SUPERTROPICAL SEMIRINGS AND SUPervaluations

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Abstract. We interpret a valuation \( v \) on a ring \( R \) as a map \( v : R \to M \) into a so called bipotent semiring \( M \) (the usual max-plus setting), and then define a supervaluation \( \varphi \) as a suitable map into a supertropical semiring \( U \) with ghost ideal \( M \) (cf. [IR1], [IR2]) covering \( v \) via the ghost map \( U \to M \). The set \( \text{Cov}(v) \) of all supervaluations covering \( v \) has a natural ordering which makes it a complete lattice. In the case that \( R \) is a field, hence for \( v \) a Krull valuation, we give a completely explicit description of \( \text{Cov}(v) \).

The theory of supertropical semirings and supervaluations aims for an algebra fitting the needs of tropical geometry better than the usual max-plus setting. We illustrate this by giving a supertropical version of Kapranov’s Lemma.

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Introduction

As explained in [MS] and [G], tropical geometry grew out of a logarithmic correspondence taking a polynomial \( f(\lambda_1, \ldots, \lambda_n) \) over the ring of Puiseux series to a corresponding polynomial \( \overline{f}(\lambda_1, \ldots, \lambda_n) \) over the max-plus algebra \( T \). A key observation is Kapranov’s
Lemma, that this correspondence sends the algebraic variety defined by $f$ into the so-called corner locus defined by $\tilde{f}$. More precisely, this correspondence involves the negative of a valuation (where the target $(T)$ is an ordered monoid rather than an ordered group), which has led researchers in tropical mathematics to utilize valuation theory. In order to avoid the introduction of the negative, some researchers, such as [SS], have used the min-plus algebra instead of the max-plus algebra. There is a deeper result which describes the image of this correspondence; several versions appear in the literature, one of which is in [P1].

Note that whereas a valuation $v$ satisfies $v(ab) = v(a) + v(b)$, one only has

$$v(a + b) = \min\{v(a), v(b)\}$$

when $v(a) \neq v(b)$; for the case that $v(a) = v(b)$, $v(a + b)$ could be any element $\geq v(a)$. From this point of view, the max-plus (or, dually, min-plus) algebra does not precisely reflect the tropical mathematics. In order to deal with this issue, as well as to enhance the algebraic structure of the max-plus algebra $T$, the first author introduced a cover of $T$, graded by the multiplicative monoid $(\mathbb{Z}_2, \cdot)$, which was dubbed the extended tropical arithmetic. Then, in [IR1] and [IR2], this structure has been amplified to the notion of supertropical semiring. A supertropical semiring $U$ is equipped with a “ghost map” $\nu := \nu_U : U \to U$, which respects addition and multiplication and is idempotent, i.e., $\nu \circ \nu = \nu$. Moreover $a + a = \nu(a)$ for every $a \in U$ (cf. §3). This rule replaces the rule $a + a = a$ in the usual max-plus (or min-plus) arithmetic. We call $\nu(a)$ the “ghost” of $a$ (often writing $a^\ast$ instead of $\nu(a)$), and we call the elements of $U$ which are not ghost “tangible”\footnote{The element 0 may be regarded both as tangible and ghost.}.

The image of the ghost map is a so-called bipotent semiring, i.e., a semiring $M$ such that $a + b \in \{a, b\}$ for every $a, b \in M$. So $M$ is a semiring typically occurring in tropical algebra. In this paper supertropical and bipotent semirings are nearly always tacitly assumed to be commutative.

It turns out that supertropical semirings allow a refinement of valuation theory to a theory of “supervaluations”. Supervaluations seem to be able to give an enriched version of tropical geometry. In the present paper we illustrate this by giving a refined and generalized version of Kapranov’s Lemma (§9, §11). Very roughly, one may say that the usual tropical algebra is present in the ghost level of our supertropical setting.

We consider valuations on rings (as defined by Bourbaki [B]) instead of just fields. We mention that these are essential for understanding families of valuations on fields, cf. e.g. [HK] and [KZ]. We use multiplicative notation, writing a valuation $v$ on a ring $R$ as a map into $\Gamma \cup \{0\}$ with $\Gamma$ a multiplicative ordered abelian group and $0 < \Gamma$, obeying the rules

$$v(0) = 0, \quad v(1) = 1, \quad v(ab) = v(a)v(b),$$

$$v(a + b) \leq \max(v(a), v(b)). \tag{*}$$

We view the ordered monoid $\Gamma \cup \{0\}$ as a bipotent semiring by introducing the addition $x + y := \max(x, y)$, cf. §1 and §2. It is then very natural to replace $\Gamma \cup \{0\}$ by any bipotent semiring $M$, and to define an $m$-valuation (= monoid valuation) $v : R \to M$ in the same way $(*)$ as before.

Given an $m$-valuation $v : R \to M$ there exist multiplicative mappings $\varphi : R \to M$ into various supertropical semirings $U$, with $\varphi(0) = 0, \varphi(1) = 1$, such that $M$ is the ghost ideal of $U$ and $\nu_U \circ \varphi = v$. These are the supervaluations covering $v$, cf. §4.

In §5 we define maps $\alpha : U \to V$ between supertropical semirings, called transmissions, which have the property that for a supervaluation $\varphi : R \to U$ the composite $\alpha \circ \varphi : R \to V$ is again a supervaluation. Given two supervaluations $\varphi : R \to U$ and $\psi : R \to V$ (not
necessarily covering the same \( m \)-valuation \( v \)), we say that \( \varphi \) dominates \( \psi \), and write \( \varphi \geq \psi \), if there exists a transmission \( \alpha : U \to V \), such that \( \psi = \alpha \circ \varphi \). {The transmission \( \alpha \) then is essentially unique.}

Restricting the dominance relation to the set of supervaluations\footnote{More precisely we should consider equivalence classes of supervaluations. We suppress this point here.} covering a fixed \( m \)-valuation \( v : R \to M \) we obtain a partially ordered set \( \text{Cov}(v) \), which turns out to be a complete lattice, as proved in §7. The bottom element of this lattice is the \( m \)-valuation \( v \), viewed as a supervaluation. The top element, denoted \( \varphi_v : R \to U(v) \), can be described explicitly in good cases. This description is already given in §4, cf. Example 4.5. The other elements of \( \text{Cov}(v) \) are obtained from \( \varphi_v \) by dividing out suitable equivalence relations on the semiring \( U(v) \), called MFCE-relations (= multiplicative fiber conserving equivalence relations). They are defined in §6. Finally in §8, we obtain an explicit description of all elements of \( \text{Cov}(v) \) in the case that \( R \) is a field, in which case \( v \) is a Krull valuation.

If \( R \) is only a ring, our results are far less complete. Nevertheless it seems to be absolutely necessary to work at least in this generality for many reasons, in particular functorial ones, cf. e.g. [IK], [KZ].

In §9 we delve deeper into the supertropical theory to pinpoint a relation, which we call the ghost surpassing relation, which seems to be a key for working in supertropical semirings. On the one hand, the ghost surpassing relation restricts to equality on tangible elements, thereby enabling us to specialize to the max-plus theory. On the other hand, the ghost surpassing relation appears in virtually every supertropical theorem proved so far, especially in supertropical matrix theory in [IR2] and [IR3].

In the present paper the ghost surpassing relation is the essential gadget for understanding and proving a general version of Kapranov’s Lemma in §11 (Theorem 11.15, preceded by Theorem 9.11), valid for any valuation \( v : R \to M \) which is “strong”. This means that \( v(a+b) = \max(v(a), v(b)) \) whenever \( v(a) \neq v(b) \). If \( R \) is a ring, every valuation on \( R \) is strong, as is very well known, but if \( R \) is only a semiring, this is a restrictive condition. On our way to Kapranov’s Lemma we employ supervaluations \( \varphi \in \text{Cov}(v) \) which are tangible, i.e., have only tangible values, and are tangibly additive, which means that \( \varphi(a+b) = \varphi(a) + \varphi(b) \) whenever \( \varphi(a) + \varphi(b) \) is tangible. We apostrophize tangibly additive supervaluations which cover strong \( m \)-valuations as strong supervaluations.

The strong tangible supervaluations in \( \text{Cov}(v) \) seem to be the most suitable ones for applications in tropical geometry also beyond Kapranov’s Lemma, as to be explained at the end of this introduction. They form a sublattice \( \text{Cov}_{\text{t,a}}(v) \) of \( \text{Cov}(v) \). In particular there exists an “initial” tangible strong valuation in \( \text{Cov}(v) \), denoted by \( \varphi_v \), which dominates all others. It gives the “best” supertropical version of Kapranov’s Lemma, cf. §9. At the end of §10 we make \( \varphi_v \) explicit in the case that \( v \) is the natural valuation of the field of formal Puiseux series in a variable \( t \) (with real or with rational exponents). We can interpret the value of \( \varphi_v(a(t)) \) of a Puiseux series \( a(t) \) as the leading term of \( a(t) \), while \( v(a(t)) \) can be seen as the \( t \)-power contained in the leading term.

Strictly speaking, Kapranov’s Lemma extends the valuation \( v \) to the polynomial ring \( R[\lambda_1, \ldots, \lambda_n] \) over \( R \), with target in the polynomial ring \( M[\lambda_1, \ldots, \lambda_n] \), which no longer is bipotent. Thus, the theory in this paper needs to be generalized if we are to deal formally with such notions. This is set forth in the last Section 11 in which the target of a valuation is replaced by a monoid with a binary sup operation.
Since the theory of tropicalization has developed recently in terms of the valuations on the field of Puiseux series, let us indicate briefly how this theory can be extended to the supertropical environment.

We recall the algebraic theory of analytification, as presented by Payne [12]. A multiplicative seminorm \( | | : R \to \mathbb{R}_{\geq 0} \) on a ring \( R \) is a multiplicative map satisfying the triangle inequality

\[ |a + b| \leq |a| + |b| \]

for all \( a, b \in R \). In particular, any \( m \)-valuation \( v : R \to \mathbb{R}_{\geq 0} \) is a seminorm. \( \{ \text{Recall that we use multiplicative notation.} \} \) If \( X \) is an affine variety over a field \( K \) (e.g. \( K \) is a field of generalized Puiseux series over an algebraically closed field) and \( v : K \to \mathbb{R}_{\geq 0} \) is a valuation, then Payne’s space \( K[X]^{an} \) is the set of all multiplicative seminorms on \( K[X] \) that extend \( v \). More generally, if \( v : K \to M \) is a valuation with \( M = G \cup \{ 0 \} \) any bipotent semifield (cf. §1), then we may define a space \( K[X]^{an} \) associated to \((K, v)\) and \( X \) in exactly the same way. But in the supertropical context we can do more. Let

\[ U := D(G) = \text{STR}(G, G, \text{id}_G), \]

as defined in Example [3,18]. This is a supertropical semifield with ghost part \( G \cup \{ 0 \} = M \).

We define a space \( K[X]^{super-an} \) as the set of all strong supervaluations \( \varphi : K[X] \to U \) such that the valuation \( w : K[X] \to M \) covered by \( \varphi \) (cf. Definition [1.1]) is an element of \( K[X]^{an} \), i.e., \( w \) extends \( v \).

We have the natural map \( K[X]^{super-an} \to K[X]^{an} \), given by \( \varphi \mapsto w \), which exhibits \( K[X]^{super-an} \) as a “covering” of Payne’s space \( K[X]^{an} \). But there is still another relation between these two spaces, which seems to be more intriguing. The set \( U \) contains a second copy of \( M \) as a multiplicative submonoid, namely the tangible part \( T(U) \cup \{ 0 \} \cong G \cup \{ 0 \} \). Interpreting the elements of \( K[X]^{an} \) as maps from \( K[X] \) to \( T(U) \cup \{ 0 \} \) we may view \( K[X]^{an} \) as the set of all tangible supervaluations \( \varphi : K[X] \to U \) (automatically strong) with \( \varphi | K \) covering \( v \). Thus \( K[X]^{an} \) can be seen as the subspace of \( K[X]^{super-an} \) consisting of the \( \varphi \in K[X]^{super-an} \) which do not have ghost values.

Of course, nothing can prevent us from replacing \( K \) by any ring, or even semiring \( R \), and \( v \) by any strong \( m \)-valuation on \( R \), and defining \( K[X]^{an} \) and \( K[X]^{super-an} \) in this generality.

The reader may ask whether valuations and supervaluations on semirings instead of just rings deserve interest apart from formal issues. They do. It is only for not making a long paper even longer that we do not give applications to semirings here. The semiring \( R = \sum A^2 \) of sum of squares of a commutative ring (or even a field) \( A \) with \(-1 \notin R \) is a case in point. Real algebra seems to be a fertile ground for studying valuations and supervaluations on semirings. The paper contains only one very small hint pointing in this direction, Example [2,4]

1. Bipotent semirings

Let \( R \) be a semiring (always with unit element \( 1 = 1_R \)). Later we will assume that \( R \) is commutative, but presently this is not necessary.

**Definition 1.1.** We call a pair \((a, b) \in R^2\) bipotent if \( a + b \in \{a, b\} \). We call the semiring \( R \) bipotent if every pair \((a, b) \in R^2\) is bipotent.

**Proposition 1.2.** Assume that \( R \) is a bipotent semiring. Then the binary relation \((a, b \in R)\)

\[ a \leq b \quad \text{iff} \quad a + b = b \quad (1.1) \]
on $R$ is a total ordering on the set $R$, compatible with addition and multiplication, i.e., for all $a, b, c \in R$

\[
    a \leq b \implies ac \leq bc, \quad ca \leq cb,
    \]

\[
    a \leq b \implies a + c \leq b + c.
    \]

Proof. A straightforward check. \qed

Remark 1.3. We can define such a binary relation $\leq$ by (1.1) in any semiring, and then obtain a partial ordering compatible with addition and multiplication. The ordering is total iff $R$ is bipotent. Clearly, $0_R \leq x$ for every $x \in R$.

Definition 1.4. We call a semiring $R$ a semidomain, if $R$ has no zero divisors, i.e., the set $R \setminus \{0\}$ is closed under multiplication. We call $R$ a semifield, if $R$ is commutative and every element $x \neq 0$ of $R$ is invertible; hence $R \setminus \{0\}$ is a group under multiplication.

Given a bipotent semidomain $R$, the set $G := R \setminus \{0\}$ is a totally ordered monoid under the multiplication of $R$.

In this way we obtain all (totally) ordered monoids. Indeed, if $G = (G, \cdot)$ is a given ordered monoid, we gain a bipotent semiring $R$ as follows: Adjoin a new element 0 to $G$ and form the set $R := G \cup \{0\}$. Extend the multiplication on $G$ to a multiplication on $R$ by the rules $0 \cdot g = g \cdot 0 = 0$ for any $g \in G$ and $0 \cdot 0 = 0$. Extend the ordering of $G$ to a total ordering on $R$ by the rule $0 < g$ for $g \in G$. Then define an addition on $R$ by the rule

\[
    x + y := \max(x, y)
    \]

for any $x, y \in R$. It is easily checked that $R$ is a bipotent semiring, and that the ordering on $R$ by the rule (1.1) is the given one. We denote this semiring $R$ by $T(G)$.

These considerations can be easily amplified to the following theorem.

Theorem 1.5. The category of (totally) ordered monoids $G$ is isomorphic\textsuperscript{3} to the category of bipotent semidomains $R$ by the assignments

\[
    G \mapsto T(G), \quad R \mapsto R \setminus \{0\}.
    \]

Here the morphisms in the first category by definition are the order preserving monoid homomorphisms $\gamma : G \rightarrow G'$ in the weak sense; i.e., $\gamma$ is multiplicative, $\gamma(1) = 1$, and $x \leq y \implies \gamma(x) \leq \gamma(y)$, while the morphisms in the second category are the semiring homomorphisms (with $1 \mapsto 1$).

In the following we regard an ordered monoid and the associated bipotent semiring as the same entity in a different disguise. Usually we prefer the semiring viewpoint.

Example 1.6. Starting with the monoid $G = (\mathbb{R}, +)$, i.e., the field of real numbers with the usual addition, we obtain a bipotent semifield

\[
    T(\mathbb{R}) := \mathbb{R} \cup \{-\infty\},
    \]

where addition $\oplus$ and multiplication $\odot$ of $T(\mathbb{R})$ are defined as follows, and the neutral element of addition is denoted by $-\infty$ instead of 0, since our monoid is now given in additive notation.

\textsuperscript{3}This is more than equivalent!
For $x, y \in \mathbb{R}$

\[
\begin{align*}
x \oplus y &= \max(x, y), \\
x \odot y &= x + y, \\
(-\infty) \oplus x &= x \oplus (-\infty) = x, \\
(-\infty) \odot x &= x \odot (-\infty) = -\infty, \\
(-\infty) \oplus (-\infty) &= -\infty, \\
(-\infty) \odot (-\infty) &= -\infty.
\end{align*}
\]

$T(\mathbb{R})$ is the “real tropical semifield” of common tropical algebra, often called the “max-plus” algebra $\mathbb{R} \cup \{-\infty\}$: cf. [IMS], or [SS] (there a “min-plus” algebra is used).

2. m-valuations

In this section we assume that all occurring semirings and monoids are commutative. Let $R$ be a semiring.

**Definition 2.1.** An m-valuation ($=\text{monoid valuation}$) on $R$ is a map $v : R \to M$ into a (commutative) bipotent semiring $M \neq \{0\}$ with the following properties:

\begin{align*}
V1 & : v(0) = 0, \\
V2 & : v(1) = 1, \\
V3 & : v(xy) = v(x)v(y) \quad \forall x, y \in R, \\
V4 & : v(x + y) \leq v(x) + v(y) \quad [= \max(v(x), v(y))] \quad \forall x, y \in R.
\end{align*}

We call the m-valuation $v$ strict, if instead of $V4$ the following stronger axiom holds:

\begin{align*}
V5 & : v(x + y) = v(x) + v(y) \quad \forall x, y \in R.
\end{align*}

Note that a strict m-valuation $v : R \to M$ is just a semiring homomorphism from $R$ to $M$.

In the special case that $M = \Gamma \cup \{0\}$ with $\Gamma$ an ordered abelian group, we call the m-valuation $v : R \to M$ a valuation. Notice that in the case that $R$ is a ring (instead of a semiring), this is exactly the notion of a valuation as defined by Bourbaki [B] (Alg. Comm. VI, §3, No.1) and studied, e.g., in [HK] and [KZ, Chap. I], except that for $\Gamma$ we have chosen the multiplicative notation instead of the additive notation.

If $v : R \to M$ is an m-valuation, we may replace $M$ by the submonoid $v(R)$. We then speak of $v$ as a surjective m-valuation.

**Definition 2.2.** A (commutative) monoid $G$ is called cancellative, if, for any $a, b, c \in G$, the equation $ac = bc$ implies $a = b$.

Notice that an ordered monoid $G$ is cancellative iff $a < b$ implies $ac < bc$ for any $a, b, c \in G$. An ordered cancellative monoid can be embedded into an ordered abelian group $\Gamma$ in the well-known way by introducing formal fractions $\frac{a}{b}$ for $a, b \in G$. Then an m-valuation $v$ from $R$ to $T(G) = G \cup \{0\}$ is essentially the same thing as an m-valuation from $R$ to $\Gamma \cup \{0\}$. For this reason, we extend the notion of “valuation” from above as follows.

**Definition 2.3.** A valuation on a semiring $R$ is an m-valuation $v : R \to G \cup \{0\}$ with $G$ a cancellative monoid.
Proposition 2.6. 

\[ \text{m-valuations on rings have been studied in } [\text{H}], \text{ and then by D. Zhang } [\text{Z}]. \]

If \( R \) is a ring, an \( m \)-valuation \( v : R \to M \) can **never** be strict, since we have an element \(-1 \in R \) with \( 1 + (-1) = 0 \), from which for \( v \) strict it would follow that \( 0_M = v(0) = \max(v(1), v(-1)) \); hence \( v(1) = 0_M \), a contradiction to axiom V2. But for \( R \) a semiring there may exist interesting strict \( m \)-valuations, even with values in a group.

**Example 2.4.** Let \( T \) be a **preprime** in a ring \( R \), by which we simply mean that \( T \) is a sub-semiring of \( R \) \( (T + T \subset T, T \cdot T \subset T, 0 \in T, 1 \in T) \). \{We do not exclude the case \(-1 \in T \) (“improper preprime”) but these will not matter.\}

We say that a valuation \( v : R \to M \) is **\( T \)-convex** if the restriction \( v|T : T \to M \) is strict. As is well-known, if \( T = \sum R^1 \) (and \( M \setminus \{0\} \) is a group) the \( T \)-convex valuations are just the real valuations on \( R \). (A valuation \( v : R \to \Gamma \cup \{0\} \) is called **real** if the residue class field \( k(v) \) is formally real.) See \([\text{KZ1}], \S5 \) for \( T \) a preordering, and \( \S2 \) for \( T = \sum R^2 \).

The entire paper \([\text{KZ1}]\) witnesses the importance of \( T \)-convex valuations for \( T \) a preordering.

If \( R \) is a ring, every \( m \)-valuation on \( R \) is strong. This can be seen by the same argument as is well-known for valuations on fields.

Semirings, even semifields, may admit valuations which are not strong.

**Example 2.5.** Let \( F \) be a totally ordered field, and \( R := \{ x \in F | x \geq 0 \} \) the subsemifield of nonnegative elements. Further let \( \Gamma := \{ x \in F | x > 0\} \), viewed as a totally ordered group, and \( M := \{0\} \cup \Gamma \) the associated bipotent semifield. The map \( v : R \to M \) with \( v(0) = 0 \), \( v(a) = \frac{1}{a} \) for \( a \neq 0 \), is a valuation on \( R \), which is not strong.

**Proposition 2.6.**

a) If \( v : R \to M \) is an \( m \)-valuation and \( M \) is a bipotent semidomain, then \( v^{-1}(0) \) is a prime ideal of \( R \) (i.e., an ideal of \( R \), whose complement in \( R \) is closed under multiplication).

b) If \( v \) is strong, then, for any \( x \in R \) and \( z \in v^{-1}(0) \),

\[ v(x + z) = v(x). \]

(2.1)

**Proof.** a): If \( v(x) = 0, v(y) = 0 \), then

\[ v(x + y) \leq \max(v(x), v(y)) = 0; \]

hence \( v(x + y) = 0 \). Thus \( v^{-1}(0) \) is closed under addition. If \( x \in R, z \in v^{-1}(0) \), then \( v(xz) = v(x)v(z) = 0 \). Thus \( v^{-1}(0) \) is closed under multiplication by elements in \( R \). If \( v(x) > 0, v(z) > 0 \), then \( v(xz) = v(x)v(z) > 0 \). Thus \( R \setminus v^{-1}(0) \) is closed under multiplication.

b): We have \( v(x + z) \leq \max(v(x), v(z)) = v(x) \). Assume that \( v \) is strong. If \( v(x) \neq 0 \) we have

\[ v(x + z) = \max(v(x), v(z)) = v(x). \]

\( \square \)

If \( v : R \to M \) is an arbitrary \( m \)-valuation, then it is still obvious that \( v^{-1}(0) \) is an ideal of \( R \).

**Definition 2.7.** We call the ideal \( v^{-1}(0) \) the **support** of the \( m \)-valuation \( v \), and write \( v^{-1}(0) = \text{supp}(v) \). We call the support of \( v \) **insensitive**, if the equality (2.1) above holds for any \( x \in R \) and \( z \in \text{supp}(v) \), **sensitive** otherwise.

Proposition 2.6b tells us that \( \text{supp}(v) \) is insensitive if \( v \) is strong. In particular, this holds if \( R \) is a ring.
Example 2.8. Let $\Gamma$ be an ordered abelian group and $H$ is a convex proper subgroup. Let $a := \{g \in \Gamma \mid g > H\} \cup \{0\}$. We regard $\Gamma \cup \{0\}$ as a bipotent semifield (cf. §1), and define a subsemiring $M$ of $\Gamma \cup \{0\}$ by

$$M := H \cup a.$$ 

Notice that we have $H \cdot a \subset a$, $a \cdot a \subset a$, and $a + a \subset a$. Thus $M$ is indeed a subsemiring of $\Gamma \cup \{0\}$, and $a$ is an ideal of $M$. We define a map $v : M \to H \cup \{0\}$ by setting $v(x) = x$ if $x \in H$, and $v(x) = 0$ if $x \in a$. It is easily checked that $v$ fulfills the axioms V1–V3 and moreover has the following “bipotency”:

If $a, b \in M$ and $v(a) \neq v(b)$, then $v(a + b) \in \{v(a), v(b)\}$.

But the support $a$ of $v$ is sensitive: For $x \in H$, $z \in a$ and $z \neq 0$, we have $v(x) > 0$, $v(z) = 0$, $x + z = z$; hence $v(x + z) = 0$.

We switch over to the problem of “comparing” different $m$-valuations on the same semiring $R$.

Definition 2.9. Let $v : R \to M$ and $w : R \to N$ be $m$-valuations.

a) We say that $v$ dominates $w$, if for any $a, b \in R$

$$v(a) \leq v(b) \Rightarrow w(a) \leq w(b).$$

b) If $v$ dominates $w$ and $v$ is surjective, there clearly exists a unique map $\gamma : M \to N$ with $w = \gamma \circ v$. We denote this map $\gamma$ by $\gamma_{w,v}$.

Clearly, $\gamma_{w,v}$ is multiplicative and sends 0 to 0 and 1 to 1. $\gamma_{w,v}$ is also order-preserving and hence is a homomorphism from the bipotent semiring $M$ to $N$.

Proposition 2.10. Assume that $M, N$ are bipotent semirings and $v : R \to M$ is a surjective $m$-valuation.

a) The $m$-valuations $w : R \to N$ dominated by $v$ correspond uniquely with the homomorphisms $\gamma : M \to N$ via $w = \gamma \circ v$, $\gamma = \gamma_{w,v}$.

b) If $v$ has one of the properties “strict”, or “strong”, and dominates $w$, then $w$ has the same property.

Proof. If $w$ is an $m$-valuation dominated by $v$ then we know already that $\gamma := \gamma_{w,v}$ is a homomorphism and $w = \gamma \circ v$. On the other hand, given a homomorphism $\gamma : M \to N$, clearly $\gamma \circ v$ is an $m$-valuation, and $\gamma \circ v$ inherits from $v$ each of the properties “strict” and “strong”. □

We mention that for strong $m$-valuations the dominance condition in Definition 2.9 can be weakened.

Proposition 2.11. Assume that $v : R \to M$ and $w : R \to N$ are strong $m$-valuations and that

$$\forall a, b \in R : \quad v(a) = v(b) \Rightarrow w(a) = w(b).$$

Then $v$ dominates $w$.

Proof. Let $a, b \in R$ and assume that $v(a) \leq v(b)$. If $v(a) < v(b)$ then $v(a + b) = v(b)$, hence $w(a + b) = w(b)$. It follows that $w(a) \leq w(b)$ since $w(a) > w(b)$ would imply $w(a + b) = w(a)$. Thus $w(a) \leq w(b)$ in both cases. □
3. Supertropical semirings

**Definition 3.1.** A **semiring with idempotent** is a pair \((R, e)\) consisting of a semiring \(R\) and a **central** idempotent \(e\). {For the moment \(R\) is allowed to be noncommutative.}

We then have an endomorphism \(\nu : R \to R\) (which usually does not map 1 to 1) defined by \(\nu(a) = ea\). It obeys the rules

\[
\begin{align*}
\nu \circ \nu &= \nu, \\
(\nu(a)b) &= \nu(\nu(a)b).
\end{align*}
\]

Conversely, if a pair \((R, \nu)\) is given consisting of a semiring \(R\) and an endomorphism \(\nu\) (not necessarily \(\nu(1) = 1\)), such that (3.1), (3.2) hold, then \(e := \nu(1)\) is a central idempotent of \(R\) and \(\nu(a) = ea\) for every \(a \in R\).

Thus such pairs \((R, \nu)\) are the same objects as semirings with idempotents.

**Definition 3.2.** A **semiring with ghosts** is a semiring with idempotent \((R, e)\) such that the following axiom holds (\(\nu(a) := ea\))

\[
\nu(a) = \nu(b) \implies a + b = \nu(a).
\]

**Remark 3.3.** This axiom implies that \(ea = e(a + b) = ea + eb\) if \(\nu(a) = \nu(b)\). We do not want to demand that then \(eb = 0\). Usually, \((R, +)\) will be a highly non-cancellative abelian semigroup.

**Terminology 3.4.** If \((R, e)\) is a semiring with ghosts, then \(\nu : x \mapsto ex, R \to R\) is called the **ghost map** of \((R, e)\). The idea is that every \(x \in R\) has an associated “ghost” \(\nu(x)\), which is thought of to be somehow “near” to the zero element 0 of \(R\), without necessarily being 0 itself. {That will happen for all \(x \in R\) only if \(e = 0\).} We call \(eR\) the **ghost ideal** of \((R, e)\).

Now observe that, if \((R, e)\) is a semiring with ghosts, the idempotent \(e\) is determined by the semiring \(R\) above, namely

\[
e = 1 + 1.
\]

Thus we may suppress the idempotent \(e\) in the notation of a semiring with ghosts and redefine these objects as follows.

**Definition 3.5.** A semiring \(R\) is called a **semiring with ghosts** if

\[
1 + 1 = 1 + 1 + 1 + 1
\]

and for all \(a, b \in R\)

\[
a + a = b + b \implies a + b = a + a.
\]

**Remark 3.6.** If (3.3) holds then \(e := 1 + 1\) is a central idempotent of \(R\). Passing from \(R\) to \((R, e) = (R, 1 + 1)\), we see that (3.3) is the previous axiom (3.3). Notice also that (3.3) implies that \(1 + 1 + 1 = 1 + 1\). (Take \(a = 1, b = e\).) Thus, \(m1 = 1 + 1\) for all natural numbers \(m \geq 2\).

**Terminology 3.7.** If \(R\) is a semiring with ghosts, we write \(e = e_R\) and \(\nu = \nu_R\) if necessary. We also introduce the notation

\[
T := T(R) := R \setminus Re,
\]

\[
G := G(R) := Re \setminus \{0\},
\]

\[
G_0 := G \cup \{0\} = Re.
\]
We call the elements of \( T \) the \textbf{tangible elements} of \( R \) and the elements of \( \mathcal{G} \) the \textbf{ghost elements} of \( R \). We do not exclude the case that \( T \) is empty, i.e., \( e = 1 \). In this case \( R \) is called a \textbf{ghost semiring}.

The ghost ideal \( \mathcal{G}_0 = eR \) of \( R \) is itself a semiring with ghosts, in fact, a ghost semiring. It has the property \( a + a = a \) for every \( a \in Re \), as follows from (3.3). \{Some people call a semiring \( T \) with \( a + a = a \) for every \( a \in T \) an “idempotent semiring”\}.

We mention a consequence of axiom (3.3) for the ghost map \( \nu : R \to Re \), \( \nu(x) := ex \).

\textbf{Remark 3.8.} If \( R \) is a semiring with ghosts, then, for any \( x \in R \),

\[ \nu(x) = 0 \iff x = 0. \]

\textbf{Proof.} (\( \Leftarrow \)): evident.

(\( \Rightarrow \)): We have \( \nu(x) = 0 = \nu(0) \); hence by (3.3) \( x + x = \nu(0) = 0 \).

We are ready for the central definition of the section.

\textbf{Definition 3.9.} A semiring \( R \) is called \textbf{supertropical} if \( R \) is a semiring with ghosts and

\[ \forall a, b \in R : a + a \neq b + b \implies a + b \in \{a, b\}. \quad (3.4) \]

In other terms, every pair \( (a, b) \) in \( R \) with \( ea \neq eb \) is bipotent.

\textbf{Remarks 3.10.}

(i) It follows that then \( \mathcal{G}(R)_0 = Re \) is a bipotent semidomain. Indeed, if \( a, b \) are different elements of \( \mathcal{G}(R) \), then \( a = ea \neq b = eb \); hence \( a + b \in \{a, b\} \) by axiom (3.4). If \( a = 0 \) or \( b = 0 \), this trivially is also true. If \( a = b \) then \( a + b = ea = a \). Thus \( a + b \in \{a, b\} \) for any \( a, b \in \mathcal{G}(R)_0 \). The set \( \mathcal{G}(R) \) is either empty (the case \( 1 + 1 = 0 \) or \( \mathcal{G}(R) \) is an ordered monoid under the multiplication of \( R \), as explained in \S 1.

(ii) The supertropical semirings without tangible elements are just the bipotent semirings.

(iii) Every subsemiring of a supertropical semiring is again supertropical.

\textbf{Theorem 3.11.} Let \( R \) be a supertropical semiring, \( e := eR \), \( \mathcal{G} := \mathcal{G}(R) \). Then the addition on \( R \) is determined by the multiplication on \( R \) and the ordering on the multiplicative submonoid \( \mathcal{G} \) of \( R \), in case \( \mathcal{G} \neq \emptyset \), as follows. For any \( a, b \in R \)

\[ a + b = \begin{cases} a & \text{if } ea > eb, \\ b & \text{if } ea < eb, \\ ea & \text{if } ea = eb, \end{cases} \]

If \( \mathcal{G} = \emptyset \) then \( a + b = 0 \) for any \( a, b \in R \).

\textbf{Proof.} We may assume that \( ea \geq eb \). If \( ea = eb \), axiom (3.3) tells us that \( a + b = ea \). Assume now that \( ea > eb \). By definition of the ordering on \( eR \) (cf. \S 1), we have

\[ e(a + b) = ea + eb = ea. \]

By axiom (3.4), \( a + b = a \) or \( a + b = b \).

Suppose that \( a + b = b \). Then \( e(a + b) = eb \). Since \( ea \neq eb \), this is a contradiction. We conclude that \( a + b = a \).

From now on, we always assume that our semirings are commutative.

\textbf{Remark 3.12.} If \( R \) is a supertropical semiring, the ghost map \( \nu_R : R \to eR \), \( x \mapsto ex \) is a strict \( m \)-valuation. Indeed, the axioms V1–V3 and V5 from \S 2 are clearly valid for \( \nu_R \).
Thus, every supertropical semiring has a natural built-in strict $m$-valuation.

There are important cases where $\nu_R$ is even a valuation (cf. Definition 2.3), as we explicate now.

**Proposition 3.13.** Assume that $R$ is a supertropical semiring and $T(R)$ is closed under multiplication. Then the submonoid $G := eT(R)$ of $G(R)$ is cancellative. (N.B. We have $eT(R) \subset G(R)$ by Remark 3.8.)

**Proof.** Let $a, b, c \in T(R)$ be given with $(ea)(ec) = (eb)(ec)$, i.e., $eac = ebc$. Suppose that $ea \neq eb$, say $ea < eb$. Then Theorem 3.11 tells us that $a + b = b$ and $ac + bc = ebc$. By assumption, $bc \in T(R)$; hence $bc \neq ebc$. But the first equation gives $ac + bc = bc$, a contradiction. Thus $ea = eb$. □

In the situation of this proposition we may omit the part $G(R) \setminus G$, consisting of “useless” ghosts, in the semiring $R$, and then obtain a “supertropical domain” $U := T(R) \cup G \cup \{0\}$, as defined below, whose ghost map $\nu_U := U \rightarrow G \cup \{0\}$ is a surjective strict valuation.

**Definition 3.14.** Let $M$ be a bipotent semiring and $R$ a supertropical semiring.

a) We say that the semiring $M$ is cancellative if for any $x, y, z \in M$

$$xz = yz, \ z \neq 0 \Rightarrow x = y.$$  

This means that $M$ is a bipotent semidomain (cf. Definition 1.4) and the multiplicative monoid $M \setminus \{0\}$ is cancellative.

b) We call $R$ a supertropical predomain, if $T(R) = R \setminus eR$ is not empty (i.e., $e \neq 1$) and is closed under multiplication, and moreover $eR$ is a cancellative bipotent semidomain.

c) We call $R$ a supertropical domain, if $T(R)$ is not empty and is closed under multiplication, and $R$ maps $T(R)$ onto $G(R)$.

Notice that the last condition in Definition 3.14.c implies that $G(R)$ is a cancellative monoid (Proposition 3.13). Thus a supertropical domain is a supertropical predomain.

Looking again at Theorem 3.11, we see that a way is opened up to construct supertropical predomains and domains. First notice that the theorem implies the following

**Remark 3.15.** If $R$ is a supertropical predomain, we have for every $a \in T(R)$ and $x \in G(R)$ the multiplication rule

$$ax = v(a)x$$

with $v := \nu_R | T(R)$. Thus the multiplication on

$$R = T(R) \cup G(R) \cup \{0\}$$

is completely determined by the triple $(T(R), G(R), v)$. We write $v = v_R$.

**Construction 3.16.** Conversely, let a triple $(T, G, v)$ be given with $T$ a monoid, $G$ an ordered cancellative monoid and $v : T \rightarrow G$ a monoid homomorphism. We define a semiring $R$ as follows. As a set

$$R = T \cup G \cup \{0\}.$$  

The multiplication on $R$ will extend the given multiplications on $T$ and $G$. If $a \in T$, $x \in G$, we decree that

$$a \cdot x = x \cdot a := v(a)x.$$  

Finally, $0 \cdot z = z \cdot 0 = 0$ for all $z \in R$. 
The addition on $R$ extends the addition on $G \cup \{0\}$ as the bipotent semiring corresponding to the ordered monoid $G$, as explained in §1. For $x, y \in T$ we decree

$$x + y := \begin{cases} 
  x & \text{if } v(x) > v(y), \\
  y & \text{if } v(x) < v(y), \\
  v(x) & \text{if } v(x) = v(y).
\end{cases}$$

Finally, for $x \in T$ and $y \in G \cup \{0\}$

$$x + y = y + x := \begin{cases} 
  x & \text{if } v(x) > y, \\
  y & \text{if } v(x) \leq y.
\end{cases}$$

It now can be checked in a straightforward way\(^4\) that $R$ is a supertropical predomain with $T(R) = T$, $G(R) = G$, $v_R = v$. Thus we have gained a description of all supertropical predomains $R$ by triples $(T, G, v)$ as above. We write

$$R = \text{STR}(T, G, v)$$

\{\text{STR = “supertropical”}\}. Notice that in this semiring $R$ every pair $(x, y) \in R^2$ is bipotent except the pairs $(a, b)$ with $a \in T$, $b \in T$ and $v(a) = v(b)$. If $v$ is onto, then $R$ is a supertropical domain.

**Definition 3.17.** A semiring $R$ is called a supertropical semifield, if $R$ is a supertropical domain, and every $x \in T(R)$ is invertible; hence both $T(R)$ and $G(R)$ are groups under multiplication.

We write down primordial examples of supertropical domains and semifields (cf. [I], [IR1]). Other examples will come up in §4.

**Examples 3.18.** Let $G$ be an ordered cancellative monoid. This given us the supertropical domain (cf. Construction 3.16)

$$D(G) := \text{STR}(G, G, \text{id}_G).$$

$D(G)$ is a supertropical semifield iff $G$ is an ordered abelian group.

We come closer to the objects and notations of usual tropical algebra if we take here for $G$ ordered monoids in additive notation, $G = (G, +)$, e.g., $G = \mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual addition. $D(G)$ contains the set $G$. For every $a \in G$ there is an element $a^\nu$ in $D(G)$ (read “$a$-ghost”), and

$$G^\nu := \{a^\nu \mid a \in G\}$$

is a copy of the additive monoid $G$ disjoint from $G$. The zero element of the semiring $D(G)$ is now written $-\infty$. Thus

$$D(G) = G \cup G^\nu \cup \{-\infty\}.$$
Denoting addition and multiplication of the semiring $D(G)$ by $\oplus$ and $\odot$, we have the following rules. For any $x \in D(G)$, $a \in G$, $b \in G$,

\begin{align*}
-\infty \oplus x &= x \oplus -\infty = x, \\
a \oplus b &= \max(a, b), \text{ if } a \neq b, \\
a \oplus a &= a^\vee, \\
a^\vee \oplus b^\vee &= \max(a, b)^\vee, \\
a \odot b &= a, \text{ if } a > b, \\
a \odot b^\vee &= b^\vee, \text{ if } a \leq b, \\
-\infty \odot x &= x \odot -\infty = -\infty, \\
a \odot b &= a + b, \\
a^\vee \odot b^\vee &= a^\vee \odot b^\vee = (a + b)^\vee.
\end{align*}

In the case $G = (\mathbb{R}, +)$ these rules can already be found in [I]. There also motivation is given for their use in tropical algebra and tropical geometry.

We now only say that the semiring $D(G)$ associated to an additive ordered cancellative monoid $G$ should be compared with the max-plus algebra $T(G) = G \cup \{-\infty\}$ introduced in §1. The ghost ideal $G^\nu \cup \{-\infty\}$ of $D(G)$ is a copy of $T(G)$.

4. Supervaluations

In this section $R$ is always a (commutative) semiring. Usually the letters $U, V$ denote supertropical (commutative) semirings. If $U$ is any such semiring, the idempotent $e_U = 1_U + 1_U$ will be often simply denoted by the letter “$e$”, regardless of which supertropical semiring is under consideration (as we write $0_U = 0$, $1_U = 1$).

Definition 4.1.

a) A supervaluation on $R$ is a map $\varphi : R \to U$ from $R$ to a supertropical semiring $U$ with the following properties.

\begin{align*}
SV1 : \varphi(0) &= 0, \\
SV2 : \varphi(1) &= 1, \\
SV3 : \forall a, b \in R : \varphi(ab) &= \varphi(a)\varphi(b), \\
SV4 : \forall a, b \in R : e\varphi(a + b) &\leq e(\varphi(a) + \varphi(b)) \quad [= \max(e\varphi(a), e\varphi(b))].
\end{align*}

b) If $\varphi : R \to U$ is a supervaluation, then the map

$$v : R \to eU, \quad v(a) := e\varphi(a)$$

is clearly an $m$-valuation. We denote this $m$-valuation $v$ by $e_U\varphi$ (or simply by $e\varphi$), and we say that $\varphi$ covers the $m$-valuation $e_U\varphi = v$.

c) We say that a supervaluation $\varphi : R \to U$ is tangible, if $\varphi(R) \subset T(U) \cup \{0\}$, and we say that $\varphi$ is ghost if $\varphi(R) \subset eU$.

N.B. A ghost supervaluation $\varphi : R \to U$ is nothing other than an $m$-valuation, after replacing the target $U$ by $eU$.

Proposition 4.2. Assume that $\varphi : R \to U$ is a supervaluation and $v : R \to e_UU =: M$ is the $m$-valuation $e_U\varphi$ covered by $\varphi$. Then

$$U' := \varphi(R) \cup e\varphi(R)$$
is a subsemiring of $U$. The semiring $U'$ is again supertropical and $e_{U'} = e_U (= e)$.

**Proof.** The set $v(R)$ is a multiplicative submonoid of the bipotent semiring $M$; hence is itself a bipotent semiring. In particular, $v(R)$ is closed under addition. If $a, b \in R$ are given with $v(a) \leq v(b)$, then either $v(a) < v(b)$, in which case

$$a + b = b, \quad v(a) + b = b, \quad a + v(b) = v(b),$$

or $v(a) = v(b)$, in which case

$$a + b = v(a) + b = a + v(b) = v(a).$$

This proves that $U' + U' \subseteq U'$. Clearly $0 \in U'$, $1 \in U'$ and $U' \cdot U' \subseteq U'$. Thus $U'$ is a subsemiring of $U$. As stated above (Remark 3.10.iii), every subsemiring of a supertropical semiring is again supertropical. Thus $U'$ is supertropical. \(\square\)

**Definition 4.3.** We say that the supervaluation $\varphi : R \to U$ is surjective if $U' = U$. We say that $\varphi$ is tangibly surjective if $\varphi(R) \supseteq T(U)$.

**Remark 4.4.** If $\varphi : R \to U$ is any supervaluation, then, replacing $U$ by $U' = \varphi(R) \cup e\varphi(R)$, we obtain a surjective supervaluation. If we only replace $U$ by $\varphi(R) \cup (eU)$, which is again a subsemiring of $U$, we obtain a tangibly surjective supervaluation.

Thus, whenever necessary we may retreat to tangibly surjective or even surjective supervaluations without loss of generality.

Recall that an $m$-valuation $v : R \to M$ is called a valuation, if the bipotent semiring $M$ is cancellative (cf. Definition 2.3, Definition 3.14.a). Every valuation can be covered by a tangible supervaluation, as the following easy but important construction shows.

**Example 4.5.** Let $v : R \to M$ be a valuation, and let $q := v^{-1}(0)$ denote the support of $v$. We then have a monoid homomorphism

$$R \setminus q \to M \setminus \{0\}, \quad a \mapsto v(a),$$

which we denote again by $v$. Let

$$U := \text{STR}(R \setminus q, M \setminus \{0\}, v),$$

the supertropical predomain given by the triple $(R \setminus q, M \setminus \{0\}, v)$, as explained in Construction 3.10. Thus, as a set,

$$U = (R \setminus q) \cup M.$$

We have $e = 1_M$, $e \cdot a = v(a)$ for $a \in R \setminus q$. The multiplication on $U$ restricts to the given multiplications on $R \setminus q$ and on $M$, and $a \cdot x = x \cdot a = v(a)x$ for $a \in R \setminus q, x \in M$. The addition on $U$ is determined by $e$ and the multiplication in the usual way (cf. Theorem 3.11). In particular, for $a, b \in R \setminus q$, we have

$$a + b = \begin{cases} a & \text{if } v(a) > v(b), \\ b & \text{if } v(a) < v(b), \\ v(a) & \text{if } v(a) = v(b). \end{cases}$$

Now define a map $\varphi : R \to U$ by

$$\varphi(a) := \begin{cases} a & \text{if } a \in R \setminus q, \\ 0 & \text{if } a \in q. \end{cases}$$

One checks immediately that $\varphi$ obeys the rules SV1–SV3. If $a \in R \setminus q$, then

$$e_U \varphi(a) = 1_M \cdot v(a) = v(a),$$
and for $x \in q$, we have
\[ e_U \varphi(a) = e_U \cdot 0 = 0 = v(a) \]
also. Thus SV4 holds, and $\varphi$ is a supervaluation covering $v$.

By construction $\varphi$ is tangible and tangibly surjective. If $v$ is surjective then $\varphi$ is surjective.

**Definition 4.6.** We denote the supertropical ring just constructed by $U(v)$ and the supervaluation $\varphi$ just constructed by $\varphi_v$. Later we will call $\varphi_v : R \to U(v)$ the initial cover of $v$, cf. Definition 5.15.

Notice that $U(v)$ is a supertropical domain iff $v$ is surjective, and that in this case the supervaluation $\varphi_v$ is surjective.

**Remark 4.7.** The supertropical predomain $U(v)$ just constructed deviates strongly in its nature from the supertropical domain $D(G)$ for $G$ an ordered monoid studied in Examples 3.18.

While for $U = D(G)$ the restriction
\[ \nu_U \mid \mathcal{T}(U) : \mathcal{T}(U) \to G(U) \]
on the ghost map $\nu_U$ is bijective, for $U = U(v)$ this map usually has big fibers.

5. **Dominance and Transmissions**

As before now all semirings are assumed to be commutative. $R$ is any semiring, and $U, V$ are bipotent semirings.

**Definition 5.1.** If $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations, we say that $\varphi$ dominates $\psi$, and write $\varphi \geq \psi$, if for any $a, b \in R$ the following holds.

1. $\varphi(a) = \varphi(b) \Rightarrow \psi(a) = \psi(b),$
2. $e\varphi(a) \leq e\varphi(b) \Rightarrow e\psi(a) \leq e\psi(b),$
3. $\varphi(a) \in eU \Rightarrow \psi(a) \in eV.$

Notice that $D3$ can be also phrased as follows:
\[ \varphi(a) = e\varphi(a) \Rightarrow \psi(a) = e\psi(a). \]

**Lemma 5.2.** Let $\varphi : R \to U$ and $\psi : R \to V$ be supervaluations. Assume that $\varphi$ dominates $\psi$, and also (without essential loss of generality) that $\varphi$ is surjective. Then there exists a unique map $\alpha : U \to V$ with $\psi = \alpha \circ \varphi$ and
\[ \forall x \in U : \alpha(e_Ux) = e_V\alpha(x) \]
(i.e., $\alpha \circ \nu_U = \nu_V \circ \alpha$).

**Proof.** By D1 and D2 we have a unique well-defined map $\beta : \varphi(R) \to \psi(R)$ with $\beta(\varphi(a)) = \psi(a)$ for all $a \in R$ and a unique well-defined map $\gamma : e\varphi(R) \to e\psi(R)$ with $\gamma(e\varphi(a)) = e\psi(a)$ for all $a \in R$. Now $U = \varphi(R) \cup e\varphi(R)$, since $\varphi$ is assumed to be surjective. Suppose that $x \in \varphi(R) \cap e\varphi(R)$. Then $x = \varphi(a)$ for some $a \in R$, and $x = ex = e\varphi(a)$. By axiom D3 we conclude that $\psi(a) = e\psi(a)$. Thus $\beta(x) = \gamma(x)$. This proves that we have a unique well-defined map $\alpha : U \to V$ with $\alpha(x) = \beta(x)$ for $x \in \varphi(R)$ and $\alpha(y) = \gamma(y)$ for $y \in e\varphi(R)$. We have $\alpha(\varphi(a)) = \psi(a)$, i.e., $\psi = \alpha \circ \varphi$. Moreover, for any $a \in R$, $\alpha(e_U \varphi(a)) = \gamma(e_U \varphi(a)) = e_V \psi(a)$.

We record that in this proof we did not use the full strength of D2 but only the weaker rule that $e\varphi(a) = e\varphi(b)$ implies $e\psi(a) = e\psi(b)$.

**Definition 5.3.** Assume that $U$ and $V$ are supertropical semirings.
a) If $\alpha$ is a map from $U$ to $V$ with $\alpha(eU) \subseteq eV$, we say that $\alpha$ covers the map $\gamma : eU \to eV$ obtained from $\alpha$ by restriction, and we write $\gamma = \alpha'$. We also say that $\gamma$ is the ghost part of $\alpha$.

b) Assume that $\varphi : R \to U$ is a surjective supervaluation and $\psi : R \to V$ is a supervaluation dominated by $\varphi$. Then we call the map $\alpha$ occurring in Lemma 5.2, which is clearly unique, the transmission from $\varphi$ to $\psi$, and we denote this map by $\alpha_{\varphi, \psi}$. Clearly, $\alpha_{\varphi, \psi}$ covers the map $\gamma_{\varphi, \psi}$ connecting the surjective $m$-valuation $v := e\varphi : R \to eU$ to the $m$-valuation $w := e\psi : R \to eV$ introduced in Definition 2.9.

**Theorem 5.4.** Let $\varphi : R \to U$ be a surjective supervaluation and $\psi : R \to V$ a supervaluation dominated by $\varphi$. The transmission $\alpha := \alpha_{\varphi, \psi}$ obeys the following rules:

- $TM1 : \alpha(0) = 0$,
- $TM2 : \alpha(1) = 1$,
- $TM3 : \forall x, y \in U : \alpha(xy) = \alpha(x)\alpha(y)$,
- $TM4 : \alpha(ev) = ev$,
- $TM5 : \forall x, y \in eU : \alpha(x + y) = \alpha(x) + \alpha(y)$.

**Proof.** $TM1$, $TM2$, and $TM4$ are obtained from the construction of $\alpha$ in the proof of Lemma 5.2. This construction tells us also that $\alpha$ sends $eU$ to $eV$. Using (again) that $U = \varphi(R) \cup e\varphi(R)$, we check easily that $TM3$ holds. The rule $D2$ (in its full strength) tells us that the map $\gamma : eU \to eV$, obtained from $\alpha$ by restriction, is order preserving. This is $TM5$. $\square$

**Definition 5.5.** If $U$ and $V$ are supertropical semirings, we call any map $\alpha : U \to V$ which the rules $TM1$–$TM5$, a transmissive map from $U$ to $V$.

The axioms $TM1$-$TM5$ tell us that a transmissive map $\alpha : U \to V$ is the same thing as a homomorphism from the monoid $(U, \cdot)$ to $(V, \cdot)$ which restricts to a semiring homomorphism from $eU$ to $eV$. It is evident that every homomorphism from the semiring $U$ to $V$ is a transmissive map, but there exist quite a few transmissive maps, which are not homomorphisms; cf. §9 below and [IKR1].

As a converse to Lemma 5.2 we have the following fact.

**Proposition 5.6.** Assume that $\varphi : R \to U$ is a supervaluation and $\alpha : U \to V$ is a transmissive map from $U$ to a supertropical semiring $V$. Then $\alpha \circ \varphi : R \to V$ is again a supervaluation. If $e\varphi$ is either “strong” or “strict”, then $e(\alpha \circ \varphi)$ has the same property.

**Proof.** Let $\psi := \alpha \circ \varphi$, $v := e\varphi$, $w := e\psi$. Clearly $\psi$ inherits the properties $SV1$–$SV3$ from $\varphi$, since $\alpha$ obeys $TM1$–$TM3$. If $a \in R$, then, by $TM4$,

$$w(a) = e\psi(a) = e(\alpha(\varphi(a))) = \alpha(e\varphi(a)) = \alpha(v(a)),$$

hence $w = \alpha' \circ v$. Now $\alpha' : N \to N$ is a semiring homomorphism, hence order preserving. Thus it is immediate that $w$ is an $m$-valuation, and $w$ is strict or strong if $v$ is strict or strong, respectively. $\square$

**Remark 5.7.** If $\varphi : R \to U$ is a surjective supervaluation (cf. Definition 4.3) and $\alpha : U \to V$ is a surjective transmissive map, then the supervaluation $\alpha \circ \varphi$ is again surjective. Conversely, if $\varphi : R \to U$ and $\psi : R \to V$ are surjective supervaluations, and $\varphi$ dominates $\psi$, then the transmission $\alpha_{\psi, \varphi} : U \to V$ is a surjective map.

Combining Theorem 5.4, Proposition 5.6 and this remark, we read off the following facts.
Scholium 5.8. Let \( U, V \) be supertropical semirings and \( \varphi : R \to U \) a surjective supervaluation.

a) The supervaluations \( \psi : R \to V \) dominated by \( \varphi \) correspond uniquely with the transmissive maps \( \alpha : U \to V \) via \( \psi = \alpha \circ \varphi \). \( \alpha = \alpha_{\psi,\varphi} \).

b) If \( P \) is one of the properties “strict” or “strong” and \( e\varphi \) has property \( P \), then \( e\psi \) has property \( P \).

c) The supervaluation \( \psi \) is surjective iff the map \( \alpha \) is surjective.

d) Given a semiring homomorphism \( \gamma : eU \to eV \), the supervaluation \( \psi \) covers the \( m \)-valuation \( \gamma \circ (e\varphi) \) iff \( \alpha^\nu = \gamma \).

\[
\begin{array}{c}
R \\
\Downarrow \psi \\
U \xrightarrow{\alpha} V \\
\Downarrow \nu_U \\
eU \\
\Downarrow \gamma \\
eV
\end{array}
\]

\( \Box \)

Example 5.9. Let \( U \) be a supertropical semiring with ghost ideal \( M := eU \). Then, as we know, the ghost map \( \nu_U : U \to M \), \( x \mapsto ex \), is a strict \( m \)-valuation on the semiring \( U \) (Remark 3.12). Clearly, the identity map \( \text{id}_U : U \to U \) is a supervaluation covering \( \nu_U \).

Assume now that \( \alpha : U \to V \) is a transmissive map. Let \( \gamma := \alpha^\nu \) denote the homomorphism from \( M \) to \( N := eV \) covered by \( \alpha \). Then \( \nu := \gamma \circ \nu_U = \nu_V \circ \alpha \) is a strict valuation on \( U \) with values in \( N \), and \( \alpha := \alpha \circ \text{id}_U \) is a supervaluation on \( U \) covering \( \nu \). Thus \( \alpha \) is the transmission from the supervaluation \( \text{id}_U : U \to U \) to the supervaluation \( \alpha : U \to V \) covering \( \nu \).

The example tells us in particular that every transmissive map is the transmission between some supervaluations. Therefore we may and will also use the shorter term “transmission” for “transmissive map”.

In general, a transmission does not behave additively; hence is not a homomorphism. We now record cases where nevertheless some additivity holds.

Proposition 5.10. Let \( \alpha : U \to V \) be a transmission and \( \gamma : eU \to eV \) denote the ghost part of \( \alpha \), \( \gamma = \alpha^\nu \) (which is a semiring homomorphism).

i) If \( x, y \in U \) and \( ex = ey \), then \( \alpha(x) + \alpha(y) = \alpha(x + y) \).

ii) If \( x, y \in U \) and \( \alpha(x) + \alpha(y) \) is tangible, then again \( \alpha(x) + \alpha(y) = \alpha(x + y) \).

iii) If \( \gamma \) is injective, then \( \alpha \) is a semiring homomorphism.

Proof. Let \( x, y \in U \) be given, and assume without loss of generality that \( ex \leq ey \). Notice that this implies

\[ ea(x) = \alpha(ex) \leq \alpha(ey) = ea(y). \]

i): If \( ex = ey \), then \( ea(x) = ea(y) \), and we have \( x + y = ex, \alpha(x) + \alpha(y) = e\alpha(x) = \alpha(ex) \); hence \( \alpha(x) + \alpha(y) = \alpha(x + y) \).

ii): If \( \alpha(x) + \alpha(y) \) is tangible, then certainly \( e\alpha(x) \neq e\alpha(y) \); hence \( e\alpha(x) < e\alpha(y) \). This implies \( ex < ey \). Thus \( x + y = y, \alpha(x) + \alpha(y) = \alpha(y) \); hence \( \alpha(x) + \alpha(y) = \alpha(x + y) \).

iii): From i) we know that \( \alpha(x + y) = \alpha(x) + \alpha(y) \) holds if \( ex = ey \). Assume now that \( ex < ey \). Since \( \gamma \) is injective this implies \( e\alpha(x) < e\alpha(y) \). Thus \( x + y = y, \alpha(x) + \alpha(y) = \alpha(y) \); hence again \( \alpha(x + y) = \alpha(x) + \alpha(y) \).
Given an $m$-valuation $v : R \to M$, we now focus on the supervaluations $\varphi : R \to U$ which cover $v$, i.e., with $eU = M$ and $e\varphi = v_U \circ \varphi = v$. We single out a class of supervaluations which will play a special role.

**Definition 5.11.** A supervaluation $\varphi : R \to U$ is called **tangibly injective** if the map $\varphi$ is injective on the set $\varphi^{-1}(\mathcal{T}(U))$, i.e.,

$$\forall a, b \in R : \varphi(a) = \varphi(b) \in \mathcal{T}(U) \Rightarrow a = b.$$

**Example 5.12.** The supervaluation $\varphi_v : R \to U(v)$ constructed in §4 (cf. Example 4.5 and Definition 4.6) is injective on the set $R \setminus v^{-1}(0)$, hence certainly tangibly injective. Notice that $\varphi^{-1}(\mathcal{T}(U(v))) = R \setminus v^{-1}(0)$, i.e., $\varphi$ is tangible. $\varphi$ is also surjective.

**Theorem 5.13.** Assume that $\varphi : R \to U$ is a tangibly injective supervaluation covering $v : R \to M$. Let $\psi : R \to V$ be another supervaluation covering $v$, in particular, $eU = eV = M$.

a) $\varphi$ dominates $\psi$ iff the following holds:

$$\forall a \in R : \varphi(a) = v(a) \Rightarrow \psi(a) = v(a), \tag{5.1}$$

in other terms, $\varphi(a) \in eU \Rightarrow \psi(a) \in eV$.

b) If, in addition, $\varphi$ is tangibly surjective (cf. Definition 4.7.c), then $\varphi$ dominates $\psi$ iff there exists a homomorphism map $\alpha : U \to V$ covering the identity of $M$ such that $\alpha \circ \varphi = \psi$. The supervaluation $\psi$ is tangibly surjective iff $\alpha$ is surjective.

**Proof.** a): In the definition of dominance in Definition 5.11 the axiom D2 holds trivially since $e\varphi(a) = e\psi(a) = v(a)$. Axiom D3 is our present condition (5.1). Axiom D1 needs only to be checked in the case $\varphi(a) = \varphi(b) \in \mathcal{T}(U)$, and then holds trivially since this implies $a = b$ by the tangible injectivity of $\varphi$.

b): Replacing $U$ by the subsemiring $\mathcal{T}(U) \cup v(R)$ we assume without loss of generality that the supervaluation $\varphi$ is surjective. A transmission $\alpha$ from $\varphi$ to $\psi$ is forced to cover the identity of $M$; hence is a semiring homomorphism, cf. Proposition 5.10.ii. We have $\alpha(U) \supset eV$. Thus $\alpha$ is surjective iff $\alpha(\mathcal{T}(U)) = \mathcal{T}(V)$. This gives us the last claim.

**Corollary 5.14.** Assume that $v : R \to M$ is a valuation. The supervaluation $\varphi_v : R \to U(v)$ dominates every supervaluation $\psi : R \to U$ covering $v$. Thus these supervaluations $\psi$ correspond uniquely with the transmissive maps $\alpha : U(v) \to U$ covering $\text{id}_M$. They are semiring homomorphisms.

**Proof.** $\varphi_v$ is tangibly injective, and (5.1) holds trivially, since $\varphi_v(a) \in eU$ only if $v(a) = 0$. Theorem 5.13 and Proposition 5.10.iii apply.

**Definition 5.15.** Due to this property of $\varphi_v$ we call $\varphi_v$ the **initial supervaluation** covering $v$ (or **initial cover** of $v$ for short).

**Remark 5.16.** We may also regard $v : R \to M$ as a cover of $v$, viewing $M$ as a ghost supertropical semiring. Clearly every supervaluation $\psi : R \to U$ covering $v$ dominates $v$ with transmission $v_U$. Thus we may view $v : R \to M$ as the **terminal supervaluation** covering $v$ (or **terminal cover** of $v$ for short).

The following proposition gives examples of dominance $\varphi \geq \psi$ where $\varphi$ is not assumed to be tangibly injective.

**Proposition 5.17.** Let $U$ be a supertropical semiring with ghost ideal $M := eU$. Assume that $L$ is a submonoid of $(M, \cdot)$ with $M \cdot (M \setminus L) \subset M \setminus L$. 

a) The map $\alpha : U \to U$, defined by

$$\alpha(x) = \begin{cases} x & \text{if } ex \in L, \\
        ex & \text{if } ex \in M \setminus L, \end{cases}$$

is an endomorphism of the semiring $U$.

b) If $\varphi : R \to U$ is any supervaluation, then the map $\varphi_L := \alpha \circ \varphi$ from $R$ to $U$ is a supervaluation dominated by $\varphi$ and covering the same $m$-valuation as $\varphi$, i.e. $e\varphi_L = e\varphi$.

Proof. a): We have $e\alpha(x) = ex$ for every $x \in U$, and $\alpha(x) = x$ for every $x \in M$. One checks in a straightforward way that $\alpha$ is multiplicative, $\alpha(0) = 0$, $\alpha(1) = 1$.

We verify additivity. Let $x, y \in U$ be given, and assume without loss of generality that $ex \leq ey$. We have $e\alpha(x) = e(\alpha(x) = \alpha(ex) = ex$ and $e\alpha(y) = ey$. If $ex = ey$ then $x + y = ex$, and $\alpha(x) + \alpha(y) = e\alpha(x) = ex = \alpha(x + y)$. If $ex < ey$ then $x + y = y$ and $\alpha(x) + \alpha(y) = \alpha(y)$; hence again $\alpha(x) + \alpha(y) = \alpha(x + y)$.

b): Now obvious. \qed

Notice that $\varphi_L = \alpha \circ \varphi$ with a map $\alpha : U \to U$ given by $\alpha(x) = x$ if $ex \in L$, and $\alpha(x) = ex$ if $ex \in M \setminus L$. Thus if $\varphi$ is surjective, $\alpha$ is the transmission from $\varphi$ to $\varphi_L$.

It is not difficult to find instances where Proposition 5.17 applies.

**Example 5.18.** Assume that $M$ is a submonoid of $\Gamma \cup \{0\}$ for $\Gamma$ an ordered abelian group. Let $H$ be a subgroup of $\Gamma$ containing the set $\{x \in M \mid x > 1\}$. Then

$$L = \{x \in M \mid \exists h \in H \text{ with } x \geq h\}$$

is a submonoid of $M \setminus \{0\}$. We claim that $M \cdot (M \setminus L) \subset M \setminus L$.

Proof. Let $x \in M$, $y \in M \setminus L$ be given. If $x \leq 1$, then $xy \leq y$; hence, clearly, $xy \in M \setminus L$. Assume now that $x > 1$. Then $x \in H$. Suppose that $xy \in L$; hence $h \leq xy$ for some $h \in H$. Then $x^{-1} \leq y$ and $x^{-1}h \in H$; hence $y \in L$, a contradiction. Thus $xy \in M \setminus L$ again. \qed

In [IKR] we will meet many transmissions which are not semiring homomorphisms.

### 6. Fiber contractions

Before we come to the main theme of this section, we write down functorial properties of the class of transmissive maps.

**Proposition 6.1.** Let $\alpha : U \to V$ and $\beta : V \to W$ be maps between supertropical semirings.

i) If $\alpha$ and $\beta$ are transmissive, then $\beta \alpha$ is transmissive.

ii) If $\alpha$ and $\beta \alpha$ are transmissive and $\alpha$ is surjective, then $\beta$ is transmissive.

Proof. a) It is evident that analogous statements hold for the class of maps between supertropical semirings obeying the axioms TM1–TM4 in §5. Thus we may assume from the beginning that $\alpha$, $\beta$ and (hence) $\beta \alpha$ obey TM1–TM4, and have only to deal with the axiom TM5 (cf. Theorem 5.4, Definition 5.5).

b) We conclude from TM3 and TM4 that $\alpha$ maps $eU$ to $eV$ and $\beta$ maps $eV$ to $eW$. TM5 demands that these restricted maps are semiring homomorphisms. Thus it is evident that $\beta \alpha$ obeys TM5 if $\alpha$ and $\beta$ do. If $\alpha$ is surjective, then also the restriction $\alpha|eU : eU \to eV$ is surjective, since for $x \in U$, $y \in eV$ with $\alpha(x) = y$ we also have $\alpha(ex) = y$. Clearly, TM5 for $\alpha$ and $\beta \alpha$ implies TM5 for $\beta$ in this case. \qed

Often we will only need the following special case of Proposition 6.1.
Corollary 6.2. Let \( U, V, W \) be supertropical semirings. Assume that \( \alpha : U \to V \) is a surjective semiring homomorphism. Then a map \( \beta : V \to W \) is transmissive iff \( \beta \alpha \) has this property. \( \square \)

In the entire section \( U \) is a supertropical semiring. We look for equivalence relations on the set \( U \) that respect the multiplication on \( U \) and the fibers of the ghost map \( \gamma_U : U \to eU \).

Definition 6.3. Let \( E \) be an equivalence relation on the set \( U \). We say that \( E \) is multiplicative if for any \( x_1, x_2, y \in U \),

\[
x_1 \sim_E x_2 \implies x_1 y \sim_E x_2 y.
\]

We say that \( E \) is fiber conserving if for any \( x_1, x_2 \in U \),

\[
x_1 \sim_E x_2 \implies e x_1 = e x_2.
\]

If \( E \) is both multiplicative and fiber conserving, we call \( E \) an MFCE-relation (multiplicative fiber conserving equivalence relation) for short.

Examples 6.4.

(i) Assume that \( \alpha : U \to V \) is a multiplicative map from \( U \) to a supertropical semiring \( V \). Then the equivalence \( E(\alpha) \), given by

\[
x_1 \sim x_2 \iff \alpha(x_1) = \alpha(x_2),
\]

is clearly multiplicative. If in addition \( \alpha(e_U) = e_V \), and if the induced map \( \gamma : eU \to eV \), \( \gamma(e x) = e \alpha(x) \), is injective, then \( E(\alpha) \) is also fiber conserving; hence an MFCE-relation. We usually denote this equivalence \( \sim \) by \( \sim_\alpha \).

In particular, we have an MFCE-relation \( E(\alpha) \) on \( U \) for any semiring homomorphism \( \alpha : U \to V \) which is injective on \( eU \).

(ii) The ghost map \( \nu = \nu_U : U \to U \) gives us an MFCE-relation \( E(\nu) \) on \( U \). Clearly

\[
x_1 \sim_\nu x_2 \iff e x_1 = e x_2.
\]

\( E(\nu) \) is the coarsest MFCE-relation on \( U \).

(iii) If \( E_1 \) and \( E_2 \) are equivalence relations on the set \( U \), then \( E_1 \cap E_2 \) is again an equivalence relation on \( U \). (As usual, we regard an equivalence relation on \( U \) as a subset of \( U \times U \).) We have

\[
x_1 \sim_{E_1 \cap E_2} x_2 \iff x_1 \sim_{E_1} x_2 \text{ and } x_1 \sim_{E_2} x_2.
\]

If \( E_1 \) is multiplicative and \( E_2 \) is an MFCE, then \( E_1 \cap E_2 \) is an MFCE.

(iv) In particular, every multiplicative equivalence relation \( E \) on \( U \) gives us an MFCE-relation \( E \cap E(\nu) \) on \( U \). This is the coarsest MFCE-relation on \( U \) which is finer than \( E \). We have

\[
x_1 \sim_{E \cap E(\nu)} x_2 \iff x_1 \sim_E x_2 \text{ and } e x_1 = e x_2.
\]

(v) We define an equivalence relation \( E_t \) (the “t” alludes to “tangible”) on \( U \) as follows, writing \( \sim_t \) for \( \sim_{E_t} \):

\[
x_1 \sim_t x_2 \iff \text{ either } x_1 = x_2 \\
\text{ or } x_1, x_2 \in T(U) \text{ and } e x_1 = e x_2.
\]

Clearly, this is an MFCE-relation iff for any tangible \( x_1, x_2, y \in E \) with \( e x_1 = e x_2 \) both \( x_1 y \) and \( x_2 y \) are tangible or equal. In particular, \( E_t \) is an MFCE if \( T(U) \) is closed under multiplication.
Let $F$ denote the equivalence relation on $U$ which has the equivalence classes $\mathcal{T}(U)$ and $cU$. It is readily checked that $E_1 = F \cap E(\nu)$.

The equivalence classes of $E_1$ contained in $\mathcal{T}(U)$ are the sets $\mathcal{T}(U) \cap \nu^{-1}_U(z)$ with $z \in M$, which are not empty. We call them the tangible fibers of $\nu_U$.

Our next goal is to prove that, given an MFCE-relation $E$ on $U$, the set $U/E$ of all $E$-equivalence classes inherits from $U$ the structure of a supertropical semiring.

Lemma 6.5. If $E$ is a fiber conserving equivalence relation on $U$, then for any $x_1, x_2, y \in U$

\[ x_1 \sim_E x_2 \implies x_1 + y \sim_E x_2 + y. \]

Proof. $ex_1 = ex_2$. If $ey < ex_1$, we have $x_1 + y = x_1, x_2 + y = x_2$. If $ey = ex_1$, we have $x_1 + y = ey = x_2 + y$. If $ey > ex_1$, we have $x_1 + y = y = x_2 + y$. Thus, in all three cases, $x_1 + y \sim_E x_2 + y$. \hfill $\square$

Notice that, as a formal consequence of the lemma, more generally

\[ x_1 \sim_E x_2, y_1 \sim_E y_2 \implies x_1 + y_1 \sim_E x_2 + y_2. \]

Theorem 6.6. Let $E$ be an MFCE-relation on a supertropical semiring $U$. On the set $\overline{U} := U/E$ of equivalence classes $[x]_E, x \in U,$ we have a unique semiring structure such that the projection map $\pi_E : U \to \overline{U}, x \mapsto [x]_E$ is a semiring homomorphism. This semiring $\overline{U}$ is supertropical, and $\pi_E$ covers a semiring isomorphism $eU \sim \overline{eU}$. (Here $\overline{\cdot} := \pi_E(\cdot)$.)

Proof. We write $\overline{x} := [x]_E$ for $x \in U$ and $\pi := \pi_E$. Thus $\pi(x) = \overline{x}$. Due to Lemma 6.5 and condition (6.1), we have a well-defined addition and multiplication on $\overline{U}$, given by the rules $(x, y \in U)$

\[ \overline{x + y} := \overline{x} + \overline{y}, \quad \overline{x \cdot y} := \overline{xy}. \]

The axioms of a commutative semiring are valid for these operations, since they hold in $U$, and the map $\pi$ is a homomorphism from $U$ onto the semiring $\overline{U}$.

We have $1 + 1 = \overline{e}$ and $\overline{eU} = \pi(eU)$. If $x, y \in eU$ and $x \sim_E y$ then $x = ex = ey = y$, since $E$ is fiber conserving. Thus the restriction $\pi|eU$ is an isomorphism from the bipotent semiring $eU$ onto the semiring $\overline{eU}$ (which thus is again bipotent).

We are ready to prove that $\overline{U}$ is supertropical, i.e. that axioms (3.3’), (3.3’’), (3.4) from $\S 3$ are valid. It is obvious that $\overline{U}$ inherit properties (3.3’) and (3.4) from $U$. Let $x, y \in E$ be given with $\overline{ex} = \overline{ey}$, i.e. $\overline{ex} = \overline{ey}$. Then $ex = ey$; hence $x + y = ex$ by axiom (3.3’’’) for $U$. Applying the homomorphism $\pi$ we obtain $\overline{x + y} = \overline{ex}$. Thus $\overline{U}$ also obeys (3.3’’’). \hfill $\square$

Remark 6.7. Theorem 6.6 tells us, in particular, that every MFCE-relation $E$ on $U$ is of the form $E(\alpha)$ for some semiring homomorphism $\alpha : U \to V$ with $\alpha|eU$ bijective, namely, $E = E(\pi_E)$.

Theorem 6.8. Assume that $\alpha : U \to V$ is a multiplicative map. Let $E$ be an MFCE-relation on $U$, which is respected by $\alpha$, i.e., $x \sim_E y$ implies $\alpha(x) = \alpha(y)$. Clearly, we have a unique multiplicative map $\overline{\alpha} : U/E \to V$ with $\overline{\alpha} \circ \pi_E = \alpha$.

Then, if $\alpha$ is a transmission (a semiring homomorphism), the map $\overline{\alpha}$ is of the same kind.

Proof. Corollary 6.2 gives us all the claims, since $\pi_E$ is a surjective homomorphism. \hfill $\square$

Definition 6.9. We call a map $\alpha : U \to V$ between supertropical semirings a fiber contraction, if $\alpha$ is transmissive and surjective, and the map $\gamma : eU \to eV$ covered by $\alpha$ is strictly order preserving.
Notice that then α is a semiring homomorphism (cf. Proposition 5.10iii) (hence α is a transmission), and γ is an isomorphism from eU to eV.

Scholium 6.10.

i) If E is an MFCE-relation on U, by Theorem 6.6 the map π_E : U → U/E is a fiber contraction. On the other hand, if a surjective fiber contraction α : U → V is given, then clearly E(α) is an MFCE-relation, and, as Theorem 6.8 tells us, α induces a semiring isomorphism \( \bar{\alpha} : U/E(\alpha) \xrightarrow{\sim} V \) with \( \alpha = \bar{\alpha} \circ \pi_{E(\alpha)} \). In short, every fiber contraction α on U is a map π_E with E an MFCE-relation on U uniquely determined by α, followed by a semiring isomorphism.

ii) If the semiring isomorphism \( \bar{\alpha} \) is the identity \( \text{id}_M \) of \( M := eU \) (in particular \( eU = eV \)), we say α is a fiber contraction over M.

If E is an equivalence relation on a set X, and Y is a subset of X, we denote the set of all equivalence classes \([x]_E\) with \( x \in Y \) by \( Y/E \).

Example 6.11. Assume that U is a supertropical domain (cf. 3.14). Then the equivalence relation \( E_t \) introduced in Example 6.4.v is MFCE, and \( \mathcal{T}(U) \) is a union of \( E_t \)-equivalence classes. The ring \( \mathcal{U} = U/E_t \) is a supertropical domain with \( \mathcal{T}(\mathcal{U}) = \mathcal{T}(U)/E_t \) and \( \mathcal{G}(\mathcal{U}) = \mathcal{G}(U) \). The ghost map of \( \mathcal{U} \) maps \( \mathcal{T}(\mathcal{U}) \) bijectively to \( \mathcal{G}(U) \); hence gives us a monoid isomorphism \( v : \mathcal{T}(\mathcal{U}) \xrightarrow{\sim} \mathcal{G}(U) \). Thus (in notation of Examples 3.18)

\[ U/E_t = D(\mathcal{G}(U)). \]

The map \( \pi_{E_t} \) is a fiber contraction over \( eU = eU/E_t \).

Example 6.12. (cf. Proposition 5.17) Let \( U \) be a supertropical semiring, \( M := eU \), and let \( L \) be a submonoid of \( (M, \cdot) \) with \( M \cdot (M \setminus L) \subseteq M \setminus L \). Then the map \( \alpha : U \rightarrow U \) with \( \alpha(x) = x \) if \( ex \in L \), \( \alpha(x) = ex \) if \( ex \in M \setminus L \), is a fiber contraction over M. The image of α is the subsemiring \( \nu_U^{-1}(L) \cup (M \setminus L) \) of U.

Example 6.13. Let again \( U \) be a supertropical semiring and \( M := eU \). But now assume only that \( L \) is a subset of \( M \) with \( M \cdot (M \setminus L) \subset M \setminus L \). We define an equivalence relation \( E(L) \) on \( U \) as follows:

\[ x \sim_{E(L)} y \iff \text{ either } x = y \text{ or } ex = ey \in M \setminus L. \]

One checks easily that \( E(L) \) is MFCE. But if \( L \) is not a submonoid of \( (M, \cdot) \), then in the supertropical semiring \( \mathcal{U} := U/E(L) \) the set \( \mathcal{T}(\mathcal{U}) \) of tangible elements is not closed under multiplication. In particular, \( \mathcal{U} \) is not isomorphic to a subsemiring of U.

For later use we introduce one more notation.

Notation 6.14. If \( \varphi : R \rightarrow U \) is a supervaluation and E is an MFCE-relation on U, let \( \varphi/E \) denote the supervaluation \( \pi_E \circ \varphi : R \rightarrow U/E \). Thus, for any \( a \in R \),

\[ (\varphi/E)(a) := [\varphi(a)]_E. \]

7. The Lattices \( C(\varphi) \) and Cov(\( v \))

Given an \( m \)-valuation \( v : R \rightarrow M \) on a semiring \( R \), we can say more about the class of all supervaluations \( \varphi \) covering \( v \). Recall that these are the supervaluations \( \varphi : R \rightarrow U \) with \( eU = M \) and \( \nu_U \circ \varphi = v \), in other words, \( ev = v \). For short, we call these supervaluations \( \varphi \) the covers of the \( m \)-valuation \( v \). It suffices to focus on covers of \( v \) which are tangibly surjective, cf. Remark 4.4. (N.B. Without loss of generality, we could even assume that \( v \) is surjective. Then a cover \( \varphi \) of \( v \) is tangibly surjective iff \( \varphi \) is surjective.)
Definition 7.1.

a) We call two covers $\varphi_1 : R \to U_1$, $\varphi_2 : R \to U_2$ of $v$ equivalent, if $\varphi_1 \geq \varphi_2$ and $\varphi_2 \geq \varphi_1$, i.e., $\varphi_1$ dominates $\varphi_2$, and $\varphi_2$ dominates $\varphi_1$. If $\varphi_1$ and $\varphi_2$ are tangibly surjective (without essential loss of generality, cf. Remark [4.4]), this means that $\varphi_2 = \alpha \circ \varphi_1$ with $\alpha : U_1 \to U_2$ a semiring isomorphism over $M$ (i.e., $e\alpha(x) = ex$ for all $x \in U_1$).

b) We denote the equivalence class of a cover $\varphi : R \to U$ of $v$ by $[\varphi]$, and we denote the set of all these equivalence classes by $\text{Cov}(v)$. {Notice that $\text{Cov}(v)$ is really a set, not just a class, since for any tangibly surjective cover $\varphi : R \to U$, we have $U = \varphi(R) \cup M$; hence the cardinality of $U$ is bounded by Card $R + \text{Card } M$.} On $\text{Cov}(v)$ we have a partial ordering: $[\varphi_1] \geq [\varphi_2]$ iff $\varphi_1$ dominates $\varphi_2$. We always regard $\text{Cov}(v)$ as a poset in this way.

c) If a covering $\varphi : R \to U$ of $v$ is given, we denote the subposet of $\text{Cov}(v)$ consisting of all $[\psi] \in \text{Cov}(v)$ with $[\varphi] \geq [\psi]$ by $C(\varphi)$. {Notice that this poset is determined by $\varphi$ alone, since $v = e\varphi$.}

In §5 we have seen that, given a tangibly surjective cover $\varphi : R \to U$ of $v$, the tangibly surjective covers $\psi : R \to V$ dominated by $\varphi$ correspond uniquely to the transmissive surjective maps $\alpha : U \to V$ which restrict to the identity on $M = eU = eV$. Scholium 6.10 from the preceding section tells us, in particular, the following.

Theorem 7.2. Assume that $\varphi : R \to U$ is a tangibly surjective covering of the $m$-valuation $v : R \to M$.

a) The elements $[\psi]$ of $C(\varphi)$ correspond uniquely to the MFCE-relations $E$ on $U$ via $[\psi] = [\varphi/E]$.

b) Let $\text{MFC}(U)$ denote the set of all MFCE-relations on $U$, ordered by the coarsening relation: $E_1 \leq E_2$ iff $E_2$ is coarser than $E_1$, i.e., $E_1 \subseteq E_2$, if the $E_i$ are viewed – as customary – as subsets of $U \times U$. The map $E \mapsto [\varphi/E]$ is an anti-isomorphism (i.e., an order reversing bijection) from the poset $\text{MFC}(U)$ to the poset $C(\varphi)$.

If $(E_i \mid i \in I)$ is a family in $\text{MFC}(U)$ then the intersection $E := \bigcap_{i \in I} E_i$ is again an MFCE-relation on $U$, and is the infimum of the family $(E_i \mid i \in I)$ in $\text{MFC}(U)$. Since $\text{MFC}(U)$ has a biggest and smallest element, namely $E(mU)$ and the diagonal of $U \times U$, it is now clear that the poset $\text{MFC}(U)$ is a complete lattice. Thus, for any cover $\varphi : R \to U$ of the $m$-valuation $v : R \to M$, also the poset $C(\varphi)$ is a complete lattice. {We easily retreat to the case that $\varphi$ is tangibly surjective.}

The supremum of a family $(E_i \mid i \in I)$ in $\text{MFC}(U)$ is the following equivalence relation $F$ on $U$. Two elements $x, y$ of $U$ are $F$-equivalent iff there exists a finite sequence $x_0 = x, x_1, \ldots, x_m = y$ in $U$ such that for each $j \in \{1, \ldots, m\}$ the element $x_{j-1}$ is $E_k$-equivalent to $x_j$ for some $k \in I$.

Construction 7.3. Assume again that $\varphi$ is tangibly surjective. The supremum $\bigvee_{i \in I} \xi_i$ of a family $(\xi_i \mid i \in I)$ in $C(\varphi)$ can be described as follows. Choose for each $i \in I$ a tangibly surjective representative $\psi_i : R \to V_i$ of $\xi_i$. Thus $eV_i = M$, and $\psi_i$ is a cover of $v$ dominated by $\varphi$. Let $\psi_i := \psi_i \ (= 1_{M})$, and let $V$ denote the set of all elements $z = (x_i \mid i \in I)$ in the semiring $\prod_{i \in I} V_i$ with $e_i x_i = e_j x_j$ for $i \neq j$. This is a subsemiring of $\prod_{i \in I} V_i$ containing the image $M'$ of $M$ in $\prod_{i \in I} V_i$ under the diagonal embedding of $M$ into $\prod V_i$. We identify $M' = M$, $5$ = partially ordered set
and then have
\[ e_U = 1_M = (e_i \mid i \in I) = 1_V + 1_V. \]

It is now a trivial matter to verify that \( V \) is a supertropical semiring by checking the axioms in §3. We have \( e_V V = e V = M' = M \). The supervaluations \( \psi_i : R \to V_i \) combine to a map \( \psi : R \to V \), given by
\[
\psi(a) := (\psi_i(a) \mid i \in I) \in V
\]
for \( a \in R \). It is a supervaluation covering \( v \), and \( \phi : R \to U \) dominates \( \psi \) (e.g., check the axioms D1–D3 in §5). The class \([\psi]\) is the supremum of the family \( (\xi_i \mid i \in I) \) in \( C(\phi) \).

Given again a family \((\xi_i \mid i \in I)\) in \( C(\phi) \) with representatives \( \psi_i : R \to V_i \) of the \( \xi_i \), we indicate how the infimum \( \bigwedge \xi_i \in C(\phi) \) can be built, without being as detailed as above for the supremum.

We assume that each supervaluation \( \psi_i \) is surjective. The transmission \( \delta_i : U \to V_i \) from \( \phi \) to \( \psi_i \) is a surjective semiring homomorphism. We form the categorical direct limit (= colimit) of the family \( (\delta_i \mid i \in I) \) in the category of semirings (cf. [Mit, Chap. II], [ML, III, §3]). Thus we have a supervaluation \( \psi \) together with a family of semiring homomorphisms \( (\alpha_i : V_i \to V \mid i \in I) \) such that \( \alpha_i \circ \delta_i = \alpha_j \circ \delta_j \) for \( i \neq j \), which is universal. This means that, given a family \( (\beta_i : V_i \to W \mid i \in I) \) of homomorphisms with \( \beta_i \circ \delta_i = \beta_j \circ \delta_j \) for \( i \neq j \), there exists a unique homomorphism \( \beta : V \to W \) with \( \beta \circ \alpha_i = \beta_i \) for every \( i \in I \). Choosing some \( i \in I \) let
\[
\varepsilon := \alpha_i \circ \delta_i : U \to V.
\]
This homomorphism, which is independent of the choice of \( i \), is surjective, due to universality, since all maps \( \delta_j : U \to V_j \) are surjective. It turns out that the restriction \( \varepsilon|eU \) maps \( eU = M \) isomorphically onto \( eV \). We identify \( M \) with \( eV \) by this isomorphism and then have \( \varepsilon|eU = 1_M \).

This can be seen as follows. Let \( \nu := \nu_U \) and \( \nu_i := \nu_{V_i} \) denote the ghost maps of \( U \) and \( V_i \). For every \( i \in I \) we have \( \nu_i \circ \delta_i = \nu_i \). By universality we obtain a homomorphism \( \mu : V \to M \) with \( \mu \circ \alpha_i = \nu_i \) for every \( i \). Let \( j_i \) denote the inclusion map from \( M \) to \( V_i \). We have \( \nu_i \circ j_i = \text{id}_M \); hence
\[
\mu \circ \alpha_i \circ j_i = \nu_i \circ j_i = \text{id}_M.
\]
The surjective homomorphism \( \alpha_i \) maps \( M = eV_i \) onto \( eV \). We conclude that the restriction \( \alpha_i|M \) gives an isomorphism from \( M \) onto \( eV \), the inverse map being given by \( \mu \).

We identify \( M \) with \( eV \) via \( \alpha_i|M \). Now \( \alpha_i : V_i \to V \) has become a surjective semiring homomorphism over \( M \) (for every \( i \)). Thus also \( \varepsilon : U \to V \) is a surjective homomorphism over \( M \). We conclude, that \( \varepsilon \) gives an MFCE-relation \( E(\varepsilon) \) and the semiring \( V \) is supertropical. The supervaluation
\[
\psi := \varepsilon \phi = \alpha_i \circ \psi_i \quad \text{is dominated by every} \quad \psi_i \quad \text{and} \quad [\psi] = \bigwedge \xi_i.
\]

Since \( V_i = \psi_i(R) \cup M \) for every \( i \), the semiring \( V \) and the \( \alpha_i \) can be described completely in terms of the \( \psi_i \) without mentioning \( U \) and the \( \delta_i \). We leave this to the interested reader.

**Definition 7.4.** We call a supervaluation \( \phi \) **initial** if \( \phi \) dominates every other supervaluation \( \psi \) with \( e\phi = e\psi \). We then also say that \( \phi \) is an **initial cover** of \( \psi := e\phi \).

If an \( m \)-valuation \( v : R \to M \) is given, a supervaluation \( \phi : R \to U \) is an initial cover of \( v \) iff \( e\phi = v \) and \([\phi]\) is the biggest element of the poset \( \text{Cov}(v) \).

Such an initial cover had been constructed explicitly in §4 in the case that \( v \) is a valuation, namely, the supervaluation \( \phi_v : R \to U(v) \), cf. Definition 4.6 and Corollary 5.1.4. We now
prove that an initial cover always exists, although in general we do not have an explicit description.

**Proposition 7.5.** Every \(m\)-valuation \(v : R \to M\) has an initial cover. The poset \(\text{Cov}(v)\) is a complete lattice.

**Proof.** Let \((\psi_i \mid i \in I)\) be a family of coverings of \(v\) which represents every element of the set \(\text{Cov}(v)\). Now repeat Construction 7.3 with this family. It gives us a covering \(\psi : R \to V\) of \(v\) which dominates all \(\psi_i\); hence is an initial covering of \(v\). Of course, \(C(\psi) = \text{Cov}(v)\), and thus \(\text{Cov}(v)\) is a complete lattice. \(\square\)

**Notation 7.6.** If \(v : R \to M\) is any \(m\)-valuation, let \(\varphi_v : R \to U(v)\), denote a fixed tangibly surjective initial supervaluation covering \(v\). If \(v\) is a valuation, we choose for \(\varphi_v\) the supervaluation constructed in Example 4.5.

Notice that \(\varphi_v\) is unique up to unique isomorphism over \(M\), i.e., if \(\psi : R \to V\) is another surjective initial cover of \(v\), there exists a unique semiring isomorphism \(\alpha : U(v) \to V\) which restricts to the identity on \(M\). We call \(\varphi_v\) “the” initial cover of \(v\). The lattice \(\text{Cov}(v)\) coincides with \(C(\varphi_v)\).

Given a supervaluation \(\varphi : R \to U\) or an \(m\)-valuation \(v : R \to M\), we view the lattice \(C(\varphi)\) and \(\text{Cov}(v)\) as a measure of complexity of \(\varphi\) and \(v\), respectively, and thus make the following formal definition.

**Definition 7.7.** We call the isomorphism class of the lattice \(C(\varphi)\) the lattice complexity of the supervaluation \(\varphi\) and denote it by \(\text{lc}(\varphi)\). In the same vein we call the isomorphism class of the lattice \(\text{Cov}(v)\) the tropical complexity of the \(m\)-valuation \(v\) and denote it by \(\text{trc}(v)\). We have \(\text{trc}(v) = \text{lc}(\varphi_v)\).

The word “complexity” in Definition 7.7 should not be taken too seriously. Usually a “measure of complexity” has values in natural numbers or, more generally, in some well understood fixed ordered set. The isomorphism classes of lattices are not values of this kind. Our idea behind the definition is that, if you are given a function \(m\) on the class of lattices which measures (part of) their complexity in some way, then \(m \circ \text{lc}\), resp. \(m \circ \text{trc}\), is such a function on the class of supervaluations, resp. \(m\)-valuations.

**Theorem 7.8.** If \(\varphi : R \to U\) and \(\varphi' : R' \to U\) are tangibly surjective supervaluations with values in the same supertropical semiring \(U\), then \(\text{lc}(\varphi) = \text{lc}(\varphi')\).

**Proof.** Both lattices \(C(\varphi)\) and \(C(\varphi')\) are anti-isomorphic to \(\text{MFC}(U)\); hence are isomorphic. \(\square\)

This result is quite remarkable, since it says that the lattice complexity of a surjective supervaluation \(\varphi : R \to U\) depends only on the isomorphism class of the target semiring \(U\).

**Example 7.9.** Let \(\varphi : R \to U\) be a tangibly surjective supervaluation. The identity \(\text{id}_U : U \to U\) is also a supervaluation. It is the initial cover of the ghost map \(\nu_U : U \to eU\). We have \(\text{lc}(\varphi) = \text{trc}(\nu_U)\).

8. **Orbital equivalence relations**

Our main goal in this section is to introduce and study a special kind of MFCE-relations on supertropical semirings, which seems to be more accessible than MFCE-relations in general. But for use in later sections, we will define more generally “orbital” equivalence relations on
supertropical semirings. They are multiplicative but not necessarily fiber conserving. The relations we are looking for here then will be the orbital MFCE-relations.

In the following $U$ is a supertropical semiring, and $M := eU$ denotes its ghost ideal. We always assume that $T(U)$ is not empty; i.e., $e \neq 1$. We introduce the set $$S(U) := \{ x \in U \mid xT(U) \subset T(U) \}.$$ This is a subset of $T(U)$ closed under multiplication and containing the unit element $1_U$; hence is a monoid.

The monoid $S(U)$ operates on the sets $U$ and $T(U)$ by multiplication. If $T(U)$ itself is closed under multiplication then $S(U) = T(U)$.

Let $G$ be a submonoid of $S(U)$. Then also $G$ operates on $U$ and on $T(U)$. For any $x \in U$ we call the set $Gx$ the orbit of $x$ under $G$ (as common at least for $G$ a group). We define a binary relation $\sim_G$ on $U$ as follows:

$$x \sim_G y \iff \exists g, h \in G : gx = hy.$$ Thus $x \sim_G y$ iff the orbits $Gx$ and $Gy$ intersect. Clearly this is an equivalence relation on $U$, which is multiplicative, i.e., obeys the rule (6.1) from §6. We denote this equivalence relation by $E(G)$.

The relation $E(G)$ on $U$ is MFCE, i.e., obeys also the rule (6.2) from §6, iff $G$ is contained in the “unit-fiber”

$$T_e(U) := \{ x \in T(U) \mid ex = e \}$$

of $T(U)$. The biggest such monoid is the unit fiber

$$S_e(U) := \{ g \in S(U) \mid eg = e \} = T_e(U) \cap S(U)$$

of $S(U)$.

Example 8.1. Assume that $R$ is a field and $v : R \to \Gamma \cup \{0\}$ is a surjective valuation on $R$. {In classical terms, $v$ is a Krull valuation on $R$ with value group $\Gamma$.} Let

$$U := U(v) = (R \setminus \{0\}) \cup \Gamma \cup \{0\},$$

cf. Definition 4.6. Then $S(U)$ is the multiplicative group $R^* = R \setminus \{0\}$ of the field $R$, and $S_e(U)$ is the group $\mathfrak{o}_v^*$ of units of the valuation domain

$$\mathfrak{o}_v := \{ x \in R \mid v(x) \leq 1 \}.$$  

Definition 8.2. We call an equivalence relation $E$ on the supertropical semiring $U$ orbital if $E = E(G)$ for some submonoid $G$ of $S(U)$. We denote the set of all orbital equivalence relations on $U$ by $\text{Orb}(U)$ and the subset $\text{Orb}(U) \cap \text{MFC}(U)$, consisting of the orbital MFCE-relations on $U$, by $\text{OFC}(U)$. {“OFC” alludes to “orbital fiber conserving”}. Consequently, we call the elements of $\text{OFC}(U)$ the orbital fiber conserving equivalence relations on $U$, or OFCE-relations for short.

Example 8.3. It is evident that $E(S(U))$ is the coarsest orbital equivalence relation and $F := E(S_e(U))$ is the coarsest OFCE-relation on $U$. Assume now that $U$ is a supertropical domain. Then $S(U) = T(U)$, $S_e(U) = T_e(U)$, and $G(U) = eT(U)$. $E(S(U))$ has just 3 equivalence classes, namely, $T(U)$, $G(U)$ and $\{0\}$. On the other hand, $F$ is finer than the MFCE-relation $E_\iota$ introduced in Example 6.4, whose equivalence classes in $T(U)$ are the tangible fibers of the ghost map $v_U$. Very often $E_\iota$ is not orbital; hence $F \subsetneq E_\iota$. 
Subexample 8.4. Let $R = k[x]$ be the polynomial ring in one variable $x$ over a field $k$. Choose a real number $\vartheta$ with $0 < \vartheta < 1$, and let $v$ be the surjective valuation on $R$ defined by

$$v(f) = \vartheta^{|\deg f|}.$$ 

Thus, $v : R \to G \cup \{0\}$ with $G$ the monoid $\{\vartheta^n \mid n \in \mathbb{N}_0\} \subset \mathbb{R}$. Finally, take

$$U := U(v) = (R \setminus \{0\}) \cup G \cup \{0\},$$

cf. Definition 4.6. We have $S(U) = R \setminus \{0\}$ and

$$S_e(U) = \{f \in R \mid \deg f = 0\} = k \setminus \{0\},$$

the set of nonzero constant polynomials. If $f, g \in T(U)$ are given with $ef = eg$, i.e.,

$$\deg f = \deg g, \text{ then } f \sim_F g \iff g = cf \text{ with } c \text{ a constant } \neq 0.$$ 

Thus, the set of $F$-equivalence classes in $T(U)$ can be identified with the set of monic polynomials in $k[x]$, while the $E$-equivalence classes are the sets $\{f \in k[x] \mid \deg f = n\}$ with $n$ running through $\mathbb{N}_0$. For $n = 0$ this $E_1$-equivalence class is also an $F$-equivalence class, while for $n > 0$ it decomposes into infinitely many $F$-equivalence classes if the field $k$ is infinite, and into $|k|^n$ $F$-equivalence classes if $k$ is finite.

The semiring $U/F$ (cf. §6) can be identified with the subsemiring $V$ of $U$, which has as tangible elements the monic polynomials in $k[x]$ and has the same ghost ideal $eV = eU$ as $U$. \qed 

Different submonoids $G, H$ of $S(U)$ may yield the same orbital equivalence relation $E(G) = E(H)$. But this ambiguity can be tamed.

**Proposition 8.5.** If $G$ is a submonoid of $S(U)$, then

$$G' := \{x \in S(U) \mid \exists g \in G : gx \in G\}$$

is a submonoid of $S(U)$ containing $G$, and $E(G) = E(G')$. If $G \subset S_e(U)$ then $G' \subset S_e(U)$.

**Proof.** a) It is immediate that $G'$ is a submonoid of $S(U)$ and that $G \subset G'$. Given $x \in G'$ we have elements $g, h \in G$ with $gx = h$. If in addition $G \subset S_e(U)$, then $e = eh = (eg)(ex) = ex$; hence $x \in S_e(U)$. Thus $G' \subset S_e(U)$. It follows from $G \subset G'$ that $E(G) \subset E(G')$.

b) Let $x, y \in U$ be given with $x \sim_{G'} y$. We have elements $g_1', g_2'$ in $G'$ with $g_1'x = g_2'y$. We furthermore have elements $h_1, h_2$ in $G$ with $h_1g_1' = g_1 \in G$ and $h_2g_2' = g_2 \in G$. Now

$$g_1h_2x = h_1h_2g_1'x = h_1h_2g_2' = h_1g_2y.$$ 

Thus $x \sim_G y$. This proves $E(G') \subset E(G)$; hence $E(G) = E(G')$. \qed 

**Definition 8.6.** We call $G'$ the **saturation** of the monoid $G$ (in $U$), and we say that $G$ is saturated if $G = G'$.

It is immediate that $(G')' = G'$. Thus $G'$ is always saturated.

**Example 8.7.** If $S(U)$ happens to be a group, then the saturation of a submonoid $G$ of $S(U)$ is just the subgroup of $S(U)$ generated by $G$. Indeed, the elements of $G'$ are the $x \in S(U)$ with $g_1x = g_2$ for some $g_1, g_2 \in G$, i.e., the elements $g_1^{-1}g_2$ with $g_1, g_2 \in G$.

**Proposition 8.8.** Let $E$ be a multiplicative equivalence relation on $U$.

a) The set

$$G_E := \{x \in S(U) \mid x \sim_E 1\}$$

is a saturated submonoid of $S(U)$.

b) If $E = E(H)$ for some submonoid $H$ of $S(U)$, then $G_E$ is the saturation $H'$ of $H$. 

c) In general, \( E(G_E) \) is the coarsest orbital equivalence relation on \( U \) which is finer than \( E \).

d) If \( E \) is MFCE then \( G_E \subset S_e(U) \), and \( E(G_E) \) is the coarsest OFCE-relation on \( U \) which is finer than \( E \).

Proof. a): If \( x, y \in G_E \) then \( x \sim_E 1, y \sim_E 1 \); hence \( xy \sim_E y \sim_E 1 \), thus \( xy \in G_E \). This proves that \( G_E \) is a submonoid of \( S(U) \). Let \( x \in G_E' \) be given. We have elements \( g, h \in G_E \) with \( hx = g \). It follows from \( g \sim_E 1, h \sim_E 1 \) that

\[
x \sim_E hx = g \sim_E 1.
\]

Thus \( x \in G_E \). This proves that \( G'_E = G_E \).

b): Assume that \( E = E(H) \) with \( H \) a submonoid of \( S(U) \). For \( x \in S(U) \) we have

\[
x \sim_E 1 \iff \exists h_1, h_2 \in H : h_1 x = h_2 \iff x \in H'.
\]

Thus \( G_E = H' \).

c): Let \( G := G_E \). If \( x \sim_G y \) then \( g_1 x = g_2 y \) with some \( g_1, g_2 \in G \). From \( g_1 \sim_E 1, g_2 \sim_E 1 \), we conclude that

\[
x \sim_E g_1 x = g_2 y \sim_E y.
\]

Thus \( E(G) \subset E \). If \( H \) is any submonoid of \( S(U) \) with \( E(H) \subset E \), then

\[
H \subset G_{E(H)} \subset G_E = G.
\]

Thus \( E(H) \subset E(G) \).

d): Assume that \( E \) is MFCE. If \( x \in G_E \) then we conclude from \( x \sim_E 1 \) that \( ex = e \). Thus \( G_E \subset S_e(U) \). Every multiplicative equivalence relation on \( U \) which is finer than \( E \) is MFCE. In particular, this holds for orbital relations. We learn from c) that \( E(G_E) \) is the coarsest OFCE-relation on \( U \) finer than \( E \).

We denote the set of saturated submonoids of \( S(U) \) by \( \text{Sat}(S(U)) \) and the set of saturated submonoids of \( S_e(U) \) by \( \text{Sat}(S_e(U)) \).

Scholium 8.9. Propositions 8.5 and 8.8 imply that we have an isomorphism of posets \( H \mapsto E(H) \) from \( \text{Sat}(S(U)) \) to \( \text{Orb}(U) \), mapping \( \text{Sat}(S_e(U)) \) onto \( \text{OFC}(U) \), with inverse map \( E \mapsto G_E \). {Here, of course, both sets \( \text{Sat}(S(U)) \) and \( \text{Orb}(U) \) are ordered by inclusion.}

It is fairly obvious that \( \text{Sat}(S(U)) \) is a complete lattice. Indeed, the supremum of a family \( (H_i \mid i \in I) \) of saturated submonoids of \( S(U) \) is the saturation \( H' \) of the submonoid of \( S(U) \) generated by the \( H_i \), while the infimum of this family is the saturation \( (\bigcap_i H_i)' \) of the intersection of the family. Thus also \( \text{Orb}(U) \) is a complete lattice. It follows that \( \text{Sat}(S_e(U)) \) and \( \text{OFC}(U) \) are complete sublattices of \( \text{Sat}(S(U)) \) and \( \text{Orb}(U) \), respectively.

Let \( \text{Mult}(U) \) denote the set of all multiplicative equivalence relations on \( U \), partially ordered by inclusion. In §7 we have seen that the subposet \( \text{MFC}(U) \) of \( \text{Mult}(U) \), consisting of the MFCE-relations on \( U \), is a complete lattice. In the same way one proves that \( \text{Mult}(U) \) itself is a complete lattice, the supremum and infimum of a family in \( \text{Mult}(U) \) being given in exactly the same way as in §7 for MFCE-relations. This makes it also evident that \( \text{MFC}(U) \) is a complete sublattice of \( \text{Mult}(U) \).

We doubt whether \( \text{Orb}(U) \) and \( \text{OFC}(U) \) are always sublattices of \( \text{Mult}(U) \) and \( \text{MFC}(U) \), respectively. But we have the following partial result.
Proposition 8.10. Let \((G_i \mid i \in I)\) be a family of submonoids of \(S(U)\), and let \(G\) denote the monoid generated by this family in \(S(U)\). Then, in the lattice \(\text{Mult}(U)\),
\[
E(G) = \bigvee_{i \in I} E(G_i).
\]

\{N.B. Thus the same holds in \(\text{MFC}(U)\), if every \(G_i \subset S_U(U)\).\}\]

Proof. Let \(F := \bigvee_{i} E(G_i)\) in \(\text{Mult}(U)\). Of course, \(F \subset E(G)\) since each \(E(G_i) \subset E(G)\). Let \(x, y \in U\) be given with \(x \sim_G y\). We want to conclude that \(x \sim_F y\), and then will be done.

We have \(gx = hy\) with elements \(g, h\) of \(G\). Now \(g\) and \(h\) are products of elements in \(\bigcup_i G_i\), and for any \(g' \in \bigcup_i G_i\) and \(z \in U\), we have \(z \sim_F g'z\). It follows that \(x \sim_F gx\) and \(y \sim_F hy\), hence \(x \sim_F y\). \(\square\)

We present an important case where \(\text{OFC}(U)\) and \(\text{MFC}(U)\) nearly coincide.

Theorem 8.11. Assume that every \(x \in T(U)\) is invertible; hence \(T(U)\) is a group under multiplication. The main case is that \(U\) is a supertropical semifield. \{Example 6.4ii\}, or \(E\) is orbital.

Proof. a) Assume that there exists some \(x_0 \in T(U)\) with \(x_0 \sim_E ex_0\). Multiplying by \(x_0^{-1}\) we obtain \(1 \sim_E e\) and then obtain \(x \sim_E ex\) for every \(x \in U\). Thus \(E = E(\nu)\). 

b) Assume now that \(x \not\sim_E ex\) for every \(x \in T(U)\) (i.e., \(E \subset E_\nu\)). Clearly \(S_\nu(U) = T_\nu(U)\). Let
\[
H := G(E) = \{x \in T(U) \mid x \sim_E 1\}.
\]
Then \(E(H) \subset E\). Given \(x, y \in U\) with \(x \sim_E y\), we want to prove that \(x \sim_H y\). We have \(ex = ey\). If \(x \in eU\) or \(y \in eU\), we conclude that \(x = y\), due to our assumption on \(E\). There remains the case that both \(x\) and \(y\) are tangible. Then we infer from \(x \sim_E y\) that
\[
1 = x^{-1}x \sim_E x^{-1}y.
\]
Thus \(x^{-1}y \in H\), which implies \(x \sim_H y\). This completes the proof that \(E = E(H)\). \(\square\)

Corollary 8.12. If every element of \(T(U)\) is invertible, then the poset \(\text{MFC}(U) \setminus \{E(\nu)\}\) is isomorphic to the lattice of subgroups of \(T_\nu(U)\).

Proposition 8.13. If \(R\) is a semifield, then every supervaluation \(\varphi : R \to U\) with \(U \neq eU\) is tangible.

This follows from Theorem 8.11 applied to the target \(U(\nu)\) of the initial supervaluation \(\varphi_\nu\) of \(v := e\varphi\), since for any orbital equivalence relation \(E\) on \(U(\nu)\) the transmission \(\pi_E\) sends tangibles to tangibles. A more direct proof runs as follows.

Proof. Let \(a \in R\), \(a \neq 0\). Then
\[
\varphi(a)\varphi(a^{-1}) = \varphi(1) = 1.
\]
Since \(1_U \neq e_U\) this forces \(\varphi(a)\) to be tangible. \(\square\)

N.B. The argument shows more generally that any supervaluation on a semiring sends units to tangible elements, provided not the whole target is ghost.

In the case that \(R\) is a field the following in now amply clear.
Scholium 8.14. *If $v$ is a Krull valuation on a field $R$ with value group $\Gamma$, then the lattice $\text{Cov}(v)$ of equivalence classes of supervaluations covering $v$ is anti-isomorphic to the lattice of subgroups of the unit group $\mathfrak{o}_v^*$ of the valuation domain $\mathfrak{o}_v := \{ x \in R \mid v(x) \leq 1 \}$, augmented by one element at the top.*

9. The ghost surpassing relation; strong supervaluations

Let $U$ be any supertropical semiring. If $x, y \in U$, it has become customary to write
\[ x = y + \text{ghost} \]
if $x$ equals $y$ plus an unspecified ghost element (including zero). In more formal terms we have a binary relation $\models_{gs}$ on $U$ defined as follows:

**Definition 9.1.**
\[ x \models_{gs} y \iff \exists z \in eU \text{ with } x = y + z. \]

We call $\models_{gs}$ the *ghost surpassing relation* on $U$ or *GS-relation*, for short.

The GS-relation seems to be at the heart of many supertropical arguments. Intuitively $x \models_{gs} y$ means that $x$ coincides with $y$ up to some “negligible” or “near-zero” element, namely a ghost element. But we have to handle the GS-relation with care, since it is not symmetric. In fact it is antisymmetric, see below.

The GS-relation is clearly transitive:
\[ x \models_{gs} y, y \models_{gs} z \Rightarrow x \models_{gs} z. \]

It is also compatible with addition and multiplication: For any $z \in U$, $x \models_{gs} y$ implies $x + z \models_{gs} y + z$, and $xz \models_{gs} yz$.

We observe the following further properties of this subtle binary relation.

**Remark 9.2.** Let $x, y \in U$.
(i) $x = y \Rightarrow x \models_{gs} y \Rightarrow \nu(x) \geq \nu(y)$.
(ii) If $x \in \mathcal{T}(U) \cup \{0\}$, then $x \models_{gs} y \Leftrightarrow x = y$.
(iii) If $x \in \mathcal{G}(U) \cup \{0\}$, then $x \models_{gs} y \Leftrightarrow \nu(x) \geq \nu(y)$.
(iv) $x \models_{gs} 0$ iff $x \in eU$.

**Lemma 9.3.** The GS-relation is antisymmetric, i.e.;
\[ x \models_{gs} y, y \models_{gs} x \Rightarrow x = y. \]

**Proof.** If $x \in \mathcal{T}(U)$ or $y \in \mathcal{T}(U)$ this is clear by Remark 9.2(ii). Assume now that both $x, y \in eU$. Then $\nu(x) \geq \nu(y)$ and $\nu(y) \geq \nu(x)$ by Remark 9.2(iii); hence $\nu(x) = \nu(y)$, i.e., $x = y$. 

**Proposition 9.4.**
Proof. (i) Assume that $\alpha : U \to V$ is a transmission. Then, for any $x, y \in U$,

$$x \triangleright^{gs} y \Rightarrow \alpha(x) \triangleright^{gs} \alpha(y).$$

(ii) Assume that $\varphi : R \to U$ and $\psi : R \to V$ are supervaluations with $\varphi \geq \psi$. Then for any $a, b \in R$

$$\varphi(a) \triangleright^{gs} \varphi(b) \Rightarrow \psi(a) \triangleright^{gs} \psi(b).$$

Proof. i): Let $x \triangleright^{gs} y$. If $x$ is tangible or zero, then $x = y$; hence $\alpha(x) = \alpha(y)$. If $x$ is ghost, then $\nu(x) \geq \nu(y)$; hence

$$\nu(\alpha(x)) = \alpha(\nu(x)) \geq \alpha(\nu(y)) = \nu(\alpha(y))$$

by rule TM5 in §5. Since $\alpha(x)$ is ghost, this means $\alpha(x) \triangleright^{gs} \alpha(y)$, cf. Remark 9.2.iii above.

ii): We may assume that the supervaluation $\varphi$ is surjective. By §5 we have a (unique) transmission $\alpha : U \to V$ with $\alpha \circ \varphi = \psi$. Thus the claim follows from part i).

We cannot resist giving a second proof of part ii) of the proposition relying only on Definition 5.1 of dominance (conditions D1-D3).

Second proof of Proposition 9.4 ii. Assume that $\varphi(a) \triangleright^{gs} \varphi(b)$. If $\varphi(a)$ is tangible or zero, then $\varphi(a) = \varphi(b)$; hence $\psi(a) = \psi(b)$ by D1; hence $\psi(a) \triangleright^{gs} \psi(b)$. If $\varphi(a)$ is ghost then $e\varphi(a) \geq e\varphi(b)$; hence $e\psi(a) \geq e\psi(b)$ by D2. By D3 the element $\psi(a)$ is ghost. Thus $\psi(a) \triangleright^{gs} \psi(b)$ again.

The GS-relation seems to be helpful for analyzing additivity properties of supervaluations.

Lemma 9.5. If $\varphi : R \to U$ is a supvaluation on a semiring $R$ with $\varphi(a) + \varphi(b) \in eU$, then

$$\varphi(a) + \varphi(b) \triangleright^{gs} \varphi(a + b).$$

(*)

Proof. Let $v : R \to eU$ denote the $m$-valuation covered by $\varphi$, $v = e\varphi$. We have $v(a + b) \leq v(a) + v(b)$; hence $e\varphi(a + b) \leq e(\varphi(a) + \varphi(b))$. If $\varphi(a) + \varphi(b) \in eU$, this shows that $\varphi(a) + \varphi(b) \triangleright^{gs} \varphi(a + b)$.

It will turn out to be desirable to have supervaluations on $R$ at hand, where the property (*) holds for all elements $a, b$ of $R$.

Definition 9.6. We call a supvaluation $\varphi : R \to U$ **tangibly additive**, if in addition to the rules SV1-SV4 from §4 the following axiom holds:

$$SV5 : \text{If } a, b \in R \text{ and } \varphi(a) + \varphi(b) \in T(U), \text{ then } \varphi(a) + \varphi(b) = \varphi(a + b).$$

Proposition 9.7. A supvaluation $\varphi : R \to U$ is tangibly additive iff for any $a, b \in R$

$$\varphi(a) + \varphi(b) \triangleright^{gs} \varphi(a + b).$$

Proof. This is clear by Lemma 9.5 and Remark 9.2 ii above. \qed
Corollary 9.8. If \( \varphi : R \to U \) is tangibly additive, then for every finite sequence \( a_1, \ldots, a_m \) of elements of \( R \)
\[
\sum_{i=1}^{m} \varphi(a_i) \models \varphi \left( \sum_{i=1}^{m} a_i \right).
\]

Proof. This holds for \( m = 2 \) by Proposition 9.7. The general case follows by an easy induction
using the transitivity of the GS-relation. \( \square \)

Comment: We elaborate what it means that a given supervaluation \( \varphi : R \to U \) is tangibly
additive in the case that the underlying \( m \)-valuation \( v = e\varphi : R \to eU \) is strong.

Let \( a, b \in R \) be given with \( \varphi(a) + \varphi(b) \in T(U) \), i.e., \( v(a) \neq v(b) \), and assume without loss of
generality that \( v(a) < v(b) \). Then \( v(a+b) = v(b) \). Hence, \( \varphi(a+b) \) is some element of
the fiber \( \nu_1^{-1}(v(b)) \); but the axioms SV1-SV4 say little about the position of \( \varphi(a+b) \) in this
fiber. SV5 demands that \( \varphi(a+b) \) has the “correct” value \( \varphi(a) + \varphi(b) = \varphi(b) \).

Concerning applications the strong \( m \)-valuations seem to be more important than the others. (Recall that any \( m \)-valuation on a ring is strong.) Thus the tangibly additive
supevaluations covering strong \( m \)-valuations deserve a name on their own.

Definition 9.9. We call a supervaluation \( \varphi : R \to U \) strong if \( \varphi \) is tangibly
additive and the covered \( m \)-valuation \( e\varphi : R \to eU \) is strong.

We exhibit an important case where a tangibly additive supervaluation is automatically
strong.

Proposition 9.10. Assume that \( \varphi : R \to U \) is a tangible (cf. Definition 4.7) and tangibly
additive supervaluation. Then \( \varphi \) is strong.

Proof. We have to verify that \( v := e\varphi \) is strong. Let \( a, b \in R \) be given with \( v(a) \neq v(b) \).
Suppose without loss of generality that \( v(a) < v(b) \). Then \( \varphi(a), \varphi(b) \in U \) and \( \varphi(b) \neq 0 \).
Since \( \varphi \) is tangible, \( \varphi(b) \in T(U) \). It follows that \( \varphi(a) + \varphi(b) \in T(U) \); hence
\[
\varphi(a) + \varphi(b) = \varphi(a + b),
\]

because \( \varphi \) is tangibly additive. Multiplying by \( e \) we obtain
\[
v(a) + v(b) = v(a + b).
\]

\( \square \)

We now are ready to aim at an application of the supervaluation theory developed so far.
We start with the polynomial semiring \( R[\lambda] = R[\lambda_1, \ldots, \lambda_n] \) in a sequence \( \lambda = (\lambda_1, \ldots, \lambda_n) \)
of \( n \) variables over a semiring \( R \). Let \( \varphi : R \to U \) be a tangibly additive valuation
with underlying \( m \)-valuation \( v : R \to M, M := eU \).

Given a polynomial
\[
f = \sum_i c_i \lambda^i \in R[\lambda] \tag{9.1}
\]
in the usual multinomial notation (\( i \) runs though the multi-indices \( i = (i_1, \ldots, i_n) \in \mathbb{N}_0^n \),
\( \lambda^i = \lambda_1^{i_1} \cdots \lambda_n^{i_n} \), only finitely many \( c_i \neq 0 \), we obtain from \( f \) polynomials
\[
\varphi(f) := \sum_i \varphi(c_i) \lambda^i \in U[\lambda],
\]
\[
\hat{v}(f) := \sum_i v(c_i) \lambda^i \in M[\lambda],
\]
by applying $\varphi$ and $v$ to the coefficients of $f$. This gives us maps

$$\tilde{\varphi} : R[\lambda] \to U[\lambda], \quad \tilde{v} : R[\lambda] \to M[\lambda].$$

Let $a = (a_1, \ldots, a_n) \in R^n$ be an $n$-tuple of elements of $R$. It gives us $n$-tuples

$$\varphi(a) = (\varphi(a_1), \ldots, \varphi(a_n)), \quad v(a) = (v(a_1), \ldots, v(a_n))$$
in $U^n$ and $M^n$, respectively. We have an evaluation map $\varepsilon_a : R[\lambda] \to R$, which sends the polynomial $f$ (notation as in (9.1)) to

$$\varepsilon_a(f) = f(a) = \sum_i c_i a^i \tag{9.2}$$

and analogous evaluation maps

$$\varepsilon_{\varphi(a)}(f) : U[\lambda] \to U, \quad \varepsilon_{v(a)}(f) : M[\lambda] \to M.$$

These evaluation maps are semiring homomorphisms. We have a diagram

$$\begin{array}{ccc}
R[\lambda] & \xrightarrow{\varepsilon_a} & R \\
\downarrow{\tilde{\varphi}} & & \uparrow{\varphi} \\
U[\lambda] & \xrightarrow{\varepsilon_{\varphi(a)}} & U
\end{array}$$

(and an analogous diagram with $v$ instead of $\varphi$) which usually does not commute. But it commutes “nearly”.

**Theorem 9.11.** For $f \in R[\lambda]$

$$\varepsilon_{\varphi(a)}(\tilde{\varphi}(f)) \mid_{gs} = \varphi(\varepsilon_a(f)).$$

**Proof.** Let again $f = \sum_i c_i \lambda^i$. Now $\varphi(\varepsilon_a(f)) = \varphi(\sum_i c_i a^i)$, while

$$\varepsilon_{\varphi(a)}(\tilde{\varphi}(f)) = \sum_i \varphi(c_i) \varphi(a)^i = \sum_i \varphi(c_i a^i).$$

Thus the claim is that

$$\sum_i \varphi(c_i a^i) \mid_{gs} = \varphi(\sum_i c_i a^i). \tag{*}$$

This follows from Corollary 9.8 above. □

We draw a consequence of this theorem. Let

$$Z(f) := \{a \in R^n \mid f(a) = 0\},$$

the zero set of $f$. Let further

$$Z_0(\tilde{\varphi}(f)) := \{b \in U^n \mid \tilde{\varphi}(f)(b) \in eU\},$$

which we call the root set of $\tilde{\varphi}(f)$. For $a \in Z(f)$ we have $\varphi(\sum_i c_i a^i) = 0$. It follows by Theorem 9.11 that $\tilde{\varphi}(f)(\varphi(a)) \mid_{gs} = 0$, i.e., $\tilde{\varphi}(f)(\varphi(a))$ is ghost.

We have proved

**Corollary 9.12.** If $\varphi : R \to U$ is tangibly additive, then, for any $f \in R[\lambda]$, 

$$\varphi(Z(f)) \subset Z_0(\tilde{\varphi}(f)).$$

□
Assume now that \( \varphi \) is tangible and tangibly additive; hence strong (cf. Proposition 9.10). Then, of course, \( \varphi(Z(f)) \subset T(U)^0 \) with \( \mathcal{T}(U)_0 := \mathcal{T}(U) \cup \{0\} \). Thus we have

\[
\varphi(Z(f)) \subset Z_0(\tilde{\varphi}(f)) \tag{**}
\]

with

\[
Z_0(\tilde{\varphi}(f)) := Z_0(\tilde{\varphi}(f)) \cap \mathcal{T}(U)^0,
\]

which we call tangible root set of \( \tilde{\varphi}(f) \). We want to translate (**) into a statement about the relation between \( Z(f) \) and the so called “corner locus”, of the polynomial \( \tilde{v}(f) \in M[\lambda] \), to be defined.

We call a polynomial \( g = \sum_i d_i \lambda^i \in M[\lambda] \) a tropical polynomial, and define the corner-locus \( \text{Corn}(g) \) of \( g \) as the set of all \( b \in M^n \) such that there exists two different multi-indices \( j, k \in \mathbb{N}_0^n \) with

\[
d_jb^j = d_kb^k \geq d_ib^i
\]

for all \( i \neq j, k \). We also say that \( \text{Corn}(g) \) is the tropical hypersurface defined by the tropical polynomial \( g \).

This is well established terminology at least in the “classical case” that \( M \) is the bipotent semiring \( T(\mathbb{R}) \) given by the order monoid \( (\mathbb{R}, +) \), the so called max-plus algebra of \( \mathbb{R} \) (cf. §1, [LMS] §1.5)). A small point here is, that we admit coordinates with value \( 0_M := -\infty \), which usually is not done in tropical geometry. On the other hand we could work as well with Laurent polynomials. Then of course we would have to discard the zero element.

Returning to our tangible strong supervaluation \( \varphi : R \to U \) and the \( m \)-valuation

\[
v = e\varphi : R \to M,
\]

we look at the tropical polynomial

\[
\tilde{v}(f) = \sum_i v(c_i)\lambda^i
\]

from above. Let \( a \in R^n \). Then

\[
\tilde{\varphi}(f)(\varphi(a)) = \sum \varphi(c_i a^i),
\]

and all summands are the right side are in \( \mathcal{T}(U)_0 \). Thus the sum is ghost iff the maximum of the \( \nu \)-values

\[
\nu(\varphi(c_i a^i)) = v(c_i)v(a^i) \quad (i \in \mathbb{N}_0^n)
\]

is attained for at least two multi-indices. This means that \( v(a) \in \text{Corn}(\tilde{v}(f)) \).

Thus (**) has the following consequence

**Corollary 9.13.** Let \( v : R \to M \) be a strong \( m \)-valuation on a semiring \( R \). Assume that there exists a tangible supervaluation \( \varphi : R \to U \) covering \( v \). Then for any polynomial \( f \in R[\lambda] \),

\[
v(Z(f)) \subset \text{Corn}(\tilde{v}(f)).
\]

We have arrived at a very general version of the Lemma of Kapranov ([EKL, Lemma 2.1.4]), as soon as we find a tangible cover \( \varphi : R \to U \) of the given \( m \)-valuation \( v : R \to M \). This turns out to be easy in the case that \( M \) is cancellative (i.e., \( v \) is a strong valuation).

**Lemma 9.14.** Suppose there is given a tangible multiplicative section of the ghost map \( v : U \to M \), i.e., a map \( s : M \to \mathcal{T}(U)_0 \) with \( s(0) = 0, s(1) = 1, s(xy) = s(x)s(y) \), and \( v(s(x)) = x \) for any \( x, y \in M \). Let \( v : R \to M \) be a strong \( m \)-valuation. Then \( s \circ v : R \to U \) is a tangible strong supervaluation covering \( v \).
Proof. Clearly \( \varphi = sv \) obeys SV1-SV4. Let \( a, b \in R \) be given with \( v(a) < v(b) \). Then \( v(a+b) = v(b) \); hence \( sv(a+b) = sv(b) \). Thus SV5 holds true. We have \( e\varphi = v \circ \varphi = v \). \( \square \)

**Example 9.15.** If \( U \) is a supertropical semifield, it is known that such a section \( s \) always exists ([IR3, Proposition 1.6]).

**Example 9.16.** Assume that \( M \) is a cancellative bipotent semiring, and \( v : R \rightarrow M \) is a strong valuation. We take \( U := D(M \setminus \{0\}) \) (Example 3.18), for which we write more briefly \( D(M) \). For every \( z \in M \) there exists a unique \( x \in T(U) \) with \( v(x) = z \). We write \( x = \hat{z} \). Clearly \( z \mapsto \hat{z} \) is a tangible multiplicative section of the ghost map, in fact the only one. By the lemma we obtain a tangible supervaluation

\[
\hat{v} : R \rightarrow U, \quad \hat{v}(z) := \overline{v(z)},
\]

which covers \( v \), in fact the only such supervaluation.

Looking again at Corollary 9.13 we now know that

\[
v(Z(f)) \subseteq \text{Corn}(\overline{v(f)}),
\]

whenever \( v : R \rightarrow M \) is a strong valuation and \( f \in R[\lambda] \).

10. The tangible strong supervaluations in \( \text{Cov}(v) \)

Given an \( m \)-valuation \( v : R \rightarrow M \), recall from §7 that the equivalence classes \([\varphi] \) of supervaluations \( \varphi \) covering \( v \) form a complete lattice \( \text{Cov}(v) \). Abusing notation, we usually will not distinguish between a supervaluation \( \varphi \) and its class \([\varphi] \), thus writing \( \varphi \in \text{Cov}(v) \) if \( \varphi \) covers \( v \). This will cause no harm in the present section. \( \{\text{N.B} \} \) If you are sceptical about this, you may always assume that \( \varphi \) is surjective, more specially, that \( \varphi = \varphi_v/E \) with \( \varphi_v \), the initial covering of \( v \) and \( E \) an MFCE-relation on \( U(v) \) (cf. Notation 6.14). These supervaluations \( \varphi \) are canonical representatives of their classes \([\varphi] \).

**Lemma 10.1.** Assume that \( \varphi : R \rightarrow U \) and \( \psi : R \rightarrow V \) are supervaluation with \( \varphi \geq \psi \).

(i) If \( \psi \) is tangible, then \( \varphi \) is tangible.

(ii) If \( \varphi \) is tangibly additive, then \( \psi \) is tangibly additive.

**Proof.** i): is clear from the axiom D3 in the definition of dominance (cf. Definition 5.11).

ii): follows from Propositions 9.7 and 9.4ii. \( \square \)

Starting from now we assume that \( v \) is a strong valuation (which means in particular that \( M \) is cancellative). Let \( q \) denotes the support of \( v \), i.e., \( q = v^{-1}(0) \).

**Notation 10.2.** \( \text{Cov}_t(v) \) denotes the set of tangible supervaluations in \( \text{Cov}(v) \), and \( \text{Cov}_a(v) \) denotes the set of strong (= tangibly additive) supervaluations in \( \text{Cov}(v) \). Finally, let

\[
\text{Cov}_{ts}(v) := \text{Cov}_t(v) \cap \text{Cov}_a(v),
\]

be the set of tangible strong supervaluations covering \( v \).

We already know by Example 9.16 that the set \( \text{Cov}_{ts}(v) \) is not empty. Lemma 10.1 tells us in particular that \( \text{Cov}_t(v) \) is an upper set and \( \text{Cov}_a(v) \) is a lower set in the poset \( \text{Cov}(v) \).

Let us study these sets more closely. We start with \( \text{Cov}_t(v) \). The initial supervaluation \( \varphi_v : R \rightarrow U(v) \) (cf. Definition 5.15) is the top (= biggest) element of \( \text{Cov}(v) \), and thus is also the top element of \( \text{Cov}_t(v) \). This can also be read off from the explicit description of \( \varphi_v \) in Example 4.5. The other elements of \( \text{Cov}(v) \) are the supervaluations \( \varphi_v/E : R \rightarrow U(v)/E \), with \( E \) running through the MFCE-relations on \( U(v) \). We have to find out which MFCE-relations \( E \) on \( U(v) \) give tangible supervaluations \( \varphi_v/E \).
Here is a definition which - for later use - is slightly more general than what we need now:

**Definition 10.3.** We call an equivalence relation $E$ on a supertropical semiring $U$ **ghost separating** if for all $x \in T(U)$, $y \in U$,

$$x \sim_E y \implies y \in T(U) \text{ or } x \sim_E 0.$$ 

If $E$ is an MFCE-relation on $U$, then $x \sim_E 0$ only if $x = 0$. Thus, $E$ is ghost separating iff $T(U)$ is a union of $E$-equivalence classes. This means that $E$ is finer than the MFCE-relation $E_t$ introduced in Examples 6.4, whose equivalence classes are the tangible fibers of $\nu_U$ and the one-point sets in $eU$.

If $\varphi : R \to U$ is a surjective tangible supervaluation and $E$ is an MFCE-relation on $U$, then it is obvious that $\varphi/E : R \to U/E$ is again tangible iff $E$ is ghost separating. Thus we see that $\varphi_v/E_t$ is the bottom (= smallest) element of $\text{Cov}_t(v)$.

Now recall from Example 6.11 that, in the notation at the end of §9 (Example 9.16),

$$U(v)/E_t = D(M);$$

hence $\varphi_v/E_t$ coincides with the only tangible cover $\hat{v}$ of $v$ with values in $D(M)$, cf. Example 9.16. We conclude that

$$\text{Cov}_t(\varphi) = \{ \psi \in \text{Cov}(v) \mid \psi \geq \hat{v} \}.$$ 

Again by Example 9.16 we know that $\hat{v}$ is strong. This $\hat{v}$ is also the bottom of the poset $\text{Cov}_{t_{\text{ts}}}(\varphi)$.

We turn to $\text{Cov}_v(v)$. We will construct a new element of this poset in a direct way. For that reason we introduce an equivalence relation on $R$.

**Definition 10.4.** Let $S(v)$ denote the equivalence relation on the set $R$ defined as follows. \{We write $\sim_v$ for $\sim_{S(v)}$.\}

If $a_1, a_2 \in R$ then

$$a_1 \sim_v a_2 \iff \text{either } v(a_1) = v(a_2) = 0,$$

or $\exists c_1, c_2 \in R$, with $v(c_1) < v(a_1), v(c_2) < v(a_2), a_1 + c_1 = a_2 + c_2$.

It is easily checked that $S(v)$ is indeed an equivalence relation on the set $R$, by making strong use if the assumption that the valuation $v$ is strong. This is the finest equivalence relation $E$ on $U$ such that $a \sim_E a + c$ if $v(c) < v(a)$. Observe also that

$$a_1 \sim_v a_2 \implies v(a_1) = v(a_2).$$

We claim that $S(v)$ is compatible with multiplication, i.e.,

$$a_1 \sim_v a_2 \implies a_1 b \sim_v a_2 b$$

for every $b \in R$. This is obvious if $a_1 \in q$ or $a_2 \in q$, or $b \in q$. Otherwise $v(b) > 0$, and we have elements $c_1, c_2 \in R$ with $v(c_1) < v(a_1), v(c_2) < v(a_2), a_1 + c_1 = a_2 + c_2$. Then $a_1 b + c_1 b = a_2 b + c_2 b$ and

$$v(c_i b) = v(c_i) v(b) < v(a_i) v(b) = v(a_i b)$$

for $i = 1, 2$, since by assumption $M$ is cancellative. Thus indeed $a_i b \sim_v a_2 b$.

We denote the $S(v)$-equivalence class of an element $a$ of $R$ by $[a]_v$. The set $R/S(v)$ is a monoid under the well defined multiplication

$$[a]_v \cdot [b]_v = [a b]_v.$$
for \(a, b \in R\). The subset \(R \setminus q\) of \(R\) is a union of \(S(v)\)-equivalence classes and the subset \(\overline{R \setminus q} := (R \setminus q)/S(v)\) of \(\overline{R}\) is a submonoid of \(R\). We have
\[
\overline{R} = \overline{R \setminus q} \cup \{0\}
\]
with \(\overline{0} = [0]_v = q\).

Since \(a_1 \sim_v a_2\) implies \(v(a_1) = v(a_2)\), we have a well defined monoid homomorphism \(\overline{R} \to M, [a]_v \mapsto v(a)\), which restricts to a monoid homomorphism
\[
\bar{v}: \overline{R \setminus q} \to M \setminus \{0\}.
\]
This map \(\bar{v}\) gives us a supertropical semiring \(U := \text{STR}(\overline{R \setminus q}, M \setminus \{0\}, \bar{v})\), cf. Construction \([3,10]\). Notice that \(T(U) = \overline{R \setminus q}\) and \(eU = M\). We identify \(T(U)_0 = \overline{R}\).

**Proposition 10.5.** The map \(\chi: R \to U\) given by
\[
\chi(a) := 0 \quad \text{if} \quad a \in q, \quad \chi(a) := [a]_v \in T(U) = \overline{R \setminus q} \quad \text{if} \quad a \notin q,
\]
is a tangible strong supervaluation covering \(v\).

**Proof.** It is obvious that \(\chi\) obeys the rules SV1-SV3 in the definition of supervaluations (Definition \([4,11]\)). Due to our construction of \(U\) we have \(\nu_U \circ \chi = v\). Thus \(\chi\) also obeys SV4, and hence is a supervaluation covering the strong valuation \(v\). It is clearly tangible.

It remains to verify that \(\chi\) is tangibly additive. Let \(a, b \in R\) be given with \(\chi(a) + \chi(b) \in T(U)\), i.e., \(v(a) \neq v(b)\). Assume without loss of generality that \(v(a) < v(b)\). Then \(a + b \sim_v b\). This means that \(\chi(a + b) = \chi(b)\), as desired. \(\Box\)

We strive for an understanding of the set of all \(\psi \in \text{Cov}(v)\) which are dominated by this supervaluation \(\chi\). We need a new definition.

**Definition 10.6.** We call a supervaluation \(\varphi: R \to V\) very strong, if
\[
\text{SV5}^* : \forall a, b \in R : e\varphi(a) < e\varphi(b) \implies \varphi(a + b) = \varphi(b).
\]

Clearly SV5* implies that the \(m\)-valuation \(v\) is strong. If we require this property only for \(a, b \in R\) with \(e\varphi(a) < e\varphi(b)\) and \(\varphi(b)\) tangible, we are back to condition SV5 given above (Definition \([9,6]\)). Thus, a very strong supervaluation is certainly strong. On the other hand, every tangible strong supervaluation is very strong.

**Lemma 10.7.** If \(\varphi: R \to V\) is very strong, then any supervaluation \(\psi: R \to W\) dominated by \(\varphi\) is again very strong.

**Proof.** Let \(a, b \in R\) be given with \(e\psi(a) < e\psi(b)\). It follows from axiom D2 that \(e\varphi(a) < e\varphi(b)\), since \(e\varphi(a) \geq e\varphi(b)\) would imply \(e\psi(a) \geq e\psi(b)\). Thus \(\varphi(a + b) = \varphi(b)\), and we obtain by D1 that \(\psi(a + b) = \psi(b)\). \(\Box\)

Returning to our given strong valuation \(v: R \to M\), let \(\text{Cov}^*_v(v)\) denote the subset of all \(\varphi \in \text{Cov}(v)\) which are very strong. Lemma \([10,7]\) tells us in particular that \(\text{Cov}^*_v(v)\) is a lower set in the poset \(\text{Cov}(v)\), and hence in \(\text{Cov}_s(v)\). We have
\[
\text{Cov}_t(v) \cap \text{Cov}^*_s(v) = \text{Cov}_1(v) \cap \text{Cov}_s(v) = \text{Cov}_{1,s}(v).
\]

**Theorem 10.8.** The tangible strong supervaluation \(\chi: R \to U\) from above (Proposition \([10,2]\)) dominates every very strong supervaluation covering \(v\), and hence is the top element of both \(\text{Cov}^*_v(v)\) and \(\text{Cov}_{1,s}(v)\).
Proof. Let $\psi : R \rightarrow V$ be a very strong strong supervaluation covering $v$ (in particular $eV = M$). We verify axioms D1-D3 for the pair $\chi$, $\psi$, and then will be done. D2 is obvious, and D3 holds trivially since $\chi$ is tangible. Concerning D1, assume that $\chi(a_1) = \chi(a_2)$. By definition of $\chi$ this means that $a_1 \sim_v a_2$.

We have to prove that $\psi(a_1) = \psi(a_2)$. Either $a_1, a_2 \in q$, or there exist $c_1, c_2 \in R$ with $v(c_1) < v(a_1)$, $v(c_2) < v(a_2)$, $c_1 + a_1 = c_2 + a_2$. In the first case $e\psi(a_1) = e\psi(a_2) = 0$ hence $\psi(a_1) = \psi(a_2) = 0$. In the second case we have $\psi(a_1) = \psi(a_1 + c_1) = \psi(a_2 + c_2) = \psi(a_2)$ since $\psi$ is very strong. Thus $\psi(a_1) = \psi(a_2)$ in both cases. \hfill \Box

Notation 10.9. We denote the semiring $U$ given above by $U(v)$ and the supervaluation $\chi$ given above by $\varphi_v$. We call
\[ \varphi_v : R \rightarrow U(v) = \text{STR}(R \setminus q, M \setminus \{0\}, v) \]
the initial very strong supervaluation covering $v$.

In this notation
\[ \text{Cov}_v^s(v) = \{ \psi \in \text{Cov}(v) | \varphi_v \geq \psi \}, \]
\[ \text{Cov}_v^t(v) = \{ \psi \in \text{Cov}(v) | \varphi_v \geq \psi \geq \hat{v} \}. \]

Let $E(v)$ denote the equivalence relation on $U(v)$ whose equivalence classes are the sets $[a]_v$ with $a \in R \setminus q = T(U(v))$ and the one point sets $\{x\}$ with $x \in M$. In other terms, the restriction $E(v)|T(U)$ coincides with $S(v)|R \setminus q$, while $E(v)|M$ is the diagonal $\text{diag}(M)$ of $M$. We identify
\[ U(v)/E(v) = \overline{U(v)} \]
in the obvious way.

Proposition 10.10. $E(v)$ is a ghost separating MFCE-relation and
\[ \varphi_v = \varphi_v/E(v). \]

Proof. It is immediate that $E(v)$ is MFCE and ghost separating. For $a \in R \setminus q$ we have
\[ \pi_{E(v)}(\varphi_v(a)) = \pi_{S(v)}(a) = [a]_v = \varphi_v(a) \]
and for $a \in q$
\[ \pi_{E(v)}(\varphi_v(a)) = \pi_{E(v)}(a) = 0 = \varphi_v(a), \]
again. Thus $\pi_{E(v)}$ is the transmission from $\varphi_v$ to $\varphi_v$. \hfill \Box

Corollary 10.11. The MFCE-relations $E$ on $U(v)$ such that $\varphi_v/E$ is very strong are precisely all $E \in \text{MFC}(U(v))$ with $E \supset E(v)$.

Proof. This is a consequence of our observations above (Lemma 10.7 Theorem 10.8 Proposition 10.10) and the theory in §7, cf. Theorem 7.2 \hfill \Box

We now focus on the special case that $R$ is a semifield. Slightly more generally we assume that every element of $R \setminus q$ is invertible, while $q$ may be different from $\{0\}$.

$T(U(v)) = R \setminus q$ is a group under multiplication. Thus the results from the end of §8 apply. We have
\[ T_e(U(v)) = \{ a \in R | v(a) = 1_M \} = o_v^s, \]
with $o_v^s$ the unit group of the subsemiring
\[ o_v := \{ a \in R | v(a) \leq 1_M \} \]
of $R$. Notice that the set
\[ m_v := o_v \setminus o_v^* = \{ a \in R \mid \psi(a) < 1_M \} \]
is an ideal of $o_v$, just as in the classical (and perhaps most important) case, where $R$ is a field and $v$ is a Krull valuation on $R$.

By Theorem 8.11 and Corollary 8.12 we know that every MFCE-relation on $U(v)$ except $E(v)$ is orbital, hence ghost separating. We have
\[ \psi_v/E(v) = v, \]
viewed as a supervaluation. The other supervaluations $\psi$ covering $v$ correspond uniquely with the subgroups $H$ of $o_v^*$ via $\psi = \psi_v/E(H)$; cf. Corollary 8.12.

Instead of $U(v)/E(H)$ and $\psi_v/E(H)$ we now write $U(v)/H$ and $\psi_v/H$ respectively. In this notation
\[ T(U(v)/H) = (R \setminus q)/H, \]
and $\psi_v/H : R \to U(v)/H$ is given by
\[ (\psi_v/H)(a) = \begin{cases} aH & \text{if } a \in R \setminus q, \\ 0 & \text{if } a \in q. \end{cases} \]

**Theorem 10.12.** Assume that every element of $R \setminus q$ is invertible (e.g. $R$ is a semifield).

(i) Every strong supervaluation covering $v$ is very strong. Except $v$ itself, viewed as a supervaluation, all these supervaluations are tangible. In other terms,
\[ \text{Cov}_v(v) = \text{Cov}_v^*(v) = \text{Cov}_{t,s}(v) \cup \{ v \}. \]

(ii) $\overline{\psi}_v = \psi_v/(1 + m_v)$, with $(1 + m_v)$ the group generated by $1 + m_v$ in $o_v^*$.\(^6\)

(iii) The tangible strong supervaluations $\psi$ covering $v$ correspond uniquely with the subgroup $H$ of $o_v^*$ containing the semigroup $1 + m_v$ via $\psi = \psi_v/H$. Thus we have an anti-isomorphism $H \mapsto \psi_v/H$ from the lattice of all subgroups $H$ of $o_v^*$ containing $1 + m_v$ to the lattice $\text{Cov}_{t,s}(v)$.

**Proof.**

(i): Every supervaluation $\psi$ covering $v$ is either tangible or $\psi = v$. Thus, if $\psi$ is strong, then $\psi$ is very strong in both cases.

(ii): We know that $\overline{\psi}_v = \psi_v/E(v)$ (Proposition 10.10). $E(v)$ is ghost separating, hence orbital. The subgroup $H$ of $o_v^*$ with $E(H) = E(v)$ has the following description (cf. Proposition 8.8): If $a \in R \setminus q = T(U(v))$, then $a \in H$ if $a \sim_v 1$. This means that there exist elements $c_1, c_2 \in m_v$ with $a + c_1 = 1 + c_2$. Now $a + c_1 = a(1 + d_1)$ with $d_1 = \frac{o_v}{a} \in m_v$. Thus $a \sim_v 1$ if $a$ is in the group $(1 + m_v)$.

(iii): Now obvious, since $\overline{\psi}_v$ is the top element of $\text{Cov}_{t,s}(v)$.

We look again at the GS-sentence
\[ \varepsilon_{\psi(a)}(\varepsilon(f)) \models \psi(\varepsilon_a(f)) \quad (\ast) \]
from §9, valid for any $\psi \in \text{Cov}_v(v)$, $f \in R[\lambda]$, $a \in R^n$, cf. Theorem 9.11. Choosing here any $\psi \in \text{Cov}_{t,s}(v)$, we learned that $(\ast)$ implies Kapranov’s Lemma (Corollary 9.13). But the statement $(\ast)$ itself has a different content for different $\psi \in \text{Cov}_{t,s}(v)$. If also $\psi \in \text{Cov}_{t,s}(v)$ and $\psi \geq \psi$, then we obtain statement $(\ast)$ for $\psi$ from the statement $(\ast)$ for $\psi$, leaving $f$ and the tuple $a$ fixed, by applying the transmission $\alpha_{\psi,\psi}$. Thus it seems that $(\ast)$ has the most content if we choose for $\psi$ the initial strong supervaluation $\overline{\psi}_v : R \to U(v)$.

---

\(^6\)If $R$ is a field, then $(1 + m_v) = 1 + m_v$. 

---
We close this section by an explicit description of $U(v)$ and $\varphi_v$ in a situation typically met in tropical geometry. Let $R := F\{t\}$ be the field of formal Puiseux series with real powers over any field $F$, cf. [IMS, p.6]. The elements of $R$ are the formal series
\[ a(t) = \sum_{j \in I} c_j t^j \]
with $c_j \in F^*$ and $I \subset \mathbb{R}$ a well ordered set, in set theoretic sense, (including $I = \emptyset$). Let further $M$ be the bipotent semifield $T(\mathbb{R}_{>0})$ (cf. Theorem 1.5), i.e.,
\[ M = \mathbb{R}_{>0} \cup \{ 0 \} = \mathbb{R}_{\geq 0}, \]
with the max-plus structure.

We define a (automatically strong) valuation $v : F\{t\} \rightarrow M$ by putting
\[ v(a(t)) := \vartheta^{\min(I)} \]
if $a(t) \neq 0$, written as above, and $v(0) := 0$. Here $\vartheta$ is a fixed real number with $0 < \vartheta < 1$ (cf. [IMS] loc. cit, but we use a multiplicative notation). Now $\ast_v$ is the group consisting of all series
\[ a(t) = c_0 + \sum_{j > 0} c_j t^j, \quad c_0 \neq 0, \]
in $F\{t\}$, and $1 + m_v$ is the subgroup consisting of these series with $c_0 = 1$.

The equivalence relation $S(v)$ on $R^* = T(U(v))$ is given by
\[ a(t) \sim_v b(t) \iff \frac{a(t)}{b(t)} \in 1 + m_v. \]
This means that the series $a(t)$ and $b(t)$ have the same leading term $\ell(a(t)) = \ell(b(t))$. Thus the group of monomials
\[ G := \{ ct^j \mid c \in F^*, \ j \in \mathbb{R} \} \]
is a system of representatives of the equivalence classes of $S(v)$. We identify
\[ G = R^*/S(v) = T(U(v))/E(v). \]
Then $U(v) = \text{STR}(G, \mathbb{R}_{>0}, v|G) = G \cup M$ in the notation of Construction 3.16, and our supervaluation $\varphi_v : R \rightarrow U(v)$ is the map $a(t) \mapsto \ell(a(t))$, which sends each formal series $a(t)$ to its leading term. \{We read $\ell(0) = 0$, of course.\}

In short, applying $v$ to a series $a(t)$ means taking its leading $t$-power and replacing $t$ by $\vartheta$, while applying $\varphi_v$ means taking its leading term.

Similarly we can interpret the bottom supervaluation $\hat{\varphi} \in \text{Cov}_{t,s}(v)$. The $t$-powers $t^j$, $j \in \mathbb{R}$, are a multiplicative set of representatives of the $E_t$-equivalence classes. Identifying
\[ U(v)/E_t = \{ t^j \mid j \in \mathbb{R} \}, \]
we can say that $\hat{\varphi}(a(t))$ is the leading $t$-power of the series $a(t)$. The ghost map from $U(v)/E_t = D(M)$ to $M$ sends $t$ to $\vartheta$.

\[ \text{For the matter of geometric applications, one usually needs } F \text{ to be algebraically closed, but here we can omit this restriction.} \]
11. IQ-VALUATIONS ON POLYNOMIAL SEMIRINGS AND RELATED SUPERVALUATIONS.

Since the semiring of polynomials over a supertropical domain is no longer supertropical (or analogously, the semiring of polynomials over a bipotent semiring is no longer bipotent), we would like a theory generalizing valuations to maps with values in these polynomial semirings. Unfortunately, the target is no longer an ordered group (and is not even an ordered monoid). In this section, we formulate some concepts of this paper in the more general context of monoids with a supremum, instead of ordered monoids, and show how this encompasses Kapranov’s Lemma.

Recall that an operation $a \lor b$ on a set $S$ is called a sup if it has a distinguished element $0$ and satisfies the following properties for all $a, b, c \in S$:

1. $0 \lor a = a$;
2. $a \lor b = b \lor a$;
3. $a \lor a = a$;
4. $a \lor (b \lor c) = (a \lor b) \lor c$.

In this case, we can define a partial order on $S$ by defining $a \leq b$ when $a \lor b = b$. Then the following properties are immediate for all $a, b, c \in S$:

(a) $0 \leq a$;
(b) $a \lor b \geq a$ and $a \lor b \geq b$;
(c) if $a \leq c$ and $b \leq c$, then $a \lor b \leq c$. (Indeed, if $a \lor c = c$ and $b \lor c = c$, then $(a \lor b) \lor c = (a \lor c) \lor (b \lor c) = c \lor c = c$.)

We also say that a given sup $x \lor y$ on a monoid $M$ is compatible with $M$ if $a(x \lor y) = ax \lor ay$ for all $a, x, y \in M$.

In order to axiomatize this in the language of semirings, we recall that an idempotent semiring $R$ satisfies the property that $x + x = x$ for all $x \in R$.

**Proposition 11.1.**

(i) Every idempotent semiring $R$ can be viewed as a multiplicative monoid with a compatible sup $\lor$ defined by

$x \lor y := x + y$.

(ii) Conversely, given a monoid $M$ with a compatible sup, we can define an idempotent semiring structure on $M$, with the same multiplication, and with addition given by $x + y := x \lor y$.

**Proof.** All of the other verifications are immediate. □

**Remark 11.2.** If $R$ is an idempotent semiring, then so is the polynomial semiring $R[\lambda]$ as well as the matrix semiring $M_n(R)$.

Both of these assertions fail when we substitute “bipotent” for “idempotent.” Thus, it makes sense to pass to idempotent semirings when studying polynomials and matrices. In the case of semifields, we actually have a lattice structure.

**Proposition 11.3.** If $R$ is a semifield, where $\lor$ is given by addition (as in Proposition 11.1), then there is a compatible inf relation $\land$ given by $x \land y := \frac{xy}{x+y}$ (taking $0 \land 0 = 0$), thereby making $(R, \lor, \land)$ a distributive lattice satisfying

$(x \lor y)(x \land y) = xy$, $\forall x, y \in R$.  \hspace{1cm} (11.1)
Proof. Property (11.1) follows at once from the definitions, and implies that \(a(x \land y) = ax \land ay\), as well as associativity of \(\land\). To check distributivity, we need to check
\[(x \land y) \lor z = (x \lor z) \land (y \lor z).\]
Since \(\leq\) is clear, we only check \(\geq\), and also may assume \(x, y, z \neq 0\). Now
\[
(x \land y) \lor z = \frac{xy}{x + y} + z
\geq \frac{xy}{x + y + z} + \frac{z(x + y)}{x + y + z}
= \frac{(x + z)(y + z)}{x + y + z} = \frac{(x + z)(y + z)}{(x + z) + (y + z)}
= (x \lor z) \land (y \lor z).
\]
\[
(11.2)
\]
□

Having the translation of the sup relation to semirings at hand, we are ready to reformulate some of the results of this paper. But first it is instructive to introduce a parallel of the ghost surpassing relation.

**Definition 11.4.** \(y \mid x \iff \exists a \in R \text{ with } y = x + a\).

Clearly, \(\mid\) is a transitive binary relation on \(R\).

**Definition 11.5.** \(R\) is an upper-bound semiring, written ub-semiring, if the relation \(\mid\) is anti-symmetric; i.e.,
\[x \mid y \text{ and } y \mid x \iff x = y.\]

The reason for this terminology is that now the relation \(\mid\) gives a partial ordering on the set \(R\)
\[a \leq b \iff b \mid a \iff \exists c \in R: a + c = b,\]
and \(x + y\) is an upper bound of \(x, y\) in this ordering\(^8\).

**Remark 11.6.**
(i) The condition that a semiring \(R\) is ub can be rephrased as follows:
For any \(a, b, x \in R\), if \(x + a + b = x\), then \(x + a = x\).
(ii) Any ub-semiring \(R\) has the property that \(a + b = 0\) implies \(a = b = 0\), by (i). (Take \(x = 0\).)

**Proposition 11.7.** Any idempotent semiring is an ub-semiring.

Proof. If \(x + a + b = x\), then
\[x + a = (x + a + b) + a = x + a + b = x.\]
\[\square\]

If \(R\) is any semiring, let \(R[\lambda] = R[\lambda_1, \ldots, \lambda_n]\) denote the polynomial semiring over \(R\) in a set of variables \(\lambda = (\lambda_1, \ldots, \lambda_n)\).

**Proposition 11.8.** Every supertropical semiring \(U\) is upper bound, and \(U[\lambda_1, \ldots, \lambda_n]\) is upper bound for every \(n\).

\(^8\)All inequalities in the following will refer to this ordering.
Proof. We have to check the condition in Remark 11.6.i. Let \( x, a, b \in U \) be given with \( x + a + b = x \). We have to verify that \( x + a = x \). Multiplying by \( e \) we obtain \( ex + ea + eb = ex \), hence \( ea \leq ex \) and \( eb \leq ex \). If \( ea < ex \), then \( x + a = x \) right away. If \( eb < ex \), then \( x + b = x \), hence \( x = x + a + b = x + a \) again. There remains the case that \( ea = eb = ex \). Now \( x + a + b = ex \), hence \( x \) is ghost, and \( x + a + b = ex = x \) again. This proves that \( U \) is ub.

Let now \( f, g, h \in U[\lambda_1, \ldots, \lambda_n] \) be given with \( f + g + h = f \). We write \( f = \sum \alpha_i \lambda^i \), \( g = \sum \beta_i \lambda^i \), \( h = \sum \gamma_i \lambda^i \). Then \( \alpha_i + \beta_i + \gamma_i = \alpha_i \) for every \( i \), and we conclude that \( \alpha_i + \beta_i = \alpha_i \) for every \( i \), hence \( f + g = f \), as desired.

The reason we want to consider the idempotent semiring \( M[\lambda] \) is that we want to extend any \( m \)-valuation \( v : R \to M \) to the map \( \tilde{v} : R[\lambda] \to M[\lambda] \), where we define

\[
\tilde{v}\left(\sum_i \alpha_i \lambda_i^1 \ldots \lambda_n^i\right) = \sum_i v(\alpha_i) \lambda_i^1 \ldots \lambda_n^i. \tag{11.3}
\]

Since \( M[\lambda] \) is no longer bipotent in the natural way, we would like to generalize Definition 2.1 to permit valuations to idempotent semirings.

Unfortunately, \( \tilde{v} \) as defined in (11.3) need not satisfy property V3 of Definition 2.1 since \( \tilde{v}(fg) \) could differ from \( \tilde{v}(f)\tilde{v}(g) \). Indeed, if \( f = \sum_i \alpha_i \lambda^i \) and \( g = \sum_j \beta_j \lambda^j \), with \( i = (i_1, \ldots, i_n) \) and \( j = (j_1, \ldots, j_n) \), then writing \( fg = \sum_k \left( \sum_{i+j=k} \alpha_i \beta_j \right) \lambda^k \), we have

\[
\tilde{v}(fg) = \sum_k v\left( \sum_{i+j=k} \alpha_i \beta_j \right) \lambda^k \\
\leq \sum_k \sum_{i+j=k} v(\alpha_i)v(\beta_j) \lambda^k \\
= \left( \sum_i v(\alpha_i) \lambda^i \right) \left( \sum_j v(\beta_j) \lambda^j \right),
\]

where there could be strict inequality. (Notice that our partial ordering on \( M[\lambda] \) extends the total ordering of \( M \).) Accordingly, we need a weaker notion:

**Definition 11.9.** An **iq-valuation** (= idempotent monoid quasi-valuation) on a semiring \( R \) is a map \( v : R \to M \) into a (commutative) idempotent semiring \( M \neq \{0\} \) with the following properties:

\[
\begin{align*}
IQV1 : & \quad v(0) = 0, \\
IQV2 : & \quad v(1) = 1, \\
IQV3 : & \quad v(xy) \leq v(x)v(y) \quad \forall x, y \in R, \\
IQV4 : & \quad v(x + y) \leq v(x) + v(y) \quad \forall x, y \in R.
\end{align*}
\]

{NB: Here as elsewhere we use the partial order introduced above following Definition 11.5.}

The following is now obvious.

**Proposition 11.10.** Suppose \( M \) is a bipotent semiring and \( v : R \to M \) is an \( m \)-valuation.

(i) Then the map \( \tilde{v} : R[\lambda] \to M[\lambda] \) given above is an iq-valuation.

(ii) For any given \( a \in M^n \), the map \( \varepsilon_a \circ \tilde{v} : R[\lambda] \to M \) is again an iq-valuation. {Here \( \varepsilon_a \) denotes the evaluation map \( f(\lambda) \mapsto f(a) \), as in the previous sections.}

\[\square\]

If \( v \) is strong we can do better.
Theorem 11.11. Assume that $v : R \to M$ is a surjective strong $m$-valuation. Then, for any $a \in M^n$, $\varepsilon_a \circ \tilde{v} : R[\lambda] \to M$ is again a strong $m$-valuation.

Proof. By an easy induction we restrict to the case of $n = 1$. Given $f = \sum_i \alpha_i \lambda^i$, $g = \sum_j \beta_j \lambda^j$ in $R[\lambda]$ we have to verify the following:

1. $\varepsilon_a \tilde{v}(fg) = \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g)$;
2. If $\varepsilon_a \tilde{v}(f) < \varepsilon_a \tilde{v}(g)$, then $\varepsilon_a \tilde{v}(f + g) = \varepsilon_a \tilde{v}(g)$.

1: We know already by Proposition 10.10 that

$$\varepsilon_a \tilde{v}(fg) \leq \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g).$$

Due to the bipotence of $M$ we have smallest indices $k$ and $\ell$ such that

$$\varepsilon_a \tilde{v}(f) = \sum_i v(\alpha_i) a^i = v(\alpha_k) a^k,$$

$$\varepsilon_a \tilde{v}(g) = \sum_j v(\beta_j) a^j = v(\beta_\ell) a^\ell.$$

We chose some $c \in R$ with $v(c) = a$. Since $v$ is strong and $k, \ell$ have been chosen minimally we have

$$v\left(\sum_{i+j=k+\ell} \alpha_i c^i \beta_j c^j\right) = v(\alpha_k c^k \beta_\ell c^\ell) = \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g).$$

Thus

$$\varepsilon_a \tilde{v}(fg) = \sum_r v\left(\sum_{i+j=r} \alpha_i \beta_j\right) v(c)^r$$

$$= \sum_r v\left(\sum_{i+j=r} \alpha_i c^i \beta_j c^j\right)$$

$$\geq \sum_{i+j=k+\ell} v\left(\sum_{i+j=k} \alpha_i c^i \beta_j c^j\right)$$

$$= \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g).$$

We conclude that

$$\varepsilon_a \tilde{v}(fg) = \varepsilon_a \tilde{v}(f) \cdot \varepsilon_a \tilde{v}(g).$$

(2): Assume that $\varepsilon_a \tilde{v}(f) < \varepsilon_a \tilde{v}(g)$. Using the same $k, \ell$, and $c$ as before we have for all $i$

$$v(\alpha_i c^i) < v(\beta_\ell c^\ell),$$

$$v(\beta_i c^i) \leq v(\beta_\ell c^\ell).$$

Now

$$\varepsilon_a \tilde{v}(f + g) = \sum_i v\left((\alpha_i + \beta_i) c^i\right),$$

and $v((\alpha_i + \beta_i) c^i) \leq v(\beta_\ell c^\ell)$ for all $i$, but

$$v\left((\alpha_i + \beta_i) c^i\right) = v(\beta_\ell c^\ell).$$

Thus,

$$\varepsilon_a \tilde{v}(f + g) = v(\beta_\ell c^\ell) = \varepsilon_a \tilde{v}(g).$$

□
In particular, we could take $v$ to be the natural valuation on the field of Puiseux series with rational exponents, as used in $\lfloor G \rfloor$, or with real exponents as introduced above in §11.

Let us formulate the analogue of Definition 4.1 in the realm of semirings with ghosts.

**Definition 11.12.** An iq-supervaluation on a semiring $R$ is a map $\varphi : R \to U$ from $R$ to a $u_b$-semiring $U$ with ghosts, satisfying the following properties.

- $\text{IQSV1} : \varphi(0) = 0$,
- $\text{IQSV2} : \varphi(1) = 1$,
- $\text{IQSV3} : \forall a, b \in R : \varphi(ab) \leq \varphi(a)\varphi(b)$,
- $\text{IQSV4} : \forall a, b \in R : e\varphi(a + b) \leq e(\varphi(a) + \varphi(b))$.

Here again we use the ordering given by the relation $\models$. The definition works in particular for $U$ a supertropical semiring and to Proposition 11.8.

We are ready for the main purpose of this section.

**Theorem 11.13.** Assume that $\varphi : R \to U$ is a surjective strong supervaluation, and $v : R \to eU = M$ is the strong $m$-valuation covered by $\varphi$. Let $a = (a_1, \ldots, a_n) \in U^n$ be given, and let $b := (ea_1, \ldots, ea_n) \in M^n$.

(i) $\varphi$ can be extended to an iq-supervaluation $\tilde{\varphi} : R[\lambda] \to U[\lambda]$ by the formula

$$\tilde{\varphi}\left(\sum_i a_i\lambda^i\right) = \sum_i \varphi(a_i)\lambda^i.$$

(ii) $\varepsilon_a \circ \tilde{\varphi} : R[\lambda] \to U$ is a strong supervaluation. It covers the (strong) valuation $\varepsilon_b \circ \tilde{v} : R[\lambda] \to M$.

**Proof.** (i): If $a, b \in R$ then we know from §9 that $\varphi(a) + \varphi(b) \models \varphi(a + b)$. This implies $\varphi(a) + \varphi(b) \models \varphi(a + b)$, i.e.

$$\varphi(a + b) \leq \varphi(a) + \varphi(b). \quad (\ast)$$

An argument parallel to the one before Definition 11.9 now tells us that for $f, g \in R[\lambda]$ we have

$$\tilde{\varphi}(f + g) \leq \tilde{\varphi}(f) \cdot \tilde{\varphi}(g).$$

Clearly $\tilde{\varphi}$ extends $\varphi$, in particular $\tilde{\varphi}(0) = 0$, $\tilde{\varphi}(1) = 1$. From (\ast) it is also obvious that $\tilde{\varphi}(f + g) \leq \tilde{\varphi}(f) + \tilde{\varphi}(g)$, hence

$$e\tilde{\varphi}(f + g) \leq e\tilde{\varphi}(f) + e\tilde{\varphi}(g).$$

Thus, $\tilde{\varphi}$ is an iq-supervaluation. Clearly $e\tilde{\varphi}(f) = \tilde{v}(f)$ for all $f \in R[\lambda]$. {By the way, this gives us again that $e\tilde{\varphi}(f + g) \leq e\tilde{\varphi}(f) + e\tilde{\varphi}(g)$.

(ii): Again we restrict to the case of $n = 1$ by an easy induction. It is pretty obvious that $\varepsilon_a \tilde{\varphi} : R[\lambda] \to U$ obeys the rules SV1, SV2, SV4 from §4 (Definition 11.1), and $e \cdot \varepsilon_a \tilde{\varphi}(f) = \varepsilon_b \tilde{v}(f)$ for every $f \in R[\lambda]$. Given $f = \sum_i a_i\lambda^i$, $g = \sum_i \beta_i\lambda^i$ in $R[\lambda]$ it remains to prove the following:

1. $\varepsilon_a \tilde{\varphi}(fg) = \varepsilon_a \tilde{\varphi}(f) \cdot \varepsilon_a \tilde{\varphi}(g)$,
2. If $\varepsilon_a \tilde{\varphi}(f) \leq \varepsilon_a \tilde{\varphi}(g)$ then $\varepsilon_a \tilde{\varphi}(f + g) = \varepsilon_a \tilde{\varphi}(g)$. 

(1): Let $k, \ell$ be the minimal indices such that
\[
e \sum_i \varphi(\alpha_i) a^i = e \varphi(\alpha_k) a^k = e \varepsilon_a \tilde{\varphi}(f),
\]
\[
e \sum_i \varphi(\beta_i) a^i = e \varphi(\beta_\ell) a^\ell = e \varepsilon_a \tilde{\varphi}(g),
\]
(\text{as in the proof of Theorem 11.11}). We know by Theorem 11.11 that
\[
e (\varepsilon_a \circ \tilde{\varphi})(fg) = e \varphi(\alpha_k) a^k \cdot e \varphi(\beta_\ell) a^\ell = e (\varepsilon_a \circ \tilde{\varphi})(f) \cdot e (\varepsilon_a \circ \tilde{\varphi})(g).
\]
We chose some $c \in R$ with $\varphi(c) = a$. Using (*) we obtain
\[
(\varepsilon_a \circ \tilde{\varphi})(fg) = \sum_r \varphi \left( \sum_{i+j=r} \alpha_i \beta_j \right) a^r
= \sum_r \varphi \left( \sum_{i+j=r} \alpha_i c^i \cdot \beta_j c^j \right)
\leq \sum_r \sum_{i+j=r} \varphi(\alpha_i c^i) \cdot \varphi(\beta_j c^j)
= \sum_{i,j} \varphi(\alpha_i) a^i \cdot \varphi(\beta_j) a^j.
\]
There is a single $\nu$-dominating term in this sum iff there is a single $\nu$-dominating term on the left of (**) and of (** *), so we conclude that
\[
\varepsilon_a \tilde{\varphi}(fg) = \varepsilon_a \tilde{\varphi}(f) \cdot \varepsilon_a \tilde{\varphi}(g)
\]
in all cases, using the fact that tangible elements $x, y$ of $U$ with $x < y$, $ex = ey$ are equal.

(2): This can be proved in the way analogous to claim (2) in the proof of Theorem 11.11.

Thus, for $U$ a supertropical semiring, the evaluation map returns us from iq-supervaluations with values in $U[\lambda]$ to the firmer ground of supervaluations.

Looking again at Theorem 9.11 we realize now that the theorem gives pleasant examples of pairs of supervaluations which obey a “GS-relation” in the following sense.

**Definition 11.14.** If $\rho : A \to V$ and $\sigma : A \to V$ are supervaluations on a semiring $A$ with values in the same supertropical semiring $V$, then we say that $\rho$ **surpasses $\sigma$ by ghost**, and write $\rho \uparrow \gamma \sigma$, if $\rho(a) \uparrow \gamma \sigma(a)$ for every $a \in A$.

In this terminology Theorem 9.11 reads as follows:

**Theorem 11.15.** Let $\varphi : R \to U$ be a strong supervaluation. Then for any $a \in R^n$ the supervaluation $\varepsilon_\varphi(a) \circ \tilde{\varphi} : R[\lambda_1, \ldots, \lambda_n] \to U$ surpasses the supervaluation $\varphi \circ \varepsilon_a : R[\lambda_1, \ldots, \lambda_n] \to U$ by ghost.

Of course, we should look for other examples of pairs of supervaluations $\rho : A \to V$ and $\sigma : A \to V$ with $\rho \uparrow \gamma \sigma$. Here the “classical” case that $A$ is a semifield, or even a field, and $eV$ is cancellative, is perhaps not the most interesting one. Indeed, for such pairs $\rho, \sigma$ we have $e \rho(a) \geq e \sigma(a)$ for every $a \in A$, and this forces $e \rho(a) = e \sigma(a)$ since for $a \neq 0$ also $e \rho(a^{-1}) \geq e \sigma(a^{-1})$. Thus $\rho$ and $\sigma$ cover the same valuation $e \rho = e \sigma : A \to eV$. But for the pairs occurring in Theorem 11.15 where $A$ is a polynomial semiring, the valuation $e \rho$ and
\(e \sigma\) usually will have even different support, and \(\rho\) can be a very interesting “perturbation” of \(\sigma\) by ghosts.

The phenomenon of “surpassing by ghost” for supervaluations shows clearly the importance of studying valuations and supervaluations on semirings instead of just semifields.

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