Liénard-Wiechert Retarded Potentials Derived from the Electrodynam

Doppler Effect

Fulvio Parmigiani

Dipartimento di Fisica, Università di Trieste, via A. Valerio 2, 34127 Trieste, Italy
Elettra Sincrotrone Trieste, I-34149 Basovizza, Trieste, Italy and
International Faculty, University of Cologne,
Albertus-Magnus-Platz, 50923 Cologne, Germany

Giovanni Perosa

Dipartimento di Fisica, Università di Trieste, via A. Valerio 2, 34127 Trieste, Italy

Simone Di Mitri

Dipartimento di Fisica, Università di Trieste,
via A. Valerio 2, 34127 Trieste, Italy and
Elettra Sincrotrone Trieste, I-34149 Basovizza, Trieste, Italy

(Dated: February 14, 2022)

Abstract

Here electrodynamic retarded potentials are derived considering the Doppler shifted electromagnetic (e.m.) field generated by a charge moving along a generic trajectory. This alternative route allows a direct physical interpretation of the Doppler term \((v/c)\) characterizing these potentials. Meanwhile, it does not incur mathematically questionable aspects or it does not hide the physical nature of these potentials when the charge volume is taken to the infinitesimal limit of a point charge or when mathematical developments derived from special relativity are used. The process outlined here is mathematically rigorous and elegant and sheds light on the physics underlying the origin of the retarded potential, making the understanding of the radiation processes by a moving charge more accessible to students, teachers, and scholars.

The following article has been submitted to American Journal of Physics. After it is published, it will be found at https://aapt.scitation.org/journal/ajp.
I. INTRODUCTION

The electrodynamics potentials associated with a charge moving along a generic trajectory are, in general, derived from arguments based on the different charge density seen by an observer at rest in the reference frame where the charge trajectory is defined, i.e. laboratory rest frame, respect to the charge density in the proper reference frame. Assuming that an electromagnetic (e.m.) signal is traveling at a constant speed, c, it will reach a point \( r \), at rest in the laboratory frame, at a later time, \( t \), respect to the starting signal time \( t_r \) (see Fig. 1). That is why the potentials related to the e.m. field, \( \psi(r, t) \) and \( A(r, t) \), are defined ”retarded potentials”. Interestingly, these potentials depend on the normalized velocity of the charge, \( v/c \equiv [\beta] \), at the retarded time \( (t_r) \) and retarded distance respect to the time \( t \) and distance \( r \) at which the signal is detected, see point in \( P \) in Fig. 1.

Most of the textbooks derive the retarded potentials following the original works of Liénard and Wiechert based on the superposition principle of the fields produced by different volume elements of the charge particle which is assumed to have a finite size volume. Otherwise, Liénard-Wiechert potentials are derived via Lorentz transformations following the hints first given by Minkowski. Indeed, some authors have interpreted, for extended charge distributions, the \( v/c \) factor in the analytical expression of the retarded potentials in the frame of special relativity.

However, the derivation of the retarded potentials using the Euclidean geometry implies some conceptual and mathematical inconsistencies. An extended physical charge cannot be reduced to a Euclidean point. Conversely, a Euclidean point cannot be expanded to a finite dimension, unless invoking more advanced geometries. As a matter of fact, a physical charge occupies a finite volume and, strictly speaking, cannot be assimilated to a Euclidean point. In this frame, the demonstrations given in most electrodynamics textbooks are acceptable but not reducible to the limit of a point charge. Accordingly, the arguments that these potentials do not depend on the charge volume, therefore they are extensible to the infinitesimal point charge limit, are arbitrary. Furthermore, these demonstrations do not provide a clear interpretation of the physics behinds the mathematical formalism.

Remarkably, Haus derived the e.m. field generated by a single charge moving along a generic trajectory by considering the Fourier-transformed components of the current density. He showed that at constant velocity \( v < c \), the charge does not irradiate because the light-like current density...
Fourier components are absent. However, while this work helps to elucidate the close relationship between current density and the radiation associated with a moving charge, thus paving the way for an elegant interpretation of the Cherenkov effect, it does not shed light on the Doppler mechanism characterizing the delayed potentials.

In these scenarios, the difficulties in understanding the retarded potentials are related to the physical origin of the Doppler-like factor \( v/c \).

Aim of this paper is to derive the retarded potentials developing the Doppler physics that their analytical formulation suggests. Clearly, these potentials are not due to relativistic effects, since only one observer (detector) is involved, nor to geometric volume effects. Actually, we show that the Lorentz transformations are unnecessary, nor are geometric arguments, except those associated with Doppler shift mechanisms.

II. DERIVATION OF THE RETARDED POTENTIALS

A. Retarded Green’s function

Let’s start from the retarded potentials in the Lorenz gauge\(^\text{14}\)

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(r, t) = \frac{\rho(r, t)}{\varepsilon_0} \tag{1}
\]

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A(r, t) = \mu_0 J(r, t) \tag{2}
\]

where \( \psi(r, t) \) is the electric scalar potential, \( A(r, t) \) is the magnetic vector potential, \( \rho(r, t) \) is the electric charge density, \( J(r, t) \) is the current density, and \( \nabla^2 \) is the Laplacian \( \left( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \).

These differential equations are often referred as wave equations\(^\text{19}\) The inhomogeneous part, i.e. the non trivial right-hand side (RHS), is regarded as the electromagnetic waves source. In a matter free space, \( c \) is the e.m. wave phase velocity. Thereafter, we focus on the case of e.m. fields generated by a moving charge \( q \), having charge and current density,

\[
\rho(r, t) = q\delta(r - r_o(t)) \quad \text{and} \quad J(r, t) = qv(t)\delta(r - r_o(t)), \tag{3}
\]

respectively, being \( r_o(t) \) the charge trajectory in the laboratory reference frame (as shown in Fig. \(\text{I}\)) and \( v(t) \) its velocity. This equations can be solved by starting from the retarded Green’s function for the wave equation\(^\text{14,19}\)

\[
G_-(r, t) = \frac{1}{4\pi r} \delta(t - r/c). \tag{4}
\]
FIG. 1: Point charge $q$ moving in the reference frame $S$, following the trajectory $r_o(t)$. The retarded potential at $P$ is computed as the superposition of outgoing spherical wave coming from $q$, schematically represented as circular wavefronts in red. The relative motion of the charge with respect to $P$ is the origin of the Doppler shift of the frequencies.

Sommerfeld also provided two derivations of the retarded potential by starting from the Green’s function, which are mathematically sophisticated, but ultimately hiding the physics of the problem. Assuming that the observer is affected only by the presence of a source that acts at an earlier time, we can discard the advanced solution, proportional to $(t + r/c)$, and use directly the retarded one. The Fourier expansion of $G_-(r, t)$ in the frequency domain describes a spherical wave propagating outward from the e.m. field source,

$$G_-(r, t) = \frac{1}{2\pi} \int d\omega \tilde{G}(r, \omega) e^{i\omega t}, \quad \text{with} \quad \tilde{G}(r, \omega) = \frac{e^{ikr}}{4\pi r}. \quad (5)$$

B. Consequences of the relative motion

Figure [1] shows that, since the signal has a finite velocity ($c = \omega/k$), the information carried by an e.m. wave will take some time to reach the point $P$. The time at which the wavefront registered in $P$ at the time $t$ was originated is called retarded time $t_r$. This can be retrieved by the identity of
the wavefront phase in $P$ at time $t$ and in the origin at time $t_r$, where the source particle’s trajectory is properly taken into account. Let us indicate with $\phi(r, t)$, the phase of a spherical wave

$$\phi(r, t) = kr - \omega t, \quad (6)$$

and the condition for the retarded time translates into

$$\phi(r, t) = \phi(r_o(t_r), t_r). \quad (7)$$

Inserting expression for the phase of a spherical phase (6), we obtain:

$$\phi(r, t) - \phi(r_o(t_r), t_r) = k|r - r_o(t_r)| - \omega(t - t_r) = -\omega \left(t - t_r - \frac{|r - r_o(t_r)|}{c}\right) = 0,$$

hence,

$$t_r = t - \frac{|r - r_o(t_r)|}{c}. \quad (8)$$

The second relevant consequence of the motion of the source in the reference frame $S$ is the Doppler shift of the wave frequency

$$\omega' = -\frac{d\phi}{dt} = -\mathbf{k} \cdot \frac{dr}{dt} + \omega = \omega \left(1 - \frac{\hat{k} \cdot \mathbf{v}}{c}\right) = \frac{\omega}{D_k(v)} \quad (9)$$

$$D_k(v) = \left(\frac{c}{c - \hat{k} \cdot \mathbf{v}}\right) \quad \text{and} \quad \hat{k} = \hat{r} \quad (10)$$

Equation (10) can also be interpreted as the change of the phase velocity due to the relative motion of the source in the laboratory reference frame, in a consistent analogy with the apparent pitch change of the sound coming from a moving emitter.

**C. Computation of the retarded potentials**

According to the superposition principle, the solution to equation (1) is given by the convolution of the Green’s function and the point charge distribution

$$\psi(r, t) = \frac{1}{\varepsilon_0} \int dt' \int d\tau' G_-(r, t|r', t') \rho(r', t'), \quad (11)$$

where $d\tau'$ is the volume three dimensional element and

$$G_-(r, t|r', t') = \frac{1}{4\pi|r - r'|} \delta(t - t' - |r - r'|/c). \quad (12)$$
In view of the interpretation that we give of the Green’s function in the frequency domain, equation (11) represents the superposition of outgoing spherical waves.

We can substitute the expression (5) and the first of equations (3) in (11) to get

\[ \psi(r, t) = \frac{q}{2\pi \epsilon_0} \int dt' \int dt' \delta(r' - r_O(t')) \int d\omega \frac{1}{4\pi |r - r'|} \exp \{i[k|r - r'| - \omega(t - t')]\} \]. \quad (13)

The first step is to simplify the inner couple of integrals (the one over \( \tau' \) and the one over \( \omega \)):

\[ \psi(r, t) = \frac{q}{2\pi \epsilon_0} \int dt' \int dt' \delta(r' - r_O(t')) \int d\omega \frac{1}{4\pi |r - r'|} \exp \{i[k|r - r'| - \omega(t - t')]\} = \]

\[ = \frac{q}{2\pi \epsilon_0} \int dt' \int d\omega \frac{1}{4\pi |r - r'|} \exp \{i[k|r - r'| - \omega(t - t')]\} = \]

\[ = \frac{q}{2\pi \epsilon_0} \int dt' \int d\omega \frac{1}{4\pi |r - r'|} \exp \{i[\phi(r, t) - \phi(r_O(t'), t')]\} \quad (14) \]

We expand the argument of the exponential, i.e. the phase difference, around \( t_r \). Actually, the only relevant contributions in the integration over \( \omega \) are the one for which this difference is zero. Clearly, the expansion of the phase difference around \( t_r \) is proportional to the phase velocity, which appears to be Doppler shifted, as shown in equation (10):

\[ \phi(r, t) - \phi(r_O(t'), t') \approx \frac{\omega}{D_k(v(t_r))}(t' - t_r) \quad \text{with} \quad \hat{k} = \frac{r - r_O(t_r)}{|r - r_O(t_r)|}. \quad (15) \]

We will omit the dependence of \( v \) from \( t_r \) hereafter.

We substitute this new expression in (14),

\[ \psi(r, t) = \frac{q}{2\pi \epsilon_0} \int dt' \int d\omega \frac{e^{i\omega(t' - t_r)/D_k(v)}}{4\pi |r - r_O(t')|} \]

\[ = \frac{q}{2\pi \epsilon_0} \int dt' \int d\omega \frac{D_k(v)e^{i\omega(t' - t_r)}}{4\pi |r - r_O(t')|} = \]

\[ = \frac{q}{4\pi \epsilon_0} \int dt' \frac{D_k(v)\delta(t' - t_r)}{|r - r_O(t')|}, \quad (16) \]

where, the first step is a change of integration variable, from \( \omega \) to \( \omega' \), while the second step exploits the following identity

\[ \delta(x - x_0) = \frac{1}{2\pi} \int dp e^{ip(x - x_0)}. \quad (17) \]

As a result of the integration over \( t' \) in eq. (16), we find that

\[ \psi(r, t) = \frac{q}{4\pi \epsilon_0} \frac{D_k(v(t_r))}{|r - r_O(t_r)|} = \frac{1}{4\pi \epsilon_0} \frac{qc}{(c - \hat{k} \cdot \mathbf{v}(t_r)) |r - r_O(t_r)|} \quad (18) \]

An analogous strategy can be applied to obtain \( \mathbf{A} \), leading to the expression

\[ \mathbf{A}(r, t) = \frac{\mathbf{v}}{c^2} \psi(r, t). \quad (19) \]
We can add a few comments to what we have derived so far. Firstly, it is possible to link our derivation with the one proposed in literature. The evaluation of the charge density at the retarded time is usually expressed as an integration over time of the Dirac delta

\[ \delta(t' - t_r) = \frac{\delta(t' - t_r)}{|g'(t_r)|} = \frac{1}{2\pi} \int \frac{e^{i\nu(t' - t_r)}}{|g'(t_r)|} = \frac{1}{2\pi} \int \frac{d\omega e^{i\omega(t' - t_r)|g'(t_r)|}}{|g'(t_r)|} \]

where we used the fact that

\[ \delta[g(x)] = \sum_n \delta(x - x_n), \quad \text{with} \quad g(x_n) = 0, g'(x_n) \neq 0. \quad (20) \]

In this case, \( g(t') = t' - t + \kappa(t')/c, \) whose zero is, by definition, the retarded time \( t_r. \) We use the notation \( \kappa(t') = |r - r'(t')| \) and find that

\[ g'(t') = \frac{d}{dt'}[t' - t + \kappa(t')/c] = 1 - v(t') \cdot \hat{r}/c = \frac{c - v(t') \cdot \hat{r}}{c} = \frac{1}{D_{\hat{r}}(v)}. \quad (21) \]

which coincides with our results.

Other authors\(^4,6,7\) introduce the Doppler term in different ways and suggest that it can be absorbed in the infinitesimal volume element, stressing with this argument its geometrical origin. In our procedure, instead, that factor is naturally embedded in the treatment and explicitly derived from a Doppler mechanism.

More expert students or teachers could have spotted the presence of a Dirac delta in expression (14) and, using equation (20), obtain the final result without passing through all the steps. Although it is undoubtedly correct and more cultivated, we think this haste becomes a lost opportunity to bring out the physics.

III. CONCLUSIONS

Retarded potentials represent a fundamental aspect of classical electrodynamics, although, sometimes the way they are derived hides the physical picture. Indeed, most of the methods reported in the literature and textbooks incurs formally questionable aspects when they deal with the limit of infinitesimal charge volume (point charge) or unclear interpretations when invoking the special relativity formalism. Nonetheless, the resulting retarded potentials are formally correct, but for a subtle and, perhaps, elusive reason. Actually, retarded potentials are due to a Doppler effect, as is evident from the characteristic correction factor \( v/c \) presents in their analytical formulation. Therefore, deriving the retarded potentials from the charge density perceived by an observer at rest
in the laboratory frame or via Lorentz transformations, obscures their origin.

In this work the retarded potentials are derived from optical physics principles taking into account the Doppler effect acting on the electromagnetic field observed at rest in the laboratory reference frame. Overall, the route followed here, besides being formally correct, makes evident the physical phenomenology from which the electrodynamics retarded potentials originate.

IV. APPENDIX

In this appendix we apply the same procedure presented above to find the equations for the retarded fields, or Jefimenko-Feynman equations.

We start from Maxwell’s equations in the form of wave equations

\[
\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) E = \frac{1}{\varepsilon_0} \nabla \rho + \mu_0 \frac{\partial J}{\partial t} \tag{22}
\]

\[
\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) B = -\nabla \times \mu_0 J \tag{23}
\]

whose solutions are

\[
E = -\int dt' \int d\tau' G_-(\mathbf{r}, \tau, t') \left[ \frac{1}{\varepsilon_0} \nabla \rho(\mathbf{r}', t') + \mu_0 \frac{\partial J(\mathbf{r}', t')}{\partial t} \right] \tag{24}
\]

\[
B = \int dt' \int d\tau' G_-(\mathbf{r}, \tau, t') \left[ \nabla \times \mu_0 J(\mathbf{r}', t') \right] \tag{25}
\]

Since the operators \(\nabla\) and \(\partial/\partial t\) act on the unprimed coordinates, for the electric field we have that

\[
E = -\frac{1}{4\pi\varepsilon_0} \nabla \left( \int dt' \int d\tau' G_-(\mathbf{r}, \tau, t') \rho(\mathbf{r}', t') \right) - \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \left( \int dt' \int d\tau' G_-(\mathbf{r}, \tau, t') J(\mathbf{r}', t') \right),
\]

while for the magnetic field,

\[
B = \mu_0 \nabla \times \left( \int dt' \int d\tau' G_-(\mathbf{r}, \tau, t') J(\mathbf{r}', t') \right).
\]

The terms in parenthesis, called \(I\) and \(\Pi\) hereafter, are exactly the same integrals already encountered, when the definition (3) is inserted. The remaining step is the evaluation of the derivatives with respect to space and time

\[
E = -\frac{1}{4\pi\varepsilon_0} \nabla I(\mathbf{r}, t) - \frac{\mu_0}{4\pi} \frac{\partial}{\partial t} \Pi(\mathbf{r}, t), \quad B = \mu_0 \nabla \times \Pi(\mathbf{r}, t) \tag{26}
\]

Following the steps proposed by Griffith, we need some useful identities to carry out the derivation. Writing equation (8) as \(\mathbf{v}(t_r) = c(t - t_r)\), it is easy to show that

\[
\nabla \mathbf{v} = -c \nabla t_r = \nabla \left( \sqrt{\mathbf{v} \cdot \mathbf{v}} \right) = \frac{1}{2} \frac{1}{\sqrt{\mathbf{v} \cdot \mathbf{v}}} \nabla (\mathbf{v} \cdot \mathbf{v}) = \frac{1}{\mathbf{v}} \mathbf{v} \cdot (\nabla \mathbf{v}),
\]
multiplying both sides for $z$ and considering the $i$-th component
\[- c \frac{\partial r_i}{\partial x_i} = \frac{\partial z_j}{\partial x_i} = z_j \left( \delta_j^i - v_i \frac{\partial t_r}{\partial x_i} \right) = \dot{z}_i - v_i \frac{\partial t_r}{\partial x_i} \]
or, in vectorial form,
\[- c \vec{z} \nabla t_r = \vec{z} - (\vec{z} \cdot \nabla) \nabla t_r, \]
hence
\[\nabla \vec{z} = -c \nabla t_r = \frac{c}{c - \hat{\vec{v}} \cdot \vec{v}} \hat{\vec{v}} = D_{\hat{\vec{v}}}(\vec{v}) \hat{\vec{v}}. \tag{27}\]

Moreover,
\[c \left( 1 - \frac{\partial t_r}{\partial t} \right) = \frac{\partial}{\partial t} c(t - t_r) = \frac{\partial}{\partial t} \sqrt{\frac{\vec{z} \cdot \vec{v}}{c}} = \frac{1}{\vec{z}} \frac{\partial \vec{z}}{\partial t} = -\frac{1}{\vec{z}} \vec{z} \cdot \frac{\partial t_r}{\partial t}, \]
and rearranging the first and last terms, we get
\[\vec{z} c = (\vec{z} c - \hat{\vec{v}} \cdot \vec{v}) \frac{\partial t_r}{\partial t}, \quad \text{hence} \quad \frac{\partial t_r}{\partial t} = D_{\hat{\vec{v}}}(\vec{v}). \tag{28}\]

Given a vectorial quantity $\vec{w}(t_r)$, evaluated at the retarded time, the directional derivative along a vector $\vec{d}$ and its curl are
\[(\vec{d} \cdot \nabla) \vec{w}(t_r) = \frac{d\vec{w}(t_r)}{dt_r} (\vec{d} \cdot \nabla t_r), \quad \nabla \times \vec{w}(t_r) = -\frac{d\vec{w}(t_r)}{dt_r} \times \nabla t_r. \tag{29}\]

and using (27),
\[(\vec{d} \cdot \nabla) \vec{w}(t_r) = -\frac{D_{\hat{\vec{v}}}(\vec{v})}{c} \frac{d\vec{w}(t_r)}{dt_r} (\vec{d} \cdot \hat{\vec{v}}), \quad \nabla \times \vec{w}(t_r) = \frac{D_{\hat{\vec{v}}}(\vec{v})}{c} \frac{d\vec{w}(t_r)}{dt_r} \times \hat{\vec{v}}. \tag{30}\]

We start from the gradient of $I$:
\[
\nabla I(\vec{r}, t) = \nabla \left[ \frac{D_z(\vec{v}(t_r))}{\vec{z}} \right] = D_z(\vec{v}(t_r)) \nabla \left( \frac{1}{\vec{z}} \right) + \frac{\nabla D_z(\vec{v}(t_r))}{\vec{z}} =
\]
\[= -D_z(\vec{v}(t_r)) \frac{\nabla \vec{z}}{\vec{z}^2} + \frac{\nabla D_z(\vec{v}(t_r))}{\vec{z}} =
\]
\[= -\frac{D_z^2(\vec{v}(t_r))}{\vec{z}^2} + \frac{\nabla D_z(\vec{v}(t_r))}{\vec{z}}.
\]
We have to simplify the term $\nabla D_z(\vec{v})$, which leads to
\[\nabla D_z(\vec{v}) = \nabla \left( \frac{c}{c - \vec{z} \cdot \vec{v}} \right) = \frac{1}{c} \left( \frac{c}{c - \vec{z} \cdot \vec{v}} \right)^2 \nabla (\vec{z} \cdot \vec{v}) = \frac{1}{c} D_z^2(\vec{v}) \nabla (\vec{z} \cdot \vec{v}). \]

Now,
\[\nabla (\vec{z} \cdot \vec{v}) = \nabla \left( \frac{\vec{z} \cdot \vec{v}}{\vec{z}} \right) = \nabla \left( \frac{1}{\vec{z}} \right) (\vec{z} \cdot \vec{v}) + \frac{1}{\vec{z}} (\nabla (\vec{z} \cdot \vec{v})) =
\]
\[= \frac{1}{\vec{z}} \left[ -\nabla \frac{\vec{z}}{\vec{z}} (\vec{z} \cdot \vec{v}) + (\nabla (\vec{z} \cdot \vec{v})) \right] =
\]
\[= \frac{1}{\vec{z}} \left[ -D_z(\vec{v}) \hat{\vec{v}} (\vec{z} \cdot \vec{v}) + (\nabla (\vec{z} \cdot \vec{v})) \right].
\]
We can exploit the vectorial identities (30) to compute $\nabla (z \cdot v)$ and get
\[
\nabla (z \cdot v) = v - \frac{1}{c} (\hat{z} \cdot a - v^2) D_z(v) \hat{z}.
\] (31)

Collecting all together the steps, we arrive at
\[
\nabla I(r, t) = -\frac{D_z^2(v)}{\hat{z}^2} \left\{ \hat{z} - \frac{1}{c} \left[ v - \frac{1}{c} (\hat{z} \cdot a - v^2) D_z(v) \hat{z} - \frac{D_z(v)}{z} (\hat{z} \cdot v) \right] \right\}.
\]

The first and last terms in the parenthesis simplify into
\[
1 + \left( \frac{z \cdot v}{cz} \right) D_z(v) = \left[ 1 + \left( \frac{\hat{z} \cdot v}{c} \right) \frac{c}{c - \hat{z} \cdot v} \right] = D_z(v)
\]
and finally,
\[
\nabla I(r, t) = -\frac{D_z^2(v)}{\hat{z}^2} \left\{ D_z(v) \hat{z} - \frac{1}{c} \left[ v - \frac{1}{c} (\hat{z} \cdot a - v^2) D_z(v) \hat{z} \right] \right\} =
\frac{D_z^2(v)}{\hat{z}^2} \left[ \frac{v}{c} - \frac{D_z(v)}{cz^2} (c^2 - v^2 + a \cdot \hat{z}) \right].
\] (32)

Similarly, one can determine $\partial I$,\[
\frac{\partial I(r, t)}{\partial t} = \frac{\partial}{\partial t} \left[ I(r, t) \frac{\partial I(r, t)}{\partial t} \right] + v \frac{\partial I(r, t)}{\partial t} = \left[ I(r, t) \frac{\partial I(r, t)}{\partial t} + v \frac{\partial I(r, t)}{\partial t} \right] =
\frac{D_z^2(v)}{\hat{z}^2} a + v \frac{\partial I(r, t)}{\partial t} = \frac{D_z^2(v)}{\hat{z}^2} a + v \frac{\partial D_z(v)}{\partial t} \left[ \frac{1}{\hat{z}} \right].
\]

For the second term in RHS,
\[
\frac{\partial D_z(v)}{\partial t} = \frac{\partial}{\partial t} \left( \frac{c}{c - v \cdot \hat{z}} \right) = \frac{1}{c} \frac{D_z^2(v)}{\hat{z}^2} \left( v \cdot \hat{z} \right) = \frac{1}{c} D_z^2(v) \frac{\partial}{\partial t} \left( \frac{1}{z} \right) + \frac{1}{c z} D_z^2(v) \frac{\partial}{\partial t} (v \cdot \hat{z}),
\]
in which
\[
\frac{\partial}{\partial t} \left( \frac{1}{z} \right) = -\frac{c}{z^2} [1 - D_z(v)], \quad \text{and} \quad \frac{\partial}{\partial t} (v \cdot \hat{z}) = (a \cdot \hat{z} - v^2) D_z(v).
\] (33)

The time derivative is
\[
\frac{\partial I(r, t)}{\partial t} = \frac{D_z^2(v)}{\hat{z}^2} \left[ \hat{z} a - cv + v D_z(v) \left( c^2 - v^2 + a \cdot \hat{z} \right) \right]
\] (34)

As explained in [II], the calculation of $\nabla \times I$ can be simplified using the expressions already presented. The final results are
\[
E = -\frac{q}{4\pi\varepsilon_0} \left( \nabla I(r, t) + \frac{1}{c^2} \frac{\partial I(r, t)}{\partial t} \right) = -\frac{q D_z^3(v)}{4\pi\varepsilon_0 \hat{z}^2} \left[ (c^2 - v^2) u + \hat{z} \times (u \times a) \right]
\] (35)
\[
B = \frac{1}{c} \hat{z} \times E(r, t),
\] (36)
where \( \mathbf{u} = \hat{\mathbf{r}} - \mathbf{v}/c \). Notice that, if we define the new directions

\[
\mathbf{u}_\parallel = (\mathbf{u} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} = D_\parallel^{-1}(\mathbf{v}) \hat{\mathbf{r}} \\
\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel = \frac{1}{c} [(\hat{\mathbf{r}} \cdot \mathbf{v}) \hat{\mathbf{r}} - \mathbf{v}],
\]

(37)

(38)

the field along \( \mathbf{u}_\parallel \) scales as \( D_\parallel^2(\mathbf{v})/\varepsilon^2 \), while in the other direction, namely \( \mathbf{u}_\perp \), is proportional to \( D_\parallel^3(\mathbf{v})/\varepsilon^2 \). The latter corresponds to the inverse direction of the velocity component perpendicular to the source position.

As suggested in\(^{22}\), it is possible to retrieve the retarded potentials from equations \(^{26}\), invoking the retarded Helmholtz theorem\(^{23}\).
14  V. N. Strel’tsov, “Liénard-Wiechert Potential as a Consequence of Lorentz Transformation of Coulomb Potential”, Technical Report JINR-D-2-93-437 (1993).
15  H. A. Haus, "On the radiation of point charge", *Am. J. Phys.* **54**, 1126 (1986).
16  J. M. Aguirregabiria, A. Hernandez, and M. Rivas, “The Liénard-Wiechert potential and the retarded shape of a moving sphere”, *Am. J. Phys.* **60**, 597 (1992).
17  C. Galeriu, ”The geometrical origin of the Doppler factor in the Liénard–Wiechert potentials”, *Eur. J. Phys.* **42**, 055204 (2021).
18  C. Galeriu, ”A derivation of the Doppler factor in the Liénard–Wiechert potentials”, *Eur. J. Phys.* **42**, 055203 (2021).
19  S. Hassani, *Mathematical Physics* (Springer, 2013) 2nd ed.
20  P. Frank and R. V. Mises, *Die Differential-and Integralgleichungen der Mechanik und Physik, Zweiter phylikalischer Teil* (Wieberg und Sohn, Braunschweig, 1935).
21  A. Robinson, *Non-standard Analysis* (Princeton University, Princeton, NJ, 1986).
22  R. Heras, ”Alternative routes to the retarded potentials”, *Eur. J. Phys.* **38**, 055203 (2017).
23  R. Heras, ”The Helmholtz theorem and retarded fields”, *Eur. J. Phys.* **37**, 065204 (2016).