Full counting statistics of chaotic cavities with many open channels

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Explicit formulas are obtained for all moments and for all cumulants of the electric current through a quantum chaotic cavity attached to two ideal leads, thus providing the full counting statistics for this type of system. The approach is based on random matrix theory, and is valid in the limit when both leads have many open channels. For an arbitrary number of open channels we present the third cumulant and an example of non-linear statistics.

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The physics of current fluctuations in mesoscopic conductors is an interesting and fundamental quantum mechanical problem, since at low temperatures they are mainly due to the discreteness of the electron charge. The study of shot noise, for example, is an active area of theoretical and experimental research involving different types of systems (e.g. quantum dots, disordered wires, quantum point contacts) and different regimes (e.g. Coulomb blockade, quantum Hall effect, localization). A relatively recent approach is the concept of full counting statistics, the study of all cumulants of the charge fluctuation, which amounts to having complete information concerning charge counting in a transport process. This approach has recently attracted much attention and has been applied in a wide variety of situations (see Refs. 3 and references therein). Experimental measurements of the third moment of these fluctuations have already been reported.

In the case of chaotic cavities the random matrix theory (RMT) approach has been very successful in reproducing different experimental observations related to quantum transport, such as weak localization and universal conductance fluctuations. If the cavity is connected to two ideal leads, supporting respectively $N_1$ and $N_2$ open channels, the conductance is given by the Landauer-Büttiker formula $g = G_0 \text{Tr}[T]$, where $G_0$ is the conductance quantum, $T = e^2\hbar t$ and $t$ is the transmission matrix. However, only very recently was an exact expression obtained within RMT for the shot noise $P = P_0 \text{Tr}[T(1-T)]$ (with $P_0 = 2eV G_0$ where $V$ is a small voltage bias) and an explicit general result is not available for higher moments of the type $\text{Tr}[T^n]$. For chaotic cavities with large $N_1, N_2$ the third and fourth cumulants of charge transfer have been obtained, as well as an expression for the cumulant generating function.

On the other hand, recent semiclassical calculations based on correlated classical trajectories that transmit through the cavity have been able to reproduce the RMT results, both for the conductance and for the shot noise. These calculations have a natural perturbative structure on the parameter $N^{-1}$, where $N = N_1 + N_2$ is the total number of channels. Initially the leading order expressions were reproduced later the full series were obtained and exactly summed. The next natural step would be to tackle $\text{Tr}[T^n]$, and it is thus of interest to have the corresponding RMT prediction of this quantity, at least to leading-order in $N^{-1}$. This is the purpose of this work.

We will be interested in the dimensionless moments defined as $\sum_{i=1}^{n} T_i^m$, where $T_i$ are the eigenvalues of the matrix $T$ and $n = \min\{N_1, N_2\}$. Within RMT the $T_i$ are correlated random numbers between 0 and 1, whose distribution depends only on the symmetries of the system (orthogonal, unitary or symplectic, labeled by $\beta = 1, 2$ or 4 respectively). The average value of the moments are then given simply by

$$M_m = n\langle T_1^m \rangle. \quad (1)$$

The distribution of transmission eigenvalues can be characterized by a density, $\rho_\beta(T)$, such that $\langle T^m \rangle = \int_0^1 \rho_\beta(T) T^m dT$, or equivalently by a joint probability distribution $P_\beta$ such that

$$\langle T^m \rangle = \int_0^1 dT_1 \cdots \int_0^1 dT_n T_1^m P_\beta(T). \quad (2)$$

The expression for $P_\beta(T)$ is

$$P_\beta(T) = \mathcal{N}_\beta^{-1} |\Delta(T)|^\beta \prod_{j=1}^n T_j^\alpha, \quad (3)$$

where $\Delta(T) = \prod_{i,j} (T_i - T_j)$ is the Vandermonde determinant, $\alpha = \beta(2(|N_2 - N_1| + 1) - 1)$ and $\mathcal{N}_\beta$ is a normalization constant. In Ref. 6 the authors used simple recurrence relations from the theory of Selberg’s integral to obtain an exact result with arbitrary $N_1, N_2$ for the second moment $M_2$ and for the shot-noise (second cumulant). Here we follow a similar approach and compute the third cumulant, sometimes called the skewness. Moreover, we then proceed to obtain explicit formulas for all moments $M_m$ and for all cumulants, valid to first order in the inverse number of channels, i.e. in the limit $N_1, N_2 \gg 1$.

We must note that in the semiclassical limit of short wavelengths some noiseless scattering states can be created leading to a breakdown of the universality implied by RMT predictions. This phenomenon is governed by the ratio $TE/T_D$ of the quantum Ehrenfest time to the classical dwell time, and its influence has
been investigated on shot-noise\cite{11,12}, the weak localization effect\cite{13} and conductance fluctuations\cite{11,12}. Our results are restricted to the universal regime $\tau_E/\tau_D \to 0$, when these system-specific corrections are neglected.

Let us consider a certain fixed sequence of $k$ positive integers, $m = [m_1, ..., m_k]$, and for any subsequence of length $q \leq k$ let us define the function

$$P^q_m(T) = \prod_{j=1}^{q} T_j^{m_j}, \quad P^0_m(T) = 1. \quad (4)$$

We take now $T_k P^k_m(T) P_\beta(T)$, derive it with respect to $T_k$ and integrate over all variables to obtain

$$F = (\alpha + m_k) \langle P^k_m(T) \rangle + \beta \sum_{j=2}^{n} \langle P^k_m(T) \rangle \frac{T_k}{T_k - T_j}, \quad (5)$$

where the constant $F$ is given by

$$F = \int_0^1 dt_1 \cdots \int_0^1 dt_n \frac{d}{dt_k} [T_k P^k_m(T) P_\beta(T)]. \quad (6)$$

We can see that $F$ is actually independent of $m_k$. Hence, we may equate the r.h.s. of (5) at different values of this variable arriving at a recurrence relation. To solve this relation in general is presently beyond reach, but armed with some patience once can compute the first moments. This is essentially what was done in Ref.\cite{6}. We take it a bit further and find the third moment. Instead of writing the lengthy expression that arises for $M_3$ we present the corresponding cumulant (assuming for simplicity $N_2 \geq N_1$),

$$\frac{Q_3}{Q_2} = \frac{(N_2 - N_1 + 1 - \frac{2}{\beta})(N_2 - N_1 - 1 + \frac{2}{\beta})}{(N - 1 + \frac{2}{\beta})(N - 3 + \frac{2}{\beta})}, \quad (7)$$

where $Q_2$ is the average shot noise in units of $P_0$,

$$\frac{\langle P \rangle}{P_0} = Q_2 = \frac{N_1 N_2 (N_1 - 1 + \frac{2}{\beta})(N_2 - 1 + \frac{2}{\beta})}{(N - 1 + \frac{2}{\beta})(N - 2 + \frac{2}{\beta})(N - 1 + \frac{1}{\beta})}. \quad (8)$$

The result for $Q_3$ agrees in the limit $N \gg 1$ with the one presented in Ref.\cite{6}.

It is also possible to go beyond linear statistics, and compute higher correlations as for example

$$\frac{n(n-1)(T_1 T_2 (1 - T_1) (1 - T_2))}{Q_2} = \frac{(N_1 - 1)(N_2 - 1)(N_1 - 2 + \frac{2}{\beta})(N_2 - 2 + \frac{2}{\beta})}{(N - 3 + \frac{2}{\beta})(N - 4 + \frac{2}{\beta})(N - 2 + \frac{1}{\beta})}, \quad (9)$$

a quantity which would be important to compute the variance of the shot noise.

To be able to arrive at a more general result, we now introduce the assumption that both leads contain a large number of open channels, $N_1, N_2 \gg 1$, and thus $n \gg k$. In this case the main contribution to the summation in (5) will come from $j > k$. We can thus approximate $F$ by

$$F \approx (\alpha + m_k) \langle P^k_m(T) \rangle + \beta n \langle P^k_m(T) \rangle \frac{T_k}{T_k - T_n}. \quad (10)$$

All the results obtained from now on should be understood as being valid to first order in $N^{-1}$. Having said that, we drop the “≈” symbol and just write equalities.

We can use the identity

$$\langle T_k^m \rangle \frac{T_k}{T_k - T_n} = \frac{1}{2} \langle T_k^m - T_n^m \rangle \quad (11)$$

to simplify our expression for $F$,

$$F = (\alpha + \beta n) \langle P^k_m(T) \rangle + \beta \frac{n}{2} \langle P^{k-1}_m(T) R^{k,n}_{m_{k-1}}(T) \rangle, \quad (12)$$

where $R^{n,q}_m(T)$ denotes the symmetric polynomial

$$R^{n,q}_m(T) = \sum_{r=1}^{n} T_p^{r-m} T_q^r, \quad (13)$$

and we have neglected $m_k$ against $\alpha + \beta n$.

Comparing (12) for $m_k$ and $m_k - 1$ we get the relations

$$\langle P^{k-1}_m(T) \rangle = A_2 \langle P^{k-1}_m(T) \rangle, \quad (14)$$

for $m_k = 1$ and more generally

$$\langle P^k_m(T) \rangle = A_2 \langle P^{k-1}_m(T) \rangle + A_1 \langle P^{k-1}_m(T) [R^{k,n}_{m_{k-2}}(T) - R^{k,n}_{m_{k-1}}(T)] \rangle, \quad (15)$$

for $m_k \geq 2$. In the previous equations $A_1$ and $A_2$ are the constants

$$A_1 = \frac{\beta n}{2(\alpha + \beta n)} = \frac{N_1}{N}, \quad A_2 = \frac{2\alpha + \beta n}{2(\alpha + \beta n)} = \frac{N_2}{N}. \quad (16)$$

Not surprisingly, the parameter $\beta$ has dropped out of the calculation since leading-order results coincide for all universality classes. Iterating (15) $k$ times we obtain

$$\langle P^k_m(T) \rangle = A_2 \langle P^{k-1}_m(T) \rangle - A_1 \langle R^{k,n}_{m_{k-2}}(T) - R^{k,n}_{m_{k-1}}(T) \rangle. \quad (17)$$

Since we are interested in moments, we consider a particular case of the previous equation which is

$$\langle T^m_1 \rangle = A_2 - A_1 \langle R^{1,2}_{m-1}(T) \rangle. \quad (18)$$

On the other hand, Eq. (17) also gives

$$\langle R^{1,2}_{m}(T) \rangle = A_2 \sum_{j=1}^{m} \langle T^j_1 \rangle - A_1 \sum_{j=1}^{m-1} \langle T^{j-1}_1 - R^{2,3}_{j}(T) \rangle. \quad (19)$$

We must remark that the exponents of the terms inside the last brackets provide all ordered partitions of $m$ into
The following equation, 
\[ \sum_{i=1}^{n} \ln\{1 + T_i[e^{\lambda} - 1]\} = \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} Q_k, \] 
relates the moments and the cumulants. These will also be given by polynomials, \( Q_k(\xi) = N \sum_{p=1}^{k} D_{kp} \xi^p \). By feeding \([23]\) with our result \([22]\) we can obtain the first few cumulants, and the coefficients \( D_{kp} \) are shown in Table II. We have found by direct inspection that these coefficients are such that

\[ Q_k(\xi) = N \sum_{p=1}^{k} (-1)^{k+p} \frac{(2p - 2)!}{p!} S(k - 1, p - 1) \xi^p, \] 

where

\[ S(k, p) = \frac{1}{p!} \sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} j^k \]

are the Stirling numbers of the second kind.\(^19\) From the cumulants we can derive the generating function

\[ \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} Q_k(\xi) = 2N\xi \int_{0}^{\lambda} \frac{dz}{1 + \sqrt{1 + 4\xi(e^{-z} - 1)}} \] 

which is in fact equal to the one obtained in Ref.\(^7\) thus implying the correctness of \([22]\).\(^20\)

In summary, we have explicitly obtained the random matrix theory prediction for all moments and all cumulants of the charge current in a chaotic cavity, in the limit of large channel numbers. Naturally, it would be desirable to obtain such explicit expressions for arbitrary channel numbers, but in this case we were able to compute only special cases such as \([7]\) and \([9]\). The moments are natural quantities to be studied in semiclassical approaches to the problem, and indeed Eq. \([22]\) has been reproduced using action-correlated trajectories in the open quantum star graph.\(^21\) The Hamiltonian case and corrections due to finite Ehrenfest time are discussed to some extent in Ref.\(^22\).

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| \( C_{mp} \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|---------|----|----|----|----|----|----|----|
| 1       | 1  |    |    |    |    |    |    |
| m       | 2  | -1 |    |    |    |    |    |
| 3       | 1  | -2 | 2  |    |    |    |    |
| 4       | 1  | -3 | 6  | -5 |    |    |    |
|         | 5  | -4 | 12 | -20| 14 |    |    |
|         | 6  | -5 | 20 | -50| 70 | -42|    |
|         | 7  | -6 | 30 | -100| 210| -252|132 |

TABLE I: The values of the moment coefficients \( C_{mp} \) for several values of \( m \) and \( p \).

3 positive integers. These two equations can now be iterated together to yield the moments \( M_m = n(T_m^n) \), which will in fact be a polynomial of degree \( m \),

\[ M_m(\xi) = N \sum_{p=1}^{m} C_{mp} \xi^p, \quad \xi = \frac{N_1 N_2}{N^2}. \] 

Finding out the coefficient \( C_{mp} \) of the power \( \xi^p \) is now an exercise in combinatorics. The first part of the problem consists in answering the following question: In how many ways can one build sequences \( \{a_1, ..., a_{2p}\} \) with \( a_j \in \{A_1, A_2\} \) such that both \( A_1 \) and \( A_2 \) appear exactly \( p \) times and in all subsequences \( \{a_1, ..., a_q\}, q < 2p \) the number of \( A_2 \)'s is not larger than the number of \( A_1 \)'s.

The solution to this classic problem are the celebrated Catalan numbers.\(^19\)

\[ c_p = \frac{1}{p + 1} \binom{2p}{p}. \] 

The power \( \xi^p \) in \( M_m(\xi) \) will thus contain a factor \((-1)^{p-1} c_{p-1} \). It will also be multiplied by another factor, which is equal to the number of ordered partitions of \( m \) into \( p \) positive integers. This is \( \binom{m-1}{p-1} \).

We thus obtain our main result, an explicit expression for all the moments, valid to first order in the inverse number of channels:

\[ M_m(\xi) = N \sum_{p=1}^{m} \binom{m-1}{p-1} (\xi)^{p-1} c_{p-1}. \] 

The first three moments agree with known results.\(^7\) We present the coefficients \( C_{mp} \) with \( m \) up to 7 in Table I.

The following coefficients \( k \) are given by polynomials, \( Q_k(\xi) = N \sum_{p=1}^{k} D_{kp} \xi^p \). By feeding \([23]\) with our result \([22]\) we can obtain the first few cumulants, and the coefficients \( D_{kp} \) are shown in Table II. We have found by direct inspection that these coefficients are such that

\[ Q_k(\xi) = N \sum_{p=1}^{k} (-1)^{k+p} \frac{(2p - 2)!}{p!} S(k - 1, p - 1) \xi^p, \] 

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| \( D_{kp} \) | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|--------|----|----|----|----|----|----|----|
| 1      | 1  |    |    |    |    |    |    |
| 2      | 0  | 1  |    |    |    |    |    |
| 3      | 0  | -1 | 4  |    |    |    |    |
| 4      | 0  | -1 | 12 | 30 |    |    |    |
| 5      | 0  | -1 | 28 | -180| 336|    |    |
| 6      | 0  | -1 | 60 | 750 | -3360| 5040|    |
| 7      | 0  | -1 | 124| -2700| 21840|-75600|95040|    |

TABLE II: The values of the cumulant coefficients \( D_{kp} \) for several values of \( k \) and \( p \).
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