ALGEBRAIC GEOMETRY CODES FROM HIGHER DIMENSIONAL VARIETIES

JOHN B. LITTLE

ABSTRACT. This paper is a general survey of work on Goppa-type codes from higher dimensional algebraic varieties. The construction and several techniques for estimating the minimum distance are described first. Codes from various classes of varieties, including Hermitian hypersurfaces, Grassmannians, flag varieties, ruled surfaces over curves, and Deligne-Lusztig varieties are considered. Connections with the theories of toric codes and order domains are also briefly indicated.

1. Introduction

The codes considered in this survey can all be understood as examples of evaluation codes produced from a finite set \( S = \{ P_1, \ldots, P_n \} \) of \( \mathbb{F}_q \)-rational points on an algebraic variety \( X \) and an \( \mathbb{F}_q \)-vector space of functions \( \mathcal{F} \) defined on \( S \). The set of codewords is the image of an evaluation mapping

\[
ev_S : \mathcal{F} \longrightarrow \mathbb{F}_q^n
\]

\[
f \mapsto (f(P_1), \ldots, f(P_n)).
\]

\( X \) will usually be assumed smooth, but in fact many of the constructions also make sense for normal varieties (much of the usual geometric theory of divisors and line bundles on normal varieties is the same as in the smooth case).

The Goppa \( C_L(D, G) \) codes from curves \( X \) where \( \mathcal{F} = L(G) \) for some divisor \( G \) on \( X \) were the first examples of codes of this type to be considered. Relatively early in the history of applications of algebraic geometry to coding theory, however, Tsfasman and Vladut proposed in Chapter 3.1 of [54] that higher dimensional varieties might also be used to construct codes. By the results of [55], every linear code can be obtained by the construction of Definition 1 below, starting from some \( S \subseteq X(\mathbb{F}_q) \) for some variety \( X \) and some line bundle \( \mathcal{L} \) on \( X \); indeed curves suffice for this (see Section 9). Hence the question is whether one can identify specific higher dimensional varieties \( X \), spaces of functions \( \mathcal{F} \), and sets of rational points \( S \) that yield particularly interesting codes using algebraic geometric constructions. There has been a fairly steady stream of articles since the 1990’s studying such codes and our first main goal here is to survey the methods that have been developed and the results that have been obtained.

In a sense, the first major difference between higher dimensional varieties and curves is that points on \( X \) of dimension \( \geq 2 \) are subvarieties of codimension \( \geq 2 \), not divisors. This means that many of the familiar tools used for Goppa codes

2000 Mathematics Subject Classification. Primary 94B27; Secondary 14G50, 14J99.

Key words and phrases. coding theory, Goppa code, quadric, Hermitian variety, Grassmannian, flag variety, Del Pezzo surface, ruled surface, Deligne-Lusztig variety.
(e.g. Riemann-Roch theorems, the theory of differentials and residues, etc.) do not apply in exactly the same way.

A second difference is the possibility of performing *birational modifications* such as blowing up points or other subvarieties on a variety of higher dimension. For instance, if \( p \) is a point in a smooth algebraic variety \( X \) of dimension \( \delta \geq 2 \), there is another smooth variety \( Y = \text{Bl}_p(X) \), a proper morphism \( \pi : Y \to X \), and an *exceptional divisor* \( E \simeq \mathbb{P}^{\delta-1} \) in \( Y \) such that \( \pi(E) = \{p\} \), and \( \pi|_{Y - E} : Y - E \simeq X - \{p\} \) as varieties. Because \( Y \) and \( X \) have isomorphic nonempty Zariski-open subsets, they have isomorphic function fields. Such varieties \( Y \) and \( X \) are said to be *birationally isomorphic*. This says that function fields in two or more variables always have many different nonisomorphic smooth models, and the connection with function fields is not as tight as in the curve case.

It must be said that the theory of Goppa-type codes from higher dimensional varieties is much less advanced at this point than the theory for Goppa codes from curves, perhaps because of these differences. There is still no clear understanding of how best to harness the properties of higher dimensional varieties in coding theory. Indeed, as we will see, most of the work that has appeared to date has been devoted to case studies of the *structural properties* of codes constructed from certain particular families of varieties \( X \) – their parameters, their weight distributions, their hierarchies of higher Hamming weights, and so forth. A few general ideas for estimating the minimum distance \( d \) have been developed. However, quite a few of the codes that we will see are rather unremarkable; in many of the cases where the exact weight distributions are known, other algebraic constructions yield better codes. In addition, the development of efficient encoding and decoding algorithms for these codes has not really begun (see Section 9 on this point, though). The theory of order domains should yield tools here as well as for codes from curves. Nevertheless, the universality of this construction offers hope that good examples can be constructed this way, and our second main goal is to encourage others to explore this area.

This survey is organized as follows. In Section 2 we give two variants of Tsfasman and Vladut’s code construction, one starting from an abstract variety \( X \) and line bundle \( \mathcal{L} \) on \( X \), the other starting from an embedded variety \( X \subset \mathbb{P}^m \). We also present some first examples. Four general methods for estimating the minimum distance are presented in Section 3. Two appeared first in S.H. Hansen’s article [26]. For the first of these, it is assumed that all of the \( \mathbb{F}_q \)-rational points of interest are contained in a family of curves on \( X \) and intersection products of divisors with those curves are used to bound \( d \). The second method is based on the Seshadri constant of the line bundle \( \mathcal{L} \) with respect to the set of \( \mathbb{F}_q \)-rational points on \( X \). A third method from [17] can be used when the set of \( \mathbb{F}_q \)-rational points is itself a complete intersection in \( \mathbb{P}^m \). Finally, we present another, more arithmetic, method based on the Weil conjectures developed by Lachaud in [37].

The next sections 4 and 5 present a selection of the examples of these codes that have appeared in the literature, codes constructed from quadric hypersurfaces, Hermitian hypersurfaces, Grassmannians and flag varieties, Del Pezzo surfaces, ruled surfaces, and Deligne-Lusztig varieties. Finally, we present some comparisons between codes in Section 7.

Where practicable, we have provided brief proofs of the results we state, in order to show the methods involved in the study of these codes.
As we proceed through these examples, the prerequisites from algebraic geometry steadily increase. Our intended audience includes both coding theorists familiar with the theory of Goppa codes on curves but not higher dimensional geometry and algebraic geometers curious about how higher dimensional varieties might be used in the coding theory context. Hence there are probably portions of what we say that might seem unnecessarily elementary to some readers. We apologize in advance.

The text [28] by Hartshorne is a good general reference for most of the algebraic geometry we need. The construction of Grassmannians via exterior algebra, Schubert varieties, and the intersection theory on Grassmannians are covered in Griffiths and Harris, [19]. A full understanding of the Deligne-Lusztig varieties also depends on the theory of reductive algebraic groups $G$ over fields of characteristic $p$ and the classification of their finite subgroups $G^F$ by root systems and Dynkin diagrams with an action of the Frobenius endomorphism, $F$. The book [5] of Carter contains all the information needed for this.

Because of space limitations, it has not been possible to discuss all the results of every paper in this area in detail. Pointers to all of the literature of which the author is aware are provided in the bibliographic notes in Section 9, the references, and their bibliographies.

Any omissions or errors are entirely due to the author. Any comments or suggestions are welcome.

1.1. Notation. We will use the following general notational and terminological conventions.

- The number of elements in a finite set $T$ will be denoted by $\#T$.
- The parameters of a linear code are denoted $[n, k, d]$ as usual, where $n$ is the block length, $k$ is the dimension, and $d$ is the minimum distance.
- The generalized Hamming weights are denoted $d_r$, $1 \leq r \leq k$. As in [57], $d_r$ is the size of the minimal support of an $r$-dimensional subcode of $C$, extending the usual minimum distance $d = d_1$.
- We denote an algebraically closed field of characteristic $p$ by $F$ and all finite fields $F_q$ for $q = p^m$ are considered as subfields of $F$.
- The projective spaces $\mathbb{P}^m$, Grassmannians $G(\ell, m)$, and so forth are considered as varieties over the algebraically closed field $\mathbb{F}$ in order to “do geometry.” The $F_q$-rational points used in the construction of the codes are finite subsets of these varieties.
- If $f$ is a homogeneous polynomial in $F_q[x_0, \ldots, x_m]$, $\mathbf{V}(f)$ is the zero locus of $f$ in $\mathbb{P}^m$.
- A line bundle is a locally free sheaf of rank one. At several points, it will be convenient to use the sheaf cohomology groups $H^i(X, \mathcal{L})$ for a line bundle $\mathcal{L}$. The space of global sections will also be written $\Gamma(X, \mathcal{L})$.

2. The General Construction

Several apparently different, but essentially equivalent, versions of the construction are commonly encountered in the literature. For instance, one description starts from a smooth projective variety $X$ defined over $F_q$, a set $S \subseteq X(F_q)$ of $F_q$-rational points of $X$, and a line bundle $\mathcal{L}$ on $X$, also defined over $F_q$. Let $P$ be
an \( \mathbb{F}_q \)-rational point of \( X \). The stalk \( \mathcal{L}_P \), modulo sections vanishing at \( P \), denoted \( \mathcal{L}_P \), is isomorphic to \( \mathbb{F}_q \) by a choice of local trivialization.

**Definition 1.** The choice of such local trivializations at each point in \( \mathcal{S} \) defines a linear mapping (called the germ map in \( \mathbb{F}_q \))

\[
\alpha : \Gamma(\mathcal{X}, \mathcal{L}) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{T}_{P_i} \cong \mathbb{F}_q^n,
\]

and the image is the code denoted \( C(\mathcal{X}, \mathcal{L}; \mathcal{S}) \), or \( C(\mathcal{X}, \mathcal{L}) \) if the set of points \( \mathcal{S} \) is understood from the context.

If \( \mathcal{L} = \mathcal{O}_X(D) \) for an \( \mathbb{F}_q \)-rational divisor \( D \) on \( X \) whose support is disjoint from \( \{P_1, \ldots, P_n\} \), then up to monomial equivalence, this is the same as the evaluation code as in (1) from the subspace \( \mathcal{F} \) of the field of rational functions of \( X \) given by

\[
\mathcal{F} = \{ f \in \mathbb{F}_q(X)^* : \text{div}(f) + D \geq 0 \} \cup \{0\}.
\]

For instance, when \( X \) is a smooth algebraic curve and \( \mathcal{L} = \mathcal{O}_X(G) \) for some divisor \( G \) defined over \( \mathbb{F}_q \) whose support is disjoint from the support of \( D = P_1 + \cdots + P_n \), then this is the same as the algebraic geometric Goppa code \( C_L(D, G) \) from \( X \).

For explicit constructions of codes from embedded varieties \( X \subseteq \mathbb{F}_q^m \), another more elementary description is also available using homogeneous coordinates \( (a_0 : a_1 : \cdots : a_m) \) for points in \( \mathbb{F}_q^m \), where \( (a_0 : a_1 : \cdots : a_m) \) and \( (\lambda a_0 : \lambda a_1 : \cdots : \lambda a_m) \) represent the same point whenever \( \lambda \in \mathbb{F}_q^* \).

**Definition 2.** Choosing any one such homogeneous coordinate vector defined over \( \mathbb{F}_q \) for each of the points \( P_i \) in the set \( \mathcal{S} \), define an evaluation map \( \nu \mid \mathcal{S} \) and a code as in (1) using the vector space \( \mathcal{F}_1 \) of linear forms (homogeneous polynomials of degree 1) in \( \mathbb{F}_q[x_0, \ldots, x_m] \). The code obtained as the image of this mapping is often denoted \( C(X) \), or \( C(X; \mathcal{S}) \) if it is important to specify the set of points. Similarly, the space of linear forms can be replaced by the vector space \( \mathcal{F}_h \) of homogeneous polynomials of any degree \( h \geq 1 \), and corresponding codes denoted \( C_h(X; \mathcal{S}) \) or \( C_h(X) \) are obtained.

**Example 1.** Let \( X = \mathbb{F}_q^m \) itself, and let \( \mathcal{S} \) be the set of affine \( \mathbb{F}_q \)-rational points of \( X \), that is, points in the complement of the hyperplane \( V(x_0) \), having homogeneous coordinate vectors of the form \( (1 : a_1 : \cdots : a_m) \). With these particular coordinate vectors, the code \( C_h(X; \mathcal{S}) \) is the well-known \( q \)-ary \( h \)th order (generalized) Reed-Muller code, denoted \( R_q(h, m) \). (When \( m = 1 \), this is the same as an extended Reed-Solomon code.) The block length is \( n = q^m \). If \( h < q \), then the monomials \( x^\beta = x_0^{\beta_0} \cdots x_m^{\beta_m} \) where \( |\beta| = \beta_0 + \cdots + \beta_m = h \) are linearly independent on \( \mathcal{S} \), so the dimension of \( \mathcal{R}_q(s, m) \) is \( k = \binom{m+h}{h} \). If \( \mathcal{S} = \mathbb{F}_q^m(\mathbb{F}_q) \), the resulting projective Reed-Muller codes have block length \( n = q^m + \cdots + q + 1 \). ☐

There is, of course, a tight connection between Definition 1 and Definition 2. If \( X \) is embedded in \( \mathbb{F}_q^m \) and \( \mathcal{L} = \mathcal{O}_X(1) \) is the hyperplane section bundle, then \( C(X, \mathcal{O}_X(1)) \) and \( C(X) \) are monomially equivalent codes (they differ at most by constant multiples in each component depending on how the isomorphisms of the fibers with \( \mathbb{F}_q \) are chosen). Similarly, \( C_h(X) \) is equivalent to \( C(X, \mathcal{O}_X(h)) \). Also, in theory it suffices to consider the \( C(\mathcal{X}) = C_1(\mathcal{X}) \) codes, since the \( C_h(X) \) code on \( X \) is the same as the \( C_1 \) code on the variety \( \nu_h(X) \), where \( \nu_h \) is the degree-\( h \)
Veronese mapping

\[ \nu_h : \mathbb{P}^m \rightarrow \mathbb{P}^{(m+h) - 1} \]
\[ (x_0 : x_1 : \cdots : x_m) \mapsto (\cdots : x^\beta : \cdots) , \]
and \( x^\beta = x_0^\beta_0 \cdots x_m^\beta_m \) ranges over all monomials of total degree \( h \). The image \( \nu_h(\mathbb{P}^m) \) has dimension \( m \), degree \( h^m \), and is isomorphic to \( \mathbb{P}^m \).

3. Estimating the Parameters

3.1. Elementary bounds. Suppose Definition 2 is used to construct a code \( C_h(X; \mathcal{S}) \) from a variety \( X \). The block length of the code is \( n = \# \mathcal{S} \). Using a standard linear algebra result, the dimension is \( k = \dim \mathcal{F}_h - \dim \ker ev_\mathcal{S} \).

Forms of degree \( h \) vanishing on \( X \) always give elements of the kernel. The dimension of the space of such forms can be computed using the long exact cohomology sequence of

\[ 0 \rightarrow I_X(h) \rightarrow \mathcal{O}_{\mathbb{P}^m}(h) \rightarrow \mathcal{O}_X(h) \rightarrow 0. \]

Since each codeword is \( ev_\mathcal{S}(f) = (f(P_1), \ldots, f(P_n)) \) for some form \( f \), the codeword weight is \( n - \#(V(f) \cap \mathcal{S}) \), the number of \( P_i \) in \( \mathcal{S} \) where \( f \) is not zero. Therefore,

\[ d = \min_{f \neq 0 \in \mathcal{F}_h} (n - \#(V(f) \cap \mathcal{S})). \]

Along similarly general lines, let \( \dim Y = \delta \) and let the degree of \( Y \) be \( s < q + 1 \) in \( \mathbb{P}^m \). Let \( E \) be an \( \mathbb{F}_q \)-rational linear subspace of dimension \( m - \delta - 1 \) with \( E \cap Y = \emptyset \). By projection from \( E \) onto a linear subspace \( L \simeq \mathbb{P}^\delta \), each \( \mathbb{F}_q \)-rational point of \( L \) corresponds to at most \( s \) such points of \( Y \), so

\[ \#Y(\mathbb{F}_q) \leq s \cdot \#\mathbb{P}^\delta(\mathbb{F}_q) = s(q^\delta + \cdots + q + 1). \]

Applying (5) to \( Y = X \cap H \) for a hyperplane, Lachaud obtains the following elementary bound in [37].

**Theorem 1.** Let \( X \) be a projective variety of dimension \( \delta \) and degree \( s < q + 1 \). Then for \( h = 1 \) the \( C(X) \) code has

\[ d \geq n - s(q^{\delta - 1} + \cdots + q + 1). \]

A more refined estimate of the number of \( \mathbb{F}_q \) rational points on a projective hypersurface establishes the following result for the projective Reed-Muller codes introduced in Example 1.

**Theorem 2.** Let \( h \leq q \). The projective Reed-Muller code of order \( h \) has parameters

\[ \left[ q^m + \cdots + q + 1, \binom{m+h}{h}, (q+1-h)q^{m-1} \right]. \]

**Proof.** Write \( \mathcal{S} = \mathbb{P}^m(\mathbb{F}_q) \). The evaluation mapping is injective and \( k = \dim \mathcal{F}_s = \binom{m+h}{h} \) provided that \( d > 0 \). By [51], if \( f \) is a homogeneous polynomial of degree \( h \leq q \), then (improving the bound of (3))

\[ \#(V(f) \cap \mathcal{S}) \leq h q^{m-1} + q^{m-2} + \cdots + q + 1. \]
Moreover, if \( V(f) \) is the union of \( h \mathbb{F}_q \)-rational hyperplanes meeting along a common \((m - 2)\)-dimensional linear subspace, this bound is attained. Hence
\[
d = (q^m + q^{m-1} + \cdots + q + 1) - (hq^{m-1} + q^{m-2} + \cdots + q + 1) = (q + 1 - h)q^{m-1}
\]
as claimed. 

In the remainder of this section, several other general techniques for estimating the minimum distance of these codes will be considered. The first three are primarily geometric, while the last is arithmetic in nature.

3.2. Bounds from covering families of curves. For the following discussion, it will be most convenient to use the code construction given in Definition 1. In many concrete cases, it can be seen that the points in the set \( S \) are distributed on a collection of curves \( C_i \) (subvarieties of dimension 1) on the variety \( X \). Since each section \( f \in \Gamma(X, L) \) on \( X \) defines a divisor of zeroes \( Z(f) \), a subvariety of codimension 1 on \( X \), determining the minimum distance of the \((X, L)\) code reduces to understanding how many times the divisors \( Z(f) \) can intersect the curves \( C_i \) at points of \( S \). To prepare, let \( C \) be any irreducible curve in \( X \). Observe that the divisors \( Z(f) \) for \( f \in \Gamma(X, L) \) all cut out divisors on \( C \) of the same degree. This degree will be denoted by \( L \cdot C \). In this situation, Hansen derives a lower bound for \( d \) in [26].

**Theorem 3.** Let \( X \) be a normal projective variety defined over \( \mathbb{F}_q \), of dimension \( \dim X \geq 2 \). Let \( S \subseteq X(\mathbb{F}_q) \) and assume \( S \subseteq \bigcup_{i=1}^a C_i \) where \( C_i \) are irreducible curves on \( X \), also defined over \( \mathbb{F}_q \). Assume that \( \#(C_i \cap S) \leq N \) for all \( i \). Let \( L \) be a line bundle on \( X \) defined over \( \mathbb{F}_q \) such that
\[
0 \leq L \cdot C_i \leq \eta \leq N
\]
for all \( i \). Let
\[
\ell = \max_{f \neq 0 \in \Gamma(X, L)} \# \{ i : Z(f) \text{ contains } C_i \}.
\]
Then the code \( C(X, L; S) \) has
\[
d \geq \#S - \ell N - (a - \ell)\eta.
\]

**Proof.** Let \( f \in \Gamma(X, L) \), let \( D = Z(f) \), and let \( E = Z(f) \cap \bigcup_{i=1}^a C_i \). Suppose \( E \) contains \( \ell' \leq \ell \) of the \( C_i \). The number points of \( S \) that are contained in \( E \) is estimated as follows:
\[
\#(E \cap S) \leq \ell' N + (a - \ell')\eta
\]
\[
\leq \ell N + (a - \ell)\eta
\]
(since by hypothesis \( \eta \leq N \)). Hence \( e_{V_2}(f) \) has at least \( \#S - \ell N - (a - \ell)\eta \) nonzero entries. 

**Example 2.** Let \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( S = X(\mathbb{F}_q) \), which consists of \((q + 1)^2\) points, equally distributed over the lines \( C_1, \ldots, C_{q+1} \) of one of the rulings. The Picard group of line bundles modulo isomorphism is \( \text{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z} \), so the lines \( C_i \) may be taken as the divisors of zeros of sections of a line bundle of type \((1, 0)\). Let \( L \) have type \((\alpha, \beta)\) where \( 0 \leq \alpha, \beta \leq q + 1 \). Apply Theorem 3 to estimate \( d \) for the \( C(X, L) \) code. Because of the description of \( S \) above, \( N = q + 1 \). The divisor \( Z(f) \)
for \( f \in \Gamma(X, \mathcal{L}) \) contains at most \( \alpha \) of the \( C_i \), so \( \ell = \alpha \). Moreover, \( \mathcal{L} \cdot C_i = \beta \) for each \( i \), so \( \eta = \beta \). The bound is

\[
d \geq (q+1)^2 - \alpha(q+1) - (q+1-\alpha)\beta = (q+1-\alpha)(q+1-\beta).
\]

It is easy to construct codewords of this weight via bihomogeneous polynomials on \( \mathbb{P}^1 \times \mathbb{P}^1 \). So this is the exact minimum distance. \( \square \)

3.3. **Bounds using Seshadri constants.** A second general method for estimating the minimum distance of the \( C(X, \mathcal{L}; \mathcal{S}) \) codes is based on the Seshadri constant of \( \mathcal{L} \) relative to the set \( \mathcal{S} \). This is potentially useful but requires some significantly more sophisticated birational geometry to state and apply. Let \( \pi : Y \to X \) be the blow up of the \( X \) at the points in \( \mathcal{S} \) and call the exceptional divisor \( E \). Then the Seshadri constant is defined as

\[
\varepsilon(\mathcal{L}, \mathcal{S}) = \sup\{ \varepsilon \in \mathbb{Q} : \pi^* \mathcal{L} - \varepsilon E \text{ is nef on } Y \}. 
\]

(Here, “nef” means *numerically effective*, that is, \( (\pi^* \mathcal{L} - \varepsilon E) \cdot C \geq 0 \) for all irreducible curves \( C \) on \( Y \).) Hansen proves the following estimate for the minimum distance of the \( C(X, \mathcal{L}; \mathcal{S}) \) codes in [26].

**Theorem 4.** Let \( X \) be a nonsingular projective variety of dimension \( \geq 2 \) over \( \mathbb{F}_q \). If \( \mathcal{L} \) is ample with Seshadri constant \( \varepsilon(\mathcal{L}, \mathcal{S}) \geq \varepsilon \in \mathbb{N} \), and \( n > e^{1-\dim(X)} \mathcal{L}^{\dim(X)} \), then \( C(X, \mathcal{L}; \mathcal{S}) \) has minimum distance \( d \geq n - e^{1-\dim(X)} \mathcal{L}^{\dim(X)} \).

This is particularly well-suited for analyzing certain codes from Deligne-Lusztig varieties to be defined in Section 5 below.

3.4. **Bounds from \( \mathcal{S} \) itself.** All of the \( C_h(X; \mathcal{S}) \) codes introduced in Section 2 can be viewed as punctures of the projective Reed-Muller code of order \( h \) on the appropriate \( \mathbb{P}^m \) (delete the components corresponding to points in the complement of \( \mathcal{S} \)). For this reason, in addition to making use of the properties of the variety \( X \), it is also possible to use properties of the 0-dimensional algebraic set (or scheme) \( \mathcal{S} \) itself to estimate \( d \). Let \( \mathcal{I}_\mathcal{P} \) be the sheaf of ideals defining any 0-dimensional \( \mathcal{P} \). From the long exact cohomology sequence of the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{I}_\mathcal{P} \longrightarrow \mathcal{O}_{\mathbb{P}^m} \longrightarrow \mathcal{O}_\mathcal{P} \longrightarrow 0,
\]

it follows that for all \( h \geq 0 \),

\[
(6) \quad 0 \to H^0(\mathcal{I}_\mathcal{P}(h)) \to H^0(\mathcal{O}_{\mathbb{P}^m}(h)) \to H^0(\mathcal{O}_\mathcal{P}(h)) \to H^1(\mathcal{I}_\mathcal{P}(h)) \to 0.
\]

The term \( H^0(\mathcal{I}_\mathcal{P}(h)) \) gives the space of homogeneous forms of degree \( h \) vanishing on \( \mathcal{P} \). The term \( H^1(\mathcal{I}_\mathcal{P}(h)) \) measures the failure of the points in \( \mathcal{P} \) to impose independent conditions on forms of degree \( h \).

In the case that \( \mathcal{S} \) is a complete intersection of hypersurfaces of degrees \( d_1, \ldots, d_m \), there are particularly nice techniques from commutative algebra and algebraic geometry related to the classical Cayley-Bacharach Theorem that apply. A modern version of this result due to Davis, Geramita, and Orecchia can be stated as follows in the situation at hand.

**Theorem 5.** Let \( \mathcal{S} \subset \mathbb{P}^m \) be a reduced complete intersection of hypersurfaces of degrees \( d_1, \ldots, d_m \). Let \( \Gamma', \Gamma'' \) be disjoint subsets of \( \mathcal{S} \) with \( \mathcal{S} = \Gamma' \cup \Gamma'' \). Let \( s = \sum_{i=1}^m d_i - m - 1 \). Then for all \( h \geq 0 \),

\[
\dim H^0(\mathcal{I}_{\Gamma'}(h)) - \dim H^0(\mathcal{I}_\mathcal{S}(h)) = \dim H^1(\mathcal{I}_{\Gamma''}(s-h)).
\]
Hence, one way to interpret Theorem 5 is that when $\Gamma' \subset S$, the difference in dimension between the space of homogeneous forms of degree $a$ vanishing on $\Gamma'$ and the subspace vanishing on $S$ is equal to the dimension of $H^1(I_{\Gamma''}((s-a)))$. Moreover by (6), this dimension measures the failure of $\Gamma''$ to impose independent conditions on homogeneous forms of degree $s-a$.

Applied to the corresponding codes from $S$ consisting of $d_1d_2\ldots d_m$ distinct $\mathbb{F}_q$-rational points, this result implies the following.

**Theorem 6.** Let $S$ be a reduced complete intersection of hypersurfaces of degrees $d_1,\ldots,d_m$ in $\mathbb{P}^m$. Let $s = \sum_{i=1}^{m} d_i - m - 1$ as in Theorem 5. If $1 \leq h \leq s$, the code $C_h(S)$ has minimum distance

$$d \geq \sum_{i=1}^{m} d_i - h - (m - 1) = s - h + 2.$$

The proof is accomplished by showing that under these hypotheses, any form of degree $h$ that is zero on a subset $\Gamma'$ that is too large must be zero at all points in $S$ because the $H^1(I_{\Gamma''}((s-h)))$ group vanishes.

The bound on $d$ given here was improved rather strikingly by Ballico and Fontanari to $d \geq m(s-h) + 2$ under the assumption that all subsets of $m+1$ of the points in $S$ span $\mathbb{P}^m$ – see [2] for this.

Bounds derived by these methods are usually interesting only for $h$ close to $s$. Moreover some, but not all, interesting examples of $S$ satisfy the complete intersection hypothesis. For instance the affine $\mathbb{F}_q$-rational points in $\mathbb{P}^m$ form a complete intersection for all $m$. The $\mathbb{F}_8$-rational points on the Klein quartic and the $\mathbb{F}_{r,2}$ points on the Hermitian curve are other examples.

### 3.5. General Weil-type bounds.

From (4) above, and the proof of Theorem 2, the minimum distance of a $C(X)$ code as in Definition 2 is determined by the numbers of $\mathbb{F}_q$-rational points on the subvarieties $Y = X \cap V(f)$. Hence, another possible approach to estimate $d$ is to apply general bounds for $\#Y(\mathbb{F}_q)$, for instance bounds derived from the statements of the Weil conjectures, or refined versions of these.

We very briefly recall the deep mathematics behind this approach. Thinking of $X$ as a variety over the algebraic closure of the finite field, the number of $\mathbb{F}_q$-rational points on $X$ can be computed by an analog of the Lefschetz trace formula for the action of the Frobenius endomorphism $F$ on the $\ell$-adic étale cohomology groups of $X$, $H^i(X)$ (where $\ell$ is any prime not dividing $q$):

$$\#X(\mathbb{F}_q) = \sum_{i=0}^{2m} (-1)^i \text{Tr}(F|H^i(X)).$$

Moreover, the eigenvalues of $F$ on $H^i(X)$ are algebraic numbers of absolute value $q^{i/2}$. When $X$ is obtained from a variety $Y$ defined over the ring of integers $R$ of some number field by reduction modulo some prime ideal in $R$, then the dimensions of the $H^i(X)$ are the same as the topological Betti numbers of the variety over $\mathbb{C}$ corresponding to $Y$. 
Thus, for instance, if \( X \) is a smooth curve of genus \( g \) which is the reduction of a smooth curve \( Y \), then
\[
\# X(\mathbb{F}_q) = 1 + q - \sum_{j=0}^{2g} \alpha_j,
\]
where \( |\alpha_j| = q^{1/2} \) for all \( j \). The Hasse-Weil bound often used in the theory of Goppa codes from curves is a direct consequence:
\[
|\# X(\mathbb{F}_q) - (1 + q)| \leq 2g\sqrt{q}.
\]

There is a correspondingly concrete Weil-type bound for hypersurfaces in \( \mathbb{P}^m \), and this can be used to derive bounds on the numbers of \( \mathbb{F}_q \)-rational points in hyperplane sections as well. A hypersurface is said to be nondegenerate if it is not contained in any linear subspace of \( \mathbb{P}^m \).

**Theorem 7.** Let \( X \) be a smooth nondegenerate hypersurface of degree \( s \) in \( \mathbb{P}^m \), \( m \geq 2 \). Then
\[
(8) \quad |\# X(\mathbb{F}_q) - (q^{m-1} + \cdots + q + 1)| \leq b(s)q^{(m-1)/2},
\]
where \( b(s) = \frac{s-1}{s}((s-1)^{2} - 1) = (s-1)(s-2) = 2g(X) \)

as expected. In order to obtain long codes over \( \mathbb{F}_q \), the maximal curves, that is, curves attaining the maximum \( \# X(\mathbb{F}_q) \) from (8), have been especially intensively studied. For instance, when \( q = r^2 \), the Hermitian curve of degree \( s = r+1 \) over \( \mathbb{F}_{r^2} \), \( X = \mathbb{V}(x_0^{r+1} + x_1^{r+1} + x_2^{r+1}) \), has \( X(\mathbb{F}_{r^2}) = r^3 + 1 = 1 + r^2 + r(r - 1)r \). ◊

**Example 3.** If \( m = 2 \) and \( X \) is a smooth curve of degree \( s \) in \( \mathbb{P}^2 \), then
\[
b(s) = \frac{s-1}{s}((s-1)^2 - 1) = (s-1)(s-2) = 2g(X)
\]
as expected. In order to obtain long codes over \( \mathbb{F}_q \), the maximal curves, that is, curves attaining the maximum \#X(\mathbb{F}_q) from (8), have been especially intensively studied. For instance, when \( q = r^2 \), the Hermitian curve of degree \( s = r+1 \) over \( \mathbb{F}_{r^2} \), \( X = \mathbb{V}(x_0^{r+1} + x_1^{r+1} + x_2^{r+1}) \) also attain the upper bound from (8), which reads
\[
\# X(\mathbb{F}_{r^2}) \leq 1 + r^2 + r^4 + \frac{r}{r+1}(r^3 + 1)r^2 = (r^2 + 1)(r^3 + 1).
\]
The Hermitian surface contains this many distinct \( \mathbb{F}_{r^2} \)-rational points because, for instance, it is possible to take the defining equation to the affine form
\[
y_1^r + y_1 = y_2^{r+1} + y_3^{r+1}
\]
by a linear change of coordinates that puts a plane tangent to the surface as the plane at infinity. Then there are \( r^3 \) affine \( \mathbb{F}_{r^2} \)-rational points (for each pair \( (y_2, y_3) \in (\mathbb{F}_{r^2})^2 \)). There are also \( (r + 1)r^2 + 1 \) rational points at infinity since the intersection of the surface with each of its tangent planes at an \( \mathbb{F}_{r^2} \)-rational point is the union of \( r + 1 \) concurrent lines in that plane. This yields \( r^5 + (r + 1)r^2 + 1 = (r^3 + 1)(r^2 + 1) \) points as claimed. ◊

The following result of Lachaud appears in [37].
Theorem 8. Let $X$ be a smooth nondegenerate hypersurface of degree $s$ in $\mathbb{P}^m$ for $m \geq 3$. Let $H = V(f)$ for a linear form in $\mathbb{F}_q[x_0, \ldots, x_m]$, and let $X_H$ denote the intersection $X \cap H$ (with the reduced scheme structure). Then

$$|\#X_H(\mathbb{F}_q) - (q^{m-2} + \cdots + q + 1)| \leq (s - 1)^{m-1}q^{(m-1)/2},$$

and

$$|q\#X_H(\mathbb{F}_q) - \#X(\mathbb{F}_q)| \leq (s - 1)^{m-1}(q + s - 1)q^{(m-1)/2}.$$  

These bounds are proved by comparing the cohomology of $X$ and $X_H$, taking into account possible singularities of $X_H$. For a proof, see Corollary 4.6 and preceding results of [37].

When $S$ is the full set of $\mathbb{F}_q$-rational points on $X$, so $n = \#S$ for the $C(X; S)$ code and $H$ is a general hyperplane, these imply the following bounds on $\#(H \cap S)$.

(10) implies

$$|n - \#(H \cap S)) - \frac{(q - 1)}{q}n| \leq (s - 1)^{m-1}(q + s - 1)q^{(m-1)/2}$$

and

$$|(n - \#(H \cap S)) - q^{m-2}| \leq s(s - 1)^{m-1}q^{(m-1)/2}.$$  

These, together with (9), give universally applicable lower bounds on $d$ by applying (10).

As is perhaps to be expected, it is often possible to derive tighter bounds in specific cases by taking the properties of $X$ into account.

4. Examples

This section will consider codes produced according to the constructions from Section 2 from various special classes of varieties. The particular varieties used here are all examples of varieties with many rational points over finite fields $\mathbb{F}_q$. The examples are ordered according to the algebraic geometric prerequisites needed for the construction.

4.1. Quadrics. First consider the $C(X)$ codes from quadric hypersurfaces $X = V(f)$ for homogeneous $f$ of degree 2 in $\mathbb{F}_q[x_0, \ldots, x_m]$. The following statements are proved, for instance, in Chapter 22 of [30]. Up to projective equivalence over $\mathbb{F}_q$, such $X$ are completely described by a positive integer called the rank and a second integer called the character, which takes values in the finite set $\{0, 1, 2\}$. The rank, denoted $\rho$, can be described as the minimum number of variables needed to express $f$ after a linear change of coordinates in $\mathbb{P}^m$. $X$ is said to be nondegenerate if $\rho = m + 1$. Nondegenerate quadrics are always smooth varieties. Degenerate quadrics are singular, but they are cones over nondegenerate quadrics in a linear subspace of $\mathbb{P}^m$. Hence in principle it suffices to study nondegenerate quadrics and we will consider only that case here. The character, denoted $w$, is most easily described by considering a finite set of possible normal forms for $f$.

If $m$ is even, then every nondegenerate quadric can be taken to the form

$$x_0^2 + x_1x_2 + x_3x_4 + \cdots + x_{m-1}x_m.$$  

$V(f)$ is called a parabolic quadric in this case, and the character $w$ is defined to be 1.
On the other hand, if $m$ is odd, there are two distinct possible forms:

$$
x_0x_1 + x_2x_3 + \cdots + x_{m-1}x_m \quad \text{or} \quad q(x_0, x_1) + x_2x_3 + \cdots + x_{m-1}x_m.
$$

In the first case, $V(f)$ is called a hyperbolic quadric and $w = 2$. In the second, $q(x_0, x_1)$ is a quadratic form in two variables which can be further reduced to slightly different normal forms depending on whether $q$ is even or odd. For both even and odd $q$, in the second case, $V(f)$ is called a elliptic quadric and $w = 0$.

**Theorem 9.** Let $X$ be a nondegenerate quadric in $\mathbb{P}^m$ with character $w$. Then

$$
\# X(\mathbb{F}_q) = \frac{(q^{(m+1-w)/2} + 1)(q^{(m-1+w)/2} - 1)}{q - 1}
= q^{m-1} + \cdots + q + 1 + (w - 1)q^{(m-1)/2}.
$$

In particular, this result says that hyperbolic and parabolic quadrics attain the upper bound from (8) with $s = 2$, and elliptic quadrics attain the lower bound.

Because each linear section of $X$ is also a quadric in a lower-dimensional space, Theorem 9 can be used to determine the full weight distributions of the $C(X)$ codes. In particular,

**Theorem 10.** The $C(X)$ code from a smooth quadric $X$ in $\mathbb{P}^m$ has $n$ given in Theorem 9, $k = m + 1$ and

$$
d = \begin{cases} 
q^{m-1} & \text{if } w = 2 \\
q^{m-1} - q^{(m-2)/2} & \text{if } w = 1 \\
q^{m-1} - q^{(m-1)/2} & \text{if } w = 0.
\end{cases}
$$

For instance, if $m$ is even, so $w = 1$ (the parabolic case), the hyperplane section of $X$ containing the most $\mathbb{F}_q$-rational points will be a hyperbolic section and $d$ is as above. When $w = 2$ (for example, for codes from hyperbolic quadrics in $\mathbb{P}^3$), the minimum weight codewords come from hyperplane sections that are degenerate quadrics.

The same sort of reasoning has also been used by Nogin and Wan to determine the complete hierarchy of generalized Hamming weights $d_1(C(X)), \ldots, d_k(C(X))$. The results are somewhat intricate to state, though, so we refer the interested reader to the articles [56, 43] and the notes in Section 9.

For the $C_h(X)$ codes with $h \geq 2$, the dimension can be estimated using (3), where $I_X(h) \simeq \mathcal{O}_{\mathbb{F}^m}(h - 2)$. This yields

$$
k \leq \binom{m + h}{h} - \binom{m + h - 2}{h - 2}.
$$

4.2. **Hermitian hypersurfaces.** For the $C(X)$ codes constructed from the Hermitian surfaces of Example 3 with $q = r^2$, (3) gives

$$
d \geq (r^2 + 1)(r^3 + 1) - (r^2 + 1 + r^4) = r^5 - r^4 + r^3.
$$

However, closer examination of the hyperplane sections of the Hermitian surface yields the following statement.

**Theorem 11.** Let $X = V(x_0^{r+1} + x_1^{r+1} + x_2^{r+1} + x_3^{r+1})$ be the Hermitian surface over $\mathbb{F}_{r^2}$. The $C(X)$ code on $S = X(\mathbb{F}_r)$ has parameters

$$
[(r^2 + 1)(r^3 + 1), 4, r^5].
$$
Proof. Every $\mathbb{F}_{r^2}$-rational plane in $\mathbb{P}^3$ intersects $X$ either in a Hermitian curve containing $r^3 + 1$ points over $\mathbb{F}_{r^2}$, or else in $r + 1$ concurrent lines containing $(r + 1)r^2 + 1$ points. Hence by (14),

$$d = n - ((r + 1) r^2 + 1) = r^5.$$ 

The $C_h(X)$ codes with $h > 1$ are more subtle here.

**Theorem 12.** Let $X$ and $S$ be as in Theorem (1) If $h < r + 1$, the $C_h(X)$ code has parameters

$$\left[(r^2 + 1)(r^3 + 1), \left(\frac{4 + h}{h}\right)d \geq n - h(r + 1)(r^2 + 1)\right].$$

Proof. This bound follows from Theorem (1) by the fact that if $f$ is a form of degree $h$, then $V(f) \cap X$ is a curve of degree $\delta = h(r + 1)$ in $\mathbb{P}^3$. The hypothesis on $h$ implies that the evaluation mapping is injective. For larger $h$, (3) would be used to determine the dimension of the space of forms of degree $h$ vanishing on the Hermitian variety.

An even tighter bound

$$(14) \quad d \geq n - (h(r^3 + r^2 - r) + r + 1)$$

has been conjectured by Sørensen for these codes in [52].

The Hermitian curve and surface codes can be generalized as follows. (see Chapter 23 of [30]). Over a field of order $q = r^2$, consider the Hermitian hypersurface in $\mathbb{P}^n$ defined by

$$(15) \quad X = V(x_0^{r+1} + x_1^{r+1} + \cdots + x_m^{r+1}).$$

The mapping $F(x) = x^r$ is an involutory field automorphism of $\mathbb{F}_{r^2}$, analogous to complex conjugation in $\mathbb{C}$, and the homogeneous polynomial defining $X$ is analogous to the usual Hermitian form on $\mathbb{C}^{m+1}$ given by $x_0 \overline{x_0} + \cdots + x_m \overline{x_m}$. The defining polynomial of $X$ may be understood as $H(x, x)$ for the mapping $H : \mathbb{F}_{r^2}^{m+1} \times \mathbb{F}_{r^2}^{m+1} \rightarrow \mathbb{F}_{r^2}$ given by

$$H(x, y) = x_0 y_0^r + \cdots + x_m y_m^r.$$ 

It is clear that $H$ is additive in each variable and satisfies $H(\lambda x, y) = \lambda H(x, y)$ and $H(x, \lambda y) = \lambda^r H(x, y) = F(\lambda) H(x, y)$ for the automorphism $F$ above. Hence $H$ is an example of what is known as a **sesquilinear form** on $\mathbb{F}_{r^2}^{m+1} \times \mathbb{F}_{r^2}^{m+1}$. It can be shown that after a linear change of coordinates defined over $\mathbb{F}_{r^2}$, any sesquilinear $H$ on $V \times V$, where $V$ is a finite-dimensional $\mathbb{F}_{r^2}$-vector space, can be expressed as

$$(16) \quad H(x, y) = x_0 y_0^\ell + \cdots + x_\ell y_\ell^r$$

for some $\ell \leq \dim V$. $H$ is said to be nondegenerate if $\ell = \dim V$ and degenerate otherwise.

It follows that every linear section $L \cap X$ of a Hermitian hypersurface is also a Hermitian variety in the linear subspace $L = PW$ for some vector subspace $W$. Moreover, if the section is degenerate (i.e. $\ell < \dim W$ in (16)), then the section is a cone over a nondegenerate Hermitian variety in a linear subspace of $L$. Thus, the properties of the codes $C(X)$ from the Hermitian hypersurfaces are formally quite similar to (and even somewhat simpler than) the properties of codes from
quadrics discussed above. The main ingredient is the following statement for the nondegenerate Hermitian hypersurfaces.

**Theorem 13.** Let $X$ be the nondegenerate Hermitian hypersurface from (13). Then

$$\#X(\mathbb{F}_q) = r^{2m-2} + \cdots + r^2 + 1 + b(r + 1)r^{m-1},$$

where $b(r + 1) = \frac{1}{r+1}(r^m - (-1)^m)$.

In other words, for all $m$, the nondegenerate Hermitian hypersurfaces meet the upper bound from (8) for a hypersurface of degree $s = r + 1$.

**Theorem 14.** Let $S = X(\mathbb{F}_q)$ for the nondegenerate Hermitian hypersurface $X$ in $\mathbb{P}^m$. The $C(X; S)$ code has $n$ given in Theorem 13 $k = m + 1$, and

$$d = \begin{cases} r^{2m-1} - r^{m-1} & \text{if } m \equiv 0 \mod 2 \\ r^{2m-1} & \text{if } m \equiv 1 \mod 2. \end{cases}$$

When $m$ is even, the minimum weight codewords of the $C(X)$ come from nondegenerate Hermitian variety hyperplane sections. On the other hand, if $m$ is odd, then the minimum weight codewords of $C(X)$ come from hyperplane sections that are degenerate Hermitian varieties. In this case, in both cases, the nonzero codewords of $C(X)$ have only two distinct weights:

$$r^{2m-1} + (-1)^{m-1}r^{m-1} \text{ and } r^{2m-1}.$$

The hierarchies of generalized Hamming weights $d_r$ are also known for the $C(X)$ codes by work of Hirschfeld, Tsfasman, and Vladut, [31]. The same sort of techniques used in Theorem 13 above can be applied to the $C_h(X)$ codes for $h \geq 2$ here. However, much less is known about the exact Hamming weights of these codes.

### 4.3. Grassmannians and flag varieties.

The Grassmannian $\mathbb{G}(\ell, m)$ is a projective variety whose points are in one-to-one correspondence with the $\ell$-dimensional vector subspaces of an $m$-dimensional vector space (or equivalently the $(\ell - 1)$-dimensional linear subspaces of $\mathbb{P}^{m-1}$). We very briefly recall the construction.

Let $\mathbb{F}$ denote an algebraic closure of $\mathbb{F}_q$. Given any basis $B = \{v_1, \ldots, v_\ell\}$ for an $\ell$-dimensional vector subspace $W$ of $\mathbb{F}^m$, form the $\ell \times m$ matrix $M(B)$ with rows $v_i$. Consider the determinants of the maximal square $(\ell \times \ell)$ submatrices of $M(B)$. There is one such maximal minor for each subset $I \subset \{1, \ldots, m\}$ with $\#I = \ell$, so writing $p_I(W)$ for the maximal minor in the columns corresponding to $I$, the Plücker coordinate vector of $W$ is the homogeneous coordinate vector

$$p(W) = (\cdots : p_I(W) : \cdots) \in \mathbb{P}^{\binom{\ell}{\ell} - 1},$$

where $I$ runs through all subsets of size $\ell$ in $\{1, \ldots, m\}$. The point $p(W)$ is a well-defined invariant of $W$ because a change of basis in $W$ multiplies the matrix $M(B)$ on the left by the change of basis matrix, an element of $\text{GL}(\ell, \mathbb{F})$. All components of the Plücker coordinate vector are multiplied by the determinant of the change of basis matrix, an element of $\mathbb{F}^*$. Hence any choice of basis in $W$ yields the same point $p(W)$ in $\mathbb{P}^{\binom{\ell}{\ell} - 1}$.

The locus of all such points (for all $W$) forms the Grassmannian $\mathbb{G}(\ell, k)$. Consider the set of $W$ such that $p_{I_0}(W) \neq 0$, so the maximal minor with $I_0 = \{1, \ldots, \ell\}$ is invertible. The set of such $W$ is one of the open subsets in the standard affine cover of $\mathbb{G}(\ell, m)$. In the row-reduced echelon form of $M(B)$, the entries in the columns
complementary to \( I_0 \) (an \( \ell \times (m - \ell) \) block) are arbitrary and uniquely determine \( W \). Hence
\[
\dim \mathcal{G}(\ell, m) = \ell(m - \ell).
\]

To construct Grassmannian codes, one uses the \( \mathbb{F}_q \)-rational points of \( \mathcal{G}(\ell, m) \), which come from subspaces \( W \) defined over \( \mathbb{F}_q \). Nogin has established the following result.

**Theorem 15.** Let \( S \) be the set of all the \( \mathbb{F}_q \)-rational points on \( X = \mathcal{G}(\ell, m) \). Then the \( C(X; S) \) code (from linear forms in the Plücker coordinates) has parameters
\[
\left[ \begin{array}{c} m \\ \ell \end{array} \right]_q, \left( \begin{array}{c} m \\ \ell \end{array} \right), q^{\ell(m - \ell)},
\]
where
\[
\left[ \begin{array}{c} m \\ \ell \end{array} \right]_q = \frac{(q^m - 1)(q^{m - q} - 1) \cdots (q^m - q^{m - \ell})}{(q^\ell - 1)(q^{\ell - q} - 1) \cdots (q^\ell - q^\ell - 1)}.
\]

**Proof.** The numerator in the formula for \( \left[ \begin{array}{c} m \\ \ell \end{array} \right]_q \) is precisely the number of ways of picking a list of \( \ell \) linearly independent vectors in \( \mathbb{F}_q^m \) (a basis for a \( W \) defined over \( \mathbb{F}_q \)). Similarly, the denominator is the number of ways of picking \( \ell \) linearly independent vectors in \( \mathbb{F}_q^\ell \), hence the order of the group \( GL(\ell, \mathbb{F}_q) \). The quotient is the number of distinct \( \ell \)-dimensional subspaces of \( \mathbb{F}_q^m \). This shows \( n = \#S = \left[ \begin{array}{c} m \\ \ell \end{array} \right]_q \).

Assuming \( d = q^{\ell(m - \ell)} \) for the moment, the fact that \( d > 0 \) says the evaluation mapping on the vector space of linear forms in \( \mathbb{P}_{\mathbb{F}_q}(\ell) \) is injective, and the formula for \( k \) follows. Finally, we must prove that \( d = q^{\ell(m - \ell)} \).

The complement of the hyperplane section \( \mathcal{G}(\ell, m) \cap V(p_{I_0}) \) contains exactly \( q^{\ell(m - \ell)} \) \( \mathbb{F}_q \)-rational points of \( \mathcal{G}(\ell, m) \). Hence \( d \leq q^{\ell(m - \ell)} \). The cleanest way to prove that this is an equality is to use the language of exterior algebra on \( \mathbb{F}_q \)-vector spaces, following Nogin in [44].

Let \( V = \mathbb{F}_q^m \) and write \( e_i \) for the standard basis vectors in \( V \). The \( \mathbb{F}_q \)-rational points of the Grassmannian \( \mathcal{G}(\ell, m) \) can be identified with the subset of \( \mathbb{P} \left( \wedge^\ell V \right) \cong \mathbb{P}^{\binom{m}{\ell} - 1} \) corresponding to the completely decomposable elements of the exterior product \( \wedge^\ell V \) (that is, nonzero elements of the form \( \omega = w_1 \wedge w_2 \wedge \cdots \wedge w_\ell \) for some \( w_i \in V \) that form a basis for the subspace they span).

The hyperplanes in \( \mathbb{P} \left( \wedge^\ell V \right) \) correspond to elements of \( \mathbb{P} \left( \wedge^{\ell - 1} V \right)^* \), hence to elements of \( \wedge^{m - \ell} V \) (up to scalars) via the nondegenerate pairing
\[
\wedge : \wedge^{m - \ell} V \times \wedge^\ell V \to \wedge^m V \cong \mathbb{F}_q.
\]

It follows that the hyperplanes in \( \mathbb{P} \left( \wedge^\ell V \right) \) all have the form
\[
H(\alpha) = \mathbb{P} \{ \omega \in \wedge^\ell V : \alpha \wedge \omega = 0 \}
\]
for some nonzero \( \alpha \in \wedge^{m - \ell} V \).

Under these identifications, each hyperplane \( V(f) \) for \( f \) a linear form in the Plücker coordinates corresponds to \( H(\alpha) \) for some \( \alpha \). For instance, \( V(p_{I_0}) \) corresponds to \( H(\alpha_0) \) for the completely decomposable element \( \alpha_0 = e_{\ell + 1} \wedge \cdots \wedge e_m \).
All completely decomposable $\alpha \in \bigwedge^{m-\ell} V$ define hyperplane sections of the Grassmannian with the same number of $F$-rational points. Call this number $N_\ell$.

What must be proved is that if $\beta \in \bigwedge^{m-\ell} V$ is arbitrary, then the linear forms $f$ in the Plücker coordinates defining the hyperplane $H(\beta)$ satisfy

$$\text{wt}(ev_S(f)) \geq N_\ell.$$  

This follows by induction on $\ell$ using the easily checked fact that if $e \in V$ and $\alpha \in \bigwedge^{m-\ell} V$, then

$$\alpha \wedge e = 0 \iff \alpha = \alpha' \wedge e$$

for some $\alpha' \in \bigwedge^{m-\ell-1} V$.

If $\ell = 1$, there is nothing to prove because every element of $\bigwedge^{m-1} V$ is completely decomposable. If $\ell > 1$, writing $[\ell]_q = \#\text{GL}(\ell, \mathbb{F}_q)$,

$$\text{wt}(ev_S(f)) = \#\{W = \text{Span}(w_1, \ldots, w_\ell) : \beta \wedge w_1 \wedge \cdots \wedge w_\ell \neq 0\}$$

$$= \#\{(w_1, \ldots, w_\ell) : \beta \wedge w_1 \wedge \cdots \wedge w_\ell \neq 0\} / [\ell]_q$$

Hence by the induction hypothesis, if $\alpha$ is completely decomposable

$$[\ell]_q \cdot \text{wt}(ev_S(f)) = \sum_{w_1 \beta \wedge w_1 \neq 0} \#\{(w_2, \ldots, w_\ell) : (\beta \wedge w_1) \wedge w_2 \wedge \cdots \wedge w_\ell \neq 0\}$$

$$\geq \sum_{w_1 \beta \wedge w_1 \neq 0} N_{\ell-1} \cdot [\ell-1]_q$$

$$= N_{\ell-1} \cdot [\ell-1]_q \cdot \#\{w_1 : \beta \wedge w_1 \neq 0\}$$

$$\geq N_{\ell-1} \cdot [\ell-1]_q \cdot \#\{w_1 : \alpha \wedge w_1 \neq 0\}$$

by (18)

$$= [\ell]_q \cdot N_{\ell}.$$  

The exterior algebra language can also be used to say more about the weight distribution of $C(\mathbb{G}(\ell, m); S)$. For instance, the number of minimum weight words of this code is equal to the number of linear forms corresponding to completely decomposable $\alpha$. This number is exactly $q - 1$ times the number of $\mathbb{F}_q$-rational points of the dual Grassmannian $G(m-\ell, m)$, or

$$(q-1) \left[ \begin{array}{c} m \\ m-\ell \end{array} \right]_q = (q-1) \left[ \begin{array}{c} m \\ \ell \end{array} \right]_q.$$  

For further information on these codes see the bibliographic notes in Section 9.

Codes on certain subvarieties of Grassmannians, the so-called Schubert varieties, have also been studied in detail by Chen, Guerra and Vincenti, and Ghorpade and Tsfasman. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{Z}^\ell$, where $1 \leq \alpha_1 \leq \cdots \leq \alpha_\ell \leq m$. If $B = \{v_1, \ldots, v_m\}$ is a fixed basis of $\mathbb{F}_q^m$, let $A_i$ be the span of the first $i$ vectors in $B$. Then the Schubert variety $\Omega_\alpha$ is defined as

$$(19) \quad \Omega_\alpha = \{p(W) \in \mathbb{G}(\ell, m) : \text{dim } W \cap A_{\alpha_i} \geq i\}.$$  

See Section 9 for some pointers to the literature here.

Just as Grassmannians parametrize linear subspaces in $\mathbb{F}^m$, the flag varieties parametrize flags of linear subspaces, that is nested sequences of subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_s,$$
where \( \dim V_i = \ell_i \) and \( 0 < \ell_1 < \ell_2 < \ldots < \ell_s < m \). The flag is said to have type \( (\ell_1, \ell_2, \ldots, \ell_s) \). Also set \( \ell_{s+1} = m \) and \( \ell_0 = 0 \) by convention. The group \( G = \text{GL}(m, \mathbb{F}) \) acts on the set of flags of each fixed type and the isotropy subgroup of a particular flag is a parabolic subgroup \( P \) conjugate to the group of block upper-triangular matrices with diagonal blocks \( M_r \) of sizes \( \ell_r - \ell_{r-1} \) for \( 1 \leq r \leq s + 1 \). Hence the quotient \( G/P \), which is denoted \( \mathcal{F}(\ell_1, \ell_2, \ldots, \ell_s; m) \), classifies flags of type \( (\ell_1, \ell_2, \ldots, \ell_s) \). The set \( G/P \) has the structure of a projective variety, which can be described as follows. Each \( V_i \) corresponds to a point of the product variety \( G \). Then the \( C \) type \( (\ell, m) \) is ample. A classical result in the theory of Segre maps is that \( Y \) is isomorphic to the variety parametrizing flags \( V_1 \subset V_2 \) consisting of a line \( V_1 \) and a hyperplane \( V_2 \) containing that line. In this case

\[
\mathcal{F}(1, m-1; m) \subset \mathbb{G}(1, m) \times \mathbb{G}(m-1, m) \simeq \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{m^2-1}.
\]

**Theorem 16.** Let \( S \) be the set of all the \( \mathbb{F}_q \)-rational points on \( \mathcal{F}(1, m-1; m) \). Then the \( C(X; S) \) code has parameters

\[
\left\lfloor \frac{(q^m - 1)(q^{m-1} - 1)}{(q - 1)^2}, m^2 - 1, q^{2m-3} - q^{m-2} \right\rfloor.
\]

The proof is due to Rodier and appears in [47]. The evaluation mapping using linear forms on \( \mathbb{P}^{m^2-1} \) is not injective in this case because the condition that \( V_1 \subset V_2 \) is expressed by a linear equation in the coordinates of the Segre embedding of \( \mathbb{P}^{m-1} \times \mathbb{P}^{m-1} \).

### 4.4 Blow-ups and Del Pezzo surfaces.

Consider the surface \( X = \mathbb{P}^2 \). Let

\[
Y_k \to Y_{k-1} \to \cdots \to Y_1 \to Y_0 = X,
\]

be a sequence of morphisms where for all \( j \), \( \pi_j : Y_j \to Y_{j-1} \) is the blow up of an \( \mathbb{F}_q \)-rational point of the surface \( Y_{j-1} \). The result will be a surface \( Y = Y_k \) containing divisors \( E_1, \ldots, E_k \) that are all contracted to a point on \( X \). Each \( E_j \) is isomorphic to \( \mathbb{P}^1 \), and each contributes \( q \) additional \( \mathbb{F}_q \)-rational points. Therefore

\[
\#Y(\mathbb{F}_q) = q^2 + q + 1 + kq,
\]

which also attains the upper Weil bound for a surface with the Betti numbers of these examples. Whether this construction gives interesting codes depends very much on the the embedding of the surface \( Y \) into \( \mathbb{P}^m \) (that is, on the linear series of divisors forming the hyperplane sections).

One famous family of examples of such surfaces are the so-called Del Pezzo surfaces. Hartshorne’s text [28] and Manin [40] are good general references for these. By definition, a Del Pezzo surface is a surface of degree \( m \) in \( \mathbb{P}^m \) on which the anticanonical line bundle \( K^{-1} \) is ample. A classical result in the theory of
algebraic surfaces is that every Del Pezzo surface over an algebraically closed field \( \mathbb{F} \) is obtained either as the degree 2 Veronese image of a quadric in \( \mathbb{P}^3 \), or as follows. Let \( \ell \) be one of the integers 0, 1, \ldots, 6, and take points \( p_1, \ldots, p_\ell \) in \( \mathbb{P}^2 \) in general position (no three collinear, and no six contained in a conic curve). The linear system of cubic curves in \( \mathbb{P}^2 \) containing the base points \( \{p_1, \ldots, p_\ell\} \) gives a rational map \( \rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^{9-\ell} \). The image is a surface \( X_\ell \) of degree \( 9-\ell \) on which the points \( p_i \) blow up to exceptional divisors \( E_i \cong \mathbb{P}^1 \) as in the composition of all the maps in \( \mathbb{P}^1 \). Since the canonical sheaf on \( \mathbb{P}^2 \) is \( K \cong \mathcal{O}_{\mathbb{P}^2}(-3) \), the anticanonical divisors are precisely the divisors in the linear system of cubics containing \( \{p_1, \ldots, p_\ell\} \). For instance, with \( \ell = 6 \), \( X_6 \) is a smooth cubic surface in \( \mathbb{P}^3 \), and every smooth cubic surface is obtained by blowing up some choice of points \( p_1, \ldots, p_6 \). With \( \ell = 0 \), the surface \( X_0 \) is the degree 3 Veronese image of \( \mathbb{P}^2 \), a surface of degree 9 in \( \mathbb{P}^9 \).

To get a Del Pezzo surface defined over \( \mathbb{F}_q \), the points \( p_i \) should be \( \mathbb{F}_q \)-rational points in \( \mathbb{P}^2 \). This means that the construction above can fail for certain small fields (there may not be enough points \( p_i \) in general position). It suffices to take \( q > 4 \), however in order to construct the Del Pezzo surfaces with \( 0 \leq \ell \leq 6 \).

By considering the possible hyperplane sections of the Del Pezzo surface Boguslavsky derives the following result in [3].

**Theorem 17.** Let \( X_\ell \) be the Del Pezzo surface constructed as above and let \( q > 4 \). The parameters of the \( C(X_\ell) \) code are

\[
\begin{align*}
 n &= q^2 + q + 1 + \ell q, \\
k &= 10 - \ell,
\end{align*}
\]

and \( d \) given in the following table.

| \( \ell \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| \( d(C(X_\ell)) \) | \( q^2-2q \) | \( q^2-2q \) | \( q^2-2q \) | \( q^2-2q+1 \) | \( q^2 \) | \( q^2+2q \) | \( q^2+4q+1^* \) |

The case \( \ell = 6 \) corresponds to the code from a cubic surface in \( \mathbb{P}^3 \). Note the asterisk in the table above. In the generic case, there are plane sections of a cubic surface consisting of three lines forming a triangle, but no sections consisting of three concurrent lines. The triangle plane sections contain the maximum number of \( \mathbb{F}_q \)-rational points, namely \( 3q \). Hence \( d(C(X_6)) = q^2 + 7q + 1 - 3q = q^2 + 4q + 1 \), as claimed in this case. For some special configurations of points \( p_i \), however, the corresponding cubic surface will have Eckardt points where there is a plane section consisting of three concurrent lines. For those surfaces, the minimum distance is \( q^2 + 4q \) rather than \( q^2 + 4q + 1 \).

4.5. **Ruled surfaces and generalizations.** A ruled surface is a surface \( X \) with a mapping \( \pi : X \to C \) to a smooth curve \( C \), whose fibers over all points of \( C \) are \( \mathbb{P}^1 \)'s. Moreover, it is usually required that \( \pi \) has a section, that is, a mapping \( \sigma : C \to X \) such that \( \pi \circ \sigma \) is the identity on \( C \). For instance, over an algebraically closed field, quadric surfaces in \( \mathbb{P}^3 \) are isomorphic to the product ruled surface \( \mathbb{P}^1 \times \mathbb{P}^1 \). For background on these varieties, Chapter V of [28] is a good reference.

Starting from a curve \( C \) and a vector bundle of rank 2 (that is, a locally free sheaf of rank 2) \( \mathcal{E} \) on \( C \), the projective space bundle \( X = \mathbb{P}(\mathcal{E}) \) is a ruled surface. Conversely, every ruled surface \( \pi : X \to C \) is isomorphic to \( \mathbb{P}(\mathcal{E}) \) for some locally free sheaf of rank 2 on \( C \). Given a curve \( C \) and two vector bundles on \( C \), the ruled surfaces \( \mathbb{P}(\mathcal{E}) \) and \( \mathbb{P}(\mathcal{E}') \) are isomorphic if and only if \( \mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L} \) for some line bundle \( \mathcal{L} \) on \( C \). By choosing \( \mathcal{L} \) appropriately, it is possible to make \( H^0(\mathcal{E}) \neq 0 \) but \( H^0(\mathcal{E} \otimes \mathcal{M}) = 0 \) whenever \( \mathcal{M} \) is a line bundle on \( C \) of negative degree and in this
case we say $E$ is normalized. Then there is a section $C_0$ of $X$ with $C_0^2 = -e$ where $e = \deg(E)$ is the degree of the divisor $E$ on $C$ corresponding to the line bundle $\mathcal{E}$. If $\mathcal{E}$ is decomposable (a direct sum of two line bundles) and normalized, then $e \geq 0$. If $\mathcal{E}$ is indecomposable, then it is known that $-g(C) \leq e \leq 2g(C) - 2$, where $g(C)$ is the genus.

Up to numerical equivalence, each divisor $D$ on $X$ is $D \sim b_1C_0 + b_2f$, where $f$ is a fiber of the mapping $\pi$ and $b_1, b_2 \in \mathbb{Z}$. The intersection product on divisors is determined by the relations $C_0^2 = -e$, $C_0 \cdot f = 1$, $f^2 = 0$. S.H. Hansen has shown the following result.

**Theorem 18.** Let $\pi : X \to C$ be a normalized ruled surface with invariant $e \geq 0$. Let $#C(\mathbb{F}_q) = a$, and let $\mathcal{S}$ be the full set of $\mathbb{F}_q$-rational points on $X$. Let $\mathcal{L} = \mathcal{O}_X(b_1C_0 + b_2f)$. Then the $C(X, \mathcal{L}, \mathcal{S})$ code has parameters

$$[a(q + 1), \dim \Gamma(X, \mathcal{L}), d \geq n - b_2(q + 1) - (a - b_2)b_1],$$

(provided that $b_2 < a$ and the bound on $d$ is positive).

**Proof.** Let $f_1, \ldots, f_a$ be the fibers of $\pi$ over the $\mathbb{F}_q$-rational points of $C$. These are disjoint curves on $X$ isomorphic to $\mathbb{P}^1$, hence contain $q + 1$ $\mathbb{F}_q$-rational points each. Every $\mathbb{F}_q$-rational point of $X$ lies on one of these lines, so $n = a(q + 1)$. As usual, the statement for $k$ follows if $d > 0$. The estimate for $d$ comes from the method of Theorem 3 applied to the covering family of curves $f_1, \ldots, f_a$. In the notation of that theorem, we have $N = q + 1$ and $\eta = (b_1C_0 + b_2f) \cdot f = b_1$. At most $\ell = b_2$ of the fibers are contained in any divisor $D$ corresponding to a global section of $\mathcal{O}_X(b_1C_0 + b_2f)$ since $D \cdot C_0 = (b_1C_0 + b_2f) \cdot C_0 = -eb_1 + b_2 \leq b_2$. The bound on $d$ follows immediately. \(\square\)

The dimension of the space of global sections of $\mathcal{L}$ can be computed via divisors on $C$ because of general facts about sheaves on the projective space bundle $\mathcal{P}(\mathcal{E})$ (see [28], Lemma V.2.4). See the bibliographic notes in Section 3 for more information about these codes and for work on codes from projective bundles of higher fiber dimension.

5. Codes from Deligne-Lusztig Varieties

Some of the most interesting varieties that have been used to produce codes by the constructions of Section 2 are the so-called Deligne-Lusztig varieties from representation theory. As we will see, their description involves several of the general processes on varieties involved in the examples above.

Let $G$ be a connected reductive affine algebraic group over the algebraically closure $\mathbb{F}$ of $\mathbb{F}_q$, a closed subgroup of $\text{GL}(n, \mathbb{F})$ for some $n$. We have the $q$-Frobenius endomorphism $F : G \to G$ whose fixed points are the $\mathbb{F}_q$-rational points of $G$.

A Borel subgroup of $G$ is a maximal connected solvable subgroup of $G$. A torus is a subgroup of $G$ isomorphic to $(\mathbb{F}_q^*)^r$ for some $s$. All Borel subgroups are conjugate, and each maximal torus $T$ is contained in some Borel subgroup. Let $N(T)$ be the normalizer of $T$ in $G$. The quotient $N(T)/T$ is a finite group called the Weyl group of $G$.

The set $\mathcal{B}$ of all Borel subgroups of $G$ can be identified with the quotient $G/B$ for any particular $B$ via the mapping $G/B \to \mathcal{B}$ given by $g \mapsto g^{-1}Bg$. If $w \in W$, then the Deligne-Lusztig variety associated to $w$ can be described as follows. Let
Let $w = s_1 \cdots s_n$ be a minimal factorization of $w$ into simple reflections in $W$, the Weyl group of $G$ as above. Then

1. $X(w)$ is a locally closed smooth variety of pure dimension $n$.
2. The variety $X(w)$ is fixed by the action of the group $G^F$ and is defined over $\mathbb{F}_{q^s}$, where $s$ is the smallest integer such that $F^s$ fixes $w$.
3. The closure of $X(w)$ in $B$ is the union of the $X(s_1, \cdots s_i)$ such that $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ and $X(e)$.

We refer to [5] for the classification of reductive $G$ in terms of Dynkin diagrams with action of $F$. In [21], J. Hansen studied the Hermitian curves over $\mathbb{F}_{q^2}$, the Suzuki curves over $\mathbb{F}_{q^{2n+1}}$ and the Ree curves over $\mathbb{F}_{q^{2n+1}}$, all well-known maximal curves, and all used to construct interesting Goppa codes with very large automorphism groups. Hansen showed that the underlying reason these particular curves are so rich in good properties is that they are the Deligne-Lusztig varieties for groups $G$ for which there is just one orbit of simple reflections in the Weyl group under the action of $F$. The Hermitian curves come from groups of type $2B_2$, the Suzuki curves come from the groups of type $2G_2$, and the Ree curves from the groups of type $2F_4$.

It is known that there are seven cases in which there are two $F$-orbits in the set of reflections in $W$, so taking $s_1, s_2$ from the distinct orbits, the Deligne-Lusztig construction with $w = s_1 s_2$ leads to algebraic surfaces:

$$A_2, C_2, G_2, 2A_3, 2A_4, 3D_4, 2F_4.$$ One of these cases is relatively uninteresting. In [46], Rodier shows that the complete, smooth Deligne-Lusztig variety $\overline{X(s_1, s_2)}$ from the group of type $A_2$ is isomorphic to the blow-up of $\mathbb{P}^2$ at all of its $\mathbb{F}_{q^2}$-rational points.

For the group of type $2A_3$, however, Rodier shows that $\overline{X(s_1, s_2)}$ is isomorphic to the blow-up of the Hermitian surface in $\mathbb{P}^3$ at its $\mathbb{F}_{q^2}$-rational points. Hence as in the discussion of the blow-ups of $\mathbb{P}^2$ above, and using Example [4] we get a surface with $(q^3 + 1)(q^2 + 1)^2$ points.

Similarly the $\overline{X(s_1, s_2)}$ from a group of type $2A_4$ is isomorphic to the blow-up of the complete intersection $Y$ of the two hypersurfaces

\begin{align*}
0 &= x_0^{q+1} + x_1^{q+1} + \cdots + x_4^{q+1} \\
0 &= x_0^{q^2+1} + x_1^{q^2+1} + \cdots + x_4^{q^2+1}
\end{align*}

in $\mathbb{P}^4$ at the $(q^3 + 1)(q^2 + 1)$ $\mathbb{F}_{q^2}$-rational points on that surface. (These are the same as the $\mathbb{F}_{q^2}$-rational points on the Hermitian 3-fold in $\mathbb{P}^4$ defined by the first equation.) It is easy to check that these points are all singular, and in fact they blow up to Hermitian curves (not $\mathbb{P}^1$'s) on the Deligne-Lusztig surface. Hence the Deligne-Lusztig surface $X$ has a very large number of $\mathbb{F}_{q^2}$-rational points in this case,

$$\#X(\mathbb{F}_{q^2}) = (q^5 + 1)(q^2 + 1)(q^3 + 1).$$

Rodier determines the structure and number of $\mathbb{F}_{q^2}$-rational points in the $G_2, 3D_4$, and $2F_4$ cases as well. Interestingly enough, his method is to realize the Deligne-Lusztig varieties as certain subsets of flag varieties as above, where the subspaces in the flags are related to each other using the Frobenius endomorphism.
Rodier and S.H. Hansen also discuss the properties of the $C_h(X)$ codes on these varieties. For instance in [26], Hansen shows the following result by relating codes on $Y$ from (21) and codes on the Deligne-Lusztig surface itself.

**Theorem 20.** Let $X$ be the Deligne-Lusztig surface of type $2A_4$ over the field $\mathbb{F}_{q^2}$. For $1 \leq h \leq q^2$, there exist codes over $\mathbb{F}_{q^2}$ with

\[
\begin{align*}
    n &= (q^5 + 1)(q^3 + 1)(q^2 + 1), \\
    k &= \left(\frac{4 + h}{h}\right) - \left(\frac{4 + h - (q + 1)}{t - (q + 1)}\right), \text{ and} \\
    d \geq n - hP(q),
\end{align*}
\]

where $P(q) = (q^3 + 1)(q^5 + 1) + (q + 1)(q^3 + 1)(q^2 - h + 1)$.

Since $P(q)$ has degree 8 in $q$, this shows that $d + k \geq n - O(n^{4/5})$ with $n = O(q^{10})$, some very long codes indeed! Hansen also considers the codes obtained from the singular points on the complete intersection from (21) (that is from the Hermitian 3-fold).

### 6. Connections with Other Code Constructions

In this section we point out some connections between the construction presented here and some other examples of algebraic geometric codes related to higher dimensional varieties in the literature. There is a close connection between the codes $C(X, \mathcal{L}; \mathcal{S})$ and the toric codes constructed from polytopes or fans in $\mathbb{R}^s$ as in [22]. A toric variety of dimension $s$ over an algebraically closed field $\mathbb{F}$ is a variety $X$ containing a Zariski-open subset isomorphic to the $s$-dimensional algebraic torus $T \simeq (\mathbb{F}^*)^s$ and on which $T$ acts in a manner compatible with the multiplicative group structure on $T$. The combinatorial data in a fan $\Sigma$ in $\mathbb{R}^s$ encodes the gluing information needed to produce a normal toric variety $X_\Sigma$ from affine open subsets of the form $\text{Spec}(\mathbb{F}[S_\sigma])$ where $\mathbb{F}[S_\sigma]$ is a semigroup algebra associated to the cone $\sigma$ in the fan $\Sigma$. A polytope $P$ in $\mathbb{R}^s$ determines a normal fan $\Sigma_P$ and line bundle $L_P$ on $X_{\Sigma_P}$. The toric codes are codes $C(X, \mathcal{L}; \mathcal{S})$ for $X = X_{\Sigma_P}$, $\mathcal{L} = L_P$ and $\mathcal{S} = T \cap \mathbb{F}^* = (\mathbb{F}^*)^s$. It is not difficult to see that toric codes are $s$-dimensional cyclic codes with certain other properties generalizing those of Reed-Solomon codes.

The study of decoding algorithms for one-point algebraic geometric Goppa codes has been unified and simplified by the theory of order domains discussed in [32, 14]. The article [38] shows how order domains can be constructed from many of the higher dimensional varieties discussed here.

### 7. Code Comparisons

It is instructive to compare codes constructed by the methods described here and the best currently known codes for the same $n, k$. We will focus on the minimum distance, although there are many other considerations too in deciding on codes for given applications.

All comparisons will be made by means of the online tables of Markus Grassl, [18]. One initial observation is that many of the varieties $X$ that we have discussed have so many $\mathbb{F}_q$-rational points that the $n$ achieved are far beyond the ranges explored to date. When no explicit codes are known, it is still possible to make comparisons with general bounds. Since the $k$ for most of the $C_h(X)$ codes we have
seen are much smaller than \( n \), the Griesmer bound yields some information. The usual form of the Griesmer bound (see [33]) says that for an \([n, k, d]\) code over \( \mathbb{F}_q \),

\[
n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor.
\]

Given \( n, k \), this inequality can also be used to derive an upper bound on realizable \( d \) for \([n, k]\) codes that, in a sense, improves the Singleton bound \( d \leq n - k + 1 \). It should be noted, however, that there are many pairs \( n, k \) for which there are no codes attaining the Griesmer upper bound on \( d \).

We begin by noting the following well-known fact.

**Theorem 21.** The projective Reed-Muller codes with \( h = 1 \) from Theorem\( \text{2} \) attain the Griesmer upper bound for all \( m \).

This follows since \( n = \# \mathbb{F}_m(\mathbb{F}_q) = q^m + \cdots + q + 1 \), \( d = q^m \), and \( k = m + 1 \).

For \( h > 1 \), however, the presence of reducible forms of degree \( h \), which can have many more \( \mathbb{F}_q \)-rational zeroes than irreducible forms (see the proof of Theorem 2), tends to reduce the minimum distance relative to other code constructions. This is true for all \( q \), although the difference shows up for smaller \( h \) the larger \( q \) is.

For instance, in the binary case, the \( h = 2 \) projective Reed-Muller code with \( m = 5 \) has parameters \([63, 21, 16]\), but there are binary \([63, 21, 18]\) codes known by [18]. Similarly, with \( q = 4 \), the \( h = 2 \) projective Reed-Muller code with \( m = 3 \) over \( \mathbb{F}_4 \) has parameters \([85, 10, 48]\), but there are \([85, 10, 52]\) codes known over \( \mathbb{F}_4 \) by [18]. In the cases that have been explored in detail, the gap between the projective Reed-Muller codes and the best known codes seems to increase with \( m \) for fixed \( h \), and also for \( h \) with fixed \( m \) (for the cases \( h < q + 1 \) considered here at least).

The minimum distance for the \( C(X) \) codes from quadrics from [13] also tend to be relatively close to the Griesmer bound for their \( n, k \), although the bounds grow slightly faster than the actual \( d \) as \( m \to \infty \) and slightly better codes are known in a number of cases. The codes from elliptic quadrics \((w = 0)\) are superior in general to those from hyperbolic quadrics \((w = 2)\) when \( m \) is odd. This is an interesting indication that perhaps the “greedy” approach of maximizing \( n = \# X(\mathbb{F}_q) \) does not always yield the best codes.

For example, over \( \mathbb{F}_8 \), the \( C(X) \) code from a hyperbolic quadric in \( \mathbb{P}^3 \) has parameters \([81, 4, 64]\), but there are \([81, 4, 68]\) codes known by [18]. (The Griesmer bound in this case gives \( d \leq 69 \).) By way of contrast, the \( C(X) \) code from an elliptic quadric has parameters \([65, 4, 56]\), and this is the best possible by the Griesmer bound. Similar patterns hold over all of the small fields where systematic exploration has been done. For larger \( m \), however, it is not always the case that the \( C(X) \) codes from elliptic quadrics meet the Griesmer bound, and there are slightly better known codes in some cases. The \( C_2(X) \) codes from quadrics seem to be similar, at least in the case \( m = 3 \), where the results of Edoukou from [12] can be applied. Over \( \mathbb{F}_8 \) for instance, the \( C_2(X) \) code from a hyperbolic quadric surface has parameters \([81, 9, 49]\), but there are \([81, 9, 58]\) codes known by [18]. On the other hand, the \( C_2(X) \) code from an elliptic quadric has parameters \([65, 9, 47]\), and this matches the best known \( d \) for this \( n, k \) over \( \mathbb{F}_8 \). (The tightest known upper bound is \( d \leq 50 \).)

The Hermitian hypersurface codes seem to be similar to those from quadrics. The \( C(X) \) codes are quite good, coming quite near the Griesmer bound. For
instance, the Hermitian surface code from Theorem 11 over \( \mathbb{F}_{16} \) has parameters \([1105, 4, 1024]\). This is far outside the range of \( n \) and fields for which tables are available, but by way of comparison, \( d \leq 1034 \) by the Griesmer bound. However, the \( C_2(X) \) codes are not as good, and the gap grows with \( h \).

The codes from Del Pezzo surfaces from Theorem 17 are interesting only for \( \ell = 0 \) (the case \( X = \mathbb{P}^2 \)) and \( \ell = 6 \) (the case of the cubic surface in \( \mathbb{P}^3 \)). The intermediate cases are quite inferior to the best known codes.

For the other families of varieties we have considered (Grassmannians, flag varieties, Deligne-Lusztig varieties), once \( q \) or \( m \) get even moderately large, \( n \) is so huge that very little is known. On the basis of rather limited evidence, the Grassmannian and flag variety codes might be especially good only over very small fields, though. For example, the \( C(X) \) code from \( X = G(2, 4) \) over \( \mathbb{F}_2 \) has parameters \([35, 6, 16]\), which attains the Griesmer bound. Over \( \mathbb{F}_3 \), the corresponding Grassmannian code has \([130, 6, 81]\), but there are \([130, 6, 84]\) codes over \( \mathbb{F}_3 \) known by [18] and the Griesmer bound gives \( d \leq 84 \) in this case.

It is unrealistic to expect every code constructed from a variety of dimension \( \geq 2 \) to be a world-beater. The examples here are offered as evidence that we still do not know how this construction can best be applied to produce good codes.

8. Conclusion

The study of error control codes constructed from higher dimensional varieties is an area where it is certainly true that we have just barely begun feeling out the lay of the land and just barely scratched the surface of what should be possible. If this survey of past work inspires further exploration, then one of its goals will have been achieved!

9. Bibliographic Notes

**Section 1.** The universality of the Goppa construction for producing linear codes is proved in [15]; specifically we are referring to Pellikaan, Shen, and van Wee’s result that every linear code is weakly algebraic-geometric: Given \( C \), there exists a smooth projective curve \( X \), a set \( S \) of \( \mathbb{F}_q \)-rational points on \( X \), and a line bundle \( L = \mathcal{O}(G) \) for some divisor \( G \) with support disjoint from \( S \), such that \( C \) is isomorphic to \( C(X, L; S) \) (with no restriction on the degree of \( G \)).

Although very little work to date has been done on decoding methods, the large groups of automorphisms of some of the varieties considered here make the permutation decoding paradigm a possibility for certain of these codes. Some work along these lines has been done by Kroll and Vincenti, [33, 35].

**Section 2.** Both forms of the construction of codes from varieties (Definitions 1 and 2) come from [54], which was the first place where this idea was described in published form. The form in Definition 2 can be made even more concrete and less algebraic-geometric by the language of projective systems of points and their associated codes.

**Section 3.** Theorem 2 is taken from [37]. It does not include the codes for \( h > q \) because the evaluation mapping is no longer injective in those cases. The parameters of the \( C_h \) codes for \( h > q \) have been studied by Lachaud in [36] and Sørensen in [53]. The generalized Hamming weights \( d_r \) for the Reed-Muller codes have been
studied by Heijnen and Pellikaan in [29]. Some ideas about finding good subcodes of the $C_2$ codes have been presented by Brouwer in [4].

Theorem 3, the following example, and the bound using Seshadri constants in Theorem 4 are all due to S.H. Hansen and are taken from [26].

The results on bounds for the minimum distance when $\mathcal{S}$ is a complete intersection come from [17] and that article’s bibliography gives several sources for the Cayley-Bacharach theorem and modern generalizations. The genesis for this was the observation that if $\mathcal{S}$ is a reduced complete intersection of two cubic curves in $\mathbb{P}^2$, and $\Gamma'$ is any subset of eight of the nine points in $\mathcal{S}$, then every cubic that contains the eight points in $\Gamma'$ also passes through the ninth point in $\mathcal{S}$. Related applications to coding theory were discussed by Duursma, Renteria and Tapia-Recillas in [10] and J. Hansen in [23]. The theorem stated here can also be extended to yield a criterion for MDS codes.

The Weil conjectures were originally stated in [58] and proved in complete generality by Deligne in [9] following three decades of work by Dwork, Serre, Artin, Grothendieck, Verdier, and many others. Weil’s paper gives a different form for middle Betti number in (8), but it can be seen that his form is equivalent to ours. The discussion of Weil-type bounds follows Lachaud’s presentation in [37]. Because of space limitations and the significantly higher prerequisites needed to work with the $\ell$-adic étale cohomology theory in any detail in higher codimension, we have focused only on the application of Lachaud’s results to codes from hypersurfaces. The discussion in [37] is considerably more general. Edoukou has verified Sørensen’s conjecture (see (14)) on the Hermitian surface codes in the case $h = 2$ in [11].

Section 4. The codes from quadrics have been intensively studied since at least the 1975 article [59] of Wolfmann. They are especially accessible because so much is known about the sets of $\mathbb{F}_q$-rational points on quadrics as finite geometries; see Hirschfeld and Thas, [30]. The complete hierarchies of generalized Hamming weights $d_r$ for the $C(X)$ codes were determined independently by Nogin in [43] and Wan in [56]. To aid in comparing these different sources, we note that Wan’s invariant $\delta$ is related to Hirschfeld and Thas’s (and our) character $w$ by $\delta = 2 - w$. The character can also be defined by $w = 2g - m + 3$ where $g$ is the dimension of the largest linear subspace of $\mathbb{P}^m$ contained in the quadric $X$. Comparatively little has appeared in the literature concerning the $C_h(X)$ codes with $h > 1$ on quadrics following the work of Aubry in [1]. One recent article studying the $C_2(X)$ codes from quadrics in $\mathbb{P}^3$ is Edoukou, [12].

Hirschfeld and Thas also contains a wealth of information related to the codes on Hermitian hypersurfaces. The parameters of the $C(X)$ codes were established by Chakravarti in [6], and the generalized Hamming weights were determined in by Hirschfeld, Tsfasman, and Vladut in [31].

Grassmannian codes were studied first in the binary case by C. Ryan and K. Ryan in [48, 49, 50]. The material on Grassmannian codes presented here is taken from [44]. In that article, Nogin also determines the complete weight distribution for the codes $G(2, m)$ and shows that the generalized weights $d_r$ of the Grassmann codes meet the generalized Griesmer bound when $r \leq \max\{\ell, m - \ell\} + 1$. More information on the generalized weights was established by Ghorpade and Lachaud in [15] and these codes are also discussed as a special case of the code construction from flag varieties by Rodier in [47]. This article also gives the proof of Theorem 10.

Codes from the Schubert varieties defined in [19] have been studied in [7, 20, 16].
The material on Del Pezzo surface codes is taken from Boguslavsky. That article also determines the complete hierarchy of generalized Hamming weights $d_r$ for these codes.

Codes from ruled surfaces were studied by S.H. Hansen in [26] as an example of how the bound from Theorem 3 could be applied. That article also addresses the cases where the invariant $e < 0$, and presents some examples involving ruled surfaces over the Hermitian elliptic curve over $\mathbb{F}_4$. Codes from ruled surfaces were also considered in Lomont’s thesis, [39]. The results for codes over ruled surfaces have been generalized to give corresponding results for codes on projective bundles $\mathbb{P}(E)$ for $E$ of all ranks $r \geq 2$ by Nakashima in [42]. Nakashima also considers codes on Grassmann, quadric, and Hermitian bundles in [41].

Other work on codes from algebraic surfaces is contained in the Ph.D. theses of Lomont, [39], and Davis, [8]. In addition, the unpublished preprint [55] of Voloch and Zarzar and the article [60] adopt the interesting approach of trying to find good surfaces for constructing codes by limiting the presence of reducible hyperplane sections through controlling the rank of the Néron-Severi group.

Section 5. Rodier’s article [46] is a gold mine of information and techniques for the Deligne-Lusztig surfaces and Deligne-Lusztig varieties more generally. The original article of Deligne and Lusztig and a number of other works devoted to this construction are referenced in the bibliography. The Picard group and other aspects of the finer structure of Deligne-Lusztig varieties have been studied by S.H. Hansen in [24, 25, 26]. Hansen’s thesis, [24] contains chapters corresponding to the other articles here.

Section 6. A standard reference for the theory of toric varieties over $\mathbb{C}$ is Fulton’s text, [13]; the construction generalizes to fields of characteristic $p$ with no difficulty.

References

[1] Y. Aubry, Reed-Muller codes associated to projective algebraic varieties, in: Coding Theory and Algebraic Geometry (Proceedings, Luminy 1991), H. Stichtenoth and M.A. Tsfasman, eds. Springer Lecture Notes in Mathematics 1518 (Springer, Berlin, 1992), 4–17.
[2] E. Ballico and C. Fontanari, The Horace method for error-correcting codes, Appl. Algebra Engrg. Comm. Comput. 17, 135–139 (2006).
[3] M.I. Boguslavsky, Sections of Del Pezzo surfaces and generalized weights, Probl. Inf. Transm. 34, 14–24 (1998).
[4] A. Brouwer, Linear spaces of quadrics and new good codes, Bull. Belg. Math. Soc. 5, 177-180 (1998).
[5] R. Carter, Finite Groups of Lie Type (Wiley, New York, 1985).
[6] I.M. Chakravarti, Families of codes with few distinct weights from singular and nonsingular Hermitian varieties and quadrics in projective geometries and Hadamard difference sets and designs associated with two-weight codes, in: Coding Theory and Design Theory, I, IMA Vol. Math Appl. 20 (Springer, New York, 1990), 35–50.
[7] H. Chen, On the minimum distance of Schubert codes, IEEE Trans. Inform. Theory, 46, 1535–1538 (2000).
[8] J. Davis, Algebraic geometric codes on anticanonical surfaces, Ph.D. thesis, University of Nebraska, 2007.
[9] P. Deligne, La conjecture de Weil, I, Publ. Math. IHES 43, 273–307 (1974).
[10] I. Duursma, C. Renteria and H. Tapia-Recillas, Reed-Muller codes on complete intersections, Algebra Engrg. Comm. Comput. 11, 455–462 (2001).
[11] F. Edoukou, Codes defined by forms of degree 2 on hermitian surfaces and Sorensen’s conjecture, Finite Fields Appl. 13, 616–627 (2007).
[12] F. Edoukou, Codes defined by forms of degree 2 on quadric surfaces, IEEE Trans. Inform. Theory 54, 860–864 (2008).
[13] W. Fulton, Introduction to toric varieties, Annals of Mathematics Studies 131, (Princeton University Press, Princeton, 1993).
[14] O. Geil and R. Pellikaan, On the structure of order domains, Finite Fields Appl. 8, 369–396 (2002).
[15] S. Ghorpade and G. Lachaud, Higher weights of Grassmann codes, in: Coding Theory, Cryptography, and Related Areas, Proceedings Guanajuato 1998, J. Buchmann, T. Høholdt, H. Stichtenoth, H. Tapia-Recillas eds. (Springer, Berlin, 2000), 122–131.
[16] S. Ghorpade and M. Tsfasman, Schubert varieties, linear codes and enumerative combinatorics, Finite Fields Appl. 11, 684–699 (2005).
[17] L. Gold, J. Little and H. Schenck, Cayley-Bacharach and evaluation codes on complete intersections, J. Pure Appl. Algebra 196, 91–99 (2005).
[18] M. Grassl, Bounds on minimum distance of linear codes, available online at http://www.codetables.de, accessed on 2008-02-02.
[19] P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley, New York, 1978).
[20] L. Guerra and R. Vincenti, On the linear codes arising from Schubert varieties, Des. Codes Cryptogr. 33, 173–180 (2004).
[21] J. Hansen, Deligne-Lusztig varieties and group codes, in: Coding Theory and Algebraic Geometry (Proceedings, Luminy 1991), H. Stichtenoth and M.A. Tsfasman, eds. Springer Lecture Notes in Mathematics 1518 (Springer, Berlin, 1992), 63–81.
[22] J. Hansen, Toric surfaces and error correcting codes, Coding Theory, Cryptography, and Related Areas, Proceedings Guanajuato 1998, J. Buchmann, T. Høholdt, H. Stichtenoth, H. Tapia-Recillas eds. (Springer, Berlin, 2000), 132–142.
[23] J. Hansen, Linkage and codes on complete intersections, Appl. Algebra Engrg. Comm. Comput. 14, 175–185 (2003).
[24] S.H. Hansen, The geometry of Deligne-Lusztig varieties: Higher dimensional AG codes, Ph.D. thesis University of Aarhus, 1999.
[25] S.H. Hansen, Canonical bundles of Deligne-Lusztig varieties. Manuscripta Math. 98 363–375 (1999).
[26] S.H. Hansen, Error-correcting codes from higher-dimensional varieties, Finite Fields Appl. 7, 530–552 (2001).
[27] S.H. Hansen, Picard groups of Deligne-Lusztig varieties—with a view toward higher codimensions, Beiträge Algebra Geom. 43, 9–26 (2002).
[28] R. Hartshorne, Algebraic Geometry (Springer, New York, 1977).
[29] P. Heijnen and R. Pellikaan, Generalized Hamming weights of q-ary Reed Muller codes, IEEE Trans. Inform. Theory 44, 181–196 (1998).
[30] J.W.P. Hirschfeld and J.A. Thas, General Galois Geometries (Oxford University Press, Oxford, 1991).
[31] J.W.P. Hirschfeld, M. Tsfasman and S.G. Vladut, The weight hierarchy of higher dimensional Hermitian codes, IEEE Trans. Inform. Theory 40, 275–278 (1994).
[32] T. Høholdt, J. van Lint and R. Pellikaan, Algebraic geometry codes, in: Handbook of Coding Theory, W. Huffman and V. Pless, eds. (Elsevier, Amsterdam, 1998), 871–962.
[33] W. Huffman and V. Pless, Fundamentals of Error-Correcting Codes, (Cambridge University Press, Cambridge, 2003).
[34] J. Little, The ubiquity of order domains for the construction of error control codes, Discrete Math. 301, 89–105 (2005).
[35] J. Little and R. Pellikaan, PD-sets for the codes related to some classical varieties, Discrete Math. 308, 408–414 (2008).
[36] G. Lachaud, The parameters of projective Reed-Muller codes, Discrete Math. 81, 217–221 (1990).
[37] G. Lachaud, Number of points of plane sections and linear codes defined on algebraic varieties, in: Arithmetic, Geometry and Coding Theory, Proceedings Luminy 1993, R. Pellikaan, M. Perret, S.G. Vladut eds. (Walter de Gruyter, Berlin, 1996), 77–104.
[38] J. Little, The ubiquity of order domains for the construction of error control codes, Adv. Math. Communications 1, 1–27 (2007).
[39] C. Lomont, Error correcting codes on algebraic surfaces, Ph.D. thesis, Purdue University, 2003, arXiv:math/0309123.
[40] Yu.I. Manin, *Cubic Forms: Algebra, Geometry, Arithmetic*, (North Holland, Amsterdam, 1986).
[41] T. Nakashima, Codes on Grassmann bundles, *J. Pure Appl. Algebra* 199, 235-244 (2005).
[42] T. Nakashima, Error-correcting codes on projective bundles, *Finite Fields Appl.* 12, 222–231 (2006).
[43] D.Yu. Nogin, Generalized Hamming weights of codes on multidimensional quadrics, *Probl. Inf. Transm.* 29, 21–30 (1993).
[44] D.Yu. Nogin, Codes associated to Grassmannians, in: *Arithmetic, Geometry and Coding Theory, Proceedings Luminy 1993*, R. Pellikaan, M. Perret, S.G. Vladut eds. (Walter de Gruyter, Berlin, 1996), 145–154.
[45] R. Pellikaan, B.-Z. Shen and G. van Wee, Which linear codes are algebraic-geometric? *IEEE Trans. Inform. Theory* IT-37, 583–602 (1991).
[46] F. Rodier, Nombre de points des surfaces de Deligne et Lusztig, *J. Algebra* 227, 706–766 (2000).
[47] F. Rodier, Codes from flag varieties over a finite field, *J. Pure Appl. Algebra* 178, 203–214 (2003).
[48] C.T. Ryan, An application of Grassmannian varieties to coding theory, *Congr. Numer.* 57, 257–271 (1987).
[49] C.T. Ryan, Projective codes based on Grassmann varieties, *Congr. Numer.* 57, 273–279 (1987).
[50] C.T. Ryan and K.M. Ryan, The minimum weight of Grassmannian codes C(k, n), *Discrete Appl. Math.* 28, 149–156 (1990).
[51] J.P. Serre, *Lettre à Tsfasman*, in: *Journées Arithmetiques, 1989 (Luminy, 1989)*, Asterisque 198-200, 351–353 (1991).
[52] A. Sørensen, Rational points on hypersurfaces, Reed-Muller codes, and algebraic-geometric codes, Ph.D. thesis, Aarhus, 1991.
[53] A. Sørensen, Projective Reed-Muller codes, *IEEE Trans. Inform. Theory* 37, 1567–1576 (1991).
[54] M.A. Tsfasman and S.G.Vladut, *Algebraic-geometric codes* (Kluwer, Dordrecht, 1991).
[55] J.Voloch and M. Zarzar, *Algebraic geometric codes on surfaces*, preprint.
[56] Z. Wan, The weight hierarchies of the projective codes from nondegenerate quadrics, *Des. Codes Cryptogr.* 4, 283–300 (1994).
[57] V.K. Wei, Generalized Hamming weights for linear codes, *IEEE Trans. Inform. Theory* 37, 1412–1418 (1991).
[58] A. Weil, Numbers of solutions of equations in finite fields, *Bull. Amer. Math. Soc.* 55, 497–508. (1949).
[59] J. Wolfmann, Codes projectifs a deux ou trois poids associés aux hyperquadriques d’une géométrie finie, *Discrete Math.* 13, 185–211 (1975).
[60] M. Zarzar, Error-correcting codes on low-rank surfaces, *Finite Fields Appl.* 13, 727–737 (2007).