A REMARK ON BLOW UP CRITERION OF
THREE-DIMENSIONAL NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. In this paper, we study the initial value problem for the three-
dimensional nematic liquid crystal flows. Blow up criterion of smooth solutions
is established by the energy method, which refines the previous result.

1. Introduction. We investigate the nematic liquid crystal flows in three space
dimensions

\[\begin{align*}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p &= -\nabla d \cdot \Delta d, \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d, \\
\nabla \cdot u &= 0,
\end{align*}\]

with the initial data

\[t = 0 : u = u_0(x), \quad d = d_0(x),\]

where \(u(t, x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) represents the velocity field \(d : \mathbb{R}^3 \rightarrow S^2\) denotes the
macroscopic average of the nematic liquid crystal orientation filed, \(p(t, x)\) denotes
the pressure and \(\nu > 0\) is the kinematic viscosity.

The materials called liquid crystals have attracted lots of scientists attention
since the end of the last century. Briefly, liquid crystals are states of matter which
are capable of flow, and in which the molecular arrangement gives rise to a pre-
ferred direction. Many of their properties in an extensive review were discussed
in \[1\]. A number of attempts have been made to formulate continuum theories to
describe properties of these peculiar liquids(see [22], [7]). Liquid crystals are di-
vided into different phases according to the behavior of the molecules. The most
important are the nematic phase, the cholesteric phase and the smectic phase. The
hydrodynamic theory of liquid crystals in the nematic case has been established by
Erickson [4] and Leslie [15] (see also [6], [8] and [16]), which has been widely used for
theoretical and experimental research(see [5]). (1) is a macroscopic continuum de-
scription of the time evolution of the materials under the influence of both the flow
field \(u(x, t)\), and the macroscopic description of the microscopic orientation config-
urations \(d(x, t)\) of rod-like liquid crystals. Recall that the Ericksen-Leslie theory
reduces to the Ossen-Frank theory in the static case, see Hardt- Lin-Kinderlehrer

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[10] and references therein. The mathematical theory is presently still under a wide development and the study of the full Ericksen-Leslie model presents relevant mathematical difficulties. Therefore, a simplified model (1) of the general Ericksen-Leslie equations (see [17], [18]) that keeps many of the mathematical difficulties of the original equations that was introduced in [17].

In [19], global weak solutions to the initial boundary value for (1) on bounded domains in two space dimensions was established (see also [11]). The uniqueness of such weak solutions is obtained by Lin and Wang [20]. Wen and Ding [27] obtained local existence and uniqueness of solution. Moreover, they also established that the uniqueness of weak solutions is obtained by Lin and Wang [20]. Wen and Ding [27] obtained local existence and uniqueness of solution. Moreover, they also established that the local existence and uniqueness of the solution with small initial data.

Local existence of smooth solutions has been announced in [12]: if \( u_0 \in H^s(\mathbb{R}^3, \mathbb{R}^3) \) with \( \nabla \cdot u_0 = 0 \) and \( d_0 \in H^{s+1}(\mathbb{R}^3, \mathbb{S}^2) \) for \( s \geq 3 \), then there is \( T_0 > 0 \) depending on \( \| u_0 \|_{H^s} \) and \( \| d_0 \|_{H^{s+1}} \) such that (1.1), (1.2) has a unique smooth solution \((u, d)\) on \( \mathbb{R}^3 \times [0, T_0) \) satisfying

\[
u \in C([0, T]; H^s(\mathbb{R}^3)) \bigcap C^1([0, T]; H^{s+1}(\mathbb{R}^3))\]

and

\[
d \in C([0, T]; H^{s+1}(\mathbb{R}^3, \mathbb{S}^2)) \bigcap C^1([0, T]; H^s(\mathbb{R}^3, \mathbb{S}^2))\]

for any \( 0 < T < T_0 \). (1.1) has distinct physical background, rich mathematical connotation and important theoretical value. The problem (1), (2) has great challenge and attracts many Mathematician’s interest. Some interesting results have been established, we may refer to [2], [9], [12], [24], [25], [28] and [29]. The existence of strong solution to liquid crystals system in critical Besov space and a criterion which is similar to Serrins criterion on regularity of weak solution to Navier-Stokes equations have been established by Hao and Liu [9]. Ladyzhenskaya-Prodi-Serrin type criterion for the breakdown of classical solutions has been established by Chen, Tan and Wu [2]. Huang and Wang [12] established a blow up criterion for classical solutions to (1.1) in three space dimensions. More precisely, they proved that \( 0 < T < +\infty \) is the maximal time interval iff \( \int_0^T (\| \nabla \times u \|_{L^\infty} + \| \nabla d \|_{L^2}^2) dt = \infty \). This blow up criterion was refined in [29]. Very recently, Zhang, Tan and Wu [28] obtained a blow up criterion of smooth solutions. Their result says \((u, d)\) is smooth up to time \( T \) provided that \( \int_0^T (\| \nabla \times u \|_{BMO} + \| \nabla d \|_{L^4}^2) dt = \infty \). The condition on \( d \) is only a special case of the Serrin-type regularity criteria. In [25], the author studied the problem (1), (2) with \( \nu = 0 \) and obtained a logarithmically improved blow up criterion of smooth solution.

In the paper, our main purpose is to refine and improve the result in [28]. More precisely, we extend the blow up criterion obtained in [28] to \( \dot{B}^{3, \infty}_{\infty, \infty} \) and \( L^p \) spaces. The blow up criterion obtained in this paper is different from blow-up criteria of smooth solutions to three-dimensional magneto-micropolar fluid equations in terms of the vorticity in a homogenous negative exponent Besov space in [26]. We overcome the difficulty which has been caused by the fact that the \( \dot{B}^{3, \infty}_{\infty, \infty} \) and \( L^p \) spaces replace the \( BMO \) and \( L^4 \) spaces. The proof is based on more sophisticated energy estimate. Now we state our results as follows.

**Theorem 1.1.** Let \( 3 < p < 6 \). Assume that \( u_0 \in H^m(\mathbb{R}^3, \mathbb{R}^3) \), \( m \geq 3 \) with \( \nabla \cdot u_0 = 0 \) and \( d_0 \in H^{m+1}(\mathbb{R}^3, \mathbb{S}^2) \). Let \((u, d)\) be a smooth solution to (1), (2) for \( 0 \leq t < T \). If

\[
\int_0^T (\| \nabla \times u(t) \|_{\dot{B}^{3, \infty}_{\infty, \infty}} + \| \nabla d(t) \|_{L^p}^2) dt < \infty,
\]

(3)
where
\[
\gamma = \begin{cases} 
6p - p^2, & 3 < p \leq 4, \\
\frac{p - 3}{p^2 + 2p}, & 4 \leq p < 6.
\end{cases}
\] (4)

then the solution \((u, d)\) can be extended beyond \(t = T\).

We have the following corollary immediately.

**Corollary 1.** Let \(3 < p < 6\) and \(\gamma\) be defined by (4). Assume that \(u_0 \in H^m(\mathbb{R}^3, \mathbb{R}^3)\), \(m \geq 3\) with \(\nabla \cdot u_0 = 0\) and \(d_0 \in H^{m+1}(\mathbb{R}^3, \mathbb{S}^2)\). Let \((u, d)\) be a smooth solution to (1), (2) for \(0 \leq t < T\). Suppose that \(T\) is the maximal existence time, then
\[
\int_0^T (\|\nabla \times u(t)\|_{\dot{B}^0_{\infty, \infty}} + \|\nabla d(t)\|_{L^p}^\gamma)dt = \infty.
\] (5)

The plan of the paper is arranged as follows. We first state some preliminary on functional settings and some important inequalities in Section 2 and then prove the blow up criterion of smooth solutions to (1), (2) in Section 3.

2. Preliminaries. Let \(\mathcal{S}(\mathbb{R}^n)\) be the Schwartz class of rapidly decreasing functions. Given \(f \in \mathcal{S}(\mathbb{R}^n)\), its Fourier transform \(\mathcal{F}f = \hat{f}\) is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx
\]
and for any given \(g \in \mathcal{S}(\mathbb{R}^n)\), its inverse Fourier transform \(\mathcal{F}^{-1}g = \check{g}\) is defined by
\[
\check{g}(x) = \int_{\mathbb{R}^n} e^{ix \cdot \xi} g(\xi) \, d\xi.
\]

Firstly, we recall the littlewood-Paley decomposition. Choose a non-negative radial functions \(\phi \in \mathcal{S}(\mathbb{R}^n)\), supported in \(\mathcal{C} = \{\xi \in \mathbb{R}^n : \frac{1}{4} \leq |\xi| \leq \frac{8}{3}\}\) such that
\[
\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.
\]
The frequency localization operator is defined by
\[
\triangle_k f = \int_{\mathbb{R}^n} \hat{\phi}(y) f(x - 2^{-k}y) \, dy.
\]

Next we recall the definition of homogeneous function spaces (see [23]). For \((p, q) \in [1, \infty]^2\) and \(s \in \mathbb{R}\), the homogeneous Besov space \(\dot{B}^s_{p,q}\) is defined as the set of \(f\) up to polynomials such that
\[
\|f\|_{\dot{B}^s_{p,q}} \triangleq \|2^{ks} \|\triangle_k f\|_{L^p}^\|_{\ell^q(\mathbb{Z})} < \infty.
\]

The following inequality is well-known Gagliardo-Nirenberg inequality.

**Lemma 2.1.** Let \(j, m\) be any integers satisfying \(0 \leq j \leq m\), and let \(1 \leq q, r \leq \infty\), and \(p \in \mathbb{R}, \frac{1}{m} \leq \theta \leq 1\) such that
\[
\frac{1}{p} - \frac{j}{n} = \theta(\frac{1}{r} - \frac{m}{n}) + (1 - \theta)\frac{1}{q}.
\]
Then for all \(f \in L^q(\mathbb{R}^n) \cap W^{m,r}(\mathbb{R}^n)\), there is a positive constant \(C\) depending only on \(n, m, j, q, r, \theta\) such that the following inequality holds:
\[
\|\nabla^j f\|_{L^p} \leq C\|f\|_{L^q}^{1 - \theta}\|\nabla^m f\|_{L^r}^\theta
\] (6)
with the following exception: if $1 < r < \infty$ and $m - j - \frac{n}{2}$ is a nonnegative integer, then (6) holds only for a satisfying $\frac{1}{m} \leq \theta < 1$.

The following Lemma comes from [13] and [21].

**Lemma 2.2.** Let $1 < p < \infty$. For $f, g \in W^{m,p}$, and $1 < q \leq \infty, 1 < r < \infty$, we have

$$
\|\nabla^\alpha (fg) - f\nabla^\alpha g\|_{L^r} \leq C (\|\nabla f\|_{L^q} \|\nabla^{\alpha-1} g\|_{L^r} + \|g\|_{L^q} \|\nabla^\alpha f\|_{L^r}),
$$

(7)

where $1 \leq \alpha \leq m$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

In what follows, we shall make use of Bernstein inequalities, which comes from [3].

**Lemma 2.3.** For any $m \in \mathbb{N}$, $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R}^n)$, then the following inequalities

$$
c 2^{km} \|\Delta_k f\|_{L^p} \leq \|\nabla^m \Delta_k f\|_{L^p} \leq C 2^{km} \|\Delta_k f\|_{L^p}
$$

(8)

and

$$
\|\Delta_k f\|_{L^q} \leq C 2^{n(\frac{1}{q} - \frac{1}{2})k} \|\Delta_k f\|_{L^p}
$$

(9)

hold, where $c$ and $C$ are positive constants independent of $f$ and $k$.

**Lemma 2.4.** There exists a uniform positive constant $C$, such that

$$
\|\nabla f\|_{L^\infty} \leq C \left(1 + \|f\|_{L^2} + \|\nabla \times f\|_{\dot{B}^0_{\infty, \infty}} \ln(\epsilon + \|f\|_{H^3})\right)
$$

(10)

holds for all vectors $f \in H^3(\mathbb{R}^n) (n = 2, 3)$ with $\nabla \cdot f = 0$.

**Proof.** The proof can be founded in [14]. For the convenience of the readers, the proof will be also sketched here. It follows from Littlewood -Paley decomposition that

$$
\nabla f = \sum_{k = -\infty}^0 \Delta_k \nabla f + \sum_{k = 1}^A \Delta_k \nabla f + \sum_{k = A+1}^\infty \Delta_k \nabla f.
$$

(11)

Using (8), (9) and (11), we obtain

$$
\|\nabla f\|_{L^\infty} \leq \sum_{k = -\infty}^0 \|\Delta_k \nabla f\|_{L^\infty} + \sum_{k = 1}^A \|\Delta_k \nabla f\|_{L^\infty} + \sum_{k = A+1}^\infty \|\Delta_k \nabla f\|_{L^\infty}
$$

$$
\leq C \sum_{k = -\infty}^0 2^{(1 + \frac{1}{2})k} \|\Delta_k f\|_{L^2} + A \max_{1 \leq k \leq A} \|\Delta_k \nabla f\|_{L^\infty} + C \sum_{k = A+1}^\infty 2^{-(2 - \frac{3}{2})k} \|\Delta_k \nabla^3 f\|_{L^2}
$$

(12)

$$
\leq C (\|f\|_{L^2} + A \|\nabla f\|_{\dot{B}^0_{\infty, \infty}} + 2^{-(2 - \frac{3}{2})A} \|\nabla^3 f\|_{L^2}).
$$

By the Biot-Savard law, we have a representation of $\nabla f$ in terms of $\nabla \times f$ as

$$
f_{x_j} = R_j (R \times \nabla f), \quad j = 1, 2, \cdots, n.
$$

where $R = (R_1, \cdots, R_n), R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since $R$ is a bounded operator in $\dot{B}^0_{\infty, \infty}$, this yields

$$
\|\nabla f\|_{\dot{B}^0_{\infty, \infty}} \leq C \|\nabla \times f\|_{\dot{B}^0_{\infty, \infty}}
$$

(13)
with \( C = C(n) \). Taking

\[
A = \left[ \frac{1}{(2 - \frac{n}{2})\ln 2} \ln(e + \|f\|_{L^3}) \right] + 1.
\]

(14)

It follows from (12), (13) and (14) that (10) holds. Thus, the lemma is proved. 

3. Proof of main results.

Proof. For the classical solutions \((u, d)\), we have

\[
\|u(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^2}^2 + \int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|\nabla^2 d(\tau)\|_{L^2}^2) d\tau
\]

\[
= \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2.
\]

(15)

Applying \( \nabla \) to the first equation of (1), multiplying the resulting equation by \( \nabla u \) and integrating with respect to \( x \) over \( \mathbb{R}^3 \), with help of integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \nu \|\nabla^2 u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla (u \cdot \nabla u) \nabla u \, dx - \int_{\mathbb{R}^3} \nabla (\nabla d \cdot \Delta d) \nabla u \, dx.
\]

(16)

Similarly, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^2 d(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2
\]

\[
= -\int_{\mathbb{R}^3} \nabla^2 (u \cdot \nabla d) \nabla^2 d \, dx + \int_{\mathbb{R}^3} \nabla^2 (|\nabla d|^2 d) \nabla^2 d \, dx.
\]

(17)

By (16), (17) and \( \nabla \cdot u = 0 \), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2 + \nu \|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2
\]

\[
= -\int_{\mathbb{R}^3} \nabla (u \cdot \nabla u) - u \cdot \nabla \nabla u) \nabla u \, dx - \int_{\mathbb{R}^3} \nabla (\nabla d \cdot \Delta d) \nabla u \, dx
\]

\[
- \int_{\mathbb{R}^3} \nabla^2 (u \cdot \nabla d) \nabla^2 d \, dx + \int_{\mathbb{R}^3} \nabla^2 (|\nabla d|^2 d) \nabla^2 d \, dx
\]

\[
\triangleq I_1 + I_2 + I_3 + I_4.
\]

(18)

It follows from Lemma 2.2 that

\[
I_1 \leq C\|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2}^2.
\]

(19)

By integration by parts, Hölder inequality, Gagliardo-Nirenberg inequality and Young inequality, we obtain

\[
I_2 \leq \|\nabla^2 u\|_{L^2} \|\nabla d\|_{L^p} \|\nabla^2 d\|_{L^\frac{2p}{p-2}}
\]

\[
\leq \frac{\nu}{4} \|\nabla^2 u\|_{L^2}^2 + C\|\nabla d\|_{L^p}^2 \|\nabla^2 d\|_{L^\frac{2p}{p-2}}^2
\]

\[
\leq \frac{\nu}{4} \|\nabla^2 u\|_{L^2}^2 + C\|\nabla d\|_{L^p}^2 \|\nabla d\|_{L^\frac{2p}{p-6}}^2 \|\nabla^3 d\|_{L^2}^2 \|\nabla^3 d\|_{L^\frac{2(12-p)}{6}}
\]

\[
\leq \frac{\nu}{4} \|\nabla^2 u\|_{L^2}^2 + \frac{1}{16} \|\nabla^3 d\|_{L^2}^2 + C\|\nabla d\|_{L^p}^\frac{3p}{2p-6}.
\]

(20)
Similarly, we have

\[ I_3 \leq \int_{\mathbb{R}^3} |\nabla^2 u \nabla \nabla^2 u| \, dx + 2 \int_{\mathbb{R}^3} |\nabla u \nabla^2 \nabla^2 u| \, dx \]
\[ \leq \|\nabla^2 u\|_{L^p} \|\nabla d\|_{L^p} \|\nabla^2 d\|_{L^{3p/2}} + 2\|\nabla u\|_{L^\infty} \|\nabla^2 d\|_{L^2} \]
\[ \leq \frac{\nu}{4} \|\nabla^2 u\|_{L^2}^2 + \frac{1}{16} \|\nabla^2 d\|_{L^2}^2 + C\|\nabla d\|_{L^p}^{3p} + C\|\nabla u\|_{L^\infty} \|\nabla^2 d\|_{L^2}^2. \] (21)

For the term \( I_4 \), we apply integration by parts and Hölder inequality. This yields

\[ I_4 = -\int_{\mathbb{R}^3} \nabla (|\nabla d|^2 d) \nabla^3 d \, dx \]
\[ = -2 \int_{\mathbb{R}^3} \nabla (\nabla d) \nabla d \nabla^3 d \, dx - \int_{\mathbb{R}^3} |\nabla d|^2 \nabla^3 d \, dx \] (22)
\[ \leq 2\|\nabla d\|_{L^p} \|\nabla^2 d\|_{L^{3p/2}} \|\nabla^3 d\|_{L^2} + \|\nabla d\|_{L^p}^3 \|\nabla^3 d\|_{L^2} \]
\[ = I_{41} + I_{42}. \]

For the term \( I_{41} \), using Gagliardo-Nirenberg inequality and Young inequality, we get

\[ I_{41} \leq C\|\nabla d\|_{L^p}^{3p} \|\nabla^3 d\|_{L^2}^{2p} \leq \frac{1}{16} \|\nabla^3 d\|_{L^2}^2 + C\|\nabla d\|_{L^p}^{3p}. \] (23)

For the term \( I_{42} \), Gagliardo-Nirenberg inequality and Young inequality give

\[ I_{42} \leq C\|\nabla d\|_{L^p}^{3p} \|\nabla^3 d\|_{L^2}^{2p-2p} \leq \frac{1}{16} \|\nabla^3 d\|_{L^2}^2 + C\|\nabla d\|_{L^p}^{3p}. \] (24)

Combining (18) - (24) yields

\[ \frac{d}{dt}(\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + \nu \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla^3 d(t)\|_{L^2}^2 \]
\[ \leq C\|\nabla u(t)\|_{L^\infty} (\|\nabla u(t)\|_{L^2}^2 + \|\nabla^2 d(t)\|_{L^2}^2) + C\|\nabla d\|_{L^p}^{3p}. \] (25)

We differentiate the second equation of (1) with respect to \( x \). This yields

\[ (\nabla d)_t - \Delta \nabla d = \nabla (|\nabla d|^2 d - u \cdot \nabla d). \] (26)

Multiplying (26) by \( |\nabla d|^{p-2} \nabla d \) and integrating with respect to \( x \) over \( \mathbb{R}^3 \), with help of integration by parts and Hölder inequality, we have

\[ \frac{1}{p} \frac{d}{dt} \|\nabla d\|_{L^p}^p + (p-1)\|\nabla^2 d\| \|\nabla d\|_{L^2}^{p-2} \|\nabla d\|_{L^2}^2 \]
\[ = \int_{\mathbb{R}^3} \nabla (|\nabla d|^2 d - u \cdot \nabla d) |\nabla d|^p \|\nabla d\|_{L^2} \]
\[ \leq \|\nabla d\|_{L^{p+2}}^{p+2} + C\|\nabla^2 d\|_{L^2} \|\nabla d\|_{L^{p+2}}^p + C\|\nabla u\|_{L^\infty} \|\nabla d\|_{L^{p+2}}. \] (27)

For the term \( \|\nabla d\|_{L^{p+2}}^{p+2} \), we apply Gagliardo-Nirenberg inequality. This yields

\[ \|\nabla d\|_{L^{p+2}}^{p+2} \leq C\|\nabla d\|_{L^{p+2}} \|\nabla^3 d\|_{L^2} \leq \frac{1}{8} \|\nabla^3 d(t)\|_{L^2}^2 + C\|\nabla d\|_{L^p}^{p+8}. \]
For the term $C \| \nabla^2 d \|_{L^2} \| \nabla d \|_{L^2}^{p^*}$, applying Gagliardo-Nirenberg inequality, we obtain

$$C \| \nabla^2 d \|_{L^2} \| \nabla d \|_{L^2}^{p^*} \leq C \| \nabla d \|_{L^{p^*/2}}^{2p^*/p} \| \nabla^3 d \|_{L^2}^{2p^*/p} \leq \frac{1}{8} \| \nabla^3 d(t) \|_{L^2}^2 + C \| \nabla d \|_{L^p}^{p^*+8}. $$

Collecting these estimates yielding

$$\frac{1}{4} \frac{d}{dt} \| \nabla d \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^2 + \frac{1}{2} \| \nabla (|\nabla d|^2) \|_{L^2}^2 \leq \frac{1}{4} \| \nabla^3 d(t) \|_{L^2}^2 + C \| \nabla d \|_{L^p}^{p^*+8} + C \| \nabla u \|_{L^\infty} \| \nabla d \|_{L^p}. $$

Adding (27) and (28) yielding

$$\frac{d}{dt} (\| \nabla u(t) \|_{L^2}^2 + \| \nabla^2 d(t) \|_{L^2}^2 + \| \nabla d \|_{L^p}^p) + \nu \| \nabla^2 u(t) \|_{L^2}^2 + \| \nabla^3 d(t) \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^{p^*+8} \leq C (\| \nabla u \|_{L^\infty} + \| \nabla d \|_{L^p}^p + \| \nabla u \|_{L^\infty} + \| \nabla d \|_{L^p}^p) $$

$$\int_{T_*}^{T} (\| \nabla \times u(t) \|_{B^0_{\infty, \infty}} + \| \nabla d \|_{L^p}^p) dt \leq \varepsilon. $$

Let

$$\Theta(t) = \sup_{T_* \leq \tau \leq t} (\| \nabla^3 u(\tau) \|_{L^2}^2 + \| \nabla^4 d(\tau) \|_{L^2}^2), \quad T_* \leq t < T. $$

It follows from (15), (29), (30), (31) and Lemma 2.4 that

$$\| \nabla u(t) \|_{L^2}^2 + \| \nabla^2 d(t) \|_{L^2}^2 + \| \nabla d(t) \|_{L^p}^p + \int_{T_*}^{T} \nu \| \nabla^2 u(\tau) \|_{L^2}^2 + \| \nabla^3 d(\tau) \|_{L^2}^2 + \| \nabla^2 d \|_{L^2}^{p^*+8} \| \nabla d \|_{L^2}^{p^*+8} \| \nabla^2 d \|_{L^2}^{p^*+8} d\tau \leq C_1 \exp \left\{ C_0 \int_{T_*}^{T} \| \nabla \times u \|_{B^0_{\infty, \infty}} \ln(e + \| u \|_{H^3}) d\tau \right\} $$

$$\leq C_1 \exp \left\{ C_0 \int_{T_*}^{T} \| \nabla \times u \|_{B^0_{\infty, \infty}} \ln(e + \| \nabla^3 u \|_{L^2}) d\tau \right\} $$

$$\leq C_1 \exp \left\{ C_0 \varepsilon \ln(e + \Theta(t)) \right\} $$

$$\leq C_1 (e + \Theta(t))^{C_0 \varepsilon}, \quad T_* \leq t < T. $$

where $C_1$ depends on $\| \nabla u(T_*) \|_{L^2}^2 + \| \nabla^2 d(T_*) \|_{L^2}^2$, while $C_0$ is an absolute positive constant.

Applying $\nabla^m$ to the first equation of (1), then taking $L^2$ inner product with $\nabla^m u$ and using integration by parts, we have

$$\frac{1}{2} \frac{d}{dt} \| \nabla^m u(t) \|_{L^2}^2 + \nu \| \nabla^{m+1} u(t) \|_{L^2}^2 \leq - \int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla u) \nabla^m u \, dx - \int_{\mathbb{R}^3} \nabla^m (\nabla d \cdot \Delta d) \nabla^m u \, dx. $$

$$\int_{\mathbb{R}^3} \nabla^m (u \cdot \nabla u) \nabla^m u \, dx \leq C \| \nabla u \|_{L^2} \| \nabla^2 d \|_{L^2} \left( \| \nabla^2 d \|_{L^2} + \| \nabla^3 d \|_{L^2} \right) \| \nabla^m u \|_{L^2}^2. $$
Likewise, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^{m+1} d(t) \|_{L^2}^2 + \| \nabla^{m+2} d(t) \|_{L^2}^2 = - \int_{\mathbb{R}^3} \nabla^{m+1} (u \cdot \nabla d) \nabla^{m+1} d \, dx + \int_{\mathbb{R}^3} \nabla^{m+1} (|\nabla d|^2) \nabla^{m+1} d \, dx. \tag{34}
\]

It follows (33), (34) and integration by parts that
\[
\frac{1}{2} \frac{d}{dt} (\| \nabla^{m} u(t) \|_{L^2}^2 + \| \nabla^{m+1} d(t) \|_{L^2}^2) + \nu \| \nabla^{m+1} u(t) \|_{L^2}^2 + \| \nabla^{m+2} d(t) \|_{L^2}^2 \]
\[
= - \int_{\mathbb{R}^3} \nabla^{m} (u \cdot \nabla u) \nabla^{m} u \, dx - \int_{\mathbb{R}^3} \nabla^{m} (\nabla d \cdot \Delta d) \nabla^{m} u \, dx - \int_{\mathbb{R}^3} \nabla^{m+1} (u \cdot \nabla d) \nabla^{m+1} d \, dx + \int_{\mathbb{R}^3} \nabla^{m+1} (|\nabla d|^2) \nabla^{m+1} d \, dx \tag{35}
\]

In what follows, for simplicity, we will set \( m = 3 \).

With help of integration by parts and Hölder inequality, we derive that
\[
\left| - \int_{\mathbb{R}^3} \nabla^{3} (u \cdot \nabla u) \nabla^{3} u \, dx \right| = \left| \int_{\mathbb{R}^3} \nabla^{2} (u \cdot \nabla u) \nabla^{4} u \, dx \right| \tag{36}
\]
\[
\leq 3 \| \nabla u \|_{L^4} \| \nabla^{2} u \|_{L^4} \| \nabla^{4} u \|_{L^2} + \| u \|_{L^6} \| \nabla^{3} u \|_{L^3} \| \nabla^{4} u \|_{L^2}
\]

It follows from Gagliardo-Nirenberg inequality and Young inequality
\[
\| \nabla u \|_{L^4} \| \nabla^{2} u \|_{L^4} \| \nabla^{4} u \|_{L^2} \leq C \| \nabla u \|_{L^2}^{\frac{7}{12}} \| \nabla^{4} u \|_{L^2}^{\frac{11}{12}} \leq \frac{\nu}{12} \| \nabla^{4} u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^{\frac{14}{3}}
\]
and
\[
\| u \|_{L^6} \| \nabla^{3} u \|_{L^3} \| \nabla^{4} u \|_{L^2} \leq C \| \nabla u \|_{L^2}^{\frac{7}{12}} \| \nabla^{4} u \|_{L^2}^{\frac{11}{12}} \leq \frac{\nu}{12} \| \nabla^{4} u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^{\frac{14}{3}}.
\]

Substituting the above two estimates into (36), we obtain
\[
\left| - \int_{\mathbb{R}^3} \nabla^{3} (u \cdot \nabla u) \nabla^{3} u \, dx \right| \leq \frac{\nu}{6} \| \nabla^{4} u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^{\frac{14}{3}}. \tag{37}
\]

Using integration by parts and Hölder inequality, we get
\[
\left| - \int_{\mathbb{R}^3} \nabla^{3} (\nabla d \cdot \Delta d) \nabla^{3} u \, dx \right| = \left| \int_{\mathbb{R}^3} \nabla^{2} (\nabla d \cdot \Delta d) \nabla^{4} u \, dx \right| \tag{38}
\]
\[
\leq 3 \| \nabla^2 d(t) \|_{L^4} \| \nabla^{3} d(t) \|_{L^4} \| \nabla^{4} u(t) \|_{L^2} + \| \nabla d(t) \|_{L^6} \| \nabla^{4} d(t) \|_{L^6} \| \nabla^{4} u(t) \|_{L^2} \leq \frac{\nu}{6} \| \nabla^{4} u(t) \|_{L^2}^2 + C \| \nabla^2 d(t) \|_{L^4}^2 \| \nabla^{3} d(t) \|_{L^4}^2 + C \| \nabla d \|_{L^6}^2 \| \nabla^{4} d \|_{L^6}^2.
Thanks to Gagliardo-Nirenberg inequality and Young inequality, we have
\[
C \|\nabla^2 d(t)\|_{L^2}^2 \|\nabla^3 d(t)\|_{L^4}^2 \leq C \|\nabla^2 d(t)\|_{L^2}^{\frac{2}{3}} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} \|\nabla^2 d(t)\|_{L^2}^{\frac{2}{3}} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} \\
\leq \frac{1}{12} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} + C \|\nabla^2 d(t)\|_{L^2}^{\frac{14}{3}}.
\]

We use Sobolev inequality, Gagliardo-Nirenberg inequality and Young inequality. This yields
\[
C \|\nabla d\|_{L^6}^2 \|\nabla^4 d\|_{L^2}^2 \leq C \|\nabla^2 d(t)\|_{L^2}^{\frac{2}{3}} \|\nabla^2 d(t)\|_{L^2}^{\frac{2}{3}} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} \\
\leq \frac{1}{12} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} + C \|\nabla^2 d(t)\|_{L^2}^{\frac{14}{3}}.
\]

We insert the above two inequality into (38), it yields
\[
\left| - \int_{\mathbb{R}^3} \nabla^3 (\nabla \cdot \Delta d) \nabla^3 u \, dx \right| \leq \frac{\nu}{6} \|\nabla^4 u(t)\|_{L^2}^{\frac{2}{3}} + \frac{1}{6} \|\nabla^5 d(t)\|_{L^2}^{\frac{2}{3}} + C \|\nabla^2 d(t)\|_{L^2}^{\frac{14}{3}}. \tag{39}
\]

Using integration by parts and Hölder inequality, we get
\[
\left| - \int_{\mathbb{R}^3} \nabla^4 (u \cdot \nabla d) \nabla^4 d \, dx \right| = \left| \int_{\mathbb{R}^3} \nabla^3 (u \cdot \nabla d) \nabla^5 d \, dx \right|
\leq \|\nabla^3 u(t)\|_{L^3} \|\nabla d\|_{L^6} \|\nabla^5 d\|_{L^2} + 3 \|\nabla^2 u(t)\|_{L^4} \|\nabla^2 d(t)\|_{L^4} \|\nabla^5 d(t)\|_{L^2} + \\
3 \|\nabla u(t)\|_{L^4} \|\nabla^3 d(t)\|_{L^4} \|\nabla^5 d(t)\|_{L^2} + \|u\|_{L^6} \|\nabla^4 d\|_{L^3} \|\nabla^5 d\|_{L^2}.
\]
\[
= J_1 + J_2 + J_3 + J_4. \tag{40}
\]

From Gagliardo-Nirenberg inequality and Young inequality, we derive that
\[
J_1 \leq C \|\nabla u(t)\|_{L^2}^{\frac{5}{9}} \|\nabla^4 u(t)\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla^5 d\|_{L^2} \\
\leq \frac{\nu}{24} \|\nabla^4 u(t)\|_{L^2}^{\frac{1}{2}} + \frac{1}{24} \|\nabla^4 d\|_{L^2}^{\frac{1}{2}} + C \|\nabla u(t)\|_{L^2} \|\nabla^5 d\|_{L^2}^{\frac{1}{2}}. \tag{41}
\]
\[
J_2 \leq C \|\nabla u(t)\|_{L^2}^{\frac{5}{9}} \|\nabla^4 u(t)\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla^5 d\|_{L^2} \\
\leq \frac{\nu}{24} \|\nabla^4 u(t)\|_{L^2}^{\frac{1}{2}} + \frac{1}{24} \|\nabla^4 d\|_{L^2}^{\frac{1}{2}} + C \|\nabla u(t)\|_{L^2} \|\nabla^5 d\|_{L^2}^{\frac{1}{2}}. \tag{42}
\]
\[
J_3 \leq C \|\nabla u(t)\|_{L^2}^{\frac{5}{9}} \|\nabla^4 u(t)\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla^5 d\|_{L^2} \\
\leq \frac{\nu}{24} \|\nabla^4 u(t)\|_{L^2}^{\frac{1}{2}} + \frac{1}{24} \|\nabla^4 d\|_{L^2}^{\frac{1}{2}} + C \|\nabla u(t)\|_{L^2} \|\nabla^5 d\|_{L^2}^{\frac{1}{2}}. \tag{43}
\]
\[
J_4 \leq C \|\nabla u(t)\|_{L^2} \|\nabla^2 d(t)\|_{L^2} \|\nabla^5 d\|_{L^2} \\
\leq \frac{\nu}{24} \|\nabla^4 u(t)\|_{L^2}^{\frac{1}{2}} + \frac{1}{24} \|\nabla^4 d\|_{L^2}^{\frac{1}{2}} + C \|\nabla u(t)\|_{L^2} \|\nabla^5 d\|_{L^2}^{\frac{1}{2}}. \tag{44}
\]

Combining the estimates (40)-(44) yields
\[
\left| - \int_{\mathbb{R}^3} \nabla^4 (u \cdot \nabla d) \nabla^4 d \, dx \right| \leq \frac{\nu}{6} \|\nabla^4 u(t)\|_{L^2}^{\frac{1}{2}} + \frac{1}{6} \|\nabla^5 d(t)\|_{L^2}^{\frac{1}{2}} + C (e + \Theta(t))^{\gamma C_0 \varepsilon}. \tag{45}
\]
We exploit integration by parts and Hölder inequality. This yields
\[
\int_{\mathbb{R}^3} \nabla^4((\nabla d^2)\nabla^4 d) \, dx
\]
\[
= \left| - \int_{\mathbb{R}^3} \nabla^3((\nabla d^2)\nabla^5 d) \, dx \right|
\]
\[
\leq 2 \| \nabla d(t) \|_{L^6} \| \nabla^4 d \|_{L^3} \| \nabla^5 d \|_{L^2} + 6 \| \nabla^2 d(t) \|_{L^4} \| \nabla^3 d(t) \|_{L^4} \| \nabla^5 d(t) \|_{L^2} + 7 \| \nabla d(t) \|_{L^6} \| \nabla^3 d(t) \|_{L^6} \| \nabla^5 d(t) \|_{L^2} + 10 \| \nabla^2 d \|_{L^6} \| \nabla^2 d \|_{L^6} \| \nabla^5 d \|_{L^2}.
\]
\[
=: K_1 + K_2 + K_3 + K_4.
\]

It follows from Gagliardo-Nirenberg inequality and Young inequality that
\[
K_1 \leq C \| \nabla^2 d(t) \|_{L^2}^{\frac{7}{2}} \| \nabla^5 d(t) \|_{L^2}^{\frac{11}{2}} \leq \frac{1}{24} \| \nabla^5 d(t) \|_{L^2}^{2} + C \| \nabla^2 d(t) \|_{L^2}^{14} \quad (47)
\]
\[
K_2 \leq C \| \nabla^2 d(t) \|_{L^2}^{\frac{7}{2}} \| \nabla^5 d(t) \|_{L^2}^{\frac{11}{2}} \leq \frac{1}{24} \| \nabla^5 d(t) \|_{L^2}^{2} + C \| \nabla^2 d(t) \|_{L^2}^{14} \quad (48)
\]
\[
K_3 \leq C \| \nabla^2 d(t) \|_{L^2}^{\frac{7}{2}} \| \nabla^5 d(t) \|_{L^2}^{\frac{5}{2}} \leq \frac{1}{24} \| \nabla^5 d(t) \|_{L^2}^{2} + C \| \nabla^2 d(t) \|_{L^2}^{14} \quad (49)
\]
and
\[
K_4 \leq C \| \nabla^2 d(t) \|_{L^2}^{\frac{7}{2}} \| \nabla^5 d(t) \|_{L^2}^{\frac{9}{2}} \leq \frac{1}{24} \| \nabla^5 d(t) \|_{L^2}^{2} + C \| \nabla d(t) \|_{L^2}^{14}. \quad (50)
\]

We substitute (47)-(50) into (46). This gives
\[
\left| - \int_{\mathbb{R}^3} \nabla^3((\nabla d^2)\nabla^5 d) \, dx \right| \leq \frac{1}{6} \| \nabla^5 d(t) \|_{L^2}^{2} + C \| \nabla^2 d(t) \|_{L^2}^{14}. \quad (51)
\]
Collecting (37), (39), (45) and (51) and using (32) yields
\[
\frac{d}{dt}(\| \nabla^3 u(t) \|_{L^2}^{2} + \| \nabla^4 d(t) \|_{L^2}^{2} + \| \nabla^5 d(t) \|_{L^2}^{2})
\]
\[
\leq C(\| \nabla^2 d(t) \|_{L^2}^{14} + \| \nabla u(t) \|_{L^2}^{14} + C(e + \Theta(t))^{7C_0\varepsilon}) \quad (52)
\]
\[
\leq C(e + \Theta(t))^{7C_0\varepsilon}. \quad T_* \leq t < T
\]
Integrating (52) with respect to time from $T_*$ to $t \in [T_*, T)$, we have
\[
e + \| \nabla^3 u(T_*) \|_{L^2}^{2} + \| \nabla^4 d(T_*) \|_{L^2}^{2} + \| \nabla^5 d(T_*) \|_{L^2}^{2} + C(e + \Theta(t))^{7C_0\varepsilon}. \quad (53)
\]
Choosing $\varepsilon$ small enough such that $7C_0\varepsilon < \frac{1}{2}$, (53) implies for all $T_* \leq t < T$
\[
e + \Theta(t) \leq C(e + \| \nabla^3 u(T_*) \|_{L^2}^{2} + \| \nabla^4 d(T_*) \|_{L^2}^{2}). \quad (54)
\]
Noting that the right hand side of (54) is independent of \( t \) for \( T_\star \leq t < T \), we know that (54) still holds for \( t = T \). So, \( u(T, \cdot) \in H^3(\mathbb{R}^3, \mathbb{R}^3) \), \( d(T, \cdot) \in H^4(\mathbb{R}^3, S^2) \). Thus, Theorem 1.1 is proved.

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