A new compactification for the scheme of moduli for Gieseker-stable vector bundles with prescribed Hilbert polynomial, on the smooth projective polarized surface \((S, L)\), is constructed. We work over the field \(k = \bar{k}\) of characteristic zero. Families of locally free sheaves on the surface \(S\) are completed with locally free sheaves on schemes which are modifications of \(S\). Gieseker – Maruyama moduli space has a birational morphism onto the new moduli space. We propose the functor for families of pairs "polarized scheme – vector bundle" with moduli space of such type.

**Keywords:** moduli space, semistable coherent sheaves, moduli functor, algebraic surface.

Bibliography: 16 items.

**Introduction**

Let \(S\) be smooth irreducible projective algebraic surface over an algebraically closed field \(k\) of characteristic zero. Fix an ample invertible sheaf \(L \in \text{Pic} S\). We will call it for brevity as polarization of the surface \(S\). In the whole of the text of the present article \(r = \text{rank } E\) is the rank, \(p_E(m) = \chi(E \otimes L^m)/r\) is reduced Hilbert polynomial of the coherent sheaf \(E\) on the scheme \(S\) with respect to the polarization \(L\). As usually, the symbol \(\chi(\cdot)\) denotes the Euler characteristic. We work with the notion of (semi)stability of a coherent sheaf \(E\) on a surface \(S\) in the sense of D.Gieseker [1].

**Definition 1.** Coherent \(\mathcal{O}_S\)-sheaf \(E\) is stable (respectively, semistable), if for any proper subsheaf \(F \subset E\) of rank \(r' = \text{rank } F\) for \(m \gg 0\)

\[
p_E(m) > p_F(m), \quad \text{(respectively, } p_E(m) \geq p_F(m) \text{)}.
\]
It is well-known [2], if the structure of the space of moduli for semistable sheaves depends strongly on the choice of polarization. Analogously, for a sheaf \( \tilde{E} \) of rank \( r \) on a projective scheme \( \tilde{S} \) with polarization \( L \) we have the notation \( p_{\tilde{E}}(m) = \chi(\tilde{E} \otimes \tilde{L}^m)/r \). The Gieseker – Maruyama moduli scheme for semistable torsion-free sheaves on the surface \( S \), with Hilbert polynomial \( rp_{\tilde{E}}(m) \) with respect to \( L \), is denoted by the symbol \( \overline{M} \). It is well-known that this is a projective scheme of finite type over \( k \). Points corresponding to the stable locally free sheaves (vector bundles), form Zariski-open subscheme \( M_0 \) in \( \overline{M} \). Let the scheme \( \overline{M} \) be a fine moduli space. Then there is a trivial product \( \Sigma := \overline{M} \times S \xrightarrow{\pi} \overline{M} \) with a universal family of stable sheaves \( E \). In [4, 5] the projective scheme \( \tilde{M} \) and non-trivial flat family of schemes \( \tilde{\Sigma} \) are constructed. The family \( \tilde{\Sigma} \) is supplied with the family of locally free sheaves \( \tilde{E} \).

Also the birational morphism of schemes \( \Phi : \tilde{\Sigma} \to \Sigma \) such that \( (\Phi^* \tilde{E})^{\vee} = E \), is defined. In [6] the analogous constructions (flat families of schemes \( \tilde{\Sigma}_i \xrightarrow{\tilde{\pi}_i} \tilde{B}_i \) with locally free sheaves \( \tilde{E}_i \) are performed over \( \tilde{B}_i \)). This is done for the case when \( \overline{M} \) carries no universal family of sheaves. In any case the scheme \( \tilde{M} \) contains Zariski-open subscheme which is isomorphic to \( M_0 \). We will call this construction as a standard resolution. To perform the procedure of standard resolution one needs a trivial family \( \Sigma = T \times S \) with reduced base \( T \) and a \( T \)-flat family \( E \) of torsion-free coherent sheaves. Generally, for standard resolution they need not to be semistable. The \( \mathcal{O}_T \times S \)-sheaf \( E \) must be of homological dimension 1. This condition is guaranteed by fibrewise torsion-freeness [9, proof of proposition 4.3].

Besides, the birational morphism \( \Phi : \tilde{\Sigma} \to \Sigma \) done in [4, 5], establishes a correspondence \( (\tilde{S}, \tilde{E}) \mapsto (S, E) \) among pairs \( (\tilde{S}, \tilde{E}) \in \tilde{M} \) and \( (S, E) \in \overline{M} \). The birational morphism \( \phi : \tilde{M} \to \overline{M} \) is also constructed there. Then when working pointwise (fibrewise) we say that a fibre \( \pi^{-1}(y) = S \) of the family \( \Sigma \) is an image of the fibre \( \tilde{\pi}^{-1}(\tilde{y}) = \tilde{S} \). The coherent sheaf \( E \) on the fibre \( S \) is an image of vector bundle \( \tilde{E} \) on the fibre \( \tilde{S} \). Scheme-theoretic description of surfaces \( \tilde{S} \) arising as fibres of flat families \( \tilde{\Sigma} \) is given in [7].

Also note for the further consideration that all the manipulations and reasoning done in [5] with the universal family of stable sheaves \( E \), hold for any its twist \( E \otimes L^m \) by fibrewise ample invertible sheaf \( L \). The same is true [6] for any flat family of (semi)stable sheaves parametrized by a smooth quasi-projective algebraic scheme.

In the present paper we work under assumption that all irreducible components of the Gieseker – Maruyama moduli scheme contain locally free sheaves. This holds asymptotically [8, Theorem 0.3], [9, Theorem D] if the discriminant of sheaves is very big: \( \Delta := c_2 - ((r - 1)/2r)c_1^2 \gg 0 \). In the most general case this is not true.

The purpose of the present article is to develop a functorial approach to the compactification of the moduli scheme for stable vector bundles on the surface, which was built up by the author in [4, 5, 6] using blowups of Fitting ideals. We will refer to the compactification constructed in these articles as to constructive
compactification and will denote it as $\tilde{M}^c$ and its morphism onto Gieseker–Maruyama scheme as $\phi^c : \tilde{M}^c \to \tilde{M}$. The constructive compactification was built up using additional blowups (smooth resolutions, partial resolutions due to Kirwan as well as flattening birational transformation at the formation of image in the Hilbert scheme). It is clear that to supply the constructive compactification with a transparent geometrical meaning as moduli space seems not to be possible. Although we will construct a birational morphism of the constructive compactification onto the projective moduli scheme for pairs polarized projective scheme – vector bundle.

Following [10] ch. 2, sect. 2.2] we recall some definitions. Let $C$ be a category, $C^\circ$ its opposite, $C' = \text{Funct}(C^\circ, \text{Sets})$ – a category of functors to the category of sets. By Yoneda lemma, the functor $\mathcal{C} \to \mathcal{C}' : F \mapsto (\mathcal{E} : X \mapsto \text{Hom}_C(X, F))$ includes $\mathcal{C}$ as a full subcategory in $\mathcal{C}'$.

**Definition 2.** [10] ch. 2, definition 2.2.1] The functor $\bar{\mathcal{F}} \in \text{Ob} C'$ is corepresented by the object $F \in \text{Ob} C$, if there exist $C'$-morphism $\psi : \bar{\mathcal{F}} \to \mathcal{E}$ such that any morphism $\phi' : \bar{\mathcal{F}} \to \mathcal{E}'$ factors through the unique morphism $\omega : \mathcal{E} \to \mathcal{E}'$.

Let $T$ be a scheme over the field $k$. Consider families of semistable pairs

$$\mathfrak{S}_T = \left\{ \begin{array}{l}
\pi : F \to T, \quad \bar{E} \subseteq \text{Pic}F, \quad \forall t \in T \quad \bar{L}_t = \bar{L} |_{\pi^{-1}(t)} \quad \text{is ample;}
\pi^{-1}(t), \bar{L}_t \quad \text{admissible scheme with distinguished polarisation;}
E \quad \text{locally free } \mathcal{O}_T \quad \text{sheaf;}
\chi(\bar{E} \otimes \bar{L}^m) |_{\pi^{-1}(t)} = \nu_{\pi E}(m);
((\pi(t), \bar{L}_t), \bar{E} |_{\pi^{-1}(t)}) \quad \text{(semi)stable pair}
\end{array} \right\},$$

and a functor $\bar{\mathcal{F}} : (\text{Schemes}_k) \to (\text{Sets})$ from the category of $k$-schemes to the category of sets. This functor assigns to any scheme $T$ the set of equivalence classes ($\mathfrak{S}_T / \sim$).

The equivalence relation $\sim$ is defined as follows. The families $((\pi : F \to T, \bar{L}, \bar{E}))$ and $((\pi' : F' \to T, \bar{L}', \bar{E}'))$ of the class $\mathfrak{S}_T$ are said to be equivalent (notation: $((\pi : F \to T, \bar{L}), \bar{E}) \sim ((\pi' : F' \to T, \bar{L}'), \bar{E}')$) if

1) there is an isomorphism $F \sim F'$ such that the diagram

$$\begin{array}{ccc}
F & \sim & F'
\downarrow \pi & \downarrow \pi'
\pi & \sim & F'
\end{array}$$

commutes.

2) There is a linear bundle $L$ on $T$ such that $\bar{E}' = \bar{E} \otimes \pi^* L$.

By technical reason (the construction of standard resolution described in §2 operates with reduced base scheme) we restrict by full subcategory $(\text{RSchemes}_k)$ of reduced schemes in $(\text{Schemes}_k)$. All the reasonings and results of this article are done for the functor $\bar{\mathcal{F}}$ restricted to this subcategory. Also we mean by $\tilde{M}$ the reduced scheme corresponding to Gieseker – Maruyama moduli scheme.
Definition 3. The scheme \( \widetilde{M} \) is a coarse moduli space of the functor \( \mathcal{F} \) if \( \mathcal{F} \) is corepresented by the scheme \( \widetilde{M} \).

Theorem 1. The functor \( \mathcal{F} \) has a coarse moduli space \( \widetilde{M} \) with the following properties:

(i) \( \widetilde{M} \) is projective Noetherian algebraic scheme;
(ii) there is a birational morphism of the union of main components of the Gieseker – Maruyama scheme: \( \kappa: \overline{M} \to \widetilde{M} \);
(iii) there is a birational morphism of the constructive compactification: \( \phi: \overline{M}^c \to \widetilde{M} \);
(iv) there is a commutative triangle of compactifications

\[
\begin{array}{ccc}
\overline{M}^c & \xrightarrow{\phi} & \widetilde{M} \\
\downarrow{\phi} & & \downarrow{\kappa} \\
\overline{M} & \xrightarrow{\kappa} & \widetilde{M}
\end{array}
\]

(v) there is a Zariski-open subscheme \( \widetilde{M}_0 \subset \widetilde{M} \) corresponding to the stable \( S \)-pairs, over which morphisms in the diagram (0.2) are isomorphisms. Namely, \( M_0 \cong \widetilde{M}_0 \cong \widetilde{M}_0 \);
(vi) there is a relation of \( M \)-equivalence defined on the class of semistable pairs, such that pairs are represented by the same point in \( \widetilde{M} \) if and only if they are \( M \)-equivalent.

All the reasoning of the present paper is applicable to any Hilbert polynomial with no relation to the value of discriminant as well as to the number and geometry of irreducible components in the corresponding Gieseker – Maruyama scheme. In general (reducible) case the theorem provides the existence of the coarse moduli space for any maximal (under inclusion) irreducible substack in \( \coprod (\mathcal{F}_T/\sim) \) if it contains pairs \( ((\pi^{-1}(t), \mathcal{L}), \mathcal{E}_{\pi^{-1}(t)}) \) such that \( (\pi^{-1}(t), \mathcal{L}) \cong (S, \mathcal{L}) \). Such pairs will be referred to as \( S \)-pairs. We mean under \( \widetilde{M} \) the moduli space of a substack containing semistable \( S \)-pairs.

Section 1 comprises some results which will be of use in the sequel. Besides, the structure of vector bundle \( E = \mathcal{E}_{\pi^{-1}(t)} \) on the scheme \( S = \pi^{-1}(t) \) is compute under the assumption that the bundle \( E \) is obtained from semistable coherent sheaf by the procedure of articles [4, 5, 6]. Pairs of such view are called as \( dS \)-pairs.

In §2 we turn to the construction of the Gieseker – Maruyama scheme \( \overline{M} \) as GIT-quotient \( \overline{M} = Q/SL(V) \) of an appropriate subscheme \( Q \) in the Grothendieck’s scheme of quotients. Let \( Q \) be the quasiprojective scheme obtained from the scheme \( Q \) by the procedure of papers [4, 5, 6]. We construct a morphism \( \mu \) of scheme \( \overline{Q} \) into the appropriate Hilbert scheme of subschemes in the Grassmann variety \( G(V, r) \). It is proven that the image \( \mu(\overline{Q}) \) is a quasiprojective \( SL(V) \)-invariant subscheme in the Hilbert scheme.
In §3 the explicit view of distinguished polarizations $\tilde{L}$ is compute on schemes $\tilde{S}$.

Section 4 is devoted to the study of the isomorphism $\nu: H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \rightarrow H^0(S, E \otimes L^m)$ of global sections, induced by the procedure of resolution of singularities of semistable sheaves. This isomorphism is used in §§5,6.

In §5 the notion of (semi)stability on the set of pairs polarized scheme – vector bundle $((\tilde{S}, \tilde{L}), \tilde{E})$ is introduced. Also we examine the relation of this new notion of (semi)stability to the classical Gieseker (semi)stability of coherent sheaves on the polarized surface $(S, L)$.

In §6 the notion of M-equivalence is introduced and motivated. Particularly it is shown that S-equivalent semistable coherent sheaves are resolved into M-equivalent semistable pairs.

Section 7 plays an auxiliary role. It contains results concerning with local freeness of subsheaves and quotient sheaves in the inverse image of Jordan – Hölder filtration for sheaves of the form $E = \sigma^* E/tors$.

In §8 we prove the boundedness of families of $dS$-pairs and show that the subscheme formed by $dS$-pairs in the Hilbert scheme coincides with $\mu(Q)$.

Section 9 is devoted to the investigation of the action of the group $PGL(V)$ upon the set of points of Hilbert scheme corresponding to $dS$-pairs. Also we verify the condition of Hilbert – Mumford criterion for the GIT-(semi)stability. The results of this section guarantee the existence, quasiprojectivity and being Noetherian for the scheme $\tilde{M}$ as GIT-quotient. Besides, the existence of the open subscheme $M_0$ in $\tilde{M}$ and its isomorphism to the open subscheme in the Gieseker – Maruyama scheme follows immediately from this section.

In §10 it is shown that there exists a birational morphism of Gieseker – Maruyama scheme onto the scheme $\tilde{M}$. This proves the projectivity of the scheme $\tilde{M}$.

In §11 we study the relation of M-equivalence of semistable pairs to GIT-equivalence of the corresponding points in Hilbert scheme.

Finally, in §12 we prove that the scheme $\tilde{M}$ constructed is indeed the moduli space for semistable pairs.

1 Coherent sheaves and their resolutions

This section plays auxiliary role. It contains results from author’s previous papers which are necessary in the sequel. Also we deduce some corollaries of these results.

Remember some formulations and results from [7]. Since not all algebraic schemes of the present paper are varieties, the ample divisor class $H$ used in [7] is replaced with the correspondent ample invertible sheaf $L$. For convenience of computations we suppose that $L$ is very ample. If this is not so, we replace the sheaf $L$ with its very ample tensor power.

**Definition 4.** [7] Artinian sheaf $\pi$ is said to be $(S, L, r, p_E(m))$-admissible if
there is an exact $\mathcal{O}_S$-triple

$$0 \to E \to E^{\vee \vee} \to \mathcal{X} \to 0,$$

(1.1)

where the coherent sheaf $E$ of rank $r$ with Hilbert polynomial $rp_E(m)$ is semistable with respect to the polarization $L$.

The relation of $(S, L, r, p_E(m))$-admissibility to Fitting ideal sheaves $\mathcal{F}itt^{0} \mathcal{E}xt^1(E, \mathcal{O}_S)$ for semistable coherent sheaves $E$ is given by the following proposition.

**Proposition 1.** [7] The class of all $\mathcal{F}itt^{0} \mathcal{E}xt^1(E, \mathcal{O}_S)$ for semistable sheaves $E$ is contained in the class of all sheaves of the view $\mathcal{F}itt^{0} \mathcal{E}xt^2(\mathcal{X}, \mathcal{O}_S)$ for all Artinian quotient sheaves $\mathcal{X}$ of length $l = l(\mathcal{X}) = h^0(S, \mathcal{X}) < c_2$ of the sheaf $\bigoplus \mathcal{O}_S$.

Let $I$ be the sheaf of ideals of some zero-dimensional subscheme $Z$ on the surface $S$ and $t$ be a symbol which is transcendental over $\mathcal{O}_S(U)$ for all open $U \subset S$. Let $\mathcal{O}_S[t]$ be the sheaf of polynomial algebras over $\mathcal{O}_S$ and $I[t]$ its subsheaf of ideals defined by the correspondence $U \to I(U)[t]$. The symbol from the right stands for the polynomial ring (without unity) over $I(U)$. Also consider the principal ideal subsheaf $(t) \subset \mathcal{O}_S[t]$ and the sum of ideal subsheaves $I[t] + (t)$.

Form $s$-th power $(I[t] + (t))^s$, $s > 0$, as a subsheaf of ideals in $\mathcal{O}_S[t]$. It contains $(t)^{s+1}$ as a submodule generated by the element $t^{s+1}$. Then the quotient module $(I[t] + (t))^s/(t)^{s+1}$ is defined. For $s = 0$ set $(I[t] + (t))^s/(t)^{s+1} := \mathcal{O}_S$. Now form a graded $\mathcal{O}_S$-algebra $\bigoplus_{s \geq 0} (I[t] + (t))^s/(t)^{s+1}$.

The $\mathcal{O}_S$-module morphism $\mathcal{O}_S \to \bigoplus_{s \geq 0} (I[t] + (t))^s/(t)^{s+1}$ leads to the morphism of projective schemes $\sigma : \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s/(t)^{s+1} \to S$. This morphism will be referred to as canonical. As shown in [7], the scheme $\text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s/(t)^{s+1}$ is obtained as a fibre of the composite map $\widetilde{T \times S} \xrightarrow{\sigma} T \times S \xrightarrow{\pi} T$ for $T = \text{Spec} k[t] \cong \mathbb{A}^1_k$ and $\sigma$ be a morphism of blowing up of the trivial family of surfaces $T \times S$ on the sheaf of ideals $I = I[t] + (t)$. This sheaf if ideals defines the subscheme $Z$ in the fibre $\pi^{-1}(0) \cong S$.

**Definition 5.** Polarized algebraic scheme $(\widetilde{S}, \tilde{L})$ is called $(S, L, r, p_E(m))$-admissible if the scheme $(S, L)$ satisfies one of the following conditions

i) $(\widetilde{S}, \tilde{L}) \cong (S, L)$,

ii) $\widetilde{S} \cong \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s/(t)^{s+1}$, where $I = \mathcal{F}itt^{0} \mathcal{E}xt^{2}(\mathcal{X}, \mathcal{O}_S)$ for Artinian quotient sheaf $q : \bigoplus \mathcal{O}_S \to \mathcal{X}$ of length $l(\mathcal{X}) \leq c_2$ and $\tilde{L} = L^m \otimes (\sigma^{-1} I \cdot \mathcal{O}_S)$ for some $m \gg 0$. In this case for any $m$ when the sheaf $L^m \otimes (\sigma^{-1} I \cdot \mathcal{O}_S)$ is very ample, the polarization $\tilde{L}^m$ is called distinguished.

In the present paper the parameters $(S, L, r, p_E(m))$ are fixed. We will call for brevity $(S, L, r, p_E(m))$-admissible schemes and $(S, L, r, p_E(m))$-admissible
Artinian sheaves as simply admissible schemes and admissible Artinian sheaves respectively.

By the constructive built up of the Fitting compactification in [4, 5, 6] admissible schemes include into one or finite collection of flat families immersed as locally closed subschemes into the projective scheme (for example, into the universal subscheme of the Hilbert scheme). Then there exists positive integer \( m_0 \) such that for all \( m \geq m_0 \) and for all isomorphism classes of schemes \( \tilde{S} \) sheaves \( L^m \otimes (\sigma^{-1} I \cdot \mathcal{O}_{\tilde{S}}) \) are very ample.

Now redenote \( L^m \) for \( L \) and \( \tilde{L}^m \) for \( \tilde{L} \). It is shown in [2] that the class of (semi)stable coherent sheaves is invariant under the change of the very ample invertible sheaf by its tensor power.

Since in the present paper the parameters \((S, L, r, p_E(m))\) are fixed then we will refer for brevity to \((S, L, r, p_E(m))\)-admissible schemes and \((S, L, r, p_E(m))\)-admissible Artinian sheaves as to simply admissible schemes and admissible sheaves respectively.

It is clear that admissible scheme of the view \( \tilde{S} = \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t)) s / (t s + 1) \) can be naturally represented as a union of irreducible components \( \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \) where the main component \( \tilde{S}_0 = \text{Proj} \bigoplus_{s \geq 0} (I) s \) is the blowup of the surface \( S \) in the sheaf of ideals \( I \) and for \( i > 0 \) \( \tilde{S}_i \) are irreducible additional components \( \bigcup_{i > 0} \tilde{S}_i \). As it is shown in [7], in this case the additional component can have a structure of nonreduced scheme. Obviously, admissible scheme consists of a single component \( S \cong \tilde{S} = \tilde{S}_0 \) if and only if it is isomorphic to the initial surface \( S \).

An admissible scheme \( \tilde{S} \) has a morphism \( \sigma : \tilde{S} \to S \). In this case the restriction \( \sigma_0 = \sigma|_{\tilde{S}_0} : \tilde{S}_0 \to S \) of the canonical morphism \( \sigma \) onto the main component \( \tilde{S}_0 \) is a blowup morphism.

A coherent torsion-free sheaf \( E \) is said to be deformation equivalent to a locally free sheaf if \( E \) can be include into a flat family of \( \mathcal{O}_S \)-sheaves \( E \) over a connected base \( T \) and restrictions of \( E \) on fibres of view \( t \times S \) for \( t \in T \) is general enough, are locally free.

**Proposition 2.** Let the coherent sheaf \( E \) is deformation equivalent to a locally free sheaf and is an image of vector bundle \( \tilde{E} \) on an admissible scheme \( \tilde{S} \). Then there is a canonically defined subsheaf tors \( \subset \sigma^* E \) such that \( \tilde{E} \cong \sigma^* E / \text{tors} \).

**Proof.** Consider a flat family \( E \) of semistable sheaves on the surface \( S \). Let \( T \) be the base of the family. By the construction developed in [4, 5] it has homological dimension equal to 1. Fix an exact sequence

\[
0 \to E_1 \to E_0 \to E \to 0,
\]

where the sheaves \( E_0 \) and \( E_1 \) are locally free. Consider the morphism of blowing up \( \sigma : \tilde{T} \times \tilde{S} \to T \times S \) of the coherent sheaf of ideals \( I = \text{Fitt}^0 \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{T \times S}) \). Applying the dualisation and inverse image under the morphism \( \sigma \) to (1.2) we define sheaves \( A = \ker (\sigma^* E_1^\vee \to \sigma^* \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_{T \times S})) \) and \( \tilde{E} = (\ker (\sigma^* E_0^\vee \to \)
A)\). According to [5] they are locally free. There is a following exact diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A' & \rightarrow & \sigma^*E_0 & \rightarrow & \hat{E} & \rightarrow & 0 \\
\tau & \uparrow & & & \downarrow & & \downarrow & & \uparrow \\
0 & \rightarrow & \sigma^*E_1 & \rightarrow & \sigma^*E_0 & \rightarrow & \sigma^*E & \rightarrow & 0 \\
0 & & & & & & & & 0
\end{array}
\]

where the symbol \(\tau\) denotes the torsion \(O_{T \times S}\)-sheaf \(\mathcal{E}xt^1(\sigma^*E, O_{T \times S}, \mathcal{O}_{\hat{T} \times \hat{S}})\). The right vertical triple in (1.3) leads to the expression \(\hat{E} = \sigma^*E/\tau\). Consider the restriction of this equality to the fibre \(pr^{-1}(t) = \hat{S}\) of the composite map \(pr : \hat{T} \times \hat{S} \rightarrow T \times S \rightarrow T\) over a closed point \(t \in T\). Let \(i_t : pr^{-1}(t) \hookrightarrow \hat{T} \times \hat{S}\) be the morphism of inclusion of the fibre. Then there are a morphism \(\sigma : \hat{S} \rightarrow S\) and isomorphisms \(i_t^*\sigma^*E = \sigma^*E\) and \(i_t^*\hat{E} = \hat{E}\). Since the sheaf \(\hat{E}\) is locally free as a sheaf of \(O_{\hat{T} \times \hat{S}}\)-modules then the sheaf \(\hat{E}\) is also locally free as a sheaf of \(O_S\)-modules.

In the case when \(S\) is a reduced scheme the restriction of right vertical triple in (1.3) on the fibre \(\hat{S}\) gives an isomorphism \(\hat{E} = \sigma^*E/tors\).

In the general case of (possibly nonreduced) scheme \(\hat{S}\) consider its decomposition into the union of irreducible components \(\hat{S} = \bigcup_{i \geq 0} \hat{S}_i\), for \(\hat{S}_0\) being its principal component. It has a structure of a reduced scheme. Additional components \(\hat{S}_i, i > 0\), can be nonreduced. Let \(U\) be a Zariski open subset of one of components \(\hat{S}_i, i \geq 0\), and let \(\sigma^*E|_{\hat{S}_i}(U)\) be the corresponding group of sections. This group is \(O_{\hat{S}_i}(U)\)-module. Consider sections \(s \in \sigma^*E|_{\hat{S}_i}(U)\) which are annihilated by prime ideals of positive codimension in \(O_{\hat{S}_i}(U)\). They constitute a submodule in \(\sigma^*E|_{\hat{S}_i}(U)\). This submodule will be referred to as \(tors_i(U)\). The correspondence \(U \rightarrow tors_i(U)\) defines the subsheaf \(tors_i \subset \sigma^*E|_{\hat{S}_i}\). Note that associated primes of positive codimension annihilating sections \(s \in \sigma^*E|_{\hat{S}_i}(U)\), correspond to subschemes supported in \(\sigma^{-1}(\text{Supp } s) = \bigcup_{i > 0} \hat{S}_i\). Since by the construction the scheme \(\hat{S}\) is connected then the sheaves \(tors_i, i \geq 0\) permit to construct a subsheaf \(tors \subset \sigma^*E\). It is defined as follows. The section \(s \in \sigma^*E|_{\hat{S}_i}(U)\) satisfies \(s \in tors|_{\hat{S}_i}(U)\) if and only if

- there is a section \(y \in O_{\hat{S}_i}(U)\) such that \(ys = 0\),
Hence we conclude that ker $\sigma \subset \tau$ is analogous to the role of torsion subsheaf in the case of reduced and irreducible basis scheme. Since there is no confusion the symbol $\tau$ is understood everywhere as described and the subsheaf $\tau$ is called a torsion subsheaf.

Consider the epimorphism $\varpi : \sigma^*E \to \hat{E}$ defined by the restriction of the epimorphism $\sigma^*E \to \hat{E}$ to the fibre $\hat{S}$. It is clear that $\tau \subset \ker \varpi$. Note that $\ker \varpi$ is annihilated by local sections which belong to ideals of positive codimension on a component $\tilde{S}_i$, $i > 0$, or when restricted to the component $S_0$. Hence we conclude that $\ker \varpi \subset \tau$ and $E = \sigma^*E/\tau$, as required. \[ \square \]

In \cite{7} it is shown that the fibre of the composite morphism $\prod \times \hat{S} \xrightarrow{\sigma} T \times S \xrightarrow{\pi} T$ at the point $t \in T$ has a structure of the scheme $\hat{S} = \text{Proj} \left( \bigoplus_{s \geq 0} (I[t] + (t)^s)/(t^{s+1}) \right)$ for $I = \text{Fit}t^0E_{xt}^1(\mathbb{E}_t, \mathcal{O}_{t\times S})$.

The behavior of vector bundles $\hat{E}$ on additional components $\tilde{S}_i \subset \hat{S}$, $i > 0$, is given by the following simple computation. The standard exact triple (1.1) is taken by the functor of the inverse image $\sigma^*_i$ to the exact sequence

$$
\cdots \to \text{Tor}_1^{\sigma^*_i \mathcal{O}_S}(\sigma^{-1} \mathcal{x}, \mathcal{O}_{\tilde{S}_i}) \to \sigma^*_i E \to \sigma^*_i E^\vee \to \sigma^*_i \mathcal{x} \to 0. \tag{1.4}
$$

In an appropriate neighborhood $U \subset \mathcal{S}$ of the support $\text{Supp} \mathcal{x}$ the locally free sheaf $E^\vee|_U$ can be replaced by its local trivialization $\mathcal{O}^{\tilde{U}}_U$. Then the exact sequence (1.4) takes the view

$$
\cdots \to \text{Tor}_1^{\sigma^*_i \mathcal{O}_S}(\sigma^{-1} \mathcal{x}, \mathcal{O}_{\tilde{S}_i}) \to \sigma^*_i E \to \sigma^*_i \mathcal{O}^{\tilde{U}} \to \sigma^*_i \mathcal{x} \to 0.
$$

Consequently for $\tilde{E}_i = \sigma^*E/\tau|_{\tilde{S}_i} = \sigma^*_i E/\tau$, we have

$$
\cdots \to \sigma^*_i E/\tau|_{\tilde{S}_i} \to \sigma^*_i \mathcal{O}^{\tilde{U}} \to \sigma^*_i \mathcal{x} \to 0, \tag{1.5}
$$

where the subsheaf $\tau$ on (possibly, nonreduced) scheme $\tilde{S}_i$ is defined as before and $\tau|_{\tilde{S}_i} = \tau|_{\tilde{S}_i}$. Dots from the left hand side indicate the terms violating exactness. These terms are not obliged to have positive codimension in $\tilde{S}_i$.

**Example 1.** Let $\mathcal{x} = k_x$, then $\tilde{S}$ consists of two reduced components: $\tilde{S}_0$ is the surface obtained by blowing up of $\mathcal{S}$ in the reduced point $x$, and $\tilde{S}_1 \cong \mathbb{P}^2$. The morphism $\sigma_1$ is a constant morphism $\sigma_1 : \mathbb{P}^2 \to x$. Then $\sigma^*_1 \mathcal{x} = \sigma^*_1 k_x = \mathcal{O}_{\mathbb{P}^2}$, and easy counting of ranks gives $\text{rank} \ker (\sigma^*_1 E/\tau|_{\tilde{S}_1} \to \mathcal{O}^{\tilde{U}}_{\tilde{S}_1}) = 1$.

Since the sheaf $\mathcal{x}$ is supported in the finite collection of points then the morphism $\mathcal{O}^{\tilde{U}}_{\tilde{S}_i} \to \mathcal{x}$ can be replaced by the morphism $\mathcal{O}^{\tilde{U}}_{\tilde{S}_i} \to \mathcal{x}$.
Let $q_0 : \mathcal{O}_S^{\oplus r} \to \pi$ be the morphism induced by the exact triple (1.1). One has
\[
\tilde{E}_i = \sigma^*_i \ker q_0 / \text{tors}.
\] (1.6)

By the Proposition 2, for all semistable coherent sheaves $E$ with fixed Hilbert polynomial $rp_E(m)$, all sheaves $\tilde{E}_i$ on additional components $\tilde{S}_i$ can be described by the relations (1.6) for appropriate $q_0 \in \coprod_{l \leq c^2} \text{Quot}^l \mathcal{O}_S^{\oplus r}$.

2 Standard resolution and Grassmannians

The procedure of transformation of a flat family $\mathcal{E}$ of coherent torsion-free sheaves on the surface $S$ over (quasi)projective base $T$ into the flat family of schemes over the base $\tilde{T}$ with a locally free sheaf $\tilde{\mathcal{E}}$, is given and motivated in [4] – [6]. The scheme $\tilde{T}$ is birational to $T$.

To proceed further it is necessary to turn to the construction of Gieseker–Maruyama scheme using geometric invariant theory. Let $E$ be a semistable coherent torsion-free sheaf with Hilbert polynomial equal to $rp_E(t)$. Also let $H^0(S, E \otimes L^m) = V$ be $k$-vector space of global sections and the sheaf $E \otimes L^m$ is assumed to be globally generated. Consider Grothendieck’s Quot-scheme $\text{Quot}^{rp_E(t)}(V \otimes L^{(-m)})$ parameterizing quotient sheaves of the form
\[
V \otimes L^{(-m)} \to E,
\] (2.1)
with Hilbert polynomial equal to $\chi(E \otimes L^t) = rp_E(t)$. Families of Gieseker-semistable coherent sheaves $E$ on the surface $S$ with fixed Hilbert polynomial $rp_E(t)$ are bounded. Then there exists integer $m_0$ such that for $m > m_0$ all the sheaves $E \otimes L^m$ are globally generated and all vector spaces $H^0(S, E \otimes L^m)$ are of dimension $rp_E(m)$. This $m_0$ is common for all $E$. Then all semistable coherent sheaves $E$ under consideration can be interpreted as quotient sheaves of the form (2.1). The projective scheme $\text{Quot}^{rp_E(t)}(V \otimes L^{(-m)})$ contains a quasiprojective subscheme $Q'$ of points corresponding to Gieseker-semistable quotient sheaves $E$ in (2.1) with an isomorphism $H^0(S, E \otimes L^m) \cong V$. The scheme $\overline{M}$ is obtained as GIT-quotient of a subset of GIT-semistable points in the quasiprojective subscheme $Q' \subset \text{Quot}^{rp_E(t)}(V \otimes L^{(-m)})$ by the action of the group $PGL(V)$. This action is induced by choices of bases in the space $V$. Let $Q$ be the component or the union of those components in $Q'$ which correspond to the components in $\overline{M}$ containing locally free sheaves. The scheme $\text{Quot}^{rp_E(t)}(V \otimes L^{(-m)})$ is supplied with the universal family of quotient sheaves $\mathcal{E}_{\text{Quot}}$. Let $E_Q := E_{\text{Quot}} | q$ be its restriction on the subscheme $Q$.

The procedures of the paper [6] are applicable to the pair $(Q, E_Q)$. In particular, set the base of the family of sheaves is taken to be the quasiprojective scheme $Q$. Repeating the proof of the proposition 1.2 in [6] and the constructions of §2 in [6] we obtain the following objects:

- $\tilde{Q}$ – quasiprojective scheme,
• $\phi : \tilde{Q} \to Q$ – birational projective morphism,
• $\tilde{\Sigma}_Q \to \tilde{Q}$ – flat family of schemes,
• $\tilde{E}_Q$ – locally free sheaf on the scheme $\tilde{\Sigma}_Q$,
• $\Phi : \tilde{\Sigma}_Q \to \Sigma_Q$ – birational projective morphism onto the product $\Sigma_Q = Q \times S$ and $(\Phi_*E_Q)^{\vee \vee} = E_Q$ (the analog of the proposition 2.9 in [1]).
• for the immersion $\tilde{\Delta} = \tilde{Q} \times S \to \tilde{Q} \times \Sigma_Q$ defined by the closure of the image of the diagonal immersion $Q_0 \times S \hookrightarrow Q_0 \times \pi^{-1}(Q_0)$, there is the following explicit description of the morphism $\phi$ (the analog of the corollary 2.10 in [1]):

$$\phi : \tilde{Q} \to Q : \tilde{y} \mapsto ((id_{\tilde{Q}} , \Phi)_* O_{\tilde{Q}} \boxtimes \tilde{E}_Q)^{\vee \vee} |_{\tilde{\Delta} \times \tilde{y} \times S}$$

Everywhere in the further text the invertible sheaf $\tilde{L}_Q \in \text{Pic} \tilde{\Sigma}_Q$ is assumed to be very ample relatively to the base $\tilde{Q}$. Also the fibres of the family $\tilde{\Sigma}_Q$ are mentioned to have constant Hilbert polynomial if it is compute relatively to $\tilde{L}_Q$. Moreover, the Euler characteristic $\chi(\tilde{E}_Q \otimes \tilde{L}_Q^m |_{\pi^{-1}(\tilde{y})})$ does not depend on the choice of the point $\tilde{q} \in \tilde{Q}$.

We assume that the sheaf $\tilde{E}_Q \otimes \tilde{L}_Q^m$ is globally generated on fibres of the morphism $\pi$. Then $\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)$ is a locally free sheaf of rank $rp(E,m)$ on the scheme $\tilde{Q}$. Form a Grassmanian bundle $\text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m),r)$ of $r$-dimensional quotient spaces in the fibres of vector bundle $\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)$. The fibre of Grassmanian bundle $\text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m),r)$ is isomorphic to the ordinary Grassmanian $G(V,r)$.

Vector bundle $\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)$ is locally trivial. Fix any finite trivializing open cover $\tilde{Q} = \bigcup_i U_i$ together with the trivializing isomorphisms

$$\tau_i : \pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)|_{U_i} \xrightarrow{\sim} V \otimes O_{U_i}.$$ 

Then there is the induced trivialization of the Grassmannian bundle $\text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m),r)$:

$$\text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m),r)|_{U_i} \cong \text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)|_{U_i},r) \xrightarrow{\sim} \text{Grass}(V \otimes O_{U_i},r) \cong G(V,r) \times U_i,$$

where the arrow is induced by the morphism $\tau_i^{-1}$. The gluing data defined on the overlaps $U_{ij} = U_i \cap U_j$ by the composite maps $\varphi_{ij} = \text{Grass}(\tau_i) \circ \text{Grass}(\tau_j)^{-1} : G(V,r) \times U_{ij} \to \text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)|_{U_{ij}},r) \to G(V,r) \times U_{ij}$ are given by the elements of group $GL(V)$. These elements act upon the space $V$ by linear
The corresponding action of the same elements on the Grassmannian $G(V, r)$ factors through the action of projective group $PGL(V)$.

For any element $U_i$ of the trivializing cover of the scheme $Q$ and for the restriction $\Sigma_i = \Sigma_Q|_{U_i}$ of the family $\Sigma_Q$ there is a morphism $\Sigma_i \to \text{Grass}(\pi_*(\mathbb{E}_Q \otimes \mathbb{L}_Q(r))|_{U_i})$. After the restriction on the fibres of the structure morphism of the Grassmannian for $m \gg 0$ it becomes an immersion. The application of the trivializing isomorphism $\text{Grass}(\tau)$ leads to the commutative diagram

\[
\begin{array}{ccc}
\Sigma_i & \xrightarrow{\pi_i} & G(V, r) \times U_i \\
\downarrow & & \downarrow^{pr_1} \\
U_i & & G(V, r)
\end{array}
\]

The horizontal composite map restricted on any fibre $\tilde{S}_q = \pi_i^{-1}(q)$ of the projection $\pi_i$, gives an immersion $j_q : \tilde{S}_q \hookrightarrow G(V, r)$ of the scheme $\tilde{S}_q$ into the Grassmannian $G(V, r)$. The image $j_q(\tilde{S}_q)$ is a closed subscheme in $G(V, r)$. Hilbert polynomial of this subscheme $\chi(j_q^*O_{G(V, r)}(I))$ is constant for all fibres, for all elements of cover and is uniform for all trivializing covers. We denote it $P(t)$.

Let $\text{Hilb}^{P(t)}(V, r)$ be the Hilbert scheme of subschemes in $G(V, r)$ with Hilbert polynomial equal to $P(t)$. For any element $U_i$ of the trivializing cover there is a map $\tilde{\mu}_i : \Sigma_i \to \text{Univ}^{P(t)}(G(V, r))$ into the universal subscheme

\[
\text{Univ}^{P(t)}(G(V, r)) \subset \text{Hilb}^{P(t)}(G(V, r) \times G(V, r).
\]

Also there is a mapping $\mu_i : U_i \to \text{Hilb}^{P(t)}(G(V, r))$ of the base $U_i$ into the Hilbert scheme. These morphisms include into the fibred square

\[
\begin{array}{ccc}
\Sigma_i & \xrightarrow{\tilde{\mu}_i} & \text{Univ}^{P(t)}(G(V, r) \\
\downarrow & & \downarrow^{\pi_H} \\
U_i & \xrightarrow{\mu_i} & \text{Hilb}^{P(t)}(G(V, r)
\end{array}
\]

**Remark 1.** Since $U_i$ are quasiprojective schemes and $\text{Hilb}^{P(t)}G(V, r)$ is a projective scheme, then $\mu_i$ are projective morphisms.

Let $T$ be a regular scheme of dimension 1. We introduce the notation $\Sigma = T \times S$. Let also $E$ be a flat family of coherent semistable torsion-free sheaves on the surface $S$ with fixed Hilbert polynomial $rP_E(t)$. We suppose that $E$ is parameterized by $T$. Let $T_0 \subset T$ be nonempty open subset such that the sheaf $E_0 = E|_{\Sigma_0}$ is locally free on the preimage $\pi^{-1}(T_0) = \Sigma_0 \subset \Sigma$. Consider a blowing up $\Sigma : \Sigma_0 \to \Sigma$ of the sheaf of ideals $I = Filt^PFilt^T(E, O_S)$. The composite map $\Sigma \xrightarrow{\pi} \Sigma \xrightarrow{\pi} T$ is a flat projective morphism because its fibres are projective schemes, $\Sigma$ is an irreducible scheme and $T$ is regular scheme of dimension 1.
It is clear that \( \Sigma_0 = \sigma^{-1}\Sigma_0 \cong \Sigma_0 \). For any \( x \in T_0 \) the sheaf \( \mathbb{E}_0|_{x \times S} = E_x \) induces an immersion \( j_x : S \hookrightarrow G(V,r) \). The composite of this immersion with a Plücker immersion \( \text{Pl} \) is very ample. It follows by the construction of the morphism \( \text{Pl} \) that \( L_E = j_x^*\mathcal{O}_{G(V,r)}(1) = j_x^*\text{Pl}^*\mathcal{O}_{P(\Lambda^r V)}(1) \). The composite map \( j_x \circ \text{Pl} : S \hookrightarrow P(\Lambda^r V) \) is given by the composite morphism of sheaves

\[
\bigwedge^r V \otimes \mathcal{O}_S \xrightarrow{\sim} \bigwedge^r H^0(S, E \otimes L^m) \otimes \mathcal{O}_S \\
\xrightarrow{\sim} \bigwedge^r H^0(S, E \otimes L^m) \otimes \mathcal{O}_S \to \bigwedge^r (E \otimes L^m).
\]

In this case, the epimorphism of two recent maps is provided by the choice of \( m \gg 0 \). The same reason guarantees the fact that the sheaf \( \bigwedge^r (E \otimes L^m) \) is very ample. It follows by the construction of the morphism \( j_x \circ \text{Pl} \) that \( L_E = \bigwedge^r (E \otimes L^m) = E \otimes L^m \otimes \det E \). In the further text we will not distinguish in the notation the first Chern class \( c_1 = c_1(E) \) and its image in the Picard group of the surface \( S \). For example we write \( L_E^{\otimes r} = L^{mr} \otimes c_1 \).

The subscheme \( \Sigma_0 = T_0 \times S \) has an immersion into the relative projective space \( P(\bigwedge^r \pi_*(\mathbb{E}_0 \otimes L^m_0)) \cong P(\bigwedge^r V \times T_0) \). This immersion is given by the sheaf \( L_E = L_E \boxtimes \mathcal{O}_{T_0} \) very ample relatively \( T_0 \), and includes in commutative diagram

\[
\Sigma_0 \xrightarrow{\pi} P(\bigwedge^r \pi_*(\mathbb{E}_0 \otimes L^m_0)) \xrightarrow{\psi} T_0
\]

Here and further \( L_0 := L \boxtimes \mathcal{O}_{T_0} \) is an invertible \( \mathcal{O}_{\Sigma_0} \)-sheaf very ample relatively \( T_0 \).

Note that the scheme \( \hat{\Sigma} \) is supplied with the locally free sheaf \( \widehat{\mathbb{E}} \). It is flat over the base \( T \).

Also note that for \( m \gg 0 \) there exists an epimorphism \( V \otimes L_0^{(-m)} \to \mathbb{E}_0 \). Fix it. Hence there is a morphism \( \nu_0 : T_0 \to \text{Quot}_{P_E(t)}(V \otimes L^{(-m)}) \) of the base scheme \( T_0 \) into the Grothendieck Quot-scheme \( \text{Quot}_{P_E(t)}(V \otimes L^{(-m)}) \). Since the sheaf \( \mathbb{E} \) is fibrewise semistable we assume without loss of generality that \( \nu_0(T_0) \subset Q \). Let \( \hat{T} = T \times_Q \hat{Q} \) and \( \nu : \hat{T} \to \hat{Q} \) be the corresponding morphism. Then for the preimage \( \Sigma = \hat{\Sigma} \times_T \hat{T} \) of the family \( \hat{\Sigma} \) by the universality of the Hilbert scheme \( \text{Hilb}^{P(t)} \text{Grass}(\pi_*(\mathbb{E}_Q \otimes \hat{L}_Q^m), r) \) one has a fibred diagram

\[
\text{Univ}^{P(t)} \text{Grass}(\pi_*(\mathbb{E}_Q \otimes \hat{L}_Q^m), r) \xrightarrow{\tilde{\mu}} \hat{\Sigma}_Q \xleftarrow{\pi} \hat{\Sigma} \xrightarrow{\nu} \hat{Q} \xrightarrow{\nu} \hat{T}
\]

Let \( \hat{L} = \tilde{\nu}^*\hat{L}_Q \). One-dimensional base \( T \) does not undergo a birational transformation, namely, \( \hat{T} \cong T \).
Since it is clear that \( L_0 \cong \tilde{L}|_{\Sigma_0} \), then the combination of the diagram (2.2) with the open immersion \( P(\Lambda^r \pi_*(E_0 \otimes L_0^m)) \hookrightarrow P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m)) \) and the formation of closure \( \Sigma_0 \) for the image of the scheme \( \Sigma_0 \) in projective bundle \( P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m)) \), lead to the commutative diagram

\[
\Sigma_0 \xrightarrow{\sim} P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m))
\]

Here the scheme \( \Sigma_0 \) coincides on the open subset with the image of the scheme \( \Sigma \). Since both schemes are irreducible and flat over the same base \( \tilde{T} \), the image \( \Sigma \) coincides with \( \Sigma_0 \). This proves that one-dimensional flat family \( \Sigma \) can be considered as the closure of the image for its open subscheme \( \Sigma_0 \) under the immersion into the projective bundle \( P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m)) \).

For the comparison of the structure of the families \( \Sigma \) and \( \Sigma_0 \) and polarizations on their fibres we construct the immersions of both families into the same relative projective space. Firstly consider the composite of the immersion into the relative Grassmannian with the Plücker imbedding, and a commutative triangle

\[
\tilde{\Sigma}_Q \xrightarrow{j} \text{Grass}(\pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m), r) \xrightarrow{P_1} P(\Lambda^r \pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m))
\]

Here \( \pi \) is flat morphism, \( \tilde{L}_Q^m \) an invertible sheaf very ample relatively \( \tilde{Q} \). Secondly, the morphism \( \nu \) induces a fibred square

\[
P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m)) \xrightarrow{\nu} P(\Lambda^r \pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m))
\]

The combination of two recent diagrams

\[
\tilde{\Sigma}_Q \xrightarrow{\text{proj}} P(\Lambda^r \pi_*(\tilde{E}_Q \otimes \tilde{L}_Q^m)) \quad (2.3)
\]

\[
\Sigma \xrightarrow{\pi} P(\Lambda^r \pi_*(\tilde{E} \otimes \tilde{L}^m)) \rightarrow \tilde{Q}
\]
shows that to compute the distinguished polarizations on fibres of the family \( \tilde{\Sigma}_Q \) one can use one-parameter families of the form \( \tilde{\Sigma} \to \tilde{T} \).

**Proposition 3.** There exist \( PGL(V) \)-invariant morphisms

\[
\tilde{\mu} : \tilde{\Sigma}_Q \to \text{Univ}^{P(t)}G(V, r), \quad \mu : \tilde{Q} \to \text{Hilb}^{P(t)}G(V, r)
\]

include into the fibred diagram

\[
\begin{array}{ccc}
\tilde{\Sigma}_Q & \xrightarrow{\tilde{\mu}} & \text{Univ}^{P(t)}G(V, r) \\
\pi \downarrow & & \downarrow \pi \\
\tilde{\Sigma}_i & \xrightarrow{\tilde{\mu}_i} & \text{Univ}^{P(t)}G(V, r) \\
\pi_i \downarrow & & \downarrow \pi_i \\
U_i & \xrightarrow{\mu_i} & \text{Hilb}^{P(t)}G(V, r)
\end{array}
\]

(2.4)

**Proof.** Refining the cover \( U_i \), if necessary, we achieve that every element \( U_i \) has an image \( U'_i = \phi(U_i) \) which is open in \( Q \) and \( U_i = \phi^{-1}(U'_i) \) where subschemes \( U'_i \) form a cover for the scheme \( Q \). As previously, the covering by the schemes \( U_i \) carries the trivialization of locally free sheaf \( \pi^*(\tilde{E}_Q \otimes \tilde{L}_Q^m) \).

Schemes \( Q \subset \text{Quot}^{rt_{P(t)}}(V \otimes L^{(-m)}) \) and \( \text{Hilb}^{P(t)}G(V, r) \) are supplied with actions of the group \( PGL(V) \). These actions are induced by linear transformations of vector space \( V \). Let

\[
\begin{align*}
\alpha : PGL(V) \times Q & \to Q \\
\beta : PGL(V) \times \text{Hilb}^{P(t)}G(V, r) & \to \text{Hilb}^{P(t)}G(V, r)
\end{align*}
\]

be corresponding morphisms. We will denote restrictions of these morphisms on any subsets by the same symbols. The abbreviation \( PGL(V) \ast U \) denotes \( \alpha(U \times PGL(V)) \) or \( \beta(U \times PGL(V)) \) respectively. It is clear that sets of the form \( PGL(V) \ast U \) are \( PGL(V) \)-invariant.

Although open subschemes \( U'_i \) and \( U_i \) a priori are not \( PGL(V) \)-invariant, they generate the following fibred diagram

\[
\begin{array}{ccc}
PGL(V) \times U'_i & \xrightarrow{\alpha} & PGL(V) \ast U'_i \\
\downarrow 1 \times \phi & & \downarrow \phi \\
PGL(V) \times U_i & \xrightarrow{\alpha'} & PGL(V) \ast U_i \\
\downarrow 1 \times \mu_i & & \downarrow \mu_i \\
PGL(V) \times \mu_i(U_i) & \xrightarrow{\beta} & PGL(V) \ast \mu_i(U_i)
\end{array}
\]

(2.5)
where $\alpha' = \beta'$ by the definition of actions $\alpha$ and $\beta$. Hence the scheme $\widetilde{Q}$ can be expressed in the form $\widetilde{Q} = \bigcup_i \text{PGL}(V) \ast \phi^{-1}(U'_i) = \bigcup_i \text{PGL}(V) \ast U_i$. By (2.4) we obtain morphisms

$$\bigcup_i \text{PGL}(V) \ast \mu_i(U_i) \xrightarrow{\mu} \bigcup_i \text{PGL}(V) \ast U_i \xrightarrow{\phi} \bigcup_i \text{PGL}(V) \ast U'_i$$

The morphism $\mu := \bigcup_i \text{PGL}(V) \ast \mu_i$ is defined by maps $\mu_i$ and their composites with group actions. By the central equality, connected components of the scheme $\widetilde{Q}$ are taken by the map $\bigcup_i \text{PGL}(V) \ast \mu_i$ to connected sets. Consequently, morphism $\mu$ is well-defined. Since morphisms $\mu_i$ are by definition equivariant, then the subscheme $\mu(\widetilde{Q})$ is $\text{PGL}(V)$-invariant in $\text{Hilb}^{\mathbb{P}(1)} G(V,r)$.

The universal property of the Hilbert scheme guarantees the existence of the morphism $\mu$ and of fibred diagram (2.4).

The following simple result provides independence of the moduli scheme $\widetilde{M}$ of the choice of the cover of the scheme $\widetilde{Q}$ trivializing the locally free sheaf $\pi^*(\widetilde{E}_Q \otimes \widetilde{L}_m^q)$.

**Proposition 4.** The subscheme $\mu(\widetilde{Q}) \subset \text{Hilb}^{\mathbb{P}(1)} G(V,r)$ does not depend on the choice of the covering $\bigcup_i U_i = \widetilde{Q}$.

**Proof.** Choose an another covering $\widetilde{Q} = \bigcup_j \widetilde{U}_j$ with trivializing isomorphisms $\tau_j : \pi_*(\widetilde{E}_Q \otimes \widetilde{L}_m^q)|_{\widetilde{U}_j} \sim V \otimes \mathcal{O}_{\widetilde{U}_j}$ and repeat the construction of subscheme $\mu(\widetilde{Q})$. Notice that

$$\bigcup_j \text{PGL}(V) \ast \widetilde{U}_j' = \bigcup_i \text{PGL}(V) \ast U'_i = Q,$$

$$\bigcup_j \text{PGL}(V) \ast \widetilde{U}_j = \bigcup_i \text{PGL}(V) \ast U_i = \widetilde{Q}.$$  

Now consider the intersection $U_i \cap \widetilde{U}_j$ and induced morphisms of trivialization

$$V \otimes \mathcal{O}_{U_i \cap \widetilde{U}_j} \xrightarrow{\tau_i} \pi_*(\widetilde{E}_Q \otimes \widetilde{L}_m^q)|_{U_i \cap \widetilde{U}_j} \xrightarrow{\tau_j} V \otimes \mathcal{O}_{U_i \cap \widetilde{U}_j}$$

Hence, on the common part $U_i \cap \widetilde{U}_j$ trivializations are identified by an appropriate $\text{GL}(V)$-transformation. Consequently, images of the induced maps of the scheme $U_i \cap \widetilde{U}_j$ in the Hilbert scheme $\text{Hilb}^{\mathbb{P}(1)} G(V,r)$ are identified by the corresponding $\text{PGL}(V)$-transformation. Then we have

$$\bigcup_i \text{PGL}(V) \ast \mu_i(U_i) = \bigcup_j \text{PGL}(V) \ast \mu_j(\widetilde{U}_j) = \mu(\widetilde{Q}).$$
Proposition 5. $\mu(\tilde{Q})$ is the quasiprojective subscheme in $\text{Hilb}^{P(t)}G(V, r)$.

Proof. Form a closure $\mu(\tilde{Q})$ of the subscheme $\mu(\tilde{Q})$ in the projective scheme $\text{Hilb}^{P(t)}G(V, r)$. It is enough to confirm that the subset $\mu(\tilde{Q})$ is open in $\mu(\tilde{Q})$.

Let $\tilde{Q}$ be the scheme-theoretic closure of the subscheme $Q$ in $\text{Quot}^{\nu_{PE}(t)}(V \otimes L(-m))$. $\tilde{Q}$ be the projective closure of the quasiprojective scheme $\tilde{Q}$. We claim that the scheme $\tilde{Q}$ can be chosen so that there is a fibred diagram

$$
\begin{array}{ccc}
\mu(\tilde{Q}) & \xrightarrow{\mu} & \tilde{Q} \\
\mu(Q) & \xrightarrow{\pi} & Q \\
\end{array}
$$

with open immersions.

Indeed, let the scheme $\tilde{Q}$ is not included in the diagram (2.6). Consider the locally closed "diagonal" embedding $\tilde{Q} \hookrightarrow \mu(\tilde{Q}) \times \tilde{Q}$. Let $Q'$ be the closure if its image. Now form a locally closed immersion $\tilde{Q} \hookrightarrow Q' \times \tilde{Q}$. Let $Q''$ be the closure of the image if this immersion. Redenoting $Q_1 := Q''$ we have the required projective closure. Morphisms $\pi$ and $\phi$ are given by the composite maps of closed immersions and projections onto the direct summand

$$
\begin{array}{c}
\pi : \tilde{Q} \rightarrow Q' \times \tilde{Q} \xrightarrow{pr_1} Q' \hookrightarrow \mu(\tilde{Q}) \times \tilde{Q} \xrightarrow{pr_2} \mu(Q), \\
\phi : \tilde{Q} \rightarrow Q' \times \tilde{Q} \xrightarrow{pr_2} \tilde{Q}.
\end{array}
$$

Then $\tilde{Q}_1$ is the required projective closure. We will denote it by the symbol $\tilde{Q}$.

Now we need a lemma to be proven later.

Lemma 1. $\mu(\tilde{Q}) \setminus \mu(\tilde{Q}) = \pi(\tilde{Q} \setminus \tilde{Q})$.

Since the scheme $\tilde{Q}$ is quasiprojective, then the "boundary" $\tilde{Q} \setminus \tilde{Q}$ is closed in $\tilde{Q}$. Since the morphism $\pi$ is projective, it is proper and takes closed subsets to closed subsets. Hence, the image $\pi(\tilde{Q} \setminus \tilde{Q})$ is closed in $\pi(\tilde{Q})$. By lemma 1 the subset $\mu(\tilde{Q}) \setminus \mu(\tilde{Q})$ is closed in $\mu(\tilde{Q})$. Hence the subset $\mu(\tilde{Q})$ is open in the projective scheme $\pi(\tilde{Q})$. Then $\mu(\tilde{Q})$ is quasiprojective scheme.

of lemma 1. It suffices to check the set-theoretical equality. The inclusion $\mu(\tilde{Q}) \setminus \mu(\tilde{Q}) \supset \pi(\tilde{Q} \setminus \tilde{Q})$ follows immediately from the construction. To prove the opposite inclusion assume that there is "non-boundary" point $x$ in the preimage $\pi^{-1}(\mu(\tilde{Q}) \setminus \mu(\tilde{Q}))$ of the "boundary". Namely, we suppose that a point $x$ does not belong to the subset $\tilde{Q} \setminus \tilde{Q}$. This means that $x \in \tilde{Q}$ but $\pi(x) \notin \mu(\tilde{Q})$. This contradiction proves the lemma. 

17
3 Distinguished polarization of the scheme $\tilde{S}$

In this section we obtain the explicit form of the distinguished polarization on the scheme $\tilde{S}$. Since this scheme can fail to be a variety we will work with very ample invertible sheaves instead divisorial classes. Properties of a morphism $\sigma : \tilde{S} \to S$ are mostly similar to ones of the blowup morphism since $\sigma$ is a structure morphism of a projective spectrum of an appropriate sheaf algebra.

**Proposition 6.** Distinguished polarizations $\tilde{L}$ of schemes of the form $\tilde{S}$ described in the definition provide a Hilbert polynomial which is constant in flat families of admissible schemes.

**Proof.** Due to (2.3), we can work over a regular one-dimensional base $T$. Let $\Sigma := T \times S$ be a trivial family of surfaces $S$ supplied with a flat family $E$ of semistable coherent torsion-free sheaves with prescribed Hilbert polynomial $rp_E(t)$. Let $I = \text{Fitt}^0 \text{Ext}^1(E, \mathcal{O}_\Sigma)$, and $\sigma : \hat{\Sigma} \to \Sigma$ be the morphism of blowing up of the scheme $\Sigma$ in the sheaf of ideals $I$. Since $\sigma$ is a projective morphism, then there exist a projective bundle $p : \mathbb{P}_\Sigma \to \Sigma$ and a closed immersion $i : \hat{\Sigma} \hookrightarrow \mathbb{P}_\Sigma$ such that the triangle

\[ \begin{array}{ccc}
\hat{\Sigma} & \xrightarrow{i} & \mathbb{P}_\Sigma \\
\sigma \downarrow & & \downarrow \pi \\
\Sigma & \xrightarrow{p} & T
\end{array} \]

commutes. Indeed, since $I$ is a finitely generated $\mathcal{O}_\Sigma$-module, then there exist a locally free $\mathcal{O}_\Sigma$-sheaf of finite rank $F$ and an epimorphism of $\mathcal{O}_\Sigma$-modules $F \twoheadrightarrow I$. Then there are the induced epimorphism of symmetric algebras $\text{Sym} F \twoheadrightarrow \bigoplus_{s \geq 0} I^s$ and, consequently, the induced morphism of immersion of projective spectra $\hat{\Sigma} = \text{Proj} \left( \bigoplus_{s \geq 0} I^s \right) \hookrightarrow \mathbb{P}_\Sigma = \text{Proj Sym} F$.

Now consider a composite $\mathbb{P}_\Sigma \xrightarrow{p} \Sigma \xrightarrow{\pi} T$ of projective morphisms. As previously, $L$ is an invertible $\mathcal{O}_\Sigma$-sheaf, very ample relatively $T$. It corresponds to the sheaf $L$ under the restriction onto any fibre of the projection $\pi$. Then by [11 ch. II, exercise 7.14 (b)], for $m \gg 0$ the sheaf $L_T := \mathcal{O}_{\mathbb{P}_\Sigma}(1) \otimes i^* L^m$ is very ample relatively $T$. The restriction of this sheaf onto the image of the immersion $i$ gives a very ample relatively $T$ sheaf $\hat{L} = i^* L_T = \mathcal{O}_{\hat{\Sigma}}(1) \otimes \sigma^* L^m$ on the scheme $\hat{\Sigma}$. Then there exist a projective bundle $p_T : \mathbb{P}_T \to T$ and a closed immersion $j : \hat{\Sigma} \hookrightarrow \mathbb{P}_T$, include into the commutative triangle

\[ \begin{array}{ccc}
\hat{\Sigma} & \xrightarrow{j} & \mathbb{P}_T \\
\pi \circ p \downarrow & & \downarrow p_T \\
T & \xrightarrow{\pi_T} & T
\end{array} \]

Here $\hat{\Sigma}$ is the closure of the image of the open subset $\Sigma_0$ under the immersion $j$. It follows [11 ch. III, theorem 9.9] that the Hilbert polynomial of the image $j(\hat{\Sigma})$ is fibrewise constant over $T$. 

18
Prove that the sheaf $\hat{L}$ provides a polarization given by $L$, on a general enough fibre which is isomorphic to the surface $S$. The restriction onto the fibre $\pi^{-1}(t) \cong S$ yields $\hat{L}|_{\pi^{-1}(t)} = i^*(\mathcal{O}_S(1) \otimes \mathbf{c}^* L^m)|_{\pi^{-1}(t)} = L^m$.

Restriction onto the special fibre $\pi^{-1}(t_0) \cong \tilde{S} \not\cong S$ results in the equalities $	ilde{L}^m := \hat{L}|_{\pi^{-1}(t_0)} = (\mathcal{O}_{\tilde{S}}(1) \otimes \mathbf{c}^* L^m)|_{\pi^{-1}(t_0)} = (\mathbf{c}^* \mathbf{c} \cdot \mathcal{O}_{\tilde{S}}) = (\mathbf{c}^* \mathbf{c} \cdot \mathcal{O}_{\tilde{S}}) \otimes \sigma^* L^m$, where $I \subset \mathcal{O}_S$ is the sheaf of ideals obtained by the restriction of the sheaf $\mathcal{I}$ on the fibre $\pi^{-1}(t_0)$.

### 4 Isomorphism $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \cong H^0(S, E \otimes L^m)$.

Obviously, such an isomorphism exists because both the spaces of global sections have equal dimensions. In this section we compute the isomorphism $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \cong H^0(S, E \otimes L^m)$ distinguished by the construction of standard resolution. This construction puts into the correspondence a pair $(\tilde{S}, \tilde{E})$ to the coherent torsion-free sheaf $E$. First we construct the required homomorphism of vector spaces and then prove that this is an isomorphism.

In the sequel we replace schemes $Q$ and $\tilde{Q}$ by their nonsingular resolutions $\xi : Q' \to Q$ and $\xi : \tilde{Q}' \to \tilde{Q}$ such that there is a birational morphism $\phi' : Q' \to Q'$. The family $\Sigma_Q$ is replaced by its preimage $\Sigma' = \Sigma_Q \times \tilde{Q}'$. Form inverse images $\Sigma'_Q = \tilde{\xi}^{-1} \Sigma_Q$ and $E'_Q = \xi^{-1} E_Q$. Since $\mu(\tilde{Q'}) = \mu(\tilde{Q})$ we preserve notations $Q$ and $Q'$ for nonsingular resolutions, $\phi : Q \to Q$ for the corresponding birational morphism, $\Sigma_Q$ for the family of schemes, $\Sigma_Q$ and $E_Q$ for families of sheaves.

**Proposition 7.** Apply the standard resolution to the family $p : Q \times S \to Q$ and to the family of sheaves $E_Q$. This induces for any pair of points $(\bar{q}, q), \bar{q} \in \tilde{Q}$, $q = \phi(\bar{q}) \in Q$, the fixed isomorphism $\nu : H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \cong H^0(S, E \otimes L^m)$.

**Proof.**

Remind that there is a map $\mu : \tilde{Q} \to \text{Hilb}^P(t, G(V, r))$, and $\Sigma = \pi^{-1}\mu(\tilde{Q})$.

The other notation is fixed in the following diagram:

\[\begin{array}{ccc}
\tilde{Q} & \xrightarrow{\tilde{\Phi}} & Q \times S \\
\Sigma_Q & \xrightarrow{\Sigma} & Q \times S \\
\mu(\tilde{Q}) \quad \mu & \xrightarrow{\phi} & Q \quad p \\
\mu(\tilde{Q}) \times S & \xrightarrow{\mu \times 1} & Q \times S \quad \phi \times 1 \\
\end{array}\]

All the parallelograms of the left hand part and the bottom parallelogram of the right hand part are fibred. All the morphisms marked with the symbol $\pi$ are inherited from the structure morphism of the universal scheme $\text{Univ}^{P(t)} G(V, r)$. 
All the morphisms marked with the symbol $p$ are projections of the corresponding direct products onto the first summands.

The scheme $Q \times S$ carries the coherent reflexive sheaf $E_Q$ with homological dimension equal to 1. It is flat over $Q$. Schemes $\Sigma$ and $\Sigma_Q$ are supplied with locally free sheaves $\tilde{E}$ and $\tilde{E}_Q$ respectively and $\tilde{E}_Q = \tilde{\mu}^* \tilde{E}$. On quasiprojective schemes $\Sigma$ and $Q \times S$ there are fixed invertible sheaves $\tilde{L}$ and $\tilde{L}$. They are very ample relatively $\mu(\tilde{Q})$ and $Q$ respectively. In this case $(\tilde{\phi}_* \tilde{\mu}^* L_m)^{\vee} = L_m$. The following notation is introduced: $L_Q := \tilde{\mu}^* \tilde{L}$. Also, according to the standard resolution, $(\tilde{\phi}_* \tilde{E}_Q)^{\vee} = E_Q$. This equality implies the following

**Lemma 2.** For appropriate very ample invertible $O_{\tilde{Q}}$-sheaf $\tilde{L}$ and $O_{\tilde{Q}}$-sheaf $\tilde{L}$ such that $(\tilde{\phi}_* \tilde{L})^{\vee} = L$, and for an appropriate integer $n \gg 0$ there is an inclusion

$$(\Phi_*(\tilde{E}_Q \otimes \tilde{L}_m \otimes \tilde{L}^{mn}))^{\vee} \hookrightarrow (\phi \times 1)^*(E_Q \otimes L_m \otimes L^{mn}).$$

**Corollary 1.** There is an inclusion

$$[(\mu \times 1)^* M_*(\tilde{E} \otimes \tilde{L}^m)]^{\vee} \hookrightarrow (\phi \times 1)^*(E_Q \otimes L_m \otimes L^{mn}).$$

Proofs of lemma 2 and corollary 1 will be done later.

The consequent base changes accordingly to (4.1) lead to the chain

$$\begin{align*}
\pi_* (\tilde{E}_Q \otimes \tilde{L}^m) &= \pi_* \mu^* (\tilde{E} \otimes \tilde{L}^m) \\
&\hookrightarrow \mu^* (\tilde{E} \otimes \tilde{L}^m) \\
&\hookrightarrow p_* [(\mu \times 1)^* M_*(\tilde{E} \otimes \tilde{L}^m)]^{\vee} \\
&\hookrightarrow p_* (\phi \times 1)^*(E_Q \otimes L_m \otimes L^{mn}). \quad (4.2)
\end{align*}$$

The first arrow in (4.2) is an isomorphism since $m \gg 0$ [12, lecture 7, 3²]. The second arrow is a morphism of locally free sheaf and both sheaves coincide on the open subset. Hence, the second arrow is a monomorphism. The sheaf morphism induced by the inclusion into the reflexive hull, is an inclusion because the sheaf from the left is torsion-free. The recent morphism is defined by the corollary 1.

For $m \gg 0$ we have the inclusion map of $O_{\tilde{Q}}$-sheaves

$$\Upsilon : \pi_* (\tilde{E}_Q \otimes \tilde{L}_m) \hookrightarrow p_* (\phi \times 1)^*(E_Q \otimes L_m \otimes L^{mn}). \quad (4.3)$$

Since the sheaf $\tilde{E}_Q \otimes \tilde{L}_m$ is flat over $\tilde{Q}$ and the sheaf $E_Q \otimes L^m$ is flat over $Q$, both sheaves in (4.3) are locally free of rank $rp_E(m)$.

Since $\pi$ is a projective morphism of Noetherian schemes and the sheaf $\tilde{E}_Q \otimes \tilde{L}_m$ is flat over $\tilde{Q}$, and for $m \gg 0$ functions $\dim H^i(\pi^{-1}(\tilde{q}), \tilde{E} \otimes \tilde{L}^m)$ are constant as functions of a point $\tilde{q} \in \tilde{Q}$, then by [11, ch. III, corollary 12.9] there is an isomorphism $\pi_* (\tilde{E}_Q \otimes \tilde{L}_m) \otimes k_{\tilde{q}} \cong H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m)$. Here we use following notations: $\tilde{S} = \pi^{-1}(\tilde{q})$, $\tilde{q} \in \tilde{Q}$, $\tilde{E} = \tilde{E}_{|\pi^{-1}(\tilde{q})}$. By the similar reason, $p_* (E_Q \otimes L^m) \otimes k_q \cong H^0(S, E \otimes L^m)$ for $S = p^{-1}(q)$, $q \in Q$, $E = E_{|p^{-1}(q)}$. Suppose that points $\tilde{q}$ and $q$ satisfy $\phi(\tilde{q}) = q$. Then we have the homomorphism of vector
spaces of global sections \( v : H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \to H^0(S, E \otimes L^m) \) induced by the inclusion (133).

Note that by the construction of the morphism \( \sigma : \tilde{S} \to S \) by means of blowing up a family of surfaces \([7]\) there is an isomorphism \( \sigma^* \mathcal{O}_S = \mathcal{O}_{\tilde{S}} \). Then, keeping in mind the equality \( \tilde{E} = \sigma^* E/tors \), consider the mapping of spaces of global sections

\[
H^0(\sigma^*) : H^0(S, E \otimes L^m) \to H^0(\tilde{S}, \sigma^*(E \otimes L^m)).
\]

It is induced by formation of the inverse image. Also consider the map

\[
\zeta : H^0(\tilde{S}, \sigma^*(E \otimes L^m)) \to H^0(\tilde{S}, \tilde{E} \otimes \sigma^* L^m),
\]

induced by the morphism \( \sigma^* E \to \tilde{E} \), and the inclusion

\[
\xi : H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \hookrightarrow H^0(\tilde{S}, \tilde{E} \otimes \sigma^* L^m),
\]

induced by the morphism of the invertible sheaves \( \tilde{L}^m \to \sigma^* L^m \). Then there is a commutative diagram of homomorphisms of vector spaces

\[
\begin{array}{ccc}
H^0(S, E \otimes L^m) & \xrightarrow{H^0(\sigma^*)} & H^0(\tilde{S}, \sigma^*(E \otimes L^m)) \\
\uparrow v & & \downarrow \zeta \\
H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) & \xrightarrow{\xi} & H^0(\tilde{S}, \tilde{E} \otimes \sigma^* L^m)
\end{array}
\]

Since \( \xi \) is a monomorphism, then \( v \) is also monomorphism. Vector spaces \( H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \) and \( H^0(S, E \otimes L^m) \) have equal dimensions hence \( v \) is an isomorphism.

\[\Box\]

of lemma 2. Note that there is a twisted analogue of the formula \( \tilde{\phi}_*, \tilde{E}_Q \rightleftharpoons E_Q \). In can be proven by the reasoning completely similar to those done in \([6]\).

For invertible sheaves \( \tilde{L}_Q \) and \( L \) such that \( \left( \phi_* \tilde{L}_Q \right) \rightleftharpoons L \), the following equality holds \( \left( \phi_* (\tilde{E}_Q \otimes \tilde{L}_Q^m) \right) \rightleftharpoons E_Q \otimes L^m \). Let \( \tilde{L} \) and \( L \) be very ample invertible sheaves as described in the formulation of lemma.

Consider a sheaf \( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q \otimes \pi^* \tilde{L}^{nm}) = \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q) \otimes \pi^* \tilde{L}^{nm} \). The first isomorphism holds by the diagram (11). the second is true by projection formula. Formation of a reflexive hull yields \( \left( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q \otimes \pi^* \tilde{L}^{nm}) \right)^{\vee} = \left( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q) \right)^{\vee} \otimes \pi^* \tilde{L}^{nm} \). Now consider another \( \mathcal{O}_{\tilde{Q} \times S} \)-sheaf \( \left( \phi \times 1 \right)^* (E_Q \otimes L^m) \). It is also reflexive on \( \tilde{Q} \times S \) \([5]\) lemmata 1.2, 1.3]. Sheaves \( \left( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q) \right)^{\vee} \) and \( \left( \phi \times 1 \right)^* (E_Q \otimes L^m) \) coincide on those open subsets of the scheme \( \tilde{Q} \times S \) where they are locally free. These subsets are obtained by excluding of closed subschemes of codimension \( \geq 3 \) from \( \tilde{Q} \times S \). By assumption, the scheme \( \tilde{Q} \times S \) is integral and normal. Then \([6] \) corollary 1.10 \( \left( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q) \right)^{\vee} = \left( \phi \times 1 \right)^* (E_Q \otimes L^m) \). Hence after tensoring by the invertible sheaf \( \pi^* \tilde{L}^{nm} \) we have the isomorphism

\[
\left( \Phi_* (\tilde{E}_Q \otimes \tilde{L}_Q \otimes \pi^* \tilde{L}^{nm}) \right)^{\vee} = \left( \phi \times 1 \right)^* (E_Q \otimes L^m) \otimes \pi^* \tilde{L}^{nm} \quad (4.4)
\]
We claim that \( p^{\ast}L^{nm} \hookrightarrow p^{\ast}L^{nm} \). Indeed, \( \phi^{\ast}L^{nm} \hookrightarrow (\phi^{\ast}L^{nm})^{\vee \vee} = L^{nm} \). Application of the inverse image results in \( \phi^{\ast}L^{nm} \rightarrow L^{nm} \). The epimorphy is provided by the condition \( nm > 0 \). Then \( \phi^{\ast}L^{nm}/\text{tors} = L^{nm} \) yields \( L^{nm} = \phi^{\ast}L^{nm}/\text{tors} \hookrightarrow \phi^{\ast}L^{nm} \).Tensoring this inclusion by the right hand side of (4.4) we get \((\phi \times 1)^{\ast}(Q_{Q} \otimes L^{nm}) \otimes p^{\ast}L^{nm} \hookrightarrow (\phi \times 1)^{\ast}(Q_{Q} \otimes L^{nm}) \otimes p^{\ast}(\phi^{\ast}L^{nm} = (\phi \times 1)^{\ast}(Q_{Q} \otimes L^{m} \otimes p^{\ast}L^{nm}) \). This completes the proof of the lemma.

Proof of corollary[1] Note that \( \widetilde{E} \otimes \widetilde{L}^{m} = \widetilde{\mu}^{\ast}(\widetilde{E} \otimes \widetilde{L}^{m}) \). Since \( \widetilde{L} \) is very ample invertible \( \mathcal{O}_{\widetilde{L}} \)-sheaf, there is an inclusion \( \Phi^{\ast}(\widetilde{E} \otimes \widetilde{L}^{m})^{\vee \vee} \hookrightarrow \Phi^{\ast}(\widetilde{\mu}^{\ast}(\widetilde{E} \otimes \widetilde{L}^{m}) \otimes \widetilde{L}^{nm})^{\vee \vee} \). The base change applied to the first sheaf, and lemma[2] yield \( \Phi^{\ast}(\widetilde{E} \otimes \widetilde{L}^{m})^{\vee \vee} = (\mu \times 1)^{\ast}M_{\widetilde{E}}(\widetilde{E} \otimes \widetilde{L}^{m})^{\vee \vee} \hookrightarrow (\phi \times 1)^{\ast}(Q_{Q} \otimes L^{m} \otimes L^{nm}) \). □

5 (Semi)stability

The notion of (semi)stability for pairs \((\widetilde{S}, \widetilde{E})\) is defined in this section.

**Definition 6.** \( S \)-(semi)stable pair \((\widetilde{S}, \widetilde{L}), \widetilde{E} \) is the following data:

- \( \widetilde{S} = \bigcup_{i \geq 0} \widetilde{S}_{i} \) - admissible scheme, \( \sigma : \widetilde{S} \rightarrow S \) - canonical morphism, \( \sigma_{i} : \widetilde{S}_{i} \rightarrow S \) - its restrictions on components \( \widetilde{S}_{i}, i \geq 0 \);
- \( \widetilde{E} \) - vector bundle on the scheme \( \widetilde{S} \);
- \( \widetilde{L} \in \text{Pic } \widetilde{S} \) - distinguished polarization;

such that

- \( \chi(\widetilde{E} \otimes \widetilde{L}^{m}) = rp_{E}(t) \);
- the sheaf \( \widetilde{E} \) is Gieseker-(semi)stable on the scheme \( \widetilde{S} \). Namely, for any proper subsheaf \( \widetilde{F} \subset \widetilde{E} \) in \( \widetilde{E} \) for \( m \gg 0 \) one has

\[
\frac{h^{0}(\widetilde{F} \otimes \widetilde{L}^{m})}{\text{rank } \widetilde{F}} < \frac{h^{0}(\widetilde{E} \otimes \widetilde{L}^{m})}{\text{rank } \widetilde{E}},
\]

(respectively,

\[
\frac{h^{0}(\widetilde{F} \otimes \widetilde{L}^{m})}{\text{rank } \widetilde{F}} \leq \frac{h^{0}(\widetilde{E} \otimes \widetilde{L}^{m})}{\text{rank } \widetilde{E}} \);

- on each of additional components \( \widetilde{S}_{i}, i > 0 \), the sheaf \( \widetilde{E}_{i} := \widetilde{E}_{|\widetilde{S}_{i}} \) is quasi-ideal sheaf, namely has a description of the form \( (\ref{1.6}) \) for some \( q_{0} \in \bigcup_{i \leq q} \text{Quot } \bigoplus F_{i}^{\ast} \mathcal{O}_{S} \).

**Remark 2.** If \( \widetilde{S} \cong S \), then (semi)stability of a pair \((\widetilde{S}, \widetilde{E})\) is equivalent to Gieseker-(semi)stability of vector bundle \( E \) on the surface \( S \) with respect to the polarization \( \widetilde{L} \in \text{Pic } \widetilde{S} \).
To investigate the relation of $\mathcal{S}$-(semi)stability of the pair $(\tilde{S}, \tilde{E})$ to Gieseker-(semi)stability of the corresponding sheaf $\tilde{E}$ on the surface $\tilde{S}$ note that for $m \gg 0$ $r_{PE}(m) = h^0(E \otimes L^m)$. For the Gieseker-stability the behavior of the Hilbert polynomial under $m \gg 0$ is important. Therefore we assume that $m$ is big enough.

**Definition 7.** The locally free sheaf $\tilde{E}$ on the admissible scheme $\tilde{S}$ is said to be obtained from the sheaf $\tilde{E}$ by its standard resolution if there exists a flat family $E$ of coherent $\mathcal{O}_S$-sheaves with base $T = \text{Spec } k[t]$, such that

(i) for $t \neq 0$ sheaves $E_t = \tilde{E}|_{t \neq 0}$ are locally free;

(ii) for $t = 0$ the sheaf $E_0 = \tilde{E}|_{t = 0}$ is isomorphic to the sheaf $\tilde{E}$;

(iii) standard resolution yields in the blowing up $\sigma : \tilde{T} \times S \rightarrow T \times S$ supplied with locally free sheaf $\tilde{E}$. The fibre of the composite map $\tilde{T} \times S \rightarrow T \times S \rightarrow T$ at the point $t = 0$ is isomorphic to $\tilde{S}$ and carries the locally free sheaf $\tilde{E}|_{t = 0} \cong \tilde{E}$.

**Remark 3.** In particular, by the proposition this definition means that for the locally free $\mathcal{O}_S$-sheaf $\tilde{E}$ there is a coherent $\mathcal{O}_S$-sheaf $E$ such that $\tilde{E} = \sigma^* E/\text{tors}$.

**Proposition 8.** Let the locally free $\mathcal{O}_S$-sheaf $\tilde{E}$ be obtained from a coherent $\mathcal{O}_S$-sheaf $E$ by its standard resolution. The sheaf $\tilde{E}$ is (semi)stable on the scheme $\tilde{S}$ if and only if the sheaf $E$ is (semi)stable.

**Proof.** Let $E$ be Gieseker-semistable on $(S, L)$ and $\tilde{E}$ be the locally free sheaf on the scheme $\tilde{S}$. Let $\tilde{E}$ be obtained from $E$ by standard resolution. Obviously, $\tilde{E}$ is quasi-ideal sheaf on additional components of $\tilde{S}$ provided it is obtained from a coherent sheaf by standard resolution. Fix any point $q \in \text{Quot } \tau_{PE}(t)(V \otimes L(-m))$ corresponding to the quotient sheaf $E$. Consider a proper subsheaf $F \subset \tilde{E}$. Since $m \gg 0$ we assume that both the sheaves $\tilde{E} \otimes \tilde{L}^m$ and $\tilde{F} \otimes \tilde{L}^m$ are globally generated. Fix an epimorphism $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \otimes \tilde{L}(-m) \rightarrow \tilde{F}$.

The subsheaf $\tilde{F}$ is generated by a subspace of global sections $V_\tilde{F} = H^0(\tilde{S}, \tilde{F} \otimes \tilde{L}^m) \subset H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m)$. Then a subspace $V_F \subset H^0(S, E \otimes L^m)$ which is isomorphic to $V_\tilde{F}$ and generates some subsheaf $F \subset E_0$ is given by the distinguished isomorphism $v : H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}^m) \sim H^0(S, E \otimes L^m)$ by the equality $V_F = v(V_\tilde{F})$. Since sheaves $\tilde{F}$ and $F$ are canonically isomorphic on the corresponding open subsets of schemes $\tilde{S}$ and $S$, then their ranks are equal. Clearly, $V_F = H^0(S, F \otimes L^m)$ and

\[
\frac{h^0(S, E \otimes L^m)}{r} - \frac{h^0(S, \tilde{F} \otimes \tilde{L}^m)}{r'} = \frac{h^0(S, E \otimes L^m)}{r} - \frac{h^0(S, F \otimes L^m)}{r'} > (\geq 0).
\]

This implies the semistability of $\tilde{E}$. The opposite implication is proven similarly.

**Remark 4.** This shows that there is a bijection among subsheaves of $\mathcal{O}_S$-sheaf $E$ and subsheaves of the corresponding $\mathcal{O}_\tilde{S}$-sheaf $\tilde{E}$. This bijection preserves Hilbert polynomials.
6 M-equivalence of semistable pairs

In this section we investigate the behavior of Jordan – Hölder filtration for semistable coherent sheaf under the standard resolution. Also the notion of M-equivalence for semistable pairs is introduced and relation of M-equivalence to S-equivalence for semistable coherent sheaves is examined. In particular it is proven that S-equivalent coherent sheaves on the surface $S$ are resolved in M-equivalent pairs of the form $(\tilde{S}, \tilde{E})$.

Remind some notions from the theory of semistable coherent sheaves.

**Definition 8.** [10, definition 1.5.1] The Jordan – Hölder filtration for semistable sheaf $E$ with reduced Hilbert polynomial $p_E(t)$ on the polarized projective scheme $X$ is a sequence of subsheaves

$$0 = F_0 \subset F_1 \subset \cdots \subset F_\ell = E,$$

such that quotient sheaves $gr_i(E) = F_i/F_{i-1}$ are stable with reduced Hilbert polynomials equal $p_E(t)$.

Denote by the symbol $gr(E)$ a polystable sheaf $\bigoplus_{i=1}^\ell gr_i(E)$. Well-known theorem [10, Prop. 1.5.2] claims that the isomorphism class of the sheaf $gr(E)$ has no dependence on a choice of Jordan – Hölder filtration of $E$.

**Definition 9.** [10, definition 1.5.3] Semistable sheaves $E$ and $E'$ are called S-equivalent if $gr(E) = gr(E')$.

**Remark 5.** Obviously, S-equivalent stable sheaves are isomorphic.

Define Jordan-Hölder filtration for S-semistable sheaf on reducible admissible polarized scheme $(\tilde{S}, \tilde{L})$. This definition will be completely analogous to the classical definition for Gieseker-semistable sheaf.

**Definition 10.** Jordan – Hölder filtration for a sheaf $\tilde{E}$ on the polarized projective reducible scheme $(\tilde{S}, \tilde{L})$ such that a pair $((\tilde{S}, \tilde{L}), E)$ is semistable in the sense of definition 7, and with reduced Hilbert polynomial $p_{E}(t)$, is a sequence of subsheaves

$$0 = \tilde{F}_0 \subset \tilde{F}_1 \subset \cdots \subset \tilde{F}_\ell = \tilde{E},$$

such that quotients $gr_i(\tilde{E}) = \tilde{F}_i/\tilde{F}_{i-1}$ are Gieseker-stable with reduced Hilbert polynomials equal to $p_{E}(t)$.

The following example shows that S-equivalent coherent sheaves can have different associated sheaves of Fitting ideals leading to non-isomorphic schemes $\tilde{S}$.

**Example 2.** Consider scheme of moduli for semistable coherent sheaves of rank 2 with Chern classes $c_1 = 0$, $c_2 = 2$ on the complex projective plane $S = \mathbb{P}^2$. As it is proven in [13], for even values of $c_1$ and $c_2 - c_1^2/4$ the moduli scheme for semistable sheaves has no universal family. This means that there is strictly semistable coherent sheaf $E$ with Jordan – Hölder filtration which
leads to the exact triple $0 \to I \to E \to I' \to 0$. Here $I, I'$ are sheaves of maximal ideals of a reduced point $x \in \mathbb{P}^2$. Note that a polystable sheaf which is $S$-equivalent to the sheaf $E$ equals $I \oplus I'$. In this case $\text{Ext}^1(I', I) \neq 0$. To prove this consider an exact $\mathcal{O}_S$-triple $0 \to I \to \mathcal{O}_S \to k_x \to 0$ and apply the functor $\text{Ext}(I', -)$. Since all extensions of the form $0 \to k_x \to A \to I' \to 0$ are trivial, then $\text{Ext}^1(I', k_x) = 0$. We have an isomorphism of groups of extensions $\text{Ext}^1(I', I) \cong \text{Ext}^1(I', \mathcal{O}_S)$. The last group is non-trivial. Indeed, it contains the class $\varepsilon$ of non-trivial extension which corresponds to the locally free resolution for the sheaf of ideals $I'$: $0 \to \mathcal{O}_S \to F_0 \to I' \to 0$. The non-trivial extension $E$ corresponding to $\varepsilon$ in $\text{Ext}^1(I', I)$ includes into the exact diagram

$$
\begin{array}{c}
\begin{array}{ccc}
0 & \to & 0 \\
\uparrow & & \uparrow \\
k_x & \to & k_x \\
0 & \to & \mathcal{O}_S \\
& \nearrow & \searrow \\
& 0 & \to F_0 \\
\downarrow & & \downarrow \\
0 & \to I & \to E \\
& \nearrow & \searrow \\
& 0 & \to I' & \to 0 \\
& \nearrow & \searrow \\
& 0 & \to 0 \\
\end{array}
\end{array}
$$

For a coherent torsion-free $\mathcal{O}_S$-sheaf $F$ we use the notation $\mathcal{E}(F) := F^\vee \vee / F$. From the middle vertical triple we have $\mathcal{E}(E) = k_x$. Also for the polystable sheaf $I \oplus I'$ holds $\mathcal{E}(I \oplus I') = k_x^{\oplus 2}$. Then $\text{Fitt}^0 \text{Ext}^2(\mathcal{E}(E), \mathcal{O}_S) = m_x$ and $\text{Fitt}^0 \text{Ext}^2(\mathcal{E}(I \oplus I'), \mathcal{O}_S) = \text{Fitt}^0 \text{Ext}^2(\mathcal{E}(I), \mathcal{O}_S) \cdot \text{Fitt}^0 \text{Ext}^2(\mathcal{E}(I'), \mathcal{O}_S) = II' = m_x^2$ is a sheaf of ideals of the first infinitesimal neighborhood of the point $x$.

The following example shows that the fibred product cannot be used to construct the notion of equivalence for semistable pairs.

**Example 3.** Consider sheaves of maximal ideals $I_1 = I_2 = m_x$ of a reduced point $x \in S$. Then corresponding schemes $S_1$ and $S_2$ have the form $S_1 = \text{Proj} \bigoplus_{t \geq 0} (I_1[t] + (t))^s / (t^{s+1})$ and $S_2 = \text{Proj} \bigoplus_{t \geq 0} (I_2[t] + (t))^s / (t^{s+1})$. As usually $\sigma_i : S_i \to S$ are canonical morphisms, $S_i = S_0 \bigcup_{\sigma_i^{-1}(x)} \mathbb{P}^2$ is the decomposition into irreducible components where $\sigma_i|_{S_0} = \sigma_0 : S_0 \to S$ is a blowing up of reduced point $x$, $\sigma_0^{-1}(x) \cong \mathbb{P}^1$ is exceptional divisor of this blowing up. Schemes $S_1$ and $S_2$ are isomorphic. Let $i : S_1 \to S_2$ be the identifying isomorphism. Let schemes $S_1$ and $S_2$ carry stable vector bundles $E_1$ and $E_2$ which are images of nonlocally free coherent sheaf $E$ on the surface $S$. Obviously, $i_* E_1 = E_2$. Obviously, in this case vector bundles $E_i$, $i = 1, 2$ are nontrivial under restriction on the exceptional divisor $\mathbb{P}^1$. Form the fibred product $S_1 \times_S S_2$, and let $\sigma'_i : S_1 \times_S S_2 \to S_i$ be its projections on factors. The
product $\tilde{S}_1 \times_S \tilde{S}_2$ contains four-dimensional component. This component is isomorphic to the product $\mathbb{P}^2 \times \mathbb{P}^2$. It contains the product of exceptional divisors of blowing ups $\sigma_{i0} : \tilde{S}_{i0} \to S$ as a closed subscheme isomorphic to a quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Then inverse images $\sigma_i^* E_i$ turn to be non-isomorphic on the fibred product $\tilde{S}_1 \times_S \tilde{S}_2$. Indeed, the restriction $\sigma_i^* E_1|_{\mathbb{P}^1 \times \mathbb{P}^1}$ is non-trivial along first factor of the product $\mathbb{P}^1 \times \mathbb{P}^1$ and trivial along the second one. The restriction $\sigma_i^* E_2|_{\mathbb{P}^1 \times \mathbb{P}^1}$ is trivial along the first factor and non-trivial along the second one.

Now consider the schemes $\tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0} (I_1[t] + (t))^s/(t)^{s+1}$ and $\tilde{S}_2 = \text{Proj} \bigoplus_{s \geq 0} (I_2[t] + (t))^s/(t)^{s+1}$ with their canonical morphisms $\sigma_1 : \tilde{S}_1 \to S$ and $\sigma_2 : \tilde{S}_2 \to S$ to the surface $S$. Form inverse images of sheaves of ideals $I'_2 = \sigma_1^{-1} I_2 : \mathcal{O}_{\tilde{S}_1} \subset \mathcal{O}_{\tilde{S}_2}$ and $I'_1 = \sigma_2^{-1} I_1 : \mathcal{O}_{\tilde{S}_2} \subset \mathcal{O}_{\tilde{S}_1}$, and projective spectra $\tilde{S}_{12} = \text{Proj} \bigoplus_{s \geq 0} (I'_1[t] + (t))^s/(t)^{s+1}$ and $\tilde{S}_{21} = \text{Proj} \bigoplus_{s \geq 0} (I'_2[t] + (t))^s/(t)^{s+1}$. There are canonical morphisms $\sigma'_2 : \tilde{S}_{12} \to \tilde{S}_1$ and $\sigma'_1 : \tilde{S}_{21} \to \tilde{S}_2$.

**Proposition 9.** $\tilde{S}_{12}$ and $\tilde{S}_{21}$ are equidimensional schemes. Moreover, $\tilde{S}_{12} \cong \tilde{S}_{21}$.

**Proof.** First we prove that $\tilde{S}_{12} \cong \tilde{S}_{21}$, and that these schemes can be include into flat families with general fibre isomorphic to $S$, or to $\tilde{S}_1$, or to $\tilde{S}_2$. This implies that all components of the scheme $\tilde{S}_{12}$ have dimension not bigger then 2. Then we will give the scheme-theoretic characterization of schemes $\tilde{S}_{12}$. It proves that $\tilde{S}_{12}$ is equidimensional scheme, namely, all reduced schemes corresponding to its components have dimension 2.

Let $T = \text{Spec} \, k[t]$. Turn to the trivial 2-parameter family of surfaces $T \times T \times S$ with projections $T \times S \xrightarrow{p_{13}} T \times T \times S \xrightarrow{p_{12}} T \times S$. Introduce the notations $\mathbb{I}_1 := \mathcal{O}_T \boxtimes I_1 \subset \mathcal{O}_{T \times S}$, $\mathbb{I}_2 := \mathcal{O}_T \boxtimes I_2 \subset \mathcal{O}_{T \times S}$. Form inverse images $p_{13}^* \mathbb{I}_1$ and $p_{23}^* \mathbb{I}_2$. These are sheaves of ideals on the scheme $T \times T \times S$. Consider the morphism $\sigma_1 \times \text{id}_T : \Sigma_1 \times T \to T \times T \times S$ with identity map on the second factor. Also consider a preimage $(\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 : \mathcal{O}_{\Sigma_1 \times T}$ on the scheme $\Sigma_1 \times T$, and the corresponding morphism of blowing up $\sigma_{12} : \Sigma_{12} \to \Sigma_1 \times T$. Now restrict the sheaf $(\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 : \mathcal{O}_{\Sigma_1 \times T}$ on the fibre of the composite map $\Sigma_1 \times T \xrightarrow{\sigma_1 \times \text{id}_T} T \times T \times S \xrightarrow{p_{12}} T \times T \quad (t_1, t_2)$. Let $\tilde{i} : \tilde{S}_1 \hookrightarrow \Sigma_1 \times T$ be the morphism of the embedding of this fibre. The commutativity of the diagram

$$
\begin{array}{cccc}
\Sigma_1 \times T & \xrightarrow{\sigma_1 \times \text{id}_T} & T \times T \times S & \xrightarrow{p_{12}} & T \times T \\
\downarrow \tilde{i} & & & & \downarrow \tilde{i} \\
\tilde{S}_1 & \xrightarrow{\sigma_1} & S & & (t_1, t_2)
\end{array}
$$

leads to $\tilde{i}^{-1}((\sigma_1 \times \text{id}_T)^{-1} p_{23}^* \mathbb{I}_2 : \mathcal{O}_{\Sigma_1 \times T}) \cdot \mathcal{O}_{\tilde{S}_1} = \sigma_1^{-1} \tilde{i}^{-1}(p_{23}^* \mathbb{I}_2 : \mathcal{O}_{\tilde{S}_1}) = \sigma_1^{-1} I_2 : \mathcal{O}_{\tilde{S}_1}$.

Now consider the embedding of the line $j_T : T \to T \times T$ fixed by the equation
If the embedding \( u \) fixes notations. If the embedding \( u \) does not correspond to the case \( b = 0 \) then \( \Sigma_{1j} \cong \hat{\Sigma}_1 \) and \( j_1^{-1}((\sigma_1 \times \id_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\hat{\Sigma}_{1T}}) \cdot \mathcal{O}_{\Sigma_{1j}} = \sigma_1^{-1}\mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_1} \). Otherwise (for \( b = 0 \)) we have \( \Sigma_{1j} \cong T \times S \).

The morphism \( \sigma_{1j} : \hat{\Sigma}_{1j} \to \Sigma_{1j} \) of the blowing up of the sheaf of ideals \( \sigma_1^{-1}\mathbb{I}_2 \cdot \mathcal{O}_{\Sigma_1} \) is include into the commutative diagram

\[
\begin{array}{ccc}
\Sigma_{12} & \xrightarrow{\sigma_{12}} & \hat{\Sigma}_1 \times T \\
\downarrow j_{12} & & \downarrow j_1 \\
\Sigma_{12j} & \xrightarrow{\sigma_{1j}} & \Sigma_{1j}
\end{array}
\]

By the universal property of the left fibred product in (6.1), there is a morphism \( u : \hat{\Sigma}_{1j} \to \Sigma_{12j} \).

The morphism of blowing up \( \sigma'_1 : \hat{\Sigma}_{12} \to \hat{\Sigma}_1 \) of the sheaf of ideals \( \sigma_1^{-1}\mathbb{I}_2 \cdot \mathcal{O}_{\hat{\Sigma}_1} \) is include into the commutative diagram

\[
\begin{array}{ccc}
\hat{\Sigma}_{12} & \xrightarrow{\sigma'_1} & \hat{\Sigma}_1 \\
\downarrow \sigma_{12} & & \downarrow \sigma_1 \\
\hat{\Sigma}_2 & \xrightarrow{\sigma_2} & T \times S
\end{array}
\]

Note that in this diagram \( \sigma'_1 \) is a morphism of blowing up of the sheaf of ideals \( \sigma_2^{-1}\mathbb{I}_1 \cdot \mathcal{O}_{\Sigma_2} \) and it follows that \( \hat{\Sigma}_{21} = \hat{\Sigma}_{12} \). Also \( \hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_{12} \) are reduced irreducible schemes. Each of them is fibred over the regular one-dimensional base \( T \) with fibres isomorphic to the projective schemes. Hence schemes \( \hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_{12} \) are flat families of projective schemes over \( T \). Each of these families has fibre isomorphic to the surface \( S \), at general enough point of \( T \). This implies that each fibre of the family \( \hat{\Sigma}_{12} \) has a form of projective spectrum \( \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t)^s/s! + 1) \) for an appropriate sheaf of ideals \( I \subset \mathcal{O}_S \). Fibres of flat family of projective schemes carry polarizations with following property. Hilbert polynomials of fibres compute with respect to these polarizations, remain constant over the base. By the construction, such polarizations on fibres of schemes \( \hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_{12} \) are exactly the same as polarizations compute in (3).

Now we prove that \( \Sigma_{12j} \) if family of schemes flat over \( T \). Consider the exact \( \mathcal{O}_{\hat{\Sigma}_1 \times T} \)-triple induced by the sheaf of ideals \( (\sigma_1 \times \id_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\hat{\Sigma}_1 \times T} \):

\[
0 \to (\sigma_1 \times \id_T)^{-1}p_{23}^*\mathbb{I}_2 \cdot \mathcal{O}_{\hat{\Sigma}_1 \times T} \to \mathcal{O}_{\hat{\Sigma}_1 \times T} \to \mathcal{O}_Z \to 0
\]
for an appropriate closed subscheme \(Z\). Apply the functor \(j^*_1\) and note that the sheaf of ideals \(j^*_1((\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1 \times T}) \cdot \mathcal{O}_{\Sigma_1}\) is isomorphic to the quotient sheaf \((\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1 \times T}/\text{tors}\), for the torsion subsheaf given by the equality \(\text{tors} = \tau_\text{or}^*_1 \mathcal{O}_{\Sigma_1 \times T}((\sigma_1 \times \text{id}_T)^{-1}\mathcal{O}_Z, \mathcal{O}_{\Sigma_1})\). Note that \(\Sigma_1 \cong \widehat{\Sigma}_1\), and \(j^*_1\mathcal{O}_{\Sigma_1 \times T} \cong \mathcal{O}_{\Sigma_1}\). With the last two isomorphisms taken into account we have \(\tau_\text{or}^*_1 \mathcal{O}_{\Sigma_1 \times T}((\sigma_1 \times \text{id}_T)^{-1}\mathcal{O}_Z, \mathcal{O}_{\Sigma_1}) = 0\). Then

\[
j^*_1((\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1 \times T}) \cdot \mathcal{O}_{\Sigma_1} = \sigma_1^{-1}\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1}.
\]

Also for blowups one has \(\Sigma_{12} = \text{Proj} \bigoplus_{s \geq 0}((\sigma_1 \times \text{id}_T)^{-1}p_{23}^*\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1})^s = \text{Proj} \bigoplus_{s \geq 0}((\sigma_1^{-1}\mathcal{I}_2 \cdot \mathcal{O}_{\Sigma_1})^s = \Sigma_{12}\). Since \(\Sigma_{12}\) is a flat family over \(T\) then the scheme \(\Sigma_{12}\) is also flat over \(T\).

Any two points on \(T \times T\) can be connected by a chain of two lines satisfying the condition \(b \neq 0\). Then Hilbert polynomials of fibres of the scheme \(\Sigma_{12} \rightarrow T \times T\) are constant over the base \(T \times T\). Hence the scheme \(\Sigma_{12}\) is flat over the base \(T \times T\).

To characterize the scheme structure of the special fibre of the scheme \(\Sigma_{12}\) (and consequently the corresponding fibre of the scheme \(\Sigma_{12}\)) it is enough to consider the embedding \(j_T\) defined by the equation \(t_2 \equiv 0\), and a subscheme \(\Sigma_1 = j_1(\Sigma_{12})\). It is a flat family of subschemes with fibre \(\Sigma_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1[t] + (t))^{s}/(t)^{s+1}\). As proven before, the preimage \(\Sigma_{12} = \sigma_1^{-1}(\Sigma_1)\) is also flat over \(j_T(T) \cong T\) with generic fibre isomorphic to \(\Sigma_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1[t] + (t))^{s}/(t)^{s+1}\). Applying in this situation the reasoning of the article \[2\] we obtain that the special fibre \(\Sigma_{12}\) of the scheme \(\Sigma_{12}\) has the following scheme-theoretic characterization: \(\Sigma_{12} = \text{Proj} \bigoplus_{s \geq 0}(I_2[t] + (t))^{s}/(t)^{s+1}\) for the sheaf of ideals \(I_2 \subset \mathcal{O}_{\Sigma_{12}}\) defined as \(I_2 = \sigma_1^{-1}I_1 \cdot \mathcal{O}_{\Sigma_{12}}\).

Hence, for any two schemes \(\tilde{S}_1 = \text{Proj} \bigoplus_{s \geq 0}(I_1[t] + (t))^{s}/(t)^{s+1}\) and \(\tilde{S}_2 = \text{Proj} \bigoplus_{s \geq 0}(I_2[t] + (t))^{s}/(t)^{s+1}\) the scheme \(\tilde{S}_{12} = \text{Proj} \bigoplus_{s \geq 0}(I_1[t] + (t))^{s}/(t)^{s+1}\) is defined together with morphisms \(\tilde{S}_1 \xleftarrow{\sigma_1} \tilde{S}_{12} \xrightarrow{\sigma_2} \tilde{S}_2\), such that the diagram

\[
\begin{array}{ccc}
\tilde{S}_{12} & \xrightarrow{\sigma_2} & \tilde{S}_2 \\
\sigma_1 \downarrow & & \downarrow \sigma_2 \\
\tilde{S}_1 & \xleftarrow{\sigma_1'} & S
\end{array}
\]

commutes. The operation \((\tilde{S}_1, \tilde{S}_2) \mapsto \tilde{S}_1 \circ \tilde{S}_2 = \tilde{S}_{12}\) defined by this way, is obviously associative. Moreover, since for any admissible morphism \(\sigma : \tilde{S} \rightarrow S\) there are equalities \(\tilde{S} \circ S = S \circ \tilde{S} = \tilde{S}\), then admissible morphisms of each class \([E]\) of \(S\)-equivalent semistable coherent sheaves generate a commutative monoid \(\Diamond [E]\) with binary operation \(\circ\) and neutral element \(\text{id}_S : S \rightarrow S\).
Note that by proposition 8 and remark 4 there is a bijective correspondence among subsheaves of coherent $O_S$-sheaf $E$ and subsheaves of the corresponding locally free $O_{\tilde{S}}$-sheaf $\tilde{E}$. This correspondence preserves Hilbert polynomials. Let there is a fixed Jordan-Hölder filtration in $E$ formed by subsheaves $F_i$. Then there is a sequence of semistable subsheaves $\tilde{F}_i$ with the same reduced Hilbert polynomial and rank $\tilde{F}_i = \text{rank} F_i$ distinguished in $\tilde{E}$ by the described correspondence.

Let $X$ be a projective scheme, $L$ be ample invertible $O_X$-sheaf, $E$ be a coherent $O_X$-sheaf. Let the sheaf $E \otimes L^m$ is globally generated, namely, there is an epimorphism $q : H^0(X, E \otimes L^m) \otimes L^{(-m)} \to E$. Fix a subspace $H \subset H^0(X, E \otimes L^m)$. The subsheaf $F \subset E$ is said to be generated by the subspace $H$ if it is an image of the composite map $H \otimes L^{(-m)} \subset H^0(X, E \otimes L^m) \otimes L^{(-m)} \xrightarrow{q} E$.

**Proposition 10.** The transformation $E \mapsto \sigma^* E/\text{tors}$ is compatible on all subsheaves $F \subset E$ with the isomorphism $\nu$ for all $m \gg 0$.

**Proof.** Take an arbitrary subsheaf $F \subset E$ of rank $r'$. It necessary to check that the subsheaf $\tilde{F} \subset \tilde{E} = \sigma^* E/\text{tors}$ generated in $\tilde{E}$ by the subspace $\nu^{-1}H^0(S, F \otimes L^m)$, coincides with the subsheaf $\sigma^* F/\text{tors}$.

It is clear that sheaves $\sigma^* F/\text{tors}$ and $\tilde{F}$ coincide on the open subset $W$ of the scheme $\tilde{S}$ where the scheme morphism $\sigma : \tilde{S} \to S$ is an isomorphism. Then it rests to check their coincidence on additional components of the scheme $\tilde{S}$. The structure of the sheaf $\tilde{E}$ on additional components is described by the data $(\ldots, 1.6)$. As earlier, $U$ is the open neighborhood of $\text{Supp} \neq$ in $S$. Coincidence of subsheaves $\tilde{F}$ and $\sigma^* F/\text{tors}$ on the open subset $\sigma^{-1}(U) \cap W$ provides (possibly after diminishing of the open subset $W$) isomorphisms $\tilde{F}|_{\sigma^{-1}(U) \cap W} = (\sigma^* F/\text{tors})|_{\sigma^{-1}(U) \cap W} = \sigma^* \bigoplus' O_{U \cup \sigma(W)}$ and the inclusion of the inverse images of trivial sheaves $\sigma^* \bigoplus' O_{U \cup \sigma(W)} \hookrightarrow \sigma^* \bigoplus' O_{U \cup \sigma(W)}$.

There is a commutative triangle

\[
\begin{array}{ccc}
\bigoplus' O_U & \xrightarrow{q_0} & \kappa \\
\uparrow & & \uparrow \\
\bigoplus' O_U & \xrightarrow{q'_0} & \kappa' \\
\end{array}
\]

where the morphism $q'_0$ is defined as composite map. Application of the functor of the inverse image $\sigma^*$ and restrictions on each of additional components lead to the expressions

$$
\tilde{E}_i = \sigma^* \ker q_0/\text{tors}_i, \quad \tilde{F}_i = \sigma^* F/\text{tors}|_{\tilde{S}_i} = \sigma^* \ker q'_0/\text{tors}_i,
$$

what completes the proof. \qed
Corollary 2. Sheaves \( \widetilde{F}_i = \sigma^* F_i/\text{tors} \) are semistable of rank \( r_i = \text{rank } F_i \) with reduced Hilbert polynomial equal to \( p_E(t) \).

Proof. The equality of ranks follows from the equality \( \widetilde{F}_i = \sigma^* F_i/\text{tors} \) and from the fact that the morphism \( \sigma \) is an isomorphism on open subscheme in \( S \). The Hilbert polynomial for all \( t \gg 0 \) is fixed by the equalities \( \chi(F_i \otimes L^t) = h^0(S, F_i \otimes \overline{L}^t) = h^0(S, F_i \otimes L^t) = \chi(F_i \otimes L^t) = r p_E(t) \).

For \( \overline{E} = \sigma^* E/\text{tors} \) consider epimorphisms \( \overline{q} : H^0(S, \overline{E} \otimes \overline{L}^m) \otimes \overline{L}(\overline{t}-m) \rightarrow \overline{E} \) and \( q : H^0(S, E \otimes L^m) \otimes L(\overline{t}-m) \rightarrow E \).

Definition 11. Subsheaves \( \overline{F} \subset \overline{E} \) \( F \subset E \) are called \( \nu \)-corresponding if there exist subspaces \( \nu(V) \subset H^0(S, \overline{E} \otimes \overline{L}^m) \) and \( V = \nu(V) \subset H^0(S, E \otimes L^m) \) such that \( \overline{q}(V \otimes \overline{L}^{-m}) = \overline{F}, q(V \otimes L^{-m}) = F \). Notation: \( F = \nu(E) \). The corresponding quotient sheaves \( \overline{E}/F \) and \( E/F \) will be also called \( \nu \)-corresponding and denoted \( E/F = \nu(E/F) \).

Proposition 11. The transformation \( E \mapsto \sigma^* E/\text{tors} \) takes saturated subsheaves to saturated subsheaves.

Proof. Let \( F_{i-1} \subset F_i \) be a saturated subsheaf. Assume that the quotient sheaf \( \overline{F}_i/\overline{F}_{i-1} \) has a subsheaf of torsion \( \nu \). This subsheaf is generated by vector subspace \( \overline{T} \subset H^0(S, \overline{F}_i \otimes \overline{L}^m) \). Let \( \overline{T}' \) be its preimage in \( H^0(S, F_i \otimes L^m) \) and \( T' \subset \overline{F}_i \) be a subsheaf generated by subspace \( \overline{T}' \). Then there is a sheaf epimorphism \( T' \rightarrow \nu \) with kernel \( T' \cap \overline{F}_{i-1} \). Let \( K = H^0(S, (T' \cap \overline{F}_{i-1}) \otimes L^m) \subset H^0(S, F_{i-1} \otimes L^m) \) be its generating subspace. Then the isomorphism \( \nu \) leads to the exact diagram of vector spaces

\[
\begin{array}{cccccc}
0 & \rightarrow & \nu(K) & \rightarrow & \nu(T') & \rightarrow & \nu(T')/\nu(K) & \rightarrow & 0 \\
& & i & & i & & i & & \\
0 & \rightarrow & \overline{K} & \rightarrow & \overline{T}' & \rightarrow & \overline{T}'/\overline{K} & \rightarrow & 0 \\
\end{array}
\]

with morphism \( \overline{\nu} \) induced by the morphism \( \nu \). Also there are exact sequences of \( \nu \)-corresponding coherent sheaves

\[
\begin{array}{cccccc}
0 & \rightarrow & \nu(T' \cap \overline{F}_{i-1}) & \rightarrow & \nu(T') & \rightarrow & \nu(\nu) & \rightarrow & 0, \\
0 & \rightarrow & \overline{T'} \cap \overline{F}_{i-1} & \rightarrow & \overline{T'} & \rightarrow & \nu & \rightarrow & 0. \\
\end{array}
\]

Sheaves \( T' \cap \overline{F}_{i-1}, \nu(T' \cap \overline{F}_{i-1}), T' \) and \( \nu(T') \) coincide under restriction on open subsets \( W \) and \( \sigma(W) \) respectively. Then if \( \nu \) is a torsion sheaf, then \( \nu(\nu) \) is also torsion sheaf. This contradicts saturatedness of the subsheaf \( F_{i-1} \). □

Corollary 3. There are isomorphisms \( \sigma^*(F_i/F_{i-1})/\text{tors} \cong \overline{F}_i/\overline{F}_{i-1} \).
Proof. Take an exact triple

$$0 \to F_{i-1} \to F_i \to F_i/F_{i-1} \to 0$$

and apply the functor $\sigma^*$. This yields

$$0 \to \sigma^* F_{i-1} \to \sigma^* F_i \to \sigma^* F_i/F_{i-1} \to 0$$

where the symbol $\tau$ denotes the subsheaf of torsion violating exactness. Factoring first two sheaves by torsion and applying the proposition 10 one has an exact diagram

$$
\begin{array}{ccccccc}
0 & \to & N & \to & \text{tors}(\sigma^* F_i) & \to & \tau' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \sigma^* F_{i-1}/\tau & \to & \sigma^* F_i & \to & \sigma^* (F_i/F_{i-1}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & F_{i-1} & \to & F_i & \to & \tilde{F}_i/F_{i-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & \\
\end{array}
$$

where the sheaf $N$ is defined as $\ker (\sigma^* F_{i-1}/\tau \to \tilde{F}_{i-1})$. It rests to note that the sheaf $\tau'$ is torsion sheaf. Also since the subsheaf $F_{i-1} \subset F_i$ is saturated then due to the proposition 11 the sheaf $\tilde{F}_i/F_{i-1}$ has no torsion. Then $\tilde{F}_i/F_{i-1} \cong \sigma^*(F_i/F_{i-1})/\text{tors}$. \qed

Corollary 4. Quotient sheaves $\tilde{F}_i/F_{i-1}$ are stable and their reduced Hilbert polynomial is equal to $p_E(t)$.

Proof. Consider a subsheaf $\tilde{R} \subset \tilde{F}_i/F_{i-1}$ and the space of global sections

$$H^0(S, \tilde{R} \otimes \tilde{L}_m) \subset H^0(S, (\tilde{F}_i/F_{i-1}) \otimes \tilde{L}_m) = H^0(S, \tilde{F}_i \otimes \tilde{L}_m)/H^0(S, \tilde{F}_{i-1} \otimes \tilde{L}_m).$$

We assume as usually $m$ to be as big as higher cohomology groups vanish. Let $\tilde{H}$ be the preimage of subspace $H^0(S, \tilde{R} \otimes \tilde{L}_m)$ in $H^0(S, \tilde{F}_i \otimes \tilde{L}_m)$, and $H := \nu(\tilde{H})$. Denote by $\mathcal{H}$ a subsheaf in $F_i$ if this subsheaf is generated by the subspace $H$.

It is clear that $\mathcal{H}/F_{i-1} \subset F_i/F_{i-1}$. From the chain of obvious equalities

$$h^0(S, \tilde{R} \otimes \tilde{L}_m) = \dim \tilde{H} - h^0(S, \tilde{F}_{i-1} \otimes \tilde{L}_m) = \dim H - h^0(S, F_{i-1} \otimes L^m)$$

$$= h^0(S, \mathcal{H} \otimes L^m) - h^0(S, F_{i-1} \otimes L^m) = h^0(S, (\mathcal{H}/F_{i-1}) \otimes L^m)$$
it follows that
\[
\frac{h^0(\tilde{S}, (\tilde{F}_i/\tilde{F}_{i-1}) \otimes \tilde{L}_m)}{\operatorname{rank}(\tilde{F}_i/\tilde{F}_{i-1})} - \frac{h^0(\tilde{S}, \tilde{R} \otimes \tilde{L}_m)}{\operatorname{rank} \tilde{R}} = \frac{h^0(\tilde{S}, (\tilde{H}/\tilde{F}_{i-1}) \otimes \tilde{L}_m)}{\operatorname{rank}(\tilde{H}/\tilde{F}_{i-1})} > 0.
\]
This proves stability of the quotient sheaf \( \tilde{F}_i/\tilde{F}_{i-1} \).

Now consider the exact triple \( 0 \to E_1 \to E \to \operatorname{gr}_1(E) \to 0 \) and the corresponding triple of spaces of global sections
\[
0 \to H^0(S, E_1 \otimes L^m) \to H^0(S, E \otimes L^m) \to H^0(S, \operatorname{gr}_1(E) \otimes L^m) \to 0.
\]
It is exact for \( m \gg 0 \). The transition to the corresponding \( \mathcal{O}_S \)-sheaf \( \tilde{E} \), to its subsheaf \( \tilde{E}_1 \), to global sections, and application of the isomorphism \( \upsilon \), lead to the commutative diagram of vector spaces
\[
\begin{array}{ccc}
0 & \longrightarrow & H^0(S, E_1 \otimes L^m) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^0(S, \tilde{E}_1 \otimes L^m)
\end{array}
\]
\[
\begin{array}{ccc}
& & H^0(S, E \otimes L^m) & \longrightarrow & H^0(S, \operatorname{gr}_1(E) \otimes L^m) & \longrightarrow & 0 \\
& & \downarrow \upsilon_i & & \downarrow \upsilon_i & & \downarrow \pi \\
& & H^0(S, \tilde{E} \otimes \tilde{L}_m) & \longrightarrow & H^0(S, \tilde{E}_1 \otimes \tilde{L}_m) & \longrightarrow & 0
\end{array}
\]
where the isomorphism \( \pi \) is induced by the isomorphism \( \upsilon \). Continuing the reasoning inductively for rest subsheaves of Jordan-Hölder filtration of the sheaf \( E \), we get that bijective correspondence of subsheaves is continued onto quotients of filtrations. Then the transition from quotient sheaves \( E_i/E_{i-1} \) to \( \tilde{E}_i/\tilde{E}_{i-1} \) preserves Hilbert polynomials and stability.

**Definition 12.** Jordan-Hölder filtration of semistable vector bundle \( \tilde{E} \) with Hilbert polynomial equal to \( \nu p(t) \), is a sequence of semistable subsheaves \( 0 \subset \tilde{F}_1 \subset \cdots \subset \tilde{F}_i \subset \tilde{E} \) with reduced Hilbert polynomials equal to \( \nu p(t) \), such that quotient sheaves \( \operatorname{gr}_i(\tilde{E}) = \tilde{F}_i/\tilde{F}_{i-1} \) are stable.

The sheaf \( \bigoplus_i \operatorname{gr}_i(\tilde{E}) \) will be called as associated polystable sheaf for the bundle \( \tilde{E} \).

Then it follows from the results of propositions\(^{10}\)\(^{11}\) and corollaries\(^2\) - \(^4\) that the transformation \( E \mapsto \sigma^* E/\text{tors} \) takes the Jordan - Hölder filtration of the sheaf \( E \) to Jordan - Hölder filtration of bundle \( \sigma^* E/\text{tors} \).

Let \( (\tilde{S}, \tilde{E}) \) and \( (\tilde{S}', \tilde{E}') \) be semistable pairs.

**Definition 13.** Semistable pairs \( (\tilde{S}, \tilde{E}) \) and \( (\tilde{S}', \tilde{E}') \) are called \( \nu \)-equivalent (monoidally equivalent) if for morphisms of \( \circ \)-product \( \tilde{S} \circ \tilde{S}' \) to factors \( \sigma' : \tilde{S} \circ \tilde{S}' \to \tilde{S} \) and \( \sigma : \tilde{S} \circ \tilde{S}' \to \tilde{S}' \) and for associated polystable sheaves \( \bigoplus_i \operatorname{gr}_i(\tilde{E}) \) and \( \bigoplus_i \operatorname{gr}_i(\tilde{E}') \) there are isomorphisms
\[
\sigma' \bigoplus_i \operatorname{gr}_i(\tilde{E})/\text{tors} \cong \bigoplus_i \operatorname{gr}_i(\tilde{E}')/\text{tors}.
\]
Proposition 12. S-equivalent semistable coherent sheaves $E$ and $E'$ correspond to M-equivalent semistable pairs $(\tilde{S}, E)$ and $(\tilde{S}', E')$.

Proof. Standard resolution takes semistable coherent sheaf $E$ to semistable pair $(\tilde{S}, \tilde{E})$. Jordan–Hölder filtration of the sheaf $E$ is taken to Jordan–Hölder filtration of the bundle $\tilde{E}$. Then the polystable sheaf $\bigoplus_i gr_i(E)$ is taken to the associated polystable sheaf $\bigoplus_i gr_i(\tilde{E})$. Hence we have for the sheaf $E$

$$\sigma^* \bigoplus_i gr_i(\tilde{E})/\text{tors} = \sigma^* \bigoplus_i gr_i(E)/\text{tors} = \sigma^* \bigoplus_i gr_i(E)/\text{tors}. \quad (6.3)$$

Analogously for a sheaf $E'$ which is S-equivalent to the sheaf $E$ one has

$$\sigma^* \bigoplus_i gr_i(E')/\text{tors} = \sigma^* \bigoplus_i gr_i(E'/\text{tors}) = \sigma^* \bigoplus_i gr_i(E')/\text{tors}. \quad (6.4)$$

Right hand sides of (6.3) and (6.4) are isomorphic by the isomorphism of polystable $\mathcal{O}_S$-sheaves $\bigoplus_i gr_i(E) \cong \bigoplus_i gr_i(E')$ and by commutativity of diagram

$$\begin{array}{ccc}
\tilde{S} \circ \tilde{S}' & \xrightarrow{\sigma} & \tilde{S}' \\
\sigma \downarrow & & \sigma' \downarrow \\
\tilde{S} & \xrightarrow{\sigma} & S
\end{array}$$

for $\circ$-product. The proposition is proven. 

□

7 Minimal resolution in the monoid $\diamondsuit[E]$

This section plays auxiliary role. We prove results concerning with local freeness of subsheaves and quotient sheaves in Jordan–Hölder filtration for sheaves of the form $E = \sigma^* E/\text{tors}$.

Definition 14. $\sigma : \tilde{S} \rightarrow S$ is the minimal resolution in the monoid $\diamondsuit[E]$ generated by canonical morphisms for the class $[E]$ of S-equivalent semistable coherent $\mathcal{O}_S$-sheaves, if the following hold:

i) (resolution) for any canonical morphisms $\sigma_i : \tilde{S}_i \rightarrow S$, $\sigma_j : \tilde{S}_j \rightarrow S$ diagrams

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\sigma_i} & \tilde{S}_i \\
\sigma \downarrow & & \sigma_i \downarrow \\
\tilde{S} & \xrightarrow{\sigma_j} & S
\end{array}$$

commute. All sheaves $\sigma^* E_i/\text{tors}$ for $E = \sigma_j \circ \sigma_i = \sigma_i \circ \sigma_j$ are locally free;

ii) (minimality) for any morphism $\sigma' \in \diamondsuit[E]$, $\sigma' : \tilde{S}' \rightarrow S$ such that i) holds,
there exists a morphism \( f : \tilde{S}' \to \tilde{S}_* \) which includes into commutative diagrams

for all \( i, j \), and \( \sigma' = \sigma \circ f \).

**Remark 6.** We naturally assume that for one-element class defined by a stable sheaf \( E \) the corresponding canonical morphism \( \sigma \) is minimal.

**Remark 7.** For a pair of semistable coherent sheaves \( E_1 \) and \( E_2 \), and for corresponding resolutions \( \sigma_1 : \tilde{S}_1 \to S \) and \( \sigma_2 : \tilde{S}_2 \to S \) the \( \circ \)-product \( \tilde{S}_1 \circ \tilde{S}_2 \) satisfies conditions of resolution and of minimality. This implies that it is the minimal resolution for the pair of sheaves \( E_1, E_2 \).

**Proposition 13.** Every class of \( S \)-equivalent coherent \( O_S \)-sheaves has the minimal resolution \( \tilde{S}_* \). This resolution corresponds to the morphism \( \sigma : \tilde{S}_* \to S \).

**Proof.** It is enough to confirm that there is a finite collection of canonical morphisms \( \sigma : \tilde{S} \to S \) corresponding to sheaves of each \( S \)-equivalence class. This means that the minimal resolution \( \tilde{S}_* \) can be constructed using the operation \( \circ \) by finite number of steps.

Indeed, every \( S \)-equivalence class consists of sheaves with singularities supported at the same points of the surface \( S \). Namely, \( \text{Supp} \mathcal{E}xt^i(E, O_S) \) is constant in the \( S \)-equivalence class. Further, colengths of zeroth Fitting ideal sheaves

\[ \text{Fitt}^0 \mathcal{E}xt^i(E, O_S) = \text{Fitt}^0 \mathcal{E}xt^2(\mathcal{K}, O_S) \]

are bounded from above globally over \( \bar{M} \). Let \( l_0 \) be the value of colength \( \text{Fitt}^0 \mathcal{E}xt^2(\mathcal{K}, O_S) \) maximal over \( \bar{M} \).

Now turn to the Grothendieck’s Quot scheme \( \text{Quot}^l \oplus \frac{r}{O_S} \) parameterizing zero-dimensional quotient sheaves of length \( l \) on the surface \( S \). It is well-known that this is a Noetherian algebraic scheme of finite type. It has natural stratification defined by isomorphism classes of zero-dimensional quotient sheaves.

**Claim 1.** The number of strata is finite.

Indeed, consider an epimorphism \( q : O_S^{p \oplus r} \to \mathcal{K} \) and an inclusion of (any) direct summand \( O_S \hookrightarrow O_S^{p \oplus r} \). Denote \( \mathcal{K}_r := \mathcal{K} \). Then \( O_{Z_1} \) is the image of this direct summand under the composite map \( O_S \hookrightarrow O_S^{p \oplus r} \to \mathcal{K}_r \). There is an exact
Now, performing the procedure for the right vertical triple and iterating the process we come to the cofiltration
\[ \kappa_r : \kappa_{r-1} : \cdots : \kappa_2 : \kappa_1 = \mathcal{O}_{Z_r} \]
with kernels \( \mathcal{O}_{Z_1}, \ldots, \mathcal{O}_{Z_{r-1}} \). Namely, we have a series of short exact sequences
\[
0 \to \mathcal{O}_{Z_1} \to \kappa_r \to \kappa_{r-1} \to 0, \\
0 \to \mathcal{O}_{Z_2} \to \kappa_{r-1} \to \kappa_{r-2} \to 0, \\
\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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in the given S-equivalence class. By Remark 7, this product satisfies minimality condition and hence \( \tilde{S}_s = S_s \). This completes the proof. \( \square \)

**Proposition 14.** If \( \eta : \tilde{S}_s \to S \) is the minimal resolution in the monoid \( \mathcal{O}[E] \) then for any \( E \in [E] \) images \( \mathcal{T}_i := \eta^*F_i / \text{tors} \) of the sheaves \( F_i \) in Jordan – Hölder filtration are locally free and \( \eta^*gr(E) = \mathcal{T}_i / \mathcal{T}_{i-1} \).

**Proof.** Consider polystable sheaf \( \bigoplus_i gr_i(E) \) which belongs to the given S-equivalence class. Its resolution on the scheme \( \tilde{S}_s \) also has a form of direct sum \( \bigoplus_i \eta^*gr_i(E) / \text{tors} \). Hence the summands \( \eta^*gr_i(E) / \text{tors} \) are also locally free. Then for any semistable sheaf \( E \) in the given S-equivalence class, with Jordan – Hölder filtration 0 \( \subset F_1 \subset \cdots \subset F_{l-1} \subset F_l = E \) the sheaves if the image of the filtration \( \mathcal{T}_i = \eta^*F_i / \text{tors} \) are locally free. Then we have for quotients of the filtration:

\[
\eta^*gr_i(E) / \text{tors} = \mathcal{T}_i / \mathcal{T}_{i-1}.
\]

We turn again to the polystable sheaf \( gr(E) = \bigoplus_i gr_i(E) \) in the given S-equivalence class. Let \( \sigma_{gr} : \tilde{S}_{gr} \to S \) be the corresponding canonical morphism defined by the sheaf of ideals \( \mathcal{I}_{gr} = \text{Fitt}^0 gr(E) \mathcal{O}_S \).

**Proposition 15.** For all \( E \in [E] \) sheaves \( \sigma_{gr}^*E / \text{tors} \) are locally free.

**Proof.** The sheaf \( \sigma_{gr}^*gr(E) / \text{tors} \) is locally free. This implies that all direct summands \( \sigma_{gr}^*gr(E) / \text{tors} \) are also locally free. Consider inductively following exact sequences

\[
0 \to \sigma_{gr}^*F_1 / \text{tors} \to \sigma_{gr}^*gr_1(E) / \text{tors} \to 0,
0 \to \sigma_{gr}^*F_1 / \text{tors} \to \sigma_{gr}^*F_2 / \text{tors} \to \sigma_{gr}^*gr_2(E) / \text{tors} \to 0,
\ldots
0 \to \sigma_{gr}^*F_{l-1} / \text{tors} \to \sigma_{gr}^*E / \text{tors} \to \sigma_{gr}^*gr_1(E) / \text{tors} \to 0,
\]

induced by Jordan – Hölder filtration of any semistable sheaf \( E \) of the given S-equivalence class. It follows that the sheaf \( \sigma_{gr}^*E / \text{tors} \) is locally free. \( \square \)

8 Boundedness of families of semistable pairs

In this section we show that pairs of the form \( (\tilde{S}, \tilde{L}, \tilde{E}) \), which are deformation equivalent to pairs with \( (\tilde{S}, \tilde{L}) \cong (S, L) \), constitute in \( \text{Hilb}^P(V, r) \) a subset equal to \( \mu(\tilde{Q}) \).

**Definition 15.** A pair \( (\tilde{S}, \tilde{L}, \tilde{E}) \) is called S-pair if \( (\tilde{S}, \tilde{L}) \cong (S, L) \).

**Definition 16.** A pair \( (\tilde{S}, \tilde{L}, \tilde{E}) \) is deformation equivalent to S-pairs if there exist

- connected algebraic scheme \( T \) of dimension 1,
- flat family of schemes \( \pi : \Sigma \to T \),
invertible sheaf $\tilde{L}$ very ample relative $\pi$, 
locally free $\mathcal{O}_\Sigma$-sheaf $\tilde{E}$
such that
for any closed point $t \in T$ the fibre $\pi^{-1}(t)$ is admissible scheme, $\tilde{L}|_{\pi^{-1}(t)}$ is its distinguished polarization, $\tilde{E}|_{\pi^{-1}(t)}$ is locally free sheaf with Hilbert polynomial equal to $\chi(\tilde{E}|_{\pi^{-1}(t)} \otimes \tilde{L}|_{\pi^{-1}(t)}) = r\rho_E(t)$,
there is a point $t_0 \in T$ such that $((\pi^{-1}(t_0), \tilde{L}|_{\pi^{-1}(t_0)}), (\tilde{S}, \tilde{L}, \tilde{E})$, 
at every general point $t \neq t_0$ $((\pi^{-1}(t), \tilde{L}|_{\pi^{-1}(t)}), (\tilde{S}, \tilde{L}, \tilde{E})$ is S-pair.

Pairs deformation equivalent to S-pairs will be called $dS$-pairs. A subset of points in $\text{Hilb}^{P(t)} G(V, r)$ corresponding to $dS$-pairs is denoted by the symbol $K^{dS}$. A subset of points corresponding to S-pairs is denoted by the symbol $K^S$.

Now we need the following well-known

**Theorem 2.** [10, 3.3.7] Let $f: X \to Y$ be projective morphism of schemes of finite type over $k$ and let $\mathcal{O}_X(1)$ be an $f$-ample line bundle. Let $P$ be a polynomial of degree $d$, and let $\mu_0$ be a rational number. Then the family of purely d-dimensional sheaves on the fibres of $f$ with Hilbert polynomial $P$ and maximal slope $\mu_{\text{max}} \geq \mu_0$ is bounded.

**Remark 8.** The theorem is formulated for more general case of purely d-dimensional sheaves and the symbol $\mu_{\text{max}}$ denotes maximal slope in the Harder – Narasimhan filtration of the sheaf $E$. Since we are interested in semistable sheaves, the Harder – Narasimhan filtration for semistable sheaf consists of this sheaf itself. Then $\mu_{\text{max}} = \mu$ is constant.

Note that by the reasonings of section 6, semistable $dS$-pairs carry locally free sheaves. Their slope is constant in Jordan-Hölder filtration and therefore is bounded. Then applying the theorem 2 in this situation we have the following

**Proposition 16.** Family of semistable $dS$-pairs is bounded.

Then there is an integer $\tilde{m}_0$ such that for $m > \tilde{m}_0$ sheaves $\tilde{E} \otimes \tilde{L}^m$ define closed immersions of the corresponding schemes $\tilde{S}$ into Grassmann variety $G(V, r)$ for $V$ being $k$-vector space of dimension $r\rho_E(m)$. The number $\tilde{m}_0$ is uniform for all semistable $dS$-pairs. Therefore in the whole construction of the moduli scheme one must take $m > \max(m_0, \tilde{m}_0)$. The number $m_0$ is characterized before, in section 2.

**Proposition 17.** S-pairs constitute an open subscheme in $K^{dS}$.

**Proof.** Note that the set of points of Hilbert scheme where fibres of the morphism

$$\pi: \text{Univ}^{P(t)} G(V, r) \to \text{Hilb}^{P(t)} G(V, r)$$

...
correspond to \(S\)-pairs, is precisely an intersection of \(K^{dS}\) with the set of points of the scheme \(\text{Hilb}^{P(t)}G(V,r)\) where the fibres of the morphism \(\pi\) are geometrically integral. The second set is open due to [13 Théorème 12.2.4(viii)]. Restriction on the subscheme \(K^{dS}\) proves the proposition.

It is clear from reasonings in the previous sections that \(\mu(\tilde{Q}) \subset K^{dS}\). Both subsets are \(PGL(V)\)-invariant.

**Proposition 18.** There is a coincidence of subsets \(\mu(\tilde{Q}) = K^{dS}\).

**Proof.** Assume that \(K^{dS} \setminus \mu(\tilde{Q}) \neq \emptyset\). Note that subsets formed by semistable \(S\)-pairs in \(\mu(\tilde{Q})\) and in \(K^{dS}\) coincide and equal \(K^{S}\). Let \(T = \text{Spec}\, k[t]\) be an affine curve in \(K^{dS}\) such that its open part \(T_0 = \{t \neq t_0\} \subset T\) belongs to the subset \(K^{S}\) but special point \(t_0 = T \setminus T_0 \in K^{dS} \setminus \mu(\tilde{Q})\). Let \(\pi_1 : \hat{\Sigma}_1 \to T\) be the flat family of subschemes defined by the fibred product \(\hat{\Sigma}_1 = T \times_{\text{Hilb}^{P(t)}G(V,r)} \text{Univ}^{P(t)}G(V,r)\), carrying the locally free sheaf \(\hat{E}_1\), and corresponding to the immersion \(T \hookrightarrow K^{dS} \subset \text{Hilb}^{P(t)}G(v,r)\). Then we have flat family of schemes \(\pi_1 : \hat{\Sigma}_1 \to T\), supplied with locally free sheaf \(\hat{E}_1\), with birational morphisms \(\xi_1 : \hat{\Sigma}_1 \to \Sigma_1\) and \(\sigma_1 : \hat{\Sigma}_1 \to T \times S\), forming a "Hironaka’s house". Denote by the symbol \(\hat{E}_2\) the reflexive sheaf \((\sigma_1, \xi_1)(\hat{E}_1)\) on the product \(T \times S\). It is locally free on the open subset off the codimension 3. Due to [13 corollary 6.3], the module over the principal ideal domain is flat if and only if it is torsion-free. Then \(\hat{E}_2\) is the sheaf of flat \(O_T\)-modules. It is a subsheaf of locally free \(O_T\) -module \(\hat{F}\) of the same rank. Namely, there is inclusion \(\hat{E}_2 \hookrightarrow \hat{F}\). Let \(\hat{\mathfrak{a}} := \hat{F}/\hat{E}_2\). It is clear that by choice of the determinant of the sheaf \(\hat{F}\) we can achieve \(\hat{\mathfrak{a}}\) to be Artinian sheaf with support on the fibre \(t_0 \times S\): \(\hat{\mathfrak{a}}|_{t_0 \times S} \cong \mathfrak{a}\).

Applying to the exact triple of \(O_{T \times S}\)-sheaves

\[
0 \to \hat{E}_2 \to \hat{F} \to \hat{\mathfrak{a}} \to 0
\]

the restriction onto the fibre \(t_0 \times S\), we get an exact triple

\[
0 \to \text{Tor}^1_{T \times S}(\mathfrak{a}, k_{t_0} \otimes O_S) \to \hat{E}_2|_{t_0 \times S} \to \hat{F}|_{t_0 \times S} \to \hat{\mathfrak{a}}|_{t_0 \times S} \to 0.
\]

Since \(\mathfrak{a}\) is a sheaf supported on the fibre \(t_0 \times S\), then \(\text{Tor}^1_{T \times S}(\mathfrak{a}, k_{t_0} \otimes O_S) = \text{Tor}^1_{T}(\mathfrak{a}, O_S) = 0\) and \(\hat{E}_2\) is a flat family of torsion-free sheaves on the smooth surface. Due to [31 proof of the proposition 4.3], \(\hat{E}_2\) is \(O_{S \times T}\)-sheaf of homological dimension equal to 1. Then it is subject to standard resolution.

**Remark 9.** By the construction of the special fibre \(\pi^{-1}(t_0) = \tilde{S}_1\) one can assume that \(\hat{\Sigma}_1 = \Sigma_1\) and \(\xi_1\) is an identity morphism.

Application of standard resolution to the sheaf \(\hat{E}_2\) leads to a family of schemes \(\pi_2 : \hat{\Sigma}_2 \to T\) which is flat over \(T\). It is supplied with the locally free sheaf \(\hat{E}_2\). Also the procedure or resolution gives the birational morphism \(\sigma_2 : \hat{\Sigma}_2 \to T \times S\) such that \((\sigma_2, \hat{E}_2)^{\vee} = \hat{E}_2\). Applying the diagram (12.2) one gets locally free sheaves \(\sigma_1^{\ast}\hat{E}_2\) and \(\sigma_2^{\ast}\hat{E}_1\). Denoting \(\sigma := \sigma_1^{\ast}\circ \sigma_2 = \sigma_2^{\ast}\circ \sigma_1\) one has obvious equalities \((\sigma, \sigma_2^{\ast}\hat{E}_1)^{\vee} = (\sigma, \sigma_1^{\ast}\hat{E}_2)^{\vee} = \hat{E}_2\).

38
Note that on open subsets \( \pi_1^{-1}t_0 \cong \pi_2^{-1}t_0 \), sheaves \( \sigma_i^*\mathcal{E}_2 \) and \( \sigma_2^*\mathcal{E}_1 \) are isomorphic. Denote \( \pi_i^{-1}(t_0) =: \tilde{S}_i \) for \( i = 1, 2 \) and, respectively, \( \tilde{E}_i := \mathcal{E}_i|_{\tilde{S}_i} \). The proof of the following lemma will be given later.

**Lemma 3.** Pairs \( (\tilde{S}_1, \tilde{E}_1) \) and \( (\tilde{S}_2, \tilde{E}_2) \) are isomorphic.

The result of the lemma contradicts the assumption \( K^{dS}\setminus \mu(\tilde{Q}) \neq \emptyset \) and completes the proof.

of the lemma. Note that locally free sheaves \( \sigma_1^*\mathcal{E}_2 \otimes \det(\sigma_1^*\mathcal{E}_2)^\vee \) and \( \sigma_2^*\mathcal{E}_1 \otimes \det(\sigma_1^*\mathcal{E}_2)^\vee \) coincide on the open subset off the codimension not less than 2. Hence they coincide on the whole of the scheme \( \tilde{S}_{12} \), namely \( \sigma_1^*\mathcal{E}_2 \otimes \det(\sigma_1^*\mathcal{E}_2)^\vee = \sigma_2^*\mathcal{E}_1 \otimes \det(\sigma_1^*\mathcal{E}_2)^\vee \). Also note that the left hand part of this equality is a locally free sheaf and it is trivial along the exceptional divisor of the morphism of blowing up \( \sigma_1^* \). Analogously, the sheaf on the right is trivial along the exceptional divisor of the morphism of blowing up \( \sigma_2^* \). This implies that there exist a scheme \( \tilde{S}_0 \), a pair of birational morphisms \( \tilde{S}_2 \rightarrow \tilde{S}_0 \sim \tilde{S}_1 \) and a locally free sheaf \( \tilde{E}_0 \) on the scheme \( \tilde{S}_0 \) such that \( \tilde{E}_i \otimes \det \mathcal{E}_i^\vee = \eta_i^*\mathcal{E}_0 \) for \( i = 1, 2 \).

We reduce our consideration to the case with trivial determinant. Introduce an auxiliary notation \( \tilde{E}_i' := \tilde{E}_i \otimes \det \mathcal{E}_i^\vee \) for \( i = 1, 2 \). Then \( \tilde{E}_i' = \eta_i^*\tilde{E}_0 \). Note that sheaves \( E \) and \( E' = E \otimes \det \mathcal{E}_1^\vee \) have equal sheaves of singularities \( \kappa = \kappa' \). Then along additional components of reducible fibres \( \tilde{S}_i \) we have

\[
\tilde{E}_i'|_{\text{add}} = \sigma^*\ker(\oplus^r\mathcal{O}_S \rightarrow \kappa)|_{\text{add/tors}}.
\]

This means that sheaves \( \tilde{E}_i' \) are nontrivial along additional components of schemes \( \tilde{S}_i \). Since \( \tilde{E}_0 \) is locally free sheaf then \( \eta_i \) are identity morphisms.

Hence morphisms \( \sigma_i \) coincide and we will use notations \( \tilde{S} := \tilde{S}_1 = \tilde{S}_2 \) and \( \sigma : \tilde{S} \rightarrow T \times S \). Also there are two locally free sheaves \( \tilde{E}_i, i = 1, 2 \), satisfying the condition \( \tilde{E}_1 \otimes \det \mathcal{E}_1^\vee = \tilde{E}_2 \otimes \det \mathcal{E}_1^\vee \). Determinants \( \det \tilde{E}_i \) coincide on the supplement of the exceptional divisor of the morphism \( \sigma \), and \( (\sigma, \det \tilde{E}_i)^\vee = \det \tilde{E}_2 \). Then sheaves \( \tilde{E}_1 = \tilde{E}_i|_{\tilde{S}_i} \) can differ only on additional components of the scheme \( \tilde{S} \). Now note that the special fibre of the projection \( \tilde{S} = \pi_1^{-1}(t_0) \) is a projective spectrum of \( \mathcal{O}_S \)-algebra. Let \( \mathcal{O}(1) \) be its twisting Serre’s sheaf. Then \( \tilde{E}_1 = \tilde{E}_2 \otimes \mathcal{O}(l_1 - l_2) \) for some integers \( l_1, l_2 \).

Let \( l_1 - l_2 \geq 0 \) (the opposite case can be considered similarly). By the definition of the distinguished polarization one has \( \tilde{L}^m = \sigma^*L^m \otimes \mathcal{O}(1) \). There is an inclusion of invertible sheaves \( \sigma^*L^m \otimes \mathcal{O}(1) \hookrightarrow \sigma^*L^m \). Then there is an inclusion \( \tilde{L}^m \otimes \mathcal{O}(l_1 - l_2) \hookrightarrow \tilde{L}^m \). By local freeness of the sheaf \( \tilde{E}_2 \) there is an exact triple

\[
0 \rightarrow \tilde{E}_2 \otimes \tilde{L}^m \otimes \mathcal{O}(l_1 - l_2) \rightarrow \tilde{E}_2 \otimes \tilde{L}^m \rightarrow Q \otimes \tilde{E}_2 \otimes \tilde{L}^m \rightarrow 0,
\]

where \( Q \) is a quotient sheaf supported on additional components of the scheme \( \tilde{S} \). For \( m \gg 0 \) the sequence of spaces of global sections associated with \( [S, 1] \) is
Then the sheaf $\det \pi \subseteq \pi$ ample. Take $X$ the sheaf $\text{Hilb} \mathbb{P}^n$ usualy following notations for projections of the universal subscheme $\text{Hilb} \mathbb{P}^{\Sigma}$ and coincide over $T_0$ then $h^0(S, E_2 \otimes L^m \otimes O(l_1 - l_2)) = h^0(S, E_2 \otimes L^m)$. This implies that $\chi(Q \otimes E_2 \otimes L^m) = 0$ for all $m \gg 0$. Hence $Q = 0$, and $l_1 = l_2$. The triple $S_1$ implies that $E_1 = E_2$.

9 $PGL(V)$-actions, GIT-stability and GIT-quotients

In this section we analyze the numerical Hilbert–Mumford criterion for an appropriate $PGL(V)$-linearized ample invertible vector bundle on $\mu(Q)$. This shows that good $PGL(V)$-quotient $\mu(Q)/PGL(V)$ is defined in the category of algebraic schemes over the field $k$. It turns out that every $PGL(V)$-orbit in $\mu(Q)$ contains at least one $PGL(V)$-semistable point.

Let $S$ be the universal quotient bundle on the Grassmannian $G(V, r)$, as usually $O_{G(V, r)}(1)$ is the positive generator in its Picard group. We use following notations for projections of the universal subscheme $\text{Hilb} P(V) G(V, r) \leftarrow \pi$ $\text{Univ} P(V) G(V, r) \rightarrow G(V, r)$. Form following sheaves on the Hilbert scheme $L = \det \pi_* \pi^* S(l)$. Since the projection $\pi : \text{Univ} P(V) G(V, r) \rightarrow \text{Hilb} P(V) G(V, r)$ is a flat morphism and sheaves $S(l)$ are locally free, then sheaves $L$ are invertible.

**Proposition 19.** Sheaves $L$ are very ample for $l \gg 0$.

**Proof.** Due to $[10]$ Proposition 2.2.5], for a projective scheme $X$, ample invertible sheaf $L$ on $X$, and Hilbert scheme $\text{Hilb} P(V) X$ with universal scheme $\text{Hilb} P(V) X \leftarrow \pi \text{Univ} P(V) X \rightarrow X$, sheaves $\det \pi_* \pi^* L^n$ are very ample whenever $n \gg 0$. Replace $L$ with its big enough tensor power so that $\det \pi_* \pi^* L$ is very ample. Take $X = G(V, r)$ and $L = \det S \otimes O_{G(V, r)}(l')$ with $l'$ so big as $L$ is ample. Then the sheaf $\det \pi_* \pi^* L = \det \pi_* \pi^* (\det S \otimes O_{G(V, r)}(l')) = \det \pi_* \pi^* S(l) = \tilde{L}^h$ for the appropriate $l \gg 0$ is very ample.

Consider the action of linear algebraic group $GL(V)$ on the vector space $V$ by its linear transformations. Then the induced actions of the group $PGL(V)$ on Grassmann variety $G(V, r)$ and on Hilbert scheme $\text{Hilb} P(V) G(V, r)$ are defined. The subscheme $\mu(Q)$ remains $GL(V)$-invariant. Fix the notation $\tilde{L} := \tilde{L}^h_{\mu(Q)}$.

We remind the following
Definition 17. \[10]\, definition 4.2.5] Let \( X \) be a k-scheme of finite type, \( G \) an algebraic k-group, and \( \alpha : X \times G \to X \) group action. \( G \)-linearization of a quasicoherent \( \mathcal{O}_X \)-sheaf \( F \) is an isomorphism of \( \mathcal{O}_X \times G \)-sheaves \( \Lambda : \alpha^* F \to pr_1^* F \) where \( pr_1 : X \times G \to X \) is the projection and the following cocycle condition holds:

\[
(id_X \times \text{mult})^* \Lambda = pr_{12}^* \Lambda \circ (\alpha \times id_G)^* \Lambda.
\]

Here \( pr_{12} : X \times G \times G \to X \times G \) is a projection onto first two factors, \( \text{mult} : G \times G \to G \) is a morphism of group multiplication in \( G \).

**Proposition 20.** Sheaves \( \tilde{L}_l \) carry \( GL(V) \)-linearization.

**Proof.** Let \( \gamma : G(V,r) \times GL(V) \to G(V,r) \) be the morphism of the action of the group \( GL(V) \). The universal quotient bundle \( S(l) \) carries \( GL(V) \)-linearization \( \Lambda : \gamma^* S(l) \to pr_1^* S(l) \). The linearization is induced by the epimorphism \( V \otimes O_{G(V,r)}(l) \to S(l) \). Now consider the morphism of \( GL(V) \)-action

\[
\alpha : \text{Hilb}^P(G(V,r)) \times GL(V) \to \text{Hilb}^P(G(V,r))
\]

induced by the action \( \gamma \). For \( l \gg 0 \) the following chain of isomorphisms holds:

\[
\alpha^* \tilde{L}_l = \det \alpha^* (\pi_* \pi'^* S(l))|_{\mu(\tilde{Q})} = \\
= \det \pi_* \pi'^* \gamma^* S(l)|_{\mu(\tilde{Q})} \overset{\det(\pi_* \pi'^* \Lambda|_{\mu(\tilde{Q})})}{\longrightarrow} \det \pi_* \pi'^* pr_1^* S(l)|_{\mu(\tilde{Q})} = \\
= \det pr_1^* \pi_* \pi'^* S(l)|_{\mu(\tilde{Q})} = pr_1^* \tilde{L}_l \quad (9.1)
\]

The central morphism in (9.1) is induced by the linearization \( \Lambda \) of the sheaf \( S(l) \) and provides the required linearization. \( \square \)

Now consider \[10\] ch. 4, sect. 4.2] an arbitrary one-parameter subgroup \( \lambda : \mathbb{A}^1 \setminus 0 \to GL(V) \). We denote the image of the point \( t \in \mathbb{A}^1 \setminus 0 \) under the morphism \( \lambda \) by the symbol \( \lambda(t) \). The composite of the morphism \( \lambda \) with the action \( \alpha \) leads to the morphism \( \alpha(\lambda) : \mathbb{A}^1 \setminus 0 \to \mu(\tilde{Q}) \) for any closed point \( \tilde{x} \in \mu(\tilde{Q}) \). This morphism is given by the correspondence \( t \mapsto \tilde{x}_t = \alpha(\lambda(t), \tilde{x}) \).

By the properness of the scheme \( \mu(\tilde{Q}) \) the morphism \( \alpha(\lambda) \) can be uniquely continued to the morphism \( \alpha(\lambda) : \mathbb{A}^1 \to \mu(\tilde{Q}) \). Then the point \( \tilde{x}_0 = \alpha(\lambda)(0) \) is a fixpoint of the action of the subgroup \( \lambda \). Notation: \( \tilde{x}_0 = \lim_{t \to 0} \alpha(\lambda(t)) \).

The subgroup \( \lambda \) acts on the fibre \( L_{\tilde{x}_0} \) of \( G \)-linearized vector bundle \( L \) with some weight \( r \). Namely, if \( \Lambda \) is the linearization on \( L \) then \( \Lambda(\tilde{x}_0, g) = g^r \cdot \text{id}_{L_{\tilde{x}_0}} \).

Define the weight of the corresponding one-dimensional representation of the group \( \lambda \) as \( w(\lambda, \lambda) = -r \).

Analogously for \( GL(V) \)-action \( \beta : Q \times GL(V) \to Q \) upon the subscheme \( Q \subset \text{Quot}^P_{\text{rk}(V) \otimes O_S} \) formed by semistable sheaves, and for the same one-parameter subgroup \( \lambda \) we have \( x_0 = \beta(\lambda)(0) = \lim_{t \to 0} \lambda(t)(x) \).

The main tool to analyze the existence of a group quotient is numerical Hilbert – Mumford criterion. Recall
Definition 18. [110, Definition 4.2.9] The point \( x \in X \) of the scheme \( X \) is semistable with respect to \( G \)-linearized ample vector bundle \( L \) if there exist an integer \( n \) and an invariant global section \( s \in H^0(X,L^n) \) such that \( s(x) \neq 0 \). The point \( x \) is stable if in addition the stabilizer \( \text{Stab}(x) \) is finite and \( G \)-orbit of the point \( x \) is closed in the open set of all semistable points in \( X \).

Theorem 3. (Hilbert – Mumford criterion) [110, Theorem 4.2.11] The point \( x \in X \) is semistable if and only if for all nontrivial one-parameter subgroups \( \lambda : \mathbb{A}^1 \setminus 0 \to G \) there is a following inequality

\[
\omega(x,\lambda) \geq 0.
\]

The point \( x \) is stable if and only if for all \( \lambda \) strict inequality holds.

Definition 19. Let \( G \) be an algebraic group, \( X,Y \) algebraic schemes, \( \alpha : Y \times G \to Y \) action of the group \( G \). The morphism \( \alpha' : X \times G \to X \) is called the action of the group \( G \) upon the scheme \( X \) induced by the action \( \alpha \) under the morphism \( \mu \) if the square

\[
\begin{array}{ccc}
X \times G & \xrightarrow{\alpha'} & X \\
\downarrow{\mu, \text{id}} & & \downarrow{\mu} \\
Y \times G & \xrightarrow{\alpha} & Y
\end{array}
\]

is cartesian.

Let \( G \) be a reductive algebraic group, \( X,Y,Z \) proper algebraic schemes, \( \mu : X \to Y \) and \( \phi : X \to Z \) scheme morphisms, \( \alpha : Y \times G \to Y \) and \( \beta : Z \times G \to Z \) actions of the group \( G \). Let also \( \alpha', \beta' : X \times G \to X \) be actions of the group \( G \) on the scheme \( X \) induced by actions \( \alpha \) and \( \beta \) respectively.

Definition 20. Actions \( \alpha \) and \( \beta \) are called \( X \)-concordant, if the following diagram commutes

\[
\begin{array}{ccc}
X \times G & \xrightarrow{\alpha'} & X \\
\downarrow{\mu, \text{id}} & & \downarrow{\mu} \\
X \times G & \xrightarrow{\beta'} & X
\end{array}
\]

Consider two morphisms \( \mu(\bar{Q}) \xrightarrow{\mu} \bar{Q} \xrightarrow{\phi} Q \). By definitions of actions of the group \( \text{GL}(V) \) upon schemes \( \mu(Q) \) and \( Q \) these actions are \( \bar{Q} \)-concordant.

Definition 21. Points \( y \in Y \) and \( z \in Z \) are called corresponding with respect to morphisms \( \mu \) and \( \phi \) if \( \mu^{-1}(y) \cap \phi^{-1}(z) \neq \emptyset \).

In the further text the symbol \( \lambda(x) \) denotes the orbit of the point \( x \) under the action of one-parameter subgroup \( \lambda \). The symbol \( \lambda(t)(x) \) denotes the point which corresponds to the given \( t \) in this orbit.
**Proposition 21.** Let $y \in Y$ and $z \in Z$ be corresponding points. For any one-parameter subgroup $\lambda$ fixedness of the point $y = y_0$ implies existence of the pair $(y_0, z_0)$ of corresponding points such that the point $z_0$ is fixed, and vice versa.

**Proof.** From the commutativity of the diagram

\[
\begin{array}{ccccccc}
y_0 \times \lambda & \rightarrow & Y \times \lambda & \leftarrow & X \times \lambda & \rightarrow & Z \times \lambda & \leftarrow & \lambda(z) \times \lambda \\
\downarrow \alpha(\lambda) & & \downarrow \alpha'(\lambda) \lor \beta'(\lambda) & & \downarrow \beta(\lambda) & & \downarrow \lambda(z) \\
y_0 & \rightarrow & Y & \leftarrow & X & \rightarrow & Z & \leftarrow & \lambda(z)
\end{array}
\]

it follows immediately that if points $y_0$ and $z$ correspond then for any $t$ points $\lambda(t)(y_0) = y_0$ and $\lambda(t)(z)$ also correspond. Now we note that by the properness of the scheme $Z$ there exist a point $z_0 = \lim_{t \to 0} \lambda(t)(z)$. Since the subset $\lambda(z)$ is open in $\mathbb{A}^1$, then we have for preimages in $X$ that $\phi^{-1}(\lambda(z))$ is open in $\phi^{-1}(\mathbb{A}^1)$. Also note that the intersection with the closed subset $\mu^{-1}(y_0)$ yields the openness of $\phi^{-1}(\lambda(z)) \cap \mu^{-1}(y_0)$ in $\phi^{-1}(\mathbb{A}^1) \cap \mu^{-1}(y_0)$. Moreover, for any $t \neq 0$ the intersection $\phi^{-1}(\lambda(t)(z)) \cap \mu^{-1}(y_0)$ is nonempty and closed in $\phi^{-1}(\mathbb{A}^1) \cap \mu^{-1}(y_0)$. Hence, $\phi^{-1}(z_0) \cap \mu^{-1}(y_0)$ is nonempty subset as required.

Let now the schemes $X, Y, Z$ be projective and morphisms $\mu$ and $\phi$ surjective. Let $L_Y$ and $L_Z$ be very ample invertible $G$-linearized vector bundles on schemes $Y$ and $Z$ respectively. Let $\Lambda_Y : \alpha^* L_Y \rightarrow pr_1^* L_Y$ and $\Lambda_Z : \beta^* L_Z \rightarrow pr_1^* L_Z$ be isomorphisms of their linearizations. For fibres of bundles $L_Y$ and $L_Z$ at closed points $y \in Y$ and $z \in Z$ respectively we introduce notations $L_{Y,y}$ and $L_{Z,z}$. Then restrictions $\Lambda_{Y,0}$ and $\Lambda_{Z,0}$ of isomorphisms of linearizations to actions of one-parameter subgroup $\lambda$ and to fibres at $\lambda$-fixpoints $y_0$ and $z_0$ has a form:

$$\Lambda_{Y,0} : \alpha(\lambda)^* L_{Y,y_0} \rightarrow pr_1^* L_{Y,y_0}, \quad \Lambda_{Z,0} : \beta(\lambda)^* L_{Z,z_0} \rightarrow pr_1^* L_{Z,z_0}.$$  

**Definition 22.** $G$-linearizations $\Lambda_Y$ and $\Lambda_Z$ are fibrewise concordant if for any two corresponding $\lambda$-fixpoints $y_0 \in Y$ and $z_0 \in Z$ there exists an isomorphism $f_0 : L_{Y,y_0} \rightarrow L_{Z,z_0}$ such that the diagram

\[
\begin{array}{ccc}
\alpha(\lambda)^* L_{Y,y_0} & \xrightarrow{\sim} & pr_1^* L_{Y,y_0} \\
\downarrow \alpha(\lambda)^* f_0 & & \downarrow \sim \phantom{pr_1^* f_0} \\
\beta(\lambda)^* L_{Z,z_0} & \xrightarrow{\sim} & pr_1^* L_{Z,z_0} \\
\end{array}
\]

commutes.

There is an obvious

**Proposition 22.** Fibrewise concordant linearizations of vector bundles $L_Y$ and $L_Z$ induce for any one-parameter subgroup $\lambda : \mathbb{A}^1 \setminus 0 \rightarrow G$ on fibres at corresponding fixpoints one-dimensional representations with equal weights.
Proof. The diagram (9.2) implies the equivalence of one-dimensional representations of multiplicative group of the field $k$. By the multiplicative property of characters equivalent representations of a group have equal characters. Hence the representations induced by morphisms $\Lambda_{Y,0}$ and $\Lambda_{Z,0}$ have equal weights. □

In our situation actions of the group $GL(V)$ on schemes $\mu(Q)$ and $Q$ are $\bar{Q}$-concordant. The reasoning in the proof of the proposition 5 and, in particular, the fibred diagram (2.6) allow to replace the schemes $\bar{Q}$, $\mu(Q)$, and $Q$ by their appropriate projective closures. Then setting $X = \bar{Q}$, $Y = \mu(Q)$, and $Z = \bar{Q}$ we apply the proposition 21.

**Proposition 23.** Bundles $\bar{L}_l$ and bundles $L_l = p_1*(E \otimes L^l)$ carry concordant linearizations.

Proof. Two corresponding points $\bar{x} \in \mu(Q)$ and $x \in Q$ define epimorphisms $V \otimes O_S \to \bar{E} \otimes \bar{L}^m$ and $V \otimes O_S \to E \otimes L^m$ respectively. Twist by $l$ and formation of $r_p E(l + m)$-th exterior power lead to epimorphisms

$$
\bigwedge^{r_p E(l + m)} (V \otimes L^l) \to \bigwedge^{r_p E(l + m)} \bar{E} \otimes \bar{L}^{l(m)}
$$

respectively. Taking of global sections for $l \gg 0$ before formation exterior powers gives

$$
\bigwedge^{r_p E(l + m)} (V \otimes H^0(S, L^l)) \to \bigwedge^{r_p E(l + m)} H^0(S, E \otimes L^{l(m)}).
$$

(9.3)

Since projections $\pi : \text{Univ}^{p(l)} G(V, r) \to \text{Hilb}^{p(l)} G(V, r)$ and $p_1 : Q \times S \to Q$ are flat morphisms, then direct images $\pi_* O_{\text{Univ}^{p(l)} G(V, r)}(l - m)$ and $p_1_! L^l$ are locally free of rank $P(l)$. By Grauert theorem [11] ch. III, corollary 12.9 one has $\pi_* O_{\text{Univ}^{p(l)} G(V, r)}(l - m) \otimes k_{\bar{x}} \cong H^0(S, L^l)$ and also $p_1_! L^l \otimes k_x \cong H^0(S, L^l)$.

Choose Zariski-open neighborhoods $\bar{U} \ni \bar{x}$ and $U \ni x$, providing local trivializations $\pi_* O_{\text{Univ}^{p(l)} G(V, r)}(l - m)|_{\bar{U}} \cong H^0(S, L^l) \otimes O_{\pi^{-1}(\bar{U})}$ and $p_1_! L^l|_U \cong H^0(S, L^l) \otimes O_{p_1^{-1}(U)}$. Without loss of generality we can assume that open subsets $\bar{U}$ and $U$ contain corresponding points representing objects $(\bar{S} \cong S, \bar{E} \cong E)$ and $E$ respectively. In this case $E$ is locally free sheaf. Then on open subsets $\bar{U}_0 \cong U_0$ formed by these points, there is identity isomorphism $H^0(S, L^l) \otimes O_{\pi^{-1}(\bar{U}_0)} \cong H^0(S, L^l) \otimes O_{p_1^{-1}(U_0)}$. By triviality of sheaves this isomorphism can be continued up to the isomorphism

$$
H^0(S, L^l) \otimes O_{\pi^{-1}(\bar{U})} \cong H^0(S, L^l) \otimes O_{p_1^{-1}(U)}.
$$

44
This fixes the isomorphism on fibres $H^0(\tilde{S}, \tilde{L}) \otimes k_x \cong H^0(S, L^1) \otimes k_x$ at points $x$ and $x$. Then epimorphisms $\mu$ can be include into the commutative diagram

$$\bigwedge^{r_{PE}(l+m)}(V \otimes H^0(\tilde{S}, \tilde{L})) \bigwedge^{r_{PE}(l+m)}(S, E \otimes L(l+m))$$

(9.4)

where the right hand side arrow is induced by the isomorphism $v$ obtained in the section 4.

Let $((\tilde{S}_{x_0}, \tilde{L}_{x_0}), \tilde{E}_{x_0})$ be a semistable pair corresponding to the point $x_0 \in \mu(\tilde{Q})$. Also $S_{x_0}$ be the fibre of the family $p_1 : Q \times S \to Q$ at the point $x_0 \in Q$, $L_{x_0} = L$ be its polarization and $E_{x_0}$ be a semistable coherent sheaf corresponding to the point $x_0 \in Q$. We introduce shorthand notations: $\tilde{W} = \bigwedge^{r_{PE}(l+m)}(V \otimes H^0(\tilde{S}_{x_0}, \tilde{L}_{x_0}))$ and $W = \bigwedge^{r_{PE}(l+m)}(V \otimes H^0(S_{x_0}, L^1_{x_0}))$. Notifying that the fibres $(\tilde{L}_l)_{x_0}$ and $(L_l)_{x_0}$ of vector bundles $\tilde{L}_l$ and $L_l$ at corresponding points $x_0$ and $x_0$ are given by the isomorphisms

$$(\tilde{L}_l)_{x_0} = \bigwedge^{r_{PE}(l+m)} H^0(\tilde{S}_{x_0}, \tilde{E}_{x_0} \otimes \tilde{L}_{x_0}^{(l+m)})$$

$$(L_l)_{x_0} = \bigwedge^{r_{PE}(l+m)} H^0(S_{x_0}, E_{x_0} \otimes L_{x_0}^{(l+m)})$$

using restrictions of linearizing isomorphisms onto fibres of vector bundles, and involving the diagram (9.4) we get the commutative diagram

$$\alpha(\lambda)^* \tilde{W} \sim \beta(\lambda)^* W$$

(9.5)

All slanted arrows are epimorphisms and the rest are isomorphisms. The front side of the diagram (9.4) proves the proposition.

**Corollary 5.** Corresponding $\lambda$-fixpoints of schemes $\mu(\tilde{Q})$ and $Q$ carry $\lambda$-actions with equal weights.

**Proof.** The result follows immediately from the previous proposition and the proposition 22. \qed
The application of numerical Hilbert–Mumford criterion yields in the following corollary.

**Corollary 6.** For any pair of corresponding points \((\tilde{x}, x)\), with \(\tilde{x} \in \mu(\tilde{Q})\) and \(x \in Q\), \(L_1\)-(semi)stability of the point \(x\) implies \(L_1\)-(semi)stability of the point \(\tilde{x}\), and vice versa.

To proceed further we need the theorem known from geometric invariant theory.

**Theorem 4.** [10, theorem 4.2.10] Let \(G\) be a reductive group acting on a projective scheme \(X\) with a \(G\)-linearized ample line bundle \(L\). Then there is a projective scheme \(Y\) and a morphism \(\pi: X^{ss}(L) \to Y\) such that \(\pi\) is a universal good \(G\)-quotient. Moreover, there is an open subset \(Y^s \subset Y\) such that \(X^s(L) = \pi^{-1}(Y^s)\) and such that \(\pi: X^s \to Y^s\) is a universal geometric quotient.

We apply this theorem in the following situation: \(X = \mu(\tilde{Q})\), \(G = \text{PGL}(V)\), \(L = \tilde{L}_l\), \(l \gg 0\). Since we do not know if the equality \(\mu(\tilde{Q})^{ss} = \mu(\tilde{Q})\) holds, by the corollary we have the following proposition.

**Proposition 24.** There is a quasiprojective algebraic scheme \(\tilde{M}\) with a morphism \(\mu(\tilde{Q}) \to \tilde{M}\), and \(\pi\) is a universal good \(\text{PGL}(V)\)-quotient. The scheme \(\tilde{M}\) contains an open subscheme \(\tilde{M}^s \supset \tilde{M}\) such that the restriction \(\pi|_{\mu(\tilde{Q})^{ss}} : \mu(\tilde{Q})^s \to \tilde{M}\) is a universal geometric quotient.

**Remark 10.** Let \(Q_0 \subset Q\) be an open subset of points corresponding to locally free quotient sheaves, \(\tilde{Q}_0 \cong Q_0\) its image under standard resolution, \(\mu(\tilde{Q}_0)\) corresponding subset in the scheme \(\text{Hilb}^q G(V, r)\). Since the morphism \(\mu\) takes distinct classes of isomorphic pairs \((\widetilde{S}, \tilde{L}), \tilde{E}\) to distinct classes of isomorphic subschemes in the Grassmannian and since \(GL(V)\)-actions on schemes \(Q\) and \(\mu(\tilde{Q})\) are concordant, we have an isomorphism of good GIT-quotients \(\tilde{M}_0 := \mu(\tilde{Q}_0)/GL(V) \cong Q_0/GL(V) =: M_0\).

**Remark 11.** Since the scheme \(\mu(\tilde{Q})\) is reduced then its quotient \(\mu(\tilde{Q})/PGL(V) = \tilde{M}\) is reduced scheme ([10, ch.0, §2, (2)]).

**Remark 12.** In our case \(\text{char}(k) = 0\) and by [10, ch.1, §2, theorem 1.1], the scheme \(\tilde{M}\) is Noetherian algebraic scheme because \(\mu(\tilde{Q})\) is Noetherian algebraic scheme.

### 10 Morphisms of compactifications and projectivity of \(\tilde{M}\).

Recall that the subset \(Q\) formed in Quot\(_{rp(l)}(V \otimes L(-m))\) by semistable coherent sheaves, is a quasiprojective algebraic scheme. The morphism of standard resolution \(\phi: Q \to Q\) is a projective morphism of algebraic schemes. This implies
that \( \widetilde{Q} \) is quasiprojective algebraic scheme. By the construction, it is supplied with flat family of \((\tilde{S},L,r,\tilde{r}p_E(m))\)-admissible schemes \( \Sigma_{\tilde{Q}} \), with locally free sheaf \( \tilde{E}_{\tilde{Q}} \). This sheaf induces a mapping \( \Sigma_{\tilde{Q}} \to G(r,V) \) which becomes a closed immersion when restricted to fibres of the family \( \Sigma_{\tilde{Q}} \). There is a morphism of the base \( \tilde{Q} \) of the family \( \Sigma_{\tilde{Q}} \) into the Hilbert scheme of subschemes in the Grassmannian \( \mu: \tilde{Q} \to \text{Hilb}^{P(t)}G(r,V) \). We denote by the symbol \( \overline{M} \) the union of components of the Gieseker – Maruyama scheme that contain locally free sheaves.

Note that there is (set-theoretical) surjective mapping \( \kappa: \overline{M} \to \tilde{M} \) given by the formula \( E \mapsto (\tilde{S},\sigma^*E/\text{tors}) \). The schemes \( \overline{M} \) and \( \tilde{M} \) contain open subschemes \( M^*_0 \subset \overline{M} \) and \( \tilde{M}^*_0 \subset \tilde{M} \). The restriction \( \kappa_0 := \kappa|_{\tilde{M}^*_0}: M^*_0 \to \tilde{M}^*_0 \) defines scheme isomorphism.

**Proposition 25.** There is a birational morphism of Noetherian schemes \( \kappa: \overline{M} \to \tilde{M} \).

**Proof.** Consider a product \( \overline{M} \times \tilde{M} \) with projections \( \overline{M} \leftarrow \overline{M} \times \tilde{M} \xrightarrow{\pi} \tilde{M} \) and a subset \( A := \{(x,\kappa(x)) \in \overline{M} \times \tilde{M} | x \in \overline{M} \} \). Also take a subset \( A_0 = \{(x,\kappa(x)) \in \overline{M} \times \tilde{M} | x \in \overline{M}^*_0 \} = A \cap M^*_0 \times \tilde{M}^*_0 \) corresponding to GIT-stable \( S \)-pairs. The inclusion \( A_0 \to \overline{M} \times \tilde{M} \) supplies the subset \( A_0 \) with structure of locally closed subscheme in the product \( \overline{M} \times \tilde{M} \). Form a closure \( \overline{A}_0 \) of subscheme \( A_0 \) in the product \( \overline{M} \times \tilde{M} \).

Note that there is an inclusion of sets \( A \subset \overline{A}_0 \). The image of the subset \( A \) coincides with the set \( \overline{A}_0 \). This follows immediately from the standard resolution of a family of semistable coherent sheaves with a base \( \mathbb{A}^1 \). The generic fibre of the family defines the point in \( A_0 \), and special fibre of the family defines the point in \( \overline{A}_0 \setminus A_0 \). In this case the special fibre corresponds to the point of the subset \( A \setminus A_0 \). Considering different immersions \( \mathbb{A}^1 \xleftarrow{A} \overline{M} \) we get a bijection \( \overline{A}_0 \cong A \). Then the subset \( A \) is supplied with a structure of a closed subscheme in the product \( \overline{M} \times \tilde{M} \). By the construction of the subset \( A \) we have \( \overline{p}(A) = \overline{M} \).

By the construction of scheme \( \tilde{M} \) also \( \tilde{p}(A) = \tilde{M} \).

Morphisms \( \overline{\pi} \) and \( \overline{\kappa} \) are defined as composite maps due to commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\kappa} & \tilde{M} \\
\downarrow{\overline{\pi}} & & \downarrow{\tilde{p}} \\
\overline{M} & \xrightarrow{\overline{\kappa}} & \tilde{M} \times \tilde{M} \\
\end{array}
\]

By the construction of the morphism \( \kappa \), morphisms \( \overline{\pi} \) and \( \overline{\kappa} \) are surjective and birational. Besides, for any closed point \( x \in \overline{M} \) the correspondence \( x \mapsto (x,\kappa(x)) \) defines set-theoretical map \( \overline{M} \to A \). This map is an inverse for the morphism \( \overline{\pi} \) if this morphism is considered as a map of sets. Then the morphism \( \overline{\pi} \) is birational and bijective on every component of the scheme \( A \). Hence \( \overline{\pi}: A \to \overline{M} \) is an isomorphism. Redenoting the composite as \( \kappa: \overline{M} \xrightarrow{\overline{\pi}^{-1}} A \xrightarrow{\kappa^{-1}} \tilde{M} \) we get the required morphism of schemes. \( \square \)
Proposition 26. \( \widetilde{M} \) is a projective scheme.

The proof of this proposition is based on the simple lemma.

Lemma 4. Quasiprojective complete scheme is projective.

Proof. Let \( X \) be a quasiprojective complete scheme. Since \( X \) is quasiprojective, then there is an appropriate projective space \( \mathbb{P} \) and an immersion \( X \hookrightarrow \mathbb{P} \). Since \( X \) is complete, then for any scheme \( Y \) the projection onto the first factor \( pr_2 : X \times Y \to Y \) takes closed subschemes to closed subschemes. Set \( Y = \mathbb{P} \) and consider the diagonal embedding \( \Delta : \mathbb{P} \hookrightarrow \mathbb{P} \times \mathbb{P} \). By the separatedness of the scheme \( \mathbb{P} \) this diagonal embedding is closed. In the commutative diagram with fibred square

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & X \times \mathbb{P} \\
\downarrow & & \downarrow^{pr_2} \\
\mathbb{P} & \xrightarrow{\Delta} & \mathbb{P} \times \mathbb{P}
\end{array}
\]

the diagonal immersion \( \Delta \) is closed. Hence the morphism \( \delta \) is closed immersion.

The image \( pr_2 \circ \delta(X) \cong X \) is closed in \( \mathbb{P} \) by the completeness of the scheme \( X \). Then \( X \) is a projective scheme. \( \square \)

Proof of the proposition 26. By the lemma it is enough to confirm that \( \widetilde{M} \) is a complete scheme. As proven before, there is a morphism \( \alpha : \overline{M} \to M \), where \( \overline{M} \) is a projective scheme. We prove that for any scheme \( Y \) and for the projection \( pr_2 : \tilde{M} \times Y \to Y \) the image \( pr_2(Z) \) of any closed subscheme \( Z \subset \tilde{M} \times Y \) is closed in \( Y \). Let \( Z' \) be a preimage of the subscheme \( Z \) in \( \overline{M} \times Y \). Then there is a commutative diagram where the square is fibred

\[
\begin{array}{ccc}
Z' & \xrightarrow{pr_2} & Y \\
\downarrow & & \downarrow \\
\widehat{M} \times Y & \xrightarrow{pr_2} & \widetilde{M} \times Y
\end{array}
\]

Since \( \overline{M} \) is complete, then the image \( pr_2(Z') = pr_2(Z) \) is closed in \( Y \). This completes the proof. \( \square \)

Remark 13. In papers [4, 5, 6] we constructed the compactification of moduli of stable vector bundles which is called as constructive compactification and denoted by \( \overline{M} \). It is shown that the constructive compactification has a birational projective morphism \( \phi^c : \overline{M} \to \overline{M} \) onto the scheme of Gieseker–Maruyama. Then the composite of this morphism with the morphism \( \kappa \) yields a birational projective morphism of schemes \( \phi : \overline{M} \to \overline{M} \to \widetilde{M} \).
11 Comparison of equivalences

The purpose of this section is to examine the relation among M-equivalence of semistable pairs and GIT-equivalence on the scheme $\mu(Q)$.

We consider the following procedure of *passing-to-the-limit*. This computation is completely parallel to that in [10] lemma 4.4.3.

Take a pair $((S, L), \tilde{E})$ and fix an epimorphism $h: H^0(S, \tilde{E} \otimes \tilde{L}) \rightarrow \tilde{E}$. If also the isomorphism $H^0(S, \tilde{E} \otimes \tilde{L}) \cong V$ is fixed then the epimorphism $h$ defines the point $h \in \text{Hilb}^{((-\nu(L), r(n))}$ and the point $h \in \text{Quot}^{((-\nu(L), r(n)))}$.

One-parameter subgroup $\lambda: \mathbb{A}^1 \setminus 0 \rightarrow SL(V)$ is defined completely by means of the decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$ of vector space $V$ into the direct sum of weight subspaces $V_n$, $n \in \mathbb{Z}$, of weight $n$. Namely, for any $v \in V_n$ the action of elements of the subgroup $\lambda$ is defined by the expression $v \cdot \lambda(T) = T^n v$. Of course, for almost all $n$ holds $V_n = 0$. Define ascending filtrations for $V$ and for the sheaf $\tilde{E}$ by the expressions $V(n) = \bigoplus_{n \leq n} V_n$, $\tilde{E}(n) = h(V_n) \otimes \tilde{L}$. Then the following epimorphisms are defined: $h_n: V_n \otimes \tilde{L} \rightarrow \tilde{E}_n$, for $\tilde{E}_n = \tilde{E}(n)/\tilde{E}(n-1)$.

Taking the sum over all weights yields in an epimorphism $\tilde{\mathcal{E}} = \bigoplus_{n} h_n: V \otimes \tilde{L} \rightarrow \bigoplus_{n} \tilde{E}_n = \text{gr}(\tilde{E})$.

Claim 2. In $\text{Quot}^{((-\nu(L), r(n)))}$ one has $\mathcal{E} = \lim_{T \rightarrow 0} h \cdot \lambda(T)$.

We construct explicitly the family $\theta: V \otimes \tilde{L} \otimes k[T] \rightarrow \mathcal{E}$ parametrized by affine line $\mathbb{A}^1 = \text{Spec} k[T]$, such that $\theta_0 = h$ and $\theta_\rho = h \cdot \lambda(\rho)$ for $\rho \neq 0$. Let $\mathcal{E} := \bigoplus_{n} \tilde{E}(n) \otimes T^n \subset \tilde{E} \otimes k[T, T^{-1}]$. Since the direct sum contains finite collection of nonzero summands, then $\mathcal{E}$ is a coherent sheaf on $\mathbb{A}^1 \times \tilde{S}$. Indeed, let $N$ be positive integer such that $V_n = 0$ and $\tilde{E}_n = 0$ for $n \leq -N$. Then $\mathcal{E} \subset \tilde{E} \otimes T^{-N} k[T]$. Similarly, define a module $\mathcal{V} := \bigoplus_{n} V(n) \otimes \tilde{L} \otimes T^n \subset V \otimes k \tilde{L} \otimes k[T, T^{-1}]$. It is clear that the epimorphism $h$ induces a surjection $h': \mathcal{V} \rightarrow \mathcal{E}$ of $\mathbb{A}^1$-flat coherent $\mathcal{O}_{\mathbb{A}^1 \times \tilde{S}}$-sheaves. Finally, define the isomorphism $\gamma: V = \bigoplus_{n} V(n) \otimes T^n \otimes k[T]$ by means of restrictions $\gamma|_{V_\nu} = T^{\nu} id_{V_\nu}$ for all $\nu$. Also define the morphism $\theta$ by the commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{n} \tilde{E}(n) \otimes T^n & \longrightarrow & \mathcal{E} \\
\gamma \downarrow & & \downarrow h \\
V \otimes \tilde{L} \otimes k[T] & \xrightarrow{\gamma} & \mathcal{V}
\end{array}
$$

Restriction to the fibre corresponding to $T = 0$ leads to

$$\mathcal{E}/T \mathcal{E} = \bigoplus_{n} \tilde{E}_n$$

and hence $\theta_0 = \bigoplus_{n} h_n$.

Restriction to the open complement $\mathbb{A}^1 \setminus 0$ corresponds to the invertibility of the element $T$. Hence tensoring by $\otimes k[T, T^{-1}]$ we get a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{n} \tilde{E}(n) \otimes T^n & \longrightarrow & \mathcal{E} \\
\gamma \downarrow & & \downarrow h \\
V \otimes \tilde{L} \otimes k[T] & \xrightarrow{\gamma} & \mathcal{V}
\end{array}
$$

49
The map $\gamma$ defines action of one-parameter subgroup $\lambda$. Then, $\theta$ is the required mapping.

**Definition 23.** Let $G$ be an algebraic group and $f : Y \to X$ a GIT-quotient. Closed points $y_1$ and $y_2$ of the scheme $Y$ are GIT-equivalent if $f(y_1) = f(y_2)$.

**Proposition 27.** M-equivalence implies GIT-equivalence, and vice versa.

**Proof.** Consider two M-equivalent semistable pairs $((\tilde{S}, \tilde{L}), \tilde{E})$ and $((\tilde{S}_{gr}, \tilde{L}_{gr}), \tilde{E}_{gr})$. Each fibre of the morphism of formation of GIT-quotient of $\mu(Q)$ contains one closed orbit. Indeed, it is known [10, Theorem 4.3.3] that points representing polystable coherent sheaves and only these points have closed orbits in Quot. By the concordance of $PGL(V)$-actions, the pairs of the form $((\tilde{S}_{gr}, \tilde{L}_{gr}), \tilde{E}_{gr})$ and only these pairs have closed orbits in Hilbert scheme.

It is enough to prove the proposition for such points that one of them has the form $((\tilde{S}_{gr}, \tilde{L}_{gr}), \tilde{E}_{gr})$.

Consider an epimorphism
\[ V \otimes \tilde{L}^\vee \to \tilde{E}. \] (11.1)

Since $K^{dS} = \mu(Q)$, then the pair $((\tilde{S}, \tilde{L}), \tilde{E})$ has a preimage in $Q$. Let this is an epimorphism $V \otimes L^\vee \to E$. The only nontrivial situation is that when $E$ is strictly semistable. Consider its $S$-equivalence class $[E]$. Let $\sigma : \mathfrak{S} \to S$ be the minimal resolution in the monoid $\triangle [E]$, $\mathcal{L}$ be a very ample invertible sheaf on the scheme $\mathfrak{S}$. The morphism of the minimal resolution includes into the commutative diagram
\[
\begin{array}{ccc}
\mathfrak{S} & \xrightarrow{\sigma'} & \tilde{S}_{gr} \\
\sigma_{gr} \downarrow & & \sigma_{gr} \downarrow \\
S & \xrightarrow{\sigma} & S
\end{array}
\]

Then the epimorphism (11.1) induces the epimorphism
\[ V \otimes \sigma'_{gr} \tilde{L}^\vee \to \sigma'_{gr} \tilde{E}. \] (11.2)

Let $\chi(\sigma'_{gr} \tilde{E} \otimes \mathcal{L}')$. By the results of section 7, if $F_i$ are subsheaves in Jordan – Hölder filtration for the sheaf $E$ then quotient sheaves $\mathcal{T}_i/\mathcal{T}_{i-1} = \sigma'(F_i/F_{i-1})/\text{tors} = \sigma'_{gr}(\tilde{F}_i/\tilde{F}_{i-1})/\text{tors}$ are locally free.

Consider the scheme of quotients $\operatorname{Quot}(\mathfrak{P}(V \otimes \mathcal{L}'))$. In this scheme the passing-to-
the-limit process in the family of locally free sheaves with general sheaf of the form (11.2) is also considered. The limit object has the form $\bigoplus_i \mathcal{F}_i \big/ \mathcal{F}_{i-1} = \bigoplus_i \mathcal{F}(F_i \big/ \mathcal{F}_{i-1}) \big/ \text{tors} = \sigma_{\text{gr}}^* \mathcal{E} \big/ \text{tors}$. The family of $\mathcal{O}_\mathcal{E}$-sheaves $\mathcal{E}$ obtained in the passing-to-the-limit process, induces the morphism of its base into Hilbert scheme $\text{Hilb}^{P(t)} G(V, r)$. Indeed, formation of direct images for sheaves on fibres at general points $T \neq 0$ under the morphism $\sigma'$ leads to $\mathcal{O}_{\tilde{\mathcal{S}}}$-sheaves. These sheaves are isomorphic to $\tilde{\mathcal{E}}$. Analogously, the direct image of the sheaf on the fibre at the point $T = 0$ under the morphism $\sigma'$ leads to $\mathcal{O}_{\tilde{\mathcal{S}}}$-sheaf $\tilde{\mathcal{E}}_{\text{gr}}$.

The reasoning done shows that the orbit of the point representing the object $((\tilde{\mathcal{S}}_{\text{gr}}, \tilde{\mathcal{L}}_{\text{gr}}), \mathcal{E}_{\text{gr}})$, belongs to the closure of the orbit of $M$-equivalent point $((\tilde{S}, \tilde{L}), E)$. This implies that GIT-equivalence is equivalent to $M$-equivalence.

\[ \square \]

12 $\tilde{M}$ as moduli space

In this section we prove that the constructed scheme $\tilde{M}$ is a coarse moduli space for the functor $f : (\text{RSchemes}_k) \to (\text{Sets})$ in the theorem 11.

It is enough to confirm that the functor $f$ is corepresented by the scheme $\tilde{M}$. Choose an object $((\tilde{S}, \tilde{L}), \mathcal{E})$. Note that the sheaf $\tilde{E} \otimes \tilde{L}$ defines the immersion $j : \tilde{S} \hookrightarrow G(V, r)$. This immersion is defined not in unique way but up to the class of the isomorphism $H^0(\tilde{S}, \tilde{E} \otimes \tilde{L}) \cong V$ modulo multiplication by nonzero scalars $\vartheta \in k^*$. Hence the point corresponding to the subscheme $j(\tilde{S}) \subset G(V, r)$, is defined in the Hilbert scheme $\text{Hilb}^{P(t)} G(V, r)$ up to the action of the group $PGL(V)$. Then the object $((\tilde{S}, \tilde{L}), \tilde{E})$ defines the morphism $h \in \text{Hom}(\text{Spec} k, \tilde{M})$.

Inversely, by the proposition 27, the morphism $h \in \text{Hom}(\text{Spec} k, \tilde{M})$ distinguishes a point representing the $M$-equivalence class of object $((\tilde{S}, \tilde{L}), \mathcal{E})$.

We construct for any scheme $B$ and for natural transformation $\psi' : f \to F'$ a unique natural transformation $\omega : \tilde{M} \to F'$ such that $\psi' = \omega \circ \psi$.

Let the transformation $\alpha$ correspond to the flat family $\pi : \tilde{\Sigma} \to B$ with fibrewise polarization $\overline{L}$, supplied with the family of locally free sheaves $\overline{E}$.

In this case the restriction onto any fibre $\pi^{-1}(y)$ of the morphism $\pi$ provides an object $((\pi^{-1}(b), \overline{E}|_{\pi^{-1}(b)}), \overline{E}|_{\pi^{-1}(b)})$. This object belongs to the class $\mathfrak{F}$. Then there is a morphism $\tilde{\Sigma} \to G(\pi_* (\overline{E} \otimes \overline{L}), r)$ such that the triangle

\[ G(\pi_* (\overline{E} \otimes \overline{L}), r) \xrightarrow{\pi} \tilde{\Sigma} \]

51
commutes. The sheaf \( \pi_*(\tilde{E} \otimes \tilde{L}) \) is locally free, hence the Grassmannian bundle \( G(\pi_*(\tilde{E} \otimes \tilde{L}), r) \) is locally trivial over \( B \). Let \( \bigcup_i B_i = B \) be the trivializing open cover. Subfamilies \( \tilde{\Sigma}_i \) are defined as fibred products

\[
\begin{array}{ccc}
\tilde{\Sigma} & \xrightarrow{\pi} & \tilde{\Sigma}_i \\
\downarrow & & \downarrow \\
B & \xrightarrow{\pi_i} & B_i
\end{array}
\]

The horizontal arrows are open immersions. Fix isomorphisms of trivializations \( \tau_i : G(\pi_*(\tilde{E} \otimes \tilde{L}), r)|_{B_i} \to G(V, r) \times B_i \). The composite map \( \tilde{\Sigma}_i \xrightarrow{\tau_i} G(\pi_*(\tilde{E} \otimes \tilde{L}), r)|_{B_i} \xrightarrow{\pi_i} G(V, r) \times B_i \xrightarrow{pr_2} G(V, r) \) provides a morphism of the base to the Hilbert scheme \( \mu_i : B_i \to \text{Hilb}^{P(t)} G(V, r) \). This morphism is defined up to \( PGL(V) \)-action. Elements of \( PGL(V) \) define gluing the elements of the trivializing cover. Then the formation of GIT-quotient leads to the morphism \( B \to \tilde{M} \). Its dual in the opposite category \((\text{Schemes}_k)^{op}\) defines the natural transformation \( \omega : \tilde{M} \to F' \).

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**References**

[1] D. Gieseker, On the moduli of vector bundles on an algebraic surface // Annals of Math., (106) 1977, pp. 45–60.

[2] G. Ellingsrud, L. Göttsche, Variation of moduli spaces and Donaldson invariants under change of polarization // J. reine angew. Math., (467) 1995, pp. 1–49.

[3] M. Maruyama, Moduli of stable sheaves, II // J. Math. Kyoto Univ. (JMKYAZ), (18-3) 1978, pp. 557–614.

[4] N. V. Timofeeva, A compactification of the moduli variety of stable vector 2-bundles on a surface in the Hilbert scheme // Mat. Zametki, (82:5) 2007, pp. 756–769 (Russian); English translation in Math. Notes, (82:5-6) 2007, pp. 677–690.

[5] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface // Mat. Sb., (199:7) 2008, pp.103–122 (Russian); English translation in Sb. Math., (199:7) 2008, pp. 1051–1070.

[6] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. II // Mat. Sb., (200:3) 2009, pp. 95 – 118 (Russian); English translation in Sb. Math., (200:3) 2009, pp. 405–427.
[7] N. V. Timofeeva, On degeneration of surface in Fitting compactification of moduli of stable vector bundles // Mat. Zametki, (90:1) 2011, pp. 144–151 (Russian); English translation in Math. Notes, (90:1) 2011; arXiv:0809.1148v3.

[8] D. Gieseker, J. Li, Moduli of high rank vector bundles over surfaces // Journal of the Amer. Math. Soc., (9:1) 1996, pp. 107–151; ArXiv:9410004v1.

[9] K. G. O'Grady, Moduli of vector bundles on projective surfaces: some basic results // Inv. Math., (123) 1996, pp. 141–207.

[10] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves. Aspects Math., E31. Vieweg, Braunschweig, 1997.

[11] R. Hartshorne, Algebraic geometry. Graduate Texts in Mathematics, 52. Springer-Verlag, New York – Heidelberg – Berlin, 1977.

[12] D. Mumford, Lectures on curves on algebraic surface. Annals of Mathematics studies, 59. Princeton Univ. Press, Princeton – New Jersey, 1966.

[13] J. Le Potier, Fibrés stables de rang 2 sur $\mathbb{P}_2(\mathbb{C})$ // Math. Ann. (241) 1979, pp. 217–256.

[14] A. Grothendieck, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes des schémas. Troisième partie. Publ. math. de I.H.É.S., tome 28. I.H.É.S., Paris, 1966.

[15] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Grad. Texts in Math., 150, Springer-Verlag, New York – Berlin, 1995.

[16] D. Mumford, J. Fogarty, Geometric Invariant Theory. Second enlarged Ed., Springer-Verlag, Berlin – Heidelberg – New York, 1982.