Abstract. This paper is meant to be an informal introduction to Quantum Groups, starting from its origins and motivations until the recent developments. We call in particular the attention on the newly discovered relationship among quantum groups, integrable models and Jordan structures.

1. INTRODUCTION

Quantum groups represent at present one of the most important topics of mathematics and of mathematical physics.

The principal aim of the paper is to make the non-expert reader acquainted with this fast developing field and motivate the investigation towards the relationship with another important, though much earlier, branch of both mathematics and mathematical physics: that of Jordan structures. In the spirit of the word quantum, to denote a construction which includes the classical case in the appropriate limit, we would like to point out that a quantum analog of the profound link existing at the classical level between Lie and Jordan structures is worth exploring.

We take this opportunity to firstly give a short introduction to quantum groups. This is done in order to make our contribution understandable, and also to introduce
in the Romanian mathematical literature this important and exciting topic in which a lot of mathematicians and mathematical physicists around the world are now working.

The reader can complete the information given below by using, for instance, the surveys by Biedenharn [1,2], Dobrev [3], Drinfeld [4], Faddeev [5], Kundu [6], Majid [7], Ruiz-Altaba [8], Smirnov [9], Takhtajan [10] - used also by us here - and, for an exhaustive information, the papers referred therein.

2. HISTORICAL ROOTS AND THE BASIC APPROACH

In this section we would like to outline the main ideas which motivated many researches, from different fields, to work on topics related to quantum groups and try to explain why quantum groups relate many branches of theoretical physics and mathematics.

The origin of quantum groups lies in the search for the mathematical background of the quantum inverse scattering method (QISM), a technique for obtaining exact solutions for integrable quantum field theories in 1+1 dimensions and classical models of statistical mechanics in dimension 2. Such solutions belong to a class of models commonly referred to as *exactly solved models*, which have a long history in theoretical and mathematical physics, starting from the work of Bethe, whose technique for diagonalizing the Hamiltonian in one dimensional quantum spin systems is known as the Bethe Ansatz Method, through the work of Onsager on two dimensional lattice models of classical statistical mechanics (method of commuting transfer matrices), to the more recent work of C.N. - C.P. Yang, Baxter, and A.B.- Al.B. Zamolodchikov in field theory, and many others.

Since all these methods share the feature of working in low dimension, let us spend few words to recall why low-dimensional Physics has had a great impulse in the last few years. Certainly an important role in this respect is played by string theory: the 2 dimensional world sheet, namely the analogue for the string of the the world-line for a relativistic point particle, is strictly related to the 4-dim world (usually called the target space). It happens that the ultraviolet limit of certain exactly solved models
are Conformal Field Theories which in turn describe string theories. Another strong physical motivation comes from Quantum Chromo-Dynamics (QCD): it has been argued that the scattering process at high energies occurs essentially in dimension 2. Moreover, to remain in the framework of quantum field theories, some of the characteristic features of Yang-Mills theories in dimension 4, like asymptotic freedom and dimensional transmutation, are also shared by the non-linear sigma model in dimension 2.

These are not the only reasons for working in low dimension. There is a branch of Physics in which low dimensional phenomena are observed and studied: Condensed Matter Physics. Superconductivity at high-temperature and quantum Hall effect are the most significant phenomena in this respect, but we may also cite technological applications like silicon chips and plastic films.

Beyond all this, low dimensional Physics has provided us with solvable models, among them the first complete examples of field theory with non-trivial interaction. This is by itself a good reason for such an investigation: models are so inherent to Physics!

With this variety of interests in low dimensional Physics, there is no wonder why a theory like QISM, which proved so powerful in yielding exact solutions for a broad class of 2-dimensional models, has attracted so much attention.

To understand how quantum groups are related to QISM and what is the origin itself of the term ”quantum group”, due to Drinfeld [4], one should start from the fact that the QISM recovers the classical inverse scattering method (ISM) in the limit $\hbar \to 0$, $\hbar$ being the quantum parameter of the theory, while in the same limit the quantum group underlying QISM goes to the Poisson-Lie group underlying ISM. Let us briefly explain what is meant by this.

The ISM is about 25 years old and has been successful in solving certain types of classical field theories and Hamiltonian mechanics, like the ones formulated in terms of the sine-Gordon equation, the non-linear Schrödinger equation, the Korteweg de
Vries equation, famous for its soliton solution and now fundamental in the matrix models of non-perturbative string theory (KdV hierarchy), the Toda systems, etc. All these models are based on highly non-linear equations and the key of the ISM is to introduce an auxiliary space and new operators (Lax pairs) in terms of which one can write a linear equation equivalent to the original one. By doing so one finds the solution to a typical inverse problem analog to that of reconstructing a potential from the scattering data. This is achieved using the technique of the commuting transfer matrices, dating back to Onsager (since 1944). We do not enter into details, since this is beyond the scope of our paper, but what we have just mentioned is a crucial point in the history of quantum groups. The method of commuting transfer matrices leads to solvability by building a number of conserved quantities in involution equal to the number of degrees of freedom, thus ensuring integrability by Liouville’s theorem. Here ”in involution” means that all these integrals of motion, taken pairwise, have vanishing Poisson bracket; in the quantum case this translates to being mutually commuting. The main equation in the Hamiltonian formulation of ISM is (in the discrete case):

\[ \{L_n(\lambda), L_m(\mu)\} = [r(\lambda - \mu), L_n(\lambda) \otimes L_n(\mu)] \delta_{nm} \]

where \( L \) is a Lax operator, that is a matrix -say \( N \times N \)- depending on the classical fields, the curly bracket is the Poisson bracket, \( \{L_n(\lambda), L_m(\mu)\} \) is a \( N^2 \times N^2 \)-matrix with elements \( \{L_n(\lambda)_{ij}, L_m(\mu)_{kl}\} \), and \( r \) (commonly called classical r-matrix) satisfies the classical Yang-Baxter equation (YBE), leading quite directly to integrability:

\[ [r_{12}(\lambda - \mu), r_{13}(\lambda - \nu)] + [r_{12}(\lambda - \mu), r_{23}(\mu - \nu)] + [r_{13}(\lambda - \nu), r_{23}(\mu - \nu)] = 0 \quad (2.1) \]

where, if \( r \) acts as a matrix on \( V \otimes V \), \( r_{\alpha\beta} \) acts on \( V \otimes V \otimes V \) as \( r \) on the \( \alpha \)-th and \( \beta \)-th components and as the identity on the remaining one.

If \( r(\lambda) \in \text{End}(V \otimes V) \) and \( V \) is an \( N \)-dimensional complex vector space, then we can regard the classical YBE as being formulated in terms of the Lie algebra structure of \( \text{End}(V) \). This suggests to recast ISM in a more general algebraic structure. Let \( \mathcal{G} \)
be a Lie algebra and $r(\lambda)$ be a $\mathcal{G} \otimes \mathcal{G}$-valued function, namely

$$r(\lambda) = r^{ij}(\lambda)X_i \otimes X_j$$

where $r^{ij}(\lambda)$ is a complex valued function. Then

$$[r_{12}(\lambda), r_{23}(\mu)] = r^{ij}(\lambda) \ r^{k\ell}(\mu)X_i \otimes [X_j, X_k] \otimes X_\ell$$

so that each term of the classical YBE belongs to $\mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$. The solution of the classical YBE, written in this way, is a universal solution, in the sense that for each triplet of representations $\{\rho_\alpha, V_\alpha\}$ ($\alpha = 1, 2, 3$) the representation $(\rho_\alpha \otimes \rho_\beta)(r_{\alpha\beta}(\lambda))$ yields a matrix solution of the classical YBE. For complex simple Lie algebras, the solutions of the classical YBE have been studied by Belavin and Drinfeld. In particular Drinfeld has shown that if $G$ is a simple and simply connected Lie group, then $G$ is a Poisson-Lie group, namely a Lie group with a compatible Poisson structure built on the functions defined on the group manifold, if and only if its Lie algebra is a Lie bialgebra (for the concept of bialgebra see below). We may say that the Lie bialgebras are the mathematical background for the theory of classical integrable models (which is essentially soliton theory in such a low dimension).

Remark 2.1. We like to point out that very recently Boldin, Safin and Sharipov [11] proved a surprising connection between Tzitzeica surfaces and ISM. The transformation that generates the family of such surfaces found by Tzitzeica [12] in 1910 and its slight generalizations obtained by Jonas [13,14] in 1921 and 1953 are known in the modern literature on integrable equations as Darboux or Bäcklund transformations. They are used to construct the soliton solutions starting from some trivial solution of the equation $u_{xy} = e^u - e^{-2u}$. It is worth mentioning that the paper [12] by Tzitzeica seems to be the first in the world where the equation $u_{xy} = e^u - e^{-2u}$ (the nearest relative of the sine-Gordon equation $u_{xy} = \sin u$) was considered.

Drinfeld’s motivation for introducing quantum groups was to quantize the structure underlying classical integrable models and classical YBE in the sense of finding a quantum analog which recovers the classical theory in the limit for a quantum
parameter going to zero. This program led him to quasi-triangular Hopf algebras in which a universal $R$-matrix is defined satisfying the quantum Yang-Baxter equation (QYBE) (see below for the definition) in each representation of the algebra. The goal of quantizing the whole structure rather than particular solutions of the classical YBE has thus been achieved.

The quantum $R$-matrix which in the limit goes to the classical $r$-matrix $r(\lambda) \in \mathcal{G} \otimes \mathcal{G}$ (solution of the classical YBE) lives in $U_q(\mathcal{G}) \otimes U_q(\mathcal{G})$, $U_q(\mathcal{G})$ being a deformation of the universal enveloping algebra of $\mathcal{G}$. After all $U_q(\mathcal{G})$ is the natural candidate for the purpose, but we would like to emphasize that one should not look only at the algebra aspect of the theory: the coalgebra structure plays a crucial role from both mathematical and physical standpoints [2] and both algebra and coalgebra are deformed in general.

Let $V$ be a complex vector space and let $R(\lambda), \lambda \in \mathbb{C}$, be an $\text{End}_\mathbb{C}(V \otimes V)$-valued function, then the equation

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu) \quad (2.2)$$

is called Quantum Yang-Baxter Equation.

**Comment.** It is possible to consider more general dependence of the $R$-matrix on the spectral parameters and solutions have been found for different cases.

We now want to show that this equation is the key for the integrability of certain 1+1 dimensional quantum systems as well as two dimensional classical statistical systems (the equivalence between the two being one of the most interesting connections between different branches of low dimensional physics).

**Remark 2.2.** In the theory of quantum groups, or quasitriangular Hopf algebras, a central role is played by the universal $R$-matrix satisfying (in all representations of the Hopf algebra) an equation like (2.2) with no dependence on the parameters $\lambda, \mu$, usually called spectral parameters. Also the equation with no parameter dependence is referred to in the literature as QYBE and to distinguish the two it is common to add "with spectral parameter" when referring to (2.2). In the QISM the spectral pa-
rameter is needed for building up the generating functional for commuting conserved quantities.

For sake of simplicity, let us focus our attention on a vertex model of statistical mechanics. We consider a two dimensional square lattice with $N \times M$ points connected by bonds which can take $n$ possible values. The partition function $Z$ gives the probability attached to each configuration of the whole structure. In order to build it one starts by setting up the Boltzmann weights at each vertex. If the vertex has bonds $ijkl$ then the weight is denoted by $R_{ijkl}^{il}$ and can be arranged in a matrix form. In physical applications the $R$-matrix should satisfy certain conditions, like unitarity, symmetry, positivity (the same for 1+1 quantum field theories) and very often periodic boundary conditions are assumed and the limit $N, M \to \infty$ is eventually taken.

Going back to the construction of the partition function, once the local properties of the model are set up, one looks for the probability for a certain configuration of $N$ lattice points in a row. Assuming periodic boundary conditions this is given by the transfer matrix $\tau = tr(\prod_{i=1}^{N} R^{(i)})$ (the trace appearing because of periodicity). Finally the partition function is obtained by repeating the procedure on the $M$ strings: $Z = tr(\prod_{j}^{M} \tau)$. The model is exactly solvable if $R$ depends on a parameter $\lambda \in \mathbb{C}$ such that eq.(2.2) is satisfied. Infact, if this is the case, one can introduce an infinite set of commuting operators as follows. First define the monodromy matrix:

$$ T(\lambda)_{ij}^{\alpha_1 \alpha_2 \ldots \alpha_N} = R(\lambda)_{k_1}^{\alpha_1} R(\lambda)_{k_2}^{\alpha_2} \ldots R(\lambda)_{k_{N-1}}^{\alpha_{N-1}} R(\lambda)_{j}^{\alpha_N} $$

that can be viewed as an $n \times n$-matrix with values in End($V^N$) whose trace $tr(T(\lambda)) \in \text{End}(V^N)$ is the transfer matrix (the partition function, we recall, is the trace in $V^N$ of the transfer matrix). The starting point of the QISM is the relation

$$ R(\lambda)_{m}^{i} T(\lambda')_{j}^{m} T(\lambda'')_{\ell}^{n} = T(\lambda'')_{m}^{i} T(\lambda')_{j}^{m} R(\lambda)_{\ell}^{n} $$

which follows from the definition of $T$ and the QYBE. This relation implies the commutativity of the transfer matrices

$$ [tr(T(\lambda')), tr(T(\lambda''))] = 0, \quad \text{for all } \lambda', \lambda'' \in \mathbb{C} $$
hence the existence of an infinite number of conserved quantities, typically regarded as the expansion coefficients \( \ln T(\lambda) = \sum_{n=1}^{\infty} C_n \lambda^n \).

For quantum field theories in \( 1 + 1 \) dimension the argument is essentially the same with the change that one of the dimensions is time. This leads to consider on a different footing the two spaces in which the bonds take values instead of having two copies of the same space \( V \) (an \( n \)-dimensional space) both "horizontally" and "vertically" along the lattice. For instance one may think of the vertical space as the space of operators describing the time evolution of the quantum mechanical system (possibly an infinite-dimensional space); the \( N \) horizontal points represent the discretization of space and the continuum limit \( N \to \infty \) yields integrable quantum field theories. If we denote by \( V \) the vertical space to distinguish it from \( V \), the horizontal space, then \( R \in \text{End}(V \otimes V) \) is a Lax operator \( L \) and the QYBE reads

\[
R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu)
\]

where \( R_{12} \) is a standard (i.e. numerical) matrix. If such a relation holds at each lattice point \((i)\) and the "ultralocality" condition

\[
[L^{(i)}(\lambda), L^{(j)}(\mu)] = 0, \quad i \neq j
\]

is satisfied then one can pursue the program as above: construct the monodromy matrix by multiplying the Lax operators and from it the commuting transfer matrices which lead to integrability.

We now introduce formally quasi-triangular Hopf algebras (quantum groups [1,15,16,17]). The property of quasi-triangularity, in particular, involves the definition of the universal \( R \)-matrix satisfying the Quantum Yang-Baxter equation without spectral parameters. The process of dressing such an \( R \)-matrix with parameters is called Yang-Baxterization, [18,19], and is not treated in the present paper.

\textit{Note.} The mathematics of quantum groups was also studied by Lusztig [20,21], Rosso [22,23], Verdier [24].
Hopf Algebras involve both an algebraic and a coalgebraic structure. Let us start by showing the algebraic structure in the following definition.

**Definition 2.1.** If $G$ is a complex Lie algebra, then the *extended enveloping algebra* $U_q(G)$ of the universal enveloping algebra $U(G)$ is the associative algebra over $\mathbb{C}$ with generators $X^\pm_i, H_i, i = 1, 2, ..., r = \text{rank } (G)$ and with relations

\[
[H_i, H_j] = 0 \ , \ [H_i, X^\pm_j] = \pm a_{ij} X^\pm_j \ , \quad (2.3)
\]

\[
[X^+_i, X^-_j] = \delta_{ij} \frac{q_i^{H_i/2} - q_i^{-H_i/2}}{q_i^{1/2} - q_i^{-1/2}} = \delta_{ij} [H_i]_{q_i} \ , \ q_i = q^{(\alpha_i, \alpha_i)/2} \ , \quad (2.4)
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k}_{q_i} (X^+_i)^k X^\pm_j (X^\pm_i)^{n-k} = 0 \ , \ i \neq j \ , \quad (2.5)
\]

where $(a_{ij}) = (2(\alpha_i, \alpha_j)/\langle \alpha_i, \alpha_i \rangle)$ is the Cartan matrix of $G$, $(\ , \ )$ is the scalar product of the roots normalized or that for the short simple roots $\alpha$ we have $\langle \alpha, \alpha \rangle = 2$, $n = 1 - a_{ij}$,

\[
\binom{n}{k}_{q} := \frac{[n]_q!}{[k]_q! [n-k]_q!} \ , \quad [m]_q! := [m]_q [m-1]_q \cdots [1]_q \ ,
\]

\[
[m]_q := \frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}} = \frac{sh(mh/2)}{sh(h/2)} = \frac{\sin(\pi m\tau)}{\sin(\pi \tau)} \ , \ q = e^h = e^{2\pi i \tau}, h, \tau \in \mathbb{C},
\]

\[
q_i^{\alpha_{ij}} = q^{(\alpha_i, \alpha_j)} = q_j^{\alpha_{ij}} \ .
\]

**Comment.** For a deeper analysis of the concepts involved in Definition 2.1 and in particular for an exhaustive discussion on the ring of functions of the Cartan generators necessary in the construction of the extended enveloping algebra we refer to Truini and Varadarajan [25].

**Remark 2.3.** The above construction works also when $G$ is an affine Kac-Moody algebra (see Drinfeld [4]).

**Convention.** The subscript $q$ in $[m]_q$ will be omitted if no confusion can arise.

**Comments.** The extended enveloping algebra $U_q(G)$ is the "$q$-deformation" of the algebra $U(G)$. Definition 2.1. may be used also for real forms (introducing the
appropriate \(*\)-involutions) namely, where \( q \in \mathbb{R} \) for the real compact forms (via the Weyl unitary trick) - e.g., for the classical compact algebras \( su(n), so(n), sp(n) \) - while for \(|q| = 1\), it may be used if \( \mathcal{G} \) is replaced by its maximally split form - e.g., for the classical complex Lie algebras these forms being \( sl(n, \mathbb{R}), so(n, n), so(n+1, n), sp(n, \mathbb{R}) \).

**Remark 2.4.** For \( q \to 1(h \to 0) \) one recovers the standard commutation relations from (2.3) and (2.4), and Serre’s relations from (2.5) in terms of generators \( H_i, X_i^{\pm} \) (for the sense in which the limit is to be understood see Drinfeld [4]).

We recall that the Serre relations allow to define the universal enveloping algebra of a Lie algebra using only the simple roots and the Cartan matrix. It is a classical result that one thus gets an equivalent structure to the one obtained by taking the quotient of the tensor algebra with generators corresponding to all the roots by the full set of commutation relations (see Varadarajan [26]). There are two cases in which the explicit introduction of the generators associated to non-simple roots is convenient. One is the ”q-analogue” of the Poincaré-Birkhoff-Witt theorem (which selects a representative element in each equivalence class of the quotient). The second is the universal \( R \)-matrix whose representations are related through Yang-Baxterization to the ”physical \( R \)-matrix” which motivated the whole subject. Let us thus spend a few words on the generators associated to non-simple roots.

**Conventions and notations.** The Cartan subalgebra, spaned by \( H_i \), will be denoted by \( \mathcal{H} \), while the subalgebras spaned by \( X_i^{\pm} \) will be denoted by \( \mathcal{G}^{\pm} \). We have the standard decompositions

\[
\mathcal{G} = \mathcal{H} \oplus_{\beta \in \Gamma} \mathcal{G}_\beta = \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- , \quad \mathcal{G}^\pm = \oplus_{\beta \in \Gamma^\pm} \mathcal{G}_\beta ,
\]

where \( \Gamma = \Gamma^+ \cup \Gamma^- \) is the root system of \( \mathcal{G} \) and \( \Gamma^+, \Gamma^- \) the sets of positive, negative roots, respectively. Let us recall that the \( H_i \)'s correspond to the simple roots \( \alpha_i \) of \( \mathcal{G} \), and if \( \beta = \sum_i n_i \alpha_i \), then to \( \beta \) correspond \( H_\beta = \sum_i n_i H_i \). The elements of \( \mathcal{G} \) which span \( \mathcal{G}_\beta \) will be denoted by \( X_\beta \). These Cartan-Weyl generators are normalized, so
that we have

\[ [X_\beta, X_{-\beta}] = [H_\beta]q_\beta \quad \text{for} \quad \beta \in \Gamma^+, \quad q_\beta = q^{(\beta, \beta)/2}. \tag{2.6} \]

**Remark 2.5.** Instead of \( H_i \), some authors prefer to use \( K^\pm_i \) defined by

\[ K_i^\pm := q^{\pm H_i/2} , \]

and then (2.3) and (2.4) become

\[
K_i, K_i^{-1} = K_i^{-1}K_i = 1, \quad [K_i, K_j] = 0, \quad K_iX_j^\pm K_i^{-1} = q_i^{\pm a_{ij}/2}X_j^\pm , \tag{2.3'}
\]

\[
[X_i^+, X_j^-] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i^{1/2} - q_i^{-1/2}} . \tag{2.4'}
\]

**Remark 2.6.** One can use, following Rosso [27], instead of \( X_i^\pm \), the generators \( E_i \) and \( F_i \) defined as follows

\[
E_i := X_i^+q_i^{-H_i/4} = X_i^+K_i^{-1/2} , \quad F_i := X_i^-q_i^{H_i/4} = X_i^-K_i^{1/2} . \tag{2.7}
\]

**Definition 2.2.** An associative algebra \( A \) with unit \( 1_A \) is called a \textit{bialgebra} if there exist two homomorphisms \( \Delta \) and \( \varepsilon \), called, \textit{comultiplication} and \textit{counit}, respectively, such that

\[
\Delta : A \rightarrow A \otimes A , \quad \Delta(1_A) = 1_A \otimes 1_A
\]

and

\[
\varepsilon : A \rightarrow C , \quad \varepsilon(1_A) = 1 ,
\]

the comultiplication \( \Delta \) satisfying the axiom of coassociativity

\[
(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta ,
\]

where both sides are maps \( A \rightarrow A \otimes A \otimes A \), the two homomorphisms fulfilling

\[
(Id \otimes \varepsilon) \circ \Delta = i_1, \quad (\varepsilon \otimes Id) \circ \Delta = i_2
\]

as maps \( A \rightarrow C \otimes A, A \rightarrow A \otimes C \), respectively, where \( i_1, i_2 \) are the maps identifying \( A \) with \( A \otimes C, C \otimes A \), respectively.
**Definition 2.3.** A bialgebra $\mathcal{A}$ is called a *Hopf algebra* if there exists an algebra antihomomorphism $S$ - called *antipode* - such that

$$S : \mathcal{A} \to \mathcal{A} \ , \ S(1_{\mathcal{A}}) = 1_{\mathcal{A}}$$

and

$$m \circ (Id \otimes S) \circ \Delta = i \circ \varepsilon \ ,$$

as maps $\mathcal{A} \to \mathcal{A}$, where $m$ is the usual product in the algebra $\mathcal{A}$ (i.e. $m(Y \otimes Z) = YZ$, $Y, Z \in \mathcal{A}$), and $i$ is the natural embedding of $\mathbb{C}$ into $\mathcal{A}$ (i.e. $i(c) = c 1_{\mathcal{A}}$, $c \in \mathbb{C}$)

**Remark 2.7.** The antipode plays the role of an inverse although there is no requirement that $S^2 = Id$.

**Remark 2.8.** Following Dobrev [3], we shall use also the *opposite comultiplication* $\Delta' := \sigma \circ \Delta$, where $\sigma$ is the permutation in $\mathcal{A} \otimes \mathcal{A}$. In case the antipode has an inverse, then one uses also the *opposite antipode* $S' := S^{-1}$ (see also Drinfeld [4] and Jimbo [16]).

The comultiplication, counit, and antipode are defined on the generators of $U_q(\mathcal{G})$ as follows:

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \ , \ \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q_i^{H_i/4} + q_i^{-H_i/4} \otimes X_i^{\pm} \ ,$$

$$\varepsilon(H_i) = \varepsilon(X_i^{\pm}) = 0 \ , \quad (2.8)$$

$$S(H_i) = -H_i \ , \ S(X_i^{\pm}) = -q_i^{\hat{\rho}/2} X_i^{\pm} q_i^{-\hat{\rho}/2} = -q_i^{\pm 1/2} X_i^{\pm} ,$$

where $\hat{\rho} \in \mathcal{H}$ corresponds to $\rho = \frac{1}{2} \sum_{\alpha \in \Gamma^+} \alpha$ , $\Gamma^+$ being the set of positive roots, $\hat{\rho} = \frac{1}{2} \sum_{\alpha \in \Gamma^+} H_\alpha$.

**Comment.** As it was remarked, for $\mathcal{G} = sl(2, \mathbb{C})$, in the paper [28] by Sklyanin, and, in general, in the papers [4,15] by Drinfeld and [16,17] by Jimbo, the algebra $U_q(\mathcal{G})$ is a Hopf algebra.

**Remark 2.9.** The above formulae (2.8) hold also for $H_\beta$, $X_{\pm \beta}$ from (2.6).

**Remark 2.10.** The opposite comultiplication and antipode from Remark 2.8. define a Hopf algebra $U_q(\mathcal{G})'$ which is related to $U_q(\mathcal{G})$ by the relation $U_q(\mathcal{G})' = U_{q^{-1}}(\mathcal{G})$. 
In terms of the generators $K_i^\pm, E_i, F_i$ from Remarks 2.5. and 2.6., the relations (2.8) become

$$
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i,
$$

$$
\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0 \quad (2.8')
$$

$$
S(K_i) = K_i^{-1}, \quad S(E_i) = -K_i E_i, \quad S(F_i) = -F_i K_i^{-1}.
$$

One can also rewrite Serre’s relations (2.5) as follows

$$(ad_q E_i)^n(E_j) = 0 = (ad'_q F_i)^n(F_j), \quad i \neq j,$$

where

$$ad_q : U_q(G^+) \to \text{End} (U_q(G^+)), \quad ad_q = (L \otimes R)(Id \otimes S)\Delta,$$

$$ad'_q : U_q(G^-) \to \text{End} (U_q(G^-)), \quad ad'_q = (L \otimes R)(Id \otimes S')\Delta', \quad (2.5')$$

and $L$ (resp. $R$) is the left (resp. right) multiplication.

**Remark 2.11.** $ad_q(E_i)$ acts as a twisted derivation.

**Definition 2.4.** A Hopf algebra $\mathcal{A}$ for which there exists an invertible element $R$ in $\mathcal{A} \otimes \mathcal{A}$ - called universal $R$-matrix (cf. Drinfeld [4,15]) - which intertwines $\Delta$ and $\Delta'$, i.e.

$$R \Delta(Y) = \Delta'(Y) R, \quad \text{for all } Y \text{ in } \mathcal{A}, \quad (2.9a)$$

and obeys also the relations

$$\Delta \otimes Id)R = R_{13} R_{23}, \quad R = R_{13}, \quad (2.9b)$$

$$Id \otimes \Delta)R = R_{13} R_{12}, \quad R = R_{12}, \quad (2.9c)$$

where the indices indicate the embeddings of $R$ into $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ is called a quasi-triangular Hopf algebra.

**Remark 2.12.** We have

$$(\varepsilon \otimes Id)R = (Id \otimes \varepsilon)R = 1_\mathcal{A}, \quad \text{and, moreover},$$
\((S \otimes \text{Id})R = R^{-1}, \quad (\text{Id} \otimes S)R^{-1} = R\).

**Definition 2.5.** A quasi-triangular Hopf algebra for which also \(\sigma R^{-1} = R\) holds is called a **triangular Hopf algebra**.

**Definition 2.6.** From (2.9a) and one of (2.9b,c) it follows

\[R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},\]

which is the Yang-Baxter equation for \(R\) without spectral parameters.

**Comments.** The universal \(R\)-matrix was given explicitly for \(\mathcal{G} = sl(2, \mathbb{C})\) by Drinfeld [15], namely,

\[R = q^{H \otimes H/4} \sum_{n \geq 0} \frac{(1 - q^{-1})^n q^{\frac{1}{4}n(n-1)}}{[n]!} \left( q^{\frac{3}{4} X^+} \right)^n \otimes \left( q^{-\frac{3}{4} X^-} \right)^n.\]

This \(R\)-matrix is not in \(U_q(sl(2, \mathbb{C})) \otimes U_q(sl(2, \mathbb{C}))\), since it contains power series involving \(X^\pm\), but it is in some completion of it (in the \(h\)-adic topology used by Drinfeld [4,15] \((q = e^h)\)). This fact is valid for the \(R\)-matrices of all \(U_q(\mathcal{G})\). Hopf algebras with such an \(R\)-matrix are called by Drinfeld [4] **pseudo quasi-triangular Hopf algebras**, and by Majid [7] **essentially quasi-triangular Hopf algebras**. For \(\mathcal{G} = sl(n, \mathbb{C})\), an explicit formula for \(R\) was given by Rosso [27].

3. **SUBSEQUENT AND RECENT DEVELOPMENTS**

At the end of eighties other approaches to quantum groups were given. The objects of these approaches - which can be called **quantum matrix groups** - are Hopf algebras in duality to quantum algebras.

**Definition 3.1.** Two Hopf algebras \(\mathcal{A}\) and \(\mathcal{A}'\) are said to be in **duality** if there exists a doubly nondegenerate bilinear form

\[<,>: \mathcal{A} \times \mathcal{A}' \to \mathbb{C}, \quad <,>(a, a') \rightarrow <a, a'>,\]

such that, for \(a, b \in \mathcal{A}\) and \(a', b' \in \mathcal{A}'\) the following relations hold

\[<a, a'b'> = <\Delta_{\mathcal{A}}(a), a' \otimes b'>, \quad <ab, a'> = <a \otimes b, \Delta_{\mathcal{A}'}(a')>,\]
\[ <1_A, a'> = \varepsilon_A(a'), \quad <a, 1_{A'}> = \varepsilon_A(a), \quad <S_A(a), a'> = <a, S_{A'}(a')>. \]

One of these approaches is due to Faddeev, Reshetikhin and Takhtajan [29,30] and it is called "R-matrix approach". It is based on the main relation of the quantum inverse scattering method. The quantum group matrices play the role of the quantum monodromy matrices (with operator-value entries) of the auxiliary linear problem and the Yang-Baxter equation is the compatibility equation.

Another approach is that of Manin [31,32,33], who considers quantum groups as symmetries of non-commutative, or quantum, spaces. The resulting objects are the same as those of the first approach.

A third approach that we would like to mention here is that of Woronowicz [34,35,36]. This approach also deals with same objects with some additional structures since Woronowicz’s starting point is the theory of C*-algebras.

*Note.* Connections between the above mentioned approaches can be found in the papers by Doebner, Hening and Lücke [37], Majid [7], and Rosso [22].

At the end of eighties, Drinfeld [15], inspired by Knizhnik-Zamolodchikov equations (see [38]), developed a theory of formal deformations and introduced a new notion of "quasi-Hopf algebras".

The matrix quantum group approaches were recently developed, in particular finding consistent multiparametric deformations. This is related to the development by Wess, Zumino and collaborators of differential calculus on quantum hyperplanes. The latter approach is actually an example of non-commutative differential geometry which in opinion of Manin [39] is different in spirit from that of Connes [40], although the exact relation is not known yet. For a detailed review of the above mentioned developments we refer the reader to §§2-5 from Dobrev’s paper [3].

Finally, we like to mention the very recent paper by Mack and Schomerus [41] on "quantum field planes". These objects are generalizations of the quantum planes which were studied by Manin, Wess and Zumino, and others, and were generalized to the quasi-associative case by the same authors [42]. Basically, the construction
of quantum field planes replaces the ground field $\mathbb{C}$ of quantum planes by the non-commutative algebra of observables of the quantum field theory in the local Lorentz frame.

**Comments.** An interesting open problem could be to reconsider Mack-Schomerus construction in case when the algebra of observables is a (non-commutative) Jordan algebra.

**Remark 3.1.** As a final note to this survey on the history of quantum groups we want to emphasize that quantum groups are involved and studied in many different fields of mathematics and physics some of which we haven’t even mentioned so far. Among them are: topological quantum field theories, 2-dimensional gravity and 3-dimensional Chern-Simons theory [43,44,45,7], rational conformal field theory [46,47,48], braid and knot theory [49,50], non-standard quantum statistics [51], quantum Hall effect [52,53].

4. RELATIONS WITH JORDAN STRUCTURES

We would like to start by recalling here the new topic proposed by Truini and Varadarajan at the end of their recent paper [54], namely: *quantization of Jordan structures*.

Let us explain why it is interesting to investigate this new topic.

It is known that the quantization of the Poincaré group is receiving more and more attention by the physicists. The main reason of this attention consists in its relationship with the non-commutative geometry of quantum Minkowski space (see [55]). It is believed that the models of non-commutative space-time, and their quantum symmetry groups, may provide a basis for building a divergence free theory of elementary particles and their fields, including gravitation, thus overcoming the difficulties arising out of the structure of conventional space-time at small distances. Consequently, Truini and Varadarajan (see [54, p.732]) intend to study quantizations of semidirect products as Hopf algebras which maintain the classical picture of a space and a set of transformations that act covariantly on it. But, this point of view
is related to the general problem of deforming Jordan structures. Indeed, the algebra of Poincaré group can be represented as the semidirect product of \( \mathcal{L} \simeq sl(2, \mathbb{C}) \) and the Jordan algebra \( J \) (representing the translations) of \( 2 \times 2 \)-Hermitian matrices over \( \mathbb{C} \),

\[
\begin{pmatrix}
\mathcal{L} & J \\
0 & -\mathcal{L}^\dagger
\end{pmatrix}
\]

the action of the algebra of the Lorentz group on the translations being determined by the commutator of the related \( 4 \times 4 \)-matrices.

Another motivation for the study of deformations of Jordan structures comes from the conformal group, in which the Poincaré group is naturally imbedded and which is a simple group, thus falling into their general theory of universal deformations. Similarly to the above representation of the Poincaré group, the algebra of the conformal group can be written as

\[
\begin{pmatrix}
\tilde{\mathcal{L}} & J_1 \\
J_2 & -\tilde{\mathcal{L}}^+ \\
\end{pmatrix}
\]

where \( J_1 \) and \( J_2 \) are Jordan algebras of \( 2 \times 2 \)-Hermitian matrices over \( \mathbb{C} \) and \( \tilde{\mathcal{L}} \simeq sl(2, \mathbb{C}) \oplus \mathbb{R} \). It is easy to show that \((J_1, J_2)\) is a Jordan pair, \( \tilde{\mathcal{L}} \) is the algebra of its automorphism group and the structural algebra of \( J \).

It is known that a Jordan pair \( V = (V^+, V^-) \) is a pair of modules acting each other through a map \( U_{x^+}x^- \) quadratic in \( x^+ \) and linear in \( x^- \) and obeying certain axioms which extend those of the quadratic formulation of the theory of Jordan algebras (where \( U_{xy} \) generalizes \( xyx \)). Roughly speaking, a Jordan pair is "a pair of spaces acting on each other like a Jordan triple system" - see McCrimmon [56, p.621]. On the other hand, let us point out that Jordan pairs are much more common than what one would expect. So, there exists a 1-1 correspondence between 3-graded Lie algebras

\[
L = L_1 \oplus L_0 \oplus L_{-1} \quad ([L_i, L_j] \subset L_{i+j})
\]
and Jordan pairs $V$ (plus the choice of $L_0$ between Der $(V)$ and Inder $(V)$) - see, for instance, McCrimmon [56, p.622]. As it is known, many Lie algebras are three-graded (e.g., $A(2), C(3), E(7)$).

In this respect, it is interesting to point out that Okubo [57] has related triple products, which are linearization of the quadratic maps, to the quantum $R$-matrix and used the relationship to find solutions to the quantum Yang-Baxter equation.

More recently, in a series of papers, Okubo [58,59,60,61] reformulated first the Yang-Baxter equation as a triple product relation and then solved it for triple systems called ”orthogonal” and ”symplectic” (see Okubo [58,59]). A superspace extension of this work was given by Okubo himself in [60,61].

In a very recent paper, Okubo [62] gave some solution of Yang-Baxter equation in terms of Jordan triple systems and of so-called ”anti-Jordan triple systems”. (see Definition 4.1. below).

**Definition 4.1.** Let $V$ be a $N$-dimensional vector space over a field $F$ and let $xyz : V \otimes V \otimes V \rightarrow V$ be a triple product in $V$ satisfying the following conditions

\[ zyx = \delta xyz , \quad (i) \]

\[ uv(xyz) = (uvx)yz - \delta x(vuy)z + xy(uvz) , \quad (ii) \]

where $\delta = \pm 1$. The case $\delta = 1$ defines the well known (linear) *Jordan triple systems* (see, for instance, Meyberg [63]), while the case $\delta = -1$ defines the *anti-Jordan triple systems*.

**Note.** As it was noticed by Koecher [64], a glimps of Jordan triple system was given by Gibbs (1839-1903) as early as 1881 (Collected Works, vol.II, p.18) in a different setting.

**Comments.** Compare the definition of ”anti-Jordan triple systems” given by Okubo [62] (see Definition 4.1.) with the definition of ”anti-Jordan pairs” given by Faulkner and Ferrar in [65].

Okubo [62] considered $V$ endowed also with a bilinear non-degenerate form $<
Let $R(\theta) \in End(V) \otimes End(V)$ be the scattering matrix with matrix elements $R^{dc}_{ab}$, defined by $R(\theta)e_a \otimes e_b = R^{dc}_{ab}e_c \otimes e_d$, with respect to a basis $\{e_j\}$ of $V$ and suppose that $R$ satisfies the QYBE

$$R_{12}(\theta)R_{13}(\theta')R_{23}(\theta'') = R_{23}(\theta'')R_{13}(\theta')R_{12}(\theta) \quad (4.1a)$$

with

$$\theta' = \theta + \theta'' \quad (4.1b)$$

Two $\theta$-dependent triple linear products $[x, y, z]_{\theta}$ and $[x, y, z]_{\theta}^*$ are defined in terms of the scattering matrix elements $R^{dc}_{ab}(\theta)$, by

$$[e^c, e_a, e_b]_{\theta} := e_d R^{dc}_{ab}(\theta)$$

$$[e^d, e_b, e_a]_{\theta}^* := R^{dc}_{ab}(\theta) e_c$$

or alternatively by

$$R^{dc}_{ab}(\theta) = <e^d|[e^c, e_a, e_b]_{\theta}> = <e^c|[e^d, e_b, e_a]_{\theta}^* >$$

where $e^d$ is given by

$$<e^d|e_c> = \delta^d_c.$$ 

The QYBE (4.1a) can be then rewritten as a triple product equation

$$\sum_{j=1}^{N} [v, [u, e_j, z]_{\theta'}, [e^j, x, y]_{\theta}^*]_{\theta''} = \sum_{j=1}^{N} [u, [v, e_j, x]_{\theta'}^*, [e^j, z, y]_{\theta''}^*]_{\theta} \quad (4.2)$$

Proposition 4.1 Let $V$ be a Jordan or anti-Jordan triple system with $\epsilon = 1$ satisfying the following conditions

i) $<u|xvy> = <v|yux>$
ii) \(< u | x v y > = \delta < x | u y v > = \delta < y | v x u >\)

iii) \((y e^\theta x) v e_j = a \{ < x | v > y + \delta < y | v > x \} + b \ y \ v \ x\)

(iv) \((y e^\theta x) v (z u e_j) - (y e^\theta z) u (x v e_j) = \alpha \{ < v | x > z u y - < u | z > x v y \} + \beta \{ < v | y > x u z - < u | y > z v x + < y | u z v > x \}

- < y | v x u > z \} + \gamma \{(y u x) v z - (y v z) u x\}

for some constants \(a, b, \alpha, \beta, \) and \(\gamma\). Then,

\([x, y, z]_\theta = P(\theta) \ y \ x \ z + B(\theta) < y | z > x + C(\theta) < z | x > y\)

for \(P(\theta) \neq 0\) is a solution of the QYBE (4.2) with

\(\frac{B(\theta)}{P(\theta)} = \delta \gamma + k \theta\), \(\frac{C(\theta)}{P(\theta)} = \frac{\beta \delta}{k \theta}\)

for an arbitrary constant \(k\), provided that we have either

(i) \(\alpha = \beta = 0\)

or

(ii) \(\alpha = \beta \neq 0\), \(b = -2 \gamma\), \(a = 2 \beta\).

Remark 4.1 The solution satisfies the unitarity condition

\(R(\theta) \ R(-\theta) = f(\theta) \ Id\)

where

\(f(\theta) = P(\theta) P(-\theta) \left\{ (a + \gamma^2) - (k\theta)^2 - \frac{\beta^2}{(k\theta)^2} \right\}\)

Proposition 4.2 Let \(V\) be the Jordan triple system defined on the vector space of the Lie-algebra \(u(n)\) by means of the product

\(x \ y \ z = x \cdot y \cdot z + \delta \ z \cdot y \cdot x\)

the dot denoting the usual associative product in \(V\) and let \(< | >\) be the trace form. Then,

\([x, y, z]_\theta = P(\theta) x \ z \ y + A(\theta) < y | z > x + C(\theta) < z | x > y\)
for $P(\theta) \neq 0$ offers solutions of the QYBE (4.2) for the following two cases:

(i) \[ \frac{A(\theta)}{P(\theta)} = -\frac{\lambda^2 e^{k\theta} - d}{\lambda(e^{k\theta} - 1)} \quad , \quad \frac{C(\theta)}{P(\theta)} = -\frac{e^{k\theta} - \lambda^2}{\lambda(e^{k\theta} - 1)} \] 

where $d$ is either $\lambda^2$ or $-\lambda^4$ and $k$ is an arbitrary constant, or

(ii) \[ \frac{A(\theta)}{P(\theta)} = -\lambda \quad , \quad \frac{C(\theta)}{P(\theta)} = -\frac{1}{\lambda} \].

In both cases $\lambda$ is given by

\[ \lambda = \frac{1}{2} \left( n \pm \sqrt{n^2 - 4} \right) \]

Remark 4.2 The first solution Eq. (4.3a) satisfies both unitarity and crossing symmetry relations:

\[ R(\theta)R(-\theta) = C(\theta)C(-\theta)Id \] (4.4a)

\[ \frac{1}{P(\theta)} [y, x, z]_{\theta} = \frac{1}{P(\theta)} [x, y, z]_{\theta} \] (4.4b)

where $\theta$ in Eq. (4.4b) is related to $\theta$ by

\[ \theta + \bar{\theta} = \frac{1}{k} \log d \] .

In view of these, the solution is likely related [66] to some exactly solvable two-dimensional quantum field theory.

Finally we want to mention a result by Svinolupov [67] which is interesting in the context of this paper. He considers systems of nonlinear equations which, in a particular case, may be reduced to the nonlinear Schrödinger equation and are therefore called generalized Schrödinger equations. A one to one correspondence between such integrable systems and Jordan Pairs is established. It turns out that irreducible systems correspond to simple Jordan Pairs.

In our opinion the general setting in which one should consider the problem of quantizing the Jordan structures is that of Jordan Pairs. We are currently investigating this possibility with the belief that a quantum analog of the classical link between Jordan and Lie structures would give a deeper insight and reveal new aspects in the theory of quantum groups.
REFERENCES

1. Biedenharn, L.C.: invited paper presented at the XVIII Int. Coll. on Group Theoretical Methods in Physics, Moscow, USSR, June 4-9 1990
2. Biedenharn, L.C.: Int. J. Theor. Phys. 32 (1993) No.10, 1789
3. Dobrev, V.K.: Invited plenary lecture at the 22nd Iranian Mathematics Conference, March 13-16 1991, Mosshad Iran
4. Drinfeld, V.G.: in Proc. Int. Congress of Math. Berkeley, California , Academic Press, New York, 1, 1986 p.798
5. Faddeev, L.D.: preprint ITP-SB-94-11 and [hep-th 9404013] (1994)
6. Kundu, A.: in ”Application of solitons in Science and Engeneering”, World Scientific, Singapore, 1994 (to appear)
7. Majid, S.: Int. J. Mod. Phys. A5 (1990), 1
8. Ruiz-Altaba, M.: preprint UGVA-DPT-1993-10-838 and [hep-th 9311069] (1993)
9. Smirnov, F.A.: in ”Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory” World Scientific, Singapore, 1990 p.1
10. Takhtajan, L.A.: in ”Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory” World Scientific, Singapore, 1990 p.69
11. Boldin, A.Yu., Safin, S.S., Sharipov, R.A.: J. Math. Phys. 34 (1993), No.12, 5801
12. Tzitzeica, G.: C.R. Acad. Sci. Paris 150 (1910), 955
13. Jonas, H.: Ann. Mat. XXX (1921), 223
14. Jonas, H.: Math. Nachr. 10 (1953), 331
15. Drinfeld, V.G.: Alg. Anal. 1 (1989)
16. Jimbo, M.: Lett. Math. Phys. 10 (1985), 63
17. Jimbo, M.: Lett. Math. Phys. 11 (1986), 247
18. Jones, V.F.R.: Int. J. Mod. Phys. B4 (1990), 701
19. Basu Mallick, B., Kundu, A.: J. Phys. A25 (1992), 4147
20. Lusztig, G.: Advances Math. 70 (1988), 237
21. Lusztig, G.: *Introduction to Quantum Groups*, Birkhäuser, Basel, 1993
22. Rosso, M.: C.R. Acad. Sci. Paris **305** (1987), Serie I, 587
23. Rosso, M.: Commun. Math. Phys. **117** (1987), 581
24. Verdier, J.-L.: in Séminaire Bourbaki, No. 685 Astérisque **152-153** (1987), 305
25. Truini, P., Varadarajan, V.S.: Rev. Math. Phys. **5** (1993), No.2, 363
26. Varadarajan, V.S.: *Lie Groups, Lie Algebras and their Representations*, Prentice Hall Inc., Englewood Cliffs, N.J. 1974
27. Rosso, M.: Commun. Math. Phys. **124** (1989), 307
28. Sklyanin, E.K.: Uspekhi Mat. Nauk. **40** (1985), 214
29. Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: Alg. Anal. **1** (1989), 178
30. Takhtajan, L.A.: Adv. Stud. Pure Mat. **19** (1989), 435
31. Manin, Yu. I.: Ann. Inst. Fourier **37** (1987), 191
32. Manin, Yu. I.: Montreal Univ. preprint, CRM-1561 (1988)
33. Manin, Yu. I.: Commun. Math. Phys. **123** (1989), 163
34. Woronowicz, S.L.: Commun. Math. Phys. **111** (1987), 613; Lett. Math. Phys. **21** (1991), 35
35. Woronowicz, S.L.: Publ. RIMS **23** (1987), 117
36. Woronowicz, S.L.: Commun. Math. Phys. **122** (1989), 125
37. Doebner, H.D., Hennig, J.D., Lücke, W.: in Proc. Quantum Groups Workshop, Clausthal 1989, Eds. H.D. Doebner and G.D. Hennig, Lecture Notes in Physics **370** (Springer-Verlag Berlin 1990) p.29
38. Knizhnik, V.G., Zamolodchikov, A.B.: Nucl. Phys. **B247** (1984), 83
39. Manin, Yu.I.: invited paper presented at the XVIII Int. Coll. on Group Theoretical Methods in Physics, Moscow, USSR, June 4-9 1990
40. Connes, A.: Publ. Mat. IHES **62** (1985), 257
41. Mack, G., Schomerous, V.: preprint HVTMP 94-B335 (1994)
42. Mack, G., Schomerous, V.: Commun. Math. Phys. **149** (1992), 513
43. Gervais, J.-L.: Commun. Math. Phys. **130** (1990), 257; Phys. Lett. **243B**
44. Witten, E.: Nucl. Phys. B330 (1990), 285
45. Guadagnini, E. et al.: Phys. Lett. B235B (1990), 275
46. Alvarez-Gaumé, L., Gomez, C., Sierra, G.: Nucl. Phys. B319B (1989), 155
47. Moore, G., Reshetikhin, N.: Nucl. Phys B328B (1989), 557
48. Gomez, C., Sierra, G.: Phys. Lett. B240B (1990), 149
49. Jones, V.F.R.: Bull. Amer. Math. Soc. 12 (1985), 103
50. Kauffman, L.: Int. J. Mod. Phys. 5A (1990), 93
51. Greenberg, O.W.: in Proc. Argonne Workshop on Quantum Groups, T. Curtright, D. Fairlie and C. Zachos Eds., World Scientific, Singapore, 1990
52. Kogan, I.I.: preprint PUPT-1439 and hep-th 9401093 (1994)
53. Sato, H.-T.: preprint OS-GE-40-93 and hep-th 9312174 (1993)
54. Truini, P., Varadarajan, V.S.: in ”Symmetries in Science VI: From the Rotation Group to Quantum Algebras”, Bregenz, Austria, August 2-7 1992, Plenum Press, New York, 1993, p.731
55. Ogievetsky, O.; Schmidke, W.B., Wess, J., Zumino, B.: Max Plank and Berkeley preprint MPI-Ph/91-98, LBL-31703, UCB 92/04 (1991)
56. McCrimmon, K.: Bull. Amer. Math. Soc. 84 (1978), No.4, 612
57. Okubo, S.: in ”Symmetries in Science VI: From the Rotation Group to Quantum Algebras”, Bregenz, Austria, August 2-7 1992, Plenum Press, New York, 1993
58. Okubo, S.: J. Math. Phys. 34 (1993), 3273
59. Okubo, S.: J. Math. Phys. 34 (1993), 3292
60. Okubo, S.: University of Rochester Report UR-1312 (1993) to appear in Proc. of the 15th Montreal-Rochester-Syracuse-Toronto Meeting for High Energy Theories
61. Okubo, S.: University of Rochester Report UR-1319 (1993)
62. Okubo, S.: University of Rochester Report UR-1334 (1993)
63. Meyberg, K.: Math. Z. 115 (1970), 58
64. Koecher, M.: in "Lectures at the Rhine-Westphalia Academy of Sciences" No. 307, Westdeutscher Verlag, Opladen, 1982, p. 53

65. Faulkner, J.R., Ferrar, J.C.: Commun. Algebra 8 (1980), No. 11, 993

66. Zamolodchikov, A.B., Zamolodchikov, A.B.: Ann. Phys. 120 (1979), 253

67. Svinolupov, S.I.: Commun. Math. Phys. 143 (1992), 559