AN ERROR ESTIMATE FOR THE FINITE DIFFERENCE APPROXIMATION TO DEGENERATE CONVECTION - DIFFUSION EQUATIONS

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Abstract. We consider semi-discrete first-order finite difference schemes for a nonlinear degenerate convection-diffusion equations in one space dimension, and prove an $L^1$ error estimate. Precisely, we show that the $L^1$ difference between the approximate solution and the unique entropy solution converges at a rate $O(\Delta x^{1/11})$, where $\Delta x$ is the spatial mesh size. If the diffusion is linear, we get the convergence rate $O(\Delta x^{1/2})$, the point being that the $O$ is independent of the size of the diffusion.

1. Introduction

In this paper, we consider semi-discrete finite difference schemes for the following Cauchy problem

$$
\begin{align*}
&u_t + f(u)_x = A(u)_{xx}, \quad (x, t) \in \Pi_T, \\
&u(0, x) = u_0(x), \quad x \in \mathbb{R},
\end{align*}
$$

where $\Pi_T = \mathbb{R} \times (0, T)$ with $T > 0$ fixed, $u : \Pi_T \to \mathbb{R}$ is the unknown function, $f$ the flux function, and $A$ the nonlinear diffusion. Regarding this, the basic assumption is that $A' \geq 0$, and thus (1.1) is a strongly degenerate parabolic problem. The scalar conservation law $u_t + f(u)_x = 0$ is a special example of this type of problems. Other examples occur in several applications, for instance in porous media flow [9] and in sedimentation processes [3].

Since $A'$ may be zero, solutions are not necessarily smooth and one must consider weak solutions. These are not necessarily uniquely determined by the initial data, and in order to get uniqueness, one considers so-called entropy solutions. The framework of entropy solutions makes the initial value problem (1.1) well posed, for a precise statement, see Section 2. For scalar conservation laws, the entropy framework (usually called entropy conditions) was introduced by Kružkov [24] and Vol’pert [30], while for degenerate parabolic equations entropy solution were first considered by Vol’pert and Hudajev [31]. Uniqueness of entropy solutions to (1.1) was first proved by Carrillo [4], see also Karlsen and Risebro [20].

For hyperbolic equations, the convergence analysis of difference schemes has a long tradition, we mention only a few references. Finite difference schemes have been studied by Oleinik [28], Harten et al. [17], Kuznetsov [26], Crandall and Majda [8], Osher and Tadmor [29], Cockburn and Gripenberg [6], Kröner and Rokyta [23], Eymard et al. [14], Noelle [32] as well as many others.
In the last decade, there has been a growing interest in numerical approximation of entropy solutions to degenerate parabolic equations. Finite difference and finite volume schemes for degenerate equations were analysed by Evje and Karlsen \[10, 11, 12, 13\] (using upwind difference schemes), Holden et al. \[18, 19\] (using operator splitting methods), Kurganov and Tadmor \[25\] (central difference schemes), Bouchut et al. \[2\] (kinetic BGK schemes), Afif and Amaziane \[1\] and Ohlberger, Gallouët et al. \[27, 16, 15\] (finite volume methods), Cockburn and Shu \[7\] (discontinuous Galerkin methods) and Karlsen and Risebro \[22, 21\] (monotone difference schemes). Many of the above papers show that the approximate solutions converge to the unique entropy solution as the discretization parameter vanishes.

Despite this relatively large body of research, to the best of our knowledge, there does not exist a result giving the convergence rate of the approximate solutions to degenerate problems. For conservation laws (very degenerate problems), the convergence rate for monotone methods has long been known to be $\Delta x^{1/2}$ \[26\], and this is also optimal for discontinuous solutions. For non-degenerate problems, the solution operator (taking initial data to the corresponding solution) has a strong smoothing effect, and truncation analysis applies. Hence difference methods produces approximations converging at the formal order of the scheme. However, all estimates depend on $A'$, and are not available if $A'$ is not bounded below by some positive number $\eta$.

Often, the viscous regularization
\[ u_\eta^t + f(u_\eta)_x = A(u_\eta)_{xx} + \eta u_\eta''_x, \tag{1.2} \]
is used to model the behaviour of first order difference schemes. The rationale behind this is that first order schemes for (1.1) are formally second order accurate for an equation resembling (1.2). If one can prove convergence, or find a convergence rate, such that $u_\eta^t \rightarrow u$, then often analogous arguments will work for appropriate difference schemes. If this convergence has a rate, it is expected that the difference scheme will have the same rate. In \[13\], Evje and Karlsen showed that
\[ \|u_\eta^t(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} = \mathcal{O}\left(\eta^{1/2}\right), \]
Also, in \[15\], Gallouët et al. showed for the boundary value problem corresponding to (1.1) that
\[ \|u_\eta^t(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbb{R})} = \mathcal{O}\left(\eta^{1/5}\right). \]
To show the same rate for a difference scheme seems remarkably difficult. The main result of this paper is that we prove a significantly lower convergence rate for a semi-discrete difference approximation $u_{\Delta x}$,
\[ \int_{-L+M}^{L-M} |u(x,t) - u_{\Delta x}(x,t)| \, dx \leq \mathcal{O}\left(\Delta x^{1/11}\right), \]
where $M$ is a constant larger than $|f'|$. The $\mathcal{O}$ symbol depends on $t$, $L$ and the initial data $u_0$, but not on the discretization parameter $\Delta x$.

The rest of this paper is organized as follows. In Section 2 we make precise the definition of a solution of (1.1), and of $u_{\Delta x}$. Then we list a number of useful properties of the (unique) weak solution and of the approximation $u_{\Delta x}$. Finally we state our main theorem. Section 3 is devoted to its proof, while we test the practical convergence properties of the scheme on a numerical example in Section 4.
2. Preliminaries

Independently of the smoothness of the initial data, due to the degeneracy of the diffusion, jumps may form in the solution $u$. Therefore we consider solutions in the weak sense, i.e.,

**Definition 2.1.** Set $\Pi_T = (0, T) \times \mathbb{R}$, a function $u(t,x) \in L^\infty ((0,T); L^1(\mathbb{R})) \cap L^\infty (\Pi_T)$ is a weak solution of the initial value problem (1.1) if it satisfies

1. $A(u)$ is continuous and $A(u)_x \in L^\infty (\Pi_T)$.
2. For all test functions $\varphi \in \mathcal{D}(\Pi_T)$

$$\int_{\Pi_T} u \varphi_t + f(u) \varphi_x + A(u) \varphi_{xx} \, dx \, dt = 0.$$  

3. The initial condition is satisfied in the $L^1$-sense

$$\lim_{t \downarrow 0} \int_{\mathbb{R}} |u(t, x) - u_0(x)| \, dx = 0.$$

In view of the existence theory, the condition D.1 is natural, and thanks to this we can replace (2.1) by

$$\int_{\Pi_T} u \varphi_t + (f(u) - A(u)_x) \varphi_x \, dx \, dt = 0.$$  

If $A$ is constant on a whole interval, then weak solutions are not uniquely determined by their initial data, and one must impose an additional entropy condition to single out the physically relevant solution. A weak solution satisfies the entropy condition if

$$\varrho(u)_t + q(u)_x + r(u)_{xx} \leq 0 \text{ in } \mathcal{D}'(\Pi_T),$$

for all convex, twice differentiable functions $\varrho : \mathbb{R} \to \mathbb{R}$, where $q$ and $r$ are defined by

$$q'(u) = \varrho'(u) f'(u), \text{ and } r'(u) = \varrho'(u) A'(u).$$

Via a standard limiting argument this implies that (2.3) holds for the Kružkov entropies $\varrho(u) = |u - c|$ for all constants $c$. We say that a weak solution satisfying the entropy condition is an entropy solution.

Let the signum function be defined as

$$\text{sign}(\sigma) = \begin{cases} -1 & \sigma < 0, \\ 0 & \sigma = 0, \\ 1 & \sigma > 0, \end{cases}$$

and its regularized counterpart, $\text{sign}_\varepsilon$, defined as

$$\text{sign}_\varepsilon(\sigma) = \begin{cases} \text{sign}(\sigma) & |\sigma| > \varepsilon, \\ \sin \left( \frac{\pi \sigma}{2\varepsilon} \right) & \text{otherwise}, \end{cases}$$

where $\varepsilon > 0$.

We collect some useful information about entropy solutions in the following, for a proof see [13].

**Theorem 2.1.** The unique entropy solution $u$ of (1.1) satisfies

$$\int_{\Pi_T} |u - c| \varphi_t + \text{sign}(u - c)(f(u) - f(c)) \varphi_x + |A(u) - A(c)| \varphi_{xx} \, dx \, dt \geq 0,$$
for all constants \( c \) and all non-negative test functions in \( \mathcal{D}'(\Pi_T) \). Furthermore, the following limits hold, provided \( A \) is strictly increasing,

\[
\int_\Pi_T |u - c| \varphi_t + \text{sign}(u - c)(f(u) - f(c) - A(u)_x) \varphi_x = \lim_{\varepsilon \downarrow 0} \int_\Pi_T |A(u)_x|^2 \text{sign}_\varepsilon(A(u) - A(c)) \varphi \, dt \, dx,
\]

(2.5) \[
\lim_{\varepsilon \downarrow 0} \int_\Pi_T (f(u) - f(c)) A(u)_x \text{sign}_\varepsilon(A(u) - A(c)) \varphi \, dt \, dx = 0,
\]

for all non-negative test functions \( \varphi \).

A common method to show existence of an entropy solution is to consider the regularized problem

\[
u^\eta + f(u^\eta)_x = (A(u^\eta) + \eta u)_x, \quad t > 0, \quad u^\eta(0, x) = u_0(x),
\]

(2.7) where \( \eta \) is some (small) positive number. This equation is not degenerate, and has a unique smooth solution for \( t > 0 \). The sequence \( \{u^\eta\}_{\eta > 0} \) is compact in \( L^1(\Pi_T) \), and converges to the entropy solution. In [13], it was established that for \( t < T \)

\[
\|u(t, \cdot) - u^\eta(t, \cdot)\|_{L^1(\mathbb{R})} \leq C\sqrt{\eta},
\]

(2.8) where the constant \( C \) only depends on \( f, A \) and the initial data \( u_0 \). Of course, \( u^\eta \) is also an entropy solution to (2.7), where the diffusion function is given by

\[
A^\eta(u) = A(u) + \eta u.
\]

Rather than discretizing (1.1), we shall discretize the regularized equation (2.7), and let \( \eta \) tend to zero in a suitable manner. Due to (2.8), it suffices to compare \( u^\eta \) and our approximate solution. To simplify our notation, we therefore, for the moment, assume that \( A \) is strictly increasing, with \( A'(u) \geq \eta > 0 \).

We consider a semi-discrete approximation, where space is discrete, but time continuous. Let \( \Delta x \) be some small parameter, and set \( x_j = j\Delta x \) and \( x_{j+1/2} = (j + 1/2)\Delta x \) for \( j \in \mathbb{Z} \). Set \( I_j = (x_{j-1/2}, x_{j+1/2}] \). The discrete derivatives \( D^\pm \) are defined by

\[
D^\pm \sigma_j = \pm \frac{\sigma_{j+1} - \sigma_j}{\Delta x}.
\]

Our scheme is defined by

\[
\begin{aligned}
\frac{d}{dt} u_j(t) + D^- F_{j+1/2} = D^- D^+ A_j, \quad t > 0, \\
u_j(0) = \frac{1}{\Delta x} \int_{I_j} u_0(x) \, dx,
\end{aligned}
\]

(2.9) for \( j \in \mathbb{Z} \). Here \( F_{i+1/2} \) is the Engquist-Osher flux and \( A_j = A(u_j) \). More precisely, for a monotone flux \( f \), the generalized upwind scheme of Engquist and Osher is defined by

\[
F_{j+\frac{1}{2}} = F(u_j, u_{j+1}) = f^+(u_j) + f^-(u_{j+1}),
\]

where

\[
f^+(u) = f(0) + \int_0^u \max(f'(s), 0) \, ds, \quad f^-(u) = \int_0^u \min(f'(s), 0) \, ds.
\]
With this, we can rewrite our scheme (2.9) as
\begin{equation}
\begin{aligned}
\frac{d}{dt}u_j(t) + (D^- f^+(u_j) + D^+ f^-(u_j)) &= D^- D^+ A_j, \quad t > 0, \\
u_j(0) &= \frac{1}{M} \int_j u_0(x) \, dx,
\end{aligned}
\end{equation}
for \( j \in \mathbb{Z} \) and \( A_j = A(u_j) \).

In order to define an approximation on the whole of \( \Pi_T \), we let \( u_{\Delta x} \) be the piecewise linear interpolant given by
\[
\begin{aligned}
u_{\Delta x}(x, t) = u_j(t) + D^+ u_j(t) (x - x_j), \quad \text{for } x \in [x_j, x_{j+1}],
\end{aligned}
\]
and with a slight abuse of notation we define \( u_j \) to be the piecewise constant (in \( x \)) function
\[
u_j(x, t) = u_j(t) \quad \text{for } x \in (x_{j-1/2}, x_{j+1/2}].
\]

We collect some useful results regarding \( u_{\Delta x} \) and the entropy solution \( u \) in the next lemma.

**Lemma 2.1.** If \( u \) is the unique entropy solution of (1.1) and \( u_j \) the function defined by the scheme (2.9). Then the following estimates hold:
\begin{align}
\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} &\leq \|u_0\|_{L^\infty(\mathbb{R})} \quad (2.11) \\
|u(\cdot, t)|_{B.V.(\mathbb{R})} &\leq |u_0|_{B.V.(\mathbb{R})} \quad (2.12) \\
\|f(u(\cdot, t)) - A(u(\cdot, t))x\|_{L^\infty(\mathbb{R})} &\leq \|f(u_0) - A(u_0)x\|_{L^\infty(\mathbb{R})} \quad (2.13) \\
\|f(u(\cdot, t)) - A(u(\cdot, t))x\|_{B.V.(\mathbb{R})} &\leq \|f(u_0) - A(u_0)x\|_{B.V.(\mathbb{R})} \quad (2.14) \\
\|u_j(t)\|_{L^\infty(\mathbb{R})} &\leq \|u_j(0)\|_{L^\infty(\mathbb{R})} \quad (2.15) \\
|u_j(t)|_{B.V.(\mathbb{R})} &\leq |u_j(0)|_{B.V.(\mathbb{R})} \quad (2.16) \\
\|F_j(1/2)(t) - D^+ A(u_j(t))\|_{L^\infty(\mathbb{R})} &\leq \|F_j(1/2)(0) - D^+ A(u_j(0))\|_{L^\infty(\mathbb{R})} \quad (2.17) \\
|F_j(1/2)(t) - D^+ A(u_j(t))|_{B.V.(\mathbb{R})} &\leq |F_j(1/2)(0) - D^+ A(u_j(0))|_{B.V.(\mathbb{R})} \quad (2.18)
\end{align}

In addition the initial error is bounded by
\[
\|u_{\Delta x}(\cdot, 0) - u_0\|_{L^1(\mathbb{R})} \leq C \Delta x, \quad (2.19)
\]
for some constant \( C \) which depends only on \( \|u_0\|_{B.V.(\mathbb{R})} \).

For a proof of this, see [13]. Note that if \( u_0 \) and \( A(u_0)_x \) (and \( u_j(0) \) and \( D^+ A(u_j(0)) \)) are bounded independently of \( \eta \) (and \( \Delta x \)), then (2.13) and (2.17) imply that
\[
\|D^+ A(u_j)\|_{L^\infty(\mathbb{R})} \leq C \quad \text{and} \quad \|A(u)_x\|_{L^\infty(\mathbb{R})} \leq C,
\]
for some constant \( C \) which is independent of \( \Delta x \) and \( \eta \).

Our main result is the following

**Main Theorem.** Let \( u \) be the unique entropy solution to (1.1) and \( u_{\Delta x} \) be as defined by (2.10). Choose a constant
\[
M > \max_{|u| < \|u_0\|_{L^\infty(\mathbb{R})}} |f'(u)|,
\]
and another constant \( L > MT \), where \( T > 0 \). Then there exists a constant \( C \), independent of \( \Delta x \), but depending on \( f \), \( L \), \( T \) and \( u_0 \), such that
\[
\int_{L+Mt}^{L-Mt} |u(t, x) - u_{\Delta x}(t, x)| \, dx \leq C \Delta x^{1/11} \quad \text{for } t \leq T.
\]

As a by-product of our method of proof we get an improved rate if the diffusion is linear. The significance of this rate is that is independent of the size of the diffusion, which in this case is $\eta$.

**Main Corollary.** Let $u$ be the unique solution to the viscous regularization
\[ u_t + f(u)_x = \eta u_{xx}, \quad t > 0, \quad u(x,0) = u_0(x), \]
and let $u_{\Delta x}$ be defined by (2.10) with $A(u) = \eta u$. Then there exists a constant $C$, independent of $\Delta x$ and $\eta$, but depending on $f$, $L$, $T$ and $u_0$, such that
\[ \int_{L-Mt}^{L+Mt} |u(t,x) - u_{\Delta x}(t,x)| \, dx \leq C\Delta x^{1/2} \quad \text{for } t \leq T. \]

In the linear case this is what we expect. In fact, in [5] Chen and Karlsen showed that for a linear flux function $f$ the expected $\eta$ independent rate of 1/2 holds. However, the methods used in [5] are not easily modified to nonlinear flux functions.

3. Proof of the main theorem

First of all, for simplicity we will prove the main theorem for $f' < 0$. Note that, in that case $F_{j+1/2} = f(u_{j+1})$. The general case treatment will be similar (see Remark 3.1). The theorem will be proved by a “doubling of the variables” argument, but we start not with the entropy condition (2.5), but in the argument leading up to this condition. Set
\[ \psi_\varepsilon(u,c) = \int_c^u \text{sign}_x (A(z) - A(c)) \, dz. \]
This is a convex entropy for all constants $c$. Set $u = u(y,s)$ and rewrite (1.1) as
\[ u_s + (f(u) - f(c))_y = (A(u) - A(c))_{yy}, \]
and multiply this with $\psi'_\varepsilon(u,c)\varphi$ where $\varphi$ is a test function with compact support in $\mathbb{R} \times (0,T)$. Remember that $A' \geq \eta > 0$, so $u$ is smooth, after a partial integration, we arrive at
\[ \int_{\Pi_T} \psi_\varepsilon(u,c)\varphi_s + Q_\varepsilon(u,c)\varphi_y \, dyds \]
\[ = \int_{\Pi_T} \text{sign}_x (A(u) - A(c)) A(u)_y \varphi_y + \text{sign}'_x (A(u) - A(c)) (A(u)_y)^2 \varphi \, dyds. \]
Where we have used $Q'_\varepsilon(u,c) = \psi'_\varepsilon(u,c)f'(u)$. Although $\psi_\varepsilon(u,c) \approx |u - c|$, $\psi_\varepsilon$ is not symmetric in $u$ and $c$. This makes it cumbersome to work with when doubling the variables, so we rewrite the above as
\[ \int_{\Pi_T} |u - c| \varphi_s + Q_\varepsilon(u,c)\varphi_y \, dyds \]
\[ = \int_{\Pi_T} \text{sign}'_x (A(u) - A(c)) (A(u)_y)^2 \varphi + \text{sign}_x (A(u) - A(c)) A(u)_y \varphi_y \]
\[ + (|u - c| - \psi_\varepsilon(u,c)) \varphi_s \, dyds. \]
In the doubling of variables argument we choose $c = u_{\Delta x}(x,t)$, and a test function $\varphi(x,y,t,s)$. Integrating the above over $(x,t) \in \Pi_T$ after an integration by parts, we end up with
(3.2) \[
\int_{\Pi_T^2} |u - u_{\Delta x}| \varphi_s + \text{sign}_e(A(u) - A(u_{\Delta x}))(f(u) - f(u_{\Delta x})) \varphi_y dX
\]
\[
= \int_{\Pi_T^2} \left[ \text{sign}'_e(A(u) - A(u_{\Delta x})) \left( (A(u)y)^2 - A(u)y A(u_{\Delta x})x \right) \varphi_y - |A(u) - A(u_{\Delta x})|_\epsilon \varphi_{xy} + (\psi_z(u, u_{\Delta x}) - |u - u_{\Delta x}|) \varphi_s 
\right.
\]
\[
+ \left( \int_{u_{\Delta x}}^u \frac{d}{dz} \left( \text{sign}_e(A(z) - A(u_{\Delta x}))(f(z) - f(u_{\Delta x})) \right) \varphi_y \right] dX,
\]
where \( dX = dydsdxdt \) and \( |u|_\epsilon = \int_0^a \text{sign}_e(z)dz \). Here we have used that
\[
0 = \int_{\Pi_T^2} (\text{sign}_e(A(u) - A(u_{\Delta x})) A(u)_y \varphi_x) dX
\]
\[
= \int_{\Pi_T^2} \text{sign}_e(A(u) - A(u_{\Delta x})) A(u)_y A(u_{\Delta x})_x \varphi - |A(u) - A(u_{\Delta x})|_\epsilon \varphi_{xy} dX,
\]
and that
\[
\int_{\Pi_T^2} \text{sign}_e(A(u) - A(u_{\Delta x}))(A(u)_y)_y \varphi_y dX = \int_{\Pi_T^2} (|A(u) - A(u_{\Delta x})|_\epsilon)_y \varphi_y dX
\]
\[
= - \int_{\Pi_T^2} |A(u) - A(u_{\Delta x})|_\epsilon \varphi_{yy} dX.
\]
Also
\[
Q_\epsilon(u, u_{\Delta x}) = \int_{u_{\Delta x}}^u \text{sign}_e(A(z) - A(u_{\Delta x})) \frac{d}{dz} (f(z) - f(u_{\Delta x})) dz
\]
\[
= - \int_{u_{\Delta x}}^u \frac{d}{dz} (\text{sign}_e(A(z) - A(u_{\Delta x}))(f(z) - f(u_{\Delta x})) dz
\]
\[
+ \text{sign}_e(A(u) - A(u_{\Delta x}))(f(u) - f(u_{\Delta x})).
\]
The next goal is to obtain an analogous estimate for the difference approximation \( u_{\Delta x} \). Set \( \varphi_j(t) = \varphi(x_j, t) \) and multiply the scheme (2.9) with \( \psi'_e(u_j, c) \varphi_j \) and do a summation by parts to get
\[
\sum_j \left[ \psi(u_j, c)_i \varphi_j + \varphi_j D^+ Q_\epsilon(u_j, c) \right]
\]
\[
= - \sum_j \psi'_e(u_j, c) D^+ A(u_j) D^+ \varphi_j + D^+ \psi'_e(u_j, c) D^+ A(u_j) \varphi_{j+1}
\]
\[
+ \sum_j \frac{\varphi_j}{\Delta x} \int_{u_j}^{u_{j+1}} \psi''_e(s, c) (f(u_{j+1}) - f(s)) ds.
\]
Where we have used the following result:
\[
0 \geq \int_{u_j}^{u_{j+1}} \psi''_e(s, c)[f(u_{j+1}) - f(s)] ds = \int_{u_j}^{u_{j+1}} \psi'_e(s, c) f'(s) ds
\]
\[
- \psi'_e(u_j)[f(u_{j+1}) - f(u_j)].
\]
Set \( \varphi^{\Delta x} = \varphi_j \) for \( x \in (x_{j-1/2}, x_{j+1/2}] \), integrating the above for \( t \in [0, T] \) and multiplying with \( \Delta x \), we obtain
\[
\int_{\Pi_T} |u_{\Delta x} - c| \varphi^{\Delta x}_T + \text{sign}_e(A(u_j) - A(c))(f(u_j) - f(c)) D^- \varphi^{\Delta x} dxdt
\]
This can be rewritten
\[
\frac{1}{2}\int_{t_0}^{t_1} \int_{\Omega} \left( \phi_x(u_j, c) - |u_{\Delta x} - c| \right) \phi_{\Delta x}^{\Delta} dx dt \\
- \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x(A(u_j) - A(c)) D^+A(u_j) D^+\varphi_{\Delta x} dx dt \\
\geq \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x'(A(\theta_{j+1/2}) - A(c)) \left[ D^+A(u_j) \right]^2 \varphi_{j+1} dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \left( \int_c^{u_j} \frac{d}{dz} \text{sign}_x(A(z) - A(c))(f(z) - f(c)) dz \right) D^-\varphi_{\Delta x} dx dt.
\]

This can be rewritten
\[
\frac{1}{2}\int_{t_0}^{t_1} \int_{\Omega} |u_{\Delta x} - c| \phi_1 + \text{sign}_x(A(u_j) - A(c))(f(u_j) - f(c)) \phi_x dx dt \\
\geq \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x(A(u_j) - A(c)) D^+A(u_j) \phi_x dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x'(A(\theta_{j+1/2}) - A(c)) \left[ D^+A(u_j) \right]^2 \varphi dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x'(A(\theta_{j+1/2}) - A(c)) \left[ D^+A(u_j) \right]^2 (\varphi_{j+1} - \varphi) dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x(A(u_j) - A(c)) D^+A(u_j) (D^+\varphi_{\Delta x} - \varphi_x) dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} (|u_{\Delta x} - c| - \psi_x(u_j, c)) \phi_{\Delta x}^{\Delta} + |u_{\Delta x} - c| (\varphi_x - \varphi_{\Delta x}) dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x(A(u_j) - A(c))(f(u_j) - f(c)) (\varphi_x - D^-\varphi_{\Delta x}) dx dt \\
+ \int_{t_0}^{t_1} \int_{\Omega} \left( \int_c^{u_j} \frac{d}{dz} \text{sign}_x(A(z) - A(c))(f(z) - f(c)) dz \right) D^-\varphi_{\Delta x} dx dt.
\]

In order to make this more compatible with the corresponding equality for the exact solution (3.1), we rewrite again to get (recall that \( u_j \) denotes the piecewise constant function taking the value \( u_j(t) \) in the cell \( (x_{j-1/2}, x_{j+1/2}] \)),

(3.3)
\[
\frac{1}{2}\int_{t_0}^{t_1} \int_{\Omega} |u_{\Delta x} - c| \phi_1 + \text{sign}_x(A(u_{\Delta x}) - A(c))(f(u_{\Delta x}) - f(c)) \phi_x dx dt
\]

(3.4)
\[
\geq \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x(A(u_{\Delta x}) - A(c)) A(u_{\Delta x}) \phi_x dx dt
\]

(3.5)
\[
+ \int_{t_0}^{t_1} \int_{\Omega} \text{sign}_x'(A(u_{\Delta x}) - A(c)) [A(u_{\Delta x})] \phi dx dt
\]

(3.6)
\[
+ \int_{t_0}^{t_1} \int_{\Omega} \left[ \text{sign}_x(A(u_{\Delta x}) - A(c)) - \text{sign}_x(A(u_j) - A(c)) \right] \times (f(u_{\Delta x}) - f(c)) \phi_x dx dt
\]

(3.7)
\[
+ \int_{t_0}^{t_1} \int_{\Omega} \left[ \text{sign}_x'(A(\theta_{j+1/2}) - A(c)) - \text{sign}_x'(A(u_{\Delta x}) - A(c)) \right] dx dt
\]

+ \int_{t_0}^{t_1} \int_{\Omega} \left[ \text{sign}_x'(A(\theta_{j+1/2}) - A(c)) - \text{sign}_x'(A(u_{\Delta x}) - A(c)) \right] dx dt.
(3.8) \[ \times \left[ D^+ A(u_j) \right]^2 \varphi \, dx \, dt \]

(3.9) \[ + \int_{\Pi_T} \sign^+ (A (u_{\Delta x}) - A(c)) \left[ (D^+ A(u_j))^2 - (A(u_{\Delta x})_c)^2 \right] \varphi \, dx \, dt \]

(3.10) \[ + \int_{\Pi_T} \sign^c (A (\theta_{j+1/2}) - A(c)) \left[ D^+ A(u_j) \right]^2 (\varphi_{j+1} - \varphi) \, dx \, dt \]

(3.11) \[ + \int_{\Pi_T} \left[ \sign^c (A (u_{\Delta x}) - A(c)) - \sign^c (A(u_j) - A(c)) \right] \times D^+ A(u_j) \varphi \, dx \, dt \]

(3.12) \[ + \int_{\Pi_T} \sign^c (A (u_{\Delta x}) - A(c)) \left[ D^+ A(u_j) - A(u_{\Delta x})_j \right] \varphi \, dx \, dt \]

(3.13) \[ + \int_{\Pi_T} \sign^c (A(u_j) - A(c)) D^+ A(u_j) (D^+ \varphi^{\Delta x} - \varphi_x) \, dx \, dt \]

(3.14) \[ + \int_{\Pi_T} (|u_{\Delta x} - c| - \psi^c (u_j, c) \varphi^{\Delta x}_t + |u_{\Delta x} - c| (\varphi_t - \varphi^{\Delta x}_t) \, dx \, dt \]

(3.15) \[ + \int_{\Pi_T} \sign^c (A(u_j) - A(c)) (f(u_j) - f(c)) (\varphi_x - D^- \varphi^{\Delta x}_x) \, dx \, dt \]

(3.16) \[ + \int_{\Pi_T} \left( \int_0^t \frac{d}{dz} (\sign^c (A(z) - A(c))(f(z) - f(c)) \, dz \right) D^- \varphi^{\Delta z} \, dx \, dt. \]

**Remark 3.1.** In the general case, we can write $F_{j+\frac{1}{2}} = f^+(u_j) + f^-(u_{j+1})$. Note that $(f^+)' \geq 0$, $(f^-)' \leq 0$, and both are Lipschitz continuous. Then if we multiply (2.10) by $\psi^c(u_j, c)$, numerical flux part can be written as $D^- Q^+_c + D^+ Q^-_c + \text{"terms having the right sign"}$. Here $(Q^\pm_c)'(u, c) = \psi^c(u, c)(f^\pm)'(u)$, and $Q^+_c + Q^-_c = Q_c$.

Let now

\[ R_1(c) = (3.6) + (3.7) + (3.8) + (3.9), \]

\[ R_2(c) = (3.10) + (3.11) + (3.12) + \cdots + (3.16). \]

Now we choose $c = u(y, s)$ and the same test function as before, and integrate the result over $(y, s) \in \Pi_T$. The result reads

(3.17) \[ \int_{\Pi_T} |u_{\Delta x} - u| \varphi_t + \sign^c (A (u_{\Delta x}) - A(u)) (f (u_{\Delta x}) - f (u)) \varphi_x \, dX \]

\[ \geq \int_{\Pi_T} \sign^c (A (u_{\Delta x}) - A(u)) \left( (A(u_{\Delta x})_x)^2 - A(u_{\Delta x})_x A(u)_y \right) \varphi \, dX \]

\[ - \int_{\Pi_T} |A(u_{\Delta x}) - A(u)| \varphi_x + \varphi_y \, dX \]

\[ + \int_{\Pi_T} R_1(u) + R_2(u) \, dy \, ds. \]

Adding this and (3.2), we get

(3.18) \[ \int_{\Pi_T} \left[ |u_{\Delta x} - u| (\varphi_t + \varphi_x) \right] \]

\[ + \sign^c (A (u_{\Delta x}) - A(u)) (f (u_{\Delta x}) - f(u)) (\varphi_x + \varphi_y) \]
\[
\begin{align*}
+ |A(u_{\Delta x}) - A(u)|_c (\varphi_{xx} + 2\varphi_{xy} + \varphi_{yy}) \] dX \\
\geq \int_{\Pi^2_T} \text{sign}'(A(u_{\Delta x}) - A(u)) (A(u_{\Delta x})_x - A(u)_y)^2 \varphi dX \\
+ \int_{\Pi^2_T} (\psi_x (u, u_{\Delta x}) - |u - u_{\Delta x}|) \varphi_x dX \\
+ \int_{\Pi^2_T} \left( \int_{u_{\Delta x}}^u \frac{d}{dz} (\text{sign}(z - A(u_{\Delta x}))(f(z) - f(u_{\Delta x})) dz \right) \varphi_y dX \\
+ \iint_{\Pi_T} R_1(u) + R_2(u) dyds \\
\geq \int_{\Pi^2_T} \left( \int_{u_{\Delta x}}^u \frac{d}{dz} (\text{sign}(z - A(u_{\Delta x}))(f(z) - f(u_{\Delta x})) dz \right) \varphi_y dX \\
+ \int_{\Pi^2_T} (\psi_s (u, u_{\Delta x}) - |u - u_{\Delta x}|) \varphi_s dX \\
+ \iint_{\Pi_T} R_1(u) + R_2(u) dyds.
\end{align*}
\]

Now we are going to specify a nonnegative test function \( \varphi = \varphi(t, x, s, y) \) defined in \( \Pi_T \times \Pi_T \). Let \( \omega \in C_0^{\infty}(\mathbb{R}) \) be a function satisfying
\[
\text{supp}(\omega) \subset [-1, 1], \quad \omega(\sigma) \geq 0, \quad \int_{\mathbb{R}} \omega(\sigma) d\sigma = 1,
\]
and define \( \omega_r(x) = \omega(x/r)/r \). Furthermore, let \( h(z) \) be defined as
\[
h(z) = \begin{cases} 
0, & z < -1, \\
z + 1, & z \in [-1, 0], \\
1, & z > 0.
\end{cases}
\]
and set \( h_\alpha(z) = h(\alpha z) \). Let \( \nu < \tau \) be two numbers in \( (0, T) \), for any \( \alpha > 0 \) define
\[
H_\alpha(t) = \int_{-\infty}^t \omega_\alpha(\xi) d\xi,
\]
\[
\Psi(x, t) = (H_\alpha(t - \nu) - H_\alpha(t - \tau)) \left( h_\alpha(x - L(t)) - h_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right)
\]
\[
= \chi_{(\nu, \tau)}(t) \chi_{(L_1, L_r)}(x, t)
\]
where the lines \( L_{1, r} \) are given by
\[
L_1(t) = -L + Mt, \quad L_r(t) = L - Mt
\]
where \( M \) and \( L \) are positive numbers, \( M \) will be specified below. With \( 0 < r < \min \{ \nu, \tau \} \) and \( \alpha_0 \in (0, \min \{ \nu - r, T - \tau - r \}) \) we set
\[
(3.19) \quad \varphi(x, t, y, s) = \Psi(x, t) \omega_r(x - y) \omega_{r_0}(t - s).
\]
We note that \( \phi \) has compact support and also that we have,
\[
\varphi_t + \varphi_s = \Psi_t(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_x + \varphi_y = \Psi_x(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_y + \varphi_{yy} = \Psi_y(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_{xx} + \varphi_{xy} + \varphi_{yy} = \Psi_{xx}(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_{yy} = \Psi_{yy}(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_{xy} = \Psi_{xy}(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi_{xx} = \Psi_{xx}(x, t) \omega_r(x - y) \omega_{r_0}(t - s),
\]
\[
\varphi = \Psi(x, t) \omega_r(x - y) \omega_{r_0}(t - s).
\]
\[ \varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} = \Psi_{xx}(x, t) \omega_r(x - y) \omega_{\alpha_0}(t - s). \]

For the record, we note that

\begin{align}
\Psi_t(x, t) &= -\chi_{(v, \tau)}^0(t) M \left( h'_\alpha(x - L_t(t)) + h''_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right) \\
&\quad + (\omega_{\alpha_0}(t - \nu) - \omega_{\alpha_0}(t - \tau)) \chi_{(L_t, L_r)}^0(x, t), \\
\Psi_x(x, t) &= \chi_{(v, \tau)}^0(t) \left( h'_\alpha(x - L_t(t)) - h'_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right), \\
\Psi_{xx}(x, t) &= \chi_{(v, \tau)}^0(t) \left( h''_\alpha(x - L_t(t)) - h''_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right).
\end{align}

We shall let all the “small parameters” \( \alpha, \alpha_0, r, r_0, \varepsilon \) \(D\alpha x\) be sufficiently small, but fixed. The goal of our manipulations is to obtain an inequality where the difference between \( u_{\Delta x} \) and \( u \) is bounded by some combination of all these parameters.

We shall repeatedly use the fact that

\[ \int \int |v(x, t) - v(y, t)| \omega_r(x - y) dxdy \leq Cr \]

(3.21)

and

\[ \int \int |v(x, s) - v(x, t)| \omega_{\alpha_0}(t - s) dxdy \leq Cr_0, \]

for \( v = u, v = u_{\Delta x}, v = f(u), v = A(u)_x \) or \( v = A(u_{\Delta x})_x \). These estimates follow from the basic bounds in Lemma 2.1. Starting the first term on the left of (3.18), we write

\[ \int_{\Omega^\delta_t} |u_{\Delta x} - u| (\varphi_s + \varphi_t) dX \leq \int_{\Omega^\delta_t} |u_{\Delta x}(x, t) - u(x, t)| \Psi_t dxdt \]

\[ + \int_{\Omega^\delta_t} \int_{\mathbb{R}} |u(x, t) - u(x, s)| |\Psi_t(x, t)| \omega_{\alpha_0}(t - s) dsdxdt \]

\[ + \int_{\Omega^\delta_t} |u(x, s) - u(y, s)| |\Psi_t(x, t)| \omega_{\alpha_0}(t - s) \omega_r(x - y) dX. \]

To estimate \( \beta \) and \( \gamma \) we use

\[ |\Psi_t| \leq (\omega_{\alpha_0}(t - \nu) + \omega_{\alpha_0}(t - \tau)) + M \left( h'_\alpha(x - L_t(t)) + h'_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right), \]

and that

\[ \int \omega_{\alpha_0}(t - \nu) dt \leq 1 \quad \text{and} \quad \left| \int M^L h'_\alpha(x - L_{t,r}(t)) dt \right| \leq C. \]

A typical term in \( \beta \) reads

\[ \int \int \left| u(x, t) - u(x, s) \right| \omega_{\alpha_0}(t - \nu) \omega_{\alpha_0}(t - s) dxdsd \]

\[ \leq C \int \int |t - s| \omega_{\alpha_0}(t - s) \omega_{\alpha_0}(t - \nu) dsdt \]

\[ \leq C r_0, \]
Hence

\[ \beta \leq C r_0. \]

Similarly a typical term in \( \gamma \) can be estimated

\[
\iint \iint |u(x, s) - u(y, s)| \omega_r(x - y) \omega_{r_0}(t - s) \omega_{\alpha_0}(t - \nu) \, dx dy dt ds \leq C r.
\]

Thus we find that

\[ (3.22) \quad \beta + \gamma \leq C (r_0 + r). \]

To continue the estimate with the first term on the left of (3.18), we split \( \delta \) as follows

\[
\delta = - \int_{\Pi_T} \chi_{(\omega, \tau)}(t) M \left( h'_\alpha(x - L(t)) + h'_\alpha(x - L_r(t) - \frac{1}{\alpha}) \right) |u_{\Delta x}(x, t) - u(x, t)| \, dx dt \\
+ \int_{\Pi_T} \chi_{(L_1, L_r)}(x, t) |u_{\Delta x}(x, t) - u(x, t)| \omega_{\alpha_0}(t - \nu) - \omega_{\alpha_0}(t - \tau) \, dx dt.
\]

The term \( \delta_1 \) will be balanced against the first order derivative term on the left hand side of (3.18). To estimate \( \delta_2 \) we set \( e(x, t) = |u_{\Delta x}(x, t) - u(x, t)| \) and proceed as follows

\[
\int_{\Pi_T} \chi_{(L_1, L_r)}(x, t) e(x, t) \omega_{\alpha_0}(t - \nu) \, dx dt \\
\leq \int \chi_{(L_1, L_r)}(x, \nu) e(x, \nu) \, dx \\
+ \int_{\Pi_T} \chi_{(L_1, L_r)}(x, t) |e(x, t) - e(x, \nu)| \omega_{\alpha_0}(t - \nu) \, dx dt \\
\leq \int \chi_{(L_1, L_r)}(x, \nu) e(x, \nu) \, dx + C \alpha_0
\]

and similarly

\[
\int_{\Pi_T} \chi_{(L_1, L_r)}(x, t) e(x, t) \omega_{\alpha_0}(t - \tau) \, dx dt \geq \int \chi_{(L_1, L_r)}(x, \tau) e(x, \tau) \, dx - C \alpha_0.
\]

Using this we get the estimate

\[ (3.23) \quad \delta_2 \leq \int \chi_{(L_1, L_r)}(x, \nu) |u_{\Delta x}(x, \nu) - u(x, \nu)| \, dx \\
- \int \chi_{(L_1, L_r)}(x, \tau) |u_{\Delta x}(x, \tau) - u(x, \tau)| \, dx + C \alpha_0.
\]

Now we rewrite the “first derivative term” on the left hand side of (3.18). Doing this, we get

\[
\int_{\Pi_T^2} \text{sgn}_r (A (u_{\Delta x}) - A(u)) (f (u_{\Delta x}) - f(u)) (\varphi_x + \varphi_y) \, dX \\
= \int_{\Pi_T^2} \text{sgn}(x, y, t, s) (f(u_{\Delta x}(x, t)) - f(u(x, t))) \Psi_x(x, t) \omega_r(x - y) \omega_{r_0}(t - s) \, dX
\]
where we have set $\text{sg}(x, y, t, s) = \text{sign}_e(A(u_{\Delta x}(x, t)) - A(u(y, s)))$. We proceed as follows

$$|\delta_4| \leq \int_{\Pi_T^2} |f(u(x, t)) - f(u(y, s))| \chi_{(\nu, \tau)}^{\alpha_0}(t)\omega_{\alpha}(t-s)\omega_r(x-y)h'_\alpha(x - L(t)) dX$$

Each of these two terms are estimated using (3.21) by

$$\int_{\Pi_T^2} |f(u(x, t)) - f(u(y, s))| \chi_{(\nu, \tau)}^{\alpha_0}(t)\omega_{\alpha}(t-s)\omega_r(x-y)h'_\alpha(x - L(t)) dX$$

$$\leq \int_{\Pi_T^2} |f(u(x, t)) - f(u(x, s))| \chi_{(\nu, \tau)}^{\alpha_0}(t)\omega_{\alpha}(t-s)\omega_r(x-y)h'_\alpha(x - L(t)) dX$$

$$+ \int_{\Pi_T^2} |f(u(x, s)) - f(u(y, s))| \chi_{(\nu, \tau)}^{\alpha_0}(t)\omega_{\alpha}(t-s)\omega_r(x-y)h'_\alpha(x - L(t)) dX$$

$$\leq C (r_0 + r)$$.

and thus

$$|\delta_4| \leq C (r_0 + r).$$

The terms $\delta_1 + \delta_3$ is bounded by choosing $M$ sufficiently large (all functions are functions of $(x, t)$).

$$\delta_1 + \delta_3$$

$$= \int_{\Pi_T} \chi_{(\nu, \tau)}^{\alpha_0}(t)h'_\alpha(x - L(t)) (-M |u_{\Delta x} - u| + \text{sg} (f(u_{\Delta x}) - f(u))) dX dt$$

$$+ \int_{\Pi_T} \chi_{(\nu, \tau)}^{\alpha_0}(t)h'_\alpha(x - L(t)) - \frac{1}{\alpha} (-M |u_{\Delta x} - u| - \text{sg} (f(u_{\Delta x}) - f(u))) dX dt.$$
Collecting (3.22), (3.23), (3.24), (3.25) and the above inequality we see that
\[
\int \chi_{(L_l, L_r)}(x, \tau) |u(\Delta \tau x) - u(x, \tau)| \, dx \
\leq \int \chi_{(L_l, L_r)}(x, \nu) |u(\Delta \tau x) - u(x, \nu)| \, dx \\
+ C (r_0 + r + \alpha_0 + \alpha) \\
+ \left| \int_{\Pi_T^2} Q_1 + Q_2 \, dX + \int_{\Pi_T} R_1(u) + R_2(u) \, dyds \right|. 
\]
(3.26)

In order to estimate the integral involving \( Q_2 \) we first observe that since \( A'(u) \geq \eta \), we get
\[
|\psi(\epsilon (a, b) - |a - b|) \leq \int_b^a |\text{sign}_\epsilon (A(z) - A(b)) - \text{sign}(A(z) - A(b))| \, dz \\
\leq \frac{1}{\eta} \int_{A(b)}^{A(a)} \chi_{|\beta| < \epsilon} \left| \sin \left( \frac{\pi \beta}{2\epsilon} \right) \right| d\beta \\
\leq C \frac{\epsilon^2}{\eta}.
\]

Using this we find that
\[
\left| \int_{\Pi_T^2} Q_2 \, dX \right| \leq C \frac{\epsilon}{r_0 \eta} \int_0^T \int_{-L+M/\alpha}^{L-(M/\alpha)t} dxdt \\
\leq C \frac{\epsilon}{r_0 \eta} \left( L + \frac{1}{\alpha} \right),
\]
(3.27)
where \( C \) is independent of the small parameters. To estimate the integral of \( Q_1 \) we use the Lipschitz continuity of \( f (A^{-1}) \). Note that \( f (A^{-1}) \) is Lipschitz continuous with Lipschitz constant \( \frac{M}{\eta} \), where \( M \) is the Lipschitz constant for \( f \).

\[
\int_{u(\Delta \tau x)}^{u} \frac{d}{dz} (\text{sign}_\epsilon (A(z) - A(u(x)))) \left| (f(z) - f(u(\Delta \tau x))) \right| \, dz \\
= \int_{A(u(x))}^{A(u(x) + \epsilon)} \text{sign}'_\epsilon (r - A(u(x))) \left| (f(A^{-1}(r)) - f(u(\Delta \tau x))) \right| \, dr \\
= \frac{1}{\epsilon} \int_{\min(A(u),A(u(\Delta \tau x)) + \epsilon)}^{\min(A(u),A(u(\Delta \tau x)) - \epsilon)} \left| (f(A^{-1}(r)) - f(u(\Delta \tau x))) \right| \, dr \\
\leq \frac{\epsilon^2 M}{\eta \epsilon}
\]
(3.28)

Using this we find that
\[
\left| \int_{\Pi_T^2} Q_1 \, dX \right| \leq \frac{M \epsilon}{\eta} \int_{\Pi_T^2} \phi \, dX \\
\leq C \frac{\epsilon}{\eta^2} \left( L + \frac{1}{\alpha} \right).
\]
(3.29)
Therefore

\[ (3.30) \quad \left| \int_{\Pi_T^2} Q_1 + Q_2 \, dX \right| \leq \frac{C \varepsilon}{\eta} \left( L + \frac{L}{r_0} + \frac{1}{r \alpha} \right). \]

Now we claim that

\[ (3.31) \quad \left| \int_{\Pi_T} R_1(u) + R_2(u) \, dy ds \right| \leq C \left[ \Delta x \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon \eta^2} + \frac{1}{r^2} + \frac{1}{r_0} + \frac{1}{r \varepsilon} + \frac{1}{r} \right) \right. \]

\[ + \varepsilon \left( \frac{1}{\eta} + \frac{1}{r_0 \eta} + \frac{1}{r \eta \alpha} + \frac{1}{\eta \alpha} \right), \]

where \( C \) depends on (among other things) \( L \) and \( T \), but not on the small parameters \( \alpha_0, \alpha, r_0, r, \eta \) or \( \varepsilon \). The proof of this claim is a tedious computation of all the terms. We start by the one from (3.6).

\[
\int_{\Pi_T} \left| (3.6) \right| \bigg|_{c=u} \, dy ds \leq \int_{\Pi_T^2} \left| \text{sign}_c \left( A(u_{\Delta x}) - A(u) \right) - \text{sign}_c \left( A(u_j) - A(u) \right) \right| \]

\[ \times \left| f(u_{\Delta x}) - f(u) \right| |\Psi_x \omega_r \omega_{\varepsilon r_0} + \Psi \omega_r \omega_{\varepsilon r_0}| \, dX \]

\[ \leq C \int_{\Pi_T} \frac{1}{\varepsilon} |u_{\Delta x} - u_j| \left( |\Psi_x| + \frac{1}{r} \right) \, dx dt \]

\[ \leq \frac{CT \Delta x}{\varepsilon r} \max_{t \in [0,T]} |u_{\Delta x}|_{B.V.(\mathbb{R})} \left( \frac{\alpha + 1}{r} \right) \]

for sufficiently small \( r \) and \( \alpha \). Now

\[
\int_{\Pi_T} \left| (3.7) \right| \bigg|_{c=u} \, dy ds \leq M \int_{\Pi_T^2} |u_{\Delta x} - u_j| \left| \Psi_x \omega_r \omega_{\varepsilon r_0} + \Psi \omega_r \omega_{\varepsilon r_0} \right| \, dX \]

\[ \leq \frac{CT \Delta x}{\varepsilon r} \max_{t \in [0,T]} |u_{\Delta x}|_{B.V.(\mathbb{R})} \left( \frac{\alpha + 1}{r} \right) \]

\[ \leq \frac{C \Delta x}{\varepsilon r}. \]

To estimate the next term we observe that \( \left| \text{sign}_c'' \right| \leq C / \varepsilon^2 \),

\[
\int_{\Pi_T} \left| (3.8) \right| \bigg|_{c=u} \, dy ds \leq \frac{C}{\varepsilon^2} \int_{\Pi_T} \left| \theta_{j+1/2} - u_{\Delta x} \right| \, dx dt \]

\[ \leq \frac{CT \Delta x}{\varepsilon^2} \max_{t \in [0,T]} |u_{\Delta x}|_{B.V.(\mathbb{R})} \]

\[ \leq \frac{C \Delta x}{\varepsilon^2}. \]

The next term involves \( D^+ A(u_j) \) and \( A(u_{\Delta x}) \), these can be written

\[ D^+ A(u_j) = A'(\alpha_{j+1/2}) D^+ u_j \] and \( A(u_{\Delta x}) = A'(\beta_{j+1/2}) D^+ u_j \),

if \( x \in [x_j, x_{j+1}) \). Here \( u_{j+1/2} \) and \( \beta_{j+1/2} \) are between \( u_j \) and \( u_{j+1} \). Furthermore, since \( A' \geq \eta \), we have that

\[ |D^+ u_j| \leq \frac{1}{\eta} |D^+ A(u_j)|. \]
\[
\int_{\Gamma_T} |(3.9)| \bigg|_{c=u} \, dyds \leq \frac{C}{\eta} \int_{\Gamma_T} |D^+ A(u_j) - A(u_{\Delta x})| \, dxdt
\]
\[
\leq \frac{C}{\eta} \int_{\Gamma_T} |\alpha_{j+1/2} - \beta_{j+1/2}| |D^+ u_j| \, dxdt
\]
\[
\leq \frac{C}{\varepsilon \eta^2} \int_{\Gamma_T} |u_{j+1} - u_j| |D^+ A(u_j)| \, dxdt
\]
\[
\leq \frac{CT}{\varepsilon \eta^2} \Delta \max_{t \in [0,T]} |u_{\Delta x}|_{B.V.(\mathbb{R})}
\]
\[
\leq \frac{C\Delta x}{\varepsilon \eta^2}.
\]

We continue with
\[
\int_{\Gamma_T} |(3.10)| \bigg|_{c=u} \, dyds
\]
\[
\leq C \int_{\Gamma_T} |D^+ A(u_j)| \omega_\rho_0(x-y)
\]
\[
\times |\Psi(x_{j+1}, t) \omega_r(x_{j+1} - y) - \Psi(x, t) \omega_r(x - y)| \, dX
\]
\[
\leq C \int \sum_j \int_{x_{j-1}}^{x_j} \int_{x_{j-1}}^{x_j} |D^+ A(u_j)|
\]
\[
\times |\Psi(z, t) \omega_r(z - y) + \Psi(z, t) \omega'_r(z - y)| \, dz \, dx \, dy \, dt
\]
\[
\leq C T \Delta x \left( \alpha + \frac{1}{r} \right) \sup_{t \in [0,T]} \|D^+ A(u_j(t))\|_{L^1(\mathbb{R})}
\]
\[
\leq \frac{C \Delta x}{r},
\]
for sufficiently small \(\alpha\). With similar arguments we show that
\[
\int_{\Gamma_T} |(3.13)| + |(3.15)| \bigg|_{c=u} \, dyds \leq \frac{C \Delta x}{r^2},
\]
and
\[
\int_{\Gamma_T} |(3.11)| + |(3.12)| \bigg|_{c=u} \, dyds \leq \frac{C \Delta x}{r \varepsilon} + \frac{C \Delta x}{r \eta}.
\]
The term \((3.14)\) consists of two parts. The first of these
\[
\int_{\Gamma_T} |\psi_\varepsilon(u_{\Delta x}, u) - |u_{\Delta x} - u|| \varphi_\varepsilon \, dX \leq \frac{C \varepsilon}{r \rho \eta} \left( L + \frac{1}{\alpha} \right),
\]
by the same arguments used to show \((3.27)\). For the second part, using the fact that \((u_{\Delta x})_i \in L^1(\mathbb{R})\), we can show that
\[
\int_{\Gamma_T} |u_{\Delta x} - u| (\varphi_\varepsilon - (\varphi_\varepsilon)_i) \, dX
\]
\[
\leq \int_{\Gamma_T} \left[ \left| (u_{\Delta x})_i (\psi(x, t) - \psi(x, t)) \omega_r(x - y) \omega_\rho_0(t - s)
\right.
\]
\[
+ \left| (u_{\Delta x})_i (\omega_r(x - y) - \omega_r(x_j - y)) \psi(x, t) \omega_\rho_0(t - s) \right] \, dX
\]
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\[ \leq C \Delta x + \frac{C \Delta x}{r}. \]

There remains the last term, (3.16). To estimate this, we follow the similar arguments to the one used in (3.29). The result is that

\[ \int \int_{\Pi_T} |(3.16)| \left|_{x=u} \right| dy ds \leq C \varepsilon \left( L + \frac{1}{\alpha} \right). \]

Now we have proved the following lemma:

**Lemma 3.1.** Assume that \( u \) and \( u_{\Delta x} \) are take values in the interval \([-K, K]\) for some positive \( K \). Let \( M > \max_{v \in [-K, K]} |f'(v)|. \) Then if \( T \geq \tau > \nu \geq 0 \) and \( L - M \tau > 0 \), we have

\[ \int_{-L+Mt}^{L-Mt} |u_{\Delta x}(x, \tau) - u(x, \tau)| \, dx \]

\[ \leq \int \int_{\Pi_T} \left[ \frac{1}{r^2} + \frac{1}{\tau r} + \frac{1}{\tau r^2} + \frac{1}{\varepsilon r} + \frac{1}{\varepsilon r^2} \right] \right] \]

\[ \leq \int_{-L+Mt}^{L-Mt} |u_{\Delta x}(x, \nu) - u(x, \nu)| \, dx \]

\[ + C \left[ \frac{1}{\eta r} + \frac{1}{\tau_0 \eta} + \frac{1}{\tau_0 \eta^2} + \frac{1}{\eta \alpha} + \frac{1}{\eta^2 \alpha} \right]. \]

This follows from (3.26) and (3.31), observing that we can send \( \alpha_0 \) to zero. Now we let \( v(x, t) \) be the unique entropy solution of (1.1), where \( A'(v) \geq 0. \) We set \( \alpha = r = r_0 = \eta^{1/2}, \) and assume that \( \alpha \) is sufficiently small, then

\[ \int_{-L+Mt}^{L-Mt} |u_{\Delta x}(x, t) - v(x, t)| \, dx \leq C \left( \alpha + \frac{\varepsilon}{\alpha^2} + \frac{\Delta x}{\varepsilon^2} \right), \]

for some constant \( C \) which is independent of the small parameters \( \alpha, \varepsilon \) and \( \Delta x. \) This follows from (3.32), (2.8) and (2.19). Setting \( \alpha = \Delta x^{1/11}, \) and \( \varepsilon = \alpha^5 \) proves the main theorem.

**Proof of the main corollary.** To prove the corollary, we retrace the proof of the main theorem, using that \( A(u) = \eta u. \) We begin by setting

\[ \psi_{\varepsilon}(u, c) = \eta \int_{c}^{u} \text{sign}_{\varepsilon}(z - c) \, dz. \]

In this case, the equation corresponding to (3.2) reads

\[ \int_{\Pi_T^2} |u - u_{\Delta x}| \varphi_s + \text{sign} (u - u_{\Delta x}) (f(u) - f(u_{\Delta x})) \varphi_y \, dX \]

\[ = \lim_{\varepsilon \to 0} \int_{\Pi_T^2} \left[ \text{sign}_{\varepsilon} (u - u_{\Delta x}) (u_y^2 - u_y (u_{\Delta x})_x) \varphi \right. \]

\[ - |u - u_{\Delta x}|_{\varepsilon} (\varphi_{yy} + \varphi_{xy}) \] dX.

On the other hand, to obtain an expression for \( u_{\Delta x} \) we can proceed as follows. In fact, in discrete set up we have the following inequality

\[ \int \int \psi_{\varepsilon}(u_j, c)(\varphi_j)_{t_j} + Q_{\varepsilon}(u_j, c)D\varphi_{j} \, dx \, dt \]
\[
\begin{align*}
\geq \eta & \int \int D^+ u_j D^+ \varphi_j \text{sgn}_x(u_j - c) \ dy ds + \eta \int \int \partial_x [\text{sgn}_x(u_{\Delta x} - c)](u_{\Delta x})_x \varphi \ dx dt \\
+ \eta & \int \int [D^+ [\text{sgn}_x(u_j - c)]D^+ u_j \varphi_j + \partial_x [\text{sgn}_x(u_{\Delta x} - c)](u_{\Delta x})_x \varphi] \ dx dt \\
= \eta & \int \int (u_{\Delta x})_x \varphi_j \text{sgn}_x(u_{\Delta x} - c) + \text{sgn}_x'(u_{\Delta x} - c)(u_{\Delta x})_x^2 \varphi \ dx dt \\
+ \eta & \int \int (u_{\Delta x})_{xx} \text{sgn}_x(u_{\Delta x} - c) \varphi - \varphi_j \text{sgn}_x(u_j - c) D^- D^+ u_j \ dx dt,
\end{align*}
\]

where we have used \( Q'_\varepsilon = \psi'_\varepsilon f' \) and the following equality:
\[
0 \geq \int_{u_j}^{u_{j+1}} \psi''_\varepsilon(s, c)[f(u_{j+1}) - f(s)] \ ds = \int_{u_j}^{u_{j+1}} \psi'_\varepsilon(s, c)f'(s) \ ds \\
- \psi'_\varepsilon(u_j)[f(u_{j+1}) - f(u_j)].
\]

By taking limit as \( \varepsilon \to 0 \), and choosing \( c = u(y, s) \) we have
\[
(3.35) \quad \int |u_j - u| (\varphi_j)_t + \text{sign}(u_j - u)(f(u_j) - f(u))D^- \varphi_j \ dX
\]
\[
\geq \lim_{\varepsilon \to 0} \int \left[ ((u_{\Delta x})_x^2 - u_y(u_{\Delta x})_x) \text{sgn}_x(u_{\Delta x} - u) \varphi - |u_{\Delta x} - u| \text{sgn}_x(\varphi_{xx} + \varphi_{yy}) \right] dX.
\]

Now adding \((3.34)\) and \((3.35)\), using \( Q(a, b) = \text{sign}(a - b)(f(a) - f(b)) \), we get
\[
\int_{\Omega_T^2} \left[ |u_{\Delta x} - u| (\varphi_x + \varphi_y) \\
+ \text{sign}(u_{\Delta x} - u)(f(u_{\Delta x}) - f(u))(\varphi_x + \varphi_y) \\
+ |u_{\Delta x} - u| (\varphi_{xx} + 2\varphi_{xy} + \varphi_{yy}) \right] dX
\]
\[
\geq \int_{\Omega_T^2} (|u_j - u| - |u_{\Delta x} - u|) (\varphi_j)_t \ dX + \int_{\Omega_T^2} |u_{\Delta x} - u| (\varphi_t - (\varphi_j)_t) \ dX \\
+ \int_{\Omega_T^2} Q(u_j, u)(\varphi_x - D^- \varphi_j) dX + \int_{\Omega_T^2} (Q(u, u_{\Delta x}) - Q(u_j, u)) \varphi_x dX.
\]

Observe that we can use similar arguments to the ones used in the nonlinear diffusion case for all the terms on the left side of the above inequality \((3.36)\). Remaining all the terms can be estimated in the following manner. We begin with
\[
\int_{\Omega_T^2} (|u_j - u| - |u_{\Delta x} - u|) (\varphi_j)_t \ dX \leq \int_{\Omega_T^2} |u_{\Delta x} - u_j| |\psi_t \omega_r \omega_{r_0} + \psi \omega_r \omega'_{r_0}| \ dX
\]
\[
\leq \frac{CaT \Delta x}{r_0} \max_{t \in [0,T]} |u_{\Delta x}|_{B.V.(\mathbb{R})} \\
\leq \frac{C \Delta x}{r_0}.
\]

Next, we continue with
\[
\int_{\Omega_T^2} Q(u_j, u)(\varphi_x - D^- \varphi_j) dX = -\int_{\Omega_T^2} Q(u_j, u)_x \varphi - D^+ Q(u_j, u) \varphi_j dX
\]
\[
= \int_{\Omega_T} \int_0^T \sum_j \int_{x_j}^{x_{j+1}} (\Delta x \varphi \delta_{x_j+1/2}) D^+ Q(u_j, u) - D^+ Q(u_j, u) \varphi_j \ dX.
\]
\[ \leq \frac{C \Delta x}{r} \]

In a similar way, we also can show that

\[ \int_{\Omega_2} \left( Q(u, u_{\Delta x}) - Q(u_j, u) \right) \varphi_x \, dX \leq \frac{C \Delta x}{r} \]

Finally, we end up with

\[ \int_{L-Mt}^{L-M\tau} |u_{\Delta x}(x, \tau) - u(x, \tau)| \, dx \]

\[ \leq \int_{L-Mt}^{L-M\tau} |u_{\Delta x}(x, \nu) - u(x, \nu)| \, dx + C \left( r_0 + r + \frac{\Delta x}{r} + \frac{\Delta x}{r_0} \right). \]

Now setting \( \alpha = r = r_0 \) yields the estimate

\[ \int_{L-Mt}^{L-Mt} |u_{\Delta x}(x, t) - v(x, t)| \, dx \leq C \left( \alpha + \frac{\Delta x}{\alpha} \right). \]

Setting \( \alpha = \Delta x^{1/2} \) proves the corollary. \( \square \)

4. A NUMERICAL TEST

In order to test the unlikely optimality of the convergence rate of our main theorem, we compute the numerical convergence rate of an example. Consider the following initial value problem

\[ \begin{cases} u_t = A(u)_{xx} & \text{for } t > 0 \text{ and } x \in (-\pi/2, \pi), \\ u(x,0) = \sin(x), & x \in [-\pi/2, \pi], \end{cases} \]

supplemented with the boundary conditions

\[ \partial_x A(u(t, x)) = 0 \quad \text{for } t > 0 \text{ and } x = -\pi/2, x = \pi. \]

In order to simplify matters, we have chosen an example without the convective term, nevertheless a discontinuity will form in the solution due to the degeneracy of \( A \). This discontinuity will take the form of a boundary on which \( u = 0 \), moving to the left.

We have used the Euler method to integrate the system of ordinary differential equations (2.9), resulting in the update formula

\[ u_j((n + 1)\Delta t) = u_j(n\Delta t) + (\Delta t)D^- A_j(n\Delta t). \]

For this to be linearly stable, the time-step \( \Delta t \) must obey the restrictive CFL condition \( \Delta t \leq 0.5 \max A'(u)\Delta x^2 \).

In Figure 4.1 we show the solution in the \((x, t)\) plane and a snapshot of \( u \) at \( t = 1 \), for an approximation using 400 grid points in the interval \((-\pi/2, \pi)\).

Finally we computed approximate errors, defined by

\[ 100 \frac{\|u_{\Delta x} - u_{\text{ref}}\|_{L_1}}{\|u_{\Delta x}\|_{L_1}} \]

for a reference solution computed by using our scheme with 4000 grid points. These errors were computed at \( t = 1 \). The result of this is shown in Table 4.1. Looking at this table, it seems the numerical rate of convergence is 1. Thus very far from the rather pessimistic lower bound proved in this paper!
Figure 4.1. An approximate solution to (4.1) using 400 grid points. Left: $u$ in the $(x,t)$ plane for $t \in [0,4]$. Right: an approximation to $u(1,x)$ using 25 grid points, a reference solution computed using 4000 grid points and the initial data.

| $N$ | 25  | 50  | 100 | 200 | 400 | 800 |
|-----|-----|-----|-----|-----|-----|-----|
| error  | 3.62 | 1.55 | 0.82 | 0.40 | 0.18 | 0.07 |
| rate    | -   | 1.22 | 0.92 | 1.02 | 1.11 | 1.42 |

Table 4.1. Numerical errors and convergence rate for the initial value problem (4.1). $N$ is the number of grid points.

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