Correction to: Explicit upper bound for the average number of divisors of irreducible quadratic polynomials

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Abstract
Consider the divisor sum \( \sum_{n \leq N} \tau(n^2 + 2bn + c) \) for integers \( b \) and \( c \). We improve the explicit upper bound of this average divisor sum in certain cases, and as an application, we give an improvement in the maximal possible number of \( D(-1) \)-quadruples. The new tool is a numerically explicit Pólya–Vinogradov inequality, which has not been formulated explicitly before but is essentially due to Frolenkov–Soundararajan.

Keywords
Number of divisors · Quadratic polynomial · Character sums

Mathematics Subject Classification
Primary 11N56; Secondary 11D09

Let \( \tau(n) \) denote the number of positive divisors of the integer \( n \). In [3], we provided an explicit upper bound for the sum \( \sum_{n=1}^{N} \tau(n^2 + 2bn + c) \) under certain conditions on the discriminant, and we gave an application for the maximal possible number of \( D(-1) \)-quadruples.

The aim of this addendum is to announce improvements in the results from [3]. We start with sharpening of Theorem 2 [3].

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Theorem 2A Let \( f(n) = n^2 + 2bn + c \) for integers \( b \) and \( c \), such that the discriminant \( \delta := b^2 - c \) is nonzero and square-free, and \( \delta \not\equiv 1 \pmod{4} \). Assume also that for \( n \geq 1 \) the function \( f(n) \) is nonnegative. Then for any \( N \geq 1 \) satisfying \( f(N) \geq f(1) \), and \( X := \sqrt{f(N)} \), we have the inequality

\[
\sum_{n=1}^{N} \tau(n^2 + 2bn + c) \leq \frac{2}{\zeta(2)} L(1, \chi) N \log X + \left( 2.332 L(1, \chi) + \frac{4M_\delta}{\zeta(2)} \right) N + \frac{2M_\delta}{\zeta(2)} X + 4\sqrt{3} \left( 1 - \frac{1}{\zeta(2)} \right) M_\delta \frac{N}{\sqrt{X}} + 2\sqrt{3} \left( 1 - \frac{1}{\zeta(2)} \right) M_\delta \sqrt{X}
\]

where \( \chi(n) = \left( \frac{4\delta}{n} \right) \) for the Kronecker symbol \( (\cdot) \) and

\[
M_\delta = \begin{cases} 
\frac{4\pi}{\delta^{1/2}} \log 4\delta + 1.8934\delta^{1/2} + 1.668, & \text{if } \delta > 0; \\
\frac{1}{\pi} |\delta|^{1/2} \log 4|\delta| + 1.6408|\delta|^{1/2} + 1.0285, & \text{if } \delta < 0.
\end{cases}
\]

In the case of the polynomial \( f(n) = n^2 + 1 \), we can give an improvement to Corollary 3 from [3].

Corollary 3A For any integer \( N \geq 1 \), we have

\[
\sum_{n \leq N} \tau(n^2 + 1) < \frac{3}{\pi} N \log N + 3.0475N + 1.3586\sqrt{N}.
\]

Just as in [2,3], we have an application of the latter inequality in estimating the maximal possible number of \( D(-1) \)-quadruples, whereas it is conjectured there are none. We can reduce this number from \( 4.7 \cdot 10^{58} \) in [2] and \( 3.713 \cdot 10^{58} \) in [3] to the following bound.

Corollary 4A There are at most \( 3.677 \cdot 10^{58} \) \( D(-1) \)-quadruples.

The improvements announced above are achieved by using more powerful explicit estimates than the ones used in [3]. More precisely, the results are obtained when instead of Lemma 2 and Lemma 3 from [3] we plug in the proof the following stronger results.

Lemma 2A For any integer \( N \geq 1 \) we have

\[
\sum_{n \leq N} \mu^2(n) = \frac{N}{\zeta(2)} + E_1(N),
\]

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where $|E_1(N)| \leq \sqrt{3} \left( 1 - \frac{1}{\xi(2)} \right) \sqrt{N} < 0.6793 \sqrt{N}$.

**Proof** This is inequality (10) from Moser and MacLeod [4]. \qed

The following numerically explicit Pólya–Vinogradov inequality is essentially proven by Frolenkov and Soundararajan [1], though it was not formulated explicitly. It supersedes the main result of Pomerance [5], which was formulated as Lemma 3 in [3].

**Lemma 3A** Let

$$M_\chi := \max_{L,P} \left| \sum_{n=L}^{P} \chi(n) \right|$$

for a primitive character $\chi$ to the modulus $q > 1$. Then

$$M_\chi \leq \begin{cases} 
\frac{2}{\pi} q^{1/2} \log q + 0.9467 q^{1/2} + 1.668, & \chi \text{ even}; \\
\frac{1}{2\pi} q^{1/2} \log q + 0.8204 q^{1/2} + 1.0285, & \chi \text{ odd}.
\end{cases}$$

**Proof** Both inequalities for $M_\chi$ are shown to hold by Frolenkov and Soundararajan in the course of the proof of their Theorem 2 [1] as long as a certain parameter $L$ satisfies $1 \leq L \leq q$ and $L = \left\lfloor \frac{\pi^2}{4\sqrt{q}} + 9.15 \right\rfloor$ for $\chi$ even, $L = \left\lfloor \pi \sqrt{q} + 9.15 \right\rfloor$ for $\chi$ odd. Thus both inequalities for $M_\chi$ hold when $q > 25$.

Then we have a look of the maximal possible values of $M_\chi$ when $q \leq 25$ from a data sheet, provided by Leo Goldmakher. It represents the same computations of Bober and Goldmakher used by Pomerance [5]. We see that the right-hand side of the bounds of Frolenkov–Soundararajan for any $q \leq 25$ is larger than the maximal value of $M_\chi$ for any primitive Dirichlet character $\chi$ of modulus $q$. This proves the lemma. \qed

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