GEOMETRIC REASONING WITH POLYMAKE

EWGENIJ GAWRILOW AND MICHAEL JOSWIG

Abstract. The mathematical software system polymake provides a wide range of functions for convex polytopes, simplicial complexes, and other objects. A large part of this paper is dedicated to a tutorial which exemplifies the usage. Later sections include a survey of research results obtained with the help of polymake so far and a short description of the technical background.

1. Introduction

The computer has been described as the mathematical machine. Therefore, it is nothing but natural to let the computer meet challenges in the area where its roots are: mathematics. In fact, the last two decades of the 20th century saw the ever faster evolution of many mathematical software systems, general and specialized. Clearly, there are areas of mathematics which are more apt to such an approach than others, but today there is none which the computer could not contribute to.

In particular, polytope theory which lies in between applied fields, such as optimization, and more pure mathematics, including commutative algebra and toric algebraic geometry, invites to write software. When the polymake project started in 1996, there were already a number of systems around which could deal with polytopes in one way or another, e.g., convex hull codes such as cdd [13], lrs [8], porta [8], and qhull [4], but also visualization classics like Geomview [1]. The basic idea to polymake was—and still is—to make interfaces between any of these programs and to continue building further on top of the combined functionality. At the same time the gory technical details which help to accomplish such a thing should be entirely hidden from the user who does not want to know about it. On the outside polymake behaves somewhat similar to an expert system for polytopes: The user once describes a polytope in one of several natural ways and afterwards he or she can issue requests to the system to compute derived properties. In particular, there is no need to program in order to work with the system. On the other hand, for those who do want to program in order to extend the functionality even further, polymake offers a variety of ways to do so.

The modular design later, since version 2.0, allowed polymake to treat other mathematical objects in the same way. Mostly guided by the research of the second author the system was augmented by the TOPAZ application which deals with finite simplicial complexes. Because of the connections between polytope theory and combinatorial topology both parts of the system now benefit from each other.

We explain the organization of the text. It begins with a quite long tutorial which should give an idea of how the system can be used. The examples are deliberately chosen to be small enough that it is possible to verify all claims while reading. What follows is an overall description of the key algorithms and methods which are available in the current version 2.1.0. The subsequent Section 4 contains several brief paragraphs dedicated to research in mathematics that was facilitated by polymake.

Previous reports on the polymake system include the two papers [15, 16]; by now they are partially outdated. polymake is open source software which can be downloaded from http://www.math.tu-berlin.de/polymake for free.

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Acknowledgments. Over the time many people made small and large contributions to the code. Most notably, Thilo Schröder and Nikolaus Witte are members of the development team since 2002.

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2. A Tutorial

This tutorial tries to give a first idea about polymake’s features by taking a look at a few small examples. We focus on computations with convex polytopes. For definitions, more explanations, and pointers to the literature see the subsequent Section 3.1.

The text contains commands to be given to the polymake system (preceded by a ‘>’) along with the output. As the environment a standard UNIX shell, such as bash, is assumed. Commands and output are displayed in typewriter type.

Most of the images shown are produced via polymake’s interface to JavaView [33] which is fully interactive.

2.1. A very simple example: the 3-cube. Suppose you have a finite set of points in the Euclidean space $\mathbb{R}^d$. Their convex hull is a polytope $P$. Now you want to know how many facets $P$ has. As an example let the set $S$ consist of the points $(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)$ in $\mathbb{R}^3$. Clearly, $S$ is the set of vertices of a cube. Employing your favorite word processor produce an ASCII text file, say cube.poly, containing precisely the following information.

```
POINTS
1 0 0 0
1 0 0 1
1 0 1 0
1 0 1 1
1 1 0 0
1 1 0 1
1 1 1 0
1 1 1 1
```

We have the keyword POINTS followed by eight lines, where each line contains the coordinates of one point. The coordinates in $\mathbb{R}^3$ are preceded by an additional 1 in the first column; this is due to the fact that polymake works with homogeneous coordinates. The solution of the initial problem can now be performed by polymake as follows. Additionally, we want to know whether $P$ is simple, that is, whether each vertex is contained in exactly 3 facets (since $\dim P = 3$).

```
> polymake cube.poly N_FACETS SIMPLE
```

While polymake is searching for the answer, let us recall what polymake has to do in order to obtain the solution. It has to compute the convex hull of the given point set in terms of an explicit list of the facet describing inequalities. Then they have to be counted, which is, of course, the easy part. Nonetheless, polymake decides on its own what has to be done, before the final —admittedly trivial— task can be performed. Checking simplicity requires to look at the vertex facet incidences. In the meantime, polymake is done. Here is the answer.

```
N_FACETS
6
```

```
SIMPLE
1
```

Simplicity is a boolean property. So the answer is either “yes” or “no”, encoded as 1 or 0, respectively. The output says that $P$ is, indeed, simple.
Depending on the individual configuration `polymake` chooses one of several convex hull computing algorithms. In the previous example `polymake` might have used the double description method from Fukuda’s `cdd` package [14]. It is possible to explicitly specify other methods. As a matter of fact, `polymake` knows quite a bit about standard constructions of polytopes. So you do have to type in your 3-cube example. You can use the following command instead. The trailing argument 0 indicates a cube with 0/1-coordinates.

```
> cube cube.poly 3 0
```

2.2. **Visualizing a Random Polytope.** But let us now try something else. How does a typical polytope look like? To be more precise: Show me an instance of the convex hull of 20 randomly distributed points on the unit sphere in $\mathbb{R}^3$. This requires one command to produce a `polymake` description of such a polytope and a second one to trigger the visualization. Again there is an implicit convex hull computation going on behind the scenes. On the way a complete combinatorial description of the polytope is obtained.

```
> rand_sphere random.poly 3 20
> polymake random.poly VISUAL
```

`polymake`’s standard tool for interactive visualization is *JavaView* by Polthier and others [33]. For instance, it allows you to rotate or zoom into your polytope. Here is a sequence of snapshots.

![Visualizing a Random Polytope](image)

**Figure 1.** Interactive visualization of a random polytope with *JavaView*: Three snapshots.

2.3. **Linear programming.** Polytopes most naturally appear as sets of feasible solutions of linear programs. Consider the following example.

Maximize
\[
x_1 + x_2 + x_3
\]
subject to
\[
0 \leq x_1, x_2, x_3 \leq 1
\]
\[
x_1 + x_2 + x_3 \leq 5/2
\]
\[
x_1 - 17x_2 \leq 8
\]

![Linear Programming](image)

**Figure 2.** Small linear program and a visualization.

A linear inequality $a_0 + a_1x_1 + \ldots + a_dx_d \geq 0$ is encoded as the inward pointing normal vector to the corresponding affine hyperplane (of suitable length). This is what `polymake` uses: the former inequality is represented as the vector $(a_0, a_1, \ldots, a_d)$. Our linear program in `polymake`’s format looks as given below. Note that we do not decide beforehand, whether we want to minimize or maximize.
People working in optimization usually prefer other file formats (such as CPLEX’s LP file format), where it is also possible to keep the names of the variables. polymake is aware of this. LP format files can be converted by calling the command lp2poly.

It is not difficult to determine the polytope forming the solution space. Assume the file linear_program.poly contains the description given above.

> polymake linear_program.poly MAXIMAL_VALUE
MAXIMAL_VALUE
5/2

This is the kind of behavior one would expect from a linear solver. Of course, usually it is also interesting to obtain a point attaining the maximum. And, in fact, polymake calls cdd’s implementation of the Simplex Method (with exact rational arithmetic). With polymake you can go one step further. It visualizes for you the polytope with directed edges. Instead of relying on the interactive JavaView visualization this time we produce postscript output directly; see Figure 2.

> polymake linear_program.poly postscript \ "VISUAL_GRAPH->DIRECTED_GRAPH->VERTEX_COLORS"

The directed edges (whose orientation is induced by the given linear objective function) are drawn as arrows. The colors of the vertices indicate the level of height with respect to the objective function. The numbering of the vertices corresponds to the order of the vertices in the file.

It is an essential feature of polytope theory that it is possible to define a polytope in one of two equivalent ways: either as the convex hull points or as the intersection of half-spaces. This is reflected in polymake: All functions can directly be applied to all polytopes regardless of how they were initially described.

2.4. Polytopes From the Combinatorial Point of View. Suppose now you are not interested in a particular coordinate representation of a polytope. But instead you want to focus on the combinatorial properties only. polymake supports this point of view, too. You can specify a polytope in terms of its vertex-facet-incidence matrix. For each facet you have a line with a list of the vertices contained in that facet. The vertices are specified by numbers. They are numbered consecutively starting from 0. In each row the vertices are listed in ascending order. The following is a valid polymake description of a square.

VERTICES_IN_FACETS

\{0 1\}
\{1 2\}
\{2 3\}
\{0 3\}

Note that in this situation polymake assumes that you actually specified a polytope in this way. Verifying realizability, albeit algorithmically possible (e.g., by means of quantifier elimination), is beyond polymake’s capabilities.
The dimension of a polytope, i.e., the dimension of its affine hull, is an intrinsic property. It does not depend on the coordinate representation. The vertex-facet-incidence matrix suffices for polymake to compute the dimension. Assume the data above was stored in a file named square.poly.

```plaintext
> polymake square.poly DIM
DIM
2
```

Often one wants to construct new polytopes from old ones. So suppose we need a prism over a triangle. This can be constructed as the wedge of a square over an arbitrary facet (e.g., the first facet, which is numbered 0). This is a simple polytope, but it is not simplicial. The program option `-noc` means “no coordinates”, and this limits the produced output to a purely combinatorial description.

```plaintext
> wedge prism.poly square.poly 0 -noc
> polymake prism.poly SIMPLE SIMPLICIAL
SIMPLE
1
```

2.5. **Checking for Combinatorial Equivalence.** Suppose the file Sharir.poly contains the following inequality description of a 3-dimensional polytope:

```plaintext
INEQUALITIES
25 -2 -25 10
-2 25 2 10
25 -2 25 10
-2 25 -2 10
0 0 -1 -1
2 0 -1 1
```

It turns out that the polytope defined this way is, in fact, combinatorially equivalent to the 3-dimensional cube that we studied above. polymake can give a proof of this as shown below; internally McKay’s nauty is called [18].

```plaintext
> polymake -v check_iso Sharir.poly cube.poly
[fixing partition]
(2 6)(3 7)(8 11)(9 10)
level 3: 10 orbits; 3 fixed; index 2
(1 3)(5 6)(8 13)(9 12)
level 2: 6 orbits; 1 fixed; index 3
(0 1)(2 3)(4 5)(6 7)(12 13)
level 1: 2 orbits; 0 fixed; index 8
2 orbits; grpsize=48; 3 gens; 10 nodes; maxlev=4
tctotal=20; canupdates=1; cpu time = 0.00 seconds
[fixing partition]
(2 4)(3 5)(10 12)(11 13)
level 3: 10 orbits; 2 fixed; index 2
(1 2)(5 6)(8 10)(9 11)
level 2: 6 orbits; 1 fixed; index 3
(0 1)(2 3)(4 5)(6 7)(8 9)
level 1: 2 orbits; 0 fixed; index 8
2 orbits; grpsize=48; 3 gens; 10 nodes; maxlev=4
tctotal=20; canupdates=1; cpu time = 0.00 seconds
h and h' are identical.
0-0 1-1 2-3 3-2 4-7 5-6 6-5 7-4 8-10 9-11 10-13 11-12 12-9 13-8
What you see is output for a nauty computation which gives you some information about the orbit structure of the automorphism groups of the polytopes involved. Essential is the fourth to last line telling us that both polytopes are, in fact, combinatorially equivalent. The third to last line describes one particular isomorphism: For each vertex (one of the nodes numbered 0 through 7) and each facet (one of the nodes numbered 8 through 13) of Sharir.poly the corresponding vertex or facet of cube.poly is listed. The final two lines are polymake's output indicating that the two polytopes are indeed combinatorially equivalent. Omitting the -v flag in the command line would have resulted in these two output lines only.

2.6. A More Detailed Look. Let us compute the volume of a polytope. And we want to know what polymake actually does in order to obtain the result. Adding the -v or -vv flag to the polymake command line call asks for (very) verbose information. We omit a couple of lines at the beginning of the output, where the system tells about its rule base.

The output below corresponds to computing the VOLUME directly from the INEQUALITIES description. It looks different if you solved that linear optimization problem before.

> polymake -vv linear_program.poly VOLUME
polymake: reading rules from ...
polymake: minimum weight rule chain constructed in 0.054 sec.
polymake: applying rule cdd.convex_hull.dual: VERTICES, POINTED, FEASIBLE :
           FACETS | INEQUALITIES
polymake: applying rule BOUNDED : VERTICES | POINTS
polymake: applying rule PRECONDITION: BOUNDED ( default.volume: VOLUME :
            VERTICES, TRIANGULATION )
polymake: applying rule beneath_beyond.convex_hull.primal, default.triangulation:
           FACETS, AFFINE_HULL, VERTICES_IN_FACETS, DUAL_GRAPH, TRIANGULATION,
          ESSENTIALLY_GENERIC : VERTICES
polymake: applying rule default.volume: VOLUME : VERTICES, TRIANGULATION
VOLUME
47/48

The program which finally produces the volume is very simple-minded: It takes any triangulation and adds up the volumes of the simplices. But before that, polymake does the following: From the rule base, the system infers that it should first call cdd to obtain the vertices from the inequalities via a (dual) convex hull computation. Then it checks whether the input polyhedron is bounded, that is, to check whether the volume is finite; otherwise polymake would abort the computation. As a third step polymake constructs a triangulation, which, in fact, is obtained from calling a second convex hull code, which differs from cdd’s double description method in that it additionally produces a triangulation.

An important feature of polymake is that all intermediate data which are computed are stored into the polytope file. Asking the program a second time for the same thing, or, for something else which had been computed before, gives an immediate answer. The result is read from the file.

> polymake -vv linear_program.poly VOLUME
polymake: reading rules from ...
VOLUME
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polymake employs a special transaction model in order to assert the consistency of the polytope data. It relies on its rule base in the sense that rules whose execution terminates properly are expected to produce valid output. Moreover, polymake has a powerful error recovery mechanism which automatically looks for an alternative way to find the answer to a user’s question in case of the failure of some external program.
3. Applications Overview

There are several different kinds of mathematical objects which polymake can deal with, most notably convex polytopes and finite simplicial complex.

As it was shown in the tutorial section above the system’s behavior is driven by rules. In fact, the part of the system which takes care of applying rules to answer user requests is entirely independent of the mathematical objects and the algorithms. Each class of objects comes with its own set of rules. We survey the more technical aspects in Section 5. Here we focus on the mathematics.

3.1. Convex Polytopes. A convex polytope is the convex hull of finitely many points in \( \mathbb{R}^d \); this is its \textit{V-description}. A basic result in this area says that this notion coincides with those intersections of finitely many affine halfspaces in \( \mathbb{R}^d \) which are bounded (\textit{H-description}). This is sometimes referred to as the \textit{Main Theorem on convex polytopes}. For an introduction to the theory, see Grünbaum [17] or Ziegler [37].

Convex polytopes arise in many mathematical fields as diverse as linear and combinatorial optimization, combinatorial topology, commutative algebra and algebraic geometry. All these areas, in their modern form, pursue algorithmic questions, and this is where polymake proved to be useful. The Section 4 discusses some of these applications in more detail.

In order to deal with polytopes algorithmically often a first step is to apply an effective version of the “Main Theorem”. While a polytope may naturally be given in its H-description (such as in linear programs) it is essential to also obtain a V-representation if one is interested in combinatorial properties. Algorithms which solve this problem are \textit{convex hull algorithms}. Many such algorithms are known and implemented. The running-time that a particular algorithm/implementation requires is known to vary strongly with the class of polytopes which it is applied to; see Avis, Bremner, and Seidel [4] and also [10]. Therefore, polymake offers three different convex hull algorithms for the user to choose. There is one which is built into the system and two more, cdd [14] and lrs [3], respectively, which are accessible via interfaces. Actually, there also interfaces to porta [8] and qhull [6] available, but these are disabled by default. For each call it is possible to specify which algorithm to choose; additionally, the system can freely be configured to generally prefer one algorithm over another.

Convex polytopes have a metric geometry look as well as a combinatorial one. Both views are supported by polymake. What follows first is a list of metric properties which can be computed with the software.

- Gale transformations
- Steiner points
- projective linear transformations
- triangulations
- Voronoi diagrams and Delaunay cell decompositions in arbitrary dimension

Combinatorial properties include:

- fast face lattice construction algorithm; due to Kaibel and Pfetsch [24]
- \(f\)-vector, \(h\)-vector, flag-\(f\)-vector, and \(cd\)-index
- various graph-theoretic properties of the vertex-edge graph
- Altshuler determinant

In addition to these features there is a wide range of standard constructions and visualization functions; e.g., see Figure 3. There is also an interface to Geomview [1].

3.2. Finite Simplicial Complexes. Given a finite vertex set \( V \), a simplicial complex on \( V \) is a subset of \( 2^V \) which is closed with respect to taking subsets. Simplicial complexes form a basic combinatorial concept to capture properties of well-behaved topological spaces. In particular, this way certain parts of topology get within reach of effective methods.

A fundamental problem in topology is to decide whether two given spaces are \textit{homeomorphic}, that is, indistinguishable from the topological point of view, or not. While it can be shown
that this is algorithmically impossible—even for finite simplicial complexes representing 4-dimensional manifolds—it remains a key task to compute algebraic (homotopy) invariants.

**polymake** offers the following:

- simplicial homology and cohomology with integer coefficients
- cup and cap products
- Stiefel-Whitney characteristic classes
- intersection forms of 4-manifolds
- flip-heuristic by Björner and Lutz [7] for detecting spheres

In particular, in view of a celebrated result of Freedman [13], **polymake** is able to solve the homeomorphism problem for combinatorial 4-manifolds which are simply connected. See the survey [20] for some example computations.

### 3.3. Extensions and Related Concepts

The whole **polymake** system is extensible in several ways. Besides adding new functionality to the applications dealing with polytopes and simplicial complexes, it is possible to define entirely new classes of objects with an entirely new set of rules. For the more technical aspects such an extension the reader is referred to Section 5.

Here we list features which are already built into the system but which go beyond standard computations with polytopes or simplicial complexes.

#### 3.3.1. Tight Spans of Finite Metric Spaces

Every tree $T$ with non-negative weights on the edges defines a metric on the nodes of $T$. Conversely, it is easy to reconstruct the tree from such a tree-like metric. The phylogenetic problem in computational biology boils down to the task to derive a sufficiently close tree from any given finite metric space. It is obvious that sometimes there is no tree at all which fits a given metric. Dress et al. [10, 9] devised tight spans as geometric objects which can be assigned to any finite metric space and which capture the deviation from a tree-like metric. Since tight spans can be described as bounded subcomplexes of unbounded polyhedra, **polymake**’s features can be exploited. See Figure 4 for an example.

Sturmfels and Yu [36] recently used TOPCOM [34] and **polymake** to classify tight spans of metric spaces with at most six points.

#### 3.3.2. Curve Reconstruction

If a sufficiently well distributed finite set $S$ of points on a sufficiently smooth planar curve $K$ is given, then it is possible to obtain a polygonal reconstruction of $K$. Amenta, Bern, and Eppstein [2] obtained a curve reconstruction procedure via an iterated Voronoi diagram computation. This beautiful algorithm is implemented in **polymake**; see Figure 5.
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|       | Athens | Berlin | Lisboa | London | Paris | Rome |
|-------|--------|--------|--------|--------|-------|------|
| Athens| 0      | 1535   | 2815   | 1988   | 1827  | 1522 |
| Berlin| 1535   | 0      | 1442   | 727    | 677   | 926  |
| Lisboa| 2815   | 1442   | 0      | 280    | 1125  |      |
| London| 1988   | 727    | 280    | 0      | 870   |      |
| Paris | 1827   | 677    | 1125   | 870    | 0     |      |
| Rome  | 1522   | 926    | 1125   | 870    | 0     |      |

**Figure 4.** Distances (in kilometers) among six European cities and a 3-dimensional visualization of their tight span, which lives in $\mathbb{R}^6$.

**Figure 5.** Planar curve(s) reconstructed from given points via the crust method of Amenta, Bern, and Eppstein [2].

4. Selected Research Projects Using polymake

We survey some projects for which polymake proved to be useful.

4.1. Extremal Combinatorial Properties of Polytopes. What originally inspired the development of the polymake system was the investigation of combinatorial properties of convex polytopes and, in particular, the construction of interesting examples. Over the years polymake experiments helped to settle a number of previously open questions. We list a few of them.

4.1.1. Cubical Polytopes and Spheres. A cubical polytope has only (combinatorial) cubes as its faces. Such objects naturally arise in computational geometry (hexahedral meshes) as well as differential geometry (via normal crossing immersions of hypersurfaces into spheres).

Algorithm complexity questions make it desirable to find out what the cubical polytopes with the most number of faces with a given number of vertices are. The main result of [23] is the construction of so-called neighborly cubical polytopes which are more complex in this respect than previously expected. A second paper on the same subject gives deeper insight and it also describes more general constructions [22].

Schwarz and Ziegler [35] construct a cubical 4-polytope with an odd number of facets. This solves a previously open question.

4.1.2. $f$-Vectors of 4-Polytopes. A classical result, due to Steinitz, characterizes all those triplets $(f_0, f_1, f_2)$ of natural numbers for which there is a 3-dimensional polytope with exactly $f_0$ vertices, $f_1$ edges, and $f_2$ facets. The corresponding question for higher dimensional polytopes is wide open. In a series of papers Ziegler et al. recently made progress as far as the understanding of such $f$-vectors of 4-polytopes is concerned. This progress, once again, is due to the construction of special classes of 4-polytopes, most of which were obtained with the help of polymake. See the survey [38] for an overview.

4.2. Representation Theory of Groups. Any representation $\nu : G \to \text{GL}(\mathbb{R}^n)$ of a finite group $G$ yields a representation polytope $P(\nu)$ as the convex hull of the image in $\mathbb{R}^{n^2}$. This notion is introduced and studied in Guralnick and Perkinson [18]. It turns out that combinatorial
properties of $P(\nu)$ are related to properties of the action of $\nu(G)$ on $\mathbb{R}^n$. For instance, it is shown \cite[Corollary 3.7]{18} that if $\nu$ is a transitive permutation representation then the diameter of the vertex-edge graph of $P(G)$ is bounded by two. This is a far generalization of previous results on the Birkhoff polytopes due to Padberg and Rao \cite{31}.

4.3. \textbf{Gröbner fans.} A key task in algorithmic commutative algebra is the computation of Gröbner bases for a given ideal in a polynomial ring. From the theoretical as well from the practical viewpoint it is essential that the resulting (reduced) Gröbner basis depends on a monomial ordering. For deeper structural investigations it is often useful to be able to understand the set of all Gröbner bases for a given ideal. The Gröbner fan is a polyhedral fan which gives a geometric structure to this set. Bahlool and Takayama \cite{5} used polymake in their quest to extend the technique of Gröbner fans to ideals in other rings, in particular, the ring of formal power series and the homogenized ring of analytic differential operators.

4.4. \textbf{Secondary Polytopes.} Triangulations of polytopes are interesting for various reasons. For instance, they are instrumental in solving certain systems of algebraic equations, via the theory of toric varieties and sparse resultants. The set of all triangulations of a fixed polytope $P$ itself is endowed with the structure of a (typically quite large) polytope, the \textit{secondary polytope} of $P$.

Pfeifle and Rambau report \cite{32} on an implementation for the construction of secondary polytopes based on TOPCOM \cite{34} and polymake.

4.5. \textbf{Enumeration of Small Triangulations of Manifolds.} The minimal number of vertices required to represent a given manifold as a simplicial complex is a quite intricate topological invariant. One approach to obtain this information for a large number of manifolds is by complete enumeration of small triangulations. This has been pursued by Lutz \cite{27} and Köhler and Lutz \cite{25}. polymake was used to compute simplicial homology groups and related invariants.

4.6. \textbf{Computational Biology.} A standard problem in computational biology is to deduce the optimal alignment of two given DNA sequences. Algorithms to solve this problem are known for some time. However, the biologically correct parameters which define what “optimal” means are usually not available. This is a serious obstacle on the way to biologically meaningful results. Typically the computations have to rely on estimates. Thus it had been suggested to keep the parameters “indeterminate” and to compute with the resulting algebraic expressions which then represent probabilities in the underlying statistical models. The impact of various choices of the parameter choices can then be studied in a post-processing step.

While this naturally requires more complicated algorithms, it turns out by work of Pachter and Sturmfels that methods from polyhedral geometry and polymake, in particular, can be employed. This approach is comprehensively covered in the forthcoming book \cite{30}.

5. \textbf{Software Design}

Since its first version from 1997 polymake was —at least partially— re-written several times. In spite of the many changes on the way, the core ideas always remained the same. The first goal was to have a flexible interface structure such that it is possible to interface to as many existing polytope processing software components (developed by other people) as possible. The second goal was scalability in the sense that the system should be useful both for programmers and mere users, and also both for students and expert scientists.

Feeling that one language is not enough for this, this resulted in an object-oriented hybrid design based on the two programming languages C++ and Perl. The borderline is roughly defined as follows: The Perl side takes care of all the object management and their interfaces, while the C++ half harbors the mathematical algorithms. Information exchange between both worlds is subject to a client-server scheme, Perl being the language of the server.
5.1. **Open Objects.** Convex polytopes are represented in the system as a class of objects which are defined by an extendible list of properties. In the current distributed version there are already more than one hundred of these properties defined; they range from the vertices (VERTICES) and facets (FACETS) of a polytope to the list of Steiner points on all the faces (STEINER_POINTS) and the information whether or not the polytope is SIMPLICIAL or CUBICAL.

The client perspective (on the C++ side) is very restricted: The client asks the server for properties of some polytope object and leaves it entirely to the server to decide how these should be obtained. The Perl-written server has a list of rules which specify how to compute properties from the already known ones. For instance, there is a rule which explains how the facets can be computed for a polytope which was initially specified as the convex hull of finitely many points; that is, there is a convex-hull algorithm rule which computes FACETS from POINTS. Actually, as in this case it is the fact, there may be several competing rules computing the same. It is the task of the server to compile admissible sequences of rules (via a Dijkstra type algorithm for determining shortest weighted paths) to fulfill the user’s (or the client’s) requests from the information initially given.

It is fundamental to the design that the set of rules as well as the list of properties known to the system can be expanded and modified. Moreover, the object management is abstract, too; this way it is possible to define entirely new classes of objects and rule bases for them. For instance, simplicial complexes are objects different from polytopes (which actually includes pointed unbounded polyhedra), while tight spans are specializations of polytope objects since they can be described as the bounded subcomplexes of certain unbounded polyhedra; see Section 3.3.1.

5.2. **Scripting.** One way of using polymake is to generate large sets of polytopes and to filter them for individual members with specific properties. Such tasks are easily accomplished by often small Perl scripts which make use of polymake’s object model.

As an example, the code below iterates through all the facets of a given polytope and offers a visualization of the vertex-edge graphs of all the facets; on the way combinatorially equivalent facets are detected and only one representative of each class is shown.

```
application 'polytope';

die "usage: polymake --script show_facets FILE\n" unless @ARGV;

my $p=load($ARGV[0]);
my @list=();

FACETS:
for (my $i=0; $i<$p->N_FACETS; ++$i) {
  my $facet=new Apps::polytope::RationalPolytope("facet #$i");
  Modules::client("facet", $facet, $p, $i, "-relabel");
  foreach my $other_facet (@list) {
    next FACETS if (check_iso($facet, $other_facet));
  }
  push @list, $facet;
}

static_javaview;
$_->VISUAL_GRAPH for @list;

This script show_facets is part of the distribution.
```

5.3. **Other Software.** There are also software packages which use polymake as a sub-system or have interfaces to polymake.

5.3.1. **convex.** Franz has a Maple package for convex polytopes which is used, e.g., for homology computations in toric varieties [12].
5.3.2. **zerOne.** Lübbecke implemented an algorithm for the enumeration of the vertices of a polytope with 01-coordinates given by its H-representation [26].

5.3.3. **OpenXM.** Noro, Ohara, and Takayama [29] are the authors of **OpenXM**, an infrastructure for mathematical communication. This is used to interface among several computer algebra systems.

5.3.4. **Electronic Geometry Models.** Electronic Geometry Models is a refereed electronic journal for the publication of geometric models [21]. Based on XML techniques these models are published in a standardized way. Some of these models describe polytopes, and these are given in **polymake** format; see the example in Figure 6.

The journal is freely accessible at [http://www.eg-models.de](http://www.eg-models.de).

![Figure 6. 3-Polytope with four edges at each vertex, that can not be placed in convex position with all vertices on the surface of a sphere.](http://www.eg-models.de)

6. **Technical Requirements**

**polymake** can be used on UNIX systems only. It has been successfully tested on Linux, Sun Solaris, FreeBSD, MacOS X, IBM AIX and Tru64 Unix. Depending on the size of your objects **polymake** can run on small machines with, say, 128 MB of RAM. Only to compile the system from the source code at least 1 GB of RAM is required.

Our website at [http://www.math.tu-berlin.de/polymake](http://www.math.tu-berlin.de/polymake) offers the full source code as well as several precompiled versions for download.

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