TENSOR PRODUCT OF COHERENT SYSTEMS

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ABSTRACT. Let $X$ be a smooth algebraic curve of genus $g \geq 2$. A stable vector bundle over $X$ of degree $d$, rank $n$ with at least $k$ sections is called a Brill-Noether bundle of type $(n, d, k)$. By tensoring coherent systems, we prove that most of the known Brill-Noether bundles define coherent systems of type $(n, d, k)$ that are $\alpha$-stables for all allowable $\alpha$.

1. Introduction

Let $X$ be a smooth projective algebraic curve over $C$ of genus $g \geq 2$. By a coherent system of type $(n, d, k)$ we mean a vector bundle of degree $d$ and rank $n$ together with a linear subspace of its space of sections of dimension $k$.

In [18], [16], [23] was introduced a notion of stability for coherent systems which permits the construction of moduli spaces. This notion depends on a real parameter $\alpha$, and thus leads to a family of moduli spaces. The different members in the family correspond to different values of $\alpha$. As $\alpha$ varies, the moduli spaces can change only when $\alpha$ passes through one of a discrete set of points, called critical values. If $k < n$, the range of parameter is a finite interval and the family has only finitely many distinct members. If $k \geq n$ the range of the parameter is infinite; however, there is only a finite number of distinct moduli spaces. Moreover, if $G(\alpha : n, d, k)$ is the moduli space of $\alpha$-stable coherent systems of type $(n, d, k)$, there is a critical value, denoted by $\alpha_L$, such that for all $\alpha, \alpha' > \alpha_L$, $G(\alpha : n, d, k) = G(\alpha' : n, d, k)$ ([10]).

Let $B(n, d, k)$ be the Brill-Noether locus of the stable vector bundles over $X$ of degree $d$, rank $n$ with at least $k$ independent sections. A triple $(n, d, k)$ is called a Brill-Noether triple if $B(n, d, k) \neq \emptyset$ and a vector bundle in $B(n, d, k)$ is called a Brill-Noether bundle of type $(n, d, k)$. It is clear that Brill-Noether bundles define coherent systems and for $\alpha > 0$ close to 0, they define $\alpha$-stable coherent systems (see [5]). However, for arbitrary choice of $\alpha$ it is not clear the relation between Brill-Noether bundles and stable coherent systems. We are interested in studying this relationship.

The case $k \leq n$ have already been considered in [5], [6] and [7] and was proved that if $n \leq d + (n - k)g$ and $(n, d, k) \neq (n, n, n)$, Brill-Noether bundles define $\alpha$-stable coherent systems for all allowable $\alpha$. Moreover, the set of such bundles is a Zariski open subset of $G(\alpha : n, d, k)$.

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For general curves, if \( k = n + 1 \) and \( d \leq g + n \) it was proved in [11] that there exist Brill-Noether bundles which are \( \alpha \)-stable for all \( \alpha > 0 \). The existence of such bundles was used to determine the structure of \( G(\alpha : n, d, k) \) and to prove that \( \alpha_L = 0 \). For \( k > n \) and \( n, d, k \) satisfying the relations in [21], Montserrat Teixidor i Bigas in [20] constructs a singular curve and a limit of coherent system such that, on a generic curve, define a coherent system that is \( \alpha \)-stable for all \( \alpha > 0 \) and the vector bundle is stable.

Our aim in this paper is to study the \( \alpha \)-stability of coherent systems defined by the Brill-Noether bundles given in [19] and [10]. Such bundles will be used to determine the structure of \( G(\alpha : n, d, k) \) and, in some cases, to determine the value of \( \alpha_L \). Our results for \( k > n \) will be for any curve and, even for the generic case, they extend beyond of those in [20]. Furthermore, our methods give another proof of non-emptiness for those values that are included.

In order to state our results we recall the following definitions and facts.

Denote by \( \mathcal{M}(n, d) \) the moduli space of stable vector bundles of rank \( n \) and degree \( d \). If \( d > n(2g - 2) \), denote by \( \text{Grass}(s) \) the Grassmannian bundle over \( \mathcal{M}(n, d) \) with fibre the Grassmannian \( \text{Grass}(s, H^0(E)) \).

From [15, Corollaire 3.14], every irreducible component of \( G(\alpha : n, d, k) \) has dimension greater or equal to the Brill-Noether number \( \beta(n, d, k) := n^2(g-1)+1-k(d-n+g-1) \). Denote by \( G_0(n, d, k) \) the first member of the moduli spaces family and by \( G_L(n, d, k) \) the last one. For coherent systems \( (E, V) \) of type \( (n, d, k) \) with \( k \geq 1 \) define \( G_g(n, d, k) \) and \( U(n, d, k) \) as

\[
G_g(n, d, k) := \{(E, V) : (E, V) \text{ is generated by } V \text{ with } H^0(E^*) = 0\},
\]

and

\[
U(n, d, k) := \{(E, V) : (E, V) \text{ is } \alpha - \text{stable for all allowable } \alpha \text{ and } E \text{ is stable}\},
\]

We are interested in studying the non-emptiness of \( U(n, d, k) \), when \( (n, d, k) \) is a Brill-Noether triple.

We now state our results.

Let \( 2 \leq gs < n \) and \( k = n + s \). If \( d < n + gs \), it is known (see [19]) that there are no semistable bundles of degree \( d \), rank \( n \) with at least \( k \) independent sections; hence \( G_0(n, d, k) = \emptyset \). Moreover, the non existence of semistable bundles also implies that (see Theorem [5.1])

- if \( d < n + gs \), \( G(\alpha : n, d, k) = \emptyset \) for all \( \alpha > 0 \).

If \( d \geq n + gs \), from [19] there are Brill-Noether bundles of type \( (n, d, k) \). In this case we prove

- If \( d = n + gs \), (see Theorem [5.6])
  
  (1) \( G(\alpha : n, d, k) \neq \emptyset \) for all \( \alpha > 0 \);
(2) \( G(n, d, k) := G(\alpha : n, d, k) = G(\alpha' : n, d, k) \) for \( \alpha, \alpha' > 0 \) i.e. \( \alpha_L = 0 \);

(3) \( U(n, d, k) \neq \emptyset \). Moreover, \( U(n, d, k) = G(n, d, k) \subset G_g(n, d, k) \);

(4) \( G(n, d, k) \) is smooth irreducible of dimension \( \beta(n, d, k) \). Moreover, \( G(n, d, k) \cong \mathcal{M}(s, d) \).

(5) For \( s \geq 1 \), \( G(\alpha : s, d, k) = G(n, d, k) \) for all \( \alpha > 0 \).

- For \( d = n + sg + s' \) with \( 0 < s' < g \) (see Theorem [6.4]):

1. \( G(\alpha : n, d, k) \neq \emptyset \) for all \( \alpha > 0 \);
2. \( U(n, d, k) \neq \emptyset \). Moreover, any \( (E, V) \in U(n, d, n + s) \) is generically generated;
3. \( G(\alpha : n, d, k) \) has a smooth irreducible component \( G^0(n, d, k) \) of dimension \( \beta \).

Moreover, \( G^0(n, d, k) \) is birationally equivalent to the Grassmanian bundle \( \text{Grass}(s') \) over \( \mathcal{M}(s, d) \).

From Theorem 5.6, 6.4 and [7] Theorem A we have that

- If \( 0 < d < 2n \), there is an open set \( Z \) of \( B(n, d, k) \) such that any \( E \in Z \) defines a coherent system that is \( \alpha \)-stable for all \( \alpha > 0 \). Moreover, for any Brill-Noether triple \( (n, d, k) \), \( U(n, d, k) \neq \emptyset \).

The idea of tensoring Brill-Noether bundles by line bundles with section was used in [10] to produce Brill-Noether bundles of degree \( d > 2n \). We use this idea and tensor coherent systems of type \( (n, d, k) \) by coherent systems of type \( (1, d', k') \).

Let \( (E, V) \) be a coherent system and \( L \) an effective line bundle. Choose a section \( s \) of \( L \) and define the coherent system \( (E \otimes L, \tilde{V}) \) where \( \tilde{V} \) is the image of \( V \) in \( H^0(E \otimes L) \) under the canonical inclusion \( H^0(E) \hookrightarrow H^0(E \otimes L) \) induced by \( s \). Raghavendra and Vishwanath in [23, Lemma 1.5] proved that \( (E, V) \) is \( \alpha \)-stable if and only if \( (E \otimes L, \tilde{V}) \) is \( \alpha \)-stable. We extend such Lemma as follows.

Let \( (L, W) \) be a coherent system of type \( (1, d', k') \) and \( K_L \) the kernel of the evaluation map \( W \otimes \mathcal{O} \rightarrow L \). Let \( (E, V) \) be a coherent system of type \( (n, d, k) \). If \( H^0(K_L \otimes E) = 0 \), we identify \( W \otimes V \) with the image of \( W \otimes V \) in \( H^0(E \otimes L) \) under the inclusions \( W \otimes V \hookrightarrow W \otimes H^0(E) \hookrightarrow H^0(E \otimes L) \). Hence, \( (E \otimes L, V \otimes W) \) is a coherent system of type \( (n, d + nd', kk') \). For such coherent systems we prove (see Lemma [7.4])

\begin{equation}
(1.1)
\end{equation}

- \( (E, V) \) is \( k'\alpha \)-stable if and only if \( (E \otimes L, V \otimes W) \) is \( \alpha \)-stable.

We use (1.1) and Theorems 5.6, 6.4 and [7] Theorem A to prove that most of the Brill-Noether bundles in [10] are \( \alpha \)-stables for all allowable \( \alpha \). Actually, we prove that

under the hypothesis that \( d = d'' + nd' \) and \( k = k'' \) with \( 0 < d'' < 2n \), \( n \leq d''+(n-k'')g \) and \( (n, d'', k'') \neq (n, n, n) \); and \( (d', k') \) satisfying one of the following conditions

1. \( d' \leq 2g \) and \( \beta(1, d', k') \geq 0 \) (see Theorem [7.4]);
2. \( d' > 2g \) and \( k' \geq 1 \) and \( nd' > (k - 1)d'' \) (see Theorem [7.5]).
• \( G(\alpha : n, d, k) \neq \emptyset \) for all allowable \( \alpha \). Moreover, \( U(n, d, k) \neq \emptyset \).

For a hyperelliptic curve, under the same hypothesis on \((n, d', k')\) and the assumptions that \( k' \geq 1 \) and \( d' = 2(k' - 1) \) we prove

• \( G(\alpha : n, d, k) \neq \emptyset \) for all allowable \( \alpha \). Moreover, \( U(n, d, k) \neq \emptyset \).

During the final stages of writing this paper, we came across the work in [8]. Our results in Theorem 5.6 and 6.4 partly coincide with some of their results.

**Notation**

We will denote by \( K \) the canonical bundle over \( X \), by \( K_E \) and \( I_E \) the kernel and image, respectively, of the evaluation map \( V \otimes O \to E, H^i(X, E) \) by \( H^i(E) \), the rank of \( E \) by \( n_E \), the degree of \( E \) by \( d_E \). If \( W \) is a vector space, we denote \( W \otimes O \otimes E \) by \( W \otimes E \). If \( \beta : V \to H \) and \( \gamma : W \to H \) are injective homomorphism, we will denote \( \beta(V) \cap \gamma(W) \subseteq H \) by \( V \cap W \).

2. Definitions and general results

We recall some definitions and facts of coherent systems we shall need. We refer the reader to [6] and [4] and references cited therein for basic properties of coherent systems on algebraic curves.

Let \( X \) be a smooth projective algebraic curve over \( \mathbb{C} \) of genus \( g \geq 2 \). A coherent system over \( X \) of type \((n, d, k)\) is a pair \((E, V)\) where \( E \) is a vector bundle over \( X \) of rank \( n \), degree \( d \) and \( V \) a linear subspace of \( H^0(X, E) \) of dimension \( k \). For any real number \( \alpha > 0 \), define the \( \alpha \)-slope of the coherent system \((E, V)\) of type \((n, d, k)\) as

\[
\mu_\alpha(E, V) := \mu(E) + \frac{\alpha k}{n},
\]

where \( \mu(E) := d/n \) is the slope of the vector bundle \( E \). A coherent subsystem \((F, W) \subseteq (E, V)\) is a coherent system such that \( F \subseteq E \) and \( W \subseteq V \cap H^0(F) \). For any \( \alpha \), a coherent system \((E, V)\) is \( \alpha \)-stable (respectively \( \alpha \)-semistable) if for all proper coherent subsystems \((F, W)\)

\[
\mu_\alpha(F, W) < \mu_\alpha(E, V) \quad (\text{respectively} \leq).
\]

Denote by \( G(\alpha : n, d, k) \) (respectively \( \tilde{G}(\alpha : n, d, k) \)) the moduli space of \( \alpha \)-stable (respectively \( \alpha \)-semistable) coherent systems of type \((n, d, k)\). For non-emptiness of \( G(\alpha : n, d, k) \) with \( k \geq 1 \) we need \( \alpha > 0 \) and \( d > 0 \). Basic properties of \( G(\alpha : n, d, k) \) have been proved in [18], [16] and [23].

Most of the detailed results known are for \( k \leq n \) (see [5], [7],[6]). For \( k = n + 1 \) and \( X \) general see [11], [3] and [6]. For \( k > n \), on a generic curve, Montserrat Teixidor i Bigas in [20] proved that, under the same relation as in [21], \( G(\alpha : n, d, k) \neq \emptyset \) and has an irreducible component of the correct dimension. For \( d >> 0 \) see [1]. It is our purpose
here to study the case $k > n$ on any curve and under different conditions then those in [20].

Every irreducible component of $G(\alpha : n, d, k)$ has dimension at least the Brill-Noether number $\beta(n, d, k) := n^2(g - 1) + 1 - k(k - d + n(g - 1))$. From the infinitesimal study of the coherent systems if $(E, V) \in G(\alpha : n, d, k)$, $G(\alpha : n, d, k)$ is smooth of dimension $\beta(n, d, k)$ in a neighbourhood of $(E, V)$ if and only if the Petri map

$$V \otimes H^0(E^* \otimes K) \to H^0(E \otimes E^* \otimes K)$$

is injective.

Given a triple $(n, d, k)$ denote by $C(n, d, k)$ the set

$$C(n, d, k) := \{ \alpha | 0 \leq \alpha = \frac{nd' - n'd}{n'k - nk'} \text{ with } 0 \leq k' \leq k, 0 < n' \leq n, \text{ and } nk' \neq n'k \}.$$ 

An element $\alpha$ in $C(n, d, k)$ is called a critical point. The set $C(n, d, k)$ defines a partition of the interval $[0, \infty)$. With the natural order on $\mathbb{R}$, label the critical points as $\alpha_i$.

It is known (see [4] and [6]) that

1. If $\alpha', \alpha'' \in (\alpha_i, \alpha_{i+1})$ then $G(\alpha' : n, d, k) = G(\alpha'' : n, d, k)$. Denote by $G_i(n, d, k)$ the moduli space $G(\alpha : n, d, k)$ for any $\alpha \in (\alpha_i, \alpha_{i+1})$.

2. For $k < n$, if $\alpha > \frac{d}{n-k}$, $G(\alpha : n, d, k) = \emptyset$.

3. For $k \geq n$, there exists $\alpha_L$ such that for any $\alpha, \alpha' > \alpha_L$, $G(\alpha : n, d, k) = G(\alpha' : n, d, k)$. Denote by $G_L(n, d, k)$ the moduli space $G(\alpha : n, d, k)$ for $\alpha > \alpha_L$.

Let $B(n, d, k)$ (respectively $\tilde{B}(n, d, k)$) be the Brill-Noether locus of stable (respectively semistable) vector bundles. There is a natural map

$$\phi : G_0(n, d, k) \to \tilde{B}(n, d, k)$$

defined by $(E, V) \mapsto E$ that is injective over $B(n, d, k) - B(n, d, k + 1)$. Moreover, if $E \in B(n, d, k)$ then for any subspace $V \subseteq H^0(E)$ of dimension $k$, $(E, V) \in G_0(n, d, k)$.

**Remark 2.1.** Let $(E, V)$ be a coherent system of type $(n, d, k)$. From the definition of $\alpha$-stability and stability of a vector bundle we have that

1. if $n = 1$, $(E, V)$ is $\alpha$-stable for all $\alpha > 0$;

2. if $(E, V) \in G(\alpha : n, d, k)$ and $E$ is stable then $(E, V)$ is $\alpha'$-stable for all $0 < \alpha' < \alpha$;

3. if $E$ is stable and for all subsystems $(F, W) \subseteq (E, V)$, $\frac{\dim W}{n_F} \leq \frac{k}{n}$ then $(E, V)$ is $\alpha$-stable for all $\alpha > 0$.

For coherent systems of type $(n, d, k)$ define $U(n, d, k)$ as

$$U(n, d, k) := \{ (E, V) : (E, V) \text{ is } \alpha\text{-stable for all allowable } \alpha \text{ and } E \text{ is stable } \}.$$ 

By `allowable` $\alpha$ we mean that if $k < n$, $\alpha < \frac{d}{n-k}$ and if $k \geq n$, $\alpha > 0$. 
Note that $G(\alpha : n, d, k)$ could be non-empty for all $\alpha > 0$, but $U(n, d, k) = \emptyset$. From the openness of $\alpha$-stability $U(n, d, k)$ is an open subset of $G_L(n, d, k)$. Moreover, if $G_L(n, d, k)$ is irreducible, $U(n, d, k)$ is irreducible.

If $k \leq n$ and $n \geq 2$, $U(n, d, k) \neq \emptyset$ if and only if $n \leq d + (n-k)g$ and $(n, d, k) \neq (n, n, n)$ (see [7, Theorem A]). Hence, the Brill-Noether bundles with $k \leq n$ are $\alpha$-stable for all allowable $\alpha$.

Our aim is to study $U(n, d, k)$ for $k > n$. In particular, the non-emptiness.

A coherent system $(E, V)$ can be defined as a triple $(E, V, \phi_{E,V})$ where $V$ is a vector space and $\phi_{E,V} : V \otimes \mathcal{O} \to E$ is a map such that the induced map $\phi^*_E : V \to H^0(E)$ is injective. Moreover, we have the exact sequence

\begin{equation}
0 \to K_E \to V \otimes \mathcal{O} \xrightarrow{\phi_{E,V}} E \to H \oplus \tau \to 0
\end{equation}

where $K_E$ and $H$ are vector bundles with $H^0(K_E) = 0$ and $\tau$ a torsion sheaf.

If $I_E$ is the image of the evaluation map $V \otimes \mathcal{O} \to E$ we split (2.1) as

\begin{equation}
0 \to K_E \to V \otimes \mathcal{O} \to I_E \to 0
\end{equation}

and

\begin{equation}
0 \to I_E \to E \to H \oplus \tau \to 0.
\end{equation}

If $K_E = 0$, the coherent system is called injective. If $H = 0$, is called generically generated and if also $\tau = 0$, generated. Note that if $(E, V)$ is generically generated, the rank of $I_E$ is $n$.

**Remark 2.2.** If $(E, V)$ is generated, the dual of the kernel of the evaluation map (i.e. the vector bundle $K_E^*$ in (2.1)) is usually denoted as $M_{V,E}$ and if $V = H^0(E)$, as $M_E$.

**Remark 2.3.** If $k \leq n$, recall from [7, Corollary 2.5] that every $\alpha$-semistable coherent system is injective, if $d \leq \min\left\{2n, n + \frac{ng}{k-1}\right\}$. Moreover, for any injective coherent system $(E, V)$, $H^0(H^*) = 0$. In particular, if $k = n$, $H = 0$.

If $k > n$, from [6, Proposition 4.4], there exists $\alpha_{gg}$ such that for any $\alpha > \alpha_{gg}$, a $\alpha$-semistable coherent system is generically generated. Actually, $\alpha_{gg} \leq \frac{d(n-1)}{k}$.

**Remark 2.4.** Note that if $k > n$ and $(E, V) \in U(n, d, k)$, $E$ is semistable and generically generated.

For any $(n, d, k)$ define $G_g(n, d, k)$ as

$G_g(n, d, k) := \{(E, V) : (E, V) \text{ is of type } (n, d, k) \text{ and it is generated with } H^0(E^*) = 0\}$

**Remark 2.5.** Let $(E, V) \in G_g(n, d, k)$. From [11, Proposition 2.5],

1. if $G_g(n, d, k) \neq \emptyset$, $k > n$;
2. any quotient coherent system $(Q, Z)$ is generated;
3. if $k = n + s$ with $s \geq 1$, for any subsystem $(F, W) \subset (E, V)$, $h^0(F) \leq n_F + s - 1$;
(4) if \( d \geq 2gn \), \( G_g(n, d, k) \neq \emptyset \) for \( k \geq n + 1 \);
(5) if \( E \) semistable and \( k = n + 1 \), \((E, V)\) is \(\alpha\)-stable for all \( \alpha > 0 \).

Given \((G, V) \in G_g(n, d, k)\), we have the exact sequence
\[
0 \to M^*_{V,G} \to V \otimes \mathcal{O} \to G \to 0
\]
with \( H^0(M^*_{V,G}) = 0 \). The vector bundle \( M_{V,G} \) is generated by \( V^* \) and \((M_{V,G}, V^*)\) is called the dual span of \((G, V)\). Moreover, if \((n, k) = 1\), \((G, V)\) is \(\alpha\)-stable for large \(\alpha\) if and only if \((M_{V,G}, V^*)\) is \(\alpha\)-stable for large \(\alpha\) (see [6, Corollary 5.10]).

**Remark 2.6.** In particular, if \((L, V) \in G_g(1, d, k)\), the dual span \((M_{V,L}, V^*)\) is \(\alpha\)-stable for large \(\alpha\), since \((L, V)\) is \(\alpha\)-stable for all \(\alpha\).

Let \( G \) be a stable of degree \( d > 2gn \). In [12] Butler proved that \( M_G \) is stable and Mercat in [19] gives an isomorphism between \( G_0(n, n + sg, n + s) \) and \( G_0(s, n + sg, n + s) \) for \( sg < n \). In section \( \S 4 \) we will study this relation for all \(\alpha > 0 \).

If \( X \) is general and \( g > n^2 - 1 \), using the dual span correspondence, Butler in [13] gave a birational map between \( G_0(n, d, n + 1) \) and \( G_0(1, d, n + 1) \). The dual span correspondence was also used in [6, Theorem 5.11] to give necessary and sufficient conditions for non-emptiness of \( G_L(n, d, n + 1) \) and in [11, Theorem 4.7] to prove non-emptiness and to describe the structure of \( G(\alpha : n, d, n + 1) \) for all \(\alpha > 0 \).

**Remark 2.7.** The vector bundles \( M_{V,G} \) have been studied from different points of view (see e.g. [14], [22] [2]). The existence of line bundle of degree \( d \) with \( M_{V,L} \) stable and \( \dim V = n + 1 \), has been proved in the following cases;

1. \( d \geq 2g \) and \( d + (1 - g) \geq n + 1 \geq 2 \) (see [19] and [12]).
2. If \( K \) is the canonical bundle and \( X \) is non-hyperelliptic (see [22]).
3. If \( X \) is general and \( n + g - \frac{q}{n+1} \leq d \leq g + n \) (see [11], [13] and [24]).
4. If \( X \) is general and; \( d \geq n + g - \frac{g}{n+1} \) and \( g \geq n^2 - 1 \) (see [11] and [13]).

**Remark 2.8.** Note that if \( G_g(n, d, k) \neq \emptyset \), from [22, Proposition 3.2], \( G_g(n, d, n + 1) \neq \emptyset \) and by the dual span correspondence \( G_g(1, d, n + 1) \neq \emptyset \).

3. Brill-Noether Bundles

In this section, we recall some facts and the construction of the Brill-Noether bundles in [9], [19] and [10].

First we recall from [19] a Proposition that will be used.

**Proposition 3.1.** [19, Proposition A.2] Let \( F \) be a vector bundle of rank \( n \) and degree \( d \) such that \( H^0(F^*) = 0 \). If the maximal semistable subbundle of \( F \) has slope \( < 2 \) then \( h^0(F) \leq n + \frac{d-n}{g} \).

Let \( E \) be a stable bundle of degree \( d \) and rank \( n \geq 2 \).
If \( d > n(2g - 2) \), \( U(n, d, k) = \emptyset \) for \( k > d + n(1 - g) \);

- if \( d \geq 2gn \), \( G_g(n, d, k) \neq \emptyset \);
- if \( 0 < d < n \) and \( k > n \), \( U(n, d, k) = \emptyset \) (see [9]).

**Remark 3.2.** If \( 0 < d < 2n \), there exist stable vector bundles of rank \( n \) and degree \( d \) with \( k \) independent sections if and only if \( n \leq d + (n - k)g \) and \( (n, d, k) \neq (n, n, n) \) (see [9] and [19]). Therefore, \( G_0(n, d, k) \neq \emptyset \) if and only if \( n \leq d + (n - k)g \) and \( (n, d, k) \neq (n, n, n) \). Moreover, if \( n > d + (n - k)g \), \( U(n, d, k) = \emptyset \).

If \( 0 < d < 2n \), stable bundles with \( k \leq n \) define injective coherent systems that are \( \alpha \)-stable for all allowable \( \alpha \) (see [6], [5] and Remark 2.3). Hence \( U(n, d, k) \neq \emptyset \).

For \( k > n \), we know from [19, 2-B1] that any such stable bundle \( A \) fits in an exact sequence

\[
0 \to G^* \to H^0(A) \otimes \mathcal{O} \to A \to 0.
\]

where \( G \) is a stable bundle of rank \( s \), slope \( > 2g \) and from Proposition 3.1, \( h^0(G) = h^0(A) = k \). Actually, \( A = M_G \) and the coherent system \( (A, H^0(A)) \) is in \( G_0(n, d, k) \) and it is generated. Moreover, \( (G, H^0(G)) \in G_0(s, d, k) \).

If \( W \) is a general subspace of \( H^0(A) \) of dim \( W = s' \), we have the exact sequence

\[
0 \to W \otimes \mathcal{O} \to A \to B \to 0.
\]

From [19, 3-B1] and its proof, we know that any such bundle \( B \) that fitting in the exact sequence (3.2) is stable.

**Remark 3.3.** Note that the condition \( d > 2gs \) with \( 0 < d < 2n \) is equivalent to \( 0 < gs < n \) and \( d \geq n + gs \). Moreover, if \( k = n + s \),

\[
d \geq n + gs \iff n \leq d + (n - k)g.
\]

**Remark 3.4.** In particular, if \( d = (n+s')+sg \), with \( 0 < s' < g \), \( h^0(G) = h^0(A) = (n+s')+s \). Moreover, \( n_B = n \), and from Proposition 3.1, \( h^0(B) = n + s \) and \( H^0(A) = W \oplus H^0(B) \). The coherent system \( (B, H^0(B)) \) is in \( G_0(n, d, k) \) and it is generated.

From the cohomology sequence of (3.2) and Remark 3.4 we have that the coboundary map \( \delta : H^0(B) \to H^1(\mathcal{O}) \otimes W \) is the zero map and hence, we have the following diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & W \otimes \mathcal{O} & \rightarrow & H^0(A) \otimes \mathcal{O} & \rightarrow & H^0(B) \otimes \mathcal{O} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & A & \rightarrow & B & \rightarrow & 0.
\end{array}
\]
Lemma 3.8. The following Lemma.

\[ k > n. \]

From the construction of the stable bundles \( M_{W,G} \) we have the following Proposition

**Proposition 3.6.** The Petri map for \((M_{W,G}, W^*)\) is injective.

*Proof.* The kernel of the Petri map

\[ W^* \otimes H^0(M_{W,G}^* \otimes K) \to H^0(M_{W,G} \otimes M_{W,G}^* \otimes K) \]

is \( H^0(G^* \otimes M_{W,G}^* \otimes K) \).

Since \( M_{W,G} \otimes G \) is semistable and \( \mu(M_{W,G} \otimes G) > 2g > 2g - 2, H^0(M_{W,G}^* \otimes G^* \otimes K) \).

Thereby, the Petri map is injective. \( \square \)

The stable vector bundles \( M_{W,G} \) in Theorem 3.5 together with the stable bundles with \( k \leq n, \) and \( 0 < d < 2n \) were tensored in [10], by line bundles with sections to produce Brill-Noether bundles with \( d > 2n \).

**Remark 3.7.** Actually, let \( d = d'' + nd' \) with \( d' \geq (k'-1)(k'+g)/k', \) \( 0 < d'' < 2n \) and \( 1 \leq k' \leq g. \) It was proved in [10] Theorem 3.2 that if \( n \leq d'' + (n-k)g \) and \( (d'', k) \neq (n, n), B(n, d, k'k) \neq \emptyset. \) Furthermore, applying the Serre duality, \( B(n, 2n(g - 1) - d, k'k - d + n(g - 1)) \neq \emptyset. \) Moreover, such bundles determine a region in the Brill-Noether map (see [9]) that extends beyond the region determined by the stable bundles in [21] (see [10] Section 5).

We want to study the coherent systems defined by the stable bundles \( M_{W,G} \otimes L \) with \( L \) a line bundle with sections.

Let \((E, V)\) be a coherent system of type \((n, d, k)\) and \((F, W)\) a subsystem of \((E, V)\). To study the stability of a coherent system we can restrict to subsystems of the form \((F, H^0(F) \cap V)\) since \( \mu_\alpha(F, W) \leq \mu_\alpha(F, H^0(F) \cap V) \) for all \( \alpha > 0. \) Assume \( E \) is stable and \( k > n. \) If \( h^0(F) \leq n_F, \mu_\alpha(F, W) < \mu_\alpha(E, V), \) for all \( \alpha > 0. \) If \( h^0(F) > n_F \) we have the following Lemma.

**Lemma 3.8.** Let \((F, W)\) be a coherent system with \( h^0(F) > n_F. \) If \( H^0(F^*) \neq 0 \) and \( F \cong \mathcal{O}^r \oplus G \) with \( r \geq 1 \) and \( H^0(G^*) = 0, \) \( \frac{h^0(F)}{n_F} < \frac{h^0(G)}{n_G}. \) Moreover, \( \mu_\alpha(F, W) < \mu_\alpha(G, H^0(G)) \) for all \( \alpha > 0. \)

*Proof.* If \( F \cong \mathcal{O}^r \oplus G \) with \( H^0(G^*) = 0, r \geq 1 \) and \( h^0(F) > n_F, \)

\[
\frac{h^0(F)}{n_F} = \frac{r + h^0(G)}{n_F} < \frac{h^0(G)}{n_F - r} = \frac{h^0(G)}{n_G}.
\]

\[
(3.4)
\]
Moreover, \( d_F = d_G \) and \( r + n_G = n_F < h^0(F) = r + h^0(G) \). Hence, \( \mu(F) < \mu(G) \) and from (3.4), \( \mu_\alpha(F, W) < \mu_\alpha(G, H^0(G)) \) for all \( \alpha > 0 \). \hfill \square

**Remark 3.9.** From Lemma 3.8, in order to prove the \( \alpha \)-stability of coherent system \( (E, V) \) with \( E \) stable we can just consider, without loss of generality, subsystems \( (F, V \cap H^0(F)) \) that satisfies \( h^0(F) > n_F, H^0(F^*) = 0 \).

### 4. Coherent systems of type \((n, d, n+1)\)

For a general curve it was proved in [11] that \( \mathcal{G}(\alpha : n, d, n+1) \neq \emptyset \) if and only if \( g \geq \beta(1,d,n+1) \geq 0 \) and if \( g = \beta \), \( n \nmid g \). Moreover, if \( \beta \geq 0 \) then \( \mathcal{G}(n,d,n+1) := \mathcal{G}(\alpha : n, d, n+1) = \mathcal{G}(\alpha' : n, d, n+1) \) for all \( \alpha, \alpha' > 0 \). For \( d > g + n \), the geometry of the \( \mathcal{G}_L(n, d, n+1) \) was described in [3].

For any curve we have the following Propositions. Note that there is an overlap between them, but the proofs illustrate different methods that could be used for \( k > n+1 \).

**Proposition 4.1.** If \( d \geq n + g \) and \( n \geq g \), \( U(n, d, n+1) \neq \emptyset \).

**Proof.** Let \( L \) be a line bundle of degree \( d \geq 2g \). Hence, \( h^0(L) = d + 1 - g \). Moreover, it is generated and from [22, Proposition 3.2] a general subspace \( V \) of dimension \( d + 1 - g \geq n + 1 \geq 2 \) generates \( L \). By Theorem 3.5, \( M_{V,L} \) is stable and from Remark 2.6 and Remark 2.4 \( (M_{V,L}, V^*) \) is \( \alpha \)-stable for all \( \alpha > 0 \). Therefore, \( U(n, d, n+1) \neq \emptyset \). \hfill \square

**Proposition 4.2.** If \( n + g \leq d < 2n \) and \( d' \geq 0 \),

- \( \mathcal{G}(\alpha : n, d + nd', n+1) \neq \emptyset \) for all \( \alpha > 0 \);  
- \( U(n, d + nd', n+1) \neq \emptyset \).

**Proof.** For any \( n + g \leq d < 2n \), there exists a generated stable bundle \( E \) (see Theorem 3.5). Moreover, from Remarks 2.8 and 2.5 there exists \( V \subset H^0(E) \) such that \( (E, V) \in G_g(n, d, n+1) \) and it is \( \alpha \)-stable for all \( \alpha > 0 \). Hence, \( U(n, d, n+1) \neq \emptyset \). The Proposition follows now from [23, Lemma 1.5]. \hfill \square

**Proposition 4.3.** For any \( n \geq 2 \) and \( d > 2gn \), there exists \( (L, W) \in G_g(1, d, n+1) \) such that \( M_{W,L} \) is stable. Moreover, \( U(n, d, n+1) \neq \emptyset \).

**Proof.** Let \( G \) be stable of rank \( n \) and degree \( d > 2gn \). Hence, \( G \) is generated and from [22, Proposition 3.4] is generated by a subspace \( W \subset H^0(G) \) of dimension \( n + 1 \).

The Proposition follows from the dual span correspondence since the dual of the kernel of the evaluation map \( M_{W,G} \) is a line bundle \( L \). Hence, \( (G, W) \) is \( \alpha_L \)-stable since \( (L, W) \) is \( \alpha \)-stable for all \( \alpha > 0 \) (see [6, Corollary 5.10]). Moreover, \( G \) is stable, and hence \( (G, W) \) is \( \alpha \)-stable for all \( \alpha > 0 \). Therefore, \( U(n, d, n+1) \neq \emptyset \). \hfill \square
5. Coherent systems of type \((n, d, n + s)\)

To study \(G(\alpha : n, d, n + s)\) with \(n + sg + s' = d < 2n\) for any \(\alpha > 0\), we shall consider three cases, depending on \(s'\), namely

1. \(d < n + sg\);
2. \(d = n + sg\);
3. \(d > n + sg\).

In this section we will give a complete description of the moduli spaces \(G(\alpha : n, d, n + s)\), for \(d \leq n + sg\) (see Theorem 5.6 and 5.1).

As we have seen (see Remark 3.2), if \(d < n + sg\), \(G_0(n, d, n + s) = \emptyset\) and the emptiness is related to the non-existence of semistable bundles of type \((n, d, n + s)\). Our object in this section is to generalize such relation to arbitrary \(\alpha > 0\) and prove

**Theorem 5.1.** If \(d < n + gs\), \(G(\alpha : n, d, n + s) = \emptyset\) for all \(\alpha > 0\).

**Remark 5.2.** For general curves, Theorem 5.1 was proved in [11, Corollary 3.10].

We shall prove Theorems 5.1 and 5.6 by means of a sequence of Propositions.

The following Lemma follows at once from Proposition 3.1 and Lemma 3.8.

**Lemma 5.3.** Let \(E\) and \(F\) be vector bundles with \(\mu(F) \leq \mu(E) < 2\). If either \(F\) is semistable or \(E\) semistable and \(F\) a subbundle of \(E\),

\[
\frac{h^0(F)}{n_F} \leq \frac{\mu(E) - 1}{g} + 1.
\]

Recall from Remark 3.9 that if \(E\) is stable, without loss of generality, we can just consider subsystems \((F, V \cap H^0(F))\) that satisfies \(h^0(F) > n_F, H^0(F^*) = 0\).

**Proposition 5.4.** Let \((E, V)\) be a coherent system of type \((n, d, n + s)\) with \(d \leq n + sg\), \(0 < sg < n\). If \(E\) is semistable, \((E, V)\) is \(\alpha\)-semistable for all \(\alpha > 0\). Moreover, if \(E\) is stable, \((E, V)\) is \(\alpha\)-stable for all \(\alpha > 0\).

**Proof.** Let \((F, W) \subset (E, V)\) be a subsystem with \(H^0(F^*) = 0\). Since \(E\) is stable and \(\mu(E) < 2\), from Lemma 5.3

\[
\frac{\dim W}{n_F} \leq \frac{h^0(F)}{n_F} \leq 1 + \frac{\mu(E) - 1}{g} \leq \frac{n + s}{n}.
\]

Therefore, from (5.2) and the semistability of \(E\), \(\mu_\alpha(F, W) \leq \mu_\alpha(E, V)\) for all \(\alpha > 0\). If \(E\) is stable \(\mu_\alpha(F, W) < \mu_\alpha(E, V)\) for all \(\alpha > 0\).

**Proposition 5.5.** If \((E, V)\) is an \(\alpha\)-stable coherent system of type \((n, d, n + s)\) with \(d \leq n + sg\), \(E\) is stable.
Proof. Suppose $Q$ is a stable quotient bundle such that $\mu(Q) \leq \mu(E) < 2$.

From Lemma 5.3,

$$\frac{h^0(Q)}{n_Q} \leq \frac{\mu(Q) - 1}{g} + 1 \leq \frac{\mu(E) - 1}{g} + 1 \leq \frac{n + s}{n}.$$ 

Hence, $\mu_\alpha(Q, W) \leq \mu_\alpha(E, V)$ which is a contradiction to the $\alpha$-stability of $(E, V)$. Therefore, $E$ is stable.

Proof of Theorem 5.1 If $(E, V) \in G(\alpha : n, d, n + s)$, from Proposition 5.5, $E$ is stable which is a contradiction (see Remark 3.2). Therefore, $G(\alpha : n, d, n + s) = \emptyset$ for all $\alpha > 0$.

For $d = n + sg$ we have the following Theorem.

**Theorem 5.6.** Let $2 \leq gs < n$ and $k = n + s$. If $d = n + gs$,

1. $G(\alpha : n, d, k) \neq \emptyset$ for all $\alpha > 0$;
2. $G(n, d, k) := G(\alpha : n, d, k) = G(\alpha' : n, d, k)$ for $\alpha, \alpha' > 0$ i.e. $\alpha_L = 0$;
3. $U(n, d, k) \neq \emptyset$. Moreover, $U(n, d, k) = G(n, d, k) \subset G_g(n, d, k)$;
4. $G(n, d, k)$ is smooth irreducible of dimension $\beta$. Moreover, $G(n, d, k) \cong \mathcal{M}(s, d)$.
5. $G(\alpha : s, d, k) = G(n, d, k)$ for all $\alpha > 0$

Proof. From Theorem 3.5 there exists a generated coherent system $(E, V)$ in $G_0(n, d, n + s)$ with $E$ stable and from Proposition 5.4 $(E, V)$ is $\alpha$-stable for all $\alpha > 0$. Therefore, $G(\alpha : n, d, n + s) \neq \emptyset$ for all $\alpha > 0$. The equality $G(\alpha; n, d, k) = G(\alpha'; n, d, k)$ for $\alpha, \alpha' > 0$ follows from Proposition 5.5. Part (4) follows from Proposition 3.6 and the dual span correspondence gives the isomorphism $G(n, d, k) \cong \mathcal{M}(s, d)$.

Part (5) follows from [6, Corollary 5.10] and the dual span correspondence. Note that in this case, for the dual span correspondence we do not need $(n, k) = 1$, since the vector bundles are stable and the case $\frac{k'}{n'} = \frac{k}{n}$ is allowed (see [6, Corollary 5.10]).

6. Case $d > n + sg$

Assume now that $d = n + sg + s'$ with $0 < s' < g$. From Remark 5.4 $G_0(n, d, n + s) \neq \emptyset$.

For any $(E, V) \in G_0(n, d, n + s)$ we have the following Proposition.

**Proposition 6.1.** Any $(E, V) \in G_0(n, d, n + s)$ is generically generated.

Proof. Let $(E, V) \in G_0(n, d, n + s)$ and $I_E$ be the image of the evaluation map $V \otimes \mathcal{O} \rightarrow E$. If $n_I$ is the rank of $I_E$, $n_I + r = n$ with $r \geq 0$. Let $I_E = \mathcal{O}^t \oplus N$ with $H^0(N^*) = 0$ and $t \geq 0$. 


Since, $E$ is semistable $\mu(N) \leq \mu(E)$. From Lemma 3.8 and Proposition 3.1
\[
\frac{n + s}{n_I} \leq \frac{h^0(I_E)}{n_I} \leq \frac{h^0(N)}{n_N}
\]
and
\[
\frac{n_I + r + s}{n_I} \leq 1 + \frac{\mu(N) - 1}{g} \leq 1 + \frac{\mu(E) - 1}{g} = 1 + \frac{s}{n} + \frac{s'}{ng}.
\]
Hence,
\[
\frac{r}{n_I} \leq \frac{s'}{ng} + \frac{s}{n} - \frac{s}{n_I} \leq \frac{s'}{ng}.
\]
Since $0 < s' < g$, $0 \leq r < 1$. Therefore, $r = 0$ and hence $(E, V)$ is generically generated. \qed

Let $(B, H^0(B)) \in G_0(n, d, n + s)$ be the coherent system defined in section §3. Let $(F, H^0(F)) \subset (B, H^0(B))$ be a coherent subsystem. Without loss of generality (see Remark 3.9) we can assume that $h^0(F) > n_F$, $H^0(F^*) = 0$. Furthermore, the following Lemma allow us to assume also that $(F, H^0(F))$ is generically generated.

**Lemma 6.2.** Let $(F, H^0(F))$ be a subsystem of $(B, H^0(B))$ with $h^0(F) > n_F$ and $H^0(F^*) = 0$. If $(F, H^0(F))$ is not generically generated then there exists a generated subsystem $(N, H^0(N)) \subset (B, H^0(B))$ with $H^0(N^*) = 0$ such $\mu_\alpha(F, H^0(F)) < \mu_\alpha(N, H^0(N))$ for all $\alpha > 0$.

**Proof.** If $h^0(F) = n_F + b$, from Proposition 3.1 $n_F + bg \leq d_F$ i.e. $d_F = n_F + bg + b'$ with $g > b' \geq 0$.

Assume $(F, H^0(F))$ is not generically generated and let $I_F$ be the image of the evaluation map $H^0(F) \otimes \mathcal{O} \to F$. If $n_I$ is the rank of $I_F$, $n_I + r = n_F$ with $r \geq 1$.

Let $I_F = \mathcal{O}^t \oplus N$ with $H^0(N^*) = 0$ and $t \geq 0$. Note that $(N, H^0(N))$ is generated.

As in the proof of Proposition 6.1 from Lemma 3.8 we have that,
\[
(6.1) \quad \frac{h^0(F)}{n_F} < \frac{n_F + b}{n_I} = \frac{h^0(I_F)}{n_I} \leq \frac{h^0(N)}{n_N}.
\]
Suppose $\mu(N) \leq \mu(F)$. From Proposition 3.1
\[
\frac{n_I + r + b}{n_I} \leq 1 + \frac{\mu(N) - 1}{g} \leq 1 + \frac{\mu(F) - 1}{g} = 1 + \frac{b}{n_F} + \frac{b'}{n_F g}.
\]
Hence,
\[
\frac{r}{n_I} < \frac{b'}{n_F g} + \frac{b}{n_F} - \frac{b}{n_I} < \frac{b'}{n_F g}.
\]
Since $b' < g$, $r$ must have to be $< 1$, which is a contradiction. Therefore, $\mu(F) < \mu(N)$. This last inequality together with (6.1) implies that $\mu_\alpha(F, H^0(F)) < \mu_\alpha(N, H^0(N))$ for all $\alpha > 0$. \qed
Therefore, from Lemmas 3.8 and 6.2 to prove the $\alpha$-stability of $(B, H^0(B))$ we can assume that the subsystems $(F, H^0(F))$ are generically generated with $h^0(F) > n$ and $H^0(F^*) = 0$.

**Proposition 6.3.** $(B, H^0(B))$ is $\alpha$-stable for all $\alpha > 0$. Moreover, $U(n, d, n + s) \neq \emptyset$.

**Proof.** Let $(F, H^0(F))$ be a generically generated subsystem of type $(n_F, n_F + bg + b', n_F + b)$ with $H^0(F^*) = 0$. Since $B$ is stable, to prove the $\alpha$-stability of $(B, H^0(B))$ we need to prove that $\frac{b}{n_F} \leq \frac{s}{n}$.

From the stability of $B$

\begin{equation}
1 + \frac{bg}{n_F} + \frac{b'}{n_F} = \mu(F) < \mu(B) = 1 + \frac{sg}{n} + \frac{s'}{n}.
\end{equation}

Thus, if $\frac{s'}{n} \leq \frac{b'}{n_F}$, from (6.2), $\frac{b}{n_F} < \frac{s}{n}$ and hence, $\mu_{\alpha}(F, H^0(F)) < \mu_{\alpha}(B, H^0(B))$ for all $\alpha > 0$.

Assume $\frac{s'}{n} > \frac{b'}{n_F}$.

Since $(I_F, H^0(F))$ is generated, the exact sequence (2.2) fits in the following diagram

\begin{equation}
\begin{array}{cccccc}
0 & 0 & 0 & \downarrow & \downarrow & \downarrow \\
0 & K_F & H^0(F) \otimes \mathcal{O} & I_F & 0 \\
& \downarrow & \downarrow & \downarrow \\
0 & M^*_B & H^0(B) \otimes \mathcal{O} & B & 0.
\end{array}
\end{equation}

By construction, $M_B$ is a stable bundle of degree $d > 2gs$ and rank $s$. From the stability of $M_B$ and diagram (6.3) we have

\begin{equation}
\frac{n + sg + s'}{s} = \mu(M_B) < \mu(K^*_F) = \frac{dI_F}{b} < \frac{n_F + bg + b'}{b}.
\end{equation}

That is,

\begin{equation}
\frac{n + s'}{s} < \frac{n_F + b'}{b}.
\end{equation}

If

\begin{equation}
\frac{b}{n_F} > \frac{s}{n},
\end{equation}

$s' > \frac{b'n_F}{n_F} > \frac{b's}{b}$ i.e. $\frac{s'}{s} > \frac{b'}{b}$. But, from (6.4),

\begin{equation}
\frac{n}{s} < \frac{n_F}{b} + \frac{b'}{b} - \frac{s'}{s}
\end{equation}

\begin{equation}
< \frac{n_F}{b},
\end{equation}

which is a contradiction to (6.5). Hence, $\frac{b}{n_F} \leq \frac{s}{n}$ and, from the stability of $B$, $\mu_{\alpha}(F, H^0(F)) < \mu_{\alpha}(B, H^0(B))$ for all $\alpha > 0$. Therefore, $(B, H^0(B))$ is $\alpha$-stable for all $\alpha > 0$ and $U(n, d, n + s) \neq \emptyset$.  

**Theorem 6.4.** If $k = n + s$ and $d = n + sg + s'$ with $0 < s' < g$ and $2 \leq gs < n$, 

\[\square\]
(1) $G(\alpha : n, d, k) \neq \emptyset$ for all $\alpha > 0$;
(2) $U(n, d, k) \neq \emptyset$. Moreover, any $(E, V) \in U(n, d, k)$ is generically generated;
(3) $G(\alpha : n, d, k)$ has a smooth irreducible component $G^0(n, d, k)$ of dimension $\beta$. Moreover, $G^0(n, d, k)$ is birationally equivalent to the Grassmanian bundle $Grass(s')$ over $M(s, d)$.

Proof. Part (1) and (2) follows from Propositions 6.3 and 6.1, respectively. The first part of (3) follows from Proposition 3.6.

For any $G \in M(s, d)$, let $U_G$ be the set

$$U_G = \{ V \in Grass(s', H^0(G)) : (G, V) \text{ is generated with } M_{V, G} \text{ stable} \}.$$  

By openness of stability, $U_G$ is an open set of $Grass(s', H^0(G))$. The open sets $U_G$ for all $G \in M(n, d)$ define an open set $Z$ in the Grassmannian bundle $Grass(s)$ over $M(n, d)$. The dual span correspondence define a coherent system in $U(n, d, n + s)$. From the universal properties of the moduli space $G_L(n, d, k)$, the map from $Z$ to $G^0(n, d, k)$ is regular and hence it gives a birational equivalence.

From Theorem 5.6 and 6.4 we conclude:

**Corollary 6.5.** For any Brill-Noether triple $(n, d, k)$ with $n < d < 2n$ and $k > n$ with $n \leq d + (n - k)g$, $U(n, d, k) \neq \emptyset$. Moreover, there is an open set $Z$ of $B(n, d, k)$ such that any $E \in Z$ defines a $\alpha$-stable coherent system for all $\alpha > 0$.

### 7. Tensoring coherent systems

Given two coherent systems $(E_i, V_i)$ of type $(n_i, d_i, k_i)$, with $i = 1, 2$, the pair $(E_1 \otimes E_2, V_1 \otimes V_2)$ need not to be a coherent system of type $(n_1n_2, d_1n_2 + d_2n_1, k_1k_2)$.

However, to get a coherent system of type $(n_1n_2, d_1n_2 + d_2n_1, k_1k_2)$ tensor the associated sequences (2.2) and (2.3) of $(E_i, V_i)$ with $E_j$ to get the following exact sequences

\begin{equation}
0 \to K_{E_i} \otimes E_j \to V_i \otimes E_j \to I_{E_i} \otimes E_j \to 0
\end{equation}

and

\begin{equation}
0 \to I_{E_i} \otimes E_j \to E_i \otimes E_j \to H \otimes E_j \oplus \tau \to 0.
\end{equation}

If $H^0(K_{E_i} \otimes E_j) = 0$,

$$(\varphi(E_i, V_i) \otimes id)^* : V_i \otimes H^0(E_j) \to H^0(E_i \otimes E_j)$$

is injective and hence we identify $V_i \otimes V_j$ with the image of $V_i \otimes V_j$ in $H^0(E_i \otimes E_j)$ under the inclusions $V_i \otimes V_j \hookrightarrow V_i \otimes H^0(E_j) \hookrightarrow H^0(E_i \otimes E_j)$. In this case, we define the tensor product of $(E_j, V_j)$ by $(E_i, V_i)$ to be the coherent system $(E_i \otimes E_j, V_i \otimes V_j)$ of type $(n_1n_2, d_1n_2 + d_2n_1, k_1k_2)$.

In this section we are interested in tensor coherent systems $(E, V)$ of type $(n, d, k)$ by coherent systems $(L, W)$ of type $(1, d', k')$. 
Given \((L, W)\) of type \((1, d', k')\) we have the exact sequences

\[
0 \to K_L \to W \otimes O \to I_L \to 0
\]

and

\[
0 \to I_L \to L \to \tau \to 0,
\]

where \(I_L\) is the image of the evaluation map \(W \otimes O \to L\).

Assume \(H^0(K_L \otimes E) = 0\) and let \((E \otimes L, V \otimes W)\) be a coherent system of type \((n, d + nd' + kk')\). To study the stability of \((E \otimes L, V \otimes W)\) we want to describe \(H^0(F) \cap (V \otimes W)\) for any subbundle \(F \subseteq E \otimes L\). The following Lemmas will do it.

First recall that any subbundle \(F \subseteq E \otimes L\) defines a subbundle \(F' \subseteq E \otimes I_L\) that fits in the following diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & F' \to F \to \tau' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & I_L \otimes E \to L \otimes E \to \tau \to 0,
\end{array}
\]

Lemma 7.1. \(H^0(F) \cap (V \otimes W) = H^0(F') \cap (V \otimes W)\).

Proof. From the diagram (7.4), \(H^0(F') = H^0(F) \cap H^0(I_L \otimes E)\). Hence,

\[
H^0(F') \cap (V \otimes W) = H^0(F) \cap H^0(I_L \otimes E) \cap (V \otimes W) = H^0(F) \cap (V \otimes W),
\]

since \(H^0(K_L \otimes E) = 0\) and \(V \otimes W \subseteq H^0(E) \otimes W \subseteq H^0(I_L \otimes E)\). \(\square\)

For the subbundle \(F' \subseteq E \otimes I_L\), we have the exact sequences \(0 \to F' \to I_L \otimes E \to Q \to 0\) and

\[
0 \to F' \otimes I_L^* \to E \overset{\pi}{\to} Q \otimes I_L^* \to 0
\]

where \(Q\) is the quotient bundle.

Tensor (7.5) by the sequence (7.3) to get the following diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & K_L \otimes F' \otimes I_L^* \to K_L \otimes E \overset{id \otimes p}{\to} K_L \otimes Q \otimes I_L^* \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & W \otimes F' \otimes I_L^* \to W \otimes E \overset{id \otimes p}{\to} W \otimes Q \otimes I_L^* \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & F' \to I_L \otimes E \overset{q}{\to} Q \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]
From the cohomology of \((7.6)\), we have the following diagram

\[
\begin{array}{ccc}
0 & \to & H^0(K_L \otimes Q \otimes I^*_L) \\
\downarrow & & \downarrow \\
W \otimes H^0(F' \otimes I^*_L) & \to & W \otimes H^0(E) \\
\downarrow & & \downarrow \text{id} \otimes p^* \\
0 & \to & H^0(F') \quad \overset{i^*}{\to} \quad H^0(I_L \otimes E) \quad \overset{q}{\to} \quad H^0(Q) \\
\end{array}
\]

since \(H^0(K_L \otimes E) = 0\).

**Lemma 7.2.** \(H^0(F') \cap (W \otimes H^0(E)) = W \otimes H^0(F' \otimes I^*_L)\).

**Proof.** From the commutativity of diagram \((7.7)\),

\[
W \otimes H^0(F' \otimes I^*_L) \subseteq (W \otimes H^0(E)) \cap H^0(F').
\]

To prove \((W \otimes H^0(E)) \cap H^0(F') \subseteq W \otimes H^0(F' \otimes I^*_L)\) take any \(0 \neq a \in W\). Hence, \(\langle a \rangle \otimes H^0(E) \subseteq W \otimes H^0(E)\). From \((7.7)\) we have the following diagram

\[
\begin{array}{ccc}
0 & \to & < a > \otimes H^0(F' \otimes I^*_L) \\
\downarrow & & \downarrow \\
0 & \to & < a > \otimes H^0(E) \\
\downarrow & & \downarrow \text{id} \otimes p^* \\
0 & \to & H^0(F') \quad \overset{i^*}{\to} \quad H^0(I_L \otimes E) \quad \overset{q^*}{\to} \quad H^0(Q) \\
\end{array}
\]

Let \(\{s_i\}\) be a basis of \(H^0(E)\). Let

\[
c = \lambda a \otimes \sum \nu_i s_i \in \langle a \rangle \otimes H^0(E)
\]

be such that \(\gamma(c) \in i^*(H^0(F')) \subseteq H^0(I_L \otimes E)\). From diagram \((7.8)\), \(q^*(\gamma(c)) = 0 \in H^0(Q)\) i.e. \(q^*(\gamma(c))\) is the zero section. Hence,

\[
q^*(\gamma(c)) = \beta(id \otimes p^*)(c) = \beta(\lambda a \otimes p^*(\sum \nu_i s_i)) = 0.
\]

That is, for \(x \in X\)

\[
x \mapsto \beta(id \otimes p^*)(c)(x) = \lambda a(x) \otimes p^*(\sum \nu_i s_i)(x) = 0.
\]

Since \(a \in W\) is a non-zero section, \(\lambda a(x) = 0\) only for a finite set of points \(x_j \in X\). Hence, \(p^*(\sum \nu_i s_i)\) must be the zero section. That is, \(\sum \nu_i s_i \in \ker(p^*) = H^0(F' \otimes I^*_L)\). Therefore, if \(\gamma(c) \in i^*(H^0(F'))\), \(c \in \langle a \rangle \otimes H^0(F' \otimes I^*_L)\). Thereby, for all \(a \in W\),

\[
\langle a \rangle \otimes H^0(E) \cap H^0(F') \subseteq \langle a \rangle \otimes H^0(F' \otimes I^*_L) \subseteq W \otimes H^0(F' \otimes I^*_L)
\]

and hence,

\[
(W \otimes H^0(E)) \cap H^0(F') \subseteq W \otimes H^0(F' \otimes I^*_L).
\]
Therefore, \( W \otimes H^0(F' \otimes I_L^*) = (W \otimes H^0(E)) \cap H^0(F') \).

Moreover, for \( V \subsetneq H^0(E) \) we have the following Lemma

**Lemma 7.3.** \( H^0(F) \cap (W \otimes V) = W \otimes (H^0(F' \otimes I_L^*) \cap V) \).

**Proof.** From the injectivity of the first two columns in diagram 7.7 it is clear that

\[
W \otimes (H^0(F' \otimes I_L^*) \cap V) \subset H^0(F') \cap (V \otimes W).
\]

Moreover, from Lemma 7.2

\[
H^0(F') \cap (V \otimes W) = H^0(F') \cap ((H^0(E) \otimes W) \cap (V \otimes W))
\]

\[
= (H^0(F') \cap (H^0(E) \otimes W)) \cap (V \otimes W)
\]

\[
= (H^0(F' \otimes I_L^*) \otimes W) \cap (V \otimes W)
\]

\[
\subseteq (H^0(F' \otimes I_L^*) \cap V) \otimes W.
\]

Therefore, \( H^0(F') \cap (V \otimes W) = W \otimes (H^0(F' \otimes I_L^*) \cap V) \) and the Proposition follows from Lemma 7.1.

For coherent systems \((L, C)\) of type \((1, d', 1)\) we know (see [23, Lemma 1.5]) that \((E, V)\) is \(\alpha\)-stable if and only if \((E \otimes L, V)\) is \(\alpha\)-stable. For coherent systems of type \((1, d', k')\) we have the following Lemma.

**Lemma 7.4.** Let \((E, V)\) be a coherent system of type \((n, d, k)\) and \((L, W)\) of type \((1, d', k')\). Assume \(H^0(K_L \otimes E) = 0\). Then \((E, V)\) is \(k'\alpha\)-stable if and only if \((E \otimes L, V \otimes W)\) is \(\alpha\)-stable. Moreover,

1. if \((E, V)\) is \(\alpha_L\)-stable, \((E \otimes L, V \otimes W)\) is \(\alpha_L\)-stable;
2. if \(U(n, d, k) \neq \emptyset, U(n, d + nd', kk') \neq \emptyset\).

**Proof.** Note that if \((F, H^0(F) \cap (V \otimes W))\) is a subsystem of \((E \otimes L, V \otimes W), (F' \otimes I_L^*, H^0(F' \otimes I_L^*) \cap V)\) is a coherent subsystem of \((E, V)\), where \(F'\) is the subbundle of \(I_L \otimes E\) that fits in 7.4.

The first part of the Lemma follows at once from the definitions and Lemmas 7.2 and 7.3 once we notice that \(\mu(F) - \mu(E \otimes L) \leq \mu(F' \otimes I_L^*) - \mu(E)\) and that for any subbundle \(F \subseteq E \otimes L\)

\[
\frac{\dim((W \otimes V) \cap H^0(F))}{n_F} = \frac{k' \dim(H^0(F' \otimes I_L^*) \cap V)}{n_F}.
\]

For the second part, if \((E, V)\) is \(\alpha_L\)-stable,

\[
0 \leq \frac{k}{n} - \frac{\dim(H^0(F' \otimes I_L^*) \cap V)}{n_F}.
\]

\[
\frac{\dim((W \otimes V) \cap H^0(F))}{n_F} = \frac{k' \dim(H^0(F' \otimes I_L^*) \cap V)}{n_F}.
\]
From Lemmas \[7.2\] and \[7.3\]

\[\mu(F) - \mu(E \otimes L) \leq \mu(F' \otimes I_L^*) - \mu(E)\]

\[< \alpha \left( \frac{k}{n} - \frac{\dim(H^0(F' \otimes I_L^*) \cap V)}{n_F} \right)\]

(7.13)

\[< \alpha k' \left( \frac{k}{n} - \frac{\dim(H^0(E \otimes I_V^*) \cap W)}{n_F} \right)\]

\[= \alpha \left( \frac{k'k}{n} - \frac{\dim(H^0(F' \otimes I_L^*) \cap V) \cap W)}{n_F} \right)\]

\[= \alpha \left( \frac{k'k}{n} - \frac{\dim(H^0(F') \cap (V \otimes W))}{n_F} \right).

Therefore, \(\mu_{\alpha}(F, H^0(F) \cap (V \otimes W)) < \mu_{\alpha}(E \otimes L, V \otimes W)\) for \(\alpha > \alpha_L\) and hence \((E \otimes L, V \otimes W)\) is \(\alpha\)-stable for all \(\alpha > \alpha_L\). It is well known that if \(E\) is stable, \(E \otimes L\) is stable. Thereby, if \((E, V) \in U(n, d, k)\), \((E \otimes L, V \otimes W) \in U(n, d + nd', kk')\).

\[\square\]

**Theorem 7.5.** Suppose \(U(n, d, k) \neq \emptyset\). If there exists a coherent system of type \((1, d', k')\) with \(K_L\) semistable and \((k' - 1)d < nd', U(n, d + nd', k'k) \neq \emptyset\).

**Proof.** Let \((E, V) \in U(n, d, k)\). By hypothesis, \(K_L \otimes E\) is semistable of slope \(0\), so \(H^0(K_L \otimes E) = 0\). The Theorem follows at once from Lemma \[7.3\] \(\square\)

**Corollary 7.6.** Let \(d' \geq 2g, k' \geq 2\) with \(d(k' - 1) < d' n\). If \(U(n, d, k) \neq \emptyset, U(n, d + nd', k'k') \neq \emptyset\).

**Proof.** Let \(L\) be a line bundle of degree \(d' \geq 2g\) and \(W\) a general subspace of \(H^0(L)\) of dimension \(k' \geq 2\) that generates \(L\). From Theorem \[3.3\], \(K_L = M_{W,L}'\) is stable and has slope \(\frac{d'}{r_{L,L}}\). Hence, if \((E, V) \in U(n, d, k), H^0(K_L \otimes E) = 0\), therefore, \((E \otimes L, V \otimes W) \in U(n, d + nd', k'k) \neq \emptyset\). \(\square\)

Assume now that \(0 < d'' < 2n\) and \(k'' \geq 1\) with \(n \leq d'' + (n - k'')g\). If \((n, d'', k'') \neq (n, n, n), U(n, d'', k'') \neq \emptyset\) (see Corollary \[6.3\] and \[7\] Theorem A). Hence, from Theorem \[7.5\] we have the following Corollary.

**Corollary 7.7.** Suppose \(0 < d'' < 2n\) and \(k'' \geq 1\) with \(n \leq d'' + (n - k'')g\). Let \(d' \geq 2g, k' \geq 2\) with \(d''(k' - 1) < d' n\). If \((n, d'', k'') \neq (n, n, n), U(n, d'' + nd', k'k'') \neq \emptyset\).

In the definition of the tensor product of coherent systems and in the proof of Lemma \[7.2\] we use that \(H^0(K_L \otimes E) = 0\). Such a condition seems very strong; however, for coherent systems \((E, V)\) with \(V = H^0(E)\) we have the following Proposition.

**Proposition 7.8.** Let \((E, V)\) be a coherent system with \(V = H^0(E)\) and \((L, W)\) a coherent system of type \((1, d', k')\).

1. If \((E, V)\) is generated and \(H^0(L \otimes M_E^*) = 0, H^0(K_L \otimes E) = 0\).
(2) If \((E, V)\) is injective, \(H^0(K_L \otimes E) = 0\).

**Proof.** Suppose \((E, V)\) is generated. Tensor the exact sequence
\[
0 \to M^*_E \to H^0(E) \otimes \mathcal{O} \to E \to 0
\]
with
\[
0 \to K_L \to W \otimes \mathcal{O} \to I_L \to 0
\]
to get the following diagram
\[
\begin{array}{cccc}
  0 & 0 & 0 \\
  \downarrow & \downarrow & \downarrow \\
  0 & K_L \otimes M^*_E & W \otimes M^*_E & I_L \otimes M^*_E \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & K_L \otimes H^0(E) & W \otimes H^0(E) & I_L \otimes H^0(E) \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & K_L \otimes E & W \otimes E & I_L \otimes E \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & 0 & 0 & 0
\end{array}
\]

The first part of the Proposition follows from the cohomology of diagram (7.16) once we notice that \(H^0(K_L) = 0 = H^0(M^*_E)\) and if \(H^0(L \otimes M^*_E) = 0, H^0(I_L \otimes M^*_E) = 0\).

If \((E, V)\) is injective, the Proposition follows at once from the cohomology of the following diagram
\[
\begin{array}{cccc}
  0 & 0 & 0 \\
  \downarrow & \downarrow & \downarrow \\
  0 & K_L \otimes H^0(E) & W \otimes H^0(E) & I_L \otimes H^0(E) \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & K_L \otimes E & W \otimes E & I_L \otimes E \\
  \downarrow & \downarrow & \downarrow & \downarrow \\
  0 & 0 & 0 \\
\end{array}
\]

since \(H^0(K_L) = 0\). \qed

Recall from [17, Lemma 2.6] that if \(B(n, d, k) \neq \emptyset\) there exists \(E \in B(n, d, k)\) with \(h^0(E) = k\). Hence, if \(G_0(n, d, k) \neq \emptyset\), there exists \((E, V) \in G_0(n, d, k)\) with \(V = H^0(E)\).

The following Theorems follow from Lemma 7.4 and Theorem 7.5. The hypothesis in each one allow us to define the tensor products of coherent systems of type \((n, d, k)\) by those of type \((1, d', k')\).

**Theorem 7.9.** Assume \(0 < d'' < 2n\) and \(k'' \geq 1\) with \(n \leq d'' + (n-k'')g\) and \((n, d'', k'') \neq (n, n, n)\). Let \(d = d'' + nd'\) and \(k = k'k''\) with \(d' < 2g\) and \(1 \leq k'\). If \(\beta(1, d', k') \geq 0\), \(U(n, d, k) \neq \emptyset\).

**Proof.** If \(\beta(1, d', k') \geq 0\), there exist coherent systems \((L, W)\) of type \((1, d', k')\). 
For $k \leq n$, from [7, Theorem A], $U(n, d'', k'') \neq \emptyset$ and any $(E, V) \in U(n, d'', k'')$ is injective. From [17, Lemma 2.6] there exists $(E, V) \in U(n, d'', k'')$ with $V = H^0(E)$ and from Proposition 7.8 $H^0(K_L \otimes E) = 0$. From Lemma 7.4, $U(n, d, k) \neq \emptyset$.

For $k > n$, from the hypothesis and the proof of Theorems 5.6 and 6.4 there exist generated coherent systems $(C, V) \in U(n, d'', k'')$ with $V = H^0(C)$ and $M_C$ stable of slope $> 2g$. Hence, for any line bundle $L$ of degree $d' \leq 2g$, $H^0(M_C \otimes L) = 0$ and from Proposition 7.8 $H^0(K_L \otimes E) = 0$. Therefore, from Lemma 7.4, $U(n, d, k) \neq \emptyset$. □

**Corollary 7.10.** Assume $0 < d'' < 2n$ and $k'' \geq 1$ with $n \leq d'' + (n - k'')g$ and $(n, d'', k'') \neq (n, n, n)$. Let $d = d'' + nd'$ and $k = k'k''$ with $0 < d' < 2g$ and $1 \leq k'$. If $\beta(1, d', k') \geq 0$, for any Brill-Noether triple $(n, d, k)$, $U(n, d, k) \neq \emptyset$. Moreover, there is an open set $Z$ of $B(n, d, k)$ such that any $E \in Z$ defines an $\alpha$-stable coherent system for all $\alpha > 0$.

For hyperelliptic curve we have the following Theorem.

**Theorem 7.11.** Let $X$ be a hyperelliptic curve of genus $g \geq 3$. Assume $0 < d'' < 2n$ and $k'' \geq 1$ with $n \leq d'' + (n - k'')g$ and $(n, d'', k'') \neq (n, n, n)$. If $d' = 2(k' - 1)$, $U(n, d + nd', k'k) \neq \emptyset$.

**Proof.** Let $L$ be the hyperelliptic line bundle over $X$. For $1 \leq k' \leq g$, $h^0(L^{\otimes (k' - 1)}) = k'$ and the degree of $L^{\otimes (k' - 1)}$ is $d' = 2(k' - 1)$. As in Theorem 7.9 we get $U(n, d + nd', k'k) \neq \emptyset$ after tensoring with $(L, H^0(L^{\otimes (k' - 1)})$). □

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