} + ||F||_{\dot{N}^k(I \times \mathbb{R}^n)} \tag{2.2}
\]

for any time \( t_0 \in I \).

For the details of proof, we refer to [14, 6]. In addition, we need some Littlewood-Paley theory. Let \( \varphi(\xi) \) be a smooth bump function which is supported in the ball \( |\xi| \leq 2 \) and equal to 1 in the ball \( |\xi| \leq 1 \). For each dyadic number \( N \in 2\mathbb{Z} \), we can define the Littlewood-Paley operators:

\[
P_{\leq N}f(\xi) := \varphi\left(\frac{\xi}{N}\right)\hat{f}(\xi),
\]
\[
P_{> N}f(\xi) := [1 - \varphi\left(\frac{\xi}{N}\right)]\hat{f}(\xi),
\]
\[
P_Nf(\xi) := \left[\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right]\hat{f}(\xi).
\]

Then by these notations, we recall a few standard Bernstein type inequalities:

**Lemma 2.3** For any \( 1 \leq p \leq q \leq \infty, s > 0 \), we have

\[
\|P_{\geq N}f\|_{L_x^p} \lesssim N^{-s}\|\nabla|^sP_{\geq N}f\|_{L_x^p},
\]
\[
\|\nabla|^sP_{\leq N}f\|_{L_x^p} \lesssim N^s\|P_{\leq N}f\|_{L_x^p},
\]
\[
\|\nabla|^{+s}P_Nf\|_{L_x^p} \sim N^{+s}\|P_Nf\|_{L_x^p},
\]
\[
\|P_{\leq N}f\|_{L_x^p} \lesssim N^{\frac{q}{p} - \frac{q}{q}}\|P_{\leq N}f\|_{L_x^p},
\]
\[
\|P_Nf\|_{L_x^p} \lesssim N^{\frac{q}{p} - \frac{q}{q}}\|P_Nf\|_{L_x^p}.
\]
Definition 2.2 Let $I \times \mathbb{R}^n$ be an arbitrary spacetime slab, we define the space
\[
\dot{X}^0(I) = \begin{cases} 
L^p_t L^q_x (I \times \mathbb{R}^n), & \text{if } q = \frac{4(p+2)}{p(n-2)}, \\
L^p_t L^{\frac{2n}{n-2}}_x (I \times \mathbb{R}^n) \cap V(I), & \text{if } p = \frac{4}{n-2},
\end{cases}
\]
where $q = \frac{4(p+2)}{p(n-2)}$, $r = \frac{n(p+2)}{n+p}$, and
\[
\dot{X}^1 := \{ u : \nabla u \in \dot{X}^0(I) \}, \quad X^1(I) := \dot{X}^0(I) \cap \dot{X}^1(I),
\]
\[
\dot{Y}^0(I) := \begin{cases} 
L^\infty_t L^2_x (I \times \mathbb{R}^n), & \text{if } 0 < \gamma \leq 2, \\
L^\infty_t L^{\frac{2n}{n-2}}_x (I \times \mathbb{R}^n) \cap L^p_t L^\sigma_x (I \times \mathbb{R}^n), & \text{if } 2 < \gamma \leq 4 \text{ and } \gamma < n.
\end{cases}
\]
where $\mu = \frac{6}{7-2\gamma}, \sigma = \frac{6\gamma}{3n+4-2\gamma}$.

Furthermore, we also need the following maximal estimate which is a direct consequence of the sharp Hardy inequality \cite{11}.

Lemma 2.4 Let $0 < \gamma < n$, we have
\[
\| |x|^{-\gamma} * |u|^2 \|_{L^\infty_x} \leq C(n, \gamma) \| u \|_{H^\frac{\gamma}{2}}^2.
\] (2.3)

Lemma 2.5 Let $I$ be a compact time interval, $0 < p \leq \frac{4}{n-2}, 0 < \gamma \leq 4$ and $\gamma < n, \lambda_1$ and $\lambda_2$ be nonzero real numbers, and $k = 0, 1$. Then
\[
\| \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \\
\| \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u - (\lambda_1 |v|^p v + \lambda_2 (|x|^{-\gamma} * |v|^2) v) \|_{\dot{N}^0(I \times \mathbb{R}^n)} \\
\| I^{1-\frac{p(n-2)}{4}} \| u \|_{X^1(I)} \| u \|_{X^k(I)} + |I| \| u \|_{Y^1(I)} \| u \|_{Y^k(I)} \\
\| I^{1-\frac{p(n-2)}{4}} \| u \|_{X^1(I)} + \| v \|_{X^1(I)} \| u - v \|_{X^0(I)} + |I| \| u \|_{Y^1(I)} + \| v \|_{Y^1(I)} \| u - v \|_{Y^0(I)},
\] (2.4)

where $\alpha = \begin{cases} 
1 & \text{if } 0 < \gamma \leq 2, \\
2 - \frac{\gamma}{2} & \text{if } 2 < \gamma \leq 4 \text{ and } \gamma < n.
\end{cases}$
(2.5)

Proof. : Using Hölder, Sobolev embedding, Hardy-Littlewood-Sobolev inequality and Lemma 2.4, we can obtain the results. \[\square\]
Lemma 2.6 Let $k = 0, 1, \frac{4}{n} < p < \frac{4}{n-2}$ and $2 < \gamma < \min\{4, n\}$. Then there exists $\theta > 0$ large enough such that on each slab $I \times \mathbb{R}^n$, we have

$$
\| |u|^p u\|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u\|_{\dot{S}^k(I \times \mathbb{R}^n)} \| u\|_{L_t^{n+1} L_x^{2(2\theta + 1)}} \| u\|_{L_t^\infty L_x^\infty},
$$

(2.6)

$$
\| (|x|^{-\gamma} * |u|^2) u\|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u\|_{\dot{S}^k(I \times \mathbb{R}^n)} \| u\|_{L_t^{n+1} L_x^{2(2\theta + 1)}} \| u\|_{L_t^\infty L_x^\infty},
$$

(2.7)

where

$$
\alpha_1(\theta) = p(1 - \frac{n}{2}) + \frac{8\theta + 1}{2(2\theta + 1)}, \quad \alpha_2(\theta) = \frac{n}{2}\left(p - \frac{n + 8\theta + 2}{n(2\theta + 1)}\right),
$$

$$
\beta_1(\theta) = (3 - \gamma) + \frac{4\theta - 1}{2(2\theta + 1)}, \quad \beta_2(\theta) = (\gamma - 1) - \frac{4\theta + n}{2(2\theta + 1)}.
$$

Proof. For the former, one can find in [24]. The same method can be used for the latter, we have

$$
\| (|x|^{-\gamma} * |u|^2) u\|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| \nabla |u|^k (|x|^{-\gamma} * |u|^2) u\|_{L_t^2 L_x^2 (I \times \mathbb{R}^n)}
$$

$$
\lesssim \| \nabla |u|^k \|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2\theta(2\theta + 1)}{2n(2\theta + 1) - \theta}}} \| u\|_{L_t^{n+1} L_x^{2(2\theta + 1)}} \| u\|_{L_t^\infty L_x^\infty},
$$

(2.8)

which is obtained by using Hölder and Hardy-Littlewood-Sobolev inequality, once $\beta_1(\theta)$ and $\beta_2(\theta)$ are positive.

Note that \(\left(2 + \frac{1}{\theta}, \frac{2n(2\theta + 1)}{2n(2\theta + 1) - \theta}\right)\) is Schrödinger-admissible. When $2 < \gamma < 4$, $\beta_1(\theta)$ and $\beta_2(\theta)$ will be positive if $\theta$ is large enough, because the above functions are increased in $\theta$, and when $\theta \to \infty$,

$$
\beta_1(\theta) \to 4 - \gamma > 0, \quad \beta_2(\theta) \to \gamma - 2 > 0.
$$

Lemma 2.7 Let $I \times \mathbb{R}^n$ be a spacetime slab. Then there exists a small constant $0 < \rho < 1$ such that

$$
\| |u|^4 |u|^{-2} u\|_{\dot{N}^0(I \times \mathbb{R}^n)} \lesssim \| u\|^p_{Z(I)} \| u\|_{S^1(I \times \mathbb{R}^n)}^{\frac{\alpha_1(\theta) - \rho}{\alpha_2(\theta)}}
$$

(2.9)

$$
\| (|x|^{-4} * |u|^2) |u|^{-2} u\|_{\dot{N}^0(I \times \mathbb{R}^n)} \lesssim \| (|x|^{-4} * |u|^2) u\|_{L_t^2 L_x^2 (I \times \mathbb{R}^n)}
$$

$$
\lesssim \| u\|_{L_t^{2+\frac{1}{\theta}} L_x^{\frac{2\theta(2\theta + 1)}{2n(2\theta + 1) - \theta}}} \| u\|_{L_t^\infty L_x^\infty} \| u\|_{Z(I)}^{\frac{\beta_1(\theta) - \rho}{\beta_2(\theta)}} \| u\|_{S^1(I \times \mathbb{R}^n)}^{\frac{\alpha_1(\theta) - \rho}{\alpha_2(\theta)}}
$$

(2.10)

where $\rho = \frac{\varepsilon(n + 1)}{2(2 + \varepsilon)}$ and $\varepsilon$ is a small constant.
Proof. The first result is proved in [24]. For the other, note that $L^{2+\epsilon}_t L^{-\frac{2n}{n+2-\epsilon}}_x$ interpolates between the $S^0$-norm $L^{2+\epsilon}_t L^{-\frac{2n}{n+2-\epsilon}}_x$ and the $S^1$-norm $L^{2+\epsilon}_t L^{-\frac{2n}{n+2-\epsilon}}_x$ provided $\epsilon$ is sufficiently small, we have

$$\| u \|_{L^{2+\epsilon}_t L^{-\frac{2n}{n+2-\epsilon}}_x} \lesssim \| u \|_{S^1(I \times \mathbb{R}^n)} .$$

Let $a(\epsilon) = \frac{\epsilon(1+\epsilon)}{2(2+\epsilon)}$, $b(\epsilon) = 2 - \frac{\epsilon(n+2+\epsilon)}{2(2+\epsilon)}$, we only need to check $a(\epsilon)$ and $b(\epsilon)$ are positive, since then the estimates is a simple consequence of Hölder inequality and Hardy-Littlewood-Sobolev inequality. As a function of $\epsilon$, $a$ is increasing and $a(0) = 0$, while $b$ is decreasing and $b(0) = 2$. Thus, taking $\epsilon > 0$ sufficient small, we have $a(\epsilon) > 0$, $b(\epsilon) > 0$. Taking $\rho = \frac{\epsilon(n+1)}{2(2+\epsilon)}$, we obtain the result. \hfill \Box

Remark 2.1 An easy consequence of the proof of Lemma 2.7 is that one can get the estimates for nonlinearities of the form $|u|^{\frac{4}{n-2}} v$ and $(|x|^{-\gamma} |u|^2) v$. More precisely, we have

$$\| |u|^{\frac{4}{n-2}} v \|_{\dot{N}^0(I \times \mathbb{R}^n)} \lesssim \| u \|_{Z(I)}^{\frac{n-2}{2}} \| u \|_{S^1(I \times \mathbb{R}^n)}^{\frac{2}{2}} \| v \|_{S^1(I \times \mathbb{R}^n)},$$

(2.11)

$$\| (|x|^{-\gamma} |u|^2) v \|_{\dot{N}^0(I \times \mathbb{R}^n)} \lesssim \| u \|_{Z(I)}^{\frac{n-2}{2}} \| u \|_{S^1(I \times \mathbb{R}^n)}^{\frac{2}{2}} \| v \|_{S^1(I \times \mathbb{R}^n)},$$

(2.12)

$$\| (|x|^{-\gamma} (w v)) v \|_{\dot{N}^0(I \times \mathbb{R}^n)} \lesssim \| u \|_{S^1(I \times \mathbb{R}^n)} \| w \|_{L^p(I \times \mathbb{R}^n)}^{\frac{\alpha}{p(n)}} \| v \|_{Z(I)}^{\frac{n}{2}} \| w \|_{L^p(I \times \mathbb{R}^n)}^{\frac{2n}{2}} .$$

(2.13)

Lemma 2.8 Let $I \times \mathbb{R}^n$ be an arbitrary spacetime slab, $\frac{4}{n} \leq p \leq \frac{4}{n-2}$, $2 \leq \gamma \leq 4$ with $\gamma < n$, and $k = 0, 1$. Then

$$\| |u|^p u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u \|_{W(I)}^{\frac{2(p-\gamma)p}{2}} \| u \|_{W(I)}^{\frac{\gamma p-n}{2}} \| |u|^p u \|_{W(I)},$$

$$\| (|x|^{-\gamma} |u|^2) u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u \|_{U(I)}^{\frac{4-\gamma}{5}} \| u \|_{U(I)}^{\frac{\gamma^2}{5}} \| |u|^p u \|_{U(I)}^{\frac{4-\gamma}{5}} \| |u|^p u \|_{U(I)}^{\frac{\gamma^2}{5}} .$$

Proof. Note that

$$\| |u|^p u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| \nabla |k| (|u|^p u) \|_{L^2(I \times \mathbb{R}^n)}^{\frac{2(p-\gamma)p}{2}} \| |u|^p u \|_{L^2(I \times \mathbb{R}^n)}^{\frac{\gamma p-n}{2}} ,$$

$$\| (|x|^{-\gamma} |u|^2) u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| \nabla |k| (|x|^{-\gamma} |u|^2) u \|_{L^2(I \times \mathbb{R}^n)}^{\frac{2n}{2}} \| |u|^p u \|_{L^2(I \times \mathbb{R}^n)}^{\frac{\gamma^2}{5}} .$$

Then using Hölder inequality and interpolation, one can get the results. \hfill \Box

3 Local Theory

Let’s show the local theory for the initial value problem (1.1). As the results are classical, we prefer to omit the proofs and refer to [13] [6] [7] [12] [8].

Proposition 3.1 (Local well-posedness for (1.1) with $H^1_x$-subcritical nonlinearities)
Let $u_0 \in H_x^1, \lambda_1$ and $\lambda_2$ be nonzero real constants, with $0 < p < \frac{4}{n-2}, 0 < \gamma < \min \{n, 4\}$. Then, there exists $T = T(\| u \|_{H^1_x})$ such that $u$ with above parameters admits a unique strong $H^1_x$-solution $u$ on $[-T, T]$. Let $(-T_{\min}, T_{\max})$ be the maximal time interval on which the solution $u$ is well-defined. For every compact time interval $I \subset (-T_{\min}, T_{\max})$, we have $u \in S^1(I \times \mathbb{R}^n)$ and the following properties hold:

- If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then
  \[ \| u(t) \|_{H^1_x} \to \infty \text{ as } t \uparrow T_{\max} \text{ (respectively, as } t \downarrow -T_{\min}). \]

- The solution depends continuously on the initial value:
  There exists $T = T(\| u \|_{H^1_x})$ such that if $u_0^{(m)} \to u_0$ in $H^1_x$ and if $u^{(m)}$ is the solution to (1.1) with initial condition $u_0^{(m)}$, then $u^{(m)}$ is defined on $[-T, T]$ for $m$ sufficiently large and $u^{(m)} \to u$ in $S^1([-T, T] \times \mathbb{R}^n)$.

**Proposition 3.2 (Local well-posedness for (1.1) with a $H^1_x$-critical nonlinearity)**

Let $u_0 \in H^1_x, \lambda_1$ and $\lambda_2$ be nonzero real constants.

- when $p = \frac{4}{n-2}$, and $0 < \gamma < \min \{n, 4\}$, for every $T > 0$, there exists $\eta = \eta(T)$ such that if
  \[ \| e^{it\Delta} u_0 \|_{\dot{X}^1([-T, T])} \leq \eta, \]
  then (1.1) with the parameters given above admits a unique strong $H^1_x$-solution $u$ defined on $[-T, T]$;

- when $0 < p < \frac{4}{n-2}$, $\gamma = 4$ and $n \geq 5$, for every $T > 0$, there exists $\eta = \eta(T)$ such that if
  \[ \| e^{it\Delta} u_0 \|_{\dot{Y}^1([-T, T])} \leq \eta, \]
  then (1.1) with the parameters given above admits a unique strong $H^1_x$-solution $u$ defined on $[-T, T]$;

- Let $(-T_{\min}, T_{\max})$ be the maximal time interval on which the solution $u$ is well-defined. Then $u \in S^1(I \times \mathbb{R}^n)$ for each compact time interval $I \subset (-T_{\min}, T_{\max})$ and the following blow up alternative hold:
  If $T_{\max} < \infty$ (respectively, if $T_{\min} < \infty$), then
  either $\| u(t) \|_{H^1_x} \to \infty$ or $\| u(t) \|_{S^1([0,T] \times \mathbb{R}^n)} \to \infty$ as $t \uparrow T_{\max}$ (respectively, as $t \downarrow -T_{\min}$).

Next, we will establish the stability results for the $H^1_x$-critical and the $L^2_x$-critical NLS with Hartree type.

**Lemma 3.1 (Short-time perturbation)**

Let $I$ be a compact interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^n$ which is a near-solution to (1.2) in the sense that
\[ (i\partial_t + \Delta)\tilde{u} = \lambda(|x|^{-4} * |\tilde{u}|^2)\tilde{u} + \epsilon \]
for some function $e$. Suppose that we have the energy bound
\[ \| \tilde{u} \|_{L^\infty_t H^1(I \times \mathbb{R}^n)} \leq E \] (3.1)
for some $E > 0$.

Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that
\[ \| u(t_0) - \tilde{u}(t_0) \|_{H^1_I} \leq E' \] (3.2)
for some $E' > 0$. Assume also that we have the smallness conditions
\[ \| \nabla \tilde{u} \|_{U(I)} \leq \epsilon_0, \] (3.3)
\[ \| e^{i(t-t_0)} \Delta (u(t_0) - \tilde{u}(t_0)) \|_{U(I)} \leq \epsilon, \] (3.4)
\[ \| e \|_{\dot{N}^1(I \times \mathbb{R}^n)} \leq \epsilon, \] (3.5)
for some $0 < \epsilon < \epsilon_0$, where $\epsilon_0$ is a small constant $\epsilon_0 = \epsilon_0(E,E') > 0$.

We conclude that there exists a solution $u$ to (1.2) on $I \times \mathbb{R}^n$ with the special initial datum $u(t_0)$ at $t_0$, and furthermore,
\[ \| u - \tilde{u} \|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim E' + \epsilon, \] (3.6)
\[ \| u \|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim E' + E, \] (3.7)
\[ \| u - \tilde{u} \|_{L^6_t L^6_x(I \times \mathbb{R}^n)} \lesssim \epsilon, \] (3.8)
\[ \| (i\partial_t + \Delta)(u - \tilde{u}) \|_{\dot{N}^1(I \times \mathbb{R}^n)} \lesssim \epsilon. \] (3.9)

**Proof.** Without loss of generality, we assume $t_0 = \inf I$. Define $z = u - \tilde{u}$, then $u = z + \tilde{u}$
\[ S(t) := \| (i\partial_t + \Delta)z \|_{\dot{N}^1([t_0,t] \times \mathbb{R}^n)}. \]

By using Hölder, Hardy-Littlewood-Sobolev inequality, we have
\[ \| (|x|^{-4} \ast (ab)) c \|_{\dot{N}^1} \lesssim \| \nabla (|x|^{-4} \ast (ab)) c \|_{L^2_t L_x^{2n}} \lesssim \| \nabla a \|_{U(I)} \| \nabla b \|_{U(I)} \| \nabla c \|_{U(I)}, \] (3.10)
and from (3.3), (3.5) and (3.10), we have
\[ S(t) \lesssim \| |x|^{-4} \ast (|z|^2 + \tilde{z} \tilde{u} + \tilde{z} \tilde{u}) \|_{\dot{N}^1} + \| (|x|^{-4} \ast |\tilde{u}|^2) z \|_{\dot{N}^1} + \| e \|_{\dot{N}^1} \]
\[ \lesssim \epsilon + \sum_{j=0}^2 \| \nabla z \|_{U(I)} \| \nabla \tilde{u} \|^{3-j}_{U(I)} \]
\[ \lesssim \epsilon + \sum_{j=0}^2 \epsilon_0^{3-j} \| \nabla z \|_{U(I)}^j. \]

On the other hand, one has
\[ \| \nabla z \|_{U(I)} \lesssim \| e^{i(t-t_0)\Delta} \nabla z(t_0) \|_{U(I)} + S(t) \lesssim S(t) + \epsilon, \] (3.11)
and

\[ S(t) \leq \epsilon + \sum_{j=0}^{2} \epsilon_{0}^{3-j}(S(t) + \epsilon)^{j} \]

By a standard continuity method, one can show that \( S(t) \leq \epsilon \), then from (3.11) and Sobolev embedding, we get

\[ \| u - \hat{u} \|_{L_{t}^{6}L_{x}^{\infty}} \leq \epsilon \]

\[ \| \hat{u} \|_{\dot{H}^{1}} \leq \| \hat{u}(t_{0}) \|_{\dot{H}^{1}} + \| \nabla \hat{u} \|_{U(I)}^{3} + \| e \|_{N^{1}} \leq E + \epsilon_{0}^{3} + \epsilon \leq E \]

\[ \| u - \hat{u} \|_{\dot{H}^{1}} \leq \| u(t_{0}) - \hat{u}(t_{0}) \|_{\dot{H}^{1}} + S(t) \leq E' + \epsilon. \]

At last, we have

\[ \| u \|_{\dot{H}^{1}} \leq \| u - \hat{u} \|_{\dot{H}^{1}} + \| \hat{u} \|_{\dot{H}^{1}} \leq E + E'. \]

\[ \square \]

**Remark 3.1** If \( \| u(t_{0}) - \hat{u}(t_{0}) \|_{\dot{H}^{1}} \leq \epsilon_{0} \), then, thanks to the Strichartz estimate, we have

\[ \| e^{i(t-t_{0})A} \nabla(u(t_{0}) - \hat{u}(t_{0})) \|_{U(I)} \leq \| u(t_{0}) - \hat{u}(t_{0}) \|_{\dot{H}^{1}} \leq \epsilon_{0}. \]

Therefore, if \( E' \) is small, then (3.14) obviously holds true.

**Lemma 3.2** (\( H^{1}_{x} \)-critical stability result for Hartree type)

Let \( I \) be a compact interval, \( t_{0} \in I \), \( \hat{u} \) be a function on \( I \times \mathbb{R}^{n} \) which is a near-solution to \( (1.2) \) in the sense that

\[ (i\partial_{t} + \Delta)\hat{u} = \lambda(|x|^{-4} * |\hat{u}|^{2})\hat{u} + e \quad \text{for some function } e, \]

and \( u(t_{0}) \) be close to \( \hat{u}(t_{0}) \) in the sense that

\[ \| u(t_{0}) - \hat{u}(t_{0}) \|_{\dot{H}^{1}} \leq E' \quad \text{for some } E' > 0. \] (3.12)

Suppose that we have the energy bound

\[ \| \hat{u} \|_{L_{t}^{\infty}H^{1}(I \times \mathbb{R}^{n})} \leq E \quad \text{for some } E > 0, \] (3.13)

and we also have the following conditions

\[ \| \nabla \hat{u} \|_{U(I)} \leq M \quad \text{for some } M > 0, \] (3.14)

\[ \| e^{i(t-t_{0})A} \nabla(u(t_{0}) - \hat{u}(t_{0})) \|_{U(I)} \leq \epsilon, \] (3.15)

\[ \| e \|_{N^{1}(I \times \mathbb{R}^{n})} \leq \epsilon \] (3.16)

for some \( 0 < \epsilon < \epsilon_{0} \), where \( \epsilon_{0} \) is a small constant \( \epsilon_{0} = \epsilon_{0}(E, E', M) > 0 \).

Then, there exists a solution \( u \) to \( (1.2) \) on \( I \times \mathbb{R}^{n} \) with the special initial datum \( u(t_{0}) \) at \( t_{0} \), satisfying

\[ \| u - \hat{u} \|_{\dot{S}^{1}(I \times \mathbb{R}^{n})} \leq C(M, E)(E' + \epsilon), \] (3.17)

\[ \| u \|_{\dot{S}^{1}(I \times \mathbb{R}^{n})} \leq C(M, E', E), \] (3.18)

\[ \| u - \hat{u} \|_{L_{t}^{6}L_{x}^{\infty}(I \times \mathbb{R}^{n})} \leq C(M, E, E') \epsilon. \] (3.19)
Proof. Without loss of generality, we assume \( t_0 = \inf I \). Split \( I \) into \( J \) intervals \( I_j \), such that on each \( I_j \) we have
\[
\| \nabla \tilde{u} \|_{U(I_j)} \leq \epsilon_0, \quad \text{then} \quad J \sim \left(1 + \frac{M}{\epsilon_0}\right)^6.
\]

Fix \( I_0 = [t_0, t_1] \), thanks to the short-time perturbation, one can get
\[
\| u - \tilde{u} \|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \lesssim E' + \epsilon,
\]
\[
\| u \|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \lesssim E + E,
\]
\[
\| u - \tilde{u} \|_{L^6_tL^\infty_x(I_0 \times \mathbb{R}^n)} \lesssim \epsilon,
\]
\[
\| (i\partial_t + \Delta)(u - \tilde{u}) \|_{\dot{N}^1(I_0 \times \mathbb{R}^n)} \lesssim \epsilon.
\]

Furthermore, we have
\[
\| u(t_1) - \tilde{u}(t_1) \|_{\dot{H}^1_x} \lesssim \| u - \tilde{u} \|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \lesssim E' + \epsilon
\]

and
\[
\| e^{i(t-t_1)\Delta} \nabla(u(t_1) - \tilde{u}(t_1)) \|_{U(I_1)} \lesssim \| e^{i(t-t_0)\Delta} \nabla(u(t_0) - \tilde{u}(t_0)) \|_{U(I_1)}
\]
\[
+ \| (i\partial_t + \Delta)(u - \tilde{u}) \|_{\dot{N}^1(I_0 \times \mathbb{R}^n)} \lesssim \epsilon.
\]

Choosing \( \epsilon \) small enough, from the short-time perturbation, we have the results also hold on \( I_1 \), continuing the inductive argument, we get the above results at last.

Remark 3.2 In our lemmas, the condition (3.15) is weaker than the condition of what stated in [19], where they require that
\[
\left(\sum_N \| P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|^2_{U(I)}\right)^{\frac{1}{2}}
\]
\[
+ \left(\sum_N \| P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|^2_{L^6_tL^\infty_x(I \times \mathbb{R}^n)}\right)^{\frac{1}{2}} \leq \epsilon
\]

In fact, for Hartree type the nonlinearity and derivatives of the nonlinearity are Lipschitz continuity.

The same method can be used to prove the perturbation theory of the \( L^2_t \)-critical NLS with Hartree type. Note that, by Hölder, Hardy-Littlewood-Sobolev inequality, we have
\[
\| (|x|^{-2} \ast (ab)) c \|_{N^0} \lesssim \| (|x|^{-2} \ast (ab)) c \|_{L^2_tL^\frac{2n}{n+2}}
\]
\[
\lesssim \| a \|_{U(I)} \| b \|_{U(I)} \| c \|_{U(I)}.
\]

Instead of (3.10), by using a similar argument as above, we can get the following result:
Lemma 3.3 \((L^2_x\)-critical stability result for Hartree type\)

Let \(I\) be a compact interval, \(t_0 \in I\), \(\tilde{u}\) be a function on \(I \times \mathbb{R}^n\) which is a near-solution to (1.4) in the sense that

\[
(i\partial_t + \Delta)\tilde{u} = \lambda |x|^{-2} \ast |\tilde{u}|^2 \tilde{u} + e
\]

for some function \(e\), and \(u(t_0)\) be close to \(\tilde{u}(t_0)\) in the sense that

\[
\| u(t_0) - \tilde{u}(t_0) \|_{L^2_x(\mathbb{R}^n)} \leq M' \quad \text{for some } M' > 0. \tag{3.21}
\]

Suppose that we have the mass bound

\[
\| \tilde{u} \|_{L^\infty_t L^2_x(I \times \mathbb{R}^n)} \leq M \quad \text{for some } M > 0 \tag{3.22}
\]

and the following conditions hold true

\[
\| \tilde{u} \|_{U(I)} \leq L \quad \text{for some } L > 0 \tag{3.23}
\]

\[
\| e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0)) \|_{U(I)} \leq \epsilon \tag{3.24}
\]

\[
\| e \|_{X^0(I \times \mathbb{R}^n)} \leq \epsilon \tag{3.25}
\]

for some \(0 < \epsilon < \epsilon_1\), where \(\epsilon_1\) is a small constant, \(\epsilon_1 = \epsilon_1(M, M', L) > 0\).

Then, there exists a solution \(u\) to (1.4) on \(I \times \mathbb{R}^n\) with the special initial datum \(u(t_0)\) at \(t_0\), and

\[
\| u - \tilde{u} \|_{H^1_x(I \times \mathbb{R}^n)} \lesssim C(L, M, M')(M' + \epsilon), \tag{3.26}
\]

\[
\| u \|_{H^1_x(I \times \mathbb{R}^n)} \lesssim C(L, M, M'), \tag{3.27}
\]

\[
\| u - \tilde{u} \|_{U(I)} \lesssim C(L, M, M')\epsilon. \tag{3.28}
\]

The corresponding stability results for the \(H^1_x\)-critical and the \(L^2_x\)-critical NLS with power type have been established by [24, 25]. However, when the dimension \(n\) is greater than 6, the case is more delicate as derivatives of the nonlinearity are merely Hölder continuous of order \(\frac{4}{n-2}\) rather than Lipschitz. One can find the details in [24, 25], we state their result below:

Lemma 3.4 \((H^1_x\)-critical stability result for power type\) Let \(I\) be a compact interval, \(t_0 \in I\), \(\tilde{u}\) be a function on \(I \times \mathbb{R}^n\) which is a near-solution to (1.3) in the sense that

\[
(i\partial_t + \Delta)\tilde{u} = \lambda \tilde{u}^4 \tilde{u} + e \quad \text{for some function } e,
\]

and \(u(t_0)\) be close to \(\tilde{u}(t_0)\) in the sense that

\[
\| u(t_0) - \tilde{u}(t_0) \|_{H^1_x} \leq E'_0 \quad \text{for some } E'_0 > 0. \tag{3.29}
\]

Suppose that we have the energy bound

\[
\| \tilde{u} \|_{L^\infty_t H^1(I \times \mathbb{R}^n)} \leq E_0 \quad \text{for some } E_0 > 0 \tag{3.30}
\]
and the following conditions to be true

\[ \| \tilde{u} \|_{W(I)} \leq M_0 \quad \text{for some } M_0 > 0 \quad (3.31) \]

\[ \left( \sum_N \| P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|^2_{L^2_t \frac{2(n+2)}{n+4} \frac{2n(n+2)}{n^2+4} (I \times \mathbb{R}^n)} \right)^{\frac{1}{2}} \leq \epsilon \quad (3.32) \]

\[ \| e \|_{N^1(I \times \mathbb{R}^n)} \leq \epsilon \quad (3.33) \]

for some \( 0 < \epsilon < \epsilon_2 \), where \( \epsilon_2 = \epsilon_2(E_0, E_0', M_0) \) is a small constant.

Then, there exists a solution \( u \) to (1.3) on \( I \times \mathbb{R}^n \) with the special initial datum \( u(t_0) \) at \( t_0 \), and

\[ \| u - \tilde{u} \|_{S^1(I \times \mathbb{R}^n)} \lesssim C(E_0, E_0', M_0)(E_0' + \epsilon + \epsilon^{(n+2)^2} \), \quad (3.34) \]

\[ \| u \|_{S^1(I \times \mathbb{R}^n)} \lesssim C(M_0, E_0', E_0), \quad (3.35) \]

\[ \| u - \tilde{u} \|_{L^2_t \frac{2(n+2)}{n+4} \frac{2n(n+2)}{n^2+4} (I \times \mathbb{R}^n)} \lesssim C(M_0, E_0, E_0')(\epsilon + \epsilon^{(n+2)^2}) \quad (3.36) \]

**Remark 3.3** From [24] by Strichartz and Plancherel, on the slab \( I \times \mathbb{R}^n \) we have

\[ \left( \sum_N \| P_N \nabla e^{i(t-t_0)\Delta} (u(t_0) - \tilde{u}(t_0)) \|^2_{L^\infty_t L^2_x (I \times \mathbb{R}^n)} \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \sum_N \| P_N (u(t_0) - \tilde{u}(t_0)) \|^2_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \]

\[ \lesssim \| \nabla (u(t_0) - \tilde{u}(t_0)) \|^2_{L^\infty_t L^2_x} \]

\[ \lesssim E_0' \]

so the hypothesis (3.32) is redundant if \( E_0' \) is small.

**Lemma 3.5** \((L^2_x\text{-critical stability result for power type})\)

Let \( I \) be a compact interval, \( t_0 \in I, \tilde{u} \) be a function on \( I \times \mathbb{R}^n \) which is a near-solution to (1.5) in the sense that

\[ (i\partial_t + \Delta)\tilde{u} = \lambda |\tilde{u}|^4 \tilde{u} + e \quad \text{for some function } e, \]

and \( u(t_0) \) be close to \( \tilde{u}(t_0) \) in the sense that

\[ \| u(t_0) - \tilde{u}(t_0) \|_{L^2_x(\mathbb{R}^n)} \leq M_0' \quad \text{for some } M_0' > 0. \quad (3.37) \]

Suppose that we have the mass bound

\[ \| \tilde{u} \|_{L^\infty_t L^2_x(I \times \mathbb{R}^n)} \leq M_0 \quad \text{for some } M_0 > 0, \quad (3.38) \]
and the following conditions to be true
\[
\| \tilde{u} \|_{V(I)} \leq L_0 \quad \text{for some } L_0 > 0, \\
\| e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0)) \|_{V(I)} \leq \epsilon, \\
\| e \|_{N^0(I \times \mathbb{R}^n)} \leq \epsilon
\]
(3.39)
(3.40)
(3.41)

for some \(0 < \epsilon < \epsilon_3\), where \(\epsilon_3\) is a small constant \(\epsilon_3 = \epsilon_3(M_0, M'_0, L_0) > 0\).

Then, there exists a solution \(u\) to (1.5) on \(I \times \mathbb{R}^n\) with the special initial datum \(u(t_0)\) at \(t_0\), and furthermore,
\[
\| e_{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0)) \|_{V(I)} \leq \epsilon, \\
\| u \|_{\dot{S}^0(I \times \mathbb{R}^n)} \leq C(L_0, M_0, M'_0), \\
\| u - \tilde{u} \|_{V(I)} \leq C(L_0, M_0, M'_0)\epsilon.
\]
(3.42)
(3.43)
(3.44)

To conclude this section, we state the results involving persistence of \(L^2\) or \(\dot{H}^1\) regularity for critical NLS with Hartree type or power type:

**Lemma 3.6 (Persistence of regularity):** Let \(k = 0, 1\), and \(I\) be a compact interval, \(t_0 \in I\).

1. **Case 1:** \(u\) is a solution to (1.2) on \(I \times \mathbb{R}^n\) obeying the bounds
\[
\| u \|_{L^6_t L^{\infty}_x(I \times \mathbb{R}^n)} \leq M.
\]

Then, if \(u(t_0) \in \dot{H}^k\), we have
\[
\| u \|_{\dot{S}^k(I \times \mathbb{R}^n)} \leq C(M) \| u(t_0) \|_{\dot{H}^k}\]

2. **Case 2:** \(u\) is a solution to (1.4) on \(I \times \mathbb{R}^n\) obeying the bounds
\[
\| u \|_{U(I)} \leq L
\]

Then, if \(u(t_0) \in \dot{H}^k\), we have
\[
\| u \|_{\dot{S}^k(I \times \mathbb{R}^n)} \leq C(L) \| u(t_0) \|_{\dot{H}^k}\]

3. **Case 3:** \(u\) is a solution to (1.3) on \(I \times \mathbb{R}^n\) obeying the bounds
\[
\| u \|_{W(I)} \leq M.
\]

Then, if \(u(t_0) \in \dot{H}^k\), we have
\[
\| u \|_{\dot{S}^k(I \times \mathbb{R}^n)} \leq C(M) \| u(t_0) \|_{\dot{H}^k}\]

4. **Case 4:** Let \(u\) be a solution to (1.5) on \(I \times \mathbb{R}^n\) obeying the bounds
\[
\| u \|_{V(I)} \leq L
\]

Then, if \(u(t_0) \in \dot{H}^k\), we have
\[
\| u \|_{\dot{S}^k(I \times \mathbb{R}^n)} \leq C(L) \| u(t_0) \|_{\dot{H}^k}.
\]
Proof. The method to prove these four cases is similar, we only consider the first case, and the others are omitted.

Subdivide the interval \( I \) into \( N \sim (1 + \frac{M}{\eta})^6 \) subintervals \( I_j = [t_j, t_{j+1}] \) such that

\[
\| u \|_{L_t^6 L_x^{6/3} (I_j \times \mathbb{R}^n)} \leq \eta
\]

where \( \eta \) is a small positive constant to be chosen later. By using Strichartz estimates, on each \( I_j \) we obtain

\[
\| u \|_{\dot{S}^k (I_j \times \mathbb{R}^n)} \lesssim \| u(t_j) \|_{\dot{H}_x^k} + \| u \|_{\dot{S}^k (I_j \times \mathbb{R}^n)} \| u \|_{L_t^6 L_x^{6/3} (I_j \times \mathbb{R}^n)}^2 \lesssim \| u(t_j) \|_{\dot{H}_x^k} + \eta^2 \| u \|_{\dot{S}^k (I_j \times \mathbb{R}^n)}
\]

Choosing \( \eta \) sufficiently small, we get

\[
\| u \|_{\dot{S}^k (I_j \times \mathbb{R}^n)} \lesssim \| u(t_j) \|_{\dot{H}_x^k}.
\]

Next, we consider the relationship between \( \| u(t_j) \|_{\dot{H}_x^k} \) and \( \| u(t_0) \|_{\dot{H}_x^k} \). For \( I_0 \), we have

\[
\| u(t_1) \|_{\dot{H}_x^k} \leq \| u \|_{\dot{S}^k (I_0 \times \mathbb{R}^n)} \leq C \| u(t_0) \|_{\dot{H}_x^k}.
\]

For \( I_1 \), we have

\[
\| u(t_2) \|_{\dot{H}_x^k} \leq \| u \|_{\dot{S}^k (I_1 \times \mathbb{R}^n)} \leq C \| u(t_1) \|_{\dot{H}_x^k} \leq C^2 \| u(t_0) \|_{\dot{H}_x^k}
\]

by using iteration arguments, for each \( I_j \) we can obtain:

\[
\| u(t_j) \|_{\dot{H}_x^k} \leq C^j \| u(t_0) \|_{\dot{H}_x^k}
\]

Adding these estimates over all the subinterval \( I_j \), we can get the results.

\[\square\]

4 Global well-posedness

The aim of this section is to prove the Theorem 1.1. For the convenience, we shall abbreviate the energy \( E(u) \) to \( E \), and the mass \( M(u) \) to \( M \). In order to prove the global well-posedness of (1.1), we should state that the blowup couldn’t hold. For (1.1) with \( \dot{H}^1_x \)–subcritical nonlinearities, we should prove that \( \| u(t) \|_{\dot{H}^1_x} \) is bounded for all time where the solution is defined. We notice the conservation of mass, so we focus on the bounds of \( \| u(t) \|_{\dot{H}^1_x} \). For (1.1) with a \( \dot{H}^1_x \)–critical nonlinearity, we view the energy-subcritical nonlinearity as a perturbation to the energy-critical NLS, which is globally well-posed. For any compact interval \( I \), \( u \) is the strong solution to (1.1) which is defined on \( I \times \mathbb{R}^n \). \( u_0 \in H^1_x \) is the initial datum.
4.1 Bound State

Let \( R(x) \) and \( W(x) \) be the positive radial Schwartz solution of the ground state to the elliptic equations respectively:

\[
\Delta R + |R|^p R = \frac{4 - (n - 2)p}{np} R,
\]

\[
\Delta W + (|x|^{-\gamma} * |w|^2) W = \frac{4 - \gamma}{\gamma} W.
\]

From the work of [1, 16, 6] and [8], we have the following characterization of \( R \) and \( W \):

\[
\| u \|_{L^{p+2}}^{p+2} \leq C_R \| \nabla u \|_{L^2} \| u \|_{L^{2}}^{4-(n-2)p}, \quad \forall u, v \in H^1_x, \quad (4.1)
\]

\[
\| (|x|^{-\gamma} * |v|^2) |v|^2 \|_{L^2} \leq C_W \| \nabla v \|_{L^2} \| v \|_{L^{2}}^{4-\gamma}, \quad (4.2)
\]

where \( C_R \) and \( C_W \) is the best constant for their respective inequality, moreover

\[
C_R = \frac{2(p+2)}{np} \| \nabla R \|_{L^2}^{-p} = \frac{2(p+2)}{np} \| R \|_{L^{p}}^{-p},
\]

\[
C_W = \frac{4}{\gamma} \| \nabla W \|_{L^2}^{-2} = \frac{4}{\gamma} \| W \|_{L^{2}}^{-2}.
\]

If we define

\[
\tilde{E}(R) := \frac{1}{2} \int |\nabla R|^2 dx - \frac{1}{p+2} \int |R|^{p+2} dx, \]

\[
\tilde{E}(W) := \frac{1}{2} \int |\nabla W|^2 dx - \frac{1}{4} \int (|x|^{-\gamma} * |w|^2) |W|^2 dx,
\]

then, we have

\[
\tilde{E}(R) = \left( \frac{1}{2} - \frac{2}{np} \right) \int |\nabla R|^2 dx = \left( \frac{1}{2} - \frac{2}{np} \right) \left( \frac{2(p+2)}{npC_R} \right)^{\frac{p}{2}},
\]

\[
\tilde{E}(W) = \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int |\nabla W|^2 dx = \frac{2(\gamma-2)}{\gamma^2C_W}.
\]

Define \( E_1 := \frac{1}{2} \int |\nabla u|^2 dx - \frac{|\lambda_1|}{p+2} \int |u|^{p+2} dx \), where \( \lambda_1 \) is the constant in (1.1).

**Lemma 4.1** Assume that

\[
\| \nabla u \|_{L^2}^2 \left( \| u \|_{L^2}^2 \right)^{\frac{4-(n-2)p}{np-4}} < |\lambda_1|^{\frac{4}{1-np}} \| \nabla R \|_{L^2}^{\frac{4p}{np-4}},
\]

\[
E_1 \cdot (\| u \|_{L^2}^2) \left( \| u \|_{L^2}^2 \right)^{\frac{4-(n-2)p}{np-4}} \leq (1 - \delta_0)|\lambda_1|^{\frac{4}{1-np}} \left( \frac{2np}{np-4} \right)^{\frac{4-(n-2)p}{np-4}} \left( \tilde{E}(R) \right)^{\frac{2p}{np-4}}, \quad \text{where} \ \delta_0 > 0.
\]

Then, when \( \frac{4}{n} < p \leq \frac{4}{n-2} \), there exists \( \bar{\delta} = \bar{\delta}(\delta_0, n) > 0 \) such that

\[
\| \nabla u \|_{L^2}^2 \left( \| u \|_{L^2}^2 \right)^{\frac{4-(n-2)p}{np-4}} \leq (1 - \bar{\delta})|\lambda_1|^{\frac{4}{1-np}} \| \nabla R \|_{L^2}^{\frac{4p}{np-4}},
\]

and

\[
E_1 \geq 0.
\]
Proof. By (4.1), we get
\[ E_1 \geq \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{|\lambda_1|}{p + 2} C_R \| \nabla u \|_{L^2}^{\frac{np}{2}} \| u \|_{L^2}^{\frac{4 - (n-2)p}{2}}. \]
Let
\[ f(x) = \frac{1}{2} x - \frac{|\lambda_1|}{p + 2} C_R \| \nabla u \|_{L^2}^{\frac{np}{2}} \frac{np}{x^\frac{np}{2}}, \]
and \( a = \int |\nabla u|^2 \, dx \). Note that
\[ f'(x) = 0 \iff x = |\lambda_1|^{\frac{4}{4-np}} \| u \|_{L^2}^{\frac{2(4-np)}{np-4}} \| \nabla R \|_{L^2}^{\frac{4p}{np-4}} \left( \tilde{E}(R) \right)^{\frac{2p}{np-4}}, \]
using the fact that \( a \in [0, x_0) \), and the condition \( E_1 \leq (1 - \delta_0) f(x_0) \), we can get that there exists \( \bar{\delta} = \bar{\delta}(\delta_0, n) \) such that
\[ a \leq (1 - \bar{\delta}) x_0 \quad \text{and} \quad E_1 \geq f(a) \geq 0. \]
\[ Q.E.D. \]

Let's define \( E_2 := \frac{1}{2} \int |\nabla v|^2 \, dx - \frac{\lambda_2}{4} \int (|x|^{-\gamma} * |v|^2) |v|^2 \, dx \), where \( \lambda_2 \) is the constant in (1.1). The same result can be gotten for \( W(x) \):

Lemma 4.2 Assume that
\[ \| \nabla v \|_{L^2} \left( \| v \|_{L^2} \right)^{\frac{4-n}{4-n-2}} < \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{2-n}}, \]
\[ E_2 \cdot (\| v \|_{L^2}^{\frac{4-n}{4-n-2}} \leq (1 - \delta_0) \left( \frac{1}{2} - \frac{1}{\gamma} \right) \left[ \frac{2\gamma \tilde{E}(W)}{|\lambda_2|(-\gamma-2)} \right]^{\frac{2}{2-n}}, \]
where \( \delta_0 > 0 \). Then, when \( 2 < \gamma \leq 4 \), there exists a \( \tilde{\delta} = \tilde{\delta}(\delta_0, n) > 0 \) such that
\[ \| \nabla v \|_{L^2} \left( \| v \|_{L^2} \right)^{\frac{4-n}{4-n-2}} \leq (1 - \tilde{\delta}) \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{2-n}} \]
and \( E_2 \geq 0 \).

4.2 Kinetic energy control

We'll get a prior control on the kinetic energy, which is bounded for all time for which the solution is defined. More precisely, the bound is only concerned with energy and mass, i.e.
\[ \| u(t) \|_{\dot{H}^1_x} \leq C(E, M). \]
We observe the energy

\[ E(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{\lambda_1}{p+2} \int |u|^{p+2} \, dx + \frac{\lambda_2}{4} \int (|x|^{-\gamma} |u|^2) |u|^2 \, dx \]

is conserved. Hence, for the case (1), we obviously obtain

\[ \| u(t) \|_{H^1} \lesssim E. \]

For the case (2), from Parseval identity, Hardy-Littlewood-Sobolev inequality and interpolation, we have

\[ \int (|x|^{-\gamma} |u|^2) |u|^2 \, dx = \int (|x|^{-(n-\gamma)} |u|^2) |u|^2 \, dx = \| \nabla |^{\frac{n-\gamma}{2}} |u|^2 \|_{L^2}^2 \]

\[ \leq \| u \|_{L^{\frac{2n+2n}{n-\gamma}}} \leq \| u \|_{L^2}^{4(1-\frac{n+\gamma}{2n})} \| u \|_{L^{\frac{2n+2n}{n-\gamma}}} \| u \|_{L^{\frac{4n+4n}{n}}}. \quad (4.4) \]

Based on the fact: for any positive constants \( a, \delta, \) and \( p_1 < p_2, \) the following inequality

\[ a^{p_1+2} \leq C(\delta) a^2 + \delta a^{p_2+2} \]

holds true, we can get

\[ \| u \|_{L^{\frac{2n+2n}{n}}} \leq C(\delta) \| u \|_{L^2}^2 + \delta \| u \|_{L^{p_2+2}}^{p_2+2} \]

if \( \frac{2n+2n}{n} < p + 2, \) i.e. \( \gamma < \frac{np_2}{2}. \) Then

\[ E(u) \geq \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{\lambda_1}{p+2} \int |u|^{p+2} \, dx - C \| u \|_{L^2}^{4(1-\frac{n+\gamma}{2n})} \delta \int |u|^{p+2} \, dx - C(M). \]

Let \( \delta \) be small enough, we have

\[ \| u(t) \|_{H^1} \leq C(E, M). \]

If \( \gamma \geq \frac{np_2}{2}, \) by using \( \lambda_1 > 0 \) and (4.2), we can obtain

\[ E \geq E_1 \geq \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{|\lambda_2|}{4} C_W \| \nabla u \|_{L^2}^2 \| u \|_{L^{\gamma}}^{4-\gamma}. \]

For the case \( \gamma < 2, \) from Young’s inequality, one has

\[ E \geq \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{|\lambda_2|}{4} \delta C_W \| \nabla u \|_{L^2}^2 - \frac{|\lambda_2|}{4} C_W C(\delta) \| u \|_{L^{\frac{2(4-\gamma)}}{2}}}^{2(4-\gamma)}. \]

Let \( \delta \) be small enough, we obtain

\[ \| u(t) \|_{H^1} \leq C(E, M). \]

When \( \gamma = 2, \) we have

\[ E \geq \left( \frac{1}{2} - \frac{|\lambda_2|}{4} C_W \| u \|_{L^2}^2 \right) \| \nabla u \|_{L^2}^2. \]

If

\[ \| u \|_{L^2}^2 < \frac{2}{C_W |\lambda_2|} = \frac{1}{|\lambda_2|} \| W \|_{L^2}^2 \]

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holds true, we can obtain
\[ \| u(t) \|_{\dot{H}^1_x} \leq C(E, M). \]

For the case \( 2 < \gamma \leq 4 \), by using Lemma 4.2 and the conservation of energy and mass, we only need to show when
\[ \| \nabla u_0 \|_{L^2}^2 (\| u_0 \|_{L^2}^2)^{\frac{4-\gamma}{2}} < \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}}, \]
we can get
\[ \| \nabla u \|_{L^2}^2 (\| u \|_{L^2}^2)^{\frac{4-\gamma}{2}} < \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}}. \]
We prove it by the continuity argument. Define
\[ \Omega = \left\{ t \in I, \| \nabla u \|_{L^2}^2 (\| u \|_{L^2}^2)^{\frac{4-\gamma}{2}} < \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}} \right\}, \]
It suffices to show \( \Omega \) is both open and closed. Note that \( t_0 \in \Omega \), the open of \( \Omega \) is obvious because of \( u \in C^0_t(I, \dot{H}^1_x) \). Therefore, we only need to prove \( \Omega \) is closed. For any \( t_n \in \Omega, T \in I \), such that \( t_n \to T \), we have
\[ \| \nabla u(t_n) \|_{L^2}^2 M^{\frac{4-\gamma}{2}} < \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}}, \]
\[ E(u(t_n)) M^{\frac{4-\gamma}{2}} \leq (1 - \delta_0)(\frac{1}{2} - \frac{1}{\gamma}) \left[ \frac{2\gamma \tilde{E}(W)}{|\lambda_2|(|\gamma - 2|)} \right]^{\frac{2}{\gamma-2}}. \]
By using Lemma 4.2, we can get
\[ \| \nabla u(t_n) \|_{L^2}^2 M^{\frac{4-\gamma}{2}} \leq (1 - \delta) \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}}, \]
Since \( u \in C^0_t(I, \dot{H}^1_x) \), the conservation of energy and mass, we get
\[ \| \nabla u(T) \|_{L^2}^2 M^{\frac{4-\gamma}{2}} \leq (1 - \delta) \left( \frac{\| \nabla W \|_{L^2}^2}{|\lambda_2|} \right)^{\frac{2}{\gamma-2}}, \]
\[ E(u(T)) M^{\frac{4-\gamma}{2}} \leq (1 - \delta_0)(\frac{1}{2} - \frac{1}{\gamma}) \left[ \frac{2\gamma \tilde{E}(W)}{|\lambda_2|(|\gamma - 2|)} \right]^{\frac{2}{\gamma-2}}. \]
This implies that \( T \in \Omega \) and \( \| u(t) \|_{\dot{H}^1_x} \leq C(E, M) \).
Remark 4.1 When \( \gamma = \frac{np}{2} \), we have

\[
E \geq \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{1}{p+2} \| u \|_{L^{p+2}}^{p+2} - C\frac{\lambda_2}{4} M^{2-p} \| u \|_{L^{p+2}}^{p+2}.
\]

The condition \( n > \gamma = \frac{np}{2} \) implies \( p < 2 \). If requiring \( \| \lambda_1 \|_{p+2} > C \frac{\lambda_2}{4} M^{2-p} \), we also can obtain \( \| u(t) \|_{\dot{H}^1} \leq C(E, M) \).

To prove the case 3, we need the following lemma before getting the prior control on the kinetic energy:

Lemma 4.3

\[
\| |\nabla|^{-\frac{n-\gamma}{4}} f \|_{L^4} \lesssim \| |\nabla|^{-\frac{n-\gamma}{2}} |f|^2 \|_{L^2}^{1/2}.
\] (4.6)

Remark 4.2 T. Tao proved the inequality for \( \gamma = 3 \) in [24]. We can use the same method to get (4.6).

Proof. It suffices to prove (4.6) for a positive Schwartz function \( f \). In fact, we only need to prove the pointwise inequality

\[
S(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) \lesssim \left[ (|\nabla|^{-\frac{n-\gamma}{2}} |f|^2(x)) \right]^{1/2},
\] (4.7)

where \( Sf := (\sum_N |P_N f|^2)^{1/2} \).

Obviously, (4.7) implies (4.6).

\[
\| |\nabla|^{-\frac{n-\gamma}{4}} f \|_{L^4} \lesssim \| S(|\nabla|^{-\frac{n-\gamma}{4}} f) \|_{L^4} \lesssim \| (|\nabla|^{-\frac{n-\gamma}{2}} |f|^2) \|_{L^4} \| |\nabla|^{-\frac{n-\gamma}{2}} |f|^2 \|_{L^2}^{1/2}.
\]

Subsequently, we’ll focus our attention to the estimate for each of the dyadic pieces

\[
P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) = \int e^{2\pi i x \cdot \xi} \hat{f}(\xi)(|\xi|^{-\frac{n-\gamma}{4}} m(\frac{\xi}{N})) d\xi,
\]

where \( m(\xi) := \varphi(\xi) - \varphi(2\xi) \) in the notation introduced in Section 2.

As \( |\xi|^{-\frac{n-\gamma}{4}} m(\frac{\xi}{N}) \sim N^{-\frac{n-\gamma}{4}} m(\frac{\xi}{N}) \), we have

\[
P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x) \sim N^{\frac{3(n-\gamma)}{4}} f \ast \hat{m}(N x) = N^{\frac{3(n-\gamma)}{4}} \int f(x-y) \hat{m}(N y) \, dy.
\]

Since \( m \) is a Schwartz function, we have

\[
|P_N(|\nabla|^{-\frac{n-\gamma}{4}} f)(x)| \lesssim N^{\frac{3(n-\gamma)}{4}} \int_{|y| \leq N^{-1}} f(x-y) \, dy + N^{\frac{3(n-\gamma)}{4}} \int_{|y| > N^{-1}} f(x-y) \frac{1}{|N y|^\beta} \, dy,
\]

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where $\beta$ is chosen later.
A simple application of Cauchy-Schwartz yields

$$S(|\nabla|^{-\frac{n-\gamma}{4}}f)(x) = \left( \sum_N |P_N(|\nabla|^{-\frac{n-\gamma}{4}}f)(x)|^2 \right)^{\frac{1}{2}}$$

$$\lesssim \left( \sum_N N^{-\frac{3n+\gamma}{2}} \int_{|y|\leq N^{-1}} f(x-y) \, dy \right)^{\frac{1}{2}}$$

$$\lesssim \left[ \sum_N N^{-\frac{3n+\gamma}{2}} N^{-n} \int_{|y|\leq N^{-1}} |f(x-y)|^2 \, dy + \sum_N N^{-\frac{3n+\gamma}{2}} \int_{|y|>N^{-1}} f(x-y) \frac{1}{|Ny|^{2\beta}} \, dy \right]^{\frac{1}{2}}$$

where $\alpha$ is decided later.

Note that

$$\sum_N N^{-\frac{3n+\gamma}{2}} \sum_{|y|\leq N^{-1}} \chi_{\{y\leq N^{-1}\}}(y) \lesssim \sum_{|y|\leq N^{-1}} |y|^{-\frac{n-\gamma}{4}}$$

$$\sum_N N^{-\frac{3n+\gamma}{2}} \left( \int_{|y|>N^{-1}} \frac{|y|^\alpha}{|Ny|^{2\beta}} \, dy \right) \chi_{\{|y|>N^{-1}\}}(y) \lesssim \sum_{|y|>N^{-1}} N^{-2\beta} N^{-(n+\alpha-2\beta)}$$

where choosing $\alpha$ and $\beta$ to satisfy that $n+\alpha-2\beta < 0, \frac{2+n}{2} - \alpha < 0$, we obtain

$$S(|\nabla|^{-\frac{n-\gamma}{4}}f)(x) \lesssim \left( \int_{|y|\leq N^{-1}} \frac{|f(x-y)|^2}{|y|^{\frac{\alpha}{2}}} \, dy + \int_{|y|>N^{-1}} |f(x-y)|^2 \frac{1}{|y|^{\frac{\alpha}{2}}} \, dy \right)^{\frac{1}{2}}$$

$$\lesssim \left( \int |f(x-y)|^2 \, dy \frac{1}{|y|^{\frac{n+\gamma}{4}}} \right)^{\frac{1}{2}}$$

$$\lesssim \left( |(\nabla)^{-\frac{n+\gamma}{4}} |f|^2(x) \right)^{\frac{1}{2}},$$

and we complete the proof. \(\Box\)

Using interpolation and Young’s inequality, we get

$$\| u \|_{L^q} \lesssim \| \nabla u \|_{L^2}^{\frac{2(n+\gamma)}{n+\gamma-q}} \| |\nabla|^{-\frac{n+\gamma}{4}} u \|_{L^4}^{\frac{8}{n+\gamma-q}} \lesssim \varepsilon \| \nabla u \|_{L^2}^2 + C(\varepsilon) \| |\nabla|^{-\frac{n+\gamma}{4}} u \|_{L^4}^4,$$

where $q = \frac{2(4+n-\gamma)}{2+n-\gamma}$. Then,

$$\| |\nabla|^{-\frac{n+\gamma}{4}} u \|_{L^4}^4 \geq c(\varepsilon) \| u \|_{L^q}^q - c(\varepsilon) \| \nabla u \|_{L^2}^2.$$

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On the other hand, In view of

$$\frac{|\lambda_2|}{4} \| |\nabla|^{\frac{n-\gamma}{2}} u|^2\|_{L^2} \geq \| |\nabla|^{-\frac{n-\gamma}{2}} u\|_{L^{q'}}^4 \geq c(\varepsilon) \| u\|_{L^4}^4 - c(\varepsilon) \| \nabla u\|_{L^2}^2$$

from (4.6), and

$$\| u\|_{L^4}^{p+2} \leq C(\delta) \| u\|_{L^2}^2 + \delta \| u\|_{L^q}^4$$

from (4.5), We have

$$E \geq \frac{1}{2} \int |\nabla u|^2 \, dx + c(\varepsilon) \| u\|_{L^4}^4 - c(\varepsilon) \| \nabla u\|_{L^2}^2 - \frac{|\lambda_1|}{p+2} \| u\|_{L^q}^q - \frac{|\lambda_1|}{p+2} C(\delta) \| u\|_{L^2}^2.$$

Choosing $\varepsilon$ and $\delta = \delta(\varepsilon)$ be small enough, we obtain

$$\| u(t) \|_{\dot{H}^1_x} \leq C(E, M).$$

If $p \geq \frac{4}{n+3-\gamma}$, notice $\lambda_2 > 0$ and (1.1), using the identical method which is applied for case (2), under the conditions of case (3) we have

$$\| u(t) \|_{\dot{H}^1_x} \leq C(E, M).$$

For the case (4), by using (4.1), (4.2) and Young’s inequality, we have

$$E \geq \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{|\lambda_1|}{p+2} C_R \| \nabla u\|_{L^2}^2 - \frac{|\lambda_2|}{4} C_W \| u\|_{L^2}^{4-\gamma} \| \nabla u\|_{L^2}^\gamma$$

$$\geq \frac{1}{2} \int |\nabla u|^2 \, dx - \frac{|\lambda_1|}{p+2} C_R \| \nabla u\|_{L^2}^2 - \frac{|\lambda_2|}{4} C_W \delta \| \nabla u\|_{L^2}^2 - C(M).$$

Chosen $\delta$ to be sufficiently small, we obtain

$$\| u(t) \|_{\dot{H}^1_x} \leq C(E, M).$$

### 4.3 Global well-posedness

In this subsection, we’ll complete the proof of Theorem 1.1. As mentioned above, when both nonlinearities are $\dot{H}^1_x$-subcritical, according to the Proposition 3.1, the prior control on the kinetic and the conservation of mass, we can conclude the unique strong solution $u$ to (1.1) is a global solution. More precisely, in this situation, we can find $T = T(\| u_0 \|_{\dot{H}^1_x})$ such that (1.1) admits a unique strong solution $u \in S^1([-T, T] \times \mathbb{R}^n)$ and

$$\| u \|_{S^1([-T, T] \times \mathbb{R}^n)} \leq C(E, M).$$

If we subdivide the interval $I$ into subintervals of length $T$, deriving the corresponding $S^1$-bounds on each of these subintervals, and at last summing these dominate together, then we can get the bound (1.6).

When one of the nonlinearities is $\dot{H}^1_x$-critical, we view the other nonlinearity as a perturbation to the energy-critical NLS, which is globally wellposed, [15] [19] [23] [8] [26] [17]. Here we only discuss the case: $p = \frac{4}{n-2}$, $0 < \gamma < \min \{n, 4\}$, the same method can be used for the other case:
0 < p < \frac{4}{n-2}, \gamma = 4 with n \geq 5. Through the proof, we can find by Strichartz estimates and Hölder inequality, we need the coefficient of subcritical nonlinearity including T = T(E, M), which will be required small in order to apply the standard continuity argument. So our approach don’t fit the case that both nonlinearities are $H_x^{1}$-critical.

Let v be the unique strong global solution to the energy-critical equation \(1.3\) with initial datum \(v_0 = u_0\) at time \(t = 0\). By the main result in [15, 23, 3, 26, 17], we know that such a \(v\) exists and

$$\| v \|_{\dot{H}^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(\| u_0 \|_{\dot{H}^1_x}).$$

Further, by Lemma 3.6 we also have

$$\| v \|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \leq C(\| u_0 \|_{\dot{H}^1_x} \| u_0 \|_{L^2} \leq C(E, M)).$$

By time reversal symmetry, it suffices to solve the problem forward in time. By (4.8), split \(\mathbb{R}^+\) into \(J = J(E, \eta)\) subintervals \(I_j = [t_j, t_{j+1}]\) such that

$$\| v \|_{\dot{B}^1(I_j)} \sim \eta$$

for some small \(\eta\) to be chosen later.

We may assume that there exits \(J' < J\) such that for any \(0 \leq j \leq J' - 1\), \([0, T] \cap I_j \neq \emptyset\).

Thus, we can write

$$[0, T] = \bigcup_{j=0}^{J'-1} ([0, T] \cap I_j).$$

According to the Strichartz estimate, Sobolev embedding and (4.9), we have the free evolution \(e^{i(t-t_j)\Delta} v(t_j)\) is small on \(I_j\)

$$\| e^{i(t-t_j)\Delta} v(t_j) \|_{\dot{B}^1(I_j)} \leq \| v \|_{\dot{B}^1(I_j)} + \| \nabla \left( |v|^{\frac{4}{n-2}} v \right) \|_{L^{2(n+2)}_{t,x} \dot{X}^{\frac{n+2}{2}}(I_j \times \mathbb{R}^n)}$$

$$\leq \| v \|_{\dot{B}^1(I_j)} + C \| v \|_{\dot{X}^{\frac{n+2}{2}}(I_j)}$$

$$\leq \| v \|_{\dot{B}^1(I_j)} + C \| v \|_{\dot{B}^1(I_j)}$$

$$\leq \eta + C\eta^{\frac{n+2}{n-2}}.$$

Thus, taking \(\eta\) sufficiently small, for any \(0 \leq j \leq J' - 1\), we obtain

$$\| e^{i(t-t_j)\Delta} v(t_j) \|_{\dot{B}^1(I_j)} \leq 2\eta$$

On the interval \(I_0\), recalling that \(u(0) = v(0) = u_0\), we estimate

$$\| u \|_{\dot{X}^1(I_0)} \leq \| e^{it\Delta} u_0 \|_{\dot{X}^1(I_0)} + C |I_0|^\alpha \| u \|_{Y^1(I_0)} + C \| u \|_{\dot{X}^1(I_0)}$$

$$\| u \|_{Y^1(I_0)} \leq \| e^{it\Delta} u_0 \|_{Y^1(I_0)} + C |I_0|^\alpha \| u \|_{\dot{X}^1(I_0)}$$

then

$$\| u \|_{\dot{B}^1(I_0)} \leq 2\eta + CT^\alpha \| u \|_{\dot{B}^1(I_0)} + C \| u \|_{\dot{X}^1(I_0)}$$
where \( \alpha = \min \{1, 2 - \frac{n}{2}\} \).

Assuming \( \eta \) and \( T \) are sufficiently small, a standard continuity argument then yields

\[
\| u \|_{\dot{B}^1(I_0)} \leq 4\eta.
\]

In order to use Lemma 3.4, we notice that (3.31) holds on \( I := I_0 \) for \( M_0 := 4C\eta \), (3.30) holds for \( E_0 := C(E, M) \). Also, (3.29) holds with \( E'_0 = 0 \). We only prove that the error, which in this case is the second nonlinearity, is sufficiently small.

In fact

\[
\| \nabla e \|_{\dot{N}^0(I_0 \times \mathbb{R}^n)} \lesssim T^\alpha \| u \|_{\dot{Y}^1(I_0)}^3 \lesssim T^\alpha \| u \|_{\dot{B}^1(I_0)}^3 \lesssim T^\alpha \eta^3.
\]

We see that by choosing \( T \) sufficiently small, we get

\[
\| \nabla e \|_{\dot{N}^0(I_0 \times \mathbb{R}^n)} < \epsilon,
\]

where \( \epsilon = \epsilon(E, M) \) is a small constant to be chosen later. Thus, taking \( \epsilon \) sufficiently small, the hypothesis of Lemma 3.4 are satisfied, which implies that

\[
\| u - v \|_{S^1(I_0 \times \mathbb{R}^n)} \leq C(E, M)\epsilon^c
\]

(4.10)

for a small positive constant \( c \) which depends only on the dimension \( n \).

Strichartz estimates and (4.10) imply

\[
\| u(t_1) - v(t_1) \|_{H^1_x} \leq C(E, M)\epsilon^c,
\]

(4.11)

\[
\| e^{i(t-t_1)\Delta} (u(t_1) - v(t_1)) \|_{\dot{B}^1(I_1)} \leq C(E, M)\epsilon^c.
\]

(4.12)

By using (4.11), (4.12) and Strichartz estimates, we can get

\[
\| u \|_{\dot{B}^1(I_1)} \leq \| e^{i(t-t_1)\Delta} u(t_1) \|_{\dot{B}^1(I_1)} + C T^\alpha \| u \|_{\dot{B}^1(I_1)} + C \| u \|_{\dot{B}^{n+2}\dot{B}^1(I_1)}
\]

\[
\leq \| e^{i(t-t_1)\Delta} v(t_1) \|_{\dot{B}^1(I_1)} + \| e^{i(t-t_1)\Delta} (u(t_1) - v(t_1)) \|_{\dot{B}^1(I_1)}
\]

\[
+ C T^\alpha \| u \|_{\dot{B}^1(I_1)} \leq 2\eta + C(E, M)\epsilon^c + C T^\alpha \| u \|_{\dot{B}^1(I_1)} + C \| u \|_{\dot{B}^{n+2}\dot{B}^1(I_1)}.
\]

A standard continuity method then yields

\[
\| u \|_{\dot{B}^1(I_0)} \leq 4\eta
\]

provided \( \epsilon \) is chosen sufficiently small depending on \( E \) and \( M \), which amounts to taking \( T \) sufficiently small depending on \( E \) and \( M \). We apply Lemma 3.4 again on \( I := I_1 \) to obtain

\[
\| u - v \|_{S^1(I_1 \times \mathbb{R}^n)} \leq C(E, M)\epsilon^c.
\]

By induction argument, for every \( 0 \leq j \leq J' - 1 \), we obtain

\[
\| u \|_{\dot{B}^1(I_j)} \leq 4\eta
\]

(4.13)
provided \(\varepsilon\) (and hence \(T\)) is sufficiently small depend on \(E\) and \(M\). Adding (4.13) over all \(0 \leq j \leq J' - 1\) and recalling that \(J' < J = J(E, M)\), we obtain

\[
\| u \|_{\dot{B}^1([0,T])} \leq 4J' \eta \leq C(E, M). \tag{4.14}
\]

Using Strichartz estimates, (2.4), (4.14) and \(T = T(E, M)\), we get

\[
\| u \|_{\dot{S}^1([0,T] \times \mathbb{R}^n)} \lesssim \| u_0 \| + T^\alpha \| u \|_{\dot{B}^1([0,T])}^2 + \| u \|_{\dot{B}^1([0,T])}^\frac{n+2}{n} \leq C(E, M). \tag{4.15}
\]

Similarly,

\[
\| u \|_{\dot{S}^0([0,T] \times \mathbb{R}^n)} \lesssim \| u_0 \| + T^\alpha \| u \|_{\dot{B}^1([0,T])}^2 + \| u \|_{\dot{X}^0([0,T])} \leq M^\frac{1}{2} + C(E, M) \| u \|_{\dot{B}^1([0,T])}^2 + \| u \|_{\dot{S}^0([0,T])} + \| u \|_{\dot{B}^1([0,T])} \| u \|_{\dot{S}^0([0,T])}.
\]

Split \([0,T]\) into \(N = N(E, M, \delta)\) subintervals \(J_k\) such that

\[
\| u \|_{\dot{B}^1(J_k)} \sim \delta
\]

for some small constant \(\delta > 0\) to be chosen later. Thus we get

\[
\| u \|_{\dot{S}^0(J_k \times \mathbb{R}^n)} \lesssim M^\frac{1}{2} + C(E, M) \delta^2 \| u \|_{\dot{S}^0(J_k \times \mathbb{R}^n)} + \delta^\frac{1}{4-n} \| u \|_{\dot{S}^0(J_k \times \mathbb{R}^n)}.
\]

Choosing \(\delta\) sufficiently small, a standard continuity method then implies

\[
\| u \|_{\dot{S}^0(J_k \times \mathbb{R}^n)} \leq C(E, M).
\]

Adding these bounds over all subintervals \(J_k\), we get

\[
\| u \|_{\dot{S}^0([0,T] \times \mathbb{R}^n)} \leq C(E, M). \tag{4.16}
\]

Combine (4.15) and (4.16), we get

\[
\| u \|_{\dot{S}^1([0,T] \times \mathbb{R}^n)} \leq C(E, M).
\]

This completes the proof of Theorem 1.1.

5 Scattering results

5.1 The interaction Morawetz inequality

Proposition 5.1 (Morawetz control) Let \(I\) be a compact interval, \(\lambda_1\) and \(\lambda_2\) are positive real numbers, and \(u\) a solution to (1.1) on the slab \(I \times \mathbb{R}^n\). Then

\[
\| u \|_{Z(I)} \lesssim \| u \|_{L^\infty T^\frac{1}{2}(I \times \mathbb{R}^n)}. \tag{5.1}
\]

We will derive Proposition 5.1 from the following:
Proposition 5.2 (General interaction Morawetz inequality)

\[-(n - 1) \int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } \Delta ( \frac{1}{|x-y|} ) |u(y)|^2 |u(x)|^2 \, dxdydt \]
\[+ 2 \int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } |u(t,y)|^2 \frac{x-y}{|x-y|} \{ N, u \}_p (t, x) \, dxdydt \]
\[\leq 4 \| u \|^3_{L^\infty_t L^2_x (I \times \mathbb{R}^n)} \| \nabla u \|^2_{L^\infty_t L^2_x (I \times \mathbb{R}^n)} \]
\[+ 4 \int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } \{ N, u \}_m (t, y) u(t, x) \nabla u(t, x) \, dxdydt, \]

where \( N := \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \), \( \{ f, g \}_p := Re (f \nabla \bar{g} - g \nabla \bar{f}) \), \( \{ f, g \}_m = Im \{ f \bar{g} \} \).

The proof can be found in [24].

Note that, in particular \( N := \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \), we have

\[\{ N, u \}_m = 0, \quad \{ N, u \}_p = -\frac{\lambda_1 p}{p+2} \nabla (|u|^{p+2}) - \lambda_2 \text{Re} \{ \nabla (|x|^{-\gamma} * |u|^2) |u|^2 \}. \]

Next we’ll show (5.2) is positive, then we obtain

\[-\int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } \Delta ( \frac{1}{|x-y|} ) |u(y)|^2 |u(x)|^2 \, dxdydt \leq \| u \|^4_{L^\infty_t H^1_x (I \times \mathbb{R}^n)}. \]

In dimension \( n = 3 \), we have \(-\Delta (\frac{1}{|x|}) = 4\pi \delta \), so (5.3) yields

\[\| u \|^4_{L^4_{t,x} (I \times \mathbb{R}^3)} \lesssim \| u \|^4_{L^\infty_t H^1_x (I \times \mathbb{R}^3)}, \]

which proves the Proposition 5.1.

In dimension \( n \geq 4 \), we have \(-\Delta (\frac{1}{|x|}) = \frac{n-3}{|x|^2} \), so (5.3) yields

\[\| \nabla |^{-\frac{n-3}{4}} u \|^2_{L^4_{t,x} (I \times \mathbb{R}^n)} \lesssim \| u \|^2_{L^\infty_t H^1_x (I \times \mathbb{R}^n)}. \]

From Lemma 4.3 and the above inequality, we have

\[\| \nabla |^{-\frac{n-3}{4}} u \|_{L^4_{t,x} (I \times \mathbb{R}^n)} \lesssim \| u \|_{L^\infty_t H^1_x (I \times \mathbb{R}^n)}. \]

Proposition 5.1 follows by interpolation between (5.5) and the bound on the kinetic energy

\[\| \nabla u \|_{L^\infty_t L^2_x} \lesssim E \frac{1}{2}, \]

which is an immediate consequence of the conservation of energy when both nonlinearities are defocusing. Note that

\[\int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } |u(t, y)|^2 \frac{x-y}{|x-y|} \{ N, u \}_p (t, x) \, dxdydt \]
\[= -\int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } |u(t,y)|^2 \frac{x-y}{|x-y|} \frac{\lambda_1 p}{p+2} \nabla (|u|^{p+2}) \, dxdydt \]
\[-\lambda_2 \text{Re} \int \int _{ \mathbb{R}^n } \int \int _{ \mathbb{R}^n } |u(t,y)|^2 \frac{x-y}{|x-y|} \{ \nabla (|x|^{-\gamma} * |u|^2) |u|^2 \} \, dxdydt \]
\[= (I) + (II). \]
For (I), we have
\[
- \int \int \int_{\mathbb{R}^n} |u(t,y)|^2 \frac{x-y}{|x-y|} \frac{\lambda_1 p}{p+2} \nabla (|u|^{p+2}) \, dxdydt = (n-1) \frac{\lambda_1 p}{p+2} \int \int_{\mathbb{R}^n} \frac{|u(t,y)|^2 |u(t,x)|^{p+2}}{|x-y|} \, dxdydt.
\]
Note that \( \lambda_1 > 0 \), we get (I) is positive.

For (II), we define \( h(x) = \int_{\mathbb{R}^n} |u(t,y)|^2 \frac{x-y}{|x-y|} \, dy \), then we have
\[
(II) \quad = \ - \lambda_2 Re \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h(x) \left\{ \nabla (|x|^{-\gamma} * |u|^2) |u|^2 \right\} \, dxdt
\]
\[
= \lambda_2 \gamma Re \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-z|^{\gamma+1}} \frac{x-z}{|x-z|} |u(t,z)|^2 |u(t,x)|^2 h(x) \, dxdzdt
\]
\[
= \frac{1}{2} \lambda_2 \gamma Re \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-z|^{\gamma+2}} |u(t,z)|^2 |u(t,x)|^2 (|x-z|(h(x) - h(z))) \, dxdzdt.
\]
Notice that
\[
(x-z)(h(x) - h(z)) = (x-z) \int_{\mathbb{R}^n} |u(t,y)|^2 \left( \frac{x-y}{|x-y|} - \frac{z-y}{|z-y|} \right) \, dy \tag{5.6}
\]
and denote \( a := x-y, \quad b := z-y \), then, we have (5.6) = \( \int_{\mathbb{R}^n} |u(t,y)|^2 (a-b)(\frac{a}{|a|} - \frac{b}{|b|}) \, dy \).

Since \( (a-b)(\frac{a}{|a|} - \frac{b}{|b|}) = (|a||b| - ab) \frac{1}{|a|} + \frac{1}{|b|} \geq 0 \) and \( \lambda_2 > 0 \), thus (II) is positive, so we show (5.2) is positive.

**Remark 5.1** When \( n = 2 \), we don’t know whether \(-\Delta(\frac{1}{|x|})\) is positive or not. However, J. Colliander, M. Grillakis and N. Tzirakis use a refined tensor product approach to prove that (5.4) also holds when \( n = 2 \). Then the corresponding (5.1) and (2.7) also exist, we can use the same approach which used in Section 5.3 to show the scattering of the power type. However, the corresponding (2.7) don’t hold. Since we need \( \gamma > 2 \), but in this case \( \gamma < n = 2 \). So the scattering of the Hartree type can’t be gotten.

### 5.2 Global bounds in the case: \( p = \frac{4}{n} \), \( 2 < \gamma < \min \{n,4\} \) and \( \lambda_1, \lambda_2 > 0 \) or \( \frac{4}{n} < p < \frac{4}{n-2} \), \( \gamma = 2 \) and \( \lambda_1, \lambda_2 > 0 \)

The approaches for both cases are the same, we settle the first case and the same method can be used for the other one. Without loss of generality, let \( \lambda_1 = \lambda_2 = 1 \).

We view the second nonlinearity as a perturbation to (1.5). By using Proposition 5.1, and the conservation of energy and mass, we get
\[
\| u \|_{Z(\mathbb{R})} \lesssim \| u \|_{L^p_t L^2_x(L^4_{x\mathbb{R}^n})} \leq C(E, M).
\]
Split \( \mathbb{R} \) into \( J = J(E, M, \varepsilon) \) subintervals \( I_j, \ 0 \leq j \leq J-1, \) such that
\[
\| u \|_{Z(I_j)} \sim \varepsilon,
\]
where \( \varepsilon \) is a small positive constant to be chosen later.

On the slab \( I \times \mathbb{R}^n \), we define:
\[
\dot{X}(I) := L_t^{2+\frac{1}{2p}} L_x^{\frac{2n(2p-1)}{n(2p-1) - dp}} (I \times \mathbb{R}^n) \cap V(I),
\]

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where \( \theta \) is introduced in Lemma 2.6. Then on each \( I_j \) \((0 \leq j \leq J - 1)\), by (2.8) we have
\[
\| \left( |x|^{-\gamma} + |u|^2 \right) u \|_{\dot{N}^0(I_j \times \mathbb{R}^n)} \lesssim \| u \|_{L^{2+\frac{2\gamma}{2(2\gamma+1)-\theta}}(I_j \times \mathbb{R}^n)} \| u \|_{L^\infty_t H^\theta_x(I_j \times \mathbb{R}^n)}^{n+1} \leq C(E, M) \varepsilon \| u \|_{\dot{X}^0(I_j)},
\] (5.7)
where \( c = \frac{n+1}{2(2\gamma+1)} \).

In what follow, we fix an interval \( I_{j_0} = [a, b] \) and prove that \( u \) obeys good Strichartz estimates on the slab \( I_{j_0} \times \mathbb{R}^n \). Let \( v \) be a solution to
\[
\%{\begin{align*}
iv_t + \Delta v &= |v|^{\frac{n}{n-1}} v, \\
v(a) &= u(a).
\end{align*}}\%
\]
As this initial value problem is globally well-posedness in \( H^1_x \), and by Assumption 1.1 and Lemma 3.6 the unique solution \( v \) satisfies
\[
\| v \|_{S^0(\mathbb{R} \times \mathbb{R}^n)} \leq C(M).
\]
Subdivide \( \mathbb{R} \) into \( K = K(M, \eta) \) subinterval \( J_k \) such that on each \( J_k \)
\[
\| v \|_{\dot{X}^0(J_k)} \sim \eta
\] (5.8)
for a small constant \( \eta > 0 \) to be chosen later.

We are only interested in the subintervals \( J_k = [t_k, t_{k+1}] \) which have a nonempty intersection with \( I_{j_0} \). Without loss of generality, assume that \( [a, b] = \bigcup_{k=0}^{k-1} J_k \), \( t_0 = a, t_{k'} = b \).

On each \( J_k \), by Strichartz estimates and (5.4), we get
\[
\| e^{i(t-t_k)\Delta} v(t_k) \|_{\dot{X}^0(J_k)} \leq \| v \|_{\dot{X}^0(J_k)} + C \| u \|_{L^1_t S^0(J_k \times \mathbb{R}^n)} \| v \|_{L^\infty_t S^0(J_k \times \mathbb{R}^n)} \| v \|_{V(J_k)} \leq \eta + C \eta^{1+\frac{1}{n}}.
\]
Choosing \( \eta \) sufficiently small, we get
\[
\| e^{i(t-t_k)\Delta} v(t_k) \|_{\dot{X}^0(J_k)} \leq 2\eta.
\] (5.9)

Next, we will use Lemma 3.5 to obtain an estimate on the \( S^1 \)-norm of \( u \) on \( I_{j_0} \times \mathbb{R}^n \). On the interval \( J_0 \), recalling that \( u(t_0) = v(t_0) \), by Strichartz estimates, (5.7) and (5.9),
\[
\| u \|_{\dot{X}^0(J_0)} \leq \| e^{i(t-t_0)\Delta} u(t_0) \|_{\dot{X}^0(J_0)} + C \| u \|_{L^\infty_t S^0(J_0)} + C(E, M) \varepsilon \| u \|_{\dot{X}^0(J_0)} \leq 2\eta + C \| u \|_{L^\infty_t S^0(J_0)} + C(E, M) \varepsilon \| u \|_{\dot{X}^0(J_0)}.
\]

By a standard continuity argument yields
\[
\| u \|_{\dot{X}^0(J_0)} \leq 4\eta
\]
provided \( \eta \) and \( \varepsilon \) are chosen sufficiently small. In order to use Lemma \([3.5]\) we notice that \([3.39]\) holds on \( I := J_0 \) for \( L_0 := 4\eta \), \([3.37]\) holds with \( M_0' = 0 \). We only show that the error is sufficiently small. In fact, from

\[
\| e \|_{\dot{H}^{0}(J_0 \times \mathbb{R}^n)} \leq C(E, M)\varepsilon^c \quad \| u \|_{\dot{H}^{\sigma}(J_0)} \leq C(E, M)\eta\varepsilon^c,
\]

and choosing \( \varepsilon \) to be sufficiently small, we obtain

\[
\| u - v \|_{\dot{H}^{0}(J_0 \times \mathbb{R}^n)} \leq \varepsilon^\frac{\sigma}{2}.
\]

From Strichartz estimates, we have

\[
\| u(t) - v(t) \|_{L^2_x} \leq \varepsilon^\frac{\sigma}{2}
\]

\[
\| e^{i(t-t_1)\Delta}(u(t_1) - v(t_1)) \|_{\dot{H}^{\sigma}(J_1)} \leq \varepsilon^\frac{\sigma}{2}.
\] (5.10)

On the other hand,

\[
\| u \|_{\dot{H}^{1}(J_0 \times \mathbb{R}^n)} \leq \| u(a) \|_{\dot{H}^{1}_x} + \| u \|_{\dot{H}^{1}(J_0 \times \mathbb{R}^n)} + \| (|x|^{-\gamma} * |u|^2)u \|_{\dot{H}^{1}(I \times \mathbb{R}^n)}
\]

\[
\leq C(E) + (4\eta)^\frac{\sigma}{2} \| u \|_{\dot{H}^{1}(J_0 \times \mathbb{R}^n)} + C(E, M)\varepsilon^c \| u \|_{\dot{H}^{1}(J_0 \times \mathbb{R}^n)}.
\]

Choosing \( \eta \) and \( \varepsilon \) sufficiently small, we have

\[
\| u \|_{\dot{H}^{1}(J_0 \times \mathbb{R}^n)} \leq C(E).
\]

On the intervals \( J_1 \), by Strichartz estimates, \([5.7], [5.10]\), we get

\[
\| u \|_{\dot{H}^{\sigma}(J_1)} \leq \| e^{i(t-t_1)\Delta}v(t_1) \|_{\dot{H}^{\sigma}(J_1)} + \| e^{i(t-t_1)\Delta}(u(t_1) - v(t_1)) \|_{\dot{H}^{\sigma}(J_1)}
\]

\[
+ C \| u \|_{\dot{H}^{\sigma}(J_1)} + C(E, M)\varepsilon^c \| u \|_{\dot{H}^{\sigma}(J_1)}
\]

\[
\leq 2\eta + \varepsilon^\frac{\sigma}{2} + C \| u \|_{\dot{H}^{\sigma}(J_1)} + C(E, M)\varepsilon^c \| u \|_{\dot{H}^{\sigma}(J_1)}.
\]

Choosing \( \eta \) and \( \varepsilon \) sufficiently small, we obtain

\[
\| u \|_{\dot{H}^{\sigma}(J_1)} \leq 4\eta.
\]

This implies that the error satisfies the condition of Lemma \([3.5]\) on \( J_1 \). Choosing \( \varepsilon \) sufficiently small, and applying Lemma \([3.5]\) to derive

\[
\| u - v \|_{\dot{H}^{\sigma}(J_1 \times \mathbb{R}^n)} \leq \varepsilon^\frac{\sigma}{2}.
\]

The same arguments as before also yields

\[
\| u \|_{\dot{H}^{1}(J_1 \times \mathbb{R}^n)} \leq C(E).
\]

By the induction argument, for each \( 0 \leq k \leq k' - 1 \), we get

\[
\| u - v \|_{\dot{H}^{\sigma}(J_k \times \mathbb{R}^n)} \leq \varepsilon^\frac{\sigma}{2^k+1},
\]

\[
\| u \|_{\dot{H}^{1}(J_k \times \mathbb{R}^n)} \leq C(E).
\]
Adding these estimates over all the intervals $J_k$ which have a nonempty intersection with $I_{j_0}$, we obtain
\[
\| u \|_{\dot{S}^0(J_{j_0} \times \mathbb{R}^n)} \leq \| v \|_{\dot{S}^0(J_{j_0} \times \mathbb{R}^n)} + \sum_{k=0}^{k'-1} \| u - v \|_{\dot{S}^0(J_k \times \mathbb{R}^n)} \leq C(E, M)
\]
\[
\| u \|_{\dot{S}^1(I_{j_0} \times \mathbb{R}^n)} \leq \sum_{k=0}^{k'-1} \| u \|_{\dot{S}^1(J_k \times \mathbb{R}^n)} \leq C(E, M).
\]

As the intervals $I_{j_0}$ was arbitrarily chosen, we obtain
\[
\| u \|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \leq \sum_{j=0}^{J-1} \| u \|_{\dot{S}^0(I_j \times \mathbb{R}^n)} \leq C(E, M)
\]
\[
\| u \|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \leq \sum_{j=0}^{J-1} \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \leq C(E, M),
\]
and hence
\[
\| u \|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M).
\]

5.3 Global bounds in the case: $\frac{4}{n} < p < \frac{4}{n-2}$, $2 < \gamma < \min \{n, 4\}$ and $\lambda_1, \lambda_2 > 0$

The results were shown in [6] with a more complicated argument. We use a simpler proof which is used in [24] that relies on the interaction Morawetz estimate.

By Proposition 5.1, we have
\[
\| u \|_{Z(\mathbb{R})} \lesssim \sum \| u \|_{L^\infty_t H^1_x(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M)
\]
Devide $\mathbb{R}$ into $J = J(E, M, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that
\[
\| u \|_{Z(I_j)} \sim \eta
\]
where $\eta > 0$ be a small constant to be chosen later.

By Strichartz estimates and Lemma 2.3, on each $I_j$, we have
\[
\| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \lesssim \| u(t_j) \|_{H^1_x} + \eta^{\frac{n+1}{2(2\theta+1)}} \| u \|_{L^\infty_t L^{2\theta}_x(I_j \times \mathbb{R}^n)} \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)}
\]
\[
+ \eta^{\frac{n+1}{2(2\theta+1)}} \| u \|_{L^\infty_t L^{2\theta}_x(I_j \times \mathbb{R}^n)} \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \leq C(E, M) + \eta^{\frac{n+1}{2(2\theta+1)}} C(E, M) \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)}
\]
\[
+ \eta^{\frac{n+1}{2(2\theta+1)}} C(E, M) \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)}.
\]

Choosing $\eta$ sufficiently small, we have
\[
\| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \leq C(E, M).
\]

Summing these bounds over all intervals $I_j$, we obtain
\[
\| u \|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \leq \sum_{j=0}^{J-1} \| u \|_{\dot{S}^1(I_j \times \mathbb{R}^n)} \leq C(E, M).
\]
5.4 Global bounds in the case: $\frac{4}{n} < p < \frac{4}{n-2}$, $\gamma = 4$ with $n \geq 5$ and $\lambda_1, \lambda_2 > 0$ or $p = \frac{4}{n-2}$, $2 < \gamma < \min\{n, 4\}$ and $\lambda_1, \lambda_2 > 0$

The approaches for both cases are the same, we show the first case and the same method can be used for the other. On the slab $I \times \mathbb{R}^n$, we define:

$$
\tilde{X}^0(I) := L^2_tL^4_x \left( L^6_tL^{\frac{6n}{n-2}}_x(I \times \mathbb{R}^n) \right)
$$

where $\theta$ is introduced in Lemma 2.6. Just replace $\tilde{X}^0(I)$ by $\tilde{Y}^0(I)$ that appears in Subsection 5.2, Lemma 3.2 replace Lemma 3.5, apply the same approach that used in Subsection 5.2, one can get

$$
\| u \|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M).
$$

5.5 Global bounds in the case: $p = \frac{4}{n-2}$, $\gamma = 2$ and $\lambda_1, \lambda_2 > 0$ or $p = \frac{4}{n}$, $\gamma = 4$ with $n \geq 5$ and $\lambda_1, \lambda_2 > 0$

The approaches for both cases are the same, we settle the first case and the same method can be used for the other one. Without loss of generality, let $\lambda_1 = \lambda_2 = 1$. The main idea is that we divide $u$ into $u_{lo}$ and $u_{hi}$ by frequency, and compare the low frequency with the $L^2_x$-critical NLS, at one time, compare the high frequency with the $H^1_x$-critical NLS. At last, we get the finite global Strichartz bounds in this case.

We will need a series of small parameters. More precisely, we will define

$$
0 < \eta_3 \ll \eta_2 \ll \eta_1 \ll 1,
$$

where any $\eta_j$ is allowed to depend on the energy and the mass as well as on any of the larger $\eta$'s.

By Proposition 5.1 and conservation of energy and mass, we have

$$
\| u \|_{Z(\mathbb{R})} \leq C(E, M).
$$

Split $\mathbb{R}$ into $K = K(E, M, \eta_3)$ subintervals $J_k$ such that on each slab $J_k \times \mathbb{R}^n$ we have

$$
\| u \|_{Z(J_k)} \sim \eta_3.
$$

Fix $J_{k_0} = [a, b]$, for every $t \in J_{k_0}$. We split $u(t) = u_{lo}(t) + u_{hi}(t)$ where $u_{lo}(t) := P_{< \eta_2^{-1}}u(t), \ u_{hi}(t) := P_{\geq \eta_2^{-1}}u(t)$.

On the slab $J_{k_0} \times \mathbb{R}^n$, we compare $u_{lo}(t)$ to the following $L^2_x$-critical Hartree NLS

$$
\begin{align*}
(i\partial_t + \Delta)v &= (|x|^{-2} * |v|^2)v \\
v(a) &= u_{lo}(a),
\end{align*}
$$

which is globally well-posedness in $H^1_x$. Moreover, by Assumption 1.2, one has

$$
\| v \|_{U(\mathbb{R})} \leq C(\| u_{lo}(a) \|_{L^2}) \leq C(M).
$$

By Lemma 3.6, we have

$$
\| v \|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(M),
$$

$$
\| v \|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M).
$$
Divide $J_{k_0} = [a, b]$ into $J = J(M, \eta_1)$ subintervals $I_j = [t_{j-1}, t_j]$ with $t_0 = a, t_J = b$, such that
\[
\| v \|_{U(I_j)} \sim \eta_1. \tag{5.14}
\]
By induction, we will establish that for each $j = 1, \cdots, J$, we have
\[
P(j) : \begin{cases}
\| u_{lo} - v \|_{\dot{S}^0([t_0, t_{j+1}])} & \leq \eta_2^{1-2\delta}, \\
\| u_{hi} \|_{\dot{S}^1(I_j)} & \leq L(E), \\
\| u \|_{S^1([t_0, t_{j+1}])} & \leq C(\eta_1, \eta_2),
\end{cases}
\tag{5.15}
\]
where $\delta > 0$ is a small constant to be chosen later, and $L(E)$ is a large quantity to be chosen later which depends only on $E$ (not on any $\eta_j$). As the method of checking that (5.15) holds for $j = 1$ is similar to that of the induction step, i.e. showing that $P(j)$ implies $P(j + 1)$, we will only prove the latter.

Assume that (5.15) is true for some $1 \leq j < J$. Then, we will show
\[
P(j) : \begin{cases}
\| u_{lo} - v \|_{\dot{S}^0([t_0, t_{j+1}])} & \leq \eta_2^{1-2\delta}, \\
\| u_{hi} \|_{\dot{S}^1(I_j)} & \leq L(E), \\
\| u \|_{S^1([t_0, t_{j+1}])} & \leq C(\eta_1, \eta_2),
\end{cases}
\tag{5.16}
\]
Let $\Omega_1$ be the set of all times $T \in I_{j+1}$ such that
\[
\begin{align*}
\| u_{lo} - v \|_{\dot{S}^0([t_0, T])} & \leq \eta_2^{1-2\delta}, \\
\| u_{hi} \|_{\dot{S}^1([T, T])} & \leq L(E), \\
\| u \|_{S^1([t_0, T])} & \leq C(\eta_1, \eta_2).
\end{align*}
\tag{5.17}
\]
In order to prove $\Omega_1 = I_{j+1}$, we notice that $\Omega_1$ is nonempty (as $t_j \in \Omega_1$) and closed (by Fatou). Let $\Omega_2$ be the set of all times $T \in I_{j+1}$ such that
\[
\begin{align*}
\| u_{lo} - v \|_{\dot{S}^0([t_0, T])} & \leq 2\eta_2^{1-2\delta}, \\
\| u_{hi} \|_{\dot{S}^1([T, T])} & \leq 2L(E), \\
\| u \|_{S^1([t_0, T])} & \leq 2C(\eta_1, \eta_2).
\end{align*}
\tag{5.18}
\]
We will show $\Omega_2 \subset \Omega_1$, which will conclude the argument.

**Lemma 5.1** Let $T \in \Omega_2$. Then, the following properties holds:
\[
\begin{align*}
\| u_{lo} \|_{U(I)} & \sim \eta_1, \\
\| u_{lo} \|_{\dot{S}^0([t_0, T] \times \mathbb{R}^n)} & \leq C(M), \\
\| u_{lo} \|_{W([t_j, T])} & \sim \eta_2, \\
\| u_{lo} \|_{\dot{S}^1(I \times \mathbb{R}^n)} & \leq E, \\
\| u_{lo} \|_{\dot{S}^1([t_0, T] \times \mathbb{R}^n)} & \leq C(\eta_1)E, \\
\| u_{hi} \|_{\dot{S}^0(I \times \mathbb{R}^n)} & \leq \eta_2 L(E), \\
\| u_{hi} \|_{\dot{S}^0([t_0, T] \times \mathbb{R}^n)} & \leq \eta C(\eta_1) L(E), \\
\| u_{hi} \|_{\dot{S}^1([t_0, T] \times \mathbb{R}^n)} & \leq C(\eta_1)L(E),
\end{align*}
\tag{5.23-5.30}
\]
where $I \in \{I_l, 1 \leq l \leq j\} \cup \{[t_j, T]\}$.
Proof. Using (5.12), (5.14), (5.20), and Bernstein inequality, we have

\[
\| u_0 \|_{U(I)} \leq \| u_0 - v \|_{U(I)} + \| v \|_{U(I)} \lesssim \eta_2^{1-2\delta} + \eta_1 \lesssim \eta_1,
\]

\[
\| u_0 \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \leq \| u_0 - v \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} + \| v \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \lesssim \eta_2^{1-2\delta} + C(M) \leq C(M),
\]

\[
\| u_{hi} \|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim \eta_2 \| u_{hi} \|_{\dot{S}^1(I \times \mathbb{R}^n)} \lesssim \eta_2 L(E).
\]

Therefore, (5.23), (5.24) and (5.28) hold. In view of \( J = O(\eta_1^{-C}) \), we get

\[
\|
\]

\[
\|
\]

\[
\|
\]

\[
\|
\]

Then, (5.25) holds. Of course, (5.27) can be obtained by (5.26), since \( J = C(\eta_1) \).

At last, we show (5.26) is true. We write \( u_0 = P_{\lesssim \eta_2} u_0 + P_{\eta_2 < \eta_2^{-1} u_0} \).
In dimension $n \geq 5$, by interpolation, Sobolev embedding, Bernstein inequality, (5.11) and (5.26), we have

\[
\| P_{\eta_2^{<\cdot}} u_{t_0} \|_{W([t_j,T])} \lesssim \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{n+1} L_x^{2(n+1)-6}([t_j,T] \times \mathbb{R}^n)} \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{2} L_x^{2n}([t_j,T] \times \mathbb{R}^n)}^{1-c} \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{2} L_x^{2n}([t_j,T] \times \mathbb{R}^n)}^{c} \\
\lesssim \| \nabla |\eta_2^{3^{-1}}|^{\frac{3}{n+1}} P_{\eta_2^{<\cdot}} u_{t_0} \|_{Z([t_j,T])} \| u_{t_0} \|_{L_t^{1} L_x^{2}(\overline{Z([t_j,T])})}^{1-c} \| S([t_j,T] \times \mathbb{R}^n)} \\
\lesssim \eta_2 \| u_{t_0} \|_{L_t^{2} L_x^{2n}([t_j,T] \times \mathbb{R}^n)}^{c} E^{1-c} \\
\lesssim \eta_2^{\frac{3}{n+1}} E^{1-c} \\
\leq \eta_2,
\]

where $c = \frac{4(n+1)}{(n-1)(n+2)}$.

In dimension $n = 4$, by using interpolation, Sobolev embedding, Bernstein inequality, the conservation of energy and (5.11), we get

\[
\| P_{\eta_2^{<\cdot}} u_{t_0} \|_{W([t_j,T])} \lesssim \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{\frac{2n}{n-1}} L_x^{\frac{2n}{n-1}}([t_j,T] \times \mathbb{R}^n)} \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{\frac{n}{n-1}} L_x^{n}([t_j,T] \times \mathbb{R}^n)} \\
\lesssim \| \nabla \| \frac{3}{n+1} P_{\eta_2^{<\cdot}} u_{t_0} \|_{Z([t_j,T])} \| E^{1-c} \\
\lesssim (\eta_2 \eta_3) \frac{3}{n} E^{\frac{3}{n}} \\
\leq \eta_2.
\]

In dimension $n = 3$, by using interpolation, Sobolev embedding, Bernstein inequality, the conservation of energy and (5.11), we get

\[
\| P_{\eta_2^{<\cdot}} u_{t_0} \|_{W([t_j,T])} \lesssim \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{\frac{2n}{n-1}} L_x^{\frac{2n}{n-1}}([t_j,T] \times \mathbb{R}^n)} \| P_{\eta_2^{<\cdot}} u_{t_0} \|_{L_t^{\frac{n}{n-1}} L_x^{n}([t_j,T] \times \mathbb{R}^n)} \\
\lesssim \| (1 + |\nabla|) \|^{\frac{3}{n+1}} P_{\eta_2^{<\cdot}} u_{t_0} \|_{Z([t_j,T])} \| E^{\frac{3}{n}} \\
\lesssim (\eta_2 \eta_3) \frac{3}{n} E^{\frac{3}{n}} \\
\leq \eta_2.
\]

Hence, in all dimension $n \geq 3$, we all have

\[
\| P_{\eta_2^{<\cdot}} u_{t_0} \|_{W([t_j,T])} \lesssim \eta_2
\]

By Sobolev embedding, Bernstein inequality and (5.24), we have

\[
\| P_{\eta_2} u_{t_0} \|_{W([t_j,T])} \lesssim \| \nabla P_{\eta_2} u_{t_0} \|_{L_t^{\frac{n}{n+1}} L_x^{2(n+1)-6}([t_j,T] \times \mathbb{R}^n)} \| u_{t_0} \|_{L_t^{2} L_x^{2n}([t_j,T] \times \mathbb{R}^n)} \lesssim \eta_2 \| u_{t_0} \|_{L_t^{2} L_x^{2n}([t_j,T] \times \mathbb{R}^n)}^{\frac{2n}{n+1}} L_x^{\frac{2n}{n+1}}([t_j,T] \times \mathbb{R}^n)}
\]

In dimension $n = 3$, by interpolation, (5.24) and the conservation of mass, we get

\[
\| P_{\eta_2} u_{t_0} \|_{W([t_j,T])} \lesssim \eta_2 \| u_{t_0} \|_{U([t_j,T])} \| u_{t_0} \|_{L_t^{2} L_x^{2}([t_j,T] \times \mathbb{R}^n)} \| u_{t_0} \|_{L_t^{2} L_x^{2}([t_j,T] \times \mathbb{R}^n)} \| u_{t_0} \|_{L_t^{2} L_x^{2}([t_j,T] \times \mathbb{R}^n)} \lesssim \eta_2 \eta_1 \| M \|^{\frac{3}{2}} \leq \eta_2
\]
provided \( \eta_1 \) is chosen sufficiently small depending on \( M \).

In dimension \( n = 4 \), because of \( L_t^{\frac{n-2}{n-4}} L_x^{n-4} = U \), then

\[
\| P_{\leq \eta_2} u_{t_0} \|_{W([t_j, T])} \lesssim \eta_2 \eta_1 \leq \eta_2
\]

In dimension \( n \geq 5 \), by interpolation, (5.23) and (5.24)

\[
\| P_{\leq \eta_2} u_{t_0} \|_{W([t_j, T])} \lesssim \eta_2 \| u_{t_0} \|_{U([t_j, T])} \| u_{t_0} \|_{L_{t,x}^{2n}([t_j, T] \times \mathbb{R}^n)}
\]

\[
\lesssim \eta_2 \eta_1 \| u_{t_0} \|_{S^1([t_j, T] \times \mathbb{R}^n)} \lesssim \eta_2 \eta_1^{\frac{\delta}{\delta}} C(M) \leq \eta_2.
\]

Hence, in all dimension \( n \geq 3 \), we get

\[
\| P_{\leq \eta_2} u_{t_0} \|_{W([t_j, T])} \leq \eta_2.
\]

Therefore, by the triangle inequality, (5.25) is true.

Now, we are ready to show \( \Omega_2 \subset \Omega_1 \). We will first show (5.15). The method is to compare \( u_{t_0} \) to \( v \) via the perturbation result of Lemma 3.3. \( u_{t_0} \) satisfies the following initial value problem on the slab \([t_0, T] \times \mathbb{R}^n\)

\[
\begin{cases}
(i\partial_t + \Delta) u_{t_0} = (|x|^{-2} * |u_{t_0}|^2) u_{t_0} + P_{t_0}(|u|^{\frac{4}{n-2}} u) \\
+ P_{t_0}([|x|^{-2} * |u|^2] u - (|x|^{-2} * |u_{t_0}|^2) u_{t_0}) - P_{t_0}(([|x|^{-2} * |u_{t_0}|^2) u_{t_0})
\end{cases}
\]

\[
u_{t_0}(t_0) = u_{t_0}(t_0).
\]

Since (5.24) and \( v(t_0) = u_{t_0}(t_0) \), in order to use Lemma 3.3 we only need to show the error term

\[
e = P_{t_0}(|u|^{\frac{4}{n-2}} u) + P_{t_0}([|x|^{-2} * |u|^2] u - (|x|^{-2} * |u_{t_0}|^2) u_{t_0}) - P_{t_0}(([|x|^{-2} * |u_{t_0}|^2) u_{t_0})
\]

is small in \( \tilde{N}^0([t_0, T] \times \mathbb{R}^n) \).

By using Lemma 2.7 (5.11) and (5.22), we have

\[
\| P_{t_0}(|u|^{\frac{4}{n-2}} u) \|_{\tilde{N}^0([t_0, T] \times \mathbb{R}^n)} \lesssim \| u \|_{U([t_0, T])} \| u \|_{S^1([t_0, T] \times \mathbb{R}^n)} \lesssim \eta_3^\theta (C(\eta_1, \eta_2))^{\frac{n-2}{n-\theta}} \leq \eta_2^{1-\delta}
\]

provided \( \eta_3 \) is chosen sufficiently small depending on \( \eta_1 \) and \( \eta_2 \). By using Bernstein inequality, Hölder inequality, Hardy–Littlewood–Sobolev inequality, (5.24) and (5.27), we have

\[
\| P_{t_0}([|x|^{-2} * |u_{t_0}|^2) u_{t_0}) \|_{\tilde{N}^0([t_0, T] \times \mathbb{R}^n)} \lesssim \eta_2 \| \nabla P_{t_0}([|x|^{-2} * |u_{t_0}|^2) u_{t_0}) \|_{\tilde{N}^0([t_0, T] \times \mathbb{R}^n)}
\]

\[
\lesssim \eta_2 \| u_{t_0} \|_{U([t_0, T])} \| u_{t_0} \|_{U([t_0, T])}
\]

\[
\lesssim \eta_2 \| u_{t_0} \|_{S^1([t_0, T] \times \mathbb{R}^n)} \| u_{t_0} \|_{S^1([t_0, T] \times \mathbb{R}^n)}
\]

\[
\lesssim \eta_2 C(M) C(\eta_1) E \leq \eta_2^{1-\delta}
\]
provided \( \eta_2 \) is sufficiently small depending on \( E, M \) and \( \eta_1 \). From Hölder inequality, Hardy-Littlewood-Sobolev inequality, \([5.24]\) and \([5.29]\), one can get

\[
\| P_{lo}([|x|^{-2} \ast |u|^2]u - (|x|^{-2} \ast |u_{lo}|^2)u_{lo} \|_{\N^0([t_0,T] \times \mathbb{R}^n)} \\
\leq \| (|x|^{-2} \ast |u_{lo}|^2)u_{hi} \|_{\N^0([t_0,T] \times \mathbb{R}^n)} \\
+ \| (|x|^{-2} \ast |u_{hi}|^2)u_{hi} \|_{\N^0([t_0,T] \times \mathbb{R}^n)} + \| (|x|^{-2} \ast |u_{hi}|^2)u_{lo} \|_{\N^0([t_0,T] \times \mathbb{R}^n)} \\
\lesssim \| u_{lo} \|^2_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \| u_{hi} \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \|
+ \| u_{hi} \|^2_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \| u_{lo} \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} + \| u_{hi} \|^3_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \\
\lesssim C(M)\eta_2 C(\eta_1)L(E) + (\eta_2 C(\eta_1)L(E))^2 C(M) + (\eta_2 C(\eta_1)L(E))^3 \\
\leq \eta_2^{1-\delta}.
\]

Therefore,

\[
\| e \|_{\N^0([t_0,T] \times \mathbb{R}^n)} \lesssim 3\eta_2^{1-\delta}
\]

and hence, taking \( \eta_2 \) sufficiently small depending on \( M \), we can apply Lemma \([3.3]\) to get

\[
\| u_{lo} - v \|_{\dot{S}^0([t_0,T] \times \mathbb{R}^n)} \lesssim C(M)\eta_2^{1-\delta} \leq \eta_2^{1-2\delta}.
\]

Thus \([5.15]\) is true. Now we turn to prove \([5.18]\) is true. The idea is to compare \( u_{hi} \) to the energy-critical NLS

\[
\begin{cases}
    i\partial_t w + \Delta w = |w|^\frac{4}{n-2} w \\
    w(t_j) = u_{hi}(t_j)
\end{cases}
(5.31)
\]

Then, citing the result in \([23, 3, 26]\), we know \([5.31]\) is globally wellposed and

\[
\| w \|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \lesssim C(E)
(5.32)
\]

Using Lemma \([3.6]\) and \([5.28]\), we also get

\[
\| w \|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \lesssim C(E) \| u_{hi}(t_j) \|_{L^2} \lesssim \eta_2 C(E)L(E).
\]

\( u_{hi} \) satisfies the following initial value problem on the slab \([t_j, T] \times \mathbb{R}^n \)

\[
\begin{cases}
    (i\partial_t + \Delta) u_{hi} = |u_{hi}|^{\frac{4}{n-2}} u_{hi} + P_{hi}(\frac{|x|^{-2} \ast |u|^2}u) \\
    + P_{hi}(\frac{|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}}u) - P_{lo}(\frac{|u_{hi}|^{\frac{4}{n-2}} u_{hi}}) \\
    u_{hi}(t_j) = u_{hi}(t_j).
\end{cases}
\]

In order to use Lemma \([3.3]\), we only need to show the error term

\[
e = P_{hi}(\frac{|x|^{-2} \ast |u|^2}u) + P_{hi}(\frac{|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}}u) - P_{lo}(\frac{|u_{hi}|^{\frac{4}{n-2}} u_{hi}})
\]

is small in \( \dot{N}^1([t_j, T] \times \mathbb{R}^n) \).
From Hölder, Hardy-Littlewood-Sobolev inequality, (5.21), (5.23), (5.26), (5.29) and (5.30), we have

\[ \| P_\alpha (|x|^{-2} \ast |u|^2) u \|_{\tilde{N}^1([t_j, T] \times \mathbb{R}^n)} \lesssim \| u \|_{L^2([0, T])} \| \nabla u \|_{L^2([0, T])} \]
\[ \lesssim \| u_{hi} \|_{S^0([t_j, T] \times \mathbb{R}^n)} \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ + \| u_{lo} \|_{S^0([t_j, T] \times \mathbb{R}^n)} \| u_{lo} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ + \| u_{hi} \|_{S^0([t_j, T] \times \mathbb{R}^n)} \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2 C(\eta_1) L(E) + \eta_2 L(E) + \eta_2^2 L(E) \]
\[ \leq \eta_2 \]

if \( \eta_2 \) is sufficiently small depending on \( E \) and \( \eta_1 \).

By using Bernstein inequality, Lemma 2.1, (5.11) and (5.22), one has

\[ \| P_\alpha (|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}) \|_{\tilde{N}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2^{-1} \| u_{hi} \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2^{-1} \| u \|_{Z([t_j, T])} \| u \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2^{-1} \eta_2^\theta C(\eta_1, \eta_2) \]
\[ \leq \eta_2 \]

if \( \eta_3 \) is sufficiently small depending on \( \eta_1 \) and \( \eta_2 \).

Now, we’ll estimate the last term \( \| P_{hi} (|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}) \|_{\tilde{N}^1([t_j, T] \times \mathbb{R}^n)} \). Since the function \( z \rightarrow |z|^{\frac{4}{n-2}} |z|^2 \) is Hölder continuous of order \( \frac{4}{n-2} \), then

\[ \| P_{hi} (|u|^{\frac{4}{n-2}} u - |u_{hi}|^{\frac{4}{n-2}} u_{hi}) \|_{\tilde{N}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \| u \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \]
\[ + \| u_{hi} \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2^\theta C(\eta_1, \eta_2) \eta_2^{-1} \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \leq \eta_2 \]

For (5.33), from Remark 2.1, Bernstein inequality, (5.11), (5.19) and (5.26), we have

\[ \| u \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \| u \|_{Z([t_j, T])} \| u \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \lesssim \eta_2^\theta C(\eta_1, \eta_2) \eta_2^{-1} \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \]
\[ \leq \eta_2 \]
if \( \eta_3 \) is chosen sufficiently small depending on \( \eta_1 \) and \( \eta_2 \).

For (5.34), when the dimension \( 3 \leq n < 6 \), by using H"older inequality, (5.21), (5.24) and (5.26), we can get

\[
\| (|u|^{\frac{4}{n-2}} - |u_{hi}|^{\frac{4}{n-2}}) \nabla u_{hi} \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \\
\lesssim \| (|u|^{\frac{6-n}{n-2}} u_{lo} \nabla u_{hi}) \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \\
\lesssim \left( \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} + \| u_{lo} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \right) \| \nabla u_{hi} \|_{\tilde{S}^0([t_j, T] \times \mathbb{R}^n)} \| u_{lo} \|_{W([t_j, T])} \\
\lesssim (L(E) + E)^{\frac{6-n}{n-2}} \eta_2 L(E) \\
\lesssim \eta_2^{\frac{3}{n-2}}
\]

provided \( \eta_2 \) is chosen sufficiently small depending on \( E \).

When the dimension \( n \geq 6 \), notice the inequality \((a + b)^p \leq a^p + b^p\) as \( a, b \geq 0, \ p \leq 1 \), we can apply Lemma 3.4 to get

\[
\| (|u|^{\frac{4}{n-2}} - |u_{hi}|^{\frac{4}{n-2}}) \nabla u_{hi} \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \\
\lesssim \| u_{lo} |^{\frac{4}{n-2}} \nabla u_{hi} \|_{\tilde{N}^0([t_j, T] \times \mathbb{R}^n)} \\
\lesssim \| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \| u_{lo} \|_{W([t_j, T])} \\
\lesssim L(E) \eta_2^{\frac{3}{n-2}} \\
\lesssim \eta_2^{\frac{3}{n-2}}
\]

(5.35) has been estimated from the above by \( \eta_2^{\frac{3}{n-2}} \).

Therefore

\[
\| e \|_{\tilde{N}^1([t_j, T] \times \mathbb{R}^n)} \leq \eta_2 + \eta_2^{\frac{1}{2}} + 2 \eta_2^{\frac{3}{n-2}} \leq \eta_2^{\frac{3}{n-2}}
\]

and hence, taking \( \eta_2 \) sufficiently small depending on \( E \), we can apply Lemma 3.4 to get

\[
\| u_{hi} - w \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \lesssim \eta_1^{\frac{1}{2}}
\]

for a small constant \( c > 0 \) depending only on the dimension \( n \). So we can obtain

\[
\| u_{hi} \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \leq \| u_{hi} - w \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} + \| w \|_{\tilde{S}^1([t_j, T] \times \mathbb{R}^n)} \lesssim \eta_1^{\frac{1}{2}} + C(E) \leq L(E)
\]

Choosing \( L(E) \) is sufficiently large.

Finally, (5.19) follows from

\[
\| u \|_{S^1([t_0, T] \times \mathbb{R}^n)} \leq \| u_{hi} \|_{S^1([t_0, T] \times \mathbb{R}^n)} + \| u_{lo} \|_{S^1([t_0, T] \times \mathbb{R}^n)} \\
\leq C(M) + C(\eta_1) E + \eta_2 C(\eta_1) L(E) + C(\eta_1) L(E) \\
\leq C(\eta_1, \eta_2).
\]

This proves that \( \Omega_2 \subset \Omega_1 \). Hence, by induction

\[
\| u \|_{S^1(J_{k_0} \times \mathbb{R}^n)} \leq C(\eta_1, \eta_2)
\]

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As $J_{k_0}$ is arbitrary and the total number of intervals $J_k$ is $K = K(E, M, \eta_3)$, put these bounds together we obtain
\[
\| u \|_{S^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(\eta_1, \eta_2, \eta_3) = C(E, M).
\]

### 5.6 Global bounds in the case: 
\[ p = \frac{4}{n-2}, \ 2 \leq \gamma < n \ \text{and} \ \lambda_1 \cdot \lambda_2 < 0 \] \[ \text{or} \quad 4 \leq p < \frac{4}{n-2}, \ \gamma = 4 \ \text{with} \ \gamma < n \ \text{and} \ \lambda_1 \cdot \lambda_2 < 0 \]

The approaches for both cases are the same, so we only prove the first case here. Without loss of generality, let $|\lambda_1| = |\lambda_2| = 1$.

In this case, we'll view $u$ the perturbation to the energy-critical problem
\[
\left\{ \begin{array}{l}
    iw_t + \Delta w = |w|^{\frac{4}{n-2}} w \\
    w(0) = u_{h_i}(0)
\end{array} \right.
\]
which is globally well-posedness by [23, 3, 26] and
\[
\| w \|_{\dot{S}^1(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M).
\]

By Lemma 5.6, (5.36) implies
\[
\| w \|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \leq C(E, M) \| u_0 \|_{L^2} \leq C(E, M)M_{\frac{1}{2}}.
\]

**Definition 5.1**
\[
\dot{D}^0(I) := V(I) \cap U(I) \cap L^{\frac{2(n+2)}{n-2}}_T L^{\frac{2(n+2)}{n^2+4}}_x.
\]

It is easy to know that
\[
\| (|x|^{-\gamma} |u|^2) u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u \|_{\dot{D}^k(I)} \| u \|_{\dot{D}^{4-\gamma}(I)} \| u \|_{\dot{D}^{\gamma-2}(I)}
\]
\[
\| u \|_{\dot{N}^k(I \times \mathbb{R}^n)} \lesssim \| u \|_{\dot{D}^k(I)} \| u \|_{\dot{D}^{4}(I)},
\]
where $k = 0, 1$.

Split $\mathbb{R}$ into $J = J(E, M, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that
\[
\| u \|_{\dot{D}^1(I_j)} \sim \eta,
\]
where $\eta > 0$ be a small constant to be chosen later.

Moreover, choosing $M$ sufficiently small depending on $E$ and $\eta$, in view of (5.37), we may assume
\[
\| w \|_{\dot{S}^0(\mathbb{R} \times \mathbb{R}^n)} \leq \eta.
\]
Then, we get
\[
\| u \|_{\dot{D}^1(I_j)} \sim \eta.
\]

In fact, on each slab $I_j \times \mathbb{R}^n$, we have
\[
\| e^{i(t-t_j)\Delta} w(t_j) \|_{\dot{D}^1(I_j)} \leq \| w \|_{\dot{D}^1(I_j)} + C \| w \|_{\dot{D}^{\frac{n+2}{2}}(I_j)} \lesssim \eta + C \eta^{\frac{n+2}{n-2}} \leq 2\eta
\]
if $\eta$ is sufficiently small.

Let $I_0 = [t_0, t_1]$. Since $w(t_0) = u(t_0) = u_0$, by using Strichartz estimates, (5.38), (5.39) and (5.41), we have
\[
\|u\|_{D^1(I_0)} \leq 2\eta + C \|w\|_{D^{\frac{n+2}{2}}(I_0)}^{\frac{n+2}{2}} + C \|w\|_{D^1(I_0)}^3.
\]
By a standard continuity argument, this yields
\[
\|u\|_{D^1(I_0)} \leq 4\eta
\] (5.42)
if $\eta$ is chosen sufficiently small.

On the other way, from Strichartz estimates, (5.38), (5.39) and (5.42), we have
\[
\|u\|_{\dot{D}^0(I_0)} \lesssim M^\frac{1}{2} + \|u\|_{\dot{D}^0(I_0)} + \|u\|_{\dot{D}^0(I_0)}^\gamma \|u\|_{\dot{D}^1(I_0)}^{\gamma - 2} \lesssim M^\frac{1}{2} + \eta^{\frac{\gamma - 2}{2}} \|u\|_{\dot{D}^0(I_0)} + \|u\|_{\dot{D}^0(I_0)}^{\gamma - 2}.\]
Therefore, choosing $\eta$ sufficiently small and $\gamma < 4$, we get
\[
\|u\|_{\dot{D}^0(I_0)} \lesssim M^\frac{1}{2}.
\]
In order to apply Lemma 3.4, we need to show the error $\|(|x|^{-\gamma} * |u|^2)u\|$ is small on the norm $N^1(I_0 \times \mathbb{R}^n)$. In fact, by
\[
\|(|x|^{-\gamma} * |u|^2)u\|_{N^1(I_0 \times \mathbb{R}^n)} \lesssim \|u\|_{\dot{D}^1(I_0)}^{\gamma - 1} \|u\|_{\dot{D}^0(I_0)}^{1 - \gamma} \lesssim \eta^{\gamma - 1} M^{2 - \frac{\gamma}{2}} \leq M^\delta_0
\]
for a small constant $\delta_0 > 0$. Then taking $M$ sufficiently small depending on $E$ and $\eta$, by Lemma 3.4, we get
\[
\|u - w\|_{\dot{S}^1(I_0 \times \mathbb{R}^n)} \lesssim M^{\delta_0}
\]
for a small constant $c > 0$ that depends only on the dimension $n$. Strichartz estimate implies
\[
\|e^{i(t - t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{S}^1(I_1 \times \mathbb{R}^n)} \lesssim M^{\delta_0}.
\] (5.43)
Now, we turn to the interval $I_1 = [t_1, t_2]$. By using Strichartz estimate, (5.38), (5.39), (5.41) and (5.43), one can get
\[
\|u\|_{D^1(I_1)} \lesssim \|e^{i(t - t_1)\Delta}u(t_1)\|_{\dot{D}^0(I_1)} + \|e^{i(t - t_1)\Delta}(u(t_1) - w(t_1))\|_{\dot{D}^1(I_1)} + \|e^{i(t - t_1)\Delta}w(t_1)\|_{\dot{D}^1(I_1)} + C \|u\|_{\dot{D}^1(I_1)}^{\frac{n+2}{2}} + C \|u\|_{\dot{D}^1(I_1)}^3.
\]
Choosing $\eta, M$ sufficiently small, by a standard continuity argument, we obtain
\[
\|u\|_{D^1(I_1)} \leq 4\eta
\]
Moreover, arguing as above, we also get
\[
\|u\|_{\dot{D}^0(I_1)} \lesssim M^\frac{1}{2}.
\]
For $M$ sufficiently small, we can apply Lemma 3.4 to obtain
\[ \| u - w \|_{S^1(I_1 \times \mathbb{R}^n)} \leq M^{c_1} \]
for a small constant $0 < \delta_1 < \delta_0$.

By using the induction argument, choosing $M$ smaller at every step, we obtain
\[ \| u \|_{D^1(I_j)} \leq 4\eta. \]
Summing these estimates over all intervals $I_j$ and for the total number of these intervals is $J = J(E, M, \eta)$, we get
\[ \| u \|_{S^1(R)} \lesssim J \eta \leq C(E, M). \]
By using Strichartz estimate, (5.38) and (5.39), we get
\[ \| u \|_{S^1(R \times R^n)} \lesssim \| u_0 \|_{H^1_x} + \| u \|_{D^1(R)} + \| u \|_{D^1(R)} \lesssim M + E + C(E) \leq C(E, M). \]

5.7 Global bounds in the case: $\frac{4}{n} \leq p < \frac{4}{n-2}$, $2 \leq \gamma < 4$ with $\gamma < n$ and $\lambda_1 \cdot \lambda_2 < 0$ or $p = \frac{4}{n}$, $\gamma = 2$ and $\lambda_1, \lambda_2 > 0$

The approaches for both cases are similar with the subsection 5.6, the only differentia is to compare $u$ to the free Schrödinger equation
\[ i\tilde{u}_t + \Delta \tilde{u} = 0, \quad \tilde{u}(0) = u_0. \]
By Strichartz estimate, the global solution $\tilde{u}$ obeys the spacetime estimates
\[ \| \tilde{u} \|_{S^1(R \times R^n)} \lesssim \| u_0 \|_{H^1_x} \leq C(E, M), \]
\[ \| \tilde{u} \|_{S^0(R \times R^n)} \lesssim \| u_0 \|_{L^2_x} \lesssim M^{\frac{1}{2}}. \]
At this time, we define

**Definition 5.2** $\dot{D}^0(I) := V(I) \cap U(I) \cap L_t^{\frac{2(n+2)}{n-2}} L_x^{\frac{2(n+2)}{n+4}}$.

By the similar method of subsection 5.6, it is not difficult to know that
\[ \| u \|_{S^1(R \times R^n)} \leq C(E, M). \]

5.8 Finite global Strichartz norms imply scattering

At last, we’ll show that finite global Strichartz norms imply scattering. For simplicity, we only construct the scattering state in the positive time direction. Similar arguments can be used to construct the scattering state in the negative time direction.

For $0 < t < \infty$, define
\[ u_+(t) = u_0 - i \int_0^t e^{-is\Delta} \left( \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} \ast |u|^2) u \right) \, ds. \]
Since $u \in S^1(\mathbb{R} \times \mathbb{R}^n)$, Strichartz estimates and Lemma 2.8 show that $u_+(t) \in H^1_x$ for all $t \in \mathbb{R}^+$, and for $0 < \tau < t$, we have

$$\| u_+(t) - u_+(-\tau) \|_{H^1_x} \lesssim \| \int_{-\tau}^{t} e^{i(t-s)\Delta} \left( \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \right) \, ds \|_{L^\infty_t H^1_x([\tau, t] \times \mathbb{R}^n)}$$

$$\lesssim \| u \|_{\dot{V}^p_W([\tau, t])} \| u \|_{\dot{W}^{-2,2}_p([\tau, t])} \| (1 + |\nabla|)u \|_{\dot{V}([\tau, t])}$$

$$+ \| u \|_{\dot{V}^{1- \gamma}_{2, 2}([\tau, t])} \| u \|_{\dot{V}^{-2,2}_p([\tau, t])} \| (1 + |\nabla|)u \|_{\dot{V}([\tau, t])},$$

and for $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\| u_+(t) - u_+(-\tau) \|_{H^1_x} \leq \varepsilon$$

for any $t, \tau > T_\varepsilon$. Thus $u_+(t)$ converges to some function $u_+ \in H^1_x$ as $t \to +\infty$. In fact

$$u_+ := u_0 - i \int_{0}^{\infty} e^{-is\Delta} \left( \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \right) \, ds.$$

At last, the scattering follows from

$$\| e^{-it\Delta} u(t) - u_+ \|_{H^1_x} = \| \int_{-\infty}^{\infty} e^{-is\Delta} \left( \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \right) \, ds \|_{H^1_x}$$

$$= \| \int_{-\infty}^{\infty} e^{i(t-s)\Delta} \left( \lambda_1 |u|^p u + \lambda_2 (|x|^{-\gamma} * |u|^2) u \right) \, ds \|_{H^1_x}$$

$$\lesssim \| u \|_{\dot{V}^p_W([t, \infty])} \| u \|_{\dot{W}^{-2,2}_p([t, \infty])} \| (1 + |\nabla|)u \|_{\dot{V}([t, \infty])}$$

$$+ \| u \|_{\dot{V}^{1- \gamma}_{2, 2}([t, \infty])} \| u \|_{\dot{V}^{-2,2}_p([t, \infty])} \| (1 + |\nabla|)u \|_{\dot{V}([t, \infty])},$$

because the right term obviously tends to 0 as $t \to +\infty$. The other properties follow from conservation of mass and energy.

6 Blowup results

From the Theorem 1.1, we can find that there are still many regions where the global well-posedness holds need a few additional conditions, for example small energy and small mass. In this section, we’ll show that on these regions, under suitable assumptions the solution of (1.1) will blow up in finite time. We follow the method of Glassey [9], which is essentially a convexity method. We consider the variance

$$f(t) = \int_{\mathbb{R}^n} |x|^2 |u(t, x)|^2 \, dx.$$

For strong $H^1_x$-solution $u$ to (1.1) with initial datum $u_0 \in \Sigma$, it is well known that $f \in C^2(-T_{\min}, T_{\max})$ and we have (see, for example the Chapter 6 of [6])

**Lemma 6.1** For all $t \in (-T_{\min}, T_{\max})$, we have

$$f'(t) = 4i \text{Im} \int \bar{u} \cdot \nabla u \, dx.$$
where
\[ \lambda \gamma \]

From (6.1), the conservation of energy and our assumption, we get
\[ f''(t) = 16E + \frac{4np - 16}{p + 2} \lambda_1 \| u \|_{L_p}^{p+2} + 2\lambda_2(\gamma - 2) \int (|x|^{-\gamma} * |u|^2)|u|^2 \, dx. \] (6.1)

If we can find, for all \( t \in (-T_{\text{min}}, T_{\text{max}}) \) there exists a constant \( A \) such that: \( f''(t) \leq A \), then we have
\[ \| xu \|_{L_2}^2 \leq \theta(t) \] (6.2)
where
\[ \theta(t) = \| x\varphi \|_{L_2}^2 + 4t \text{Im} \int \bar{\varphi} x \cdot \nabla \varphi \, dx + \frac{1}{2} t^2 A. \]

If assume \( A \) is negative, observe that \( \theta(t) \) is a second-degree polynomial, then \( \theta(t) < 0 \) for \( |t| \) large enough. Since \( \| xu \|_{L_2}^2 \geq 0 \), we deduce from (6.1) that both \( T_{\text{min}} \) and \( T_{\text{max}} \) are finite. However, it is not a necessary and sufficient condition so that \( \theta(t) \) takes negative values that \( A \) is negative. A necessary and sufficient condition so that \( \theta(t) \) takes negative values is that
\[ 8(\text{Im} \int \bar{\varphi} x \cdot \nabla \varphi \, dx)^2 > A \| x\varphi \|_{L_2}^2. \]

But in many states, we can’t get both \( T_{\text{min}} \) and \( T_{\text{max}} \) are finite. People who are interested in it can see the Chapter 6 of [5].

In the following, we’ll find the negative constant \( A \) such that \( f''(t) \leq A \):

- **case (1):** \( \lambda_1 < 0, \lambda_2 > 0, \frac{4}{n} \leq p \leq \frac{4}{n-2}, 0 < \gamma \leq \frac{np}{2} \) and \( E < 0 \).

By using (6.1), the conservation of energy and our assumption, we get
\[ f''(t) = 16E + (4np - 16)\{E - \frac{1}{2} \| \nabla u \|_{L_2}^2 - \frac{\lambda_2}{4} \int (|x|^{-\gamma} * |u|^2)|u|^2 \, dx \}
+ 2\lambda_2(\gamma - 2) \int (|x|^{-\gamma} * |u|^2)|u|^2 \, dx \]
\[ = 4npE - (2np - 8) \| \nabla u \|_{L_2}^2 - (np - 2\gamma)\lambda_2 \int (|x|^{-\gamma} * |u|^2)|u|^2 \, dx \]
\[ \leq 4npE. \] (6.3)

Let \( A := 4npE < 0 \), then we find the negative constant \( A \).

- **case (2):** \( \lambda_1 > 0, \lambda_2 < 0, \frac{2\gamma}{n} \leq p \leq \frac{4}{n-2}, 2 \leq \gamma \leq 4 \) and \( E < 0 \).

From (6.1), the conservation of energy and our assumption, we get
\[ f''(t) = 16E + \frac{4np - 16}{p + 2} \lambda_1 \| u \|_{L_p}^{p+2}
+ 8(\gamma - 2)\{E - \frac{1}{2} \| \nabla u \|_{L_2}^2 - \frac{\lambda_1}{p + 2} \| u \|_{L_p}^{p+2} \}
= 8\gamma E - 4(\gamma - 2) \| \nabla u \|_{L_2}^2 + \frac{4np - 8\gamma}{p + 2} \lambda_1 \| u \|_{L_p}^{p+2} \]
\[ \leq 8\gamma E. \] (6.4)

Let \( A := 8\gamma E < 0 \), then we find the negative constant \( A \).

- **case (3):** \( \lambda_1 < 0, \lambda_2 < 0, \frac{4}{n} \leq p \leq \frac{4}{n-2}, 2 \leq \gamma \leq 4 \) and \( E < 0 \).

When \( \gamma \geq \frac{np}{2} \), using (6.3) and our assumption, we have
\[ f''(t) \leq 4npE. \]
When $\gamma < \frac{np}{2}$, from (6.4) and our assumption, we have

$$f''(t) \leq 8\gamma E.$$ 

So we also find the negative constant $A$.

**case (4)** $\lambda_1 < 0$, $\lambda_2 < 0$, $0 < \gamma < 2$, $\frac{4}{n} < p \leq \frac{4}{n-2}$ and $4npE + C(M) < 0$

By using (4.2) and Young’s inequality, we have, when $\gamma < 2$,

$$\| \nabla u \|^2_{L^2} \leq \delta \| \nabla u \|^2_{L^2} + C(\delta).$$

From (6.3) and our assumption, we have

$$f''(t) \leq 4npE + [C(np - 2\gamma)|\lambda_2|\delta - (2np - 8)] \| \nabla u \|^2_{L^2} + C(np - 2\gamma)|\lambda_2|C(\delta) \| u \|^4_{L^2}. $$

Choosing $\delta$ sufficiently small, then

$$f''(t) \leq 4npE + C(M).$$

Let $A := 4npE + C(M) < 0$, then we find the negative constant $A$.

**case (5)** $\lambda_1 < 0$, $\lambda_2 < 0$, $2 < \gamma \leq 4$, $0 < p < \frac{4}{n}$ and $8\gamma E + C(M) < 0$.

From (4.1) and Young’s inequality, we have, when $p < \frac{4}{n}$,

$$\| \nabla u \|^2_{L^2} \leq \delta \| \nabla u \|^2_{L^2} + C(\delta).$$

From (6.3) and our assumption, we have

$$f''(t) \leq 8\gamma E - 4(\gamma - 2) \| \nabla u \|^2_{L^2} + \left(\frac{4np - 8\gamma}{p + 2} \lambda_1 C\delta \| \nabla u \|^2_{L^2} + \frac{4np - 8\gamma}{p + 2} \lambda_1 C(\delta)\right) \| u \|^4_{L^2}. $$

Choosing $\delta$ sufficiently small, then

$$f''(t) \leq 8\gamma E + C(M).$$

Let $A := 8\gamma E + C(M) < 0$, then we find the negative constant $A$.

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