A superspace description of 
Friedmann–Robertson–Walker models

Sudhaker Upadhyay*

Department of Physics, Indian Institute of Technology Kanpur, Kanpur 208016, India
*E-mail: sudhakerupadhyay@gmail.com, sudhaker@iitk.ac.in

Received July 15, 2015; Accepted July 22, 2015; Published September 14, 2015

We illustrate the cosmological Friedmann–Robertson–Walker (FRW) models realized as gauge theory in the extended configuration space with its Becchi–Rouet–Stora–Tyutin (BRST) invariance up to total derivative. To investigate the model in Batalin–Vilkovisky (BV) formalism the Lagrangian density of the models is further extended by introducing shifted fields corresponding to all fields. The extended Lagrangian density having shifted fields admits more general BRST symmetry (including shift symmetry), called “extended BRST symmetry.” In this framework the antighost fields corresponding to the shift symmetry get identified with antifields of BV formulation. Further, we analyse the models on the supermanifold with the help of the additional super (Grassmannian) coordinate \( \theta \). Remarkably, we observe that the \( \theta \) components of superfields naturally produce the gauge-fixing term in tandem with the ghost term of the effective Lagrangian density. Furthermore, we show that the quantum master equation of the BV quantization method can be translated to have a superfield structure for the FRW models.

Subject Index  B05, B10

1. Introduction

Quantum cosmology is a consequence of the efforts toward the development of a quantum theory of gravity, i.e., unification of quantum mechanics and general relativity [1,2]. According to the cosmological principle, both the spatial homogeneity and the isotropy of universe, which was originally stated for the large scale, is actually valid for the very large large scale of the universe. The study about homogeneous and isotropic spacetime symmetry was originally made by Friedmann, Robertson, and Walker (FRW), see Refs. [3–8], and in honour of them such universe models are called FRW models. FRW models play a central role in modern cosmology. Most of the works on quantum cosmology are based on FRW universe models, although some authors have also studied anisotropic models (see, for instance, Ref. [9]). In particular, almost all popular theoretical models of dark energy have relevance in FRW spacetime. Nevertheless, it is worth mentioning that almost all the models of dark energy meet some difficulties like cosmological constant problems, fine-tuning problems, and so on. One result of such difficulties in modern cosmology is the necessity of more careful investigation of the basics of FRW cosmology.

The realization of FRW models as a gauge theory is well established. When it comes to the quantization of general gauge theories in a Lagrangian formalism, the Batalin–Vilkovisky (BV; also called field/antifield) approach [10,11] appears to be more prevalent and rigorous than the other available schemes. In this formulation the solution of a so-called master equation provides...
the configuration-space counterpart of the Batalin–Fradkin–Vilkovisky (BFV) phase-space quantum action [12–15]. The BV formulation encompasses the Faddeev–Popov quantization and uses the Becchi–Rouet–Stora–Tyutin (BRST) symmetry, which plays a prominent role in the standard paradigm of fundamental interactions, to build on it [16,17]. The BV formalism, which is based on an action that contains both fields and antifields, can be thought of as a vast generalization of the original BRST formalism for pure Yang–Mills theory to an arbitrary Lagrangian gauge theory.

A superspace description for the non-cosmological modes in the BV formulation has been analyzed extensively [18–23]. In particular, Yang–Mills theory [21], higher derivative theory [24], and higher-form gauge theories [25] have been studied in this context. Recently, a similar formulation has been established for the theory of perturbative gravity at one-loop order [26]. However, for the cosmological models such analysis has not yet been made, although the BRST symmetry has been analyzed for FRW models [27,28]. This provides our motivation to make the analysis in cosmological models.

In this paper, we study the extended BRST symmetry of the FRW models which incorporates a shift symmetry in tandem with the usual BRST symmetry. Within the analysis we need ghost and antighost fields corresponding to shift symmetry. Further, we choose a differential gauge-fixing term to fix the shift symmetry in such a manner that it removes the shift in fields and we recover our original theory. By doing so, the antighost fields are identified with the antifields of the BV formulation. Further, we discuss the extended BRST invariant models in superspace. To make such an analysis in superspace we need one more coordinate with a Grassmannian nature. Finally, we show that the gauge-fixing and ghost terms of the extended BRST invariant models are the outcome of the $\theta$ components of superfields. The superspace description of the quantum master equation at one-loop order for the FRW models is also analyzed.

This paper is presented in the following manner. In Sect. 2, a mathematical formulation of the FRW models is given where the outline of Hamiltonian dynamics in extended phase space is presented. Further, in Sect. 3, we demonstrate the extended FRW models possessing extended BRST transformations (including shift symmetry), where we derive the antifields in a more natural way. The extended BRST-invariant description of the models on the supermanifold is discussed in Sect. 4. We discuss the results and future problems in the final section.

**2. FRW models and BRST symmetry**

In this section, we analyze the BRST symmetry of the cosmological FRW models describing a homogeneous and isotropic universe. The metric tensor for FRW models in spherical coordinates is given by

$$ds^2 = N^2 dt^2 + a^2(t) \left( \frac{1}{1 - kr^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \quad (2.1)$$

where $N$ and $a(t)$ are the lapse function and the scale factor, respectively, depending on time only, and the values of $k = 1, 0, -1$ correspond to a space of positive, negative, and zero curvature respectively. The classical Lagrangian density of the models traditionally described in Arnowitt–Deser–Misner (ADM) variables is given by [28,29]

$$L_{\text{inv}} = -\frac{1}{2} \frac{a \dot{a}^2}{N} + \frac{k}{2} Na, \quad (2.2)$$

$$2/10$$
where the parameter $k = 1, 0, -1$ refers to a closed, flat, and open universe, respectively. The conjugate momenta corresponding to the lapse function $N$ and the scale factor $a$ read
\begin{align}
\pi_N &= 0, \\
\pi_a &= -\frac{a\dot{a}}{N},
\end{align}
(2.3)
(2.4)
where condition (2.3) describes the primary constraint of the theory. Using the Legendre transform, the canonical Hamiltonian density is calculated as [28,29]
\[ H_c = \pi_a \dot{a} - L_{\text{inv}} = -\frac{N\pi_a^2}{2a} - \frac{k}{2} Na. \]
(2.5)
Conservation of the primary constraint (2.3) with respect to time yields the secondary constraint of the theory as follows:
\[ \frac{\pi_a^2}{2a} + \frac{k}{2} a = 0. \]
(2.6)
Now, it is easy to verify that both the constraints are first class (as they commute with each other). Therefore, we can say with certainty that the FRW models admit gauge invariance. The gauge transformation, under which the Lagrangian density (2.2) remains invariant, is given by [28]
\[ \delta N = -N\dot{\eta} - \dot{N}\eta, \quad \delta a = -\dot{a}\eta, \]
(2.7)
where $\eta(t)$ is an infinitesimal time-dependent transformation parameter. Before quantizing the theory, it is necessary to impose a gauge-fixing condition which breaks the gauge invariance. This gauge-fixing condition must satisfy the following requirements: (i) it must fix the gauge completely, i.e., there must be no residual gauge freedom; (ii) using the transformations it must be possible to bring any configuration specified by $N$ and $a$ into one satisfying the gauge condition. We choose the following gauge condition satisfying the abovementioned requirements [29]:
\[ \dot{N} = \frac{d}{dt} f(a). \]
(2.8)
To employ the gauge condition (2.8) in the theory at quantum level we add the following gauge-fixing term in the invariant Lagrangian density (2.2) [28]:
\[ L_{\text{gf}} = \lambda \left( \dot{N} - \frac{d}{dt} f(a) \right), \]
(2.9)
where $\lambda$ is the multiplier (auxiliary) field.

Further, to compensate for the effect of the above gauge-fixing term from the functional integral, we add the following ghost term corresponding to the above gauge-fixing condition in the effective Lagrangian density [28]:
\[ L_{\text{gh}} = \bar{c} \left( \dot{N} - \frac{d}{dt} f(a) \right) c + \dot{c} N\dot{c}, \]
(2.10)
where the fields $c$ and $\bar{c}$ are Faddeev–Popov ghost and antghost respectively. Now, by adding (2.2), (2.9), and (2.10), the complete extended Lagrangian density reads [28]
\[ L_{\text{ext}} = L_{\text{inv}} + L_{\text{gf}} + L_{\text{gh}}. \]
(2.11)
The nilpotent BRST transformations are constructed by replacing the parameter $\eta$ of (2.7) by ghost field $c$ as follows [28]:

\begin{align*}
    s_b N &= -(\dot{N} c + N \dot{c}), \\
    s_b a &= -\dot{a} c, \\
    s_b c &= 0, \\
    s_b \bar{c} &= -\lambda, \\
    s_b \lambda &= 0,
\end{align*}

(2.12)

under which the extended Lagrangian density $L_{\text{ext}}$ is invariant. Since the combination $L_{gf} + L_{gh}$ is BRST exact, we can express it in terms of BRST variation of gauge-fixing fermion $\Psi$ as follows:

\begin{align*}
    L_{gf} + L_{gh} &= s_b \Psi = -s_b \left[ \bar{c} \left( \dot{N} - \frac{d}{dt} f(a) \right) \right],
\end{align*}

(2.13)

where the gauge-fixing fermion is defined as $\Psi = -\bar{c} \left( \dot{N} - \frac{d}{dt} f(a) \right)$.

3. Extended BRST invariant FRW model

In this section, we analyze the extended BRST transformations for FRW models. To do so, let us start by shifting the fields from their original values as follows [21]:

\begin{align*}
    N &\rightarrow N - \tilde{N} \\
    a &\rightarrow a - \tilde{a} \\
    \bar{c} &\rightarrow \bar{c} - \tilde{c} \\
    c &\rightarrow c - \tilde{c} \\
    \lambda &\rightarrow \lambda - \tilde{\lambda}.
\end{align*}

(3.1)

With these shifts in fields, the extended Lagrangian density (2.11) is given by

\begin{align*}
    \tilde{L}_{\text{ext}} &= L_{\text{ext}} \left( N - \tilde{N}, a - \tilde{a}, \bar{c} - \tilde{c}, c - \tilde{c}, \lambda - \tilde{\lambda} \right) \\
    &= -\frac{1}{2} \left( a - \tilde{a} \right) \left( \frac{\dot{a} - \dot{\tilde{a}}}{N - \tilde{N}} \right)^2 + \frac{k}{2} \left( N - \tilde{N} \right) \left( a - \tilde{a} \right) + \left( \lambda - \tilde{\lambda} \right) \left( \dot{N} - \tilde{N} - \frac{d}{dt} f(a - \tilde{a}) \right) \\
    &\quad + \left( \dot{\bar{c}} - \dot{\tilde{c}} \right) \left( \dot{N} - \tilde{N} - \frac{d}{dt} f(a - \tilde{a}) \right) \left( \dot{\bar{c}} - \dot{\tilde{c}} \right) + \left( \dot{c} - \dot{\tilde{c}} \right) \left( N - \tilde{N} \right) \left( \dot{c} - \dot{\tilde{c}} \right).
\end{align*}

(3.2)

This extended Lagrangian density is invariant under the same BRST structure (3.1) but for shifted fields. Here, we notice that the extended Lagrangian density is also invariant under the shift symmetry [21]

\begin{align*}
    s \Phi(x) &= \alpha(x), \\
    s \tilde{\Phi}(x) &= \alpha(x),
\end{align*}

(3.3)

where $\Phi$ and $\tilde{\Phi}$ denote the original fields and the shift in fields collectively. The BRST symmetry together with the shift symmetry manifest the extended BRST symmetry. Hence, the extended BRST transformation can be defined, compactly, as [21]

\begin{align*}
    s_b \Phi(x) &= \alpha(x), \\
    s_b \tilde{\Phi}(x) &= \alpha(x) - \beta(x).
\end{align*}

(3.4)

Here, $\beta(x)$ refers to the original BRST variation, whereas $\alpha(x)$ denotes the change under shift symmetry.
The extended BRST symmetry transformation, under which the Lagrangian density (3.2) remains invariant, is constructed explicitly as

\[ s_b N = \zeta, \quad s_b \tilde{N} = \zeta + (\dot{N} - \dot{\tilde{N}}), \]
\[ s_b a = \epsilon, \quad s_b \dot{a} = \epsilon - \dot{a}, \]
\[ s_b c = \vartheta, \quad s_b \dot{c} = \vartheta, \]
\[ s_b \tilde{c} = \zeta, \quad s_b \dot{\tilde{c}} = \zeta + \lambda - \dot{\lambda}, \]
\[ s_b \lambda = \varrho, \quad s_b \dot{\lambda} = \varrho. \] (3.5)

where \( \zeta, \epsilon, \vartheta, \zeta, \) and \( \varrho \) are the introduced ghost fields for the fields \( N, a, c, \tilde{c}, \) and \( \lambda, \) respectively, having the following ghost numbers:

\[ \text{gh}(\zeta) = 1, \quad \text{gh}(\epsilon) = 1, \quad \text{gh}(\vartheta) = 2, \quad \text{gh}(\zeta) = 0, \quad \text{gh}(\varrho) = 1. \] (3.6)

The nilpotency property of extended BRST symmetry transformations (3.5) restrict the ghost fields \( \zeta, \epsilon, \vartheta, \zeta, \) and \( \varrho \) in such a manner that these vanish under BRST transformation, i.e.

\[ s_b \zeta = 0, \quad s_b \epsilon = 0, \quad s_b \vartheta = 0, \quad s_b \zeta = 0, \quad s_b \varrho = 0. \] (3.7)

However, the requirement of the physical theory is that the ghost number of the Lagrangian density must be zero. So, to make the theory ghost free we incorporate the antighost fields \( N^*, a^*, \tilde{c}^*, c^*, \) and \( \lambda^* \) having ghost numbers opposite to that of the respective ghost fields. Now, we propose that these antighost fields transform under the BRST transformation as follows:

\[ s_b N^* = -K, \quad s_b a^* = -l, \quad s_b \tilde{c}^* = -\tilde{m}, \quad s_b c^* = -m, \quad s_b \lambda^* = -n. \] (3.8)

where \( K, l, \) and \( n \) are the bosonic Nakanishi–Lautrup-type auxiliary fields and \( \tilde{m}, m \) are fermionic auxiliary fields. The nilpotency property of BRST transformation (3.8) again gives:

\[ s_b K = 0, \quad s_b l = 0, \quad s_b \tilde{m} = 0, \quad s_b m = 0, \quad s_b n = 0. \] (3.9)

Now, one possible way to recover the original FRW models described by the Lagrangian density (2.2) is to make a suitable gauge condition which fixes the shift symmetry such that all the tilde fields...
vanish. The suitable gauge-fixing term which fixes the shift symmetry is constructed as follows:

$$\tilde{L}_{gf} \, + \, \tilde{L}_{gh} \, = \, -K \tilde{N} \, - \, N^* \left[ \zeta \, + \, (\tilde{N}c \, - \, \tilde{N}\tilde{c} \, + \, Nc \, - \, N\tilde{c}) \right] \, - \, \tilde{m} \tilde{a} \, - \, a^* (\epsilon \, + \, \tilde{a}c \, - \, \tilde{a}\tilde{c}) \, - \, m\tilde{c} \\
+ \, c^* [\zeta \, + \, \lambda \, - \, \tilde{\lambda}] \, - \, \tilde{m}\tilde{c} \, + \, \tilde{c}^* \tilde{\theta} \, - \, n\tilde{\lambda} \, - \, \lambda^* \varrho. \quad (3.10)$$

After performing integration over the auxiliary fields (within the functional integral) the above expression (3.10) reduces to

$$\tilde{L}_{gf} \, + \, \tilde{L}_{gh} \, = \, -N^* [\zeta \, + \, (\tilde{N}c \, + \, N\tilde{c})] \, - \, a^* (\epsilon \, + \, \tilde{a}c) \\
+ \, c^* [\zeta \, + \, \lambda] \, - \, \tilde{c}^* \tilde{\theta} \, - \, \lambda^* \varrho. \quad (3.11)$$

The original gauge-fixing and ghost terms of the model in terms of the general gauge-fixing fermion $\Psi$ can be described by

$$L_{gf} \, + \, L_{gh} \, = \, s_b \Psi = \left[ s_b N \frac{\delta \Psi}{\delta N} + s_b a \frac{\delta \Psi}{\delta a} + s_b \tilde{c} \frac{\delta \Psi}{\delta \tilde{c}} + s_b c \frac{\delta \Psi}{\delta c} + s_b \lambda \frac{\delta \Psi}{\delta \lambda} \right].$$

$$= \left[ \zeta \frac{\delta \Psi}{\delta N} + \epsilon \frac{\delta \Psi}{\delta a} + \tilde{c} \frac{\delta \Psi}{\delta \tilde{c}} + \tilde{\theta} \frac{\delta \Psi}{\delta c} + \varrho \frac{\delta \Psi}{\delta \lambda} \right]. \quad (3.12)$$

Now, the complete gauge-fixing and ghost terms [the sum of (3.11) and (3.12)] which fix the extended BRST symmetry are given by

$$\tilde{L}_{gf} \, + \, \tilde{L}_{gh} \, + \, L_{gf} \, + \, L_{gh} \, = \, \left( -N^* \, - \, \frac{\delta \Psi}{\delta N} \right) \zeta \, + \, \left( -a^* \, - \, \frac{\delta \Psi}{\delta a} \right) \epsilon \, + \, \left( c^* \, + \, \frac{\delta \Psi}{\delta \tilde{c}} \right) \tilde{\theta} \\
+ \, \left( \tilde{c}^* \, + \, \frac{\delta \Psi}{\delta c} \right) \tilde{\theta} \, - \, \left( \lambda^* \, + \, \frac{\delta \Psi}{\delta \lambda} \right) \varrho \, - \, N^* (\tilde{N}c \, + \, N\tilde{c}) \\
- \, a^* \tilde{a}c \, + \, c^* \lambda. \quad (3.13)$$

If we integrate out the ghost fields $\zeta$, $\epsilon$, $\tilde{c}$, $\theta$, and $\varrho$ associated with the shift symmetry, the above expression becomes

$$\tilde{L}_{gf} \, + \, \tilde{L}_{gh} \, + \, L_{gf} \, + \, L_{gh} \, = \, -N^* (\tilde{N}c \, + \, N\tilde{c}) \, - \, a^* \tilde{a}c \, + \, c^* \lambda, \quad (3.14)$$

leading to following constraints:

$$N^* = - \frac{\delta \Psi}{\delta N}, \quad a^* = - \frac{\delta \Psi}{\delta a},$$

$$c^* = - \frac{\delta \Psi}{\delta \tilde{c}}, \quad \tilde{c}^* = - \frac{\delta \Psi}{\delta c},$$

$$\lambda^* = - \frac{\delta \Psi}{\delta \lambda}. \quad (3.15)$$

Here we observe that the antighosts of the theory are identified with the antifields of the BV formulation. The consistency of the result can be checked as follows: for the theory of the FRW model the expression for the gauge-fixing fermion $\Psi$ is given in (2.13). For that $\Psi$, the antighost fields
established in (3.15) get the following identification:

\[
\begin{align*}
N^* &= -\dot{c}, \\
a^* &= \dot{c} \frac{d}{da} f(a), \\
c^* &= \left( \dot{N} - \frac{d}{dt} f(a) \right), \\
\lambda^* &= 0, \\
\bar{c}^* &= 0.
\end{align*}
\]

Plugging these specific values of antighosts back into (3.14), we recover the sum of our original gauge-fixing and ghost terms of the FRW models given in (2.9) and (2.10). Remarkably, we notice that the original FRW theory is recovered from the models in extended configuration space possessing extended BRST symmetry where antighosts are identified with antifields (of BV formalism) naturally.

4. FRW model in superspace

In this section, we analyze the extended BRST-invariant FRW models on the five-dimensional supermanifold. To describe the model in superspace, we first define the superfields, which depend on five super-coordinates \((x, \theta)\) such that at vanishing \(\theta\) these superfields identify with the original fields as follows:

\[
\begin{align*}
\mathcal{N}(x, \theta) &= N(x) + \theta \zeta, \\
\tilde{\mathcal{N}}(x, \theta) &= \tilde{N}(x) + \theta \left[ \zeta + (\tilde{N} \dot{c} - \tilde{N} \ddot{c} + N \dot{\tilde{c}} - N \ddot{\tilde{c}}) \right], \\
\mathcal{A}(x, \theta) &= a(x) + \theta \epsilon, \\
\tilde{\mathcal{A}}(x, \theta) &= \tilde{a}(x) + \theta \epsilon + \dot{a} \dot{c} - \dot{\tilde{a}} \dot{\tilde{c}}, \\
\mathcal{C}(x, \theta) &= c(x) + \theta \vartheta, \\
\tilde{\mathcal{C}}(x, \theta) &= \tilde{c}(x) + \theta \vartheta, \\
\mathcal{\tilde{C}}(\xi, \theta) &= \tilde{\tilde{c}}(x) + \theta \left[ \vartheta + \lambda - \tilde{\lambda} \right], \\
\Lambda(x, \theta) &= \lambda(x) + \theta \varphi, \\
\tilde{\Lambda}(x, \theta) &= \tilde{\lambda}(x) + \theta \varphi,
\end{align*}
\]

where \(\theta\) is the Grassmannian coordinate. Similarly, we define the super-antifields which depend on super-coordinates \((x, \theta)\) whose vanishing \(\theta\) components yield the original four-dimensional local antifields:

\[
\begin{align*}
\mathcal{N}^*(x, \theta) &= N^*(x) - \theta K, \\
\mathcal{A}^*(x, \theta) &= a^*(x) - \theta l, \\
\mathcal{C}^*(x, \theta) &= c^*(x) - \theta m, \\
\tilde{\mathcal{C}}^*(x, \theta) &= \tilde{c}^*(x) - \theta \tilde{m}, \\
\Lambda^*(x, \theta) &= \lambda^*(x) - \theta n.
\end{align*}
\]
From the above expressions of superfields and super-antifields given in (4.1) and (4.2), respectively, we compute the following relations:

\[
\frac{\delta}{\delta \theta}(\mathcal{N}^* \tilde{\mathcal{N}}) = -K \tilde{N} - N^*[\zeta + (\dot{N}c - \dot{\tilde{N}}\tilde{c} + N\tilde{c} - \tilde{N}\tilde{c})],
\]

\[
\frac{\delta}{\delta \theta} (\mathcal{A}^* \tilde{\mathcal{A}}) = -l\tilde{a} - a^*(\epsilon + \dot{a}c - \dot{\tilde{a}}\tilde{c}),
\]

\[
\frac{\delta}{\delta \theta}(\tilde{\mathcal{C}}\mathcal{C}^*) = -m\tilde{c} + c^*[\zeta + \lambda - \tilde{\lambda}],
\]

\[
\frac{\delta}{\delta \theta}(\mathcal{C}^* \tilde{\mathcal{C}}) = -\tilde{m}\tilde{c} + \tilde{c}^*\theta,
\]

\[
\frac{\delta}{\delta \theta}(\Lambda^* \tilde{\Lambda}) = -n\tilde{\lambda} - \lambda^*\theta.
\]  

(4.3)

where derivatives with respect to \(\theta\) are considered from the left side. Now, from expression (4.3) it is easy to derive the following relation:

\[
\frac{\delta}{\delta \theta}(\mathcal{N}^* \tilde{\mathcal{N}} + \mathcal{A}^* \tilde{\mathcal{A}} + \tilde{\mathcal{C}}\mathcal{C}^* + \mathcal{C}^* \tilde{\mathcal{C}} + *\tilde{*}) = -K \tilde{N} - N^*[\zeta + (\dot{N}c - \dot{\tilde{N}}\tilde{c} + N\tilde{c} - \tilde{N}\tilde{c})] - l\tilde{a}
\]

\[
- a^*(\epsilon + \dot{a}c - \dot{\tilde{a}}\tilde{c}) - m\tilde{c} + c^*[\zeta + \lambda - \tilde{\lambda}] - \tilde{m}\tilde{c}
\]

\[
+ \tilde{c}^*\theta - \tilde{\lambda} - \lambda^*\theta.
\]  

(4.4)

Here we observe that the right-hand side of above relation exactly coincides with the shifted gauge-fixed Lagrangian density given in (3.10). Further, we define the gauge-fixing fermion on the supermanifold as

\[
\Gamma(x, \theta) = \Psi(x) + \theta(s_b\Psi).
\]  

(4.5)

The expression (4.5) suggests that the original gauge-fixing and ghost Lagrangian densities for FRW models can be acquired from left derivation of the gauge-fixing fermion in superspace with respect to \(\theta\) as follows:

\[
L_{gf} + L_{gh} = \frac{\delta \Gamma(x, \theta)}{\delta \theta}.
\]  

(4.6)

From relations (4.4) and (4.6), we therefore conclude that the complete gauge-fixed Lagrangian density of extended BRST-invariant FRW models can simply be derived from the \(\theta\) components as follows:

\[
L_{gf} + L_{gh} + \tilde{L}_{gf} + \tilde{L}_{gh} = \frac{\delta}{\delta \theta} \left(\mathcal{N}^* \tilde{\mathcal{N}} + \mathcal{A}^* \tilde{\mathcal{A}} + \tilde{\mathcal{C}}\mathcal{C}^* + \mathcal{C}^* \tilde{\mathcal{C}} + *\tilde{*} + - \right).
\]  

(4.7)

Being the \(\theta\) component of superfields it is obvious that the complete gauge-fixed Lagrangian density of extended BRST-invariant FRW models in superspace remains intact under extended BRST transformations.

Furthermore, to investigate the BRST variation of the quantum action in the standard BV quantization method, we first define the vacuum functional for FRW models in BV formulation as

\[
Z_P = \int \prod D\Phi \exp \left[ \frac{i}{\hbar} W \left( \Phi, \Phi^* = \frac{\partial \Phi}{\partial \bar{\Phi}} \right) \right],
\]  

(4.8)

where \(\Phi\) and \(\Phi^*\) are the generic fields and corresponding antifields of the theory. However, \(W\) is an extended quantum action of the theory.
The condition of gauge independence of the generating functional, which translates into the so-called quantum master equation, is given by [26]

$$\frac{1}{2} \left( \frac{\partial_r W}{\partial \Phi} \frac{\partial_l W}{\partial \Phi^*} - \frac{\partial_r W}{\partial \Phi^*} \frac{\partial_l W}{\partial \Phi} \right) = i \hbar \Delta W,$$

(4.9)

where the operator $\Delta$ is defined by

$$\Delta = \frac{\partial_r}{\partial \Phi} \frac{\partial_l}{\partial \Phi^*}.$$

(4.10)

The extended quantum action can be written up to the one-loop order correction by

$$W(\Phi, \Phi^*) = S_{\text{ext}}(\Phi, \Phi^*) + \hbar M_1(\Phi, \Phi^*),$$

(4.11)

where $S_{\text{ext}}$ is the action corresponding to (2.11) and $M_1$ appears from nontrivial measure factors.

The gauge theory having no anomaly up to first-order correction $M_1$ does not depend on the antifields and, consequently, the BRST transformations of the action $S_{\text{ext}}$ and $M_1$ are given by

$$s_b S_{\text{ext}} = 0, \quad s_b M_1 = i \Delta S_{\text{ext}}.$$

(4.12)

Now, we apply the $\Delta$ operator on total action $S_T$ (having both the original and shifted actions) as

$$\Delta S_{\text{ext}} = \Delta S_T = \frac{\partial_r}{\partial \Phi} \frac{\partial_l}{\partial \Phi^*} S_T.$$

(4.13)

Here we note that the generic fields $\Phi$ and $\Phi^*$ include all the fields, shifted fields, ghosts, and corresponding antighost fields. Therefore, at the one-loop order, it is logical to define a superfield as

$$\mathcal{M}_1(x, \theta) = M_1(x) + \theta i \Delta S_T.$$

(4.14)

However, in the superspace, the extended quantum action is described by

$$\mathcal{W}(x, \theta) = W(x) + \theta \hbar \Delta W.$$

(4.15)

Therefore, the quantum master equation for FRW models can simply be derived from the above relation as the following:

$$\frac{\partial}{\partial \theta} \mathcal{W} = i \hbar \tilde{\Delta} \mathcal{W},$$

(4.16)

where the $\tilde{\Delta}$ operator has the following expression:

$$\tilde{\Delta} = \frac{\partial_r}{\partial \Phi(x, \theta)} \frac{\partial_l}{\partial \Phi^*(x, \theta)}.$$

(4.17)

Therefore, by enlarging the configuration space with the variable $\theta$, the quantum master equation is equipped with Grassmannian translations that reproduce the effect of the antibrackets given in (4.9).

5. Conclusion

In modern cosmology the FRW models play a central role. These cosmological models assume a zero cosmological constant. The only force acting is gravity. In this paper we have considered FRW models describing closed, flat, and open universes. Further, we have demonstrated the BRST symmetry of the models and have analyzed the extended BRST symmetry in which we have made a linear shift in all the fields. We have recovered the original theory from the shifted Lagrangian density by adding a suitable gauge-fixing term. The BV procedure represents a very powerful framework for the quantization of general gauge theories. The advantage of extending the phase space by shifting the fields is that the antifields of the BV formulation get their identifications naturally. Furthermore, it is worth
analyzing the model on the supermanifold. We have described the models in five-dimensional superspace with coordinates \((x, \theta)\). For a general gauge-fixing fermion, we have shown that the shifted Lagrangian density can be written in a manifestly extended BRST-invariant manner in a superspace with one Grassmann coordinate. We have shown that the quantum master equation of the standard BV formalism can be represented as the requirement of a superspace structure for the extended quantum action. We hope this formulation will be helpful in explaining the FRW models systematically.

It would be interesting to develop an anti-BRST transformation for the FRW models. With the help of such an anti-BRST transformation (where the role of ghosts and antighosts are interchanged with some coefficients), we can analyze the models on a six-dimensional superspace. However, without an anti-BRST transformation, we are constrained to analyzing the models up to the five-dimensional superspace.

References
[1] B. S. DeWitt, Phys. Rev. 160, 1113 (1967).
[2] D. L. Wiltshire, *Cosmology: The Physics of the Universe* (World Scientific, Singapore, 1996).
[3] A. Friedmann, Zeit. f. Phys. 10, 377 (1922).
[4] A. Friedmann, Zeit. f. Phys. 21, 326 (1924).
[5] H. P. Robertson, Astrophys. J. 82, 284 (1935).
[6] H. P. Robertson, Astrophys. J. 83, 187 (1935).
[7] H. P. Robertson, Astrophys. J. 83, 257 (1936).
[8] A. G. Walker, Proc. Lond. Math. Soc. 42, 90 (1937).
[9] J. Louko, Ann. Phys. 181, 318 (1988).
[10] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B 102, 27 (1981).
[11] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D 28, 2567 (1983); 30, 508 (1984). [erratum]
[12] E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B 55, 224 (1975).
[13] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B 69, 309 (1977).
[14] E. S. Fradkin and T. E. Fradkina, Phys. Lett. B 72, 343 (1978).
[15] I. A. Batalin and E. S. Fradkin, Phys. Lett. B 122, 157 (1983).
[16] C. Becchi, A. Rouet, and R. Stora, Commun. Math. Phys. 42, 127 (1975).
[17] C. Becchi, A. Rouet, and R. Stora, Ann. Phys. 98, 287 (1976).
[18] J. Alfaro and P. H.Damgaard, Phys. Lett. B 222, 425 (1989).
[19] J. Alfaro, P. H. Damgaard, J. I. Latorre, and D. Montano, Phys. Lett. B 233, 153 (1989).
[20] J. Alfaro and P. H. Damgaard, Nucl. Phys. B 404, 751 (1993).
[21] N. R. F. Braga and A. Das, Nucl. Phys. B 442, 655 (1995).
[22] P. M. Lavrov, Phys. Lett. B 366, 160 (1996).
[23] P. M. Lavrov and P. Yu. Moshin, Phys. Lett. B 508, 127 (2001).
[24] M. Faizal and M. Khan, Eur. Phys. J. C 71, 1603 (2011).
[25] S. Upadhyay and B. P. Mandal, Eur. Phys. J. C 72, 2059 (2012).
[26] S. Upadhyay, Phys. Lett. B 723, 470 (2013).
[27] J. J. Halliwell, Phys. Rev. D 38, 2468 (1988).
[28] F. Cianfrani and G. Montani, [arXiv:1301.4122 [gr-qc]] [Search INSPIRE].
[29] T. P. Shestakova, Class. Quantum Grav. 28, 055009 (2011).