COMBINATORIAL METHODS: FROM GROUPS TO POLYNOMIAL ALGEBRAS

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Abstract. Combinatorial methods (or methods of elementary transformations) came to group theory from low-dimensional topology in the beginning of the century. Soon after that, combinatorial group theory became an independent area with its own powerful techniques. On the other hand, combinatorial commutative algebra emerged in the sixties, after Buchberger introduced what is now known as Gröbner bases. The purpose of this survey is to show how ideas from one of those areas contribute to the other.

1. Introduction

Let \( F = F_n \) be the free group of a finite rank \( n \geq 2 \) with a set \( X = \{ x_1, ..., x_n \} \) of free generators. Let \( Y = \{ y_1, ..., y_m \} \) and \( \tilde{Y} = \{ \tilde{y}_1, ..., \tilde{y}_m \} \) be arbitrary finite sets of elements of the group \( F \). Consider the following elementary transformations that can be applied to \( Y \):

(N1) \( y_i \) is replaced by \( y_i y_j \) or by \( y_j y_i \) for some \( j \neq i \);

(N2) \( y_i \) is replaced by \( y_i^{-1} \).

(N3) \( y_i \) is replaced by some \( y_j \), and at the same time \( y_j \) is replaced by \( y_i \).

It is understood that \( y_j \) doesn’t change if \( j \neq i \).

One might notice that some of these transformations are redundant, i.e., are compositions of other ones. There is a reason behind that which we are going to explain a little later.

We say that two sets \( Y \) and \( \tilde{Y} \) are Nielsen equivalent if one of them can be obtained from another by applying a sequence of transformations (N1)–(N3). It was proved by Nielsen that two sets \( Y \) and \( \tilde{Y} \) generate the same subgroup of the group \( F \) if and only if they are Nielsen equivalent. This result is now one of the central points in combinatorial group theory.

Note however that this result alone does not give an algorithm for deciding whether or not \( Y \) and \( \tilde{Y} \) generate the same subgroup of \( F \). To obtain an algorithm, we need to somehow define the complexity of a given set of elements, and then to show that a sequence of Nielsen transformations (N1)–(N3) can be arranged so that this complexity decreases.
(or, at least, does not increase) at every step (this is where we may need “redundant” elementary transformations!).

This was also done by Nielsen; the complexity of a given set $Y = \{y_1, \ldots, y_m\}$ is just the sum of the lengths of the words $y_1, \ldots, y_m$. We refer to [15] for details.

Nielsen’s method therefore yields (in particular) an algorithm for deciding whether or not a given endomorphism of a free group of finite rank is actually an automorphism.

A somewhat more difficult problem is, given a pair of elements of a free group $F$, to find out if one of them can be taken to another by an automorphism of $F$. We call this problem the automorphic conjugacy problem. It was addressed by Whitehead who came up with another kind of elementary transformations in a free group:

(W1) For some $j$, every $x_i$, $i \neq j$, is replaced by one of the elements $x_i x_j$, $x_j^{-1} x_i$, $x_j^{-1} x_i x_j$, or $x_i$.

(W2) $x_i$ is replaced by $x_i^{-1}$.

(W3) $x_i$ is replaced by some $x_j$, and at the same time $x_j$ is replaced by $x_i$.

One might notice a similarity of Nielsen and Whitehead transformations. However, they differ in one essential detail: Nielsen transformations are applied to arbitrary sets of elements, whereas Whitehead transformations are applied to a fixed basis of the group $F$.

Using (informally) matrix language, we can say that Nielsen transformations correspond to elementary rows transformations of a matrix (this correspondence can actually be made quite formal – see [17]), whereas Whitehead transformations correspond to conjugations (via changing the basis). This latter type of matrix transformation is known to be more complex, and the corresponding structural results are deeper.

There is very much the same relation between Nielsen and Whitehead transformations in a free group.

Note also that Whitehead transformation (W1) is somewhat more complex than its analog (N1). This is – again – in order to be able to arrange a sequence of elementary transformations so that the complexity of a given element (in this case, just the lexicographic length of a cyclically reduced word) would decrease (or, at least, not increase) at every step – see [14].

This arrangement still leaves us with a difficult problem - to find out if one of two elements of the same complexity (= of the same length) can be taken to another by an automorphism of $F$. This is actually the most difficult part of Whitehead’s algorithm.

In one special case however this problem does not arise, namely, when one of the elements is primitive, i.e., is an automorphic image of $x_1$. If we have managed to reduce an element of a free group (by Whitehead transformations) to an element of length 1, we immediately conclude that it is primitive; no further analysis is needed.
Thus, the problem of distinguishing primitive elements of a free group is a relatively easy case of the automorphic conjugacy problem. As we shall see in Section 2, this is also the situation in a polynomial algebra.

In Section 2, we review the results of various attempts to create something similar to Nielsen’s and Whitehead’s methods for a polynomial algebra in two variables. For a polynomial algebra in more than two variables, these problems are unapproachable so far, since we don’t even know what the generators of the automorphism group of an algebra like that look like.

In Section 3, we talk about retracts of a polynomial algebra in two variables. Basic properties of retracts of a free group are given in [15]. Since then, retracts have not been getting much attention until very recently, when Turner [21] and, independently, Bergman [4] brought them back to life by employing them in various interesting research projects in combinatorial group theory. Here we show the relevance of polynomial retracts to several well-known problems about polynomial mappings, in particular, to the notorious Jacobian conjecture.

In the concluding Section 4, we have gathered some open combinatorial problems about polynomial mappings that are motivated by similar issues in combinatorial group theory.

2. Elementary transformations in polynomial algebras

Let \( P_n = K[x_1, ..., x_n] \) be the polynomial algebra in \( n \) variables over a field \( K \) of characteristic 0. We are going to concentrate here mainly on the algebra \( P_2 \).

The first description of the group \( \text{Aut}(P_2) \) was given by Jung [12] back in 1942, but it was limited to the case \( K = \mathbb{C} \) since he was using methods of algebraic geometry. Later on, van der Kulk extended Jung’s result to arbitrary ground fields. In the form we give it here, the result appears as Theorem 8.5 in P.M.Cohn’s book [6]; this form is consistent with the idea of elementary transformations as described in the Introduction.

**Theorem 2.1.** [6] Every automorphism of \( K[x_1, x_2] \) is a product of linear automorphisms and automorphisms of the form \( x_1 \to x_1 + f(x_2); \ x_2 \to x_2 \). More precisely, if \((g_1, g_2)\) is an automorphism of \( K[x_1, x_2] \) such that \( \deg(g_1) \geq \deg(g_2) \), say, then either \((g_1, g_2)\) is a linear automorphism, or there exists a unique \( \mu \in K^* \) and a positive integer \( d \) such that \( \deg(g_1 - \mu g_2^d) < \deg(g_1) \).

The proof given in [6] is attributed to Makar-Limanov (unpublished), with simplifications by Dicks [8].

Note that the “More precisely, ...” statement serves the algorithmic purposes: upon defining the complexity of a given pair of polynomials \((g_1, g_2)\) as the sum \( \deg(g_1) + \deg(g_2) \),
we see that Theorem 2.1 allows one to arrange a sequence of elementary transformations (these are linear automorphisms and automorphisms of the form \( x_1 \rightarrow x_1 + f(x_2); \ x_2 \rightarrow x_2 \)) so that this complexity decreases at every step, until we either get a pair of polynomials that represents a linear automorphism, or conclude that \((g_1, g_2)\) was not an automorphism of \(K[x_1, x_2]\). The parallel with Nielsen’s method described in the Introduction is obvious.

We also mention here another proof of this result (in case \(\text{char} \ K = 0\)) due to Abhyankar and Moh [1]. In fact, their method is even more similar to Nielsen’s method in a free group. Many of their results are based on the following fundamental theorem which we give here only in the characteristic 0 case (it will also play an essential role in our Section 3):

**Theorem 2.2.** [1] Let \(u(t), v(t) \in K[t]\) be two one-variable polynomials of degree \(n \geq 1\) and \(m \geq 1\). Suppose \(K[t] = K[u, v]\). Then either \(n\) divides \(m\), or \(m\) divides \(n\).

Now let’s see how one can adopt a more sophisticated Whitehead’s method in a polynomial algebra situation. It appears that elementary basis transformations (see Theorem 2.1), when applied to a polynomial \(p(x_1, x_2)\), are mimicked by Gröbner transformations of a basis of the ideal of \(P_2\) generated by partial derivatives of this polynomial. To be more specific, we have to give some background material first.

In the course of constructing a Gröbner basis of a given ideal of \(P_n\), one uses “reductions”, i.e., transformations of the following type (see [2], p.39-43): given a pair \((p, q)\) of polynomials, set \(S(p, q) = \frac{l.t.(p)}{l.t.(q)} \cdot p - \frac{L}{l.t.(q)} \cdot q\), where \(l.t.(p)\) is the leading term of \(p\), i.e., the leading monomial together with its coefficient; \(L = \text{l.c.m.}(l.m.(p), l.m.(q))\) (here, as usual, \(\text{l.c.m.}\) means the least common multiple, and \(l.m.(p)\) denotes the leading monomial of \(p\)). In this paper, we’ll always consider what is called “deglex ordering” in [2] - where monomials are ordered first by total degree, then lexicographically with \(x_1 > x_2 > \ldots > x_n\).

Now a crucial observation is as follows. These Gröbner reductions appear to be of two essentially different types:

(i) **regular**, or **elementary**, transformations. These are of the form \(S(p, q) = \alpha \cdot p - r \cdot q\) or \(S(p, q) = \alpha \cdot q - r \cdot p\) for some polynomial \(r\) and scalar \(\alpha \in K^*\). This happens when the leading monomial of \(p\) is divisible by the leading monomial of \(q\) (or vice versa). The reason why we call these transformations **elementary** is that they can be written in the form \((p, q) \rightarrow (\alpha_1 p, \alpha_2 q) \cdot M\), where \(M\) is an **elementary matrix**, i.e., a matrix which (possibly) differs from the identity matrix by a single element outside the diagonal. In case when we have more than 2 polynomials \((p_1, \ldots, p_k)\), we also can write \((p_1, \ldots, p_k) \rightarrow (\alpha_1 p_1, \ldots, \alpha_k p_k) \cdot M\), where \(M\) is a \(k \times k\) elementary matrix; elementary reduction here is actually applied to a pair of polynomials (as usual) while the other ones are kept
fixed. Sometimes, it is more convenient for us to get rid of the coefficients $\alpha_i$ and write 
$(p_1, \ldots, p_k) \rightarrow (p_1, \ldots, p_k) \cdot M$, where $M$ belongs to the group $GE_k(P_n)$ generated by all elementary and diagonal matrices from $GL_k(P_n)$. It is known \cite{20} that $GE_k(P_n) = GL_k(P_n)$ if $k \geq 3$, and $GE_2(P_n) \neq GL_2(P_n)$ if $n \geq 2$ - see \cite{5}.

(ii) **singular** transformations – these are non-regular ones.

Denote by $I_{d(p)}$ the ideal of $P_2$ generated by partial derivatives of $p$. We say that a polynomial $p \in P_n$ has a **unimodular gradient** if $I_{d(p)} = P_n$ (in particular, the ideal $I_{d(p)}$ has rank 1 in this case). Note that if the ground field $K$ is algebraically closed, then this is equivalent, by Hilbert’s Nullstellensatz, to the gradient being nowhere-vanishing.

Furthermore, define the **outer rank** of a polynomial $p \in P_n$ to be the minimal number of generators $x_i$ on which an automorphic image of $p$ can depend.

Then we have:

**Theorem 2.3.** \cite{18} Let a polynomial $p \in P_2$ have unimodular gradient. Then the outer rank of $p$ equals 1 if and only if one can get from $(d_1(p), d_2(p))$ to $(1, 0)$ by using only elementary transformations. Or, in the matrix form: if and only if $(d_1(p), d_2(p)) \cdot M = (1, 0)$ for some matrix $M \in GE_2(P_2)$.

The proof \cite{18} of Theorem 2.3 is based on a generalization of Wright’s Weak Jacobian Theorem \cite{22}.

**Remark 2.4.** Elementary transformations that reduce $(d_1(p), d_2(p))$ to $(1, 0)$, can be actually chosen to be Gröbner reductions, i.e., to decrease the maximum degree of monomials at every step – the proof \cite{18} is based on a recent result of Park \cite{16}.

Now we show how one can apply this result to the study of so-called coordinate polynomials.

We call a polynomial $p \in P_n$ coordinate if it can be included in a generating set of cardinality $n$ of the algebra $P_n$. It is clear that the outer rank of a coordinate polynomial equals 1 (the converse is not true!). It is easy to show that a coordinate polynomial has a unimodular gradient, and again – the converse is not true! On the other hand, we have:

**Proposition 2.5.** \cite{18} A polynomial $p \in P_n$ is coordinate if and only if it has outer rank 1 and a unimodular gradient.

Combining this proposition with Theorem 2.3 yields the following

**Theorem 2.6.** \cite{18} A polynomial $p \in P_2$ is coordinate if and only if one can get from $(d_1(p), d_2(p))$ to $(1, 0)$ by using only elementary Gröbner reductions.

This immediately yields an algorithm for detecting coordinate polynomials in $P_2$ (see \cite{18}) which is similar to Whitehead’s algorithm for detecting primitive elements in a free
This algorithm is very simple and fast: it has quadratic growth with respect to the degree of a polynomial. In case $p$ is revealed to be a coordinate polynomial, the algorithm also gives a polynomial which completes $p$ to a basis of $P_2$.

In the case when $K = \mathbb{C}$, the field of complex numbers, an alternative, somewhat more complicated algorithm, has been recently reported in [9]. It is not known whether or not there is an algorithm for detecting coordinate polynomials in $P_n$ if $n \geq 3$.

Theorems 2.3 and 2.6 also suggest the following conjecture which is relevant to an important problem known as “effective Hilbert’s Nullstellensatz”:

Conjecture “G”. Let a polynomial $p \in P_2$ have a unimodular gradient. Then one can get from $(d_1(p), d_2(p))$ to $(1, 0)$ by using at most one singular Gröbner reduction.

Remark 2.7. For $n \geq 3$, Theorem 2.3 is no longer valid since in this case, by a result of Suslin [20], the group $GL_n(P_n) = GE_n(P_n)$ acts transitively on the set of all unimodular polynomial vectors of dimension $n$, yet there are polynomials with unimodular gradient, but of the outer rank 2, for example, $p = x_1 + x_1^2 x_2$. The “only if” part however is valid for an arbitrary $n \geq 2$ - see [18]. It is also easy to show that one always has $orank \ p \geq rank(I_{d(p)})$.

Finally, we mention that our method also yields an algorithm which, given a coordinate polynomial $p \in P_2$, finds a sequence of elementary automorphisms (i.e., automorphisms of the form $x_1 \to x_1 + f(x_2); \ x_2 \to x_2$ together with linear automorphisms) that reduces $p$ to $x_1$.

3. POLYNOMIAL RETRACTS

Let $K[x, y]$ be the polynomial algebra in two variables over a field $K$ of characteristic 0. A subalgebra $R$ of $K[x, y]$ is called a retract if it satisfies any of the following equivalent conditions:

(R1) There is an idempotent homomorphism (a retraction, or projection) $\varphi : K[x, y] \to K[x, y]$ such that $\varphi(K[x, y]) = R$.

(R2) There is a homomorphism $\varphi : K[x, y] \to R$ that fixes every element of $R$.

(R3) $K[x, y] = R \oplus I$ for some ideal $I$ of the algebra $K[x, y]$.

(R4) $K[x, y]$ is a projective extension of $R$ in the category of $K$-algebras. In other words, there is a splitting exact sequence $1 \to I \to K[x, y] \to R \to 1$, where $I$ is the same ideal as in (R3) above.

Examples: $K$; $K[x, y]$; any subalgebra of the form $K[p]$, where $p \in K[x, y]$ is a coordinate polynomial (i.e., $K[p, q] = K[x, y]$ for some polynomial $q \in K[x, y]$). There are
other, less obvious, examples of retracts: if \( p = x + x^2y \), then \( K[p] \) is a retract of \( K[x, y] \), but \( p \) is not coordinate since it has a fiber \( \{ p = 0 \} \) which is reducible, and therefore is not isomorphic to a line.

The very presence of several equivalent definitions of retracts shows how natural these objects are.

In [7], Costa has proved that every proper retract of \( K[x, y] \) (i.e., a one different from \( K \) and \( K[x, y] \)) has the form \( K[p] \) for some polynomial \( p \in K[x, y] \), i.e., is isomorphic to a polynomial \( K \)-algebra in one variable. A natural problem now is to characterize somehow those polynomials \( p \in K[x, y] \) that generate a retract of \( K[x, y] \). Since the image of a retract under any automorphism of \( K[x, y] \) is again a retract, it would be reasonable to characterize retracts up to an automorphism of \( K[x, y] \), i.e., up to a “change of coordinates”. We give an answer to this problem in the following

**Theorem 3.1.**[19] Let \( K[p] \) be a retract of \( K[x, y] \). There is an automorphism \( \psi \) of \( K[x, y] \) that takes the polynomial \( p \) to \( x + y \cdot q \) for some polynomial \( q = q(x, y) \). A retraction for \( K[\psi(p)] \) is given then by \( x \to x + y \cdot q; \ y \to 0 \).

Geometrically, Theorem 3.1 says that (in case \( K = \mathbb{C} \)) every polynomial retraction of a plane is a “parallel” projection (sliding) on a fiber of a coordinate polynomial (which is isomorphic to a line) along the fibers of another polynomial (which generates a retract of \( K[x, y] \)).

Our proof of this result is based on the Abhyankar-Moh theorem (see Theorem 2.2).

Theorem 3.1 yields another characterization of retracts of \( K[x, y] \):

**Corollary 3.2.**[19] A polynomial \( p \in K[x, y] \) generates a retract of \( K[x, y] \) if and only if there is a polynomial mapping of \( K[x, y] \) that takes \( p \) to \( x \). The “if” part is actually valid for a polynomial algebra in arbitrarily many variables.

We also note that if a mapping described in Corollary 3.2 is injective, then \( p \) is a coordinate polynomial – this follows from the Embedding theorem of Abhyankar and Moh [1].

Theorem 3.1 has several interesting applications, in particular, to the notorious

**Jacobian conjecture.** If for a pair of polynomials \( p, q \in K[x, y] \), the corresponding Jacobian matrix is invertible, then \( K[p, q] = K[x, y] \).

This problem was introduced in [13], and is still unsettled. For a survey and background, the reader is referred to [3].

Now we establish a link between retracts of \( K[x, y] \) and the Jacobian conjecture by means of the following
**Conjecture “R”**. If for a pair of polynomials $p, q \in K[x, y]$, the corresponding Jacobian matrix is invertible, then $K[p]$ is a retract of $K[x, y]$.

This statement is formally much weaker than the Jacobian conjecture since, instead of asking for $p$ to be a coordinate polynomial, we only ask for $p$ to generate a retract, and this property is much less restrictive as can be seen from Theorem 3.1. However, the point is that these conjectures are actually equivalent:

**Theorem 3.3.** [19] Conjecture “R” implies the Jacobian conjecture.

Another application of retracts to the Jacobian conjecture (somewhat indirect though) is based on the “$\varphi^\infty$-trick” familiar in combinatorial group theory (see [21]). For a polynomial mapping $\varphi : K[x, y] \to K[x, y]$, denote by $\varphi^\infty(K[x, y]) = \bigcap_{k=1}^{\infty} \varphi^k(K[x, y])$ the stable image of $\varphi$. Then we have:

**Theorem 3.4.** [19] Let $\varphi$ be a polynomial mapping of $K[x, y]$. If the Jacobian matrix of $\varphi$ is invertible, then either $\varphi$ is an automorphism, or $\varphi^\infty(K[x, y]) = K$.

The proof [19] of Theorem 3.4 is based on a recent result of Formanek [11].

Obviously, if $\varphi$ fixes a polynomial $p \in K[x, y]$, then $p \in \varphi^\infty(K[x, y])$. Therefore, we have:

**Corollary 3.5.** [19] Suppose $\varphi$ is a polynomial mapping of $K[x, y]$ with invertible Jacobian matrix. If $\varphi(p) = p$ for some non-constant polynomial $p \in K[x, y]$, then $\varphi$ is an automorphism.

This yields the following promising re-formulation of the Jacobian conjecture: if $\varphi$ is a polynomial mapping of $K[x, y]$ with invertible Jacobian matrix, then for some automorphism $\alpha$, the mapping $\alpha \cdot \varphi$ fixes a non-constant polynomial.

### 4. Some open problems

In this section, we have gathered a few combinatorial problems about polynomial mappings that are motivated by similar issues in combinatorial group theory. Two most important problems however – Conjectures “G” and “R” – appear earlier in the text (in Sections 2 and 3, respectively).

Throughout, $P_n = K[x_1, ..., x_n]$ is the polynomial algebra in $n$ variables, $n \geq 2$, over a field $K$ of characteristic 0.

(1) [10] Is it true that every endomorphism of $P_n$ taking any coordinate polynomial to a coordinate one, is actually an automorphism? (It is true for $n = 2$ – see [10]).

(2) Is there a polynomial $p \in P_n$ with the following property: whenever $\varphi(p) = \psi(p)$ for some non-constant-valued endomorphisms $\varphi, \psi$ of $P_n$, it follows that $\varphi = \psi$? (In
other words, every non-constant-valued endomorphism of $P_n$ is completely determined by its value on just 1 polynomial).

(3) Suppose $\varphi(p) = x_1$ for some monomorphism (i.e., injective endomorphism) $\varphi$ of the algebra $P_n$. Is it true that $p$ is a coordinate polynomial? (It is true for $n = 2$ – see [19]).

(4) Let $p \in P_n$ be a polynomial such that $K[p]$ is a retract of $P_n$. Is it true that $\varphi(p) = x_1$ for some endomorphism $\varphi$ of the algebra $P_n$? (It is true for $n = 2$ – see [19]).

(5) Is it true that for any endomorphism $\varphi$ of the algebra $P_n$, its stable image $\varphi^\infty(P_n)$ is a retract of $P_n$? (The answer to this question might depend on the properties of the ground field $K$).

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