Numerical analysis of the bifurcation to wavy Taylor vortex flow with a small aspect ratio

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Abstract. We present a numerical analysis of the bifurcation to wavy Taylor vortex flow from Taylor vortex flow with a small aspect ratio and compare the experimental results for the instability of the Taylor vortex flow and for the critical Reynolds number to the wavy Taylor vortex flow. In this work, flow patterns and spatially averaged energy and spatially averaged enstrophy are shown by 3-dimensional numerical analysis. Comparison of the critical Reynolds numbers between numerical results and experimental results shows broad agreement about qualitative variation.

1. Introduction

One of the representative models with the bifurcation in the non-linear dynamical system is evidently the Taylor vortex flow between concentric cylinders with a rotating inner cylinder and a stationary outer cylinder. The Taylor vortex flow having the non-uniqueness has multiple solutions against a condition. The studies about the Taylor vortex flow, which has multiple modes depending on the dynamical parameter expressed as a Reynolds number and on the geometric parameter as an aspect ratio were reported [1], [2]. The simplest example of the Taylor vortex flow for the bifurcation in the non-linear dynamics should be the relationship between the Taylor vortex flow and the wavy Taylor vortex flow. The many experimental studies for the wavy Taylor flow were reported [3]. However the numerical analysis for the wavy mode and then for the comparison with the experimental result is not necessary reported enough in our knowledge.

The numerical studies for the Taylor vortex flow with the finite annulus were reported by Cliffe et al. [4] and Anson et al. [5]. They found the normal two-cell mode and the anomalous mode, and they compared the numerical results from the experimental results. And they also clarified the ten-cell mode. The studies for the development of the cell mode in 2-dimensional numerical analysis for the non-linear development of the flow pattern were reported by Watanabe et al. [6] and Furukawa et.al. [7]. And the 3-dimensional numerical studies were reported by Liao et al. [8] and Toya et al.[9]. However there are not enough studies to indicate the bifurcation problems between the modes.

In this work we present a 3-dimensional numerical analysis of the bifurcation from the Taylor vortex flow to the wavy Taylor vortex with the aspect ratio. The numerical result shows the formation processes of the Taylor vortex flow and the wavy Taylor vortex flows. They show the characteristic
flow patterns with the boundary surface individually. The variations of the spatially averaged energy and the spatially averaged enstrophy are shown. The critical Reynolds number at which the Taylor vortex flow bifurcates to the wavy Taylor vortex flow is estimated. Comparison of the critical Reynolds numbers between the numerical results and the experimental results are broad agreement about qualitative variation.

2. Basic Equation
Governing equations are Navier-Stokes equation and continuity equation in the cylindrical coordinate system \((r, \theta, z)\) as shown in the followings but in the vectorial notation.

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} \quad \nabla \cdot \mathbf{u} = 0. \tag{1-a}
\]

Where \(\mathbf{u} = (u, v, w)\). A kind of stream functions \(\Psi_1\) and \(\Psi_2\) which are similar to the Stokes’ stream function are adopted for flow visualization. Radius component \(u\) and axial component \(w\) are expressed as a function \(\Psi_1\) and another function \(\Psi_2\) respectively as follows

\[
u = -\frac{1}{r} \frac{\partial \Psi_1}{\partial z}, \quad w = \frac{1}{r} \frac{\partial \Psi_2}{\partial r}. \tag{2-a}
\]

Spatially averaged energy and enstrophy are defined as follows

\[
E = \frac{1}{V} \int_V \frac{1}{2} \mathbf{u}^2 \, dV, \quad \Omega = \frac{1}{V} \int_V \frac{1}{2} \mathbf{\omega}^2 \, dV. \tag{3-a}
\]

3. Numerical methods
The MAC method was used as a basic solution procedure. The time integration was explicit method and spatial differentiation was the QUICK method for the convection terms and the second-order central difference method for other terms. A hybrid method of SOR and ILUCGS was used to solve the Poisson equation. The staggered grid was adopted and the grid interval was uniform in each direction. The numbers of staggered grid points were constant of 21 in the radial direction, 20 in the azimuthal direction but the number of grid point in the axial direction was adopted according to the aspect ratio, which is the ratio of the height of the flow and the radial distance of the annulus, from 1 to 7.5 based on the 84 at \(\Gamma = 4.0\). These intervals of the grid points were checked \((21, 20, 84), (41, 38, 168), (21, 20, 120), (21, 20, 160), (21, 20, 200)\) and \((21, 20, 240)\) for the aspect ratio 4.0 in the previous trial. It was confirmed that the set of the \((21, 20, 84)\) had the highest agreement 75% compared with the experimental results.

The Reynolds number, \(Re\) is defined by the velocity of the inner cylinder, the gap between the cylinders, and the kinetic viscosity of the fluid. The radius ratio, \(Ri / Ro\), \(Ri\) is the radius of the inner cylinder, \(Ro\) is the radius of the outer cylinder, was 0.667. The way of the increment of the Reynolds number in all cases was adopted as shown in Fig.1. Reynolds number was increased from zero up to the provided value quasi-statically with a constant ratio.

![Figure 1. Increment of the Reynolds number](image)
4. Results

4.1 Formation processes of the Taylor vortex flow and the wavy Taylor vortex flow

Sectional vector diagrams of the Taylor vortex flow (TVF) and the wavy Taylor vortex flow (WTVF) are shown in Fig. 2. The Fig. 2 (a) is TVF at \( \text{Re}=900 \) and the (b) is WTVF at \( \text{Re}=1600 \) for the normal 4-cell mode. The Fig. 2 (c) is TVF at \( \text{Re}=1000 \) and the (d) is WTVF at \( \text{Re}=1600 \) for the normal 6-cell mode. In each diagram, the left side indicates the inner cylinder and the right side indicates the outer cylinder. Diagrams were selected in any interval time because they should be shown the much more characteristic flow patterns. The formation processes were qualitatively similar to the observations in our experiment \([10]\).

We can see that the boundary surfaces between the cells of the TVF for the normal 4-cell mode and the 6-cell mode show the horizontal lines stably in the diagrams (a) and (c), and the surfaces between the cells of the WTVF in the diagrams (b) and (d) show to be upward and downward one after the other. These diagrams for WTVF obviously show that the cells are unstable and wavy. So these sectional vector diagrams can distinguish the difference between the TVF and the WTVF.

Figure 2. Development of the Taylor vortex flow and Wavy Taylor vortex flow
4.2 Spatially averaged energy on the Taylor vortex flow and the wavy Taylor vortex flow

Figure 3 shows variations of the spatial averaged energy in the axial direction on the TVF and the WTVF. The Fig. 3 (a) shows the energy of the 2-cell mode on the TVF at Re=800 and the (b) shows that of the 2-cell mode on the WTVF at Re=1500 both at $\Gamma = 1.5$. As the Reynolds number was increased, the energy for the TVF was unsteady until the dimensional time 300, and after then the energy was constant finally. This unsteady state indicates that the Taylor cells would make and collapse in the initial state. And when the Taylor cells became to be steady, the energy became to be constant. This series of the energy indicates the formation process of the TVF. On the other hand, for the WTVF, though the unsteady state in the initial state was similar to that of TVF, the final value of the energy maintained to be oscillated. The amplitude of the oscillation wave could be estimated to be $1.3 \times 10^{-7}$. The degree of the intensity of the oscillation was quite small because it would be assumed that the energy due to the oscillation was averaged by the total volume of the flow. Beside Re=1500 was a little bit above the critical Reynolds number Re=1360 of WTVF at $\Gamma = 1.5$. The final values of the energy on TVF and WTVF would depend to the difference of the Reynolds number.

Figure 3 (c) shows the energy of the 4-cell mode on the TVF at Re=900 and the (d) shows that of the 4-cell mode on the WTVF at Re=1600 both at $\Gamma = 4.0$. The unsteady states of the 4-cell mode were a little bit different between TVF and WTVF but it would depend to the final value of the Reynolds number. Obviously the final value of the energy of TVF became constant but that of WTVF became to be unsteady and oscillated. The amplitude of the oscillation wave of the WTVF could be estimated to be $1.8 \times 10^{-5}$. The amplitude of the (d) was bigger than that of the (b). Because in this case, Re=1600 is 660 larger than Re=940 of the WTVF at $\Gamma = 4.0$.

Figure 3 (e) shows the energy of the 6-cell mode on the TVF at Re=1000 and the (f) shows that of the 6-cell mode on the WTVF at Re=1600 both at $\Gamma = 6.5$. The start of the variation are much similar and the variations of the final value are different. As same as the energy, the intensities of the oscillation of the variations in the (d) and the (f) are small because the enstrophy also would be averaged by over volume of the flow field.

In this numerical analysis, there are three methods of the decision of the critical Reynolds number that the Taylor vortex flow bifurcates to the wavy Taylor vortex flow. One is the vector diagram of the sectional flow and the others are the variations of the energy and the enstrophy. We could estimate the critical Reynolds number by the combination of three methods.
(a) Energy of the TVF with 2-cell mode
\[ \Gamma = 1.5 \quad Re = 800 \]

(b) Energy of the WTVF with 2-cell mode
\[ \Gamma = 1.5 \quad Re = 1500 \]

(c) Energy of the TVF with 4-cell mode
\[ \Gamma = 4.0 \quad Re = 900 \]

(d) Energy of the WTVF with 4-cell mode
\[ \Gamma = 4.0 \quad Re = 1600 \]

(e) Energy of the TVF with 6-cell mode
\[ \Gamma = 6.5 \quad Re = 1000 \]

(f) Energy of the WTVF with 6-cell mode
\[ \Gamma = 6.5 \quad Re = 1600 \]

Figure 3. Spatially averaged Energy in the axial direction
Figure 4. Spatially averaged Enstrophy in the axial direction

(a) Enstrophy of the TVF with 2-cell mode  
\[ \Gamma = 1.5 \quad \text{Re}=800 \]

(b) Enstrophy of the WTVF with 2-cell mode  
\[ \Gamma = 1.5 \quad \text{Re}=1500 \]

(c) Enstrophy of the TVF with 4-cell mode  
\[ \Gamma = 4.0 \quad \text{Re}=900 \]

(d) Enstrophy of the WTVF with 4-cell mode  
\[ \Gamma = 4.0 \quad \text{Re}=1600 \]

(e) Enstrophy of the TVF with 6-cell mode  
\[ \Gamma = 6.5 \quad \text{Re}=1000 \]

(f) Enstrophy of the WTVF with 6-cell mode  
\[ \Gamma = 6.5 \quad \text{Re}=1600 \]
4.4 Critical Reynolds number to the wavy Taylor vortex flow

Critical Reynolds number at which the Taylor vortex flow bifurcates to the wavy Taylor vortex flow has been clarified in the experiment. However the verification of the critical Reynolds number between the numerical analysis and experiment is not necessary enough. In our experimental study, the final modes made by some ways of the increment of the Reynolds number have been clarified [10]. And this numerical analysis distinguishes the difference between The TVF and the WTVF. In this study, the methods of the decision were adopted the combination of three methods. One is the sectional vector diagram, which shows that the boundary surface is wavy or not. Other two methods are the spatially averaged energy and the spatially averaged enstrophy, which show the oscillation in the final value on the WTVF.

Some numerical simulations around the expected value estimated a final critical Reynolds number by trial and error. Aspect ratio, which is a ratio between the height of the working fluid and the radial distance of the annulus of the cylinders was changed from one to 7.5.

Figure 5 shows the critical Reynolds number with the aspect ratio. Solid marks indicate the numerical results and the empty marks indicate the experimental results. Circles indicate for the 2-cell mode, quadrangles indicate for the 4-cell mode, triangles indicate for the 6-cell mode and diamonds indicate for the 8-cell mode. These are not completely agreement each other quantitatively. However they are qualitatively similar expect the tops of the curved line are shifted to the direction of the higher value of the $\Gamma$.

5. Conclusions

The numerical analysis for the small aspect ratio Taylor vortex flow clarifies the difference of the characteristics between the Taylor vortex flow and the wavy Taylor vortex flow in the 3-dimensional systems.

The spatially averaged energy and the spatially averaged enstrophy for the Taylor vortex flow and wavy Taylor vortex flow were estimated with the aspect ratio. These variations showed the difference clearly for the Taylor flow and Wavy Taylor flow.
The critical Reynolds number form the Taylor flow to the wavy Taylor vortex flow was estimated and was compared with the experimental result. Though numerical result is not completely agreement with the experimental result but is similar to it qualitatively.

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