Lyapunov function selection for saturated constrained system

Cui Zhang

College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong, 266590, China
*Corresponding author’s e-mail: 15762182962@163.com.

Abstract. For a nonlinear system which is not globally asymptotically stable, it is important to study its attractive domain. Saturation constrained control system is one of the most important nonlinear systems. The estimation of attractive domain of saturated constrained control system involves the selection of Lyapunov function. This paper summarizes the different Lyapunov functions in the estimation of attractive domain of saturated constrained control system and illustrates them with numerical examples.

1. Introduction
In the nonlinear system, the saturation constrained control system is a kind of basic system. Saturation phenomenon is often seen in practical applications, so it is of great significance to study the characteristics of saturated constrained systems for nonlinear control theory. For a locally stable nonlinear system, it is a research hotspot to analyze its attractive domain. In recent decades, scholars in the field of control have conducted systematic research on the saturated constrained control system, including the selection of Lyapunov function.

Because of the simplicity of quadratic Lyapunov function, it is widely used in stability analysis and controller design of saturated constrained control system. However, the quadratic Lyapunov function is very conservative in the process of system analysis and synthesis. In order to reduce conservatism, scholars have made a lot of attempts. For example, the compound quadratic Lyapunov function [1], saturation-dependent Lyapunov function [2] and piecewise quadratic Lyapunov function [3]. In this paper, two Lyapunov functions are summarized and explained with numerical examples.

2. Lyapunov function selection for saturated constrained system

2.1 Quadratic Lyapunov function
The local sector condition and the convex hull representation are the main tools to deal with the saturation constraint, which can be used to express the time derivative of the quadratic function along the system trajectory into a negative definite quadratic function, so as to obtain the linear matrix inequality condition that makes the time derivative of the quadratic function negative. Based on this condition, a corresponding optimization problem is constructed, and the optimal contraction invariant ellipsoid is obtained.

It is worth studying whether the optimal shrinkage invariant ellipsoid obtained by solving the optimization problem is the maximum.

Definition 1. For $\mu_c := \sup \{\mu : E(P, \mu)\}$, an ellipsoid $E(P, \mu_c)$ is the largest contracting invariant ellipsoid.
From the above definitions, two conclusions can be drawn: 1) For any \( x \in E^c(P, \mu_c) \setminus \{0\} \), \( \dot{V}(x) < 0 \); 2) There are some \( x_0 \in \partial E(P, \mu) \) such that \( \dot{V}(x_0) = 0 \), where \( E(P, \mu) = \{ x : x^T P x < \mu \} \), \( \partial E(P, \mu) = \{ x : x^T P x = \mu \} \).

Consider a single input saturation constraint control system
\[ \dot{x} = Ax + Bsat(u), \quad u = Fx \] (1) sat\( (Fx) \in \text{co}\{Fx, Hx\}, \quad x \in L(H) \). Where \( H \) is the auxiliary feedback matrix, given a positive definite matrix that satisfies \( He(P(A + BF)) < 0 \).

Ellipsoids \( E(P, \mu) \) that satisfy the sum of conditions are invariant in contraction. Set up the following optimization problem to solve a shrinkage invariant ellipsoid as large as possible:
\[
\begin{align*}
\max_{H \in \mathbb{R}^{n \times n}} & \quad \mu \\
\text{s.t.} & \quad (A + BH)^T P + P(A + BH) \leq 0; \\
& \quad \mu \text{HP}^{-1}H^T \leq 1.
\end{align*}
\] (2)

For \( \mu^* \) is the optimal solution to the optimization problem (2).

**Example 1.** Consider a single-input saturation constraint control system with the following parameters (1):
\[ A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad F = \begin{bmatrix} 1.2231 \\ -2.2468 \end{bmatrix} \]
For \( P = \begin{bmatrix} 2.4628 & -1.5372 \\ -1.5372 & 1.3307 \end{bmatrix} \), solve the optimization problem, get \( \mu^* = 1.4686 \).

![figure 1. \( \dot{V}(x) \) along the ellipsoid \( \partial E(P, \mu^*) \) boundary (Example 1)](image1)

![figure 2. \( \dot{V}(x) \) along the ellipsoid \( \partial E(P, \mu^*) \) boundary (Example 2)](image2)

In order to verify whether the optimal ellipsoid \( E(P, \mu^*) \) is the maximum shrinkage invariant ellipsoid, the time derivative of the quadratic Lyapunov function \( V(x) = x^T P x \) is drawn in figure 1 along the ellipsoid boundary \( \partial E(P, \mu^*) \). The x-coordinate \( \theta \) in figure 1 represents the angle \( x \) on \( \partial E(P, \mu^*) \).

Obviously, the maximum value \( \dot{V}(x) \) at \( \partial E(P, \mu^*) \) is 0. That existence \( x_0 \in \partial E(P, \mu^*) \) such that \( \dot{V}(x_0) = 0 \). Therefore, the optimal ellipsoid \( E(P, \mu^*) \) is the maximum shrinkage invariant ellipsoid.

For multi-input systems, sufficient conditions to guarantee the invariance of ellipsoidal contraction are not necessarily necessary. Construct the following optimization problem[4,5]:

\[
\begin{align*}
\max_{H \in \mathbb{R}^{n \times n}} & \quad \mu \\
\text{s.t.} & \quad (A + BH)^T P + P(A + BH) \leq 0; \\
& \quad \mu \text{HP}^{-1}H^T \leq 1.
\end{align*}
\]
\[
\begin{align*}
\max_{\mu \in \mathbb{R}^{m \times n}} & \quad \mu \\
\text{s.t.} & \quad \begin{bmatrix} P(A + BD_j F + BD_j^T H) \end{bmatrix} \leq 0, \quad i \in \{1, m\} \\
& \quad \mu_j h_j^T I_j \leq 1, \quad j \in \{1, m\}
\end{align*}
\]

\text{(3)}

\textbf{Example 2.} Considers the following system parameters:

\[
A = \begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix}, \quad B = \begin{pmatrix} 0.8828 & -0.1455 \\ 0.2842 & -0.0896 \end{pmatrix}, \quad F = \begin{pmatrix} 2.6921 & -9.1511 \\ -0.9778 & -18.2487 \end{pmatrix}
\]

For \( P = \begin{pmatrix} 0.2773 & -0.3815 \\ -0.3815 & 7.8606 \end{pmatrix} \), solve the optimization problem, get \( \mu^* = 0.0342 \).

As shown in the dotted line in figure 2, the maximum value \( \breve{V}(x) \) on \( \partial E(P, \mu^*) \) is less than 0, which means that \( \mu^* \neq \mu_c \).

\textbf{2.2 Convex hull Lyapunov function}

Because of the simplicity of quadratic Lyapunov function, it is widely used in stability analysis and controller design of saturated constrained control system. However, the quadratic Lyapunov function is very conservative in the process of system analysis and synthesis. As shown in figure 2, there is a big difference between the estimation of attractor domain using quadratic Lyapunov function and the real attractor domain. Scholars have proposed a variety of non-quadratic functions to reduce the conservatism brought by quadratic Lyapunov function, among which the convex hull Lyapunov function [6] with good characteristics has attracted much attention.

Given a positive definite matrix \( K_j \in \mathbb{R}^{n \times n}, j \in \{1, J\} \).

For \( O = \{ \rho = [\rho_1, \rho_2, \ldots, \rho_J] \in \mathbb{R}^J : \rho_j \geq 0, \sum_{j=1}^J \rho_j = 1 \} \), the convex hull Lyapunov function is defined as

\[
V_c(x) := \min_{\rho \in O} x^T \left( \sum_{j=1}^J \rho_j K_j \right) x.
\]

\text{(4)}

And, of course, \( V_c(x) \) is quadratic homogeneous, immediate \( V_c(\alpha x) = \alpha^2 V_c(x) \). \( V_c(x) \) is convex and continuous and differentiable.

A level set defined as \( V_c(x) \) is \( L_{V_c}(1) = \{ x \in \mathbb{R}^n : V_c(x) \leq 1 \} \). Obviously, ellipsoids \( E(O_j) \subset L_{V_c}(1) \), and \( L_{V_c}(1) = \cap_{j=1}^J E(O_j^1) : j \in \{1, J\} = \left\{ \sum_{j=1}^J \rho_j x_j : x_j \in E(O_j) \right\} \).

Using traditional convex hull representation to deal with the saturation constraint, the following theorem 1 is obtained:

\textbf{Theorem 1.} [7] Considering the saturation constraint control system (1), if there are matrix groups \( Q_j \in \mathbb{R}^{m \times n} \) and non-negative scalar groups \( \beta_{jk}, \quad i \in \{1, 2, \ldots \}, \quad j, \quad k \in \{1, J\} \), such that

\[
He\left(\left[(A + B(D_j F + D_j^T Q_j))Q_j\right) - \sum_{j=1}^J \beta_{jk} (O_j - O_j^1)\right) = \begin{pmatrix} q_{lj} - q_{lj}^1 \end{pmatrix} \leq 1, \quad l \in \{1, m\}, \quad j \in \{1, J\}.
\]

\text{(5)}

Where \( q_{lj} \) is the \( l \)th row of the matrix \( Q_j \), then the level set \( L_{V_c}(1) \) is invariable in contraction.

In order to obtain the largest possible estimation of the attractive domain, the following optimization problem is constructed:


\[ \alpha \max_{o_{i,j}, z_{i,j}, \beta_{j,k,m}, \nu_{j,m}} \left\{ I \alpha \sum_{j=1}^{r} \nu_{j,m} O_{j} \right\} \geq 0, \quad p \in I[1, N] \]

b) \( \text{He}(\mathbf{A} + BD, \mathbf{F}) O_{j} + BD_{r} Z_{j} \prec \sum_{k \in I, J} \beta_{j,k,m}(O_{k} - O_{j}) \), \( i \in I[1, 2^{m}] \) \quad (7)

c) \( \frac{1}{z_{j}} O_{j} \geq 0, \quad l \in I[1, m], \quad j \in I[1, J] \)

d) \( \sum_{j=1}^{J} \rho_{j} = 1, \quad p \in I[1, N] \)

Where \( z_{j} \) is the \( l \)th row of the matrix \( Z_{j} = H_{j} O_{j} \).

Using the improved convex hull representation to deal with the saturation constraint, the following theorem 2 is obtained.

**Theorem 2**[8]. Considering the saturation constraint control system (1), if there are matrix groups \( Q_{y} \in \mathbb{R}^{m \times m} \) and non-negative scalar groups \( \beta_{j,k,m}, \quad i \in I[1, 2^{m}], \quad j, k \in I[1, J] \), such that

\[ \text{He}(\mathbf{A} + B(D, F + D^{T} Q_{y}) O_{j}) \prec \sum_{k \in I, J} \beta_{j,k,m}(O_{k} - O_{j}) \), \quad i \in I[1, J] \]

\[ q_{i,j} O_{j} q_{i,j}^{T} \leq 1, \quad l \in I[1, m], \quad j \in I[1, J] \] \quad (8)

Where \( q_{i,j} \) is the \( l \)th row of the matrix \( Q_{y} \), then the level set \( L_{i}^{t} \) is invariable in contraction.

### 3. Summary

In this paper, we introduce the different Lyapunov functions dealing with saturation constraints, and summarize their applications in attractor region estimation of saturated control systems. These Lyapunov functions have also been successfully applied in the robust stability analysis of saturated control system and the design of anti-integrator overflow compensator.

### References

[1] Chaves M., Eissing T., Allgower F. (2008) Bistable biological systems: A characterization through local compact input-to-state stability. IEEE Transactions on Automatic Control, 53: 87 - 100.

[2] Da Silva J., Tarbouriech S., Reginatto R. (2002) Analysis of regions of stability for linear systems with saturating inputs through an anti-windup scheme. Proceedings of the International Conference on Control Applications, 2: 1106 - 1111.

[3] Cao YY., Lin Z. (2003) Stability analysis of discrete-time systems with actuator saturation by a saturation-dependent Lyapunov function. Automatica, 39(7):1235 - 1241.

[4] Hu T., Lin Z. (2001) Control systems with actuator saturation: Analysis and design. Boston: Birkhauser, 55-67.

[5] Hu T., Lin Z., Chen B. (2002) Analysis and design method for linear systems subject to actuator saturation and disturbances. Automatica, 38(2):351-359.

[6] Hu T., Lin Z. (2003) Composite quadratic Lyapunov functions for constrained control systems. IEEE Trans on Automatic Control, 48(3):440-452.

[7] Hu T., Teel AR., Zaccarian L. (2006) Stability and performance for saturated systems via quadratic and non quadratic Lyapunov functions. IEEE Trans on Automatic Control, 51(11):1770-1785.
[8] Li Y., Lin Z. (2017) The maximal contractively invariant ellipsoids for discrete-time linear systems under saturated linear feedback. Automatica, 76:336-344.