Submodular Maximization under Fading Model:
Building Online Quizzes for Better Customer Segmentation

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E-Commerce personalization aims to provide individualized offers, product recommendations, and other content to customers based on their interests. The foundation of any personalization effort is customer segmentation. The idea of customer segmentation is to group customers together according to identifiable segmentation attributes including geolocation, gender, age, and interests. Personality quiz turns out to be a powerful tool that enables customer segmentation by actively asking them questions, and marketers are using it as an effective method of generating leads and increasing e-commerce sales. In this paper, we study the problem of how to select and sequence a group of quiz questions so as to optimize the quality of customer segmentation. In particular, we use conditional entropy to measure the utility of a given group of quiz questions. Our objective is to compute a sequence of quiz questions that lead to the maximum utility. We model the user behavior when interacting with a sequence of quiz questions as a Markov process. Then we develop a series of question selection and sequencing strategies with provable performance bound.

Key words: active learning, consumer segmentation, online quizzes, personalization.

History:

1. Introduction

E-Commerce personalization has been recognized as one of the most effective methods in increasing sales (Yang and Padmanabhan 2005, Ansari and Mela 2003). For example, Amazon and other retail giants provide personalized product recommendations based on their customers’ interest. Gartner predicts that by 2020, those who successfully handle personalization in E-Commerce will increase their profits by up to 15%. The starting point of any personalization effort is to obtain a clearer picture of individual customers, this is often done by customer segmentation, e.g., partition a customer base into groups of individuals with similar characteristics that are relevant to marketing, such as a geographic location, interests, time of visit, etc. Since customer segmentation relies on both the quality and quantity of data collected from customers, it is critical to decide what data will be collected
and how it will be collected. For returning customers, we can use their past behavior such as browsing history to perform customer segmentation and further personalize her current experience. But how to decide the segment a new customer belongs to without knowing her browsing history? One popular approach to gather data from first-time customers is personality quiz. The purpose of personality quiz is to segment every potential and current customer by actively asking them questions. After answering a few questions, customers are matched with the type of recommendations or product that best suit their responses. Marketers are starting to use it as an effective method of generating leads and increasing e-commerce sales. For example, several websites such as Warby Parker and ipsy are using personality quizzes to determine users’ interest profiles and make recommendations. As compared with other preference elicitation methods, personality quiz-based system requires significantly less effort from users and they expressed stronger intention to reuse this system and introduce it to others (Hu and Pu 2009a,b). Although the benefit of personality quiz has been well recognized, it is not clear, in general, how to optimize the quiz design so as to maximize this benefit. In this paper, we formulate the quiz design problem as a combinatorial optimization problem. We aim at selecting and further sequencing a group of questions from a given pool so as to maximize the quality of customer segmentation. While the design of each individual question such as formatting, coloring, the way the question is asked can be complex and important (Couper et al. 2001), that topic is out of scope of this paper. We exclusively focus on the question selection and sequencing problem by assuming that all candidate questions are pre-given.

The input of our problem is a set of attributes and a pool of candidate questions. We say a question covers an attribute if the answer to that question reveals the value of that attribute. For example, question “Where are you living?” covers attribute “location”. Intuitively, a group of “good” questions should cover as many important attributes as possible so as to minimize the uncertainty about the user. Given answers to a group of questions, we measure its uncertainty using the conditional entropy of the uncovered attributes of the user. Our ultimate goal is to select and sequence a group of questions so as to minimize the uncertainty subject to a set of practical constraints.

1https://www.warbyparker.com/
2https://www.ippsy.com/
In general, our problem falls into the category of non-adaptive active learning based user profiling. The idea of most existing studies is to actively select a group of items, e.g., movies or cars, and asking for users’ feedback on them. This feedback, in turn, can help to enhance the performance of recommendation in the future. However, they often assume that the users are willing to provide feedback on all selected items, irrespective of which items are selected and in what sequence. As a consequence, their problem is reduced to a subset selection problem. We argue that this assumption may not hold in our problem, e.g., it has been shown that not all users are willing to share their personal information with a site. According to the survey conducted by Culnan (2001), two of three users abandon sites that asks for personal information and one of five users has provided false information to a site. This motivates us to consider a realistic but significantly more complicated user behavior model. Our model captures the externality of a question by allowing the user to “opt-out” of answering a question or even quit the quiz prematurely after answering some questions. A more detailed comparison between our work and related work is presented in Section 2. We next give a brief overview to some important constraints considered in this paper.

1.1. Cardinality Constraint
We can select up to $b$ questions to include in the quiz where $b$ is some positive integer. For example, it has been shown that 6-8 questions per quiz could be an appropriate setting since it maximizes completions and leads generated (https://socialmediaexplorer.com/content-sections/tools-and-tips/how-to-make-a-personalized-quiz-to-drive-sales/).

1.2. User Behavior
Our setting considers that the user behavior during a personality quiz can be described as a Markov process (a detailed description of this model is presented in Section 3.1). The user interacts with a sequence of questions in order, after reading a question, she decides probabilistically whether or not to answer it with some question specific probability, called answer-through-rate. In principle, this probability could depend on many factors including the cognitive efforts required for understanding and answering the question, and the sensitivity of the question, etc. Our model also allows the user to select “Prefer Not to Answer” (PNA) option, if any, to avoid answering a particular question. A more detailed discussion on PNA option is provided in the next subsection. In addition, each question has
a *continuation probability*, representing the likelihood that the user is willing to continue the quiz after interacting with the current one. This continuation probability captures the *externality* of a question, e.g., a very sensitive or lengthy question could cause the user to exit the quiz prematurely. The existence of such externality makes our problem even more complicated, e.g., the ordering of selected questions matters. For example, Typeform\(^3\), an online software as a service (SaaS) company that specializes in online form building and online surveys, suggests that it is better to put sensitive and demographic questions at the end of a quiz or survey: “Starting a survey with intimidating or demographic questions like age and income can put people off. Your first survey question should be interesting, light, and easy to answer. Once they've started, they’re more likely to finish and answer more sensitive questions.”

### 1.3. PNA option

Regarding the role played by PNA option in a quiz, there exist two contradicting arguments. On the one hand, several studies (Schuman and Presser 1996, Hawkins and Coney 1981) empirically demonstrate that the data and subsequent analyses will better off by including a PNA option due to it decreases the proportion of uninformed responses. On the other hand, opponents believe that providing a PNA option could negatively impact the quality of the answer because some users tend not to answer the question so as to minimize the effort required to complete the quiz (Poe et al. 1988, Sanchez and Morchio 1992). Since both arguments are empirically validated by previous studies, we decide to cover both cases in this work.

We next summarize the contributions made in this paper. We first show (in Section 3.2) that our problem subject to the above constraints is NP-hard. Then we develop a series of effective solutions with provable performance bound. For the case where PNA is not an option (in Section 4), our algorithm achieves an approximation ratio that is arbitrarily close to \( \frac{1}{4(1-1/e)} \) where \( e \) is a constant whose value is arbitrarily close to 2.718. For the case where PNA is available (in Section 5), our algorithm achieves an approximation ratio that is arbitrarily close to \( \frac{1}{4(1-1/e)^2} \). We subsequently consider a extension of the basic model by taking into account the slot-dependent decay factor (in Section 6). We assume that the answer-through-rate of a question does not only depend on its intrinsic quality, but

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3 [https://www.typeform.com/surveys/question-types/](https://www.typeform.com/surveys/question-types/)
also depends on its position. For the question selection and sequencing problem under this model, our algorithm achieves an approximation ratio that is arbitrarily close to \( \frac{1}{4(1-1/e)} \) when PNA is not an option (resp. is an option).

Most of notations are listed in Table 1.

| Notation | Description |
|----------|-------------|
| \( Q \)  | A set of questions. |
| \( \hat{Q} \) | A sorted sequence of \( Q \). |
| \( q(Q) \) | The utility of answers to \( Q \). |
| \( f(Q) \) | The expected utility of displaying \( Q \) to the user. |
| \( p_q^+ \) (resp. \( p_q^- \)) | Probability of answering (resp. PNA) \( q \) after reading it. |
| \( c_q^+ \) (resp. \( c_q^- \)) | Probability of continuing to read the next question after answering (resp. PNA) \( q \). |
| \( c_q \) | Aggregated continuation probability of \( q \): \( p_q^+ c_q^+ + p_q^- c_q^- \). |
| \( C_q^Q \) | Reachability of \( q \) given \( Q \) is displayed to the user. |
| \( Q_1 \oplus Q_2 \) | Concatenation of two sequences \( Q_1 \) and \( Q_2 \). |
| \( Q_{\leq i} \) (resp. \( Q_{<i} \), \( Q_{\geq i} \)) | The subsequence of \( Q \) which is scheduled no later than (resp. before, after, no earlier than) slot \( i \). |
| \( \mathcal{J}(Q) \) | A random set of questions that are answered by the user given \( Q \). |
| \( \mathcal{R}(Q) \) or \( \mathcal{R}(Q) \) | A random set obtained by including each question \( q \in Q \) independently with probability \( p_q^+ \). |

2. Literature Review

Our paper falls into the general category of non-adaptive active learning supported personalization. This section reviews the literature on three topics that are closely related to our research.

2.1. Active Learning based Recommender System

Active learning, as a subfield of machine learning \cite{Bishop2006}, has been widely used in the design of effective recommender systems. For the purpose of acquiring training data, active learning based recommender system often actively solicits customer feedback on a group of carefully selected items \cite{Kohrs2001, Rubens2007, Golbandi2011, Chang2015}. Existing systems can be further classified into two categories: adaptive learning and non-adaptive learning. Non-adaptive learning refers to those learning strategies who require all users to rate the same set of items while adaptive learning \cite{Boutilier2002, Golbandi2011, Rubens2007} could propose different items to different users to rate. Since our paper belongs to the category of non-adaptive learning, we next give a detailed review to the state-of-art of
non-adaptive learning based recommender system. Depending on the item selection rule, there are three types of strategies: uncertainty-reduction, error reduction and attention based. The goal of uncertainty-reduction based systems is to reduce the uncertainty about the users’ opinion about new items, they achieve this by selecting items with highest variance (Kohrs 2001, Teixeira et al. 2002) or highest entropy (Rashid et al. 2002) or highest entropy (Rashid et al. 2008). The goal of error reduction based systems is to minimize the system error (Golbandi et al. 2010, Liu et al. 2011, Cremonesi et al. 2010). The idea of attention-based strategy is to select the items that are most popular among the users (Golbandi et al. 2010, Cremonesi et al. 2010). Our problem is largely different from all aforementioned studies in terms of both application context and problem formulation: (1) Instead of investigating a particular recommender system, we study a general customer segmentation problem whose solution serves as the foundation of any personalized service; (2) We are dealing with a significantly more complicated user behavior model where the user is allowed to pick PNA option or terminate the quiz prematurely. All aforementioned studies assume that the users are guaranteed to rate all selected items, regardless of the sequence of those items, thus their problem is reduced to a subset selection problem; (3) Most of existing studies are developing heuristics without provable performance bound, we develop the first series of algorithms that achieve bounded approximation ratios.

2.2. Learning Offer Set and Consideration Set

The other two related topics are “offer set” (Atahan and Sarkar 2011) and “consideration sets” (Roberts and Lattin 1991). Our problem differs from both “offer set” and “consideration sets” in fundamental ways. The focus of offer set is to investigate how the profile learning process can be accelerated by carefully selecting the links to display to the user. In (Atahan and Sarkar 2011), users implicitly compare alternative links and reveal their preferences based on the set of links offered, this is different from our model where users are explicitly asked to answer questions. The literature on consideration sets aims at determining the subset of brands/products that a customer may evaluate when making purchase decision. Their model did not capture the externality of a question e.g., the users are forced to answer all questions, thus the sequence of questions did not play a role in their non-adaptive solution. In addition, most of aforementioned studies did not provide any theoretical bounds on their proposed solutions. We consider a joint question selection and sequencing problem which is proved to be NP-hard (in Section 3.2), and theoretically bound the gap between our solution and the optimal solution.
2.3. Submodular Optimization

We later show that our problem is a submodular maximization problem. Although submodular maximization has been extensively studied in the literature (Nemhauser and Wolsey 1978, Nemhauser et al. 1978, Nemhauser and Wolsey 1981, Kawahara et al. 2009, Calinescu et al. 2011), most of them focus on subset selection problem where the ordering of selected elements does not affect its utility. Our work differs from theirs in that we consider a joint selection and sequencing problem. The only study that considers sequencing problem is (Tschiatschek et al. 2017), however their model and problem formulation are largely different from ours. They use a directed graph to model the ordered preferences and their objective is to find a sequence of nodes that covers as many edges in the directed graph as possible. Their objective function is not always submodular, and their formulation does not involve any subset selection, because, by default, they can select all elements. Although we restrict our attention to the question selection and sequencing problem in this paper, our research contributes fundamentally to the field of submodular subset selection and sequencing maximization.

3. Preliminaries and Problem Formulation

3.1. Preliminaries

3.1.1. Submodular Function and Correlation Gap. A set function $h(\mathcal{Y})$ that maps subsets of a finite ground set $\Omega$ to non-negative real numbers is said to be submodular if for every $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \Omega$ with $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$ and every $y \in \Omega \setminus \mathcal{Y}_2$, we have that

$$h(\mathcal{Y}_1 \cup \{v\}) - h(\mathcal{Y}_1) \geq h(\mathcal{Y}_2 \cup \{y\}) - h(\mathcal{Y}_2)$$

A submodular function $h$ is said to be monotone if $h(\mathcal{Y}_1) \leq h(\mathcal{Y}_2)$ whenever $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$.

We next present a useful result about any submodular function. For any distribution $\theta$ on $2^\mathcal{N}$, let $\theta_y$ be the marginal probability that $y$ is included and let $\mathcal{R}(\theta)$ be a random set independently containing each element $y$ with probability $\theta_y$. The correlation gap of $h$ is

$$\inf_{\theta} \frac{\mathbb{E}_{y \sim \theta}[h(\mathcal{Y})]}{\mathbb{E}[h(\mathcal{R}(\theta))]}$$

Intuitively, the correlation gap is the maximum ratio between the expected value of a function when the random variables are correlated to its expected value when the random variables are independent.

**Lemma 1** (Agrawal et al. 2012) The correlation gap of a monotone and submodular function is upper bounded by $1/(1 - 1/e)$. 

3.1.2. Utility of Answered Questions Consider any group of questions $S \subseteq \Psi$, we use $g(S)$ to represent the utility of $S$ given $S$ has been answered by the user. Intuitively, obtaining answers to a group of “good” questions should reduce the uncertainty and provide better insights on the user.

**Assumption 1** *In this work, we assume that $g(S)$ is non-decreasing and submodular.*

We next give a concrete example to show that an entropy-like utility function $g(S)$ is indeed non-decreasing and submodular.

**An Example of Entropy-like Utility Function.** Assume there are $m$ attributes $\Phi$ and $n$ questions $\Psi$. We say question $q \in \Psi$ covers attribute $a \in \Phi$ if the answer to $q$ reveals the value of $a$. We say a group of questions $S \subseteq \Psi$ covers $a$ if $a$ can be covered by at least one question from $S$. We use $A(S)$ to denote the set of all attributes that can be covered by $S$. One common notation of uncertainty is the conditional entropy of the unobserved attributes of a user after answering $S$.

$$H(X_{A\setminus A(S)}|X_{A(S)}) = - \sum_{x_{A\setminus A(S)} \in \text{dom}X_{A\setminus A(S)}} P(x_{A\setminus A(S)}; x_{A(S)}) \log P(x_{A\setminus A(S)}|x_{A(S)})$$ (1)

where we use $X_{A\setminus A(S)}$ and $X_{A(S)}$ to denote sets of random variables associated with attributes in $A \setminus A(S)$ and $A(S)$. Intuitively, a group of “good” questions $S$ would minimize Eq. (1). Based on the chain-rule of entropies, we have $H(X_{A\setminus A(S)}|X_{A(S)}) = H(X_A) - H(X_{A(S)})$. Due to $H(X_A)$ is fixed, minimizing Eq. (1) is reduced to maximizing $H(X_{A(S)})$. Therefore, it is reasonable to define the utility of $S$ as $g(S) = H(X_{A(S)})$ and we next show that $H(X_{A(S)})$ is non-decreasing and submodular.

**Lemma 2** $H(X_{A(S)})$ is non-decreasing and submodular.

The proof is provided in the appendix.

3.1.3. Question Scanning Process We use a Markov process to model the user’s behavior when interacting with a sequence of quiz questions. Our model is similar to the Cascade Model (Craswell et al. 2008) that provides the best explanation for position bias of organic search results. We define the *answer-through-rate* $p_q^+ \in [0, 1]$ of a question $q \in \Psi$ as the probability that the user chooses to answer $q$ after reading it. In principle, this probability could depend on many factors including the cognitive efforts required for understanding
and answering the question, question sensitivity, etc. Instead of answering \( q \), the user may also select “Prefer Not to Answer” (PNA) option, if any, with probability \( p_q^- \) to avoid answering \( q \), or simply exit the quiz with probability \( 1 - (p_q^+ + p_q^-) \). In addition to the intrinsic \( p_q^+ \) and \( p_q^- \), each question \( q \) is also associated with a continuation probability \( c_q^+ \) and \( c_q^- \): \( c_q^+ \) (resp. \( c_q^- \)) represents the probability that the user will continue to read the next question after answering (resp. PNA) \( q \).

We summarize the question scanning process of a user as follows.

- Starting with question \( q_1 \) placed at the first slot.
- After reading \( q_i \), the user chooses one of the following five actions to take:
  1. Answer \( q_i \) and
     (a) continue to read the next question with probability \( p_q^+ c_q^+ \);
     (b) exit the quiz with probability \( p_q^+ (1 - c_q^+) \).
  2. PNA \( q_i \) and
     (a) continue to read the next question with probability \( p_q^- c_q^- \);
     (b) exit the quiz with probability \( p_q^- (1 - c_q^-) \).
  3. Exit the quiz with probability \( 1 - (p_q^+ + p_q^-) \).
- The above process repeats until the user exits the quiz or no more questions remain.

Some Basics: Throughout this paper, we use capital letter to denote sequence and calligraphic letter to denote set. For example, \( Q \) denotes a set of questions and \( Q \) denotes a sorted sequence of \( Q \). For a given sequence of questions \( Q \), let \( q_i \) denote the question scheduled at slot \( i \), we use \( Q_{\leq i} \) (resp. \( Q_{<i} \), \( Q_{>i} \), \( Q_{\geq i} \)) to denote the subsequence of \( Q \) which is scheduled no later than (resp. before, after, no earlier than) slot \( i \). Given two sequences \( Q_1 \) and \( Q_2 \), we define \( Q_1 \oplus Q_2 \) as a new sequence by first displaying \( Q_1 \) and then displaying \( Q_2 \). For notational simplicity, we define \( c_q = p_q^+ c_q^+ + p_q^- c_q^- \) as the aggregated continuation probability of \( q \).

We next introduce an important definition.

**Definition 1 (Reachability of a Question)** Given that a sequence of questions \( Q \) is displayed to the user, we define the reachability \( C_q^Q \) of \( q_i \in Q \) as the probability that \( q_i \) will be read:

\[
C_q^Q = \prod_{q \in Q_{<i}} c_q
\]

In Section 6 we also consider an advanced model by assuming slot-dependent answer-through-rate, e.g., assume \( q \) is scheduled at slot \( i \), the answer-through-rate of question \( q \) is \( \lambda_i p_q^+ \) where \( \lambda_i \) is a slot-dependent decay factor.
3.2. Problem Formulation

Given any sequence of questions \( Q \), we define its expected utility as

\[
    f(Q) = \sum_{S \subseteq \Omega} \Pr[S|Q]g(S)
\]

where \( \Pr[S|Q] \) denotes the probability that we can receive answers to \( S \) given that \( Q \) is displayed to the user. Our objective is to identify the best sequence of questions subject to a cardinality constraint. We next present the formal definition of our problem.

\[
\text{P1 max } f(Q) \\
\text{subject to: } |Q| \leq b;
\]

The following theorem states that this problem is intractable in general.

**Theorem 1** Problem P1 is NP-hard.

The proof is provided in the appendix.

4. Warming Up: Question Selection and Sequencing with No PNA Option

We first study the case where “PNA” is not an option. In other words, the user is left with two options after reading the current question: either answer it or exist the quiz. The reason for investigating this restricted case is twofold: (1) Although the benefit of including PNA option has been discussed in many existing work (Schuman and Presser 1996, Hawkins and Coney 1981), opponents believe that providing a PNA option could have negative impact on the quality of the answer because some users tend not to answer the question so as to minimize the effort required to complete the quiz by simply ticking PNA option (Poe et al. 1988, Sanchez and Morchio 1992). Since both arguments are empirically validated by previous studies, we decide to study both cases in this work. (2) Technically speaking, the case with no PNA option is a special case of the original problem (by setting \( p_q^- = 0 \) for every \( q \) in the original problem), starting with this simplified case makes it easier to explain our approach used to solve the general case.

We first present a simplified question scanning process under this restricted case as follows:
• Starting with question $q_1$ placed at the first slot.
• After reading question $q_i$, the user chooses one of the following three actions to take:
  1. Answer $q_i$ and
     (a) continue to read the next question with probability $p^+ c^+_{q_i}$;
     (b) exit the quiz with probability $p^+ (1 - c^+_{q_i})$.
  2. Exit the quiz with probability $1 - p^+$.  
• The above process repeats until the user exits the quiz or no more questions remain.

4.1. Algorithm Design

The general framework for our method is inspired by the early work of (Kempe and Mahdian 2008), however, their approach only works for linear objective function. Before presenting our algorithm, we first introduce a useful property of any optimal solution. In particular, given an optimal solution $Q^*$, we show that little is lost by discarding those questions whose reachability is sufficiently small.

Lemma 3 For any $\rho \in [0, 1]$, there is a solution $Q_\rho$ of value at least $(1 - \rho) f(Q^*)$ such that $|Q_\rho| \leq b$ and $\forall q \in Q_\rho : C_{Q_\rho}^q \geq \rho$.

Proof: Let $q_i^*$ denote the $i$-th question in $Q^*$. Assume $q_k^*$ is the last question in $Q^*$ whose reachability is no smaller than $\rho$, e.g., $k = \text{arg max}_i (C_{Q^*_\leq k}^{q_i^*} \geq \rho)$. Recall that we use $Q_{>k}^*$ (resp. $Q_{\leq k}^*$) to denote the sequence of questions scheduled after (resp. no later than) slot $k$. Therefore the reachability of every question in $Q_{\leq k}^*$ is no smaller than $\rho$.

We first show that $C_{q_k^*}^{Q^*} c_{q_k^*} f(Q_{>k}^*) \geq f(Q^*) - f(Q_{\leq k}^*)$. Let $e_1$ denote the event that $S \subseteq Q_{\leq k}^*$ is answered by the user. Let $e_2$ denote the event that the first question of $Q_{>k}^*$ has been read and $A \subseteq Q_{>k}^*$ is answered by the user.

$$f(Q^*) = \sum_{e_1, e_2} \Pr[e_1](f(S) + \Pr[e_2|e_1](f(S \cup A) - f(S)))$$

$$\leq \sum_{e_1, e_2} \Pr[e_1](f(S) + \Pr[e_2|e_1]f(A))$$

$$= \sum_{e_1, e_2} \Pr[e_1] f(S) + \sum_{e_1, e_2} \Pr[e_2|e_1] f(A)$$

$$\leq f(Q_{\leq k}^*) + C_{q_k^*}^{Q^*} c_{q_k^*} f(Q_{>k}^*)$$

Inequality (2) is due to $f$ is a submodular function and Inequality (3) is due to given $Q^*$, the first question of $Q_{>k}^*$ can be reached with probability $C_{q_k^*}^{Q^*} p^+ c^+_{q_k^*}$. 

It follows that \( f(Q^*_{\leq k}) \geq f(Q^*) - C_{q^*_{\leq k}} c_{q^*_{\leq k}} f(Q^*_{> k}) \). Since \( C_{q^*_{\leq k}} c_{q^*_{\leq k}} < \rho \) due to the definition of \( k \), and \( f(Q^*_{> k}) \leq f(Q^*) \) due to \( Q^* \) is the optimal solution, we have \( f(Q^*_{\leq k}) \geq (1 - \rho) f(Q^*) \). Since every question in \( Q^*_{\leq k} \) can be reached with probability at least \( \rho \) and \( |Q^*_{\leq k}| \leq b \), \( Q^*_{\leq k} \) is a valid \( Q_{\rho} \).

Lemma 3 allows us to ignore those questions whose reachability is small, at the expense of a bounded decrease in utility. This motivates us to introduce a new problem \( P_2 \) by only considering those questions whose reachability is sufficiently high. The objective function of \( P_2 \) is

\[
u(q, S) = p^+_q g(S \cup \{q\}) + (1 - p^+_q) g(S)
\]

The goal of \( P_2 \) is to find a solution \((q, S)\) that maximizes \( u(q, S) \) subject to three constraints. After solving \( P_2 \) (approximately) and obtaining a solution \((q', S')\), we build the final solution to the original problem based on \((q', S')\).

**Lemma 4** For any fixed \( q \), \( u(q, S) \) is a submodular function of \( S \).

**Proof:** We first show that for any fixed \( q \), \( g(S \cup \{q\}) \) is submodular as a function of \( S \). For any \( S_1 \subseteq S_2 \subseteq \Psi \) and \( q' \notin S_2 \), we have \( g(S_1 \cup \{q'\} \cup \{q\}) - g(S_1 \cup \{q\}) \geq g(S_2 \cup \{q'\} \cup \{q\}) - g(S_2 \cup \{q\}) \) due to \( g(S) \) is submodular and \( S_1 \cup \{q\} \subseteq S_2 \cup \{q\} \). Thus, \( g(S \cup \{q\}) \) is a submodular function of \( S \). It follows that for any fixed \( q \), \( u(q, S) = p^+_q g(S \cup \{q\}) + (1 - p^+_q) g(S) \) is submodular due to the linear combination of two submodular functions is submodular. □
As a consequence of Lemma 4, for any fixed $q$, $P_2$ is a submodular maximization problem subject to two linear constraints (constraints (C1) and (C2)), and there exists a $\left(1-1/e-\epsilon\right)$ approximate algorithm to this problem (Kulik et al. 2009) where $e$ is a constant whose value is arbitrarily close to 2.718. In order to solve $P_2$, we exhaustively try all questions $q$ which will be scheduled at the last slot, for each $q$, we run a $\left(1-1/e-\epsilon\right)$ approximate algorithm to obtain a candidate solution $S$. Among all candidate solutions, assume $(q', S')$ has the largest utility, $S' \oplus q'$ is returned as the final solution to the original problem where $S'$ is an arbitrary sequence of $S'$.

We present the detailed description of our solution in Algorithm 1.

**Algorithm 1** Question Selection and Sequencing with No PNA option

**Input:** $\rho, b, A$.

**Output:** $Q_{\text{Alg1}}$.

1. Set $S' = \emptyset, q' = \emptyset$.
2. for $q \in A$ do
   3. Fix $q$, apply a $\left(1-1/e-\epsilon\right)$ approximate algorithm (Kulik et al. 2009) to solve $P_2$ and obtain $S$
   4. if $u(q, S) > u(q', S')$ then
      5. $S' \leftarrow S, q' \leftarrow q$
   6. $Q_{\text{Alg1}} \leftarrow S' \oplus \{q'\}$ where $S'$ is an arbitrary sequence of $S'$
3. return $Q_{\text{Alg1}}$

**4.2. Performance Analysis**

We next analyze the performance bound of Algorithm 1. We first present some preparatory lemmas. Since for each $q$, we find a $\left(1-1/e-\epsilon\right)$ approximate solution and $(q', S')$ has the maximum utility among all returned solutions, the following lemma holds.

**Lemma 5** $(q', S')$ is a $\left(1-1/e-\epsilon\right)$ approximate solution to $P_2$.

Now we are ready to provide a performance bound on the final solution $Q_{\text{Alg1}}$. We first show that for any $\rho \in [0, 1]$, $f(Q_{\text{Alg1}}) \geq \rho u(q', S')$, i.e., the utility of $Q_{\text{Alg1}}$ is close to the value of $u(q', S')$.

**Lemma 6** For any $\rho \in [0, 1]$, $f(Q_{\text{Alg1}}) \geq \rho u(q', S')$ where $Q_{\text{Alg1}}$ is composed of $S'$, which is an arbitrary sequence of $S'$, and $q'$ (refer to Line 6 of Algorithm 1 for details).
Proof: Due to $S'$ satisfies constraint (C2) in problem P2, we have $\prod_{q \in S'} p^+_q c^+_q \geq \rho$. It follows that with probability at least $\rho$, all questions in $S'$ will be answered and $q'$ will be read. Moreover, the probability that $q'$ will be answered by the user is $p^+_q$ conditioned on all questions in $S'$ are answered and $q'$ is read. It follows that $f(Q^{\text{Alg1}}) \geq \rho p^+_q g(S' \cup \{q'\}) + \rho(1 - p^+_q)g(S') = \rho u(q', S')$.

We present the main theorem as follows.

**Theorem 2** For any $\rho \in [0, 1]$, $f(Q^{\text{Alg1}}) \geq \rho(1 - \rho)(1 - 1/e - \epsilon)f(Q^*)$

Proof: For any $\rho > 0$, let $Q^\rho$ denote the optimal solution subject to $\forall q \in Q : C^Q_q \geq \rho$ and $|Q| \leq b$. Lemma 3 implies $f(Q^\rho) \geq (1 - \rho)f(Q^*)$. Therefore, in order to prove this theorem, it suffices to prove $f(Q^{\text{Alg1}}) \geq \rho(1 - 1/e - \epsilon)f(Q^\rho)$.

Assume $|Q^\rho| = z$, let $Q^\rho \setminus \{q^\rho_z\}$ denote the subsequence of $Q^\rho$ by excluding the last question $q^\rho_z$. Because $\forall q \in Q^\rho : C^Q_q \geq \rho$ and $|Q^\rho| < b$, $(q^\rho_z, Q^\rho \setminus \{q^\rho_z\})$ is a valid solution to problem P2. Therefore, $u(q', S') \geq (1 - 1/e - \epsilon)u(q^\rho_z, Q^\rho \setminus \{q^\rho_z\})$ due to Algorithm 1 finds a $(1 - 1/e - \epsilon)$ approximate solution to P2 (Lemma 3). We next prove that $f(Q^\rho) \leq u(q^\rho_z, Q^\rho \setminus \{q^\rho_z\})$.

\[
\begin{align*}
f(Q^\rho) &= f(Q^\rho \setminus \{q^\rho_z\}) + f(Q^\rho \setminus \{q^\rho_z\}) \\
&= f(Q^\rho \setminus \{q^\rho_z\}) + C^Q_{q^\rho_z} p^+_q (g(Q^\rho) - g(Q^\rho \setminus \{q^\rho_z\})) \\
&\leq g(Q^\rho \setminus \{q^\rho_z\}) + p^+_q (g(Q^\rho) - g(Q^\rho \setminus \{q^\rho_z\})) \\
&= u(q^\rho_z, Q^\rho \setminus \{q^\rho_z\})
\end{align*}
\]

Inequality (1) is due to $f(Q^\rho \setminus \{q^\rho_z\}) \leq g(Q^\rho \setminus \{q^\rho_z\})$ and $C^Q_{q^\rho_z} \leq 1$. Eq. (4) is due to the definition of $u(q^\rho_z, Q^\rho \setminus \{q^\rho_z\})$. Together with Lemma 6, we have $f(Q^{\text{Alg1}}) \geq \rho u(q', S') \geq \rho(1 - 1/e - \epsilon)u(q^\rho_z, Q^\rho \setminus \{q^\rho_z\}) \geq \rho(1 - 1/e - \epsilon)f(Q^\rho)$.

**Corollary 3** By choosing $\rho = 1/2$, we have $f(Q^{\text{Alg1}}) \geq \frac{(1 - 1/e - \epsilon)}{4}f(Q^*)$

5. Question Selection and Sequencing under General Model

We now add PNA option to our model. The workflow of our solution is similar in structure to Algorithm 1. We first introduce a new problem, then build the final solution based on the solution to that new problem. However, the way we define the new problem as well as the analysis of our solution are largely different from the one used in the previous model.
5.1. Algorithm Design

For any given sequence of questions $Q$, we use $\mathcal{R}(Q)$ and $\mathcal{R}(Q)$ interchangeably to denote a random set obtained by including each question $q \in Q$ independently with probability $p_q^+$. We first introduce a new problem $\textbf{P3}$ whose objective function is

$$v(q, S) = \mathbb{E}[g(\mathcal{R}(S \cup \{q\}))]$$

The goal of $\textbf{P3}$ is to find a solution $(q, S)$ that maximizes function $v$. Similar to constraints (C1) and (C2) used in $\textbf{P2}$, we use constraint (C3) (resp. constraint (C4)) to ensure that all selected questions can be reached with high probability (resp. the size of the solution is upper bounded by $b$).

$$\textbf{P3}$$

Maximize$_{q, S} v(q, S)$

subject to:

$$\begin{align*}
- \sum_{q' \in S} \log c_{q'} &\leq - \log \rho \quad \text{(C3)} \\
|S| &< b \quad \text{(C4)} \\
S &\subseteq \Psi \setminus \{q\}
\end{align*}$$

In the following lemma we show that if $q$ is fixed, then $v(q, S)$, as a function of $S$, is submodular.

**Lemma 7** For any fixed $q$, $v(q, S)$ is a submodular function of $S$.

**Proof:** Assume $r$ is a (random) realization of $\mathcal{R}(\Psi \setminus \{q\})$, let $\Pr[r]$ denote the probability that $r$ is realized, we have

$$v(q, S) = \mathbb{E}[g(\mathcal{R}(S \cup \{q\}))] = p_q^+ \sum_{r \subseteq \Psi \setminus \{q\}} \Pr[r]g(r \cap S \cup \{q\}) + (1 - p_q^+) \sum_{r \subseteq \Psi \setminus \{q\}} \Pr[r]g(r \cap S)$$

We next prove that for any fixed $r$ and $q$, $g(r \cap S \cup \{q\})$ as a function of $S$ is submodular. For any $S_1 \subseteq S_2 \subseteq \Psi$ and $q' \notin S_2$, we have $g(r \cap (S_1 \cup \{q'\}) \cup \{q\}) - g(r \cap S_1 \cup \{q\}) = g((r \cap S_1) \cup (r \cap \{q'\}) \cup \{q\}) - g(r \cap S_1 \cup \{q\}) \geq g((r \cap S_2) \cup (r \cap \{q'\}) \cup \{q\}) - g(r \cap S_2 \cup \{q\})$. The inequality is due to $(r \cap S_1) \subseteq (r \cap S_2)$ and $g$ is submodular. Thus $g(r \cap S \cup \{q\})$ is a submodular function of $S$. By a similar proof, we can show that $g(r \cap S)$ is also a submodular function of $S$. It follows that $v(q, S)$ is a submodular function of $S$ due to linear combination of submodular functions is submodular. $\Box$
Algorithm 2 Question Selection and Sequencing with PNA option

**Input:** $\rho, b, A$

**Output:** $Q_{\text{Alg2}}$

1. Set $S' = \emptyset, q' = \emptyset$.
2. for $q \in A$ do
   3. Fix $q$, apply a $(1 - 1/e - \epsilon)$ approximate algorithm [Kulik et al. 2009] to solve $P3$ and obtain $S$.
   4. if $v(q, S) \geq v(q', S')$ then
      5. $S' \leftarrow S, q' \leftarrow q$
   6. $Q_{\text{Alg2}} \leftarrow S' \cup \{q'\}$ where $S'$ is an arbitrary sequence of $S'$.
7. return $Q_{\text{Alg2}}$

5.2. Performance Analysis

Recall that we use $Q^*_{\leq k}$ to denote the prefix of $Q^*$ whose reachability is no smaller than $\rho$, e.g., $k = \arg \max_i (C_{Q^*_{\leq k}}^q \geq \rho)$. We first show that the expected utility of random set $R(Q^*_{\leq k})$ is at least $(1 - 1/e)f(Q^*_{\leq k})$.

**Lemma 8** Assume $Q^*$ is the optimal solution, $(1 - 1/e)f(Q^*_{\leq k}) \leq \mathbb{E}[g(R(Q^*_{\leq k}))]$.

**Proof:** Consider any question $q^*_i \in Q^*_{\leq k}$ that is scheduled at slot $i$ of $Q^*$, the probability that $q^*_i$ is read by the user is $\prod_{q \in Q^*_{<i}} c_q$, thus the probability that $q^*_i$ is answered by the user is $(\prod_{q \in Q^*_{<i}} c_q)p^+_{q^*_i}$. Let $Z(Q^*_{\leq k})$ denote a random set obtained by including each question $p_q^*$ with independently probability $(\prod_{q \in Q^*_{<i}} c_q)p^+_{q^*_i}$.

First of all, we show that $f(Z(Q^*_{\leq k})) \leq (1 - 1/e)f(Q^*_{\leq k})$. This is based on two observations: (1) for each question $q^*_i \in Q^*_{\leq k}$, the marginal probability that $q^*_i$ is included in $Z(Q^*_{\leq k})$ is identical to the probability that $q^*_i$ is answered by the user, (2) $f$ is a submodular function and the correlation gap of a submodular function as defined in [6.1] is bounded by $1/(1 - 1/e)$.

Thus $f(Z(Q^*_{\leq k})) \leq f(Q^*_{\leq k})$, which implies $f(Z(Q^*_{\leq k})) \leq (1 - 1/e)f(Q^*_{\leq k})$. This is because for every $q^*_i$, we have $(\prod_{q \in Q^*_{<i}} c_q)p^+_{q^*_i} \leq p^+_{q^*_i}$, it follows that $q^*_i$ has larger probability to be included in $R(Q^*_{\leq k})$ than in $Z(Q^*_{\leq k})$. Then we have $\mathbb{E}[g(R(Q^*_{\leq k}))] \geq \mathbb{E}[g(Z(Q^*_{\leq k}))]$.

Thus $f(Z(Q^*_{\leq k})) \leq (1 - 1/e)f(Q^*_{\leq k})$ and $\mathbb{E}[g(R(Q^*_{\leq k}))] \geq \mathbb{E}[g(Z(Q^*_{\leq k}))]$, together imply that $(1 - 1/e)f(Q^*_{\leq k}) \leq \mathbb{E}[g(R(Q^*_{\leq k}))]$.

We next prove that the utility of $Q_{\text{Alg2}}$ is close to the expected utility of a random set $R(Q_{\text{Alg2}})$.
Lemma 9 For any $\rho \in [0, 1]$,

$$f(Q_{Alg2}) \geq \rho E[g(R(Q_{Alg2}))]$$

Proof: We first introduce some useful notations. Given the solution $Q_{Alg2}$ that is returned from Algorithm 2, let $J(Q_{Alg2})$ denote the (random) set of questions answered by the user given $Q_{Alg2}$. Then we have $f(Q_{Alg2}) = E[g(J(Q_{Alg2}))]$. For notational simplicity, we use $J$ (resp. $R$) to denote $J(Q_{Alg2})$ (resp. $R(Q_{Alg2})$) for short in the rest of this proof. Because $f(Q_{Alg2}) = E[g(J)]$, we focus on proving $E[g(J)] \geq \rho E[g(R)]$.

Define $J_{\geq i} = J \cap Q_{\geq i}^{Alg2}$ and $R_{\geq i} = R \cap Q_{\geq i}^{Alg2}$. The main result that we will prove is that for any fixed $i \in \{1, 2, \ldots |Q_{Alg2}|\}$,

$$E[g(J_{\geq i}) - g(J_{\geq i+1})] \geq \rho E[g(R_{\geq i}) - g(R_{\geq i+1})]$$ (8)

Then the theorem follows from Eq. (8) since

$$E[g(J)] = f(\emptyset) + \sum_{i=1}^{m} E[g(J_{\geq i}) - f(J_{\geq i+1})] \geq g(\emptyset) + \rho \sum_{i=1}^{m} E[f(R_{\geq i}) - g(R_{\geq i+1})] = \rho E[g(R)]$$

Based on this observation, we next prove Eq. (8).

Notice that the distribution of $J$ is determined by the Markov process defined in the previous section. For ease of analysis, for any fixed slot $i$, we next introduce an alternative way to generate the distribution of $J_{\geq i}$: For every $q \in Q_{Alg2}$, (1) we first determine whether $q$ will be answered or not given that $q$ has been read by the user, and (2) then determine whether $q$ will be read or not. In particular, we first construct a random set $R$ by including each question $q \in Q_{Alg2}$ independently with probability $p_q^\dagger$. Let $q_i$ denote the $i$-th question in $Q_{Alg2}$, we generate another random set $U$ based on $R$ as follows:

- Initially, $U = q_1$.
- Starting from $i = 2$, if $q_{i-1} \in R \land q_{i-1} \in U$ (resp. $q_{i-1} \notin R \land q_{i-1} \in U$), add $q_i$ to $U$ with probability $c_{q_{i-1}}^+$ (resp. $c_{q_{i-1}}^-$), repeat this step with $i = i+1$; otherwise, return $U$.

Return $U$ also once no more questions remain.

Intuitively, $R$ includes those questions which are answered by the user given that they have been read, and $U$ includes those questions which can be read by the user. It is easy to verify that $J$ has the same distribution of $R \cap U$. We use $R_{\geq i}$ (resp. $U_{\geq i}$) to denote $R \cap Q_{\geq i}^{Alg2}$ (resp. $U \cap Q_{\geq i}^{Alg2}$), it follows that $J_{\geq i}$ has the same distribution of $R_{\geq i} \cap U_{\geq i}$. 

We next focus on bounding the value of $\mathbb{E}[f(J_{\geq i}) - f(J_{>i+1})]$. Let $e_3$ (resp. $e_4$) denote the event that $q_i$ is included in $R_{\geq i}$ (resp. $U_{>i}$). For notational simplicity, define $g_S(q_i) = g(S \cup \{q_i\}) - g(S)$ as the marginal benefit of $q_i$ given $S$. For any fixed $i$, we have

$$
\begin{align*}
\mathbb{E}[f(J_{\geq i}) - f(J_{>i+1})] &= \Pr[e_3] \mathbb{E}_{R_{\geq i}}[\Pr[e_4] \mathbb{E}_{U_{>i}}[g_{R_{\geq i+1} \cup U_{>i+1}}(q_i) | e_4] | e_3] \\
&\geq \Pr[e_3] \mathbb{E}_{R_{\geq i}}[\Pr[e_4] g_{R_{\geq i+1}}(q_i) | e_3] \\
&\geq \Pr[e_3] \mathbb{E}_{R_{\geq i}}[\Pr[e_4] g_{R_{\geq i+1}}(q_i) | e_3] \\
&= \Pr[e_3] \mathbb{E}_{R_{\geq i}}[g_{R_{\geq i+1}}(q_i) | e_3] \\
&\geq \rho \Pr[e_3] \mathbb{E}_{R_{\geq i}}[g_{R_{\geq i+1}}(q_i) | e_3] = \rho \mathbb{E}[g(R_{\geq i}) - g(R_{\geq i+1})]
\end{align*}
$$

Eq. (9) is due to the observation that $J_{\geq i}$ has the same distribution of $R_{\geq i} \cap U_{>i}$. Inequality (10) is due to $f$ is a submodular function. Inequality (11) and (12) are due to both $e_4$ and $e_3$ are independent of the realization of $R_{\geq i+1}$. Inequality (13) is due to the fact that every question in $Q_{\text{Alg}2}$ has reachability no less than $\rho$, e.g., $\Pr[e_4] \geq \rho$.

Now we are ready to present the main theorem of this paper.

**Theorem 4** For any $\rho \in [0, 1]$, $f(Q_{\text{Alg}2}) \geq \rho(1 - \rho)(1 - 1/e)(1 - 1/e - \epsilon)f(Q^\ast)$. 

*Proof:* Similar to the proof of Lemma 3 we can show that $f(Q_{\leq k}^\ast) \geq (1 - \rho)f(Q^\ast)$. Together with Lemma 8 we have $\mathbb{E}[g(R(Q_{\leq k}^\ast))] \geq (1 - 1/e - \epsilon)f(Q_{\leq k}^\ast) \geq (1 - \rho)(1 - 1/e - \epsilon)f(Q^\ast)$. We next focus on proving

$$
\mathbb{E}[g(R(Q_{\text{Alg}2}))] \geq (1 - 1/e - \epsilon)\mathbb{E}[g(R(Q_{\leq k}^\ast))]
$$

Then this theorem follows from Inequality (14) and Lemma 8.

Note that for a fixed $q$, Line 3 of Algorithm 2 finds a $(1 - 1/e - \epsilon)$ approximate solution to P.3. Since we enumerate all possible $q$ and return the best solution $(q', S')$, it is easy to verify that $v(q', S') \geq (1 - 1/e - \epsilon)v^\ast$ where $v^\ast$ denotes the optimal solution to P3. In addition, $(q^\ast_{k+1}, Q_{\leq k}^\ast)$ is a valid solution to P3. Thus $(1 - 1/e - \epsilon)v(q^\ast_{k+1}, Q_{\leq k+1}^\ast) \leq (1 - 1/e - \epsilon)v^\ast \leq v(q', S')$. According to the definition of $u$, we have $\mathbb{E}[g(R(Q_{\leq k}^\ast))] = v(q^\ast_{k+1}, Q_{\leq k+1}^\ast)$ and $\mathbb{E}[g(R(Q_{\text{Alg}2}))] = v(q', S')$, it follows that $\mathbb{E}[g(R(Q_{\text{Alg}2}))] \geq (1 - 1/e - \epsilon)\mathbb{E}[g(R(Q_{\leq k}^\ast))].$  

**Corollary 5** By choosing $\rho = 1/2$, we have $f(Q_{\text{Alg}2}) \geq \frac{1}{4}(1 - 1/e)(1 - 1/e - \epsilon)f(Q^\ast)$. 

6. Extension: Incorporating Slot-Dependent Decay Factor

In this section, we take into account the slot-dependent decay factor, e.g., the answer-through-rate of a question could be influenced by its position. In the extended model, each slot $i$ has slot-dependent decay factor $\lambda_i \leq 1$. Given a question $q$ that is placed in slot $i$, the probability that $q$ is answered, conditioned on it has been read, is $\lambda_i p^+_q c^+_q$. We assume that $\forall i \leq j : \lambda_i \geq \lambda_j$, i.e., one would typically expect the answer-through-rate to decrease with slot. For ease of presentation, we assume $\lambda_1 = 1$, e.g., slot-dependent decay effect does not apply to the first slot.

- Starting with question $q_1$ placed at the first slot.
- After reading $q_i$, the user chooses one of the following five actions to take:
  1. Answer $q_i$ and continue to read the next question (resp. exit the quiz) with probability $\lambda_i p^+_q c^+_q$ (resp. $\lambda_i p^+_q (1 - c^+_q)$).
  2. PNA $q_i$ and continue to read the next question (resp. exit the quiz) with probability $p^-_q c^-_q$ (resp. $p^-_q (1 - c^-_q)$).
  3. Exit the quiz with probability $1 - (\lambda_i p^+_q + p^-_q)$.
- The above process repeats until the user exits the quiz or no more questions remain.

We revise the definition of reachability to incorporate slot-dependent decay effect.

**Definition 2 (Reachability of a Question)** For any given sequence of questions $Q$, the reachability of $q_i$ is:

$$ C^{Q}_{q_i} = \prod_{1 \leq j < i} (\lambda_j p^+_q c^+_q + p^-_q c^-_q) $$

6.1. Question Selection and Sequencing with No PNA option

We first study the case when PNA is not an option. A simplified question scanning process is presented as follows.

- Starting with question $q_1$ placed at the first slot.
- After reading $q_i$, the user chooses one of the following five actions to take:
  1. Answer $q_i$ and continue to read the next question (resp. exit the quiz) with probability $\lambda_i p^+_q c^+_q$ (resp. $\lambda_i p^+_q (1 - c^+_q)$).
  2. Exit the quiz with probability $1 - \lambda_i p^+_q$.
- The above process repeats until the user exits the quiz or no more questions remain.
P2.1 Maximize \(u(t,q,S)\) subject to:

\[
\begin{align*}
-\left(\log \Lambda_t + \sum_{q' \in S} \log(p_{q'}^+ c_{q'}^+)\right) & \leq -\log \rho & (C1.1) \\
|S| & < t & (C2.1) \\
S & \subseteq \Psi \setminus \{q\} \\
0 & \leq t & \leq b
\end{align*}
\]

Define \(\Lambda_t = \prod_{1 \leq j \leq t} \lambda_j\). By setting \(p_q^- = 0\) for every \(q \in \Psi\), we derive a simplified form of \(C_{q_i}^Q\) as follows:

\[
C_{q_i}^Q = \prod_{1 \leq j < i} (\lambda_j p_{q_j}^+ c_{q_j}^+) = \Lambda_i \prod_{1 \leq j < i} p_{q_j}^+ c_{q_j}^+
\]

We first introduce a new problem P2.1. The formulation of P2.1 is similar to P2 except that there is one additional decision variable \(t\), which specifies the index of the last slot occupied by our solution. The reason why we introduce this additional decision variable is because \(\lambda_i\) is slot-dependent, by fixing the index of the last slot enables us to separate the slot-dependent decay effect from other question-dependent factors such as answer-through-rate and continuation probability. The basic idea of our solution is similar to Algorithm \[\text{Algorithm}\] after solving P2.1 and obtain a solution \((t',q',S')\), we build the final solution to the original problem based on \((t',q',S')\).

We next give a detailed description of P2.1. The objective function of P2.1 is

\[
u(t,q,S) = \lambda_t p_q^+ g(S \cup \{q\}) + (1 - \lambda_t p_q^+) g(S)
\]

Constraint (1.1) ensures that the reachability of every question, after taking into account the slot-dependent decay effect \(\Lambda_i\), is no smaller than \(\rho\). Constraint (2.1) ensures that our solution occupies up to \(t\) slots. Note that for any fixed \(t\) and \(q\), P2.1 is a submodular maximization problem subject to two linear constraint. In order to solve P2.1, we exhaustedly try all possible \(t\) and \(q\). For each \(t\) and \(q\), we run the \((1 - 1/e - \epsilon)\) approximate algorithm to obtain a candidate solution \((t,q,S)\). Among all candidate solutions, assume \((t',q',S')\) has the largest utility, \(Q_{\text{Alg3}} = S' \oplus q'\) is returned as the final solution to the original problem.

We present the detailed description of our solution in Algorithm \[\text{Algorithm}\].

To provide a performance bound to our solution, we first present four preparatory lemmas as follows.

Lemma 10 Given \(Q\), consider any question \(q_i \in Q\), \(f(Q)\) is a non-decreasing function of \(\lambda_i\) and \(C_{q_i}^Q\).
Algorithm 3 Question Selection and Sequencing with No PNA option

Input: $\rho, b, \Psi$

Output: $Q_{\text{Alg}3}$.

1. Set $S' = \emptyset, q' = \emptyset, t' = 0$.
2. for $t \in [1, b]$ do
3.   for $q \in \Psi$ do
4.     Fix $t$ and $q$, apply a $(1 - 1/e - \epsilon)$ approximate algorithm [Kulik et al. 2009] to solve $\text{P2.1}$ and obtain $S$
5.     if $u(t, q, S) > u(t', q', S')$ then
6.         $t' \leftarrow t, S' \leftarrow S, q' \leftarrow q$
7.     end if
8. end for
9. end for
10. $Q_{\text{Alg}3} \leftarrow S' \oplus \{q'\}$ where $S'$ is an arbitrary sequence of $S'$
11. end for
12. return $Q_{\text{Alg}3}$

Proof: Let $\Delta^+$ (resp. $\Delta^-$) denote the marginal benefit of all questions scheduled at and after slot $i$ conditioned on $q_i$ has been answered (resp. PNA), we have $f(Q) = f(Q_{<i}) + C^Q_i (\lambda_i p^+_{q_i} \Delta^+ + p^-_{q_i} \Delta^-)$. Since both $p^+_{q_i} \Delta^+$ and $\lambda_i p^+_{q_i} \Delta^+ + p^-_{q_i} \Delta^-$ are non-negative, $f(Q)$ is a non-decreasing function of $\lambda_i$ and $C^Q_{q_i}$.

Similar to Lemma 3, we can show that ignoring those questions with small reachability does not affect the utility much.

Lemma 11 For any $\rho \in [0, 1]$, there is a solution $Q_{\rho}$ of value at least $(1 - \rho)f(Q^*)$ such that $|Q_{\rho}| \leq b$ and $\forall q \in Q_{\rho}: C^Q_q \geq \rho$.

Proof: We still use $q^*_i$ to denote the $i$-th question in $Q^*$ and assume $q^*_k$ is the last question in $Q^*$ whose reachability is no smaller than $\rho$. The main result that we will prove is that

$$C^Q_{q^*_k} c_{q^*_k} f(Q^*_{> k}) \geq f(Q^*) - f(Q^*_{\leq k})$$  \hspace{1cm} (15)

We use $f(Q|Q' \oplus Q)$ to denote the conditional utility of $Q$ given that $Q'$ is scheduled ahead of $Q$. We use $f(Q)$ to denote $f(Q|\emptyset \oplus Q)$ for short. It follows that $C^Q_{q^*_k} c_{q^*_k} f(Q^*_{> k}|Q^*_{\leq k} \oplus Q^*_{> k}) \geq f(Q^*) - f(Q^*_{\leq k})$ due to for any given $Q^*$, the first question of $Q^*_{> k}$ can be reached with probability $C^Q_{q^*_k} p^+_{q^*_k} c^+_k$ and $f$ is a submodular function. In order to prove Inequality \hspace{1cm} (15), it remains to prove that $f(Q^*_{> k}) \geq f(Q^*_{> k}|Q^*_{\leq k} \oplus Q^*_{> k})$, i.e., we need to show that moving $Q^*_{> k} k$ slots earlier does not decrease its utility. Because $\forall i \geq j: \lambda_i \leq \lambda_j$, it implies that moving a question to some earlier slot does not decrease its reachability and answer-through-rate, then we have $f(Q^*_{> k}) \geq f(Q^*_{> k}|Q^*_{\leq k} \oplus Q^*_{> k})$ due to Lemma 10.

□
Lemma 12 \((t', q', S')\) is a \((1 - 1/e - \epsilon)\) approximate solution to \(P_{2.1}\).

Proof: According to Algorithm 3, for any fixed \(t\) and \(q\), we are able to find a \((1 - 1/e - \epsilon)\) approximate solution to \(P_{2.1}\). Then this lemma follows from the fact that \((t', q', S')\) is returned as the best solutions after exhaustively trying all possible \(t\) and \(q\). \(\square\)

Lemma 13 For any \(\rho \in [0, 1]\), \(f(Q_{\text{Alg3}}) \geq \rho u(t', q', S')\).

Proof: Due to \(S'\) satisfies constraint (C2.1) in problem \(P_{2.1}\), we have \(\Lambda v_{t-1} \prod_{q \in S'} p^{+}_q c^{+}_q \geq \rho\). It follows that with probability at least \(\rho\), all questions in \(S'\) will be answered and \(q'\) will be read. Moreover, the probability that \(q'\) will be answered by the user is \(\lambda v p^{+}_q\) conditioned on all questions in \(S'\) are answered and \(q'\) is read. It follows that \(f(Q_{\text{Alg3}}) \geq \rho \lambda v p^{+}_q g(S' \cup \{q'\}) + \rho (1 - \lambda v p^{+}_q) g(S') = \rho u(t', q', S')\). \(\square\)

Lemma 11, Lemma 12, and Lemma 13 together imply the following main theorem.

Theorem 6 For any \(\rho \in [0, 1]\), \(f(Q_{\text{Alg3}}) \geq \rho (1 - \rho)(1 - 1/e - \epsilon)f(Q^*)\).

Corollary 7 By choosing \(\rho = 1/2\), we have \(f(Q_{\text{Alg3}}) \geq \frac{1}{4}(1 - 1/e - \epsilon)f(Q^*)\).

6.2. Question Selection and Sequencing under General Model

We next study this extended problem under general model where PNA is an option. The basic idea of our approach is to covert the original joint selection and sequencing problem to a simplified selection problem. For each question \(q\), we create \(b\) copies of virtual questions \(\Psi_q = \{q^1, \cdots, q^b\}\). Let \(\Psi' = \bigcup_{q \in \Psi} \Psi_q^v\) denote the expanded ground set that is composed of virtual questions. We next focus on selecting a group of virtual questions. Intuitively, selecting a virtual question \(q^i\) translates to placing \(q\) at slot \(i\).

We next introduce some important notations. Define \(\Psi_i^v = \{q^i | q \in \Psi\}\). For every \(q^i\), let \(c_q = \lambda v p^{+}_q c^{+}_q + p_q^- c^-_q\). Given a set of virtual questions \(S^v \subseteq \Psi^v\), we use \(S \subseteq \Psi\) to denote the set of its actual copies. For example, given \(S^v = \{a^2, a^4, b^3\}\), we have \(S = \{a, b\}\). Then we are ready to define the utility function \(g\) over \(\Psi^v\): \(g(S^v) = g(S)\).

We next introduce problem \(P_{3.1}\) whose objective function is

\[v(t, q, S^v) = \mathbb{E}[g(\mathcal{R}(S^v \cup \{q^i\}))]\]

where \(\mathcal{R}(S^v \cup \{q^i\})\) is (redefined as) a random set obtained by including each virtual question \(q^i \in S^v \cup \{q^i\}\) with probability \(\lambda v p^{+}_q\). The goal of \(P_{3.1}\) is to find a solution that maximizes function \(v\). After solving \(P_{3.1}\) approximately and obtain \((t', q', S'^v)\), we build the final solution based on \((t', q'', S'^v)\).
The formulation of $P3.1$ is similar to $P3$, except that now we are dealing with virtual questions. We next explain how to covert a solution to $P3.1$ to a solution to the original problem: given a solution $(t, q, S^v)$ to $P3.1$, we place $q$ at slot $i$ if and only if $q^i \in S^v \cup q^i$. For example, $(4, d^4, \{a^1, b^3, c^2\})$ translates to placing $a$ (resp. $b, c, d$) at the first (resp. third, second, forth) slot. To ensure the feasibility of the solution, we employ condition (4.1) to avoid assigning multiple questions to the same slot. Similar to Lemma [7], we can prove that for any fixed $t$ and $q$, $(t, q, S^v)$ is submodular as a function of $S^v$. Together with the fact that (C4.1) is a (partition) matroid constraint, we have that for any fixed $t$ and $q$, $P3.1$ is a submodular maximization problem subject to two linear and one matroid constraints. There exists a 0.38 approximate solution (Vondrakov et al. 2011) to this problem.

**Remark:** Notice that a feasible solution to $P3.1$ could select multiple virtual questions that are created from the same actual question, this is clearly unacceptable since we are not allowed to display the same question more than once. Fortunately, this redundancy issue can be easily resolved by keeping any one of those copies in the solution. This will not affect the utility of our solution due to the submodularity of the utility function (according to Eq. (1), the marginal utility of any redundant question is zero). Another potential issue is that our solution may contain some “gap”, i.e., there is at least one empty slot between two scheduled questions. This gap issue can also be easily resolved by simply removing those gaps from the final solution, e.g., this can be done by moving all questions to its earliest possible slot while respecting their original ordering. Due to Lemma [10] moving questions to some earlier slot will not decrease its utility.

We present the detailed description of our solution in Algorithm [4].

In the rest of this paper, we redefine $\mathcal{R}(Q)$ as a random set obtained by including each question $q_i \in Q$ with probability $\lambda_i p_{q_i}^+$, then the proof of Lemma [14] (resp. Lemma [15]) is similar to the proof of Lemma [8] (resp. Lemma [9]).

**Lemma 14** $(1 - 1/e)f(Q^*_k) \leq \mathbb{E}[g(\mathcal{R}(Q^*_k))]$. 

We next present the main theorem.

**Theorem 8** For any \( \rho \in [0, 1] \), \( f(Q^{Alg4}) \geq \rho \mathbb{E}[g(\mathcal{R}(Q^{Alg4}))] \).

Proof: Similar to the proof of Lemma 11 we can show that \( f(Q_{s\leq k}^*) \geq (1-\rho) f(Q^*) \). Together with Lemma 8 we have

\[ \mathbb{E}[g(\mathcal{R}(Q_{s\leq k}^*))] \geq (1 - 1/e) f(Q_{s\leq k}^*) \geq (1 - \rho) (1 - 1/e) f(Q^*) \tag{16} \]

We next focus on proving

\[ \mathbb{E}[g(\mathcal{R}(Q^{Alg4}))] \geq 0.38 \mathbb{E}[g(\mathcal{R}(Q_{s\leq k}^*))] \tag{17} \]

Then this theorem follows from Inequality (17), (16) and Lemma 15.

Note that for a fixed \( t \) and \( q \), Line 4 of Algorithm 4 finds a 0.38 approximate solution to problem P3.1. Since \((t', q', S^{v'})\) is the best candidate solution after enumerating all possible \( t \) and \( q \), it is easy to verify that \( v(t', q', S^{v'}) \geq 0.38 v^* \) where \( v^* \) denotes the utility of the optimal solution to P3.1. In addition, because \((k, q^*_k, Q_{s\leq k-1}^*)\) is a valid solution to P3.1, we have \( 0.38 v(k, q^*_k, Q_{s\leq k-1}^*) \leq 0.38 v^* \leq v(t', q', S^{v'}) \). According to the definition of \( v \), we have \( \mathbb{E}[g(\mathcal{R}(Q_{s\leq k}^*))] = v(k, q^*_k, Q_{s\leq k-1}^*) \) and \( \mathbb{E}[g(\mathcal{R}(Q^{Alg4}))] = v(t', q', S^{v'}) \), it follows that \( \mathbb{E}[g(\mathcal{R}(Q^{Alg4}))] \geq 0.38 \mathbb{E}[g(\mathcal{R}(Q_{s\leq k}^*))] \). \( \square \)

**Corollary 9** By choosing \( \rho = 1/2 \), we have \( f(Q^{Alg4}) \geq \frac{0.38}{4} (1 - 1/e) f(Q^*) \).
7. Conclusion

In this paper, we study the optimal quiz design problem. We assume the utility function of a group of answered questions is submodular and our objective is to select and sequence of a group of questions so as to maximize the expected utility. We model the user behavior as a Markov process. Then we develop a series of question allocation strategies with provable performance bound. Although we restrict our attention to quiz design problem in this paper, our results apply to a broad range of applications which can be formulated as a submodular maximization problem under fading model.

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### Appendix: Missing Proofs

#### Proof of Lemma 2

Proof: $g(S)$ is clearly non-decreasing according to *information never hurts principle* (Krause and Guestrin 2005). Moreover, consider any $S_1 \subseteq S_2 \subseteq \Psi$, we have $A(S_1) \subseteq A(S_2)$, it follows that $g(S_1 \cup \{q\}) - g(S_1 \cup \{q\}) = H(X_{A(S_1) \cup \{q\}}) - H(X_{A(S_1)}) = H(X_{A(S_1) \cup \{q\}}) - H(X_{Z(A)}) \geq H(X_{A(S_2) \cup \{q\}}) - H(X_{A(S_2)}) = g(S_2 \cup \{q\}) - g(S_2 \cup \{q\})$. The inequality is due to $H(X_{A})$ as a function of $A$ is submodular (Krause and Guestrin 2005). Therefore, $g(S)$ is also non-decreasing and submodular. □
Proof of Theorem 1

Proof: Consider a special case of problem $P_1$ where (1) $g(S) = H(X_{A(S)})$, i.e., we assume an entropy-like utility function as defined in Section 3.1.2 (2) $p^+_{q_i} = 1$ and $c^+_{q_i} = 1$, i.e., the user is guaranteed to answer all questions, and (3) $\forall a \in \Phi : X_a \in \{0,1\}$ and $X_\Phi$ follows uniform distribution i.e., each attribute has binary value and $\forall x \in \{0,1\}^{|\Phi|}, \Pr[X_\Phi = x] = \frac{1}{2^{|\Phi|}}$. It is easy to see that finding a solution to this special case is reduced to selecting a subset of questions that covers the largest number of attributes subject to a cardinality constraint $b$. Next, using a reduction to the maximum coverage problem (Feige 1998), a known NP-hard problem, we show that $P_1$ is NP-hard. Given sets $\{Y_1, \cdots, Y_n\}$ and a ground set $\{w_1, \cdots, w_m\}$ of elements to cover, the goal of the maximum coverage problem is to find a group of at most of $l$ sets so as the cover the largest number of elements. We next construct an equivalent instance of $P_1$. We first set $b = l$. There is a attribute $a_i$ for each element $w_i$, and there is a question $q_j$ for each set $Y_j$, and we define $q_j$ covers $a_i$ if and only if $Y_j$ covers $w_i$. Then finding an optimal solution to the maximum coverage problem is equivalent to solving the special case of $P_1$ optimally. This finishes the proof of this theorem.  \[\square\]