Spike-Train Level Backpropagation for Training Deep Recurrent Spiking Neural Networks

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Abstract

Spiking neural networks (SNNs) are more biologically plausible than conventional artificial neural networks (ANNs). SNNs well support spatiotemporal learning and energy-efficient event-driven hardware neuromorphic processors. As an important class of SNNs, recurrent spiking neural networks (RSNNs) possess great computational power. However, the practical application of RSNNs is severely limited by challenges in training. Biologically-inspired unsupervised learning has limited capability in boosting the performance of RSNNs. On the other hand, existing backpropagation (BP) methods suffer from high complexity of unrolling in time, vanishing and exploding gradients, and approximate differentiation of discontinuous spiking activities when applied to RSNNs. To enable supervised training of RSNNs under a well-defined loss function, we present a novel Spike-Train level RSNNs Backpropagation (ST-RSBP) algorithm for training deep RSNNs. The proposed ST-RSBP directly computes the gradient of a rate-coded loss function defined at the output layer of the network w.r.t tunable parameters. The scalability of ST-RSBP is achieved by the proposed spike-train level computation during which temporal effects of the SNN is captured in both the forward and backward pass of BP. Our ST-RSBP algorithm can be broadly applied to RSNNs with a single recurrent layer or deep RSNNs with multiple feedforward and recurrent layers. Based upon challenging speech and image datasets including TI46 [25], N-TIDIGITS [3] and Fashion-MNIST [39], ST-RSBP is able to train RSNNs with an accuracy surpassing that of the current state-of-art SNN BP algorithms and conventional non-spiking deep learning models.

1 Introduction

In recent years, deep neural networks (DNNs) have demonstrated outstanding performance in natural language processing, speech recognition, visual object recognition, object detection, and many other domains [6, 14, 21, 35, 13]. On the other hand, it is believed that biological brains operate rather differently [17]. Neurons in artificial neural networks (ANNs) are characterized by a single, static, and continuous-valued activation function. More biologically plausible spiking neural networks (SNNs) compute based upon discrete spike events and spatio-temporal patterns while enjoying rich coding mechanisms including rate and temporal codes [11]. There is theoretical evidence supporting that SNNs possess greater computational power over traditional ANNs [11]. Moreover, the event-driven nature of SNNs enables ultra-low-power hardware neuromorphic computing devices [7, 2, 10, 28].

Backpropagation (BP) is the workhorse for training deep ANNs [22]. Its success in the ANN world has made BP a target of intensive research for SNNs. Nevertheless, applying BP to biologically more plausible SNNs is nontrivial due to the necessity in dealing with complex neural dynamics and non-differentiability of discrete spike events. It is possible to train an ANN and then convert...
it to an SNN \[9, 10, 16]\); however this suffers from conversion approximations and gives up the opportunity in exploring SNNs’ temporal learning capability. One of the earliest attempts to bridge the gap between discontinuity of SNNs and BP is the SpikeProp algorithm \[5]\); however, SpikeProp is restricted to single-spike learning and has not yet been successful in solving real-world tasks.

Recently, training SNNs using BP under a firing rate (or activity level) coded loss function has been shown to deliver competitive performances \[23, 38, 4, 33]\). Nevertheless, \[23]\) does not consider the temporal correlations of neural activities and deals with spiking discontinuities by treating them as noise. \[33]\) gets around the non-differentiability of spike events by approximating the spiking process via a probability density function of spike state change. \[38, 4, 15]\) capture the temporal effects by performing backpropagation through time (BPTT) \[36]\). Among these, \[15]\) adopts a smoothed spiking threshold and a continuous differentiable synaptic model for gradient computation, which is not applicable to widely used spiking neuron models such as the leaky integrate-and-fire (LIF) model. Similar to \[23, 38\) and \[4\) compute the error gradient based on the continuous membrane waveforms resulted from smoothing out all spikes. In these approaches, computing the error gradient by smoothing the microscopic membrane waveforms may lose the sight of the all-or-none firing characteristics of the SNN that defines the higher-level loss function and lead to inconsistency between the computed gradient and target loss, potentially degrading training performance \[19]\).

Most existing SNN training algorithms including the aforementioned BP works focus on feedforward networks. Recurrent spiking neural networks (RSNNs), which are an important class of SNNs and are especially competent for processing temporal signals such as time series or speech data \[12]\), deserve equal attention. The liquid State Machine (LSM) \[27]\) is a special RSNN which has a single recurrent reservoir layer followed by one readout layer. To mitigate training challenges, the reservoir weights are either fixed or trained with unsupervised learning like spike-timing-dependent plasticity (STDP) \[29]\) with only the readout layer trained with supervision \[31, 40, 18]\). The inability in training the entire network with supervision and its architectural constraints, e.g. only admitting one reservoir and one readout, limit the performance of LSM. \[4]\) proposes an architecture called long short-term memory SNNs (LSNNs) and trains it using BPTT with the aforementioned issue on approximate gradient computation. When dealing with training of general RSNNs, in addition to the difficulties encountered in feedforward SNNs, one has to cope with added challenges incurred by recurrent connections and potential vanishing/exploding gradients.

This work is motivated by: 1) lack of powerful supervised training of general recurrent SNNs (RSNNs), and 2) an immediate outcome of 1), i.e. the existing SNN research has limited scope in exploring sophisticated learning architectures like deep RSNNs with multiple feedforward and recurrent layers hybridized together. We address these challenges by proposing a novel Spike-Train level RSNNs Backpropagation (ST-RSBP) algorithm applicable to RSNNs with an arbitrary network topology. The proposed ST-RSBP employs spike-train level computation similar to what is adopted in the recent hybrid macro/micro level BP (HM2-BP) method for feedforward SNNs \[19]\), which demonstrates encouraging performances and outperforms BPTT such as the one implemented in \[38]\).

ST-RSBP is rigorously derived and can handle arbitrary recurrent connections in various RSNNs. While capturing the temporal behavior of the RSNN at the spike-train level, ST-RSBP directly computes the gradient of a rate-coded loss function w.r.t tunable parameters without incurring approximations resulted from altering and smoothing the underlying spiking behaviors. ST-RSBP is able to train RSNNs without costly unrolling the network through time and performing BP time point by time point, offering faster training and avoiding vanishing/exploding gradients for general RSNNs. We apply ST-RSBP to train several deep RSNNs with multiple feedforward and recurrent layers to demonstrate the best performances on several widely adopted datasets. Based upon challenging speech and image datasets including TI46 \[25]\), N-TIDIGITS \[4]\) and Fashion-MNIST \[39]\), ST-RSBP trains RSNNs with an accuracy noticeably surpassing that of the current state-of-art SNN BP algorithms and conventional non-spiking deep learning models and algorithms.

2 Background

2.1 SNN Architectures and Training Challenges

Fig.\[1]\) A shows two SNN architectures often explored in neuroscience: single layer (top) and liquid state machine (bottom) networks for which different mechanisms have been adopted for training. However, typically spike timing dependent plasticity (STDP) \[29]\) and winner-take-all (WTA) \[8]\) are
only for unsupervised training and have limited performance. WTA and other supervised learning rules \cite{31, 40, 18} can only be applied to the output layer, obstructing adoption of more sophisticated deep architectures.

While bio-inspired learning mechanisms are yet to demonstrate competitive performance for challenging real-life tasks, there has been much recent effort aiming at improving SNN performance with supervised BP. Most existing SNN BP methods are only applicable to multi-layer feedforward networks as shown in Fig. 1B. Several such methods have demonstrated promising results \cite{23, 38, 19, 33}. Nevertheless, these methods are not applicable to complex deep RSNNs such as the hybrid feedforward/recurrent networks shown in Fig. 1C, which are the target of this work. Backpropagation through time (BPTT) in principle may be applied to training RSNNs \cite{4}, but bottlenecked with several challenges in: (1) unrolling the recurrent connections through time, (2) back propagating errors over both time and space, and (3) back propagating errors over non-differentiable spike events.

Fig. 2 compares BPTT and ST-RSBP, where we focus on a recurrent layer since feedforward layer can be viewed as a simplified recurrent layer. To apply BPTT, one shall first unroll the entire RSNN through time to convert it into a larger feedforward network without recurrent connections. The total number of layers in the feedforward network is increased by a factor equal to the number of times the RSNN is unrolled, and hence can be very large. Then, this unrolled network is integrated in time with a sufficiently small time step to capture dynamics of the spiking behavior. BP is then performed spatiotemporally layer-by-layer across the unrolled network based on the same time stepsize used for integration as shown in Fig. 2. In contrast, the proposed ST-RSBP does not convert the RSNN...
We outline the key component of derivation of ST-RSBP: the back propagated errors. The full derivation of ST-RSBP is presented in Section 2 of the Supplementary Materials.

2.2 Spike-train Level Post-synaptic Potential (S-PSP)

S-PSP captures the spike-train level interactions between a pair of pre/post-synaptic neurons. Note that each neuron fires whenever its post-synaptic potential reaches the firing threshold. The accumulated contributions of the pre-synaptic neuron's spike train to the (normalized) post-synaptic potential of the neuron right before all the neuron's firing times is defined as the (normalized) S-PSP from the neuron to the neuron as in (6) in the Supplementary Materials. The S-PSP characterizes the aggregated effect of the spike train of the neuron on the membrane potential of the neuron and its firing activities. S-PSPs allow consideration of the temporal dynamics and recurrent connections of an RSNN across all firing events at the spike-train level without expensive unrolling through time and backpropagation time point by time point.

The sum of the weighted S-PSPs from all pre-synaptic neurons of the neuron is defined as the total post-synaptic potential (T-PSP) . is the post-synaptic membrane potential accumulated right before all firing times and relates to the firing count via the firing threshold .

where and are analogous to the pre-activation and activation in the traditional ANNs, respectively, and can be considered as an activation function converting the T-PSP to the output firing count.

A detailed description of S-PSP and T-PSP can be found in Section 1 in the Supplementary Materials.

3 Proposed Spike-Train level Recurrent SNNs Backpropagation (ST-RSBP)

We use the generic recurrent spiking neural network with a combination of feedforward and recurrent layers of Fig. 2 to derive ST-RSBP. For the spike-train level activation of each neuron in the layer , is modified to include the recurrent connections explicitly if necessary:

where

and

The rate-coded loss is defined at the output layer as:

where and are vectors of the desired output neuron firing counts (labels) and actual firing counts, and the T-PSPs of the output neurons, respectively. Differentiating with respect to each trainable weight incident upon the layer leads to:

where and are referred to as the back propagated error and differentiation of activation, respectively, for the neuron . ST-RSBP updates by , where is a learning rate.

We outline the key component of derivation of ST-RSBP: the back propagated errors. The full derivation of ST-RSBP is presented in Section 2 of the Supplementary Materials.
3.1 Outline of the Derivation of Back Propagated Errors

3.1.1 Output Layer

If the layer \( k \) is the output, the back propagated error of the neuron \( i \) is given by differentiating (3):

\[
\delta^k_i = \frac{\partial E}{\partial o^k_i} = \left( \frac{o^k_i - y^k_i}{\nu^k} \right),
\]

where \( o^k_i \) is the actual firing count, \( y^k_i \) the desired firing count (label), and \( \nu^k \) the T-PSP.

3.1.2 Hidden Layers

At each hidden layer \( k \), the chain rule is applied to determine the error \( \delta^k_i \) for the neuron \( i \):

\[
\delta^k_i = \frac{\partial E}{\partial a^k_i} = \sum_{l=1}^{N_{k+1}} \frac{\partial E}{\partial a^{k+1}_l} \frac{\partial a^{k+1}_l}{\partial a^k_i} = \sum_{l=1}^{N_{k+1}} \delta^{k+1}_l \frac{\partial a^{k+1}_l}{\partial a^k_i}.
\]

(6)

Define two error vectors \( \delta^{k+1} \) and \( \delta^k \) for the layers \( k + 1 \) and \( k \): \( \delta^{k+1} = [\delta^{k+1}_1, \ldots, \delta^{k+1}_{N_{k+1}}] \), and \( \delta^k = [\delta^k_1, \ldots, \delta^k_{N_k}] \), respectively. Assuming \( \delta^{k+1}_i \) is given, which is the case for the output layer, the goal is to back propagate from \( \delta^{k+1} \) to \( \delta^k \). This entails to compute \( \frac{\partial a^{k+1}_l}{\partial a^k_i} \) in (6).

[Backpropagation from a Hidden Recurrent Layer] Now consider the case that the errors are back propagated from a recurrent layer \( k + 1 \) to its preceding layer \( k \). Note that the S-PSP \( e_{lj} \) from any pre-synaptic neuron \( j \) to a post-synaptic neuron \( l \) is a function of both the rate and temporal information of the pre/post-synaptic spike trains, which can be made explicitly via some function \( f \):

\[
e_{lj} = f(o_j, o_l(t^j_{(f)}, t^l_{(f)})),
\]

(7)

where \( o_j, o_l, t^j_{(f)}, t^l_{(f)} \) are the pre/post-synaptic firing counts and firing times, respectively.

Now based on (2), \( \frac{\partial a^{k+1}_l}{\partial a^k_i} \) is split also into two summations:

\[
\frac{\partial a^{k+1}_l}{\partial a^k_i} = \sum_j N_{k+1} w^{k+1}_{lj} \frac{de^{k+1}_{lj}}{da^k_i} + \sum_p N_{k+1} w^{k+1}_{lp} \frac{de^{k+1}_{lp}}{da^k_i},
\]

(8)

where the first summation sums over all pre-synaptic neurons in the previous layer \( k \) while the second sums over the pre-synaptic neurons in the current recurrent layer as illustrated in Fig. 3.

On the right side of (8), \( \frac{de^{k+1}_{lj}}{da^k_i} \) is given by:

\[
\frac{de^{k+1}_{lj}}{da^k_i} = \begin{cases} \frac{1}{\nu^k} \frac{\partial o^{k+1}_l}{\partial a^k_i} + \frac{1}{\nu^{k+1}} \frac{\partial o^{k+1}_l}{\partial a^k_i} \frac{\partial a^{k+1}_l}{\partial o^k_j}, & j = i \\ \frac{1}{\nu^{k+1}} \frac{\partial o^{k+1}_l}{\partial a^k_i}, & j \neq i. \end{cases}
\]

(9)

\( \nu^k \) and \( \nu^{k+1} \) are the firing threshold voltages for the layers \( k \) and \( k + 1 \), respectively, and we have used that \( o^k_l = a^k_l/\nu^k \) and \( o^{k+1}_l = a^{k+1}_l/\nu^{k+1} \) from (1). Importantly, the last term on the right side of (6) exists due to \( e^{k+1}_{lj} \)'s dependency on the post-synaptic firing rate \( a^{k+1}_l \) per \( \delta^{k+1}_l \) and \( o^{k+1}_l \)'s further dependency on the pre-synaptic activation \( a^k_l \) (hence pre-activation \( a^k_i \)), as shown in Fig. 3.

On the right side of (8), \( \frac{de^{k+1}_{lp}}{da^k_i} \) is due to the recurrent connections within the layer \( k + 1 \):

\[
\frac{de^{k+1}_{lp}}{da^k_i} = \frac{1}{\nu^{k+1}} \frac{\partial e^{k+1}_{lp}}{\partial o^{k+1}_l} \frac{\partial o^{k+1}_l}{\partial a^k_i} + \frac{1}{\nu^{k+1}} \frac{\partial e^{k+1}_{lp}}{\partial o^{k+1}_p} \frac{\partial o^{k+1}_p}{\partial a^k_i}.
\]

(10)
All reported experiments below are conducted on an NVIDIA Titan XP GPU. The experimented SNNs where

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The complete ST-RSBP algorithm is summarized in Section 2.4 in the Supplementary Materials.
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The first term on the right side of (10) is due to $e_{l+1}^{k+1}$'s dependency on the post-synaptic firing rate $o_{i}^{k+1}$ per (7) and $o_{i}^{k+1}$'s further dependence on the pre-synaptic activation $o_{i}^{k}$ (hence pre-activation $a_{i}^{k}$). Per (7), it is important to note that the second term exists because $e_{l+1}^{k+1}$'s dependency on the pre-synaptic firing rate $o_{p}^{k+1}$, which further depends on $o_{i}^{k}$ (hence pre-activation $a_{i}^{k}$), as shown in Fig. 3

Putting (8), (9), and (10) together leads to:

```
\begin{align*}
\left(1 - \frac{1}{\nu(k+1)} \left( \sum_{j} w_{lj}^{k+1} \frac{\partial e_{l}^{k+1}}{\partial o_{j}^{k+1}} + \sum_{p} w_{lp}^{k+1} \frac{\partial e_{p}^{k+1}}{\partial o_{i}^{k+1}} \right) \right) \frac{\partial o_{i}^{k+1}}{\partial a_{i}^{k}}
= w_{li}^{k+1} \frac{1}{\nu(k+1)} \frac{\partial e_{l}^{k+1}}{\partial o_{i}^{k}} + \sum_{p} w_{lp}^{k+1} \left( \frac{1}{\nu(k+1)} \frac{\partial e_{p}^{k+1}}{\partial o_{p}^{k}} + \frac{\partial e_{p}^{k+1}}{\partial o_{i}^{k}} \right) h_{p}^{k+1}.
\end{align*}
```

Putting (8), (9), and (10) together leads to:

```
\begin{align*}
\left(1 - \frac{1}{\nu(k+1)} \left( \sum_{j} w_{lj}^{k+1} \frac{\partial e_{l}^{k+1}}{\partial o_{j}^{k+1}} + \sum_{p} w_{lp}^{k+1} \frac{\partial e_{p}^{k+1}}{\partial o_{i}^{k+1}} \right) \right) \frac{\partial o_{i}^{k+1}}{\partial a_{i}^{k}}
= w_{li}^{k+1} \frac{1}{\nu(k+1)} \frac{\partial e_{l}^{k+1}}{\partial o_{i}^{k}} + \sum_{p} w_{lp}^{k+1} \left( \frac{1}{\nu(k+1)} \frac{\partial e_{p}^{k+1}}{\partial o_{p}^{k}} + \frac{\partial e_{p}^{k+1}}{\partial o_{i}^{k}} \right) h_{p}^{k+1}.
\end{align*}
```

It is evident that all $N_{k+1} \times N_{k}$ partial derivatives involving the recurrent layer $k+1$ and its preceding layer $k$, i.e. $\frac{\partial e_{I}^{k+1}}{\partial o_{I}^{k}}$, $I = \{1, N_{k+1}\}$, $i = \{1, N_{k}\}$, form a coupled linear system via (11), which is written in a matrix form as:

```
\begin{align*}
\Omega^{k+1, k} \cdot P^{k+1, k} = \Phi^{k+1, k} + \Theta^{k+1, k} \cdot P^{k+1, k}.
\end{align*}
```

where $P^{k+1, k} \in \mathbb{R}^{N_{k+1} \times N_{k}}$ contains all the desired partial derivatives, $\Omega^{k+1, k} \in \mathbb{R}^{N_{k+1} \times N_{k+1}}$ is diagonal, $\Theta^{k+1, k} \in \mathbb{R}^{N_{k+1} \times N_{k+1}}$, $\Phi^{k+1, k} \in \mathbb{R}^{N_{k} \times N_{k}}$, and the detailed definitions of all these matrices can be found in Section 2.1 of the Supplementary Materials.

Solving the linear system in (12) gives all $\frac{\partial o_{I}^{k+1}}{\partial a_{i}^{k}}$:

```
\begin{align*}
P^{k+1, k} = \left( \Omega^{k+1, k} - \Theta^{k+1, k} \right)^{-1} \cdot \Phi^{k+1, k}.
\end{align*}
```

Note that since $\Omega$ is a diagonal matrix, the cost in factoring the above linear system can be reduced by approximating the matrix inversion using a first-order Taylor’s expansion without matrix factorization.

Error propagation from the layer $k+1$ to layer $k$ of (6) is cast in the matrix form: $\delta^{k} = P^{T} \cdot \delta^{k+1}$.

[Backpropagation from a Hidden Feedforward Layer] The much simpler case of backpropagating errors from a feedforward layer $k+1$ to its preceding layer $k$ is described in Section 2.1 of the Supplementary Materials.

The complete ST-RSBP algorithm is summarized in Section 2.4 in the Supplementary Materials.

4 Experiments and Results

4.1 Experimental Settings

All reported experiments below are conducted on an NVIDIA Titan XP GPU. The experimented SNNs are based on the LIF model and weights are randomly initialized by following the uniform distribution $U[-1, 1]$. Fixed firing thresholds are used in the range of 5mV to 20mV depending on the layer. Exponential weight regularization [23], lateral inhibition in the output layer [23] and Adam [20] as the optimizer are adopted. We will release the source code of our CUDA implementation on GitHub in the future. More detailed training settings are demonstrated in the source code. The best results and settings for each datasets are also included in the “Result” fold.

Using three speech datasets and one image dataset, we compare the proposed ST-RSBP with several other methods which either have the best previously reported results on the same datasets or represent the current state-of-the-art for training SNNs. Among these, HM2-BP [19] is the best reported BP algorithm for feedforward SNNs based on LIF neurons. ST-RSBP is evaluated using RSNNs of multiple feedforward and recurrent layers with full connections between adjacent layers and sparse connections inside the recurrent layers. The network models of all other BP methods we compare with are fully connected feedforward networks. The liquid state machine (LSM) networks demonstrated below have sparse input connections, sparse reservoir connections, and a fully connected readout layer. Since HM2-BP cannot train recurrent networks, we compare ST-RSBP with HM2-BP using models of a similar number of tunable weights. Each experiment reported below is repeated five times to obtain the mean and standard deviation (stddev) of the accuracy.
4.2 TI46-Alpha Speech Dataset

TI46-Alpha is the full alphabets subset of the TI46 Speech corpus \[25\] and contains spoken English alphabets from 16 speakers. There are 4,142 and 6,628 spoken English examples in 26 classes for training and testing, respectively. The continuous temporal speech waveforms are first preprocessed by the Lyon’s ear model \[26\] and then encoded into 78 spike trains using the BSA algorithm \[32\].

Table 1: Comparison of different SNN models on TI46-Alpha

| Algorithm      | Hidden Layers | # Params | # Epochs | Mean  | Stddev | Best     |
|----------------|---------------|----------|----------|-------|--------|----------|
| HM2-BP         | 800           | 83,200   | 138      | 89.36%| 0.30%  | 89.92%   |
| HM2-BP         | 400-400       | 201,600  | 163      | 89.83%| 0.71%  | 90.60%   |
| HM2-BP         | 800-800       | 723,200  | 174      | 90.50%| 0.45%  | 90.98%   |
| Non-spiking BP | LSM: R2000    | 52,000   |          |       |        |          |
| ST-RSBP        | R800          | 86,280   | 75       | 91.57%| 0.20%  | 91.85%   |
| ST-RSBP        | 400-R400-400  | 363,313  | 57       | 93.06%| 0.21%  | 93.35%   |

* We show the number of neurons in each feedforward/recurrent hidden layer. R represent recurrent layer.

An LSM model. The state vector of the reservoir is used to train the single readout layer using BP.

Table 1 compares ST-RSBP with several other algorithms on TI46-Alpha. The result from \[37\] shows that only training the single readout layer of a recurrent LSM is inadequate for this challenging task, demonstrating the necessity of training all layers of a recurrent network using techniques such as ST-RSBP. ST-RSBP outperforms all other methods. In particular, ST-RSBP is able to train a three-hidden-layer RSNN with 363,313 weights to increase the accuracy from 90.98% to 93.35% when compared with the feedforward SNN with 723,200 weights trained by HM2-BP.

4.3 TI46-Digits Speech Dataset

TI46-Digits is the full digits subset of the TI46 Speech corpus \[25\]. It contains 1,594 training examples and 2,542 testing examples of 10 utterances for each of digits “0” to “9” spoken by 16 different speakers. The same preprocessing used for TI46-Alpha is adopted. Table 2 shows that the proposed ST-RSBP delivers a high accuracy of 99.39% while outperforming all other methods including HM2-BP. On recurrent network training, ST-RSBP produces large improvements over two other methods. For instance, with 19,057 tunable weights, ST-RSBP delivers an accuracy of 98.77% while \[34\] has an accuracy of 86.66% with 32,000 tunable weights.

Table 2: Comparison of different SNN models on TI46-Digits

| Algorithm      | Hidden Layers | # Params | # Epochs | Mean  | Stddev | Best     |
|----------------|---------------|----------|----------|-------|--------|----------|
| HM2-BP         | 100-100       | 18,800   | 22       | 98.42%|        |          |
| HM2-BP         | 200-200       | 57,600   | 21       | 98.50%|        |          |
| Non-spiking BP | LSM: R500     | 5,000    |          | 78%   |        |          |
| SpiLinC        | LSM: R3200    | 32,000   |          | 86.66%|        |          |
| ST-RSBP        | R100-100      | 19,057   | 75       | 98.77%| 0.13%  | 98.95%   |
| ST-RSBP        | R200-200-200  | 58,230   | 28       | 99.16%| 0.11%  | 99.27%   |
| ST-RSBP        | R200-200-200  | 98,230   | 23       | 99.25%| 0.13%  | 99.39%   |

* An LSM with multiple reservoirs in parallel. Weights between input and reservoirs are trained using STDP.

The excitatory neurons in the reservoir are tagged with the classes for which they spiked at a highest rate during training and are grouped accordingly. During inference, for a test pattern, the average spike count of every group of neurons tagged is examined and the tag with the highest average spike count represents the predicted class.

4.4 N-Tidigits Neuromorphic Speech Dataset

The N-Tidigits \[3\] is the neuromorphic version of the well-known speech dataset Tidigits, and consists of recorded spike responses of a 64-channel CochleaAMS1b sensor in response to audio waveforms from the original Tidigits dataset \[24\]. 2,475 single digit examples are used for training and the same number of examples are used for testing. There are 55 male and 56 female speakers and each of them speaks two examples for each of the 11 single digits including “oh,” “zero”, and the digits “1” to “9”. Table 3 shows that proposed ST-RSBP achieves excellent accuracies up to 93.90%, which is significantly better than that of HM2-BP and the non-spiking GRN and LSTM in \[3\]. With a similar/less number of tunable weights, ST-RSBP outperforms all other methods rather significantly.
### Table 3: Comparison of different models on N-Tidigits

| Algorithm                        | Hidden Layers | # Params | # Epochs | Mean     | Stddev   | Best    |
|----------------------------------|---------------|----------|----------|----------|----------|---------|
| HM2-BP [19]                      | 250-250       | 81,250   |          | 89.69%   |          |         |
| GRN(non-spiking) [3]             | 2 × G200-100a | 109,200  |          | 90.90%   |          |         |
| Phased-LSTM(non-spiking) [3]     | 2 × 250Lb     | 610,500  |          | 91.25%   |          |         |
| ST-RSBP                          | 250-R250      | 82,050   | 268      | 92.94%   | 0.20%    | 93.13%  |
| ST-RSBP                          | 400-R400-400  | 351,241  | 287      | 93.63%   | 0.27%    | 93.90%  |

*a* G represents a GRN layer; *b* L represents an LSTM layer.

### 4.5 Fashion-MNIST Image Dataset

The Fashion-MNIST dataset [39] contains 28x28 grey-scale images of clothing items, meant to serve as a much more difficult drop-in replacement for the well-known MNIST dataset. It contains 60,000 training examples and 10,000 testing examples with each image falling under one of the 10 classes. Using Poisson sampling, we encode each 28 × 28 image into a 2D 784 × L binary matrix, where L = 400 represents the duration of each spike sequence in ms, and a 1 in the matrix represents a spike. The simulation time step is set to be 1ms. No other preprocessing or data augmentation is applied. Table 4 shows that ST-RSBP outperforms all other SNN and non-spiking BP methods.

### Table 4: Comparison of different models on Fashion-MNIST

| Algorithm          | Hidden Layers | # Params    | # Epochs | Mean     | Stddev   | Best    |
|--------------------|---------------|-------------|----------|----------|----------|---------|
| HM2-BP [19]        | 400-400       | 477,600     | 15       | 88.99%   |          |         |
| BP [30]            | 5 × 256       | 465,408     |          | 87.02%   |          |         |
| LRA-E [30]         | 5 × 256       | 465,408     |          | 87.69%   |          |         |
| DL BP [1]          | 3 × 512       | 662,026     |          | 89.06%   |          |         |
| Keras BP [3]       | 512-512       | 669706      | 50       | 89.01%   |          |         |
| ST-RSBP            | 400-R400      | 478,841     | 36       | 90.00%   | 0.14%    | 90.13%  |

*a* Fully connected ANN trained with the BP algorithm.

*b* Fully connected ANN with locally defined errors trained using gradient descent. Loss functions are L2 norm for hidden layers and categorical cross-entropy for the output layer.

*c* Fully connected ANN trained using the Keras package with RELU activation, categorical cross-entropy loss, and RMSProp optimizer; a dropout layer applied between each dense layer with rate of 0.2.

### 5 Discussions and Conclusion

In this paper, we present the novel spike-train level backpropagation algorithm ST-RSBP, which can be directly applied to different types of RSNNs or SNNs. ST-RSBP addresses key challenges of RSNN training in handling of temporal effects, gradient computation of loss functions with inherent discontinuities, and high cost as well as the vanishing/exploding gradients problem of unrolling network through time using BPTT. More specifically, in ST-RSBP, the given rate-coded errors can be efficiently computed and backpropagated through layers without costly unrolling the network in time and through expensive time point by time point computation. Moreover, ST-RSBP handles the discontinuity of spikes during BP without altering and smoothing the microscopic spiking behaviors. The problem of network unrolling is dealt with using accurate spike-train level BP such that the effect of all spikes are captured and propagated in an aggregated manner to achieve accurate and fast training. As such, both rate and temporal information in the SNN are well exploited during the training process, leading to the state-of-the-art performances.

Using the efficient GPU implementation of ST-RSBP, we demonstrate the best performances for both feedforward SNNs and RSNNs over the speech datasets TI46-Alpha, TI46-Digits, and N-Tidigits and the image dataset Fashion-MNIST, outperforming the current state-of-the-art SNN training techniques. Moreover, ST-RSBP outperforms conventional deep learning models like LSTM, GRN, and traditional non-spiking BP on the same datasets. By releasing the GPU implementation code, we expect this work would advance the research on spiking neural networks and neuromorphic computing.
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1 Detailed Description of Spike-train Level Post-synaptic Potential (S-PSP) and Total PSP (T-PSP)

S-PSP captures the spike-train level interactions between a pair of pre/post-synaptic neurons and can be defined for any neural models with an all-or-none spiking characteristics and any synaptic models [2]. Without loss of generality, we describe S-PSP using the widely adopted leaky integrate-and-fire (LIF) model of spiking neurons and a first-order synaptic model [1]:

\[
\tau_m \frac{d u_i(t)}{dt} = -u_i(t) + R \alpha_i(t),
\]

(1)

with

\[
\tau_s \frac{d \alpha_i(t)}{dt} = -\alpha_i(t) + \sum_j w_{ij} \sum_{t_j} D(t - t_{j(t)}) ,
\]

(2)

where \( u_i(t) \) is the membrane potential of the neuron \( i \), \( \alpha_i(t) \) the total synaptic current input based on a first order synaptic model with time constant \( \tau_s \) and \( \tau_m \) the time constant of membrane potential with value \( \tau_m = RC \). \( R \) and \( C \) are the effective leaky resistance and effective membrane capacitance and \( R \) is set to 1 since it can be absorbed into synaptic weights. \( w_{ij} \) is the weight of the synapse from the pre-synaptic neuron \( j \) to the neuron \( i \). \( t_{j(t)} \) denotes a particular firing time of the neuron \( j \). \( D(t) \) is the Dirac delta function.

The integration of (1) and (2) leads to the spike response model (SRM) [1]:

\[
u_i(t) = \sum_j w_{ij} \sum_{t_j} \epsilon(t - t_{j(t)}, t - t_{j(t)}),
\]

(3)

where \( \epsilon(t, s, t) \) specifies the normalized time course of the post-synaptic potential evoked by a single firing spike of the pre-synaptic neuron:

\[
\epsilon(s, t) = \frac{1}{C} \int_0^s \exp\left(-\frac{t'}{\tau_m}\right) \alpha_i(t - t') \ dt'.
\]

(4)

Through integration, (4) can be re-written as:

\[
\epsilon(s, t) = \frac{\epsilon\left(-\frac{\max(t-s,0)}{\tau_s}\right)}{1 - \frac{s}{\tau_m}} \left[ \epsilon\left(-\frac{\min(t,s)}{\tau_m}\right) - \epsilon\left(-\frac{\min(t,s)}{\tau_s}\right) \right] H(s) H(t),
\]

(5)

where \( H(t) \) is the Heaviside step function.

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Note that each neuron fires whenever its post-synaptic potential reaches the firing threshold. We now sum up the contributions of the pre-synaptic neuron \( j \)'s spike train to the (normalized) post-synaptic potential of the neuron \( i \) right before all the neuron \( i \)'s firing times as illustrated in Fig. 1:

\[
e_{ij} = \sum_{t_i^{(f)}, t_j^{(f)}} e(t_i^{(f)} - \hat{t}_i^{(f)}, t_i^{(f)} - t_j^{(f)}),
\]

defining the (normalized) spike-train level post-synaptic potential (S-PSP) from the neuron \( j \) to the neuron \( i \).

The significance of S-PSPs lies on that it characterizes the aggregated effect of the spike train of the pre-synaptic neuron \( j \) on the membrane potential of the post-synaptic neuron \( i \) and its firing activities. Employing S-PSPs in the proposed ST-RSBP algorithm is beneficial; it allows efficient consideration of the temporal dynamics and recurrent connections of an RSNN across all firing events at the spike-train level without expensive unrolling in time and backpropagation time point by time point, which are required by BPTT.

The sum of the weighted S-PSPs from all pre-synaptic neurons of the neuron \( i \) is defined as the total post-synaptic potential (T-PSP) \( a_i \), relating to the neuron \( i \)'s firing count \( o_i \) via the firing threshold \( \nu \):

\[
a_i = \sum_j w_{ij} e_{ij}, \quad o_i = g(a_i) \approx \frac{a_i}{\nu}.
\]

\( a_i \) and \( o_i \) are analogous to the pre-activation and activation in the traditional ANNs, respectively, and \( g(\cdot) \) can be considered as an activation function converting the T-PSP to the output firing count.

## 2 Detailed Derivation of the ST-RSBP Algorithm

The rate-coded loss is defined at the output layer as:

\[
E = \frac{1}{2} \| o - y \|^2 = \frac{1}{2} \| a - \nu y \|^2,
\]

where \( y \), \( o \) and \( a \) are vectors of the desired output neuron firing counts (labels), actual firing counts, and the T-PSPs of the output neurons, respectively. Differentiating (8) with respect to each trainable weight \( w_{ij}^k \) incident upon the layer \( k \) leads to:

\[
\frac{\partial E}{\partial w_{ij}^k} = \frac{\partial E}{\partial a_i^k} \frac{\partial a_i^k}{\partial w_{ij}^k} = \delta_i^k \frac{\partial a_i^k}{\partial w_{ij}^k}, \quad \text{with} \quad \delta_i^k = \frac{\partial E}{\partial a_i^k},
\]

where \( \delta_i^k \) and \( \frac{\partial a_i^k}{\partial w_{ij}^k} \) are referred to as the back propagated error and differentiation of activation, respectively, for the neuron \( i \). ST-RSBP updates \( w_{ij}^k \) by \( \Delta w_{ij}^k = \eta \frac{\partial E}{\partial a_i^k} \), where \( \eta \) is a learning rate.
2.1 Back Propagated Errors

2.1.1 Output Layer

When the layer \( k \) is the output layer, the back propagated error at the \( i \)th neuron of the layer is given by differentiating the loss defined in (8):

\[
\delta_i^k = \frac{\partial E}{\partial a_i^k} = \frac{(o_i^k - y_i^k)}{\nu^k},
\]

(10)

where \( o_i^k \) is the actual firing count, \( y_i^k \) the desired firing count (label), and \( a_i^k \) the corresponding T-PSP.

2.1.2 Hidden Layers

At each hidden layer \( k \), by applying the chain rule, the back propagated error \( \delta_i^k \) for the neuron \( i \) can be expressed as:

\[
\delta_i^k = \frac{\partial E}{\partial a_i^k} = \sum_{l=1}^{N_{k+1}} \frac{\partial E}{\partial a_i^{k+1}} \frac{\partial a_i^{k+1}}{\partial a_i^k} = \sum_{l=1}^{N_{k+1}} \delta_l^{k+1} \frac{\partial a_i^{k+1}}{\partial a_i^k}.
\]

(11)

\( N_{k+1} \) is the number of neurons in the layer \( k+1 \). Define two error vectors \( \delta^{k+1} \) and \( \delta^k \) for the two layers: \( \delta^{k+1} = [\delta_1^{k+1}, \ldots, \delta_{N_{k+1}}^{k+1}] \) and \( \delta^k = [\delta_1^k, \ldots, \delta_{N_k}^k] \), respectively for the layers \( k+1 \) and \( k \), where \( N_k \) is the number of the neurons in the layer \( k \). Assuming \( \delta^{k+1} \) is given, which is the case for the output layer based on (10), the goal is to back propagate from \( \delta^{k+1} \) to \( \delta^k \). Clearly, this entails to compute \( \frac{\partial a_i^{k+1}}{\partial a_i^k} \) in (11).

[Backpropagation from a Hidden Recurrent Layer] Now consider that the errors are back propagated from a recurrent layer \( k+1 \) to its preceding layer \( k \). Note that the S-PSP \( e_{lj} \) from any pre-synaptic neuron \( j \) to a post-synaptic neuron \( l \) is a function of both the rate and temporal information of the pre/post spike trains, which can be made explicitly via some function \( f \):

\[
e_{lj} = f(o_j, a_l, t_j^{(f)}, t_l^{(f)}),
\]

(12)

where \( o_j, a_l, t_j^{(f)}, t_l^{(f)} \) are the pre-synaptic/post-synaptic firing counts and firing times, respectively.

Now based on (2) of the main manuscript, \( \frac{\partial a_l^{k+1}}{\partial a_i^k} \) is split also into two summations:

\[
\frac{\partial a_l^{k+1}}{\partial a_i^k} = \sum_j w_{lj}^{k+1} \frac{d e_{lj}^{k+1}}{d a_i^k} + \sum_p w_{lp}^{k+1} \frac{d e_{lp}^{k+1}}{d a_i^k},
\]

(13)

where the first summation sums over all pre-synaptic neurons in the previous layer \( k \) while the second sums over the pre-synaptic neurons in the current recurrent layer as illustrated in Fig. 2.

On the right side of (13), \( \frac{d e_{lj}^{k+1}}{d a_i^k} \) is given by:

\[
\frac{d e_{lj}^{k+1}}{d a_i^k} = \begin{cases} 
\frac{1}{\nu^k} \frac{\partial a_{lj}^{k+1}}{\partial a_i^k} + \frac{1}{\nu^{k+1}} \frac{\partial a_{lj}^{k+1}}{\partial a_i^k} \frac{\partial a_{lj}^{k+1}}{\partial a_i^k} & j = i \\
\frac{1}{\nu^{k+1}} \frac{\partial a_{lj}^{k+1}}{\partial a_i^k} \frac{\partial a_{lj}^{k+1}}{\partial a_i^k} & j \neq i,
\end{cases}
\]

(14)

where \( \nu^k \) and \( \nu^{k+1} \) are the firing threshold voltages for the layers \( k \) and \( k+1 \), respectively, and we have used that \( a_{lj}^k \approx a_{lj}^k / \nu^k \) and \( a_{lj}^{k+1} \approx a_{lj}^{k+1} / \nu^{k+1} \) from (7). Importantly, the last term on the right side of (14) exists due to \( e_{lj}^{k+1} \)'s dependency on the post-synaptic firing rate \( o_l^{k+1} \) per (12) and \( o_l^{k+1} \)'s further dependency on the pre-synaptic activation \( a_l^{k+1} \) (hence pre-activation \( a_l^k \)), as shown in Fig. 2.

On the right side of (13), \( \frac{d e_{lp}^{k+1}}{d a_i^k} \) is due to the recurrent connections within the layer \( k+1 \) and is given by:

\[
\frac{d e_{lp}^{k+1}}{d a_i^k} = \frac{1}{\nu^{k+1}} \frac{\partial a_{lp}^{k+1}}{\partial a_i^k} \frac{\partial a_{lp}^{k+1}}{\partial a_i^k} + \frac{1}{\nu^{k+1}} \frac{\partial a_{lp}^{k+1}}{\partial a_i^k} \frac{\partial a_{lp}^{k+1}}{\partial a_i^k}.
\]

(15)
The first term on the right side of (15) is due to \( e_{l_p}^{k+1} \)'s dependency on the post-synaptic firing rate \( o_{i}^{k+1} \) per (12) and \( o_{i}^{k+1} \)'s further dependence on the pre-synaptic activation \( o_{i}^{k} \) (hence pre-activation \( a_{i}^{k} \)). Per (12), it is important to note that the second term exists because \( e_{l_p}^{k+1} \)'s dependency on the pre-synaptic firing rate \( o_{p}^{k+1} \), which further depends on \( o_{i}^{k} \) (hence pre-activation \( a_{i}^{k} \)), as shown in Fig. 2.

Putting (13), (14), and (15) together leads to:

\[
\frac{\partial a_{i}^{k+1}}{\partial a_{i}^{k}} = w_{l_i}^{k+1} \frac{1}{\nu^{k}} \frac{\partial e_{l_i}^{k+1}}{\partial o_{i}^{k+1}} + \frac{1}{\nu^{k+1}} \frac{\partial a_{i}^{k+1}}{\partial o_{i}^{k+1}} \left( \sum_{j} w_{l_j}^{k+1} \frac{\partial e_{l_j}^{k+1}}{\partial o_{i}^{k+1}} + \sum_{p} w_{l_p}^{k+1} \frac{\partial e_{l_p}^{k+1}}{\partial o_{i}^{k+1}} \right) + \sum_{p} w_{l_p}^{k+1} \frac{1}{\nu^{k+1}} \frac{\partial e_{l_p}^{k+1}}{\partial o_{p}^{k+1}} \frac{\partial o_{p}^{k+1}}{\partial a_{i}^{k}}. \tag{16}
\]

Now, (16) is rearranged to:

\[
\left( 1 - \frac{1}{\nu^{k+1}} \right) \left( \sum_{j} w_{l_j}^{k+1} \frac{\partial e_{l_j}^{k+1}}{\partial o_{i}^{k+1}} + \sum_{p} w_{l_p}^{k+1} \frac{\partial e_{l_p}^{k+1}}{\partial o_{i}^{k+1}} \right) \frac{\partial a_{i}^{k+1}}{\partial a_{i}^{k}} = w_{l_i}^{k+1} \frac{1}{\nu^{k}} \frac{\partial e_{l_i}^{k+1}}{\partial o_{i}^{k+1}} + \sum_{p} w_{l_p}^{k+1} \frac{1}{\nu^{k+1}} \frac{\partial e_{l_p}^{k+1}}{\partial o_{p}^{k+1}} \frac{\partial o_{p}^{k+1}}{\partial a_{i}^{k}}. \tag{17}
\]

It is evident that all \( N_{k+1} \times N_{k} \) partial derivatives involving the recurrent layer \( k+1 \) and its preceding layer \( k \), i.e. \( \frac{\partial a_{i}^{k+1}}{\partial a_{i}^{l}}, i = [1, N_{k}], l = [1, N_{k+1}] \), form a coupled linear system via (17), which is written in a matrix form as:

\[
\Omega^{k+1,k} \cdot P^{k+1,k} = \Phi^{k+1,k} + \Theta^{k+1,k} \cdot P^{k+1,k}, \quad \tag{18}
\]
where $P^{k+1,k} \in \mathbb{R}^{N_{k+1} \times N_k}$ contains all the desired partial derivatives, $\Omega^{k+1,k} \in \mathbb{R}^{N_{k+1} \times N_{k+1}}$ is diagonal, $\Theta^{k+1,k} \in \mathbb{R}^{N_{k+1} \times N_{k+1}}$, and $\Phi^{k+1,k} \in \mathbb{R}^{N_{k+1} \times N_k}$, and

$$
\Omega^{k+1,k}_{ij} = \begin{cases} 
1 - \frac{1}{\nu^k} \left( \sum_{m} N_k w_{k+1}^{m} \frac{\partial e_{k+1}^{m}}{\partial o_{i}^{m}} + \sum_{p} N_{k+1} w_{k+1}^{p} \frac{\partial e_{k+1}^{p}}{\partial o_{i}^{p}} \right) & i = j \\
0 & i \neq j 
\end{cases} 
$$

(19)

$$
P^{k+1,k}_{ij} = \frac{\partial a_{k+1}^{i}}{\partial o_{j}^{k}} \Phi_{ij}^{k+1,k} = w_{ij}^{k+1} \frac{1}{\nu^k} \frac{\partial e_{k+1}^{i}}{\partial o_{j}^{k}}, \quad \Theta_{ij}^{k+1,k} = w_{ij}^{k+1} \frac{1}{\nu^k} \frac{\partial e_{k+1}^{i}}{\partial o_{j}^{k+1}}. 
$$

The partial derivatives of the S-PSP with respect to the pre-synaptic and post-synaptic firing counts, i.e. $\frac{\partial e_{k+1}^{i}}{\partial o_{j}^{k}}$ and $\frac{\partial e_{k+1}^{i}}{\partial o_{j}^{k+1}}$ as needed in (19) will be determined in Section 2.3. Solving the linear system in (18) gives all $\frac{\partial a_{k+1}^{i}}{\partial o_{j}^{k}}$:

$$
P^{k+1,k} = (\Omega^{k+1,k} - \Theta^{k+1,k})^{-1} \cdot \Phi^{k+1,k}. 
$$

(20)

Note that since $\Omega$ is a diagonal matrix, the cost in factoring the above linear system can be reduced by approximating the matrix inversion using a first-order Taylor’s expansion without performing any matrix factorization.

All $N_{k}$ errors at the layer $k$ back propagated from the layer $k + 1$ per (11) is put into a vector form: $\delta^{k} = [\delta_{1}^{k}, \ldots, \delta_{N_{k}}^{k}]$, and is given by:

$$
\delta^{k} = (P^{k+1,k})^T \cdot \delta^{k+1}, 
$$

(21)

where $\delta^{k+1}$ is the error vector at the layer $k + 1$.

**[Backpropagation from a Hidden Feedforward Layer]** Consider the much simpler case of back-propagating errors from a feedforward layer $k + 1$ to its preceding layer $k$. Due to non-existence of recurrent connections in the layer $k + 1$, (20) is simplified to:

$$
P^{k+1,k} = (\Omega^{k+1,k})^{-1} \cdot \Phi^{k+1,k}. 
$$

(22)

Since $\Omega^{k+1,k}$ is diagonal, each $\frac{\partial a_{k+1}^{i}}{\partial o_{j}^{k}}$ can be directly computed:

$$
\frac{\partial a_{k+1}^{i}}{\partial o_{j}^{k}} = \frac{1}{\nu^k} w_{ij}^{k+1} \frac{\partial e_{k+1}^{i}}{\partial o_{j}^{k}}, 
$$

(23)

\[2.2\] Differentiation of Activation

Per (9), we derive the differentiation of activation $\frac{\partial a_{k}^{i}}{\partial w_{ij}^{k}}$ under two cases.

**2.2.1 Feedforward Layers**

For a feedforward layer $k$ and based on (2) of the main paper, differentiation of each activation is given by:

$$
\frac{\partial a_{k}^{i}}{\partial w_{ij}^{k}} = \frac{\partial}{\partial w_{ij}^{k}} \left( \sum_{l} w_{il}^{k} e_{l}^{k} \right) = e_{ij}^{k} + \frac{1}{\nu^k} \frac{\partial a_{k}^{i}}{\partial w_{ij}^{k}} \sum_{l} w_{il}^{k} \frac{\partial e_{l}^{k}}{\partial o_{j}^{k}}. 
$$

(24)

The first term on the right side of (24) reflects the direct dependency of $a_{k}^{i}$ on $w_{ij}^{k}$ while the second term captures the dependency of each S-PSP $e_{l}^{k}$ on the post-synaptic firing count $o_{j}^{k}$, which further depends on $w_{ij}$ according to (12). (24) gives the desired differentiation of activation as:

$$
\frac{\partial a_{k}^{i}}{\partial w_{ij}^{k}} = \frac{1}{\nu^k} \sum_{l} N_{k-1} w_{il}^{k} \frac{\partial e_{l}^{k}}{\partial o_{j}^{k}}, 
$$

(25)
2.2.2 Recurrent Layers

For the activation $a_i^k$ of the neuron $i$ at the recurrent layer $k$, we further consider the recurrent connections and get

$$
\frac{\partial a_i^k}{\partial w_{ij}^k} = \frac{\partial}{\partial w_{ij}^k} \left( \sum_{l} w_{il}^k e_i^l + \sum_p w_{ip}^k e_p^k \right) = e_i^k + \frac{\partial a_i^k}{\partial w_{ij}^k} \frac{1}{\tau_k} \left( \sum_{l} w_{il}^k \frac{\partial e_i^k}{\partial a_i^l} + \sum_p w_{ip}^k \frac{\partial e_i^k}{\partial a_p^k} \right),
$$

leading to:

$$
\frac{\partial a_i^k}{\partial w_{ij}^k} = \frac{1}{1 - \frac{1}{\tau_k}} \left( \sum_{l} w_{il}^k \frac{\partial e_i^k}{\partial a_i^l} + \sum_p w_{ip}^k \frac{\partial e_i^k}{\partial a_p^k} \right).
$$

(26)

2.3 Differentiation of S-PSP w.r.t Pre/Post-Synaptic Firing Counts

Before presenting the final ST-RSBP algorithm, we shall determine the partial derivatives $\frac{\partial e_{ij}}{\partial o_i^j}$ and $\frac{\partial e_{ij}}{\partial o_j^i}$ of an S-PSP $e_{ij}$ with respect to the firing counts of the pre-synaptic neuron $j$ and post-synaptic neuron $i$, respectively, as needed in (19), (23), (25), and (26). As discussed in Section 1, S-PSPs serve as a bridge between neuron-level firing timings and spike-train level firing count and allow backpropagating errors defined for a rate-coded loss at the spike-train level.

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[2] computes the two partial derivatives by assuming that each S-PSP $e_{ij}$ is approximately linear in both $o_i$ and $o_j$. To examine this assumption, we evaluate the S-PSP from the neuron $j$ to neuron $i$ via a synapse. The LIF neuron model of (1) and the synaptic model of (2) with $\tau_m = 64 ms$, $\tau_e = 8 ms$ are adopted in this analysis. The simulation duration is set to $600 ms$ and the first-order Euler method with a fixed stepsize of $1 ms$ is used for simulation. To cover a wide range of interactions between the two neurons, we consider all combinations of the firing rates of two neurons $o_i$ and $o_j$ when they are swept widely from 1 to 50. For each combination of $o_i$ and $o_j$ values, we generate the spike trains of the two neurons by randomly choosing $o_i$ and $o_j$ numbers of random spiking times, respectively, and compute the S-PSP $e_{ij}$ according to (6). We repeat this process 500 times and take the average value of $e_{ij}$.

We plot the relation between the pre/post-synaptic firing counts $o_j$ and $o_i$ and the average $e_{ij}$ in Fig. 3A. Fig. 3B shows that with $o_i$ fixed $e_{ij}$ increases rather linearly in $o_j$, consistent with [2], and hence we have:

$$
\frac{\partial e_{ij}}{\partial o_i} \approx \frac{e_{ij}}{o_j}.
$$

(27)

However, Fig. 3C shows that with $o_j$ fixed, $e_{ij}$ is not linear in a wide range of $o_i$, suggesting that the assumption made in [2] can lead to errors when the postsynaptic firing rates vary a lot. Based on the data collected for Fig. 3A, for each fixed $o_j$, we instead fit $e_{ij}$ as a third-order polynomial in $o_i$ to obtain the corresponding values for the derivative $\frac{\partial e_{ij}}{\partial o_i}$. The characterization of $\frac{\partial e_{ij}}{\partial o_i}$ occurs offline prior to the training process.

![Figure 3](image)

Figure 3: (A) The average S-PSP value vs. pre and post-synaptic firing counts; (B) The average $e_{ij}$ vs. $o_j$ when the post-synaptic firing count is fixed ($o_i = 10$); (C) The average $e_{ij}$ vs. $o_i$ when the pre-synaptic firing count is fixed ($o_j = 10$).

2.4 The Final Proposed ST-RSBP Algorithm

For each layer $k$, denote the error vector by $\delta^k \in \mathbb{R}^{N_k}$, the matrix of differentiation of activation by $F^{k,k-1} \in \mathbb{R}^{N_k \times N_{k-1}}$, and the weight matrix from the layer $k-1$ to layer $k$ by $W^{k,k-1} \in \mathbb{R}^{N_k \times N_{k-1}}$, and...
respectively. \( P^{k+1,k} \in \mathbb{R}^{N_{k+1} \times N_k} \) contains all derivatives of \( \frac{\partial a^{k+1}}{\partial o^i} \) obtained from (20) or (23). If the layer \( k \) is recurrent layer, we additionally use \( F^{k,k} \in \mathbb{R}^{N_k \times N_k} \) and \( W^{k,k} \in \mathbb{R}^{N_k \times N_k} \) to denote the matrix of differentiation of activation and the weight matrix of recurrent connections within the layer \( k \). Putting everything together, the complete ST-RSBP algorithm with a learning rate \( \eta \) is as follows:

\[
\begin{align*}
\Delta W^{k,k-1} &= \eta \frac{\nabla^k \mathcal{E}^k}{\nabla^k \mathcal{W}^k} = \eta \cdot \text{diag}(\delta^k) \cdot F^{k,k-1} \quad \text{for feedforward connections} \\
\Delta W^{k,k} &= \eta \frac{\nabla^k \mathcal{E}^k}{\nabla^k \mathcal{W}^k} = \eta \cdot \text{diag}(\delta^k) \cdot F^{k,k} \quad \text{for recurrent connections}
\end{align*}
\]

\[
F_{ij}^{k,k} = \left \{ \begin{array}{ll}
\frac{e^k_{ij}}{1 - \frac{1}{\nu^k} \sum_{l} \eta^{l+1} \frac{\partial a^k_{ij}}{\partial o^l_{ij}}} & \\
\frac{e^k_{ij}}{1 - \frac{1}{\nu^k} \left( \sum_{l} \eta^{l+1} \frac{\partial a^k_{ij}}{\partial o^l_{ij}} \right)} & \text{if layer } k \text{ is the output}
\end{array} \right.
\]

\[
\begin{align*}
\delta_k &= \phi_{k+1,k}^T \cdot \delta^{k+1} \\
\delta^{k} &= ((\Omega^{k+1,k} - \Theta^{k+1,k})^{-1} \cdot \Phi^{k+1,k})^T \cdot \delta^{k+1}
\end{align*}
\]

\[
\Omega_{ij}^{k+1,k} = \left \{ \begin{array}{ll}
1 - \frac{1}{\nu^{k+1}} \left( \sum_{l} \eta^{k+1} w^{k+1}_{lp} \frac{\partial a^{k+1}_{ij}}{\partial o^{l+1}_{ip}} + \sum_{l} \eta^{k+1} w^{k+1}_{lp} \frac{\partial a^{k+1}_{ij}}{\partial o^{l+1}_{ip}} \right) & i = j \\
0 & i \neq j
\end{array} \right.
\]

\[
\begin{align*}
\phi_{ij}^{k+1} &= w_{ij}^{k+1} \frac{1}{\nu^{k+1}} \frac{\partial e_{ij}^{k+1}}{\partial o_{ij}^{k+1}} \\
\theta_{ij}^{k+1} &= \phi_{ij}^{k+1} \frac{1}{\nu^{k+1}} \frac{\partial e_{ij}^{k+1}}{\partial o_{ij}^{k+1}} \\
\Phi_{ij}^{k+1} &= \left \{ \begin{array}{ll}
\frac{1}{\nu^{k+1}} w_{ij}^{k+1} \frac{\partial e_{ij}^{k+1}}{\partial o_{ij}^{k+1}} & \\
\frac{1}{\nu^{k+1}} w_{ij}^{k+1} \frac{\partial e_{ij}^{k+1}}{\partial o_{ij}^{k+1}} & \text{if layer } k+1 \text{ is recurrent}
\end{array} \right.
\end{align*}
\]

The application of ST-RSBP follows the typical backpropagation steps. First, the SNN is simulated layer-by-layer based on chosen synaptic/neural models such as the LIF model (1). Second, the firing counts of the output layer are compared with the desirable firing labels to compute the output error \( \delta^k \). After that, the error vector in the output layer is propagated backwards to determine the gradient, based on which both the recurrent synapses weights and the weights between layers are trained.

**References**

[1] Wulfram Gerstner and Werner M Kistler. *Spiking neuron models: Single neurons, populations, plasticity*. Cambridge university press, 2002.

[2] Yingyezhe Jin, Wenrui Zhang, and Peng Li. Hybrid macro/micro level backpropagation for training deep spiking neural networks. *In Advances in Neural Information Processing Systems*, pages 7005–7015, 2018.