BERRY-ESSEEN TYPE ESTIMATE AND RETURN SEQUENCE FOR PARABOLIC ITERATION IN THE UPPER HALF-PLANE

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Abstract. We answer the question of finding a Berry-Esseen type theorem for the convergence rate in monotone central limit theorem. When the underlying measure is singular to Lebesgue measure, this central limit process is viewed as an infinite measure-preserving dynamical system and we prove that it has a regularly varying return sequence of index $1/2$.

The main tool in our proofs is the complex analysis techniques from free probability.

1. Introduction

Let $\mathbb{C}^+ = \{x + iy : y > 0\}$, and let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic map satisfying $F(iy)/iy \to 1$ as $y \to \infty$. This paper studies the dynamics of the parabolic iteration $\{F^{\circ n}\}_{n=1}^\infty$ from two different perspectives.

First, by Proposition 5.1 in [6], there is a one-to-one correspondence between the set of all analytic self-maps $F$ on $\mathbb{C}^+$ with $\lim_{y \to \infty} F(iy)/iy = 1$ and that of all probability measures $\mu$ on $\mathbb{R}$ in the sense that

$$\frac{1}{F(z)} = \int_{\mathbb{R}} \frac{1}{z-t} \, d\mu(t), \quad z \in \mathbb{C}^+. \tag{1}$$

Accordingly, we will denote the function $F$ by $F_\mu$ and call it the $F$-transform of $\mu$ from now on. Moreover, by Muraki’s monotone probability theory [10, 11], each $n$-fold iteration $F_\mu^{\circ n}$ corresponds to a unique probability measure $\mu^{\circ n}$ (called the $n$-th monotone convolution power of $\mu$) on $\mathbb{R}$ such that

$$\frac{1}{F_\mu^{\circ n}(z)} = \int_{\mathbb{R}} \frac{1}{z-t} \, d\mu^{\circ n}(t), \quad z \in \mathbb{C}^+, \quad n \geq 1. \tag{2}$$

Indeed, the measure $\mu^{\circ n}$ may be interpreted as the spectral distribution of a sum of $n$ self-adjoint operators that are monotonically independent and identically distributed according to the measure $\mu$. (See Section 2.1 for a review of the construction of monotonically independent random variables.) Of course, at the level $n = 1$ one has $\mu^{\circ 1} = \mu$ and the equation (2) reduces to (1), the definition of $F_\mu$.

We assume the measure $\mu$ in (1) has zero mean and unit variance throughout this paper. Let $\mu_n$ be the dilation defined by $\mu_n(A) = \mu^{\circ n}(\sqrt{n}A)$ for all sets $A$ in the Borel $\sigma$-field $\mathcal{B}$ on $\mathbb{R}$.
Then the measures $\mu_n$ converge weakly on $\mathbb{R}$ to the absolutely continuous probability measure

$$d\gamma(t) = \frac{1}{\pi \sqrt{2 - t^2}} \, d\lambda(t), \quad -\sqrt{2} < t < \sqrt{2},$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. This is known as the monotone central limit theorem (Monotone CL T) [11]. Since weak convergence to an absolutely continuous measure implies the uniform convergence of the corresponding cumulative distribution functions, it is natural to prove the following quantitative version of Monotone CL T:

**Theorem 1.1.** We write $F_\mu$ in its Nevanlinna integral representation

$$F_\mu(z) = z + \int_\mathbb{R} \frac{1}{t - z} \, d\nu(t), \quad z \in \mathbb{C}^+.$$

If the measure $\nu$ has the first absolute moment $c \in (0, +\infty)$, then one has the Kolmogorov distance

$$\sup_{x \in \mathbb{R}} |\mu_n (((-\infty, x))) - \gamma ((-\infty, x))] | \leq 71 \sqrt{c} n^{-1/8}$$

for sufficiently large $n$.

This result is an analogue of the Berry-Esseen estimate for the classical CL T. Here the first absolute moment $c$ is acting like the third absolute moment in the classical situation, and $c < +\infty$ if $\mu$ is further assumed to have finite fourth moment. Unlike the classical Berry-Esseen theorem, the best convergence rate here is at the order of $O(n^{-1/4})$ under a finite sixth moment condition and it cannot be improved without additional assumptions on the measure $\mu$ (see Theorem 3.1 and Remark 3.2). The case of $c = 0$ (i.e., $\nu = \delta_0$, the point mass at zero) corresponds to $\mu = (\delta_{-1} + \delta_1)/2$ and the convergence rate is also $O(n^{-1/4})$.

Our second perspective on $\{ F_\mu^n \}_{n=1}^\infty$ comes from the fact that when $\mu$ is singular to the Lebesgue measure $\lambda$ on $\mathbb{R}$, the boundary limit

$$T_\mu x = \lim_{y \to 0^+} F_\mu(x + iy)$$

exists in $\mathbb{R}$ for $\lambda$-almost every $x \in \mathbb{R}$ and the transformation $T_\mu$ preserves the measure $\lambda$ in the sense that $\lambda \circ T_\mu^{-1} = \lambda$ on the $\sigma$-field $\mathcal{B}$ [8]. Hence the dynamical system $(\mathbb{R}, \mathcal{B}, T_\mu)$ is an object of study in infinite ergodic theory [2].

It was proved in [12] that $T_\mu$ is conservative in the sense that for any $A \in \mathcal{B}$ with $\lambda(A) > 0$, one has the limit

$$S_n^A(x) = \# \left\{ k \in \mathbb{Z} \cap [0, n - 1] : T_\mu^k x \in A \right\} \to +\infty \quad (n \to \infty)$$

for $\lambda$-a.e. $x \in A$. In other words, $T_\mu$ has the recurrence behavior that any typical orbit $\{ T_\mu^n x \}_{n=1}^\infty$ starting from $x \in A$ will return to $A$ infinitely often.

Given such a nature of the system $(\mathbb{R}, \mathcal{B}, T_\mu)$, it is reasonable to investigate the asymptotic growth rate for the occupation times $S_n^A(x)$, and to ask further how this growth rate may depend on the set $A$, the point $x$, or the underlying measure $\mu$ (cf. [2]).
Aaronson proved that a normalization sequence for $S_n^A$, or, more generally, for the ergodic sum

$$S_n(f) = \sum_{k=0}^{n-1} f \circ T^o_k$$

of any $f \in L_1(\lambda)$ exists only at the dual level [1]. His result implies that there are constants $a_n(T_\mu) > 0$ satisfying a weak form of Birkhoff type ergodic theorem:

$$\frac{1}{a_n(T_\mu)} \sum_{k=0}^{n-1} \widehat{T}_\mu^k f \to \int f \, d\lambda \quad \text{a.e. as } n \to \infty, \quad \forall f \in L_1(\lambda),$$

where the Perron-Frobenius operator $\widehat{T}_\mu : L_1(\lambda) \to L_1(\lambda)$ is defined through the duality

$$\int \widehat{T}_\mu f \cdot g \, d\lambda = \int f \cdot g \circ T_\mu \, d\lambda, \quad f \in L_1(\lambda), \ g \in L_\infty(\lambda).$$

The sequence $a_n(T_\mu)$ is called a return sequence of $T_\mu$ and is unique up to asymptotic equivalence. Aaronson also showed that $a_n(T_\mu) \sim \pi^{-1} \sqrt{2n}$ when $\mu$ has bounded support (see also [9]). We obtain an extension of this result, saying that the $\pi^{-1} \sqrt{2n}$ growth rate is in fact universal in the set of measures attracted to the arcsine law $\gamma$ in Monotone CLT. (This set is the same as the normal domain of attraction of the standard Gaussian law in the classical CLT [12]).

**Theorem 1.2.** We have $a_n(T_\mu) \sim \pi^{-1} \sqrt{2n}$ among all singular distributions $\mu$ with zero mean and unit variance.

In particular, the Darling-Kac type theorem from [2 Corollary 3.7.3] implies that for any absolutely continuous probability measure $P$ on $\mathbb{R}$ and for any integrable $f : \mathbb{R} \to [0, +\infty]$ with $\|f\|_1 = 1$, one has the weak limit

$$\lim_{n \to \infty} P \left( \left[ S_n(f) \leq \pi^{-1} \sqrt{2nt} \right] \right) = \frac{2}{\pi} \int_0^t e^{-x^2/\pi} \, dx, \quad t \geq 0.$$ We then obtain a good understanding for the growth of the random variable $S_n^A = S_n(I_A)$ in the distributional sense. (For the pointwise behavior of $S_n^A$, we refer to the book [2] for various rational ergodic theorems.)

For the organization of this paper, we first collect the preliminary materials in Section 2, then we prove the Berry-Esseen estimate and the asymptotics of $a_n(T_\mu)$ in Sections 3 and 4, respectively.

2. Preliminaries

2.1. Realization of monotone convolution and $F$-transform. Given two probability measures $\nu_1$ and $\nu_2$ on $\mathbb{R}$, we consider the Hilbert space $H = L_2(\mathbb{R}^2, \nu_1 \otimes \nu_2)$ and two self-adjoint operators $X_1$ and $X_2$ on $H$ defined by

$$X_1 h_1(x, y) = x \int_\mathbb{R} h_1(x, t) \, d\nu_2(t), \quad X_2 h_2(x, y) = y h_2(x, y),$$
where the domain of $X_1$ consists of all $h_1 \in H$ satisfying
\[
\int_{\mathbb{R}} x^2 \left| \int_{\mathbb{R}} h_1(x, t) \, d\nu_2(t) \right|^2 \, d\nu_1(x) < +\infty,
\]
and that of $X_2$ consists of all $h_2 \in H$ such that
\[
\int_{\mathbb{R}^2} |y h_2(x, y)|^2 \, d\nu_1 \otimes \nu_2(x, y) < +\infty.
\]

It follows that the sum $X_1 + X_2$ is essentially self-adjoint, and hence the spectral theorem yields a unique probability measure $\nu_1 \triangleright \nu_2$ on $\mathbb{R}$ such that
\[
\varphi(\psi(X_1 + X_2)) = \int_{\mathbb{R}} \psi \, d\nu_1 \triangleright \nu_2
\]
for all bounded Borel functions $\psi$ on $\mathbb{R}$. That is, $\nu_1 \triangleright \nu_2$ is the distribution of the (noncommutative) random variable $X_1 + X_2$. Here $\varphi$ denotes the vector state associated with the constant function one on $\mathbb{R}^2$, and $\psi(X_1 + X_2)$ is defined through the functional calculus.

The measure $\nu_1 \triangleright \nu_2$ is called the monotone convolution of $\nu_1$ and $\nu_2$. It was shown in [7] that the algebras $A_i = \{ f(X_i) : f \in C_b(\mathbb{R}), f(0) = 0 \}, i \in \{1, 2\}$, are monotonically independent in the $C^*$-probability space $(B(H), \varphi)$ in the sense that for every mixed moment $\varphi(a_1 a_2 \cdots a_n)$ where $a_j \in A_j$, $i_j \in \{1, 2\}$, and $i_1 \neq i_2 \neq \cdots \neq i_n$, one has that
\[
\varphi(a_1 a_2 \cdots a_n) = \varphi(a_j) \varphi(a_1 \cdots a_{j-1} a_{j+1} \cdots a_n),
\]
whenever $a_j \in A_2$.

An important feature of the monotone convolution is that
\[
F_{\nu_1 \triangleright \nu_2}(z) = F_{\nu_1} \circ F_{\nu_2}(z), \quad z \in \mathbb{C}^+.
\]
Thus, the existence of the monotone convolution powers $\mu^{*n}$ may be proved by applying the above construction inductively to the measure $\mu$. To reiterate, we now have that $\mu^{*1} = \mu$, $\mu^{*n} = \mu \triangleright \mu^{*(n-1)}$ for $n \geq 2$, and $F_{\mu^{*n}} = F_{\mu^{*n}}^{\mu}$ in $\mathbb{C}^+$. As for the dilation $\mu_n$, it follows from the spectral theorem that
\[
F_{\mu_n}(z) = \frac{1}{\sqrt{n}} F_{\mu^{*n}} \left( \sqrt{n} z \right) = \frac{1}{\sqrt{n}} F_{\mu}^{\mu} \left( \sqrt{n} z \right), \quad z \in \mathbb{C}^+.
\]

We also refer to Muraki’s original papers [10, 11] for a more general monotone product construction, by which a sequence $X_1, X_2, \cdots, X_n$ of monotonically independent self-adjoint operators with the same distribution $\mu$ can be constructed at once on a Hilbert space so that the measure $\mu^{*n}$ is the distribution of the sum $X_1 + X_2 + \cdots + X_n$.

We conclude this subsection by gathering some well-known facts about the $F$-transform. First, $\mu$ has finite second moment if and only if the Nevanlinna integral form of $F_{\mu}$ is given by
\[
F_{\mu}(z) = z - m(\mu) + \int_{\mathbb{R}} \frac{1}{t - z} \, d\nu(t), \quad z \in \mathbb{C}^+,
\]
where $m(\mu)$ denotes the mean of $\mu$ and $\nu$ is a finite Borel measure on $\mathbb{R}$ with $\nu(\mathbb{R}) = \text{var}(\mu)$, the variance of $\mu$. 

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We also note that if the measure $\mu$ has finite moment $m_{2n}(\mu)$ of order $2n$ then $m_{2n-2}(\nu) < +\infty$, see [4].

The integral form (4) shows that $\exists F_{\mu}(z) \geq 3z$ for $z \in \mathbb{C}^+$, and the equality holds at some point in $\mathbb{C}^+$ if and only if $\mu = \delta_a$ for some $a \in \mathbb{R}$.

2.2. Bai’s inequality and the arcsine measure. For notational convenience, we use $dx$ or $dt$ to denote the Lebesgue measure $\lambda$ in the sequel.

Let $C_{\mu}(x) = \mu((-\infty, x])$, $x \in \mathbb{R}$, be the cumulative distribution function of a probability measure $\mu$. Recall the Bai’s inequality from [5] as follows: If two probability measures $\mu$ and $\nu$ satisfy

$$\int_{\mathbb{R}} |C_{\mu}(x) - C_{\nu}(x)| \, dx < +\infty,$$

then one has the supremum norm

$$\|C_{\mu} - C_{\nu}\|_{\infty} \leq \int_{\mathbb{R}} \left| \frac{1}{F_{\mu}(x + iy)} - \frac{1}{F_{\nu}(x + iy)} \right| \, dx + \frac{1}{y} \sup_{r \in \mathbb{R}} \int_{|t| \leq 2\sqrt{3}y} |C_{\nu}(r + t) - C_{\nu}(t)| \, dt$$

for all $y > 0$. We examine this inequality when $\nu$ is the arcsine measure $\gamma$.

**Lemma 2.1.** For any probability measure $\mu$ with finite variance and for any $0 < y < (4\sqrt{3})^{-1}$, one has the Kolmogorov distance

$$\|C_{\mu} - C_{\gamma}\|_{\infty} \leq \int_{\mathbb{R}} \left| \frac{1}{F_{\mu}(x + iy)} - \frac{1}{F_{\gamma}(x + iy)} \right| \, dx + 11\sqrt{y}.$$

**Proof.** We first show that $\mu$ and $\gamma$ satisfy the integrability requirement in Bai’s inequality. Indeed, the Chebyshev’s inequality yields

$$\int_{\mathbb{R}} |C_{\mu}(x) - C_{\gamma}(x)| \, dx \leq \int_{|x| \geq 2} \mu(\{t : |t| \geq x\}) \, dx + \int_{-\sqrt{2}}^{\sqrt{2}} |C_{\mu}(x) - C_{\gamma}(x)| \, dx$$

$$\leq \text{var}(\mu) \int_{|x| \geq \sqrt{2}} x^{-2} \, dx + 4\sqrt{2} < +\infty.$$

Next, we show

$$\frac{1}{y} \sup_{r \in \mathbb{R}} \int_{|t| \leq 2\sqrt{3}y} |C_{\gamma}(r + t) - C_{\gamma}(t)| \, dt \leq 11\sqrt{y}$$

(5) to conclude the proof of this lemma.

We first examine the case $0 \leq t \leq 2\sqrt{3}y$, in which our assumption on $y$ implies $t < \sqrt{2} - t$. Note that the difference $\Delta C(r) = C_{\gamma}(r + t) - C_{\gamma}(t)$ satisfies $\Delta C(r) = 0$ for $r \in (-\infty, -\sqrt{2} - t] \cup [\sqrt{2}, +\infty)$, because the arcsine density $f(x) = \pi^{-1}(2 - x^2)^{-1/2}$ is supported on $(-\sqrt{2}, \sqrt{2})$. Also, from the graph of $f$, we observe that

$$\Delta C(r) \leq \Delta C(\sqrt{2} - t) = \Delta C(-\sqrt{2}) = \int_{\sqrt{2} - t}^{\sqrt{2}} f(x) \, dx$$

for $r \in (-\sqrt{2} - t, -\sqrt{2}] \cup [\sqrt{2} - t, \sqrt{2})$. 

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If \( r \in [0, \sqrt{2} - t] \), we have the derivative \( d\Delta C(r)/dr > 0 \), and therefore the monotonicity of \( \Delta C \) implies that \( \Delta C(r) \leq \Delta C(\sqrt{2} - t) \) for such \( r \). For \( r \in (-\sqrt{2}, -t] \), the symmetry of \( \Delta C \) shows that \( \Delta C(r) = \Delta C(-r - t) \leq \Delta C(\sqrt{2} - t) \) as well.

It remains to estimate \( \Delta C(r) \) for \( r \in (-t, 0) \). In this case we note that

\[
\Delta C(r) = \int_0^r f(x) \, dx + \int_t^{r+t} f(x) \, dx
\]

\[
\leq \int_0^r f(x) \, dx + \int_0^t f(x) \, dx
\]

\[
= \Delta C(-t) + \Delta C(0) \leq 2\Delta C(\sqrt{2} - t)
\]

by our previous discussions.

We have shown that \( \Delta C(r) \leq 2\Delta C(\sqrt{2} - t) \) when \( 0 \leq t \leq 2\sqrt{3}y \). By the same method, one can also prove that \( |\Delta C(r)| \leq 2\Delta C(\sqrt{2} + t) \) for \( 0 > t \geq -2\sqrt{3}y \). In summary, we have \( |\Delta C(r)| \leq 2 [C(\sqrt{2}) - C(\sqrt{2} - |t|)] \) for any \( r \in \mathbb{R} \) and \( |t| \leq 2\sqrt{3}y \).

For \( r \in \mathbb{R} \), we now compute

\[
\int_{|t| \leq 2\sqrt{3}y} |\Delta C(r)| \, dt \leq 2 \int_{|t| \leq 2\sqrt{3}y} C(\sqrt{2}) - C(\sqrt{2} - |t|) \, dt
\]

\[
= 4 \int_0^{2\sqrt{3}y} C(\sqrt{2}) - C(\sqrt{2} - t) \, dt
\]

\[
= 4 \int_0^{2\sqrt{3}y} \int_{\sqrt{2}-t}^{\sqrt{2}} \frac{1}{\pi \sqrt{2-x^2}} \, dx \, dt
\]

\[
\leq 4 \frac{\pi}{\sqrt{2}} \int_0^{2\sqrt{3}y} \int_{\sqrt{2}-t}^{\sqrt{2}} \frac{1}{\pi \sqrt{2-x}} \, dx \, dt
\]

\[
= 4 \frac{\pi}{\sqrt{2}} \int_0^{2\sqrt{3}y} \int_0^t \frac{1}{\sqrt{x}} \, dx \, dt = \frac{32\sqrt{2}}{\pi \sqrt{3}} y \sqrt{y},
\]

from which the estimate (5) follows. \( \square \)

The \( F \)-transform of \( \gamma \) is known to be

\[
F_\gamma(z) = \sqrt{z^2 - 2}, \quad z \in \mathbb{C}^+,
\]

where the branch of the square root is chosen to be analytic on \( \mathbb{C} \setminus [0, +\infty) \) and \( \sqrt{-1} = i \).

Our next lemma deals with small, uniform perturbations of \( F_\gamma \) in Bai’s inequality.

**Lemma 2.2.** If \( y > 0 \) is given and \( \varepsilon \) is a continuous function on the horizontal line \( L = \{x + iy : x \in \mathbb{R}\} \) such that \( |\varepsilon(z)| < 3y/2 \) for all \( z = x + iy \in L \), then

\[
\int_{\mathbb{R}} \left| \frac{1}{\sqrt{z^2 - 2 + \varepsilon(z)}} - \frac{1}{\sqrt{z^2 - 2}} \right| \, dx \leq 39\sqrt{3}z, \quad z \in L.
\]

**Proof.** The function \( z \mapsto z^2 - 2 \) maps the horizontal line \( L \) bijectively to the parabola \( \{(u,v) : u = v^2/4y^2 - 2 - y^2\} \). For all \( z \in L \), let \( \ell(z) \) be the line segment connecting the
points $z^2 - 2$ and $z^2 - 2 + \varepsilon(z)$. Our hypothesis on $\varepsilon(z)$ implies that $\ell(z)$ is contained in $\mathbb{C} \setminus [0, +\infty)$, and hence the fundamental theorem of calculus for complex line integral yields

$$\sqrt{z^2 - 2 + \varepsilon(z)} - \sqrt{z^2 - 2} = \int_{w \in \ell(z)} \frac{1}{2\sqrt{w}} \, dw.$$ 

On account of the parametrization $w(t) = z^2 - 2 + t\varepsilon(z)$, $t \in [0, 1]$, and the fact

$$|z^2 - 2 + t\varepsilon(z)| \geq |z^2 - 2| - \frac{3}{2} |z^2 - 2| - \frac{3}{4} |z^2 - 2| = \frac{|z^2 - 2|}{4},$$

we get the estimate

$$\left| \int_{w \in \ell(z)} \frac{1}{2\sqrt{w}} \, dw \right| \leq \frac{\text{length}(\ell(z))}{2} \cdot \sup_{w \in \ell(z)} \frac{1}{\sqrt{|w|}} < \frac{|\varepsilon(z)|}{|z^2 - 2|^{1/2}} < \frac{3y}{2|z^2 - 2|^{1/2}},$$

which leads further to the inequality

$$\left| \sqrt{z^2 - 2 + \varepsilon(z)} - \sqrt{z^2 - 2} \right| \leq \frac{3y}{|z^2 - 2|^{3/2}}.$$

Therefore, we conclude that

$$\int_{\mathbb{R}} \left| \frac{1}{\sqrt{z^2 - 2 + \varepsilon(z)}} - \frac{1}{\sqrt{z^2 - 2}} \right| \, dx \leq \int_{\mathbb{R}} \frac{3y}{|z^2 - 2|^{3/2}} \, dx$$

$$= \int_{0}^{\infty} \frac{6y}{|z^2 - 2|^{3/2}} \, dx$$

$$= \int_{0}^{\infty} \frac{6y}{|z + \sqrt{2}|^{3/2} |z - \sqrt{2}|^{3/2}} \, dx$$

$$+ \int_{2}^{\infty} \frac{6y}{|z^2 - 2|^{3/2}} \, dx$$

$$\leq \int_{0}^{2} \frac{6y}{|z - \sqrt{2}|^{3/2}} \, dx + \int_{2}^{\infty} \frac{6y}{(x - 1)^3} \, dx$$

$$= \int_{-\sqrt{2}}^{2\sqrt{y}} \frac{6y}{(x^2 + y^2)^{3/4}} \, dx + 3y$$

$$\leq 6\sqrt{y} \int_{\mathbb{R}} \frac{1}{(u^2 + 1)^{3/4}} \, du + 3\sqrt{y} < 39\sqrt{y}.$$

\[\square\]

3. Berry-Esseen Estimates

Recall that $\mu_n$ is the normalization of $\mu^{\otimes n}$ in Monotone CLT. From the characterization of the domain of attraction of $\gamma$ in [12], each monotone convolution power $\mu^{\otimes n}$ has variance $n$.
and zero mean, implying further that $\text{var}(\mu_n) = 1$ and $m(\mu_n) = 0$. In particular, $\mu_n$ satisfies the integrability hypothesis of Bai’s inequality from the proof of Lemma 2.1.

By (3) and (4), we have

$$F_{\mu_n}(z) = \frac{F_{\mu(\sqrt{n}z)}}{\sqrt{n}} = z + \frac{1}{\sqrt{n}} \int_\mathbb{R} \frac{1}{t - \sqrt{n}z} \, d\nu(t), \quad z \in \mathbb{C}^+,$$

where $\nu(\mathbb{R}) = \text{var}(\mu) = 1$, and $F_{\mu_n} = F_{\nu_n}^\mu$ in $\mathbb{C}^+$. Note that $\Im F_{\nu_n}(z) > \Im z$ for $z \in \mathbb{C}^+$.

We introduce two conjugacy functions $\psi_1(z) = z^2$ and $\psi_2(z) = \sqrt{z}$, as well as the auxiliary function

$$r_n(z) = \frac{2}{n} \int_\mathbb{R} \frac{t}{t - \sqrt{n}z} \, d\nu(t) + \frac{1}{n} \left( \int_\mathbb{R} \frac{1}{t - \sqrt{n}z} \, d\nu(t) \right)^2, \quad z \in \mathbb{C}^+,$$

so that

$$F_{\mu_n}(\sqrt{z})^2 = \psi_1 \circ F_{\mu_n} \circ \psi_2(z) = (\psi_1 \circ F_n \circ \psi_2)^\mu(z) = \left[ z - \frac{2}{n} + r_n(\sqrt{z}) \right]^\mu = z - 2 + \sum_{j=0}^{n-1} r_n(F_{\nu_n}(\sqrt{z}))$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. Note that the maps $r_n$ have the following uniform bound in $\mathbb{C}^+$:

$$|r_n(z)| \leq \frac{2c}{n \sqrt{n} \Im z} + \frac{1}{n^2 (\Im z)^2}, \quad z \in \mathbb{C}^+, \quad n \geq 1,$$

if we assume

$$c = \int_\mathbb{R} |t| \, d\nu(t) \in [0, +\infty).$$

Below is the main result of this section, and Theorem 1.1 is the part (1) in it.

**Theorem 3.1.** Let $\mu$ be a probability measure on $\mathbb{R}$ with $m(\mu) = 0$ and $\text{var}(\mu) = 1$, and let $\mu_n$ be the normalization of $\mu^\mu$ by the factor of $n^{-1/2}$. Assume the first absolute moment $c$ of the measure $\nu$ is finite.

(1) If $c > 0$ then

$$\|C_{\mu_n} - C_{\gamma}\|_\infty \leq \frac{71 \sqrt{c}}{n^{1/8}}, \quad n > \max\left\{ (8\sqrt{3}c)^4, (8c\sqrt{c})^{-4} \right\}.$$

In particular, this convergence rate holds when the fourth moment $m_4(\mu) < +\infty$.

(2) If $c > 0$ and if the $F$-transform of $\nu$ has the Nevanlinna form

$$F_{\nu}(z) = z - m(\nu) + \int_\mathbb{R} \frac{1}{t - z} \, dp(t)$$

where

$$d = \int_\mathbb{R} |t| \, dp(t) \in [0, +\infty),$$
then the estimate
\[ \|C_{\mu_n} - C_\gamma\|_\infty \leq \frac{200\sqrt{d + 3(1 + m_2(\nu))^2}}{n^{1/4}} \]
holds for \( n > \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2, 12288 \left[ d + 3(1 + m_2(\nu))^2 \right]^2 \right\} \). In particular, this convergence rate holds if \( m_0(\mu) < +\infty \).

(3) The case \( c = 0 \) corresponds to \( \mu = (\delta_{-1} + \delta_1)/2 \), and we have
\[
\|C_{\mu_n} - C_\gamma\|_\infty \leq \frac{200}{n^{1/4}}, \quad n > 12288.
\]

Proof. We first prove (1). Define the error function
\[ \varepsilon_n(z) = \sum_{j=0}^{n-1} r_n \left( F_n^{\nu,j}(z) \right), \quad z \in \mathbb{C}^+, \]
and assume \( n > \max \left\{ (8\sqrt{3}c)^4, (8c\sqrt{c})^{-4} \right\} \). From the uniform bound (7), we know that \( |\varepsilon_n(z)| \leq 2cn^{-1/2}y^{-1} + n^{-1}y^{-2} < 3y/2 \) for all \( z \) on the line
\[ \left\{ x + iy : x \in \mathbb{R}, \ y = 2\sqrt{cn}^{-1/4} \right\}, \]
and hence the square root \( \sqrt{z^2 - 2 + \varepsilon_n(z)} \) is well-defined for such \( n \) and \( z \). Furthermore, the computation (6) yields the formula
\[ F_{\mu_n}(z) = \sqrt{z^2 - 2 + \varepsilon_n(z)}. \]

Therefore, Lemmas 2.1 and 2.2 allow us to conclude that
\[ \|C_{\mu_n} - C_\gamma\|_\infty \leq 50\sqrt{y} \leq 71\sqrt{c}n^{-1/8}. \]

Next, we prove (2) by improving the bound of \( \varepsilon_n \) on the horizontal line
\[ L_n = \left\{ x + iy : x \in \mathbb{R}, \ y = 16 \left[ d + 3(1 + m_2(\nu))^2 \right] n^{-1/2} \right\}. \]

For computational convenience, we introduce the notation
\[ G_\tau(z) = \int_\mathbb{R} \frac{1}{z - t} d\tau(t) \]
for any finite Borel measure \( \tau \) on \( \mathbb{R} \), and recall the two Nevanlinna representaions
\[ F_\mu(z) = z - G_\nu(z) \quad \text{and} \quad F_\nu(z) = z - m(\nu) - G_\rho(z), \]
where \( \rho(\mathbb{R}) = \text{var}(\nu) \). Furthermore, we set \( \widehat{G}_\rho(z) = m(\nu) + G_\rho(z) \). Note that although the map \( G_\rho \) may be constantly zero on \( \mathbb{C}^+ \) when \( \text{var}(\nu) = 0 \), the map \( \widehat{G}_\rho \) never vanishes in \( \mathbb{C}^+ \) because \( c > 0 \). Accordingly, the error \( \varepsilon_n(z) \) now becomes
\[
\varepsilon_n(z) = -\frac{2}{n} \sum_{j=0}^{n-1} \frac{\widehat{G}_\rho \left( \sqrt{n}F_n^{\nu,j}(z) \right)}{\sqrt{n}F_n^{\nu,j}(z) - G_\rho \left( \sqrt{n}F_n^{\nu,j}(z) \right)} + \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\sqrt{n}F_n^{\nu,j}(z) - \widehat{G}_\rho \left( \sqrt{n}F_n^{\nu,j}(z) \right)^2}. \]
We first estimate $B_n(z)$. Note that

$$|F_n(z) - z| = \left| \frac{1}{\sqrt{n}} \int_{\mathbb{R}} \frac{1}{t - \sqrt{n}z} \, d\nu(t) \right| \leq 1, \quad \exists z \geq n^{-1}, \quad n \geq 1. \quad (8)$$

Meanwhile, recall the dilation equation

$$F_n(z) = z - \frac{1}{\sqrt{n}} G_\nu(\sqrt{n}z) = z - \frac{1}{n} \frac{z - m(\nu)/\sqrt{n} - G_\rho(\sqrt{n}z)/\sqrt{n}}{|z|},$$

and observe from which that

$$|F_n(z) - z| \leq n^{-1}, \quad \exists z \geq n^{-1/2}, \quad |z| \geq 2, \quad n \geq \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\}. \quad (9)$$

Now for any $z \in L_n$ with $|z| \geq 3$, the inequality (9) yields $|F_n(z) - z| \leq n^{-1}$ for sufficiently large $n$. It follows that $|F_n(z)| \geq |z| - n^{-1} > 2$ and $\exists F_n(z) > \exists z \geq n^{-1/2}$. Therefore one can apply (9) to $F_n(z)$ to get

$$|F_n^{\circ 2}(z) - z| \leq |F_n(F_n(z)) - F_n(z)| + |F_n(z) - z| \leq 2n^{-1}.$$

Proceeding inductively, we obtain

$$|F_n^{\circ n}(z) - z| \leq 1, \quad z \in L_n, \quad |z| \geq 3, \quad n \geq \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\}.$$

As for $z \in L_n$, $|z| < 3$, we introduce the finite set $J = \{ j \in \mathbb{N} \cap [1, n] : |F_n^{\circ j}(z)| \geq 3 \}$ and observe that if $J = \emptyset$, then the orbit $\{ F_n^{\circ j}(z) \}_{j=0}^{n}$ stays within the ball $\{|z| < 3\}$ all the time, whence

$$|F_n^{\circ n}(z) - z| \leq |F_n^{\circ n}(z)| + |z| < 6.$$

If $J \neq \emptyset$, we take $k = \min J$, the first exit time of the orbit $\{ F_n^{\circ j}(z) \}_{j=0}^{n}$ out of the ball, then (8) and (9) together imply that

$$|F_n^{\circ n}(z) - z| \leq \left| F_n^{\circ (n-k)}(F_n^{\circ k}(z)) - F_n^{\circ k}(z) \right|$$
$$+ \left| F_n \left( F_n^{\circ (k-1)}(z) \right) - F_n^{\circ (k-1)}(z) \right| + \left| F_n^{\circ (k-1)}(z) - z \right| \leq \frac{n-k}{n} + 1 + 6 < 8$$

Thus, in all cases, we always have

$$|F_n^{\circ n}(z) - z| < 8, \quad z \in L_n, \quad n \geq \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\}.$$  

By running an induction argument on $F_n(z) = z - n^{-1/2} G_\nu(\sqrt{n}z)$, we get

$$F_n^{\circ n}(z) = z - n^{-1/2} \sum_{j=0}^{n-1} G_\nu \left( \sqrt{n} F_n^{\circ j}(z) \right).$$

Hence, the preceding estimate on $F_n^{\circ n}(z)$ leads to

$$\left| \sum_{j=0}^{n-1} G_\nu \left( \sqrt{n} F_n^{\circ j}(z) \right) \right| < 8 \sqrt{n}.$$
Since \( G_\nu(z) = 1/[z - \hat{G}_\rho(z)] \), \( \Im \hat{G}_\rho(z) = \Im G_\rho(z) \leq 0 \), and \( \Im F_n(z) > \Im z \), we conclude further that

\[
(10) \quad \sqrt{n} \geq \sum_{j=0}^{n-1} G_\nu \left( \sqrt{n} F_n^{o_j}(z) \right) \geq \Im \sum_{j=0}^{n-1} G_\nu \left( \sqrt{n} F_n^{o_j}(z) \right)
\]

\[
= \sum_{j=0}^{n-1} \frac{\sqrt{n} F_n^{o_j}(z) - \Im G_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}{\sqrt{n} F_n^{o_j}(z) - \hat{G}_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}
\]

\[
\geq \sum_{j=0}^{n-1} \frac{1}{\sqrt{n} F_n^{o_j}(z) - \hat{G}_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}
\]

\[
\geq n |B_n(z)|.
\]

In summary, we have shown that

\[
|B_n(z)| < 8n^{-1/2}, \quad z \in L_n, \quad n > \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\}.
\]

Next, we turn to \( A_n(z) \). For \( z \in L_n \) and \( n > \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\} \), (10) and the following two inequalities

\[
|G_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)| \leq \frac{\rho(\mathbb{R})}{\sqrt{n} F_n^{o_j}(z)} \left[ |m(\nu)| + \frac{\rho(\mathbb{R})}{\sqrt{n} F_n^{o_j}(z)} \right]
\]

\[
\leq m_2(\nu) |m(\nu)| + m_2(\nu)^2,
\]

\[
|\sqrt{n} F_n^{o_j}(z)| \geq \sqrt{n} F_n^{o_j}(z) \geq \int_\mathbb{R}\frac{t}{\sqrt{n} F_n^{o_j}(z) - t} \left\{ d\rho(t) + \rho(\mathbb{R}) \right\} \leq d + m_2(\nu)
\]

imply that

\[
\left| \frac{n}{2} A_n(z) \right| \leq \sum_{j=0}^{n-1} \frac{m(\nu)}{\sqrt{n} F_n^{o_j}(z) - \hat{G}_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}
\]

\[
+ \sum_{j=0}^{n-1} \frac{G_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}{\sqrt{n} F_n^{o_j}(z) - \hat{G}_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}
\]

\[
= |m(\nu)| \sum_{j=0}^{n-1} G_\nu \left( \sqrt{n} F_n^{o_j}(z) \right)
\]

\[
+ \sum_{j=0}^{n-1} \frac{G_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}{\sqrt{n} F_n^{o_j}(z) - \hat{G}_\rho \left( \sqrt{n} F_n^{o_j}(z) \right)}
\]

\[
\leq 8 |m(\nu)| \sqrt{n} + 8 \left[ d + m_2(\nu) + m_2(\nu) |m(\nu)| + m_2(\nu)^2 \right] \sqrt{n}
\]

\[
\leq 8 \left[ d + 3(1 + m_2(\nu))^2 \right] \sqrt{n}.
\]
So we have

$$|A_n(z)| \leq 16 \left[ d + 3(1 + m_2(\nu))^2 \right] n^{-1/2}, \quad z \in L_n, \quad n > \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2 \right\}.$$  

Finally, the hypotheses

$$|\varepsilon_n(z)| < 24 \left[ d + 3(1 + m_2(\nu))^2 \right] n^{-1/2} = 3y/2$$

and \( y < (4\sqrt{3})^{-1} \) are satisfied if \( z \in L_n \) and

$$n > \max \left\{ 4m(\nu)^2, 4m_2(\nu)^2, 12288 \left[ d + 3(1 + m_2(\nu))^2 \right] \right\}.$$  

Therefore, the convergence rate in (2) follows from Lemmas 2.1 and 2.2 again.

Finally, (3) can be proved by examining the values of \( |\varepsilon_n(z)| \) on the line

$$L_n = \left\{ x + iy : x \in \mathbb{R}, \quad y = 16n^{-1/2} \right\}.$$  

Indeed, in this case we have \( \nu = \delta_0 \) and

$$\varepsilon_n(z) = \frac{1}{n^2} \sum_{j=0}^{n-1} \frac{1}{F_n^{zy}(z)};$$

so that a similar argument as in the proof of (2) shows that \( |\varepsilon_n| < 8n^{-1/2} \) on \( L_n \) for all \( n \geq 1 \).

We omit the details to avoid repetition. \( \square \)

Remark 3.2. We associate each parameter \( \Gamma > 0 \) with a probability measure \( \nu_\Gamma \) whose \( F \)-transform is given by

$$F_{\nu_\Gamma}(z) = r + \sqrt{(z - r)^2 - 2}.$$  

Since \(-\Im G_{\nu_\Gamma}(x + iy)/\pi\) is the Poisson integral of \( \nu_\Gamma \) in \( \mathbb{C}^+ \), it is easy to see from the boundary values \( \lim_{y \to 0^+} \Re G_{\nu_\Gamma}(x + iy) \) that \( \nu_\Gamma \) is supported on the disjoint union

$$\left\{ -\sqrt{2 + r^2} + r \right\} \cup [-\sqrt{2} + r, \sqrt{2} + r],$$

and the mass of \( \nu_\Gamma \) at \(-\sqrt{2} + r^2 + r \) is \( |r|/(2 + r^2) \). We now consider the measure \( \mu = \nu_1 \) and note, by induction, that

$$F_{\mu_\Gamma}(z) = n^{-1/2} + \sqrt{(z - n^{-1/2})^2 - 2} = F_{\nu_{1/\sqrt{n}}}(z).$$  

Since \(-\sqrt{2} + (4n)^{-1/2} < -\sqrt{2 + n^{-1} + n^{-1/2}} \), it follows that the interval \(( -\infty, -\sqrt{2} + (4n)^{-1/2} \) is disjoint from the support of \( \mu_n \) and hence

$$\|C_{\mu_n} - C_\gamma\|_\infty \geq \left| \mu_n \left( ( -\infty, -\sqrt{2} + (4n)^{-1/2}) \right) - \gamma \left( ( -\infty, -\sqrt{2} + (4n)^{-1/2}) \right) \right|$$

$$= \left| \gamma \left( [-\sqrt{2}, -\sqrt{2} + (4n)^{-1/2}] \right) \right|$$

$$= \int_{-\sqrt{2} + (4n)^{-1/2}}^{\sqrt{2} + (4n)^{-1/2}} \frac{dt}{\pi \sqrt{2 - t^2}} \geq \frac{1}{2\pi} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{dt}{\sqrt{2 + t}} = \frac{1}{2\pi} \frac{1}{n^{1/4}}.$$  

This shows that the convergence rate in Theorem 3.1 (2) cannot be improved without further assumptions on the measure \( \mu \).
4. Return Sequence

In addition to having zero mean and unit variance, we now assume that $\mu$ is also singular to the Lebesgue measure $\lambda$, so that the boundary restriction $T_\mu x = \lim_{y \to 0^+} F_\mu(x + iy) \in \mathbb{R}$ for $\lambda$-a.e. $x \in \mathbb{R}$. In [1], Aaronson proved the following formula

$$a_n(T_\mu) \sim \frac{1}{\pi} \sum_{j=1}^{n} \Im \frac{-1}{F_\mu^j(z)}, \quad z \in \mathbb{C}^+.$$ 

By taking an appropriate $z$ on the $y$-axis, we will show that the series on the right side is asymptotically equivalent to $\sqrt{2n/\pi}$ as $n \to \infty$.

Toward this end, we first prove a result that is of some interest in complex dynamics. It is rather obvious that the sequence $\{F_\mu^j\}_{j=1}^\infty$ tends to its Denjoy-Wolff point $\infty$ pointwisely on $\mathbb{C}^+$. Under our assumptions on mean and variance of $\mu$, this convergence is in fact non-tangential for points that are sufficiently far from the real line.

**Proposition 4.1.** Recall the Nevanlinna representation $F_\mu(z) = z - G_\nu(z)$. Let $k$ be a positive integer such that $\nu([-k,k]) \geq 0.9$ and define the truncated cone

$$\Gamma = \{x + iy : |x| \leq y, \ y \geq 2k + 2\}.$$ 

Then $\Gamma$ is an invariant set under the map $F_\mu$, that is, $F_\mu(\Gamma) \subset \Gamma$. Consequently, the parabolic iterations $F_\mu^j(z)$ tend to $\infty$ non-tangentially for every $z \in \Gamma$.

**Proof.** We decompose the cone $\Gamma$ into a disjoint union $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where

$$\Gamma_- = \{x + iy \in \Gamma : -y \leq x \leq -y + 1\}$$

and

$$\Gamma_+ = \{x + iy \in \Gamma : y - 1 \leq x \leq y\}.$$ 

It follows that $\Re G_\nu(z) > 0$ for $z \in \Gamma_+$ and that $\Re G_\nu(z) < 0$ for $z \in \Gamma_-$. Indeed, since

$$\frac{1}{2} \leq \frac{\theta(t)}{y} \leq \frac{3}{2}$$

holds for any $z = x + iy \in \Gamma_+$ and $t \in [-k,k]$, we have

$$\Re G_\nu(z) = \frac{1}{y} \int_{|t| \leq k} \frac{\theta(t)}{\theta(t)^2 + 1} \, d\nu(t) + \frac{1}{y} \int_{|t| > k} \frac{\theta(t)}{\theta(t)^2 + 1} \, d\nu(t)$$

$$\geq \frac{1}{y} \int_{|t| \leq k} \frac{2}{5} \, d\nu(t) - \frac{1}{y} \int_{|t| > k} \frac{1}{2} \, d\nu(t)$$

$$= \frac{1}{y} [0.9\nu([-k,k]) - 0.5] > 0.$$ 

The proof of $\Re G_\nu(z) < 0$ for $z \in \Gamma_-$ is similar.

We now consider any $z = x + iy \in \Gamma$. If $z \in \Gamma_0$ then the distance from $z$ to the exterior $\{w \in \mathbb{C}^+ : \Im w \geq 2k + 2, \ w \notin \Gamma\}$ is at least $1/\sqrt{2}$. Since $|F_\mu(z) - z| = |G_\nu(z)| \leq y^{-1} < 2^{-1}$,
we have $F_\mu(z) \in \Gamma$. Next, if $z \in \Gamma_+$, we observe that
\[
\Re F_\mu(z) = x - \Re G_\nu(z) < x \leq y \leq \Im F_\mu(z)
\]
and
\[
\Re F_\mu(z) = x - \Re G_\nu(z) \geq y - 1 - \frac{1}{y} > \frac{1}{2} > -\Im F_\mu(z).
\]
So we also have $F_\mu(z) \in \Gamma$ in this case. Finally, the case of $z \in \Gamma_-$ follows similarly from the fact that $\Re G_\nu < 0$ on $\Gamma_-$.  

We proceed to the main result, from which Theorem 1.2 follows.

**Theorem 4.2.** For $z = (2k + 2)i$, we have
\[
\sum_{j=1}^n \frac{1}{\Im F_{\mu}^{(j)}(z)} \sim \sqrt{2n} \quad (n \to \infty),
\]
whence $a_n(T_\mu) \sim \sqrt{2n}/\pi$.

**Proof.** Denoting $z_j = F_{\mu}^{(j)}(z)$ for $j \geq 1$, the Nevanlinna representation of $F_\mu$ implies
\[
z_{j+1} - z_j = \int_{\mathbb{R}} \frac{1}{t - z_j} \, d\nu(t),
\]
so that
\[
\left| \frac{z_{j+1}}{z_j} - 1 \right| \leq \frac{1}{|z_j|} \int_{\mathbb{R}} \frac{1}{|t - z_j|} \, d\nu(t) \leq \frac{1}{|z_j|} \to 0 \quad (j \to \infty).
\]
Proposition 4.1 yields
\[
\left| \frac{z_j}{t - z_j} \right| \leq \sqrt{1 + \left( \frac{\Re z_j}{\Im z_j} \right)^2} \leq \sqrt{2}, \quad t \in \mathbb{R},
\]
and hence the dominated convergence theorem shows further that
\[
\lim_{j \to \infty} \int_{\mathbb{R}} \frac{z_j}{t - z_j} \, d\nu(t) = -1
\]
Now we have
\[
\frac{z_{j+1}^2}{z_j^2} = \left[ \frac{z_{j+1}}{z_j} + 1 \right] \int_{\mathbb{R}} \frac{z_j}{t - z_j} \, d\nu(t) \to -2
\]
as $j \to \infty$. Consequently, we obtain the convergence of the averages
\[
\frac{1}{j - 1} \sum_{k=1}^{j-1} \left( \frac{z_{k+1}^2}{z_k^2} \right) \to -2 \quad (j \to \infty).
\]
Observe next that
\[
\frac{z_j^2}{j} = \frac{z_j^2}{j} + \frac{j - 1}{j} \left[ \frac{1}{j - 1} \sum_{k=1}^{j-1} \left( \frac{z_{k+1}^2}{z_k^2} \right) \right], \quad j \geq 2,
\]
and so we finally get
\[
\lim_{j \to \infty} \frac{z_j^2}{j} = -2.
\]
Therefore, for sufficiently large $j$, we can apply the analytic square root $\sqrt{\cdot}$ to $z_j^2/j$, and the continuity of $\sqrt{\cdot}$ on $(-\infty, 0)$ says that

$$\lim_{j \to \infty} \frac{z_j}{\sqrt{j}} = i\sqrt{2}.$$

This shows that

$$\Im\left(-\frac{1}{z_j}\right) \sim \frac{1}{\sqrt{2j}} \quad (j \to \infty),$$

and naturally,

$$\sum_{j=1}^{n} \Im\left(-\frac{1}{z_j}\right) \sim \sqrt{2n} \quad (n \to \infty).$$

We conclude this paper with the following remarks. First, a simple normalization argument shows that

$$\sum_{j=1}^{n} \frac{-1}{F_{\mu}^{(2j)}((2k+2)i)} \sim \sqrt{\frac{2n}{\text{var}(\mu)}} \quad (n \to \infty)$$

for sufficiently large $k \in \mathbb{N}$, as long as $m(\mu) = 0$ and $\text{var}(\mu) \in (0, +\infty)$. Also, the singularity of $\mu$ to the Lebesgue measure plays no role in our proofs of Proposition 4.1 and Theorem 4.2. Finally, we mention that Theorem 4.2 also implies other ergodic properties of $T_{\mu}$ such as log-lower boundedness and quasi-finiteness (cf. [3, Proposition 3.1]).

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