Almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds

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Abstract

As a generalization of Riemannian submersions, horizontally conformal submersions, semi-invariant submersions, h-semi-invariant submersions, almost h-semi-invariant submersions, conformal semi-invariant submersions, we introduce h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We study their properties: the geometry of foliations, the conditions for total manifolds to be locally product manifolds, the conditions for such maps to be totally geodesic. Finally, we give some examples of such maps.

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1. Introduction

Riemannian submersions were independently introduced by B. O’Neill [20] and A. Gray [11] in 1960s. Using the notion of almost Hermitian submersions, B. Watson [29] obtained some differential geometric properties among fibers, base manifolds, and total manifolds. After that, many geometers study this area and there are a lot of results on this topic.

As a generalization of Riemannian submersions, a horizontally conformal submersion was introduced independently by B. Fuglede [14] and T. Ishihara [18] in 1970s and it is a particular type of conformal maps.

Given a $C^\infty$-submersion $F$ from a Riemannian manifold $(M, g_M)$ onto a Riemannian manifold $(N, g_N)$, according to the conditions on the map $F : (M, g_M) \rightarrow (N, g_N)$, we have the following types of submersions: a Riemannian submersion ([11,13,20]), an almost Hermitian submersion [29], an invariant submersion [27], an anti-invariant submersion [24], a slant submersion ([9,25]), a semi-invariant submersion [26], a semi-slant submersion [23], a quaternionic submersion [15], an h-anti-invariant submersion and an almost h-anti-invariant submersion [22], an h-semi-invariant submersion and an almost h-semi-invariant submersion [21], a horizontally conformal submersion ([4,12]), a conformal anti-invariant submersion [1], a conformal semi-invariant submersion [2], etc.

It is well-known that Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([7,30]), Kaluza-Klein theory ([8,16]), Supergravity and superstring theories ([17,19]), etc. And the quaternionic Kähler manifolds have
applications in physics as the target spaces for nonlinear $\sigma-$models with supersymmetry [10].

The paper is organized as follows. In Section 2 we remind some notions, which are needed in the following sections. In Section 3 we give the definitions of $h$-conformal semi-invariant submersions and almost $h$-conformal semi-invariant submersions and obtain some properties on them: the characterizations of such maps, the harmonicity of such maps, the conditions for such maps to be totally geodesic, the integrability of distributions, the geometry of foliations, etc. In Section 4 we give some examples of $h$-conformal semi-invariant submersions and almost $h$-conformal semi-invariant submersions.

2. Preliminaries

Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds, where $g_M$ and $g_N$ are Riemannian metrics on $C^\infty$-manifolds $M$ and $N$, respectively.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a $C^\infty$-map.

We call the map $F$ a $C^\infty$-submersion if $F$ is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

Then the map $F$ is said to be a Riemannian submersion ([13, 20]) if $F$ is a $C^\infty$-submersion and

\[
(F_*)_p : ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})
\]

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space $T_p M$ to $M$ at $p$.

The map $F$ is called horizontally weakly conformal at $p \in M$ if it satisfies either (i) $(F_*)_p = 0$ or (ii) $(F_*)_p$ is surjective and there exists a positive number $\lambda(p) > 0$ such that

\[
g_N((F_*)_p X, (F_*)_p Y) = \lambda^2 g_M(X, Y) \quad \text{for } X, Y \in (\ker(F_*)_p)^\perp.
\]  

We call the point $p$ a critical point if it satisfies the type (i) and call the point $p$ a regular point if it satisfies the type (ii). And the positive number $\lambda(p)$ is said to be dilation of $F$ at $p$. The map $F$ is called horizontally weakly conformal if it is horizontally weakly conformal at any point of $M$. If the map $F$ is horizontally weakly conformal and it has no critical points, then we call the map $F$ a horizontally conformal submersion. The horizontally conformal submersion $F$ is said to be horizontally homothetic if $X(\lambda) = 0$ for $X \in \Gamma((\ker(F_*)_p)^\perp)$.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion.

Given any vector field $U \in \Gamma(TM)$, we write

\[
U = \nabla U + \mathcal{H} U,
\]

where $\nabla U \in \Gamma(\ker(F_*)_p)$ and $\mathcal{H} U \in \Gamma((\ker(F_*)_p)^\perp)$.

Define the (O’Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

\[
\mathcal{A}_E F = \mathcal{H}(\nabla_{\mathcal{H} E} V) F + \nabla_{\mathcal{H} E} \mathcal{H} F 
\]

\[
\mathcal{T}_E F = \mathcal{H}(\nabla_{V} V) F + \nabla_{V} \mathcal{H} F 
\]

for vector fields $E, F \in \Gamma(TM)$, where $\nabla$ is the Levi-Civita connection of $g_M$ ([13, 20]).

Then it is well-known that

\[
g_M(\mathcal{T}_U V, W) = -g_M(V, \mathcal{T}_U W)
\]

\[
g_M(\mathcal{A}_U V, W) = -g_M(V, \mathcal{A}_U W)
\]

for $U, V, W \in \Gamma(TM)$.

Let $F : (M, g_M) \mapsto (N, g_N)$ be a $C^\infty$-map.

Then the second fundamental form of $F$ is given by

\[
(\nabla F_*)(X, Y) := \nabla_X^F F_*(Y) - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),
\]
where $\nabla^F$ is the pullback connection and we denote conveniently by $\nabla$ the Levi-Civita connections of the metrics $g_M$ and $g_N$, [4].

Recall that $F$ is said to be harmonic if the tension field $\tau(F) = \text{trace}(\nabla F_x) = 0$ and $F$ is called a totally geodesic map if $(\nabla F_x)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$, [4].

**Lemma 2.1** ([28]). Let $(M, g_M)$ and $(N, g_N)$ be Riemannian manifolds and $T : (M, g_M) \rightarrow (N, g_N)$ a $C^\infty$-map. Then we have

$$\nabla^F_X F_y Y - \nabla^F_X F_y Y - F_g ([X, Y]) = 0 \quad (2.7)$$

for $X, Y \in \Gamma(TM)$.

**Remark 2.2.** (1) From (2.7), we see that the second fundamental form $\nabla F$ is symmetric.
(2) By (2.7), we obtain

$$[V, X] \in \Gamma(\ker F) \quad (2.8)$$

for $V \in \Gamma(\ker F)$ and $X \in \Gamma((\ker F)^\perp)$. Let $F : (M, g_M) \rightarrow (N, g_N)$ be a horizontally conformal submersion with dilation $\lambda$.

We call a vector field $X \in \Gamma(TM)$ basic if (i) $X \in \Gamma((\ker F)^\perp)$ and (ii) $X$ is $F$-related with some vector field $X^\prime \in \Gamma(TM)$. (i.e., $(F^*_x)pX(p) = X^\prime(F(p))$ for any $p \in M$.)

Given any fiber $F^{-1}(y)$, $y \in N$, and any basic vector fields $X, Y \in \Gamma((\ker F)^\perp)$, we have

$$\lambda(x)^2 g_M(X, Y)(x) = g_N(F_*X, F_*Y)(y) \quad \text{constant}$$

for any $x \in F^{-1}(y)$ so that

$$V(\lambda^2 g_M(X, Y)) = V(g_N(F_*X, F_*Y)) = 0 \quad \text{for} \ V \in \Gamma(\ker F). \quad (2.9)$$

Then we get

**Proposition 2.3** ([12]). Let $F : (M, g_M) \rightarrow (N, g_N)$ be a horizontally conformal submersion with dilation $\lambda$. Then we obtain

$$A_X Y = \frac{1}{2} \{[X, Y] - \lambda^2 g_M(X, Y) \nabla_\lambda(\frac{1}{\lambda^2})\} \quad (2.10)$$

for $X, Y \in \Gamma((\ker F)^\perp)$. Here, $\nabla_\lambda$ denotes the gradient vector field in the distribution $\ker F \subset TM$. (i.e., $\nabla_\lambda f = \sum_{i=1}^m V_i(f)V_i$ for $f \in C^\infty(M)$ and a local orthonormal frame $\{V_1, \cdots, V_m\}$ of $\ker F$.)

**Lemma 2.4** ([4]). Let $F : (M, g_M) \rightarrow (N, g_N)$ be a horizontally conformal submersion with dilation $\lambda$. Then we have

$$\begin{align*}
(\nabla F)_x (X, Y) &= X(\ln \lambda) F_y Y + Y(\ln \lambda) F_x X - g_M(X, Y) F_\lambda(\nabla_\lambda \ln \lambda), \\
(\nabla F)_x (V, W) &= -F_\lambda (\nabla_\lambda W), \\
(\nabla F)_x (X, V) &= -F_\lambda (\nabla_\lambda V) = -F_\lambda (\nabla_\lambda V) = -F_\lambda (A_X V) \\
\n\end{align*}$$

(2.11) to (2.13) for $X, Y \in \Gamma((\ker F)^\perp)$ and $V, W \in \Gamma(\ker F)$. Let $(M, g_M, J)$ be an almost Hermitian manifold, where $J$ is a compatible almost complex structure on $M$ (i.e., $J^2 = -id$, $g_M(JX, JY) = g_M(X, Y)$ for $X, Y \in \Gamma(TM)$).

We call a horizontally conformal submersion $F : (M, g_M, J) \rightarrow (N, g_N)$ a conformal anti-invariant submersion [1] if $J(\ker F) \subset (\ker F)^\perp$.

A horizontally conformal submersion $F : (M, g_M, J) \rightarrow (N, g_N)$ is called a conformal semi-invariant submersion [2] if there is a distribution $\mathcal{D}_1 \subset \ker F$ such that

$$\ker F = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1, \quad J(\mathcal{D}_2) \subset (\ker F)^\perp,$$

where $\mathcal{D}_2$ is the orthogonal complement of $\mathcal{D}_1$ in $\ker F$. 
Let \( M \) be a \( 4m \)-dimensional \( C^\infty \)-manifold and let \( E \) be a rank 3 subbundle of \( \text{End}(TM) \) such that for any point \( p \in M \) with a neighborhood \( U \), there exists a local basis \( \{ J_1, J_2, J_3 \} \) of sections of \( E \) on \( U \) satisfying for all \( \alpha \in \{ 1, 2, 3 \} \)
\[
J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},
\]
where the indices are taken from \( \{ 1, 2, 3 \} \) modulo 3.

Then we call \( E \) an almost quaternionic structure on \( M \) and \((M, E)\) an almost quaternionic manifold \cite{3}.

Moreover, let \( g \) be a Riemannian metric on \( M \) such that for any point \( p \in M \) with a neighborhood \( U \), there exists a local basis \( \{ J_1, J_2, J_3 \} \) of sections of \( E \) on \( U \) satisfying for all \( \alpha \in \{ 1, 2, 3 \} \)
\[
J_\alpha^2 = -\text{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},
\]
where the indices are taken from \( \{ 1, 2, 3 \} \) modulo 3.

Then we call \((M, E, g)\) an almost quaternionic Hermitian manifold \cite{15}.

For convenience, the above basis \( \{ J_1, J_2, J_3 \} \) satisfying (2.14) and (2.15) is said to be a quaternionic Hermitian basis.

Let \((M, E, g)\) be an almost quaternionic Hermitian manifold.

We call \((M, E, g)\) a quaternionic Kähler manifold if there exist locally defined 1-forms \( \omega_1, \omega_2, \omega_3 \) such that for \( \alpha \in \{ 1, 2, 3 \} \)
\[
\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_\alpha + 2 \\text{id}
\]
for any vector field \( X \in \Gamma(TM) \), where the indices are taken from \( \{ 1, 2, 3 \} \) modulo 3 \cite{15}.

If there exists a global parallel quaternionic Hermitian basis \( \{ J_1, J_2, J_3 \} \) of sections of \( E \) on \( M \) (i.e., \( \nabla J_\alpha = 0 \) for \( \alpha \in \{ 1, 2, 3 \} \), where \( \nabla \) is the Levi-Civita connection of the metric \( g \)), then \((M, E, g)\) is said to be a hyperkähler manifold. Furthermore, we call \((J_1, J_2, J_3)\) a hyperkähler structure on \( M \) and \( g \) a hyperkähler metric \cite{6}.

Let \((M, E_M, g_M)\) and \((N, E_N, g_N)\) be almost quaternionic Hermitian manifolds.

A map \( F : M \to N \) is called a \((E_M, E_N)\)-holomorphic map if given a point \( x \in M \), for any \( J \in (E_M)_x \) there exists \( J' \in (E_N)_{F(x)} \) such that
\[
F_* \circ J = J' \circ F_*.
\]
A Riemannian submersion \( F : M \to N \) which is a \((E_M, E_N)\)-holomorphic map is called a quaternionic submersion \cite{15}.

Moreover, if \((M, E_M, g_M)\) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that \( F \) is a quaternionic Kähler submersion (or a hyperkähler submersion) \cite{15}.

Then we know that any quaternionic Kähler submersion is a harmonic map \cite{15}.

Let \((M, E, g_M)\) be an almost quaternionic Hermitian manifold and \((N, g_N)\) a Riemannian manifold.

A Riemannian submersion \( F : (M, E_M) \to (N, g_N) \) is called an \( h \)-semi-invariant submersion if given a point \( p \in M \) with a neighborhood \( U \), there exists a quaternionic Hermitian basis \( \{ I, J, K \} \) of sections of \( E \) on \( U \) such that for any \( R \in \{ I, J, K \} \), there is a distribution \( \mathcal{D}_1 \subset \ker F_* \) on \( U \) such that
\[
\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad R(\mathcal{D}_1) = \mathcal{D}_1, \quad R(\mathcal{D}_2) \subset (\ker F_*)^\perp,
\]
where \( \mathcal{D}_2 \) is the orthogonal complement of \( \mathcal{D}_1 \) in \( \ker F_* \) \cite{21}.

We call such a basis \( \{ I, J, K \} \) an \( h \)-semi-invariant basis.

A Riemannian submersion \( F : (M, E, g_M) \to (N, g_N) \) is called an almost \( h \)-semi-invariant submersion if given a point \( p \in M \) with a neighborhood \( U \), there exists a quaternionic Hermitian basis \( \{ I, J, K \} \) of sections of \( E \) on \( U \) such that for each \( R \in \{ I, J, K \} \), there is a distribution \( \mathcal{D}_1^R \subset \ker F_* \) on \( U \) such that
\[
\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \quad R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,
\]
where \( \mathcal{D}_2^R \) is the orthogonal complement of \( \mathcal{D}_1^R \) in \( \ker F_* \) [21].

We call such a basis \( \{I, J, K\} \) an *almost h-semi-invariant basis*.

Throughout this paper, we will use the above notations.

### 3. Almost h-conformal semi-invariant submersions

In this section, we define h-conformal semi-invariant submersions and almost h-conformal semi-invariant submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. And we study their properties: the integrability of distributions, the geometry of foliations, the conditions for such maps to be totally geodesic, etc.

**Definition 3.1.** Let \((M, E, g_M)\) be an almost quaternionic Hermitian manifold and \((N, g_N)\) a Riemannian manifold. A horizontally conformal submersion \(F: (M, E, g_M) \rightarrow (N, g_N)\) is called an *h-conformal semi-invariant submersion* if given a point \(p \in M\) with a neighborhood \(U\), there exists a quaternionic Hermitian basis \(\{I, J, K\}\) of sections of \(E\) on \(U\) such that for any \(R \in \{I, J, K\}\), there is a distribution \(\mathcal{D}_1 \subset \ker F_*\) on \(U\) such that

\[
\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad R(\mathcal{D}_1) = \mathcal{D}_1, \quad R(\mathcal{D}_2) \subset (\ker F_*)^\perp,
\]

where \(\mathcal{D}_2\) is the orthogonal complement of \(\mathcal{D}_1\) in \(\ker F_*\).

We call such a basis \(\{I, J, K\}\) an *h-conformal semi-invariant basis*.

**Definition 3.2.** Let \((M, E, g_M)\) be an almost quaternionic Hermitian manifold and \((N, g_N)\) a Riemannian manifold. A horizontally conformal submersion \(F: (M, E, g_M) \rightarrow (N, g_N)\) is called an *almost h-conformal semi-invariant submersion* if given a point \(p \in M\) with a neighborhood \(U\), there exists a quaternionic Hermitian basis \(\{I, J, K\}\) of sections of \(E\) on \(U\) such that for each \(R \in \{I, J, K\}\), there is a distribution \(\mathcal{D}_1^R \subset \ker F_*\) on \(U\) such that

\[
\ker F_* = \mathcal{D}_1^R \oplus \mathcal{D}_2^R, \quad R(\mathcal{D}_1^R) = \mathcal{D}_1^R, \quad R(\mathcal{D}_2^R) \subset (\ker F_*)^\perp,
\]

where \(\mathcal{D}_2^R\) is the orthogonal complement of \(\mathcal{D}_1^R\) in \(\ker F_*\).

We call such a basis \(\{I, J, K\}\) an *almost h-conformal semi-invariant basis*.

**Remark 3.3.** (1) Let \(F\) be an h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \(\{I, J, K\}\) is an h-conformal semi-invariant basis. Then the fibers of the map \(F\) are quaternionic CR-submanifolds [5].

(2) Let \(F: (M, E, g_M) \rightarrow (N, g_N)\) be an h-conformal semi-invariant submersion. Then the map \(F\) is also an almost h-conformal semi-invariant submersion.

Let \(F: (M, E, g_M) \rightarrow (N, g_N)\) be an almost h-conformal semi-invariant submersion with an almost h-conformal semi-invariant basis \(\{I, J, K\}\). Denote the orthogonal complement of \(R\mathcal{D}_2^R\) in \((\ker F_*)^\perp\) by \(\mu^R\) for \(R \in \{I, J, K\}\). We easily see that \(\mu^R\) is \(R\)-invariant for \(R \in \{I, J, K\}\).

Then given \(X \in \Gamma(\ker F_*)\), we write

\[
RX = \phi_RX + \omega_RX, \quad (3.1)
\]

where \(\phi_RX \in \Gamma(\mathcal{D}_1^R)\) and \(\omega_RX \in \Gamma(\mathcal{D}_2^R)\) for \(R \in \{I, J, K\}\).

Given \(Z \in \Gamma((\ker F_*)^\perp)\), we get

\[
RZ = B_RZ + C_RZ, \quad (3.2)
\]

where \(B_RZ \in \Gamma(\mathcal{D}_2^R)\) and \(C_RZ \in \Gamma(\mu^R)\) for \(R \in \{I, J, K\}\).

We see that

\[
(\ker F_*)^\perp = R\mathcal{D}_2^R \oplus \mu^R \quad \text{for } R \in \{I, J, K\} \quad (3.3)
\]

and

\[
g_M(C_RX, RV) = 0 \quad (3.4)
\]
for \( X \in \Gamma((\ker F_\ast)^\perp) \) and \( V \in \Gamma(D_2^R) \).

Define \( \nabla_X Y := \nabla_X Y \) for \( X, Y \in \Gamma(\ker F_\ast) \).

We also define
\[
(\nabla_X \phi_R)Y := \nabla_X \phi_R Y - \phi_R \nabla_X Y
\]
and
\[
(\nabla_X \omega_R)Y := \nabla_X \omega_R Y - \omega_R \nabla_X Y
\]
for \( X, Y \in \Gamma(\ker F_\ast) \) and \( R \in \{I, J, K\} \).

Then we easily obtain

**Lemma 3.4.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Then we get

1. \[
\nabla_X \phi_R Y + \tau_X \omega_R Y = \phi_R \nabla_X Y + B_R \tau_X Y
\]
   \[
\tau_X \phi_R Y + \kappa(\nabla_X \omega_R Y) = \omega_R \nabla_X Y + C_R \tau_X Y
\]
   for \( X, Y \in \Gamma(\ker F_\ast) \) and \( R \in \{I, J, K\} \).

2. \[
\nabla_Y Z B_R W + A_Z C_R W = \phi_R A_Z W + B_R \kappa(\nabla_Y Z W)
\]
   \[
A_Z B_R W + \kappa(\nabla_Y Z C_R W) = \omega_R A_Z W + C_R \kappa(\nabla_Y Z W)
\]
   for \( Z, W \in \Gamma((\ker F_\ast)^\perp) \) and \( R \in \{I, J, K\} \).

3. \[
\nabla_X B_R Z + \tau_X C_R Z = \phi_R \tau_X Z + B_R \kappa(\nabla_X Z)
\]
   \[
\tau_X B_R Z + \kappa(\nabla_X C_R Z) = \omega_R \tau_X Z + C_R \kappa(\nabla_X Z)
\]
   for \( X \in \Gamma(\ker F_\ast), Z \in \Gamma((\ker F_\ast)^\perp) \), and \( R \in \{I, J, K\} \).

**Remark 3.5.** By (3.5), (3.6), and Lemma 3.4 (1), we have
\[
(\nabla_X \omega_R)Y = B_R \tau_X Y - \tau_X \omega_R Y
\]
(3.7)
\[
(\nabla_X \omega_R)Y = C_R \tau_X Y - \tau_X \phi_R Y
\]
(3.8)
for \( X, Y \in \Gamma(\ker F_\ast) \) and \( R \in \{I, J, K\} \).

Now, we investigate the integrability of some distributions.

**Lemma 3.6.** Let \( F \) be an h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an h-conformal semi-invariant basis. Then we have the followings:

(i) The distribution \( D_2 \) is always integrable.

(ii) The following conditions are equivalent:

(a) The distribution \( D_1 \) is integrable.

(b) \( (\nabla F_\ast)(W, IV) - (\nabla F_\ast)(V, IW) \in \Gamma(F_\ast \mu^l) \) for \( V, W \in \Gamma(D_1) \).

(c) \( (\nabla F_\ast)(W, JV) - (\nabla F_\ast)(V, JW) \in \Gamma(F_\ast \mu^l) \) for \( V, W \in \Gamma(D_1) \).

(d) \( (\nabla F_\ast)(W, KV) - (\nabla F_\ast)(V KW) \in \Gamma(F_\ast \mu^k) \) for \( V, W \in \Gamma(D_1) \).

**Proof.** By (2.7), we have \([V, W] \in \Gamma(\ker F_\ast)\) for \( V, W \in \Gamma(\ker F_\ast) \).

We claim that \( \tau_V = \tau_W \) for \( V, W \in \Gamma(D_2) \) and \( R \in \{I, J, K\} \).

Given \( X \in \Gamma(\ker F_\ast) \), we get
\[
g_M(\tau_V W, X) = -g_M(\tau_W W, \nabla_V X) = -g_M(\tau_W V, \nabla_X V) = g_M(\nabla_X RW, V)
\]
\[
= -g_M(\nabla_X W, RV) = -g_M(\nabla_W X, RV) = g_M(X, \tau_W RV),
\]
which means our claim.

Given $V, W \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we obtain
\[
g_M([V, W], Z) = g_M(\nabla_V W - \nabla_W V, Z) = g_M(\mathcal{T}_V RW - \mathcal{T}_W RV, RZ) = 0,
\]
which implies (i).

For (ii), given $V, W \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$, and $R \in \{I, J, K\}$, we have
\[
g_M([V, W], Z) = \frac{1}{\lambda^2} g_N(F_\ast \nabla_V RW - F_\ast \nabla_W RV, F_\ast RZ)
\]
\[
= \frac{1}{\lambda^2} g_N((\nabla F_\ast)(W, RV) - (\nabla F_\ast)(V, RW), F_\ast RZ)
\]
so that we get $(a) \iff (b)$, $(a) \iff (c)$, $(a) \iff (d)$.

Therefore, the result follows. \hfill \Box

**Theorem 3.7.** Let $F$ be an almost $h$-conformal semi-invariant submersion from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost $h$-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution $(\ker F_\ast)^\perp$ is integrable.

(b) $A_Y \omega_I B_I X - A_X \omega_I B_I Y + \phi_I (A_Y C_I X - A_X C_I Y) \in \Gamma(D_2^I)$ and
\[
\frac{1}{\lambda^2} g_N(\nabla^F F_\ast C_I X - \nabla^F F_\ast C_I Y, F_\ast IV)
\]
\[
= g_M(A_Y B_I X - A_X B_I Y - C_I Y(\ln \lambda) X + C_I X(\ln \lambda) Y
\]
\[
+ 2g_M(X, C_I Y)\nabla(\ln \lambda), IV)
\]
for $X, Y \in \Gamma((\ker F_\ast)^\perp)$ and $V \in \Gamma(D_2^I)$.

(c) $A_Y \omega_I B_J X - A_X \omega_I B_J Y + \phi_I (A_Y C_J X - A_X C_J Y) \in \Gamma(D_2^J)$ and
\[
\frac{1}{\lambda^2} g_N(\nabla^F F_\ast C_J X - \nabla^F F_\ast C_J Y, F_\ast JV)
\]
\[
= g_M(A_Y B_J X - A_X B_J Y - C_J Y(\ln \lambda) X + C_J X(\ln \lambda) Y
\]
\[
+ 2g_M(X, C_J Y)\nabla(\ln \lambda), JV)
\]
for $X, Y \in \Gamma((\ker F_\ast)^\perp)$ and $V \in \Gamma(D_2^J)$.

(d) $A_Y \omega_K B_K X - A_X \omega_K B_K Y + \phi_K (A_Y C_K X - A_X C_K Y) \in \Gamma(D_2^K)$ and
\[
\frac{1}{\lambda^2} g_N(\nabla^F F_\ast C_K X - \nabla^F F_\ast C_K Y, F_\ast KV)
\]
\[
= g_M(A_Y B_K X - A_X B_K Y - C_K Y(\ln \lambda) X + C_K X(\ln \lambda) Y
\]
\[
+ 2g_M(X, C_K Y)\nabla(\ln \lambda), KV)
\]
for $X, Y \in \Gamma((\ker F_\ast)^\perp)$ and $V \in \Gamma(D_2^K)$.

**Proof.** Given $X, Y \in \Gamma((\ker F_\ast)^\perp)$, $W \in \Gamma(D_2^R)$, and $R \in \{I, J, K\}$, we have
\[
g_M([X, Y], W) = g_M(\nabla_X B_R Y, RW) + g_M(\nabla_X C_R Y, RW)
\]
\[
- g_M(\nabla_Y B_R X, RX) - g_M(\nabla_Y C_R X, RX)
\]
\[
= -g_M(\nabla_X RB_R Y, W) + g_M(A_X C_R Y, RW)
\]
\[
+ g_M(\nabla_Y RB_R X, W) - g_M(A_Y C_R X, RW)
\]
\[
= -g_M(\nabla_X \omega_R B_R Y, W) - g_M(\phi_R A_X C_R Y, W)
\]
\[
+ g_M(\nabla_Y \omega_R B_R X, W) + g_M(\phi_R A_Y C_R X, W) \text{ (since } \phi_R B_R = 0)
\]
\[
= g_M(A_Y \omega_R B_R X - A_X \omega_R B_R Y + \phi_R A_Y C_R X - \phi_R A_X C_R Y, W)
\]
so that
\[ g_M([X,Y], W) = 0 \quad \text{for } W \in \Gamma(D^R) \] (3.9)
\[ \iff A_Y\omega_R B_R X - A_X\omega_R B_R Y + \phi_R A_Y C_R X - \phi_R A_X C_R Y \in \Gamma(D^R_2). \]

Given \( V \in \Gamma(D^R_2) \), by using (2.11) and (3.4), we get
\[ g_M([X,Y], V) = g_M(\nabla_X B_R Y, RV) + g_M(\nabla_X C_R Y, RV) - g_M(\nabla_Y B_R X, RV) - g_M(\nabla_Y C_R X, RV) = g_M(A_X B_R Y - A_Y B_R X, RV) + \frac{1}{\lambda^2}g_N(-X(\ln \lambda)F_* C_R Y - C_R Y(\ln \lambda) F_* X) + g_M(X, C_R Y)F_* \nabla(\ln \lambda) + \nabla^F_X F_* C_R Y + Y(\ln \lambda) F_* C_R X + C_R X(\ln \lambda) Y - g_M(Y, C_R X) F_* \nabla(\ln \lambda) - \nabla^F_Y F_* C_R X, F_* RV) = g_M(A_X B_R Y - A_Y B_R X + C_R X(\ln \lambda) Y - C_R Y(\ln \lambda) X + 2g_M(X, C_R Y) \nabla(\ln \lambda), RV) - \frac{1}{\lambda^2}g_N(\nabla^F_Y F_* C_R X - \nabla^F_X F_* C_R Y, F_* RV) \]
so that
\[ g_M([X,Y], V) = 0 \quad \text{for } V \in \Gamma(D^R_2) \] (3.10)
\[ \iff \frac{1}{\lambda^2}g_N(\nabla^F_Y F_* C_R X - \nabla^F_X F_* C_R Y, F_* RV) = g_M(A_X B_R Y - A_Y B_R X + C_R X(\ln \lambda) Y - C_R Y(\ln \lambda) X + 2g_M(X, C_R Y) \nabla(\ln \lambda), RV). \]

Using (3.9) and (3.10), we obtain (a) \( \iff \) (b), (a) \( \iff \) (c), (a) \( \iff \) (d).

Therefore, we have the result. \( \square \)

**Theorem 3.8.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Assume that the distribution \((\ker F_*)^\perp\) is integrable. Then the following conditions are equivalent:

(a) The map \( F \) is horizontally homothetic.

(b) \( \lambda^2 g_M(A_Y B_I X - A_X B_I Y, IV) = g_N(\nabla^F_Y F_* C_I X - \nabla^F_X F_* C_I Y, F_* IV) \) for \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R). \)

(c) \( \lambda^2 g_M(A_Y B_J X - A_X B_J Y, JV) = g_N(\nabla^F_Y F_* C_J X - \nabla^F_X F_* C_J Y, F_* JV) \) for \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R). \)

(d) \( \lambda^2 g_M(A_Y B_K X - A_X B_K Y, KV) = g_N(\nabla^F_Y F_* C_K X - \nabla^F_X F_* C_K Y, F_* KV) \) for \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R). \)

**Proof.** Given \( X, Y \in \Gamma((\ker F_*)^\perp) \), \( V \in \Gamma(D^R) \), and \( R \in \{I, J, K\} \), from the proof of Theorem 3.7, we have
\[ g_M([X,Y], V) = g_M(A_X B_R Y - A_Y B_R X + C_R X(\ln \lambda) Y - C_R Y(\ln \lambda) X + 2g_M(X, C_R Y) \nabla(\ln \lambda), RV) = 0 \] (3.11)

Using (3.11), it is easy to see (a) \( \Rightarrow \) (b), (a) \( \Rightarrow \) (c), (a) \( \Rightarrow \) (d).

Conversely, from (3.11), we get
\[ g_M(C_R X(\ln \lambda) Y - C_R Y(\ln \lambda) X + 2g_M(X, C_R Y) \nabla(\ln \lambda), RV) = 0 \] (3.12)
Applying $Y = RV$ at (3.12), we obtain
\[ g_M(\nabla(\ln \lambda), C_R X)g_M(RV, RV) = 0, \]
which implies
\[ g_M(\nabla(\lambda), X) = 0 \quad \text{for } X \in \Gamma(\mu^R). \]  
(3.13)

Applying $Y = C_R X$, $X \in \Gamma(\mu^R)$, at (3.12), we have
\[ 2g_M(X, C_R^2 X)g_M(\nabla(\ln \lambda), RV) = -2g_M(X, X)g_M(\nabla(\ln \lambda), RV) = 0, \]
which implies
\[ g_M(\nabla(\lambda), RV) = 0 \quad \text{for } V \in \Gamma(D_2^R). \]  
(3.14)

By (3.13) and (3.14), we get (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a).

Therefore, the result follows. 

We deal with some particular type of conformal submersions.

**Definition 3.9.** Let $F$ be an almost h-conformal semi-invariant submersion from an almost quaternionic Hermitian manifold $(M, E, g_M)$ onto a Riemannian manifold $(N, g_N)$. If $R(D_2^R) = (\ker F_*)^\perp$ for $R \in \{I, K\}$ and $J(\ker F_*)^\perp = \ker F_*$ (i.e., $D_2^R = \{0\}$), then we call the map $F$ an almost h-conformal anti-holomorphic semi-invariant submersion.

We call such a basis $\{I, J, K\}$ an almost h-conformal anti-holomorphic semi-invariant basis.

**Remark 3.10.** (1) We easily see that $J(\ker F_*)^\perp = \ker F_*$ implies $J((\ker F_*)^\perp) = (\ker F_*)^\perp$.

(2) Let $F : (M, E, g_M) \rightarrow (N, g_N)$ be an h-conformal semi-invariant submersion. Then it is not possible to get $R(D_2^R) = (\ker F_*)^\perp$ for $R \in \{I, J, K\}$. If not, then $K(D_2^I) = (\ker F_*)^\perp$ and $K(D_2^I) = IJ(D_2^I) = I((\ker F_*)^\perp) = D_2^I$, contradiction!

So, our definition makes sense for this case. See Example 4.7.

**Corollary 3.11.** Let $F$ be an almost h-conformal anti-holomorphic semi-invariant submersion from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution $(\ker F_*)^\perp$ is integrable.

(b) $\mathcal{A}_{IV_1}IV_2 = \mathcal{A}_{IV_2}IV_1$ for $V_1, V_2 \in \Gamma(D_2^I)$.

(c) $\mathcal{A}_{KV_1}KV_2 = 0$ for $V_1, V_2 \in \Gamma(D_2^K)$.

**Proof.** We see that $C_R = 0$, $B_R = R$ on $(\ker F_*)^\perp$ and $\omega_R = R$ on $D_2^R$ for $R \in \{I, K\}$.

Applying $X = RV_1$ and $Y = RV_2$, $V_1, V_2 \in \Gamma(D_2^R)$, at Theorem 3.7, we have
\[ \mathcal{A}_{RV_1}RV_2 - \mathcal{A}_{RV_2}RV_1 \in \Gamma(D_2^R) \]
and
\[ 0 = g_M(\mathcal{A}_{RV_2}RV_1 - \mathcal{A}_{RV_1}RV_2, V) \quad \text{for } V \in \Gamma(D_2^R), \]
which are equivalent to
\[ \mathcal{A}_{RV_1}RV_2 = \mathcal{A}_{RV_2}RV_1 \quad \text{for } V_1, V_2 \in \Gamma(D_2^R). \]

Hence, we get (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c).

Therefore, we obtain the result. 

We consider the geometry of foliations and the condition for such maps to be horizontally homothetic throughout this section.

**Theorem 3.12.** Let $F$ be an almost h-conformal semi-invariant submersion from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on $M$. 

(b) \( A_X C_1 Y + \nabla_X B_1 Y \in \Gamma(\mathcal{D}_2^k) \) and
\[
g_N(\nabla_X^F F_i V, F_i C_1 V) = \lambda^2 g_M(A_X B_1 Y - C_1 Y (\ln \lambda) X + g_M(X, C_1 Y) \nabla(\ln \lambda), IV)
\]
for \( X, Y \in \Gamma((\ker F_s)^\perp) \) and \( V \in \Gamma(\mathcal{D}_2^k) \).

(c) \( A_X C_2 Y + \nabla_X B_2 Y \in \Gamma(\mathcal{D}_2^k) \) and
\[
g_N(\nabla_X^F F_i V, F_i C_2 Y) = \lambda^2 g_M(A_X B_2 Y - C_2 Y (\ln \lambda) X + g_M(X, C_2 Y) \nabla(\ln \lambda), IV)
\]
for \( X, Y \in \Gamma((\ker F_s)^\perp) \) and \( V \in \Gamma(\mathcal{D}_2^k) \).

(d) \( A_X C_K Y + \nabla_X B_K Y \in \Gamma(\mathcal{D}_2^k) \) and
\[
g_N(\nabla_X^F F_i C_K V, F_i C_K Y) = \lambda^2 g_M(A_X B_K Y - C_K Y (\ln \lambda) X + g_M(X, C_K Y) \nabla(\ln \lambda), KV)
\]
for \( X, Y \in \Gamma((\ker F_s)^\perp) \) and \( V \in \Gamma(\mathcal{D}_2^k) \).

**Proof.** Given \( X, Y \in \Gamma((\ker F_s)^\perp) \), \( W \in \Gamma(\mathcal{D}_2^k) \), and \( R \in \{I, J, K\} \), we obtain
\[
g_M(\nabla_X Y, W) = -g_M(\phi(A_X C_R Y + \nabla_X B_R Y), W)
\]
so that
\[
g_M(\nabla_X Y, W) = 0 \iff A_X C_R Y + \nabla_X B_R Y \in \Gamma(\mathcal{D}_2^k).
\]

Given \( V \in \Gamma(\mathcal{D}_2^k) \), by using (2.11) and (3.4), we have
\[
g_M(\nabla_X Y, V) = g_M(A_X B_R Y, RV) - g_M(C_R Y, \nabla_X RV)
\]
\[
= g_M(A_X B_R Y, RV) + \frac{1}{\lambda^2} g_N(F_s C_R Y, RV (\ln \lambda) F_s X
\]
\[
- g_M(X, RV) F_s \nabla(\ln \lambda) - \nabla_X^F F_s RV
\]
\[
= g_M(A_X B_R Y + g_M(C_R Y, X) \nabla(\ln \lambda) - C_R Y (\ln \lambda) X, RV)
\]
\[
- \frac{1}{\lambda^2} g_N(F_s C_R Y, \nabla_X^F F_s RV)
\]
so that
\[
g_M(\nabla_X Y, V) = 0 \iff g_N(F_s C_R Y, \nabla_X^F F_s RV) = \lambda^2 g_M(A_X B_R Y
\]
\[
+ g_M(C_R Y, X) \nabla(\ln \lambda) - C_R Y (\ln \lambda) X, RV).
\]

By (3.15) and (3.16), we get (a) \( \iff \) (b), (a) \( \iff \) (c), (a) \( \iff \) (d).

Therefore, the result follows. \qed

We introduce another notion on distributions and investigate it.

**Definition 3.13.** Let \( F \) be an almost \( h \)-conformal semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost \( h \)-conformal semi-invariant basis. Given \( R \in \{I, J, K\} \), we call the distribution \( \mathcal{D}_2^R \) parallel along \( (\ker F_s)^\perp \) if \( \nabla_X V \in \Gamma(\mathcal{D}_2^R) \) for \( X \in \Gamma((\ker F_s)^\perp) \) and \( V \in \Gamma(\mathcal{D}_2^R) \).

**Lemma 3.14.** Let \( F \) be an almost \( h \)-conformal semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost \( h \)-conformal semi-invariant basis. Assume that the distribution \( \mathcal{D}_2^R \) is parallel along \( (\ker F_s)^\perp \) for \( R \in \{I, J, K\} \). Then the following conditions are equivalent:

(a) The map \( F \) is horizontally homothetic.

(b) \[
\lambda^2 g_M(A_X B_1 Y, IV) = g_N(\nabla_X^F F_i IV, F_i C_1 Y)
\]
for \( X, Y \in \Gamma((\ker F_s)^\perp) \) and \( V \in \Gamma(\mathcal{D}_2^k) \).
(c) 
\[ \lambda^2 g_M(A_X B_J Y, J V) = g_N(\nabla_X F_* J V, F_* C_J Y) \]
for \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R_2) \).

(d) 
\[ \lambda^2 g_M(A_X B_K Y, K V) = g_N(\nabla_X F_* K V, F_* C_K Y) \]
for \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R_2) \).

**Proof.** Given \( X, Y \in \Gamma((\ker F_*)^\perp), V \in \Gamma(D^R_2) \), and \( R \in \{ I, J, K \} \), by the proof of Theorem 3.12, we have

\[
g_M(\nabla_X Y, V) = g_M(A_X B_R Y + g_M(C_R Y, X) \nabla(\ln \lambda) - C_R Y(\ln \lambda) X, R V) - \frac{1}{\lambda^2} g_N(F_* C_R Y, \nabla_X F_* R V). \tag{3.17}
\]

Since \( g_M(\nabla_X Y, V) = -g_M(Y, \nabla_X V) = 0 \), from (3.17), we get \((a) \Rightarrow (b), (a) \Rightarrow (c), (a) \Rightarrow (d)\).

Conversely, from (3.17), we obtain

\[
-g_M(C_R Y, \nabla(\ln \lambda)) g_M(X, R V) + g_M(X, C_R Y) g_M(\nabla(\ln \lambda), R V) = 0. \tag{3.18}
\]

Applying \( X = R V \) at (3.18), we have

\[
-g_M(C_R Y, \nabla(\ln \lambda)) g_M(R V, R V) = 0,
\]

which implies

\[
g_M(X, \nabla(\lambda)) = 0 \quad \text{for} \quad X \in \Gamma(\mu^R). \tag{3.19}
\]

Applying \( X = C_R Y \) at (3.18), we get

\[
g_M(C_R Y, C_R Y) g_M(\nabla(\ln \lambda), R V) = 0,
\]

which implies

\[
g_M(\nabla(\lambda), R V) = 0 \quad \text{for} \quad V \in \Gamma(D^R_2). \tag{3.20}
\]

Using (3.19) and (3.20), we obtain \((b) \Rightarrow (a), (c) \Rightarrow (a), (d) \Rightarrow (a)\).

Therefore, the result follows. \(\square\)

**Lemma 3.15.** Let \( F \) be an almost h-conformal anti-holomorphic semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost h-conformal anti-holomorphic semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution \( (\ker F_*)^\perp \) defines a totally geodesic foliation on \( M \).

(b) The distribution \( D^R_2 \) is parallel along \( (\ker F_*)^\perp \).

(c) The distribution \( D^R_2 \) is parallel along \( (\ker F_*)^\perp \).

**Proof.** We see that \( B_R = R \) and \( C_R = 0 \) on \( (\ker F_*)^\perp \) for \( R \in \{ I, K \} \).

Given \( X, Y \in \Gamma((\ker F_*)^\perp) \) and \( V \in \Gamma(D^R_2) \), from Theorem 3.12, we have

\[
(a) \iff \nabla_X R Y \in \Gamma(D^R_2) \quad \text{and} \quad g_M(A_X R Y, R V) = 0
\]

\[
\iff \nabla_X R Y \in \Gamma(D^R_2) \tag{3.20}
\]

Hence, we get \((a) \iff (b), (a) \iff (c)\).

Therefore, we obtain the result. \(\square\)

**Theorem 3.16.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution \( \ker F_* \) defines a totally geodesic foliation on \( M \).
Almost h-conformal semi-invariant submersions

(b) $\mathcal{T}_{V\omega}U + \hat{\nabla}_{V\phi}U \in \Gamma(D^1_\mu)$ and

\[ g_N(\nabla^F_{\omega}V^F_{*}U, F_{*}U) = \lambda^2 g_M(C_{I}T_{U}^{\phi}V^F_{*} + A_{\omega} \nabla_{V}^{\phi}U + g_M(\omega_{V}, \omega_{U}) \nabla(\ln \lambda), X) \]

for $U, V \in \Gamma(\text{ker} F_\ast)$ and $X \in \Gamma(\mu^I)$.

(c) $\mathcal{T}_{V\omega}U + \hat{\nabla}_{V\phi}U \in \Gamma(D^2_\mu)$ and

\[ g_N(\nabla^F_{\omega}V^F_{*}U, F_{*}U) = \lambda^2 g_M(C_{I}T_{U}^{\phi}V^F_{*} + A_{\omega} \nabla_{V}^{\phi}U + g_M(\omega_{V}, \omega_{U}) \nabla(\ln \lambda), X) \]

for $U, V \in \Gamma(\text{ker} F_\ast)$ and $X \in \Gamma(\mu^I)$.

(d) $\mathcal{T}_{V\omega}U + \hat{\nabla}_{V\phi}U \in \Gamma(D^2_\mu)$ and

\[ g_N(\nabla^F_{\omega}V^F_{*}U, F_{*}U) = \lambda^2 g_M(C_{I}T_{U}^{\phi}V^F_{*} + A_{\omega} \nabla_{V}^{\phi}U + g_M(\omega_{V}, \omega_{U}) \nabla(\ln \lambda), X) \]

for $U, V \in \Gamma(\text{ker} F_\ast)$ and $X \in \Gamma(\mu^I)$.

**Proof.** Given $U, V \in \Gamma(\text{ker} F_\ast)$, $W \in \Gamma(D^2_\mu)$, and $R \in \{I, J, K\}$, by using (3.4), we have

\[ g_M(\nabla_V U, RW) = -g_M(\omega_R(\nabla_V \phi U + \mathcal{T}_V \omega_R U), RW) \]

so that

\[ g_M(\nabla_V U, RW) = 0 \iff \hat{\nabla}_V \phi U + \mathcal{T}_V \omega_R U \in \Gamma(D^2_\mu) \]

(3.21)

Given $X \in \Gamma(\mu^R)$, by using (2.8) and (3.3), we get

\[
g_M(\nabla_V U, X) = g_M(\nabla_U \phi V, RX) + g_M(\phi R U, \nabla_{\omega V} X) + g_M(\omega_V, \nabla_{\omega V} X) = g_M(\mathcal{T}_U \phi V, RX) + g_M(\phi R U, A_{\omega V} X)
\]

\[
= -\frac{1}{\lambda^2} g_M(\nabla(\ln \lambda), X) g_N(F_{*} \omega R V, F_{*} \omega R U) + \frac{1}{\lambda^2} g_N(\nabla^F_{\omega R} F_{*} X, F_{*} \omega R U)
\]

\[
= g_M(\mathcal{T}_U \phi V, X) - A_{\omega V} \phi R U - g_M(\omega V, \omega R U) \nabla(\ln \lambda), X)
\]

so that

\[ g_M(\nabla_V U, X) = 0 \]

(3.22)

\[ \iff g_N(\nabla^F_{\omega R} F_{*} X, F_{*} \omega R U) = \lambda^2 g_M(C_{I}T_{U}^{\phi}V^F_{*} + A_{\omega R} \nabla_{V}^{\phi}U + g_M(\omega R, \omega U) \nabla(\ln \lambda), X). \]

Using (3.21) and (3.22), we obtain (a) $\iff$ (b), (a) $\iff$ (c), (a) $\iff$ (d).

Therefore, the result follows. \(\square\)

**Definition 3.17.** Let $F$ be an almost h-conformal semi-invariant submersion from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(\mathcal{N}, g_N)$ such that $(I, J, K)$ is an almost h-conformal semi-invariant basis. Then given $R \in \{I, J, K\}$, we call the distribution $\mu^R$ parallel along $\text{ker} F_\ast$ if $\nabla_U X \in \Gamma(\mu^R)$ for $X \in \Gamma(\mu^R)$ and $U \in \Gamma(\text{ker} F_\ast)$.

**Lemma 3.18.** Let $F$ be an almost h-conformal semi-invariant submersion from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(\mathcal{N}, g_N)$ such that $(I, J, K)$ is an almost h-conformal semi-invariant basis. Assume that the distribution $\mu^R$ is parallel along ker $F_\ast$ for any $R \in \{I, J, K\}$.

Then given $R \in \{I, J, K\}$, the following conditions are equivalent:

(a) Dilation $\lambda$ is constant on $\mu^R$.

(b) \[ g_N(\nabla^F_{\omega R} F_{*} X, F_{*} \omega R U) = \lambda^2 g_M(C_{I}T_{U}^{\phi}V^F_{*} + A_{\omega R} \nabla_{V}^{\phi}U, X) \]

for $X \in \Gamma(\mu^R)$ and $U, V \in \Gamma(\text{ker} F_\ast)$.
Proof. Given \( X \in \Gamma(\mu^R) \) and \( U, V \in \Gamma(\ker F_\ast) \), by using the proof of Theorem 3.16 and (3.4), we have

\[
g_M(\nabla_U V, X) = g_M(-C_R \nabla_U \phi_V - A_w \nabla_U \phi_R - g_M(\omega_R V, \omega_R U) \nabla(\ln \lambda), X)
\]

so that since \( g_M(\nabla_U V, X) = -g_M(V, \nabla_U X) = 0 \), it is easy to get (a) \( \iff \) (b).  \( \square \)

Denote by \( M_{\ker F_\ast} \) and \( M_{(\ker F_\ast)^\perp} \) the integral manifolds of the distributions \( \ker F_\ast \) and \( (\ker F_\ast)^\perp \), respectively.

Using Theorem 3.12 and Theorem 3.16, we have

**Theorem 3.19.** Let \( F \) be an almost \( h \)-conformal semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an almost \( h \)-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) \( M \) is locally a product Riemannian manifold \( M_{\ker F_\ast} \times M_{(\ker F_\ast)^\perp} \).

(b) \( A_X C_{1} Y + \nabla_X B_{1} Y \in \Gamma(D_1^2) \),

\[
g_N(\nabla^F_X F_{1} V, F_{1} C_{1} Y) = \lambda^2 g_M(A_X B_{1} Y - C_{1} Y(\ln \lambda) X + g_M(X, C_{1} Y) \nabla(\ln \lambda), IV)
\]

for \( X, Y \in \Gamma(\ker F_\ast)^\perp \), \( V \in \Gamma(D_1^2) \).

(c) \( A_X C_{1} Y + \nabla_X B_{1} Y \in \Gamma(D_2^2) \),

\[
g_N(\nabla^F_X F_{1} V, F_{1} C_{1} Y) = \lambda^2 g_M(A_X B_{1} Y - C_{1} Y(\ln \lambda) X + g_M(X, C_{1} Y) \nabla(\ln \lambda), JV)
\]

for \( X, Y \in \Gamma(\ker F_\ast)^\perp \) and \( V \in \Gamma(D_2^2) \).

(d) \( A_X C_{1} Y + \nabla_X B_{1} Y \in \Gamma(D_3^2) \),

\[
g_N(\nabla^F_X F_{1} V, F_{1} C_{1} Y) = \lambda^2 g_M(A_X B_{1} Y - C_{1} Y(\ln \lambda) X + g_M(X, C_{1} Y) \nabla(\ln \lambda), K V)
\]

for \( X, Y \in \Gamma(\ker F_\ast)^\perp \) and \( V \in \Gamma(D_3^2) \).

**Theorem 3.20.** Let \( F \) be an \( h \)-conformal semi-invariant submersion from a hyperkähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \( (I, J, K) \) is an \( h \)-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution \( \mathcal{D}_1 \) defines a totally geodesic foliation on \( M \).

(b) \( (\nabla F_\ast)(V, IW) \in \Gamma(F_\ast \mu^I) \),

\[
g_N((\nabla F_\ast)(V, IW), F_{1} C_{1} X) = \lambda^2 g_M(W, \nabla_{\omega_1} B_{1} X)
\]

for \( V, W \in \Gamma(\mathcal{D}_1) \) and \( X \in \Gamma((\ker F_\ast)^\perp) \).

(c) \( (\nabla F_\ast)(V, JW) \in \Gamma(F_\ast \mu^J) \),

\[
g_N((\nabla F_\ast)(V, JW), F_{1} C_{1} X) = \lambda^2 g_M(W, \nabla_{\omega_1} B_{1} X)
\]

for \( V, W \in \Gamma(\mathcal{D}_1) \) and \( X \in \Gamma((\ker F_\ast)^\perp) \).
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(d) 
\[(\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K),\]
\[g_N(\nabla F_*)(V, KW, F_*C_KX) = \lambda^2 g_M(W, \nabla \omega KB_KX)\]

for \(V, W \in \Gamma(D_1)\) and \(X \in \Gamma((\ker F_*)^\bot)\).

**Proof.** Given \(U, V \in \Gamma(D_1), W \in \Gamma(D_2)\), and \(R \in \{I, J, K\}\), we get
\[g_M(\nabla V U, W) = g_M(\nabla_V RU, RW)\]
\[= -\frac{1}{\lambda^2} g_N((\nabla F_*)(V, RU), F_*RW)\]

so that
\[g_M(\nabla V U, W) = 0 \iff (\nabla F_*)(V, RU) \in \Gamma(F_*\mu^R). \quad (3.23)\]

Given \(X \in \Gamma((\ker F_*)^\bot)\), we obtain
\[g_M(\nabla V U, X) = g_M(U, \nabla V RB_KX) + g_M(\nabla_V RU, C_RX)\]
\[= g_M(U, \nabla \omega KB_KX) - \frac{1}{\lambda^2} g_N((\nabla F_*)(V, RU), F_*C_RX)\]

so that
\[g_M(\nabla V U, X) = 0 \iff g_N((\nabla F_*)(V, RU), F_*C_RX) = \lambda^2 g_M(U, \nabla \omega KB_KX). \quad (3.24)\]

Using (3.23) and (3.24), we have (a) \(\iff\) (b), (a) \(\iff\) (c), (a) \(\iff\) (d).

Therefore, we obtain the result. \(\square\)

**Theorem 3.21.** Let \(F\) be an h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an h-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The distribution \(D_2\) defines a totally geodesic foliation on \(M\).

(b) \((\nabla F_*)(V, IW) \in \Gamma(F_*\mu^I)\),
\[-\frac{1}{\lambda^2} g_N(\nabla^F_{I, V} F_I U, F_*IC_IX) = g_M(V, B_I \nabla B_I X)\]
\[+ g_M(U, V) g_M(\nabla (\ln \lambda), IC_IX)\]

for \(U, V \in \Gamma(D_2), W \in \Gamma(D_1),\) and \(X \in \Gamma((\ker F_*)^\bot)\).

(c) \((\nabla F_*)(V, JW) \in \Gamma(F_*\mu^J)\),
\[-\frac{1}{\lambda^2} g_N(\nabla^F_{J, V} F_J U, F_*JC_JX) = g_M(V, B_J \nabla B_J X)\]
\[+ g_M(U, V) g_M(\nabla (\ln \lambda), JC_JX)\]

for \(U, V \in \Gamma(D_2), W \in \Gamma(D_1),\) and \(X \in \Gamma((\ker F_*)^\bot)\).

(d) \((\nabla F_*)(V, KW) \in \Gamma(F_*\mu^K)\),
\[-\frac{1}{\lambda^2} g_N(\nabla^F_{K, V} F_K U, F_*KC_KX) = g_M(V, B_K \nabla B_K X)\]
\[+ g_M(U, V) g_M(\nabla (\ln \lambda), KC_KX)\]

for \(U, V \in \Gamma(D_2), W \in \Gamma(D_1),\) and \(X \in \Gamma((\ker F_*)^\bot)\).

**Proof.** Given \(U, V \in \Gamma(D_2), W \in \Gamma(D_1), R \in \{I, J, K\}\), we get
\[g_M(\nabla V U, W) = \frac{1}{\lambda^2} g_N((\nabla F_*)(U, RW), F_*RW)\]

so that
\[g_M(\nabla U V, W) = 0 \iff (\nabla F_*)(U, RW) \in \Gamma(F_*\mu^R). \quad (3.25)\]
Given \( X \in \Gamma((\ker F_*)^\perp) \), by using (2.8), (2.11), (3.4), we obtain
\[
g_M(\nabla_U X) = -g_M(RV, T_U B_RX) + g_M(\nabla_R U, C_R X) - g_M(RV, T_U B_RX) + g_M(\nabla_R U, RC_R X) = g_M(V, B_R T_U B_RX) + g_M(U, V)g_M(3\n\n(\ln \lambda), RC_R X) + \frac{1}{\lambda^2}g_N(\nabla_{RV} F_* RU, F_* RC_R X)
\]
so that
\[
g_M(\nabla_U X) = 0
\]
\[
\Leftrightarrow -\frac{1}{\lambda^2}g_N(\nabla_{RV} F_* RU, F_* RC_R X) = g_M(V, B_R T_U B_RX) + g_M(U, V)g_M(3\n\n(\ln \lambda), RC_R X).
\]
Using (3.25) and (3.26), we have \((a) \Leftrightarrow (b), (a) \Leftrightarrow (c), (a) \Leftrightarrow (d)\).
Therefore, the result follows.

Using Theorem 3.20 and Theorem 3.21, we obtain

**Theorem 3.22.** Let \( F \) be an \( h \)-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an \( h \)-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The fibers of \( F \) are locally product Riemannian manifolds \( M_{D_1} \times M_{D_2} \).

(b)
\[
(\nabla F_*)(V, IW) \in \Gamma(F_* \mu^I),
g_N((\nabla F_*)(V, IW), F_* C_I X) = \lambda^2 g_M(W, T_V \omega J B_I X)
\]
for \( V, W \in \Gamma(D_1) \) and \( X \in \Gamma((\ker F_*)^\perp) \).

(c)
\[
(\nabla F_*)(V, JW) \in \Gamma(F_* \mu^J),
g_N((\nabla F_*)(V, JW), F_* C_J X) = \lambda^2 g_M(W, T_V \omega J B_J X)
\]
for \( V, W \in \Gamma(D_1) \) and \( X \in \Gamma((\ker F_*)^\perp) \).

(d)
\[
(\nabla F_*)(V, KW) \in \Gamma(F_* \mu^K),
g_N((\nabla F_*)(V, KW), F_* C_K X) = \lambda^2 g_M(W, T_V \omega K B_K X)
\]
for \( V, W \in \Gamma(D_1) \) and \( X \in \Gamma((\ker F_*)^\perp) \).
Let \( F \) be a horizontally conformal submersion from a Riemannian manifold \((M, g_M)\) onto a Riemannian manifold \((N, g_N)\) with dilation \( \lambda \).

Then the tension field \( \tau(F) \) of \( F \) is given by

\[
\tau(F) = -mF_*H + (2 - n)F_*\langle \nabla(\ln \lambda) \rangle,
\]

(3.27)

where \( H \) is the mean curvature vector field of the distribution \( \ker F_* \), \( m = \dim \ker F_* \), \( n = \dim N \).

Using Lemma 3.23, we easily get

**Corollary 3.24.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Assume that \( F \) is harmonic with \( \dim \ker F_* > 0 \) and \( \dim N > 2 \). Then the following conditions are equivalent:

(a) All the fibers of \( F \) are minimal.

(b) The map \( F \) is horizontally homothetic.

**Corollary 3.25.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Assume that \( \dim \ker F_* > 0 \) and \( \dim N = 2 \). Then the following conditions are equivalent:

(a) All the fibers of \( F \) are minimal.

(b) The map \( F \) is harmonic.

We introduce another notion and investigate the condition for such a map to be totally geodesic.

**Definition 3.26.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Then given \( R \in \{I, J, K\} \), we call the map \( F \) a \((R\mathbb{D}_2^R, \mu^R)\)-totally geodesic map if \( (\nabla F_*)(RU, X) = 0 \) for \( U \in \Gamma(\mathbb{D}_2^R) \) and \( X \in \Gamma(\mu^R) \).

**Theorem 3.27.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyperkähler manifold \((M, I, J, K, g_M)\) onto a Riemannian manifold \((N, g_N)\) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent:

(a) The map \( F \) is horizontally homothetic.

(b) The map \( F \) is a \((I\mathbb{D}_2^I, \mu^I)\)-totally geodesic map.

(c) The map \( F \) is a \((J\mathbb{D}_2^J, \mu^J)\)-totally geodesic map.

(d) The map \( F \) is a \((K\mathbb{D}_2^K, \mu^K)\)-totally geodesic map.

**Proof.** Given \( U \in \Gamma(\mathbb{D}_2^R) \), \( X \in \Gamma(\mu^R) \), and \( R \in \{I, J, K\} \), we have

\[
(\nabla F_*)(RU, X) = RU(\ln \lambda)F_*X + X(\ln \lambda)F_*RU - g_M(RU, X)F_*\nabla(\ln \lambda)
\]

(3.28)

so that we easily get \((a) \Rightarrow (b), (a) \Rightarrow (c), (a) \Rightarrow (d)\).
Conversely, from (3.28), we obtain
\[ RU(\ln \lambda)F_*X + X(\ln \lambda)F_*RU = 0. \]
Since \( \{F_*X, F_*RU\} \) is linearly independent for nonzero \( X, U \), we have \( RU(\ln \lambda) = 0 \) and \( X(\ln \lambda) = 0 \), which means \((a) \iff (b), (a) \iff (c), (a) \iff (d)\).
Therefore, the result follows.

\[ \square \]

**Theorem 3.28.** Let \( F \) be an almost h-conformal semi-invariant submersion from a hyper-kähler manifold \( (M, I, J, K, g_M) \) onto a Riemannian manifold \( (N, g_N) \) such that \((I, J, K)\) is an almost h-conformal semi-invariant basis. Then the following conditions are equivalent: Then the following conditions are equivalent:

(a) The map \( F \) is totally geodesic.
(b) \((i)\) \( C^*T_*UV + \omega J^*V^*U^* = 0 \) for \( U, V \in \Gamma(D^1_2) \).
(ii) \( C^*T_*UV + \omega J^*V^*U^* = 0 \) for \( U \in \Gamma(\ker F_*) \) and \( W \in \Gamma(D^2_2) \).
(iii) The map \( F \) is horizontally homothetic.
(iv) \( T_*B^1X + \mathcal{H}V^*C^1X \in \Gamma(D^1_1) \) and \( J^*V^*B^1X + T_*C^1X \in \Gamma(D^1_1) \) for \( U \in \Gamma(\ker F_*) \) and \( X \in \Gamma((\ker F_*)^1) \).
(c) \((i)\) \( C^*T_*UV + \omega K^*V^*U^* = 0 \) for \( U, V \in \Gamma(D^1_2) \).
(ii) \( C^*T_*UV + \omega K^*V^*U^* = 0 \) for \( U \in \Gamma(\ker F_*) \) and \( W \in \Gamma(D^2_2) \).
(iii) The map \( F \) is horizontally homothetic.
(iv) \( T_*B^1X + \mathcal{H}V^*C^1X \in \Gamma(D^1_1) \) and \( J^*V^*B^1X + T_*C^1X \in \Gamma(D^1_1) \) for \( U \in \Gamma(\ker F_*) \) and \( X \in \Gamma((\ker F_*)^1) \).

**Proof.** Given \( U, V \in \Gamma(D^1_2) \) and \( R \in \{I, J, K\} \), we have
\[
(\nabla F_*)(U, V) = F_* (R(T_*UW + J^*V^*U^*))
= F_* (B^R_TU^*R + C^R_TU^*R + \phi^R_J^*V^*U^*R + \omega^R_J^*V^*U^*)
= F_* (C^R_TU^*R + \omega^R_J^*V^*U^*)
\]
so that
\[
(\nabla F_*)(U, V) = 0 \iff C^R_TU^*R + \omega^R_J^*V^*U^* = 0. \tag{3.29}
\]
Given \( U \in \Gamma(\ker F_*) \) and \( W \in \Gamma(D^2_2) \), we get
\[
(\nabla F_*)(U, W) = F_* (R(\nabla U^*RW))
= F_* (R(T_*U^*RW + J^*V^*U^*RW))
= F_* (C^R_TU^*RW + \omega^R_J^*V^*U^*RW)
\]
so that
\[
(\nabla F_*)(U, W) = 0 \iff C^R_TU^*RW + \omega^R_J^*V^*U^*RW = 0. \tag{3.30}
\]
We claim that
\[
(\nabla F_*)(X, Y) = 0 \text{ for } X, Y \in \Gamma((\ker F_*)^1) \iff F \text{ is horizontally homothetic.} \tag{3.31}
\]
Given \( X, Y \in \Gamma((\ker F_*)^1) \), by (2.11), we obtain
\[
(\nabla F_*)(X, Y) = X(\ln \lambda)F_*Y + Y(\ln \lambda)F_*X - g_M(X, Y)F_*\nabla(\ln \lambda) \tag{3.32}
\]
so that the part from right to left immediately follows.

Conversely, we have
\[
0 = X(\ln \lambda)F_*Y + Y(\ln \lambda)F_*X - g_M(X, Y)F_*\nabla(\ln \lambda). \tag{3.33}
\]
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Applying $Y = RX$, $X \in \Gamma(\mu^R)$, at (3.33), we get

$$0 = X(\ln \lambda)F_*RX + RX(\ln \lambda)F_*X - g_M(X, RX)F_*\nabla(\ln \lambda)$$

so that since $\{F_*RX, F_*X\}$ is linearly independent for nonzero $X$, we obtain $X(\ln \lambda) = 0$ and $RX(\ln \lambda) = 0$, which implies

$$X(\lambda) = 0 \quad \text{for} \quad X \in \Gamma(\mu^R).$$

(3.34)

Applying $X = Y = RU$, $U \in \Gamma(D^B_2)$, at (3.33), we obtain

$$0 = 2RU(\ln \lambda)F_*RU - g_M(RU, RU)F_*\nabla(\ln \lambda).$$

(3.35)

Taking inner product with $F_*RU$ at (3.35), we have

$$0 = 2g_M(RU, \nabla(\ln \lambda))g_N(F_*RU, F_*RU) - g_M(RU, RU)g_N(F_*\nabla(\ln \lambda), F_*RU)$$

$$= \lambda g_M(RU, RU)g_M(RU, \nabla(\ln \lambda)),$$

which implies

$$RU(\lambda) = 0 \quad \text{for} \quad U \in \Gamma(D^B_2).$$

(3.36)

By (3.34) and (3.36), we get the part from left to right.

Given $U \in \Gamma(\ker F_\ast)$ and $X \in \Gamma((\ker F_\ast)^\perp)$, we obtain

$$(\nabla F_\ast)(U, X) = F_\ast(R(\nabla U RX))$$

$$= F_\ast(R(\nabla U BX + \nabla BX) + R(\nabla C R X + \nabla C R X))$$

$$= F_\ast(C_R(\nabla U BX + \nabla C R X) + \omega_R(\nabla BX) + \nabla C R X))$$

so that

$$(\nabla F_\ast)(U, X) = 0 \iff \nabla U BX + \nabla C R X \in \Gamma(RD^B_2)$$

(3.37)

By (3.29), (3.30), (3.31), (3.37), we have $(a) \iff (b)$, $(a) \iff (c)$, $(a) \iff (d)$.

Therefore, we get the result. \qed

Let $F : (M, g_M) \to (N, g_N)$ be a horizontally conformal submersion. The map $F$ is called a horizontally conformal submersion with totally umbilical fibers if

$$\nabla_X Y = g_M(X, Y)H \quad \text{for} \quad X, Y \in \Gamma(\ker F_\ast),$$

(3.38)

where $H$ is the mean curvature vector field of the distribution $\ker F_\ast$.

**Lemma 3.29.** Let $F$ be an almost h-conformal semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an almost h-conformal semi-invariant basis. Then

$$H \in \Gamma(RD^B_2) \quad \text{for} \quad R \in \{I, J, K\}.$$  

(3.39)

**Proof.** Given $X, Y \in \Gamma(D^B_2)$, $W \in \Gamma(\mu^R)$, and $R \in \{I, J, K\}$, we have

$$\nabla_X RY + \nabla_X RY = \nabla_X RY = R\nabla_X Y$$

$$= B_R\nabla_X Y + C_R\nabla_X Y + \phi_R\nabla_X Y + \omega_R\nabla_X Y$$

so that

$$g_M(\nabla_X RY, W) = g_M(C_R\nabla_X Y, W) = -g_M(\nabla_X Y, RW).$$

Using (3.38), we obtain

$$g_M(X, RY)g_M(H, W) = -g_M(X, Y)g_M(H, RW).$$
Interchanging the role of $X$ and $Y$, we get
\[ g_M(Y, RX)g_M(H, W) = -g_M(Y, X)g_M(H, RW). \]
Combining the above two equations, we have
\[ g_M(X, Y)g_M(H, RW) = 0, \]
which implies $H \in \Gamma(RD_2^R)$ (since $R \mu^R = \mu^R$).

**Theorem 3.30.** Let $F$ be an h-conformal semi-invariant submersion with totally umbilical fibers from a hyperkähler manifold $(M, I, J, K, g_M)$ onto a Riemannian manifold $(N, g_N)$ such that $(I, J, K)$ is an h-conformal semi-invariant basis. Then all the fibers of $F$ are totally geodesic.

**Proof.** By Lemma 3.29, we have
\[ H \in \Gamma(RD_2) \quad \text{for} \quad R \in \{I, J, K\} \]
so that
\[ \{IH, JH, KH\} \subset \Gamma(D_2). \]
But
\[ KH = IJH = I(JH) \in \Gamma(D_2) \quad \text{with} \quad JH \in \Gamma(D_2). \]
Since $ID_2 \subset (\ker F_\ast)^\perp$, we must have $H = 0$. By (3.38), we obtain the result. \qed

4. Examples

Note that given a Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \ldots, x_{4m})$, we can canonically choose complex structures $I, J, K$ on $\mathbb{R}^{4m}$ as follows:
\[
I(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+2}}, \quad I(\frac{\partial}{\partial x_{4k+2}}) = -\frac{\partial}{\partial x_{4k+3}}, \quad I(\frac{\partial}{\partial x_{4k+3}}) = \frac{\partial}{\partial x_{4k+4}}, \quad I(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+1}},
\]
\[
J(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+3}}, \quad J(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+2}}, \quad J(\frac{\partial}{\partial x_{4k+2}}) = \frac{\partial}{\partial x_{4k+4}}, \quad J(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+1}},
\]
\[
K(\frac{\partial}{\partial x_{4k+1}}) = \frac{\partial}{\partial x_{4k+4}}, \quad K(\frac{\partial}{\partial x_{4k+4}}) = -\frac{\partial}{\partial x_{4k+2}}, \quad K(\frac{\partial}{\partial x_{4k+2}}) = \frac{\partial}{\partial x_{4k+3}}, \quad K(\frac{\partial}{\partial x_{4k+3}}) = -\frac{\partial}{\partial x_{4k+1}},
\]
for $k \in \{0, 1, \ldots, m - 1\}$.

Then we easily check that $(I, J, K, \{\ , \})$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $(\ , \)$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

**Example 4.1.** Let $(M, E, g)$ be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection [15]. Then the map $\pi$ is an h-conformal semi-invariant submersion such that $D_1 = \ker \pi_\ast$ and dilation $\lambda = 1$.

**Example 4.2.** Let $(M, E_M, g_M)$ and $(N, E_N, g_N)$ be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion [15]. Then the map $F$ is an h-conformal semi-invariant submersion such that $D_1 = \ker F_\ast$ and dilation $\lambda = 1$.

**Example 4.3.** Let $(M, E, g_M)$ be an almost quaternionic Hermitian manifold and $(N, g_N)$ a Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an h-semi-invariant submersion [21]. Then the map $F$ is an h-conformal semi-invariant submersion with dilation $\lambda = 1$.

**Example 4.4.** Let $(M, E, g_M)$ be an almost quaternionic Hermitian manifold and $(N, g_N)$ a Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-semi-invariant submersion [21]. Then the map $F$ is an almost h-conformal semi-invariant submersion with dilation $\lambda = 1$.

**Example 4.5.** Let $(M, E, g_M)$ be a $4n$-dimensional almost quaternionic Hermitian manifold and $(N, g_N)$ a $(4n - 1)$-dimensional Riemannian manifold. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation $\lambda$. Then the map $F$ is an h-conformal semi-invariant submersion such that $D_2 = \ker F_\ast$ and dilation $\lambda$. 

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Example 4.6. Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be a horizontally conformal submersion with dilation $\lambda$. Then the map $F$ is an $h$-conformal semi-invariant submersion such that $D_2 = \ker F_x$ and dilation $\lambda$.

Example 4.7. Define a map $F : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x_1, \cdots, x_4) = e^{1934}(x_1, x_2).$$

Then the map $F$ is an almost $h$-conformal anti-holomorphic semi-invariant submersion such that $I(\ker F_x) = \ker F_x$, $J(\ker F_x) = (\ker F_x)^\perp$, $K(\ker F_x) = (\ker F_x)^\perp$, and dilation $\lambda = e^{1934}$.

Here, $(K, I, J)$ is an almost $h$-conformal anti-holomorphic semi-invariant basis.

Example 4.8. Define a map $F : \mathbb{R}^8 \to \mathbb{R}^6$ by

$$F(x_1, \cdots, x_8) = \pi^{1934}(x_3, \cdots, x_8).$$

Then the map $F$ is an almost $h$-conformal semi-invariant submersion such that $I(\ker F_x) = \ker F_x$, $J(\ker F_x) \subset (\ker F_x)^\perp$, $K(\ker F_x) \subset (\ker F_x)^\perp$, and dilation $\lambda = \pi^{1934}$.

Example 4.9. Define a map $F : \mathbb{R}^8 \to \mathbb{R}^4$ by

$$F(x_1, \cdots, x_8) = e^{1968}(x_1, x_2, x_5, x_7).$$

Then the map $F$ is an almost $h$-conformal semi-invariant submersion such that $\mathcal{D}_1 = \mathcal{D}_2 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle$, $\mathcal{D}_1 = \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} \rangle$, $K(\ker F_x) = (\ker F_x)^\perp$, and dilation $\lambda = \pi^{1968}$.

Example 4.10. Define a map $F : \mathbb{R}^8 \to \mathbb{R}^3$ by

$$F(x_1, \cdots, x_8) = \pi^{1978}(x_6, x_7, x_8).$$

Then the map $F$ is a $h$-conformal semi-invariant submersion such that $\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_4} \rangle$, $\mathcal{D}_2 = \langle \frac{\partial}{\partial x_5} \rangle$, and dilation $\lambda = \pi^{1978}$.

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