About long-range interaction of spheroidal solitons in scalar field nonlinear model

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Abstract. The nonlinear scalar field model of space-time film (Born – Infeld type nonlinear scalar field model) is considered. Its spherically symmetrical solution is obtained. This solution gives the class of moving solitary solutions or solitons with the Lorentz transformation. We consider the distant interaction between such spheroidal solitons or spherons. This interaction is caused by the nonlinearity of the model. Starting from the static configuration with two spherons we show that the interaction under investigation is similar to electromagnetic one.

1. Introduction
We consider the nonlinear scalar field model that can be called the extremal space-time film one [1, 2]. The investigation of this field model is interesting from the various point of view.

The variational formulation of this field model is similar to the appropriate formulation for Born – Infeld nonlinear electrodynamics [3, 4]. What is more, the scalar nonlinear equation under consideration has the same spherically symmetrical solution that we have in Born – Infeld electrodynamics for zero component of the electromagnetic potential and which is sometimes called the Born’s electron.

The Lorentz transformations give the appropriate moving solitary solutions or solitons. Such solution will be called spheroidal soliton or spheron.

As it is known the nonlinearity of the field model account for the interaction between the solitons. The appropriate methods for investigation for BI electrodynamics [5, 4] can be applied to the scalar field model under consideration.

Here we show that the interaction between the spherons in the scalar field model under consideration looks like the electromagnetic one.

2. Extremal space-time film and energy-momentum conservation law
The equation of the model under consideration has the following form in orthogonal coordinates

\[ \frac{\partial \Upsilon^\mu}{\partial x^\mu} = 0, \quad \Upsilon^\mu = \frac{\Phi^\mu}{\mathcal{E}}, \quad \Phi_\nu = \frac{\partial \Phi}{\partial x^\nu}, \quad \mathcal{E} = \sqrt{1 + \chi^2 m^{\mu\nu} \Phi_\mu \Phi_\nu}, \] (1)
where \( \Phi \) is the scalar real field function, \( \chi \) is the dimensional constant, \( m_{\mu \nu} \) are the components of the Minkowski metric. The Greek indices take values \{0, 1, 2, 3\}.

Customary method gives the following differential conservation law for energy-momentum tensor in the area outside of singularities:

\[
\frac{\partial \mathcal{T}^{\mu \nu}}{\partial x^\mu} = 0, \quad \mathcal{T}^{\mu \nu} = \frac{1}{4\pi} \left( \Phi^{\mu} \Phi^{\nu} - \frac{m^{\mu \nu}}{\chi^2} \left( \mathcal{L} - 1 \right) \right),
\]  

(2)

Let us consider the integral from the energy-momentum differential conservation law (2) over the four-volume \( \mathcal{V} \) including the three-volume \( \mathcal{V} \) and the time interval \([x^0 - \Delta x^0/2, x^0 + \Delta x^0/2]\). Integration by parts in this integral gives the appropriate integral conservation law

\[
\int \frac{\partial \mathcal{T}^{\mu \nu}}{\partial x^\nu} dx^0 dx^1 dx^2 dx^3 = 0 \quad \Rightarrow \quad \int_\Sigma \mathcal{T}^{\mu \nu} d\Sigma_{\nu} = 0, \tag{3}
\]

where \( \Sigma \) is the three-dimensional boundary hypersurface for the four-volume \( \mathcal{V} \), \( d\Sigma_{\nu} \) are components of the outer hypersurface element four-vector such that \( d\Sigma_0 = \pm dx^1 dx^2 dx^3 \), \( d\Sigma_1 = \pm dx^2 dx^3 dx^0 \), \( d\Sigma_2 = \pm dx^3 dx^1 dx^0 \), \( d\Sigma_3 = \pm dx^1 dx^2 dx^0 \).

Let us divide the three-dimensional closed hypersurface \( \Sigma \) into two unclosed hypersurfaces in some coordinate system such that \( \Sigma = \Sigma^F \cup \Sigma^P \):

\[
\Sigma = \mathcal{V} \bigg|_{x^0 = x^0 - \Delta x^0/2} \cup \mathcal{V} \bigg|_{x^0 = x^0 + \Delta x^0/2}, \quad \Sigma^P = \sigma \cup \left[ x^0 - \Delta x^0/2, x^0 + \Delta x^0/2 \right]. \tag{4}
\]

where \( \sigma \) is the closed two-surface bounding the three-volume \( \mathcal{V} \), \( \Sigma^P \) will be called the momentum hypersurface and \( \Sigma^F \) will be called the force hypersurface. The three-dimensional sections for these hypersurfaces are shown on Fig. 1.

\[ \text{Figure 1. Section for } \Sigma^P \text{ in three-dimensional space } \{x^0, x^1, x^2\} \text{ (a) and section for } \Sigma^F \text{ in three-dimensional space } \{x^1, x^2, x^3\} \text{ (b).} \]

Thus we have the obtained integral conservation law in the following integral force dynamic law form:

\[
\Delta F^P_\chi = \Delta F^F_\chi, \quad \Delta F^P_\chi = \mathcal{F}^{\mu}_P \bigg|_{x^0 = x^0 + \Delta x^0/2} - \mathcal{F}^{\mu}_P \bigg|_{x^0 = x^0 - \Delta x^0/2}, \quad \Delta F^P_\sigma = \int_{x^0 - \Delta x^0/2}^{x^0 + \Delta x^0/2} F^{\mu}_\sigma dx^0, \tag{5}
\]
where \( P^\mu_{\nabla} \) is the momentum of field into the three-dimensional volume \( \nabla \), \( d^{\mu}_{\sigma} \) is the integral force on the two-dimensional surface \( \sigma \). The momentum and the force are defined as follows:

\[
P^\mu_{\nabla} \triangleq \frac{1}{\sqrt{\epsilon}} \int_\nabla F^{\mu 0} \, d\nabla, \quad P^\mu_{\sigma} \triangleq - \int_\sigma F^{\mu i} \, d\sigma_i . \tag{6}
\]

where \( d\nabla = \pm dx^1 dx^2 dx^3 \), \( \{d\sigma_i\} = \{\pm dx^2 dx^3, \pm dx^1 dx^3, \pm dx^1 dx^2\} \) are the outer normal components of the two-dimensional surface \( \sigma \) bounded the three-dimensional volume \( \nabla \).

3. Spheroidal soliton

In the case when the field function depends on the radial coordinate \( r \) only we have the following simplest solution

\[
\frac{\partial \Phi}{\partial r} = - \frac{q}{\sqrt{\varepsilon^4}} \pm \frac{q}{r}, \quad \bar{r} \triangleq \sqrt{|q\chi|} \quad \Rightarrow \quad \Phi = \frac{q}{r} F^j_{\bar{r} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}} \left( + \bar{r}^4 \right) , \tag{7}
\]

where \( F_{\bar{r}j;\bar{r}j}(z) \) is hypergeometric function, top and bottom sings are correspond to different signatures of metric in the model action: \( \{+, -, -, -\} \) (top) and \( \{-, +, +, +\} \) (bottom).

The obtained solution (7) give birth to the class of the moving soliton solutions with the Lorentz transformations:

\[
x^\mu = L^0_\mu \, x^\nu, \quad x^\mu = L^i_\mu \, x^\nu, \quad L^\mu_\nu L^\nu_\rho = \delta^\mu_\rho . \tag{8}
\]

Here \( \{x^\mu\} \) is the intrinsic coordinate system of soliton, thus we make the substitution \( r \rightarrow \bar{r} = \sqrt{x^i x^i} \) in (7). The components \( L^\mu_\nu \) (the point designates that the index is second) are

\[
L^0_0 = \sqrt{1 - \varepsilon^2}, \quad L^0_i = \sqrt{1 - \varepsilon^2}, \quad L^i_0 = -\varepsilon^i + \frac{\varepsilon^i \varepsilon^j}{1 + \varepsilon^2} \quad \Rightarrow \quad \left| m_{\mu \nu} \varepsilon^\mu \varepsilon^\nu \right| = 1 , \tag{9}
\]

where \( \varepsilon^i \) are the components of three-dimensional velocity of the soliton, \( \varepsilon^\mu \) are the components of its four-dimensional velocity.

Energy-momentum four-vector components for spheroidal soliton are

\[
P^\mu = L^\mu_0 \, \varepsilon = \varepsilon^\mu \, m , \tag{10}
\]

where \( m \) the energy of the soliton in its intrinsic coordinate system. Here it should be noted that the transformation (10) is true if only we consider the moving tree-volume \( \nabla \) in definition (6) and transformation of its element as the zero component of four-vector \( d\Sigma_0 \).

The spheroidal soliton under consideration will be called spheron for short.
4. Interacting spherons and electromagnetism

Let us consider the sum of the moving spherons which are distant from each other as the first approximation to an unknown multisoliton solution. But we assume that the velocities of these solitons can depend on time. To investigate the trajectories of the spherons in this first approximation we will use the integral conservation law for momentum (5). This method was used for the similar problem in nonlinear electrodynamics [5, 4].

The calculation gives that the interaction of two spherons at rest fully conforms with the electrostatic interaction of two charged point particles. Then we can transform this static configuration to a moving reference frame with the Lorentz transformations (8). The appropriate transformation for acceleration or force is known [6, 7]. Thus for this case we obtain also the magnetic component of the force in the moving reference frame.

Let us consider the internal tensor structure of the four-vector of integral force $\Delta F^\mu_\sigma$ (5).

Assuming that the time interval is vanishingly small ($\Delta x^0 \to 0$) and designating the symbols related to intrinsic coordinate system of the two spherons with point under letter we can write

$$d^n F^\mu_\nu = n \sigma \delta^\mu_\nu , \quad d^n F^0_0 = 0 .$$

(11)

Here we must consider $dx^0$ as zero component for four-vector with component of trajectory differentials:

$$dx^\mu = \{dx^0, 0, 0, 0\} \implies dx^\mu = \{dx^0, dx^1, dx^2, dx^3\} .$$

(12)

Thus to obtain the right tensor law of motion we must consider the electrostatic force components $\vec{n} F^\mu_\nu$ as components of a second-rank tensor such that

$$d^n F^\mu_\nu = q n F^\mu_\nu d^\sigma d^\nu , \quad n F^\mu_\nu = 0 , \quad q n F^\mu_\nu = n F^\mu_\nu \implies m d^n \sigma = q n F^\mu_\nu \sigma \nu \sigma d^n d^n ,$$

(13)

where $d^n d^n = dx^0 dx^0 - d^n d^n$ is invariant parameter of trajectory for the $n$-th spheron.

Thus we have obtained that the dynamical equation for interacting spherons looks like the equation of motion for point charged particle in electromagnetic field. But here we based on the static configuration of spherons where the electrostatic interaction was obtained. In a conventional manner for electrostatic interaction we can introduce the appropriate electrostatic potential satisfying the Laplace’s equation with a charge density as the source. The appropriate relativistic generalization gives the four-vector electromagnetic potential components satisfying the wave equations with the appropriate four-vector current density as the source. It is obvious that the rotating configuration with the spherons has the angular momentum or spin the dynamics of which is investigated with the help of the appropriate integral conservation law.

5. Conclusions

Thus we have the similarity between the interacting spherons of the scalar field and the classical electrodynamics of the point charged particles. But because the starting configuration of spherons is statical we must keep in reserve the investigation of additional effects for the interacting spherons with large relative velocities.

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