Stochastic Differential Equations Driven by Fractional Brownian Motion and Poisson Point Process

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Abstract. In this paper we study a class of stochastic differential equations with additive noise that contains a fractional Brownian motion and a Poisson point process of class (QL). The differential equation of this kind is motivated by the reserve processes in a general insurance model, in which the long term dependence between the claim payment and the past history of liability becomes the main focus. We establish some new fractional calculus on the fractional Wiener-Poisson space, from which we define the weak solution of the SDE and prove its existence and uniqueness. Using a extended form of Krylov-type estimate for the combined noise of fBM and compound Poisson, we prove the existence of the strong solution, along the lines of Gyöngy and Pardoux [12]. Our result in particular extends a recent work of Mishura-Nualart [17].

Keywords: fractional Brownian motion, Poisson point process, fractional Wiener-Poisson space, stochastic differential equations, discontinuous fractional calculus, Krylov estimates.

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1 Introduction

In this paper we are interested in the following stochastic differential equation:

$$X_t = x + \int_0^t b(s, X_s)ds + \sigma B_t^H - L_t, \quad t \in [0, T],$$

(1.1)

where $B^H = \{B_t^H : t \geq 0\}$ is a fractional Brownian motion (fBM) with Hurst parameter $H \in (0, 1)$, defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, with $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ being a filtration that satisfies the usual hypotheses (cf. e.g., [23]); and $L = \{L_t : t \geq 0\}$ is a Poisson point process of class (QL), independent of $B^H$. More precisely, we assume that $L$ takes the form

$$L_t = \int_0^t \int_{\mathbb{R}} f(s, x) N_p(dsdx), \quad t \geq 0,$$

(1.2)

where $f$ is a deterministic function, and $p$ is a stationary Poisson point process whose counting measure $N_p$ is a Poisson random measure with Lévy measure $\nu$.

One of the motivations for our study is to consider a general reserve process of an insurance company, perturbed by an additive noise that has long term dependency. A commonly seen perturbed reserve (or surplus) model is of the following form:

$$U_t = x + c(1 + \rho)t + \varepsilon W_t - L_t, \quad t \in [0, T].$$

(1.3)

Here $x \geq 0$ denotes the initial surplus, $T > 0$ is a fixed time horizon, and $L_t$ denotes cumulated claims up to time $t$, $c > 0$ is the premium rate, and $\rho > 0$ is the “safety” (or expense) loading; and finally, $W = \{W_t : t \geq 0\}$ is a Brownian motion, which represents an additional uncertainty coming from either the aggregate claims or the premium income, and $\varepsilon > 0$ is the perturbation parameter. We refer to the well-referred book [24, Chapter 13] and the references cited therein for more explanations of such models.

In this paper we are particularly interested in the case where the diffusion perturbation term possesses long-range dependency. Such a phenomenon has been noted in insurance models based on the observations that the claims often display long memories due to extreme weather, natural disasters, and also noted in casualty insurance such as automobile third-party liability (cf. e.g., [6, 9, 10, 11, 14, 18, 20] and references cited therein). A reasonable refinement that reflects the long memory but also retain the original features of the aggregated claims is to assume that the Brownian motion $W$ in (1.3) is replaced by a fractional Brownian motion $B^H$, for a certain Hurst parameter $H \in (0, 1)$. In fact, if we assume further that in addition to the premium income, the company also receives interest of its reserves at time with interest rate $r > 0$, and that the safety loading $\rho$ also depends on the current reserve value, one can argue that the reserve process $X$
should satisfy an SDE of the form of (1.1) with
\[ b(t, x) = r x + c(1 + \rho(t, x)), \quad (t, x) \in [0, T] \times \mathbb{R}. \]

The main purpose of this paper is to find the minimum conditions on the function \( b \) under which the SDE (1.1) is well posed, in both weak and strong sense. In the case when \( L \equiv 0 \), the SDE (1.1) becomes one driven by an (additive) fBM and the similar issues was investigated by Nualart-Ouknine [22] and Hu-Nualart-Song [13]. One of the main results is that, unlike the ordinary differential equation case, the well-posedness of the SDE can be established under only some integrability conditions, and in particular, no Lipschitz continuity is required for uniqueness. The main idea is to use a Krylov-type estimate to obtain a comparison theorem, whence the pathwise uniqueness. Such a scheme was utilized by Gyöngy-Pardoux [12] when studying the quasi-linear SPDEs, and has been a frequently used tool to treat the SDEs with non-Lipschitz coefficients, as an alternative to the well known Yamada-Watanabe Theorem. In fact, this method is even more crucial in the current case, as the usual Yamada-Watanabe Theorem type of argument does not seem to work due to the lack of independent increment property of an fBM.

The main difficulty in the study of SDE (1.1), however, is the presence of the jumps. In the case when \( H > 1/2 \), Mishura and Nualart [17] studied the existence of weak solution of SDE (1.1) with \( L \equiv 0 \), but the coefficient \( b \) has finitely many discontinuous points, but otherwise Hölder continuous. Our result, among other things, in a sense extends their result to a more general case with countably many discontinuities. More importantly, we remove the extra assumption that \( H < (1+\sqrt{5})/4 \) in [17] when the number of jumps is finite. To our best knowledge, the fractional calculus applying to SDE driven by both fBM and Poisson point process is new.

The rest of the paper is organized as follows. In section 2 we review briefly the basics on fBM and some fractional calculus that is needed in this paper. In section 3 we prove a Girsanov theorem and in section 4 we apply it to study the existence of the weak solution. In section 5 we address the uniqueness issue, in both weak and strong forms, and in section 6 we study the existence of the strong solution.

## 2 Preliminaries

In this section we review some of the basic concepts in fractional calculus and introduce the notion of (canonical) fractional Wiener-Poisson spaces which will be the basis of our study. Throughout this paper we denote \( \mathbf{E} \) (also \( \mathbf{E}_1, \cdots \)) for a generic Euclidean space, whose inner products and norms will be denoted as the same ones \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \), respectively; and denote \( \| \cdot \| \) to be the norm of a generic Banach space. Let \( \mathcal{U} \subset \mathbf{E} \) be a bounded measurable subset, we shall denote
Let \( L^p(\mathcal{U}; \mathbb{E}_1) \), \( 0 \leq p < \infty \) to be the space of all \( \mathbb{E}_1 \)-valued measurable function \( \phi(\cdot) \) defined on \( \mathbb{E} \) such that \( \int_\mathcal{U} |\phi(t)|^p dt < \infty \); and \( \mathcal{C}^n(\mathcal{U}; \mathbb{E}_1) \), \( n \geq 0 \), denotes all the \( \mathbb{E}_1 \)-valued, \( n \)-th continuously differentiable functions on \( \mathcal{U} \), with the usual sup-norm.

1. Fractional calculus

We begin by a brief review of the deterministic fractional calculus. We refer to the book Samko-Kilbas-Mariachev [25] for an exhaustive survey on the subject. We first recall some basic definitions.

Let \(-\infty < a < b < \infty\), and \( \varphi \in L^1([a, b]) \). The integrals

\[
(I^\alpha_a \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \quad x > a, 
\]

\[
(I^\alpha_b \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}} dt, \quad x < b, 
\]

where \( \Gamma(\cdot) \) is the Gamma-function and \( \alpha > 0 \), are called \textit{fractional integrals of order} \( \alpha \). Both \( I^\alpha_a \) and \( I^\alpha_b \) are the so-called Riemann-Liouville fractional integrals, and they are often called “left” and “right” fractional integrals, respectively. We shall denote the image of \( L^p([a, b]) \) under the fractional integration operator \( I^\alpha_{a,+} \) (resp. \( I^\alpha_{b,-} \)) by \( L^p(\mathcal{U}; \mathbb{E}_1) \) defined on \( \mathcal{U} \), with the usual sup-norm.

The (Riemann-Liouville) fractional derivatives are defined, naturally, as the inverse operator of the fractional integration. To wit, for any function \( f \in L^0([a, b]) \), we define

\[
(D^\alpha_{a,+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt; 
\]

\[
(D^\alpha_{b,-} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt. 
\]

We call \( D^\alpha_{a,+} f \) (resp. \( D^\alpha_{b,-} f \)) the \textit{left} (resp. \textit{right}) fractional derivative of order \( \alpha \), \( 0 < \alpha < 1 \). We note that if \( f(t) \in \mathcal{C}^1([a, b]) \), then it is easy to verify that (see, [25, p224])

\[
D^\alpha_{a,+} f = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x) - f(t)}{(x-t)^{1+\alpha}} dt \triangleq D^\alpha_{a,+} f. 
\]

The derivative \( D^\alpha_{a,+} f \) is called Marchaud fractional derivative. We should note that the right-hand side of (2.6) is not only well-defined for differentiable functions, but for example, for function \( f(x) \) that is Hölder-\( \beta \) continuous, with \( \beta > \alpha \). For more general functions, the fractional Marchaud derivative (2.6) should be understood as (cf. [25])

\[
D^\alpha_{a,+} f \triangleq \lim_{\varepsilon \to 0} D^\alpha_{a,+} f. 
\]
where the limit is in the space $L^p$, and

$$
[D_{a+}^\alpha f](x) \triangleq \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha)} \frac{f(x)}{(x-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(s) - f(t)}{(x-t)^{1+\alpha}} dt. \tag{2.8}
$$

We collect some of the important properties of the fractional integral and derivative in the following theorem. The proofs can be found in [25].

**Theorem 2.1**

(i) For any $\varphi \in L^1([a,b])$ and $0 < \alpha < 1$, it holds that

$$
D_{a+}^\alpha I_{a+}^\alpha \varphi = \lim_{\varepsilon \to 0} D_{a+}^\alpha I_{a+}^\alpha \varphi = D_{a+}^\alpha I_{a+}^\alpha \varphi = \varphi. \tag{2.9}
$$

(ii) For any $f \in I_{a+}^\alpha (L^1([a,b]))$ and $\alpha > 0$, it holds that

$$
I_{a+}^\alpha D_{a+}^\alpha f = I_{a+}^\alpha D_{a+}^\alpha f = f. \tag{2.10}
$$

(iii) Let $\psi \in L^p([0,b]), b > 0, 1 < p < \infty$. Then $\psi$ has the representation $\psi(x) = I_{0+}^\alpha x^\mu f(x)$, a.e. $x \in [0,b]$, for some $f \in L^p([0,b]), \alpha > 0$, and $p(1+\mu) > 1$ if and only if $\psi$ takes one of the following two forms:

(a) $\psi(x) = x^\mu [I_{0+}^\alpha g](x)$, a.e. $x \in [0,b], g \in L^p([0,b])$;

(b) $\psi(x) = x^{\mu-\varepsilon} [I_{0+}^\alpha x^\varepsilon g_1](x)$, a.e. $x \in [0,b], g_1 \in L^2([0,b]), p(1+\varepsilon) > 1$. 

\[ \blacksquare \]

2. **Fractional Wiener-Poisson space**

We recall that a stochastic process $B^H = \{B^H_t, t \in [0,T]\}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0})$, is called an $\mathbb{F}$-fractional Brownian motion (fBM for short) with Hurst parameter $H \in (0,1)$ if

(i) $B^H$ is a Gaussian process with continuous paths and $B^H_0 = 0$;

(ii) for each $t \geq 0$, $B^H$ is an $\mathbb{F}$-adapted process satisfying $\mathbb{E}B^H_t = 0, \forall t \geq 0$;

(iii) for all $s, t \geq 0$, it holds that

$$
\mathbb{E}(B^H_t B^H_s) = R_H(t,s) = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}). \tag{2.11}
$$

It follows from (2.11) that $\mathbb{E}|B^H_t - B^H_s|^2 = |t-s|^{2H}$, that is, $B^H$ has stationary increments. Furthermore, by Kolmogorov’s continuity criterion, $B^H_t$ has $\alpha$-Hölder continuous paths for all $\alpha < H$. In particular, if $H = 1/2$, then $B^H$ becomes a standard Brownian motion; and if $H = 1$, then we have $\{B^H_t; t \geq 0\} \sim \{\xi_t; t \geq 0\}$, where $\xi \sim N(0,1)$.

In what follows we shall consider the canonical space with respect an fBM or the fractional Wiener space, which we now describe. Let $\Omega = \mathbb{C}_0([0,T])$, the space of all continuous functions,
null at zero, and endowed with the usual sup-norm. Let $\mathcal{F}_t \triangleq \sigma\{\omega(\cdot \wedge t) | \omega \in \Omega\}$, $t \geq 0$, $\mathcal{F} \triangleq \mathcal{F}_T$, and $\mathbb{P}_{B^H}$ is the probability measure on $(\Omega, \mathcal{F})$ under which the canonical process

$$B^H_t(\omega) \triangleq \omega(t), \quad (t, \omega) \in [0, T] \times \Omega,$$

is an fBM of Hurst parameter $H$.

For any $H \in (0, 1)$ we define

$$R_H(t, s) = \int_0^{t \wedge s} K_H(t, r)K_H(s, r)dr,$$  \hspace{1cm} (2.12)

where $K_H$ is the square integrable kernel given by

$$K_H(t, s) \triangleq \Gamma(H + \frac{1}{2})^{-1}(t-s)^{H-1/2}F(H - \frac{1}{2}, \frac{1}{2} - H, H + \frac{1}{2}, 1 - \frac{t}{s}),$$  \hspace{1cm} (2.13)

and $F(a, b, c, z)$ is the Gauss hypergeometric function:

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}du.$$

Now, let $\mathcal{E}$ be the set of all step functions on $[0, T]$, and let $\mathcal{H}$ be the so-called Reproducing Kernel Hilbert space, defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle I_{[0,t]}, I_{[0,s]} \rangle_{\mathcal{H}} = R_H(t, s), \quad s, t \in [0, T].$$  \hspace{1cm} (2.14)

For any $H \in (0, 1)$ and $f \in L^2([0, T])$ we define a linear operator:

$$(K_H f)(t) \triangleq \int_0^t K_H(t, s)f(s)ds, \quad t \in [0, T].$$  \hspace{1cm} (2.15)

Also, for any $f \in L^0([0, T])$ and $\beta > 0$, we shall denote

$$\lVert f \rVert^\beta(t) \triangleq t^{\beta}f(t), \quad t \in [0, T],$$  \hspace{1cm} (2.16)

and $I^\alpha_{0+}L^p([0, T])) = \{f \in L^0([0, T]) : \lVert f \rVert^\beta \in I^\alpha_{0+}L^p([0, T])\}$. Then we have the following result (cf. e.g., [7, Theorem 2.1] or [25, Theorem 10.4]).

**Theorem 2.2** For each $H \in (0, 1)$, the operator $K_H$ is an isomorphism between $L^2([0, T])$ and $I^{H+1/2}_{0+}L^2([0, T])$. Furthermore, it holds that

$$[K_H f](s) = \begin{cases} 
I^{2H}_{0+}\left[I^{1/2-H}_{0+}\lVert f \rVert^{H-1/2}\right]^{1/2-H}(s), & H < 1/2, \\
I_{0+}^{1/2-H}\left[I^{H-1/2}_{0+}\lVert f \rVert^{1/2-H}\right]^{H-1/2}(s), & H > 1/2.
\end{cases}$$  \hspace{1cm} (2.17)
From (2.17) it is easy to check that the inverse operator $K_H^{-1}$ is given by

$$K_H^{-1}h = \begin{cases} 
\left[ \frac{1}{2} - H \right]^{1/2} I_{0^+} h' 
&\text{if } H < 1/2, \\
\left[ D_{0^+} \right]^{1/2} - H I_{1/2 - H} h' 
&\text{if } H > 1/2.
\end{cases}$$  

(2.18)

(cf. e.g., [25, Theorem 10.6] and [22]).

Next, let $K_H^*$ be the adjoint of $K_H$ on $L^2([0,T])$, that is, for any $f, g \in L^2([0,T])$,

$$\int_0^T [K_H f](t) g(t) dt = \int_0^T f(t) [K_H^* g](t) dt.$$  

Then, it can be shown by Fubini and integration by parts that for any $f \in L^2([0,T])$,

$$[K_H^* f](t) = K_H(T,t) \varphi(t) + \int_0^T (f(s) - f(t)) \frac{\partial K_H}{\partial s}(s,t) ds, \quad t \in [0,T].$$

In particular, for $\varphi, \psi \in \mathcal{E}$, we have (see, e.g., [1])

$$\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,T])} = \langle \varphi, \psi \rangle_H.$$  

Consequently, the operator $K_H^*$ is an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0,T])$. Furthermore, it can be shown that the process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t = B_H((K_H^*)^{-1}(I_{[0,t]}))$$  

(2.19)

is a Wiener process, and the process $B_H$ has an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s) dW_s, \quad t \in [0,T].$$  

(2.20)

We now turn our attention to the Poisson part. We first consider a Poisson random measure $N(\cdot, \cdot)$ on $[0,T] \times \mathbb{R}$, defined on a given probabiliy space $(\Omega, \mathcal{F}, P)$, with Lévy measure $\nu$ that satisfies the standard integrability condition:

$$\int_{\mathbb{R}\setminus [0]} (1 \land |x|^2) \nu(dx) < +\infty.$$  

The compensator of $N$ is thus the deterministic measure $\acute{N}(dt dx) = dt \nu(dx)$, on $[0,T] \times \mathbb{R}$. In this paper we shall be interested in Poisson point process of class (QL), namely a point process whose counting measure has deterministic and continuous compensator (cf. [15]). More precisely, in light of the representation theorem of the Poisson point process [15, Theorem II-7.4], we shall assume that $L$ is a pure jump process of the following form:

$$L_t = \int_0^t \int_{\mathbb{R}\setminus [0]} f(s,z) N(ds dz), \quad t \geq 0,$$
where \( f \in L^1(dt d\nu) \) is a given deterministic function so that the counting measure of \( L \), denoted by \( N_L(dt dx) \) takes the form

\[
N_L((0, t] \times A) = \int_0^t \int_{\mathbb{R}} 1_A(f(s, x))N(ds dx) = \sum_{0 < s \leq t} 1_{\{\Delta L_s \in A\}},
\]

(2.21)

and its compensator is therefore \( \widehat{N}_L(dt dx) = f(t, x)N(dt dx) \), and hence deterministic and continuous. Furthermore, if \( f(s, x) \equiv g(x) \), then \( L \) is a stationary Poisson point process. In particular, if we assume that \( g(x) \equiv x \) and \( \nu(dz) = \lambda F(dz) \), where \( F(\cdot) \) is a finite probability measure on \( \mathbb{R} \), then \( L \) is a compound Poisson process \( L_t = \sum_{i=1}^{N_t} U_i \), where \( N \) is a standard Poisson process with intensity \( \lambda \), and \( \{U_i\}_{i=1}^{\infty} \) is a sequence of i.i.d. random variables, independent of \( N \), with common distribution \( F \). Moreover, we assume that for any \( t \geq 0 \), it holds that

\[
\mathbb{E}\left\{ \int_0^T |L|^2 dt + e^{\beta|L|_T} \right\} < \infty, \quad \forall \beta > 0,
\]

(2.22)

where \( |L_t|_t = \sum_{0 \leq s \leq t} |\Delta L_s|, \quad t \in [0, T]. \)

**Remark 2.3** We note that the second assumption in (2.22) contains in particular the compound Poisson case. Indeed, if \( L_t = \sum_{i=1}^{N_t} U_i \), where \( N \) is a standard Poisson process with intensity \( \lambda > 0 \), then we can easily calculate that

\[
\mathbb{E}\left\{ \int_0^T |L|^2 dt + e^{\beta|L|_T} \right\} = \frac{(\lambda \mathbb{E}[U_1])^2 T^3}{3} + \frac{\lambda \mathbb{E}[|U_1|^2] T^2}{2} + \sum_{k=1}^{\infty} \mathbb{E}\left\{ e^{\beta \sum_{i=1}^k |U_i|} |N_T = k\right\} \frac{(\lambda T)^k}{k!} e^{-\lambda T} < \infty.
\]

(2.23)

We can also consider the canonical space for a given Poisson point process of class (QL). Let \( \Omega = \mathbb{D}([0, T]) \), the space of all real-valued, càdlàg (right-continuous with left limit) functions, endowed with the Skorohod topology, and let \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) and \( \mathcal{F} \) be defined as the same as before. Let \( \mathbb{P}_L \) be the law of the process \( L \) on \( \mathbb{D}([0, T]) \). Then, the coordinate process, by a slight abuse of notations,

\[
L_t(\omega) = \omega(t), \quad (t, \omega) \in [0, T] \times \Omega,
\]

is a Poisson point process, defined on \( (\Omega, \mathcal{F}, \mathbb{P}_L) \), whose compensated counting measure is \( \widehat{N}_L(dt dz) = \mathbb{E}[N_L(dt dz)] = f(t, z)\nu(dz)dt \), where \( \nu \) is a Lévy measure and (2.22) holds.
Combining the discussions above, we now consider two canonical spaces \((\Omega_1, \mathcal{F}_1, \mathbb{P}_B; \mathbb{F}_1)\) and \((\Omega_2, \mathcal{F}_2, \mathbb{P}_L; \mathbb{F}_2)\), where \(\Omega_1 = \mathbb{C}([0,T])\) and \(\Omega_2 = \mathbb{D}([0,T])\). We define the fractional Wiener-Poisson space to simply be the product space:

\[
\Omega \triangleq \Omega_1 \times \Omega_2; \quad \mathcal{F} \triangleq \mathcal{F}_1 \otimes \mathcal{F}_2; \quad \mathbb{P} \triangleq \mathbb{P}_B \otimes \mathbb{P}_L; \quad \mathcal{F}_t \triangleq \mathcal{F}_{t_1} \otimes \mathcal{F}_{t_2}, \quad t \in [0,T].
\]

(2.24)

We write the generic element of \(\Omega\) as \(\omega = (\omega^1, \omega^2) \in \Omega\). Then, the two marginal coordinate processes defined by

\[
B_t^H(\omega) \overset{\triangle}{=} \omega^1(t), \quad L_t(\omega) \overset{\triangle}{=} \omega^2(t), \quad (t, \omega) \times [0,T] \times \Omega,
\]

(2.25)

will be the fractional Brownian motion and Poisson point process, respectively, with the given law. Note that under our assumptions \(B^H\) and \(L\) are always independent (cf. e.g., [15, Theorem II-6.3]). Also, we can assume without loss of generality that the filtration \(\mathbb{F}\) is right continuous, and is augmented by all the \(\mathbb{P}\)-null sets so that it satisfies the usual hypotheses.

To end this section, we recall that if \(X\) is a metric space valued Gaussian process, and \(X \mapsto g(X)\) is a seminorm, such that and \(\mathbb{P}(g(X) < \infty) > 0\). Then it follows from the Fernique Theorem (cf. [8]) that there exists \(\varepsilon > 0\) such that \(\mathbb{E}[\exp(\lambda g^2(X))] < \infty\), for all \(0 < \lambda < \varepsilon\). It is then easy to see that for all \(0 < \rho < 2\), one has

\[
\mathbb{E}[\exp(\lambda g^\rho(X))] < \infty, \quad \forall \lambda > 0.
\]

(2.26)

This fact is useful in the our analysis, similar to, e.g., [22].

3 The Problem

In this paper we are interested in the following stochastic differential equation with additive noise:

\[
X_t = x + \int_0^t b(s, X_s)ds + B^H_t - L_t, \quad t \in [0,T],
\]

(3.1)

where \(b\) is a Borel function on \([0,T] \times \mathbb{R}\), \(B^H\) is an fBM with Hurst parameter \(H \in (0,1)\) and \(L\) is a Poisson point process of class (QL), both defined on some filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})\). We assume that \(B^H\) and \(L\) are both \(\mathbb{F}\)-adapted, and they are independent. We often consider the filtration generated by \((B^H, L)\), denoted by \(\mathbb{F}^{(B^H, L)} = \{\mathcal{F}_t^{(B^H, L)} : t \geq 0\}\) where

\[
\mathcal{F}_t^{(B^H, L)} \overset{\triangle}{=} \sigma\{(B^H_s, L_s) : 0 \leq s \leq t\}, \quad t \geq 0,
\]

(3.2)

and we assume that \(\mathbb{F}^{(B^H, L)}\) is augmented by all the \(\mathbb{P}\)-null sets so that it satisfies the usual hypotheses. As usual, we have the following definitions of solutions to the SDE (3.1).
**Definition 3.1** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space on which are defined an fBM \(B^H\), \(H \in (0, 1)\), and a Poisson point process \(L\), independent of \(B^H\) and of class (QL). A process \(X\) defined on \((\Omega, \mathcal{F}, \mathbb{P})\) is called a strong solution to (3.1) if

(i) \(X\) is \(\mathbb{F}^{(B^H, L)}\)-adapted;

(ii) \(X\) satisfies (3.1), \(\mathbb{P}\)-almost surely.

**Definition 3.2** A seven-tuple \((\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}, X, B^H, L)\) is called a weak solution to (3.1) if

(i) \((\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F})\) is a filtered probability space;

(ii) \(B^H\) is an \(\mathcal{F}\)-fBM, and \(L\) is an \(\mathcal{F}\)-Poisson point process of class (QL):

(iii) \((X, B^H, L)\) satisfies (3.1), \(\mathbb{P}\)-almost surely.

For simplicity, we often say that \((X, B^H, L)\) (or simply \(X\)) is a weak solution to (3.1) without specifying the associated probability space \((\Omega, \mathcal{F}, \mathbb{P}; \mathcal{F})\) when the context is clear. It is readily seen from (3.1) that if \((X, B^H, L)\) is a weak solution, then \(\mathbb{F}^{(B^H, L)} \subseteq \mathbb{F}^X\). The well-known example of Tanaka indicates that the converse is not necessarily true, even in the case when \(H = 1/2\) and \(L \equiv 0\).

Throughout this paper we shall make use of the following **Standing Assumptions**:

**Assumption 3.3** The function \(b : [0, T] \times \mathbb{R} \mapsto \mathbb{R}\) satisfies the following assumptions for \(H \in (0, 1/2)\) and \(H \in (1/2, 1)\), respectively:

(i) If \(H < 1/2\), then for some \(0 < \rho < 1/2\), it holds that

\[
|b(t, x)| \leq k(1 + |x|^\rho), \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\]

(ii) If \(H > 1/2\), then \(b\) is Hölder-\(\gamma\) continuous in \(t\) and Hölder-\(\alpha\) in \(x\), where \(\gamma > H - 1/2\), and \(1 - \frac{1}{2H} < \alpha < 1\). That is,

\[
|b(t, x) - b(s, y)| \leq k(|x - y|^\alpha + |t - s|^\gamma), \quad \forall (t, x), (s, y) \in [0, T] \times \mathbb{R}.
\]

**Remark 3.4** We note that in the case when \(H < 1/2\) we do not require any regularity on the coefficient \(b\). To discuss the well-posedness under such a weak condition on the coefficient is only possible due to the presence of the “noises” \(B^H\) and \(L\) (see also [22] for the case when \(L \equiv 0\)), and it is quite different from the theory of ordinary differential equations, for example.
We begin by making the following observation. Denote $\tilde{X} = X + L$, and

$$\tilde{b}(t, x, \omega) \triangleq b(t, x - L_t(\omega)), \quad (t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega.$$ 

Then the SDE (3.1) becomes

$$\tilde{X}_t = x + \int_0^t \tilde{b}(s, \tilde{X}_s)ds + B^H_t, \quad t \in [0, T],$$

(3.5)

Thus the problem is reduced to the case studied by [22], except that the coefficient $\tilde{b}$ is now random. However, if we consider the problem on the canonical (Wiener-Poisson) space in which $(B^H_t(\omega), L_t(\omega)) = (\omega^1(t), \omega^2(t)), t \in [0, T]$, then we can formally consider the SDE (3.5) as one on $(\Omega_1, \mathcal{F}_1, \mathbb{P})$:

$$\tilde{X}_t = x + \int_0^t b^{\omega^2}(s, \tilde{X}_s)ds + B^H_t, \quad t \in [0, T],$$

(3.6)

where $b^{\omega^2}(t, x) \triangleq b(t, x - \omega^2(t)) = \tilde{b}(t, x, \omega^2)$, for each fixed $\omega^1 \in \Omega_1$. In other words, we can apply the result of [22] to obtain the well-posedness for each $\omega^2 \in \Omega^2$, provided that the coefficient $b^{\omega^2}$ satisfies the assumptions in [22]. We should note, however, that such a seemingly simple argument is actually rather difficult to implement, especially for the weak solution case, due to some subtle measurability issues caused by the lack of regularity of $b$ in the case $H < 1/2$, and the discontinuity of the paths of $L$ (whence $\tilde{b}$ in the temporal variable $t$), in the case $H > 1/2$.

In the rest of this section we shall validate this argument directly in the case $H < 1/2$. Namely, we shall prove that the SDE (3.5) possesses a weak solution, by mimicking the solution scheme proposed in [22]. The case $H > 1/2$ will be investigated separately in the next section. Recall from Assumption 3.3 that in the case $H < 1/2$ the function $b$ satisfies (3.3). Consider the canonical Wiener-Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}^B \otimes \mathbb{P}^L$, with a given Hurst parameter $H \in (0, 1/2)$, a Lévy measure $\nu(dz)$, and a deterministic function $f : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ so that $\tilde{N}_L(dtdz) = \mathbb{E}[N_L(dtdz)] = f(t, z)\nu(dz)dt$ satisfies (2.22). Let $(B^H, L)$ be the canonical process. Define $u_t \triangleq -b(B^H_t - L_t + x)$ and

$$v_t \triangleq -K^{-1}_H \left( \int_0^t b(r, B^H_r - L_r + x)dr \right)(t) = K^{-1}_H \left( \int_0^t u_rdr \right)(t), \quad t \in [0, T],$$

(3.7)

where $K^{-1}_H$ is defined by (2.18). We have the following lemma.

**Lemma 3.5** Assume $H < 1/2$ and (3.3) is in force. Then the process $v$ defined by (3.7) enjoys the following properties:

1. $\mathbb{P}\{v \in L^2([0, T])\} = 1;
(2) $v$ satisfies the Novikov condition:

$$
\mathbb{E}\left\{ \exp\left( \frac{1}{2} \int_0^T |v_t|^2 dt \right) \right\} < \infty.
$$

(3.8)

**Proof.**  (1) In what follows we denote $C > 0$ to be a generic constant depending only the coefficient $b$, the constants in Assumption [3.3] and the Hurst parameter $H$; and is allowed to vary from line to line. Since $H < 1/2$, and (3.3) holds, some simple computation, together with assumption [22], shows that

$$
\mathbb{E} \int_0^T |u_t|^2 dt = \mathbb{E} \int_0^T |b(t, B_t^H - L_t + x)|^2 ds \leq C \mathbb{E} \int_0^T (1 + |B_t^H - L_t + x)|^2 dt
$$

$$
\leq C \left[ (1 + |x|)^2 T + \mathbb{E} \int_0^T |B_t^H|^2 dt + \mathbb{E} \int_0^T |L_t|^2 dt \right]
$$

$$
= C \left[ (1 + |x|)^2 T + \frac{T^{2H+1}}{2H+1} + \mathbb{E} \int_0^T |L_t|^2 dt \right] < \infty.
$$

Therefore, $\int_0^T |u_t|^2 ds < \infty$, $\mathbb{P}$-a.s. Since $H < 1/2$, $[u]^{1/2-H}$ belongs to $L^2([0, T])$, $\mathbb{P}$-a.s. as well. Thus, applying [25, Theorem 5.3] $I_{0+}^{1/2-H} [u]^{1/2-H} \in L^q([0, T])$, $\mathbb{P}$-a.s., for all $q > 2$. In particular $I_{0+}^{1/2-H} [u]^{1/2-H} \in L^2([0, T])$, $\mathbb{P}$-a.s. Let $N \subset \Omega$ be the exceptional $\mathbb{P}$-null set. Then for any $\omega \notin N$, we can apply Theorem [2.1] (iii-a) to find $h^{\omega} \in L^2([0, T])$ such that

$$
[I_{0+}^{1/2-H} [u]^{1/2-H} (\omega)](t) = t^{1/2-H} [I_{0+}^{1/2-H} h^{\omega}](t), \quad \omega \notin N.
$$

Now recall from (2.18) we see that this implies that for each $\omega \notin N$, it holds that

$$
K_{H}^{-1} \left( \int_0^t u_r(\omega) dr \right) = I_{0+}^{1/2-H} h^{\omega}.
$$

Thus, applying [25, Theorem 5.3] again we have $K_{H}^{-1} (\int_0^t u_r(\cdot) dr) \in L^q([0, T])$, $\mathbb{P}$-a.s., for all $q > 2$. In particular, (1) holds.

(2) Using the Assumption [3.3] again we have, $\mathbb{P}$-almost surely,

$$
|v_s| = |s^{H-1/2} [u]^{1/2-H}_0 + \int_0^s (s - r)^{-1/2-H} b(r, B_r^H - L_r + x) dr |
$$

$$
\leq C T^{1/2-H} (1 + |x|^q + \|B^H\|_\infty + |L_t|^q),
$$

where $\|B^H\|_\infty \triangleq \sup_{0 \leq s \leq T} |B_s^H|$. Note that $L$ and $B^H$ are independent we have

$$
\mathbb{E}\left\{ \exp\left( \frac{1}{2} \int_0^T |v_t|^2 dt \right) \right\} \leq e^{C T^{2-2H} (1 + |x|^{2q})} \mathbb{E}\left\{ \exp\left( C T^{2-2H} \|B^H\|_\infty^{2q} \right) \right\} \mathbb{E}\left\{ e^{C T^{2-2H}|L_t|^2q} \right\}. \quad (3.9)
$$
Note that $2\rho < 1$ by (3.3) in Assumption 3.3, we have
\[ \mathbb{E}\{e^{CT^{2-2H}|L|^{2\rho}}\} \leq \mathbb{E}\{e^{CT^{2-2H}(|L|T+1)}\} < \infty, \] (3.10)
thanks to (2.22). Finally, since $\rho < 1/2$ also guarantees that (2.26) holds for all $T > 0$, with $X = B^H$ and $g(\cdot) = \|\cdot\|_\infty$. This, together with (3.9) and (3.10), proves (3.8), whence the lemma.

**Remark 3.6** We note that the assumption $\rho < 1/2$ is slightly stronger than that in [22], where only $\rho < 1$ was assumed. This is to guarantee the finiteness of $\mathbb{E}\{e^{L^{2\rho}}\}$. In fact, if $\rho > 1/2$, then even in the simplest standard Poisson case $L_t \equiv N_t$ we have $\mathbb{E}e^{(N_t)^{2\rho}} = \infty$.

If we denote $a_n = e^{n^{2\rho} \lambda^n/n!}$, then $\ln a_n = n^{2\rho} + n \ln \lambda - \ln n!$. Since $\ln n! < n \ln n$, and $\lim_{n \to \infty} \frac{n \ln n}{n^{2\rho} + n \ln \lambda} = 0$, a simple calculation then shows that
\[
\lim_{n \to \infty} \ln a_n = \lim_{n \to \infty} \{n^{2\rho} + n \ln \lambda - \ln n!\} = \lim_{n \to \infty} \{n^{2\rho} + n \ln \lambda\} \left(1 - \frac{\ln n!}{n^{2\rho} + n \ln \lambda}\right) = +\infty.
\]
That is, $a_n \to +\infty$, and consequently $\mathbb{E}e^{(N_t)^{2\rho}} = \infty$.

We can now construct a weak solution to (3.1), in the case $H < 1/2$, as follows. Define
\[ \tilde{B}_t^H \triangleq B_t^H - \int_0^t b(s, B_s^H, L_s + x)ds = B_t^H + \int_0^t u_s ds, \quad t \in [0, T]. \] (3.11)
Using the representation (2.20) we can write
\[ \tilde{B}_t^H = B_t^H + \int_0^t u_s ds = \int_0^t K_H(t, s)dW_s + \int_0^t u_s ds = \int_0^t K_H(t, s)d\tilde{W}_s, \]
where
\[ \tilde{W}_t = W_t + \int_0^t (K_H^{-1}(\int_0^r u_s ds)(r))dr = W_t + \int_0^t v_r dr. \] (3.12)
By Lemma 3.5, the process $v$ satisfies the Novikov condition (3.8). Thus, if we define a new probability measure $\tilde{P}$ on the canonical fractional Wiener-Poisson space $(\Omega, \mathcal{F})$ by
\[
\frac{d\tilde{P}}{dP} \triangleq \exp \left\{ - \int_0^T v_s dW_s - \frac{1}{2} \int_0^T v_s^2 ds \right\}. \] (3.13)
then, under $\tilde{P}$, $\tilde{W}$ is an $\mathcal{F}$-Brownian motion, and $\tilde{B}^H$ is an $\mathcal{F}$-fractional Brownian motion with Hurst parameter $H$ (cf. Decreusefond and Üstunel [7]).

Furthermore, since $B^H$ and $L$ are independent, we can easily check, by following the arguments of Brownian case (cf. e.g., [26, Theorem 124], [15, Theorem II-6.3]) that $L_t$ is still a Poisson point process of class (QL) with same parameters, and is independent of $\tilde{B}^H$. We now define

$$X_t = x + B^H_t - L_t, \quad t \in [0, T].$$

Then, it follows from (3.11) that

$$\tilde{B}^H_t = (X_t - x + L_t) - \int_0^t b(t, X_s)ds, \quad t \in [0, T].$$

(3.14)

In other words, $(\Omega, \mathcal{F}, \tilde{P}, \mathbb{P}, X, \tilde{B}^H, L)$ is a weak solution of (3.1). That is, we have proved the following theorem.

**Theorem 3.7** Assume $H < 1/2$ and that Assumption 3.3-(i) holds. Then for any $T > 0$, the SDE (3.1) has at least one weak solution on $[0, T]$. $\blacksquare$

### 4 Existence of a weak solution ($H > 1/2$)

In this section we study the existence of the weak solution in the case when $H > 1/2$, and the coefficient $b$ satisfies the Assumption 3.3-(ii) (3.4). We note that in this case the coefficient $\tilde{b}$ of the reduced SDE (3.5) will have discontinuity on the variable $t$, thus the Assumption 3.3-(ii) is no longer valid for $\tilde{b}$, and therefore one should not even attempt to apply the results of [22] directly. We shall, however, using the same scheme as in the last section to prove the existence of the weak solution, although the arguments is much more involved.

Let us again start from the canonical fractional Wiener-Poisson space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and let $(B^H, L)$ be the canonical process. For fixed $x \in \mathbb{R}$, consider again the process

$$u_t(\omega) = -b(t, B^H_t(\omega) - L_t(\omega) + x) = -b(t, \omega^1(t) - \omega^2(t) + x), \quad (t, \omega) \in [0, T] \times \Omega,$$

and define $v_t(\omega) = K^{-1}_H \left( \int_0^T u_r(\omega)dr \right)(t)$, $(t, \omega) \in [0, T] \times \Omega$, where $K^{-1}_H$ is given by (2.18) in the case $H > 1/2$. As in the previous section, we shall again argue that Lemma 3.5 holds. The main difference is that now the paths of $u$ are no longer continuous, the fractional calculus will need to be modified.

We first note that, by Fubini,

$$\mathbb{P}\{v \in L^2([0, T])\} = \int_{\Omega^2} \mathbb{P}^{B^H} \left\{ \int_0^T |v_s(\omega^1, \omega^2)|^2 ds < \infty \right\} \mathbb{P}^L(d\omega^2).$$

Thus to show $\mathbb{P}\{v \in L^2([0, T])\} = 1$, it suffices to show that, for $\mathbb{P}^L$-a.e., $\omega^2 \in \Omega^2$, it holds that

$$\mathbb{P}^{B^H} \left\{ \int_0^T |v_s(\omega^1)|^2 ds < \infty \right\} = 1,$$
where \( u_t^2(\omega^1) \triangleq v_t(\omega^1, \omega^2) \) is the “\( \omega^2 \)-section” of \( v_t \). But in light of (2.18), we need first show that, for \( \mathbb{P}^L \)-a.e. \( \omega^2 \in \Omega^2 \), \( u_t^2 \in L^{H-1/2,1/2-H}(L^1([0,T])) \cap L^1([0,T]), \mathbb{P}^B^H \)-a.s., where

\[
\begin{align*}
u_t^2(\cdot) &\triangleq \nu_t(\cdot, \omega^2) = -b^{x_2}(t, B_t^H(\cdot)), \quad (t, \omega^1) \in [0,T] \times \Omega^1. 
\end{align*}
\] (4.1)

and

\[
\begin{align*}
b^{x_2}(t, y) &\triangleq b(t, y - \omega^2(t) + x), \quad (t, y) \in [0,T] \times \Omega^1. 
\end{align*}
\] (4.2)

Now for each fixed \( \omega^2 \in \Omega^2 \), denote \( 0 < \sigma_1(\omega^2) < \cdots < \sigma_n(\omega^2) < T \) be the jump times of \( \omega^2 \) in \( (0,T) \), and set \( \sigma_0(\omega^2) = 0, \sigma_{n+1}(\omega^2) = T \). Since we are considering only the canonical process \( L(\omega^2) = \omega_2 \), we can, modulo a \( \mathbb{P}^L \)-null set, assume without of generality that \( \omega^2 \) is piece constant. Then by Assumption 3.3(ii) we see that \( t \mapsto b^{x_2}(t, B_t^H) \) is \( \gamma \)-Hölder continuous on every interval \( (\sigma_i, \sigma_{i+1}), i = 0, 1, \cdots, n \), with \( \gamma = H - \frac{1}{2} + \varepsilon \) for some \( \varepsilon > 0 \). Thus, by virtue of Theorem 6.5 in (25), \( u_t^2 \in L^{H-1/2}_i(L^2(\sigma_i, \sigma_{i+1})), \mathbb{P}^B^H \)-a.s., for all \( i = 0, \cdots, n \). It then follows from Theorem 13.11 of (25) that \( u_t^2 \in L^{H-1/2}(L^2([0,T])), \mathbb{P}^B^H \)-a.s. Therefore, there exists a \( \mathbb{P}^B^H \)-null set \( N \subset \Omega^1 \), so that for any \( \omega^1 \notin N \), we can apply Theorem 2.1 (iii)-(a) or Lemma 3.2 in (25) to find a function \( h^{\omega_1, \omega^2} \in L^2([0,T]) \), such that:

\[
[u_t^2]^{1/2-H}(t, \omega^1) = t^{1/2-H}u_t^2(\omega^1) = t^{H-1/2}t^{1/2-H}h^{\omega_1, \omega^2}(t), \quad t \in [0,T].
\]

That is, \( u_t^2 \in L^{H-1/2,1/2-H}(L^1([0,T])), \mathbb{P}^B^H \)-a.s. On the other hand, since \( u_t^2 \in L^{H-1/2}_i(L^2([0,T])) \) implies \( u_t^2 \in L^2([0,T]) \), thanks to Theorem 5.3 of (25), we conclude that (2.18) holds with \( h(\cdot) = \int_0^\gamma u_t dr, \mathbb{P}^B^H \)-a.s. That is, \( v_t = K_H^1(\int_0^\gamma u_t dr)(t), t \in [0,T] \) belongs to \( L^2([0,T]), \mathbb{P}^B^H \)-a.s. Note that the argument is valid for \( \mathbb{P}^L \)-a.e. \( \omega^2 \in \Omega^2 \), we obtain that \( \mathbb{P}\{v \in L^2([0,T])\} = 1 \).

It remains to prove that the process \( v \) also satisfies the Novikov condition (3.8), whence part(2) of Lemma 3.3 We first note that on the canonical space \( \Omega^2 = \mathbb{D}([0,T]) \), and under the probability \( \mathbb{P}^L \), the canonical process \( L(\omega^2) = \omega^2 \) is a Poisson point process of class (QL). Now, for fixed \( T > 0 \), denote \( \Omega_2^2 \triangleq \{ \omega^2 : N_L([0,T] \times \mathbb{R})(\omega^2) = n \} \); and for \( \omega^2 \in \Omega_2^2 \), again denote \( 0 < \sigma_1(\omega^2) < \cdots < \sigma_n(\omega^2) < T \) be the jump times of \( N(\omega^2) \), and \( \sigma_0(\omega^2) = 0, \sigma_{n+1}(\omega^2) = T \). Finally, denote \( S_k(\omega^2) \triangleq \sum_{i=1}^k \Delta L_{\omega^1}(\omega^2), k = 1, 2, \cdots \) and \( S_0(\omega^2) = 0 \). In what follows we often suppress the variable \( \omega^2 \) when the context is clear.

Now recall from (2.18) that, for \( H > 1/2 \),

\[
\begin{align*}
u_t^2 &\triangleq K_{H}^{-1}\left( \int_0^\gamma \nu_r^2 dr \right)(t) = t^{H-1/2}D_{0+}^{H-1/2}[u_t^2]^{1/2-H}(t), \quad t \in [0,T]. 
\end{align*}
\] (4.3)

We shall calculate \( D_{0+}^{H-1/2}[u_t^2]^{1/2-H} \) for \( \omega^2 \in \Omega_2^2 \), for each \( n = 1, 2, \cdots \). To see this, fix \( n \in \mathbb{N} \), and let \( \omega^2 \in \Omega_2^2 \). For notational simplicity, in what follows we denote

\[
\begin{align*}
u_t^{\omega^2,k}(\omega^1) &\triangleq -b(t, B_t^H(\omega^1) - S_{k-1}(\omega^2) + x), \quad (t, \omega^1) \in [0,T] \times \Omega^1, \quad k \geq 1, 
\end{align*}
\] (4.4)
so that
\[ u_t^{\omega^2} = \sum_{k=1}^{N_t(\omega^2)} u_t^{\omega^2,k} 1_{[\sigma_{k-1}(\omega^2), \sigma_k(\omega^2))}(t), \quad t \in [0, T], \ \mathbb{P}^1 \text{-a.s.} \]

Then, for \( t \in [0, \sigma_1(\omega^2)) \), by definition \((2.7)\) and \((2.8)\) with \( p = 2 \) we have

\[
D_{0+}^{H-1/2}[u^{\omega^2}]^{1/2-H}(t) = \frac{1}{\Gamma(3/2 - H)} \left[ \frac{[u^{\omega^2}]^{1/2-H}(t)}{t^{H-1/2}} \right] + \lim_{\epsilon \to 0} \frac{H - 1/2}{\Gamma(3/2 - H)} \int_0^t \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
= \frac{1}{\Gamma(3/2 - H)} \left[ \frac{[u^{\omega^2}]^{1/2-H}(t)}{t^{H-1/2}} \right] + \frac{H - 1/2}{\Gamma(3/2 - H)} \int_0^t \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
\triangleq \Phi_1(t). \quad (4.5)
\]

Similarly, for \( \sigma_{k-1}(\omega^2) \leq t < \sigma_k(\omega^2) \) with \( 1 < k \leq n + 1 \), we have

\[
D_{0+}^{H-1/2}[u^{\omega^2}]^{1/2-H}(t) = \frac{1}{\Gamma(3/2 - H)} \left[ \frac{[u^{\omega^2}]^{1/2-H}(t)}{t^{H-1/2}} \right] + \lim_{\epsilon \to 0} \frac{H - 1/2}{\Gamma(3/2 - H)} \int_0^t \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
= \frac{1}{\Gamma(3/2 - H)} \left[ \frac{[u^{\omega^2}]^{1/2-H}(t)}{t^{H-1/2}} \right] + \frac{H - 1/2}{\Gamma(3/2 - H)} \int_0^t \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
+ \frac{H - 1/2}{\Gamma(3/2 - H)} \sum_{i=1}^{k-1} \int_{\sigma_{i+1}}^{\sigma_i} \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
+ \frac{H - 1/2}{\Gamma(3/2 - H)} \int_{\sigma_{k-1}}^t \left[ \frac{[u^{\omega^2}]^{1/2-H}(r) - [u^{\omega^2}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr \\
\triangleq \Phi_k(t). \quad (4.6)
\]

Consequently, we obtain the following formula:

\[
D_{0+}^{H-1/2}[u^{\omega^2}]^{1/2-H}(t) = \sum_{k=1}^{n+1} \Phi_k(t) 1_{[\sigma_{k-1}(\omega^2), \sigma_k(\omega^2))}(t), \quad t \in [0, T), \ \mathbb{P}^1 \text{-a.s.}, \quad (4.7)
\]

where \( \Phi_k \)'s are defined by \((4.5)\) and \((4.6)\). We can now prove Lemma 3.3 for the case \( H > 1/2 \).

Lemma 4.1 Assume that \( H > 1/2 \), and Assumption 3.3 holds. Then the conclusion of Lemma 3.3 remains valid.

Proof. We have already proved Part (1) in the beginning of this section. We shall now check the Novikov condition \((3.8)\).
To this end, we first fix $n \in \mathbb{N}$, and $\omega^2 \in \Omega^2_n$. Then combining (4.3) and (4.7) we have

$$v_t^\omega = t^{H-1/2}D_0^{H-1/2}[u^\omega^2]^{1/2-H}(t) = t^{H-1/2} \sum_{i=1}^{n+1} \Phi_k(t)1_{[\sigma_{k-1}(\omega^2)<t\leq \sigma_k(\omega^2)}(t),$$  

(4.8)

where $\Phi_k$’s are defined by (4.6). But note that for each $k = 1, 2, \ldots, n + 1, \frac{H - \frac{1}{2}}{\Gamma(\frac{3}{2} - H)} \sum_{i=1}^{\infty} \int_{\sigma_{i-1}}^{\sigma_i} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}}{(t-r)^{H+1/2}} \right] dr = \frac{1}{\Gamma(\frac{3}{2} - H)} \left\{ \frac{1}{(t-\sigma_{k-1})^{H-\frac{1}{2}}} - \frac{1}{t^{H-\frac{1}{2}}} \right\} [u^{\omega^2,k}]^{1/2-H}(t),$

we see from (4.6) that, for $t \in [\sigma_{k-1}, \sigma_k)$ it holds that

$$t^{H-1/2} \Phi_k(t)$$

$$= t^{H-1/2} \left\{ \frac{1}{\Gamma(3/2 - H)} \frac{[u^{\omega^2,k}]^{1/2-H}}{t^{H-1/2}} \right\}$$

$$+ \frac{H - \frac{1}{2}}{\Gamma(3/2 - H)} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}}{(t-r)^{H+1/2}} \right] dr$$

$$+ \frac{H - \frac{1}{2}}{\Gamma(3/2 - H)} \int_{\sigma_{k-1}}^{t} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}}{(t-r)^{H+1/2}} \right] dr$$

$$= C^H_1 \frac{t^{H-1/2}[u^{\omega^2,k}]^{1/2-H}(t)}{(t-\sigma_{k-1})^{H-1/2}} - C^H_2 t^{H-1/2} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr$$

$$+ C^H_2 t^{H-1/2} \int_{\sigma_{k-1}}^{t} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr$$

$$= A^k(t) + B^k(t),$$

where $C^H_1 \triangleq \frac{1}{\Gamma(3/2-H)}$, $C^H_2 \triangleq \frac{H-1/2}{\Gamma(3/2-H)} = (H - 1/2)C^H_1$, and

$$A^k(t) \triangleq C^H_1 \frac{t^{H-1/2}[u^{\omega^2,k}]^{1/2-H}(t)}{(t-\sigma_{k-1})^{H-1/2}} - C^H_2 t^{H-1/2} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr;$$

(4.9)

$$B^k(t) \triangleq C^H_2 t^{H-1/2} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr - C^H_2 t^{H-1/2} \int_{\sigma_{k-1}}^{t} \left[ \frac{[u^{\omega^2,k}]^{1/2-H}(t)}{(t-r)^{H+1/2}} \right] dr.$$  

(4.10)
It is readily seen that (suppressing \( \omega = (\omega^1, \omega^2)'s\))

\[
|A^k(t)| = \left| C_1^H \sum_{i=1}^{k-1} b(t, B_t^H - S_{i-1} + x) \left[ \frac{1}{(t - \sigma_i - 1)^{H-1/2}} - \frac{1}{(t - \sigma_i)^{H-1/2}} \right] + C_1^H \frac{b(t, B_t - S_{k-1} + x)}{(t - \sigma_{k-1})^{H-1/2}} \right| \\
\leq \left| C_1^H \sum_{i=1}^{k-1} \left[ b(t, B_t^H - S_{i-1} + x) - b(t, B_t^H + x) \right] \left[ \frac{1}{(t - \sigma_i - 1)^{H-1/2}} - \frac{1}{(t - \sigma_i)^{H-1/2}} \right] + C_1^H \frac{b(t, B_t + x)}{(t - \sigma_k - 1)^{H-1/2}} \right| \\
+ \left| C_1^H \sum_{i=1}^{k-1} \left[ b(t, B_t^H + x) \left[ \frac{1}{(t - \sigma_i - 1)^{H-1/2}} - \frac{1}{(t - \sigma_i)^{H-1/2}} \right] \right] + C_1^H t^{1/2-H} \right| b(t, B_t^H + x) \right| \\
\leq C \max_{1 \leq i \leq k} |b(t, B_t^H - S_{i-1} + x) - b(t, B_t^H + x)| \times \\
\sum_{i=1}^{k-1} \left[ \frac{1}{(t - \sigma_i - 1)^{H-1/2}} - \frac{1}{(t - \sigma_i)^{H-1/2}} \right] + \frac{1}{(t - \sigma_k - 1)^{H-1/2}} + C_1^H t^{1/2-H} |b(t, B_t^H + x)|
\]

where \( C \) is a generic constant depending on \( H, \alpha \), and \( k \), thanks to Assumption [3.3]. On the other hand, we write \( B^k(t) = -C(B_1^k(t) + B_2^k(t)) \), where

\[
B_1^k(t) = t^{H-1/2} \sum_{i=1}^{k-1} \left[ b(t, B_t^H - S_{i-1} + x) \right] \int_{\sigma_{i-1}}^{\tau} \frac{t^{1/2-H} - r^{1/2-H}}{(t - r)^{1/2+H}} dr \\
+ \int_{\sigma_{i-1}}^{\tau} b(t, B_t^H - S_{i-1} + x) - b(r, B_r^H - S_{i-1} + x) \frac{r^{1/2-H}}{(t - r)^{1/2+H}} dr \\
+ t^{H-1/2} b(t, B_t^H - S_{k-1} + x) \int_{\sigma_{k-1}}^{t} \frac{t^{1/2-H} - r^{1/2-H}}{(t - r)^{1/2+H}} dr \\
+ t^{H-1/2} \int_{\sigma_{k-1}}^{t} b(t, B_t^H - S_{k-1} + x) - b(r, B_r^H - S_{k-1} + x) \frac{r^{1/2-H}}{(t - r)^{1/2+H}} dr
\]

and

\[
B_2^k(t) = t^{H-1/2} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\tau} b(r, B_r^H - S_{i-1} + x) - b(r, B_r^H - S_{i-1} + x) \frac{r^{1/2-H}}{(t - r)^{1/2+H}} dr \\
+ t^{H-1/2} \int_{\sigma_{k-1}}^{t} b(r, B_r^H - S_{k-1} + x) - b(r, B_r^H - S_{k-1} + x) \frac{r^{1/2-H}}{(t - r)^{1/2+H}} dr.
\]

Then, it is easy to see that, for each fixed \( 0 < \varepsilon < H - \frac{H-1/2}{\alpha} \), and denoting

\[
G = \sup_{0 \leq t < r \leq T} \frac{|B_t^H - B_r^H|}{|t - r|^{H-\varepsilon}},
\]

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we have

\[ |B_2^k(t)| \leq t^{H-1/2} \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} |B_t^H - B_r^H|^\alpha (t-r)^{1/2-H} dr + t^{H-1/2} \int_{\sigma_{k-1}}^{t} |B_t^H - B_r^H|^\alpha (t-r)^{1/2-H} dr \]

\[ = t^{H-1/2} \int_{0}^{t} |B_t^H - B_r^H|^\alpha (t-r)^{1/2-H} dr \leq C t^{1/2-H + \alpha(H-\varepsilon)} G^\alpha. \quad (4.12) \]

Furthermore, by the same argument as in (4.11) we also have

\[ |B_1^k(t)| = t^{H-1/2} \max_{1 \leq i \leq k} |b(t, B_t^H - S_{i-1} + x)| \]

\[ \leq \left[ \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \frac{t^{1/2-H} - t^{1/2-H}}{(t-r)^{1/2+H}} dr + \int_{\sigma_{k-1}}^{t} \frac{t^{1/2-H} - t^{1/2-H}}{(t-r)^{1/2+H}} dr \right] \]

\[ + t^{H-1/2} \left[ \sum_{i=1}^{k-1} \int_{\sigma_{i-1}}^{\sigma_i} \frac{|t-r|^\gamma (t-r)^{1/2-H}}{(t-r)^{1/2+H}} dr + \int_{\sigma_{k-1}}^{t} \frac{|t-r|^\gamma (t-r)^{1/2-H}}{(t-r)^{1/2+H}} dr \right] \]

\[ \leq \int_{0}^{t} \frac{|t-r|^\gamma}{(t-r)^{1/2+H}} dr \]

\[ \leq C [b(0,x)] + |t|^\gamma + |B_t^H|^\alpha + |L_T|^\alpha \int_{0}^{t} \frac{r^{1/2-H} - r^{1/2-H}}{(t-r)^{1/2+H}} dr \quad (4.13) \]

\[ \leq C t^{1/2-H} [b(0,x)] + |t|^\gamma + \|B_t^H\|^\alpha + |L_T|^\alpha. \]

Combining (4.12) and (4.13) we have for any \( t \in [0,T] \),

\[ |B^k(t)| \leq C t^{1/2-H} [b(0,x)] + |t|^\gamma + \|B_t^H\|^\alpha + |L_T|^\alpha + t^{\alpha(H-\varepsilon)} G^\alpha. \quad (4.14) \]

Now we are ready to verify Novikov condition. Combining (4.11) and (4.14), and denoting \( \Sigma_n[\cdot] = \)
\[ \mathbb{E}[|N_T = n|] \text{, we have} \]
\[
\mathbb{E}\left\{ \exp\left\{ \frac{1}{2} \int_0^T v^2(t) dt \right\} \right\} 
= \sum_{n=1}^{\infty} \mathbb{E}_n \left\{ \exp\left\{ \frac{1}{2} \sum_{k=1}^{n} \int_{\sigma_{k-1}}^{\sigma_k} t^{2H-1} \Phi_k^2(t) dt + \frac{1}{2} \int_{\sigma_{n+1}}^{T} t^{2H-1} \Phi_{n+1}^2(t) dt \right\} \right\} P(N_T = n)
\]
\[
= \sum_{n=1}^{\infty} \mathbb{E}_n \left\{ C \left( \sum_{k=1}^{n} \int_{\sigma_{k-1}}^{\sigma_k} (|A_k(t)| + |B_k(t)|)^2 dt \right) \right\} P(N_T = n)
\]
\[
\leq \sum_{n=1}^{\infty} \mathbb{E}_n \left\{ \exp\left\{ \sum_{k=1}^{n} \int_{\sigma_{k-1}}^{\sigma_k} (t - \sigma_{k-1})^{1-2H} |L|^{2\alpha} dt 
+ C \int_0^T t^{1-2H} (|b^2(0, x)| + |t|^{2\gamma} + \|B^H\|_{2\alpha} + t^{2\alpha(H-\varepsilon)} G^{2\alpha}) dt \right\} \right\} P(N_T = n)
\]
\[
\leq \sum_{n=1}^{\infty} \mathbb{E}_n \left\{ \exp\left\{ C \left( n + 1 \right)^{2H-1} |L|^{2\alpha} + C \left[ 1 + \|B^H\|_{2\alpha} + G^{2\alpha} \right] \right\} \right\} P(N_T = n)
\]
\[
\leq \mathbb{E} \left\{ \exp\left\{ C \left[ 1 + \|B^H\|_{2\alpha} + G^{2\alpha} \right] \right\} \right\} \mathbb{E} \left\{ \exp\left\{ C(N_T + 1)^{2H-1} |L|^{2\alpha} \right\} \right\},
\]
where we used the fact
\[
\sum_{i=1}^{n+1} x_i^{2-2H} \leq \left( \sum_{i=1}^{n+1} x_i \right)^{2-2H}
\]
in the third inequality. By the same argument as Lemma 3.5, it is easy to prove that \( \mathbb{E} \left\{ \exp\left\{ C \|B^H\|_{2\alpha} + G^{2\alpha} \right\} \right\} < \infty \), hence \( \mathbb{E} \left\{ \exp\left\{ \frac{1}{2} \int_0^T v^2(s) ds \right\} \right\} < \infty \).

We need show that \( \mathbb{E} \left\{ \exp\left\{ C_T(N_T + 1)^{2H-1} |L|^{2\alpha} \right\} \right\} < \infty \). Note that \( \alpha < 1 - H \) in Assumption (C3) implies that \( 2H - 1 + 2\alpha < 1 \), we have
\[
\mathbb{E} \exp\left\{ C(N_T + 1)^{2H-1} |L|^{2\alpha} \right\} \leq \mathbb{E} \exp\left\{ C \left( \sum_{i=1}^{N_T} \tilde{U}_i \sqrt{1 + 1} \right)^{2H-1+2\alpha} \right\}
\]
\[
\leq \mathbb{E} \exp\left\{ C \left( \sum_{i=1}^{N_T} \tilde{U}_i \sqrt{1 + 2} \right) \right\}
\]
\[
eq e^{2C} \sum_{k=1}^{\infty} \mathbb{E} \left\{ \exp\left\{ C \sum_{i=1}^{k} \left( \tilde{U}_i \sqrt{1+1} \right) \right\} \right\} \mathbb{E}[N_T = k] \frac{(\lambda T)^k}{k!} e^{-\lambda T}
\]
\[
= e^{2C} \sum_{k=1}^{\infty} \left( \lambda T \mathbb{E}[\mathbb{E}[C(\tilde{U}_1 \sqrt{1+1})^k]} \right) \frac{1}{k!} e^{-\lambda T} = e^{2C+\lambda T(\mathbb{E}[\mathbb{E}[C(\tilde{U}_1 \sqrt{1+1})^k}] - 1) < \infty,
\]
We have the following analogues of Theorem 3.7.

**Theorem 4.2** Assume $H > 1/2$ and that Assumption 3.3-(ii) holds. Then the SDE (3.1) has at least one weak solution on $[0, T]$.

5 Uniqueness in law and pathwise uniqueness

In this section we study the uniqueness of the weak solution. We shall first show that the weak solutions to (3.1) are uniqueness in law. The argument is very similar to that of [22], we describe it briefly.

Let $(X, B^H, L)$ be a weak solutions of (3.1), defined on some probability space $(\Omega, \mathcal{F}, P; F)$, with the existence interval $[0, T]$. Let $W$ be the $\mathcal{F}$-Brownian motion such that

$$B_t^H = \int_0^t K_H(t, s)dW_s, \quad t \in [0, T].$$

(5.1)

Define

$$v_t = K^{-1}_H \left( \int_0^t b(r, X_r)dr \right)(t), \quad t \in [0, T],$$

(5.2)

and let us assume that $v$ satisfies the assumption (1) and (2) in Lemma 3.5. Then applying the Girsanov theorem we see that the process $\tilde{W}_t = W_t + \int_0^t v_s ds, t \in [0, T]$, is an $\mathcal{F}$-Brownian motion under the new probability measure $\tilde{P}$, defined by

$$\frac{d\tilde{P}}{dP} = \xi_T(X) \triangleq \exp \left\{ - \int_0^T v_t dW_t - \frac{1}{2} \int_0^T |v_t|^2 dt \right\}. \quad (5.3)$$

Thus $\tilde{B}_t^H \triangleq \int_0^t K_H(t, s)d\tilde{W}_s, t \in [0, T]$ is an fBM under $\tilde{P}$, and it holds that

$$X_t + L_t - x = \int_0^t b(s, X_s)ds + B_t^H = \int_0^t K_H(t, s)d\tilde{W}_s = \tilde{B}_t^H, \quad t \in [0, T].$$

Since under the Girsanov transformation the process $L$ remains a Poisson point process with the same parameters, and is automatically independent of the Brownian motion $\tilde{W}$ under $\tilde{P}$ (cf. [15, Theorem II-6.3]), we can then write $X$ as the independent sum of $\tilde{B}^H$ and $-L$:

$$X_t = x + \tilde{B}_t^H - L_t, \quad t \in [0, T].$$

Since the argument above can be applied to any weak solution, we have essentially proved the following weak uniqueness result.
\textbf{Theorem 5.1} Suppose that Assumption 3.3 holds. Then two weak solutions of SDE (5.1) must have the same law, over their common existence interval \([0,T]\).

\textbf{Proof.} We need only show that the adapted process \(v\) defined by (5.2) satisfies (1) and (2) in Lemma 3.5.\footnote{In what follows we let \(C\) denote a generic constant depending on \(H, \lambda, k,\) and \(T,\) and is allowed to vary from line to line. In the case \(H < \frac{1}{2},\) denoting \(u = b(\cdot, X,\), for any \(t \in [0,T]\) we have

\[
\mathbb{E} \int_0^t |u_r|^2 dr = \mathbb{E} \int_0^t |b(r, X_r)|^2 dr \leq C \mathbb{E} \int_0^t (1 + |X_r|^2) dr
\]

\[
\leq C \mathbb{E} \int_0^t \left[ 1 + |x|^2 + \int_0^r b(s, X_s)^2 ds \right] dr
\]

\[
\leq C \left\{ \mathbb{E} \int_0^t \int_0^r |u_s|^2 ds dr + (1 + |x|^2) t + \frac{2^{H+1}}{2H+1} + \mathbb{E} \int_0^T |L|^2 dr \right\}
\]

By Grownall’s inequality we obtain

\[
\mathbb{E} \int_0^T |u_s|^2 ds = \mathbb{E} \int_0^T |b(s, X_s)|^2 ds \leq C(1 + |x|^2)e^{CT} < \infty.
\]

Then, by the same argument as Lemma 3.5 we can check that \(v = K H^{-1}(\int_0 u_r dr)\) satisfies (1) of Lemma 3.5. Furthermore, similarly to the proof Lemma 3.5 we can obtain that

\[
|v_s| \leq CT^{1/2-H}(1 + \|X\|_\infty^\rho),
\]

where \(\|X\|_\infty \triangleq \sup_{0 \leq s \leq T} |X_s|\). Applying Grownall’s inequality again it is easy to show that

\[
\|X\|_\infty \leq (|x| + \|B_H\|_\infty + CT + |L|) e^{CT},
\]

which then leads to (2) of Lemma 3.5.

We now assume \(H > \frac{1}{2}\). Following the same argument of Lemma 4.1 it suffices to show that between two jump times of \(L\), the process \(u = b(\cdot, X,\) \(\in I_{\sigma k+1}^{H-1/2}(L^2(\sigma k-1, \sigma k)),\) \(\mathbb{P}\)-almost surely. But note that between two jumps we have, by Assumption 3.3.

\[
|b(t, X_t) - b(s, X_s)| \leq C\{t - s}^\gamma + |X_t - X_s|\}
\]

\[
\leq C\{|t - s}^\gamma + \int_s^t b(u, X_u) du\|_{\alpha} + |B_t^H - B_s^H|\}
\]

\[
\leq C\{|t - s}^\gamma + \int_s^t (|b(0, x)| + |x|^\gamma + |X_u - x|^\gamma) du\|_{\alpha} + |B_t^H - B_s^H|\}
\]

\[
\leq C\{|t - s}^\gamma + (|b(0, x)| + |T|\gamma + \|X\|_\infty^\alpha + |x|\}|t - s}^\alpha + |B_t^H - B_s^H|\alpha\}.\]
Since $\gamma > H - \frac{1}{2}$ and $\alpha > 1 - \frac{1}{2H} > H - \frac{1}{2}$, we see that between jumps the paths $t \mapsto b(t, X_t)$ are Hölder continuous of order $H - \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$. By the same argument as in Section 4, it can be checked that $\mathbb{P}\{v \in L^2([0, T])\} = 1$. Using the estimates

$$
|b(t, X_t)| \leq C(|b(0, x)| + t^\gamma + |X_t - x|^{\alpha})
$$

and $\|X\|_\infty \leq C(1 + |x| + \|H^2\|_\infty + |L_T|)$, where $C$ is a generic constant depending only on $\alpha, H, \gamma$, and $T$, which may vary from line to line, we deduce that

$$
\left| \int_0^t u_s ds \right|^\alpha \leq C(|b(0, x)| + t^\gamma + |x|^{\alpha} + \|X\|_\infty^{\alpha} t^{\alpha} \leq C(1 + |b(0, x)| + t^\gamma + |x|^{\alpha} + \|B^H\|_\infty + |L_T|)^{\alpha} T^{\alpha} \leq C[1 + |b(0, x)|^{\alpha} + t^{\alpha \gamma} + |x|^{\alpha} + \|B^H\|_\infty^{\alpha} + |L_T|^\alpha].
$$

Furthermore, denoting $u = b(\cdot, X_t)$ again, one can also check that

$$
|A^k(t)| \leq C \max_{1 \leq i \leq k} |b(t, B^H_t + \int_0^t u_s ds - S_{i-1} + x) - b(t, B^H_t + \int_0^t u_s ds + x)| \times
\left| \sum_{i=1}^{k-1} \frac{1}{(t - \sigma_i)^{H-1/2}} - \frac{1}{(t - \sigma_{i-1})^{H-1/2}} \right| + \frac{1}{(t - \sigma_k)^{H-1/2}} \right|
+C^H t^{1/2 - H} \left| b(t, B^H_t + \int_0^t u_s ds + x) \right|
$$

(5.5)

$$
\leq C \left\{ (t - \sigma_{k-1})^{1/2 - H} |L|_{T}^{\alpha} + t^{1/2 - H} \left| b(0, x) \right| + t^{\gamma} + \|B^H\|_\infty^{\alpha} + \int_0^t |u_s ds|^{\alpha} \right\}
\leq C \left\{ (t - \sigma_{k-1})^{1/2 - H} |L|_{T}^{\alpha} + t^{1/2 - H} \|B^H\|_\infty^{\alpha} + t^{1/2 - H} [1 + |x| + |b(0, x)| + |t|^{\gamma} + |L_T|] \right\}
\leq C \left\{ (t - \sigma_{k-1})^{1/2 - H} |L|_{T}^{\alpha} + t^{1/2 - H} \|B^H\|_\infty^{\alpha} + t^{1/2 - H} (1 + |x| + |b(0, x)| + |t|^{\gamma}) \right\};
$$

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We can follow the same arguments of Lemma 4.1 to show that

\[ v = \text{max}_{1 \leq i \leq k} |b(t, B^H_t + \int_0^t u_s ds - S_{i-1} + x)|^{1/2 - H} + t^{\gamma + 1/2 - H} \]

Next, we show that the pathwise uniqueness holds for solutions to (3.1). The proof is more or less standard, see [19] or [26], we provide a sketch for completeness.

**Theorem 5.2** Suppose that Assumption 3.3 holds. Then two weak solutions of SDE (3.1) defined on the same filtered probability space with the same driving fBM \( B^H \) and Poisson point process \( L \) must coincide almost surely on their common existence interval.

**Proof.** Let \( X^1 \) and \( X^2 \) be two weak solutions defined on the same filtered probability space with the same driving \( B^H \) and \( L \). Define \( Y^+ \triangleq X^1 \lor X^2 \), and \( Y^- \triangleq X^1 \land X^2 \). One shows that both \( Y^+ \) and \( Y^- \) both satisfy (3.1). In fact, note that \( X^1 - X^2 \) involves only Lebesgue integral, the occupation density formula yields that the local time of \( X^1 - X^2 \) at 0 is identically zero. Thus, by Tanaka formula,

\[(X^1_t - X^2_t)^+ = \int_0^t (b(s, X^1_s) - b(s, X^2_s))I_{\{X^1_t - X^2_t > 0\}} ds.\]

Then, note that \( Y^+ = X^2 + (X^1 - X^2)^+ \), we have

\[ Y^+_t = x + \int_0^t b(s, X^2_s) ds + B^H_t - L_t + \int_0^t (b(s, X^1_s) - b(s, X^2_s))I_{\{X^1_t - X^2_t > 0\}} ds \]

\[ = x + \int_0^t b(s, Y^+_s) ds + B^H_t - L_t. \]

We can follow the same arguments of Lemma 4.1 to show that \( v \) also satisfies the Novikov condition (3.3), proving the theorem.
Similarly one shows that \( Y_t^- \) satisfies SDE (3.1) as well. We claim that
\[
P\left\{ \sup_{0 \leq t \leq T} (Y_t^+ - Y_t^-) = 0 \right\} = 1. \tag{5.8}
\]
Indeed, if not, then \( P\left\{ \sup_{0 \leq t \leq T} (Y_t^+ - Y_t^-) > 0 \right\} > 0 \). Then, there exists a rational number \( r \) and \( t > 0 \) such that \( P(Y_t^+ > r > Y_t^-) > 0 \). But since \( \{Y_t^+ > r\} = \{Y_t^- > r\} \cup \{Y_t^+ > r \geq Y_t^-\} \), we have
\[
P(Y_t^+ > r) = P(Y_t^- > r) + P(Y_t^+ > r > Y_t^-) > P(Y_t^- > r).
\]
This contradicts with the fact that \( Y_t^+ \) and \( Y_t^- \) have the same law, thanks to Theorem \( 5.1 \). Thus, (5.8) holds, and consequently, \( X^1 \equiv X^2 \), \( P \)-a.s., proving the theorem.

6 Existence of strong solutions

Having proved the existence of the weak solution and pathwise uniqueness, it is rather tempting to invoke the well-known Yamada-Watanabe Theorem to conclude the existence of the strong solution. But there seem to be some fundamental difficulties in the proof of such a result, mainly because of the lack of the independent increment property for an fBM, which is crucial in the proof. However, it is also well-known that, unlike an ODE, in the case of stochastic differential equations, the existence of the strong solution could be argued with assumptions on the coefficients being much weaker than Lipschitz, due to the presence of the “noise”. We note that the argument in this section is quite similar to \( [12] \) and \( [22] \), with some necessary adjustments for the presence of the jumps.

We begin by observing that the SDE (3.1) can be solved pathwisely, as an ODE, when the coefficient \( b \) is regular enough (e.g., continuous in \((t, x)\), and uniformly Lipschitz in \( x \)). Second, we claim that, under Assumption 3.3 it suffices to prove the existence of the strong solution when the coefficient \( b \) is uniformly bounded. Indeed, if we consider the following family of SDEs:
\[
X_t = x + \int_0^t b_R(s, X_s)ds + B_t^H - L_t, \quad t \in [0, T], \quad R > 0, \tag{6.1}
\]
where \( b_R \) is the truncated version of \( b \): \( b_R(t, x) = b(t, (x \wedge R) \lor (-R)) \), \((t, x) \in [0, T] \times \mathbb{R}\), then for each \( R \), \( b_R \) is bounded, hence (6.1) has a strong solution, denoted by \( X^R \), defined on \([0, T]\), and we can now assume that they all live on a common probability space. Now note that for \( R_1 < R_2 \), one has \( b_{R_1} \equiv b_{R_2} \) whenever \( |x| \leq R_1 \), thus by the pathwise uniqueness, it is easy to see that \( X_t^{R_1} \equiv X_t^{R_2} \), for \( t \in [0, \tau_{R_1}] \), \( P \)-a.s., where \( \tau_{R} \triangleq \inf\{t > 0 : |X_t^R| \geq R\} \wedge T \). Therefore we can almost surely extend the solution to \([0, \tau]\), where \( \tau \triangleq \lim_{R \to \infty} \tau_R \). Furthermore, it was shown (see, e.g., (5.4)) that \( X \) will never explode on \([0, \tau]\). Consequently, we must have \( \tau = T \), \( P \)-a.s.
We now give our main result of this section.

**Theorem 6.1** Assume that \( b(t, x) \) satisfies Assumption [3.3]. Then there exists a unique strong solution SDE [3.1].

The proof of Theorem 6.1 follows an argument by Gyöngy-Pardoux [12], using the so-called Krylov estimate (cf. [16]). We note that by the argument preceding the Theorem we need only consider the case when the coefficient \( b \) is bounded. The following lemma is thus crucial.

**Lemma 6.2** Suppose that the coefficient \( b \) satisfies Assumption [3.3] and is uniformly bounded by a constant \( C > 0 \). Suppose also that \( X \) is a strong solution to SDE [3.1]. Then, there exist \( \beta > 1 \) and \( \gamma > 1 + H \) such that for any measurable nonnegative function \( g : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \), it holds that

\[
E \int_0^T g(t, X_t) dt \leq G \left( \int_0^T \int_{\mathbb{R}} g^{\beta \gamma}(t, x) dx dt \right)^{1/\beta \gamma},
\]

where \( 1/\alpha + 1/\beta = 1, 1/\gamma + 1/\gamma' = 1 \), and \( G \) is a constant defined by

\[
G \overset{\triangle}{=} J^{1/\gamma' \beta} K^{1/\alpha},
\]

in which

\[
K \overset{\triangle}{=} \left\{ \mathbb{E} \exp \left\{ 2\alpha^2 \int_0^T v_t^2 dt \right\} \right\}^{1/2}, \quad J \overset{\triangle}{=} \frac{(2\pi)^{1/2-\gamma'/2} T^{1+(1-\gamma')H}}{\sqrt{\gamma'}}.
\]

**Proof.** Let \( (\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F}) \) be a filtered probability space on which are defined a fBM \( B^H \), a Poisson point process \( L \) of class (QL) and independent of \( B^H \), and \( X \) is the strong solution to the corresponding SDE [3.1]. Let \( W \) be an \( \mathbb{F} \)-Brownian motion such that \( B^H = \int_0^\cdot K_H(t, s) dW_s \). Recall from [5.2] the process \( v = K^{-1}_H \left( \int_0^\cdot b(r, X_r) dr \right) \), and define a new measure \( \tilde{\mathbb{P}} \) by

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp \left\{ -\int_0^T v_t dW_t - \frac{1}{2} \int_0^T v_t^2 dt \right\} \overset{\triangle}{=} Z_T(v).
\]

Then, in light of Lemmas [3.5] and [4.1] we know that \( \tilde{\mathbb{P}} \) is a probability measure under which \( \tilde{W}_t = W_t + \int_0^t v_r dr \) is a Brownian motion, \( \tilde{B}^H_t = \int_0^t K_H(t, s) d\tilde{W}_s \) is a fBM, and \( L \) remains a Poisson point process with same parameters and is independent of \( \tilde{B}^H \). Hence, under \( \tilde{\mathbb{P}} \), \( X_t = x + \tilde{B}^H_t - L_t \) has the density function:

\[
p_t(y) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{(y+z-x)^2}{2t^H}} \sum_{n=1}^{\infty} f^n(z) \frac{(\lambda t)^n}{n!} e^{-\lambda t} dz, \quad (t, y) \in [0, T] \times \mathbb{R},
\]

where \( f \) is the density of the jump size distribution, and \( f^n \) denotes the \( n \)-th convolution of \( f \).
Now, applying Hölder’s inequality we have
\[ \mathbb{E} \int_0^T g(t, X_t) dt = \tilde{\mathbb{E}} \left\{ Z_T \int_0^T g(t, X_t) dt \right\} \leq \left\{ \tilde{\mathbb{E}} [Z_T^n] \right\}^{1/\alpha} \left\{ \tilde{\mathbb{E}} \int_0^T g^{\beta}(t, X_t) dt \right\}^{1/\beta}, \quad (6.5) \]
where \( 1/\alpha + 1/\beta = 1. \) Rewriting \( v_t \) as \( v_t = K_{H}^{-1} \left( \int_0^t b(r, \tilde{B}_r^H - L_r + x) dr \right) (t), \) we can follow the same argument as the proof of Lemma 3.5 and 4.1 to get, \( \tilde{\mathbb{E}} e^{2\alpha \int_0^T v_t^2 dt} < \infty. \) Therefore, \( \exp \left\{ 2\alpha \int_0^T v_t d\tilde{W}_t - 2\alpha^2 \int_0^T v_t^2 dt \right\} \) is a \( \tilde{\mathbb{P}} \)-martingale, and consequently, applying Hölder’s inequality we obtain
\[
\tilde{\mathbb{E}} [Z_T^n] = \tilde{\mathbb{E}} \exp \left\{ \alpha \int_0^T v_t dW_t + \frac{\alpha}{2} \int_0^T v_t^2 dt \right\} \\
= \tilde{\mathbb{E}} \exp \left\{ \alpha \int_0^T v_t dW_t - \frac{\alpha}{2} \int_0^T v_t^2 dt \right\} \\
= \tilde{\mathbb{E}} \exp \left\{ \alpha \int_0^T v_t dW_t - 2\alpha^2 \int_0^T v_t^2 dt + (\alpha^2 - \frac{\alpha}{2}) \int_0^T v_t^2 dt \right\}, \quad (6.6) \\
\leq \left( \tilde{\mathbb{E}} \exp \left\{ 2\alpha \int_0^T v_t dW_t - 2\alpha^2 \int_0^T v_t^2 dt \right\} \right)^{1/2} \left( \tilde{\mathbb{E}} \exp \left\{ (2\alpha^2 - \alpha) \int_0^T v_t^2 dt \right\} \right)^{1/2} \\
\leq \left( \tilde{\mathbb{E}} \exp \left\{ 2\alpha^2 \int_0^T v_t^2 dt \right\} \right)^{1/2} < \infty. 
\]
On the other hand, applying Hölder’s inequality with \( 1/\gamma + 1/\gamma' = 1, \gamma > H + 1 \) yields
\[
\tilde{\mathbb{E}} \int_0^T g^\beta(t, X_t) dt = \int_0^T \int_{\mathbb{R}} g^\beta(t, y)p_t(y) dy dt \leq \|g^\beta\|_{L^\gamma([0,T] \times \mathbb{R})} \|p_t(\cdot)\|_{L^{\gamma'}([0,T] \times \mathbb{R})}. \quad (6.7) 
\]
Now, by the generalized Minkowski inequality (cf. e.g., [23] (1.33)), we have
\[
\int_{\mathbb{R}} [p_t(y)]^{\gamma'} dy = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{(y+z-x)^2}{2t^{2H}}} \sum_{n=1}^\infty f^n(z) \left( \frac{\lambda t}{n!} \right)^n e^{-\lambda t} dz \right\}^{\gamma'} dy \\
\leq \left\{ \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{(y+z-x)^2}{2t^{2H}}} \sum_{n=1}^\infty f^n(z) \left( \frac{\lambda t}{n!} \right)^n e^{-\lambda t} dy \right]^{1/\gamma'} dy \right\}^{\gamma'} \\
= \left\{ \int_{\mathbb{R}} \sum_{n=1}^\infty f^n(z) \left( \frac{\lambda t}{n!} \right)^n e^{-\lambda t} \left[ \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{(y+x-z)^2}{2t^{2H}}} dy \right]^{1/\gamma'} dz \right\}^{\gamma'}. \quad (6.8) 
\]
The direct calculation gives
\[
\int_{\mathbb{R}} \left( \frac{1}{\sqrt{2\pi t^H}} e^{-\frac{(y+z-x)^2}{2t^{2H}}} \right)^{\gamma'} dy = (2\pi)^{1/2} \gamma'/2 (\gamma')^{-1/2} t^{1-\gamma'} H. 
\]
Plugging into (6.8) we obtain
\[
\int_{\mathbb{R}} [p_t(y)]^{\gamma'} dy = (2\pi)^{1/2} \gamma'/2 (\gamma')^{-1/2} t^{1-\gamma'} H \left( \int_{\mathbb{R}} \sum_{n=1}^\infty f^n(z) \left( \frac{\lambda t}{n!} \right)^n e^{-\lambda t} dz \right)^{\gamma'} \\
\quad = (2\pi)^{1/2} \gamma'/2 (\gamma')^{-1/2} t^{1-\gamma'} H. 
\]
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Since $\gamma > H + 1$, this leads to that
\[
\|p(.\|_L^{\gamma'}([0,T] \times \mathbb{R}) \leq J^{1/\gamma'},
\] (6.9)
where $J$ is defined by (6.3). Finally, noting that $\|g_1^{\beta}\|_L^{1/\gamma'} = \|g\|_L^{\beta/\gamma'}([0,T] \times \mathbb{R})$, the estimate (6.2) then follows from (6.5), (6.6), (6.7), and (6.9).

[Proof of Theorem 6.1] Since the proof is more or less standard, we only give a sketch for the completeness. We refer to [16], [12] and/or [22] for more details.

We need only prove the existence. We assume that the coefficient $b$ is bounded (by $C > 0$) and satisfies Assumption 3.3. Let $\{b_n(\cdot, \cdot)\}_{n=1}^{\infty}$ be a sequence of the mollifiers of $b$, so that all $b_n$’s are smooth, have the same bound $C$, and satisfy Assumption 3.3 with the same parameters.

Next, for $n \leq k$ we define $\tilde{b}_{n,k} \triangleq \bigwedge_{j=n}^{k} b_j$ and $\tilde{b}_n \triangleq \bigwedge_{j=n}^{\infty} b_j$. Then clearly, each $\tilde{b}_{n,k}$ is continuous, and uniformly Lipschitz in $x$, uniformly with respect to $t$. Further, it holds that
\[
\tilde{b}_{n,k} \downarrow \tilde{b}_n, \text{ as } k \to \infty,
\]
\[
\tilde{b}_n \uparrow b, \text{ as } n \to \infty,
\]
for almost all $x$. Now for fixed $n, k$, consider SDE
\[
X_t = x + \int_{0}^{t} \tilde{b}_{n,k}(s, X_s)ds + B_t^H - L_t, \quad t \geq 0.
\] (6.10)
As a pathwise ODE, (6.10) has a unique strong solution $\tilde{X}^{n,k}$, and comparison theorem holds. That is, $\{\tilde{X}^{n,k}\}$ decrease with $k$. Furthermore, since $\tilde{b}_{n,k}$’s are uniformly bounded by $C$, the solutions $\tilde{X}^{n,k}$ are pathwisely uniformly bounded, uniformly in $n$ and $k$. Thus $X^n_t \triangleq \lim_{k \to \infty} \tilde{X}^{n,k}_t$ exists, for all $t \in [0,T]$, $\mathbb{P}$-a.s. Since $b_n$’s are still Lipschitz, the standard stability result of ODE then implies that $\tilde{X}^n$ solves
\[
X_t = x + \int_{0}^{t} \tilde{b}_n(s, X_s)ds + B_t^H - L_t, \quad t \in [0,T].
\]
Furthermore, the Dominated Convergence Theorem leads to that the estimate (6.2) holds for all $X^n$’s, for any bounded measurable function $g$.

Next, since $\tilde{X}^{n,k} \leq \tilde{X}^{m,k}$, for $n \leq m \leq k$, we see that $\tilde{X}_n$ increases as $n$ increases, thus $\tilde{X}^n$ converges, $\mathbb{P}$-almost surely, to some process $X$. The main task remaining is to show that $X$ solves SDE (3.1), as $b$ is no longer Lipschitz. In other words, we shall prove that
\[
\lim_{n \to \infty} \mathbb{E} \int_{0}^{T} |\tilde{b}_n(t, X^n_t) - b(t, X_t)|dt = 0.
\] (6.11)
To see this, we first note that
\[
\mathbb{E} \int_{0}^{T} |\tilde{b}_n(t, X^n_t) - b(t, X_t)|ds \leq I^n_1 + I^n_2,
\] (6.12)
where
\[
I_1^n \triangleq \sup_k \mathbb{E} \int_0^T |\tilde{b}_k(t, X_t^n) - \tilde{b}_k(t, X_t)|dt, \quad I_2^n \triangleq \mathbb{E} \int_0^T |\tilde{b}_n(t, X_t) - b(t, X_t)|dt. \tag{6.13}
\]

Let \(\kappa : \mathbb{R} \to \mathbb{R}\) be a smooth truncation function satisfying \(0 \leq \kappa(z) \leq 1\) for every \(z, \kappa(z) = 0\) for \(|z| \geq 1\) and \(\kappa(0) = 1\). Then by Bounded Convergence Theorem one has
\[
\lim_{R \to \infty} \mathbb{E} \int_0^T (1 - \kappa(X_t/R))dt = 0. \tag{6.14}
\]

Now for any \(R > 0\), we apply Lemma 6.2 and note that both \(\tilde{b}_n\) and \(b\) are bounded by \(C\) to get
\[
I_2^n = \mathbb{E} \int_0^T \kappa(X_t/R)|\tilde{b}_n(t, X_t) - b(t, X_t)|dt + \mathbb{E} \int_0^T (1 - \kappa(X_t/R))|\tilde{b}_n(t, X_t) - b(t, X_t)|dt
\leq G \left( \int_0^T \int_{-R}^R |\tilde{b}_n(t, x) - b(t, x)|^2 dx dt \right)^{1/2} + 2CE \int_0^T (1 - \kappa(X_t/R))dt. \tag{6.15}
\]

First letting \(n \to \infty\) and then letting \(R \to \infty\) we get \(\lim_{n \to \infty} I_2^n = 0\).

To show that \(\lim_{n \to \infty} I_1^n = 0\), we first note that by (6.14), for any \(\varepsilon > 0\), there exists \(R_0\) such that
\[
\mathbb{E} \int_0^T |1 - \kappa(X_t/R_0)|dt < \varepsilon. \tag{6.16}
\]

Second, since \(\{b_n\}\) converge to \(b\) almost everywhere, the Bounded Convergence Theorem then shows that \(\tilde{b}_n\) converges to \(b\) in \(L^2_{T,R_0} \triangleq L^2([0, T] \times [-R_0, R_0])\), hence \(\{b_n, b\}_{n \geq 1}\) is a compact set in \(L^2_{T,R_0}\). Thus, we can find finitely many bounded smooth function \(H_1, \cdots, H_N\) such that for each \(k\), there is a \(H_{ik}\) so that
\[
\left( \int_0^T \int_{-R_0}^{R_0} |\tilde{b}_k(t, x) - H_{ik}(t, x)|^2 dr dt \right)^{1/2} < \varepsilon. \tag{6.17}
\]

Now, we write
\[
I_1^n = \mathbb{E} \int_0^T |\tilde{b}_k(t, X_t^n) - \tilde{b}_k(t, X_t)|dt \leq I_1(n, k) + I_2(n) + I_3(k),
\]
where
\[
\begin{align*}
I_1(n, k) &= \mathbb{E} \int_0^T |\tilde{b}_k(t, X_t^n) - H_{ik}(t, X_t^n)|dt; \\
I_2(n) &= \sum_{j=1}^N \mathbb{E} \int_0^T |H_j(t, X_t^n) - H_j(t, X_t)|dt; \\
I_3(k) &= \mathbb{E} \int_0^T |\tilde{b}_k(t, X_t) - H_{ik}(t, X_t)|dt.
\end{align*}
\]
It is obvious that \( \lim_{n \to \infty} I_2(n) = 0 \). Further, since the estimate \([6.2]\) holds for all \( X^n \)'s, similar to \([6.15]\) we have

\[
I_1(n, k) \leq G \left( \int_0^T \int_{-R_0}^{R_0} \left| \tilde{b}_k(t, x) - H_k(t, x) \right|^2 dx dt \right)^{1/2} + C_1 \mathbb{E} \int_0^T \left( 1 - \kappa \left( X_t^{(n)} / R_0 \right) \right) dt,
\]

where \( G \) is defined by \([6.3]\) with \( \beta \gamma = 2 \), and \( C_1 \) is a constant depending on \( C \) and \( \max_{1 \leq i \leq N} \| H_i \|_{\infty} \).

Hence, by \([6.16]\), \([6.17]\), and the Dominated Convergence Theorem again we have

\[
\lim_{n \to \infty} \sup_k I_1(n, k) \leq G \varepsilon + C_1 \mathbb{E} \int_0^T \left( 1 - \kappa \left( X_t / R_0 \right) \right) dt \leq (G + C_1) \varepsilon.
\]

Similarly, we have \( \sup_k I_3(k) \leq (G + C_1) \varepsilon \). Letting \( \varepsilon \to 0 \) we obtain \( \lim_{n \to \infty} I^n_1 = 0 \). The proof is now complete.

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