A Comment on Budach’s Mouse-in-an-Octant Problem

Amir M. Ben-Amram∗

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The decidability of Budach’s Mouse-in-an-Octant Problem [2] [1] is apparently still open. This note sketches a proof that an extended version of the problem (a super-mouse) is undecidable.

The Problem

In the original formulation of the problem, the Budach mouse is a kind of finite automaton with a very simple program, a cycle of states, $q_0 \overset{d_1}{\rightarrow} q_1 \overset{d_2}{\rightarrow} \ldots q_{n-1} \overset{d_n}{\rightarrow} q_0$, with $d_1, \ldots, d_n \in \{N, E\}$. The mouse moves among integer points in the octant of the Cartesian plane delimited by the lines $y = 0$ and $y = x$. It begins on $(1, 0)$, at state $q_0$, and advances in steps which increase the state number (cyclically) and move the mouse, one unit at a time, in the direction indicated in the program (North or East). If the mouse hits the line $y = x$, its position is reset from $(x, x)$ to $(x, 0)$, without affecting the control state. Operation proceeds as usual. The computational problem is to decide, given the mouse’s program and a distinguished state number $k$, whether the mouse will ever visit the line $y = x$ while at state $q_k$. We may refer to this event as a stopping condition, giving the problem the flavour of a halting problem. Whether this problem is decidable is not known.

In this note, we increase the power of the mouse by allowing several cycles, with states denoted $q_{c,0}, \ldots, q_{c,n_c-1}$ where $c$ is the index of the cycle. A function $\delta$ maps states to cycles, and its use is that if the mouse hits the line $y = x$ in state $q_{c,i}$, not only is it returned to the $x$ axis, but also its state is set to the initial state of the cycle $\delta(q_{c,i})$.

We shall prove that mortality of this mouse is undecidable, by reducing from the problem: does a given 2CM halt, when started with null counters?

2-Counter Machines

A machine is specified by a list of $N$ instructions. Instruction $s$, where $0 \leq s < N$ is the instruction number (can be seen as the control state of the machine), is a pair $(D_1, D_2)$, such that $D_j$ is either $+1$, $-1$ or $0$ and represents a requested change to $R_j$, followed by a list of 3-tuples of the form $(b_1, b_2, k)$, where $b_j \in \{Z, P\}$ represents

∗amirben@mta.ac.il
whether register $R_j$ is zero or positive (there should be precisely four such 3-tuples, one for each possibility), and $k$ is the label of the next instruction, or $N$, which signifies halting. Instructions have to be valid: they never decrement a null register. A configuration of the machine is written as $(i, r_1, r_2)$ where $i$ is the program counter and $r_j$ the contents of $R_j$.

It is quite easy to see that the set of possible instructions can be limited without weakening the model. In particular, the following set of three combinations of $(D_1, D_2)$ suffices: $(+1, 0), (0, -1), (-1, +1)$.

**The Reduction**

Let a 2-counter machine $M$ be given. The idea is to encode the register values $(r_1, r_2)$ by an $x$ value satisfying $x = 2^{r_1}3^{r_2}q$, for some $q$ not divisible by 2, 3. The control state $i$ will be simulated by the $i$th cycle. Observe that $x \mod 6$ determines whether each register is null or not.

Each cycle of the mouse represents an instruction, one of the types $(+1, 0), (0, -1)$ and $(-1, +1)$. They all have 6 “essential” states (states moving north and not preceded by an $E$), so that when the mouse hits the diagonal $y = x$, its state indicates $x \mod 6$. As explained by van Emde Boas and Karpinski, the next value of $x$ depends on the previous one as $x + \lfloor x/6 \rfloor \cdot b + t$, where $b$ is fixed by the design of the cycle, and $t$ depends on the design of the cycle and the value of $x \mod 6$.

The cycle $(ENNENN)^6$ implements the update $(+1, 0)$. In fact, it multiplies $x$ by 2.

The cycle $((EN)^5NN)^2$ implements the update $(0, -1)$. It multiplies $x$ by $10/6 = 5/3$, so that $2^{r_1}3^{r_2}q$ (with $r_2 > 0$) becomes $2^{r_1}3^{r_2-1}(5q)$.

The cycle $((EN)^3NN)^3$ implements the update $(-1, +1)$. In fact, it multiplies $x$ by $9/6 = 3/2$, so that $2^{r_1}3^{r_2}q$ (with $r_1 > 0$) becomes $2^{r_1-1}3^{r_2+1}q$.

The update of the program counter is implemented in a simple way by the transition function $\delta$ of the super-mouse.

It is possible to restrict super-mice so that all cycles have equal length. This is achieved by choosing the length as the least common multiple of the lengths of the cycles above (30, 26 and 24): each cycle of the above construction is “pumped up” to the common length by simply repeating the string several times. It is easy to verify that the mouse can still carry the same computation.

**A Variation**

It is also possible to prove that the undecidability already holds for a certain (large enough, but bounded) number of cycles. For this purpose, we use a universal CM. The first cycle will create an initial value of $x$ in such a way as to “load” the machine’s register with some encoding of a machine to be simulated. It then jumps into a fixed set of cycles, representing the part of the machine that does the simulation.
Acknowledgement

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References

[1] A. Pultr and J. Úlehla. On two problems of mice. *Proceedings of the 10th Winter School on Abstract Analysis*, pages 249–262, 1982.

[2] P. van Emde Boas and M. Karpinski. A number theoretic problem arising from a problem in automata theory. *Bulletin of the EATCS*, 12:50–53, 1980.