SHARP NON-UNIQUENESS OF WEAK SOLUTIONS TO 3D MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. We prove the non-uniqueness of weak solutions to 3D hyper viscous and resistive MHD in the class $L_t^\gamma W_x^{s,p}$, where the exponents $(s, \gamma, p)$ lie in two supercritical regimes. The result reveals that the scaling-invariant Ladyženskaja-Prodi-Serrin (LPS) condition is the right criterion to detect non-uniqueness, even in the highly viscous and resistive regime beyond the Lions exponent. In particular, for the classical viscous and resistive MHD, the non-uniqueness is sharp near the endpoint $(0, 2, \infty)$ of the LPS condition. Moreover, the constructed weak solutions admit the partial regularity outside a small fractal singular set in time with zero $H^{\eta^*}$-Hausdorff dimension, where $\eta^*$ can be any given small positive constant. Furthermore, we prove the strong vanishing viscosity and resistivity result, which yields the failure of Taylor’s conjecture along some subsequence of weak solutions to the hyper viscous and resistive MHD beyond the Lions exponent.

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1. Introduction and main results

1.1. Introduction. We are concerned with the three-dimensional magnetohydrodynamic (MHD for short) system on the torus $T^3 := [-\pi, \pi]^3$,

$$\begin{align*}
\partial_t u + \nu_1 \Delta u + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla P &= 0, \\
\partial_t B + \nu_2 \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\
\text{div} u &= 0, \quad \text{div} B = 0,
\end{align*}$$

(1.1)

where $u = (u_1, u_2, u_3)^\top(t, x) \in \mathbb{R}^3$, $B = (B_1, B_2, B_3)^\top(t, x) \in \mathbb{R}^3$ and $P = P(t, x) \in \mathbb{R}$ correspond to the velocity field, magnetic field and pressure of the fluid, respectively, and $\nu_1, \nu_2 \geq 0$ are the viscous and resistive coefficients, respectively. In particular, in the case without magnetic fields, equations (1.1) reduce to the classical incompressible Navier-Stokes equation (NSE for short).

In the celebrated papers [47], Leray proved the existence of global solutions in $L_t^\infty L_x^2 \cap L_t^3 H^1_x$, which now refers to Leray-Hopf solutions, also due to the important contribution of Hopf [40] in the case of bounded domains. Leray-Hopf solutions enjoy several nice properties, including the partial regularity and weak-strong uniqueness. This kind of solutions also have been obtained for MHD by Serrin (LPS for short) condition.

It is well-known that, weak solutions in the (sub)critical spaces $L_t^\infty L_x^p \cap L_t^3 H^1_x$ has been proved in [1.1].

Introduction.

On the flexible side, in the recent remarkable paper [19], Cheskidov-Luo achieved the non-uniqueness of weak solutions to NSE in the spaces $L_t^\gamma L_x^p$ for any $1 \leq \gamma < 2$. The non-uniqueness is sharp as $L_t^2 L_x^\infty$ is the critical endpoint space in view of the LPS condition. Moreover, in another endpoint case where $(\gamma, p) = (\infty, 2)$ when $d = 2$, the sharp non-uniqueness in $L_t^\gamma L_x^p$, $1 \leq p < 2$, has been proved for 2D NSE in [20]. Very recently, for the 3D hyperdissipative NSE with viscosity beyond the Lions exponent $5/4$, the sharp non-uniqueness at two endpoints of generalized LPS conditions has been proved in [50].

In general, the scaling exponent is a very helpful criterion for the well-posedness/ill-posedness. We refer to the papers by Klainerman [45, 46] for comprehensive discussions of scaling exponents for general nonlinear PDEs.

Another significant progress is that, Buckmaster-Vicol [12] proved that weak/distributional solutions to 3D NSE are not unique in the space $C_t L_x^2$. It was proved there even more that, there are indeed infinitely many weak solutions with the same initial data and with prescribed energy profiles. The intermittent convex integration, developed in [12], has been further applied to many other models, including, for instance, 3D hyperdissipative NSE up to the Lions exponent [7, 53], hypodissipative NSE [22, 31, 52], the stationary NSE [55], SQG equation [11] and transport equations [18, 21, 56]. Remarkably, new intermittent convex integration has been recently developed in [15, 57] for Euler equations, which, particularly, enables to construct non-conservative weak solutions to 3D Euler equations in the class $C_t(H^\beta \cap L_t^{1/(1-2\beta)})$ with $\beta < 1/2$, hence by interpolation also in the class $C_t B^s_{3, \infty}$ for $s < 1/3$. We refer to [13, 14, 23, 30] for comprehensive surveys on convex integration method. We also refer to [42, 43] for non-uniqueness of Leray-Hopf solutions based on spectral assumptions and the recent work [1] for non-unique Leray-Hopf solutions to forced 3D NSE.

For the NSE, one important critical scaling in the mixed Lebesgue space $L_t^\gamma L_x^p$ is the Ladyženskaja-Prodi-Serrin (LPS for short) condition

$$\frac{2}{\gamma} + \frac{d}{p} = 1$$

(1.2)

with $d$ being the underlying spatial dimension, which is critical under the scaling

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad P(t, x) \mapsto \lambda^2 P(\lambda^2 t, \lambda x).$$

(1.3)

It is well-known that, weak solutions in the (sub)critical spaces $L_t^\gamma L_x^p$ with $2/\gamma + 3/p \leq 1$ are unique and even Leray-Hopf. See [19, 32, 33] and references therein.

On the flexible side, in the recent remarkable paper [19], Cheskidov-Luo achieved the non-uniqueness of weak solutions to NSE in the spaces $L_t^\gamma L_x^p$ for any $1 \leq \gamma < 2$. The non-uniqueness is sharp as $L_t^2 L_x^\infty$ is the critical endpoint space in view of the LPS condition. Moreover, in another endpoint case where $(\gamma, p) = (\infty, 2)$ when $d = 2$, the sharp non-uniqueness in $L_t^\gamma L_x^p$, $1 \leq p < 2$, has been proved for 2D NSE in [20]. Very recently, for the 3D hyperdissipative NSE with viscosity beyond the Lions exponent $5/4$, the sharp non-uniqueness at two endpoints of generalized LPS conditions has been proved in [50].
Regarding MHD equations, the corresponding non-uniqueness and related turbulence have attracted significant interests in literature. In particular, several progresses have been made for the ideal MHD, which corresponds to (1.1) when the viscous and resistive coefficients vanish, that is,

\[
\begin{align*}
\partial_t u + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla P &= 0, \\
\partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\
\text{div} u &= 0, \quad \text{div} B = 0.
\end{align*}
\] (1.4)

The lack of uniqueness of regular weak solutions is in agreement with the natural stochasticity observed in turbulent regimes ([36]).

A first attempt at constructing non-vanishing smooth strict subsolutions to 3D ideal MHD is the work by Faraco-Lindberg [34]. Afterwards, bounded weak solutions to (1.4) were constructed by Faraco-Lindberg-Székelyhidi [37], which violate the energy conservation while preserve the magnetic helicity (see §1.2 below). Bounded solutions with prescribed total energy and cross helicity also have been constructed by Faraco-Lindberg-Székelyhidi [37, 39]. For weak solutions with non-conserved magnetic helicity, the first example was constructed by Beekie-Buckmaster-Vicol [3], based on the intermittent convex integration scheme. We also would like to mention the recent work by Faraco-Lindberg-Székelyhidi [39], where the conjecture that \( L^{3}_{t,x} \) is the threshold for magnetic helicity conservation is solved, based on the convex integration via staircase laminates.

In contrast, to the best of our knowledge, there seem not many works on the non-uniqueness problem for the viscous and resistive MHD. One major difficulty here lies in the strong viscosity and resistivity but the limitation of spatial intermittency restricted by the specific geometry of MHD (see §1.2 below for detailed discussions). Dai [24] constructed non-unique weak solutions to Hall MHD equations, where the Hall nonlinearity takes the dominant effect. Moreover, Beekie-Buckmaster-Vicol [3] constructed the intermittent shear flows with 1D intermittency, which actually permit to control hypo viscosity and resistivity \(( -\Delta)^\alpha \) for any \( \alpha \in (0, 1/2) \). Recently, the authors [49] proved the non-uniqueness for MHD type equations with the viscosity and resistivity up to the Lions exponent, by constructing intermittent flows featuring both 2D spatial intermittency and 1D temporal intermittency.

1.2. Main results. The main interest of this paper is to further study the non-uniqueness problem for MHD in the following two aspects:

- Sharp non-uniqueness of weak solutions to MHD (1.1) in the spaces \( L^\gamma_t L^\infty_x \), \( 1 \leq \gamma < 2 \), which approach the endpoint space \( L^2_t L^\infty_x \) with respect to the critical LPS condition.
- Non-uniqueness of weak solutions to MHD type equations (see (1.5) below) in the highly viscous and resistive regime, particularly, beyond the Lions exponent.

We would consider a general formulation of MHD equations with hyper viscosity and resistivity:

\[
\begin{align*}
\partial_t u + \nu_1 (-\Delta)^\alpha u + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla P &= 0, \\
\partial_t B + \nu_2 (-\Delta)^\alpha B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\
\text{div} u &= 0, \quad \text{div} B = 0.
\end{align*}
\] (1.5)

where \( \nu_1, \nu_2 \geq 0 \) and \( \alpha \in [1, 3/2] \). Note that, the viscosity and resistivity exponents can be larger than the Lions exponent 5/4.

It should be mentioned that, one major obstruction at constructing non-unique weak solutions is that, the specific geometry of MHD restricts the choice of oscillations directions, and so, in particular, limits the spatial intermittency of building blocks in the convex integration scheme. Thus, the control of viscosity and resistivity in (1.1) becomes significantly hard.

On the other hand, high dissipations usually help to establish well-posedness of PDEs. One well-known result due to Lions [51] is that, for any divergence-free \( L^2 \) initial data, there exist unique Leray-Hopf solutions to 3D NSE when the viscosity exponent \( \alpha \geq 5/4 \). This kind of well-posedness also holds for hyper viscous and resistive MHD equations, due to Wu [63]. Hence, it is non-trivial to find non-unique weak solutions particularly in the very high viscous and resistive regime. New appropriate criterion, rather than dissipation, is required to detect non-unique weak solutions here.
We will show that, the scaling-invariant LPS condition is the right criterion to detect non-uniqueness: the uniqueness still would fail even in the high viscous and resistive regime beyond the Lions exponent, if the (sub)criticality of state space is violated.

As a matter of fact, the hyper viscous and resistive MHD (1.5) is invariant under the scaling
\[ u(t, x) \mapsto \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \quad B(t, x) \mapsto \lambda^{2\alpha-1} B(\lambda^{2\alpha} t, \lambda x), \]
and \( P(t, x) \mapsto \lambda^{4\alpha-2} P(\lambda^{2\alpha} t, \lambda x). \) The critical exponent \((s, \gamma, p)\) of the mixed Lebesgue spaces \( L^s_t W^{s,p}_x \) satisfies the generalized Ladyženskaja-Prodi-Serrin condition
\[ \frac{2\alpha}{\gamma} + \frac{3}{p} = 2\alpha - 1 + s. \]

We show that the non-uniqueness of weak solutions do exhibit in the spaces \( L^s_t W^{s,p}_x \), where the exponents \((s, \gamma, p)\) lie in the following two supercritical regimes, with respect to the scaling (1.6):

\[ S_1 := \left\{ (s, \gamma, p) \in [0, 3] \times [1, \infty] \times [1, \infty] : 0 \leq s < \frac{2\alpha}{\gamma} + \frac{2\alpha - 2}{p} + 1 - 2\alpha \right\}, \tag{1.8} \]
and
\[ S_2 := \left\{ (s, \gamma, p) \in [0, 3] \times [1, \infty] \times [1, \infty] : 0 \leq s < \frac{4\alpha - 4}{\gamma} + \frac{2}{p} + 1 - 2\alpha \right\}. \tag{1.9} \]

Before stating the main results, let us mention that the weak solutions to (1.5) is taken in the standard distributional sense.

**Definition 1.1 (Weak solutions).** We say that \((u, B) \in L^2([0, T]; L^2(\mathbb{T}^3))\) is a weak solution to the MHD equations (1.5) if
- For all \( t \in [0, T] \), \((u(t, \cdot), B(t, \cdot))\) are divergence free in the sense of distributions and have zero spatial mean.
- Equations (1.5) hold in the sense of distributions, i.e., for any divergence-free test functions \( \varphi \in C_0^\infty([0, T] \times \mathbb{T}^3) \),
\[
\int_{\mathbb{T}^3} u_0 \varphi(0, x) dx + \int_0^T \int_{\mathbb{T}^3} \partial_t \varphi \cdot u + \nu_1 (-\Delta)^\alpha \varphi \cdot u + \nabla \varphi : (u \otimes u - B \otimes B) dx dt = 0,
\]
\[
\int_{\mathbb{T}^3} B_0 \varphi(0, x) dx + \int_0^T \int_{\mathbb{T}^3} \partial_t \varphi \cdot B - \nu_2 (-\Delta)^\alpha \varphi \cdot B + \nabla \varphi : (B \otimes u - u \otimes B) dx dt = 0.
\]

The main result of this paper is formulated in Theorem 1.2 below.

**Theorem 1.2.** Let \( \alpha \in [1, 3/2] \) and \((\overline{u}, \overline{B})\) be any smooth, divergence-free and mean-free vector fields on \([0, T] \times \mathbb{T}^3\). Then, there exists \( \beta' \in (0, 1) \), such that for any \( \varepsilon_* \eta_* > 0 \) and for any \((s, p, \gamma) \in S_1 \cup S_2\), there exist vector fields \( u, B \) and a set
\[ G = \bigcup_{i=1}^\infty (a_i, b_i) \subseteq [0, T], \]
such that the following holds:

(i) Weak solution: \((u, B)\) is a weak solution to (1.5) with the initial datum \((\overline{u}(0), \overline{B}(0))\) and has zero spatial mean.

(ii) Regularity: \( u, B \in H^{\beta'}_{t,x} \cap L^s_t W^{s,p}_x \), and \((u, B)|_{G \times \mathbb{T}^3} \in C^\infty(G \times \mathbb{T}^3) \times C^\infty(G \times \mathbb{T}^3). \)

Moreover, if there exists \( t_0 \in (0, T) \) such that \((\overline{u}, \overline{B})\) is the solution to (1.5) on \([0, t_0]\), then \((u, B)\) agrees with \((\overline{u}, \overline{B})\) near \( t = 0 \).

(iii) The Hausdorff dimension of the singular set \( \mathcal{B} = [0, T] \setminus G \) satisfies
\[ d_H(\mathcal{B}) < \eta_* \]
In particular, the singular set \( \mathcal{B} \) has zero Hausdorff \( H^{\beta_*} \) measure, i.e., \( H^{\beta_*}(\mathcal{B}) = 0 \).
(iv) Small deviations of temporal support:

\[ \text{supp}_t(u, B) \subseteq N_{\varepsilon_*}(\text{supp}_t(\tilde{u}, \tilde{B})). \]

(v) Small deviations on average:

\[ \|u - \tilde{u}\|_{L_t^2 L_x^2} + \|u - \tilde{u}\|_{L_t^2 W_x^{2,p}} \leq \varepsilon_* \quad \|B - \tilde{B}\|_{L_t^2 L_x^2} + \|B - \tilde{B}\|_{L_t^2 W_x^{2,p}} \leq \varepsilon_* . \]

We note that, the borderline of \( S_1 \) regime in particular contains the endpoint \((s, \gamma, p) = (0, 2, \infty)\) of the LPS condition \((1.7)\) when \( \alpha = 1 \). Hence, by virtue of Theorem 1.2, we obtain the non-uniqueness for the classical viscous and resistive MHD \((1.1)\), which is sharp in view of the uniqueness result in the critical endpoint space \( L_t^2 L_x^\infty \) in Theorem B.1 of Appendix.

**Corollary 1.3** (Sharp non-uniqueness for MHD). Consider the viscous and resistive MHD \((1.1)\). Then, for any \( 1 \leq \gamma < 2 \), there exist infinitely many weak solutions in \( L_t^2 L_x^\infty \) with the same initial data.

Our last result is concerned with the strong vanishing viscosity and resistivity limits, which relates both the ideal MHD \((1.4)\) and hyper viscous, resistive MHD equations \((1.5)\). Its interesting relationship to the Taylor conjecture will be discussed in the comments below.

**Theorem 1.4** (Strong vanishing viscosity and resistivity limit). Let \( \alpha \in [1, 3/2] \) and \( \tilde{\beta} > 0 \). Let \((u, B) \in H_{t,x}^{1} \times H_{t,x}^{\tilde{\beta}} \) be any mean-free weak solution to the ideal MHD \((1.1)\). Then, there exist \( \beta' \in (0, \tilde{\beta}) \) and a sequence of weak solutions \((u^{(\nu_n)}, B^{(\nu_n)}) \in H_{t,x}^{\beta'} \times H_{t,x}^{\beta'} \) to the hyper viscous and resistive MHD \((1.5)\), where \( \nu_n = (\nu_{1,n}, \nu_{2,n}) \) and \( \nu_{1,n}, \nu_{2,n} \) are the viscosity and resistivity coefficients, respectively, such that as \( \nu_n \to 0 \),

\[ (u^{(\nu_n)}, B^{(\nu_n)}) \to (u, B) \quad \text{strongly in } H_{t,x}^{\beta'} \times H_{t,x}^{\beta'}. \]

**Comments on main results.** We present some comments on the main results below.

(i) Sharp non-uniqueness of MHD near the endpoint space \( L_t^2 L_x^\infty \). In the recent remarkable paper \([19]\), Cheskidov-Luo proved the non-uniqueness of weak solutions to NSE in the spaces \( L_t^2 L_x^\infty \) for any \( 1 \leq \gamma < 2 \) and any dimensions. This result is sharp because weak solutions are unique in the endpoint space \( L_t^2 L_x^\infty \), which is critical with respect to the LPS condition \((1.2)\). Sharp non-uniqueness for the 2D NSE near another endpoint space was proved by Cheskidov-Luo \([20]\). Recently, for the 3D hyper-dissipative NSE, sharp non-uniqueness near both endpoints of the generalized Ladyženskaja-Prodi-Serrin condition was proved in \([50]\).

To the best of our knowledge, Corollary 1.3 provides the first non-uniqueness result for the viscous and resistive MHD near the endpoint space \( L_t^2 L_x^\infty \). In view of Theorem B.1, i.e., the uniqueness of weak solutions to MHD in \( L_t^2 L_x^\infty \), the non-uniqueness result in Corollary 1.3 is sharp.

We would expect the non-uniqueness also exhibit in the remaining supercritical regimes. Actually, the second set \( S_2 \) includes a part of the supercritical regime. One difficulty of the current MHD situation lies in the specific geometry of MHD, which restricts the choice of oscillating directions, and makes it very hard to construct building blocks with 3D spatial intermittency as in the NSE context. Another difficulty is due to the \( L^2 \)-critical nature of the current convex integration scheme, as observed by Cheskidov-Luo \([19]\). In view of the LPS condition \((1.7)\), there exists some regime of exponents \((\gamma, p)\) both larger than two, which seem out of reach of the current method.

(ii) Non-uniqueness in the highly viscous and resistive regime: beyond the Lions exponent.

As mentioned above, the high dissipation usually helps to establish well-posedness of PDEs, and so is the obstacle for the ill-posedness. For instance, in the case of 3D NSE, the viscosity \((-\Delta\) was the major obstruction at constructing non-unique weak solutions. The crucial novelty in \([12]\) to overcome this difficulty is the construction of \( L^2 \)-based building blocks with 3D spatial intermittency. However, due to the MHD geometry, it is still open to construct 3D spatial intermittent building blocks.

Another thing that deserves attention is that, it is indeed impossible to construct non-unique weak solutions in \( C_t L_x^2 \) in the very high viscous and resistive regime beyond the Lions exponent, due to the well-posedness results by Lions \([51]\) in the NSE case and by Wu \([63]\) in the MHD case.

Theorem 1.2 shows that, even in the high dissipative regime beyond the Lions exponent, the LPS condition serves as the compass to find non-unique weak solutions, which do exist in the supercritical mixed Lebesgue spaces.
(iii) Partial regularity of weak solutions. To our best knowledge, Theorem 1.2 provides the first example for weak solutions to MHD, whose singular sets in time have small Hausdorff dimensions. The interesting phenomenon here is that, though weak solutions are not unique in the supercritical space $L^2_t W^{s,p}_x$, they seem not so “bad”: they can be very close to Leray-Hopf solutions and are smooth in a majority of time, of which the complement has zero Hausdorff measure with $\eta_*$ being any given small positive constant.

Let us mention that, the delicate partial regularity of non-unique weak solutions was first discovered by Buckmaster-Colombo-Vicol [7] in $C_t L^2_x$ for the hyperdissipative NSE, up to the Lions exponent $5/4$, whose singular set in time has Hausdorff dimension strictly less than one. Weak solutions with singular temporal sets of small Hausdorff dimension also have been constructed in $L^2_t L^p_x$ for NSE by Cheskidov-Luo [19], and in $L^2_t W^{s,p}_x$ for hyperdissipative NSE with viscosity beyond the Lions exponent $5/4$ [50]. The crucial ingredient here is the use of gluing technique, developed in a series of works [6, 8–10, 27–29, 41] in the resolution of Onsager’s conjecture, combined with suitable regularity and stability estimates for MHD equations.

(iv) Strong vanishing viscosity and resistivity limit and Taylor’s conjecture. It would be interesting to see the relationship between the vanishing viscosity and resistivity limit in Theorem 1.4 and the Taylor conjecture.

The ideal MHD (1.4) has several global invariants:

- The total energy: $E(t) = \frac{1}{2} \int_{\mathbb{T}^3} |u(t,x)|^2 + |B(t,x)|^2 dx$;
- The cross helicity: $\mathcal{H}_{\omega,B}(t) = \int_{\mathbb{T}^3} u(t,x) \cdot B(t,x) dx$;
- The magnetic helicity: $\mathcal{H}_{B,B}(t) := \int_{\mathbb{T}^3} A(t,x) \cdot B(t,x) dx$.

Here $A$ is a mean-free periodic vector field satisfying $\text{curl}A = B$. The famed Taylor conjecture, commonly accepted in the plasma physics literature, is that the magnetic helicity is expected to be conserved in the infinite conductivity limit [60, 61]. This conjecture is valid under weak ideal limits, namely, the weak limits of Leray-Hopf solutions to MHD (1.1). It has been proved by Faraco-Lindberg [35] in simply connected, magnetically closed domain, and by Faraco-Lindberg-MacTaggart-Valli [38] in multiply connected domains. On the other hand, Beckie-Buckmaster-Vicol [3] constructed weak solutions with non-conserved magnetic helicity, which shows that the ideal MHD version of Taylor’s conjecture is false. The delicate point here is that, the conservation of magnetic helicity requires much milder regularity than that of total energy and cross helicity. Actually, the energy and cross helicity are conserved by weak solutions in $B^{3/2}_{t,\infty}$ with $\alpha > 1/3$ or $B^{3/3}_{t,\infty}$, while the magnetic helicity is conserved in the less regular space $B^{3/2}_{3,\infty}$ with $\alpha > 0$ or in the endpoint space $L^{3}_{t,\gamma}$. See [2, 16, 34, 44].

Theorem 1.4 provides another viewpoint that, in contrast to weak ideal limits, there indeed exists certain sequence of weak (non-Leray-Hopf) solutions even for the hyper viscous and resistive MHD beyond Lions’ exponent, such that the Taylor conjecture fail in the limit along this sequence.

Distinctions between NSE and MHD in intermittent convex integration schemes. As we have already seen above, one major distinction between NSE and MHD is that, the strong coupling between the velocity and magnetic fields in MHD restricts the choice of oscillating directions. This makes it quite difficult to construct spatial building blocks with strong spatial intermittency adapted to the specific geometric structure of MHD.

Actually, in the context of NSE, spatial building blocks with 3D intermittency (known as intermittent Beltrami flows in [12], and intermittent jets in [7]) can be constructed. Such strong intermittent building blocks enable to construct non-unique weak solutions in $C_t L^2_x$ for the 3D NSE [12], and even for the hyperdissipative NSE up to the Lions exponent [7, 53]. Moreover, the intermittent jets also enable us to achieve the sharp non-uniqueness at the endpoint $(\gamma, p) = (\infty, \frac{3}{2m-1})$ of the LPS condition, for the hyperdissipative NSE beyond the Lions exponent (50]).

In contrast, due to the geometry of MHD equations, the shear flows constructed in [3] have 1D spatial intermittency, and the intermittent flows constructed in [49] have 2D spatial intermittency, both are less than 3D spatial intermittency in the context of NSE. Hence, unlike the NSE case [50], these spatial intermittencies
are not able to control the viscosity and resistivity \((-\Delta)\) of MHD. The construction of non-unique weak solutions to MHD in the endpoint space \(C_tL^2_x\) is still open.

Another delicate fact is that, the intermittent jets for NSE have disjoint supports, while the intermittent flows in [3, 49] for MHD may interact with each other. This in particular gives rise to the interaction errors in the Reynolds and magnetic stresses. See §3 and §5.4 below. Stronger intermittency needs to be gained to control these interaction errors. The keypoint here is to exploit the geometric nature of the interactions of intermittent flows. As a matter of fact, the interaction of intermittent shear flows in [3] is given by thickened lines which give 2D intermittency, and the interaction of intermittent flows in [49] concentrates on small cuboids which have 2D intermittency (see [49, Figure 3]).

Furthermore, the distinction between two equations also lies in the construction of the amplitudes of velocity and magnetic perturbations. Because of the anti-symmetry of magnetic nonlinearity, the construction of magnetic amplitudes requires a second geometric lemma in a small neighborhood of the null matrix (i.e., Lemma A.2 below), which is different from the geometric lemma near the identity matrix (i.e., Lemma A.1) for velocity amplitudes. Moreover, when constructing velocity amplitudes, a new matrix \(\hat{G}^B\) (see (3.22) below) also will be required, which does not appear in the NSE case. These distinctions lead to different algebraic identities (3.30) and (3.31), respectively, for the nonlinear effects of magnetic perturbations and velocity perturbations.

Cancellations provided by the specific structure of MHD nonlinearities also would be used towards the derivation of regularity and stability estimates, when constructing the concentrated velocity and magnetic fields in the gluing stage.

1.3. Outline of the proof. The proof of Theorem 1.2 proceeds in two main stages: the gluing stage and the convex integration stage.

The starting point of analysis is to consider approximate solutions to the following relaxation system with Reynolds and magnetic stresses: for each integer \(q \in \mathbb{N}\),

\[
\begin{align*}
\partial_t u_q + \nu_1(-\Delta)^\alpha u_q + \text{div}(u_q \otimes u_q - B_q \otimes B_q) + \nabla P_q &= \text{div} \hat{R}_q^u, \\
\partial_t B_q + \nu_2(-\Delta)^\alpha B_q + \text{div}(B_q \otimes u_q - u_q \otimes B_q) &= \text{div} \hat{R}_q^B, \\
\text{div} u_q &= 0, \quad \text{div} B_q = 0,
\end{align*}
\]

where the Reynolds stress \(\hat{R}_q^u\) is a symmetric traceless \(3 \times 3\) matrix, and the magnetic stress \(\hat{R}_q^B\) is a skew-symmetric \(3 \times 3\) matrix.

The gluing stage aims to construct new Reynolds and magnetic stresses, which, particularly, have small disjoint temporal supports, and, simultaneously, maintain small amplitudes. Then, in the convex integration stage, the crucial point is to construct appropriate velocity and magnetic perturbations, featuring both the temporal and spatial intermittency, to decrease the effect of Reynolds and magnetic stresses and to control the hyper viscosity and resistivity \((-\Delta)^\alpha\) where \(\alpha\) can be larger than \(5/4\).

In both stages, the fundamental quantities are the frequency \(\lambda_q\) and the amplitude \(\delta_q\) below:

\[
\lambda_q = a^q, \quad \delta_q = \lambda_1^{3\beta} \lambda_q^{-2\beta}.
\]

Here \(a \in \mathbb{N}\) is a large integer to be determined later, \(\beta > 0\) is the regularity parameter, \(b \in 2\mathbb{N}\) is a large integer of multiple 2 such that

\[
b > \frac{1500}{\varepsilon \eta_*}, \quad 0 < \beta < \frac{1}{100b^2},
\]

where \(\varepsilon \in \mathbb{Q}_+\) is sufficiently small such that, for the given \((s, \gamma, p) \in \mathcal{S}_1\)

\[
\varepsilon \leq \frac{1}{20} \min\left\{\frac{3}{2} - \alpha, \frac{2\alpha}{\gamma} + \frac{2\alpha - 2}{p} - (2\alpha - 1) - s\right\} \quad \text{and} \quad b(2\alpha - 8\varepsilon) \in \mathbb{N}.
\]

and for the given \((s, \gamma, p) \in \mathcal{S}_2\)

\[
\varepsilon \leq \frac{1}{20} \min\left\{\frac{3}{2} - \alpha, \frac{4\alpha - 4}{\gamma} + \frac{2}{p} - (2\alpha - 1) - s\right\} \quad \text{and} \quad b\varepsilon \in \mathbb{N},
\]

The objectives of both the gluing stage and the convex integration stage are quantified in the iterative estimates in Theorems 1.6 and 1.7 below.
• Gluing stage. Let us first introduce the notion of well-preparedness as in the NSE context [19, 50].

**Definition 1.5. (Well-preparedness)** We say that the smooth solution \((u_q, B_q, \tilde{R}_q^u, \tilde{R}_q^B)\) to (1.11) is well-prepared if there exist a set \([I \subseteq [0, T]\), a length scale \(\theta > 0\) and \(\eta > 0\), such that \(I\) is a union of at most \(\theta^{-N}\) many closed intervals of length scale \(5\theta\) and

\[
\tilde{R}_q^u(t) = 0, \quad \tilde{R}_q^B(t) = 0, \quad \text{if dist}(t, I^c) \leq \theta.
\]

We would divide the whole interval \([0, T]\) into \(m_{q+1}\) many subintervals \([t_i, t_{i+1}]\), and denote by \(\theta_{q+1}\) the length scale of bad sets supporting new concentrated Reynolds and magnetic stresses. The parameters \(m_{q+1}\) and \(\theta_{q+1}\) are chosen in the way

\[
T/m_{q+1} = \lambda_q^{-12}, \quad \theta_{q+1} := (T/m_{q+1})^{1/n} \approx \lambda_q^{-\frac{12}{n}},
\]

where \(\eta\) is a small constant such that

\[
0 < \frac{\eta}{2} < \eta < \eta_* < 1.
\]

Without loss of generality, we assume \(m_{q+1}\) is an integer such that the time interval is perfectly divided.

The main result in this stage is that, we can construct new concentrated solutions to (1.11) supported on \(m_{q+1}\) intervals of length \(\theta_{q+1}\), and, importantly, maintain the \(L^1_t L^\infty_x\)-decay of Reynolds and magnetic stresses. We note that, the \(L^1_t L^\infty_x\)-decay property is not destroyed in the concentrating procedure. This is the content of Theorem 1.6 below.

**Theorem 1.6 (Well-preparedness).** Let \(\alpha \in [1, 3/2)\) and \((s, p, \gamma) \in S_1 \cup S_2\). If \((u_q, B_q, \tilde{R}_q^u, \tilde{R}_q^B)\) is a well-prepared smooth solution to (1.11) for some set \(I_q\) and a length scale \(\theta_q\), then there exists another well-prepared solution \((\tilde{u}_q, \tilde{B}_q, \tilde{R}_q^u, \tilde{R}_q^B)\) to (1.11) for a new set \(I_{q+1} \subseteq I_q, 0 \neq I_{q+1}\) and the smaller length scale \(\theta_{q+1}(\leq \theta_q/2)\), satisfying:

\[
\|(\tilde{u}_q, \tilde{B}_q)\|_{L^\infty_t H^s_x} \lesssim \lambda_q^5,
\]

\[
\|(\tilde{u}_q - u_q, \tilde{B}_q - B_q)\|_{L^\infty_t L^2_x} \lesssim \lambda_q^{-3},
\]

\[
\|\langle \tilde{R}_q^u, \tilde{R}_q^B \rangle\|_{L^1_t L^1_x} \lesssim \lambda_q^{\frac{12}{n}} \delta_{q+1},
\]

\[
\|\partial_t^{M} \nabla^N \tilde{R}_q^u, \partial_t^{M} \nabla^N \tilde{R}_q^B\|_{L^\infty_t H^s_x} \lesssim \theta_{q+1}^{-M-N-1} \lambda_q^5,
\]

where \(0 \leq M \leq 1, 0 \leq N \leq 9\) and the implicit constants are independent of \(q\). Moreover, we have

\[
\langle \tilde{R}_q^u(t), \tilde{R}_q^B(t) \rangle = 0 \quad \text{if dist}(t, I_{q+1}^c) \leq \frac{3}{2} \theta_{q+1},
\]

\[
\text{supp}_t(\tilde{u}_q, \tilde{B}_q) \subseteq N_{2T/m_{q+1}}(\text{supp}_t(u_q, B_q)).
\]

The proof of Theorem 1.6 is based on the gluing technique. Precisely, in every small neighborhood of length \(\theta_{q+1}\) at \(t_i(=iT/m_{q+1})\), \(0 \leq i \leq m_{q+1} - 1\), we generate the velocity and magnetic fields to MHD equations (1.5). Then, we glue all these local solutions together, by using a partition of unity, to get new concentrated velocity and magnetic fields. The important roles here are played by the regularity and stability estimates of solutions to (relaxation) MHD equations, which are contained in Propositions 2.1 and 2.2, respectively.

• Convex integration stage. In this stage, the crucial inductive estimates of well-prepared solutions to (1.11) at level \(q \in \mathbb{N}\) are as follows

\[
\|(u_q, B_q)\|_{L^\infty_t H^{N+5}_x} \lesssim \lambda_q^{N+5},
\]

\[
\|\partial_t u_q, \partial_t B_q\|_{L^\infty_t H^5_x} \lesssim \lambda_q^{N+5},
\]

\[
\|\tilde{R}_q^u, \tilde{R}_q^B\|_{L^\infty_t H^5_x} \lesssim \lambda_q^{N+6},
\]

\[
\|\tilde{R}_q^u, \tilde{R}_q^B\|_{L^1_t L^1_x} \lesssim \lambda_q^{-\varepsilon_R} \delta_{q+1},
\]

where \(0 \leq N \leq 4, \tilde{N} = 3, 4\), the implicit constants are independent of \(q\), and \(\varepsilon_R > 0\) is a small parameter such that

\[
\varepsilon_R < \frac{b \delta}{10} < \frac{\varepsilon \eta_*}{1.5 \times 10^6}.
\]
We note that, by (1.27),
\[ \lambda_q^{-\varepsilon} \delta_{q+1} \ll 1 \quad \text{for all} \quad q \geq 1, \]
and for \( q = 0 \),
\[ \lambda_0^{-\varepsilon} \delta_1 = a^{b_2-\varepsilon} \gg 1, \]
by choosing \( a \) large enough.

The main iteration estimates in the convex integration stage are formulated in Theorem 1.7 below.

**Theorem 1.7** (Main iteration). Let \( \alpha \in [1, 3/2] \) and \( (s, p, \gamma) \in S_1 \cup S_2 \). Then, there exist \( \beta \in (0, 1) \), \( M^* \) and \( a_0 = a_0(\beta, M^*) \) large enough, such that for any integer \( a \geq a_0 \), the following holds:

Suppose that \( (u_q, B_q, \tilde{R}_q^u, \tilde{R}_q^B) \) is a well-prepared solution to (1.11) for some set \( I_q \) and the length scale \( \theta_q \) and satisfies (1.23)-(1.26). Then, there exists another well-prepared solution \( (u_{q+1}, B_{q+1}, \tilde{R}_{q+1}^u, \tilde{R}_{q+1}^B) \) to (1.11) for some set \( I_{q+1} \subseteq I_q \), \( 0 \notin I_{q+1} \), and the length scale \( \theta_{q+1} \) is less than \( \theta_q / 2 \), and \( (u_{q+1}, B_{q+1}, \tilde{R}_{q+1}^u, \tilde{R}_{q+1}^B) \) satisfies (1.23)-(1.26) with \( q+1 \) replacing \( q \).

In addition, we have
\[
\begin{align*}
\| (u_{q+1} - u_q, B_{q+1} - B_q) \|_{L_t^2 L_x^p} & \leq M^* \delta_{q+1}^{\frac{1}{2}}, \\
\| (u_{q+1} - u_q, B_{q+1} - B_q) \|_{L_t^1 L_x^p} & \leq \delta_{q+2}, \\
\| (u_{q+1} - u_q, B_{q+1} - B_q) \|_{L_t^1 W_x^{s,p}} & \leq \delta_{q+2}^{\gamma},
\end{align*}
\]

and
\[
\text{supp}_t (u_{q+1}, B_{q+1}, \tilde{R}_{q+1}^u, \tilde{R}_{q+1}^B) \subseteq N_{\delta_{q+2}^{\gamma+2}} \left( \text{supp}_t (u_q, B_q, \tilde{R}_q^u, \tilde{R}_q^B) \right).
\]

The proof of Theorem 1.7 relies crucially on the construction of appropriate velocity and magnetic perturbations, adapted to the MHD geometry. As a matter of fact, different building blocks would be used in the supercritical regimes \( S_1 \) and \( S_2 \).

In the supercritical regime \( S_1 \), the spatial integrability can be close to infinity, while the temporal integrability is less than 2. This fact indicates that, the building blocks may have acceptable few spatial intermittency, but require strong temporal intermittency. The intermittent shear flows, constructed by Beekie-Buckmaster-Vicol [3], give the desired spatial intermittency, and the temporal building blocks in [19, 49, 50] are able to provide sufficiently strong temporal intermittency.

Hence, besides the parameters \( \lambda_q \) and \( \delta_q \), three more parameters \( r_\perp, \tau, \sigma \) are needed in the building blocks, which parameterize the spatial concentration, temporal oscillation and temporal concentration, respectively. It should be mentioned that, all these parameters shall be balanced with each other, such that the following leading order constrains are satisfied
\[
\begin{align*}
\lambda^s r_\perp^\frac{1}{2} \tau^\frac{1}{2} \sigma^\frac{1}{2} & \ll 1 \quad \text{for} \quad (w_{q+1}^{(p)}, \phi_{q+1}^{(p)} \in L_t^s W_x^{s,p}) \tag{1.32a} \\
\sigma \lambda^{-1} r_\perp^\frac{1}{2} \tau^\frac{1}{2} & \ll 1 \quad \text{(Time derivative error for} \quad w_{q+1}^{(p)}, \phi_{q+1}^{(p)}) \tag{1.32b} \\
\lambda^{2s-1} r_\perp^\frac{1}{2} \tau^{-\frac{1}{2}} & \ll 1 \quad \text{(Hyperdissipativity error for} \quad w_{q+1}^{(p)}, \phi_{q+1}^{(p)}) \tag{1.32c} \\
\lambda^{-1} r_\perp & \ll 1 \quad \text{(Oscillation error for} \quad w_{q+1}^{(p)}, \phi_{q+1}^{(p)}) \tag{1.32d}
\end{align*}
\]

It turns out that, there do exist appropriate parameters to fulfill the above constrains. The specific choice is given by (3.1) below.

In contrast, in the supercritical regime \( S_2 \), the spatial integrability may be less than two, while the temporal integrability can be close to infinity. Hence, stronger spatial intermittency is required, in order to compensate the possibly weak temporal intermittency. In this case, we use the spatial building blocks constructed in [49], which provide 2D spatial intermittency. This requires five parameters \( r_\parallel, r_\perp, \mu, \tau, \sigma \) to parameterize the spatial-temporal building blocks. Note that, the extra two parameters \( r_\parallel, \mu \), respectively, parameterize the further spatial concentration and balance high temporal oscillations. The crucial constrains in the convex integration scheme are captured by
\[
\begin{align*}
\lambda^s r_\perp^\frac{1}{2} \tau^\frac{1}{2} \sigma^\frac{1}{2} & \ll 1 \quad \text{for} \quad (w_{q+1}^{(p)}, \phi_{q+1}^{(p)} \in L_t^s W_x^{s,p}) \tag{1.33a}
\end{align*}
\]
where

\[ \mu \|\tau\|^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \ll 1 \] (Time derivative error for \( u^{(p)}_{q+1}, q^{(p)}_{q+1} \)) \hfill (1.33b)

\[ \lambda^{2\alpha-1} \|\tau\|^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \ll 1 \] (Hyperdissipativity error for \( u^{(p)}_{q+1}, q^{(p)}_{q+1} \)) \hfill (1.33c)

\[ \lambda^{2\alpha-1} \mu^{-1} \ll 1 \] (Hyperdissipativity error for \( u^{(t)}_{q+1}, d^{(t)}_{q+1} \)) \hfill (1.33d)

\[ \lambda^{-1} \|\tau\|^{-\frac{1}{2}} \ll 1 \] (Oscillation error for \( u^{(p)}_{q+1}, q^{(p)}_{q+1} \)) \hfill (1.33e)

\[ \mu^{-1} \sigma_{\tau} \ll 1 \] (Oscillation error for \( u^{(t)}_{q+1}, d^{(t)}_{q+1} \)) \hfill (1.33f)

Obviously, the admissible set of the parameters (1.33a)-(1.33f) shall be different from that of (1.32a)-(1.32d).

The precise choice of these five parameters will be given by (5.1) below.

It should be mentioned that, in both supercritical regimes \( S_1 \) and \( S_2 \), it is crucial to exploit the temporal intermittency, particularly, to handle the hyper viscosity and resistivity beyond the Lions exponent. Actually, as we have already seen above, the uniqueness of weak solutions in \( C^1 \) holds in the high-dissipative regime when \( \alpha \geq 5/4 \). Hence, it is impossible to construct non-unique weak solutions in \( C^1 \) with merely spatial intermittency. Below we shall see that, the temporal building blocks would provide 2D intermittency in the supercritical regime \( S_2 \), the temporal building blocks in \( S_1 \) can even provide the stronger 3D intermittency, which in particular compensates the weak spatial intermittency and enables to control the hyper viscosity and resistivity beyond the Lions exponent.

**Organization of paper.** The remaining part of this paper is structured as follows. First, §2 is devoted to the gluing stage. We prove the regularity and stability results for the hyper viscous and resistive MHD equations and then use the gluing technique to prove Theorem 1.6. Then, in §3 and §4, we mainly treat the supercritical regime \( S_1 \) by using the intermittent convex integration approach. The other supercritical regime \( S_2 \) is later treated in §5. Then, in §6, we prove the main results, namely, Theorems 1.2 and 1.7. At last, some standard tools of convex integration method and the uniqueness of weak solutions are collected in Appendices A and B.

**Notations.** The mean of \( u \in L^1(\mathbb{T}^n) \) is given by \( \int_{\mathbb{T}^n} u dx = |\mathbb{T}^n|^{-1} \int_{\mathbb{T}^n} u dx \), where \( |\cdot| \) denotes the Lebesgue measure. For \( p \in [1, +\infty] \) and \( s \in \mathbb{R} \), we use the following shorthand notations

\[ L^p_x := L^p_x(\mathbb{T}^3), \quad H^s_x := H^s_x(\mathbb{T}^3), \quad W^{s,p}_x := W^{s,p}_x(\mathbb{T}^3). \]

Moreover, let

\[ \|u\|_{W^{\infty,s}_x} := \sum_{0 \leq m + |\zeta| \leq N} \|\partial_t^m \nabla^\zeta u\|_{L^p_x}, \quad \|u\|_{C^{\infty}_{t,x}} := \sum_{0 \leq m + |\zeta| \leq N} \|\partial_t^m \nabla^\zeta u\|_{C_{t,x}}, \]

where \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \) is the multi-index and \( \nabla^\zeta := \partial_{\zeta_1} \partial_{\zeta_2} \partial_{\zeta_3} \). We also consider the product space \( L^p_x := L^p_x \times L^p_x \), equipped with the norm \( \|(u, v)\|_{L^p_x} := \|u\|_{L^p_x} + \|v\|_{L^p_x} \) if \( (u, v) \in L^p_x \). Similar notations also apply to \( H^s_x \) and \( W^{s,p}_x \).

Given any Banach space \( X, \gamma \in [1, \infty], L^0_\gamma X \) denotes the space of integrable functions from \( [0, T] \) to \( X \), equipped with the norm \( \|u\|_{L^0_\gamma X} := \left( \int_0^T \|u(t)\| X dt \right)^{1/\gamma} \) (with the usual adaptation when \( \gamma = \infty \)). In particular, we write \( L^0_\gamma L^p_x :\equiv L^0_\gamma (\mathbb{I} \times L^p_x(T^3)), L^{p}_{x} := L^p_x L^p_x \) and \( C_{t,x} := C_{t,x} \).

We also adopt the notations from [48]. Let \( u, v \) be two vector fields, the corresponding second order tensor product is defined by

\[ u \otimes v := (u_i v_j)_{1 \leq i, j \leq 3}. \]

For any second-order tensor \( A = (a_{ij})_{1 \leq i, j \leq 3} \), set

\[ \text{div}A := \left( \sum_{j=1}^3 \partial_{x_j} a_{1j}, \sum_{j=1}^3 \partial_{x_j} a_{2j}, \sum_{j=1}^3 \partial_{x_j} a_{3j} \right)^T. \]
The right product of a vector field $v = (v_1, v_2, v_3)^	op$ to a second-order tensor $A = (a_{ij})_{1 \leq i,j \leq 3}$ is defined by

$$Av := \left( \sum_{j=1}^{3} a_{1j} v_j, \sum_{j=1}^{3} a_{2j} v_j, \sum_{j=1}^{3} a_{3j} v_j \right)^	op.$$

In particular, for any scalar function $f$ and second-order tensor $A = (a_{ij})_{1 \leq i,j \leq 3}$, one has the Leibniz rule

$$\text{div}(fA) = f \text{div}A + A \nabla f.$$

For any $3 \times 3$ matrices $A = (A_{ij})$ and $S = (S_{ij})$, let $A : S = \sum_{i,j=1}^{3} A_{ij} S_{ij}$.

Let $N_{\epsilon}(A)$ denote the $\epsilon$-neighborhood of $A \subseteq [0, T]$, i.e.,

$$N_{\epsilon}(A) := \{ t \in [0, T] : \exists s \in A, s.t. |t - s| \leq \epsilon \}.$$

Let $\mathbb{P}_H$ denote the Helmholtz-Leray projector, i.e., $\mathbb{P}_H = \text{Id} - \nabla \Delta^{-1} \text{div}$.

The notation $a \lesssim b$ means that $a \leq Cb$ for some constant $C > 0$.

### 2. Concentrating the Reynolds and Magnetic Stresses

Let us start with the gluing stage. This section mainly consists of four parts: the regularity estimates for hyper viscous and resistive MHD, the stability estimates for relaxed MHD equations, the temporal gluing of local solutions and the proof of Theorem 1.6.

#### 2.1. Regularity estimates

Let us first establish the regularity estimates of local strong solutions to (1.5) when $\alpha \in [1, 3/2)$. This is the content of Proposition 2.1 below.

**Proposition 2.1** (Regularity estimates). Let $\alpha \in [1, 3/2)$, $v_0, B_0 \in H_x^3$ be any mean-free and divergence-free fields. Consider the Cauchy problem for (1.5) with initial condition $(u, B)|_{t=0} = (u_0, B_0)$. Then, there exists $c > 0$ sufficiently small such that the following holds:

(i) If

$$0 < t_* - t_0 \leq \frac{c}{\|(u_0, B_0)\|_{H_x^3}},$$

then there exists a unique strong solution $(u, B)$ to (1.5) on $[t_0, t_*)$ satisfying

$$\|u(t)\|_{C([t_0, t_*], \mathbb{L}_x^2)}^{2} + \int_{t_0}^{t_*} \|u(t), B(t)\|_{H_x^2}^{2} \, dt \lesssim \|(u_0, B_0)\|_{C([t_0, t_*], \mathbb{H}_x^3)}^{2}.$$

(ii) If

$$0 < t_* - t_0 \leq \text{min} \left\{ 1, \frac{c}{\|(u_0, B_0)\|_{H_x^3}(1 + \|(u_0, B_0)\|_{\mathbb{L}_x^2})} \right\},$$

then it holds that for any $N \geq 0$ and $M \in \{0, 1\}$,

$$\sup_{t \in [t_0, t_*)} (|t - t_0|^{M+\frac{1}{2}} \|\partial_t^M \nabla^N u(t), \partial_t^M \nabla^N B(t)\|_{\mathbb{H}_x^N}) \lesssim \|(u_0, B_0)\|_{\mathbb{H}_x^N},$$

where the implicit constant depends on $\alpha$, $N$ and $M$.

**Proof.** (i). By the energy inequality, one has

$$\frac{1}{2} \frac{d}{dt} (\|u(t)\|_{\mathbb{H}_x^2}^2 + \|B(t)\|_{\mathbb{H}_x^2}^2) \leq -\nu_1 \|u\|_{H_x^2}^2 - \nu_2 \|B\|_{H_x^2}^2,$$

which directly yields (2.2) by Gronwall’s inequality.

Next, taking the $H_x^3$ inner product of the velocity equation in (1.5) with $u$ and the magnetic equation with $B$, since $\nabla \cdot u = \nabla \cdot B = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|(u(t))\|_{H_x^2}^2 + \|B(t)\|_{H_x^2}^2 + \nu_1 \|u(t)\|_{H_x^{2+\alpha}}^2 + \nu_2 \|B(t)\|_{H_x^{2+\alpha}}^2$$

$$= -((u \cdot \nabla)u, u)_{H_x^3} + ((B \cdot \nabla)B, u)_{H_x^3} - ((u \cdot \nabla)B, B)_{H_x^3} + ((B \cdot \nabla)u, B)_{H_x^3}.$$

(2.6)
By Kato’s inequality (cf. [58, P.155]),
\[
((u \cdot \nabla)u, u)_{\dot{H}^2_x} \lesssim \|\nabla u\|_{L^\infty_x} \|u\|_{\dot{H}^2_x}^2,
\]
and similarly,
\[
((u \cdot \nabla)B, B)_{\dot{H}^2_x} \lesssim \|\nabla u, \nabla B\|_{L^\infty_x} \|(u, B)\|_{\dot{H}^2_x}^2.
\]
For the remaining two terms on the right-hand-side of (2.6), it is important to use the cancellation of the fourth-order derivative terms, which permits to obtain the desirable \(\dot{H}^2_x\) estimate. Actually, we compute
\[
((B \cdot \nabla)B, u)_{\dot{H}^2_x} + ((B \cdot \nabla)u, B)_{\dot{H}^2_x} = \sum_{|\alpha|\geq 3} \sum_{i,j} \int_{\mathbb{T}^3} \nabla^\alpha (B_i \partial_{x_i} B_j) \nabla^\alpha u_j + \nabla^\alpha (B_i \partial_{x_i} u_j) \nabla^\alpha B_j dx.
\]
(2.9)
Since \(\nabla \cdot u = \nabla \cdot B = 0\), using the integration-by-parts formula we obtain the cancellation for the terms of the highest fourth order derivatives
\[
\sum_{i,j} \int_{\mathbb{T}^3} B_i \nabla^3 \partial_{x_i} B_j \nabla^3 u_j + B_i \nabla^3 \partial_{x_i} u_j \nabla^3 B_j dx = 0.
\]
(2.10)
Moreover, Hölder’s inequality gives
\[
\sum_{i,j} \int_{\mathbb{T}^3} \nabla^3 B_i \partial_{x_i} B_j \nabla^3 u_j + \nabla^3 B_i \partial_{x_i} u_j \nabla^3 B_j dx \lesssim \|\nabla B\|_{L^\infty_x} \|B\|_{\dot{H}^2_x} \|u\|_{\dot{H}^2_x} + \|\nabla u\|_{L^\infty_x} \|B\|_{\dot{H}^2_x}^2,
\]
(2.11)
and
\[
\sum_{i,j} \int_{\mathbb{T}^3} \nabla B_i \nabla^3 \partial_{x_i} B_j \nabla^3 u_j + \nabla B_i \nabla^3 \partial_{x_i} u_j \nabla^3 B_j dx \lesssim \|\nabla B\|_{L^\infty_x} \|B\|_{\dot{H}^2_x} \|u\|_{\dot{H}^2_x}.
\]
(2.12)
We also derive from the Hölder’s inequality and the Gagliardo-Nirenberg inequality that
\[
\sum_{i,j} \int_{\mathbb{T}^3} \nabla^2 B_i \nabla^1 \partial_{x_i} B_j \nabla^1 u_j + \nabla^2 B_i \nabla^1 \partial_{x_i} u_j \nabla^1 B_j dx \lesssim \|\nabla^2 B\|_{L^1_x} \|u\|_{\dot{H}^2_x} + \|\nabla^2 B\|_{L^1_x} \|\nabla^1 u\|_{\dot{H}^2_x} \|B\|_{\dot{H}^2_x}
\lesssim \|\nabla B\|_{L^\infty_x} \|B\|_{\dot{H}^2_x} \|u\|_{\dot{H}^2_x} + \|\nabla B\|_{L^\infty_x} \|\nabla^1 u\|_{\dot{H}^2_x} \|\nabla^1 B\|_{\dot{H}^2_x}.
\]
(2.13)
Thus, combining (2.6)-(2.13) together and using the Sobolev embedding \(\dot{H}^2_x \hookrightarrow L^\infty_x\) we come to
\[
\frac{d}{dt} \|(u(t), B(t))\|_{\dot{H}^2_x}^2 + \|(u(t), B(t))\|_{\dot{H}^2_x}^2 \lesssim \|(u(t), B(t))\|_{\dot{H}^2_{x+\alpha}}^2,
\]
(2.14)
which, via Gronwall’s inequality, yields
\[
\|(u(t), B(t))\|_{\dot{H}^2_x}^2 \leq C \|(u_0, B_0)\|_{\dot{H}^2_x}^2 \exp \left\{ \int_0^t \|(u(s), B(s))\|_{\dot{H}^2_x}^2 ds \right\}.
\]
(2.15)

We claim that
\[
\sup_{t \in [0, t_\ast]} \|(u(t), B(t))\|_{\dot{H}^2_x} \leq 2 \|(u_0, B_0)\|_{\dot{H}^2_x}.
\]
(2.16)
To this end, plugging (2.16) into (2.15), we get
\[
\|(u(t), B(t))\|_{\dot{H}^2_x}^2 \leq C \|(u_0, B_0)\|_{\dot{H}^2_x}^2 \exp \left\{ 2t \|(u(t), B(t))\|_{\dot{H}^2_x}^2 \right\}.
\]
(2.17)
In view of (2.1), we deduce that for \(c\) sufficiently small such that \(\exp \{ 2t \|(u_0, B_0)\|_{\dot{H}^2_x} \} \leq e^{2c} \leq \frac{3}{2},
\[
\|(u(t), B(t))\|_{\dot{H}^2_x}^2 \leq \frac{3}{2} C \|(u_0, B_0)\|_{\dot{H}^2_x}^2,
\]
which, via bootstrap arguments, yields (2.16) for any \(t \in [t_0, t_\ast]\), thereby proving (2.3).

(ii). Now we prove estimate (2.5). For this purpose, we reformulate (1.5) as
\[
u \frac{d}{dt} e^{-(t-t_0)\nu} u_0 - \int_{t_0}^t e^{-(t-s)\nu} \mathbb{P}_H \text{div}(u(s) \otimes u(s) - B(s) \otimes B(s)) ds, \]
(2.18a)
\[ B(t) = e^{-(t-t_0)\nu_2(-\Delta)^{\alpha}}B_0 - \int_{t_0}^t e^{-(t-s)\nu_2(-\Delta)^{\alpha}}\mathbb{P}_H \text{div}(B(s) \otimes u(s) - u(s) \otimes B(s))ds, \tag{2.18b} \]

Without loss of generality, we consider only the case \( t_0 = 0 \) below. We will frequently use the following estimates, via the Gagliardo-Nirenberg inequality, for any \( f, g \in L^2 \),

\[ \|f\|_{H^s_2} \leq \|f\|_{H^s_2} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^{-s}_2}, \quad \|f\|_{L^\infty} \leq \|f\|_{H^s_2} \|f\|_{L^2}^{\frac{1}{2}}. \tag{2.19} \]

First note that, in the case where \( N = M = 0, (2.5) \) follows immediately from (2.3).

\( (ii.1) \). The case where \( M = 0 \) and \( N = 1 \). Applying Lemma A.5 to equations (2.18a)-(2.18b) and using the boundedness of \( \mathbb{P}_H \) on \( L^2 \), we get

\[ t^{\frac{1}{2s}} \|\nabla u(t), \nabla B(t)\|_{H^s_2} \leq C_s \|u_0, B_0\|_{H^s_2}. \tag{2.21} \]

We prove (2.21) by using bootstrap arguments. More precisely, applying (2.21), Gagliardo-Nirenberg inequality, (2.3), (2.5) with \( N = 0, M = 0 \), and the inequality (2.19) to the nonlinearities of (2.20), we obtain

\[ \|\text{div}(B \otimes u)\|_{H^s_2} \leq \|u\|_{H^s_2} \|\nabla B\|_{L^2}^{\frac{1}{2}} \|\nabla B\|_{H^s_2} + \|u\|_{H^s_2} \|u\|_{L^2}^{\frac{1}{2}} \|\nabla B\|_{H^s_2} \]

\[ \leq C_1 C_s \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} + s^{-\frac{1}{2}} \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}}, \tag{2.22} \]

which, along with Young’s inequality, yields that

\[ \|\text{div}(B \otimes u)\|_{H^s_2} \leq C_2 C_s \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} + \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}}. \tag{2.23} \]

Similarly, \( \|\text{div}(u \otimes u)\|_{H^s_2}, \|\text{div}(B \otimes B)\|_{H^s_2} \) and \( \|\text{div}(u \otimes B)\|_{H^s_2} \) obey the same upper bound. Thus, turning back to (2.20), we get

\[ t^{\frac{1}{s}} \|\nabla u(t), \nabla B(t)\|_{H^s_2} \leq C_s \|u_0, B_0\|_{H^s_2} + C_{2s} \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2s}} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} ds \]

\[ + C_{2s} \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2s}} \|u_0, B_0\|_{L^2}^{\frac{1}{2}} ds \]

\[ \leq C_s \|u_0, B_0\|_{H^s_2} (1 + C_4 C_1 t^{1+\frac{1}{2s}} \|u_0, B_0\|_{H^s_2} + C_4 C_1 t^{1-\frac{1}{2s}} \|u_0, B_0\|_{H^s_2} \|u_0, B_0\|_{L^2}^{\frac{1}{2}}) \tag{2.24} \]

where \( C_4 \geq 1 \) is a universal constant. Thus, choosing \( t_* \) sufficiently small such that

\[ 0 < t_* \leq \min \left\{ 1, \frac{1}{16C_4^2 C_1^2 \|u_0, B_0\|_{H^s_2} (1 + \|u_0, B_0\|_{L^2}^{\frac{1}{2}}) \right\}, \]

we arrive at

\[ t^{\frac{1}{s}} \|\nabla u(t), \nabla B(t)\|_{H^s_2} \leq \frac{3}{2} C_s \|u_0, B_0\|_{H^s_2}, \tag{2.25} \]

which, via bootstrap argument, yields (2.21), as claimed. Thus, (2.5) is proved when \( N = 1 \) and \( M = 0 \).

\( (ii.2) \). The case where \( N = 2 \) and \( M = 0 \). We apply Lemma A.5 again to (2.18b) and then use (2.19) to get

\[ \begin{align*}
  t^{\frac{1}{2}} \|\nabla^2 B(t)\|_{H^2_2} & \leq C_s \|B_0\|_{H^2_2} + C_s t^{\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2s}} \|u(s), B(s)\|_{H^s_2} \|u(s), B(s)\|_{L^2}^{\frac{1}{2}} ds \tag{2.26}
\end{align*} \]
We claim that
\[ t^{\frac{3}{2}} \| (\nabla^2 u(t), \nabla^2 B(t)) \|_{H^2_2} \leq 2C_4 \|(u_0, B_0)\|_{H^2_2}, \]
holds for all \( t \in [0, t_*] \).

In order to prove (2.28), let us estimate the right-hand-side of (2.26). By the Leibniz rule we have
\[
\|\nabla^2 (B \otimes u)\|_{H^2_2} + \|\nabla^2 (u \otimes B)\|_{H^2_2} \leq 2 \sum_{j=0}^{2} (\|\nabla^j u\|_{H^2_2} \|\nabla^{n-j} B\|_{L^\infty_2} + \|\nabla^{n-j} B\|_{H^2_2} \|\nabla^j u\|_{L^\infty_2}).
\]

Note that, by the Gagliardo-Nirenberg inequality, the Young inequality (2.3) and (2.19),
\[
\sum_{j=0}^{2} \|\nabla^j u\|_{H^2_2} \|\nabla^{n-j} B\|_{L^\infty_2} \leq \|u\|_{H^2_2} \|\nabla^2 B\|_{H^2_2} \|\nabla B\|_{L^\infty_2} + \|u\|_{H^2_2} \|\nabla^2 u\|_{H^2_2} \|B\|_{L^\infty_2} \|B\|_{L^\infty_2} \\
\leq C_3 C_4 \left( s^{-\frac{1}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla B\|_{L^\infty_2} \|\nabla u\|_{H^2_2} + s^{-\frac{1}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla u\|_{H^2_2} \|\nabla B\|_{L^\infty_2} \|\nabla B\|_{L^\infty_2} \right) \\
\leq C_3 C_4 \left( s^{-\frac{1}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla B\|_{L^\infty_2} \|\nabla u\|_{H^2_2} + \|(u_0, B_0)\|_{H^2_2} \right).
\]

One can estimate the other terms in (2.29) in a similar manner. It follows that
\[
\|\nabla^2 (u \otimes B)\|_{H^2_2} \leq 4C_3 C_4 s^{-\frac{1}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla^2 u\|_{H^2_2} + \|(u_0, B_0)\|_{H^2_2} \right).
\]

Plugging this into (2.26) and using (2.2) and (2.3) lead to
\[
t^{\frac{3}{2}} \|\nabla^2 B(t)\|_{H^2_2} \leq C_4 \|B_0\|_{H^2_2} + C_4 t^{\frac{3}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla B\|_{L^\infty_2} \int_0^t (t-s)^{-\frac{1}{2}} ds \\
+ C_4 C_4 t^{\frac{3}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla u\|_{H^2_2} \|\nabla B\|_{L^\infty_2} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \left( \|\nabla u\|_{H^2_2} \|\nabla B\|_{L^\infty_2} + \|(u_0, B_0)\|_{H^2_2} \right) ds \\
\leq C_4 \|B_0\|_{H^2_2} + C_4 C_4 t^{\frac{3}{2}} \|(u_0, B_0)\|_{H^2_2} \|\nabla u\|_{H^2_2} + C_4 C_4 t^{\frac{3}{2}} \|(u_0, B_0)\|_{H^2_2} 
\]
for some universal constant \( C_4 \). Hence, taking \( t_* \) possibly smaller such that
\[
0 < t_* \leq \min \left\{ 1, \frac{1}{16C_4^2 C_4^2 \|(u_0, B_0)\|_{H^2_2} (1 + \|(u_0, B_0)\|_{H^2_2})} \right\}
\]
we deduce from (2.32) that for all \( t \in [0, t_*] \),
\[
t^{\frac{3}{2}} \|\nabla^2 B(t)\|_{H^2_2} \leq \frac{3}{2} C_4 \|B_0\|_{H^2_2}.
\]

The estimate of \( \|\nabla^2 u\|_{H^2_2} \) can be proved in a similar manner. Hence, using bootstrap arguments we obtain (2.28), as claimed. This verifies (2.5) when \( N = 2 \) and \( M = 0 \).

(ii.3). The general case where \( N \geq 3 \) and \( M = 0 \). In order to treat the general case we use the induction arguments. Suppose that (2.5) is valid for all \( N' \leq N \) and \( M = 0 \). By (2.18), we have
\[
\int_0^t (t-s)^{-\frac{N+1}{N}} \left( \|\nabla^N (u \otimes u)\|_{H^2_2} + \|\nabla^N (B \otimes B)\|_{H^2_2} \right) ds \\
+ \int_0^t (t-s)^{-\frac{1}{2}} \left( \|\nabla^N (u \otimes u)\|_{H^2_2} + \|\nabla^N (B \otimes B)\|_{H^2_2} \right) ds.
\]
As in (2.21) and (2.28), we claim that
\[
 t^{\frac{N}{2n}} \| \nabla^N u(t), \nabla^N B(t) \|_{H^3} \leq 2C_* \|(u_0, B_0)\|_{H^3}, \quad \forall \ t \in [0, t_*].
\]
(2.36)

To this end, concerning the right-hand side of (2.35), by the Leibniz rule,
\[
 \| \nabla^N (B \otimes u) \|_{H^3} + \| \nabla^N (u \otimes B) \|_{H^3} \lesssim \sum_{j=0}^{N} \| \nabla^j u \|_{H^3} \| \nabla^{N-j} B \|_{L^\infty} + \| \nabla^{N-j} B \|_{H^3} \| \nabla^j u \|_{L^\infty}. \tag{2.37}
\]

Using (2.19), (2.36) and the induction assumption, we get
\[
 \sum_{j=0}^{N} \| \nabla^j u \|_{H^3} \| \nabla^{N-j} B \|_{L^\infty} \lesssim \sum_{j=0}^{N-3} \| \nabla^j u \|_{H^3} \| \nabla^{N-j} B \|_{H^3} + \| \nabla^{N-2} u \|_{H^3} \| \nabla^{N-1} B \|_{H^3} + \| \nabla^{N-1} u \|_{H^3} \| \nabla B \|_{H^3} + \| \nabla^N u \|_{H^3} \| B \|_{H^3} + \| \nabla^N u \|_{H^3} \| B \|_{H^3} + \| \nabla^{N-1} u \|_{H^3} \| \nabla B \|_{H^3} + \| \nabla^N u \|_{H^3} \| B \|_{H^3}
\]
\[
\leq C_1 \left( s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|^2_{H^3} + s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} \right)
\]
\[
+ s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} + s^{-\frac{N}{2n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} \right),
\]
which yields
\[
 \| \nabla^N (B \otimes u) \|_{H^3} + \| \nabla^N (u \otimes B) \|_{H^3} \leq 4C_1 \left( s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|^2_{H^3} + s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} \right)
\]
\[
+ s^{-\frac{N}{2n} + \frac{1}{8n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} + s^{-\frac{N}{2n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} \right). \tag{2.38}
\]

The other terms \| \nabla^N (u \otimes u) \|_{H^3} and \| \nabla^N (B \otimes B) \|_{H^3} can be estimated in the same manner. Hence, using Young’s inequality, we obtain
\[
 \| \nabla^N (u \otimes u) \|_{H^3} + \| \nabla^N (B \otimes B) \|_{H^3} + \| \nabla^N (B \otimes u) \|_{H^3} + \| \nabla^N (u \otimes B) \|_{H^3} \leq 4C_1 \left( s^{-\frac{N}{2n}} \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} + \|(u_0, B_0)\|_{H^3}^{\frac{2}{3}} \|(u_0, B_0)\|_{L^2}^{\frac{1}{3}} \right). \tag{2.39}
\]

Thus, we derive from (2.2), (2.3), (2.19), (2.37) and (2.39) that
\[
 t^{\frac{N}{2n}} \| (\nabla^N u(t), \nabla^N B(t)) \|_{H^3} \leq C_* \| (u_0, B_0) \|_{H^3} + \tilde{C}_3 C_* t^{\frac{N}{2n}} \| (u_0, B_0) \|_{H^3}^{\frac{2}{3}} \int_0^t (t-s)^{-\frac{N}{2n}} ds
\]
\[
+ \tilde{C}_3 C_* t^{\frac{N}{2n}} \| (u_0, B_0) \|_{H^3}^{\frac{2}{3}} \int_0^t (t-s)^{-\frac{N}{2n}} (s^{-\frac{N}{2n}} \| (u_0, B_0) \|_{H^3}^{\frac{2}{3}} + \| (u_0, B_0) \|_{L^2}) ds
\]
\[
\leq C_* \| (u_0, B_0) \|_{H^3} \left( 1 + \tilde{C}_3 C_* t^{1-\frac{N}{2n}} \| (u_0, B_0) \|_{H^3}^{\frac{2}{3}} \| (u_0, B_0) \|_{L^2}^{\frac{1}{3}} + \tilde{C}_4 C_* t^{1+\frac{N}{2n}} \| (u_0, B_0) \|_{H^3}^{\frac{2}{3}} \right),
\]

which, along with the smallness condition (2.33) with \( c \) small enough, yields that (2.36) still holds but with the improved constant \( 3C_* / 2 \). Then, using bootstrap arguments we prove (2.36), as claimed. Finally, the induction arguments give (2.5) for any \( N \geq 3 \) and \( M = 0 \).

(ii.4). The case where \( N \geq 0 \) and \( M = 1 \). We deduce from (1.5), Lemma A.5 and estimate (2.39) that
\[
 \| \partial_\alpha \nabla^N u, \partial_\alpha \nabla^N B \|_{H^3} \lesssim t^{-\frac{N+2}{2n}} \| (u_0, B_0) \|_{H^3} + t^{-\frac{N}{2n}} \| (u, B) \|_{H^3} \left( t^{\frac{N}{2n}} \| (u, B) \|_{H^3}^{\frac{2}{3}} + \| (u, B) \|_{L^2} \right),
\]
which, along with (2.4), yields
\[
 \| \partial_\alpha \nabla^N u, \partial_\alpha \nabla^N B \|_{H^3} \lesssim t^{-\frac{N}{2n}-1} \| (u_0, B_0) \|_{H^3}.
\]

Therefore, the proof of Proposition 2.1 is complete.
2.2. Stability estimates. Next, we will generate the local solutions $\{v_i, D_i\}$ to MHD equations near every $t_i, 0 \leq i \leq m_{q+1} - 1$. Precisely, for every $0 \leq i \leq m_{q+1} - 1$, let $\{v_i, D_i\}$ solve the following MHD equations on the small subinterval $[t_i, t_{i+1} + \theta_{q+1}]$.

\[
\begin{align*}
\partial_t v_i + v_i (\Delta) v_i + (v_i \cdot \nabla) v_i - (D_i \cdot \nabla) D_i + \nabla p_i &= 0, \\
\partial_t D_i + v_i (\Delta) D_i + (v_i \cdot \nabla) D_i - (D_i \cdot \nabla) v_i &= 0, \\
\text{div} v_i &= 0, \\
\text{div} D_i &= 0, \\
v_i |_{t=t_i} &= u_q(t_i), \\
D_i |_{t=t_i} &= B_q(t_i).
\end{align*}
\] (2.40)

Note that, by the iterative estimate (1.23) with $p_i$ unique solution $(\|\nabla\|_{L^2} \leq \alpha u_i)$, let $\tilde{H}_i, \tilde{D}_i$ be the well-prepared solution to (2.41) at level $q$, let $\{w_i, H_i\}$ solve (2.41), and set $s_{i+1} := t_{i+1} + \theta_{q+1}$, $0 \leq i \leq m_{q+1} - 1$. Then, $\{w_i, H_i\}$ satisfies the linearized equations, via (1.11) and (2.40),

\[
\begin{align*}
\partial_t w_i + v_i (\Delta) w_i + \text{div}(v_i \otimes w_i + w_i \otimes u_q - D_i \otimes H_i - H_i \otimes B_q) + \nabla p_i &= \text{div} \tilde{R}^w_q, \\
\partial_t H_i + v_i (\Delta) H_i + \text{div}(H_i \otimes u_q + D_i \otimes w_i - w_i \otimes B_q - v_i \otimes H_i) &= \text{div} \tilde{R}^B_q, \\
\text{div} w_i &= 0, \\
\text{div} H_i &= 0, \\
w_i |_{t=t_i} &= 0, \\
D_i |_{t=t_i} &= 0,
\end{align*}
\] (2.41)

for some pressure $p_i : [t_i, t_{i+1}] \times \mathbb{T}^3 \to \mathbb{R}$.

Proposition 2.2 below contains the key stability estimates for the solutions to (2.41).

**Proposition 2.2 (Stability estimates).** Let $\alpha \in [1, 3/2), 1 < \rho \leq 2$. Let $\{u_q, B_q, \tilde{R}^w_q, \tilde{R}^B_q\}$ be the well-prepared solution to (1.11) at level $q$, let $\{w_i, H_i\}$ solve (2.41), and set $s_{i+1} := t_{i+1} + \theta_{q+1}, 0 \leq i \leq m_{q+1} - 1$. Then, the following estimates hold:

\[
\begin{align*}
\|\{w_i, H_i\}\|_{L^\infty([t_i, s_{i+1} + \theta_{q+1}]; L^p_\nu)} &\lesssim \int_{t_i}^{s_{i+1}} \|\nabla \tilde{R}^w_q(s), \nabla \tilde{R}^B_q(s)\|_{L^p_\nu} ds, \\
\|\{w_i, H_i\}\|_{L^\infty([t_i, s_{i+1} + \theta_{q+1}]; H^\rho_2)} &\lesssim \int_{t_i}^{s_{i+1}} \|\nabla \tilde{R}^w_q(s), \nabla \tilde{R}^B_q(s)\|_{H^\rho_2} ds, \\
\|\{R^a w_i, R^B H_i\}\|_{L^\infty([t_i, s_{i+1} + \theta_{q+1}]; L^p_\nu)} &\lesssim \int_{t_i}^{s_{i+1}} \|\tilde{R}^w_q(s), \tilde{R}^B_q(s)\|_{L^p_\nu} ds,
\end{align*}
\] (2.42) (2.43) (2.44)

where $0 \leq i \leq m_{q+1} - 1$, $R^a$ and $R^B$ are the inverse divergence operators given by (A.5) in Appendix A, and the implicit constants depend only on $\alpha$ and $\rho$.

**Proof.** Without loss of generality, we may consider the case $t_i = 0$. We reformulate (2.41) as

\[
\begin{align*}
w_i(t) &= \int_0^t e^{-(t-s)\alpha (\Delta)} \mathbb{P}_H \text{div}(\tilde{R}^w_q - v_i \otimes w_i - w_i \otimes u_q + D_i \otimes H_i + H_i \otimes B_q) ds, \\
H_i(t) &= \int_0^t e^{-(t-s)\alpha (\Delta)} \mathbb{P}_H \text{div}(\tilde{R}^B_q - H_i \otimes u_q - D_i \otimes w_i + w_i \otimes B_q + v_i \otimes H_i) ds.
\end{align*}
\] (2.45a) (2.45b)

Applying Lemma A.5 to (2.45a)-(2.45b) yields

\[
\begin{align*}
\|\{w_i(t), H_i(t)\}\|_{L^p_\nu} &\leq C_* \int_0^t \|\nabla \tilde{R}^w_q, \nabla \tilde{R}^B_q\|_{L^p_\nu} ds \\
&\quad + C_* \int_0^t (t-s)^{-\frac{\alpha}{2}} \|\{u_q, v_i\}\|_{L^p_\nu} + \|\{B_i, D_i\}\|_{L^p_\nu} (\|w_i\|_{L^p_\nu} + \|H_i\|_{L^p_\nu}) ds
\end{align*}
\] (2.46)

for some universal constant $C_*$ depending only on $\rho$ and $\alpha$.

We claim that for all $t \in [0, t_1 + \theta_{q+1}]

\[
\|w_i(t)\|_{L^p_\nu} + \|H_i(t)\|_{L^p_\nu} \leq 2C_* \int_0^t \|\nabla \tilde{R}^w_q, \nabla \tilde{R}^B_q\|_{L^p_\nu} ds.
\] (2.47)
The proof of (2.47) is based on bootstrap arguments. First we note that, (2.47) is valid for \( t = 0 \). Moreover, by assuming (2.47), we would prove that the same estimate holds but with the constant \( 2C_* \) replaced by a smaller constant \( 3C_* / 2 \). To see this, plugging (2.47) into (2.46) we get

\[
\begin{align*}
\|(w_i(t), H_i(t))\|_{L^\infty_x} & \leq 2C_* \left( \frac{1}{2} + C_* \int_0^t (t-s)^{-\frac{1}{2n}} \left( \|(u_q, v_i)\|_{L^\infty_x L^2_t} + \|(B_i, D_i)\|_{L^\infty_x L^\infty_t} \right) ds \right) \\
& \leq 2C_* \left( \frac{1}{2} + 2C_* t^{1-\frac{2}{n}} \left( \|(u_q, v_i)\|_{L^\infty_x L^2_t} + \|(B_i, D_i)\|_{L^\infty_x L^\infty_t} \right) \right) \int_0^t \|(\nabla |\tilde{R}_q^u|, \nabla |\tilde{R}_q^B|)\|_{L^\infty_x} ds.
\end{align*}
\]

Thus, using the Sobolev embedding \( H^3 \rightarrow L^\infty_x \), (1.23), (1.16) and (2.3) we bound

\[
\begin{align*}
2C_* (t_1 + \theta_{q+1})^{-\frac{1}{2n}} & \left( \|(u_q, v_i)\|_{L^\infty_x L^2_t} + \|(B_i, D_i)\|_{L^\infty_x L^\infty_t} \right) \\
\leq & \ 2CC_* (T/m_{q+1})^{-\frac{1}{2n}} \left( \|(u_q, v_i)\|_{L^\infty_H^3} + \|(B_i, D_i)\|_{L^\infty_H^3} \right) \\
\leq & \ 6CC_* (\lambda_q^{-12})^{\frac{1}{2} + \frac{2n}{n-1}} \|(u_q, B_q)\|_{L^\infty_H^3} \\
\leq & \ C'(\lambda_q^{-12})^{\frac{1}{2} + \frac{2n}{n-1}} \leq \frac{1}{4},
\end{align*}
\]

where \( C' \) is some universal constant, and \( \alpha \) is sufficiently large such that \( C' \lambda_q^{-1} \leq 1 / 4 \). Hence, we obtain (2.47) with the improved constant \( 3C_* / 2 \), which, via bootstrap arguments, yields (2.47) and so (2.42).

Estimate (2.43) can be proved in the fashion of the proof of (2.47). We claim that for all \( t \in [0, t_1 + \theta_{q+1}] \),

\[
\|(w_i(t), H_i(t))\|_{H^2_x} \leq 2C_* \int_0^t \|(\nabla |\tilde{R}_q^u|, \nabla |\tilde{R}_q^B|)\|_{H^2_x} ds.
\]

To this end, we apply Lemma A.5 to (2.41) to get

\[
\begin{align*}
\|(w_i(t), H_i(t))\|_{H^2_x} & \leq C_* \int_0^t \|(\nabla |\tilde{R}_q^u(s)|, \nabla |\tilde{R}_q^B(s)|)\|_{H^2_x} ds \\
& + C_* \int_0^t (t-s)^{-\frac{1}{2n}} \left( \|(u_q, v_i)\|_{H^2_x} + \|(B_i, D_i)\|_{H^2_x} \right) \|(w_i, H_i)\|_{H^2_x} ds.
\end{align*}
\]

Plugging (2.50) into (2.51) and estimating as in (2.48) and (2.49) we come to

\[
\begin{align*}
\|w_i(t), H_i(t)\|_{H^2_x} & \leq 2C_* \left( \frac{1}{2} + 2C_* t^{1-\frac{2}{n}} \left( \|(u_q, v_i)\|_{L^\infty H^2_x} + \|(B_i, D_i)\|_{L^\infty H^2_x} \right) \right) \\
& \times \int_0^t \|(\nabla |\tilde{R}_q^u(s)|, \nabla |\tilde{R}_q^B(s)|)\|_{H^2_x} ds \leq \frac{3}{2} C_*.
\end{align*}
\]

This gives (2.50) with the improved constant \( 3C_* / 2 \). Hence, using bootstrap arguments once more we prove (2.50) and so (2.43) for all \( t \in [0, t_1 + \theta_{q+1}] \).

It remains to prove (2.44). Let

\[
y_i := \Delta^{-1} \text{curl} w_i, \quad J_i := \Delta^{-1} \text{curl} H_i.
\]

Note that \( \text{curl} y_i = -w_i \) and \( \text{curl} J_i = -H_i \), as \( w_i \) and \( H_i \) are divergence free. By the boundedness of Calderón-Zygmund operators \( R^\alpha \text{curl} \) and \( R^B \text{curl} \) in \( L^p_x \), \( 1 < p < 2 \), we have, for any \( t \in [t_i, t_{i+1}] \),

\[
\| R^\alpha w_i \|_{L^p_x} \leq C \| y_i \|_{L^p_x}, \quad \| R^B H_i \|_{L^p_x} \leq C \| J_i \|_{L^p_x},
\]

where \( C \) is a universal constant.

Moreover, straightforward computations show that \( y_i \) satisfies the equation

\[
\begin{align*}
\partial_t y_i + u_q \cdot \nabla y_i - B_q \cdot \nabla J_i + \nu_i (-\Delta)^{\alpha} y_i = & \ \Delta^{-1} \text{curl} \text{div} \tilde{R}_q^u + \Delta^{-1} \text{curl} \text{div} \left( ((y_i \times \nabla) v_i)^T \right) \\
& + \Delta^{-1} \text{curl} \left( (y_i \times \nabla) u_q \right) + \Delta^{-1} \text{div} \left( (y_i \cdot \nabla) u_q \right) \\
& - \Delta^{-1} \text{curl} \left( ((J_i \times \nabla) D_i)^T \right) - \Delta^{-1} \text{curl} \left( (J_i \cdot \nabla) B_q \right) \\
& - \Delta^{-1} \text{div} \left( (J_i \cdot \nabla) B_q \right),
\end{align*}
\]
and \( J_i \) satisfies
\[
\partial_t J_i + u_q \cdot \nabla J_i - B_q \cdot \nabla y_i + \nu_2 (-\Delta)^\alpha J_i = \Delta^{-1} \text{curl} \mathrm{div} \tilde{J}_q^B + \Delta^{-1} \text{curl} \mathrm{div} \left( \left((y_i \times \nabla)D_i\right)^T \right) \\
+ \Delta^{-1} \text{curl} \mathrm{div} \left( (J_i \times \nabla)u_q \right) + \Delta^{-1} \text{div} \left( (J_i \cdot \nabla)u_q \right) \\
- \Delta^{-1} \text{curl} \mathrm{div} \left( ((y_i \times \nabla)D_i)^T \right) - \Delta^{-1} \text{curl} \mathrm{div} \left( (y_i \cdot \nabla)B_q \right) \\
- \Delta^{-1} \text{div} \left( (y_i \cdot \nabla)B_q \right).
\quad (2.56)
\]

Then, we can apply Lemma A.5 and use the boundedness of Calderón-Zygmund operators \( \Delta^{-1} \text{curl} \mathrm{div} \) and \( \Delta^{-1} \text{div} \) in \( L^p_t \) to derive
\[
\|(y_i(t), J_i(t))\|_{L_t^p} \leq C_* \int_0^t \|\tilde{R}_q^u(s), \tilde{R}_q^B(s)\|_{L_t^p} ds \\
+ C_* \|(u_q, B_q)\|_{L_t^p L_x^\infty} \int_0^t (t-s)^{-\frac{\alpha}{2}} \|(y_i(s), J_i(s))\|_{L_t^p} ds \\
+ C_* \left( \left\| (\nabla u_q, \nabla v_i) \right\|_{L_t^p L_x^\infty} + \left\| (\nabla B_q, \nabla D_i) \right\|_{L_t^p L_x^\infty} \right) \int_0^t \|(y_i(s), J_i(s))\|_{L_t^p} ds,
\quad (2.57)
\]
where \( C_* \) is a universal constant depending only on \( \rho \) and \( \alpha \).

We claim that for any \( t \in [0, t_1 + \theta_{q+1}] \),
\[
\|(y_i(t), J_i(t))\|_{L_t^p} \leq 2C_* \int_0^t \|\tilde{R}_q^u(s), \tilde{R}_q^B(s)\|_{L_t^p} ds.
\quad (2.58)
\]

In order to prove (2.58), inserting (2.58) into (2.57), we get
\[
\|(y_i(t), J_i(t))\|_{L_t^p} \leq 2C_* \int_0^t \|\tilde{R}_q^u(s), \tilde{R}_q^B(s)\|_{L_t^p} ds \times \left( \frac{1}{2} + 2C_* t^{1-\frac{\alpha}{2}} \|(u_q, B_q)\|_{L_t^p L_x^\infty} + C_* t \left( \left\| (\nabla u_q, \nabla v_i) \right\|_{L_t^p L_x^\infty} + \left\| (\nabla B_q, \nabla D_i) \right\|_{L_t^p L_x^\infty} \right) \right).
\]

Then, by the Sobolev embedding \( H^2_x \hookrightarrow L^\infty_x, \) (1.23), (1.16) and (2.3),
\[
2C_* t^{1-\frac{\alpha}{2}} + C_* t \left( \left\| (\nabla u_q, \nabla v_i) \right\|_{L_t^p L_x^\infty} + \left\| (\nabla B_q, \nabla D_i) \right\|_{L_t^p L_x^\infty} \right) \|(u_q, B_q)\|_{L_t^p L_x^\infty} \\
\leq 2C_* t^{1-\frac{\alpha}{2}} \|(u_q, B_q)\|_{L_t^p H^2_x} + C_* t \left( \|(u_q, v_i)\|_{L_t^p H^2_x} + \|(B_q, D_i)\|_{L_t^p H^2_x} \right) \\
\leq CC_* (3(t_1 + \theta_{q+1}) + 2(t_1 + \theta_{q+1}) t^{1-\frac{\alpha}{2}}) \|(u_q, B_q)\|_{L_t^p H^2_x} \\
\leq C' \left( \lambda^{-12} - \lambda^{-12} + \frac{2\alpha}{2+\alpha} \right) \lambda^5 \leq \frac{1}{4}
\quad (2.59)
\]
for \( \alpha \) sufficiently large. It follows that the constant in (2.58) can be improved by \( 3C_* / 2 \). This yields (2.58) for all \( t \in [0, t_1 + \theta_{q+1}] \) by bootstrap arguments, as claimed. In view of (2.54), we consequently prove (2.44). Therefore, the proof is complete.

\section{Temporal gluing of local solutions.}

From the previous section, we see that, for every \( 0 \leq i \leq m_{q+1} - 1 \), \( (v_i, D_i) \) is exactly the solution to MHD equations (1.5) on the small subinterval \([t_i, t_{i+1} + \theta_{q+1}]\), and its difference from the old well-prepared solution \( (u_q, B_q) \) can be controlled by using the stability estimates in Proposition 2.2.

In this subsection, we will glue all these local solutions together by using a partition of unity \( \{ \chi_i \} \), such that \( \sum_i \chi_i v_i, \sum_i \chi_i D_i \) solves equations (1.5) in a majority part of the time interval \([0, T]\). The important outcome is that, the new Reynolds and magnetic stresses would have disjoint temporal supports with smaller length, and, simultaneously, maintain the decay of amplitudes.

To be precise, let \( \{ \chi_i \}_{i=0}^{m_{q+1}-1} \) be a \( C^\infty \) partition of unity on \([0, T]\) such that
\[
0 \leq \chi_i(t) \leq 1, \quad \text{for} \quad t \in [0, T],
\]
for \( 0 < i < m_{q+1} - 1 \),
\[
\chi_i = \begin{cases} 
1 & \text{if } t_i + \theta_{q+1} \leq t \leq t_{i+1}, \\
0 & \text{if } t \leq t_i, \text{ or } t \geq t_{i+1} + \theta_{q+1},
\end{cases}
\quad (2.60)
\]
and for $i = 0$,  
\[ \chi_0 = \begin{cases} 1 & \text{if } 0 \leq t \leq t_{i+1}, \\ 0 & \text{if } t \geq t_{i+1} + \theta_{q+1}, \end{cases} \]  
(2.61)
and for $i = m_{q+1} - 1$,  
\[ \chi_{m_{q+1} - 1} = \begin{cases} 1 & \text{if } t_i + \theta_{q+1} \leq t \leq T, \\ 0 & \text{if } t \leq t_i, \end{cases} \]  
(2.62)
Furthermore,  
\[ \|\partial_t^M \chi_i\|_{L_t^\infty} \lesssim \theta_{q+1}^{-M}, \quad 0 \leq i \leq m_{q+1} - 1, \]  
(2.63)
where the implicit constant is independent of $\theta_{q+1}, i$ and $M \geq 0$.

Then, we define the gluing solutions by  
\[ \tilde{u}_q := \sum_{i=0}^{m_{q+1} - 1} \chi_i v_i, \quad \tilde{B}_q := \sum_{i=0}^{m_{q+1} - 1} \chi_i D_i, \]  
(2.64)
Note that, $\tilde{u}_q, \tilde{B}_q : [0, T] \times \mathbb{T}^3 \to \mathbb{R}^3$ are divergence free and mean free. Moreover, we have  
\[ \tilde{u}_q = (1 - \chi_i) v_{i-1} + \chi_i v_i, \quad \tilde{B}_q = (1 - \chi_i) D_{i-1} + \chi_i D_i, \quad t \in [t_i, t_{i+1}], \]  
and the glued solutions $(\tilde{u}_q, \tilde{B}_q)$ satisfy the following equations on $[t_i, t_{i+1}]$:  
\[ \begin{cases} \partial_t \tilde{u}_q + \nu(-\Delta) \tilde{u}_q + \text{div}(\tilde{u}_q \otimes \tilde{u}_q - \tilde{B}_q \otimes \tilde{B}_q) + \nabla \tilde{p} = \text{div} \tilde{R}_q^u, \\ \partial_t \tilde{B}_q + \nu(-\Delta) \tilde{B}_q + \text{div}(\tilde{B}_q \otimes \tilde{u}_q - \tilde{B}_q \otimes \tilde{B}_q) = \text{div} \tilde{R}_q^B, \end{cases} \]  
(2.65)
for some pressure term $\tilde{p}$ and the new stresses  
\[ \tilde{R}_q^u = \partial_t \chi_i \mathcal{R}^u(v_i - v_{i-1}) - \chi_i \nu(-\Delta)^a (v_i - v_{i-1}) - (D_i - D_{i-1}) \otimes (D_i - D_{i-1}), \]  
(2.66a)
\[ \tilde{R}_q^B = \partial_t \chi_i \mathcal{R}^B(D_i - D_{i-1}) - \chi_i \nu(-\Delta)^a (D_i - D_{i-1}) \otimes (v_i - v_{i-1}) - (v_i - v_{i-1}) \otimes (D_i - D_{i-1}). \]  
(2.66b)

### 2.4. Proof of well-preparedness in Theorem 1.6.

Using the finite overlaps of the supports of $\{\chi_i\}$, (1.23) and (2.3), we get  
\[ \|\tilde{u}_q\|_{L_t^\infty H_x^3} \leq \|\sum_i \chi_i v_i\|_{L_t^\infty H_x^3} \leq \sup_i \left( \|(1 - \chi_i) v_{i-1}\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} + \|\chi_i v_i\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} \right) \lesssim \|u_q\|_{L_t^\infty H_x^3} \lesssim \lambda_q^5, \]  
(2.67)
and similarly,  
\[ \|\tilde{B}_q\|_{L_t^\infty H_x^3} \leq \sup_i \left( \|(1 - \chi_i) D_{i-1}\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} + \|\chi_i D_i\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} \right) \lesssim \lambda_q^5, \]  
(2.68)
which yields (1.17).

Regarding estimate (1.18), using (1.16), (1.25) and (2.42), we get  
\[ \|\tilde{u}_q - u_q, \tilde{B}_q - B_q\|_{L_t^\infty L_x^2} \leq \left\| \sum_i \chi_i(v_i - u_i, D_i - B_q) \right\|_{L_t^\infty L_x^2} \leq \sup_i \left( \|(v_i - u_i, D_i - B_q)\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} + \|(v_i - u_i, D_i - B_q)\|_{L_t^\infty(\text{supp}(\chi_i; H_x^3))} \right) \lesssim m_{q+1}^{-1} \|\tilde{R}_q^u, \tilde{R}_q^B\|_{L_t^\infty H_x^3} \lesssim m_{q+1}^{-1} \lambda_q^9 \lesssim \lambda_q^{-3}. \]  
(2.69)
Hence, estimate (1.18) is verified.

Concerning the $L_{t,x}^1$ estimate of the new Reynolds stress $\tilde{R}_q^u$, the expression (2.66) yields that  
\[ \|\tilde{R}_q^u\|_{L_t^1 L_x^\infty} \leq \|\partial_t \chi_i \mathcal{R}(v_i - v_{i-1})\|_{L_t^1 L_x^\infty} \]
+ \|\chi_i(1 - \chi_i)((v_i - v_i -1)\hat{\Delta}(v_i - v_i -1) - (D_i - D_i -1)\hat{\Delta}(D_i - D_i -1))\|_{L^1_{t,x}}
=:K_1 + K_2. 

(2.70)

In order to estimate the right-hand-side above, for the first term $K_1$, we choose

\[ 1 < \rho < \frac{4\varepsilon_R + 36 + 8\beta b}{\varepsilon_R + 36 + 8\beta b}. \]

(2.71)

and use Hölder’s inequality to obtain

\[
K_1 \leq \sum_i \|\partial_i \chi_i\|_{L^1_t}\|R((v_i - v_i -1))\|_{L^\infty_t L^2_x} 
\lesssim \sum_i \|R((w_i - w_i -1))\|_{L^\infty_t (\text{supp } (\chi_i(1 - \chi_i); L^2_x))}. 
\]

(2.72)

Note that, the uniform bound $\|\partial_i \chi_i\|_{L^1_t} \lesssim 1$ was used in the last step. Then, by the Gagliardo-Nirenberg inequality, (1.25), (1.26), (2.44) and (2.71),

\[
K_1 \lesssim \sum_i \int_{t_i}^{t_{i+1} + \theta_{q+1}} \|(\hat{R}_q^u(s), \hat{R}_q^B(s))\|_{L^2_x} ds 
\lesssim \|(\hat{R}_q^u(s), \hat{R}_q^B(s))\|_{L^1_{t} L^2_x}^{1 - \frac{2(n-1)}{2(n-1)}} \|(\hat{R}_q^u(s), \hat{R}_q^B(s))\|_{L^n_t H^2_x}^{\frac{2(n-1)}{2(n-1)}} 
\lesssim \delta_{q+1} \lambda_q \lesssim \lambda_q \delta_{q+1}, \tag{2.73}
\]

where in the last step we used $\lambda_q^{-\varepsilon R/8}$ (or a sufficiently large) to absorb the implicit constant.

For the second term $K_2$ of (2.70), we use (2.42) to estimate

\[
K_2 \lesssim \sum_i \|\text{supp } \chi_i(1 - \chi_i)\|_{L^\infty_t} \left(\|\hat{R}_q^u(s)\|_{L^2_x} \left|\|\hat{R}_q^B(s)\|_{L^2_x}\right|^2\right)^{\theta_{q+1}} 
\lesssim \theta_{q+1}\left(\int_{t_i}^{t_{i+1}} \left|\|\nabla |\hat{R}_q^u(s)\|_{L^2_x} \left|\|\hat{R}_q^B(s)\|_{L^2_x}\right|\right|^2 ds\right)^{\theta_{q+1}} \lesssim \theta_{q+1} \left(\left|\|\nabla |\hat{R}_q^u(s)\|_{L^2_x} \left|\|\hat{R}_q^B(s)\|_{L^2_x}\right|\right|^2\right)^{\theta_{q+1}}. \tag{2.74}
\]

Using the interpolation estimate, (1.25) with $N = 3$, (1.26) and the facts that $\lambda_q^{-1} \ll \delta_q^{1/9}$ and $0 < \eta < 1$, we obtain

\[
K_2 \lesssim m_{q+1} \left(\int_{t_i}^{t_{i+1}} \left|\|\nabla |\hat{R}_q^u(s)\|_{L^2_x} \left|\|\hat{R}_q^B(s)\|_{L^2_x}\right|\right|^2 ds\right)^{\theta_{q+1}} \lesssim \lambda_q^{-12/\eta} \lambda_q^{-\varepsilon R/8} \delta_{q+1} \lambda_q^{10} \lesssim \lambda_q^{-\varepsilon R/8} \delta_{q+1}, \tag{2.75}
\]

where we also chose $a$ sufficiently large and used $\lambda_q^{-\varepsilon R/4}$ to absorb the implicit constant.

Thus, it follows from (2.70), (2.73) and (2.75) that

\[
\left|\|\hat{R}_q^u\|_{L^1_{t,x}}\right| \lesssim \lambda_q^{-\varepsilon R/8} \delta_{q+1}. \tag{2.76}
\]

The $L^1_{t,x}$-estimate of the magnetic stress $\hat{R}_q^B$ can be obtained by using analogous arguments. Hence, (1.19) is verified.

For estimate (1.20), we infer from (2.66) that, for any $t \in [t_i, t_{i+1}]$, $0 \leq i \leq m_{q+1} - 1$,

\[
\|\hat{\partial}_t^N \nabla R \hat{R}_q^B\|_{L^n_t H^2_x} \lesssim \|\hat{\partial}_t^N \nabla \chi_i R((v_i - v_i -1))\|_{L^n_t H^2_x} 
+ \|\hat{\partial}_t^N \nabla \chi_i(1 - \chi_i)((v_i - v_i -1)\hat{\Delta}(v_i - v_i -1) - (D_i - D_i -1)\hat{\Delta}(D_i - D_i -1)))\|_{L^n_t H^2_x} 
=: \hat{K}_1 + \hat{K}_2. \tag{2.77}
\]
Note that, by (1.23) with \( N = 3 \), (2.5) and (2.63),
\[
\widetilde{K}_1 \lesssim \sum_{M_1 + M_2 = M} \| \partial_t^{M_1+1} \chi_i \|_{L^\infty_t} \left( \| \partial_t^{M_2} \nabla^N v_i \|_{L^\infty_t(\text{supp}(\chi); H^2)} + \| \partial_t^{M_2} \nabla v_i \|_{L^\infty_t(\text{supp}(\chi); H^2)} \right)
\]
\[
\lesssim \sum_{M_1 + M_2 = M} \theta^{-M_1}_{q+1} m^{\infty}_{q+1} \lambda_q^5 \lesssim \theta^{-M-1}_{q+1} m^{\infty}_{q+1} \lambda_q^5,
\]
(2.78)
where the last step is due to \( m_{q+1} \leq \theta^{-1}_{q+1} \) and the implicit constants are independent of \( i \) and \( q \). Moreover, by (1.23), (2.5),
\[
\widetilde{K}_2 \lesssim \sum_{M_1 + M_2 = M} \left( \| \partial_t^{M_1} (\chi_i (1 - \chi_i)) \|_{L^\infty_t} \| \partial_t^{M_2} \nabla^N ((v_i - v_i) \partial_t (v_i - u_i - 1)) \|_{L^\infty_t(\text{supp}(\chi); H^2)} 
\right.
\]
\[
+ \| \partial_t^{M_1} (\chi_i (1 - \chi_i)) \|_{L^\infty_t} \| \partial_t^{M_2} \nabla^N ((D_i - D_i - 1) \partial_t (D_i - D_i - 1)) \|_{L^\infty_t(\text{supp}(\chi); H^2)}
\left. \right) \lesssim \sum_{M_1 + M_2 = M} \theta^{-M_1}_{q+1} \left( \| \partial_t^{M_2} \nabla^N ((v_i - v_i) \partial_t (v_i - u_i - 1)) \|_{L^\infty_t(\text{supp}(\chi); H^2)} 
\right.
\]
\[
+ \| \partial_t^{M_2} \nabla^N ((D_i - D_i - 1) \partial_t (D_i - D_i - 1)) \|_{L^\infty_t(\text{supp}(\chi); H^2)} \right) \lesssim \sum_{M_1 + M_2 = M} \theta^{-M_1}_{q+1} m^{\infty}_{q+1} \lambda_q^{10} \lesssim \theta^{-M-1}_{q+1} m^{\infty}_{q+1} \lambda_q^5,
\]
(2.79)
where the last step is due to \( \theta^{-1}_{q+1} \geq m_{q+1} \geq \lambda_q^5 \), and the implicit constants are independent of \( i \) and \( q \). Thus, it follows from (2.77), (2.78) and (2.79) that
\[
\| \partial_t^M \nabla^N \tilde{R}^B_q \|_{L^\infty_t H^3_q} \lesssim \theta^{-M-1}_{q+1} m^{\infty}_{q+1} \lambda_q^5 \lesssim \theta^{-M-N-1}_{q+1} \lambda_q^5.
\]
(2.80)
Similar arguments also yield the same upper bound of the magnetic stress \( \tilde{R}^B_q \). Thus, (1.20) is verified.

Finally, we are left to prove the remaining well-preparedness of \( (\tilde{u}_q, \tilde{B}_q, \tilde{R}^u_q, \tilde{R}^B_q) \), (1.21) and (1.22), which can be proved by using analogous arguments in [50]. For the reader’s convenience, we sketch the main arguments below.

Let
\[
\mathcal{C} := \left\{ i \in \mathbb{Z} : 1 \leq i \leq m_{q+1} - 1 \text{ and } (\tilde{R}^u_q, \tilde{R}^B_q) \neq 0 \text{ on } [t_{i-1}, t_i + \theta_{q+1}] \cap [0,T] \right\},
\]
(8.21)
and
\[
I_{q+1} := \bigcup_{i \in \mathcal{C}} [t_i - 2\theta_{q+1}, t_i + 3\theta_{q+1}].
\]
(8.22)
Such constructions guarantee that, for any \( q \geq 0 \),
\[
I_{q+1} \subseteq I_q.
\]
(8.23)
In order to prove (1.21) when \( \text{dist}(t, I_{q+1}) \leq \theta_{q+1}/2 \), we see that if \( t \in I_{q+1} \), then by (8.22), we have for some \( i \in \mathcal{C} \),
\[
t \in [t_i - 2\theta_{q+1}, t_i + \theta_{q+1} / 2] \quad \text{or} \quad t \in [t_i + \theta_{q+1} / 2, t_i + 3\theta_{q+1}].
\]
(8.24)
But in both cases, \( \partial_t \chi_j \) and \( 1 - \chi_j \) vanish, \( j = i - 1 \) or \( i \). In view of (2.66), (1.21) then follows.

Moreover, if \( t \in I_{q+1} \), then \( t \in [t_j, t_{q+1}] \) for some \( 0 \leq j \leq m_{q+1} - 1 \). If \( t \in [t_j + \theta_{q+1}, t_j + 1] \), reasoning as above yields (1.21). If \( t \in [t_j, t_j + \theta_{q+1}] \), then \( j \notin \mathcal{C} \), otherwise \( t \in [t_j - 2\theta_{q+1}, t_j + 3\theta_{q+1}] \subseteq I_{q+1} \). Hence, by the definition of \( \mathcal{C} \) in (8.21),
\[
(\tilde{R}^u_q, \tilde{R}^B_q) \equiv 0 \ \text{on} \ [t_{j-1}, t_j + \theta_{q+1}],
\]
and so \( (u_q, B_q) \) solves equations (1.5) on \( [t_{j-1}, t_j + \theta_{q+1}] \). But since \( (v_{j-1}, D_{j-1}) \) also solves (1.5) with the same condition \( (u_q(t_{j-1}), B_q(t_{j-1})) \) at \( t_{j-1} \), so does \( (v_j, D_j) \) on \( [t_j, t_j + \theta_{q+1}] \). Thus, in the overlapped regime \( [t_j, t_j + \theta_{q+1}] \), the uniqueness in Proposition 2.1 then yields
\[
(v_{j-1}, D_{j-1}) = (v_j, D_j) = (u_q, B_q) \ \text{on} \ [t_j, t_j + \theta_{q+1}].
\]
Plugging this into (2.66) with \( j \) replacing \( i \) we thus obtain (1.21).

Regarding (1.22), we take any \( t \in [0, T] \) such that \((\tilde{u}_q(t), \tilde{B}_q(t)) \neq 0\). Then, \( t \in [t_i, t_{i+1}] \) for some \( 0 \leq i \leq m_{q+1}-1 \). If \( t \in [t_i, t_{i+1}] \), then we have \((u_q(t), B_q(t)) \neq 0\), otherwise, by (2.40), \((\tilde{u}_q(t), \tilde{B}_q(t)) = (v_i(t), H_i(t)) = 0\). Since \(|t-t_i| \leq T/m_{q+1}\), we obtain (1.22). Moreover, if \( t \in [t_i, t_i + \theta_{q+1}] \), we have \((u_q(t), B_q(t)) \neq 0\) or \((u_q(t_{i-1}), B_q(t_{i-1})) \neq 0\). Otherwise by the uniqueness of (2.40), \((v_i, D_i) = (v_{i-1}, D_{i-1}) = 0\), which by (2.64) leads to \((\tilde{u}_q(t), \tilde{B}_q(t)) = 0\), a contradiction. Hence, we get \(|t-t_i| \leq T/m_{q+1}\) and \(|t-t_{i-1}| \leq 2(T/m_{q+1})\), and so (1.22) follows.

Therefore, the proof of Theorem 1.6 is complete. \( \square \)

3. Velocity and Magnetic Perturbations

We start to treat the supercritical regime \( \mathcal{S}_1 \), whose borderline in particular includes the endpoint \((s, \gamma, p) = (2\alpha/\gamma + 1 - 2\alpha, \gamma, \infty)\).

The aim of this section is to first construct suitable velocity and magnetic perturbations and then to verify the inductive estimates (1.23), (1.24), (1.28)-(1.30) for the new velocity and magnetic fields at level \( q + 1 \). For this purpose, let us first introduce the appropriate intermittent velocity and magnetic flows adapted to the supercritical regime \( \mathcal{S}_1 \), which in particular feature both the spatial and temporal intermittency.

3.1. Intermittent velocity and magnetic flows. The intermittent flows mainly contain the spatial building blocks and the temporal building blocks, indexed by four parameters \( r_\perp, \lambda, \tau \) and \( \sigma \):

\[
\begin{align*}
    r_\perp &:= \lambda_{q+1}^2 - 2\alpha - 10\varepsilon, \\
    \lambda &:= \lambda_{q+1}, \\
    \tau &:= \lambda_{q+1}^{2\alpha}, \\
    \sigma &:= \lambda_{q+1}^{2\alpha},
\end{align*}
\]

where \( \varepsilon \) is given by (1.14).

- Spatial building blocks. The appropriate spatial building blocks in the supercritical regime \( \mathcal{S}_1 \) would be the intermittent shear flows in [3]. In the following we mainly recall from [3] their constructions and properties.

Let \( \Phi : \mathbb{R} \to \mathbb{R} \) be a smooth cut-off function supported on the interval \([-1, 1]\) and normalize \( \Phi \) such that \( \phi := -\frac{d^2}{dx^2}\Phi \) satisfies

\[
\frac{1}{2\pi} \int_{\mathbb{R}} \phi^2(x) dx = 1.
\]

The corresponding rescaled cut-off functions are defined by

\[
\phi_{r_\perp}(x) := \frac{1}{r_\perp} \phi\left(\frac{x}{r_\perp}\right), \quad \Phi_{r_\perp}(x) := r_\perp^2 \phi\left(\frac{x}{r_\perp}\right).
\]

Note that, \( \phi_{r_\perp} \) is supported in the ball of radius \( r_\perp \), in \( \mathbb{R} \). By an abuse of notation, we periodize \( \phi_{r_\perp} \) and \( \Phi_{r_\perp} \), so that they are treated as periodic functions defined on \( \mathbb{T} \).

The intermittent velocity shear flows in [3] are defined by

\[
W_{(k)} := \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x) k_1, \quad k \in \Lambda_u \cup \Lambda_B,
\]

and the intermittent magnetic shear flows are defined by

\[
D_{(k)} := \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x) k_2, \quad k \in \Lambda_B.
\]

Here, \( N_A \) is given by (A.3), \( (k, k_1, k_2) \) are the orthonormal bases in \( \mathbb{R}^3 \) in Geometric Lemmas A.1 and A.2, \( \Lambda_u, \Lambda_B \) are the wave vector sets of finite cardinality, and the parameter \( r_\perp \) parameterizes the concentration of the flows. In particular, \( \{W_{(k)}, D_{(k)}\} \) are \((\mathbb{T}/(\lambda r_\perp))^{3}\)-periodic, supported on thin plane with thickness \( \sim 1/\lambda \) in each periodic domain. See [3].

For brevity of notations, we set

\[
\phi_{(k)}(x) := \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x), \quad \Phi_{(k)}(x) := \Phi_{r_\perp}(\lambda r_\perp N_A k \cdot x),
\]

and rewrite

\[
W_{(k)} = \phi_{(k)} k_1, \quad k \in \Lambda_u \cup \Lambda_B, \quad (3.4a)
\]

\[
D_{(k)} = \phi_{(k)} k_2, \quad k \in \Lambda_B. \quad (3.4b)
\]
Then, $W(k)$ and $D(k)$ are mean zero on $T^3$. Moreover, the corresponding potentials are defined by

$$W^c(k) := \frac{1}{\lambda^2 N_A^2} \Phi(k_1), \quad k \in \Lambda_u \cup \Lambda_B, \tag{3.5a}$$

$$D^c(k) := \frac{1}{\lambda^2 N_A^2} \Phi(k_2), \quad k \in \Lambda_B. \tag{3.5b}$$

The intermittent flows $W(k)$ and $D(k)$ will be used in the construction of the crucial principal parts of the perturbations $w^{(p)}_{q+1}$ and $d^{(p)}_{q+1}$ in (3.29a) and (3.29b) below, and the corresponding potentials will be used in the construction of the incompressible correctors $w^{(c)}_{q+1}$ and $d^{(c)}_{q+1}$ in (3.32a) and (3.32b).

Lemma 3.1 below contains their crucial analytic estimates, which in particular provide 1D spatial intermittency.

**Lemma 3.1 ([3] Estimates of intermittent shear flows).** For any $p \in [1, \infty]$ and $N \in \mathbb{N}$, we have

$$\|\nabla^N \phi(k)\|_{L^p_\tau} + \|\nabla^N \Phi(k)\|_{L^p_\tau} \lesssim r_\perp^{-\frac{1}{2}} \lambda^N. \tag{3.6}$$

In particular,

$$\|\nabla^N W(k)\|_{C_1 L^p_\tau} + \lambda^2 \|\nabla^N W^c\|_{C_1 L^p_\tau} \lesssim r_\perp^{-\frac{1}{2}} \lambda^N, \quad k \in \Lambda_u \cup \Lambda_B, \tag{3.7}$$

$$\|\nabla^N D(k)\|_{C_1 L^p_\tau} + \lambda^2 \|\nabla^N D^c\|_{C_1 L^p_\tau} \lesssim r_\perp^{-\frac{1}{2}} \lambda^N, \quad k \in \Lambda_B. \tag{3.8}$$

Moreover, for every $k \neq k' \in \Lambda_u \cup \Lambda_B$, $N \in \mathbb{N}$ and $p \in [1, \infty]$, the following product estimate holds

$$\|\nabla^N (\phi(k)\phi(k'))\|_{C_1 L^p_\tau} \lesssim \lambda^N r_\perp^{-1}. \tag{3.9}$$

The implicit constants above are independent of the parameters $r_\perp$ and $\lambda$.

**Temporal building blocks.** The intermittent shear flows provide 1D spatial intermittency and so permit to control the velocity and resistivity ($-\Delta^\alpha$ with $\alpha \in [0, 1/2]$). In order to control the stronger velocity and resistivity, particularly, when $\alpha$ is beyond the Lions exponent $5/4$, it is crucial to introduce the temporal intermittency in the building blocks.

We adopt the notations as in [18, 19, 49]. Let $g \in C^\infty_c([0, T])$ be any cut-off function such that

$$\int_0^T g^2(t) dt = 1,$$ 

and rescale the cut-off function $g$ by

$$g_{\tau}(t) := \tau^{-\gamma} g(\tau t), \tag{3.10}$$

where $\tau \in \mathbb{N}_+$. By an abuse of notation, we periodize $g_{\tau}$ such that it is treated as a periodic function defined on $[0, T]$. Moreover, we define $h_{\tau} : [0, T] \to \mathbb{R}$ by

$$h_{\tau}(t) := \int_0^t (g^2_{\tau}(s) - 1) \, ds, \tag{3.11}$$

and set

$$g(\tau) := g_{\tau}(\tau t), \quad h(\tau)(t) := h_{\tau}(\tau t). \tag{3.12}$$

The function $h(\tau)$ will be used later in the construction of the temporal correctors $w^{(c)}_{q+1}$ and $d^{(c)}_{q+1}$ (see (3.34a)-(3.34b)), which permit to balance the high temporal oscillations caused by $g_{\tau}$.

We have the following estimates of the temporal building blocks.

**Lemma 3.2 ([49] Estimates of temporal intermittency).** For $\gamma \in [1, +\infty]$, $M \in \mathbb{N}$, we have

$$\|\partial^M g(\tau)\|_{L^\gamma_\tau} \lesssim \sigma^M \tau^{M+\frac{1}{2} - \frac{1}{\gamma}}, \tag{3.13}$$

where the implicit constants are independent of $\tau$ and $\sigma$. Moreover, we have

$$\|h(\tau)\|_{L^\infty_\tau} \leq 1. \tag{3.14}$$

In the next §3.2, we define the amplitudes of the velocity and magnetic flows adapted to the geometry of MHD equations.
3.2. Amplitudes of velocity and magnetic perturbations. The amplitudes of the magnetic and velocity perturbations are constructed mainly to decrease the effects of the old Reynolds and magnetic stresses, based on Geometric Lemmas A.1 and A.2, so that the desirable \( L^1 L^\infty \)-decay property (1.26) can be achieved for the new stresses at level \( q + 1 \).

- **The magnetic amplitudes.** Let \( \chi : [0, +\infty) \to \mathbb{R} \) be a smooth cut-off function such that
  \[
  \chi(z) = \begin{cases} 
    1, & 0 \leq z \leq 1, \\
    z, & z \geq 2,
  \end{cases}
  \quad (3.15)
\]
  and
  \[
  \frac{1}{2} z \leq \chi(z) \leq 2z \quad \text{for} \quad z \in (1, 2).
  \quad (3.16)
\]
  Set
  \[
  \varrho_B(t, x) := 2\varepsilon_B^{-1} \lambda_q^{-2} \delta_{q+1} \chi \left( \frac{\tilde{R}^B_q(t, x)}{\lambda_q \delta_{q+1}} \right),
  \quad (3.17)
\]
  where \( \varepsilon_B \) is the small radius in Geometric Lemma A.2, and \( \tilde{R}^B_q \) is the well-prepared magnetic stress in the last section. Then, one has
  \[
  \left| \frac{\tilde{R}^B_q}{\varrho_B} \right| = \left| \frac{\tilde{R}^B_q}{2\varepsilon_B^{-1} \lambda_q^{-2} \delta_{q+1} \chi \left( \frac{\tilde{R}^B_q(t, x)}{\lambda_q \delta_{q+1}} \right)} \right| \leq \varepsilon_B,
  \quad (3.18)
\]
  Moreover, choose the smooth temporal cut-off function \( f_B \), adapted to the support of \( \tilde{R}^B_q \), such that
  - \( 0 \leq f_B \leq 1 \) and \( f_B \equiv 1 \) on \( \text{supp} \tilde{R}^B_q \);
  - \( \text{supp}_t f_B \subseteq N_{\theta_{q+1}/2}(\text{supp} \tilde{R}^B_q) \);
  - \( \|f_B\|_{C^9} \lesssim \theta_{q+1}^{-N}, \quad 1 \leq N \leq 9 \).

  Then, the amplitudes of the magnetic perturbations are defined by
  \[
  a_{(k)}(t, x) := a_{k, B}(t, x) = \varrho_B^*(t, x) f_B(t) \gamma_{(k)} \left( \frac{-\tilde{R}^B_q(t, x)}{\varrho_B(t, x)} \right), \quad k \in \Lambda_B,
  \quad (3.19)
\]
  where \( \gamma_{(k)} \) is the smooth function in Geometric Lemma A.2.

  The important algebraic and analytic properties of the amplitudes \( \{a_{(k)}, k \in \Lambda_B\} \) are summarized in Lemma 3.3 below. The proof of (3.21) below is similar to that in [49] with slight modifications and replacing \( \ell \) by \( \theta_{q+1} \), we omit the details here.

**Lemma 3.3 (Magnetic amplitudes).** We have that
  \[
  \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\gamma)}^2(D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)})
  = -\tilde{R}^B_q + \sum_{k \in \Lambda_B} a_{(k)}^2 g_{(\gamma)}^2 \mathbb{P}_{\neq 0} (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)})
  + \sum_{k \in \Lambda_B} a_{(k)}^2 (g_{(\gamma)}^2 - 1) \int_{T^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} dx,
  \quad (3.20)
\]
  where \( \mathbb{P}_{\neq 0} \) denotes the spatial projection onto nonzero Fourier modes. Moreover, for \( 1 \leq N \leq 9, k \in \Lambda_B \),
  \[
  \|a_{(k)}\|_{L^4_{t,x}} \lesssim \theta_{q+1}^{\frac{N}{2}}, \quad \|a_{(k)}\|_{C^{1}_{t,x}} \lesssim \theta_{q+1}^{-1}, \quad \|a_{(k)}\|_{C^{N}_{t,x}} \lesssim \theta_{q+1}^{-7N}.
  \quad (3.21)
\]

- **The velocity amplitudes.** In contrast with the NSE, because of the strong coupling between the velocity and magnetic fields, the construction of the velocity amplitudes would reply on the magnetic shear flows and wave vectors. Moreover, we also would need an additional matrix \( \tilde{G}^B \) defined by
  \[
  \tilde{G}^B := \sum_{k \in \Lambda_B} a_{(k)}^2 \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx.
  \quad (3.22)
\]
It holds that, for \( N \geq 1 \),
\[
\| \dot{G}^B \|_{\mathcal{L}_{t,x}} \lesssim \theta_{q+1}^{-2}, \quad \| \dot{G}^B \|_{C_{t,x}^N} \lesssim \varepsilon^{-7N-1}, \quad \| \dot{G}^B \|_{L_{t,x}^1} \lesssim \delta_{q+1}.
\]  
(3.23)

Set
\[
\varrho_u(t, x) := 2\varepsilon_u^{-1} \delta_{q+1}^2 \left( \frac{\tilde{R}^B_u(t, x) + \dot{G}^B(t, x)}{\delta_{q+1}} \right).
\]
(3.24)

By (3.15) and (3.16), it holds that
\[
\left| \frac{\tilde{R}^B_u + \dot{G}^B}{\varrho_u} \right| \leq \varepsilon_u,
\]
(3.25)

where \( \varepsilon_u \) is the small radius in Geometric Lemma A.1, and \( \tilde{R}^B_u \) is the well-prepared Reynolds stress in the last section. Then, we choose the smooth temporal cut-off function \( f_u \), adapted to the support of \( (\tilde{R}^B_u, \dot{G}^B) \), such that

- \( 0 \leq f_u \leq 1 \), \( f_u \equiv 1 \) on \( \text{supp}(\tilde{R}^B_u, \dot{G}^B) \);
- \( \text{supp}(f_u) \subseteq N_{\theta_{q+1}/2}(\text{supp}((\tilde{R}^B_u, \dot{G}^B))) \);
- \( \| f_u \|_{C_{t,x}^N} \lesssim \theta_{q+1}^{-N}, 1 \leq N \leq 9 \).

The amplitudes of the velocity perturbations are defined by
\[
a^{(k)}(t, x) := a^{(k)} u(t, x) = \frac{1}{\varrho_u} f_u(t) \gamma(k) \left( \text{Id} - \frac{\tilde{R}^B(t, x) + \dot{G}^B(t, x)}{\varrho_u(t, x)} \right), \quad k \in \Lambda_u.
\]
(3.26)

We collect the algebraic and analytic properties of the velocity amplitudes \( \{a^{(k)}, k \in \Lambda_u\} \) in the following lemma.

**Lemma 3.4** ([49] Estimates of velocity amplitudes). We have that
\[
\sum_{k \in \Lambda_u} a^2 (k) g_{(\tau)}^2 W(k) \otimes W(k) = \rho_u f^2 u \text{Id} - \tilde{R}^B_u - \dot{G}^B + \sum_{k \in \Lambda_u} a^2 (k) g_{(\tau)}^2 L_{\neq 0}(W(k) \otimes W(k))
\]
\[
+ \sum_{k \in \Lambda_u} a^2 (k) \left( g_{(\tau)}^2 - 1 \right) \int_{T^3} W(k) \otimes W(k) dx.
\]
(3.27)

Moreover, for \( 1 \leq N \leq 9 \), \( k \in \Lambda_u \), we have
\[
\| a^{(k)} \|_{L^2_{t,x}} \lesssim \delta_{q+1}^2, \quad \| a^{(k)} \|_{C_{t,x}^1} \lesssim \theta_{q+1}^{-1}, \quad \| a^{(k)} \|_{C_{t,x}^N} \lesssim \theta_{q+1}^{-14N}.
\]
(3.28)

### 3.3. Velocity and magnetic perturbations

We are now in stage to define the crucial velocity and magnetic perturbations, mainly consisting of three parts: the principal parts, the incompressible parts and the temporal correctors.

- **Principal parts.** Let us first define the principal parts \( w^{(p)}_{q+1} \) and \( d^{(p)}_{q+1} \), respectively, of the velocity and magnetic perturbations by
\[
w^{(p)}_{q+1} := \sum_{k \in \Lambda_{u \cup \Lambda_B}} a^{(k)} g_{(\tau)} W(k),
\]
(3.29a)
\[
d^{(p)}_{q+1} := \sum_{k \in \Lambda_B} a^{(k)} g_{(\tau)} D(k).
\]
(3.29b)

The key fact here is that, the MHD type nonlinearity of the principal parts would decrease the effects of the old stresses in the convex integration scheme, as shown in the following algebraic identities:
\[
d^{(p)}_{q+1} \otimes w^{(p)}_{q+1} - w^{(p)}_{q+1} \otimes d^{(p)}_{q+1} + \tilde{R}^B
\]
\[
= \sum_{k \in \Lambda_B} a^2 (k) g_{(\tau)}^2 L_{\neq 0}(D(k) \otimes W(k) - W(k) \otimes D(k))
\]
+ \sum_{k \in \Lambda_B} a_k^2 (g_k^2 (\tau) - 1) \int_{T^3} D_k \otimes W_k - W_k \otimes D_k \, dx \\
+ \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_u, k' \in \Lambda_B} \right) a_k a_{k'} g_{k'}^2 (D_{k'} \otimes W_k - W_k \otimes D_{k'}) ,
\tag{3.30}
\end{align}

and
\begin{align*}
w_{q+1}^{(p)} + w_{q+1}^{(c)} - d_{q+1}^{(p)} + \tilde{d}_{q+1}^{(c)} &= \rho u f_t \mathbb{I} + \sum_{k \in \Lambda_u} a_k^2 g_k^2 (\tau) \mathbb{P}_{\not= 0} (W_k \otimes W_k) \\
+ \sum_{k \in \Lambda_B} a_k^2 (g_k^2 (\tau) - 1) \int_{T^3} W_k \otimes W_k \, dx \\
+ \sum_{k \in \Lambda_B} a_k^2 (g_k^2 (\tau) - 1) \int_{T^3} W_k \otimes W_k - D_k \otimes D_k \, dx \\
+ \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} a_k a_{k'} g_{k'}^2 (W_k \otimes W_k - \sum_{k \neq k' \in \Lambda_B} a_k a_{k'} g_{k'}^2 D_k \otimes D_{k'}).
\tag{3.31}
\end{align*}

- **Incompressibility correctors.** The corresponding incompressibility correctors are defined by
\begin{align*}
w_{q+1}^{(c)} := & \sum_{k \in \Lambda_u \cup \Lambda_B} g_k (\nabla a_k \times \text{curl} W_k) + \text{curl} (\nabla a_k \times W_k) ,
\tag{3.32a}
d_{q+1}^{(c)} := \sum_{k \in \Lambda_B} g_k (\nabla a_k \times \text{curl} D_k) + \text{curl} (\nabla a_k \times D_k) .
\tag{3.32b}
\end{align*}

One has that (see [3, (5.28), (5.29)])
\begin{align*}
w_{q+1}^{(p)} + w_{q+1}^{(c)} &= \sum_{k \in \Lambda_u \cup \Lambda_B} \text{curl} \text{curl} (a_k g_k (\tau) W_k) ,
\tag{3.33a}
d_{q+1}^{(p)} + d_{q+1}^{(c)} &= \sum_{k \in \Lambda_B} \text{curl} \text{curl} (a_k g_k (\tau) D_k) ,
\tag{3.33b}
\end{align*}

and so
\[ \text{div} (w_{q+1}^{(p)} + w_{q+1}^{(c)}) = \text{div} (d_{q+1}^{(p)} + d_{q+1}^{(c)}) = 0, \]
thereby justifying the definition of the incompressible correctors.

- **Temporal correctors.** We also need the temporal correctors $w_{q+1}^{(o)}$ and $d_{q+1}^{(o)}$ in order to balance the high temporal frequency oscillations in (3.30) and (3.31):
\begin{align*}
w_{q+1}^{(o)} := & -\sigma^{-1} \sum_{k \in \Lambda_u} \mathbb{P}_H \mathbb{P}_{\not= 0} \left( h(\tau) \int_{T^3} W_k \otimes W_k \, dx \nabla (a_k^2) \right) \\
& - \sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\not= 0} \left( h(\tau) \int_{T^3} W_k \otimes W_k - D_k \otimes D_k \, dx \nabla (a_k^2) \right) ,
\tag{3.34a}
d_{q+1}^{(o)} := -\sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\not= 0} \left( h(\tau) \int_{T^3} D_k \otimes W_k - W_k \otimes D_k \, dx \nabla (a_k^2) \right) .
\tag{3.34b}
\end{align*}

Recall that $h(\tau)$ is given by (3.11).

The effects of these temporal correctors to balance high temporal oscillations are encoded in the following key algebraic identities:
\[ \partial_t w_{q+1}^{(o)} + \sum_{k \in \Lambda_u} \mathbb{P}_{\not= 0} \left( (g_k^2 (\tau) - 1) \int_{T^3} W_k \otimes W_k \, dx \nabla (a_k^2) \right) \]
\[ + \sum_{k \in \Lambda_B} P_{\neq 0} \left( (g_{(\tau)}^2 - 1) \int_{T^3} W_k \otimes W_k - D_k \otimes D_k d\mathcal{N} (a_k^2) \right) \]

\[ = (\nabla \Delta^{-1} \text{div}) \sigma^{-1} \sum_{k \in \Lambda_B} P_{\neq 0} \partial_t \left( h_{(\tau)} \int_{T^3} W_k \otimes W_k d\mathcal{N} (a_k^2) \right) \]

\[ + (\nabla \Delta^{-1} \text{div}) \sigma^{-1} \sum_{k \in \Lambda_B} P_{\neq 0} \partial_t \left( h_{(\tau)} \int_{T^3} W_k \otimes W_k - D_k \otimes D_k d\mathcal{N} (a_k^2) \right) \]

\[ - \sigma^{-1} \sum_{k \in \Lambda_B} P_{\neq 0} \left( h_{(\tau)} \int_{T^3} W_k \otimes W_k - D_k \otimes D_k d\mathcal{N} (a_k^2) \right), \]

and

\[ \partial_t d_{q+1}^{(c)} + \sum_{k \in \Lambda_B} P_{\neq 0} \left( (g_{(\tau)}^2 - 1) \int_{T^3} D_k \otimes W_k - W_k \otimes D_k d\mathcal{N} (a_k^2) \right) \]

\[ = (\nabla \Delta^{-1} \text{div}) \sigma^{-1} \sum_{k \in \Lambda_B} P_{\neq 0} \partial_t \left( h_{(\tau)} \int_{T^3} D_k \otimes W_k - W_k \otimes D_k d\mathcal{N} (a_k^2) \right) \]

\[ - \sigma^{-1} \sum_{k \in \Lambda_B} P_{\neq 0} \left( h_{(\tau)} \int_{T^3} D_k \otimes W_k - W_k \otimes D_k d\mathcal{N} (a_k^2) \right). \]

**Velocity and magnetic perturbations.** Now we are ready to define the velocity and magnetic perturbations \( w_{q+1} \) and \( d_{q+1} \) at level \( q+1 \) by

\[ w_{q+1} := w_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(o)}, \]

\[ d_{q+1} := d_{q+1}^{(p)} + d_{q+1}^{(c)} + w_{q+1}^{(o)}, \]

and the velocity and magnetic fields at level \( q+1 \) by

\[ u_{q+1} := \bar{u}_q + w_{q+1}, \]

\[ B_{q+1} := \bar{B}_q + d_{q+1}, \]

where \( (\bar{u}_q, \bar{B}_q) \) is the well-prepared velocity and magnetic fields in the previous gluing stage in §2. Note that, by the above constructions, \( w_{q+1} \) and \( d_{q+1} \) are mean free and divergence free.

The main estimates of the velocity and magnetic perturbations are summarized in Lemma 3.5 below.

**Lemma 3.5 (Estimates of perturbations).** For any \( \rho \in (1, \infty), \gamma \in [1, \infty] \) and every integer \( 0 \leq N \leq 7 \), we have the following estimates:

\[ \| \nabla^N w_{q+1}^{(p)} \|_{L^N_t L^\gamma_x} + \| \nabla^N d_{q+1}^{(p)} \|_{L^N_t L^\gamma_x} \lesssim \theta_{q+1}^{-1} \lambda^{N-1} \rho_{q+1}^{1/2} \gamma^{1/2}, \]

\[ \| \nabla^N w_{q+1}^{(c)} \|_{L^N_t L^\gamma_x} + \| \nabla^N d_{q+1}^{(c)} \|_{L^N_t L^\gamma_x} \lesssim \theta_{q+1}^{-14} \lambda^{N-1} \rho_{q+1}^{1/2} \gamma^{1/2}, \]

\[ \| \nabla^N w_{q+1}^{(o)} \|_{L^N_t L^\gamma_x} + \| \nabla^N d_{q+1}^{(o)} \|_{L^N_t L^\gamma_x} \lesssim \theta_{q+1}^{-14} \lambda^{N-1} \gamma^{1/2 - 1/2}, \]

where the implicit constants depend only on \( N, \gamma \) and \( \rho \). In particular, for integers \( 1 \leq N \leq 7 \),

\[ \| w_{q+1}^{(p)} \|_{L^N_t H^N_x} + \| w_{q+1}^{(c)} \|_{L^N_t H^N_x} + \| w_{q+1}^{(o)} \|_{L^N_t H^N_x} \lesssim \lambda^{N+2}, \]

\[ \| d_{q+1}^{(p)} \|_{L^N_t H^N_x} + \| d_{q+1}^{(c)} \|_{L^N_t H^N_x} + \| d_{q+1}^{(o)} \|_{L^N_t H^N_x} \lesssim \lambda^{N+2}. \]

Moreover, for the temporal derivatives, we have, for \( 1 \leq N \leq 7 \),

\[ \| \partial_t w_{q+1}^{(p)} \|_{L^N_t H^N_x} + \| \partial_t w_{q+1}^{(c)} \|_{L^N_t H^N_x} + \| \partial_t w_{q+1}^{(o)} \|_{L^N_t H^N_x} \lesssim \lambda^{N+5}, \]

\[ \| \partial_t d_{q+1}^{(p)} \|_{L^N_t H^N_x} + \| \partial_t d_{q+1}^{(c)} \|_{L^N_t H^N_x} + \| \partial_t d_{q+1}^{(o)} \|_{L^N_t H^N_x} \lesssim \lambda^{N+5}, \]

where the implicit constants are independent of \( \lambda \).
Proof. First, using (3.7), (3.13), (3.29a), (3.29b) and Lemmas 3.3 and 3.4 we get, for any \( \rho \in (1, \infty) \), that
\[
\|\nabla^N u_{q+1}^{(p)}\|_{L_t^\infty L_x^\infty} + \|\nabla^N d_{q+1}^{(c)}\|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \sum_{N_1+N_2=N} \|a(k)\|_{C_{t,x}^{N_1+1}} \|g(\tau)\|_{L_t^\infty} \|\nabla^2 W_k\|_{C_{t,x}^1} + \sum_{k \in \Lambda_B} \sum_{N_1+N_2=N} \|a(k)\|_{C_{t,x}^{N_1+1}} \|g(\tau)\|_{L_t^\infty} \|\nabla^2 D_k\|_{C_{t,x}^1}
\lesssim \theta_{q+1}^{-1} \lambda^N r_{1+\frac{1}{2}} \tau^\frac{1}{2} - \frac{1}{4},
\]
where we also used \( \theta_{q+1}^{-1} \ll \lambda \) in the last step, due to (1.13) and (1.16). Thus, (3.39) follows.

Similarly, we deduce
\[
\|\nabla^N u_{q+1}^{(o)}\|_{L_t^\infty L_x^\infty} + \|\nabla^N d_{q+1}^{(o)}\|_{L_t^\infty L_x^\infty} \\
\lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \sum_{N_1+N_2=N} \|g(\tau)\|_{L_t^\infty} \sum_{N_1+N_2=N} \left( \|a(k)\|_{C_{t,x}^{N_1+1}} \|\nabla^2 W_k^c\|_{C_{t,x}^1} + \|a(k)\|_{C_{t,x}^{N_1+1}} \|\nabla^2 W_k^c\|_{C_{t,x}^1} \right)
\lesssim \theta_{q+1}^{-14} \lambda^{N-1} r_{1+\frac{1}{2}} \tau^\frac{1}{2} - \frac{1}{4},
\]
which yields (3.40).

For the temporal correctors, using (3.34), (3.14) and Lemmas 3.3 and 3.4, we obtain
\[
\|\nabla^N u_{q+1}^{(o)}\|_{L_t^\infty L_x^\infty} + \|\nabla^N d_{q+1}^{(o)}\|_{L_t^\infty L_x^\infty} \lesssim \sigma^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \|h(\tau)\|_{C_{t,x}^1} \|\nabla^{N+1} (a^2_k)\|_{C_{t,x}^1} \lesssim \theta_{q+1}^{-14} \lambda^{N-16} \sigma^{-1},
\]
which yields (3.41).

The \( L_t^{\infty} H_x^N \)-estimates (3.42) and (3.43) then follow directly from estimates (3.39)-(3.41) with \( \gamma = \infty \) and \( \rho = 2 \), and the choice of parameters in (1.13), (1.16) and (3.1).

Concerning estimate (3.44), by virtue of (3.1) and Lemmas 3.1-3.4, we get
\[
\|\partial_t u_{q+1}^{(p)}\|_{L_t^{\infty} H_x^N} \lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \|a(k)\|_{C_{t,x}^{N_1+1}} \|g(\tau)\|_{W^{1,\infty}} \|W_k\|_{L_t^{\infty} H_x^N} \lesssim \theta_{q+1}^{-14} \lambda^{N-14} \sigma^\frac{3}{2}
\]
and
\[
\|\partial_t u_{q+1}^{(c)}\|_{L_t^{\infty} H_x^N} \lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \|a(k)\|_{C_{t,x}^{N_1+1}} \|g(\tau)\|_{W^{1,\infty}} \left( \|W_k\|_{L_t^{\infty} H_x^N} + \|\nabla W_k\|_{L_t^{\infty} H_x^N} \right)
\lesssim \theta_{q+1}^{-14} \lambda^{N-14} \sigma^\frac{3}{2}.
\]
Since \( \mathbb{P}_x \mathbb{P}_{x \neq 0} \) is bounded in \( H_x^N \), and \( \partial_t h(\tau) = \sigma (g^2(\tau) - 1) \), we deduce
\[
\|\partial_t u_{q+1}^{(o)}\|_{L_t^{\infty} H_x^N} \lesssim \sigma^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \|\partial_t (h(\tau) (a_k^2))\|_{L_t^{\infty} H_x^N} \lesssim \theta_{q+1}^{-14} \sigma^{-1} \tau.
\]
Thus, taking into account (3.1), \( \theta_{q+1}^{-14} \leq \lambda^2 \) for \( 1 \leq N \leq 7 \), \( 0 < \varepsilon \leq 1/20 \) and \( \alpha < 3/2 \), we conclude that
\[
\|\partial_t u_{q+1}^{(p)}\|_{L_t^{\infty} H_x^N} + \|\partial_t u_{q+1}^{(c)}\|_{L_t^{\infty} H_x^N} + \|\partial_t u_{q+1}^{(o)}\|_{L_t^{\infty} H_x^N} \\
\lesssim \theta_{q+1}^{-14} \lambda^{N-14} \sigma^\frac{3}{2} + \theta_{q+1}^{-14} \lambda^{N-1} \sigma^\frac{3}{2} + \theta_{q+1}^{-14} \lambda^{N-16} \sigma^\frac{3}{2}
\lesssim \theta_{q+1}^{-14} \lambda^{N-14} \lambda^{3N+N} + \theta_{q+1}^{-14} \lambda^{N-14} \lambda^{3N+N-1} + \theta_{q+1}^{-14} \lambda^{N-16} \lambda^2 \lesssim \lambda^{N+5},
\]
thereby yielding (3.44). Analogous arguments also apply to the temporal derivatives of the magnetic perturbations and give estimate (3.45). Therefore, the proof of Lemma 3.5 is complete. \( \square \)
3.4. Verification of inductive estimates for velocity and magnetic fields. We are now in stage to verify the inductive estimates (1.23), (1.24), (1.28)-(1.30) for the velocity and magnetic fields.

First, in view of (1.23), (2.5), (3.38) and (3.42), for $0 \leq N \leq 4$ we have that,

$$
\| (u_{q+1}, B_{q+1}) \|_{L^\infty_t H^N_x} \lesssim \| (\tilde{u}_{q}, \tilde{B}_{q}) \|_{L^\infty_t H^N_x} + \| (w_{q+1}, d_{q+1}) \|_{L^\infty_t H^N_x} \\
\lesssim \sup_i \| (\tilde{v}_i, H_i) \|_{L^\infty_t (\text{supp}(\chi_i); H^N_x)} + \lambda_{q+1}^{N+5} \\
\lesssim m_{q+1}^{N} \| (u_{q}, B_{q}) \|_{H^N_x} + \lambda_{q+1}^{N+5} \\
\lesssim m_{q+1}^{N} \lambda_{q}^5 + \lambda_{q+1}^{N+5} \lesssim \lambda_{q+1}^{N+5},
$$

(3.50)

where we also used $m_{q+1}^{N} \lambda_{q}^5 \lesssim \lambda_{q+1}^{N+6}$, due to (1.12), (1.13) and (1.16). Similarly,

$$
\| (\partial_t u_{q+1}, \partial_t B_{q+1}) \|_{L^\infty_t H^N_x} \\
\lesssim \| (\partial_t \tilde{u}_{q}, \partial_t \tilde{B}_{q}) \|_{L^\infty_t H^N_x} + \| (\partial_t w_{q+1}, \partial_t d_{q+1}) \|_{L^\infty_t H^N_x} \\
\lesssim \sup_i \| (\nabla (\chi_i v_i), \nabla (\chi_i H_i)) \|_{L^\infty_t (\text{supp}(\chi_i); H^N_x)} + \lambda_{q+1}^{N+5} \\
\lesssim \sup_i \| (\partial_t \chi_i v_i, \partial_t \chi_i H_i) \|_{L^\infty_t (\text{supp}(\chi_i); H^N_x)} + \| \chi_i |\partial_t v_i, \partial_t H_i) \|_{L^\infty_t (\text{supp}(\chi_i); H^N_x)} + \lambda_{q+1}^{N+5} \\
\lesssim m_{q+1}^{N} \| (u_{q}, B_{q}) \|_{L^\infty_t H^N_x} + m_{q+1}^{N+1} \| (u_{q}, B_{q}) \|_{L^\infty_t H^N_x} + \lambda_{q+1}^{N+5} \\
\lesssim m_{q+1}^{N} \lambda_{q}^5 + \lambda_{q+1}^{N+5} \lesssim \lambda_{q+1}^{N+5}.
$$

(3.51)

Hence, estimates (1.23) and (1.24) are verified at level $q+1$.

Next, we consider estimates (1.28) and (1.29). It should be mentioned that, the derivation towards $L^2_t L^2_x$-decay estimates of principal parts requires to exploit the decoupling between the high and low frequency parts. To be precise, we apply the $L^p$ decorrelation Lemma A.3 with $f = a(k)$, $g = g(\tau)\phi(\kappa)$ and $\sigma = \lambda^2 \zeta$ and then using (1.12), (1.13) and Lemmas 3.1-3.4 to derive

$$
\| (w^{(p)}_{q+1}, d^{(p)}_{q+1}) \|_{L^2_t L^2_x} \lesssim \sum_{k \in \Lambda_{A \cup \Lambda_B}} \left( \| a(k) \|_{L^2_x} \| g(\tau) \|_{L^2_x} \| \phi(k) \|_{C_t L^2_x} + \sigma^{-\frac{1}{2}} \| a(k) \|_{C^1_{t,x}} \| g(\tau) \|_{L^2_x} \| \phi(k) \|_{C_t L^2_x} \right) \\
\lesssim \delta_{q+1}^{\frac{1}{2}} + \theta_{q+1}^{-14} \lambda_{q+1}^{-1} + \theta_{q+1}^{-16} \sigma^{-1} \lesssim \delta_{q+1}^{\frac{1}{2}},
$$

(3.52)

Taking into account (3.40) and (3.41), we obtain

$$
\| (w_{q+1}, d_{q+1}) \|_{L^2_t L^2_x} \lesssim \delta_{q+1}^{\frac{1}{2}} + \theta_{q+1}^{-14} \lambda_{q+1}^{-1} + \theta_{q+1}^{-16} \sigma^{-1} \lesssim \delta_{q+1}^{\frac{1}{2}},
$$

(3.53)

which, along with (1.18), yields

$$
\| (w_{q+1} - u_{q}, B_{q+1} - B_{q}) \|_{L^2_t L^2_x} \lesssim \| (\tilde{u}_q - u_{q}, \tilde{B}_q - B_{q}) \|_{L^2_t L^2_x} + \| (w_{q+1}, d_{q+1}) \|_{L^2_t L^2_x} \\
\lesssim \lambda_{q+1}^{-3} + \delta_{q+1}^{\frac{1}{2}} \lesssim M^* \delta_{q+1}^{\frac{1}{2}}
$$

(3.54)

for $M^*$ sufficiently large. This verifies the $L^2_t L^2_x$-decay estimate (1.28) at level $q+1$.

The $L^1_t L^2_x$-estimate (1.29) can be verified easier as follows: by Lemma 3.5 and (3.1),

$$
\| (w_{q+1}, d_{q+1}) \|_{L^1_t L^2_x} \lesssim \| (w^{(c)}_{q+1}, d^{(c)}_{q+1}) \|_{L^1_t L^2_x} + \| (w^{(o)}_{q+1}, d^{(o)}_{q+1}) \|_{L^1_t L^2_x} \\
\lesssim \theta_{q+1}^{-1} \lambda_{q+1}^{-\frac{1}{2}} + \theta_{q+1}^{-14} \lambda_{q+1}^{-1} \tau^{-\frac{1}{2}} + \theta_{q+1}^{-16} \sigma^{-1} \lesssim \lambda_{q+1}^{-\varepsilon},
$$

(3.55)

which along with (1.18) yields that

$$
\| (u_{q+1} - u_{q}, B_{q+1} - B_{q}) \|_{L^1_t L^2_x} \lesssim \| (\tilde{u}_q - u_{q}, \tilde{B}_q - B_{q}) \|_{L^1_t L^2_x} + \| (w_{q+1}, d_{q+1}) \|_{L^1_t L^2_x} \\
\lesssim \lambda_{q+1}^{-3} + \lambda_{q+1}^{-\varepsilon} \lesssim \delta_{q+2}^{\frac{1}{2}}.
$$

(3.56)

Thus, the $L^1_t L^2_x$-estimate (1.29) is verified at level $q+1$.

At last, concerning estimate (1.30), since $(s, \gamma, p) \in S_1$, one has the embedding (see [50, (6.32)])

$$
H^3_x \hookrightarrow W^{s,p}_x.
$$

(3.57)
By virtue of (1.25) and (2.43), we have

\[
\left\| (\bar{u}_q - u_q, \bar{B}_q - B_q) \right\|_{L^\gamma_t W^{s,p}_x} \lesssim \left\| \sum_i \chi_i (v_i - u_q), \sum_i \chi_i (H_i - B_q) \right\|_{L^\gamma_t H^{s}_x}
\]

\[
\lesssim \sup_i \int_t^{t+1} \left\| \left( \left| \nabla \bar{R}_q^a(s), |\nabla | \bar{R}_q^B(s) \right| \right) \right\|_{H^{s}_x} ds
\]

\[
\lesssim m_q^{s+1} \left\| (\left| \nabla | \bar{R}_q^a(s), |\nabla | \bar{R}_q^B(s) \right| \right) \right\|_{H^{s}_x}
\]

\[
\lesssim m_q^{s+1} \lambda_0^s \lesssim \lambda_q^{-2},
\]

where \( s_{i+1} = t_{i+1} + \theta_q \). Taking into account (1.13), (3.1), (3.57) and Lemma 3.5, we derive

\[
\left\| (u_{q+1} - u_q, B_{q+1} - B_q) \right\|_{L^\gamma_t W^{s,p}_x} \lesssim \left\| (\bar{u}_q - u_q, \bar{B}_q - B_q) \right\|_{L^\gamma_t W^{s,p}_x} + \left\| (w_{q+1}, d_{q+1}) \right\|_{L^\gamma_t W^{s,p}_x}
\]

\[
\lesssim \lambda_q^{-2} + \lambda_q^{-1} \lambda_q^{s+1} \lambda_q^{s+1} \lesssim \lambda_q^{-2} + \lambda_q^{-10} \lesssim \lambda_q^{-2},
\]

\[
\lesssim \lambda_q^{-2} + \lambda_q^{-10} \lesssim \lambda_q^{-2},
\]

(3.58)

Since by (1.14),

\[
s + 2\alpha - 1 - \frac{2\alpha}{\gamma} - \frac{2\alpha}{p} + \varepsilon (6 - \frac{16}{p}) < -10\varepsilon,
\]

we thus obtain

\[
\left\| (u_{q+1} - u_q, B_{q+1} - B_q) \right\|_{L^\gamma_t W^{s,p}_x} \leq \lambda_q^{s+2},
\]

which verifies the \( L^\gamma_t W^{s,p}_x \)-estimate (1.30) at level \( q + 1 \).

4. ROYDENS AND MAGNETIC STRESSES

In this section we aim to choose suitable velocity and magnetic stresses in the new relaxation system (1.11) at level \( q + 1 \) and to verify the inductive estimates (1.25) and (1.26) in the supercritical regime \( S_1 \).

4.1. Decomposition of magnetic and Reynolds stresses. Let us first consider the magnetic stress. Using (1.11) with \( q + 1 \) replacing \( q \), (3.37) and (3.38) we derive the equation for the magnetic stress:

\[
\text{div} \bar{R}_q^{\gamma+1} = \partial_t (d_{q+1}^{(p)} + d_{q+1}^{(c)}) + \nu_2 (-\Delta)^\alpha d_{q+1} + \text{div} (d_{q+1} \otimes \bar{u}_q - \bar{u}_q \otimes d_{q+1} + \bar{B}_q \otimes w_{q+1} - w_{q+1} \otimes \bar{B}_q)
\]

\[
\text{div} \bar{R}_q^{\gamma+1} + \text{div} (\bar{R}_q^{\gamma+1} - \bar{R}_q^{\gamma+1} + \bar{R}_q^{\gamma+1})
\]

\[
+ \text{div} \left( d_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(o)}) = (w_{q+1}^{(c)} + w_{q+1}^{(o)}) \otimes d_{q+1} + d_{q+1}^{(c)} + d_{q+1}^{(o)} \right),
\]

(4.1)

Based on this fact, we define the new magnetic stress by

\[
\bar{R}_q^{\gamma+1} := \bar{R}_q^{\gamma+1} + \bar{R}_q^{\gamma+1} + \bar{R}_q^{\gamma+1},
\]

(4.2)

where \( \bar{R}^{\gamma+1} \) is the inverse divergence operator given by (A.5b) in the Appendix, the linear error

\[
\hat{R}_q^{\gamma+1} := \bar{R}_q^{\gamma+1} \left( \partial_t (d_{q+1}^{(p)} + d_{q+1}^{(c)}) + \nu_2 (-\Delta)^\alpha d_{q+1}
\]

\[
+ \bar{R}_q^{\gamma+1} \bar{P}_H \text{div} \left( d_{q+1} \otimes \bar{u}_q - \bar{u}_q \otimes d_{q+1} + \bar{B}_q \otimes w_{q+1} - w_{q+1} \otimes \bar{B}_q \right),
\]

(4.3)

the oscillation error

\[
\bar{R}_q^{\gamma+1} := \sum_{k \in \Lambda_k} \bar{R}_q^{\gamma+1} \bar{P}_H \bar{P}_{k} \left( \bar{g}_{(r)}^{(s)} (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \nabla (u_{k}^2) \right)
\]

(4.4)
\[-\sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u \mathbb{P} H \mathbb{P} \neq 0 \left( h(\tau) \int_{\mathbb{T}^3} D(k) \otimes W(k) - W(k) \otimes D(k) d\mathcal{H} \right) \]
\[+ \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_n, k' \in \Lambda_B} \right) \mathcal{R}^u \mathbb{P} H \text{div} \left( a(k) a(k') g(\tau)^2 (D(k) \otimes W(k) - W(k) \otimes D(k')) \right), \quad (4.4)\]

and the corrector error
\[\mathcal{R}^u_{\text{cor}} := \mathcal{R}^u \mathbb{P} H \text{div} \left( \frac{d(\rho)}{q+1} \otimes (w^{(c)} q+1 + w^{(o)} q+1) - (w^{(c)} q+1 + w^{(o)} q+1) \otimes d(q+1) \right) \]
\[+ \left( \frac{d(\rho)}{q+1} \otimes w^{(c)} q+1 - w^{(p)} q+1 \otimes (\frac{d(\rho)}{q+1} + d(q+1)) \right). \quad (4.5)\]

Moreover, we also compute for the Reynolds stress:
\[\text{div} \mathcal{R}^u q+1 = \frac{\partial (w q+1 + w^{(c)} q+1)}{\partial q+1} + \nu_1(-\Delta) a q+1 + \text{div} \left( \frac{u q+1 + w^{(c)} q+1 - \frac{d(\rho)}{q+1} \otimes d(q+1)}{\partial q+1} + \frac{d(\rho)}{q+1} \otimes w^{(o)} q+1 \right) \]
\[- (\frac{d(\rho)}{q+1} + d(q+1) \otimes d(q+1) - \frac{d(\rho)}{q+1} \otimes (\frac{d(\rho)}{q+1} + d(q+1)) + \text{div} P_{q+1}. \quad (4.6)\]

Hence, using the inverse divergence operator \( \mathcal{R}^u \) given by (A.5) we define the new Reynolds stress by
\[\mathcal{R}^u q+1 := \mathcal{R}^u_{\text{lin}} + \mathcal{R}^u_{\text{osc}} + \mathcal{R}^u_{\text{cor}}, \quad (4.7)\]

where the linear error
\[\mathcal{R}^u_{\text{lin}} := \mathcal{R}^u \left( \frac{\partial (w q+1 + w^{(c)} q+1)}{\partial q+1} + \nu_1 (-\Delta) a q+1 \right) + \mathcal{R}^u \mathbb{P} H \text{div} \left( \frac{u q+1 + w^{(c)} q+1 - \frac{d(\rho)}{q+1} \otimes d(q+1)}{\partial q+1} + \frac{d(\rho)}{q+1} \otimes w^{(o)} q+1 \right) - B \otimes d(q+1) - d(q+1) \otimes B q \right), \quad (4.8)\]

the oscillation error
\[\mathcal{R}^u_{\text{osc}} := \sum_{k \in \Lambda_n} \mathcal{R}^u \mathbb{P} H \mathbb{P} \neq 0 \left( g(\tau)^2 \mathbb{P} \neq 0 (W(k) \otimes W(k)) \otimes (a^2(k)) \right) \]
\[+ \sum_{k \in \Lambda_B} \mathcal{R}^u \mathbb{P} H \mathbb{P} \neq 0 \left( g(\tau)^2 \mathbb{P} \neq 0 (W(k) \otimes W(k) - D(k) \otimes D(k)) \otimes (a^2(k)) \right) \]
\[- \sigma^{-1} \sum_{k \in \Lambda_n} \mathcal{R}^u \mathbb{P} H \mathbb{P} \neq 0 \left( h(\tau) \int_{\mathbb{T}^3} W(k) \otimes W(k) d\mathcal{H} \right) \]
\[- \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u \mathbb{P} H \mathbb{P} \neq 0 \left( h(\tau) \int_{\mathbb{T}^3} W(k) \otimes W(k) - D(k) \otimes D(k) d\mathcal{H} \right) \]
\[+ \sum_{k \neq k' \in \Lambda_n \cup \Lambda_B} \mathcal{R}^u \mathbb{P} H \text{div} \left( a(k) a(k') g(\tau)^2 (W(k) \otimes W(k')) \right) \]
\[- \sum_{k \neq k' \in \Lambda_n \cup \Lambda_B} \mathcal{R}^u \mathbb{P} H \text{div} \left( a(k) a(k') g(\tau)^2 D(k) \otimes D(k') \right), \quad (4.9)\]

and the corrector error
\[\mathcal{R}^u \text{cor} := \mathcal{R}^u \mathbb{P} H \text{div} \left( w^{(p)} q+1 \otimes (w^{(c)} q+1 + w^{(o)} q+1) + (w^{(c)} q+1 + w^{(o)} q+1) \otimes w^{(c)} q+1 \right) \]
\(-d^{(p)}_{q+1} \odot (d^{(c)}_{q+1} + d^{(o)}_{q+1}) - (d^{(c)}_{q+1} + d^{(o)}_{q+1}) \odot d_{q+1}\).

(4.10)

**Remark 4.1.** We note that, by the algebraic identities (3.30)-(3.31), (3.35)-(3.36), the new magnetic and Reynolds stresses at level \(1 + 1\) satisfy equations (4.1) and (4.6), respectively. Moreover, one also has (see, e.g., [3, 49])

\[
\begin{align*}
\dot{R}^u_{q+1} &= \mathcal{R}^u \mathcal{P}_H \text{div} \dot{R}^u_{q+1}, \\
\dot{R}^B_{q+1} &= \mathcal{R}^B \mathcal{P}_H \text{div} \dot{R}^B_{q+1}.
\end{align*}
\]

(4.11)

4.2. Verification of inductive estimates of magnetic stress. In the following we verify the inductive estimates (1.25) and (1.26) for the new Reynolds and magnetic stresses.

4.2.1. Verification of \(L^\infty_t H^\tilde{N}_x\)-estimates of magnetic stress. Using the identity (4.11b) and equations (1.11) at level \(q + 1\) we get that for \(\tilde{N} = 3, 4\),

\[
\|\dot{R}^B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} \leq \|\mathcal{R}^B \mathcal{P}_H (\text{div} \dot{R}^B_{q+1})\|_{L^\infty_t H^{\tilde{N}}_x} \\
\lesssim \|\partial_t B_{q+1} + \text{div}(B_{q+1} \otimes u_{q+1} - u_{q+1} \otimes B_{q+1}) + \nu(-\Delta)\alpha B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} \\
\lesssim \|\partial_t B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} + \|B_{q+1} \otimes u_{q+1} - u_{q+1} \otimes B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} + \|B_{q+1}\|_{L^\infty_t H^{\tilde{N} + 2\alpha - 1}_x} \\
\lesssim \|\partial_t B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} + \|u_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} + \|B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} + \|B_{q+1}\|_{L^\infty_t H^{\tilde{N} + 2\alpha - 1}_x}.
\]

(4.12)

Concerning the \(L^\infty_t H^2_x\) estimates of \(u_{q+1}\) and \(B_{q+1}\) in (4.12), we use (1.17), (3.38), the Sobolev embedding \(H^2_x \hookrightarrow L^\infty_x\) and Lemma 3.5 to derive

\[
\|(u_{q+1}, B_{q+1})\|_{L^\infty_t L^\infty_x} \lesssim \|(\tilde{u}_{q}, \tilde{B}_{q})\|_{L^\infty_t L^\infty_x} + \|(w_{q+1}, d_{q+1})\|_{L^\infty_t H^2_x} \\
\lesssim \lambda^4_q + \lambda^{4}_q \lesssim \lambda^4_q + 1,
\]

(4.13)

where the implicit constant is independent of \(q\).

Thus, inserting (1.23), (1.24) and (4.13) into (4.12), we obtain

\[
\|\dot{R}^B_{q+1}\|_{L^\infty_t H^{\tilde{N}}_x} \lesssim \lambda^{\tilde{N} + 4}_q + \lambda^{\tilde{N} + 6}_q + \lambda^{\tilde{N} + 5}_q + 1 \lesssim \lambda^{\tilde{N} + 6}_q
\]

(4.14)

for some universal constant, which verifies (1.25) of \(\dot{R}^B_{q+1}\) at level \(q + 1\).

4.2.2. Verification of \(L^1_{t,x}\)-decay of magnetic stress. We aim to verify the \(L^1_{t,x}\)-decay (1.26) of the magnetic stress \(\dot{R}^B_{q+1}\) at level \(q + 1\). For this purpose, we choose

\[
\rho := \frac{2\alpha - 2 + 10\varepsilon}{2\alpha - 2 + 9\varepsilon} \in (1, 2),
\]

(4.15)

where \(\varepsilon\) is given by (1.14), such that

\[
(2 - 2\alpha - 10\varepsilon)(\frac{1}{\rho} - \frac{1}{2}) = 1 - \alpha - 4\varepsilon,
\]

(4.16)

and

\[
r^{\frac{1}{\alpha} - 1} = \lambda^{1-\alpha-4\varepsilon}.
\]

(4.17)

Below we estimate the three parts of the magnetic stress \(\dot{R}^B_{q+1}\) separately.
(i) **Linear error.** Note that, by Lemmas 3.1, 3.2, 3.3 and 3.5, (1.14), (3.1), (3.33b) and (4.17),
\[
\| R^B \partial_t (d^{(c)}_{q+1} + d^{(p)}_{q+1}) \|_{L^1_t L^p_x} \\
\lesssim \sum_{k \in \Lambda_B} \left( \| g(\tau) \|_{L^1_t} \| a(k) \|_{C^{1+\eta}_{t,x}} \| D^*_{k} \|_{C^1_t L^p_x} + \| \partial_t g(\tau) \|_{L^1_t} \| a(k) \|_{C^{1+\eta}_{t,x}} \| D^*_{k} \|_{C^1_t L^p_x} \right) \\
\lesssim \theta_{q+1}^{-14} \frac{1}{q_1^\frac{1}{2}} + \frac{1}{r_1^\frac{1}{2}} \lambda^{-1} + \theta_{q+1}^{-14} \frac{1}{r_1^\frac{1}{2}} \sigma r_1^\frac{1}{2} \lambda^{-1} \lesssim \theta_{q+1}^{-14} \lambda^{-2\varepsilon}. \quad (4.18)
\]

Next, we control the hyper-resistivity \((-\Delta)^\alpha\). It is important here to exploit the temporal intermittency in order to control the resistivity beyond the Lions exponent 5/4.

More precisely, using the interpolation estimate, Lemma 3.5, (3.1), (3.39), (4.17) and the fact that \(2 - \alpha \geq 5\varepsilon\), we estimate
\[
\| R (-\Delta)^\alpha d^{(p)}_{q+1} \|_{L^1_t L^p_x} \lesssim \| \nabla [2\alpha - 1] d^{(p)}_{q+1} \|_{L^1_t L^p_x} \lesssim \| d^{(p)}_{q+1} \|_{L^1_t L^p_x} \| \nabla \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-14} \lambda^{2\alpha - 2} r_1^\frac{1}{2} + \frac{1}{q_1^\frac{1}{2}} \lambda^{-1} \lesssim \theta_{q+1}^{-14} \lambda^{-1 - 4\varepsilon}, \quad (4.19)
\]
and
\[
\| R (-\Delta)^\alpha d^{(c)}_{q+1} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-14} \lambda^{2\alpha - 2} r_1^\frac{1}{2} + \frac{1}{q_1^\frac{1}{2}} \lambda^{-1} \lesssim \theta_{q+1}^{-14} \lambda^{1 - 4\varepsilon}, \quad (4.20)
\]

Hence, it follows that
\[
\| R (-\Delta)^\alpha d^{(p)}_{q+1} \|_{L^1_t L^p_x} \lesssim \| R (-\Delta)^\alpha d^{(p)}_{q+1} \|_{L^1_t L^p_x} + \| R (-\Delta)^\alpha d^{(c)}_{q+1} \|_{L^1_t L^p_x} + \| R (-\Delta)^\alpha d^{(o)}_{q+1} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-14} \lambda^{-2\varepsilon}. \quad (4.22)
\]

At last, for the remaining terms in (4.3), using (1.17), Lemma 3.5 and (4.17), we have
\[
\| R^B \mathbb{P}_H \text{div} \left( d_{q+1} \otimes \bar{u}_q - \bar{u}_q \otimes d_{q+1} + \bar{B}_q \otimes w_{q+1} - w_{q+1} \otimes \bar{B}_q \right) \|_{L^1_t L^p_x} \lesssim \| \bar{u}_q \|_{L^1_t L^p_x} \| d_{q+1} \|_{L^1_t L^p_x} + \| \bar{B}_q \|_{L^1_t L^p_x} \| w_{q+1} \|_{L^1_t L^p_x} \lesssim \lambda_1^\alpha \theta_{q+1}^{-14} \lambda^{-2\varepsilon} + \theta_{q+1}^{-14} \lambda^{-2\varepsilon}. \quad (4.23)
\]

Therefore, we conclude from (4.18), (4.22) and (4.33) that the linear error can be bounded by
\[
\| R_{\text{lin}} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-14} \lambda^{-2\varepsilon} + \theta_{q+1}^{-14} \lambda^{-2\varepsilon} + \theta_{q+1}^{-17} \lambda^{-2\varepsilon} \lesssim \theta_{q+1}^{-14} \lambda^{-2\varepsilon}. \quad (4.24)
\]

(ii) **Oscillation error.** In view of (4.4), we decompose the oscillation error into three parts
\[
\hat{R}_{\text{osc}} = \hat{R}_{\text{osc.1}} + \hat{R}_{\text{osc.2}} + \hat{R}_{\text{osc.3}},
\]
where the high-low spatial oscillation error
\[
\hat{R}_{\text{osc.1}} := \sum_{k \in \Lambda_B} \mathbb{P}_H R^B \left( g^2_{\tau(\cdot)} \mathbb{P}_0 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \nabla (a^2_{(k)}) \right),
\]
the low frequency oscillation error
\[
\hat{R}_{\text{osc.2}} := -\sigma^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H R^B \left( h(\tau) \int_{\mathbb{T}^3} D_{(k')} \otimes W_{(k)} - W_{(k)} \otimes D_{(k')} dx \partial_t \nabla (a^2_{(k)}) \right),
\]
and the interaction error
\[
\hat{R}_{\text{osc.3}} := \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_u, k' \in \Lambda_B} \right) \mathbb{P}_H R^B \text{div} \left( a_{(k')} g^2_{\tau(\cdot)} (D_{(k')} \otimes W_{(k)} - W_{(k)} \otimes D_{(k')}) \right).
\]

First, the control of \(\hat{R}_{\text{osc.1}}\) relies on the key fact that the velocity and magnetic flows are of high oscillations
\[
\mathbb{P}_0 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) = \mathbb{P}_{\geq \lambda r_{\perp}} (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}),
\]
while the amplitudes $a_{(k)}$ are slowly varying. Hence, one may gain an extra factor $(\lambda r_\perp)^{-1}$ by using the inver-divergence operator $R^B$. This leads to, via Lemma A.4 with $a = \nabla (a_{(k)}^2)$, $f = \phi_{(k)}^2$ and $k = \lambda r_\perp / 2$,

$$\| \dot{R}^B_{osc,1} \|_{L^1_t L^6_x} \lesssim \sum_{k \in \Lambda_B} \| g_{(r)} \|_{L^2_{k'}}^2 \| \nabla^{1-\Psi \neq 0} \left( \hat{F}_{(r)} (\lambda r_\perp / 2) (D_{(k')} \otimes W_{(k)} - W_{(k)} \otimes D_{(k')} \nabla (a_{(k)}^2)) \right) \|_{C_t L^6_x}$$

$$\lesssim \sum_{k \in \Lambda_B} \lambda^{-1} r_\perp^{-1} \| \nabla (a_{(k)}^2) \|_{C_{t,x}} \| \phi_{(k)}^2 \|_{L^6_x}$$

$$\lesssim \theta_{q+1}^{-23} \lambda^{-1} r_\perp^{-1},$$

(4.25)

where we also used Lemmas 3.1 and 3.3 in the last step.

Moreover, the low frequency part $\dot{R}^B_{osc,2}$ can be estimated by using (3.14) and (3.21):

$$\| \dot{R}^B_{osc,2} \|_{L^1_t L^6_x} \lesssim \sigma^{-1} \sum_{k \in \Lambda_B} \| h_{(r)} \|_{C_t} \left( \| a_{(k)} \|_{C_{t,x}} \| a_{(k)} \|_{C^2_{t,x} + \| a_{(k)} \|_{C^2_{t,x}}} \right) \lesssim \theta_{q+1}^{-15} \sigma^{-1}.$$  

(4.26)

Finally, thanks to the small interactions between different intermittent spatial building blocks, the interaction error $\dot{R}^B_{osc,3}$ can be controlled by the product estimate in Lemma 3.1,

$$\| \dot{R}^B_{osc,3} \|_{L^1_t L^6_x} \lesssim \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_{B', k' \in \Lambda_B}} \right) \| a_{(k)} a_{(k')} g_{(r)}^{(2)} (D_{(k')} \otimes W_{(k)} - W_{(k)} \otimes D_{(k')}) \|_{L^1_t L^6_x}$$

$$\lesssim \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_{B', k' \in \Lambda_B}} \right) \| a_{(k)} \|_{C_{t,x}} \| a_{(k')} \|_{C_{t,x}} \| g_{(r)}^{(2)} \|_{L^1_t} \| \phi_{(k')} \|_{C_{t,x}}$$

$$\lesssim \theta_{q+1}^{-2} r_\perp^{-1}.$$  

(4.27)

Therefore, combing (4.25), (4.26) and (4.27) altogether and using (3.1) and (4.17) we arrive at

$$\| \dot{R}_{osc} \|_{L^1_t L^6_x} \lesssim \theta_{q+1}^{-23} \lambda^{-1} r_\perp^{-1} + \theta_{q+1}^{-15} + \theta_{q+1}^{-2} r_\perp^{-1}$$

$$\lesssim \theta_{q+1}^{-23} \lambda^{-1} r_\perp^{-1} + \theta_{q+1}^{-15} + \theta_{q+1}^{-2} r_\perp^{-1}$$

$$\lesssim \theta_{q+1}^{-15} \lambda^{-1} - 2\epsilon,$$

(4.28)

where the last step is due to $2\alpha - 3 \leq -20\epsilon$ implied by (1.14).

(iii) Corrector error. It is relatively easier to control the corrector error $\dot{R}^B_{cor}$. Using Hölder’s inequality, Lemma 3.5, (1.14), (3.1) and (4.17), we get

$$\| \dot{R}^B_{cor} \|_{L^1_t L^6_x} \lesssim \| a_{(k)}^{(c)} \|_{L^2_{t,x}} + \| a_{(k)}^{(o)} \|_{L^2_{t,x}}$$

$$+ (\| a_{(k)} \|_{L^2_{t,x}} + \| a_{(k)} \|_{L^2_{t,x}}) \| d_{(k)}^{(c)} \|_{L^2_{t,x}} + \| d_{(k)}^{(o)} \|_{L^2_{t,x}}$$

$$\lesssim \left( \theta_{q+1}^{-14} + \theta_{q+1}^{-16} \right) \left( \theta_{q+1}^{-1} + \theta_{q+1}^{-14} \right) \left( \theta_{q+1}^{-1} + \theta_{q+1}^{-14} \right)$$

$$\lesssim \theta_{q+1}^{-32} \lambda^{-4\epsilon}.$$  

(4.29)

Therefore, from estimates (4.24), (4.28), (4.29) of the three parts of magnetic stress we conclude that

$$\| \dot{R}^B_{q+1} \|_{L^1_{t,x}} \lesssim \| \dot{R}^B_{in} \|_{L^1_t L^6_x} + \| \dot{R}^B_{osc} \|_{L^1_t L^6_x} + \| \dot{R}^B_{cor} \|_{L^1_t L^6_x}$$

$$\lesssim \theta_{q+1}^{-44} \lambda^{-2\epsilon} + \theta_{q+1}^{-14} \lambda^{-2\epsilon} + \theta_{q+1}^{-32} \lambda^{-4\epsilon}$$

$$\lesssim \lambda^{-\epsilon R} \delta_{q+2},$$

(4.30)

where the last step is due to (1.13) and $\varepsilon < \varepsilon / 10$. This justifies the inductive estimate (1.26) for the $L^1_{t,x}$-decay of the magnetic stress $R^B_{q+1}$. 
4.3. Verification of inductive estimates of Reynolds stress. We now verify the inductive estimates (1.25) and (1.26) for the Reynolds stress \( R_{q+1} \) given by (4.7) at level \( q + 1 \).

The verification of estimate (1.25) can be proved in a similar fashion as the proof of (4.14), by using estimates (3.42)-(3.45). Below we will focus on the proof of \( L_{1,\infty}^1 \)-decay estimate (1.26) of the three parts \( R_{\text{hess}}, R_{\text{osc}} \) and \( R_{\text{corr}} \) for the Reynolds stress \( R_{q+1} \) at level \( q + 1 \).

(i) Linear error. Arguing as in the proof of (4.18), but with (5.21a) instead, we get

\[
\| R_u \partial_t (w_q^{(p)} + w_q^{(c)}) \|_{L_x^1 L_t^\infty} \lesssim \sum_{k \in \Lambda_u} \| R_u \text{curl} \partial_t (g(\rho) a(k) W^c) \|_{L_x^1 L_t^\infty} \lesssim \theta_{q+1}^{-28} \lambda^{-2e}. \tag{4.31}
\]

For the hyper-viscosity term, similarly to (4.22), we use the temporal and spatial intermittency to derive

\[
\| R_u (-\Delta)^a w_q + \|_{L_x^1 L_t^\infty} \lesssim \theta_{q+1}^{-44} \lambda^{-2e}. \tag{4.32}
\]

Moreover, using (1.17), Lemma 3.5 and (1.17), we get

\[
\| R_u \mathbb{P}_H \partial_t (\tilde{u}_q \otimes w_q + w_q \otimes \bar{u}_q - \bar{B}_q \otimes d_q + d_q \otimes \bar{B}_q) \|_{L_x^1 L_t^\infty} \lesssim \theta_{q+1}^{-28} \lambda^{-2e}. \tag{4.33}
\]

Therefore, estimates (4.31), (4.32) and (4.33) together yield that

\[
\| R_u \|_{L_x^1 L_t^\infty} \lesssim \theta_{q+1}^{-28} \lambda^{-4e} + \theta_{q+1}^{-44} \lambda^{-2e} + \theta_{q+1}^{-44} \lambda^{-2e} \lesssim \theta_{q+1}^{-44} \lambda^{-2e}. \tag{4.34}
\]

(ii) Oscillation error. We infer from (4.9) that the oscillation error consists of three parts

\[
\hat{R}_{\text{osc}} = \hat{R}_{\text{osc.1}} + \hat{R}_{\text{osc.2}} + \hat{R}_{\text{osc.3}},
\]

where the low-high spatial oscillation error

\[
\hat{R}_{\text{osc.1}} := \sum_{k \in \Lambda_u} R_u \mathbb{P}_H \mathbb{P}_H f(k, W(k) \otimes W(k)) \nabla(a^2_k) + \sum_{k \in \Lambda_B} R_u \mathbb{P}_H \mathbb{P}_H f(k, W(k) \otimes W(k) - D(k) \otimes D(k)) \nabla(a^2_k),
\]

the low frequency error

\[
\hat{R}_{\text{osc.2}} := -\sigma^{-1} \sum_{k \in \Lambda_u} R_u \mathbb{P}_H \mathbb{P}_H f(k, W(k) \otimes W(k)) \nabla(a^2_k) + \sum_{k \in \Lambda_B} R_u \mathbb{P}_H \mathbb{P}_H f(k, W(k) \otimes W(k) - D(k) \otimes D(k)) \nabla(a^2_k),
\]

and the interaction error

\[
\hat{R}_{\text{osc.3}} := \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} R_u \mathbb{P}_H \mathbb{P}_H f(k, a_k a_{k'} W(k) \otimes W(k')) - \sum_{k \neq k' \in \Lambda_B} R_u \mathbb{P}_H \mathbb{P}_H f(k, a_k a_{k'} D(k) \otimes D(k)).
\]

Then, applying the decoupling Lemma A.4 again with \( a = \nabla(a^2_k) \) and \( f = \phi^2_k \) and using Lemmas 3.1, 3.3 and 3.4, we estimate

\[
\| \hat{R}_{\text{osc.1}} \|_{L_x^1 L_t^\infty} \lesssim \sum_{k \in \Lambda_u} \| g(\rho) \|_{L_x^2} \| \nabla^{-1} \mathbb{P}_H f(k, \nabla(a^2_k)) \|_{L_x^1 L_t^\infty} \| C_{\Lambda} \|_{L_x^\infty} + \sum_{k \in \Lambda_B} \| g(\rho) \|_{L_x^2} \| \nabla^{-1} \mathbb{P}_H f(k, \nabla(a^2_k)) \|_{L_x^1 L_t^\infty} \| C_{\Lambda} \|_{L_x^\infty} \lesssim \theta_{q+1}^{-44} \lambda^{-2e}.
\]

(4.35)
Moreover, (3.11) and Lemma 3.5 yield that
\[ \| \hat{R}_{\text{osc}, 2} \| L^1_t L^s_x \lesssim \sigma^{-1} \sum_{k \in \Lambda_{\text{osc}} \cup \Lambda_B} \| h(\tau) \| C_i \left( \| a(k) \| C_i \| a(k) \| C_i^{2} \right) \lesssim \theta_{q+1}^{-2} \sigma^{-1}. \]  

Regarding the interaction error \( \hat{R}_{\text{osc}, 3} \), we use the product estimate (3.9) to estimate
\[
\| \hat{R}_{\text{osc}, 3} \| L^1_t L^s_x \lesssim \sum_{k \neq k' \in \Lambda_{\text{osc}} \cup \Lambda_B} \| a(k) a(k') \| g^2(\tau) W(k) \otimes W(k') \| L^1_t L^s_x \\
+ \sum_{k \neq k' \in \Lambda_{\text{osc}} \cup \Lambda_B} \| a(k) a(k') \| g^2(\tau) D(k) \otimes D(k') \| L^1_t L^s_x \\
\lesssim \sum_{k \neq k' \in \Lambda_{\text{osc}} \cup \Lambda_B} \| a(k) \| C_i a(k') \| C_i \| g^2(\tau) \| L^1_t \| \phi(k) \phi(k') \| C_i L^s_x \\
\lesssim \theta_{q+1}^{-2} \theta_1^{-1} \sigma^{-1}. \]  

Therefore, combining (4.35), (4.36) and (4.37) altogether and using (1.14) and (3.1), we arrive at
\[
\| \hat{R}_{\text{osc}} \| L^1_t L^s_x \lesssim \theta_{q+1}^{-44} \lambda^{-1} r_\perp^{-2} + \theta_{q+1}^{-29} \sigma^{-1} + \theta_{q+1}^{-2} r_\perp^{-1} \\
\lesssim \theta_{q+1}^{-44} \lambda^{2-3+11\ve} + \theta_{q+1}^{-29} \lambda^{-2-\ve} + \theta_{q+1}^{-2} \lambda^{-2-2a-8\ve} \\
\lesssim \theta_{q+1}^{-29} \lambda^{-2-\ve}. \]  

Therefore, we conclude from estimates (4.34), (4.38) and (4.39) that
\[
\| \tilde{R}_{q+1} \| L^1_{t,x} \leq \| \tilde{R}_{\text{lin}} \| L^1_t L^s_x + \| \tilde{R}_{\text{osc}} \| L^1_t L^s_x + \| \tilde{R}_{\text{cor}} \| L^1_t L^s_x \\
\lesssim \theta_{q+1}^{-44} \lambda^{-2-\ve} + \theta_{q+1}^{-29} \lambda^{-2-\ve} + \theta_{q+1}^{-32} \lambda^{-4-\ve} \\
\leq \lambda^{-\ve} R_{q+2}. \]  

which verifies the inductive \( L^1_{t,x} \)-decay estimate (1.26) of the new Reynolds stress \( \hat{R}_{q+1} \) at level \( q+1 \).

5. The supercritical regime \( S_2 \)

In this section, we treat the other supercritical regime \( S_2 \) when \( \alpha \in [1,3/2] \). Quite differently from \( S_1 \), we shall use the building blocks with stronger spatial intermittency, in order to achieve the possibly low spatial integrability in \( S_2 \). Hence, unlike in §3, we choose the intermittent flows constructed in [49], which in particular provide 2D spatial intermittency.

The main parameters here for the intermittent flows will be indexed by six parameters \( r_\perp, r_\parallel, \lambda, \mu, \tau \) and \( \sigma \), chosen in the following way:
\[
 r_\perp := \lambda_{q+1}^{-1+2\ve}, \quad r_\parallel := \lambda_{q+1}^{-1+6\ve}, \quad \lambda := \lambda_{q+1}, \quad \tau := \lambda_{q+1}^{4a-4+12\ve}, \quad \mu := \lambda_{q+1}^{2a-1+3\ve}, \quad \sigma := \lambda_{q+1}^{2\ve}, \]

where \( \ve \) is the small constant satisfying (1.15). Note that, two new parameters \( r_\parallel \) and \( \mu \), respectively, are introduced here, in order to further concentrate the flows along direction \( k_1 \) and to balance high temporal oscillations.

5.1. Spatial-temporal building blocks. Let \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth and mean-free function, supported on the interval \([-1,1]\), satisfying
\[
\frac{1}{2\pi} \int_{\mathbb{R}} \psi^2(x) \, dx = 1. \]  

(5.2)
The corresponding rescaled cut-off functions are defined by
\[ \psi_{r_\parallel}(x) := r_\parallel^{-\frac{1}{2}} \psi\left(\frac{x}{r_\parallel}\right). \]

Note that, \( \psi_{r_\parallel} \) is supported in the ball of radius \( r_\parallel \) in \( \mathbb{R} \). By an abuse of notation, we periodize \( \psi_{r_\parallel} \) so that it is treated as a periodic function defined on \( \mathbb{T} \). We also keep using the same rescaled function \( \psi_{r_\perp} \) and \( \Psi_{r_\parallel} \) as in §3.1.

The new intermittent velocity and magnetic flows are defined by
\[ W_{(k)} := \psi_{r_\parallel}(\lambda r_\perp N_A (k_1 \cdot x + \mu t)) \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x) k_1, \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ D_{(k)} := \psi_{r_\parallel}(\lambda r_\perp N_A (k_1 \cdot x + \mu t)) \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x) k_2, \quad k \in \Lambda_B. \]

Here, \( N_A, (k_1, k_2, k), \Lambda_u \) and \( \Lambda_B \) are as in §3.1. We note that, compared to the previous spatial building blocks in §3.1, the current building blocks \( \{W_{(k)}, D_{(k)}\} \) are supported on thinner cuboids with length \( \sim 1/(\lambda r_\perp) \), width \( \sim r_\parallel/(\lambda r_\perp) \) and height \( \sim 1/\lambda \). See [49, Figure 2]. Moreover, by choosing \( k_2 \neq k_2' \) if \( k \neq k' \), one has much smaller volume of the intersections of distinct intermittent flows, see Lemma 5.1 below.

For brevity of notations, we let
\[ \phi_{(k)}(x) := \phi_{r_\perp}(\lambda r_\perp N_A k \cdot x), \quad \Phi_{(k)}(x) := \Phi_{r_\perp}(\lambda r_\perp N_A k \cdot x), \]
\[ \psi_{(k_1)}(x) := \psi_{r_\parallel}(\lambda r_\perp N_A (k_1 \cdot x + \mu t)), \]
and rewrite
\[ W_{(k)} = \psi_{(k)} \phi_{(k_1)}, \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ D_{(k)} = \psi_{(k)} \phi_{(k_2)}, \quad k \in \Lambda_B. \]

The corresponding incompressible correctors are defined by
\[ \tilde{W}_{(k)}^c := \frac{1}{\lambda^2 N_A} \nabla \psi_{(k)} \times \text{curl}(\Phi_{(k)} k_1), \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ \tilde{D}_{(k)}^c := \frac{1}{\lambda^2 N_A} \Delta \psi_{(k)} \Phi_{(k)} k_2, \quad k \in \Lambda_B. \]

Let
\[ W_{(k)}^c := \frac{1}{\lambda^2 N_A} \psi_{(k)} \Phi_{(k)} k_1, \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ D_{(k)}^c := \frac{1}{\lambda^2 N_A} \psi_{(k)} \Phi_{(k)} k_2, \quad k \in \Lambda_B. \]

Then, it holds that (see [49, (3.18), (3.22)])
\[ W_{(k)} + \tilde{W}_{(k)}^c = \text{curl} \text{curl} W_{(k)}^c, \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ D_{(k)} + \tilde{D}_{(k)}^c = \text{curl} \text{curl} D_{(k)}^c, \quad k \in \Lambda_B, \]
and thus
\[ \text{div}(W_{(k)} + \tilde{W}_{(k)}^c) = 0, \quad k \in \Lambda_u \cup \Lambda_B, \]
\[ \text{div}(D_{(k)} + \tilde{D}_{(k)}^c) = 0, \quad k \in \Lambda_B. \]

Besides the above algebraic identities adapted to the geometry of MHD equations, another nice feature of the current spatial building blocks is that, they provide the 2D spatial intermittency and permit to control the hypo-dissipativity and hypo-resistivity \( (-\Delta)^{\alpha_i} \), for any \( \alpha_i \in [0, 1), i = 1, 2 \). Moreover, stronger intermittency also can be gained for the interactions between different spatial building blocks. This is the content of Lemma 5.1 below.

**Lemma 5.1** ([49] Estimates of spatial intermittency). For \( p \in [1, +\infty] \), \( N, M \in \mathbb{N} \), we have
\[ \|\nabla^N \partial_1^M \psi_{(k_1)}\|_{C^1 L^p_x} \lesssim r_\parallel^{\frac{1}{2}-\frac{1}{2}} \left(\frac{r_\perp \lambda}{r_\parallel}\right)^N \left(\frac{r_\perp \lambda \mu}{r_\parallel}\right)^M, \]
\[ \|\nabla^N \Phi_{(k)}\|_{L^p_x} + \|\nabla^N \Phi_{(k)}\|_{L^p_x} \lesssim r_\perp^{\frac{1}{2}+\frac{1}{2}} \lambda^N. \]
where the implicit constants are independent of \(r_\perp, r_\parallel, \lambda\) and \(\mu\). In particular, we have
\[
\begin{align*}
\left\| \nabla^N \partial_t^M \mathcal{W}(k) \right\|_{C^1 L^p_t} &+ \frac{r_\parallel}{r_\perp} \left\| \nabla^N \partial_t^M \tilde{\mathcal{W}}(k) \right\|_{C^1 L^p_t} + \lambda^2 \left\| \nabla^N \partial_t^M \mathcal{W}_c(k) \right\|_{C^1 L^p_t} \\
\leq r_\perp^{\frac{1}{2}-\frac{1}{2}} r_\parallel^{\frac{1}{2}} \lambda^N \left( \frac{r_\perp}{r_\parallel} \right)^M, \quad k \in \Lambda_u \cup \Lambda_B, \quad (5.12) \\
\left\| \nabla^N \partial_t^M \mathcal{D}(k) \right\|_{C^1 L^p_t} &+ \frac{r_\parallel}{r_\perp} \left\| \nabla^N \partial_t^M \tilde{\mathcal{D}}(k) \right\|_{C^1 L^p_t} + \lambda^2 \left\| \nabla^N \partial_t^M \mathcal{D}_c(k) \right\|_{C^1 L^p_t} \\
\leq r_\perp^{\frac{1}{2}-\frac{1}{2}} r_\parallel^{\frac{1}{2}} \lambda^N \left( \frac{r_\perp}{r_\parallel} \right)^M, \quad k \in \Lambda_B. \quad (5.13)
\end{align*}
\]
Moreover, for every \(k \neq k' \in \Lambda_u \cup \Lambda_B\) and \(p \in [1, \infty)\), we have
\[
\left\| \psi(k) \phi(k') \psi(k') \right\|_{C^1 L^p_t} \lesssim r_\perp^{\frac{1}{2}-1} r_\parallel^{\frac{3}{2}-1}, \quad (5.14)
\]
where the implicit constant is independent of the parameters \(r_\perp, r_\parallel\) and \(\lambda\).

At last, in order to control the hyper-viscosity and hyper-resistivity \((-\Delta)\alpha\) when \(\alpha \geq 1\), again it is crucial to use the temporal intermittency. We shall use the temporal building blocks as in §3.1, that is, the same temporal building blocks \(g(t)\) and \(h(t)\) given by (3.12), but with the new parameters \(\tau, \sigma\) given by (5.1), which turns out to be effective to treat the current supercritical regime \(S_2\).

5.2. Velocity and magnetic perturbations. Let us first define the amplitude functions.
- **Amplitudes.** As in §3.2, the amplitudes of the magnetic perturbations are defined by
\[
a_{(k)}(t, x) := \gamma_k - \frac{\tilde{g}^{B}(t, x)}{g^{B}(t, x)} \parallel \mathcal{W}(k) \parallel_{L^\infty}, \quad k \in \Lambda_B, \quad (5.15)
\]
and the amplitudes of the velocity perturbations are defined by
\[
a_{(k)}(t, x) := \gamma_k - \frac{\tilde{g}^{u}(t, x) + \tilde{G}^{B}(t, x)}{g^{u}(t, x)} \parallel \mathcal{W}(k) \parallel_{L^\infty}, \quad k \in \Lambda_u, \quad (5.16)
\]
where \(g^{u}, g^{B}, f^{u}, f^{B}, \gamma_k\) and \(\tilde{G}^{B}\) are defined as in §3.1. Note that, the amplitudes obey the same estimates as in Lemmas 3.3 and 3.4. Namely, one has

**Lemma 5.2.** For \(1 \leq N \leq 9\) we have
\[
\| a_{(k)} \|_{L^\infty_t} \lesssim \delta^\frac{1}{2}_{q+1}, \quad \| a_{(k)} \|_{C^1_t L^\infty_x} \lesssim \theta^\frac{1}{2}_{q+1}, \quad \| a_{(k)} \|_{C^1_t L^\infty_x} \lesssim \theta^{-7N}_{q+1}, \quad k \in \Lambda_B, \quad (5.17)
\]
and
\[
\| a_{(k)} \|_{L^\infty_t} \lesssim \delta^\frac{1}{2}_{q+1}, \quad \| a_{(k)} \|_{C^1_t L^\infty_x} \lesssim \theta^\frac{1}{2}_{q+1}, \quad \| a_{(k)} \|_{C^1_t L^\infty_x} \lesssim \theta^{-14N}_{q+1}, \quad k \in \Lambda_u, \quad (5.18)
\]
where the implicit constants are independent of \(q\).

Next, we are going to construct the velocity and magnetic perturbations, which consist of the principal parts, the incompressible correctors and the temporal correctors. It should be mentioned that, the incompressibility correctors in (5.20) below are different from the previous ones in (3.32) in the supercritical regime \(S_1\). Moreover, the current supercritical regime \(S_2\) also would require a new type of temporal correctors \(w_{q+1}^{(t)}\) and \(d_{q+1}^{(t)}\), in order to balance the high spatial oscillations caused by the concentration function \(\psi_{r_\parallel}\), that did not appear in the previous supercritical regime \(S_1\).

- **Principal parts.** The principal parts of the velocity and magnetic perturbations are defined by
\[
w_{q+1}^{(p)} := \sum_{k \in \Lambda_u \cup \Lambda_B} a_{(k)} g_{(\tau)} \mathcal{W}(k), \quad (5.19a)
\]
\[
d_{q+1}^{(p)} := \sum_{k \in \Lambda_B} a_{(k)} g_{(\tau)} \mathcal{D}(k). \quad (5.19b)
\]
Note that, the algebraic identities (3.30) and (3.31) still hold (see [49]).
\begin{itemize}

- **Incompressibility correctors.** The corresponding incompressibility correctors are defined by

\begin{align}
\hat{w}^{(c)}_{q+1} &:= \sum_{k \in \Lambda_u \cup \Lambda_B} g(\tau) \left( \text{curl}(\nabla a(k) \times W^c_{(k)}) + \nabla a(k) \times \text{curl}W^c_{(k)} + a(k)\tilde{W}^c_{(k)} \right), \\
\hat{d}^{(c)}_{q+1} &:= \sum_{k \in \Lambda_B} g(\tau) \left( \text{curl}(\nabla a(k) \times D^c_{(k)}) + \nabla a(k) \times \text{curl}D^c_{(k)} + a(k)\tilde{D}^c_{(k)} \right),
\end{align}

where $W^c_{(k)}$ and $D^c_{(k)}$ are given by (5.6) and $\tilde{W}^c_{(k)}$ and $\tilde{D}^c_{(k)}$ are as in (5.5). One has that (see [49, (4.35a), (4.35b)])

\begin{align}
\hat{w}^{(p)}_{q+1} + \hat{w}^{(c)}_{q+1} &= \text{curl} \left( \sum_{k \in \Lambda_u \cup \Lambda_B} a(k)g(\tau) W^c_{(k)} \right), \\
\hat{d}^{(p)}_{q+1} + \hat{d}^{(c)}_{q+1} &= \text{curl} \left( \sum_{k \in \Lambda_B} a(k)g(\tau) D^c_{(k)} \right).
\end{align}

In particular,

\begin{equation}
\text{div}(\hat{w}^{(p)}_{q+1} + \hat{w}^{(c)}_{q+1}) = \text{div}(\hat{d}^{(p)}_{q+1} + \hat{d}^{(c)}_{q+1}) = 0.
\end{equation}

- **Two types of temporal correctors to balance spatial-temporal oscillations.** As in the regime $S_1$, the temporal correctors $\hat{w}^{(p)}_{q+1}$ and $\hat{d}^{(p)}_{q+1}$ of the same expressions as in (3.34a) and (3.34b), respectively, also will be used in the current situation, in order to balance the high temporal oscillations. We note that the algebraic identities (3.35) and (3.36) still hold.

However, unlike in the previous supercritical regime $S_1$ in §3.3, we also need to introduce a new type of temporal corrections, particularly, to balance the high spatial oscillations.

More precisely, we define the temporal correctors $\hat{w}^{(t)}_{q+1}$ and $\hat{d}^{(t)}_{q+1}$ by

\begin{align}
\hat{w}^{(t)}_{q+1} &= -\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_1 \right), \\
\hat{d}^{(t)}_{q+1} &= -\mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_H \mathbb{P}_{\neq 0} \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_2 \right)
\end{align}

where $\mathbb{P}_H$ denotes the Helmholtz-Leray projector, i.e., $\mathbb{P}_H = 1d - \nabla \Delta^{-1} \text{div}$.

The important algebraic identities to balance high spatial oscillations are stated below (see [49, (4.38), (4.39)]):

\begin{align}
\partial_t \hat{w}^{(t)}_{q+1} + \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \left( a^2_{(k)}g^2_{(\tau)} \text{div}(W_{(k)} \otimes W_{(k)}) \right) &= (\nabla \Delta^{-1} \text{div})\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \partial_t \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_1 \right) \\
&\quad - \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathbb{P}_{\neq 0} \left( \partial_t \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_1 \right) \right),
\end{align}

and

\begin{align}
\partial_t \hat{d}^{(t)}_{q+1} + \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \left( a^2_{(k)}g^2_{(\tau)} \text{div}(D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \right) &= (\nabla \Delta^{-1} \text{div})\mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \partial_t \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_2 \right) \\
&\quad - \mu^{-1} \sum_{k \in \Lambda_B} \mathbb{P}_{\neq 0} \left( \partial_t \left( a^2_{(k)}g^2_{(\tau)}\psi^2_{(k)}\phi^2_{(k)}k_2 \right) \right).
\end{align}

- **Velocity and magnetic perturbations.** Now we define the velocity and magnetic perturbations $\hat{w}_{q+1}$ and $\hat{d}_{q+1}$ at level $q + 1$:

\begin{equation}
\hat{w}_{q+1} := \hat{w}^{(p)}_{q+1} + \hat{w}^{(c)}_{q+1} + \hat{w}^{(t)}_{q+1} + \hat{w}^{(o)}_{q+1},
\end{equation}
\[ d_{q+1} := d_{q+1}^{(p)} + d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}. \]  

The corresponding velocity and magnetic fields at level \( q + 1 \) are defined by

\[ u_{q+1} := \tilde{u}_q + w_{q+1}, \]

\[ B_{q+1} := \tilde{B}_q + d_{q+1}, \]

where \( \tilde{u}_q, \tilde{B}_q \) are the well-prepared velocity and magnetic fields in \( \S 2 \).

Lemma 5.3 below contains the key analytic estimates of the velocity and magnetic perturbations.

**Lemma 5.3 (Estimates of perturbations).** Let \( \rho \in (1, \infty), \gamma \in [1, +\infty] \) and integer \( 0 \leq N \leq 7 \). Then, the following estimates hold:

\[ \| \nabla^N w_{q+1}^{(p)} \|_{L_t^\infty L_x^\infty} + \| \nabla^N d_{q+1}^{(p)} \|_{L_t^\infty L_x^\infty} \lesssim \theta_{q+1}^{-1} \lambda^N r_\perp^{-\frac{1}{2} - \frac{1}{2}} \tau^{-\frac{1}{4}}, \]

\[ \| \nabla^N w_{q+1}^{(c)} \|_{L_t^\infty L_x^\infty} + \| \nabla^N d_{q+1}^{(c)} \|_{L_t^\infty L_x^\infty} \lesssim \theta_{q+1}^{-1} \lambda^N r_\perp^{\frac{1}{2} + \frac{1}{2}} \tau^{-\frac{1}{4} - \frac{1}{4}}, \]

\[ \| \nabla^N w_{q+1}^{(t)} \|_{L_t^\infty L_x^\infty} + \| \nabla^N d_{q+1}^{(t)} \|_{L_t^\infty L_x^\infty} \lesssim \theta_{q+1}^{-1} \mu^{-1} r_\perp^{-\frac{1}{4}} \tau^{-1 - \frac{1}{4}}, \]

\[ \| \nabla^N w_{q+1}^{(o)} \|_{L_t^\infty L_x^\infty} + \| \nabla^N d_{q+1}^{(o)} \|_{L_t^\infty L_x^\infty} \lesssim \theta_{q+1}^{-1} \lambda^{4N-16} r_\perp^{-1}, \]

where the implicit constants depend only on \( N, \gamma \) and \( \rho \). In particular, for integrals \( 1 \leq N \leq 7 \),

\[ \| w_{q+1}^{(p)} \|_{L_t^\infty H_x^N} + \| w_{q+1}^{(c)} \|_{L_t^\infty H_x^N} + \| w_{q+1}^{(t)} \|_{L_t^\infty H_x^N} + \| w_{q+1}^{(o)} \|_{L_t^\infty H_x^N} \lesssim \lambda^{N+2}. \]

Moreover, for the temporal derivatives, we have that for integers \( 1 \leq N \leq 7 \),

\[ \| \partial_t w_{q+1}^{(p)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(c)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(t)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(o)} \|_{L_t^\infty H_x^N} \lesssim \lambda^{N+5}, \]

\[ \| \partial_t d_{q+1}^{(p)} \|_{L_t^\infty H_x^N} + \| \partial_t d_{q+1}^{(c)} \|_{L_t^\infty H_x^N} + \| \partial_t d_{q+1}^{(t)} \|_{L_t^\infty H_x^N} + \| \partial_t d_{q+1}^{(o)} \|_{L_t^\infty H_x^N} \lesssim \lambda^{N+5}, \]

where the implicit constants are independent of \( \lambda \).

**Proof.** Estimates (5.28), (5.29) and (5.31) can be proved in the same fashion of (3.39)-(3.41). Regarding the new temporal correctors \( w_{q+1}^{(t)} \) and \( d_{q+1}^{(t)} \), by (5.23a), (5.23b), Lemmas 3.2, 5.1, 5.2 and the boundedness of operators \( P_{\neq 0} \) and \( P_{H} \) in \( L_t^N, \rho \in (1, \infty) \),

\[ \| \nabla^N w_{q+1}^{(t)} \|_{L_t^\infty L_x^\infty} \leq \mu^{-1} \sum_{k \in \Lambda_{\rho, 1} \cup \Lambda_{\rho, \mu}} \| g(\tau) \|_{L_t^2}^2 \prod_{N_1 + N_2 + N_3 = N} \| \nabla^N_1 (a^2_k) \|_{C_{t,x}} \| \nabla^N_2 (\psi^2_k) \|_{C_{t,x}} \| \nabla^N_3 (\tilde{\psi}^2_k) \|_{L_t^\infty} \]

\[ \lesssim \mu^{-1} r_\perp^{-\frac{1}{4}} \sum_{N_1 + N_2 + N_3 = N} \theta_{q+1}^{-1} \lambda^N r_\perp^{-\frac{1}{2}} - 1 \lambda^{N_3} r_\perp^{-\frac{1}{4}} \lesssim \theta_{q+1}^{-1} \lambda^N r_\perp^{-\frac{1}{4}} - 1 \lambda^{N_3} r_\perp^{-\frac{1}{4}}, \]

which yield (5.30).

The \( L_t^\infty H_x^N \) estimates of the velocity and magnetic perturbations in (5.32) and (5.33) then follow from estimates (5.28)-(5.31) with \( \gamma = \infty \) and \( \rho = 2 \).

It remains to prove estimates (5.34) and (5.35) of the temporal derivatives. Using (1.13), (5.1) and Lemmas 3.2, 5.1 and 5.2 we get

\[ \| \partial_t w_{q+1}^{(p)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(c)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(t)} \|_{L_t^\infty H_x^N} + \| \partial_t w_{q+1}^{(o)} \|_{L_t^\infty H_x^N} \]

\[ \lesssim \theta_{q+1}^{-1} \lambda^N \mu^{-1} r_\perp^{-\frac{1}{2}} + \theta_{q+1}^{-14N-42} \lambda^N r_\perp^{\frac{1}{2} - \frac{1}{2}} + \theta_{q+1}^{-14N-16} \lambda^N r_\perp^{-\frac{1}{4}} - 1 \tau^{-\frac{1}{4}} \leq \theta_{q+1}^{-14N-16} \lambda^N r_\perp^{-\frac{1}{4}} - 1 \tau^{-\frac{1}{4}}, \]

\[ \lesssim \theta_{q+1}^{-14N-14} \lambda^N r_\perp^{-1} \mu^{-1} r_\perp^{-\frac{1}{4}} - 1 \tau^{-\frac{1}{4}}, \]

\[ \lesssim \lambda^{N+5}, \]

where we also used the fact that \( \theta_{q+1}^{-14N-42} \ll \lambda^3 \) and \( 0 < \varepsilon \leq 1/10 \). This yields (5.34).
Arguing in a similar manner, we see that the derivative of magnetic perturbations obey the same upper bound and so get (5.35). Therefore, the proof of Lemma 5.3 is complete.

5.3. Verification of inductive estimates for velocity and magnetic fields. We now verify the inductive estimates (1.23), (1.24), (1.28)-(1.30) for the velocity and magnetic fields.

First, we derive from (1.23), (5.32) and (5.33) that, for \(0 \leq N \leq 4\),

\[
\| (u_{q+1}, B_{q+1}) \|_{L^\infty H^N_x} \lesssim \| (\tilde{u}_q, \tilde{B}_q) \|_{L^\infty H^N_x} + \| (w_{q+1}, d_{q+1}) \|_{L^\infty H^N_x} \\
\lesssim m^{N}_{q+1} \lambda_q^5 + \lambda_{q+1}^{N+5} \lesssim \lambda_{q+1}^N ,
\]

(5.38)

which verifies (1.23) at level \(q+1\).

Moreover, by (1.24), (5.34) and (5.35),

\[
\| (\partial_t u_{q+1}, \partial_t B_{q+1}) \|_{L^\infty H^N_x} \lesssim \| (\partial_t \tilde{u}_q, \partial_t \tilde{B}_q) \|_{L^\infty H^N_x} + \| (\partial_t w_{q+1}, \partial_t d_{q+1}) \|_{L^\infty H^N_x} \\
\lesssim \theta_{q+1}^{-\frac{3}{2}} \lambda_q^{N+5} \lesssim \lambda_{q+1}^N ,
\]

(5.39)

and thus (1.24) is verified at level \(q+1\).

In order to obtain the \(L^2 L^2_x\)-decay estimate (1.28), we apply Lemma A.3 with \(f = a(k), g = g(\tau)\psi(k)\phi(k)\) and \(\theta = \lambda_q\) and Lemma 5.2 to derive

\[
\| (u_{q+1}^{(p)}, d_{q+1}^{(p)}) \|_{L^2_x} \lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} \left( \| a(k) \|_{L^2_x} \| g(\tau) \|_{L^2_x} \| \psi(k) \|_{L^2_x} \| \phi(k) \|_{L^2_x} \right) \\
\lesssim \delta_{q+1}^\lambda + \theta_{q+1}^{-1} \lambda_q^{-1} \lesssim \delta_{q+1}^\lambda ,
\]

(5.40)

Then, in view of (5.1), (5.26) and Lemma 5.3, we get

\[
\| w_{q+1} \|_{L^2_x} \lesssim \| u_{q+1}^{(p)} \|_{L^2_x} + \| u_{q+1}^{(e)} \|_{L^2_x} + \| w_{q+1}^{(t)} \|_{L^2_x} + \| w_{q+1}^{(s)} \|_{L^2_x} \\
\lesssim \delta_{q+1}^\lambda + \theta_{q+1}^{-1} \mu^{-1} r_\perp^{-\frac{1}{2}} r_\parallel^{-\frac{1}{2}} + \theta_{q+1}^{-2} \sigma^{-1} \lesssim \delta_{q+1}^\lambda ,
\]

(5.41)

Similar upper bound also holds for the magnetic perturbation \(d_{q+1}\). Hence, taking into account (1.18) and (5.27) we obtain

\[
\| (u_{q+1} - u_q, B_{q+1} - B_q) \|_{L^2_x} \lesssim \| (u_{q+1} - \tilde{u}_q, B_{q+1} - \tilde{B}_q) \|_{L^2_x} + \| (w_{q+1}, d_{q+1}) \|_{L^2_x} \\
\lesssim \lambda_q^{-3} + \delta_{q+1}^\lambda \leq M^* \delta_{q+1}^\lambda ,
\]

(5.42)

where \(M^*\) is a universal large constant. Hence, we prove the \(L^2 L^2_x\)-decay estimate (1.28) at level \(q+1\).

Furthermore, we derive from Lemma 5.3 and (5.1) that,

\[
\| (w_{q+1}, d_{q+1}) \|_{L^2_x} \lesssim \theta_{q+1}^{-1} \tau^{-\frac{1}{2}} + \theta_{q+1}^{-1} \mu^{-1} r_\perp^{-\frac{1}{2}} + \theta_{q+1}^{-2} \sigma^{-1} \lesssim \lambda_q^{\tau+1} .
\]

This along with (1.18) yields that

\[
\| (u_{q+1} - u_q, B_{q+1} - B_q) \|_{L^2_x} \lesssim \| (u_{q+1} - \tilde{u}_q, B_{q+1} - \tilde{B}_q) \|_{L^2_x} + \| (w_{q+1}, d_{q+1}) \|_{L^2_x} \\
\lesssim \lambda_q^{-3} + \lambda_q^{\tau+1} \leq \delta_{q+1}^\frac{1}{2} ,
\]

(5.43)

where the last step is due to \(\lambda_q^{-3} \ll \delta_{q+1}^\frac{1}{2}\). Hence, estimate (1.29) is justified.

At last, regarding estimate (1.30), since \((s, \gamma, p) \in \Delta_s\), we use the Sobolev embedding (cf. [50])

\[
H^3_x \hookrightarrow W^{s,p}_x ,
\]

(5.44)

together with (1.16), (1.25), (2.43) and (2.64), to deduce

\[
\| (\tilde{u}_q - u_q, \tilde{B}_q - B_q) \|_{L^2_x W^{s,p}_x} \lesssim \| (\sum_i \chi_i (v_i - u_q), \sum_i \chi_i (D_i - B_q)) \|_{L^\infty H^3_x} \\
\lesssim \sup_i \| (v_i - u_q, D_i - B_q) \|_{L^\infty (\text{supp}(\chi_i); H^3_x)} \\
\lesssim \sup_i |t_{i+1} + \theta q_{i+1} - t_i| \| (\nabla \tilde{R}_q, \nabla \tilde{R}_q^B) \|_{L^\infty H^3_x} \\
\lesssim m_{q+1}^{-1} \lambda_q^{10} \lesssim \lambda_q^-2 .
\]

(5.45)
Taking into account Lemma 5.3 we deduce
\[
\left\|(u_{q+1} - u_q, B_{q+1} - B_q)\right\|_{L^2_t W^s_{x,p}} \lesssim \left\|(\tilde{u}_q - u_q, \tilde{B}_q - B_q)\right\|_{L^2_t W^s_{x,p}} + \left\|(w_{q+1}, d_{q+1})\right\|_{L^2_t W^s_{x,p}}
\lesssim \lambda_q^{-2} + \theta^{-1}\lambda_{q+1}^\alpha \left\|\tilde{u}_q - u_q\right\|_{L^2_t W^s_{x,p}} + \theta^{-1}\lambda_{q+1}^\alpha - \frac{4\alpha - 4\gamma}{p} + \varepsilon(2 + \gamma - \frac{12}{p}) + \lambda_{q+1}^\alpha \lesssim \delta_{q+1}^\frac{1}{2},
\]
where in the last inequality we also used (1.13), (5.1) and the fact that
\[
s + 2\alpha - 1 \leq 2 + \frac{4\alpha - 4}{p} + \varepsilon(2 + \gamma - \frac{12}{p}) - 10\varepsilon.
\]
Therefore, the inductive estimate (1.30) is also verified at level \(q + 1\).

5.4. Reynolds and magnetic stresses. We now treat the new Reynolds and magnetic stresses at level \(q + 1\) and prove the corresponding inductive estimates (1.25) and (1.26) in the supercritical regime \(S_2\).

5.4.1. Decomposition of magnetic and Reynolds stresses. Using (1.11) with \(q + 1\) replacing \(q\), (5.26) and (5.27) we derive
\[
\text{div}\tilde{R}^B_{q+1} = \partial_t\left(\int_{t_{q+1}}^{t_{q+1}} + d_{q+1}) + \text{div}(d_{q+1} \otimes \tilde{u}_q - \tilde{u}_q \otimes d_{q+1} + \tilde{B}_q \otimes w_{q+1} - w_{q+1} \otimes \tilde{B}_q)
\]
\[
\text{div}\tilde{R}^B_{\text{lin}} + \text{div}\tilde{R}^B_{\text{osc}} + \text{div}\tilde{R}^B_{\text{cor}}.
\]
Using the inverse divergence operator \(R^B\) given by (A.5b) we may choose the magnetic stress
\[
\tilde{R}^B_{q+1} := \tilde{R}^B_{\text{lin}} + \tilde{R}^B_{\text{osc}} + \tilde{R}^B_{\text{cor}},
\]
where the linear error
\[
\text{div}\tilde{R}^B_{\text{lin}} := \text{div} R^B\left(\partial_t (d_{q+1} + d_{q+1}) + \nu_2 R^B(-\Delta)\right) + \nu_2 R^B(-\Delta)\right) + \nu_2 R^B(-\Delta)\right)
\]
the oscillation error
\[
\tilde{R}^B_{\text{osc}} := \sum_{k \in \Lambda_B} R^B H \{ (g_{(r)}^2(k) \otimes (h_{(r)}^2(k) k_2)
\]
and the corrector error
\[
\tilde{R}^B_{\text{cor}} := \text{div} R^B\left((d_{q+1} \otimes w_{q+1} + w_{q+1} \otimes d_{q+1} + \tilde{B}_q \otimes w_{q+1} - w_{q+1} \otimes \tilde{B}_q)
\]
\[
+ \text{div}\left((d_{q+1} + d_{q+1} \otimes w_{q+1} - w_{q+1} \otimes d_{q+1} \right)
\]
\[
(5.48)
\]
Furthermore, regarding the Reynolds stress we compute
\[
\text{div}\hat{R}_{q+1}^u = \partial_t(u_{q+1}^{(p)} + u_{q+1}^{(c)}) + \nu_1(-\Delta)^\alpha w_{q+1} + \text{div}\left(\tilde{u}_q \otimes w_{q+1} + w_{q+1} \otimes \tilde{u}_q - \tilde{B}_q \otimes d_{q+1} - d_{q+1} \otimes \tilde{B}_q\right)
\]
\[
+ \text{div}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - \phi_{q+1}^{(p)} \otimes \phi_{q+1}^{(p)} + \hat{R}_{q}^u) + \partial_t w_{q+1}^{(i)} + \partial_t w_{q+1}^{(o)}
\]
\[
+ \text{div}\left((w_{q+1}^{(c)} + w_{q+1}^{(t)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(o)})
\right.
\]
\[
- (d_{q+1}^{(c)} + d_{q+1}^{(t)}) \otimes d_{q+1} - d_{q+1}^{(p)} \otimes (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}) + \nu_1 \text{div}(\hat{R}_{q+1}^\text{osc} + \nabla P_{\text{osc}})
\]
\[
+ \text{div}(w_{q+1}^{(1)} + w_{q+1}^{(o)}) \otimes w_{q+1} + w_{q+1}^{(p)} \otimes (w_{q+1}^{(1)} + w_{q+1}^{(o)})
\]
\[
- (d_{q+1}^{(1)} + d_{q+1}^{(o)}) \otimes d_{q+1} - d_{q+1}^{(p)} \otimes (d_{q+1}^{(1)} + d_{q+1}^{(o)} + d_{q+1}^{(o)}) + \nu_1 \text{div}(\hat{R}_{q+1}^\text{cor} + \nabla P_{\text{cor}})
\]  
\Rightarrow (5.53)

Then, using the inverse divergence operator $\mathcal{R}^u$ given by (A.5a) we choose the Reynolds stress
\[
\hat{R}_{q+1}^u := \hat{R}_\text{lin}^u + \hat{R}_{\text{osc}}^u + \hat{R}_{\text{cor}}^u,
\]  
\Rightarrow (5.54)

where the linear error
\[
\hat{R}_\text{lin}^u := \mathcal{R}^u \left(\partial_t(u_{q+1}^{(p)} + u_{q+1}^{(c)}) + \nu_1 \mathcal{R}^u(-\Delta)^\alpha w_{q+1}
\right.
\[
+ \mathcal{R}^u P_H \text{div}\left(\tilde{u}_q \otimes w_{q+1} + w_{q+1} \otimes \tilde{u}_q - \tilde{B}_q \otimes d_{q+1} - d_{q+1} \otimes \tilde{B}_q\right),
\]
\Rightarrow (5.55)

the oscillation error
\[
\hat{R}_{\text{osc}}^u := \sum_{k \in \Lambda_u} \mathcal{R}^u P_H \mathcal{P} \neq 0 \left(g_{(r)}^2 \mathcal{P} \neq 0 (W_k \otimes W_k) \nabla(a_{(k)}^2)\right)
\]
\[
+ \sum_{k \in \Lambda_B} \mathcal{R}^u P_H \mathcal{P} \neq 0 \left(g_{(r)}^2 \mathcal{P} \neq 0 (W_k \otimes W_k) - D(k) \otimes D(k) \nabla(a_{(k)}^2)\right)
\]
\[
- \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u P_H \mathcal{P} \neq 0 \left(\partial_t (a_{(k)}^2 g_{(r)}^2) \phi_{(k)}^2 \phi_{(k)}^1\right)
\]
\[
- \sigma^{-1} \sum_{k \in \Lambda_u} \mathcal{R}^u P_H \mathcal{P} \neq 0 \left(h_{(r)} \int_{\mathcal{T}} W_k \otimes W_k d\mathcal{T} \nabla(a_{(k)}^2)\right)
\]
\[
- \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u P_H \mathcal{P} \neq 0 \left(h_{(r)} \int_{\mathcal{T}} W_k \otimes W_k - D(k) \otimes D(k) d\mathcal{T} \nabla(a_{(k)}^2)\right)
\]
\[
+ \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u P_H \text{div}\left(a_{(k)} a_{(k')} g_{(r)}^2 W_k \otimes W_{k'}\right)
\]
\[
- \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u P_H \text{div}\left(a_{(k)} a_{(k')} g_{(r)}^2 D(k) \otimes D(k')\right),
\]  
\Rightarrow (5.56)

and the corrector error
\[
\hat{R}_{\text{cor}}^u := \mathcal{R}^u P_H \text{div}\left(u_{q+1}^{(p)} \otimes u_{q+1}^{(p)} + w_{q+1}^{(t)} + u_{q+1}^{(o)} + (u_{q+1}^{(c)} + w_{q+1}^{(t)} + u_{q+1}^{(o)}) \otimes w_{q+1}
\right.
\[
- d_{q+1}^{(p)} \otimes d_{q+1}^{(p)} + d_{q+1}^{(t)} + u_{q+1}^{(o)} - (d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(o)}) \otimes d_{q+1} + d_{q+1}^{(o)}\).
\]  
\Rightarrow (5.57)

As in Remark 4.1, the new magnetic and Reynolds stresses at level $q + 1$ satisfy equations (5.48) and (5.53), respectively, and the algebraic identities (4.11a) and (4.11b) hold.

5.5. Verification of inductive estimates for velocity and magnetic stresses. Since by (5.32)-(5.35), the velocity and magnetic perturbations obey the same upper bounds as in (3.42)-(3.45), we can argue as in the same fashion as the proof of (4.12)-(4.14) to derive estimate (1.25) in the supercritical regime $S_2$. Hence, we focus on the verification of the $\mathcal{L}_1^1 \mathcal{L}_1^1$-decay estimate (1.26) below.
5.5.1. **Verification of $L_{1,\infty}^\alpha$-decay of magnetic stress.** We choose
\[ p := \frac{2 - 8\varepsilon}{2 - 9\varepsilon} \in (1, 2), \]
(5.58)
where $\varepsilon$ is given by (1.15), such that
\[ (1 - 4\varepsilon)(1 - \frac{1}{p}) = \frac{\varepsilon}{2}, \quad (2 - 8\varepsilon)(\frac{1}{p} - \frac{1}{2}) = 1 - 5\varepsilon, \]
(5.59)
and, via (5.1),
\[ r_\perp^{\frac{1}{p} - 1} r_\perp^{\frac{1}{p} - 1} = \lambda^\varepsilon, \quad r_\perp^{\frac{1}{2} + \frac{1}{p}} r_\perp^{\frac{1}{2} - \frac{1}{p}} = \lambda^{-1 + 5\varepsilon}. \]
(5.60)
Let us estimate the three parts of the magnetic stress $\tilde{R}^B_{q+1}$ separately below.

(i) **Linear error.** Note that, by (5.21b) and the boundedness of $R^B$ curl in $L^p$,
\[
\|R^B \partial_t (d_{q+1}^{(p)} + d_{q+1}^{(c)})\|_{L^p_t L^p_x} \lesssim \sum_{k \in \Lambda_B} \|\text{curl} \partial_t (g_1(t) \eta_1(k) D_{C}^c(k))\|_{L^p_t L^p_x}
\]
\[
\lesssim \sum_{k \in \Lambda_B} \left( \|g_1(t)\|_{L^1_t} \left( \|a(k)\|_{C^1_t, \infty} \|D_1^c(k)\|_{C_t W^{1, p}_x} + \|a(k)\|_{C^1_t, \infty} \|\partial_t D_1^c(k)\|_{C_t W^{1, p}_x} \right) + \|\partial_t g_1(t)\|_{L^1_t} \|a(k)\|_{C^1_t, \infty} \|D_1^c(k)\|_{C_t W^{1, p}_x} \right). \]
(5.61)
Then, in view of Lemmas 5.1 and 5.2, (1.15), (5.1) and (5.60), we obtain
\[
\|R^B \partial_t (d_{q+1}^{(p)} + d_{q+1}^{(c)})\|_{L^p_t L^p_x} \lesssim \tau^{\frac{1}{2} - \frac{1}{2}(\theta_{q+1} - L^p_t W^{1, p}_x)} + \theta_{q+1}^2 \|D_{q+1}^{(c)}\|_{C_t W^{1, p}_x} + \|\partial_t D_{q+1}^{(c)}\|_{C_t W^{1, p}_x}
\]
(5.62)
For the hyper-resistivity term, using the interpolation inequality and Lemma 5.3 we derive
\[
\|R^B (-\Delta)^\alpha d_{q+1}^{(p)}\|_{L^p_t L^p_x} \lesssim \|\nabla^{2\alpha - 1} d_{q+1}^{(p)}\|_{L^p_t L^p_x}
\]
\[
\lesssim \|d_{q+1}^{(p)}\|_{L^p_t L^p_x} \|d_{q+1}^{(p)}\|_{L^{\frac{2\alpha}{\alpha-1}}_t L^{\frac{2\alpha}{\alpha-1}}_x}
\]
\[
\lesssim \theta_{q+1}^{-1} \lambda^{2\alpha - 1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} \tau^{-\frac{1}{2}}, \]
(5.63)
and
\[
\|R^B (-\Delta)^\alpha d_{q+1}^{(c)}\|_{L^p_t L^p_x} \lesssim \theta_{q+1}^{-2} \lambda^{2\alpha - 1} \mu^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}}, \]
(5.64)
\[
\|R^B (-\Delta)^\alpha d_{q+1}^{(t)}\|_{L^p_t L^p_x} \lesssim \theta_{q+1}^{-2} \lambda^{2\alpha - 1} \mu^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}}, \]
(5.65)
\[
\|R^B (-\Delta)^\alpha d_{q+1}^{(o)}\|_{L^p_t L^p_x} \lesssim \theta_{q+1}^{-4} \mu^{-1}. \]
(5.66)
Hence, taking into account (5.1), (5.26b) and (5.60) we obtain
\[
\|R^B (-\Delta)^\alpha d_{q+1}\|_{L^p_t L^p_x} \lesssim \|R^B (-\Delta)^\alpha d_{q+1}\|_{L^p_t L^p_x} + \|R^B (-\Delta)^\alpha d_{q+1}\|_{L^p_t L^p_x} + \|R^B (-\Delta)^\alpha d_{q+1}\|_{L^p_t L^p_x} + \|R^B (-\Delta)^\alpha d_{q+1}\|_{L^p_t L^p_x}
\]
\[
\lesssim \theta_{q+1}^{-1} \lambda^{2\alpha - 1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} \tau^{-\frac{1}{2}} + \theta_{q+1}^{-1} \lambda^{2\alpha - 1} \mu^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} + \theta_{q+1}^{-2} \lambda^{2\alpha - 1} \mu^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} + \theta_{q+1}^{-4} \mu^{-1}
\]
\[
\lesssim \theta_{q+1}^{-1} \lambda^{-\varepsilon}. \]
(5.67)
For the remaining terms in (5.50), again Lemma 5.3 together with (5.1) and (5.60) yield that
\[
\|R^B \text{Hdiv}(d_{q+1} \otimes \tilde{u}_q - \tilde{u}_q \otimes d_{q+1} + \tilde{B}_q \otimes w_{q+1} - w_{q+1} \otimes \tilde{B}_q)\|_{L^p_t L^p_x}
\]
\[
\lesssim \|d_{q+1} \otimes \tilde{u}_q - \tilde{u}_q \otimes d_{q+1} + \tilde{B}_q \otimes w_{q+1} - w_{q+1} \otimes \tilde{B}_q\|_{L^1_t L^1_x}
\]
\[
\lesssim \|\tilde{u}_q\|_{L^\infty_t H^2_x} \|d_{q+1}\|_{L^1_t L^1_x} + \|\tilde{B}_q\|_{L^\infty_t H^2_x} \|w_{q+1}\|_{L^1_t L^1_x}
\]
\[
\lesssim \lambda^5 (\theta_{q+1}^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} \tau^{-\frac{1}{2}} + \theta_{q+1}^{-2} \lambda^{2\alpha - 1} \mu^{-1} r_\perp^{\frac{1}{\alpha-1}} r_\parallel^{\frac{1}{\alpha-1}} + \theta_{q+1}^{-4} \mu^{-1}) \lesssim \theta_{q+1}^{-17} \lambda^{-2\varepsilon}. \]
(5.68)
Therefore, combining (5.62), (5.67), (5.68) altogether and using (1.13) we arrive at
\[ \| \tilde{R}_{\text{osc}}^B \|_{L^1_t L^p_x} \lesssim \frac{\theta^{-14} \lambda^{-2e} + \theta^{-1} \lambda^{-e} + \theta^{-17} \lambda^{-2e}}{q + 1} \lesssim \frac{\theta^{-1} \lambda^{-e}}{q + 1}. \] (5.69)

(ii) Oscillation error. In contrast to the previous regime \( S_1 \), because of the presence of the temporal correctors \( \hat{v}_{q+1}^{(o)} \) and \( \hat{q}_{q+1}^{(o)} \), the magnetic oscillation error consists of four parts:
\[ \hat{R}_{\text{osc}}^B = \hat{R}_{\text{osc},1}^B + \hat{R}_{\text{osc},2}^B + \hat{R}_{\text{osc},3}^B + \hat{R}_{\text{osc},4}^B, \]
where \( \hat{R}_{\text{osc},1}^B \) contains the high-low spatial oscillations
\[ \hat{R}_{\text{osc},1}^B := \sum_{k \in \Lambda_B} R^B \mathbb{P}_{H}^\perp \mathbb{P} \neq 0 \left( g_{(r)}^2 \mathbb{P} \neq 0 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \nabla(a^2_{(k)}) \right), \]
\( \hat{R}_{\text{osc},2}^B \) contains the high temporal oscillation
\[ \hat{R}_{\text{osc},2}^B := -\mu^{-1} \sum_{k \in \Lambda_B} R^B \mathbb{P}_{H}^\perp \mathbb{P} \neq 0 \left( \partial_t (a^2_{(k)} g_{(r)}^2) \psi_{(k)}^2 \phi_{(k)}^2 k_2 \right), \]
\( \hat{R}_{\text{osc},3}^B \) is of low frequency
\[ \hat{R}_{\text{osc},3}^B := -\sigma^{-1} \sum_{k \in \Lambda_B} R^B \mathbb{P}_{H}^\perp \mathbb{P} \neq 0 \left( h_{(r)} \int_{\mathbb{T}^3} D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)} d x \partial_t \nabla(a^2_{(k)}) \right), \]
and \( \hat{R}_{\text{osc},4}^B \) contains the the interactions
\[ \hat{R}_{\text{osc},4}^B := \left( \sum_{k \neq k' \in \Lambda_B} + \sum_{k \in \Lambda_B, k' \in \Lambda_B} \right) R^B \mathbb{P}_{H}^\perp \text{div} \left( a_{(k)} a_{(k')} g_{(r)}^2 (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \right). \]

Let us separate the four parts separately in the following.

The decoupling Lemma A.4 with \( a = \nabla(a^2_{(k)}), f = \psi_{(k)}^2 \phi_{(k)}^2 \) and \( k = \lambda r_{\perp} / 2 \) permits to control the high-low oscillations error:
\[ \| \hat{R}_{\text{osc},1}^B \|_{L^1_t L^p_x} \lesssim \sum_{k \in \Lambda_B} \| g_{(r)} \|_{L^2_t} \| \nabla |^{-1} \mathbb{P} \neq 0 \left( \mathbb{P} \mathbb{P}_{(\lambda r_{\perp} / 2)} (D_{(k)} \otimes W_{(k)} - W_{(k)} \otimes D_{(k)}) \nabla(a^2_{(k)}) \right) \| \mathbb{P}_{C^1 L^p_x} \]
\[ \lesssim \sum_{k \in \Lambda_B} \lambda^{-1} r_{\perp}^{-1} \| \nabla |^{-1} a^2_{(k)} \| \mathbb{P}_{C^1 L^p_x} \| C^1 L^p_x \]
\[ \lesssim \sum_{k \in \Lambda_B} \theta^{-23} q_{q+1} \lambda^{-1} r_{\perp}^{-1} \| \psi_{(k)}^2 \phi_{(k)}^2 \| \mathbb{P}_{C^1 L^p_x} \]
\[ \lesssim \theta^{-23} q_{q+1} \lambda^{-1} r_{\perp}^{-2} \frac{1}{r_{\perp}^{\sigma-1}}, \] (5.70)
where we also used Lemmas 5.1 and 5.2 in the last two steps.

Regarding the second part \( \hat{R}_{\text{osc},2}^B \), we note that the derivative landing on \( g_{(r)} \) gives rise to high temporal oscillations. The point here is to balance the high temporal oscillations by the large parameter \( \mu \). Precisely, we apply Fubini’s theorem, Lemmas 3.2, 3.1 and 5.2 to derive
\[ \| \hat{R}_{\text{osc},2}^B \|_{L^1_t L^p_x} \lesssim \mu^{-1} \sum_{k \in \Lambda_B} \| \nabla |^{-1} \mathbb{P} \neq 0 \left( \partial_t (a_{(k)}^2 g_{(r)}^2) \psi_{(k)}^2 \phi_{(k)}^2 k_2 \right) \| \mathbb{P}_{C^1 L^p_x} \]
\[ \lesssim \mu^{-1} \sum_{k \in \Lambda_B} \left( \| \partial_t (a_{(k)}^2) \|_{C^1 L^p_x} \| g_{(r)}^2 \|_{L^1_t L^p_x} + a_{(k)} \|_{C^1 L^p_x} \| \partial_t (g_{(r)}^2) \|_{L^1_t L^p_x} \| \psi_{(k)}^2 \phi_{(k)}^2 \|_{C^1 L^p_x} \right) \]
\[ \lesssim \left( (\theta^{-9} + \theta^{-2} \tau \sigma) \mu^{-1} \frac{1}{r_{\perp}^{\sigma-1}} \right) \frac{1}{r_{\perp}^{\sigma-1}} \lesssim \theta^{-9} \tau \sigma \mu^{-1} r_{\perp}^{\sigma-1} \frac{1}{r_{\perp}^{\sigma-1}}. \] (5.71)
The low frequency part \( \hat{R}_{\text{osc},3}^B \) can be estimated easily by (3.14) and (5.17),
\[ \| \hat{R}_{\text{osc},3}^B \|_{L^1_t L^p_x} \lesssim \sigma^{-1} \sum_{k \in \Lambda_B} \| h_{(r)} (k_2 \otimes k_1 - k_1 \otimes k_2) \partial_t \nabla(a_{(k)}^2) \| \mathbb{P}_{L^1 L^p_x} \]
\[ \lesssim \sigma^{-1} \sum_{k \in \Lambda_B} \| h_{(r)} \| C^1 \| a_{(k)} \| C^1 \| a_{(k)} \| C^2_{\tau, x} + \| a_{(k)} \| C^1_{\tau, x} \). \]
\[ \lesssim \theta_{q+1}^{-15} \sigma^{-1}. \]  

Finally, for the interaction errors \( \tilde{R}^{B}_{osc, 4} \), it can be controlled by the product estimate in Lemma 5.1, due to the small intersections between different spatial intermittent flows:

\[ \| \tilde{R}^{B}_{osc, 4} \|_{L^1_t L^p_x} \lesssim \left( \sum_{k \neq k' \in A_B} + \sum_{k \in A_B} \right) \| a(k)a(k') \theta_{2r}(D_{k'}) \otimes W_{k} - W_{k} \otimes D_{k'} \|_{L^1_t L^p_x} \]

\[ \lesssim \left( \sum_{k \neq k' \in A_B} + \sum_{k \in A_B} \right) \| a(k) \|_{C_{t,x}} \| a(k') \|_{C_{t,x}} \| \theta_{2r} \psi_{k} \phi_{k'} \|_{L^1_t L^p_x} \]

\[ \lesssim \theta_{q+1}^{-2} \frac{1}{r} \frac{1}{r_\perp}. \]  

(5.73)

Therefore, putting estimates (5.70)-(5.73) altogether and using (5.1), (5.60) we arrive at

\[ \| R^{B}_{osc} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-23} \lambda^{-1} \frac{1}{r} \frac{1}{r_\perp} + \theta_{q+1}^{-9} \tau \sigma \mu^{-1} \frac{1}{r} \frac{1}{r_\perp} + \theta_{q+1}^{-15} \lambda^{-2} + \theta_{q+1}^{-2} \frac{1}{r} \frac{1}{r_\perp} \]

\[ \lesssim \theta_{q+1}^{-23} \lambda^{-1} \varepsilon + \theta_{q+1}^{-9} \lambda^{2n-3+12\varepsilon} + \theta_{q+1}^{-15} \lambda^{-2\varepsilon} + \theta_{q+1}^{-2} \lambda^{-1+9\varepsilon} \]

\[ \lesssim \theta_{q+1}^{-23} \lambda^{-1} \varepsilon, \]  

(5.74)

where the last step is due to \( \varepsilon < 1/10 \).

(iii) **Corrector error.** Let \( p_1, p_2 \in (1, \infty) \) be such that \( 1/p_1 = 1 - \eta \) and \( 1/p_1 = 1/p_2 + 1/2 \) where \( \eta \leq \varepsilon/(2(2-8\varepsilon)) \). Then, using Hölder’s inequality, applying Lemma 5.3 and using (5.40) we get

\[ \| \tilde{R}^{B}_{cor} \|_{L^1_t L^p_x} \lesssim \| \tilde{w}^{(c)}_{q+1} \|_{L^1_t L^2_x} + \| \tilde{w}^{(t)}_{q+1} \|_{L^2_t L^2_x} \| \tilde{w}^{(o)}_{q+1} \|_{L^2_t L^2_x} + \| \tilde{d}^{(c)}_{q+1} \|_{L^1_t L^2_x} + \| \tilde{d}^{(t)}_{q+1} \|_{L^2_t L^2_x} \]

\[ \lesssim \| w_{q+1}^{(c)} \|_{L^1_t L^2_x} + \| w_{q+1}^{(t)} \|_{L^2_t L^2_x} \| w_{q+1}^{(o)} \|_{L^2_t L^2_x} \| d_{q+1}^{(c)} \|_{L^1_t L^2_x} + \| d_{q+1}^{(t)} \|_{L^2_t L^2_x} \]

\[ \lesssim \theta_{q+1}^{-22} \lambda^{-\varepsilon} + \theta_{q+1}^{-9} \lambda^{2n-3+12\varepsilon} + \theta_{q+1}^{-15} \lambda^{-2\varepsilon} + \theta_{q+1}^{-2} \lambda^{-1+9\varepsilon} \lesssim \theta_{q+1}^{-22} \lambda^{-\varepsilon}. \]  

(5.75)

Now, combining estimates (5.69), (5.74) and (5.75) altogether of the three types of the magnetic stress we get for the magnetic stress

\[ \| \tilde{R}^{B}_{q} \|_{L^1_t L^p_x} \lesssim \| \tilde{R}^{B}_{inj} \|_{L^1_t L^p_x} + \| \tilde{R}^{B}_{osc} \|_{L^1_t L^p_x} + \| \tilde{R}^{B}_{cor} \|_{L^1_t L^p_x} \]

\[ \lesssim \theta_{q+1}^{-21} \lambda^{-\varepsilon} - \theta_{q+1}^{-23} \lambda^{-\varepsilon} + \theta_{q+1}^{-9} \lambda^{2n-3+12\varepsilon} + \theta_{q+1}^{-15} \lambda^{-2\varepsilon} + \theta_{q+1}^{-2} \lambda^{-1+9\varepsilon} \lesssim \theta_{q+1}^{-22} \lambda^{-\varepsilon}. \]  

(5.76)

where we also used (1.13) in the last step. This verifies the \( L^1_t L^p_x \)-decay estimate (1.26) for the magnetic stress at level \( q + 1 \).

5.5.2. **Verification of \( L^1_t L^p_x \)-decay of Reynolds stress.** We shall verify the \( L^1_t L^p_x \)-decay estimate (1.26) of the three parts \( \tilde{R}^{u}_{inj} \), \( \tilde{R}^{u}_{osc} \) and \( \tilde{R}^{u}_{cor} \) for the Reynolds stress \( \tilde{R}^{u}_{q+1} \) at level \( q + 1 \).

(i) **Linear error.** The linear error \( \tilde{R}^{u}_{inj} \) can be estimated in the same fashion as the proof of (5.69) and so we also have

\[ \| \tilde{R}^{u}_{inj} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-28} \lambda^{-2\varepsilon} + \theta_{q+1}^{-9} \lambda^{-\varepsilon} + \theta_{q+1}^{-16} \lambda^{-2\varepsilon} \lesssim \theta_{q+1}^{-21} \lambda^{-\varepsilon}. \]  

(5.77)

(ii) **Oscillation error.** The velocity oscillation error \( \tilde{R}^{u}_{osc} \) consists of the following four parts, via (5.56),

\[ \tilde{R}^{u}_{osc} = \tilde{R}^{u}_{osc, 1} + \tilde{R}^{u}_{osc, 2} + \tilde{R}^{u}_{osc, 3} + \tilde{R}^{u}_{osc, 4} \]

where \( \tilde{R}^{u}_{osc, 1} \) is the high-low spatial frequency part

\[ \tilde{R}^{u}_{osc, 1} := \sum_{k \in A_u} R^{u} \not\equiv 0 \left( g_{(r)}^{2} \not\equiv 0 (W_{k} \otimes W_{k}) \nabla (a_{(k)}^{2}) \right) \]

\[ + \sum_{k \in A_B} R^{u} \not\equiv 0 \left( g_{(r)}^{2} \not\equiv 0 (W_{k} \otimes W_{k} - D_{k} \otimes D_{k}) \nabla (a_{(k)}^{2}) \right); \]  

(5.78)
\( \hat{R}_{\text{osc,2}} \) contains the high temporal oscillations

\[
\hat{R}_{\text{osc,2}}^u := -\mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u\mathbb{P}_{\neq 0} \left( \partial_k(a_k^2 \hat{g}_{(\tau)}^2) \psi_{(k)}^2 \phi_{(k)}^2 k_l \right),
\]

(5.79)

\( \hat{R}_{\text{osc,3}}^u \) is of low frequency

\[
\hat{R}_{\text{osc,3}}^u := -\sigma^{-1} \sum_{k \in \Lambda_u} \mathcal{R}^u\mathbb{P}_{\neq 0} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} dx \partial_t \nabla (a_k^2_{(k)}) \right)

- \sigma^{-1} \sum_{k \in \Lambda_B} \mathcal{R}^u\mathbb{P}_{\neq 0} \left( h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \partial_t \nabla (a_k^2_{(k)}) \right),
\]

(5.80)

and \( \hat{R}_{\text{osc,4}}^u \) contains the interaction error

\[
\hat{R}_{\text{osc,4}}^u := \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} \mathcal{R}^u\mathbb{P}_{\neq 0} \left( a_k(a_k')g_{(\tau)}^2 W_{(k)} \otimes W_{(k')} \right)

- \sum_{k \neq k' \in \Lambda_B} \mathcal{R}^u\mathbb{P}_{\neq 0} \left( a_k(a_k')g_{(\tau)}^2 D_{(k)} \otimes D_{(k')} \right).
\]

Let us treat \( \hat{R}_{\text{osc,1}}^u \), \( \hat{R}_{\text{osc,2}}^u \), \( \hat{R}_{\text{osc,3}}^u \) and \( \hat{R}_{\text{osc,4}}^u \) separately below. Again we shall apply Lemma A.4 to decouple the high-low interactions to get

\[
\| \hat{R}_{\text{osc,1}}^u \|_{L^p_t L^2_x} \lesssim \sum_{k \in \Lambda_u} ||g_{(\tau)}^2||_{L^1_t} ||\nabla||^{-1} \mathbb{P}_{\neq 0} (\mathbb{P}_{\lambda \tau / 2} (W_{(k)} \otimes W_{(k)}) \nabla (a_k^2))||_{L^2_t L^p_x}

+ \sum_{k \in \Lambda_B} ||g_{(\tau)}^2||_{L^1_t} ||\nabla||^{-1} \mathbb{P}_{\neq 0} (W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)}) \nabla (a_k^2))||_{L^2_t L^p_x}

\lesssim \sum_{k \in \Lambda_u \cup \Lambda_B} (\lambda \tau)^{-1} ||\nabla^3(a_k^2)\|_{C_{r,x}} ||\psi_{(k)}^2\|_{C_{r,x}} \cdot \|\phi_{(k)}^2\|_{C_{r,x}}

\lesssim \theta^{-2} q \lambda^{-1} p^{-2} \frac{4}{p} \| \mathbb{P}_{\neq 0} \|_{L^p_x},
\]

(5.81)

where we also used Lemmas 5.1 and 5.2 in the last step.

Regarding the high temporal oscillation error \( \hat{R}_{\text{osc,2}}^u \), estimating as in the proof of (5.71) we get

\[
\| \hat{R}_{\text{osc,2}}^u \|_{L^1_t L^p_x} \lesssim \mu^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} ||\partial_k(a_k^2)||_{C_{r,x}} ||g_{(\tau)}^2||_{L^1_t} + ||a_k||_{C_{r,x}} ||\partial_k(g_{(\tau)}^2)||_{L^1_t} ||\psi_{(k)}||_{C_{r,x}} \cdot ||\phi_{(k)}||_{L^p_x}

\lesssim \theta^{-2} q \sigma \mu^{-1} \| \mathbb{P}_{\neq 0} \|_{L^p_x},
\]

(5.82)

Moreover, we bound the low frequency part \( \hat{R}_{\text{osc,3}}^u \) by

\[
\| \hat{R}_{\text{osc,3}}^u \|_{L^1_t L^p_x} \lesssim \sigma^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \|\nabla||^{-1} \mathbb{P}_{\neq 0} (h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} dx \partial_t \nabla (a_k^2))||_{L^2_t L^p_x}

+ \sigma^{-1} \sum_{k \in \Lambda_B} \|\nabla||^{-1} \mathbb{P}_{\neq 0} (h_{(\tau)} \int_{T^3} W_{(k)} \otimes W_{(k)} - D_{(k)} \otimes D_{(k)} dx \partial_t \nabla (a_k^2))||_{L^2_t L^p_x}

\lesssim \sigma^{-1} \sum_{k \in \Lambda_u \cup \Lambda_B} \|h_{(\tau)}\|_{C_{r,x}} \cdot ||a_k||_{C_{r,x}} ||a_k||_{C_{r,x}} + ||a_k||_{C_{r,x}^2}

\lesssim \theta^{-2} q \sigma^{-1} \| \mathbb{P}_{\neq 0} \|_{L^p_x},
\]

(5.83)

Finally, for the interaction oscillation \( \hat{R}_{\text{osc,4}}^u \), we apply Lemma 5.1 to get

\[
\| \hat{R}_{\text{osc,4}}^u \|_{L^1_t L^p_x} \lesssim \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} ||a_k(a_k')g_{(\tau)}^2 W_{(k)} \otimes W_{(k')}||_{L^1_t L^2_x} + \sum_{k \neq k' \in \Lambda_B} ||a_k(a_k')g_{(\tau)}^2 D_{(k)} \otimes D_{(k')}||_{L^1_t L^2_x}

\lesssim \sum_{k \neq k' \in \Lambda_u \cup \Lambda_B} ||a_k||_{C_{r,x}} ||a_k||_{C_{r,x}} ||g_{(\tau)}^2||_{L^1_t} ||\psi_{(k)}\| \cdot ||\phi_{(k')}||_{C_{r,x}} \cdot ||\phi_{(k')}||_{C_{r,x}}

\lesssim \theta^{-2} q \sigma \mu^{-1} \| \mathbb{P}_{\neq 0} \|_{L^p_x}.
\]

(5.84)
Thus, putting estimates (5.81)-(5.84) altogether we arrive at
\[
\| \tilde{R}^u_{osc} \|_{L^1_t L^p_x} \lesssim \theta_{q+1}^{-44 \lambda - \varepsilon} + \theta_{q+1}^{\frac{1}{2} - \frac{1}{p} - 1} + \theta_{q+1}^{\frac{1}{2} - \frac{1}{p} - 1} + \theta_{q+1}^{-29 \sigma - 1} + \theta_{q+1}^{\frac{1}{2} - \frac{1}{p} - 1}
\]
\[
\lesssim \theta_{q+1}^{-44 \lambda - \varepsilon}.
\]  
(5.85)

(iii) Corrector error. Taking \( p_1, p_2 \in (1, \infty) \) as in the proof of (5.75), applying Lemma 5.3 and using (5.40) we have that, similarly to (5.75),
\[
\| \tilde{R}^u_{cor} \|_{L^1_t L^p_x} \lesssim \| w^{(c)}_{q+1} + w^{(t)}_{q+1} + w^{(o)}_{q+1} \|_{L^1_t L^{p_2}_x} (\| u_q^{(p)} \|_{L^1_t L^2_x} + \| w_{q+1} \|_{L^1_t L^{p_2}_x})
\]
\[
+ (\| d_q^{(p)} \|_{L^1_t L^{p_2}_x} + \| d_{q+1} \|_{L^1_t L^{p_2}_x}) (\| d_q^{(c)} + d_q^{(t)}_{q+1} + d_q^{(o)}_{q+1} \|_{L^1_t L^{p_2}_x})
\]
\[
\lesssim \theta_{q+1}^{-32 \lambda - \varepsilon}.
\]  
(5.86)

Therefore, combining estimates (5.77), (5.85) and (5.86) we obtain the estimates of Reynolds stress
\[
\| \tilde{R}^u_{q+1} \|_{L^1_t L^{p_2}_x} \leq \| \tilde{R}^u_{tin} \|_{L^1_t L^{p_2}_x} + \| \tilde{R}^u_{osc} \|_{L^1_t L^{p_2}_x} + \| \tilde{R}^u_{cor} \|_{L^1_t L^{p_2}_x}
\]
\[
\lesssim \theta_{q+1}^{-1} \lambda - \varepsilon + \theta_{q+1}^{-44 \lambda - \varepsilon} + \theta_{q+1}^{-32 \lambda - \varepsilon}
\]
\[
\leq \lambda^{\frac{1}{q+1}} q^{2 q+2}_+. \hspace{1cm} (5.87)
\]

where \( p \) and \( p_1 \) are as in (4.15) and (5.86). Thus, the \( L^1_{t,x} \)-decay estimate (1.26) for the Reynolds stress \( \tilde{R}^u_{q+1} \) is verified at level \( q+1 \).

6. PROOF OF MAIN RESULTS

We are now in position to prove the main results, mainly for the iteration estimates in Theorem 1.7 and the non-uniqueness in Theorem 1.2. Theorem 1.4 can be proved in a similar fashion as the proof of [49, Theorem 1.5], based on the present convex integration stage in §3-§5, and on certain mollification procedure as in [49] instead of the gluing stage. We note that, the mollification procedure only involves an extra parameter \( \ell \), which is of negligible order as the role of \( \theta_{q+1} \) and hence does not affect the main iteration estimates as in Theorem 1.7. Hence, for the simplicity of exposition we omit the details here. Below we focus on the proof of Theorems 1.7 and 1.2.

6.1. Proof of main iteration in Theorem 1.7. The inductive estimates (1.23)-(1.30) have been verified in the previous sections. Hence, it remains to verify the well-preparedness of \((u_{q+1}, B_{q+1}, \tilde{R}^u_{q+1}, \tilde{R}^B_{q+1})\) and the inductive inclusion (1.31) for the temporal supports.

For the well-preparedness of \((u_{q+1}, B_{q+1}, \tilde{R}^u_{q+1}, \tilde{R}^B_{q+1})\), because the temporal cut-off functions \( f_B \) and \( f_u \) respect the support of \((\tilde{R}^u_{q}, \tilde{R}^B_{q})\), in view of (1.21), we deduce
\[
\| w_{q+1} (t) = d_{q+1} (t) = 0 \| \leq \theta_{q+1}.
\]

Then, by (3.38) and (5.27),
\[
(u_{q+1} (t), B_{q+1} (t)) = (\bar{w}_{q+1} (t), \bar{B}_{q+1} (t)) \| \leq \theta_{q+1}.
\]

Thus, using (1.21) again we lead to
\[
(\tilde{R}^u_{q+1} (t), \tilde{R}^B_{q+1} (t)) = (\tilde{R}^u_{q+1} (t), \tilde{R}^B_{q+1} (t)) \| \leq \theta_{q+1},
\]
which verifies the well-preparedness of \((u_{q+1}, B_{q+1}, \tilde{R}^u_{q+1}, \tilde{R}^B_{q+1})\).

The proof of the temporal inductive inclusion (1.31) is similar to that in [50]. For the reader’s convenience, we sketch the main arguments below. By the construction of perturbations and amplitudes in §3 and §5 and (1.21),
\[
\| supp_{t}(w_{q+1}, d_{q+1}) \| \leq \bigcup_{k \in \Lambda_n \cup \Lambda_f} \| supp_{t}(a_{(k)}) \| \leq N_{2q} (supp_{t}(\tilde{R}^u_{q}, \tilde{R}^B_{q})) \| \leq N_{2q} (I_{q+1}). \hspace{1cm} (6.1)
\]

But, by the construction of \( I_{q+1} \) in (2.82), one has,
\[
I_{q+1} \| \leq N_{4T/m_{q+1}} (supp_{t}(\tilde{R}^u_{q}, \tilde{R}^B_{q})). \hspace{1cm} (6.2)
\]
Hence, it follows from (6.1) and (6.2) that
\[
\text{supp}_t(w_{q+1}, d_{q+1}) \subseteq N_{6T/m_{q+1}}(\text{supp}_t(\tilde{R}^u_{q}, \tilde{R}^B_{q})).
\] (6.3)

Thus, taking into account (3.38) and (5.27) we arrive at
\[
\text{supp}_t(u_{q+1}, B_{q+1}) \subseteq \text{supp}_t(\tilde{u}_q, \tilde{B}_q) \cup \text{supp}_t(w_{q+1}, d_{q+1}) \subseteq N_{6T/m_{q+1}}(\text{supp}_t(u_q, B_q, \tilde{R}^u_q, \tilde{R}^B_q)) \subseteq N_{\frac{1}{\delta_{q+2}}} (\text{supp}_t(u_q, B_q, \tilde{R}^u_q, \tilde{R}^B_q)),
\] (6.4)

where the last step is due to $6T/m_{q+1} \ll \delta_{q+2}^{1/2}$.

By the construction of stresses in §4 and §5, (6.1), (6.3) and (1.22), we also have
\[
\text{supp}_t(\tilde{R}^u_{q+1}, \tilde{R}^B_{q+1}) \subseteq \bigcup_{k \in \Lambda_0} \text{supp}_t(\alpha(k)) \cup \text{supp}_t(\tilde{u}_q, \tilde{B}_q) \subseteq N_{\frac{1}{\delta_{q+2}}} (\text{supp}_t(u_q, B_q, \tilde{R}^u_q, \tilde{R}^B_q)).
\] (6.5)

Finally, the inductive inclusion (1.31) follows from (6.4) and (6.5). The proof is complete. □

6.2. Proof of Theorem 1.2. We prove the statements (i)-(v) in Theorem 1.2 below.

(i). In the initial step $q = 0$, we take the relaxed solution for equation (1.11) as follows:
\[
(u_0, B_0) = (\tilde{u}, \tilde{B}),
\] (6.6)
\[
\tilde{R}^u_0 = R^n (\partial_t u_0 + \nu_1 (-\Delta)^n u_0) + u_0 \otimes u_0 - B_0 \otimes B_0,
\] (6.7)
\[
\tilde{R}^B_0 = R^B (\partial_t B_0 + \nu_2 (-\Delta)^n B_0) + B_0 \otimes u_0 - u_0 \otimes B_0,
\] (6.8)

together with $P_0 = -\frac{1}{T}((u_0)^2 - |B_0|^2)$.

Then, $(u_0, B_0)$ is a well-prepared solution to (1.11) with the set $I_0 = [0, T]$ and $\theta_0 = T$. Moreover, we choose $a$ sufficiently large such that (1.23)-(1.26) are satisfied at level $q = 0$. Thus, we apply Theorem 1.7 to obtain a sequence of relaxed solutions $(u_{q+1}, B_{q+1}, \tilde{R}^u_{q+1}, \tilde{R}^B_{q+1})$ to (1.11) obeying estimates (1.23)-(1.31) for all $q \geq 0$.

Below we show that $\{(u_{q+1}, B_{q+1})\}_q$ is a Cauchy sequence in $H^{\beta'}_{t,x}$, $\beta' \in \left(0, \frac{2}{1+\beta}\right)$, and the limit solves (1.5) in the sense of Definition 1.1.

Actually, by (1.23), (1.24) and the Sobolev embedding $H^{\frac{3}{2}} \hookrightarrow L^{\infty}$,
\[
\|u_{q+1} - u_q\|_{H^{\frac{3}{2}}_{t,x}} \leq \|\partial_t (u_{q+1} - u_q)\|_{L^{\infty} H^2_{t,x}} + \|u_{q+1} - u_q\|_{L^\infty H^2_{t,x}} \leq \|\partial_t u_{q+1}\|_{L^\infty H^2_{t,x}} + \|u_{q+1}\|_{L^\infty H^2_{t,x}} \leq \lambda_{q+1}^7 + \lambda_1^7 + \lambda_{q+1}^7 \lesssim \lambda_{q+1}^7,
\] (6.9)

and similarly,
\[
\|B_{q+1} - B_q\|_{H^{\beta'}_{t,x}} \leq \|\partial_t (B_{q+1} - B_q)\|_{L^\infty H^2_{t,x}} + \|B_{q+1} - B_q\|_{L^\infty H^2_{t,x}} \lesssim \lambda_{q+1}^7.
\] (6.10)

Then, by the interpolation inequality and (1.12), (1.28), (6.9) and (6.10), for any $\beta' \in (0, \frac{2}{1+\beta})$,
\[
\sum_{q \geq 0} \|[(u_{q+1} - u_q, B_{q+1} - B_q)]_{H^{\beta'}_{t,x}} \leq \sum_{q \geq 0} \|[(u_{q+1} - u_q, B_{q+1} - B_q)]_{L^{1-\beta'}_{t,x}}\|_{H^{\beta'}_{t,x}} \lesssim \sum_{q \geq 1} M^{1-\beta'} \lambda_{q+1}^{\frac{1-\beta'}{2}} \lambda_1^{\beta'} + \sum_{q \geq 1} M^{1-\beta'} \lambda_{q+1}^{\beta'(1-\beta') + \beta'} < \infty.
\] (6.11)

Thus, $\{(u_q, B_q)\}_{q \geq 0}$ is a Cauchy sequence in $H^{\beta'}_{t,x}$ and so, there exist $u, B \in H^{\beta'}_{t,x}$ such that
\[
\lim_{q \to +\infty} (u_q, B_q) = (u, B) \text{ in } H^{\beta'}_{t,x}.
\] (6.12)
Since by (1.26), \( \lim_{q \to +\infty} (\tilde{R}_q, \tilde{\tilde{R}}_q) = 0 \) in \( L^1_t L^\infty_x \), it follows that \((u, B)\) indeed solves (1.5) in the sense of Definition 1.1.

(ii). By virtue of the inductive decay estimate (1.30), we have

\[
\sum_{q \geq 0} \|(u_{q+1} - u_q, B_{q+1} - B_q)\|_{L^1_t W^s_x} < \infty. \tag{6.13}
\]

This yields that \( \{(u_q, B_q)\}_{q \geq 0} \) is also a Cauchy sequence in \( L^1_t W^s_x \). Thus, taking into account (6.12) we infer that

\[
u, B \in H^{s}_t \cap L^1_t W^s_x,
\]

thereby proving the regularity statement (ii).

(iii). Regarding the Hausdorff measure of the singular set, we set

\[
\mathcal{G} := \bigcup_{q \geq 0} I^c_q \setminus \{0, T\}, \quad \mathcal{B} := [0, T] \setminus \mathcal{G}.
\]

By construction, \((u_q, B_q)\) is a smooth solution to the MHD equations (1.11) on \( \mathcal{G} \), and \((u, B) \equiv (u_q, B_q)\) on \( I^c_q \) for each \( q \). Thus, the complement set \( \mathcal{B} \) contains the singular set of time. Since by (2.82), each \( I_q \) is covered by at most \( m_q = \theta_q^{-\eta} \) many balls of radius \( 5\theta_q \), i.e.,

\[
\mathcal{B} \subseteq \bigcup_{i=0}^{m_{q+1}-1} [t_i - 2\theta_{q+1}, t_i + 3\theta_{q+1}]. \tag{6.14}
\]

Note that, for any \( \kappa > \eta \),

\[
\sum_{i=0}^{m_{q+1}-1} (5\theta_{q+1})^\kappa \leq m_{q+1} \theta_{q+1}^\kappa \leq \theta_{q+1}^{\kappa-\eta} \to 0, \quad \text{as } q \to \infty.
\]

This yields that the Hausdorff measure \( \mathcal{H}^\kappa(\mathcal{B}) = 0 \) for any \( \kappa > \eta \). Thus, we deduce

\[
d_\mathcal{H}(\mathcal{B}) \leq \eta < \eta_*.
\]

(iv). Concerning the temporal support, let

\[
K_q := \text{supp}_t (u_q, B_q, \tilde{R}_q, \tilde{\tilde{R}}_q), \quad q \geq 1. \tag{6.15}
\]

By (6.7) and (6.8),

\[
\text{supp}_t (\tilde{R}_0, \tilde{\tilde{R}}_0) \subseteq K_0 := \text{supp}_t (u_0, B_0) = \text{supp}_t (\tilde{\tilde{u}}, \tilde{\tilde{B}}). \tag{6.16}
\]

Moreover, by the inductive inclusion (1.31),

\[
K_{q+1} \subseteq N_{\frac{1}{3}} \bigcup_{j=2}^{q+1} \bigcup_{q \geq 2}^{m_q+1-2} K_0. \tag{6.17}
\]

Note that, for \( a \) large enough,

\[
\sum_{q \geq 0} \beta_{q+2} \leq \sum_{q \geq 2} a^{-\beta q^\eta} \leq \sum_{q \geq 2} a^{-\beta q} = \frac{a^{-2\beta}}{1-a^{-\beta}} \leq \frac{1}{2} \varepsilon_*.\]

Thus, it follows that

\[
\text{supp}_t (u, B) \subseteq \bigcup_{q \geq 0} K_q \subseteq N_{\varepsilon} (\text{supp}_t (\tilde{\tilde{u}}, \tilde{\tilde{B}})), \tag{6.18}
\]

thereby yielding the temporal support statement (iv).

(v). Finally, the small deviations on average follow from (1.29) and (1.30):

\[
\|(u - \tilde{\tilde{u}}, B - \tilde{\tilde{B}})\|_{L^1_t L^2_x} \leq \sum_{q \geq 0} \|(u_{q+1} - u_q, B_{q+1} - B_q)\|_{L^1_t L^2_x} \leq 2 \sum_{q \geq 0} \delta_{q+2} \leq \frac{2a^{-2\beta}}{1-a^{-\beta}} \leq \varepsilon_*,
\]

and

\[
\|(u - \tilde{\tilde{u}}, B - \tilde{\tilde{B}})\|_{L^1_t W^{s,p}_x} \leq \sum_{q \geq 0} \|(u_{q+1} - u_q, B_{q+1} - B_q)\|_{L^1_t W^{s,p}_x} \leq 2 \sum_{q \geq 0} \delta_{q+2} \leq \varepsilon_*.\]
Therefore, the proof of Theorem 1.2 is complete. \hfill \Box

APPENDIX A. STANDARD TOOLS IN CONVEX INTEGRATION

To begin with, let us first recall two geometric lemmas in [3].

Lemma A.1. **(First Geometric Lemma, [3, Lemma 4.2])** There exists a set \( \Lambda_u \subset S^2 \cap \mathbb{Q}^3 \) that consists of vectors \( k \) with associated orthonormal bases \( (k, k_1, k_2) \), \( \varepsilon_u > 0 \), and smooth positive functions \( \gamma(k) : \mathbb{B}_{\varepsilon_u}(\text{Id}) \to \mathbb{R} \), where \( \mathbb{B}_{\varepsilon_u}(\text{Id}) \) is the ball of radius \( \varepsilon_u \) centered at the identity in the space of \( 3 \times 3 \) symmetric matrices, such that for \( S \in \mathbb{B}_{\varepsilon_u}(\text{Id}) \) we have the following identity:

\[
S = \sum_{k \in \Lambda_u} \gamma^2_{(k)}(S) k_1 \otimes k_1. \tag{A.1}
\]

Furthermore, we may choose \( \Lambda_u \) such that \( \Lambda_B \cap \Lambda_u = \emptyset \).

Lemma A.2. **(Second Geometric Lemma, [3, Lemma 4.1])** There exists a set \( \Lambda_B \subset S^2 \cap \mathbb{Q}^3 \) that consists of vectors \( k \) with associated orthonormal bases \( (k, k_1, k_2) \), \( \varepsilon_B > 0 \), and smooth positive functions \( \gamma(k) : \mathbb{B}_{\varepsilon_B}(0) \to \mathbb{R} \), where \( \mathbb{B}_{\varepsilon_B}(0) \) is the ball of radius \( \varepsilon_B \) centered at \( 0 \) in the space of \( 3 \times 3 \) skew-symmetric matrices, such that for \( A \in \mathbb{B}_{\varepsilon_B}(0) \) we have the following identity:

\[
A = \sum_{k \in \Lambda_B} \gamma^2_{(k)}(A) (k_2 \otimes k_1 - k_1 \otimes k_2). \tag{A.2}
\]

As pointed out in [3], there exists \( N_A \in \mathbb{N} \) such that

\[
\{ N_A k, N_A k_1, N_A k_2 \} \subset N_A S^2 \cap \mathbb{Z}^3. \tag{A.3}
\]

For instance, as in [3], choosing

\[
\Lambda_u = \left\{ \frac{5}{13} e_1 + \frac{12}{13} e_2, \frac{12}{13} e_1 + \frac{5}{13} e_3, \frac{5}{13} e_2 + \frac{12}{13} e_3 \right\},
\]

and

\[
\Lambda_B = \left\{ e_1, e_2, e_3, \frac{3}{5} e_1 + \frac{4}{5} e_2, -\frac{4}{5} e_2 - \frac{3}{5} e_3 \right\},
\]

\( N_A = 65 \) suffices.

We denote by \( M_* \) the geometric constant such that

\[
\sum_{k \in \Lambda_u} \| \gamma(k) \| C^4(\mathbb{B}_{\varepsilon_u}(\text{Id})) + \sum_{k \in \Lambda_B} \| \gamma(k) \| C^4(\mathbb{B}_{\varepsilon_B}(0)) \leq M_* \tag{A.4}
\]

This parameter is universal and will be used later in the estimates of the size of perturbations.

We recall from [3, 28] the inverse-divergence operator \( R^u \) and \( R^B \), defined by

\[
(R^u v)^{kl} := \partial_k \Delta^{-1} v^l + \partial_l \Delta^{-1} v^k - \frac{1}{2}(\delta_{kl} + \partial_k \partial_l \Delta^{-1})\text{div}\Delta^{-1} v, \tag{A.5a}
\]

\[
(R^B f)_{ij} := \varepsilon_{ijk} (\Delta^{-1})^{-1}(\text{curl} f)_k, \tag{A.5b}
\]

where \( v \) is mean-free, i.e., \( \int_{\mathbb{T}^d} v \, dx = 0 \), \( \varepsilon_{ijk} \) is the Levi-Civita tensor, \( i, j, k, l \in \{1, 2, 3\} \). Note that, the inverse-divergence operator \( R \) maps mean-free functions to symmetric and trace-free matrices, while the operator \( R^B \) returns skew-symmetric matrices. Moreover, one has the algebraic identities

\[
\text{div} R^u(v) = v, \quad \text{div} R^B(f) = f.
\]

Both \( |\nabla| R^u \) and \( |\nabla| R^B \) are Calderon-Zygmund operators and thus they are bounded in the spaces \( L^p \), \( 1 < p < +\infty \). See [3, 28] for more details.

Lemma A.3 ( [18, Lemma 2.4; see also [12, Lemma 3.7]. Let \( \theta \in \mathbb{N} \) and \( f, g : \mathbb{T}^d \to \mathbb{R} \) be smooth functions. Then for every \( p \in [1, +\infty] \),

\[
\| fg(\theta) \|_{L^p(\mathbb{T}^d)} - \| f \|_{L^p(\mathbb{T}^d)} \| g \|_{L^p(\mathbb{T}^d)} \| \lesssim \theta^{-\frac{1}{p}} \| f \|_{C^0(\mathbb{T}^d)} \| g \|_{L^p(\mathbb{T}^d)}. \tag{A.6}
\]
Lemma A.4 ([53], Lemma 6; see also [12], Lemma B.1). Let $a \in C^2(\mathbb{T}^3)$. For all $1 < p < +\infty$ we have
\[
\|\nabla^{-1} P_{\neq 0} (aP_{\geq k} f)\|_{L^p(\mathbb{T}^3)} \lesssim k^{-1} \|\nabla^2 a\|_{L^\infty(\mathbb{T}^3)} \|f\|_{L^p(\mathbb{T}^3)},
\]
holds for any smooth function $f \in L^p(\mathbb{T}^3)$.

Lemma A.5 (Semigroup estimates). Let $\alpha \in [1, 2)$. For any $1 \leq \rho_2 \leq \rho_1 \leq \infty$ and $n \in \mathbb{N}$, we have
\begin{align}
\|e^{-\nu t(-\Delta)^s} v\|_{L^p_t L^s_x} &\leq C t^{\frac{s}{2}} \left(\frac{3}{2} - \frac{3}{p}\right) \|v\|_{L^2}, \\
\|\nabla^2 e^{-\nu t(-\Delta)^s} v\|_{L^p_t L^{s+1}_x} &\leq C t^{\frac{s}{2}} \left(\frac{3}{2} - \frac{3}{p}\right) \|v\|_{L^2}.
\end{align}

(A.7) hold for any function $v \in L^{p_2}(\mathbb{T}^3)$.

Lemma B.3 (Uniqueness of weak solutions)

In this section, we consider the uniqueness of weak solutions to (1.1) in the critical mixed Lebesgue space $L^p_t W^{s,p}_x$, where $(s, \gamma, p)$ lies in some suitable part of the generalized Ladyženskaja-Prodi-Serrin condition (1.7).

Let us define the space $X^{s,\gamma,p}_{(0,t)}$, $0 \leq t \leq T$, by
\[
X^{s,\gamma,p}_{(0,t)} = \begin{cases}
L^\gamma(0,t; W^{s,p}_x), & \text{if $\frac{2\alpha}{\gamma} + \frac{3}{p} = 2\alpha - 1 + 1 \leq \gamma < \infty$, $1 \leq p \leq \infty$, $s \geq 0$,} \\
C([0,t]; W^{s,p}_x), & \text{if $\gamma = \infty$, $1 \leq p \leq \infty$, $s \geq 0$.}
\end{cases}
\]

(B.1)
The main result is formulated in Theorem B.1 below.

Theorem B.1 (Uniqueness of weak solutions). Let $(u, B) \in X^{s,\gamma,p}_{(0,T)} \times X^{s,\gamma,p}_{(0,T)}$ and $(s, \gamma, p)$ satisfy (1.7) with $\alpha \geq 1$, $s \geq 0$, $2 \leq \gamma \leq \infty$, $1 \leq p \leq \infty$ and $0 \leq \frac{1}{p} - \frac{s}{2} < \frac{1}{2}$. If $(u, B)$ is a weak solution to (1.5) with $\nu_1 = \nu_2$ in the sense of Definition 1.1, then $(u, B)$ is the unique Leray-Hopf solution.

In particular, for the MHD equations (1.1), we have the uniqueness of weak solutions in $L^p_t L^p_x$ when $(\gamma,p)$ satisfies the Ladyženskaja-Prodi-Serrin condition
\[
\frac{2}{\gamma} + \frac{3}{p} = 1.
\]

(B.2)

Corollary B.2. Let $(u, B) \in X^{s,\gamma,p}_{(0,T)} \times X^{s,\gamma,p}_{(0,T)}$ and $(\gamma,p)$ satisfy (B.2) with $2 \leq \gamma \leq \infty$. If $(u, B)$ is a weak solution to (1.1) in the sense of Definition 1.1 with $\alpha = 1$ and $\nu_1 = \nu_2$, then $(u, B)$ is the unique Leray-Hopf solution.

As in the context of NSE [19, 50], let us first present the following existence lemma of linearized generalized MHD equations, which can be proved via the classical Galerkin method.

Lemma B.3. Let $(u, B) \in X^{s,\gamma,p}_{(0,T)} \times X^{s,\gamma,p}_{(0,T)}$ be a weak solution to (1.5) with $\nu_1 = \nu_2 = \nu$ and $(s, \gamma, p)$ satisfy (1.7) with $\alpha \geq 1$, $s \geq 0$, $2 \leq \gamma \leq \infty$, $1 \leq p \leq \infty$ and $0 \leq \frac{1}{p} - \frac{s}{2} < \frac{1}{2}$. Given any divergence-free datum $(v_0, H_0) \in L^2(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$, there exists weak solutions $v, H \in C_w([0,T]; L^2_x) \cap L^2(0,T; H^2_o)$ to the linearized equations
\[
\begin{aligned}
\partial_t v + \nu (-\Delta)^s v + u \cdot \nabla v - B \cdot \nabla H + \nabla P &= 0, \\
\partial_t H + \nu (-\Delta)^s H + u \cdot \nabla H - B \cdot \nabla v &= 0, \\
\text{div} v &= 0, \quad \text{div} H = 0, \\
v(0) &= v_0, \quad H(0) = H_0,
\end{aligned}
\]

(B.3)

which satisfies the energy inequality
\[
\frac{1}{2} \|v(t), H(t)\|_{L^2_x}^2 + \nu \int_{t_0}^t \|\nabla^\alpha v(s), \nabla^\alpha H(s)\|_{L^2_x}^2 \, ds \leq \frac{1}{2} \|v(t_0), H(t_0)\|_{L^2_x}^2,
\]
for all $t \in [t_0,T]$, a.e. $t_0 \in [0,T]$ (including $t_0 = 0$).
Next, let \((w, D) := (u - v, B - H).\) We derive from (B.3) that
\[
\begin{align*}
\frac{\partial_t w + \nu(-\Delta)^\gamma w + u \cdot \nabla w - B \cdot \nabla D + \nabla P = 0,}{\partial_t D + \nu(-\Delta)^\gamma D + u \cdot \nabla D - B \cdot \nabla w = 0,} \\
\text{div} w = 0, \quad \text{div} D = 0, \\
w(0) = 0, \quad D(0) = 0.
\end{align*}
\] (B.4)

Moreover, by virtue of linearity, using \((\tilde{w}, \tilde{D}) := (w + D, w - D)\) we may decouple equations (B.4) to get equations of \(\tilde{w}\) and \(\tilde{D}\) as follows:
\[
\begin{align*}
\frac{\partial_t \tilde{w} + \nu(-\Delta)^\gamma \tilde{w} + (u - B) \cdot \nabla \tilde{w} + \nabla \tilde{P} = 0,}{\text{div} \tilde{w} = 0,} \\
\tilde{w}(0) = 0,
\end{align*}
\] (B.5)

and
\[
\begin{align*}
\frac{\partial_t \tilde{D} + \nu(-\Delta)^\gamma \tilde{D} + (u + B) \cdot \nabla \tilde{D} + \nabla \tilde{P} = 0,}{\text{div} \tilde{D} = 0,} \\
\tilde{D}(0) = 0.
\end{align*}
\] (B.6)

Since \(u \pm B \in L^2_t W^{s,p}\), arguing as in the proof of [50, (B.13)] we infer that \(\tilde{w} = 0, \tilde{D} = 0,\) and thus
\[u \equiv v, \quad B \equiv H.\] (B.7)

This yields that the weak solutions \(u, B\) to (1.5) are indeed Leray-Hopf solutions.

Finally, the uniqueness of weak solutions in Theorem B.1 will be a consequence of the following weak-strong uniqueness.

**Lemma B.4** (Weak-strong uniqueness). Let \((u, B), (v, H)\) be two Leray-Hopf weak solutions to (1.5) with the same initial datum \((u_0, B_0)\). If \(u, B \in X^{\gamma, p}_{(0,t_*)}\) and \((s, \gamma, p)\) satisfy (1.7) with \(1 \leq \alpha \leq \frac{\gamma}{\gamma - 2}, s \geq 0, 2 \leq \gamma \leq \infty, 1 \leq p \leq \infty\) and \(0 \leq \frac{1}{p} - \frac{\gamma}{\gamma - 2} \leq \frac{1}{2},\) then \(u \equiv v\) and \(B \equiv H.\)

**Proof.** We only prove that \(u \equiv v, B \equiv H\) on \([0, t_*]\) for some \(t_*\) sufficiently small, since the general case can be proved by continuation arguments (see e.g., [62]).

Let \(w := u - v, D := B - H,\) we derive that
\[
\begin{align*}
\frac{\partial_t w + \nu(-\Delta)^\gamma w + w \cdot \nabla u + ((u - w) \cdot \nabla) w - D \cdot \nabla B - ((B - D) \cdot \nabla) B + \nabla \tilde{P} = 0,}{\partial_t D + \nu(-\Delta)^\gamma D + w \cdot \nabla D + ((u - w) \cdot \nabla) D - D \cdot \nabla u + ((B - D) \cdot \nabla) w = 0,} \\
\text{div} w = 0, \quad \text{div} D = 0, \\
w(0) = 0, \quad D(0) = 0.
\end{align*}
\]

Using mollification arguments as in the proof of [62, Theorem 4.4] we obtain the energy inequality
\[
\begin{align*}
\frac{1}{2}(\|w(t)\|_{L^2_t}^2 + \|D(t)\|_{L^2_t}^2) + \int_0^t \nu_1\|\nabla^\alpha w\|_{L^2}^2 + \nu_2\|\nabla^\gamma D\|_{L^2}^2 ds \\
\leq \int_0^t \int_{\mathbb{T}^3} u \cdot (w \cdot \nabla) w - B \cdot (D \cdot \nabla) w + B \cdot (w \cdot \nabla) D - u \cdot (D \cdot \nabla) D dx ds.
\end{align*}
\] (B.8)

Concerning the nonlinear terms on the right-hand side above, when \(2 \leq \gamma < \infty,\) using Hölder’s inequality, Sobolev’s embedding and Young’s inequality we obtain
\[
\begin{align*}
\int_0^t \int_{\mathbb{T}^3} B \cdot (w \cdot \nabla) D dx ds \\
\lesssim \|B\|_{L^\gamma(0,t;L^\infty)} \|w\|_{L^\infty(0,t;L^2)}^{\theta} \|w\|_{L^2(0,t;L^2)}^{1-\theta} \|\nabla D\|_{L^2(0,t;L^2)} \\
\lesssim \|B\|_{X^{\gamma, p}_{(0,t)}} \|w\|_{L^\infty(0,t;L^2)}^{\theta} \|\nabla^\alpha w\|_{L^2(0,t;L^2)}^{1-\theta} \|\nabla^\gamma D\|_{L^2(0,t;L^2)} \\
\lesssim \|B\|_{X^{\gamma, p}_{(0,t)}} \|w\|_{L^\infty(0,t;L^2)}^{\theta} \|\nabla^\alpha w, \nabla^\gamma D\|_{L^2(0,t;L^2)}^{2-\theta} \\
\lesssim \|(u, B)\|_{X^{\gamma, p}_{(0,t)}} \|w\|_{L^\infty(0,t;L^2)} + \|D\|_{L^\infty(0,t;L^2)} + \|\nabla^\alpha w\|_{L^2(0,t;L^2)} + \|\nabla^\gamma D\|_{L^2(0,t;L^2)}.
\end{align*}
\] (B.9)
where \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{4} \), \( \frac{3}{p_1} = \frac{3}{2} - \alpha \), \( \frac{3}{p_2} = \frac{3}{2} - \alpha \), \( \frac{1}{p_1} + \frac{\theta}{2} = \frac{1}{q} \). The other terms on the right-hand side of (B.8) can be estimated in the same manner. Hence, we have

\[
\|u\|_{L^{\infty}(0,t;L^q)}^2 + \|D\|_{L^{\infty}(0,t;L^2)}^2 + \nu_1 \|\nabla^\alpha w\|_{L^2(0,t;L^q)}^2 + \nu_2 \|\nabla^\alpha D\|_{L^2(0,t;L^2)}^2 \\
\leq C\|(u, B)\|_{X^{s,\gamma,p}_x} \|u\|_{L^{\infty}(0,t;L^q)}^2 + \|D\|_{L^{\infty}(0,t;L^2)}^2 + \|\nabla^\alpha w\|_{L^2(0,t;L^q)}^2 + \|\nabla^\alpha D\|_{L^2(0,t;L^2)}^2,
\]

for some universal constant \( C \).

Thus, taking \( t_* \) sufficiently close to 0, such that \( C\|(u, B)\|_{X^{s,\gamma,p}_x} < 1 \) for \( t \in [0, t_*] \), we obtain \( w \equiv D \equiv 0 \) on \([0, t_*] \).

The other terms on the right-hand side of (B.8) obey the same upper bound. Thus, taking \( t_* \) small enough and decomposing \( u = u_1 + u_2 \) (see e.g., [62, P.174]) such that

\[
\|(u_1, B_1)\|_{X^{s,\gamma,p}_x} \leq \varepsilon \quad \text{and} \quad u_2, B_2 \in X^{s,\gamma,p}_x, \quad \text{for} \quad t \in [0, t_*].
\]

Then, Hölder’s inequality, Sobolev’s embedding and Young’s inequality yield

\[
\left| \int_0^t \int_{\mathbb{T}^3} B \cdot (w \cdot \nabla) D \, dx \, dz \right| \leq \varepsilon \|w(t)\|_{L^{2, p}(0,t;L^2)} \|\nabla D\|_{L^2(0,t;L^2)} + \varepsilon \|w\|_{L^2(0,t;L^2)} \|\nabla D\|_{L^2(0,t;L^2)} + \varepsilon \|\nabla^\alpha w\|_{L^2(0,t;L^2)} \|\nabla^\alpha D\|_{L^2(0,t;L^2)} + \varepsilon \|\nabla^\alpha D\|_{L^2(0,t;L^2)} + \|w\|_{L^2(0,t;L^2)} \|\nabla^\alpha w\|_{L^2(0,t;L^2)} + \|w\|_{L^2(0,t;L^2)} \|\nabla^\alpha D\|_{L^2(0,t;L^2)}.
\]

Similarly, the other terms on the right-hand side of (B.8) obey the same upper bound. Thus, taking \( \varepsilon \) small enough, it follows that

\[
\|w(t)\|_{L^2}^2 + \|D(t)\|_{L^2}^2 \leq C \int_0^t \|w(s)\|_{L^2}^2 + \|D(s)\|_{L^2}^2 \, ds, \quad \forall t \in [0, t_*],
\]

which, via Gronwall’s inequality, yields that \( w \equiv D \equiv 0 \) on \([0, t_*] \). The proof of Lemma B.4 is complete. \( \square \)

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