A review of fatigue crack propagation modelling techniques using FEM and XFEM

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Abstract. Fatigue is one of the main causes of failures in mechanical and structural systems. Offshore installations, in particular, are susceptible to fatigue failure due to their exposure to the combination of wind loads, wave loads and currents. In order to assess the safety of the components of these installations, the expected lifetime of the component needs to be estimated. The fatigue life is the sum of the number of loading cycles required for a fatigue crack to initiate, and the number of cycles required for the crack to propagate before sudden fracture occurs. Since analytical determination of the fatigue crack propagation life in real geometries is rarely viable, crack propagation problems are normally solved using some computational method. In this review the use of the finite element method (FEM) and the extended finite element method (XFEM) to model fatigue crack propagation is discussed. The basic techniques are presented, together with some of the recent developments.

1. Introduction
Components which are subjected to fluctuating loads are found virtually everywhere: Vehicles and other machinery contain rotating axles and gears, pressure vessels and piping may be subjected to pressure fluctuations (e.g. water hammer) or repeated temperature changes, structural members in bridges are subjected to traffic loads and wind loads, while those in ships and offshore structures are subjected to the combination of wind loads, wave loads and currents. If the components are subjected to a fluctuating load of a certain magnitude for a sufficient amount of time, small cracks will nucleate in the material. Over time, the cracks will propagate, up to the point where the remaining cross-section of the component is not able to carry the load, at which the component will be subjected to sudden fracture [1]. This process is called fatigue, and is one of the main causes of failures in structural and mechanical components [2]. In order to assess the safety of the component, engineers need to estimate its expected lifetime. The fatigue life is the sum of the number of loading cycles required for a fatigue crack to nucleate/initiate, and the number of cycles required for the crack to propagate until its critical size has been reached [2, 3]. In this paper, computational methods to estimate the lifespan of a propagating crack whose initial geometry is known will be considered.

Estimations of the fatigue crack propagation rate, $da/dN$, are normally based on a relation with the range of the stress intensity factor, $\Delta K$, which is a linear elastic fracture mechanics (LEFM) parameter for quantifying the load and geometry of the crack. Paris, Gomez and Anderson [4] first proposed the existence of such a relation in 1961, and its simplest form is the Paris law [5]:

$$da/dN = C(\Delta K)^m$$
where $a$ is the length of an edge crack (or half the length of an internal crack), $N$ is the number of loading cycles, and $C$ and $m$ are scaling constants. Due to its simplicity, the Paris law is only applicable for intermediate values of $\Delta K$ (region II in Figure 1), under constant amplitude cyclic loading [6]. Furthermore, the constants $C$ and $m$ are influenced by the applied stress ratio $R = K_{\text{min}}/K_{\text{max}}$, and crack closure effects are not taken into account. Therefore, a number of different crack propagation laws have been proposed, each one taking different factors into account [7]. One of the most detailed and commonly used is the so-called NASGRO equation [8]:

\[
\frac{da}{dN} = C [\left(1 - f\right) \cdot \Delta K]\left[\frac{1 - \frac{\Delta K_{\text{th}}}{\Delta K}}{1 - \frac{K_{\text{max}}}{K_c}}\right]^m \Delta K^p
\]

where $\Delta K_{\text{th}}$ is the threshold value for fatigue crack propagation, $K_c$ is the critical stress intensity factor at which fracture occurs, $\Delta K = K_{\text{max}} - K_{\text{min}}$, $R = K_{\text{min}}/K_{\text{max}}$, $f$ is a crack opening function and $C$, $m$, $p$ and $q$ are empirical constants. Note that for a given material, $C$ and $m$ do not have the same numerical values in different crack propagation laws.

![Fracture Region](image)

**Figure 1.** Typical fatigue crack growth curve.

The loading and displacement of a crack can be described by the three modes of fracture, each with its own stress intensity factor; mode I (tensile opening, $K_I$), mode II (in-plane sliding, $K_{II}$) and mode III (tearing/out-of-plane shear, $K_{III}$) [6]. The different modes require different values for the constants in the crack propagation law. In the case of mixed-mode fatigue, it may be necessary to use an effective mixed-mode stress intensity factor [9], for instance as given in [10], or a modified crack propagation law [11].

The crack propagation life can be estimated by integrating equation (1) or (2). However, the stress intensity factors $K_{\text{max}}$ and $K_{\text{min}}$ are normally functions of the crack length $a$, and depend on the geometry of the structure. Analytical integration of equations (1) and (2) is therefore rarely viable for complicated geometries. Instead, crack propagation problems are normally solved using some computational method, e.g. the Finite Element Method (FEM) [12]. The crack propagation process is then solved in a step-wise manner. For each step, the crack is advanced a small length, and the number of cycles required for the next crack increment is estimated using one of the crack propagation laws. In
order to accomplish this, the computational method needs to perform the following tasks within each step [12, 13]:

1. Computation of the minimum and maximum stress and displacement fields within the cracked component.
2. Evaluation of the minimum and maximum stress intensity factors for the crack.
3. Evaluation of the direction for further crack propagation.
4. Generation of a representation of the crack advancement.

This process is repeated until the critical stress intensity factor is reached; $K_{\text{max}} = K_c$, and the number of experienced cycles are summed to obtain the crack propagation life.

The main challenge when using the finite element method to estimate the fatigue crack propagation, lies in the fourth task in the list above [12]. In order to evaluate stress intensity factors for the advanced crack, the finite element mesh needs to be updated, a process which has been shown to be challenging [6]. This has led to the development of alternative computational methods to handle propagating cracks, among them the eXtended Finite Element Method (XFEM) [14, 15], the Boundary Element Method (BEM) [16], hybrids between finite and boundary element methods, e.g. the Symmetric Galerkin Boundary Element Method – Finite Element Method (SGBEM-FEM) [17] and the Scaled Boundary Finite Element Method (SBFEM) [18], and meshless methods [19].

In this review, the use of the finite element method (FEM) and the extended finite element method (XFEM) to model fatigue crack propagation will be discussed. The basic techniques will be presented, together with some of the recent developments. The review will focus on the modelling techniques, and only to a less extent on their applications.

2. Crack propagation by the finite element method

2.1. Computation of stress and displacement fields, and the stress intensity factor

The first issue when estimating fatigue crack propagation rates using the finite element method is the computation of sufficiently accurate values for the stress intensity factor for the crack at the maximum and minimum applied loads during each cycle. In order to compute the stress intensity factor, the stress and displacement fields of the whole component are also needed. Three simple methods for computing the stress intensity factors of mode I cracks from a finite element stress field were presented by Chan, et al. [20] already in 1970. These methods were called the stress method, the displacement method and the line integral method, and were all computed using ordinary linear (constant-strain) triangle elements, with a high degree of refinement at the crack tip.

2.1.1. The stress method. In the stress method, nodal stress values are extrapolated to the crack tip. It is often easiest to do this extrapolation along the crack plane, in which case the stress intensity factor is related to the stress normal to the crack plane, $\sigma_{yy}$, by [6]:

$$K_I = \lim_{r \to 0} \sigma_{yy} \sqrt{2\pi r}$$  \hspace{1cm} (3)

where $r$ is the distance from the crack tip. The nodal values of $K_I$ are plotted as a function of $r$, and extrapolated to $r = 0$. As ordinary finite elements are not able to represent the stress singularity at the crack tip, the nodal values of $K_I$ closest to the tip should be omitted when performing the extrapolation [20]. The singularity of the stress field also causes this method to be one of the least accurate methods available [6]. A similar approach is also possible for mode II and mode III cracks.

2.1.2. The displacement method. The second method is called the displacement method, and involves a relation between the stress intensity factor and the crack-opening displacement $u_y$ [20]. For mode I loading, this relation is given as [6, 20]:
\[ K_I = \lim_{r \to 0} \left( \frac{E' u_y}{4} \sqrt{\frac{2\pi}{r}} \right) \]  

where:

\[ E' = \begin{cases} E, & \text{plane stress} \\ \frac{E}{1 - \nu^2}, & \text{plane strain} \end{cases} \]  

\( E \) is the elastic modulus, while \( \nu \) is Poisson's ratio. As the opening displacement \( u_y \) is equal to zero at the crack tip, the nodal values of \( u_y \) near the crack tip must be disregarded when performing the extrapolation of \( K_I \) to \( r = 0 \). The displacement method does generally give more accurate estimations than the stress method [6].

2.1.3. Finite elements at the crack tip. In order to reduce the mesh quality required for the displacement method, special crack tip elements were developed in the 1970’s, which were able to describe the singularity which exists in the near-crack stress field. One of them was the isoparametric bilinear rectangle (quadrilateral) 4-node element with special shape functions, proposed by Tracey [21] in 1971, which should be used at the crack tip. Four years later, Henshell and Shaw [22] declared in the title of their paper that “Crack tip finite elements are unnecessary”. They could show that the crack tip singularity could be represented by 8-node isoparametric quadratic rectangle elements, if the mid-side nodes closest to the crack tip were moved to the “quarter points,” i.e. \( \frac{1}{4} \) of the element length away from the crack tip. Similar 2D elements, as well as three-dimensional elements, were independently developed by Barsoum, and presented in the same journal, two issues later [23].

When used at the crack tip, these so-called "quarter-point elements" [24] have a reasonably accurate stiffness, but the local values of stress and displacement within the quarter-point elements are poor [22]. The nodal displacements for these elements should therefore be omitted when calculating \( K \). Recommendations for generating a suitable mesh with quarter-point elements for evaluating \( K_I \) by the displacement method have been given by Menandro, et al. [25] and Guinea, et al. [26], and a typical example of the crack tip mesh is shown in Figure 2. Note that the quarter-point (8-node quadratic rectangle) elements are collapsed down to triangles, where each element has three nodes located at the crack tip.

**Figure 2.** (a) Typical rosette pattern for the FEM mesh at a LEFM crack tip, and (b) detail of the two inner rings, with nodes shown. Note the use of quarter-tip elements only in the innermost ring.

2.1.4. Energy release rate methods. The potential energy decrease per unit crack advance is called the energy release rate, \( G \), and may be used to characterise crack growth in a linear or nonlinear elastic body [27-29]. Rice [27] showed that the energy release rate could be computed by a path independent line integral, which for a two-dimensional problem is defined by:

\[ J = \int W \, dy - T \cdot \frac{\partial u}{\partial x} \, dx \]  

(a)  

(b)
where $\Gamma$ is a curve surrounding the crack tip, $W(x, y)$ is the strain-energy density field, and $x, y$ are the Cartesian coordinates being parallel and normal to the crack tip, respectively. $T$ is the traction vector associated to the outward normal to $\Gamma$, $u$ is the displacement vector, and $ds$ is an incremental arc length along $\Gamma$.

The third method to compute the stress intensity factors mentioned by Chan, et al. [20] is the line integral method. This method consists of numerically evaluating the $J$ integral, equation (6), for the finite element solution over an arbitrary path surrounding the crack tip. In the case of small-scale yielding, the energy release rate may be related to the mode I stress intensity factor by the following equation [20, 27]:

$$K_I = \sqrt{GE' = \sqrt{JE''}}$$

(7)

The $J$ integral may be evaluated at a remote contour, like the outer boundary of the geometry [20], which improves its numerical accuracy, compared with the stress method and the displacement method. The application of this method to three-dimensional problems is more difficult, however, because the integral becomes a surface integral. It is difficult both to define the surface and to perform the numerical integration [29]. This led to the development of alternative methods to evaluate the energy release rate, e.g. the virtual crack extension methods developed by Parks [28], Hellen [30] and deLorenzi [29]. The most accurate and efficient [6] method for the numerical evaluation of the energy release rate seems to be the domain integral method, formulated by Shih, et al. in 1986 [31]. They compute the $J$ integral by using the divergence theorem, in which the three-dimensional surface integral is transformed into a volume domain integral, which is evaluated using Gaussian quadrature.

For mixed mode I+II loading with small-scale yielding, the following relationship exists between the energy release rate and the stress intensity factors [14, 27]:

$$J = G = \frac{1}{E'} \left( K_{I}^2 + K_{II}^2 \right)$$

(8)

In order to extract the stress intensity factors from the $J$ integral, Yau, et al. [32] developed a technique using an interaction integral. The combination of the interaction integral with the domain integral method is shown in [14, 33].

The main advantage of the energy release rate methods is that accurate estimations for the stress intensity factor may be obtained even with a relatively coarse mesh [26]. A finer mesh, with quarter-point elements at the crack tip, is required if determination of the stress field is part of the objective. The domain integral is often recommended for practical use. Some find it easier to implement in certain FE codes (because quarter-point elements are not needed) [34], and others have found it to be more stable than the displacement method [24]. On the other hand, if quarter-point elements may easily be used in a FE code, the displacement method does not need any specialised post-processing routine to obtain the stress intensity factor. Both the displacement method and the domain integral method are able to accurately predict the stress intensity factor [24]. The displacement method is still widely used [9, 26, 35], and Guinea, et al. [26] question whether the domain integral actually is the most efficient.

2.2. Evaluation of the direction for further crack growth

If a crack is subjected to a mixed-mode loading, the propagating crack seeks the path of least resistance. Several theories have been proposed to choose this path. The three most used [35] are the criteria of maximum tangential (circumferential) stress [36], maximum energy release rate [37] and minimum strain energy density [38]. Other criteria include the criterion of maximum dilatational strain energy density [39] and the criterion of minimum accumulated strain energy [11]. There is no general agreement about which criterion should be used for a given material, but Bittencourt, et al. [24] have shown that the three former criteria predict basically the same crack growth trajectory for poly(methyl methacrylate) (PMMA), which is a brittle material. The criterion of maximum tangential stress is often applied in FEM simulations of fatigue crack growth, because it is simple to implement, as it has an approximate explicit solution for the crack growth direction $\theta$ as a function of $K_I$ and $K_{II}$ [9, 14, 35].
To implement the evaluation of the direction for further crack propagation in FE codes is generally not a problem [12].

2.3. Representation of the crack advancement
When the stress intensity range and the crack growth direction have been found, the number of cycles required for the crack to propagate a distance Δa may be estimated by the crack propagation law. As the crack propagation law is not linear with respect to Δa, the stress intensity factor range ΔK needs to be re-evaluated after the increment Δa. Reducing this increment will increase the accuracy of the solution, at the cost of increased computational effort.

In order to re-evaluate ΔK, the crack increment needs to be represented by the finite element mesh. As noted in section 2.1, some mesh refinement at the crack tip is normally needed. When the crack tip moves due to the crack increment, the focused region of the mesh should follow. The most common technique when modelling propagating fatigue cracks under LEFM conditions is to perform global or local re-meshing [9, 24, 35, 40]. Local re-meshing is generally preferred, due to the lower computational effort, compared to global re-meshing.

Local re-meshing was employed already in 1986 by Højfeldt and Østervig [40] to predict the shape and crack propagation life of fatigue cracks in shafts with a diameter transition. They employed the displacement method and quarter-point elements to estimate the stress intensity factors of the three-dimensional crack, and the criterion of minimum strain energy density to estimate the crack growth direction. Bittencourt, et al. [24] and Miranda, et al. [35] used the local re-meshing technique, together with quarter-point elements and the displacement technique, in their methodology for assessment of fatigue crack propagation in two-dimensional components. The re-meshing technique was also used by Alegre and Cuesta [9] to model mode I+II crack propagation in a valve.

The local re-meshing technique generally consists of four steps [24], as illustrated in Figure 3; (a) removing the existing mesh around the crack, (b) advancing the length of the crack, (c) applying quarter-point elements in a uniform rosette pattern at the crack tip, and (d) generating a new mesh in the open area.

Figure 3. Local re-meshing technique. (a) Existing mesh. (b) The mesh around the crack tip is removed, and the crack is advanced. (c) Rosette of elements are applied at the crack tip. (d) The remaining area is meshed.

Recent research in crack propagation modelling by FEM often deals with plasticity induced crack closure, e.g. [41-43], in which the material plasticity is explicitly taken into account in the finite element analysis. Re-meshing is then highly cumbersome, with respect to both computational and programming effort, because the plastic strain history needs to be mapped from the old mesh to the new one [6]. Furthermore, accuracy is lost during this mapping process. As an alternative, it is possible to create a single mesh which accommodates crack growth, by being refined along the assumed crack propagation path. The disadvantage of this approach is that the crack path and shape are predetermined by the mesh [6, 12]. In addition, the crack increment Δa has to correspond to the
element size, making the crack growth response mesh dependent. By employing micrometre-size elements, this approach is nevertheless widely used [42-43].

In order to advance the crack through the refined mesh, the following techniques may be used [6, 44]: (a) Removing elements along the crack front once a failure criterion is reached, (b) releasing the nodes at the crack tip at specific load steps or according to a failure criterion, so that these elements are no longer connected, or (c) using cohesive elements. The cohesive elements are zero-thickness elements which are placed in between the ordinary elements, and for which a certain force-displacement law is specified. When computing fatigue crack propagation with plasticity induced crack closure, the node release technique is the most common to use [45], but cohesive elements are also used by some researchers [42, 43, 46]. It should be noted that the plastic crack tip field does not contain the \( \frac{1}{\sqrt{r}} \) singularity [6], making quarter point elements unnecessary when studying plasticity induced crack closure.

Colombo and Giglio [12] recently developed a crack advancement technique for LEFM conditions which is a combination of the local re-meshing and node release techniques. Their technique may be described by four steps: (a) the crack increment is projected on the FE mesh, (b) the elements touched by the crack increment are deleted, (c) the open area is re-meshed by debondable triangular elements which conform to the crack increment, and (d) the nodes along the crack increment are released. This technique limits the re-meshing to the absolute minimum, while still keeping the crack growth mesh-independent. The resulting mesh is inadequate for evaluating the stress intensity factors using the displacement technique, however. Therefore, the submodelling technique [47, 48] is used, in which the displacements from this mesh are imposed on an independent focused mesh which represents the crack tip using quarter-point elements (as shown in Figure 2(a)). The displacement field of this focused mesh is then used to estimate the stress intensity factor. The submodelling technique has also been used by Schöllmann, et al. [49] in their fatigue crack growth simulation software.

2.4. Current trends

In recent years the research focus seems to have shifted from simple geometries modelled under LEFM conditions towards three-dimensional cracks in complex geometries or with plasticity induced crack closure. A review of the literature on finite element analysis of three-dimensional fatigue cracks is found in [50], while an overview of the finite element analysis of plasticity induced crack closure is found in [45]. We will here limit our discussion to three of the latest works.

Aguilar Espinosa, et al. [41] have used four-node quadrilateral elements with the node release technique in Abaqus, in order to estimate the crack opening and closing stress intensity factors for a crack in a four-point bending specimen subjected to fatigue loading. The contact of the crack flanks was simulated by using Abaqus’ surface contact boundary condition. However, even with a mesh refined to 4 µm elements at the crack tip, the estimated crack opening and closing loads are significantly lower than the experimentally determined values.

Solanki, et al. [45] stated that the physics of fatigue crack growth is not taken into account when the node release technique is used. García-Collado, et al. [42] illustrate some of the differences between the results obtained by using a physics-based cohesive element technique, and those obtained by using the node release technique, like differences in the plastic strain field around the crack tip. They used an element size of 15 µm at the crack tip, with 13 213 nodes in total to model a compact-tension specimen. In order to reduce the mesh refinement required at the crack tip, Hu, et al. [43] propose a special singular element to be used at the crack tip when considering plasticity induced crack closure in two-dimensional geometries. This single element covers the whole crack tip, and its radius should be equal to the plastic zone length. The element is based on a cohesive zone model, and is used to simulate variable amplitude propagation of a mode I fatigue crack.

3. Crack propagation by the extended finite element method

As indicated in section 2.3, the methods for evaluating fatigue crack growth by the ordinary finite element method do normally contain one of two shortcomings: The methods are either mesh-
dependent, or requires re-meshing. In order to avoid both these shortcomings, Belytschko and Black published a mesh-independent method with minimal re-meshing in 1999 [14]. This method was further developed by Moës, Dolbow and Belytschko [51] into a mesh-independent method without any re-meshing. The method has later become known as the eXtended Finite Element Method (XFEM) [15, 44], and has become widely popular [52] for solving continuum mechanics problems containing discontinuities like cracks and material interfaces. In addition to crack propagation problems, XFEM has been employed to solve two-phase flows and fluid-structure interaction problems [15], and has even been proposed as an applicable tool to predict the deformation of a brain subjected to a surgeon's cut [53].

As illustrated in Figure 4, XFEM uses a non-conforming mesh to model the crack (or other discontinuities), i.e. the cracks are modelled independently of the mesh [44]. This is made possible by "enriching" the elements cut by the crack, by adding special shape functions to take care of the local discontinuities and singularities around the crack [15]. The mesh-independency makes it possible to use the same mesh for all stages of a growing fatigue crack. Even though the modelling of the crack is said to be independent of the mesh, Ren and Guan [54] clearly illustrate that the level of mesh refinement at the crack influences the accuracy of the representation of a three-dimensional crack. The mesh-independency does rather imply that the crack growth increment and orientation may be chosen independently of the mesh.

Figure 4. Illustrative sketches of (a) a conforming FEM mesh and (b) a non-conforming XFEM mesh. ▲ indicates tip enriched nodes, while ● indicates step enriched nodes. Element sizes not to scale.

We will here briefly review XFEM for the purpose of modelling fatigue cracks. An extensive review of XFEM in general is found in [15], and some of its numerous applications are reviewed in [55].

3.1. Representation of the crack

As the extended finite element method mesh does not normally conform to the crack, it is necessary to formulate the position and shape of the crack face and the position of the crack tips mathematically. This is normally (but not necessarily) done using the level set method [15, 44, 56, 57]. An arbitrary crack may be described by two level set curves. The first level set curve, $\phi(x, t)$, is the signed-distance function, which is equal to zero along the whole crack surface, as well as at its tangential extensions at its ends, as shown in Figure 5(a) [15, 56]:

$$
\phi(x, t) = \pm \min_{x_{\epsilon(t)}} \| x - x_{\epsilon(t)} \|, \quad \forall x \in \Omega
$$

where the sign is positive and negative on opposite sides of the crack. $x_{\epsilon}$ is the closest point to $x$ on the crack face $\Gamma(t)$. $\Omega$ represents the domain of the solid.

While $\phi(x, t)$ represents the position and geometry of the crack face, it does not specify where the crack ends. This is accomplished by a second level set curve, $\gamma(x, t)$, which is orthogonal to $\Gamma(t)$ and equal to zero at the crack front [15], as shown in Figure 5(b). The crack front corresponds to the crack tip for a two-dimensional crack, or to the perimeter of a three-dimensional crack [57].
Figure 5. Level set curves (a) $\phi(x,t)$ and (b) $\gamma(x,t)$ for a 2D crack.

Given these two level sets, the geometry and position of the crack is given by [15, 56]:

$$\Gamma(t) = \{ x : \phi(x,t) = 0 \land \gamma(x,t) \leq 0 \}$$  \hspace{1cm} (10)

As the crack grows, $\Gamma(t)$ is updated, based on the crack increment length and direction. One of the possible ways to update the crack representation is the fast marching method used by Sukumar, et al. [57]. Alternative methods are mentioned in [15].

The values of the level sets are normally saved at the element nodes, and interpolated within the finite elements using standard shape functions [15]. It should be noted that this interpolation makes the accuracy of the crack representation dependable on the element size and the number of nodes per element, as illustrated by Ren and Guan [54].

3.2. The stress and displacement fields

The fundamental requirement of any finite element method is its ability to represent the stress and displacement fields of the loaded solid. With a crack being defined by the level set method, the extended finite element method needs to be able to compute these fields with sufficient accuracy. In order to take the discontinuity and the singularity around the crack into account, the XFEM introduces an enrichment to the finite elements which are cut by the crack [15, 44]. More precisely, "a node is enriched if its support is cut by the crack" [51]. The nodes are enriched by the introduction of an additional set of degrees of freedom, $q$. The nodes do still contain their traditional degrees of freedom, $u$, meaning that the enriched nodes have a higher number of degrees of freedom than the ordinary nodes. The displacement field within the elements is then approximated by the following expression:

$$u^e(x) = \sum_{i=1}^{I} N_i(x) u_i + \sum_{i \in I^e} N_i^e(x) \left[ \psi(x) - \psi(x_i) \right] q_i,$$ \hspace{1cm} (11)

which is the standard XFEM approximation [15]. The enriched nodes $I^e$ are a subset of all the nodes $I$, $N_i(x)$ are the standard FEM shape functions and $\psi(x)$ is called the enrichment function. $N_i^e(x)$ are some functions which have the partition of unity property [15]:

$$\sum_{i \in I^e} N_i^e(x) = 1$$  \hspace{1cm} (12)
As the standard FEM shape functions $N_i(x)$ do have the partition of unity property [44], $N_i^*(x)$ are usually chosen to be equal to the $N_i(x)$. The term $-\psi_i(x)$ is called shifting, and is used to remove the effects of $q_i$ on the nodes and ensure compatibility across elements, i.e. to ensure that $u_h^j(x_i) = u_i$.

The enrichment function is responsible for introducing discontinuities and singularities to the displacement field. Belytschko and Black [14] proposed the enrichment functions that describe the singular displacement field around the crack tip under LEFM conditions in 1999:

$$\psi_{tip}^j(x) = \sqrt{r} \left\{ \cos \frac{\theta}{2}, \sin \frac{\theta}{2}, \sin \sin \frac{\theta}{2}, \cos \sin \frac{\theta}{2} \right\} \tag{13}$$

Here, $r$ and $\theta$ are the polar coordinates with the origin at the crack tip, and $\theta = 0$ along the parallel extension of the crack into the material.

The above enrichment is based on linear elastic fracture mechanics (LEFM), and is therefore only representative for cases where the crack tip plastic zone is considered to be small [44]. For the case of significant plasticity, an alternative set of crack tip enrichment functions was developed by Elguedj, et al. [58] in 2006. They still assume that the plasticity is confined to a region near the crack tip, and apply the Hutchinson-Rice-Rosengren (HRR) solution for a power-law hardening material, proposing the following crack tip enrichment functions:

$$\psi_{tip}^j(x) = r^{n+1} \left\{ \sin \frac{\theta}{2}, \cos \frac{\theta}{2}, \sin \sin \frac{\theta}{2}, \cos \sin \frac{\theta}{2}, \sin \sin 3\frac{\theta}{2}, \cos \sin 3\frac{\theta}{2} \right\} \tag{14}$$

$n$ is here the hardening exponent of the material.

Even though the enrichment functions, equations (13) & (14), are discontinuous along $\theta = \pm \pi$ [14], i.e. throughout the crack, they are not readily applicable to describe long, severely curved or three-dimensional cracks [14, 51]. Hence, Belytschko and Black stated their method to require "minimal re-meshing." Moës, et al. [51] removed the need of re-meshing altogether, by introducing an additional enrichment, to be used for the enriched nodes along the crack, away from the crack tip. This enrichment models the discontinuous displacement field over the crack by using the sign function [15, 51]:

$$\psi_{step} = \text{sign}(\phi(x,t)) = \begin{cases} -1 & \text{if } \phi(x,t) < 0, \\ 0 & \text{if } \phi(x,t) = 0, \\ 1 & \text{if } \phi(x,t) > 0. \end{cases} \tag{15}$$

This enrichment function is equal to $-1$ and $+1$ on opposite sides of the crack, which is normally represented by the level set $\phi(x,t)$. Moës, et al. [51] used the symbol $H(x)$ for this enrichment function, and subsequent authors have often referred to it as the Heaviside function, e.g. [53, 55, 57, 59], even though the Heaviside step function formally has the property $H(\phi(x)) = 0$ if $\phi(x) \leq 0$ [15]. However, the sign and the Heaviside step functions do actually lead to identical results because they span the same approximation space [15].

By using the crack tip and step enrichments, an appropriate approximation for the stress and displacement fields may be obtained, both for two-dimensional [51, 60] and three-dimensional [57, 59] cracks. If we divide the additional degrees of freedom, $q_i$, into those around the crack tip, $b_i$ (indicated by $\bigtriangleup$ in Figure 4(b)), and those remaining, $a_i$ (indicated by $\bullet$ in Figure 4(b)), we can rewrite equation (11) [15]:

$$u_h^j(x) = \sum_{i \in I} N_i(x) u_i + \sum_{i \in I_{tip}} N_i^*(x) \left[ \psi_{step}^j(x) - \psi_{step}^j(x_i) \right] a_i + \sum_{i \in I_{tip}} N_i^*(x) \left[ \psi_{tip}^j(x) - \psi_{tip}^j(x_i) \right] b_i \tag{16}$$

It should be noted that integration of the weak form of the finite element formulation is not straightforward when the elements contain singularities and discontinuities, because the standard Gauss
quadrature requires a smooth integrand and a finite order polynomial \cite{44}. Special integration techniques have therefore been developed for this purpose \cite{15, 44, 51, 58}.

3.3. Evaluation of the stress intensity factor and the direction for further crack growth
The stress intensity factor is commonly extracted from the extended finite element solution by employing the domain form of the interaction integral \cite{14, 51, 54, 57, 58}, as explained in section 2.1.4. It is likely that this method is preferred over the displacement method, because the crack tip displacements are less readily available in the XFEM results than in corresponding FEM results. Moës, et al. \cite{51} used a proper mesh and demonstrated computation of the stress intensity factors for two-dimensional mode I+II cracks within 1–2 % deviation from the analytical values. The XFEM enrichment takes care of the near-crack singularity, and linear elements can therefore be used even for the LEFM analyses.

The direction for further crack growth is often evaluated using the criterion of maximum tangential stress \cite{36}, e.g. in \cite{14, 51, 54, 60}, just like in fatigue analyses using the ordinary FEM. The angle of propagation is estimated using a function of $K_I$ and $K_{II}$, and the level sets describing the crack are updated accordingly. The relation between the angle of propagation and the level set updates are illustrated in \cite{44}.

3.4. Fatigue crack propagation and current trends
The first application of the extended finite element method to fatigue crack growth propagation was illustrated already by Moës, et al. \cite{51} in 1999. They modelled the propagation of mode I+II fatigue cracks propagating from two holes in a two-dimensional plate with 2650 nodes. Two different values for the crack increment size are used, but the numerical results are not compared to any experimental ones.

Sukumar, et al. \cite{57} presented the first XFEM simulation of planar fatigue cracks propagating through a three-dimensional solid (i.e. pure mode I cracks) in 2003. They use the level set method to track the position of the crack front, and the fast marching method to advance the crack front. Using three-dimensional meshes of 24x24x24 eight-node (linear) hexahedral elements, they compute the stress intensity factors for a planar penny-shaped crack with errors between 0.4 % and 2.9 %, while for a planar elliptical crack the errors are between 0.6 % and 3.7 %. The fatigue simulations correctly predicted an initially elliptical crack to develop into a penny-shaped crack, but the crack propagation life is not compared to any experimental results.

Comparisons of XFEM fatigue simulations to experimental results have recently been published by Bergara, et al. \cite{59}. In this work, they have used the XFEM-based LEFM approach which is implemented in Abaqus to simulate the growth of a semi-elliptical crack located at the side of a beam specimen subjected to four-point bending. The specimen is completely modelled in three dimensions by 8-node hexahedral elements, with refinements around the crack and the boundary condition sites. With 173 892 nodes in total, approximately one week was required to run one fatigue crack propagation simulation using three 3.40 GHz processors. Bergara, et al. report excellent correspondence between the numerical and experimental crack propagation histories, as well as good agreement for the evolution of the crack geometry. However, there are large differences between the experimental and numerical stress intensity factor ranges; up to approximately 40 %. This seems remarkable given the fine mesh and good correspondence in the crack propagation histories.

Zhan, et al. \cite{3} have proposed an entire framework for the fatigue life prediction of metallic components, where the crack initiation life is evaluated using FEM with continuum damage mechanics, while the crack propagation life is evaluated using XFEM. The initiated crack is set to be 0.1 mm long. Computational and experimental results have been compared for a mode I crack in a fuselage structure, modelled by 8-node hexahedral elements, using 20 106 nodes for half the model. The predicted crack propagation life was 28 % shorter than the experimental one, indicating a potential for improving the technique.
A new method for reducing the required structural mesh density near the crack has recently been proposed by Ren and Guan [54]. Their method is intended for the crack growth analysis of three-dimensional and arbitrarily shaped cracks subjected to mixed mode loading. They note that the level set method is not able to fully represent the crack front surface if the crack has a complicated geometry or if the mesh is relatively coarse. It is therefore proposed to use an individual mesh for the crack, and let the crack geometry in the XFEM model of the structure be described by this mesh, instead of the level set method. It is shown that this method makes the values for the stress intensity factor range more uniform along the crack front, and they are also less dependent on the structural mesh. However, the crack mesh is shown to be very fine in the illustrations, and necessarily needs to be updated (re-meshed) as the crack propagates. It has not been shown whether the computation time saved by using a coarser overall mesh makes up for the time required to re-mesh the crack for each propagation increment. Still, it is possible that this technique is more efficient for analysing fatigue crack propagation in large and complex components.

4. Discussion

The studies reviewed in section 3.4 indicate that even though XFEM is capable of accurately predicting stress intensity factors for simple “handbook” geometries, further development is required in order to make accurate predictions of stress intensity factors and fatigue crack propagation lives for more complex geometries. It should be noted that the accuracy of the predicted crack propagation life will always be limited by the fracture mechanics assumptions and the crack propagation law which is used, however. On the other hand, even though the XFEM was developed to reduce the computational effort required to simulate fatigue crack propagation, none of the studies reviewed do actually compare computation times between FEM and XFEM. This is remarkable.

In 1986 Højfeldt and Østervig [40] were able to predict the fatigue crack propagation life of shafts with shoulder fillets, with errors below 20% in just 20 400 s (CPU-time). These results were obtained using the ordinary FEM with a three-dimensional mesh and the local re-meshing technique. By exploiting symmetry, only half the shaft was modelled, using only 1160 nodes. One would expect that after 30 years of research, either the accuracy of the computational results would increase, or the computation time would decrease, especially with the introduction of XFEM. The works considered in this review are inconclusive in this regard, mainly because computation times (with processor frequencies) are rarely stated. In order to choose between the various computational techniques for modelling fatigue crack propagation, and their variants, such information is essential.

Yazid, et al. state that “about the only drawback” of the present XFEM “is the need for a variable number of degrees of freedom per node” [55]. This is mainly a challenge when XFEM is incorporated into existing FEM codes [15]. A more important drawback, as indicated above, for both FEM and XFEM, seems to be the large number of nodes and long computation time required. FEM and XFEM are challenged by methods requiring significantly fewer nodes, like the boundary element method (BEM) [16] and the SGBEM-FEM by Nikishkov, Park and Atluri [17]. Recent comparisons by Dong and Atluri [52, 61] indicate that the SGBEM-FEM is both more accurate and more efficient than XFEM to analyse propagating fatigue cracks under LEFM conditions.

5. Conclusions

Both the ordinary Finite Element Method (FEM) and the eXtended Finite Element Method (XFEM) are able to provide good predictions for the fatigue crack growth propagation in simple geometries. While the FEM mesh conforms to the fatigue crack, the crack is only implicitly modelled in XFEM, independently of the mesh.

In FEM, the domain integral method is often recommended for evaluating the stress intensity factors, while the displacement method is still often used in practice. The domain integral method is commonly used also in XFEM. The criterion of maximum tangential stress is chosen for determining the crack propagation path in the majority of the analyses. Under linear elastic fracture mechanics (LEFM) conditions, local re-meshing is used to advance the crack in FEM, whereas either the node
release technique or cohesive elements are used under elastic-plastic fracture mechanics conditions. In XFEM, it is common to use the level set method to describe the crack geometry.

As the complexity of the problem increases, the accuracy of both methods decreases. The latest research therefore focuses on applying and improving these techniques to complex problems, like three-dimensional crack propagation under mixed-mode loading. In the current literature, there is also an increasing focus on modelling plasticity induced closure and fatigue propagation under variable amplitude loading using FEM.

For both methods, there is also a focus on improving their efficiency, by introducing new techniques. Even though the XFEM was developed to reduce the computational effort required for crack propagation problems, none of the studies reviewed do actually compare computation times between FEM and XFEM. This is remarkable, and should be considered for later studies.

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