Analytical approximations to the spectra of quark-antiquark potentials

Paolo Amore
E-mail: paolo@ucol.mx
Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima,
Colima, Mexico

Arturo De Pace
Istituto Nazionale di Fisica Nucleare, Sezione di Torino, via Giuria 1, I-10125 Torino,
Italy

Jorge Lopez
Physics Department, University of Texas at El Paso, El Paso, Texas, USA

Abstract.
A method, recently devised to obtain analytical approximations to certain classes of integrals, is used in combination with the WKB expansion to derive accurate analytical expressions for the spectrum of quantum potentials. The accuracy of our results is verified by comparing them both with the literature on the subject and with the numerical results obtained with a Fortran code. As an application of the method that we propose, we consider the meson spectroscopy with various phenomenological potentials.

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1. Introduction

Recently, in Refs. [1, 2] a method has been developed that allows one to obtain analytical approximations with arbitrary precision to a large class of integrals appearing in many diverse physical problems. The method is based on a carefully chosen expansion, which has been shown to converge uniformly and gives excellent results already at first order. In Refs. [1, 2] such method has been applied to a few problems in Classical Mechanics and General Relativity, allowing one to obtain simple analytical formulas that work even in the non-perturbative regime. In this paper we consider a further application of the method, showing that it is possible to obtain accurate analytical expressions for the spectrum of a quantum potential by using it in conjunction with the WKB expansion.

WKB is a semi-classical approximation, whose accuracy improves for large quantum numbers, based on an expansion in \( \hbar \) of the Schrödinger equation: it has been widely used in the past to get approximate evaluations of the spectra of quantum potentials and it is generally employed at the leading order in \( \hbar \), although its extension at higher orders has allowed one to recover, in special cases, the exact energy eigenvalues [3]. On the other hand, for a generic potential function \( V(x) \) the calculation of the WKB correction to a given order can be done only numerically and, in general, no analytical expression can be found for the WKB integrals and, as a consequence, for the spectrum.

The WKB method has already been employed also in the specific case of the phenomenological quark-antiquark potentials, using both non-relativistic [4] and relativistic [5] kinematics, showing a remarkably good accuracy even for the lowest quantum numbers. However, the solution of the WKB equation, although simpler than the original Schrödinger equation, has been again done numerically and only in the limit of large quantum numbers it has been possible to obtain the spectrum in a closed form.

In this paper, by applying the method of Ref. [1] to the calculation of the WKB integrals we show that one can obtain an accurate analytical approximation to the full spectrum of the potential (see also Ref. [6]). The accuracy of the approximation depends both on the order to which the WKB expansion is considered and on the order to which our method is applied. We apply the method to a few phenomenological quark-antiquark potentials that have been used in the literature to calculate the spectra of heavy mesons.

In Sect. 2 we discuss our method and in Sect. 3 we first apply it to the case of the non-relativistic Cornell potential [7] and then to the case of a linear potential with relativistic kinematics. Finally, in Sect. 4 we briefly summarize our results.

2. Formalism

The standard WKB approximation is obtained by solving for the energy the following equation:

\[
\int_{x_{-}}^{x_{+}} dx \sqrt{E - V(x)} = \pi (n + \frac{1}{2}),
\]  

(1)
where $x_\pm$ are the classical turning points of the potential $V(x)$ and $E = V(x_\pm)$ is the energy.

Let us concentrate on the integral
\[ J = \int_{x_-}^{x_+} dx \sqrt{E - V(x)}. \] (2)

In the spirit of the Linear Delta Expansion \[8\] we interpolate the potential $V(x)$ as
\[ V_\delta(x) = V_0(x) + \delta(V(x) - V_0(x)), \] (3)
where $V_0(x)$ is a suitably chosen potential that depends on one or more arbitrary parameters $\lambda_1, \lambda_2, \ldots$ that we collectively call $\lambda$ in the following. $V_\delta(x)$ reduces to the full potential for $\delta = 1$. We want to perform an expansion in $\delta$ without moving the inversion points: for this reason we impose that $x_\pm$ be the inversion points also of the potential $V_0(x)$. As a result, the energy $E_0$ that the particle would possess if it were moving only in the potential $V_0(x)$ is given by $E_0 = V_0(x_\pm)$.

We therefore introduce a new integral
\[ J(\delta) = \int_{x_-}^{x_+} dx \sqrt{E_0 - V_0(x)} \left[ 1 + \delta \Delta(x) \right]^{1/2}, \] (4)
where
\[ \Delta(x) \equiv \frac{E - E_0 - V(x) + V_0(x)}{E_0 - V_0(x)}. \] (5)

Of course, one has $J(\delta=1) \equiv J$.

We then treat the term proportional to $\delta$ as a perturbation and perform an expansion in $\delta$,
\[ J(\delta) = \sum_{n=0}^{\infty} \delta^n J_n. \] (6)

Such expansion will converge uniformly when $|\Delta(x)| < 1$ in the region $x_- \leq x \leq x_+$. This condition selects a particular region in the parameter space $\lambda$; however, maximal convergence is achieved when the Principle of Minimal Sensitivity (PMS) \[9\] is used, i.e. when, given the $N$-th order approximation to $J$,
\[ J(N) = \sum_{n=0}^{N} J_n, \] (7)
the condition
\[ \partial J(N)/\partial \lambda = 0 \] (8)
is enforced. Eq. \[8\] fixes the parameters $\lambda$, which are then inserted back in Eq. \[7\] yielding the approximant of $J$.

Notice that if the potential $V_0(x)$ is chosen appropriately it will be possible to calculate analytically each term in the expansion.
3. Applications: meson spectroscopy

3.1. Non-relativistic kinematics: the Cornell potential

Following [4] we write

$$\int_{r_-}^{r_+} \sqrt{2\mu \left[ E + \frac{\kappa}{r} - ar - \frac{L^2}{2\mu r^2} \right]} \, dr = \pi \left( n + \frac{1}{2} \right)$$

(9)

The effective potential in this case is given by

$$V_{\text{eff}}(r) = -\frac{\kappa}{r} + ar + \frac{L^2}{2\mu r^2}$$

(10)

where the last term is the centrifugal barrier and $r_\pm$ are the classical inversion points.

As one can see from Fig. 1, this potential is not symmetrical with respect to the minimum, which we call $r_0$. We therefore write

$$J \equiv \int_{r_-}^{r_+} \sqrt{2\mu \left[ E + \frac{\kappa}{r} - ar - \frac{L^2}{2\mu r^2} \right]} \, dr = \pi \left( n + \frac{1}{2} \right)$$

and evaluate the two integrals separately.

Following the procedure outlined in the previous section, we can write

$$J_+ = + \int_{r_+}^{r_0} \sqrt{2\mu \left[ E + \frac{\kappa}{r} - ar - \frac{L^2}{2\mu r^2} \right]} \, dr + \int_{r_0}^{r_+} \sqrt{2\mu \left[ E + \frac{\kappa}{r} - ar - \frac{L^2}{2\mu r^2} \right]} \, dr$$

$$\equiv J_+ + J_-$$

(11)
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Figure 2. % error of $J_{(1)+} + J_{(1)-}$ for $a = \kappa = L = \mu = 1$ as a function of $E$

where

$$\Delta_{\pm}(r) \equiv \frac{E - E_{0\pm} + \frac{\kappa}{r} - ar - \frac{L^2}{2\mu r^2} + V_{0\pm}(r)}{E_{0\pm} - V_{0\pm}(r)}$$

and

$$V_{0\pm}(r) = \frac{\lambda_{\pm}^2}{2} (r - r_0)^2, \quad E_{0\pm} = V_{0\pm}(r_\pm),$$

$\lambda_{\pm}$ being arbitrary parameters.

Provided that $|\Delta_{\pm}(r)| < 1$ for all $r \in (r_\pm, r_0)$, one can expand the expression for $J_{\pm}$ in powers of $\Delta_{\pm}(r)$. At first order one obtains

$$J_{(1)\pm} = \sqrt{\mu \lambda_{\pm}} \left( \frac{1}{2} \mathcal{H}_{0\pm} + \frac{\mathcal{H}_{1\pm}}{\lambda_{\pm}^2} \right),$$

where

$$\mathcal{H}_{1\pm} = \pm \left[ a\mathcal{H}_{1\pm}^a + \frac{1}{r_{\pm}} \left( \kappa - \frac{L^2}{2\mu r_{\pm}^2} \right) \mathcal{H}_{1\pm}^b - \frac{L^2}{2\mu r_{\pm}^2} \mathcal{H}_{1\pm}^c \right]$$

$\mathcal{H}_{0\pm}$, $\mathcal{H}_{1\pm}^a$, $\mathcal{H}_{1\pm}^b$ and $\mathcal{H}_{1\pm}^c$ being analytic functions of $r_{\pm}$ and $r_0$ given in the Appendix.

By minimizing $J_{(1)\pm}$ with respect to the corresponding parameter $\lambda_{\pm}$ one finds the optimal values,

$$\lambda_{\pm} = \sqrt{\frac{2\mathcal{H}_{1\pm}}{\mathcal{H}_{0\pm}}},$$

and, as a consequence, the optimal first order approximation to the integrals in Eq. (11) turns out to be

$$J_{(1)\pm} = \sqrt{2\mu} \sqrt{\mathcal{H}_{0\pm} \mathcal{H}_{1\pm}}.$$

In Fig. 2 one can see the typical accuracy one gets using this approximation.

One has now to impose the WKB condition

$$J_{(1)+} + J_{(1)-} = \pi \left( n + \frac{1}{2} \right)$$
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and solve for $E \equiv E_n$. In order to simplify this equation we expand the left hand side in $1/E_n$ to leading order: by inspecting the resulting expression one can easily check that the WKB condition can be satisfied by expanding the energy as a series in $1/n$, i.e. assuming the following ansatz

$$E_n = \sum_{n=0}^{\infty} c_k \left( n + 1/2 + \frac{L(8 + 17\pi)}{32\sqrt{2\pi}} \right)^{2/3-k/6}$$

(20)

and determining the coefficients $c_k$ so that the quantization condition is obeyed. These coefficients can be recursively calculated using any computer algebra program: stopping at order 10 in the $1/n$ expansion one finds

$$c_0 = \frac{2^{2/3}a^{2/3} \sqrt{\pi}}{\sqrt[3]{\mu}(\pi - 2)}$$

(21a)

$$c_1 = c_2 = c_3 = 0$$

(21b)

$$c_4 = a r_0 \left( 1 - \frac{2}{3} \sqrt{\frac{\pi}{\pi - 2}} \right)$$

(21c)

$$c_5 = \frac{a \left( \frac{\pi}{\pi - 2} \right)^{5/12} \sqrt{Lr_0}}{32^{11/12} \sqrt{a\mu}}$$

(21d)

$$c_6 = 0$$

(21e)

$$c_7 = -\frac{\sqrt{a} \left[ (3\pi - 8)L^2 + 40\kappa\mu\pi r_0 \right]}{96\sqrt{\pi - 2}(2\pi)^{3/4} \sqrt{L}\mu r_0}$$

(21f)

$$c_8 = \frac{4608\mu^{2/3}(2(\pi - 2))^{2/3}\pi^{11/6}r_0}{\sqrt{a}} \left\{ 3\sqrt{\pi - 2}(64 + 21(\pi - 16))L^2 
+ 32\mu\pi r_0 \left[ 3\kappa \left[ (8 - 21\pi)\sqrt{\pi - 2} - 48(\pi - 2)\sqrt{\pi} \right] 
- 4a \left[ 24\sqrt{\pi - 2} + (18 - 13\pi)\sqrt{\pi} r_0^2 \right] \right] \right\}$$

(21g)

$$c_9 = \frac{\sqrt{a}}{18432L^{3/2}\mu^{5/6}(2(\pi - 2))^{7/12}\pi^{23/12}r_0^{3/2}} \left[ -256aL^2\mu\pi \left( 3\sqrt{\pi - 2}(4 - 3\pi) + 2\pi^{3/2} \right) r_0^3 
- 3\sqrt{\pi - 2} \left[ (-192 + \pi(112 + 97\pi))L^4 - 16\kappa\mu\pi(77\pi - 24)r_0L^2 
+ 448\kappa^2\mu^2\pi^2 r_0^2 \right] \right]$$

(21h)

$$c_{10} = \frac{1}{49152\sqrt{2L}a\mu\pi^{3/2}r_0^2} \left\{ (8 + \pi)(64 + 21(181\pi - 144))L^4 + 32\mu\pi r_0 \left[ \kappa[64 - 3\pi(32 + 3\pi)] 
- 16a[16 + \pi(13\pi - 66)]r_0^2 \right] L^2 + 256\kappa^2\mu^2\pi^2(8 + 25\pi) r_0^2 \right\}.$$ 

(21i)

In Table 1 we compare the analytic spectrum of the $c\bar{c}$ and $b\bar{b}$ systems — obtained from Eqs. (20) and (3.1) — to the spectra obtained from the exact solutions of the Schrödinger and WKB equations, respectively [4].

As one can see from the table, in the case of the $c\bar{c}$ mesons the error in the analytic expressions is at most 1% for the lowest state and it is rapidly decreasing with the excitation energy; in the case of the $b\bar{b}$ mesons one goes from a 2.8% error in the ground state up to a 0.06% error for the 2P state. Note that the accuracy of the analytic approximation can be arbitrarily improved simply by extending the order in the $1/E_n$
and $1/n$ expansions and that similar analytic formulae can also be obtained for any observable by expanding its WKB expression.

It is also quite straightforward to get simple formulae in the asymptotic regimes. For instance, if we assume $n \gg L \gg 1$, the energy is approximated by our formula as

$$E_n \approx c_0 n^{2/3} \approx 2.2245 \frac{a^{2/3}}{\mu^{1/3}} n^{2/3},$$  

(22)

which is extremely close to the result of Eq. (10) of Ref. [4]:

$$E_n \approx 2.2309 \frac{a^{2/3}}{\mu^{1/3}} n^{2/3}.$$  

(23)

On the other hand, in the regime $L \gg n \gg 1$ we have to make explicit the dependence of $r_0$ upon $L$ and then expand around $L = \infty$; we obtain:

$$E_n \approx 1.62176 \frac{a^{2/3}}{\mu^{1/3}} L^{2/3},$$  

(24)

which again can be compared with the formula (9) of Ref. [4]:

$$E_n \approx 1.5 \frac{a^{2/3}}{\mu^{1/3}} L^{2/3}.$$  

(25)

### 3.2. Relativistic kinematics: linear potential

A generalization of the Schrödinger equation to include relativistic effects can be done by starting with the Bethe-Salpeter equation and making a few approximations, namely assuming an instantaneous local potential and neglecting spin and the coupling of “large” to “small” components in the relativistic wave function. The resulting equation

| States | Exact | WKB\text{exact} | WKB\text{analytic} |
|--------|-------|-----------------|--------------------|
| $cc$   |       |                 |                    |
| 1S     | 3067  | 3062            | 3097               |
| 2S     | 3693  | 3691            | 3722               |
| 1P     | 3497  | 3497            | 3463               |
| 2P     | 3991  | 3991            | 3980               |
| 1D     | 3806  | 3806            | 3791               |
| 2D     | 4242  | 4242            | 4235               |
| $bb$   |       |                 |                    |
| 1S     | 9448  | 9439            | 9705               |
| 2S     | 10007 | 10003           | 10131              |
| 1P     | 9901  | 9900            | 9865               |
| 2P     | 10261 | 10261           | 10255              |
| 1D     | 10148 | 10148           | 10098              |
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— also known as spinless Salpeter equation — has exactly the form of a Schrödinger equation with the kinetic term replaced with \( \sqrt{-\nabla^2 + m_1^2 + \sqrt{-\nabla^2 + m_2^2}} \).

Although the relativistic Schrödinger equation has been used in the literature with many realistic phenomenological potentials, in the following we shall consider for simplicity the linear potential \( V(r) = \mu^2 r \). In the relativistic case the WKB condition turns out to be slightly modified and it can be written as [5]

\[
J = \pi \left( n + \frac{l}{2} + \frac{3}{4} \right),
\]

where

\[
J = \int_0^{r_+} dr \sqrt{\frac{(E - \mu^2 r)^2}{4} - \frac{m_1^2 + m_2^2}{2} + \frac{(m_1^2 - m_2^2)^2}{4(E - \mu^2 r)^2}},
\]

with

\[
r_+ \equiv \frac{E - m_1 - m_2}{\mu^2}.
\]

In the limit \( m_1 = m_2 = m \), Eq. (26) can be integrated exactly to give [5]

\[
\frac{E}{2} \sqrt{\frac{E^2}{4} - m^2 - m^2 \log \left[ \frac{E}{2m} + \sqrt{\frac{E^2}{4m^2} - 1} \right]} = \mu^2 \pi \left( n + \frac{l}{2} + \frac{3}{4} \right).
\]

Before dealing with the general case \( m_1 \neq m_2 \), we first want to show that Eq. (29) can be used to obtain an analytical approximation to the spectrum. We make the ansatz:

\[
E \approx \sum_{k=0}^{\infty} \left\{ c_k \left( n + \frac{l}{2} + \frac{3}{4} \right)^{1/2-k} + d_k \left( n + \frac{l}{2} + \frac{3}{4} \right)^{-1/2-k} \log \left( n + \frac{l}{2} + \frac{3}{4} \right) \right\}.
\]

The unknown coefficients \( c_k \) and \( d_k \) in this expansion are determined by inserting Eq. (30) into Eq. (29) and by Taylor expanding in \( 1/(n+l/2+3/4) \ll 1 \).

\[
E_{n,l} \approx \frac{m^2 + \mu^2 (3 + 2l + 4n) \pi + m^2 \log \left( \frac{4m\pi}{\mu} \right)}{\mu \sqrt{3 + 2l + 4n} \sqrt{\pi}} + \frac{m^2 \log \left( \frac{3}{4} + \frac{l}{2} + n \right) \left[ m^2 + \mu^2 (3 + 2l + 4n) \pi - m^2 \log \left( \frac{4m\pi}{\mu} \right) \right]}{\mu^3(3 + 2l + 4n)^{3/2} \pi^{3/2}} + \ldots.
\]

This formula has been obtained truncating the sum into Eq. (30) to \( k = 2 \). In Fig. 3 we compare the numerical solutions of Eq. (26) with the analytical approximation of Eq. (31), using \( m = \mu = 1 \) and \( l = 0 \) (solid curve). As one can see the accuracy is very good.

We now want to extend our previous results to cases where the quantization integral cannot be performed analytically, such as for \( m_1 \neq m_2 \). Following the procedure outlined in Sect. 2 we can rewrite Eq. (27) as

\[
\mathcal{J}(\delta) = \int_0^{r_+} dr \sqrt{\frac{(E_0 - \lambda^2 r)^2}{4} + \delta \left[ \frac{(E - \mu^2 r)^2}{4} - \frac{m_1^2 + m_2^2}{2} + \frac{(m_1^2 - m_2^2)^2}{4(E - \mu^2 r)^2} - \frac{(E_0 - \lambda^2 r)^2}{4} \right]},
\]
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where \( \lambda \) is an arbitrary parameter and \( \delta \) is used for power counting. For \( \delta = 1 \) the expression above reduces to the one in Eq. (25). \( E_0 \) is fixed by asking that the zero-th order term in \( \delta \) vanishes at the inversion point and its value is

\[
E_0 = \lambda^2 r_+ .
\]

If we expand Eq. (32) in powers of \( \delta \) at order \( N \) and then set \( \delta = 1 \), we obtain an expression that depends upon \( \lambda \). We again take the approximant \( J_{(N)} \) and impose \( dJ_{(N)}/d\lambda = 0 \) to minimize such dependence. To leading order in \( \delta \) one obtains

\[
\lambda^4 = \mu^4 \frac{-8(\Delta m)^2 (E + 2\overline{m}) + E (E - 2\overline{m})(E + 6\overline{m}) + 8(\Delta m)^2 E \log(2E\overline{m})}{E(E-2\overline{m})^2} ,
\]

where \( \Delta m = (m_1 - m_2)/2 \) and \( \overline{m} = (m_1 + m_2)/2 \).

Thus, to order \( \delta \) we find

\[
J_{(1)} = \frac{E - 2\overline{m}}{4\sqrt{E\mu^2}} \left\{ E (E - 2\overline{m})(E + 6\overline{m}) - 8(\Delta m)^2 [E + 2\overline{m} - E \log(2)] + 8(\Delta m)^2 E [\log(E) + \log(\overline{m})] \right\}^{1/2} .
\]

In Fig. 4 we compare the numerical result for \( J \) obtained using \( m_1 = 1, m_2 = 2 \) and \( \mu = 1 \), with the one of Eq. (35), as a function of the energy \( E \). Eq. (35) provides a quite simple and precise approximation to the exact integral.

By introducing the ansatz

\[
E = \sum_{k=0}^{\infty} \left\{ c_k \left( n + \frac{l}{2} + \frac{3}{4} \right)^{1/2-k/2} + d_k \left( n + \frac{l}{2} + \frac{3}{4} \right)^{-1/2-k/2} \log \left( n + \frac{l}{2} + \frac{3}{4} \right) \right\}
\]

into Eq. (35) and Taylor expanding the WKB condition in \( 1/(n + l/2 + 3/4) \ll 1 \) one obtains to leading order

\[
E_{n,l} \approx \frac{6\overline{m}^2 + 3\mu^2 \pi + 2l\mu^2 \pi + 4\mu^2 n\pi - 2(\Delta m)^2 \log(\overline{m}\mu) - (\Delta m)^2 (-2 + \log(16\pi))}{\mu \sqrt{3} + 2l + 4n \sqrt{\pi}}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{Error over the spectrum defined as \( \Sigma \equiv (E_{n}^{\text{exact}} - E_{n}^{\text{approx}})/E_{n}^{\text{exact}} \) using the analytical formulas obtained and taking \( m = \mu = 1 \) and \( l = 0 \).}
\end{figure}
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Figure 4. Comparison between the exact numerical value and the analytical approximation at first order (Eq. (35)) of the quantization integral $J$, taking $m_1 = 1$, $m_2 = 2$ and $\mu = 1$.

\[
\begin{align*}
&\frac{(\Delta m)^2}{\mu^2(3 + 2l + 4n)^{3/2}\pi} \left(4\pi\sqrt{3 + 2l + 4n} - \mu(3 + 2l + 4n)\sqrt{\pi}\right) \log\left(\frac{3}{4} + \frac{l}{2} + n\right) \quad + \ldots.
\end{align*}
\]

(37)

The dashed curve of Fig. 3 displays the error over the spectrum obtained using eq. (37), showing that accurate predictions can be obtained with this simple formula.

4. Conclusions

In this paper we have applied to the integrals occurring in the application of the WKB a simple method, which allows one to express them analytically and to arbitrary precision in terms of elementary functions (a formal proof of the convergence of such method has been previously given in Ref. [1]). This procedure is advantageous even in the case (not so frequent) in which the integral can be performed analytically, since the exact result will in general contain special functions, which are usually more difficult to handle.

Successively, the approximate WKB quantization condition obtained in the first stage has been inverted, thus providing analytical formulas for the spectrum of a particle in a given potential.

In this paper we have considered two physical applications, which are relevant in meson spectroscopy, showing that in both cases excellent accuracy is obtained. The precision of our results essentially depends upon the intrinsic precision of the leading order WKB approximation, since the WKB integrals can be approximated analytically to arbitrary precision using our method. As a matter of fact we have also applied our method to one-dimensional anharmonic potentials (not discussed here) — using a higher order WKB calculation — and obtaining highly accurate formulas for the spectrum. Given the simplicity of our method we believe that it can be applied to a wide class of problems.
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5. Acknowledgments

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Appendix A.

In this Appendix we provide the explicit expressions of the turning points and of the minimum of the Cornell potential (10) and also the functions \( \mathcal{H}_{0 \pm}, \mathcal{H}_{1 \pm}^a, \mathcal{H}_{1 \pm}^b \) and \( \mathcal{H}_{1 \pm}^c \) introduced in Sect. 3.1.

For the turning points we follow Ref. [4]:

\[
\begin{align*}
    r_+ &= \frac{1}{3a} \left\{ E + 2\sqrt{E^2 + 3ak} \cos \left[ \frac{1}{3} \sec^{-1} \left[ \frac{4\mu(E^2 + 3ak)^{3/2}}{4E^3\mu + 9a(-3aL^2 + 2E\kappa\mu)} \right] \right] \right\}, \quad (1.1a) \\
    r_- &= \frac{1}{3a} \left\{ E + 2\sqrt{E^2 + 3ak} \cos \left[ \frac{4\pi}{3} + \frac{1}{3} \sec^{-1} \left[ \frac{4\mu(E^2 + 3ak)^{3/2}}{4E^3\mu + 9a(-3aL^2 + 2E\kappa\mu)} \right] \right] \right\}. \quad (1.1b)
\end{align*}
\]

The position of the minimum of the effective Cornell potential is:

\[
    r_0 = \frac{\sqrt[3]{2} \left( 9a^2L^2\mu^2 + \sqrt{3}a^3\mu^4 (27aL^4 + 4k^3\mu^2) \right)^{2/3} - 2\sqrt[3]{5}ak\mu^2}{6^{2/3}a\mu\sqrt{9a^2L^2\mu^2 + \sqrt{3}a^3\mu^4 (27aL^4 + 4k^3\mu^2)}}. \quad (1.2)
\]

The functions \( \mathcal{H} \) entering the first order approximation to the WKB integral of Eq. (1.3) are defined as follows:

\[
\begin{align*}
    \mathcal{H}_{0 \pm} &= \pm \int_{r_0}^{r_\pm} dr \sqrt{(r_\pm - r)(r + r_\pm - 2r_0)} = \frac{\pi}{4} (r_\pm - r_0)^2 \quad (1.3a) \\
    \mathcal{H}_{1 \pm}^a &= \pm \int_{r_0}^{r_\pm} dr \sqrt{\frac{r_\pm - r}{r + r_\pm - 2r_0}} = \pm \left( \frac{\pi}{2} - 1 \right) (r_\pm - r_0) \quad (1.3b) \\
    \mathcal{H}_{1 \pm}^b &= \int_{r_0}^{r_\pm} dr \sqrt{\frac{r_\pm - r}{r + r_\pm - 2r_0}} \frac{1}{r + r_\pm - 2r_0} = -\frac{\pi}{2} + \frac{1}{\sqrt{1 - \frac{2r_0}{r_\pm}}} \log \left[ \frac{r_\pm}{r_0} \left( 1 + \sqrt{1 - \frac{2r_0}{r_\pm}} \right) - 1 \right] \quad (1.3c) \\
    \mathcal{H}_{1 \pm}^c &= \int_{r_0}^{r_\pm} dr \sqrt{\frac{r_\pm - r}{r + r_\pm - 2r_0} \frac{1}{r + r_\pm - 2r_0}} \\
    &= \frac{r_\pm - r_0}{r_\pm - 2r_0} \left\{ \frac{1}{r_0} - \frac{1}{r_\pm} \frac{1}{\sqrt{1 - \frac{2r_0}{r_\pm}}} \log \left[ \frac{r_\pm}{r_0} \left( 1 + \sqrt{1 - \frac{2r_0}{r_\pm}} \right) - 1 \right] \right\} \quad (1.3d) \\
    \mathcal{H}_{1 \pm}^\prime &= \int_{r_0}^{r_\pm} dr \sqrt{\frac{r_\pm - r}{r + r_\pm - 2r_0} \frac{1}{r + r_\pm - 2r_0}} = \frac{r_\pm - r_0}{r_\pm - 2r_0} \left[ \frac{1}{r_0} + \frac{2}{r_\pm} \frac{1}{\sqrt{1 - \frac{2r_0}{r_\pm}}} \arctan \sqrt{\frac{2r_0}{r_\pm} - 1} \right] \quad (1.3e)
\end{align*}
\]
Analytical approximations to the spectra of quark-antiquark potentials

[1] Amore P and Sáenz R A 2005 Europhys. Lett. 70 425
[2] Amore P, Aranda A, Fernandez F and Sáenz R A 2005 Phys. Rev. E 71 016704
[3] Bender C M, Olausen K and Wang P S 1977 Phys. Rev. D 16 1740
[4] Brau F 2000 Phys. Rev. D 62 014005
[5] Cea P, Colangelo P, Nardulli G, Paiano G and Preparata G 1982 Phys. Rev. D 26 1157
[6] Seetharaman M, Raghavan S and Vasan S S 1983 J. Phys. A 16 455
[7] Eichten E et al. 1975 Phys. Rev. Lett. 34 369
[8] Okopińska A 1987 Phys. Rev. D 35 1835; Duncan A and Moshe M 1988 Phys. Lett. B 215 352
[9] Stevenson P M 1981 Phys. Rev. D 23 2916
[10] Fulcher L P 1994 Phys. Rev. D 50 447