Non-Commutativity and the Theta Term

O Obregón\textsuperscript{1} and R Santos-Silva\textsuperscript{2}

Departamento de Física, Universidad de Guanajuato
C.P. 37150, León, Guanajuato, México.

E-mail: \textsuperscript{1}octavio@fisica.ugto.mx
E-mail: \textsuperscript{2}rsantos@fisica.ugto.mx

Abstract. In the present paper we will study the non-commutativity in the space-time through the map developed by N. Seiberg and E. Witten, and explore the topological implications due to the introduction of non-commutativity. We will explore this topological properties using a non-commutative extension of the $\theta$-term (second Chern class) on the 4-dimensional sphere.

1. Introduction
Non-commutativity in the space-time took N. Seiberg y E. Witten to construction of the well known Seiberg-Witten map \cite{1}. Through the Moyal product they introduced non-commutativity on the space-time and fields; thereby the non-commutative gauge fields are an infinity serie on the non-commutative parameter $\theta_{\mu\nu} \ (\left[x_\mu, x_\nu\right] = \theta_{\mu\nu}$ where $x_\mu$ are the coordinates on the space-time). But on the other hand assuming gauge invariance, it could be determined every order on the non-commutative field in terms of the usual gauge field (commutative field). This construction was realized for $U(1)$ gauge fields.

Such construction was extending by B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess for non-abelian gauge fields coupled with matter \cite{2}, this proposal had been studied widely in the literature; in particular H. García-Compeán, O. Obregón, C. Ramírez and M. Sabido used this ideas to propose non-commutativity gravity, in a gauge field framework \cite{4}, also they extended the Euler characteristic and the Signature to their non-commutative counterparts \cite{5}.

Motivated by the idea of extending topological invariants on this non-commutative structure, the aim of this paper is to study the solution of instantons in Yang-Mills theories in the context of non-commutativity, i.e. we will focus on the study and definition of the non-commutative instanton number and which is the contribution or correction to the usual instanton number.

The structure of this paper is the following. We give two reviews: on the construction of the Seiberg-Witten map, continued by a review on the instanton solution on the $S^4$ equivalently $\mathbb{R}^4$ with topological charge equal 1. Then we give the definition of non-commutative $\theta$-term and the computation up to second order, finally we present the conclusion and a discussion about the results.
2. Non-Commutativity Via The Seiberg-Witten map

We are interested in non-commutativity in the space-time via the Seiberg-Witten map [1]. Such theory was extended by J. Wess [2, 3] for any gauge field coupled with matter. Below we present a brief description of the construction.

The central idea is to deform the algebraic structure of continuous spaces in particular the polynomials in \( N \) variables generated by powers of \( x^I \) where \( I = 0, \ldots, N \), which is a freely generated algebra \( \mathbb{C}[x^1, \ldots, x^N] \). Now consider the usual commutations relations between the coordinates

\[
[x^\mu, x^\nu] = 0,
\]

this algebraic structure will be deformed assuming

\[
[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}
\]

where \( \theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R} \) i.e. \( \theta^{\mu\nu} \) is the non-commutative parameter and \( \hat{x} \) are the non-commutative coordinates (as we can see this algebra is similar to the Heisenberg algebra in the phase space) for a formal description see [3].

Explicitly this modifies the way we multiply polynomials and in general functions over the non-conmutative variables in terms of the commutative variables through the \( \star \) (Moyal) product defined by:

\[
f \star g(x) = \mu(e^{\frac{i}{2} \Theta^{\alpha\beta} \partial_\alpha \otimes \partial_\beta} f \otimes g).
\]

In this context the gauge transformations of a matter field \( \Psi(x) \) is defined as

\[
\delta_\Lambda \Psi(x) = i\Lambda \star \Psi(x),
\]

\( \Lambda(x) \) is the non-commutative gauge parameter which is Lie algebra valued i.e. \( \Lambda(x) = \Lambda^a(x)T^a \).

Now let us compute explicitly the following variation of the field \( \Psi \)

\[
(\delta_{\Lambda_1}, \delta_{\Lambda_2}, -\delta_{\Lambda_1}, \delta_{\Lambda_2})\Psi = (\Lambda_1 \star \Lambda_2 - \Lambda_2 \star \Lambda_1) \star \Psi = \frac{1}{2} \left( [\Lambda_1^a, \Lambda_2^b] [T^a, T^b] + [\Lambda_1^a, \Lambda_2^b] [T^b, T^a] \right) \star \Psi.
\]

It is worth mentioning that the fields are not Lie algebra valued because not only we have commutators but also we have anticommutators. So the algebra that close both operations (commutators and anticommutators) is known as universal enveloping algebra.

The covariant derivative is defined by

\[
D^\mu_\star \Psi(x) = \partial_\mu \Psi(x) - i\hat{A}_\mu \star \Psi(x),
\]

where \( \hat{A}_\mu \) is the non-commutative gauge field which transform as

\[
\delta_\Lambda \hat{A}_\mu = \partial_\mu \Lambda + i[\Lambda, \hat{A}_\mu].
\]

We can see that these terms have an infinity degrees of freedom, but N. Seiberg and E. Witten [1] showed that all the higher order terms depend only on the degrees at zeroth order (the commutative term) i.e. of the gauge parameter \( \Lambda^{0a}T^a \) and the gauge field \( A^{\mu a}_1 T^a \). Let us assume that the gauge parameter \( \Lambda_\alpha \) depends only on \( \alpha \) and \( A_\mu \) i.e. the gauge parameter and
the gauge field respectively. With these assumptions bearing in mind, let us substitute in the expression (5)

\[ \Lambda_\alpha \ast \Lambda_\beta - \Lambda_\beta \ast \Lambda_\alpha + i(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) = i\Lambda_{-i[\alpha, \beta]}, \]

This expression could be solved perturbatively assuming an expansion in the parameter \( \theta \) as \( \Lambda_\alpha = \Lambda_\alpha^0 + \Lambda_\alpha^1 + \cdots \), where \( \Lambda_\alpha^0 = \alpha = \alpha^a T^a \).

For example up to first order in \( \theta \), we find the following expression:

\[ \delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha - i[\alpha, \Lambda_\beta] - i[\Lambda_\alpha, \beta] - \Lambda_{-i[\alpha, \beta]} = -\frac{1}{2} \theta^{\mu\nu} \{ \partial_\mu \alpha, \partial_\nu \beta \}. \]

whose solution is \( \Lambda_\alpha = \alpha - \frac{1}{4} \theta^{\mu\nu} \{ A_\mu, \partial_\nu \alpha \} \). With this expression we can compute in a similar way the expression for matter fields assuming the transformation at zeroth order \( \delta_\alpha \Psi^0 = i\alpha \Psi^0 \).

For the gauge field we expand again in orders of \( \theta \) as \( \hat{A} = A^0 + A^1 + A^2 + \cdots \) and substituting in (7), up to first order in \( \theta \), we obtain

\[ A^1_\mu = \frac{1}{4} \theta^{\rho\sigma} (\{ F_{\rho\mu}, A_\sigma \} - \{ A_\rho, \partial_\sigma A_\mu \}), \]

finally the field strength tensor is given by \( F^*_{\mu\nu} = i[D^*_{\mu}, d^*_\nu] \), whose solution up to first order is

\[ F^*_{\mu\nu} = F_{\mu\nu} + \frac{1}{4} \theta^{\rho\sigma} (2 \{ F_{\rho\mu}, F_{\sigma\nu} \} + \{ D_\rho F_{\mu\nu}, A_\sigma \} - \{ A_\rho, \partial_\sigma F_{\mu\nu} \}). \]

Where we can identify \( F^0_{\mu\nu} = F_{\mu\nu} \) and \( F^1_{\mu\nu} = \frac{1}{2} \theta^{\rho\sigma} (2 \{ F_{\rho\mu}, F_{\sigma\nu} \} + \{ D_\rho F_{\mu\nu}, A_\sigma \} - \{ A_\rho, \partial_\sigma F_{\mu\nu} \}) \).

3. Instantons

In this section we will review briefly the solution of instantons in \( S^4 \) (4-sphere) or equivalently in \( \mathbb{R}^4 \) in Yang-Mills theory (since \( S^4 \) is the one point compactification) we will follow closely the references [6, 7].

An instanton is a solution to the classical equations of motion of finite action namely, if \( A \) is an instanton solution which satisfies the boundary conditions \( F_{\mu\nu} \to 0 \) as \( |x| \to \infty \) such that \( S = \int_{S^4} F \wedge *F \) remains finitely, where \( * \) is the hodge dual\(^1\).

This implies from the mathematical point of view that the potential \( A \) is a self-dual or anti-self-dual connection (i.e. satisfies the instantons equations \( *F = \pm F \)) in a principal bundle over a 4-manifold.

For our case of interest the \( S^4 \) we will compute explicitly the potential that satisfies the mentioned boundary conditions and also under the anzats that the potential is purely gauge we assume that the potential has the following form

\[ A_\mu(x) = f(x^2)g^{-1}(x)\partial_\mu g(x) \]

where \( f \) satisfies

\(^1\) In physics \(*F\) sometimes is called the dual of \( F \) and usually is denoted by \( \tilde{F} \).
we consider the following choice \( g(x) = g_\alpha(x^2)^\tau^\alpha \), where \( \tau^\alpha \)'s are defined as

\[
\tau^\alpha = \begin{cases} I, & \tau_j = i\sigma_j \Rightarrow \tau^0 = \tau^0, \quad \tau^\dagger_j = -i\sigma_j \\
\end{cases}
\]

with \( j = 1, 2, 3 \) and \( \sigma_j \)'s are the Pauli matrices and \( I \) is the identity matrix \( 2 \times 2 \). Since \( g(x) \) is a \( SU(2) \)-valued function such that

\[
g_\alpha \tau^\alpha g_\beta \tau^\beta = I.
\]

Now consider \( g_\alpha(x^2) = \partial_\alpha h(x^2) = 2x^2 h'(x^2) \), substituting in the previous expression and under the anzats we find

\[
h(x^2) = (x^2)^{1/2} + c \quad \Rightarrow \quad g(x) = \frac{x^\alpha \tau^\alpha}{(x^2)^{1/2}},
\]

finally the potential has the following expression

\[
A_\mu = \frac{f(x^2)}{x^2} x^\alpha \tau^\alpha \tau^\mu.
\]

Substituting the potential into the field strength we find the condition for self-duality:

\[
\partial_\mu \left( \frac{f(x^2)}{x^2} \right) + 2x_\mu \left( \frac{f(x^2)}{x^2} \right)^2 = 0,
\]

solving the equation we find that

\[
f(x^2) = \frac{x^2}{x^2 + \lambda^2} \quad \text{where} \quad \lambda \in \mathbb{R},
\]

finally substituting in (17) we get the desired solution.

\[
A_\mu = \frac{x^2}{x^2 + \lambda^2} x^\alpha \tau^\alpha \tau^\mu.
\]

3.1. Instanton Number

The instanton number or the first Chern class, also known in quantum field theory as the \( \theta \)-term which is a topological term, is given by the following expression

\[
k = -\frac{1}{8\pi^2} \text{tr} \int_{\mathbb{R}^4} F \wedge F,
\]

so from the self-dual connection (20) we compute the field strength given explicitly by

\[
F_{\mu\nu} = \frac{\lambda^2}{(x^2 + \lambda^2)^2} (\tau^\dagger_\mu \tau_\nu - \tau^\dagger_\nu \tau_\mu)
\]

substituting in the previous expression we obtain \( k = 1 \), where we made use of the following relations

\[
[\tau^\dagger_\mu, \tau_\nu] = 2\epsilon_{\mu\nu\rho\sigma} \tau_\rho, \quad \{\tau^\dagger_\mu, \tau_\nu\} = 2\delta_{\mu\nu} I + 2(\delta_{0\mu} \tau_\nu - \delta_{0\nu} \tau_\mu).
\]

The instanton number also is known as topological charge, since we can add to the Yang-Mills action and does not contribute to the dynamics, but it carries topological information about the bundle, but at quantum level it is relevant (the path integral has their maximum contribution located at the instanton space).
4. Non-Commutative Instanton Number

The aim of this section is explore the modification to the topology when we introduce non-commutativity (through the Seiberg-Witten map), in particular we are interested on the information provided by the instanton number. Since the non-commutative extension gives new terms, our goal is to check if this information is topological, new, relevant or any of previously mentioned.

The non-commutative $\theta$-term is defined in similar way as the Yang-Mills action, redefining the action via the Moyal product

$$ \int_M F \wedge F = \int_M e^{i\theta} \hat{F}_{\mu\nu} \hat{F}_{\rho\theta}, $$

where $\hat{F}_{\mu\nu} = F^0_{\mu\nu} + F^1_{\mu\nu} + F^2_{\mu\nu} + \cdots$ is the non-commutative field strength, where every term in the expansion is determined by the zeroth order, which are determined as in the previous section through the Seiberg-Witten map. Explicitly up to second order the $A^i$’s are given (for arbitrary order expansion check \[8\])

$$ A^0 = A_\mu, $$
$$ A^1_\mu = -\frac{1}{4} g^{\kappa\lambda} \{ A_\kappa, \partial_\lambda A_\mu + F_{\lambda\mu} \}, \quad (25) $$
$$ A^2_\mu = -\frac{1}{8} g^{\kappa\lambda} \{ \{ A_\kappa, \partial_\lambda A_\mu + F_{\lambda\mu} \} + \{ A_\kappa, \partial_\lambda A^1_\mu + F^1_{\lambda\mu} \} \} $$
$$ -\frac{i}{16} g^{\kappa\lambda\mu\nu} \{ \partial_\rho A_\kappa, \partial_\lambda (\partial_\sigma A_\mu + F_{\lambda\mu}) \}, \quad (26) $$

and the $F^i$’s are

$$ F^0_{\mu\nu} = F_{\mu\nu}, $$
$$ F^1_{\mu\nu} = -\frac{1}{4} g^{\kappa\lambda} \{ A_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F_{\mu\nu} \} - 2\{ F_{\mu\kappa}, F_{\nu\lambda} \}, \quad (28) $$
$$ F^2_{\mu\nu} = -\frac{1}{8} g^{\kappa\lambda} \{ \{ A_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F^1_{\mu\nu} \} - i[A_\lambda, F_{\mu\nu}] + \frac{i}{2} \theta^{\kappa\lambda\mu\nu} \{ \partial_\delta A_\kappa, \partial_\sigma F_{\mu\nu} \} \} $$
$$ + \{ A^1_\kappa, \partial_\lambda F_{\mu\nu} + D_\lambda F^1_{\mu\nu} \} - 2\{ F_{\mu\kappa}, F_{\nu\lambda} \} - 2\{ F^1_{\mu\kappa}, F_{\nu\lambda} \} \} $$
$$ -\frac{i}{16} g^{\kappa\lambda\mu\nu} \{ \partial_\rho A_\kappa, \partial_\lambda (\partial_\sigma F_{\mu\nu} + D_\lambda F_{\mu\nu}) \} - 2[\partial_\rho F_{\mu\kappa}, \partial_\delta F_{\nu\lambda}]. \quad (29) $$

the covariant derivative is $D_\lambda \Phi = \partial_\lambda \Phi - i[A_\lambda, \Phi]$ where $\Phi$ is a Lie algebra valued field.

Now consider the expansion of the $\theta$-term (24) up to second order,

$$ S = \int tr e^{i\theta} \left( F^0_{\mu\nu} F_{\rho\theta} + F^1_{\mu\nu} F^0_{\rho\theta} + F^1_{\rho\theta} F^0_{\mu\nu} + F^2_{\mu\nu} F^0_{\rho\theta} + F^1_{\mu\nu} F^1_{\rho\theta} + F^2_{\mu\nu} F^1_{\rho\theta} + F^2_{\rho\theta} F^0_{\mu\nu} \right) d^4x, \quad (31) $$

using the expressions for the potential (self-dual connection) (20) and the field strength (22) with the aid of the expression for the different orders for the potential and field strength, we can compute the second order action for a $k = 1$ instanton over $S^4$.

For this case of $SU(2)$, we can check from the non-commutative action and the expression for the different orders; in general only the even orders contribute, since the even orders fields
strength are always Lie algebra valued i.e. any even term will be a product of two Pauli matrices \( \sigma_a \sigma_b = i \epsilon_{abc} \sigma_c + \delta_{ab} I \), hence when we take the trace this terms in general are not zero.

For the odd terms we deduce that the field strength of one of the products is proportional to an anticommutator between Pauli matrices i.e. in general always will be proportional to the identity \( \{ \sigma_a, \sigma_b \} = 2 \delta_{ab} I \), and when we multiply by the other Pauli matrices, this always be proportional to another Pauli matrices; hence the product in general will be always a Pauli matrix, therefore when we trace an odd term this term will be zero.

Thus the action (31) reduces to:

\[
S = \int_{S^4} tr \epsilon^{\mu \nu \rho \theta} \left( F_0^{\mu \nu} F_0^{\rho \theta} + F_0^{\mu \nu} F_1^{\rho \theta} + F_1^{\mu \nu} F_1^{\rho \theta} + F_2^{\mu \nu} F_0^{\rho \theta} \right) d^4 x, \tag{32}
\]

where the first term is the usual commutative term \( \int_{S^4} tr \epsilon^{\mu \nu \rho \theta} F_0^{\mu \nu} F_0^{\rho \theta} = 1 \).

To compute the last three terms we use the equations (25)-(30) and substituting the expression for the potential and the field strength at zeroth order (20) and (22) for the solution of instanton \( (k = 1) \) respectively. It turns out that these terms (second order terms) are zero. Therefore there is no contribution up to second order, these means that the commutative \( \theta \)-term and the non-commutative \( \theta \)-term at least up to second order give us the same information.

5. Conclusions
We computed up to second order on \( S^4 \) (equivalently for \( \mathbb{R}^4 \)) the non-commutative and the commutative \( \theta \)-term and we observe that both have the same information; in this special case considered (up to second order), the instanton number was 1 in both cases.

It is worth mentioning that if we integrate the \( k = 1 \) solution for higher orders in \( \theta \), regardless the terms of the algebra it is clearly that the value of the integral over \( S^4 \) depends on \( \lambda \) (the instanton size), this suggest that these terms must be zero because they are not topological (since it depends on \( \lambda \)). Thus we expect that for orders larger than two each term must be zero, in the particular case \( S^4 \) there is no new topological information given through the non-commutative \( \theta \)-term. This is in agreement with some results and discussion presented in [9].

It will be interesting to show formally in general for \( S^4 \) that the contribution of the non-commutativity \( \theta \)-term is the same as the usual. By the other hand it could be more interesting compute the instanton number in a torus or in a different topological configuration.

References
[1] Seiberg N. and Witten E. String Theory and Noncommutative Geometry, JHEP 9909:032 (1999).
[2] Jurco B., Moller L., Schraml S., Schupp P. and Wess J., Construction of non-Abelian gauge theories on noncommutative spaces Eur. Phys. J. C 21 (2001).
[3] Aschieri P., Dimitrijevic M., Kulish P., Lizzi F., Wess J., Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory, Lect. Notes Phys. 774 (Springer, Belin Heidelberg 2009).
[4] Garcia-Compean H., Obregon O., Ramirez C., Sabido M., Noncommutative Selfdual Gravity, Phys.Rev. D68 (2003).
[5] Garcia-Compean H., Obregon O., Ramirez C., Sabido M., Noncommutative Topological Theories of Gravity, Phys.Rev. D68 (2003).
[6] Rajaraman R., Solitons and Instantons, North-Holland (1989).
[7] Nash C. and Sen S., Topology and Geometry for Physicist, Academic Press (1987).
[8] Ulker K. and Yapiskan B., Seiberg-Witten maps to all orders, Phys. Rev. D77 (2008).
[9] Madea Y., and Sako A., Noncommutativity Deformation of Instantons J. Geom. Phys. Vol. 58(12) (2008).