LEMNISCATE ENSEMBLES WITH SPECTRAL SINGULARITY

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Abstract. We consider a family of random normal matrix models whose eigenvalues tend to occupy lemniscate type droplets as the size of the matrix increases. Under the insertion of a point charge, we derive the scaling limit at the singular boundary point, which is expressed in terms of the solution to the model Painlevé IV Riemann-Hilbert problem. For this, we introduce a version of the Christoffel-Darboux identity and combine it with the strong asymptotics of the associated orthogonal polynomials due to Bertola, Elias Rebelo and Grava.

1. Introduction and main results

In the theory of random normal matrices [31, 51], one usually starts with a suitable real-valued function \( W \) called the external potential and consider a normal matrix of size \( N \) picked randomly with respect to the measure proportional to \( e^{-N\text{Tr} W(M)} dM \). Here \( dM \) is the induced surface measure on the space of normal matrices \( \{ M \in \mathbb{C}^{N^2} : MM^* = M^*M \} \). Then its eigenvalues \( \{ \lambda_j \}_{j=1}^N \) behave like equally charged Coulomb particles [33,45,50] in the external field \( NW \) at specific inverse temperature \( \beta = 2 \), namely, the joint probability distribution of the system is proportional to

\[
\prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_{j=1}^N e^{-NW(\lambda_j)} dA(\lambda_j), \quad (dA(\lambda) := d^2\lambda/\pi).
\]

We refer to [23, Section 5] for a recent review.

As the size of the matrix increases, the eigenvalue ensemble tends to minimise the weighted logarithmic energy functional [48], which can be recognised as the continuum limit of its discrete Hamiltonian, see e.g. [8, 26]. In particular the support of the limiting empirical distribution is given by a certain compact set called the droplet. Due to Sakai’s regularity theory [49], it is well known that for a real analytic potential \( W \), all but finitely many boundary points of the droplet are “regular” in a proper sense. Furthermore in the case that there exists a local Schwarz function near the prescribed boundary point, the possible types of singularities are classified. On the other hand, the construction of a droplet containing singular boundary points requires a separate analysis, see [2, 13, 15, 20, 21, 29, 40] and references therein for recent works in this direction.

The detailed statistical information about the joint intensity functions of the eigenvalue system can be effectively analysed by the correlation (reproducing) kernel of the orthogonal polynomials with respect to the weighted Lebesgue measure \( e^{-NW} dA \). Recently, for a quite general class of the potentials, the asymptotic behaviours of the associated planar orthogonal polynomials were obtained by Hedenmalm and Wennman [35]. As a consequence, they derived the boundary scaling limit of the
correlation kernel, which leads to the local universality at regular boundary points of the droplet. (We also refer to [10] for an earlier work on the local universality at regular bulk points.)

On the other hand, it is intuitively clear that different kinds of scaling limits should appear at singular boundary points. However the description of such scaling limits remains open in general and we aim to contribute to this problem. In particular we shall consider two types of singularities; one is the lemniscate type singularity arising from the local geometric structure of the droplet (see Figure 1) and the other is the spectral singularity arising from an insertion of a point charge.

Figure 1. An illustration of a lemniscate ensemble

1.1. Setup. Let us be more precise now in introducing our model that we call the lemniscate ensemble following [11]. First we consider the (shifted) Gaussian potential $Q$ of the form

\begin{equation}
Q(\zeta) := |\zeta - a|^2, \quad a \geq 0.
\end{equation}

This is a building block to define

\begin{equation}
V(\zeta) := \frac{1}{d}Q(\zeta^d) = \frac{1}{d} |\zeta^d - a|^2,
\end{equation}

where $d > 1$ is a fixed integer. We remark that even though $Q$ can be realised as a special case of $V$ with $d = 1$, we intentionally distinguish this case for our purpose described below. For a given point charge $c > -1$, let

\begin{equation}
Q_c(\zeta) := Q(\zeta) - \frac{2c}{N} \log |\zeta|, \quad V_c(\zeta) := V(\zeta) - \frac{2c}{N} \log |\zeta|.
\end{equation}

Such an extra logarithmic factor is often referred to as a spectral singularity, see e.g. [1, Chapter 6]. Here the condition $c > -1$ is required to guarantee that the partition functions $Z_N, \hat{Z}_N$ below are finite.

We shall study random normal matrix ensembles \{\zeta_j\}_1^N, \{\hat{\zeta}_j\}_1^N associated with the potentials $V_c, Q_c$ respectively. By definition, their joint probability distributions $P_N, \hat{P}_N$ are given by

\begin{equation}
dP_N(\zeta_1, \ldots, \zeta_N) = \frac{1}{Z_N} \prod_{j<k} |\zeta_j - \zeta_k|^2 \prod_{j=1}^N |\zeta_j|^{2c e^{-NV(\zeta_j)}} dA(\zeta_j),
\end{equation}

\begin{equation}
d\hat{P}_N(\hat{\zeta}_1, \ldots, \hat{\zeta}_N) = \frac{1}{\hat{Z}_N} \prod_{j<k} |\hat{\zeta}_j - \hat{\zeta}_k|^2 \prod_{j=1}^N |\hat{\zeta}_j|^{2c e^{-NQ(\hat{\zeta}_j)}} dA(\hat{\zeta}_j),
\end{equation}
where \( Z_N, \hat{Z}_N \) are normalisation constants which turn \( P_N, \hat{P}_N \) into probability measures, see \([30,53]\) for asymptotics of the partition function \( \hat{Z}_N \). We also mention that for an integer-valued \( c \), the system \( \{\hat{\zeta}_j\}_1^N \) has an alternative realisation as eigenvalues of the \textit{induced Ginibre ensemble} \([32]\), an extension of the Ginibre ensemble to include zero eigenvalues.

The well-known circular law \([34]\) asserts that as \( N \) increases, the eigenvalues \( \{\hat{\zeta}_j\}_1^N \) tend to be uniformly distributed on the disc \( \hat{S} := \{\zeta \in \mathbb{C} : |\zeta - a|^2 \leq 1\} \). As a consequence, it is easy to observe that \( \{\zeta_j\}_1^N \) tend to occupy the droplet

\[
S := \{\zeta \in \mathbb{C} : |\zeta^d - a|^2 \leq 1\}
\]

and that the limiting density on \( S \) with respect to the area measure \( dA \) is given by

\[
\Delta V(\zeta) = d |\zeta|^{2d-2}, \quad (\Delta := \partial \bar{\partial}),
\]

see \([15, \text{Lemma 1}]\). Note that the topology of \( S \) reveals a phase transition at the value \( a = 1 \), where the droplet \( S \) is of lemniscate type having \( d \)-fold symmetry, see Figure 2.

![Plots of \( \partial S \) for a few values of \( d \) and \( a \)](image)

We denote by \( p_{j,N}^c, q_{j,N}^c \) the orthonormal polynomials of degree \( j \) with respect to the weighted measure \( e^{-NQ_c} \, dA, e^{-NV_c} \, dA \), respectively, i.e.

\[
\int_{\mathbb{C}} p_{j,N}^c(\zeta) \overline{p_{k,N}^c(\zeta)} |\zeta|^{2c} e^{-NQ(\zeta)} \, dA(\zeta) = \int_{\mathbb{C}} q_{j,N}^c(\zeta) \overline{q_{k,N}^c(\zeta)} |\zeta|^{2c} e^{-NV(\zeta)} \, dA(\zeta) = \delta_{jk}.
\]

Here \( \delta_{jk} \) is the Kronecker delta. The strong asymptotics of \( p_{j,N}^c \) were extensively studied in \([13,14,17,42]\), see also recent works \([16,43,44]\) on the case with multiple point charges. We also refer the reader to \([20,36–39,46]\) for the strong asymptotics of planar orthogonal polynomials associated with some other classes of potentials.
Let us write $K_N^c, \hat{K}_N^c$ for the correlation kernels of the point processes $\{\zeta_j\}_1^N, \{\hat{\zeta}_j\}_1^N$, respectively. Due to Dyson’s determinantal formula, we have the canonical expressions

\begin{align}
K_N^c(\zeta, \eta) &= (\zeta \overline{\eta})^c e^{-\frac{N}{d} (V(\zeta) + V(\eta))} \sum_{j=0}^{N-1} q_{j,N}^c(\zeta) \overline{q_{j,N}^c(\eta)}, \\
\hat{K}_N^c(\zeta, \eta) &= (\zeta \overline{\eta})^c e^{-\frac{N}{d} (Q(\zeta) + Q(\eta))} \sum_{j=0}^{N-1} p_{j,N}^c(\zeta) \overline{p_{j,N}^c(\eta)}.
\end{align}

The joint intensity (correlation) functions are then given in terms of the determinant of such correlation kernels, see e.g. [23, 33].

To describe the local statistics of $\{\zeta_j\}_1^N, \{\hat{\zeta}_j\}_1^N$ at the origin, it is convenient to define the rescaled point processes $\{z_j\}_1^N, \{\hat{z}_j\}_1^N$ as

\begin{equation}
z_j := (N/d)^{\frac{1}{2}} \cdot \zeta_j, \qquad \hat{z}_j := N^{\frac{1}{2}} \cdot \hat{\zeta}_j,
\end{equation}

see Figure 1. Here the rescaling order $N^{\frac{1}{2}}$ is chosen according to the mean eigenvalue density (1.8) at the origin. By definition, the correlation kernels $K_N^c, \hat{K}_N^c$ associated with the point processes $\{z_j\}_1^N, \{\hat{z}_j\}_1^N$ are given by

\begin{align}
K_N^c(z, w) &= \frac{1}{(N/d)^d} K_N^c \left( \frac{z}{(N/d)^{\frac{d}{2}}}, \frac{w}{(N/d)^{\frac{d}{2}}} \right), \\
\hat{K}_N^c(z, w) &= \frac{1}{N} \hat{K}_N^c \left( \frac{z}{\sqrt{N}}, \frac{w}{\sqrt{N}} \right).
\end{align}

We aim to derive the large-$N$ limits

\begin{equation}
K^c := \lim_{N \to \infty} K_N^c, \qquad \hat{K}^c := \lim_{N \to \infty} \hat{K}_N^c
\end{equation}

of the correlation kernels, where the convergence is uniform on compact subsets of $\mathbb{C}$. The existence of the large-$N$ limits can be found in [12, Theorem 1.1] and [11, Lemma 3]. Let us also stress here that by [11, Lemma 1], the limiting point processes are indeed determined by their 1-point densities

\begin{equation}
R^c(z) := K^c(z, z), \qquad \hat{R}^c(z) := \hat{K}^c(z, z).
\end{equation}

1.2. Finite-$N$ analysis. The main ingredient to analyse the correlation kernel is a version of the Christoffel-Darboux identity. This can be applied to various situations for instance to the case studied in [13]; cf. see [25] for a recent implementation.

To describe the Christoffel-Darboux formula, let $P_j$ be the monic orthogonal polynomial of degree $j$ satisfying

\begin{equation}
\int_\mathbb{C} P_j(\zeta) \overline{P_k(\zeta)} |\zeta - a|^2 e^{-N|\zeta|^2} dA(\zeta) = h_j \delta_{jk},
\end{equation}

where $h_j$ is the orthogonal norm. We denote

\begin{align}
\psi_j(\zeta) := (\zeta - a)^c P_j(\zeta), \\
\phi_j(\zeta) := (\zeta - a)^c \frac{P_j(\zeta)}{h_j}.
\end{align}

and define

\begin{equation}
\overline{K}_N^c(\zeta, \eta) := e^{-N\zeta \overline{\eta}} \sum_{j=0}^{N-1} \overline{\phi_j(\eta)} \psi_j(\zeta) = |\zeta - a|^2 e^{-N\zeta \overline{\eta}} \sum_{j=0}^{N-1} P_j(\zeta) \overline{P_j(\eta)}.
\end{equation}

Note that it is related to $\hat{K}_N^c$ in (1.11) as

\begin{equation}
\hat{K}_N^c(\zeta, \zeta) = \overline{K}_N^c(a - \zeta, a - \zeta).
\end{equation}

We obtain the following theorem.
Theorem 1.1 (Christoffel-Darboux formula). Suppose that $a \neq 0$. Then we have
\[
\partial_{\eta} \overline{K}_n^c(\zeta, \eta) = e^{-N \zeta \bar{\eta}} \frac{1}{N h_{n-1} - h_n} \partial_{\eta} \psi_n(\eta) \left( \psi_n(\zeta) - \zeta \psi_{n-1}(\zeta) \right)
\]
(1.20)
\[
- e^{-N \zeta \bar{\eta}} \frac{P_{n+1}(a)}{P_n(a)} \frac{N h_{N-1}/h_N - h_{n+1}}{h_{n-1}} \psi_{n-1}(\eta) \left( \psi_{n+1}(\zeta) - \zeta \psi_n(\zeta) \right).
\]

Contrary to the classical Christoffel-Darboux formula for the orthogonal polynomial kernel on the real axis, the identity (1.20) shows that the summation in (1.18) can be expressed in terms of the three last orthogonal polynomials.

Remark 1.2. For the radially symmetric case when $a = 0$, we have
\[
P_j(\zeta) = \zeta^j, \quad h_j = \frac{\Gamma(j + c + 1)}{N^{j+c+1}}.
\]
Thus Theorem 1.1 cannot be directly applied to this case since
\[
\frac{N^{+c} h_{N-1} - h_N}{h_{n-1}} = \psi_N(\zeta) - \zeta \psi_{N-1}(\zeta) = 0.
\]
On the one hand, for $a \neq 0$, it was shown in [13, Appendix D] that $\frac{N^{+c} h_{N-1} - h_N}$ does not vanish.

Remark 1.3 (Three-term recurrence relation). In Subsection 3.2, we also show that the orthogonal polynomial $P_k$ satisfies the (non-standard) three-term recurrence relation of the form
\[
z P_k(z) = P_{k+1}(z) + b_k P_k(z) + c_k z P_{k-1}(z),
\]
where
\[
b_k := -\frac{P_{k+1}(0)}{P_k(0)}, \quad c_k := \frac{P_k(0) - P_{k+1}(0)}{P_{k-1}(0)}.
\]
This relation (1.23) plays an important role in the proof of Theorem 1.1. We mention that it is shown in [6, Corollary 5.3] that $P_k$ does not satisfy the standard three-term recurrence relation, (i.e. the relation of the form (1.23) with $c_k z$ replaced by $c_k$).

Example 1.4. (Exactly solvable case: $c = 1$) For an integer-valued point charge $c$, one can explicitly express the associated orthogonal polynomials using the well-known special functions. For instance, when $c = 1$, we have
\[
P_k(\zeta) = \sum_{j=0}^{k} a^{k-j} k! \frac{\Gamma(j+1, Na^2)}{j! \Gamma(k+1, Na^2)} \zeta^j = \frac{1}{\zeta - a} \left( e^{k+1} - e^{a N(\zeta - a)} \frac{Q(k+1, Na \zeta)}{Q(k+1, Na^2)} a^{k+1} \right)
\]
and
\[
h_k = \frac{(k+1)!}{N^{k+2}} \frac{Q(k+2, Na^2)}{Q(k+1, Na^2)}
\]
see [7, Section 3]. Here $Q$ is the regularised incomplete Gamma function. Then by using some basic properties of the incomplete Gamma function, one can directly check the Christoffel-Darboux formula (1.20) as well as the three-term recurrence relation (1.23).

Due to the relation (1.19), one can notice that the use of Theorem 1.1 can be made to derive the asymptotic behaviours of $\bar{K}_N^c$ in (1.13). Furthermore, the behaviours of $K_N^c$ follows from the following proposition.
Proposition 1.5 (Multi-fold transform). For each \( c > -1 \) and \( d \in \mathbb{N} \), we have

\[
K_{dN}^c(\zeta, \eta) = d(\zeta \bar{\eta})^{d-1} \sum_{l=0}^{d-1} \tilde{K}_{N}^{c+l+1}(\zeta^d, \eta^d),
\]

(1.27)

\[
K_{dN}^c(z, w) = d(z \bar{w})^{d-1} \sum_{l=0}^{d-1} \tilde{K}_{N}^{c+l+1}(z^d, w^d).
\]

(1.28)

We refer to [27, Proposition 2.1] and [5, Appendix B] for related statements on Hermitian matrix models. This proposition follows from a simple relation (2.17) between the orthogonal polynomials.

1.3. Scaling limits. We now focus on the critical regime when \( a \to 1 \) in a way that the scaled parameter

\[
S := 2\sqrt{N}(a - 1)
\]

remains bounded. An analogue of such regime in the Hermitian random matrix theory is called multi-criticality [19,28], see also [2] for the chiral counterpart.

We first introduce the Riemann-Hilbert problem for \( \tilde{\Psi}^c \) that describes the special solution of the Painlevé IV appeared in [17]. See [17, Subsection 2.2] for more details. For a given real parameter \( s \), the matrix \( \tilde{\Psi}^c(\zeta; s) \) of size 2 is analytic in \( \mathbb{C} \setminus (\Gamma_1 \cup \Gamma_\infty \cup \mathbb{R}_-) \) and admits non-tangential boundary values. Here \( \Gamma_\infty = i\mathbb{R} \) and \( \Gamma_1 \) is a contour in the left-half plane crossing the origin as shown in Figure 3. One may simply assume that \( \Gamma_1 \) comes from the infinity straight to the origin in an angle between \( \pi \) and \( 3\pi/2 \) and going straight back to the infinity in an angle between \( \pi/2 \) and \( \pi \).

![Figure 3. The jump contours of \( \tilde{\Psi}^c(\zeta; s) \).](image)

The jump conditions and the asymptotic behaviours of \( \tilde{\Psi}^c \) are given as follows:

- The jump condition is given by

\[
\tilde{\Psi}^c_\pm(\zeta; s) = \tilde{\Psi}^c(\zeta; s) \begin{cases} 
1 & -1 \\
0 & 1 
\end{cases}, \quad \zeta \in \Gamma_1,
\]

(1.30)

\[
\begin{cases} 
1 & 0 \\
1 & 1 
\end{cases}, \quad \zeta \in \Gamma_\infty,
\]

\[
e^{-c\pi i \sigma_3} \begin{cases} 
1 & 0 \\
1 & 1 
\end{cases}, \quad \zeta \in \mathbb{R}_-,
\]

where \( \sigma_3 \) is the third Pauli matrix. Here \( \tilde{\Psi}^c_\pm(\zeta, s) \) are continuous boundary values on the left and right of the jumping contours;
In terms of the function \( W \), we also write
\[
Z = \left( I + \frac{\Psi_1(s)}{\xi} + \frac{\Psi_2(s)}{\xi^2} + O\left(\frac{1}{\xi^3}\right)\right) \xi^{-\frac{5}{2}} e^{-\left(\frac{c_2}{2} + \frac{1}{2}\zeta\right)\sigma_3}.
\]

Here
\[
\Psi_1(s) = \begin{pmatrix} H(s) & Z(s) \\ U(s) & -H(s) \end{pmatrix},
\]
\[
\Psi_2(s) = \begin{pmatrix} \frac{1}{2}(H(s)^2 + Z(z) - sH(s)) & \frac{Z(s)(Z(s) + c - Y(s)s - H(s)Y(s))}{U(s)Y(s)} \\ U(s)(H(s) + Y(s) - s) & \frac{1}{2}(H(s)^2 + Z(z) + sH(s)) \end{pmatrix}.
\]

As \( \zeta \to \infty \),
\[
\tilde{\Psi}^c(\zeta; s) = \left( I + \frac{\Psi_1(s)}{\xi} + \frac{\Psi_2(s)}{\xi^2} + O\left(\frac{1}{\xi^3}\right)\right) \xi^{-\frac{5}{2}} e^{-\left(\frac{c_2}{2} + \frac{1}{2}\zeta\right)\sigma_3}.
\]

As \( \zeta \to 0 \) in the region \( \Omega_\infty \),
\[
\tilde{\Psi}^c(\zeta; s) = \zeta^{-\frac{5}{2}} e^{-O(1)}.
\]

By [17] and the references therein, the unique solution to the above Riemann-Hilbert problem is related to the Painlevé IV equation by the following lax pair,
\[
\frac{d}{d\zeta} \tilde{\Psi}^c = A \tilde{\Psi}^c, \quad \frac{d}{ds} \tilde{\Psi}^c = B \tilde{\Psi}^c,
\]
where
\[
A = A(\zeta; s) = -\frac{\zeta + s}{2} \sigma_3 + \begin{bmatrix} -\frac{c}{\xi} & Z \\ -U & 0 \end{bmatrix} - \frac{1}{\xi} \begin{bmatrix} -\frac{c}{\xi} & Z \\ -U & Z + \frac{5}{2} \end{bmatrix} Z + \frac{c}{2} Z + cZ
\]
and
\[
B = B(\zeta; s) = -\frac{\zeta}{2} \sigma_3 + \begin{bmatrix} -\frac{c}{\xi} & Z \\ -U & 0 \end{bmatrix}.
\]
The compatibility condition of the linear system (1.32) gives \( A_s - B_\zeta + [A, B] = 0 \). It follows that
\[
U' = U(Y - s), \quad Z' = ZY - \frac{Z^2 + cZ}{Y}, \quad Y' = sY - Y^2 - 2Z - c.
\]

Using the above differential equations, one can observe that \( Y \) satisfies the Painlevé IV equation
\[
Y'' = \frac{1}{2} \frac{(Y')^2}{Y} + \frac{3}{2} Y^3 - 2sY^2 + \left(1 + \frac{s^2}{2} + c\right) Y - \frac{c^2}{2Y}.
\]
The functions \( Y(s), Z(s), \) and \( H(s) \) are interrelated through
\[
Z = \frac{1}{2}(sY - c - Y' - Y^2), \quad H = \left(s - \frac{c}{Y} - Y\right)Z - \frac{Z^2}{Y}.
\]
We also write
\[
W := Z/U.
\]
In terms of the function \( W \), we define
\[
A \equiv A(s) := \left[ \frac{W''(s)}{W(s)} - \left( \frac{W'(s)}{W(s)} \right)^2 \right]^{-1}
\]
and
\[
B \equiv B(s) = 2 \frac{W'(s)}{W(s)A(s)} + \frac{W'(s)}{W(s)} \left( \frac{W''(s)}{W(s)} - \frac{W'''(s)}{W'(s)} \right),
\]
\text{cf. (4.30) and (4.31).}

We write \( R_{\text{edge}}^c \), \( R_{\text{edge}}^c \) for the associated limiting 1-point functions when \( a \) is given by (1.29). Let us also denote by \( \tilde{R}_{\text{edge}}^c \), \( R_{\text{edge}}^c \) the corresponding correlation kernels. Let us define an analytic continuation, \( \Psi^c(\zeta; s) \), of the matrix function \( \tilde{\Psi}^c(\zeta; s) \) by
where \( \Omega_0, \Omega_2 \) and \( \Omega_\infty \) are specified in Figure 3.

**Theorem 1.6.** Let \( \Psi^c_{jk} \) be the \((j, k)\) entry of \( \Psi^c \) defined in (1.41).

- **(Induced Ginibre ensemble with a point charge at a boundary point)** For each \( c \in (-1, 0) \), we have
  \[
  \hat{R}^c_{\text{edge}}(z) = \frac{2\mathfrak{A}(S)(-1)^c}{\sqrt{2\pi}} e^{-\frac{(w-S)^2}{4}} z^c \Psi^c_{21}(-z; S) W(S) \]
  \[
  \times \int_{-\infty}^{\tilde{z}} e^{-z w - \frac{(w-S)^2}{4}} w^z \left[ \Psi^c_{11}(-w; S) \chi + \Psi^c_{21}(-w; S) \psi + Z(S) \right] dw,
  \]
  where
  \[
  \chi \equiv \chi(S) = \frac{Z}{w} - z + \mathfrak{A} \mathfrak{B} - \frac{Z + c}{Y} + Y, \quad \psi \equiv \psi(S) = -\frac{W Z + c}{w Y}.
  \]

- **(Recursive formula)** For each \( c > -1 \), we have
  \[
  \hat{R}^{c+1}_{\text{edge}}(z) = \hat{R}^c_{\text{edge}}(z) - \frac{|\hat{R}^c_{\text{edge}}(0, z)|^2}{\hat{R}^c_{\text{edge}}(0)}.
  \]

**Remark 1.7.** It is known [17, Theorem 2.5] that the solution \( \hat{\Psi}^c(\zeta; S) \) exists for \( S < -0.7701449782 \). Also it is expected that there are discrete values of \( S \) where the solution does not exist. Hence the above theorem makes sense for all \( S \) except those discrete values. The asymptotic behaviours of the 1-point function in (1.42) will be discussed in Appendix C.

As an immediate consequence of Proposition 1.5 and Theorem 1.6, we obtain the scaling limit of the lemniscate ensembles.

**Theorem 1.8 (Lemniscate ensemble with d-fold symmetry).** Under the same assumptions of Theorem 1.6, we have that for each \( d > 1 \) and \( c > -1 \),

\[
R^c_{\text{edge}}(z) = d |z|^{2d-2} \sum_{l=0}^{d-1} \frac{\mathfrak{A}(S)(-1)^{c-l}}{\sqrt{2\pi}} e^{-\frac{(w-S)^2}{4}} z^{c-l} \Psi^{c-l}_{21}(-z; S) W(S) \]

\[
\times \int_{-\infty}^{\tilde{z}} e^{-z w - \frac{(w-S)^2}{4}} w^z \left[ \Psi^{c-l}_{11}(-w; S) \chi + \Psi^{c-l}_{21}(-w; S) \psi + Z(S) \right] dw.
\]

**Remark 1.9.** For a fixed \( a \in [0, 1) \), thus when the origin is inside of the droplet \( S \) (see Figure 2 (A) and (D)), it was shown in [12] that the limiting 1-point functions \( \hat{R}^c_{\text{bulk}}, \hat{R}^c_{\text{bulk}} \) are given by

\[
\hat{R}^c_{\text{bulk}}(z) = |z|^{2c} e^{-|z|^2} E_{1,1+c}(|z|^2),
\]

\[
R^c_{\text{bulk}}(z) = d |z|^{2c} e^{-|z|^2} E_{\frac{1}{2}, 1+c}(|z|^2),
\]

where \( E_{a,b} \) is the two-parametric Mittag-Leffler function

\[
E_{a,b}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}.
\]
We mention that the approach in \cite{12} using Ward’s equation relies on the fact that the limiting 1-point functions \((1.46)\) are rotationally symmetric. Our approach can also be applied to the bulk case when \(a \in (0, 1)\), which provides an alternative derivation of the limiting one-point functions \((1.46)\), see Theorem D.1.

An additional advantage of our approach using the Christoffel-Darboux identity lies in the fact that both in Theorems 1.6 and D.1, it indeed allows to compute not only the leading order asymptotic but also its fine asymptotic as long as the detailed strong asymptotics of the associated orthogonal polynomial are provided. We refer to \cite{22, 41} for previous works in this direction on exactly solvable models.

The rest of this paper is organised as follows.

- In Section 2, we derive \((1.44)\) and \((1.45)\) by showing the recursive formula and the multi-fold transformation of correlation kernels in a general context.
- In Section 3, we present the Christoffel-Darboux identities for some class of planar orthogonal polynomials and show Theorem 1.1.
- In Section 4, we derive the large-\(N\) limit of the correlation kernel and complete the proof of Theorem 1.6.
- This article contains several appendices. In appendices A and B, we compile detailed computations used in the proofs of Theorem 1.6. In Appendix C, we discuss the asymptotic behaviours of the 1-point function in Theorem 1.6. In Appendix D, we re-derive the bulk scaling limits in Remark 1.9 based on our strategy of using the Christoffel-Darboux formula.

2. Recursive formula and multi-fold transformation

In this section, we present the recursive formula and multi-fold transformation of correlation kernels. First let us recall some well-known facts.

Note that \(E^{1,c}(z) = z^{-c} e^{z} P(c, z)\), where \(P(c, z) := \frac{1}{\Gamma(c)} \Gamma(c, z)\) is the regularised incomplete Gamma function. For \(a \in [0, 1)\) fixed, we write \(\hat{R}^{c}_{\text{bulk}}, K^{c}_{\text{bulk}}\) for the corresponding correlation kernels. It then follows from \((1.46)\) that

\[
\hat{R}^{c}_{\text{bulk}}(z) = P(c, |z|^2), \quad \hat{K}^{c}_{\text{bulk}}(z, w) = G(z, w) P(c, zw),
\]

where

\[
G(z, w) := e^{z \bar{w} - |z|^2/2 - |w|^2/2}
\]

is the bulk Ginibre kernel. On the other hand when \(c = 0\), we have the boundary Ginibre kernel

\[
\hat{R}^{0}_{\text{edge}}(z) = \frac{1}{2} \text{erfc}(\frac{-z \pm \bar{z} - S}{\sqrt{2}}), \quad \hat{K}^{0}_{\text{edge}}(z, w) = G(z, w) \frac{1}{2} \text{erfc}(\frac{-z \pm \bar{w} - S}{\sqrt{2}}).
\]

2.1. Recursive formula. Let us define the Berezin kernel

\[
\hat{B}^{c}_{N}(z, w) := \frac{|\hat{K}^{c}_{N}(z, w)|^2}{\hat{R}^{c}_{N}(z)}.
\]

We now derive the following recursive formula for \(\hat{R}^{c}_{N}\), see \cite[Lemma 7.6.2]{10} for a similar statement.

**Lemma 2.1.** For any \(a \geq 0\) and \(c > -1\), we have

\[
\hat{R}^{c+1}_{N}(z) = \hat{R}^{c+1}_{N+1}(z) - \hat{B}^{c+1}_{N+1}(0, z).
\]

As an immediate consequence, by letting \(N \to \infty\), we obtain \((1.44)\). Before the proof, let us present some examples.
Example 2.2. (Bulk case) By (2.1) and [47, Eq.(8.7.1)], we have

\[
\frac{\hat{K}_c^{\text{bulk}}(0, z)^2}{\hat{R}_c^{\text{bulk}}(z)} = e^{-|z|^2} \frac{|z|^{2c}}{\Gamma(c + 1)}.
\]

Then it follows from the recurrence relation of the regularised Gamma function (see [47, Eq.(8.8.5)]) that

\[
\hat{R}_{c+1}^{\text{bulk}}(z) = \hat{R}_c^{\text{bulk}}(z) - \frac{\hat{K}_c^{\text{bulk}}(0, z)^2}{\hat{R}_c^{\text{bulk}}(z)} = \Phi(c + 1, |z|^2).
\]

Example 2.3. (Edge case) For \( a = 1 \), by (2.3), we have

\[
\hat{R}_c^{\text{edge}}(z) = \frac{1}{2} \mathrm{erfc}(-\frac{z + \bar{z}}{\sqrt{2}}) - \frac{1}{2} e^{-|z|^2} \mathrm{erfc}(-\frac{z}{\sqrt{2}})^2.
\]

Similarly, we have

\[
\hat{R}_c^{\text{edge}}(z) = \frac{1}{2} \mathrm{erfc}(-\frac{z + \bar{z}}{\sqrt{2}}) - \frac{1}{2} e^{-|z|^2} \mathrm{erfc}(-\frac{z}{\sqrt{2}})^2 - \frac{1}{\pi} - 2 e^{-|z|^2} \left( \sqrt{\frac{\pi}{2}} - 1 \right) \mathrm{erfc}(-\frac{z}{\sqrt{2}}) + e^{-z^2/2}.
\]

See Figure 4 below for the graphs of \( \hat{R}_c^{\text{edge}} \).

![Figure 4](image.png)

**Figure 4.** The plots display the graphs of \( \hat{R}_c^{\text{edge}} \) for a few values of \( c \).

We now prove Lemma 2.1.

**Proof of Lemma 2.1.** Let us write

\[
\hat{R}_{N,k}^{c}(\hat{\zeta}_1, \ldots, \hat{\zeta}_k) := \frac{1}{Z_N} \frac{N!}{(N-k)!} \int_{\mathbb{C}^{N-k}} \prod_{j<k} |\hat{\zeta}_j - \hat{\zeta}_k|^2 \frac{e^{-N \sum_j Q_c(\hat{\zeta}_j)}}{\prod_{j=k+1}^N d\hat{A}(\hat{\zeta}_j)}
\]

for the \( k \)-point correlation (joint intensity) function. Recall that we have

\[
\hat{R}_{N,k}^{c}(\hat{\zeta}_1, \ldots, \hat{\zeta}_k) = \det \left[ \hat{K}_N(\hat{\zeta}_j, \hat{\zeta}_l) \right]_{j,l=1}^k,
\]

see e.g. [33].

The main idea of the proof is the following simple observation

\[
|\Delta(\hat{\zeta}_1, \ldots, \hat{\zeta}_N)|^2 \prod_{j=1}^N |\hat{\zeta}_j|^2 = |\Delta(\hat{\zeta}_1, \ldots, \hat{\zeta}_N, 0)|^2,
\]
where $\Delta$ is the Vandermonde determinant. Using this, we have

$$
\hat{R}^{e+1}_{N,1}(\zeta_1) = \frac{1}{N} \frac{N!}{(N-1)!} \int_{C^{N-1}} |\Delta(\zeta_1, \ldots, \zeta_N)|^2 e^{-N\sum_j Q_c(\zeta_j)} \prod_{j=2}^N dA(\zeta_j)
$$

$$
= \frac{1}{N+1} \frac{1}{N+1} \frac{(N+1)!}{(N-1)!} \int_{C^{N-1}} |\Delta(\zeta_1, \ldots, \zeta_N, 0)|^2 e^{-N\sum_j Q_c(\zeta_j)} \prod_{j=2}^N dA(\zeta_j)
$$

$$
= \frac{1}{N+1} \hat{R}^{c}_{N+1,2}(\zeta_1, 0),
$$

which leads to

$$
\hat{R}^{c+1}_{N,1}(\zeta) = \frac{N}{N+1} \hat{R}^{c+1}_{N+1,2}(\zeta_0)/\hat{R}^{c+1}_{N+1,2}(\zeta_0).
$$

Therefore we obtain

$$
\hat{R}^{c+1}_{N,1}(z) = \frac{1}{N} \hat{R}^{c+1}_{N,1}(\frac{z}{\sqrt{N}}) = \frac{1}{N+1} \hat{R}^{c+1}_{N+1,2}(\frac{z}{\sqrt{N}}, 0).
$$

Since

$$
\hat{R}^{c}_{N+1,1}(0) = \frac{1}{N+1} \hat{R}^{c}_{N+1,1}(0), \quad \hat{R}^{c}_{N+1,2}(0, z) = \frac{1}{(N+1)^2} \hat{R}^{c}_{N+1,2}(\frac{z}{\sqrt{N}}, 0),
$$

we conclude

$$
\hat{R}^{c+1}_{N,1}(z) = \frac{\hat{R}^{c+1}_{N+1,2}(0, z)}{\hat{R}^{c+1}_{N+1,1}(0)} = \hat{R}^{c}_{N+1,1}(z) - \hat{B}_{N+1}(0, z).
$$

This completes the proof. 

2.2. Multi-fold transformations. In this subsection, we show Proposition 1.5.

We first note that by taking $N \to \infty$ of (1.28), we have

$$
(2.12) \quad K^c(z, w) = d(z\bar{w})^{d-1} \sum_{l=0}^{d-1} \hat{K}^{c+l+1}_l(z^d, w^d).
$$

**Remark 2.4.** In the opposite direction, one can also express $\hat{K}^c$ in terms of $K^c$. For instance when $d = 2$, we have the relations

$$
(2.13) \quad K^c(z, w) + K^c(z, -w) = 4z\bar{w} \hat{K}^{c+1}_2(z^2, w^2),
$$

$$
(2.14) \quad K^c(z, w) - K^c(z, -w) = 4z\bar{w} \hat{K}^{c+1}_2(z^2, w^2).
$$

Summing these two equations, we obtain (2.12) with $d = 2$:

$$
(2.15) \quad K^c(z, w) = 2z\bar{w} \left( \hat{K}^{c+1}_2(z^2, w^2) + \hat{K}^{c+1}_2(z^2, w^2) \right).
$$

We remark that when $c \in \{0, 1\}$, the term $2z\bar{w} \hat{K}^{0}_2(z^2, w^2)$ in the right-hand side of the above equation corresponds to the kernel appearing in the context of chiral Ginibre ensembles, see [2, Theorem 3].

Before the proof, we interpret $K^c_{\text{bulk}}$ for general $d > 1$ from the viewpoint of Proposition 1.5.

**Example 2.5.** (Bulk case) It follows from the definition (1.47) that

$$
\sum_{l=0}^{d-1} x^l E_{1, l+1+1}(x^d) = \sum_{l=0}^{d-1} \sum_{k=0}^{\infty} \frac{x^{dk+l}}{\Gamma(k + \frac{c+l+1}{d})} = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\frac{j}{d} + \frac{1+c}{d})} = E_1^{\frac{1}{d} + \frac{1+c}{d}}(x).
$$
Then by (2.12), one can obtain $K^c_{\text{bulk}}$ from $\hat{K}^c_{\text{bulk}}$ as

$$K^c_{\text{bulk}}(z, w) = d(z\bar{w})^{d-1} \sum_{l=0}^{d-1} \hat{K}^c_{\text{bulk}}^{\frac{c+l+1}{d}-1}(z^d, w^d)$$

(2.16)

$$= d(z\bar{w})^c e^{-|z|^{2d}/2 - |w|^{2d}/2} \sum_{l=0}^{d-1} \int E_{1, \frac{c+l+1}{d}}(z\bar{w})^d = d(z\bar{w})^c e^{-|z|^{2d}/2 - |w|^{2d}/2} E_{1, \frac{c+l+1}{d}}(z\bar{w}).$$

Proof of Proposition 1.5. By the change of variable $\zeta \mapsto \zeta^d$, it is easy to observe that $p_j^c$ and $q_j^c$ enjoy the intimate relation

(2.17)

$$q_{dj+l,dN}(\zeta) = \sqrt{d} \zeta^d \frac{c+l+1}{d} \left(\zeta^d\right).$$

This property is also discussed in [17, Section 3] but it is easy enough to recall a proof. By definition, we have

$$\delta_{jk} = \int_C q_{j,dN}(\zeta) \overline{q_{k,dN}(\zeta)} |\zeta|^{2c} e^{-dNV(\zeta)} dA(\zeta) = \int_C p_{j,N}(\zeta) \overline{p_{k,N}(\zeta)} |\zeta|^{2c} e^{-NQ(\zeta)} dA(\zeta).$$

Since $V_c(\zeta)$ is invariant under the discrete rotation $\zeta \mapsto e^{2\pi i/d} \cdot \zeta$, there exists a polynomial $p_j$ such that $q_{dj+l,dN}(\zeta) = \zeta^l p_j(\zeta^d)$. By the change of variable $\eta = \zeta^d$, we have

$$\delta_{jk} = \frac{1}{d^2} \int_{0<\arg \zeta<\frac{2\pi}{d}} p_j(\zeta^d) \overline{p_k(\zeta^d)} |\zeta|^{2c+2l} e^{-NQ(\zeta^d)} dA(\zeta) = \frac{1}{d} \int_C p_j(\eta) \overline{p_k(\eta)} |\eta|^{2c+2l} e^{-NQ(\eta)} dA(\eta).$$

Thus we obtain $p_j(\zeta) = \sqrt{d} \frac{c+l+1}{d} \left(\zeta^d\right)$, which leads to (2.17).

By (1.10) and (1.11) we have

$$K^c_{dN}(\zeta, \eta) = (\zeta\bar{\eta})^{c-\frac{dN}{2}} e^{-\frac{\bar{\eta}}{2} V^c(\zeta) - \frac{\bar{\eta}}{2} V^c(\eta)} \sum_{j=0}^{dN-1} q_{j,dN}(\zeta) \overline{q_{j,dN}(\eta)},$$

(2.18)

$$\hat{K}^c_{dN}(\zeta, \eta) = (\zeta\bar{\eta})^{c-\frac{\bar{\zeta}}{2} Q^c(\zeta) - \frac{\bar{\zeta}}{2} Q^c(\eta)} \sum_{j=0}^{dN-1} p_{j,N}(\zeta) \overline{p_{j,N}(\eta)}.$$ 

(2.19)

Observe here that by (2.17),

$$\sum_{j=0}^{dN-1} q_{j,dN}(\zeta) \overline{q_{j,dN}(\eta)} = \sum_{j=0}^{dN-1} q_{dj+l,dN}(\zeta) \overline{q_{dj+l,dN}(\eta)} = d \sum_{l=0}^{d-1} (\zeta\bar{\eta})^{c-l} \sum_{j=0}^{N-1} p_{j,N}^{c+l+1}(\zeta^d) \overline{p_{j,N}^{c+l+1}(\eta^d)}.$$

Thus we obtain

$$K^c_{dN}(\zeta, \eta) = d \sum_{j=0}^{d-1} (\zeta\bar{\eta})^{c-l} \sum_{j=0}^{N-1} p_{j,N}^{c+l+1}(\zeta^d) \overline{p_{j,N}^{c+l+1}(\eta^d)}.$$

$$= d(\zeta\bar{\eta})^{d-1} \sum_{l=0}^{d-1} \hat{K}^c_{dN}^{\frac{c+l+1}{d}-1}(\zeta^d, \eta^d).$$
Remark 3.2. Note in particular that the

Proposition 3.1. We have

The following form of the Christoffel-Darboux identity.

\[
\int_K(z,w) = \frac{1}{N^{d/2}} K_N \left( \frac{z}{N^{1/2}}, \frac{w}{N^{1/2}} \right) = d(z\bar{w})^{d-1} \sum_{l=0}^{d-1} K_N^{\frac{-d+l+1}{d}}(\frac{z^d}{\sqrt{N}}, \frac{w^d}{\sqrt{N}}) = d(z\bar{w})^{d-1} \sum_{l=0}^{d-1} \tilde{K}_N^{\frac{-d+l+1}{d}}(z^d, w^d),
\]

which completes the proof. \(\square\)

\section{Christoffel-Darboux identity for planar orthogonal polynomials}

This section is devoted to proving the Christoffel-Darboux identity, Theorem 1.1, see \cite{18} for a similar method of deriving such identity in the context of bi-orthogonal polynomials. We also refer to \cite[Subsection 4.1]{3} for a version of the Christoffel-Darboux which involves differential operators.

\subsection{Elliptic potential revisited}

To better introduce the general strategy of deriving the Christoffel-Darboux identity for planar orthogonal polynomials, let us first consider the elliptic potential

\[
Q(\zeta) := \frac{1}{1-\tau^2}(|\zeta|^2 - \tau \Re \zeta^2), \quad \tau \in [0,1).
\]

The random normal matrix ensemble associated with such a potential is equivalent to the elliptic Ginibre ensemble. It is well known that the orthogonal polynomial with respect to the measure \(e^{-NQ} dA\) can be expressed in terms of the Hermite polynomial \(H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}\). More precisely, the monic orthogonal polynomial

\[
P_j(\zeta) = \frac{1}{N^{j/2}} H_j(\sqrt{\frac{N}{2\pi}} \zeta)
\]

satisfies the orthogonality relation

\[
\int_{\mathbb{C}} P_j(\zeta) \overline{P_k(\zeta)} e^{-NQ(\zeta)} dA(\zeta) = h_j \delta_{jk}, \quad h_j = \sqrt{1-\tau^2} \frac{j!}{N^{j/2}},
\]

see e.g. \cite{52} or \cite[Lemma 7]{4}.

Let us write

\[
W(\zeta) := e^{-\frac{N}{2(1-\tau^2)} \zeta^2}, \quad \psi_j(\zeta) := W(\zeta) P_j(\zeta), \quad \phi_j(\zeta) := W(\zeta) \frac{P_j(\zeta)}{h_j}.
\]

Then the associated correlation kernel \(\tilde{K}_N\) is written as

\[
\tilde{K}_N(\zeta, \eta) = e^{-\frac{N}{2(1-\tau^2)}} \sum_{j=0}^{N-1} \phi_j(\eta) \psi_j(\zeta).
\]

We have the following form of the Christoffel-Darboux identity.

\begin{proposition}
We have
\end{proposition}

\[
\partial_{\eta} \tilde{K}_N(\zeta, \eta) = \frac{N}{1-\tau^2} e^{-\frac{N}{2(1-\tau^2)} \zeta^2} \left( \tau \phi_N(\eta) \psi_{N-1}(\zeta) - \overline{\phi_{N-1}(\eta)} \psi_N(\zeta) \right).
\]

Remark 3.2. Note in particular that the 1-point function \(\tilde{R}_N(\zeta) = \tilde{K}_N(\zeta, \zeta)\) satisfies

\[
\partial_{\eta} \tilde{R}_N(x+iy) = -\frac{2}{N-1} \frac{e^{-NQ(x+iy)}}{(N-1)! (1-\tau^2)(1+\tau)} \Re \left[ P_N(x-iy) P_{N-1}(x+iy) \right].
\]

We refer to \cite[Proposition 2.3]{41} for a direct proof of (3.6) using the three-term recurrence relation and differentiation rule of Hermite polynomials. (See also \cite[Lemma 4.1]{9} for a related statement.) Together with Plancherel-Rotach type strong asymptotics of Hermite polynomials, the identities (3.6),
(3.7) were used in [9, 41] to derive the associated limiting local kernels in various situations. Beyond the study of determinantal point processes, the identity (3.7) was also utilized to analyse real eigenvalue distributions of real elliptic random matrices, see [24].

Proof of Proposition 3.1. Let us define semi-infinite dimensional vectors
\[ \Psi := [\psi_0, \psi_1, \cdots]^t, \quad \Phi := [\phi_0, \phi_1, \cdots]^t, \]
where \( t \) is the transpose of a matrix and write
\[ \Pi_N := \text{diag}(1, \cdots, 1, 0, \cdots) \]
for the projection (truncation) operator. Then the kernel \( \tilde{K}_N \) can be rewritten as
\[ \tilde{K}_N(\zeta, \eta) = e^{-\frac{N}{1-\tau^2} \zeta \eta} \Phi^*(\eta) \Pi_N \Psi(\zeta), \]
where * denotes the Hermitian transpose of the matrix. We also define
\[ \langle T|S \rangle := \int T S e^{-\frac{N}{1-\tau^2} |\zeta|^2} dA(\zeta) = \langle S|T \rangle. \]

By definition, there exist semi-infinite dimensional matrices \( A, B \) such that
\[ \partial \phi_j = \sum_k A_{jk} \phi_k, \quad \partial \Phi = A \Phi, \]
\[ \zeta \psi_j(\zeta) = \sum_k B_{jk} \psi_k(\zeta), \quad \zeta \Psi(\zeta) = B \Psi. \]

Notice here that integration by parts gives rise to
\[ B_{jk} = \int \overline{\phi_k(\zeta)} e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} \psi_j(\zeta) dA(\zeta) = -\frac{1-\tau^2}{N} \int \overline{\phi_k(\zeta)} \partial \left( e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} \right) \psi_j(\zeta) dA(\zeta) \]
\[ = \frac{1-\tau^2}{N} \int \overline{\partial \phi_k(\zeta)} \psi_j(\zeta) e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} dA(\zeta) = \frac{1-\tau^2}{N} A_{kj}. \]

In other words, we have
\[ B = \frac{1-\tau^2}{N} A^*. \]

Using this, we obtain
\[ (\zeta - \frac{1-\tau^2}{N} \partial_\eta) \Phi^*(\eta) \Pi_N \Psi(\zeta) = \Phi^*(\eta) \Pi_N B \Psi(\zeta) - \frac{1-\tau^2}{N} \Phi^*(\eta) A^* \Pi_N \Psi(\zeta) \]
\[ = \Phi^*(\eta) \left( \Pi_N B - B \Pi_N \right) \Psi(\zeta). \]

Thus we have
\[ \partial_\eta \tilde{K}_N(\zeta, \eta) = \partial_\eta \left[ e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} \Phi^*(\eta) \Pi_N \Psi(\zeta) \right] \]
\[ = -\frac{N}{1-\tau^2} e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} (\zeta - \frac{1-\tau^2}{N} \partial_\eta) \Phi^*(\eta) \Pi_N \Psi(\zeta) \]
\[ = -\frac{N}{1-\tau^2} e^{-\frac{N}{1-\tau^2} \zeta \bar{\eta}} \Phi^*(\eta) \left( \Pi_N B - B \Pi_N \right) \Psi(\zeta). \]

Now let us determine the matrix \( B \). It follows from the three-term recurrence relation of Hermite polynomials that
\[ \zeta P_j(\zeta) = P_{j+1}(\zeta) + \frac{j}{N} P_{j-1}(\zeta). \]
Thus we obtain
\begin{equation}
B_{j,k} = \begin{cases} 
1 & \text{if } k = j + 1, \\
\frac{j}{N} & \text{if } k = j - 1, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}
Using this, we conclude
\begin{equation}
\Phi^*(\eta)(\Pi_N B - B \Pi_N) \Phi(\zeta) = \phi_{N-1}(\eta) \psi_N(\zeta) - \phi_N(\eta) \psi_{N-1}(\zeta).
\end{equation}
This completes the proof.

3.2. Gaussian potential with an insertion of a point charge. In this subsection, we derive the Christoffel-Darboux identity (Theorem 1.1) for the orthogonal polynomials associated with the potential of the form (1.4). The overall strategy is similar to the one presented in the previous subsection. However, it requires some modifications due to the lack of standard three-term recurrence relation. We now prove Theorem 1.1.

Proof of Theorem 1.1. As in Subsection 3.1, we write \( \Psi := [\psi_0, \psi_1, \cdots]^t, \Phi := [\phi_0, \phi_1, \cdots]^t. \)

Using the projection operator \( \Pi_N \) in (3.8), we write
\begin{equation}
\tilde{K}_N(\zeta, \eta) = e^{-N|\zeta|^2} \Phi^*(\eta) \Pi_n \Phi(\zeta).
\end{equation}
Then we have
\begin{equation}
\partial_\eta \tilde{K}_N(\zeta, \eta) = -Ne^{-N|\zeta|^2} (\zeta - \frac{1}{N}\partial_\eta) \Phi^*(\eta) \Pi_n \Phi(\zeta).
\end{equation}
For each \( j \), let
\begin{equation}
L_{j,j-1} = -\int \frac{\zeta \psi_j(\zeta) \phi_0(\zeta) e^{-N|\zeta|^2} dA(\zeta)}{\int \zeta \psi_{j-1}(\zeta) \phi_0(\zeta) e^{-N|\zeta|^2} dA(\zeta)}.
\end{equation}
Notice that the denominator does not vanish. Note also that
\[ \zeta (\psi_j(\zeta) + L_{j,j-1} \psi_{j-1}(\zeta)) \perp \phi_0 \]
with respect to the inner product
\begin{equation}
\langle T | S \rangle := \int T S e^{-N|\zeta|^2} dA(\zeta) = \langle S | T \rangle.
\end{equation}
The numbers \( L_{j,j-1} \) are building blocks to define the lower diagonal matrix
\[ L := \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ L_{2,1} & 0 & 0 & 0 & \cdots \\ 0 & L_{3,2} & 0 & 0 & \cdots \\ 0 & 0 & L_{4,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}. \]
Write
\begin{equation}
\tilde{\psi}_j := \psi_j + L_{j,j-1} \psi_{j-1}.
\end{equation}
Then if
\[ \phi(\zeta) = \text{(polynomials of deg } \leq j - 2) \cdot (\zeta - a) \cdot (\zeta - a) \cdot \}
we have
\begin{equation}
\langle \phi | \zeta \tilde{\psi}_j \rangle = \langle \partial \phi | \tilde{\psi}_j \rangle = 0.
\end{equation}
In other words, we have
\[
\text{span}\{\phi_0, \phi_1, \cdots, \phi_{j-1}\} \perp \zeta \tilde{\psi}_j,
\]
which leads to
\[
\zeta \tilde{\psi}_j(\zeta) = \psi_{j+1}(\zeta) + B_{j,j} \psi_j(\zeta)
\]
for some $B_{j,j}$. Thus we obtain
\[
(3.24) \quad \zeta(I + L)\Psi = B \Psi, \quad B := \begin{pmatrix}
B_{1,1} & 1 & 0 & 0 & \ldots \\
0 & B_{2,2} & 1 & 0 & \ldots \\
0 & 0 & B_{3,3} & 1 & \ldots \\
0 & 0 & 0 & B_{4,4} & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Let us also write
\[
(3.25) \quad U_{j,j+1} = -\frac{h_{j+1}}{h_j} \frac{P_j(a)}{P_{j+1}(a)},
\]
and define the upper diagonal matrix
\[
U := \begin{pmatrix}
0 & U_{1,2} & 0 & 0 & \ldots \\
0 & 0 & U_{2,3} & 0 & \ldots \\
0 & 0 & 0 & U_{3,4} & \ldots \\
0 & 0 & 0 & 0 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then the function
\[
(3.26) \quad \hat{\phi}_j(\zeta) := \phi_j(\zeta) + U_{j,j+1} \phi_{j+1}(\zeta) = \text{(polynomials of deg } \leq j) \cdot (\zeta - a) \cdot (\zeta - a)^c
\]
satisfies
\[
(3.27) \quad \langle \partial \hat{\phi}_j | \psi_k \rangle = \langle \hat{\phi}_j | \zeta \psi_k \rangle = 0 \quad \text{if } k \leq j - 2.
\]

Thus we have
\[
(3.28) \quad \partial \hat{\phi}_j = A_{2,j} \phi_j + A_{1,j-1} \phi_{j-1}
\]
for some $A_{j,k}$, equivalently,
\[
(3.29) \quad \partial(I + U)\Phi = A \Phi, \quad A := \begin{pmatrix}
A_{1,1} & 0 & 0 & 0 & \ldots \\
A_{2,1} & A_{2,2} & 0 & 0 & \ldots \\
0 & A_{3,2} & A_{3,3} & 0 & \ldots \\
0 & 0 & A_{4,3} & A_{4,4} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

We now determine $A_{j,j-1}$ and $A_{j,j}$. Note that integration by parts gives
\[
\overline{B}(I + U)^t = \overline{B}(\phi | \Phi^t)(I + U)^t = \langle B \Psi | \Phi^t(I + U)^t \rangle
\]
\[
= \langle (\zeta(I + L))\Psi | \Phi^t(I + U)^t \rangle = \frac{1}{N} \langle (I + L)\Psi | \partial \Phi^t(I + U)^t \rangle
\]
\[
= \frac{1}{N} \langle (I + L)\Psi | \Phi^t A^t \rangle = \frac{1}{N}(I + L)A^t.
\]

Thus we obtain the relation
\[
(3.31) \quad \frac{1}{N}A(I + L)^* = (I + U)B^*, \quad B = \frac{1}{N}(I + L)A^*(I + U^*)^{-1}.
\]
Comparing the terms involving $A_{j,j-1}$, one can observe that
\begin{equation}
A_{j,j-1} = N.
\end{equation}

To determine $A_{j,j}$, note that
\[
\partial \hat{\phi}_j(\zeta) = \partial \left( \phi_j + U_{j,j+1} \phi_{j+1} \right) = \frac{1}{h_j} \partial \left( (\zeta - a) \partial \right) + \frac{U_{j,j+1}}{h_{j+1}} \partial \left( (\zeta - a)^c P_{j+1} \right)
\]
\[
= (\zeta - a)^{c-1} \frac{1}{h_j} \left[ \left( cP_j + (\zeta - a) P'_j \right) - \frac{P_j(a)}{P_{j+1}(a)} \left( cP_{j+1} + (\zeta - a) P'_{j+1} \right) \right]
\]
\[
= (\zeta - a)^{c-1} \frac{1}{h_j} \left[ \frac{c}{\zeta - a} \left( P_j - \frac{P_j(a)}{P_{j+1}(a)} P_{j+1} \right) + P'_j - \frac{P_j(a)}{P_{j+1}(a)} P'_{j+1} \right].
\]

This gives
\[
A_{j,j} P_j + N \frac{h_j}{h_{j+1}} P_{j-1} = \frac{c}{\zeta - a} \left( P_j - \frac{P_j(a)}{P_{j+1}(a)} P_{j+1} \right) + P'_j - \frac{P_j(a)}{P_{j+1}(a)} P'_{j+1}.
\]

Comparing the coefficient of $\zeta^j$ term of this identity, we obtain
\begin{equation}
A_{j,j} = - \frac{P_j(a)}{P_{j+1}(a)} (c + j + 1).
\end{equation}

Notice in particular that $A_{j,k}$'s are real.

Now let us consider the decomposition
\begin{equation}
A = NT_+ + A_0,
\end{equation}
where
\[
T_- := \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{pmatrix}, \quad A_0 := \text{diag}(A_{1,1}, A_{2,2}, \ldots)
\]
are the translation and the diagonal part respectively. Write
\begin{equation}
A^* = NT_+ + A_0^*, \quad T_+ := T_+^*.
\end{equation}

Note also that we have
\[
(T_+ - \zeta) \Psi = (T_+ - (I+L)^{-1}B) \Psi = (T_+ - (I+L)^{-1} \frac{1}{N}(I+L)A^* (I+U^*)^{-1}) \Psi
\]
\[
= (T_+ - \frac{1}{N} A^* (I+U^*)^{-1}) \Psi = (T_+ (I+U^*) - \frac{1}{N} A^*) (I+U^*)^{-1} \Psi = (T_+ U^* - \frac{1}{N} A_0^*) (I+U^*)^{-1} \Psi,
\]
where the second and the fourth identity follow from (3.31) and (3.34) respectively.

We pause here to observe that $(T_+ U^* - \frac{1}{N} A_0^*)$ is invertible. Suppose that this is not the case. Then there exists some $k$ such that $U_{k-1,k}^* - \frac{1}{N} A_{k-1,k}^* = 0$. Consequently, we have $\psi_k(\zeta) = \zeta \psi_{k-1}(\zeta)$. This is a contradiction due to the assumption $a \neq 0$, see Remark 1.2. Therefore we have shown that $(T_+ U^* - \frac{1}{N} A_0^*)$ is invertible.

By letting
\begin{equation}
\hat{\Psi} := (T_+ U^* - \frac{1}{N} A_0^*)^{-1} (T_+ - \zeta) \Psi,
\end{equation}
we have
\begin{equation}
(I + U^*) \hat{\Psi} = \Psi.
\end{equation}
Similarly, we obtain
\[(3.39)\]
\[\zeta \] which leads to
\[(3.38)\]
\[\hat{\zeta} \]
Thus we have
\[(3.41)\]
As an immediate consequence of Theorem 1.1, we have the following corollary.

Combining (3.39), (3.29) and (3.31), we obtain
\[\begin{align*}
\zeta (I + U^*) \tilde{\Psi} &= (I + L)\zeta \tilde{\Psi} = B \Psi = B(I + U^*) \tilde{\Psi} = \frac{1}{N} (I + L) A^* \tilde{\Psi}, \\
\end{align*}\]
which leads to
\[(3.39)\]
\[\zeta (I + U^*) \tilde{\Psi} = \frac{1}{N} A^* \tilde{\Psi}.\]

Moreover by (3.32) and (3.38), we have
\[\Phi^*(\eta)[\Pi_n, A^*] \tilde{\Psi}(\zeta) = \frac{1}{N} \Phi^*(\eta)[\Pi_n, A^*] \tilde{\Psi}(\zeta) - \frac{1}{N} \hat{\eta} \Phi^*(\eta)[\Pi_n, I + U^*] \tilde{\Psi}(\zeta).\]

Similarly, we obtain
\[\begin{align*}
\hat{\eta} \Phi^*(\eta)[\Pi_n, I + U^*] \tilde{\Psi}(\zeta) &= \frac{1}{N} \Phi^*(\eta)[\Pi_n, A^*] \tilde{\Psi}(\zeta) - \frac{1}{N} \hat{\eta} \Phi^*(\eta)[\Pi_n, I + U^*] \tilde{\Psi}(\zeta). \\
\end{align*}\]

Combining all of the above identities with (3.18), the proof is complete. \[\square\]

To our purpose, let us define
\[(3.40)\]
\[\tilde{K}^c_N(z, w) := \frac{1}{N} \tilde{K}^c_N(\zeta, \eta), \quad \begin{cases} \zeta = a + \frac{z}{\sqrt{N}}, \\ \eta = a + \frac{w}{\sqrt{N}}, \end{cases}\]
and write \(\tilde{R}^c_N(z) := \tilde{K}^c_N(z, z)\). Notice that by (1.19), the function \(\tilde{R}^c_N\) is related to \(\tilde{R}_N\) as
\[(3.41)\]
\[\tilde{R}^c_N(z) = \tilde{R}^e_N(-z).\]

As an immediate consequence of Theorem 1.1, we have the following corollary.

**Corollary 3.3.** We have
\[(3.42)\]
\[\hat{\zeta} \tilde{R}^c_N(z) = e^{-|z|^2} (I^c_N(z) - \Pi_N^c(z)),\]
where
\[(3.43)\]
\[I^c_N(z) = \frac{1}{N \sqrt{N}} \frac{1}{h_{N-1}^N - h_N^N} \psi_N(\zeta) \left( \psi_N(\zeta) - \zeta \psi_{N-1}(\zeta) \right),\]
and
\begin{equation}
\Pi_N(z) = e^{-a^2 N - a \sqrt{N} (z + \bar{z})} \frac{P_{N+1}(a)}{P_N(a)} \frac{h_N/h_{N-1}}{N^{c+1} h_N - h_{N+1}} \psi_{N-1}(\xi) \left( \psi_{N+1}(\xi) - \xi \psi_N(\xi) \right).
\end{equation}
Here, $\xi = a + \frac{z}{\sqrt{N}}$.

4. LARGE-$N$ LIMIT OF THE ONE-POINT FUNCTION

In this section, we shall prove Theorem 1.6. Let us write $\tilde{R}_c^\text{bulk}$ for the large-$N$ limit of $\tilde{R}_c^N$ when $a \in [0, 1)$ is fixed. Similarly, we write $\tilde{R}_c^\text{edge}$ for the large-$N$ limit of $\tilde{R}_c^N$ when $a$ is given by (1.29).

We consider the case that $a$ is given by (1.29) and prove Theorem 1.6. We need to compute the asymptotic behaviours of the right-hand side of (3.42). This consists of the terms involving orthogonal polynomials and norms which are presented in Lemmas 4.3 and 4.4 respectively. Combining these, we obtain Theorem 1.6.

To derive Lemmas 4.3 and 4.4, we shall use the Riemann-Hilbert analysis for $P_j$ by Bertola, Elias Rebelo and Grava [14]. We postpone to Appendices A and B the proofs of most of the lemmas used during the proof of the main theorem.

Following [17, Proposition 4.5], let
\begin{equation}
\phi(z) \equiv \phi(z; z_0) = \frac{z - 1}{z_0} - \log z, \quad z_0 = \frac{1}{a^2} N
\end{equation}
and
\begin{equation}
A(z_0)^2 = 2 \log z_0 - 2 \frac{z_0 - 1}{z_0}, \quad \xi(z) \equiv \xi(z; z_0) = -A(z_0) + \sqrt{2 \phi(z; z_0) + A(z_0)^2}.
\end{equation}
We also write
\begin{equation}
S_r := 2\sqrt{N} \left( \sqrt{\frac{N}{N + r}} a - 1 \right) = S - \frac{r S}{2N} + O(N^{-2}).
\end{equation}
Recall that $\Psi$ is a solution to the Riemann-Hilbert problem in Section 1 and that $H$ and $W = Z/U$ are given by (1.37). We shall use the fine asymptotic behaviour of the orthogonal polynomial $P_k$.

**Proposition 4.1.** Let $z \in D$ and $k = N + r$ with $r$ fixed. Then as $N \to \infty$, we have
\begin{equation}
P_k(z) = z^k \left( \frac{z}{z - a} \right)^c e^{\frac{z}{2} \phi(z/a)} \left( \sqrt{k} \xi(z/a) \right)^\xi \psi_{11}(\sqrt{k} \xi(z/a); \sqrt{k} A(z_0)) \left( 1 + \frac{p_k(z)}{k^2} + O \left( \frac{1}{k^{1+\epsilon}} \right) \right),
\end{equation}
where
\begin{equation}
p_k(z) = H(S_r) \left( \frac{a}{z - a} - \frac{1}{\xi(z/a)} \right) + W(S_r) \left( \frac{a}{z - a} \left( \frac{z - a}{z} \right)^c \frac{1}{\xi(z/a)^c} - \frac{1}{\xi(z/a)} \right) \psi_{21}(\sqrt{k} \xi(z/a); \sqrt{k} A(z_0)),
\end{equation}
Here, $\phi$, $A$ and $S_r$ are given by (4.1), (4.2) and (4.3).

The leading order asymptotic of Proposition 4.1 is given in [17, Theorem 1.3]. Furthermore the authors presented a constructive way to derive the subleading correction terms albeit it requires long (but straightforward) computations. We defer the detailed computations to Appendix A.

As a direct consequence of Proposition 4.1, we obtain the following. To lighten notations, we sometimes omit the argument and write for instance
\begin{equation}
\psi_{11} \equiv \psi_{11}(z; S), \quad H = H(S), \quad \partial_s \psi_{11} = \partial_s \psi_{11}(z; s)|_{s=S}.
\end{equation}

**Lemma 4.2.** Let $z \in D$. Then as $N \to \infty$, the following holds.
• *Asymptotics of the difference* We have

\[
\psi_N + 1 \left( a + \frac{z}{\sqrt{N}} \right) - \left( a + \frac{z}{\sqrt{N}} \right) \psi_N \left( a + \frac{z}{\sqrt{N}} \right) = \left( a + \frac{z}{\sqrt{N}} \right)^{N+1+c} e^{\frac{z(z+2S)}{4}} z^s \Psi_{11}(z; S) \\
\times \left[ - \left( \frac{z}{2} + \frac{\partial_a \Psi_{11}}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + \left[ \phi(z) + \Phi(z) + \delta(z) + \left( \frac{z}{2} \frac{\partial \Psi_{11}}{\Psi_{11}} + \frac{z^2}{8} + \frac{1}{2} \frac{\partial^2 \Psi_{11}}{\Psi_{11}} \right) \right] \right] \frac{1}{N} + O(N^{-\frac{3}{2}})
\]

and

\[
\psi_N \left( a + \frac{z}{\sqrt{N}} \right) - \left( a + \frac{z}{\sqrt{N}} \right) \psi_{N-1} \left( a + \frac{z}{\sqrt{N}} \right) = \left( a + \frac{z}{\sqrt{N}} \right)^{N+c} e^{\frac{z(z+2S)}{4}} z^s \Psi_{11}(z; S) \\
\times \left[ - \left( \frac{z}{2} + \frac{\partial_a \Psi_{11}}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + \left[ \phi(z) + \Phi(z) + \delta(z) - \left( \frac{z}{2} \frac{\partial \Psi_{11}}{\Psi_{11}} + \frac{z^2}{8} + \frac{1}{2} \frac{\partial^2 \Psi_{11}}{\Psi_{11}} \right) \right] \right] \frac{1}{N} + O(N^{-\frac{3}{2}})
\]

where

\[
\phi(z) = \frac{z^2(2z + S)}{24} \frac{1}{2} \frac{\psi(z(2z + S)) \frac{\partial \Psi_{11}}{\Psi_{11}} + \frac{S^2}{12} \frac{\partial^2 \Psi_{11}}{\Psi_{11}}} \\
+ \frac{z(z + S)}{4} \frac{1}{12} + \frac{z^2 \Psi_{11}}{12} + \frac{1}{12} \frac{\partial^2 \Psi_{11}}{\Psi_{11}} + \frac{z(z + S)}{6} \frac{\partial \Psi_{11}}{\Psi_{11}},
\]

(4.7)

\[
\Phi(z) = -\frac{S - z}{3z} \left( \frac{H + W \Psi_{21}}{\Psi_{11}} \right) - \frac{H}{3z} + W \frac{\Psi_{21}}{\Psi_{11}} \frac{c - 1}{3z} + W \frac{\Psi_{21}}{\Psi_{11}} \frac{S - z}{3z} \left( \frac{\partial \Psi_{11}}{\Psi_{11}} - \frac{\partial_a \Psi_{21}}{\Psi_{21}} \right),
\]

(4.8)

\[
\delta(z) = -\left( \frac{z}{2} + \frac{\partial_a \Psi_{11}}{\Psi_{11}} \right) \frac{S - z}{3z} \left( H + W \frac{\Psi_{21}}{\Psi_{11}} \right),
\]

(4.9)

• *Asymptotics of the derivative* We have

\[
\frac{1}{N} \psi'_N \left( a + \frac{z}{\sqrt{N}} \right) = \left( a + \frac{z}{\sqrt{N}} \right)^{N+c-1} e^{\frac{z(z+2S)}{4}} z^s \Psi_{11}(z; S) \\
\times \left[ 1 + \left( \frac{z + S}{2} + \frac{c}{2z} \frac{1}{2} \frac{\partial \Psi_{11}}{\Psi_{11}} + \frac{F_N(z) + G_N(z)}{\frac{1}{\sqrt{N}}} + O \left( \frac{1}{N^{1+\frac{s}{2}}} \right) \right) \right],
\]

(4.10)

where

\[
F_N^{(1)}(z) = -\left( \frac{z^2 + 6Sz + 3S^2}{24} \right) - \frac{c}{2z} \frac{1}{2} \frac{\partial \Psi_{11}}{\Psi_{11}} + \left[ - \frac{z(z + S)}{6} \frac{\partial \Psi_{11}}{\Psi_{11}} - \frac{(S^2 + r)}{12} \frac{\partial_a \Psi_{11}}{\Psi_{11}} \right]
\]

and

\[
G_N^{(1)}(z) = \frac{S - z}{3z} \left( H + W \frac{\Psi_{21}}{\Psi_{11}} \right).
\]

(4.11)

Lemma 4.3 (Asymptotics of the terms involving orthogonal polynomials). Let \( z \in D \). Then as \( N \to \infty \), we have

\[
\frac{1}{N} \psi'_N \left( a + \frac{z}{\sqrt{N}} \right) \left[ \psi_N \left( a + \frac{z}{\sqrt{N}} \right) - \left( a + \frac{z}{\sqrt{N}} \right) \psi_{N-1} \left( a + \frac{z}{\sqrt{N}} \right) \right] = e^{(S+z+z)\sqrt{N}} \frac{z^2}{4} \frac{z^2}{4} \frac{\Psi_{11}(z; S)^2}{N^2} \\
\times \left[ - \left( \frac{z}{2} + \frac{\partial_a \Psi_{11}}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + \frac{1}{N} \right] + \frac{1}{N} + O(N^{\frac{3}{2} - \frac{c}{2}}) \left[ 1 + \frac{c(z)}{\sqrt{1 - N}} + O \left( \frac{1}{N} \right) \right]
\]

(4.12)
and
\[
\psi_{N-1}(a + \frac{z}{\sqrt{N}}) \psi_{N+1}(a + \frac{z}{\sqrt{N}}) - (a + \frac{z}{\sqrt{N}}) \psi_N(a + \frac{z}{\sqrt{N}})
\]
\[
= e^{(S+z+\bar{z})\sqrt{N}-\frac{z^2+\bar{z}^2}{2}} \frac{\partial_z}{\Psi_{11}} |\Psi_{11}(z; S)|^2
\]
\[
\times \left[ -\left( \frac{z}{2} + \frac{\partial_z}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + \frac{\Theta(z)}{N} + O(N^{-\frac{3}{2}-\frac{\bar{z}}{2}}) \right] [1 + C(z) \frac{1}{\sqrt{N}} + O(\frac{1}{N})],
\]
where
\[
\Theta(z) = -\left( \frac{z}{2} + \frac{\partial_z}{\Psi_{11}} \right) \left( \frac{z + S}{2} + \frac{c}{2} \frac{\partial_z}{\Psi_{11}} + F_N^{(1)}(z) + G_N^{(1)} \right)
\]
\[
+ \overline{\Theta}(z) + \Phi(z) + \overline{\Phi}(z) - \left( \frac{z}{2} \frac{\partial_z}{\Psi_{11}} + \frac{z^2}{8} + \frac{1}{2} \frac{\partial_z^2}{\Psi_{11}} \right),
\]
\[
\Theta(z) = -\left( \frac{z}{2} + \frac{\partial_z}{\Psi_{11}} \right) \left( \frac{2z + S}{2} + F_{N-1}^{(1)}(z) + G_{N-1}^{(1)}(z) \right)
\]
\[
+ \overline{\Theta}(z) + \Phi(z) + \overline{\Phi}(z) + \left( \frac{z}{2} \frac{\partial_z}{\Psi_{11}} + \frac{z^2}{8} + \frac{1}{2} \frac{\partial_z^2}{\Psi_{11}} \right),
\]
and
\[
C(z) = \frac{1}{24} \left( (S+2z)(2c+(S+2z)^2) + (S+2\bar{z})(2c-1) + (S+2\bar{z})^2 \right).
\]
Here \( \Theta, \Phi, \Phi \) are given by (4.7), (4.8), (4.9) and \( F_k^{(1)}, G_k^{(1)} \) are given by (4.10), (4.11).

Proof. This directly follows from Lemma 4.2. Note that the term (4.16) originates from
\[
\left( a + \frac{z}{\sqrt{N}} \right)^{N+c-1} \left( a + \frac{z}{\sqrt{N}} \right)^{N+c} = e^{(S+z+\bar{z})\sqrt{N}-\frac{1}{2}(S+2z)^2-\frac{1}{2}(S+2\bar{z})^2}
\]
\[
\times \left[ 1 + \frac{1}{24} \left( (S+2z)(2c+(S+2z)^2) + (S+2\bar{z})(2c-1) + (S+2\bar{z})^2 \right) \frac{1}{\sqrt{N}} + O(\frac{1}{N}) \right].
\]

\section*{Lemma 4.4 (Asymptotics of the terms involving orthogonal norms).}
As \( N \to \infty \), we have
\[
e^{-a^2/N+\sqrt{N}} \frac{1}{\sqrt{N}} \frac{N+c}{N} h_{N-1} - h_N = e^{-S^2/4} \mathcal{A} \left( a - a \mathcal{B}_0 \sqrt{N} + o(\sqrt{N}) \right)
\]
and
\[
e^{-a^2/N+\sqrt{N}} \frac{h_N}{\sqrt{N}} \frac{N+c+1}{N} h_{N-1} - h_{N+1} = e^{-S^2/4} \mathcal{A} \left( a - a \mathcal{B}_0 \sqrt{N} + o(\sqrt{N}) \right)
\]
where
\[
\mathcal{B}_0 = \frac{3}{2} \frac{W'(S)}{W(S)} - S \left( \frac{W'(S)}{W(S)} \right)^2 - \left( \frac{W'(S)}{W(S)} \right)^3 + S \frac{W''(S)}{W(S)}.
\]
Here \( \mathcal{A} \) and \( \mathcal{B} \) are given by (1.39) and (1.40).

\section*{Proposition 4.5.}
As \( N \to \infty \), we have
\[
\overline{\partial_z} \overline{R}_N^c(z) = \frac{\mathcal{A}}{\sqrt{2\pi}} e^{-|z|^2 - \frac{(z+S)^2+(\bar{z}+S)^2}{4}} |z|^c \psi_{21}(z; S) W
\]
\[
\times \left[ \overline{\psi}_{11}(z; S) \left( \frac{Z}{z} - z - a \mathcal{B} + \frac{Z+c}{Y} - \frac{W}{z} \frac{Z+c}{Y} \psi_{21}(z; S) + Z + o(1) \right) \right].
\]
Proof. Recall that $\Gamma_N^c$ and $\Pi_N^c$ are given by (3.43) and (3.44). Note that by (A.21), we have
\[
\frac{P_{N+1}(a)}{P_N(a)} = a \left( 1 - \left( \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + O\left( \frac{1}{N^{1+\varepsilon}} \right) \right) = 1 + \left( \frac{S}{2} - \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \frac{1}{\sqrt{N}} + O\left( \frac{1}{N^{1+\varepsilon}} \right).
\]
Combining this with Lemmas 4.3 and 4.4, after simplifications, we obtain
\[
\Gamma_N^c(z) = -\sqrt{N} e^{-\frac{(z+S)^2+(x+S)^2}{4}} |z|^c |\Psi_{11}(z;S)|^2 \left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \mathfrak{A} \times \left[ 1 + \left( -\mathfrak{A} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right)^{-1} \mathfrak{A} \mathfrak{A} + C(z) \frac{1}{\sqrt{N}} + O\left( \frac{1}{N} \right) \right]
\]
and
\[
\Pi_N^c(z) = -\sqrt{N} e^{-\frac{(z+S)^2+(x+S)^2}{4}} |z|^c |\Psi_{11}(z;S)|^2 \left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \mathfrak{A} \times \left[ 1 + \left( \mathfrak{A} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right)^{-1} \mathfrak{A} \mathfrak{A} + C(z) \frac{1}{\sqrt{N}} + O\left( \frac{1}{N} \right) \right].
\]
This gives
\[
\Gamma_N^c(z) - \Pi_N^c(z) = e^{-\frac{(z+S)^2+(x+S)^2}{4}} |z|^c |\Psi_{11}(z;S)|^2 \mathfrak{A} \times \left[ \mathfrak{A} \mathfrak{A} + C(z) \frac{1}{\sqrt{N}} + O\left( \frac{1}{N} \right) \right].
\]
(4.23)
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\frac{\partial_z \Psi_{11}(z;S)}{\Psi_{11}(z;S)} = -\frac{z+S}{2} - \frac{c+Z}{z} + W \left( 1 + \frac{c+Z}{Y} \right) \frac{\Psi_{21}(z;S)}{\Psi_{11}(z;S)}.
\]
(4.25)
Also we have
\[
\frac{\partial_z \Psi_{11}(z;S)}{\Psi_{11}(z;S)} = -\frac{z}{2} + W \frac{\Psi_{21}(z;S)}{\Psi_{11}(z;S)},
\]
(4.26)
\[
\frac{\partial_z^2 \Psi_{11}(z;S)}{\Psi_{11}(z;S)} = \frac{z^2}{4} - Z(s) + W'(s) \frac{\Psi_{21}(z;S)}{\Psi_{11}(z;S)}.
\]
(4.27)
Combining (4.25), (4.26) and (4.27), after some computations, we obtain
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} - \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
By the Lax pair given in [17, Eq.(2.1)], we have
\[
\mathfrak{A} \mathfrak{A} = -\left( \frac{z}{2} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) \left( -z + \frac{c}{z} + \frac{1}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{\partial_z \Psi_{11}}{\Psi_{11}} \right) - \left( \frac{z}{2} \frac{\partial_z \Psi_{11}}{\Psi_{11}} + \frac{z^2}{4} + \frac{\partial_z^2 \Psi_{11}}{\Psi_{11}} \right).
\]
Recall that $W$ is given by (1.38). Using (1.35) and (1.37), it is straightforward to check

\begin{equation}
\frac{W'}{W} = s - \frac{Z + c}{Y}, \quad \frac{W''}{W} = 1 - c + s^2 - 2Z - s\frac{Z + c}{Y},
\end{equation}

\begin{equation}
\frac{W'''}{W} = s(3 - 2c - 4Z + s^2) - 2YZ + \frac{c^2 - c(2 + s^2 - 5Z) + Z(4Z - s^2 - 2)}{Y}.
\end{equation}

Now it follows from (3.42), (4.30) and

\begin{equation}
\frac{\partial_x \Psi_{11}(0, s)|_{s=1}}{\Psi_{11}(0, s)} = W(S)\frac{\Psi_{21}(0, S)}{\Psi_{11}(0, S)} = Y\frac{Z}{Z + c}
\end{equation}

that

\begin{equation}
\tilde{R}_N(z) = \frac{2i}{\sqrt{2\pi}} e^{-\frac{(z+c)^2}{4}} |z|^c \Psi_{11}(z; S) \frac{\partial_x R_N}{\partial x} = 0 \quad \text{as } x \to +\infty, \quad \text{as } x \to -\infty, \quad (x \in \mathbb{R}).
\end{equation}

Using Proposition 4.5, the first limit gives

\begin{equation}
\tilde{R}_c(x + iy) \to \begin{cases} 0 & \text{as } x \to +\infty, \\ 1 & \text{as } x \to -\infty, \end{cases} \quad (x \in \mathbb{R}).
\end{equation}

**Appendix A. Fine asymptotic behaviours of the orthogonal polynomials**

In this section, we show Proposition 4.1 and Lemma 4.2.

**Proof of Proposition 4.1.** Let $z \in D \cap \Omega_{\infty}$. We also write $\Psi \equiv \Psi^c$ and

\begin{equation}
\tilde{\Psi}(z) \equiv \tilde{\Psi}(z; s) := \Psi(z; s) \begin{pmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{xL}, \quad \theta = \frac{z^2}{4} + \frac{s}{2},
\end{equation}

where $\chi_L$ is the characteristic function whose support is the left of the contour $\hat{\Gamma}_{r=1}$, see [17, Eqs.(2.6),(3.2), p.27]. We also denote

\begin{align*}
\tilde{P}(z) := & \begin{pmatrix} (\sqrt{k}\zeta)^\frac{z}{2} U_{\frac{z-1}{z}} \frac{U_{\frac{z+1}{z}}}{z-1} & 0 \\ (\sqrt{k}\zeta)^\frac{z}{2} \frac{U_{\frac{z+1}{z}}}{z-1} & (\sqrt{k}\zeta)^\frac{z}{2} \frac{U_{\frac{z-1}{z}}}{z-1} \end{pmatrix} \tilde{\Psi}(\sqrt{k}\zeta; \sqrt{k}A), \\
\tilde{P}(z)_{11} = & (\sqrt{k}\zeta)^\frac{z}{2} \left( \frac{z-1}{z} \right)^{-\frac{z}{2}} \tilde{\Psi}_{11}, \\
\tilde{P}(z)_{21} = & \left( U_{\frac{z-1}{z}} \frac{U_{\frac{z+1}{z}}}{z-1} \left( \frac{z-1}{z} \right)^{-\frac{z}{2}} \frac{U_{\frac{z+1}{z}}}{z-1} \right) \tilde{\Psi}_{11} + (\sqrt{k}\zeta)^{-\frac{z}{2}} \left( \frac{z-1}{z} \right)^{\frac{z}{2}} \tilde{\Psi}_{21}.
\end{align*}
Let $E$ be the error matrix given in [17, Subsection 4.3.6]. It satisfies the asymptotic expansion
\begin{equation}
E(z) = I + \frac{E^{(1)}}{k^{1/2}} + \frac{E^{(2)}}{k^{3/2}} + \frac{E^{(3)}}{k^{1+1/2}} + O\left(\frac{1}{k}\right),
\end{equation}
where
\begin{align*}
E^{(1)} &= \left( -\frac{H}{\zeta} + \frac{H}{z-1} \right) \sigma_3, \\
E^{(2)} &= \left[ -\left( \frac{z-1}{z\zeta} \right)^c \left( \frac{Z}{U\zeta} \right) + \frac{Z/U}{z-1} \right] \sigma_+,
\end{align*}
\begin{equation*}
E^{(3)} = \left[ \left( \frac{z-1}{z\zeta} \right)^c (s - Y) \frac{HU}{\zeta^2} - 2 \frac{HU}{(z-1)\zeta} + \frac{2c^2(2H - c - s) + H - c - s}{z - 1} + \frac{U(2H + Y - s)}{z - 1} \right] \sigma_-.
\end{equation*}
Here
\begin{equation}
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{equation}
see [17, Eqs.(4.18),(4.22)]. Therefore we have
\begin{equation}
E(z) = \begin{pmatrix} 1 + \left( -\frac{H}{\zeta} + \frac{H}{z-1} \right) \frac{1}{k^{1/2}} & E^{(2)} \frac{1}{k^{1/2}} \\ E^{(3)} \frac{1}{k^{1+1/2}} & 1 - \left( -\frac{H}{\zeta} + \frac{H}{z-1} \right) \frac{1}{k^{1/2}} \end{pmatrix} + O\left(\frac{1}{k}\right).
\end{equation}

The function $\pi_k$ in [17] is related to $P_k$ as
\begin{equation}
P_k(z) = a^k \pi_k(z/a).
\end{equation}
By the asymptotic result in [17, Subsection 4.4], we have
\begin{equation}
\pi_k(z) = z^k \left( 1 - \frac{1}{z} \right)^{-\frac{n}{2}} [E(z)\tilde{P}(z)]_{11}.
\end{equation}
Combining the above asymptotic behaviours of $E$ and $\tilde{P}$, after long but straightforward computations, we obtain
\begin{equation}
\pi_k(z) = \frac{z^k}{2} \left( \frac{z-1}{z} \right)^c k^{1/2} \zeta(z) \frac{1}{\sqrt{k}} \left[ \Psi_{11} + \left[ H \left( \frac{1}{z-1} - \frac{1}{\zeta(z)} \right) \Psi_{11} + W \left( \frac{1}{z-1} \left( \frac{z-1}{z} \right)^c \zeta(z) - \frac{1}{\zeta(z)} \right) \Psi_{21} \right] \frac{1}{k^{1/2}} + O\left(\frac{1}{k^{1+1/2}}\right) \right].
\end{equation}

Note that by (4.2), we have
\begin{equation}
\sqrt{k} \zeta(z) = \left( z - 1 - \frac{(z-1)^2}{3} \right) \sqrt{k} + O(1).
\end{equation}
Recall that
\begin{equation}
\phi(z; z_0) = \frac{z - 1}{z_0} - \log z, \quad \left( z_0 = \frac{t_c^2}{f^2} \right) \quad \hat{\phi}(\lambda) = \log(t_c - \lambda^d) - \frac{\lambda^d}{t_c} - \log t_c
\end{equation}
and
\begin{equation}
\phi(z; z_0) = \frac{1}{2} \zeta^2(z; z_0) + A(z_0) \zeta(z; z_0), \quad \zeta(1; z_0) \equiv 0, \quad A(1) = 0,
\end{equation}
see [17, Proposition 4.5].

By (4.2) and (A.1), we have
\begin{equation}
e^\theta = \exp \left( \frac{k}{4} \zeta^2 + \frac{\sqrt{k} A}{2} \sqrt{k} \zeta \right) = e^{\frac{k}{4} \phi(z; z_0)} \quad \Psi_{11}(\sqrt{k} \zeta; \sqrt{k} A) = e^{\frac{k}{4} \phi(z; z_0)} \Psi_{11}(\sqrt{k} \zeta; \sqrt{k} A).
\end{equation}
Therefore by (A.5) we obtain
\[
\psi_k(z) = z^{k+c} e^{k \phi(z/a; z_0)} \left( \sqrt{k} \zeta(z/a) \right)^{k \Psi_{11}} (\sqrt{k} \zeta(z/a; z_0); \sqrt{k} A(z_0))
\]
(A.12)
\[
times \left[ 1 + \left( H \left( \frac{a}{z-a} - \frac{1}{\zeta(z/a)} \right) + W \left( \frac{a}{z-a} \left( \frac{z-a}{z} \right)^{c} \frac{1}{\zeta(z/a)^{c}} - \frac{1}{\zeta(z/a)^{c}} \right) \Psi_{21} \right) \frac{1}{k^{\frac{r}{2}}} + O(\frac{1}{k^{1+\frac{r}{2}}} \right)
\]
A similar computation can be done for \( z \) in other regions of \( D \). Note that
\[
P_{n,N}(z; a) = \left( \frac{n}{N} \right)^{\frac{3}{2}} P_{n,n} \left( \sqrt{\frac{N}{n} z}, \sqrt{\frac{N}{n} a} \right)
\]
see e.g. [42, p.304]. Now Proposition 4.1 follows from (4.3).
\[
\square
\]
Lemma A.1. Let
\[
F_k(z) := e^{\frac{1}{2} \phi(z/a; z_0)} \left( \sqrt{k} \zeta \left( 1 + \frac{z/a}{\sqrt{N}} \right) \right)^{k \Psi_{11}} \left( \sqrt{k} \zeta \left( 1 + \frac{z/a}{\sqrt{N}}; z_0 \right); \sqrt{k} A(z_0) \right).
\]
Then we have
\[
F_k(z) = e^{\frac{(z/a) + 2 \Psi_{11}(z; S)}{4}} \left( 1 + \frac{F_k^{(1)}(z)}{\sqrt{N}} + \frac{F_k^{(2)}(z)}{N} + O(N^{-\frac{3}{2}}) \right),
\]
(A.15)
where \( F_k^{(1)} \) is given by (4.10) and
\[
F_k^{(2)}(z) = \frac{z(4z^2 + 6Sz + 3S^2)}{24} \left[ \frac{c}{2} \frac{2z + S}{6} \right] + \frac{1}{2} \left( \frac{z(4z^2 + 6Sz + 3S^2)}{24} + \frac{r}{2} \right)^2 + \frac{1}{2} \left( \frac{z(2z + S)}{6} + \frac{r}{2} \right)^2 + \frac{1}{2} \left( \frac{z(2z + S)}{6} + \frac{r}{2} \right)^2
\]
\[
+ \frac{c(c-2)}{8} \left( \frac{2z + S}{6} \right)^2 + \frac{c}{2} \left( \frac{7z^2 + 9Sz + 3S^2}{36} + \frac{r}{6} \right)
\]
\[
+ \left( \frac{7z^2 + 9Sz + 3S^2}{36} + \frac{r}{6} \right) \frac{\partial_2 \Psi_{11}}{\Psi_{11}} + \left( \frac{S^3}{36} + \frac{r}{6} \right) \frac{\partial_3 \Psi_{11}}{\Psi_{11}}
\]
\[
+ \left( \frac{z(2z + S)}{6} \right)^2 \frac{\partial_2^2 \Psi_{11}}{\Psi_{11}} + \left( \frac{2z + S}{12} + \frac{r}{2} \right)^2 \frac{\partial_3^2 \Psi_{11}}{\Psi_{11}} + \frac{z(2z + S)}{6} \left( \frac{S^2}{12} + \frac{r}{2} \right) \frac{\partial_2 \Psi_{11}}{\Psi_{11}}.
\]
Proof. Let \( k = N + r \). By (4.2), we have
\[
\sqrt{N + r} \zeta \left( 1 + \frac{z/a}{\sqrt{N}}; z_0 \right) = z \left[ 1 - \frac{2z + S}{6} + \frac{r}{6} \right] \frac{1}{N} + O(N^{-\frac{3}{2}})
\]
and
\[
\sqrt{N + r} A(z_0) = S + \frac{1}{\sqrt{N}} + \frac{1}{N} + O(N^{-\frac{3}{2}}).
\]
Note also that
\[
\frac{N + r}{2} \phi \left( 1 + \frac{z/a}{\sqrt{N}}; z_0 \right) = \frac{z(2z + S)}{4} + \frac{z(4z^2 + 6Sz + 3S^2 + r)}{24} \frac{1}{\sqrt{N}}
\]
\[
+ \left( \frac{7z^2 + 9Sz + 3S^2}{36} + \frac{r}{6} \right) \frac{1}{N} + O(N^{-\frac{3}{2}}).
\]
\[
(A.19)
\]
This gives
\[
\frac{1}{2} \phi(1 + \frac{z/a}{\sqrt{N}}) e^{-\frac{z(4z^2 + 6Sz + 3S^2)}{24}} = 1 - \left( \frac{z(4z^2 + 6Sz + 3S^2)}{24} + \frac{r}{2z} \right) \frac{1}{\sqrt{N}} + \frac{1}{2} \left( \frac{z(4z^2 + 6Sz + 3S^2)}{24} + \frac{r}{2z} \right)^2 + \left( \frac{z(z + S)(2z^2 + 2Sz + S^2)}{16} + r \frac{z(z + S)}{4} \right) \frac{1}{N} + O(N^{-\frac{3}{2}}).
\]
We have
\[
\left( \sqrt{\kappa} \zeta \left( 1 + \frac{z/a}{\sqrt{N}} \right) \right) \frac{\dot{z}}{z} = 1 - \frac{c}{2} \frac{2z + S}{6} \frac{1}{\sqrt{N}} + \left[ \frac{c(c - 2)}{8} \left( \frac{2z + S}{6} \right)^2 + \frac{c}{2} \left( \frac{7z^2 + 9Sz + 3S^2}{36} + \frac{r}{6} \right) \right] \frac{1}{N} + O(N^{-\frac{3}{2}}).
\]
Finally, we have
\[
\Psi_{11} \left( \sqrt{\kappa} \zeta \left( 1 + \frac{z/a}{\sqrt{N}} ; z_0 \right) ; \sqrt{\kappa} A(z_0) \right) = \Psi_{11}(z) + \frac{z(2z + S)}{6} \partial_z \Psi_{11} + \left( \frac{S^2}{12} + r \right) \partial_s \Psi_{11} \right) \frac{1}{N} + O(N^{-\frac{3}{2}}).
\]
Combining all of the above, lemma follows.

**Lemma A.2.** Let
\[
G_k(z) := 1 + \left[ H(S) \left( \frac{a}{z} \sqrt{N} - \frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})} \right) + \frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})} \right] \frac{1}{\sqrt{N}} + \frac{1}{N} + O(N^{-\frac{3}{2}}),
\]
Then we have
\[
G_k(z) = 1 + \frac{G_k^{(1)}(z)}{\sqrt{N}} + \frac{G_k^{(2)}(z)}{N} + O(N^{-\frac{3}{2}}),
\]
where $G_k^{(1)}$ is given by (4.11) and (A.20)
\[
G_k^{(2)}(z) = \frac{S - z}{3z} \left( H(S) + W(S) \frac{\Psi_{21}}{\Psi_{11}} \right) + \frac{H(S)}{36z} \left( \frac{3z^2 + 5Sz + 2S^2}{36z} - \frac{r}{3z} \right) \left[ \frac{z(2z + S)}{6} \left( \frac{\partial_z \Psi_{11}}{\Psi_{11}} - \frac{\partial_z \Psi_{21}}{\Psi_{21}} \right) + \left( \frac{S^2}{12} + r \right) \left( \frac{\partial_s \Psi_{11}}{\Psi_{11}} - \frac{\partial_s \Psi_{21}}{\Psi_{21}} \right) \right].
\]

**Proof.** Since
\[
\frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})} = \sqrt{N} + \frac{2z + S}{6z} + \left( \frac{3z^2 + 5Sz + 2S^2}{36z} + \frac{r}{3z} \right) \frac{1}{\sqrt{N}} + O(N^{-1}),
\]
we have
\[
\left(\frac{a}{z} \sqrt{N} - \frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})}\right) \frac{1}{k^{1/2}} = \frac{S - z}{3z} \frac{1}{\sqrt{N}} + \left(\frac{3z^2 + 5Sz + 2S^2}{36z} - \frac{r}{3z}\right) \frac{1}{\sqrt{N}} + O(N^{-\frac{3}{2}})
\]
and
\[
\frac{a}{z} \sqrt{N} \left(\frac{z}{a\sqrt{N} + z}\right)^{c} \frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})^c} - \frac{1}{\zeta(1 + \frac{z/a}{\sqrt{N}})} = \frac{S - z}{3z} + \left[ \frac{c}{z} \left(\frac{21z^2 + 7Sz - 2S^2}{36} + \frac{r}{3}\right) + \frac{c(c - 1)}{2z} \left(\frac{2z + S}{3}\right)^2 + \frac{3z^2 + 5Sz + 2S^2}{36z} - \frac{r}{3z}\right] \frac{1}{\sqrt{N}} + O(N^{-1}).
\]

We also have
\[
\frac{\Psi_{21}}{\Psi_{11}} \left(\frac{\sqrt{k}\zeta\left(1 + \frac{z/a}{\sqrt{N}}; z_0\right)}{\sqrt{k}A(z_0)}\right) = \frac{\Psi_{21}(z; S)}{\Psi_{11}(z; S)} + \frac{\Psi_{21}(z; S)}{\Psi_{11}(z; S)} \left[ \frac{\sqrt{2} + y}{6} \left(\frac{\partial_{\Psi_{11}}}{\Psi_{11}} - \frac{\partial_{\Psi_{21}}}{\Psi_{21}}\right) + \left(\frac{S^2}{12} - \frac{r}{2}\right) \left(\frac{\partial_{\Psi_{11}}}{\Psi_{11}} - \frac{\partial_{\Psi_{21}}}{\Psi_{21}}\right)\right] \frac{1}{\sqrt{N}} + O(N^{-1}).
\]

Combining above asymptotic behaviours, we conclude the lemma.

**Proof of Lemma 4.2.** Combining Proposition 4.1 with Lemmas A.1 and A.2, we have
\[
\psi_k \left(a + \frac{z}{\sqrt{N}}\right) = \left(a + \frac{z}{\sqrt{N}}\right)^{k+c} e^{\frac{z(z+2S)}{4}} \frac{1}{\sqrt{N}} \psi_{11}(z; S)
\]
(A.21)

\[
\times \left(1 + \frac{F_k^{(1)}(z) + G_k^{(1)}(z)}{\sqrt{N}} + \frac{F_k^{(1)}(z)G_k^{(1)}(z) + F_k^{(2)}(z)G_k^{(2)}(z) + G_k^{(2)}(z) + O(\frac{1}{N^{1+\frac{1}{2}}} + \frac{1}{N})}{N}\right),
\]

where the $O(\frac{1}{N^{1+\frac{1}{2}}} + \frac{1}{N})$ term does not depend on $r$. Here, $F_k^{(j)}$ and $G_k^{(j)}$ ($j = 1, 2$) are defined by (A.16) and (A.20). Then lemma follows from straightforward computations. □

**APPENDIX B. FINE ASYMPTOTIC BEHAVIOURS OF THE ORTHOGONAL NORMS**

By [13, Proposition 7.1], we have
\[
h_k = -\frac{1}{2\pi i} \frac{\Gamma(c + k + 1)}{N^{c+k+1}} \tilde{h}_k, \quad \tilde{h}_k := \int_{\Gamma} P_k(z)^2 \tilde{w}_k(z) \, dz, \quad \tilde{w}_k(z) := \left(\frac{z - a}{z}\right)^c e^{-Naz} z^{-k}.
\]

**Lemma B.1.** As $N \to \infty$, we have
\[
P_{N+r,N}(0) = a^{N+r} e^{-Na^2} \left(\frac{1}{N^{-r}} W(S_r) + O(\frac{1}{N})\right).
\]

**Proof of Lemma B.1.** For $z \in \Omega_0 \setminus \mathbb{D}$,
\[
\pi_k(z) = e^{kg(z)} \left(\frac{z - 1}{z}\right)^{-\frac{k}{2}} [E(z)\tilde{N}(z)]_{11}, \quad \tilde{N}(z) = \begin{pmatrix}
0 & (\frac{z-1}{z})^{-\frac{k}{2}} \\
-(\frac{z-1}{z})^{-\frac{k}{2}} & \frac{UK^{-\frac{k}{2}}}{1-\frac{1}{z}} (\frac{z-1}{z})^{-\frac{k}{2}}
\end{pmatrix},
\]
see [17, p.30]. By (A.4), we have
\[
[E(z)\tilde{N}(z)]_{11} = -\left(W \frac{1}{z - 1} \frac{1}{k^{1/2}} + O(\frac{1}{k})\right) \left(\frac{z - 1}{z}\right)^{\frac{k}{2}} = -\left(\frac{z - 1}{z}\right)^{\frac{k}{2}} W \frac{1}{z - 1} \frac{1}{k^{1/2}} + O(\frac{1}{k}).
\]

Therefore we have
\[
\pi_N(z) = -e^{kg(z)} \left(W(S) \frac{1}{z - 1} \frac{1}{N^{-r}} + O(\frac{1}{k})\right).
\]
In particular,
(B.6) \[ \pi_k(0) = e^{-k/\omega} \left( W(S)_{k/\omega} + O\left( \frac{1}{k} \right) \right) = e^{-Na^2} \left( W(S)_{k/\omega} + O\left( \frac{1}{k} \right) \right), \]

Using (A.13), this gives
(B.7) \[ P_{N+r,N}(0) = \left( \frac{N + r}{N} \right)^{\frac{N+r}{2}} P_{N+r,N+r}(0, \sqrt{\frac{N}{N+r}} a) = a^{N+r} e^{-Na^2} \left( W(S_r)_{N^{1/2}} + O\left( \frac{1}{N} \right) \right). \]

**Lemma B.2.** As \( N \to \infty \), we have
(B.8) \[ \tilde{h}_{N+r,N} = -2\pi i e^{-Na^2} a^{N+r+1} \left( \frac{1}{N^{1/2}} W(S_r) + O\left( \frac{1}{N} \right) \right). \]

**Proof of Lemma B.2.** Let \( \tilde{Y} \) be the matrix \( Y \) in [13]. Then we have
(B.9) \[ \tilde{h}_k = -2\pi i \lim_{z \to \infty} z^{k+1} [\tilde{Y}(z)]_{12}, \]

see [13, Eq.(7.2)]. Let
(B.10) \[ \omega_k(z) := \left( \frac{z-1}{z} \right)^c e^{-Na^2} = a^k \tilde{w}_k(az), \]

see [17, p.18]. Therefore by the change of variables, we have
(B.11) \[ [Y(z)]_{12} := \frac{1}{2\pi i} \int \frac{\pi_k(z')}{z' - z} \omega_k(z') dz' = \frac{1}{2\pi i} \int \frac{P_k(z')}{z' - az} \tilde{w}_k(az) dz' = [\tilde{Y}(az)]_{12}, \]

where \( Y \) is a matrix given in [17, p.19].

Note that by the transforms in [17], we have
(B.12) \[ Y(z) = e^{\frac{i}{2} \sigma_3} E(z) \left( \frac{z-1}{z} \right)^{-\frac{3}{2}} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) e^{-\frac{i}{2} \sigma_3} e^{kg(z)} \sigma_3 \left( \frac{z-1}{z} \right)^{-\frac{3}{2}}. \]

Thus by (A.4) we have
(B.13) \[ [Y(z)]_{12} = \left( \frac{1}{k^{1/2}} \frac{W}{z-1} + O(k^{-1}) \right) \left( \frac{z-1}{z} \right)^c e^{kl-kg(z)}, \]

which gives
(B.14) \[ [\tilde{Y}(z)]_{12} = [Y(z/a)]_{12} = \left( \frac{1}{k^{1/2}} \frac{W}{z/a - 1} + O(k^{-1}) \right) \left( \frac{z-a}{z} \right)^c e^{kl-kg(z/a)}. \]

Therefore we obtain
(B.15) \[ \tilde{h}_k = -2\pi i \lim_{z \to \infty} z^{k+1} [\tilde{Y}(z)]_{12} = -2\pi i \lim_{z \to \infty} z^{k+1} \left( \frac{1}{k^{1/2}} \frac{W}{z/a - 1} + O(k^{-1}) \right) \left( \frac{z-a}{z} \right)^c e^{kl} \left( \frac{a}{z} \right)^k \]

\[ = -2\pi i \lim_{z \to \infty} \left( \frac{1}{k^{1/2}} \frac{W}{1/a - 1/\omega} + O(k^{-1}) \right) \left( \frac{z-a}{z} \right)^c e^{kl a^k} = -2\pi i \left( \frac{W}{k^{1/2}} + O(k^{-1}) \right) e^{kl a^k+1}. \]

We have shown that
(B.16) \[ \tilde{h}_{N,N} = -2\pi i e^{-Na^2} a^{N+1} \left( \frac{1}{N^{1/2}} W(S) + O(k^{-1}) \right) . \]

Note also that by (A.13),
(B.17) \[ \tilde{h}_{n,n} = \left( \frac{n}{N} \right)^{\frac{n+1}{2}} \tilde{h}_{n,n} \left( \sqrt{\frac{N}{a}} \right). \]
Thus
\[
\bar{h}_{N+r,N} = \left(\frac{N + r}{N}\right)^{\frac{N+r+1}{2}} h_{N+r,N} \left(\sqrt{\frac{N}{N+r}}a\right) = -2\pi i e^{-Na^2} a^{N+r+1} \left(\frac{1}{Nz^{-2r}} W(S_r) + O(N^{-1})\right).
\]

(B.17)

Proof of Lemma 4.4. By combining (B.1) with Lemmas B.1 and B.2, we have
\[
h_{N+r} = \frac{\Gamma(c + N + r + 1)}{N^{c+N+r+1}} W(S_r) + O(N^{-1/2}) W(S_{r+1}) + O(N^{-1/2}), \quad h_N/h_{N-1} = 1 + O(1/N).
\]

We also have
\[
\frac{N + c}{N} h_{N-1} - h_N = \frac{\Gamma(N + c)}{N^{N+c}} \left( W(S_{r-1}) + O(N^{-1/2}) - W(S_{r}) + O(N^{-1/2}) - W(S_{r+1}) + O(N^{-1/2}) \right),
\]

(B.19)

\[
\frac{N + c}{N} h_{N} - h_{N+1} = \frac{\Gamma(N + c + 1)}{N^{N+c+1}} \left( W(S_{r}) + O(N^{-1/2}) - W(S_{r+1}) + O(N^{-1/2}) - W(S_{r+2}) + O(N^{-1/2}) \right).
\]

(B.20)

Note here that
\[
\frac{W(S_{r-1}) - W(S_{r})}{W(S_{r+1})} = \frac{W''(S)}{W(S)} - \left(\frac{W'(S)}{W(S)}\right)^2 \cdot \frac{1}{N} + \frac{3 W''}{2 W} - S \left(\frac{W'}{W}\right)^3 + S W'' \frac{W'}{W} + r \left(3 \frac{W''W'}{W^2} - 2 \left(\frac{W'}{W}\right)^3 - \frac{W''}{W}\right) \cdot \frac{1}{N^{1/2}} + O(N^{-2}).
\]

(B.21)

Then lemma follows from the Stirling’s formula. □

Appendix C. Asymptotic behaviours of the 1-point function

Let us recall that
\[
\bar{R}^0(z) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 + \bar{z}^2)},
\]

(C.1)

\[
\bar{R}^1(z) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 + \bar{z}^2)} + \frac{1}{\sqrt{2\pi}} \text{erfc}(\frac{z}{\sqrt{2}}) e^{-|z|^2} \left( e^{-\frac{z^2}{2}} + \sqrt{\frac{\pi}{2}} z \text{erfc}(\frac{\bar{z}}{\sqrt{2}}) \right).
\]

(C.2)

Note that as \( z \to \infty \),
\[
\Psi_{11}(z; s) = z^{-\frac{c}{2}} e^{-\frac{z^2}{2} - \frac{\bar{z}^2}{2}} \left(1 + O\left(\frac{1}{z}\right)\right), \quad \Psi_{21}(z; s) = z^{-\frac{c}{2}} e^{-\frac{z^2}{2} - \frac{\bar{z}^2}{2}} O\left(\frac{1}{z}\right).
\]

(C.3)

see [17, p.9]. Using these, we obtain that as \( z \to \infty \),
\[
e^{-|z|^2} |z|^c |\Psi_{11}(z; S)|^2 = e^{-\frac{1}{2}(z^2 + \bar{z}^2)} \left(1 + O\left(\frac{1}{z}\right)\right)
\]

and
\[
W_{\Psi_{21}(z; s)} (S + \frac{Z}{\bar{z}} - z - \frac{W'}{W} - \frac{\partial_z \Psi_{11}(0, S)}{\Psi_{11}(0, S)} - \frac{W^2 c + Z}{\bar{z}Y} |\Psi_{21}(z; s)|^2 + Z = O(1).
\]

Therefore by Proposition 4.5 we obtain
\[
\bar{R}^c(z) = O(1) e^{-\frac{1}{2}(z^2 + \bar{z}^2)} \quad z \to \infty.
\]

We mention that this asymptotic behaviour can be directly checked for \( c = 0, 1 \).
We now consider the asymptotic behaviour near the origin. Using the Lax pair given in [17, Eq.(2.1)], we have
\[
\frac{d}{dz} \Psi = (A_0 \frac{1}{z} + \ldots) \Psi, \quad A_0 = \begin{pmatrix} Z + c & Z \\ UY & UY \end{pmatrix} \begin{pmatrix} -\frac{c}{2} & 0 \\ 0 & \frac{c}{2} \end{pmatrix} \begin{pmatrix} Z + c & Z \\ UY & UY \end{pmatrix}^{-1}.
\]
Then it follows from
\[
\Psi = \begin{pmatrix} Z + c & Z \\ UY & UY \end{pmatrix} \left( I + O(z) \right) \begin{pmatrix} z^{-\frac{c}{2}} & 0 \\ 0 & z^{\frac{c}{2}} \end{pmatrix}. \text{(Regular), \quad (z \to 0)}
\]
that
\[
\Psi_{11}(z; S) = \frac{Z + c}{z^{c/2}}(1 + O(z)), \quad \Psi_{21}(z; S) = \frac{UY}{z^{c/2}}(1 + O(z)), \quad (z \to 0).
\]
Therefore
\[
\Psi_{21}(0; S) = \frac{UY}{Z + c} = \frac{Y}{W} \frac{Z}{Z + c}
\]
By [17, Eq.(2.1)], we have
\[
\Psi'_{11} = -\left( \frac{z + s}{2} + \frac{Z + c}{z} \right) \Psi_{11} + \left( \frac{Z}{U} + \frac{1}{z} \frac{Z(Z + c)}{UY} \right) \Psi_{21},
\]
\[
\Psi'_{21} = -(U + \frac{UY}{z}) \Psi_{11} + \left( \frac{z + s}{2} + \frac{Z + c}{z} \right) \Psi_{21}.
\]
This gives
\[
\frac{\Psi_{21} \Psi_{11} - \Psi_{11} \Psi_{21}}{\Psi_{11}^2} = \left( \frac{Z}{U} + \frac{1}{z} \frac{Z(Z + c)}{UY} \right) \frac{\Psi_{21}^2}{\Psi_{11}} - 2 \left( \frac{z + s}{2} + \frac{Z + c}{z} \right) \frac{\Psi_{21}}{\Psi_{11}} + \left( U + \frac{UY}{z} \right).
\]
Combining the above, after simplifications, we obtain that as \( z \to 0, \)
\[
\frac{\Psi_{21}(z; S)}{\Psi_{11}(z; S)} = \frac{\Psi_{21}(0; S)}{\Psi_{11}(0; S)} + \frac{\Psi_{21}(0; S) \Psi'_{11}(0; S) - \Psi_{11}(0; S) \Psi'_{21}(0; S)}{\Psi_{11}(0; S)^2} z + O(z^2)
\]
\[
\frac{\Psi_{21}(z; S)}{\Psi_{11}(z; S)} = \frac{UY}{Z + c} + z \left[ \frac{Z}{U} \left( \frac{UY}{Z + c} \right)^2 - s \frac{UY}{Z + c} + U \right] + O(z^2).
\]
On the other hand, by (4.32), we have
\[
W(\frac{\Psi_{21}(z; S)}{\Psi_{11}(z; S)}) \left( \frac{Z}{\bar{z}} - \frac{W C + Z \Psi_{21}(z; S)}{\bar{Y} \Psi_{11}(z; S)} \right) = \Psi_{21}(z; S) - \frac{Z + c}{Y} \bar{\zeta} \Psi_{11}(z; S) - \frac{\partial_s \Psi_{11}(0; S)}{\Psi_{11}(0; S)} + Z
\]
\[
= \frac{1}{\bar{z}} W(\frac{Z \Psi_{21}(z; S)}{\Psi_{11}(z; S)} - \frac{Z + c}{Y} \left| \Psi_{21}(z; S) \right|^2) + \frac{W \Psi_{21}(z; S)}{\Psi_{11}(z; S)} \left( - z + \frac{Z + c}{Y} \bar{\zeta} \Psi_{11}(z; S) - \frac{\partial_s \Psi_{11}(0; S)}{\Psi_{11}(0; S)} - \frac{Z + c}{Y} \right) + Z
\]
Note that
\[
W(\frac{Z \Psi_{21}(0; S)}{\Psi_{11}(0; S)} - \frac{Z + c}{Y} \left| \Psi_{21}(0; S) \right|^2) = W \frac{Y}{W Z + c} - W^2 \frac{Z + c}{Y} \left( \frac{Y}{W Z + c} \right)^2 = 0.
\]
Thus we have
\[
\frac{Z \Psi_{21}(z; S)}{\Psi_{11}(z; S)} - \frac{Z + c}{Y} \left| \Psi_{21}(z; S) \right|^2 = - \frac{Z}{U} \left( \frac{UY}{Z + c} \right)^2 - s \frac{UY}{Z + c} + U \bar{\zeta} + O(z^2).
\]
Therefore we obtain
\[
\frac{1}{\bar{z}} W(\frac{Z \Psi_{21}(z; S)}{\Psi_{11}(z; S)} - \frac{Z + c}{Y} \left| \Psi_{21}(z; S) \right|^2) = -ZW \left( \frac{UY}{Z + c} \right)^2 - s \frac{UY}{Z + c} + U + O(z).
\]
On the other hand,
\[ W \frac{\Psi_{21}(0; S)}{\Psi_{11}(0; S)} \left( \frac{Z + c}{Y} - \mathfrak{A}B - Y \frac{Z}{Z + c} \right) + Z = 2Z - Y \frac{Z}{Z + c} \left( \mathfrak{A}B + Y \frac{Z}{Z + c} \right). \]

By Proposition 4.5, this gives that as \( z \to 0 \),
\[ (D.15) \quad \partial_x \tilde{R}^c(z) = \frac{\mathfrak{A}D}{\sqrt{2\pi}} e^{-\frac{1}{2}(z + S)^2} + o(1). \]

where
\[ (D.16) \quad \mathcal{D} = -ZW \left( \frac{UY}{Z + c} \right)^2 - S \frac{UY}{Z + c} + U \right) + 2Z - Y \frac{Z}{Z + c} \left( \mathfrak{A}B + Y \frac{Z}{Z + c} \right). \]

**APPENDIX D. SCALING LIMITS FOR THE BULK CASE**

Here, we consider the case that \( a \in (0, 1) \). Let us first briefly recall the strong asymptotics of \( P_k \) from [42]. Let
\[ (D.1) \quad \phi_A(\zeta) := a(\zeta - a) - \log \frac{\zeta}{a} = \frac{a}{\sqrt{N}} z - \log \left( 1 + \frac{z}{a} \right). \]

Note that as \( N \to \infty \),
\[ (D.2) \quad \phi_A(\zeta) = \frac{a^2 - 1}{a} \frac{z}{\sqrt{N}} + \frac{z^2}{2a^2} \frac{1}{N} + O(N^{-\frac{3}{2}}). \]

Let us also write
\[ (D.3) \quad \hat{\phi}_A(\zeta) := \frac{N}{N - 1} a(\zeta - a) - \log \frac{\zeta}{a} = \phi_A(\zeta) + \frac{1}{N - 1} a(\zeta - a). \]

By [42, Theorem 3], for \( \zeta \) in a neighbourhood of \( a \), we have
\[ (D.4) \quad P_N(\zeta) = \zeta^N \left( \frac{\zeta}{\zeta - a} \right)^c \left[ 1 - (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \hat{f}(-N\phi_A(\zeta)) + O(\frac{1}{N}) \right) + O(\frac{1}{N^\infty}) \right] \]

and
\[ (D.5) \quad P_{N-1}(\zeta) = \zeta^{N-1} \left( \frac{\zeta}{\zeta - a} \right)^c \left[ 1 - ((N-1)\hat{\phi}_A(\zeta))^c e^{(N-1)\hat{\phi}_A(\zeta)} \left( \hat{f}(-(N-1)\hat{\phi}_A(\zeta)) + O(\frac{1}{N}) \right) + O(\frac{1}{N^\infty}) \right], \]

where the error bound \( O(\frac{1}{N^\infty}) \) means that \( O(\frac{1}{N^{k+1}}) \) for all \( k > 0 \). Here
\[ (D.6) \quad \hat{f}(z) := -\frac{1}{2\pi i} \int_L e^{s^c(s - z)} ds, \]

where the integration contour \( L \) begins at \( -\infty \), encircles the origin once in the counter-clockwise direction and returns to \( -\infty \). Note that as \( z \to \infty \),
\[ (D.7) \quad \hat{f}(z) = \frac{1}{2\pi i} \int_L e^{s^c} ds \cdot \frac{1}{z} + O(|z|^{-2}) = \frac{1}{\Gamma(c)} \frac{1}{z} + O(|z|^{-2}). \]

For general \( c > -1 \), we present an alternative derivation of (1.46) and (2.1) by virtue of the Christoffel-Darboux identity. Recall that \( P \) denotes the regularised incomplete gamma function.

**Theorem D.1.** (Large-\( N \) limit for the bulk case) For each \( c > -1 \), we have
\[ (D.8) \quad \tilde{R}^c_{\text{bulk}}(z) = P(c, |z|^2). \]
Proof. By (D.4), we have
\[\psi_N(\zeta) = \zeta^{N+c} \left[ 1 - (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \hat{f}(-N\phi_A(\zeta)) + O\left(\frac{1}{N}\right) \right) + O\left(\frac{1}{N^2}\right) \right].\]
Differentiating (D.9), we have
\[
\psi_N'(\zeta) = (N + c)\zeta^{N+c-1} \left[ 1 + O\left(\frac{1}{N}\right) \right] - (N + c)\zeta^{N+c-1} (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \hat{f}(-N\phi_A(\zeta)) + O\left(\frac{1}{N}\right) \right)
+ N\zeta^{N+c-1} (N + c) (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \frac{e^{-c}}{\phi_A(\zeta)} - 1 \right) \left( \hat{f}(-N\phi_A(\zeta)) + O\left(\frac{1}{N}\right) \right) + \hat{f}'(-N\phi_A(\zeta)),
\]
where we have used \(\phi_A'(\zeta) = (a\zeta - 1)/\zeta\). Rearranging the terms using (D.2) and (D.7), we have
\[
\psi_N'(\zeta) = (N + c)\zeta^{N+c-1} \left[ 1 + O\left(\frac{1}{N}\right) \right] - Na\zeta^{N+c} (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \hat{f}(-N\phi_A(\zeta)) + O\left(\frac{1}{N}\right) \right).
\]
Then it follows from
\[\zeta^{N+c} = a^{N+c} e^{\frac{1}{2} \sqrt{N} \frac{a^2}{2N}} \cdot (1 + o(1))\]
and (D.2) that
\[
\psi_N'(\zeta) = a^{N+c-1} e^{\frac{1}{2} \sqrt{N} \frac{a^2}{2N}} \cdot (1 + o(1)) - a^2 \left( \frac{1-a^2}{a} \sqrt{N} \right)^c \cdot (1 + o(1)) \cdot (1 + o(1)).
\]
Now let us compute the asymptotic of \(\psi_N(\zeta) - \zeta \psi_{N-1}(\zeta)\). By (D.9) and (D.5),
\[
\psi_N(\zeta) - \zeta \psi_{N-1}(\zeta) = \zeta^{N+c} (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \left( \hat{f}(-N\phi_A(\zeta)) + O\left(\frac{1}{N}\right) \right)
- \zeta^{N+c} (-N-1 \hat{\phi}_A(\zeta))^c e^{(N-1)\hat{\phi}_A(\zeta)} \left( \hat{f}(-(N-1)\hat{\phi}_A(\zeta)) + O\left(\frac{1}{N}\right) \right).
\]
Note that by (D.3),
\[
(N-1) \hat{\phi}_A(\zeta) - N \phi_A(\zeta) = a(\zeta - a) - \phi_A(\zeta) = \frac{z}{a} \frac{1}{\sqrt{N}} + O\left(\frac{1}{N}\right).
\]
Using this, we have
\[
\left( \frac{N - 1}{N} \hat{\phi}_A(\zeta) \right)^c e^{(N-1)\hat{\phi}_A(\zeta)} \frac{\hat{f}(-(N-1)\hat{\phi}_A(\zeta))}{\hat{f}(-N\phi_A(\zeta))} = 1 + \frac{z}{a} \frac{1}{\sqrt{N}} + O\left(\frac{1}{N}\right).
\]
This gives that
\[
\psi_N(\zeta) - \zeta \psi_{N-1}(\zeta) = \zeta^{N+c} (-N\phi_A(\zeta))^c e^{N\phi_A(\zeta)} \hat{f}(-N\phi_A(\zeta)) \frac{1}{a} \frac{1}{\sqrt{N}} \cdot (z + O\left(\frac{1}{\sqrt{N}}\right)).
\]
Then it again follows from (D.10) and (D.2) that
\[\psi_N(\zeta) - \zeta \psi_{N-1}(\zeta) = a^{N+c-1} e^{\frac{1}{2} \sqrt{N} \frac{a^2}{2N}} \cdot (1 + o(1)).\]

\[
\frac{P_{N+1}(a)}{P_N(a)} = a + o(1).
\]
Using the above asymptotic behaviours, we obtain
\[
I_N(z) = \frac{e^{-Na^2} a^{2N+2c-2} \left( \frac{1-a^2}{a} \sqrt{N} \right)^c}{N} \cdot \frac{N+1}{h_{N-1} - h_N}
\times \left( e^{\frac{z}{a} - a} \sqrt{N} \frac{a^2}{2N} - \hat{f}(-N\phi_A(\zeta)) a^2 \left( \frac{1-a^2}{a} \sqrt{N} \right)^c \right) \cdot (z^c + o(1)).
\]
and
\[
\Pi_N^c(z) = \frac{e^{-Na^2} h_N/h_{N-1}}{N} a^{2N+2c} \left( \frac{1-a^2}{a} \sqrt{N} \right)^{c-1} \times \left( e^{\left( \frac{1-a^2}{a} \sqrt{N} z - \frac{z^2}{2a^2} \right)} - \hat{f}(-N\phi_A(\zeta))( \frac{1-a^2}{a} \sqrt{N} \bar{z} ) \right) \cdot (z^c + o(1)).
\]
\[(D.14)\]

Combining above equations with
\[
\hat{f}(-N\phi_A(\zeta)) = \frac{1}{\Gamma(c)} \left( \frac{1}{a^2 \sqrt{N} z} + o(1) \right),
\]
\[(D.15)\]
we obtain that for Re $z < 0$,
\[
\partial_z \tilde{R}_N(z) = C_N(a) \left( \frac{\bar{z}^2 - 2e^{-|z|^2}}{\Gamma(c)} + o(1) \right),
\]
where
\[
C_N(a) := \frac{e^{-Na^2} h_N/h_{N-1}}{N} a^{2N+2c} \left( \frac{1-a^2}{a} \sqrt{N} \right)^{2c-2} \left( \frac{h_N/h_{N-1}}{h_N - h_{N+1}} - \frac{1}{N^c h_{N-1} - h_N} \right).
\]
We remark here that the case Re $z > 0$ follows from the case Re $z < 0$ since in the end, the limiting point process has the rotation invariance, see [12, Section 5].

Since $\tilde{R}_N$ has a non-trivial limit $\tilde{R}$, the existence of the limit
\[
C(a) := \lim_{N \to \infty} C_N(a), \quad (C(a) \neq 0)
\]
follows. Therefore we obtain that
\[
\tilde{R}_N^{c_{\text{bulk}}}(z) = C(a) P(c, |z|^2) + \tilde{C}(a),
\]
where $\tilde{C}(a) \in \mathbb{R}$ is some constant.

We now recall from the general theory on determinantal point process that
\[
\tilde{R}_N^{c_{\text{bulk}}}(z) \to 1, \quad (z \to \infty),
\]
\[
\tilde{R}_N^{c_{\text{bulk}}}(z) = O(|z|^{2c}), \quad (z \to 0),
\]
see [12, Theorem 1.4]. This behaviour implies that $C(a) = 1$ and $\tilde{C}(a) = 0$, which completes the proof.

\[\square\]

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