Compact complex surfaces with geometric structures related to split quaternions

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Abstract

We study the problem of existence of geometric structures on compact complex surfaces that are related to split quaternions. These structures, called para-hypercomplex, para-hyperhermitian and para-hyperkähler, are analogs of the hypercomplex, hyperhermitian and hyperkähler structures in the definite case. We show that a compact 4-manifold carries a para-hyperkähler structure iff it has a metric of split signature together with two parallel, null, orthogonal, pointwise linearly independent vector fields. Every compact complex surface admitting a para-hyperhermitian structure has vanishing first Chern class and we show that, unlike the definite case, many of these surfaces carry infinite dimensional families of such structures. We provide also compact examples of complex surfaces with para-hyperhermitian structures which are not locally conformally para-hyperkähler. Finally, we discuss the problem of non-existence of para-hyperhermitian structures on Inoue surfaces of type $S^0$ and provide a list of compact complex surfaces which could carry para-hypercomplex structures.

1 Introduction

It was noticed long ago [34] that many integrable systems arise as reductions of self-dual Yang-Mills equations in signature $(2,2)$ and it is known that they allow approaches via Lax pairs and twistor theory (see [13] for a recent survey). The geometry of a superstring with $N=2$ supersymmetry was shown in [28], [29] to be described by a space-time with a pseudo-Kähler metric of signature $(2,2)$, whose curvature satisfies the (anti) self-duality equations. As noticed in [28] this space-time also admits a (local) holomorphic $(2,0)$-form, parallel with respect to the Levi-Civita connection. The structures obtained in this way define a holonomy reduction to the group $SU(1,1) \cong SL(2, \mathbb{R})$ and are an indefinite analog of hyperkähler structures which have holonomy $SU(2) \cong Sp(1)$. Mathematically, these structures are described by quadruples $(g, I, S, T)$ where $g$ is a signature
metric and $I, S, T$ are parallel endomorphisms of the tangent bundle such that:

$$I^2 = -S^2 = -1, \quad T = IS = -SI, \quad g(IX, IY) = -g(SX, SY) = g(X, Y) \quad (1)$$

In the literature such structures are called hypersymplectic [18], neutral hyperkähler [22], para-hyperkähler [8, 12], pseudo-hyperkähler [13], etc. They are not preserved by a conformal change of the metric and a natural conformally invariant generalization is to relax the condition for covariant constancy of $I, S, T$ to their integrability (see Section 2). Such structures are an indefinite analog of the hyperhermitian structures and are called para-hyperhermitian [12] or neutral hyperhermitian [22, 23]. In dimension 4, they are self-dual and, similarly to the positive definite case, always admit connections with skew-symmetric torsion and holonomy $\text{SL}(2, \mathbb{R})$ [21]. This geometry was considered in [17], where it was argued that the $(2,2)$ supersymmetric string based on chiral multiplets is a theory of self-dual gravity. If we forget the metric $g$ and consider only the triples $(I, S, T)$ of integrable endomorphisms of the tangent bundle satisfying the algebraic conditions in (1), the structures are called para-hypercomplex [8, 12], neutral hypercomplex [22, 23] or complex product [2, 3]. They provide examples of geometric structures with special holonomy of a non-metric connection.

Due to the non-elliptic nature of the self-duality equations in the split signature case, their solutions are more flexible. For example, a conformal self-dual or anti-self-dual structure is not necessary analytic unlike the definite case. As a consequence, most of the research deals with local properties of the structures. However, topological information like the Kodaira classification of compact complex surfaces allows one to study global properties. Important examples in this direction are the classifications of compact pseudo-Kähler Einstein and para-hyperkähler surfaces obtained by Petean [30] and Kamada [22, 23], respectively.

In this paper, we study the compact 4-manifolds admitting para-hyperhermitian or para-hypercomplex structures and our first aim is to relate the existence of para-hyperkähler structures to the existence of parallel null orthogonal vector fields. More precisely, in Section 3 we show that if a compact 4-manifold with a $(2,2)$-signature metric admits two parallel, null, orthogonal, pointwise linearly independent vector fields, then it is a torus or a primary Kodaira surface and we notice that these surfaces do admit such vector fields (Theorem 7).

A drastic difference between the definite and the split signature case is that some compact para-hypercomplex 4-manifolds do not admit compatible $(2,2)$-signature metrics, unlike the usual hypercomplex manifolds [12]. We showed however that every compact para-hypercomplex 4-manifold (para-hypercomplex surface) has a
double cover which admits a compatible para-hyperhermitian metric \cite{12}. Heuristically, this is due to the fact that $GL(1, \mathbb{H}'\mathbb{H}')/SU(1, 1) = \mathbb{R}\{0\}$ has two connected components, where $\mathbb{H}'$ is the algebra of split quaternions. Using the fact that the canonical bundle of a complex surface with a para-hyperhermitian structure has a nowhere-vanishing smooth section, we list in Section 4 the possible candidates for para-hyperhermitian surfaces (Theorem 8). A main observation in Theorem 9 is that most of these surfaces do admit para-hyperhermitian structures which come in infinite dimensional families. This shows that the para-hyperhermitian structures are much more flexible than the hyperhermitian ones.

The considerations in Section 5 are motivated by the fact that, unlike the positive definite case, there are compact para-hyperhermitian surfaces which are not locally conformally para-hyperkähler. In Theorem 10 we obtain a descrition of compact complex surfaces admitting locally conformally para-hyperkähler structures. To do this we first reduce the list of possible candidates to those considered in Theorem 9 and then notice that the structures constructed there, are in fact locally conformally para-hyperkähler. An additional restriction comes from the observation that the canonical bundle of such a surface is of real type in the sense of [4]. Moreover we give a construction leading to an infinite dimensional family of para-hyperhermitian structures which are not locally conformally para-hyperkähler.

In Section 6 we provide a list of possible compact para-hypercomplex surfaces by using Theorem 8 and the fact that up to a double cover every para-hypercomplex surface is para-hyperhermitian. Moreover, we construct a para-hypercomplex structure on a surface in this list which does not admit a compatible para-hyperhermitian metric.

Finally, in Section 7 we study the Inoue surfaces of type $S^0$ which, as is well-known \cite{16}, are solvmanifolds. We prove in Theorem 12 that they do not admit para-hyperhermitian structures with left-invariant canonical (2,0)-forms. This is a slight generalization of the well-known result \cite{8} that these surfaces have no para-hyperhermitian structures induced by left-invariant ones. This observation makes reasonable the conjecture that the Inoue surfaces of type $S^0$ do not admit para-hyperhermitian structures at all.

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2 Preliminaries

Denote by $\mathbb{H}'$ the algebra of split quaternions, i.e.

$$\mathbb{H}' = \{ q = a + bi + cs + dt \in \mathbb{R}^4 \mid i^2 = -1, s^2 = t^2 = 1, t = is = -si \}.$$ 

They are associated with a natural scalar product of split signature (2,2) such that $|q|^2 = a^2 + b^2 - c^2 - d^2$. Based on the algebra $\mathbb{H}'$, one defines an almost para-hypercomplex structure on a manifold $M$ as a triple $(I, S, T)$ of anti-commuting endomorphisms of the tangent bundle $TM$ with $I^2 = -Id$ and $S^2 = T^2 = Id$, $T = IS$. Such a structure is called para-hypercomplex if $I, S, T$ satisfy the integrability condition $N_I = N_S = N_T = 0$, where

$$N_A(X, Y) = A^2[A X, A Y] + [X, Y] - A[A X, Y] - A[X, A Y]$$

is the Nijenhuis tensor associated with $A = I, S, T$. The para-hypercomplex structures are the "split analog" of hypercomplex structures.

An almost product structure $S$ is integrable if and only if the eigenbundles $T^\pm = \{ X \in TM : SX = \pm X \}$ are involutive [33]. Therefore, if $(I, S, T)$ is a para-hypercomplex structure, then $T^\pm$ are two transversal involutive distributions mapped to each other by the complex structure $I$. Conversely, if a complex manifold $(M, I)$ admits such distributions $T^\pm$, then we can define a para-hypercomplex structure setting $S = Id$ on $T^+$, $S = -Id$ on $T^-$ and $T = IS$.

A pseudo-Riemannian metric $g$ for which the endomorphisms $I, S, T$ are skew-symmetric is called para-hyperhermitian. Such a metric necessarily has split signature and is also called neutral hyperhermitian. Every para-hypercomplex structure on a 4-manifold locally admits a para-hyperhermitian metric but a globally defined one may not exist. More precisely, the following proposition is true [12].

**Proposition 1** Every para-hypercomplex structure on a 4-manifold $M$ determines a conformal class of para-hyperhermitian metrics up to a double cover of $M$.

Examples of para-hypercomplex structures that do not admit para-hyperhermitian metrics are given in [12]. We provide another example in Section 6.

Para-hypercomplex four-manifolds can also be characterized in the following way:
Proposition 2  A four-dimensional smooth manifold admits a para-hypercomplex structure if and only if it admits two complex structures $I_1$ and $I_2$ yielding the same orientation and such that $I_1 I_2 + I_2 I_1 = 2pId$ for a constant $p$ with $|p| > 1$.

Proof. Suppose $I_1$ and $I_2$ are two complex structures such that $I_1 I_2 + I_2 I_1 = 2pId$ with $|p| > 1$. Then
\[ I = I_1, \quad S = \frac{1}{2\sqrt{p^2 - 1}} [I_1, I_2], \quad T = -\frac{1}{\sqrt{p^2 - 1}} (I_1 + pI_2) \]
form an almost para-hypercomplex structure [15]. The integrability of the structures $I$, $S$, $T$ is proved in [12] Lemma 1] based on the fact [12] that, for each point, there is a locally defined para-hyperhermitian metric $g$.

Conversely, suppose that we a given are para-hypercomplex structure $(I, S, T)$. Take any real number $p$ with $|p| > 1$ and set
\[ I_1 = I, \quad I_2 = -pI - \sqrt{p^2 - 1} T. \]
Then $[I_1, T] = 2\sqrt{p^2 - 1} S$ and $I_1 I_2 + I_2 I_1 = 2pId$. It is well-known that there is a unique torsion-free connection $\nabla$ such that $\nabla I = \nabla S = \nabla T = 0$ (an analog of the Obata connection) [2] Theorem 3.1]. Clearly, $\nabla I_1 = \nabla I_2 = 0$. Since $\nabla$ is torsion-free, this implies that the Nijenhuis tensors of $I_1$ and $I_2$ vanish, thus $I_1$ and $I_2$ are integrable. Take a point $x$ of the manifold and a metric on the tangent space at $x$ which is compatible with $(I, S, T)$. Let $E$ be a non-isotropic tangent vector. Then $E_1 = E$, $E_2 = IE$, $E_3 = SE$, $E_4 = TE$ is an orthogonal basis with $E_2 = I_1 E_1$, $E_4 = I_1 E_3$. Moreover
\[ I_2 E_1 = -pE_2 - \sqrt{p^2 - 1} E_4, \quad I_2 E_3 = -\sqrt{p^2 - 1} E_2 - pE_4. \]
Therefore $I_1$ and $I_2$ determine the same orientation. Q.E.D.

Remark 1 If $I_1 I_2 + I_2 I_1 = 2pId$ for a constant $p$ with $|p| < 1$, then $I_1$, $I_2$ determine a usual hypercomplex structure and vice versa.

Let $(g, I, S, T)$ be an almost para-hyperhermitian structure on a 4-manifold. Then we can define three fundamental 2-forms $\Omega_i$, $i = 1, 2, 3$, setting
\[ \Omega_1(X, Y) = g(I X, Y), \quad \Omega_2(X, Y) = g(S X, Y), \quad \Omega_3(X, Y) = g(T X, Y). \]
Note that the form $\Omega = \Omega_2 + i\Omega_3$ is of type $(2, 0)$ with respect to $I$. As in the definite case, the corresponding Lee forms are defined by
\[ \theta_1 = \delta \Omega_1 \circ I, \quad \theta_2 = \delta \Omega_2 \circ S, \quad \theta_3 = \delta \Omega_3 \circ T, \]
where $\delta$ is the codifferential with respect to $g$. It is well-known \cite{10, 14, 20, 23} that $I, S, T$ are integrable if and only if $\theta_1 = \theta_2 = \theta_3$. Thus, for a para-hyperhermitian structure, we have just one Lee form $\theta$; it satisfies the identities

$$d\Omega_i = \theta \wedge \Omega_i, \quad i = 1, 2, 3.$$  

When additionally the three 2-forms $\Omega_i$ are closed, i.e. $\theta = 0$, the para-hyperhermitian structure is called \textit{para-hyperkahler} (also hypersymplectic or neutral hyperkahler). When $d\theta = 0$ the structure is called \textit{locally conformally para-hyperkahler}. We note that, in dimension 4, the para-hyperhermitian metrics are self-dual and the para-hyperkahler metrics are self-dual and Ricci-flat \cite{23}. It is well-known that every hyperhermitian structure on a 4-dimensional compact manifold is locally conformally hyperkahler \cite{10}, but we shall see in Theorem 9, that this is not true in the indefinite case.

A para-hyperhermitian 4-manifold can be characterized by means of the forms $\Omega_i$ and $\theta$ in the following way \cite{18, 23}.

**Proposition 3** Every para-hyperhermitian structure on a 4-manifold is uniquely determined by three non-degenerate 2-forms $(\Omega_1, \Omega_2, \Omega_3)$ and a 1-form $\theta$ such that

$$-\Omega_1^2 = \Omega_2^2 = \Omega_3^2, \quad \Omega_l \wedge \Omega_m = 0, \quad 1 \leq l \neq m \leq 3, \quad d\Omega_l = \theta \wedge \Omega_l.$$  

**Proposition 4** Let $(M, J)$ be a simply connected complex surface that carries a para-hyperhermitian structure $\{g, I, S, T\}$ with $I = J$. If $\theta$ is the Lie form of this structure, then $dd^c\theta = 0$ and the class of $\theta$ in

$$\frac{\ker(dd^c)}{\text{Im}(d) + \text{Im}(d^c)}$$

depends only on $J$.

**Proof.** The form $\Omega = \Omega_2 + i\Omega_3$ is of type $(2, 0)$ with respect to $I = J$ and nowhere-vanishing. Moreover $d\Omega = \theta \wedge \Omega$, hence $d\theta \wedge \Omega = 0$ which implies $(d\theta)^{(0,2)} = 0$. Then $(d\theta)^{(2,0)} = 0$ since $\theta$ is real-valued. Therefore $d\theta$ is a $(1,1)$-form. It follows that $\partial \theta^{(1,0)} = 0$ and $\overline{\partial} \theta^{(0,1)} = 0$. Then $dd^c\theta = -i\partial\overline{\partial}\theta = -i\partial\overline{\partial}(\theta^{(1,0)}) = 0$.

Let now $\{g', I', S', T'\}$ be another para-hyperhermitian structure on $(M, J)$ with $I' = J$. Denote by $\Omega'_1, \Omega'_2, \Omega'_3$ the 2-forms determined by this structure and let $\theta'$ be the corresponding Lee form. The form $\Omega' = \Omega'_2 + i\Omega'_3$ is of type $(2, 0)$ with respect to $J$, hence $\Omega' = F\Omega$ for a nowhere-vanishing complex-valued smooth function $F$. Since $M$ is simply connected, there is a smooth function $\varphi$ such that $F = |F|e^{i\varphi}$. Then $(\theta' - \theta - d\ln|F| - id\varphi) \wedge \Omega = 0$ which implies $(\theta' - \theta - d\ln|F| - id\varphi) \wedge$
It follows that \( \theta' = \theta + d \ln |F| + d^c \varphi \), so \( \theta \) and \( \theta' \) determine the same class in
\[
\frac{\text{Ker}(d^c)}{\text{Im}(d) + \text{Im}(d^c)}.
\]
Q.E.D.

Note that the cohomology class in Proposition 4 is related to the Aeppli cohomology groups (see [1]):
\[
H^{p,q}_A = \frac{\text{Ker}(d^c) \cap \Omega^{p,q}}{(\text{Im}(d) + \text{Im}(d^c)) \cap \Omega^{p,q}}.
\]

We can say a little bit more for locally conformally para-hyperkähler structures. The first Chern class of a holomorphic line bundle is determined by the coboundary map (the Bockstein map)
\[
\delta : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}),
\]
where \( \mathcal{O}^* \) is the sheaf of non-vanishing holomorphic functions. The equivalence classes of topologically trivial holomorphic line bundles are in the kernel \( H^1_0(M, \mathcal{O}^*) \) of the map \( \delta \). Then following [4] we consider the sequence of natural morphisms
\[
H^1(M, \mathbb{R}) \to H^1(M, \mathbb{R}_+) \to H^1(M, \mathbb{C}^*) \to H^1_0(M, \mathcal{O}^*),
\]
where the first morphism is induced by the exponential map \( \mathbb{R} \to \mathbb{R}_+ \) and we say that a bundle \( L \in H^1_0(M, \mathcal{O}^*) \) is of real type if its class is in the image of \( H^1(M, \mathbb{R}_+) \).

**Lemma 5** If \( M \) carries a locally conformally para-hyperkähler structure, then its canonical bundle is of real type.

**Proof.** Let \( \Omega = \Omega_2 + i \Omega_3 \) be the \((2,0)\)-form and \( \theta \) be the Lie form of the given structure. Cover \( M \) by open sets \( \{U_\alpha\} \) such that the intersections \( U_\alpha \cap U_\beta \) are connected and \( \theta|U_\alpha = d \varphi_\alpha \) for a smooth function \( \varphi_\alpha \). Then \( \varphi_\alpha = \varphi_\beta + c_{\alpha\beta} \) on \( U_\alpha \cap U_\beta \), where \( c_{\alpha\beta} \) are constants. We have \( d(e^{-\varphi_\alpha} \Omega) = 0 \), so \( e^{-\varphi_\alpha} \Omega \) are local holomorphic sections of the canonical bundle. These sections determine the transition functions \( \psi_{\alpha\beta} = e^{c_{\alpha\beta}} \). Q.E.D.

### 3 Para-hyperkähler surfaces and parallel null vector fields

It has been shown by H. Kamada [22, 23] that the only compact complex surfaces admitting para-hyperkähler structures are the primary Kodaira surfaces and the
complex tori. Moreover, he has described all such structures on these surfaces in
terms of the solutions of non-linear PDE’s for a scalar function [22, 23]. The aim
of this section is to find another characterization of para-hyperkähler surfaces
by showing that they coincide with the compact 4-manifolds admitting metrics
of signature (2, 2) and pairs of parallel and orthogonal null vector fields. Before
stating our main result in this direction we shall prove an auxiliary lemma.

Lemma 6 Let $M$ be a 4-manifold with a metric $g$ of signature $(2, 2)$ and let $X$
and $Y$ be orthogonal null vector fields which are linearly independent at every
point of $M$. Then the triple $(g, X, Y)$ determines an orientation and a unique and orientation compatible almost complex structure $J$ on $M$ such that $JX = Y$.

Proof. We first show that in a neighbourhood of every point of $M$, there exist vector fields $Z, T$ such that:

(i) $(X, Y, Z, T)$ is a local frame of the tangent bundle $TM$;

(ii) $g(X, Z) = 1, g(X, T) = 0; \quad g(Y, Z) = 0, g(Y, T) = 1.$

Indeed, by the Witt theorem, for every $p \in M$, there exist isotropic tangent vectors $u, v \in T_p M$ such that $(X_p, Y_p, u, v)$ is a basis of $T_p M$ and $g(X_p, u) = 1, g(X_p, v) = 0, g(Y_p, u) = 0, g(Y_p, v) = 1$. Extend $u, v$ to vector fields $U, V$ in a neighbourhood of $p$ and consider the system

$$1 = \alpha g(U, X) + \beta g(V, X), \quad 0 = \alpha g(U, Y) + \beta g(V, Y)$$

with respect to the unknown functions $\alpha, \beta$. The determinant of this system at the point $p$ is equal to 1, hence in a neighbourhood of $p$ it has a (unique) solution of smooth functions $\alpha, \beta$. Similarly for the system

$$0 = \phi g(U, X) + \psi g(V, X), \quad 1 = \phi g(U, Y) + \psi g(V, Y).$$

Set

$$Z = \alpha U + \beta V, \quad T = \phi U + \psi V.$$

We have $\alpha(p) = 1, \beta(p) = 0$, thus $Z_p = u$; similarly $T_p = v$. Therefore $X, Y, Z, T$ form a frame of vector fields in a neighbourhood of $p$.

Now let $\tilde{Z}, \tilde{T}$ be another pair of vector fields around $p$ having the properties (i) and (ii) stated above. Then they have the form

$$\tilde{Z} = a X + b Y + Z, \quad \tilde{T} = c X + d Y + T,$$

where $a, b, c, d$ are smooth functions. It follows that the frames $(X, Y, Z, T)$ and $(X, Y, \tilde{Z}, \tilde{T})$ determine the same orientation. Thus, the orientation determined
by \((X,Y,Z,T)\) does not depend on the choice of the vector fields \(Z,T\) and we shall say that it is determined by the triple \((g,X,Y)\).

Next following [11], set

\[ a = g(Z,Z), \quad b = g(T,T), \quad c = g(Z,T) \]

and

\[ E_1 = \frac{1-a}{2}X + Z, \quad E_2 = \frac{1-b}{2}Y + T - cX, \]

\[ E_3 = -\frac{1+a}{2}X + Z, \quad E_4 = -\frac{1+b}{2}Y + T - cX. \]

Then \((E_1,E_2,E_3,E_4)\) is an orthogonal frame, positively oriented with respect to the orientation determined by \((g,X,Y)\) and such that \(g(E_1,E_1) = g(E_2,E_2) = 1, g(E_3,E_3) = g(E_4,E_4) = -1\). The almost complex structure \(J\) for which \(JE_1 = E_2, JE_3 = E_4\) has the required properties. Let \(K\) be another complex structure on \(T_pM\) with these properties. Define endomorphisms \(S\) and \(T\) of \(T_pM\) such that \(S^2 = T^2 = Id\) and \(SE_1 = E_3, SE_2 = -E_4, TE_1 = E_4, TE_2 = E_3\). Set \(I = J\).

Then \(IS = -SI = T\). Since \(K\) is compatible with the metric it can be written as \(K = \lambda_1 I + \lambda_2 S + \lambda_3 T\), where \(\lambda_1, \lambda_2, \lambda_3\) are real numbers with \(\lambda_1^2 - \lambda_2^2 - \lambda_3^2 = 1\). Moreover, in view of (2), the identity \(KX = Y\) is equivalent to \(KE_1 - KE_3 = E_2 - E_4\) which implies that \(\lambda_1 = 1, \lambda_2 = \lambda_3 = 0\). Thus \(K = J\) which proves the lemma. Q.E.D.

A complex structure compatible with a split signature metric and preserving a null distribution of dimension 2 is called proper for the null distribution [27].

Lemma 6 shows that there is a unique proper complex structure if we fix two orthogonal vector fields of the distribution.

**Theorem 7** Let \((M,g)\) be a compact 4-manifold with a metric \(g\) of signature \((2,2)\). Suppose that \(M\) admits two parallel and orthogonal null vector fields \(X,Y\), linearly independent at every point of \(M\) and let \(J\) be the almost complex structure determined by \((g,X,Y)\) as in Lemma[7]. Then:

(i) The structure \((g,J)\) is (pseudo) Kähler.

(ii) The metric \(g\) is Ricci-flat.

(iii) \((M,J)\) is either a torus or a primary Kodaira surface.

(iv) \(M\) admits a para-hyperkähler structure with metric \(g\) and complex structure \(I = J\).

Conversely, every torus and every primary Kodaira surface \((M,J)\) admits a metric \(g\) of signature \((2,2)\) and vector fields \(X, Y = JX\) which are parallel, orthogonal, null and linearly independent at every point of \(M\).
Proof. Since $X$ and $Y$ are parallel, the proof of [33, Theorem 3] shows that, around every point of $M$, there are local coordinates $(x, y, z, t)$, such that $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$ and the metric $g$ in these coordinates has the form

$$g(x,y,z,t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}$$

where $a, b, c$ are smooth functions independent of the coordinates $x$ and $y$. According to Lemma 6, the manifold $M$ admits a unique almost complex structure $J$ compatible with $g$ and such that $J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$. Since $a, b, c$ do not depend on $x, y$, it follows from [11, Corollary 14] that the structure $(g, J)$ is (pseudo) Kähler and Ricci-flat. Then, by [30, Corollary 2], $M$ is one of the following: a torus, a primary Kodaira surface or a hyperelliptic surface. To prove the result we need first to exclude the last option. Note that $X - iY$ is a parallel, isotropic, nowhere-vanishing, holomorphic vector field on $M$. Every hyperelliptic surface $M$ is the quotient of a product $E \times F$ of two elliptic curves by a finite fixed-point-free abelian group of automorphisms. We can take $E$ of the form $C/\Lambda$ where $\Lambda$ is the lattice generated by 1 and a complex number $\tau$ with $\text{Im}\tau > 0$. Then $M$ is the quotient of $E \times F$ by the group generated by certain translations of $F$ and a map of the type

$$\varphi(z, w) = (z + \tau/m, e^{2k\pi i/m}w),$$

where $m$ is one of the numbers 2, 3, 4, 6 and $k \in \{1, ..., m - 1\}$ [7, VI.19, VI.20]. The quotient map $E \times F \to M$ is a (finite) covering and we take the pull-back $g'$ of the metric $g$, then lift $g'$ to the universal covering $\mathbb{C}^2$ of $E \times F$. In this way we get a (pseudo) Kähler, Ricci-flat metric $\tilde{g}$ on $\mathbb{C}^2$. It is of the form

$$\tilde{g} = \alpha dzd\overline{z} + 2\text{Re}(\gamma dzd\overline{w}) + \beta dwd\overline{w}$$

for real smooth functions $\alpha, \beta$ and a complex smooth function $\gamma$. The lift $\tilde{U} = \lambda \frac{\partial}{\partial z} + \mu \frac{\partial}{\partial w}$ of a holomorphic vector field $U$ on $M$ satisfies the identity $\varphi_* \tilde{U} = \tilde{U} \circ \varphi$. This implies $\mu = 0$ since $\varphi_*(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}$ and $\varphi_*(\frac{\partial}{\partial w}) = e^{2k\pi i/m} \frac{\partial}{\partial w}$. Therefore, if $Z$ is the holomorphic vector field on $M$ given in the local coordinates $(z, w)$ as $\frac{\partial}{\partial z}$, we have $U = f Z$ for a function $f$. The function $f$ is holomorphic on the compact manifold $M$, hence it is a constant. Thus, every holomorphic vector field on $M$ is proportional to $Z$. It follows that the vector field $\frac{\partial}{\partial z}$ on $\mathbb{C}^2$ is
parallel and null with respect to $\tilde{g}$. The fact that this field is null implies $\alpha = 0$. Since it is parallel, the Lie derivative of the Kähler form
\[
\Omega = -i(\alpha \, dz \wedge d\overline{z} + \gamma \, dz \wedge dw + \gamma \, dw \wedge d\overline{z} + \beta \, dw \wedge d\overline{w})
\]
vanishes. Thus, it follows by Cartan’s formula that
\[
0 = \mathcal{L}_{\partial_z} \Omega = d \circ i_{\partial_z} \Omega = -i(d(\gamma \, dw))
\]
since $d\Omega = 0$ and $\alpha = 0$. This implies that the derivatives of $\gamma$ with respect to $z, \overline{z}, w$ vanish. Thus $\gamma$ depends only on the variable $w$ and $\gamma(w)$ is an anti-holomorphic function. Then, since $\tilde{g}$ is the lift of a metric on $E \times F$, $\gamma$ descends to an anti-holomorphic function $\gamma'$ on $F$. By the maximum principle, $\gamma' \equiv const$, therefore $\gamma$ is a constant. This constant is not zero since $\tilde{g}$ is non-degenerate. Then $\tilde{g}$ is not invariant under the map $\varphi$, so it does not descend to $M$, a contradiction.

Next we show that the primary Kodaira surfaces and 4-tori do admit para-hyperkähler structures, compatible with $g$ and $I = J$. Suppose that $M$ is a primary Kodaira surface. Then it can be obtained in the following way. Consider the affine transformations $\varphi_k(z, w)$ of $\mathbb{C}^2$ given by
\[
\varphi_k(z, w) = (z + a_k, w + b_k),
\]
where $a_k, b_k, k = 1, 2, 3, 4$, are complex numbers such that
\[
a_1 = a_2 = 0, \quad Im(a_3\overline{a_4}) = b_1 \neq 0, \quad b_2 \neq 0.
\]

They generate a group $G$ of affine transformations acting freely and properly discontinuously on $\mathbb{C}^2$ and $M$ is the quotient space $\mathbb{C}^2/G$ for a suitable choice of $a_k$ and $b_k$ [24, p.786]. Taking into account the identities
\[
\varphi_k(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z} + \overline{a}_k \frac{\partial}{\partial w}, \quad \varphi_k(\frac{\partial}{\partial w}) = \frac{\partial}{\partial w}
\]
we see that every holomorphic vector field on $M$ is proportional to the vector field $W$ given in the local coordinates $(z, w)$ as $\frac{\partial}{\partial w}$. Therefore the vector field $\partial_w = \frac{\partial}{\partial w}$ on $\mathbb{C}^2$ is parallel and null with respect to the lift
\[
\tilde{g} = \alpha \, dz d\overline{z} + 2Re(\gamma \, dz d\overline{w}) + \beta \, dw d\overline{w}
\]
of the metric $g$. Arguments similar to that above show that $\beta = 0$ and $\gamma = const \neq 0$. Then the Kähler form of the Kähler metric $\tilde{g}$ is given by
\[
\tilde{\Omega}_1 = -i(\alpha(z) \, dz \wedge d\overline{z} + \gamma \, dz \wedge d\overline{w} + \gamma \, dw \wedge d\overline{z}),
\]
where $\alpha(z)$ is a smooth function depending only of $z$. Set
$$\tilde{\Omega}_2 = \gamma \, dz \wedge dw + \gamma \, d\bar{z} \wedge d\bar{\tau}, \quad \tilde{\Omega}_3 = -i(\gamma \, dz \wedge dw - \gamma \, d\bar{z} \wedge d\bar{\tau}).$$

By Proposition 3, the forms $\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3$ determine a para-hyperkähler structure on $\mathbb{C}^2$. Since these 2-forms are invariant under the action of the group $G$, we obtain a para-hyperkähler structure on $M$ with metric $g$ and complex structure identical to the complex structure of $M$.

Now consider the case when $M$ is a complex torus. Let $\partial_z$ and $\partial_w$ be the global holomorphic vector fields given in the standard local coordinates $(z, w)$ on the torus by $\partial_z = \frac{\partial}{\partial z}, \partial_w = \frac{\partial}{\partial w}$. Then the holomorphic vector field $U = X - iY$ on $M$ is a linear combination $U = \lambda \partial_z + \mu \partial_w$, where $\lambda$ and $\mu$ are constants. The Lie derivative with respect to $U$ of the Kähler form $\Omega_1 = -i(\alpha \, dz \wedge d\bar{z} + \gamma \, dz \wedge d\bar{w} + \bar{\gamma} \, dw \wedge d\bar{\tau} + \beta \, dw \wedge d\bar{w})$ of the metric $g$ vanishes. It follows that the derivatives of $\lambda \alpha + \mu \gamma$ with respect to $z, w, \bar{w}$ vanish and the derivatives of $\lambda \gamma + \mu \beta$ with respect to $z, \overline{z}, w$ also vanish. Then $f^\pm dz \wedge dw$ are globally defined closed forms. We have $|f|^2 = |\lambda|^2 - \alpha \beta$. Note that either $\lambda + \mu \neq 0$ or $\lambda - \mu \neq 0$ since $U \neq 0$. In the first case we define real-valued 2-forms $\Omega_2$ and $\Omega_3$ by
$$\Omega_2 + i\Omega_3 = \frac{2f^-}{|\lambda + \mu|} dz \wedge dw$$
and in the second case we set
$$\Omega_2 + i\Omega_3 = \frac{2f^+}{|\lambda - \mu|} dz \wedge dw.$$

In both cases the forms $\Omega_1, \Omega_2, \Omega_3$ determine a para-hyperkähler structure on $M$ with metric $g$ and complex structure $I = J$.

Finally we show that the primary Kodaira surfaces and 4-tori admit metrics with 2 parallel orthogonal null vector fields. Let $M$ be a primary Kodaira surface represented as $\mathbb{C}^2/G$, where the group $G$ has been described above. As in [30], set $\alpha(z) = f(z) - \gamma z - \bar{\gamma} \bar{z}$, where $f(z)$ is a smooth function on $\mathbb{C}$ satisfying the identities $f(z + a_3) = f(z), f(z + a_4) = f(z)$. Then the metric $\tilde{g} = \alpha \, dzd\bar{z} + 2RRe(dzd\bar{w})$ on $\mathbb{C}^2$ descends to a split signature Kähler, Ricci flat metric $g$ on $M$ for which the holomorphic vector field $W$ is parallel and null (the metric $g$ is flat if $f \equiv const$). Hence the real and imaginary parts of the vector field $W$ have the required properties.
The case of a complex torus $M$ is similar. Let $M$ be the quotient of $\mathbb{C}^2$ by a lattice $<a_1, a_2, a_3, a_4>$. Take a smooth function $\alpha$ on $\mathbb{C}$ such that $\alpha(z + a_3) = \alpha(z)$, $\alpha(z + a_4) = \alpha(z)$. Then the metric
\[
\tilde{g} = \alpha \, dz \overline{dw} + 2 \text{Re} (dz \overline{dw})
\]
on $\mathbb{C}^2$ descends to a a split signature Ricci flat, Kähler metric $g$ on the torus $M$ [30]. For this metric, the holomorphic vector field $\frac{\partial}{\partial w}$ on $M$ is parallel and null. Q.E.D.

4 Para-hyperhermitian structures and compact complex surfaces with vanishing first Chern class

Let $(g, I, S, T)$ be a para-hyperhermitian structure on a compact 4-manifold $M$ with fundamental 2-forms $\Omega_i$, $i = 1, 2, 3$. Then $\Omega = \Omega_2 + i \Omega_3$ is of type $(2,0)$ w.r.t. the complex structure $I$, thus the canonical bundle of the complex manifold $(M, I)$ is smoothly trivial. For a para-hyperkähler structure the $(2,0)$-form $\Omega$ is holomorphic, hence the canonical bundle of $(M, I)$ is holomorphically trivial. Thus a compact complex surface $(M, J)$ admits a para-hyperhermitian structure with $I = J$ only if its integral first Chern class $c_1(J)$ vanishes. A weaker condition is the vanishing of the real first Chern class $c_1^R(J)$. If $c_1^R(J) = 0$, then $c_1(J)$ is torsion and the canonical bundle of $(M, J)$ is flat as well as the principle circle bundle corresponding to it. Since the flat principal $G$-bundles for any Lie group $G$ are in bijection with the conjugacy classes of $G$-representations of the fundamental group of $M$, there is a finite covering $(\tilde{M}, \tilde{J})$ of $(M, J)$ with $c_1(\tilde{J}) = 0$.

The classification of surfaces with topologically trivial canonical bundle seems to be known to the experts, but we were not able to find an explicit proof. So we provide a short proof below for the sake of completeness. The primary Kodaira surfaces from the list were defined in Theorem 7 while the Hopf surfaces, the Inoue surfaces, and the minimal properly elliptic surfaces of odd first Betti number are discussed in Theorems 9 and 10. Note that we use the notations for Inoue surfaces from [32] and $S^0$ corresponds to $S_M$ and $S^\pm$ corresponds to $S^\pm_N$ in [19] [4].

**Theorem 8** Let $(M, J)$ be a compact complex surface with topologically trivial canonical bundle. Then $(M, J)$ is one of the following: a complex torus, a $K3$ surface, a primary Kodaira surface, a Hopf surface, an Inoue surface of type $S^0$ or $S^\pm$ without curves, or a minimal properly elliptic surface of odd first Betti number.

**Proof.** The proof is based on [32] and makes use of the completed proof (c.f.
and [25]) of a result due to Bogomolov [9] about surfaces of class VII with vanishing second Betti number.

It is well-known that if the first Betti number $b_1$ of $M$ is even, then it admits a Kähler metric. In this case the canonical bundle $\mathcal{K}$ of $M$ is not just topologically but also holomorphically trivial. Indeed, by [9, Théorème 1], $M$ admits a finite holomorphic covering with holomorphically trivial canonical bundle. Then a tensor power $\mathcal{K}^d$ of $\mathcal{K}$ is also holomorphically trivial (c.f., for example, [5, (16.2) Lemma, p.54]). Since $\mathcal{K}$ is topologically trivial, it follows that it is holomorphically trivial. Then $M$ is either a torus or a $K3$ surface by the Kodaira-Enriques classification. So we further consider only the case $b_1$ odd.

Notice that, since $c_1(M) = 0$, the adjunction formula shows that $M$ is minimal. Hence it also satisfies $c_1^2(M) = 0$. Such surfaces are easy to identify in the Kodaira-Enriques classification. In particular their Kodaira dimension $k$ is one the numbers $-\infty, 0, 1$. We start with $k = -\infty$. In this case it follows from [32, Sec. 6] that $c_1^2 = -b_2(M)$, so $b_2(M) = 0$. Then $c_1^2(M) = 0$ and by the above mentioned result of Bogomolov, $(M, J)$ is either a Hopf surface or Inoue surface. The second case is $k = 0$. Then we see, again from the Kodaira-Enriques classification, that $(M, J)$ is a Kodaira surface. Lastly, if $k = 1$, then $(M, J)$ is a properly elliptic surface. Q.E.D.

**Remark 2** It is well known that every torus, $K3$ surface, primary Kodaira surface and primary Hopf surface has a topologically trivial canonical bundle [2].

Some of the non-primary Hopf surfaces have trivial canonical bundle, but some of them do not. For example, the quotient of a quaternionic Hopf surface by a finite subgroup of $SU(2)$ admits a hypercomplex structure, hence it has a non-vanishing section of its canonical bundle, so the latter is trivial. We can define a Hopf surface with topologically non-trivial canonical bundle as follows. Let $(z, w)$ be the standard coordinates on $\mathbb{C}^2 \setminus \{(0, 0)\}$ and $G$ the group of transformations of $\mathbb{C}^2 \setminus \{(0, 0)\}$ generated by $g_0(z, w) = (\frac{1}{2}z, \frac{1}{2}w)$, $g_1(z, w) = (w, z)$. Consider the secondary Hopf surface $M = (\mathbb{C}^2 \setminus \{(0, 0)\})/G$. Let $\mu$ be the representation of the fundamental group $G$ of $M$ yielding its canonical bundle $\mathcal{K}$. According to [32, p. 271], the integral first Chern class of $\mathcal{K}$ vanishes if and only if $\mu|\text{Tor}(G/[G,G]) = 1$. It is clear that $g_0$ is the free generator of $G$, $g_1^2 = 1$, $g_0g_1g_0^{-1}g_1^{-1} = Id$ and $g_1 \text{mod}([G,G]) \in \text{Tor}(G/[G,G])$. We have $\mu(g_1) = -1$, hence $\mathcal{K}$ is not topologically trivial (but note that $\mathcal{K}^2$ is trivial).

The integral first Chern class of a properly elliptic surface depends on the invariant $c(\eta)$ defined in [32, p. 140]. The first Chern class is proportional to a generator $c$ of the second cohomology of the base of the elliptic fibration. In particular it vanishes iff $c(\eta)$ is primitive, i.e. $c(\eta) = \pm c$. 

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The Inoue surfaces also have topologically trivial canonical bundle - the surfaces of type $S^+$ and $S^0$ admit non-vanishing smooth $(2,0)$-forms - see, for example, the proofs of Theorems 9 and 10 while to see that the integral first Chern class of an Inoue surface of type $S^-$ vanishes we can use the condition given in [10, p. 271]. Every Inoue surface $S$ of type $S^-$ is the quotient of $\mathbb{C} \times H$ by a group $G$ of transformations acting freely and properly discontinuous (see the proof of Theorem 9 for a detailed description). The canonical bundle of $S$ is the associated bundle $(\mathbb{C} \times H) \times_\mu \mathbb{C}$, where the representation $\mu : G \to \text{End}(\mathbb{C})$ is defined by $g^* (dz \wedge dw) = \mu(g)(dz \wedge dw)$, $g \in G$, $(z, w)$ being the standard coordinates on $\mathbb{C} \times H$. By [10, p. 279], $\text{Tor}(G/\{G, G\})$ is generated by the transformations $g_k(z, w) = (z + b_k w + c_k, w + a_k)$, $k = 1, 2, 3$, $g_3(z, w) = (z + \frac{b_1 a_2 - b_2 a_1}{r}, w)$ where $a_k, b_k, c_k, r$ are certain numbers. These transformations leave invariant the form $dz \wedge dw$, thus $\mu|\text{Tor}(G/\{G, G\}) = 1$. Therfore the canonical bundle of $S$ is topologically trivial by [10, p. 271].

In the next theorem we show that most of the surfaces listed in Theorem 8 do admit para-hyperhermitian structures. Moreover, they vary in infinite dimensional families.

**Theorem 9** The following compact complex surfaces admit infinite dimensional families of para-hyperhermitian structures: complex tori, primary Kodaira surfaces, Inoue surfaces of type $S^+$, a special type of minimal properly elliptic surfaces with odd first Betti number and quaternionic primary Hopf surfaces. The complex tori and the primary Kodaira surfaces admit an infinite dimensional family of non-locally conformally para-hyperkähler structures.

**Proof.** The proof is case by case.

1. **Complex tori.**

As we have mentioned, every complex torus of dimension 2 admits a para-hyperkähler structure [23]. Here we shall construct an infinite dimensional family of para-hyperhermitian structures which are not locally conformally para-hyperkähler.

Let $M = \mathbb{C}^2/\Lambda$ be a complex torus with lattice $\Lambda$ generated by vectors $\tau_1, ..., \tau_4$ where $\tau_1 = (a_1, 0)$ and $\tau_2 = (a_2, 0)$. Take a non-constant real-valued smooth doubly-periodic function $\varphi$ on $\mathbb{C}$ with periods $a_1$ and $a_2$. Let $(z, w)$ be the standard coordinates on $\mathbb{C}^2$. Set

$$
\Omega_1 = \text{Im} \left( e^{i\varphi} dz \wedge d\overline{w} \right), \quad \Omega_2 + i\Omega_3 = e^{i\varphi} dz \wedge dw, \quad \theta = i \frac{\partial \varphi}{\partial \overline{z}} dz - i \frac{\partial \varphi}{\partial z} d\overline{z}.
$$

These forms descend to $M$ and, in view of Proposition 9, they determine a para-hyperhermitian structure on $M$ which is not para-hyperkähler. We have $d\theta = 0$
exactly when the function $\varphi$ is harmonic. In this case $\varphi$ is constant since it is bounded. Hence the para-hyperhermitian structure on $M$ defined above is not locally conformally para-hyperkähler.

(2) Primary Kodaira surfaces.

Every such a surface admits a para-hyperkähler structure [22, 23]. Here we shall use the description of primary Kodaira surfaces as quotients $M = \mathbb{C}^2/G$ given in the proof of Theorem 7. Note first that the complex numbers $a_3$ and $a_4$ are linearly independent over $\mathbb{R}$ since $\text{Im}(a_3a_4) \neq 0$. Now take a non-constant real-valued doubly-periodic function $\varphi$ on $\mathbb{C}$ with periods $a_3$ and $a_4$.

Set
$$\Omega_1 = \text{Im}(e^{i\varphi}dz\wedge d\bar{z}) + i\text{Re}(e^{i\varphi}z)dz\wedge d\bar{z}, \quad \Omega_2 + i\Omega_3 = e^{i\varphi}dz\wedge dw,$$
$$\theta = i\frac{\partial \varphi}{\partial \bar{z}}dz - i\frac{\partial \varphi}{\partial z}d\bar{z},$$

where $(z, w)$ are the standard coordinates on $\mathbb{C}^2$. Then these forms satisfy the identities of Proposition 3. Moreover, the forms $\Omega_1, \Omega_2, \Omega_3, \theta$ are invariant under the action of the group $G$, so they define a para-hyperhermitian structure on $M$ which is not locally conformally para-hyperkähler.

(3) Quaternionic Hopf surfaces.

These surfaces are the quotient spaces $M = (\mathbb{H}\setminus\{0\})/\mathbb{Z}$, the action of $\mathbb{Z}$ being generated by $L_a : q \rightarrow aq$, where $a$ is a fixed complex number with $|a| > 1$. If $q = z_1 + s\bar{z}_2$, the action is $(z_1, z_2) \rightarrow (az_1, \bar{a}z_2)$. Then the following forms define a conformally para-hyperkähler structure on $\mathbb{H}\setminus\{0\}$ which descends to a locally conformal para-hyperkähler structure on $M$:

$$\Omega_1 = i\frac{dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2}{|z_1|^2 + |z_2|^2}, \quad \Omega_2 + i\Omega_3 = \frac{dz_1 \wedge dz_2}{|z_1|^2 + |z_2|^2},$$
$$\theta = -\frac{1}{|z_1|^2 + |z_2|^2}(\bar{z}_1 dz_1 + z_1 d\bar{z}_1 + \bar{z}_2 dz_2 + z_2 d\bar{z}_2).$$

We can also take
$$\Omega_1 = i\frac{dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2}{|z_1|^2 + |z_2|^2} + i\bar{\partial}\partial \varphi,$$

with a smooth real-valued function $\varphi$ depending only on $z_1$.

(4) Inoue surfaces of type $S^\pm$

We first recall the construction of the Inoue surfaces of type $S^\pm$ [19]. Set $\varepsilon = \pm 1$ and take a matrix $N = (n_{ij}) \in GL(2, \mathbb{Z})$ with $\text{det} N = \varepsilon$ having two real eigenvalues $\alpha > 1$ and $\varepsilon\alpha$. Note that $\alpha$ is an irrational number. Choose real eigenvectors $(a_1, a_2)$ and $(b_1, b_2)$ corresponding to $\alpha$ and $\varepsilon\alpha$, respectively. Take
integers \( p, q, r \), \( r \neq 0 \) and a complex number \( t \). Let \((c_1, c_2)\) be the solution of the equation
\[
\varepsilon(c_1, c_2) = (c_1, c_2) N^{tr} + (e_1, e_2) + \frac{b_1a_2 - b_2a_1}{r}(p, q)
\]
where \( N^{tr} \) is the transpose matrix of \( N \) and
\[
e_k = \frac{1}{2} n_{k1}(n_{k1} - 1)a_1b_1 + \frac{1}{2} n_{k2}(n_{k2} - 1)a_2b_2 + n_{k1}n_{k2}a_1b_2, \quad k = 1, 2.
\]

Let \( G^\varepsilon \) be the group generated by the following automorphisms of \( \mathbb{C} \times \mathbb{H} \), \( \mathbb{H} \) being the upper half-plane:
\[
g_0 = (z, w) = (\varepsilon z + \frac{1}{2}(1 + \varepsilon)t, \alpha w)
\]
\[
g_k(z, w) = (z + b_kw + c_k, w + a_k), \quad k = 1, 2,
\]
\[
g_3(z, w) = (z + \frac{b_1a_2 - b_2a_1}{r}, w).
\]

The group \( G^\varepsilon \) acts properly discontinuously and without fixed points in view of (4) and the fact that \((a_1, b_1)\) and \((a_2, b_2)\) are linearly independent vectors. The quotient \((\mathbb{C} \times \mathbb{H})/G^\varepsilon\) is a compact complex surface, known as an Inoue surface of type \( S^\varepsilon \).

Given an Inoue surface of type \( S^+ \), we set \( t_2 = \text{Im} t \) and
\[
\alpha_1 = dx - \frac{1}{v}(y - t_2 \ln v) du, \quad \alpha_2 = dy - \frac{1}{v}(y - t_2 \ln v) dv, \quad \alpha_3 = \frac{du}{v}, \quad \alpha_4 = \frac{dv}{v},
\]
where \( z = x + iy \) and \( w = u + iv \). These forms are linearly independent and invariant under the action of the group \( G^+ \). Moreover
\[
d\alpha_1 = \alpha_3 \wedge \alpha_2 - \frac{t_2}{v^2 \ln \alpha} du \wedge dv, \quad d\alpha_2 = \alpha_4 \wedge \alpha_2, \quad d\alpha_3 = \alpha_3 \wedge \alpha_4, \quad d\alpha_4 = 0.
\]

Set
\[
\Omega_1 = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4, \quad \Omega_2 = \alpha_1 \wedge \alpha_3 - \alpha_2 \wedge \alpha_4, \quad \Omega_3 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.
\]

Then
\[
-\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = 2\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4, \quad \Omega_l \wedge \Omega_m = 0, \quad 1 \leq l \neq m \leq 3, \quad d\Omega_l = -\alpha_4 \wedge \Omega_l.
\]

Therefore, by Proposition 3, \( \Omega_1, \Omega_2, \Omega_3 \) define an \( G^+ \)-invariant para-hyperhermitian structure on \( \mathbb{C} \times \mathbb{H} \) which is locally conformally para-hyperkähler since its Lie form \( \theta = -\alpha_4 \) is closed. This structure descends to a para-hyperhermitian structure on the Inoue surface \( S^+ \).
We can deform $\Omega_1$ to $\Omega_1 + \frac{i \partial \bar{\partial} \varphi}{\operatorname{Im}(w)}$ for arbitrary function $\varphi$ depending only on $\operatorname{Im}(w)$ and satisfying $\varphi(ax) = \varphi(x)$. These functions are in one-to-one correspondence with the functions on the circle $S^1 = \mathbb{R}^+ / \langle \alpha \rangle$ and form an infinite dimensional family.

(5) **Minimal properly elliptic surfaces of odd first Betti number**

A properly elliptic surface is, by definition, a compact complex surface admitting a fibration $\pi : M \to B$ onto a complex orbifold curve of genus $g \geq 2$ and generic fiber an elliptic curve. Every such a surface is of Kodaira dimension 1 and has universal cover $\mathbb{C} \times \mathbb{H}$. Among these surfaces, the ones with vanishing first Chern class are precisely those with odd first Betti number. In this case $M$ has no singular fibers \cite[Lemma 7.2]{32} and has a good orbifold base - see the considerations preceding Theorem 7.4 in \cite{32}.

It is convenient to use here the description of the minimal properly elliptic surfaces $M$ with odd first Betti number given in \cite{26}. Set

$$D = \{(x, y) \in \mathbb{C}^2 | \operatorname{Im}(x/y) > 0\}.$$ 

According to \cite[Theorem 1]{29}, every minimal elliptic surface of odd first Betti number is a quotient of this (non-simply connected) domain by a discrete group $\Gamma$ generated by a finite number of linear transformations of $\mathbb{C}^2$ of the form $L = \lambda M$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $M \in \text{SL}(2, \mathbb{R})$. The matrices $L$ satisfy a number of relations and the elliptic fibration is determined by the map $\pi : (x, y) \to \frac{x}{y}$ on the covering space $D$.

Take a transformation $L(x, y) = (\lambda(ax + by), \lambda(cx + dy))$ of $D$ where $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R})$$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then

$$L^*(dx \wedge dy) = |\lambda|^2 dx \wedge dy, \quad \operatorname{Im}(\lambda(ax + by) \overline{\lambda(cx + dy)}) = |\lambda|^2 \operatorname{Im}(x \overline{y}).$$

Thus, when the number $\lambda$ is real, or equivalently when the group $\Gamma$ is a subgroup of $\text{GL}(2, \mathbb{R})$, the forms

$$\Omega_1 = \operatorname{Im}(\frac{dx \wedge d\overline{y}}{\operatorname{Im}(xy)}), \quad \Omega_2 + i \Omega_3 = \frac{dx \wedge dy}{\operatorname{Im}(x \overline{y})}, \quad \theta = \frac{1}{2i \operatorname{Im}(x \overline{y})} (ydx - \overline{y}dx + \overline{x}dy - xdy).$$

descend to $M$. It is easy to check that they determine a locally conformally para-hyperkähler structure.

Taking a smooth real-valued function $\varphi$ such that $\varphi(|\lambda|^2 z) = \varphi(z)$ for all $z \in \mathbb{H}$, we obtain an infinite-dimensional family of para-hyperhermitian structures on $M$ by changing $\Omega_1$ to

$$\Omega_1 = \operatorname{Im}(\frac{dx \wedge d\overline{y}}{\operatorname{Im}(xy)}) + dd^c \varphi.$$
Note that every function $\varphi$ depending only on $\arg(z)$ satisfies the above condition. \(Q.E.D.\)

Every minimal properly elliptic surface with good orbifold base, odd first Betti number and Kodaira dimension 1 which does not have singular fibres is the quotient of $\widetilde{SL(2,\mathbb R)} \times \mathbb R$ by a discrete subgroup \cite{32} Theorem 7.4. Also, every Inoue surface $S^+$ defined by means of a real parameter $t$ is the quotient of the group $Sol^1_4$ by a discrete subgroup \cite{32} Proposition 9.1. Left-invariant para-hypercomplex structures descending to these types of elliptic and Inoue surfaces have been constructed in \cite{3,8}. Compatible metrics have been given in \cite{20} where it is shown that the respective para-hyperhermitian structures are locally, but not globally, conformally para-hyperkähler, the metric on the elliptic surfaces being locally conformally flat. The flat para-hyperkähler structures on compact complex surfaces have been described in \cite{22,23}; they exist only on complex tori and primary Kodaira surfaces. Notice that the structures constructed in Theorem \cite{9} on the quaternionic Hopf surfaces and elliptic surfaces are locally conformally flat for $\varphi = \text{const}$.

5 Locally conformally para-hyperherkähler surfaces

We have seen in the proof of Theorem \cite{9} that all surfaces listed there admit locally conformally para-hyperkähler structures. The next result shows that these are the only compact complex surfaces admitting such structures.

**Theorem 10** If a compact complex surface $(M,J)$ admits a locally conformally para-hyperkähler structure $(g,I,S,T)$ with $I = J$ it is one of the following: a complex torus, a primary Kodaira surface, an Inoue surface of type $S^+$, a properly elliptic surface of real type with odd first Betti number or a Hopf surface of real type.

**Proof.** We have to show that some of the surfaces in Theorem \cite{9} do not admit locally conformally para-hyperkähler structures. We first exclude the K3 surfaces. All K3 surfaces are simply connected, so any locally conformally para-hyperkähler structure on a K3 surface is globally conformally para-hyperkähler and after a conformal change it becomes para-hyperkähler. However the K3 surfaces do not admit such structures as proven by Kamada \cite{22,23}.

Let $M$ be an Inoue surface of type $S^-$ defined via $g_0(z,w) = (-z,\alpha w)$ and $g_i$, $i = 1,2,3$, as in the proof of Theorem \cite{9} Assume that $M$ admits a locally conformally para-hyperkähler structure and denote by $\Omega'$ and $\theta'$ the $(2,0)$-form and the Lee form of this structure. For suitable $(p,q) \in \mathbb Z^2$ in the definition of Inoue surfaces, the group generated by $g_0^p, g_1, g_2, g_3$ defines an Inoue surface $M$
of type \( S^+ \) such that \( M \) is the quotient of \( \tilde{M} \) by the fixed point free involution \( \sigma \) determined by \( g_0 \) [19, p. 279]. Denote by \( \pi : \tilde{M} \rightarrow M \) the projection and set \( \tilde{\Omega} = \pi^* \Omega' \) and \( \tilde{\theta} = \pi^* \theta' \). Then \( d\tilde{\Omega} = \tilde{\theta} \wedge \tilde{\Omega} \) and \( d\tilde{\theta} = 0 \). Lift the forms \( \tilde{\Omega} \) and \( \tilde{\theta} \) to the universal covering \( \mathbb{C} \times \mathbb{H} \) of \( \tilde{M} \) and denote the forms obtained by the same symbols. If \( \alpha_1, \ldots, \alpha_4 \) are the \( G^+ \)-invariant 1-forms defined by (6), then \( \Omega = (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4) \) is a nowhere-vanishing \( G^+ \)-invariant \((2,0)\)-form on \( \mathbb{C} \times \mathbb{H} \). Hence \( \tilde{\Omega} = f \Omega \) for a complex-valued nowhere-vanishing smooth function. Then

\[
d\tilde{\Omega} = df \wedge \Omega + f d\Omega = \left( \frac{df}{f} - \alpha_4 \right) \wedge \tilde{\Omega}
\]

Since \( \mathbb{C} \times \mathbb{H} \) is simply connected, there is a smooth function \( g \) such that \( e^g = f \). If we set \( \psi = Im g \), then \( g = \ln |f| + i\psi \) and \( \frac{df}{f} = d\ln |f| + i\partial \psi \). We have \( \partial \psi \wedge \tilde{\Omega} = 0 \) since \( \tilde{\Omega} \) is of type \((2,0)\), thus

\[
\frac{df}{f} \wedge \tilde{\Omega} = (d \ln |f| + i\partial \psi - i\partial \psi) \wedge \tilde{\Omega} = (d \ln |f| + d^c \psi) \wedge \tilde{\Omega}
\]

Therefore

\[
\tilde{\theta} \wedge \tilde{\Omega} = (d \ln |f| + d^c \psi - \alpha_4) \wedge \tilde{\Omega}
\]

and it follows that

\[
\tilde{\theta} - (d \ln |f| + d^c \psi - \alpha_4) = 0 \quad (7)
\]

since the 1-form on the left-hand side of (7) is real-valued and \( \tilde{\Omega} \) is of type \((2,0)\). The function \( f \) is \( G^+ \)-invariant since \( \tilde{\Omega} \) and \( \Omega \) are invariant. Hence the function \( \ln |f| \) and the form \( d\psi \) are also \( G^+ \)-invariant as well as \( d^c \psi = -Jd\psi \), \( J \) being the complex structure (but the function \( \psi \) is not necessarily invariant). Consider \( \ln |f| \) and \( d^c \psi \) on the surface \( \tilde{M} \) and note that the form \( \tilde{\theta} - d \ln |f| \) is closed on \( \tilde{M} \). It is shown in [19] that the first Betti number of \( \tilde{M} \) is equal to 1. The form \( \alpha_4 \) considered on \( \tilde{M} \) is closed and nowhere-vanishing, hence not exact. Therefore there are a real constant \( C \) and a real-valued smooth function \( \eta \) on \( \tilde{M} \) such that

\[
\tilde{\theta} - d \ln |f| = C\alpha_4 + d\eta \quad (8)
\]

Then, by (7),

\[
d^c \psi = (C + 1)\alpha_4 + d\eta \quad (9)
\]

Applying the operator \( d^c \) to both sides, we obtain

\[
0 = (C + 1)d^c \alpha_4 + d^c d\eta.
\]

It follows from (6) that \( d^c \alpha_4 = \alpha_3 \wedge \alpha_4 \). Hence

\[
d^c d\eta = -(C + 1)\alpha_3 \wedge \alpha_4.
\]
Let \( h \) be an Hermitian metric on \( \tilde{M} \) and denote by \( \omega \) the fundamental 2-form of the Hermitian manifold \((\tilde{M}, h)\). Then
\[
h(d^\omega \eta, \omega) = -(C + 1)h(\alpha_3 \wedge \alpha_4, \omega).
\]
Extend \( J \) on 1-forms by \( J(\alpha)(X) = -\alpha(JX) \). Then \( J\alpha_3 = \alpha_4 \) and we get
\[
\Box_h \eta = -(C + 1)|\alpha_4|^2_h
\]
where \( \Box_h \) is the complex Laplacian. The right-hand side of the latter identity has a constant sign, hence \( \eta = \text{const} \) by the maximum principle. Therefore \( C = -1 \) and identity \((9)\) becomes \( d^\omega \psi = 0 \) on \( \tilde{M} \), hence \( d^\omega \psi = 0 \) on \( \mathbb{C} \times \mathbb{H} \). Therefore \( \psi = \text{const} = c \) and we have \( \tilde{\Omega} = |f|e^{i\phi} \Omega \). Consider the latter identity on \( \tilde{M} \). The form \( \tilde{\Omega} \) is \( \sigma \)-invariant, while \( \sigma^*(\Omega) = -\Omega \). It follows that \( |f \circ \sigma| = -|f| \). However \( |f| \) is positive everywhere, a contradiction.

Now we shall discuss the Inoue surfaces of type \( S^0 \) in a similar way. First recall their construction. Let \( A \in SL(3, \mathbb{Z}) \) be a matrix with two complex eigenvalues \( \alpha \) and \( \overline{\alpha} \), and a real eigenvalue \( c > 1 \). Choose eigenvectors \((\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 \) and \((c_1, c_2, c_3) \in \mathbb{R}^3 \) corresponding to \( \alpha \) and \( c \), respectively. Then the vectors \((\alpha_1, \alpha_2, \alpha_3), (\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}) \) and \((c_1, c_2, c_3) \) are \( \mathbb{C} \)-linearly independent. Let \( \Gamma \) be the group of automorphisms of \( \mathbb{C} \times \mathbb{H} \) generated by
\[
g_0: (z, w) \rightarrow (\alpha z, cw), \quad g_i: (z, w) \rightarrow (z + \alpha_i, w + c_i), \quad i = 1, 2, 3.
\]
Then \( S = (\mathbb{C} \times \mathbb{H})/\Gamma \) is an Inoue surface of type \( S^0 \) \([19]\).

Set \( w = u + iv, \quad a = \ln |\alpha|, \quad b = -\text{Arg} \alpha, \quad 0 < \text{Arg} \alpha \leq 2\pi, \) and \( t = \frac{\ln v}{\ln c} \). Define real vector fields \( E_1, \ldots, E_4 \) on \( \mathbb{C} \times \mathbb{H} \) by
\[
E_1 - iE_2 = 2\alpha i \frac{\partial}{\partial z}, \quad E_3 - iE_4 = 2v \ln c \frac{\partial}{\partial w}.
\]
These vector fields are \( \Gamma \)-invariant, hence they define \((1, 0)-\) vector fields on \( S \) which we denote by the same symbols. Thus, if \( J \) is the complex structure of \( S \), then \( JE_1 = E_2 \) and \( JE_3 = E_4 \). It is easy to check that
\[
[E_4, E_1] = aE_1 - bE_2, \quad [E_4, E_2] = bE_1 + aE_2, \quad [E_4, E_3] = -2aE_3
\]
and all other brackets vanish. Denote by \( \alpha_1, \ldots, \alpha_4 \) the dual frame of \( E_1, \ldots, E_4 \). Clearly, the 1-forms \( \alpha_i \) are \( \Gamma \)-invariant. Moreover, \((11)\) implies that
\[
da_1 = a\alpha_1 \wedge \alpha_4 + b\alpha_2 \wedge \alpha_4, \quad da_2 = -b\alpha_1 \wedge \alpha_4 + a\alpha_2 \wedge \alpha_4,
\]
\[
da_3 = -2a\alpha_3 \wedge \alpha_4, \quad da_4 = 0.
\]
Suppose that $S$ admits a locally conformally para-hyperkähler structure. Let $\Omega'$ and $\theta'$ be the $(2,0)$-form and the Lie form of this structure. Then $d\Omega' = \theta' \wedge \Omega'$ and $d\theta' = 0$. We lift $\Omega'$ and $\theta'$ to the universal covering $\mathbb{C} \times H$ of $S$ and denote the lifts by the same symbols. Set $\Omega = (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4)$. Then $\Omega' = f\Omega$ for a complex-valued nowhere-vanishing smooth function $f$ on $\mathbb{C} \times H$. We have $d\Omega' = df \wedge \Omega + f d\Omega$ where, in view of (12), $d\Omega = (-b\alpha_3 + a\alpha_4) \wedge \Omega$. Thus

$$d\Omega' = \left( \frac{df}{f} - b\alpha_3 + a\alpha_4 \right) \wedge \Omega'. \quad (13)$$

Since $\mathbb{C} \times H$ is simply connected, there is a smooth function $\psi$ such that $f = |f|e^{i\psi}$. We have $\partial\psi \wedge \Omega' = 0$ since $\Omega'$ is of type $(2,0)$. Then

$$\frac{df}{f} \wedge \Omega' = \left( d\ln |f| + i\partial\psi - i\partial\psi \right) \wedge \Omega' = \left( d\ln |f| + d^c\psi \right) \wedge \Omega'$$

and it follows from (13) that

$$\theta' \wedge \Omega' = \left( d\ln |f| + d^c\psi - b\alpha_3 + a\alpha_4 \right) \wedge \Omega'. \quad (14)$$

This implies

$$\theta' - d\ln |f| + d^c\psi - b\alpha_3 + a\alpha_4 = 0. \quad (15)$$

The form $\theta' - d\ln |f|$ is $\Gamma$-invariant, so we can consider it on the surface $S$. According to [19], $b_1(S) = 1$. The form $\alpha_4$ considered on $S$ is closed and is not exact. Therefore there are a real constant $C$ and a real-valued smooth function $\eta$ on $S$ such that

$$\theta' - d\ln |f| = C\alpha_4 + d\eta. \quad (15)$$

Identities (14) and (15) imply that

$$d^c\psi = b\alpha_3 + (C - a)\alpha_4 + d\eta. \quad (16)$$

Applying the operator $d^c$ to both sides, we obtain

$$0 = bd^c\alpha_3 + (C - a)d^c\alpha_4 + d^c d\eta.$$ 

It follows from (10) and (11) that $d^c\alpha_3 = 0$ and $d^c\alpha_4 = -2a\alpha_3 \wedge \alpha_4$. Thus

$$d^c d\eta = 2a(C - a)\alpha_3 \wedge \alpha_4.$$ 

Take an Hermitian metric $h$ on $S$ (for example that for which $E_1, \ldots, E_4$ is an orthonormal frame) and let $\omega$ be the fundamental 2-form of the Hermitian manifold $(S, h)$. Then

$$h(d^c d\eta, \omega) = 2a(C - a)h(\alpha_3 \wedge \alpha_4, \omega).$$
and, since $J\alpha_3 = \alpha_4$, and we get
\[ \Box h \eta = 2a(C - a)|\alpha_4|^2. \]
The latter identity implies $\eta = \text{const}$. Therefore $C = a$ and identity (16) takes the form $d^*\psi = b\alpha_3$. Since $d^*\psi = Jd\psi$, we get $d\psi = -b\alpha_4$. According to (16), $\alpha_4 = \frac{dv}{v \ln c}$, hence $d\psi = d(b \frac{\ln v}{\ln c})$. Therefore there is a constant $C'$ such that
\[ \exp(i\psi) = C' \exp(ib \frac{\ln v}{\ln c}). \]
Since $\exp(i\psi) = \frac{f}{|f|}$, the function on the right-hand side of the latter identity is $\Gamma$-invariant. In particular, this function is invariant under the transformation $g_0$, hence
\[ \exp(ib \frac{\ln(cv)}{\ln c}) = \exp(ib \frac{\ln v}{\ln c}). \]
This gives $\exp(ib) = 1$, hence $b = 2k\pi$ for an integer $k$. But this means that the eigenvalue $\alpha$ of the matrix $A$ is a real number, a contradiction. $Q.E.D.$

6 Para-hypercomplex surfaces

Now we shall use Theorem 8 to provide a list of the compact complex surfaces that could admit a para-hypercomplex structure.

**Theorem 11** Let $(M, J)$ be a compact complex surface admitting a para-hypercomplex structure. Then it is one of the surfaces listed in Theorem 8: a hyperelliptic surface, a secondary Kodaira surface, or an Enriques surface.

**Proof.** Every complex surface with a para-hypercomplex structure which does not admit a para-hyperhermitian structure has a double cover which admits a para-hyperhermitian structure compatible with the pull-back of the para-hypercomplex structure (Proposition 1). Then it follows from the list of possible para-hyperhermitian surfaces in Theorem 8 that we have to consider only that admitting holomorphic involutions and to identify the corresponding quotient surfaces. It is well known that a smooth quotient of a torus, a K3 surface, or a primary Kodaira surface is, respectively, a hyperelliptic surface, an Enriques surface, or a secondary Kodaira surface. Also the quotient of an Inoue surface with $b_2 = 0$ by a holomorphic involution is a surface of the same type. Note also that such a quotient of a Hopf surface is a Hopf surface since it has the same universal cover.

Let $\pi : M \to C$ be a properly elliptic surface with odd $b_1$ and base $C$ of genus $g > 1$. Then $M$ does not have singular fibers and multisections. If $M$ admits an
involution $\tau$, then $\tau$ transforms a fixed fiber $E$ into a curve $C'$. Any curve in $M$ which is not a fiber should project onto the whole base and hence should be a multisection, a contradiction. So the projection $\pi(C')$ is a point, hence the image $\tau(E) = C'$ is contained in a fiber. Since all fibers are irreducible smooth elliptic curves, $C'$ is again a fiber, possibly with different multiplicity. Then $\tau$ induces an involution $\tau'$ of the base $C$ and $M/\tau$ is elliptically fibred over $C/\tau'$ without singular fibers. It should have vanishing real first Chern class. Then it is either a properly elliptic surface, a Hopf surface or a Kodaira surface depending on the genus of $C/\tau'$. Q.E.D.

In [12] we have shown that every Inoue surface of type $S^-$ has a para-hypercomplex structure which does not admit a compatible para-hyperhermitian metric. Here we construct such a structure on a hypereleptic surface.

**Example.** Let $T^2 = \mathbb{C}/<1,i>$ be the complex torus with lattice generated by 1 and $i$. Denote by $\varphi$ the holomorphic involution of $T^2 \times T^2$ defined by $(z, w) \mapsto (z + \frac{1}{2}, -w)$. Then the quotient $M$ of $N = T^2 \times T^2$ by the group generated by $\varphi$ is a hypereleptic surfaces. Let $z = x + iy$, resp. $w = u + iv$ be the local coordinate on the first, resp. the second factor of $N$ induced by the standard complex coordinate of $\mathbb{C}$. Then

$$K^+ = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial u}\}, \quad K^- = \text{span}\{\frac{\partial}{\partial y}, \frac{\partial}{\partial v}\}$$

are $\varphi$-invariant and involutive subbundles of the tangent bundle $TN$. Define an isomorphism $S$ of $TN$ setting $S = +\text{Id}$ on $K^+$ and $S = -\text{Id}$ on $K^-$. Let $I$ be the complex structure of $N$ and set $T = IS$. In this way we obtain a para-hypercomplex structure on $N$ which descends to $M = N/\langle \varphi \rangle$. This structure does not admit a compatible metric since otherwise, as we have seen, the canonical bundle of the hypereleptic surface $M$ would be topologically trivial, a contradiction with Theorem 8.

7 Nonexistence of para-hyperhermitian structures on Inoue surfaces of type $S^0$

It is well-known [13] that every Inoue surface of type $S^0$ is a solvmanifold, i.e. the quotient of a solvable Lie group by a cocompact subgroup. Note also that the 4-dimensional solvable Lie algebras admitting para-hypercomplex structures have been classified in [8]. This together with the identities (11) above implies that the Inoue surfaces of type $S^0$ do not admit para-hyperhermitian structures induced by left invariant ones. In this section we shall slightly generalize this observation by showing that these surfaces have no para-hyperhermitian structures whose $(2,0)$-forms are defined by left invariant 2-forms. This leads to the natural conjecture
that the Inoue surfaces of type \( S^0 \) do not admit para-hyperhermitian structures at all.

**Theorem 12** Let \( S \) be an Inoue surface of type \( S^0 \) which is a quotient of a solvable Lie group \( G \). Then \( S \) has no para-hyperhermitian structure whose \((2,0)\)-form is defined by a left invariant 2-form on \( G \).

**Proof.** We shall use the notation introduced in the proof of Theorem 10. To prove the theorem, we have to consider the class of para-hyperhermitian structures on \( S \) whose \((2,0)\)-form \( \Omega_2 + i\Omega_3 \) is given (up to a constant) by

\[
\Omega_2 + i\Omega_3 = (\alpha_1 + i\alpha_2) \wedge (\alpha_3 + i\alpha_4),
\]

i.e.

\[
\Omega_2 = \alpha_1 \wedge \alpha_3 - \alpha_2 \wedge \alpha_4, \quad \Omega_3 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3.
\]

Since \( \Omega_1 \) is a real \((1,1)\)-form with respect to \( J \) it has the form

\[
\Omega_1 = p\alpha_1 \wedge \alpha_2 + q(\alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4) + r(\alpha_1 \wedge \alpha_4 - \alpha_2 \wedge \alpha_3) + s\alpha_3 \wedge \alpha_4,
\]

where \( p, q, r, s \) are smooth functions on \( S \). Further on, we shall consider the smooth functions on \( S \) as smooth \( \Gamma \)-invariant functions on \( \mathbb{C} \times \mathbb{H} \). Denote by \( \theta \) the Lie form of the para-hyperhermitian structure on \( S \) and set \( f = q + ir \). Then a direct but long computation using (10) and (12) shows that the identities in Proposition 9 are satisfied if and only if \( \theta = -b\alpha_3 + a\alpha_4 \), \( p = 0 \), \( |f|^2 = 1 \) and

\[
\frac{\partial f}{\partial z} = 0, \quad d\mathbb{I}(c^l \frac{\partial}{\partial w}) = -2bf - ids(\alpha^l \frac{\partial}{\partial z}).
\]

Now differentiating the identity \( |f|^2 = 1 \) with respect to \( \bar{z} \) we get \( \frac{\partial f}{\partial \bar{z}} = 0 \). Hence the function \( f \) depends only on \( u \) and \( v \) and satisfies the identity

\[
\ln c \frac{\partial f}{\partial \bar{w}} = -bf - ids(\alpha^l \frac{\partial}{\partial z}). \quad (17)
\]

Next we shall need the following

**Lemma 13** Let \( F \) be a continuous function on \( \mathbb{C} \times \mathbb{H} \) which is invariant under the action of \( \Gamma \) and depends only on \( u \) and \( v \). Then \( F \) depends only on \( v \).

**Proof.** The invariance of \( F \) implies that

\[
F(u + xc_1 + yc_2 + zc_3, v) = F(u, v) \quad (18)
\]
for arbitrary \( x, y, z \in \mathbb{Z} \). Note that at least two of the numbers \( c_1, c_2, c_3 \) are nonzero and since the eigenvalue \( c \) is irrational we may assume that the ratio \( \frac{c_1}{c_2} \) is irrational too. Then the Kronecker lemma implies that the set \( \{xc_1 + yc_2 \mid x, y \in \mathbb{Z}\} \) is dense in \( \mathbb{R} \) and by continuity we get that \( F(u + w, v) = F(u, v) \) for arbitrary \( u, v, w \in \mathbb{R} \). Hence \( F \) depends only on \( v \). Q. E. D.

The lemma above implies that the function \( f \) depends only on \( v \) and it follows from (17) that the same is also true for \( \partial s \). Since \( s \) is a real function we have

\[
\frac{\partial s}{\partial z} = \overline{A} \quad \text{and therefore}
\]

\[
s = zA + \overline{A} + \gamma(u, v). \tag{19}
\]

The invariance of \( s \) implies that

\[
s(z + x\alpha_1 + y\alpha_2 + t\alpha_3, z + x\alpha_1 + y\alpha_2 + t\alpha_3, u + xc_1 + yc_2 + tc_3, v) = s(z, z, u, v)
\]

for all \( x, y, t \in \mathbb{Z} \) and it follows from (19) that

\[
xA_1 + yA_2 + tA_3 = \gamma(u, v) - \gamma(u + xc_1 + yc_2 + tc_3, v), \tag{20}
\]

where \( A_i = \alpha_i A + \overline{\alpha_i}A, \ i = 1, 2, 3 \).

We shall show now that \( A = 0 \). To do this we consider two cases.

**Case 1.** \( c_1c_2c_3 \neq 0 \).

Take sequences \( \{x_n\} \) and \( \{y_n\} \) of integers such that \( x_n \frac{\alpha_1}{\alpha_2} + y_n \) tends to a rational number \( \omega \). Then setting \( x = x_n, y = y_n, t = 0 \) and \( u = 0 \) in (20) gives

\[
x_nA_1 + y_nA_2 = \gamma(0, v) - \gamma(x_nc_1 + y_n\alpha_2, v).
\]

Hence \( x_nA_1 + y_nA_2 \) tends to \( B = \gamma(0, v) - \gamma(c_2\omega, v) \). We may assume without loss of generality that \( A_2 > 0 \). Then there exists \( N \) such that for every \( n > N \) we have

\[
|x_n \frac{\alpha_1}{\alpha_2} + y_n - \omega| < 1.
\]

On the other hand there exists \( M \) such that for every \( n > M \) we have

\[
|x_nA_1 + y_nA_2 - B| < 1.
\]

The last two inequalities imply that for \( n > \max(M, N) \) we have

\[
B + 1 - A_2(1 + \omega) < x_n(A_1 - \frac{A_2\alpha_1}{\alpha_2}) < B + 1 - A_2(1 - \omega).
\]
Suppose that $A_1 - \frac{A_2 c_1}{c_2} \neq 0$. Since $x_n$ are integers this inequality implies that $x_n$ take a finite number of values and the same is also true for $y_n$. But the sequence $\{x_n c_1 + y_n\}$ is convergent and therefore its limit is equal to a term of it, a contradiction since the number $\frac{c_1}{c_2}$ is irrational. Thus $A_1(\omega) = \frac{c_1}{c_2} A_2(\omega)$ for every $\omega \in \mathbb{Q}$, hence for every $\omega \in \mathbb{R}$. The same reasoning shows that the vectors $(A_1, A_2, A_3)$ and $(c_1, c_2, c_3)$ are collinear and therefore the vectors $(A_1, A_2, A_3), (\bar{A}_1, \bar{A}_2, \bar{A}_3)$ and $(c_1, c_2, c_3)$ are $C$-linearly dependent. Hence $A = 0$.

Case 2. $c_1 c_2 c_3 = 0$.

We may assume that $c_3 = 0$, $c_1 c_2 \neq 0$. Then, applying (20) for $x = y = 0$, $t = 1$, we get $A_3 = 0$. Since $c_2 \neq 0$ the same reasoning as in Case 1 implies that the vectors $(A_1, A_2, A_3)$ and $(c_1, c_2, c_3)$ are collinear and we get again that $A = 0$.

Now the equation (17) takes the form

$$v \ln c \frac{\partial f}{\partial v} = 2i\beta f.$$

Since $|f| = 1$ and $f$ depends only on $v$, it follows that $f = e^{i\varphi}$, where $g$ is a smooth real-valued function on $\mathbb{R}_+$. Then the latter equation takes the form

$$\frac{\partial g}{\partial v} = \frac{2b}{v \ln c}$$

which shows that

$$g = \frac{2b \ln v}{\ln c} + g_0,$$

where $g_0$ is a constant. Therefore

$$f(v) = f_0 \exp\left(\frac{2ib \ln v}{\ln c}\right),$$

where $f_0$ is a constant with $|f_0| = 1$. Now the invariance of $f$ under $\Gamma$ implies that $\beta = k\pi, k \in \mathbb{Z}$. Hence $\alpha$ is a real number, a contradiction. Q.E.D.

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