CONLEY CONJECTURE FOR NEGATIVE MONOTONE
SYMPLECTIC MANIFOLDS

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Dedicated to Edi Zehnder on the occasion of his seventieth birthday

Abstract. We prove the Conley conjecture for negative monotone, closed symplectic manifolds, i.e., the existence of infinitely many periodic orbits for Hamiltonian diffeomorphisms of such manifolds.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper, we establish the Conley conjecture for negative monotone, closed symplectic manifolds. More specifically, we show that a Hamiltonian diffeomorphism of such a manifold has infinitely many periodic orbits.

The Conley conjecture was formulated by Charles Conley in 1984 for the torus (see [Co]), and since then the conjecture has been a subject of active research focusing on establishing the existence of infinitely many periodic orbits for broader and broader class of symplectic manifolds or Hamiltonian diffeomorphisms. Note
that the example of an irrational rotation of $S^2$ shows that the Conley conjecture does not hold unconditionally – some assumptions on the manifold or the Hamiltonian diffeomorphism are required – in contrast with, say, the Arnold conjecture; see [GK] for more examples of this type.

The conjecture was proved for the so-called weakly non-degenerate Hamiltonian diffeomorphisms in [SZ] (see also [CZ]) and for all Hamiltonian diffeomorphisms of surfaces other than $S^2$ in [FrHa] (see also [LeC]). In its original form, as stated in [Co] for the torus, the conjecture was established in [Hi] and the case of an arbitrary closed, symplectically aspherical manifold was settled in [Gi2]. The proof was extended to rational, closed symplectic manifolds $M$ with $c_1(TM)|_{\pi_2(M)} = 0$ in [GG1] and the rationality requirement was eliminated in [He1]. Thus, after [SZ], the main difficulty in establishing the Conley conjecture for more and more general manifolds with spherically vanishing first Chern class, overcome in this series of works, lied in proving the conjecture for totally degenerate Hamiltonian diffeomorphisms not covered by [SZ]. (The internal logic in [FrHa, LeC], relying on low-dimensional dynamics methods, is somewhat different.)

Two other variants of the Conley conjecture have also been investigated. One is the Conley conjecture for Hamiltonian diffeomorphisms with displaceable support; see, e.g., [FS, Gi, HZ, Sc, Vi]. Here the manifold $M$ is required to be symplectically aspherical, but not necessarily closed. The second one is the Lagrangian Conley conjecture or more generally the Conley conjecture for Hamiltonians with controlled behavior at infinity on cotangent bundles; see [He2, Lo2, Lu, Ma].

As has been pointed out above, some requirement on $M$ is necessary for the Conley conjecture to hold, but this requirement need not necessarily be the condition that $c_1(TM)|_{\pi_2(M)} = 0$. One hypothetical replacement of this condition, conjectured by the second author of this paper, is that the minimal Chern number $N$ of $M$ is sufficiently large, e.g., $N > \dim M$. (The condition $c_1(TM)|_{\pi_2(M)} = 0$ corresponds to $N = \infty$.) More generally, it might be sufficient to require that the Gromov–Witten invariants of $M$ vanish, as suggested by Michael Chance and Dusa McDuff, or even that the quantum product is undeformed. No results in this direction have been proved to date. Alternatively, one may require $M$ to be negative monotone and this is the case we consider in the present paper. (Our result almost, but not quite, fits within the scope of the Chance-McDuff conjecture: it is easy to see that the Gromov–Witten invariants of a negative monotone manifold $M$ vanish when $N > \dim M/2$; cf. [LO].)

A major difficulty in proving the conjecture when $c_1(TM)|_{\pi_2(M)} \neq 0$ lies in establishing the weakly non-degenerate, or even non-degenerate, case. For the totally degenerate case is settled in [GG1] without any requirements on the Chern class for rational manifolds. (Interestingly, once the rationality condition is dropped, vanishing of the first Chern class becomes again essential for the proof, [He1].) Our proof relies on keeping track of the behavior of both the index and the action under iterations along the lines of the reasoning from [GG1], in contrast with the argument from [SZ] making use only of the index, and on the fact that for a negative monotone manifold the index and action change in the opposite ways under recapping. The proof also crucially relies on the subadditivity of the action selector with respect to the pair-of-pants product. This part of the proof is reminiscent of the argument for Hamiltonians with displaceable support in [Sc, Vi] and has no corresponding counterpart in other proofs of the Conley conjecture; cf. [SZ, Hi, Gi2, GG1, He1].
On the technical level, the present work draws heavily on [GG1] and can be thought of as a follow up of that paper. We also use the machinery developed in [GG1] to relate, in the symplectically aspherical case, the growth of orbits to the decay of mean indices for a certain type of periodic orbits.

1.2. Results. Recall that a symplectic manifold \((M, \omega)\) is called negative monotone if \([\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)}\) for some \(\lambda < 0\). Negative monotone symplectic manifolds exist in abundance. A standard example is the hypersurface \(z^m_0 + \ldots + z^m_n = 0\) in \(\mathbb{C}P^n\), where \(m > n + 1\); see, e.g., [MS, pp. 429–430] for the case of \(n = 4\).

**Theorem 1.1.** Let \(\varphi\) be a Hamiltonian diffeomorphism of a closed, negative monotone symplectic manifold. Then \(\varphi\) has infinitely many periodic orbits.

**Remark 1.2.** Note that this theorem is somewhat weaker than the versions of the Conley conjecture established in the case where \(c_1(TM)|_{\pi_2(M)} = 0\). Namely, while here we prove only the existence of infinitely many periodic orbits, the results for \(c_1(TM)|_{\pi_2(M)} = 0\) assert the existence of simple periodic orbits with arbitrarily large period, provided that the fixed point set is finite.

Furthermore, the proof of Theorem 1.1 utilizes Hamiltonian Floer theory. Hence, unless \(M\) is required to be weakly monotone, the argument ultimately, although not explicitly, relies on the machinery of multi-valued perturbations and virtual cycles; see Remark 2.1 for further discussion.

As has been pointed out above, the other result of this paper, also established using some of the machinery utilized in the proof of Theorem 1.1, relates the growth of periodic orbits to the decay of the mean indices for a certain type of periodic orbits (the so-called carriers of the action selector) for symplectically aspherical manifolds. In particular, we show that whenever the mean indices of the carriers are bounded away from zero, the number of simple periodic orbits grows linearly with the order of iteration.

The paper is organized as follows. In Section 2, we set our conventions and notation and discuss some standard notions and results from symplectic topology, needed for the proof of the main theorem. These include the mean index, the filtered and local Floer homology, the action selector, and their properties. The main objective of Section 3 is to prove Theorem 1.1. The proof hinges on the notion of a carrier of the action selector discussed in Section 3.1. The relation between the growth of orbits and the decay of mean indices is established in Section 3.3.

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2. Preliminaries

The goal of this section is to set notation and conventions and to give a brief review of Floer homology and several other notions used in the paper.
2.1. Conventions and notation. Let \((M^{2n}, \omega)\) be a closed symplectic manifold. Throughout the paper we will usually assume, for the sake of simplicity, that \(M\) is rational, i.e., the group \(\langle [\omega], \pi_2(M) \rangle \subset \mathbb{R}\) formed by the integrals of \(\omega\) over the spheres in \(M\) is discrete. This condition is obviously satisfied when \(M\) is negative monotone as in Theorem 1.1 or monotone, i.e., \([\omega]\mid_{\pi_2(M)} = \lambda c_1(M)\mid_{\pi_2(M)}\) for some \(\lambda < 0\) in the former case or \(\lambda \geq 0\) in the latter. Recall also that \(M\) is called symplectically aspherical if \([\omega]\mid_{\pi_2(M)} = 0 = c_1(M)\mid_{\pi_2(M)}\).

All Hamiltonians \(H\) on \(M\) considered in this paper are assumed to be \(k\)-periodic in time, i.e., \(H : S^1_k \times M \to \mathbb{R}\), where \(S^1_k = \mathbb{R}/k\mathbb{Z}\), and the period \(k\) is always a positive integer. When the period is not specified, it is equal to one, which is the default period in this paper. We set \(H_t = H(t, \cdot)\) for \(t \in S^1 = \mathbb{R}/\mathbb{Z}\). The Hamiltonian vector field \(X_H\) of \(H\) is defined by \(i_{X_H} \omega = -dH\). The (time-dependent) flow of \(X_H\) will be denoted by \(\varphi^t_H\) and its time-one map by \(\varphi_H\). Such time-one maps are referred to as Hamiltonian diffeomorphisms. A one-periodic Hamiltonian \(H\) can always be treated as \(k\)-periodic. In this case, we will use the notation \(H^\# k\) and, abusing terminology, call \(H^\# k\) the \(k\)th iteration of \(H\).

Let \(K\) and \(H\) be one-periodic Hamiltonians such that \(K_1 = H_0\) and \(H_1 = K_0\). We denote by \(K\# H\) the two-periodic Hamiltonian equal to \(K_t\) for \(t \in [0, 1]\) and \(H_{t-1}\) for \(t \in [1, 2]\). Thus, \(H^\# k = H \# \ldots \# H\) \((k\) times).

Let \(x : S^1_k \to W\) be a contractible loop. A capping of \(x\) is a map \(u : D^2 \to M\) such that \(u|_{S^1_k} = x\). Two cappings \(u\) and \(v\) of \(x\) are considered to be equivalent if the integrals of \(\omega\) and \(c_1(TM)\) over the sphere obtained by attaching \(u\) to \(v\) are equal to zero. A capped closed curve \(\bar{x}\) is, by definition, a closed curve \(x\) equipped with an equivalence class of cappings. In what follows, the presence of capping is always indicated by a bar.

The action of a one-periodic Hamiltonian \(H\) on a capped closed curve \(\bar{x} = (x, u)\) is defined by

\[
A_H(\bar{x}) = -\int_u \omega + \int_{S^1} H_t(x(t)) \, dt.
\]

The space of capped closed curves is a covering space of the space of contractible loops and the critical points of \(A_H\) on the covering space are exactly capped one-periodic orbits of \(X_H\). The action spectrum \(S(H)\) of \(H\) is the set of critical values of \(A_H\). This is a zero measure set; see, e.g., [HZ, Sc]. When \(M\) is rational, \(S(H)\) is a closed, and hence nowhere dense, set. Otherwise, \(S(H)\) is dense in \(\mathbb{R}\). These definitions extend to \(k\)-periodic orbits and Hamiltonians in an obvious way. Clearly, the action functional is homogenous with respect to iteration:

\[
A_{H^\# k}(\bar{x}^k) = k A_H(\bar{x}).
\]

Here \(\bar{x}^k\) stands for the \(k\)th iteration of the capped orbit \(\bar{x}\).

All results of this paper concern only contractible periodic orbits and throughout of the paper a periodic orbit is always assumed to be contractible, even if this is not explicitly stated.

A periodic orbit \(x\) of \(H\) is said to be non-degenerate if the linearized return map \(d\varphi_H : T_{x(0)}W \to T_{x(0)}W\) has no eigenvalues equal to one. Following [SZ], we call \(x\) weakly non-degenerate if at least one of the eigenvalues is different from one. A Hamiltonian is non-degenerate if all its one-periodic orbits are non-degenerate.

Let \(\bar{x}\) be a non-degenerate (capped) periodic orbit. The Conley–Zehnder index \(\mu_{CZ}(\bar{x}) \in \mathbb{Z}\) is defined, up to a sign, as in [Sa, SZ]. (Sometimes, we will also use the notation \(\mu_{CZ}(H, \bar{x})\).) More specifically, in this paper, the Conley–Zehnder index is
the negative of that in [Sa]. In other words, we normalize $\mu_{cz}$ so that $\mu_{cz}(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian. The mean index $\Delta_H(\bar{x}) \in \mathbb{R}$ measures, roughly speaking, the total angle swept by certain eigenvalues with absolute value one of the linearized flow $dF^t_H$ along $x$ with respect to the trivialization associated with the capping; see [Lo1, SZ]. The mean index is defined regardless of whether $x$ is degenerate or not and $\Delta_H(\bar{x})$ depends continuously on $H$ and $\bar{x}$ in the obvious sense. When $x$ is non-degenerate, we have

$$0 < |\Delta_H(\bar{x}) - \mu_{cz}(H, \bar{x})| < n.$$  \hspace{1cm} (2.1)

Furthermore, the mean index is homogeneous with respect to iteration:

$$\Delta_{H^{\pm k}}(\bar{x}^k) = k\Delta_H(\bar{x}).$$

2.2. Floer homology. In this subsection, we very briefly recall, mainly to set notation, the construction of the filtered Floer homology. We refer the reader to, e.g., [HS, MS, Sa, SZ] and also [FO, LT] for detailed accounts and additional references.

Fix a ground field $\mathbb{F}$. Let $H$ be a non-degenerate Hamiltonian on $M$. Denote by $\text{CF}_k^{(-\infty, b)}(H)$, where $b \in (-\infty, \infty]$ is not in $S(H)$, the vector space of formal sums

$$\sigma = \sum_{\bar{x} \in \mathcal{P}(H)} \alpha_{\bar{x}} \bar{x}.$$ 

Here $\alpha_{\bar{x}} \in \mathbb{F}$ and $\mu_{cz}(\bar{x}) = k$ and $\mathcal{A}_H(\bar{x}) < b$. Furthermore, we require, for every $a \in \mathbb{R}$, the number of terms in this sum with $\alpha_{\bar{x}} \neq 0$ and $\mathcal{A}_H(\bar{x}) > a$ to be finite. The graded $\mathbb{F}$-vector space $\text{CF}_k^{(-\infty, b)}(H)$ is endowed with the Floer differential counting the anti-gradient trajectories of the action functional; see, e.g., [HS, MS, On, Sa] and also [FO, LT]. Thus, we obtain a filtration of the total Floer complex $\text{CF}_*(H) := \text{CF}_k^{(-\infty, \infty)}(H)$. Furthermore, we set $\text{CF}_{(a, b)}(H) := \text{CF}_k^{(-\infty, b)}(H) / \text{CF}_k^{(-\infty, a)}(H)$, where $-\infty < a < b \leq \infty$ are not in $S(H)$. The resulting homology, the filtered Floer homology of $H$, is denoted by $\text{HF}^{(a, b)}_*(H)$ and by $\text{HF}_*(H)$ when $(a, b) = (-\infty, \infty)$.

Working over $\mathbb{F} = \mathbb{Z}_2$, we can view a chain $\sigma = \sum \alpha_{\bar{x}} \bar{x} \in \text{CF}_*(H)$ as simply a collection of periodic orbits $\bar{x}$ for which $\alpha_{\bar{x}} \neq 0$. In general, will say that $\bar{x}$ enters the chain $\sigma$ when $\alpha_{\bar{x}} \neq 0$. Note also that every $\mathbb{F}$-vector space $\text{CF}_k(H)$ is finite-dimensional when $M$ is negative monotone or monotone with $\lambda > 0$.

The total Floer complex and homology are modules over the Novikov ring $\Lambda$. In this paper, the latter is defined as follows. Let $\omega(A)$ and $\langle c_1(TM), A \rangle$ denote the integrals of $\omega$ and, respectively, $c_1(TM)$ over a cycle $A$. Set

$$I_\omega(A) = -\omega(A) \text{ and } I_{c_1}(A) = -2 \langle c_1(TM), A \rangle,$$

where $A \in \pi_2(M)$. For instance,

$$I_\omega = \frac{\lambda}{2} I_{c_1}$$

when $M$ is monotone or negative monotone. In particular, $I_\omega(A)$ and $I_{c_1}(A)$ have opposite signs when $M$ is negative monotone. Let

$$\Gamma = \frac{\pi_2(M)}{\ker I_\omega \cap \ker I_{c_1}}.$$ 

Thus, $\Gamma$ is the quotient of $\pi_2(M)$ by the equivalence relation where the two spheres $A$ and $A'$ are considered to be equivalent if $\omega(A) = \omega(A')$ and $\langle c_1(TM), A \rangle = \langle c_1(TM), A' \rangle$. 


\[ \langle c_1(TM), A' \rangle \). For example, \( \Gamma \simeq \mathbb{Z} \) when \( M \) is negative monotone or monotone with \( \lambda \neq 0 \). The homomorphisms \( I_x \) and \( I_{c_1} \) descend to \( \Gamma \) from \( \pi_2(M) \).

The group \( \Gamma \) acts on \( CF_*(H) \) and on \( HF_*(H) \) via recapping: an element \( A \in \Gamma \) acts on a capped one-periodic orbit \( \bar{x} \) of \( H \) by attaching the sphere \( A \) to the original capping. We denote the resulting capped orbit by \( \bar{x}#A \). Then,

\[ \mu_{cz}(\bar{x}#A) = \mu_{cz}(\bar{x}) + I_{c_1}(A) \text{ and } \mathcal{A}_H(\bar{x}#A) = \mathcal{A}_H(\bar{x}) + I_\omega(A). \]

Note that in a similar vein we also have

\[ \Delta_H(\bar{x}#A) = \Delta_H(\bar{x}) + I_{c_1}(A), \]

regardless of whether \( x \) is non-degenerate or not.

The Novikov ring \( \Lambda \) is a certain completion of the group ring of \( \Gamma \) over \( \mathbb{F} \). Namely, \( \Lambda \) is comprised of formal linear combinations \( \sum \alpha_A e^A \), where \( \alpha_A \in \mathbb{F} \) and \( A \in \Gamma \), such that for every \( a \in \mathbb{R} \) the sum contains only finitely many terms with \( I_\omega(A) > a \) and, of course, \( \alpha_A \neq 0 \). The Novikov ring \( \Lambda \) is graded by setting \( \deg(e^A) = I_{c_1}(A) \) for \( A \in \Gamma \). The action of \( \Gamma \) turns \( CF_*(H) \) and \( HF_*(H) \) into \( \Lambda \)-modules.

The definition of Floer homology extends to all, not necessarily non-degenerate, Hamiltonians by continuity. Let \( H \) be an arbitrary (one-periodic in time) Hamiltonian on \( M \) and let the end points \( a \) and \( b \) of the action interval be outside \( S(H) \). By definition, we set

\[ HF_*^{(a,b)}(H) = HF_*^{(a,b)}(\hat{H}), \]

where \( \hat{H} \) is a non-degenerate, small perturbation of \( H \). It is well known that here the right hand side is independent of \( \hat{H} \) as long as the latter is sufficiently close to \( H \). Working with filtered Floer homology, we will always assume that the end points of the action interval are not in the action spectrum. (At this point the background assumption that \( M \) is rational becomes essential; see \([He1]\) for the irrational case and also \([GG1, \text{Remark 2.3}]\).)

The total Floer homology is independent of the Hamiltonian and, up to a shift of the grading and the effect of recapping, is isomorphic to the homology of \( M \). More precisely, we have

\[ HF_*(H) \cong H_{*+n}(M; \mathbb{F}) \otimes \Lambda \]
as graded \( \Lambda \)-modules.

**Remark 2.1.** We conclude this discussion by recalling that in order for the Floer differential to be defined certain regularity conditions must be satisfied generically. To ensure this, we have to either require \( M \) to be weakly monotone (see \([HS, MS, On, Sa]\)) or utilize the machinery of virtual cycles (see \([FO, FOOO, LT]\) or, for the polyfold approach, \([HWZ1, HWZ2]\) and references therein). In the latter case the ground field \( \mathbb{F} \) is required to have zero characteristic. Here we are primarily interested in negative monotone manifolds. Such a manifold is weakly monotone if and only if \( N \geq n - 2 \), where \( N \) is the minimal Chern number, i.e., the positive generator of \( \langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z} \).

### 2.3. Local Floer homology

In this section, we briefly recall the definition and basic properties of local Floer homology, following mainly \([GG1]\) although this notion goes back to the original work of Floer (see, e.g., \([Fl1, Fl2]\)) and has been revisited a number of times since then; see, e.g., \([Po, \text{Section 3.3.4}]\) and \([Gi2, GG2, He1]\).
Let $\bar{x}$ be a capped isolated one-periodic orbit of a Hamiltonian $H: S^1 \times M \to \mathbb{R}$. Pick a sufficiently small tubular neighborhood $U$ of $x$ and consider a non-degenerate $C^2$-small perturbation $\tilde{H}$ of $H$. (Strictly speaking $U$ is a neighborhood of the graph of the orbit in the extended phase space $S^1 \times M$.) The orbit $x$ splits into non-degenerate one-periodic orbits of $\tilde{H}$, which are $C^1$-close to $x$ and hence contained in $U$. The capping of $\bar{x}$ gives rise to the cappings of these orbits in an obvious way and the action of $\tilde{H}$ on the resulting capped orbits is close to $A_H(\bar{x})$.

Every (anti-gradient) Floer trajectory $u$ connecting such capped one-periodic orbits of $\tilde{H}$ lying in $U$ is also contained in $U$, provided that $\|H - \tilde{H}\|_{C^2}$ is small enough. Thus, by the compactness and gluing theorems, every broken anti-gradient trajectory connecting two such orbits also lies entirely in $U$. Similarly to the definition of the ordinary Floer homology, consider the complex $\text{CF}_*(\tilde{H}, \bar{x})$ over $\mathbb{F}$ generated by the capped one-periodic orbits of $\tilde{H}$ in $U$ with cappings inherited from $\bar{x}$, graded by the Conley–Zehnder index and equipped with the Floer differential defined in the standard way. The continuation argument (see, e.g., [MS, SZ]) shows that the homology of this complex is independent of the choice of $\tilde{H}$ and of other auxiliary data (e.g., an almost complex structure). We refer to the resulting homology group $HF_*(\tilde{H}, \bar{x})$ as the local Floer homology of $H$ at $\bar{x}$. This definition is a Floer theoretic analogue of the local Morse homology or critical modules; see [GM] and also, e.g., [Lo1, Section 14.1] and references therein.

**Example 2.2.** Assume that $\bar{x}$ is non-degenerate and $\mu_{cz}(\bar{x}) = k$. Then $HF_l(H, \bar{x}) = \mathbb{F}$ when $l = k$ and $HF_l(H, \bar{x}) = 0$ otherwise.

**Remark 2.3.** In [Gi2, GG1, GG2], the perturbation $\tilde{H}$ is required to be supported in $U$ or, more precisely, $\text{supp}(H - \tilde{H}) \subset U$. This requirement is immaterial and we omit it in this paper. Furthermore we would like to emphasize that the above construction is genuinely local. At no point it depends on global properties of $M$ (such as weak monotonicity) or requires virtual cycles. In fact, it suffices to have the Hamiltonian $H$ defined only on a neighborhood of the orbit. However, a capping of the orbit is still essential or, at least, a trivialization of $TM$ along the orbit must be fixed.

Let us now briefly review the basic properties of local Floer homology that are essential for what follows, assuming for the sake of simplicity that $M$ is rational.

Just as the ordinary Floer homology, the local Floer homology is homotopy invariant. To be more precise, let $H_s$, $s \in [0, 1]$, be a family of Hamiltonians such that $x$ is a uniformly isolated one-periodic orbit for all $H_s$, i.e., $x$ is the only periodic orbit of $H_s$ for all $s$, in some open set independent of $s$. Then $HF_*(H^s, \bar{x})$ is constant throughout the family: $HF_*(H^0, \bar{x}) = HF_*(H^1, \bar{x})$.

Local Floer homology spaces are building blocks for filtered Floer homology. Namely, assume that all one-periodic orbits $x$ of $H$ are isolated and $HF_k(H, \bar{x}) = 0$ for some $k$ and all $\bar{x}$. Then $HF_k(H) = 0$. Moreover, let $c \in \mathbb{R}$ be such that all capped one-periodic orbits $\bar{x}_i$ of $H$ with action $c$ are isolated. (As a consequence, there are only finitely many orbits with action close to $c$.) Then, if $c > 0$ is small enough,

$$HF_*(H^{c-\epsilon, c+\epsilon})(H) = \bigoplus_i HF_*(H, \bar{x}_i).$$

By definition, the support of $HF_*(H, \bar{x})$ is the collection of integers $k$ such that $HF_k(H, \bar{x}) \neq 0$. By (2.1) and continuity of the mean index, $HF_*(H, \bar{x})$ is supported
in the range \([\Delta_H(\bar{x}) - n, \Delta_H(\bar{x}) + n]\). Moreover, when \(x\) is weakly non-degenerate, the support is contained in the open interval \((\Delta_H(\bar{x}) - n, \Delta_H(\bar{x}) + n)\); cf. [SZ].

2.4. Action selectors. The theory of Hamiltonian action selectors or spectral invariants, as they are usually referred to, was developed in its present Floer–theoretic form in [Oh, Sc] although the first versions of the theory go back to [HZ, Vi]. (See also, e.g., [EP, FS, FGS, Gi1, Gi2, MS, U1, U2] and references therein for other versions, some of the applications and a detailed discussion of the theory; here, we mainly follow [GG1].)

Let \(M\) be a closed symplectic manifold and let \(H\) be a Hamiltonian on \(M\). We assume that \(M\) is rational – this assumption greatly simplifies the theory (cf. [U1]) and is obviously satisfied for negative monotone manifolds.

The action selector \(c\) associated with the fundamental class \([M] \in H_{2n}(M; \mathbb{F}) \subset HF_n(H)\) is defined as

\[
c(H) = \inf\{a \in \mathbb{R} \setminus S(H) \mid u \in \text{im}(i^a)\} = \inf\{a \in \mathbb{R} \setminus S(H) \mid j^a(u) = 0\},
\]

where \(i^a: HF(\mathbb{R}, \mathbb{R}) \to HF(a, H)\) and \(j^a: HF(a, H) \to HF(\mathbb{R}, \mathbb{R})\) are the natural “inclusion” and “quotient” maps. Then \(c(H) > -\infty\) as is easy to see; [Oh].

The action selector \(c\) has the following properties:

- (AS1) Normalization: \(c(H) = \max H\) if \(H\) is autonomous and \(C^2\)-small.
- (AS2) Continuity: \(c\) is Lipschitz in \(H\) in the \(C^0\)-topology.
- (AS3) Monotonicity: \(c(H) \geq c(K)\) whenever \(H \geq K\) pointwise.
- (AS4) Hamiltonian shift: \(c(H + a(t)) = c(H) + \int_0^1 a(t) dt\), where \(a: S^1 \to \mathbb{R}\).
- (AS5) Symplectic invariance: \(c(H) = c(\varphi^*H)\) for any symplectomorphism \(\varphi\).
- (AS6) Homotopy invariance: \(c(H) = c(K)\) when \(\varphi_H = \varphi_K\) in the universal covering of the group of Hamiltonian diffeomorphisms and both \(H\) and \(K\) are normalized to have zero mean.
- (AS7) Triangle inequality or sub-additivity: \(c(H\#K) \leq c(H) + c(K)\).
- (AS8) Spectrality: \(c(H) \in S(H)\). More specifically, there exists a capped one-periodic orbit \(\bar{x}\) of \(H\) such that \(c(H) = \mathcal{A}_H(\bar{x})\).

This list of the properties of \(c\) is far from exhaustive, but it is more than sufficient for our purposes. Most of the properties are implicitly used in the proof of Lemma 3.2 below and the sub-additivity in the form

\[
c(H\#k) \leq kc(H)
\]

is crucial for the proof of Theorem 1.1. It is worth emphasizing the rationality assumption plays an important role in the proofs of the homotopy invariance and spectrality; see [Oh, Sc] and also [EP] for a simple proof. (The latter property also holds in general for non-degenerate Hamiltonians. This is a non-trivial result; [U1].) Finally note that for the triangle inequality to hold one has to work with a suitable definition of the pair-of-pants product in Floer homology; cf. [AS, U2]. We refer the reader to [U2] for a very detailed treatment of action selectors in generality more than sufficient for our purposes.

3. Proof of Theorem 1.1

As most of the proofs of the Conley conjecture type results, the proof of Theorem 1.1 amounts to dealing with two cases. The so-called degenerate case is established for all rational symplectic manifolds in [GG1]. Here we consider the second, the non-degenerate case. (This terminology should be taken with a grain of salt. For,
for instance, in the non-degenerate case only some orbits of the Hamiltonian must be non-degenerate, and in fact only weakly non-degenerate. Note also that, when $c_1(TM)_{\pi_2(M)} = 0$, this case was settled already in [SZ].)

To illustrate the idea of the proof, let us consider the following situation. Assume that an orbit $x$ and its sufficiently large iteration $x^k$, equipped with suitable cappings, “represent” the fundamental class in the Floer homology. Denote the corresponding capped orbits by $\bar{x}$ and $\bar{y} = \bar{x}^k \# A$. To represent the fundamental class, the orbits $\bar{x}$ and $\bar{y}$ must at least have non-vanishing Floer homology in degree $n$. When $x$ is weakly non-degenerate, we have $\Delta_H(\bar{x}) > 0$. Hence, $HF_n(H, \bar{y}) \neq 0$ implies, by (2.2), that $I_{c_1}(A) < 0$ for large $k$. On the other hand, due to the sub-additivity of the action selector, we have

$$kA_H(\bar{x}) + I_\omega(A) = A_H(\bar{y}) = c(H^k) \leq k c(H) = kA_H(\bar{x}).$$

Hence, $I_\omega(A) \leq 0$, which is impossible since for negative monotone manifolds $I_{c_1}$ and $I_\omega$ have opposite signs. The remaining case where $x$ is strongly degenerate and $\Delta_H(\bar{x}) = 0$ is considered in [GG1].

To make this argument rigorous, we need to make sense of the statement that the orbits represent the fundamental class. This is done using the notion of a carrier of the action selector, discussed in the next section.

### 3.1. Carrier of the action selector

When $H$ is non-degenerate, the action selector can also be evaluated as

$$c(H) = \inf_{[\sigma] = [M]} A_H(\sigma),$$

where we set

$$A_H(\sigma) = \max\{A_H(\bar{x}) | \alpha_{\bar{x}} \neq 0\} \text{ for } \sigma = \sum \alpha_{\bar{x}} \bar{x} \in CF_n(H).$$

The infimum here is obviously, when $M$ is rational, attained. Hence there exists a cycle $\sigma = \sum \alpha_{\bar{x}} \bar{x} \in CF_n(H)$, representing the fundamental class $[M]$, such that $c(H) = A_H(\bar{x})$ for an orbit $\bar{x}$ entering $\sigma$. In other words, $\bar{x}$ maximizes the action on $\sigma$ and the cycle $\sigma$ minimizes the action over all cycles in the homology class $[M]$. We call such an orbit $\bar{x}$ a carrier of the action selector. Note that this is a stronger requirement than just the equality $c(H) = A_H(\bar{x})$. A carrier is not in general unique, but it becomes unique when all one-periodic orbits of $H$ have distinct action values.

Our goal is to generalize this definition to the case where all one-periodic orbits of $H$ are isolated but possibly degenerate. Under a $C^2$-small, non-degenerate perturbation $\tilde{H}$ of $H$, every such orbit $x$ splits into several non-degenerate orbits, which are close to $x$. Furthermore, a capping of $x$ naturally gives rise to a capping of each of these orbits; cf. Section 2.3.

**Definition 3.1.** A capped one-periodic orbit $\bar{x}$ of $H$ is a carrier of the action selector for $H$ if there exists a sequence of $C^2$-small, non-degenerate perturbations $\tilde{H}_i \subseteq H$ such that one of the capped orbits $\bar{x}$ splits into is a carrier for $\tilde{H}_i$. An orbit (without capping) is said to be a carrier if it turns into one for a suitable choice of capping.

It is easy to see that a carrier necessarily exists, provided that $M$ is rational and all one-periodic orbits of $H$ are isolated. As in the non-degenerate case, a carrier is of course not unique in general – different choices of sequences $\tilde{H}_i$ can
lead to different carriers. However, it becomes unique when all one-periodic orbits of $H$ have distinct action values. In other words, under the latter requirement, the carrier is independent of the choice of the sequence $\hat{H}_i$.

Note also that one can always arrange for the sequence $\hat{H}_i$ to satisfy the additional condition

$$c(H) = c(\hat{H}_i) \quad (3.1)$$

by adding a small constant to $\hat{H}_i$. We can further assume that all one-periodic orbits of $\hat{H}_i$ have distinct action values. (This is a consequence of the fact that the Floer complex is stable under small perturbations of the Hamiltonian and auxiliary structures when all regularity requirements are met.) In what follows, we will always pick $\hat{H}_i$ such that (3.1) holds and the latter condition is satisfied.

As an immediate consequence of the definition of the carrier and continuity of the action and the mean index, we have

$$c(H) = A_H(x) \text{ and } 0 \leq \Delta_H(x) \leq 2n, \quad (3.2)$$

and the inequalities are strict when $x$ is weakly non-degenerate.

We will need the following result asserting that a carrier of the action selector is in some sense homologically essential.

**Lemma 3.2.** Assume that all one-periodic orbits of $H$ are isolated and let $x$ be a carrier of the action selector. Then $\text{HF}_n(H, \bar{x}) \neq 0$.

**Proof.** Let $\bar{H}$ be a one of the Hamiltonians in the sequence $\hat{H}_i$ and let $\bar{y}$ be a carrier of the action selector for $\bar{H}$ in the collection $Y$ of non-degenerate orbits which $\bar{x}$ splits into. Thus, $\bar{y} \in Y$ is an action maximizing orbit in an action minimizing cycle $\sigma = \bar{y} + \ldots \in \text{CF}_n(\bar{H})$. As has been pointed out above, we may assume that $\bar{y}$ is a unique action maximizer in $\sigma$ and that $A_{\bar{H}}(\bar{y}) = c(\bar{H})$.

At this point it is convenient to specify how close $\bar{H}$ must be to $H$. Fix a small, isolating neighborhood $U$ of $x$ and let $V$ be the union of small neighborhoods of the remaining one-periodic orbits of $H$. (In particular, $U$ and $V$ are disjoint.) Thus, for instance, the group $Y$ comprises one periodic orbits of $\bar{H}$ contained in $U$ and equipped with the cappings inherited from $\bar{x}$. (Strictly speaking, here, as in the definition of the local Floer homology, one should work with tubular neighborhoods of periodic orbits in the extended phase-space $S^1 \times M$.) It is easy to show that for every $\bar{H}$, which is $C^2$-close to $H$, all one-periodic orbits of $\bar{H}$ are contained in $U \cup V$ and, moreover, every solution $u$ of the Floer equation connecting an orbit in $U$ and an orbit in $V$ must have energy $E(u) > \epsilon$ for some constant $\epsilon > 0$ independent of $u$ and $\bar{H}$; see, e.g., [Sa, Section 1.5] or [FO, Lemma 19.8]. This is our first closeness requirement. Furthermore, we pick $\bar{H}$ so $C^2$-close to $H$ that every orbit from $Y$ has action in the range $(c(H) - \epsilon/2, c(H) + \epsilon/2)$.

Denote by $\sigma_Y$ the chain formed by the orbits from $Y$ entering the cycle $\sigma$. For instance, $\bar{y}$ enters the chain $\sigma_Y$. Our goal is to prove that $\sigma_Y \in \text{CF}_n(\bar{H}, \bar{x})$ is closed but not exact.

First let us show that $\sigma_Y$ is in fact a cycle in $\text{CF}_n(\bar{H}, \bar{x})$. To see this, denote by $\partial_Y \sigma_Y$ the part of the cycle $\partial \sigma_Y$ formed by the orbits that also belong to $Y$. We need to show that $\partial_Y \sigma_Y = 0$. Assume the contrary: an orbit $\bar{z}'$ enters this cycle. Then, since $\sigma$ is closed, there must be an orbit $\bar{z} \not\in Y$ in $\sigma$ connected to $\bar{z}'$ by a Floer trajectory $u$. In particular, $u$ connects an orbit in $V$ to an orbit in $U$ and
hence }E(u) > \epsilon.\) Thus
\[
A_{\bar{H}}(\bar{z}) \geq A_{\bar{H}}(\bar{z}') + E(u) > c(H) - \frac{\epsilon}{2} + \epsilon > c(H) = A_{\bar{H}}(\bar{y}).
\]
This is impossible since \(\bar{y}\) is an action maximizing orbit in the cycle \(\sigma\).

To finish the proof, it remains to show that \(\sigma_Y\) is not exact in \(\text{CF}_*(\bar{H}, \bar{z})\). Assume the contrary. Then there exists a chain \(\eta\) formed by some orbits from \(Y\) such that \(\partial Y \eta = \sigma_Y\) in obvious notation. We have \(\partial \eta = \partial Y \eta + \lambda\), where all orbits entering \(\lambda\) are contained in \(V\), while those from \(\partial Y \eta\) are contained in \(U\). By the definition of the Floer differential, any orbit \(\bar{z}\) entering \(\lambda\) is connected to an orbit entering \(\eta\) by a Floer trajectory \(u\) beginning in \(U\) and ending in \(V\). Hence,
\[
A_{\bar{H}}(\bar{z}) \leq c(H) + \epsilon/2 - \epsilon < c(H) = A_{\bar{H}}(\sigma).
\]
As a consequence, every orbit in the cycle
\[
\sigma' = \sigma - \partial \eta = (\sigma - \sigma_Y) - \lambda
\]
has action strictly smaller than \(A_{\bar{H}}(\sigma)\). Indeed, an orbit from \(\sigma'\) either enters \(\sigma\), but not \(Y\), or enters \(\lambda\). In the latter case, the action does not exceed \(A_{\bar{H}}(\sigma) - \epsilon/2\) as we have just shown. In the former case, the action is again smaller than \(A_{\bar{H}}(\sigma) = A_{\bar{H}}(\bar{y})\). For \(\bar{y}\) is a unique action maximizing orbit in \(\sigma\).) To summarize, we have \([\sigma'] = [M]\) and \(A_{\bar{H}}(\sigma') < A_{\bar{H}}(\sigma)\). This contradicts our choice of \(\sigma\) as an action minimizing cycle.

\[\square\]

**Remark 3.3.** We finish this discussion with one minor, fairly standard, technical point. Namely, recall that the Floer complex of a non-degenerate Hamiltonian \(H\) depends not only on \(H\) but also on an auxiliary structure \(J\), e.g., an almost complex structure when \(M\) is weakly monotone. Moreover, the complex is defined only when suitable regularity requirements are met. As a consequence, an action carrier is in reality assigned to the pair \((H, J)\) rather than to just a Hamiltonian \(H\) in either non-degenerate or degenerate case. Thus, in Definition 3.1, we tacitly assumed the presence of an auxiliary structure \(J\) in the background and that the regularity requirements are satisfied for the sequence of perturbations. This can be achieved by either considering regular pairs \((\bar{H}_i, J_i)\) with \(J_i \to J\) or even by setting \(J_i = J\); cf. [FHS, SZ].

**3.2. Proof.** Arguing by contradiction, assume that \(H\) has finitely many simple, i.e., non-iterated, periodic orbits. By passing if necessary to an iteration (and “adjusting the time”), we can also assume that all periodic orbits are one-periodic.

**3.2.1. A common action carrier.** Our next goal is to show that after passing if necessary to an iteration we can assume that there exists a one-periodic orbit \(x\) (not capped) of \(H\) such that its iterations \(x^{k_i}\) are action carriers for \(H^{\#k_i}\) for some infinite sequence \(k_1 = 1, k_2, k_3, \ldots\).

Indeed, let \(x_1, \ldots, x_m\) be all periodic (and thus in fact one-periodic) orbits of \(H\). Let us break down the set of positive integers \(\mathbb{N}\) into \(m\) groups \(Z_1, \ldots, Z_m\) by assigning \(k \in \mathbb{N}\) to \(Z_j\) if \(x_j\) is an action selector carrier for \(H^{\#k}\). If the action carrier for \(H^{\#k}\) is not unique, we pick it in an arbitrary fashion. Thus, we have
\[
\mathbb{N} = \bigcup_{j=1}^{m} Z_j
\]
3.2.1. \textbf{Growth of orbits vs. decay of the mean indexes.} To state this result, we need to change slightly our notation and recall some standard terminology. Let $\phi$ be the Hamiltonian diffeomorphism and $\tau$ be a time parameter. Consider the minimal mean index $\lambda(I)$. Then $\lambda(I) = 0$ is not in the support of $H$. For some non-trivial recapping by some sphere $A$. We have

$$2n + I_{c_1}(A) < k\Delta_H(\bar{x}) + I_{c_1}(A) = \Delta_{H^{\#k}}(\bar{y}) \leq 2n.$$ 

Hence,

$$I_{c_1}(A) < 0.$$ 

On the other hand, using the sub-additivity (2.3) of the action selector, we obtain

$$kA_H(\bar{x}) + I_\omega(A) = A_{H^{\#k}}(\bar{y}) = c(H^{\#k}) \leq k c(H) = kA_H(\bar{x}).$$

Therefore,

$$I_\omega(A) \leq 0.$$ 

However, we cannot have $I_{c_1}(A) < 0$ and $I_\omega(A) \leq 0$ simultaneously for a negative monotone manifold $M$, since for $M$, as has been mentioned above, $I_\omega = \lambda I_{c_1}$ with $\lambda < 0$. This contradiction completes the proof.

3.2.2. \textbf{Two alternatives and the degenerate case.} Let $\bar{x}$ be the orbit $x$ capped as a carrier of $H$. Then $\Delta_H(\bar{x}) \geq 0$ and $HF(H, \bar{x}) \neq 0$ by (3.2) and Lemma 3.2. When $\Delta_H(\bar{x}) = 0$, the capped orbit $\bar{x}$ is the so-called \textit{symplectically degenerate maximum} and in the presence of such an orbit the Conley conjecture is proved for rational symplectic manifolds in [GG1]. (See also [Gi2, GG2, He1] for the definition, a detailed discussion and applications of this notion, which originates from Hingston’s proof of the Conley conjecture for tori; see [Hi].)

Thus, it remains to prove the theorem in the case where $\Delta_H(\bar{x}) > 0$ which is done in the next subsection. Here we only mention that this is the so-called \textit{weakly non-degenerate case}. For, as is shown in [GG2], whenever $\Delta_H(\bar{x}) > 0$ and $HF(H, \bar{x}) \neq 0$, the orbit $x$ is weakly non-degenerate.

3.2.3. \textbf{Weakly non-degenerate case.} Pick a sufficiently large entree $k$ in the sequence $k_1$ from Section 3.2.1 so that $k\Delta_H(\bar{x}) > 2n$. Then $n$ is not in the support of $HF(H, \bar{x}^k)$. The capped orbit $\bar{x}^k$ cannot be a carrier of the action selector for $H^{\#k}$ by (3.2) and thus it becomes one after non-trivial recapping by some sphere $A$. Denote by $\bar{y} = \bar{x}^k \# A$ the resulting carrier for $H^{\#k}$. Applying (2.2) and (3.2) to $\bar{y}$, we have

$$2n + I_{c_1}(A) < k\Delta_H(\bar{x}) + I_{c_1}(A) = \Delta_{H^{\#k}}(\bar{y}) \leq 2n.$$ 

Hence,

$$I_{c_1}(A) < 0.$$ 

On the other hand, using the sub-additivity (2.3) of the action selector, we obtain

$$kA_H(\bar{x}) + I_\omega(A) = A_{H^{\#k}}(\bar{y}) = c(H^{\#k}) \leq k c(H) = kA_H(\bar{x}).$$

Therefore,

$$I_\omega(A) \leq 0.$$ 

However, we cannot have $I_{c_1}(A) < 0$ and $I_\omega(A) \leq 0$ simultaneously for a negative monotone manifold $M$, since for $M$, as has been mentioned above, $I_\omega = \lambda I_{c_1}$ with $\lambda < 0$. This contradiction completes the proof.

3.3. \textbf{Growth of orbits vs. decay of the mean indexes.} The notion of an action carrier also lends itself readily to the proof of a simple result relating, in the symplectically aspherical case, the growth of the number of orbits of a certain type to the decay of the minimal mean index. To state this result, we need to change slightly our notation and recall some standard terminology. Let $x$ be a simple $\tau$-periodic orbit of the Hamiltonian diffeomorphism $\varphi_H$. Thus $x$ comprises exactly $\tau$ fixed points of $\varphi_H^\tau$ and as many $\tau$-periodic orbits of $H$. In what follows, we do not distinguish these orbits of $H$. In other words, we differentiate only \textit{geometrically distinct} simple periodic orbits of $H$ or $\varphi_H$. Furthermore, we restrict our attention only to $x$ such that the corresponding orbits of $H$ are contractible. Set $\Delta(x) := \Delta_{H^{\#x}}(x)$ and $HF_*(H, x) := HF_*(H^{\#x}, x)$, where on the right hand side we can take any periodic orbit of $H$ representing $x$. 
Proposition 3.4. Let $H$ be a Hamiltonian on a closed, symplectically aspherical manifold $M^{2n}$. Assume that all periodic orbits of $H$ are isolated. Then

$$2n \sum_x \frac{1}{\Delta(x)} \geq k,$$

where the sum is taken over all (geometrically distinct) simple periodic orbits $x$ with period less than or equal to $k$ and $HF_n(H, x) \neq 0$ (and hence $0 \leq \Delta(x) \leq 2n$).

Here we use the convention that when $\Delta(x) = 0$ the left hand side is infinite and the inequality is automatically satisfied. Thus the proposition essentially concerns only the weakly non-degenerate case. For, when there is a symplectically degenerate maximum ($\Delta(x) = 0$ and $HF_n(H, x) \neq 0$), the result holds trivially and gives no information.

Example 3.5. Assume that $\Delta(x) > \delta > 0$ for all simple periodic orbits with $HF_n(H, x) \neq 0$. Then the number of such (geometrically distinct) orbits of period up to $k$, which we denote by $P(k)$, grows at least linearly with $k$. More precisely, $P(k) \geq \delta k/2n$.

Proof of Proposition 3.4. Fix a positive integer $k$. For a simple periodic orbit $x$ with period $\tau \leq k$, let $Z(x)$ be the collection of positive integers $\ell \leq k$, divisible by $\tau$, such that the iteration $x^{\ell/\tau}$ is a carrier of the action selector for $H^{\#\ell/\tau}$. By definition,

$$\bigcup_x Z(x) = \{1, 2, \ldots, k\}.$$

Note that $|Z(x)| \leq 2n/\Delta(x)$. Indeed, $\Delta_{H^{\#r/\tau}}(x^r) > 2n$ when $r > 2n/\Delta(x)$, and hence $x^r$ cannot be a carrier of the action selector. Thus

$$\sum_x \frac{2n}{\Delta(x)} \geq \sum_x |Z(x)| \geq k$$

and the proposition follows.

Remark 3.6. In the context of symplectic topology, little is known about the growth of orbits for sufficiently general Hamiltonian diffeomorphisms $\varphi$ of symplectically aspherical manifolds. (Numerous growth results obtained by dynamical systems methods usually rely on restrictive additional requirements on $\varphi$ such as, for instance, hyperbolicity or topological requirements; see, e.g., [KH] and references therein.) In the setting considered here, the strongest growth result is probably the one from [SZ] asserting the existence of a simple periodic orbit for every sufficiently large prime period when $H$ is weakly non-degenerate. Thus, in this case, $P(k)$ grows at least as fast as $k/\ln k$. The only linear growth result the authors are aware of is that of Viterbo, [Vi], according to which the number of simple periodic orbits with positive action and period less than or equal to $k$ grows at least linearly with $k$ when $H \geq 0$ is a compactly supported, non-zero Hamiltonian on $\mathbb{R}^{2n}$ (or on any so-called wide manifold – see [Gi]).

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