Second-order Conditional Gradients

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Abstract

Constrained second-order convex optimization algorithms are the method of choice when a high accuracy solution to a problem is needed, due to the quadratic convergence rates these methods enjoy when close to the optimum. These algorithms require the solution of a constrained quadratic subproblem at every iteration. In the case where the feasible region can only be accessed efficiently through a linear optimization oracle, and computing first-order information about the function, although possible, is costly, the coupling of constrained second-order and conditional gradient algorithms leads to competitive algorithms with solid theoretical guarantees and good numerical performance.

1 Introduction

We focus on the optimization problem defined as

$$\min_{x \in \mathcal{X}} f(x),$$

where $f : \mathcal{X} \to \mathbb{R}$ is strongly convex, has Lipschitz continuous gradients and is twice differentiable.

1.1 Newton and Quasi-Newton Methods

An immensely powerful approach when $\mathcal{X} = \mathbb{R}^n$ is to construct a second-order approximation to $f(x)$ at the current iterate using first and second order information, denoted by $\hat{f}(x)$, and move in the direction that minimizes this approximation, giving rise to a family of methods known as Newton methods (Kantorovich, 1948). A damped variant of the former applied to the minimization of a self-concordant function, converges globally, and shows quadratic local convergence when the iterates are close enough to the optimum (Nesterov & Nemirovskii, 1994). The global convergence of this method also extends to strongly convex and smooth function (Nesterov & Nemirovskii, 1994; Nesterov, 2013). Using a cubic regularized version of Newton’s method, the global convergence of the method can also be extended to a broader class of functions than that of self-concordant or strongly convex and smooth functions (Nesterov & Polyak, 2006).

When $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex set, one can use a constrained analog of these methods (Levitin & Polyak, 1966), where a quadratic approximation to the function is minimized over $\mathcal{X}$ at each iteration. There are two shortcomings to these methods. First, computing second-order information about $f(x)$, especially in high dimensions, is computationally expensive, this has led to the development of Quasi-Newton methods, that build an approximation to the hessian using gradient information acquired from past iterates (see Nocedal & Wright (2006) for an overview). Secondly, in many cases solving the approximate quadratic subproblem to optimality is not possible, and so inexact variants are used. This has resulted in numerous inexact Quasi-Newton methods, which inherit many of the favorable properties of Newton methods (Scheinberg & Tang, 2016; Lee et al., 2014). The global convergence analysis of these methods can be recast in terms of a notion related to the multiplicative stability of the Hessian, allowing for elegant proofs of global convergence for several variants, both in the unconstrained, and constrained case (Karimireddy et al., 2018a).
1.2 The Conditional Gradient Algorithms

The conditional gradient algorithm (Levitin & Polyak, 1966) (also known as the Frank-Wolfe algorithm (Frank & Wolfe, 1956)), is to be found on the other side of the computational cost spectrum. Instead of requiring the resolution of a constrained quadratic problem at each iteration, it requires the resolution of a constrained linear problem, and only uses first-order information about \( f(x) \). As the conditional gradient algorithm maintains its iterates as convex combinations of extremal points of \( \mathcal{X} \) obtained from the linear optimization problem it is \emph{projection-free}. The conditional gradient algorithm is a robust convex optimization algorithm that produces sparse solutions and, e.g., its Away-Step and Pairwise-Step variants enjoy a global linear convergence for strongly convex and smooth functions over polytopes (Lacoste-Julien & Jaggi, 2015); improved up to linear rates of convergence also hold for sharp functions (Kerdreux et al., 2019). It has been the method of choice in many machine learning applications, where projecting onto \( \mathcal{X} \) is computationally prohibitive, such as, e.g., in video co-localization (Joulin et al., 2014), greedy particle optimization in Bayesian inference (Futami et al., 2019) or in structural SVMs (Lacoste-Julien et al., 2013).

For problems where first-order information about \( f(x) \) is hard to compute, using second-order methods seems counter-intuitive, yet it allows the construction of a quadratic approximation \( \hat{f}(x) \) to the function \( f(x) \) whose gradients are much cheaper to compute (another alternative is to replace the first-order oracle by a stochastic first order oracle to combat the computational cost of the first-order oracle; see e.g., (Robbins & Monro, 1951)). The observation that the progress made by moving towards the minimizer of \( \hat{f}(x) \) is often much greater than the progress made by an iteration of any first-order method allows these algorithms to be competitive in practice (Schmidt et al., 2009) although working with much more expensive second-order approximations. We consider the case where both the first-order oracle for \( f(x) \) and projection oracle onto \( \mathcal{X} \) are computationally expensive, but linear programming oracles over \( \mathcal{X} \) are relatively cheap. This is a setup that is often found in many applications. In this setting, we show how conditional gradient algorithms can be coupled with Quasi-Newton methods to obtain competitive algorithms both in progress per iteration and in progress per unit time; while the former is expected the second is not.

1.3 Quadratic Subproblem Stopping Criterion

The algorithm presented in this work uses a conditional gradient algorithm to approximately minimize the quadratic approximation \( \hat{f}(x) \) over \( \mathcal{X} \). The basic idea is similar to the Conditional Gradient Sliding algorithm (Lan & Zhou, 2016) where quadratic subproblems appearing in accelerated methods are solved via conditional gradients, yet our analysis is quite different.

The stopping criterion for our subproblems relies on knowledge of the primal gap \( f(x) - \hat{f}(x^\ast) \) (or any lower bound on the primal gap bounded away from zero). Knowledge of the value of \( f(x^\ast) \) has been used to obtain optimal step sizes, known as Polyak steps, for first-order methods (Polyak, 1969; Nedic, 2002; Hazan & Kakade, 2019) as well as to obtain local estimates of the strong convexity parameter of a function for accelerated algorithms (Barré & d’Aspremont, 2019). This setup is well-known and important in the context of machine learning applications where the loss function is usually lower bounded by 0, i.e., \( f(x^\ast) \geq 0 \). A similar assumption is made in Asi & Duchi (2019), to build optimization algorithms that have a robust performance with respect to different step sizes and algorithm parameters. Another important example is the approximate Carathéodory problem (Mirrokni et al., 2017; Combettes & Pokutta, 2019) where we want to find the decomposition of a point \( x_0 \) in a polytope \( \mathcal{P} \) as a convex combination of the vertices of \( \mathcal{P} \) and where the objective \( f(x) = \|x - x_0\|^2 \geq 0 \).

We stress that while we opted for the primal gap here as stopping criterion, as done in the aforementioned recent works, other choices together with the necessary changes to the argument are possible.

1.4 Contributions

The contribution can be summarized as follows:

**Projection-free Quasi-Newton Method.** We provide an inexact projection-free Quasi-Newton method, denoted as Second-order Conditional Gradients (SOG) with a stopping criterion that relies on knowledge of the primal gap and achieves global linear convergence and quadratic local convergence with properly tuned parameters when close to the optimum. Note that (Ochs & Malitsky, 2019)[Example 11] also proposed a conditional gradient-based Newton method via their model function-based conditional gradient algorithm, however their approach is markedly different.
from ours: the steps performed in their algorithm are projected onto the feasible region using the Euclidean norm while the projections in our algorithm use a norm defined by a positive semi-definite matrix that approximates the Hessian.

We also discuss how the algorithm can be used when the primal gap is not known to obtain an algorithm that retains the attractive numerical performance of the original algorithm; see Section 5.2.

Computational Experiments. We compare our algorithm with other state-of-the-art projection free optimization algorithms and provide numerical evidence that the proposed algorithm is able to significantly outperform other algorithms in terms of per-iteration and per-unit-time progress.

2 Preliminaries

We denote the unique minimizer of Problem 1.1 by \( x^* \), and we define the scaled projection of \( x \) onto \( \mathcal{X} \), denoted by \( \Pi^H_{\mathcal{X}}(x) : \mathbb{R}^n \to \mathcal{X} \), as:

\[
\Pi^H_{\mathcal{X}}(x) = \arg\min_{y \in \mathcal{X}} \frac{1}{2} \| y - x \|^2_H
\]

(2.1)

and:

\[
= \arg\min_{y \in \mathcal{X}} \frac{1}{2} \| H^{1/2} (y - x) \|^2,
\]

(2.2)

where \( \| \cdot \|_H \) is the matrix norm defined by \( H \in \mathbb{R}^{n \times n} \) such that \( H \succeq 0 \) and \( \| \cdot \| \) is the Euclidean norm. As \( \| H^{1/2} (y - x) \|^2 \) is strongly-convex with respect to \( y \), Problem 2.2 has a unique minimizer. Let \( \hat{f}_k(x) \) denote the quadratic approximation of the function \( f(x) \) around the point \( x_k \), i.e.:

\[
\hat{f}_k(x) = f(x_k) + \nabla f(x_k)(x - x_k) + \frac{1}{2} \| x - x_k \|^2_{H_k}.
\]

Where \( H_k \) is an approximation of the true Hessian of \( f \) at \( x_k \), i.e., \( \nabla^2 f(x_k) \) that satisfies the following assumption:

Assumption 1. There exists an \( \eta_k \geq 1 \) such that for \( y \in \mathcal{X} \) and \( k \geq 0 \):

\[
\frac{1}{\eta_k} \| y - x_k \|^2_{H_k} \leq \| y - x_k \|^2_{\nabla^2 f(x_k)} \leq \eta_k \| y - x_k \|^2_{H_k}.
\]

(2.3)

Moreover, as \( k \to \infty \) we have that \( \eta_k \to 1 \). This condition is similar in spirit to Condition C in (Karimireddy et al., 2018a).

Assumption 2. For all \( k \geq 0 \), there exists a \( \tilde{\mu}_k > 0 \) such that \( H_k \succeq \tilde{\mu}_k I \).

3 Second-order Conditional Gradients Algorithm

The exact SOCG algorithm (Algorithm 1) defines iterates as \( x_{k+1} = x_k + \gamma_k \left( x^*_{k+1} - x_k \right) \) where \( \gamma_k > 0 \) is a step size and:

\[
x^*_{k+1} = \Pi^H_{\mathcal{X}} \left( x_k - H_k^{-1} \nabla f(x_k) \right)
\]

(3.1)

\[
= \arg\min_{y \in \mathcal{X}} \frac{1}{2} \| y - (x_k - H_k^{-1} \nabla f(x_k)) \|^2_{H_k},
\]

where the minimization problem is solved with a Conditional Gradients algorithm.

Remark 3.1. The minimization problem shown in Equation 3.1 is equivalent to minimizing the quadratic approximation of \( f(x) \) over \( \mathcal{X} \), that is:

\[
x^*_{k+1} = \arg\min_{x \in \mathcal{X}} \hat{f}_k(x).
\]

(3.2)
We first assume that the scaled projection problem in Line 3 of Algorithm 1 is solved to optimality, i.e. 
\( \varepsilon \)
Algorithm 1
\( f \)
where computing first-order information of
And for the scaled projection in Equation 3.1
\( \varepsilon \)
3.1 Exact projections:
\( L \)
for\( \varepsilon \)
\( \Omega (\cdot, \cdot, \cdot) : S^n_{++} \times [\mathbb{R}^n]^p \times [\mathbb{X}^n]^p \rightarrow S^n_{++} \) a procedure that updates the current approximation of the Hessian using information from \( p \) past gradients and iterates.
\[
\text{Algorithm 1 Second-order Conditional Gradients (SOCG) Algorithm}
\]
\begin{enumerate}
\item \textbf{Input:} Initial point \( x_0 \in \mathcal{X} \), accuracies \( \{\varepsilon_0, \ldots, \varepsilon_{N-1}\} \) such that \( \varepsilon_i \geq 0 \) for \( k \in [0, N-1] \).
\item \textbf{for} \( k = 0 \) to \( N-1 \) \textbf{do}
\item \textbf{Find} \( \varepsilon_k \)-approximate solution to:
\[
\hat{x}_{k+1} \leftarrow \arg\min_{y \in \mathcal{X}} \hat{f}_k(y)
\] (3.6)
\item using a Conditional Gradients algorithm.
\item \textbf{Choose} \( \gamma_k \) with an exact line search between 0 and 1. Find \( \gamma_k \) such that:
\[
\gamma_k = \arg\min_{\gamma \in [0,1]} f(x_k + \gamma(\hat{x}_{k+1} - x_k)).
\] (3.7)
\item \( x_{k+1} \leftarrow x_k + \gamma_k (\hat{x}_{k+1} - x_k) \)
\item \( H_{k+1} \leftarrow \Omega \left( H_k, \{\nabla f(x_i)\}^{k+1}_{i=k+1-p}, \{x_{k+1}\}^{k+1}_{i=k+1-p} \right) \)
\item \textbf{end for}
\item \textbf{Output:} Point \( x_N \in \mathcal{X} \).
\end{enumerate}

3.1 Exact projections: \( \varepsilon_k = 0 \) for \( k \in [0, N-1] \)
We first assume that the scaled projection problem in Line 3 of Algorithm 1 is solved to optimality, i.e. \( \varepsilon_k = 0 \) for \( k \in [0, N-1] \), and thus \( x^*_k = x_k \).

3.1.1 Global Convergence
The global convergence of Projected Newton Methods (a class of algorithms to which SOCG belongs to) is well-known for \( L \)-smooth and \( \mu \)-strongly convex functions when \( \varepsilon_k = 0 \), we will simply restate the most important results. Proofs of these can be found in Appendix B and in the references cited.
Theorem 3.2. (Karimireddy et al., 2018a) Given an \( L \)-smooth and \( \mu \)-strongly convex function and a convex set \( \mathcal{X} \), the step size sequence \( \gamma_k \leq \frac{\mu}{L \eta_k} \) guarantees a primal gap contraction in Algorithm 1 such that:

\[
f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu \gamma_k^2}{L \eta_k} \right) (f(x_k) - f(x^*)).
\]

Corollary 3.3. If we pick \( \gamma_k \) using the exact bounded line search strategy shown in Equation 3.7, the SOCG algorithm (Algorithm 1) will make as least as much progress as it will make by choosing \( \gamma_k = \frac{\mu}{\eta_k L} \), therefore if we pick perform a bounded exact line search the primal gap will contract as:

\[
f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu^3}{\eta_k L^3} \right) (f(x_k) - f(x^*)).
\]

Which follows by particularizing the results from Theorem 3.2 to \( \gamma_k = \frac{\mu}{\eta_k L} \).

3.1.2 Local convergence

When the iterates are close enough to \( x^* \) a quadratic convergence rate in the distance to \( x^* \) can be obtained.

Theorem 3.4. Given an \( L \)-smooth and \( \mu \)-strongly convex function with \( L^2 \)-Lipschitz Hessian and a convex set \( \mathcal{X} \), if the subproblems are solved until \( \varepsilon_k = 0 \) and Assumption 1 is satisfied then the SOCG algorithm (Algorithm 1) has iterates that satisfy for \( k \geq 0 \):

\[
\|x_{k+1} - x^*\| \leq (1 - \gamma_k) \|x_k - x^*\| + \frac{L^2 \gamma_k}{2 \mu} \|x_k - x^*\|^2.
\]

Corollary 3.5. If the assumptions of Theorem 3.4 are met, then the iterates converge quadratically to \( x^* \).

Proof. The claim follows from the application of Lemma C.2 in Appendix C and Theorem 3.4.

3.2 Inexact projections: \( \varepsilon_k > 0 \) for \( k \in [0, N - 1] \)

Solving the Problem in Equation 3.7 to optimality is computationally prohibitive, and because of this an \( \varepsilon \)-optimal solution is often used. Usually two approaches are used, either the problems are solved up to a certain multiplicative error or they are solved up to an additive error. We focus on the latter case, where we require knowledge of the primal gap of the Problem in Equation 3.7.

3.2.1 Global convergence: Inexact SOCG

Theorem 3.6. Assume \( f \) is \( L \)-smooth and \( \mu \)-strongly convex function, \( \mathcal{X} \) is convex, the step size sequence satisfies \( \gamma_k \leq \frac{\mu}{L \eta_k} \) and the subproblems are solved to an accuracy \( \varepsilon_k = f_k(\tilde{x}_{k+1}) - f_k(x^*_{k+1}) \) such that:

\[
\varepsilon_k \leq \kappa_k^2 \left( \frac{f(x_k) - f(x^*)}{\|\nabla f(x_k)\|} \right)^2,
\]

where \( \kappa_k \) is a, to be chosen, forcing parameter. Then the following bound on the primal gap of the iterates \( x_k \) of the SOCG algorithm (Algorithm 1) holds:

\[
f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu^3}{L \eta_k} + \frac{\gamma_k \kappa_k^2}{2 \mu} \right) (f(x_k) - f(x^*))
\]

Corollary 3.7. Assume the conditions shown in Theorem 3.6 are verified. If we pick a forcing parameter \( \kappa_k \leq \frac{\sqrt{2\mu^3/2}}{L \eta_k} \), then the SOCG algorithm (Algorithm 1) with exact bounded line search will have a global linear convergence in the primal gap.
3.2.2 Local convergence: Inexact SOCG

Theorem 3.8. If the scaled projection subproblems in the SOCG algorithm (Algorithm 1) are solved to an accuracy such that \( \varepsilon_k \leq \kappa_k^2 \left( \frac{f(x_k) - f(x^*)}{\|x_k - x^*\|} \right)^2 \) and Assumptions 1 and 2 are satisfied then the distance to the optimum evolves as:

\[
\|x_{k+1} - x^*\| \leq 1 - \gamma_k + \kappa_k \gamma_k \sqrt{\frac{2}{\mu_k}} \|x_k - x^*\| + \frac{L^2 \gamma_k^2}{2 \mu} \|x_k - x^*\|^2.
\]

Corollary 3.9. If the assumptions of Theorem 3.8 are satisfied, then there are three options: If we choose a parameter \( \kappa_k \leq \sqrt{\frac{\mu_k}{2}} \) then the SOCG algorithm (Algorithm 1) will generate iterates that converge linearly to \( x^* \). If we choose \( \kappa_k \to 0 \) as \( k \to \infty \) the distance from the iterates to \( x^* \) will converge superlinearly. Lastly, if \( \lim_{k \to \infty} \|x_k - x^*\| = \xi \), where \( 0 < \xi < \infty \) is a constant, the distance of the iterates to \( x^* \) will converge quadratically.

Proof. The claim follows by applying Lemma C.2 in Appendix C and Theorem 3.8 with the aforementioned step sizes.

4 Limited-memory BFGS

Several methods \( \Omega (\cdot, \cdot, \cdot) \) exist to obtain an approximation \( H_k \) to the hessian \( \nabla^2 f(x_k) \) from first-order information. Notable among them are the BFGS (Broyden, Fletcher, Goldfarb, and Shanno) and DFP (Davidon, Fletcher, and Powell) algorithms, which update \( H_k \) with a rank-2 matrix at each iteration. However these methods explicitly require the storage of a dense matrix, so the memory requirements scale as \( O(n^2) \), the updating requires \( O(n^2) \) operations and computing the gradient of \( \hat{f} \) using a dense \( H_k \) requires \( O(n^2) \) multiplications. To avoid these requirements a Limited-memory BFGS (L-BFGS) algorithm that uses only the information from the \( p < n \) past iterates can be used. This lowers the memory requirements to \( O(np) \) and requires only \( O(np + p^2) \) operations to update at each iteration. Moreover computing the gradient of \( \hat{f} \) with an L-BFGS reconstruction of the hessian requires \( O(np + p^2) \) multiplications (Nocedal & Wright, 2006).

The L-BFGS algorithm produces a positive definite \( H_k \) if \( f \) is \( \mu \)-strongly convex and \( L \)-smooth, or if the descent directions of the SOCG algorithm (Algorithm 1) are descent directions. We use the compact representation of L-BFGS shown Chapter 7.2 in Nocedal & Wright (2006).

5 Computations

The purpose of this computational section is to demonstrate the numerical advantage of using second-order approximations to the original function \( f \) when minimizing functions with expensive first-order oracles, in the context of projection-free optimization.

5.1 Sparse Coding over the Birkhoff Polytope

Given a set of \( m \) input data points \( Y = [y_1, \cdots, y_m] \) with \( y_i \in \mathbb{R}^d \), sparse dictionary learning attempts to find a dictionary \( X \in \mathbb{R}^{d \times n} \) and a sparse representation \( Z = [z_1, \cdots, z_m] \) with \( z_i \in \mathbb{R}^n \) that minimizes:

\[
\min_{X \in \mathbb{R}^{d \times n}, Z \in \mathbb{R}^{n \times m}} \sum_{i=1}^m \|y_i - Xz_i\|^2 + \lambda \|z_i\|_1.
\]  

(5.1)

Where \( \mathcal{C} = \left\{ X \in \mathbb{R}^{d \times n} \mid \sum_{j=1}^n X^2_{j,i} \leq 1, \forall i \in [1, d] \right\} \) is the set of matrices with columns with \( \ell_2 \) norm less than one. This problem is of interest as many signal processing see performance boosts when given a learned dictionary \( X \) that is able to give a sparse representation (see Mairal et al. (2010) for an overview), as opposed to a predefined dictionary obtained from Fourier or wavelet transforms. The elements in this learned dictionary are not required to be orthogonal, and they can form an undercomplete or an overcomplete dictionary.
The problem in Equation 5.1 is convex with respect to \( X \) when \( Z \) is fixed, and vice-versa, and so it can be solved by alternating between minimizing with respect to \( Z \) with fixed \( X \), and minimizing with respect to \( X \) with fixed \( Z \) (Lee et al., 2007) (see (Mairal et al., 2010) for an online learning approach). The latter problem is typically solved with a stochastic projected gradient descent (Aharony et al., 2006). We focus on a variation of the minimization with respect to \( X \) with fixed \( Z \), more concretely, we additionally require the rows of \( X \) to have norm bounded below 1, the elements of \( X \) be non-negative, and \( d = n \). A natural way to impose this is to solve the problem over the Birkhoff polytope. Given a set of vectors \( Y = \{y_1, \cdots, y_m\} \) and \( Z = \{z_1, \cdots, z_m\} \), such that \( y_i, z_i \in \mathbb{R}^n \) for all \( i \in \{1, m\} \), we aim to solve the problem \( \min_{X \in \mathcal{X}} f(X) \) where \( \mathcal{X} \) is the Birkhoff polytope and \( f(X) \) has the form:

\[
f(X) = \sum_{i=1}^{m} \|y_i - Xz_i\|^2,
\]

The gradient of \( f(X) \) amounts to computing:

\[
\nabla f(X) = \sum_{i=1}^{m} -2(y_i - Xz_i) z_i^T.
\]

The hessian of \( f(X) \) is given by the block diagonal matrix \( \nabla^2 f(X) \in \mathbb{R}^{n^2 \times n^2} \) with \( \nabla^2 f(X) = \text{diag}[B, \cdots, B] \) where \( B \in \mathbb{R}^{n \times n} \) has the form \( B = \sum_{i=1}^{m} z_i z_i^T \). As the eigenvalues of a block-diagonal matrix are the eigenvalues of the blocks that form the diagonal, and \( B \) is positive definite, the function \( f(X) \) is \( \mu \)-strongly convex and \( L \)-smooth. The complexity of the gradient computation scales as \( O(n^3) \). Solving a linear program over the Birkhoff polytope using the Hungarian method has a complexity of \( O(n^5) \). Thus it is more expensive to compute a gradient than it is to solve an LP over \( \mathcal{X} \) if \( m \) is large. We generate synthetic data by creating a matrix \( B \in \mathbb{R}^{n \times n} \) with \( n = 80 \) and entries sampled from a standard normal distribution, and \( m = 100000 \) vectors \( x \in \mathbb{R}^n \), with entries sampled from a standard normal distribution, in order to form \( Z = \{z_1, \cdots, z_m\} \). The set of vectors \( Y = \{y_1, \cdots, y_m\} \) is generated by computing \( y_i = Bz_i \) for all \( i \in \{1, m\} \).

In order to use the subproblem accuracy described in Theorems 3.6 and 3.8 (with forcing parameter \( \kappa_k \leq \sqrt{\mu_k/2} \)), the value of \( f(x^*) \) is first computed to high accuracy, and used in the experiments. Figure 1 shows the performance of Algorithm 1 when the DICG algorithm (Garber & Meshi, 2016) is used, and the matrix exact hessian is used, i.e., \( H_k = \nabla^2 f(x_k) \) (denoted by SOCG DICG), and also when an L-BFGS and BFGS method is used to generate \( H_k \) (denoted by SOCG DICG (L-BFGS) and SOCG DICG (BFGS) respectively). These algorithms are compared to the vanilla Conditional Gradients algorithm (denoted by FW), the Away-Step and Pairwise-Step Conditional Gradients algorithms (AFW and PFW), the Lazy Away-Step Conditional Gradients algorithm (Braun et al., 2017) (AFW (L)), the aforementioned DICG algorithm, the Stochastic Variance-Reduced Conditional Gradients algorithm (Hazan & Luo, 2016) and the Conditional Gradient Sliding algorithm (Lan & Zhou, 2016). All algorithms use exact line search bounded between 0 and 1.

The results in Figure 1, show that the SOCG algorithm (Algorithm 1) significantly outperforms other projection-free algorithms in this setting both in per-iteration and per-unit time progress. This stems from the fact that the calls to the first-order oracle to \( f_k(x) \) are much cheaper than the first-order oracle calls to \( f(x) \) in this example. With respect to SVRFW, we highlight that the cost of computing a stochastic gradient is the same as computing the non-stochastic gradient of \( f_k(x) \), and we expect more progress with the latter.

### 5.2 Inverse covariance estimation over spectrahedron

In many applications the relationships between variables can be modeled with the use of undirected graphical models, such is the case for example in gene expression problems, where the goal is to find out which groups of genes are responsible for producing a certain outcome, given a gene dataset. When the underlying distribution of these variables is Gaussian, the problem of determining the relationship between variables boils down to finding patterns of zeros in the inverse covariance matrix \( \Sigma^{-1} \) of the distribution. A common approach to solving this problem relies on finding a \( \ell_1 \)-regularized maximum likelihood estimator of \( \Sigma^{-1} \), so as to encourage sparsity, over the positive definite cone (Banerjee et al., 2008; Friedman et al., 2008), this is often called the Graphical Lasso.

Several optimization algorithms have been used to tackle this problem, such as interior point methods (Yuan & Lin, 2007), block coordinate descent or accelerated first-order algorithms (Banerjee et al., 2008), coordinate descent
algorithms (Friedman et al., 2008) and even projected limited-memory quasi-Newton algorithms (Schmidt et al., 2009). We solve the same problem over the space of bounded trace positive semidefinite matrices, that is:

\[
\min_{X \succeq 0 \text{ trace}(X) = 1} f(X) = \min_{X \succeq 0 \text{ trace}(X) = 1} - \log \det X + \text{trace} (S X) + \lambda \|X\|_1.
\]

Where \( S = \sum_{i=1}^{N} (z_i - \mu)(z_i - \mu)^T \) is the empirical covariance matrix of a set of datapoints \( Z = \{ z_1, \cdots, z_N \} \) drawn from a Gaussian distribution in \( \mathbb{R}^n \), \( \lambda > 0 \) is a regularization parameter and \( \|Y\|_1 = \sum_{i,j=1}^{n} |Y_{i,j}| \). Note that the problem is convex and non-smooth. Evaluating \( f(X) \) has complexity \( \mathcal{O}(n^3) \) if we compute the determinant with a LU decomposition, and evaluating the gradient is equal to \( \nabla f(X) = -X^{-1} + S + \lambda X \otimes |X| \), where \( \otimes \) represents Hadamard division, has complexity \( \mathcal{O}(n^2) \), dominated by the matrix inversion. Solving the linear program \( \min_{Y \in \mathbb{X}} \nabla f(X) \otimes Y \), where \( \otimes \) denotes Hadamard product, amounts to computing the largest eigenvector of \( -\nabla f(X) \), with a complexity of \( \mathcal{O}(n^2) \). The matrix \( S \) is generated by computing a random orthonormal basis in \( \mathbb{R}^m \) \( B = \{ v_1, \cdots, v_m \} \) and computing \( S = \sum_{i=1}^{m} \sigma_i v_i v_i^T \), where \( \sigma_i \) is uniformly distributed between 0.5 and 1 for \( i \in [1, m] \).

When the primal gap of the problem is not known, one can substitute the subproblem accuracy rule in Theorem 3.6 and 3.8 by one that will still ensure that the local superlinear convergence is achieved. If we pick \( \epsilon_k = \rho^k \max_{x \in \mathbb{X}} \langle \nabla f(x_0), x_0 - x \rangle \) with \( \rho \in (0, 1) \) (and we check afterwards that the directions obtained are descent directions, otherwise increase the accuracy of the subproblem solution) we can ensure this convergence by Lemma C.6 as \( k \to \infty \). The performance of this strategy can be seen in Figure 2 for \( \rho_1 = 0.99, \rho_2 = 0.9 \) and \( \rho_3 = 0.8 \), when the Lazy Away-Step Conditional Gradients algorithm is used in the SOCG algorithm (Algorithm 1) with the L-BFGS strategy, denoted by SOCG AFW (L) (L-BFGS). All algorithms use a golden-section bounded line search between 0 and 1 in this comparison.

5.3 Structured Logistic Regression over \( \ell_1 \) unit ball

Given a binary classification task with \( m \) labels \( Y = \{ y_1, \cdots, y_m \} \) and \( m \) samples \( Z = \{ z_1, \cdots, z_m \} \) with \( y_i \in \{-1, 1\} \) and \( z_i \in \mathbb{R}^n \) for all \( i \in [1, m] \), we wish to solve:

\[
\min_{x \in \mathbb{X}} f(x) = \min_{x \in \mathbb{X}} \frac{1}{m} \sum_{i=1}^{m} \log \left( 1 + e^{-y_i \langle x, z_i \rangle} \right),
\]

where \( \mathbb{X} \) is the \( \ell_1 \) unit ball centered at the origin. Although projecting into any \( \ell_1 \) ball has complexity \( \mathcal{O}(n) \), and so projections are cheap, this feasible region is often used to compare the performance of projection-free algorithms between each other (see Lacoste-Julien & Jaggi (2015); Rao et al. (2015); Braun et al. (2019)). Solving a linear program program over the \( \ell_1 \) ball also has complexity \( \mathcal{O}(n) \). This experiment was also considered in Ghanbari & Scheinberg (2018) and Scheinberg & Tang (2016) to compare the performance of several Proximal Quasi-Newton methods in the context of minimization with a projection oracle.

The labels and samples used are taken from the training set of the gisette (Guyon et al., 2007) dataset, where \( n = 5000 \) and \( m = 6000 \). Figure 3 shows the performance of Algorithm 1 with the Lazy Away-Step Conditional Gradient algorithm (Braun et al., 2019) and the L-BFGS strategy, denoted by SOCG AFW (L) (L-BFGS). All algorithms use a golden-section bounded line search between 0 and 1 in this comparison, and we use \( \rho_1 = 0.9 \) and \( \rho_2 = 0.8 \) for the SOCG algorithm.

The results in Figure 2 and 3 show that this simple modification to the subproblem accuracy strategy results in good performance both in per-iteration and per-unit-time progress.

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Figure 1: **Sparse coding over Birkhoff polytope** Primal gap, $\ell_2$ distance to $x^*$ and Frank-Wolfe gap in terms of iteration (top-row) and wall-clock time (bottom-row).

Figure 2: **Graphical Lasso over spectrahedron**: Primal gap, $\ell_2$ distance to $x^*$ and Frank-Wolfe gap in terms of iteration (top-row) and wall-clock time (bottom-row)
Figure 3: Logistic regression over $\ell_1$ ball: Primal gap, $\ell_2$ distance to $x^*$ and Frank-Wolfe gap in terms of iteration (top-row) and wall-clock time (bottom-row).
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A Auxiliary Results

Lemma A.1. Given a positive definite matrix $H$, an $x \in \mathcal{X}$ satisfies:
$$x = \Pi_{\mathcal{X}}^H (x - H^{-1} \nabla f(x)),$$  \hfill (A.1)
if and only if $x = x^*$.

Proof. ($\Rightarrow$) Using the first order optimality conditions for the scaled projection, $\forall z \in \mathcal{X}$:
$$\langle H (x - x) + \nabla f(x), z - x \rangle = \langle \nabla f(x), z - x \rangle \geq 0.$$  \hfill (A.2)
Which hold true if and only if $x = x^*$, as Equation A.1 represents the first-order optimality conditions for Problem 1.1, of which $x^*$ is the unique solution.

($\Leftarrow$) Assume that $x = x^*$. As $H$ is positive definite the objective function in the optimization problem in Equation A.1 is strongly convex, therefore the solution to the scaled projection problem $x = \Pi_{\mathcal{X}}^H (x^* - H^{-1} \nabla f(x^*))$ is unique, and must verify the first order optimality conditions:
$$\langle H (x - x^*) + \nabla f(x^*), z - x^* \rangle \geq 0 \quad \forall z \in \mathcal{X}.$$  \hfill (A.3)


Lemma A.2. Assume that $H_k$ is positive definite and $x_k \neq x^*$. Then the directions given by $x_{k+1}^* - x_k$, where $x_{k+1}^* = \text{argmin}_{y \in \mathcal{X}} \langle \nabla f(x_k), y - x_k \rangle + \frac{1}{2} \| y - x_k \|_{H_k}^2$ are descent directions at point $x_k$, i.e., they satisfy $\langle -\nabla f(x_k), x_{k+1}^* - x_k \rangle > 0$.

Proof. Using the first-order optimality conditions for $x_{k+1}^*$ in Equation 3.3, particularizing for $z = x_k$ and reordering:
$$\langle -\nabla f(x_k), x_{k+1}^* - x_k \rangle \geq \| x_{k+1}^* - x_k \|_{H_k}^2 > 0.$$  \hfill (B.2)


B Global Convergence

B.1 Exact projections: $\varepsilon_k = 0$ for $k \in [0, N - 1]$

We state some auxiliary results that will be used in the convergence proof of Algorithm 1 when $\varepsilon_k = 0$ for $k \in [0, N - 1]$.

Lemma B.1. (Karimireddy et al., 2018a;b) Given a convex domain $\mathcal{X}$ and a convex function $Q_k^\alpha = (\nabla f(x_k), x - x_k) + \frac{\alpha}{2} \| x - x_k \|_{H_k}^2$, then for constants $\alpha \beta \geq 1$:
$$\min_{x \in \mathcal{X}} Q_k^\alpha \leq \frac{1}{\alpha \beta} \min_{x \in \mathcal{X}} Q_k^{1/\beta}.$$  \hfill (B.1)

Theorem B.2. (Karimireddy et al., 2018a) Given an $L$-smooth and $\mu$-strongly convex function and a convex set $\mathcal{X}$, if Assumption 1 is satisfied, then the step size sequence $\gamma_k \leq \frac{\mu}{L}\eta_k$ guarantees a primal gap contraction at every iteration such that:
$$f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu\gamma_k^2}{L\eta_k} \right) (f(x_k) - f(x^*)).$$  \hfill (B.2)

Proof. Consider the iterates $x_{k+1} = (1 - \gamma_k)x_k + \gamma_k x_{k+1}^*$, the iterate $x_{k+1}$ can be written as:
$$x_{k+1} = \text{argmin}_{x \in (1-\gamma_k)x_k + \gamma_k x^*} \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\gamma_k} \| x - x_k \|_{H_k}^2$$  \hfill (B.2)
Using the $L$-smoothness and the $\mu$-strong convexity of the function $f$ we can write:

$$f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2\mu} \|x_{k+1} - x_k\|^2_{\nabla^2 f(x_k)}$$  \hfill (B.3)

$$\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L_\eta}{2\mu} \|x_{k+1} - x_k\|^2_H$$  \hfill (B.4)

$$\leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{\gamma_k}{2} \|x_{k+1} - x_k\|^2_{\nabla^2 f(x_k)}$$  \hfill (B.5)

$$= \min_{x \in (1-\gamma_k)x_k + \gamma_k X} \left( \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\gamma_k} \|x - x_k\|^2_H \right)$$  \hfill (B.6)

$$\leq \frac{\mu \gamma_k}{L_\eta \kappa} \min_{x \in (1-\gamma_k)x_k + \gamma_k X} \left( \langle \nabla f(x_k), x - x_k \rangle + \frac{\mu}{2L_\eta} \|x - x_k\|^2_H \right)$$  \hfill (B.7)

$$\leq \frac{\mu \gamma_k}{L_\eta \kappa} \min_{x \in (1-\gamma_k)x_k + \gamma_k X} \left( \langle \nabla f(x_k), x - x_k \rangle + \frac{\mu}{2L} \|x - x_k\|^2_{\nabla^2 f(x_k)} \right)$$  \hfill (B.8)

$$\leq \frac{\mu \gamma_k^2}{L_\eta \kappa} \left( \langle \nabla f(x_k), x^* - x_k \rangle + \frac{\gamma_k}{2L} \\|x^* - x_k\|^2_{\nabla^2 f(x_k)} \right)$$  \hfill (B.9)

$$\leq \frac{\mu \gamma_k^2}{L_\eta \kappa} \left( \langle \nabla f(x_k), x^* - x_k \rangle + \frac{\mu}{2L} \\|x^* - x_k\|^2_{\nabla^2 f(x_k)} \right)$$  \hfill (B.10)

$$\leq \frac{\mu \gamma_k^2}{L_\eta \kappa} \left( f(x^*) - f(x_k) \right).$$  \hfill (B.11)

Where the second and fifth inequality come from Assumption 1, the third inequality follows from the fact that $\gamma_k \leq \frac{\mu}{L_\eta \kappa}$, the fourth inequality from Lemma B.1, the sixth from the fact that we particularize for $x = x_k + \gamma_k (x^* - x_k)$, the seventh from the fact that $\gamma_k \leq 1$, and the last inequality from the $\mu$-strong convexity and $L$-smoothness of the function $f$. Reordering the previous expression leads to:

$$f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu \gamma_k^2}{L_\eta \kappa} \right) \left( f(x_k) - f(x^*) \right).$$

\hfill \Box

### B.2 Inexact projections: $\varepsilon_k > 0$ for $k \in [0, N-1]$

**Theorem B.3.** Given an $L$-smooth and $\mu$-strongly convex function and a convex set $X$, if Assumption 1 is satisfied, the step size sequence satisfies $\gamma_k \leq \frac{\mu}{L_\eta \kappa}$ and the subproblems are solved to an additive accuracy $\varepsilon_k = f(\tilde{x}_{k+1}) - f(x_{k+1})$ such that:

$$\varepsilon_k \leq \kappa_k^2 \left( \frac{f(x_k) - f(x^*)}{\|\nabla f(x_k)\|} \right)^2,$$

where $\kappa_k$ is a forcing term, then the following bound on the primal gap of the iterates $x_k$ of Algorithm 1 holds:

$$f(x_{k+1}) - f(x^*) \leq \left( 1 - \frac{\mu \gamma_k^2}{L_\eta \kappa} \right) \left( f(x_k) - f(x^*) \right)$$

**Proof.** Using the $L$-smoothness and $\mu$-strong convexity of $f$ and noting that $x_{k+1} = x_k + \gamma_k (\tilde{x}_{k+1} - x_k)$ we have
that:

\[ f(x_{k+1}) - f(x_k) \leq \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2\mu} \| x_{k+1} - x_k \|^2 \]  
\[ \leq \gamma_k \left( \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \gamma_k \frac{L}{2\mu} \| x_{k+1} - x_k \|^2 \right) \]  
\[ \leq \gamma_k \left( \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \gamma_k \frac{L\eta_k}{2\mu} \| x_{k+1} - x_k \|^2 \right) \]  
\[ \leq \gamma_k \left( \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{1}{2} \| x_{k+1} - x_k \|^2 \right) \]  
\[ = \gamma_k f_k(x_{k+1}) \]  
\[ = \gamma_k \left( \varepsilon_k + f_k(x_k^\ast) \right) \]  
\[ = \gamma_k \varepsilon_k + \langle \nabla f(x_k), \gamma_k (x_{k+1}^\ast - x_k) \rangle + \frac{1}{2\gamma_k} \| \gamma_k (x_{k+1}^\ast - x_k) \|^2_{H_k}. \]

Where the second inequality comes from the definition of \( x_{k+1} \), the third from Assumption 1, the fourth from the fact that \( \gamma_k \leq \frac{L}{2\mu} \), and the fifth from Lemma B.1. If we denote by \( x_{k+1}^\ast \) the iterate generated if the problem had been solved to optimality, where \( x_{k+1}^\ast = \arg\min_{x \in \Omega} \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\gamma_k} \| x - x_k \|^2_{L^2f(x_k)} = x_k + \gamma_k (x_{k+1}^\ast - x_k) \) (see Equation B.2), we can see that \( \gamma_k (x_{k+1}^\ast - x_k) = x_{k+1}^\ast - x_k \). Plugging this into Equation B.18 we have:

\[ f(x_{k+1}) - f(x_k) \leq \gamma_k \varepsilon_k + \langle \nabla f(x_k), x_{k+1}^\ast - x_k \rangle + \frac{1}{2\gamma_k} \| x_{k+1}^\ast - x_k \|^2_{H_k} \]  
\[ = \gamma_k \varepsilon_k + \min_{x \in (1-\gamma_k)x_k + \gamma_k x^\ast} \left( \langle \nabla f(x_k), x - x_k \rangle + \frac{1}{2\gamma_k} \| x - x_k \|^2_{H_k} \right). \]

Where the last equality follows from the definition of \( x_{k+1}^\ast \). We can bound the second term in the previous expression using the arguments in the proof from Theorem 3.2 (note that this is the same term as the one shown in Equation B.6), which leads to:

\[ f(x_{k+1}) - f(x_k) \leq \gamma_k \varepsilon_k + \frac{\mu \gamma_k^2}{L\eta_k} (f(x^\ast) - f(x_k)) \]  
\[ \leq \frac{\gamma_k \kappa_k^2}{2\mu} \left( \frac{f(x_k) - f(x^\ast)}{\| \nabla f(x_k) \|^2} \right)^2 + \frac{\mu \gamma_k^2}{L\eta_k} \| \nabla f(x_k) \| \]  
\[ \leq \left( \frac{\gamma_k \kappa_k^2}{2\mu} - \frac{\mu \gamma_k^2}{L\eta_k} \right) \| f(x_k) - f(x^\ast) \| \]  
\[ \leq \left( \frac{\gamma_k \kappa_k^2}{2\mu} - \frac{\mu \gamma_k^2}{L\eta_k} \right) \| f(x_k) - f(x^\ast) \|. \]

Where the second inequality follows from using the subproblem additive accuracy described, and the last inequality from \( f(x_k) - f(x^\ast) \leq \frac{\| \nabla f(x_k) \|^2}{2\mu} \) by the strong-convexity of \( \mu \). Reordering the previous expression leads to:

\[ f(x_{k+1}) - f(x^\ast) \leq \left( 1 - \frac{\mu \gamma_k^2}{L\eta_k} + \frac{\gamma_k \kappa_k^2}{2\mu} \right) (f(x_k) - f(x^\ast)) \]

Since the bounded line search between 0 and 1 will make at least as much progress as any step size \( \gamma_k \leq \frac{\mu}{L\eta_k} \), the bound also holds for the bounded exact line search rule.

### C Local Convergence

#### C.1 Exact projections: \( \varepsilon_k = 0 \) for \( k \in [0, N - 1] \)

When the iterates are close enough to \( x^\ast \) a quadratic convergence rate in the distance to \( x^\ast \) can be obtained. We first review a series of key lemmas.
Lemma C.1. (Nesterov, 2018) [Lemma 4.1.1] If a twice differentiable function $f$ has $L_2$-Lipschitz continuous Hessian over $X$ then for all $x, y \in X$:
\[
\|\nabla f(y) - \nabla f(x) - \nabla^2 f(x) (y - x)\| \leq \frac{L_2}{2} \|y - x\|^2.
\]
(C.1)

Lemma C.2. (Ben-Tal & Nemirovskii, 2020) Given an $L$-smooth and $\mu$-strongly convex function with $L_2$-Lipschitz Hessian, if the subproblems are solved until $\varepsilon_k = 0$ and Assumption 1 is satisfied then the stepsizes chosen by the SOCG algorithm (Algorithm 1) when using the exact line search will satisfy $\lim_{k \to \infty} (1 - \gamma_k) / \|x_k - x^*\| < \nu$, where $\nu$ is a positive constant.

Lemma C.3. (Beck, 2017) [Theorem 6.42] Given a positive definite $H$ and a convex set $X$, the scaled projection is a contraction mapping (it is firmly-nonexpansive) in the $H$-norm:
\[
(x - y)^T H(\Pi^H_X (x) - \Pi^H_X (y)) \geq \|\Pi^H_X (x) - \Pi^H_X (y)\|^2_H
\]
(C.2)

Using the Cauchy-Schwarz inequality this leads to $\|x - y\|_H \geq \|\Pi^H_X (x) - \Pi^H_X (y)\|_H$.

With these results at hand we can move to prove the convergence in distance to the optimum of the SOCG algorithm (Algorithm 1).

Theorem C.4. Given an $L$-smooth and $\mu$-strongly convex function with $L_2$-Lipschitz Hessian and a convex set $X$, if the subproblems are solved until $\varepsilon_k = 0$ and Assumption 1 is satisfied then the SOCG algorithm (Algorithm 1) has iterates that satisfy for $k \geq 0$:
\[
\|x_{k+1} - x^*\| \leq (1 - \gamma_k) \|x_k - x^*\| + \frac{L_2 \gamma_k}{2\mu} \|x_k - x^*\|^2.
\]
(C.3)

Proof. The steps taken by the exact SOCG algorithm satisfy:
\[
\|x_{k+1} - x^*\| = \|x_k + \gamma_k (x_{k+1} - x_k) - x^*\| \\
\leq (1 - \gamma_k) \|x_k - x^*\| + \gamma_k \|x_{k+1} - x^*\| \\
\leq (1 - \gamma_k) \|x_k - x^*\| + \frac{\gamma_k}{\sqrt{\mu}} \|x_{k+1} - x^*\| \|
\]
(C.4)
(C.5)
(C.6)

Where the first equality follows from the definition of $x_{k+1}$, the first inequality from the triangle inequality and the second inequality from the $\mu$-strong convexity of $f(x)$. We can bound the second term in Equation C.6 as follows:
\[
\|x_{k+1}^* - x^*\|^2_{\nabla^2 f(x_k)} = \||x_{k+1} - x^*\|_{\nabla^2 f(x_k)}^2 \\
\leq \|x_k - x^*\| - \nabla^2 f(x_k)^{-1} (\nabla f(x_k) - \nabla f(x^*)) \|
\]
(C.7)
(C.8)
(C.9)
(C.10)
(C.11)

Where the first equality comes from the definition of $x_{k+1}$ in Equation 3.1 and Lemma A.1 and the first inequality from Lemma C.3. The second inequality comes from the fact that $f$ is $\mu$-strongly convex and therefore we have that for any $d \in \mathbb{R}^n$ the inequality $d^T [\nabla^2 f]^{-1} d \leq \frac{1}{\mu} \|d\|^2$ holds true (or conversely $d^T [\nabla^2 f] d \geq \mu \|d\|^2$). The last inequality is due to the $L_2$-Lipschitz Hessian (Lemma C.1). Plugging this bound into Equation C.6:
\[
\|x_{k+1} - x^*\| \leq (1 - \gamma_k) \|x_k - x^*\| + \frac{\gamma_k}{\sqrt{\mu}} \|x_{k+1} - x^*\| \|
\]
(C.12)
\[
\leq (1 - \gamma_k) \|x_k - x^*\| + \frac{L_2 \gamma_k}{2\mu} \|x_k - x^*\|^2.
\]
(C.13)

Which completes the proof.  


C.2 Inexact projections: $\varepsilon_k > 0$ for $k \in [0, N - 1]$

We first state some auxiliary results that will be useful in the proof.

Lemma C.5. If the scaled projection subproblems in Equation 3.1 and Equation 3.2 are solved to an accuracy $\varepsilon_k = f_k(\hat{x}_{k+1}) - f_k(x_{k+1}^*) > 0$ and Assumption 2 is satisfied then:

$$\| \hat{x}_{k+1} - x_{k+1}^* \|^2 \leq \frac{2}{\hat{\mu}_k} \varepsilon_k$$

Proof. By the $\hat{\mu}_k$-strong convexity of $f$:

$$\hat{f}_k(\hat{x}_{k+1}) - \hat{f}_k(x_{k+1}^*) \geq \langle \nabla \hat{f}_k(x_{k+1}^*), \hat{x}_{k+1} - x_{k+1}^* \rangle + \frac{\hat{\mu}_k}{2} \| \hat{x}_{k+1} - x_{k+1}^* \|^2$$

The second inequality in the previous equations follows from the first order optimality conditions for the scaled projection problem, of which $x_{k+1}^*$ is the exact solution. Rearranging the previous expression gives the desired result.

Lemma C.6. Given a $\mu$-strongly convex and $L$-smooth function $f$ with $L_2$-Lipschitz continuous Hessian and a convex set $\mathcal{X}$, if Assumptions 1 and 2 are satisfied the SOCG algorithm (Algorithm 1) generates iterates for $k \geq 0$ such that:

$$\| x_{k+1} - x^* \| \leq (1 - \gamma_k) \| x_k - x^* \| + \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \varepsilon_k} + \frac{L_2 \gamma_k}{2\mu} \| x_k - x^* \|^2$$

(C.14)

Proof. The steps taken by the inexact SOCG algorithm satisfy:

$$\| x_{k+1} - x^* \| = \| x_k + \gamma_k (\hat{x}_{k+1} - x_k) - x^* \|$$

(C.15)

$$\leq (1 - \gamma_k) \| x_k - x^* \| + \gamma_k \| \hat{x}_{k+1} - x_{k+1}^* \| + \gamma_k \| x_{k+1} - x^* \|$$

(C.16)

The steps taken by the inexact SOCG algorithm satisfy:

$$\| x_{k+1} - x^* \| \leq (1 - \gamma_k) \| x_k - x^* \| + \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \varepsilon_k} + \frac{L_2 \gamma_k}{2\mu} \| x_k - x^* \|^2$$

(C.17)

Where the inequality comes from applying the triangle inequality. The second term in Equation C.17 can be bound using Lemma C.5 and the third term using a similar reasoning as the one used in Lemma 3.4 (see Equation C.11). This leads to:

$$\| x_{k+1} - x^* \| \leq (1 - \gamma_k) \| x_k - x^* \| + \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \varepsilon_k} + \frac{L_2 \gamma_k}{2\mu} \| x_k - x^* \|^2$$

(C.18)

And therefore the proof is complete.

Theorem C.7. If the scaled projection subproblems in the SOCG algorithm (Algorithm 1) are solved to an accuracy such that $\varepsilon_k \leq \kappa^2 \left( \frac{f(x_k) - f(x^*)}{\| \nabla f(x_k) \|} \right)^2$ and Assumptions 1 and 2 are satisfied then the distance to the optimum evolves as:

$$\| x_{k+1} - x^* \| \leq \left( 1 - \gamma_k + \kappa \gamma_k \sqrt{\frac{2}{\hat{\mu}_k}} \right) \| x_k - x^* \| + \frac{L_2 \gamma_k}{2\mu} \| x_k - x^* \|^2$$

Proof. Using the stopping criterion for the scaled projection subproblems the second term in Equation C.18 can be bounded as follows:

$$\gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \varepsilon_k} \leq \kappa \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \frac{f(x_k) - f(x^*)}{\| \nabla f(x_k) \|}}$$

$$\leq \kappa \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \langle \nabla f(x_k), x_k - x^* \rangle}$$

$$\leq \kappa \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \| \nabla f(x_k) \| \| x_k - x^* \|}$$

$$\leq \kappa \gamma_k \sqrt{\frac{2}{\hat{\mu}_k} \| x_k - x^* \|}.$$
Where the second inequality follows from the convexity of $f$, and the third from the Cauchy-Schwartz inequality. Plugging this into the results from Lemma C.6 leads to:

$$\|x_{k+1} - x^*\| \leq (1 - \gamma_k) \|x_k - x^*\| + \gamma_k \sqrt{\frac{2}{\mu_k} \varepsilon_k} + \frac{L^2 \gamma_k}{2\mu} \|x_k - x^*\|^2 \quad (C.19)$$

$$\leq (1 - \gamma_k) \|x_k - x^*\| + \kappa_k \gamma_k \sqrt{\frac{2}{\mu_k} \|x_k - x^*\|} + \frac{L^2 \gamma_k}{2\mu} \|x_k - x^*\|^2 \quad (C.20)$$

$$= \left(1 - \gamma_k + \kappa_k \gamma_k \sqrt{\frac{2}{\mu_k}}\right) \|x_k - x^*\| + \frac{L^2 \gamma_k}{2\mu} \|x_k - x^*\|^2 . \quad (C.21)$$

Which completes the proof. □