EXISTENCE OF REGULAR SOLUTIONS FOR A CERTAIN TYPE OF NON-NEWTONIAN NAVIER-STOKES EQUATIONS

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Abstract. We are concerned with existence of regular solutions for non-Newtonian fluids in dimension three. For a certain type of non-Newtonian fluids we prove local existence of unique regular solutions, provided that the initial data are sufficiently smooth. Moreover, if the $H^3$-norm of initial data is sufficiently small, then the regular solution exists globally in time.

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1. Introduction

We consider the Cauchy problem of non-Newtonian fluids in three dimensions

\[
\begin{aligned}
  u_t - \nabla \cdot (G(\|Du\|^2)Du) + (u \cdot \nabla)u + \nabla p &= 0, \\
  \text{div } u &= 0,
\end{aligned}
\]

in $\mathbb{R}^3 \times (0, T)$ (1.1)

with the viscous part of the stress tensor, $G(\|Du\|^2)$, such that $G : [0, \infty) \to [0, \infty)$ satisfies for any $s \in [0, \infty)$

\[
G[s] \geq m_0 > 0, \quad G[s] + 2G'[s]s \geq m_0 > 0,
\]

\[
|G^{(k)}[s]s^\alpha| \leq C_k |G^{(k-1)}[s]| \quad \alpha \in \{0, 1\}.
\]

Here, $G^{(k)}[\cdot]$ is the $k$-th derivative of $G$, $m_0$ and $C_k$ are positive constants, and $Du$ denote the symmetric part of the velocity gradient, i.e.

\[
Du = D_{ij}u := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2, 3.
\]

In (1.1), $u : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times (0, T) \to \mathbb{R}$ represent the flow velocity vector and the scalar pressure, respectively. We study Cauchy problem of (1.1), which requires initial conditions

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3, \quad \text{div } u_0 = 0.
\]

We note that a typical type of the viscous part of the stress tensor satisfying the property (1.2) is of some power-law models, for example, $G(\|Du\|^2) = (m_0^{\frac{2}{q-2}} + \|Du\|^2)^{\frac{q-2}{2}}$ with $2 < q < \infty$ or $G(\|Du\|^2) = m_0 + (\sigma + \|Du\|^2)^{\frac{q-2}{2}}$ with $1 < q < \infty$, $\sigma > 0$.

It is said that a fluid is Newtonian if the viscous stress tensor is a linear function of the rate of deformation tensor. On the contrary, for some fluids such as blood, paint and starch, it is observed that the relation between the shear stress and the shear strain rate is non-linear, and we commonly call those to the non-Newtonian fluids (see [4]). One class of non-Newtonian fluids is defined by $S = \mu(|Du|)Du$, where $\mu(\cdot)$ is a nonnegative real function defined on $[0; \infty)$. In this paper, we establish the existence of unique regular solution for the...
incompressible non-Newtonian Navier-Stokes equations (1.1), (1.2) and (1.3); in particular, by estimating higher derivatives in \( H^l(\mathbb{R}^3) \), \( l \geq 3 \).

We report shortly some known results related to existence of solutions. In the case of \( \mu(s) = (\mu_0 + \mu_1 s^2)^{1/2} \), Málek, Nečas, Rokyta and Růžička proved in [10] that a strong solution exists globally in time in periodic domains for \( q \geq \frac{11}{5} \) in dimension three and for \( q > 1 \) in dimension two, respectively (see [11] for the whole space case in dimension two or three). Also, they established local existence of strong solution in time for \( q > \frac{5}{3} \) in three dimensional periodic domains (refer to [3] for shear thinning case and [2] for shear thickening case). Here by strong solutions we mean solutions solving the equations a. e. and satisfying the following energy estimate:

\[
\sup_{0 \leq t \leq T} \|u(t)\|^2_{H^1(\mathbb{R}^3)} + \int_0^T \|u(t)\|^2_{H^2(\mathbb{R}^3)} + \int_0^T \int_{\mathbb{R}^3} |Du|^{q+2} |\nabla Du|^2 \leq C \|u_0\|^2_{H^1(\mathbb{R}^3)}.
\]

On the other hand, weak solutions are meant to solve the equations in the sense of distributions and hold

\[
\sup_{0 \leq t \leq T} \|u(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^T (\|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla u(t)\|^q_{L^q(\mathbb{R}^3)}) dt \leq C \|u_0\|^2_{L^2(\mathbb{R}^3)}.
\]

In any dimension \( n \geq 2 \) and \( \mu_0 \geq 0 \), the existence of weak solutions was firstly shown in [5] for \( \frac{2n+2}{n+2} \leq q \), and later, its existence was improved up to \( \frac{2n}{n+2} < q \) in [5] (see also [12] and other related references therein).

In [7], Kaplický, Malek and Stará consider a generalized non-Newtonian Navier-Stokes equations with a stress tensor \( S \) of the form

\[
S = S_{ij} = 2U^\prime(\|Du\|^2) Du_{ij}, \tag{1.4}
\]

where \( U : [0, \infty) \to [0, \infty) \) is \( C^2 \)-function such that

\[
\frac{\partial^2 U(\|Du\|^2)}{\partial D_{ij} \partial D_{kl}} D_{ij} D_{kl} \geq C_1 (1 + \|Du\|^2)^{\frac{q-2}{2}} |Du|^2, \quad \left| \frac{\partial^2 U(\|Du\|^2)}{\partial D_{ij} \partial D_{kl}} \right| \leq C_2 (1 + \|Du\|^2)^{\frac{q-2}{2}}. \tag{1.5}
\]

A typical example of \( S \) satisfying above assumptions is \((1+|Du|^2)^{\frac{q-2}{2}} Du \) and the corresponding potential \( U \) becomes \( \frac{1}{q} (1 + |Du|^2)^{\frac{q}{2}} \) (compared to our notation, \( G(\|Du\|^2) = 2U^\prime(\|Du\|^2) \)).

It was shown in [7] that in case that \( q > \frac{4}{3} \), \( C^{1, \alpha} \) regularity solution exists for the non-Newtonian fluid flows satisfying (1.4)-(1.5) in two dimensional periodic domain \( \mathbb{T}^2 \). More specifically, the authors deal with the equations (1.1)-(1.3) involving the stress tensor \( S \) with (1.4)-(1.5) for the case of periodic domains \( \mathbb{T}^2 \), and when \( q > \frac{4}{3} \), they established the global-in-time existence of a Hölder continuous solution, namely,

\[
u \in C^{1, \alpha}(\mathbb{T}^2 \times (0, T)) \quad \text{and} \quad p \in C^{0, \alpha}(\mathbb{T}^2 \times (0, T)), \quad 0 < \alpha < 1.
\]

As far as we know, it is, however, unknown for the existence of \( C^{k, \alpha}, k \geq 1 \), solution in three dimensions. The method of proof in [7] seems to work on only two dimensions, and thus its extension to three dimensions is our main motivation. Our main results are two-fold. Firstly, if initial data \( u_0 \) belongs to \( H^l(\mathbb{R}^3) \) with \( l \geq 3 \), we establish a local regular solution for some time \( T \) in the class

\[
\mathcal{X}_l([0, T]; \mathbb{R}^3) := L^\infty([0, T]; H^l(\mathbb{R}^3)) \cap L^2([0, T]; H^{l+1}(\mathbb{R}^3)),
\]

and furthermore, such solution is unique (see Theorem 1.1). A consequence for local existence of regular solutions is that \( \nabla^{l-2} u \) and \( \partial_t \frac{1}{l} u \) become Hölder continuous for an even
integer \( l > 3 \) (see Corollary 1.2). Secondly, we can obtain a global regular solution, provided that initial data is sufficiently small. To be more precise, if initial data \( u_0 \in H^l(\mathbb{R}^3) \), \( l \geq 3 \) such that \( \|u_0\|_{H^l(\mathbb{R}^3)} \) is sufficiently small, then the local solution in Theorem 1.1 exists in fact globally in time (see Theorem 1.3).

One of main observations is that there are two good terms caused by energy estimates of higher derivatives, provided that the condition (1.2) is satisfied. More specifically, in case \( l \geq 3 \), by taking the derivative \( \partial^l \) on (1.1), multiplying it by \( \partial^l u \), and integrating it by parts, we can see that the following two integrals appear

\[
\int_{\mathbb{R}^3} G([Du]^2) |\partial^l Du|^2 \, dx, \quad \int_{\mathbb{R}^3} G'(|Du|^2) |Du : \partial^l Du|^2 \, dx.
\]

(1.6)

It turns out that, due to the hypothesis \( G(s) \geq m_0 \) and \( G(s) + 2G'(s)s \geq m_0 \) in (1.2), the sum of two integrals in (1.6) is bounded below by \( m_0 \int_{\mathbb{R}^3} |\partial^l u|^2 \, dx \), which plays an important role for local existence of solutions.

Now we are ready to state our first main result.

**Theorem 1.1.** Let \( u_0 \in H^l(\mathbb{R}^3) \), \( l \geq 3 \). There exists \( T_l := T(\|u_0\|_{H^l}) > 0 \) such that the equation (1.1) - (1.3) has a unique solution \( u \) in the class \( X_l([0,T_l];\mathbb{R}^3) \). Furthermore, the solution \( u \) satisfies

\[
\sup_{0 \leq t \leq T_l} \|u(t)\|_{H^l}^2 + \int_0^{T_l} \|u(t)\|_{H^{l+1}}^2 < C = C(\|u_0\|_{H^l}).
\]

A consequence of Theorem 1.1 is the Hölder continuity of the regular solutions until the time of existence.

**Corollary 1.2.** Let \( l \) be an even positive integer. Under the assumption of Theorem 1.1, the solution \( u \) of (1.1) - (1.3) belongs to \( C^{l-2+\frac{l}{2}}_x C^{\frac{l-2+\frac{l}{2}}{2}}_t (\mathbb{R}^3 \times (0, T_l)) \).

Another main result is the global existence of regular solutions, in case that initial data are sufficiently small. More precisely, our second result reads as follows:

**Theorem 1.3.** Let \( u_0 \in H^l(\mathbb{R}^3) \), \( l \geq 3 \). There exists \( \epsilon_0 > 0 \) such that if \( \|u_0\|_{H^3(\mathbb{R}^3)} < \epsilon \) for any \( \epsilon > 0 \), then the unique regular solution \( u \) in Theorem 1.1 is extended globally in time, i.e. \( T_l = \infty \).

This paper is structured as follows: In Section 2, we state a key lemma, whose proof is given in the Appendix. In Section 3, the proof of Theorem 1.1 is presented. Section 4 is devoted to proving Corollary 1.2. In Section 5, we provide the proof of Theorem 1.3.

2. Preliminaries

We introduce some notations. For \( 1 \leq q \leq \infty \), we denote by \( W^{k,q}(\mathbb{R}^3) \) the standard Sobolev space. Let \( (X, \| \cdot \|_X) \) be a normed space and by \( L^q(0, T; X) \) we mean the space of all Bochner measurable functions \( \varphi : (0, T) \to X \) such that

\[
\|\varphi\|_{L^q(0,T;X)} := \left( \int_0^T \|\varphi(t)\|_X^q \, dt \right)^{\frac{1}{q}} < \infty \quad \text{if} \quad 1 \leq q < \infty,
\]

\[
\|\varphi\|_{L^\infty(0,T;X)} := \sup_{t \in (0,T)} \|\varphi(t)\|_X < \infty \quad \text{if} \quad q = \infty.
\]

We denote by \( C^\alpha_{x,t} \) (or \( C^\alpha_{x,t} \)) the space of Hölder continuous functions with an exponent \( \alpha \in (0,1) \). For a non-negative integer \( k \) we mean, in general, by \( C^{2k,\alpha}_{x,t} \) (or \( C^{2k,\alpha}_{x,t} \)) the
space of functions whose mixed derivative \( \nabla_x^{2(i-j)} \nabla_t^j u \) belongs to \( C^\alpha \) for all integers \( i, j \) with \( 0 \leq j \leq i \leq k \). Let \( a_{ij} \) and \( b_{ij} \) with \( i, j = 1, 2, 3 \) be scalar functions, and for \( 3 \times 3 \) matrices \( A = (a_{ij})_{i,j=1}^3 \) and \( B = (b_{ij})_{i,j=1}^3 \) we write

\[
A : B = \sum_{i,j=1}^3 a_{ij} b_{ij}, \quad \nabla A : \nabla B = \sum_{i,j=1}^3 \nabla a_{ij} \cdot \nabla b_{ij}, \quad \nabla^2 A : \nabla^2 B = \sum_{i,j=1}^3 \nabla^2 a_{ij} : \nabla^2 b_{ij}.
\]

The letter \( C \) is used to represent a generic constant, which may change from line to line.

Next lemma is a key observation for our analysis, which shows some estimates of higher derivatives for the viscous part of the stress tensor.

**Lemma 2.1.** Let \( l \) be a positive integer, \( \tilde{\sigma}_l : \{1, 2, \ldots, l\} \to \{1, 2, \ldots, l\} \) a permutation of \( \{1, 2, \ldots, l\} \), and \( \sigma_l \) a mapping from \( \{1, 2, \ldots, l\} \) to \( \{1, 2, 3\} \). Suppose that \( u \in C^\infty(\mathbb{R}^3) \cap H^l(\mathbb{R}^3) \). Assume further that \( G : [0, \infty) \to [0, \infty) \) is infinitely differentiable and satisfies properties given in \( (1.2) \). Then, the multi-derivative of \( G \) can be rewritten as the following decomposition:

\[
\partial_{x_{\sigma_l(1)}} \partial_{x_{\sigma_l(1)}(i)} \cdots \partial_{x_{\sigma_l(1)}(l)} G[|Du|^2] = 2(G'[|Du|^2]Du : \partial_{x_{\sigma_l(1)}} \partial_{x_{\sigma_l(1)}(i-1)} \cdots \partial_{x_{\sigma_l(1)}} Du) + E_l,
\]

where \( \sigma_l := \pi_l \circ \tilde{\sigma}_l \) and

\[
E_l = 2(\partial_{x_{\sigma_l(1)}} G'[|Du|^2]Du : \partial^{l-1} Du) + \partial_{x_{\sigma_l(1)}} E_{l-1}, \quad E_1 = 0,
\]

where \( \partial^{l-1} := \partial_{x_{\sigma_l(1)}}(i) \cdots \partial_{x_{\sigma_l(1)}}(l-1) \). Furthermore, we obtain the following.

1. \( E_2 \) and \( E_3 \) satisfy

\[
|E_2| \leq CG[|Du|^2] |\nabla Du|^2, \quad |E_3| \leq CG[|Du|^2] (|\nabla Du|^3 + |\nabla^2 Du| |\nabla Du|).
\]

2. For \( 1 \leq \alpha \leq l \)

\[
\|\partial^{\alpha} G[|Du|^2] \partial^{\ell-\alpha} Du \|_{L^2} + \|E_\alpha \partial^{\ell-\alpha} Du \|_{L^2} \\
\leq C\|G[|Du|^2]\|_{L^\infty} (\|Du\|_{L^\infty} + \|Du\|_{L^\infty}^\alpha) \|\nabla^\ell Du\|_{L^2}.
\]

(2.2)

3. In case that \( l \geq 4 \), there exists \( \beta \) with \( 0 < \beta \leq l \) such that the following is satisfied:

   (i) If \( \alpha \leq l - 1 \), then

\[
\|\partial^{\alpha} G[|Du|^2] \partial^{\ell-\alpha} Du \|_{L^2} \leq C\|G[|Du|^2]\|_{L^\infty} \|Du\|_{L^\infty}^\beta \|\nabla^{l-1} Du\|_{L^2}^{\frac{2l-3}{2l-5}}.
\]

(2.3)

(ii) If \( \alpha \leq l \), then

\[
\|E_\alpha \partial^{\ell-\alpha} Du \|_{L^2} \leq C\|G[|Du|^2]\|_{L^\infty} \|Du\|_{L^\infty}^\beta \|\nabla^{l-1} Du\|_{L^2}^{\frac{2l-3}{2l-5}}.
\]

(2.4)

The proof of Lemma \( (2.1) \) will be given at the Appendix, since it is a bit lengthy.

Next, we estimate the difference of the viscous part of the stress tensor, which is useful for uniqueness of regular solutions. Although it seems elementary, we give the details for clarity.

**Lemma 2.2.** Let \( v, w \in W^{1,2}(\mathbb{R}^3) \). Under the assumptions on \( G \) given in \( (1.2) \), we have

\[
m_0 \|Dv - Dw\|_{L^2(\mathbb{R}^3)}^2 \leq \int_{\mathbb{R}^3} (G[|Dv|^2]Dv - G[|Dw|^2]Dw) : (Dv - Dw) \, dx,
\]

where \( m_0 \) is a positive constant in \( (1.2) \).
Proof. We note that
\[
\int_{\mathbb{R}^3} \left( G[|Dv|^2]Dv - G[|Dw|^2]Dw \right) : (Dv - Dw).
\]
\[
= \int_{\mathbb{R}^3} \left( \int_0^1 \frac{d}{d\theta} \left( G[|\theta Dv + (1 - \theta) Dw|^2] (\theta Dv + (1 - \theta) Dw) \right) d\theta \right) : (Dv - Dw)
\]
\[
= 2 \int_{\mathbb{R}^3} \left( \int_0^1 G'[|\theta Dv + (1 - \theta) Dw|^2] (\theta Dv + (1 - \theta) Dw) (\theta Dv + (1 - \theta) Dw) d\theta \right) : (Dv - Dw)
\]
\[
+ \int_{\mathbb{R}^3} \left( \int_0^1 G[|\theta Dv + (1 - \theta) Dw|^2] (Dv - Dw) d\theta \right) : (Dv - Dw)
\]
\[
= 2 \int_{\mathbb{R}^3} \int_0^1 G'[|\theta Dv + (1 - \theta) Dw|^2] (\theta Dv + (1 - \theta) Dw : Dv - Dw)^2 d\theta
\]
\[
+ \int_{\mathbb{R}^3} \int_0^1 G'[|\theta Dv + (1 - \theta) Dw|^2] (Dv - Dw)^2 d\theta.
\]
\[
\geq m_0 \int_{\mathbb{R}^3} |Dv - Dw|^2 d\theta \ dx. \tag{2.5}
\]
Due to the properties in (1.2) for G, namely \( G[s] \geq m_0 \) and \( G[s] + 2G'[s]s \geq m_0 \) for any \( s \in [0, \infty) \), we deduce the inequality (2.5). Indeed, for any \( 3 \times 3 \) matrices \( A \) and \( B \), we have
\[
G[|A|^2]|B|^2 + 2G'[|A|^2](A : B)^2 \geq m_0|B|^2. \tag{2.6}
\]
Since, if \( G'[|A|^2] \geq 0 \) then
\[
G[|A|^2]|B|^2 + 2G'[|A|^2](A : B)^2 \geq G[|A|^2]|B|^2 \geq m_0|B|^2. \tag{2.7}
\]
In case that \( G'[|A|^2] < 0 \), we note that
\[
G[|A|^2]|B|^2 + 2G'[|A|^2](A : B)^2 \geq (G[|A|^2] + 2G'[|A|^2]|A|^2)|B|^2 \geq m_0|B|^2. \tag{2.8}
\]
We combine (2.7) and (2.8) to conclude (2.6). We exploit (2.6) with \( A = \theta Dv + (1 - \theta) Dw \) and \( B = Dv - Dw \) to get (2.5). This completes the proof. \( \square \)

3. Proof of Theorems 1.1

In this section, we prove the existence of a local solution to the equation (1.1)–(1.3). We first obtain a priori estimates and we then justify the estimates by using Galerkin method.

3.1. A priori estimate. We suppose that \( u \) is regular. We then compute certain a priori estimates.

\( ||u||_{L^2} \)-estimate) We multiply \( u \) to (1.1) and integrate it by parts to get
\[
\frac{1}{2} \frac{d}{dt} ||u||_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G[|Du|^2]|Du|^2 dx = 0. \tag{3.1}
\]

\( ||\nabla u||_{L^2} \)-estimate) Taking derivative \( \partial_x u \) to (1.1) and multiplying \( \partial_x u \),
\[
\frac{1}{2} \frac{d}{dt} ||\partial_x u||_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \partial_x u (G[|Du|^2]Du) : \partial_x Du dx = - \int_{\mathbb{R}^3} \partial_x (u \cdot \nabla u) \cdot \partial_x u dx. \]
Noting that
\[ \partial_{x_i}(G ||Du||^2Du) : \partial_{x_i}Du = \left[ \partial_{x_i}G ||Du||^2Du + G ||Du||^2 \partial_{x_i}Du \right] : \partial_{x_i}Du \]
\[ = 2G' ||Du||^2(\partial_{x_i}Du) : \partial_{x_i}Du + G ||Du||^2 ||\partial_{x_i}Du||^2 \]
\[ = 2G' ||Du||^2 ||\partial_{x_i}Du||^2 + G ||Du||^2 ||\partial_{x_i}Du||^2, \]
we have
\[ \frac{1}{2} \frac{d}{dt} ||\partial_{x_i}u||^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} G ||Du||^2 ||\partial_{x_i}Du||^2 dx + \int_{\mathbb{R}^3} 2G' ||Du||^2 ||\partial_{x_i}Du||^2 dx \]
\[ = - \int_{\mathbb{R}^3} \partial_{x_i} ((u \cdot \nabla)u) \cdot \partial_{x_i}u dx. \]  

Using \( A = Du \) and \( B = \partial_{x_i}Du \), we apply the inequality (2.5) to (3.2), and get
\[ \frac{1}{2} \frac{d}{dt} ||\partial_{x_i}u||^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} m_0 ||\partial_{x_i}Du||^2 dx \leq - \int_{\mathbb{R}^3} \partial_{x_i} ((u \cdot \nabla)u) \cdot \partial_{x_i}u dx. \]  

We will treat the term in righthand side caused by convection together later.

- (\( ||\nabla^2 u||_{L^2}\)-estimate) Taking the derivative \( \partial_{x_j} \partial_{x_i} \) on (1.1) and multiplying it by \( \partial_{x_j} \partial_{x_i}u \),
\[ \frac{1}{2} \frac{d}{dt} ||\partial_{x_j} \partial_{x_i}u||^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \partial_{x_j} \partial_{x_i} \left[ G ||Du||^2 Du \right] : \partial_{x_j} \partial_{x_i}Du dx \]
\[ = - \int_{\mathbb{R}^3} \partial_{x_j} \partial_{x_i} ((u \cdot \nabla)u) \cdot \partial_{x_j} \partial_{x_i}u dx. \]  

We observe that
\[ \int_{\mathbb{R}^3} \partial_{x_j} \partial_{x_i} \left[ G ||Du||^2 Du \right] : \partial_{x_j} \partial_{x_i}Du dx \]
\[ = \int_{\mathbb{R}^3} \left( G ||Du||^2 \partial_{x_j} \partial_{x_i}Du \right)^2 + \sum_{\sigma} \int_{\mathbb{R}^3} \partial_{x_{\sigma(i)}} G ||Du||^2 (\partial_{x_{\sigma(j)}} Du : \partial_{x_j} \partial_{x_i}Du) dx \]
\[ + \int_{\mathbb{R}^3} \partial_{x_j} \partial_{x_i} G ||Du||^2 (Du : \partial_{x_j} \partial_{x_i}Du) dx =: I_{21} + I_{22} + I_{23}, \]
where \( \sigma : \{i, j\} \rightarrow \{i, j\} \) is a permutation of \( \{i, j\} \). We separately estimate terms \( I_{22} \) and \( I_{23} \) in (3.5). Using Hölder, Young’s and Gagliardo-Nirenberg inequalities, we have for \( I_{22} \)
\[ |I_{22}| = \int_{\mathbb{R}^3} 2G' ||Du||^2 (Du : \partial_{x_{\sigma(i)}} Du)(\partial_{x_{\sigma(j)}} Du : \partial_{x_j} \partial_{x_i}Du) dx \]
\[ \leq C ||G ||Du||^2 ||\nabla Du||^2_{L^4} ||\nabla^2 Du||_{L^2} \]
\[ \leq C ||G ||Du||^2 ||Du||_{L^\infty} ||\nabla^2 Du||^2_{L^2}, \]
where we used the condition (1.2).

For \( I_{23} \), using Lemma [2.1], we compute
\[ I_{23} = \int_{\mathbb{R}^3} 2(G' ||Du||^2)(Du : \partial_{x_j} \partial_{x_i}Du) + E_2) (Du : \partial_{x_j} \partial_{x_i}Du) dx \]
\[ = \int_{\mathbb{R}^3} E_2 (Du : \partial_{x_j} \partial_{x_i}Du) dx + 2 \int_{\mathbb{R}^3} G' ||Du||^2 ||Du : \partial_{x_j} \partial_{x_i}Du||^2 dx \]
\[ := I_{231} + I_{232}. \]
The term $I_{231}$ is estimated as

\[
|I_{231}| \leq C\|G[Du]\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla Du\|_{L^1} \|\nabla^2 Du\|_{L^2}
\leq C\|G[Du]\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^2 Du\|_{L^2},
\]

(3.6)

where we used the first inequality of (2.1). We combine estimates (3.4)-(3.6) to get

\[
\frac{1}{2} \frac{d}{dt} \|\partial_{x_j} x_i u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G[|Du|^2] \partial_{x_j} x_i Du^2 + \int_{\mathbb{R}^3} 2G'(|Du|^2) |Du : \partial_{x_j} x_i Du|^2
\leq C\|G[Du]\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^2 Du\|_{L^2}
\leq \int_{\mathbb{R}^3} \partial_{x_j} x_i \left( (u \cdot \nabla u) \right) \cdot \partial_{x_j} x_i u. \tag{3.7}
\]

Similarly as in (3.3), we have

\[
\frac{1}{2} \frac{d}{dt} \|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0 |\partial_{x_j} x_i Du|^2
\leq C\|G[Du]\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^2 Du\|_{L^2}
\leq \int_{\mathbb{R}^3} \partial_{x_j} x_i \left( (u \cdot \nabla u) \right) \cdot \partial_{x_j} x_i u. \tag{3.8}
\]

\[ \bullet \] \( (\nabla^3 u \|_{L^2} \)-estimate) For convenience, we denote \( \partial^3 := \partial_{x_k} x_j x_i \). Similarly as before, taking the derivative \( \partial^3 \) on (1.1) and multiplying it by \( \partial^3 u \),

\[
\frac{1}{2} \frac{d}{dt} \|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \partial^3 \left( G[|Du|^2] Du \right) : \partial^3 Du dx = - \int_{\mathbb{R}^3} \partial^3 \left( (u \cdot \nabla u) \right) \cdot \partial^3 u dx. \tag{3.9}
\]

Direct computations show that

\[
\int_{\mathbb{R}^3} \partial^3 \left( G[|Du|^2] Du \right) : \partial^3 Du dx = \int_{\mathbb{R}^3} G[|Du|^2] |\partial^3 Du|^2 dx
\]

\[
+ \sum_{\sigma_3} \int_{\mathbb{R}^3} \partial_{x_{\sigma_3(i)}} G[|Du|^2] \partial_{x_{\sigma_3(k)}} \partial_{x_{\sigma_3(j)}} Du : \partial^3 Du dx
\]

\[
+ \sum_{\sigma_3} \int_{\mathbb{R}^3} \partial_{x_{\sigma_3(j)}} \partial_{x_{\sigma_3(i)}} G[|Du|^2] \partial_{x_{\sigma_3(k)}} Du : \partial^3 Du dx
\]

\[
+ \int_{\mathbb{R}^3} \partial^3 G[|Du|^2] (Du : \partial^3 Du) dx = I_{31} + I_{32} + I_{33} + I_{34}, \tag{3.10}
\]

where \( \sigma_3 = \pi_3 \circ \tilde{\sigma}_3 \) such that \( \tilde{\sigma}_3 : \{i, j, k\} \rightarrow \{i, j, k\} \) is a permutation of \( \{i, j, k\} \) and \( \pi_3 \) is a mapping from \( \{i, j, k\} \) to \( \{1, 2, 3\} \).

We separately estimate terms \( I_{32}, I_{33} \) and \( I_{34} \). We note first that

\[
|I_{32}| \leq \int_{\mathbb{R}^3} \left( 2G'(|Du|^2) |Du| |\partial_{x_{\sigma_3(i)}} Du| |\partial_{x_{\sigma_3(k)}} \partial_{x_{\sigma_3(j)}} Du| |\partial^3 Du| \right) dx
\]

\[
\leq C\|G[|Du|^2]\|_{L^\infty} \|\nabla Du\|_{L^6} \|\nabla^2 Du\|_{L^3} \|\nabla^3 Du\|_{L^2} \tag{3.11}
\]

\[
\leq C\|G[|Du|^2]\|_{L^\infty} \|\nabla^2 Du\|_{L^6} \|Du\|_{L^\infty}^{\frac{1}{3}} \|\nabla^3 Du\|_{L^2}^{\frac{2}{3}} \|\nabla^3 Du\|_{L^2}
\]

\[
\leq C\|G[|Du|^2]\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^3 Du\|_{L^2}^2 + \epsilon \|\nabla^3 Du\|_{L^2}^2.
\]

For \( I_{33} \), we have

\[
|I_{33}| = \left| \int_{\mathbb{R}^3} \left( 2G'(|Du|^2) |Du| \partial_{x_{\sigma_3(j)}} \partial_{x_{\sigma_3(i)}} Du + E_2 \right) (\partial_{x_{\sigma_3(k)}} Du : \partial^3 Du) dx \right|
\]

\[
\leq \int_{\mathbb{R}^3} 2(G'(|Du|^2) |Du| |\nabla^2 Du| + G(|Du|^2) |\nabla Du|^2) |\nabla Du| |\nabla^3 Du| dx
\]
\[
\leq C \|G'[|Du|^2]Du\|_{L^\infty} \|\nabla^2 Du\|_{L^3} \|\nabla^3 Du\|_{L^5}^2 + \|G[|Du|^2]\|_{L^\infty} \|\nabla^3 Du\|_{L^2}
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty}^6 \|Du\|_{L^\infty}^6 \|\nabla^2 Du\|_{L^5}^2 + C \|G[|Du|^2]\|_{L^\infty}^6 \|\nabla^2 Du\|_{L^2}^2 + 2\epsilon \|\nabla^3 Du\|_{L^2}^2
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty}^6 \|Du\|_{L^\infty}^6 + \|G(|Du|)|\|\|\nabla^2 Du\|_{L^\infty}^6 + 2\epsilon \|\nabla^3 Du\|_{L^2}^2,
\]  

(3.12)

where we use same argument as \ref{3.11} in the fourth inequality. Finally, for \(I_{34}\), using Lemma \ref{2.1}, we note that

\[
I_{34} = \int_{\mathbb{R}^3} (2G'[|Du|^2]Du : \partial^3 Du) + E_3(Du : \partial^3 Du) \, dx
\]

\[
= 2 \int_{\mathbb{R}^3} G'[|Du|^2]Du : \partial^3 Du \, dx + \int_{\mathbb{R}^3} E_3(Du : \partial^3 Du) \, dx. \tag{3.13}
\]

The second term in \ref{3.13} is estimated as follows:

\[
\int_{\mathbb{R}^3} E_3(Du : \partial^3 Du) \, dx \leq \int_{\mathbb{R}^3} |E_3| |Du| |\nabla^3 Du| \, dx
\]

\[
\leq C \int_{\mathbb{R}^3} G[|Du|^2]|(|\nabla Du| + |\nabla^2 Du|)|Du| |\nabla^3 Du| \, dx
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty} \|Du\|_{L^\infty}^3 \|\nabla Du\|_{L^3}^3 \|\nabla^2 Du\|_{L^5} \|\nabla^3 Du\|_{L^2}
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty}^6 \|Du\|_{L^\infty}^6 \|\nabla^2 Du\|_{L^5}^2 + C \|G[|Du|^2]\|_{L^\infty}^6 \|\nabla^2 Du\|_{L^2}^2 + 2\epsilon \|\nabla^3 Du\|_{L^2}^2,
\]  

(3.14)

where we use same argument as \ref{3.12} in the third inequality. Adding up the estimates \ref{3.9}–\ref{3.14}, we obtain

\[
\frac{d}{dt} \|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G[|Du|^2]|\partial^3 Du|^2 \, dx + \int_{\mathbb{R}^3} 2G'[|Du|^2]Du : \partial^3 Du \, dx
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty}^2 + \|G[|Du|^2]\|_{L^\infty}^2 \|Du\|_{L^\infty}^2 \|\nabla^2 Du\|_{L^2}^2 + 5\epsilon \|\nabla^3 Du\|_{L^2}^2
\]

\[
- \int_{\mathbb{R}^3} \partial^3 ((u \cdot \nabla u)) \cdot \partial^3 u \, dx. \tag{3.15}
\]

Hence, we have

\[
\frac{d}{dt} \|\partial^3 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0|\partial^3 Du|^2 \, dx
\]

\[
\leq C \|G[|Du|^2]\|_{L^\infty}^2 + \|G[|Du|^2]\|_{L^\infty}^2 \|Du\|_{L^\infty}^2 \|\nabla^2 Du\|_{L^2}^2 + 5\epsilon \|\nabla^3 Du\|_{L^2}^2
\]

\[
- \int_{\mathbb{R}^3} \partial^3 ((u \cdot \nabla u)) \cdot \partial^3 u \, dx. \tag{3.16}
\]

Next, we estimate the terms caused by convection terms in \ref{3.3}, \ref{3.8} and \ref{3.16}.

\[
\sum_{1 \leq |\alpha| \leq 3} \int_{\mathbb{R}^3} \partial^\alpha ((u \cdot \nabla) u) \cdot \partial^\alpha u \, dx = \sum_{1 \leq |\alpha| \leq 3} \int_{\mathbb{R}^3} [\partial^\alpha ((u \cdot \nabla) u) - u \cdot \nabla \partial^\alpha u] \partial^\alpha u \, dx
\]

\[
\leq \sum_{1 \leq |\alpha| \leq 3} \|\partial^\alpha ((u \cdot \nabla) u) - u \cdot \nabla \partial^\alpha u\|_{L^2} \|\partial^\alpha u\|_{L^2}
\]

\[
\leq \sum_{1 \leq |\alpha| \leq 3} \|\nabla u\|_{L^\infty} \|u\|_{H^3} \|\partial^\alpha u\|_{L^2}
\]

\[
\leq C \|\nabla u\|_{L^\infty} \|u\|_{H^3}^2,
\]  

(3.17)
where we use the following inequality:
\[
\sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} \|\nabla^\alpha (fg) - (\nabla^\alpha f)g\|_{L^2} \leq C(\|f\|_{H^{m-1}} \|\nabla g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^m}).
\] (3.18)

We combine (3.1), (3.3), (3.8) and (3.16) with (3.17) to conclude
\[
\frac{d}{dt} \|u\|_{H^3(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} (m_0 - 5\epsilon)(|\nabla^2 Du|^2 + |\nabla^2 D^2 u|^2 + |\nabla D u|^2 + |D u|^2) \, dx
\leq C\|\nabla u\|_{L^\infty} \|u\|_{H^3}^2 + C\|G[D u|^2]\|_{L^\infty}(\|D u\|_{L^\infty}^2 + \|D u\|_{L^\infty})\|\nabla^2 D u\|_{L^2}^2
+ C(\|G[D u|^2]\|_{L^\infty}^2 + \|G[D u|^2]\|_{L^\infty}(\|D u\|_{L^\infty}^2 + \|D u\|_{L^\infty}))\|\nabla^2 D u\|_{L^2}^2.
\] (3.19)

Here we choose a sufficiently small $\epsilon > 0$ such that $m_0 - 5\epsilon > 0$. Furthermore, we have
\[
\|G[D u|^2]\|_{L^\infty} \leq \max_{0 \leq s \leq \|u\|_{H^3}} G[s] \leq \max_{0 \leq s \leq C\|u\|_{H^3}} G[s] \coloneqq g(\|u\|_{H^3}),
\] (3.20)

where $g : [0, \infty) \to [0, \infty)$ is a nondecreasing function. We set $X(t) \coloneqq \|u(t)\|_{H^3(\mathbb{R}^3)}$ and it then follows from (3.19) and (3.20) that
\[
\frac{d}{dt} X^2 \leq f_3(X)X^2
\] (3.21)

for some nondecreasing continuous function $f_3$, which immediately implies that there exists $T_3 > 0$ such that
\[
\sup_{0 \leq t \leq T_3} X(t) < \infty.
\]

- (\|\nabla^4 u\|_{L^2}-estimate) For simplicity, we denote $\partial^4 \coloneqq \partial_{x_1} \partial_{x_2} \partial_{x_3} \partial_{x_4}$, $i,j,k,l = 1,2,3$. Similarly as in (3.9) and (3.10), we have
\[
\frac{1}{2}\frac{d}{dt}\|\partial^4 u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \partial^4 [G[D u|^2]D u] : \partial^4 D u \, dx = -\int_{\mathbb{R}^3} \partial^4 ((u \cdot \nabla u)) : \partial^4 u \, dx.
\] (3.22)

We note that
\[
\int_{\mathbb{R}^3} \partial^4 [G[D u|^2]D u] : \partial^4 D u \, dx = \int_{\mathbb{R}^3} G[D u|^2]\|\partial^4 D u\|^2 \, dx
+ \sum_{\sigma_4} \int_{\mathbb{R}^3} \partial_{\sigma_4(i)} G[D u|^2](\partial_{\sigma_4(i)} \partial_{\sigma_4(k)} D u : \partial^4 D u) \, dx
+ \sum_{\sigma_4} \int_{\mathbb{R}^3} \partial_{\sigma_4(j)} \partial_{\sigma_4(i)} G[D u|^2](\partial_{\sigma_4(i)} \partial_{\sigma_4(k)} D u : \partial^4 D u) \, dx
+ \sum_{\sigma_4} \int_{\mathbb{R}^3} \partial_{\sigma_4(k)} \partial_{\sigma_4(j)} \partial_{\sigma_4(i)} G[D u|^2](\partial_{\sigma_4(i)} \partial_{\sigma_4(k)} D u : \partial^4 D u) \, dx
+ \int_{\mathbb{R}^3} \partial^4 G[D u|^2](D u : \partial^4 D u) \, dx := I_{41} + I_{42} + I_{43} + I_{44} + I_{45},
\] (3.23)

where $\sigma_4 = \pi_3 \circ \sigma_3$ such that $\sigma_4 : \{i,j,k,l\} \to \{i,j,k,l\}$ is a permutation of $\{i,j,k,l\}$ and $\pi_3$ is a mapping from $\{i,j,k,l\}$ to $\{1,2,3\}$.
We first estimate $I_{42}$. Exploiting (2.3) with $l = 4$, $\alpha = 1$, we get
\[
|I_{42}| \leq C\|\partial G[Du]^2\| \partial^4 Du\|L^2\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 + \epsilon \|\nabla^4 D u\|L^2
\]
for some $0 < \beta_1 \leq 4$. Similarly, using (2.3) with $l = 4$, $\alpha = 2$, we have
\[
|I_{43}| \leq C\|\partial^2 G[Du^2]\| \partial^2 D u\|L^2\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 + \epsilon \|\nabla^4 D u\|L^2
\]
for some $0 < \beta_2 \leq 4$. Again, due (2.3) with $l = 4$, $\alpha = 3$, we obtain for some $0 < \beta_3 \leq 4$
\[
|I_{44}| \leq C\|\partial^3 G[Du^2]\| \partial^4 D u\|L^2\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 + \epsilon \|\nabla^4 D u\|L^2.
\]
For the term $I_{45}$, we note that
\[
I_{45} = \int_{\mathbb{R}^3} \left[2G'[|Du|^2](Du : \partial^4 Du) + E_4(Du : \partial^4 Du)\right] dx
\]
\[
= \int_{\mathbb{R}^3} 2G'[|Du|^2]Du : \partial^4 Du^2 dx + \int_{\mathbb{R}^3} E_4(Du : \partial^4 Du) dx. \quad (3.24)
\]
Owing to (2.4), we see that
\[
\int_{\mathbb{R}^3} E_4(Du : \partial^4 Du) dx \leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 \\
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \partial^3 Du\|L^5/3\| \partial^4 D u\|L^2 + \epsilon \|\nabla^4 D u\|L^2 \quad (3.25)
\]
for some $0 < \beta_4 \leq 4$. We combine (3.22)-(3.25) to have
\[
\frac{1}{2} \frac{d}{dt}\|\partial^4 u\|L^2(\mathbb{R}^3)^2 + \int_{\mathbb{R}^3} G[Du^2]\| \partial^4 D u\|L^2 dx + \int_{\mathbb{R}^3} 2G'[|Du|^2]Du : \partial^4 Du^2 dx
\]
\[
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \nabla^3 Du\|L^2 + 4\epsilon \|\nabla^4 D u\|L^2 - \int_{\mathbb{R}^3} \partial^4 ((u \cdot \nabla u)) \cdot \partial^4 u dx, \quad (3.26)
\]
where $\beta = \beta_1 + \beta_2 + \beta_3 + \beta_4$. Hence, as before, we have
\[
\frac{1}{2} \frac{d}{dt}\|\partial^4 u\|L^2(\mathbb{R}^3)^2 + \int_{\mathbb{R}^3} m_0 \|\partial^4 D u\|L^2 dx
\]
\[
\leq C\|G[Du^2]\|L^\infty \| Du\|L^\infty \| \nabla^3 Du\|L^2 + 4\epsilon \|\nabla^4 D u\|L^2 - \int_{\mathbb{R}^3} \partial^4 ((u \cdot \nabla u)) \cdot \partial^4 u dx. \quad (3.27)
\]
Using (3.18), we estimate the convection term
\[
\sum_{|\alpha|=4} \int_{\mathbb{R}^3} \partial^\alpha \left[ (u \cdot \nabla) u \right] \cdot \partial^\alpha u \, dx = \sum_{|\alpha|=4} \int_{\mathbb{R}^3} \left[ \partial^\alpha \left( (u \cdot \nabla) u \right) \right] - u \cdot \nabla \partial^\alpha u \, dx
\leq \sum_{|\alpha|=4} \left\| \partial^\alpha \left( (u \cdot \nabla) u \right) \right\|_{L^2} \left\| \partial^\alpha u \right\|_{L^2} \tag{3.28}
\leq C \sum_{|\alpha|=4} \left\| \nabla u \right\|_{L^\infty} \left\| u \right\|_{H^4} \left\| \partial^\alpha u \right\|_{L^2}
\leq C \left\| \nabla u \right\|_{L^\infty} \left\| u \right\|_{H^4}^2.
\]

Finally, we combine (3.1), (3.3), (3.8), (3.16) and (3.27) with (3.28) to conclude
\[
\frac{1}{2} \frac{d}{dt} \left\| u \right\|_{H^4(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \left( m_0 - 9\varepsilon \right) \left( \| \nabla^4 u \|^2 + \| \nabla^3 u \|^2 + \| \nabla^2 u \|^2 + \| \nabla u \|^2 + \| u \|^2 \right) \, dx
\leq C \left( \| G[\| D u \|^2] \|_{L^\infty}^2 + \| G[\| D u \|^2] \|_{L^\infty}^2 \left( \| D u \|_{L^\infty} + \| D u \|_{L^2} + \| D u \|_{L^2}^2 + \| D u \|_{L^2}^2 \right) + C \left\| \nabla u \right\|_{L^\infty} \left\| u \right\|_{H^4}^2 \right)
\times \left( \| \nabla^3 u \|_{L^2}^{10/3} + \| \nabla^2 u \|_{L^2}^6 + \| \nabla^2 u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) + C \left\| \nabla u \right\|_{L^\infty} \left\| u \right\|_{H^4}^2.
\]
Let us denote \( X(t) := \| u(t) \|_{H^4(\mathbb{R}^3)} \). Similarly as in (3.24), we have
\[
\frac{d}{dt} X^2 \leq f_4(X) X^2.
\]
for some non-decreasing continuous function \( f_4 \). Therefore, there exists \( T_4 > 0 \) such that
\[
\sup_{0 \leq t \leq T_4} X(t) < \infty.
\]
So far, we have proven for some non-decreasing continuous function \( f_k, k = 3, 4 \)
\[
\frac{1}{2} \frac{d}{dt} \left\| u \right\|_{H^k}^2 + \int_{\mathbb{R}^3} \left( m_0 - \varepsilon \right) \left( \| \nabla^k u \|^2 + \cdots + \| \nabla u \|^2 + \| u \|^2 \right) \, dx \leq f_k(\| u \|_{H^k}) \left\| u \right\|_{H^k}^2. \tag{3.29}
\]
Next, we will show that (3.29) holds for general \( k \geq 3 \) by the induction argument. Suppose (3.29) is true for \( k = l - 1 \) for some \( l \geq 4 \). We then prove that (3.29) is true for \( k = l \).

Indeed, let \( \sigma_l = \pi_l \circ \tilde{\sigma}_l \) such that \( \tilde{\sigma}_l : \{1, 2, \cdots, l\} \rightarrow \{1, 2, \cdots, l\} \) is a permutation of \( \{1, 2, \cdots, l\} \) and \( \pi_l \) is a mapping from \( \{1, 2, \cdots, l\} \) to \( \{1, 2, 3\} \). For simplicity, we denote \( \partial^l := \partial_{x_{\sigma(1)}} \partial_{x_{\sigma(l-1)}} \cdots \partial_{x_{\sigma(l)}} \). Similar computations as before leads to
\[
\frac{1}{2} \frac{d}{dt} \left\| \partial^l u \right\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} G[\| D u \|^2] |\partial^l D u|^2 \, dx + \int \left( 2 G' \| D u \|^2 \right) |D u : \partial^l D u|^2 \, dx
\leq \sum_{|\alpha|=1} \int_{\mathbb{R}^3} |\partial^\alpha G[\| D u \|^2]| |\partial^{l-\alpha} D u| |\partial^l D u| \, dx + \int_{\mathbb{R}^3} |E_l| |D u| |\partial^l D u| \, dx
- \int_{\mathbb{R}^3} \partial^l (u \cdot \nabla u) \cdot \partial^l u \, dx := I_{l1} + I_{l2} + I_{l3}. \tag{3.30}
\]
We exploit (2.3) for $I_{11}$ to get
\[
|I_{11}| \leq \sum_{\alpha=1}^{l-1} \|\partial^\alpha G(\|Du\|^2)\|_{L^\infty} \sum_{\alpha=1}^{l-1} \|Du\|_{L^\infty}^{\frac{3\alpha}{2}} \|\nabla^{l-1} Du\|_{L^2} \|\nabla^l Du\|_{L^2}
\]
\[
\leq C \|G(\|Du\|^2)\|_{L^\infty} \sum_{\alpha=1}^{l-1} \|Du\|_{L^\infty}^{\frac{3\alpha}{2}} \|\nabla^{l-1} Du\|_{L^2} \|\nabla^l Du\|_{L^2}
\]  \hspace{1cm} (3.31)
\[
\leq f_1(\|u\|_{H^{l-1}}) \|u\|_{H^{l-1}}^{\frac{3\alpha}{2}} \|\nabla^l Du\|_{L^2}
\]
for some function $f_1$. For $I_{12}$, due to (2.4), we obtain
\[
|I_{12}| \leq \|E_t Du\|_{L^2} \|\nabla^l Du\|_{L^2}
\]
\[
\leq C \|G(\|Du\|^2)\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^{l-1} Du\|_{L^2} \|\nabla^l Du\|_{L^2}
\]  \hspace{1cm} (3.32)
\[
\leq f_1(\|u\|_{H^{l-1}}) \|u\|_{H^{l-1}}^{\frac{3\alpha}{2}} \|\nabla^l Du\|_{L^2}
\]
for some function $f_1$. Lastly, we estimate $I_{13}$ similarly as before.
\[
\sum_{1 \leq |\alpha| \leq l} \int_{\mathbb{R}^3} \partial^\alpha[(u \cdot \nabla)u] \cdot \partial^\alpha u \, dx = \sum_{1 \leq |\alpha| \leq l} \int_{\mathbb{R}^3} [\partial^\alpha((u \cdot \nabla)u) - u \cdot \nabla \partial^\alpha u] \partial^\alpha u \, dx
\]
\[
\leq \sum_{1 \leq |\alpha| \leq l} \|\nabla u\|_{L^\infty} \|u\|_{H^l} \|\partial^\alpha u\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|u\|_{H^l}^2.
\]  \hspace{1cm} (3.33)
Combining (3.30), (3.31), (3.32) and (3.33), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^l u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0 |\nabla^l Du|^2 \, dx
\]
\[
\leq f_1(\|u\|_{H^{l-1}}) \|u\|_{H^{l-1}}^{\frac{3\alpha}{2}} \|\nabla^l Du\|_{L^2} + C \|\nabla u\|_{L^\infty} \|u\|_{H^l}^2
\]
\[
\leq f_1(\|u\|_{H^{l-1}}) \|u\|_{H^l}^{\frac{3\alpha}{2}} + C \|\nabla u\|_{L^\infty} \|u\|_{H^l}^2 + \epsilon \|\nabla^l Du\|_{L^2}^2
\]
for some nondecreasing continuous function $f_1$. Hence, we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^l u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} (m_0 - \epsilon)|\nabla^l Du|^2 \, dx \leq f_1(\|u\|_{H^l}) \|u\|_{H^l}^2.
\]
Since (3.29) is true for $k = l - 1$, we conclude that
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^l}^2 + \int_{\mathbb{R}^3} (m_0 - \epsilon)(|\nabla^l Du|^2 + \cdots |\nabla^2 Du|^2 + |Du|^2) \leq f_1(\|u\|_{H^l}) \|u\|_{H^l}^2.
\]
Choosing $\epsilon > 0$ so small and letting $X(t) := \|u(t)\|_{H^l}$, we obtain
\[
\frac{d}{dt} X^2 \leq f_1(X) X^2,
\]
which yields that there exists $T_1 > 0$ such that
\[
\sup_{0 \leq t \leq T_1} \|u(t)\|_{H^l} < \infty.
\]
We complete the a priori estimates. \qed
Remark 3.1. We note that $\partial_t u \in L^2((0,T); L^2(\mathbb{R}^3))$. Indeed, we introduce the antiderivative of $G$, denoted by $\tilde{G}$, i.e., $\tilde{G}[s] = \int_0^s G[\tau]d\tau$. Multiplying $\partial_t u$ to (3.1), integrating it by parts, and using Hölder and Young’s inequalities, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} |\partial_t u|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \tilde{G}[|Du|^2] \, dx \leq C \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2. \quad (3.34)$$

Again, integrating the estimate (3.34) over the time interval $[0,T_i]$, we obtain

$$\int_0^{T_i} \int_{\mathbb{R}^3} |\partial_t u|^2 \, dxdt + \int_{\mathbb{R}^3} \tilde{G}[|Du(\cdot,T_i)|^2] \, dx \leq \int_{\mathbb{R}^3} \tilde{G}[|Du(0)|^2] \, dx + C \int_0^{T_i} \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dxdt. \quad (3.35)$$

Using Sobolev embedding, the second term in (3.35) is estimated as follows:

$$\int_0^{T_i} \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 \, dxdt \leq \int_0^{T_i} \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \, dt \leq C \sup_{0<\tau\leq T_i} \|u(\tau)\|_{W^{1,2}}^2 \int_0^{T_i} \|\nabla u\|_{L^2}^2 \, dt < C.$$

Therefore, we obtain $\partial_t u \in L^2(0,T; L^2(\mathbb{R}^3))$. \hfill \Box

Using the method of Galerkin approximation, we construct the regular solution satisfying the a priori estimates above.

**Proof of Theorem 1.1** To precisely justify existence of a regular solution, we proceed by a Galerkin method. In view of [11, Lemma 3.10], there exists a countable subset $\{w_i\}_{i=1}^\infty$ of the space $V := \{\varphi \in D(\mathbb{R}^3)^3 : \nabla \cdot \varphi = 0\}$ that is dense in $H^1(\mathbb{R}^3) \cap V$. We consider Galerkin approximate equation for $u_m(t) = \sum_{i=1}^m g_i^m(t)w_i$,

$$\int_{\mathbb{R}^3} (\partial_t u_m(t) \cdot w + G[|Du_m(t)|^2]Du_m(t)) : Dw + (u_m \otimes u_m)(t) : \nabla w \, dx = 0,$$

where $w \in \text{span}\{w_1, w_2, \cdots, w_m\}$ and $\|u_m(0) - u_0\|_{H^1} \to 0$ as $m \to \infty$. From a priori estimate above, we obtain

$$\|u_m\|_{L^\infty(0,T; H^1(\mathbb{R}^3))} \leq C,$$

$$\|u_m\|_{L^2(0,T; H^{1+1}(\mathbb{R}^3))} \leq C,$$

$$\|\partial_t u_m\|_{L^2(0,T; L^2(\mathbb{R}^3))} \leq C,$$

where $C$ is independent on $m$. Due to the uniform boundedness above, we can choose a subsequence $u_{m_k}$ of $u_m$ such that

$$u_{m_k} \to u \quad \text{weakly-* in } L^\infty(0,T; H^1(\mathbb{R}^3)).$$

$$u_{m_k} \to u \quad \text{weakly in } L^2(0,T; H^{1+1}(\mathbb{R}^3)).$$

$$\partial_t u_{m_k} \to \partial_t u \quad \text{weakly in } L^2(0,T; L^2(\mathbb{R}^3)),$$

as $k \to \infty$. Using the Aubin-Lions Lemma, we obtain

$$u_{m_k} \to u \quad \text{strongly in } L^2(0,T; H^{1}_{\text{loc}}(\mathbb{R}^3)), \quad k \to \infty. \quad (3.36)$$

By the standard argument (see e.g., [10]), we see that $u \in L^\infty(0,T; H^1(\mathbb{R}^3)) \cap L^2(0,T; H^{1+1}(\mathbb{R}^3))$ is a solution of the following equation in a weak sense

$$\partial_t u - \nabla \cdot \tilde{G} + (u \cdot \nabla)u + \nabla p = 0,$$
that is, for any \( \psi \in C^1([\mathbb{R}^3 \times [0,T]) \) such that \( \nabla \cdot \psi = 0 \), we have
\[
\int_{\mathbb{R}^3} u(t)\psi dx + \int_0^t \int_{\mathbb{R}^3} -u \cdot \partial_t \psi + [\tilde{G} + (u \otimes u)] : \nabla \psi \, dx \, ds = \int_{\mathbb{R}^3} u_0 \cdot \psi(x,\cdot) \, dx
\]
for a.e. \( t \in [0,T] \). Here, \( \tilde{G} \) is a weak limit of \( G[D\eta_k]D\eta_k \) in \( L^q((0,T) \times \mathbb{R}^3) \). Due to the strong convergence \( \text{(3.36)} \), it follows that
\[
\tilde{G} = G[Du]Du \quad \text{a.e. in } \mathbb{R}^3 \times (0,T_0).
\]
As \( m \to \infty \), we conclude the existence of a solution of \( \text{(1.1)-(1.3)} \) in \( \mathcal{X}_1 := L^\infty([0,T] ; H^l(\mathbb{R}^3)) \cap L^2([0,T] ; H^{l+1}(\mathbb{R}^3)) \), \( l \geq 3 \). Next, we show uniqueness of the solution of \( \text{(1.1)-(1.3)} \) until the time \( t \leq T_1 \). More precisely, we prove uniqueness of weak and regular solutions, in case that initial data are the same. Let \((u^1,p^1)\) be a weak solution and \((u^2,p^2)\) a solution constructed above of the equation \( \text{(1.1)-(1.3)} \). We consider the equation for \( \tilde{u} = u_1 - u_2 \) and \( \tilde{p} = p_1 - p_2 \).
\[
\partial_t \tilde{u} - \nabla \cdot (G[D\eta_1]^2)D\eta_1 - G[D\eta_2]^2D\eta_2 + (u_1 \cdot \nabla)\tilde{u} + (\tilde{u} \cdot \nabla)u_2 + \nabla \tilde{p} = 0, \quad \text{div } \tilde{u} = 0, \quad (3.37)
\]
Testing \( \tilde{u} \) to the difference equation \( \text{(3.37)} \) and integrating it by parts, we have
\[
\frac{1}{2} \frac{d}{dt} \| \tilde{u} \|^2_{L^2(\mathbb{R}^3)} + m_0 \| \nabla \tilde{u} \|^2_{L^2(\mathbb{R}^3)} \leq \int_{\mathbb{R}^3} |\tilde{u}| \| \nabla u_2 \| \tilde{u} \, dx := J_1, \quad (3.38)
\]
where we use the divergence free condition and Lemma 2.2. Since \( u_2 \in L^\infty(0,T_1 ; H^3(\mathbb{R}^3)) \cap L^2(0,T_1 ; H^4(\mathbb{R}^3)) \), we can estimate the term \( J_1 \) as follows. Using Hölder’s inequality and Young’s inequality, we have
\[
|J_1| \leq \| \tilde{u} \|^2_{L^2(\mathbb{R}^3)} \| \nabla u_2 \|_{L^\infty(\mathbb{R}^3)}.
\]
Thus, the estimate \( \text{(3.38)} \) becomes
\[
\frac{d}{dt} \| \tilde{u} \|^2_{L^2(\mathbb{R}^3)} + m_0 \| \nabla \tilde{u} \|^2_{L^2(\mathbb{R}^3)} \leq C \| \tilde{u} \|^2_{L^2(\mathbb{R}^3)} \| u_2 \|_{H^4(\mathbb{R}^3)}.
\]
Due to Grownwall’s inequality and \( \tilde{u}(x,0) = 0 \), we conclude that \( \tilde{u} = 0 \), i.e., \( u_1 = u_2 \).

4. PROOF OF COROLLARY 1.2

For the proof of corollary 1.2 we need the following lemma, which is a simpler case of [6] Lemma 2.2. For clarity, we present its proof. For notational convention, we write the average of \( f \) on \( E \) as \( f_E \), that is \( f_E = \frac{1}{|E|} \int_E f \).

**Lemma 4.1.** Let \( \alpha \in (0,\frac{1}{2}) \). Suppose that \( v \in L^\infty([0,T];C^\alpha(\mathbb{R}^3)) \) and \( v_t \in L^\infty([0,T];L^2(\mathbb{R}^3)) \). Then \( v \in C^\alpha_C(\mathbb{R}^3 \times [0,T]) \).

**Proof.** To show \( v \in C^\alpha_C(\mathbb{R}^3 \times (0,T_0)) \), it suffices to show that \( |v(x,t_1) - v(x,t_2)| \leq C|t_1 - t_2|^{\frac{\alpha}{2}} \). For \( x \in \mathbb{R}^3 \) and \( \rho > 0 \) we define
\[
v_{\rho}(x,t) = \int_{B_{x,\rho}} v(y,t) \, dy.
\]
We note that
\[
|v(x,t_1) - v(x,t_2)| \leq |v(x,t_1) - v_{\rho}(x,t_1)| + |v_{\rho}(x,t_1) - v_{\rho}(x,t_2)| + |v_{\rho}(x,t_2) - v(x,t_2)|. \quad (4.1)
\]
By the hypothesis, first and third terms in (4.1) are estimated easily as
\[ |v(x, t_i) - v(x, t_i)| = \left| \int_{B_i} v(x, t_i) - v(y, t_i) \, dy \right| \leq C \rho^\alpha, \quad i = 1, 2. \]
For second term in (4.1), since \( v_t \in L^\infty([0, T]; L^2(\mathbb{R}^3)) \), we obtain
\[ |v_\rho(x, t_2) - v_\rho(x, t_1)| = \left| \int_{B_{t_i}} v(x, t_2) - v(x, t_1) \, dy \right| = \left| \int_{B_{t_i}} \int_0^1 \frac{d}{d\theta} v(x, \theta t_2 + (1 - \theta)t_1) \, d\theta \, dy \right| \leq \int_{B_{t_i}} \int_0^1 |v_t(x, \theta t_2 + (1 - \theta)t_1)| |t_2 - t_1| \, d\theta \, dy \leq C \rho^{-\frac{3}{2}} |v_t|_{L^\infty(0, T; L^2(\mathbb{R}^3))} |t_2 - t_1|. \]
Putting \( \rho = |t_2 - t_1|^{1/\mu} \), we have
\[ |v(x, t_1) - v(x, t_2)| \leq C(\rho^\alpha + \rho^{-\frac{3}{2} + \mu}). \]
Choosing \( \mu = \alpha + \frac{3}{2} \), we see that
\[ |v(x, t_1) - v(x, t_2)| \leq C|t_2 - t_1|^{1 - \frac{3}{2\mu}}, \]
which implies \( v \in C_x^\alpha C_t^{1 - \frac{3}{2\mu}} \). It follows from \( \alpha \in (0, \frac{1}{2}] \) that \( 1 - \frac{3}{2\mu} \geq \alpha/2 \). Therefore, we conclude that \( v \in C_x^\alpha C_t^{\frac{1}{2}} \). This completes the proof. \( \Box \)

Next, we control mixed derivatives of \( G||Du||^2 Du \) for spatial and temporal variables, in case that some mixed derivatives of \( u \) are bounded. Since the proof is similar to that of Lemma 2.1, likewise, the details are put off in the Appendix.

**Lemma 4.2.** Let \( l \) be a positive integer. If there exist constants \( A, B > 0 \) such that
\[ \| \partial_x^{l-2m} \partial_t^b u \|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \leq A, \quad \| \partial_x \partial_t^b u \|_{L^\infty(0, T; L^\infty(\mathbb{R}^3))} \leq B \]
for any nonnegative integer \( m \) and \( b \) with \( 0 \leq b \leq m \leq \frac{l}{2} \), then
\[ \| \partial_x^{l-2m-1} \partial_t^b (G Du) \|_{L^\infty(0, T; L^2(\mathbb{R}^3))} \leq C A (B + B^{l-m-1}). \]

**Proof.** See Appendix for the proof. \( \Box \)

Now we are ready to provide the proof of Corollary 1.2.

**Proof of Corollary 1.2** First, we show the following statement:
Let \( l \) and \( m \) be integers with \( 0 \leq m \leq \frac{l}{2} \) and \( l \geq 4 \). If \( u \in L^\infty(0, T : H^l(\mathbb{R}^3)) \), then
\[ \nabla_x^{l-2m} \partial_t^b u \in L^\infty(0, T : L^2(\mathbb{R}^3)) \quad (4.2) \]
for any integer \( b \) satisfying \( 0 \leq b \leq m \).

Indeed, using mathematical induction for \( m \), we will prove (4.2).

\begin{itemize}
  \item (Case \( m = 1 \)) Since \( u \in L^\infty(0, T : H^l(\mathbb{R}^3)) \), we have
    \[ \partial_x^{l-2} u \in L^\infty(0, T : L^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}^3). \]
  
  Next, we show
  \[ \partial_x^{l-2} \partial_t u \in L^\infty(0, T : L^2(\mathbb{R}^3)). \]
  
  From the equation (1.1), we have
  \[ \partial_x^{l-2} \partial_t u = -\partial_x^{l-2} (\nabla \cdot (G||Du||^2 Du) + \nabla \cdot (u \otimes u) + \nabla p). \]
\end{itemize}
Hence, we are going to show
\[ \partial_x^{l-1}(G[|Du|^2]Du), \quad \partial_x^{l-1}(u \otimes u), \quad \partial_x^{l-1}p \in L^\infty(0, T : L^2(\mathbb{R}^3)). \]

Taking the divergence operator to (1.1), we note that
\[ -\Delta p = \text{div}[\text{div}(G[|Du|^2]Du)] + \text{div}(u \otimes u). \]

From the standard elliptic theory, it is enough to show
\[ \partial_x^{l-1}(G[|Du|^2]Du), \quad \partial_x^{l-1}(u \otimes u) \in L^\infty(0, T : L^2(\mathbb{R}^3)). \]

First, we note
\[
\|\partial_x^{l-1}(G[|Du|^2]Du)\|_{L^2} \leq \|G[|Du|^2]\partial_x^{l-1}(Du)\|_{L^2} + \sum_{\alpha=1}^{\alpha=l-1} \|\partial_x^\alpha G[|Du|^2] \partial_x^{l-1-\alpha} Du\|_{L^2} \\
\leq \|G[|Du|^2]\|_{L^\infty} \|\nabla^{l-1} Du\|_{L^2} + C\|G[|Du|^2]\|_{L^\infty} (\| Du\|_{L^2}^2 + \| Du\|_{L^2}^{l-1}) \|\nabla^{l-1} Du\|_{L^2} \\
\leq C\|G[|Du|^2]\|_{L^\infty} (1 + \| Du\|_{L^2(\mathbb{R}^3)}^{l-1}) \|\nabla^{l} Du\|_{L^2(\mathbb{R}^3)},
\]
where we used (2.2) in the second inequality.

Next, we have
\[
\|\partial_x^{l-1}(u \otimes u)\|_{L^2} \leq \sum_{\alpha=0}^{\alpha=l-1} \|\partial_x^\alpha u \otimes \partial_x^{l-1-\alpha} u\|_{L^2} \\
\leq \sum_{\alpha=0}^{\alpha=l-2} \|\partial_x^\alpha u\|_{L^\infty} \|\partial_x^{l-1-\alpha} u\|_{L^2} + \| u\|_{L^\infty} \|\partial_x^{l-1} u\|_{L^2} \leq C\|u\|_{H^l}^2.
\]

• (Case \(m = n + 1\)) Assume (4.2) holds for the case \(m = n\), that is
\[
\nabla_x^{l-2n} \partial_t^b u \in L^\infty(0, T : L^2) \tag{4.3}
\]
for any integer \(b\) such that \(0 \leq b \leq n\). We then show the case \(m = n + 1\), that is,
\[
\nabla_x^{l-2(n+1)} \partial_t^b u \in L^\infty(0, T : L^2), \quad 0 \leq b \leq n + 1. \tag{4.4}
\]

Since the other cases \(0 \leq b \leq n\) can be shown similarly, we only prove (4.4) for \(b = n + 1\), which is
\[
\nabla_x^{l-2(n+1)} \partial_t^{n+1} u \in L^\infty(0, T : L^2).
\]

Note that, similarly to the case \(m = 1\), once we have
\[
\nabla_x^{l-2(n+1)+1} \partial_t^n (G[|Du|^2]Du), \quad \nabla_x^{l-2(n+1)+1} \partial_t^n (u \otimes u) \in L^\infty(0, T : L^2(\mathbb{R}^3)) \tag{4.5},
\]
we conclude
\[
\nabla_x^{l-2(n+1)+1} \partial_t^{n+1} p \in L^\infty(0, T : L^2(\mathbb{R}^3)). \tag{4.6}
\]

Combining (4.5) and (4.6) with the equation (1.1), we get
\[
\nabla_x^{l-2(n+1)} \partial_t^{n+1} u \in L^\infty(0, T : L^2).
\]

Therefore, it remains to prove (4.5) to conclude the proof. Let us prove (4.5). First of all, it follows from Lemma 1.2 that
\[
\|\partial_x^{l-2n} \partial_t^n (G[|Du|^2]Du)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C\mathcal{A}(\mathcal{B} + \mathcal{B}^{-n-1}),
\]
where
\[
\|\partial_x^{l-2n} \partial_t^n u\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq \mathcal{A}, \quad \|\partial_x \partial_t^n u\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \leq \mathcal{B}.
\]
Due to (4.3), we note that $A, B < \infty$, which implies
\[
\|\partial_x^{-2n-1} \partial_t^m (G |Du|^2)Du\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} < \infty.
\]
Next, we can also observe from the assumption (4.3) that
\[
\|\partial_x^{-2n-1} \partial_t^m (u \otimes u)\|_{L^2(\mathbb{R}^3)} < \infty.
\]
This completes the proof of (4.2).

Now, we finish the proof of the Corollary 1.2. For some integers $b$ and $m$ satisfying $0 \leq b \leq m \leq \frac{1}{2} - 1$, we define
\[
v := \partial_x^{-2(m+1)} \partial_t^b u.
\]
Then, via the estimate (4.2), we have
\[
v \in L^\infty(0,T:H^2(\mathbb{R}^3)) \subset L^\infty(0,T:C^{\frac{1}{2}}(\mathbb{R}^3)).
\]
Furthermore, we also have
\[
\partial_t v = \partial_x^{-2(m+1)} \partial_t^{b+1} u \in L^\infty(0,T:L^2(\mathbb{R}^3))
\]
from the estimate (4.2) due to the fact $0 \leq b + 1 \leq m + 1 \leq \frac{1}{2}$. We plug (4.7) and (4.8) into Lemma 4.1 and conclude
\[
\partial_x^{-2(m+1)} \partial_t^b u \in C^b_C \left( \mathbb{R}^3 \times [0,T] \right), \quad \forall \ 0 \leq b \leq m \leq \frac{1}{2} - 1,
\]
which completes the proof of Corollary 1.2.

5. Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3. We recall (3.1), (3.3) and (3.8).

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} G(|Du|^2)|Du|^2 = 0,
\]
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} m_0 |\partial_x Du|^2 \leq -\int_{\mathbb{R}^3} \partial_x ((u \cdot \nabla) u) \cdot \partial_x u,
\]
\[
\frac{1}{2} \frac{d}{dt} \|\partial_x \partial_x u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} m_0 |\partial_x \partial_x Du|^2 \leq C \|G(|Du|^2)\|_{L^\infty} \|Du\|_{L^\infty} + \|Du\|^2_{L^\infty} \|\nabla^2 Du\|^2_{L^2}
\]
\[
- \int_{\mathbb{R}^3} \partial_x \partial_x ((u \cdot \nabla u)) \cdot \partial_x \partial_x u.
\]

Via (3.9), (3.10), and (3.13), we also remind that
\[
\frac{1}{2} \frac{d}{dt} \|\partial^3 u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} G(|Du|^2)|\partial^3 u|^2 \, dx + 2 \int_{\mathbb{R}^3} G'(|Du|^2)|Du : \partial^3 Du|^2 \, dx
\]
\[
\leq |I_{32}| + |I_{33}| - \int_{\mathbb{R}^3} E_3(Du : \partial^3 Du) \, dx - \int_{\mathbb{R}^3} \partial^3 ((u \cdot \nabla u)) \cdot \partial^3 u \, dx,
\]
where $I_{32}$ and $I_{33}$ are given in (3.10). Now we estimate each term on the right hand side of (5.4) differently than we did in Theorem 1.1. We note first that
\[
|I_{32}| \leq \int_{\mathbb{R}^3} |\partial G| |Du|^2| |\partial^3 Du| \, dx
\]
\[
\leq \|\partial G\|_{L^2} \|\partial^2 Du\|_{L^2} \|\nabla^3 Du\|_{L^2}
\]
\[
\leq C \|G\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^3 Du\|_{L^2},
\]
where we exploit (2.2) in the last inequality. For $I_{33}$, again due to (2.2), we have
\[ |I_{33}| \leq \int_{\mathbb{R}^3} |\partial^2 G||Du|^2|\partial Du|d|\partial^3 Du| dx \]
\[ \leq ||\partial^2 G||Du|^2||\partial Du||L^2||\nabla^3 Du||L^2 \]
\[ \leq C||G||Du|^2||L^\infty||Du||L^\infty||Du|^2||L^2. \]

Using again estimate (2.2), we obtain
\[ \int_{\mathbb{R}^3} E_3(Du : \partial^3 Du) dx \leq ||E_3 Du||L^2||\nabla^3 Du||L^2 \]
\[ \leq C||G||Du|^2||L^\infty||Du||L^\infty||Du|^2||L^2. \] (5.5)

Next, we estimate the terms caused by convection term. Using div $u = 0$, we have
\[ \int_{\mathbb{R}^3} \partial((u \cdot \nabla) u) \cdot \partial u dx = \int_{\mathbb{R}^3} [(\partial u \cdot \nabla) u] \cdot \partial u dx \leq C||\nabla u||L^\infty||\nabla u||L^2, \] (5.6)
\[ \int_{\mathbb{R}^3} \partial^2((u \cdot \nabla) u) \cdot \partial^2 u dx = \int_{\mathbb{R}^3} [(\partial u \cdot \nabla) \partial u] \cdot \partial^2 u dx + \int_{\mathbb{R}^3} [(\partial^2 u \cdot \nabla) u] \cdot \partial^2 u dx \]
\[ \leq C||\nabla u||L^\infty||\nabla^2 u||L^2, \] (5.7)
and
\[ \int_{\mathbb{R}^3} \partial^3((u \cdot \nabla) u) \cdot \partial^3 u dx = \int_{\mathbb{R}^3} [(\partial^3 u \cdot \nabla) u] \cdot \partial^3 u dx + \int_{\mathbb{R}^3} [(\partial^2 u \cdot \nabla) \partial u] \cdot \partial^3 u dx \]
\[ + \int_{\mathbb{R}^3} [(\partial u \cdot \nabla) \partial^2 u] \cdot \partial^3 u dx \]
\[ \leq C||\nabla u||L^\infty||\nabla^3 u||L^2 + C||\nabla^2 u||L^2||\nabla^3 u||L^2 \]
\[ \leq C||\nabla u||L^\infty||\nabla^3 u||L^2. \] (5.8)

Combining (5.4)–(5.5) and (5.8), we have
\[ \frac{1}{2} \frac{d}{dt} ||\partial^3 u||L^2(\mathbb{R}^3)^2 + \int_{\mathbb{R}^3} G||Du|^2||\partial^3 Du||^2 dx + 2G||Du|^2||Du : \partial^3 Du||^2 dx \]
\[ \leq C||G||Du|^2||L^\infty||Du||L^\infty||Du|^2||L^2 + C||\nabla u||L^\infty||\nabla^3 u||L^2. \]

Again, as before, we have
\[ \frac{1}{2} \frac{d}{dt} ||\partial^3 u||L^2(\mathbb{R}^3)^2 + \int_{\mathbb{R}^3} m_0 ||\partial^3 Du||^2 dx \]
\[ \leq C||G||Du|^2||L^\infty||Du||L^\infty||\nabla^3 Du||L^2 + C||\nabla u||L^\infty||\nabla^3 u||L^2. \] (5.9)

Finally, we combine (5.1), (5.2), (5.3), (5.6), (5.7) and (5.9) to conclude
\[ \frac{1}{2} \frac{d}{dt} ||u||_{H^3(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0(||\nabla^3 Du||^2 + ||\nabla^2 Du||^2 + ||\nabla Du||^2 + ||Du||^2) dx \]
\[ \leq C||G||Du|^2||L^\infty||Du||L^\infty||\nabla^3 Du||L^2 + C||\nabla u||L^\infty||\nabla^3 u||L^2 + C||\nabla^2 u||L^2 + C||\nabla u||L^2, \]
\[ \leq Cg(||u||_{H^3}(||u||_{H^3}^3 + ||u||_{H^3})||\nabla^3 Du||L^2 + ||\nabla^2 Du||L^2 \]
\[ + C||u||_{H^3}||\nabla^3 u||L^2 + ||\nabla^2 u||L^2 + ||\nabla u||L^2), \] (5.10)
where \( g \) is a nondecreasing function defined in (3.20). Since \( \|u_0\|_{H^3} \leq \epsilon_0 \), it follows from local existence of solution that there exists a time \( t^* > 0 \) such that \( \|u(t)\|_{H^3} \leq 2\epsilon_0 \) for all \( t \leq t^* \). Thus, due to estimate (5.10), we obtain for any \( t \leq t^* \)

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^3(\mathbb{R}^3)} + \int_{\mathbb{R}^3} (m_0 - 2\epsilon_0)(|\nabla^3 Du|^2 + |\nabla^2 Du|^2 + |\nabla Du|^2 + |Du|^2) \, dx \leq 0,
\]

which implies that after integrating it in time,

\[
\|u(t)\|^2_{H^3(\mathbb{R}^3)} \leq \|u_0\|^2_{H^3(\mathbb{R}^3)} \leq \epsilon_0, \quad t \leq t^*.
\]

Repeating this procedure at \( t = t^* \), we extend the solution in \([t^*, 2t^*]\), which immediately implies that \( T_3 \) in Theorem 1.1 becomes infinity, i.e. \( T_3 = \infty \).

5.1. Estimation of \( \|u\|_{H^l}, l \geq 4 \). In case that \( l = 4 \), via (3.22), (3.23) and (3.24), we remind that

\[
\frac{1}{2} \frac{d}{dt} \|\partial^4 u\|^2_{L^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} m_0|\partial^4 Du|^2 \, dx \leq |I_{42}| + |I_{43}| + |I_{44}| - \int_{\mathbb{R}^3} E_4(Du : \partial^4 Du) \, dx - \int_{\mathbb{R}^3} \partial^4((u \cdot \nabla u)) \cdot \partial^4 u \, dx,
\]

where \( I_{42}, I_{43} \) and \( I_{44} \) are as defined in (3.23). We estimate the right hand side of (5.11). For \( I_{42} \), we exploit (2.2) to get

\[
|I_{42}| \leq \int_{\mathbb{R}^3} |\partial G||Du|^2||\partial^3 Du||\partial^4 Du| \, dx \\
\leq C\|\partial G||Du|^2\| \|\partial^3 Du\| \|\nabla^4 Du\|_{L^2} \leq C\|G||Du|^2\|_{L^\infty} \|Du\|_{L^\infty} \|\nabla^4 Du\|_{L^2}^2.
\]

Similarly, we have

\[
|I_{43}| \leq \int_{\mathbb{R}^3} |\partial^2 G||Du|^2||\partial^2 Du||\partial^4 Du| \, dx \\
\leq C\|\partial^2 G||Du|^2\| \|\partial^2 Du\| \|\nabla^4 Du\|_{L^2} \leq C\|G||Du|^2\|_{L^\infty} \|\nabla^4 Du\|_{L^2} \|\nabla^4 Du\|_{L^2}^2.
\]

For \( I_{44} \), again due to (2.2), we obtain

\[
|I_{44}| \leq \int_{\mathbb{R}^3} |\partial^3 G||Du|^2||\partial Du||\partial^4 Du| \, dx \\
\leq C\|\partial^3 G||Du|^2\| \|\partial Du\| \|\nabla^4 Du\|_{L^2} \leq C\|G||Du|^2\|_{L^\infty} \|\nabla^4 Du\|_{L^2} \|\nabla^4 Du\|_{L^2}^2.
\]

It follows from (2.2) that

\[
\int_{\mathbb{R}^3} |E_4(Du : \partial^4 Du)| \, dx \leq \|E_4 Du\|_{L^2} \|\nabla^4 Du\|_{L^2} \leq C\|G||Du|^2\|_{L^\infty} \|\nabla^4 Du\|_{L^2} \|\nabla^4 Du\|_{L^2}^2.
\]
Lastly, for the convection term, using \( \text{div} u = 0 \), we have

\[
\int_{\mathbb{R}^3} \partial^4[(u \cdot \nabla)u] \cdot \partial^4 u \, dx = \int_{\mathbb{R}^3} \sum_{\alpha=1}^{4} (\partial^\alpha u \cdot \nabla) \partial^{4-\alpha} u \cdot \partial^4 u \, dx
\]

\[
\leq C(\|\nabla u\|_{L^\infty}\|\nabla^4 u\|_{L^2} + \|\nabla^2 u\|_{L^\infty}\|\nabla^3 u\|_{L^2})\|\nabla^4 u\|_{L^2}
\]

\[
\leq C(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty})(\|\nabla^4 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2).
\]

In general, for \( l \geq 4 \), we note that

\[
\int_{\mathbb{R}^3} \partial^l[(u \cdot \nabla)u] \cdot \partial^l u \, dx \leq C \left( \sum_{\alpha=1}^{\lfloor \frac{l+1}{2} \rfloor} \|\nabla^\alpha u\|_{L^\infty} \|\nabla^{l+1-\alpha} u\|_{L^2} \right)\|\nabla^4 u\|_{L^2}
\]

\[
\leq C \left( \sum_{\alpha=1}^{\lfloor \frac{l+1}{2} \rfloor} \|\nabla^\alpha u\|_{L^\infty} \right) \left( \sum_{\alpha=1}^{l} \|\nabla^\alpha u\|_{L^2}^2 \right).
\]

Finally, we combine (5.10), (5.11), (5.12), (5.13), (5.14), (5.15) and (5.16) to conclude

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^4(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0(\|\nabla^4 Du\|^2 + \|\nabla^3 Du\|^2 + \|\nabla^2 Du\|^2 + \|\nabla Du\|^2 + \|Du\|^2) \, dx
\]

\[
\leq C(G\|Du\|_{L^\infty})(\|Du\|_{L^2} + \|Du\|_{L^\infty})(\|\nabla^4 Du\|_{L^2}^2 + \|\nabla^3 Du\|_{L^2}^2 + \|\nabla^2 Du\|_{L^2}^2 + \|\nabla Du\|_{L^2}^2 + \|Du\|_{L^2}^2)
\]

\[
+ C(\|\nabla u\|_{L^\infty} + \|u\|_{L^\infty})(\|\nabla^4 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)
\]

\[
\leq C(\|u\|_{H^3+1})(\|u\|_{H^3} + \|u\|_{H^3})(\|\nabla^4 Du\|_{L^2}^2 + \|\nabla^3 Du\|_{L^2}^2 + \|\nabla^2 Du\|_{L^2}^2 + \|\nabla Du\|_{L^2}^2 + \|Du\|_{L^2}^2).
\]

Since \( \|u(t)\|_{H^3} \leq \epsilon_0 \ll 1 \) for \( t \in [0, \infty) \), we have

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^4(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0(\|\nabla^4 Du\|^2 + \|\nabla^3 Du\|^2 + \|\nabla^2 Du\|^2 + \|\nabla Du\|^2 + \|Du\|^2) \, dx
\]

\[
\leq C\epsilon_0(\|\nabla^4 Du\|_{L^2}^2 + \|\nabla^3 Du\|_{L^2}^2 + \|\nabla^2 Du\|_{L^2}^2 + \|\nabla Du\|_{L^2}^2 + \|Du\|_{L^2}^2),
\]

and, therefore, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^4(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} (m_0 - C\epsilon_0)(\|\nabla^4 Du\|^2 + \|\nabla^3 Du\|^2 + \|\nabla^2 Du\|^2 + \|\nabla Du\|^2 + \|Du\|^2) \, dx \leq 0.
\]

This implies the global existence of solution in the class \( L^\infty(0, \infty : H^4) \).

For general \( l > 4 \), it follows from (5.30) that

\[
\frac{1}{2} \frac{d}{dt} \|\partial_{x_{\sigma(1)}(i)} \cdots \partial_{x_{\sigma(l)}(i)} u\|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} m_0(\|\partial_{x_{\sigma(1)}(i)} \partial_{x_{\sigma(2)}(i)} \cdots \partial_{x_{\sigma(l)}(i)} Du\|^2 \, dx
\]

\[
\leq |I_{11}| + |I_{12}| + |I_{13}|,
\]

(5.17)
where $I_{l1}, I_{l2}$ and $I_{l3}$ are given in (3.30). We then first estimate $I_{l1}$. 

\[
|I_{l1}| \leq \sum_{\alpha=1}^{l-1} \int_{\mathbb{R}^3} \left| \partial^{\alpha} G[|Du|^2] \right| \left| \partial^{\alpha} Du \right| \left| \partial^l Du \right| dx \\
\leq \sum_{\alpha=1}^{l-1} \left| \partial^{\alpha} G[|Du|^2] \right| \|\partial^{\alpha} Du\|_{L^2} \|\nabla^l Du\|_{L^2} \\
\leq C \|G[|Du|^2]\|_{L^\infty} \sum_{\alpha=1}^{l-1} \left( \|Du\|_{L^\infty} + \|Du\|_L^{\alpha} \right) \|\nabla^l Du\|_{L^2} \\
\leq C \|G[|Du|^2]\|_{L^\infty} \left( \|Du\|_{L^\infty} + \|Du\|_L^{l-1} \right) \|\nabla^l Du\|_{L^2} \\
\leq C g(\|u\|_{H^3}) \left( \|u\|_{H^3} + \|u\|_{H^3}^{l-1} \right) \|\nabla^l Du\|_{L^2}^2, 
\]

where we exploit (2.2) in the third inequality. Similarly, the term $|I_{l2}|$ is estimated

\[
|I_{l2}| \leq C \int_{\mathbb{R}^3} |E_l| \left| \partial^l Du \right| dx \\
\leq C \|E_l Du\|_{L^2} \|\nabla^l Du\|_{L^2} \\
\leq C \|G[|Du|^2]\|_{L^\infty} \left( \|Du\|_{L^\infty} + \|Du\|_L^l \right) \|\nabla^l Du\|_{L^2} \\
\leq C g(\|u\|_{H^3}) \left( \|u\|_{H^3} + \|u\|_{H^3}^l \right) \|\nabla^l Du\|_{L^2}^2. 
\]

Lastly, we estimate the convection term: Using divergence free condition

\[
\int_{\mathbb{R}^3} \partial^\alpha [(u \cdot \nabla)u] \cdot \partial^\alpha u dx = \sum_{k=1}^{\alpha} \int_{\mathbb{R}^3} \left[ (\partial^k u \cdot \nabla) \partial^{\alpha-k} u \right] \cdot \partial^\alpha u dx \\
\leq C \sum_{k=1}^{\alpha} \|\partial^k u\|_{L^p} \|\partial^{\alpha-k} \nabla u\|_{L^q} \|\partial^\alpha u\|_{L^2} \\
\leq C \|\partial^\alpha u\|_{L^2} \|\nabla u\|_{L^\infty} \|\partial^\alpha u\|_{L^2},
\]

where we apply Gagliardo-Nirenberg to the third inequality. This gives us

\[
\sum_{\alpha=1}^{l} \int_{\mathbb{R}^3} \partial^\alpha [(u \cdot \nabla)u] \cdot \partial^\alpha u dx \leq C \|\nabla u\|_{L^\infty} \sum_{\alpha=1}^{l} \|\partial^\alpha u\|_{L^2}^2 \leq C \|u\|_{H^3} \sum_{\alpha=1}^{l} \|\partial^\alpha u\|_{L^2}^2. 
\]

Plugging (5.18), (5.19) and (5.20) into (5.17), since $\|u(t)\|_{H^3} \leq \epsilon$ for $t \in [0, \infty)$, we obtain

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^1(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \left( m_0 - \epsilon \right) \left( |\nabla^l Du|^2 + |\nabla^{l-1} Du|^2 + \cdots + |Du|^2 \right) dx \leq 0.
\]

The above estimate implies that solution exists globally in time. \n
\[\square\]

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Appendix A.

In this Appendix, we provide the proofs of Lemma 2.1 and Lemma 4.2.

Proof of Lemma 2.1. We note first that
\[ \partial_{x_i} G([Du]^2) = 2G'([Du]^2)(Du : \partial_{x_i} Du), \quad i = 1, 2, 3, \]
From now on, we write \( G([Du]^2) \) and \( G'([Du]^2) \) as \( G \) and \( G' \), respectively, unless any confusion is to be expected. For convenience, we set \( E_1 = 0 \). Direct computations show that
\[ \partial_{x_j} \partial_{x_i} G = 2G' (Du : \partial_{x_j} \partial_{x_i} Du) + 2(\partial_{x_j} (G' Du) : \partial_{x_i} Du), \quad i, j = 1, 2, 3. \quad (A.1) \]
We define the second term of the righthand side in (A.1) by \( E_2 \), namely
\[ E_2 := 2(\partial_{x_j} (G' Du) : \partial_{x_i} Du). \]
Next, we define \( E_l, l \geq 3 \), inductively via multi-derivatives of \( G \).
\[ \partial_{x_k} \partial_{x_j} \partial_{x_i} G = 2G'' (Du : \partial_{x_k} \partial_{x_j} \partial_{x_i} Du) + 2(\partial_{x_k} (G'' Du) : \partial_{x_j} \partial_{x_i} Du) + \partial_{x_k} E_2 \]
= \[ 2G'' (Du : \partial_{x_j} \partial_{x_i} Du) + E_3, \quad i, j, k = 1, 2, 3, \]
where
\[ E_3 = 2(\partial_{x_k} (G' Du) : \partial_{x_j} \partial_{x_i} Du) + \partial_{x_k} E_2. \]
For \( l \geq 4 \) we define \( E_l \) inductively as follows:
\[ \partial_{x_{\sigma (l)}} \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} G([Du]^2) = 2\left[G''([Du]^2)Du : \partial_{x_{\sigma (l)}} \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} Du\right] + E_l, \quad (A.2) \]
where
\[ E_l = 2(\partial(G' Du) : \partial^{l-1} Du) + \partial E_{l-1}. \]
More precisely, we have
\[ \begin{align*}
E_l &= 2\partial_{x_{\sigma (l)}} G' (Du : \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} Du) + 2G' (Du : \partial_{x_{\sigma (l)}} \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} Du) + \partial_{x_{\sigma (l)}} E_{l-1} \\
&= 2G'' (Du : \partial_{x_{\sigma (l)}} Du)(Du : \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} Du) \\
&+ 2G' (\partial_{x_{\sigma (l)}} Du : \partial_{x_{\sigma (l-1)}} \ldots \partial_{x_{\sigma (1)}} Du) + \partial_{x_{\sigma (l)}} E_{l-1}. \quad (A.3)
\end{align*} \]
Next, we estimate \( E_2 \) and \( E_3 \). From (A.3), we have
\[ E_2 \simeq G'' (Du : \partial Du)(Du : \partial Du) + G' (\partial Du : \partial Du) \]
\[ \leq (|G''| |Du|^2 + |G'|) \|\nabla Du\|^2 \leq C G \|\nabla Du\|^2, \quad (A.4) \]
where we used the properties of \( G \) to get the last inequality. Combining (A.3) and (A.4), we have
\[ \begin{align*}
E_3 &\simeq G'' (Du : \partial^2 Du)(Du : \partial^2 Du) + G' (\partial^2 Du : \partial^2 Du) \\
&+ \partial G'' (Du : \partial Du)(Du : \partial Du) + G'' (Du : \partial Du)(\partial \partial Du : \partial DU) + (Du : \partial^2 Du) \\
&+ \partial G' (\partial Du : \partial Du) + G' (\partial^2 Du : \partial Du) \\
&\simeq G'' (Du : \partial Du)(Du : \partial^2 Du) + G' (\partial Du : \partial^2 Du) \\
&+ G'' (Du : \partial Du)(Du : \partial Du)(\partial Du : \partial Du) + G'' (Du : \partial Du)(\partial Du : \partial Du).
\end{align*} \]
Let \( P_n(G, Du) \) stand for the linear combination of \( G^{(k)}(Du)^l \) with \( 0 \leq l \leq k \leq n \), where \( G^{(k)} \) is the \( k \)-th derivative of \( G \). Using this notation, we rewrite \( E_3 \) as follows
\[ E_3 \simeq P_3(G, Du)\left[\partial (\partial Du)(\partial^2 Du) + (\partial Du)^3\right]. \quad (A.5) \]
Due to the property (1.2), we note that
\[ |G^{(k)}(Du)^l| |Du|^l \leq C|G^{(k-1)}(Du)^l| |Du|^{l-1} \leq \cdots \leq CG|Du|^2 \]
for any \(0 \leq l \leq k\). This implies
\[ |P_n(G, Du)| \leq CG|Du|^2, \quad \forall n \geq 1. \tag{A.6} \]
Hence, we have
\[ |E_3| \leq CG|Du|^2 (|\nabla Du| |\nabla^2 Du| + |\nabla Du|^3). \]
This completes the proof of \((2.1)\) in Lemma 2.1.

It remains to prove \((2.2)-(2.4)\) in Lemma 2.1. For notational convention, we denote \(k\)-th order spatial derivative operators by \(\partial^k\), unless any confusion is to be expected. We then introduce \(R_m(Du)\) as a linear combination of
\[(\partial^{a_1} Du)^{i_1} (\partial^{a_2} Du)^{i_2} \cdots (\partial^{a_k} Du)^{i_k}, \quad a_1, i_1, \ldots, a_k, i_k \in \mathbb{N},\]
such that
\[a_1i_1 + a_2i_2 + \cdots + a_ki_k = m \quad \text{and} \quad 1 \leq a_j \leq m - 1, \quad \forall j = 1, \ldots, k.\]
Note that, for example, \(R_3(Du) = (\partial Du)(\partial^2 Du) + (\partial Du)^3\). We then rewrite \((A.5)\) as
\[E_3 \simeq P_3(G, Du)R_3(Du).\]
In general, we will show that
\[E_n \simeq P_n(G, Du)R_n(Du), \quad n \geq 1. \tag{A.7}\]
(Proof of (A.7)) We prove \((A.7)\) by inductive argument. It is already shown that \((A.7)\) holds for \(n = 1, 2, 3\). Now we suppose \((A.7)\) is true for \(n = m \geq 3\). It follows from \((A.3)\) that
\[E_{m+1} = (G''(Du)(\partial Du) + G'(Du)\partial^m Du) + \partial E_m \]
\[\simeq (G''(Du) + G') (\partial Du)(\partial^m Du) + \partial (P_m(G, Du))R_m(Du) \tag{A.8} \]
\[+ P_m(G, Du)\partial (R_m(Du)).\]
We first show that
\[\partial (P_m(G, Du)) = P_{m+1}(G, Du)\partial Du. \tag{A.9}\]
Indeed, for \(0 \leq l \leq k \leq m\), we have
\[\partial (G^{(k)}(Du)^l) = \partial (G^{(k)})(Du)^l + G^{(k)}\partial ((Du)^l) \]
\[= G^{(k+1)}(Du: \partial Du)(Du)^l + G^{(k)}(Du)^l-1\partial Du \]
\[\simeq (G^{(k+1)}(Du)^{l+1} + G^{(k)}(Du)^{l-1})\partial Du \]
\[\simeq (G^{(k)}(Du)^{l'})\partial Du,\]
where \(0 \leq l' \leq k' \leq m + 1\). This proves the identity \((A.9)\). Using the definition of \(R_m\), the direct computations show that
\[\partial R_m(Du) \simeq \partial R_{m+1}(Du). \tag{A.10}\]
Plugging \((A.9)\) and \((A.10)\) into \((A.8)\), we obtain
\[E_{m+1} \simeq (G''(Du) + G') (\partial Du)(\partial^m Du) + P_{m+1}(G, Du)(\partial Du)R_m(Du) \]
\[+ P_m(G, Du)R_{m+1}(Du) \]
\[\simeq P_{m+1}(G, Du)R_{m+1}(Du), \]
where we used that 
\[ (G''(Du) + G'), \ P_m(G, Du) \simeq P_{m+1}(G, Du), \]
\[ (\partial Du) (\partial^m Du), \ (\partial Du) R_m(Du) \simeq R_{m+1}(Du). \]

This completes the proof of (A.7). \( \Box \)

Next, we will prove that for any \( 1 \leq \alpha \leq l \)
\[ \| E_\alpha \partial^{\alpha} Du \|_{L^2} \leq C \| G[\|Du\|^2]\|_{L^\infty} \| \nabla^l Du \|_{L^2}(\|Du\|_{L^\infty} + \|Du\|_{L^\infty}^2), \quad (A.11) \]
and
\[ \| \partial^\alpha G \partial^{\alpha} Du \|_{L^2} \leq C \| G[\|Du\|^2]\|_{L^\infty} \| \nabla^l Du \|_{L^2}(\|Du\|_{L^\infty} + \|Du\|_{L^\infty}^2), \quad (A.12) \]

- (Proofs of (A.11) and (A.12)) We note from (A.6) that
\[ |P_n(G, Du)| \leq CG[\|Du\|^2], \quad \forall \ n \geq 1. \quad (A.13) \]

Next, for some \( a_1, i_1, \ldots, a_k, i_k \in \mathbb{N} \) such that \( a_1 i_1 + a_2 i_2 + \cdots + a_k i_k = \alpha \), we observe that
\[ \| (\partial^{a_1} Du)^{i_1} (\partial^{a_2} Du)^{i_2} \cdots (\partial^{a_k} Du)^{i_k} (\partial^{\alpha} Du) \|_{L^2} \]
\[ \leq C \| \partial^{a_1} Du \|_{L^{p_{i_1}}}^{i_1} \cdots \| \partial^{a_k} Du \|_{L^{p_{i_k}}}^{i_k} \| \partial^{\alpha} Du \|_{L^q}, \quad (A.14) \]

where
\[ \frac{1}{p_1} + \cdots + \frac{1}{p_k} + \frac{1}{q} = \frac{1}{2}. \quad (A.15) \]

With the aid of Gagliardo-Nirenberg inequality, we note that
\[ \| \partial^{a_j} Du \|_{L^{p_{i_j}}} \leq C \| \partial^j Du \|_{L^2}^{\theta_j} \| Du \|_{L^\infty}^{1-\theta_j}, \quad 1 \leq j \leq k \]
\[ \| \partial^{\alpha} Du \|_{L^q} \leq C \| \partial^\alpha Du \|_{L^2}^{\theta_q} \| Du \|_{L^\infty}^{1-\theta_q}, \quad (A.16) \]
such that
\[ \frac{1}{p_j i_j} = \frac{a_j}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_j, \quad 1 \leq j \leq k \]
\[ \frac{1}{q} = \frac{l - \alpha}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_q. \quad (A.17) \]

It follows from (A.17) that \( 1 = \sum_{j=1}^k \theta_j i_j + \theta_q \). Indeed,
\[ \left( \sum_{j=1}^k \frac{1}{p_j} \right) + \frac{1}{q} = \sum_{j=1}^k \left[ \frac{a_j i_j}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_j i_j \right] + \frac{l - \alpha}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_q \]
\[ \implies \frac{1}{2} = \frac{\alpha}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \sum_{j=1}^k \theta_j i_j + \frac{l - \alpha}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_q \]
\[ \implies \frac{1}{2} - \frac{l}{3} = \left( \frac{1}{2} - \frac{l}{3} \right) \left[ \sum_{j=1}^k \theta_j i_j + \theta_q \right] \quad (A.18) \]
\[ \implies 1 = \sum_{j=1}^k \theta_j i_j + \theta_q, \]
where we used $\sum_{j=1}^{k} a_j i_j = \alpha$. We plug (A.16) and (A.18) into (A.14) to get

$$
\| (\partial^\alpha \theta \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta ) \|_{L^2} \\
\leq C \| \partial^\delta \theta \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} \| \frac{1}{(1-\theta_1)^{i_1} \cdots (1-\theta_k)^{i_k} + 1 - \theta_k} \|_{L^\infty} \\
\leq C \| \partial^\delta \theta \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{(i_1 + \cdots + i_k)} ,
$$

which immediately implies that

$$
\| R_\alpha ( Du ) ( \partial^\delta \theta \partial^\alpha \partial^\alpha \theta ) \|_{L^2} \leq C \| \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} \| \frac{1}{(1-\theta_1)^{i_1} \cdots (1-\theta_k)^{i_k} + 1 - \theta_k} \|_{L^\infty} + \| Du \|_{L^\infty}^\alpha . \tag{A.19}
$$

We combine (A.7), (A.13), and (A.19) to conclude (A.11).

To get (A.12), we first note

$$
\| G'( [ Du ]^2 ) ( Du : \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta ) \|_{L^2} \leq \| G'( [ Du ]^2 ) Du \|_{L^\infty} \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} \\
\leq C \| G'[ Du ]^2 \|_{L^\infty} \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} ,
$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Using Gagliardo-Nirenberg inequality, we have

$$
\| \partial^\alpha \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} \leq C \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{1-\theta_2} \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{\theta_2} \\
\leq C \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{\theta_2 + \theta_3} \| Du \|_{L^\infty}^{2-(\theta_2 + \theta_3)} ,
$$

with

$$
\frac{1}{p} = \frac{\alpha}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_2 \quad \text{and} \quad \frac{1}{q} = \frac{l}{3} + \left( \frac{1}{2} - \frac{l}{3} \right) \theta_3 . \tag{A.20}
$$

Due to (A.20) and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we get $\theta_2 + \theta_3 = 1$. Hence, we have

$$
\| G'( [ Du ]^2 ) ( Du : \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta ) \|_{L^2} \leq C \| G'[ Du ]^2 \|_{L^\infty} \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} . \tag{A.21}
$$

We combine (A.2), (A.11) and (A.21) to conclude

$$
\| \partial^\alpha G'[ Du ]^2 \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^2} \leq \| G'[ Du ]^2 \|_{L^\infty} \| \partial^\beta \theta \partial^\gamma \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\delta \theta \partial^\epsilon \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty} + \| E_\alpha \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^2} \\
\leq C \| G'[ Du ]^2 \|_{L^\infty} \| \nabla^l Du \|_{L^2} \left( \| Du \|_{L^\infty} + \| Du \|_{L^\infty}^\alpha \right) ,
$$

which gives us (A.12).

Finally, we show that for $l \geq 4$ there exist some $\beta_1 > 1$ and $\beta_2$, $\beta_3 > 0$ such that

$$
\| E_\alpha \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^2} \leq C \| G'[ Du ]^2 \|_{L^\infty} \| \nabla^{l-1} Du \|_{L^\infty}^{\beta_1} \| Du \|_{L^\infty}^{\beta_2} , \quad 1 \leq \alpha \leq l , \tag{A.22}
$$

$$
\| \partial^\alpha G \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^2} \leq C \| G'[ Du ]^2 \|_{L^\infty} \| \nabla^{l-1} Du \|_{L^\infty}^{\beta_1} \| Du \|_{L^\infty}^{\beta_2} , \quad 1 \leq \alpha \leq l - 1 . \tag{A.23}
$$

Proofs of (A.22) and (A.23) are exactly same as those of (A.11) and (A.12) except for following inequalities:

$$
\| \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^p(i_j)} \leq C \| \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{1-\theta_j} \| Du \|_{L^\infty}^{1-\theta_j} , \quad 1 \leq j \leq k
$$

and

$$
\| \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^q} \leq C \| \partial^\delta \theta \partial^\alpha \partial^\alpha \theta \|_{L^\infty}^{1-\theta_j} \| Du \|_{L^\infty}^{1-\theta_j} , \quad 1 \leq j \leq k . \tag{A.24}
$$
Proof of Lemma 4.2

We note that
\[ \square \]
We plug this into (A.25) and (A.26) to conclude the proof.

Let us first estimate
\[ \| \partial^l Du \|_{L^2} \leq C\| G \|_{L^\infty} \| \nabla^{l-1} Du \|_{L^2} \| Du \|_{L^\infty}^{1+i_1+\cdots+i_k-\beta_1}. \]  
(A.25)

Similarly, if \( 1 \leq \alpha \leq l-1 \), we have
\[ \| G \|_{L^\infty} \| Du \|_{L^2} \| \partial^\alpha Du \|_{L^2}^{l-1} \| Du \|_{L^\infty}^{1+i_1+\cdots+i_k-\beta_1}. \]  
(A.26)

Since \( l \geq 4 \), we have
\[ 1 < \beta_1 < 2 \quad \text{and} \quad 0 < 1 + i_1 + \cdots + i_k - \beta_1, \quad 2 - \beta_1 < \alpha. \]

We plug this into (A.25) and (A.26) to conclude the proof. \( \square \)

Next we provide the proof of Lemma 4.2

Proof of Lemma 4.2

We note that
\[ \partial^l Du = \sum_{\alpha=0}^{l-2m-1} \sum_{\beta=0}^{m} (\partial_x^\alpha \partial_t^\beta G) (\partial_x^{l-\alpha} \partial_t^{m-\beta} Du), \]  
(A.27)

where \( l := l - 2m - 1 \). Similar to the proof of Lemma 2.1 we have
\[ \partial_x^\alpha \partial_t^\beta G = 2(G \partial_x^\alpha \partial_t^\beta Du) + E_{\alpha+\beta}, \]
where
\[ E_{\alpha+\beta} \simeq P_{\alpha+\beta}(G, Du) \tilde{R}_{\alpha+\beta}(Du). \]  
(A.28)

Here, \( P_{\alpha+\beta}(G, Du) \) is a linear combination of \( G^{(i)}(Du)^j \) with \( 0 \leq i \leq j \leq \alpha + \beta \) as before and \( \tilde{R}_{\alpha+\beta}(Du) \) has a form such that
\[ \tilde{R}_{\alpha+\beta}(Du) = (\partial_x^{a_1} \partial_t^{b_1} Du)^{n_1} \cdots (\partial_x^{a_k} \partial_t^{b_k} Du)^{n_k} \]  
(A.29)

with \( a_1n_1 + \cdots + a_kn_k = \alpha \) and \( b_1n_1 + \cdots + b_kn_k = \beta \).

\[ \| (\partial_x^\alpha \partial_t^\beta G) (\partial_x^{l-\alpha} \partial_t^{m-\beta} Du) \|_{L^2(\mathbb{R}^3)} \leq \| (G \partial_x^\alpha \partial_t^\beta Du)(\partial_x^{l-\alpha} \partial_t^{m-\beta} Du) \|_{L^2(\mathbb{R}^3)} + \| E_{\alpha+\beta}(\partial_x^{l-\alpha} \partial_t^{m-\beta} Du) \|_{L^2(\mathbb{R}^3)} := I + II. \]  
(A.30)

Let us first estimate \( II \). We combine (A.28) and (A.29), and use the property (A.6) to get
\[ II \leq C\| G \|_{L^\infty} \| \partial_x^{l-\alpha} \partial_t^{m-\beta} Du \|_{L^2} \| \partial_x^{n_1} \partial_t^{b_1} Du \|_{L^p} \cdots \| \partial_x^{n_k} \partial_t^{b_k} Du \|_{L^p} \| \tilde{R}_{\alpha+\beta}(Du) \|_{L^2(\mathbb{R}^3)} \]  
(A.31)

where \( \frac{1}{p_1} + \cdots + \frac{1}{p_k} + \frac{1}{q} = \frac{1}{2} \). We exploit Gagliardo-Nirenberg as in (A.16) and have
\[ \| \partial_x^\theta \partial_t^b Du \|_{L^{p_j}} \leq C\| \partial_x^{l-\alpha} \partial_t^{m-\beta} Du \|_{L^2} \| \tilde{R}_{\alpha+\beta}(Du) \|^{1-\theta_j}_{L^\infty}, \quad 1 \leq j \leq k \]
\[ \| \partial_x^{l-\alpha} \partial_t^{m-\beta} Du \|_{L^q} \leq C\| \partial_x^\theta \partial_t^{m-\beta} Du \|_{L^2} \| \tilde{R}_{\alpha+\beta}(Du) \|^{1-\theta_j}_{L^\infty}, \]
where
\[ \frac{1}{p_j n_j} = \frac{a_j}{3} + \left( \frac{1}{2} - \frac{i}{3} \right) \theta_j, \quad 1 \leq j \leq k \]
(A.32)
\[ \frac{1}{q} = \frac{\tilde{l} - \alpha}{3} + \left( \frac{1}{2} - \frac{i}{3} \right) \theta_q. \]

Note that \( \tilde{l} = l - 2m - 1 \) and \( b_j \leq m \), for all \( j = 1, \ldots, k \). Hence, we have
\[
\| \partial_x^a \partial_t^b Du \|_{L^{\tilde{l}, n_j}} \leq C \| \partial_x^{l-2m} (\partial_t^{b_j} Du) \|_{L^2} \| \partial_t^{b_j} Du \|_{L^{\tilde{l} - (\theta_j)n_j}} \leq C A^{\theta_j n_j} B^{(1-\theta_j)n_j}, \quad 1 \leq j \leq k.
\]  
(A.33)

Similarly, we obtain
\[
\| \partial_t^{-\alpha} \partial_t^\beta Du \|_{L^q} \leq C A^{\theta_q} B^{1-\theta_q}. \]  
(A.34)

We combine (A.33) and (A.34) to get
\[
II \leq C \| G(Du)^2 \|_{L^\infty} A^{\theta_1 n_1 + \cdots + \theta_k n_k + \theta_q} B^{(1-\theta_1)n_1 + \cdots + (1-\theta_k)n_k + (1-\theta_q)}. 
\]  
(A.35)

We plug \( a_1 n_1 + \cdots + a_k n_k = \alpha \) into (A.32) and get \( \theta_1 n_1 + \cdots + \theta_k n_k + \theta_q = 1 \). Hence, we finally have
\[
II \leq C \| G(Du)^2 \|_{L^\infty} A B^{n_1 + \cdots + n_k}. \]  
(A.35)

Similarly, we have estimates for \( I \)
\[
I \leq C \| G(Du)^2 \|_{L^\infty} \| (\partial_x^a \partial_t^\beta Du) (\partial_x^{-\alpha} \partial_t^{m-\beta} Du) \|_{L^2(\mathbb{R}^3)} \leq C \| G(Du)^2 \|_{L^\infty} A B. \]  
(A.36)

We first note
\[ n_1 + \cdots + n_k \leq \alpha + \beta \leq l - m - 1, \]
and combine (A.27), (A.30), (A.35) and (A.36) to conclude
\[
\| \partial_x^{l-2m-1} \partial_t^m (GDu) \|_{L^2(\mathbb{R}^3)} \leq C \| G(Du)^2 \|_{L^\infty} A(B + B^{l-m-1}).
\]

This completes the proof. \( \square \)

**References**

[1] Bae, H-O.: Regularity criterion for generalized Newtonian fluids in bounded domains. J. Math. Anal. Appl. 421, 489–500 (2015)
[2] Bae, H-O., Wolf, J.: Existence of strong solutions to the equations of unsteady motion of shear thickening incompressible fluids. Nonlinear Anal. Real World Appl. 23, 160–182 (2015)
[3] Berselli, L.C., Diening, L., Růžička, M.: Existence of strong solutions for incompressible fluids with shear dependent viscosities. J. Math. Fluid Mech. 12 101–132 (2010)
[4] Bohme, G.: Non-Newtonian fluid mechanics, North-Holland Series in Applied Mathematics and Mechanics, 1987
[5] Diening, L., Růžička, M., Wolf, J.: Existence of weak solutions for unsteady motions of generalized Newtonian fluids. Ann. Sc. Norm. Super. Pisa Cl. Sci. 5, 1–46 (2010)
[6] John, O., Stará, J.: On the regularity of weak solutions to parabolic systems in two spatial dimensions. Comm. Partial Differential Equations 23, 1159–1170 (1998)
[7] Kaplický, P., Malek, J. Stará, J.: Global-in-time Holder continuity of the velocity gradients for fluids with shear-dependent viscosities. NoDEA Nonlinear Differential Equations Appl. 9, 175–195 (2002)
[8] Ladyzhenskaya, O. A.: New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems, Trudy Mat. Inst. Steklov. 102, 85–104 (1967)
[9] Ladyzhenskaya, O. A.: The mathematical theory of viscous incompressible flow, Gordon and Breach, New York, 2nd edition (1969)
[10] Málek, J., Nečas, M., Rokyta, M., Růžička, M.: Weak and Measure-valued Solutions to Evolutionary PDEs, Chapman & Hall (1996)

[11] Pokorný, M.: Cauchy problem for the non-Newtonian viscous incompressible fluid. Appl. Math. 41, 169–201 (1996)

[12] Wolf, J.: Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity, J. Math. Fluid Mech. 9, 104–138 (2007)

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