A comparative study of the anelastic and subseismic approximations for low-frequency gravity modes in stars

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ABSTRACT

A comparative study of the properties of the anelastic and subseismic approximations is presented. The anelastic approximation is commonly used in astrophysics in compressible convection studies whereas the subseismic approximation comes from geophysics where it is used to describe long-period seismic oscillations propagating in the Earth's outer fluid core. Both approximations aim at filtering out the acoustic waves while retaining the density variations of the equilibrium configuration. However, they disagree on the form of the equation of mass conservation. We show here that the anelastic approximation is in fact the only consistent approximation as far as stellar low-frequency oscillations are concerned. We also show that this approximation implies Cowling's approximation which neglects perturbations of the gravity field. Examples are given to assert the efficiency of the anelastic approximation.

Key words: stars: oscillations

1 INTRODUCTION

Initiated with the work on atmospheric convection by Ogura \& Phillips (1962) and Gough (1969), the anelastic approximation has been widely used in astrophysics to describe compressible convection. For example, it has been first applied by Latour et al. (1976) and Toomre et al. (1976) to investigate the non-rotating and non-magnetized convective envelopes. Then, Gilman \& Glatzmaier (1981) and Glatzmaier \& Gilman (1981a,b,c) extended these studies to the rotating case whereas the MHD case has been considered by Glatzmaier (1984,1985a,b), Lantz \& Sudan (1995) and Lantz \& Fan (1999).

In all these studies, this approximation has been preferred to another famous one in fluid dynamics, the so-called Boussinesq approximation where the velocity is assumed to be solenoidal, i.e. $\nabla \cdot \vec{v} = 0$ (Spiegel \& Veronis 1960). As the density varies on several orders of magnitude in a convection zone, using the anelastic approximation is indeed more relevant and leads to $\nabla \cdot (\rho \vec{v}) = 0$. However, both approximations filter out short-period acoustic oscillations by assuming that the Mach number of the convection is small so that ‘the task of numerical solution will not be complicated by the need to resolve very rapid time variations’ (Latour et al. 1976).

More recently, the anelastic approximation has been used in the field of stellar oscillations. Waves can propagate over long distances in star interiors and large variations of density need also to be taken into account. Thus, as in stellar convection studies, the Boussinesq approximation is too restrictive and not adapted for this problem. However, since acoustic waves are filtered out, only the low-frequency part of the oscillations spectrum is captured. As this part of the spectrum also corresponds to the most perturbed one when rotation acts, this approximation is very attractive when low-frequency modes of a rotating star are considered.

Understanding the spectrum of rotating stars is indeed a difficult problem when the rotation period is of the same order than the oscillation one. In this case, the usual perturbative theory of Ledoux (1951) reaches its limits and the eigenvalue problem needs to be solved non-perturbatively (Dintrans \& Rieutord 2000, hereafter referred to as Paper I). But rotation introduces new computational challenges: (i) the strong Coriolis coupling between normal modes of different degrees ($\ell, \ell \pm 1$) leads to a large system of coupled differential equations; (ii) some singularities due to the presence of wave attractors emerge in the region $\sigma \approx 2\Omega$, $\sigma$ being the wave frequency and $\Omega$ the uniform rotation rate of the star. Applying the anelastic approximation does not remove these singularities but decreases the size of the numerical problem since acoustic quantities disappear (see Paper I). We note that Berthomieu et al. (1978) used the same arguments when considering a hierarchy between the displacement and pressure components, this hierarchy leading to the same result as the direct use of the anelastic approximation.

Contrary to stars, the mass of the Earth is not strongly
concentrated near its centre. Only 30% of the whole mass is indeed inside the half radius whereas this ratio is around 90% for the Sun (see e.g. Stix 1989). It means that the variations in density in the Earth’s outer fluid core generated by the propagation of seismic P-waves may produce non negligible variations in the gravitational potential \( \phi \) and explains why geophysicists do not wish to use the Cowling approximation in their description of P-waves. To take into account this effect but to simplify the equations for low frequency modes Smylie & Rochester (1981), Smylie, Szeto & Rochester (1984) introduced the ‘subseismic’ approximation where self-gravity is included.

Recently, De Boek, Van Hoolst & Sneyers (1992) applied this approximation to describe the low-frequency g-modes of non-rotating stars. Starting from the same equations as Smylie & Rochester (1981), they found an analytic expression for the asymptotic low frequencies which slightly differs from the standard one deduced by Tassoul (1980) in the Cowling approximation. By applying this expression to the low-frequency oscillations of both homogeneous and polytropic \((n = 3)\) star models, De Boek et al. (1992) concluded that the subseismic approximation is more accurate than the standard one in the polytropic case but less accurate in the homogeneous case.

However, as will be shown below, the subseismic approximation is not a consistent approximation of the equations of motion; we will show indeed that the proper way to simplify the equations in order to deal with the low frequency modes is to use the anelastic approximation and that this approximation also implies Cowling’s approximation.

Hence, after establishing the complete set of equations describing linear oscillations of a stratified compressible fluid and the appropriate boundary conditions (Section 2), we focus on the similarities and differences of the anelastic and subseismic approximations (Section 3); in particular, we show that both imply Cowling’s approximation and demonstrate the inconsistency of the subseismic approximation. Then we compare the two approximations on the test cases used by De Boek et al. (1992), namely, the homogeneous and polytropic star models (Section 4). Comparison of the resulting approximate eigenfrequencies with the exact ones (computed numerically for the polytrope but analytically for the homogeneous model) shows clearly the better efficiency of the anelastic approximation. Finally, we conclude in Section 5 with some outlooks of our results.

## 2 The Basic Equations

Assuming a time-dependence of the form \( \exp(it\tau) \), the governing equations that describe the adiabatic oscillations of a spherically non-rotating star are given by

\[
\rho' + \vec{\nabla} \cdot (\rho \vec{\rho}) = 0, \tag{1}
\]

\[
\sigma^2 \vec{\rho} = \vec{\nabla} \left( \frac{P'}{\rho} + \phi' \right) - \frac{N^2}{\rho g} \delta P \xi, \tag{2}
\]

\[
\delta P = c^2 \delta \rho, \tag{3}
\]

\[
\nabla^2 \phi' = 4\pi G \rho', \tag{4}
\]

where \( P', \rho' \) and \( \phi' \) respectively denote the Eulerian fluctuations of pressure, density and gravitational potential whereas \( \delta P, \delta \rho, \xi \) are the Lagrangian fluctuations of pressure, density and displacement such that

\[
\delta P = P' + \frac{dP}{dr} \xi, \quad \delta \rho = \rho' + \frac{d\rho}{dr} \xi, \tag{5}
\]

with a pressure gradient satisfying the hydrostatic equilibrium \( dP/dr = -\rho g \). Also, \( \rho \) is the equilibrium density, \( g \) the gravity and \( \gamma = (\partial \ln P/\partial \ln \rho)_S \) the first adiabatic exponent. Finally, \( c^2 \) and \( N^2 \) respectively denote the squares of sound speed velocity and Brunt-Väisälä frequency such as

\[
c^2 = \frac{\gamma P}{\rho}, \quad N^2 = \frac{g}{\rho} \left( \frac{1}{\gamma} \frac{d \ln P}{dr} - \frac{d \ln \rho}{dr} \right). \tag{6}
\]

In order to obtain a well-posed eigenvalue problem with an eigenvalue \( \sigma' \), boundary conditions should be stipulated at the star centre and surface. At the centre, we impose the regularity of all pulsation quantities such as (see e.g. Unno et al. 1989)

\[
\xi \propto r^{-1}, \quad P' \propto r^{\ell}, \quad \phi' \propto r^{\ell}, \tag{7}
\]

where \( \ell \) denotes the spherical harmonics degree of the eigenmode. At the surface, which is assumed to be free, we impose the following condition coupling \( \xi \) and \( \phi' \) (Ledoux and Walraven 1958)

\[
\left( \frac{d\phi'}{dr} \right)_R + \frac{\ell + 1}{R} \phi'(R) = -4\pi G(\rho \xi)_R. \tag{8}
\]

Finally, we adopt the classical mechanical outer condition for the pressure which reads

\[
\delta P = 0 \quad \text{at} \quad r = R. \tag{9}
\]

## 3 The Subseismic and Anelastic Approximations Applied to Stellar Oscillations

### 3.1 The common properties

Since the work of Cowling (1941), it is well known that g-modes are mainly transverse whereas p-modes are predominantly radial. The Eulerian pressure and density perturbations are thus small for a g-mode and both subseismic and anelastic approximations take advantage of that by assuming that the Lagrangian pressure fluctuation \( \delta P \) is only due to the radial displacement one; i.e. the Eulerian fluctuation \( P' \) does not contribute and we have

\[
\delta P = P' + \frac{dP}{dr} \xi \sim \frac{dP}{dr} \xi = -\rho g \xi. \tag{10}
\]

In this case, Eqs. (2), (3) and (4) read now

\[
\sigma^2 \xi = \nabla \left( \frac{P'}{\rho} + \phi' \right) + N^2 \xi \xi, \tag{11}
\]

\[
\rho' = \frac{N^2}{g} \rho \xi, \tag{12}
\]

\[
\nabla^2 \phi' = 4\pi G \frac{N^2}{g} \xi. \tag{13}
\]

At this stage, two important consequences appear:
the radial Lagrangian displacement necessarily vanishes at the surface. To show this, we take the radial component of Eq. (10)

\[ \sigma^2 \xi_r = \frac{\partial}{\partial r} \left( \frac{P'}{\rho} + \phi' \right) + N^2 \xi_r. \]

The problem arises from the fact that the Brunt-Väisälä frequency diverges as 1/c² at the star surface (see Eq. 13). Thus, as \( N^2 \) diverges to infinity at \( r = R \), \( \xi_r \) must vanish at this point to avoid a singular radial displacement. In the complete case (i.e. when acoustic waves are included), we note that it is the mechanical condition \( \delta P = 0 \) which permits a finite radial displacement at the surface, as shown by the radial component of Eq. (13).

\[ \sigma^2 \xi_r = \frac{\partial}{\partial r} \left( \frac{P'}{\rho} + \phi' \right) + N^2 \delta P, \]

where the term \( N^2/(\rho g) \) diverges as \( 1/(\rho c^2) \). Thus, the subseismic and anelastic \( \xi_r \)-eigenfunctions always have one node less than their corresponding unapproximated eigenfunctions for which \( \xi_r(R) \) is finite.

2. The second consequence, which is not straightforward, is that the use of the subseismic or anelastic approximation necessarily implies Cowling’s approximation where the perturbations \( \phi' \) are neglected (Cowling 1941). To show this, we first take the curl of Eq. (13) and obtain

\[ \sigma^2 \nabla \times \xi = \nabla \times (N^2 \xi_r, \xi_r). \]

Furthermore, combining Eqs. (13) and (14) allows us to find the following subseismic form for the equation of mass conservation

\[ \frac{N^2}{g} \rho \xi_r + \nabla \cdot (\rho \xi_r) = 0 \quad \Rightarrow \quad \nabla \cdot \xi = \frac{g}{\gamma} \xi_r, \]

whereas, as shown in the introduction, the anelastic counterpart of this equation reads

\[ \nabla \cdot (\rho \xi) = 0 \quad \Rightarrow \quad \nabla \cdot \xi = -\frac{d \ln \rho}{d \xi_r}. \]

The two different forms (13) and (14) may be formally written \( L(\xi) = 0 \) where \( L \) denotes a differential operator depending on the approximation we use. We have thus to deal with the new following system

\[ \sigma^2 \nabla \times \xi = \nabla \times (N^2 \xi_r, \xi_r), \]

\[ \frac{N^2}{g} \rho \xi_r + \nabla \cdot (\rho \xi_r) = 0 \quad \Rightarrow \quad \nabla \cdot \xi = \frac{g}{\gamma} \xi_r, \]

whereas the surface boundary conditions now read

\[ \xi_r(R) = 0 \quad \text{and} \quad \left( \frac{d \phi'}{d r} \right) + \frac{\ell + 1}{R} \phi'(R) = 0. \]

It is thus clear that the Poisson equation (17) decouples from Eqs. (13) and (14). In other words, the knowledge of Eqs. (15) and (16) with the boundary condition \( \xi_r(R) = 0 \) is sufficient to find the set of eigenfrequencies \( \sigma^2 \). The gravitational potential perturbation \( \phi' \) does not act on the determination of \( \sigma^2 \) and can be seen as ‘slaved’ to the \( \xi \), via Eq. (15). We note that Smylie \\& Rochester (1981) do not clearly emphasize this point and just noticed that the \( \phi' \)-perturbations can be determined after solving their subseismic equations for velocity and reduced pressure.

3.2 The main difference

As shown above, the subseismic and anelastic approximations do not lead to the same equation of mass conservation since the term stemming from the density perturbations is kept in the subseismic case (see Eqs. 13 and 14). We will now show that the subseismic form of this equation is in fact not correct because it is not consistent with the basic assumption from Eq. (10).

First of all, we recall that both approximations assume that the Eulerian pressure perturbation does not contribute to the Lagrangian one, that is

\[ \frac{P'}{P} \ll -\xi_r \frac{d \ln P}{d r} \quad \Leftrightarrow \quad \frac{P'}{P} \ll \frac{\xi_r}{H}, \]

where \( H = (-d \ln P/dr)^{-1} \) denotes the pressure scale height. Moreover, we can relate the Eulerian pressure and density perturbations using Eq. (1) as

\[ \frac{P'}{P} = \gamma \left( \frac{\rho'}{\rho} - \frac{N^2}{g} \xi_r \right), \quad \frac{N^2}{g} = \frac{1}{H} - \frac{1}{\gamma H}. \]

where \( H = (-d \ln \rho/dr)^{-1} \) denotes the density scale height and where Eq. (10) has also been used. Therefore the basic assumption from Eq. (8) leads to

\[ \frac{\rho'}{\rho} - \left( \frac{1}{H} - \frac{1}{\gamma H} \right) \xi_r \ll \frac{\xi_r}{\gamma H} \Rightarrow \frac{\rho'}{\rho} \ll \frac{\xi_r}{\gamma H}. \]

This last equation is important since it shows that we can also neglect the Eulerian contribution \( \rho' \) in the Lagrangian perturbation \( \delta \rho \) when Eq. (10) is assumed, i.e. we have

\[ \delta \rho = \rho' + \xi_r \frac{d \rho}{d r} \ll \xi_r \frac{d \rho}{d r}. \]

In other words, if we just take into account the contribution stemming from the equilibrium pressure gradient in \( \delta P \), we should also do the same with the contribution coming from the equilibrium density gradient in \( \delta \rho \). Hence

\[ \delta P \simeq \xi_r \frac{d P}{d r} \quad \text{is equivalent to} \quad \delta \rho \simeq \xi_r \frac{d \rho}{d r}. \]

We can now easily understand why the subseismic equation of mass conservation is not correct. Eq. (10) indeed gives

\[ \rho' + \nabla \cdot (\rho \xi) = 0 \quad \Rightarrow \quad \delta \rho + \rho \nabla \cdot \xi = 0. \]

In order to be consistent with the basic assumption from Eq. (20), the equation of mass conservation now reads

\[ \frac{d \xi_r}{d r} + \rho \nabla \cdot \xi = 0 \quad \Rightarrow \quad \nabla \cdot \xi = -\frac{d \ln \rho}{d r} \xi_r, \]

and we recover the good anelastic form (14).

The subseismic form of mass conservation (13) of Smylie \\& Rochester (1981) is therefore clearly inconsistent because

* It explains why Smeyers et al. (1995), starting from the complete case, found the same asymptotic development of \( \sigma^2 \) as those derived by Tassoul (1980) under the Cowling approximation.
of its incompatibility with the hypothesis \( \Gamma \). As a consequence, the subseismic approximation is necessarily less accurate than the anelastic one as we will now show.

## 4 RESULTS

In order to test the efficiency of both approximations, we applied the following procedure to the oscillations of the homogeneous compressible star (i.e. a polytrope of order 0) and the polytrope \( n = 3 \):

1. We first calculated the exact eigenfrequencies of the full case, that is we solved the set of equations (1-4) using boundary conditions (6-8). This has been achieved analytically for the polytrope \( n = 0 \) (Pekeris 1938) and numerically for the polytrope \( n = 3 \).

2. We next calculated the subseismic and anelastic eigenfrequencies by solving the system consisting in Eq. (15), the equation of mass conservation (13) or (14) and finally the boundary condition \( \xi_\ell (R) = 0 \) discussed in Section 3.1. Once again, analytical expressions have been used in the homogeneous case whereas the polytrope \( n = 3 \) has been computed numerically. The efficiency of both approximations has been deduced by a direct comparison between these approximated eigenfrequencies and the previous exact ones.

### 4.1 The homogeneous polytrope

As a first example, we concentrate on the unstable second-degree \( g^{-} \)-modes of the polytrope \( n = 0 \). Pekeris (1938) first studied the nonradial oscillations of this model and found the exact \( g^{-} \)-eigenfrequencies as

\[
\omega^2 = \Delta_{\ell k} - \sqrt{\Delta_{\ell k}^2 + \Lambda}, \quad (k = 0, 1, 2, \ldots),
\]

where \( \Lambda = \ell(\ell + 1) \) and

\[
\Delta_{\ell k} = \gamma \left[ k \left( \ell + \frac{5}{2} \right) + \ell + \frac{3}{2} \right] - 2.
\]

Here \( \omega^2 \) denote the dimensionless eigenfrequencies, which for a star of mass \( M \) and radius \( R \), are related to the \( \sigma^2 \)-ones by

\[
\sigma^2 = \frac{GM}{R^3} \omega^2.
\]

Using an equivalent method as the one used by Pekeris (i.e. the search of a power-serie solution), we calculated the analytic expressions of the subseismic and anelastic eigenfrequencies of the homogeneous polytrope and found (see appendix A)

\[
\omega_{\text{sub}}^2 = \frac{2}{\gamma} \frac{\Lambda}{\Lambda - (\ell + 2k') \left( \ell + 2k' + 1 + \frac{3}{2} \right)}, \quad (21)
\]

\[
\omega_{\text{anel}}^2 = \frac{2}{\gamma} \frac{\Lambda}{\Lambda - (\ell + 2k') \left( \ell + 2k' + 1 \right)}, \quad (22)
\]

where \( (k' = 1, 2, \ldots) \).

We summarized in Table 1 the exact eigenfrequencies calculated for the homogeneous star with \( \gamma = 5/3 \), \( \ell = 2 \) and various orders \( k \), whereas Fig. 1 gives the relative errors (in percents) made with both approximations for the first thirty \( g^{-} \)-modes. The anelastic approximation clearly appears to be more precise than the subseismic one since, for example, the anelastic eigenfrequency at \( k = 30 \) is about twenty times more precise than the subseismic one.

| \( \omega^2 \times 10^2 \) | \( \omega_{\text{sub}}^2 \times 10^2 \) | \( \omega_{\text{anel}}^2 \times 10^2 \) |
|-----------------|-----------------|-----------------|
| \( g_3^- \) | -7.251703 | -6.206896 | -6.923077 |
| \( g_5^- \) | -3.613671 | -3.260869 | -3.529411 |
| \( g_{10}^- \) | -1.221965 | -1.156069 | -1.212121 |
| \( g_{15}^- \) | -0.610579 | -0.587851 | -0.608108 |
| \( g_{20}^- \) | -0.365629 | -0.355239 | -0.364741 |
| \( g_{25}^- \) | -0.243308 | -0.237718 | -0.242914 |
| \( g_{30}^- \) | -0.173527 | -0.170180 | -0.173327 |

### 4.2 The polytrope \( n = 3 \)

#### 4.2.1 Numerics

We next calculated the second-degree \( g^{-} \)-mode of the polytrope \( n = 3 \). As already mentioned, analytic expressions of eigenfrequencies are not known for this model then we computed numerically the eigenfrequencies of \( g^{-} \)-modes with the complete set of equations and their approximated subseismic and anelastic counterparts. As in our preceding work (Dintrans, Rieutord & Valdettaro 1999; paper I), we used an iterative eigenvalue solver based on the incomplete Arnoldi-Chelyshev algorithm. This numerical method differs from the classical ones where eigenfrequencies are determined by a direct integration of the governing equations using either relaxation methods (Osaki 1975) or shooting methods (Hansen & Kawaler 1994). Our eigenvalue formulation is presented in appendix B, with a discussion on the tricky problem of the degeneracy of the eigenvalue equations at the star surface when the subseismic or anelastic approximation are used.

Eigenfrequencies \( \lambda^2 \) are solutions of a generalized eigenvalue problem.
This case, we found an analytic expression for the subseismic modes of the homogeneous star. In the anelastic case, the anelastic approximation appears to be superior to the subseismic one. This is illustrated by Fig. 2 where we show that the anelastic relative errors are always smaller than the subseismic ones. Still for \( k = 30 \), the agreement with the true eigenvalue is of about five times better with the anelastic approximation than with the subseismic one.

We note, however, that this accuracy difference tends to be less pronounced than for the homogeneous model. In fact, the comparison of Fig. 2 and 3 shows that the subseismic relative errors are the same in both cases whereas the anelastic ones increase in the polytropic case. Therefore dealing with a model with important density variations attenuates the differences between the two approximations.

We studied the properties of the subseismic and anelastic approximations devised for low-frequency g-modes of stars. These approximations both assume that the Eulerian perturbations of pressure do not contribute to their Lagrangian parts. As a consequence, acoustic waves are filtered out. However, the equation of mass conservation differs since perturbations of self-gravity will automatically decouple from the set of equations and will not influence the eigenfrequencies.

As already mentioned in the introduction, an obvious application of the anelastic approximation concerns the low-frequency g^-modes of a rapidly rotating \( \gamma \) Doradus-type star, for which periods of oscillations are of the same order as that of rotation (i.e. around one day).

ACKNOWLEDGEMENTS

BD has been supported by an ATER position at Université Paul Sabatier and now by the European Commission under Marie-Curie grant no. HPMF-CT-1999-00411 which are gratefully acknowledged.

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APPENDIX A: ANALYTIC EXPRESSIONS FOR THE HOMOGENEOUS STAR MODEL

A1 The subseismic case

We start from Eqs. (13) and (13)

\[
\begin{align*}
\sigma^2 \vec{\nabla} \times \xi &= \vec{\nabla} \times (N^2 \xi \bar{e_r}), \\
\vec{\nabla} \cdot \xi &= \frac{g}{c^2} \xi_r,
\end{align*}
\]

where \( g \), \( c^2 \) and \( N^2 \) are given by

\[
g(r) = \frac{4\pi}{3} G \rho r, \quad c^2 = \frac{2\pi}{3} G \rho R^2 \left(1 - \frac{r^2}{R^2}\right), \]

\[
N^2(r) = \frac{g}{\gamma} \frac{d \ln P}{dr} = \frac{g}{c^2}.
\]

The star radius is the length scale and the dynamical time \( T_{\text{dyn}} = (R^3/GM)^{1/2} \) is the time scale. In addition, we expand the spheroidal eigenvectors \( \bar{e}_r \) on spherical harmonics as (Rieutord 1987)

\[
\bar{e}_r = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m}^{(r)}(r) Y_{\ell m}(\theta, \phi) \bar{e}_r + \frac{v_{\ell m}^{(r)}}{r} \nabla Y_{\ell m}^m,
\]

where \( Y_{\ell m}(\theta, \phi) \) denotes the normalized spherical harmonics. We thus obtain the same radial equations coupling \( u_{\ell m}(r) \) and \( v_{\ell m}(r) \) as De Boeck et al. (1992), namely

\[
\begin{align*}
\frac{du_{\ell m}^r}{dr} &= \frac{N^2}{\omega^2} u_{\ell m}^r, \\
\frac{dv_{\ell m}^r}{dr} - \frac{g}{c^2} u_{\ell m}^r - \Lambda v_{\ell m}^r &= 0,
\end{align*}
\]

(A2)

where \( \Lambda = \ell (\ell + 1) \) and

\[
N^2 = -\frac{2}{\gamma - 1} \left(1 - \frac{r^2}{R^2}\right), \quad \frac{g}{c^2} = \frac{2}{\gamma - 1} \frac{r}{1 - r^2}.
\]

Eliminating \( v_{\ell m}^r \) leads to the equation for \( u_{\ell m}^r \) alone (for clarity, we drop the subscripts \( \ell \) and \( m \))

\[
Au + Av^2 u + Br^2 \frac{du}{dr} + \frac{2}{\gamma} \frac{r^3}{\gamma - 1} \frac{du}{dr} = 0,
\]

where

\[
A = \frac{2}{\gamma} \left(1 + \frac{1}{\omega^2}\right), \quad B = \frac{2}{\gamma} \left(1 - \frac{1}{\omega^2}\right) + A.
\]

We now expand \( u(r) \) in a power-series of \( r \) such as

\[
u(r) = \sum_{q=0}^{\infty} c_q r^q, \quad (q = \ell + 1, \ell + 3, \ldots),
\]

where we took into account that \( u(r) = r^2 \xi_r \propto r^{\ell+1} \) near the centre. We thus obtain the following difference equation for the \( c_q \) coefficients

\[
\begin{align*}
B - \frac{2}{\gamma} q(q - 1) c_q + \\
&\left[ A + \frac{2}{\gamma} (q + 2) + 2(q + 2)(q + 1) \right] c_{q+2} + \\
&\left[ \Lambda - (q + 4)(q + 3) \right] c_{q+4} = 0.
\end{align*}
\]

The convergence of the series implies that

\[
B - \frac{2}{\gamma} q(q - 1) = 0,
\]

which leads to the following expression of the subseismic eigenfrequencies

\[
\omega^2_{\text{sub}} = \frac{2}{\gamma} \left(\Lambda - (q - 1)(q + 2)\right).
\]

In order to have an easy connection with the eigenfrequencies of the unapproximated case, we adopt \( q = \ell + 1 + 2k' \) and finally obtain

\[
\omega^2_{\text{sub}} = \frac{2}{\gamma} \left(\Lambda - (\ell + 2k')(\ell + 2k' + 1 + \frac{2}{\gamma})\right),
\]

with \( (k' = 1, 2, \ldots) \).
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A2 The anelastic case

We start now from Eqs. (15) and (14)

\[
\begin{aligned}
\sigma^2 \nabla \times \tilde{\xi} &= \nabla \times \left( N^2 \xi_r \tilde{e}_r \right), \\
\nabla \cdot \tilde{\xi} &= - \frac{d \ln \rho}{dr} \xi_r = 0,
\end{aligned}
\]

where we used the fact that \( \rho \) is a constant for the homogeneous star. Thus, by applying the same formalism as above, we obtain the following anelastic radial equations for \( u \) and \( v \)

\[
\begin{aligned}
&u - r^2 \frac{dv}{dr} = \frac{N^2 \omega^2 u}{\omega^2}, \\
&\frac{du}{dr} - \Lambda v = 0,
\end{aligned}
\]

and the equation for \( u \) alone now reads

\[
\Lambda u - \Lambda r^2 u \left( 1 - \frac{2}{\gamma \omega^2} \right) - r^2 \frac{d^2 u}{dr^2} + r^4 \frac{d^2 u}{dr^4} = 0.
\]

As in the subseismic case, we look for a power-serie solution \( u(r) = \sum_{q} c_q r^q \) which leads to the following difference equation

\[
\left[ q(q - 1) + \Lambda \left( 1 - \frac{2}{\gamma \omega^2} \right) \right] c_q + \left[ \Lambda - (q + 2)(q + 1) \right] c_{q+2} = 0.
\]

The convergence of the serie requires that

\[
q(q - 1) + \Lambda \left( 1 - \frac{2}{\gamma \omega^2} \right) = 0,
\]

and we deduce the anelastic eigenfrequencies of the homogeneous star as

\[
\omega_{\text{anel}}^2 = \frac{2}{\gamma \Lambda} \frac{\Lambda}{q(q - 1)}.
\]

As before, we adopt \( q = \ell + 1 + 2k' \) and obtain

\[
\omega_{\text{anel}}^2 = \frac{2}{\gamma \Lambda} \frac{\Lambda}{(\ell + 2k')(\ell + 2k' + 1)},
\]

with \( (k' = 1, 2, \ldots) \).

APPENDIX B: THE EIGENVALUE PROBLEM FOR THE POLYTROPE \( N = 3 \)

In this appendix, we formulate the oscillation equations as a generalized eigenvalue problem and discuss the ‘degeneracy’ of these equations at the star surface when the subseismic or anelastic approximation is applied.

B1 The complete case

The governing equations we need to solve are Eqs. (1-4) with boundary conditions (6-8). The aim is to derive a generalized eigenvalue problem of the form

\[
\mathcal{M}_A \tilde{\xi} = \sigma^2 \mathcal{M}_B \tilde{\xi},
\]

where \( \mathcal{M}_A \) and \( \mathcal{M}_B \) denote two differential operators. Here \( \tilde{\xi} \) is the eigenvector associated with the eigenvalue \( \sigma^2 \); it may read, for instance,

\[
\tilde{\xi} = \begin{bmatrix}
\rho' \\
\delta P \\
\xi_r \\
\phi'
\end{bmatrix}
\]

These equations are clearly not well adapted for an eigenvalue problem formulation since Eqs. (1), (3) and (4) lead to three lines of zeros in the matrix \( \mathcal{M}_B \) making the system not well-conditioned for iterative solvers.

We therefore prefer to use the oscillation equations of Pekeris (1938) who obtained, after judicious substitutions, the following reduced system (its equations 12, 14 and 15; see also Hurley, Roberts & Wright 1966)

\[
\begin{aligned}
\sigma^2 w &= g \frac{dw}{dr} + \frac{dg}{dr} w - \epsilon^2 \frac{dX}{dr} - g(1 - \gamma) X - \frac{d\Psi}{dr}, \\
\sigma^2 w &= \frac{d^2 \Psi}{dr^2} + \frac{2}{r} \frac{d\Psi}{dr} - \Lambda \frac{\Psi}{r^2} - 4\pi G \left( \frac{w}{dr} + \rho X \right) = 0,
\end{aligned}
\]

where \( \epsilon = \nabla \cdot \tilde{e}_r \) (should not be confused with the dimensionless eigenfrequencies \( \omega \)), \( X = \nabla \cdot \tilde{v} \) and \( \Psi = i \sigma \psi \), \( \sigma \) being the gravitational potential perturbation.

In order to obtain the simplest dimensionless polytropic equations, we choose two new length and time scales. As length scale, we take \( R/x_1 \) whereas \( T_0 = (4\pi G \rho_c)^{-1/2} \) is our time scale. Here \( x_1 \) is related to \( P_c \) and \( \rho_c \) by (see e.g. Hansen & Kawaler 1994)

\[
\left( \frac{R}{x_1} \right)^2 = \frac{n + 1}{4\pi G \rho_c^2},
\]

where \( P_c \) and \( \rho_c \) respectively denote the central pressure and density of the polytrope. With these scales, we have for example the following relations

\[
\begin{aligned}
\sigma^2 &= 4\pi G \rho_c \Lambda^2 = \frac{x_1 GM}{q R^3} \omega^2, \\
\gamma^2 &= \left( \frac{R}{x_1} \right)^2 4\pi G \rho_c \frac{\gamma - 1}{n + 1},
\end{aligned}
\]

We note that for the polytrope \( n = 3 \), \( x_1 \approx 6.89685 \) and \( q \approx 4.24297 \times 10^{-2} \). Moreover, we recall that the function \( \theta(x) \) is solution of the Lane-Emden equation

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d\theta}{dx} \right) = -\theta^\nu, \quad \theta(0) = 1, \quad \theta(x_1) = 0.
\]
B2 The subseismic and anelastic cases

We have shown in Section 3.1 that the Cowling approximation is necessary when the subseismic or anelastic approximation are used; thus we now have to deal with

\[
\sigma^2 \tilde{\xi} = \nabla' \left( \frac{\nabla'}{\rho} \right) + N^2 \xi_r \hat{e}_r = \nabla \chi + N^2 \xi_r \hat{e}_r,
\]

\[
\nabla \cdot \tilde{\xi} = \frac{\partial \xi}{\partial r} \text{ or } \nabla \cdot \tilde{\xi} = -\frac{\partial \ln \rho}{\partial r} \xi_r,
\]

where we defined \( \chi = P'/\rho \). By using the same scales as above and eliminating \( v \) with Eq. (B9), we obtain the following dimensionless system for \( u \) and \( p \)

\[
\lambda^2 u = x^2 p' + \left( n - \frac{n + 1}{\gamma} \right) \frac{\theta^2}{\theta} u,
\]

\[
\lambda^2 \left( u' + \alpha \frac{\theta'}{\theta} u \right) = \Lambda p.
\]

Here the parameter \( \alpha \) differs according to the approximation used, namely

\[
\alpha = \frac{n + 1}{\gamma} : \text{subseismic}, \quad \alpha = \frac{n}{\gamma} : \text{anelastic},
\]

whereas \( p \equiv p'_{\text{eq}} \) is related to the projection of \( \chi \) on the spherical harmonics as

\[
\chi = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} p'_{\text{eq}} Y_{\ell}^m(\theta, \phi).
\]

At the surface, we have \( \theta(x_1) = 0 \) thus regular singularities appear in Eqs. (B10) and (B11). To remove them, a solution would consist in multiplying these two equations by \( \theta \) to obtain the new system

\[
\lambda^2 \theta u = x^2 p' + \left( n - \frac{n + 1}{\gamma} \right) \frac{\theta^2}{\theta} u,
\]

\[
\lambda^2 \left( u' + \alpha \frac{\theta'}{\theta} u \right) = \Lambda \theta p.
\]

The singularities indeed disappear but are now replaced by a surface degeneracy since each equation gives the same solution \( u = 0 \) at \( x = x_1 \) (which is however physically correct when using the subseismic or anelastic approximation; see Section 3.1). This is however making the discretized eigenvalue problem singular and this should be avoided.

Following Hurley et al. (1966), we preferred to use the Frobenius method. We first develop \( \theta(x) \) near \( x = x_1 \) as

\[
\theta(x) = \theta(x_1) + (x - x_1) \theta'(x_1) + O(x - x_1)^2,
\]

and, as \( \theta(x_1) = 0 \),

\[
\frac{\theta'}{\theta} \sim \frac{1}{x - x_1} \text{ for } x \rightarrow x_1.
\]

We next expand \( u(x) \) and \( p(x) \) in series of the form

\[
u(x) = \sum_{q=0}^{+\infty} a_q (x - x_1)^q, \quad p(x) = \sum_{q=0}^{+\infty} b_q (x - x_1)^q,
\]

and obtain, using Einstein’s notations,

\[
u' = qa_q (x - x_1)^{q-1}.
\]

Thus Eq. (B11) reads now

\[
\lambda^2 \left[ qa_q (x - x_1)^{q-1} + \alpha a_q (x - x_1)^{q-1} \right] = \Lambda b_q (x - x_1)^q.
\]

Equating the lowest power of \( (x - x_1) \), which is here \( (x - x_1)^{-1} \), gives again

\[
a_0 = 0 \iff u = 0,
\]

whereas the next power of \( (x - x_1) \) gives

\[
\lambda^2 (a_1 + \alpha a_1/a) = \Lambda b_1 \iff \lambda^2 (1 + \alpha)u' = \Lambda p.
\]

We therefore obtain the second boundary condition and the system we need to solve is

\[
\begin{cases}
\lambda^2 u = x^2 p' + \left( n - \frac{n + 1}{\gamma} \right) \frac{\theta^2}{\theta} u, \\
\lambda^2 \left( u' + \alpha \frac{\theta'}{\theta} u \right) = \Lambda p,
\end{cases}
\]

\[
\begin{aligned}
u = 0 \quad \text{and} \quad \lambda^2 (1 + \alpha)u' = \Lambda p & \quad \text{at} \quad x = x_1,
\end{aligned}
\]

which can be rewritten as (B15) with

\[
\tilde{\xi} = \begin{vmatrix} u \\ p \end{vmatrix}
\]