SINGULAR INTEGRALS WITH ANGULAR INTEGRABILITY

FEDERICO CACCIAFESTA AND RENATO LUCA

Abstract. In this note we prove a class of sharp inequalities for singular integral operators in weighted Lebesgue spaces with angular integrability.

1. Introduction

We consider singular integral operators

\[ Tf(x) := \text{P.V.} \int_{\mathbb{R}^n} f(x - y)K(y) \, dy, \]

where the kernel \( K \) satisfies the following conditions:

\[ |y|^n |K| \leq C, \quad |y|^{n+1} |\nabla K| \leq C, \quad |\hat{K}| \leq C. \]

Here \( C > 0 \) is a constant and \( \hat{\cdot} \) denotes the Fourier transform. The main example we have in mind is the directional Riesz transform, which corresponds to the choice \( K(y) := |y|^{-(n+1)}y \cdot \theta, \theta \in S^{n-1}. \)

The study of the boundedness of these operators in weighted Lebesgue spaces \( L^p(w(x)dx), \) for \( 1 < p < \infty \) and \( 0 < w \in L^1_{\text{loc}}(\mathbb{R}^n) \), is a classical problem in harmonic analysis: in particular, Stein [13] proved it for the (sharp range of) homogeneous weights \( w(x) = |x|^\alpha, \) \( -n/p < \alpha < n - n/p. \) The result was later extended by Coifman and Fefferman [2] to any \( A_p \) weight.

While the weighted \( L^p \)-theory has been extensively studied, less is known in the case of Lebesgue norms with different integrability in the radial and angular directions, namely

\[ \|f\|_{L^p_{|x|}L^\tilde{p}_{\tilde{\theta}}} := \left( \int_0^{+\infty} \|f(\rho \cdot)\|_{L^\tilde{p}(S^{n-1})}^\rho \rho^{n-1} d\rho \right)^{\frac{1}{\tilde{p}}}. \]

These mixed radial-angular spaces have been successfully used in recent years to improve several results in the framework of partial differential equations; see e.g. [1,8,12,14,15]. Notice that when \( p = \tilde{p} \) the norms reduce to the usual \( L^p \) norms.

Notice also that, neglecting the constants, they are increasing in \( \tilde{p} \), and that they behave as the \( L^p \) norms under homogeneous rescaling, namely \( f(\cdot) \rightarrow f(\lambda \cdot), \lambda > 0. \)

In a recent paper A. Córdoba [3] proved, among other things, the \( L^p_{|x|}L^\tilde{p}_{\tilde{\theta}} \) boundedness for operators of the form \( (1.1). \). Here we give an extension of this result to the weighted setting.

©2016 American Mathematical Society
\textbf{Theorem 1.1.} Let \( n \geq 2, 1 < p < \infty, 1 < \tilde{p} < \infty \) and \(-n/p < \alpha < n - n/p\). Then
\begin{equation}
\| |x|^{\alpha}T\phi \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}} \leq C\| |x|^{\alpha} \phi \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}},
\end{equation}
where \( C \) is a constant depending only on \( \alpha, p, \tilde{p}, n \).

Let us point out that the case \( \alpha = 0, 1 < \tilde{p} < \infty \) in inequality (1.3) may be deduced by the application of Córdoba’s argument; see [6, Theorem 2.6]. Therefore the novelty of Theorem 1.1 is that it covers all the possible homogeneous weights of the kind \( |x|^{\alpha}p \).

\textbf{Remark 1.1.} Condition \(-n/p < \alpha \) turns out to be necessary by testing the inequality on functions \( \phi \) such that \( \phi = 0 \) and \( T\phi > 0 \) in a neighborhood of the origin. On the other hand, condition \( \alpha < n - n/p \) turns out to be necessary for the same reason by considering the dual inequality.

\textbf{Remark 1.2.} One of the estimates (1.3) has been used in [6, Theorem 1.5] to deduce information about the regularity of weak solutions of the 3d Navier–Stokes problem with initial velocities satisfying good angular integrability properties.

We write \( A \lesssim B \) if \( A \leq CB \) with a constant \( C \) depending only on \( \alpha, p, \tilde{p}, n \). We write \( A \simeq B \) if both \( A \lesssim B \) and \( B \lesssim A \).

\section{Proof}

We know by [6, Theorem 2.6] that inequality (1.3) is true in the case \( \alpha = 0 \), that is,
\begin{equation}
\| Tg \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}} \lesssim \| g \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}}.
\end{equation}

Following Stein [13], we now show that the weighted case (1.3) can be then deduced by the unweighted one. The next lemma represents the core of the proof.

\textbf{Lemma 2.1.} Let \( n \geq 2, 1 < p < \infty, 1 \leq \tilde{p} \leq \infty, -n/p < \alpha < n - n/p \) and
\begin{equation}
F(x, y) := \left| 1 - \left( |x|/|y| \right)^{\alpha} \right| / |x - y|^n;
\end{equation}
then
\begin{equation}
\left\| \int_{\mathbb{R}^n} F(x, y)\phi(y)dy \right\|_{L^p_{|x|}L^{\tilde{p}}_{\theta}} \leq C\| \phi \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}}.
\end{equation}

Assume indeed this has been proved and first apply inequality (2.1) with the choice \( g := | \cdot |^{\alpha}f \) to have
\begin{equation}
\| T(|x|^{\alpha}f) \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}} \lesssim \| |x|^{\alpha}f \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}}.
\end{equation}

Then notice that
\begin{equation}
\begin{aligned}
&\left| T(|x|^{\alpha}f) - |x|^{\alpha}Tf \right| \leq \int_{\mathbb{R}^n} |K(x - y)(|y|^{\alpha} - |x|^{\alpha})f(y)|dy \\
&\lesssim \int_{\mathbb{R}^n} \left| y^{\alpha} - |x|^{\alpha} \right| / |x - y|^n |f(y)|dy = \int_{\mathbb{R}^n} \left| 1 - \left( |x|/|y| \right)^{\alpha} \right| / |x - y|^n |y|^{\alpha} |f(y)|dy,
\end{aligned}
\end{equation}
so that by using Lemma 2.1 with \( \phi = | \cdot |^{\alpha}f \) we obtain
\begin{equation}
\| T(|x|^{\alpha}f) - |x|^{\alpha}Tf \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}} \lesssim \| |x|^{\alpha}f \|_{L^p_{|x|}L^{\tilde{p}}_{\theta}}.
\end{equation}
Then, the desired estimate (1.3) follows by (2.4) and (2.5) and triangle inequality. Thus it only remains to prove Lemma 2.1.

The idea is to use a change of variables which resembles the standard polar coordinates. In this variant the integration over the sphere is replaced by integration over the special orthogonal group $SO(n)$ and the radial integration is replaced by integration over the multiplicative group of the positive real numbers. This method works efficiently when homogeneous power weights are involved; see e.g. [5,7].

Proof of Lemma 2.1 By using the isomorphism

$$S^{n-1} \simeq SO(n)/SO(n-1)$$

we can rewrite integrals on $S^{n-1}$ as follows:

$$\int_{S^{n-1}} g(y) dS(y) \simeq \int_{SO(n)} g(Ae) dA, \quad n \geq 2,$$

where $dA$ is the left Haar measure on $SO(n)$, and $e \in S^{n-1}$ is a fixed unit vector. Thus, via polar coordinates, a generic integral can be rewritten as

$$\int_{\mathbb{R}^n} F(x,y) \phi(x) dy = \int_0^\infty \int_{S^{n-1}} F(x,\rho \omega) \phi(\rho \omega) dS_\omega \rho^{n-1} d\rho \simeq \int_0^\infty \int_{SO(n)} F(x,\rho Be) \phi(\rho Be) dB \rho^{n-1} d\rho.$$ 

Hence the $L_\theta^p$ norm can be written as

$$\left\| \int_{\mathbb{R}^n} F(|x| \theta, y) \phi(y) dy \right\|_{L_\theta^p(S^{n-1})} \simeq \left\| \int_{\mathbb{R}^n} F(|x|Ae, y) \phi(y) dy \right\|_{L_\theta^p(SO(n))} \leq \int_0^\infty \left\| \int_{SO(n)} F(|x|Ae, \rho Be) \phi(\rho Be) dB \right\|_{L_\theta^p(SO(n))} \rho^{n-1} d\rho$$

where $e$ is any fixed unit vector. We choose $F$ as in (2.2) and we change variables $B \rightarrow AB^{-1}$ in the inner integral. By the invariance of the measure this is equivalent to

$$= \int_0^\infty \left\| \int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|x|Be - pe} | \phi(\rho AB^{-1} e) dB \right\|_{L_\theta^p(SO(n))} \rho^{n-1} d\rho$$

$$= \int_0^\infty \left\| \int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|x|Be - pe} | \phi(\rho AB^{-1} e) dB \right\|_{L_\theta^p(SO(n))} \rho^{n-1} d\rho.$$ 

Notice that the integral

$$\int_{SO(n)} \frac{|1 - (|x|/\rho)^\beta|}{|x|Be - pe} | \phi(\rho AB^{-1} e) dB = G * \phi(A)$$

is a convolution on $SO(n)$ of the functions

$$G(A) = \frac{|1 - (|x|/\rho)^\beta|}{|x|A e - pe}$$.  

$H(A) = |\phi(\rho A e)|.$
We can thus apply Young’s inequality on $SO(n)$ (see for instance [9, Theorem 1.2.12]) to obtain, for any $1 \leq \tilde{p} \leq \infty$, the estimate

\[(2.7)\]

\[
\left\| \int_{\mathbb{R}^n} F(|x|, y) \phi(y) dy \right\|_{L_{\rho}^p(S^{n-1})} \lesssim \int_0^\infty \left\| \frac{1 - (|x|/\rho)^\beta}{(|x| e - \rho \theta)^n} \right\|_{L_{\rho}^p(S^{n-1})} \left\| \phi(\rho \theta) \right\|_{L_{\rho}^p(S^{n-1})^\beta} d\rho \]

\[
= \int_0^\infty \left\| \frac{1 - (|x|/\rho)^\beta}{(|x| e - \rho \theta)^n} \right\|_{L_{\rho}^p(S^{n-1})} \left\| \phi(\rho \theta) \right\|_{L_{\rho}^p(S^{n-1})} \frac{d\rho}{\rho},
\]

where we switched back to the coordinates of $S^{n-1}$. Then we notice

\[(2.8)\]

\[
\left\| \int_{\mathbb{R}^n} F(x, y) \phi(y) dy \right\|_{L_{|x|}^p} \lesssim \left\| \int_{\mathbb{R}^n} F(|x|, y) \phi(y) dy \right\|_{L_{\rho}^p(S^{n-1})} \left\| dS_\theta |x|^{\frac{n}{p}} \right\|_{L^p(\mathbb{R}^+ \cdot d|x|/|x|)},
\]

where $\mathbb{R}^+ (\cdot)$ is the multiplicative group of positive real numbers equipped with its Haar measure $d\rho/\rho$. Using (2.7) allows one to estimate (2.8) with

\[\lesssim \left\| \int_0^\infty \left( \frac{|x|}{\rho} \right)^\beta \left\| \frac{1 - (|x|/\rho)^\beta}{(|x| e - \rho \theta)^n} \right\|_{L_{\rho}^p(S^{n-1})} \left\| \phi(\rho \theta) \right\|_{L_{\rho}^p(S^{n-1})} \frac{d\rho}{\rho} \right\|_{L^p(\mathbb{R}^+ \cdot d|x|/|x|)}.
\]

Notice that the inner term is a convolution on $\mathbb{R}^+ (\cdot)$ of the functions

\[g(\rho) = \rho^{\frac{n}{p}} \left\| \frac{1 - \rho^\beta}{(|\rho e - \theta|)^n} \right\|_{L_{\rho}^p(S^{n-1})}, \quad h(\rho) = \rho^{\frac{n}{p}} \left\| \phi(\rho \theta) \right\|_{L_{\rho}^p(S^{n-1})}.
\]

Thus we can apply again Young’s inequality to estimate further (2.8) with

\[(2.9)\]

\[\lesssim \left\| g(\rho) \right\|_{L_{\rho}^p(S^{n-1})} \left\| h(\rho) \right\|_{L_{\rho}^p(S^{n-1})} \left\| dS_\theta (|x|) \right\|_{L^p(\mathbb{R}^+ \cdot d\rho/\rho)} \]

for all $1 \leq p \leq \infty$. Once we have noticed that

\[\left\| h(\rho) \right\|_{L_{\rho}^p(S^{n-1})} \left\| dS_\theta (|x|) \right\|_{L^p(\mathbb{R}^+ \cdot d\rho/\rho)} = \left\| \phi \right\|_{L_{|x|}^p} \left\| dS_\theta \right\|_{L^p(\mathbb{R}^+ \cdot d\rho/\rho)} \]

the concluding step of the proof of the lemma is represented by showing that the first term of (2.9) is bounded. We split the integral

\[
\int_0^{+\infty} \rho^{\frac{n}{p}} \int_{S^{n-1}} \frac{|1 - \rho^\beta|}{(|\rho e - \theta|)^n} dS_\theta \frac{d\rho}{\rho} = \int_0^{\frac{1}{2}} (\cdot) + \int_{\frac{1}{2}}^{2} (\cdot) + \int_2^{+\infty} (\cdot) =: I + II + III
\]

and we bound separately the three terms.

If $0 < \rho < 1/2$, then $|\rho e - \theta| \geq 1/2$. Thus, since $|1 - \rho^\beta| < 1 + \rho^\beta$,

\[I \lesssim \int_0^{\frac{1}{2}} (\rho^{\frac{n}{p} - 1} + \rho^{\frac{n}{p} - 1 + \beta}) d\rho < \infty
\]

provided that $p < \infty$ and $\beta > -n/p$.

If $1/2 \leq \rho \leq 2$, we notice that $|1 - \rho^\beta| \lesssim |1 - \rho|$ and we use that (see for instance Lemma 2.1 in [5])

\[
\int_{S^{n-1}} |\rho e - \theta|^{-n} dS_\theta \simeq \frac{1}{|1 - \rho|}
\]
to bound
\begin{equation}
II \simeq \int_{\frac{1}{2}}^{2} \rho^{n-1} \frac{|1-\rho^\beta|}{|1-\rho|} d\rho \lesssim \int_{\frac{1}{2}}^{2} \rho^{n-1} d\rho < \infty.
\end{equation}

Finally, if $2 < \rho < +\infty$, then $|\rho e - \theta| \geq |\rho| - |\theta| \geq |\rho|/2$. Thus, since $|1-\rho^\beta| < 1 + \rho^\beta$,
\begin{equation}
III \lesssim \int_{\frac{1}{2}}^{+\infty} \left( \rho^{\frac{n}{p}-1-n} + \rho^{n-1+\beta-n} \right) d\rho < \infty
\end{equation}
provided that $p > 1$ and $\beta < n - n/p$ and that concludes the proof. \qed

ACKNOWLEDGEMENTS

The first author was supported by the FIRB 2012 ‘Dispersive dynamics, Fourier analysis and variational methods’. Part of the work was done during the first author’s visit at MSRI, Berkeley (CA), within the program ‘New Challenges in PDE: Deterministic Dynamics and Randomness in High and Infinite Dimensional Systems’, which he acknowledges for the wonderful working conditions.

The second author was supported by the ERC grant 277778 and MINECO grant SEV-2011-0087 (Spain).

REFERENCES

[1] Federico Cacciafesta and Piero D’Ancona, Endpoint estimates and global existence for the nonlinear Dirac equation with potential, J. Differential Equations 254 (2013), no. 5, 2233–2260, DOI 10.1016/j.jde.2012.12.002. MR3007110

[2] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974), 241–250. MR0358205 (50 #10670)

[3] A. Córdoba, Singular integrals and maximal functions: The disk multiplier revisited, arXiv:1310.6276.

[4] A. Cordoba and C. Fefferman, A weighted norm inequality for singular integrals, Studia Math. 57 (1976), no. 1, 97–101. MR0420115 (54 #8132)

[5] Piero D’Ancona and Renato Luca’, Stein-Weiss and Caffarelli-Kohn-Nirenberg inequalities with angular integrability, J. Math. Anal. Appl. 388 (2012), no. 2, 1061–1079, DOI 10.1016/j.jmaa.2011.10.051. MR2869807 (2012m:46070)

[6] P. D’Ancona and R. Lucà, On the regularity set and angular integrability for the Navier–Stokes equation, ArXiv:1501.07780.

[7] Pablo L. De Nápoli, Irene Drelichman, and Ricardo G. Durán, On weighted inequalities for fractional integrals of radial functions, Illinois J. Math. 55 (2011), no. 2, 575–587 (2012). MR3020697

[8] Daoyuan Fang and Chengbo Wang, Weighted Strichartz estimates with angular regularity and their applications, Forum Math. 23 (2011), no. 1, 181–205, DOI 10.1515/FORM.2011.009. MR2769870 (2012b:35208)

[9] Loukas Grafakos, Classical Fourier analysis, 2nd ed., Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008. MR2445437 (2011c:42001)

[10] Renato Lucà, Regularity criteria with angular integrability for the Navier-Stokes equation, Nonlinear Anal. 105 (2014), 24–40, DOI 10.1016/j.na.2014.04.004. MR3200738

[11] Shuji Machihara, Makoto Nakamura, Kenji Nakanishi, and Tohru Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, J. Funct. Anal. 219 (2005), no. 1, 1–20, DOI 10.1016/j.jfa.2004.07.005. MR2108356 (2006b:35109)

[12] Tohru Ozawa and Keith M. Rogers, Sharp Morawetz estimates, J. Anal. Math. 121 (2013), 163–175, DOI 10.1007/s11854-013-0031-0. MR3127381

[13] E. M. Stein, Note on singular integrals, Proc. Amer. Math. Soc. 8 (1957), 250–254. MR0088666 (19,547b)

[14] Jacob Sterbenz, Angular regularity and Strichartz estimates for the wave equation, Int. Math. Res. Not. 4 (2005), 187–231, DOI 10.1155/IMRN.2005.187. With an appendix by Igor Rodnianski. MR2128434 (2006i:35212)
[15] Terence Tao, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*, Comm. Partial Differential Equations 25 (2000), no. 7-8, 1471–1485, DOI 10.1080/03605300008821556, MR1765155 [2001h:35038]

Dipartimento di Matematica, SAPIENZA — Università di Roma, Piazzale A. Moro 2, I-00185 Roma, Italy

*E-mail address*: cacciafe@mat.uniroma1.it

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Madrid, 28049, Spain

*E-mail address*: renato.luca@icmat.es