Dynamical Instability of Shear-free Collapsing Star in Extended Teleparallel Gravity

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Abstract

We study the spherically symmetric collapsing star in terms of dynamical instability. We take the framework of extended teleparallel gravity with non-diagonal tetrad, power-law form of model presenting torsion and matter distribution as non-dissipative anisotropic fluid. The vanishing shear scalar condition is adopted to search the insights of collapsing star. We apply first order linear perturbation scheme to metric, matter and \( f(T) \) functions. The dynamical equations are formulated under this perturbation scheme to develop collapsing equation for finding dynamical instability limits in two regimes such as Newtonian and post-Newtonian. We obtain constraint free solution of perturbed time dependent part with the help of vanishing shear scalar. The adiabatic index exhibits the instability ranges through second dynamical equation which depend on physical quantities such as density, pressure components, perturbed parts of symmetry of star, etc. We also develop some constraints on positivity of these quantities and obtain instability ranges to satisfy the dynamical instability condition.

Keywords: \( f(T) \) gravity; Instability; Shear-free; Newtonian and post-Newtonian regimes.

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1 Introduction

The gravitational collapse of self-gravitating objects has become widely discussed phenomena in general relativity (GR) as well as in modified theories of gravity. This contains the evolutionary development and constancy of these objects during collapse process and rests importantly at the center of structure formation. This process occurs when a stable matter becomes unbalanced and ultimately undergoes a collapse which results different structures like stars, stellar groups and planets. In this way, self-gravitating objects go across various dynamical states which may be analyzed through dynamical equations. The dynamical instability was firstly investigated by Chandrasekhar [1] with the help of adiabatic index $\Gamma$ of a spherical star with isotropic pressure. This index depicts the consequences of various structural quantities of a fluid on the instability ranges. For instability ranges in Newtonian and post-Newtonian regimes, Herrera et al. [2] explored dissipative, non-adiabatic spherically symmetric collapsing star.

The adiabatic index develops the instability ranges in GR as well as in modified theories of gravity which induces that these ranges depending on dark source terms in addition to usual terms. Under different conditions for cylindrically and spherically symmetric collapsing matters in $f(R)$ gravity, the instability ranges have been explored through adiabatic index[3, 4]. Taking expansion-free condition, Skripkin [5] developed a model for non-dissipative spherically symmetric fluid distribution with isotropy and constant energy density and remarked that a Minkowskian cavity is observed at a center of fluid. Under this condition, the instability for spherically and cylindrically symmetric anisotropic fluids in Newtonian, post-Newtonian regimes is explored in GR [6] as well as $f(R)$ gravity [7]. In Brans-Dicke gravity, Sharif and Manzoor [8] explored the instability ranges of spherically symmetric collapsing star.

The physical aspects such as isotropy, radiation, anisotropy, shear, dissipation, expansion are main sources of cause for the gravitational evolution. Among these factors, the shear leads to the formation of naked singularities. That is, it contribute to the formation of an apparent horizon which results in a black hole of the evolving cloud. Thus, the shear tensor occupies a direction of well-motivation to study structure formation and its consequences on the dynamical instability range of a self-gravitating body.

In context of extended teleparallel gravity (ETG) (or $f(T)$ gravity) which is the generalization of teleparallel gravity, the gravitational collapse is dis-
cussed with and without expansion scalar by Sharif and Rani [9, 10]. They found that the physical properties invade a vast impact of dynamical instability in studying the self-gravitating objects with expansion. Without expansion, they obtain the instability ranges for Newtonian (N) and post-Newtonian (pN) regimes. In this paper, we assume shear-free condition instead expansion-free and explore the instability ranges of a collapsing star in ETG.

The scheme of the paper is given by: In section 2, we give the basics of ETG and provide the construction of field equations in two ways, simple and covariant form. Section 3 contains the basic equations for the static spherically symmetry. Also, junction conditions are given for dynamical instability of a spherically symmetric collapsing star in the context of ETG gravity. In the next section, we represent perturbation scheme and ETG model and apply to all matter, metric and \( f(T) \) functions. In section 5, we formulate dynamical collapsing equation and found the instability ranges in N and pN regimes. The last section summarizes the results and elaborate the comparison.

## 2 Extended Teleparallel Gravity

In this section, we provide the basics such as tetrad field and the Weitzenböck connection of ETG. We give field equations in simple form as well as its covariant construction.

### 2.1 Tetrad Field

The geometry of ETG is unambiguously described through an orthonormal set having three spacelike and one timelike fields called tetrad field. The trivial tetrad field has the form \( e_a = \delta^\mu_a \partial_\mu \), \( e^b = \delta^b_\mu dx^\mu \), where \( \delta^a_\mu \) named as the Kronecker delta. This is the simplest field and less important due to zero torsion. The non-trivial tetrad field allows non-zero torsion and grants the construction of teleparallel as well as ETG theory. It is given by

\[
h_a = h^a_\mu \partial_\mu, \quad h^b = h^b_\nu dx^\nu
\]

satisfying the following properties

\[
h^a_\mu h^b_\mu = \delta_b^a, \quad h^a_\mu h_\nu^a = \delta^\nu_\mu.
\]

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The metric tensor is demonstrated as a by product of this field which is as follows
\[ g_{\mu\nu} = \eta_{ab} h^a_{\mu} h^b_{\nu}. \]  
(3)

2.2 The Weitzenböck Connection

The basic phenomenon of teleparallel gravity and ETG is the parallel transport of tetrad field in Weitzenböck spacetime which is carried out by the significant component Weitzenböck connection.

By applying covariant derivative w.r.t spacetime of tetrad field, we get
\[ \Delta_{\nu}h^a_{\mu} = \partial_{\nu}h^a_{\mu} - \tilde{\Gamma}^a_{\alpha\mu\nu} h^a_{\alpha} \equiv 0, \]  
(4)
where \( \tilde{\Gamma}^a_{\mu\nu} = h^a_{\alpha\beta} \partial_{\nu}h^a_{\mu} \) is the Weitzenböck connection. We obtain torsion tensor by the antisymmetric part of this connection as follows
\[ T^a_{\mu\nu} = \tilde{\Gamma}^a_{\nu\mu} - \tilde{\Gamma}^a_{\mu\nu} = h^a_{\alpha}(\partial_{\nu}h^a_{\mu} - \partial_{\mu}h^a_{\nu}), \]  
(5)
which is antisymmetric in its lower indices, i.e., \( T^a_{\mu\nu} = -T^a_{\nu\mu} \). This absolute parallelism passed away rapidly the curvature of the Weitzenböck connection identically. The following relation is also satisfied by the Weitzenböck connection
\[ \tilde{\Gamma}^a_{\mu\nu} = \Gamma^a_{\mu\nu} + K^a_{\mu\nu}, \]  
(6)
here \( \Gamma^a_{\mu\nu} \), \( K^a_{\mu\nu} \) appear as the usual Levi-Civita connection (torsionless) and the contorsion tensor, respectively, which can be defined as follows
\[ \Gamma^a_{\mu\nu} = \frac{1}{2} g^{a\rho}[g_{\mu\nu,\rho} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}], \]  
(7)
\[ K^a_{\mu\nu} = \frac{1}{2} [T^a_{\mu\nu} + T^a_{\nu\mu} - T^a_{\mu\nu}]. \]  
(8)

2.3 Field Equations

To generalize the action of teleparallel gravity, we just substitute a general function of torsion scalar by itself as follows [13]–[15]
\[ S = \frac{1}{2\kappa^2} \int h(f(T) + \mathcal{L}_m) d^4x. \]  
(9)
where, \( h = \det(h^a_\lambda) \), \( \mathcal{L}_m \) is the matter Lagrangian and \( f \) is the function of torsion scalar. The torsion scalar is defined as

\[
T = S^\alpha_{\mu\nu} T^\alpha_{\mu\nu}.
\]  

(10)

where

\[
S^\alpha_{\mu\nu} = \frac{1}{2}[ - \frac{1}{2}(T_{\mu\nu}^\alpha - T_{\nu\mu}^\alpha - T_{\alpha\mu}^\nu) + \delta^\mu_\alpha T_{\theta\nu}^\theta - \delta^\nu_\alpha T_{\theta\mu}^\theta].
\]  

(11)

called the superpotential tensor. Applying the variation of action (9) w.r.t tetrad field, we will get field equations as follows

\[
\frac{1}{h}[ \partial_\mu(h_a^a S^\alpha_{\mu\nu}) + h_a^a T^\lambda_{\mu\sigma} S_\lambda^{\nu\mu}] f_\tau + h_a^a S^\alpha_{\mu\nu} \partial_\mu T f_{\tau\tau} + \frac{1}{4} h_a^a f = \frac{1}{2} \kappa^2 h_a^a T^\alpha_{\tau},
\]  

(12)

where \( f_\tau \) is the first order derivative and \( f_{\tau\tau} \) represent second order derivative of \( f \) with respect to \( T \).

The field equations (12) turn out to be extremely different from Einstein’s equations on account of tetrad components and partial derivatives. Since tetrad are not completely eradicated which causes difficulty to compare teleparallel gravity (and ETG) with GR. To obtain equivalent description of field equations (12) with the other modified theories, we will apply covariant formalism [16]. We replace all partial derivatives in Eqs.(5), (8), (11) by covariant derivatives using the condition on metric tensor, \( \nabla_\sigma g_{\mu\nu} = 0 \), i.e.,

the compatibility of the metric tensor, we get

\[
T_{\mu\nu} = h_a^a (\partial_\mu h_\nu^a - \Gamma_\mu^\sigma h_\nu^\sigma + \partial_\nu h_\mu^a),
\]

\[
K_{\mu\nu} = h_a^a \nabla_\nu h_\mu^a, \quad S^\mu_{\nu\alpha} = \eta^{ab} h_\mu^\alpha \nabla_\nu h_\alpha^b + \delta^\nu_\mu \eta^{ab} h_\alpha^b \nabla_\sigma h_\alpha^a
\]  

where we have applied the following relations

\[
S^{\mu(\nu\alpha)} = T^{(\mu\nu)} = K^{(\mu\nu)} = 0. \tag{13}
\]

In this case, the Weitzenböck connection becomes zero, while the Riemann tensor turns out to be

\[
R_{\mu\nu} = \partial_\rho \Gamma_\mu^\rho_{\nu} - \partial_\rho \Gamma_\nu^\rho_{\mu} + \Gamma_\mu^\alpha \Gamma_\rho^\sigma_{\nu} - \Gamma_\nu^\alpha \Gamma_\sigma^\rho_{\mu} - \Gamma_\rho^\alpha_{\sigma\mu} \Gamma_\nu^\sigma_{\rho}
\]

\[
= \nabla_\nu K_{\mu\rho} - \nabla_\rho K_{\mu\nu} + K_{\mu\nu} K_{\rho\lambda} - K_{\mu\rho} K_{\nu\lambda}.
\]

The corresponding Ricci tensor becomes

\[
R_{\mu\nu} = \nabla_\nu K_{\mu\rho} - \nabla_\rho K_{\mu\nu} + K_{\rho\mu} K_{\nu\alpha} - K_{\rho\nu} K_{\mu\alpha}.
\]
Using relations (13) along with $S^\nu_{\alpha \nu} = -2T^\nu_{\alpha \nu} = 2K^\nu_{\alpha \nu}$, we have
\[
R_{\mu \nu} = -\nabla^\alpha S_{\nu \alpha \mu} - g_{\mu \nu} \nabla^\alpha T^\rho_{\alpha \rho} - S^\alpha_{\mu} K_{\rho \alpha \nu}, \quad (14)
\]
\[
R = -T - 2\nabla^\alpha T^\nu_{\alpha \nu}. \quad (15)
\]

The covariant derivative of torsion tensor in the last equation shows the only difference between Ricci and torsion scalars.

After some calculations, we will get
\[
G_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T = -\nabla^\alpha S_{\nu \alpha \mu} - S^\alpha_{\mu} K_{\alpha \sigma \nu}, \quad (16)
\]
where $G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$ appears as the Einstein tensor. By using this in (12), we attain the required field equations in $f(T)$ gravity
\[
f_T G_{\mu \nu} + \frac{1}{2} g_{\mu \nu} (f - T f_T) + D_{\mu \nu} = \kappa^2 T_{\mu \nu}, \quad (17)
\]
here $D_{\mu \nu} = S^\nu_{\mu \alpha} \nabla_\alpha T$. It can be observed that (17) has an equivalent structure such as $f(R)$ gravity and reduces to GR for $f(T) = T$. Here, the trace of the above equation is
\[
Df_T T_{\mu \nu} - (R + 2T)f_T + 2f = \kappa^2 T, \quad (18)
\]
with $D = D^\nu_{\nu}$ and $T = T^\nu_{\nu}$. The $f(T)$ field equations can also defined as
\[
G_{\mu \nu} = \frac{\kappa^2}{f_T} (T^m_{\mu \nu} + T^T_{\mu \nu}). \quad (19)
\]
Here $T^m_{\mu \nu}$ represents the matter fluid and torsion contribution is
\[
T^T_{\mu \nu} = \frac{1}{\kappa^2} [-D_{\mu \nu} f_T - \frac{1}{4} g_{\mu \nu} (T - D f_{TT} + R f_T)]. \quad (20)
\]

3 Basic Equations

The general spherically symmetric metric in the interior region is
\[
ds_+^2 = X^2 dt^2 - Y^2 dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (21)
\]
where $X$, $Y$ and $R$ are functions of $t$ and $r$. The line element for exterior spacetime (the Schwarzschild metric) is

$$ ds_+^2 = \left( 1 - \frac{2M}{r} \right) dv^2 + 2rdr - r^2(d\theta^2 + \sin^2 \theta d\phi^2), $$

(22)

where $v$ is the retarded time and $M$ represents the total mass of the bounded surface. Also, the anisotropic energy-momentum tensor can be defined as follows

$$ T^\mu_\nu = (\rho + p_\perp)u^\mu u_\nu - p_\perp \delta^\mu_\nu + (p_r - p_\perp)v^\mu v_\nu, $$

(23)

in the interior region while $\rho = \rho(t, r)$, $p_r = p_r(t, r)$, $p_\perp = p_\perp(t, r)$. The four velocity $u_\mu = \frac{1}{X} \delta^0_\mu$ and unit four vector directed towards radial component $v_\mu = \frac{1}{Y} \delta^1_\mu$ satisfy the relations $u_\mu u^\mu = 1$, $v_\mu v^\mu = -1$, $u_\mu v^\mu = 0$. We take the non-diagonal tetrad for the interior spacetime as

$$ h^i_\mu = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & Y \sin \theta \cos \phi & R \cos \theta \cos \phi & -R \sin \theta \sin \phi \\ 0 & Y \sin \theta \sin \phi & R \cos \theta \sin \phi & R \sin \theta \cos \phi \\ 0 & Y \cos \theta & -R \sin \theta & 0 \end{pmatrix}. $$

The description of gravitational collapsing star takes kinematics of the dynamical equations of spherically symmetric models. This needs acceleration, expansion, rotation and distortion or shear. Due to shearfree condition, there have been many interesting results such as this condition depicts the physical aspects of compact bodies in the relativistic astrophysics phenomena. The shear scalar and tensor are defined by

$$ \sigma^2 = \frac{1}{2} \sigma^{\alpha \beta} \sigma_{\alpha \beta}, $$

(24)

where

$$ \sigma_{\alpha \beta} = V_{(\alpha ; \beta)} + a_{(\alpha} V_{\beta)}, \quad \Theta = V^\alpha_{(\alpha}, \quad a = V_{(\alpha ; \beta)} V^\beta. $$

(25)

For interior spacetime, this tensor yields the following scalar

$$ \sigma = \frac{1}{X} \left( \frac{\dot{X}}{X} - \frac{\dot{R}}{R} \right). $$

(26)
Using the tetrad along with Eq. (21) in (19), the field equations are

\[
\left( \frac{2\dot{Y}}{Y} + \frac{\dot{R}}{R} \right) \frac{\ddot{R}}{R} - \left( \frac{X}{Y} \right)^2 \left[ \frac{2R''}{R} + \left( \frac{R'}{R} \right)^2 - 2Y'R'Y - \left( \frac{Y}{R} \right)^2 \right]
\]

\[
= \frac{\kappa^2}{f_T} \left[ \rho X^2 + \frac{X^2}{\kappa^2} \left\{ \frac{T f_T - f}{2} - \frac{1}{Y^2} \left( \frac{R'}{R} - \frac{Y}{R} \right) f_T \right\} \right], \tag{27}
\]

\[
2 \left( \frac{\dot{R}}{R} - \frac{\dot{R}X'}{RX} - \frac{Y'R'}{YR} \right) = \frac{\dot{R}f_T}{R f_T}, \tag{28}
\]

\[
2 \left( \frac{\dot{R}}{R} - \frac{\dot{R}X'}{RX} - \frac{Y'R'}{YR} \right) = \frac{\dot{T}}{T'} \left( \frac{R'}{R} - \frac{Y}{R} \right) f_T, \tag{29}
\]

\[
- \left( \frac{Y}{X} \right)^2 \left[ \frac{2\ddot{R}}{R} - \left( \frac{2\dot{X}}{X} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} + \left( \frac{2X'}{X} + \frac{R'}{R} \right) \frac{R'}{R} - \left( \frac{Y}{R} \right)^2 \right]
\]

\[
= \frac{\kappa^2}{f_T} \left[ p_r Y^2 - \frac{Y^2}{\kappa^2} \left\{ \frac{T f_T - f}{2} + \frac{\dot{R}f_T}{X^2 R T'} \right\} \right], \tag{30}
\]

\[
- \left( \frac{R}{X} \right)^2 \left[ \frac{\ddot{Y}}{Y} + \frac{\dot{R}}{R} - \frac{\dot{X}}{X} \left( \frac{\dot{Y}}{Y} + \frac{\dot{R}}{R} \right) + \frac{\dot{Y}}{Y} R' \right] + \left( \frac{R}{Y} \right)^2 \left[ \frac{X''}{X} + \frac{R''}{R} \right]
\]

\[
- \frac{XX'}{XY} + \left( \frac{X'}{X} - \frac{Y'}{Y} \right) \frac{R'}{R} = \frac{\kappa^2}{f_T} \left[ p_\perp R^2 - \frac{R^2}{\kappa^2} \left\{ \frac{T f_T - f}{2} + \frac{1}{2} \right\}
\]

\[
\times \left( \frac{1}{X^2} \left( \frac{\ddot{Y}}{Y} + \frac{\dot{R}}{R} \right) \frac{\dot{T}}{T'} - \frac{1}{Y^2} \left( \frac{X'}{X} + \frac{R'}{R} - \frac{Y}{R} \right) \right) f_T \right\} \right]. \tag{31}
\]

The torsion scalar takes the form

\[
T = 2 \left[ \frac{2}{XY R} \left( \frac{XR'}{Y'} - \frac{\dot{Y} R}{X} \right) - \frac{1}{X^2} \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{Y^2} \left( \frac{R'}{R} \right)^2 + \frac{1}{R^2}
\]

\[
- \frac{2}{Y R} \left( \frac{X'}{X} + \frac{R'}{R} \right) \right]. \tag{32}
\]

Also, from Eqs. (28) and (29), we obtain a relationship as follows

\[
\frac{\dot{R}}{R} = \frac{\dot{T}}{T'} \left( \frac{R'}{R} + \frac{Y}{R} \right). \tag{33}
\]
To analyze the properties of collapsing star, the non-trivial contracted identities yield the dynamical equations which are very useful. These are given by
\[
\left( \mathcal{T}^{\mu \nu} + \mathcal{T}^{\nu \mu} \right)_{,\mu} u_\mu = 0, \quad \left( \mathcal{T}^{\mu \nu} + \mathcal{T}^{\nu \mu} \right)_{,\nu} v_\mu = 0.
\] (34)

Using these equation, the dynamical equations become
\[
\begin{align*}
\left[ \frac{\dot{\rho}}{\rho} + (\rho + p_r) \frac{\dot{Y}}{XY} + 2(\rho + p_\perp) \frac{\dot{R}}{XR} \right] + \frac{X}{\kappa^2} \left[ \left( \frac{\dot{T} f_T - f}{2X^2} - \frac{\dot{RT} f_T'}{X^2 Y^2 R T} \right)_{,0} 
\right. \\
+ \frac{\dot{X}}{X^3} (T f_T - f) + \left( \frac{\dot{R} f_T'}{X^2 Y^2 R} \right)_{,1} - \frac{1}{X^2 R} \left\{ \frac{2\dot{R}}{XY^2} \left( \frac{\dot{X} T'}{T} - 2X' \right) + \frac{\dot{R}}{X^2} \times \left( \frac{2\dot{Y}}{Y} + \frac{\dot{R}}{R} \right) \frac{T'}{T} + \frac{\dot{R}}{Y^2} \left( \frac{\dot{R}}{R} + \frac{\dot{Y}}{Y} \right) \frac{T'}{T} - \frac{Y'}{Y} - \frac{2R'}{R} \right\} f_T' \right] &= 0, \\
\left[ \frac{p_r'}{Y} + (\rho + p_r) \frac{X'}{XY} + 2(p_r - p_\perp) \frac{R'}{RY} \right] + \frac{Y}{\kappa^2} \left[ \left( \frac{\dot{R} f_T'}{X^2 Y^2 R} \right)_{,0} - \frac{1}{Y^2} 
\right. \\
\times \left( \frac{Y'}{Y} + \frac{R'}{R} \right) (T f_T - f) + \left( \frac{f - T f_T}{2Y^2} - \frac{\dot{RT} f_T'}{X^2 Y^2 RT} \right)_{,1} + \frac{1}{XY^2 R} \\
\times \left\{ -\frac{X' \dot{R} T'}{Y^2 T} + \frac{\dot{R}}{X} \left( \frac{3\dot{Y}}{Y} + \frac{\dot{X}}{X} \right) + 2\frac{\dot{R}^2}{XR} \left( 1 - \frac{1}{Y^2} \right) - \frac{\dot{R}}{XY} 
\times \left( \frac{Y'}{Y^2} + \frac{X'}{XY} \right) \frac{T'}{T} \right\} f_T' \right] &= 0.
\end{align*}
\] (35)

\[\text{(36)}\]

3.1 Junction Conditions

In order to join smoothly the interior and exterior spacetimes over a hypersurface \(\Sigma^{(e)}\), we apply junction conditions. The collapse problems are dealt by Darmois junction conditions in appropriate manner. These conditions require the continuity of intrinsic and extrinsic curvatures over the hypersurface, i.e., \((ds^2)_\Sigma = (ds^2)^+\) and \(K_{ab} = K_{ab}^- = K_{ab}^+,\) respectively. The Misner-Sharp mass function is given by
\[
m(t, r) = \frac{R}{2} (1 + g^{\mu \nu} R_{,\mu} R_{,\nu}),
\]
where a spherical object of radius $R$ contributes total energy and contributes to study Darmois junction conditions. For Eq. (21), it takes the form

$$m(t, r) = \frac{R}{2} \left( 1 + \frac{\dot{R}^2}{X^2} - \frac{R'^2}{Y^2} \right). \quad (37)$$

In order to match exterior region with interior, it requires that $r = r^{(e)} = \text{constant}$ on the boundary surface $\Sigma^{(e)}$ \[7\] \[17\] which results

$$M \Sigma^{(e)} = m(t, r), \quad (38)$$

$$2 \left( \frac{\dot{R}}{R} - \frac{\dot{X}}{X} \frac{\dot{R}'}{R'} - \frac{\dot{Y}}{Y} \frac{\dot{R}'}{R'} \right) \Sigma^{(e)} = \frac{Y}{X} \left[ \frac{2\dot{R}}{R} - \left( \frac{2\dot{X}}{X} - \frac{\dot{R}}{R} \right) \frac{\dot{R}}{R} \right]$$

$$+ \frac{X}{Y} \left[ \left( \frac{2X'}{X} + \frac{R'}{R} \right) \frac{R'}{R} - \left( \frac{Y}{R} \right)^2 \right]. \quad (39)$$

Substituting the field equations (28) and (30) in the above equation, we get

$$- p_r \Sigma^{(e)} = \frac{T_{11}}{Y^2} - \frac{T_{01}}{XY} = \frac{f(T_c)}{2}, \quad (40)$$

where $f(T_c)$ represents a constant value for constant torsion scalar $T_c$.

4 Linear Perturbation Strategy and Power-law $f(T)$ Model

In $f(R)$ gravity, the gravitational collapse is widely discussed taking power-law form of model which is simply generalizes GR. We take particular $f(T)$ model in power-law form analogy to $f(R)$ model \[7\] like $f(R) = R + \gamma R^2$ to analyze the evolution of collapsing star. The power-law $f(T)$ model has contributed as a most viable model due to its simple form and we may directly compare our results with GR. We assume the ETG model as follows

$$f(T) = T + \delta T^2, \quad (41)$$

where $\delta$ is an arbitrary constant. For this model, we obtain accelerated expansion universe in phantom phase, possibility of realistic wormhole solutions
and instability conditions for a collapsing star. We assume the linear perturbation strategy to construct the dynamical equations in order to explore instability ranges for the underlying scenario. For this purpose, we assume the system initially in static equilibrium. That is, metric and matter parts are at zero order perturbation are only radial dependent only which also become time dependent for the first order perturbations. These perturbations are described as follows [3]-[10], [17]-[20]

\[ X(t, r) = X_0(r) + \epsilon \Pi(t) x(r), \] (42)

\[ Y(t, r) = Y_0(r) + \epsilon \Pi(t) y(r), \] (43)

\[ R(t, r) = R_0(r) + \epsilon \Pi(t) c(r), \] (44)

\[ \rho(t, r) = \rho_0(r) + \epsilon \hat{\rho}(t, r), \] (45)

\[ p_r(t, r) = p_{r0}(r) + \epsilon \hat{p}_r(t, r), \] (46)

\[ p_\perp(t, r) = p_{\perp0}(r) + \epsilon \hat{p}_\perp(t, r), \] (47)

\[ m(t, r) = m_0(r) + \epsilon \hat{m}(t, r), \] (48)

\[ T(t, r) = T_0(r) + \epsilon \Pi(t) e(r), \] (49)

where the quantities with zero subscript denotes static parts of corresponding functions and \(0 < \epsilon \ll 1\). The perturbed vanishing shear scalar and \(f(T)\) model takes the form

\[ f(T) = T_0(1 + \delta T_0) + \epsilon \Pi e(1 + 2\delta T_0), \] (50)

\[ f_r(T) = 1 + 2\delta T_0 + 2\epsilon \delta \Pi e, \] (51)

\[ \frac{y}{Y_0} = \frac{c}{R_0}. \] (52)

Taking into account shear-free condition (\(\sigma = 0\)), Eq.[26] yields \(\frac{Y}{Y_0} = \frac{R}{R_0}\). The solution of this equation turns out as \(Y = \alpha R\) where \(\alpha\) is an arbitrary function of \(r\) taken as 1 without loss of generality and using the freedom to rescale the radial coordinate, we take \(R_0 = r\) which is also the Schwarzschild coordinate. The condition under which an initially shear-free flow remains shear-free all along the evolution, has been studied by Herrera et al. [21]. One of the consequences of such a study is that the pressure anisotropy may affect the propagation in time, of the shear-free condition. The shear-free condition is unstable, in particular, against the presence of pressure anisotropy. An expansion scalar and a scalar function insured the departures from the shearfree condition for the geodesic case. These scalars are defined
in purely physical variables such as in terms of the Weyl tensor, anisotropy of pressure and the shear viscosity. It is remarked that one can consider such a case that pressure anisotropy and density inhomogeneity are present in a way that the scalar function appearing in orthogonal splitting of Riemann tensor vanishes, implying non-homogeneous anisotropic stable shear-free flow. Since we are dealing with fluid evolving under shearfree condition, so we shall make use of this condition while evaluating the components of field equations and also in conservation equations.

Now we evaluate zero order as well as first order configurations. The zero order perturbation of the field equations (27)-(31) is given by

\[
\frac{1}{1 + 2\delta T_0} \left[ \kappa^2 \rho_0 + \delta \left( \frac{T_0^2}{2} - \frac{2cT_0'^2}{erY_0^2} \right) \right] = \frac{1}{(Y_0r)^2} \left( \frac{2rY_0'}{Y_0} + Y_0^2 - 1 \right), \tag{53}
\]

\[
\frac{1}{1 + 2\delta T_0} \left[ \kappa^2 p_{r0} - \delta \frac{T_0'^2}{2} \right] = \frac{1}{(Y_0r)^2} \left( 2r \frac{X_0'}{X_0} - Y_0^2 + 1 \right), \tag{54}
\]

\[
\frac{1}{1 + 2\delta T_0} \left[ \kappa^2 p_{\perp0} - \delta \frac{T_0'^2}{2} + \delta T_0'' \frac{X_0'}{X_0} \right] = \frac{1}{Y_0^2} \left[ \frac{X_0''}{X_0} - \frac{X_0' Y_0'}{X_0 Y_0} \right], \tag{55}
\]

The first dynamical equation (35) fulfills the zero order perturbation identically while second dynamical equation (36) becomes

\[
\frac{2}{r} (p_{r0} - p_{\perp0}) + p_{r0}' + \frac{X_0'}{X_0} (p_{r0} + p_{\perp0}) - \frac{\delta}{\kappa^2} \left[ T_0 T_0'' + \frac{T_0'^2}{r} + \frac{2cX_0'T_0'^2}{eX_0Y_0'^2} \right] = 0. \tag{56}
\]

In static background, the matching condition, mass function and torsion scalar contribute for static equilibrium as

\[
p_{r0} \Delta^{(e)} = \frac{\delta T_0^2}{2\kappa^2}, \quad m_0(r) = \frac{r}{2} \left[ 1 - \frac{1}{Y_0^2} \right], \tag{57}
\]

\[
T_0 = \frac{2}{r} \left[ \frac{1}{r} \left( 1 - \frac{3}{Y_0} \right) - \frac{1}{Y_0} \left( \frac{2X_0'}{X_0} + \frac{1}{r} \right) \left( 1 - \frac{1}{Y_0} \right) \right]. \tag{58}
\]

Applying the perturbed quantities to the field equations, we obtain

\[
- \frac{2\Pi}{Y_0^2} \left[ \left( \frac{c}{r} \right)'' - \frac{1}{r} \left( \frac{y}{Y_0} \right)' - \left( \frac{Y_0'}{Y_0} - \frac{3}{r} \right) \left( \frac{c}{r} \right)' \right].
\]

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Applying Eqs. (42)-(47) to the Bianchi identities, it follows that

\[
\begin{align*}
\text{(59)} & \quad = \frac{2\Pi c}{Y_0^3} \left( \frac{r}{2} \frac{Y_0'}{Y_0} + Y_0^2 - 1 \right) + \frac{\kappa^2 \dot{\rho}}{1 + 2\delta T_0} + \frac{\Pi \delta}{(1 + 2\delta T_0)} \left[ T_0 e - \frac{2\kappa^2 \rho_0 y}{1 + 2\delta T_0} \right] \\
\text{(60)} & \quad - \frac{\delta T_0^2 y}{1 + 2\delta T_0} - \frac{cT_0'}{eY_0^2 r} \left( 2e' - \frac{4\delta T_0' e}{1 + 2\delta T_0} - \frac{4cT_0'}{r} \right), \\
\text{(61)} & \quad \left( \frac{c}{r} \right)' - \left( \frac{X_0'}{X_0} - \frac{1}{r} \right) c - \frac{c}{r^2} = \frac{\delta T_0' e}{r(1 + 2\delta T_0)}, \\
\text{(62)} & \quad \left( \frac{c}{r} \right)' - \left( \frac{X_0'}{X_0} - \frac{1}{r} \right) c - \frac{c}{r^2} = \frac{\kappa^2 \rho_{0 e}}{1 + 2\delta T_0},
\end{align*}
\]

\[
\begin{align*}
\text{(63)} & \quad = \frac{\kappa^2 Y_0^2 \dot{\rho}_r}{1 + 2\delta T_0} + \frac{2\kappa^2 \Pi Y_0^2}{1 + 2\delta T_0} \left( \frac{c}{r} - \frac{\delta e}{1 + 2\delta T_0} \right) p_{r0} - \frac{\delta \Pi T_0 Y_0^2}{1 + 2\delta T_0} \left( e + \frac{cT_0}{r} \right) \\
& \quad - \frac{\delta Y_0 T_0 e}{1 + 2\delta T_0},
\end{align*}
\]

Applying Eqs. (42)-(47) to the Bianchi identities, it follows that

\[
\dot{\rho} + \left[ (3\rho_0 + p_{r0} + 2p_{\perp 0}) \frac{c}{r} + J_0 \right] \dot{\Pi} = 0,
\]

where

\[
J_0 = \frac{1}{\kappa^2} \left[ \delta T_0 e - \frac{4cT_0' \delta}{eY_0^2} \left( e' - T_0' \left( \frac{x}{X_0} + \frac{2e}{r} \right) \right) + \frac{2\delta}{X_0^2} \left( \frac{cT_0'}{X_0^2 Y_0^2 r} \right) \right] \\
- \frac{4c\delta T_0'}{X_0 Y_0^2 r} \left( \frac{xT_0'}{e} - 2X_0' \right) + \frac{2c\delta T_0'}{r Y_0^2} \left\{ \frac{2}{r} + \frac{Y_0'}{Y_0} - \frac{2cT_0'}{er} \right\}.
\]
Integrating Eq.(64) with respect to time, we have
\[ \hat{\rho} = - \left( 3\rho_0 + p_{r0} + 2p_{\perp0} \right) \frac{c}{r} + J_0 \] \tag{65}

The perturbed second Bianchi identity is given by
\[ p'_{r0} - (\rho_0 + p_{r0}) \left( \frac{x}{X_0} \right)' \frac{r}{c} + (\rho_0 + p_{r0}) \frac{X_0'}{X_0} - (\hat{\rho} + \hat{p}_r) \frac{X_0'r}{X_0} + J_1 \]
\[ - 2(p_{r0} - p_{\perp0}) \left( \frac{r}{c} \left( \frac{c}{r} \right)' - \frac{1}{r} \right) - (\hat{p}_r - \hat{p}_\perp) \frac{2}{\Pi c} - \frac{c}{\Pi r} \hat{\beta}'_r = 0, \] \tag{66}

where
\[ J_1 = \frac{r \delta}{c \kappa^2} \left[ Y_0^2 \left( \frac{T_0 e}{Y_0^2} - \frac{T_0^2 c}{rY_0^2} \right) \right]_1 - \frac{2T_0^2 c}{rX_0^2} \hat{\Pi} + \frac{2T_0Y'_0e}{Y_0} - \frac{3cY'_0T^2_0}{rY_0}, \]
\[ + \left( \frac{cY_0}{r} \right)' \frac{T_0^2}{Y_0} - \frac{2cT_0^2 e}{r^2} + \frac{2T_0e}{r} + T_0 \left( \frac{c}{r} \right)' + \frac{2cx'T^2_0}{erX_0Y_0^2} \]
\[ + \frac{2cX'_0T'_0}{erX_0Y_0^2} \left( 1 - \frac{m_0}{r} \right)' \left( 1 - \frac{m_0}{r} \right) \left( 2e' - \frac{xT'_0}{X_0} - \frac{4cT'_0}{r} \right). \]

The junction condition, torsion scalar and mass function are
\[ \hat{p}_r \Sigma^{(c)} = \frac{\delta}{\kappa^2} \left( \Pi T_0 e - \frac{2T_0^2 \hat{\Pi} c}{rX_0Y_0} \right), \] \tag{67}
\[ e = \frac{4}{rY_0^2} \left[ \frac{1}{X_0Y_0} \left\{ x' + X_0'c' - \frac{X'_0}{Y_0} \left( y + \frac{x}{X_0} + \frac{y}{Y_0} + \frac{c}{r} \right) \right\} \right.
\[ - \frac{cY_0}{r^2} + \frac{xX'_0}{X_0^2} - \left( \frac{c}{r} \right)' + \left( \frac{y}{Y_0} + \frac{c}{r} \right) \left( \frac{X'_0}{X_0} + \frac{2}{r} \right) \left( \right), \] \tag{68}
\[ \bar{m} = - \frac{\Pi}{Y_0^2} \left[ r \left( c' - \frac{y}{Y_0} \right) + \frac{c}{2} (1 - Y_0^2) \right]. \] \tag{69}

Substituting zero order and first order matching conditions in Eq.(62), we have
\[ \hat{\Pi} \Sigma^{(c)} = 0, \] \tag{70}

yields the general solution of this equation is
\[ \Pi(t) = h_1 t + h_2, \] \tag{71}
where \(h_1\) and \(h_2\) are arbitrary constants. It is remarked that we do not need to impose any extra condition on this solution due to vanishing shear scalar condition for collapse to occur. In [3, 4, 7-10], we need to apply some constraint for the static solution in order to discuss instability analysis.

5 Collapse Equation and Dynamical Instability

In this section, we construct the collapse equation in order to work for dynamical instability in different regimes with shear-free condition. The Harison-Wheeler equation of state is used in this regard which is given by [22]

\[
\hat{p}_{ir} = \hat{\rho} \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\perp0}} \Gamma.
\]

We use this index in order to examine instability ranges in the context of ETG with \(\sigma = 0\). The adiabatic index \(\Gamma\) finds the rigidity of the fluid and evaluates the change of pressure to corresponding density. Substituting the value of \(\hat{\rho}\), it follows that

\[
\hat{p}_r = -\Pi \left[ \frac{2c}{r} \rho_0 + p_{\perp0} + \frac{c}{r} p_{r0} + \frac{p_{r0}}{\rho_0 + p_{r0}} J_0 \right] \Gamma, \tag{73}
\]

\[
\hat{p}_\perp = -\Pi \left[ \frac{c}{r} \rho_0 + p_{r0} \frac{p_{\perp0}}{\rho_0 + p_{\perp0}} + \frac{2c}{r} p_{\perp0} + \frac{p_{\perp0}}{\rho_0 + p_{\perp0}} J_0 \right] \Gamma. \tag{74}
\]

Finally, the collapse equation used to analyze instability ranges of collapsing star in N and pN regimes, we insert all the corresponding values in Eq.(66), we have

\[
\frac{\Delta T_0 T_0'}{\kappa^2} - (\rho_0 + p_{r0}) \left[ \frac{X_0'}{cX_0} - \frac{X_0'}{X_0} \right] + (\Gamma + 1) \left( \frac{X_0'}{cX_0} \right) \left[ \frac{2c}{r} (\rho_0 + p_{\perp0}) + J_0 \right.
\]

\[
+ \frac{c}{r} (\rho_0 + p_{r0}) \right] + J_1 - 2(p_{r0} - p_{\perp0}) \left( \frac{r}{c} \left( \frac{c}{r} \right)' - \frac{1}{r} \right) + \Gamma \left[ \frac{c}{r} (p_{r0}
\]

\[
- \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\perp0}} p_{\perp0} \right] + \frac{2c}{r} \left( \frac{\rho_0 + p_{\perp0}}{\rho_0 + p_{r0}} p_{r0} - p_{\perp0} \right) + \left( \frac{p_{r0}}{\rho_0 + p_{r0}} - \frac{p_{\perp0}}{\rho_0 + p_{\perp0}} \right)
\]

\[
\times J_0 \] + \frac{\Gamma r}{c} \left[ \frac{c}{r} p_{r0} + \frac{2c}{r} \frac{\rho_0 + p_{\perp0}}{\rho_0 + p_{r0}} p_{r0} + \frac{p_{r0}}{\rho_0 + p_{r0}} J_0 \right] = 0. \tag{75}
\]

We consider the N and pN approximations to obtain the instability ranges in the following.
Newtonian limit

The N approximations are given by

\[ X_0 = 1 = Y_0, \quad \rho_0 \gg p_{\perp 0}, \quad \rho_0 \gg p_{\perp 0}, \quad p_{\perp 0} \gg p_{\perp 0}. \]

Inserting these conditions as well as Eq. (71) in (75), the collapse equation yields

\[ \frac{\delta T_0}{\kappa^2} - \frac{\delta r}{c} + J_{1(N)} = 0, \]

where

\[ J_{1(N)} = \frac{r}{c^2} \left[ \left( eT_0 - \frac{e T_0^2}{r} \right)' + 2 \left( \frac{e}{r} \right)' T_0^2 - \frac{2 e T_0^2}{r'} + \frac{2 T_0 e}{r'} + \frac{2 c x' T_0^2}{e r} \right]. \]

To check the instability range for N approximation, we obtain

\[ \Gamma < \frac{1}{I_0} \left[ -\frac{\delta T_0}{\kappa^2} + \frac{\delta r}{c} - J_{1(N)} + 2 p_{\perp 0} \left( \frac{\delta}{c} - \frac{1}{r} \right) \right], \]

where \( I_0 = \frac{3 c p_{\perp 0}}{r} + \frac{3 c}{e} \left( \frac{p_{\perp 0}}{e} \right)' \). This equation expresses the dependence of adiabatic index on torsion terms along with physical properties such as anisotropic pressure and energy density. As long as the above inequality maintains, the collapsing system stays unstable. To be hold for dynamical instability condition the terms in collapse equation should be positive in the inequality. It requires that

\[ -\frac{\delta T_0}{\kappa^2} + \frac{\delta r}{c} - J_{1(N)} + 2 p_{\perp 0} \left( \frac{\delta}{c} - \frac{1}{r} \right) > 0, \]

where \( I_0 > 0 \) holds. In GR [1], the adiabatic index represents a numerical value corresponding to dynamical instability of an isotropic sphere. This is given by

- It is found that the dynamical stability is achieved for \( \Gamma > \frac{4}{3} \) when the weight of outer layers is weaker in contrast to the pressure in star.
- The case \( \Gamma = \frac{4}{3} \) corresponds to the hydrostatic equilibrium condition.
When the weight of outer layers increases very fast as compared to pressure inside the star gives the collapse and $\Gamma < \frac{4}{3}$ constitutes the dynamical instability.

The expressions in Eq.(77) constitute the following possibilities

\[
(i) \quad \left[ -\frac{\delta T_0 T_0'}{\kappa^2} + \frac{x r \rho_0}{c} - J_{1(N)} + 2 p_{r0} \left( \frac{r}{c} \left( \frac{c}{r} \right)' \right) \right] = I_0,
\]

\[
(ii) \quad \left[ -\frac{\delta T_0 T_0'}{\kappa^2} + \frac{x r \rho_0}{c} - J_{1(N)} + 2 p_{r0} \left( \frac{r}{c} \left( \frac{c}{r} \right)' \right) \right] < I_0,
\]

\[
(iii) \quad \left[ -\frac{\delta T_0 T_0'}{\kappa^2} + \frac{x r \rho_0}{c} - J_{1(N)} + 2 p_{r0} \left( \frac{r}{c} \left( \frac{c}{r} \right)' \right) \right] > I_0.
\]

In the first possibility together with Eq.(77), we obtain the instability range as $0 < \Gamma < 1$. The case $(ii)$ yields that faction in Eq.(77) always less than 1 but depending on the values of dynamical terms. The third case $(iii)$ represents the different numerical values for different values of dynamical terms. Thus, it shows that the collapsing star remains unstable for $\Gamma > 1$. Also, Eq.(76) depicts that the adiabatic index contains the physical quantities like $f(R)$ gravity [4, 7].

**Post-Newtonian limit**

We consider the following pN approximations in terms of metric component expressions given by

\[
X_0 = 1 - \frac{m_0}{r}, \quad Y_0 = 1 + \frac{m_0}{r}, \quad \text{(78)}
\]

upto order $\frac{m_0}{r}$. Introducing these approximations in Eq.(75), we obtain

\[
\frac{\delta T_0 T_0'}{\kappa^2} - \left[ \frac{x r}{c} \left( 1 + \frac{m_0}{r} \right) - \frac{m_0}{r^2} \right] \left( \rho_0 + p_{r0} \right) + J_{1(pN)} - 2(p_{r0} - p_{\perp0})
\]

\[
\times \left[ \frac{r}{c} \left( \frac{c}{r} \right)' \right] + \left[ \frac{\Gamma + 1) m_0}{rc} \right] \left[ \frac{r}{c} \left( 3 \rho_0 + p_{r0} + 2 p_{\perp0} \right) + J_{0(pN)} \right]
\]

\[
+ \Gamma \left[ \frac{c}{r} \left( p_{r0} - \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\perp0}} p_{\perp0} \right) \right] + \frac{2 c}{r} \left( \frac{\rho_0 + p_{\perp0}}{\rho_0 + p_{r0}} p_{r0} - p_{\perp0} \right)
\]

\[
+ \left( \frac{p_{r0}}{\rho_0 + p_{r0}} - \frac{p_{\perp0}}{\rho_0 + p_{\perp0}} \right) J_{0(pN)} \right] \right] + \frac{\Gamma r}{c} \left[ \frac{c}{r} p_{r0} + \frac{2 c p_{\perp0} + \rho_0}{r} p_{r0} \right]
\]

\[
= I_0.
\]
where $J_{1(pN)}$ and $J_{0(pN)}$ are those terms which belong to pN regime in $J_1$ and $J_0$ expressions are given below.

\[
J_{0(pN)} = \frac{1}{\kappa^2} \left[ \delta T_0 e - \frac{4c\delta T_0'}{er} \left( 1 - \frac{2m_0}{r} \right) \left\{ e' - T_0' \left( x \left( 1 + \frac{m_0}{r} \right) + 2 \frac{c}{r} \right) \right\} \right. \\
+ 2 \delta \left( 1 + \frac{2m_0}{r} \right) \left( \frac{cT_0'}{r} \right)_{,1} - 4c\delta T_0' \left( 1 - \frac{m_0}{r} \right) \left( \frac{xT_0'}{e} + \frac{2m_0}{r^2} \right) \\
+ \frac{2c\delta T_0'}{r} \left( 1 - \frac{2m_0}{r} \right) \left\{ 2 \frac{m_0}{r} - 2cT_0' \right\}.
\]

\[
J_{1(pN)} = \frac{r\delta}{ck^2} \left[ \left( 1 + \frac{2m_0}{r} \right) \left\{ T_0 \left( e - \frac{T_0 c'}{r} \right) \left( 1 - \frac{2m_0}{r} \right) \right\} \right. \\
+ \left( 1 + \frac{m_0}{r} \right)' \left( 1 - \frac{m_0}{r} \right) \left( 2T_0 e - 3c\delta T_0' + \frac{c}{r} T_0'^2 \right) - \frac{2c\delta T_0'^2}{r^2} + 2T_0 e \\
+ \frac{2c\delta T_0'^2}{er} \left( 1 - \frac{m_0}{r} \right) + \frac{2c\delta T_0'}{er} \left( 2e' - xT_0' \left( 1 + \frac{m_0}{r} \right) - \frac{4cT_0'}{r} \right) \left( 1 + \frac{m_0}{r} \right).
\]

Similar to the N approximation strategy, the instability range is given by

\[
\Gamma < \frac{A}{B}, \tag{80}
\]

where

\[
A = -\frac{\delta T_0 T_0'}{\kappa^2} + \left[ \frac{xr}{c} \left( 1 + \frac{m_0}{r} \right) - \frac{m_0}{r^2} \right] \left( \rho_0 + p_{r0} \right) - J_{1(pN)} + 2(p_{r0} - p_{\perp0}) \\
\times \left[ \frac{r}{c} \left( \frac{c}{r} \right)' - \frac{1}{r} \right] - \frac{m_0}{rc} \left[ \frac{c}{r} \left( 3p_0 + p_{r0} + 2p_{\perp0} \right) + J_{0(pN)} \right],
\]

\[
B = \frac{m_0}{rc} \left[ \frac{c}{r} \left( 3p_0 + p_{r0} + 2p_{\perp0} \right) + J_{0(pN)} \right] + \left[ \frac{c}{r} \left( p_{r0} - \frac{\rho_0 + p_{r0}}{\rho_0 + p_{\perp0}} p_{\perp0} \right) \right. \\
+ \frac{2c}{r} \left( \frac{p_{r0} + p_{\perp0}}{\rho_0 + p_{r0}} p_{r0} - p_{\perp0} \right) \right] + \left( \frac{p_{r0}}{\rho_0 + p_{r0}} - \frac{p_{\perp0}}{\rho_0 + p_{\perp0}} \right) J_{0(pN)} \\
+ \frac{r}{c} \left[ \frac{c}{r} p_{r0} + \frac{2c}{r} \left( \frac{p_{r0} + p_{\perp0}}{\rho_0 + p_{r0}} \right) p_{r0} + \frac{p_{r0}}{\rho_0 + p_{r0}} J_{0(pN)} \right]_{,1}.
\]

We obtain instability ranges of collapsing star with vanishing shear as long as this inequality holds. We analyze that the adiabatic index evinces the
dependence of instability ranges on relativistic and torsion terms under zero order configuration.

For the dynamical instability condition, it is required that the right hand side (expressions $A$ and $B$) in inequality (80) remains positive. Similarly to the N regime, we have three cases such as $A = B$, $A < B$, $A > B$. These cases constitute the instability ranges as $0 < \Gamma < 1$ for the first and second case while $\Gamma > 1$ for the last case.

6 Concluding Remarks

The collapse in a star occurs due to state of disequilibrium between inwardly acting gravitational pull and outwardly drawn pressure in it. In modified theories of gravity, the ranges of dynamical instability depends on dark source terms in addition to usual terms (that is terms of GR) determined by the adiabatic index. We have analyzed the dynamical instability of a collapsing star taking vanishing shear scalar in ETG gravity. We have taken interior metric as general spherically symmetric metric while Schwarzschild metric is considered in exterior region to $\Sigma^{(e)}$ in anisotropic matter distribution. The contracted Bianchi identities are used to find two dynamical equations corresponding to a star experiencing collapse process. A well-known power-law ETG model is considered to analyze these ranges. We have used perturbation strategy on all functions such that metric components, energy density, pressure components, mass, torsion, shear scalar to examine the evolution of collapse with respect to time. We have applied zero order and first order perturbed configurations on the field and dynamical equations. The collapse equations has been constructed through second dynamical equation. The results are given as follows.

The adiabatic index ($\Gamma$) plays a vital role to examine the instability ranges for a collapsing star with shear-free condition. We can analyze the instability regions through this index by applying N and pN approximations on collapsing equation. In both cases of approximations, we have observed that this index depends upon on various quantities such as energy density, anisotropy of pressure and on some other constraints. This index shows that how we can modify the instability range of collapsing star under relativistic as well as torsion terms. Hence, the physical properties play a crucial role in analyzing the self-gravitating objects in dynamical instability. Similar to the case of $f(R)$ gravity, the results of present paper contain physical quantities. It is
pointed out that the instability ranges are found as $0 < \Gamma < 1$ and $\Gamma > 1$ for both approximation i.e., system yields unstable behavior and meets $\Gamma < \frac{4}{3}$ in the N and pN perfect fluid limit [1].

We would like to point out here that some authors have discussed the instability analysis by imposing some constraints for finding the solutions [3, 4, 7-10]. However, we have found the solutions without imposing any extra constraints.

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