CLASSIFICATION OF SINGULAR SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS

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Abstract. In this paper, we mainly investigate the classification of singular sets of solutions to elliptic equations. Firstly, we define the j-symmetric singular set $S_j(u)$ of solution $u$, and show that the Hausdorff dimension of the j-symmetric singular set $S_j(u)$ is not more than $j$. Then we prove the generalized $\varepsilon$-regularity lemma for j-symmetric homogeneous harmonic polynomial $P$ with origin 0 as the isolated critical point in $\mathbb{R}^{n-j}$, and by the generalized $\varepsilon$-regularity lemma, we show the Hausdorff measure estimate of the j-symmetric singular set $S_j(u)$. Moreover, we study the geometric structure of interior singular points of solutions $u$ in a planar bounded domain.

1. Introduction. In this paper we mainly consider the classification of singular sets of solutions to the following linear elliptic equations in the unit ball $B_1(0) \subset \mathbb{R}^n$:

$$Lu(x) := \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x) u(x) = 0,$$

(1)

where the coefficients $a_{ij}(x), b_i(x)$ and $c(x)$ satisfy the following assumptions:

$$\begin{align*}
\sum_{i,j=1}^{n} a_{ij}(x) \eta_i \eta_j & \geq \lambda |\eta|^2 \text{ for any } \eta \in \mathbb{R}^n, x \in B_1(0); \\
\sum_{i,j=1}^{n} |a_{ij}(x)| + \sum_{i=1}^{n} |b_i(x)| + |c(x)| & \leq \kappa \text{ for any } x \in B_1(0); \\
\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)| & \leq l|x - y| \text{ for any } x, y \in B_1(0)
\end{align*}$$

(2)

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for some positive constants $\lambda, \kappa$ and $l$.

Assume that $u$ is a $W^{2,2}(B_1)$ solution of (1) in $B_1$ such that

$$\frac{\int_{B_1(0)} |\nabla u|^2 dx}{\int_{\partial B_1(0)} u^2 ds} \leq N_0$$

for some positive constant $N_0$. By Theorem 1.3 in [12], we have the following doubling estimate

$$\int_{B_{2r}(x_0)} u^2 dx \leq 4^N \int_{B_r(x_0)} u^2 dx \text{ for any } x_0 \in B_{2/3}, r < r_0,$$

where $N = C(n, \lambda, \kappa, l)N_0$ and $r_0 = r_0(n, \lambda, \kappa, l)$.

The subject of geometric measure and structure of singular and critical sets is a significant research topic for solutions to elliptic equations. Until now, there are many results about the geometric measure and structure of singular and critical set, respectively.

In 1987 Alessandrini [1] investigated the geometric structure of the critical set of solutions to linear elliptic equations without zero order term in a bounded simply connected domain in $\mathbb{R}^2$. In 1994, Sakaguchi [30] considered the critical points of solutions to an obstacle problem in a planar, bounded, smooth and simply connected domain. He showed that if the number of critical points of the obstacle is finite and the obstacle has only $N$ local (global) maximum points, then the inequality $\sum_{i=1}^{k} m_i + 1 \leq N$ (the equality $\sum_{i=1}^{k} m_i + 1 = N$) holds for the critical points of one solution in the noncoincidence set, where $m_1, m_2, \ldots, m_k$ are the multiplicities of critical points $x_1, x_2, \ldots, x_k$ respectively. Recently, the authors [5] investigated the geometric structure of interior critical points of solutions $u$ to a quasilinear elliptic equation with nonhomogeneous Dirichlet boundary conditions in a simply connected or multiply connected domain $\Omega$ in $\mathbb{R}^2$. For the related research work, we refer, for example, [2, 6, 7, 8, 10, 20, 28].

In 1989, Hardt and Simon [17] proved that the singular sets for classical solutions to a linear elliptic equation are countable unions of subsets of correspondingly smooth $(n-2)$ dimensional submanifolds. It was generalized to weak solution by Han in 1994 (see [13]). In 1996, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof and N. Nadirashvili [19] proved that the singular sets of smooth solutions for the second order elliptic equation in three dimensional space have locally finite one dimensional Hausdorff measure. This result was generalized to higher dimensional space in 1999 (see [18]). In 1998, Han, Hardt and Lin [14] concerned with the geometric measure of singular sets of weak solutions of elliptic second-order equations. They gave a uniform estimate on the measure of singular sets in terms of the frequency of solutions. In 2003, Han, Hardt and Lin [15] studied the singular sets of solutions to arbitrary order elliptic equations. In 2012, Laurent [21] considered the eigenfunctions of the Laplacian on a compact, analytic, $n$-dimensional Riemannian manifold and proved that the $(n-1)$-dimensional Hausdorff measure of the critical set of each eigenfunction is bounded above by $C\sqrt{\lambda}$, where $C$ is a constant that only depends on the manifold and $\lambda$ is the corresponding eigenvalue. In 2015, Cheeger, Naber and Valtorta [4] introduced some techniques for giving improved estimates of the critical set, including the Minkowski type estimates on the effective critical set $C_r(u)$, which roughly consists of points $x$ such that the gradient of $u$ is small somewhere on $B_r(x)$ compared to the nonconstancy of $u$. In 2017, Naber and Valtorta [29] introduced some new techniques for estimating the critical set and singular set, which avoids
Theorem 1.2. Let \( S_{N,r_0}(\lambda, M, \alpha, K) \) such that \( u \in S_{N,r_0}(\lambda, M, \alpha, K) \) and \( u \) is a \( W^{2,2}(B_1) \) solution of (1) satisfying (3), which the coefficients \( a_{ij}(x), b_i(x), c(x) \) satisfy

\[
\begin{align*}
\sum_{i,j=1}^{n} a_{ij} \eta_i \eta_j &\geq \lambda |\eta|^2 \quad \text{for any } \eta \in \mathbb{R}^n, x \in B_1(0); \\
\sum_{i,j=1}^{n} ||a_{ij}||_{C^{M,\alpha}(B_1)} + \sum_{i=1}^{n} ||b_i||_{C^{M,\alpha}(B_1)} + ||c||_{C^{M,\alpha}(B_1)} &\leq K,
\end{align*}
\]

where \( \lambda, \alpha \in (0,1) \), \( K > 0 \) and \( M \) is a nonnegative integer.

Our main results are as follows:

**Theorem 1.1.** Let \( P \) be a \((n-j)\)-symmetric harmonic polynomial with degree \( d \) in \( \mathbb{R}^n \) and origin \( 0 \) be an isolated critical point of \( P \) in \( \mathbb{R}^j \), where integer \( 2 \leq j \leq n \). Then there exist positive constants \( \delta \) and \( r \), depending on \( P \), such that for any \( u \in C^{2d\epsilon}(B_1) \), if

\[ |u - P|_{C^{2d\epsilon}(B_1)} < \delta, \]

then

\[ \mathcal{H}^{n-j}(|\nabla u|^{-1}(0) \cap B_r) \leq c(n,j)(d-1)^{j}r^{n-j}, \]

where \( \mathcal{H}^{n-j} \) denotes the \((n-j)\)-dimensional Hausdorff measure.

**Remark 1.** The result of case \( j = 2 \) in Theorem 1.1 was proved in [14]. Theorem 1.1 is the generalization of the main result in [14] to lower stratus of the singular sets and the proof of Theorem 1.1 follows closely the proof in [14].

**Theorem 1.2.** Let \( u \) be a \( W^{2,2} \)-solution of (1) in \( B_1(0) \). For any \( \epsilon > 0 \) and any \( u \in S_{N,r_0}(\lambda, 2N^2, \alpha, K) \), we have

\[ \mathcal{H}^{j}\{S^j(u) \cap B_{1/2}\} \leq C(j, \epsilon), \]

where \( \mathcal{H}^{j} \) is as in Theorem 1.1 and positive constants \( C(j, \epsilon) \) also depends on \( N, \lambda, \alpha \) \( K \) and \( n \).

Moreover, we further study the geometric structure of interior singular points of solutions \( u \) in a planar domain.

**Theorem 1.3.** Let \( \Omega \) be a bounded, smooth and simply connected domain in \( \mathbb{R}^2 \). Suppose that \( \psi(x) \in C^1(\overline{\Omega}) \) is sign-changing and that \( \psi \) has \( N \) isolated zero points on \( \partial \Omega \). In addition, suppose that \( a_{ij}(x), b_i(x), c(x) \) are smooth and \( c(x) \leq 0 \). Let \( u \) be a non-constant solution of the following boundary value problem

\[
\begin{align*}
\sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{2} b_i(x)u_{x_i}(x) + c(x)u(x) &= 0 \text{ in } \Omega, \\
u &= \psi(x) \text{ on } \partial \Omega,
\end{align*}
\]

Then \( u \) has finite interior singular points, that is, the points such that \( u(x) = |\nabla u(x)| = 0 \), and

\[ S \leq \frac{n}{2} - 1, \]

where \( S := \sharp\{x \in \Omega; u(x) = |\nabla u(x)| = 0\} \).
The rest of this paper is written as follows. In Section 2, we recall the \( j \)-symmetric function in [4] and define the \( j \)-symmetric singular set \( S^j(u) \) of solution \( u \), and prove that the Hausdorff dimension of the \( j \)-symmetric singular set \( S^j(u) \) is not more than \( j \). In Section 3, we prove the generalized \( \varepsilon \)-regularity lemma for \( j \)-symmetric homogeneous harmonic polynomial \( P \) with origin 0 as the isolated critical point in \( \mathbb{R}^{n-j} \), and then show the Hausdorff measure estimate of the \( j \)-symmetric singular set \( S^j(u) \). In Section 4, we study the geometric structure of interior singular points of solutions \( u \) to linear elliptic equations with nonhomogeneous Dirichlet boundary conditions in a planar bounded domain.

2. Hausdorff dimension of the \( j \)-symmetric singular set. In this section, we mainly consider the Hausdorff dimension of the \( j \)-symmetric singular set \( S^j(u) \) of a solution \( u \). Firstly, we need introduce the following definitions:

**Definition 2.1** (\( j \)-symmetric, see [4], Definition 1.4). Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a smooth function:

(i) We say \( u \) is 0-symmetric if \( u \) is a homogeneous polynomial;

(ii) We say \( u \) is \( j \)-symmetric if \( u \) is 0-symmetric and there exists a largest \( j \)-dimensional subspace \( U \) such that, for every \( x \in \mathbb{R}^n \) and \( z \in U \), we have that \( u(x + z) = u(x) \).

According to Definition 2.1, we know that if a function is \( j \)-symmetric, then it is not \((j-1)\)-symmetric. For example, \( u(x, y, z, w) = xy \) in 4-dimensional space, it is only a 2-symmetric polynomial (not 1-symmetric).

**Definition 2.2** (tangent maps, see [4], Definition 1.2). Let \( u : B_1(0) \to \mathbb{R} \) be a smooth nonconstant function and \( r > 0 \), then

(i) For \( y \in B_{1-r}(0) \), we define

\[
T_{y,r}u(x) := \frac{u(y + rx) - u(y)}{\left(\int_{\partial B_r(0)} (u(y + rx) - u(y))^2 \, ds\right)^{\frac{1}{2}}}. 
\]

If the denominator vanishes, we set \( T_{y,r} = \infty \).

(ii) For \( y \in B_1(0) \), we define

\[
T_{y,0}u(x) := T_yu(x) = \lim_{r \to 0} T_{y,r}u(x). 
\]

According to the definition of \( T_yu(x) \), we know that \( T_yu \) is just the leading polynomial of the Taylor expansion of \( u - u(y) \) at \( y \). In particular, \( T_yu \) is a homogeneous polynomial, and if \( u \) satisfies a second-order elliptic equation, then this polynomial is a homogeneous solution to the constant-coefficient equation \( \sum_{i,j=1}^n a_{ij}(y)D_{x,x_j}T_yu = 0 \). Then, by a change of coordinates, it is a homogeneous harmonic polynomial.

**Definition 2.3** (\( j \)-symmetric singular set). For any integer \( j \leq n-3 \), we define the following \( j \)-symmetric singular set of \( u \)

\[
S^j(u) = \{ p \in S(u) : T_pu(x) \text{ is } j \text{-symmetric and origin } 0 \text{ as the isolated critical point in } \mathbb{R}^{n-j} \} \tag{8}
\]

and

\[
S^{n-2}(u) = \{ p \in S(u) : p \text{ is not in } S^j(u) \text{ for any } j \leq n-3 \}, \tag{9}
\]

where \( S(u) \) is the singular set of \( u \), that is, \( S(u) = \{ x \in B_1(0) : u(x) = |\nabla u(x)| = 0 \} \).
According to the above definition of $S^j(u)$, we can easily know that singular sets

$$S(u) = \bigcup_{j \leq n-2} S^j(u).$$

**Definition 2.4** (see[13, 16]). For any $p \in B_1(0)$, we define the vanishing order $O(p)$ of $u$ at $p$ by

$$O(p) = O_u(p) := \{ k : \partial^\alpha u(p) = 0 \text{ for any } |\alpha| < k, \partial^\alpha u(p) \neq 0 \text{ for some } |\alpha| = k \}.$$  

The rest of this section is aimed to give the estimate of Hausdorff dimension of the $j$-symmetric singular set $S^j(u)$ of solutions to elliptic equations (1). In order to prove the dimensional estimate of the $j$-symmetric singular set, we need the following lemma.

**Lemma 2.5** (see[13], see also [16], Lemma 4.3.1). Suppose that $\{L_l\}_{l=0}^\infty$ is a family of elliptic operators in $B_1$ of the form (1), with coefficients satisfying (2), and that $u_l$ is a $W^{2, p}$ solution of $L_l u_l = 0$ in $B_1$ for $l = 0, 1, \cdots$, and for some $p > n/2$.

Assume that $L_l \to L_0$ in the sense that the corresponding coefficients converge uniformly and that $u_l \to u_0$ in $L^p(B_1)$. Then

$$\limsup_{l \to \infty} O_{u_l}(0) \leq O_{u_0}(0).$$

If, in addition, $O_{u_l}(0) = k$ and $P_l$ is the leading polynomial of $u_l$ at $0$ for $l = 1, 2, \cdots$, then the following conclusions hold:

(i) If $O_{u_0}(0) > k$, then

$$P_l \to 0 \text{ uniformly in } B_1 \text{ as } l \to \infty;$$

(ii) If $O_{u_0}(0) = k$, then

$$P_l \to P_0 \text{ uniformly in } B_1 \text{ as } l \to \infty,$$

where $P_0$ is the leading polynomial of $u_0$ at 0.

Now we give the dimensional estimate of the $j$-symmetric singular set. The proof of next theorem follows closely the proof in [13].

**Theorem 2.6.** Let $u$ be a $W^{2, 2}$-solution of (1) with (2) in $B_1(0)$. Then we have

$$\dim S^j(u) \leq j,$$

where the integer $j \leq n - 2$.

**Proof.** We divide the proof into two steps.

For any $k \geq 2$, we define

$$S_k(u) := \{ p \in S(u) : O_u(p) = k \}.$$  

(10)

Step 1, we study the local behaviors at each singular point and prove that $\dim S_k(P_p) \leq n-2$, where $P_p$ is the leading polynomial of the Taylor expansion of $u$ at $p$ and $p$ is a singular point of $u$. For any $p \in \partial B_1 \cap S_k(u)$ and any $r \in (0, \frac{1}{2}(1-|p|))$, by the Definition 2.2, we know

$$T_{p, r} u(x) := \frac{u(p + r x)}{\left( \int_{\partial B_1(0)} |u|^2 \right)^{\frac{1}{2}}} \text{ for any } x \in B_{4/3}.$$  

(11)

For some $p > n/2$, by Definition 2.2 and Lemma 2.5, then we have

$$T_{p, r} u(x) \to P \text{ in } L^p(B_{4/3}) \text{ as } r \to 0,$$
where $P = P_p$ is a nonconstant homogeneous polynomial of degree $k$ such that \( \sum_{i,j=1}^{n} a_{ij}(p)D_{x_i x_j}P = 0 \) and \( ||P||_{L^2(\partial B_1)} = 1 \). In fact, since $p \in B_{3/4} \cap S_k(u)$, we set $u(p + rx) = P(rx) + o(|rx|^k)$. Then

\[
T_{p,r}u(x) = \frac{P(rx) + o(|rx|^k)}{\left( f_{\partial B_1}(0)(P(rx) + o(|rx|^k))^2 \right)^{\frac{1}{2}}} \\
= \frac{r^k P(x) + r^k o(|x|^k)}{r^k (f_{\partial B_1}(0)(P(x) + o(|x|^k))^2)^{\frac{1}{2}}} \\
\to \frac{P(x)}{|P|^2} \ as \ r \to 0 \\
= P(x).
\]

Since $P$ is a nonconstant homogeneous polynomial of degree $k$, we know that

\[
S_k(P) = \{ x : \partial^\alpha P(x) = 0 \ for \ any \ |\alpha| \leq k-1 \}.
\]

Obviously, $0 \in S_k(P)$. Moreover, we claim that

$S_k(P)$ is a linear subspace and $P(x) = P(x + p)$ for any $x \in \mathbb{R}^n$ and $p \in S_k(P)$.

In fact, by the homogeneity of $P$, we assume that

\[
P(x) = \sum_{|\alpha| = k} a_{\alpha} x^\alpha.
\]

For any $p \in S_k(P)$, then we have

\[
P(x) = \sum_{|\alpha| = k} a_{\alpha} (x - p)^\alpha.
\]

This implies that $P(x) = P(x + p)$ for any $x \in \mathbb{R}^n$ and $p \in S_k(P)$. Then, for any $p_1, p_2 \in S_k(P)$, we have

\[
\partial^\alpha P(p_1 + p_2) = \partial^\alpha P(p_1) = 0 \ for \ any \ |\alpha| \leq k-1.
\]

(14)

So $(p_1 + p_2) \in S_k(P)$.

Since $P$ is a homogeneous polynomial of degree $k$, then we have

\[
P(x) = P(x + \tau p) \ and \ \partial^\alpha P(\tau p) = 0
\]

for any $x \in \mathbb{R}^n, \tau \in \mathbb{R} \ and \ |\alpha| \leq k-1$. Therefore, $\tau p \in S_k(P)$ for any $\tau \in \mathbb{R}$. Hence (14) and (15) imply (13).

Next, we claim that

\[
dim S_k(P_p) \leq n - 2 \ for \ any \ k \geq 2.
\]

(16)

In fact, suppose by contradiction that $\dim S_k(P_p) = n - 1$. By (13), we have that $P$ is a function of $(n - \dim S_k(P))$ variables, i.e., $P$ is a degree $k$ monomial of one variable such that $\sum_{i,j=1}^{n} a_{ij}(p)D_{x_i x_j}P = 0$. Then $k < 2$, this contradicts with $k \geq 2$. This means (16).

Step 2, for any $j \leq n - 3$, we define

\[
S_k^j(u) = \{ p \in S_k(u) : P_p \ is \ j\text{-symmetric} \ \text{and} \ \text{origin} \ 0 \ as \ the \ isolated \ critical \ point \ in \ \mathbb{R}^{n-j} \}.
\]
and

\[ S_k^{n-2}(u) = \{ p \in S_k(u) : p \text{ is not in } S_k^j(u) \text{ for any } j \leq n - 3 \}. \]

By the results in [13, 14, 16], we know that \( \dim S_k^{n-2}(u) \leq n - 2 \) for any \( k \geq 2 \).

Next, we will prove that \( \dim S_k^j(u) \leq j \) for any \( j \leq n - 3 \) and \( k \geq 2 \). Firstly we claim that \( S_k^j(u) \) is on a countable union of \( j \)-dimensional \( C^1 \) graphs. In other words, we should show that there exists a \( r = r(p) \) such that \( S_k^j(u) \cap B_r(p) \) is contained in a \( j \)-dimensional \( C^1 \) graph for any \( p \in S_k^j(u) \).

In order to prove this, we use \( \gamma_p \) to denote the \( j \)-dimensional linear subspace \( S_k(P_p) \) for any \( p \in S_k^j(u) \). Firstly, for any \( \{ p_l \} \subset S_k^j(u) \) such that \( p_l \to p \), we show that

\[
\text{Angle } (\overrightarrow{pp_l}, \gamma_p) \to 0. 
\tag{17}
\]

In order to show (17), without loss of generality, we suppose \( p = 0 \) and \( q_l = \frac{p_l}{|p_l|} \to \rho \in S^{n-1} \). By (11), we know \( q_l \in S_k(T_0,|p_l|u) \).

By an easy calculation, we have that

\[
L_i T_{0,|p_l|u} = \sum_{i,j=1}^n a_{ij}(|p_l|x) D_{x_i,x_j} u(|p_l|x) + |p_l| \sum_{i=1}^n b_i(|p_l|x) D_{x_i} u(|p_l|x)
= |p_l|^2 c(|p_l|x) u(|p_l|x)
= 0,
\]

where \( L_i \) is some second order elliptic operator with a similar structure as \( L \) in (1).

Furthermore, we have

\[
L_i \to L_0 = \sum_{i,j=1}^n a_{ij}(0) D_{x_i,x_j},
\]

in the sense that corresponding coefficients converge uniformly in \( L^p(B_1) \) for some \( p > n/2 \). Therefore, by Lemma 2.5, we know that the vanishing order of \( P_p \) at \( \rho \) is at least \( k \), i.e., \( O_{P_p}(\rho) \geq k \). Since \( P_p \) is a homogeneous polynomial of degree \( k \), hence \( O_{P_p}(\rho) = k \) and \( \rho \in S_k(P_p) = \gamma_p \). This implies (17).

Secondly, for any \( p \in S_k^j(u) \) and any small \( \delta > 0 \), by (17), there exists an \( r = r(p, \delta) \) satisfying

\[
S_k^j(u) \cap B_r(p) \subset R_\delta(\gamma_p) \cap B_r(p),
\tag{18}
\]

where \( R_\delta(\gamma_p) := \{ y \in \mathbb{R}^n : \text{dist}(y, \gamma_p) \leq \delta \} \) is a cone domain.

Next we assume that \( P_l \) and \( P \) are the leading polynomials of \( u \) at \( p_l \) and \( p = 0 \) respectively. According to Lemma 2.5, we obtain

\[
P_l \to P \text{ uniformly in } L^k(B_1).
\]

This implies \( \gamma_{p_l} \to \gamma_p \) as \( l \to \infty \), as subspaces in \( \mathbb{R}^n \). By a similar argument in proving (17), we prove that the constant \( r \) in (18) can be chosen uniformly for any point \( y \in S_k^j(u) \) in a neighborhood of \( p \). In other words, for any \( p \in S_k^j(u) \) and any small \( \delta > 0 \), there exists an \( r = r(\delta, y) \) satisfying

\[
S_k^j(u) \cap B_r(y) \subset R_\delta(\gamma_y) \cap B_r(y) \text{ for any } y \in S_k^j(u) \cap B_r(p).
\]

For small enough \( \delta \), this implies that \( S_k^j(u) \cap B_r(p) \) is contained in a \( j \)-dimensional Lipschitz graph and (17) implies this graph is \( C^1 \). \( \blacksquare \)
3. Hausdorff measure estimate of the \( j \)-symmetric singular set. In this section, we investigate the Hausdorff measure estimate of singular sets. For the sake of need, for any integer \( d \geq 1 \), we use \( \mathcal{H}_d(\mathbb{R}^n) \) to denote the collection of all homogeneous harmonic polynomials of degree \( d \) in \( \mathbb{R}^n \). Obviously, \( \mathcal{H}_d(\mathbb{R}^n) \) is a linear space. In the next lemma, we will give the explicit bases of \( \mathcal{H}_d(\mathbb{R}^n) \).

Lemma 3.1 (see\cite{[3]}, Theorem 5.25 and \cite{[27]}). (1) If \( n = 2 \), by the transformation of polar coordinate \( x = r \cos \theta, y = r \sin \theta \), then the set
\[
\{ r^d \cos(d\theta), r^d \sin(d\theta) \}
\]
is a vector space basis of \( \mathcal{H}_d(\mathbb{R}^2) \):

(2) If \( n = 3 \), by the transformation of polar coordinate \( x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta \), then the set
\[
\{ r^d g(\varphi) f_{d,k}(\cos \theta) \}
\]
is a vector space basis of \( \mathcal{H}_d(\mathbb{R}^3) \), where \( g(\varphi) = A \cos(k\varphi) + B \sin(k\varphi), f_{d,k}(\sigma) = (1 - \sigma^2)^{\frac{d+k}{2}} (1 - \sigma^2)^d, \sigma = \cos \theta \) and \( k = 0, 1, \ldots, d \).

(3) If \( n > 3 \), then the set
\[
\{ K[D^\alpha |x|^{2-n}] = c_d(x^\alpha - |x|^2q) : |\alpha| = d \text{ and } \alpha_1 \leq 1 \}
\]
is a vector space basis of \( \mathcal{H}_d(\mathbb{R}^n) \), where the function \( K[u] \) is the Kelvin transformation of \( u, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) is a multiindex, constant \( c_d = \prod_{k=0}^{d-1} (2 - n - 2k) \) and \( q \in \mathcal{P}_{d-2}(\mathbb{R}^n) \).

Moreover, if \( d \geq 2 \), then
\[
\dim \mathcal{H}_d(\mathbb{R}^n) = \dim \mathcal{P}_d(\mathbb{R}^n) - \dim \mathcal{P}_{d-2}(\mathbb{R}^n) = \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1},
\]
where \( \mathcal{P}_d(\mathbb{R}^n) \) is the complex vector space of all homogeneous polynomials of degree \( d \) in \( \mathbb{R}^n \).

Next we will give the generalized \( \varepsilon \)-regularity lemma.

Theorem 3.2 (generalized \( \varepsilon \)-regularity lemma). Let \( P \) be a \((n-j)\)-symmetric harmonic polynomial with degree \( d \) in \( \mathbb{R}^n \) and origin 0 be an isolated critical point of \( P \) in \( \mathbb{R}^n \), where the integer \( 2 \leq j \leq n \). Then there exists positive constants \( \delta \) and \( r \), depending on \( P \), such that for any \( u \in C^{2d+1}(B_1) \), if
\[
|u - P|_{C^{2d+1}(B_1)} < \delta,
\]
then
\[
\mathcal{H}^{n-j}(|\nabla u|^{-1} \{0\} \cap B_r) \leq c(n,j)(d-1)^j r^{n-j}.
\]

In order to prove Theorem 3.2, we need the following lemma.

Lemma 3.3 (see\cite{[14]}, Theorem 4.1). Let \( f \) be a smooth map from \( B_1 \subset \mathbb{R}^n \) to \( \mathbb{R}^n \) with \( f(0) = 0 \). Assume that the extended map \( f \) from the unit ball in \( C^n \) to \( C^n \) is holomorphic and that the origin is its isolated zero in \( C^n \). Then there exist positive constants \( \delta, M, r, \) and \( k \), all depending on \( f \), such that for any \( u \in C^M(B_1; \mathbb{R}^n) \) with
\[
|u - f|_{C^M(B_1)} < \delta,
\]
then
\[
\sharp(u^{-1} \{0\} \cap B_r) \leq k,
\]
where \( k = M/2 \).
Proof of Theorem 3.2. Let \( \bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \) be the coordinate in \( \mathbb{R}^n \). Since \( P \) is a \((n-j)\)-symmetric harmonic polynomial with degree \( d \) in \( \mathbb{R}^n \), then \( P \) is harmonic polynomial of degree \( d \) and of \( j \) variables in \( \mathbb{R}^j = \{(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_j)\} \). Now, for any \( P \in \mathcal{H}_d(\mathbb{R}^j) \) (the space \( \mathcal{H}_d(\mathbb{R}^j) \) see Lemma 3.1), then
\[
\nabla P = (D_{\bar{x}_1}P, D_{\bar{x}_2}P, \cdots, D_{\bar{x}_j}P).
\]
Therefore all \( D_{\bar{x}_1}P, D_{\bar{x}_2}P, \cdots, D_{\bar{x}_j}P \) are products of \((d-1)\) different homogeneous linear functions. In fact, there exist \( j(d-1) \) nonzero vectors \( \xi_k^i \in \mathbb{R}^j \) such that
\[
D_{\bar{x}_i}P(\bar{x}) = \prod_{k=1}^{d-1} (\xi_k^i \cdot (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_j)), \quad i = 1, 2, \cdots, j.
\]
(19)
By the harmonicity of \( P \), we have that
\[
\xi_k^i \neq c\xi_k^i \quad \text{for any} \ c \in \mathbb{R}^1 \text{ and } (i, k) \neq (\bar{i}, \bar{k}),
\]
(20)
where \( \bar{i}, \bar{k} = 1, 2, \cdots, j \), and \( \bar{k}, \bar{k} = 1, 2, \cdots, d-1 \). Obviously, \( D_{\bar{x}_i}P = 0 \) for \( i = j + 1, \cdots, n \).
Now we make a change of coordinate \( \bar{x} = Tx \), where \( T = (t_{ij}) \) is an orthogonal matrix. Next we calculate the \( \nabla P \) in the new coordinate. Put \( \beta_i = (t_{1i}, t_{2i}, \cdots, t_{ji}) \in \mathbb{R}^j \). Then, for each \( i = 1, 2, \cdots, n \), we have
\[
D_{\bar{x}}P(x) = t_{1i} \prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_n \cdot \xi_k^i)x_n)
\]
\[
+ t_{2i} \prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_n \cdot \xi_k^i)x_n)
\]
\[
+ \cdots + t_{ji} \prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_n \cdot \xi_k^i)x_n).
\]
(21)
If \( D_{\bar{x}}P(x) \) is not zero, then it is a homogeneous polynomial of degree \((d-1)\). Note that only the first \( j \) rows of the matrix \( T \) appear in the expression (21).
For any \( a \in \mathbb{R}^n \) and any \( 1 \leq i_1 < i_2 < \cdots < i_j \leq n \), we assume that \( \mathcal{A}_{i_1 \cdots i_j}(a) \) be the \( j \)-dimensional hyperplane
\[
\{(a_1, \cdots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \cdots, a_{i_2-1}, x_{i_2}, a_{i_2+1}, \cdots, a_{i_3-1}, x_{i_3}, a_{i_3+1}, \cdots, a_n)\}
\]
(22)
and simply assume that \( \mathcal{A}_{i_1 \cdots i_j}(a) = (x_{i_1}, x_{i_2}, \cdots, x_{i_j}) \).
Put
\[
\mathcal{A}_{i_1 \cdots i_j} := \mathcal{A}_{i_1 \cdots i_j}(0) = (0, \cdots, 0, x_{i_1}, 0, \cdots, 0, x_{i_2}, 0, \cdots, 0, x_{i_3}, 0, \cdots, 0).
\]
For any \( 1 \leq i_1 < i_2 < \cdots < i_j \leq n \), then \( D_{x_{i_1}}P(x), \cdots, D_{x_{i_j}}P(x) \) restricted on \( \mathcal{A}_{i_1 \cdots i_j} \) are given by
\[
\begin{pmatrix}
D_{x_{i_1}}P|_{\mathcal{A}_{i_1 \cdots i_j}} \\
D_{x_{i_2}}P|_{\mathcal{A}_{i_1 \cdots i_j}} \\
\vdots \\
D_{x_{i_j}}P|_{\mathcal{A}_{i_1 \cdots i_j}}
\end{pmatrix} =
\begin{pmatrix}
t_{i_{11}} & t_{i_{12}} & \cdots & t_{i_{1j}} \\
t_{i_{21}} & t_{i_{22}} & \cdots & t_{i_{2j}} \\
\vdots & \vdots & \ddots & \vdots \\
t_{i_{j1}} & t_{i_{j2}} & \cdots & t_{i_{jj}}
\end{pmatrix}
\begin{pmatrix}
\prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_j \cdot \xi_k^i)x_{i_j}) \\
\prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_j \cdot \xi_k^i)x_{i_j}) \\
\vdots \\
\prod_{k=1}^{d-1} ((\beta_1 \cdot \xi_k^i)x_1 + \cdots + (\beta_j \cdot \xi_k^i)x_{i_j})
\end{pmatrix}.
\]
(23)
By the assumption of origin 0 is an isolated critical point of \( P \) in \( \mathbb{R}^j \). We have that the origin 0 is an isolated zero of the map \( f_{i_1 \cdots i_j} = (D_{x_{i_1}}P(x), \cdots, D_{x_{i_j}}P(x))|_{\mathcal{A}_{i_1 \cdots i_j}} \).
where $f_{i_1 \cdots i_j}$ is a map from $\mathbb{R}^j$ to $\mathbb{R}^j$. In fact, if $f_{i_1 \cdots i_j}$ is viewed as a map from $\mathbb{C}^j$ to $\mathbb{C}^j$, with $x \in \mathbb{R}^n$ replaced by $z \in \mathbb{C}^n$, the origin is also its isolated zero. This is easy to see since each expression of $D_{x_{i_1}} P(x) \mathcal{A}_{i_1 \cdots i_j}, \cdots, D_{x_{i_j}} P(x) \mathcal{A}_{i_1 \cdots i_j}$ in (23) is a product of $d - 1$ homogeneous linear functions with real valued coefficients. Hence $f_{i_1 \cdots i_j}$ satisfies the assumption of Lemma 3.3 for $n = j$. Then there exist positive constants $\delta_{i_1 \cdots i_j}$ and $r_{i_1 \cdots i_j}$ such that for any $v \in C^M(B^{j}_{1/2}; \mathbb{R}^j)$ with
\[ |v - f_{i_1 \cdots i_j}|_{C^M(B^{j}_{1/2})} < \delta_{i_1 \cdots i_j}, \]
then
\[ \sharp(v^{-1}(0) \cap B^{j}_{i_1 \cdots i_j}) \leq k, \] (24)
where $M = 2(d - 1)^j$ and $k = (d - 1)^j$ independently on $i_1, \ldots, i_j$, and $B^{j}_{r}$ is the ball centered at the origin with radius $r$ in $\mathbb{R}^j$.

Next we set
\[ \delta = \frac{1}{2} \min_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} \delta_{i_1 \cdots i_j}, \quad r = \min_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} r_{i_1 \cdots i_j}. \]

Now we study any $u \in C^{M+1}(B^j_1)$ with
\[ |u - P|_{C^{M+1}(B^j_1)} < \delta. \]
We assume that
\[ f_{i_1 \cdots i_j} = (D_{x_{i_1}} P(x), \cdots, D_{x_{n-j}} P(x)) \mathcal{A}_{i_1 \cdots i_j}, \]
and
\[ v_{i_1 \cdots i_j, p} = (D_{x_{i_1}} u(x), \cdots, D_{x_{n-j}} u(x)) \mathcal{A}_{i_1 \cdots i_j}(p) \]
for any $p \in \mathbb{R}^n$ and $1 \leq i_1 < i_2 < \cdots < i_j \leq n$. For any $p \in B_r$, We can take $r$ small such that
\[ |v_{i_1 \cdots i_j, p} - f_{i_1 \cdots i_j}|_{C^{M+1}(B^{j}_{1/2})} < 2\delta \leq \delta_{i_1 \cdots i_j}. \]
Then, by (24), we have
\[ \sharp(v_{i_1 \cdots i_j, p}^{-1}(0) \cap B^{j}_{i_1 \cdots i_j}) \leq k. \] (25)
According to the assumption, we know
\[ |\nabla u|^{-1}(0) \cap \mathcal{A}_{i_1 \cdots i_j}(p) \subset v_{i_1 \cdots i_j, p}^{-1}(0). \] (26)
Now we suppose that $\omega_{i_1 \cdots i_j}$ is a projection such that
\[ \omega_{i_1 \cdots i_j}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i_j-1}, x_{i_j+1}, \ldots, x_{i_1-1}, x_{i_2+1}, \ldots, x_{i_j-1}, x_{i_j+1}, \ldots, x_n) \in \mathbb{R}^{n-j}. \]
By (25) and (26), for any $q \in B^n_{-j} \subset \mathbb{R}^{n-j}$, we have
\[ \sharp(|\nabla u|^{-1}(0) \cap \omega_{i_1 \cdots i_j}^{-1}(q) \cap B_r) \leq k. \]
Therefore, by the integral geometric formula (see [11, 3.2.22] and [23]), we have
\[ \mathcal{H}^{n-j}(|\nabla u|^{-1}(0) \cap B_r) \leq \sum_{1 \leq i_1 < \cdots < i_j \leq n} \int_{B^n_{-j}} \sharp(|\nabla u|^{-1}(0) \cap \omega_{i_1 \cdots i_j}^{-1}(q) \cap B_r) d\mathcal{H}^{n-j} q \]
\[ \leq c(n, j)kr^{n-j}. \] (27)
In the end, we note that $M + 1 = 2(d - 1)^j + 1 \leq 2d^j$. \qed
Remark 2. The result of case $j = 2$ in Theorem 3.2 was proved in [14] and the proof of Theorem 3.3 follows closely the proof in [14]. Theorem 3.2 shows that if function $u$ is very close to the $(n-j)$-symmetric homogeneous harmonic polynomial $P$ in the sense of $C^{2d^j}(B_1)$ norm, where $P$ with degree $d$ in $\mathbb{R}^n$ and origin $0$ as the isolated critical point of $P$ in $\mathbb{R}^d$, then the Hausdorff dimension of the critical set of $u$ does not exceed $n - j$. In fact, if $0$ is not an isolated critical point of $P$ in $\mathbb{R}^d$, the result of Theorem 3.2 does not hold in general. For example,

$$P(x, y, z) = xyz, \quad u(x, y, z) = xyz - x^2y, \quad (x, y, z) \in \mathbb{R}^3.$$  

We know that $P$ is $0$-isometric (not $1$-isometric) and origin $0$ is not the isolated critical point of $P$ in $\mathbb{R}^3$, while the Hausdorff dimension of the critical set of $u$ is $1$.

By the generalized $\varepsilon$-symmetric lemma, we will give the Hausdorff measure estimate of the $j$-symmetric singular set of a solution $u$. The proof of next theorem follows closely the proof in [14].

**Theorem 3.4.** Let $u$ be a $W^{2,2}(B_1)$ solution of (1) in $B_1(0)$. Then for any $\varepsilon > 0$, there exist positive constants $C(j, \varepsilon)$ also depending on $N, \lambda, \alpha, K$ and $n$. In addition, for any $u \in S_{N, r_0}(\lambda, 2N^2, \alpha, K)$, there exists a finite collection of balls $B_j = \{B_{r_j}(x_j)\}$ with $S^j(u) \subset B_j$, $\{x_j\}_{i=1}^k \subset S^j(u)$ and $r_j \leq \varepsilon$ such that

$$\mathcal{H}^j\{S^j(u) \cap B_{1/2}\} \leq C(j, \varepsilon). \tag{28}$$

**Proof.** Suppose that for any $u_0 \in S_{N, r_0}(\lambda, 2N^2, \alpha, K)$. By Definitions 2.3, for any $j \leq n - 3$, we have

$$S^j(u_0) := \{p \in S(u_0) : \text{the leading polynomial of } u_0 \text{ at } p \text{ is a homogeneous polynomial of } (n-j) \text{ variables and origin as the isolated critical point in } \mathbb{R}^{n-j} \text{ by an appropriate rotation}\}.$$  

By Theorem 2.6, we know that the $\dim S^j(u_0) \leq j$ for any $j \leq n - 2$.

(i) Case 1: If $j = n - 2$, the result was proved in [14].

(ii) Case 2: If $j \leq n - 3$, we need divide the proof of Case 2 into three steps. Step 1, we prove the claim, that is, for any $u \in S^j(u_0) \cap B_{3/4}$, there exist

$$R = R(y, u_0), \quad r = r(y, u_0), \quad \delta = \delta(y, u_0) \quad \text{and} \quad c = c(y, u_0) \quad \text{with} \quad r < R \quad \text{such that}$$

$$|u - u_0|_{C^{M+\varepsilon}(B_R(y))} < \delta, \tag{29}$$

then

$$\mathcal{H}^j\{S(u) \cap B_{r}(y)\} \leq cr^j. \tag{30}$$

In fact, for any $y \in S^j(u_0) \cap B_{3/4}$, we have

$$u_0(x + y) = P(x) + \omega(x),$$

where $P$ is a nonzero homogeneous polynomial of $(n-j)$ variables and of degree $d$ ($2 \leq d \leq N$) with $a_{ij}(y)D_{x_ix_j}P = 0$ in $\mathbb{R}^n$. In addition, by a suitable linear change of coordinates, Definitions 2.2 and 2.3 show that $P$ is a $j$-symmetric homogeneous harmonic polynomial and origin $0$ is an isolated critical point of $P$ in $\mathbb{R}^{n-j}$, and $\omega(x)$ satisfies

$$|\omega(x)| \leq C|x|^{d+\alpha} \quad \text{for any } |x| < \frac{1}{8},$$

where $C$ is a positive constant depending only on $N, \lambda, \alpha, K$ and $n$. Then it follows from interior Schauder estimates that for any $|x| < \frac{1}{8}$,

$$\left\{ \begin{array}{l}
|D^i\omega(x)| \leq C|x|^{d-i+\alpha} \quad \text{for } i = 1, \cdots, d, \\
|D^i\omega(x)| \leq C \quad \text{for } i = d + 1, \cdots, M + 1.
\end{array} \right. \tag{31}$$
Since $P$ is a homogeneous harmonic polynomial with origin as the isolated critical point in $\mathbb{R}^{n-1}$, we can apply Theorem 3.2 to $P$. Suppose that $\delta$ and $\tilde{r}$ are the constants which given in Theorem 3.2 for $P$. According to (31), then there exists a positive constant $\tilde{R} = R(y, u_0) < \frac{1}{2}$ satisfying
\[
\|\frac{1}{\tilde{R}^\omega}|||u|||_{C^{M+1}(B_{\tilde{R}})} < \frac{1}{2}\delta,
\]
where $||| \cdot |||_{C^{M+1}(B_{\tilde{R}})}$ denotes the $C^{M+1}$ norm weighted with the radius $\tilde{R}$, that is
\[
|||u|||_{C^{M+1}(B_{\tilde{R}})} = \sum_{i=0}^{M+1} \tilde{R}^i \sup_{x \in B_{\tilde{R}}} |D^i u(x)|.
\]
In (29), we take $\delta = \delta(R, \tilde{\delta})$ enough small, such that (29) implies that
\[
\|\frac{1}{R^\omega}(u - u_0)|||_{C^{M+1}(B_{R}(y))} < \frac{1}{2}\delta.
\]
By (32), (33) and $u_0(x+y) = P(x) + \omega(x)$, we have $\|\frac{1}{\tilde{R}^\omega}(u - P)|||_{C^{M+1}} \leq \|\frac{1}{\tilde{R}^\omega}(u - P - \omega)|||_{C^{M+1}, \tilde{R}}$, that implies
\[
\|\frac{1}{\tilde{R}^\omega}(u - P(\cdot - y))|||_{C^{M+1}(B_{R}(y))} < \delta.
\]
We consider the transformation $x \mapsto y + Rx$, then we have
\[
\|\frac{1}{\tilde{R}^\omega}(u(y + R \cdot)) - P|||_{C^{M+1}(B_{1}(0))} < \delta.
\]
Then we can apply Theorem 3.2 to $P$. Now we transform $B_1(0)$ to $B_R(y)$, we obtain (30) for $r \leq R\tilde{r}$. We require that $r \leq \varepsilon$. This completes the proof of step 1.

Step 2, we prove that there exists a covering of $S^j(u)$, which made of open balls in the collection $\mathcal{B}_j = \{B_{r_{j, i}}(x_{j, i})\}$ with $\{x_{j, i}\}_{i=1}^{j_k} \subset S^j(u_0)$ such that
\[
\mathcal{H}^{j}(S(u) \cap \mathcal{B}_j \cap B_{1/2}) \leq C \sum_{i=1}^{j_k} r_{j, i},
\]
where $r_{j, i} \leq \varepsilon$.

In fact, by the compactness of $S(u_0)$, there exist $\{x_{j, i}\}_{i=1}^{j_k} \subset S^j(u_0)$ and $r_{j, i} \leq \varepsilon$ such that
\[
S(u_0) \cap B_{3/4} \subset \bigcup_{j=0}^{n-2} B_j := \bigcup_{j=0}^{n-2} \{B_{r_{j, i}}(x_{j, i})\}.
\]
Since $S(u_0)$ is closed, then there exists a positive constant $r_0 = r_0(u_0, \varepsilon)$ with
\[
\{x \in B_{3/4} : \text{dist}(x, S(u_0)) < r_0 \} \subset \bigcup_{j=0}^{n-2} B_j.
\]
For such $r_0$, there exists $\eta_1 = \eta_1(u_0, \varepsilon)$ with $|u - u_0|_{C^{1}(B_{3/4})} < \eta_1$ in $B_1$, which implies that
\[
S(u) \cap B_{1/2} = \bigcup_{j=0}^{n-2} S^j(u) \cap B_{1/2} \subset \{x \in B_{3/4} : \text{dist}(x, S(u_0)) < r_0 \} \subset \bigcup_{j=0}^{n-2} B_j.
\]
If $u \in S_{N, r_0}(\lambda, 2N^2, \alpha, K)$, then $|u - u_0|_{L^\infty(B_{r/2})} < \eta_0$ follows from interior elliptic estimate that
\[
|u - u_0|_{C^{M+1}(B_{3/4})} < C(\eta_0),
\]
where $C(\eta_0) \to 0$ as $\eta_0 \to 0$.

We apply step 1 to $\{x_{j, i}\}_{i=1}^{j_k}$ and take $\eta_0 = \eta_0(u_0, \varepsilon)$ small such that
\[
C(\eta_0) \leq \min\{\eta_1, \delta(x_{j, 1}), \delta(x_{j, 2}), \cdots, \delta(x_{j, n-2}), \cdots\}.
\]
Then for any $u \in S_{N,r_0}(\lambda, 2N^2, \alpha, K)$ with $|u - u_0|_{L^\infty(B_{1/8})} < \eta_0$, by (39) and step 1, we have
\[
\|u - u_0\|_{L^\infty(B_{1/8})} < \eta_0,
\]
where $\eta_0$ is a positive constant. This completes the proof of step 2.

Step 3. We prove (28) holds for any $u \in S_{N,r_0}(\lambda, 2N^2, \alpha, K)$. In fact, by the compactness of the class $S_{N,r_0}(\lambda, 2N^2, \alpha, K)$ under the local $L^\infty$-metric (see [14, Lemma 2.1]). For any $u \in S_{N,r_0}(\lambda, 2N^2, \alpha, K)$, then there exist a subset $\{u_1, u_2, \cdots, u_s\} \subset S_{N,r_0}(\lambda, 2N^2, \alpha, K)$ and a set $\{\eta_1 = \eta_1(u_1, \epsilon), \eta_2 = \eta_2(u_2, \epsilon), \cdots, \eta_s = \eta_s(u_s, \epsilon)\}$ such that
\[
|u - u_k|_{L^\infty(B_{1/8})} \leq \eta_k
\]
for some $k$ ($1 \leq k \leq s$). By interior elliptic estimate, there exists a $C(\eta_k)$ such that
\[
|u - u_k|_{C^{M+1}(B_{3/4})} < C(\eta_k) = C(u_k, \epsilon).
\]
Set
\[
C(\epsilon) = \max\{C(u_1, \epsilon), C(u_2, \epsilon), \cdots, C(u_s, \epsilon)\}.
\]
This $C$ also depend on $N, \lambda, \alpha, K$ and $n$. It is obvious that $C(\epsilon) < \infty$. This completes the proof of Theorem 3.4.

4. The geometric structure of interior singular points in planar domains.

In this section, we will study the geometric structure of interior singular points of a solutions $u$ to linear elliptic equations with nonhomogeneous Dirichlet boundary conditions in a planar, bounded, smooth connected domain $\Omega$ such that $u|_{\partial \Omega} = \psi(x)$ and $\psi$ is sign-changing and has $N$ zero points on $\partial \Omega$.

In order to prove Theorem 1.3, we need the following basic lemmas.

**Lemma 4.1.** Let $u$ be a non-constant solution of (6). For any $t \in (\min_{\partial \Omega} \psi(x), \max_{\partial \Omega} \psi(x))$, we have that any connected component of $\{x \in \Omega : u(x) > t\}$ and $\{x \in \Omega : u(x) < t\}$ is simply connected, which has to meet the boundary $\partial \Omega$.

**Proof.** Let $A$ be a connected component of $\{x \in \Omega : u(x) > t\}$ and $\alpha$ be a non-equivalent simple closed curve in $A$. By the Jordan curve theorem there exists a bounded domain $B$ with $\partial B = \alpha$. Since $\Omega$ is simply connected, then $B$ is contained in $\Omega$. The strong maximum principle implies that $u > t$ in domain $B$. It shows that $B$ is contained in $\Omega$, namely $A$ is simply connected. The strong maximum principle shows that $u$ attains its maximum points and minimal points on boundary $\partial \Omega$, therefore the connected component $A$ has to meet the boundary $\partial \Omega$. The proof of the case of $\{x \in \Omega : u(x) < t\}$ is similar.

**Lemma 4.2.** Suppose that $x_0$ is an interior singular point of $u$ in $\Omega$ and that $m$ is the multiplicity of $x_0$. Then $m+1$ distinct connected components of $\{x \in \Omega : u(x) > 0\}$ and $\{x \in \Omega : u(x) < 0\}$ cluster around the point $x_0$ respectively.

**Proof.** According to the results of Remark 1.2 in [1], in a neighborhood of $x_0$ the level line $\{x \in \Omega : u(x) = 0\}$ consists of $m+1$ simple arcs intersecting at $x_0$. By the results of Lemma 4.1, there exist $m+1$ distinct connected components of $\{x \in \Omega : u(x) > 0\}$ and $\{x \in \Omega : u(x) < 0\}$ clustering around the point $x_0$ respectively. This completes the proof.

Now we are prepare to prove Theorem 1.3.
Proof of Theorem 1.3. We need divide the proof into two cases.

Firstly, we should show that $u$ has finite singular points in $\Omega$, denoting by $x_1, \ldots, x_k$, and we set $m_i$ as the multiplicity of corresponding singular point $x_i$ $(i = 1, \ldots, k)$. Suppose by contradiction that $u$ has infinite singular points in $\Omega$. According to the results of Lemma 4.1 and Lemma 4.2, we know that every zero level line and $\partial \Omega$ have two intersection points. In addition, Theorem 1.1 in [14] shows that the interior critical points of $u$ are isolated. Then there exist infinite zero level lines across these singular points and there are infinite zero points on $\partial \Omega$, this contradicts with the assumption of Theorem 1.3.

(i) Case 1: If all the singular points $x_1, \ldots, x_k$ together with the corresponding zero level lines $\{x \in \Omega : u(x) = 0\}$ clustering round these points form one connected set. Then there are just $(\sum_{i=1}^{k} m_i + 1)$ zero level lines across these singular points. In addition, by the results of Lemma 4.1, we know that every zero level line must have two points of intersection with $\partial \Omega$. Then we have

$$2(\sum_{i=1}^{k} m_i + 1) \leq N,$$

that is

$$\sum_{i=1}^{k} m_i \leq \frac{N}{2} - 1. \quad (40)$$

Then we know that when all $m_i = 1$, $i = 1, \ldots, k$, the largest number of interior critical points is $\frac{N}{2} - 1$. Hence we have

$$S \leq \frac{N}{2} - 1. \quad (41)$$

(ii) Case 2: If all the singular points $x_1, \ldots, x_k$ together with the corresponding zero level lines $\{x \in \Omega : u(x) = 0\}$ clustering round these points at least form two connected sets. Then there are at least $(\sum_{i=1}^{k} m_i + 2)$ zero level lines across these singular points. According to the analysis of above Case 1, then we have

$$S \leq \frac{N}{2} - 2.$$

According to Theorem 1.3 and Theorem 1.5 in [5] and Theorem 1.3, we can easily have the following results.

**Corollary 1.** Let $\Omega$ be a bounded, smooth and multiply connected domain with an interior boundary $\gamma_I$ and the external boundary $\gamma_E$ in $\mathbb{R}^2$. Suppose that $a_{ij}(x), b_i(x), c(x)$ are as in Theorem 1.3, $\psi(x) \in C^1(\overline{\Omega})$ is sign-changing, $H$ is a given constant and that $\psi$ has $N$ isolated zero points on $\gamma_E$. Let $u$ be a non-constant solution of the following boundary value problem

$$\begin{aligned}
\sum_{i,j=1}^{2} a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^{2} b_i(x) u_{x_i}(x) + c(x) u(x) &= 0 \text{ in } \Omega, \\
u|_{\gamma_I} = H, \enspace u|_{\gamma_E} = \psi(x).
\end{aligned} \quad (42)$$

Then $u$ has finite interior singular points and

$$S \leq \frac{N}{2}, \quad (43)$$

where $S$ is as in Theorem 1.3.
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