APPENDIX TO “TWISTING OF SIEGEL PARAMODULAR FORMS”

JENNIFER JOHNSON-LEUNG
BROOKS ROBERTS

Abstract. In this appendix we present an expanded version of Section 4 of our paper [JR], including the proofs of all of the technical lemmas.

1. Introduction

This paper is an appendix to our paper [JR]. Here, we present an expanded version Section 4 of [JR] including detailed proofs of the all of the technical lemmas. Throughout this appendix, we will use the following notation. Let \( F \) be a nonarchimedean local field of characteristic zero, with ring of integers \( \mathfrak{o} \) and generator \( \varpi \) of the maximal ideal \( p \) of \( \mathfrak{o} \). We let \( q \) be the number of elements of \( \mathfrak{o}/p \) and use the absolute value on \( F \) such that \( |\varpi| = q^{-1} \). We use the Haar measure on the additive group \( F \) that assigns \( \mathfrak{o} \) measure 1 and the Haar measure on the multiplicative group \( F^\times \) that assigns \( \mathfrak{o}^\times \) measure \( 1 - q^{-1} \). We \( \chi \) be a quadratic character of \( F^\times \) of conductor \( p \). Let

\[
J' = \begin{bmatrix}
1 & 1 \\
-1 & 1 \\
-1 & 1
\end{bmatrix}.
\]

For this section only, we define \( \text{GSp}(4,F) \) as the subgroup of all \( g \in \text{GL}(4,F) \) such that \( gJ'g = \lambda(g)J' \) for some \( \lambda(g) \in F^\times \) called the multiplier of \( g \). For \( n \) a non-negative integer, we let \( K(p^n) \) be the subgroup of \( k \in \text{GSp}(4,F) \) such that \( \lambda(k) \in \mathfrak{o}^\times \) and

\[
k \in \begin{bmatrix}
0 & 0 & 0 & p^{-n} \\
p^n & 0 & 0 & 0 \\
p^n & 0 & 0 & 0 \\
p^n & p^n & p^n & 0
\end{bmatrix}.
\]

Throughout this section, \( (\pi, V) \) is a smooth representation of \( \text{GSp}(4,F) \) for which the center acts trivially. If \( n \) is a non-negative integer, then \( V(n) \) is the subspace of vectors fixed by the paramodular subgroup \( K(p^n) \); also, we let \( V(n, \chi) \) be the subspace of vectors \( v \in V \) such that \( \pi(k)v = \chi(\lambda(k))v \) for \( k \in K(p^n) \). Finally, let

\[
\eta = \begin{bmatrix}
\varpi^{-1} & 0 \\
1 & 1 \\
\varpi & 1
\end{bmatrix}, \quad \tau = \begin{bmatrix}
1 & \varpi^{-1} \\
\varpi & \varpi \\
1 & 1
\end{bmatrix}, \quad t_4 = \begin{bmatrix}
1 & -\varpi^{-4} \\
\varpi^4 & 1
\end{bmatrix}.
\]

Usually, we will write \( \eta \) and \( \tau \) for \( \pi(\eta) \) and \( \pi(\tau) \), respectively.

In [JR1] we constructed a twisting map,

\[
T_\chi : V(0) \to V(4, \chi),
\]

(1)
given by

$$T_\chi(v) = q^3 \int \int \int \int \chi(ab) \pi(\begin{bmatrix} 1 & -a & 0 & bw^{-2} \\ x & 1 & -z & bw^{-2} \\ 0 & 0 & 1 & a^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}) \tau v \, da \, db \, dx \, dz$$ \quad (P1)

$$+ q^3 \int \int \int \int \chi(ab) \pi(\begin{bmatrix} 1 & 0 & 0 & bw^{-2} \\ -1 & 1 & -z & bw^{-1} \\ 0 & 0 & 1 & a^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}) \tau v \, da \, db \, dy \, dz$$ \quad (P2)

$$+ q^2 \int \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix} 1 & 0 & 0 & bw^{-2} \\ x & 1 & -z & bw^{-1} \\ 0 & 0 & 1 & a^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tau v \, da \, db \, dx \, dz$$ \quad (P3)

$$+ q^2 \int \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix} 1 & 0 & 0 & bw^{-2} \\ -1 & 1 & -z & bw^{-1} \\ 0 & 0 & 1 & a^{-1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tau v \, da \, db \, dy \, dz.$$ \quad (P4)

**Remark 1.1.** The Iwasawa decomposition asserts that $GSp(4, F) = B \cdot GSp(4, \mathfrak{o})$ where $B$ is the Borel subgroup of upper-triangular matrices in $GSp(4, F)$. Hence, if $v \in V(0)$ so that $v$ is invariant under $GSp(4, \mathfrak{o})$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices. The remainder of this section will be devoted to calculating formulas for the terms (P1), (P2), (P3), (P4) involving only upper-triangular matrices. The resulting formula for $T_\chi(v)$ is given in the following theorem. The proof of this theorem is spread over four sections of technical lemmas.

In some cases, we directly provide an Iwasawa identity $g = bk$ where $g \in GSp(4, F)$, $b \in B$, and $k \in GSp(4, \mathfrak{o})$. In many cases, we are able to obtain an appropriate Iwasawa identity by using the following formal matrix identity

$$\begin{bmatrix} 1 & \times^{-1} \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & \times^{-1} \\ -\times^{-1} & -1 \end{bmatrix} \begin{bmatrix} 1 & \times^{-1} \\ -1 & 1 \end{bmatrix}.$$

Both methods will require that we decompose the domains of integration in an advantageous manner. The assumptions on the character, $\chi \neq 1$, $\chi^2 = 1$ and $\chi(1 + p) = 1$, also play a significant role in the computations.

**Theorem 1.2.** Let $v \in V(0)$. Then the twisting operator (1) is given by the formula

$$T_\chi(v) = \sum_{k=1}^{14} T_{\chi}^k(v)$$

where

$$T_{\chi}^1(v) = q^2 \int \int \int \int \chi(ab) \pi(\begin{bmatrix} 1 & -(a + xb)w^{-1} \\ 1 & (a + xb)w^{-1} \end{bmatrix})$$
\begin{align*}
T_2^2(v) &= q\eta \int \int \int \int \chi(abxy)\pi\left( \begin{bmatrix}
1 & b\varpi^2 & z\varpi^{-4} \\
1 & x^{-1}\varpi^{-1} & b\varpi^{-2} \\
1 & 1 & 1
\end{bmatrix} \right) v da db dx dz, \\
T_3^3(v) &= \eta \int \int \int \int \chi(b(1-x))\pi\left( \begin{bmatrix}
1 & a\varpi^2 & b\varpi^{-3} \\
1 & a^2b^{-1}z\varpi^{-1} & a\varpi^{-2} \\
1 & 1 & 1
\end{bmatrix} \right) v da db dz, \\
T_4^4(v) &= q\eta \int \int \int \chi(b)\pi\left( \begin{bmatrix}
1 & x\varpi^{-2} & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \right) v da db dx, \\
T_5^5(v) &= \eta \int \int \int \int \chi(b)\pi\left( \begin{bmatrix}
1 & x\varpi^{-1} & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix} \right) v da db dx, \\
T_6^6(v) &= q^{-1}\eta^2 \int \int \int \chi(bx)\pi\left( \begin{bmatrix}
1 & a\varpi^2 & b(1-x\varpi)\varpi^{-2} \\
1 & a^2b^{-1}\varpi^{-2} & a\varpi^{-2} \\
1 & 1 & 1
\end{bmatrix} \right) v da db dx, \\
T_7^7(v) &= q\tau \int \int \int \chi(ab)\pi\left( \begin{bmatrix}
1 & -a\varpi^{-2} & 1 \\
1 & a\varpi^{-2} & 1 \\
1 & 1 & 1
\end{bmatrix} \right) v da db dz, \\
T_8^8(v) &= \eta\tau \int \int \int \chi(abz(1-z))\pi\left( \begin{bmatrix}
1 & b\varpi^{-2} & z\varpi^{-4} \\
1 & b\varpi^{-1} & b\varpi^{-1} \\
1 & 1 & 1
\end{bmatrix} \right) v da db dz,
\end{align*}
Proof. Substituting the formulas from Lemmas 2.1, 3.1, 4.8 and 5.1 we have that the twisting operator is given by the formula

\[
T_{\chi}^2(v) = q^{-2} \eta^2 \tau \int_{\no} \int \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-1} & 1 \\ 1 & 1 & -a \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & x \omega^{-1} & -b \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) v \, db \, dx,
\]

\[
T_{\chi}^{10}(v) = q^{-3} \eta^2 \tau^2 \int_{\no} \int \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & -b \omega^{-1} \\ 1 & 1 & -a \omega^{-2} \\ 1 & 1 & 1 \end{array} \right) v \, db,
\]

\[
T_{\chi}^{11}(v) = q^3 \tau^{-1} \int_{\no} \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b \omega^{-1} & 1 \\ 1 & 1 & -b \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & (xb + a) \omega^{-3} & z \omega^{-4} \\ 1 & -x \omega^{-2} & (xb + a) \omega^{-3} \\ 1 & 1 & 1 \end{array} \right) v \, db \, dx \, dz,
\]

\[
T_{\chi}^{12}(v) = q^2 \eta \tau^{-1} \int_{\no} \int \int \chi(abz(1 - z)) \pi \left( \begin{array}{ccc} 1 & a \omega^{-1} & 1 \\ 1 & 1 & -a \omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -a^{-1} & (a + x z) \omega^{-2} \\ 1 & 1 & (a + x z) \omega^{-2} \\ 1 & 1 & 1 \end{array} \right) v \, db \, dy \, dz,
\]

\[
T_{\chi}^{13}(v) = \eta^2 \tau^{-1} \int_{\no} \int \int \chi(bx) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & b(1 - x) \omega^{-1} \\ 1 & a^2 b^{-1} \omega^{-3} & a \omega^{-2} \\ 1 & 1 & 1 \end{array} \right) v \, db \, dx,
\]

\[
T_{\chi}^{14}(v) = q \eta^2 \tau^{-2} \int_{\no} \int \int \chi(b) \pi \left( \begin{array}{ccc} 1 & a \omega^{-2} & -b \omega^{-1} \\ 1 & x \omega^{-4} & a \omega^{-2} \\ 1 & 1 & 1 \end{array} \right) v \, db \, dx \, dz.
\]
\[ + q\chi(-1)\eta \int_{\mathfrak{o}^x-(1+p)\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(abx) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} b\mathfrak{w}^{-1} \\ 1 \\ -b\mathfrak{w}^{-1} \\ 1 \end{array} \right) v \, dx \, da \, db \, dz \]

\[ + \chi(-1)\eta \int_{\mathfrak{o}^x-(1+p)\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(b(1-z)) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} a\mathfrak{w}^{-2} \\ 1 \\ a\mathfrak{w}^{-2} \\ 1 \end{array} \right) v \, da \, db \, dz \]

\[ + q\eta \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(b) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} x\mathfrak{w}^{-2} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx \]

\[ + q\eta \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(ab) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right) v \, da \, db \, dy \]

\[ + q^{-1}\chi(-1)\eta^2 \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(bx) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ a^2 b^{-1} \mathfrak{w}^{-2} \\ b(1+x) \mathfrak{w}^{-2} \\ a \mathfrak{w}^{-2} \end{array} \right) v \, da \, db \, dx \]

\[ + q\tau \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(ab) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right) v \, da \, db \, dz \]

\[ + \eta\tau \int_{\mathfrak{o}^x-(1+p)\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(abz(1-z)) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ 1 \end{array} \right) v \, da \, db \, dz \]

\[ + q^{-1}\chi(-1)\eta^2 \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(b) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ a^2 b^{-1} \mathfrak{w}^{-2} \\ b(1+x) \mathfrak{w}^{-2} \\ a \mathfrak{w}^{-2} \end{array} \right) v \, da \, db \, dx \]

\[ + q^{-1}\chi(-1)\eta^2 \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \int_{\mathfrak{o}^x} \chi(b) \pi \left( \begin{array}{c} 1 \\ 1 \end{array} \begin{array}{c} 1 \\ a^2 b^{-1} \mathfrak{w}^{-2} \\ b(1+x) \mathfrak{w}^{-2} \\ a \mathfrak{w}^{-2} \end{array} \right) v \, da \, db \, dx \]
\[ \begin{align*}
+ q^{-3} \chi(-1) \eta^2 \tau^2 & \int_{\partial^\times} \int_{\partial^\times} \chi(b) \pi \left( \begin{bmatrix}
1 & a \varpi^{-2} & b \varpi^{-1} \\
1 & 1 & -a \varpi^{-2} \\
1 & 1 & 1
\end{bmatrix} \right) v \, da \, db \\
+ q^3 \tau^{-1} & \int_{\partial^\times} \int_{\partial^\times} \int_{\partial^\times} \chi(ab) \pi \left( \begin{bmatrix}
1 & b \varpi^{-1} \\
1 & 1 & -b \varpi^{-1} \\
1 & 1 & 1
\end{bmatrix} \right) v \, da \, db \, dx \\
+ q^2 \tau^{-1} & \int_{\partial^\times} \int_{\partial^\times} \int_{\partial^\times} \chi(ab) \pi \left( \begin{bmatrix}
1 & b \varpi^{-1} \\
1 & 1 & -b \varpi^{-1} \\
1 & 1 & 1
\end{bmatrix} \right) v \, da \, db \, dy \\
+ q \eta \tau^{-1} & \int_{\partial^\times-(1+p)} \int_{\partial^\times} \int_{\partial^\times} \int_{\partial^\times} \chi(ab) \pi \left( \begin{bmatrix}
1 & -a \varpi^{-1} \\
1 & 1 & a \varpi^{-1} \\
1 & 1 & 1
\end{bmatrix} \right) v \, da \, db \, dy \\
+ q^2 \chi(-1) \eta \tau^{-1} & \int_{\partial^\times-(1+p)} \int_{\partial^\times} \int_{\partial^\times} \int_{\partial^\times} \chi(ba(1-z)z) \pi \left( \begin{bmatrix}
1 & b \varpi^{-1} \\
1 & 1 & -b \varpi^{-1} \\
1 & 1 & 1
\end{bmatrix} \right) v \, dx \, da \, db \\
+ \chi(-1) \eta^2 \tau^{-1} & \int_{\partial^\times} \int_{\partial^\times} \int_{\partial^\times} \chi(bx) \pi \left( \begin{bmatrix}
1 & \frac{a \varpi^{-2}}{1 + a^{-1} b} & \frac{b(x - 1) \varpi^{-1}}{1 + a^{-1} b} \\
1 & 1 & \frac{a \varpi^{-2}}{1 + a^{-1} b} \\
1 & 1 & 1
\end{bmatrix} \right) v \, da \, db \, dx
\end{align*} \]
\[
q(-1)\eta^2\tau^{-2} \int \int \int \chi(b) \pi \left( \begin{array}{ccc}
1 & a\varpi^{-2} & b\varpi^{-1} \\
1 & x\varpi^{-4} & a\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dx \\
+ \eta^2\tau^{-2} \int \int \int \chi(a) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-2} & -a\varpi^{-1} \\
1 & y\varpi^{-3} & b\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dy \\
\]

For the remainder of the proof, we will simplify by combining pairs of terms and rewriting certain domains. First we combine the terms involving \(\eta^2\tau^{-2}\).

\[
q(-1)\eta^2\tau^{-2} \int \int \int \chi(b) \pi \left( \begin{array}{ccc}
1 & a\varpi^{-2} & b\varpi^{-1} \\
1 & x\varpi^{-4} & a\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dx \\
+ \eta^2\tau^{-2} \int \int \int \chi(a) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-2} & -a\varpi^{-1} \\
1 & y\varpi^{-3} & b\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dy \\
= q(-1)\eta^2\tau^{-2} \int \int \int \chi(b) \pi \left( \begin{array}{ccc}
1 & a\varpi^{-2} & b\varpi^{-1} \\
1 & x\varpi^{-4} & a\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dx \\
+ q(-1)\eta^2\tau^{-2} \int \int \int \chi(a) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-2} & a\varpi^{-1} \\
1 & y\varpi^{-4} & b\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dy \\
= q(-1)\eta^2\tau^{-2} \int \int \int \chi(b) \pi \left( \begin{array}{ccc}
1 & a\varpi^{-2} & b\varpi^{-1} \\
1 & x\varpi^{-4} & a\varpi^{-2} \\
1 & \ \\
\end{array} \right) v \, da \, db \, dx.
\]

Similarly, we combine the terms involving \(\eta\tau^{-1}\),

\[
q\eta\tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & -a\varpi^{-1} \\
1 & \ \\
1 & \ \\
\end{array} \right) v \, da \, db \, dy \, dz \\
+ q^2\chi(-1)\eta\tau^{-1} \int \int \int \int \chi(ba(1-z)z) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-1} \\
1 & \ \\
1 & \ \\
\end{array} \right)
\]
\[
\begin{bmatrix}
1 & -x\omega^{-3} & b(x - za\omega)\omega^{-4} \\
1 & b^{-1}(x + a\omega)\omega^{-2} & -x\omega^{-3} \\
1 & 1 & 1 \\
\end{bmatrix}v \, dx \, da \, db \, dz
\]

\[
= q^{2}\chi(-1)\eta^{-1} \int \int \int \int \chi(\omega)\left(\begin{array}{c}
1 & b\omega^{-1} \\
1 & 1 - b\omega^{-1} \\
1 & 1 \\
\end{array}\right) \left(\begin{array}{c}
\omega^{-1} \\
1 \\
1 \\
\end{array}\right) \, dv \, da \, db \, dy \, dz,
\]

the two terms involving \( \tau^{-1} \),

\[
q^{2}\tau^{-1} \int \int \int \int \chi(\omega)\left(\begin{array}{c}
1 & b\omega^{-1} \\
1 & 1 - b\omega^{-1} \\
1 & 1 \\
\end{array}\right) \left(\begin{array}{c}
\omega^{-1} \\
1 \\
1 \\
\end{array}\right) \, dv \, da \, db \, dx \, dz
\]

and two of the terms that involve the \( \eta \) operator,

\[
q\eta \int \int \int \chi(\omega)\left(\begin{array}{c}
1 & x\omega^{-2} \\
1 & 1 - x\omega^{-2} \\
1 & 1 \\
\end{array}\right) \left(\begin{array}{c}
\omega^{-2} \\
1 \\
1 \\
\end{array}\right) \, dv \, da \, db \, dx
\]
We rewrite one of the terms involving $\eta$ after making the observation that if $z \in \mathfrak{o}^\times - (1 + p)$ and $f$ is a locally constant function on $\mathfrak{o}^\times$, then

$$\int_{\mathfrak{o}^\times - A(z)} f(x) dx = \int_{\mathfrak{o}^\times - (1 + p)} f((z^{-1} - 1)(w^{-1} - 1)^{-1}) dw.$$ 

Hence

$$q\chi(-1)\eta \int \int \int \int \chi(abx)\pi(\begin{bmatrix} 1 & b\varpi^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -b\varpi^{-1} \end{bmatrix}) v dx da db dz$$

$$= q\chi(-1)\eta \int \int \int \int \chi(abz(1 - z)w(1 - w))\pi(\begin{bmatrix} 1 & b\varpi^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -b\varpi^{-1} \end{bmatrix}) v dw da db dz$$

$$= q\chi(-1)\eta \int \int \int \int \chi(az^{-1}(1 - z)bw^{-1}(1 - w))\pi(\begin{bmatrix} 1 & b\varpi^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -b\varpi^{-1} \end{bmatrix}) v dw da db dz$$
\[
q \chi(-1) \eta \int \int \int \int \chi(a(z^{-1} - 1)b(w^{-1} - 1)) \pi \begin{bmatrix}
1 & b\omega^{-1} \\
1 & 1 & -b\omega^{-1}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \omega
\end{bmatrix}
\] 
\[
= q \chi(-1) \eta \int \int \int \int \chi(a(z - 1)b(w - 1)) \pi \begin{bmatrix}
1 & b\omega^{-1} \\
1 & 1 & -b\omega^{-1}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \omega
\end{bmatrix}
\] 
\[
= q \chi(-1) \eta \int \int \int \int \chi(abxy) \pi \begin{bmatrix}
1 & b\omega^{-1} \\
1 & 1 & -b\omega^{-1}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 
\end{bmatrix}
\] 
Finally, we are able to eliminate the factor \( \chi(-1) \) from all terms using an appropriate change of variables. Substituting the simplified terms into the formula for \( T_\chi(v) \), we obtain the result. \( \square \)

2. Calculation of the First Part (P1)

Lemma 2.1. If \( v \in V(0) \), then we have that (P1) is given by

\[
q^3 \int \int \int \int \chi(ab) \pi \begin{bmatrix}
1 & b\omega^{-1} \\
1 & x & 1
\end{bmatrix}
\begin{bmatrix}
1 & -a\omega^{-1} & b\omega^{-2} & z\omega^{-4} \\
1 & 1 & b\omega^{-2} & a\omega^{-1}
\end{bmatrix}
\]
Proof. In this proof we use the methods described in Remark 1.1 to obtain an upper triangular form of summand (P1) of the operator $T\chi$. 

\[
q^3 \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & -a\omega^{-2} & 1 \\ 1 & a\omega^{-2} & 1 \\ 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz + q^3 \tau^{-1} \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & b\omega^{-1} & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz \\
+ q^2 \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & -(a + xb)\omega^{-1} & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz.
\]

\[
q^3 \tau \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & -a\omega^{-1} & b\omega^{-2} & z\omega^{-4} \\ x & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz + q^3 \tau \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & a\omega^{-2} & b\omega^{-1} & z\omega^{-4} \\ 1 & x\omega^{-2} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz \\
+ q^3 \tau \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & -(a + bx\omega^{-1})\omega^{-2} & b\omega^{-1} & z\omega^{-4} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz \\
+ q^3 \tau \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & x^{-1}\omega^2 & -x\omega^{-2} & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz \\
+ q^3 \tau \int \int \int \chi(ab)\pi(\begin{bmatrix} 1 & (b + ax^{-1}\omega)\omega^{-1} & a\omega^{-2} & z\omega^{-4} \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix})v \, da \, db \, dz.
\]
\[ q \int_0^\infty \int_0^\infty \pi \left( \begin{array}{ccc}
-\alpha \omega^{-2} & b \omega^{-1} & z \omega^{-4} \\
1 & 1 & b \omega^{-1} \\
1 & a \omega^{-2} & 1
\end{array} \right) v \, da \, db \, dz + q^3 \int_0^\infty \int_0^\infty \int_0^\infty \pi \left( \begin{array}{ccc}
1 & x^{-1} \omega^2 & 1 \\
1 & 1 & -x^{-1} \omega \\
1 & 1 & -x \omega^{-2}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ + q^3 \int_0^\omega \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & x^{-1} \omega^2 & 1 \\
1 & 1 & -x^{-1} \omega \\
1 & 1 & -x \omega^{-2}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ + q^3 \int_0^\infty \int_0^\infty \int_0^\infty \pi \left( \begin{array}{ccc}
1 & x^{-1} \omega^2 & 1 \\
1 & 1 & -x^{-1} \omega \\
1 & 1 & -x \omega^{-2}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ = q \int_0^\infty \int_0^\infty \pi \left( \begin{array}{ccc}
1 & -a \omega^{-2} & 1 \\
1 & 1 & a \omega^{-2} \\
1 & 1 & 1
\end{array} \right) v \, da \, db \, dz + q^3 \int_0^\omega \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & b \omega^{-1} & z \omega^{-4} \\
1 & 1 & b \omega^{-1} \\
1 & 1 & b \omega^{-1}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ + q^3 \int_0^\infty \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & b \omega^{-1} & z \omega^{-4} \\
1 & 1 & b \omega^{-1} \\
1 & 1 & b \omega^{-1}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ + q^2 \int_0^\infty \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & x^{-1} \omega & 1 \\
1 & 1 & -x^{-1} \omega \\
1 & 1 & -x \omega^{-1}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ + q^2 \int_0^\omega \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & x^{-1} \omega & 1 \\
1 & 1 & -x^{-1} \omega \\
1 & 1 & -x \omega^{-1}
\end{array} \right) v \, da \, db \, dx \, dz 
\]

\[ = q \int_0^\infty \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & -a \omega^{-2} & 1 \\
1 & 1 & a \omega^{-2} \\
1 & 1 & 1
\end{array} \right) v \, da \, db \, dx \, dz + q^3 \int_0^\omega \int_0^\omega \int_0^\omega \pi \left( \begin{array}{ccc}
1 & b \omega^{-1} & z \omega^{-4} \\
1 & 1 & b \omega^{-1} \\
1 & 1 & b \omega^{-1}
\end{array} \right) v \, da \, db \, dx \, dz 
\]
\[
q \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -a \varpi -2 \\
1 & a \varpi -2 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
b \varpi -1 & z \varpi -4 \\
1 & b \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dz
\]

\[
q^3 \tau -1 \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -x \varpi -2 \\
1 & x \varpi -2 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
z \varpi -4 \\
-x \varpi -2 \\
1
\end{array} \right] \left[ \begin{array}{ccc}
xb \varpi -1 \\
1 & xb \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dx \, dz
\]

\[
q^2 \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -x \varpi -1 \\
1 & x \varpi -1 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
z \varpi -4 \\
-x \varpi -2 \\
1
\end{array} \right] \left[ \begin{array}{ccc}
ya \varpi -1 \\
1 & ya \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dx \, dz
\]

\[
q \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -a \varpi -2 \\
1 & a \varpi -2 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
b \varpi -1 & z \varpi -4 \\
1 & b \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dz
\]

\[
q^3 \tau -1 \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -x \varpi -2 \\
1 & x \varpi -2 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
z \varpi -4 \\
-x \varpi -2 \\
1
\end{array} \right] \left[ \begin{array}{ccc}
ya \varpi -1 \\
1 & ya \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dx \, dz
\]

\[
q^2 \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -x \varpi -1 \\
1 & x \varpi -1 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
z \varpi -4 \\
-x \varpi -2 \\
1
\end{array} \right] \left[ \begin{array}{ccc}
ya \varpi -1 \\
1 & ya \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dx \, dz
\]

\[
q \int \int \int \int \chi(ab) \pi \left[ \begin{array}{ccc}
1 & -a \varpi -2 \\
1 & a \varpi -2 \\
1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
b \varpi -1 & z \varpi -4 \\
1 & b \varpi -1 \\
1 & 1
\end{array} \right] v da \, db \, dz
\]
\[+
q^3 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-1} & 1 \\
1 & 1 & -b\varpi^{-1} \\
1 & 1 & 1 \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
(b - xa\varpi)x^{-1}\varpi^{-3} (z + x^{-1}b^2 - ba\varpi)\varpi^{-4} \\
-(b - xa\varpi)x^{-1}\varpi^{-3} \\
1 \\
\end{array} \right) \] 
v \, da \, db \, dx \, dz \]

\[+
q^2 \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & 1 & (xb + a)\varpi^{-1} \\
1 & 1 & 1 \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
(b\varpi^{-2} z\varpi^{-4}) \\
(x - x^{-1}\varpi^{-2} b\varpi^{-2}) \\
1 \\
\end{array} \right) \] 
v \, da \, db \, dx \, dz \]

\[= q\tau \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & -(a\varpi^{-2}) & 1 \\
1 & 1 & a\varpi^{-2} \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
1 & b\varpi^{-1} & z\varpi^{-4} \\
1 & 1 & b\varpi^{-1} \\
\end{array} \right) \] 
v \, da \, db \, dz \]

\[+
q^3 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & b\varpi^{-1} & 1 \\
1 & 1 & -b\varpi^{-1} \\
1 & 1 & 1 \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
-(b - xa\varpi)x^{-1}\varpi^{-3} (z + x^{-1}b^2 - ba\varpi)\varpi^{-4} \\
-(b - xa\varpi)x^{-1}\varpi^{-3} \\
1 \\
\end{array} \right) \] 
v \, da \, db \, dx \, dz \]

\[+
q^2 \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & 1 & (xb + a)\varpi^{-1} \\
1 & 1 & 1 \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
(b\varpi^{-2} z\varpi^{-4}) \\
(x - x^{-1}\varpi^{-2} b\varpi^{-2}) \\
1 \\
\end{array} \right) \] 
v \, da \, db \, dx \, dz \]

\[= q\tau \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc}
1 & -(a\varpi^{-2}) & 1 \\
1 & 1 & a\varpi^{-2} \\
\end{array} \right) \] 
\[ \left( \begin{array}{c}
1 & b\varpi^{-1} & z\varpi^{-4} \\
1 & 1 & b\varpi^{-1} \\
\end{array} \right) \] 
v \, da \, db \, dz \]
\[+ q^3 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b\omega^{-1} & 1 \\ 1 & 1 & -b\omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dx \, dz\]

\[+ q^2 \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & -(a+xb)\omega^{-1} & 1 \\ 1 & 1 & (a+xb)\omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dx \, dz\]

This completes the calculation of the first term (P1).

\[\square\]

3. CALCULATION OF THE SECOND PART (P2)

Lemma 3.1. If \( v \in V(0) \), then we have that (P2) is given by

\[q^3 \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b\omega^{-1} & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dy \, dz\]

\[= q^2 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b\omega^{-1} & 1 \\ 1 & 1 & -b\omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dy \, dz\]

Proof. This calculation is straightforward and uses only the invariance of \( v \) under GSp(4, \( \mathfrak{o} \)). We have

\[q^3 \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b\omega^{-2} & \omega^{-4} \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dy \, dz\]

\[= q^2 \tau^{-1} \int \int \int \int \chi(ab) \pi \left( \begin{array}{ccc} 1 & b\omega^{-2} & \omega^{-4} \\ 1 & 1 & a\omega^{-1} \\ 1 & 1 & 1 \end{array} \right) \tau \, da \, db \, dy \, dz\]
This completes the calculation of the second term \((P2)\). □

4. Calculation of the Third Part \((P3)\)

The calculation of the \((P3)\) term is the most delicate. We begin this calculation with a preparatory lemma which breaks the term into four pieces. The majority of this section will be devoted to handling the first of these terms.

**Lemma 4.1.** If \(v \in V(0)\), then we have that \((P3)\) is given by

\[
q^2 \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix} 1 & -a \omega^{-2} & b \omega^{-1} & z \omega^{-4} \\ 1 & b \omega^{-1} & a \omega^{-2} & 1 \\ 1 & a \omega^{-1} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tau v \, da \, db \, dx \, dz.
\]
\[\begin{align*}
q^2 \int \int \int \chi(b) \eta \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz \\
+ q \int \int \int \int \chi(b) \eta \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz \\
+ \int \int \int \int \chi(b) \eta \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz \\
+ q^{-1} \int \int \int \int \chi(b) \eta \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz.
\end{align*}\]

**Proof.** In the following computation, we again use the invariance of \(v\) under GSp(4, \(\mathfrak{o}\)) together with the useful identity (2).

\[\begin{align*}
q^2 \int \int \int \chi(ab) \pi(t_4) & \vdash da \ db \ dx \ dz \\
= q^2 \int \int \int \int \chi(ab) \eta^4 \tau \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz \\
= q^2 \int \int \int \int \chi(ab) \eta^4 \tau \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz \\
= q^2 \int \int \int \int \chi(ab) \eta^4 \tau \pi(1, 1, 1, 1, 1, 1, 1, 1) & \vdash da \ db \ dx \ dz.
\end{align*}\]
\[
\begin{align*}
\begin{bmatrix}
1 & b^{-1} \bar{\omega} \\
1 & 1 & -b^{-1} \bar{\omega} \\
-1 & 1 & -b^{-1} \bar{\omega} \\
1 & 1 & 1 & 1
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
q^2 \int \int \int \int \chi(ab) \eta^3 \tau^2 \pi \left( \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} \\
1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} & \bar{\omega}^{-1} \\
1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} & \bar{\omega}^{-1}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} \\
1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} & \bar{\omega}^{-1} \\
1 & \bar{\omega}^{-1} & \bar{\omega}^{-1} & \bar{\omega}^{-1}
\end{bmatrix}
\begin{bmatrix}
a^{-1} \bar{\omega}^2 & a^{-1} \bar{\omega}^2 \\
a^{-1} \bar{\omega}^2 & a^{-1} \bar{\omega}^2 \\
a^{-1} \bar{\omega}^2 & a^{-1} \bar{\omega}^2 \\
a^{-1} \bar{\omega}^2 & a^{-1} \bar{\omega}^2
\end{bmatrix}
\right)
\end{align*}
\]

\[
\begin{align*}
v da db dx dz
\end{align*}
\]
\[
= q^2 \int \int \int \chi(ab) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^3 & 1 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
= q^2 \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ \omega^3 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
= q^2 \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^4 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
= q^2 \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^5 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
+ q \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^4 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
+ \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^5 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
+ q^{-1} \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^6 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz.
\]

This completes the preliminary calculation of the third term. 

The following three lemmas calculate the straightforward terms from the decomposition in Lemma 4.1. The strategy in each case is to conjugate by the integral lower triangular matrix, use the invariance of \( v \) under \( \text{GSp}(4, \mathfrak{o}) \), and apply appropriate changes of variables.

**Lemma 4.2.** If \( v \in V(0) \), then we have that (4) is given by

\[
q \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^4 & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
= (q - 1) \int \int \int \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 \\ \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 \end{array} \right) v \, da \, db \, dx.
\]
Proof. We calculate (4) as follows:

\[ q \int_0^\infty \int_0^\infty \int_0^\infty \chi(b) \eta \pi \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & a^{\frac{1}{2}} & b^{\frac{1}{2}} & -x^{\frac{1}{2}} \\
\begin{bmatrix}
1 & b
\end{bmatrix} & \begin{bmatrix}
a & b & a & b
\end{bmatrix} & \begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
\end{bmatrix} v \, da \, db \, dx \, dz 
\]

\[ = q \int_0^\infty \int_0^\infty \int_0^\infty \chi(b) \eta \pi \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & a^{\frac{1}{2}} & b^{\frac{1}{2}} & -x^{\frac{1}{2}} \\
\begin{bmatrix}
1 & b
\end{bmatrix} & \begin{bmatrix}
a & b & a & b
\end{bmatrix} & \begin{bmatrix}
1 & 1 & 1 & 1
\end{bmatrix}
\end{bmatrix} v \, da \, db \, dx \, dz 
\]
Lemma 4.3. If \( v \in V(0) \), then we have that (5) is given by
\[
\int \int \int \int \chi(b) \eta \pi(\begin{bmatrix} 1 & x(1 + z \varpi)^{-2} & a(1 + z \varpi)^{-2} & b(1 + z \varpi)^{-3} \\ 1 & 1 & a(1 + z \varpi)^{-2} & -x(1 + z \varpi)^{-2} \\ 1 & 1 & a(1 + z \varpi)^{-2} & -x(1 + z \varpi)^{-2} \\ 1 & 1 & a(1 + z \varpi)^{-2} & -x(1 + z \varpi)^{-2} \end{bmatrix}) v \ d a \ d b \ d x \ d z
\]
\[
= (q - 1) \int \int \int \chi(b) \eta \pi(\begin{bmatrix} 1 & x \varpi^{-2} & a \varpi^{-2} & b \varpi^{-3} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \end{bmatrix}) v \ d a \ d b \ d x.
\]

This completes the calculation. \( \square \)

Proof. We calculate (5) as follows:
\[
\int \int \int \int \chi(b) \eta \pi(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}) v \ d a \ d b \ d x \ d z
\]
\[
= \int \int \int \int \chi(b) \eta \pi(\begin{bmatrix} 1 & x \varpi^{-2} & a \varpi^{-2} & b \varpi^{-3} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \end{bmatrix}) v \ d a \ d b \ d x \ d z
\]
\[
= (1 - q^{-1}) \int \int \int \chi(b) \eta \pi(\begin{bmatrix} 1 & x \varpi^{-2} & a \varpi^{-2} & b \varpi^{-3} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \\ 1 & 1 & a \varpi^{-2} & -x \varpi^{-2} \end{bmatrix}) v \ d a \ d b \ d x.
\]
This completes the calculation.

**Lemma 4.4.** If $v \in V(0)$, then we have that (6) is given by

\[
q^{-1} \int_{\partial \phi} \int_{\partial \phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, db \, dx \, dz
\]

\[
=q^{-1} \int_{\partial \phi} \int_{\partial \phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, da \, db \, dx.
\]

**Proof.** We calculate (6) as follows:

\[
q^{-1} \int_{\partial \phi} \int_{\partial \phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
=q^{-1} \int_{\partial \phi} \int_{\partial \phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, da \, db \, dx \, dz
\]

\[
=q^{-1} \int_{\partial \phi} \int_{\partial \phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, da \, db \, dx.
\]

This completes the calculation.

Finally, to calculate the remaining term (3) of (P3) in Lemma 4.1, let

\[
D(a, b) = \begin{bmatrix} 1 & ab^{-1} \\ ab^{-1} & a^{-1} \\ a^{-1} & b^{-1} \end{bmatrix},
\]

for $a, b \in F^\times$. Then, we decompose (3) into two further terms as follows:

\[
q^2 \int_{\phi} \int_{\phi} \int_{\phi} \chi(b) \eta \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ x^{-2} & a^{-2} & b^{-3} \\ \frac{a^{-2}}{1} & \frac{b^{-3}}{1} & \frac{a^{-2}}{1} \end{array} \right) v \, da \, db \, dx \, dz
\]
\[= q^2 \int \int \int \chi(-b) \eta \pi \left( \begin{array}{cc} 1 & 1 \\ \bar{z} \bar{w}^{3} & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ b \bar{w}^{3} & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]

\[= q^2 \chi(-1) \int \int \int \chi(b) \eta \pi(D(a, b)) \left( \begin{array}{cc} 1 & 1 \\ b \bar{z} \bar{w}^{3} & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ -ab^{-1} x \bar{w}^{-2} & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]

\[= \chi(-1) q^2 \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left( \begin{array}{ccc} 1 & 1 & 1 \\ \bar{z} \bar{w}^{3} & 1 & \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ -ab^{-1} x \bar{w}^{-2} & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]

\[= q^2 \chi(-1) \int \int \int \chi(b) \eta \pi(D(a, b)) \left( \begin{array}{ccc} 1 & 1 & 1 \\ \bar{z} \bar{w}^{3} & 1 & \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ -ab^{-1} x \bar{w}^{-2} & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]

\[+ q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left( \begin{array}{ccc} 1 & 1 & 1 \\ \bar{z} \bar{w}^{3} & 1 & \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ 1 & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]

**Lemma 4.5.** If \( v \in V(0) \), then we have that (7) is given by

\[q^2 \chi(-1) \int \int \int \chi(b) \eta \pi(D(a, b)) \left( \begin{array}{ccc} 1 & 1 & 1 \\ \bar{z} \bar{w}^{3} & 1 & \end{array} \right) \left( \begin{array}{ccc} 1 & x \bar{w}^{-2} & -b \bar{w}^{-3} \\ \bar{w}^{2} & a \bar{w}^{-2} & \frac{a \bar{w}^{-2}}{1} \\ 1 & 1 & \end{array} \right) v \, \text{d}a \, \text{d}b \, \text{d}x \, \text{d}z \]
\[
q^2 \chi(-1) \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \chi(ba(1-z)z) \eta \pi^{-1} \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
b_\omega^{-1} & 1 \\
b_\omega^{-1} & 1 \\
-a_\omega^{-1} & 1 \\
-a_\omega^{-1} & 1
\end{array} \right)v \, dx \, da \, db \, dz \\
+ \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \chi(abz(1-z)) \eta \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
a_\omega^{-1} & 1 \\
a_\omega^{-1} & 1 \\
ab^{-1} & 1 \\
ab^{-1} & 1
\end{array} \right)v \, dx \, da \, db \, dz \\
+ \chi(-1) \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \int_{\sigma^x} \chi(b(1-z)) \eta \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
a_\omega^{-2} & 1 \\
a_\omega^{-2} & 1 \\
-a_\omega^{-1}(1+x-z) & 1 \\
-a_\omega^{-1}(1+x-z) & 1
\end{array} \right)v \, dx \, da \, db \, dz \\
+
\]

\[= q^2 \int_{\sigma^x} \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
x_\omega^{-2} & 1 \\
x_\omega^{-2} & 1 \\
-x_\omega^{-2} & 1 \\
-x_\omega^{-2} & 1
\end{array} \right)v \, dx
\]

\[= q^2 \int_{\sigma^x} \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
x_\omega^{-1} & 1 \\
x_\omega^{-1} & 1 \\
-x_\omega^{-1} & 1 \\
-x_\omega^{-1} & 1
\end{array} \right)v \, dx
\]

\[+ q \int_{\sigma^x} \pi \left( \begin{array}{c}
1 \\
1 \\
1 \\
1 \\
x_\omega^{-3} & 1 \\
x_\omega^{-3} & 1 \\
-x_\omega^{-3} & 1 \\
-x_\omega^{-3} & 1
\end{array} \right)v \, dx.
\]

\[\text{Proof.}\] We first consider only the integration over the \( x \) variable and break the integral into several pieces.
We now consider the integral (9):

\[
q^2 \int_{\phi^\times} \pi \left( \begin{bmatrix} 1 & 1 \\ z \tau^3 & 1 \end{bmatrix} \right) \left[ \begin{bmatrix} 1 & x \tau^{-2} & -\tau^{-3} \\ 1 & x \tau^{-2} & -\tau^{-3} \\ \tau^{-2} & \tau^{-2} & \tau^{-2} \end{bmatrix} \right] v \, dx
\]

\[
= q^2 \int_{\phi^\times} \pi \left( \begin{bmatrix} 1 & 1 \\ z \tau^3 & 1 \end{bmatrix} \right) \left[ \begin{bmatrix} 1 & (zx + (1 - z)\tau)^{-1}\tau & -(1 - (z)\tau)^{-1}\tau & (zx + (1 - z)\tau)^{-1}\tau^{-3} \\ 1 & z(x + (1 - z)\tau)(1 - z)^{-2}\tau^2 & 1 \end{bmatrix} \right] v \, dx
\]

\[
= q^2 \int_{\phi^\times} \pi \left( \begin{bmatrix} 1 & 1 \\ z \tau^3 & 1 \end{bmatrix} \right) \left[ \begin{bmatrix} 1 & (zx + (1 - z)\tau)^{-1}\tau & -(1 - (z)\tau)^{-1}\tau & (zx + (1 - z)\tau)^{-1}\tau^{-3} \\ 1 & z(x + (1 - z)\tau)(1 - z)^{-2}\tau^2 & 1 \end{bmatrix} \right] v \, dx
\]

\[
= q^2 \int_{\phi^\times} \pi \left( \begin{bmatrix} 1 & 1 \\ z \tau^3 & 1 \end{bmatrix} \right) \left[ \begin{bmatrix} 1 & (zx + (1 - z)\tau)^{-1}\tau & -(1 - (z)\tau)^{-1}\tau & (zx + (1 - z)\tau)^{-1}\tau^{-3} \\ 1 & z(x + (1 - z)\tau)(1 - z)^{-2}\tau^2 & 1 \end{bmatrix} \right] v \, dx
\]
Next, we consider the integral (10). For \( z \in o^\infty - (1 + p) \), it will be helpful to consider the following set \( A(z) = -z^{-1}(1 - z) + p \) in order to apply the useful identity (2). We calculate (10) as follows:

\[
q \int_{o^\infty} \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} x & -z^{-2} \omega^3 & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-3} \\ 1 & -x \omega^{-1} & 1 \end{array} \right) vdx \\
= q \int_{A(z)} \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} x & -z^{-2} \omega^3 & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-3} \\ 1 & -x \omega^{-1} & 1 \end{array} \right) vdx \\
+ q \int_{o^\infty - A(z)} \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} x & -z^{-2} \omega^3 & \omega^{-3} \\ 1 & \omega^{-2} & \omega^{-3} \\ 1 & -x \omega^{-1} & 1 \end{array} \right) vdx \\
= \int_{o} \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -z^{-2}(1 - z) \omega^{-1} & \omega^{-3} \\ 1 & \omega^{-2} & -(1 - x \omega) \omega^{-3} \\ 1 & z^{-1}(1 - z) \omega^{-1} & 1 \end{array} \right) vdx \\
+ q \int_{o^\infty - A(z)} \pi \left( \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & -z(1 - z + zx)^{-1} \omega^{-2} & \omega^{-3} \\ 1 & \omega^{-2} & (x - 1)(1 - z + z x)^{-1} \omega^{-3} \\ 1 & (1 - z + zx)^{-1} \omega^{-1} & 1 \end{array} \right) vdx \\
\left[ \begin{array}{ccc} 1 & (1 - z)^{-1} \omega^{-1} \\ 1 & 1 - (1 - z)^{-1} \omega^{-1} \\ 1 & 1 - (1 - z)^{-1} \omega^{-1} \\ \omega^{-1} & 1 & (1 - z)^{-1} \\ 1 & 1 - (1 - z)^{-1} \omega^{-1} \\ 1 & 1 - (1 - z)^{-1} \omega^{-1} \end{array} \right] vdx
\]
$$\int_0^\pi \pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$

$$+ q \int_0^\pi \pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} (1-z+zx)^{-1} & (x-1)(1-z+zx)^{-1} & 1 \\ \frac{1}{\omega^3} & (1-z+zx)^{-1} & \omega^{-1} \end{array} \right] v dx$$

$$= \int_0^\pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$

$$= \int_0^\pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$

$$+ q \int_0^\pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$

$$+ q \int_0^\pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$

$$+ q \int_0^\pi \left[ \begin{array}{ccc} 1 & 1 & \frac{1}{\omega^3} \\ \frac{1}{\omega} & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & -z^{-1}(1-z) & 1 \\ 1 & \omega^{-2} & -\frac{1}{\omega^2} \\ 1 & z^{-1}(1-z) & 1 \end{array} \right] v dx$$
\[
\int_0^\pi \frac{1}{(1 - z)^{-1} x_\omega} \left( 1 - (1 - z)^{-1} x_\omega^{-1} \right) v dx
\]

\[
= \int_0^\pi \pi(0) \left( \begin{array}{ccc}
1 & -z^{-1}(1 - z)^{-1} & -z(1 - x_\omega)^{-2} \\
1 & -z^{-1}(1 - z)^{-1} & x^{-1} - (1 - x_\omega)^{-3} \\
1 & -z^{-1}(1 - z)^{-1} & 1
\end{array} \right) v dx
\]

\[
+ q \int_{\phi^*-A(z)} \pi(0) \left( \begin{array}{ccc}
1 & (1 - z)^{-1} x_\omega^{-1} \\
1 & -z^{-1}(1 - z)^{-1} x_\omega^{-1} \\
1 & -z^{-1}(1 - z)^{-1} x_\omega^{-1} \\
\end{array} \right) v dx
\]

\[
= \int_0^\pi \pi(\tau) \left( \begin{array}{ccc}
1 & (1 - z)(x_\omega - 1) \frac{z(x_\omega - 1)}{(1 - z)^2} \frac{-z(x_\omega - 1)^2}{(1 - z^2)(x_\omega - 1)} \\
1 & z(x_\omega - 1) \frac{(1 - z) x_\omega}{(1 - z)^2} \frac{-z(x_\omega - 1)^2}{(1 - z^2)(x_\omega - 1)} \\
1 & (1 - z)(x_\omega - 1) \frac{z(x_\omega - 1)}{(1 - z)^2} \frac{-z(x_\omega - 1)^2}{(1 - z^2)(x_\omega - 1)} \\
\end{array} \right) v dx
\]

\[
+ q \int_{\phi^*-A(z)} \pi(0) \left( \begin{array}{ccc}
1 & (1 - z + z x)^{-1} x^{-2} \\
1 & -z(1 - z + z x)^{-1} x^{-1} \\
1 & (1 - z + z x)^{-1} x^{-1} \\
\end{array} \right) v dx
\]
\[= \int_0^\pi \pi(\tau) \begin{bmatrix} 1 & -(1 - z)z^{-1}(1 - z + xz\varpi)^{-1} \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 - z)z^{-1}(1 - z + xz\varpi)^{-1} \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 - z + xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 + z^2(x\varpi - 1)^2 - 2z + xz\varpi)(1 - z + xz\varpi)^{-2} \omega^{-3} \\ 1 & 1 \end{bmatrix} v \, dx \]

\[+ q \int_{\sigma^x - A(z)} \pi(\tau) \begin{bmatrix} 1 & (1 - z - xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z(1 - z + xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} v \, dx \]

\[= \int_0^\pi \pi(\tau) \begin{bmatrix} 1 & -(1 - z)z^{-1}(1 - z + xz\varpi)^{-1} \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 - z)z^{-1}(1 - z + xz\varpi)^{-1} \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 - z + xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & (1 + z^2(x\varpi - 1)^2 - 2z + xz\varpi)(1 - z + xz\varpi)^{-2} \omega^{-3} \\ 1 & 1 \end{bmatrix} v \, dx \]

\[+ q \int_{\sigma^x - A(z)} \pi(\tau) \begin{bmatrix} 1 & (1 - z - xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -z(1 - z + xz\varpi)^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} v \, dx.\]
\[
\int \pi \left( \begin{array}{ccc}
1 & (1 - z - z x \omega)^{-1} \omega^{-2} & -(1 + x \omega)(1 - z - z x \omega)^{-1} \omega^{-3} \\
1 & -z(1 - z - z x \omega)^{-1} \omega^{-1} & (1 - z - z x \omega)^{-1} \omega^{-2}
\end{array} \right) v dx
\]

Returning to the integral (7), we have

\[
q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & z \omega^3 & 1
\end{array} \right] v d a d b d x d z
\]

\[
q^2 \chi(-1) \int \int \int \int \chi(b) \eta^{-1} \pi(D(a, b)) \left[ \begin{array}{ccc}
1 & (z x + (1 - z) \omega)^{-1} \omega^{-2} & (x - \omega)(z x + (1 - z) \omega)^{-1} \omega^{-3} \\
1 & -z(z x + (1 - z) \omega)^{-1} \omega^{-2} & (z x + (1 - z) \omega)^{-1} \omega^{-2}
\end{array} \right]
\]

\[
\int \pi \left( \begin{array}{ccc}
1 & x(1 - z)^{-1} \omega^{-1} & 1 \\
1 & -x(1 - z)^{-1} \omega^{-1} & 1
\end{array} \right) v d x d a d b d z
\]

\[
+ \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b))
\]

where \( \omega = x - y \), \( a = y - z \), and \( b = z + y \).
\[
\begin{pmatrix}
1 & -(1 - z) z^{-1} (1 - z + x z \overline{w})^{-1} \overline{w}^{-2} \\
1 & (1 - z) z^{-1} (1 - z + x z \overline{w})^{-1} \overline{w}^{-2} \\
1 & \overline{w}^{-1} (1 + z^2 (x \overline{w} - 1)^2 - 2 z + x z \overline{w}) (1 - z + x z \overline{w})^{-2} \overline{w}^{-3} \\
1 & \overline{w}^{-1} (1 - z + x z \overline{w})^{-1} \overline{w}^{-1}
\end{pmatrix} v \, dx \, da \, db \, dz
\]

\[+ q \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \begin{pmatrix}
1 & (1 - z) z^{-1} \overline{w}^{-1} \\
1 & -z (1 - z + x z) \overline{w}^{-1} \\
1 & -z (1 - z - z x \overline{w})^{-1} \overline{w}^{-1} \\
1 & (1 - z) \overline{w}^{-2}
\end{pmatrix} v \, dx \, da \, db \, dz
\]

\[+ \chi(-1) \int \int \int \int \chi(b) \int \eta \pi(D(a, b)) \begin{pmatrix}
1 & (1 - z - z x \overline{w})^{-1} \overline{w}^{-2} \\
1 & -(1 + x \overline{w}) (1 - z - z x \overline{w})^{-1} \overline{w}^{-3} \\
1 & -(1 - z - z x \overline{w})^{-1} \overline{w}^{-2} \\
1 & (1 - z)^{-1} \overline{w}^{-2}
\end{pmatrix} v \, dx \, da \, db \, dz
\]

\[= q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi^{-1} \begin{pmatrix}
1 & a (z x + (1 - z) \overline{w})^{-1} \overline{w}^{-2} \\
1 & -a^2 b^{-1} z (z x + (1 - z) \overline{w})^{-1} \overline{w}^{-2} \\
1 & a (z x + (1 - z) \overline{w})^{-1} \overline{w}^{-2} \\
1 & -a^{-1} b \overline{w} (1 - z)^{-1} \overline{w}^{-1}
\end{pmatrix} v \, dx \, da \, db \, dz
\]

\[+ \chi(-1) \int \int \int \int \chi(b) \eta \pi \begin{pmatrix}
1 & -a^{-1} b \overline{w} (1 - z)^{-1} \overline{w}^{-1} \\
1 & 1
\end{pmatrix} v \, dx \, da \, db \, dz
\]

\[+ \chi(-1) \int \int \int \int \chi(b) \eta \pi \begin{pmatrix}
1 & -a^{-1} b (1 - z) \overline{w}^{-1} (1 - z + x z \overline{w})^{-1} \overline{w}^{-2} \\
1 & 1 \overline{a^2} b (1 - z)^{-1} \overline{w}^{-2}
\end{pmatrix}
\]
\[
\begin{align*}
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\int \int \int \int \chi(b)\eta\pi(1 - a^{-1}b(1 - z)^{-1}x^{-1})v dx \, da \, db \, dz \\
\end{align*}
\]
\[\begin{align*}
&+ q\chi(-1) \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \chi(b) \eta \pi(\begin{bmatrix} 1 & a^{-1}b(1-z)^{-2}x\omega^{-1} & 1 & -a^{-1}b(1-z)^{-2}x\omega^{-1} \\ 1 & -a^2(1-z)^2b^{-1}z(1-z + zx)^{-1}\omega^{-1} & a\omega^{-2} & b(z - x - 1)(1-z)^{-2}\omega^{-3} \end{bmatrix}) v \, dx \, da \, db \, dz \\
&+ \chi(-1) \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \chi(b) \eta \pi(\begin{bmatrix} 1 & a\omega^{-2} & -b(1 + x\omega)(1-z - zx\omega)^{-1}\omega^{-3} \\ 1 & -a^2b^{-1}z(1-z - zx\omega)\omega^{-1} & a\omega^{-2} \\ 1 & -ab^{-1}(1-z)^{-1}zx\omega^{-2} & a\omega^{-2} \end{bmatrix}) v \, dx \, da \, db \, dz \\
&= q^2\chi(-1) \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \chi(ba(1-z)z) \eta \pi^{-1} \pi(\begin{bmatrix} 1 & a\omega^{-2} & ba(1-z)x^{-1}(x - \omega)\omega^{-3} \\ 1 & -a^2b^{-1}z(1-z - zx\omega)\omega^{-1} & b(1 + x\omega)(1-z - zx\omega)^{-1}\omega^{-3} \\ 1 & -ab^{-1}(1-z)^{-1}zx\omega^{-2} & -ab^{-1}(1-z)^{-1}\omega^{-3} \end{bmatrix}) v \, dx \, da \, db \, dz \\
&+ \chi(-1) \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \chi(b) \eta \pi(\begin{bmatrix} 1 & -a^{-1}b(1-z)z^{-1}\omega^{-2} \\ 1 & -ab^{-1}(1-z)^{-1}zx\omega^{-2} \\ 1 & a^{-1}b(1-z)^{-1}\omega^{-2} \end{bmatrix}) v \, dx \, da \, db \, dz \\
&+ q\chi(-1) \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \int_{\phi \times -A(z)} \chi(b) \eta \pi(\begin{bmatrix} 1 & a^{-1}bx\omega^{-1} \\ 1 & -a^2b^{-1}z(1-z)^{-2}b^{-1}z(1-z + zx)^{-1}\omega^{-1} & a\omega^{-2} & b(z - x - 1)(1-z)^{-2}\omega^{-3} \end{bmatrix}) v \, dx \, da \, db \, dz
\end{align*}\]
\[
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & \frac{b(z-x-1)\varpi^{-3}}{a\varpi^{-2}} \\
1 & -a^2 b^{-1} z (1 - z + zx)^{-1} \varpi^{-1} & 1 \\
1 & \varpi^{-2} & 1
\end{bmatrix}
\int v \, dx \, da \, db \, dz
\]

\[+ \chi(-1) \int \int \int \chi(b(1-z)) \eta \pi \left( 
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & -b(1+x\varpi)\varpi^{-3} \\
1 & -a^2 b^{-1} z \varpi^{-1} \varpi^{-3} & 1 \\
1 & -b\varpi^{-1} & 1
\end{bmatrix}
\right) \int v \, dx \, da \, db \, dz
\]

\[= q^2 \chi(-1) \int \int \int \int \chi(ba(1-z)) \eta \pi \left( 
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & \frac{ba(1-z-x^{-1}\varpi)\varpi^{-3}}{a\varpi^{-2}} \\
1 & -ab^{-1} x z \varpi^{-2} \varpi^{-3} & 1 \\
1 & -b\varpi^{-1} & 1
\end{bmatrix}
\right) \int v \, dx \, da \, db \, dz
\]

\[+ \chi(-1) \int \int \int \int \chi(b(1-z)) \eta \pi \left( 
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & \frac{abx^{-1}(z-x-1)\varpi^{-3}}{a\varpi^{-2}} \\
1 & -ab^{-1} x z (1 - z + zx)^{-1} \varpi^{-1} \varpi^{-3} & 1 \\
1 & -b\varpi^{-1} & 1
\end{bmatrix}
\right) \int v \, dx \, da \, db \, dz
\]

\[+ q \chi(-1) \int \int \int \int \chi(abx) \eta \pi \left( 
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & \frac{a^2 b^{-1} z \varpi^{-1} \varpi^{-3}}{a\varpi^{-2}} \\
1 & -a^2 b^{-1} z \varpi^{-1} \varpi^{-3} & 1 \\
1 & -b\varpi^{-1} & 1
\end{bmatrix}
\right) \int v \, da \, db \, dz
\]

\[= q^2 \chi(-1) \int \int \int \int \chi(ba(1-z)z) \eta \pi \left( 
\begin{bmatrix}
1 & \frac{a\varpi^{-2}}{} & \frac{ba(1-z)z \varpi^{-1} \varpi^{-3}}{a\varpi^{-2}} \\
1 & -a^2 b^{-1} z \varpi^{-1} \varpi^{-3} & 1 \\
1 & -b\varpi^{-1} & 1
\end{bmatrix}
\right) \int v \, da \, db \, dz
\]
\[
\begin{aligned}
&\left[ \begin{array}{ccc}
1 & a(xz + \wp)\wp^{-3} & -ba(1 + x^{-1}\wp)(xz + \wp)\wp^{-4} \\
1 & -ab^{-1}xz\wp^{-2} & a(xz + \wp)\wp^{-3} \\
1 & 1 & 1 \\
\end{array} \right] v \ dx \ da \ db \ dz \\
+ & \int_{\wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \chi(abz(1 - z)) \eta \tau \pi \left( \begin{array}{ccc}
1 & b\wp^{-2} & 1 \\
1 & 1 & -b\wp^{-2} \\
1 & 1 & 1 \\
\end{array} \right) v \ dx \ da \ db \ dz \\
& \left[ \begin{array}{ccc}
a\wp^{-1} & -ab(1 - z)^{-1}(1 - z)^2 + (x^2z^2\wp - 2xz^2 + xz)\wp^{-3} \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right] v \ dx \ da \ db \ dz \\
+ & q\chi(-1) \int_{\wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \chi(abx) \eta \pi \left( \begin{array}{ccc}
1 & b\wp^{-1} & 1 \\
1 & 1 & -b\wp^{-1} \\
1 & 1 & 1 \\
\end{array} \right) v \ dx \ da \ db \ dz \\
& \left[ \begin{array}{ccc}
a\wp^{-2} & -ab^{-1}(1 + x - z)\wp^{-3} \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right] v \ dx \ da \ db \ dz \\
+ & \chi(-1) \int_{\wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \chi(b(1 - z)) \eta \pi \left( \begin{array}{ccc}
1 & a\wp^{-2} & -b\wp^{-3} \\
1 & -a^2b^{-1}z\wp^{-1} & a\wp^{-2} \\
1 & 1 & 1 \\
\end{array} \right) v \ dx \ da \ db \ dz \\
= & q^2\chi(-1) \int_{\wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \chi(ba(1 - z)z) \eta \tau^{-1} \pi \left( \begin{array}{ccc}
1 & b\wp^{-1} & 1 \\
1 & 1 & -b\wp^{-1} \\
1 & 1 & 1 \\
\end{array} \right) v \ dx \ da \ db \ dz \\
& \left[ \begin{array}{ccc}
-x\wp^{-3} & -bx(1 + az(x - a\wp)^{-1}\wp)\wp^{-4} \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right] v \ dx \ da \ db \ dz \\
+ & \int_{\wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \int_{a^{-1} \wp^{-1}} \chi(abz(1 - z)) \eta \tau \pi \left( \begin{array}{ccc}
1 & b\wp^{-2} & 1 \\
1 & 1 & -b\wp^{-2} \\
1 & 1 & 1 \\
\end{array} \right) v \ dx \ da \ db \ dz \\
& \left[ \begin{array}{ccc}
a(1 - z)^{-2}\wp^{-1} & -ab(1 - z)^{-1}\wp^{-3} \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \right] v \ dx \ da \ db \ dz
\end{aligned}
\]
\[
+ q\chi(-1) \int_{\omega^x-(1+p)} \int_{\omega^x} \int_{\omega^x} \int_{\omega^x} \chi(abx) \eta \pi\left( \begin{array}{cc} 1 & b\omega^{-1} \\ 1 & -b\omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & \omega^{-3} \\ 1 & \omega^{-4} \end{array} \right) v \, dx \, da \, db \, dz
\]

\[
= q^2\chi(-1) \int_{\omega^x-(1+p)} \int_{\omega^x} \int_{\omega^x} \int_{\omega^x} \chi(b(1-z)z) \eta \tau \pi\left( \begin{array}{cc} 1 & b\omega^{-1} \\ 1 & -b\omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & \omega^{-3} \\ 1 & \omega^{-4} \end{array} \right) v \, dx \, da \, db \, dz
\]

\[
+ \int_{\omega^x-(1+p)} \int_{\omega^x} \int_{\omega^x} \int_{\omega^x} \chi(abz(1-z)) \eta \tau \pi\left( \begin{array}{cc} 1 & b\omega^{-2} \\ 1 & -b\omega^{-2} \end{array} \right) \left( \begin{array}{cc} 1 & \omega^{-3} \\ 1 & \omega^{-4} \end{array} \right) v \, dx \, da \, db \, dz
\]

\[
+ q\chi(-1) \int_{\omega^x-(1+p)} \int_{\omega^x} \int_{\omega^x} \int_{\omega^x} \chi(abx) \eta \pi\left( \begin{array}{cc} 1 & b\omega^{-1} \\ 1 & -b\omega^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & \omega^{-3} \\ 1 & \omega^{-4} \end{array} \right) v \, dx \, da \, db \, dz
\]

\[
+ \chi(-1) \int_{\omega^x-(1+p)} \int_{\omega^x} \int_{\omega^x} \int_{\omega^x} \chi(b(1-z)) \eta \pi\left( \begin{array}{cc} 1 & a\omega^{-2} \\ 1 & -a^2b^{-1}\omega^{-2} \end{array} \right) \left( \begin{array}{cc} 1 & \omega^{-3} \\ 1 & \omega^{-4} \end{array} \right) v \, da \, db \, dz
\]
\[ q^2 \chi(-1) \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(ba(1-z)z)\eta\tau^{-1}\pi \begin{bmatrix} \frac{1}{b^3} & 1 & 1-b^3 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi \begin{bmatrix} -x^3 & b(x-za^3) & -x^3 \\ 1 & b^3(x+a^3) & -x^3 \\ 1 & 1 & 1 \end{bmatrix} \] \[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abz(1-z))\eta\tau\pi \begin{bmatrix} \frac{1}{b^2} & 1 & 1-b^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abx)\eta\pi \begin{bmatrix} \frac{1}{b^2} & 1 & 1-b^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi \begin{bmatrix} -x^3 & b(x-za^3) & -x^3 \\ 1 & b^3(x+a^3) & -x^3 \\ 1 & 1 & 1 \end{bmatrix} \] \[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abz(1-z))\eta\tau\pi \begin{bmatrix} \frac{1}{a^2} & 1 & 1-a^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abx)\eta\pi \begin{bmatrix} \frac{1}{a^2} & 1 & 1-a^2 \end{bmatrix} \]

This completes the calculation. \[ \square \]

The identities from the next lemma will be used in the calculation of the remaining term (8).

**Lemma 4.6.** Let \( v \in V(0) \) and \( z \in 1+p \).

1. Assume that \( x \in \text{o}^\times \). Then:

\[ \pi \begin{bmatrix} \frac{1}{x^3} & 1 & 1 \\ \frac{1}{x^3} & 1 & 1 \end{bmatrix} \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi -(1)z)\eta\pi \begin{bmatrix} \frac{1}{x^2} & 1 & 1-a^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abx)\eta\pi \begin{bmatrix} \frac{1}{x^2} & 1 & 1-a^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi \begin{bmatrix} \frac{1}{x^3} & 1 & 1 \\ \frac{1}{x^3} & 1 & 1 \end{bmatrix} \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi -(1)z)\eta\pi \begin{bmatrix} \frac{1}{a^2} & 1 & 1-a^2 \end{bmatrix} \]

\[ \int_{\text{o}^\times - (1+p) \text{o}^\times \text{o}^\times \text{o}^\times} \chi(abx)\eta\pi \begin{bmatrix} \frac{1}{a^2} & 1 & 1-a^2 \end{bmatrix} \]
(2) Assume that \( x \in \mathfrak{o}^\times \). Then:
\[
\pi \left( \begin{bmatrix} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \omega^{-1} & \omega^{-2} & -\omega^{-3} \\ 1 & 1 & -x & -\omega^{-1} \end{bmatrix} \right) v = \pi \left( \begin{bmatrix} \omega^{-1} & 1 \\ \omega & \omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & (1 - z + zx)^{-1} \omega^{-2} & -z^{-2}(1 - z + x)^{-1} \omega^{-1} + z^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \right) v.
\]

(3) Assume that \( x \in (1 - z)^{-1} \omega^{-1} + \mathfrak{o}^\times \). Then:
\[
\pi \left( \begin{bmatrix} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \omega^{-2} & -\omega^{-3} \\ 1 & 1 & -x \omega^{-1} \end{bmatrix} \right) v = \pi \left( \begin{bmatrix} \omega^{-1} & 1 \\ \omega & \omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} w^{-1} \omega^{-2} & (1 + zw)z^{-2} w^{-1} \omega^{-2} \\ 1 & 1 \end{bmatrix} \right) v
\]
where \( w = (1 - x \omega - z^{-1}) \omega^{-1} \in \mathfrak{o}^\times \).

(4) Assume that \( x \in (1 - z)^{-1} \omega^{-1} + \omega \mathfrak{o}^\times \). Then:
\[
\pi \left( \begin{bmatrix} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \omega^{-2} & -\omega^{-3} \\ 1 & 1 & -x \omega^{-1} \end{bmatrix} \right) v = \pi \left( \begin{bmatrix} \omega^{-1} & 1 \\ \omega & \omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & w^{-1} \omega^{-1} & z^{-1} \omega^{-1} \\ 1 & 1 \end{bmatrix} \right) v
\]
where \( w = (1 - x \omega - z^{-1}) \omega^{-2} \in \mathfrak{o}^\times \).

(5) Assume that \( x \in (1 - z)^{-1} \omega^{-1} + \mathfrak{p}^2 \). Then:
\[
\pi \left( \begin{bmatrix} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & x \omega^{-2} & -\omega^{-3} \\ 1 & 1 & -x \omega^{-1} \end{bmatrix} \right) v = \pi \left( \begin{bmatrix} \omega^{-1} & 1 \\ \omega & \omega^{-1} \end{bmatrix} \begin{bmatrix} 1 & -z^{-1} \omega^{-2} & \omega^{-1} \\ 1 & 1 \end{bmatrix} \right) v.
\]
Proof. To prove the first assertion, we note that we have the matrix identity

\[
\begin{bmatrix}
1 & 1 & \bar{x}^{-2} & \bar{x}^{-3} \\
1 & 1 & \bar{x}^{-2} & -1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(\bar{x}z)^{-1} & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & -z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(\bar{x}z)^{-1} & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & -z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(\bar{x}z)^{-1} & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & -z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(\bar{x}z)^{-1} & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & -z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(\bar{x}z + (1 - z)\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\end{bmatrix}
\]

where the last matrix is in GSp(4, \sigma).

For the second assertion, we have a similar identity

\[
\begin{bmatrix}
1 & 1 & \bar{x}^{-1} & \bar{x}^{-3} \\
1 & 1 & \bar{x}^{-2} & -1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(1 - z + x\bar{x})^{-1} & 1 & \bar{x}^{-2} & \bar{x}^{-3} \\
1 & -z(1 - z + x\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(1 - z + x\bar{x})^{-1} & 1 & \bar{x}^{-2} & \bar{x}^{-3} \\
1 & -z(1 - z + x\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
(1 - z + x\bar{x})^{-1} & 1 & \bar{x}^{-2} & \bar{x}^{-3} \\
1 & -z(1 - z + x\bar{x})^{-1} & \bar{x}^{-4} & \bar{x}^{-2} \\
(1 - z)^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
z(1 - z)(1 - z + x\bar{x})^{-1} & \bar{x}^{-1} & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\end{bmatrix}
\]

where the last matrix is in GSp(4, \sigma).

For the third assertion, recall that \( x \in (1 - z^{-1})\bar{x}^{-1} + \sigma^\times \) and \( w = (1 - x\bar{x} - z^{-1})\bar{x}^{-1} \in \sigma^\times \).

Then we have the identity

\[
\begin{bmatrix}
1 & 1 & \bar{x}^{-2} & \bar{x}^{-3} \\
1 & 1 & \bar{x}^{-2} & -1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & \bar{x}^{-2} & -1 & 1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & \bar{x}^{-2} & -1 & 1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & \bar{x}^{-2} & -1 & 1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & \bar{x}^{-2} & \bar{x}^{-3} & \bar{x}^{-2} \\
1 & \bar{x}^{-2} & -1 & 1 \\
\bar{x}^{-3} & 1 & \bar{x}^{-2} & 1 \\
z \bar{x}^{-2} & 1 & \bar{x}^{-2} & 1 \\
z & 1 & \bar{x}^{-2} & 1 \\
\end{bmatrix}
\end{bmatrix}
\]
where the last matrix is again in $\text{GSp}(4, \mathfrak{o})$.

For the fourth assertion, recall that $x \in (1 - z^{-1}) \mathfrak{o}^{-1} + \mathfrak{o}^\times$ and $w = (1 - x \mathfrak{o} - z^{-1}) \mathfrak{o}^{-2} \in \mathfrak{o}^\times$. Then we have the identity

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & x & \mathfrak{o}^{-2} & \mathfrak{o}^{-3} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2}
\end{bmatrix}
\begin{bmatrix}
1 & w^{-1} \mathfrak{o}^{-1} & w^{-1} \mathfrak{o}^{-1} & w^{-1} \mathfrak{o}^{-1} \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
$$

where the last matrix is again in $\text{GSp}(4, \mathfrak{o})$.

For the final assertion, recall that $x \in (1 - z^{-1}) \mathfrak{o}^{-1} + \mathfrak{p}^2$. Then we have the identity

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & x & \mathfrak{o}^{-2} & \mathfrak{o}^{-3} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2}
\end{bmatrix}
\begin{bmatrix}
1 & z^{-1} \mathfrak{o}^{-1} & z^{-1} \mathfrak{o}^{-1} & z^{-1} \mathfrak{o}^{-1} \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
$$

where again the last matrix is in $\text{GSp}(4, \mathfrak{o})$. This completes the proof of the lemma. $\square$

**Lemma 4.7.** If $v \in V(0)$, then we have that (8) is given by

$$
q^2 \chi(-1) \int \int \int \int_{1+\mathfrak{p}} \int \int \int \int_{\mathfrak{o}^\times \mathfrak{o}^\times \mathfrak{o}^\times \mathfrak{o}^\times} \chi(b) \eta \pi(D(a, b) \left[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array} \right] \left[ \begin{array}{cccc}
1 & x \mathfrak{o}^{-2} & \mathfrak{o}^{-2} & \mathfrak{o}^{-3} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2} \\
1 & 1 & \mathfrak{o}^{-2} & \mathfrak{o}^{-2}
\end{array} \right] v \, da \, db \, dx \, dz
$$

$$
= q \chi(-1) \int \int \int \int_{\mathfrak{o}^\times \mathfrak{o}^\times \mathfrak{o}^\times \mathfrak{o}^\times} \chi(b) \eta^2 \tau^{-2} \pi \left[ \begin{array}{cccc}
1 & a \mathfrak{o}^{-2} & b \mathfrak{o}^{-1} \\
1 & 1 & x \mathfrak{o}^{-4} & a \mathfrak{o}^{-2} \\
1 & 1 & 1 & 1
\end{array} \right] v \, da \, db \, dx
$$
Proof. Using the identities from Lemma 4.6, we calculate

\[ + \chi(-1) \int \int \int \chi(bx) \eta^2 \tau^{-1} \pi(1, a, \omega^2, b(x - 1) \omega^{-1}) v \, da \, db \, dx \]

\[ + q^{-1} \chi(-1) \int \int \int \chi(bx) \eta^2 \tau^{-2} \pi(1, a^2 \omega^{-2}, b(1 + x) \omega^{-2}) v \, da \, db \, dx \]

\[ + q^{-2} \chi(-1) \int \int \int \chi(b) \eta^2 \tau^{-1} \pi(1, a \omega^{-1}, b \omega^{-1}) v \, da \, db \, dx \]

\[ + q^{-3} \chi(-1) \int \int \int \chi(b) \eta^2 \tau^{-2} \pi(1, a \omega^{-2}, b \omega^{-2}) v \, da \, db \, dx \]

\[ q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ z \omega^3 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]

\[ = q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ x \omega^{-2} & 1 & 1 \\ -x \omega^{-2} & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]

\[ + q \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ x \omega^{-1} & 1 & 1 \\ -x \omega^{-1} & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]

\[ + \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ x \omega^{-1} & 1 & 1 \\ -x \omega^{-1} & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]

\[ = q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ x \omega^{-2} & 1 & 1 \\ -x \omega^{-2} & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]

\[ + q \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b)) \left[ \begin{array}{ccc} 1 & 1 & 1 \\ x \omega^{-1} & 1 & 1 \\ -x \omega^{-1} & 1 & 1 \end{array} \right] v \, da \, db \, dx \, dz \]
\[
+ \chi(-1) \int \frac{1}{1+p((1-z^{-1})w^{-1}+w^p)^{o^x_o^x}} \int \chi(b) \eta \pi(D(a,b) \left[ \begin{array}{ccc}
1 & 1 & 1 \\
z & w^{-3} & 1 \\
\end{array} \right] )v \ da \ db \ dx \ dz \\
\left[ \begin{array}{ccc}
1 & x & w^{-2} \\
1 & 1 & w^{-2} \\
1 & -z & 1 \\
\end{array} \right] \]

\[
+ \chi(-1) \int \frac{1}{1+p((1-z^{-1})w^{-1}+w^p)^{o^x_o^x}} \int \chi(b) \eta \pi(D(a,b) \left[ \begin{array}{ccc}
1 & 1 & 1 \\
z & w^{-3} & 1 \\
\end{array} \right] )v \ da \ db \ dx \ dz \\
\left[ \begin{array}{ccc}
1 & x & w^{-2} \\
1 & 1 & w^{-2} \\
1 & -z & 1 \\
\end{array} \right] \]

\[
+ \chi(-1) \int \frac{1}{1+p((1-z^{-1})w^{-1}+w^p)^{o^x_o^x}} \int \chi(b) \eta \pi(D(a,b) \left[ \begin{array}{ccc}
1 & 1 & 1 \\
z & w^{-3} & 1 \\
\end{array} \right] )v \ da \ db \ dx \ dz \\
\left[ \begin{array}{ccc}
1 & x & w^{-2} \\
1 & 1 & w^{-2} \\
1 & -z & 1 \\
\end{array} \right] \]

\[
= q^2 \chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a,b) \pi\left[ \begin{array}{ccc}
-w^{-1} & w^2 & 1 \\
w^{-1} & w^{-2} & w \\
\end{array} \right] )v \ da \ db \ dx \ dz \\
\left[ \begin{array}{ccc}
1 & (xz)^{-1}w^{-2} & 1 - z(xz + (1 - z)w)^{-1}w^{-4} \ (xz)^{-1}w^{-2} \\
1 & 1 & 1 \\
\end{array} \right] \]

\[
+ q\chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a,b) \left[ \begin{array}{ccc}
-w^{-1} & w & 1 \\
w & w^{-1} & w \\
\end{array} \right] )v \ da \ db \ dx \ dz \\
\left[ \begin{array}{ccc}
1 & (1 - z + zx)^{-1}w^{-2} & -z^{-2}(1 - z + x)^{-1}w^{-1} + z^{-1}w^{-1} \\
1 & 1 & 1 \\
\end{array} \right] \]
\]
\[
\chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ 1 \\ 1 \end{bmatrix}) v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ \infty \end{bmatrix}) v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ \infty^{-2} \\ \infty^2 \end{bmatrix}) v \, da \, db \, dx \, dz
\]

\[
q^2 \chi(-1) \int \int \int \int \chi(b) \eta^2 \tau^{-2} \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ \infty^{-2} \\ \infty^{-3} \end{bmatrix}) v \, da \, db \, dx \, dz
\]

\[
q \chi(-1) \int \int \int \int \chi(b) \eta^2 \tau^{-1} \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ \infty^{-2} \end{bmatrix}) v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta^2 \pi(D(a, b) \begin{bmatrix} \infty^{-1} \\ \infty^{-1} \end{bmatrix}) v \, da \, db \, dx \, dz
\]
\[
\begin{bmatrix}
1 & -z^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1} & \frac{(1 + z(1 - x \omega - z^{-1})) \omega^{-1}}{z^2(1 - x \omega - z^{-1})} \\
1 & (1 - x \omega - z^{-1})^{-1} \omega^{-1} & -z^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1}
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta^2 \tau \pi(D(a, b))
\]

\[
\begin{bmatrix}
1 & -z^{-1} \omega^{-1} & 1 \\
1 & z^{-1} \omega^{-1} & 1 \\
1 & z^{-1} \omega^{-2} & 1
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]

\[
q^2 \chi(-1) \int \int \int \int \chi(b) \eta^2 \tau^2 \pi(\ )
\]

\[
\begin{bmatrix}
1 & a(xz)^{-1} \omega^{-2} & bz^{-1} \omega^{-1} \\
1 & -a^2 b^{-1} z(xz + (1 - z) \omega)^{-1} \omega^{-4} & a(xz)^{-1} \omega^{-2}
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]

\[
q \chi(-1) \int \int \int \int \chi(b) \eta^2 \tau^{-1} \pi(\ )
\]

\[
\begin{bmatrix}
1 & a(1 - z + z x)^{-1} \omega^{-2} & -b z^{-2}(1 - z + x)^{-1} \omega^{-1} + bz^{-1} \omega^{-1} \\
1 & -a^2 b^{-1}(1 - z + z x)^{-1} \omega^{-3} & a(1 - z + z x)^{-1} \omega^{-2}
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta^2 \pi(\ )
\]

\[
\begin{bmatrix}
1 & -az^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1} & \frac{b(1 + z(1 - x \omega - z^{-1})) \omega^{-1}}{z^2(1 - x \omega - z^{-1})} \\
1 & a^2 b^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1} & -az^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1}
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]

\[
\chi(-1) \int \int \int \int \chi(b) \eta^2 \tau \pi(\ )
\]

\[
\begin{bmatrix}
1 & -az^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1} & \frac{b(1 + z(1 - x \omega - z^{-1})) \omega^{-1}}{z^2(1 - x \omega - z^{-1})} \\
1 & a^2 b^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1} & -az^{-1}(1 - x \omega - z^{-1})^{-1} \omega^{-1}
\end{bmatrix}
\int v \, da \, db \, dx \, dz
\]
\[
\begin{pmatrix}
1 & -a^{-1}b_{z-1}\omega^{-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a^2b^{-1}(1-x\omega -z^{-1})^{-1}\omega \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & b_{z-1}\omega^{-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
vda\ db\ dx\ dz
\end{pmatrix}
\]

\[+
\chi(-1)\int\int\int\int\chi(b)\eta^2\tau^2\pi(1)
\begin{pmatrix}
1 & -a^{-1}b_{z-1}\omega^{-2} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & ax^{-1}\omega^{-2} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & b_{z-1}\omega^{-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
vda\ db\ dx\ dz
\end{pmatrix}
\]

\[+
q\chi(-1)\int\int\int\int\chi(b)\eta^2\tau^{-1}\pi(1)
\begin{pmatrix}
1 & -a^2b^{-1}(1-(z+x))\omega^{-3} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -b_{z-2}(1-(z+x))^{-1}\omega^{-1} + b_{z-1}\omega^{-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & ax^{-1}\omega^{-2} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
vda\ db\ dx\ dz
\end{pmatrix}
\]

\[+
\chi(-1)\int\int\int\int\chi(b)\eta^2\pi(1)
\begin{pmatrix}
1 & -a^{-1}b_{\omega-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & b(1+z(1-x\omega -z^{-1})) \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
b_{\omega-1} \\
1
\end{pmatrix}
\begin{pmatrix}
vda\ db\ dx\ dz
\end{pmatrix}
\]

\[+
\chi(-1)\int\int\int\int\chi(b)\eta^2\tau\pi(1)
\begin{pmatrix}
1 & -a^{-1}b_{\omega-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a^{2b^{-1}z^{-1}(1-x\omega -z^{-1})^{-1}\omega} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & b_{\omega-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
vda\ db\ dx\ dz
\end{pmatrix}
\]

\[+
q^{-3}\chi(-1)\int\int\int\chi(b)\eta^2\tau^2\pi(1)
\begin{pmatrix}
1 & a_{\omega-2} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & b_{\omega-1} \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
vda\ db
\end{pmatrix}
\]
\[
\begin{align*}
&= q^2 \chi(-1) \int_{1+\rho \times \rho \times \rho} \chi(b) \eta^2 \tau^{-2} \pi \left[ \begin{array}{ccc}
1 & a x^{-1} \omega^{-2} & b \omega^{-1} \\
1 & -a^2 b^{-1} (x + (1 - z) \omega)^{-4} & a x^{-1} \omega^{-2} \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ q \chi(-1) \int_{1+\rho \times \rho \times \rho} \chi(b) \eta^2 \tau^{-1} \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b(1 + z(1 - x \omega - z^{-1})) \\
1 & -a^2 b^{-1}(1 - z + zx) \omega^{-3} & \frac{b(1 - x \omega - z^{-1})}{z^2(1 - x \omega - z^{-1})} \omega^{-1} \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ \chi(-1) \int_{1+\rho ((1 - z^{-1}) \omega^{-1} + \phi \times \phi)} \int \int \int \chi(b) \eta^2 \tau \pi \left[ \begin{array}{ccc}
1 & a \omega^{-1} & b \omega^{-1} \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ q^{-3} \chi(-1) \int_{\phi \times \phi} \int \chi(b) \eta^2 \tau^{-2} \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b \omega^{-1} \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] v \, da \, db \\
&= q^2 \chi(-1) \int_{1+\rho \times \rho \times \rho} \chi(b) \eta^2 \tau^{-2} \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b \omega^{-1} \\
1 & -a^2 b^{-1} (1 - x \omega - z^{-1})^{-4} & a x^{-1} \omega^{-2} \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ q \chi(-1) \int_{1+\rho \times \rho \times \rho} \chi(b) \eta^2 \tau^{-1} \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b(1 - x^{-1}) \omega^{-1} \\
1 & -a^2 b^{-1}(1 - z + zx) \omega^{-3} & \frac{b(1 - x^{-1}) \omega^2}{x^2 \omega^2} \omega^{-2} \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ \chi(-1) \int_{1+\rho \times \rho \times \rho} \int \int \int \chi(b) \eta^2 \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b \omega^{-1} \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz \\
&+ q^{-1} \chi(-1) \int_{1+\rho \times \rho \times \rho} \int \int \int \chi(b) \eta^2 \tau \pi \left[ \begin{array}{ccc}
1 & a \omega^{-2} & b \omega^{-1} \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] v \, da \, db \, dx \, dz
\end{align*}
\]
\[
\begin{align*}
&= q^2 \chi(-1) \int_1^1 \int_0^{\infty} \chi(bx) \eta^2 \pi(1 - a^{2b-1}x^{-4} b(x-1)^{b-1})v \, db \, dx \\
&+ q^{-2} \chi(-1) \int_1^1 \int_0^{\infty} \chi(bx) \eta^2 \pi(1 - a^{2b-1}x^{-2} b(x-1)^{b-1})v \, db \, dx \\
&+ q^3 \chi(-1) \int_0^{\infty} \int_0^{\infty} \chi(bx) \eta^2 \pi(1 - a^{2b-1}x^{-2} b(x-1)^{b-1})v \, db \, dx
\end{align*}
\]
Lemma 4.8. If 
\[
+ q^{-2} \chi(-1) \int \int \int \chi(b) \eta^2 \tau \pi \left( \begin{array}{c} 1 \\ a \eta^{-1} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx 
\]
\[
+ q^{-3} \chi(-1) \int \int \int \chi(b) \eta^2 \tau^2 \pi \left( \begin{array}{c} 1 \\ a \eta^{-2} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db 
\]
\[
= q \chi(-1) \int \int \int \chi(b) \eta^2 \tau^{-2} \pi \left( \begin{array}{c} 1 \\ a \eta^{-4} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx 
\]
\[
+ \chi(-1) \int \int \int \chi(bx) \eta^2 \tau^{-1} \pi \left( \begin{array}{c} 1 \\ a \eta^{-2} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx 
\]
\[
+ q^{-1} \chi(-1) \int \int \int \chi(bx) \eta^2 \pi \left( \begin{array}{c} 1 \\ a \eta^{-2} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx 
\]
\[
+ q^{-2} \chi(-1) \int \int \int \chi(b) \eta^2 \tau \pi \left( \begin{array}{c} 1 \\ a \eta^{-3} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db \, dx 
\]
\[
+ q^{-3} \chi(-1) \int \int \int \chi(b) \eta^2 \tau^2 \pi \left( \begin{array}{c} 1 \\ a \eta^{-3} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a^{-2} b \tau^{-1} \\ 1 \\ 1 \end{array} \right) v \, da \, db 
\]

This completes the calculation. 

Finally, we are able to calculate term (P3)

Lemma 4.8. If \( v \in V(0) \), then we have that (P3) is given by

\[
q^2 \int \int \int \int \chi(ab) \pi(t_4) \left( \begin{array}{c} 1 \\ a \eta^{-1} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ a \eta^{-1} \\ 1 \\ 1 \end{array} \right) \tau v \, da \, db \, dx \, dz 
\]
\[
= q^2 \chi(-1) \int \int \int \int \chi(abx(1-z)) \eta \tau^{-1} \pi \left( \begin{array}{c} 1 \\ b \eta^{-1} \\ 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ b \eta^{-1} \\ 1 \\ 1 \end{array} \right) v \, dx \, db \, dz 
\]
\[
\begin{align*}
&\quad + \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(abz(1-z)) \eta \tau(\pi) \begin{bmatrix}
1 & b \tau^{-2} \\
1 & 1 \\
1 & -b \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, dx \, da \, db \, dz \\
&= q \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(abx) \eta \tau(\pi) \begin{bmatrix}
1 & b \tau^{-1} \\
1 & 1 \\
1 & -b \tau^{-1} \\
1 & 1 \\
\end{bmatrix} v \, dx \, da \, db \, dz \\
&+ \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(b(1-z)) \eta \tau(\pi) \begin{bmatrix}
1 & a \tau^{-2} \\
1 & 1 \\
1 & -a \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, dx \, da \, db \, dz \\
&+ q \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta^2 \tau^{-2}(\pi) \begin{bmatrix}
1 & a \tau^{-2} b \tau^{-2} \\
1 & 1 \\
1 & a \tau^{-2} b \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \, dx \\
&+ \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta^2 \tau^{-2}(\pi) \begin{bmatrix}
1 & a \tau^{-1} b \tau^{-1} \\
1 & 1 \\
1 & a \tau^{-1} b \tau^{-1} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \, dx \\
&+ q^{-2} \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta^2 \tau^{-2}(\pi) \begin{bmatrix}
1 & a \tau^{-1} \tau^{-1} \\
1 & 1 \\
1 & a \tau^{-1} \tau^{-1} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \, dx \\
&+ q^{-3} \chi(-1) \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta^2 \tau^{-2}(\pi) \begin{bmatrix}
1 & a \tau^{-2} b \tau^{-2} \\
1 & 1 \\
1 & a \tau^{-2} b \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \\
&+ q \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta \tau(\pi) \begin{bmatrix}
1 & a \tau^{-2} \\
1 & 1 \\
1 & -a \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \\
&+ q^{1} \int_{\phi^{-1}(1+p)} \int_{\phi^{-1}} \int_{\phi} \int_{\phi^{-1}} \chi(bx) \eta^2 \tau^{-2}(\pi) \begin{bmatrix}
1 & a \tau^{-2} \tau^{-2} \\
1 & 1 \\
1 & a \tau^{-2} \tau^{-2} \\
1 & 1 \\
\end{bmatrix} v \, da \, db \, dx.
\end{align*}
\]

**Proof.** The proof follows by combining the formulas in Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, and 4.7. □
5. Calculation of the Fourth Part (P4)

In this section, we compute the final summand of the operator defined in [JR1]. We continue to use the same techniques as in the previous sections, in particular, the invariance of \( v \) under GSp(4, \( \sigma \)) and the useful identity (2).

Lemma 5.1. If \( v \in V(0) \), then we have that (P4) is given by

\[
q^2 \int \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix} 1 & -1 & 1 & -a \omega^{-1} \\ -1 & 1 & 1 & b \omega^{-2} \\ 1 & y & 1 & z \omega^{-3} \\ 1 & 1 & 1 & a \omega^{-1} \end{bmatrix} \tau v \, da \, db \, dy \, dz
\]

\[
= \int \int \int \chi(ab) \eta \pi \begin{bmatrix} 1 & -a \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \omega^{-1} & a(y + b) \omega^{-3} \\ 1 & y \omega^{-2} \end{bmatrix} \tau v \, da \, db \, dy
\]

\[
+ q \int \int \int \int \chi(ab) \eta \pi^{-1} \begin{bmatrix} 1 & -a \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \omega^{-1} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \omega^{-2} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & a \omega^{-2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \omega^{-1} \\ 1 \end{bmatrix} \, da \, db \, dy \, dz
\]

Proof. For \( a, b \in \sigma^\times \), define the diagonal matrix

\[
C(a, b) = \begin{bmatrix} 1 & a \\ b & ab \end{bmatrix}
\]

Then we calculate as follows:

\[
q^2 \int \int \int \int \chi(ab) \pi(t_4) \begin{bmatrix} 1 & -1 & 1 & -a \omega^{-1} \\ -1 & 1 & 1 & b \omega^{-2} \\ 1 & y & 1 & z \omega^{-3} \\ 1 & 1 & 1 & a \omega^{-1} \end{bmatrix} \tau v \, da \, db \, dy \, dz
\]

\[
= q^2 \int \int \int \int \chi(ab) \pi^{-1}(C(a, b) t_4) \begin{bmatrix} 1 & -1 & 1 & -a \omega^{-1} \\ -1 & 1 & 1 & b \omega^{-2} \\ 1 & y \omega^{-2} & 1 & z \omega^{-3} \\ 1 & 1 & 1 & a \omega^{-1} \end{bmatrix} \tau v \, da \, db \, dy \, dz
\]

\[
= q^2 \int \int \int \int \chi(ab) \pi^{-1}(C(a, b) t_4) \begin{bmatrix} 1 & -1 & 1 & -a \omega^{-1} \\ -1 & 1 & 1 & b \omega^{-2} \\ 1 & y \omega^{-2} & 1 & z \omega^{-3} \\ 1 & 1 & 1 & a \omega^{-1} \end{bmatrix} \tau v \, da \, db \, dy \, dz
\]
\[ q^2 \int \int \int \chi(ab)q^4\tau^{-1} \pi(C(a,b) \begin{bmatrix} 1 & -y \omega^{-2} & \frac{1}{y} \\ -\omega^{-2} & 1 & 1 \\ \omega^{-1} & \omega^{-1} & \omega^{-2} \end{bmatrix} v \, da \, db \, dy \, dz. \]

We first focus on the integration over \( z \).

\[ \int \pi \left[ \begin{array}{ccc} 1 & 1 & 1 \\ -\omega^{-2} & 1 & \omega^{-1} \\ \omega^{-3} & \omega^{-1} & \omega^{-2} \end{array} \right] v \, dz \]

\[ = \int \pi \left[ \begin{array}{ccc} 1 & 1 & 1 \\ -\omega^{-2} & 1 & \omega^{-1} \\ \omega^{-2} & 1 & \omega^{-1} \end{array} \right] v \, dz \]

\[ = \int \pi \left[ \begin{array}{ccc} 1 & 1 & 1 \\ -\omega^{-2} & 1 & \omega^{-1} \\ \omega^{-2} & 1 & \omega^{-1} \end{array} \right] v \, dz \]

\[ = \int \pi \left( -\omega^{-2} \right) v \, dz \]

\[ = \int \pi \left( 1 \right) v \, dz \]

\[ = \int \pi \left( \frac{1}{1+\omega} \right) v \, dz \]

\[ = \int \pi \left( \frac{1}{1 + \omega} \right) v \, dz \]

\[ = \int \pi \left( \frac{1}{1+\omega} \right) v \, dz \]

\[ = \int \pi \left( \frac{1}{1+\omega} \right) v \, dz \]

\[ = \int \pi \left( \frac{1}{1+\omega} \right) v \, dz \]
\[
\int_0^{-(1+p)} \pi \left( \begin{array}{ccc}
-\omega & 1 & 1 \\
(z-1)\omega^{-2} & 1 & (z-1)\omega^{-2} \\
(z-1)\omega^{-1} & -\omega & -\omega^{-1} \\
\end{array} \right) v \, dz
\]

\[
+ \int_0^{-(1+p)} \pi \left( \begin{array}{ccc}
-\omega & 1 & 1 \\
(z-1)\omega^{-2} & 1 & (z-1)\omega^{-2} \\
(z-1)\omega^{-1} & -\omega & -\omega^{-1} \\
\end{array} \right) v \, dz
\]

\[
= \int_0^{1+p} \pi \left( \begin{array}{ccc}
\omega & 1 & 1 \\
\omega^{-1} & 1 & (z-1)\omega^{-1} \\
\omega^{-1} & 1 & (z-1)\omega^{-1} \\
\end{array} \right) v \, dz
\]

\[
+ \int_0^{-(1+p)} \pi \left( \begin{array}{ccc}
-\omega & 1 & 1 \\
(z-1)\omega^{-2} & 1 & (z-1)\omega^{-2} \\
(z-1)\omega^{-1} & -\omega & -\omega^{-1} \\
\end{array} \right) v \, dz
\]

\[
= \int_{1+p}^{1} \pi \left( \begin{array}{ccc}
\omega & 1 & 1 \\
\omega^{-1} & 1 & (z-1)\omega^{-1} \\
\omega^{-1} & 1 & (z-1)\omega^{-1} \\
\end{array} \right) v \, dz
\]
\[ + \int_{-(1+p)} \pi \left[ \begin{array}{cc} 1 & \omega \\ -\omega^{-2} & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & -z \omega^{-1} \\ 1 & \omega^{-1} \end{array} \right] v \, dz \]

\[ \left[ \begin{array}{cc} -\omega & -\omega^{-1} \\ (z - 1)\omega^{-1} & -\omega^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 & (z - 1)^{-1} \omega \\ -z^{-1} & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & -\omega^{-1} \\ 1 & \omega^{-1} \end{array} \right] v \, dz \]

\[ = \int_{1+p} \pi \left[ \begin{array}{cc} \omega & \omega^{-1} \\ \omega^{-2} & \omega^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ -\omega^{-1} & -z \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & (z - 1)^{-1} \omega^{-1} \\ -z^{-1} & 1 \end{array} \right] v \, dz \]

\[ + \int_{-(1+p)} \pi \left[ \begin{array}{cc} \omega^2 & \omega \\ \omega^{-1} & \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ -\omega^{-1} & -z \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & (z - 1)^{-1} \omega^{-1} \\ -z^{-1} & 1 \end{array} \right] v \, dz \]

\[ = \int_{1+p} \pi \left[ \begin{array}{cc} \omega & \omega^{-1} \\ \omega^{-2} & \omega^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ -\omega^{-1} & -z \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & (z - 1)^{-1} \omega^{-1} \\ -z^{-1} & 1 \end{array} \right] v \, dz \]

\[ + \int_{-(1+p)} \pi \left[ \begin{array}{cc} \omega^2 & \omega \\ \omega^{-1} & \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & 1 \\ -\omega^{-1} & -z \omega^{-2} \end{array} \right] \left[ \begin{array}{cc} 1 & (z - 1)^{-1} \omega^{-1} \\ -z^{-1} & 1 \end{array} \right] v \, dz \]
\[
\int \frac{1}{1+p} \left[ \begin{array}{cc}
\nu & \nu \\
\nu^{-1} & \nu^{-1}
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{1+z\nu^{-2}} & 1 & \frac{1}{z\nu^{-2}} \\
1 & \frac{1}{z\nu^{-2}} & 1
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, dz \\
+ \int \frac{1}{\nu^{-2}} \left[ \begin{array}{ccc}
\frac{1}{1+z\nu^{-2}} & 1 & \frac{1}{z\nu^{-2}} \\
1 & \frac{1}{z\nu^{-2}} & 1
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, dz.
\]

Substituting into the full integral, we have

\[
q^2 \int \int \int \chi(ab)\eta_4^2 \pi(C(a,b)) \left[ \begin{array}{ccc}
1 & -y\nu^{-2} & \nu \\
\nu^{-1} & 1 & \nu^{-1}
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q^2 \int \int \int \chi(ab)\eta_4^2 \pi(C(a,b)) \left[ \begin{array}{ccc}
1 & -y\nu^{-2} & \nu \\
\nu^{-1} & 1 & \nu^{-1}
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q^2 \int \int \int \chi(ab)\eta_4^3 \tau^{-2} \pi(C(a,b)) \left[ \begin{array}{ccc}
1 & -y\nu^{-4} & \nu \\
\nu^{-1} & 1 & \nu^{-1}
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q^2 \int \int \int \chi(ab)\eta_4^3 \tau^{-2} \pi(C(a,b)) \left[ \begin{array}{ccc}
1 & -y\nu^{-4} & \nu \\
\nu^{-1} & 1 & \nu^{-1}
\end{array} \right] \left[ \begin{array}{ccc}
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}} \\
\frac{1}{\nu^{-1}} & \frac{(z-1)\nu^{-3}}{\nu^{-1}}
\end{array} \right] v \, da \, db \, dy \, dz.
\]
\[
\begin{bmatrix}
1 & -1 \\
-z^{\infty} & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-2} \\
1 & (z-1)^{-1} \infty^{-1}
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-3} \\
1 & \infty^{-2}
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
\begin{bmatrix}
1 & -1 \\
-z^{\infty} & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-2} \\
1 & (z-1)^{-1} \infty^{-1}
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-3} \\
1 & \infty^{-2}
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
= q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
= q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
+ q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
= q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
+ q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]

\[
+ q \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b) \begin{bmatrix}
1 & \infty^{-3} \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \infty^{-1} \\
1 & 1
\end{bmatrix} v da db dy dz
\]
\[
\begin{bmatrix}
1 & -z\varpi^{-1} & 1 \\
-1 & \varpi^{-1} & 1 \\
\varpi^{-1} & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
= q \int \int \int \int \chi(ab)\eta^{2}\varpi^{2}\pi(C(a,b))
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
+ q \int \int \int \int \chi(ab)\eta^{2}\varpi^{2}\pi(C(a,b))
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
= q \int \int \int \int \chi(ab)\eta^{2}\varpi^{2}\pi(C(a,b))
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
+ q \int \int \int \int \chi(ab)\eta^{2}\varpi^{2}\pi(C(a,b))
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
= q \int \int \int \int \chi(ab)\eta^{2}\varpi^{2}\pi(C(a,b))
\begin{bmatrix}
1 & \frac{\varpi^{2}}{1} & (z-1)^{-1}\varpi^{-1} \\
\varpi^{-2} & 1 & 1 \\
\varpi^{-2} & 1 & 1
\end{bmatrix}
\]
\[vd\, db\, dy\, dz\]
\[
\begin{bmatrix}
  z^{-1} \omega^-1 & z \omega^-1 & 1 \\
  -z^{-1} \omega^-1 & -z \omega^-1 & -1 \\
 1 & -z^{-1} \omega^-1 & 1 \\
 1 & z^{-1} \omega^-1 & 1
\end{bmatrix}
\begin{bmatrix}
  1 \\
  -1 \\
 1 \\
 1
\end{bmatrix}
\]
\]
\[
\left( z^{-1} \omega^-1 \right) v \, db \, dy \, dz
\]
\[
+ q \int \int \int \int \chi(ab)\eta^2 \tau^{-2} \pi(C(a, b))
\]
\[
\begin{bmatrix}
  1 & -(z - 1)^{-1} \omega^-2 \\
  -1 & -1 \\
 1 & 1
\end{bmatrix}
\]
\[
\left( (z - 1)^{-1} \omega^-2 \right) v \, db \, dy \, dz
\]
\[
= q \int \int \int \int \chi(ab)\eta \pi(C(a, b))
\]
\[
\begin{bmatrix}
  1 & z^{-1} \omega^-1 \\
  1 & 1 \\
 1 & 1
\end{bmatrix}
\]
\[
\left( (z - 1)^{-1} \omega^-1 \right) v \, db \, dy \, dz
\]
\[
+ q \int \int \int \int \chi(ab)\eta^3 \tau^{-2} \pi(C(a, b))
\]
\[
\begin{bmatrix}
  1 & -(z - 1)^{-1} \omega^-2 \\
  -1 & -1 \\
 1 & 1
\end{bmatrix}
\]
\[
\left( (z - 1)^{-1} \omega^-2 \right) v \, db \, dy \, dz
\]
\[
= q \int \int \int \int \chi(ab)\eta^3 \pi(C(a, b))
\]
\[
\begin{bmatrix}
  1 & -a^{-1} z^{-1} \omega^-2 \\
  1 & 1 \\
 1 & 1
\end{bmatrix}
\]
\[
\left( -a^{-1} z^{-1} \omega^-2 \right) v \, db \, dy \, dz
\]
\[
+ q \int \int \int \chi(ab) \eta \tau^{-1} \pi \left[ \begin{array}{ccc}
1 & -a^{-1}z^{-1}w^{-1} & 1 \\
1 & 1 & a^{-1}z^{-1}w^{-1} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q \int \int \int \chi(ab) \eta^2 \tau^{-2} \pi \left[ \begin{array}{ccc}
1 & -b^{-1}(z-1)^{-1}w^{-1} & 1 \\
1 & a^{-1}b^{-1}(z-1)^{-1}w^{-1} & 1 \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
= q \int \int \int \int \chi(ab) \eta \pi \left[ \begin{array}{ccc}
1 & -a^{-1}w^{-2} & 1 \\
1 & 1 & a^{-1}w^{-2} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q \int \int \int \int \chi(abz) \eta \tau^{-1} \pi \left[ \begin{array}{ccc}
1 & -a^{-1}w^{-1} & 1 \\
1 & 1 & a^{-1}w^{-1} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
+ q \int \int \int \int \chi(ab) \eta^2 \tau^{-2} \pi \left[ \begin{array}{ccc}
1 & b^{-1}w^{-2} & -a^{-1}b^{-1}w^{-1} \\
1 & yw^{-3} & b^{-1}w^{-2} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
= q \int \int \int \int \chi(ab) \eta \pi \left[ \begin{array}{ccc}
1 & -a^{-1}w^{-2} & 1 \\
1 & 1 & a^{-1}w^{-2} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
\]

\[
\left[ \begin{array}{ccc}
1 & b^{-1}(y-z)w^{-1} & a^{-1}b^{-1}y^{-1}w^{-3} \\
1 & 1 & b^{-1}(y-z)w^{-1} \\
1 & 1 & 1 \\
\end{array} \right] v \, da \, db \, dy \, dz \\
\]
\[ + q \int \int \int \int \chi(ab)\eta \tau^{-1}\pi(\begin{bmatrix} 1 & -a^{-1}\tau^{-1} \\ 1 & a^{-1}\tau^{-1} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ + q \int \int \int \int \chi(ab)\eta^2 \tau^{-2}\pi(\begin{bmatrix} 1 & b\tau^{-2} \\ 1 & -ab\tau^{-2} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ = \int \int \int \chi(ab)\eta \pi(\begin{bmatrix} 1 & -a^{-1}\tau^{-1} \\ 1 & a^{-1}\tau^{-1} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ + q \int \int \int \int \chi(ab)\eta \tau^{-1}\pi(\begin{bmatrix} 1 & b^{-1}y\tau^{-2} \\ 1 & ab^{-1}(y + z)\tau^{-3} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ + q \int \int \int \int \chi(ab)\eta^2 \tau^{-2}\pi(\begin{bmatrix} 1 & b\tau^{-2} \\ 1 & -ab\tau^{-2} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ = \int \int \int \chi(ab)\eta \pi(\begin{bmatrix} 1 & -a\tau^{-2} \\ 1 & a\tau^{-2} \end{bmatrix}) v \, da \, db \, dy \, dz \]

\[ + q \int \int \int \int \chi(ab)\eta \tau^{-1}\pi(\begin{bmatrix} 1 & b\tau^{-2} \\ 1 & ab\tau^{-2} \end{bmatrix}) v \, da \, db \, dy \, dz \]
\[ + q \int \int \int \chi(a) \eta^2 \tau^{-2} \pi \left( \begin{array}{ccc} 1 & \frac{b}{\omega} & -\frac{a}{\omega} \\ \frac{y}{\omega^3} & \frac{b}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) v \, da \, db \, dy \, dz \]

\[ = \int \int \int \chi(ab) \eta \pi \left( \begin{array}{ccc} 1 & -\frac{a}{\omega} & 1 \\ \frac{1}{\omega} & \frac{a}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} 1 & \frac{a(y + b)}{\omega^3} & \frac{y}{\omega^2} \\ \frac{a(y + b)}{\omega^3} & \frac{y}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) v \, da \, db \, dy \]

\[ + q \int \int \int \chi(ab) \eta \pi(-1+(1+p)) \left( \begin{array}{ccc} 1 & -\frac{a}{\omega} & 1 \\ \frac{1}{\omega} & \frac{a}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) \left( \begin{array}{ccc} \frac{y}{\omega^2} & \frac{a(y + b)}{\omega^3} & \frac{y}{\omega^2} \\ \frac{a(y + b)}{\omega^3} & \frac{y}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) v \, da \, db \, dy \, dz \]

\[ + \int \int \int \chi(a) \eta^2 \tau^{-2} \pi \left( \begin{array}{ccc} 1 & \frac{b}{\omega} & -\frac{a}{\omega} \\ \frac{y}{\omega^3} & \frac{b}{\omega^2} & 1 \\ 1 & 1 & 1 \end{array} \right) v \, da \, db \, dy. \]

This completes the calculation. \qed

References

[JR] Johnson-Leung, J., Roberts, B.: Twisting of Siegel paramodular forms, arXiv:1404.4596 (2014).

[JR1] Johnson-Leung, J., Roberts, B.: Twists of paramodular vectors, Int. J. Number Theory, doi:10.1142/S1793042114500146, (2014).