Geometry of phase-covariant qubit channels

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Abstract

We analyze the geometry on the space of non-unital phase-covariant qubit maps. Using the corresponding Choi-Jamiołkowski states, we derive the Hilbert-Schmidt line and volume elements using the channel eigenvalues together with the parameter that characterizes non-unitality. We find the shapes and analytically compute the volumes of phase-covariant channels, in particular entanglement breaking and obtainable with time-local generators.

1. Introduction

In the recent years, non-unital channels have been receiving increasing attention. Of particular interest are phase-covariant qubit channels, where the non-unitality property is controlled with a single parameter. First master equations for phase-covariant dynamical maps were phenomenological in nature, and they were used to describe thermalization and dephasing processes beyond the Markovian approximation [1]. Later, a microscopic derivation was presented, modelled with weakly-coupled spin-bosons under the secular approximation [2]. During further research, a connection was found between the population monotonicity, coherence monotonicity, and Markovianity of evolution [3]. Due to high symmetries of phase-covariant channels, it was possible to estimate the upper and lower bounds of their classical capacity [4, 5]. In the theory of open quantum systems, the evolution they provide was analyzed in terms of non-Markovianity [6–8] and quantum speed limit [9, 10].

However, many questions that have been answered for unital qubit maps are still open for non-unital channels. Among such open problems, there is their geometrical characterization, like the shape and size of the space of non-unital maps, or the analysis of the underlying geometrical structures. For the Pauli channels, which are unital qubit maps, Jagadish et al proposed a measure, under which the volumes of positive and completely positive, trace-preserving maps were analyzed [11]. However, it was also stated that the same measure cannot be used for non-unital channels [12]. More results for unital channels were obtained using the Lebesgue measure [13] and the Hilbert-Schmidt metric [14–16]. The corresponding dynamical maps have been also studied. The geometry of Markovian semigroups was considered [17–19], as well as the volumes of non-Markovian dynamical maps under mixing semigroups [20] and their generalizations [21]. Recently, fractions of invertible maps obtained through mixing non-invertible maps have also been analyzed [22].

The motivation behind this paper is gaining some intuition on expected properties of randomly selected quantum channels. One example of such instance would be a noisy evolution whose parameters are not precisely defined. Through analytical volume formulas, one can determine geometrical regions occupied by specific subsets of quantum maps. Relative volumes can be interpreted as probabilities that random channels have certain physical properties. Random quantum channels find applications in qubit encryption [23] and superdense coding [24]. The authors prove that even incomplete information about quantum evolution can be sufficient to determine its characteristics, such as bounds on minimal output entropy and its asymptotic behavior [25–28]. Their generation methods, spectral properties, and state transformations were analyzed in [13, 29, 30].

Another question concerns the compatibility problem between the formalisms of quantum channels and dynamical maps that arise as solutions to master equations. For example, it has already been shown that a
significant amount of legitimate quantum channels does not arise from regular (and therefore physical) time-
local generators [14]. Going a step further, Wolf et al [31] investigated the channels that reflect Markovian open quantum systems evolution. These results prove how important it is to further develop the memory kernel approach, where quantum evolution is given by integro-differential dynamical equations.

In this paper, we analyze geometrical properties of phase-covariant channels. Using the Choi-Jamiokowski isomorphism, we introduce the Hilbert-Schmidt metric on the space of trace-preserving phase-covariant qubit maps. Through integration of the associated volume elements, we calculate the relative volumes of completely positive maps, entanglement breaking channels, and quantum maps obtainable with time-local generators. This is accomplished for the general case, characterized by three distinct parameters, as well as the maps that are non-invertible or have degenerate eigenvalues due to symmetries. The shapes and sizes of the regions corresponding to specific classes of channels are graphically presented, which allows for better understanding of their geometry. Finally, we list interesting open problems that have arisen during our research.

2. Phase-covariant qubit channels

Consider a qubit evolution that combines pure dephasing, energy emission, and energy absorption. It can be described by phase-covariant channels, which are completely positive, trace-preserving maps \( \Lambda \) covariant with respect to a unitary transformation \( U(\phi) = \exp(-i\sigma_3\phi), \phi \in \mathbb{R} \). In other words,

\[
\forall \phi \in \mathbb{R}, \Lambda[U(\phi)XU^\dagger(\phi)] = U(\phi)\Lambda[X]U^\dagger(\phi).
\]

The most general form of a quantum map that satisfies equation (1) reads [6, 32]

\[
\Lambda[X] = \frac{1}{2} \left[ (I + \lambda_3 \sigma_3) \text{Tr}X + \lambda_1 \sigma_1 \text{Tr}(\sigma_1X) + \lambda_2 \sigma_2 \text{Tr}(\sigma_2X) + \lambda_3 \sigma_3 \text{Tr}(\sigma_3X) \right],
\]

where \( \sigma_3, \alpha = 0, 1, 2, 3 \), denote the Pauli matrices. The real numbers \( \lambda_1 \) and \( \lambda_3 \), which together with \( \lambda_3 \) are parameters characterizing the channel, are the eigenvalues of \( \Lambda \) to the eigenvectors given by the eigenvalue equations,

\[
\Lambda[\sigma_1] = \lambda_1 \sigma_1, \quad \Lambda[\sigma_2] = \lambda_2 \sigma_2, \quad \Lambda[\sigma_3] = \lambda_3 \sigma_3.
\]

Note that \( \Lambda \) is in general non-unital (\( \Lambda[I] \neq I \)). Instead of the maximally mixed state \( I \), it preserves the state

\[
\rho_* = \frac{1}{2} \left[ I + \frac{\lambda_3}{1 - \lambda_3} \sigma_3 \right],
\]

meaning that \( \Lambda[\rho_*] = \rho_* \). Therefore, the real number \( \lambda_3 \) determines the invariant state of \( \Lambda \).

The positivity of the phase-covariant map is determined by its action on an arbitrary density matrix

\[
\rho = \frac{1}{2} \left[ 1 + x_3 \left( x_1 - i x_2 \right) \right], \quad x_1^2 + x_2^2 + x_3^2 \leq 1.
\]

The output state is represented via

\[
\Lambda[\rho] = \frac{1}{2} \begin{pmatrix}
1 + \lambda_3 x_3 & \lambda_1 (x_1 - i x_2) \\
\lambda_1 (x_1 + i x_2) & 1 - \lambda_3 x_3
\end{pmatrix},
\]

whose eigenvalues

\[
\mu_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{\lambda_1^2 (x_1^2 + x_2^2) + (\lambda_3 + \lambda_3 x_3)^2} \right]
\]

are positive for any input state if and only if

\[
|\lambda_1| \leq 1, \quad |\lambda_3| + |\lambda_3| \leq 1,
\]

which give the positivity conditions for \( \Lambda \). Now, \( \Lambda \) is completely positive (and therefore, a quantum channel) provided that the associated Choi matrix

\[
\rho_\Lambda = \frac{1}{2} \sum_{k, \ell = 0}^{1} |k\rangle \langle \ell| \otimes \Lambda[|k\rangle \langle \ell|] = \frac{1}{4} \begin{pmatrix}
1 + \lambda_3 & 0 & 0 & 2 \lambda_1 \\
0 & 1 - \lambda_3 - \lambda_3 & 0 & 0 \\
0 & 0 & 1 + \lambda_3 - \lambda_3 & 0 \\
2 \lambda_1 & 0 & 0 & 1 - \lambda_3 + \lambda_3
\end{pmatrix}
\]

is positive-semidefinite. From the form of its eigenvalues

\[
\alpha_\pm = \frac{1}{4} (1 - \lambda_3 \pm \lambda_3), \quad \beta_\pm = \frac{1}{4} (1 + \lambda_3 \pm \sqrt{\lambda_3^2 + 4 \lambda_3^2}),
\]

is positive-semidefinite.
it is straightforward to show that complete positivity of $\Lambda$ is equivalent to [17]
\[ |\lambda_3| + |\lambda_4| \leq 1, \quad 4\lambda_1^2 + \lambda_3^2 \leq (1 + \lambda_3)^2. \tag{11} \]
An important class of quantum maps are entanglement breaking channels, which produce separable states when acting on a composite system. It was shown that a qubit channel is entanglement breaking if and only if its Choi matrix satisfies [33]
\[ \rho_{\Lambda} \leq \frac{1}{2} I \otimes I. \tag{12} \]
For phase-covariant channels, the above condition reduces to
\[ 4\lambda_1^2 + \lambda_3^2 \leq (1 - \lambda_3)^2. \tag{13} \]
Quantum channels are used not only to provide discrete evolution of physical systems, which occurs e.g. in quantum processing and quantum measurements [34]. Continuous models describe time evolution of open quantum systems by applying dynamical maps—that is, families of time-parameterized quantum channels $\{\Lambda(t) | t \geq 0, \Lambda(0) = 1\}$ [35]. Dynamical maps are the solutions of master equations, the simplest being the dynamical equation for the Markovian semigroup $\Lambda(t) = e^{t \mathcal{L}} (\Lambda(0)) = 1$, with the time-independent Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) generator $\mathcal{L}$ [36, 37]. For the phase-covariant dynamical maps [17],
\[ \mathcal{L} = \gamma_1 \mathcal{L}_+ + \gamma_2 \mathcal{L}_- + \gamma_3 \mathcal{L}_3 \tag{14} \]
with the decoherence rates $\gamma_{\pm}, \gamma_3$ and
\[ \mathcal{L}_\pm[X] = \sigma_+ X \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, X \}, \quad \mathcal{L}_3[X] = \frac{1}{4} (\sigma_3 X \sigma_3 - X). \tag{15} \]
Memory effects of quantum evolution are often included by replacing the constant generator $\mathcal{L}$ with the time-local generator $\mathcal{L}(t)$ of the same form but with time-dependent (not necessarily positive) rates $\gamma_{\pm}(t), \gamma_3(t)$. The corresponding dynamical map is characterized by time-dependent eigenvalues [17]
\[ \lambda_1(t) = \exp \left( -\frac{1}{2} \Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t) \right), \quad \lambda_2(t) = \exp \left( -\Gamma_1(t) - \Gamma_2(t) \right), \tag{16} \]
\[ \lambda_{\pm}(t) = \exp \left[ -\Gamma_1(t) - \Gamma_2(t) \right] \int_0^t [\gamma_{\pm}(\tau) - \gamma_3(\tau)] \exp \left[ \Gamma_3(\tau) + \Gamma_2(\tau) \right] d\tau, \tag{17} \]
where $\Gamma_1(t) = \int_0^t \gamma_1(\tau) d\tau, k = \pm, 3$. Note that a vanishing eigenvalue corresponds to a singular generator with at least one infinite decoherence rate, which is unphysical. Physical time-local generators always produce dynamical maps with
\[ \lambda_1(t) > 0, \quad \lambda_3(t) > 0. \tag{18} \]
Finally, quantum maps $\Lambda$ arise from dynamical maps $\Lambda(t)$ with a fixed time $t = t_*$ $\geq 0$. We refer to such channels as obtainable with time-local generators.

3. Line and volume elements

In this section, we follow the method from [14] to introduce geometrical structures in the space of Choi states associated with the phase-covariant maps. Recall that, due to the Choi-Jamiolkowski isomorphism [38, 39], there exists a one-to-one correspondence between quantum maps $\Lambda$ and quantum states $\rho_{\Lambda}$ (see equation (9)). From now on, we work on the space of the Choi states. We equip it with the Hilbert-Schmidt metric
\[ g = \frac{1}{4} \text{diag}(2, 1, 1, 1), \]
defined by the line element
\[ ds^2 = \text{Tr}(d\rho_{\Lambda}^2) = \frac{1}{4} (2 d\lambda_1^2 + d\lambda_2^2 + d\lambda_3^2). \tag{19} \]
This allows us to introduce the corresponding volume element
\[ dV = \sqrt{\det g} d\lambda_0 d\lambda_2 d\lambda_3 = \sqrt{\frac{\lambda_0}{8}} (d\lambda_0 d\lambda_3 - d\lambda_3 d\lambda_0). \tag{20} \]
Interestingly, on the level of geometrical structures, the parameter $\lambda_0$ that is responsible for non-unitality of $\Lambda$ behaves just like the channel eigenvalues $\lambda_1$ and $\lambda_3$.

Comparing our results to the geometry of Pauli channels
\[ \Lambda[\sigma_{\alpha}] = \lambda_0 \sigma_{\alpha}, \quad \alpha = 0, 1, 2, 3, \quad \lambda_0 = 1, \tag{21} \]
for which
\[ ds^2 = \frac{1}{4} (d\lambda_1^2 + d\lambda_2^2 + d\lambda_3^2), \quad dV = \frac{1}{8} d\lambda_1 d\lambda_2 d\lambda_3, \]
one observes certain similarities. Both metrics are diagonal in \( \lambda_i \)’s. The difference in the first term of \( g \) comes from the fact that \( \lambda_1 \) for the phase-covariant channels is two-times degenerate, while all the other parameters are non-degenerate. This also influences the final formula for the volume element. By fixing \( \lambda_* = 0 \) in equation (19) and \( \lambda_2 = \lambda_1 \) in equation (22), we recover the common subspace for the Pauli channels and phase-covariant channels.

Now, the volume element from equation (20) can be used to determine volumes of various classes of the phase-covariant maps. These volumes
\[ V = \int_{C_{\text{TP}}} dV \]
are calculated by integrating \( dV \) over the corresponding regions. In particular, we are interested in the following regions of integration:

(i) the positivity region given by equation (8),
\[ C_{\text{PT}} = \{ \lambda_1, \lambda_3, \lambda_3^* : |\lambda_3| \leq 1 \land |\lambda_3| \leq 1 \land |\lambda_3 + |\lambda_*| \leq 1 \}, \]

(ii) the complete positivity region from equation (11),
\[ C_{\text{CPT}} = \{ \lambda_1, \lambda_3, \lambda_3^* : |\lambda_3| \leq 1 \land |\lambda_3| + |\lambda_*| \leq 1 \land 4\lambda_1^2 + \lambda_3^2 \leq (1 + \lambda_3)^2 \}, \]

(iii) the entanglement breaking region in equation (13),
\[ C_{\text{EBC}} = \{ \lambda_1, \lambda_3, \lambda_3^* : |\lambda_3| \leq 1 \land |\lambda_3| + |\lambda_*| \leq 1 \land 4\lambda_1^2 + \lambda_3^2 \leq (1 \pm \lambda_3)^2 \}, \]

(iv) the region of positive eigenvalues that corresponds to maps obtainable with physical time-local generators (see equation (18)),
\[ C_{\text{TLG}} = \{ \lambda_1, \lambda_3, \lambda_3^* : \lambda_1 > 0 \land \lambda_3 > 0 \land |\lambda_*| \leq 1 \}. \]

4. Volume of non-unital maps

Integrating the volume element \( dV \) over the positivity \( C_{\text{PT}} \) and complete positivity \( C_{\text{CPT}} \) regions, one arrives at the volumes
\[ V(C_{\text{PT}}) = \frac{\sqrt{2}}{2}, \quad V(C_{\text{CPT}}) = \frac{2\sqrt{2}}{9} \]
of the positive, trace-preserving phase-covariant maps (PT) and channels (CPT), respectively. Note that \( V(C_{\text{CPT}})/V(C_{\text{PT}}) = 4/9 \), so less than a half of PT maps are CPT. This ratio is close to the relative volume for the Pauli maps with \( \lambda_2 = \lambda_1 \) (or phase-covariant channels with \( \lambda_* = 0 \)), where the channels amount to exactly a half of the positive, trace-preserving maps [16]. The shapes of \( C_{\text{PT}} \) and \( C_{\text{CPT}} \) regions are plotted in figure 1. It is easy to see that they are more complicated than the PT and CPT regions for the Pauli channels (a cube and a tetrahedron, respectively). In particular, \( C_{\text{PT}} \) resembles a house with a square floor, while \( C_{\text{CPT}} \) is an elliptic cone with a vertex at \( \lambda_1, \lambda_3, \lambda_3^* = (0, -1, 0) \), cut off by the house’s roof.

Among the phase-covariant maps, there are some that can be obtained with time-local generators (TLG). For positive, trace-preserving maps, the relative volume reads
\[ \frac{V(C_{\text{PT}} \cap C_{\text{TLG}})}{V(C_{\text{PT}})} = \frac{1}{4}, \]
whereas for the quantum channels
\[ \frac{V(C_{\text{CPT}} \cap C_{\text{TLG}})}{V(C_{\text{CPT}})} = \frac{1}{2} - \frac{3\pi}{64} \approx 0.35, \]
We graphically represent the ranges of \( \lambda_1, \lambda_3, \lambda_* \) that correspond to these regions in figure 2. The region \( C_{\text{PT}} \cap C_{\text{TLG}} \) is a triangular prism (light gray), whereas \( C_{\text{CPT}} \cap C_{\text{TLG}} \) is a cutoff of the elliptic cone (dark gray inside the prism, meshed on its faces). The shape in the \( \lambda_1, \lambda_* \)-plane is a half-ellipse, and in the \( \lambda_3, \lambda_* \)-plane, there is an
isosceles triangle. The remaining fraction of maps—that is,
\[ 1 - \frac{V(C_{\text{CPT}} \cap C_{\text{TLC}})}{V(C_{\text{CPT}})} = \frac{1}{2} + \frac{3\pi}{64} \approx 0.65, \]  
(31)
corresponds to the volume ratio of the phase-covariant channels that are obtainable only by considering the non-local master equations \([40, 41]\)
with memory kernels $K(t, \tau)$. As only around a third of all channels arise from time-local master equations with regular generators, these results show us how important it is to further develop the theories of non-invertible dynamical maps, admissible memory kernels, and singular generators.

Among all the phase-covariant channels, there are channels that break quantum entanglement, which interestingly is more than $1/2$ for the Pauli maps with $\lambda_3 = \lambda_1 [16]$. Finally, we can ask how much of the phase-covariant channels with positive eigenvalues break quantum entanglement. The corresponding volume ratio is

$$\frac{V(C_{\text{EBC}})}{V(C_{\text{CPT}})} = \frac{3\pi}{32} \approx 0.59$$

Moreover, among all the entanglement breaking channels,

$$\frac{V(C_{\text{EBC}} \cap C_{\text{TLG}})}{V(C_{\text{EBC}})} = \frac{1}{4}$$

can be obtained with time-local generators. Our results are presented in figure 3. It shows that the entanglement breaking region (dark gray) is inscribed inside the complete positivity region (light gray). Observe that $C_{\text{EBC}}$ is in the shape of an elliptic bicone, which bears certain symmetry resemblance to the EBC region of the Pauli channels (octahedron). The lower half of the bicone is coplanar with the CPT region, whereas its upper half has common boundary with $C_{\text{CPT}}$ at $\lambda_1 = 0$.

The results of this section are summarized in figure 4. The rectangle divided into $8 \times 17$ identical squares represents the volume of positive, trace-preserving phase-covariant maps $V(C_{\text{PT}})$. The gray area shows the part occupied by the quantum channels, whereas the white area corresponds to the positive but not completely positive maps. The hatched regions are associated with the volumes of entanglement breaking channels (left-to-right) and the positive maps obtainable with time-local generators (right-to-left). The smallest part of the rectangle corresponds to $C_{\text{EBC}} \cap C_{\text{TLG}}$, which is not surprising, as these maps satisfy the most restrictive conditions. Interestingly, the next smallest regions are $(C_{\text{CPT}} \cap C_{\text{TLG}}) \backslash C_{\text{EBC}}$ and $(C_{\text{CPT}} \backslash C_{\text{TLG}}) \backslash C_{\text{EBC}}$, respectively. On the other hand, the biggest area is occupied by positive, trace-preserving maps obtainable only with memory kernels.

\[
\Lambda(t) = \int_0^\infty K(t, \tau)\Lambda(\tau)\,d\tau, \quad \Lambda(0) = I, \tag{32}
\]
Even though the formula for the line element is general, the volume element is not. For example, the volume for a fixed $\lambda_*$ is zero—unless we restrict the manifold to a hypersurface. This way, we can calculate relative volumes for unital maps ($\lambda_*=0$), quantum maps with even more degenerated eigenvalues ($\lambda_1=\lambda_3$), or even non-invertible mappings ($\lambda_1=0$ or $\lambda_3=0$). For a full analysis of unital qubit maps ($\lambda_*=0$), refer to our previous works [14, 16].

4.1. Symmetric maps
By introducing additional symmetries, we consider the isotropic maps with a three-times degenerated eigenvalue $\lambda_1=\lambda_3$. In this case, the phase-covariant maps are fully determined by only two real parameters: $\lambda_3$ and $\lambda_*$. Now, the line and volume elements are given by

$$
\frac{1}{2}(3 \, d\lambda_1^2 + d\lambda_3^2), \quad dV = \frac{\sqrt{3}}{4} \, d\lambda_3 \, d\lambda_*.
$$

The regions corresponding to special classes of phase-covariant maps are shown in figure 5(a). Observe that there are as many channels obtainable with time-local generators that are not entanglement breaking as positive but not completely positive TP maps. Moreover, exactly half of entanglement breaking channels is only obtainable with memory kernels. Relatively to positive, trace-preserving maps, there are

$$
\frac{V(C_{\text{CPT}})}{V(C_{\text{PT}})} = \frac{9 + 2\sqrt{3}}{27} \approx 0.74
$$

quantum channels, whereas entanglement breaking channels constitute

$$
\frac{V(C_{\text{EBC}})}{V(C_{\text{CPT}})} = \frac{4\sqrt{3} - 9}{2\sqrt{3} + 9} \approx 0.20
$$

of all phase-covariant channels.

If we include one last symmetry constraint, which is taking $\lambda_* = \lambda$, then $d\lambda^2 = d\lambda^2$ and $dV = d\lambda$. In this case, the regions of integration reduce to

$$
C_{\text{PT}} = \left\{ \lambda : |\lambda| \leq \frac{1}{2} \right\}, \quad C_{\text{EBC}} = \left\{ \lambda : |\lambda| \leq \frac{1}{4}(\sqrt{5} - 1) \right\},
$$

$$
C_{\text{CPT}} = \left\{ \lambda : -\frac{1}{4}(\sqrt{5} - 1) \leq \lambda \leq \frac{1}{2} \right\}, \quad C_{\text{PT}} \cap C_{\text{TLG}} = C_{\text{CPT}} \cap C_{\text{TLG}} = \left\{ \lambda : 0 \leq \lambda \leq \frac{1}{2} \right\}.
$$

Figure 4. An approximate quantitative representation of the volumes for special classes of trace-preserving phase-covariant maps.
Now, nearly three out of four positive TP maps are completely positive. Exactly a half of PTP maps and over two thirds of channels are obtainable with time-local generators. Almost one in every five channels breaks quantum entanglement. Similarly to the less symmetric case, there are as many entanglement breaking channels obtainable with time-local generators as with only memory kernels.

4.2. Non-invertible maps

For a map to be non-invertible, at least one of its eigenvalues has to vanish. However, if $\lambda_1 = \lambda_3 = 0$, then every positive TP map is trivially an entanglement breaking channel not obtainable with time-local generators. Therefore, we first focus on $\lambda_1 = 0$ and $\lambda_3 \neq 0$. Now, the geometry of the hypersurface is determined by

$$\begin{align*}
\text{d}s^2 &= \frac{1}{4} \left( d\lambda_1^2 + d\lambda_3^2 \right), \\
\text{d}V &= \frac{1}{4} d\lambda_1 d\lambda_3.
\end{align*}$$

By examining the regions of integration, we see that

$$\mathcal{C}_{\text{PT}} = \mathcal{C}_{\text{CPT}} = \mathcal{C}_{\text{EBC}} = \{ \lambda_3, \lambda_6 : |\lambda_3| \leq 1, |\lambda_6| + |\lambda_4| \leq 1 \},$$

$$\mathcal{C}_{\text{PT}} \cap \mathcal{C}_{\text{TLG}} = \{ \lambda_3, \lambda_6 : 0 \leq |\lambda_3| \leq 1, |\lambda_6| \leq 1 - |\lambda_3| \}.$$

Our observations agree with figure 5(b), where there are only two visible, equinumerous regions of PTP maps: entanglement breaking with positive and negative eigenvalue $\lambda_3$.

Let us observe what happens if one instead considers the maps with $\lambda_3 = 0$ and $\lambda_1 \neq 0$. Unlike in the case of Pauli channels, the positivity, complete positivity, and entanglement breaking conditions are not symmetric with respect to $\lambda_1 \leftrightarrow \lambda_3$. This is reflected in the geometry of the manifold, as now we have

$$\begin{align*}
\text{d}s^2 &= \frac{1}{4} (2 d\lambda_1^2 + d\lambda_3^2), \\
\text{d}V &= \frac{\sqrt{2}}{4} d\lambda_1 d\lambda_3,
\end{align*}$$

and also

$$\mathcal{C}_{\text{PT}} = \{ \lambda_1, \lambda_6 : |\lambda_6| \leq 1, |\lambda_4| \leq 1 \},$$

$$\mathcal{C}_{\text{CPT}} = \mathcal{C}_{\text{EBC}} = \left\{ \lambda_1, \lambda_6 : |\lambda_6| \leq 1, |\lambda_4| \leq 1 - \frac{\lambda_1^2}{2} \right\},$$

$$\mathcal{C}_{\text{TLG}} = \{ \lambda_1, \lambda_6 : 0 \leq |\lambda_1| \leq 1, |\lambda_6| \leq 1 \}.$$

The integration regions are presented in figure 5(c). Finally, the volume of positive, trace-preserving maps $V(\mathcal{C}_{\text{PT}}) = 4$ is bigger than that of quantum channels $V(\mathcal{C}_{\text{CPT}}) = \pi/2$, similarly to the case of three-parameter phase-covariant maps in equation (28). Just like in figure 5(b), we again have the complete positivity region that has two axes of symmetry. Once more, every single channel is entanglement breaking.

5. Conclusions

In this paper, we considered the geometrical aspects of the space of phase-covariant maps. We analytically derived the regions corresponding to positive, completely positive, entanglement breaking, and time-local generated maps. For these special classes of maps and channels, we used the Hilbert-Schmidt metric on the manifold of Choi-Jamiolkowski states to analytically calculate their volumes. We compared our results with the...
The geometry of non-unital channels is still a relatively unexplored area of research. Therefore, there are many open questions that require further study. First of all, it would be interesting to consider more general quantum evolution.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Lankinen J, Lyra H, Sokolov B, Teitinen J, Ziaei B and Maniscalco S 2016 Phys. Rev. A 93 052103
[2] Haase J F, Smirne A, Kolodyński J, Demkowicz-Dobrzański R and Huelga S F 2018 New J. Phys. 20 053009
[3] Haase J F, Smirne A and Huelga S F 2019 Non-monotonic population and coherence evolution in Markovian open-system dynamics Advances in Open Systems and Fundamental Tests of Quantum Mechanics (Germany: Bad Honnef)
[4] Filippow S N 2018 Evaluation of non-unital qubit channel capacities Uchenye Zapisi Kazanskogo Universiteta. Seriya Fiziko-Matematicheskie Nauki 160 258–65 Kazan Federal University, Kazan
[5] Ciampini M A, Cuevas A, Mataloni P, Macchiavello C and Sacchi M F 2021 Phys. Rev. A 103 062414
[6] Filippow S N, Glinov A N and Leppäjärvi L 2020 Lobachevskii J. Math. 41 617–30
[7] Shrikant U, Srikanth R and Petruccione F 2020 On the eternal non-Markovianity of non-unital quantum channels arXiv:2003.10625 [quant-ph]
[8] Siudzińska K 2022 J. Phys. A: Math. Theor. 55 405303
[9] Baruah R, Paulson K G and Banerjee S 2022 Phase covariant channel: quantum speed limit of evolution Ann. Phys. 533 2200199
[10] Teitinen J and Maniscalco S 2021 Entropy 23 331
[11] Jagadish V, Srikanth R and Petruccione F 2019 Phys. Rev. A 99 022321
[12] Jagadish V, Srikanth R and Petruccione F 2019 Phys. Rev. A 100 012336
[13] Lovas A and Andai A 2018 Rev. Math. Phys. 30 1850019
[14] Siudzińska K 2019 Phys. Rev. A 100 062331
[15] Siudzińska K 2020 Phys. Rev. A 101 062323
[16] Siudzińska K 2020 Phys. Rev. A 102 062615
[17] Filippow S N, Pülo J, Maniscalco S and Ziman M 2017 Phys. Rev. A 96 032111
[18] Puchala Z, Rudnicki and Zyczkowski K 2019 Phys. Lett. A 383 2376–81
[19] Shahbeigi F, Amaro-Alcalá D, Puchala Z and Zyczkowski K 2021 Log-Convex set of Lindblad semigroups acting on N-level system J. Math. Phys. 62 072105
[20] Jagadish V, Srikanth R and Petruccione F 2020 Phys. Rev. A 101 062304
[21] Jagadish V, Srikanth R and Petruccione F 2020 Phys. Lett. A 384 126907
[22] Jagadish V, Srikanth R and Petruccione F 2022 Phys. Rev. A 106 012338
[23] Bouda J, Koniorczyk M and Varga A 2009 Eur. Phys. J. D 53 365–72
[24] Harrow A, Hayden P and Leung D 2004 Phys. Rev. Lett. 92 187901
[25] Collins B and Nechita I 2010 Comm. Math. Phys. 297 345–70
[26] Collins B and Nechita I 2011 Adv. Math. 226 1181–201
[27] Collins B and Nechita I 2011 Ann. Appl. Probab. 21 1136–79
[28] Fukuda M and Nechita I 2018 IEEE Trans. Inf. Theory 64 17472877
[29] Bruzda W, Cappellini V, Sommers H-J and Zyczkowski K 2009 Phys. Lett. A 373 320–4
[30] Kukulski R, Nechita I, Pawela Ł, Puchała Z and Życzkowski K 2021 J. Math. Phys. 62 062201
[31] Wolf M M, Eisert J, Cubitt T S and Cirac J I 2008 Phys. Rev. Lett. 101 150402
[32] Smirne A, Kołodyński J, Huelga S F and Demkowicz-Dobrzański R 2016 Phys. Rev. Lett. 116 120801
[33] Ruskai M B 2003 Rev. Math. Phys. 15 643–62
[34] Watrous J 2018 The Theory of Quantum Information (Cambridge: Cambridge University Press)
[35] Breuer H-P and Petruccione F 2003 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[36] Gorini V, Kossakowski A and Sudarshan E 1976 J. Math. Phys. 17 821
[37] Lindblad G 1976 Comm. Math. Phys. 48 119
[38] Choi M-D 1975 Linear Algebra Appl. 10 285–90
[39] Jamiołkowski A 1972 Rep. Math. Phys. 3 275–8
[40] Nakajima S 1958 Prog. Theor. Phys. 20 948
[41] Zwanzig R 1960 J. Chern. Phys. 33 1338