A QUICK PROOF OF NONVANISHING FOR ASYMPTOTIC SYZYGIES

LAWRENCE EIN, DANIEL ERMAN, AND ROBERT LAZARSFELD

Introduction

The purpose of this note is to give a very quick new approach to the main cases of the nonvanishing theorems of [5] concerning the asymptotic behavior of the syzygies of a projective variety as the positivity of the embedding line bundle grows. In particular, we present a surprisingly elementary and concrete approach to the asymptotic nonvanishing of Veronese syzygies, and we obtain effective statements for arithmetically Cohen-Macaulay varieties.

Let \( X \) be an irreducible projective of dimension \( n \) over an algebraically closed field \( k \), and let \( L \) be a very ample divisor on \( X \), defining an embedding \( X \subseteq \mathbb{P}H^0(L) = \mathbb{P}^r \).

Write \( S = \text{Sym} H^0(L) \) for the homogeneous coordinate ring of \( \mathbb{P}^r \), and for a fixed divisor \( B \) on \( X \) consider the \( S \)-module \( M = M(B; L) = \bigoplus_m H^0(B + mL) \).

We are interested in the minimal graded free resolution \( E_\bullet = E_\bullet(B; L) \) of \( M \) over \( S \):

\[
0 \longrightarrow E_r \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow M \longrightarrow 0,
\]

with \( E_p = \bigoplus S(-a_{p,j}) \). Denote by

\[
K_{p,q}(B; L) = K_{p,q}(X, B; L)
\]

the finite dimensional vector space of degree \( p + q \) minimal generators of the \( p^{th} \) module of syzygies of \( M \), so that

\[
E_p(B; L) = \bigoplus_q K_{p,q}(B; L) \otimes_k S(-p - q).
\]

(When \( B = \mathcal{O}_X \), we write simply \( K_{p,q}(X; L) \) or \( K_{p,q}(L) \) if no confusion seems likely.) It is elementary that if \( L \) is very positive compared to \( B \) then non-zero syzygies can only appear in weights \( 0 \leq q \leq n + 1 \), and it turns out that the extremal cases \( q = 0 \) and \( q = n + 1 \) are easy to control. So the first interesting question is to fix \( B \) and \( 1 \leq q \leq n \), and to ask which groups \( K_{p,q}(B; L) \) are nonvanishing when \( L \) becomes very positive. The main result of [5] asserts in effect that – contrary to what one might have expected by extrapolating from the case of curves – these groups are eventually non-zero for almost all values of \( p \in [1, r] \).

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Perhaps the most natural instance of these matters occurs when $X = \mathbb{P}^n$, $B = \mathcal{O}_{\mathbb{P}^n}(b)$ and $L = L_d = \mathcal{O}_{\mathbb{P}^n}(d)$, so that one is looking at the syzygies of Veronese varieties. It was established in [5] that if one fixes $q \in [1, n]$ and $b \geq 0$, then for $d \gg 0$ one has
\begin{equation}
K_{p,q}(\mathbb{P}^n, B; L_d) \neq 0
\end{equation}
for every value of $p$ satisfying
\begin{equation}
\left(\frac{d+q}{q}\right) - \left(\frac{d-b-1}{q}\right) - q \leq p \leq \left(\frac{d+n}{n}\right) - \left(\frac{d+n-q}{n-q}\right) + \left(\frac{n+b}{n-q}\right) - q - 1.
\end{equation}
For example, when $n = 2$ and $b = 0$, this asserts that

\begin{equation}
K_{p,2}(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(d)) \neq 0 \text{ for } 3d - 2 \leq p \leq \left(\frac{d+2}{2}\right) - 3,
\end{equation}

which was the main result of the interesting paper [7] of Ottaviani and Paoletti. The proof in [5] of the Veronese nonvanishing theorem involved a rather elaborate induction on $n$ to show that certain well-chosen secant planes to the Veronese variety force the presence of non-zero syzygies. For $b = 0$ the same statement was obtained independently in characteristic zero by Weyman, who identified certain representations of $SL(n+1)$ that appear non-trivially in the $K_{p,q}$. Some other work concerning Veronese syzygies appears in [9], [1], [2], and [6], and a simplicial analogue of the results of [5] is given in [3].

The goal of the present paper is to present a much simpler and more elementary approach to the nonvanishing of Veronese syzygies, and to use this method to establish effective statements for arithmetically Cohen-Macaulay varieties. The idea is that one can reduce the question to elementary computations with monomials by modding out by a suitable regular sequence. In order to explain how this goes, consider the problem of proving the first case of the Ottaviani-Paoletti statement [3], namely that if $d \geq 3$ then
\begin{equation}
(*) \quad K_{3d-2,2}(\mathbb{P}^2; \mathcal{O}_{\mathbb{P}^2}(d)) \neq 0.
\end{equation}
Writing $S_k$ for the degree $k$ piece of the polynomial ring $S = k[x, y, z]$, it is well-known that the group in question can be computed as the cohomology at the middle term of the Koszul-type complex
\[ \cdots \longrightarrow \Lambda^{3d-1}S_d \otimes S_d \longrightarrow \Lambda^{3d-2}S_d \otimes S_{2d} \longrightarrow \Lambda^{3d-3}S_d \otimes S_{3d} \longrightarrow \cdots .\]

The most naive approach to (*) would be to write down explicitly a cocycle representing a non-zero element in $K_{3d-2,2}$, but we do not know how to do this on the other hand, consider the ring
\[ \mathcal{S} = S/(x^d, y^d, z^d).\]
As $x^d, y^d, z^d$ form a regular sequence in $S$, the dimensions of the Koszul cohomology groups of $\mathcal{S}$ are the same as those of $S$, and hence the question is equivalent to proving the nonvanishing of the cohomology of
\begin{equation}
(**) \quad \cdots \longrightarrow \Lambda^{3d-1}\mathcal{S}_d \otimes \mathcal{S}_d \longrightarrow \Lambda^{3d-2}\mathcal{S}_d \otimes \mathcal{S}_{2d} \longrightarrow \Lambda^{3d-3}\mathcal{S}_d \otimes \mathcal{S}_{3d} \longrightarrow \cdots .
\end{equation}
Now view $\mathcal{S}$ as the ring spanned by monomials in which no variable appears with exponent $\geq d$, with multiplication governed by the vanishing of the $d^{\text{th}}$ powers of each variable. The plentiful presence of zero-divisors in $\mathcal{S}$ means that one can write down by hand many monomial...
Koszul cycles: for instance if $m_1, \ldots, m_{3d-2}$ are monomials of degree $d$ each divisible by $x$ or $y$, then
\[
c = m_1 \wedge \ldots \wedge m_{3d-2} \otimes x^{d-1} y^{d-1} z^2
\]
gives a cycle for the complex (**). Note next that $x^{d-1} y^{d-1} z^2$ has exactly $3d - 2$ monomial divisors of degree $d$ with exponents $\leq d - 1$, viz:
\[
x^{d-1} y, x^{d-2} y^2, \ldots, x^2 y^{d-2}, xy^{d-1}
\]
\[
x^{d-1} z, x^{d-2} yz, \ldots, xy^{d-2} z, y^{d-1} z
\]
\[
x^{d-2} z^2, x^{d-3} yz^2, \ldots, xy^{d-3} z^2, y^{d-2} z^2.
\]
Taking these as the $m_i$, we claim that the resulting cycle $c$ represents a non-zero Koszul cohomology class. In fact, suppose that $c$ appears even as a term in the Koszul boundary of an element
\[
e = n_0 \wedge n_1 \ldots \wedge n_{3d-2} \otimes g,
\]
where the $n_i$ and $g$ are monomials of degree $d$. After re-indexing and introducing a sign we can suppose that
\[
c = n_1 \wedge \ldots \wedge n_{3d-2} \wedge n_0 g.
\]
Then the $\{n_j\}$ with $j \geq 1$ must be a re-ordering of the monomials $\{m_i\}$ dividing $x^{d-1} y^{d-1} z^2$. On the other hand $n_0 g = x^{d-1} y^{d-1} z^2$, so $n_0$ is also such a divisor. Therefore $n_0$ coincides with one of $n_1, \ldots, n_{3d-2}$, and hence $e = 0$, a contradiction.

We show that this sort of argument gives the full nonvanishing of Veronese syzygies appearing in equation (2), assuming now only that $d \geq q + 1 + b$: it was conjectured in [5] that the statement should hold in this range. We also obtain a new statement, for values of $b$ close to $d$ (Theorem 1.2). More interestingly, whereas the results of [5] for varieties other than $\mathbb{P}^n$ were ineffective, we are able here to give effective statements for a large class of general varieties.

Specifically, consider an arithmetically Cohen-Macaulay variety $X \subseteq \mathbb{P}^m$ of dimension $n$, and for $d > 0, b \geq 0$ write
\[
L_d = \mathcal{O}_X(d), \quad B = \mathcal{O}_X(b).
\]
Put $c(X) = \min \{k \mid H^n(X, \mathcal{O}_X(k - n)) = 0\}$, and write
\[
r_d = \dim H^0(X, \mathcal{O}_X(d)), \quad r'_d = r_d - (\deg X)(n + 1).
\]
We prove:

**Theorem.** Assume that $q \in [1, n - 1]$, and fix $d \geq b + q + c(X) + 1$. Then
\[
K_{p,q}(X, B; L_d) \neq 0
\]
for every value of $p$ satisfying
\[
\deg(X)(q + b + 1) \binom{d + q - 1}{q - 1} \leq p \leq r'_d - \deg(X)(d - q - b) \binom{d + n - q - 1}{n - q - 1}.
\]
Analogous statements hold, with slightly different numbers, when $q = 0$ and $q = n$; see Theorem 2.1 below. We note that Zhou [10] has given effective results for adjoint-type (and in particular, for very positive) line bundles $B$ on an arbitrary smooth complex projective
variety. It would be interesting to know whether one could recover his statement by the present techniques: see Remark 2.7.

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1. Nonvanishing Results for \( \mathbb{P}^n \)

This section is devoted to the nonvanishing results for Veronese syzygies.

Let \( k \) be any field, and consider the polynomial ring \( S = k[x_0, \ldots, x_n] \). Given \( d \geq 1 \) we denote by \( S^{(d)} \subseteq S \) the Veronese subring

\[
S^{(d)} = \bigoplus_{j \in \mathbb{Z}} S_{jd} \subseteq S
\]

of \( S \). For an \( S \)-module \( M \), we write \( M^{(b)}_{d} \) for the \( S^{(d)} \)-module \( \bigoplus_{j \in \mathbb{Z}} M_{b+jd} \). Note that \( M^{(b)}_{d} \) is also naturally a \( \text{Sym}(S^{(d)}) \)-module. We denote by \( K_{p,q}(n, b; d) = K^{\text{Sym}(S^{(d)})}_p(S^{(b)}_{d}) \) the Koszul cohomology group of \( S^{(b)}_{d} \), where \( S^{(b)}_{d} \) is considered as a \( \text{Sym}(S^{(d)}) \)-module.

Thus \( K_{p,q}(n, b; d) \) is the cohomology of the Koszul-type complex

\[
\cdots \rightarrow \Lambda^{p+1} S_{d} \otimes S_{(q-1)d+b} \rightarrow \Lambda^{p} S_{d} \otimes S_{qd+b} \rightarrow \Lambda^{p-1} S_{d} \otimes S_{(q+1)d+b} \rightarrow \cdots
\]

and

\[
K_{p,q}(n, b; d) = K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b); \mathcal{O}_{\mathbb{P}^n}(d)).
\]

Since

\[
K_{p,q}(n, b; d) = K_{p,q+1}(n, b - d; d),
\]

we will always assume that \( 0 \leq b \leq d - 1 \).

Our nonvanishing results break into two cases, depending on the value of \( b \) relative to \( d \). These results are more precise than those in [5], since in that paper, \( b \) was always fixed and \( d \rightarrow \infty \). In particular, the results in Theorem 1.2 are new.

**Theorem 1.1.** Fix any \( d \), any \( q \in [0, n] \), and any \( b \in [0, d - q - 1] \). Then:

\[
K_{p,q}(n, b; d) \neq 0
\]

for all \( p \) in the range

\[
\left( \frac{q + d}{q} \right) - \left( d - b - 1 \right) - q \leq p \leq \left( \frac{n + d}{n} \right) - \left( d + n - q \right) + \left( \frac{n + b}{q + b} \right) - q - 1.
\]

**Theorem 1.2.** Fix any positive integer \( d \), any \( q \in [0, n - 1] \) and any \( b \in [d - q, d - 1] \). Then:

\[
K_{p,q}(n, b; d) \neq 0
\]

for all \( p \) in the range

\[
\left( \frac{q + d}{q + 1} \right) - \left( \frac{2d - b - 1}{q + 1} \right) - q - 1 \leq p \leq \left( \frac{n + d}{n} \right) - \left( d + n - q - 1 \right) + \left( \frac{n + b - d}{n - q - 1} \right) - q - 1.
\]
For the proofs, the idea is to mod out by a regular sequence to arrive at a situation where we can work by hand with monomials. Specifically, by the technique of Artinian reduction, we can compute syzygies after modding out by a linear sequence. Having fixed \( d > 0 \), we put

\[
S = \defeq S/(x_0^d, \ldots, x_n^d).
\]

Slightly abusively, we view \( S \) as the graded ring spanned by monomials in the \( x_i \) in which no variable appears with exponent \( \geq d \), with multiplication determined by the vanishing of the \( d \)th power of each variable.

Since \( x_0^d, \ldots, x_n^d \) is a regular sequence in \( \text{Sym} S_d \), modding out by these powers does not affect the Koszul cohomology groups. In other words:

\[
K^{\text{Sym}(S_d)}_{p,q}(S(b)^{(d)}) \cong K^{\text{Sym}(S_d)}_{p,q}(S(b)^{(d)} \otimes \text{Sym}(S_d)) \cong K^{\text{Sym}(S_d)}_{p,q}(\overline{S}(b)^{(d)}).
\]

It thus suffices to compute this last group, which is the homology at the middle of

\[
(1.1) \quad \bigwedge^{p+1} S_d \otimes \overline{S}_{(q-1)d+b} \xrightarrow{\partial_{p+1}} \bigwedge^p S_d \otimes \overline{S}_{qd+b} \xrightarrow{\partial_p} \bigwedge^{p-1} S_d \otimes \overline{S}_{(q+1)d+b}.
\]

In particular, \( K_{p,q}(n, b; d) \neq 0 \) if and only if this complex has non-trivial homology, and we are therefore reduced to studying cycles and boundaries in \((1.1)\).

We start with some notation that will prove useful. Fix a finite set of elements \( P \subseteq S \) (which in practice will be a collection of monomials).

**Definition 1.3.** We write \( \zeta \in \bigwedge P \) (or \( \zeta \in \bigwedge^s P \)) if

\[
\zeta = m_1 \wedge \cdots \wedge m_s
\]

with \( m_i \in P \) for all \( i \). We write \( \zeta = \det P \) if \( \zeta \) is the wedge product of all elements in \( P \) (in some fixed order). We say that a wedge product \( m_1 \wedge \cdots \wedge m_s \) is a monomial if each \( m_i \) is a monomial.

The following lemma guarantees the existence of many non-zero monomial classes in the cohomology of \((1.1)\). It systematizes the computations worked out for a special case in the Introduction.

**Lemma 1.4.** Fix a nonzero monomial \( f \in \overline{S}_{qd+b} \), and denote by

\[
Z_f, \ D_f \subseteq \overline{S}_d
\]

respectively the set of degree \( d \) monomials that annihilate or divide \( f \).

(i). If \( \zeta \in \bigwedge^p Z_f \), then \( \zeta \otimes f \in \ker \partial_p \).

(ii). Let \( \zeta \in \bigwedge^s \overline{S}_d \) be any monomial such that such that \( \det D_f \wedge \zeta \otimes f \) is nonzero. Then

\[
(\det D_f \wedge \zeta) \otimes f \notin \im \partial_{(\lvert D_f \rvert + s)}.
\]

**Proof.** For (i), write \( \zeta = m_1 \wedge \cdots \wedge m_s \) with \( m_i \in Z_f \). Since \( m_if = 0 \in \overline{S} \) for all \( i = 1, \ldots, s \), the assertion is immediate. Turning to (ii), assume that

\[
\partial\left( \sum \xi_j \otimes g_j \right) = (\det D_f \wedge \zeta) \otimes f
\]
Then there must be some index $j$ and some monomial appearing in $\xi_j \otimes g_j$ that maps to the monomial $(\det D_f \wedge \zeta) \otimes f$. In particular, $\xi_j \otimes g_j$ must contain a non-zero monomial of the form $(m \wedge \det D_f \wedge \zeta) \otimes g$ where $mg = f$. But then $m \in D_f$ and hence $m \wedge \det D_f = 0$, a contradiction. \hfill \Box

**Corollary 1.5.** Given $q, d$ and $b$, let $f \in S_{qd+b}$ be a monomial such that $D_f \subseteq Z_f$. Then any non-zero monomial of the form

$$(\det D_f \wedge \zeta) \otimes f,$$

where $\zeta \in \bigwedge Z_f$, represents a nonzero element of the cohomology of (1.1). In particular,

$$K_{p,q}(n, b; d) \neq 0$$

for every $p$ satisfying

$$|D_f| \leq p \leq |Z_f|. \hfill \Box$$

**Remark 1.6.** The Koszul classes just constructed are linearly independent. In fact, keeping the notation of the corollary, and with an appropriate degree twist, there is a natural map from the Koszul complex on the linear forms in $Z_f$ to the minimal free resolution of $S(b)^{(d)}$ over $\text{Sym} S_d$ given by sending $1 \mapsto f$. This induced map yields an inclusion of the Koszul subcomplex on the linear forms $Z_f \setminus D_f \subseteq \text{Sym}(S_d)$ spanning homological degrees $p = |D_f|, |D_f| + 1, \ldots, |Z_f|$. In Conjecture B from [4], we conjectured that each row of the Betti table of a high degree Veronese looks roughly like the Betti table of a Koszul complex. Although this result has a similar flavor, the lower bound on the size of the Koszul cohomology groups constructed via this method is far too small to verify Conjecture B [4].

Theorems 1.1 and 1.2 now follow from Corollary 1.5 by choosing a convenient monomial $f$ and computing the number of elements in the resulting sets $Z_f$ and $D_f$.

**Proof of Theorem 1.1.** Fix $1 \leq q \leq n$, $b \leq d - q - 1$, and put

$$s_d = \dim \overline{S} = \binom{n + d}{d} - (n + 1).$$

Let $f$ be the “leftmost” monomial of $\overline{S}$ having degree $dq + b$, ie:

$$f = x_0^{d-1} \cdot x_1^{d-1} \cdot \ldots \cdot x_{q-1}^{d-1} \cdot x_q^{q+b}.$$

In order to establish the theorem, it suffices to prove three assertions:

(i). $s_d - |Z_f| = \binom{d+n-q}{d} - \binom{n+b}{q+b} - (n - q)$.

(ii). $|D_f| = \binom{d+q}{q} - \binom{d-b-1}{q} - q$.

(iii). $D_f \subseteq Z_f$. 

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For (i), observe that $Z_f = (0 : S \cdot f)_d$ contains all monomials of degree $d$ that are divisible by any of $x_0, \ldots, x_{q-1}$ as well as those divisible by $x_q^{d-q-b}$. Hence among the $s_d$ monomials in $S_d$, the ones not lying in $Z_f$ are the monomials of degree $d$ appearing in the quotient $S/(x_0, \ldots, x_{q-1}, x_q^{d-q-b})$.

We can compute this via the resolution:

$$\ldots \rightarrow \frac{S(-d)}{(x_0, \ldots, x_{q-1})} \rightarrow \frac{S(-d + q + b)}{(x_0, \ldots, x_{q-1})} \rightarrow \frac{S}{(x_0, \ldots, x_{q-1})}.$$ 

Therefore

$$s_d - |Z_f| = \dim_k \left( \frac{S}{(x_0, \ldots, x_{q-1}, x_q^{d-q-b})} \right)_d$$

$$= \dim \left( \frac{S}{(x_0, \ldots, x_{q-1})} \right)_d - \dim \left( \frac{S}{(x_0, \ldots, x_{q-1})}_{q+b} \right) + \dim \left( \frac{S}{(x_0, \ldots, x_{q-1})}_0 \right)$$

$$= \binom{d + n - q}{d} - n + q - 1 - \binom{n + b}{q + b} + 1.$$

For (ii), note that $D_f$ can be identified with the degree $d$ monomials of $S/(x_q^{b+1}, x_{q+1}, \ldots, x_n)$. A similar computation yields

$$|D_f| = \dim \left( \frac{S}{(x_q^{b+1}, x_{q+1}, \ldots, x_n)} \right)_d$$

$$= \dim \left( \frac{S}{(x_{q+1}, \ldots, x_n)} \right)_d - \dim \left( \frac{S}{(x_{q+1}, \ldots, x_n)}_{d-b-1} \right) + \dim \left( \frac{S}{(x_{q+1}, \ldots, x_n)}_0 \right)$$

$$= \binom{q + d}{d} - q - 1 - \binom{d - b - 1}{q} + 1.$$

Finally, since the exponent of $x_q$ in $f$ is $q + b \leq d - 1$, it follows that any element of $D_f$ is divisible at least by one of $x_0, \ldots, x_{q-1}$, and hence any such element annihilates $f$. $\square$

**Proof of Theorem 1.2.** When $d - q \leq b \leq d - 1$ we follow the outline of the previous proof, this time taking:

$$f = x_0^{d-1} \cdot x_1^{d-1} \cdots x_q^{d-1} \cdot x_{q+1}^{q+b-d+1}.$$

The corresponding computations are:

$$s_d - |Z_f| = \dim_k \left( \frac{S}{(x_0, \ldots, x_q, x_{q+1}^{2d-1-q-b})} \right)_d$$

$$= \dim \left( \frac{S}{(x_0, \ldots, x_q)} \right)_d - \dim \left( \frac{S}{(x_0, \ldots, x_q)}_{q+b+1-d} \right) + \dim \left( \frac{S}{(x_0, \ldots, x_q)}_0 \right)$$

$$= \binom{d + n - q - 1}{d} - n + q - 1 - \binom{n + b - d}{q + b + 1 - d} + 1.$$

To count the number of monomials in $D_f$ we observe:

$$|D_f| = \dim \left( \frac{S}{(x_q^{b-d+2}, x_{q+2}, \ldots, x_n)} \right)_d$$

$$= \dim \left( \frac{S}{(x_q+2, \ldots, x_n)} \right)_d - \dim \left( \frac{S}{(x_q+2, \ldots, x_n)}_{2d-b-2} \right) + \dim \left( \frac{S}{(x_q+2, \ldots, x_n)}_0 \right)$$

$$= \binom{q + 1 + d}{d} - q - 2 - \binom{2d - b - 1}{q + 1} + 1.$$

Finally, we note that $D_f \subseteq Z_f$: since the exponent of $x_{q+1}$ in $f$ is strictly less than $d$, it follows that any element of $D_f$ is divisible at least by one of $x_0, \ldots, x_q$, as required. $\square$
Remark 1.7. Zhou [11] has recently established some results about the asymptotic distribution of torus weights appearing in the $K_{p,q}$ of toric varieties. It would be interesting to know if the present arguments can be used to give more refined information in the case $X = \mathbb{P}^n$.

Remark 1.8. It is conjectured in [5, Conjecture 7.5] that the assertion of Theorem 1.1 is optimal in the sense that the $K_{p,q}$ in question vanish outside the stated range, and we conjecture that the bounds in Theorem 1.2 are optimal as well. It would be exceedingly interesting to know whether one can use the approach introduced here to make progress on this conjecture, at least in the case $d \gg 0$ as in [5, Problem 7.7]. Unfortunately it seems that one can’t work only with monomials – it’s possible for instance that a monomial Koszul cocycle appears as the boundary of non-monomial elements. It is tempting to wonder whether there are Gröbner-like techniques that could be used to study the issue systematically. We note that Raicu [8] has reduced the general vanishing conjecture [5, Conjecture 7.1] to the case of Veronese syzygies.

2. Nonvanishing for arithmetically Cohen-Macaulay schemes

In this section we extend the results of the previous section to the setting of arithmetically Cohen-Macaulay schemes.

Consider an arithmetically Cohen-Macaulay scheme $X \subseteq \mathbb{P}^m$ of dimension $n$ over the field $k$, and let

$$R = \oplus H^0(X, \mathcal{O}_X(k))$$

be the homogeneous coordinate ring of $X$. Setting $L_d = \mathcal{O}_X(d)$ and $B = \mathcal{O}_X(b)$, we are interested in the syzygies

$$K_{p,q}(X, B; L_d) = K_{p,q}(R(b)^{(d)})$$

of $B$ with respect to $L_d$ for $d \gg 0$. Put

$$c = c(X) = \min \{ k \mid H^n(X, \mathcal{O}_X(k-n)) = 0 \}$$

and write

$$r_d = \dim H^0(X, \mathcal{O}_X(d)) = \dim R_d, \quad r'_d = r_d - (\deg X)(n+1).$$

Our first result holds when $d \geq b + q + c + 1$.

Theorem 2.1. Fix $b \in [0, d - q - 1 - c]$.

(i) If $q \in [1, n - 1]$, then $K_{p,q}(X, B; L_d) \neq 0$ for

$$(\deg X)(q + b + 1) \binom{d + q - 1}{q - 1} \leq p \leq r'_d - (\deg X)(d - q - b) \binom{d + n - q - 1}{n - q - 1}.$$ 

(ii) When $q = n$, one has $K_{p,n}(X, B; L_d) \neq 0$ when

$$(\deg X)(n + b + 1) \binom{n - 2 - d}{n - 1} \leq p \leq r'_d - \deg X.$$ 

\footnote{Equivalently, $c(X) = \text{reg}(\mathcal{O}_X)$ is the Castelnuovo-Mumford regularity of $\mathcal{O}_X$ as a sheaf on $\mathbb{P}^m$.}
(iii). When $q = 0$ one has $K_{p,0}(X, B; L_d) \neq 0$ when

$$\deg X \leq p \leq r'_d - (d - b) \binom{n - 1 + d}{n - 1}.$$

A somewhat more complicated but sharper statement appears in Remark 2.4 below.

**Remark 2.2.** If we fix $b$ and $q$, we can interpret these bounds as asymptotic statements as $d \to \infty$. Under these assumptions, we are saying that $K_{p,q}(X, B; L_d) \neq 0$ for all $p$ in the range

$$\frac{\deg(X)(q + b + 1)}{(q - 1)!} d^{q-1} + O(d^{q-2}) \leq p \leq r'_d - \left(\frac{\deg(X)}{(n - q - 1)!} d^{n-q} + O(d^{n-q-1})\right).$$

Conjecture 7.1 in [5] states that one should have $K_{p,q} = 0$ for $p \leq O(d^{q-1})$; it would be interesting to understand the optimal leading coefficients as well. In the ACM case this implies that $K_{p,q} = 0$ also for $p > r_d - O(d^{n-q})$, but in the non-ACM case the groups in question can be nonvanishing for $p \approx r_d$ [5, Remark 5.3].

For the proofs of the Theorems, let $I_X \subseteq k[x_0, \ldots, x_m]$ be the defining ideal of $X$, so that $R = k[x_0, \ldots, x_m]/I_X$. The statement is independent of the ground field, so we may assume that $k$ is infinite. Then, after a general change of coordinates, we may assume that $x_0, \ldots, x_n$ form a system of parameters for $R$. To help clarify the following arguments, we will relabel the variables $x_{n+1}, \ldots, x_m$ as $y_{n+1}, \ldots, y_m$.

Let $S = k[x_0, \ldots, x_n] \subseteq R$, which is a Noether normalization since $x_0, \ldots, x_n$ is a system of parameters. As $R$ is Cohen-Macaulay of dimension $n + 1$, it follows that it is a maximal Cohen-Macaulay $S$-module, and hence a free $S$-module. We may choose a set $\Lambda$ of monomials of the form $y^\beta \in R$ which form a basis for $R$ as an $S$-module, so that

$$R = \bigoplus_{y^\beta \in \Lambda} S \cdot y^\beta.$$

We assume that $1 \in \Lambda$. Thus $\deg(X) = \# \Lambda$ and we observe that $c(X) = \max\{\deg y^\beta\}$.

Fix $q \in [0, n]$, $d > 0$ and $b \geq 0$. Set

$$\overline{R} = R/(x_0^d, \ldots, x_n^d),$$

and define $\overline{S}$ as in the previous section. Thus $\overline{R} = R \otimes_S \overline{S}$, and $\overline{R}$ is a free $\overline{S}$-module with basis $\Lambda$. Since $R$ is Cohen-Macaulay, we have

$$\dim K_{p,q}(R(b)^{(d)}) = \dim K_{p,q}(\overline{R}(b)^{(d)})$$

for all $p$ and $q$, where the group on the right is computed as the cohomology of the complex

$$\wedge^{p+1} \overline{R}_d \otimes \overline{R}_{(q-1)d+b} \overset{\partial}{\longrightarrow} \wedge^p \overline{R}_d \otimes \overline{R}_{qd+b} \overset{\partial}{\longrightarrow} \wedge^{p-1} \overline{R}_d \otimes \overline{R}_{(q+1)d+b}.$$

In the natural way, we can speak of monomials in $\overline{R}$: these are (the images in $\overline{R}$ of) elements of the form $x^\alpha y^\beta$ where $y^\beta \in \Lambda$, and the degree of such a monomial is $|\alpha| + |\beta|$. Given a monomial $f \in \overline{R}$, we denote by

$$Z_f, E_f \subseteq \overline{R}_d$$


respectively the set of degree $d$ monomials that annihilate $f$ and the collection of degree $d$ monomials of the form $x^\alpha y^\beta$ where $x^\alpha$ divides $f$ and $y^\beta \in \Lambda$.

We start with an analogue of Lemma 1.4.

**Lemma 2.3.** Let $$f \in S_{qd+b} \subseteq \overline{R}_{qd+b}$$ be a monomial such that $E_f \subseteq Z_f$. Then any non-zero monomial element $$m = (\det E_f \wedge \zeta) \otimes f$$ with $\zeta \in \bigwedge Z_f$ represents a non-zero Koszul cohomology class. In particular $$K_{p,q}(X,\mathcal{O}_X(b);\mathcal{O}_X(d)) \neq 0$$ for every $p$ with $$|E_f| \leq p \leq |Z_f|.$$

**Proof.** Since $E_f \subseteq Z_f$, $m$ is evidently a Koszul cycle. It remains to prove that it is not cohomologous to zero. In fact, we’ll show that $m$ cannot occur as a monomial appearing in the expansion (with respect to the chosen basis of $\overline{R}$) of $\partial(\xi \otimes g)$ for any monomials $\xi \in \Lambda^{b+1}R_d$ and $g \in S_{(q-1)d+b}$. Suppose to the contrary that $m$ appears as a term in $\partial(\xi_0 \wedge \ldots \wedge \xi_p \otimes g)$. Then after possibly reindexing and introducing a sign, we can suppose $$\xi_1 \wedge \ldots \wedge \xi_p = \det(E_f) \wedge \zeta,$$ and that $f$ appears as a term in the expansion of $\xi_0 g$ in terms of the basis $\Lambda$. Suppose that $$\xi_0 = x^\alpha y^\beta, \quad g = x^\gamma y^\delta$$ where $y^\beta, y^\delta \in \Lambda$. Then in $\overline{R}$ we can rewrite $$y^{\beta+\delta} = h_0 \cdot 1 + \sum_{\lambda \in \Lambda} h_\lambda \cdot y^\lambda$$ where $h_\lambda \in S_{|\beta|+|\delta|-|\lambda|}$. Therefore $f = x^{\alpha+\gamma} h_0$, and consequently $x^\alpha y^\beta \in E_f$. In particular $\xi_0$ also appears as one of $\xi_1, \ldots, \xi_p$, and hence $m = 0$. \ \ \ \ \ \Box$

We now turn to the

**Proof of Theorem 2.1.** As before, we take $f$ to be the be the leftmost non-zero monomial of $S$ of degree $dq+b$: $$f = x_0^{d-1} \cdot x_1^{d-1} \cdots x_{q-1}^{d-1} \cdot x_q^{q+b}.$$ We claim first of all that $E_f \subseteq Z_f$ provided that $d \geq b + q + c + 1$. In fact, suppose that $$w = x_0^{a_0} \cdots x_q^{a_q} \cdot y^\beta \in E_f.$$ Then $a_q \leq q + b$, and hence $$a_0 + \ldots + a_{q-1} = d - a_q - |\beta| \geq d - (q + b) - c > 0.$$ Therefore at least one of $a_0, \ldots, a_{q-1}$ is strictly positive, and consequently $w \in Z_f$. 10
In order to apply Lemma 2.3, it remains to estimate the sizes of \( E_f \) and \( Z_f \). Writing \( r_d = \dim R_d \), we start by giving an upper bound on \( r_d - |Z_f| \). Assume first that \( q \in [1, n-1] \), and consider a monomial \( x^\alpha = x_0^a_0 \cdot \ldots \cdot x_n^a_n \). Then a degree \( d \) monomial \( x^\alpha y^\beta \) lies in the complement of \( Z_f \) if and only if
\[
a_0 = \ldots = a_{q-1} = 0 , \ a_q \leq d - b - q.
\]
The number of possibilities for \( x^\alpha \) is (rather wastefully) bounded above simply by the number of degree \( d \) monomials in \( k[x_{q+1}, \ldots, x_n] \), times the number of choices for \( a_q \), times the number of choices for \( y^\beta \). Since \( |\Lambda| = \deg X \), this leads to the lower bound
\[
\bar{r}_d - (\deg X)(d - q - b) \left( \frac{d + n - q - 1}{n - q - 1} \right) \leq |Z_f|.
\]
Turning to an upper bound on \( |E_f| \), observe that \( x^\alpha y^\beta \in E_f \) if and only if
\[
a_0, \ldots, a_{q-1} \leq d - 1 , \ a_q \leq q + b \ \text{and} \ a_{q+1} = \ldots = a_n = 0
\]
We can bound this (again wastefully) by the number of monomials of degree \( d \) in \( k[x_0, \ldots, x_{q-1}] \), times the number of choices for \( a_q \), times the number of choices for \( y^\beta \). This leads to:
\[
(q + b + 1) \left( \frac{q - 1 + d}{q - 1} \right) \geq |E_f|.
\]
So to obtain assertion (i) of Theorem 2.1 it remains only to observe that
\[
\bar{r}_d = \sum_{y^\beta \in \Lambda} \dim S_{d-|\beta|} \geq \sum_{y^\beta \in \Lambda} \left( \dim S_{d-|\beta|} - (n + 1) \right)
\]
\[
= \dim R_d - |\Lambda|(n + 1)
\]
\[
= r'_d.
\]
When \( q = n \) we get the same bound on \( |E_f| \), but now we find that
\[
\bar{r}_d - (\deg X) \leq |Z_f|,
\]
and this yields statement (ii) of the Theorem. Finally, when \( q = 0 \) we get the same lower bound on \( |Z_f| \) as above, but now we use \( \deg X \leq |E_f| \). \( \square \)

**Remark 2.4.** By separating the estimates into two terms depending on whether \( \beta \) is equal to zero or not, one gets a slightly better upper bound on the size of \( E_f \), when \( q \in [1, n-1] \):
\[
(deg X - 1)(q + b + 1) \left( \frac{q - 1 + d - 1}{q - 1} \right) + \left( \frac{q + d}{q} \right) - \left( \frac{d - b - 1}{q} \right) - q \geq |E_f|.
\]
In particular, this reduces to the statements obtained for \( \mathbb{P}^n \) in the previous sections.

**Remark 2.5.** In the spirit of Theorem 1.2, one can use similar arguments that hold for \( q \in [0, n-1] \) and \( b \in [d - q, d - 1] \), provided that \( 2d \geq q + b + c + 2 \). Here one works with the monomial
\[
f = x_0^{d-1} \cdot x_1^{d-1} \cdot \ldots \cdot x_q^{d-1} \cdot x_{q+1}^{q+b-d+1}.
\]
Assuming for instance that \( q \in [0, n - 2] \), one finds that
\[
K_{p,q}(X, B; L_d) \neq 0 \text{ for } LB \leq p \leq UB,
\]
where
\[
LB = (\deg X)(q + b + 2 - d)\binom{q + d}{q},
\]
\[
UB = r_d' - (\deg X)(2d - (q + b + 1))\binom{n - q - 2 + d}{n - q - 2}.
\]

**Remark 2.6.** The bounds for \(|E_f|\) and \(\tau_d - |Z_f|\) appearing in the proof of Theorem 2.1 could be improved by a more precise count of the relevant possibilities, in particular taking into account the degrees of the \(y^\beta\). This amounts to certain Hilbert function computations, and confronted with a specific example, it is often quite easy to use directly the method of the proof to get stronger statements. For example, let \( X \subseteq \mathbb{P}^5 \) be the hypersurface
\[
x_0^3 + \ldots + x_5^3 = 0.
\]
Then \( \Lambda = \{1, x_5, x_5^2\} \), so \( c = 2 \). We take \( (q, b, d) = (3, 0, 8) \) and
\[
f = x_0^7x_1^7x_2^7x_3^3.
\]
Then \( R = \mathbb{k}[x_0, \ldots, x_5]/(x_0^3 + \ldots + x_5^3) \) and \( \overline{R} = R/(x_0^8, \ldots, x_4^8) \). The bounds from Theorem 2.1 and Remark 2.5 yield the nonvanishing result \( K_{p,3}(X; \mathcal{O}_X(8)) \neq 0 \) for \( p \) between 540 and 1005.

However, if we follow the method of the proof, we can compute the size of \( E_f \) directly. Let \( A := \mathbb{k}[x_0, \ldots, x_q]/(x_0^d, \ldots, x_q^d, x_{q-b-1}^d) = \mathbb{k}[x_0, \ldots, x_3]/(x_0^8, x_1^8, x_2^8, x_3^4) \). Then
\[
\sum_{y^\beta \in \Lambda} \dim A_{d-\deg y^\beta} = \dim A_8 + \dim A_7 + \dim A_6 = 301.
\]

A similar computation shows that there are 14 monomials in the complement of \( Z_f \) and so \(|Z_f| = 1030 - 14 = 1016\), and the nonvanishing statement can be extended to all values of \( p \) between 301 and 1016.

**Remark 2.7.** Let \( X \subseteq \mathbb{P}^m \) be an arbitrary variety of dimension \( n \), and suppose that \( B \) is a line or vector bundle on \( X \) with the property that
\[
H^i(X, B \otimes \mathcal{O}_X(k)) = 0
\]
for all \( k \in \mathbb{Z} \) and \( 0 < i < n \): in other words, \( M = \oplus H^0(X, B \otimes \mathcal{O}_X(k)) \) is a Cohen-Macaulay module over the homogeneous coordinate ring of \( \mathbb{P}^m \). Replacing \( B \) by a twist, one can assume without loss of generality that \( M_{-1} = 0 \) but \( M_0 \neq 0 \). Then one can use the methods of this section to obtain effective nonvanishing statements for the syzygies \( K_{p,q}(X, B; \mathcal{O}_X(d)) \). In fact, the hypotheses on \( M \) imply that it has a generator in degree 0, and then in the arguments above one can replace \( R \) by \( M \). We leave details to the interested reader. It would be interesting to compare the resulting statements with the results [10] of Zhou which fall under this rubric.
Finally, we expect that nonvanishing statements similar to Theorem 2.1 hold for any finitely generated, graded $k$-algebra $R$. More precisely, we conjecture the following analogue of part (i) of Theorem 2.1.

**Conjecture 2.8.** Fix $b$ and $R$ and $q \in [1, \dim R - 1]$. Then there exist constants $c$ and $C$ such that if $d \gg 0$ then

$$K_{p,q}(R(b)^{(d)}) \neq 0 \text{ for all } cd^{q-1} \leq p \leq r_d - Cd^{n-q}$$

and for all $d \gg 0$.

We expect similar analogues of parts (ii) and (iii) of Theorem 2.1 as well as analogues of the cases where $b$ is close to $d$, as in Remark 2.5.

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Department of Mathematics, University Illinois at Chicago, 851 South Morgan St., Chicago, IL 60607

Department of Mathematics, University of Wisconsin, Madison, WI 53706

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794