Influence of the material substructure on crack propagation: a unified treatment.

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Abstract
The influence of the material texture (substructure) on the force driving the crack tip in complex materials admitting Ginzburg-Landau-like energies is analyzed in a three-dimensional continuum setting. The theory proposed accounts for finite deformations and general coarse-grained order parameters. A modified expression of the J-integral is obtained together with other path-integrals which are necessary to treat cases where the process zone around the tip has finite size. The results can be applied to a wide class of material substructures. As examples, cracks in ferroelectrics and in materials with strain-gradient effects are discussed: in these cases the specializations of the general results fit reasonably experimental data.

1 Introduction
To analyze the behavior of cracks, one needs to understand how the interactions in the material cooperate to drive the crack tip. In the common setting of deformable simple bodies (Cauchy’s model) the question has been clarified in many aspects from theoretical and computational points of view \(^1\). It is not so for complex bodies where experiments suggest that the influence of material texture (substructure) on the expression of the ‘force’ driving the crack tip may be prominent. For instance, for materials that fail by decohesion or cleavage

\(^1\)From the pioneer works of Griffith (1920), Atkinson & Eshelby (1968), Barenblatt (1972), Freund (1972), Rice (1968), to contributions like (Adda-Bedia, Arias, Ben Amar & Lund, 1999; Dolbow, Moës & Belytschko, 2001; Freund, 1990; Gurtin & Shvartsman, 1997; Heino & Kaski, 1997; Moës & Belytschko, 2002; Obrezanova, Movchan & Willis, 2002; Oleaga, 2003; Slepian, 2002).
at the atomic scale, the predictions of the standard theory of fracture are not satisfactory\(^2\), and the same inaccuracy occurs in the case of cracks propagating along metal-ceramic interfaces. Moreover, e.g., the spontaneous polarization in ferroelectrics influences rather strongly the crack growth\(^3\).

The investigation is matter of continuum theories more intricated than the standard elasticity where only some global aspects of the material texture are considered through material symmetries, and a direct description of the configuration of the substructure (together with its changes) and the related interactions is absent.

However, if one would like to determine the appropriate expression of the force driving the crack tip in each special case of complex material, one would have to construct only a zoo of possible modifications of the standard theory of fracture for small and large strains, obscuring the basic fact that there exists a common unifying physical mechanism. The aim of the present paper is just to show such a mechanism common to all complex materials admitting Ginzburg-Landau-like energies and to provide accordingly a general expression of the driving force which can be specified and tested in special cases. We consider large strains to encompass cases in which they are prominent as, e.g., in presence of elastomers.

In what follows, an order parameter \(\nu\) is assigned to each material element as coarse grained geometrical descriptor of the substructure. To maintain generality, we require only that \(\nu\) belongs to a differentiable manifold \(M\). Each special choice of \(M\) characterizes the model of each complex material. Interactions developing extra power on the rate of the order parameter, i.e., on the rate of change of the substructure, are considered and satisfy appropriate balance equations. Models of fiber-reinforced composites, ferroelectric and magnetoelastic solids, interacting elastomers, microcracked and multiphase solids fall, e.g., within such a general approach.

In what follows, first we derive balance equations at the crack tip for the interactions generated by the substructure, considering possible substructural inertia effects at the tip. Then we determine the contribution of substructural interactions to the expression of the driving force at the tip of the crack. It results in a modified expression of the \(J\)-integral:

\[
J = n \cdot \int_{\text{tip}} \left( \rho \left( \frac{1}{2} \| \dot{x} \|^2 + k (\nu, \dot{\nu}) \right) I + P \right) n,
\]

where \(n\) is the direction of propagation of the crack, the second-order tensor \(P\) is given by \(P = \psi I - F^T T - \nabla \nu^T S\), with \(\psi\) the free energy density, \(F\) the gradient of deformation, \(T\) the first Piola-Kirchhoff stress, \(S\) a measure of substructural interactions called microstress, \(I\) the unit tensor, \(\rho\) the density of mass, while the other terms account for standard and substructural kinetic energy (if existent). \(\int_{\text{tip}}\) indicates a special limit process which consists in evaluating an integral on
the boundary of a disc centered at the tip in a plane orthogonal to the tangent of the tip at a given point and in shrinking the disc up to the tip. The product $\star$ is defined below. Special expressions for the $J$-integral follow once the order parameter is specified. Here, to suggest examples, ferroelectrics and materials with strain-gradient effects are considered: the relevant $J$-integrals provide values of the driving force close to experimental data. Other path-integrals besides $J$-integral are obtained: they allow us to express the energy dissipated during the evolution of the crack in the case in which the process zone around the tip has finite size. In the expression of $J$, two elements mark the difference with the standard theory of fracture, namely the densities $\rho k (\nu, \dot{\nu})$ and $\nabla \nu^T \star S$ that underline the explicit influence of the material substructure. Whilst the former is negligible unless the substructure oscillates at very high frequencies, the latter may be crucial to justify the discrepancies between experimental data and the previsions of the standard fracture mechanics. (To render the formulas as concise as possible, in the integrals we do not write explicitly line, area and volume differentials: the kind of integration is clear looking at the domain of integration directly.)

2 Order parameters for the substructure and the geometry of the crack

Let $\mathcal{B}$ be the regular region of the three-dimensional Euclidean point space $E^3$ occupied by the body in its reference place. A generic point of $\mathcal{B}$ is indicated with $X$. A ‘standard’ deformation is described by a sufficiently smooth bijection $\mathcal{B} \ni X \mapsto x(X) \in E^3$ (with the current place $x(B)$ a regular region) which is also orientation-preserving in the sense that the gradient $\nabla x$ of $x$ with respect to $X$ (indicated with $F$) is such that $\det F > 0$.

As anticipated above, information on the substructure of each material element are given through the assignment of an order parameter $\nu$, by means of a sufficiently smooth mapping $\mathcal{B} \ni X \mapsto \nu(X) \in \mathcal{M}$, where $\mathcal{M}$ is a finite dimensional differentiable paracompact manifold without boundary endowed with metric and connection (Capriz, 1989). The choice of $\mathcal{M}$ determines the characteristic features of each special model of substructure. Vector order parameters

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4 Notations. Some standard notations are summarized here. Non-standard notations are introduced in detail in the rest of the paper. Let $A$ and $B$ be tensors of type $(p, q)$ of components e.g. $A_{i_1 \ldots i_p}^{j_1 \ldots j_q}$. We denote with $A \cdot B$ the standard scalar product given by $A_{i_1 \ldots i_p}^{j_1 \ldots j_q} B_{i_1 \ldots i_p}^{j_1 \ldots j_q}$. In particular, if $A$ and $B$ are second order tensors, we denote with $AB$ the product which contracts only one index and bears a second order tensor; for example, we have $(AB)_{ij} = A_{ik} B_{kj}$. If $A$ and $B$ are third order tensors, we indicate with $A : B$ the product contracting two indices and bearing a second order tensor. If $A$ is a tensor of the type $(p, q)$, with $p, q > 0$, and $B$ is another tensor of the type $(r, s)$, with $r, s > 0$ and $r < p$, $s < q$, or $(r = p, s < q)$ or $(r < p, s = q)$, we indicate with $AB$ (with some slight abuse of notation with respect to the product between second order tensors) the product which contracts all the indices of $B$; in particular, if $p = 0$ or $q = 0$ we take $r = 0$ and $s = 0$ respectively. Given two vectors $a$ and $b$, $a \otimes b$ denotes their tensor product. In particular, if $A$ and $B$ are second order tensors we have $AB \cdot (a \otimes b) = A^T a \cdot Bb$. For any region $b$ of the space, $\partial b$ represents its boundary.
with unit length may represent stiff microfibers in composites with a softer matrix or magnetostrictive materials. Vector order parameters not constrained to have unit length are used for ferroelectrics, elastic microcracked bodies, nematic elastomers. Second order tensor valued order parameters may also serve as descriptors of families of polymeric chains or polymer stars.

Let $\dot{x}(X,t)$ and $\dot{\nu}(X,t)$ be the velocity and the rate of the order parameter (in their referential description) evaluated by a given observer. After a change in observers ruled by $SO(3)$ new rates $\dot{x}^*$ and $\dot{\nu}^*$ are measured: they are given by

$$\dot{x}^* = \dot{x} + c(t) + \dot{q}(t) \times (x - x_0),$$  
$$\dot{\nu}^* = \dot{\nu} + A\dot{q},$$

where $c$ and $\dot{q}$ are translational and rotational velocities, respectively; $A\dot{q}$ is the infinitesimal generator of the action of $SO(3)$ on $M$ and $A = \frac{d\nu_q}{dq}\big|_{q=0}$ (a linear operator mapping vectors of $\mathbb{R}^3$ into elements of the tangent space of $M$) is represented by a matrix with three columns and a number of lines equal to dim $M$, being $\nu_q$ the order parameter measured after the action of $SO(3)$.

2.1 Cracks

We imagine that the reference place $B$ be free of cracks. When a crack is generated in the current configuration of the body, the mapping $X \mapsto \mathbf{x}(X)$ is pointwise one-to-one except a surface $C \equiv \{X \in B, f(X) = 0\}$, with $f$ a smooth function (Fig. 1). The assumption of smoothness for $f$ is only of convenience. Notice that $C$ is only a geometrical (non-material) picture in the reference configuration of the real crack occurring in $\mathbf{x}(B)$.

The intersection of $C$ with the boundary of $B$ is a regular curve $\partial B \cap C$ endowed with unit normal $m$ such that $m(X)$ belongs to the tangent plane of $C$ at each $X \in \partial B$. Subsets $b$ of $B$ are called ‘parts’ here when they are regular regions. When we consider any part $b_C$ intersecting $C$, the intersection $\partial b_C \cap C$ is a regular curve whose normal in the tangent space of $C$ is also indicated with $m$.

The normal $m$ to $C$ is defined by $m = \frac{\nabla f}{|\nabla f|}$; the opposite of its surface gradient, namely $L = -\nabla_C m$, is the curvature tensor, its trace is the opposite of the overall curvature $K$. In the case treated here, $C$ does not cross completely $B$. The image in $B$ of the real tip of the crack is thus the margin $J$ of $C$ within the interior of $B$. We assume that $J$ is a simple regular curve parametrized by arc length $s \in [0, \bar{s}]$ and represented by a point-valued mapping $Z : [0, \bar{s}] \rightarrow B$ so that the derivative $Z_s(s)$ of $Z$ with respect to $s$ is the tangent vector $t(s)$ at $Z(s)$, while $h = -Z_{ss}$ is the curvature vector. A normal vector field $n$ is chosen along $J$ to be at each $Z(s)$ an element of the tangent plane of $C$ at $Z(s)$ outward $C$ (Fig. 1).

When the crack evolves in the current configuration, its picture in the reference configuration is a surface $C(t)$ growing in a certain time interval $[0, \bar{t}]$; $J$ has then an intrinsic fictitious relative motion with respect to the rest of the body, while any piece of $C(t)$ far from $J$ remains at rest.
The intuitive behavior of the crack during the motion is simply described in the reference place by the monotonicity of $C(t)$, namely $C(t_1) \subseteq C(t_2), \forall t_1 \leq t_2$. We assume also that during the time interval in which we study the motion of the crack, it does not cut completely the body. In $B$ the velocity of $J$ is

$$v_{tip} = \frac{\partial Z(s,t)}{\partial t}. \quad (3)$$

Only its normal component $V = v_{tip} \cdot n$ is independent of the parametrization $s$, and in what follows, we shall consider only $v_{tip} = V n$.

For any field $e(X,t)$ continuous in time and space except $C$, where it suffers bounded discontinuities, its jump $[e]$ there is defined by the difference between the outer and the inner trace, i.e., $[e] = e^+ - e^-$ (when the difference makes sense), while the mean value $\langle e \rangle$ is given by $\langle e \rangle = \frac{1}{2} (e^+ + e^-)$. When the crack is closed in the current configuration, the requirement that its sides do not penetrate one into another during the deformation is then $[x] \cdot m = 0$. Some special choices of the order parameter require the continuity of it across $C$. Without loss of generality we can consider $\nu$ continuous across $C$, while its rate may suffer bounded discontinuities. When it is not so, since $\mathcal{M}$ is a non-linear manifold, the jump $\nu^+ - \nu^-$ could not make sense and to define $[\nu]$ it should be necessary to embed $\mathcal{M}$ into an appropriate linear space (a procedure based on Whitney’s or Nash’s theorems of embedding). In that case, since the embedding is not unique, one should select the one convenient to maintain the gauge properties of the underlying physics.

When there exists any field $\hat{e}(Z,t)$ defined along $J(t)$ and such that $e(X,t) \rightarrow \hat{e}(Z(s,t),t)$ as $X \rightarrow Z(s,t)$ uniformly in time, we say that $e$ has uniform limit.
at the tip and confuse \( \dot{e}(Z(s,t),t) \) with \( e(Z(s,t),t) \). In this sense, we indicate the tip rate of change of the order parameter as \( w_{\text{tip}} \) by considering it at each \( Z \in J \) as the uniform limit \( \lim_{X \to Z} \dot{\nu}(X,t) \).

Rates following the crack tip may be defined (by chain rule) as

\[
x^{\diamond} = \dot{x} + Fv_{\text{tip}}, \quad \nu^{\diamond} = \dot{\nu} + (\nabla \nu) v_{\text{tip}},
\]

and of course the derivatives \( x^{\diamond} \) and \( \nu^{\diamond} \) are meant for points away from the tip, being rates perceived by observers sitting on the crack tip. Their uniform limit at the crack tip are indicated with \( \tilde{v}_{\text{tip}} \) and \( \tilde{w}_{\text{tip}} \) respectively, being \( \tilde{v}_{\text{tip}} \) the velocity of the deformed tip and \( \tilde{w}_{\text{tip}} \) the rate of \( \nu \) at the deformed tip.

Moreover, let \( b \) be any part of \( B \). The boundary \( \partial b \) of \( b \) is a two-dimensional surface (of normal \( n \)) and may be parametrized by parameters \( u_1 \) and \( u_2 \). If we consider \( b \) varying in time, i.e., \( b(t) \), points \( X \) of the boundary \( \partial b(t) \) are identified by \( X(u_1,u_2,t) \) so that the velocity \( u \) of \( \partial b(t) \) is given by

\[
u = \frac{\partial X(u_1,u_2,t)}{\partial t}.
\]

Only the normal component \( U = u \cdot n \) is independent of the parametrization \( (u_1,u_2) \). Rates following the moving boundary \( \partial b(t) \) are then given by

\[
x^{\circ} = \dot{x} + Fu, \quad \nu^{\circ} = \dot{\nu} + (\nabla \varphi) u.
\]

3 Balance of standard and substructural interactions via an invariance argument

In the standard mechanics of deformable bodies, common stresses and bulk forces are the sole measures of interaction. When the material substructure is accounted for, the picture of the interactions become more articulated because we must consider objects measuring the extra power developed in the rate \( \dot{\nu} \).

Below, we summarize balance equations for standard and substructural interactions in the bulk, at the lateral margins of the crack and at the tip (and the attention is focused on the substructural ones because the others are well known). To obtain them we could follow two ways: (i) we could use variational arguments involving a Lagrangian and its invariance with respect to general Lie groups (underlining in this way their covariance\(^5\)) or (ii) we could require \( SO(3) \) invariance of the power of ‘external’ interactions. Though we prefer the former, we sketch here the latter because, in this way, we may clearly underline the distinction between balance equations and the constitutive structure of the interactions involved (two aspects that are mixed in an Hamiltonian approach).

Such a distinction clarifies our aim to obtain relations valid even for general irreversible processes in deformable solids in different circumstances, although our treatment deals mainly with non-linear elastic processes, unless otherwise stated.

\(^5\)Following in this way the guidelines of the basic proof of covariance developed for simple materials in (Marsden & Hughes, 1983).
3.1 Balance of interactions in the bulk

Let \( b \) be any arbitrary part of \( B \) (of boundary \( \partial b \)) far from the crack. The external power \( P_{\text{ext}}^{b} \) of the standard and substructural interactions on \( b \) is a linear real functional on the pairs \((\dot{x}, \dot{\nu})\) represented by

\[
P_{\text{ext}}^{b}(\dot{x}, \dot{\nu}) = \int_{b} (b \cdot \dot{x} + \beta \cdot \dot{\nu}) + \int_{\partial b} (Tn \cdot \dot{x} + S \cdot \dot{\nu}),
\]

where the substructural interactions are measured through volume \( \beta \) and surface \( S \) densities as in the case of standard interactions. The bulk density \( \beta \) may account for both possible substructural inertia effects and interactions due, e.g., to electromagnetic fields acting on the substructure (as in the case of ferroelectrics). Both \( b \) and \( \beta \) are continuous on \( b \).

Roughly speaking, \( \tau = Sn \) is a 'generalized traction'; the product \( \tau \cdot \dot{\nu} \) is the power exchanged between two adjacent parts at \( X \) through a surface of normal \( n \), as a consequence of the change of the substructure at the same point. Since our analysis is developed in the reference place \( B \), the first Piola-Kirchhoff stress \( T \) is used: it associates tensions in \( x(B) \) to vectors in \( B \) and is the pull-back in \( B \) of the standard (Cauchy) stress \( \sigma \) measuring "true" tensions in the current place \( x(B) \) of the body: in fact, \( T = (\det F) \sigma F^{-T} \).

Accordingly, the co-vector \( b \) is the pull back of the body forces (including inertia) living in the current place \( x(B) \).

We impose that \( P_{\text{ext}}^{b} \) is invariant under \( SO(3) \), for any \( b \), i.e.

\[
P_{\text{ext}}^{b}(\dot{x}^*, \dot{\nu}^*) = P_{\text{ext}}^{b}(\dot{x}, \dot{\nu}),
\]

for any choice of \( c(t), \dot{q}(t) \) and \( b \).

By using (1) and (2), thanks to the arbitrariness of \( c \) and \( \dot{q} \), we obtain the standard integral balance of forces

\[
\int_{b} b + \int_{\partial b} Tn = 0,
\]

and a generalized integral balance of moments:

\[
\int_{b} ((x - x_0) \times b + A^T \beta) + \int_{\partial b} ((x - x_0) \times Tn + A^T Sn) = 0.
\]

From (9) the common pointwise balance

\[
b + \text{Div} T = 0 \quad \text{in } B
\]

follows thanks to the arbitrariness of \( b \), while, from (11), we get

\[
A^T (\text{Div} S + \beta) = cT F^T - (\nabla A^T) S.
\]
with Ricci’s alternator. The condition (12) implies that the co-vector $e^{ABC} T_i B F^C_i - (\nabla A_i B_j) S_{ij}^B$ belongs to the range of the linear operator $A^T$ at each $\nu$. However, $A$ is not one-to-one, then at each $\nu$ there may exist an element $z$ of the cotangent space of $M$ at $\nu$ such that

$$A^T z = e T F^T - (\nabla A^T) S,$$

which implies

$$\text{Div} S - z + \beta = 0 \quad \text{in } B. \quad (14)$$

Equation (14) is the pointwise balance of substructural interactions (Capriz 1989) and $z$ is an internal self-force. Equation (13) states that the presence of substructural interactions renders unsymmetrical the Cauchy stress $\sigma$ given by $\sigma = (\text{det } F)^{-1} T F^T$. The case of scalar order parameters seems to be pathological for the procedure used here because $A$ vanishes. Formally, one may circumvent the problem by making use of spherical second order tensors, obtaining the balance (14), then reducing it to the scalar case (which would be in any case the Euler-Lagrange equation of some Lagrangian). If one is skeptical about such an interpretation, one could accept the point of view discussed here to derive (14) only for other types of order parameters and postulate (14) a priori in the scalar case.

### 3.2 Balance of interactions along the sides of the crack

Let $b_C$ be any arbitrary part intersecting the crack $C$ far from the tip. If one writes $P_{b_C}^{\text{ext}}$ and requires $SO(3)$ invariance, the integral balances (9) and (10) follow but $b_C$ is now the domain of integration. If we shrink $b_C$ to $b_C \cap C$, since the integrands of the volume integrals are continuous while the stress does not, we obtain

$$\int_{b_C} b \to 0 \quad \text{as } b_C \to b_C \cap C, \quad (15)$$

$$\int_{\partial b_C} T n \to \int_{b_C \cap C} [T] m \quad \text{as } b_C \to b_C \cap C. \quad (16)$$

Consequently, the arbitrariness of $b_C$ implies from (9) the common pointwise balance

$$[T] m = 0 \quad \text{along } C, \quad (17)$$

An analogous reasoning can be applied to (10) and leads to

$$\int_{\partial b_C} A^T [S] n = 0 \quad (18)$$

as $b_C \to b_C \cap C$. The arbitrariness of $b_C$ implies the pointwise balance $A^T [S] m = 0$ which is tantamount to write

$$[S] m = z'_C \quad \text{along } C, \quad \text{with } A^T z'_C = 0. \quad (19)$$

\(^6A, B, \ldots\) denote components in $B$, $i, j, \ldots$ components in $x(B)$ and $\alpha, \ldots$ components over the atlas on $M$. 

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(balance of substructural interactions along the sides of the crack). \( z'_C \) vanishes when at each \( \nu \) the range of \( \mathcal{A} \) covers the whole tangent space of \( \mathcal{M} \) there.

### 3.3 Balance of interactions at the tip of the crack

Take a part of \( \mathcal{B} \) with the form of a ‘curved cylinder’ (see, e.g., Fig. 1) \( b^*_R \) obtained by translating a disc \( D_R \) of diameter \( R \) from \( Z(s_1) \) to \( Z(s_2) \) (two arbitrary points of \( J \) with \( s_1 < s_2 \)) maintaining \( D_R \) orthogonal to \( t(s) \) at each \( Z(s) \) and the centre of \( D_R \) coincident with \( Z(s) \) at each \( s \). The external power of all interactions acting over \( b^*_R \) is

\[
P^\text{ext}_{b^*_R} (\dot{x}, \dot{\nu}, \ddot{v}_{\text{tip}}, \ddot{w}_{\text{tip}}) = \int_{b^*_R} (b \cdot \dot{x} + \beta \cdot \dot{\nu}) + \\
+ \int_{\partial b^*_R} (Tn \cdot \dot{x} + Sn \cdot \dot{\nu}) + \int_{s_1}^{s_2} (b_{\text{tip}} \cdot \ddot{v}_{\text{tip}} + \beta_{\text{tip}} \cdot \ddot{w}_{\text{tip}}),
\]

(20)

where \( b_{\text{tip}} \) collects only the inertia effects at the tip, and \( \beta_{\text{tip}} \) accounts formally for possible substructural inertia effects at the tip.

The changes of observers described in (1) and (2) can be written for the velocities at the tip as

\[
\ddot{v}_{\text{tip}}^* = \ddot{v}_{\text{tip}} + c(t) + \dot{q}(t) \times (x_{\text{tip}} - x_0),
\]

(21)

\[
\ddot{w}_{\text{tip}}^* = \ddot{w}_{\text{tip}} + A_{\text{tip}} \dot{q},
\]

(22)

where \( A_{\text{tip}} = \frac{d\omega_{\text{tip}}(Z)}{dq} \big|_{q=0} \) in the sense of uniform limit. We impose here \( SO(3) \) invariance requiring that

\[
P^\text{ext}_{b^*_R} (\ddot{x}^*, \ddot{\nu}^*, \ddot{v}_{\text{tip}}^*, \ddot{w}_{\text{tip}}^*) = P^\text{ext}_{b^*_R} (\dot{x}, \dot{\nu}, \ddot{v}_{\text{tip}}, \ddot{w}_{\text{tip}}),
\]

(23)

for any choice of \( c(t), \dot{q}(t) \) and \( b^*_R \). From (23), thanks to the arbitrariness of \([s_1, s_2]\), by shrinking \( b^*_R \) to the tip \((R \to 0)\), we obtain

\[
b_{\text{tip}} + \int_{\text{tip}} Tn = 0
\]

(24)

(balance of standard forces at the tip),

\[
\beta_{\text{tip}} + \int_{\text{tip}} Sn = z'_{\text{tip}}, \quad \text{with} \quad A_{\text{tip}}^T z'_{\text{tip}} = 0
\]

(25)

(balance of substructural interactions at the tip), where we use the notation \( \int_{\text{tip}} (\cdot) \) for \( \lim_{R \to 0} \int_{D_R} (\cdot) \), interpreting it in the sense of uniform limits.

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3.4 Standard and substructural inertia effects

3.4.1 Bulk terms

The bulk interactions $b$ and $\beta$ contain both inertial and non-inertial terms, namely $b = b^n + b^{ni}$ and $\beta = \beta^n + \beta^{ni}$. The overall power of the inertial terms over any stationary part $b$ far from the crack is equal to the opposite of the rate of kinetic energy of $b$, the density of which is commonly chosen to be the sum $\rho \left( \frac{1}{2} \| \dot{x} \|^2 + k (\nu, \dot{\nu}) \right)$, where $k (\nu, \dot{\nu})$ the kinetic contribution of the substructure, if there is some experimental evidence of it. From the arbitrariness of $b$, the common identification of the inertial terms follows (Capriz, 1989):

$$b^{in} = -\rho \ddot{x}, \quad \beta^{in} = -\frac{d}{dt} \rho \partial_\nu \chi + \rho \partial_\nu \chi,$$

(26)

where $\chi$ is the substructural kinetic co-energy density: its Legendre transform with respect to the rate coincides with the kinetic energy $k (\nu, \dot{\nu})$. In common cases, the experiments show that the term $\frac{d}{dt} \rho \partial_\nu \chi - \rho \partial_\nu \chi$ is in general negligible, unless the substructure oscillates at very high frequencies.

3.4.2 Tip effects

The standard results before allow us to identify explicitly the tip inertial terms. We consider a part around the tip the ‘curved cylinder’ $b^*_R$, varying it in time, $b^*_R(t)$, to follow the growth of the crack. We then write the integral versions of (26) over $b^*_R$ adding not only a tip inertial term $b_{tip}$, as in the common treatments, but also a substructural tip inertial term $\beta_{tip}$, and obtaining then

$$\int_{b^*_R(t)} b^{in} + \int_{s_1}^{s_2} b_{tip} = -\frac{d}{dt} \int_{b^*_R(t)} \rho \dot{x} + \int_{\partial b^*_R(t)} \rho \dot{x} U,$$

(27)

$$\int_{b^*_R(t)} \beta^{in} + \int_{s_1}^{s_2} \beta_{tip} = -\frac{d}{dt} \int_{b^*_R(t)} \rho \partial_\nu \chi +$$

$$+ \int_{b^*_R(t)} \rho \partial_\nu \chi + \int_{\partial b^*_R(t)} \rho \partial_\nu \chi U.$$

(28)

The last integral in (27) is the inflow of standard momentum through the boundary $\partial b^*_R(t)$ of $b^*_R$, due to the ‘virtual’ (non-material) movement of $b^*_R$ in time; an analogous meaning has the last integral in (28). By shrinking $b^*_R$ at the tip $\mathcal{I}$ (letting $R \to 0$), we get

$$b_{tip} = \int_{\mathcal{I}} \rho \dot{x} (\nu_{tip} \cdot n),$$

(29)

$$\beta_{tip} = \int_{\mathcal{I}} \rho \partial_\nu \chi (\nu, \dot{\nu}) (\nu_{tip} \cdot n),$$

(30)

thanks to the arbitrariness of $[s_1, s_2]$. The obvious changes in (24) and (25) follow. The result (29) is standard in the theory of dynamic fracture in simple bodies.
3.5 Stresses at the tip

The speed of the crack growth is finite, so we may assume that \( \int_{\partial D_R} \mathbf{n} \otimes \rho \dot{\mathbf{x}} \) be bounded up to the tip as \( R \to 0 \). Such an assumption (which is in certain sense on the behavior of the solution) implies that

\[
\int_{\text{tip}} \mathbf{n} \otimes \rho \dot{\mathbf{x}} = 0. \tag{31}
\]

Moreover, we assume also that the tip flow of substructural momentum be bounded up to the tip, i.e., \( \int_{\partial D_R} \rho \partial_{\nu} \chi \dot{U} \) is bounded as \( R \to 0 \) for any choice of the order parameter. This implies that \( \partial_{\nu} \chi \) is bounded as \( R \to 0 \) (because \( U \) is bounded) and then

\[
\int_{\text{tip}} \mathbf{n} \otimes \rho \partial_{\nu} \chi = 0. \tag{32}
\]

If the range of \( A_{\text{tip}} \) covers the whole tangent space of \( \mathcal{M} \) at \( \nu (Z, t) \), we get \( z'_{\text{tip}} = 0 \). Consequently, from (24), (25), (29), (30), we get

\[
\int_{\text{tip}} T_{\mathbf{n}} = 0, \quad \int_{\text{tip}} S_{\mathbf{n}} = 0. \tag{33}
\]

We could reverse the point of view, following a remark of Landau and Lifšits, and we could say that since we allow the crack to evolve, the stresses are bounded up to the tip. This circumstance together with the assumption (32) would imply (33) directly and (31) and \( z'_{\text{tip}} = 0 \) as further consequences.

4 Interactions due to the growth of the crack

The evolution of the crack is represented by the ‘fictitious’ growth of \( \mathcal{C} \) in the reference configuration \( \mathcal{B} \) (which on the contrary would remain fixed once and for all). Such a growth generates an independent kinematics in \( \mathcal{B} \) and interactions power-conjugated with it in the bulk, at the lateral sides and at the tip of the crack. They should satisfy appropriate balances. These interactions have a twofold nature: from one hand they live in \( \mathcal{B} \) and are thus different from the standard and substructural interactions (notice, e.g., that (11) is the balance of forces living in \( \mathbf{x}(\mathcal{B}) \)); from the other hand, since the kinematics of \( \mathcal{C} \) is only ‘apparent’ (non-material) in the reference configuration, the new ‘forces’ may be expressed in terms of standard and substructural interactions and the free energy. We list below these ‘fictitious forces’ and their balances that are commonly justified in various manners and used to describe different cases of mutations in bodies. However, we do not give contribution to the current discussion about the attribution of such balances \(^7\). In the identification of them in terms of standard and substructural interactions, our contribution relies in the deduction of the substructural components, and this step is crucial toward the main result of the present paper.

\(^7\)See the results and the theoretical discussions in (Abeyaratne & Knowles, 1990; Eshelby, 1975; Epstein, 2002; Gurtin, 1995; James, 2002; Segev, 1996; Šilhavý, 1997).
4.1 Balance in the bulk of forces due to the crack growth

For any time varying part \( b(t) \) (remind that the evolution of \( b \) is not material) we consider bulk internal and external forces, namely the vectors \( g \) and \( e \) respectively, and a stress \( P \), a second order tensor that maps at each \( X \) the tangent space there onto the corresponding cotangent space. It is commonly postulated that they satisfy the integral balance

\[
\int_{\partial b(t)} P \mathbf{n} + \int_{b(t)} (g + e) = 0, \tag{34}
\]

for any choice of \( b(t) \). The pointwise balance

\[
\text{Div} P + g + e = 0, \quad \text{in } B, \tag{35}
\]

then follows.

In subsequent steps, the strategy foresees the identification of the various elements of (35) in terms of the true stresses by making use of a mechanical dissipation inequality (a mechanical version of the second law). We use such a procedure having in mind the need to manage an approach valid even in fully dissipative processes like viscosity or plastic flows. However, we develop the identification of the terms of (35) just in non-linear elasticity (suggesting also the necessary developments in other circumstances) to put in evidence the basic fact that in that case, the balance (35) reduces to one of the conservation laws that can be deduced from a Nöther-like theorem, precisely the one associated with the invariance of the Lagrangian with respect to diffeomorphisms altering \( B \).

4.2 Balance along the sides of the crack of forces due to the crack growth

For any arbitrary part \( b_{C}(t) \) intersecting the crack far from the tip, we write the balance (34) adding surface measures of interaction along the margins of the crack, namely a surface stress \( \bar{\sigma} (I - m \otimes m) \), with \( I \) the unit second order tensor and \( \bar{\sigma} (X) \) a scalar function continuous up to the tip where it is indicated with \( \bar{\sigma}_{\text{tip}} \), and an internal surface force \( g_{C} \) (vector). So that we obtain

\[
\int_{\partial b_{C}(t)} P \mathbf{n} + \int_{b_{C}(t)} (g + e) + \int_{(\partial b_{C}(t)) \cap C(t)} \bar{\sigma} m + \int_{b_{C}(t) \cap \mathcal{C}(t)} g_{C}. \tag{36}
\]

The arbitrariness of \( b_{C} \) and the bulk balance (35) imply

\[
[P] m + g_{C} + \nabla_{C} \bar{\sigma} + \bar{\sigma} (kI - L) m = 0 \quad \text{along } \mathcal{C}, \tag{37}
\]

where \( \nabla_{C} \) denotes the surface gradient (see Section 2).
4.3 Balance of tip forces due to the crack growth

Along the tip we consider a line tension \( \lambda_{\text{tip}} t \), a tip internal force \( g_{\text{tip}} \) (vector), and a tip external inertial force \( e_{\text{tip}} \) (vector). If we consider a ‘curved cylinder’ \( b_R^* \) of the type used above) around the tip, intersecting \( J \) in two points, say \( Z(s_1) \) and \( Z(s_2) \), in writing on \( b_R^* \) the balance of interactions power-conjugated with the ‘fictitious’ kinematics of \( C(t) \), we add a line term

\[
(\lambda_{\text{tip}} t(s_2) - \lambda_{\text{tip}} t(s_1)) + \int_{s_1}^{s_2} (g_{\text{tip}} + e_{\text{tip}}).
\]

(38)

to the bulk and surface terms used in (36). By shrinking \( b_R^* \) up to the tip letting \( R \to 0 \) (uniformly in time), the arbitrariness of \( [s_1, s_2] \) implies

\[
g_{\text{tip}} + e_{\text{tip}} - \bar{\sigma}_{\text{tip}} n - \lambda_{\text{tip}} h + \int_{\text{tip}} \mathbb{P} n = 0 \quad \text{along } J.
\]

(39)

4.4 Identification of the inertial term \( e_{\text{tip}} \)

To identify explicitly \( e_{\text{tip}} \) in terms of the standard and substructural measures of interaction, it suffices to consider a ‘curved cylinder’ \( b_R^* \) wrapped around the tip and to write an inertial balance of the type

\[
\mathbb{K}^{\text{rate}} (b_R^*(t)) + \int_{b_R^*(t)} (b^{\text{in}} \cdot \dot{x} + \beta^{\text{in}} \cdot \dot{\nu}) + \\
\int_{s_1}^{s_2} (b_{\text{tip}} \cdot \bar{v}_{\text{tip}} + \beta_{\text{tip}} \cdot \bar{w}_{\text{tip}} + e_{\text{tip}} \cdot v_{\text{tip}}) = 0,
\]

(40)

where \( \mathbb{K}^{\text{rate}} (b_R^*(t)) \) is the rate of the kinetic energy of \( b_R^*(t) \) and, of course, \( b_{\text{tip}} \) and \( \beta_{\text{tip}} \) develop power in the actual rates. Since \( b_R^* \) varies in time, \( \mathbb{K}^{\text{rate}} \) is the difference between the time derivative of the integral of \( \rho \left( \frac{1}{2} ||\dot{x}||^2 + k(\nu, \dot{\nu}) \right) \) and its inflow through the moving boundary \( \partial b_R^*(t) \). By shrinking \( b_R^* \) up to the tip (uniformly in time), the arbitrariness of \( [s_1, s_2] \) and the use of the identities (26), (29), (30) and (32) imply

\[
e_{\text{tip}} = \int_{\text{tip}} \rho \left( \frac{1}{2} ||\dot{x}||^2 + k(\nu, \dot{\nu}) \right) n.
\]

(41)

5 The mechanical dissipation inequality and its consequences

Consider \( \nabla \nu \) and \( S \). We define the product \( \pi \) by

\[
(\nabla \nu \cdot S) \cdot n = \mathcal{S} n \cdot (\nabla \nu) u,
\]

(42)

\( \text{The terms } \nabla \bar{\sigma} + \bar{\sigma} (K-I) \mathbf{m} \text{ and } e_{\text{tip}} \text{ mark the difference with the equations similar to (35), (37) and (39) discussed for the quasi-static evolution of planar three-dimensional cracks in (Gurtin & Shvartsman, 1997).} \)
for any pair of vectors \( n \) and \( u \). Some special cases are the following: when \( \nu \) is scalar, \( \nabla \nu^T \mathcal{S} = \nabla \nu \otimes \mathcal{S} \), when \( \varphi \) is a vector, \( \nabla \nu^T \mathcal{S} = \nabla \nu^T \mathcal{S} \), while when \( \nu \) is a second order tensor, \( \nabla \nu^T \mathcal{S} = \nabla \nu^T : \mathcal{S} \).

5.1 The formal statement of the mechanical dissipation inequality

An isothermal version of such an inequality is the following:

\[
\Psi_{b \text{ rate}}^\text{rate} - P_{b \text{ ext}}^\text{ext} \leq 0, \tag{43}
\]

where \( \Psi_{b \text{ rate}}^\text{rate} \) is the rate of the Helmholtz free energy of \( b \) the power of interactions over \( b \). The standard assumption is that the (here purely mechanical) inequality (43) holds for any choice of the rates involved and for any part \( b \).

Below we account for time varying parts \( b (t) \) of \( B \) to follow in \( B_0 \) the growth of the crack in \( B \), then the external power \( P_{b \text{ ext}}^\text{ext} \) must account for the interactions associated with such a growth.

5.2 The mechanical dissipation inequality in the bulk. Consequences.

Let \( b (t) \) be any time varying part of \( B \) far from the crack. \( \Psi_{b} \) is expressed only by means of a bulk free energy density \( \psi \) and we have

\[
\int_{b(t)} \dot{\psi} + \int_{\partial b(t)} \psi (u \cdot n) - \int_{b(t)} (b \cdot \dot{x} + \beta \cdot \dot{\nu}) -
\int_{\partial b(t)} (Tn \cdot \dot{x} + Sn \cdot \dot{\nu} + (\mathbb{P} + F^T T + \nabla \nu^T \mathcal{S}) n \cdot u) \leq 0. \tag{44}
\]

where we have used (6) to transform the original integrand \( Tn \cdot x^2 + Sn \cdot \nu^2 + Fn \cdot u \) appearing in the last integral. Since only the component of \( u \) normal to the surface \( \partial b (t) \) is independent of the parametrization of \( \partial b (t) \), a natural invariance requirement with respect to such a parametrization implies that the vector \( (\mathbb{P} + F^T T + \nabla \nu^T \mathcal{S}) n \) must be purely normal to \( \partial b \); then there exists a scalar \( \varpi \) such that \( \mathbb{P} + F^T T + \nabla \nu^T \mathcal{S} = \varpi \mathbb{I} \). By substituting \( \varpi \mathbb{I} \) within the previous inequality, we find the term \( (\psi - \varpi) (u \cdot n) \). However, since (44) is assumed to hold for any choice of the velocity fields, we get \( \psi = \varpi \), then \( \mathbb{P} = \psi \mathbb{I} - F^T T - \nabla \nu^T \mathcal{S} \). Notice that, in absence of prominent effects of the material substructure described by \( \nu \), the second order tensor \( \mathbb{P} \) reduces to the well-known Eshelby tensor \( \psi \mathbb{I} - F^T T \).

We restrict our analysis to the non-homogeneous purely non-linear elastic case and assume constitutive expressions of the form \( T = \hat{T} (X, F, \nu, \nabla \nu) \) for the Piola-Kirchhoff stress, \( z = \hat{z} (X, F, \nu, \nabla \nu) \) for the self-force and \( \mathcal{S} = \hat{S} (X, F, \nu, \nabla \nu) \) for the microstress. If we take the free energy as \( \psi = \hat{\psi} (X, F, \nu, \nabla \nu) \),
and assume that it admits partial derivatives with respect to its entries, with the use of previous results, the mechanical dissipation inequality reduces to
\[
\int_{b(t)} \left((\partial_x \psi - T) \cdot \hat{F} + (\partial_y \psi - z) \cdot \hat{\nu}\right) + \int_{b(t)} ((\rho \partial_{w,\nu} \psi - S) \cdot \nabla \hat{\nu}) \leq 0, \tag{45}
\]
where \(\partial_x \psi\) means partial derivative of \(\psi\) with respect to the argument \(y\). Its validity for any choice of the rates implies
\[
T = \partial_x \psi \quad ; \quad z = \partial_y \psi \quad ; \quad S = \partial_{w,\nu} \psi. \tag{46}
\]
As a consequence, taking into account the explicit expression of \(P\), from (35) we get \(g = -\partial_x \psi\) and \(e = -F^T b - (\nabla \nu^T) \beta\).

Special expressions of \(\psi\) are of Ginzburg-Landau type. In most cases, in fact, it seems to be natural to assume \(\psi = \psi(X, F, \nu) + \frac{1}{2} a(X) ||\nabla \nu||^2\). In particular, if we have \(\psi(X, F, \nu) = \psi_1(X, F) + \psi_2(\nu)\), a constant and \(\psi_2(\nu)\) a coarse-grained (perhaps multivell) energy, the balance of substructural interactions (14) coincides with the well known Ginzburg-Landau equation.

Viscous effects may occur at the gross scale and at the substructural level. In this case, the measures of interaction, \(T, z, S\) may depend on the rates of the fields and their gradients. We assume, as a prototype example, that only the self-force \(z\) depends on the sole rate \(\hat{\nu}\). The self-force \(z\) may be thus decomposed into its viscous \(\nu\) and non-viscous \(nv\) parts, namely \(z = z^\nu + z^{nv}\), with \(z^{nv} = z^{nv}(X, F, \nu, \nabla \nu)\) and \(z^\nu = z^\nu(X, F, \nu, \nabla \nu, \hat{\nu})\), with \(z^{\nu} \cdot \hat{\nu} \geq 0\) for any choice of \(\hat{\nu}\), which implies \(z^\nu = \lambda(X, F, \nu, \nabla \nu) \hat{\nu}\), with \(\lambda\) some positive scalar function and \(z^{nv}\) satisfying (46b).

### 5.3 The mechanical dissipation inequality along the sides of the crack. Consequences.

For a part \(b_C(t)\) crossing the crack away the tip, we consider an additional surface free energy density \(\phi(X)\) along the margins of the crack; it is continuous up to the tip where it is indicated with \(\phi_{\text{tip}}\). As a consequence, to the bulk terms, the ones in (44), we must add the piece
\[
\int_{b_C(t) \cap C(t)} \hat{\phi} + \int_{(\partial b_C(t) \cap C(t))} \phi (m \cdot u) - \int_{(\partial b_C(t) \cap C(t))} \bar{\sigma} (m \cdot u). \tag{47}
\]
We may then reduce the resulting inequality by shrinking \(b_C(t)\) to \(C(t)\), and taking the limit uniformly in time. In this case we get
\[
\int_{(\partial b_C(t) \cap C(t))} (\phi - \bar{\sigma}) (m \cdot u) - \int_{b_C(t) \cap C} ([T m \cdot \hat{x}] + [S m \cdot \hat{\nu}]) \leq 0. \tag{48}
\]
The validity of such an inequality for any choice of velocity fields implies \(\phi = \bar{\sigma}\) and the local dissipation inequality \([T m \cdot \hat{x}] + [S m \cdot \hat{\nu}] \leq 0\).
5.4 The mechanical dissipation inequality at the tip of the crack. Consequences.

Let $b^*_R(t)$ a ‘curved cylinder’ wrapped around the tip as used in previous sections; its boundary intersects the tip in two points $Z(s_1(t))$ and $Z(s_2(t))$. In writing the mechanical dissipation inequality on $b^*_R(t)$, we consider, in addition to bulk and surface energies, a line energy density $\zeta$; moreover, since the tip moves, we must account also for the power of $\lambda_{tip} t$ and $e_{tip}$ ($g_{tip}$ is excluded because it is internal). Consequently, to bulk and surface contributions we add the term

$$\frac{d}{dt} \left( \int_{s_2(t)}^{s_1(t)} \zeta \right) \left( \lambda_{tip} t \cdot v_{tip} \big|_{s=s_2} - \lambda_{tip} t \cdot v_{tip} \big|_{s=s_1} \right). \quad (49)$$

By shrinking $b^*_R(t)$ up to the tip, taking the limit uniformly in time and making use of the line balance (39), we get

$$\left( \zeta - \lambda_{tip} \right) \left( t \cdot v_{tip} \big|_{s=s_2} - t \cdot v_{tip} \big|_{s=s_1} \right) +
\int_{s_1(t)}^{s_2(t)} \zeta h \cdot v_{tip} + \int_{s_1(t)}^{s_2(t)} v_{tip} \cdot g_{tip} \leq 0 \quad (50)$$

Since the resulting inequality must be valid for any choice of the velocity fields, then of $v_{tip}$, we obtain $\zeta = \lambda_{tip}$ and $g_{tip} \cdot v_{tip} \leq 0$, which reduces to $V g_{tip} \cdot n \leq 0$ since $v_{tip} = V n$, implying then that the component of $g_{tip}$ along the motion of $J$, namely $g_{tip} = g_{tip} \cdot n$, must have a structure of the type $g_{tip} = g_{tip} V$, where $g_{tip}$ is a negative ‘diffusion’ coefficient that must be assigned constitutively.

6 Driving the tip of the crack

6.1 The driving force

The explicit expressions of $\Phi$, $g$, $\bar{\sigma}$, $\lambda_{tip}$ allow us to write the tip balance (39) as

$$- \phi_{tip} n - \lambda_{tip} h + \int_{tip} \left( \rho \left( \frac{1}{2} \| \dot{x} \|^2 + k (\nu, \dot{\nu}) \right) I - \Phi \right) n = -g_{tip}. \quad (51)$$

We indicate with $j$ the vector

$$j = \int_{tip} \left( \rho \left( \frac{1}{2} \| \dot{x} \|^2 + k (\nu, \dot{\nu}) \right) I - \Phi \right) n \quad (52)$$

It represents the tip traction exerted by the bulk material on an infinitesimal neighborhood around the tip. Let $n(s)$ be the direction of propagation of the crack at the point $Z(s)$ of the tip, the component of (51) along $n(s)$ is given by

$$- \phi_{tip} - \lambda_{tip} R + n \cdot j = -g_{tip} V. \quad (53)$$
By indicating with $J$ (the J-integral) the product $n \cdot j$, we interpret the difference

$$\dot{f} = J - \phi_{\text{tip}} - \lambda_{\text{tip}} \cdot \mathbf{R}$$

(54)

as the *force driving the tip of the crack*; it accounts directly for the influence of the material substructure. Since $g_{\text{tip}} \cdot v_{\text{tip}} \leq 0$, we get $\dot{f} V \geq 0$ (we remind that $v_{\text{tip}}$ is of the form $V n$). When the crack grows, i.e., when $V > 0$, the driving force must be non-negative, i.e., $\dot{f} \geq 0$.

### 6.2 Dynamic energy release rate at the tip.

The energy release rate at the tip is given by the power $fV$ developed by the driving force along the normal motion of the crack tip. Previous results allow us to express the product $fV$ in terms of the power of standard and substructural interactions and of the free energy, and we obtain

$$\int_{\text{tip}} \rho \left( \psi + \frac{1}{2} \| \dot{x} \|^2 + k (\nu, \nu) \right) V + T n \cdot \dot{x} + S n \cdot \dot{\nu} - \phi_{\text{tip}} V - \lambda_{\text{tip}} \cdot \mathbf{R} V = f V,$$

(55)

which represents the *balance of energy at the tip*.

### 6.3 Quasi-static extended J-integral and its path independence

When inertial effects are negligible, the evolution of the crack is “quasi static”. The J-integral ($J = n \cdot j$) reduces to its quasi static counterpart $J_{qs}$:

$$J_{qs} = n \cdot \int_{\text{tip}} F n.$$

(56)

It reduces to the standard J-integral given by $n \cdot \int_{\text{tip}} (\psi I - F^T T) n$ when the substructure is absent or its gross effects are negligible.

*If the material is homogeneous, $\mathcal{C}$ is planar (i.e. the crack is straight), the crack has the margins free of standard and substructural tractions (in the sense that $T^\perp m = 0$ and $S^\perp m = 0$), $J_{qs}$ is path-independent.*

The hypotheses of homogeneity of the material and absence of inertial effects imply $g = 0$ and $e = 0$. Then, the bulk balance (35) reduces to $\text{Div} \mathbf{P} = 0$. Moreover, since the margins of the crack are free of standard and substructural tractions, we get $[\mathbf{F}] \mathbf{m} = [\psi] \mathbf{m}$.

With these premises, we take in $\mathcal{B}$ an arbitrary ‘curved cylinder’ $b_{\mathcal{R}}^*$ wrapped around the tip where we have

$$n \cdot \int_{\partial b_{\mathcal{R}}^*(t)} \mathbf{P} n = n \cdot \int_{S^2} \int_{\partial D_R} \mathbf{P} n.$$

(57)
$D_R$ is arbitrary, then we need only to evaluate the difference

$$n(s) \cdot \int_{D_R} Pn - n(s) \cdot \int_{\text{tip}} Pn,$$

that is equal to $n \cdot \int_{D_R \cap C} [\psi] m$, which vanishes because the crack is straight, i.e. $m \perp n$. The path-independence of $J_{qs}$ follows.

7 The energy dissipated into a process zone of finite size around the crack tip

When a crack propagates, a material part $P_z$ around the tip becomes highly unstable, in certain sense ‘fragmented’ (Aoki, Kishimoto & Sakata, 1981; 1984). Usually, $P_z$ is called process zone. In brittle fracture, the process zone may be considered practically coincident with the tip, while in ductile fracture $P_z$ has finite size. For the latter case we obtain new path-integrals which allow us to evaluate the energy dissipated during the evolution of the crack.

We assume that $P_z$ can be approximated reasonably by a ‘curved cylinder’ $P$ wrapped around $J$ (basically, $P$ has geometrical properties analogous to $b_R^*$ used in previous sections); the intersection of $P$ with the plane orthogonal to the tangent $t(s)$ of $J$ at $s \in [0, \bar{s}]$ is a disc $P^\pi$ with the centre on $J$ (being the centre the sole intersection of $J$ with $P$). The approximation of $P_z$ with $P$ is rather rough but it does not influence the basic structure of the results obtained in the present section. During the evolution of the crack, $P$ varies in time and is $P(t)$. The boundary $\partial P$ is a surface (with outward unit normal indicated with $n$) parametrized by $v_1, v_2$ and points $\bar{X}(v_1, v_2, t) \in \partial P(t)$ have an intrinsic velocity $\dot{\bar{u}}$ given by $\dot{\bar{u}} = \partial \bar{X}/\partial t(v_1, v_2, t)$. Consequently, rates following $\partial P(t)$ are $\dot{x} = \dot{\bar{X}} + F\dot{\bar{u}}$ and $\dot{\nu} = \dot{\bar{v}} + (\nabla \nu) \dot{\bar{u}}$.

The velocity $\dot{\bar{u}}$ at the boundary $\partial P$ is decomposed as

\[
\dot{\bar{u}} = \dot{u}_{tr}(t) + \dot{q}(t) \times (X - X_0) + \alpha(t)(X - X_0) + \dot{u}_d(X,t),
\]

$\dot{u}_{tr}(t)$ denotes the component of rigid translation; $\dot{q}(t) \times (X - X_0)$ the rotational component, with $X_0$ an arbitrary fixed point (note that the presence of $(X - X_0)$ instead of $(x - x_0)$ is due to the circumstance that $\bar{u}$ is a material velocity in $B$); $\alpha(t)(X - X_0)$ the velocity associated with the self-similar expansion of $P$; $\dot{u}_d(X,t)$ the component of the velocity due to the distortion.

Let now $b_R^*$ be another ‘curved cylinder’ wrapped around the tip, fixed in time and containing a piece $P$ of the process zone. We analyze the behavior of $P(t)$ in a time interval in which $\partial P(t)$ does not intersect $\partial b_R^*$.

With reference to the situation described before, we indicate with $\dot{\Phi}(\bar{P})$ the rate of energy dissipated in $P$ during the evolution of the crack and assume that all the dissipation is concentrated in $P$ during the evolution of the crack. Such an assumption implies that

\[
\frac{d}{dt} \int_{b_R^* \setminus \bar{P}(t)} \psi = \int_{b_R^* \setminus \bar{P}(t)} \left( T \cdot \dot{\Phi} + z \cdot \dot{\nu} + S \cdot \nabla \dot{\nu} \right).
\]
In other words, the mechanical dissipation inequality in $b^*_R \setminus \bar{P}(t)$ reduces to an equation because no dissipation mechanism occur outside $\bar{P}$.

With these premises, the balance of the energy takes the form

$$
\frac{d}{dt} \int_{b^*_R \setminus \bar{P}(t)} \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) + \int_{\partial \bar{P}(t)} \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) (\dot{u} \cdot n) + \\
+ \frac{d}{dt} \int_{b^*_R \setminus \bar{P}(t)} \psi + \int_{\partial \bar{P}(t)} \psi (\dot{u} \cdot n) - \dot{\Phi}(\bar{P}) = \\
= \int_{b^*_R \setminus \bar{P}(t)} (b^{ni} \cdot \dot{x} + \beta^{ni} \cdot \dot{\nu}) + \int_{\partial b^*_R} (Tn \cdot \dot{x} + Sn \cdot \dot{\nu}) \ .
$$

By making use of the weak form over $b^*_R \setminus \bar{P}(t)$ of the balances of standard and substructural interactions (11) and (14), and of the integral identity

$$
\int_{\partial b^*_R} (Tn \cdot \dot{x} + Sn \cdot \dot{\nu}) = \int_{\partial \bar{P}(t)} (Tn \cdot \dot{x} + Sn \cdot \nu) + \\
+ \int_{b^*_R \setminus \bar{P}(t)} (Div (\dot{x}T) + Div(\dot{\nu}S)),
$$

from the arbitrariness of the piece of $\mathcal{J}$ considered, we obtain

$$
\Phi (P^\pi) = \dot{u}_r (t) \cdot j (P^\pi) + \dot{q} (t) \cdot L + \alpha(t) M + I,
$$

where

$$
j (P^\pi) = \int_{\partial P^\pi(t)} \left( \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) I-P \right) n
$$

$$
L = \int_{\partial P^\pi(t)} (X - X_0) \times \left( \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) I-P \right) n
$$

$$
M = \int_{\partial P^\pi(t)} \left( \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) I-P \right) n \cdot (X - X_0)
$$

$$
I = \int_{\partial P^\pi(t)} \left( \rho \left( \frac{1}{2} \|\dot{x}\|^2 + k(\nu, \dot{\nu}) \right) I-P \right) n \cdot \hat{u}_d - \int_{\partial P^\pi(t)} (Tn \cdot \dot{x} + Sn \cdot \nu) .
$$

8 Special cases

The theory discussed in previous sections allows us to describe the behavior of cracks in several cases of complex materials. Two essential ingredients are necessary to apply the results: (i) the choice of an order parameter $\nu$ (hence of $\mathcal{M}$) describing appropriately the material substructure; (ii) an explicit expression of the free energy. In what follows, we indicate two possible spheres of application: ferroelectric solids and solids exhibiting strain gradient effects.
8.1 Cracks in ferroelectrics

8.1.1 Preliminary remarks

The predictions of Griffith’s theory fall for cracks propagating in ferroelectric solids subjected to the action of external electric fields: there is a discrepancy between the driving force predicted theoretically and the experimental data. Basically, we regard this discrepancy as due to the circumstance that Griffith’s theory does not account for the substructural interactions associated with the occurrence of spontaneous polarization that can be induced by strain, variation of temperature and applied electric fields. The polarization is indicated with \( p \) and we take as order parameter the vector \( \vec{p} = \rho \vec{p} - \frac{1}{m} \vec{p} \) such that \( 0 \leq |\vec{p}| \leq \rho m \), with \( \rho m \) a material constant. Then \( M \) is the ball of radius \( \rho m \) in \( \mathbb{R}^3 \). We also consider the body as subjected to an external electric field \( E \).

Balance equations are formally identical to (11), (13) and (14). In this case, \( A \) is the second order tensor \( \vec{p} \times \) (in components \( (\vec{p} \times)_{ij} = e_{ijk} p_k \), with \( e_{ijk} \) the alternating symbol). As a consequence, the relation (5) becomes \( \dot{\vec{p}} = \dot{\vec{p}} + \vec{p} \times \dot{\vec{q}} \).

Here, the microstress \( S \) accounts for interactions between neighboring crystals with different polarizations; \( z \) measures self-interactions within each polarized crystal. To include the effects of the applied electric field in the balance equations we assume that the bulk interactions \( b \) and \( \beta \) and the boundary ‘tractions’ \( t = Tn \) and \( \tau = Sn \) can be decomposed additively in electromechanical parts (em) and purely electric parts (el):

\[
b = b_{em} + b_{el} ; \quad \beta = \beta_{em} + \beta_{el};
\]

\[
t = t_{em} + t_{el} ; \quad \tau = \tau_{em} + \tau_{el}.
\]

For any arbitrary part \( b \) of \( B \), the purely electric parts are characterized by the balance

\[
\frac{d}{dt} D(b) + \int_b (b_{el} \dot{\vec{x}} + \beta_{el} \cdot \hat{\rho}) + \int_{\partial b} (t_{el} \dot{\vec{x}} + \tau_{el} \cdot \hat{\rho}) = 0,
\]

where \( D(b) = \frac{1}{2} \int_b |\vec{E}| = -\frac{1}{2} \int_b \rho \vec{E} \cdot \vec{p} \) is the electric energy of \( b \).

A theorem of Tiersten (1964) give us the explicit expression of the rate of \( D \) in the current configuration. We indicate the material version of Tiersten’s formula by pulling it back in the reference configuration:

\[
\frac{d}{dt} D(b) = -\int_b \rho (\text{grad} \vec{E}) \cdot \vec{p} - \int_{\partial b} \frac{1}{2} (\det F) p_n^2 F^{-T} n \cdot \hat{\vec{x}} - \int_b \rho \vec{E} \cdot \hat{\vec{p}},
\]

where \( p_n \) is the normal component of \( p \), \( \text{grad} \) indicates the gradient with respect to \( x \). It follows that

\[
b_{el} = \rho (\text{grad} \vec{E}) \vec{p} ; \quad \beta_{el} = \rho \vec{E};
\]

\[
t_{el} = \frac{1}{2} (\det F) p_n^2 F^{-T} n ; \quad \tau_{el} = 0.
\]
Consequently, the balance equations (11) and (14) become

\[ b_{em} + \text{Div} \mathbf{T} + \rho (\text{grad} \mathbf{E}) = 0, \] (74)

\[ \beta_{em} - z + \text{Div} \mathbf{S} + \rho \mathbf{E} = 0, \] (75)

where \( b_{em} \) and \( \beta_{em} \) include the inertial terms as ever (see also Davi, 2001).

### 8.1.2 The driving force in ferroelectrics

Taking in mind the identification of \( \nu \) with \( p \), all the general results on cracks presented before apply and for non-linear elastic ferroelectrics the quasi-static \( J \)-integral is given by

\[ J_{qs} = n \cdot \int_{\text{tip}} \left( \psi I - F^T \partial_F \psi - \nabla p^T \partial_V \psi \right) n \] (76)

An expression of this type has been used in (Wei & Hutchinson, 1997) for the special case of infinitesimal strains and fits reasonably the experimental data about the driving force.

### 8.2 Cracks in materials with strain gradient effects

Size effects are well recognized in the behavior of crystalline solids even during phase transitions. To describe these experimental evidences, models involving the second gradient of deformation have been considered (Dunn & Serrin, 1985). Strain gradient effects are induced by a latent substructure. Following the thermodynamically consistent theory of Capriz (1985), latency is induced here by an internal constraint obtained by identifying \( \nu \) with \( F \), in absence of external actions on the substructure (\( \beta = 0 \)). As a consequence, if one assumes constitutive equations of the type \( \mathbf{T} = \hat{T} (\mathbf{F}, \nabla \mathbf{F}) \), \( \mathbf{S} = \hat{S} (\mathbf{F}, \nabla \mathbf{F}) \) and \( \psi = \hat{\psi} (\mathbf{F}, \nabla \mathbf{F}) \), it follows that \( \mathbf{T} = \partial_F \psi \), \( \mathbf{S} = \partial_{\nabla F} \psi \) and the balances of standard and substructural interactions merge one into the other and reduce to the sole balance

\[ \text{Div} (\mathbf{T} - \text{Div} \mathbf{S}) + \mathbf{b} = 0. \] (77)

The quasi static \( J \)-integral \( J_{qs} \) takes the special form

\[ J_{qs} = n \cdot \int_{\text{tip}} \left( \psi I - F^T \partial_F \psi - \nabla F^T : \partial_{\nabla F} \psi \right) n; \] (78)

its reduced version in infinitesimal deformations has been used in (Fulton & Gao, 2001) to calculate the driving force in crystalline materials that fail by decohesion at atomic scale: the results fit reasonably experimental data.
9 Discussion

We have presented a fully non-linear three-dimensional description of the crack growth under large strains in complex bodies suffering a prominent influence of the material substructure on the gross behavior. We have used ‘abstract’ order parameter fields as coarse-grained descriptors of the morphology of the substructure. From one hand such a point of view allows us to unify some existing preliminary tentative to modify the standard theory of fracture in special cases of complex bodies, while, on the other hand, it furnishes a general tool able to provide directly the expression of the driving force in all cases of materials admitting Ginzburg-Landau-like energies. The present paper extends and renders more perspicuous preliminary two-dimensional results presented in a section of the article (Mariano, 2001).

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10 References

Abeyaratne, R., Knowles, J. K. 1990 On the diving traction acting on a surface of strain discontinuity in a continuum. J. Mech. Phys. Solids 38, 345-360.

Adda-Bedia, M., Arias, R., Ben Amar, M. & Lund, F. 1999 Generalized Griffith criterion for dynamic fracture and the stability of crack motion at high velocities. Phys. Rev. E 60, 2366-2376.

Atkinson, C. & Eshelby, J. D. 1968 The flow of energy into the tip of a moving crack. Int. J. Fracture 4, 3-8.

Aoki, S., Kishimoto, K. & Sakata, M. 1981 Energy release rate in elastic plastic fracture problems. ASME - J. Appl. Mech. 48, 825-829.

Aoki, S., Kishimoto, K. & Sakata, M. 1984 Energy flux into process region in elastic plastic fracture problems. Engng Fracture Mech. 19, 827-836.

Barenblatt, G. I. 1972 The mathematical theory of equilibrium cracks in brittle fracture. Adv. Appl. Mech. 7, 55-129.

Capriz, G. 1985 Continua with latent microstructure. Arch. Rational Mech. Anal. 90, 43-56.

Capriz, G. 1989 Continua with microstructure. Springer.

Davì, F. 2001 On domain switching in deformable ferroelectrics, seen as continua with microstructure. Z. angew. Math. Phys. ZAMP 52, 966-989.

Dunn, J. E. & Serrin, J. 1985 On the thermodynamics of interstitial working. Arch. Rational Mech. Anal. 88, 95-133.
Dolbow, J., Moës, M. & Belytschko, T. 2001 An extended finite element method for modeling crack growth with frictional contact. *Comp. Meth. Appl. Mech. Eng.* **190**, 6825-6846.

Epstein, M. 2002 The Eshelby tensor and the theory of continuous distributions of inhomogeneities. *Mech. Res. Comm.* **29**, 501-506.

Eshelby, J. D. 1975 The elastic energy-momentum tensor. *J. Elasticity* **5**, 321-335.

Freund, L. B. 1972 Energy flux into the tip of an extending crack in elastic solids. *J. Elasticity* **2**, 321-349.

Freund, L. B. 1990 *Dynamic fracture mechanics*. Cambridge University Press.

Fulton, C. C. & Gao, H. 2001 Microstructural modeling of ferroelectric fracture. *Acta Materialia* **49**, 2039-2054.

Griffith, A. A. 1920 The phenomenon of rupture and flow in solids. *Phil. Trans. Roy. Soc. London Ser. A* **221**, 163-198.

Gurtin, M. E. 1995 The nature of configurational forces. *Arch. Rational Mech. Anal.* **131**, 67-100.

Gurtin, M. E. & Shvartsman, M. M. 1997 Configurational forces and the dynamics of planar cracks in three-dimensional bodies. *J. Elasticity* **48**, 167-191.

Heino, P. & Kaski, K. 1997 Dynamic fracture of disordered viscoelastic solids. *Phys. Rev. E* **56**, 4364-4370.

James, R. D. 2002 Configurational forces in magnetism with application to the dynamics of a small scale ferromagnetic shape-memory cantilever. *Cont. Mech. Therm.* **14**, 56-86.

Mariano, P. M. 2001 Multifield theories in mechanics of solids. *Adv. Appl. Mech.* **38**, 1-93.

Moës, M. & Belytschko, T. 2002 Extended finite element method for cohesive crack growth. *Eng. Frac. Mech.* **69**, 813-833.

Obrezanova, O., Movchan, A. B. & Willis, J. R. 2003 Dynamic stability of a propagating crack. *J. Mech. Phys. Solids* **50**, 2637-2668.

Oleaga, G. E. 2003 On the dynamics of cracks in three dimensions. *J. Mech. Phys. Solids* **51**, 169-185.

Rice, J. R. 1968 Mathematical analysis in the mechanics of fracture, in *Fracture* **2** (H. Liebowitz ed.) 191-311, Academic Press.

Segev, R. 1996 On smoothly growing bodies and the Eshelby tensor. *Meccanica* **31**, 507-518.
ˇSilhavý, M. 1997 *The mechanics and thermodynamics of continuous media.* Springer.

Slepyan, L. I. 2002 *Models and phenomena in fracture mechanics.* Springer.

Tiersten, H. F. 1964 Coupled magnetomechanical equations for magnetically saturated insulators. *J. Math. Phys.* 5, 1298-1318.

Wei, Y. & Hutchinson, J. W. 1997 Steady-state crack growth and work of fracture for solids characterized by strain gradient plasticity. *J. Mech. Phys. Solids* 45, 1253-1273.