HOMOGENEOUS SPECTRUM, DISJOINTNESS OF CONVOLUTIONS, AND MIXING PROPERTIES OF DYNAMICAL SYSTEMS
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Abstract

In connection with Rokhlin’s question on an automorphism with a homogeneous nonsimple spectrum, we indicate a class of measure-preserving maps $T$ such that $T \times T$ has a homogeneous spectrum of multiplicity 2. The automorphisms in question satisfy the condition $\sigma * \sigma \perp \sigma$, where $\sigma$ is the spectral measure of $T$. We also show that there is a mixing automorphism possessing the above properties and their higher order analogs.

1 Introduction

Let $T$ be an automorphism of a Lebesgue space $(X, \mu)$, $\mu(X) = 1$. This automorphism induces a unitary operator $\hat{T} : L_2(\mu) \to L_2(\mu)$, $\hat{T}f(x) = f(Tx)$, and one can speak about spectral measure of the operator $\hat{T}$ and the multiplicity function of the spectrum.

Rokhlin posed the question on the existence of an automorphism with a nonsimple homogeneous spectrum of finite multiplicity. Automorphisms having nonsimple spectra of finite multiplicity were constructed by Oseledets [8]. Katok [7] obtained the following result: for a generic collection of maps $T$, the essential range of the multiplicity function of $T \times T$ is either $\mathcal{M}_{T \times T} = \{2\}$ or $\mathcal{M}_{T \times T} = \{2, 4\}$. (Goodson and Lemanczyk noted [4] that for $T \times T$ the spectrum multiplicity function does not take odd values.) Katok conjectured that, for a generic $T$, the automorphism $T \times T$ has a homogeneous spectrum of multiplicity 2. This conjecture was confirmed by Ageev and the author, see [2], [11].

In this paper we study automorphisms $T$ for which $\mathcal{M}_{T \times T} = \{2\}$, i.e. $T \times T$ has a homogeneous spectrum of multiplicity 2. Namely, we consider automorphisms $T$ of simple spectrum such that for $a \in (0, 1)$, the operator $(aI + (1 - a)\hat{T})$ (or the operator $(1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots)$) belongs to the weak closure of the powers of $\hat{T}$ (§§2, 3). Using certain staircase constructions [11], one can obtain similar spectral properties for mixing automorphisms (§4). In §5 we discuss disjointness of higher order convolutions.

For definitions, we refer the reader to [3], [12]. We shall denote by $\sigma = \sigma_T$ the maximal spectral type of the unitary operator $\hat{T}$ acting on the space $H = \{f \in L_2(\mu) : \int f \, d\mu = 0\}$. It is well known that the spectral type of $\hat{T} \otimes \hat{T}$ is the measure $\sigma + \sigma * \sigma$. Note also that the convolution $\sigma * \sigma$ is the maximal spectral type of the restriction of $\hat{T} \otimes \hat{T}$ on the space $H \otimes H$.

Throughout we use the fact that the disjointness of $\sigma$ and $\sigma * \sigma$ is equivalent to the absence of nonzero operators intertwining $\hat{T}|H$ with $\hat{T} \otimes \hat{T}|H \otimes H$. We note that the
conditions

\[(\hat{T} \otimes \hat{T})J = J\hat{T}, \quad J \neq 0\]

imply the existence of a complex measure \(\lambda \neq 0\) such that

\[\hat{\lambda}(n) = \langle \hat{T}^n f | J^* g \rangle_{L^2(\mu)} = \langle \hat{T}^n \otimes \hat{T}^n Jf | g \rangle_{L^2(\mu \times \mu)} ,\]

hence \(\lambda\) is a common component of the measures \(\sigma\) and \(\sigma \ast \sigma\).

2 The case \(\hat{T}^{k_i} \rightarrow (aI + (1 - a)\hat{T})\).

Goodson \[3\] noted that, given an automorphism \(T\), the simplicity of the spectrum of the map \(R: X \times X \rightarrow X \times X\),

\[R(x, y) = (y, Tx)\],

implies that \(T \times T\) has a homogeneous spectrum of multiplicity 2. We shall use this in the proofs below.

**Theorem 2.1.** Let \(T\) be an ergodic automorphism and let the weak convergence

\[\hat{T}^{k_i} \rightarrow (aI + (1 - a)\hat{T})\]

hold for some sequence \(k_i \rightarrow \infty\) and some \(a \in (0, 1)\). Then

(1) the spectral measure \(\sigma\) of \(\hat{T}\) and the convolution \(\sigma \ast \sigma\) are disjoint;

(2) if, in addition, \(T\) has a simple spectrum, then \((T \times T)\) has a homogeneous spectrum of multiplicity 2.

Proof. (1) Suppose that a bounded operator \(J\) intertwines \(\hat{T}\) and \((\hat{T} \otimes \hat{T})\), i.e.,

\[J\hat{T} = (\hat{T} \otimes \hat{T})J\].

Setting \(b = 1 - a\) we obtain

\[J(aI + b\hat{T}) = ((aI + b\hat{T}) \otimes (aI + b\hat{T}))J,\]

\[(a(I \otimes I) + b(\hat{T} \otimes \hat{T}))J = (a^2(I \otimes I) + ab(\hat{T} \otimes I) + ab(I \otimes \hat{T}) + b^2(\hat{T} \otimes \hat{T}))J,\]

\[J + (\hat{T} \otimes \hat{T})J = (I \otimes \hat{T})J + (\hat{T} \otimes I)J.\]  (1)

This implies that, for all \(i, j\), we have

\[(\hat{T}^i \otimes \hat{T}^j)J + (\hat{T}^{i+1} \otimes \hat{T}^{j+1})J - (\hat{T}^i \otimes \hat{T}^{j+1})J - (\hat{T}^{i+1} \otimes \hat{T}^j)J = 0.\]

Now we obtain

\[\sum_{0 \leq i, j < n} (\hat{T}^i \otimes \hat{T}^j)J + (\hat{T}^{i+1} \otimes \hat{T}^{j+1})J - (\hat{T}^i \otimes \hat{T}^{j+1})J - (\hat{T}^{i+1} \otimes \hat{T}^j)J = \]
\[ 0 = J + (\widehat{T}^n \otimes \widehat{T}^m)J - (I \otimes \widehat{T}^n)J - (\widehat{T}^m \otimes I)J. \] (2)

Since \( T \) is weakly mixing (it is not hard to check that an eigenfunction of operator \( \widehat{T} \) must be a constant function), we have \( \widehat{T}^n \to \Theta \), where \( \Theta \) is the orthogonal projection onto the space of the constant functions. Hence, from (2) we conclude

\[ J + (\Theta \otimes \Theta)J = (I \otimes \Theta)J + (\Theta \otimes I)J. \]

Thus, \( \text{Im}(J) \perp H \otimes H \), which is equivalent to the assertion that the zero operator is a unique operator intertwining \( \widehat{T}|H \) and \( \widehat{T} \otimes \widehat{T}|H \otimes H \). It is a well-known fact that the weak closure of the powers \( \widehat{T}^m \) is a semigroup, hence it contains the operators \( (aI + b\widehat{T})^p \). It follows that the vector \( U_p = [(aI + b\widehat{T})^p \otimes (aI + b\widehat{T})^p]V_{0,0} \) belongs to the space \( L \). We write

\[ U_p = (a^p b^p V_{p,0} + a^p b^p V_{0,p}) + \sum_{m,n:|m-n|<p} c_{m,n} V_{m,n}, \]

where all \( V_{m,n} \) belong to \( L \) as \(|m-n|<p\). So we get \((V_{p,0} + V_{0,p}) \in L\). This implies that

\[ V_{p,0} \in L, \quad V_{0,p+1} \in \widehat{R}V_{p,0} \in L. \]

We shall assume that \( V_{i,i} \in L \) for \( i = 0, 1, \ldots, p \) and prove that \( V_{0,p+1} \in L \). It is a well-known fact that the weak closure of the powers \( \widehat{T}^n \) is a semigroup, hence it contains the operators \( (aI + b\widehat{T})^p \). It follows that the vector \( U_p = [(aI + b\widehat{T})^p \otimes (aI + b\widehat{T})^p]V_{0,0} \) belongs to the space \( L \). We write

\[ U_p = (a^p b^p V_{p,0} + a^p b^p V_{0,p}) + \sum_{m,n:|m-n|<p} c_{m,n} V_{m,n}, \]

where all \( V_{m,n} \) belong to \( L \) as \(|m-n|<p\). So we get \((V_{p,0} + V_{0,p}) \in L\). This implies that

\[ V_{p,0} \in L, \quad V_{0,p+1} \in \widehat{R}V_{p,0} \in L. \]

Thus we have proved that, for all \( m, n \), \( V_{m,n} = R^{2m} V_{0,n-m} \in L \), i.e., \( L = H \otimes H \), the restriction of \( \widehat{R} \) to \( H \otimes H \), has a simple spectrum. Note that the restriction of \( \widehat{R} \) to \((1 \otimes H) + (H \otimes 1)\) has also a simple spectrum \((1 \otimes f \text{ is a cyclic vector})\). Since the action
of the operator $\hat{\mathcal{R}}^2$ on $(1 \otimes H) + (H \otimes 1)$ and the action of $\hat{\mathcal{R}}^2$ on $(H \otimes H)$ are disjoint, the same is true for $\hat{R}$ and we obtain that $R$ has a simple spectrum.

**Remarks.**
(i) Theorem 2.1 was proved in part independently by Ageev, see [2]. He also gave a solution to Katok’s conjecture concerning the sets of spectral multiplicities of $T \times T \times \ldots \times T$ for generic automorphisms $T$.

(ii) Katok and Stepin pointed out that there is a three interval exchange transformation $T$ (in fact an automorphism of the half-circle, induced by some rotation of the unit circle) possessing the property of the $(n, n+1)$-type approximation (see [7], [5] for the definition). This property implies, for some sequence $h(i) \to \infty$ and any $p > 0$, the weak convergence

$T^{p h(i)} \to \frac{1}{2}(I + \Theta).$  

Thus, we have interval exchange transformations satisfying the conditions of Theorem 2.1. In addition, we obtain $\kappa$-mixing property for our $T$ for $\kappa = \frac{1}{2}$, i.e., for some sequence $k_i \to \infty$, one has

$\hat{T}^{k_i} \to \frac{1}{2}(I + \Theta),$  

where $\Theta$ is the orthogonal projection onto the space of the constant functions. This confirms the corresponding Oseledets conjecture from [9] (see also [3]). To see that (*) implies (**) we use the fact that, for some sequence $p_j \to \infty$, we have $\hat{T}^{p_j} \to \Theta$, since $T$ is weakly mixing.

### 3 The case $\hat{T}^{k_i} \to (1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots).$

In this section we consider automorphisms which can be close to the class of mixing automorphisms: if $a$ is close to 1, then, for ergodic $T$, the operator $(1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots)$ will be close to $\Theta$.

**Theorem 3.1.** Suppose that $T$ is an ergodic automorphism such that, for some sequence $k_i \to \infty$ and some $a \in (0, 1)$, one has the weak convergence

$\hat{T}^{k_i} \to (1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots).$

Then

(1) the spectral measure $\sigma$ of $\hat{T}$ and the convolution $\sigma * \sigma$ are disjoint;

(2) if $T$ has a simple spectrum, then the automorphism $(T \times T)$ has a homogeneous spectrum of multiplicity 2.

**Proof.** (1) We denote

$P = (1 - a)(I - a\hat{T})^{-1} = (1 - a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots).$

Let an operator $J: H \to H \otimes H$ satisfy the condition

$J\hat{T} = (\hat{T} \otimes \hat{T})J.$
Since $\hat{T}^k \rightarrow P$, we have

$$JP = (P \otimes P)J,$$

$$J(1-a)(I + a\hat{T} + a^2\hat{T}^2 + \ldots) = (P \otimes P)J,$$

$$V \in C \setminus V \in \bigcup_{1}^{\infty} R = \ldots$$

The equality $A^{-1}J = B^{-1}J$ implies $A = B$, hence we obtain

$$V \in C \setminus V \in \bigcup_{1}^{\infty} R = \ldots$$

$$A^{-1}J = B^{-1}J.$$
4 The case of mixing $T$.

A generalization of the above methods enables us to prove the existence of a mixing operator $T$ with the following properties:

1. Spectrum of the symmetric product $T \circ T$ is simple: $\mathcal{M}_{T \circ T} = \{1\}$,
2. $\mathcal{M}_{T \times T} = \{2\}$,
3. $\sigma_T \ast \sigma_T \perp \sigma_T$.

We recall that $T \circ T$ denotes the restriction of $T \times T$ to the factor of $S$-fixed subsets of $X \times X$, where the map $S$ is defined as $S(x, y) = (y, x)$. Note also that Property 1 implies Property 2 and Property 3, which is readily seen.

Adams [1] proved the property of mixing for a large class of rank 1 staircase constructions. For some special automorphisms of the Adams class, we are able to prove Property 1. This is a positive answer to the corresponding question of J.-P. Thouvenot.

Let us recall the definition of a staircase construction. Let an automorphism $T$ admit, for any $n$, a partition $\xi_n$ of $X$ into sets $B_n^1$, $TB_n^1$, $\ldots$, $T^{h_n-1}B_n^1$, $B_n^2$, $TB_n^2$, $\ldots$, $T^{h_n-1}B_n^2$, $T^{h_n}B_n^2$, $B_n^3$, $TB_n^3$, $\ldots$, $T^{h_n-1}B_n^3$, $T^{h_n}B_n^3$, $T^{h_n+1}B_n^3$, $\ldots$, $B_n^r$, $TB_n^r$, $\ldots$, $T^{h_n+r_n-1}B_n^r$, $Y_n$ such that

$$B_n^2 = T^{h_n}B_n^1, B_n^3 = T^{h_n+1}B_n^2, \ldots, B_n^r = T^{h_n+r_n-1}B_n^r$$

for all $n$, and the sequence of the partitions $\xi_n$ tends to the partitions into singletons ($\xi_n \to \varepsilon$). If, in addition, for all $n$ we have

$$B_{n-1}^1 = B_n^1 \sqcup B_n^2 \sqcup \ldots \sqcup B_n^r,$$

we say that such an automorphism $T$ is a staircase construction. One can see that this construction is defined by $h_1$ and the sequence $\{r_n\}$.

It is easily seen that $h_n + 1$ is the number of atoms in the partition $\xi_{n-1}$. Note also that the set $Y_{n-1}$ is the union of the sets

$$T^{h_n}B_n^2, T^{h_n}B_n^3, T^{h_n+1}B_n^3, \ldots, T^{h_n+r_n-2}B_n^r, Y_n.$$

The Adams theorem asserts that in the case where $r_n \to \infty$ and $\frac{(r_n)^2}{h_n} \to 0$, the corresponding staircase construction $T$ is mixing.

We can prove that there is a sequence $r_n \to \infty$ such that $\frac{(r_n)^2}{h_n} \to 0$, and the corresponding operator $T$ possesses the property $\mathcal{M}_{T \circ T} = \{1\}$.

**Theorem 4.1.** There is a staircase mixing construction $T$ possessing the property $\mathcal{M}_{T \circ T} = \{1\}$. 

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We introduce special classes \( \text{St.C.}(p_j, h_j) \) of staircase constructions, and prove that the property \( M_{T_j \otimes T_j} = \{1\} \) holds for any \( T_j \in \text{St.C.}(p_j, h_j) \). Then we find a sequence of corresponding \( T_j \) which approximates sufficiently well some staircase construction \( T \) satisfying the conditions of the Adams theorem. Due to the approximation procedure, the property \( M_{T \otimes T} = \{1\} \) will be preserved.

**Definition.** We say that \( T \) is in the class \( \text{St.C.}(p, h) \) if \( T \) is a staircase construction with a sequence \( r_n \) satisfying the following conditions:

1. \( \lim \inf_{n \to \infty} r_n = p \);
2. \( \forall q, p \leq q \leq h, \exists n_i \to \infty \) such that \( r_{n_i+1} \to \infty \) and \( \forall i \; r_{n_i} = q \).

It follows from Condition 2 that the operators

\[
P_q = \frac{1}{q}(I + \hat{T} + \hat{T}^2 + \ldots + \hat{T}^{q-2} + \Theta)
\]

are in \( WCl(T) \) as \( p \leq q \leq h \). (In fact the powers \( \hat{T}^{-h_n} \) converge weakly to \( P_q \) as \( r_n = q \) for all \( i \).)

**Lemma.** Let \( T \) be in \( \text{St.C.}(p, h + 2) \), and let \( 3p < h + 2 \). Suppose that \( B, TB, \ldots, T^h B \) are disjoint measurable sets, and denote by \( C_{B \times B} \) the cyclic space generated by vector \( \chi_B \otimes \chi_B \) under the action of the operator \( \hat{T} \otimes \hat{T} \). Then the functions

\[
F_{i,j} = \chi_{T^i B} \otimes \chi_{T^j B} + \chi_{T^i B} \otimes \chi_{T^j B}, \quad 0 \leq i, j \leq h,
\]

belong to the cyclic space \( C_{B \times B} \).

**Proof.** Since

\[
F_{q,0} \in C_{B \times B} \implies F_{q+i,i} \in C_{B \times B}
\]

we only show that \( F_{q,0} \in C_{B \times B} \). We have

\[
P_{q+2} \chi_B \otimes P_{q+2} \chi_B, \quad P_{q+1} \chi_B \otimes P_{q+1} \chi_B, \quad P_q \chi_B \otimes P_q \chi_B \in C_{B \times B}.
\]

Let

\[
G_m = \frac{m^2}{(1 + \mu(B))^2} P_m \chi_B \otimes P_m \chi_B.
\]

One can check that

\[
F_{q,0} = \text{Const} \left[ G_{q+2} - G_{q+1} - (\hat{T} \otimes \hat{T}) G_{q+1} + (\hat{T} \otimes \hat{T}) G_q \right].
\]

Thus, for all \( q \geq p \), we have \( F_{q,0} \in C_{B \times B} \), hence, all the functions \( F_{i,j} \) are in \( C_{B \times B} \).

Now we prove that, for all \( q < p \), we have \( F_{q,0} \in C_{B \times B} \) too. Let us show this for \( q = 1 \). Since

\[
F_{p,0} = [(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)] \chi_{B \times B}
\]

and \( C_{B \times B} \) contains \( F_{p+1,0} \), we obtain that the cyclic space generated by \( F_{p,0} \) contains

\[
[(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)] F_{p+1,0} = [(\hat{T}^p \otimes I) + (I \otimes \hat{T}^p)][(\hat{T}^{p+1} \otimes I) + (I \otimes \hat{T}^{p+1})] \chi_{B \times B}.
\]
The latter can be represented as the sum $F_{2p+1,0} + F_{p+1,p}$, where $F_{2p+1,0} \in C_{B \times B}$. Since this sum is also in $C_{B \times B}$, we get $F_{p+1,p} \in C_{B \times B}$ and, hence $F_{1,0}$ belongs to $C_{B \times B}$.

**Theorem 4.2.** Let $T$ belong to $St.C.(p, \infty)$. Then $\mathcal{M}_{T \otimes T} = \{1\}$.

Proof. We consider the sequence $\xi_n$ and the corresponding sequence of the sets $B_n = B_n^1$ (see the above definition of a staircase construction). Let us consider the sequence of the cyclic spaces $C_{B_n \times B_n}$ for the operator $T \otimes \hat{T}$. Each symmetric function $F(x, y)$ can be approximated by linear combinations of $F_{i,j}$, where the functions $F_{i,j}$ depend on $n$ and $0 \leq i, j \leq b_n$. From the above lemma we obtain that the symmetric product $L_2(\mu) \otimes L_2(\mu)$ can be approximated by the cyclic spaces $C_{B_n \times B_n}$. The Katok–Oseledets–Stepin approach yields that $L_2(\mu) \otimes L_2(\mu)$ is a cyclic space as well. Thus $\mathcal{M}_{T \otimes T} = \{1\}$.

How to construct a mixing operator $T$ with the property $\mathcal{M}_{T \otimes T} = \{1\}$? In fact we prove only the existence. A non-constructive description is this: we consider a sequence \{r_n\} such that, for all $k$, we have $r_{2k+1} = 2k + 1$, and the sequence $r_{2k}$ extremely slowly tends to the infinity as $k \to \infty$. Now we explain why such a sequence \{r_n\} can give the desired staircase construction.

We shall consider a sequence of automorphisms $T_j$ such that $T_j$ is of class $St.C.(p_j, \infty)$ on $X_j = \{x : T_j(x) \neq x\}$, where $\mu(X_j) \to 1$ and $p_j \to \infty$. Given an automorphism $T_j$ with the corresponding sequence $r_n^{(j)}$, we choose a sufficiently large number $N_j$ and change this sequence only for $n > N_j$. We obtain a new sequence $r_n^{(j+1)}$ and the corresponding construction $T_{j+1}$ such that $T_j$ differs from $T_{j+1}$ only on the very small set $Y_{N_j}$ (here $Y_{N_j}$ corresponds to the automorphism $T_{j+1}$). Our operator $T$ will be a limit automorphism, in fact a staircase construction with $r_n \to \infty$. We can choose $r_n \leq n$, then we obtain $\frac{r_n^2}{\log n} \to 0$, which guarantees the property of mixing.

The sequence \{T_j\} is organized so that, for any symmetric functions $F(x, y)$, the distance between $F$ and the space $C_{B_j \times B_j}$ (here the cyclic space is considered for the operator $\hat{T} \otimes \hat{T}$) tends to zero. However, it is possible to obtain the same property for the sequence of the spaces $C'_{B_j \times B_j}$, the cyclic spaces of the operator $\hat{T} \otimes \hat{T}$. Indeed, let for some $F$ ($F$ will be taken from a fixed finite collection of symmetric functions) we have

$$\left\| F - \sum_{k=-N}^{N} a_k U_j^k \chi_{B_j \times B_j} \right\| < \varepsilon,$$

where $U_j = \hat{T}_j \otimes \hat{T}_j$. However, we can ensure that

$$\left\| F - \sum_{k=-N}^{N} a_k U^k \chi_{B_j \times B_j} \right\| < 2\varepsilon$$

for $U = \hat{T} \otimes \hat{T}$, since the measure of the set $\{x : T_j(x) \neq T(x)\}$ can be made as small as desired by an appropriate choice of $T_{j+1}, T_{j+2}, \ldots$.

Thus, it is possible to find a construction such that, for any fixed countable set of symmetric functions $F$ and, hence, for all symmetric functions (due to the separability of $L_2 \otimes L_2$), the distance between $F$ and the space $C'_{B_j \times B_j}$ tends to 0 as $j \to \infty$. 

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Thus we conserve the property $M_{T \circ T} = \{1\}$ for some mixing staircase construction $T$.

**Remark.** It is worth noting that for mixing $T$ the property $M_{T \circ T} = \{1\}$ implies mixing of all orders. This follows from Host’s theorem: if $\sigma T \ast \sigma T \perp \sigma T$, then the mixing automorphism $T$ possesses the multiple mixing property [6].

5 On higher order properties.

Disjointness of $\sigma \ast \sigma$ and $\sigma$ in the case $2\hat{T}^k \to (I + \hat{T})$ was obtained also by M. Lemanczyk and generalized by F. Parreau to $\sigma \ast n \perp \sigma$. The problem “$\sigma \ast m \perp \sigma \ast n$?” as $m > n > 1$ remains open. In some cases, we can obtain a bit more than $\sigma \ast n \perp \sigma$.

**Theorem 5.1.**

If, for an ergodic automorphism $T$, there is a sequence $k_i \to \infty$ such that

$\hat{T}^{k_i} \to P = (aI + b\hat{T})$, \hspace{1cm} 1 > a > b > 0,

then $\sigma \ast \sigma \perp \sigma \ast \sigma \ast \sigma$, where $\sigma$ is the spectral measure of the automorphism $T$.

**Proof.** Suppose that some operator $J: L_2 \otimes L_2 \to L_2 \otimes L_2 \otimes L_2$ satisfies the intertwining condition

$J(\hat{T} \otimes \hat{T}) = (\hat{T} \otimes \hat{T} \otimes \hat{T})J$.

We have to show that $J = 0$. Since

$\hat{T}^{-k_i+1} \to Q = (bI + a\hat{T})$,

we obtain

$J[P \otimes P \otimes Q \otimes Q] = [P \otimes P \otimes P \otimes Q \otimes Q \otimes Q]J, J(a^2 - b^2)[I \otimes I \otimes \hat{T} \otimes \hat{T}] = [P \otimes P \otimes P \otimes Q \otimes Q \otimes Q]J,$

$J(a^2 - b^2) = [P \otimes P \otimes P \otimes P \otimes P \otimes Q \otimes Q \otimes Q \otimes Q]J,$

$(a^2 - b^2 - a^3 + b^3)J = [\ldots + (a^2 - b^2 - a^3 + b^3)\hat{T} \otimes \hat{T} \otimes \hat{T}]J,$

$J = [\ldots + \hat{T} \otimes \hat{T} \otimes \hat{T}]J,$

where $(\ldots)$ is a linear combination of the operators $\hat{T} \otimes I \otimes I, \ldots, I \otimes \hat{T} \otimes \hat{T}$. Let, for $h \in H \otimes H$, we have

$Jh = [\ldots + \hat{T} \otimes \hat{T} \otimes \hat{T}]Jh.$

We rewrite this for the spectral representation of $\hat{T}$ as follows:

$f(x, y, z) = [\ldots + xyz]f(x, y, z),$
where \( f \) is the image of \( Jh \) in the spectral representation, and \( x, y, z \) belong to the unit circle. We can see that, for any fixed \( x, y \), there is a unique point \( z \) such that

\[
0 \neq f(x, y, z) = [(\ldots) + xyz]f(x, y, z).
\]

Since the measure \( \sigma \) is continuous, one has \( \sigma \otimes \sigma \otimes \sigma(\text{support}(f)) = 0 \), i.e., \( f = 0 \) in the space \( L_2(\sigma \otimes \sigma \otimes \sigma) \). Thus we obtain

\[
Jh = 0, \quad J = 0, \quad \sigma \ast \sigma \perp \sigma \ast \sigma \ast \sigma.
\]

The following assertion is a natural generalization of Theorem 4.1.

**Theorem 5.2.** There is a mixing automorphism \( T \) with the following properties:

1. \( \mathcal{M}_{T \otimes n} = \{1\} \),
2. \( \mathcal{M}_{T \times n} = \{n, n(n-1), \ldots, n!\} \),
3. \( \sigma^{*k} \perp \sigma^{*m} \) for all \( k > m > 0 \).

The proof of Theorem 5.2 will be published in a separate paper.\(^2\)

**Remark.** Ageev\(^2\) proved property (2) for generic (non-mixing) automorphisms. In [10], property (3) has been established for the well-known Chacon automorphism. We conjecture that the Chacon automorphism has properties (1) and (2) as well.

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