Intermittency in the Enstrophy Cascade of
Two-dimensional Fully-developed Turbulence: Universal Features

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Abstract

Intermittency (externally induced) in the two-dimensional (2D) enstrophy cascade is shown to be able to maintain a finite enstrophy along with a vorticity conservation anomaly. Intermittency mechanisms of three-dimensional (3D) energy cascade and 2D enstrophy cascade in fully-developed turbulence (FDT) seem to have some universal features. The parabolic-profile approximation (PPA) for the singularity spectrum \( f(\alpha) \) in multi-fractal model is used and extended to the appropriate microscale regimes to exhibit these features. The PPA is also shown to afford, unlike the generic multi-fractal model, an analytical calculation of probability distribution functions (PDF) of flow-variable gradients in these FDT cases and to describe intermittency corrections that complement those provided by the homogeneous fractal model.

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1 Introduction

Spatial intermittency is a common feature of fully developed turbulence (FDT) and implies that turbulence activity at small scales is not distributed uniformly throughout space. This leads to a violation of an assumption (Landau [1]) in the Kolmogorov [2] theory that the statistical quantities show no dependence in the inertial range \( L \gg \ell \gg \eta \) on the large scale \( L \) (where the external stirring mechanisms are influential) and the Kolmogorov microscale \( \eta = (\nu^3/\varepsilon)^{1/4} \) (where the viscous effects become important), \( \varepsilon \) being the mean energy dissipation rate. Spatial intermittency effects can be very conveniently imagined to be related to the fractal aspects of the geometry of FDT (Mandelbrot [3]). The energy dissipation field may then be assumed to be a multi-fractal (Parisi and Frisch [4], Mandelbrot [5]). The latter idea has received experimental support (Meneveau and Sreenivasan [6]).

In the multi-fractal model one stipulates that the fine-scale regime of FDT possesses a range of scaling exponents \( \alpha \in I \equiv [\alpha_{\min}, \alpha_{\max}] \). Each \( \alpha \in I \) has the support set \( S(\alpha) \subset \mathbb{R}^3 \) of fractal dimension (also called the singularity spectrum) \( f(\alpha) \) such that, as \( \ell \to 0 \), the velocity increment has the scaling behavior \( \delta v(\ell) \sim \ell^\alpha \). The sets \( S(\alpha) \) are nested so that \( S(\alpha') \subset S(\alpha) \) for \( \alpha' < \alpha \).

Experimental data on three-dimensional (3D) FDT (Meneveau and Sreenivasan [7]) suggested that the singularity spectrum function \( f(\alpha) \) around its maximum may be expanded up to second order via the parabolic-profile approximation (PPA) [7, 8, 9]:

\[
\begin{align}
\frac{f'(\alpha)}{f''(\alpha)} &= f(\alpha_0) + \frac{1}{2} f''(\alpha_0) (\alpha - \alpha_0)^2 \\
&= f(\alpha_0) + 1
\end{align}
\]

where,

\[
f(\alpha_0) = 3.
\]

On the other hand, spatial intermittency in two-dimensional (2D) FDT has been a controversial issue. Indeed, even the whole theory of 2D FDT (Kraichnan [10] and Batchelor [11]) had, until recently, remained almost an academic exercise, notwithstanding its possible connections with atmospheric and oceanic large-scale flows. 2D FDT has now been produced to a close approximation

\[\text{\footnotesize{\textsuperscript{2}The PPA is equivalent to the log-normal model (Monin and Yaglom [8]).}}\]

\[\text{\footnotesize{\textsuperscript{3}The PPA was shown to be a good fit for Lagrangian velocity statistics as well (Chevillard et al. [9]).}}\]
in a variety of laboratory experiments (Couder [12], Kellay et al. [13], Martin et al. [14], Rutgers [15], Rivera et al. [16], Vorobieff et al. [17], Rivera et al. [18] and [19]). However, the Batchelor-Kraichnan theory for the enstrophy cascade in 2D FDT (with the energy spectrum $E(k) \sim k^{-3}$) has essential differences with the Kolmogorov theory for the energy cascade in 3D FDT. The Batchelor-Kraichnan theory corresponds to flows with infinite enstrophy so there is no need for a nonlinearity-sustained cascade to transfer enstrophy across the inertial range to small scales to counter the dissipative action of viscosity there (Lopes Filho et. al. [21]). Consequently, the usual cascade mechanism is not operational in the Batchelor-Kraichnan theory for the enstrophy regime.

Direct numerical simulations (DNS) of freely-decaying 2D FDT (McWilliams [22], Benzi et al. [23], Brachet et al. [24] and [25], Kida [26], Ohkitani [27], Schneider and Farge [28]) and forced-dissipative 2D FDT (Basdevant et al. [29], Legras et al. [30] and Tsang et al. [31]) showed intermittency caused by the presence of coherent structures. These structures inhibit the local inertial transfer of enstrophy via phase correlations across many length scales and produce energy spectra which are steeper than the Kraichnan-Batchelor spectrum $E(k) \sim k^{-3}$ for the enstrophy cascade.

The theoretical issue of intermittency in the 2D enstrophy cascade in view of the regular behavior of 2D Navier-Stokes solutions is a delicate one. In addition to coherent structures, another cause of intermittency in the 2D enstrophy cascade can be contamination from the 3D effects in any real flow situation. In the geophysical context, it can be caused by the Ekman drag ([31]) which simulates the frictional planetary boundary layer (Pedlosky [33]). The Ekman drag acts also as a sink at low wave numbers to take out the 'condensate' from the inverse energy cascade. It is of interest to note that irrespective of the origin, an externally-induced intermittency is able to restore the usual nonlinearity-sustained cascade mechanism in the Batchelor-Kraichnan theory. Intermittency renders enstrophy of the flow finite so a nonlinearity-sustained cascade is very much needed to transfer enstrophy across the inertial range to small scales to counter the dissipative action.

4The Batchelor-Kraichnan energy spectrum was recently shown (Eyink [24]) to correspond to the inviscid limit of the Leray solutions of 2D Navier-Stokes equations. These solutions, however, have vorticity fields which exist only as distributions.

5There is, therefore, some speculation whether different universality classes exist in the enstrophy cascade depending on the particular initial conditions involved ([23], [27] and [30]). On the other hand, the sensitivity of the 2D flow dynamics to the initial non-regularity of the flow raises questions about the whole universality concept in 2D FDT (Farge and Holschneider [32]).
Paladin and Vulpiani [34] pointed out that the intermittency in the enstrophy cascade may be described in terms of a multi-fractal probability measure for the vorticity gradient. The multi-fractal characterization of the intermittent enstrophy dissipation field was given by Mizutani and Nakano [37], Benzi and Scardovelli [38], and Shivamoggi [39]. In the intermittent case, the vorticity at very small length scales (where viscous effects are strong and nonlinearities are weak) behaves like a passive scalar (Weiss [40], Falkovich and Lebedev [41] and Nam et al. [42]) and is advected by the large-scale flow structures [30]. This leads to thin sheets with large vorticity gradients (Saffman [43]) and selective rapid decay of vorticity in these layers because such regions experience typically stronger viscous diffusion than other regions. Conversely, vorticity-gradient layers (or divorticity sheets) are more likely to occur near vortex nulls. (This is very akin to the vortex-sheet formation near velocity nulls and current-sheet formation near magnetic nulls in magnetohydrodynamics (Shivamoggi et al. [44]-[46])). On the other hand, the consequent statistical dependence between the vorticity and the vorticity gradient at a point may be expected in turn to lead to non-gaussian statistics. Indeed, probability density functions (PDF) of enstrophy flux was measured in a freely-decaying 2D FDT by Kellay et al. [47] which was found to be highly non-gaussian. On the otherhand, DNS (Herring and McWilliams [48], Borue [49], Maltrud and Vallis [50], Oetzel and Vallis [51]), soap-film experiments (Gharib and Derango [52], Kellay et al. [53], Martin et al. [14], Rutgers [15]), electromagnetically driven flow experiments (Paret al. [54]) showed restoration of the Kraichnan-Batchelor scaling behavior when the coherent structures are suppressed.

The issue of whether intermittency (albeit externally-induced) in the 2D enstrophy cascade can maintain a finite enstrophy along with a vorticity conservation anomaly is of great interest - this is addressed in this paper. The possibility of common universal features (other than energy spectra) in the intermittency mechanisms of 2D and 3D FDT was raised by Dubrulle [55] and is also a topic of great interest. We explore here universal features in the intermittency mechanisms of 3D energy cascade and 2D enstrophy cascade (when it is intermittent via external induction). The PPA (1) is

\[ \text{PPA (1)} \]

\[ \text{Indeed, experimental work of Jun and Wu [35] showed that large intermittency can be accounted for by the non-uniform distribution of saddle points in the flow (which are responsible for the energy transfer/dissipation (Daniel and Rutgers [36])).} \]
used and extended to the appropriate *microscale* regimes for this purpose. We will also show that the PPA also affords, unlike the generic multi-fractal model, an analytical calculation of PDF’s of flow-variable gradients in these FDT cases.

## 2 3D FDT: Energy Cascade

(i) **Inertial Regime**

Let us briefly review and then extend the PPA applied to the inertial regime (Meneveau and Sreenivasan [7], Benzi and Biferale [56]).

According to the multi-fractal model for the $p$th order velocity structure function (Parisi and Frisch [4]), we have

$$A_p \equiv \langle |\delta v|^p \rangle \int \ell^{p\alpha + 3 - f(\alpha)} d\mu(\alpha) \sim \xi_p^{(1)}$$

where,

$$\xi_p^{(1)} = \inf_{\alpha} [p\alpha + 3 - f(\alpha)] \quad (2b)$$

and this minimum occurs for $\alpha = \alpha^*$, which, according to the saddle-point method, is given by

$$f'(\alpha^*) = p. \quad (3)$$

Writing (1a) and (1b) in the form -

$$3 - f(\alpha^*) = a(\alpha^* - \alpha_0)^2, \quad a > 0 \quad (4)$$

(3) yields,

$$\alpha^*(p) = \alpha_0 - \frac{p}{2a}. \quad (5)$$

Using (4) and (5), (2a) and (2b) become

$$\xi_p^{(1)} = p\alpha_0 - \frac{p^2}{4a}. \quad (6)$$

The parameter $\alpha_0$ may now be determined (Benzi and Biferale [56]) by using the exact 3D...
Navier-Stokes result (Kolmogorov [57]) -

\[ \xi^{(1)}_3 = 1 \]  

which, on application to (6), yields

\[ \alpha_0 = \frac{1}{3} + \frac{3}{4a} \]  

Using (8), (6) becomes

\[ \xi^{(1)}_p = \frac{p}{3} + \frac{1}{4a}(3 - p)p \]  

while (5) becomes

\[ \alpha_*(p) = \frac{1}{3} + \frac{(3 - 2p)}{4a} \]  

Next, using (5) and (8), (4) becomes

\[ 3 - f(\alpha_*) = \frac{p^2}{4a} \]  

Upon interpreting \( \varepsilon \) as the energy transfer rate (in the inertial regime, as Kraichnan [58] pointed out, the energy transfer rate rather than the energy dissipation rate is the dynamically important parameter), we have from (9),

\[ \langle \varepsilon^2 \rangle \sim \ell^{-\frac{9}{2a}} \]  

Comparing (12) with the log-normal result [8] -

\[ \langle \varepsilon^2 \rangle \sim \ell^{-\mu} \]  

we have

\[ \mu = \frac{9}{2a} \]  

Kolmogorov [59] and Obukhov [60] hypothesized that \( \mu \) is a universal constant.

It is of interest to note the sense in which the PPA (1) mimics a multi-fractal in describing the intermittency aspects of FDT. Comparing (9) with the multi-fractal result (Meneveau and Sreenivasan [6]) for the inertial regime -

\[ \xi^{(1)}_p = \frac{p}{3} + \frac{1}{3}(3 - p)(3 - D_{p/3}) \]
we obtain
\[ D_{p/3} = 3 \left( 1 - \frac{p}{4a} \right) \tag{16a} \]

(16a) implies
\[ D_0 = 3 \tag{16b} \]

which is also confirmed by (11a) that yields
\[ f(\alpha_*(0)) = 3 \tag{11b} \]
f(\(\alpha_*(0)\)) being the fractal dimension of the support of the measure, namely, \(D_0\). It should be noted therefore that in the PPA the support of the measure is not a fractal; consequently, in the PPA the multi-fractality manifests itself via the way the measure is distributed rather than the geometrical properties like the support of the set.\(^7\) In this sense the PPA is complementary to the homogeneous-fractal model (Frisch et al. [61]) in describing the intermittency aspects of FDT. This also implies that the failure to recognize the fractality of the support of the measure is apparently the cause of the well-known inability of PPA (See Castaing et al. [62]) to capture the quantitative aspects of intermittency adequately.

(ii) Kolmogorov-Microscale Regime
In order to determine universal features in the intermittency mechanisms of the various FDT cases, it is necessary to extend considerations to the Kolmogorov-microscale regime. Extension of the PPA to the latter regime has not been done. Let us now proceed to give this formulation.

On extending the multi-fractal scaling to the Kolmogorov-microscale \(\eta_1\), where,
\[ \eta_1 \sim \left( \frac{\nu^3}{\varepsilon} \right)^{1/4} \tag{17} \]
we have (Sreenivasan and Meneveau [63] and Nelkin [64]) -
\[ B_p \equiv \left\langle \left| \frac{\partial v}{\partial x} \right|^p \right\rangle \sim \int R_1^{[p\alpha - p + 3 - f(\alpha)]/\alpha} d\mu(\alpha), \tag{18} \]

\(^7\)A multi-fractal generalizes, as Mandelbrot [5] clarified, the notion of self-similarity from sets to measures.
where \( R_1 \) is the Reynolds number -

\[
R_1 \sim \frac{(\varepsilon L^4)^{1/3}}{\nu}.
\]

Saddle-point evaluation of the integral in (18) yields

\[
(1 + \alpha_*) [p - f'(\alpha_*)] = p\alpha_* - p + 3 - f(\alpha_*).
\]  

(19)

Using (4), (19) leads to

\[
a\alpha_*^2 + 2a\alpha_* + (2p - 2a\alpha_0 - a\alpha_0^2) = 0
\]

(20)

from which,

\[
\alpha_*(p) = -1 \pm \sqrt{(\alpha_0 + 1)^2 - \frac{2p}{a}}.
\]

(21)

Imposing the condition -

\[B_0 \sim 1\]

(22)

which, from (18), implies

\[
\alpha_*(0) = \alpha_0
\]

(23)

we see from (21) that the negative root needs to be discarded, and we obtain

\[
\alpha_*(p) = -1 + \sqrt{(\alpha_0 + 1)^2 - \frac{2p}{a}}.
\]

(24)

On the other hand, using (4) and (19), (18) yields

\[
B_p \sim R_1^{\gamma_p^{(1)}}
\]

(25)

where,

\[
\gamma_p^{(1)} \equiv [p + 2a\{\alpha_*(p) - \alpha_0}\}]
\]

(26)

In order to determine the parameter \( \alpha_0 \), the most pertinent framework for the Kolmogorov-microscale regime appears to be imposing the physical condition of inviscid dissipation of energy
This implies

\[ \nu B_2 \sim R_1^{(1)} - 1 \sim \text{constant} \]  

(27)

from which,

\[ \gamma^{(1)}_2 - 1 = 0. \]  

(28)

Using (24) and (26), (28) yields

\[ \alpha_0 = \frac{1}{3} + \frac{3}{4a} \]  

(29)

which is identical to (28) that was obtained by imposing the exact 3D Navier-Stokes result (7) in the inertial regime! This is of course to be expected because the IDE is incorporated into the exact 3D Navier-Stokes result (7).

Using (29), (24) yields

\[ \alpha^*(p) = -1 + \sqrt{\left( \frac{4}{3} + \frac{3}{4a} \right)^2 - \frac{2p}{a}} \]  

(30)

while (26) then gives

\[ \gamma^{(1)}_p = \left[ p + 2a \sqrt{\left( \frac{16a + 9}{12a} \right)^2 - \frac{2p}{a} - \frac{16a + 9}{6}} \right]. \]  

(31)

For large \( a \), (30) and (4) give the following asymptotic results -

\[ \alpha^*(p) = \frac{1}{3} + (1 - p) \frac{3}{4a} + O \left( \frac{1}{a^2} \right) \]  

(32)

\[ 3 - f(\alpha^*) = \frac{9p^2}{16a} + O \left( \frac{1}{a^2} \right). \]  

(33)

(32) and (33) show that the zero-intermittency limit corresponds to \( a \to \infty \), as before.

(32) shows (as (10) does) that

\[ \alpha^* < \frac{1}{3}, \quad \forall p \geq 2, \]  

(34)

implying of course the strengthening of the velocity-field singularities by intermittency in the Kolmogorov-microscale regime!
(iii) Probability Distribution Function for the Velocity Gradient

The multi-fractal model is known not to afford an analytical calculation of PDF of velocity gradient in intermittent FDT (Benzi et al. [65]). We now wish to show that the PPA is fruitful on this aspect. The physical principle underlying the calculation of the intermittency correction to the PDF of velocity gradient in the PPA turns out to be however the same as the one (namely, IDE) underlying the homogeneous-fractal model used in [65].

Noting the scaling behavior of the velocity gradient (Frisch and She [66]) -

\[ s \sim \frac{v}{\eta_1} \sim v_0^\frac{2}{\eta_1^2} \nu_0^\frac{2-\alpha^*}{\eta_1^2} \tag{35} \]

\( v_0 \) being the velocity increment characterizing large scales, and assuming \( v_0 \) to be gaussian distributed, i.e.,

\[ P(v_0) \sim e^{-\frac{v_0^2}{2\langle v_0^2 \rangle}} \tag{36} \]

we observe

\[ v_0^2 \sim s^{1+\alpha^*}. \tag{37} \]

So, \( \alpha^*(p) \) corresponds to \( \alpha^*(\tilde{p}) \) where \( \tilde{p} \) is the solution of

\[ \tilde{p} = 1 + \alpha^*(\tilde{p}). \tag{38} \]

Using (38), and assuming \( a \) to be large to simplify the calculations, we have from (32),

\[ \alpha^*(\tilde{p}) = \frac{1}{3} - \frac{1}{4a} + O\left(\frac{1}{a^2}\right). \tag{39} \]

Using (39), the PDF of the velocity gradient [66] -

\[ P(s, \alpha^*(\tilde{p})) \sim \left(\frac{\nu}{|s|}\right)^\frac{1-\alpha^*(\tilde{p})}{2} e^{\left[ \frac{\nu(1-\alpha^*(\tilde{p}))}{2\langle v_0^2 \rangle} \right]} \tag{40} \]

becomes

\[ P(s, \alpha^*(\tilde{p})) \sim \left(\frac{\nu}{|s|}\right)^{(\frac{1}{3}+\frac{1}{\pi a})} e^{\left[ \frac{\nu(\frac{4}{3}+\frac{1}{\pi a})}{2\langle v_0^2 \rangle} \right]}. \tag{41} \]
Incidentally, using (39), (38) gives

$$\tilde{p} = \frac{4}{3} - \frac{1}{4a} + O\left(\frac{1}{a^2}\right)$$

(42)

which is of course the exponent of $|s|$ in the argument of the exponential in (41) as to be expected from (37). Note the accentuation of the non-gaussianity of the PDF due to intermittency, as also indicated by the homogeneous-fractal model [65] which, as pointed out before, is however complementary to the PPA in describing the intermittency aspects of FDT.

3 2D FDT: Enstrophy Cascade

The evolution of vorticity in a 2D fluid flow is governed by

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\omega = \nu \nabla^2 \omega$$

(43a)

and is based on the competition between the viscous diffusion and the advection processes.

Writing (43a) in the form

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\mathbf{v} \omega) = \nu \nabla^2 \omega$$

(43b)

we observe that vorticity is globally conserved in the absence of viscous diffusion, so the local maxima of vorticity can grow and possibilities of generation of non-Gaussian statistics for the vorticity PDF exist.

On the other hand, taking the gradient of (43a), we obtain

$$\frac{\partial}{\partial t}(\nabla \omega) + (\mathbf{v} \cdot \nabla)(\nabla \omega) = -\left(\nabla v_x \frac{\partial \omega}{\partial x} + \nabla v_y \frac{\partial \omega}{\partial y}\right) + \nu \nabla^2 (\nabla \omega).$$

(44)

(44) shows that the vorticity gradient is not conserved along the Lagrangian trajectories even in the absense of viscous diffusion, so the local maxima of the vorticity gradient field can grow. This would imply that the statistics of the vorticity gradient can also become non-Gaussian.

We therefore consider intermittency (albeit externally-induced) in the enstrophy cascade of 2D FDT and extend the PPA to formulate this.
(i) Inertial Regime

The multi-fractal model for the $p$th order velocity structure function gives (Shivamoggi [39])

$$A_p \sim \ell^{\xi_p(2)}$$

where,

$$\xi_p^{(2)} = \inf_{\alpha} [p\alpha + 2 - f(\alpha)]$$

(46)

and this minimum $\alpha = \alpha_*$, according to the saddle-point method, corresponds to

$$f'(\alpha_*) = p.$$  

(47)

Assuming a PPA for the 2D case, we have -

$$2 - f(\alpha_*) = a(\alpha_* - \alpha_0)^2, \quad a > 0$$

(48)

Using (48), (47) yields,

$$\alpha_*(p) = \alpha_0 - \frac{p}{2a}.$$  

(49)

Using (48) and (49), (46) becomes

$$\xi_p^{(2)} = p\alpha_0 - \frac{p^2}{4a}.$$  

(50)

Using the exact result for the enstrophy cascade in 2D FDT (Eyink [67] and Lindborg [68])

$$\xi_3^{(2)} = 3$$

(51)

the parameter $\alpha_0$ may be determined -

$$\alpha_0 = 1 + \frac{3}{4a}.$$  

(52)

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8 Equation (51) is also predicted by the multi-fractal model for the enstrophy cascade in 2D FDT (Shivamoggi [39] and [69]) and is in agreement with the rigorous inequalities established by Eyink [70] and has been verified by DNS [49].
Using (55), (53) becomes
\[ \xi_p^{(2)} = p + \frac{1}{4a}(3 - p)p \]  
while (52) becomes
\[ \alpha_s(p) = 1 + \frac{(3 - 2p)}{4a}. \]  
(54) shows that
\[ \alpha_s(p) < 1, \quad \forall p \geq 2 \]  
implying of course the strengthening of the vorticity-field singularities by intermittency! Indeed, (53) predicts an energy spectrum -
\[ E(k) \sim k^{-3 - \frac{1}{2a}} \]  
which is steeper than \( k^{-3} \), as required.

Observe that the intermittency corrections in (50) and (51) are identical to those for the energy cascade in 3D FDT, namely, (8) and (9)!

On the other hand, (53) implies for the structure function of the vorticity field, the scaling behavior -
\[ \langle |\delta\Omega|^p \rangle \sim \xi_p^{(2)} \]  
where,
\[ \xi_p^{(2)} = -\frac{3p}{4a}(p - 3). \]  
Comparing (57) with the multi-fractal result (Benzi and Scardovelli [38])\(^9\)
\[ \zeta_p^{(2)} = -\frac{1}{3}(p - 3)(2 - D_{p/3}) \]  
which implies that the energy spectrum in the enstrophy cascade cannot be steeper than \( k^{-11/3} \) (corresponding to \( D_{p/3} = 0 \)) in agreement with Sulem and Frisch [71] and Pouquet [72]. Further, the result \( E(k) \sim k^{-11/3} \) agrees with the result of Gilbert [73] which considered the dissipative structures to be line elements (with zero fractal dimension for their projections on the plane) - these lines are centers of accumulation of singularities associated with spiral vortex sheets in Gilbert’s model.

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\(^9\) (58) is equivalent to [39]
\[ \xi_p^{(2)} = p - \frac{1}{3}(p - 3)(2 - D_{p/3}) \]
we obtain

\[ D_{p/3} = 2 - \frac{3p}{4a}. \]  

(60a)

Observe that the intermittency correction in (60a) is again identical to that for the energy cascade in 3D FDT, namely, (16a)! Incidentally, the necessity of negative Hölder singularities of the vorticity field to produce intermittency in the enstrophy cascade and to preclude enstrophy conservation in the inviscid limit was noted by Eyink [67].

On the other hand, (60a) implies

\[ D_0 = 2 \]  

(60b)

which signifies that in the PPA the support of the enstrophy-dissipation field in 2D FDT is not a fractal. This has been confirmed for 2D FDT however by the DNS [30].

These results appear to signify universal features in the intermittency mechanisms of 3D energy cascade and 2D enstrophy cascade.

(ii) Kraichnan Microscale Regime

On extending the multi-fractal scaling to the Kraichnan microscale \( \eta_2 \) (Shivamoggi [74]), where,

\[ \eta_2 \sim \left( \frac{\nu^3}{\tau} \right)^{1/6} \]  

(61)

\( \tau \) being the mean enstrophy dissipation rate, we have [39] -

\[ C_p \equiv \left\langle \left| \frac{\partial^2 v}{\partial x^2} \right|^p \right\rangle \sim \int R_2^{\frac{p\alpha - 2p + 2 - f(\alpha)}{1+\alpha}} d\mu(\alpha) \]  

(62)

where \( R_2 \) is the Reynolds number for the 2D FDT -

\[ R_2 \sim \frac{(\tau L^6)^{1/3}}{\nu}. \]

Saddle-point evaluation of the integral in (62) yields

\[ (1 + \alpha_*)[p - f'(\alpha_*)] = p\alpha_* - 2p + 2 - f(\alpha_*). \]  

(63)
Using (48), (63) leads to

\[ a\alpha^2 + 2a\alpha + (3p - 2a\alpha_0 - a\alpha_0^2) = 0 \quad (64) \]

from which,

\[ \alpha(p) = -1 \pm \sqrt{(\alpha_0 + 1)^2 - \frac{3p}{a}}. \quad (65) \]

Imposing the condition -

\[ C_0 \sim 1 \quad (66) \]
on (62), we have,

\[ \alpha(0) = \alpha_0. \quad (67) \]

Using (67), we see that the negative root in (65) is to be discarded, and we obtain

\[ \alpha(p) = -1 + \sqrt{(\alpha_0 + 1)^2 - \frac{3p}{a}}. \quad (68) \]

On the other hand, using (48) and (63), (67) yields

\[ C_p \sim R_2^{(2)} \gamma_p \quad (69) \]

where,

\[ \gamma_p^{(2)} \equiv -[p + 2a(\alpha(p) - \alpha_0)]. \quad (70) \]

In order to determine the parameter \( \alpha_0 \), the most pertinent framework for the Kraichnan-microscale regime in the 2D enstrophy cascade appears to be imposing the physical condition of \textit{inviscid dissipation of enstrophy} (IDÉ) (39) \footnote{Indeed, Polyakov [75] argued that the existence of the enstrophy cascade is predicated on the enstrophy conservation law anomaly, while recent numerical computations (Dmitruk and Montgomery [76]) showed that the tendency, if any, of enstrophy dissipation to go to zero in the inviscid limit is a very weak one.}. This implies

\[ \nu C_2 \sim R_2^{(2) - 1} \sim \text{const} \quad (71) \]
from which,
\[ \gamma_2^{(2)} - 1 = 0. \]  

(72)

Using (68) and (70), (72) yields
\[ \alpha_0 = 1 + \frac{3}{4a} \]  

(73)

which is identical to (52) that was obtained by imposing the 2D multi-fractal model result (51) in the inertial regime! This appears to indicate that the ID\( \hat{E} \) has been incorporated into the result (51) in a manner similar to the case with the Kolmogorov exact result in the 3D incompressible case. Thus, intermittency (albeit externally-induced) can maintain a finite enstrophy along with a vorticity conservation anomaly without contradicting the rigorous result in [20] and [21].

Using (73), (68) yields
\[ \alpha_s(p) = -1 + \sqrt{\left( 2 + \frac{3}{4a} \right)^2 - \frac{3p}{a}} \]  

(74)

while (70) then gives
\[ \gamma_{p}^{(2)} = - \left[ p + 2a \sqrt{\left( \frac{3 + 8a}{4a} \right)^2 - \frac{3p}{a} - \frac{3 + 8a}{2}} \right]. \]  

(75)

The intermittency corrections \( \Delta \alpha_s(p) \) to the scaling exponent \( \alpha_s(p) \) for the 3D (given by (30)) and the 2D (given by (74)) are sketched in Figures 1a and 1b for different values of the intermittency parameter \( a \). Observe that the velocity-field singularities are strengthened in the microscale regimes by the intermittency effects. Intermittency corrections \( \Delta \alpha_s(p) \) for 2D cases are smaller than those for 3D cases. Further, in the weak-intermittency limit \( (a \text{ large}) \), the intermittency corrections \( \Delta \alpha_s(p) \) for the 3D and 2D cases are almost identical.

For large \( a \), (74) and (48) give the following asymptotic results -
\[ \alpha_s(p) = 1 + (1 - p) \frac{3}{4a} + O \left( \frac{1}{a^2} \right) \]  

(76)

\[ 2 - f(\alpha_s) = \frac{9p^2}{16a} + O \left( \frac{1}{a^2} \right). \]  

(77)

Observe that the intermittency corrections in (76) and (77) for the 2D enstrophy cascade microscale regime are identical to those in (32) and (33) for the 3D energy cascade microscale regime! This
appears to confirm further the universal features in the intermittency mechanisms of 3D energy cascade and 2D enstrophy cascade pointed out earlier.

(iii) Probability Density Function for the Vorticity-Gradient

Noting the scaling behavior of the vorticity gradient (Shivamoggi [39]) -

\[ r \sim \frac{v}{\eta^2} \sim v_0^{1+\alpha_s} \nu_0^{\alpha_s-2} \]

(78)

and assuming again, we observe

\[ v_0^2 \sim r^{3(1+\alpha_s)} \]

(79)

So, \( \alpha_s(p) \) corresponds to \( \alpha_s(\tilde{p}) \) where \( \tilde{p} \) is now the solution of

\[ \tilde{p} = \frac{2}{3}[1 + \alpha_s(\tilde{p})] \]

(80)

Using (80), and assuming again \( a \) to be large, we have from (76),

\[ \alpha_s(\tilde{p}) = 1 - \frac{1}{4a} + O \left( \frac{1}{a^2} \right) \]

(81)

Using (81), the PDF of the vorticity gradient [39] -

\[ P(r, \alpha_s(\tilde{p})) \sim \left( \frac{\nu}{|r|} \right)^{\frac{1}{3}(1+\frac{1}{4a})} e^{\left[ \frac{\nu^2(2-\alpha_s(\tilde{p}))}{|r|^2} \right]^{\frac{2}{3}(1+\alpha_s(\tilde{p}))} \frac{\nu^2}{2\langle v_0^2 \rangle}} \]

(82)

becomes

\[ P(r, \alpha_s(\tilde{p})) \sim \left( \frac{\nu}{|r|} \right)^{\frac{1}{3}(1+\frac{1}{4a})} e^{\left[ \frac{\nu^2(1+\frac{1}{4a})}{|r|^2} \right]^{\frac{2}{3}(2-\frac{1}{4a})} \frac{\nu^2}{2\langle v_0^2 \rangle}} \]

(83)

Using (81), (80) gives

\[ \tilde{p} = \frac{2}{3} \left( 2 - \frac{1}{4a} \right) + O \left( \frac{1}{a^2} \right) \]

(84)

which is again the exponent of \( |r| \) in the argument of the exponential in (83), as to be expected from (79). Note again the accentuation of the non-gaussianity of the PDF due to intermittency, as also indicated by the homogeneous-fractal model [74] which is however complementary to the PPA.
4 Conclusions

The theoretical issue of intermittency in the 2D enstrophy cascade in view of the regular behavior of 2D Navier-Stokes solutions is a delicate one. It is of interest to note that, irrespective of the origin, an externally-induced intermittency is able to restore the usual nonlinearity-sustained cascade mechanism in the Batchelor-Kraichnan theory. Intermittency renders enstrophy of the flow finite so a nonlinearity sustained cascade is very much needed to transfer enstrophy across the inertial range to small scales to counter the dissipative action of viscosity there. Further, as we have shown in Section 3 (ii), intermittency can maintain a finite enstrophy along with a vorticity conservation anomaly without contradicting the rigorous result in [20] and [21].

Intermittency mechanisms of 3D energy cascade and 2D enstrophy cascade appear to have certain universal features notwithstanding very different physics underlying these FDT cases. This probably results, following Kadanoff’s [77] speculation, because of consideration of the limit $R \Rightarrow \infty$, which happens to be the critical point for FDT. Indeed, universal features for very diverse FDT cases become apparent at the critical point (Shivamoggi [78]). On the other hand, via (14) this appears to support the universality of the log-normal exponent $\mu$ hypothesized by Kolmogorov [59] and Obukhov [60]. It remains to be pointed out, however, though the PPA appears to have the capacity to provide considerable insight into the qualitative aspects of intermittent FDT quantitative aspects have been a different story for the 3D FDT problem (see Castaing et al. [62]). This is probably traceable to the failure of PPA to recognize the fractality of the support of the measure, as shown in this paper.

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Figure 1a. Intermittency correction $\Delta \alpha_n(p)$ vs. $p$ for the intermittency parameter $a = 10$ (— 3D, - - - 2D).

Figure 1b. Intermittency correction $\Delta \alpha_n(p)$ vs. $p$ for the intermittency parameter $a = 100$ (— 3D, - - - 2D).