On the global version of Euler-Lagrange equations.

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Abstract. The introduction of a covariant derivative on the velocity phase space is needed for a global expression of Euler-Lagrange equations. The aim of this paper is to show how its torsion tensor turns out to be involved in such a version.

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The introduction of numbers as coordinates... is an act of violence

Hermann Weyl

An increasing attention has been recently paid to coordinate free formulations of motion equations in Classical Mechanics (see for instance [1, 2] and references therein). In this work we write down intrinsic Euler-Lagrange equations and show the appearance of a torsion term. Furthermore, we shall see that this term should also be present in the horizontal Lagrange-Poincaré equations considered in [1, 2], if the torsion of the chosen derivative does not vanish.

It is worth noticing that covariant derivatives with non vanishing torsion naturally arise in several branches of Physics; namely, dynamics with nonholonomic constraints [3, 4], E. Cartan’s Theory of Gravity (see for instance [5]) and modern string theories (see for example [6]), among others.

Let us consider a physical system with configuration manifold $Q$ and Lagrangian $L(q, \dot{q}): TQ \rightarrow \mathbb{R}$ (for this geometrical setting see for instance [7]).

If a coordinate free characterization of the Euler-Lagrange Equations associated to the system is required a covariant derivative $D$ must be introduced on $TQ$, for $\frac{\partial L}{\partial \dot{q}}$ is involved (see for instance [8]). Once such $D$ is chosen, $\frac{DL}{Dq}$ is defined in the standard way

$$\frac{DL}{Dq}(q_0, \dot{q}_0) = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} L \circ \gamma(\lambda),$$

(1)
On the global version of Euler-Lagrange equations.

with \( \gamma(\lambda) = (q(\lambda), \dot{q}_0(\lambda)) \), \( q(0) = q_0 \), \( \dot{q}(0) = \dot{q}_0 \) and \( \dot{q}_0(\lambda) \) the parallel transport of \( \dot{q}_0 \) along \( q(\lambda) \).

Moreover, an associated covariant derivative on \( T^*Q \), that we will also denote by \( D \), is naturally defined through Leibnitz rule: for any curves \( \alpha(t) \) and \( v(t) \) in \( T^*Q \) and \( TQ \) respectively

\[
\frac{d}{dt} (\alpha(t), v(t)) = \left( \frac{D\alpha(t)}{dt}, v(t) \right) + (\alpha(t), \frac{Dv(t)}{dt}) ,
\]

where \( \langle , \rangle \) denotes the pairing between \( T^*Q \) and \( TQ \).

It is worth noticing that \( \frac{\partial}{\partial q} \) has a coordinate free sense: it is the derivative along the fibre.

**PROPOSITION 1.** Let \( D \) an arbitrary covariant derivative on \( TQ \). Then the coordinate free expression of the Euler-Lagrange equations is

\[
\frac{D}{Dt} \left( \frac{\partial L}{\partial q} \right) - \frac{DL}{Dq} = \frac{\partial L}{\partial q} T(q(t), v(t)) ,
\]

where \( T( , , ) \) is the torsion tensor of \( D \).

**Proof.** The curve \( q(t) \) is a solution of the Euler-Lagrange equations if and only if it is a critical point for the action

\[
S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt ,
\]

for variations of the curves such that \( q_0 \) and \( q_1 \) remain fixed. That is, for each \( q(t, \lambda) : [t_0, t_1] \times (-\varepsilon, \varepsilon) \to Q \) such that \( q(t, 0) = q(t), q(t_0, \lambda) = q(t_0) \) and \( q(t_1, \lambda) = q(t_1) \),

\[
\left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} \int_{t_0}^{t_1} L(q(t, \lambda), \dot{q}(t, \lambda)) dt = \int_{t_0}^{t_1} \delta L(q(t), \dot{q}(t)) dt = 0 ,
\]

where \( \delta L(q(t), \dot{q}(t)) = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} L(q(t, \lambda), \dot{q}(t, \lambda)) \).

But

\[
\delta L(q(t), \dot{q}(t)) = \lim_{\lambda \to 0} \frac{L(q(t, \lambda), \dot{q}(t, \lambda)) - L(q(t, 0), \dot{q}(0))}{\lambda}
\]

\[
= \lim_{\lambda \to 0} \frac{L(q(t, \lambda), \dot{q}(t, \lambda)) - L(q(t, \lambda), \dot{q}(t, \lambda))}{\lambda}
\]

\[
+ \lim_{\lambda \to 0} \frac{L(q(t, \lambda), \dot{q}_0(\lambda)) - L(q(t_0, 0), \dot{q}(0))}{\lambda} ,
\]

where \( \dot{q}_0(\lambda) \) is the parallel translated of the vector \( \dot{q}(t, 0) \) along the curve \( q(t, \lambda) \) fixed \( t \), see Figure 1.

Then

\[
\delta L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial q} D_{\delta q(t)} \dot{q}(t) + \frac{DL}{Dq} \delta q(t) ,
\]

where we have denoted

\[
\delta q(t) = \left. \frac{\partial q(t, \lambda)}{\partial \lambda} \right|_{\lambda=0} \quad \text{and} \quad \dot{q}(t) = \left. \frac{\partial q(t, \lambda)}{\partial t} \right|_{\lambda=0} .
\]
On the global version of Euler-Lagrange equations.

By definition of the torsion tensor $T(\ , \ )$, we have
\begin{equation}
T(\dot{q}(t), \delta q(t)) = D_{\dot{q}(t)} \delta q(t) - D_{\delta q(t)} \dot{q}(t) - [\dot{q}(t), \delta q(t)] .
\end{equation}

Thus, by using (2) and taking into account that $[\dot{q}(t), \delta q(t)]$ vanishes, we have
\begin{equation}
\delta L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial \dot{q}} D_{\dot{q}(t)} \delta q(t) + \frac{DL}{Dq} \dot{q}(t) - \frac{\partial L}{\partial q} T(\dot{q}(t), \delta q(t))
\end{equation}

Now integrating along the curve $q(t)$ we finally get
\begin{equation}
\frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \delta q(t) - \frac{DL}{Dq} \dot{q}(t) = \frac{\partial L}{\partial q} T(\dot{q}(t), \delta q(t)) . \quad \Box
\end{equation}

REMARK 1. Of course, regardless the covariant derivative $D$ we introduced, in any coordinate patch Euler-Lagrange equations always read
\begin{equation}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 .
\end{equation}

REMARK 2. As well known (e.g. [8]), a torsion free $D$ can always be chosen. It is obvious that for such a connection, global Euler-Lagrange equations read
\begin{equation}
\frac{D}{Dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{DL}{Dq} = 0 .
\end{equation}

So in this case, the global expression can be obtained merely by replacing the usual derivatives by $D$ in (12).

REMARK 3. A similar result holds for the horizontal Lagrange-Poincaré equations considered in [11, 12]. One of the goals of these references is to analyze the intrinsic meaning of motion equations for constrained systems with symmetries. Let us recall that, in such a system, the lagrangian $L$ and the constrains remain invariant under the lifting to $TQ$ of a suitable action of a Lie group $G$ on $Q$. A connection $A$, related to the constrains, is introduced on the principal bundle $Q \rightarrow Q/G$. If $\tilde{g}$ is the adjoint bundle to the principal bundle $Q$, the connection $A$ yields an isomorphism $\alpha$ between $TQ/G$ and the Whitney sum $T(Q/G) \oplus \tilde{g}$ in the following way
\begin{equation}
\alpha_A[q, \dot{q}] = (x, \dot{x}, \dot{\tilde{v}}) = \pi_*(q, \dot{q}) \oplus [q, A(q, \dot{q})] .
\end{equation}
On the global version of Euler-Lagrange equations.

Now, one can define the reduced Lagrangian $\ell : T(Q/G) \oplus \tilde{g} \to \mathbb{R}$ as

$$\ell(x, \dot{x}, \tilde{v}) = L(q, \dot{q}).$$

(15)

A variation $\delta q$ of a curve in $Q$ is said to be horizontal if $A(\delta q) = 0$. In this case, the corresponding variation $\alpha(\delta q(t))$ of the curve $\alpha(q(t))$ in $T(Q/G) \oplus \tilde{g}$ is

$$\alpha(\delta q(t)) = \delta x \oplus \tilde{B}(\delta x, \dot{x}),$$

(16)

where $\tilde{B}$ is the $\tilde{g}$-valued two-form on $Q/G$ defined by

$$\tilde{B}([q]_G)(X, Y) = [q, B(X^h(q), Y^h(q))]_G,$$

(17)

with $X^h, Y^h$ the horizontal lifts to $Q$ of $X$ and $Y$, and $B$ the curvature of the connection $A$.

The horizontal Lagrange-Poincaré equations for $L$ are defined as the Euler-Lagrange ones for $\ell$ restricted to horizontal variations $\delta q$. A coordinate free version of them can be written down by introducing an arbitrary covariant derivative $D$ on $T(Q/G)$ and using the covariant derivative $\tilde{D}$ induced by $A$ on $\tilde{g}$.

Under the implicit assumption that the torsion of $D$ vanishes, it is shown in [1, 2] that, for horizontal variations $\delta q$,

$$\delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt = 0$$

(18)

if and only if the following horizontal Lagrange-Poincaré equations hold

$$\frac{D}{Dt} \left( \frac{\partial \ell}{\partial \dot{x}} \right)(x, \dot{x}, \tilde{v}) - \frac{D\ell}{Dx}(x, \dot{x}, \tilde{v}) = -\left< \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, \cdot) \right>. $$

(19)

It is easy to see that, for an arbitrary covariant derivative $D$ on $T(Q/G)$, its torsion tensor $T$ must be taken into account in the previous equations. Arguing as above one gets

$$\frac{D}{Dt} \left( \frac{\partial \ell}{\partial \dot{x}} \right)(x, \dot{x}, \tilde{v}) - \frac{D\ell}{Dx}(x, \dot{x}, \tilde{v}) = -\left< \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, \cdot) \right> + T(\dot{q}, \cdot).$$

(20)

Assuming as in [1, 2] the torsion free requirement for $D$, the last term clearly vanishes and we recover the horizontal Lagrange-Poincaré equations found in those references.

Again, in any coordinate patch, the expression of horizontal Lagrange-Poincaré equations is independent of the choice of the covariant derivative $D$.

**Remark 4.** When considered as a map of the second order tangent bundle to the cotangent bundle, the Euler-Lagrange operator turns out to be intrinsic without any choice of connection (e.g., [3]). The need for connections appears if one wants to stay in the framework of tangent bundles, as it is usually done, and not to deal with second order ones.

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