Upper bound for the first non-zero eigenvalue of the $p$-Laplacian

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Abstract  
Let $M$ be a closed hypersurface in $\mathbb{R}^n$ and $\Omega$ be a bounded domain such that $M = \partial \Omega$. In this article, we obtain an upper bound for the first non-zero eigenvalue of the following problems.

• Closed eigenvalue problem:
  \[ \Delta_p u = \lambda_p |u|^{p-2} u \quad \text{on} \quad M. \]

• Steklov eigenvalue problem:
  \[ \Delta_p u = 0 \quad \text{in} \quad \Omega, \]
  \[ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \mu_p |u|^{p-2} u \quad \text{on} \quad M. \]

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1 Introduction

The $p$-Laplace operator, defined as $\Delta_p u := -\text{div} \left( |\nabla u|^{p-2} \nabla u \right)$, is the nonlinear generalization of the usual Laplace operator.

Many interesting results, providing the sharp upper bounds for the first non zero eigenvalue of the usual Laplacian ($p = 2$) have been obtained. In [4], Bleecker and Weiner obtained a sharp upper bound of the first non-zero eigenvalue of Laplacian in terms of the second fundamental form on a hypersurface $M$ in $\mathbb{R}^n$. In [5], Reilly gave an upper bound for the first non-zero eigenvalue in terms of higher order mean curvatures for a compact $n$-dimensional manifold isometrically immersed in $\mathbb{R}^{n+p}$, which improves the earlier estimate. This result was later extended to submanifolds of simply connected space forms in various ways ( see [6, 10]). These upper bounds are extrinsic in the sense that they depend either on the length of the second fundamental form or the higher order mean curvatures of $M$.

Let $M$ be a hypersurface in a rank-1 symmetric space. In [11], an upper bound for the first non-zero eigenvalue of $M$ was obtained in terms of the integral of the first non-zero eigenvalue of the geodesic spheres centered at the centre of gravity of $M$.

For a closed hypersurface $M$ contained in a ball of radius less than $\frac{i(M(k))}{4}$ and bounding a convex domain such that $\partial \Omega = M$ in the simply connected space form $\mathbb{M}(k)$, $k = 0$ or 1, Santhanam [12] proved that

\[ \frac{\lambda_1(M)}{\lambda_1(S(R))} \leq \frac{\text{Vol}(M)}{\text{Vol}(S(R))}, \]
where \( S(R) = \partial B(R) \) is the geodesic sphere of radius \( R > 0 \) such that \( \text{Vol}(B(R)) = \text{Vol}(\Omega) \). A similar result was also obtained for \( k = -1 \).

In this article, we extend the results in [12] to \( p \)-Laplacian for a closed hypersurface \( M \subset \mathbb{R}^n \). In particular, we consider the closed eigenvalue problem

\[
\Delta_p u = \lambda_p |u|^{p-2} u \quad \text{on } M,
\]

where \( M \) is a closed hypersurface in \( \mathbb{R}^n \) and find an upper bound for the first non-zero eigenvalue of this problem.

Let \( M \) be a closed hypersurface in \( \mathbb{R}^n \) and \( \Omega \) be the bounded domain such that \( M = \partial \Omega \). Consider the following problem

\[
\Delta f = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial f}{\partial \nu} = \mu f \quad \text{on } \partial \Omega,
\]

where \( \nu \) is the outward unit normal on the boundary \( \partial \Omega \) and \( \mu \) is a real number. This problem is known as the Steklov eigenvalue problem and was introduced by Steklov [1] for bounded domains in the plane in 1902. This problem is important as the set of eigenvalues of the Steklov problem is the same as the set of eigenvalues of the well-known Dirichlet-Neumann map.

There are several results which estimate the first non-zero eigenvalue \( \mu_1 \) of the Steklov eigenvalue problem [3, 7, 8, 9, 13]. The first isoperimetric upper bound for \( \mu_1 \) was given by Weinstock [2] in 1954. He proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes \( \mu_1 \). In [3], Payne obtained a two sided bound for the first non-zero Steklov eigenvalue on a convex planar domain in terms of minimum and maximum curvature. The lower bound in [3] has been generalized by Escobar [7] to 2-dimensional compact manifolds with non-negative Gaussian curvature. Using the Weinstock inequality, Escobar [8] proved that for a fixed volume, among all bounded simply connected domain in 2-dimensional simply connected space forms, geodesic balls maximize the first non-zero Steklov eigenvalue. This result has been extended to non-compact rank-1 symmetric spaces in [13]. We prove the similar result for the first non-zero eigenvalue of the eigenvalue problem

\[
\Delta_p u = 0 \quad \text{in } \Omega,
\]

\[
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \mu_p |u|^{p-2} u \quad \text{on } M,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) such that \( M = \partial \Omega \) and \( \nu \) is outward unit normal on \( M \).

In Section 2, we state our main results. In section 3, we state some basic facts about the first non-zero eigenvalues of problem (1) and (2), and prove some results which will be required in the later sections. Followed by this, in section 4, 5 and 6, we provide the proof of results stated in section 2.

### 2 Statement of the results

We state a variation of centre of mass theorem. This is crucial for our proof of main results.

**Theorem 1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( M = \partial \Omega \). Then for every real number \( 1 < p < \infty \), there exists a point \( t \in \Omega \) depending on \( p \) and normal coordinate system \((X_1, X_2, \ldots, X_n)\) centered at \( t \) such that for \( 1 \leq i \leq n \),

\[
\int_M |X_i|^{p-2} X_i = 0.
\]

Now we state our main results.

The following theorem provides an upper bound for the first non-zero eigenvalue \( \lambda_{1,p} \) of the closed eigenvalue problem (1).
Theorem 2. Let $M$ be a closed hypersurface in $\mathbb{R}^n$ bounding a bounded domain $\Omega$. Let $R > 0$ be such that $\text{Vol}(\Omega) = \text{Vol}(B(R))$, where $B(R)$ is a ball of radius $R$. Then the first non-zero eigenvalue $\lambda_{1,p}$ of the closed eigenvalue problem (1) satisfies

$$
\lambda_{1,p} \leq n^{\frac{p}{p-2}} \lambda_1(S(R)) \left( \frac{\text{Vol}(M)}{\text{Vol}(S(R))} \right)^{\frac{p}{p-2}}.
$$

Furthermore, for $p = 2$, the upper bound (3) is sharp and the equality holds if and only if $M$ is a geodesic sphere of radius $R$ (see [12]).

If equality holds in (3) then $M$ is a geodesic sphere and $p = 2$.

In case of Steklov eigenvalue problem, we have the following upper bound for the first non-zero eigenvalue $\mu_{1,p}$.

Theorem 3. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $M$ and $R > 0$ be such that $\text{Vol}(\Omega) = \text{Vol}(B(R))$, where $B(R)$ is a ball of radius $R$. Then the first non-zero eigenvalue $\mu_{1,p}$ of problem (2) satisfies the following inequality.

- For $1 < p < 2$,

$$
\mu_{1,p} \leq \frac{1}{R^{p-1}}.
$$

- For $p \geq 2$,

$$
\mu_{1,p} \leq \frac{n^{p-2}}{R^{p-1}}.
$$

Furthermore, for $p = 2$, equality holds in (4) and (5) iff $M$ is a geodesic sphere of radius $R$ (see [13]).

If equality holds in (4) and (5) then $M$ is a geodesic sphere of radius $R$ and $p = 2$.

3 Preliminaries

In this section, we state some basic facts about the first non-zero eigenvalue of the eigenvalue problems (1) and (2). We will also prove some results that are needed in subsequent sections.

Let $u_1$ be an eigenfunction corresponding to the eigenvalue $\lambda_p$ of closed eigenvalue problem (1) and $u_2$ be an eigenfunction corresponding to the eigenvalue $\mu_p$ of the Steklov eigenvalue problem (2). Then $\lambda_p$ and $\mu_p$ satisfy

$$
\lambda_p \int_M |u_1|^p = \int_M \|\nabla_M u_1\|^p,
$$

$$
\mu_p \int_M |u_1|^p = \int_\Omega \|\nabla u_1\|^p.
$$

This shows that all eigenvalues of problems (1) and (2) are non-negative.

Let $\lambda_{1,p}$ and $\mu_{1,p}$ be the first non-zero eigenvalues of the closed and steklov eigenvalue problems, respectively. Then the variational characterization for $\lambda_{1,p}$ and $\mu_{1,p}$ is given by

$$
\lambda_{1,p} = \inf \left\{ \frac{\int_M \|\nabla_M u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2} u = 0, u(\neq 0) \in C^1(M) \right\},
$$

$$
\mu_{1,p} = \inf \left\{ \frac{\int_\Omega \|\nabla u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2} u = 0, u(\neq 0) \in C^1(\Omega) \right\}.
$$
Remark 4. If \( p = 2 \), then the condition \( \int_M |u|^{p-2} u = \int_M u = 0 \) is equivalent to say that the test function must be orthogonal to the constant function in \( L^2 \)-norm.

Let \( M \) be a closed hypersurface in \( \mathbb{R}^n \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) such that \( M = \partial \Omega \). Fix a point \( q \in \Omega \). Then for every point \( s \in M \), the line joining \( q \) and \( s \) may intersect \( M \) at some points other than \( s \). For every point \( s \in M \), let \( r(s) := d(q, s) \) and for every \( u \in \mathbb{S}^{n-1} \), let \( \beta(s) := \max \{ \beta > 0 \mid q + \beta u \in M, \beta \in \mathbb{R} \} \). Let \( A := \{ q + \beta(u) u \mid u \in \mathbb{S}^{n-1} \} \). Then \( A \subseteq M \).

Lemma 1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( M \) and \( R > 0 \) be such that \( \text{Vol} (\Omega) = \text{Vol} (B(R)) \), where \( B(R) \) is a ball of radius \( R \). Fix a point \( q \in \Omega \), then

\[
\int_M r^p(s) \, ds \geq R^p \text{Vol} (S(R)).
\]

Further, equality holds in (6) iff \( M \) is a geodesic sphere of radius \( R \) centered at \( q \).

Proof. For a point \( s \in A \), let \( \gamma_s \) be the unique unit speed geodesic joining \( q \) and \( s \) with \( \gamma_s(0) = q \). Let \( u = \gamma'_s(0) \) and \( t_s(u) = d(q, s) \). Let \( \theta(s) \) be the angle between the outward unit normal \( \nu(s) \) to \( M \) and the radial vector \( \partial r(s) \). Let \( du \) be the spherical volume density of the unit sphere \( \mathbb{S}^{n-1} \). Then

\[
\int_M r^p(s) \, ds \geq \int_A r^p(s) \, ds
= \int_{\mathbb{S}^{n-1}} (t_s(u))^p \sec \theta(s) (t_s(u))^{n-1} \, du
\geq \int_{\mathbb{S}^{n-1}} (t_s(u))^{n+p-1} \, du
= (n + p - 1) \int_{\mathbb{S}^{n-1}} \int_0^{t_s(u)} r^{n+p-2} \, dr \, du
\geq (n + p - 1) \int_{\Omega} r^{p-1} \, dV
\]

and

\[
\int_{\Omega} r^{p-1} \, dV = \int_{\Omega \setminus B(R)} r^{p-1} \, dV + \int_{\Omega \setminus (\Omega \cap B(R))} r^{p-1} \, dV
= \int_{B(R)} r^{p-1} \, dV - \int_{B(R) \setminus (\Omega \cap B(R))} r^{p-1} \, dV + \int_{\Omega \setminus (\Omega \cap B(R))} r^{p-1} \, dV
\geq \int_{B(R)} r^{p-1} \, dV - \int_{B(R) \setminus (\Omega \cap B(R))} r^{p-1} \, dV + \int_{\Omega \setminus (\Omega \cap B(R))} R^{p-1} \, dV
= \int_{B(R)} r^{p-1} \, dV + \int_{B(R) \setminus (\Omega \cap B(R))} (R^{p-1} - r^{p-1}) \, dV
\geq \int_{B(R)} r^{p-1} \, dV
= \int_{\mathbb{S}^{n-1}} \int_0^R r^{n+p-2} \, dr \, du
= \int_{\mathbb{S}^{n-1}} \frac{R^{n+p-1}}{n + p - 1} \, du
= \frac{R^p}{n + p - 1} \text{Vol} (S(R)).
\]

We have used the fact that \( R \leq r \in (\Omega \setminus (\Omega \cap B(R))) \) in (7).

Further, equality holds in (6) iff \( \sec \theta(s) = 1 \) for all points \( s \in M \) and \( \text{Vol} (B(R) \setminus (\Omega \cap B(R))) = 0 \). Note that \( \sec \theta(s) = 1 \) iff \( \theta(s) = 0 \) for all points \( s \in M \). Therefore outward unit normal
\( \nu(s) = \partial r(s) \) for all points \( s \in M \). This shows that \( \Omega = B(q, R) \) and \( M \) is a geodesic sphere of radius \( R \).

Above lemma is the generalization of the Lemma 1 in [12].

**Lemma 2.** Let \( n \in \mathbb{N} \) and \( y_1, y_2, \ldots, y_n \) be non-negative real numbers. Then for every real number \( \gamma \geq 1 \), the following inequality holds.

\[
(y_1 + y_2 + \cdots + y_n)^\gamma \geq y_1^\gamma + y_2^\gamma + \cdots + y_n^\gamma. \tag{8}
\]

**Proof.** Let \( n \in \mathbb{N} \) and \( y_1, y_2, \ldots, y_n \) be non-negative real numbers. Let \( \gamma \geq 1 \). Then inequality (8) can be written as

\[
\left( \frac{y_1}{y_1 + y_2 + \cdots + y_n} \right)^\gamma + \left( \frac{y_2}{y_1 + y_2 + \cdots + y_n} \right)^\gamma + \cdots + \left( \frac{y_n}{y_1 + y_2 + \cdots + y_n} \right)^\gamma \leq 1.
\]

Therefore, it is enough to show that \( a_1^\gamma + a_2^\gamma + \cdots + a_n^\gamma \leq 1 \) for non-negative real numbers \( a_i \) such that \( a_1 + a_2 + \cdots + a_n = 1 \).

Since \( 0 \leq a_i \leq 1 \) and \( \gamma \geq 1 \), then \( a_i^\gamma \leq a_i \). Therefore, \( a_1^\gamma + a_2^\gamma + \cdots + a_n^\gamma \leq a_1 + a_2 + \cdots + a_n = 1 \).

This proves the Lemma. \( \Box \)

Next we estimate \( \sum_{i=1}^n \| \nabla^M x_i \|^2 \).

**Lemma 3.** Let \( M \) be a closed hypersurface in \( \mathbb{R}^n \) and \( \Omega \) be a bounded domain such that \( M = \partial \Omega \). For a fixed point \( t \in \Omega \), let \((x_1, x_2, \ldots, x_n)\) be the normal coordinate system centered at \( t \). Then

\[
\sum_{i=1}^n \| \nabla^M x_i \|^2 = (n-1).
\]

**Proof.** Observe that \( \| \nabla x_i(p) \| = 1 \) for \( 1 \leq i \leq n \) and a point \( p \in \mathbb{R}^n \). Let \( \nu \) be the outward unit normal on \( M \). Then

\[
\sum_{i=1}^n \| \nabla^M x_i \|^2 = \sum_{i=1}^n \left( \| \nabla x_i \|^2 - \langle \nabla x_i, \nu \rangle^2 \right)
= \sum_{i=1}^n \| \nabla x_i \|^2 - \| \nu \|^2
= n - 1. \tag{9}
\]

For a Riemannian geometric proof of above lemma, see [6].

### 4 Proof of Theorem 1

**Proof.** Given a point \( x \in \mathbb{R}^n \), we write \((x_1, \ldots, x_n)\), the standard Euclidean coordinate system centered at origin. For \( 1 < p < \infty \), define a function \( f : \overline{\Omega} \to \mathbb{R} \) by

\[
f(t_1, \ldots, t_n) = \frac{1}{p} \int_M \sum_{i=1}^n |x_i - t_i|^p \, dx_1 \cdots dx_n.
\]

The function \( f \) is non-negative on \( \overline{\Omega} \). Let \( \alpha \) be its infimum. Then there exists a sequence \((t_1^j, \ldots, t_n^j)\) in \( \overline{\Omega} \) such that

\[
\frac{1}{p} \int_M \sum_{i=1}^n |x_i - t_i^j|^p \to \alpha \quad \text{in } \mathbb{R} \quad \text{as } j \to \infty. \tag{9}
\]
Observe that the sequence \((t^1_j, \ldots, t^n_j)\) is bounded. Therefore it has a convergent subsequence, without loss of generality, we denote it by \((t^1_j, \ldots, t^n_j)\) itself, which converges to \(t = (t_1, \ldots, t_n) \in \mathbb{R}^n\). Thus \(t \in \overline{\Omega}\). Then

\[
\sum_{i=1}^{n} |x_i - t^i_j|^p \rightarrow \sum_{i=1}^{n} |x_i - t_i|^p \quad \text{as} \quad j \rightarrow \infty
\]

and

\[
\frac{1}{p} \int_M \sum_{i=1}^{n} |x_i - t^i_j|^p \rightarrow \frac{1}{p} \int_M \sum_{i=1}^{n} |x_i - t_i|^p \quad \text{as} \quad j \rightarrow \infty.
\]

Therefore,

\[
\frac{1}{p} \int_M \sum_{i=1}^{n} |x_i - t_i|^p = \alpha \quad \text{and} \quad f(t_1, \ldots, t_n) = \alpha.
\]

Since \(f\) attains its minimum at \(t = (t_1, \ldots, t_n)\), we have \((\nabla f)_i = 0\). Therefore for each \(1 \leq i \leq n\),

\[
\langle \nabla f, e_i \rangle_{(t_1, \ldots, t_n)} = \int_M |X_i|^{p-2} X_i = 0,
\]

where \(\{e_i, 1 \leq i \leq n\}\) is the standard orthonormal basis of \(\mathbb{R}^n\) and \(X_i := (x_i - t_i), 1 \leq i \leq n\). This proves the theorem.

We will use the above theorem to show the existence of a point \(t = (t_1, \ldots, t_n) \in \overline{\Omega}\), such that the coordinate functions with respect to \(t\) are test functions for the eigenvalue problems (1) and (2).

### 5 Proof of Theorem 2

**Proof.** Let \(M\) be a closed hypersurface in \(\mathbb{R}^n\) and \(\Omega\) be the bounded domain such that \(M = \partial \Omega\). Let \(R > 0\) be such that \(\text{Vol} (\Omega) = \text{Vol} (B(R))\). The variational characterization for \(\lambda_{1,p}\) is given by

\[
\lambda_{1,p} = \inf \left\{ \frac{\int_M \|\nabla^M u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2} u = 0, u(\neq 0) \in C^1(M) \right\}.
\]

By Theorem 1 there exists a point \(t \in \overline{\Omega}\) such that

\[
\int_M |x_i|^{p-2} x_i = 0 \quad \text{for} \quad 1 \leq i \leq n,
\]

where \((x_1, \ldots, x_n)\) denotes the normal coordinate system centered at \(t\). Therefore, for all \(p > 1\),

\[
\lambda_{1,p} \int_M \sum_{i=1}^{n} |x_i|^p \leq \int_M \sum_{i=1}^{n} \|\nabla^M x_i\|^p \quad \text{for} \quad 1 \leq i \leq n.
\]

(10)

Now, we divide the proof of the theorem into the following two cases.

**Case 1.** \(1 < p \leq 2\).

Since \(|\frac{x_i}{r}| \leq 1\), it follows that

\[
|x_i|^p = r^p \left|\frac{x_i}{r}\right|^p \geq r^p \left|\frac{x_i}{r}\right|^2 \quad \text{for} \quad 1 \leq i \leq n.
\]

(11)
Therefore,

\[ r^p = r^p \sum_{i=1}^{n} \left| \frac{x_i}{r} \right|^2 \leq r^p \sum_{i=1}^{n} \left| \frac{x_i}{r} \right|^p = \sum_{i=1}^{n} |x_i|^p. \]

For \( 1 < p < 2 \), using Hölder’s inequality, we obtain

\[ \sum_{i=1}^{n} \| \nabla^M x_i \|^p \leq \left( \sum_{i=1}^{n} \| \nabla^M x_i \|^2 \right)^{\frac{p}{2}} n^{\frac{2-p}{2}}. \]

This combining with Lemma 3 gives

\[ \sum_{i=1}^{n} \| \nabla^M x_i \|^p \leq (n-1)^{\frac{p}{2}} n^{\frac{2-p}{2}}. \]

Observe that the above inequality is also true for \( p = 2 \). By substituting above values in inequality (10), we get

\[ \lambda_{1,p} \int_M r^p \leq (n-1)^{\frac{p}{2}} n^{\frac{2-p}{2}} \text{Vol}(M). \] (12)

By substituting \( \int_M r^p \geq R^p \text{Vol}(S(R)) \) in above equation, we get

\[ \lambda_{1,p} R^p \text{Vol}(S(R)) \leq (n-1)^{\frac{p}{2}} n^{\frac{2-p}{2}} \text{Vol}(M). \]

As a consequence, we have

\[ \lambda_{1,p} \leq n^{\frac{2-p}{2}} \lambda_1(S(R))^{\frac{p}{2}} \left( \frac{\text{Vol}(M)}{\text{Vol}(S(R))} \right). \]

This proves Theorem 2 for \( 1 < p \leq 2 \).

Equality in (3) implies equality in Lemma 1 and equality in (11), which implies that \( M \) is a geodesic sphere of radius \( R \) and \( p = 2 \).

**Case 2.** \( p \geq 2 \).

For \( p > 2 \), by Hölder’s inequality, we have

\[ \sum_{i=1}^{n} |x_i|^2 \leq \left( \sum_{i=1}^{n} (|x_i|^2)^{\frac{p}{2}} \right)^{\frac{2}{p}} n^{\frac{p-2}{p}}. \]

Therefore,

\[ n^{\frac{2-p}{2}} r_p \leq \sum_{i=1}^{n} |x_i|^p. \] (13)

Observe that equality holds in the above inequality for \( p = 2 \), so (13) holds for \( p \geq 2 \). Now we estimate \( \sum_{i=1}^{n} \| \nabla^M x_i \|^p \). Since \( \frac{p}{2} \geq 1 \) and \( \| \nabla^M x_i \|^2 \geq 0 \), for each \( 1 \leq i \leq n \), it follows from Lemma 2 that

\[ \sum_{i=1}^{n} \| \nabla^M x_i \|^p = \sum_{i=1}^{n} \left( \| \nabla^M x_i \|^2 \right)^{\frac{p}{2}} \leq \left( \sum_{i=1}^{n} \| \nabla^M x_i \|^2 \right)^{\frac{p}{2}} \leq (n-1)^{\frac{p}{2}}. \] (14)
The last inequality follows from Lemma 3. By substituting values from (13) and (14) in (10), we get
\[
\lambda_{1, p} n^{\frac{2-p}{2}} \int_M r^p \leq (n-1)^{\frac{2}{p}} \operatorname{Vol}(M).
\]  
(15)

By substituting \( \int_M r^p \geq R^p \operatorname{Vol}(S(R)) \) from Lemma 1 in above inequality, we have
\[
\lambda_{1, p} n^{\frac{2-p}{2}} R^p \operatorname{Vol}(S(R)) \leq (n-1)^{\frac{2}{p}} \operatorname{Vol}(M).
\]
Therefore,
\[
\lambda_{1, p} \leq n^{\frac{p-2}{2}} \lambda_1(S(R)) \left( \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S(R))} \right). 
\]

This proves Theorem 2 for \( p \geq 2 \).

If equality holds in (3), then equality holds in (6) and also in (13). Equality in (6) implies that \( M \) is a geodesic sphere of radius \( R \) and equality in (13) holds iff \( p = 2 \). Otherwise, \( p > 2 \) and equality in (13) implies that \( |x_i| = c \), for some constant \( c \) and \( 1 \leq i \leq n \). Therefore, each point of \( M \) is of the form \( (\pm c, \pm c, \pm c, \ldots, \pm c) \), for some constant \( c \). This contradicts our assumption that \( M \) is the boundary of a bounded domain \( \Omega \).

6 Proof of Theorem 3

**Proof.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega = M \) and \( R > 0 \) be such that \( \operatorname{Vol}(\Omega) = \operatorname{Vol}(B(R)) \), where \( B(R) \) is a ball of radius \( R \). The variational characterization for \( \mu_{1, p} \) is given by
\[
\mu_{1, p} = \inf \left\{ \frac{\int_M \|
abla u\|^p}{\int_M |u|^p} : \int_M |u|^{p-2} u = 0, u(\neq 0) \in C^1(\Omega) \right\}. 
\]
By Theorem 1, there exists a point \( t \in \overline{\Omega} \) such that
\[
\int_M |x_i|^{p-2} x_i = 0, \quad \text{for all } 1 \leq i \leq n, 
\]
where \( (x_1, x_2, x_3, \ldots, x_n) \) denotes the normal coordinate system centered at \( t \). By considering each \( x_i \) as test function, we have
\[
\mu_{1, p} \int_M \sum_{i=1}^n |x_i|^p \leq \int_{\Omega} \sum_{i=1}^n \|
abla x_i\|^p.  
\]  
(16)

Now we consider the following two cases to prove the theorem.

**Case 1.** \( 1 < p \leq 2 \).

By similar argument as in (14), we get
\[
r^p \leq \sum_{i=1}^n |x_i|^p. 
\]
By H"older’s inequality,
\[
\sum_{i=1}^n \|
abla x_i\|^p \leq \left( \sum_{i=1}^n \|
abla x_i\|^2 \right)^{\frac{p}{2}} n^{\frac{2-p}{2}} = n. 
\]
By substituting above values in (16), we get
\[
\mu_{1, p} \int_M r^p \leq n \operatorname{Vol}(\Omega). 
\]
By substituting $\int_M r^p \geq R^p \text{Vol}(S(R))$ from Lemma 1, we have

$$\mu_{1,p} R^p \text{Vol}(S(R)) \leq n \text{Vol}(\Omega).$$

Since $\text{Vol}(\Omega) = \text{Vol}(B(R))$ and $\frac{\text{Vol}(B(R))}{\text{Vol}(S(R))} = \frac{R}{n}$, we get

$$\mu_{1,p} \leq \frac{1}{R^{p-1}}.$$

**Case 2.** $p \geq 2$.

From (13), we have

$$n^{\frac{2-p}{2}} r^p \leq \sum_{i=1}^{n} |x_i|^p \quad \text{for all } p \geq 2.$$

By Lemma 2, we have

$$\sum_{i=1}^{n} \frac{\|\nabla x_i\|^p}{\|\nabla x_i\|^2} \leq \left( \sum_{i=1}^{n} \|\nabla x_i\|^2 \right)^{\frac{p}{2}} \leq n^{\frac{p}{2}}.$$

By substituting above values in (16), we get

$$\mu_{1,p} n^{\frac{2-p}{2}} \int_M r^p \leq n^{\frac{p}{2}} \text{Vol}(\Omega).$$

We use Lemma 1 again to get

$$\mu_{1,p} n^{\frac{2-p}{2}} R^p \text{Vol}(S(R)) \leq n^{\frac{p}{2}} \text{Vol}(\Omega).$$

Since $\text{Vol}(\Omega) = \text{Vol}(B(R))$ and $\frac{\text{Vol}(B(R))}{\text{Vol}(S(R))} = \frac{R}{n}$, above equation becomes

$$\mu_{1,p} \leq \frac{n^{p-2}}{R^{p-1}}.$$

Equality case will follow same as in Theorem 2. This completes the proof.

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