Octions: An $E_8$ description of the Standard Model

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We interpret the elements of the exceptional Lie algebra $e_8$ as objects in the Standard Model, including lepton and quark spinors with the usual properties, the Standard Model Lie algebra $su(6) \oplus su(2) \oplus u(1)$, and the Lorentz Lie algebra $so(3,1)$. Our construction relies on identifying a complex structure on spinors and then working in the enveloping algebra. The resulting model naturally contains GUTs based on SO(10) (Georgi–Glashow), SU(5) (Georgi), and SU(4) $\times$ SU(2) $\times$ SU(2) (Pati–Salam). We then briefly speculate on the role of the remaining elements of $e_8$, and propose a mechanism leading to exactly three generations of particles.

I. INTRODUCTION

From the earliest days of quantum mechanics, mathematical physicists have been searching for a mathematical structure rich enough to describe the physical world. Exceptional structures involving the octonions, as the largest of the normed division algebras, the Albert algebra, as the only exceptional arena for quantum mechanics, the exceptional Lie algebra $e_8$, describing the symmetries of the Albert algebra, its larger cousins $e_7$, and $e_8$, and tensor products of division algebras such as $R \otimes C \otimes H \otimes O$, have all received attention.

Beginning in 1933, Jordan and colleagues [1–3] showed that there was just one exceptional, nonassociative algebraic system, now known as the Albert algebra, in which the postulates of quantum mechanics can be realized. However, as described by Gürsey and Tze [4, pp. 213–214] (see also [5]), it was not until the 1960s that these structures began to be used seriously in quantum mechanics. In the early 1980s, there was an explosion of interest in using octonionic structures to describe supersymmetry, starting with [6, 7], and superstring theory [8–13]; a short history is given in [14]. These constructions led us to explore models based on $E_6$ [e.g. 15, 16].

Meanwhile, in the 1970s, Grand Unified Theories (GUTs) using octonions and exceptional Lie algebras began to appear, such as the work of Gürsey and Günaydin [e.g. 19, 20]. To the best of our knowledge, Bars and Günaydin [21] were the first to propose a GUT using $e_8$. The beautiful and extensive work by Gürsey and collaborators, as summarized in [4], is particularly noteworthy.

More recently, Lisi [22] attempted to fit the Standard Model into $e_8$, and Chester et al. [23] proposed a model based on decomposing $e_8$ (and provided extensive references to other approaches). Each of these models interprets the elements of $e_8$ somewhat differently. In the 1990s, Dixon (see [24] and the references cited therein) used matrices over the tensor product of the four division algebras, $R \otimes C \otimes H \otimes O$, to represent particle states. More recently, taking a different tack, Furey and Hughes [25–29] have built a model using only $R \otimes C \otimes H \otimes O$ to represent particle states. Other recent work along similar lines includes [30–32].

The Lie group $E_8$ is the largest of the exceptional Lie groups, all of which are naturally associated with the octonions, the largest of the normed division algebras. The Freudenthal–Tits magic square [33, 34] provides a unified description of 16 Lie algebras, parameterized by two division algebras (or their split cousins). The magic square is remarkable for containing four of the five exceptional Lie algebras, culminating in $e_8$. Building on Sudbery’s early work [35, 36] using the octonions to describe Lie algebras over the octonions, Barton and Sudbery [37] gave a unified presentation of the magic square based on Vinberg’s symmetric construction [38]. Along the way, they gave explicit matrix interpretations of the Lie algebras in the first three rows. In recent work [39], we provided a new description of $e_8$ in terms of $3 \times 3$ matrices over (two copies of) the octonions. We summarize that description in Appendix A.

Here, we build up to $e_8$ by working our way through the magic square, offering a physical interpretation at each step that leads naturally to a description of the physical world that shares many features with the Standard Model.

Although we work here with the Lie algebra $e_8$, earlier efforts by ourselves [40, 43] and others [4, 13] to lift the Barton and Sudbery construction to the group level, combined with the techniques used in [39], allow our results to be applied directly at the group level.

The “half-split” version of the magic square [37] is shown in Table I parameterized by a split division alge-
TABLE I. The “half-split” Freudenthal–Tits magic square of Lie algebras. The shaded cells show our path through the magic square, as discussed in Section II.

| Algebra      | Maximal Subalgebra | Centralizer |
|--------------|--------------------|-------------|
| $\mathfrak{so}(3)$ | $\mathfrak{so}(2)$ | $\mathfrak{g}_2 \oplus \mathfrak{g}_{2(2)}$ |
| $\mathfrak{sl}(3, \mathbb{R})$ | $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(1, 1)$ | $\mathfrak{g}_2 \oplus \mathfrak{sl}(3, \mathbb{R})$ |
| $\mathfrak{su}(3, \mathbb{H})$ | $\mathfrak{so}(5, 1) \oplus \mathfrak{so}(1, 1) \oplus \mathfrak{su}(2)$ | $\mathfrak{su}(2) \oplus \mathfrak{sl}(3, \mathbb{R})$ |
| $\mathfrak{e}_6$ | $\mathfrak{so}(9, 1) \oplus \mathfrak{so}(1, 1)$ | $\mathfrak{sl}(3, \mathbb{R})$ |
| $\mathfrak{e}_7$ | $\mathfrak{so}(8, 4) \oplus \mathfrak{su}(2)$ | $\mathfrak{su}(2)$ |
| $\mathfrak{e}_8$ | $\mathfrak{so}(12, 4)$ | — |

TABLE II. Selected Lie algebras in the magic square, with their maximal (proper, reductive) subalgebra in $\mathfrak{so}(12, 4)$ and their centralizer in $\mathfrak{e}_8$.

In terms of physical interpretation, our journey starts with a choice of complex structure, that is, an operator that acts as a complex unit, thus allowing us to treat certain subsets of these real algebras as complex representations. We then add a $u(1)$ that will turn out to be closely related to hypercharge, then the Lorentz algebra and – separately – two $\mathfrak{su}(2)$ subalgebras, then finally $\mathfrak{su}(3)$. At the end of the journey, we obtain precisely the Lorentz group, an $\mathfrak{SO}(10)$ GUT, and just enough spinors to define one generation of quarks and leptons (and their adjoints). We provide in Section VI a possible mechanism for generalizing this construction to exactly three generations, and in Section VII a possible identification of mediators (vector bosons). All without leaving $\mathfrak{e}_8$!

We refer to the particles obtained in this unified physical interpretation of (all of) the elements of $\mathfrak{e}_8$ as actions. Let’s begin.

II. CONSTRUCTING $\mathfrak{e}_8$

In this section, we describe our preferred path through the magic square, discussing which new physical features are added at each step. Since our ultimate goal is to present a unified description in terms of $\mathfrak{e}_8$, each step builds on the previous ones. Among other things, this leads us to construct the Cartan subalgebra of $\mathfrak{e}_8$ as we go.

A. $\mathfrak{so}(3)$

We enter the magic square in the top left corner with the real Lie algebra $\mathfrak{so}(3)$, generated by the three antisymmetric matrices $X_1, Y_1, Z_1$ (which are defined in Appendix A). The Lie algebra $\mathfrak{so}(3)$ can be identified with $\mathfrak{su}(3, \mathbb{R}' \otimes \mathbb{R})$, using labels $1 \in \mathbb{R}, U \in \mathbb{R}'$; the subscripts “1” above are shorthand for “1U.” The single element $X_1$ generates $\mathfrak{so}(2)$, which we choose as the Cartan subalgebra. Choosing a Cartan element breaks the symmetry, yielding a preferred $2 \times 2$ block structure; as discussed above, we refer to $X_1$ as an infinitesimal rotation in the $1U$-plane.

Explicitly,

$$x X_1 + y Y_1 + z Z_1 = \begin{pmatrix} 0 & x & -z \\ -x & 0 & y \\ z & -y & 0 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$, so that $X_1$ can be identified with the $2 \times 2$ matrix in the upper left of this matrix, and the remaining elements can be identified with the 2-component column to its right; see Appendix A and especially A4.

The elements, $\{Y_1, Z_1\}$ are a basis for a representation of $\mathfrak{so}(2)$. However, since

$$[X_1, [X_1, y Y_1 + z Z_1]] = -(y Y_1 + z Z_1)$$

form spinor representations.
we have \( X_1 \circ X_1 = -1 \) in the enveloping algebra on this representation (as discussed in Appendix B), and can therefore interpret \( X_1 \) as a complex structure \( i \) by defining
\[
\iota(\psi) = [X_1, \psi]
\]
for any linear combination \( \psi \) of \( Y \)'s and \( Z \)'s, thus converting the \( YZ \)-subspace of \( \mathfrak{so}(3) \) into a (complex) spinor representation of \( \mathfrak{so}(2) \cong \mathfrak{spin}(2) \cong \mathfrak{u}(1) \).

More generally, a complex structure on any representation \( V \) of a Lie algebra \( \mathfrak{h} \) is a linear map \( \iota \) from \( V \) to itself satisfying
\[
\iota^2 = -1, \quad [X, \iota(\psi)] = \iota([X, \psi])
\]
for any \( X \in \mathfrak{h} \) and \( \psi \in V \). In writing the action of \( \mathfrak{h} \) on \( V \) in \[ \text{Appendix A} \] as a commutator, we are implicitly assuming that both \( \mathfrak{h} \) and \( V \) are contained in a single, larger Lie algebra, such as \( \mathfrak{e}_8 \), an assumption that underlies our entire program.

### B. \( \mathfrak{sl}(3, \mathbb{R}) \)

We next move down a row in the magic square by adding the label \( L \in \mathbb{C}' \), thus arriving at \( \mathfrak{sl}(3, \mathbb{R}) \), a real form of \( \mathfrak{a}_2 \) (whose compact real form is \( \mathfrak{su}(3) \)), consisting of the split, antihermitean cousins of the Gell-Mann matrices. Explicitly, \( \mathfrak{sl}(3, \mathbb{R}) \) is generated by \( \{X_1, Y_1, Z_1, X_L, Y_L, Z_L, D_L, S_L\} \). As discussed in Appendix A, the five basis elements labeled by \( L \) are boosts, and the remaining three are rotations. The \( 2 \times 2 \) subalgebra is therefore \( \mathfrak{so}(2,1) \), generated by \( \{X_1, X_L, D_L\} \), and implementing rotations and boosts on the labels \( \{1, U, L\} \). The centralizer of \( \mathfrak{so}(2,1) \) in \( \mathfrak{sl}(3, \mathbb{R}) \) is generated by \( S_L \), so we can expand the Cartan subalgebra by including \( S_L \), which is boost-like.

In this case, the \( Y, Z \)-subspace splits into two representations of \( \mathfrak{so}(2,1) \), namely the eigenspaces of \( S_L \), labeled by \( \pm L \). The eight-dimensional Lie algebra \( \mathfrak{sl}(3, \mathbb{R}) \) therefore decomposes as \( \mathfrak{so}(2,1) \oplus \langle S_L \rangle \oplus (2 \times 2) \). Explicitly, the two spinor \( 2 \)'s are
\[
\mathbf{S}_\pm = \{y_{\pm} Y_{1 \pm L} + z_{\pm} Z_{1 \mp L}\}
\]
with \( y_{\pm}, z_{\pm} \in \mathbb{R} \). (Since \( \mathfrak{so}(p,q) \cong \mathfrak{spin}(p,q) \) as Lie algebras, we use the former notation even when acting, as here, as infinitesimal Spin\((p,q)\) transformations.) Each of \( \mathbf{S}_\pm \) is a copy of the complex plane, since
\[
\iota(Z_{1 \mp L}) = [X_1, Z_{1 \mp L}] = Y_{1 \pm L}.
\]

Furthermore, the Killing form \( B \) (see Appendix C) is degenerate on each vector space \( \mathbf{S}_\pm \); in each case, the basis consists of two orthogonal null elements. The dual basis of \( \mathbf{S}_+ \) is in \( \mathbf{S}_- \), and vice versa, since \( B(Y_{1 \pm L}, Y_{1 \mp L}) = 2 \), and similarly for \( Z \). Explicitly, the Killing dual of any element in \( \mathbf{S}_\pm \) is obtained by reversing the sign of \( L \). This operation \( \varphi \) of “\( L \) conjugation” can be realized on \( \mathbf{S}_\pm \) in the enveloping algebra of \( \mathfrak{so}(3,3) \), as discussed in Appendix B see also Appendix C. We interpret Killing duality as an adjoint operation, so that pairs of dual elements do not represent separate physical degrees of freedom.

### C. \( \mathfrak{sl}(3, \mathbb{H}) \)

Our next stop in the magic square adds the labels \( \{i,j,k\} \), thus arriving at \( \mathfrak{sl}(3, \mathbb{H}) \cong \mathfrak{su}(3, \mathbb{C}' \otimes \mathbb{H}) \), a real form of \( \mathfrak{a}_3 \) (whose compact real form is \( \mathfrak{su}(6) \)), with dimension 35. The \( 2 \times 2 \) structure now consists of \( \mathfrak{so}(5,1) \), acting on the six labels \( \{1, i,j,k, U, L\} \) (the \( D \)'s and \( X \)'s with labels in \( \mathbb{C}' \) and/or \( \mathbb{H} \)), and there are now 16 spinors (the \( Y \)'s and \( Z \)'s with labels in \( \mathbb{C}' \otimes \mathbb{H} \)). The \( 35-15=16 \) remaining elements of this basis centralize \( \mathfrak{so}(5,1) \) in \( \mathfrak{sl}(3, \mathbb{H}) \); these elements include \( S_L \), of course, but also three new elements, forming a copy of \( \mathfrak{su}(2) \) that we henceforth refer to as \( \mathfrak{su}_2(\mathbb{R}) \).

Our previous work \[39\], summarized in Appendix A, shows that most (but not all) elements of \( \mathfrak{e}_8 \) admit an explicit matrix form. Here, since the underlying algebras \( \mathbb{C}' \) and \( \mathbb{H} \) are associative, we can realize every element of \( \mathfrak{sl}(3, \mathbb{H}) \) as a matrix. In particular, we can multiply out the nested double-index \( D \)'s, so that, for example
\[
D_{i,j} \equiv \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
where the dot over the equals sign is to remind us that this expression is not valid in all of \( \mathfrak{e}_8 \).

Similarly, although the elements of \( \mathfrak{su}_2(\mathbb{R}) \) must be represented in \( \mathfrak{e}_8 \) as (linear combinations of) double-index \( D \) with labels in \( \mathbb{H}_\perp = \ell \mathbb{H} \), within \( \mathfrak{sl}(3, \mathbb{H}) \) they take the form
\[
GS_q \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q \\ 0 & q & 0 \end{pmatrix},
\]
with \( q \in \mathbb{R} \). These two descriptions of \( \mathfrak{su}_2(\mathbb{R}) \) can be shown to be equivalent using triality \[39\]. Thus, \( \mathfrak{sl}(3, \mathbb{H}) \) can be represented as the \( 3 \times 3 \) antihermitean matrices over \( \mathbb{C}' \otimes \mathbb{H} \), where a nonzero (imaginary) quaternionic trace is allowed.

The 16 spinor degrees of freedom form two real, 8-dimensional representations, namely the eigenspaces of \( S_\ell \), labeled by \( \pm L \), where we have reused the names introduced in Section II B. Since \( L^2 = 1 \),
\[
L(\pm L) = \pm(\pm L),
\]
that is, \( \pm L \) are eigenvectors in \( \mathbb{C}' \) of multiplication by \( L \). Since \( S_L \) multiplies spinors by (some multiple of) \( L \), these eigenspaces have the matrix form
\[
\psi_\pm = \begin{pmatrix} -\overline{q} \\ q \end{pmatrix} (\pm L),
\]
TABLE III. Half of the spinors in \( su(3, \mathbb{H}) \), showing one of the two complex 4s of \( so(3, 1) \oplus so(4) \), along with their eigenvalues under \( i \circ GS_k \) and \( i \circ Di_{i,j} \) and their traditional name, with their eigenvalue under hypercharge shown in parentheses.

| Type/Spin | Re(\( \psi \)) | Im(\( \psi \)) | Name |
|-----------|----------------|----------------|------|
| ++        | \( Y_{1,1L} + Z_{1,1L} \) | \( -j_{1,1L} \) | \( \nu_{3R} \) |
| ++        | \( -Y_{1,1L} + Z_{1,1L} \) | \( j_{1,1L} \) | \( \nu_{3L} \) |
| --        | \( -Y_{1,1L} + Z_{1,1L} \) | \( j_{1,1L} \) | \( \epsilon_{6R} \) |
| --        | \( Y_{1,1L} + Z_{1,1L} \) | \( -j_{1,1L} \) | \( \epsilon_{6L} \) |

We can decompose the Lie algebra \( so(5, 1) \) as \( so(2) \oplus so(3, 1) \oplus 2 \times 4 \), interpreting \( so(3, 1) \) as the Lorentz group, acting on \( \{i, j, k, L\} \), and where the last term refers to two vector representations of \( so(3, 1) \), labeled by \( \{1, U\} \), each of which is 4-dimensional. Since \( S_L \) commutes with both \( so(3, 1) \) and \( su2_R \), each eigenspace \( S_\pm \) is also a representation of \( so(3, 1) \oplus su2_R \). Since \( X_1 \) commutes with both \( so(3, 1) \) and \( su2_R \), each eigenspace is in fact a complex representation of each of these algebras. Although the representations are reducible over any one of these algebras, each of \( S_\pm \) is an irreducible representation of \( so(3, 1) \oplus su2_R \) over the sum. (For convenience, we refer to these representations as “pots”; see Appendix [A]).

The Cartan basis can now be expanded to, say, \( \{X_1, S_L, Di_{i,j}, X_{kL}, GS_k\} \), where it is important to notice that the choice of Cartan elements in \( so(3, 1) \) is independent of the choice of Cartan element in \( su2_R \). If we now interpret \( su2_R \) as the right-handed weak symmetry algebra, then each of these irreducible pots of spinors is an \( 8 \) containing a (right-handed) weak doublet of Lorentz (Weyl) spinors. We can therefore interpret the eigenvalues of \( i \circ Di_{i,j} \) as spin around the \( z \) axis and of \( i \circ GS_k \) as (weak) particle type, and thus identify spinor eigenstates of these Cartan elements with the Weyl spinors of physical particles, as shown in Table [III] of Appendix [B]. The (independent) identifications of particular elements of a pot having spin up or down, or as being an “electron” or “neutrino”, depend on the Cartan elements chosen.

D. Interlude: Hypercharge

We now have enough ingredients to introduce hypercharge. There are four Weyl spinors of \( so(3, 1) \) in this theory, labeled by their eigenvalues under \( S_L \) (\( \pm 3 \)) and \( i \circ GS_k \) (\( \pm 2 \)). We can combine \( S_L \) and \( GS_k \) into the two-parameter family of operators

\[
\eta_{gs} = s S_L + g i \circ GS_k \tag{12}
\]

with \( s, g \in \mathbb{R} \), where the presence of the complex structure \( i \) is required since \( S_L \) has real eigenvalues whereas \( GS_k \) has imaginary eigenvalues. Each operator \( \eta_{gs} \) admits the four real eigenvalues \( \{-3s - 2g, -3s + 2g, 3s - 2g, 3s + 2g\} \), which are distinct so long as \( 3s \neq \pm 2g \). Assuming without loss of generality that \( g, s > 0 \), there is a unique (up to scale) operator \( \eta \) that does admit the degenerate eigenvalue \( 0 \), namely

\[
\eta = h(2 S_L + 3 i \circ GS_k). \tag{13}
\]

If \( \eta \) does not distinguish all four Weyl spinors, we need a second operator to distinguish them. The natural choice is the unique (again, up to scale) linear combination of the two Cartan elements \( S_L \) and \( GS_k \) that is orthogonal to \( \eta \) using the Killing form, namely

\[
\eta_{pert} = i \circ GS_k - S_L. \tag{14}
\]

We therefore use the eigenvalues of \( \eta_{pert} \) to distinguish the four Weyl spinors. The operator \( \eta_{pert} \) is orthogonal not only to \( \eta \), but also to the Lorentz group, \( so(3, 1) \). As noted in Section [III] \( \eta_{pert} \) generates the centralizer of \( su(5) \) in \( so(10) \).

Physically, we have identified a preferred Cartan element \( \eta \) that admits the degenerate eigenvalue \( 0 \). We identify \( \eta \) as hypercharge, generating the (complex) \( u_1 \) of the Standard Model; setting \( h = -\frac{1}{12} \) will reproduce the correct values on the eigenstates shown in the last column in Table [III].

E. \( c_6(-26) \)

We now continue horizontally in the magic square from \( sl(3, \mathbb{H}) \) to \( c_6(-26) \), henceforth referred to simply as \( c_6 \). We add the four remaining labels in \( \mathbb{O} \), thus expanding \( so(5, 1) \) to \( so(9, 1) \), as discussed in more detail in [H]. Thanks to triality, \( su2_R \) is absorbed into the new \( so(4) \); its centralizer in \( so(4) \) is a second copy of \( su2 \), which we call \( su2_L \). The algebra \( su2_L \) is in fact precisely the subalgebra of \( g_2 \), the automorphism algebra of \( \mathbb{O} \), that fixes \( \mathbb{H} \). We have also expanded the Cartan basis by the addition of any single element from \( su2_R \); although the choice is arbitrary, we choose \( A_k \), defined in Appendix [A] in order to simplify later calculations.

As well as adding a second \( su2 \) subalgebra, going from \( sl(3, \mathbb{H}) \) to \( c_6 \) also doubles the number of spinors, adding spinors labeled by \( \mathbb{H}_L = \ell \mathbb{H} \) to those labeled by \( \mathbb{H} \). Not only does \( su2_L \) commute with the \( \mathbb{H}-\)labeled spinors (since it centralizes \( sl(3, \mathbb{H}) \) in \( c_6 \)), it also turns out that \( su2_R \) commutes with the \( \mathbb{H}_L \)-labeled spinors. Furthermore, the 32 spinor degrees of freedom in \( c_6 \) form two Majorana–Weyl representations of \( so(9, 1) \), again called \( S_\pm \), distinguished as usual as eigenspaces of \( S_L \) and \( GS_k \), labeled by \( \pm 1 \). Elements of \( S_\pm \) are again Killing duals of each other. Each such eigenspace splits naturally into two “pots”, each of which is an \( 8 \) that is a simultaneous complex representation of \( so(4) =

su2R ⊕ su2L and so(3, 1). One pot is left-handed, consisting of doublets of su2L and singlets of su2R; the other is right-handed, consisting of doublets of su2R and singlets of su2L. As before, the identification of (weak) particle type and spin depend on the choice of Cartan elements. However, the pots themselves are independent of these choices, as discussed further in Appendix D We can again identify eigenstates of the Cartan elements with physical particles, as shown in Table IV (compare Tables III and VII).

We now have a theory containing a complex structure, so(3, 1), su2L, and su2R, as well as an additional so(1, 1) generated by S_L, along with two sets of (complex!) spinors, each consisting of both a left- and a right-handed weak doublet of Lorentz (Weyl) spinors. As we will see in Sections III and IV suitable combinations of the spinors in these two sets represent leptons and their antiparticles, respectively. Having introduced hypercharge in Section III D and su2L here, it is now straightforward to construct the charge operator

$$ q = -\frac{1}{6} S_L - \frac{1}{2} D_{\ell \ell} $$

(15)

by combining hypercharge with the suitably-scaled Cartan element of su2L. The resulting charge eigenvalues are shown in the last column in Table IV.

F. c8(−24)

Our final stop is c8 itself. We add the labels in C_3, thus expanding so(9, 1) to so(12, 4). Thanks to triality, S_L is absorbed in the new so(3, 3), where it is centralized by sl(3, R), which, as already noted in Section II B is a real form of complexified a8 = sl(3). We have added a color symmetry! As discussed in Appendix D the single Majorana–Weyl spinor of so(12, 4) in c8 splits into two 64s, denoted S_±. Each of S_± is an irreducible representation of so(9, 1) ⊕ so(3, 3), generalizing the language used in the previous sections.

We have thus added three new sets of “colored” spinors – quarks – labeled by two new Cartan elements in sl(3, R), which we choose to be GL and AL (see Appendix A). Since so(3, 3) commutes with the complex structure X_1, its sl(3, R) subalgebra acts as complex su(3) on these spinors, justifying the name su3, for this copy of sl(3, R). The algebra su3 is in fact precisely the subalgebra of g_2, the automorphism algebra of O', that fixes L, and hence C'. (To the best of our knowledge, the first people to use this split form of su(3) to describe the color symmetry of quarks were Gunaydin and Gursey [14].)

Apart from the action of su3, colored spinors can be divided into electroweak pots (representations of so(3, 1) ⊕ so(4)) exactly as we did for the original, colorless pots. However, the eigenvalues of S_L are different on colored spinors, ±1 rather than ±3, thus ensuring that quarks have the correct fractional values of hypercharge and charge.

When moving to c8, we have not merely added an so(3, 3), but expanded so(7, 1) ⊂ so(9, 1) to so(10, 4) ⊂ so(12, 4). Since X_1 commutes with all of so(10, 4), the spinors of c8 are simultaneous complex representations of Lorentz so(3, 1) and the remaining so(7, 3). Thus, our theory contains all of the spinor pieces of the Georgi–Glashow SO(10) GUT [45], along with its subtheories based on SU(5) and SU(4) × SU(2) × SU(2), due to Georgi [46] and Pati–Salam [47], respectively.

Explicitly, this so(7, 3) contains so(4), acting on labels in H_L, as well as so(3, 3), acting on labels in C_3. The weak algebra so(4) of course splits as su2R ⊕ su2L, where it is noteworthy that su2L ⊂ g_2 (which su2R is not), but that su2R ⊂ sl(3, R) (which su2L is not). Similarly, so(3, 3) contains both color sl(3, R) and S_L, but sl(3, R) ⊂ g_2 (which {S_L} is not), whereas {S_L} ⊂ sl(3, R) (which this copy of sl(3, R) is not).

In this signature, so(7, 3) itself does not contain any real form of su(5). Rather, on any spinor representation of so(7, 3) we work with the complexification of so(7, 3) in the enveloping algebra, using ι, yielding (complex) so(10). It is now straightforward to combine the right-handed weak Cartan element (which commutes with su2L) with S_L (which commutes with su3c) into the “pure trace” element η_{perp} introduced in Section III D A (complex) su(5) ⊂ so(10) can now be defined as the subset of so(10) that commutes with η_{perp}. This su(5) will include not only su2L and su3c, but also the orthogonal linear combination η introduced in Section III D which we recognize (when suitably normalized) as the hypercharge operator of the su(5) GUT; ι ∞ η generates the u_1 of the Standard Model.

III. SPINORS AND THE DIRAC EQUATION

Having constructed c8 in Section III by traveling through the magic square, we have seen how c8 consists of so(12, 4) together with a single Majorana–Weyl spinor representation. Along the way, we have constructed so(12, 4) from its Lorentz so(3, 1), weak so(4) = su2R ⊕ su2L), and color {sl(3, R) ⊂ so(3, 3)} sectors, together with a complex structure given by the remaining
$\mathfrak{so}(2)$. This construction has allowed us to interpret the 128 spinor degrees of freedom as complex Weyl spinors of $\mathfrak{so}(3,1)$ and their Killing duals. How do they combine to form the Dirac spinors used to represent particles and antiparticles?

We begin with the lepton sector, with labels in $\mathbb{C}^* \otimes \Omega$, which lives in $\epsilon_6 \subset \epsilon_8$. The $2 \times 2$ subalgebra of $\epsilon_6$ is $\mathfrak{so}(9,1)$, which decomposes into the complex structure, $\mathfrak{so}(7,1)$ and two vector $\mathfrak{s}s$. Either of these vector representations can be used to generate $\mathcal{C}(7,1)$ in the enveloping algebra of $\mathfrak{so}(9,1)$, acting on spinors; $\mathcal{C}(7,1)$, of course, contains the Lorentz Clifford algebra, $\mathcal{C}(3,1)$. The chosen representation can then be interpreted as the degree-1 elements (gamma matrices) of $\mathcal{C}(7,1)$. Since the commutator of the two representations is proportional to the complex structure, which is also the volume element $\Omega_{7,1}$ (see Appendix D), the other representation corresponds to pseudovectors, that is, degree-7 elements of $\mathcal{C}(7,1)$. We make the arbitrary choice to generate $\mathcal{C}(7,1)$ with the vector representation that contains the energy basis element $D_L$.

As discussed in Appendix D, the lepton spinors divide naturally into two irreducible representations $\mathbf{S}_L$ of $\mathfrak{so}(9,1)$. Each of these subspaces is totally null under the Killing form $B$, and there is a natural pairing between them given by $L$ conjugation ($\varphi$), defined in the enveloping algebra in Appendix B.

As we proposed in [13], we interpret the additional spacelike dimensions as possible masses, thus converting the massive Dirac equation in four spacetime dimensions to the massless Weyl equation in higher dimensions. So we seek spinor solutions $\psi$ of the equation

$$[Q, \psi] = 0$$

where $Q$ lies in our chosen vector representation of $\mathfrak{so}(7,1)$, and therefore has the form

$$Q = ED_L + p_a X_a + m_a X_b$$

with $a = i, j, k \in \text{Im} \mathbb{H}$ and $b = k \ell, j \ell, i \ell, \ell \in \mathbb{H}_L$. We further write $Q = P + M$, where $P$ is the Lorentz vector, consisting of the first four terms of $Q$, and $M$ is the mass vector, consisting of the remaining four terms.

Recall that each of $\mathbf{S}_L$ divides naturally into a right-handed pot with labels in $\mathbb{H}$ (and hence inside $\mathfrak{a}_5$) and a left-handed pot with labels involving $\ell$. Since $P \in \mathfrak{a}_5$ and $M \notin \mathfrak{a}_5$, it is clear that $P$ preserves pots whereas $M$ maps between pots. Writing $\psi = \psi_L + \psi_R$, [13] becomes

$$[P, \psi_L] + [M, \psi_R] = 0, \tag{18}$$

$$[P, \psi_R] + [M, \psi_L] = 0. \tag{19}$$

Since $Q$ is a degree-1 element of $\mathcal{C}(7,1)$, we can use the Clifford identity [B3] to solve [18] for $\psi_L$ in terms of $\psi_R$, obtaining

$$|P|^2 \psi_L = +[P, [M, \psi_R]] \tag{20}$$

which we can then substitute into [19], resulting in

$$|P|^2 [P, \psi_R] + [M, [P, [M, \psi_R]]] = 0. \tag{21}$$

Since $P$ and $M$ anticommute, and once again using the Clifford identity, we have

$$[M, [P, [M, \psi_R]]] = -[M, [M, [P, \psi_R]]] = |M|^2 [P, \psi_R] \tag{22}$$

and thus finally obtain

$$|Q|^2 = |P|^2 + |M|^2 = 0 \tag{23}$$

as expected. That is, solutions to the eight-dimensional Weyl equation [16] only exist if $Q$ is null. Furthermore, for given $Q$, any right-handed spinor $\psi_R$ determines a unique left-handed partner $\psi_L$, given by [B8], such that the resulting Dirac spinor $\psi$ satisfies [16]. We henceforth refer to [16] as the generalized Dirac equation, and its solutions as Dirac spinors.

This construction can be reversed; we can recover the 8-momentum $Q$ (up to scale) from Dirac spinors as the vector bilinear

$$Q_\psi = [\varphi(\psi), D_L], \psi]. \tag{24}$$

The generalized Dirac equation [16] is thus a special case of the “3-$\psi$ rule” [10, 32, 48, 49], an identity on (Dirac) spinors. Thanks to the Jacobi identity, both [16] and [24] are covariant under $\mathfrak{so}(12,4)$, and in fact under all of $\epsilon_8$.

We have already labeled (four-dimensional) Weyl spinors by their eigenvalues with respect to our chosen Cartan basis; see Tables III, IV and VIII. These eigenvalues encode physical properties such as spin and particle type. We seek Dirac spinors that combine two Weyl spinors with the same physical properties. Our ability to achieve this goal is constrained by [20]. If $\psi_R$ is an eigenvector of a particular Cartan element, does $\psi_L$ have the same eigenvalue?

We must first deal with the potential complication that the complex structure anticommutes with $Q$, that is, with both $P$ and $M$. Nonetheless, the complex structure commutes with the combination $P \circ M$ that appears in [20], thus ensuring that solutions of the Dirac equation are complex, that is, if $\psi$ satisfies [16] for given $Q$, so does $\varphi(\psi)$.

Turning next to our spin Cartan element, which we have chosen to be $L_z = i \circ D_{i,j}$, we discover that $L_z$ only commutes with $P \circ M$ if the momenta $p_a$ vanish for $a = i, j$. This is the expected result: the eigenvalue of $L_z$ only corresponds to spin (or helicity) if the $x$- and $y$-momenta vanish.

We finally consider particle type. Since each chiral $\mathfrak{su}(2)$ acts only on one of the Weyl spinors $\psi_L$ and $\psi_R$, we must add the two Cartan elements to get an operator whose eigenvalues will be the same for both of these spinors, a property that we have already used in [15]. As discussed more fully in Section VI, we have chosen these Cartan elements so that the diagonal subalgebra of $\mathfrak{so}(4)$ obtained by adding corresponding elements of $\mathfrak{su}_2$ and $\mathfrak{su}_2$ is the $\mathfrak{so}_3$ that fixes $\ell$ in $\mathfrak{so}(4)$. With this choice, the combined weak Cartan element which forms part of
the charge operator, is \( R_z = i \circ D_{ij,kl} \). Similarly to the
analysis above for spin, in order for a Dirac spinor, satisfying \( \Phi \), to have a well-defined weak eigenvalue, that
is, be an eigenstate of \( R_z \), two of the mass components
must vanish, namely \( m_b = 0 \) for \( b = i \ell, j \ell \).

Extending the above description of leptons in \( \ell \) to
quarks in \( q \) is straightforward. As pointed out in Appendix
\[ D \] the spinor eigenspaces \( S_{\pm} \) are now irreducible
representations of \( \mathfrak{so}(9,1) \oplus \mathfrak{so}(3,3) \), and we obtain
colored versions of the spinor eigenstates in Tables \[ III \]
and \[ IV \] with null labels \( I \pm JL, J \pm JL, K \pm KL \)
instead of \( 1 \pm L \). (The correct signs can be determined by
commuting the lepton spinors with elements of \( \mathfrak{so}(3,3) \)
such as \( D_{ij,kl} \).) As discussed in Appendix \[ VIII \] three addi-
tional digits can be added to the binary code to account
for color.

IV. PARTICLES AND ANTIPARTICLES

For massive particles, the eigenvalues of the Cartan
element \( L_z \) correspond to spin at rest, that is, with
\( p_k = 0 \) (in addition to the assumptions above). Using
our principle that \( \ell \) should be “special,” we will also
make the simplifying assumption that \( m_{i\ell} = 0 \). Tables
\[ III \], \[ IV \] and \[ VIII \] were constructed with these
conventions in mind; when combined into Dirac spinors by
addition across tables, all of the resulting particles are at
rest, and the only nonzero mass is \( m_{\ell} \). A discussion of
the more general case appears in Section \[ IX \].

At rest, our general 8-momentum \( \Gamma \) has been re-
duced to

\[
Q = ED_L + mX_\ell
\]

and we arbitrarily assume that \( E, m > 0 \) corresponds to
particles. The Dirac equation for a particle with mass \( m \)
and energy \( E \) is therefore

\[
[ED_L + mX_\ell, \psi] = 0
\]

with \( E^2 = m^2 \) following from \( \[ 23 \] \). What equation does
the Killing dual \( \varphi(\psi) \) satisfy? The operation \( \varphi \) commutes
with the spatial momenta and masses \( X_q \) for \( q \in \text{Im}\mathbb{C} \),
but \textit{anticommutes} with the energy \( D_L \), so we have

\[
[-ED_L + mX_\ell, \psi] = 0
\]

that is, \( \varphi(\psi) \) satisfies a Dirac equation with negative
energy. What about its charge? The hypercharge opera-
tor defined in Section \[ IID \] contains \( S_L \), which \textit{anticom-
mutes} with \( \varphi \), so the possible charges for \( \varphi(\psi) \) are op-
posite those of \( \psi \). However, both the spin Cartan element
\( L_z \) and the combined weak Cartan element \( R_z \) \textit{commute}
with \( \varphi \), so \( \varphi(\psi) \) doesn’t quite have the opposite charge
from \( \psi \).

These issues with the mass and charge can both be
resolved at once by defining

\[
a(\psi) = [\varphi(\psi), D_L]
\]

since \( D_L \) \textit{anticommutes} with both \( \varphi \) and \( X_\ell \). Thus, \( a(\psi) \) satisfies the same (uncoupled) Dirac equation as \( \psi \),
namely \( \[ 26 \] \), but has the opposite charge. We therefore
identify \( a(\psi) \) as the \textit{antiparticle} of \( \psi \). Thus, particles
live in \( S_+ \), and antiparticles live in \( S_- \). However, nega-
tive energy solutions of \( \[ 15 \] \) also exist in \( S_\pm \), namely the
Killing duals of antiparticles and particles, respectively.
We reiterate that, as pointed out in Section \[ IID \] we con-
sider Killing duals to represent adjoints and not separate
physical degrees of freedom.

We can perform a Lorentz transformation on solutions of
the Dirac equation at rest to obtain particle solutions of
the Dirac equation with arbitrary momentum.

We can now finally interpret the two spinor spaces \( S_{\pm} \)
first introduced in Section \[ IID \] and generalized in Appendix
\[ D \] as discussed in Section \[ IID \] elements of these
spaces have opposite eigenvalues under \( S_L \), and hence op-
posite charges. By the reasoning just given above, each
of these spaces contains both particles and antiparticles.
We have shown that \( S_+ \) contains particles and the Killing
duals of antiparticles, while the antiparticles themselves
live in \( S_- \), along with the Killing duals of particles, as
illustrated in Table \[ IV \]. As this table makes clear, the tradi-
tional names used in Tables \[ III \] and \[ IV \] for Weyl spinors
are ambiguous until they are combined into Dirac spinors
with the correct relative sign.

V. DECOMPOSITIONS AND MEDIATORS

We have chosen to construct \( \ell \) from smaller Lie al-
gebras, as we feel that the construction process most
clearly demonstrates the fundamental nature played by
each constituent. Our preferred path, outlined in Sec-
tion \[ IID \] introduces the complex structure first, then the
timelike coordinate \( L \), then both the Lorentz algebra \( \mathfrak{so}(3,1) \)
and the right-handed weak algebra \( \mathfrak{su}(2)_R \), and only then the
left-handed weak algebra \( \mathfrak{su}(2)_L \) followed by color \( \mathfrak{su}(3_c) \). This process can of course be reversed, by
starting with \( \ell \) and decomposing it into smaller Lie al-
gebras together with their representations.

The first step in this decomposition is to choose a pre-

| Type/Spin | Dirac Spinor | Name |
|-----------|--------------|------|
| ++        | \( \nu_L + \nu_R \in S_+ \) | \( \nu_1 \) (0) |
| ++        | \( \nu_L - \nu_R \in S_+ \) | \( \varphi(a(\nu_1)) \) (0) |
| ++        | \( \varphi(\nu_L) - \varphi(\nu_R) \in S_- \) | \( a(\nu_1) \) (1) |
| ++        | \( \varphi(\nu_L) + \varphi(\nu_R) \in S_\pm \) | \( \varphi(\nu_1) \) (1) |
| --        | \( e_L + e_R \in S_+ \) | \( e_\ell (-1) \) |
| --        | \( e_L - e_R \in S_+ \) | \( \varphi(a(e_\ell)) \) (−1) |
| --        | \( \varphi(e_L) - \varphi(e_R) \in S_- \) | \( a(e_\ell) \) (0) |
| --        | \( \varphi(e_L) + \varphi(e_R) \in S_\pm \) | \( \varphi(e_\ell) \) (0) |

TABLE V. Combinations of spinors from Tables \[ III \] and \[ IV \]
that correspond to particles (\( \psi \)), antiparticles (\( a(\psi) \)), and
their Killing duals (using \( \varphi \)), with charge eigenvalues (us-
ing \[ 15 \] ) in parentheses. Only spin-up eigenstates are shown;
the spin-down case is similar.
ferred step of $\mathfrak{so}(12, 4)$ in $\mathfrak{e}_8$; as outlined in Appendix A this step amounts to choosing a preferred $2 \times 2$ block structure in the $3 \times 3$ matrix representation. The remaining $248 - 120 = 128$ elements form a single Majorana–Weyl representation of $\mathfrak{so}(12, 4)$, thus dividing $\mathfrak{e}_8$ into "adjoint" and "spinor" sectors.

This remarkable division of an exceptional Lie algebra into adjoint and spinor representations of a smaller Lie algebra is in fact a unifying feature of all of the Lie algebras in the magic square. In each case, the Cartan subalgebra of the maximal subalgebra (as given in Table [1]) is also the Cartan subalgebra of the full $(3 \times 3)$ Lie algebra. It is well known that any (complex) simple Lie algebra can be expressed in terms of its chosen Cartan subalgebra and its eigenstates, which serve as raising and lowering operators for any Cartan basis. Since the maximal subalgebra and the full algebra have the same Cartan subalgebra, we obtain three types of elements: the Cartan subalgebra, its simultaneous eigenstates in the maximal subalgebra, and the remaining simultaneous eigenstates in the $3 \times 3$ block. The latter elements are spinor eigenstates of the chosen Cartan subalgebra, on which the eigenstates in the maximal subalgebra act as raising and lowering operators.

There is one further subtlety: constructing eigenstates requires a complex structure, and our Lie algebras are real. However, we can interpret one of the "rotation-like" Cartan elements as a complex structure, thus complexifying the spinor representation at the cost of a slight reduction in the size of the adjoint piece. The remaining elements of the original adjoint can be identified as the degree-1 and degree-$(n - 1)$ elements of $C\ell(n)$, as discussed in Appendix [D].

We therefore decompose $\mathfrak{so}(12, 4)$ as $\mathfrak{so}(10, 4) \oplus \mathfrak{so}(2) \oplus (2 \times 14)$. The $\mathfrak{so}(2)$ acts as a complex structure for the action of $\mathfrak{so}(10, 4)$ on the $128$ spinor representation, realizing it as a single Weyl spinor representation of $\mathfrak{so}(10, 4)$, that is, $64\mathfrak{c}_8$. We have therefore decomposed the real Lie algebra $\mathfrak{e}_8(-24)$ into complex $\mathfrak{so}(14)$, a single Weyl spinor of $\mathfrak{so}(14)$, and a complex 14-vector that can be used to represent the degree-1 and degree-13 elements of $C\ell(10, 4)$.

The stage is now set to decompose complex $\mathfrak{so}(14)$ (in the enveloping algebra) into the desired physical symmetry algebras, and the spinors into representations of those algebras. As outlined in Section II only in reverse, $\mathfrak{so}(10, 4)$ contains Lorentz $\mathfrak{so}(3, 1)$ together with $\mathfrak{so}(7, 1)$; the latter, when complexified, yields $\mathfrak{so}(10)$, leading directly to the Georgi–Glashow $\mathfrak{so}(10)$ GUT [13], along with its subalgebras $\mathfrak{su}(5)$ and $\mathfrak{su}(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and hence their corresponding GUTs, due to Georgi [16] and Pati–Salam [17], respectively.

Within $\mathfrak{e}_8$, our spinor pots are irreducible representations of $\mathfrak{so}(3, 1) \oplus \mathfrak{so}(4) \subset \mathfrak{so}(7, 1)$. What role is played by the remaining 16 elements of $\mathfrak{so}(7, 1)$? These elements comprise four Lorentz vectors, with weak labels $i\ell$, $j\ell$, $k\ell$, $l\ell$, suggesting their interpretation as the electroweak gauge potentials.

Since the complex structure $i$ commutes with $\mathfrak{so}(7, 1)$, we can find an eigenbasis of this $4 \times 4$, that is, a basis of simultaneous raising and lowering operators for the four Cartan elements of $\mathfrak{so}(3, 1) \oplus \mathfrak{so}(4)$ (when acting on spinors). Half of these basis elements commute with spin $(L_z)$; the other half raise or lower the spin eigenvalue. Similarly half commute with the (combined) weak Cartan element $R_z$; the other half change the weak eigenvalue and hence the particle type, as expected. We are puzzled by the fact that, since each element of the $4 \times 4$ necessarily anticommutes with the volume elements $\omega$ and $\Omega$ of $\mathfrak{so}(3, 1)$ and $\mathfrak{so}(4)$, that is, with $\gamma_5$, these weak mediators mix right- and left-handed spinors.

Similarly, when adding color, we extended $\mathfrak{so}(9, 1)$ to $\mathfrak{so}(9, 1) \oplus \mathfrak{so}(3, 3)$, thus also adding, among other things, six (not 8!) new vectors of $\mathfrak{so}(3, 1)$. Applying an analogous analysis leads to six possible gauge potentials, with particular behavior regarding spin and color eigenvalues when acting on spinors.

If, as suggested by the structure of $\mathfrak{e}_8$, these $4 + 6 = 10$ Lorentz vectors are indeed mediators, we are led to weak mediators that form a vector $3$ of the combined weak algebra $\mathfrak{so}(3)$, and to color mediators that form a $3 \oplus 3$ of $\mathfrak{su}_c$, rather than the adjoint representations predicted by the Standard Model.

VI. GENERATIONS

The decomposition of $\mathfrak{so}(4)$ into two copies of $\mathfrak{su}(2)$ is well known, as is the fact that adding appropriate commuting elements of these subalgebras yields the rotation subalgebra $\mathfrak{so}(3)$. However, an underappreciated property of this decomposition is that the two $\mathfrak{su}(2)$ subalgebras are in fact uniquely determined, which follows from the uniqueness of the decomposition of semisimple Lie groups into commuting simple Lie groups. In particular, the $\mathfrak{su}(2)$ subalgebras do not depend on a choice of $\mathfrak{so}(3)$ subalgebra. Which $\mathfrak{so}(3)$ subalgebra one gets depends on the chosen correspondence between the two $\mathfrak{su}(2)$ subalgebras.

Without loss of generality, we chose $\ell \in \mathbb{H}_\perp$ to be the axis fixed by $\mathfrak{so}(3)$. However, even with the pairing fixed between elements of $\mathfrak{su}_L$ and $\mathfrak{su}_R$, we still have the choice of which pair to choose as the basis of the Cartan subalgebra of $\mathfrak{so}(4)$. We chose $\{A_k, G_{5k}\}$ above, but we could equally well have replaced $k$ with $i$ or $j$. Equivalently, we have the choice of Cartan element in $\mathfrak{so}(3)$; the choice made above is $D_{ii,jj}$, where we could cycle $i\ell$, $j\ell$, $k\ell$.

What effect would such a change in the Cartan element have on the particle eigenstates? In quantum mechanics, it is well understood how to combine spin eigenstates along one axis in order to get spin eigenstates along another, corresponding to a different choice of angular momentum Cartan element. Here, we have the analogous ability to combine weak eigenstates of one “type” in order to get weak eigenstates of another “type,” corresponding
to a different choice of weak Cartan element. We propose interpreting these different types as generations.

In this interpretation, the generations sit on top of each other, in the sense that they belong to a single pot. However, each generation corresponds to a particular choice of Cartan elements in $\mathfrak{so}(4)$, leading to a particular division of the pots into Weyl spinors of $\mathfrak{so}(3,1)$, representing the weak eigenstates of that generation.

Is this generation structure compatible with observation? Answering that question would require a mechanism to break the continuous symmetry, presumably also accounting for the different masses associated with each generation. Since the charge operator $[\mathfrak{g}_3]$ incorporates the $\mathfrak{so}(3)$ Cartan element, it would be generation dependent, although the resulting charge eigenvalues would be the same. Just as there are three spin axes, this model has three natural “generation” axes. Furthermore, the overlapping nature of the generations in this model suggests a natural framework for the observed mixing between generations.

VII. CHIRALITY

Although there are several notions of “chiral” theories in the literature, the real question is whether a theory describes the chiral asymmetry seen in nature. Since the theory presented here does not (yet) describe interactions, that question can not (yet) be answered directly.

Distler and Garibaldi define a chiral $E_8$ theory to be, essentially, one in which the spinors do not have a self-conjugate structure, then argue that such a theory exists. However, they assume both that the GUT group is compact, and that $E_8$ has been complexified, neither of which holds for our model.

Interestingly, there are several senses in which our model is fundamentally “chiral.” First of all, $E_8(-24)$ contains a single Weyl spinor of $\mathfrak{so}(10,4)$. As a consequence, the seven “bits” of information in the spinor binary code are not independent, as discussed in Appendix VII. Thus, “Lorentz handedness” (eigenvalue of the $\mathfrak{so}(3,1)$ volume element $\omega$) and “weak handedness” (eigenvalue of the $\mathfrak{so}(4)$ volume element $\Omega$) are correlated. If we were to complexify $E_8$, as required by $E_8(-24)$, then this complexified $E_8$ would contain spinors of both $\mathfrak{so}(10,4)$ handednesses. We reiterate, however, that we work throughout with a real form of $E_8$.

Furthermore, our route through the magic square has a stop at $\mathfrak{g}_3$, which contains $\mathfrak{su}_2^R$ but not $\mathfrak{su}_2^L$. Even though these algebras appear symmetrically as $\mathfrak{so}(4) \subset \mathfrak{so}(12,4)$, that symmetry is misleading, as $\mathfrak{su}_2^L$, but not $\mathfrak{su}_2^R$, is in $\mathfrak{g}_2$, as was pointed out in Section IV. Is this asymmetry enough to result in a chiral theory?

VIII. CONCLUSION

We cannot overemphasize the extent to which our construction is driven by the underlying mathematical structure. We make two key assumptions:

- We work with a real Lie algebra, $E_8(-24)$, without complexifying it.

- All objects in the theory are constructed from the same copy of $E_8(-24)$.

We find not only the Standard Model symmetry algebras, including Lorentz symmetry, as well as the appropriate notion of complexification, but also the spinors on which they act, their momenta, and possibly the mediators and a version of the Higgs particle, all within a particular real form of $E_8$. We refer to the particles among these objects collectively as actions. Furthermore, as emphasized by our chosen path through the magic square, by taking the division algebra structure seriously we see that the $\mathfrak{so}(4)$ and $\mathfrak{so}(3,3)$ subalgebras of $\mathfrak{so}(12,4)$ break naturally into $\mathfrak{su}_2^L \oplus \mathfrak{su}_2^R$, and $\mathfrak{so}(1,1) \oplus \mathfrak{so}(3,\mathbb{R})$, respectively. Recall from Table III that the centralizer of $\mathfrak{su}(3,\mathbb{H})$ is precisely $\mathfrak{su}_2 \oplus \mathfrak{so}(3,\mathbb{R})$, corresponding to the weak and strong interactions. Thus, the unitary groups in the standard model are a natural feature of our construction, even though triality later reveals them to be subalgebras of the orthogonal algebra $\mathfrak{so}(12,4)$.

We reiterate that the action of $\mathfrak{so}(7,3) \oplus \mathfrak{so}(3,1) \subset \mathfrak{so}(12,4)$ on spinors is complex, using the complex structure in the enveloping algebra. In particular, $\mathfrak{so}(7,3)$ acts as $\mathfrak{so}(10)$, and $\mathfrak{su}(3,\mathbb{R}) \subset \mathfrak{so}(3,3)$ acts as color $\mathfrak{su}(3)$ on spinors as usual.

It is remarkable that $E_8$, the largest exceptional Lie algebra, turns out to be precisely the right size to capture the essential content of the Standard Model. There is no wasted space: Using two of the 16 degrees of freedom for the complex structure and 10 for the internal symmetries of the Standard Model requires the external world to be 4-dimensional, and described by special relativity. Furthermore, as we have shown, the 128 spinor degrees of freedom correspond precisely to one generation of leptons and quarks (and their Killing adjoints). We have also described how this description naturally generalizes to three overlapping generations.

For several decades, a number of authors have been exploring the relationship between division algebras and the Standard Model and, more recently, GUTs based on $\mathfrak{so}(10)$ in ways that parallel our identification here of $\mathfrak{so}(7,3)$ inside of $E_8$. We believe that these authors have been seeing the same physical structures that we have described, but without the guidance of our two key assumptions.

IX. FUTURE WORK

We speculate here on several unanswered questions that we hope to address in the future.
1. It would be immensely helpful to have an action for this theory. The spinor interaction terms in the Standard Model action involve contractions between elements of spinor and vector representations of \( \mathfrak{so}(3,1) \) and elements of \( \mathfrak{cl}(3,1) \). Working in momentum space, all of these objects can be represented in the enveloping algebra of \( \mathfrak{c}_8 \), with the Killing form playing the role of the (real part of the) contraction. But a straightforward translation of the Standard Model action in this way turns out to be cumbersome and unilluminating.

The more interesting question is whether the result can be written nicely in terms of our division algebra description of \( \mathfrak{c}_8 \). Preliminary calculations suggest that this is at least partially the case, with \( \mathfrak{so}(7,1) \) momenta playing the roles of \( \mathfrak{so}(3,1) \) momenta and mass, and a sum over pots as a sum both over particle types and over generations. To complete this program, it is necessary to better understand the proposed electroweak mediators (described in Section VI) that always link right handed spinors to left-handed ones. Is such a theory chiral, and if so, why?

2. In the absence of a full, interacting theory, it is difficult to discuss whether that theory will satisfy the requirements of the Coleman–Mandula theorem [51, 52]. For instance, the theorem assumes Poincaré invariance of massive particles, but the Weyl spinors in our unbroken theory are massless. By the time the symmetry is broken so that (massive) Dirac spinors can be identified, the symmetry group has been reduced to a direct product of the Lorentz group and the internal symmetry group, as the theorem demands. Ultimately, we will need an action that has this direct product as its symmetry group.

3. Does the prediction of only six gluons, transforming as a \( 3 \) and \( \overline{3} \) of \( \mathfrak{sl}(3,\mathbb{R}) \) agree with experiment?

4. It is possible to add dimensions and scales to a Lie algebra by multiplying the Cartan elements by dimensionful constants. Can this be done in a consistent way within \( \mathfrak{c}_8 \) that explains why the fundamental constants of nature are what they are?

5. In Section IV we assumed that the only nonzero mass was \( m_L \). For completeness, we note here that choosing \( m_{kL} \neq 0 \) still preserves weak eigenvalues \( \psi_L \) under the action of a weak raising or lowering operator. However, the resulting Dirac spinor satisfies a slightly different Dirac equation, namely one in which the sign of \( m_{kL} \) has flipped. Can such a mechanism be used, for instance, to provide different masses to electrons and neutrinos?

6. Can the description of the three overlapping generations be used to explain the role of the CKM (quark) and PMNS (lepton) mixing matrices in the Standard Model?

7. When breaking \( \mathfrak{so}(12,4) \) to \( \mathfrak{so}(10,4) \) in Section V the degree-1 elements of \( \mathfrak{cl}(3,1) \subset \mathfrak{cl}(10,4) \) were interpreted, in Section III as the energy and momentum of the fermions. The \( 2 \times 10 \) elements of \( \mathfrak{cl}(10,4) \) that are simultaneously Lorentz scalars and a vector representation of the internal symmetries suggest an identification as generalized Higgs degrees of freedom. In Section III when we broke weak \( \mathfrak{so}(4) \) into \( \mathfrak{so}(3) \) by choosing \( \ell \) to be special, we added the corresponding degree-1 element of the Clifford algebra to the energy/momentum degrees of freedom in order to provide a mass for the fermions. Could this process be used to describe the usual Higgs symmetry breaking?

8. The canonical commutation relations between position and momentum closely parallels that between vectors and pseudovectors as discussed in Appendix D. We have used the degree-1 elements of \( \mathfrak{cl}(10,4) \) to describe (generalized) momenta. Could the degree-13 elements of \( \mathfrak{cl}(10,4) \) corresponding to Lorentz \( \mathfrak{so}(3,1) \), which are also degree-3 elements of \( \mathfrak{cl}(3,1) \), represent physical space-time?

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**Appendix A: The Structure of \( \mathfrak{c}_8 \)**

We summarize here the construction of the adjoint representation of \( \mathfrak{c}_8 \) given in [39]. We work throughout with the “half-split” real form \( \mathfrak{c}_8^{(24)} \), referred to throughout as \( \mathfrak{c}_8 \), although only minor changes are required to handle the other real forms. The octonions \( O \) are the real algebra spanned by the identity element 1 and seven square roots of −1 that we denote \( i, j, k, k', j', i', l, \ell \), whose multiplication table is neatly described by the oriented
Fano geometry shown in Figure 1 and given explicitly in Table VII. The split octonions \( \mathbb{O}' \) are the real algebra spanned by the identity element \( U \), three square roots of \(-1\) that we denote \( I, J, K \), and four square roots of \(+1\) that we denote \( k, j, l, f \). The multiplication table is given in Table VII. We will often write \( U \) as 1 when there is no ambiguity, as in Table VII.

Although there are many complex and quaternionic subalgebras of \( \mathbb{O} \) and \( \mathbb{O}' \), we normally take the quaternionic \( \mathbb{H} \) to be spanned by \( \{1, i, j, k\} \), and the split complex numbers \( \mathbb{C}' \) to be spanned by \( \{U, L\} \). The orthogonal complements of these algebras will be denoted as \( \mathbb{H}_1 \subset \mathbb{O} \) and \( \mathbb{C}'_1 \subset \mathbb{O}' \), respectively.

As shown in [32], almost all of \( \mathfrak{e}_8 \) can be represented as \( 3 \times 3 \) antihermitian tracefree matrices over \( \mathbb{O}' \otimes \mathbb{O} \), thus justifying the alternate name \( \mathfrak{su}(3, \mathbb{O}' \otimes \mathbb{O}) \) for \( \mathfrak{e}_8 \). There are \( 3 \times 8 \times 8 + 2 \times (7 + 7) = 220 \) independent such matrices, which take the form

\[
D_q = \begin{pmatrix}
q & 0 & 0 \\
0 & -q & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad X_p = \begin{pmatrix}
0 & p & 0 \\
0 & 0 & 0 \\
-\bar{p} & 0 & 0
\end{pmatrix},
\]

\[
Y_p = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & p \\
0 & -\bar{p} & 0
\end{pmatrix}, \quad Z_p = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
p & 0 & 0
\end{pmatrix},
\]

\[
S_q = \begin{pmatrix}
q & 0 & 0 \\
0 & q & 0 \\
0 & 0 & -2q
\end{pmatrix},
\]

where \( p \in \mathbb{O} \otimes \mathbb{O} \) and \( q \in \text{Im}\mathbb{O} + \text{Im}\mathbb{O}' \). It is straightforward to work out the commutators of these elements of \( \mathfrak{e}_8 \) using matrix multiplication except for commutators involving two imaginary labels, both from \( \mathbb{O} \), or both from \( \mathbb{O}' \). As described in [40], and discussed further in [39], composition of imaginary elements of \( \mathbb{O} \) requires nesting due to the lack of associativity. We must therefore introduce the additional elements

\[
D_{p,q} = \frac{1}{2} [D_p, D_q] = \frac{1}{2} [X_p, X_q],
\]

where \( p, q \in \text{Im}\mathbb{O} \) or \( p, q \in \text{Im}\mathbb{O}' \). We will normally assume that \( p, q \) are orthogonal and have unit norm. There are \( 21 + 21 = 42 \) such elements, generating \( \mathfrak{so}(7) \oplus \mathfrak{so}(3, 4) \), but only the \( 14 + 14 = 28 \) elements of \( \mathfrak{g}_2 + \mathfrak{g}_2(2) \) are independent of the matrices given in \( \text{A1} \). Specifically, due to triality, the \( S_q \) can be expressed in terms of the \( D_{p,q} \). We choose a particular basis \( \{G_q, A_q\} \) for the remaining nested elements, with \( \{A_1, ..., A_4, G_k\} \) generating \( \mathfrak{su}(3) \subset \mathfrak{g}_2 \) that fixes \( k \). These elements can be identified with the usual Gell-Mann matrices, with \( \{A_k, G_k\} \) corresponding (up to scale and a factor of \( i \)) to \( \{\lambda_3, \lambda_8\} \), respectively. A similar construction holds in \( \mathbb{O}' \), where however we fix \( L \). As shown in [32], the resulting algebra closes; the 248 elements given above form a basis for \( \mathfrak{e}_8 \) (using either the double-index \( Ds \), or the \( Ss, Gs \), and \( As \)).

As suggested by the form of the \( Ds \), we treat the upper left \( 2 \times 2 \) block as special; the \( 64 + 14 + 42 = 120 \) \( Xs, Ds \), and double-index \( Ds \) generate \( \mathfrak{so}(12, 4) \). The remaining \( 64 + 64 = 128 \) \( Ys \) and \( Zs \) form a single Majorana–Weyl representation of \( \mathfrak{so}(12, 4) \) under the adjoint action in \( \mathfrak{e}_8 \), and will be called spinors of \( \mathfrak{e}_8 \). That is, we have the decomposition

\[
\mathfrak{e}_8 = \mathfrak{so}(12, 4) \oplus 128.
\]

The commutator action of \( \mathfrak{so}(12, 4) \) on spinors, regarding both as elements of \( \mathfrak{e}_8 \), is equivalent to (possibly nested) matrix multiplication, with the \( 2 \times 2 \) blocks of the \( \mathfrak{so}(12, 4) \) matrices above acting on 2-component spinors of the form

\[
\alpha_\psi = \begin{pmatrix}
-\bar{q} \\
p
\end{pmatrix},
\]

corresponding to the \( \mathfrak{e}_8 \) element \( \psi = Y_p + Z_q \), with \( p, q \in \mathbb{O}' \otimes \mathbb{O} \).

---

**TABLE VI.** The octonionic multiplication table.

| \( i \) | \( j \) | \( k \) | \( k\ell \) | \( j\ell \) | \( i\ell \) | \( \ell \) |
|---|---|---|---|---|---|---|
| \(-1\) | \(-j\) | \(-k\) | \(-j\ell\) | \(-i\ell\) | \(-\ell\) | \(-\ell\) |
| \(-k\) | \(-1\) | \(-i\) | \(-j\ell\) | \(-i\ell\) | \(-\ell\) | \(-\ell\) |
| \(-j\) | \(-i\) | \(-1\) | \(-i\ell\) | \(-\ell\) | \(-\ell\) | \(-\ell\) |
| \(-\ell\) | \(-j\ell\) | \(-i\ell\) | \(-1\) | \(-i\) | \(-j\) | \(-k\) |
| \(-\ell\) | \(-j\ell\) | \(-i\ell\) | \(-1\) | \(-i\) | \(-j\) | \(-k\) |
| \(-\ell\) | \(-j\ell\) | \(-i\ell\) | \(-1\) | \(-i\) | \(-j\) | \(-k\) |

**TABLE VII.** The split octonionic multiplication table.
A similar choice of preferred $2 \times 2$ block inside $3 \times 3$ matrices is a feature of our earlier work \cite{17,18,48} in a somewhat different context, based on ideas suggested to us by Fairlie and Corrigan \cite{53}. The essential idea is that vectors are squares of spinors ($X = \psi\psi^\dagger$), so the actions of spin groups on spinors ($\psi \mapsto M\psi$) and on vectors ($X \mapsto MXM^\dagger$) use the same matrices. Embedding these $2 \times 2$ matrices as above, and reinterpreting both vectors ($2 \times 2$) and spinors ($2 \times 1$) as pieces of $3 \times 3$ matrices results in a single $3 \times 3$ action that correctly reproduces both the spinor and vector actions.

Appendix B: The Enveloping Algebra

We work throughout in the enveloping algebra of $\mathfrak{e}_8$, that is, the algebra of compositions of Lie algebra elements. Such compositions must always act on some representation of the given Lie algebra. In many cases, we restrict the adjoint action of $\mathfrak{e}_8$ (on itself) to that of $\mathfrak{so}(12,4)$ on the spinor $128$. Thus, for any elements $P, Q \in \mathfrak{so}(12,4)$ and spinor $\psi$, we define $P \circ Q$ by

$$ (P \circ Q)[\psi] = [P, [Q, \psi]]. \quad (B1) $$

The Jacobi identity now ensures that

$$ (P \circ Q - Q \circ P)[\psi] = ([P, Q], \psi), \quad (B2) $$

so that $P \circ Q = Q \circ P$ if $[P, Q] = 0$.

We emphasize that composition $(\circ)$ provides a product on the enveloping algebra; we can multiply elements, not merely take their commutators. This product is associative, since nested operations are always evaluated from the inside out. So we can also construct anticommutators in the enveloping algebra. Thus, given any proper orthogonal subalgebra $\mathfrak{so}(p,q) \subset \mathfrak{so}(12,4)$, we can, first of all, find a vector representation $V$ of $\mathfrak{so}(p,q)$ in its orthogonal complement in $\mathfrak{so}(12,4)$, and then construct the full (real) Clifford algebra $\mathbb{C}\ell(p,q)$ as $\mathbb{C}\ell(V)$ acting on the $128$. The Clifford identity $\{\gamma_m, \gamma_n\} = 2g_{mn}$ then takes the form $Q \circ Q = -|Q|^2$; that is,

$$ [Q, [Q, \psi]] = -|Q|^2\psi \quad (B3) $$

for any $Q \in V$ and spinor $\psi$, with $|Q|^2$ denoting the Lorentz norm. (The minus sign comes from our chosen Lorentz signature.)

The enveloping algebra can be used to construct the operation $\varphi$, first introduced in Section \ref{LA} which maps $L$ to $-L$ and also $IL$ to $-IL$, etc. This operation $\varphi$ of “$L$ conjugation” can be realized on $S_\pm$ in the enveloping algebra of $\mathfrak{so}(4,4) \subset \mathfrak{e}_8$ as

$$ \varphi = D_{KL} \circ D_{JL} \circ D_{IL} \circ D_L. \quad (B4) $$

However, since $\varphi$ maps between different representations of $\mathfrak{so}(9,1)$, it can not be represented within (the enveloping algebra of) $\mathfrak{so}(9,1)$ itself.

Appendix C: The Killing Form

Every semisimple Lie algebra possesses a nondegenerate, symmetric inner product, the Killing form $B$. For matrix Lie algebras, $B(\mathbb{M}, \mathbb{N})$ can be taken to be $\text{tr}(\mathbb{M}\mathbb{N})$. The Killing form is unique up to an overall scale, which we choose so that normalized boosts and rotations in $\mathfrak{so}(12,4)$ square to $\pm 1$, respectively, when acting on spinors. The basis for $\mathfrak{e}_8$ given by omitting the $\mathfrak{so}(128)$ from $\mathfrak{A1}$ and $\mathfrak{A2}$ is orthonormal (so long as $p, q$ are normalized), as is the alternative basis using $(S_q, A_q, G_q)$ instead of $\mathfrak{A2}$, apart from the conventional normalizations $B(S_q, S_q) = \pm 6, B(G_q, G_q) = \pm 3, B(A_q, A_q) = \pm 2$. The signature of (a real form of) a Lie algebra is the number of boosts minus the number of rotations in any orthonormal basis. For example, the 26 boosts in (the half-split form of) $\mathfrak{e}_8 \subset \mathfrak{e}_7$ are the elements containing $L$ in their label $(X_{aL}; Y_{aL}; Z_{aL}; D_L; S_L)$, so the signature is $26 - 52 = -26$. In $\mathfrak{e}_8$, the boosts are the $4 \times 8 \times 3$ $X$s, $Y$s, and $Z$s labeled by $IL$, $JL$, $KL$, $L$, as well as the $4 \times 4$ boosts in the $\mathfrak{so}(4,4)$ that act on labels in $\mathbb{O}$; linear combinations of the latter include $D_L$ and $S_L$. Thus, there are $4 \times 8 \times 3 + 4 \times 4 = 112$ boosts in $\mathfrak{e}_8$, so the signature is

Given a nondegenerate, symmetric inner product $g$ and a complex structure $\iota$ which is compatible with $g$ in the sense that

$$ g(\iota(\alpha), \iota(\beta)) = g(\alpha, \beta), \quad (C1) $$

the inner product extends naturally to a hermitian product, of which it is the real part. Explicitly, the associated hermitian product is given by

$$ (\alpha, \beta) = g(\alpha, \beta) - ig(\alpha, \iota(\beta)) \in \mathbb{C} \quad (C2) $$

with hermiticity following from \ref{C1}, which is equivalent to

$$ g(\alpha, \iota(\beta)) = -g(\iota(\alpha), \beta). \quad (C3) $$

(The complex unit $i$ in \ref{C2} is unrelated to the division algebra structure used elsewhere in this paper.) We could of course set $g = B$, but this construction also applies to the inner product

$$ g(\alpha, \beta) = B(\varphi(\alpha), \beta) \quad (C4) $$

where $\varphi$ denotes $L$ conjugation, as introduced in Section \ref{LA}. The compatibility condition is still satisfied so long as $\varphi$ commutes with $\iota$, which it does for the complex structure $\iota = X_1$ introduced in Section \ref{LA} when acting on spinors. In this case, the pairing given by $\varphi$ is analogous to the dagger operation in traditional language, with $\psi^\dagger \psi = (\psi, \psi) = B(\varphi(\psi), \psi)$.

Appendix D: Volume Elements and Spinors

In this section, we use volume elements to describe how to decompose spinors of $\mathfrak{e}_8$ into spinor representations of particular subalgebras of $\mathfrak{so}(12,4)$.
Every orthogonal Lie algebra $\mathfrak{so}(m, n)$ is associated with an (abstract) Clifford algebra $\text{Cl}(m, n)$, whose degree-1 elements are the physicist’s gamma matrices, and whose degree-2 elements can be identified with $\mathfrak{so}(m, n)$ itself. When $m + n$ is even, the Clifford product of any basis of Cartan elements in $\mathfrak{so}(m, n)$ is, up to an overall scale, independent of the elements used to construct it; the resulting Clifford algebra element can be thought of as the product of all of the gamma matrices. We refer to this normalized product as the Casimir operator, as it commutes with all of $\mathfrak{so}(m, n)$.

For the Lorentz Lie algebra $\mathfrak{so}(3, 1)$ and the weak Lie algebra $\mathfrak{so}(4)$, the volume elements $\omega_{3,1}$ and $\omega_{4,0}$ square to $\mp 1$ on spinors, respectively. In order to obtain real eigenvalues, we therefore define

$$\omega = \iota \circ \omega_{3,1}, \quad \Omega = \omega_{4,0}; \quad (D1)$$

$\omega$ can be thought of as $\gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3$. The eigenvalues of the pair $(\omega, \Omega)$ now provide binary labels for the spinor bases of $\mathfrak{so}(3, 1) \oplus \mathfrak{so}(4)$. These irreducible bases are the “pots” of spinors introduced in Section III. Furthermore, due to the presence of $\iota$ in $\omega$, we have

$$\omega_{9,1} = \omega \circ \Omega \quad (D2)$$

so that spinor representations of $\mathfrak{so}(9, 1)$ (such as $\mathbf{S}_L$ in Section III) can be distinguished by the sign of $\omega \circ \Omega$.

We emphasize that these representations depend only on the decomposition of $\iota_\pm$ described in Section III independent of any choice of a maximal set of commuting elements, $\iota$, that is, independent of the choice of Cartan subalgebras.

This construction extends to $\iota_8$, where each representation of $\mathfrak{so}(3, 1) \oplus \mathfrak{so}(4)$ will also carry color labels, according to their eigenvalues under $G_L$ and $A_L$, the chosen Cartan elements in $\mathfrak{su}_k$. We interpret spinor representations for which both color eigenvalues are 0 as leptons; these representations can be distinguished using their eigenvalues under either $\omega_{9,1}$ or $\mathbf{S}_L$. An alternate labeling is discussed in Appendix E.

Equivalently, the volume element $\omega \circ \Omega$ of $\mathfrak{so}(9, 1)$ divides the 128 of $\mathfrak{so}(12, 4)$ into two 64s, which we denote $\mathbf{S}_L$ (thus further generalizing the notation in Section III). Since there is only one Majorana–Weyl representation of $\mathfrak{so}(12, 4)$ in $\iota_8$, the $\mathfrak{so}(12, 4)$ volume element $\Omega_{12,4}$ must act as a constant on this representation, and hence on all spinors. Similarly to (D2), we also have

$$\Omega_{12,4} = \Omega_{9,1} \circ \Omega_{3,3} \quad (D3)$$

so that the actions of $\Omega_{9,1}$ and $\Omega_{3,3}$ on spinors are the same. Thus, $\mathbf{S}_L$ are also representations of $\mathfrak{so}(3, 3)$; they are, in fact, irreducible representations of $\mathfrak{so}(9, 1) \oplus \mathfrak{so}(3, 3)$.

A further application of these volume elements arises when removing a complex structure $(\iota \in \mathfrak{so}(2))$ from $\mathfrak{so}(m, n)$ yielding both $\mathfrak{so}(m-2, n)$ and two vector representations of $\mathfrak{so}(m-2, n)$, related by $\iota$. On any representation of $\mathfrak{so}(m, n)$, the volume element $\omega_{m,n} = \omega_{m-2,n} \omega_{2,0}$ must be constant, so that $\omega_{m-2,n} = \pm \omega_{2,0}$ is just the complex structure itself. In other words, either one of the two vector representations can be chosen as the degree-1 elements that generate $\text{Cl}(m-2, n)$, with the other representation then being the pseudovectors, that is, the elements of degree $m + n - 3$.

### Appendix E: A Spinor Binary Code

We now choose specific Cartan elements in $\mathfrak{so}(3, 1)$ and $\mathfrak{so}(4)$, then use them to provide a binary code for the spinors, analogous to that of Zee [54].

For $\mathfrak{so}(3, 1)$, we associate the label $k$ with the $z$ direction. The resulting Cartan elements are the $xy$-rotation $(D_{i,j})$ and the $zt$-boost $(B_z)$. When acting on spinors, we replace $D_{i,j}$ with $L_z = \iota \circ D_{i,j}$ to ensure real eigenvalues. We interpret the sign of the eigenvalue of $L_z$ as giving the spin in the $z$ direction, as usual. The choice of $z$-direction to determine spin is, of course, arbitrary. We also have

$$\omega = L_z \circ B_z. \quad (E1)$$

We repeat this construction for $\mathfrak{so}(4)$, with a few small but important differences. We begin by decomposing $\mathfrak{so}(4)$ as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Choosing $\mathfrak{so}(3) \subset \mathfrak{so}(4)$ to act only on the labels $\{i\ell,j\ell,k\ell\}$, and pairing each resulting rotation (e.g. $D_{i\ell,j\ell}$) with the unique generator of $\mathfrak{so}(4)$ with which it commutes, (e.g. $D_{i\ell,k\ell}$), the sum and difference of these two sets each generate an $\mathfrak{su}(2)$.

As noted in Section VII, an underappreciated feature of this construction is that it does not depend on the choice of $\mathfrak{so}(3)$; the two $\mathfrak{su}(2)$ subalgebras are in fact unique, and are precisely the subalgebras $\mathfrak{su}_R$ and $\mathfrak{su}_L$ introduced in Sections II and III. We interpret $\mathfrak{su}_R$ as the electro-weak symmetry group, and, in the context of GUTs, $\mathfrak{su}_2$ as its right-handed counterpart.

We now finally choose Cartan elements in $\mathfrak{su}_L$ and $\mathfrak{su}_R$, or, equivalently, their sum and difference in $\mathfrak{so}(4)$. Since $\mathfrak{so}(4)$ is compact, every Cartan element will be a rotation, so we compose with $\iota$ when acting on spinors. Our choice for the resulting Cartan elements of (complexified) $\mathfrak{so}(4)$ is

$$R_a = \iota \circ D_{i\ell,j\ell}, \quad R_b = \iota \circ D_{k\ell,l\ell}; \quad (E2)$$

and our conventions are such that $R_a = R_b - R_b \in \mathfrak{su}_L \subset \mathfrak{g}_2$. We can not overemphasize that this association of $\iota$ with $k\ell$ is completely independent of the association of $L$ with $k$ in $\mathfrak{so}(3, 1)$, although this combination does simplify the notation later. We also reiterate that the decomposition $\mathfrak{so}(4) = \mathfrak{su}_L \oplus \mathfrak{su}_R$ is unique; it is unaffected by the choice of Cartan elements in $\mathfrak{so}(4)$. In any case, we have

$$\Omega = -R_a \circ R_b = D_{i\ell,j\ell} \circ D_{k\ell,l\ell}. \quad (E3)$$

Each pot now admits a basis of simultaneous eigenstates of the Cartan elements $\{R_a, R_b, L_z, B_z\}$, with
Table VIII. The binary code for the spinors in the 16 of $\mathfrak{so}(9, 1)$ with $\omega \circ \Omega = 1$, corresponding to two complex 4s of $\mathfrak{so}(3, 1) \oplus \mathfrak{so}(4)$ (above and below the double line), along with their interpretation.

| Code    | $Y_{i}(1+L) + Z_{j}(1-L)$, $-Y_{i}(1+L) = Z_{j}(1-L)$ | $Y_{j}(1+L) + Z_{i}(1-L)$, $-Y_{j}(1+L) = Z_{i}(1-L)$ | $Y_{i}(1+L) - Z_{j}(1-L)$, $-Y_{i}(1+L) - Z_{j}(1-L)$ | $Y_{j}(1+L) - Z_{i}(1-L)$, $-Y_{j}(1+L) - Z_{i}(1-L)$ |
|---------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| $++-+$  | $\nu_{R}$                                       | $\nu_{R}$                                       | $\nu_{R}$                                       | $\nu_{R}$                                       |
| $+-+$   | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |
| $-++$   | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |
| $-+$    | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |
| $+-+$   | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |
| $-++$   | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |
| $-+$    | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       | $\nu_{L}$                                       |

We therefore have a four-digit binary code $b_{1}b_{2}b_{3}b_{4}$ labeling the 16 independent (complex!) spinor degrees of freedom, with $b_{m} = \pm 1$. The product of all four digits is constant within a “double pot” $S_{\pm}$, and either the product of the first two or last two digits can be used to distinguish the two pots within a given double pot.

A table listing the binary codes of the $\epsilon_{6}$ spinors in $S_{+}$ (so $\omega \circ \Omega = 1$) is given in Table VIII along with their interpretation. For each pair of basis elements given in the table, $i(\text{Re}(\psi)) = \text{Im}(\psi)$, so that the complex spinor $(a + bi)\psi$ becomes $a\text{Re}(\psi) + b\text{Im}(\psi)$ when represented within $\epsilon_{6}$.

This binary code is similar in spirit to the five-digit code of Zee [54]. The spinors shown in Table VIII are all leptons, so no color labels are needed. As discussed in Appendix III, color is determined by the eigenvalues of $G_{L}$ and $A_{L}$, although $S_{L}$ is also needed to distinguish leptons from antileptons. Using the equivalent Cartan basis $\{D_{1, IL}, D_{J, JL}, D_{K, KL}\}$, each with eigenvalues $\pm 1$, reproduces the first three digits of the Zee binary code. Our resulting seven-digit code then matches that of Zee, with an additional two digits at the end associated with the two Lorentz Cartan elements.

This seven-digit code labels 128 possible states, yet there are only 64 (complex!) spinor states available to us. How is this possible? The seven-digit code corresponds to seven of the eight Cartan elements of $\mathfrak{so}(12, 4)$. However, three of the seven were originally rotations, requiring composition with $X_{1}$, the eighth Cartan element. Thus, the product of the digits of our seven-digit code yields (minus) the eigenvalue under the product of all eight Cartan elements, that is, of the volume element $\Omega_{12, 4}$. Since we have a single, Majorana–Weyl representation of $\mathfrak{so}(12, 4)$, this eigenvalue must be constant; with our conventions, it is +1. Thus, the seventh digit can in fact be dropped, and we recover exactly the Zee five-digit code with one additional digit for spin.
