Ascending Chain Conditions in Free Baxter Algebras *

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Abstract

In this paper we study ascending chain conditions in a free Baxter algebra by making use of explicit constructions of free Baxter algebras that were obtained recently. We investigate ascending chain conditions both for ideals and for Baxter ideals. The free Baxter algebras under consideration include free Baxter algebras on sets and free Baxter algebras on algebras. We also consider complete free Baxter algebras.

1 Introduction

Let $C$ be a commutative ring and let $\lambda$ be an element of $C$. A Baxter $C$-algebra of weight $\lambda$ is a commutative $C$-algebra $R$ with a $C$-linear operator $P$ that satisfies the Baxter identity

$$P(x)P(y) = P(xP(y)) + P(yP(x)) + \lambda P(xy), \forall x, y \in R.$$ (1)

The study of Baxter algebras was started by Baxter in 1963 [2]. He was motivated by problems from fluctuation theory. In 1968, Rota [11] began a systematic study of Baxter algebras from an algebraic point of view. Since then Baxter algebras have been related to hypergeometric functions, combinatorics, statistics, incidence algebras and theory of symmetric functions [12, 13].

Free Baxter algebras play a fundamental role in the study of Baxter algebras. Explicit descriptions of free Baxter algebras were first considered by Rota [11] and Cartier [3]. In two recent papers [7, 8], William Keigher and the author furthered the work of Cartier and Rota, giving the explicit descriptions in complete generality. Using these constructions, further properties of Baxter algebras, in particular the zero divisors, were studied [5], Baxter algebras were related to Hopf algebras [11] and were applied to the umbral calculus [6].

In this paper, we study ascending chain conditions in free Baxter algebras. Other than considering the noetherian ring property, we also consider modified noetherian

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properties, such as the ascending chain condition for Baxter ideals. Let \( X \) be a set. Denote \( F_C(X) \) for the free Baxter \( C \)-algebra on \( X \). The following is a summary of the main results on \( F_C(X) \).

1. \( F_C(\phi) \) is a noetherian ring if and only if \( C \) is a noetherian \( \mathbb{Q} \)-algebra (Theorem 3.1).

2. If \( X \) is not the empty set, then \( F_C(X) \) is not noetherian (Theorem 3.3).

3. If \( C \) is a noetherian ring, then \( F_C(\phi) \) satisfies the ascending chain condition for Baxter ideals (Theorem 4.2).

4. If \( X \) is not empty, then \( F_C(X) \) of weight 0 does not satisfy the ascending chain condition for Baxter ideals. If \( X \) is infinite, then \( F_C(X) \) of any weight does not satisfy the ascending chain condition for Baxter ideals (Corollary 4.5).

As a generalization of free Baxter algebras on sets that were studied in [11] and [3], free Baxter algebras on \( C \)-algebras were introduced in [7] (see §2 for more details). Ascending chain conditions in free Baxter algebras on \( C \)-algebras are also studied in this paper.

In [8], we showed how one could complete a free Baxter algebra and get a complete free Baxter algebra. A summary of this construction is given in §2. This construction is similar to completing a free \( C \)-algebra (i.e., a polynomial ring with coefficients in \( C \)) and obtain a complete \( C \)-algebra (i.e, a power series ring with coefficients in \( C \)). In the current paper we also consider the ascending chain conditions in a complete free Baxter algebra.

We will provide some background on Baxter algebras in §2. In §3 the ascending chain condition for ideals will be studied and Theorems 3.1 and 3.3 will be proved. The ascending chain condition for Baxter ideals will be studied in §4. Theorem 4.2, Theorem 4.4 and Corollary 4.5 are the main results in this section.

## 2 Notations and background

We review concepts and results on Baxter algebras that will be needed later in this paper. See [7] [8] [5] for detail.

### 2.1 General notations

We write \( \mathbb{N} \) for the set of natural numbers and \( \mathbb{N}_+ = \{ n \in \mathbb{N} \mid n > 0 \} \) for the positive integers.

In this paper, every ring \( C \) is commutative with identity element \( 1_C \), and every ring homomorphism preserves the identity elements. For any \( C \)-modules \( M \) and \( N \), the tensor product \( M \otimes N \) is taken over \( C \) unless otherwise indicated. For \( n \in \mathbb{N} \), denote the tensor power \( M \otimes \ldots \otimes M \) by \( M^{\otimes n} \) with the convention that \( M^{\otimes 0} = C \).

This applies in particular if \( M \) is a \( C \)-algebra. Let \( 1 \) be the identity element in a \( C \)-algebra \( A \). We also use the notation \( 1^{\otimes n} = 1 \otimes \ldots \otimes 1 \).
2.2 Free Baxter algebras

Let \((R, P)\) be a Baxter \(C\)-algebra of weight \(\lambda\) with Baxter operator \(P\). So \(P\) satisfies the identity \([1]\). A Baxter ideal of \((R, P)\) is an ideal \(I\) of \(R\) such that \(P(I) \subseteq I\). The concepts of sub-Baxter algebras, quotient Baxter algebras and homomorphisms of Baxter algebras can be easily defined.

Let \(A\) be a \(C\)-algebra. A free Baxter algebra on \(A\) is a Baxter algebra \((F_C(A), P_A)\) with a \(C\)-algebra homomorphism \(j_A : A \to F_C(A)\) that satisfies the following universal property. For any Baxter \(C\)-algebra \((R, P)\) and any \(C\)-algebra homomorphism \(\varphi : A \to R\), there exists a unique Baxter \(C\)-algebra homomorphism \(\tilde{\varphi} : (F_C(A), P_A) \to (R, P)\) such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{j_A} & F_C(A) \\
\varphi & \downarrow & \downarrow \tilde{\varphi} \\
\quad & R
\end{array}
\]

commutes. Let \(X\) be a set and let \(A = C[X]\). Then \(F_C(A)\) is the free Baxter algebra on \(X\) in the usual sense. The existence of free Baxter algebras follows from the general theory of universal algebras. In order to get a good understanding of free Baxter algebras and Baxter algebras in general, it is desirable to find more explicit descriptions of free Baxter algebras.

2.3 Shuffle Baxter algebras

Motivated by the shuffle product of iterated integrals \([10]\), an explicit description of free Baxter algebras was given in \([7]\). This generalizes earlier construction of free Baxter algebras by Cartier \([3]\). The resulting free Baxter algebras are called shuffle Baxter algebras. We summarize the construction.

For \(m, n \in \mathbb{N}_+\), define the set of \((m, n)\)-shuffles by

\[
S(m, n) = \left\{ \sigma \in S_{m+n} \mid \begin{array}{l}
\sigma^{-1}(1) < \sigma^{-1}(2) < \ldots < \sigma^{-1}(m), \\
\sigma^{-1}(m+1) < \sigma^{-1}(m+2) < \ldots < \sigma^{-1}(m+n) 
\end{array} \right\}.
\]

Given an \((m, n)\)-shuffle \(\sigma \in S(m, n)\), a pair of indices \((k, k+1), 1 \leq k < m+n\), is called an admissible pair for \(\sigma\) if \(\sigma(k) \leq m < \sigma(k+1)\). Denote \(T^\sigma\) for the set of admissible pairs for \(\sigma\). For a subset \(T\) of \(T^\sigma\), we call the pair \((\sigma, T)\) a mixable \((m, n)\)-shuffle. Let \(|T|\) be the cardinality of \(T\). We will identify \((\sigma, T)\) with \(\sigma\) if \(T\) is the empty set. Denote

\[
\tilde{S}(m, n) = \{(\sigma, T) \mid \sigma \in S(m, n), \ T \subset T^\sigma\}
\]

for the set of \((m, n)\)-mixable shuffles.

Let \(A\) be a \(C\)-algebra. For \(x = x_1 \otimes \ldots \otimes x_m \in A^\otimes m\), \(y = y_1 \otimes \ldots \otimes y_n \in A^\otimes n\) and \((\sigma, T) \in \tilde{S}(m, n)\), the element

\[
\sigma(x \otimes y) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \ldots \otimes u_{\sigma(m+n)} \in A^\otimes (m+n),
\]

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Remark 2.1

Because of this theorem, we will use \( \lambda \) together with the natural embedding \( j \) the following conventions in the rest of this paper.

\[ u_k = \begin{cases} x_k, & 1 \leq k \leq m, \\ y_{k-m}, & m + 1 \leq k \leq m + n, \end{cases} \]

is called a shuffle of \( x \) and \( y \); the element

\[ \sigma(x \otimes y; T) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(m+n)} \in A^{\otimes(m+n-|T|)}, \]

where for each pair \((k, k+1), 1 \leq k < m+n,\)

\[ u_{\sigma(k)} \otimes u_{\sigma(k+1)} = \begin{cases} u_{\sigma(k)} u_{\sigma(k+1)}, & (k, k+1) \in T \\ u_{\sigma(k)} \otimes u_{\sigma(k+1)}, & (k, k+1) \notin T, \end{cases} \]

is called a mixable shuffle of \( x \) and \( y \).

Fix a \( \lambda \in C \) and a \( C \)-algebra \( A \). There is a Baxter \( C \)-algebra of weight \( \lambda \)

\[ \Pi C(A) = \Pi C,\lambda(A) = \bigoplus_{k \in \mathbb{N}} A^{\otimes(k+1)} = A \oplus A^{\otimes2} \oplus \ldots \]

in which

- the \( C \)-module structure is the natural one,
- the multiplication is the mixed shuffle product, defined by

\[ x \otimes y = \sum_{(\sigma,T) \in S(m,n)} \lambda^{|T|} x_0 y_0 \otimes \sigma(x^+ \otimes y^+; T) \in \bigoplus_{k \leq m+n+1} A^{\otimes k} \]

for \( x = x_0 \otimes x_1 \otimes \ldots \otimes x_m \in A^{\otimes(m+1)} \) and \( y = y_0 \otimes y_1 \otimes \ldots \otimes y_n \in A^{\otimes(m+1)} \), where \( x^+ = x_1 \otimes \ldots \otimes x_m \) and \( y^+ = y_1 \otimes \ldots \otimes y_n \),
- the Baxter operator \( P_A \) on \( \Pi C(A) \) is obtained by assigning

\[ P_A(x_0 \otimes x_1 \otimes \ldots \otimes x_n) = 1_A \otimes x_0 \otimes x_1 \otimes \ldots \otimes x_n, \]

for all \( x_0 \otimes x_1 \otimes \ldots \otimes x_n \in A^{\otimes(n+1)} \).

\( (\Pi C(A), P_A) \) is called the shuffle Baxter \( C \)-algebra on \( A \) of weight \( \lambda \).

For a given set \( S \), we also let \((\Pi C(S), P_S)\) denote the shuffle Baxter \( C \)-algebra \((\Pi C(C[S]), P_C(S))\), called the shuffle Baxter \( C \)-algebra on \( S \) (of weight \( \lambda \)).

Theorem 2.1 [3, 7] The pair \((\Pi C(A), P_A)\), together with the natural embedding \( j_A \), is a free Baxter \( C \)-algebra on \( A \) of weight \( \lambda \). Similarly, the pair \((\Pi C(S), P_S)\), together with the natural embedding \( j_S \), is a free Baxter \( C \)-algebra on \( S \) of weight \( \lambda \).

We will use the following conventions in the rest of this paper.

Remark 2.1

1. Because of this theorem, we will use \( \Pi C(S) \) instead of \( F_C(S) \) to denote the free Baxter \( C \)-algebra on \( S \).
2. From the definition of the mixed shuffle product, we have

\[ x \triangleright y = \begin{cases} 
  x_0 y_0, & \text{if } x_0, y_0 \in A, \\
  x_0 (y_0 \otimes y_1 \otimes \ldots \otimes y_n), & \text{if } x = x_0 \in A, y = y_0 \otimes \ldots \otimes y_n \in A^{\otimes (n+1)}, \\
  (x_0 \otimes x_1 \otimes \ldots \otimes x_n) y_0, & \text{if } x = x_0 \otimes \ldots \otimes x_m \in A^{\otimes (m+1)}, y = y_0 \in A.
\end{cases} \]

This shows that the mixed shuffle product is compatible with the product in \( A \). Thus we will suppress the symbol \( \triangleright \) in the mixed shuffle product unless there is the risk of confusion.

3. Unless otherwise specified, we use \( A^{\otimes k} \) to denote the \( C \)-submodule of \( \Pi C(A) \) instead of the tensor product algebra.

4. For \( k \in \mathbb{N} \), we denote \( \text{Fil}^k \Pi C(A) \) for \( \bigoplus_{n \geq k} A^{\otimes (n+1)} \).

2.4 Complete shuffle Baxter algebras

We now take the completion of \( \Pi C(A) \) in a manner similar to taking the completion of a polynomial ring to get a power series ring.

Given \( k \in \mathbb{N}_+ \), \( \text{Fil}^k \Pi C(A) \) is a Baxter ideal of \( \Pi C(A) \). On the other hand, consider the infinite product of \( C \)-modules \( \prod_{k \in \mathbb{N}} A^{\otimes (k+1)} \). It contains \( \Pi C(A) \) as a dense subset with respect to the topology defined by the filtration \( \text{Fil}^k \Pi C(A), \ k \geq 0 \). All operations of the Baxter \( C \)-algebra \( \Pi C(A) \) are continuous with respect to this topology, hence extend uniquely to operations on \( \prod_{k \in \mathbb{N}} A^{\otimes (k+1)} \), making \( \prod_{k \in \mathbb{N}} A^{\otimes (k+1)} \) into a Baxter algebra of weight \( \lambda \). We denote this Baxter algebra by \( \hat{\Pi} C(A) \) and denote the Baxter operator by \( \hat{P} \). The pair \( (\hat{\Pi} C(A), \hat{P}) \) is called the complete shuffle Baxter algebra on \( A \). It has been shown that \( \hat{\Pi} C(A) \) is a free object in the category of Baxter algebras that are complete with respect to a canonical filtration defined by the Baxter operator \( \hat{P} \).

When \( A = C \), we have

\[ \Pi C(C) = \bigoplus_{n \in \mathbb{N}} C \mathbf{1}^{\otimes (n+1)}, \quad \hat{\Pi} C(C) = \prod_{n \in \mathbb{N}} C \mathbf{1}^{\otimes (n+1)}, \]

where \( \mathbf{1}^{\otimes (n+1)} = \mathbf{1}_C \otimes \ldots \otimes \mathbf{1}_C \). In this case the mixable shuffle product formula (2) gives

\[ \mathbf{1}^{\otimes (m+1)} \circ \mathbf{1}^{\otimes (n+1)} = \sum_{k=0}^{m} \binom{m+n-k}{n} \binom{n}{k} \lambda^k \mathbf{1}^{\otimes (m+n+1-k)}, \ \forall \ m, n \in \mathbb{N}. \] (3)

This holds in both \( \Pi C(C) \) and \( \hat{\Pi} C(C) \).
2.5 The internal construction

Later in the paper will use another construction of free Baxter algebras \cite{8}, generalizing the work of Rota \cite{11}. Since we will only need this construction in the special case when \( A = C \), we will give a simplified description here. See \cite{8, 5} for details.

Define \( \mathfrak{A}(C) = \prod_{n=1}^{\infty} C \) with componentwise addition and multiplication. Then \( \mathfrak{A}(C) \) is a \( C \)-algebra. It is in fact a Baxter \( C \)-algebra.

**Proposition 2.2** \cite{5} Let \( \lambda \in C \) be a non-zero divisor. Define \( \Phi: \Pi C(C) \to \mathfrak{A}(C) \) by sending \( b = \sum_{m=0}^{\infty} b_m 1 \otimes (m+1) \in \Pi C(C) \) to \( \left( \sum_{i=0}^{n-1} \binom{n-1}{i} \lambda^i b_i \right)_{n \in \mathbb{N}} \in \mathfrak{A}(C) \). Then \( \Phi \) is an injective \( C \)-algebra homomorphism. Further, \( \Phi \) extends to an injective \( C \)-algebra homomorphism \( \hat{\Phi}: \hat{\Pi} C(C) \to \hat{\mathfrak{A}}(C) \).

3 Ascending chain condition for ideals

In this section we prove the two theorems (Theorem \ref{3.1} and Theorem \ref{3.3}) on the ascending chain condition for ideals in a free Baxter algebra.

3.1 The case when \( A = C \)

**Theorem 3.1**

1. If \( C \) is a noetherian \( \mathbb{Q} \)-algebra, then \( \Pi C(C) \) is a noetherian ring for every \( \lambda \in C \).

2. If \( C \) is a noetherian \( \mathbb{Q} \)-algebra and if \( \lambda = 0 \), then \( \hat{\Pi} C(C) \) is a noetherian ring.

3. If \( C \) is a \( \mathbb{Q} \)-algebra, \( \lambda \in C \) is a not a zero divisor and \( \cap_{n \in \mathbb{N}} \lambda^n C \neq 0 \), then \( \hat{\Pi} C(C) \) is not a noetherian ring.

4. If \( C \) is not a \( \mathbb{Q} \)-algebra, then \( \Pi C(C) \) and \( \hat{\Pi} C(C) \) are not noetherian rings.

**Proof:** (1). Let \( C \) be a \( \mathbb{Q} \)-algebra. It is well-known that \( R \) is a noetherian ring if and only if every ideal \( I \) of \( R \) is finitely generated. So we only need to prove that any ideal of \( \Pi C(C) \) is finitely generated. The idea of the proof is the same as that of the Hilbert basis theorem for \( C[x] \). Let \( I \subseteq \Pi C(C) \) be an ideal. For each \( j \in \mathbb{N} \), let

\[
\Sigma_j = \{ b_j \in C \mid \exists f_j \in I \text{ such that } f_j = \sum_{k=0}^{j} b_k 1 \otimes (k+1) \}.
\]

Then for any \( b_j, c_j \in \Sigma_j \), there are \( f_j \) and \( g_j \) in \( I \) such that \( f_j = \sum_{k=0}^{j} b_k 1 \otimes (k+1) \) and \( g_j = \sum_{k=0}^{j} c_k 1 \otimes (k+1) \). So \( f_j - g_j = \sum_{k=0}^{j} (b_k - c_k) 1 \otimes (k+1) \) is in \( I \). Thus \( b_j - c_j \) is in \( \Sigma_j \). Also, for any \( b_j \in \Sigma_j \) and \( c \in C \), there is \( f_j \) in \( I \) such that \( f_j = \sum_{k=0}^{j} b_k 1 \otimes (k+1) \). So \( c f_j = \sum_{k=0}^{j} cb_k 1 \otimes (k+1) \) is in \( I \). Thus \( c b_j \) is in \( \Sigma_j \). Therefore \( \Sigma_j \) is an ideal of \( C \). Further, \( b_j \in \Sigma_j \) implies that there exists \( f_j \) in \( I \) such that \( f_j = \sum_{k=0}^{j} b_k 1 \otimes (k+1) \).
Thus \(1 \otimes^2 f_j\) is in \(I\). By equation (3),

\[
1 \otimes^2 f_j = \sum_{k=0}^{j} b_k 1 \otimes^2 1 \otimes^{(k+1)}
\]

\[
= \sum_{k=0}^{j} b_k((k+1)1 \otimes^{(k+2)} + k1 \otimes^{(k+1)})
\]

\[
= b_j(j+1)1 \otimes^{(j+2)} + \text{lower degree terms}.
\]

Here we define \(\deg f = n\) if \(f = \sum_{i=0}^{\infty} c_i 1 \otimes^{(i+1)}\) with \(c_n \neq 0\) and \(c_i = 0\) for \(i > n\) and define \(\deg 0 = \infty\). Thus \(b_j(j+1)\) is in \(\Sigma_j+1\). Since \(\Sigma_j+1\) is an ideal and \(C\) is a \(\mathbb{Q}\)-algebra, we have \(b_j = (j+1)^{-1} b_j(j+1) \in \Sigma_j+1\). Thus \(\Sigma_j \subseteq \Sigma_j+1\). Since \(C\) is noetherian, this chain of ideals stabilizes, say at \(j = m\). Then \(\Sigma_m = \cup_{j=1}^{\infty} \Sigma_j\), and is finitely generated. Let \(f_1^{(m)}, \ldots, f_{k_m}^{(m)}\) be a set of generators of \(\Sigma_m\). Then we have \(f_i^{(m)} \in I, 1 \leq i \leq k_m\), with \(f_i^{(m)} = b_i^{(m)} 1 \otimes^{(m+1)} + g_i^{(m)}, \deg g_i^{(m)} < m\). For each \(j < m\), \(\Sigma_j\) is also finitely generated with a set of generators \(b_1^{(j)}, \ldots, b_{k_j}^{(j)}\). Then there are \(f_i^{(j)} \in I\) such that \(f_i^{(j)} = b_i^{(j)} 1 \otimes^{(j+1)} + g_i^{(j)} \in I\) with \(\deg g_i^{(j)} < j\). To prove the theorem, we only need to prove that \(I\) is the ideal generated by

\[
\{f_1^{(0)}, \ldots, f_{k_0}^{(0)}, f_1^{(1)}, \ldots, f_{k_1}^{(1)}, \ldots, f_1^{(m)}, \ldots, f_{k_m}^{(m)}\}.
\]

Let \(I'\) be the ideal generated by this set. Clearly 0 is in \(I'\). For \(f \in I\) with \(f \neq 0\), we use induction on \(\deg f\) to show that \(f\) is in \(I'\). If \(f \in I\) with \(\deg f = 0\), then \(f \in \Sigma_0 1 = \Sigma_0\), so can be expressed as a \(C\)-linear combination of \(f_1^{(0)}, \ldots, f_{k_0}^{(0)}\). Thus \(f \in I'\). Now for any \(n > 0\). Assume that all \(f \in I\) with \(\deg f < n\) are in \(I'\) and take \(f \in I\) with \(\deg f = n\). Write \(f = b_n 1 \otimes^{(n+1)} + g, b_n \neq 0, \deg g < n\). Then \(b_n\) is in \(\Sigma_n\). If \(n \geq m\), then by the definition of \(m, \Sigma_n = \Sigma_m\). So \(b_n = \sum_i a_i b_i^{(m)}\) for some \(a_i \in C\). Consider \(h = 1 \otimes^{(n-m+1)} \sum_i a_i f_i^{(m)}\). From \(f_i^{(m)} \in I\) we see that \(h\) is in \(I'\). Also, by equation (3)

\[
1 \otimes^{(n-m+1)} f_i^{(m)} = b_i^{(m)} 1 \otimes^{(n-m+1)} 1 \otimes^{(m+1)} + 1 \otimes^{(n-m+1)} g_i^{(m)}
\]

\[
= b_i^{(m)} \binom{n}{m} 1 \otimes^{(n+1)} + \text{lower degree terms}.
\]

Thus

\[
h = \sum_i a_i 1 \otimes^{(n-m+1)} f_i^{(m)} = \left(\binom{n}{m}\right) \sum_i a_i b_i^{(m)} 1 \otimes^{(n+1)} + \text{lower degree terms}.
\]

Therefore \(\left(\binom{n}{m}\right)^{-1} h\), still in \(I'\) since \(C\) is a \(\mathbb{Q}\)-algebra, has the same leading coefficient as \(f\). Hence \(f - \left(\binom{n}{m}\right)^{-1} h\) has degree less then \(n\). Since \(f - \left(\binom{n}{m}\right)^{-1} h\) is in \(I\), it is in \(I'\) by induction. Then \(f\) is in \(I'\). If \(n < m\), then \(b_n = \sum_i a_i b_i^{(n)}, a_i \in C\). So
\[ \sum_i a_i f_i^{(n)} \text{ is in } I' \text{ and has the same leading coefficient as } f. \text{ Then } f - \sum_i a_i f_i^{(n)} \in I \text{ with } \deg(f - \sum_i a_i f_i^{(n)}) < n. \text{ By induction, } f - \sum_i a_i f_i^{(n)} \text{ is in } I'. \text{ Hence } f \text{ is in } I'. \]

(2) We now consider \( \hat{\Pi}_C(C) \). Since \( \lambda = 0 \), equation (3) becomes
\[
1 \otimes (m+1) 1 \otimes (n+1) = \binom{m+n}{n} 1^{m+n+1}.
\]
Then we can use an argument that is similar to the previous part of the proof. Just replace \( \Sigma_j \) by
\[
\Omega_j = \{ b_j \in C \mid \exists f_j \in I \text{ such that } f_j = \sum_{k=j}^{\infty} b_k 1^{\otimes (k+1)} \}
\]
and follow the well-known argument in proving that \( C[[x]] \) is a noetherian ring.

(3) We only need to find ideals \( I_n, n \geq 1, \) of \( \hat{\Pi}_C(C) \) such that, for each \( n, \)
\[
I_n \subseteq I_{n+1}.
\]
For this purpose, we will construct a sequence \( d^{(k)} \), \( k \geq 1 \) of elements in \( \hat{\Pi}_C(C) \)
with the property that, for each \( m \geq 1, \)
\[
d^{(k)} \text{Fil}^m \hat{\Pi}_C(C) \begin{cases} 
\neq 0, & \text{if } m = k - 1, \\
= 0, & \text{if } m = k.
\end{cases}
\]
(4) We then let \( I_n \) be the ideal of \( \hat{\Pi}_C(C) \) generated by \( d^{(k)}, 1 \leq k \leq n. \) Then clearly \( I_n \subseteq I_{n+1} \). From equation (4) we have
\[
I_n \text{Fil}^n \hat{\Pi}_C(C) \begin{cases} 
\neq 0, & \text{if } m = n - 1, \\
= 0, & \text{if } m = n.
\end{cases}
\]
In particular, \( I_n \text{Fil}^n \hat{\Pi}_C(C) = 0 \) while \( I_{n+1} \text{Fil}^n \hat{\Pi}_C(C) \neq 0 \). Therefore, \( I_n \neq I_{n+1}, \)
as is desired. The rest of the proof will be devoted to the construction of such a sequence \( d^{(k)}, k \geq 1. \)

We first assume that \( C = \mathbb{Q}(x) = \mathbb{Q}[x, x^{-1}] \) and assume that the weight of \( \hat{\Pi}_{\mathbb{Q}(x)}(\mathbb{Q}(x)) \) is \( x \). For a fixed \( k \in \mathbb{N}_+, \) we want to find a solution \( b = b^{(k)} \in \hat{\Pi}_{\mathbb{Q}(x)}(\mathbb{Q}(x)) \) to the equation
\[
1^{\otimes (k+1)} b = 0.
\]
(5) Write \( b = \sum_{n=0}^{\infty} b_n 1^{\otimes (n+1)}, b_n \in \mathbb{Q}(x) \). We have
\[
1^{\otimes (k+1)} b = 1^{\otimes (k+1)} (\sum_{n=0}^{\infty} b_n 1^{\otimes (n+1)})
\]
\[
= \sum_{n=0}^{\infty} b_n 1^{\otimes (k+1)} 1^{\otimes (n+1)}
\]
\[
= \sum_{n=0}^{\infty} b_n \left( \sum_{i=0}^{k} \binom{n+k-i}{k} x^i 1^{\otimes (n+k-i+1)} \right)
\]
(equation (3))
\[
\begin{align*}
&= \sum_{i=0}^{k} \sum_{n=0}^{\infty} \binom{n+k-i}{k} \binom{k}{i} x^n b_n 1 \otimes (m+k-i+1) \quad \text{(exchanging the order of summation)} \\
&= \sum_{i=0}^{k} \sum_{m=k-i}^{\infty} \binom{m}{k} \binom{k}{i} x^i b_{m-k+i} 1 \otimes (m+1) \quad \text{(replacing } n \text{ by } m-k+i) \\
&= \sum_{i=0}^{k} \sum_{m=k}^{\infty} \binom{m}{k} \binom{k}{i} x^i b_{m-k+i} 1 \otimes (m+1) \quad \left( \binom{m}{k} = 0 \text{ for } m < k \right) \\
&= \sum_{m=k}^{\infty} \binom{m}{k} \left( \sum_{i=0}^{k} \binom{k}{i} x^i b_{m-k+i} \right) 1 \otimes (m+1) \quad \text{(exchanging the order of summation)}.
\end{align*}
\]

Thus finding a solution \( b \in \hat{\Pi}_{Q(x)}(Q(x)) \) of equation (5) is equivalent to finding solutions \( b_n \in Q(x) \) of the system of equations

\[
\left( \binom{m}{k} \sum_{i=0}^{k} \binom{k}{i} x^i b_{m-k+i} = 0, \ m \geq k. \right) \quad (6)
\]

Since \( Q(x) \) has characteristic zero, solving system (5) is equivalent to solving the system

\[
\sum_{i=0}^{k} \binom{k}{i} x^i b_{m-k+i} = 0, \ m \geq k
\]

in \( Q(x) \). This last system of equations can be rewritten as

\[
b_m = -x^{-k} \left( \sum_{i=0}^{k-1} \binom{k}{i} x^i b_{m-k+i} \right), \ m \geq k.
\]

For \( m = k \), we have

\[
b_k = -x^{-k} \left( \sum_{i=0}^{k-1} \binom{k}{i} x^i b_i \right), \ m \geq k.
\]

Choosing \( b_0 = 1 \) and \( b_i = 0, \ 1 \leq i \leq k-1 \), we have \( b_k = b_k(x) = -x^{-k} \). Inductively, these values of \( b_0, \ldots, b_{k-1} \) and equation (8) uniquely determine a rational function \( b_m^{(k)}(x) \in Q(x) \) for each \( m \geq k \), giving us a non-zero solution

\[
\{b_m = b_m^{(k)}(x), \ m \geq 0\}
\]

of the linear system (7) with values in \( Q(x) \). Hence we obtain a non-zero solution

\[
b^{(k)} = \sum_{n=0}^{\infty} b_n^{(k)}(x) 1 \otimes (n+1)
\]

to equation (5) in \( \hat{\Pi}_{Q(x)}(Q(x)) \). Note that \( b^{(k)} \) is a function of \( x \). We denote it by \( b^{(k)}(x) \).
Using Proposition 2.2 from $1^\otimes(k+1)b^{(k)}(x) = 0$ we have
\[ \Phi(1^\otimes(k+1))\Phi(b^{(k)}(x)) = \Phi(1^\otimes(k+1)b^{(k)}(x)) = 0 \] (9)
in $\mathfrak{A}(C)$. But from the definition of $\Phi$, we have
\[ \Phi(1^\otimes(k+1)) = \left( \binom{n-1}{k} x^k \right)_n \]
and
\[ \Phi(b^{(k)}(x)) = \left( \sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i \right)_n. \]
So
\[ \Phi(1^\otimes(k+1))\Phi(b^{(k)}(x)) = \left( \binom{n-1}{k} x^k \left( \sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i \right) \right)_n, \]
and equation (9) becomes
\[ \left( \binom{n-1}{k} x^k \left( \sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i \right) \right)_n = 0. \]
Therefore,
\[ \left( \binom{n-1}{k} x^k \left( \sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i \right) \right)_n = 0, \ n \geq 1. \]
Since \( \binom{n-1}{k} \neq 0 \) for $n \geq k + 1$, we must have
\[ \sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i = 0, \ n \geq k + 1. \] (10)

We now let $C$ be any $\mathbb{Q}$-algebra and let $\lambda$ be a non-zero divisor in $C$. Let $S = \{ \lambda^n, \ n \geq 0 \}$ and consider the localization $S^{-1}C$. Since $\lambda$ is not a zero divisor, the assignment $x \mapsto \lambda$ induces a ring homomorphism
\[ C[x, x^{-1}] \to S^{-1}C. \]
Let $b_i^{(k)}(\lambda)$ be the image of $b_i^{(k)}(x)$ under this homomorphism. Then from equation (10) we have the equations
\[ \sum_{i=0}^{n-1} b_i^{(k)}(\lambda) \binom{n-1}{i} \lambda^i = 0, \ n \geq k + 1 \]
in $S^{-1}C$. This shows that, for
\[ b^{(k)}(\lambda) = \sum_{n=0}^{\infty} b_n^{(k)}(\lambda) 1^\otimes(n+1), \]
the $n$-th component of $\Phi(b^{(k)}(\lambda))$ is zero for $n \geq k + 1$. On the other hand, from the definition of $\Phi$, for any $\alpha \in \text{Fil}^k \widehat{\Pi}_{S^{-1}C}(S^{-1}C)$, the $n$-th component of $\Phi(\alpha)$ is zero for $n \leq k$. Since the product in $\mathfrak{A}(C)$ is defined componentwise, we further have, for $\alpha \in \text{Fil}^k \widehat{\Pi}_{S^{-1}C}(S^{-1}C)$,

$$\Phi(\alpha b^{(k)}(\lambda)) = \Phi(\alpha) \Phi(b^{(k)}(\lambda)) = 0.$$  

Since $\Phi$ is injective, we have

$$b^{(k)}(\lambda) \text{Fil}^k \widehat{\Pi}_{S^{-1}C}(S^{-1}C) = 0. \quad (11)$$

By the assumption of the theorem, there is a non-zero element $c$ in $\cap_{n=0}^{\infty} \lambda^n C$. Fix such a $c$ and define

$$d^{(k)} = cb^{(k)}(\lambda), \quad k \geq 1.$$  

Then $d^{(k)}$ is in $\widehat{\Pi}_{S^{-1}C}(S^{-1}C)$. To finish the proof, we only need to show that each $d^{(k)}$ is in $\widehat{\Pi}_C(C)$ and satisfies equation (11). Here we regard $\widehat{\Pi}_C(C)$ as the subalgebra of $\widehat{\Pi}_{S^{-1}C}(S^{-1}C)$ consisting of sequences $\sum_{n=0}^{\infty} a_n \lambda^n$ with $a_n \in C$, $n \geq 0$. This is justified because $C$ can be identified with a subalgebra of $S^{-1}C$ since $\lambda$ is not a zero divisor, and because $\widehat{\Pi}_C(C) = \prod_{n=0}^{\infty} C1^{\otimes (n+1)}$ and $\widehat{\Pi}_{S^{-1}C}(S^{-1}C) = \prod_{n=0}^{\infty} S^{-1}C1^{\otimes (n+1)}$, as we have seen in §2.3.

Since $c$ is in $\cap_{n=0}^{\infty} \lambda^n C$, for each $n \in \mathbb{N}$, there is $c_n \in C$ such that $c = \lambda^n c_n$. Further, for each $n \geq 0$, the rational function $x^n b_n^{(k)}(x)$ in $\mathbb{Q}(x)$ is a polynomial in $\mathbb{Q}[x]$. This is clear for $0 \leq n \leq k$ and the general case follows by induction on $n$. Thus for each $n \geq 0$, $cb_n^{(k)}(\lambda) = c_n \lambda^n b_n^{(k)}(\lambda)$ is an element in $C$. Therefore,

$$d^{(k)} = cb^{(k)}(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n b^{(k)}(\lambda) 1^{\otimes (n+1)}$$

is an element in $\widehat{\Pi}_C(C)$.

Further,

$$\Phi(b^{(k)}(\lambda)) = c(b^{(k)}(\lambda) \text{Fil}^k \widehat{\Pi}_C(C)) = 0$$

since we have proved that $b^{(k)}(\lambda) \text{Fil}^k \widehat{\Pi}_{S^{-1}C}(S^{-1}C) = 0$ in equation (11). On the other hand, since we have chosen $b_0^{(k)}(x) = 1$ and $b_1^{(k)}(x) = \ldots = b_{k-1}^{(k)}(x) = 0$ in $b^{(k)} = b^{(k)}(x) = \sum_{n=0}^{\infty} b_n^{(k)}(x) 1^{\otimes (n+1)}$ we see that, for $1 \leq n \leq k$, the $n$-th component of $\Phi(b^{(k)})$ is

$$\sum_{i=0}^{n-1} b_i^{(k)}(x) \binom{n-1}{i} x^i = 1.$$  

Thus, the $n$-th component of $\Phi(cb^{(k)}(\lambda))$ for $1 \leq n \leq k$ is $c$. For $1 \otimes k \in C^{\otimes k}$, the $k$-th component of $\Phi(1 \otimes k) \in \mathfrak{A}(C)$ is $\lambda^{k-1}$. Thus the $k$-th component of $\Phi(cb^{(k)}(\lambda)1 \otimes k)$ is $c\lambda^{k-1}$. It is not zero, since $c$ is not zero and $\lambda$ is not a zero divisor. Thus $cb^{(k)}(\lambda) \text{Fil}^{k-1} \widehat{\Pi}_C(C)$ is not zero. Thus we have shown that the elements $d^{(k)} =
Let $c^b(\lambda), k \geq 1,$ of $\hat{\Pi}_C(C)$ satisfy equation (4). This completes the proof of part 3 of the theorem.

(4) If $C$ is not a $\mathbb{Q}$-algebra, then there is a prime number $p$ such that $p \cdot 1_C$ is not a unit in $C$. Thus there is a maximal ideal $M$ of $C$ containing $p \cdot 1_C$. Let $F = C/M$ be the residue field. Then $F$ is an algebra over the finite field $\mathbb{F}_p$. Let $\hat{M}$ be the Baxter ideal of $\hat{\Pi}_C(C)$ generated by $M$. Then by Proposition 3.3 in [5],

$$\hat{\Pi}_C(C)/\hat{M} \cong \hat{\Pi}_C(F) \cong \hat{\Pi}_F(F).$$

If $\hat{\Pi}_C(C)$ were noetherian, then its quotient $\hat{\Pi}_C(F)$ would also be noetherian. Thus the theorem follows from the following lemma.

**Lemma 3.2** If $F$ is a field of non-zero characteristic $p$, then $\hat{\Pi}_F(F)$ is not a noetherian ring.

**Proof:** For each $k \geq 1$, define

$$I_k = \sum_n F 1^\otimes (n+1) \subseteq \hat{\Pi}_F(F),$$

where the sum is over all $n \in \mathbb{N}$ with $p^k \nmid n$. We prove that each $I_k$ is an ideal of $\hat{\Pi}_F(F)$. For this we only need to show that $1^\otimes (m+1) 1^\otimes (n+1) \in I_n$ for $m \in \mathbb{N}$ and $p^k \nmid n$. We have

$$1^\otimes (m+1) 1^\otimes (n+1) = \sum_{i=0}^n \binom{m+n-i}{n} \binom{n}{i} 1^\otimes (m+n-i+1).$$

For each $0 \leq i \leq n$, if $p^k \nmid m + n - i$, then $1^\otimes (m+n-i+1) \in I_n$; if $p^k \mid m + n - i$, then from $p^n \nmid n$ we have $\binom{m+n-i}{n} \equiv 0 \pmod{p}$ [9, p.68]. So $1^\otimes (m+1) 1^\otimes (n+1) \in I_n$, and $I_n$ is an ideal. By definition we have $1^\otimes (p^n+1) \in I_{n+1}$ but $1^\otimes (p^n+1) \not\in I_n$ for each $n \geq 1$. Therefore, $I_n$ is a strictly increasing sequence of ideals, as needed.

### 3.2 The general case

**Theorem 3.3** Let $C$ be a ring of characteristic zero. For any non-empty set $X$, the free Baxter algebra $\hat{\Pi}_C(X)$ is not a noetherian algebra.

**Proof:** We start with the case when $X$ is a singleton $\{x\}$. For each integer $n \geq 1$, let $\Sigma_n$ be the ideal of $\hat{\Pi}_C(X)$ generated by $1 \otimes x^i$, $1 \leq i \leq n$. To prove the theorem, it suffices to show that $\Sigma_n \not\subseteq \Sigma_{n+1}$ for each $n \geq 1$. We prove this by contradiction. Assume that $\Sigma_{n+1} = \Sigma_n$ for some $n$. Then in particular, $1 \otimes x^{n+1} \in \Sigma_n$. Thus $1 \otimes x^{n+1}$ can be expressed in the form

$$\sum_{k=1}^n (1 \otimes x^k)G_k, \ G_k \in \hat{\Pi}_C(X).$$

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The construction of $\mathcal{III}_C(X)$ shows that $\mathcal{III}_C(X)$ is a free $C[x]$-module on the set

$$\mathcal{X} = \{1\} \cup \{1 \otimes x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} \mid i_j \in \mathbb{N} \text{ for } 1 \leq j \leq m, \ m \geq 1\}.$$

Define

$$\mathcal{I} = \{\phi\} \cup \left(\bigcup_{m=1}^{\infty} \mathbb{N}^m\right)$$

and, for $I \in \mathcal{I}$, denote

$$x^I = \begin{cases} 1, & \text{if } I = \phi, \\ 1 \otimes x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m}, & \text{if } I = (i_1, \ldots, i_m). \end{cases}$$

Then $\mathcal{X} = \{x^I \mid I \in \mathcal{I}\}$. Thus each $G_k$ above, $1 \leq k \leq n$, can be written as $\sum_{I \in \mathcal{I}} g^{(k)}_I x^I$ for unique $g^{(k)}_I \in C[x]$ and we have

$$1 \otimes x^{n+1} = \sum_{k=1}^{n} (1 \otimes x^k)(\sum_{I \in \mathcal{I}} g^{(k)}_I x^I) = \sum_{k=1}^{n} \sum_{I \in \mathcal{I}} g^{(k)}_I (1 \otimes x^k) x^I. \quad (12)$$

We will derive a contradiction from this equation.

Since elements in $\mathcal{X}$ form a basis for the free $C[x]$-module $\mathcal{III}_C(X)$, we can write

$$\sum_{k=1}^{n} \sum_{I \in \mathcal{I}} g^{(k)}_I (1 \otimes x^k) x^I = \sum_{J \in \mathcal{I}} h_J x^J \quad (13)$$

for unique $h_J \in C[x]$, $J \in \mathcal{I}$. Comparing this with equation (12), we see that $h_J = 1$ if $J = n+1$ and $h_J = 0$ for all other $J \in \mathcal{I}$. For $J \in \mathcal{I}$, define

$$|J| = \begin{cases} 0, & \text{if } J = \phi, \\ j_1 + \ldots + j_m, & \text{if } J = (j_1, \ldots, j_m). \end{cases}$$

Then we in particular have $h_J = 0$ for $|J| \neq n+1$. Thus equation (13) becomes

$$\sum_{k=1}^{n} \sum_{I \in \mathcal{I}} g^{(k)}_I (1 \otimes x^k) x^I = \sum_{|J|=n+1} h_J x^J \quad (14)$$

and equation (12) becomes

$$1 \otimes x^{n+1} = \sum_{|J|=n+1} h_J x^J. \quad (15)$$

Next we will study the relation between the coefficients $g^{(k)}_I$ and $h_J$ more carefully.

Fix a $k \in \mathbb{N}$ and an $I \in \mathcal{I}$. From the definition of the mixable shuffle product in equation (2), we have

$$(1 \otimes x^k)x^I = 1 \otimes x^k$$
when $I = \phi$; while when $I = (i_1, \ldots, i_m) \in \mathcal{I}$, we have

$$(1 \otimes x^k)x^I = 1 \otimes (x^k \otimes x^{i_1} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^k \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes \ldots \otimes x^{i_m} \otimes x^k + \lambda(x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^{i+k+1} \otimes x^{i_3} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes \ldots \otimes x^{i_m} \otimes x^{k+i_m})).$$

Note that for each of the basis elements $x^I \in \mathcal{X}$ that occurs on the right hand side of the equation, we have $|J| = k + |I|$. This shows that in equation (14), a coefficient $h_J$ on the right hand side must be a sum of the coefficients $g_I^{(k)}$ on the left hand side with the property $|J| = k + |I|$. Thus equation (14) becomes

$$\sum_{k=1}^{n} \sum_{|I|=n+1-k} g_I^{(k)}(1 \otimes x^k)x^I = \sum_{|J|=n+1} h_J x^J.$$ 

Exchanging the order of summation on the left hand side, we have

$$\sum_{m=1}^{n} \sum_{|I|=m,k=n+1-m} g_I^{(k)}(1 \otimes x^k)x^I = \sum_{|J|=n+1} h_J x^J \quad (16)$$

Since $|I| = n + 1 - k$ and $1 \leq k \leq n$, we have $|I| \neq 0$ for any $I$ in this equation. So $I \neq \phi$ and hence $I = (i_1, \ldots, i_m)$ for $i_j \in \mathbb{N}$ and $m \geq 1$. Then we have

$$\sum_{|I|=m,k=n+1-m} g_I^{(k)}(1 \otimes x^k)x^I$$

$$= \sum_{k+i_1+\ldots+i_m=n+1} g_{i_1,\ldots,i_m}^{(k)}(1 \otimes x^k)(1 \otimes x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m})$$

$$= \sum_{k+i_1+\ldots+i_m=n+1} g_{i_1,\ldots,i_m}^{(k)} \otimes \left( x^k \otimes x^{i_1} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^k \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} \otimes x^k + \lambda(x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^{i+k+1} \otimes x^{i_3} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes \ldots \otimes x^{i_m} \otimes x^{k+i_m}) \right).$$

Let $G_m^{(2)}$ (resp. $G_m^{(1)}$) be the sum of the terms on the right hand side in which the tensor product has $m + 2$ (resp. $m + 1$) tensor factors. More precisely,

$$G_m^{(2)} = \sum_{k+i_1+\ldots+i_m=n+1} g_{i_1,\ldots,i_m}^{(k)} \otimes \left( x^k \otimes x^{i_1} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^k \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} \otimes x^k + \lambda(x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^{i+k+1} \otimes x^{i_3} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes \ldots \otimes x^{i_m} \otimes x^{k+i_m}) \right),$$

$$G_m^{(1)} = \sum_{k+i_1+\ldots+i_m=n+1} \lambda g_{i_1,\ldots,i_m}^{(k)} \otimes \left( x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + x^{i_1} \otimes x^{i_2} \otimes \ldots \otimes x^{i_m} + \ldots + x^{i_1} \otimes \ldots \otimes x^{i_m} \otimes x^{k+i_m} \right).$$
Then from equation (15) and equation (16), we have

\[ 1 \otimes x^{n+1} = \sum_{m=1}^{n} (G_m^{(2)} + G_m^{(1)}). \]  

(17)

Thus for each \( r \geq 2 \), the sum of the terms on the right hand side of equation (17) with \( r \) tensor factors is given by

\[
\begin{align*}
0, & \quad \text{when } r < 2, \\
G_1^{(1)}, & \quad \text{when } r = 2, \\
G_{r-2}^{(2)} + G_{r-1}^{(1)}, & \quad \text{when } 3 \leq r \leq n + 1, \\
G_n^{(2)}, & \quad \text{when } r = n + 2, \\
0, & \quad \text{when } r > n + 2.
\end{align*}
\]

Therefore from equation (17) we have

\[
\begin{align*}
G_1^{(1)} = 1 \otimes x^{n+1}, \\
G_{r-2}^{(2)} + G_{r-1}^{(1)} = 0, & \quad 3 \leq r \leq n + 1, \\
G_n^{(2)} = 0.
\end{align*}
\]  

(18)

From the definition of \( G_m^{(2)} \), we see that the sum of the coefficients of all the basis elements \( x^I \in X \) in \( G_m^{(2)} \) is \((m + 1)g_m\) where

\[ g_m = \sum_{k+i_1+...+i_m=n+1} j_{i_1,..,i_m}^{(k)}. \]

Similarly, the sum of the coefficients of all the basis elements \( x^I \in X \) in \( G_m^{(1)} \) is \( \lambda mg_m \). Therefore, the sum of the coefficients of all monomials in \( G_{r-2}^{(2)} + G_{r-1}^{(1)} \) is

\[
\begin{align*}
\lambda g_1, & \quad \text{when } r = 2, \\
(r - 1)g_{r-2} + \lambda(r - 1)g_{r-1}, & \quad 3 \leq r \leq n + 1, \\
(n + 1)g_n, & \quad \text{when } r = n + 2.
\end{align*}
\]  

(19)

Recall that III\(_C(X)\) is a free \( C[x] \)-module on the set \( X \). So combining equation (18) and (19), we obtain

\[
\begin{align*}
\lambda g_1 = 1, \\
(r - 1)(g_{r-2} + \lambda g_{r-1}) = 0, & \quad 3 \leq r \leq n + 1, \\
(n + 1)g_n = 0.
\end{align*}
\]

Since the characteristic of \( C \) is zero by assumption, we have

\[
\begin{align*}
\lambda g_1 = 1, \\
g_{r-2} + \lambda g_{r-1} = 0, & \quad 3 \leq r \leq n + 1, \\
g_n = 0.
\end{align*}
\]
Thus we have $g_n = 0, g_{n-1} = -\lambda g_n = 0, \ldots, g_1 = -\lambda g_2 = 0$. This contradicts with $\lambda g_1 = 1$, proving Theorem 3.3 when $X = \{x\}$.

Let $X$ be any non-empty set. Fix an element $x_0 \in X$. Then the surjective map $X \to \{x_0\}$ sending all $x \in X$ to $x_0$ induces a surjective homomorphism $\text{III}_C(X) \to \text{III}_C(\{x_0\})$ of Baxter algebras. In fact, the homomorphism $\text{III}_C(\{x_0\}) \to \text{III}_C(X)$ induced by $\{x_0\} \to X, x_0 \mapsto x_0$ provides a section of the first homomorphism. If $\text{III}_C(X)$ were a noetherian, then its surjective image $\text{III}_C(\{x_0\})$ would have to be noetherian also. We have already shown above that this is impossible. So $\text{III}_C(X)$ is not noetherian. ■

4 Ascending chain condition for Baxter ideals

We now consider $\text{III}_C(C)$ in the category of Baxter algebras. We first give some definitions.

Definition 4.1 1. A Baxter algebra $(R, P)$ is called a noetherian Baxter algebra if the set of Baxter ideals of $(R, P)$ satisfies the ascending chain condition.

2. A Baxter ideal $I$ of $(R, P)$ is called Baxter finitely generated if there are finitely many elements $f_1, \ldots, f_r$ of $R$ such that $I$ is the smallest Baxter ideal of $R$ containing $f_1, \ldots, f_r$.

4.1 The case when $A = C$

Theorem 4.2 If $C$ is a noetherian ring, then $\text{III}_C(C)$ and $\text{\hat{III}}_C(C)$ are noetherian Baxter algebras.

Corollary 4.3 If $C$ is a noetherian ring, then any irreducible Baxter $C$-algebra is a noetherian $C$-algebra.

Proof: This follows from Theorem 4.2 since any irreducible Baxter $C$-algebra is a quotient of the free Baxter algebra $\text{III}_C(C)$. ■

Proof of Theorem 4.2 It is easy to see that $R$ is a noetherian Baxter algebra if and only if every Baxter ideal $I$ of $R$ is Baxter finitely generated. So we only need to prove that any Baxter ideal of $\text{III}_C(C)$ is Baxter finitely generated. The idea of the proof is the same as that of Theorem 3.3 following the Hilbert basis theorem, except that multiplying by $x$ is replaced by applying the Baxter operator $P_C$. Let $I \subseteq \text{III}_C(C)$ be an Baxter ideal. For each $j \in \mathbb{N}$, let

$$\Sigma_j = \{b_j \in C \mid \exists f_j \in I, \text{ such that } f_j = \sum_{k=0}^j b_k 1_k\}.$$ 

Then the same argument as in Theorem 3.3 shows that $\Sigma_j$ is an ideal of $C$. Also $b_j \in \Sigma_j$ implies that there exists a $f_j \in I$ such that $f_j = \sum_{k=0}^j b_k 1_k$. Then $P_C(f_j) =$
\[ \sum_{k=0}^{j} b_k 1_{k+1} \text{ and so } b_j \text{ is in } \Sigma_{j+1}. \] Thus \( \Sigma_j \subseteq \Sigma_{j+1} \). Since \( C \) is noetherian, this chain of ideals stabilizes, say at \( j = m \). Then \( \Sigma_m = \cup_{j=1}^{\infty} \Sigma_j \), and is finitely generated. Then as in the proof of Theorem 3.1, we construct from this a set of elements of \( I \{ f^{(0)}(0), \ldots, f^{(0)}(k_0), f^{(1)}(1), \ldots, f^{(1)}(k_1), \ldots, f^{(m)}(m) \} \)

and prove that \( I \) is the Baxter ideal generated by this set.

The statement for \( \hat{X} \) is proved in the same way, replacing \( \Sigma_j \) by \( \Omega_j = \{ b_j \in C \mid \exists f_j \in I, \text{ such that } f_j = \sum_{k=j}^{\infty} b_k 1_k \} \).

4.2 The general case

Because of Theorem 4.2, it is natural to ask whether other Baxter algebras are noetherian, and in particular, whether the free Baxter algebras are noetherian. Theorem 4.2 shows that if \( C \) is noetherian, then \( \Pi_C(X) \) is noetherian if \( X \) is the empty set. We will prove a theorem on free Baxter algebra \( \Pi_C(A) \). Consequences on \( \Pi_C(X) \) will be given in the corollary.

Consider the \( A \)-module \( A \otimes A \) with \( A \) acting on the left tensor factor. Let \( M \) be an \( A \)-submodule of \( A \otimes A \). We use \( \overline{A \otimes M} \) to denote the subgroup of \( A \otimes A \) generated by elements of the form \( a \otimes b, \ a \in A, \ b \in M \). It is an \( A \)-submodule of \( A \otimes A \). It is easy to see that \( \overline{A \otimes M} \) is the image of \( A \otimes M \) in \( A \otimes A \) under the natural map \( A \otimes M \rightarrow A \otimes A \) induced by \( M \hookrightarrow A \).

Theorem 4.4 Let \( A \) be a \( C \)-algebra.

1. Let \( M \) be the partially ordered set consisting of \( A \)-submodules of \( A \otimes A \) of the form \( \overline{A \otimes M} \), where \( M \) runs through \( C \)-submodules of \( A \). If \( M \) does not satisfy the ascending chain condition, then \( \Pi_C(A) \) of weight zero does not satisfy the ascending chain condition for Baxter ideals.

2. If \( A \) is not a noetherian ring, then \( \Pi_C(A) \) of any weight does not satisfy the ascending chain condition for Baxter ideals.

Remark 4.1 The condition in (1) implies that \( A \) is not a noetherian \( C \)-module. The following example shows that if the condition in (1) is weakened to the condition that \( A \) is not a noetherian \( C \)-module, then the conclusion of (1) might not hold. To see what extra restriction is needed, see (1) in Corollary 4.5.

Example 4.1 Let \( C = \mathbb{Z} \) and \( A = \mathbb{Q} \). An infinite ascending chain of \( \mathbb{Z} \)-modules of \( \mathbb{Q} \) is given by

\[ \mathbb{Z} \subset 2^{-1} \mathbb{Z} \subset \ldots \subset 2^{-n} \mathbb{Z} \subset \ldots. \]

So the \( \mathbb{Z} \)-submodules of \( \mathbb{Q} \) does not satisfy the ascending chain condition. On the other hand, it is easy to verify that \( \mathbb{Q} \otimes \mathbb{Q} = \mathbb{Q} \otimes 1 \) and it is the only \( \mathbb{Q} \)-submodule.
of \( Q \otimes Q \). Thus \( M = \{ Q \otimes Q \} \), trivially satisfying the ascending chain condition. Now for each \( n \geq 1 \),
\[
Q^\otimes n = Q^n \cong \underbrace{Q \otimes Q \ldots \otimes Q}_{n \text{-factors}}
\]
as \( Q \)-modules. This implies that \( \mathfrak{III}_n(Q) \) is isomorphic to \( \mathfrak{III}_n(Q) \) as rings. Since \( \mathfrak{III}_n(Q) \) is a noetherian ring by Theorem 3.7, \( \mathfrak{III}_n(Q) \) is a noetherian ring. In particular, it has the ascending chain condition for Baxter ideals.

Applying Theorem 4.4 to the case when \( A = C[X] \), we obtain

Corollary 4.5

1. Let \( A \) be a \( C \)-algebra. If \( A \) is not a noetherian \( C \)-module, and if there is a \( C \)-linear homomorphism \( A \to C \), then \( \mathfrak{III}_C(X) \) of weight zero does not have the ascending chain condition for Baxter ideals.

2. If \( X \) is not empty, then \( \mathfrak{III}_C(X) \) of weight zero does not have the ascending chain condition for Baxter ideals.

3. If \( X \) is infinite, then \( \mathfrak{III}_C(X) \) of any weight does not have the ascending chain condition for Baxter ideals.

Proof: (1) Denote the \( C \)-linear homomorphism \( A \to C \) by \( f \). By its \( C \)-linearity, \( f \) must be surjective. Since the tensor product functor is right exact, for any \( C \)-module \( M \), the surjective \( C \)-linear map \( f : A \to C \) induces surjective abelian group homomorphism \( f \otimes \text{id}_M : A \otimes M \to C \otimes M \cong M \). Since \( A \) is not a noetherian \( C \)-module, there are \( C \)-modules \( M_n, n \geq 1 \) such that \( M_n \subseteq M_{n+1} \) for all \( n \). Suppose \( A \otimes M_n = A \otimes M_{n+1} \) for some \( n \). Let \( j_{n,n+1} : M_n \to M_{n+1} \) and \( j_{n+1} : M_{n+1} \to A \) be the natural embeddings. Consider the commutative diagram

\[
\begin{array}{ccc}
A \otimes M_n & \longrightarrow & C \otimes M_n & \longrightarrow & M_n \\
\downarrow \text{id}_A \otimes j_{n,n+1} & & \downarrow \text{id}_C \otimes j_{n,n+1} & & \downarrow j_{n,n+1} \\
A \otimes M_{n+1} & \longrightarrow & C \otimes M_{n+1} & \longrightarrow & M_{n+1} \\
\downarrow \text{id}_A \otimes j_{n+1} & & \downarrow \text{id}_C \otimes j_{n+1} & & \downarrow j_{n+1} \\
A \otimes A & \longrightarrow & C \otimes A & \longrightarrow & A
\end{array}
\]

where all horizontal maps on the left column are surjective and all horizontal maps on the right column are isomorphisms. From \( A \otimes M_n = A \otimes M_{n+1} \) we have
\[
(id_A \otimes j_{n+1})(A \otimes M_{n+1}) = A \otimes M_{n+1}
\]
\[
= A \otimes M_n
\]
\[
= (id_A \otimes j_n)(A \otimes M_n)
\]
\[
= ((id_A \otimes j_{n+1}) \circ (id_A \otimes j_{n,n+1}))(A \otimes M_n).
\]

Since all the horizontal maps in the commutative diagrams are surjective, we further have
\[
M_{n+1} = j_{n+1}(M_{n+1}) = (j_{n+1} \circ j_{n,n+1})(M_n) = M_n.
\]
If $\lambda I$ is in $I$, we have an infinite ascending chain $n$ since, for each $n \geq 1$, $M_n$ is a submodule of $M_{n+1}$ and $x^{n+1}$ of $M_{n+1}$ is not in $M_n$, we have an infinite ascending chain $M_n \subsetneq M_{n+1}$. So $C[X]$ is not a noetherian $C$-module. On the other hand, the $C$-algebra map $f : C[X] \to C$ induced by sending $x \in X$ to 0 is clearly the $C$-linear map we want. Therefore (1) applies.

(3) This follows from the second statement of Theorem 4.4 since $C[X]$ is not a noetherian ring when $X$ is infinite. ■

Before the proof of Theorem 4.4, we first prove a lemma.

**Lemma 4.6** Let $M$ be a $C$-submodule of $A$. Define

$$S = \bigcup_{k=1}^{\infty} \left\{ \otimes_{i=0}^{k} x_i \in A^{\otimes(k+1)} \mid x_{i_0} \in M \text{ for some } 1 \leq i_0 \leq k \right\}.$$ 

Let $I$ be the abelian subgroup of $\text{III}_C(A)$ generated by $S$. If either the weight of $\text{III}_C(A)$ is zero or $M$ is an ideal of $A$, then $I$ is a Baxter ideal of $\text{III}_C(X)$. In fact, $I = I'$, the Baxter ideal of $\text{III}_C(A)$ generated by $P_A(M)$.

**Proof:** We only need to prove the last statement. We first prove that $I \subseteq I'$. For this we only need to show that, for each $x = \otimes_{i=0}^{k} x_i \in S$, we have $x \in I'$. We prove by induction on $k \geq 1$. When $k = 1$, we have $x = x_0 \otimes x_1$ with $x_1 \in M$. Then $x = x_0(1 \otimes x_1) \in I_n'$. Now let $x = \otimes_{i=0}^{k+1} x_i \in S$, so $x_i \in A$ and $x_{i_0} \in M$ for some $1 \leq i_0 \leq k+1$. If $i_0 > 1$, then $x_1 \otimes \ldots \otimes x_{k+1}$ is in $S$, and hence by the induction hypothesis, is in $I'$. Then $x = x_0P_A(x_1 \otimes \ldots \otimes x_{k+1})$ is in $I'$. If $i_0 = 1$, consider the equation obtained by the definition of the mixable shuffle product (2)

$$(x_0 \otimes x_2 \otimes \ldots \otimes x_{k+1})(1_A \otimes x_1) = x_0 \otimes x_1 \otimes \ldots \otimes x_{k+1} + \sum_{j=2}^{k} x_0 \otimes x_2 \otimes \ldots \otimes x_j \otimes x_{j+1} \otimes \ldots \otimes x_{k+1} + x_0 \otimes x_2 \otimes \ldots \otimes x_{k+1} \otimes x_1 + \lambda \sum_{j=2}^{k+1} x_0 \otimes x_2 \otimes \ldots \otimes x_j \otimes x_{j+1} \otimes \ldots \otimes x_{k+1}.$$

Since $x_1$ is in $M$, we see that $1_A \otimes x_1$ is in $I'$. So the left hand side of the equation is in $I'$. Again because $x_1$ is in $M$, the induction hypothesis implies that every term on the right hand side except the first term and the terms in the last sum are in $I'$. If $\lambda = 0$, then the last sum disappears. So the first term is also in $I'$. On the other
hand, if \( M \) is an ideal of \( A \), then \( x_j x_1 \) is in \( M \) for \( 2 \leq j \leq k + 1 \). Hence by induction hypothesis, every term in the last sum is in \( I' \). So again the first term is in \( I' \). This completes the induction, proving that \( I \subseteq I' \).

We next prove that \( I \) contains \( I' \). For this we only need to show that \( I \) is a Baxter ideal of \( \mathbb{III}_C(A) \) since \( I \) clearly contains \( M \). By the definition of \( S \) we have \( P_A(S) \subseteq S \). So we get \( P_A(I) \subseteq I \). Since \( I \) is clearly closed under addition, it remains to verify that, if \( x \in I \) and \( y \in \mathbb{III}_C(A) \), then \( xy \) is in \( I \). For this we only need to verify this property for \( x = \otimes_{i=0}^k x_i \in S \) and \( y = \otimes_{j=0}^m y_j \in A^\otimes(m + 1) \), \( m \geq 0 \). By definition,

\[
x y = x_0 y_0 \otimes \sum_{(\sigma, T) \in S(m, n)} \lambda^{[T]} \sigma(x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n; T).
\]

For each \((\sigma, T) \in S(m, n)\), the set of \((m, n)\)-mixable shuffles defined in \( \S 2.3 \) write

\[
\sigma(x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n; T) = z_1 \hat{\otimes} \ldots \hat{\otimes} z_{m+n}.
\]

Then \((z_1, \ldots, z_{m+n})\) is a permutation of \((x_1, \ldots, x_m, y_1, \ldots, y_n)\). Since \( x_{i_0} \) is in \( M \) for some \( 1 \leq i_0 \leq n \), one of \( z_i \), \( 1 \leq i \leq m + n \), is in \( M \). If \( \lambda = 0 \), then the only non-zero terms in the sum on the right hand side of equation (20) are of the form \( z_1 \hat{\otimes} \ldots \hat{\otimes} z_{m+n} \). So the right hand side is in \( I \). Now assume that \( M \) is an ideal of \( A \). For any \((\sigma, T) \in S(m, n)\), by the definition of \( \sigma(x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n; T) \), either \( x_{i_0} \) or \( x_{i_0} y_j \) for some \( 1 \leq j \leq n \) occurs as one of the tensor factors in

\[
\sigma(x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n; T) = z_1 \hat{\otimes} \ldots \hat{\otimes} z_{m+n}.
\]

Since \( x_{i_0} \) and \( x_{i_0} y_j \) are in \( M \), we see that

\[
\lambda^{[T]} x_{i_0} y_0 \otimes \sigma(x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n; T)
\]

is in \( S \). Thus \( xy \) is in \( I \). Therefore \( I \) is an Baxter ideal of \( \mathbb{III}_C(A) \). Consequently, \( I' \subseteq I \). The lemma is now proved. 

**Proof of Theorem 4.4**

(1) By assumption, there are \( C \)-submodules \( M_n, n \in \mathbb{N}_+ \), of \( A \) such that \( \overline{A} \otimes M_n \) is a proper submodule of \( \overline{A} \otimes M_{n+1} \) for all \( n \). Define

\[
S_n = \bigcup_{k=1}^{\infty} \left\{ \otimes_{i=0}^k x_i \in A^\otimes(k+1) \mid x_{i_0} \in M_n \text{ for some } 1 \leq i_0 \leq k \right\}.
\]

Let \( I_n \) be the abelian subgroup of \( \mathbb{III}_C(A) \) generated by \( S_n \). Since we assume that \( \lambda \) is zero, by Lemma 4.3, \( I_n \) is a Baxter ideal of \( \mathbb{III}_C(X) \). Suppose \( \mathbb{III}_C(X) \) satisfies the ascending chain condition for Baxter ideals. Then the ascending chain \( I_n, n \geq 1 \) stabilizes for large \( n \). In particular, \( I_n = I_{n+1} \) for some \( n \). Let

\[
p : \mathbb{III}_C(A) = \oplus_{m=1}^{\infty} \overline{A}^\otimes m \to \overline{A} \otimes \overline{A}
\]

be the natural projection from \( \mathbb{III}_C(A) \) onto the summand with \( m = 2 \). Then from the construction of \( I_n \) we have \( p(I_n) = \overline{A} \otimes M_n \). Thus \( I_n = I_{n+1} \) implies that
\[ A \otimes M_n = A \otimes M_{n+1} \]. This contradicts the choice of \( M_n \). So \( \text{III}_C(A) \) does not have the ascending chain condition for Baxter ideals.

The proof of (2) is similar. ■

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References

[1] G. E. Andrews, L. Guo, W. Keigher and K. Ono, Baxter Algebras and Hopf Algebras, preprint.

[2] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, Pacific J. Math. 10 (1960), 731-742.

[3] P. Cartier, On the structure of free Baxter algebras, Adv. in Math. 9 (1972), 253-265.

[4] K.T. Chen, Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula, Ann. of Math. 65 (1957), 163-178.

[5] L. Guo, Properties of free Baxter algebras, Adv. in Math. 151 (2000), 346-374.

[6] L. Guo, Baxter algebras and the umbral calculus, to appear in Adv. in Appl. Math.

[7] L. Guo and W. Keigher, Baxter algebras and shuffle products, Adv. in Math. 150 (2000), no.1, 117-149.

[8] L. Guo and W. Keigher, On free Baxter algebras: completions and interior constructions, Adv. in Math. 151 (2000), 101-127.

[9] D. Knuth, The Art of Computing, Second printing, Addison-Wesley Publishing Co., Reading, MA, 1969.

[10] R. Ree, Lie elements and an algebra associated with shuffles, Ann. Math. 68 (1958), 210-220.

[11] G. Rota, Baxter algebras and combinatorial identities I, Bull. AMS 5 (1969), 325-329.

[12] G. Rota, Baxter operators, an introduction, In: “Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries”, ed. Joseph P.S. Kung, Birkhäuser, Boston, 1995.
[13] G. Rota, *Ten mathematics problems I will never solve*, Invited address at the joint meeting of the American Mathematical Society and the Mexican Mathematical Society, Oaxaca, Mexico, December 6, 1997. DMV Mittellungen Heft 2, 1998, 45-52.