Expansive homoclinic classes

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Abstract
We prove that for \(C^1\) generic diffeomorphisms, every expansive homoclinic class is hyperbolic.

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1. Introduction

Let \(M\) be a \(d\)-dimensional boundaryless Riemannian manifold. Denote by \(\text{Diff}^1(M)\) the space of \(C^1\) diffeomorphisms on \(M\), endowed with the usual \(C^1\) topology. For \(f \in \text{Diff}^1(M)\), a compact invariant set \(\Lambda\) of \(f\) is called hyperbolic if there is a continuous invariant splitting \(T_\Lambda M = E^s \oplus E^u\) on \(\Lambda\), and two constants \(C \geq 1, \lambda \in (0, 1)\), such that for any \(x \in \Lambda\) and any \(n \in \mathbb{N}\), we have

\[
\|Df^n|_{E^s(x)}\| \leq C\lambda^n, \quad \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n.
\]

A periodic point \(p\) of \(f\) is hyperbolic if the orbit \(\text{Orb}(p) = \text{Orb}_f(p)\) of \(p\) is a hyperbolic set. Denote by \(\pi(p)\) the period of \(p\).

A compact invariant set \(\Lambda\) of \(f \in \text{Diff}^1(M)\) is called expansive if there is an \(\alpha > 0\), such that for any \(x, y \in \Lambda\), if \(d(f^n(x), f^n(y)) < \alpha\) for any \(n \in \mathbb{Z}\), then \(x = y\). It is well known that every hyperbolic set is expansive.

A subset \(\mathcal{R} \subset \text{Diff}^1(M)\) is called residual if it contains a countable intersection of open and dense subsets of \(\text{Diff}^1(M)\). A property is called \((C^1)\) generic if it holds in a residual subset of \(\text{Diff}^1(M)\). We use the terminology ‘for \(C^1\) generic \(f\)’ to express ‘there is a residual subset \(\mathcal{R} \subset \text{Diff}^1(M)\) and \(f \in \mathcal{R}\).

Given \(f \in \text{Diff}^1(M)\), two hyperbolic periodic points \(p\) and \(q\) of \(f\) are called homoclinically related if \(W^s(\text{Orb}(p)) \cap W^u(\text{Orb}(q)) \neq \emptyset\) and \(W^u(\text{Orb}(p)) \cap W^s(\text{Orb}(q)) \neq \emptyset\), denoted as \(p \sim_f q\), or simply \(p \sim q\). If \(p\) and \(q\) are homoclinically related, their stable manifolds must have the same dimension. The homoclinic class of \(p\) is defined as \(H(p) = \{q: q \sim p\}\), which is a transitive compact invariant set.
of \( f \). The study of homoclinic classes is an important topic in smooth dynamical systems, for instance,

- If \( f \) is axiom A, Smale’s spectral decomposition theorem says that the non-wandering set can be decomposed into finitely many basic sets, and each basic set is a homoclinic class.
- The chain recurrent set can be divided into (maybe infinitely many) chain recurrent classes.

It was proved [1] that for \( C^1 \) generic \( f \), a chain recurrent class containing a periodic point \( p \) is the homoclinic class \( H(p) \).

The following proposition is our main technical result.

**Proposition A.** For \( C^1 \) generic \( f \), if a homoclinic class \( H(p) \) is expansive, then there are three constants \( i \in \mathbb{N}, K \geq 1 \) and \( \lambda \in (0, 1) \) such that for any periodic point \( q \sim_f p \) with period \( \pi(q) > i \), and for any \( x \in \text{Orb}_f(q) \), we have

\[
\prod_{j=0}^{\lfloor \pi(q)/i \rfloor - 1} \left\| Df^j|_{E^s(f^j(x))} \right\| \leq K \lambda^{\lfloor \pi(q)/i \rfloor},
\]

\[
\prod_{j=0}^{\lfloor \pi(q)/i \rfloor - 1} \left\| Df^{-j}|_{E^u(f^{-j}(x))} \right\| \leq K \lambda^{\lfloor \pi(q)/i \rfloor},
\]

\[
\left\| Df^i|_{E^s(x)} \right\| \cdot \left\| Df^{-i}|_{E^u(f^i(x))} \right\| \leq \lambda.
\]

The relation of expansiveness and hyperbolicity of homoclinic classes has been discussed in [8–11]. And it is essentially proved in [11] that under the assumptions and conclusions of proposition A, the homoclinic class \( H(p) \) is hyperbolic. So we get the following theorem:

**Theorem B.** For \( C^1 \) generic \( f \), every expansive homoclinic class of \( f \) is hyperbolic.

**Remark 1.1.**

1. One may notice that not every expansive homoclinic class is hyperbolic. For example, the critical saddle-node horseshoe constructed in [9, section 2.2] is not hyperbolic.
2. Although theorem B is essentially proved in [11], for the convenience of reader, we would like to give an outline of the proof, which is slightly different from [11]. According to the last inequality of proposition A, there exists a dominated splitting \( T_{H(p)}M = E^s \oplus E^u \).

Since \( H(p) \) is expansive and for \( C^1 \) generic diffeomorphism \( f \), \( H(p) \) is the chain component containing \( p \), the local invariant manifolds corresponding to \( E^s/\pi \) are dynamically defined. Then \( H(p) \) is shadowable. According to [12, proposition 3.3], \( H(p) \) is hyperbolic.

**2. Proof of proposition A**

We will first prove some new generic properties and then use these generic properties and some known results to prove proposition A. Let us introduce some terminologies first.

For \( \eta > 0 \) and \( f \in \text{Diff}^1(M) \), a \( C^1 \) curve \( \gamma \) is called \( \eta \)-simply periodic curve of \( f \) if

- \( \gamma \) is diffeomorphic to \([0, 1]\), and its two endpoints are hyperbolic periodic points of \( f \);
- \( \gamma \) is periodic with period \( \pi(\gamma) \), i.e. \( f^{\pi(\gamma)}(\gamma) = \gamma \), and \( I(f^i(\gamma)) < \eta \) for any \( 0 \leq i \leq \pi(\gamma) - 1 \), where \( l(\gamma) \) denotes the length of \( \gamma \);
- \( \gamma \) is normally hyperbolic. (See [5] for the definition of normal hyperbolicity.)
Let \( p \) be a periodic point of \( f \). For \( \delta \in (0, 1) \), we say \( p \) has a \( \delta \)-weak eigenvalue, if \( D_f^{\sigma(p)}(p) \) has an eigenvalue \( \sigma \) such that \((1 - \delta)^{\tau(p)} < |\sigma| < (1 + \delta)^{\tau(p)}\).

The following lemma gives three generic properties, which says (roughly) that if an arbitrary small perturbation of \( f \) has some ‘stable’ property, then \((C^1 \text{ generic}) \ f \) itself has this property.

**Lemma 2.1.** For \( C^1 \) generic \( f \) and any hyperbolic periodic point \( p \) of \( f \),

1. for any \( \eta > 0 \), if for any \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \), some \( g \in \mathcal{U} \) has an \( \eta \)-simply periodic curve \( \gamma \), such that the two endpoints of \( \gamma \) are homoclinically related with \( p_g \), then \( f \) has a \( 2\eta \)-simply periodic curve \( \alpha \) such that the two endpoints of \( \alpha \) are homoclinically related to \( p \).

2. for any \( \delta > 0 \), if for any \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \), some \( g \in \mathcal{U} \) has a periodic point \( q \sim_g p_g \) with \( \delta \)-weak eigenvalue, then \( f \) has a periodic point \( q' \sim_f p \) with \( 2\delta \)-weak eigenvalue.

3. for any \( \delta > 0 \), if for any \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \), some \( g \in \mathcal{U} \) has a periodic point \( q \sim_g p_g \) with \( \delta \)-weak eigenvalue and every eigenvalue of \( q \) is real, then \( f \) has a periodic point \( q' \sim_f p \) with \( 2\delta \)-weak eigenvalue and every eigenvalue of \( q' \) is real.

**Proof.** Let \( \mathcal{C} \) be the space of all compact subsets of \( M \), endowed with the Hausdorff distance. Then \( \mathcal{C} \) is a compact separable metric space. Let \( \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n, \ldots \) be a countable base of \( \mathcal{C} \).

We prove item 1 first. We would like to introduce the terminologies of two types of compact invariant sets so that items 2 and 3 can be proved similarly. For any \( \eta > 0 \), we call hyperbolic periodic orbits of type (I), and orbits of \( \eta \)-simply periodic curves of type (II). Both these types of compact invariant sets have continuations \([5]\): if \( \Lambda \) is a compact invariant set of \( f \) of type (I) or (II), then there is a neighbourhood \( \mathcal{U} \) of \( f \), a neighbourhood \( \mathcal{V} \) of \( \Lambda \) in \( \mathcal{C} \), such that for any \( g \in \mathcal{U} \), \( \Lambda \) has a unique continuation \( \Lambda_g \in \mathcal{V} \) of types (I) or (II), and moreover, \( \Lambda_g \rightarrow \Lambda \) as \( g \rightarrow f \). We say that a type (I) set \( \Lambda^1 \) and a type (II) set \( \Lambda^2 \) have a relation \( \rightsquigarrow_f \), if \( \Lambda^1 \) is homoclinically related to the two endpoints of \( \Lambda^2 \), denoted by \( \Lambda^1 \rightsquigarrow_f \Lambda^2 \).

This relation is stable: if \( \Lambda^1 \rightsquigarrow_f \Lambda^2 \), then there is a neighbourhood \( \mathcal{U} \) of \( f \), such that for any \( g \in \mathcal{U} \), we have \( \Lambda^1_g \rightsquigarrow_g \Lambda^2_g \).

Let \( \mathcal{H}_n(\eta) \) be the set of \( C^1 \) diffeomorphisms \( f \in \text{Diff}^1(M) \), such that \( f \) has a type (I) set \( \Lambda^1 \in \mathcal{Y}_n \) and a type (II) set \( \Lambda^2 \) verifying that \( \Lambda^1 \rightsquigarrow_f \Lambda^2 \). From the stability of the relation \( \rightsquigarrow \), \( \mathcal{H}_n(\eta) \) is open. Let \( \mathcal{N}_n(\eta) = \text{Diff}^1(M) - \mathcal{H}_n(\eta) \).

Since \( \mathcal{H}_n(\eta) \cup \mathcal{N}_n(\eta) \) is open and dense in \( \text{Diff}^1(M) \) from their definitions, \( \mathcal{R}(\eta) = \cap_{n \in \mathbb{N}}(\mathcal{H}_n(\eta) \cup \mathcal{N}_n(\eta)) \) is residual. And let \( \mathcal{R} = \cap \{ R(\eta) : \eta > 0 \} \), which is also a residual subset. We will prove that if \( f \in \mathcal{R} \), \( f \) has the properties in item 1.

For any \( \eta > 0 \), take \( r \in \mathbb{Q} \) such that \( \eta < r < 2\eta \). For any hyperbolic periodic point \( p \) of \( f \), take a neighbourhood \( \mathcal{Y}_n \) of \( \text{Orb}_f(p) \) in \( \mathcal{C} \), such that \( \text{Orb}_f(p) \) is the unique compact invariant set belonging to \( \mathcal{Y}_n \). Since \( \eta < r \), according to the assumption of item 1, for any \( C^1 \) neighbourhood \( \mathcal{U} \) of \( f \), some \( g \in \mathcal{U} \) has an \( r \)-simply periodic curve \( \gamma \), such that the two endpoints of \( \gamma \) are homoclinically related to \( p_g \). If the neighbourhood \( \mathcal{U} \) of \( f \) is small enough, the orbit of the continuation \( p_g \) is contained in \( \mathcal{Y}_n \) for any \( g \in \mathcal{U} \). This implies that \( f \notin \mathcal{N}_n(r) \) and hence \( f \in \mathcal{H}_n(r) \subseteq \mathcal{H}_n(2\eta) \), which means that \( f \) has a periodic orbit \( \mathcal{O} \in \mathcal{Y}_n \) and a \( 2\eta \)-simply periodic curve, such that its endpoints are homoclinically related to \( \mathcal{O} \). By the choice of \( \mathcal{Y}_n \), this periodic orbit \( \mathcal{O} \) is \( \text{Orb}_f(p) \). This completes the proof of item 1.

We can prove item 2 (and item 3) similarly by defining the type (II) sets to be periodic orbits homoclinically related to \( p \) which have \( \delta \)-weak eigenvalue (and every eigenvalue is real).
Proof. Assume that for some periodic point \( q \sim_f p \) with \( \delta \)-weak eigenvalue, \( Df^{\pi(q)}(q) \) has some complex eigenvalues. As in the proof of \([2, \text{lemma 4.16}]\), an arbitrarily small perturbation \( g \) of \( f \) has a periodic point \( p_\delta \sim_f p \) with \( \delta \)-weak eigenvalue, whose eigenvalues are all real. Since \( p_\delta \sim_f q_\delta \) and \( q_\delta \sim_f p_\delta \) imply that \( p_\delta \sim_f p_\delta \), according to item 3 of lemma 2.1, \( C^1 \) generic \( f \) has this property itself.

Proof of proposition A. For \( C^1 \) generic \( f \), assume that \( H(p) \) is an expansive homoclinic class. We first claim that there is a \( \delta_0 > 0 \), such that for any periodic point \( q \sim_p q \), \( q \) has no \( \delta_0 \)-weak eigenvalue. Otherwise, for any \( \delta > 0 \), \( H(p) \) contains a periodic point \( q_\delta \sim_f p \) with \( \delta \)-weak eigenvalue, whose eigenvalues are all real. From \([10, \text{section 4}]\), for any \( \eta > 0 \), for any \( C^1 \) neighbourhood \( U \) of \( f \), some \( \eta \)-simply periodic curve \( \gamma \), whose endpoints are homoclinically related to \( p_\delta \). By item 1 of lemma 2.1, for any \( \eta > 0 \), \( f \) itself has a 2-\( \eta \)-simply periodic curve \( \alpha \) that the endpoints of \( \alpha \) are homoclinically related to \( p_\delta \). Then all iterates of the two endpoints of \( \alpha \) have distance \(< 2\eta \), which contradicts the expansiveness of \( H(p) \).

According to remark 2.2 (for \( C^1 \) generic \( f \)), there is a \( C^1 \) neighbourhood \( U \) of \( f \), such that for any \( g \in U \), any periodic point \( q \sim_g p_\delta \) has no \( \delta_0/2 \)-weak eigenvalue.

Gourmelon proved \([4, \text{theorem 2.1}]\) an extension of Franks’ lemma \([3, 7]\), which preserves the strong (un)stable manifolds outside any given neighbourhood. A simple case (for (un)stable manifolds) can be stated as follows: for any neighbourhood \( U \) of \( f \), there is an \( \varepsilon > 0 \), such that for any hyperbolic periodic point \( q \) of \( f \), for any sequence of linear isomorphisms \( \{L_i : T_{f^i(q)}M \to T_{f^i(q)}M\}_{i=0}^{\pi(q)-1} \) verifying \( \|Df^{\pi(q)}(q) - L_i\| < \varepsilon \) for \( 0 \leq i \leq \pi(q)-1 \), for any neighbourhood \( U \) of \( \text{Orb}(q) \), there exists \( g \in U \) such that \( f^i(q) = g^i(q) \), \( Dg(f^i(q)) = L_i \) and \( g = f \) outside \( U \); moreover, if \( y \in W^s(f^i(q)) \) for some \( 0 \leq i \leq \pi(q)-1 \) and \( f^k y \in U \) for some \( k > 0 \) implies \( f^k y \in W^s(f^i(q)) \), then \( y \in W^s(g^i(q), g) \), where

\[
W^s_f(f^i(q)) = \{ z \in M | \forall n \geq 0, f^n z \in U \}
\]

is the local stable manifold of \( f^i(q) \) with respect to \( f \) and \( W^s(x, g) \) is the stable manifold of \( x \) with respect to \( g \); a similar conclusion also holds for the unstable manifolds. So for any small perturbation of the derivatives along a periodic orbit, by choosing a suitable neighbourhood of the periodic orbit, there exists a small perturbation of the diffeomorphism which preserves the homoclinic relation simultaneously.

Since there is a \( C^1 \) neighbourhood \( U \) of \( f \), such that for any \( g \in U \), any periodic point \( q \sim_g p_\delta \) has no \( \delta_0/2 \)-weak eigenvalue, according to the extension of Franks’ lemma described above,

\[
\{Df(q), Df(f(q)), \ldots, Df(f^{\pi(q)-1}(q)) : q \sim_f p\}
\]

is a uniformly hyperbolic family of periodic sequences of isomorphisms of \( \mathbb{R}^d \) (see \([7, \text{pp 524–5}]\) for more details). By \([7, \text{lemma II.3}]\), there are three constants \( \iota \in \mathbb{N}, K \geq 1 \) and \( \lambda \in (0, 1) \), such that for any periodic point \( q \sim_f p \) with period \( \pi(q) > \iota \), for any
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$\pi(q)/\iota$, we have

$$\prod_{j=0}^{[\pi(q)/\iota]-1} \|Df^j|_{E_s(f^j(q))}\| \leq K\lambda^{[\pi(q)/\iota]},$$

$$\prod_{j=0}^{[\pi(q)/\iota]-1} \|Df^{-j}|_{E_s(f^{-j}(q))}\| \leq K\lambda^{[\pi(q)/\iota]},$$

$$\|Df^j|_{E^s(q)}\| \cdot \|Df^{-j}|_{E^u(f^{-j}(q))}\| \leq \lambda. \square$$

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References

[1] Bonatti C and Crovisier S 2004 Récurrence et génériqueité Invent. Math. 158 33–104
[2] Bonatti C, Díaz L and Pujals E 2003 A $C^1$-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources Ann. Math. 158 355–418
[3] Franks J 1971 Necessary conditions for the stability of diffeomorphisms Trans. Am. Math. Soc. 158 301–8
[4] Gourmelon N 2008 A Franks’ lemma that preserves invariant manifolds Preprint http://www.preprint.impa.br/
[5] Hirsch M, Pugh C and Shub M 1977 Invariant Manifolds (Lecture Notes in Mathematics vol 583) (Berlin: Springer)
[6] Mañé R 1975 Expansive diffeomorphisms Proc. Symp. on Dynamical Systems (University of Warwick, 1974) (Lecture Notes in Mathematics vol 468) pp 162–74
[7] Mañé R 1982 An ergodic closing lemma Ann. Math. 116 503–40
[8] Pacifico M, Pujals E, Sambarino M and Vieitez J 2009 Robustly expansive codimension-one homoclinic classes are hyperbolic Ergod. Theory Dyn. Syst. 29 179–200
[9] Pacifico M, Pujals E and Vieitez J 2005 Robustly expansive homoclinic classes Ergod. Theory Dyn. Syst. 25 271–300
[10] Sambarino M and Vieitez J 2006 On $C^1$-persistent expansivity Ergod. Theory Dyn. Syst. 25 465–81
[11] Sambarino M and Vieitez J 2008 $C^1$-robustly expansive homoclinic classes are generically hyperbolic Preprint
[12] Wen X, Gan S and Wen L 2009 $C^1$-stably shadowable chain component is hyperbolic J. Diff. Eqns 246 340–57