An Anomalous Dimension for Small $x$ Evolution

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Abstract

We construct an anomalous dimension for small $x$ evolution which goes beyond standard fixed order perturbative evolution by including resummed small $x$ logarithms deduced from the leading order BFKL equation with running coupling. Surprisingly, we find that once running coupling effects are properly taken into account, the leading approximation is very close to standard perturbative evolution in the range of $x$ accessible at HERA, in overall agreement with the data, with no need for phenomenological parameters to summarise subleading effects. We also show that further corrections due to subleading small $x$ logarithms derived from the Fadin-Lipatov kernel can be kept under control, but that they involve substantial resummation ambiguities which limit their practical usefulness.
1. Introduction

In recent years the theory of scaling violations for deep inelastic structure functions at small $x$ has attracted considerable interest, prompted by the experimental information coming from HERA [1,2]. New effects beyond the low–order perturbative approximation [3–5] to anomalous dimensions or splitting functions should become important at small-$x$. The BFKL approach [6–10] provides in principle a tool for the determination of the small-$x$ improvements of the anomalous dimensions [11,12]. However, no major deviation of the data from a standard next–to–leading order perturbative treatment of scaling violations [13,14,1,2] has been found, and a straightforward [15] inclusion of small-$x$ logarithms is in fact ruled out by the data [16,17]. By now the origin of this situation has been mostly understood [18–21].

The BFKL kernel $\chi(\alpha_s, M)$, as is well known, has been computed to next-to-leading accuracy (NLO):

$$\chi(M, \alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \ldots. \quad (1.1)$$

The problem is how to use the information contained in $\chi_0$ and $\chi_1$ in order to improve the splitting function derived from the perturbative leading singlet anomalous dimension function $\gamma(\alpha_s, N)$ which is known [4] up to NLO in $\alpha_s$:

$$\gamma(N, \alpha_s) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \ldots, \quad (1.2)$$

in such a way that the improved splitting function remains a good approximation down to small values of $x$. This can be accomplished by exploiting the fact [22] that the solutions of the BFKL and GLAP equations coincide at leading twist if their respective evolution kernels are related by a “duality” relation. In the fixed coupling limit the duality relation is simply given by:

$$\chi(\gamma(N, \alpha_s), \alpha_s) = N. \quad (1.3)$$

The splitting function then will contain all relative corrections of order $(\alpha_s \log 1/x)^n$, derived from $\chi_0(M)$, and of order $\alpha_s (\alpha_s \log 1/x)^n$, derived from $\chi_1(M)$.

The early wisdom on how to implement the information from $\chi_0$ was completely shaken by the much softer behaviour of the data at small $x$ and by the computation of $\chi_1$ [8–10], which showed that the naive expansion for the improved anomalous dimension had a hopelessly bad behaviour. In refs. [20,19] we have shown that, as a consequence of the physical requirement of momentum conservation, these problems are cured if the small-$x$ resummation is combined with the standard resummation of collinear singularities, by constructing a ‘double-leading’ perturbative expansion. However, the next-to-leading correction remains large, and it qualitatively changes the asymptotic small-$x$ behaviour of structure functions by changing it from $x^{-\lambda_0}$ to $x^{-\lambda} = x^{-\lambda_0} e^{\Delta \lambda \xi} \approx x^{-\lambda_0} [1 + \Delta \lambda \xi + \ldots]$, with:

$$\lambda = \lambda_0 + \Delta \lambda, \quad \lambda_0 = \alpha_s \chi_0(\frac{1}{2}) = \alpha_s c_0, \quad \Delta \lambda = \alpha_s^2 \chi_1(\frac{1}{2}) + \ldots. \quad (1.4)$$

The effects of this correction can be resummed, but there are ambiguities in the procedure which entail the dependence of the results on free parameters, or, alternatively, model assumptions.
A traditionally delicate point of the small-\(x\) resummation is the treatment of the running of the coupling [23–28]. While it has been known for some time [29] that running coupling effects can be included perturbatively order by order at small \(x\), in a recent paper [30] we have shown that an all-order resummation of running coupling effects at small \(x\) is desirable and can in fact be accomplished. Indeed, in a perturbative approach, running coupling terms can be included by adding effective subleading \(\Delta \chi_i\) contributions to the BFKL kernels \(\chi_1, \chi_2, \ldots\), which turn out to have singularities at \(M = 1/2\). These singularities correspond [29,30] to an enhancement by powers of \(\ln 1/x\) of the associated splitting functions, which may offset the perturbative suppression by powers of \(\alpha_s\). In ref. [30] we have shown that the all-order resummation of these contributions is fully compatible (up to higher twist corrections) with the standard factorized perturbative evolution of parton distributions, and can thus be performed at the level of splitting functions. The resummed splitting functions remain smooth in the small-\(x\) limit, despite the well-known [23–26] fact that the naive running coupling BFKL solution instead displays oscillatory instabilities at small \(x\).

The purpose of this paper is to discuss the physical impact of this running coupling resummation, and describe in detail its phenomenological implementation. We will argue that we now know the way the information contained in \(\chi_0(M)\) should be used in order to construct a better first approximation for the improved anomalous dimension. Indeed, we find that, once running coupling effects are properly included in the improved anomalous dimension, the asymptotic behavior near \(x = 0\) is much softened with respect to the naive Lipatov exponent. Hence, the corresponding dramatic rise of structure functions at small \(x\), which is phenomenologically ruled out, is replaced by a milder rise, whose steepness is determined by the Lipatov exponent. It then follows that a leading-order approximation based on the standard BFKL kernel \(\chi_0\) is phenomenologically viable.

Our proposed leading approximation is of a reasonably simple explicit form, directly suitable for the determination of scaling violations and fits to the data, contains no free parameters, and is in agreement with the general trend of the data. We find it remarkable that such a first approximation indeed exists and apparently works so well, so that we devote a good fraction of this article to describing it and its physical foundation. We then construct and discuss a perturbative expansion based upon it, and show in particular that the next order correction, which includes the effects of the subleading kernel \(\chi_1\), leads to small corrections for a reasonable range of the resummation parameters. However, some parameter dependence and resummation ambiguities are necessarily present at this level, and in fact at next-to-leading order the associated ambiguity is of the same size as the correction itself. The fact that the corrections are small but ambiguous justifies the use of the simple leading-order approximation for practical purposes.

2. Notation and Basic Formalism

The behaviour of structure functions at small \(x\) is dominated by the large eigenvalue of evolution in the singlet sector. Thus we consider the singlet parton density

\[
G(\xi,t) = x[g(x,Q^2) + k_q \otimes q(x,Q^2)],
\]

(2.1)
where $\xi = \log 1/x$, $t = \log Q^2/\mu^2$, $g(x,Q^2)$ and $q(x,Q^2)$ are the gluon and singlet quark parton densities, respectively, and $k_q$ is such that, for each moment

$$G(N,t) = \int_0^\infty d\xi e^{-N\xi}G(\xi,t),$$

(2.2)

the associated anomalous dimension $\gamma(\alpha_s(t),N)$ corresponds to the largest eigenvalue in the singlet sector. The generalization to the full two–by–two matrix of anomalous dimensions is discussed in detail in ref. [20].

At large $t$ and fixed $\xi$ the evolution equation in $N$-moment space is then

$$\frac{d}{dt}G(N,t) = \gamma(\alpha_s(t),N)G(N,t),$$

(2.3)

where $\alpha_s(t)$ is the running coupling. The anomalous dimension is completely known at one– and two–loop level, as given in eq. (1.2). The corresponding splitting function is related by a Mellin transform to $\gamma(\alpha_s,N)$:

$$\gamma(\alpha_s,N) = \int_0^1 dx x^N P(\alpha_s, x).$$

(2.4)

Small $x$ for the splitting function corresponds to small $N$ for the anomalous dimension: more precisely $P \sim 1/x(\log(1/x))^n$ corresponds to $\gamma \sim n!/N^{n+1}$. Even assuming that a leading twist description of scaling violations is still valid in some range of small $x$, as soon as $x$ is small enough that $\alpha_s\xi \sim 1$, with $\xi = \log 1/x$, all terms of order $(\alpha_s/x)(\alpha_s\xi)^n$ (LLx) and $\alpha_s(\alpha_s/x)(\alpha_s\xi)^n$ (NLLx) which are present in the splitting functions must be considered in order to achieve an accuracy up to terms of order $\alpha_s^2(\alpha_s/x)(\alpha_s\xi)^n$ (NNLLx).

As is well known, these terms can be derived from the knowledge of the kernel $\chi(\alpha_s,M)$, given in NLO approximation in eq. (1.1), of the BFKL $\xi$–evolution equation

$$\frac{d}{d\xi}G(\xi,M) = \chi(\alpha_s,M)G(\xi,M),$$

(2.5)

which is satisfied at large $\xi$ by the inverse Mellin transform of the parton distribution

$$G(\xi,M) = \int_{-\infty}^\infty dt e^{-Mt}G(\xi,t).$$

(2.6)

This derivation was originally performed [11] at LLx by assuming the common validity of eq. (2.5) and eq. (2.3) in the region where $Q^2$ and $\xi$ are both large. However, it was more recently realized [22,19,20] that the solution of eq. (2.5) coincides generally with that of eq. (2.3), up to higher twist corrections, provided only that the kernel of the former is related to that of the latter by a ‘duality’ relation, and boundary conditions are suitably matched. This implies that the domains of validity of these two equations are in fact the same in perturbation theory, up to power–suppressed corrections.
The derivation of duality is simplest when the coupling does not run, in which case the relation between the kernels of the two equations is given by eq. (1.3). A simple proof of this statement was given in ref. [19]. Expanding $\gamma(\alpha_s, N)$ in powers of $\alpha_s$ at fixed $\alpha_s/N$

$$\gamma(\alpha_s, N) = \gamma_s(\alpha_s/N) + \alpha_s \gamma_{ss}(\alpha_s/N) + \ldots,$$

and $\chi(\alpha_s, M)$ in powers of $\alpha_s$ at fixed $M$

$$\chi(\alpha_s, M) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \ldots,$$

we then find that $\chi_0$ determines $\gamma_s(\alpha_s/N)$, while $\alpha_s \chi_1$ leads to $\alpha_s \gamma_{ss}(\alpha_s/N)$:

$$\chi_0(\gamma_s(\alpha_s/N)) = \frac{N}{\alpha_s},$$

$$\gamma_{ss}(\alpha_s/N) = -\frac{\chi_1(\gamma_s(\alpha_s/N))}{\chi'_0(\gamma_s(\alpha_s/N))}.$$

Upon Mellin inversion, $\gamma_s(\alpha_s/N)$ and $\alpha_s \gamma_{ss}(\alpha_s/N)$ correspond respectively to all terms of order $(\alpha_s/x)(\alpha_s \xi)^n$ and $\alpha_s(\alpha_s/x)(\alpha_s \xi)^n$ in the splitting functions.

When one goes beyond LLx, i.e. beyond the leading–order approximation for $\chi$, the running of the coupling cannot be neglected, and the duality relation must be re–examined. Indeed, in $M$ space the usual running coupling $\alpha_s(t)$ becomes a differential operator and, by taking only the one-loop beta function into account, one has:

$$\hat{\alpha}_s = \frac{\alpha_s}{1 - \beta_0 \alpha_s \frac{d}{dM}} + \ldots,$$

where $\beta_0$ is the first coefficient of the $\beta$-function (so $\beta = -\beta_0 \alpha_s^2 + \cdots$ and $\beta_0 = 0.663146...$ for 4 flavours), with the obvious generalization to higher orders. Hence, assuming the coupling to run in the usual way with $Q^2$, the $\xi$-evolution equation eq. (2.5) becomes [31]

$$\frac{d}{d\xi}G(\xi, M) = \chi(\hat{\alpha}_s, M)G(\xi, M),$$

where the derivative with respect to $M$ acts on everything to the right, and $\chi$ may be expanded as in eq. (1.1) keeping the powers of $\hat{\alpha}_s$ on the left. In ref. [30] we have shown that eq. (2.12) can indeed be viewed as an alternative representation of the standard renormalization–group equation.

It is clear from eq. (2.12) that running coupling effects begin at NLLx. To NLLx one finds [29] that the solution of eq. (2.12) is again the same as that of a dual $t$–evolution equation (2.3), provided the fixed–coupling duality relation eq. (1.3) is modified by letting $\alpha_s \to \alpha_s(t)$, and then by adding to $\gamma_{ss}$ eq. (2.10) an extra term [23] $\Delta \gamma_{ss}$ proportional to $\beta_0$:

$$\Delta \gamma_{ss}(\alpha_s) = -\beta_0 \frac{\partial^2 \gamma_s(\alpha_s)}{\partial \alpha_s^2} \frac{\chi_0(\gamma_s)}{2 \chi_0^2(\gamma_s)}.$$

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Equivalently, the duality relation eq. (1.3) can be formally preserved, provided that \( \alpha_s \to \alpha_s(t) \) and the function \( \chi \) used in it is no longer identified with the BFKL kernel, but rather given by an ‘effective’ \( \chi \) function

\[
\chi_{\text{eff}}(\alpha_s, M) = \chi(\alpha_s, M) + \Delta \chi(\alpha_s, M),
\]

\[
\Delta \chi(\alpha_s, M) = \alpha_s^2 \Delta \chi_1(M) + \ldots,
\]

where \( \chi(\alpha_s, M) \) is obtained letting \( \hat{\alpha}_s \to \alpha_s \) in the kernel of eq. (2.12), and the correction term \( \Delta \chi \) to NLLx is given by

\[
\Delta \chi_1(M) = \beta_0 \frac{\chi''(M) \chi_0(M)}{2 \chi'(M)}.
\]

The problem with this perturbative approach is that the correction terms which must be included in the effective \( \chi \) eq. (2.14) have an unphysical singularity: \( \chi_0(M) \) has a minimum at \( M = \frac{1}{2} \), so the denominator of \( \Delta \chi_1(M) \) vanishes, resulting in a simple pole in the NLLx correction \( \Delta \chi_1 \) eq. (2.15) at \( M = \frac{1}{2} \). The NNLLx correction \( \Delta \chi_2 \) turns out to have a fourth-order pole, and in fact at each extra order three extra powers of \( (\chi_0')^{-1} \) appear. As explained in ref. [29], this leads to a perturbative instability: as a consequence of the singularity, the splitting function \( \Delta P_{ss} \) eq. (2.4) associated with the anomalous dimension \( \Delta \gamma_{ss} \) behaves as

\[
\frac{\Delta P_{ss}(\alpha_s, \xi)}{P_0(\alpha_s, \xi)} \sim (\alpha_s \xi)^2.
\]

This problem was completely solved in ref. [30], where we have shown that the \( M = \frac{1}{2} \) singularity and the corresponding ones that appear at higher orders in \( \alpha_s \beta_0 \) are an artifact of the expansion, are not present in the all-order solution and can thus be eliminated, thanks to the possibility of matching the all-order solution to the perturbative one. The corresponding resummation, which will be discussed in detail in the next section, is obtained by explicitly solving eq. (2.12) for \( \chi(\hat{\alpha}_s, M) \sim \hat{\alpha}_s \chi_0 \), with \( \chi_0 \) replaced by its quadratic approximation near \( M = \frac{1}{2} \). As is well known, the solution is given in terms of an Airy function. This solution is added to the anomalous dimension with subtraction of the NNLx expansion terms to avoid double counting. The \( \chi \) function dual to the resulting expression for \( \gamma \) is now regular near \( M = \frac{1}{2} \).

In the sequel, we will discuss in detail the form and the properties of the improved anomalous dimension. In the next section we describe the leading approximation, where as input we only use the one-loop perturbative anomalous dimension \( \alpha_s \gamma_0(N) \) and the leading order BFKL kernel \( \alpha_s \chi_0(M) \) but with the running coupling effects taken into account. Subleading corrections, including those from \( \chi_1 \), will then be discussed in section 4.

3. The Improved Anomalous Dimension: First Iteration

Assuming that one only knows \( \gamma_0(N) \) and \( \chi_0(M) \), we argue that the optimal use that one can make of these inputs is to write down for the improved anomalous dimension the
The following expression:

\[
\gamma_I(\alpha_s, N) = \left[\alpha_s \gamma_0(N) + \gamma_s(\frac{\alpha_s}{N}) - \frac{n_c \alpha_s}{\pi N}\right] + \\
+ \left[\gamma_A(c_0, \alpha_s, N) - \frac{1}{2} + \sqrt{\frac{2}{\kappa_0 \alpha_s}}[N - \alpha_s c_0] + \frac{1}{4} \beta_0 \alpha_s\right] - \text{mom. sub.} \tag{3.1}
\]

The first line on the right-hand side, within square brackets, is the usual double-leading [19] expression for the improved anomalous dimension at this level of accuracy, made up of the first order perturbative term \(\alpha_s \gamma_0\) plus the power series of terms \(\frac{\alpha_s}{N}\) contained in \(\gamma_s\), obtained from \(\chi_0\) using eq. (2.9), with subtraction of the order \(\alpha_s\) term to avoid double counting \((c_A = n_c = 3)\). In the second line, the second pair of square brackets contain the running coupling Airy term: \(\gamma_A(c_0, \alpha_s, N)\) is the Airy anomalous dimension, and the remaining terms subtract the contributions to \(\gamma_A(c_0, \alpha_s, N)\) which are already contained in \(\gamma_s\) and \(\gamma_0\) as we shall explain shortly. Finally “mom. sub.” is a subleading subtraction that ensures momentum conservation \(\gamma_I(\alpha_s, N = 1) = 0\). We now specify \(\gamma_A(c_0, \alpha_s, N)\) and “mom. sub.”.

For \(\gamma_A(c, \alpha_s, N)\) \(^1\) we start from the Mellin transform of eq. (2.12), namely [30]

\[
NG(N, M) = \chi(\hat{\alpha}_s, M)G(N, M) + F(M), \tag{3.2}
\]

with kernel

\[
\chi_q(\hat{\alpha}_s, M) = \hat{\alpha}_s[c + \frac{1}{2} \kappa(M - M_s)^2]. \tag{3.3}
\]

The expression within squared brackets is the quadratic approximation of \(\chi\) near its minimum \(M = M_s\). At leading order, \(M_s\), \(c\) (the Lipatov exponent) and \(\kappa\) are given by:

\[
M_s = M_{s}^0 = \frac{1}{2}, \tag{3.4}
\]

\[
c = c_0 = \frac{4n_c}{\pi} \log 2 = 2.64763..., \tag{3.5}
\]

\[
\kappa = \kappa_0 = -\frac{2n_c}{\pi} \psi''(\frac{1}{2}) = 32.1406..., \tag{3.6}
\]

where \(\psi(x)\) is the digamma function. The Airy anomalous dimension is found from the (Mellin transformed) solution \(G_q(N, t)\) of this equation:

\[
\gamma_A(c, \alpha_s(t), N) \equiv \frac{d}{dt} \ln G_q(N, t) = M_s + \left(\frac{2\beta_0 N}{\kappa}\right)^{1/3} \frac{\text{Ai}'[z(\alpha_s(t), N)]}{\text{Ai}[z(\alpha_s(t), N)]}, \tag{3.7}
\]

where \(\text{Ai}(z)\) is the Airy function, which satisfies

\[
\text{Ai}''(z) - z \text{Ai}(z) = 0, \tag{3.8}
\]

\(^1\) For convenience we only include \(c\) (and not \(\kappa\), see below) among the explicit arguments of \(\gamma_A\).
with \( \text{Ai}(0) = 3^{-2/3}/\Gamma(2/3) \),

\[
z(\alpha_s(t), N) = \left( \frac{2\beta_0 N}{\kappa} \right)^{1/3} \frac{1}{\beta_0} \left[ \frac{1}{\alpha_s(t)} - \frac{c}{N} \right]. \tag{3.9}
\]

Along the positive real axis, the Airy function is a positive definite, monotonically decreasing function of its argument, and it behaves asymptotically as

\[
\text{Ai}(z) = \frac{1}{2\pi^2/3} z^{-1/4} \exp\left(-\frac{2}{3}z^{3/2}\right)(1 + O(z^{-3/2})). \tag{3.10}
\]

Thus the large-\( z \) behaviour of \( \gamma_A \), which corresponds to \( N \) large and/or \( \alpha_s(t) \) small, is given by

\[
\gamma_A(c, \alpha_s, N) = \gamma_A^s(\alpha_s/N) + O\left(\frac{\alpha_s^2}{N}\right)
\]

\[
\gamma_A^s(\alpha_s/N) = M_s - \sqrt{\frac{2N}{\kappa}} \left[ \frac{N}{\alpha_s} - c \right]. \tag{3.11}
\]

The leading \( \gamma_A^s \) behaviour coincides with the ‘naive’, i.e. fixed-coupling dual anomalous dimension: namely, the anomalous dimension which is found from \( \chi_q \) eq. (3.3) using the leading-order duality relation (1.3). Furthermore, the leading \( O(\alpha_s^2/N) \) correction is simply obtained from eq. (3.10):

\[
\alpha_s \gamma_A^s(\alpha_s/N) = -\frac{1}{4}\beta_0 \alpha_s \left( 1 - \frac{N}{\alpha_s} c \right), \tag{3.12}
\]

and it coincides with the next-to-leading running coupling correction eq. (2.13) computed for the quadratic kernel eq. (3.3). Subsequent terms in the asymptotic expansion eq. (3.11) correspond to the further \( \alpha_s^2 \gamma_{sss}, \ldots \) corrections which are found when the perturbative solution of the running coupling eq. (2.12) is pursued to higher orders. The asymptotic series is (Borel) resummed [30] by the Airy anomalous dimension. At leading order, the double-counting between the double-leading terms and Airy anomalous dimension is removed by subtracting the leading term of the asymptotic expansion eq. (3.12), as well as the term \( \alpha_s \gamma_A^s(0) = -\frac{1}{4}\beta_0 \alpha_s \), which is \( O(\alpha_s) \) and thus must be subtracted because it is already in \( \alpha_s \gamma_0 \).

The momentum conservation subtraction “mom. sub.” is an \( O(\alpha_s^2) \) term which can be defined as:

\[
\text{mom. sub.} = g(N)\bar{\gamma}_I(\alpha_s, 1), \tag{3.13}
\]

where \( g(N) \) is a weight function with \( g(1) = 1 \), with no singularities for \( N > 0 \) and \( \bar{\gamma}_I(\alpha_s, N) \) is given by

\[
\gamma_I(\alpha_s, N) = \bar{\gamma}_I(\alpha_s, N) - \text{mom. sub.}. \tag{3.14}
\]

We can take \( g(N) = 1 \) or, for instance,

\[
g(N) = \frac{1 + r}{N + r} \tag{3.15}
\]
Figure 1: The improved anomalous dimension \( \gamma_I(\alpha_s, N) \) for \( c = c_0, \kappa = \kappa_0, M_s = 1/2 \) (i.e. the values in eqs. (3.5), (3.6), (3.4) that are obtained from \( \chi_0(M) \)), \( g(N) = 1 \) and \( \alpha_s = 0.2 \) (solid), compared with \( \gamma_{\text{DL}-\text{LO}} \) (the curve with a step) and GLAP LO and NLO (dashed and dotted).

where \( r \) is a positive real number. The possible advantage of this choice is to have a damped large \( N \) behaviour. These different choices are all equivalent up to subleading terms.

We now discuss the properties of the improved anomalous dimension in this approximation. In the limit \( \alpha_s \to 0 \) with arbitrary \( N \), \( \gamma_I(\alpha_s, N) \) reduces to \( \alpha_s \gamma_0(N) + O(\alpha_s^2) \): in particular as \( N \to \infty \) the Airy term reduces to an \( O(\alpha_s^2) \) constant (the \( N \to \infty \) limit of the next term, \( \gamma_{\text{A}_s} \), in the expansion of eqs. (3.10)–(3.12) up to an \( O(N^{-1}) \) correction). For \( \alpha_s \to 0 \) with \( \alpha_s/N \) fixed, \( \gamma_I(\alpha_s, N) \) reduces to \( \gamma_{\text{DL}-\text{LO}} = \alpha_s \gamma_0(N) + \gamma_s(\frac{\alpha_s}{N}) - \frac{\alpha_s}{\pi N} \), i.e. the leading term of the double-leading expansion. Thus the Airy term is subleading in both limits. In spite of this, its role is very significant because of the singularity structure of the different terms in eq. (3.1). In fact, \( \gamma_0(N) \) has a pole at \( N = 0 \), \( \gamma_s \) has a branch cut at \( N = \alpha_s c_0 \), and \( \gamma_A \) has a pole at \( N = N_0 < \alpha_s c_0 \), where \( N_0 \) is the position of the rightmost zero of the Airy function. The importance of the Airy term is that the square root term subtracted from \( \gamma_A \) cancels, within the stated accuracy, the branch cut of \( \gamma_s \) at \( N = \alpha_s c_0 \) and replaces the corresponding asymptotic behaviour at small \( x \) with the much softer one from \( \gamma_A \).

The effect of this replacement is clearly seen from figure 1, where we compare the plot of \( \gamma_I \) (solid curve) as a function of \( N \) with those of \( \gamma_{\text{DL}-\text{LO}} \) (the curve with a step, starting at the cut branch point) and of GLAP LO and NLO (which are nearly superimposed). The curves are computed with \( c = c_0, \kappa = \kappa_0, M_s = 1/2 \) (i.e. the values in eqs. (3.5)–(3.4) that are obtained from \( \chi_0(M) \)), to the momentum subtraction with \( g(N) = 1 \) eq. (3.15) and \( \alpha_s = 0.2 \), a value which is roughly appropriate for the HERA kinematic range. Since \( N_0 \) is significantly smaller than \( \alpha_s c_0 \), the improved anomalous dimension \( \gamma_I \) remains close to the GLAP curves down to much smaller values of \( N \) than \( \gamma_{\text{DL}-\text{LO}} \). It is only at \( N \lesssim 0.3 \),
where $\gamma_I > 0.5$, that the improved anomalous dimension becomes sizably different than the NLO perturbative anomalous dimension (for the chosen value of $\alpha_s$ the pole of $\gamma_I$ is at $N_0 = 0.211237...$ while $\alpha_s c_0 = 0.529525...$).

The deviation of $\gamma_I$ from GLAP at small values of $N$ has a moderate effect on the splitting function. In fig. 2 we plot $P_{gg}(x)$ corresponding to $\gamma_I$ (dashed) compared to the perturbative LO (solid), while the rising curve at small $x$ corresponds to $\gamma_{DL-LO}$. This plot makes particularly clear that the improvement obtained by implementing the Airy asymptotics is really important at small $x$, especially when we take into account that the data seem to follow rather closely the unresummed perturbative evolution: the rapid small $x$ rise of the splitting function is removed by the resummation, and in fact the resummed splitting function closely follows the perturbative one down to $x \sim 10^{-3}$.

The improvement of the agreement with the leading order GLAP anomalous dimension when the double–leading anomalous dimension is supplemented by running coupling resummation according to fig. 1 follows from three reasons. First, the location of the rightmost singularity is lowered from $\alpha_s c_0$ to $N_0$. Second, the double–leading branch cut is replaced by a pole, which is the same type of singularity as in GLAP. Finally, the residue of the pole in $\gamma_A$ is more than an order of magnitude smaller than the residue of the GLAP pole. Indeed, a straightforward calculation using eq. (3.7) shows that

$$
\gamma_A(\alpha_s(t), N) = \frac{1}{N - N_0} r_A + O[(N - N_0)^0]
$$

$$
r_A = \frac{3\beta_0 N_0^2 \alpha_s(t)}{N_0 + 2c_0 \alpha_s(t)}.
$$

(3.16)

When $\alpha_s = 0.2$, $r_A = 0.014...$ to be compared with the value $\gamma_0 = 3\alpha_s/\pi + O[N^0] = 0.191.../N + O[N^0]$ of the residue of the leading-order GLAP pole. The combination of
these facts implies that the singularity in $\gamma_A$ only kicks in rather late, i.e. for rather small values of $N$. This is apparent in fig. 2, where the rise $r_A x^{-N_0}$ due to the Airy pole is only visible at very small $x$ on top of the constant behaviour related to the GLAP pole. Due to momentum conservation, which fixes the integral of $x P$, the rise must be compensated by a small depletion of the splitting function in the intermediate $x$ range, also visible in the figure. While all these properties are generic, the quantitative agreement of the resummed result with leading order GLAP is only good when the intercept is close to the leading order value $\alpha_s c_0$ [eq. (3.5)] determined from the BFKL kernel $\chi_0$. Running coupling resummation is the essential ingredient that reconciles the BFKL intercept with the GLAP behaviour of the data GLAP [14,32].

In our previous work in refs. [19,20] we had adopted a more phenomenological approach in order to cope with the fact that the rise of the singlet structure function implied by the Lipatov exponent (corresponding to the anomalous dimension with the cut at $\alpha_s c_0$ in fig.1) is too sharp in comparison to what is seen in the data. Noting that the correct asymptotic exponent is not reliably determined by the two known terms of its perturbative expansion (in fact the computed correction to the leading result is quite large), we had treated it as a parameter to be fitted from the data. To this effect, we interpreted the parameter $\Delta \lambda$ in eq. (1.4), formally of order $\alpha_s^2$, as a parameter to be fixed empirically. The lack of any indication of a power-rise in the data suggests then that $\Delta \lambda$ should be negative and possibly numerically as large as, say, $O(\alpha_s)$. In leading order, the corresponding improved anomalous dimension becomes

$$
\gamma_{Iold}(\alpha_s, N) = [\alpha_s \gamma_0(N) + \gamma_s (\frac{\alpha_s}{N - \Delta \lambda}) - \frac{n_c \alpha_s}{\pi N}] - \text{mom. sub.} \quad (3.17)
$$

The shift $N \rightarrow (N - \Delta \lambda)$ in $\gamma_s$ provides the change from $x^{-\lambda_0}$ to $x^{-\lambda} = x^{-\lambda_0} e^{\Delta \lambda \xi}$. As discussed in ref. [19], there is an ambiguity in the subtraction $n_c \alpha_s / (\pi N)$ because there too we could make the replacement $N \rightarrow (N - \Delta \lambda)$. In eq. (3.17) we have adopted the subtraction with $N$, denoted as R-subtraction in ref. [20], because it is more suitable for the present purpose of comparing the old approach with $\gamma_I$ in eq. (3.1), as we shall see shortly.

What we have accomplished in this section is to show that the resummation through the Airy asymptotics of higher–order contributions to the anomalous dimension related to the running coupling corrections is by itself sufficient to produce the required softening of the behaviour at small $x$. Thus it is no longer necessary to introduce $\Delta \lambda$ by hand as a free parameter (at least at leading accuracy). In fact, we had already found in ref. [19] that the value of $\Delta \lambda$ that provides the best matching to GLAP is exactly such that $\lambda = \alpha_s c_0 + \Delta \lambda \sim 0.21$ (for $\alpha_s \sim 0.2$) in remarkable agreement with the value $N_0 = 0.211237$ determined here from the Airy asymptotics. The fact that the data are in good agreement with unresummed perturbative evolution suggests that the true all-order value of $\lambda$ is likely to be quite close to the leading–order value, so $\Delta \lambda$ is presumably rather small and there is perhaps no need to introduce it as a free parameter in order to fit the data.

The sensitivity of our new approach to the value at the minimum $\lambda = \alpha_s c = \alpha_s c_0 + \Delta \lambda$ can be demonstrated by going back to eq. (3.1), replacing $N$ by $N - \Delta \lambda$ in $\gamma_s$ and $c_0$ by...
Figure 3: The effect on the improved anomalous dimension of varying the value of $\chi$ at the minimum: $\lambda = \alpha_s c = \alpha_s c_0 + \Delta \lambda$, as described in the text, with $\alpha_s = 0.2$ and $\alpha_s c = 0.4, 0.529525... , 0.6$. The dashed curve is the LO GLAP curve, the solid curve corresponds to the BFKL value $\alpha_s c = 0.529525...$ while the upper and lower (at small $N$) dot-dashed curves refer to $\alpha_s c = 0.6, 0.4$, respectively.

Note that we have not performed the $\Delta \lambda$ shift in the double-counting subtraction term, corresponding to the R-subtraction of ref. [20]. This is necessary, otherwise the cancellation of singularities at $N = \alpha_s c$ between the double-leading anomalous dimension and the Airy term would be spoiled. The resulting improved anomalous dimension for $\alpha_s = 0.2$ and $\alpha_s c = 0.4, 0.529525...$ (the BFKL value, see eq. (3.5)), and 0.6 are shown in figure 3. The corresponding splitting functions are shown in figure 4. We see that changing $c$ in this range does not alter the overall trend of the anomalous dimension. However the impact on the splitting function at small $x$ is quite noticeable and, in fact, the observed approximate validity of the perturbative fits suggests that the viable range of $\Delta \lambda$ is inside the interval presented in figure 4. This possibility of varying $c$ in the leading term could be adopted as a phenomenological way to optimize the leading formula on the data. We have also checked that moderate variations of $\kappa_0$ and $M_s$ eqs. (3.4),(3.6), do not alter the general picture.

Since the reference perturbative fits are based on the NLO GLAP it may be desirable to modify our improved anomalous dimension in such a way that at small $\alpha_s$ with $\alpha_s/N$ fixed it reduces to NLO GLAP given by eq. (1.2): $\gamma(N, \alpha_s) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N)$. This
Figure 4: The effect on the improved splitting function of varying the value of $\chi$ at the minimum: $\lambda = \alpha_s c = \alpha_s c_0 + \Delta \lambda$, as described in the text, with $\alpha_s = 0.2$ and $\alpha_s = 0.4, 0.529525...$, 0.6. The dashed curve is the LO GLAP curve, the solid curve corresponds to the BFKL value $\alpha_s c = 0.529525...$ (same as the solid curve of Fig. 1) while the upper and lower (at large $1/x$) dotdashed curves refer to $\alpha_s c = 0.6, 0.4$, respectively.

can readily be obtained by modifying eq. (3.1) in the following way:

$$\gamma_I^{NL}(\alpha_s, N) = [\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \gamma_s(\frac{\alpha_s}{N}) - \frac{n_c \alpha_s}{\pi N}] + 
\gamma_A(c_0, \alpha_s, N) - \frac{1}{2} + \sqrt{\frac{2}{\kappa_0 \alpha_s}} [N - \alpha_s c_0] 
+ \frac{1}{3} \beta_0 \alpha_s (1 + \frac{\alpha_s}{N} c_0) - \text{mom. sub.}$$

(3.19)

Here we used the fact that in the expansion of $\gamma_s(\alpha_s/N)$ in powers of its argument the quadratic term is absent, so that it is sufficient to only add a subtraction to the Airy term in order to guarantee that in the perturbative limit the NLO GLAP anomalous dimension is exactly recovered. This subtraction (the first term in the third line of eq. (3.19)) is obtained by expanding $\alpha_s \gamma_s^A$ eq. (3.12) to order $\alpha_s^2$ at fixed $N$. This formula improves the NLO GLAP with the leading terms of the BFKL corrections with running coupling. The result is shown in figure 5 and compared to the GLAP anomalous dimension and the resummed LO result of Figure 1. It appears that, thanks to the extra next-to-leading subtraction in eq. (3.19), the effect of the resummation is further reduced, in particular correcting the slight distortion of the LO resummed result in the $0.5 \lesssim N \lesssim 1.5$ region seen in the figure (dotdashed curve).

In conclusion we have shown that, given the tendency of the data to closely follow the GLAP evolution down to the smallest values of $N$ accessible to HERA, the most effective way to implement the information contained in the leading BFKL kernel $\chi_0$ is to use duality, the double-leading prescription (that improves $\chi_0$ near $M = 0$ in agreement
Figure 5: The improved anomalous dimension $\gamma^{NL}_I$ eq. (3.19)(solid), LO (dotted) and NLO GLAP (dashed), which is the corresponding perturbative limit in this case, for $c = c_0, \kappa = \kappa_0, M_s = 1/2$ (i.e. the values in eqs. (3.5), (3.6), (3.4) that are obtained from $\chi_0(M)$), $g(N) = 1$ and $\alpha_s = 0.2$. The resummed LO curve (solid curve of Figure 1) is also shown for comparison (dotted).

with momentum conservation), and the Airy asymptotics to get just the right amount of softening of the small $x$ exponent. In the next section we will show that the non leading corrections, including those arising from $\chi_1$, do not spoil this overall picture.

4. Perturbative expansion of the resummed anomalous dimension

In the previous section we have seen that at leading order the resummed anomalous dimension eq. (3.1) and the GLAP anomalous dimension are in very good agreement. However, this result is phenomenologically significant only if the resummed anomalous dimension eq. (3.1) is stable upon higher order corrections. To investigate whether this is the case we must discuss how eq. (3.1) can be viewed as the leading order of a resummed perturbative expansion of the anomalous dimension. As suggested in ref. [30], this can be done by using the resummation of running coupling effects by means of the Airy asymptotics to improve the expansion eq. (2.7) of the anomalous dimension in powers of $\alpha_s$ at fixed $\alpha_s/N$, and then combining this improved expansion with the standard GLAP anomalous dimension, to obtain a running-coupling improvement of the double leading expansion, thereby generalizing to all orders eq. (3.1).

Hence, we must consider the running-coupling resummation of the small-$x$ expansion of the anomalous dimension eq. (2.7) beyond leading order. Higher order corrections $\chi_i$ to the kernel $\chi_0$ will change in general the values of all the parameters (3.4)–(3.6) which characterize the quadratic approximation to $\chi$ near its minimum. Whenever the value of $\chi$ at its minimum differs from the leading–order value, the perturbative expansion of the kernel $\chi$ must be reorganized if we wish to obtain a stable expansion of the splitting
function, i.e. such that the ratios $P_{ss}/P$, $P_{sss}/P$, ... all remain finite as $\xi \to \infty$. A necessary and sufficient condition for this to be the case is \cite{29,19} that to each order a constant is subtracted from $\chi_i$ and added back into $\chi_0$, in such a way that the subtracted $\chi_i$ no longer lead to a shift in the minimum of $\chi_0$. At leading order, this amounts to performing the shift $N \to N - \Delta \lambda$ as in eq. (3.18).

In view of a running coupling resummation, however, it is convenient to include in $\chi_0$ the full leading asymptotic behaviour, by subtracting not only a constant, but also the leading linear and quadratic terms in the Taylor expansion of $\chi$ about its all–order minimum. Namely, assume that the all-order $\chi$ has a minimum in the vicinity of which it can be approximated by the quadratic expression of $\chi_q$, as given in eq. (3.3), where now

$$M_s = M_s^0 + \alpha_s M_s^{(1)} + \ldots,$$

$$c = c_0 + \alpha_sc_1 + \ldots,$$

$$\kappa = \kappa_0 + \alpha_s\kappa_1 + \ldots.\quad (4.3)$$

We then reorganize the expansion of $\chi$ eq. (1.1) as follows:

$$\chi(M, \alpha_s) = \alpha_s \chi_0^R + \alpha_s^2 \chi_1^R + \ldots;$$

$$\chi_0^R = \chi_0 + \chi_q - \chi_q^{(0)};$$

$$\chi_i^R = \chi_i - \chi_q^{(i)}, \quad i > 0;$$

$$\chi_q^{(i)} = \chi_i(M_s) + (M - M_s) \chi'_i(M_s) + \frac{1}{2}(M - M_s)^2 \chi''_i(M_s), \quad i \geq 0.\quad (4.4)$$

To any finite order, the expansion eq. (4.4) of $\chi$ differs from the standard one eq. (1.1) by subleading terms:

$$\sum_{i=0}^{n} \alpha_s^{i+1} \chi_i(M) = \sum_{i=0}^{n} \alpha_s^{i+1} \chi_R^{(i)}(M) + O[\alpha_s^{n+1}].\quad (4.5)$$

Therefore, eq. (4.4) is a bona fide resummed version of the original expansion. We may then use the resummed expansion to determine order by order $\gamma^R_s$, $\gamma^R_{ss}$ and so forth, through perturbative duality with the suitable running coupling corrections. In particular, the next-to-leading order running coupling contribution to $\gamma^R_{ss}$ is still given by eq. (2.13), but with $\chi_0$ replaced by its resummed counterpart $\chi_0^R$ eq. (4.4). This leads to a resummed expansion of $\gamma$ in powers of $\alpha_s$ at fixed $\alpha_s/N$ which, to any finite order $n$, differs by the standard expansion eq. (2.7) by subleading $O(\alpha_s^{n+1})$ terms. Because, by construction, $\chi_q^{(i)}$ do not shift the minimum of $\chi_q^{(0)}$, the ensuing resummed expansion of $\gamma$ is stable, just like the expansion discussed in our previous work refs. \cite{29,19,20}.

This new reorganized expansion differs from that discussed in refs. \cite{29,19} because now not only the value of $\chi$ at the minimum, but also its curvature is reshuffled into the leading order, and furthermore, the expansion is performed about the all–order minimum $M_s$ instead of the leading order minimum $M = \frac{1}{2}$. The leading order resummed $\chi_0^R$ has in general not only a value at the minimum $c \neq c_0$, like the leading order resummed $\bar{\chi}_0$,
of refs. \([29,19]\) but also a minimum at \(M_s \neq \frac{1}{2}\) and curvature \(\kappa \neq \kappa_0\). Only if \(\kappa = \kappa_0\) and \(M_s = \frac{1}{2}\), then \(\chi_0^R\) would be of the same form as the leading order resummed \(\tilde{\chi}_0\) of refs. \([29,19]\). Note, however, that even so, the subsequent terms \(\chi_i^R\) with \(i > 1\) in the expansion of \(\chi\) would not be the same as the corresponding \(\tilde{\chi}_i\) of refs. \([29,19]\) (compare eq. (4.4)).

The resummed \(\gamma\) at any desired order can now be combined with the Airy anomalous dimension \(\gamma_A\) determined from the quadratic kernel with generalized \((4.1)-(4.3)\) values of the parameters. However, if the location of the minimum \(M_s\) differs from its leading order value, \(M_s \neq \frac{1}{2}\), already at next-to-leading order this is no longer sufficient to resum all singularities related to the running coupling term \(\Delta \gamma_{ss}\) eq. (2.13). Indeed, if \(M_s \neq \frac{1}{2}\), \(\chi_0^R\) is no longer symmetric about \(M = M_s\) and therefore its Taylor expansion about \(M = M_s\) also contains in general a cubic term. It is straightforward to see that in such case \(\Delta \gamma_{ss}\) acquires a new singularity.

Consider the case of a generic kernel \(\chi\), with a minimum at \(M = M_s\), and containing terms up to fourth order:

\[
\chi^{(4)}(\hat{\alpha}_s,M) = \chi_q + \hat{\alpha} \left[ \frac{\delta}{3!} (M - M_s)^3 + \frac{\epsilon}{4!} (M - M_s)^4 \right]. \tag{4.6}
\]

Expanding the anomalous dimension \(\gamma^{(4)}(\alpha_s/N)\) which is dual (eq. (1.3)) to this \(\chi^{(4)}\) about its rightmost singularity \(\alpha_s/N = c\) yields

\[
\gamma^{(4)}_s(\alpha_s/N) = M_s - \sqrt{\frac{2}{\kappa} \left( \frac{N}{\alpha_s} - c \right)} \left\{ 1 - \frac{\delta}{\kappa 3!} \sqrt{\frac{2}{\kappa} \left( \frac{N}{\alpha_s} - c \right)} \right. \\
\left. - \frac{\epsilon}{\kappa 4!} \left[ \frac{2}{\kappa} \left( \frac{N}{\alpha_s} - c \right) \right] + O [\delta^2] + O [\epsilon^2] \right\}, \tag{4.7}
\]

so higher order terms in the expansion of \(\chi\) in powers of \(M - M_s\) correspond to higher order contributions in the expansion of its dual anomalous dimension in powers of \(\sqrt{N/\alpha_s - c}\). Substituting \(\gamma^{(4)}_s\) (eq. (4.7)) in the expression eq. (2.13) of the running coupling contribution to \(\gamma_{ss}\) leads to

\[
\gamma^{(4)}_{ss} \equiv \frac{\chi'^0_0(\gamma^{(4)}_s)\chi_0(\gamma^{(4)}_s)}{2\chi'^2_0(\gamma^{(4)}_s)} = \gamma_A^{ss} + \frac{\beta_0 \delta}{24} \left( \frac{2}{\kappa} \right)^{3/2} \sqrt{\frac{N}{\alpha_s}} \frac{1}{\sqrt{1 - \frac{N}{\alpha_s}c}} + O \left[ \left( \frac{N}{\alpha_s} - c \right)^0 \right], \tag{4.8}
\]

where \(\gamma_A^{ss}\) is the running coupling contribution eq. (3.12) which is resummed by the Airy anomalous dimension. The cubic term induces the square-root singularity proportional to \(\delta\) in eq. (4.8), which must also be resummed. Because the singularity is linear in \(\delta\), this resummation can be performed by treating the cubic term as a leading-order perturbation. Note that the quartic term does not induce any further singularities to this order.

In order to resum running-coupling singularities which are associated to the cubic term in the expansion of \(\chi\) about its minimum, we determine the (Mellin transformed)
solution $G^{(3)}(N, t)$ to eq. (2.12) when the cubic term is also included in the kernel:

$$G^{(3)}(N, t) = \left(\frac{2N\beta_0}{\kappa}\right)^{1/3} \exp \frac{M_s}{\beta_0\alpha_s(t)} \exp \left\{ -\frac{\delta}{12\kappa} \left(\frac{2N\beta_0}{\kappa}\right)^{1/3} \frac{d^4}{dz^4} \right\} \text{Ai} \left[z(\alpha_s(t), N)\right].$$

Using the identity $Ai''(z) = zAi(z)$ and expanding in powers of $\delta$ we get the anomalous dimension

$$\gamma^{(3)}_A(\alpha_s(t), N) \equiv \frac{d}{dt} \ln G^{(3)}_q(N, t) = \alpha_s \gamma^{(3)}_{A, ss}(\alpha_s/N) + O(\alpha^2_s),$$

where $\gamma_A$ is given by eq. (3.7).

The last term inside the square bracket in eq. (4.10) has a double pole at $N = N_0$, where $Ai(z(\alpha_s, N))$ vanishes. This is due to the fact that the location of the zero of $G^{(3)}(N, t)$ eq. (4.9) is shifted away from $N_0$ by the $\delta$ correction, and therefore the pole in the anomalous dimension is also shifted. To first order in $\delta$, the zero moves from $N_0$ to

$$N^\delta_0 = N_0 + \frac{\delta}{6\kappa} r_A,$$

with $r_A$ given by eq. (3.16). Because $r_A = O(\alpha_s)$, the ensuing correction to the asymptotic behaviour is subleading [30].

It follows that the linearized expression eq. (4.10) of the $\delta$ correction to the anomalous dimension has a double pole in $N^\delta_0$, and is correct provided if $N - N_0$ is not too small, i.e. if $x$ is not too small. For the typical values of the parameters ($\alpha_s c \sim 0.4$), $N^\delta_0 - N_0 \sim 10^{-2}$, so the correction to the asymptotic behaviour $x^{-N_0} \to x^{-N^\delta_0}$ is approximately $\sim 10\%$ when $x \sim 10^{-5}$, and only becomes of order one when $x$ is extremely small. Hence, even though the linearized anomalous dimension eq. (4.10) does not reproduce the correct leading singularity of the full anomalous dimension determined from the solution eq. (4.9), it is a good approximation to it for all practical purposes.

We can now determine the expansion of $\gamma^{(3)}_A$ in powers of $\alpha_s$ at fixed $\alpha_s/N$

$$\gamma^{(3)}_A = \gamma^{(3)}_{A, ss}(\alpha_s/N) + \alpha_s \gamma^{(3)}_{A, sss}(\alpha_s/N) + O(\alpha^2_s),$$

by using the asymptotic expansion eq. (3.10):

$$\gamma^{(3)}_{A, ss}(\alpha_s/N) = \gamma_s^A - \frac{\delta}{12} \left(\frac{2}{\kappa}\right)^2 \left(\frac{N}{\alpha_s - c}\right),$$

Note that the explicit expressions of the correction to the solution and the location of the pole in the anomalous dimension given in eq. (5.2) and (5.3) of ref. [30] are incorrect, and should be respectively replaced by eq. (4.9) and eq. (4.11). The main conclusion that the modification of the asymptotic behaviour due to cubic and higher corrections is subleading remains however unaffected.
\[
\gamma^{(3)}_{A,ss}(\alpha_s/N) = \gamma^A_{ss} + \frac{\beta_0 \delta}{24} \left(\frac{2}{\kappa}\right)^{3/2} \frac{N/\alpha_s}{\sqrt{N/\alpha_s - c}},
\] (4.14)

where \(\gamma^A_{ss}\) and \(\gamma^A_s\) are given by eq. (3.11) and eq. (3.12) respectively. This verifies explicitly that the \(O(\delta)\) running coupling contributions are correctly resummed by the correction term in eq. (4.10): the leading-order term (4.13) in the expansion eq. (4.12) matches the naive (fixed coupling) dual eq. (4.7), while the first subleading correction matches the next-to-leading running coupling contribution eq. (4.8).

We can thus write down the next-to-leading version of the resummed expression eq. (3.1):

\[
\gamma(\alpha_s, N) = \gamma_I(\alpha_s, N) + \alpha_s\gamma_{II}(\alpha_s, N) + O(\alpha_s^2) - \text{mom. sub.}
\] (4.15)

The leading-order term \(\gamma_I\) is now given by

\[
\gamma_I(\alpha_s, N) = \left[\alpha_s \gamma_0(N) + \gamma_s^R\left(\frac{\alpha_s}{N}\right) - \frac{n_c}{\pi N}\right] + \\
\frac{\gamma_{(3)}(\alpha_s, N)}{N} - M_s + \sqrt{\frac{2}{\kappa \alpha_s}}\left[N - \alpha_s c\right] - \frac{1}{4}\beta_0 \alpha_s + \frac{\delta}{12}\left(\frac{2}{\kappa}\right)^2 \left(\frac{N}{\alpha_s} - c\right),
\] (4.16)

where \(\gamma_s^R\) is determined from \(\chi_0^R\) eq. (4.4) using duality eq. (1.3), \(\gamma_{(3)}^{(3)}\) is given by eq. (4.10), and the same values of the parameters \(M_s, \kappa\) and \(c\) are used in these two contributions. Because, as discussed above, if \(\kappa = \kappa_0\) and \(M_s = \frac{1}{2}\) the resummed \(\chi_0^R\) eq. (4.4) reduces to the resummed \(\tilde{\chi}_0\) of refs. [29,19], it follows that in such case \(\gamma_s^R(\alpha_s/N)\) coincides with \(\gamma_s[\alpha_s/(N - \Delta\lambda)]\) with \(\Delta\lambda = (c - c_0)/\alpha_s\). In other words, this is the same as eq. (3.1), but with generic values of the parameters which characterize the behaviour of \(\chi\) about its minimum, eq. (3.3), and the cubic contributions to the Airy anomalous dimension accordingly included.

The next-to-leading correction is

\[
\gamma_{II}(\alpha_s, N) = \left[\alpha_s \gamma_1(N) + \gamma_{ss}^R\left(\frac{\alpha_s}{N}\right) - e_0 - \alpha_s \left(\frac{e_1}{N} + \frac{e_2}{N^2}\right)\right] \\
+ \alpha_s \frac{\beta_0 c}{N - \alpha_s c} - \frac{\beta_0 \delta}{24} \left(\frac{2}{\kappa}\right)^{3/2} \frac{N/\alpha_s}{\sqrt{N/\alpha_s - c}}.
\] (4.17)

Here, the term in square brackets is the analogue of the usual [19,20] next-to-leading order term in the double-leading expansion (but determined from the new resummed \(\chi_1^R\) eq. (4.4)) while the last two terms are the \(O(\alpha_s^2/N)\) terms eq. (4.8) in the expansion of the Airy anomalous dimension.

Let us now discuss the properties of this next-to-leading order expression. First, it is easy to see that the anomalous dimension eq. (4.15) only differs from the standard next-to-leading double-leading expansion of refs. [19-20] by sub-subleading terms. Indeed, we have seen that because of eq. (4.5) the reorganized expansion of the anomalous dimension in terms of \(\gamma_r^R, \gamma_{ss}^R\ldots\) differs from the usual one by subleading terms. Furthermore, terms of order up to \(\alpha_s^2\) in the expansion eq. (4.12) of the Airy anomalous dimension are explicitly
subtracted: note in particular that the sum of the order $\beta_0$ term in $\gamma_I$ and the order $\beta_0 c$ term in $\gamma_{II}$ exactly matches the subleading term eq. (3.12) in the expansion of $\gamma_A$.

The small $N$ behaviour of both $\gamma_I$ and $\gamma_{II}$ is controlled by the pole of $\gamma_A^{(3)}$ at $N = N_0$, whose residue receives a $O(\delta)$ correction (compare eq. (4.10)), while singularities at $N = \alpha_s c$ in the expansion eq. (2.7) of the anomalous dimension are subtracted to the stated accuracy. As it is apparent from eq. (4.7), terms of increasingly higher order in the expansion of $\chi$ about its minimum correspond to singular contributions to increasingly higher order derivatives of $\gamma$ at the point $N = \alpha_s c$. Namely, the quadratic term in $\chi$ leads to a singularity in the first derivative of $\gamma$, the cubic and quartic term to a singularity in the second derivative, and so on. Hence, at the leading ($\gamma_s$) level, the square-root cut at $N = \alpha_s c$ is subtracted so that the anomalous dimension is continuous and has a continuous first derivative at $N = \alpha_s c$, though it has discontinuous second derivative, which would be made continuous by including the quartic $\epsilon$ term eq. (4.7) in the Airy resummation.

At the next-to-leading ($\gamma_{ss}$) level, thanks to the reorganization of the expansion eq. (4.4) there are no singularities in the fixed-coupling dual anomalous dimension nor in its first derivative. There are however singularities related to the running coupling term eq. (2.13): the pole (related to the quadratic contribution) and square-root singularity (related to the cubic contribution) are subtracted, while the first derivative has a discontinuity that would be removed by including cubic corrections eq. (4.10) to $O(\delta^2)$.

At large $N$, the $\sqrt{N}$ growth of $\gamma_A$ is canceled by the $\gamma_s^A$ subtraction, while the remaining constant large $N$ limit is canceled by the $\gamma_{ss}^A$ subtraction, leaving a $1/\sqrt{N}$ drop. The linear and $\sqrt{N}$ growth of $\gamma_A^{(3)}$ are canceled by the $\gamma_{A,s}^{(3)}$ and $\gamma_{A,ss}^{(3)}$ subtractions respectively, leaving an $O(\alpha_s^2)$ constant:

$$\lim_{N \to \infty} \left( \gamma_A^{(3)} - \gamma_{A,s}^{(3)} - \gamma_{A,ss}^{(3)} \right) = (\alpha_s \beta_0)^2 \frac{\delta}{24 \kappa}.$$  \hspace{1cm} (4.18)

This constant may be included in the momentum subtraction term. This leaves then an $O(1/\sqrt{N})$ drop. Therefore, at large $N$ the next-to-leading resummed anomalous dimension eq. (4.15) coincides with the standard GLAP anomalous dimension, since the subtracted small $x$ eq. (2.7) terms drop at least as $1/N$ and the subtracted Airy terms drop at least as $1/\sqrt{N}$, while the GLAP anomalous dimension, as well known, grows (in modulus) as a power of $\ln N$. This property is preserved if the momentum subtraction is performed by subtracting the constant eq. (4.18), and then subtracting the remaining momentum non-conserving terms through eqs. (3.13)-(3.15). If momentum is instead enforced by simply subtracting a constant, the asymptotic large $N$ resummed anomalous dimension will differ from the GLAP one by a fixed $O(\alpha_s^2)$ constant.

We conclude that to next-to-leading order the analytic properties of the leading order result eq. (3.1) are preserved: the branch cut at $N = \alpha_s c$ is subtracted from the small $x$ anomalous dimension $\gamma_s + \alpha_s \gamma_{ss}$ and replaced by the Airy pole, while the large $N$ behaviour of the perturbative anomalous dimension $\gamma_0 + \alpha_s \gamma_1$ is preserved.

We can now check that the next-to-leading order anomalous dimension eq. (4.15) differs from the leading order one eq. (3.1) displayed in figure 1 by a small correction. We expect this on the grounds that at large $N$ they reduce to the corresponding GLAP anomalous dimension, while at small $N$ for $N \gtrsim N_0^\delta$ they are controlled by the Airy
Figure 6: The improved anomalous dimension eq. (4.15) for $\alpha_s c = 0.3$ (lower dot-dashed), 0.4 (solid), 0.5 (upper dot-dashed), $\kappa = \kappa_0$, $M_s = 0.4$, $g(N)$ eq. (3.15) with $r = 1$ and $\alpha_s = 0.2$ compared with the NLO GLAP curve (dashed). The small kinks are due to cuts at $N = \alpha_s c$ which reappear at this accuracy through $\gamma_{ss}$.

pole, whose location depends very little on the input value of $c$. At next-to-leading order, however, we cannot use the perturbative values of the parameters $c$, $\kappa$ and $M_s$, because the next-to-leading order $\chi$ does not have a minimum. This is due to the fact that, whereas the small $M \sim 0$ drop of $\chi_1$ is removed by the double-leading resummation, fixed by momentum conservation, the large $M \sim 1$ drop of $\chi_1$ can only be removed through model-dependent assumptions, such as those of refs. [18,21]. Because the series of small $M$ poles has alternating signs, this problem would disappear at the next-to-next-to-leading-log level, while at this order $c$, $\kappa$ and $M_s$ must be treated as free parameters, upon the assumption that the minimum is restored by higher order corrections.

The anomalous dimension eq. (4.15) is compared to its perturbative counterpart in figs. 6-8, for typical values of the parameters. It is seen that a reasonable stability of the results is obtained for $0.35 \lesssim M_s \lesssim 0.45$, $0.3 \lesssim \alpha_s c \lesssim 0.5$, $3/4\kappa_0 \lesssim \kappa \lesssim 5/4\kappa_0$, though at the extremes of the parameter range some instability due to large subleading corrections starts to appear. For instance, the discontinuity at $N = \alpha_s c$ in the first derivative of the running coupling term eq. (2.13) starts showing up (figure 8) when $\kappa$ is lowered, since its coefficient is proportional to $1/\kappa$.

This shows that the leading order resummed expression eq. (3.1) is stable upon perturbative corrections for a range of values of the parameters which are reasonably close to the leading order values eq. (3.4)-(3.6). However, we must conclude that the subleading correction eq. (4.17) cannot be used to improve the leading order resummation, because the values of the parameters which characterize it cannot be calculated in a reliable way. In fact, as already mentioned, the NLO perturbative $\chi = \chi_0 + \alpha_s \chi_1$ has no minimum. As a consequence, the uncertainty on this correction is of the same order as its size.

Hence, the available information on the parameters $c$, $\kappa$ and $M_s$ from the next-to-
leading order kernel $\chi_1$ is insufficient to improve the leading order resummed result. The only viable procedure, without further theoretical input, is to combine the leading-order resummation eq. (3.1) with the standard next-to-leading order correction to the GLAP equation, as we did in eq. (3.19) (figure 5).
5. Conclusion

The main result of this work is the improved anomalous dimension given in eq. (3.1) (and its generalizations eqs. (3.18) and (3.19)). These relatively manageable expressions have the correct perturbative limit for small \( \alpha_s \) with \( N \) fixed (which corresponds to large \( x \) for the splitting functions) and include the leading BFKL corrections with running coupling effects for small \( \alpha_s \) with \( \alpha_s/N \) fixed. It is quite remarkable that the improved anomalous dimension in eq. (3.1), which does not contain free parameters, leads to a splitting function which is surprisingly close to the perturbative result down to small values of \( x \) (figure 2) in agreement with the data. The running coupling effects are essential to soften the small-\( x \) asymptotics, but \textit{a priori} one would not expect that the corrected and the perturbative anomalous dimensions should be so close over an extended range at small \( x \). The typical BFKL rise is softened and delayed to very small values of \( x \), while in the intermediate small-\( x \) region the splitting function is flat and close to the perturbative one.

As is by now well known, a flat perturbative splitting function reproduces by evolution a rising structure function, and predicts a double logarithmic slope in quantitative agreement with that seen in the data [33]. However, the data suggest that the optimal value of the slope is contained in a region which is up to 15% smaller than the leading order GLAP value. It follows that the value of the splitting function in the flat region, being proportional to the square of the slope, can be at most 30% below its leading order GLAP value. Thus it is quite non trivial that our improved anomalous dimension directly fulfills this empirical constraint in the flat region (where it lies below the perturbative splitting function) down to \( x \sim 10^{-3} \).

In comparison to the approach of refs. [18,21] we share the general physical framework, but there are two main differences. The first concerns the resummation procedure for \( \chi(\alpha_s, M) \) near \( M = 0 \). We match systematically the perturbative expansion of \( \chi \) to that of the usual anomalous dimension, thereby finding a shift of order \( \alpha_s \) of the poles of \( \chi \) at \( M = 0 \) in a way which is made unambiguous by the constraint of momentum conservation. They instead, based on the change of scale variables in going from the BFKL symmetric into the DIS asymmetric configuration, displace the pole from \( M = 0 \) to \( M + N/2 = 0 \). In the relevant limit, \( \alpha_s \) small with \( \alpha_s/N \) fixed, the two displacements are of the same order, but in refs. [18,21] the input of momentum conservation, which controls our double-leading resummation, is not imposed although a posteriori the results are numerically compatible with it. Second, they determine the asymptotic small-\( x \) behaviour by making some assumptions on the way to implement a symmetrisation \( M \rightarrow (1 - M) \) of the BFKL kernel \( \chi(\alpha_s, M) \), which we prefer not to do because of the model dependence it entails. For this we have to pay the price of some parameter dependence in the non leading terms. However, the fact that we find that after running coupling resummation the parameter dependence is weak gives support to their statement that the model dependence is not important.

However, an advantage of our approach is that we end up with an explicit formal expression for the improved anomalous dimension which can be used in the conventional evolution equations and thus matched to standard perturbative evolution. In refs. [18,21] instead only the resummed BFKL kernel is determined analytically, so anomalous dimensions must be evaluated numerically from the solution. This makes the possibility of fitting
the data and matching to usual evolution equations more difficult. The recent work [34] by
the same authors, which appeared while the present work was being completed, adopts a
somewhat different symmetrisation procedure, such that, for example, momentum conser-
vation is not automatic and is therefore imposed. A new result for the splitting function
is found with a dip in the intermediate region at small \( x \) before the onset of the BFKL
rise at even smaller values of \( x \). This confirms the model dependence of the approach of
refs. [18,21], which corresponds to the parameter dependence that we find. An advantage
of our approach in this respect, however, is that the parameter dependence can be cleanly
separated into the non-leading corrections.

Indeed, the simple leading order expression of the improved anomalous dimension eq.
(3.1) is only valid when non leading corrections in both the perturbative and the BFKL
expansions can be neglected. We have discussed a general procedure for a systematic
treatment of these higher order corrections, of which the NLO term is available, and we
have considered it in detail. Following a general approach already used in our previous
papers, our resummation strategy is to reabsorb in the leading term the parameters that
describe the behaviour of \( \chi(M) \) near \( M = 1/2 \). In refs. [19,20] we resummed the value
\( \lambda = \alpha_s c \) determining the small-\( x \) asymptotic behaviour as \( x^{-\lambda} \). Here we include in the
leading \( \chi(M) \) the first few terms of its expansion near the minimum at \( M = M_s \) that
are needed for the Airy resummation. From the known non-leading correction we remove
the perturbative expansion of the parameters included in the leading term. While the
non leading correction is large before the shift we have shown that it reduces to a small
correction after the subtraction. Thus we have shown that the non leading corrections are
in principle under control.

The difficulty is that the negative infinite behaviour of \( \chi_1(M) \) near \( M = 1 \) makes
the treatment of the non leading contribution in the central region of \( M \) ambiguous. For
example, the perturbative curvature \( \kappa \) becomes negative in the central region when the
\( \chi_1 \) term is added, while we assume that the true curvature is positive (corresponding
to a minimum). On the basis of the closeness of the leading–order result to standard
perturbative evolution, and the phenomenological success of the latter, we expect that the
subleading correction after subtraction is small. However, this also implies that it contains
ambiguities of the same order as the correction itself.

Thus, in practice, the known next-to-leading order correction is mainly useful as an
estimate of the error on the leading approximation. The uncertainty on the non leading
correction can be phenomenologically described in practice by replacing in the leading
formula the lowest order perturbative values of the parameters describing the central region
of \( \chi \) with generic values to be optimized to the data. For example, we have explicitly
examined the variation of our results when the intercept \( c \) of the anomalous dimension
is varied [eq. (3.18) and figure 3], and we also studied variations of the location of the
minimum \( M_s \) and the curvature at the minimum \( \kappa \).

In conclusion, we have achieved a precise quantitative understanding of the resum-
mation of BFKL logarithms, which explains the empirical observation of the validity of
GLAP evolution down to rather smaller values of \( x \) than might be naively expected. It
will be interesting to compare parton distribution functions evolved with our resummed
anomalous dimension with the latest HERA data to look for evidence [35] for the small
departures from GLAP evolution that we predict.
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