DETERIORATING INVENTORY WITH PRESERVATION TECHNOLOGY UNDER PRICE- AND STOCK-SENSITIVE DEMAND

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\textbf{Abstract.} In this paper, we formulate and solve an Economic Production Quantity inventory model with deteriorating items. To reduce the rate of deterioration, we apply a preservation technology and calculate the amount for preservation technology investment. The demand function is dependent on stock-level and price. We assume that the production rate is linearly dependent on time, based on customer demand. Shortages are allowed in our consideration, and the shortages amount is partially backlogged for the interested customers for the next slot. The effect of inflation is incorporated, which indicates a critical factor in modern days. Our main objective is to find the optimal cycle length and the optimal amount of preservation technology investment by adjusting the inflation rate with maximizing the profit. A numerical example is provided to illustrate the features and advances of the model. A sensitivity analysis with respect to major parameters is performed in order to assess the stability of our model. The paper ends with a conclusion and an outlook at possible future directions.

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1. Introduction. In classical inventory models, it is assumed that the items can be preserved for an infinite time without any change of their physical status. However, in reality, many products become partially or totally unusable after a certain time period. Deterioration is regarded as a natural phenomenon for inventories which has been demonstrated very extensively for agricultural products, volatile liquids, food items, pharmaceutical products, perfumes, radioactive substances, gasoline, electronic components and photographic films. In general, it is found that items always deteriorate continuously with respect to time, but deterioration can be controlled by applying some suitable preservation technology. For example, the rate of deterioration of fish can be reduced by storing the fish in a deep fridge or by using ice. Applying cool supply-chain policy, the rate of deterioration for fruits becomes less. Though the cost of preservation technology may be high, it will be our attempt to reduce the total cost which is expended for preservation technology, and for calculating the amount of preservation technology investment to diminish the deterioration cost and to maximize the profit.

Inflation and time value of money play a significant role in an inventory model for an optimal ordering policy. Inflation always influences the demand for a certain product. It is found that many companies are suffering from high inflation, which means a huge loss of time value of money. As a result, dependent countries tolerate loss for purchasing oil or power. When the time value of money collapses, spending on luxury items or peripherals always increases the demand. Since the resources of a company or enterprize are highly dependent on time, inflation always leads the company to a better relation with the customers in the return of investment. Hence, inflation and time value of money cannot be ignored when forecasting and investment are conducted.

In this competitive era, variable demand is of an utmost importance. Here, we assume stock- and price-dependent demand which both have positive as well as negative impacts on customers. As an example, a huge amount on rack can evoke in customers the idea of freshness, whereas some customers can feel different. In fact, they can imagine that there is a large stock level because of a possible low quality of items. Moreover, most of the managers have the practical experience that Displayed Stock Level (DSL) can increase demand, as customers think that a small stock is definitely defective or unusable. Hence, DSL has an attractive importance in formulating an inventory problem. Similarly, a low price can appeal strongly to a large number of customers. A sudden price discount on items can draw the attention of customers to buy more (e.g., seasonal discount; also known as Chaitra Sale), but some customers can questions about the quality of the items and they may not be interested to purchase. Market price is always highly depending on market demand. Hence, stock and price discount on demand have a joint effect in formulating an inventory model. The main contributions of the paper are summarized as follows:

- An Economic Production Quantity (EPQ) model for deteriorating items with preservation technology investment is formulated here.
- The amount of preservation technology is calculated here, for the first time.
- The model is allowed for stock and price sensitive demand with partial backlogging.
- Rate of inflation is included to reflect the model in a more realistic sense.
- We have established four results and procedures, and demonstrated them to find the global concavity of the model.
- Implicit Function Theorem has to be addressed to prove the results.
• Total profit of the model is calculated and global concavity is provided.
• The models are illustrated by a numerical example, and optimal results are investigated by performing a sensitivity analysis.

The organization of this paper is as follows:
• A detailed literature survey on previous researches is provided in Section 2 to identify yet existing drawbacks and problems in their works.
• In Section 3, some preliminary notations and assumptions are given.
• A clear mathematical model, allowing for graphic illustration, and calculation of the profit function are presented in Section 4.
• To solve our mathematical model, we claim four results and computer proofs (supported computationally) to ensure the global concavity of the profit function in Section 5.
• A numerical example is provided to confirm the reliability of our formulated model in Section 6.
• Section 7 deals with computational experiments and a statistical analysis to identify the important ones among all with 6 exclusive figures.
• Finally, Section 8 presents final conclusions and further research avenues.

2. Review on research. Deteriorating inventories have gained the interest of many researchers in previous decades. For example, Dave and Patel [6], Wee [33], Papachristos and Skouri [19], Dye [7] and Bhunia et al. [3] focused on variable and constant deterioration. Goyal and Giri [9] and Baker et al. [1] gave a detailed survey over the last twenty years on deteriorating inventory. Deterioration was introduced into inventory management by Ghare and Schader [8], but Murr and Morris [16] first stated that a lower temperature rate can decrease the deterioration rate and increase the storage life. To reduce the deterioration rate for food and medicine, vacuum technology was introduced by Zauberman et al. [35]. Ouyang et al. [17] incorporated the phenomenon of “non-instantaneous deterioration” into an inventory model. Maihami and Abadi [15] extended the model [17] by including time- and price-dependent demand. Shah et al. [29] described time-varying deterioration and holding cost rates where shortages were allowed. Pervin et al. [21] introduced a deteriorating Economic Order Quantity (EOQ) model with variable demand and holding costs under the effect of trade-credit policy. Hsu et al. [11] studied the effect of preservation technology investment for deteriorating inventory. Recently, Roy et al. [26] proposed a two-warehouse with price discount on backorders and trade-credit policy where the demand function was depicted in probabilistic sense. Khedlekar et al. [12] established an inventory model with declining demand under preservation technology investment. In our model, we reduce an important gap in the literature by calculating the amount of preservation technology investment.

Production by many countries varies very strongly under the effect of inflation. So, while calculating the optimal inventory policy, the impact of inflation and time value of money cannot be ignored. Many researchers have engaged themselves to find the effect of inflation on an inventory model. Among them, Buzacott [4] was the first who tried to assess the influence of inflation on EOQ model. Gupta and Vrat [10] constructed a multi-item inventory model with a variable inflation rate. Ray and Chaudhuri [25] developed an EOQ model with shortage, inflation, time discounting and stock-dependent demand. Liao et al. [14] investigated such a model with delay in payments under inflation. An EPQ model with defective manufacturing quality and with the impact of inflation was offered by Sarkar and
Moon [28]. Shah et al. [30] developed a model with imperfect manufacturing system and quadratic demand. Our model is derived to find the effect of inflation in an EPQ model under preservation technology investment.

In a supermarket, one can see that not only the amount of stock but, especially, the price of the items affects the demand strongly. Levin et al. [13] observed that substantial heaps of customer products shown in a grocery store will lead the customer to purchase more. Later, Teng and Chang [31] derived an EPQ model where they also allowed stock- and price-dependent demand. Thereafter, Pal et al. [18] derived a deteriorating-inventory model with stock- and price-sensitive demand, where they assumed inflation and delay in payment. Recently, Pervin et al. [20] presented a deteriorating and decaying inventory model under trade-credit policy. Tsao [32] studied on a deteriorating model with price adjustments and trade credits. Widyadana and Wee [34] developed an EPQ model of deteriorating items for a stochastic demand function. Pervin et al. [22] formulated and solved a deteriorating inventory model with stock-dependent demand and variable holding cost. Recently, Pervin et al. [23] derived an integrated model with variable holding cost under trade-credit policy. A multi-item deteriorating inventory model with trade-credit policy was elaborated by Pervin et al. [24]. In our model, we formulate a deteriorating inventory model with preservation technology investment under stock-price sensitive demand. The studies made by research groups related on our topic are surveyed in Table 1.

| Authors                          | Deteriorations | Preservation Technology | Stock-dependent demand | Price-dependent demand | Inflation | Partial backorder |
|----------------------------------|----------------|-------------------------|------------------------|------------------------|-----------|-------------------|
| Bhunia et al. (2009)             | √              |                         |                        |                        |           |                   |
| Dave & Patel (1981)              | √              |                         |                        |                        |           |                   |
| Papachristos & Skouri (2003)     | √              |                         |                        |                        |           |                   |
| Perng et al. (1998)              | √              |                         |                        |                        |           |                   |
| Hsu et al. (2010)                | √              | √                       | √                      |                        |           |                   |
| Maihami and Abadi (2012)         | √              |                         |                        |                        |           |                   |
| Keelilekar et al. (2016)         | √              | √                       |                        |                        |           |                   |
| Muri and Morris (1975)           | √              | √                       |                        |                        |           |                   |
| Buzacott (1975)                  | √              | √                       |                        |                        |           |                   |
| Gupta and Vrat (1986)            | √              |                         |                        |                        |           |                   |
| Liao et al. (2000)               | √              | √                       |                        |                        |           |                   |
| Ray and Chandrakar (1997)        | √              | √                       |                        |                        |           |                   |
| Pal et al. (2014)                | √              | √                       |                        |                        |           |                   |
| Sarkar and Moon (2011)           | √              |                         |                        | √                      |           |                   |
| Shah et al. (2017)               | √              | √                       |                        |                        |           |                   |
| Teng and Chang (2005)            | √              | √                       |                        | √                      |           |                   |
| Widyadana and Wee (2010)         | √              | √                       | √                      | √                      |           |                   |
| This work                        | √              | √                       | √                      | √                      | √         |                   |

3. Notations & assumptions. For formulating our deteriorating optimized inventory model, we introduce the following notations and assumptions, to execute a comprehensive form of it.

**Notations**

- $c$: Unit purchasing cost per item;
- $r_1$: Market price per unit item, where $r_1 > c$ (a decision variable);
- $r_2$: Rate of inflation per unit time, where $0 \leq r_2 < 1$ (a decision variable);
- $\theta$: Deterioration rate, where $0 < \theta < 1$;
- $x$: Preservation technology investment amount per unit time to reduce the deterioration rate (a decision variable);
$P(t)$: Unit production rate per unit time;
$Q$: Order quantity;
$s_1$: Unit shortage cost per item and per unit time;
$s_2$: Unit lost sale cost per item and per unit time;
$h_1$: Unit holding cost for deteriorating items per unit and per time unit;
$I(t)$: Inventory level with respect to time $t$;
$I_1(t)$: Inventory level that changes with time $t$ during production period;
$I_2(t)$: Inventory level that changes with time $t$ during non-production period;
$I_3(t)$: Inventory level that changes with time $t$ during shortage period;
$I_4(t)$: Inventory level that changes with time $t$ during reproduction period;
$T$: Length of cycle time (a decision variable);
$q_1$: Unit transportation cost of a shipment from supplier to retailer;
$TP$: Total profit of the system per unit time (a decision variable).

Assumptions.

3.1. The demand rate is dependent on time and price and expressed as
$D(t) = a_1 + b_1t + c_1r_1$, where $a_1$ is the initial demand, $a_1 > 0$, $c_1 > 0$, is the price sensitive parameter; $b_1$ is the constant increase rate of the demand, and $t$ is the time variable.

3.2. The production rate is linearly dependent on time and expressed as
$P(t) = a_2 + b_2t$, where $a_2 > 0$ and $b_2 > 0$ are constants.

3.3. Shortages are partially backlogged with a backlogging rate $S(t) = \frac{1}{1+\delta t}$, where $0 < \delta < 1$ is the backlogging parameter.

3.4. The function $g(x) = 1 - e^{-x}$ is the proportion of reduced deterioration rate, where $0 \leq g(x) \leq 1$, if $x \geq 0$.

4. Mathematical formulation. Based on our aforementioned assumptions, the inventory system is detailed as follows: Initially (at time $t = 0$), the cycle starts with a stock level of 0 at production rate $P(t)$. The replenishment or supply continues up to time $t_1$. During the time period $[0, t_1]$, inventory piles up by adjusting the demand and deterioration in the market and stops at time $t = t_1$. Then, this accumulated inventory level at time $t_1$ gradually diminishes due to demand and deterioration during the period $[t_1, t_2]$ and ultimately falls to 0 at time $t = t_2$. During the period $[t_2, t_3]$, shortages occur and continue up to time $t = t_3$. After that time, the production is restarted to recover both demand and backordered items with a rate of $S(t)$, until the inventory level reaches the level 0 by time $t = t_4$. Following the scheduling period, the cycle begins again.

Now, the differential equations in the interval $[0, T]$ involving the subsequent initial and boundary value problems are:

$$\frac{dI_1(t)}{dt} + (\theta - g(x))I_1(t) = P(t) - D(t), \ t \in [0, t_1],$$  \hspace{1cm} (1)

with the initial condition $I_1(0) = 0$;

$$\frac{dI_2(t)}{dt} + (\theta - g(x))I_2(t) = -D(t), \ t \in [t_1, t_2],$$  \hspace{1cm} (2)

with the terminal condition $I_2(t_2) = 0$;

$$\frac{dI_3(t)}{dt} = -D(t), \ t \in [t_2, t_3],$$  \hspace{1cm} (3)

with the initial condition $I_3(t_2) = 0$;
with the terminal condition \( I_4(t_4) = 0 \).

Inserting the values of \( P(t), D(t) \) and \( S(t) \) into the corresponding equations and using the boundary conditions, the solutions of the above equations are given in the following.

The particular solution of equation 1 becomes:

\[
I_1(t) = e^{-(\theta - g(x))t} \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_1 - b_2}{(\theta - g(x))^2} \\
+ \frac{1}{\theta - g(x)} \left[ (a_2 - a_1 + c_1r_1) + (b_2 - b_1)T \right];
\]

the solution of equation 2 results as

\[
I_2(t) = e^{-(\theta - g(x))t_2-t} \left[ \frac{a_1 + c_1r_1 + b_1t_2}{\theta - g(x)} - \frac{b_1}{(\theta - g(x))^2} \right] \\
- \frac{a_1 + c_1r_1 + b_1t}{\theta - g(x)} + \frac{b_1}{(\theta - g(x))^2};
\]

the solution of equation 3 will be

\[
I_3(t) = [(a_1 + c_1r_1)(t_2-t) + \frac{b_1}{2}(t_2^2-t^2)];
\]

finally, the solution of equation 4 is

\[
I_4(t) \\
= a_2(t_4-t) + \frac{b_2}{2}(t^2_4-t^2) - \frac{a_1 + c_1r_1}{\delta} \log(1 + \delta(t_4-t)) - \frac{b_1}{2\delta^2}(\delta^2(t_4^2-t^2) - 1).
\]

Consequently, the maximum inventory level is

\[
Q = e^{-\frac{t^2}{\delta}} \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_1 - b_2}{(\theta - g(x))^2} \\
+ \frac{1}{\theta - g(x)} \left[ (a_2 - a_1 + c_1r_1) + (b_2 - b_1)T \right].
\]

Now, by a suitable choice of \( t_1 \), where \( I_1(t_1) = I_2(t_1) \), the entire function \( I(t) \) will be guaranteed to be continuous at \( t = t_1 \) indeed. Similarly, for a suitable selection of \( t_2, t_3 \) and \( t_4 \), the entire function \( I(t) \) will be guaranteed to be continuous, respectively.

Figure 1 gives a graphical presentation of the system.

Now, to calculate the supplier's profit function, we calculate the following terms:

4.1. Annual sales revenue is equal to \( r_1(a_1T + \frac{b_2}{2}T^2 + c_1r_1T) \).

4.2. Annual producing cost be

\[
c(Q + S(t)) = e^{-(\theta - g(x))T} \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \\
+ \frac{b_1 - b_2}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)} \left[ (a_2 - a_1 + c_1r_1) + (b_2 - b_1)T \right] \\
+ \frac{1}{1 + \delta t} \left( a_1(t_4 - t_3) + \frac{b_1}{2}(t_4^2 - t_3^2) + c_1r_1(t_4 - t_3) \right).
\]
4.3. The total holding cost is computed for the time intervals \([0, t_1]\) and \([t_1, t_2]\), because only during these periods, inventory is available in the system. So, the annual stock holding cost is represented as follows:

\[
\begin{align*}
    h_1 & \left[ \int_0^{t_1} I_1(t)e^{-r_2t}dt + \int_{t_1}^{t_2} I_2(t)e^{-r_2t}dt \right] \\
    &= h_1 \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \left[ \frac{1}{\theta - g(x) - r_2} (1 - e^{-(\theta - g(x) - r_2)t_1}) \right] \\
    &+ \frac{b_2 - b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2} (1 - e^{-r_2 t_1}) \right] + \frac{1}{\theta - g(x)} \left[ (a_2 - a_1 + c_1 r_1) \frac{1}{r_2} (1 - e^{-r_2 t_1}) \right] \\
    &+ (b_2 - b_1) \left( \frac{1}{r_2^2} - \frac{t_1 e^{-r_2 t_1}}{r_2} - \frac{e^{-r_2 t_1}}{r_2^2} \right) \\
    &+ \frac{e^{-(\theta - g(x))t_2}}{\theta - g(x) - r_2} \left[ \frac{a_1 + c_1 r_1 + b_1 t_2}{(\theta - g(x))^2} \right] - \frac{b_1}{(\theta - g(x))^2} \\
    &\left[ e^{(\theta - g(x)-r_2)t_2} - e^{-(\theta - g(x)-r_2)t_1} \right] - \frac{b_1}{r_2} \left[ \frac{t_1 e^{-r_2 t_1}}{r_2^2} + \frac{e^{-r_2 t_1}}{r_2^2} \right] \\
    &- \frac{t_2 e^{-r_2 t_2}}{r_2} - \frac{e^{-r_2 t_1}}{r_2^2} \\
    &+ \frac{a_1 + c_1 r_1}{r_2} \left( e^{-r_2 t_1} - e^{-r_2 t_2} \right) + \frac{b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2} (e^{-r_2 t_1} - e^{-r_2 t_2}) \right].
\end{align*}
\]

4.4. Annual preservation technology cost depends on the cycle time. Hence, it is given by

\[
x Q T \int_0^T e^{-r_2 t} dt = \frac{x T}{r_2} e^{-(\theta - g(x))T} \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_1 - b_2}{(\theta - g(x))^2}
\]
4.5. The shortage cost is accumulated during the time interval $[t_2, t_3]$; so, it is expressed as:

$$-s_1 \left[ \int_{t_2}^{t_3} I_3(t) e^{-r_2 t} dt \right]$$

$$= s_1 \left[ \left( a_1 + c_1 r_1 \right) \frac{1}{r_2^2} e^{-r_2 (t_3 - t_2)} + \frac{1}{r_2^2} (t_3 - t_2) e^{-r_2 (t_3 - t_2)} + b_1 \left( \frac{1}{r_2} (r_3^2 - t_2) e^{-r_2 (t_3 - t_2)} \right) \right] + \frac{2}{r_2^2} \left( \frac{1}{r_2} e^{-r_2 (t_3 - t_2)} + \frac{t_3 - t_2}{r_2} e^{-r_2 (t_3 - t_2)} \right) \right].$$

4.6. Due to shortage during the time interval $[t_2, t_3]$, some customers are interested to wait for the coming lot size to arrive. But some impatient or needy customers leave the system, which may cause a loss in profit. Hence, the lost sale cost is:

$$s_2 S(t) \left[ \int_{t_2}^{t_3} I_3(t) e^{-r_2 t} dt \right]$$

$$= -s_2 \left[ \frac{1}{1 + \delta t} \left( a_1 + c_1 r_1 \right) \frac{1}{r_2^2} e^{-r_2 (t_3 - t_2)} + \frac{1}{r_2^2} (t_3 - t_2) e^{-r_2 (t_3 - t_2)} + b_1 \left( \frac{1}{r_2} (r_3^2 - t_2) e^{-r_2 (t_3 - t_2)} \right) \right] + \frac{2}{r_2^2} \left( \frac{1}{r_2} e^{-r_2 (t_3 - t_2)} + \frac{t_3 - t_2}{r_2} e^{-r_2 (t_3 - t_2)} \right) \right].$$

4.7. Backorder cost between $[t_3, t_4]$ is expressed as follows:

$$S(t) r_1 \left[ \int_{t_3}^{t_4} I_3(t) e^{-r_2 t} dt \right]$$

$$= -r_1 \left[ \frac{1}{1 + \delta t} \left( a_2 e^{-r_2 (t_4 - t_3)} + a_3 e^{-r_3 (t_4 - t_3)} + \frac{b_2}{2} \left( \frac{1}{r_2} (r_3^2 - t_2) e^{-r_2 (t_3 - t_2)} \right) \right) + \frac{2}{r_2^2} \left( \frac{1}{r_2} e^{-r_2 (t_3 - t_2)} + \frac{t_3 - t_2}{r_2} e^{-r_2 (t_3 - t_2)} \right) + \frac{a_1 + c_1 r_1}{\delta} \left( \frac{r_2}{2} (r_3^2 - t_2) e^{-r_2 (t_3 - t_2)} \right) + \frac{1}{\delta} \log(1 + \delta (t_4 - t_3)) \right] - b_1 \frac{1}{2 \delta^2} \left( \delta^2 \left( \frac{1}{r_2} (r_3^2 - t_2) e^{-r_2 (t_3 - t_2)} \right) \right) \right].$$

4.8. Annual transportation cost is

$$q_1 Q = q_1 \left[ e^{-(\theta - g(x)) T} \left( a_1 - a_2 - c_1 r_1 \right) + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_1 - b_2}{(\theta - g(x))^2}$$

$$+ \left[ (a_2 - a_1 + c_1 r_1 + (b_2 - b_1) T) \right].$$

Therefore, the total profit of the supplier is

$$TP = Sales Revenue - Production cost - Holding cost - Preservation technology.$$
cost — Shortage cost — Lost sale cost — Backorder cost — Transportation cost.

Now, by inserting the values of the aforementioned cost factors into equation 13, we obtain:

\[
TP = r_1(a_1T + \frac{b_1}{2}T^2 + c_1r_1T) - e^{-(\theta-g(x))T} \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_1 - b_2}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)}[(a_2 - a_1 + c_1r_1) + (b_2 - b_1)T] + \frac{1}{1 + \delta t}(a_1(t_4 - t_3) + \frac{b_1(t_2^2 - t_3^2) + c_1r_1(t_4 - t_3)}{2}) - h_1 \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \]

\[
\left[ \frac{1}{\theta - g(x) - r_2} \left( 1 - e^{-(\theta-g(x)-r_2)t_1} \right) \right] + \frac{b_2 - b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2^2} \left( 1 - e^{-r_2t_1} \right) \right] + \frac{1}{\theta - g(x)} \left[ \left( a_2 - a_1 + c_1r_1 \right) \frac{1}{r_2} \left( 1 - e^{-r_2t_1} \right) + \left( b_2 - b_1 \right) \left( \frac{1}{r_2^2} - \frac{t_1e^{-r_2t_1} - e^{-r_2t_{t_1}}}{r_2^2} \right) \right] + \frac{e^{-(\theta-g(x))t_2}}{\theta - g(x) - r_2} \left[ \frac{a_1 + c_1r_1 + b_1t_2}{\theta - g(x)} - \frac{b_1}{(\theta - g(x))^2} \right] \]

\[
\left[ e^{-(\theta-g(x)-r_2)t_1} - e^{-(\theta-g(x)-r_2)t_{t_1}} \right] - \frac{b_1}{\theta - g(x)} \left[ \frac{1}{r_2} \left( e^{-r_2t_{t_1}} - e^{-r_2t_{t_2}} \right) \right] + \frac{b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2^2} \left( e^{-r_2t_{t_2}} - e^{-r_2t_{t_1}} \right) \right] - \frac{xT}{r_2^2} \left[ \frac{a_1 - a_2 - c_1r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_2 - b_1}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)} \left[ \left( a_2 - a_1 + c_1r_1 \right) + (b_2 - b_1)T \right] \cdot (1 - e^{-r_2T}) - s_1 \left[ \frac{a_1 + c_1r_1}{r_2^2} e^{-r_2t_1} + \frac{1}{r_2^2} (t_3 - t_2)e^{-r_2(t_3-t_2)} + \frac{b_1}{2} \left( \frac{1}{r_2} \right) \right] \]

\[
\left[ \frac{1}{r_2^2} (t_3^2 - t_2^2) e^{-r_2(t_3-t_2)} + \frac{2}{r_2} \left( \frac{1}{r_2^2} e^{-r_2(t_3-t_2)} + \frac{t_3 - t_2}{r_2} e^{-r_2(t_3-t_2)} \right) \right] + \frac{1}{r_2^2} \left[ \left( a_1 + c_1r_1 \right) e^{-r_2(t_3-t_2)} + \frac{1}{r_2} (t_3 - t_2) e^{-r_2(t_3-t_2)} + \frac{b_1}{2} \left( \frac{1}{r_2} \right) \right] \]

\[
e^{-r_2(t_3-t_2)} + \frac{2}{r_2} \left( \frac{1}{r_2^2} e^{-r_2(t_3-t_2)} + \frac{t_3 - t_2}{r_2} e^{-r_2(t_3-t_2)} \right) \right] + \frac{r_1}{(1 + \delta t)} \left[ \frac{a_2}{r_2^2} e^{-r_2(t_4-t_3)} + \frac{a_2}{r_2^2} e^{-r_2(t_4-t_3)} + \frac{b_2}{2} \left( \frac{1}{r_2} \right) \right] \]

\[
e^{-r_2(t_4-t_3)} + \frac{t_3 - t_2}{r_2} e^{-r_2(t_3-t_2)} + \frac{1}{r_2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]

\[
e^{-r_2(t_4-t_3)} + \frac{1}{r_2^2} \left( \frac{1}{r_2^2} \right) \]
Let $\theta$ be a continuous function with continuous partial derivatives defined on an open set $T$ containing the point $P_1 = (x_0, y_0, z_0)$. If $\frac{\partial F}{\partial r} \neq 0$ at $P_1$, then there exists a neighborhood $N$ around $(x_0, y_0)$ such that for any $(x, y)$ in $N$, there is a unique solution of the Karush-Kuhn-Tucker (KKT) conditions such that $F(x_0, y_0, z_0) = 0$.

Let us now construct our results and prove them by using the above all.

Result 1: The profit function $TP(T, x, r_1, r_2)$ is a strictly pseudoconcave function in $T$ and hence, exists a unique maximum solution, denoted as $T^*$, for any $x$, $r_1$ and $r_2$.

Proof (computer supported). Let us define nonnegative function $p_1(T)$ and $q_1(T)$ for any given $(x, r_1, r_2)$:

$$p_1(T) = r_1 T(a_1 T + \frac{b_1}{2} T^2 + c_1 r_1 T) - cT \left[ e^{-(\theta-g(x))} T \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \right]$$

$$+ \frac{b_1 - b_2}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)} \left[ (a_2 - a_1 + c_1 r_1) + (b_2 - b_1) T \right] + \frac{1}{1 + \delta t} (a_1 (t_4 - t_3))$$

$$+ \frac{b_1}{2} (r_4^2 - t_3^2) + c_1 r_1 (t_4 - t_3) \right) - b_1 T \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right]$$

$$\left[ \frac{1}{\theta - g(x) - r_2} \left( 1 - e^{-(\theta-g(x)-r_2)} T \left[ 1 \right] \right) \right] + \frac{b_2 - b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2} \left( 1 - e^{-r_2 T} \left[ 1 \right] \right) \right] + \frac{1}{\theta - g(x)}$$

$$\left[ (a_2 - a_1 + c_1 r_1) \frac{1}{r_2} \left( 1 - e^{-r_2 T} \left[ 1 \right] \right) + (b_2 - b_1) \left( \frac{1}{r_2} - \frac{t_1 e^{-r_2 T} \left[ 1 \right]}{r_2} - \frac{e^{-r_2 T} \left[ 1 \right]}{r_2^2} \right) \right]$$
Now, taking the first- and second-order partial derivatives of $p_1(T)$, with respect to $T$, we get

\[
\frac{\partial p_1(T)}{\partial T} = r_1 T(a_1 + b_1 T + c_1 r_1) + c \left[ e^{-(\theta-g(x))T} \left( \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right) + \frac{b_2 - b_1}{\theta - g(x)} \right] - e^{-(\theta-g(x))T} \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{T}{1 + \delta T} \left[ (a_2 - a_1 + c_1 r_1)(\frac{dt_4}{dT} - \frac{dt_3}{dT}) \right],
\]
\[+ b_1 \left( \frac{dt_4}{dT} - t_3 \frac{dt_3}{dT} \right) - h_1 \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \frac{1}{\theta - g(x) - r_2} \]

\[(1 - e^{-(\theta - g(x) - r_2)t_1}) + \frac{b_2 - b_1}{(\theta - g(x))^2} \frac{1}{r_2} \left( 1 - e^{-r_2 t_1} \right) + \frac{1}{\theta - g(x)} \]

\[\left[ (a_2 - a_1 + c_1 r_1) \frac{1}{r_2} (1 - e^{-r_2 t_1}) + (b_2 - b_1) \left( \frac{1}{r_2^2} - \frac{t_1 e^{-r_2 t_1}}{r_2} - \frac{e^{-r_2 t_1}}{r_2} \right) \right] \]

\[e^{-(\theta - g(x)) t_2} \theta \frac{b_1}{(\theta - g(x))^2} \frac{1}{r_2} \left[ e^{(\theta - g(x) - r_2) t_2} - e^{-(\theta - g(x) - r_2) t_1} \right] \]

\[- \frac{b_1}{\theta - g(x)} \left[ t_1 e^{-r_2 t_1} - \frac{t_2 e^{-r_2 t_2}}{r_2} - \frac{e^{-r_2 t_2}}{r_2} \right] \]

\[+ \frac{1}{\theta - g(x)} \frac{1}{r_2} \left[ (a_1 + c_1 r_1 + b_1 t_2) \frac{1}{r_2} \left( e^{-(r_2 + \theta - g(x)) t_2} - t_1 e^{-(r_2 + \theta - g(x)) t_1} \right) \right] \]

\[- \frac{T}{(\theta - g(x) - r_2)^2} \]

\[e^{-(\theta - g(x)) t_1} \frac{dt_1}{dT} + \frac{b_2 - b_1}{r_2^2 (\theta - g(x))^2} e^{-r_2 t_1} \frac{dt_1}{dT} + \left( a_2 - a_1 + c_1 r_1 \right) \frac{dt_1}{dT} \]

\[- e^{-(\theta - g(x)) t_2} \frac{dt_2}{dT} + \frac{b_2 - b_1}{(\theta - g(x))^2} \frac{dt_2}{dT} + \frac{1}{\theta - g(x) - r_2} \left( a_1 + c_1 r_1 + b_1 t_2 \right) \frac{dt_2}{dT} \]

\[+ \left( \frac{t_2 e^{-r_2 t_2}}{r_2^2} + \frac{e^{-r_2 t_2}}{r_2^2} \right) \left( \frac{a_1 + c_1 r_1}{\theta - g(x)} - \frac{b_1}{(\theta - g(x))^2} \right) \]

\[\left[ e^{-r_2 t_1} \frac{dt_1}{dT} + \frac{e^{-r_2 t_2}}{r_2^2} \frac{dt_2}{dT} \right] \]

\[- \frac{2 x T}{r_2} e^{-(\theta - g(x)) T} \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x) + \frac{b_2 - b_1}{(\theta - g(x))^2}} \frac{b_2 - b_1}{r_2 (\theta - g(x))^2} e^{-r_2 T} \]

\[+ s_1 T \frac{a_1 + c_1 r_1}{r_2} e^{-r_2 (t_3 - t_2)} \frac{dt_3}{dT} - \frac{dt_3}{dT} \frac{dt_2}{dT} \]

\[e^{-r_2 (t_3 - t_2)} \frac{dt_3}{dT} - \frac{dt_2}{dT} \frac{dt_3}{dT} \]

\[\frac{dt_2}{dT} \left( \frac{b_1}{(t_3 - t_2)^2} e^{-r_2 (t_3 - t_2)} \frac{dt_3}{dT} - \frac{dt_2}{dT} \right) + \frac{T r_1}{1 + \delta t} \left( \frac{a_2 t_4 - t_3}{r_4 (t_4 - t_3)} e^{-r_2 (t_4 - t_3)} \right) \]

\[\left( \frac{dt_4}{dT} \frac{dt_3}{dT} + e^{-r_2 (t_3 - t_2)} \frac{dt_3}{dT} \right) \left( \frac{t_3^2 - t_2^2}{r_2^2} + \frac{a_1 + c_1 r_1}{\delta} (r_2 (t_4 - t_3)) \right) \]
We see by the computer use of Mathematica that

\[
\left(\frac{dt_4}{dT} - \frac{dt_3}{dT}\right) + \frac{1}{\delta(1 + \delta(t_4 - t_3))} \left(\frac{dt_4}{dT} - \frac{dt_3}{dT}\right) - b_2 \frac{2\delta^2}{r_2} (t_3 - t_2) \left(\frac{dt_3}{dT} - \frac{dt_2}{dT}\right) e^{-r_2(t_3 - t_2) - e^{-r_2(t_3 - t_2)}} (t_2 - t_2) (\frac{dt_3}{dT} - \frac{dt_2}{dT}) - e^{-r_2(t_4 - t_3)} (\frac{dt_4}{dT} - \frac{dt_3}{dT})
\]

\[
+ q_1 \left[ e^{-\frac{(\theta - g(x))^T}{\theta - g(x)}} \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} + \frac{b_2 - b_1}{\theta - g(x)} \right].
\]

\[
\frac{\partial^2 p_1(T)}{\partial T^2} = r_1 a_1 + 2b_1 r_1 T + c_1 r_1^2 - e^{\frac{e^{-\frac{(\theta - g(x))^T}{\theta - g(x)}}}{\theta - g(x)}} \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} \right] + b_2 - b_1 \frac{1}{(\theta - g(x))^2} + \frac{1}{1 + \delta t} \left[ (a_1 + c_1 r_1) (\frac{d^2 t_3}{dT^2} - \frac{d^2 t_4}{dT^2}) \right] + b_1 \left( (\frac{dt_4}{dT})^2 + t_4 \frac{d^2 t_4}{dT^2} \right)
\]

\[
- \frac{dt_3}{dT} (t_3 - t_3) \frac{d^3 t_3}{dT^2} + t_1 \frac{d^2 t_1}{dT^2} - \frac{2}{t_1} (t_2 - t_2) \frac{d^2 t_2}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_2}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2}
\]

\[
- t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2} + t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2}
\]

\[
\frac{e^{-r_2(t_3 - t_2)}}{r_2} \frac{d^2 t_2}{dT^2} - \frac{e^{-r_2(t_3 - t_2)}}{r_2} \frac{d^2 t_2}{dT^2} - \frac{b_2 - b_1}{(\theta - g(x))^2} \frac{a_1 + c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2}
\]

\[
+ \frac{b_2 - b_1}{(\theta - g(x))^2} e^{-r_2 t_1} \frac{dt_1}{dT} - \frac{2}{t_2} (t_2 - t_2) \frac{d^2 t_2}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_2}{dT^2} - t_1 (t_3 - t_3) \frac{d^2 t_3}{dT^2}
\]

\[
+ \frac{1}{1 + \delta t} \left[ a_2 (t_4 - t_3) + \frac{b_1}{(\theta - g(x))^2} \frac{a_1 + c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2}
\]

\[
+ \frac{1}{1 + \delta t} \left[ \frac{a_1 + c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right]
\]

We see by the computer use of Mathematica that

\[
\frac{\partial^2 p_1(T)}{\partial T^2} < 0, \quad \forall T > 0.
\]
Here, $TP(T, x, r_1, r_2)$ is a continuously differentiable real function; therefore, there exists a continuous gradient of the function $TP(T, x, r_1, r_2)$. Therefore, by computational verification (including usage of the Intermediate Value Theorem) we demonstrate the existence of one zero point of $F$ (i.e., $F = 0$ is fulfilled). By Implicit Function Theorem, there exists a point solution going through the point $(T_0, x_0, r_{10}, r_{20})$ of the problem $F = 0$ with $F = (\frac{\partial TP}{\partial T})|_{(x, r_1, r_2)}$, a local solution, not vanishing at the point solution, locally unique, and with a local parametrization, called as implicit function, $T^* = T^*(x, r_1, r_2)$.

Here, $p_1(T)$ is differentiable and strictly concave, and $q_1(T)$ is positive, differentiable and convex. Hence, by Lemma 1 we say (in compact notation) that

$$\overline{TP}(T, x, r_1, r_2) = \frac{p_1(T)}{q_1(T)} < 0$$

is a pseudoconcave function in $T$ for any $x, r_1, r_2$.

Now, to find the optimal implicit function, $T^*(x, r_1, r_2)$, explicitly, let us take the first-order partial derivative of $\overline{TP} := TP(T, x, r_1, r_2)$ with respect to $T$, for equating that derivative with respect to $T$, we compute it first:

$$\frac{\partial \overline{TP}}{\partial T} = r_1 T (a_1 + b_1 T + c_1 r_1)$$

$$+ e \left[ T e^{-\theta g(x)} \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{b_2 - b_1}{\theta - g(x)} \right]$$

$$- e^{-\theta g(x)} \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] + \frac{T}{1 + \delta t} \left[ \frac{a_1 + c_1 r_1}{(\theta - g(x))^2} - \frac{b_2 - b_1}{\theta - g(x)} - r_2 \right]$$

$$+ b_1 T \left( t_4 \frac{dt_4}{dT} - t_3 \frac{dt_3}{dT} \right) - h_1 \left[ T \left( a_1 - a_2 - c_1 r_1 \right) + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \frac{1}{\theta - g(x)} - r_2$$

$$\left[ (a_2 - a_1 + c_1 r_1) \frac{1}{r_2} (1 - e^{-r_2 t_1}) + (b_2 - b_1) \left( \frac{1}{r_2^2} - \frac{t_1 e^{-r_2 t_1}}{r_2} - \frac{e^{-r_2 t_1}}{r_2^2} \right) \right]$$

$$+ e^{-(\theta g(x))t_2} \left[ \frac{a_1 + c_1 r_1 + b_1 t_2}{\theta - g(x)} - \frac{b_1}{(\theta - g(x))^2} \right] \left[ e^{-(\theta g(x))t_2} - 1 - e^{-(\theta g(x))t_2} \right]$$

$$- e^{-(\theta g(x))t_2} \left[ \frac{a_2 - a_1 + c_1 r_1}{\theta - g(x)} + \frac{b_2 - b_1}{(\theta - g(x))^2} \right] \left[ e^{-(\theta g(x))t_2} - 1 - e^{-(\theta g(x))t_2} \right]$$

$$+ a_1 + c_1 r_1 \frac{1}{\theta - g(x)} e^{-(r_2 t_1) - e^{-r_2 t_2}}$$

$$+ T \left( \frac{b_1}{(\theta - g(x))^2} \left[ \frac{1}{r_2} (e^{-r_2 t_1} - e^{-r_2 t_2}) \right] \right) - \frac{x T^2}{r_2} e^{-(\theta g(x))T} \left[ \frac{a_1 - a_2 - c_1 r_1}{\theta - g(x)} + \right.$$
The first-order partial derivative of the profit function \( TP \) with respect to \( T \) is given by:

\[
\frac{\partial TP}{\partial r_1} := a_1 T + b_1 T^2 + c r_1 T
\]

and

\[
\frac{\partial TP}{\partial r_2} := a_2 (t_4 - t_3) e^{-r_2(t_4 - t_3)} \left( \frac{dt_3}{dT} - \frac{dt_2}{dT} \right) - \frac{T r_1}{1+\delta t} \left[ a_2 (t_4 - t_3) e^{-r_2(t_4 - t_3)} \right] + \left( \frac{dt_4}{dT} \right) \left( \frac{dt_3}{dT} - \frac{dt_2}{dT} \right)
\]

Now, by employing computational power and by applying Intermediate Value Theorem to the Lagrangian system, with respect to \( T \), we can conclude the existence of a solution of the KKT condition. Then, by using Lemma 2, we say that there exists a unique global maximum solution, \( T^*(x, r_1, r_2) \). This proves the result.

**Result 2:** For any feasible \( T \), there exist a unique \( r_1^* \) that maximizes the profit function \( TP := TP(T, x, r_1, r_2) \) for any \( r_2 \) and \( x \).
The second-order partial derivative of the profit function \( \tilde{r}_1 \) with respect to \( r_1 \) is:

\[
\frac{\partial^2 \tilde{TP}}{\partial r_1^2} = -2c_1 T.
\]

Now, we check with the help of Mathematica for any \( c_1, T > 0 \) that
\[
\frac{\partial^2 \tilde{TP}}{\partial r_2^2} = 2c_1 T - \frac{1}{(1+\delta t)} \left[ \frac{c_1}{\delta} \left( \frac{r_2}{2} (t_1^2 - t_3^2) + \frac{1}{\delta} \log(1 + \delta(t_4 - t_3)) \right) \right.
- \frac{a_1 + c_1 r_1}{\delta^2} \log(1 + \delta(t_4 - t_3)) \left. \right] - \frac{c_1}{(1+\delta t)\delta} \log(1 + \delta(t_4 - t_3)) < 0.
\]

Thus, \( r_1^*(r_2, x) \) is the optimal market price that maximizes the profit function \( \tilde{TP}(T, r_1^*(r_2, x), r_2, x) \) for any \( r_2 \) and \( x \).

**Result 3:** For any feasible \( r_1 \) and \( T \), there exists an \( r_2^* \) that maximizes the profit function \( \tilde{TP}(T, x, r_1, r_2) \) for any \( x \).

**Proof (computer supported).** The first- and second-order partial derivatives of the profit function \( \tilde{TP}(T, x, r_1, r_2) \) with respect to \( r_2 \) can be stated as:

\[
\frac{\partial \tilde{TP}}{\partial r_2} = \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_1 - b_2}{(\theta - g(x))^2} \right] \frac{2e^{-(r_2^* + \theta - g(x))t_1}}{\theta - g(x)^2} + \frac{b_2 - b_1}{r_2^2 (\theta - g(x))^2} + \frac{1}{\theta - g(x)}
\]

\[
\left[ \frac{a_2 - a_1 + c_1 r_1}{r_2^2} \right] + \left[ \frac{b_1}{(\theta - g(x))^2} \right] - \left[ \frac{1}{\theta - g(x)^2} \right] - \left[ \frac{2e^{-(\theta - g(x))t_2}}{r_2} \right] - \left[ \frac{e^{-(\theta - g(x))t_1}}{r_2} \right] + \frac{a_1 + c_1 r_1 + b_1 t_2}{(\theta - g(x))^2} \left[ \frac{2e^{-r_2 t_2}}{r_2^2} - \frac{2e^{-r_2 t_1}}{r_2^2} \right]
\]

\[
\frac{\partial^2 \tilde{TP}}{\partial r_2^2} = -h_1 \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_1 - b_2}{(\theta - g(x))^2} \right] \frac{4e^{-(r_2^* + \theta - g(x))t_1}}{\theta - g(x)^2} + \frac{b_2 - b_1}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)}
\]

\[
\frac{\partial^2 \tilde{TP}}{\partial r_2^2} = -h_1 \left[ \frac{a_2 - a_1 - c_1 r_1}{\theta - g(x)} + \frac{b_1 - b_2}{(\theta - g(x))^2} \right] \frac{4e^{-(r_2^* + \theta - g(x))t_1}}{\theta - g(x)^2} + \frac{b_2 - b_1}{(\theta - g(x))^2} + \frac{1}{\theta - g(x)}
\]
Now, we check with the help of Mathematica for any $x$ that
\[
\frac{\partial^2 \bar{TP}}{\partial r_2^2} < 0.
\]
Therefore, it is concluded that $\bar{TP}(T, r_1, r_2^*(x), x)$ is a concave function with respect to $r_2^*(x)$. The problem is a concave fractional program. Then, we can say from Lemma 2 and Implicit Function Theorem, that $\bar{TP}(T, r_1, r_2^*(x), x)$ attains a local maximum which is the global maximum also due to the pseudoconcavity of $\bar{TP}(T, r_1, r_2^*(x), x)$. Therefore, it is sufficient to establish the KKT conditions. This completes the proof of the result. □

**Result 4:** There exists a unique $x^*$ for which the profit function $\bar{TP} := TP(T, x^*, r_1, r_2)$ is concave, for any feasible $r_1, r_2$ and $T$.

**Proof (computer supported).** The first- and second-order partial derivatives of the profit function $\bar{TP}(T, x, r_1, r_2)$ with respect to $x$ is calculated as:

\[
\frac{\partial \bar{TP}}{\partial x} = -c \left[ (a_2 - a_1 + c_1 r_1) e^{-x} \left( e^{-\theta g(x)} T - e^{-\theta g(x)} T \right) + (b_2 - b_1) e^{-x} \left( e^{-\theta g(x)} - \left( \theta - g(x) \right)^2 T \right) \right]
\]

\[
\frac{\partial^2 \bar{TP}}{\partial x^2} = -c^2 \left[ (a_2 - a_1 + c_1 r_1) e^{-x} \left( e^{-\theta g(x)} - \left( \theta - g(x) \right)^2 T \right) + (b_2 - b_1) e^{-x} \left( e^{-\theta g(x)} - \left( \theta - g(x) \right)^2 T \right) \right].
\]
\[-2(b_1 - b_2) \frac{e^{-x}}{(\theta - g(x))^3} - \frac{e^{-x}}{(\theta - g(x))^2} \left[(a_2 - a_1 + c_1 r_1) + (b_2 - b_1)T\right]
\]
\[-h_1 \frac{e^{-x}(a_2 - a_1 - c_1 r_1)}{(\theta - g(x))^2} - 2e^{-x}(b_2 - b_1) + \left(\frac{e^{-x}}{(\theta - g(x))^3}\right)\]
\[-e^{-x}e^{-(\theta - g(x) - r_2)T_1} + \frac{e^{-x}(1 - e^{-(\theta - g(x) - r_2)T_1})}{(\theta - g(x) - r_2)^2}\]
\[-(b_2 - b_1) \frac{2e^{-x}}{r_2(\theta - g(x))^3}(1 - e^{-r_2T_1}) - \frac{e^{-x}}{(\theta - g(x))^2}\]
\[
[(a_2 - a_1 + c_1 r_1) \frac{1}{r_2}(1 - e^{-r_2T_1}) + (b_2 - b_1)\left(\frac{1}{r_2} - \frac{t_1 e^{-r_2T_1}}{r_2} - \frac{e^{-r_2T_1}}{r_2^2}\right)]
\]
\[-\left[\frac{e^{-x}(a_2 + c_1 r_1 + b_1 T_2)}{(\theta - g(x))^2} - 2e^{-x} \frac{e^{-(\theta - g(x))T_2}}{\theta - g(x) - r_2} + \frac{e^{-(\theta - g(x))T_2}}{\theta - g(x)}\right]\]
\[-b_1 \frac{e^{-x}e^{-(\theta - g(x))T_2}}{(\theta - g(x))^2}\left[\frac{1}{\theta - g(x) - r_2}(\theta + 1)T_2 - \frac{(\theta - g(x) - r_2)^2}{\theta - g(x)}\right]\]
\[-\left[\frac{e^{-x}e^{-(\theta - g(x))T_2}}{(\theta - g(x))^2}\left(\theta + 1 - r_2\right)T_2 - \frac{(\theta + 1 - r_2)T_2}{\theta - g(x) - r_2}\right]\]
\[-\frac{b_1 e^{-x}}{r_2} \left[\frac{t_1 e^{-r_2T_1}}{r_2} + \frac{r_2 e^{-r_2T_1}}{r_2} - \frac{e^{-r_2T_1}}{r_2^2}\right]\]
\[+ \frac{(a_1 + c_1 r_1 e^{-x}}{(\theta - g(x))^2} \left[\frac{1}{r_2}(e^{-r_2T_1} - e^{-r_2T_2}) - \frac{2b_1 e^{-x}}{r_2} \left(\frac{1}{r_2}(e^{-r_2T_1} - e^{-r_2T_2})\right)\right]\]
\[-\frac{T}{r_2} e^{-(\theta - g(x))T} e^{x} \frac{(a_2 - a_1 - c_1 r_1)}{\theta - g(x)}\]
\[+ \frac{b_2 - b_1}{(\theta - g(x))^2} + \frac{xT}{r_2} e^{-(\theta - g(x))T} \left[\frac{(a_2 - a_1 - c_1 r_1)e^{-x}}{(\theta - g(x))^2} - 2(b_2 - b_1)e^{-x}\right]\]
\[-q_1 \left[\frac{e^{-x}e^{-(\theta - g(x))T}}{(\theta + 1)T}\left[\frac{(a_2 - a_1 - c_1 r_1)}{\theta - g(x)}\right] + \frac{b_2 - b_1}{\theta - g(x)}\right]\]
\[+ \frac{e^{-(\theta - g(x))T} e^{-x}}{(\theta - g(x))^2} \left[\frac{a_2 - a_1 + c_1 r_1}{(\theta - g(x))^2} - \frac{2(b_2 - b_1)e^{-x}}{(\theta - g(x))^3} - \frac{1}{(\theta - g(x))^2}\right] \left[(a_2 - a_1 + c_1 r_1) + (b_2 - b_1)T\right].\]

Setting up our optimality condition again,
\[\frac{\partial T P}{\partial x} = 0,\]

we compute a value of \(x\), denoted by \(x^*\), with the help of Mathematica. Now, we calculate
\[
\frac{\partial^2 T P}{\partial x^2} = -c \left[(a_2 - a_1 + c_1 r_1)\left(e^{-(\theta - g(x))T} e^{-2x} - \frac{e^{-(\theta - g(x))T} e^{-x}}{(\theta + 1)^2T^2} + \frac{e^{-(\theta - g(x))T} e^{-x}}{(\theta - g(x))^2}\right)
\]
\[-\frac{e^{-(\theta - g(x))T} e^{-2x}}{(\theta + 1)T(\theta - g(x))^2} + \frac{b_2 - b_1}{(\theta - g(x))^3} + \frac{b_2 - b_1}{(\theta + 1)T(\theta - g(x))^2}\]
\[-\frac{e^{-(\theta - g(x))T} e^{-2x}}{(\theta + 1)T(\theta - g(x))^2} + \frac{e^{-(\theta - g(x))T} e^{-2x}}{(\theta - g(x))^3} - \frac{b_2 - b_1}{(\theta - g(x))^4(\theta - g(x))^2}\].
\[-\frac{e^{-(\theta-g(x))}T + e^{-(\theta-g(x))}T 2e^{-2x}}{\theta + 1} \frac{e^{-2x}}{(\theta - g(x))^3} + \frac{e^{-(\theta-g(x))}T 2e^{-2x}}{(\theta - g(x))^3}
\]
\[+ 2(b_1 - b_2) \frac{e^{-2x}}{(\theta - g(x))^3} + 3e^{-2x}
\]
\[+ [(a_2 - a_1 + c_1 r_1) + (b_2 - b_1)T] \left( \frac{e^{-x}}{(\theta - g(x))^2} + \frac{2e^{-2x}}{\theta - g(x)^3} \right)
\]
\[- h_1 [(a_2 - a_1 - c_1 r_1) \frac{e^{-x}}{(\theta - g(x))^3 - \theta - g(x)^3} - \frac{e^{-x}}{(\theta - g(x))^3}
\]
\[- 2(b_2 - b_1) \frac{e^{-2x}}{(\theta - g(x))^3} - \frac{3e^{-2x}}{(\theta - g(x))^3} - \frac{(\theta - g(x) - r_2)^3}{(\theta - g(x))^3} + 2e^{-2x} e^{-(\theta-g(x))} T T_1
\]
\[- \frac{e^{-x}}{(\theta - g(x))^3 + \frac{3e^{-2x}}{(\theta - g(x))^3}} (1 - e^{-r_2 T_1} - \frac{e^{-x}}{(\theta - g(x))^3}
\]
\[+ \frac{2e^{-2x}}{(\theta - g(x))^3} [(a_2 - a_1 + c_1 r_1) \frac{1}{r_2} (1 - e^{-r_2 T_1})
\]
\[+ (b_2 - b_1) \left( \frac{1}{r_2} - \frac{t_1 e^{-r_2 T_1}}{r_2} - \frac{e^{-r_2 T_1}}{r_2^2} \right) - [(a_1 + c_1 r_1 + b_1 T_2)
\]
\[- \frac{e^{-x}}{(\theta - g(x))^2} - \frac{2e^{-2x}}{(\theta - g(x))^4} - 2(\theta - g(x))^3
\]
\[- 3e^{-2x} \frac{e^{-(\theta-g(x))} t_2}{(\theta - g(x))^2} e^{-x} e^{-(\theta-g(x))} T_1
\]
\[- 2b_1 \frac{e^{-x}}{(\theta - g(x))^3} [\frac{(\theta - g(x) - r_2)(\theta - g(x) + 1)T_2}{(\theta - g(x) - r_2)^3} - e^{-x} e^{-(\theta-g(x))} T_2]
\]
\[- 2b_1 \left( \frac{e^{-x}}{(\theta - g(x))^2} + \frac{2e^{-2x}}{(\theta - g(x))^3} \right)
\]
\[\left[ \frac{t_1 e^{-r_2 T_1}}{r_2} + \frac{e^{-r_2 T_1}}{r_2^2} - \frac{t_2 e^{-r_2 T_2}}{r_2} - \frac{e^{-r_2 T_2}}{r_2^2} \right] - (2b_1 + a_1 + c_1 r_1)
\]
\[\frac{\left[ a_1 - a_2 - c_1 r_1 \right] e^{-x}}{(\theta - g(x))^2} + \frac{b_2 - b_1}{(\theta - g(x))^2} \frac{x T e^{-x} e^{-(\theta-g(x))} T}{r_2^2} (\theta + 1)T
\]
\[\left[ \frac{1}{(\theta + 1)T} \frac{e^{-(\theta-g(x))} T e^{-x}}{(\theta - g(x))^3} - \frac{e^{-(\theta-g(x))} T e^{-2x}}{(\theta + 1)T (\theta - g(x))^3} \right]
\]
per unit time is $T P$, investment per unit per month is $x$. The computation is performed based on the example of Section 6, and the single parameter as varying at a time while treating the remaining parameters as changing the major parameters with +50%, +20%, -20% and -50%, regarding a sensitivity analysis.

3. and Figure 4. Hence, these figures demonstrate the concavity of the total profit, the optimal inflation rate as $c = 0$ compatible with our assumptions.

Numerical example. In this section, to illustrate the solution procedure, we present a numerical example with reduced deterioration rate $g(x) = 1 - e^{-x}$. We consider the following parametric values: $a_2 = 100$, $b_2 = 150$, $c = 15$ per unit, $\theta = 0.02$, $s_1 = 3$ per unit and per item, $s_2 = 4$ per unit and per item, $h_1 = 10$ per unit and per month, $q_1 = 20$ per shipment and per time, $a_1 = 1000$, $b_1 = 200$, $c_1 = 0.9$, $\delta = 0.07$. Then, with the help of Mathematica, we obtain the optimal cycle time as $T^* = 0.804$ years, the optimal selling price per unit time as $r_1^* = 17.12508$, the optimal inflation rate as $r_2^* = 11.208$, and the amount of preservation technology investment per unit per month is $x^* = 2.68$. Hence, the total profit of the system per unit time is $TP(T, x^*, r_1^*, r_2^*) = 2553.208$. Now, the concavity properties of the total profit are shown in Figure 2, Figure 3 and Figure 4. Hence, these figures demonstrate the concavity of the total profit, which is compatible with our assumptions.

7. Sensitivity analysis. In this section, we perform a sensitivity analysis by changing the major parameters with +50%, +20%, -20% and -50%, regarding a single parameter as varying at a time while treating the remaining parameters as fixed. The computation is performed based on the example of Section 6, and the computational results are displayed by Table 2.

Figure 5, Figure 6 and Figure 7 are graphical representations of variables depending on parameters, reflected in our sensitivity analysis.

The following observations are made from Table 2, disclosing the importance of our model:

7.1. When the initial demand $a_1$ increases, the optimal total profit $TP(T^*, x^*, r_1^*, r_2^*)$, the optimal selling price $r_1^*$ and the inflation rate $r_2^*$ grow, but the optimal replenishment time $T^*$ and the amount of preservation technology investment
Figure 2. Concavity of the profit function; including are $x^*$, $r_1^*$ and $TP$, as $x$-axis, $y$-axis and $z$-axis, respectively.

Figure 3. Concavity of the profit function; including are $r_1^*$, $r_2^*$ and $TP$, as $x$-axis, $y$-axis and $z$-axis, respectively.
Figure 4. Concavity of the profit function; including are $x^*$, $r_2^*$ and $TP$, as $x$-axis, $y$-axis and $z$-axis, respectively.

Figure 5. Effect of $\theta$ on total profit $TP$. 
Figure 6. Effect of $\theta$ on selling price $r_1$.

Figure 7. Effect of demand parameter $b_1$ on preservation amount $x$.

$x^*$ decrease. So, to maintain the initial demand rate as high, the retailer has to order more quantity per replenishment cycle. Moreover, the retailer will shorten the replenishment cycle, and if the initial demand rate is quite low, then it will be better for the retailer to terminate the order.
7.2. When the price sensitive parameter $c_1$ expands, then the optimal profit $TP(T^*, x^*, r_1^*, r_2^*)$, the optimal selling price $r_1^*$ and the inflation rate $r_2^*$, will diminish, whereas the optimal cycle length $T^*$ and the amount of preservation technology investment $x^*$ will rise. This fact illustrates that when the market price increases, the demand automatically declines. Hence, to maintain the demand of the original items, the retailer should reduce the optimal selling price.

7.3. When the holding cost $h_1$, and the purchasing cost $c$ increase, then the optimal replenishment cycle $T^*$ and the total profit $TP(T^*, x^*, r_1^*, r_2^*)$ decrease, which demonstrates the fact that for a minimum holding cost, the profit will be maximal.

7.4. A high deterioration rate $\theta$ and a high preservation technology investment $x^*$ decrease the optimal replenishment cycle $T^*$ and the total profit $TP(T^*, x^*, r_1^*, r_2^*)$. But, if the initial deterioration is marginal or less in value, then investment on preservation technology is unnecessary.

7.5. For a higher rate of the increasing demand $b_1$, the optimal replenishment cycle $T^*$ is decreasing. However, the total profit $TP(T^*, x^*, r_1^*, r_2^*)$ is increasing and the preservation technology investment amount $x^*$ is decreasing. Therefore, a high amount of demand will enrich the profit and also the possibility of shortage.

7.6. A higher rate of deterioration $\theta$, preservation technology investment $x^*$, holding cost $h_1$ and purchasing cost $c$ will increase the inflation rate $r_2^*$. This is not suitable for the system, and it will also diminish the total profit $TP(T^*, x^*, r_1^*, r_2^*)$ of the system.

8. **Concluding remarks and future research directions.** A main objective of inventory management is to enrich the profit by reducing the unnecessary cost. In this paper, we have presented a deteriorating inventory model with stock- and price-sensitive demand by calculating the preservation investment amount. Numerical examples have signified the importance of preservation technology investment. Our sensitivity analysis has made clear that when the capital investment on preservation technology increases, the profit of the system also increases. Furthermore, it has been shown that if the deterioration rate will be raised, the amount of investment also grows. But, any investment is not required, if the deterioration is marginal or less in value at the initial level.

Main contributions of our work to literature and to managerial practice are summarized as follows: (i) We have determined the amount of preservation technology investment for deteriorated items. Therefore we have employed calculus, convex analysis and, for overcoming the problem nonlinearity and complexity, computational power. (ii) We have provided some useful theorems and have proved them for ensuring the unique optimal solution. (iii) We have found that by applying preservation technology, one can reduce the deterioration rate; this extends the total profit for the system. (iv) The model has allowed us for a conclusion that consideration of inflation rate provides an interesting new result. (v) Lastly, the model has shown that stock- and price-sensitive demand exercises a remarkable effect on an inventory system.

Regarding further research, we may take care of multiple items instead of single item under stochastic demand constraint. For addressing a realistic situation, we can design an extended model by introducing warehouses, quantity discounts, stochastic inflation, deteriorating cost, time-dependent deterioration rate, permissible
delay in payments and uncertainty of demand. Taking into account a model with
dynamic preservation technology and limited capacity of shelf space will mean an-
other potential enrichment of our investigation. This research can also be continued
by inserting unit purchase cost, inventory holding cost and other related factors as
time-dependent rather than constant.

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Table 2: Sensitivity analysis on decision variables with respect to major
parameters.

| Input Parameters | Output Variables and Parameters | -50% | -20% | 0% | +20% | +50% |
|------------------|---------------------------------|------|------|----|------|------|
| \( a \)          | \( T \)                          | 1.213| 1.120| 0.932| 0.740| 0.691|
|                  | \( c \)                          | 4.5  | 4.0  | 3.5 | 2.7  | 2.1  |
|                  | \( r_1 \)                        | 12.7212| 15.1139| 16.2681| 17.2527| 18.1462|
|                  | \( r_2 \)                        | 11.58 | 12.86 | 13.42| 14.23| 15.70 |
|                  | \( TP \)                         | 1980.21| 2378.00| 2549.00| 2645.72| 2868.43|
| \( b \)          | \( T \)                          | 0.647| 0.720| 0.839| 0.986| 0.987|
|                  | \( x \)                          | 1.8  | 2.2  | 2.6 | 2.9  | 3.3  |
|                  | \( r_1 \)                        | 19.3510| 18.7220| 17.5701| 16.0148| 14.7370|
|                  | \( r_2 \)                        | 15.65 | 13.76 | 12.59| 11.38| 10.44 |
|                  | \( TP \)                         | 3376.30| 3048.00| 2849.21| 2432.43| 2155.67|
| \( \delta \)     | \( T \)                          | 1.211| 1.053| 0.916| 0.848| 0.720|
|                  | \( x \)                          | 3.4  | 3.0  | 2.9 | 2.6  | 2.1  |
|                  | \( r_1 \)                        | 15.5108| 16.7233| 17.3604| 17.9451| 18.4817|
|                  | \( r_2 \)                        | 11.55 | 11.89 | 12.10| 12.93| 13.41 |
|                  | \( TP \)                         | 3142.31| 2991.00| 2635.21| 2307.10| 2069.38|
| \( \theta \)     | \( T \)                          | 1.120| 0.984| 0.764| 0.798| 0.807|
|                  | \( x \)                          | 3.6  | 3.4  | 3.1 | 2.8  | 2.5  |
|                  | \( r_1 \)                        | 16.4707| 16.8574| 17.3612| 18.7185| 20.7354|
|                  | \( r_2 \)                        | 10.73 | 11.81 | 12.46| 12.79| 13.99 |
|                  | \( TP \)                         | 2376.00| 2578.82| 2854.06| 3075.56| 3243.61|
| \( c \)          | \( T \)                          | 0.867| 0.867| 0.867| 0.867| 0.835|
|                  | \( x \)                          | 2.7  | 2.8  | 2.9 | 3.0  | 3.2  |
|                  | \( r_1 \)                        | 19.4603| 18.8746| 18.2163| 17.7563| 17.3356|
|                  | \( r_2 \)                        | 11.77 | 12.13 | 12.85| 13.23| 13.79 |
|                  | \( TP \)                         | 2649.91| 2546.24| 2480.23| 2465.35| 2455.31|

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