EIGENVALUES OF TOEPLITZ OPERATORS ON THE ANNULUS AND NEIL ALGEBRA

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Abstract. By working with all collection of all the Sarason Hilbert Hardy spaces for the annulus algebra an improvement to the results of Aryana and Clancey on eigenvalues of self-adjoint Toeplitz operators on an annulus is obtained. The ideas are applied to Toeplitz operators on the Neil algebra. These examples may provide a template for a general theory of Toeplitz operators with respect to an algebra.

1. Introduction

In this article, eigenvalues for self-adjoint Toeplitz operators with real symbols associated to the Neil algebra and the algebra of bounded analytic functions on an annulus are investigated.

The Neil algebra \( \mathcal{A} \) is the subalgebra of \( H^\infty \) consisting of those \( f \) whose derivative at 0 is 0. Pick interpolation in this, and other related more elaborate subalgebras of \( H^\infty \), is a current active area of research with [DP], [DPRS], [BBtH], [BH1][BH2][JKM], and [K] among the references.

For the algebra \( A(\mathcal{A}) \) of functions analytic on the annulus \( \mathcal{A} \) and continuous on the closure of \( \mathcal{A} \) the results obtained here give finer detail than those of Aryana and Clancey [Ar1], [AC] (see also [Ar2] and [C]) in their generalization of a result of Abrahamse [A1]. The proofs are accessible to readers familiar with basic functional analysis and function theory on the annulus as found in either [F] or [S1]; in particular, they make no use of theta functions.

The approach used and structure exposed here applies to many other algebras, including \( H^\infty(\mathbb{R}) \) for a (nice) multiply connected domain in \( \mathbb{C} \) and finite codimension subalgebras of \( H^\infty \), though the details would necessarily be more complicated and less concrete than for the two algebras mentioned above.

The article proper is organized as follows. The algebras \( A(\mathcal{A}) \) and \( \mathcal{A} \) are treated in Sections 2 and 3 respectively. These sections can be read independently. Only the standard theory of \( H^2 \) is needed for Section 3. The article concludes with Section 4; it provides an additional rationale for

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considering families of representations when studying Toeplitz operators associated to the algebras $\mathcal{A}$ and $A(\mathcal{A})$.

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2. Toeplitz Operators on the Annulus

Fix $0 < q < 1$ and let $\mathbb{A}$ denote the annulus,

$$\mathbb{A} = \mathbb{A}_q = \{ z \in \mathbb{C} : q < |z| < 1 \}.$$ 

The boundary $B$ of $\mathbb{A}$ has two components

$$B_q = \{ z \in \mathbb{C} : |z| = q \}$$

and

$$B_1 = \{ z \in \mathbb{C} : |z| = 1 \}.$$ 

It is well known that for $\mathbb{A}$ the analog of the classical Hilbert Hardy space $H^2$ on the disc is a one parameter family of Hilbert spaces that can be described in several different ways $[AD]$, $[S1]$, or $[A1]$. For our purposes the following is convenient. Following $[S1]$ we will use the universal covering space of the annulus, $\hat{\mathbb{A}} = \{(r, t) \in \mathbb{R}^2 : q < r < 1 \text{ and } -\infty < t < \infty \}$ with locally conformal coordinates given by the map $\phi(r, t) \mapsto re^{it}$, to define modulus automorphic functions.

A **Modulus Automorphic** function, $F$, on $\hat{\mathbb{A}}$ is a meromorphic function on $\hat{\mathbb{A}}$ that satisfies

$$|F(r, t)| = |F(r, t + 2n\pi)| \text{ for all } q < r < 1, \ 0 \leq t < 2\pi \text{ and } n \in \mathbb{Z}.$$ 

So, although $f := F \circ \phi^{-1}$ may be multivalued on $\mathbb{A}$, the function $|f|$ is single valued. Because an analytic function is determined, up to a unimodular constant, by its modulus, if $F$ is modulus automorphic, then there exists a unimodular constant, $\lambda_F$, such that $F(r, t + 2\pi) \equiv \lambda_F F(r, t)$. The **index** of $F$, denoted by $\text{index}(F)$, is the unique $\alpha \in [0, 1)$ such that $\alpha = (2\pi i)^{-1} \log \lambda_F$. Let $\mu_j$ denote the multiple of arclength measure on $B_j$ weighted so that $\mu_j(B_j) = 2\pi$ and let $\mu = \mu_1 + \mu_q$.

Given $\alpha \in [0, 1)$, define an analog of $H^2(\mathbb{D})$ in the following way

$$H^2_\alpha(\mathbb{A}) := \{ F \circ \phi^{-1} : \text{index}(F) = \alpha \text{ and } \int_B |F \circ \phi^{-1}|^2 \ d\mu < \infty \}.$$ 

In $[S1$, Section 7] Sarason established the following important properties of $H^2_\alpha(\mathbb{A})$:

$$H^2_\alpha(\mathbb{A}) \subseteq L^2(\mathbb{A}) \text{ for all } \alpha \in [0, 1)$$

and, letting $\chi(z) = z$,

$$H^2_\alpha(\mathbb{A}) = \{ \chi^\alpha f : f \in H^2_0(\mathbb{A}) \}.$$ 

Moreover, Sarason showed that the Laurent polynomials are dense in $H^2_0(\mathbb{A})$ and thus $H^2_0(\mathbb{A})$ admits an analog to Fourier Analysis on the disk.
Turning to multiplication and Toeplitz operators on the $H^2_\alpha$ spaces, let $C(\mathbb{A})$ denote the Banach algebra (in the uniform norm) of continuous functions on the closure of $\mathbb{A}$. The annulus algebra, $A(\mathbb{A})$, is the (Banach) subalgebra of $C(\mathbb{A})$ consisting of those $f$ which are analytic in $\mathbb{A}$. It is easy to see that each $H^2_\alpha$ space is invariant for $A(\mathbb{A})$ in the sense that each $a \in A(\mathbb{A})$ determines a bounded linear operator $M^\alpha_a$ on $H^2_\alpha$ defined by

$$M^\alpha_a f = af.$$

Moreover, the mapping $\pi_\alpha : A(\mathbb{A}) \rightarrow B(H^2_\alpha)$ defined by $\pi_\alpha(a) = M^\alpha_a$ is a unital representation of the algebra $A(\mathbb{A})$ into the the space $B(H^2_\alpha)$ of bounded linear operators on the Hilbert space $H^2_\alpha$.

Next let $\phi \in L^\infty$ denote a real-valued function on $B$. The symbol $\phi$ determines a family, one for each $\alpha$, of Toeplitz operators. Specifically, let $T^\alpha_\phi$ denote the Toeplitz operator on $H^2_\alpha$ defined by

$$H^2_\alpha \ni f \mapsto P_\alpha \phi f,$$

where $P_\alpha$ is the projection of $L^2(B)$ onto $H^2_\alpha$. A function $g \in H^2_\alpha$ is outer if $\{ag \in H^2_\alpha : a \in A(\mathbb{A})\}$ is dense in the Hilbert space $H^2_\alpha$ (see [S1, Theorem 14]).

The following is the main result on the existence of eigenvalues for Toeplitz operators on $\mathbb{A}$.

**Theorem 2.1.** Fix a real-valued $\phi \in L^\infty$. Let $\alpha \in [0,1)$ and a nonzero $g \in H^2_\alpha$ be given. If $T^\alpha_\phi g = 0$, then $g$ is outer and moreover there exists a nonzero $c \in \mathbb{R}$ such that

$$\phi |g|^2 = c \log \left| \frac{1}{q} \right| .$$

If there is an $\alpha$ and an outer function $g \in H^2_\alpha$ such that Equation (1) holds, then $T^\alpha_\phi g = 0$, where $\alpha$ is necessarily the index of $g$. Thus $\alpha$ is congruent modulo 1 to,

$$\frac{1}{4\pi \log q} \left( \int_{B_1} \log |\phi| \ d\mu_1 - \int_{B_q} \log |\phi| \ d\mu_q \right).$$

In particular, there exists at most one $\alpha$ such that $T^\alpha_\phi$ has eigenvalue 0 and the dimension of this eigenspace is at most one.

Before we prove Theorem 2.1, we pause to collect two corollaries. We say that $\lambda$ is an eigenvalue of $\phi$ relative to $A(\mathbb{A})$ if there exists an $\alpha \in [0,1)$ and nontrivial solution $g$ to $T^\alpha_\phi g = \lambda g$.

**Corollary 2.2.** If

$$\text{ess sup} \{|\phi(z)| : z \in B_q\} = m < 0 < M = \text{ess inf} \{|\phi(z)| : z \in B_1\}$$

or

$$\text{ess sup} \{|\phi(z)| : z \in B_1\} = m < 0 < M = \text{ess inf} \{|\phi(z)| : z \in B_q\},$$

then each $\lambda \in (m, M)$ is an eigenvalue of $\phi$ relative to $A(\mathbb{A})$, the latter case only happening when the $c$ from Theorem 2.1 is negative. Further, $M$ (resp. $m$) is an eigenvalue if and only if $\frac{\log |q^{-1/2}|}{\phi-M} \in L^1$ (resp. $\frac{\log |q^{-1/2}|}{\phi-m} \in L^1$).
Corollary 2.3. The set of eigenvalues of $\phi$ relative to $A(\mathcal{A})$ is either empty, a point, or an interval.

The following Corollary of Theorem 2.1 and Corollary 2.2 generalizes the main result of [AC] for the annulus.

Corollary 2.4. With the hypotheses of Corollary 2.2, if either

$$\int_B \log |\phi - M| \, d\mu = -\infty$$

or

$$\int_B \log |\phi - m| \, d\mu = -\infty,$$

then for each $\alpha$ the Toeplitz operator $T^\alpha_\phi$ has infinitely many eigenvalues in the interval $(-m, M)$.

The remainder of this section is organized as follows. Subsection 2.1 contains the proof of Theorem 2.1. The corollaries are proved in Subsection 2.2.

2.1. Proof of Theorem 2.1. Let

$$A(\mathcal{A})^* := \{ f^* : f \in A(\mathcal{A}) \}.$$

In the context of Theorem 2.1, suppose $T^\alpha_\phi g = 0$. Using the fact that, if $a \in A(\mathcal{A})$, then $ag \in H^2_\alpha$ it follows that

$$0 = \langle T^\alpha_\phi g, ag \rangle = \int_B \phi |g|^2 a^* \, d\mu.$$

Since $\phi |g|^2$ is real-valued

$$\int_B \phi |g|^2 a \, d\mu = 0$$

too. Thus $\phi |g|^2$ annihilates $A(\mathcal{A}) \oplus A(\mathcal{A})^*$. In fact we know the following about measures that annihilate $A(\mathcal{A}) \oplus A(\mathcal{A})^*$.

Proposition 2.5. A measure $\nu << \mu$ who’s Radon-Nikodym derivative with respect to $\mu$ is in $L^1(B)$ annihilates $A(\mathcal{A}) \oplus A(\mathcal{A})^*$ if and only if there exists a $c \in \mathbb{C}$ such that

$$\frac{d\nu}{d\mu} = c \log \left( \chi q^{-\frac{1}{2}} \right).$$

Proof. If $\nu$ annihilates $A(\mathcal{A}) \oplus A(\mathcal{A})^*$, then for each $n \in \mathbb{Z}$

$$\int_B \chi^n \, d\nu = \int_B \chi^n \, \frac{d\nu}{d\mu} \, d\mu = \sum_{j=0}^{2\pi} \int_0^{2\pi} \frac{d\nu}{d\mu}(q^j e^{it})q^j e^{-int} \, dt$$

and

$$\int_B \overline{\chi^n} \, d\nu = \int_B \overline{\chi^n} \, \frac{d\nu}{d\mu} \, d\mu = \sum_{j=0}^{2\pi} \int_0^{2\pi} \frac{d\nu}{d\mu}(q^j e^{it})q^j e^{int} \, dt.$$
Hence
\[ \int_0^{2\pi} \frac{d\nu}{d\mu}(e^{it})e^{int} \, dt = -\int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})q^n e^{int} \, dt \] and
\[ \int_0^{2\pi} \frac{d\nu}{d\mu}(e^{it})e^{-int} \, dt = -\int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})q^n e^{-int} \, dt. \]

By replacing \( n \) with \(-n\) in the last equation we can see that
\[ q^n \int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})e^{int} \, dt = q^{-n} \int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})e^{int} \, dt. \]

Hence for all \( n \in \mathbb{Z} \)
\[ (q^n - q^{-n}) \int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})e^{int} \, dt = 0. \]

That means that for all \( 0 \neq m \in \mathbb{Z} \) and \( j = 0, 1 \)
\[ \int_0^{2\pi} \frac{d\nu}{d\mu}(qe^{it})e^{int} \, dt = 0. \]

So \( \frac{d\nu}{d\mu} \) must be equal to a constant, \( a \), almost everywhere on each boundary with \( a := \frac{d\nu}{d\mu}\big|_{B_1} = -\frac{d\nu}{d\mu}\big|_{B_q} \). If we choose \( c \in \mathbb{R} \) to be \( \frac{a}{\log q^2} \), then \( \frac{d\nu}{d\mu} = c \log |z q^{-\frac{1}{2}}| \) almost everywhere on \( B \).

Now let \( \nu \) be a measure such that \( \frac{d\nu}{d\mu} = c \log |z q^{-\frac{1}{2}}| \) almost everywhere on \( B \). Since \( \text{span}\{z^n, \bar{z}^n \mid n \in \mathbb{Z}\} \) is dense in \( A(\mathbb{A}) \oplus A(\mathbb{A})^* \) it will suffice to show that for \( n \in \mathbb{Z} \)
\[ \int_B \chi^n \, d\nu = \int_B \chi^n c \log |\chi q^{\frac{1}{2}}| \, d\mu = 0 \] and
\[ \int_B \bar{\chi}^n \, d\nu = \int_B \bar{\chi}^n c \log |\bar{\chi} q^{\frac{1}{2}}| \, d\mu = 0 \]
to prove that \( \nu \) annihilates \( A(\mathbb{A}) \oplus A(\mathbb{A})^* \). If \( n = 0 \), then
\[ \int_B c \log |\chi q^{\frac{1}{2}}| \, d\mu = \int_{B_1} c \log \left(q^{\frac{1}{2}}\right) \, d\mu + \int_{B_q} c \log \left(q^{\frac{1}{2}}\right) \, d\mu = 0. \]
If \( n \neq 0 \), then for \( j = 1, q \),
\[ \int_{B_j} c \log |\chi q^{\frac{1}{2}}| = c \log (jq^{-\frac{1}{2}}) \int_{B_j} \chi^n \, d\mu_1 = 0. \]

A similar computation shows that \( \int_B \bar{\chi}^n c \log |\chi q^{\frac{1}{2}}| \, d\mu = 0 \).

Combining the fact that if \( T^*_q g = 0 \), then \( \phi |g|^2 \) annihilates \( A(\mathbb{A}) \oplus A(\mathbb{A})^* \) and the above proposition we see that if \( T^*_q g = 0 \), then there exists some \( c \in \mathbb{R} \) such that \( \phi |g|^2 = c \log |\chi q^{-\frac{1}{2}}| \).

The next objective is to show that \( g \) is outer. First we need the following definition. A function \( \theta \) in \( H^2_\beta \) is **inner** if \( |	heta| = 1 \) on \( B \). Sarason in \([S1, \text{Theorem 7}]\) proved a version of inner-outer...
factorization for the annulus: given \( f \in H^2_a \), there is a \( \beta \) and an inner function \( \psi \in H^2_\beta \) and outer function \( F \in H^2_{a-\beta} \) such that

\[
f = \psi F.
\]

Let \( g = \psi F \) denote the inner-outer factorization of \( g \) as an \( H^2_a \) function as in Equation (3) and let \( \beta \in [0, 1) \) be the index of \( \psi \). Since \( \text{index}(\psi) + \text{index}(F) = \text{index}(g) \), we have that \( \chi^\beta F \in H^2_a \). Similarly we have that \( C := \chi^{-\beta} \psi \in H^2 \) which means that its restrictions to each of the boundary components \( B_1 \) and \( B_q \) is representable as a Fourier series whose coefficients we will denote by \( \hat{C}_q(n) \) and \( \hat{C}_1(n) \) respectively. Moreover by [S1, Lemma 1.1] we know that \( \hat{C}_1(n) = q^{-n} \hat{C}_q(n) \). Since we showed above that \( \phi|g|^2 = c \log |\chi q^{-\frac{n}{2}}| \) for some \( c \in \mathbb{R} \), for any \( n \in \mathbb{Z} \)

\[
0 = \left< T^a_\phi g, \chi^n \chi^\beta F \right> \\
= \int_B \phi |g|^2 |\chi|^{2\beta} \chi^{\beta} \psi \tilde{\chi} \ d\mu \\
= \int_B c \log (|\chi q^{-1/2}|) |\chi|^{2\beta} \tilde{\chi} \ d\mu \\
= c \log(q^{1/2}) \left( q^{n+2\beta} \hat{C}_q(n) - \hat{C}_1(n) \right) \\
= c \log(q) \hat{C}_q(n) \left( q^{n+2\beta} - q^{-n} \right).
\]

From Equation (4) it follows that, for each \( n \), either \( \hat{C}_q(n) = 0 \) or \( n + \beta = 0 \). Since \( \beta \in [0, 1) \) and \( \hat{C}_q(m) \neq 0 \) for some \( m \), it follows that \( \hat{C}_q(n) = 0 \) for \( n \neq 0 \) and \( \beta = 0 \). Thus \( \psi \) is a unitary constant and \( g \) is outer.

Next assume that \( g \in H^2_a \) is outer and equation (1) holds. By Proposition 2.5,

\[
\left< T^a_\phi g, ag \right> = \int_B \phi |g|^2 \tilde{a} \ d\mu = \int_B \left| \chi q^{-\frac{n}{2}} \right| \tilde{a} = 0,
\]

for every \( a \in \mathbb{A}(\mathbb{D}) \). Further since \( g \) is outer \( \{ag \mid a \in \mathbb{A}(\mathbb{D})\} \) is dense in \( H^2_a \). Thus \( T^a_\phi g = 0 \) which proves the second part.

To prove the third part of the Theorem, simply choose \( \alpha \) to be the index of \( g \). From [S1, Theorem 6] we know that, modulo one, the index of \( g \) is

\[
\frac{-1}{2\pi \log q} \left( \int_{B_1} \log |g| \ d\mu_1 - \int_{B_q} \log |g| \ d\mu_q \right).
\]

Applying the fact that \( |g| = \left( \frac{c \log |\chi q^{-1/2}|}{\phi} \right)^{1/2} \) the above expression simplifies to

\[
\frac{1}{4\pi \log q} \left( \int_{B_1} \log |\phi| \ d\mu_1 - \int_{B_q} \log |\phi| \ d\mu_q \right).
\]
Finally, suppose that $T_\theta^*g = 0$ and also $T_\theta^*h = 0$. From what has already been proved $g$ and $h$ are outer and there exists nonzero $c, d \in \mathbb{R}$ such that
\[
\phi |g|^2 = c \log |\chi q^{-1/2}|, \quad \phi |h|^2 = d \log |\chi q^{-1/2}|
\]
on $B$. Since $\phi$ is almost everywhere nonzero we have that $|g|^2 = \frac{c}{d} |h|^2$ and because $g$ and $h$ are outer, they are equal up to a complex scalar multiple, see [S1, Theorem 7.9].

2.2. Proofs of the corollaries. To prove Corollary 2.2, observe that the first (resp. second) displayed inequality implies, for $m < \lambda < M$, that
\[
(5) \quad \psi = \frac{\log |\chi q^{-1/2}|}{\phi - \lambda}
\]
takes nonnegative (resp. nonpositive) values and is essentially bounded above (resp. below) and below (resp. above) away from zero. Hence by [S1, Theorem 9] there exists an outer function $g$ such that such that $|g|^2 = \psi$ (resp. $|g|^2 = -\psi$). From Theorem 2.1 there is a $\alpha$ such that $T_\theta^*g = \lambda g$. The case $\lambda = M$ (resp. $\lambda = m$) is similar, but now, while $\psi$ is still essentially bounded below away from zero, it need not be integrable. If $\psi$ is integrable, than the argument above shows it is an eigenvector with eigenvalue $M$ (resp. $m$). On the other hand, if $M$ (resp. $m$) is an eigenvalue, then there is an outer function $g$ so that $\psi = |g|^2$ and hence $\psi$ is integrable.

It suffices to prove Corollary 2.3 for $M = \text{ess inf} \{\phi(z) : z \in B_1\}$ and $m = \text{ess sup} \{\phi(z) : z \in B_q\}$. By Corollary 2.2 if $m < M$ then the set of $\phi$ relative to $A(\mathbb{A})$ contains the interval $(m, M)$ and otherwise 2.1 says that the set of eigenvalues is at most a point. So we must show that if $\lambda > M$ or $\lambda < m$ then $\lambda$ is not an eigenvalue. To this end, suppose $\lambda > M$. Since $\lambda > M = \text{ess inf} \{\phi(z) : z \in B_1\}$ we have that $\mu(\{(\phi(z) - \lambda < 0 : z \in B_1\})) > 0$. On the other hand, if $\lambda$ is an eigenvalue, then there is a nonzero $d \in \mathbb{R}$ and multivalued outer function $h \in L^2(B)$ such that $(\phi - \lambda) |h|^2 = d \log |\chi q^{-1/2}|$ which implies that either $(\phi - \lambda)|_{B_1}$ is positive almost everywhere or $(\phi - \lambda)|_{B_q}$ is positive almost everywhere. This is a contradiction since we know that $(\phi - \lambda)$ is negative on the inner boundary of the annulus and not positive almost everywhere on the outer boundary. This proves Corollary 2.3 for $\lambda > M$. The proof for the case $\lambda < m$ proceeds analogously. The details are omitted.

It suffices to prove Corollary 2.4 when $M = \text{ess inf} \{\phi(z) : z \in B_1\}$ and $m = \text{ess sup} \{\phi(z) : z \in B_q\}$. Assume that $\int_B \log |\phi - M| \, d\mu = -\infty$. Given $m < \lambda < M$, by Corollary 2.2, there is a unique, up to scalar multiple, outer function $g_{\lambda}$ of $T_{\phi_{\lambda}}$ whose modulus squared is given by $\psi$ in equation (5). By Theorem 2.1, the index $\alpha_{\lambda} \in \{0, 1\}$ of $g_{\lambda}$ is congruent, modulo one, to
\[
\beta_{\lambda} := \frac{1}{4\pi \log q} \left( \int_{B_1} \log |\phi - \lambda| \, d\mu_1 - \int_{B_q} \log |\phi - \lambda| \, d\mu_q \right).
\]
Notice that as $\lambda$ approaches $M$ on $B_q$ we have that $\int_{B_q} \log |\phi - \lambda| < \infty$ since $\phi \in L^\infty$ and $\text{ess sup} \{\phi(z) : z \in B_q\} < M$. It follows from the monotone convergence theorem that $\beta_{\lambda}$ approaches $-\infty$.
as $\lambda$ approaches $M$. Hence $\alpha_\lambda$ takes every value in the interval $[r, 1)$ infinitely often for every choice of $0 \leq r < 1$. Which completes the proof in the case $M = \text{ess inf} \{\phi(z) : z \in B_1\}$ and $\int_B \log |\phi - M| \, d\mu = -\infty$. The proof for the case $m = \text{ess sup} \{\phi(z) : z \in B_1\}$ and $\int_B \log |\phi - m| \, d\mu = -\infty$ proceeds similarly.

3. Toeplitz Operators on the Neil Parabola

Let $\mathcal{A}$ denote the Neil Algebra; i.e., $\mathcal{A}$ is the unital subalgebra of the disc algebra $A(D)$ consisting of those $f$ with $f'(0) = 0$. Each subspace $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ determines a subspace $H^2_\mathcal{V} = H^2 \ominus \mathcal{V}$ of the classical Hardy space $H^2$ which is invariant for $\mathcal{A}$ in the following sense. Each $a \in \mathcal{A}$ determines a bounded linear operator $M^\mathcal{V}_a$ on $H^2_\mathcal{V}$ defined by $M^\mathcal{V}_a f = af$. Moreover, the mapping $\pi_\mathcal{V} : \mathcal{A} \to B(H^2_\mathcal{V})$ defined by $\pi_\mathcal{V}(a) = M^\mathcal{V}_a$ is a unital representation. Here $B(H^2_\mathcal{V})$ is the algebra of bounded operators on $H^2_\mathcal{V}$. A further discussion of the collection of representations $\pi_\mathcal{V}$ can be found in Section 4.

Let $\phi$ denote a real-valued function on the unit circle $\mathbb{T}$. The symbol $\phi$ determines a family, one for each $\mathcal{V}$, of Toeplitz operators. Specifically, let $T^\mathcal{V}_\phi$ denote the Toeplitz operator on $H^2_\mathcal{V}$ defined by $H^2_\mathcal{V} \ni f \mapsto P_\mathcal{V}\phi f$, where $P_\mathcal{V}$ is the projection of $L^2(\mathbb{T})$ onto $H^2_\mathcal{V}$.

Letting $\chi(z) = z$ we get the following as the main result on the existence of eigenvalues for Toeplitz operators on $H^2_\mathcal{V}$.

**Theorem 3.1.** Fix a real-valued $\phi \in L^\infty$ and let $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ and nonzero $g \in H^2_\mathcal{V}$ be given. If $T^\mathcal{V}_\phi g = 0$, then $g$ is outer and moreover there is a $c \in \mathbb{C}$ such that, on $\mathbb{T}$,

$$
\phi |g|^2 = c\chi + (c\chi)^*.
$$

Conversely, if there is a $c \in \mathbb{C}$ and outer function $g \in H^2$ such that Equation (6) holds, then $T^\mathcal{V}_\phi g = 0$, where $\mathcal{V}$ is uniquely determined by the values $g(0)$ and $g'(0)$.

In particular, there exists at most one $\mathcal{V}$ such that $T^\mathcal{V}_\phi$ has eigenvalue 0 and the dimension of this eigenspace is at most one.

Before turning to the proof of Theorem 3.1, we pause to state the analogs of corollaries 2.2, and 2.3. By analogy with the case of the annulus, we say that $\lambda$ is an **eigenvalue of $\phi$ relative to $\mathcal{A}$** if there exists a $\mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z$ and nontrivial solution to $T^\mathcal{V}_\phi g = \lambda g$. 
Corollary 3.2. If there is a $c \in \mathbb{C}$ such that
\[
\text{ess sup}\{\phi(z) : cz + (cz)^* < 0\} = m < 0 < M = \text{ess inf}\{\phi(z) : cz + (cz)^* > 0\},
\]
then each $\lambda \in (m, M)$ is an eigenvalue of $\phi$ relative to $\mathcal{A}$. Further, for each such $\lambda$ there is an essentially unique outer function $f_\lambda$ such that
\[
(\phi - \lambda)|f_\lambda|^2 = c\chi + (c\chi)^*.
\]
(Here $c$ is independent of $\lambda$.)
Moreover, $M$ (resp. $m$) is an eigenvalue if and only if $\frac{c\chi^*}{\phi - M}$ (resp. $\frac{c\chi^*}{\phi - m}$) is in $L^1$.

Corollary 3.3. The set of eigenvalues of $\phi$ relative to $\mathcal{A}$ is either empty, a point, or an interval.

If we are given a $\phi$ that satisfies Corollary 3.2 and a $\lambda \in (m, M)$ the following corollary allows us to determine exactly what $H^2_\nu$ space the resulting outer function $f_\lambda$ is in. But first for $z \in \mathbb{D}$ and $t$ real, let
\[
H(z, t) = \frac{e^{it} + z}{e^{it} - z}.
\]

Corollary 3.4. Under the hypotheses of corollary 3.2 let
\[
h_c(z) := \exp\left(\frac{1}{2\pi} \int_{-\pi}^\pi H(z, t) \log |ce^{it} + c^*e^{-it}|^{1/2} \, dt\right),
\]
and
\[
g_\lambda(z) := \exp\left(\frac{1}{2\pi} \int_{-\pi}^\pi H(z, t) \log |\phi(t) - \lambda|^{-1/2} \, dt\right).
\]
The eigenvector $f_\lambda$ of $T^\nu_\phi$ associated with the eigenvalue $\lambda$ is in the $H^2_\nu$ space where (nontrivial) $\nu \subseteq \mathbb{C} \oplus \mathbb{C}_z$ is orthogonal to
\[
h_c(0)g_\lambda(0) + (h_c(0)g'_\lambda(0) + h'_c(0)g_\lambda(0))z
\]
Moreover since $h_c$ and $g_\lambda$ are outer functions neither $h_c(0)$ or $g_\lambda(0)$ are zero.

There is no analog of Corollary 2.4 or of the main result of [AC] for the Neil parabola. In fact Corollary 3.4 implies that if $e \subseteq \mathbb{C} \oplus \mathbb{C}_z$ is spanned by 1, then no Toeplitz operator on $H^2_e$ has eigenvalues. Although an easier way to see that no Toeplitz operator on $H^2_e$ has eigenvalues is to note that $H^2_e = zH^2$. Moreover for similar reasons no Toeplitz operator on $H^2_\nu$ has eigenvalues if $\nu = \{0\}$ or $\nu = \mathbb{C} \oplus \mathbb{C}_z$. In fact the following corollary says that there are many nonzero proper $\nu$, not just $e$, such that $T^\nu_\phi$ has no eigenvalues. To prove this we identify each nonzero proper $\nu$ with an element of the complex projective line, $\mathbb{P}^1(\mathbb{C})$, which is $X = \mathbb{C}^2 \setminus \{0\}$ modulo the equivalence relation $v \sim w$ if and only if there is a complex number $\lambda$ such that $v = \lambda w$. Let
\[
\pi : X \to \mathbb{P}^1(\mathbb{C})
\]
denote the quotient mapping of $v \in \mathbb{C}^2 \setminus \{0\}$. The space $\mathbb{P}^1(\mathbb{C})$ can be realized as a Riemann surface by the charts $\Phi_j : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ defined by $\Phi_0(\zeta) = \left(\zeta, 1\right)^T$ and $\Phi_1(\xi) = \left(1, \xi\right)^T$. 

**EIGENVALUES FOR TOEPLITZ OPERATORS**
Indeed, the transition mappings between these charts are \( \zeta = \frac{1}{\xi} \) and \( \xi = \frac{1}{\zeta} \). A map \( F : \mathbb{R} \to \mathbb{P}^1(\mathbb{C}) \) is differentiable if the maps \( \Phi_0^{-1} \circ F \) and \( \Phi_1^{-1} \circ F \) are differentiable where defined. Finally, if \( I \) is an interval in \( \mathbb{R} \) and \( g : I \to X \) is twice differentiable, then so is \( \pi \circ g \) and in this case the Hausdorff dimension of the range of \( \pi \circ g \) is at most one and in this sense the range is a relatively small subset of \( \mathbb{P}^1(\mathbb{C}) \). For a discussion of properties of the Hausdorff dimension see [Sc].

**Corollary 3.5.** The function, \( \Lambda \), from \((m, M)\) to \( \mathbb{P}^1(\mathbb{C}) \) defined by

\[
\Lambda : \lambda \mapsto \pi \left( \begin{bmatrix} h_c(0)g_{\lambda}(0) \\ h'_c(0)g_{\lambda}(0) + h_c(0)g'_{\lambda}(0) \end{bmatrix} \right)
\]

is locally Lipschitz with respect to \( \lambda \) on \((m, M)\). Thus in addition to \( \mathcal{V} = \text{span}[1] \) there exist nonzero proper \( \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C} \zeta \) such that \( T^\mathcal{V}_\phi \) has no eigenvalues.

The remainder of this section is organized as follows. Subsection 3.1 contains the proof of Theorem 3.1. The corollaries are proved in Subsection 3.2.

### 3.1. Proof of Theorem 3.1.

Let \( \mu \) denote normalized arclength measure on \( \mathbb{T} \) and let

\[
\mathcal{A}^* := \{ f^* : f \in \mathcal{A} \}.
\]

In the context of Theorem 3.1, suppose \( T^\mathcal{V}_\phi g = 0 \). Using the fact that, if \( a \in \mathcal{A} \), then \( ag \in \text{H}^2_\mathcal{V} \), it follows that

\[
0 = \langle T^\mathcal{V}_\phi g, ag \rangle = \int_{\mathbb{T}} \phi |g|^2 a \, d\mu.
\]

Since \( \phi |g|^2 \) is real-valued it is also the case that

\[
\int_{\mathbb{T}} \phi |g|^2 a \, d\mu = 0.
\]

Thus \( \phi |g|^2 \) annihilates \( \mathcal{A} \oplus \mathcal{A}^* \). In fact we know the following about measures that annihilate \( \mathcal{A} \oplus \mathcal{A}^* \).

**Proposition 3.6.** A measure \( \nu \ll \mu \) who’s Radon-Nikodym derivative with respect to \( \mu \) is in \( L^1(\mathbb{T}) \) annihilates \( \mathcal{A} \oplus \mathcal{A}^* \) if and only if there exists a \( c \in \mathbb{C} \) such that

\[
\frac{d\nu}{d\mu} = c\chi + (c\chi)^*.
\]

**Proof.** Assume that \( \nu \) annihilates \( \mathcal{A} \oplus \mathcal{A}^* \), then for each \( n \in \mathbb{Z} \setminus \{1, -1\} \)

\[
\int_{\mathbb{T}} \chi^n \, d\nu = \int_{\mathbb{T}} \chi^n \frac{d\nu}{d\mu} \, d\mu = 0
\]
Hence for any \( n \) different from \( \pm 1 \) the corresponding Fourier coefficient of \( \frac{dv}{d\mu} \) is zero. Thus there exists \( c \in \mathbb{C} \) such that all the Fourier coefficients of \( \frac{dv}{d\mu} - c\chi + (c\chi)^* \) are 0, which implies that \( \frac{dv}{d\mu} = c\chi + (c\chi)^* \) almost everywhere on \( \mathbb{T} \).

Now let \( \nu \) be a measure such that \( \frac{dv}{d\mu} = c\chi + (c\chi)^* \) almost everywhere on \( \mathbb{T} \). Since the span of the set \( \{z^n \mid n \in \mathbb{Z} \setminus \{1, -1\}\} \) is dense in \( \mathcal{A} \oplus \mathcal{A}^* \) it will suffice to show that for \( n \in \mathbb{Z} \setminus \{1, -1\} \)

\[
\int_{\mathbb{T}} x^n \, dv = \int_{\mathbb{T}} x^n (c\chi + (c\chi)^*) \, d\mu = 0
\]

to prove that \( \nu \) annihilates \( \mathcal{A} \oplus \mathcal{A}^* \). But for \( n \in \mathbb{Z} \setminus \{1, -1\} \)

\[
\int_{\mathbb{T}} x^n (c\chi + (c\chi)^*) \, d\mu = \int_{\mathbb{T}} c\chi^{n+1} + (c\chi^{n-1})^* = 0.
\]

Combining the fact that if \( T_\phi^V g = 0 \), then \( \phi |g|^2 \) annihilates \( \mathcal{A} \oplus \mathcal{A}^* \) and the above proposition we see that if \( T_\phi^V g = 0 \), then there exists some \( c \in \mathbb{C} \) such that \( \phi |g|^2 = c\chi + (c\chi)^* \).

The next objective is to show \( g \) is outer. To this end, let \( g = \Psi F \) denote the inner-outer factorization of \( g \) as an \( H^2 \) function. Observe that, \( z^n F \in H^2_\mathcal{V} \) for integers \( n \geq 2 \). Thus, for such \( n \),

\[
0 = \left\langle T_\phi^V g, Fz^n \right\rangle = \int_{\mathbb{T}} \phi |g|^2 \Psi z^n \, d\mu = \left\langle (cz + (cz)^*)\Psi, z^n \right\rangle_{L^2}.
\]

It follows, writing \( \Psi = \sum_{k=0}^{\infty} \Psi_k z^k \), that

\[
c\Psi_{n-1} + c^*\Psi_{n+1} = 0
\]

for \( n \geq 2 \). In particular, \( \Psi_{2k+1} = (-\frac{c}{c^*})^k \Psi_1 \) for \( k \geq 1 \) and likewise, \( \Psi_{2k+2} = (-\frac{c}{c^*})^k \Psi_2 \).

Because \( \Psi \in H^2 \) these last two equations imply that \( \Psi_k = 0 \) for \( k \geq 1 \); i.e., \( \Psi \) is a unimodular constant and thus \( g \) is outer, and the first part of the Theorem is established.

The proof of the converse uses the following lemma.

**Lemma 3.7.** Given a nonzero \( \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z \), if \( g \) is outer and in \( H^2_\mathcal{V} \), then the set

\[
\{ag : a \in \mathcal{A}\}
\]

is dense in \( H^2_\mathcal{V} \).
Proof. Let $f \in H^2_V$ be given. Since $g$ is outer there exists a sequence of functions $\{a_n\} \subseteq H^\infty(D)$ such that $a_ng$ converges to $f$ in $H^2(D)$. Let $b_n := a_n - a_n'(0)$, so each $b_n \in \mathcal{A}$. To show that $b_ng$ also converges to $f$ it suffices to show that $a'_n(0)$ converges to zero. To do this note that for each nonzero proper $V$ there exists a pair $(\alpha, \beta) \in \mathbb{C}^2$ not both zero such that if $h \in H^2_V$, then there exists a $\zeta \in \mathbb{C}$ and $q \in H^2(D)$ such that $h = \zeta \alpha + \zeta \beta z + z^2 q$. Since $g(0) \neq 0$ this means that

$$\frac{f(0)}{g(0)} \cdot g'(0) = f'(0).$$

Because $a_n(0)g(0)$ converges to $f(0)$ and $(a_ng)'(0)$ converges to $f'(0)$ we have that

$$\lim_{n \to \infty} a_n(0) = \frac{f(0)}{g(0)}$$
$$\lim_{n \to \infty} a'_n(0) = \frac{f'(0) - \frac{f(0)}{g(0)} \cdot g'(0)}{g(0)} = 0$$

To prove the converse, suppose that $g$ is outer and there is a $c \in \mathbb{C}$ such that Equation (6) holds. Let $V$ be the nonzero subspace of $\mathbb{C} \oplus \mathbb{C}z$ such that $g(0) + g'(0)z \in V^\perp$. In particular, $g \in H^2_V$. It follows that, for any $a \in \mathcal{A}$,

$$\langle T^V_\phi g, ag \rangle = \int_\mathbb{T} \phi |g|^2 a^* \ d\mu = 0$$

and thus, in view of Lemma 3.7, $T^V_\phi g = 0$.

Finally, suppose that $T^V_\phi g = 0$ and $T^W_\phi h = 0$. From what has already been proved $g$ and $h$ are outer and there exists $c, d \in \mathbb{C}$ such that

$$\phi |g|^2 = c\chi + (c\chi)^* \quad \text{and} \quad \phi |h|^2 = d\chi + (d\chi)^*$$
on $\mathbb{T}$. It follows that $\phi$ is positive almost everywhere both where $c\chi + (c\chi)^*$ and $(d\chi) + (d\chi)^*$ are positive. Hence $c = td$ for some positive real number $t$. But then, $t |g| = |h|$ and because $g$ and $h$ are outer, they are equal up to a (complex) scalar multiple.

3.2. Proofs of the corollaries. To prove Corollary 3.2, observe that the hypotheses imply, for $m < \lambda < M$, that

$$\psi = \frac{c\chi + (c\chi)^*}{\phi - \lambda}$$
is nonnegative, in $L^1$ and moreover

$$\int_\mathbb{T} \log |\psi| \ d\mu > -\infty$$

(7) because the same is true with $\psi$ replaced by $c\chi + (c\chi)^*$, $\phi$ is essentially bounded and $\text{sgn} (c\chi + (c\chi)^*) = \text{sgn}(\phi)$. Hence there is an outer function $g \in H^2$ such that

$$(\phi - \lambda) |g|^2 = c\chi + (c\chi)^*.$$
From Theorem 3.1 there is a nonzero proper \( V \) such that

\[
T^V \phi \ g = \lambda \ g.
\]

The case \( \lambda = M \) (resp. \( \lambda = m \)) are similar, with the only issue being that a hypothesis is needed to guarantee that \( \psi \), as defined above, is integrable.

Turning to the proof of Corollary 3.3, because Corollary 3.2 implies the interval \((m, M)\) is contained in the set of eigenvalues of \( \phi \) with respect to \( \mathcal{A} \), it suffices to show if \( \lambda > M \) or \( \lambda < m \), then \( \lambda \) is not an eigenvalue. Accordingly suppose \( \lambda > M \). In this case the measure of the set \( S = \{ z \in \mathbb{T} : \phi(z) > \lambda \} \) is less than \( \frac{\pi}{2} \). On the other hand, if \( \lambda \) is an eigenvalue, then there is a non-zero \( c \) and outer function \( h \in H^2 \) such that

\[
(\phi - \lambda) |h|^2 = c\chi + (c\chi)^*.
\]

But then the measure of the set \( S \) is \( \frac{\pi}{2} \), a contradiction. Which proves the corollary when \( \lambda > M \).

The proof of the case \( \lambda < m \) proceeds analogously. It now follows that set of eigenvalues contains \((m, M)\) and is contained in \([m, M]\) and the proof of the corollary is complete.

To prove Corollary 3.4 use (6) and the fact that \( f_\lambda \) is outer to see that

\[
f_\lambda(z) = \exp \left( \int_{\mathbb{T}} H(z, \cdot) \log \left( \frac{c\chi + (c\chi)^*}{\phi - \lambda} \right)^{1/2} \ d\mu \right)
\]

\[
= \exp \left( \int_{\mathbb{T}} H(z, \cdot) \log |c\chi + (c\chi)^*|^{1/2} \ d\mu \right) \exp \left( \int_{\mathbb{T}} H(z, \cdot) \log |\phi - \lambda|^{-1/2} \ d\mu \right)
\]

Thus \( f_\lambda(0) = h_c(0)g_\lambda(0) \) and \( f'_\lambda(0) = h_c(0)g'_\lambda(0) + h'_c(0)g_\lambda(0) \) and the conclusion follows.

To prove Corollary 3.5 we will first show that the maps

\[
\lambda \mapsto g_\lambda(0)\text{ and } \lambda \mapsto g'_\lambda(0)
\]

are twice differentiable with respect to \( \lambda \) on \((m, M)\). Those questions boil down to checking if

\[
\lambda \mapsto \int_{\mathbb{T}} H(0, \cdot) \log |\phi - \lambda| \ d\mu \text{ and }
\lambda \mapsto \int_{\mathbb{T}} H'(0, \cdot) \log |\phi - \lambda| \ d\mu
\]

are twice differentiable with respect to \( \lambda \) on \((m, M)\). For a given \( \lambda_0 \in (m, M) \) there is a \( \delta > 0 \) such that \( \phi - \lambda \) is essentially bounded above and away from zero for \( |\lambda - \lambda_0| < \delta \). It follows that for such \( \lambda \), the functions \( H(0, t) \log(|\phi(t) - \lambda|) \) and \( H'(0) \log(|\phi(t) - \lambda|) \) as well as \( (\phi(t) - \lambda)^{-1} \) are all bounded above and below. Thus a standard application of the dominated convergence theorem establishes the desired differentiability. A similar argument shows that in fact both functions are infinitely
differentiable. Since $\Phi_1 \circ \Lambda$ is twice differentiable on $(m, M)$ it is locally Lipschitz. Because $(m, M)$ can be written as a countable union of intervals with $\Phi_1 \circ \Lambda$ Lipschitz on each interval the Hausdorff dimension of $\Phi_1 \circ \Lambda((m, M))$ is at most 1. So $\Lambda((m, M))$ cannot be all of $\mathbb{P}^1(\mathbb{C}) \setminus \{[0, 1]\}$ since $\Phi_1$ is injective on its range. Let $\mathcal{L} = \{\mathcal{V} \mid \mathcal{V}$ is a nonzero proper subspace of $\mathbb{C} + \mathbb{C}z\}$. Finally for each nonzero proper $\mathcal{V}$ choose a $f \in H_1^\mathcal{V}$ with $f(0)$ and $f'(0)$ not both zero and let $\tau : \mathcal{L} \to \mathbb{P}^1(\mathbb{C})$ be defined by map $\tau : \mathcal{V} \mapsto [f(0), f'(0)]$. The map $\tau$ is a bijection between $\mathcal{L}$ and $\mathbb{P}^1(\mathbb{C})$, thus if $\tau(\mathcal{V}) \notin \Lambda((m, M))$, then by 3.3 and 3.4 we have that $T_\phi^\mathcal{V}$ has no eigenvalues. Additionally if $m$ (resp. $M$) is an eigenvalue of $\phi$ relative to $\mathcal{A}$ then we need to add the condition that $\tau(\mathcal{V}) \neq \Lambda(M)$ (resp. $\tau(\mathcal{V}) \neq \Lambda(M)$) for the above conclusion to hold.

4. Bundle shifts

It is natural to ask what distinguishes the families of representations $\{\pi_\alpha : 0 \leq \alpha < 1\}$ and $\{\pi_\mathcal{V} : \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z\}$ of the algebras $\Lambda(\mathbb{A})$ and $\mathcal{A}$ as multiplication operators on the spaces $\{H_\alpha^2 : 0 \leq \alpha < 1\}$ and $\{H_\alpha^\mathcal{V} : \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z\}$ respectively.

For the annulus, an answer is that Sarason recognized that the collection of representations $(\pi_\alpha, H_\alpha^2)$ played the same role on the annulus as the single representation determined by the shift operator $S$ given by

\begin{equation}
\Lambda(\mathbb{A}) \ni f \mapsto f(S)
\end{equation}

plays for the disc algebra $\Lambda(\mathbb{D})$. For $\mathcal{A}$ the representations $\pi_\mathcal{V}$ generate a family of positivity conditions sufficient for Pick interpolation in $\mathcal{A}$ [DPRS]. Likely it is a minimal set of conditions too. Corollary 4.2 below can be interpreted as saying that the representations $(\pi_\mathcal{V}, H_\alpha^\mathcal{V}; \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z)$ should play the role of the rank one bundle shifts for the Neil algebra $\mathcal{A}$. In a dual direction [DP] found a minimal set of test functions for $\mathcal{A}$. For similar results on multiply connected domains see [BH1] and [BH2].

For positive integers $n$, the algebra $M_n(B(H))$ of $n \times n$ matrices with entries from $B(H)$ is naturally identified with $B(\mathbb{C}^n \otimes H)$, the operators on the Hilbert space $\mathbb{C}^n \otimes H \equiv \oplus^n H$. In particular, it is then natural to give an element $X \in M_n(B(H))$ the norm $\|X\|_n$ it inherits as an operator on $\mathbb{C}^n \otimes H$. If $A$ is a subalgebra of $B(H)$, then the norms $\|\cdot\|_n$ of course restrict to $M_n(A)$, the $n \times n$ matrices with entries from $A$, and $A$ together with this sequence of norms is a concrete operator algebra.

Turning to the disc algebra, an element $F \in M_n(\Lambda(\mathbb{D}))$ takes the form $F = (F_{j,k})_{j,k=1}^n$ for $F_{j,k} \in \Lambda(\mathbb{D})$. In particular, $M_n(\Lambda(\mathbb{D}))$ is itself an algebra and comes naturally equipped with the norm,

$$
\|F\|_n = \sup\{\|F(z)\| : z \in \mathbb{D}\}.
$$
where \( \|F(z)\| \) is the usual operator norm of the \( n \times n \) matrix \( F(z) \). The representation \( \pi : \mathbb{A}(\mathbb{D}) \to B(H^2) \) given by \( \pi(a) = M_a \) extends naturally to \( M_n(\mathbb{A}(\mathbb{D})) \) as \( 1_n \otimes \pi : M_n(\mathbb{A}(\mathbb{D})) \to B(\otimes^n H^2) \) by

\[
1_n \otimes \pi(F) = \left( \pi(F_{j,k}) \right)_{j,k=1}^n.
\]

Moreover, the maps \( 1_n \otimes \pi \) are isometric. Thus the algebra \( \mathbb{A}(\mathbb{D}) \) can be viewed as an operator algebra by identifying \( \mathbb{A}(\mathbb{D}) \) together with the sequence of norms \( (\|\cdot\|_n) \) with its image in \( B(H^2) \) under the mappings \( 1_n \otimes \pi \). Of course, any subalgebra of \( \mathbb{A}(\mathbb{D}) \) can also then be viewed as an operator algebra by inclusion.

Given an operator algebra \( A \), a representation \( \rho : A \to B(H) \) is completely contractive if \( \|1_n \otimes \rho(F)\|_n \leq \|F\|_n \) for each \( n \) and \( F \in M_n(A) \). If \( A \) and \( B \) are unital, then \( \rho \) is a unital representation if \( \rho(1) = 1 \). The representation \( \rho \) on \( B(H) \) is pure if

\[
\bigcap_{a \in A} \rho(a)H = \{0\}.
\]

It is immediate that the representations of \( \mathbb{A}(\mathbb{D}) \) determined by \( S \) as well as the representations \( \pi_\alpha \) of \( A(\mathbb{A}) \) and \( \pi_V \) of \( \mathcal{A} \) are unital, completely contractive, and pure.

Following Agler [Ag], a completely contractive (unital) representation \( \pi : \mathcal{A} \to B(H) \) of \( \mathcal{A} \) on the Hilbert space \( H \) is extremal if whenever \( \rho : \mathcal{A} \to B(K) \) is a completely contractive representation on the Hilbert space \( K \) and \( V : H \to K \) is an isometry such that

\[
\pi(a) = V^* \rho(a) V
\]

then in fact

\[
V \pi(a) V = \rho(a) V.
\]

Given a Hilbert space \( \mathcal{N} \), let \( H^2_\mathcal{N} \) denote the Hilbert Hardy space of \( \mathcal{N} \)-valued analytic functions on the disc with square integrable boundary values. Associated to \( \mathcal{N} \) is the representation \( \rho : \mathbb{A}(\mathbb{D}) \to B(H^2_\mathcal{N}) \) defined by

\[
\rho(\varphi)f = \varphi f.
\]

Thus, \( \rho(\varphi) \) is multiplication by the scalar-valued \( \varphi \) on the vector-valued \( H^2 \) space \( H^2_\mathcal{N} \). Of course, \( H^2_\mathcal{N} \) is naturally identified with \( \mathcal{N} \otimes H^2 \) and the representation \( \rho \) is then the identity on \( \mathcal{N} \) tensored with the representation of \( \mathbb{A}(\mathbb{D}) \) in Equation (8). If \( \rho \) is a completely contractive unital pure extremal representation of \( \mathbb{A}(\mathbb{D}) \), then there exists a Hilbert space \( \mathcal{N} \) so that, up to unitary equivalence, \( \rho : \mathbb{A}(\mathbb{D}) \to B(H^2_\mathcal{N}) \) is given by \( \rho(\varphi)f = \varphi f \).

For \( \mathcal{A} \) it turns out that the subspaces of \( H^2_\mathcal{N} \) identified in [R] give rise to the extremal representations. Indeed, given a Hilbert space \( \mathcal{N} \) and a subspace \( \mathcal{V} \) of the subspace \( \mathcal{N} \oplus z\mathcal{N} \) of \( H^2_\mathcal{N} \), the mapping \( \pi_\mathcal{V} : \mathcal{A} \to B(H^2_\mathcal{N} \oplus \mathcal{V}) \) defined by

\[
\pi_\mathcal{V}(a)f = af,
\]

is easily seen to be a unital pure completely contractive representation of \( \mathcal{A} \).
Theorem 4.1. The representations $\pi_V$ are unital pure completely contractive extremal representations. Moreover, if $\nu$ is a unital pure extremal completely contractive representation of $\mathcal{A}$, then $\nu$ is unitarily equivalent to $\pi_V$ for some Hilbert space $N$ and $V \subseteq N \oplus N_z$.

Finally we will say a representation has rank one if there does not exist a nontrivial orthogonal pair of subspaces invariant for the representation.

Corollary 4.2. The representations $\pi_V$ for $V \subseteq \mathbb{C} \oplus \mathbb{C} z$ have rank one. Moreover if the representation $\pi$ is a unital pure extremal completely contractive rank one representation of $\mathcal{A}$, then there is a $V \subseteq \mathbb{C} \oplus \mathbb{C} z$ such that $\pi$ is unitarily equivalent to $\pi_V$.

The remainder of the section is organized as follows. Subsection 4.1 proves that the representations $\pi_V$ are extremal. Subsection 4.2 contains the proof of the remainder of Theorem 4.1. The corollary is proved in 4.3.

4.1. The Extremal Representations of $\mathcal{A}$. While it is easy to see that the representations $\pi_V : \mathcal{A} \to B(H^2_V)$ are unital, pure, and completely contractive showing that they are also extremal is a bit harder. To prove they are extremal we will first prove a proposition which gives us an easy to verify sufficient condition for a representation to be extremal.

The first lemma we need is a well known generalization of Sarason’s Lemma [S2]. Given a representation $\rho : A \to B(K)$, a subspace $M$ of $K$ is invariant for $\rho$ if $\rho(a)M \subseteq M$ for all $a \in A$. A subspace $H$ of $K$ is semi-invariant for $\rho$ if there exist invariant subspaces $M$ and $N$ such that $H = N \ominus M$. Note that, letting $V : H \to K$ denote the inclusion, the mapping $A \ni a \mapsto V^*\rho(a)V$ is also a representation of $A$.

Lemma 4.3. Let $\nu : A \to B(H)$ be a representation of $A$ in $B(H)$ and $\rho : A \to B(K)$ be a representation of $A$ in $B(K)$ and $V : H \to K$ an isometry. If $\nu(a) = V^*\rho(a)V$ for all $a \in A$, then $VH$ is a semi-invariant for $\rho$.

Proof. Let

$$N := \bigvee_{a \in A} \rho(a)VH,$$

the smallest (closed) subspace of $K$ containing all of the spaces $\rho(a)VH$. Notice that the elements of the form

$$\sum_{i=0}^{N} \rho(a_i)Vh_i$$

where $\{h_i\} \subseteq H$, $\{a_i\} \subseteq A$, and $N > 0$ form a dense subset of $N$. Since $\rho$ is a representation, for any $a \in A$, $\{h_i\} \subseteq H$, $\{a_i\} \subseteq A$, and $N > 0$ we have that

$$\rho(a)\left(\sum_{i=0}^{N} \rho(a_i)Vh_i\right) = \sum_{i=0}^{N} \rho(a \cdot a_i)Vh_i \in N$$
and thus \( N \) is \( \rho(a) \) invariant. To complete this direction of the proof we only need to show that \( M := N \oplus VH \) is also invariant for \( \rho(a) \). Notice that \( \nu(a) = V^*\rho(a)V \) implies

\[
V^*\rho(a) \left( \sum_{i=0}^{N} \rho(a_i)Vh_i \right) = \sum_{i=0}^{N} V^*\rho(a \cdot a_i)Vh_i
\]

\[
= \sum_{i=0}^{N} \nu(aa_j)h_j
\]

\[
= \sum_{i=0}^{N} \nu(a)a_i h_i
\]

\[
= \nu(a) \sum_{i=0}^{N} V^*\rho(a_i)Vh_i.
\]

Thus \( V^*\rho(a)|_N = \nu(a)V^*|_N \). If \( m \in M \), then \( m \in N \) and by the Fredholm alternative \( V^*m = 0 \). Thus, if \( a \in A \), then \( V^*\rho(a)m = \nu(a)V^*m = 0 \), which, again by the Fredholm alternative, implies \( \rho(a)m \in M \).

The next lemma allows us to improve semi-invariance to invariance if \( \nu(a) \) is an isometry and \( \|\rho(a)\| = 1 \).

**Lemma 4.4.** If \( H \subseteq K \) is a semi-invariant subspace for a contraction \( T \) and \( S := P_H T|_H \) is an isometry, then \( H \) is an invariant subspace for \( T \).

*Proof.* Since \( H \) is semi-invariant for \( T \) and \( S = P_H T|_H \) we know that there exists two \( T \) invariant spaces \( N \) and \( M \) such that \( N = N \oplus H \) and

\[
T = \begin{bmatrix} A & B & C \\ 0 & S & F \\ 0 & 0 & K \end{bmatrix}
\]

Where \( A : M \rightarrow M, B : M \rightarrow H, C : M \rightarrow N^\perp, F : H \rightarrow N^\perp, \) and \( K : N^\perp \rightarrow N^\perp \). Since \( T \) is a contraction we have \( I - T^*T \geq 0 \) thus, for all \( h \in H \),

\[
0 \leq \begin{pmatrix} (I - T^*T)h \mid 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -A^*Bh \\ (I - B^*B - S^*S)h \end{pmatrix} \begin{pmatrix} 0 \\ h \end{pmatrix} = ||h||^2 - ||Bh||^2 - ||S h||^2 = -||Bh||^2
\]

so \( Bh = 0 \) for all \( h \in H \). Thus \( H \) is a \( T \) invariant subspace. ■

Combining Lemmas 4.3 and 4.4 yields the following proposition.

**Proposition 4.5.** Let \( \nu : A \rightarrow B(H) \) be a contractive representation of \( A \) in \( B(H) \) and \( \{a_i\}_{i \in J} \subseteq A \) be a set that generates a dense subalgebra of \( A \) with \( ||a_i|| = 1 \) for all \( i \in J \). If \( \nu(a_i) \) is an isometry for all \( i \in J \), then \( \nu \) is extremal.
Proof. Let \( \rho : A \to B(K) \) be a contractive representation of \( A \) in \( B(K) \) and \( V : H \to K \) be an isometry such that \( \nu(a) = V^*\rho(a)V \) for all \( a \in A \). By lemma 4.3, \( VH \) is a semi-invariant subspace of \( K \) for \( \rho \). By lemma 4.4, \( VH \) is invariant for \( \rho(a_i) \) for each \( i \in J \). Because \( \rho \) is a representation we have that \( VH \) is invariant for \( \rho(a) \) for each \( a \) in the algebra generated by the set \( \{a_i\}_{i \in J} \). Thus \( VH \) is invariant for \( \rho \).

Now since \( VH \) is \( \rho(a) \) invariant and \( \nu(a) = V^*\rho(a)V \) for all \( a \in A \) we have that

\[
V\nu(a) = VV^*\rho(a)V = P_{VH}\rho(a)V = \rho(a)V \quad \text{for all} \quad a \in A.
\]

Thus \( \nu \) is extremal.

Now it is easy to show that all of the \( \pi_V \)'s are extremal representations of \( \mathcal{A} \).

**Corollary 4.6.** The representation \( \pi_V \) is an extremal representation of \( \mathcal{A} \).

**Proof.** Since \( \pi_V(1), \pi_V(z^2), \) and \( \pi_V(z^3) \) are isometries and \( 1, z^2, \) and \( z^3 \) generate \( \mathcal{A} \), by proposition 4.5 we know that \( \pi_V \) is extremal.

### 4.2. Proof of Theorem 4.1

Let \( \nu : \mathcal{A} \to B(H) \) be a pure extremal representation of \( \mathcal{A} \) on some separable Hilbert space \( H \). By [P, Corollary 7.7], the representation \( \nu \) has a \( C(\mathbb{T}) \)-dilation; i.e., there exists a completely contractive representation \( \rho : L^\infty(\mathbb{D}) \to B(K) \) and an isometry \( V : H \to K \) such that \( \nu(a) = V^*\rho(a)V \) for all \( a \in \mathcal{A} \). Moreover since \( \nu \) is extremal \( V\nu(a) = \rho(a)V \) for all \( a \in \mathcal{A} \) and \( VH \) is invariant for \( \rho \). Finally let

\[
E = \bigvee_{i=0}^{\infty} \rho(z^i)VH \subseteq K.
\]

Since \( z^i \in \mathcal{A} \) for all \( i \in \mathbb{N} \) and \( i \neq 1 \),

\[
\bigvee_{i \neq 1}^{\infty} \rho(z^i)VH = \bigvee_{i \neq 1}^{\infty} V\nu(z^i)H = VH.
\]

In particular, \( E = \rho(z)VH \vee VH \).

First we will show that \( S = \rho(z)|_E \) is a pure isometry on \( E \); if \( f, g \in K \), then

\[
\langle \rho(z)f, \rho(z)g \rangle = \langle \rho(z)^*\rho(z)f, g \rangle = \langle \rho(z\overline{z})f, g \rangle = \langle f, g \rangle.
\]

Since \( S \) is the restriction of an isometry to an invariant subspace \( S \) is an isometry. To show that \( S \) is pure note that

\[
\rho(z^2)E = \rho(z^2)(\rho(z)VH \vee VH) = \rho(z^3)VH \vee \rho(z^3)VH \subseteq VH.
\]
Since \( \nu \) is pure we have

\[
\bigcap_{b \in \mathcal{A}(\mathbb{D})} \rho(b)E \subseteq \bigcap_{b \in \mathbb{C}, a \in \mathcal{A}} \rho(a)\rho(z^2)E \\
\subseteq \bigcap_{a \in \mathcal{A}} \rho(a)VH \\
= \bigcap_{a \in \mathcal{A}} V\nu(a)H = \{0\}.
\]

Thus \( S \) is a pure shift on \( E \).

Since \( S \) is a pure shift there is a Hilbert space \( N \) and a unitary map \( W : E \to H^2_N \) such that \( WS = M_zW \). Since the subspace \( S^2E \) lies in \( VH \) and \( VH \) is a subspace of \( E \), there exists a subspace \( \mathcal{V} \) of \( N \oplus zN \) such that \( WVH = H^2_{\mathcal{V}} = H^2_N \Theta \mathcal{V} \). Let \( U : H \to H^2_{\mathcal{V}} \) be defined by \( U = WV \), this is a unitary map such that

\[
U^*\pi_{\mathcal{V}}(a)Uh = U^*M_aUh = \nu(a)h \text{ for all } h \in H,
\]

i.e. \( \pi_{\mathcal{V}} \) is unitarily equivalent to \( \nu \).

4.3. **Proof of Corollary 4.2.** Suppose \( \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z \) and \( M \) and \( N \) are orthogonal subspaces of \( H^2_{\mathcal{V}} \) invariant for \( \mathcal{A} \). Choosing non-zero \( \varphi \) and \( \psi \) from \( M \) and \( N \) respectively, it follows that \( \langle z^m\varphi, z^n\psi \rangle = 0 \) for natural numbers \( m \neq 1 \neq n \). Hence if \( \mu \) is normalized arclength measure on \( \mathbb{T} \) and \( \chi(z) = z \),

\[
0 = \int_{\mathbb{T}} \overline{\varphi}\overline{\psi}\chi^j \, d\mu
\]

for all \( j \) and therefore \( \varphi \overline{\psi} = 0 \). Since both \( \varphi \) and \( \psi \) are in \( H^2 \), each is non-zero almost everywhere whenever it is not the zero function. Thus at least one must be zero, which is a contradiction. So if \( \mathcal{V} \subseteq \mathbb{C} \oplus \mathbb{C}z \), then \( \pi_{\mathcal{V}} \) is rank one.

By theorem 4.1 it suffices to check the second part of the corollary for \( \pi_{\mathcal{V}} \) where \( \mathcal{V} \subseteq N \oplus Nz \). If \( N \) is one dimensional then \( N \) is unitarily equivalent to \( \mathbb{C} \) and we are done. If \( N \) is not one dimensional, then choose a pair of non-zero vectors \( e \) and \( f \) in \( N \) such that \( \langle e, f \rangle = 0 \) and let \( \mathcal{E} = z^2H^2e \) and \( \mathcal{F} = z^2H^2f \).

Both \( \mathcal{E} \) and \( \mathcal{F} \) are non-trivial subspaces of \( H^2_{\mathcal{V}} \) for any \( \mathcal{V} \) and are \( \mathcal{A} \) invariant. They are also orthogonal by construction. Hence \( \pi_{\mathcal{V}} \) is not rank one.

**References**

[A1] Abrahamse, M. B., *Toeplitz operators in multiply connected regions*. Bull. Amer. Math. Soc. **77** (1971) 449–454.

[A2] Abrahamse, M. B., *The Pick interpolation theorem for finitely connected domains*. Michigan Math. J. **26** (1979), no. 2, 195–203.
[AD] Abrahamse, M. B.; Douglas, R. G., A class of subnormal operators related to multiply-connected domains. Advances in Math. 19 (1976), no. 1, 106–148.

[Ag] Agler, J., An abstract approach to model theory. (pp. 1–23) Surveys of some recent results in operator theory. Vol. II. Edited by John B. Conway and Bernard B. Morrel. Pitman Research Notes in Mathematics Series, 192. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1988. vi+266 pp. ISBN: 0-582-00518-3.

[Ar1] Aryana, C. P., Self-adjoint Toeplitz Operators Associated with Representing Measure on Multiply Connected Planar Regions and their Eigenvalues. Complex Anal. Oper. Theory, to appear.

[Ar2] Akbari Estahbanati, Gholamreza On the spectral character of Toeplitz operators on planar regions. Proc. Amer. Math. Soc. 124 (1996), no. 9, 2737–2744.

[AC] Aryana, C. P.; Clancey, K. F., On the existence of eigenvalues of Toeplitz operators on planar regions. Proc. Amer. Math. Soc. 132 (2004), no. 10, 3007–3018.

[BH1] Ball, J. A.; Guerra Huamán, M. D., Convexity analysis and matrix-valued Schur class over finitely connected planar domains. arXiv:1109.3793

[BH2] Ball, J. A.; Guerra Huamán, M. D., Test functions, Schur-Agler classes and transfer-function realizations: The matrix-valued setting. arXiv:1109.3795

[C] Clancey, Kevin F. Toeplitz operators on multiply connected domains and theta functions. Contributions to operator theory and its applications (Mesa, AZ, 1987), 311–355, Oper. Theory Adv. Appl., 35, Birkhuser, Basel, 1988.

[DP] Dritschel, M. A.; Pickering, J., Test functions in constrained interpolation. Trans. Amer. Math. Soc. 364 (2012) no. 11, 5589–5604.

[DPRS] Davidson, K. R.; Paulsen, V. I.; Raghupathi, M.; Singh, D., A constrained Nevanlinna-Pick interpolation problem. Indiana Univ. Math. J. 58 (2009), no. 2, 709–732.

[F] Fisher, S. D., Function theory on planar domains. A second course in complex analysis. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1983. xiii+269 pp. ISBN: 0-471-87314-4.

[JKM] Jury, M. T.; Knese, G.; McCullough, S., Nevanlinna-Pick interpolation on distinguished varieties in the bidisk. J. Funct. Anal. 262 (2012), no. 9, 3812–3838.

[R] Raghupathi, M., Nevanlinna-Pick interpolation for \( \mathbb{C} + BH^\infty \). Integral Equations Operator Theory 63 (2009), no. 1, 103–125.

[K] Knese, G., Function theory on the Neil parabola. Michigan Math. J. 55 (2007), no. 1, 139–154.

[P] Paulsen, V. I., Completely bounded maps and operator algebras. Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002. xii+300 pp. ISBN: 0-521-81669-6.

[S1] Sarason, D., The \( H^p \) spaces of an annulus. Mem. Amer. Math. Soc. No. 56 (1965) 78 pp.

[S2] Sarason, D., On spectral sets having connected complement, Acta Sci. Math. (Szeged) 26 (1965) 289–299.
EIGENVALUES FOR TOEPLITZ OPERATORS

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