Wedge domains in compactly causal symmetric spaces

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Dedicated to the memory of Ottmar Loos

Abstract

Motivated by construction in Algebraic Quantum Field Theory we introduce wedge domains in compactly causal symmetric spaces $M = G/H$, which includes in particular anti de Sitter space in all dimensions and its coverings. Our wedge domains generalize Rindler wedges in Minkowski space. The key geometric structure we use is the modular flow on $M$ defined by an Euler element in the Lie algebra of $G$. Our main geometric result asserts that three seemingly different characterizations of these domains coincide: the positivity domain of the modular vector field; the domain specified by a KMS like analytic extension condition for the modular flow; and the domain specified by a polar decomposition in terms of certain cones.

In the second half of the article we show that our wedge domains share important properties with wedge domains in Minkowski space. If $G$ is semisimple, there exist unitary representations $(U, H)$ of $G$ and isotope covariant nets of real subspaces $H(O) \subseteq H$, defined for any open subset $O \subseteq M$, which assign to connected components of the wedge domains a standard subspace whose modular group corresponds to the modular flow on $M$. This corresponds to the Bisognano–Wichmann property in Quantum Field Theory. We also show that the set of $G$-translates of the connected components of the wedge domain provides a geometric realization of the abstract wedge space introduced by the first author and V. Morinelli.

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Let $M = G/H$ be a homogeneous space of the Lie group $G$ and $(U, H)$ a unitary representation of $G$. Motivated by their occurrence in Algebraic Quantum Field Theory (AQFT), in the sense of Haag–Kastler ([Ha96]), one considers nets of real subspaces on $M$, i.e., to each open subset $O \subseteq M$, we assign a closed real subspace $H(O) \subseteq H$ satisfying

- Isotony: $O_1 \subseteq O_2 \Rightarrow H(O_1) \subseteq H(O_2)$,
- Equivariance: $H(gO) = U(g)H(O)$ for $g \in G$.

If $(W(\tau))_{\tau \in \mathbb{H}}$ are the Weyl operators on bosonic Fock space $\mathcal{F}_+(\mathcal{H})$, then we assign to each closed real subspace $E \subseteq H$ the von Neumann algebra $\mathcal{R}(E) := W(E)'$ ([Si74]). We thus obtain on $M$ the isotone covariant net $\mathcal{R}(\mathcal{H}(O))_{O \subseteq M}$ of von Neumann algebras, where covariance refers to the canonical representation of $G$ on $\mathcal{F}_+(\mathcal{H})$. This method has been developed by Araki and Woods in the context of free bosonic quantum fields ([Ar63, Ar64, AW63, AW68]), but it can be modified to deal with fermionic Fock spaces (cf. [EO73, BJL02]), and there are variants for other statistics (anyons) ([Schr97, Le15, §3]).

We are interested in structures on homogeneous spaces which resemble space-times in AQFT. In this context the von Neumann algebra associated to an open subset $O \subseteq M$ corresponds to observables measurable in the “laboratory” $O$ ([Ha96]). One often focuses on those
von Neumann algebras $R(V)$ for which the “vacuum vector” $\Omega \in F_+(H)$ is cyclic and separating, so that the Tomita–Takesaki Theorem applies to $(R(V), \Omega)$ [BR87, Thm. 2.5.14]. It is not hard to see that $\Omega$ is.

- cyclic for $R(V)$ if and only if $V + iV$ is dense in $H$,
- separating for $R(V)$ if and only if $V \cap iV = \{0\}$.

A closed real subspace $V \subseteq H$ with both properties is called standard (cf. [Lo18] for the basic theory of standard subspaces). It is therefore of particular interest to understand how to construct nets for which the subspaces $H(\mathcal{O})$ are standard. The main feature of Tomita–Takesaki theory is that it provides a modular conjugation and a modular automorphism group, and these can already be associated to a standard subspace $V \subseteq H$ (and pass through second quantization to the von Neumann context). Concretely, we associate to $V$ a pair of modular objects $(\Delta_V, J_V)$:

- the modular operator $\Delta_V$ is a positive selfadjoint operator,
- $J_V$ is a conjugation (an antiunitary involution),

and these two operators satisfy the modular relation $J_V \Delta_V J_V = \Delta_V^{-1}$. The pair $(\Delta_V, J_V)$ is obtained by the polar decomposition $\sigma_V = J_V \Delta_V^{1/2}$ of the closed operator

$$\sigma_V : V + iV \to H, \quad x + iy \mapsto x - iy$$

with $V = \text{Fix}(\sigma_V)$. Key properties of these operators are that

$$J_V V = V' := \{ w \in H : (\forall v \in V) \text{ Im}(v, w) = 0 \} \quad \text{and} \quad \Delta_V^t V = V \quad \text{for} \quad t \in \mathbb{R}.$$

So we obtain a one-parameter group of automorphisms of $V$ (the modular group) and a symmetry between $V$ and its commutant $V'$, implemented by $J_V$.

The current interest in standard subspaces arose in the 1990s from the work of Borchers and Wiesbrock [Bn92, W93]. This led to the concept of modular localization in Quantum Field Theory introduced by Brunetti, Guido and Longo [BGL02, BGL03]; see also [BDFS00] and [Le15, LL15] for important applications of this technique.

For a net on a homogeneous space $G/H$, the domains $\mathcal{O}$ for which $H(\mathcal{O})$ is standard are of particular relevance. Here one would like to know when the modular group $(\Delta^t_G)_{t \in \mathbb{R}}$ is “geometric” in the sense that it is implemented by a one-parameter subgroup of $G$, hence corresponds to a one-parameter group of symmetries of $M$ preserving the domain $\mathcal{O}$. For the modular conjugation $J_\mathcal{O}$, we may likewise ask for the existence of an involution $\tau^M$ on $M$ satisfying $H(\tau^M \mathcal{O}) = H(\mathcal{O})'$. Building on [NÔ21a], which deals with the case $H = \{e\}$, i.e., left invariant nets on Lie groups, we turn in this paper to nets on a class of symmetric spaces to which the tools and methods developed in [NÔ21a] apply particularly well; see also the companion paper [NÔ21b].

First we explain how to find natural standard subspaces for unitary representations. So let $(U, H)$ be a unitary representation of $G$ which extends to an antiunitary representation of the extended group $G_e = G \rtimes \{\text{id}_G, \tau^G\}$, where $\tau^G$ is an involutive automorphism of $G$. Then $J := U(\tau^G)$ is a conjugation satisfying $J U(g) J = U(\tau^G(g))$ for $g \in G$. We then obtain for each pair $(h, \tau^G)$, for which $h$ is fixed by the Lie algebra involution $\tau$ induced by $\tau^G$, a standard subspace $V := V(h, \tau^G, U) = \text{Fix}(J_U \Delta^1_U)$, specified by

$$J_U = U(\tau^G) \quad \text{and} \quad \Delta_V^{-it^2/2} = U(\exp t\mathbf{h}) \quad \text{for} \quad t \in \mathbb{R}. \quad (1)$$

This assignment is called the Brunetti–Guido–Longo (BGL) construction (see [BGL02]). As a consequence, standard subspaces can be associated to antiunitary representations in abundance, but only a few of them carry interesting geometric information. In particular, we would like
to understand when a standard subspace of the form $V_{(h,\tau,C,U)}$ arises from a net on some homogeneous space $G/H$.

A structural property with strong impact in this context is the spectrum condition on the infinitesimal generators $\partial U(x)$ of the one-parameter groups $(U(\exp tx))_{t\in\mathbb{R}}$. This is the requirement that the closed convex cone

$$C_U := \{x \in \mathfrak{g} : -i\partial U(x) \geq 0\}$$

(the positive cone of $U$) is “large” in the sense that the ideal $\mathfrak{g}_C := C_U - C_U$ satisfies $\mathfrak{g} = \mathfrak{g}_C U + \mathbb{R} h$. Let $V = V_{(h,\tau,C,U)}$ be a standard subspace obtained from the BGL construction as in [1]. Then it is a natural requirement that the semigroup

$$S_U = \{g \in G : U(g)V \subseteq V\}$$

of endomorphisms of $V$ is large in the sense that its Lie wedge

$$L(S_U) = \{x \in \mathfrak{g} : \exp(\mathbb{R} x) \subseteq S_U\}$$

(the set of infinitesimal generators of one-parameter subsemigroups of $S_U$) spans the Lie algebra $\mathfrak{g}$. In [Ne21] it is shown that if $L(S_U)$ spans $\mathfrak{g}$, then $\text{ad} h$ defines a 3-grading in the sense that

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h) \quad \text{for} \quad \mathfrak{g}_j(h) := \ker(\text{ad} h - j \text{id}_\mathfrak{g}),$$

and

$$\text{Ad}(\tau^C) = \tau_h,$$

so that $\mathfrak{g}_0(h) = \tau_h^\mathfrak{g}$ and $\mathfrak{g}_1(h) + \mathfrak{g}_{-1}(h) = \mathfrak{g}^{-\tau_h}$. We call elements $h$ with [3] Euler elements because they correspond to the Euler vector field on an open subset of the homogeneous spaces with tangent space $\mathfrak{g}/(\mathfrak{g}_0(h) + \mathfrak{g}_{-1}(h))$. Assuming that $h$ is an Euler element and $\tau = \tau_h$, the semigroup $S_U$ has been completely determined in [Ne19]. To describe the structure of $S_U$, let

$$C_{\pm} := \pm C_U \cap \mathfrak{g}_{\pm 1}(h) \quad \text{and write} \quad G_V = \{g \in G : U(g)V = V\}$$

for the stabilizer group of $V$ in $G$. Then

$$S_U = \exp(C_+)G_V \exp(C_-) = G_V \exp(C_+ + C_-).$$

Combining Euler elements with symmetric spaces leads to the following concept: A modular causal symmetric Lie algebra is a quadruple $(\mathfrak{g},\tau,C,h)$, where

- $(\mathfrak{g},\tau)$ is a symmetric Lie algebra, i.e., $\mathfrak{g}$ is a finite-dimensional real Lie algebra and $\tau$ an involutive automorphism of $\mathfrak{g}$.
- $(\mathfrak{g},\tau,C)$ is a causal symmetric Lie algebra, i.e., $C \subseteq \mathfrak{g}^{-\tau}$ is a pointed generating closed convex cone invariant under the group $\text{Inn}_\mathfrak{g}(h) = \langle e^{\text{ad} h} \rangle$.
- $h \in \mathfrak{g}^r$ is an Euler element, i.e., $\text{ad} h$ is diagonalizable with eigenvalues $\{-1,0,1\}$, and the involution $\tau_h := e^{\pi i \text{ad} h} \in \text{Aut}(\mathfrak{g})$ satisfies $\tau_h(C) = -C$.

A causal symmetric Lie algebra $(\mathfrak{g},\tau,C)$ is called compactly causal (cc for short) if the cone $C$ is elliptic, i.e., if its interior consists of elements $x$ which are elliptic in the sense that $\text{ad} x$ is semisimple with purely imaginary spectrum. Typical examples arise from invariant pointed generating cones $C_0 \subseteq \mathfrak{g}$. Then

$$(\mathfrak{g} \oplus \mathfrak{g},\tau_{\text{up}},C_0) \quad \text{with} \quad \tau_{\text{up}}(x,y) = (y,x),$$

is a compactly causal symmetric Lie algebra; called of group type (GT). Another interesting class of compactly causal symmetric Lie algebras arises from the involution $\tau_h = e^{\pi i \text{ad} h}$ defined by an Euler element. If $C_0 \subseteq \mathfrak{g}$ satisfies $-\tau_h(C_0) = C_0$, then $(\mathfrak{g},\tau_h,C_0^{\tau_h},h)$ is a modular compactly causal symmetric Lie algebra; called of Cayley type (CT).
Irreducible compactly causal symmetric Lie algebras \((\mathfrak{g}, \tau, C)\) are either of group type with \(\mathfrak{g}^+\) simple hermitian, or \(\mathfrak{g}\) is simple hermitian, and \(\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}^+\) holds for a \(\tau\)-invariant Cartan decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\). We refer to [H"O97] for a classification. The additional assumption that \(\mathfrak{h}\) contains an Euler element leaves us with the following four types:

(GT) Group type: \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}\), \(\mathfrak{h}\) simple hermitian of tube type

\[ \mathfrak{h} = \mathfrak{su}_{r,r}(\mathbb{C}), \ \mathfrak{sp}_{2r}(\mathbb{R}), \ \mathfrak{so}_{2,d}(\mathbb{R}), \ \mathfrak{so}^+(4r), \ \mathfrak{e}_7(-25). \]

(CT) The Euler element \(h\) is central in \(\mathfrak{h}\) and \(\tau = \tau_h\):

\[
\begin{array}{|c|c|c|c|c|}
\hline
\mathfrak{g} & \mathfrak{su}_{r,r}(\mathbb{C}) & \mathfrak{sp}_{2r}(\mathbb{R}) & \mathfrak{so}_{2,d}(\mathbb{R}), d > 2 & \mathfrak{so}^+(4r) & \mathfrak{e}_7(-25) \\
\hline
\mathfrak{h} & \mathbb{R} \oplus \mathfrak{sl}(\mathbb{C}) & \mathbb{R} \oplus \mathfrak{sl}(\mathbb{R}) & \mathbb{R} \oplus \mathfrak{so}_{1,d-1}(\mathbb{R}) & \mathbb{R} \oplus \mathfrak{sl}(\mathbb{H}) & \mathbb{R} \oplus \mathfrak{e}_6(-26) \\
\hline
\end{array}
\]

(ST) Split types: \(\tau \neq \tau_h\) and \(\text{rk}_{\mathfrak{h}} \mathfrak{h} = \text{rk}_{\mathfrak{g}} \mathfrak{g}\)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\mathfrak{g} & \mathfrak{su}_{r,r}(\mathbb{C}) & \mathfrak{so}_{2,p+q}(\mathbb{R}) & \mathfrak{so}^+(4r) & \mathfrak{e}_7(-25) \\
\hline
\mathfrak{h} & \mathfrak{so}_{r,r}(\mathbb{R}) & \mathfrak{so}_{1,p}(\mathbb{R}) \oplus \mathfrak{so}_{1,q}(\mathbb{R}) & \mathfrak{so}_{2d}(\mathbb{C}) & \mathfrak{so}_{4}(\mathbb{H}) \\
\hline
\end{array}
\]

(NST) Non-split types: \(\tau \neq \tau_h\), \(\text{rank}_{\mathfrak{h}} \mathfrak{h} = \frac{\text{rank}_{\mathfrak{g}} \mathfrak{g}}{2}\)

\[
\begin{array}{|c|c|c|}
\hline
\mathfrak{g} & \mathfrak{su}_{2s,2s}(\mathbb{C}) & \mathfrak{sp}_{2s}(\mathbb{C}) \\
\hline
\mathfrak{h} & \mathfrak{su}_{s,s}(\mathbb{C}) & \mathfrak{sp}_{s}(\mathbb{C}) \\
\hline
\end{array}
\]

We introduce “wedge domains” in compactly causal symmetric spaces \(M = G/H\), specified by a modular causal symmetric Lie algebra \((\mathfrak{g}, \tau, C, h)\) and a connected Lie group \(G\) with Lie algebra \(\mathfrak{g}\) on which \(\tau\) induces an involutive automorphism \(\tau^G\) for which \(H \subseteq G^\tau\) is an open subgroup. We then obtain a one-parameter group \(\alpha_t := \exp(\text{ad} h)\) of automorphisms of \(\mathfrak{g}\), inducing automorphisms of \(G\) and \(M\).

Let \(V_+(gH) := g.C^0 \subseteq T_{gH}(M)\) denote the open cones defining the causal structure on \(M\) and let \(X^M_h \in \mathcal{V}(M)\) be the modular vector field defined by

\[
X^M_h(gH) := \left. \frac{d}{dt} \right|_{t=0} \alpha_t(g)H = \left. \frac{d}{dt} \right|_{t=0} \exp(th)gH.
\]

We introduce the following domains associated to this data:

- The positivity domain of the vector filed \(X^M_h\) in \(M\):

\[
W^+(M)(h) := \{ m \in M : X^M_h(m) \in V_+(m) \}.
\]

- The KMS wedge domain

\[
W^{\text{KMS}}_M(h) := \{ m \in M : (\forall z \in S_\tau) \alpha_z(m) \in T_M \},
\]

where \((\alpha_z)_{z \in \mathbb{C}}\) denotes the holomorphic extension of the modular flow to the complex symmetric space \(M_C\) and

\[
T_M := G \times_H iC^0
\]

is the tube domain of \(M\), a complex manifold containing \(M \cong G \times_H \{0\}\) in its “boundary”.

---

1The real rank \(\text{rank}_{\mathfrak{h}}(\mathfrak{g})\) of a reductive Lie algebra \(\mathfrak{g}\) is the dimension of a maximal abelian ad-diagonalizable subspace.
• The polar wedge domain

\[ W_M(h) := \bigcup_{m \in M^a} \text{Exp}_m((C_m)^0), \]

where \( M^a \) is the submanifold of fixed points of the modular flow on \( M \),

\[ C_m := V_+(m) \subseteq T_m(M), \]

and

\[ C_m^0 := C_m \cap T_m(M)_{-1} \subseteq T_m(M), \]

where \( T_m(M), j = -1, 0, 1 \) denote the eigenspaces of the generator of the modular flow on \( T_m(M) \).

Our main geometric result, which is proved after various preparations in Section 6 (Theorem 6.6), asserts that these three open subsets coincide:

\[ W_M^+(h) = W_M^{\text{KMS}}(h) = W_M(h). \]

For this result we assume that the causal symmetric Lie algebra \((\mathfrak{g},\tau,C)\) is extendable, i.e., there exists a pointed closed convex invariant cone \( C_\mathfrak{g} \subseteq \mathfrak{g} \) satisfying

\[ C_\mathfrak{g} \cap \mathfrak{q} = C \quad \text{and} \quad -\tau(C_\mathfrak{g}) = C_\mathfrak{g}. \]

If \( \mathfrak{g} \) is reductive and \( \mathfrak{h} \) contains no non-zero ideals (which is a very natural assumption as these ideals act trivially on \( M \)), then \((\mathfrak{g},\tau,C)\) is extendable (Theorem 2.4), but there are also interesting non-extendible examples. As we show in [NÔ21], the companion paper dealing with wedge domains in non-compactly causal symmetric spaces, the geometry of wedge domains in a non-compactly causal space is more complicated because these domains are generated from pieces of closed geodesics, the corresponding polar map has singularities and it is defined only on a subset of an open cone.

To connect this geometric insight with nets of standard subspaces, we show in our second main result (Theorem 7.5) that for any antiunitary representation \((U,\mathcal{H})\) of \( G_\mathfrak{h} = G \rtimes \{1,\tau_\mathfrak{h}^0\} \) whose positive cone \( C_U \) is pointed and generating and satisfies \( C = C_U \cap \mathfrak{q} \), every cyclic distribution vector \( \eta \in \mathcal{H}^{-\infty} \) fixed by \( H \) and the conjugation \( J = U(\tau_\mathfrak{h}^0) \) leads to a natural covariant isotone net \( H^M \) on \( M \) assigning to the connected component \( W_M(h)_{\mathcal{H}} \) of the wedge domain the standard subspace \( V_{(h,\tau_\mathfrak{h}^0,\mathcal{H})} \). For classification results concerning this class of representations, we refer to [KNÔ97].

Our third main result (Theorem 8.32) provides a concrete construction of such representations and thus implies that such nets exist on all reductive compactly causal modular symmetric spaces. Starting with a \( C_\mathfrak{g} \)-positive antiunitary representation \((\rho,K)\) of \( G_\mathfrak{h} \) which is a finite sum of irreducible ones, we obtain on the space \( B_2(K) \) of Hilbert–Schmidt operators on \( K \) a representation by \( U(g)A := \rho(g)AP(\tau_\mathfrak{h}^0(g))^{-1} \) and the trace defines a distribution vector which generates a subspace \( \mathcal{H}_\rho \), so that Theorem 8.3 applies to \((U,\mathcal{H}_\rho)\). We even obtain a rather direct description of \( \mathcal{V} \) as the closed real subspace generated by \( \rho(C_\mathfrak{g}^{\infty}(S_{\mathfrak{g},\mathbb{R}})) \subseteq B_1(K) \subseteq B_2(K) \).

Here we use that \( \rho \) is a trace class representation (see Section 8 for details).

Motivated by the analysis of abstract wedge spaces in [MN21], we prove in Section 9 that the wedge space \( W(M,h) \) of all \( G \)-translates of the connected component \( W := W_M(h)_{\mathcal{H}} \) actually is a non-compactly causal parahermitian symmetric space. It is a covering of the adjoint orbit \( O_h = \text{Ad}(G)h \subseteq \mathfrak{g} \). The most difficult part of the proof is to get hold of the stabilizer group \( G_W = \{ g \in G : gW = W \} \). It is an interesting open problem to connect the covering theory and the corresponding twisted duality developed in [MN21] to the geometry of coverings of compactly causal symmetric spaces \( G/H \).

We conclude this paper with a section on the anti de Sitter space \( \text{AdS}^d \), the prototypical Lorentzian example of a compactly causal symmetric space. This section can be read independently. For this example we evaluate all geometric data explicitly and show directly that the
three types of wedge domains coincide, without referring to the elaborate structure theory used in the rest of the paper.

**Structure of this paper:** We start in Section 2 by recalling basic facts on causal symmetric spaces on the Lie algebra level (Subsections 2.1 to 2.3) and also on the global level (Subsection 2.4).

In Subsection 3.1 we classify for a reductive compactly causal symmetric reductive Lie algebra \((g, \tau)\) the orbits of \(\text{Im}_h(h)\) in the set of Euler elements in \(h\). As this set describes the different choices of the Euler element \(h \in h\), this can be seen as a classification of all "modular structures" on the associated symmetric spaces. In Subsection 3.2 we show that the orbits of the centralizer group \(G^h\) in \(M^h\) are open and that this leads to a natural bijection between the orbit spaces \(M^h/G^h\) and \((O^h \cap h)/H\). Actually both sets correspond to the set of double cosets \(G^h \cap H \subseteq G\) for which \(\text{Ad}(g)^{-1}h \in h\).

In Section 4 we introduce the three types of wedge domains in a compactly causal symmetric space \(M = G/H\). In our analysis of wedge domains we follow the strategy to first study spaces of group type and then use embeddings into these spaces to derive corresponding results in general. In Section 5 we start with causal symmetric spaces of group type, i.e., pairs \((G, C_g)\) of a connected Lie group \(G\) and a pointed generating invariant cone \(C_g \subseteq g\).

In Section 6 we turn to wedge domains in compactly causal symmetric spaces. In this context we assume that \((g, \tau, C)\) is extendable in the sense of Subsection 2.1, keeping in mind that this is always the case if \(g\) is reductive. The main result of this section is Theorem 6.5 asserting that the three wedge domains in \(M\) coincide. This is first proved for the special case \(H = G^G\) in Theorem 6.3. The proof of this theorem builds heavily on the group case (Theorem 5.2). We conclude this section with a brief discussion of the assumption that the Euler element \(h\) is contained in \(\mathfrak{h}\), i.e., that the corresponding modular flow on \(M\) has a fixed point. Note that the definition of \(W_M(h)\) makes no sense if \(M^h = \emptyset\).

In Section 7 we turn to representation theoretic aspects of compactly causal symmetric spaces. We start with introducing standard subspaces, the Brunetti–Guido–Longo construction and recall the construction of nets of closed real subspaces from distribution vectors, introduced in [NO21a]. In Subsection 7.3 we describe in the general Theorem 7.5 how the methods from [NO21a] can be used to construct covariant nets of standard subspaces on compactly causal symmetric spaces. In Section 8 we construct such representations explicitly in spaces of Hilbert–Schmidt operators \(\mathcal{H}_\rho \subseteq B_2(K)\), where \((\rho, K)\) is an antiunitary representation of \(G^G\) which is a finite sum of irreducible representations. This is done in three steps: First we recall from Ne00 [Ne19] some results on the representations \((\rho, K)\) of \(G^h = G \times \{1, \tau^0\}\), then we use these representations to construct nets of standard subspaces on the symmetric space of group type \(G \cong (G \times G)/\Delta_G\), and finally we use the twisted embedding \(G \rightarrow G \times G, g \mapsto (g, \tau^0(g))\) to obtain pullbacks representations of \(G^h\) that can be used to obtain with Theorem 7.5 nets of standard subspaces on \(M = G/H\).

In Section 9 we return to a geometric topic. We show that, under rather natural assumptions, the wedge space \(W(M, h)\) of all \(G\)-translates of the connected component \(W_M(h)_{<0}\) of the wedge domain in \(M\) carries the structure of an ordered symmetric space. We conclude this paper with a short section discussing some open problems and two appendices on some useful facts on symmetric Lie groups and on distribution vectors of unitary representations.

**Notation:**
- \(g, \mathfrak{h}\) and \(\mathfrak{k}\) will denote Lie algebras of the Lie groups \(G, H, K\).
- An element \(h \in g\) is called an Euler element if \(\text{ad} h\) is non-zero and diagonalizable with \(\text{Spec}(\text{ad} h) \subseteq \{-1, 0, 1\}\). The set of Euler elements in \(g\) is denoted by \(\mathcal{E}(g)\).
- For \(h \in \mathcal{E}(g)\) we write \(\tau_h := e^{\pi \text{ad} h}\) for the corresponding involution on \(g\) and note that \(\kappa_h := e^{-\pi \text{ad} h}\) is an automorphism of \(g_\mathcal{C}\) of order 4 with \(\kappa_h^2 = \tau_h\) which implements the
“Wick rotation” (partial Cayley transform).

- For $h \in \mathfrak{g}$, $\lambda \in \mathbb{R}$, and $I \subseteq \mathfrak{g}$ an ad $h$ invariant subspace we write $I,(h) := \ker(\ad_h|_I - \lambda 1)$ for the corresponding eigenspace in the adjoint representation.

- We write $e \in G$ for the identity element in the Lie group $G$, $G_e$ for its identity component and $Z(G)$ for its center.

- For the left and right translations on the tangent bundle $T(G)$, we write $e.g$ and $v.g$ for $g \in G$, $v \in T(G)$, respectively.

- For $x \in \mathfrak{g}$, we write $G^x := \{ g \in G : \Ad(g)x = x \}$ for the stabilizer of $x$ in the adjoint representation and $G^x_e := (G^x)_e$ for its identity component.

- If $\tau^G \in \text{Aut}(G)$ is an involution, then we write $G_{\tau} := G \times \{ \text{id}_G, \tau^G \}$, $g^\tau := \tau^G(g)^{-1}$, and $G^\tau := \{ g \in G : g^\tau = g \}$.

- For a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we write $\text{Inn}_\mathfrak{g}(\mathfrak{h}) = (e^{ad h}) \subseteq \text{Aut}(\mathfrak{g})$ for the corresponding group of inner automorphism and put $\text{Inn}(\mathfrak{g}) := (e^{ad \theta})$.

- For a (continuous) unitary representation $(\mathcal{U}, \mathcal{H})$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$, we write $\partial \mathcal{U}(x)$ for the skew-adjoint infinitesimal generator of the unitary one-parameter group $\mathcal{U}(\exp tx)$, so that we have $\mathcal{U}(\exp tx) = e^{i \partial \mathcal{U}(x)}$ for $t \in \mathbb{R}$. The positive cone of $U$ is the $\text{Ad}(G)$-invariant closed convex cone

$$C_U := \{ x \in \mathfrak{g} : -i \partial \mathcal{U}(x) \geq 0 \}. \quad (8)$$

- For a complex Hilbert space, we write $\text{AU}(\mathcal{H})$ for the group of unitary and antiunitary operators.

- For a function $f : G \to \mathbb{C}$, we write $f^\tau(g) := f(g^{-1})$.

## 2 Causal symmetric Lie algebras

In this section we collect some basic facts and definitions concerning causal symmetric spaces. Our main reference for these spaces is [HÖ97] for the semisimple situation and [HÖ93] for the general case.

**Definition 2.1.** For a symmetric Lie algebra $(\mathfrak{g}, \tau)$, we write

$$\mathfrak{h} := \mathfrak{g}^\tau = \{ x \in \mathfrak{g} : \tau(x) = x \} \quad \text{and} \quad \mathfrak{q} := \mathfrak{g}^{-\tau} = \{ x \in \mathfrak{g} : \tau(x) = -x \}.$$

If $\mathfrak{g}$ is semisimple, then there exists a Cartan involution $\theta$ commuting with $\tau$, and all these Cartan involutions are mutually conjugate under $\text{Inn}_\mathfrak{g}(\mathfrak{h})$ ([KN90 Prop. I.5(iii)]). If $\theta$ is a Cartan involution commuting with $\tau$, then we write $\mathfrak{t} := \mathfrak{g}^\theta$ and $\mathfrak{p} := \mathfrak{g}^{-\theta}$ for the $\theta$-eigenspaces.

Now $\mathfrak{g}$ decomposes as follows into simultaneous $(\tau, \theta)$-eigenspaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{t} \oplus \mathfrak{p} = \mathfrak{h}_t \oplus \mathfrak{h}_p \oplus \mathfrak{q}_t \oplus \mathfrak{q}_p \quad (9)$$

with

$$\mathfrak{h}_t := \mathfrak{h} \cap \mathfrak{t}, \quad \mathfrak{h}_p := \mathfrak{h} \cap \mathfrak{p}, \quad \mathfrak{q}_t := \mathfrak{q} \cap \mathfrak{t}, \quad \mathfrak{q}_p := \mathfrak{q} \cap \mathfrak{p}.$$

We call $(\mathfrak{g}^\tau, \tau^\circ)$ with $\mathfrak{g}^\tau = \mathfrak{h} + i \mathfrak{q}$ and $\tau^\circ(x + iy) = x - iy$ for $x \in \mathfrak{h}, y \in \mathfrak{q}$, the $c$-dual symmetric Lie algebra.

**Definition 2.2.** Let $(\mathfrak{g}, \tau)$ be a symmetric Lie algebra.  

---

2Here we extend concepts typically used for irreducible semisimple symmetric Lie algebras to a more general context. This is motivated by our heavy use of embeddings, which is most nicely done in a suitable functorial framework.
(a) A triple \((\mathfrak{g}, \tau_h, h)\) is called a \emph{parahermitian symmetric Lie algebra} if \(h \in \mathcal{E}(\mathfrak{g})\) is an Euler element and \(\tau_h = e^{x_1 \text{ad } h}\).

(b) A triple \((\mathfrak{g}, \tau, C)\) is called a \emph{compactly causal (cc) symmetric Lie algebra} if \(C \subseteq \mathfrak{q}\) is a pointed generating elliptic cone invariant under \(\text{Inn}_\mathfrak{g}(h) := (e^{\text{ad } h})\).

(c) A triple \((\mathfrak{g}, \tau, C)\) is called a \emph{non-compactly causal (ncc) symmetric Lie algebra} if \(C \subseteq \mathfrak{q}\) is a pointed generating hyperbolic cone invariant under \(\text{Inn}_\mathfrak{g}(h)\), i.e., the interior of \(C\) consists of ad-diagonalizable elements.

(d) A triple \((\mathfrak{g}, \tau, C)\) is called a \emph{causal symmetric Lie algebra} if it is either a cc or an ncc symmetric Lie algebra.

(e) A \emph{modular causal symmetric Lie algebra} is a quadruple \((\mathfrak{g}, \tau, C, h)\), where \((\mathfrak{g}, \tau, C)\) is a causal symmetric Lie algebra, \(h \in \mathfrak{g}^*\) is an Euler element, and the involution \(\tau_h = e^{x_1 \text{ad } h} \in \text{Aut}(\mathfrak{g})\) satisfies \(\tau_h(C) = -C\).

Note that the \(c\)-dual of a cc symmetric Lie algebra \((\mathfrak{g}, \tau, C)\) is ncc and vice versa.

For the classification of \(\text{Inn}(h)\)-invariant elliptic cones \(C \subseteq \mathfrak{q}\), i.e., causal structures on the symmetric Lie algebra \((\mathfrak{g}, \tau)\), see Sections 4.4 and 4.5 in \cite{HO97} and \cite{KN99} Thm. X.3. Note that the classification in the cc case and ncc case are the same via \(c\)-duality.

\textbf{Remark 2.3.} Suppose that \(C \subseteq \mathfrak{q}\) is a closed convex \(\text{Inn}_\mathfrak{g}(h)\)-invariant cone with \(-\tau_h(C) = C\).

Since \(\text{ad } h\) preserves \(\mathfrak{q}\), the space \(\mathfrak{q}\) decomposes as

\[
\mathfrak{q} = q_{-1}(h) \oplus q_0(h) \oplus q_1(h).
\]

As \(C\) is invariant under \(e^{t \text{ad } h}\) and \(-\tau_h\), we see that \(x = x_1 + x_0 + x_{-1} \in C\) with \(x_j \in \mathfrak{q}_j(h)\) implies

\[
e^{t \text{ad } h}x = e^tx_1 + x_0 + e^{-t}x_{-1} \in C.
\]

This shows that \(x_{\pm 1} = \lim_{t \to \infty} e^{-t}e^{t \text{ad } h}x \in C\). Therefore the two closed convex cones

\[
C_{\pm} := \pm C \cap q_{\pm 1}(h)
\]

satisfy

\[
p_{q_{-\tau_h}}(C) = C^{-\tau_h} = C_+ - C_-.
\]

Furthermore, the automorphism \(\kappa_h = e^{-\frac{i\pi}{2} \text{ad } h} \in \text{Aut}(\mathfrak{g}_C)\) takes the form

\[
\kappa_h(z_{-1} + z_0 + z_1) = iz_{-1} + z_0 - iz_1
\]

with respect to the \(3\)-grading defined by \(\text{ad } h\) and we have

\[
i\kappa_h(C^{-\tau_h}) = C_+ + C_- =: C^c
\]

\[1\] Embedding into group type

In this section we discuss the embedding of compactly causal triples \((\mathfrak{g}, \tau, C)\) into a natural causal triple of group type \(((\mathfrak{g} \oplus \mathfrak{g})_{C}, \tau_{\text{trip}}, \mathcal{C}_g)\). This embedding will play a crucial role in the following.

We call a compactly causal symmetric Lie algebra \((\mathfrak{g}, \tau, C)\) \emph{extendable} if there exists a pointed closed convex invariant cone \(\mathcal{C}_g \subseteq \mathfrak{g}\) satisfying

\[
\mathcal{C}_g \cap \mathfrak{q} = C \quad \text{and} \quad -\tau(\mathcal{C}_g) = \mathcal{C}_g.
\]

We then call \((\mathfrak{g}, \tau, \mathcal{C}_g)\) an \emph{extension of \((\mathfrak{g}, \tau, C)\). The following theorem \cite[Thm. X.7]{KN99}, \cite[Thm. 4.5.8]{HO97}, \cite[Thm. 7.8]{O01} shows that, for reductive Lie algebras, extensions always exist, even with generating cones \(\mathcal{C}_g\).
Theorem 2.4. (Extension Theorem) Let \((\mathfrak{g}, \tau, C)\) be a reductive compactly causal symmetric Lie algebra for which \(\mathfrak{h} \) contains no non-zero ideals of \(\mathfrak{g}\). Then there exists a pointed generating invariant closed convex cone \(C' \subseteq \mathfrak{g}\) with
\[
-\tau(C') = C' \quad \text{and} \quad C' \cap q = C.
\]

Suppose that \((\mathfrak{g}, \tau, C)\) is an extension of the compactly causal symmetric Lie algebra \((\mathfrak{g}, \tau, C)\). If \(C'\) is not generating, then
\[
\mathfrak{g}_C := C' \subseteq \mathfrak{g}
\]
is a \(\tau\)-invariant ideal containing \(C - C = q\), so that \(\mathfrak{g}_C = \mathfrak{h} \oplus q\) for \(\mathfrak{h} = \mathfrak{h} \cap \mathfrak{g}_C\). Then
\[
(\mathfrak{g} \oplus \mathfrak{g})_C = \{(x, y) \in \mathfrak{g} \oplus \mathfrak{g} : x - y \in \mathfrak{g}_C\} \quad \text{with} \quad \tau(x, y) = (y, x)
\]
is a symmetric Lie algebra with
\[
(\mathfrak{g} \oplus \mathfrak{g})_C^\tau = \{(x, x) : x \in \mathfrak{g}\} \cong \mathfrak{g} \quad \text{and} \quad (\mathfrak{g} \oplus \mathfrak{g})_C^{-\tau} = \{(x, -x) : x \in \mathfrak{g}_C\} \cong \mathfrak{g}_C.
\]
We thus obtain an embedding of causal symmetric Lie algebras
\[
(\mathfrak{g}, \tau, C) \hookrightarrow ((\mathfrak{g} \oplus \mathfrak{g})_{C'}, \tau_c, C'), \quad x \mapsto (x, \tau x) \quad \text{for} \quad x \in \mathfrak{g}.
\]
This embedding is also compatible with the corresponding Euler elements. Any Euler element \(h \in \mathfrak{h} = \mathfrak{g}^*\) is mapped to the Euler element \((h, h) \in (\mathfrak{g} \oplus \mathfrak{g})_{C'}^\tau\).

On the global level, the embedding \((14)\) corresponds to the quadratic representation of the symmetric space \(G/H\) in the group \(G_C\) with Lie algebra \(\mathfrak{g}_C\) (see \((14)\) below). We shall use such embeddings to derive properties of compactly causal symmetric spaces from those of spaces of group type.

Since \(\tau\) commutes with \(\text{ad} \ h\) on \(\mathfrak{g}\), it preserves all eigenspaces \(\mathfrak{g}_j(h), j = -1, 0, 1\). We conclude from \((18)\) that the cones
\[
C_{\theta, \pm} := \pm C' \cap \mathfrak{g}_{\pm 1}(h)
\]
satisfy \(-\tau(C_{\theta, \pm}) = C_{\theta, \pm}\), which shows in particular that the cone
\[
C'_{\theta} := C_{\theta, +} \oplus C_{\theta, -}
\]
is \(-\tau\)-invariant with
\[
(C'_{\theta})^{-\tau} = (C_{\theta, +} \oplus C_{\theta, -})^{-\tau} = C_{\theta, +} \oplus C_{\theta, -} = C'_{\theta}.
\]

Remark 2.5. In \((21)\) one finds examples of compactly causal symmetric Lie algebras \((\mathfrak{g}, \tau, C)\) which are not extendable, but \((21)\) Thm. 7 describes sufficient conditions for general Lie algebras that cover all cases where \(C'\) is generating.

We also note that, if \(\mathfrak{g}\) is not reductive and \(C' \subset \mathfrak{g}\) is a pointed generating invariant cone, then \(C' \cap \mathfrak{j}(\mathfrak{g}) \neq \{0\}\). As \(\tau_h\) fixes the center pointwise, this condition is incompatible with \(-\tau_h(C') = C'\). Therefore it would be too much to hope for extensions of modular compactly causal symmetric Lie algebras \((\mathfrak{g}, \tau, C, h)\) for which \(C'\) is also generating. We refer to \((21)\) for more details on the structure of non-reductive modular causal symmetric Lie algebras \((\mathfrak{g}, \tau, C, h)\) and corresponding classification results.

Example 2.6. (A non-reductive example; cf. \((21)\) Ex. 3.7]) We consider the Lie algebra
\[
\mathfrak{g} = \mathfrak{heis}(V, \omega) := \mathfrak{heis}(V, \omega) \times \mathfrak{sp}(V, \omega),
\]
where \((V, \omega)\) is a symplectic vector space, \(\mathfrak{heis}(V; \omega) = \mathbb{R} \oplus V\) is the corresponding Heisenberg algebra with the bracket \([\cdot, \cdot], (z', v') = (\omega(v, v'), 0)\), and
\[
\mathfrak{sp}(V, \omega) := \mathfrak{sp}(V, \omega) \oplus \mathbb{R} \text{id}_V
\]
is the **conformal symplectic Lie algebra** of \( (V, \omega) \).

The hyperplane ideal \( j := \mathfrak{h} \mathfrak{is}(V, \omega) \ltimes \mathfrak{sp}(V, \omega) \) (the **Jacobi algebra**) can be identified by the linear isomorphism

\[
\varphi: j \to \text{Pol}_{\leq 2}(V), \quad \varphi(z, v, x)(\xi) := z + \omega(v, \xi) + \frac{1}{2} \omega(x, \xi), \quad \xi \in V
\]

with the Lie algebra of real polynomials \( \text{Pol}_{\leq 2}(V) \) of degree \( \leq 2 \) on \( V \), endowed with the Poisson bracket (\cite[Prop. A.IV.15]{Ne00}). The set

\[
C := \{ f \in \text{Pol}_{\leq 2}(V): f \geq 0 \}
\]

is a pointed invariant cone in \( \mathfrak{g} \) generating the ideal \( \mathfrak{g}_0 = \mathfrak{j} \). The element \( h_0 := \text{id}_V \) defines a derivation on \( j \) by \( (\text{ad} h_0)(z, v, x) = (2z, v, 0) \) for \( z \in \mathbb{R}, v \in V, x \in \mathfrak{sp}(V, \omega) \). Any involution \( \tau_V \) on \( V \) satisfying \( \tau_V^* \omega = -\omega \) defines by

\[
\tau_V(z, v, x) := (-z, \tau_V(v), \tau_V x \tau_V)
\]

an involution on \( \mathfrak{g} \) with \( \tau_V(h_0) = h_0 \), and \( -\tau_V(C) = C \) follows from

\[
\varphi(\tau_V(z, v, x)) = -\varphi(z, v, x) \circ \tau_V.
\]

Considering \( \tau_V \) as an element of \( \mathfrak{sp}(V, \omega) \), we obtain an Euler element

\[
h := \frac{1}{2}(\text{id}_V + \tau_V) \in \mathfrak{sp}(V, \omega).
\]

Writing \( V = V_1 \oplus V_{-1} \) for the \( \tau_V \)-eigenspace decomposition, the ad \( h \)-eigenspaces in \( \mathfrak{g} \) are

\[
\mathfrak{g}_{-1} = 0 \oplus 0 \oplus \mathfrak{sp}(V, \omega)_{-1}, \quad \mathfrak{g}_0 = 0 \oplus V_{-1} \oplus \mathfrak{sp}(V, \omega)_0 \cong V_{-1} \ltimes \mathfrak{gl}(V_{-1}), \quad \mathfrak{g}_1 = \mathbb{R} \oplus V_1 \oplus \mathfrak{sp}(V, \omega)_1.
\]

The eigenspace \( \mathfrak{g}_1 \) can be identified with the space \( \text{Pol}_{\leq 2}(V_{-1}) \) of polynomials of degree \( \leq 2 \) on \( V_{-1} \) and

\[
C_+ = C \cap \mathfrak{g}_1 = \{ f \in \text{Pol}_{\leq 2}(V_{-1}): f \geq 0 \}.
\]

This cone is invariant under the natural action of the affine group

\[
C^0 \cong \text{Aff}(V_{-1})_c \cong V_{-1} \ltimes \mathfrak{gl}(V_{-1})_c
\]

whose Lie algebra is \( \mathfrak{g}_0 \). We also note that

\[
\mathfrak{g}_{-1} = \mathfrak{sp}(V, \omega)_{-1} \cong \text{Pol}_2(V_1) \quad \text{and} \quad C_{-} = -C \cap \mathfrak{g}_{-1} = \{ f \in \text{Pol}_2(V_1): f \leq 0 \}.
\]

Finally, we observe that \( \tau_h = e^{x \text{ad} h} = (\tau_V)^{-1} \) implies in particular \( -\tau_h(C) = C \), so that

\[
(\mathfrak{g}, \tau_h, C^e, h)
\]

is a modular causal symmetric Lie algebra for which \( (\mathfrak{g}, \tau_h, C^e) \) is compactly causal and extendable to \( (\mathfrak{g}, \tau_h, C) \).

### 2.2 Hermitian simple Lie algebras

A real simple Lie algebra \( \mathfrak{g} \) contains a pointed generating invariant cone \( C_0 \) if and only if \( \mathfrak{g} \) is hermitian (\cite{Vi80}). If this is the case, then there exist two such cones \( C_0^\text{min} \leq C_0^\text{max} \) with the property that any other pointed generating invariant cone \( C_0 \) satisfies

\[
C_0^\text{min} \leq C_0 \leq C_0^\text{max} \quad \text{or} \quad C_0^\text{min} \leq -C_0 \leq C_0^\text{max}.
\]

We start by collecting some information on the relevant class of simple Lie algebras.
Proposition 2.7. Let \( g \) be a simple hermitian Lie algebra. Then \( g \) contains an Euler element if and only if \( g \) is of tube type. If this is the case, then the following assertions hold:

(a) \( \text{Im}(g) \) acts transitively on the set \( E(g) \) of Euler elements.
(b) If \((g, \tau, C)\) is compactly causal, then \( E(g) \cap h \neq \emptyset \).
(c) For every Euler element \( h \in g \), the cone \( C_h^{\text{max}} \subseteq g \) satisfies \( \tau(h)(C_h^{\text{max}}) = -C_h^{\text{max}} \). In particular \((g, \tau_h)\) is compactly causal.
(d) For any pointed generating invariant cone \( C_0 \subseteq C_h^{\text{max}} \), we have \( C_0 \cap g^{-\tau_h} = C_h^{\text{max}} \cap g^{-\tau_h} \).

Proof. (a) follows from [MN21 Prop. 3.11].
(b) Assume that \((g, \tau)\) is compactly causal. In [NO21b App. D] we show that \( \tau \) preserves a 3-grading of \( g \), and this means that the corresponding Euler element \( h \) is fixed by \( \tau \), i.e., \( h \in h \).
(c) Since \( \tau(h)(C_h^{\text{max}}) \) is one of the two maximal invariant cones \( \pm C_h^{\text{max}} \), (c) follows from [Oeh20 Lemma 2.28].
(d) Clearly, \( C_h^{\text{max}} \cap g^{-\tau_h} \supseteq C_0 \cap g^{-\tau_h} \) by (18). Writing \( g^{-\tau_h} = g_1(h) \oplus g_{-1}(h) \) and using the invariance under \( e^{\text{ad} h} \), it follows as in Remark 2.3 that
\[
C_0 \cap g^{-\tau_h} = C_{\phi,+} + -C_{\phi,-}, \quad \text{where} \quad C_{\phi,\pm} = \pm C_0 \cap g_{\pm 1}(h). \tag{19}
\]
The cones \( C_{\phi,\pm} \) coincide by [Oeh20 Lemma 2.28], so that (d) follows from (19). \( \blacksquare \)

Proposition 2.8. Let \( g \) be simple and \( h \in E(g) \). Then \((g, \tau_h)\) is causal if and only if it is of Cayley type, i.e., \( g \) is simple hermitian of tube type.

In this case any causal structure represented by a cone \( C \subseteq q = g^{-\tau_h} \) leads to two cones \( C = C_+ - C_- \) and \( C^C = C_+ + C_- \). One is hyperbolic and the other is elliptic.

Proof. We choose a Cartan involution \( \theta \) with \( \theta(h) = -h \).

If \((g, \tau_h)\) is causal and \( C \subseteq q \) is a pointed generating \( \text{Im}_q(h) \)-invariant cone, then \( C^0 \) contains fixed points of the compact group \( \text{Im}_q(h) \), so that \( q^{\text{ad} h} \neq \{0\} \). Hence the conclusion follows from [HO97 Thm. 1.3.11].

If, conversely, \( g \) is simple hermitian and \( h \in E(g) \), then Proposition 2.7(c) implies that \((g, \tau_h)\) is causal. \( \blacksquare \)

2.3 Reductive Lie algebras

The last section was devoted to the semisimple case. In this section we discuss the reductive case. **We assume that \( h \) does not contain any non-zero ideal.** This is no restriction because the integral subgroup corresponding to such an ideal would act trivially on \( G/H \) and thus can be factorized out. This in particular implies that \( \mathfrak{z}(g) \cap h = \{0\} \) because \( \mathfrak{z}(g) \cap h \) is an ideal of \( g \). Thus \( H \cap Z(G) \) is discrete.

The following proposition extends Proposition 2.7(c) to the larger class of reductive Lie algebras.

Proposition 2.9. If \((g, \tau)\) is a reductive compactly causal symmetric Lie algebra containing an Euler element and \( h \) contains no non-zero ideal, then \( E(g) \cap h \neq \emptyset \).

Proof. If \( h \in g \) is an Euler element and \( h = h_0 + h_1 \) with \( h_0 \in \mathfrak{z}(g) \) and \( h_1 \in [g, g] \), then \( h_1 \in E(\mathfrak{z}(g)) \). As \((g, \tau)\) is compactly causal and \( h \) contains no non-zero ideal, all simple ideals of \( g \) are either hermitian or compact. In fact, the Extension Theorem 2.4 implies the existence of a pointed generating invariant cone in \( g \), hence \( g \) is quasihermitian by [Nc01 Cor. VII.3.9], i.e., \( \mathfrak{z}g(\mathfrak{h}(\mathfrak{e})) \) holds for a maximal compactly embedded subalgebra. Since \( \mathfrak{e} \) is adapted to the decomposition into simple ideals \( \mathfrak{g}_j \), the ideal \( \mathfrak{g}_j \) is compact if \( \mathfrak{z}(\mathfrak{e}_j) = \{0\} \) and it is hermitian if \( \mathfrak{z}(\mathfrak{e}_j) \neq \{0\} \) (by definition).
As \( \text{ad} h_1 \) is diagonalizable, \( h_1 \) is contained in the sum of all hermitian simple ideals. Any such ideal is either \( \tau \)-invariant, or corresponds to an irreducible summand of group type. For group type Lie algebras \( (g \oplus \mathfrak{h}, \tau_{\text{min}}) \) (cf. \( \mathfrak{g} \)), the assertion is trivial because \( h = (h_1, h_2) \in \mathcal{E}(g \oplus \mathfrak{g}) \) implies \( (h_j, h_j) \in \mathfrak{h} \) for \( j = 1, 2 \), and at least one of these two elements is non-central, hence an Euler element.  

For the \( \tau \)-invariant ideals, the assertion follows from Proposition 2.7(b).

Let \( (\mathfrak{g}, \tau, C) \) be a reductive compactly causal symmetric Lie algebra with \( \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g} \) (which follows from \( \mathfrak{h} \cap \mathfrak{z}(\mathfrak{g}) = \{0\} \)). Then \( \mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{j=1}^n \mathfrak{g}_j \), where the \( \mathfrak{g}_j \) are simple ideals that are either compact or simple hermitian. If \( \mathfrak{g}_j \) is compact, we put \( C_{\mathfrak{g}_j}^{\min} = \{0\} \) and \( C_{\mathfrak{g}_j}^{\max} = \mathfrak{g}_j \). With \( \mathfrak{z} \) this leads to invariant cones

\[
C_{\mathfrak{g}}^{\min} := \bigoplus_{j=1}^n C_{\mathfrak{g}_j}^{\min} \subseteq C_{\mathfrak{g}}^{\max} := \mathfrak{z}(\mathfrak{g}) \oplus \bigoplus_{j=1}^n C_{\mathfrak{g}_j}^{\max}.
\]  

Every Euler element \( h \in \mathfrak{g} \) decomposes as \( h = h_0 + \sum_{j=1}^n h_j \) with \( h_0 \in \mathfrak{z}(\mathfrak{g}) \) and \( h_j = 0 \) if \( \mathfrak{g}_j \) is compact, and if \( h_j \neq 0 \), then it is an Euler element in \( \mathfrak{g}_j \). From Proposition 2.7 we now obtain:

**Corollary 2.10.** Let \( \mathfrak{g} \) be a reductive quasihermitian Lie algebra, i.e., all simple ideals are compact or hermitian, and \( h \in \mathcal{E}(\mathfrak{g}) \). Then the following assertions hold:

(a) If \( \mathfrak{g}^{\text{th}} = \ker(\text{ad} h) \) contains no simple hermitian ideal, then \( \tau_h(C_{\mathfrak{g}}^{\max}) = -C_{\mathfrak{g}}^{\max} \).

(b) For any pointed generating invariant cone \( C_{\mathfrak{g}} \subseteq C_{\mathfrak{g}}^{\max} \), we have \( C_{\mathfrak{g}} \cap \mathfrak{g}^{-\tau_h} = C_{\mathfrak{g}}^{\max} \cap \mathfrak{g}^{-\tau_h} \).

**Proof.** (a) Our assumption on \( h \) means that \( h_j \in \mathcal{E}(\mathfrak{g}_j) \) for every simple hermitian ideal \( \mathfrak{g}_j \) because \( h_j \neq 0 \). Hence (a) follows immediately from (b) and the corresponding assertion for hermitian simple Lie algebras (Proposition 2.9).

(b) As \( \mathfrak{g}^{\text{th}} \) contains the center and all compact ideals, \( \mathfrak{g}^{-\tau_h} \) is contained in the sum

\[
\mathfrak{g}_{\text{herm}} := \mathfrak{g}_{j_1} \oplus \cdots \oplus \mathfrak{g}_{j_k}
\]

of all hermitian simple ideals. Further, the invariance of all ideals under \( \tau_h \) shows that

\[
\mathfrak{g}^{-\tau_h} = \mathfrak{g}_{j_1}^{-\tau_h} \oplus \cdots \oplus \mathfrak{g}_{j_k}^{-\tau_h}.
\]

The classification of invariant cones (see [Ne00] Thm. VIII.3.21) for more details) implies that \( C_{\mathfrak{g},\text{herm}} := C_{\mathfrak{g}} \cap \mathfrak{g}_{\text{herm}} \) is a pointed generating invariant cone in \( \mathfrak{g}_{\text{herm}} \) with

\[
\sum_{\ell=1}^k C_{\mathfrak{g}_{j_\ell}}^{\min} \subseteq C_{\mathfrak{g},\text{herm}} \subseteq C_{\mathfrak{g}_{j_\ell}}^{\max} = \sum_{\ell=1}^k C_{\mathfrak{g}_{j_\ell}}^{\max}.
\]

Therefore Proposition 2.7(d) leads to

\[
C_{\mathfrak{g}}^{\max} \cap \mathfrak{g}^{-\tau_h} = C_{\mathfrak{g}}^{\max} \cap \mathfrak{g}^{-\tau_h} = \mathfrak{g}^{-\tau_h}.
\]

### 2.4 The global setting for symmetric spaces

In the following \( G \) denotes a connected real Lie group and \( \eta_G: G \to G_C \) its universal complexification. We assume that \( \ker(\eta_G) \) is discrete, which is always the case if \( \mathfrak{g} \) is semisimple (because \( \ker(\eta_G) \subseteq \mathfrak{z}(G) \)), if \( G \) is simply connected, or if \( G \) is a matrix group. The groups \( G \) and \( G_C \) carry natural structures of Loos symmetric spaces (\[Lo69\]), defined by

\[
g \bullet h := s_g(h) := gh^{-1} g,
\]

(21)
Then automorphisms, antiautomorphisms and left and right translations define automorphisms of the symmetric space \((G, \cdot)\). In particular, we obtain a transitive action of the product group \(G \times G\) on \(G\) by automorphisms of \((G, \cdot)\) via

\[(g_1, g_2) : g = g_1 g_2^{-1}.\]

Then \(T_e(G) \cong \{(x, -x) : x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}\), and the exponential function of the pointed symmetric space \((G, \cdot, e)\) is given by

\[
\text{Exp}_e : T_e(G) \to G, \quad (x, -x) \mapsto \exp(2x) = (\exp x, \exp -x).e. \tag{22}
\]

If \(\tau^G\) is an involutive automorphism of \(G\) and \(H \subset G^{rG}\) an open subgroup, then

\[
M := G/H, \quad g_1 H \bullet g_2 H := g_1 \tau^G(g_2)^{-1} g_1 H
\]

is the corresponding symmetric space. Its quadratic representation is the map

\[
Q : M \to G, \quad Q(gH) := gg^\tau, \quad g^\tau := \tau^G(g)^{-1}. \tag{23}
\]

It is a covering of the identity component \(G^0 := G/G^{rG}\) of the symmetric subspace

\[
G^\circ := \{g \in G : g^2 = g\} \subset G
\]

(see Lemma A.1 in Appendix A).

We write the natural action of \(G\) on the tangent bundle \(T(M)\) by \((g, v) \mapsto g.v\) and identify the tangent space \(T_e(M)\) with \(\mathfrak{g}\). If \((g, \tau, C)\) is a causal symmetric Lie algebra for which \(\text{Ad}(H)C = \tau C\) then

\[
V_{\tau}(gH) = g.C^0 \subset T_{gH}(M)
\]

defines on \(M = G/H\) a \(G\)-invariant cone field, i.e., a causal structure.

- We call the pair \((M, C)\) a causal symmetric space. Note that \(H\) need not be connected, so that the requirement \(\text{Ad}(H)C = \tau C\) is stronger than the invariance under \(\text{Ad}(H \varepsilon) = \text{Im} \mathfrak{h}_0\).

- We call \((M, C)\) compactly causal if \((g, \tau, C)\) is compactly causal.

**Example 2.11.** The following example shows that the causality of \(G/H\) can depend on \(\tau_0(H)\).

For that let \(G = \text{Ad}(\text{SL}_2(\mathbb{R})) \cong \text{SO}_{2,1}(\mathbb{R})\). Let

\[
h = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})\]

Then \(h\) is an Euler element and the involution \(\tau_h\) is given by

\[
\tau_h = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} x & -y \\ -z & -x \end{pmatrix}.
\]

Hence

\[
h = Rh \quad \text{and} \quad q = \left\{ \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} : y, z \in \mathbb{R} \right\} \cong \mathbb{R} x_+ + \mathbb{R} x_- \quad \text{with} \quad x_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

We also have \(g_{+1}(h) = \mathbb{R} x_+, g_{-1}(h) = \mathbb{R} x_-\) and \(g_0(h) = \mathbb{R} h\). The map \(\theta(x) = -x^\top\) is a Cartan involution commuting with \(\tau_h\). Let

\[
C_+ := [0, \infty) x_+ \subset g_1 \quad \text{and} \quad C_- := [0, \infty) x_- \subset g_-.\]

Then \((g, \tau_h, C_+ - C_-)\) is compactly causal and \((\mathfrak{g}, \tau_h, C^0 = C_+ + C_-)\) is non-compactly causal. Let \(z = x_+ - x_- \in \mathfrak{t} = \mathfrak{so}_2(\mathbb{R})\). Then \(\theta = e^\theta = e^{z \mathfrak{h}} \in G^{r\mathfrak{h}} =: H\) as \(\theta\) commutes with \(\tau_h\), but \(\theta(C^0) = -C^0\). As \(H = (H \cap K)H_\varepsilon = \{1, \theta\}H_\varepsilon\) it follows that \(C\) is \(H\)-invariant but \(\text{Ad}(H)C^0 \supseteq -C^0\), so that \(C^0\) is not. It is only invariant under the connected group \(H_\varepsilon\).

In general, if \(g\) is simple hermitian of tube type with Euler element \(h\) and involution \(\tau = \tau_h\), then if \(C = C_+ - C_-\) defines a compactly causal structure, then the cone \(C^0 = C_+ + C_-\) defines a non-compactly causal structure on \((g, \tau_h)\). As above we see that \(C^0\) is invariant under \(\text{Ad}(G)_h\) but not under \(\text{Ad}(G)^{r\mathfrak{h}}\). On the other hand \(C\) is invariant under the full group \(\text{Ad}(G)^{r\mathfrak{h}}\).
If \((g, \tau, C, h)\) is a modular compactly causal symmetric Lie algebra, then we further assume that the involution \(\tau_h \in \text{Aut}(g)\) integrates to an involution \(\tau_h^G\) of \(G\) satisfying
\[
\tau_h^G(H) = H,
\]
so that it induces an involution \(\tau_h^M\) on \(M\).

**Remark 2.12.** (Integrability of \(\tau_h\)) If \(G\) is a connected Lie group with Lie algebra \(g\) and \(h \in \mathfrak{e}(g)\) an Euler element, then \(\tau_h\) does not always integrate to an automorphism of \(G\).

Here is an example where \(g\) is simple. Let \(G := SU_{p,p}(\mathbb{C})\) and \(K := S(U_p(\mathbb{C}) \times U_p(\mathbb{C}))\) be the canonical maximal compact subgroup. For the Euler element \(h := \frac{1}{2} \begin{pmatrix} 0 & 1_p \\ 1_p & 0 \end{pmatrix}\), the corresponding involution \(\tau_h\) acts on \(g\) by
\[
\tau_h \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix},
\]
so that we have on the group level \(\tau_h^G(k_1, k_2) = (k_2, k_1)\) on \(K \subseteq U_p(\mathbb{C}) \times U_p(\mathbb{C})\). On
\[
Z(\hat{G}) = \{(n, z_1, z_2) \in \mathbb{Z} \times C_p \times C_p : e^{2\pi i/n} z_1 = z_2\} \cong \mathbb{Z} \times C_p
\]
(see \cite[p. 28]{Ti67}), this leads to the non-trivial involution on \(\mathbb{Z} \times C_p\), given by
\[
\tau_h^G(n, z_1) = (-n, z_1^n), \quad \text{where} \quad z_\rho = e^{2\pi i/n}.
\]
It fixes the subgroup \(\{0\} \times C_p\) pointwise and maps \((-2, z_\rho)\) to its inverse. In particular, the cyclic subgroup \(\Gamma = \langle (1, 1) \rangle\) is not invariant, so that \(\tau_h\) does not integrate to the group \(\hat{G}/\Gamma\).

**Remark 2.13.** (Symmetric space structures) Let \(G\) be a connected Lie group and \(H \subseteq G\) be a closed subgroup with \(L(H) = h\), where \((g, \tau)\) is a symmetric Lie algebra. Let \(N \subseteq H\) be the largest normal subgroup of \(G\), contained in \(H\), i.e., the kernel of the \(G\)-action on \(M\). We put
\[
G_1 := G/N \quad \text{and} \quad H_1 := H/N.
\]
(a) \(M = G/H \cong G_1/H_1\) carries a symmetric space structure if and only if \(\tau\) integrates to an involution \(\tau^G_1\) on \(G_1\) leaving \(H_1\) invariant. Then \(H_1 \subseteq G_1^G\) is an open subgroup because it preserves the base point and the corresponding involution \(\tau^M\) on \(M\) is determined by its differential in the base point. If \(\tau\) integrates to an involution \(\tau^G\) on \(G\), then we need not have \(H \subseteq G^G\), we only have the weaker condition
\[
H \subseteq \{g \in G : g\tau^G(g)^{-1} \in N\}.
\]
(b) If \(g\) is semisimple, then \(Z(G) \cap H\) acts trivially on \(M\), hence is contained in \(N\). It follows that, for \(N = \{e\}\), the adjoint action restricts to an injective representation of \(H\).

If, in addition, \(h\) contains no non-zero ideal of \(g\), then \(N\) is a discrete normal subgroup of the connected group \(G\), hence central, and thus
\[
N = H \cap Z(G).
\]
(c) From \(L(H) = h\) we obtain \(\text{Ad}(H)h = h\). If \(g\) is semisimple, so that \(q = h^+\) with respect to the Cartan–Killing form implies \(\text{Ad}(H)q = q\). This already implies
\[
H \subseteq \text{Ad}^{-1}(\text{Ad}(G)^+) = \{g \in G : gg^* \in Z(G)\}.
\]
(d) If \(\tau^G(H) = H\) and \(\text{Ad}\) is faithful on \(H\), then \(\text{Ad}(H) \subseteq \text{Ad}(G)^+\) implies \(H \subseteq G^G\).

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3 Fixed points of the modular flow in \( M \)

In Subsection 3.1 we classify for a reductive compactly causal symmetric reductive Lie algebra \((g, \tau)\) the orbits of \( \text{Im}_q(h) \) in the set \( \mathcal{E}(g) \cap h \) of Euler elements in \( h \). As this set describes the choices of the Euler element \( h \in h \), this provides a classification of all modular structures on the associated symmetric spaces. The classification is done by reduction to the simple case.

On \( M = G/H \), an element \( m = gH \) is fixed by the modular flow

\[
\alpha^M_t(gH) = \exp(th)gH = g\exp(t\text{Ad}(g)^{-1}h)H
\]

if and only if \( \text{Ad}(g)^{-1}h \in h \). Therefore elements of the fixed point set \( M^\alpha \) correspond to Euler elements in \( O_h \cap h \), where \( O_h = \text{Ad}(G)h \) is the adjoint orbit of \( h \).

In Subsection 3.2 we show that the orbits of the group \( G^h \) in \( M^{\alpha} \) are open and that this leads to a natural bijection between the orbit spaces \( M^{\alpha}/G^h \) and \( (O^h \cap h)/H \). Both sets correspond to the set of double cosets \( G^h \neq gH \subseteq G \) for which \( \text{Ad}(g)^{-1}h \in h \).

### 3.1 \( \text{Im}_q(h) \)-orbits in \( \mathcal{E}(g) \cap h \)

Let \((g, \tau, C)\) be a reductive compactly causal symmetric Lie algebra. Then \((g, \tau)\) decomposes as

\[
(g, \tau) \cong (g_0, \tau_0) \oplus \bigoplus_{j=1}^{N} (g_j, \tau_j),
\]

where \( g_0 \) contains the center and all compact ideals and the \( \tau \)-invariant ideals \( g_j, j \geq 1 \), are either simple hermitian or irreducible of group type. This decomposition leads to a corresponding decomposition for \( h \):

\[
h = h_0 \oplus \bigoplus_{j=1}^{N} h_j,
\]

and an element \( h \in g \setminus \mathfrak{g}(\mathfrak{g}) \) is an Euler element in \( g \) if and only if all its components \( h_0, h_1, \ldots, h_N \) either vanish or are Euler elements in \( g_j \). In particular, if \( h_j \neq 0 \), then \( g_j \) is simple hermitian of tube type (Proposition 2.7). Furthermore, \( h_0 \in \mathfrak{g}(\mathfrak{g}) \).

The group \( \text{Im}_q(h) \) is a product of the subgroups \( \text{Im}_q(h_j), j = 0, \ldots, N, \) acting on the ideals \( g_j \). This reduces the problem to classify \( \text{Im}_q(h) \)-orbits in the subset \( \mathcal{E}(g) \cap h \) to the case where \((g, \tau)\) is irreducible. In the group case \( g = h \oplus h \), and \( \text{Im}(h) \) acts transitively on \( \mathcal{E}(g) \cap h = \mathcal{E}(h) \) (Proposition 2.7). Therefore it suffices to analyze the case where \( g \) is simple hermitian.

We assume for the moment that \( g \) is simple hermitian and that \((g, \tau, C)\) is compactly causal. We do not assume that \( \mathcal{E}(g) \neq \emptyset \). Let \( g = t \oplus p \) a Cartan decomposition invariant under \( \tau \). Then the compact causality of \((g, \tau)\) is equivalent to \( \mathfrak{z}(t) \subseteq \mathfrak{q} \). Let \( \mathfrak{c} \subseteq \mathfrak{h} \) be maximal abelian and \( \mathfrak{a} \supseteq \mathfrak{c} \) maximal abelian in \( p \), so that

\[
s := \dim \mathfrak{c} = \text{rank}_g(h) \quad \text{and} \quad r := \dim \mathfrak{a} = \text{rank}_p(g).
\]

If \( h \in \mathcal{E}(g) \cap h \), then \( \tau \) preserves the 3-grading defined by \( h \). On the euclidean simple Jordan algebra \( E := g_1(h) \) ([O1920 Rem. 2.24]), \( \gamma := -\tau|_E \) is a Jordan automorphism ([O1920 Prop. 3.12]). If \( \gamma = \text{id}_E \), then \((g, \tau)\) is of Cayley type (CT) and \( \tau = \tau_h \), and if this is not the case, then \( \gamma \) is non-trivial and there are two cases. For split type (ST) we have \( r = s \), and for non-split type (NST) we have \( r = 2s \) (cf. [O191 §8], [NO21]§ 2.2, Lemma D.3], [BH98]). If \( r = s \), then \( \Delta(g, c) \cong C_s \), by Moore’s Theorem as \( g \) is of tube type.

To understand the \( \text{Im}_q(h) \)-orbits in \( \mathcal{E}(g) \cap h \), we have to compare the root systems

\[
\Delta(h, c) \subseteq \Delta(g, c)
\]
and the inclusions $\mathcal{W}(h, c) \hookrightarrow \mathcal{W}(g, c)$ of the corresponding Weyl groups. Any Euler element $h \in \mathcal{E}(g) \cap h$ is conjugate under $\text{Inn}_g(h)$ to one in $c$. The space $\mathcal{E}(g)$ consists of a single $\text{Inn}(g)$-orbit (Proposition 3.2\textsuperscript{a}), but $\text{Inn}_g(h)$ may not act transitively on $\mathcal{E}(g) \cap h$ (see Theorem 3.2).

For the table below, we need the following two examples:

**Example 3.1.** (a) Assume that $\tau = \tau_0$ and that $(g, \tau, C)$ is compactly causal of Cayley type. Then $a = c$, and the root system $\Delta(g, a)$ is of type $C_r$:

$$\Delta(g, a) \cong C_r = \{\pm 2\varepsilon_i, \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq r\}$$

and

$$\Delta(h, a) \cong A_{r-1} = \{\pm (\varepsilon_i - \varepsilon_j) : 1 \leq i \neq j \leq r\}.$$

The Weyl group $\mathcal{W}(C_r)$ consists of all products of permutations of $\{1, \ldots, r\}$ with sign changes. It contains the Weyl group $\mathcal{W}(A_{r-1})$ as the subgroup of permutations. Let $w_0 = 1$ and $w_j(x_1, \ldots, x_r) = (-x_1, \ldots, -x_j, x_{j+1}, \ldots, x_r)$. Then

$$\mathcal{W}(A_{r-1})/\mathcal{W}(C_r) \cong \{w_0, \ldots, w_r\}.$$ 

It follows that, up the conjugation by $\text{Inn}_g(h)$, the Euler elements are given by

$$h_k = \frac{1}{2}(1, \ldots, 1, -1, \ldots, -1) \quad \text{for } 0 \leq k \leq r.$$

To see this, we first note that $h = (x_1, \ldots, x_r) \in a$ is an Euler element if and only if $x_j \in \{\frac{1}{2}, -\frac{1}{2}\}$. Applying a suitable permutation we can always order the $x_j$ so that $x_1, \ldots, x_k = 1/2$ and $x_{k+1}, \ldots, x_r = -1/2$. Note that all of those elements are conjugate under $\mathcal{W}(C_r)$ as this group also contains the sign changes.

(b) For $g = \mathfrak{su}_{2s,2s}(\mathbb{C})$ and $h = \mathfrak{u}_{s,s}(\mathbb{H})$ we have $\Delta(g, a) \cong C_{2s}$ and $\Delta(h, c) \cong C_s$. To determine $\Delta(g, c)$, we realize $g$ with respect to the form defined by the hermitian $2 \times 2$-block matrix

$$B = \begin{pmatrix} 0 & 1_{2s} \\ 1_{2s} & 0 \end{pmatrix} \quad \text{as} \quad g = \{X \in \mathfrak{sl}_{2s}(\mathbb{C}) : X^* B = -BX \} \cong \mathfrak{su}_{s,s}(\mathbb{C}).$$

Then $a \subseteq g$ can be realized by diagonal matrices of the form

$$\text{diag}(x_1, \ldots, x_{2s}, -x_1, \ldots, -x_{2s}),$$

and $c \subseteq a$ corresponds to diagonal matrices satisfying $x_1 = x_2, x_3 = x_4, \ldots, x_{2s-1} = x_{2s}$. In these coordinates,

$$\Delta(g, a) \cong C_{2s} = \{\pm 2\varepsilon_j, \pm \varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq 2s\} \quad \text{and} \quad \Delta(g, c) \cong C_s.$$

Restricting from $a$ to $c$, maps $\varepsilon_{2j-1}$ and $\varepsilon_{2j}$ (in $C_{2s}$) to $\varepsilon_j$ (in $C_s$) for $j = 1, \ldots, s$. Now it is easy to see that

$$\Delta(g, c) = \Delta(h, c) \cup \{0\}.$$

(c) For $g = \mathfrak{sp}_{4s}(\mathbb{R})$ and $h = \mathfrak{sp}_{2s}(\mathbb{C})$ we have exactly the same pattern and therefore

$$\Delta(g, c) = \Delta(h, c) \cup \{0\}.$$
| $g$ | $g' = h + iq$ | $h$ | type | $\Delta(g, a)$ | $\Delta(g, c)$ | $\Delta(h, c)$ |
|-----|---------------|-----|------|----------------|----------------|----------------|
| $su_r, r(C) \oplus 2$ | $su_r, r(C)$ | $su_r, r(C)$ | (GT) | $C_r \oplus C_r$ | $C_r$ | $C_r$ |
| $sp_2, r(C) \oplus 2$ | $sp_2, r(C)$ | $sp_2, r(C)$ | (GT) | $C_r \oplus C_r$ | $C_r$ | $C_r$ |
| $so_{2d}(R), d > 2$ | $so_{2d}(R)$ | $so_{2d}(R)$ | (GT) | $C_2 \oplus C_2$ | $C_2$ | $C_2$ |
| $so^*(4r) \oplus 2$ | $so^*(4r)$ | $so^*(4r)$ | (GT) | $C_r \oplus C_r$ | $C_r$ | $C_r$ |
| $\tau_{7 \mid 25}$ | $\tau_r(C)$ | $\tau_{7 \mid 25}$ | (GT) | $C_3 \oplus C_3$ | $C_3$ | $C_5$ |
| $su_r, r(C)$ | $su_r, r(C)$ | $R \oplus sl_r(R)$ | (CT) | $C_r$ | $C_r$ | $A_{r-1}$ |
| $sp_2, (R)$ | $sp_2, (R)$ | $R \oplus sl_r(R)$ | (CT) | $C_r$ | $C_r$ | $A_{r-1}$ |
| $so_{2d}(R), d > 2$ | $so_{2d}(R)$ | $R \oplus so_{1,d-1}(R)$ | (CT) | $C_2$ | $C_2$ | $A_1$ |
| $so^*(4r)$ | $so^*(4r)$ | $R \oplus sl_r(R)$ | (CT) | $C_r$ | $C_r$ | $A_{r-1}$ |
| $\tau_{7 \mid 25}$ | $\tau_r(C)$ | $\tau_{7 \mid 25}$ | (CT) | $C_3$ | $C_3$ | $C_2$ |
| $su_{r, r}(C)$ | $su_{r, r}(C)$ | $R \oplus sl_r(C)$ | (CT) | $C_r$ | $C_r$ | $D_r$ |
| $so_{2d}(R)$ | $so_{2d}(R)$ | $R \oplus so_{1,d-1}(R)$ | (CT) | $C_2$ | $C_2$ | $D_2$ |
| $\tau_{7 \mid 25}$ | $\tau_{7 \mid 25}$ | $R \oplus \tau_{6 \mid 20}$ | (ST) | $C_2$ | $C_2$ | $A_1 \oplus A_1 = D_2$ |
| $so_{2d}(R), d = p + q > 2$ | $so_{p+1,q+1}(R)$ | $so_{p, r}(R) \oplus so_{1,q}(R)$ | (ST) | $C_2$ | $C_2$ | $A_1 \oplus A_1 = D_2$ |
| $su_{2p, 2r}(C)$ | $su_{2p, 2r}(C)$ | $su_{2p, 2r}(C)$ | (NST) | $C_2s$ | $C_s$ | $C_s$ |
| $sp_{2p, r}(R)$ | $sp_{2p, r}(R)$ | $sp_{2p, r}(R)$ | (NST) | $C_2s$ | $C_s$ | $C_s$ |
| $so_{2d}(R)$ | $so_{2d}(R)$ | $so_{2d}(R)$ | (NST) | $A_1$ | $A_1$ | $A_1$ |

Table 1: Irreducible compactly causal symmetric Lie algebras $(g, \tau)$ with $E(g) \cap h \neq \emptyset$.

For non-split type

From Table 1 we get the following classification theorem. Here we use that, as in [MN21, Thm. 3.10], orbits of Euler elements can be classified by representatives in a positive Weyl chamber in terms of a root basis. They correspond to 3-gradings of the root system $\Delta(g, a)$ for $h \in a \cap E(g)$ and to 3-gradings of $\Delta(g, c)$ for $h \in c \cap E(g)$. We also recall from Proposition 2.7 that the existence of an Euler element in a simple hermitian Lie algebra $g$ implies that $g$ is of tube type.

**Theorem 3.2.** (Classification of $\text{Inn}_g(h)$-orbits in $E(g) \cap h$) Let $(g, \tau)$ be an irreducible compactly causal symmetric Lie algebra. Then we have the following situations:

1. **(GT) For group type** $\Delta(g, c) = \Delta(h, c) \cong C_r$, and $\text{Inn}_g(h)$ is transitive on $E(g) \cap h$.

2. **(CT) For Cayley type**

   \[ \Delta(h, c) \cong A_{r-1} \subseteq C_r = \Delta(g, c). \]

   In the canonical identification of $\epsilon$ with $R^r$, we have

   \[ E(g) \cap \epsilon = \{ \frac{1}{2}(\pm 1, \ldots, \pm 1) \} \]

   and the orbits of $\text{Inn}_g(h)$ in $E(g) \cap h$ are represented by the elements

   \[ h_k = \frac{1}{2}(1, \ldots, 1, -1, \ldots, -1), \quad k = 0, 1, \ldots, r. \]  

3. **(ST) For split type**

   \[ \Delta(h, c) \cong D_r \subseteq C_r = \Delta(g, c). \]

   where we identify $D_r \cong A_2$ and $D_2 \cong A_1 \oplus A_1$. Then $\text{Inn}_g(h)^r$ acts transitively on $E(g) \cap h$ and its identity component $\text{Inn}_g(h)$ has two orbits represented by $h_{r-1}$ and $h_r$.

4. **(NST) For non-split type**

   \[ \Delta(h, c) = \Delta(g, c) = C_r \]

   and $\text{Inn}_g(h)$ acts transitively on $E(g) \cap h$.  

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Proof. (GT) Here \((g, \tau) \cong (h \oplus h, \tau_{\text{hyp}})\), where \(h\) is simple hermitian of tube type. Accordingly \(a = e \oplus c\), and the assertion follows.

(CT) The first assertion follows from Table 1 and the remainder from the discussion in Example 3.1(a).

(ST) For split type a look at Table 1 shows that

\[
\Delta(h, c) \cong D_\tau \subseteq C_\tau = \Delta(g, c),
\]

where we identify \(D_\tau \cong A_1\) and \(D_\tau \cong A_1 \oplus A_1\). Then \(\mathcal{W}(D_\tau) \leq \mathcal{W}(C_\tau)\) is an index 2 subgroup and \(\mathcal{W}(C_\tau)\) acts by automorphisms on \(D_\tau\). Therefore \(\text{Inn}(g)^\tau\) induces on \(c\) the full group \(\mathcal{W}(C_\tau)\). Hence the \(\mathcal{W}(D_\tau)\)-orbits in \(\mathcal{E}(g) \cap c\) are represented by \(h_{\tau-1}\) and \(h_\tau\), and both are conjugate under \(\mathcal{W}(C_\tau)\).

(NST) Table 1 shows that, for non-split type, we have \(\Delta(h, c) = \Delta(g, c) = C_\tau\). As \(\mathcal{W}(C_\tau)\) acts transitively on \(\mathcal{E}(g) \cap c\), it follows that \(\text{Inn}(c)\) acts transitively on \(\mathcal{E}(g) \cap h\).

**Corollary 3.3.** If \((g, \tau)\) is irreducible and not of Cayley type, then \(\text{Inn}(g)^\tau\) acts transitively on \(\mathcal{E}(g) \cap h\).

### 3.2 Connected components of \(M^\alpha\)

Recall our global context from Subsection 2.4. Our goal in this section is to prove the following proposition:

**Proposition 3.4.** The orbits of the identity component \(G^h\) on the fixed point set

\[
M^\alpha := \{gH \in M : \text{Ad}(g)^{-1}h \in h\} = \{m \in M : X^M_h(m) = 0\}
\]

of the modular flow coincide with its connected components.

Example 3.6 below shows that, for Cayley type involutions on hermitian simple Lie algebras, the connected components of \(M^\alpha\) may have different dimensions. Our discussion of de Sitter space in Subsection 11.1 reveals a situation where \(G^h\) does not preserve the connected components of \(M^\alpha\).

**Proof.** Clearly, \(M^\alpha\) is invariant under the action of the group \(G^h\) which commutes with \(\alpha\). Therefore \(G^h\) preserves all connected components of \(M^\alpha\). We now show that all \(G^h\)-orbits in \(M^\alpha\) are open, hence coincide with its connected components.

First we consider the base point \(eH\) which is contained in \(M^\alpha\) because \(h \in h\). Using the exponential function \(\text{Exp}_{eH} : q \to M\) as a chart around \(eH\), its equivariance with respect to \(\alpha\) and the one-parameter group \(e^{\tau t}h\) on \(q\), it follows that \(M^\alpha\) is a symmetric subspace of \(M\) with \(T_{eH}(M^\alpha) = q_0(h) \subseteq g^h\). Therefore \(G^h_{eH}eH \geq \text{Exp}_{eH}(q_0(h))\) contains a neighborhood of \(eH\) in \(M^\alpha\), and this implies that the \(G^h_{eH}\)-orbit of \(eH\) in \(M^\alpha\) is open.

Now let \(m = gH \in M^\alpha\) be arbitrary. Then

\[
\alpha_t^M(gH) = \exp(th)gH = g \exp(t \text{Ad}(g)^{-1}h)H
\]

shows that \(m \in M^\alpha\) is equivalent to \(\text{Ad}(g)^{-1}h \in h\) and

\[
\Phi : q \to M, \quad \Phi(x) := g \text{Exp}_{eH}(x)
\]

defines a chart around \(m\) satisfying

\[
\alpha_t \Phi(x) = \exp(th)g \text{Exp}_{eH}(x) = \Phi(e^{t \text{Ad}(g)^{-1}h}x).
\]

Hence \(\Phi\) maps

\[
\{x \in q : [\text{Ad}(g)^{-1}h, x] = 0\} = q \cap \text{Ad}(g)^{-1}h
\]

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onto a neighborhood of $m$ in $M^\alpha$. Finally
\[
\Phi(q \cap \text{Ad}(g)^{-1}g^h) \subseteq g \exp(q \cap \text{Ad}(g)^{-1}g^h)H \subseteq \exp(g^h)gH \subseteq G^h_m
\]
shows that $G^h_m$ is open in $M^\alpha$. ■

**Example 3.5.** (The group case) In the group case the set $M^\alpha = G^\alpha$ is rather simple to describe. Here $M = G, h = (h_0, h_0)$ and $\text{Ad}(g_1, g_2)h = (\text{Ad}(g_1)h_0, \text{Ad}(g_2)h_0) \in h$ is equivalent to $\text{Ad}(g_1)h_0 = \text{Ad}(g_2)h_0$, i.e., $g_1^{-1}g_2 \in G^{h_0}$. If this is the case, then $\text{Ad}(g_1, g_2)h = (\text{Ad}(g_1, g_1)h \in \text{Ad}(\Delta_G)h$.

We also note that $(G \times G)^h = G^{h_0} \times G^{h_0}$ acts transitively on the submanifold $M^\alpha = G^{h_0}$.

**Example 3.6.** (An example of Cayley type) As in Example 3.1, we consider the hermitian form on $C^{2r}$ defined by the block diagonal matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that
\[
g := \{ X \in \mathfrak{s}(C) : X^*B = -BX \} \cong \mathfrak{s}(C) \quad \text{and} \quad \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.
\]
Then
\[
h = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X^* \end{pmatrix} : \text{tr}(X) \in \mathbb{R} \right\} \cong \mathbb{R} \oplus \mathfrak{s}(C).
\]
The corresponding groups are
\[
G \cong \text{SU}_{r,r}(C) \supseteq H \cong \{ g \in \text{GL}_r(C) : \det(g) \in \mathbb{R} \}.
\]
Then
\[
a = \{ \text{diag}(x_1, \ldots, x_r, -x_1, \ldots, -x_r) : x_j \in \mathbb{R} \} \cong \mathbb{R}^r,
\]
and in these coordinates, we obtain Euler elements
\[
h_k = \frac{1}{2}(1_k, -1_{r-k}, -1_k, 1_{r-k}), \quad k = 0, \ldots, r
\]
(Theorem 3.2). Then
\[
\dim G^H = \dim[h_k, h] = \dim(M_{k,r-k}(C) \oplus M_{r-k,k}(C)) = 2k(r-k)
\]
depends on $k$. For $k = 0, r$, the orbit is trivial. This corresponds to the fact that the base point $eH$ in $M = G/H \cong G^H$ is an isolated fixed point of the modular flow. The components
\[
M^\alpha := \{ gH : \text{Ad}(g)^{-1}h \in G^H \}, \quad k = 0, \ldots, r,
\]
are of different dimensions (for $r > 1$).

**Lemma 3.7.** We have a bijection
\[
\Gamma : M^\alpha / G^h \rightarrow (O_h \cap h) / H, \quad G^h gH \mapsto \text{Ad}(H) \text{Ad}(g)^{-1}h.
\]
Proof. As $gH \in M^\alpha$ is equivalent to $\text{Ad}(g)^{-1}h \in h$, we obtain a well-defined surjective map sending the $G^h$-orbit $G^h gH$ to the $H$-orbit $\text{Ad}(H) \text{Ad}(g)^{-1}h \subseteq \mathcal{E}(g) \cap h$.

If $g_1H, g_2H \in M^\alpha$ map to the same $H$-orbit in $\mathcal{E}(g) \cap h$, then there exists an element $k \in H$ with $\text{Ad}(g_1)^{-1}h = \text{Ad}(g_1k)^{-1} \text{Ad}(g_2)^{-1}h$, i.e., $g_2kg_1^{-1} \in G^h$. This implies that $g_2H = g_2kH \in G^h g_1H$. As the surjectivity of $\Gamma$ follows from the fact that $gH \in M^\alpha$ is equivalent to $\text{Ad}(g)^{-1}h \in h$, $\Gamma$ is bijective. ■

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Remark 3.8. (a) If $H$ is connected, then the preceding lemma shows that the orbits of $\text{Ad}(H) = \text{Inn}_g(h)$ in $E(g) \cap h$ are in one-to-one correspondence with the $G^h$-orbits in $M^a$ for $M = G/H$. Therefore Theorem \ref{thm:classification} provides a classification of $G^h$-orbits in $M^a$ whenever $H$ is connected.

(b) As $G^h$-orbits in $M^a$ are open by Proposition \ref{prop:open}, we may also think of $\pi_0(G^h)$-orbits in $\pi_0(M^a)$ being classified by $(O_h \cap h)/H$.

Remark 3.9. The involution $\tau^M_h$ acts on the fixed point manifold $M^a$, but not necessarily trivially. A typical example is the group type space $M = \tilde{\text{SL}}_2(\mathbb{R})$, where $Z(G) \subseteq M^a$ and $Z(G) \cap M^a = \{e\}$.

Remark 3.10. ($G^h$ and $G^{aG}$ for simple Lie algebras) As $G^h = \{g \in G : \text{Ad}(g)h = h\}$, we have

$$\text{Ad}(G^h) = \text{Ad}(G)^h = \text{Ad}(G)^{adh} \subseteq \text{Ad}(G)^{aG}.$$ 

We claim that, if $g$ is simple, then $\text{Ad}(G^h)$ is of index 2 in $\text{Ad}(G)^{aG}$. In fact, in this case $g^{-\tau_h}$ is a direct sum of two irreducible $g_0(h)$-modules $g_{\pm h}(h)$ (\cite[Cor. 1.3.13]{H}, and any element $\text{Ad}(g)$ commuting with $\tau_h$ either preserves both subspaces or exchanges them. If it preserves both, then $\text{Ad}(g)$ commutes with $\text{ad} h$, i.e., $g \in G^h$. If not, then $\text{Ad}(g)h = -h$, and as such an element exists (Proposition \ref{prop:existence}(a)), the index of $\text{Ad}(G^h)$ in $\text{Ad}(G)^{aG}$ is two.

4 Wedge domains in compactly causal symmetric spaces

In this section we introduce the wedge domains $W_M^+(h)$, $W_M^+(h)$ and $W_M^+(h)$ in a compactly causal symmetric space $M = G/H$ with the infinitesimal data $(g, \tau, C, h)$ (under the global assumption from Subsection 2.4 that the universal complexification $\eta_G$ has discrete kernel). One of our main results asserts that these three subsets coincide, which is far from obvious from their definitions. Moreover, their connected components are orbits of an open real Olshanski subsemigroup of $G$. We therefore start recalling the construction of Olshanski subsemigroups.

4.1 Tube domains of compactly causal symmetric spaces

Definition 4.1. (Complex Olshanski semigroups) Let $G$ be a connected Lie group with simply connected covering group $q_G : \tilde{G} \rightarrow G$. For a pointed closed convex $\text{Ad}(G)$-invariant cone $C_\theta \subseteq g$, Lawson’s Theorem (\cite[Thm. IX.1.10]{La}) implies the existence of a semigroup $\Gamma_{C_\theta}(C_\theta)$ which is a covering of the subsemigroup $\eta_G(\tilde{G})\exp(iC_\theta)$ of the universal complexification $\tilde{G}$ (the simply connected group with Lie algebra $q_G$). Then the exponential function $\exp : g + iC_\theta \rightarrow \tilde{G}$ lifts to an exponential function $\text{Exp} : g + iC_\theta \rightarrow \Gamma_{C_\theta}(C_\theta)$, and the polar map

$$\tilde{G} \times C_\theta \rightarrow \Gamma_{C_\theta}(C_\theta), \quad (g, x) \mapsto g \text{Exp}(ix)$$

is a homeomorphism. We now define the \textit{closed complex Olshanski semigroup} corresponding to the pair $(G, C_\theta)$ by

$$\Gamma_C(C_\theta) := \Gamma_{C_\theta}(C_\theta)/(\ker q_C), \quad (32)$$

where $q_C : \tilde{G} \rightarrow G$ is the universal covering map (cf. \cite[Thm. XI.1.12]{Ne}). Then the polar map $G \times C_\theta \rightarrow \Gamma_{C_\theta}(C_\theta)$ is a homeomorphism, and if $C_\theta$ has interior points, then it restricts to a diffeomorphism from $G \times C_\theta$ onto the open subsemigroup

$$\Gamma_{C_\theta}(C_\theta) := \Gamma_{C_\theta}(C_\theta)/(\ker q_C) \subseteq \Gamma_{C_\theta}(C_\theta). \quad (33)$$

Complex Olshanski semigroups are non-abelian generalizations of complex tube domains defined by open cones in real vector spaces; the domains $g + iC_\theta$ which are the tangent objects of the open complex Olshanski semigroups $\Gamma_{C_\theta}(C_\theta)$ are typical examples. We now turn to similar objects for symmetric spaces, which on the tangent level correspond to the tube domain $q + iC_\theta \subseteq q_C$. 

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Definition 4.2. (The complex tube domain of a compactly causal symmetric space) Let \((G, \tau, H, C)\) be a compactly causal symmetric Lie group, i.e., \(G\) is a connected Lie group, \(\tau^G\) an involutive automorphism of \(G\), \(H \subseteq G^G\) an open subgroup, and \(C \subseteq \mathfrak{g}\) a pointed generating \(\text{Ad}(H)\)-invariant closed convex cone such that \(C^0\) consists of elliptic elements. We assume that the universal complexification \(\eta_G : G \rightarrow G_C\) has discrete kernel (which is the case if \(G\) is simply connected or semisimple). The involution \(\tau^G \in \text{Aut}(G)\) induces by the universal property a holomorphic automorphism \(\tau^{G_C}\) of \(G_C\) with \(\eta_G(H) \subseteq G^{G_C}_C\), and
\[
G^1_C = \{ g \in G_C : \tau^{G_C}(g) = g^{-1} \}
\]
is a complex symmetric subspace of \(G_C\).

We call the fiber product
\[
T_M(C) := G \times_H iC^0
\]
the tube domain of the pair \((M, C)\). To obtain a complex manifold structure on this domain, we observe that the quadratic representation \(Q : M = G/H \rightarrow G\) extends to a map
\[
Q_T : T_M(C) = G \times_H iC^0 \rightarrow G^1_C, \quad (g, ix) \mapsto g \exp(2ix) g^\dagger
\]
which is a covering of an open subset. Therefore \(T_M(C)\) carries a unique complex manifold structure for which \(Q_T\) is holomorphic. Clearly, \(T_M(C) \subseteq G \times H iC^0\) contains \(M\) as \(G \times H \{0\}\) in its “boundary” (cf. [HØ91, KNØ97, Ne99]).

Remark 4.3. (a) Suppose that \((\mathfrak{g}, \tau, C)\) is extendable and that \(C_0 \subseteq \mathfrak{g}\) is a \(-\tau\)-invariant \(\text{Ad}(G)\)-invariant pointed cone with \(C = C_0 \cap \mathfrak{q}\) (see Subsection 2.1). Let \(G_C \leq G\) be the normal integral subgroup with Lie algebra \(\mathfrak{g}_C = C_0 - C_0\). Then we have the corresponding open complex Olshanski semigroup
\[
\mathcal{T}_{G_C}(C_0) := G_C \exp(iC^0_0)
\]
and the quadratic representation \(Q : M \rightarrow G_C\) extends to a map
\[
Q : T_M(C) = G_C \times_H iC^0 \rightarrow \mathcal{T}_{G_C}(C_0), \quad (g, ix) \mapsto g \exp(2ix) g^\dagger.
\]
By Lemma A.2 and Remark A.3, its range is the identity component of the complex submanifold \(\mathcal{T}_{G_C}(C_0)^1\):
\[
Q(T_M(C)) = \mathcal{T}_{G_C}(C_0)^1 = \{ ss^2 : s \in \mathcal{T}_{G_C}(C_0) \}.
\]
Since \(Q : T_M(C) \rightarrow Q(T_M(C))\) is a covering map and the \(G_C\)-action on its range lifts to the \(G_C\)-action on \(M\), the holomorphic action of the semigroup \(\mathcal{T}_{G_C}(C_0)\) on \(Q(T_M(C))\) by \(s.m = s m s^2\) lifts to a holomorphic action
\[
\mathcal{T}_{G_C}(C_0) \times T_M(C) \rightarrow T_M(C),
\]
extending the \(G\)-action.

(b) Let \(\tau^{G_C} \in \text{Aut}(G_C)\) be the holomorphic involution with \(\tau^{G_C} \circ \eta_G = \eta_{G_C} \circ \tau^G\) and \(H_C \subseteq G^{G_C}_C\) be an open subgroup with \(\eta_{G_C}^{-1}(H_C) = H\). Then we have a natural embedding of \(G/H\) as the \(G\)-orbit of the base point \(eH_C \in M_C := G_C/H_C\), and
\[
T_M(C) = G \cdot \exp(iC^0)
\]
can be identified with the orbit \(\Gamma_G(C_0^C) \cdot eH_C\) of the base point under the action of the open complex Olshanski semigroup \(\mathcal{T}_{G_C}(C_0)\), which is an open subset of \(M_C\) (cf. [HO091, Lemma 1.3]).

Example 4.4. (a) Let \((G, \tau)\) be a connected symmetric Lie group and \((\mathfrak{g}, \tau, C)\) be a corresponding compactly causal symmetric Lie algebra. Suppose that \(C_0 \subseteq \mathfrak{g}\) is a pointed \(\text{Ad}(G)\)-invariant generating cone satisfying \(-\tau(C_0) = C_0\) and \(C := C_0 \cap \mathfrak{q}\) (cf. Theorem 2.4). Then we have
a corresponding open complex Olshanski semigroup \( S = \mathcal{T}_G(C_\theta) \) (Definition 4.1) and thus an embedding
\[
\mathcal{T}_{G/G^{\alpha}} (C) = G \times_{G^{\alpha}} iC^0 \cong \bigcup_{g \in G} g \exp(iC^0)g^{-1} = \mathbb{S}^1 \to S
\]
(Lemma 4.2 (4)).

(b) Consider a symmetric space \( M = (G, \bullet) \) of group type and an \( \text{Ad}(G) \)-invariant cone \( C_{\theta} \subseteq g \).

The corresponding symmetric Lie group is \( G \times G \) with
\[
\tau_{G \times G}(g_1, g_2) = (g_2, g_1), \quad (G \times G)^{G \times G} = \Delta_G, \quad q = \{(x, -x) : x \in g\},
\]
and the cone
\[
C = \{(x, -x) : x \in C_{\theta}\} = (C_{\theta} \oplus -C_{\theta}) \cap q
\]
is \( \text{Ad}(\Delta_G) \)-invariant. We have the open complex Olshanski semigroup
\[
\Gamma_{G \times G}(C_{\theta} \oplus -C_{\theta}) = \Gamma_G(C_{\theta}) \times \Gamma_G(-C_{\theta}).
\]
Here \((g_1, g_2)^2 = (g_2^{-1}, g_1^{-1})\) is the \( \sharp \)-operation in \( G \times G \), so that the \( \sharp \)-fixed points in the open complex Olshanski semigroup
\[
S := \Gamma_{G \times G}(C_{\theta}^0 \oplus -C_{\theta}^0)
\]
are the pairs \((s, s^{-1}), s \in \Gamma_G(C_{\theta}^0)\). We thus obtain
\[
\mathcal{T}_G(C) = (G \times G) \times_{\Delta_G} iC^0 \cong G \times iC^0 \cong \Gamma_G(C_{\theta}^0) = \{(s, s^{-1}) : s \in \Gamma_G(C_{\theta})\} \cong \mathbb{S}^2.
\]
As a complex manifold, this is a copy of the open complex Olshanski semigroup \( \Gamma_G(C_{\theta}) \).

4.2 The modular flow and three types of wedge domains

We are now ready to introduce the three different type of wedge domains in \( M \). We already introduced the modular flow in (35) but repeat the definition here:

**Definition 4.5.** (The modular flow) The Euler element \( h \in \mathfrak{h} \) defines an \( \mathbb{R} \)-action by automorphisms on \( G \) via
\[
\alpha_t(g) = \exp(th)g \exp(-th), \quad g \in G.
\]
Then \( \alpha \) preserves all connected components of the subgroup \( G^{\alpha} \), hence in particular \( H \).

Therefore \( \alpha_t \) induces a flow
\[
\alpha_t^M(gH) = \exp(th)gH = g \exp(t \text{Ad}(g^{-1}h)H)
\]
on \( M = G/H \). This flow is generated by the modular vector field
\[
X^M_h \in \mathcal{V}(M), \quad X^M_h(m) = \left. \frac{d}{dt} \right|_{t=0} \alpha_t(m).
\]

The modular flow extends to \( \mathbb{R} \)-actions by holomorphic maps on the complex tube domains \( \mathcal{T}_G(C_{\theta}) \) and on \( \mathcal{T}_M(C) \) via
\[
\alpha_t(g \exp(ix)) = \alpha_t(g) \exp(ie^{t \text{ad}_h}x) \quad \text{and} \quad \alpha_t^M([g, ix]) := [\alpha_t(g), ie^{t \text{ad}_h}x].
\]
Their infinitesimal generators are denoted \( X^G_h \) and \( X^M_h \), respectively.

On \( G \) we even obtain a holomorphic flow by \( \alpha_z(g) = \exp(zh)g \exp(-zh) \), \( z \in \mathbb{C} \), but on \( \mathcal{T}_G(C_{\theta}) \) and \( \mathcal{T}_M(C) \), the real flow does not extend to all of \( \mathbb{C} \). However, for \( s \) in the closed complex semigroup \( \Gamma_G(C_{\theta}) \), we consider the orbit map
\[
\alpha^s : \mathbb{R} \to \Gamma_G(C_{\theta}), \quad t \mapsto \alpha_t(s)
\]
and define
\[ \alpha_{x+iy}(s) := \alpha^2(x + iy) \quad \text{for} \quad x + iy \in \mathbb{C}, \]
whenever the maximal local flow of the vector field \( iX^T_{\mathcal{M}} \) with initial value \( \alpha_x(s)|_{s=0} \) is defined in \( y \in \mathbb{R} \). This implies that \( \alpha^s \) extends to a continuous map on the closed strip between \( \mathbb{R} \) and \( \mathbb{R} + iy \) which is holomorphic on the interior in the sense that its composition with the natural map \( \Gamma_G(C_g) \rightarrow G, g \exp(ix) \rightarrow \gamma_G(g) \exp(ix) \) is a holomorphic \( GC \)-valued map (cf. Definition 4.1).

Recall that we assume that \( \tau \) integrates to an involution \( \tau^G_0 \) on \( G \) which leaves \( H \) invariant, so that it induces an involution \( \tau^M_0 \) on \( M = G/H \) (see the end of Subsection 2.4).

**Definition 4.6.** We consider the following types of wedge domains:

- The positivity domain of the modular vector field \( X^K_{\mathcal{M}} \) in \( M \) is
  \[ W^+_\mathcal{M}(h) := \{ m \in M : X^K_{\mathcal{M}}(m) \in V_+(m) \}, \]
  where \( V_+(m) \subseteq T_m(M) \) is the open cone corresponding to the \( G \)-invariant cone field with \( V_+(eH) = C^0 \) in \( T_{eH}(M) \cong_q q \). If \( M \) is Lorentzian and \( V_+(m) \) is the future light cone in \( T_m(M) \), then this is the domain where the modular vector field is future oriented timelike.

- The KMS wedge domain is
  \[ W^K_{\mathcal{M}}(h) := \{ m \in M : (\forall z \in \mathcal{S}_z) \alpha_z(m) \in T_m(C) \} \]
  \[ = \{ m \in M : (\forall y \in (0, \pi)) \alpha_{iy}(m) \in T_m(C) \}, \]
  where \( \alpha_z(m) \) is assumed to be defined in \( m \) in as in Definition 4.5 and
  \[ \mathcal{S}_x = \{ z \in \mathbb{C} : 0 \leq \text{Im} z < \pi \}. \]
  This is the set of all points \( m \in M \) whose \( \alpha \)-orbit map extends analytically to a map \( \mathcal{S}_x \rightarrow T_m(C) \). Comparing boundary values on \( \mathbb{R} \) and \( \pi i + \mathbb{R} \) resembles KMS conditions; hence the name (see also [NO021 App. A.2]).

- Let \( m \in M^a \) be a fixed point of the modular flow and \( C_m = V_+(m) \subseteq T_m(M) \). Then \( \alpha \) induces a 1-parameter group of linear automorphisms on \( T_m(M) \) whose infinitesimal generator has the eigenvalues -1, 0, 1. We write \( T_m(M)_j, j = -1, 0, 1 \), for the corresponding eigenspaces and consider the pointed cone
  \[ C^m_m := C_m \cap T_m(M)_1 - C_m \cap T_m(M)_{-1} \subseteq T_m(M). \]
  Note that \( C^m_m - C^m_m \subseteq T_m(M) \) is a vector space complement of \( T_m(M^a) = T_m(M)_0 \). Then the wedge domain in \( m \) is defined as
  \[ W_M(h)_m := C^m_{+}, \exp_m(C^{m,0}_{m}), \]
  where \( C^{m,0} := (C_m)^0 \) is the relative interior of \( C_m \) in \( T_m(M)_1 + T_m(M)_{-1} \) (cf. [10]) and \( G^H_m = (G^H)_m \) is the connected subgroup with Lie algebra \( g^H = g_0(h) \). On the infinitesimal level, this domain corresponds to the wedge \( T_m(M)_0 + C^m_m \subseteq T_m(M) \).

Identifying \( C \) with \( C_{eH} \subseteq T_{eH}(M) \cong q \), the wedge domain in the base point is
  \[ W_M(H)_{eH} := C^m_{+}, \exp_{eH}((C_+ + C_-)^0), \quad \text{where} \quad C_{\pm} = \pm C \cap q_\pm(h). \]

We define the polar wedge domain of \((M, h)\) as
  \[ W_M(h) := \bigcup_{m \in M^a} W_M(h)_m = \bigcup_{m \in M^a} \exp_m(C^{m,0}_m). \quad (38) \]

In complex analysis “domains” are assumed connected. Here we use the term “domain” for an open subset.
Remark 4.7. For \( g \in G \) and \( v \in T_m(M) \), the fact that \( G \) acts by automorphisms on the symmetric space \( M \) implies that

\[
g. \exp_{m}(v) = \exp_{gm}(g.v) \quad \text{for} \quad g \in G, m \in M, v \in T_m(M).
\]

Now let \( m = gh \in M^\alpha \) and identify \( T_{\alpha h}(M) \) with \( q \). For \( h' := \text{Ad}(g)^{-1}h \in h \), we then have

\[
C_m = g.C \quad \text{and} \quad T(\alpha_t)(g.x) = g.(e^{t \text{ad} h'} x) \quad \text{for} \quad t \in \mathbb{R}, x \in q,
\]

so that

\[
C_m^c = g.C^c(h') \quad \text{for} \quad C^c(h') := C \cap q_1(h') - C \cap q_{-1}(h').
\]

This leads to the relation

\[
\exp_{m}(C_m^c) = \exp_{g\cdot h}(g.C^c(h')^0) = g.\exp_{eH}(C^c(h')^0),
\]

which in turn entails with \( G^h g = gG^h \)

\[
W_M(h)gH = C^e_g \cdot \exp_{gH}(C_m^c) = g.\exp_{eH}(C^c(h')^0) = gC^h \cdot \exp_{eH}(C^c(h')^0)
\]

\[
= g.W_M(h)^{\tau h} \quad \text{for} \quad gH \in M^\alpha.
\]

Example 3.6 shows that the dimensions of the eigenspaces of \( \text{ad} h' \) on \( q \) are not always the same.

Remark 4.8. (On the assumption \( h \in h \))

(a) The domains \( W^\alpha_M(h) \) and \( W^\text{KMS}_M(h) \) are defined for any Euler element \( h \in \mathcal{E}(g) \). They only require the corresponding flows in \( M \) and \( T_M(C) \). However, these domains may be trivial if \( h \notin h \) (cf. Proposition 4.6).

(b) If the vector field \( X^M_{\mathfrak{h}} \) has a zero, then we may choose the base point accordingly to obtain \( h \in h \). An Euler element \( h \in \mathcal{E}(g) \) has this property if and only if its adjoint orbit \( \mathcal{O}_h \) intersects \( h \) (cf. 19 in Definition 2.2).

For a compactly causal symmetric Lie algebra \((\mathfrak{g}, \tau, C)\), there may be many different cones \( C' \) for which \((\mathfrak{g}, \tau, C')\) is compactly causal, but there is a rather explicit classification of all these cones which is described in [NO21a, Thm. 3.6] in some detail in the c-dual context of non-compactly causal spaces. It implies in particular that \( C \) is contained in a uniquely determined maximal \( \text{Inn}_{\mathfrak{h}}(\mathfrak{h}) \)-invariant elliptic cone \( C_{\mathfrak{q}}^{\max} \) (which need not be pointed; see also [20] and the discussion in Subsection 2.3).

Recall the definition of \( C_{\pm} := (\pm C) \cap q_{\pm} \), from Remark 2.3. The following proposition follows immediately by c-duality from [NO21b, Prop. 3.8]. It has the interesting consequence that, if \( g \) is reductive, then the cones \( C_{\pm} \) remain the same when we replace \( C \) by the maximal cone \( C_{\mathfrak{q}}^{\max} \subseteq q \) containing \( C \) (see also Corollary 2.10(b) for spaces of group type).

Proposition 4.9. Let \((\mathfrak{g}, \tau, C, h)\) be a reductive modular compactly causal symmetric Lie algebra. Then

\[
C_+ - C_- = C \cap q^{-\tau h} = C_{\mathfrak{q}}^{\max} \cap q^{-\tau h}.
\]

This proposition implies in particular that the wedge domain \( W_M(h) \) remains the same if we replace \( C \) by \( C_{\mathfrak{q}}^{\max} \). So it only depends on the “direction” of the cone \( C \), not on its specific shape. This has interesting consequences for the global structure of \( W_M(h) \).

Remark 4.10. (Factorization of wedge domains) Let \((\mathfrak{g}, \tau, C)\) be a reductive compactly causal symmetric Lie algebra. Then \((\mathfrak{g}, \tau)\) decomposes as

\[
(\mathfrak{g}, \tau) \cong (\mathfrak{g}_0, \tau_0) \oplus \bigoplus_{j=1}^{N}(\mathfrak{g}_j, \tau_j),
\]

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where \(g_0\) contains the center and all compact ideals and the \(\tau\)-invariant ideals \(g_j, j \geq 1\), are either simple hermitian or irreducible of group type (cf. [NO21a] Prop. 2.5 and Subsection 2.3). Decomposing \(q = g_0 \oplus \bigoplus_{j=1}^{N} q_j\) accordingly,

\[
C_j^{\max} = g_0 + \sum_{j=1}^{N} C_j^{\max} \quad \text{with} \quad C_j^{\max} = C^{\max} \cap q_j,
\]

where the cones \(C_j^{\max}\) are pointed. If \(M\) is the simply connected symmetric space corresponding to \((g, \tau)\), it follows that

\[
M \cong M_0 \times M_1 \times \cdots \times M_N,
\]

and

\[
W_M(h) = M_0 \times \prod_{j=1}^{N} W_{M_j}(h_j),
\]

where \(h = h_0 + \sum_{j=1}^{N} h_j\) with \(h_j \in g_j\) and \(h_0 \in \mathfrak{z}(g)\).

We have a similar global decomposition if \(G\) is semisimple with \(Z(G) = \{e\}\). Then

\[
G = \text{Aut}(G)_e \cong \prod_{j=0}^{N} \text{Aut}(\mathfrak{g}_j), \quad H = G^\tau \cong \prod_{j=0}^{N} \text{Inn}(\mathfrak{g}_j)^{\tau_j}, \quad M \cong \prod_{j=0}^{N} M_j.
\]

**Example 4.11.** (The case of compact Lie algebras) If \(g\) is a compact Lie algebra, then \(\text{ad} \, x\) has purely imaginary spectrum for every \(x \in g\), so that \(\mathcal{E}(g) = \emptyset\). However, we may consider central elements \(h \in \mathfrak{z}(g)\) as a degenerate kind of Euler elements with \(\text{ad} \, h = 0\) and \(g = g_0(h)\).

There are many compactly causal symmetric spaces for which \(g\) is a compact Lie algebra. For instance, a compact Lie algebra \(g\) contains pointed generating invariant cones \(C\) if and only if \(\mathfrak{z}(g) \neq \{0\}\), and then \(\mathfrak{z}(g) \cap C^0 \neq \emptyset\) holds for any such cone. The Lie algebra

\[
g = \mathfrak{u}_n(\mathbb{C}) \quad \text{with} \quad C := \{X \in g: -iX \geq 0\}
\]

is an important example. The corresponding causal group \((\mathfrak{u}_n(\mathbb{C}), C)\) is the conformal compactification of the euclidean Jordan algebra \(\text{Herm}_n(\mathbb{C})\), which, for \(n = 2\), is isomorphic to the 4-dimensional Minkowski space \([\text{PK91}]\). Involutions \(\tau\) of \(\mathfrak{u}_n(\mathbb{C})\) with \(\tau(C) = -C\) are obtained from involutive automorphisms of \(\mathfrak{u}_n(\mathbb{C})\), extending by \(\tau(1) = -i1\).

For any compactly causal symmetric Lie algebra \((g, \tau, C)\) and \(h \in \mathfrak{z}(g)\), we have \(C_h = \{0\}\), so that \(W_M(h) = \emptyset\). Further, the triviality of the modular flow implies \(W_{M_{\text{KMS}}}^K(h) = \emptyset\), but if \(h \in \mathfrak{z}(g)^{-}\cap C^0\), then \(W_{M}(h) = M\).

### 5 Wedge domains in spaces of group type

For our analysis of wedge domains in compactly causal symmetric spaces, we shall follow the strategy to first study spaces of group type \(G \cong (G \times G)/\Delta G\), and then use embeddings into these spaces implemented by the quadratic representation to derive corresponding results in general.

Let \((G, C)\) be a causal symmetric spaces of group type, i.e., \(G\) is a connected Lie group and \(C \subseteq g\) a pointed generating invariant cone. We further assume that the universal complexification \(g_C\); \(G \to G_C\) has discrete kernel. We consider \(G\) as the symmetric space \((G \times G)/\Delta G\) with \(\tau^{G \times G}(g_1, g_2) = (g_2, g_1)\) and \(G \times G\) acting by \((g_1, g_2) \cdot g = g_1 g_2^{-1}\). The associated causal symmetric Lie algebra is

\[
(g \oplus g, \tau, C), \quad \text{where} \quad \tau(x, y) = (y, x),
\]

and

\[
C \subseteq g \cong q = \{(x, -x): x \in g\}
\]
is an $\text{Ad}(G)$-invariant pointed generating invariant cone. The Euler element $h \in \mathcal{E}(g)$ defines the Euler element $(h, h) \in g \times g$ which generates the flow $\alpha_t(g) = \exp(th)g\exp(-th)$ on $G$. If $\tau^0_h$ is an involution on $G$ integrating $\tau_h = e^{\text{tr} \text{ad} h}$ (cf. Remark \ref{rem:adh}), then this involution extends to the involution $\tau^0_h \times \tau^0_h$ on $G \times G$.

We recall the closed/open complex Olshanski semigroup $\Gamma_G(C)$ from Definition \ref{def:olshanski}. We start with the preparation of the analysis of wedge domains in modular symmetric spaces of group type.

**Proposition 5.1.** Let $G$ be a connected Lie group with Lie algebra $g$, $h \in \mathcal{E}(g)$ an Euler element and $C \subseteq g$ an $\text{Ad}(G)$-invariant pointed closed convex cone such that:

(i) $-\tau_h(C) = C$.

(ii) $\tau_h$ integrates to an automorphism $\tau^0_h$ of $G$.

(iii) The kernel of the universal complexification $\eta_G: G \rightarrow G_C$ is discrete.

Let $g^s = \tau(s)^{-1}, g \in G$. Then the following assertions hold:

(a) Let $C^o := C_+ + C_-$ with $C_\pm := \pm C \cap g_{\pm 1}(h)$. Then the convex cones $C^o, C_+$ and $C_-$ are $G^h$-invariant and the subset

$$\Gamma_{G^h}(C^o) := G^h \exp(C^o)$$

is a real Olshanski semigroup, associated to the symmetric Lie group $(G, \tau^0_h)$. In particular, the polar map $G^h \times C^o \rightarrow \Gamma_{G^h}(C^o)$ is a homeomorphism and $\Gamma_{G^h}(C^o)$ is invariant under $s \mapsto s^t$.

(b) The semigroup

$$S(C, h) := \{ g \in G: h - \text{Ad}(g)h \in C \}$$

is invariant under $s \mapsto s^t$ and satisfies

$$S(C, h) = \exp(C_+)G^h \exp(C_-) = G^h \exp(C_+) \exp(C_-) = \exp(C_-) \exp(C_+)G^h = \exp(C_-)G^h \exp(C_+) = \Gamma_{G^h}(C^o).$$

If $C$ is generating, $S(C, h)^0 = \Gamma_{G^h}(C^{o, 0}) = S(C^o, h) := \{ g \in G: h - \text{Ad}(g)h \in C^o \}$. (48)

It is invariant under $g \mapsto g^s$.

(c) The subset of those elements $g \in G$ for which the orbit map

$$\alpha^g: \mathbb{R} \rightarrow G, \quad \alpha^g(t) = \exp(th)g\exp(-th)$$

extends to a continuous map $\alpha^g: S_g \rightarrow \Gamma_G(C)$, which is holomorphic on $S_g$ when composed with the natural map $\Gamma_G(C) \rightarrow G_C$, coincides with the closed semigroup $\Gamma_{G^h}(C^o)$.

If $C$ has interior points, then

$$W^\text{KMS}_G(h) = \Gamma_{G^h}(C^{o, 0}).$$

(d) Let $X^G_h \in \mathcal{V}(G)$ be the vector field defined by $X^G_h(g) := \frac{d}{dt} \big|_{t=0} \alpha_t(g)$. We write

$$G \times TG \rightarrow TG, \quad (g, x) \mapsto g.x$$

for left translation of vectors on $G$. Then

$$S(C, h) = \{ g \in G: X^G_h(g) \in g.C \}, \quad \text{and} \quad S(C, h)^0 = \{ g \in G: X^G_h(g) \in g.C^o \}$$

if $C$ is generating.

\footnote{If $g$ is not reductive and $C$ is pointed and generating, then $C \cap g \neq \{0\}$ (Fr Vo Thm. VII.3.10)). As $\tau_0 = \text{id}_g$, the condition $-\tau_0(C) = C$ can only be satisfied if $C$ is not generating (see Example 5.5).}
(e) On $\Gamma_{G}(C)$ we consider the antiholomorphic involution defined by
\[
\varphi_{h}(g \exp(ix)) = \tau_{h}^{G}(g) \exp(-i\tau_{h}(x)). \tag{51}
\]
Its set of fixed points coincides with
\[
\Gamma_{G}(C)\varphi_{h} = G^{h} \exp(iG^{-r_{h}}) \tag{52}
\]
and, if $C$ has interior points, then the interior of the identity component $\Gamma_{G}(C)\varphi_{h}$ satisfies
\[
S(C, h)^{0} = \alpha_{-\pi/2}((\Gamma_{G}(C)\varphi_{h})^{0}).
\]

Proof. Similar results are stated in [Ne19] for simply connected Lie groups $G$. If \( q_{G} : \tilde{G} \to G \) is the universal covering group of $G$, they apply to the group $\tilde{G}$. It therefore suffices to derive everything from [Ne19].

(a) That $C^{0}, C_{+}$ and $C_{-}$ are $G^{h}$ invariant follows from $\text{Ad}(G^{h})q_{G}(h) = q_{G}(h)$ and the invariance of $C$. The remainder of (a) follows by applying [Ne19, Prop. 2.6] to $\tilde{G}$ because $\ker(q_{G}) \subseteq G^{h}$ and $G^{h} = q_{G}(\tilde{G})$. (b) First we note that for $g \in G$ we have
\[
-\tau(h - \text{Ad}(g^{+}h)) = \text{Ad}(g^{-1}h - h) = \text{Ad}(g^{-1})(h - \text{Ad}(g)h) \in C
\]
As $C$ is $G$-invariant. Thus $h - \text{Ad}(g^{+}h) \in -\tau(C) = C$. That the four middle sets in (47) agree now follows from the $\text{Ad}(G^{h})$ invariance of $C_{\pm}$.

For $\tilde{S}(C, h) := \{g \in \tilde{G} : h = \text{Ad}(g)h \in C\}$ we have $\tilde{S}(C, h) = \tilde{q}_{G}^{-1}(S(C, h))$, so that the corresponding result for $\tilde{G}$ ([Ne19, Thm. 2.16]) and the invariance of $G^{h}, S(C, h)$ and $\Gamma_{G^{h}}(C^{0}) := G^{h} \exp(C^{0})$ under multiplication with the central subgroup $\ker(q_{G})$ show that it also holds for $G$.

Assume, in addition, that $C$ is generating, so that $C^{0} \neq \emptyset$. Then $S(C^{0}, h) \subseteq S(C, h)$ is an open subsemigroup, hence contained in
\[
S(C, h)^{0} = G^{h} \exp(C^{0}) = G^{h} \exp(C_{+}^{0}) \exp(C_{-}^{0}) = \exp(C_{+}^{0}) \exp(C_{-}^{0})G^{h}.
\]
To show equality, write an element $g \in S(C, h)^{0}$ as
\[
g = \exp(x_{-}) \exp(x_{+})g' \quad \text{with} \quad g' \in G^{h}_{C}, x_{\pm} \in C_{\pm}^{0}.
\]
Then
\[
\begin{align*}
\text{Ad}(g)h &= e^{ad x_{+}}e^{ad x_{-}}h = e^{ad x_{-}}(h + [x_{+}, h]) = e^{ad x_{-}}(h - x_{+}) \\
&= h - x_{+} + [x_{+}, h] - [x_{-} - [x_{-}, x_{+}]] = h - \text{Ad}([x_{+}, x_{-}, [x_{-}, x_{+}]]).
\end{align*}
\]
as $C^{0}$ is $G$-invariant. It follows that $h - \text{Ad}(g)h \in C^{0}$, and hence $g \in S(C^{0}, h)$.

(c) For $\tilde{G}$ and the closed cone, this is [Ne19, Thm. 2.21]. As $\tilde{G}^{h}$ and the specified subset of $G$ are $\ker(q_{G})$-invariant, the assertion also holds for $G$.

For $z = x + iy \in \mathcal{C}$ with $x, y \in \mathfrak{g}$, we write $x = \text{Re} z$ and $y = \text{Im} z$. To verify [49], for $g = g_{0} \exp(x_{+} + x_{-})$ with $g_{0} \in G^{h}, x_{\pm} \in C_{\pm} = C \cap \mathfrak{g}_{\pm 1}(h)$, and $0 < t < \pi$, we note that $\sin(t) > 0$ and
\[
\begin{align*}
\alpha_{it}(g_{0} \exp(x_{+} + x_{-})) &= g_{0} \exp(\exp(it x_{+} + e^{-it} x_{-}) \\
&= g_{0} \exp((\cos t + i \sin t)x_{+} + (\cos t - i \sin t)x_{-}).
\end{align*}
\]
Thus
\[
\text{Im}(e^{it}x_{+}e^{-it}x_{-}) = \sin(t)(x_{+} - x_{-}).
\]
We therefore obtain (40).

(d) With the notation for left and right translations by \( G \) on \( TG \) (see the notation introduced in the introduction), we have

\[
X_h^C(g) = h.g - g.h = g.(\mathrm{Ad}(g)^{-1}h - h).
\]

Hence (50) follows immediately from \( \mathrm{Ad}(g)C = C \). (47) and (48).

(e) Equation (52) follows from (51). We observe that

\[
\alpha_{\pi i/2}(S(C, h), e) = (C^h)_{e} \exp (e^{\frac{\pi i}{2} \mathrm{ad} h} (C_{+} + C_{-})) = (G^h)_{e} \exp (i(C_{+} - C_{-})),
\]

so that \( \alpha_{\pi i/2}(S(C, h), e) \) is the identity component of the fixed point set \( \Gamma G(C)^{\sim} \) in the closed Olshanski semi-group \( \Gamma G(C) \). If \( C \) has interior points, this argument shows that \( \alpha_{\pi i/3}(S(C, h)^{0}) \) is the connected component of the fixed point submanifold in the open semi-group \( \Gamma G(C)^{0} \) whose closure contains the identity. This proves (e). \( \square \)

To provide a good context to deal also with group type spaces that are not reductive, we have to deal with invariant cones \( C \subseteq g \) which are not generating. This is mainly due to the fact that the requirement of \( C \) being generating is not compatible with \( -\tau(h)C = C \). As in Subsection 2.1, this problem can be overcome by observing that

\[
g_{C} = C - C \subseteq g
\]

is an ideal of the Lie algebra \( g \). We consider the corresponding integral subgroup \( G_{C} \subseteq G \), endowed with its natural Lie group structure (it is closed if \( G \) is simply connected). Then

\[
(G \times G)C := \{(g_{1}, g_{2}) \in G : g_{1}g_{2}^{-1} \in G_{C}\}
\]

is a Lie group containing the diagonal \( \Delta_{G} \) and isomorphic to \( G_{C} \times G \), where \( G \) acts by conjugation on \( G_{C} \). Then

\[
G_{C} \cong (G \times G)C/\Delta_{G}
\]

is a symmetric space on which

\[
V_{+}(g) := g.C^{0}
\]

defines a \((G \times G)C\)-invariant field of pointed open cones (a causal structure). The corresponding symmetric Lie algebra is

\[
(g \oplus g)C = \{(x, y) \in g \oplus g : x - y \in g_{C}\} \quad \text{with} \quad \tau(x, y) = (y, x).
\]

Here \( q = \{(x, -x) : x \in g_{C}\} \cong g_{C} \) contains the pointed generating cone \( \{(x, -x) : x \in C\} \cong C \). We now determine the wedge domains in the causal symmetric space \( M = G_{C} \) for the structure defined by \((g \oplus g)C, \tau, C\).

We do not assume that \( C \) is generating in \( g \), but it is natural to assume that the two cones \( C_{\pm} \) generate \( g_{\pm 1}(h) \) to ensure that the semigroup \( S(C, h) \) has interior points.

**Theorem 5.2.** (Wedge domains in spaces of group type) Suppose that \( C_{\pm} \) generate \( g_{\pm 1}(h) \). Then the wedge domains in the compactly causal symmetric space \((G_{C}, C)\) corresponding to the modular compactly causal symmetric Lie algebra \(( (g \oplus g)C, \tau, C, (h, h) ) \) are

\[
W_{G_{C}}^{KMS}(h) = W_{G_{C}}^{KM}(h) = W_{G_{C}}^{\text{ad}}(h) = G_{C} \exp (C_{+}^{0} + C_{-}^{0}).
\]
Proof. The fixed point set of the modular flow in \( M = G_C \) is the centralizer \( G_C^h \) of \( G \). For a pair \((g_1, g_2) \in (G \times G)_C\), we have \( m := (g_1, g_2).e = g_1 g_2^{-1} \in G_C^h \) if and only if \( \text{Ad}(g_1)^{-1} h = \text{Ad}(g_2)^{-1} h \), i.e., \( \text{Ad}(g_1, g_2)^{-1}(h, h) \in h = \Delta_g \). The invariance of \( C \) under \( \text{Ad}(G) \) implies the invariance of the cones \( C_\pm = \pm C \cap g_{\pm 1}(h) \) under \( \text{Ad}(G^h) \). Therefore \( C^c = C^+ + C^- \subseteq g^{-\tau h} \) is invariant under \( \text{Ad}(G^h) \).

For \( m = g_1 g_2^{-1} \in G_C^h \), the modular flow acts on \( T_m(G_C) = g_1 g_C g_2^{-1} = m.g_C \) by

\[
\alpha_t(m.x) = m.e^{t \text{ad} h} x,
\]

so that \( C^c_m := m.C^c \) (here we use the dot notation for left and right translations on \( T(G) \)). Therefore the polar wedge domain in \( M = G_C \) is

\[
W_G^C(h) = \bigcup_{m \in G_C^h} m \exp(C^c, 0) = G_C^h \exp(C^c, 0).
\]

Next we derive from Proposition 5.1(c) that

\[
W_{\text{KMS}}^C(h) = G_C^h \exp(C_0^+ + C_0^-) = W_G^C(h).
\]

Finally, we observe that Proposition 5.1(a),(d) imply that

\[
W_{G_C}^+(h) = G_C^h \exp(C_0^+ + C_0^-) = W_G^C(h).
\]

The assertion now follows from (53) and (55).

The open subset \( W_G^C(h) \) need not be connected, but its connected components are parametrized by the group \( \pi_0(G_C^h) \) of connected components of \( G_C^h \).

Remark 5.3. (Some information on \( \pi_0(G^h) \))

(a) As the example \( G = \tilde{\text{SL}}_2(\mathbb{R}) \) with the Cayley type involution

\[
\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}
\]

shows, \( G^h \) is in general not connected. It contains \( Z(G) \cong \mathbb{Z} \) and

\[
(G^h)_e = \exp(\mathbb{R}h) \text{ for } h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

This shows that \( \pi_0(G^h) \cong \mathbb{Z} \). This example also shows that, in general

\[
G^h \not\subseteq G^{\tau G},
\]

because here \( \tau^G \) acts on \( Z(G) \) by inversion (cf. [MN21], Ex. 2.10(d)).

(b) If \( G \) is contained in \( G_C \), then \( (G_C)^h \) coincides with the centralizer of the circle group \( \exp(\mathbb{R}h) \), hence is connected ([HN12, Cor. 14.3.10]). Therefore \( G^h = (G_C)^h \cap G \) is the fixed point group of the complex conjugation (with respect to \( G \)) on the connected complex group \( (G_C)^h \). Thus \( \pi_0(G^h) \) is a finite elementary abelian 2-group, i.e., isomorphic to \( \mathbb{Z}_2^k \) for some \( k \in \mathbb{N}_0 \) ([Lo69, Thm. IV.3.4]).
6 Wedge domains in extendable causal symmetric spaces

In this section we turn to the wedge domains in compactly causal symmetric spaces. We assume that \((\mathfrak{g}, \tau, C)\) is extendable in the sense of Subsection 2.1, i.e., there exists a \(G\)-invariant convex cone \(C \subseteq \mathfrak{g}\) such that \(-\tau C = C\) and \(C \cap \mathfrak{q} = C\). This always holds if \(\mathfrak{g}\) is reductive by the Extension Theorem 2.4. We write \(\mathfrak{g}_C = C - C \triangleq \mathfrak{g}\) for the ideal generated by \(C\) and \(G_C \leq G\) for the connected normal subgroup whose Lie algebra is \(\mathfrak{g}_C\) (see [24]).

The main result of this section is Theorem 6.3 asserting that all three wedge domains in \(M = G/H\) are the same. This is first proved for the special case \(H = G^C\) in Theorem 6.4. The proof of this theorem builds heavily on the group case (Theorem 5.2). We conclude this section with a brief discussion of the assumption that the Euler element \(h\) is contained in \(\mathfrak{h}\), i.e., that the corresponding modular flow on \(M\) has a fixed point.

The following proposition identifies the basic connected components of the wedge domains in \(G^C\).

**Proposition 6.1.** Let \((G, \tau^G)\) be a connected symmetric Lie group corresponding to the modular compactly causal symmetric Lie algebra \((\mathfrak{g}, \tau, C, h)\),

\[
M_G := G^G = G^C = (G^C)_e,
\]

and let \(C \subseteq \mathfrak{g}\) be a pointed invariant closed convex cone with

\[
C \cap \mathfrak{q} = C \quad \text{and} \quad -\tau(C) = -\tau_h(C) = C.
\]

For the real Olshanski semigroup \(S(C, h) = G^C_0 \exp(C^e_\theta)\), the following assertions hold:

(a) The basic connected component of the wedge domains in \(M_G\) coincide:

\[
W_{M_G}(h)e = W_{\mathfrak{M}_G}(h)e = W_{\mathfrak{KMS}}(h)e,
\]

and this domain can be obtained from the semigroup \(S(C, h) = G^C_0 \exp(C^e_\theta)\) as

\[
S(C^0_\theta, h)^e = \bigcup_{g \in (G^C_\theta)e} g \exp(C^0_+ + C^0_0)g^2 \cong (G^C_\theta)e \times (G^C_\theta)_{e, e} (C^0_+ + C^0_0).
\]

(b) \(W_{M_G}(h)e\) is the orbit \(S(C^0_\theta, h)^e\) of the base point under the action of the open semigroup \(S(C^0_\theta, h)^e\) on \(M_G = (G^C_\theta)_{e, e}\) by \(g.x := gxg^\tau\).

**Proof.** The flow \(\alpha_t\) commutes with the involution \(\tau\). If \(m \in M_G \cong (G^C_0)_e \subseteq G^C_\theta\), is contained in \(W_{\mathfrak{KMS}}(h)\) (Definition 1.6), then

\[
\alpha_z(m) \in \mathcal{T}_{M_G}(C) \subseteq \mathcal{T}_{G^C}(C) = \Gamma_{G^C}(C^0_\theta) \quad \text{for} \quad z \in S
\]

(cf. Remark 5.5), so that Theorem 5.3 implies that

\[
W_{\mathfrak{KMS}}(h) \subseteq M_G \cap W_{\mathfrak{KMS}}(h) = M_G \cap S(C^0_\theta, h) \subseteq S(C^0_\theta, h)^\tau,
\]

and thus

\[
W_{\mathfrak{KMS}}(h) \subseteq S(C^0_\theta, h)^e = (G^C_\theta)e. \exp((C^0_\theta)^{-\tau})
\]

\[
\cong (G^C_\theta)e. \exp((C^0_+ + C^0_0)^\tau) = W_{M_G}(h)e.
\]

The invariance of both sides under \((G^C_\theta)e\) reduces the verification of the converse inclusion to showing that

\[
\exp(C^0_+ + C^0_0) \subseteq W_{\mathfrak{KMS}}(h).
\]
For $x_+ \in C_\pm$ and $0 < t < \pi$, we have $\sin t > 0$ and hence
\[
\text{Im}(e^{it}x_+ + e^{-it}x_-) = \sin(t)(x_+ - x_-) \in C_0.
\]
Now (59) follows from $\alpha_M \text{Exp}(x_+ + x_-) = \text{Exp}(e^{it}x_+ + e^{-it}x_-)$, $\text{Exp}(e^{it}x_+ + e^{-it}x_-) \in \Gamma_G(C_0^\alpha)$ and
\[
T_{M_G}(C) = \{ss^2 : s \in \Gamma_G(C_0^\alpha)\} \subseteq \Gamma_G(C_0^\alpha) = T_G(C_0).
\]
Next we observe that $V_+(m) = g.C_0^\alpha.g^\gamma = T_{m(G)} \cap g.C_0^\alpha.g^\gamma = T_{m(G)} \cap V_{M_G}(g^\gamma)$.
\[
(\text{Proposition 5.1}). \text{ Further, } e\gamma : \gamma \in C_0^\alpha, \gamma \cap G \text{ is } -\gamma \text{-invariant.}
\]
We thus conclude with (60) and Lemma A.2(3,4) that
\[
W_{M_G}^{\text{KMS}}(h)_e = (G_\gamma)_{c}. \text{Exp}(C_0^\alpha + C_0^\alpha) = W_{M_G}(h)_e.
\]
In view of Theorem 6.3, this shows that
\[
W_{M_G}^{\pm}(h) = W_{G_G}^{\pm}(h) \cap M_G = S(C_\gamma,h)_e \cap M_G.
\]
The semigroup $S(C_\gamma,h) = G_\gamma \text{exp}(C_\gamma)$ is a real Olshanski semigroup corresponding to the involution $\tau_h$ on $g$ (Proposition 3.4). Further, $\tau$ and $\tau_h$ commute and the cone $C_\alpha$ is $-\tau$-invariant (cf. (16)), so that Lemma A.2(3) shows that
\[
S(C_\gamma, h) = (G_\gamma)_{c}. \text{exp}(C_\gamma^\alpha) = (G_\gamma)_{c}. \text{exp}(C_\gamma) = (G_\gamma)_{c}. \text{exp}(C_+ + C_-).
\]
We thus conclude with (62) and Lemma A.2(3,4) that
\[
W_{M_G}^{\pm}(h)_c \overset{\text{A.2}}{=} S(C_\gamma,h)_{e} \overset{\text{60}}{=} W_{M_G}^{\text{KMS}}(h)_e.
\]
Now (a) follows from (62), so that (b) is a direct consequence of (13).

The preceding proposition identifies the basic components of the wedge domains in the compactly causal symmetric space $M_G$. We now prepare the ground for the analysis of the other connected components. We recall the polar wedge domain
\[
W_M(h) = \bigcup_{m \in M_0} \text{Exp}_m(C_0^\alpha(h)_0^0) \overset{\text{63}}{=} \bigcup_{g \in H \in M_0} g. \text{Exp}_H(C_0^\alpha(\text{Ad}(g)^{-1}h)_0^0) \subseteq M.
\]
Comparing with Definition 3.1 and using that $M_0^\alpha = G_\gamma.eH$ (Proposition 5.4), we now see that
\[
W_M(h)_eH = G_\gamma.eH(C_0^\alpha(h)_0^0) = \bigcup_{m \in M_0^\alpha} \text{Exp}_m(C_0^\alpha(h)_0).\]

**Remark 6.2.** As the connected components of $M_0$ may have different dimensions (Example 5.3) and the cones $C_0^\alpha \subseteq T_m(M)$ span complements to $T_m(M_0)$, they may also be of different dimension.

**Theorem 6.3.** (Wedge domains in extendable compactly causal symmetric spaces) Let $(G, \tau^G)$ be a connected symmetric Lie group corresponding to the modular compactly causal symmetric Lie algebra $(g, \tau, C, h)$,
\[
M = M_G = G_\gamma \cong G/G^\tau \cong (G_\gamma)_e,
\]
and let $C_\gamma \subseteq g$ be a pointed invariant closed convex cone with
\[
C_\gamma \cap g = C \quad \text{and} \quad -\tau(C_\gamma) = -\tau_h(C_\gamma) = C_\gamma.
\]
Then
\[
W_M(h) = W_M^+(h) = W_M^{\text{KMS}}(h).
\]
Proof. We start with the discussion of the positivity domain $W_M^+(h)$. From (62) in the proof of Proposition 6.1 we know that, for

$$S_C := S(C_0, h) = G_C^h \exp(C_0^h) \quad \text{and} \quad S_C^0 = G_C^h \exp(C_0^c, 0),$$

we have

$$W_M^+(h) = S_C^0 \cap M = S_C^0 \cap G_c^h \subseteq (S_C^0)^2$$

is the union of all connected components of $(S_C^0)^2$ which are contained in $M = G_C^h$. Information on these connected components comes from Lemma A.2 which shows that each connected component of $(S_C^0)^2$ intersects $(G_C^h)^2$, the $\alpha$-fixed point set in $G_C^h$. So let $g_0 \in M^\alpha \cap S_C$ and consider the involution $\gamma := g_0^{-1} \tau(g) g_0$ of $G$ and the Lie algebra involution

$$\gamma := \Ad(g_0)^{-1} \tau = \Ad(\tau(g)) \tau \Ad(\tau(g))^{-1} \in \Aut(g).$$

By Lemma A.2 (5)(c), the connected component of $S_C^0$ containing $g_0$ is

$$g_0 \{ s \gamma^G(s) : s \in S_{C, e} \} = g_0 \bigcup_{g_1 \in (G_C^h)} g_1 \exp(C_0^c, -\gamma) g_0^{-1} g_1^4 g_0. \quad (65)$$

We write $g_0 = gg^4$ with $g \in G$ and observe that $h' = \Ad(g)^{-1} h \in \mathfrak{h}$, so that we also have $h' = \tau(h') = \Ad(\tau(g))^{-1} h$. The Lie algebra involution $\gamma$ satisfies

$$\Ad(g^4) g^{-\gamma} = \Ad(\tau(g)^{-1} g^{-\gamma} = g = q. \quad (66)$$

Therefore the cone

$$C_0^c, -\gamma = C_0^c \cap g^{-\gamma} = (C_0 \cap g_1(h) - C_0 \cap g_1^{-1}(h)) \cap g^{-\gamma}$$

satisfies

$$\Ad(g^4) C_0^c, -\gamma = (C_0 \cap g_1(h') - C_0 \cap g_1^{-1}(h')) \cap q = C \cap g_1(h') - C \cap g_1^{-1}(h') =: C_0^c(h'), \quad (67)$$

where we have used that the projection $p_\mathfrak{q} : \mathfrak{g} \to \mathfrak{q}$ commutes with $\ad h'$ and maps $C_0^c$ into itself.

We now arrive with (65) at

$$(S_C)^2_{g_0} = \bigcup_{g_1 \in (G_C^h)_c} g_0 g_1 \exp(C_0^c, -\gamma) g_0^{-1} g_1^4 g_0
\begin{align*}
= & \bigcup_{g_1 \in (G_C^h)_c} (g_0 g_1 g_0^{-1}) g_3 \exp(C_0^c, -\gamma) (g_0 g_1 g_0^{-1})^2 \\
= & \bigcup_{g_2 \in (G_C^h)_c} g_2 g_0 \exp(C_0^c, -\gamma) g_2^4 = (G_C^h)_{c}. g_3 \exp(C_0^c, -\gamma)) \\
= & (G_C^h)_{c}. (gg^4 \exp(C_0^c, -\gamma)) = (G_C^h)_{c}. \left( g \exp \left( \Ad(g^4) C_0^c, -\gamma \right) g^4 \right)
\end{align*}$$

(64)

For the interior of this connected component, we obtain

$$(S_C^0)^2_{g_0} = (G_C^h)_{c}. g. \exp(C_0^c, h')^0. \quad (68)$$

The boundary of this domain contains in particular the connected component

$$M_0^\alpha = (G_C^h)_{c}. g.e = \{ g_1 g g^4 g_1^4 : g_1 \in (G_C^h)_c \}$$

(cf. Proposition 3.4),

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We are now ready to use this information to identify the positivity domain. As $X^M_h(g.m) \in V_+(g.m) = g.V_+(m)$ is equivalent to $g^{-1}X^M_h(g.m) = X^M_h(m) \in V_+(m)$, it follows that

$$g^{-1}W^+_M(h) = W^+_M(h') = W^+_M(\text{Ad}(g)^{-1}h). \quad (69)$$

Combining Remark 6.4 with Proposition 6.1, we conclude that

$$W_M(h) = \bigcup_{m \in M^\alpha} W_M(h)_m \bigcup_{gH \in M^\alpha} g.W_M(\text{Ad}(g)^{-1}h)_h$$

$$\subseteq \bigcup_{gH \in M^\alpha} g.W^+_M(\text{Ad}(g)^{-1}h) \subseteq W^+_M(h). \quad (70)$$

To obtain the converse inclusion, we use (62) to obtain

$$W^+_M(h) \subseteq S^0_C \cap M = S^0_C \cap G^2_x = \bigcup_{g \in G, g \notin G^h} (S^0_C)_{gg^h}$$

$$\subseteq \bigcup_{g \in G, \text{Ad}(g)^{-1}h \notin M^\alpha} (S^0_C)_{gg^h}. \quad (71)$$

This shows that

$$g^{-1}W^{KMS}_M(h) = W^{KMS}_M(h') = W^{KMS}_M(\text{Ad}(g)^{-1}h). \quad (72)$$

As for the positivity domain, we now obtain with Proposition 6.1

$$W_M(h) = \bigcup_{m \in M^\alpha} W_M(h)_m = \bigcup_{gH \in M^\alpha} g.W_M(\text{Ad}(g)^{-1}h)_h$$

$$\subseteq \bigcup_{gH \in M^\alpha} g.W^{KMS}_M(\text{Ad}(g)^{-1}h) \subseteq W^{KMS}_M(h). \quad (73)$$

Combining (70) with (73), we finally get

$$W^{KMS}_M(h) \subseteq \bigcup_{g \in G, \text{Ad}(g)^{-1}h \notin M^\alpha} (S^0_C)_{gg^h}$$

and therefore the equality $W_M(h) = W^{KMS}_M(h)$.

\[\square\]

**Remark 6.4.** The relation in (69) shows that any $h' = \text{Ad}(g)^{-1}h \in O_h$ leads to a wedge domain $W_M(h') = g^{-1}W_M(h)$ that is a translate of the wedge domain $W_M(h)$. In this sense the geometric structure of the set $W_M(h)$ does not depend on the choice of the Euler element in $O_h \cap \mathfrak{h}$. Nevertheless the structure of the basic connected component $W_M(h)_h$ of the wedge domain $W_M(h)$ does (Example 5.3).

Eventually, we turn to the general case, where $M = G/H$ does not necessarily embed into $G$.

**Theorem 6.5.** (Wedge domains in extendable compactly causal symmetric spaces) Let $(G, \gamma^G)$ be a connected symmetric Lie group corresponding to the modular compactly causal symmetric Lie algebra $(\mathfrak{g}, \gamma, C, h)$, assume that $\eta_G: G \rightarrow G_c$ has discrete kernel, let $H \subseteq \gamma^G_c$ be an open subgroup, $M := G/H$, and let $C^g \subseteq \mathfrak{g}$ be a pointed invariant closed convex cone with

$$C^g \cap \mathfrak{h} = C \quad \text{and} \quad -\tau(C^g) = -\tau_h(C^g) = C^g.$$

Then

$$W_M(h) = W^+_M(h) = W^{KMS}_M(h). \quad (74)$$
Proof. Let \( \tau \in \text{Aut}(\tilde{G}) \) be the involution integrating \( \tau \). In the simply connected covering \( \tilde{G} \) of \( G \), the subgroup \( \tilde{G}^\tau \) is connected by [Lo60] Thm. IV.3.4, so that [HM12] Cor. 11.1.14 implies that
\[
M_{\tilde{G}} := \tilde{G}_e^\tau \cong \tilde{G}/\tilde{G}^\tau
\]
is simply connected, hence can be identified with the simply connected covering \( \tilde{M} \) of \( M \). Accordingly, the universal covering map is
\[
q_M: \tilde{M} = \tilde{G}/\tilde{G}^\tau \to M = G/H, \quad g\tilde{G}^\tau \mapsto q_G(g)H,
\]
where \( q_G: \tilde{G} \to G \) is the universal covering of \( G \). From the isomorphism \( \tilde{M} \cong M_{\tilde{G}} \) and Theorem 6.3, we obtain the equalities
\[
W^+_M(h) = W^{KMS}_{M\tilde{G}}(h) = W_{\tilde{M}}(h).
\] (75)

The equivariance of \( q_M \) with respect to the modular flow implies that
\[
W^+_M(h) = q^{-1}_M(W^+_M(h)), \quad \text{hence} \quad W^+_M(h) = q_M(W^+_M(h)).
\] (76)

We also have coverings of the tube domains \( T_{\tilde{M}}(C) \to T_M(C) \) which are equivariant with respect to the modular flow. This entails
\[
q^{-1}_M(W^{KMS}_{\tilde{M}}(h)) = W^{KMS}_M(h), \quad \text{and thus} \quad W^{KMS}_M(h) = q_M(W^{KMS}_{\tilde{M}}(h)).
\] (77)

In fact, the inclusion follows directly from the equivariance. For the converse inclusions we use the existence of lifts of trajectories of the imaginary vector field \( iX^\tau_h \) in \( T_{\tilde{M}}(C) \) to \( T^{\tau}_{\tilde{M}}(C) \).

Next we observe that
\[
\tilde{M}^\alpha = \{ m \in \tilde{M} : X^\tau_h(m) = 0 \} = q^{-1}_M(M^\alpha).
\] (78)

As \( T_m(q_M)V^+_m(m) = V^+_m(q_M(m)) \) and \( q_M \) is \( \alpha \)-equivariant, we have
\[
T_m(q_M)C^c_m = C^{q_M}_m,
\]
and thus, for \( m \in \tilde{M}^\alpha \),
\[
q_M(W_{\tilde{M}}(h)_m) = q_M(G^h_e \cdot \text{Exp}_m(C^c_m)) = G^h_e \cdot \text{Exp}_{q_M}(T_m(q_M)C^c_m).
\]

Taking the union over all \( \alpha \)-fixed points, we arrive with (78) at
\[
q_M(W_{\tilde{M}}(h)) = W_M(h).
\] (79)

Applying \( q_M \) to the equalities in (75) now leads with (76), (77) and (79) to (74).

Assumptions on the Euler element

We assume throughout that the Euler element \( h \) generating the modular flow on \( M = G/H \) is contained in \( h = \mathfrak{g}^r \) (see in particular Proposition 2.7(b)).

For group type spaces \( (\mathfrak{g} \oplus \mathfrak{g}, \tau_{\mathfrak{g} \oplus \mathfrak{g}}) \), Euler elements in \( \mathfrak{g} \oplus \mathfrak{g} \) are of the form \( h = (h_1, h_2) \), where \( h_j \in \mathfrak{e}(\mathfrak{g}) \) are Euler elements. The following proposition shows that, for \( h \not\in \mathfrak{h} \), the wedge domains may degenerate drastically.\footnote{If \( \mathfrak{g} \) is simple hermitian, then \( \text{Inn}(\mathfrak{g}) \) acts transitively on \( \mathfrak{e}(\mathfrak{g}) \) by Proposition 2.7, so that there are only 3 orbits of \( \text{Inn}(\mathfrak{g} \oplus \mathfrak{g}) \) on \( \mathfrak{e}(\mathfrak{g} \oplus \mathfrak{g}) \), represented by elements of the form \( (h, h), (h, 0), (0, h) \) with \( h \in \mathfrak{e}(\mathfrak{g}) \).}
Proposition 6.6. Let \((\mathfrak{g} \oplus \mathfrak{g}, \tau_{\text{fix}}, C)\) be a causal symmetric Lie algebra of group type. If \(h_0 \in \mathcal{E}(\mathfrak{g})\), then \(h := (h_0, 0) \in \mathcal{E}(\mathfrak{g} \oplus \mathfrak{g})\) and

\[ W_G(h) = W^+_G(h) = W^{\text{KMS}}_G(h) = \emptyset. \]

Proof. We write

\[ C = \{ (x, -x) : x \in C_\mathfrak{g} \} \]

for an invariant cone \(C_\mathfrak{g} \subseteq \mathfrak{g}\). The modular flow acts on \(G\) by left translations \(\alpha_t(g) = \exp(th_0)g\). For \(g \in G\), the inclusion

\[ \exp(\mathbf{t}h_0)g \subseteq G \exp(iC_\mathfrak{g}) \]

implies \(h_0 \in C_\mathfrak{g}\), contradicting the fact that \(\text{ad} h_0\) is diagonalizable and the elements of \(C_\mathfrak{g}\) have purely imaginary spectrum. This shows that \(W^{\text{KMS}}_G(h) = \emptyset\).

Moreover,

\[ (\mathfrak{g} \oplus \mathfrak{g})_{\pm 1}(h) = \mathfrak{g}_{\pm 1}(h_0) \oplus \{0\} \]

implies that \(C_{\pm 1} = C \cap (\mathfrak{g} \oplus \mathfrak{g})_{\pm 1}(h) = \{0\}\), so that \(W_G(h) = \emptyset\).

Finally, we observe that

\[ p_0(\text{Ad}(g_1, g_2)^{-1}(h_0, 0)) = \frac{1}{2}(\text{Ad}(g_1^{-1})h_0, - \text{Ad}(g_1)^{-1}h_0) \in C^0_\mathfrak{g} \]

is equivalent to \(\text{Ad}(g_1^{-1})h \in C^0_\mathfrak{g}\), which is never the case because \(C^0_\mathfrak{g}\) consists of elliptic elements. This shows that \(W^+_G(h) = \emptyset\).

For right translations one argues similarly. On the other hand, the Euler elements of the form \((h, h)\) correspond to non-trivial wedge domains, such as

\[ W_G(h) = G^h \exp(C^0_\mathfrak{g}) \]

(Theorem 5.2).

7 Nets of standard subspaces

In this section we turn to representation theoretic aspects of compactly causal symmetric spaces. We start with introducing standard subspaces, the Brunetti–Guido–Longo construction and recall the construction of nets of closed real subspaces based on distribution vectors from [NO21a]. In Subsection 7.4 we show in the general Theorem 7.5 how the methods from [NO21a] can be used to construct covariant nets of standard subspaces on compactly causal symmetric spaces.

7.1 Standard subspaces

In this section we recall the notion of standard subspaces and modular objects and how those objects are related to anti-unitary representation of \(\mathbb{R}^\infty\).

Let \(\mathcal{H}\) be a complex Hilbert space. A closed real subspace \(\mathcal{V} \subseteq \mathcal{H}\) is called standard if

\[ \mathcal{V} \cap i\mathcal{V} = \{0\} \quad \text{and} \quad \mathcal{H} = \mathcal{V} + i\mathcal{V} \tag{80} \]

(see [Lo08] or [NO02] for basic facts on standard subspaces). Associated to every standard subspace is the closed densely defined conjugate linear operator

\[ \sigma_{\mathcal{V}} : \mathcal{V} + i\mathcal{V} \to \mathcal{H}, \quad x + iy \mapsto x - iy \]

with polar decomposition

\[ \sigma_{\mathcal{V}} = J_{\mathcal{V}} \Delta^{1/2}_{\mathcal{V}}. \]
The operator $J_t$ is an everywhere defined conjugation (an antiunitary involution), and the modular operator $\Delta_t$ is a positive selfadjoint operator. These two operators satisfy the modular relation $J_t\Delta_tJ_t = \Delta_t^{-1}$.

Conversely, every pair $(\Delta, J)$, consisting of a positive selfadjoint operator $\Delta$ and a conjugation satisfying the modular relation $J\Delta J = \Delta^{-1}$ defines a standard subspace via

$$V := \text{Fix}(J\Delta^{1/2}).$$

In this sense standard subspaces are in bijection with pairs of modular objects.

We also note that, for any pair $(\Delta, J)$ of modular objects,

$$U(e^t) := \Delta^{-it/2\pi} \quad \text{and} \quad U(-1) := J$$

(81)

define a continuous homomorphism $U : \mathbb{R}^\times \to \text{AU}(\mathcal{H})$ mapping negative numbers to antiunitary operators. Conversely, every such homomorphism is obtained for

$$\Delta := e^{2\pi i \cdot \partial U(1)} \quad \text{and} \quad J := U(-1).$$

(82)

### 7.2 The BGL construction

In this section we generalize the above construction for $\mathbb{R}^\times$ to more general Lie groups. For an involutive automorphism $\tau^G$ on a connected Lie group $G$, we consider the group

$$G_\tau = G \times \{\text{id}_G, \tau^G\}.$$  

An antiunitary representation of $G_\tau$ is a homomorphism $U : G_\tau \to \text{AU}(\mathcal{H})$ (the group of unitary and antiunitary operators on $\mathcal{H}$) such that $U$ is strongly continuous and $J := U(\tau^G)$ is a conjugation. We then have

$$JU(g)J = U(\tau^G(g)) \quad \text{for} \quad g \in G.$$

For every $h \in \mathfrak{g}$ fixed by $\tau := L(\tau^G)$, we then obtain a standard subspace

$$V := V_{(h, \tau^G, U)} \subseteq \mathcal{H},$$

specified by

$$J_t = U(\tau^G) \quad \text{and} \quad \Delta_t^{-it/2\pi} = U(\exp th) \quad \text{for} \quad t \in \mathbb{R}$$

(83)

as in [G]. Here the fact that $U(\tau^G)$ commutes with $U(\exp \mathbb{R}h)$ implies the modular relation, so that $V = \text{Fix}(J_t\Delta_t^{1/2})$ is a standard subspace of $\mathcal{H}$. This assignment is called the Brunetti-Guido-Longo (BGL) construction (see [BGL2]).

### 7.3 Nets of real subspaces on homogeneous spaces

With the BGL construction, standard subspaces can be associated to antiunitary representations in abundance, but only a few of them carry interesting geometric information. In particular, we would like to understand when a standard subspace of the form $V_{(h, \tau^G, U)}$ arises from a natural family $V(O)$ of real subspaces associated to open subsets of a homogeneous space $M = G/P$ and for which domains $O \subseteq M$ the subspace $V(O)$ is standard.

Covariant families of real subspaces of $\mathcal{H}$ which are not necessarily standard are easy to construct in any unitary representation $(U, \mathcal{H})$ of $G$ from distribution vectors (see Appendix [B] for definitions and basic properties). To a real linear subspace $E \subseteq \mathcal{H}^{-\infty}$ and an open subset $O \subseteq G$, we associate a closed real subspace of $\mathcal{H}$ by

$$H_E(O) := \text{span}_\mathbb{R} \left\{ U^{-\infty}(C^\infty(O, \mathbb{R}))E \right\} \subseteq \mathcal{H}.$$  

(84)

With the projection map $q_P : G \to G/P$, we then obtain on $G/P$ a net of real subspaces by

$$H_E^{G/P}(O) := H_E(q_P^{-1}(O)).$$

If $E$ is $P$-invariant, then we actually have $H_E(O) = H_E(PO)$ ([NÓ21a Lemma 2.11]), so that any subspace $H_E(O)$ attached to an open subset of $G$ also corresponds to an open subset of $G/P$.  

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7.4 Connecting wedge domains and standard subspaces

We now connect the BGL construction with the construction based on real subspaces of distribution vectors. For an Euler element $h \in \mathfrak{g}$ we consider the involution $\tau_h = e^{\pi i ad h}$ and assume that the Lie algebra involution $\tau_h$ integrates to an involutive automorphism $\tau_h^G$ of the connected Lie group $G$ with Lie algebra $\mathfrak{g}$ (cf. Remark 2.12). Suppose that $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h} = G \times \{1, \tau_h^G\}$ with $C_U$ pointed and that $\mathcal{V} \subseteq \mathcal{H}$ is the standard subspace specified by the triple $(h, \tau_h^G, U)$ as in [S3] by the BGL construction. We also assume that $\mathfrak{g} = \mathfrak{g}_C + \mathfrak{h} \mathfrak{r}$ for $\mathfrak{g}_C = C_U - C_U$ (cf. [NO21a §3]). Then [Ne19] Thms. 2.16, 3.4 imply that, for $C_{\mathfrak{g}} = C_U$,

$$S_{\mathfrak{g}} = G_{\mathfrak{g}} \exp(C_{\mathfrak{g}_+, \mathfrak{g}_-}), \quad \text{where} \quad L(G_{\mathfrak{g}}) = \mathfrak{g}^\ast h, \quad C_{\mathfrak{g}_\pm} = \pm C_U \cap \mathfrak{g}_{\pm 1}(h). \quad (85)$$

**Definition 7.1.** For an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ and $J := U(\tau_h^G)$, we write $\mathcal{H}_{\text{ext}, J}$ for the real linear space of all distribution vectors $\eta \in \mathcal{H}^{\ast -\infty}$, for which the orbit map

$$\alpha^\eta; \mathbb{R} \to \mathcal{H}^{\ast -\infty}, \quad \alpha^\eta(t) := \eta \circ U(\exp(-th))$$

extends to a map on the closed strip $\mathcal{S}_\pi$, which is continuous with respect to the weak-$s$-topology on the space of distribution vectors, weak-$s$-holomorphic on the open strip $\mathcal{S}_\pi$ and satisfies

$$\alpha^\eta(\pi i) = J\eta \quad (86)$$

(cf. [NO21a Def. 3.6]).

**Theorem 7.2.** ([NO21a Thm. 3.5, Prop. 3.13]) Let $\mathcal{E} \subseteq \mathcal{H}^{\ast -\infty}$ be a real subspace invariant under $U(\exp(\mathbb{R}\mathfrak{h}))$ such that $\mathcal{H}_\mathcal{E}(G)$ is total in $\mathcal{H}$. Then the following assertions hold:

(a) If $\emptyset \neq \mathcal{O} \subseteq G$ is open, then $\mathcal{H}_\mathcal{E}(\mathcal{O})$ is total in $\mathcal{H}$ (Reeh–Schlieder property).

(b) If $\mathcal{E} \subseteq \mathcal{H}_{\text{ext}, J}^{\ast -\infty}$, then $\mathcal{H}_\mathcal{E}(S_\mathcal{E}^0) = \mathcal{V}$ is the standard subspace from [S3].

We shall use the following consequence of this theorem:

**Corollary 7.3.** For any non-empty open subset $\emptyset \neq \mathcal{O} \subseteq S_{\mathfrak{g}}$ with $\exp(\mathbb{R}\mathfrak{h})\mathcal{O} = \mathcal{O}$, we have

$$\mathcal{H}_\mathcal{E}(\mathcal{O}) = \mathcal{V}. \quad (90)$$

In particular, $\mathcal{H}_\mathcal{E}(S_{\mathfrak{g}}^0) = \mathcal{V}$.

**Proof.** From Theorem 7.2 we obtain $\mathcal{H}_\mathcal{E}(\mathcal{O}) \subseteq \mathcal{H}_\mathcal{E}(S_\mathcal{E}^0) = \mathcal{V}$, so that

$$\mathcal{H}_\mathcal{E}(\mathcal{O}) \cap \pi \mathcal{H}_\mathcal{E}(\mathcal{O}) \subseteq \mathcal{V} \cap \pi \mathcal{V} = \{0\}. \quad (91)$$

Further, Theorem 7.2(a) implies that $\mathcal{H}_\mathcal{E}(\mathcal{O})$ is total, hence standard. Now the invariance of $\mathcal{H}_\mathcal{E}(G)$ under the modular group $U(\exp(\mathbb{R}h))$ of $\mathcal{V}$ shows equality ([NO21a Lemma 3.4]).

**Corollary 7.4.** Let $P \subseteq G$ be a closed subgroup with $h \in \mathfrak{L}(P)$ and $\eta \in \mathcal{H}^{\ast -\infty}$ fixed by $U(P)$ and $J$ which is cyclic in the sense that $H_{\tau_h}(G)$ is total in $\mathcal{H}$. Then

$$\mathcal{H}_\mathcal{E}(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0) = \mathcal{H}^{G/P}_{\mathcal{E}}(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0 P/P) = \mathcal{V}. \quad (92)$$

**Proof.** We put $E := R\eta$. As $\eta$ is fixed by $P$, it is $U(P)$-invariant, hence in particular invariant under $U(\exp(\mathbb{R}h))$. As $J\eta = \eta$, we have $\mathcal{E} \subseteq \mathcal{H}_{\text{ext}, J}^{\ast -\infty}$, so that Corollary 7.3 shows that $\mathcal{H}_\mathcal{E}(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0) = \mathcal{V}$. For the second assertion we use [NO21a Lemma 2.11] to obtain

$$\mathcal{V} = \mathcal{H}_E(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0) = \mathcal{H}_E(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0 P) = \mathcal{H}_E^{G/P}(S_{\mathfrak{g}_-, \mathfrak{g}_+}^0 P/P). \quad (93)$$
The preceding corollary shows that the orbit \( S^0_{q,e}P/P = S^0_{q,e}eP \) of the base point \( eP \in G/P \) under the open subsemigroup \( S^0_{q,e} \) is a natural domain to which the standard subspace can be associated.

For compactly causal symmetric spaces, we thus obtain the following concretization:

**Theorem 7.5.** Let \( M = G/H \) be a compactly causal modular symmetric space with infinitesimal data \((g, \tau, C, h)\). Further, let \((U, H)\) be an anti-unitary representation of \( G_{\sigma} \) whose positive cone \( C_U \) is pointed, \( g = C_U - C_U + \mathbb{R}h \), and \( C = C_U \cap q \). Let \( \eta \in \mathcal{H}^{-\infty} \) be fixed by \( U(H) \) and \( J = U(\pi^G_\eta) \) such that \( H_{\eta_0}(G) \) is total in \( H \). Then

\[
H_{\eta_0}^G/H \left( W \right) = V.
\]

**Proof.** In view of Corollary 7.4, it suffices to show that \( S^0_{q,e}eH = W \). As \( C_U \) is pointed, the kernel of \( U \) is discrete. Therefore \( C^h \) and \( G^h \subseteq G^{C^{\infty}} \) imply

\[
S^0_{q,e} = G^h \exp(C_U^h).
\]

Now Proposition 6.1(b) leads to

\[
S^0_{q,e}eH = (G^h)^e \exp(eH)^{C^{\infty}} = W M(h) e H.
\]

This completes the proof.  

\[ \square \]

# 8 Nets of standard subspaces on compactly causal symmetric spaces

The main result of the preceding section (Theorem 7.5) describes how certain antiunitary representations of \( G_{\sigma} \) lead to interesting nets of standard subspaces on compactly causal symmetric spaces. These representations where supposed to have a positive cone \( C_U \) which is pointed and satisfies \( g = C_U - C_U + \mathbb{R}h \), and \( C = C_U \cap q \). Further, let \( \eta \in \mathcal{H}^{-\infty} \) be fixed by \( U(H) \) and \( J = U(\pi^G_\eta) \) such that \( H_{\eta_0}(G) \) is total in \( H \). Then

\[
H_{\eta_0}^G/H \left( W \right) = V.
\]

In this section we construct such representations explicitly in spaces of Hilbert–Schmidt operators \( \mathcal{H}_p \subseteq B_2(k) \), where \( (p, K) \) is an antiunitary representation of \( G_{\sigma} \) which is a finite sum of irreducible representations. This is done in three steps: First we recall from \([\text{Ne00, Ne19}]\) some results on the representations \((p, K)\) of \( G_{\sigma} \), and then we use these representations to construct nets of standard subspaces on the symmetric space of group type \( G \cong (G \times G)/\Delta \), and finally we use the twisted embedding \( G \rightarrow G \times G, g \mapsto (g, \pi^G_\eta(g)) \) to obtain pullback representations of \( G_{\sigma} \) that can be used to obtain with Theorem 7.5 nets of standard subspaces on \( M = G/H \).

In this section we fix the following notation: \( g \) is a semisimple Lie algebra and \( C_q \subseteq g \) is a pointed generating invariant cone. Further, \( G \) is a connected Lie group with Lie algebra \( g \), \( \tau^G \) an involutive automorphism of \( G \), and the corresponding automorphism \( \tau = L(\tau^G) \) of \( g \) satisfies \( -\tau(C_q) = C_q \). This implies that \( C_\eta := C_q \cap \eta, \eta = g^{-\tau} \), has interior points, so that \( (g, \tau, C_\eta) \) is a compactly causal symmetric Lie algebra. We also fix an Euler element \( h \in \eta = g^{-\tau} \) and assume that the Lie algebra involution \( \pi_\eta = e^{i \ad h} \) integrates to an involutive automorphism \( \tau^G_\eta \) of \( G \) and that \( -\tau_\eta(C_q) = C_q \) and \( \tau^G_\eta(H) = H \).

## 8.1 \( C_q \)-positive representations and standard subspaces

To prepare our construction of nets of standard subspaces on compactly causal symmetric spaces, we first collect some information on \( C_q \)-positive representations of \( G \). We thus obtain an interface to \([\text{Ne19, NO021, NO21a}]\) from which some results and constructions will be used below.
Let \((ρ, K)\) be an antiunitary \(C_0\)-positive representation of \(G_{τ_h} := G \rtimes \{\text{id}_G, τ_h^G\}\) and write \(J_K := ρ(τ_h^K)\) for the corresponding conjugation on \(K\). Then there exists a unique standard subspace \(V_K \subseteq \mathcal{V}\) with

\[
J_{\mathcal{V}_K} = J_K \quad \text{and} \quad Δ^{−s/t/2n}_{\mathcal{V}_K} = ρ(\exp t h) \quad \text{for} \quad t \in \mathbb{R}.
\]

Let

\[G_{\mathcal{V}_K} = \{g \in G : ρ(g)\mathcal{V}_K = \mathcal{V}\}\]

Assume that \(ρ\) has a discrete kernel. Then the derived representation \(dρ\) is injective, so that \(ρ(G_{\mathcal{V}_K})\) commutes with \(dρ(h)\) and thus \(G_{\mathcal{V}_K} \subseteq G^h\). More specifically,

\[
G_{\mathcal{V}_K} = \{g \in G^h : ρ(g)J_Kρ(g)^{-1} = J_K\} = \{g \in G^h : gτ_h^G(g)^{-1} ∈ \ker(ρ)\}.
\]

This is an open subgroup of \(G^h\) that only depends on the kernel of \(ρ\).

Furthermore, the discreteness of the kernel of \(ρ\) implies that its positive cone \(C_ρ \subseteq \mathfrak{g}\) is pointed. It is also generating because it contains \(C_θ\). Now \([\text{Ne19}]\) Thms. 2.16, 3.4 imply that

\[
S_{\mathcal{V}_K} := \{g \in G : ρ(g)\mathcal{V}_K \subseteq \mathcal{V}_K\} = G_{\mathcal{V}_K} \exp(C_ρ^v),
\]

where

\[
C_ρ^v := C_ρ^+ + C_ρ^- \quad \text{with} \quad C_ρ^± := ±C_ρ \cap \mathfrak{g}^±(h).
\]

Note that \(C_ρ^v\) is a \(G_{\mathcal{V}_K}\)-invariant hyperbolic convex cone in \(\mathfrak{g}^{−τ_h}\). Since \(C_ρ^v\) coincides with \(C_θ^v\) by Corollary 2.10 and \(−τ_h(C_ρ) = C_ρ\), the interior of this cone intersects \(\mathfrak{g}^{−τ_h}\), so that \(C_ρ^v\) generates \(\mathfrak{g}^{−τ_h}\). Therefore

\[
S_{\mathcal{V}_K} = G_{\mathcal{V}_K} \exp(C_θ^v),
\]

and this semigroup only depends on \(ρ\) through its unit group \(G_{\mathcal{V}_K}\). We conclude that the semigroup \(S_{\mathcal{V}_K}\) only depends on the representation \(ρ\) through the discrete subgroup \(\ker(ρ)\).

For a more detailed discussion of irreducible antiunitary representations of \(G_{τ_h}\), we refer to Proposition 4.6 and Remark 4.10 in \([\text{NÔ21a}]\). They restrict to representations of \(G\) which are either irreducible or a direct sum of two inequivalent representations exchanged by twisting with \(τ_h\).

For \(1 ≤ p < ∞\) we denote by \(B_p(\mathcal{H})\) the space of linear operators with finite \(p\)-Schatten norm:

\[
B_p(\mathcal{H}) = \{T \in B(\mathcal{H}) : \|T\|_p = \text{tr}(\|T_p\|^{1/p}) < ∞\}.
\]

Then \(B_1(\mathcal{H})\) is the space of trace class operators and \(B_2(\mathcal{H})\) is the space of Hilbert–Schmidt operators.

**Proposition 8.1.** There exists an injective antiunitary representation \((ρ, K)\) of \(G_{τ_h}\) such that

(a) \(C_ρ \supseteq C_θ\)

(b) \(ρ\) is a finite direct sum of irreducible representations.

(c) \(ρ\) is a trace-class representation, i.e., \(ρ(C_n^∞(G)) \subseteq B_1(\mathcal{H})\).

**Proof.** From \([\text{Ne00}]\) Thm. XI.5.2 we know that there exist finitely many irreducible representations \((π_j, K_j)\) of \(G\) with \(C_π_j \supseteq C_θ\) and \(\bigcap_{j \in F} \ker(π_j) = \{e\}\). Then \(π := \bigoplus_{j \in F} π_j\) is injective and

\[
ρ := π \ominus (π^* \circ τ_h^G)
\]

extends to an injective antiunitary representation of \(G_{τ_h}\). As \(−τ_h(C_θ) = C_θ\), we also have \(C_θ \subseteq C_ρ\). This proves (a) and (b). That \(ρ\) is a trace class representation follows from the fact that all representations \(π_j\) are trace class \(([\text{Ne00}]\) Thm. X.4.10). \(\square\)
8.2 The group case: biinvariant nets on Lie groups

On $G$ we obtain by the assignment

$$g \mapsto V_+(g) := g.C^0_\theta \subseteq T_\theta(G) \quad \text{for} \quad g \in G,$$

a biinvariant cone field, turning $G$ into a compactly causal symmetric space on which the group $G \times G$ acts by left and right translations. Its infinitesimal data is represented by the compactly causal symmetric Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}, \tau, C_{\mathfrak{g} \oplus \mathfrak{g}})$ of group type, where

$$\tau(x, y) = (y, x) \quad \text{and} \quad C = \{(y, -y) : y \in C_\theta \} \subset (\mathfrak{g} \oplus \mathfrak{g})^{-\tau} \quad (90)$$

We also fix an Euler element $h \in \mathcal{E}(\mathfrak{g})$.

We want to apply \cite{NO21} to representations of $G \times G$, so we assume that $\mathfrak{g}$ is semisimple\footnote{For generalization of some constructions in \cite{NO21} to non-reductive Lie algebras, we refer to \cite{Oeh21}.}

To this end, we consider in $\mathfrak{g} \oplus \mathfrak{g}$ the invariant cone

$$C_{\mathfrak{g} \oplus \mathfrak{g}} := C_\theta \oplus -C_\theta \quad \text{which satisfies} \quad -\tau(C_{\mathfrak{g} \oplus \mathfrak{g}}) = C_{\mathfrak{g} \oplus \mathfrak{g}}.$$  

**Remark 8.2.** (Irreducible representations) As $C_\mathfrak{g}$-positive representations of $G$ are type I \cite[Thm. X.6.21]{Ne00}, irreducible $C_\mathfrak{g}$-positive representations of $G \times G$ have the form

$$\pi = \rho_1 \boxtimes \rho_2^*, \quad \pi(g_1, g_2) = \rho_1(g_1) \otimes \rho_2^*(g_2)$$

where $(\rho_1, K_1)$ and $(\rho_2, K_2)$ are irreducible $C_\mathfrak{g}$-positive representations of $G$.

For the existence of an antiunitary extension to

$$(G \times G)_{\tau_h \times \tau_h} = (G \times G) \times \{\text{id}, \tau_h^G \times \tau_h^G\},$$

we need that

$$\pi^* \cong \pi \circ (\tau_h^G \times \tau_h^G) = (\rho_1 \circ \tau_h^G) \boxtimes (\rho_2 \circ \tau_h^G)^* \quad (91)$$

and this is equivalent to

$$\rho_j^* \cong \rho_j \circ \tau_h^G \quad \text{for} \quad j = 1, 2. \quad (92)$$

If these two conditions are satisfied, then $\rho_1$ and $\rho_2$ extend to antiunitary representations of $G_{\tau_h}$ and $\pi$ extends to $(G \times G)_{\tau_h \times \tau_h}$ by

$$\pi(\tau_h^G \times \tau_h^G) := \rho_1(\tau_h^G) \otimes \rho_2(\tau_h^G).$$

If \ref{92} is not satisfied, then the canonical antiunitary extension of the representation

$$\pi \oplus (\pi^* \circ (\tau_h^G \times \tau_h^G))$$

is irreducible, although its restriction to $G \times G$ decomposes into two inequivalent irreducible constituents.

Let $(\rho_j, K_j), \ j = 1, 2,$ be two antiunitary $C_\mathfrak{g}$-positive representations of $G_{\tau_h}$ which need not be irreducible (cf. Subsection 8.1). Then

$$\mathcal{H} := B_2(K_2, K_1) \cong K_1 \otimes K_2^*$$

carries the unitary representation $(\mathcal{H}, \pi)$ of $G \times G$ defined by

$$\pi(g_1, g_2)A = \rho_1(g_1)A \rho_2(g_2)^{-1},$$

and we extend this representation to an antiunitary representation of $(G \times G)_{\tau_h \times \tau_h}$ by

$$\pi(\tau_h^G \times \tau_h^G) := J, \quad J(A) := J_{K_1}AJ_{K_2}, \quad \text{where} \quad J_{K_j} := \rho_j(\tau_h^G).$$
Lemma 8.3. Let $V_j \subseteq \mathcal{H}_j$, $j = 1, 2$, be standard subspaces with the modular objects $(\Delta_j, J_j)$, $j = 1, 2$ where $\Delta_j = \Delta_{V_j}$, $J_j = J_{V_j}$. Then

$$V_1 \otimes V_2 \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2$$

is the standard subspace $V$ corresponding to the modular group defined by

$$\Delta^{it} = \Delta_1^{it} \otimes \Delta_2^{it} \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and the conjugation} \quad J = J_1 \otimes J_2. \quad (93)$$

Proof. The tensor product $H := V_1 \otimes V_2$ is a closed real subspace of $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ which we want to compare with the standard subspace $V := \text{Fix}(J\Delta^{1/2})$ defined by the pair $(\Delta, J)$ from (93). For $\xi_1 \in V_1$ and $\xi_2 \in V_2$, we have

$$\Delta_1^{1/2}\xi_1 = J_1\xi_1 \quad \text{and} \quad \Delta_2^{1/2}\xi_2 = J_2\xi_2.$$ 

For $\xi := \xi_1 \otimes \xi_2$ the orbit map

$$\mathbb{R} \to \mathcal{H}, \quad t \mapsto \Delta^{-it/2\pi}\xi = \Delta_1^{-it/2\pi}\xi_1 \otimes \Delta_2^{-it/2\pi}\xi_2$$

extends holomorphically to the closure of the strip $S_\xi \subseteq \mathbb{C}$ by

$$z \mapsto \Delta_1^{-iz/2\pi}\xi_1 \otimes \Delta_2^{-iz/2\pi}\xi_2$$

(cf. [NO02] Prop. 2.1]). This implies that

$$\Delta^{1/2}\xi = \Delta_1^{1/2}\xi_1 \otimes \Delta_2^{1/2}\xi_2 = J_1\xi_1 \otimes J_2\xi_2 = J\xi.$$ 

Therefore $\xi \in V$ (cf. [NO02] Prop. 2.1]). This argument shows that $H \subseteq V$. As $H$ is generating in $\mathcal{H}$ and invariant under the modular group $\Delta^{-it/2\pi} = \Delta_1^{-it/2\pi} \otimes \Delta_2^{-it/2\pi}$, we have $H \subseteq V$. \hfill \square

Corollary 8.4. Write $K_2^\text{op} \cong K_2^*$ for $K_2$, endowed with the opposite complex structure. Then the standard subspace $V \subseteq K_1 \otimes K_2^\text{op}$ specified by the Lie algebra element $(h, h) \in g \otimes g$ and $J$ is the tensor product $V = V_1 \otimes V_2$, constructed from the corresponding standard subspaces $V_j \subseteq K_j$.

Proof. In view of the preceding lemma, we only have to observe that, changing the complex structure on $K_2$, corresponds to replacing the modular operator $\Delta_{V_2} = e^{2\pi i \partial_\rho(h)}$ by its inverse. \hfill \square

We now assume that $\rho := \rho_1 = \rho_2$ is a sum of finitely many irreducible $C_0$-positive antiunitary representations of $G_{\text{reg}}$. Then

$$\pi(g_1, g_2)A = \rho(g_1)A\rho(g_2)^{-1} \quad \text{and} \quad J(A) = Je A J_e.$$ 

By [Ne08] Thm. X.4.10 $\rho$ is a trace class representation, i.e., $\rho(C^\infty(G)) \subseteq B_1(\mathcal{H})$. This implies that the integrated representation defines a continuous linear map $\rho: C^\infty_c(G) \to B_1(K)$ (DNSZ16 Thm. 1.3)).

Lemma 8.5. The following assertions hold:

(a) For the left multiplication representation of $G$ on $B_2(K)$,

$$B_2(K)^\infty = \{ A \in B(K) : AK \subseteq K^\infty \} \subseteq B_1(K) \quad (94)$$

and the inclusion $B_2(K)^\infty \to B_1(K)$ is continuous with respect to the norm topology on $B_1(K)$ and the natural Fréchet topology on $B_2(K)^\infty$.

(b) We have a $(G \times G)$-equivariant map

$$\rho^* : B(K) \to C^\infty_c(G), \quad \rho^*(A)(\xi) := \text{tr}(\rho(\xi)^* A), \quad \xi \in C^\infty_c(G).$$
(c) The space of smooth vectors for the representation $\pi$ of $G \times G$ on $B_2(K)$ is
$$B_2(K)^{\infty, r} = \{ A \in B(K) : AK \subseteq K^{\infty}, A^*K \subseteq K^{\infty} \}.$$  

(d) The map $B(K) \to B_2(K)^{\infty, r}$, $A \mapsto \eta_A$, with $\eta_A(B) = \text{tr}(B^*A)$ is injective and equivariant with respect to the representation of $(G \times G)_{\tau_{h_1} \times \tau_{h_1}}$ on $B(H)$, given by
$$(g_1, g_2).A := \rho(g_1)A\rho(g_2)^{-1} \quad \text{for} \quad g_1, g_2 \in G \quad \text{and} \quad (\tau_{h_1}^G \times \tau_{h_1}^G).A = J_KA_J.$$  

Note that (a) implies in particular that each bounded operator on $K$ defines a distribution vector for the left multiplication representation on $B_2(K)$, hence also for the representation $\pi$ of $G \times G$. The smooth vectors for $G \times G$ coincide with the algebra of so-called Schwartz operators, i.e., operators which remain bounded when composed on the left and the right with elements of the enveloping algebra acting on smooth vectors (cf. [DNSZ16, KKW16]).

**Proof.** (a) The first equality in [18] follows from [DNSZ16 Prop. 1.7].

Let $A \in B_2(K)$ be a smooth vector for the left multiplication representation of $G$. By the Dixmier–Malliavin Theorem, $A$ is a finite sum of operators of the form $\rho(\xi_j)A_j$, $A_j \in B_2(K)$, $\xi_j \in C_c^\infty(G)$, and these are trace class because $(\rho, K)$ is a trace class representation.

For the last assertion, we have to show that the inclusion map
$$\Gamma : B_2(K)^\infty \to B_1(K)$$

is continuous. This is a linear map from a Fréchet space to a Banach space. Therefore it suffices to write it as a pointwise limit of a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of continuous linear maps ([DNSZ16 Lemma 1.2]). So let $\xi_n \in C_c^\infty(G)$ be a $\delta$-sequence and $\Gamma_n(\xi) := \rho(\xi)B$. These maps are continuous from $B_2(K) \to B_1(K)$, hence in particular from $B_2(K)^\infty \to B_1(K)$. Since the left multiplication representation of $G$ on the Banach space $B_1(K)$ is continuous, $\|\Gamma_n(\xi) - B\|_1 \to 0$ holds for every $B \in B_1(H)$.

(b) From the continuous linear map $\rho : C_c^\infty(G) \to B_1(K)$, we obtain the map $\rho^*$ by taking adjoints. The equivariance for the action of $G \times G$ on both sides follows from
$$\rho^*(\rho(g_1)A\rho(g_2)) = \text{tr}((\rho(g_1)^{-1}\rho(\xi)\rho(g_2))^*A) = \langle \rho(g_1, g_2)^*A, \rho^*(\rho(\xi))\rangle.$$  

(c) follows from [DNSZ16 Cor. 1.8].

(d) That every functional $\eta_A$ defines a distribution vector follows from (a). For the injectivity of the assignment, we first observe that all rank one operators $P_{\xi, \eta} = [\xi] \langle \eta \rangle$, defined by smooth vectors $\xi, \eta \in K^\infty$, are contained in $B_2(K)^\infty$. For these we have
$$\eta_A(P_{\xi, \eta}) = \text{tr}(P_{\xi, \eta}A) = \text{tr}(P_{\eta, \xi}A) = \langle \xi, A\eta \rangle,$$  

and since $K^\infty$ is dense in $K$, any bounded operator $A$ is determined by $\eta_A$.

For the equivariance, we calculate
$$\eta_A(\pi(g_1, g_2)^{-1}B) = \text{tr}((\rho(g_1)^*B\rho(g_2))^*A) = \text{tr}(\rho(g_2)^*B^*\rho(g_1)A) = \eta_{(g_1, g_2)^{-1}}(B).$$  

Further
$$\eta_{J_KA_J}(B) = \text{tr}(B^*J_KA_J) = \text{tr}(J_KB^*J_KA) = \text{tr}((J_KB^*J_K)^*A) = \pi^{-\infty}(\tau_h^G \times \tau_h^G)(\eta_A)(B).$$

We want exhibit concrete contexts, where the assumptions of Theorem 7.3 are satisfied. This will be achieved in Theorem 8.7 and the ground is prepared by the following proposition. Recall that we assume in this section that $g$ is semisimple. In the following proposition this is crucial to use Corollary 2.10 in the proof (d).
Proposition 8.6. Assume that $\ker \rho$ is discrete. Let $\mathcal{H}_\rho \subseteq \mathcal{H} = B_2(\mathbb{K})$ denote the closed subspace generated by the image $\rho(C_c^\infty(G))$ of the integrated representation of the convolution algebra of test functions and denote the corresponding subrepresentation of $\pi$ by $\pi_\rho$. Then the following assertions hold:

(a) $\eta^0 := \text{tr}|_{\mathcal{H}_\rho^\infty} \in \mathcal{H}_\rho^{-\infty}$ is a cyclic distribution vector invariant under the diagonal subgroup $\Delta G = \{(g, g) : g \in G\}$ and $J(A) = J_KAJ_K$.

(b) $\mathcal{H}_\rho$ is $J$-invariant and $E := \mathbb{R}\eta^0 \subseteq (\mathcal{H}_\rho^{-\infty})_{\text{ext}, J}$.

(c) $\ker(\pi_\rho)$ is discrete.

(d) Let $V_\rho \subseteq \mathcal{H}_\rho$ be the standard subspace with the modular data $J_{V_\rho} = J|_{\mathcal{H}_\rho}$ and $\Delta_{V_\rho} = e^{2\pi i \omega(h, h)}$. Then

$$S_{V_\rho} = \{g \in G \times G : \pi_\rho(g)V_\rho \subseteq V_\rho\} = (G \times G)_{V_\rho} \exp(C^e_\rho \oplus -C^e_\rho).$$

Proof. (a), (b) By Lemma 8.5(d), the trace $\eta_1 = \text{tr} \in \mathcal{H}^{-\infty} = B_2(\mathbb{K})^{-\infty}$ is invariant under $J$ and the diagonal subgroup $\Delta G$. In particular, it is fixed under the modular group $\exp(\mathbb{R}(h, h))$, which implies (b).

If we consider $1 \in B(\mathcal{H})$ as a distribution vector, we have for $\xi_1, \xi_2 \in C_c^\infty(G)$ the relation

$$\pi^{-\infty}(\xi_1 \otimes \xi_2)1 = \rho(\xi_1) \int_G \xi_2(g)\rho(g^{-1}) \, dg = \rho(\xi_1) \int_G \xi_2(g^{-1})\rho(g) \, dg$$

$$= \rho(\xi_1)\rho(\xi_2^*) = \rho(\xi_1 * \xi_2^*).$$

(95)

As tensor products of test functions on $G$ span a dense subspace of $C_c^\infty(G \times G)$, this calculation shows that the closed subspace of $\mathcal{H}$ generated by $\pi^{-\infty}(C_c^\infty(G \times G))E$ coincides with $\mathcal{H}_\rho$. This subspace is also $J$-invariant because

$$J\rho(\xi)J = \int_G \xi(g_1, g_2)\pi(\rho_1^G(g_1), \rho_2^G(g_2)) \, dg_1 \, dg_2 = \int_G \xi(\rho_1^G(g_1), \rho_2^G(g_2))\pi(g_1, g_2) \, dg_1 \, dg_2.$$

(c) As $\mathcal{H}_\rho$ is generated by $E$, the kernel of $\pi_\rho$ is the largest normal subgroup of $G \times G$ acting trivially on $E$. A pair $(g_1, g_2) \in G \times G$ fixes $1 \in B(\mathbb{K})$ if and only if $\rho(g_1g_2^{-1}) = 1$, i.e., $g_1g_2^{-1} \in \ker(\rho)$. If $(g_1, g_2) \in \ker(\pi_\rho)$, then we have for all $a, b \in G$ the relation

$$ag_1a^{-1}bg_2^{-1}b^{-1} \in \ker(\rho),$$

and since $G$ is connected and $\ker(\rho)$ discrete, it follows that $g_1, g_2 \in Z(G)$. This shows that

$$\ker(\pi_\rho) = \{(g_1, g_2) \in Z(G) \times Z(G) : g_1g_2^{-1} \in \ker(\rho)\} = \Delta_{Z(G)} \cdot (\ker(\rho) \times \{e\}),$$

which is a discrete subgroup of $G \times G$.

(d) As the kernel of $\pi_\rho$ is discrete by (c), the description of the semigroup $S_{V_\rho}$ follows from [Ne19 Thms. 2.16, 3.4]. Note that the discreteness of $\ker(\pi_\rho)$ implies that $C^e_{\pi_\rho} = C^e_{\rho \otimes \rho} = C^e_\rho \oplus -C^e_\rho$ (Corollary 24.11).

The following theorem shows that, for symmetric spaces of group type, there exist biinvariant nets of real subspaces assigning to the wedge domain $W_G(h)_x = G^h_k \exp(C^e_{\rho \otimes \rho})$ the standard subspace associated to the pair $(h, \tau_0^G)$ by the BGL construction (Subsection 7.2).

Theorem 8.7. Let $(\rho, K)$ be a $C_0^\infty$-positive antiunitary trace class representation of $G_{\text{ch}}$ with discrete kernel and consider the representation $(\pi_\rho, \mathcal{H}_\rho)$ of $G \times G$ on the closed subspace

$$\mathcal{H}_\rho := \rho(C_c^\infty(G)) \subseteq B_2(\mathbb{K}), \quad \text{given by} \quad \pi_\rho(g_1, g_2)A = \rho(g_1)A\rho(g_2)^{-1}. $$

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extended to an antiunitary representation of \((G \times G)_{\tau_h \times \tau_h}\) by

\[
J(A) := \pi_\rho(C^G_{\tau_h} \times C^G_{\tau_h})(A) := \rho(\tau_h^G)A\rho(\tau_h^G).
\]

For \(E := \mathbb{R} \eta^0\), \(\eta^0(A) := \text{tr}(A)\), and \(q: G \times G \to G, q(g_1, g_2) := g_1g_2^{-1}\),

\[
H^G_\rho(\mathcal{O}) := H^G_{\times G}(q^{-1}(\mathcal{O})) = \rho(C^G_{\tau_h^G}(q^{-1}(\mathcal{O})), \mathbb{R}) \subseteq H_{\rho}
\]

(97)
defines a \((G \times G)\)-covariant net of closed real subspaces in \(H_{\rho}\) on \(G \cong (G \times G)/\Delta_G\). The wedge domain

\[
W_{\rho}(h) = G^h_0 \exp(C^G_{\tau_h^G}(0))
\]
satisfies

\[
H^G_\rho(W_{\rho}(h)) = \mathcal{V}_{\rho} = \span_{\mathbb{R}} \rho(C^\infty(W_{\rho}(h)), \mathbb{R}),
\]

where \(\mathcal{V}_{\rho}\) is the standard subspace corresponding to the pair \((e^{2\pi i \Delta_{\tau_h^G}^h}, J)\) of modular data.

**Proof.** By Proposition 5.6(c), the kernel of \(\pi_\rho\) is discrete. Therefore the invariance of \(\eta^0\) under the modular group and \(J\) and Theorem 7.2(b) imply that

\[
\mathcal{V}_{\rho} = H^G_{\times G}(S^0_{\rho}) := \pi^{-\infty}(C^\infty(S^0_{\rho}, \mathbb{R}))\eta^0
\]

holds for the open subsemigroup

\[
S^0_{\rho} = (G \times G)_{\rho} \exp(C^G_{\tau_h^G} + C^G_{\tau_h^G}), \text{ where } C^G_{\tau_h^G} = C_{\tau^G_{\rho} + C_{\tau^G_{\rho}} - C_{\tau^G_{\rho}}}.\]

(Proposition 5.6(d)). With Corollary 5.6 we even obtain

\[
\mathcal{V}_{\rho} = H_{\rho}(S^0_{\rho, e}) \text{ with } S^0_{\rho, e} = (G^h_0 \times G^h_0) \exp(C^G_{\tau^G_{\rho}} e - C^G_{\tau^G_{\rho}} e) = (S(C^0_{\rho}, h)_e \times S(C^0_{\rho}, h)^{-1}_e),\]

(99)

where

\[
S(C^0_{\rho}, h)^0_0 = S(C^0_{\rho}, h)_e = W_{\rho}(h)_e
\]

(Theorem 5.2). With

\[
\pi^{-\infty}(S(C^0_{\rho}, h)_e \times S(C^0_{\rho}, h)^{-1}_e)\eta^0 = \rho(S(C^0_{\rho}, h)_e),
\]

we further derive from 99

\[
\mathcal{V}_{\rho} = \rho(C^\infty(S(C^0_{\rho}, h)_e)), \quad (100)
\]

As \(\eta^0\) is \(\Delta_G\)-invariant (Proposition 5.6) \(\text{[NÖ21a] Lemma 2.11(a)]}\) shows that the net defined in 97 satisfies

\[
\mathcal{V}_{\rho} = H_{\rho}^{G \times G}(S^0_{\rho, e}) = \rho(C^\infty(S^0_{\rho, e}, \Delta_G)).\]

(101)

Now \(q^{-1}(W_{\rho}(h)_e) = (W_{\rho}(h)_e \times W_{\rho}(h)^{-1})_e\Delta_G = S^0_{\rho, e, \Delta_G}\) leads to

\[
H^G_{\rho}(W_{\rho}(h)_e) = H^{G \times G}_{\rho}(q^{-1}(W_{\rho}(h)_e)) \equiv \mathcal{V}_{\rho}.\]

**Remark 8.8.** For the stabilizer group of \(\mathcal{V}_{\rho}\), we have

\[
(G \times G)_{\rho} = \{(g_1, g_2) \in G^h \times G^h : \pi_\rho(g_1, g_2)J = J\pi_\rho(g_1, g_2) \supseteq G_{\tau^G_{\rho}} \times G_{\tau^G_{\rho}}\}.
\]

By 99, the relation \(\pi_\rho(g_1, g_2)J = J\pi_\rho(g_1, g_2)\) is equivalent to

\[
(g_1\tau^G_{\rho}(g_1)^{-1}, g_2\tau^G_{\rho}(g_2)^{-1}) \in \ker(\pi_\rho) = \Delta_{Z(G)}(\ker(\rho) \times \{e\}) \subseteq Z(G) \times Z(G).
\]

(102)

From Remark 5.10 we know that

\[
Ad(G^h) = Ad(G^h) \subseteq Ad(G)^{\tau_h}
\]

is of index two if \(g\) is simple (otherwise of index \(2^k\), where \(k\) is the number of simple hermitian ideals of \(g\)). Equation 102 implies in particular that

\[
g_1, g_2 \in Ad^{-1}(Ad(G)^{\tau_h}) = \{g \in G : g\tau^G_{\rho}(g)^{-1} \in Z(G)\}.
\]

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Lemma 8.9. \( \mathcal{H}_\rho = B_2(\mathcal{K}) \) if and only if \((\rho, \mathcal{K}) \) is an irreducible representation of \( G \).

Proof. If \( \mathcal{H}_\rho = B_2(\mathcal{K}) \), then \( \rho(C^\infty_c(G)) \) is dense in \( B_2(\mathcal{K}) \), so that the von Neumann algebra generated by this subspace is \( B(\mathcal{K}) \), which by Schur’s Lemma implies that \((\rho, \mathcal{K}) \) is irreducible.

If, conversely, \((\rho, \mathcal{K}) \) is irreducible, then \( \rho(C^\infty_c(G)) \subseteq B_1(\mathcal{K}) \subseteq B_2(\mathcal{K}) \) is weakly dense in \( B(\mathcal{K}) \), i.e., its annihilator in \( B_1(\mathcal{K}) \) is trivial. Therefore the orthogonal space \( \rho(C^\infty_c(G))^\perp \subseteq B_2(\mathcal{K}) \) has trivial intersection with \( B_1(\mathcal{K}) \). As it is invariant under multiplication with \( \rho(C^\infty_c(G)) \), which consists of trace class operators, this can only happen if \( \rho(C^\infty_c(G))^\perp = \{0\} \), i.e., if \( \mathcal{H} = B_2(\mathcal{K}) \).

Remark 8.10. (a) If \( \rho \) is reducible and extends to an irreducible antiunitary representation of \( G_{\tau_h} \), then \( G \) preserves a decomposition \( \mathcal{K} = K_1 \oplus K_2 \) and \( \rho(C^\infty_c(G)) \) generates \( B(K_1) \oplus B(K_2) \). Further \( J = \rho(\tau_h^G) \) is a conjugation with \( J K_1 = K_2 \). Accordingly, \( \mathcal{H}_\rho \subseteq B_2(\mathcal{K}) \) is isomorphic to \( B_2(K_1) \oplus B_2(K_2) \) (diagonal matrices).
(b) The trace class representation \((\rho, \mathcal{K})\) of \( G \) on \( \mathcal{K} \) decomposes with finite multiplicities as a direct sum
\[
(\rho, \mathcal{K}) \cong \bigoplus_{j \in J}(\rho_j, K_j)^{\oplus m_j}.
\]
Identifying \( B(K_j) \) with diagonal operators in \( M_{m_j}(B(K_j)) \cong B(\mathcal{K}_j^{\oplus m_j}) \), we then have
\[
\rho(C^\infty_c(G)) \subseteq \bigoplus_{j \in J} B_2(K_j) \cong \mathcal{H}_\rho.
\]

8.3 The general case: Nets on compactly causal symmetric spaces

Let \((\mathfrak{g}, \tau, C)\) be a semisimple compactly causal symmetric Lie algebra for which \( \mathfrak{h} = \mathfrak{g}^* \) contains no non-zero ideal. We recall from (14) the canonical embedding
\[
i : \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}, \ i(x) := (x, \tau(x)) \quad \text{for} \quad x \in \mathfrak{g}
\]
into a symmetric Lie algebra of group type. The Extension Theorem 2.4 implies that the elliptic cone \( C \subseteq \mathfrak{q} \) extends to a \( -\tau \)-invariant cone \( \tilde{C}_\mathfrak{g} \subseteq \mathfrak{g} \), so that
\[
\tilde{C}_\mathfrak{g} := \{ (y, -y) : y \in \mathfrak{g} \}
\]
leads to the causal symmetric Lie algebra \((\mathfrak{g} \oplus \mathfrak{g}, \tau_{\mathfrak{h}\mathfrak{g}}, \tilde{C}_\mathfrak{g})\) of group type. On the level of global symmetric spaces \( M = G/H, \ H \subseteq G^G \) open, the infinitesimal embedding \( \iota \) corresponds to the quadratic representation
\[
Q : M \rightarrow M_G = G^2_c \subseteq G, \quad gH \mapsto g g^\sharp,
\]
which is a covering morphism of symmetric spaces.

The following theorem is an important tool to analyze smooth vectors, resp., distribution vectors for subgroups.

Theorem 8.11. (Zellner’s Smooth Vector Theorem; [NSZ17] Thm. 3.1) If \((\pi, \mathcal{H})\) is a unitary representation of a Lie group and \( x_0 \in C^0_\mathfrak{g} \) in the interior of its positive cone, then the inclusion \( \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty(\partial \pi(x_0)) \) is an isomorphism of Fréchet spaces. In particular every smooth vector for the single operator \( \partial \pi(x_0) \) is smooth for \( G \).

We now consider a \( C^\infty_c \)-positive antiunitary trace class representation \((\rho, \mathcal{K})\) of \( G_{\tau_h} \) with discrete kernel and use Theorem 8.7 to obtain a representation \((\pi_\rho, \mathcal{H}_\rho)\) of \((G \times G)_{\tau_h \times \tau_h}\) on \( \mathcal{H}_\rho \subseteq B_2(\mathcal{K}) \) by
\[
\pi_\rho(g_1, g_2)A = \rho(g_1) A \rho(g_2)^{-1} \quad \text{and} \quad \pi_\rho(\tau_h \times \tau_h)A = J \mathcal{K} A J \mathcal{K},
\]
We introduce the morphism of graded Lie groups
\[ \iota_G : G_{r_h} \to (G \times G)_{r_h \times r_h}, \quad g \mapsto (g, \tau^G(g)), \quad \tau^G_h \mapsto \tau^G_h \times \tau^G_h, \]
corresponding to the inclusion \( \iota : g \to g \times g \) from (103). We thus obtain the antiunitary representation of \( G_{r_h} \), defined by
\[ U := \pi_{\rho} \circ \iota_G, \quad U(g)A = \rho(g) \rho(g)^2 \] (104)

**Theorem 8.12.** The representations \( U \) and \( \pi_{\rho} \) on \( H_{\rho} \) have the same smooth vectors. Let
\[ E = \mathbb{R} \eta^0 \subseteq H^{-\infty}(\pi_{\rho}) = H^{-\infty}(U) \] with \( \eta_0 = \text{tr} \)
be as in Proposition S.6 If \( q_M : G \to M, g \mapsto gH \) denotes the canonical projection, then
\[ H^M_E(\mathcal{O}) := H^U_E(\iota^{-1}(\mathcal{O})) := U^{-\infty}(C^\infty(\iota^{-1}(\mathcal{O})), \mathbb{R})E \] (105)
defines a \( G \)-covariant net of closed real subspaces of \( H_{\rho} \) on the symmetric space \( M = G/H \) which satisfies
\[ V_{(h, \tau^G_{r_h}, U)} = H^M_E(W_{M(h)_{eH}}). \] (106)

**Proof.** For any \( x_0 \in C^0\subseteq q \) we have \( \iota(x_0) = (x_0, -x_0) \in \tilde{C}_0^q \subseteq C_0^q \oplus -C_0^q \) (see (90)). Therefore Zel’ner’s Theorem [S.11] implies that smooth vectors of the single operator \( \partial U(x_0) = \partial \pi_{\rho}(x_0, -x_0) \) are smooth for \( G \times G \). This shows that
\[ H^\infty(\pi_{\rho}) = H^\infty(U) \quad \text{and therefore} \quad H^{-\infty}(\pi_{\rho}) = H^{-\infty}(U). \] (107)
We conclude in particular that \( E \subseteq H^{-\infty}(U) \), so that (103) defines a \( G \)-covariant net of closed real subspaces of \( H_{\rho} \) on the symmetric space \( M = G/H \).

The positive cone \( C_U \) of \( U \) contains \( \iota^{-1}(C_{\pi_{\rho}}) \supseteq \iota^{-1}(C^q_0 \oplus -C^q_0) \), hence is a closed generating invariant cone whose intersection with \( q \) contains \( C \). As \( \iota(h) = (h, h) \), the standard subspace \( V_\rho \subseteq H_{\rho} \) from Theorem S.7 coincides with the standard subspace \( V_U := V_{(h, \tau^G_{r_h}, U)} \) associated to the pair \( (h, \tau^G_h) \) and the representation \( U \) of \( G \) by the BGL construction.

The discreteness of the kernel of \( \pi_{\rho} \) (Proposition S.6) implies that \( U \) has discrete kernel. Therefore [Ne19] Thms. 2.16, 3.4] implies that
\[ S_{V_{\rho}} = (G^0)_{V_{\rho}} \exp(C^0_U). \] (108)
We claim that
\[ C^0_U = C^0_0. \] (109)
In fact, we have
\[ C_U \cap q = \{ x \in q : (x, -x) \in C_{\pi_{\rho}} \} \supseteq C_0 \cap q, \]
so that \( C_0 \) and \( C_U \) are invariant cones in \( q \) with intersecting interiors, and this implies that \( C^0_0 = C^0_U \) by Corollary S.10. With (108) and (109) we now obtain
\[ q_M(S^0_{V_{\rho}, e}) = G^0_0 \exp(C^0_{V_{\rho}}) \cap q = W_{M(h)_{eH}}, \]
and thus by (108) in Theorem S.7
\[ V_{\rho} = H^G_U(S^0_{V_{\rho}, e}) = H^G_U(S^0_{V_{\rho}, e} H) = H^G_U(q^{-1}_M(q_M(S^0_{V_{\rho}, e}))) = H^G_U(q^{-1}_M(W_{M(h)_{eH}})) = H^M_E(W_{M(h)_{eH}}). \]
Remark 8.13. (The kernel of $U$) (a) For a representation $(\rho, K)$ of $G$, the representation $\pi_\rho$ on $\mathcal{H}_\rho \subseteq B_2(K)$ has the property that $\Delta_{Z(G)}$ acts trivially. In fact, the subspace $\mathcal{H}_\rho$ is generated by $\rho(C^g(G, \mathbb{R})) \subseteq B_2(K)$ which commutes with $\rho(Z(G))$. We therefore have for $A \in \mathcal{H}_\rho$ and $z \in Z(G)$

$$
\pi_\rho(z, z)A = \rho(z)A\rho(z)^{-1} = A.
$$

For the representation $U(g) := \pi_\rho(g, \tau^G(g))$ of $G$, we obtain for $z \in Z(G)$ that

$$
U(z)A = \rho(z)A\rho(z)^2 = \rho(z^2)A,
$$

so that $zz^2 \in \ker(\rho)$ is equivalent to $z \in \ker(U)$. We conclude that

$$
\ker U = \{z \in Z(G) : zz^2 \in \ker \rho \}. \tag{110}
$$

(b) If $\ker(\rho) = Z(G)$ (Proposition 8.1), the representation $(U, \mathcal{H}_\rho)$ factors through $G/Z(G)$. We then have

$$
G_\rho = \{g \in G^h : gg^2 \in \ker(U)\} = \{g \in G^h : gg^2 \in Z(G)\} = \Ad^{-1}(\Ad(G)^r).
$$

(c) If $\rho$ is faithful (Proposition 8.1), then

$$
\ker U = \{z \in Z(G) : zz^2 = e\} = Z(G)^{\tau^G}
$$

and thus

$$
G_\rho = \{g \in G^h : gg^2 \in \ker(U)\} = \{g \in G^h : gg^2 \in Z(G)^{\tau^G}\}
$$

$$
= \{g \in G^h : gg^2 \in Z(G), (gg^2)^2 = e\}. \tag{111}
$$

9 The wedge space as an ordered symmetric space

In this section we return to a geometric topic. We show that, under rather natural assumptions, the wedge space $W(M, h)$, consisting of all $G$-translates of the connected component $W_M(h)_{eH}$ of the wedge domain in $M$, carries the structure of an ordered symmetric space. We start in Subsection 9.1 with the order structure and the determination of the stabilizer group of $W(M, h)_{eH}$. This turns out to be rather hard to do directly, so we use the representation theoretic results from Subsection 8.3 on the correspondence between wedge domains and standard subspaces to obtain all required information. The symmetric space structure is discussed in Subsection 9.2. This result establishes a bridge to the abstract wedge spaces introduced in [MN21].

9.1 The order structure

Let $(G, \tau^G)$ be a connected symmetric Lie group corresponding to the modular compactly causal symmetric Lie algebra $(\mathfrak{g}, \tau, C, h)$, let $H \subseteq G^G$ be an open subgroup, $M = G/H$ the corresponding compactly causal symmetric space and

$$
W := W_M(h)_{eH} \subseteq M
$$

the connected component of the wedge domain specified by $h$ corresponding to the base point $eH$. As in Theorem 6.3 we assume that $C = C_g \cap \mathfrak{q}$ for a pointed invariant cone $C_g \subseteq \mathfrak{g}$ with $-\tau(C_g) = C_g$. We further assume the context of Theorem 8.12 where $(U, \mathcal{H})$ is an antiunitary representation of $G \tau_h$ with discrete kernel and there exists a distribution vector $\eta_0$ with

$$
\mathcal{H}^M_{\eta_0}(W) = \mathcal{V} := \Fix(U(\tau^G_h)e^{\pi i \partial U(h)}).
$$
The wedge space
\[ W(M, h) := \{ gW : g \in G \} \]
is the G-orbit of \( W = W_M(h)eH \) in the set of subsets of \( M \). The order on the wedge space is determined by the subsemigroup
\[ S_W := \{ g \in G : gW \subseteq W \} \quad \text{with} \quad S_W \cap S_W^{-1} = G_W. \] (112)
Below we abbreviate \( G^h_e := (G^h)_e \).

**Theorem 9.1.** The compression semigroup of \( W \) is
\[ S_W = G_W \exp(C_{g}^0) \quad \text{with} \quad G_W = G^h_e H^h. \]
Furthermore, \( G_W \) is open in \( G^h \).

**Proof.** By definition,
\[ W = W_M(h)eH = G^h_e. \exp(C_{g}^0 - C_{g}^0) = S(C_{g}^0, h)_e.eH \]
is the orbit of the base point under the interior of the semigroup \( S(C_{g}^0, h)_e = G^h_e \exp(C_{g}^0) \) (cf. Proposition 6.1(b)). This shows that \( S(C_{g}^0, h)_e \subseteq S_W \), and in particular \( G^h_e \subseteq G_W \). For \( s \in S_W \), the inclusion \( sW \subseteq W \) implies
\[ U(s)\mathcal{V} = U(s)H^M(W) = H^M_e(sW) \subseteq H^M_e(W) = \mathcal{V}, \]
so that \( S_W \subseteq S_V \). We thus obtain with (113)
\[ G^h_e \exp(C_{g}^0) \subseteq S_W \subseteq S_V \subseteq G^h_e \exp(C_{g}^0). \] (114)
It follows in particular that \( G^h_e \subseteq G_W \subseteq G^h_e \). This implies in particular that \( G_W \) is open in \( G^h \). As \( G^h_e.eH = M^\alpha H \) by Proposition 3.4 and this submanifold equals \( M^\alpha \cap \mathcal{W} \), it follows that
\[ G_W = \{ g \in G^h : gM^\alpha H = M^\alpha H \} = G^h_e(G^h \cap H) = G^h_e H^h. \]
By (113), \( \exp(C_{g}^0) \subseteq S_W \), so that \( G_W \exp(C_{g}^0) \subseteq S_W \). For the converse, let \( s = g \exp(x) \in S_W \subseteq G^h \exp(C_{g}^0) \). Then
\[ s.eH = g \exp(x).eH \in gW_M(h)eH \subseteq W_M(h) \]
implies \( g \in G_W \).

The preceding proof makes heavy use of representation theoretic results to determine \( S_W \). It would be nice to have a more direct geometric proof, but we presently do not see how to do that.

### 9.2 The symmetric space structure

We now turn to the structure of a symmetric space on the wedge space \( \mathcal{W}(M, h) \).

**Proposition 9.2.** Let \( (g, \tau) \) be a semisimple symmetric Lie algebra, \( G \) a connected Lie group with Lie algebra \( \mathfrak{g} \), \( \tau^G \) an involutive automorphisms of \( G \) integrating \( \tau \) and \( H \subseteq G \) a closed subgroup satisfying
\[ L(H) = \mathfrak{h} \quad \text{and} \quad \tau^G(H) = H. \]
Then \( M = G/H \) carries the structure of a symmetric space.
Proof. In view of Remark 2.13(b), we may factor the subgroup $H \cap Z(G)$ which acts trivially on $M$ and thus assume that $H = G^G$. Then Remark 2.13(d) implies that $H \subseteq G^G$, and since $L(H) = \mathfrak{h} = \mathfrak{g}^r = L(G^G)$, the subgroup $H$ is open in $G^G$. Therefore $M = G/H$ is a symmetric space.

Proposition 9.3. (The wedge space as a symmetric space) Let $(g, \tau, C, h)$ be a non-compactly causal semisimple symmetric Lie algebra, $G$ a connected Lie group with Lie algebra $\mathfrak{g}$ and assume that

- $\tau$ integrates to an involutive automorphism $\tau^G_h$ of $G$,
- $H \subseteq G^G$ is an open subgroup,
- $\tau_h = e^{i\mathfrak{ad}h}$ integrates to an involutive automorphism $\tau^G_h$ of $G$,
- $\tau^G_h(H) = H$, i.e., $\tau_h$ induces an involution $\tau^M_h$ on $M := G/H$.

Then the wedge space $W(G, h) = G/W \cong G/G_W$ is a symmetric space.

Proof. Recall that $G_W = G_h^G H^h \subseteq G^h$ (Theorem 9.1). First we observe that $\tau^G_h(G^h) = G^h$ follows from $\tau(h) = h$. With $\tau^G_h(H) = H$, we therefore get

$$\tau^G_h(G_W) = \tau^G_h(G_h^G H^h) = G_h^G H^h = G_W.$$ 

Hence the assertion follows from Proposition 9.2.

Remark 9.4. In the context of Subsection 9.1 we have four subgroups with the Lie algebra $\mathfrak{g}_0(h) = \ker(\mathfrak{ad} h)$:

$$G_h^h \subseteq G_W \subseteq G_V \subseteq G^h.$$ 

Accordingly, we have a sequence of coverings of homogeneous spaces

$$G/G^h_0 \to W(M, h) \cong G/G_W \to U(G)V \cong G/G_V \to O_h = \text{Ad}(G)h \cong G/C^h.$$ 

Here $O_h \cong \text{Ad}(G).h$ always is a symmetric space because $\tau_h$ defines an involution on $\text{Ad}(G)$ and $\text{Ad}(G)h \subseteq \text{Ad}(G)^G h$ is an open subgroup.

Likewise, $G/G_V \cong O_V := U(G)V$ carries a natural symmetric space structure. The point reflection in the base point $V \in O_V$ is given by $\mathfrak{w} \mapsto J_V\mathfrak{w}$. We refer to [Ne19] for more on the geometry of the space $\text{Stand}(\mathcal{H})$ of standard subspaces of $\mathcal{H}$.

Remark 9.5. (a) The three conditions in Proposition 9.3 are in particular satisfied for spaces of group type $M = G \cong (G \times G)/\Delta_G$, where $\tau^{G \times G}(g_1, g_2) = (g_2, g_1)$ and $\tau^G$ exists on $G$, so that $\tau^{G \times G}_h = \tau^G_h \times \tau^G_h$.

(b) For Cayley type spaces we have $\tau = \tau_h$, so that $G_W = G_h^h H^h \subseteq G^G$ follows from $H \subseteq G^G$.

(c) If $Z(G) = \{e\}$ and $G \cong \text{Ad}(G)$, then

$$M_G \cong \{\varphi \in \text{Aut}(G)_e : \tau \varphi \tau = \varphi^{-1}\}, \quad \tau^G_h(\varphi) = \tau_h \varphi \tau_h, \quad \tau^G(\varphi) = \tau \varphi \tau.$$ 

Further, $G_W \subseteq G^h \subseteq G^G$ (Remark 9.11), so that $W(M_G, h)$ is a symmetric space.

d) For $M = M_G$ we have $H = G^G$, which is $\tau^G_h$-invariant. If $G$ is simply connected, then $H = G^G$ is connected, so that we have in particular $M = G/H \cong M_G$.

e) If the representation $U$ in Theorem 8.12 is faithful, then

$$G_W \subseteq \mathfrak{g}_V^G \tau^G_h \subseteq \mathfrak{g}^G_h,$$

which obviously implies the invariance of $G_W$ under $\tau^G_h$.  

10 Open problems

In this section we briefly discuss some interesting open problems that we plan to address in the future.

**Problem 10.1.** For a compactly causal symmetric space \( M = G/H \) with the infinitesimal data \((g, \tau, C, h)\), characterize the antiunitary representations \((U, \mathcal{H})\) of \( G \) for which there exists a real subspace \( E \subseteq H^- \) satisfying
\[
H^G_G (W_M(h)eH) = V_{(h, \tau, G, U)}.
\]

We conjecture that \( C_g \)-positivity should be enough, but this is not easy to see.

**Problem 10.2.** The symmetric spaces \( W(M, h) \) of wedge domains in \( M \) are particular examples of non-compactly causal parahermitian symmetric spaces. The corresponding causal symmetric Lie algebra is \((g, \tau, h, C_c g)\) (Theorem 9.1). Up to coverings, we thus obtain interesting realizations of non-compactly causal symmetric spaces of Cayley type.

However, there are much more parahermitian symmetric spaces (cf. [Kan00], [MN21]), and we expect that they can likewise be realized as wedge spaces \( W(M, h) \) of a non-compactly causal symmetric space \( M \) (see [NÓ21b] for details and results in this direction). If \((g, \tau, C)\) is non-compactly causal and \( g \) is simple, \( h \in E(g) \), then \((g, \tau, h)\) can be any simple parahermitian symmetric Lie algebra by [´O91, Lem. 2.6]. In this case \( G/G_h \) carries no natural order structure, which is also reflected in the geometry of \( W(M, h) \), respectively the fact that the inclusion order on this set is trivial.

For example, if \( M = G_C/G \) is irreducible non-compactly causal of complex type, corresponding to \((g_C, \tau, iC_g, h)\), where \( g \) is simple hermitian (not necessarily of tube type), and \( h \) a causal Euler element, then \( G_C^h \cong K_C \) is a connected subgroup, and
\[
W(G_C/G, h) \cong G_C/K_C
\]
is the complexification of the Riemannian symmetric space \( G/K \).

11 Anti-de Sitter space \( \text{AdS}^d \)

In this section we discuss the \( d \)-dimensional anti-de Sitter space \( \text{AdS}^d \) as an example of a compactly causal symmetric space. As we verify all assertions for this example directly, it can be read independently of the other sections. It illustrates our results for this concrete example and it is a guiding example for the general theory. From the perspective of physics it is of particular importance because anti-de Sitter spaces are precisely the irreducible compactly causal symmetric spaces which are Lorentzian. There are more reducible examples, even compact ones, such as the conformal compactification of Minkowski space, considered as a symmetric space of the compact group \( \text{SO}_2(\mathbb{R}) \times \text{SO}_d(\mathbb{R}) \).

We endow \( V := \mathbb{R}^{d+1} \) with the symmetric bilinear form
\[
\beta(x, y) := x_0 y_0 + x_1 y_1 - x_2 y_2 - \cdots - x_d y_d.
\]

We consider the connected group \( G := \text{SO}_{2,d-1}(\mathbb{R})_e \) and consider the anti-de Sitter space
\[
M := \text{AdS}^d := \{ x \in V : \beta(x, x) = 1 \} = G.e_0 \cong G/\text{G}^{e_0}
\]
and note that
\[
H := G^{e_0} \cong \text{SO}_{1,d-1}(\mathbb{R})
\]
is connected. By [NÓ21b] Prop. B.3], the exponential function of the symmetric space \( \text{AdS}^d \) is given by
\[
\text{Exp}_p(v) = C(\beta(v, v))p + S(\beta(v, v))v,
\]

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where the functions \( C, S : C \to C \) are defined by
\[
C(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^k \quad \text{and} \quad S(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^k.
\]

Then \( C(z^2) = \cos(z) \) and \( S(z^2) = \frac{\sin(z)}{z} \). It follows in particular that \( \beta \)-positive vectors \( v \in T_{\nu}(M) \) (timelike vectors) generate closed geodesics.

The domain \( V_{>0} := \{ v \in V : \beta(v, v) > 0 \} \) intersects \( T_{e_0}(\text{AdS}^d) \cong e_0^+ \) in a double light cone, and we write \( V_+(e_0) := \{ x \in T_{e_0}(\text{AdS}^d) = e_0^+ : \beta(x, x) > 0, \beta(x, e_1) > 0 \} \) for the “positive” cone containing \( e_1 \). As this cone is \( G_0 \)-invariant by (115), we obtain a causal structure on \( M \) by \( V_+(g.e_0) := g.V_+(e_0) \) for \( g \in G \).

The tangent bundle of \( M \) is \( T(M) = \{ (x, y) \in V \times V : \beta(x, x) = 1, \beta(x, y) = 0 \} \). As \( G \) preserves the causal orientation on \( M \), we call a pair \( (x, y) \in T(M) \)
- • positive if \( G.(x, y) \cap V_+(e_0) \neq \emptyset \), and
- • negative if \( G.(x, y) \cap V_-(e_0) \neq \emptyset \).

Then \( T(M)_\pm = \bigcup_{m \in M} \pm V_+(m) \) is the open subset of positive/negative vectors in the tangent bundle. The reflection \( r_{01}(x) = (-x_0, -x_1, x_2, \ldots, x_d) \) is contained in \( G \), so that \( V_+(-e_0) = r_{01}(V_+(e_0)) = -V_+(e_0) \).

We consider the boost vector field \( X_h(v) = hv \), where \( h \in \mathfrak{so}_{2,d-1}(\mathbb{R}) \) is defined by
\[
hx = (0, x_2, x_1, 0, \ldots, 0).
\]
It generates the flow
\[
(\alpha_t(x) = e^{th} x = (x_0, \cosh t \cdot x_1 + \sinh t \cdot x_2, \cosh t \cdot x_2 + \sinh t \cdot x_1, x_3, \ldots, x_d).
\]

It also leads to the involutions
\[
(\tau_h(x) = e^{\pi i h} x = (x_0, -x_1, -x_2, x_3, \ldots, x_d) \text{ on } \mathbb{R}^{d+1} \quad \text{and} \quad \tau_h^G(g) = \tau_h g \tau_h \text{ on } G,
\]
and the Wick rotation
\[
(\kappa_h(x) = e^{-\frac{\pi i}{2}h} x = (x_0, -ix_2, -ix_1, x_3, \ldots, x_d)
\]
with \( \tau_h = \kappa_h^2 \). Note that \( \tau_h \not\in G \) by (115). This element reverses the causal structure on \( \text{AdS}^d \).
11.1 Subgroup data

We now describe the groups \( G^r_k \) and \( G^h \) in more detail. The subgroup \( G^h \) consists of all elements \( g \) in \( G \) such that \( gh = hg \). Hence \( G^h \) can also be described as the group of all elements \( g \in G \) leaving all \( h \)-eigenspaces invariant:

\[
G^h = \{ g \in G : g(e_1 \pm e_2) \in \mathbb{R}(e_1 \pm e_2), g(e_1, e_2)^\perp = \{ e_1, e_2 \}^\perp \}.
\]

Writing \( V_h := hV = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \cong \mathbb{R}^{1,1} \), we have

\[
G^h = \{ g \in G : gV_h = V_h \} = G \cap (O_{1,1}(\mathbb{R}) \times O_{d-2}(\mathbb{R}))
\]

\[
\supseteq G^h \cong SO^+_1(\mathbb{R}) \times SO^+_1(d-2)(\mathbb{R}).
\]

A maximal compact subgroup \( K \) of \( G \) is

\[
K = G \cap O_{d+1}(\mathbb{R}) \cong SO_d(\mathbb{R}) \times SO_{d-1}(\mathbb{R}).
\]

As

\[
K^r = \{ \pm 1 \} \times (O(\mathbb{R}) \times O_{d-2}(\mathbb{R})) \cong \{ \pm 1 \} \times O_{d-2}(\mathbb{R})
\]

has two connected component, polar decomposition implies

\[
\pi_0(G^h) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } d > 2 \\ \mathbb{Z}_2 & \text{for } d = 2. \end{cases}
\]

(119)

The group \( G^h \) acts on the Minkowski plane \( V_h \cong \mathbb{R}^{1,1} \) by a homomorphism

\[
\gamma : G^h \rightarrow O_{1,1}(\mathbb{R})
\]

which is surjective for \( d > 2 \), but not for \( d = 2 \). We have

\[
G^h = G^h \cup r_0 G^h, \quad \text{where } r_0 = \text{diag}(-1, 1_{d-1}).
\]

We also note that

\[
\text{Ad}(G^h)h = \{ \pm h \} \quad \text{and} \quad G^h \setminus G^h = \{ g \in G : \text{Ad}(g)h = -h \}.
\]

It follows in particular that \( G^h \) is connected if \( d = 2 \) and has two connected components if \( d > 2 \):

\[
G^h = G^h \cup rG^h, \quad \text{where } r = \text{diag}(-1, 1_{d-3})
\]

(120)

The subgroup \( H = G^{\text{\alpha}} \cong SO^+_1(d-1)(\mathbb{R}) \) satisfies

\[
H^r = \text{SO}_1^+(\mathbb{R}) \times O_{d-2}(\mathbb{R}),
\]

which has two connected components for \( d > 2 \) and one for \( d = 2 \).

The submanifold of \( \alpha \)-fixed points in \( M \) is

\[
M^\alpha = \{ (x_0, 0, 0, x_3, \ldots, x_d) : x_0^2 - x_3^2 - \cdots - x_d^2 = 1 \} \cong \mathbb{H}^{d-2} \cup -\mathbb{H}^{d-2}.
\]

Proposition \( \text{3.4} \) implies that \( G^h_v(\pm \mathbb{H}^{d-2}) = \pm \mathbb{H}^{d-2} \). The element \( r(e_0) = -e_0 \), so that \( G^h \) acts transitively on \( M^\alpha \).

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11.2 The tube domain

The tube domain

$$\mathcal{T}_V := V + i V_{>0} \subseteq V_{c}$$

is connected, but its intersection with

$$M_{c} := \{ z = x + iy : \beta(x, x) - \beta(y, y) = 1, \beta(x, y) = 0 \}$$

decomposes into two connected components. In fact, \( z \in M_{c} \cap \mathcal{T}_V \) implies

$$\beta(x, x) = 1 + \beta(y, y) > 1,$$

so that

$$\Gamma(z) := \frac{1}{\sqrt{\beta(x, x)}}(x, y) \in T(M),$$

and the map \( \Gamma : M_{c} \cap \mathcal{T}_V \rightarrow T(M) \) is \( G \)-equivariant. The tangent vectors \( \Gamma(z) = (\tilde{x}, \tilde{y}) \) are \( \beta \)-positive with \( \beta(\tilde{y}, \tilde{y}) < 1 \). If, conversely, \( (\tilde{x}, \tilde{y}) \in T(M) \) satisfies \( 0 < \beta(\tilde{y}, \tilde{y}) < 1 \), then

$$c := \beta(\tilde{x}, \tilde{x}) - \beta(\tilde{y}, \tilde{y}) = 1 - \beta(\tilde{y}, \tilde{y}) > 0,$$

so that \( z := c^{-1/2}(\tilde{x} + i \tilde{y}) \in M_{c} \cap \mathcal{T}_V \). This shows that \( \Gamma \) is a diffeomorphism

$$\Gamma : M_{c} \cap \mathcal{T}_V \rightarrow \{(x, y) \in T(M) : 0 < \beta(y, y) < 1\}.$$

In particular, \( M_{c} \cap \mathcal{T}_V \) has two \( G \)-invariant connected components

$$\mathcal{T}_M^\pm := \Gamma^{-1}(T(M)_\pm),$$

which are called the tube domains of \( M \).

The following lemma describes how the positive tube domain \( \mathcal{T}_M^+ \) can be parametrized by the exponential function of \( M_{c} \).

**Lemma 11.1.** We have

$$\mathcal{T}_M^+ = G \cdot \text{Exp}_{e_0}(iV_+(e_0)) = \text{Exp}(iT(M)_+),$$

and the fixed point set of the antilinear extension \( \nabla_h \) of \( \tau_h \) in the complex manifold \( \mathcal{T}_M^+ \) is

$$(T_M^+)^{\nabla_h} = G^{\tau_h} \cdot \text{Exp}_{e_0}(iV_+(e_0)^{-\tau_h}) = G^{\tau_h} \cdot \text{Exp}_{e_0}(i{\mathbb{R}}_+e_1).$$

**Proof.** The second equality in (122) follows from the \( G \)-equivariance of the exponential function of \( M_{c} \). For the first equality in (122), we start with the observation that both sides are \( G \)-invariant.

“\( \supseteq \)”: We have to show that any \( x \in V_+(e_0) \) satisfies \( z := \text{Exp}_{e_0}(ix) \in T_M^+ \). Since the orbit of \( x \) under \( G^{\tau_h} \) contains an element in \( \mathbb{R}_+e_1 \), we may assume that \( x = x_1 e_1 \) with \( x_1 > 0 \). Then

$$z = \text{Exp}_{e_0}(ix) = \text{Exp}_{e_0}(ix_1 e_1) = C(-x_1^2)e_0 + S(-x_1^2)x_1 i e_1 = \cosh(x_1)e_0 + \sinh(x_1) i e_1$$

and \( x_1 > 0 \) imply

$$\Gamma(z) = (e_0, \tanh(x_1)e_1) \in T(M)_+,$$

hence that \( z \in T_M^+ \).

“\( \subseteq \)”: Conversely, let \( z = x + iy \in T_M^+ \). Acting with \( G \), we may thus assume that \( y = y_1 e_1 \) with \( y_1 > 0 \). Then \( x \in y^{-1} + e_1 \) follows from \( \text{Im} \beta(z, z) = 0 \), so that (acting with \( G^{\tau_h} \)) we may further assume that \( x = x_0 e_0 \) for some \( x_0 \neq 0 \). Now \( z = x_0 e_0 + iy_1 e_1 \in \text{AdS}_c^4 \) implies that

$$1 = \beta(z, z) = x_0^2 - y_1^2.$$
Since
$$\Gamma(z) = \frac{1}{|x_0|}(x_0e_0, y_1e_1) \in T(M)_+,$$
and $y_1 > 0$, we also have $x_0 > 0$. Hence there exists a $t > 0$ with $y_1 = \sinh t$ and $x_0 = \cosh t$. We then have
$$\text{Exp}_{e_0}(ite_1) = \cosh(t)e_0 + \sinh(t)ie_1 = x_0e_1 + y_1ie_1 = z.$$  
This proves (122).

For the second assertion, we start from $T^+_M = G \cdot \text{Exp}_{e_0}(iV_+(e_0))$, which immediately shows that
$$(T^+_M)^{G^h} \supseteq G^{G^h} \cdot \text{Exp}_{e_0}(iV_+(e_0)^{-G^h}),$$
where
$$V_+(e_0)^{-G^h} = V_+(e_0) \cap (\mathbb{R}e_1 + \mathbb{R}e_2) = \{x_1e_1 + x_2e_2 : x_1 > 0, x_1 > |x_2|\}.$$  
We now show that we actually have equality. So let $z = x + iy \in (T^+_M)^{G^h}$. Then $y = y_1e_1 + y_2e_2 \in V^{-G^h}$ is $\beta$-positive, i.e., $y_1^2 > y_2^2$. Acting with $G^h$, we may thus assume that $x = x_0e_0$ and $y = y_1e_1$. From $1 = \beta(z, z) = x_0^2 - y_0^2$ and $z = x + iy = x_0e_0 + iy_1e_1 \in T^+_M$ it now follows that $x_0y_1 > 0$. Acting with the rotation $r_{01}(x) = (-x_0, -x_1, x_2, \ldots, x_d)$, which is contained in $G^{G^h}$, we may further assume that $x_0 > 0$, so that $y_1 > 0$ holds as well. Hence there exists a $t > 0$ with $y_0 = \sinh(t)$ and $x_0 = \cosh(t)$. As above, we now obtain
$$\text{Exp}_{e_0}(ite_1) = \cosh(t)e_0 + \sinh(t)ie_1 = z.$$  
This implies the second assertion.

\section{The wedge domains}

For the closed convex $H$-invariant cone $C := \overline{V_+(e_0)} \subseteq T_{e_0}(M) \cong \mathfrak{g}$, we obtain
$$C_+ = [0, \infty)(e_2 + e_1) \quad \text{and} \quad C_- = [0, \infty)(e_2 - e_1).$$

Lemma 11.2. We have
$$W_M(h)e_0 = G^h_e \cdot \text{Exp}_{e_0}(C^0_+ + C^0_-) \quad \text{and} \quad W_M(h)-e_0 = G^h_e \cdot \text{Exp}_{-e_0}(-C^0_+ - C^0_-).$$

Proof. First we recall that
$$W_M(h)e_0 = G^h_e \cdot \text{Exp}_{e_0}(C^0_+ + C^0_-) = G^h_e \cdot \text{Exp}_{e_0}(\mathbb{R}_+(e_1 + e_2) - \mathbb{R}_+(e_1 - e_2)).$$
For $g = r_{01} = \text{diag}(-1, -1, 1, \ldots, 1) \in G$ we have $\text{Ad}(g)h = -h$, $ge_0 = -e_0$ and $gC_+ = C_-$. With Remark 47 we further obtain
$$W_M(h)-e_0 = W_M(h)g_e_0 = gW_M(-h)e_0 = gG^h_e \cdot \text{Exp}_{e_0}(-(C^0_+ + C^0_-))$$
$$= G^h_e \cdot \text{Exp}_{e_0}(-(C^0_+ + C^0_-)) = G^h_e \cdot \text{Exp}_{-e_0}(-g(C^0_+ + C^0_-))$$
$$= G^h_e \cdot \text{Exp}_{-e_0}(-(C^0_+ + C^0_-)).$$

Lemma 11.3. The positivity domain
$$W^+_M(h) = \{x \in M : X_h(x) \in V_+(x)\}$$
of the vector field $X_h$ satisfies
$$W^+_M(h) = G^h \{x_0e_0 + x_2e_2 \in M : x_0x_2 > 0\}.$$ (123)
Proof. If $X_h(x) \in V_+(x)$, then $x_3^2 > x_1^2$ implies that
\[
x_0^2 = 1 + x_2^2 - x_1^2 + x_3^2 + \cdots + x_d^2 > 1 + x_2^2 + \cdots + x_d^2 > 0.
\]
Acting with the centralizer $G^h$, which contains the group $SO_{1,d-2}^+(\mathbb{R})$, acting on
\[
\ker(h) = \text{span}(e_0, e_3, \ldots, e_d),
\]
we may assume that $x_3 = \cdots = x_d = 0$, so that $x = (x_0, x_1, x_2, 0, \ldots, 0)$ is an element of the
2-dimensional anti-de Sitter space $AdS^2$. Next we observe that the modular flow $\alpha_t$ moves
any element $x$ with $x_3^2 > x_1^2$ to an element with $x_1 = 0$. For $x = (x_0, 0, x_2, 0, \ldots, 0) \in AdS^d$ we
have $x_3^2 > x_1^2$. Now $X_h(x) = x_0 e_1 \in V_+(x)$ is equivalent to $x_0 x_2 > 0$ (see [118]). As $W^+_M(h)$ is
invariant under $G^h \supset \exp(Rh)$, the assertion follows.

Lemma 11.4. The subset
\[
W^K_M(h) := \{ x \in M : (\forall t \in (0, \pi)) \alpha_{it}(x) \in \mathcal{T}_M^+ \}
\]
coincides with $W^+_M(h)$.

Proof. For $x \in W^K_M(h)$ and $0 < t < \pi$ we have
\[
\alpha_{it}(x) = (x_0, \cos t \cdot x_1 + \sin t \cdot ix_2, \cos t \cdot x_2 + \sin t \cdot ix_1, x_3, \ldots, x_d),
\]
so that
\[
\text{Im}(\alpha_{it}(x)) = \sin t \cdot (0, x_2, x_1, 0, \ldots, 0).
\]
This element is $\beta$-positive for every $t \in (0, \pi)$ if and only if $x_3^2 > x_1^2$. Using the $G^h$-invariance of
both sides, we see as in the proof of Lemma 11.3 that we may assume that $x = (x_0, 0, x_2, 0, \ldots, 0)$, so that $x_0^2 - x_2^2 = 1$ and $x_2 \neq 0$. Then
\[
z_t := x_t + iy_t := \alpha_{it}(x) = (x_0, \sin t \cdot ix_2, \cos t \cdot x_2, 0, \ldots, 0) = x_0 e_0 + \cos t \cdot x_2 e_2 + \sin t \cdot x_2 e_1
\]
is $G$-conjugate to
\[
\text{sgn}(x_0) \sqrt{x_0^2 - \cos^2(t)x_2^2} e_0 + \sin t \cdot ix_2 e_1.
\]
Hence $\Gamma(z_t) \in T(M)_+$ for $0 < t < \pi$ is equivalent to $x_0 x_2 > 0$. In view of [128], this is
equivalent to $x \in W^K_M(h)$. We conclude that $W^K_M(h) = W^+_M(h)$.

Lemma 11.5. $W^K_M(h) = \kappa_h((\mathcal{T}^+_M)^\tau h)$.

Proof. For $m \in W^K_M(h)$ we have
\[
\tau_h(\alpha_{m}^+(m)) = \alpha_{-\frac{m}{2}}^+(\tau_h(m)) = \alpha_{-\frac{m}{2}}^+ \alpha_{s1}(m) = \alpha_{m}^+(m),
\]
so that
\[
\alpha_{m}^+(m) \in (\mathcal{T}^+_M)^{\tau h}, \quad \text{and thus} \quad W^K_M(h) \subseteq \kappa_h((\mathcal{T}^+_M)^{\tau h}).
\]
To show that we actually have equality, let $z = x + iy \in (\mathcal{T}^+_M)^{\tau h}$. Then $y = y_1 e_1 + y_2 e_2 \in V^{-\tau h}$ is $\beta$-positive, i.e., $y_1^2 > y_2^2$. Further, $\beta(x, x) > \beta(y, y) > 0$. Acting with $G^h$, we may thus assume that $x = x_0 e_0$ and $y = y_1 e_1$. From $1 = \beta(z, z) = x_0^2 - y_2^2$ and $z = x + iy = x_0 e_0 + iy_1 e_1 \in \mathcal{T}^+_M$ it now follows that $x_0 y_1 > 0$. So
\[
\kappa_h(z) = (x_0, 0, y_1, 0, \ldots, 0) \in W^+_M(h)
\]
follows from $x_0 y_1 > 0$ and [128]. This proves the assertion.

\footnote{As a manifold $AdS^2$ identifies naturally with $dS^2$, but the causal structure is different.}
The following theorem represents four different characterizations of the wedge domain in $M$.

**Theorem 11.6.** For anti-de Sitter space $M = \text{AdS}^d$, we have the equalities

$$W^+_M(h) = W^\text{KMS}_M(h) = \kappa_h((T_M^+)_{e_0}) = W_M(h).$$

**Proof.** The proof follows from the preceding five lemmas. It only remains to show that

$$\kappa_h((T_M^{+})_{e_0}) = W_M(h)_{e_0} \quad \text{and} \quad \kappa_h((T_M^{+})_{-e_0}) = W_M(h)_{-e_0}.$$ 

The component

$$(T_M^{+})_{e_0} = G^h_{G} \cdot \text{Exp}_{e_0}(iV_+ (e_0)^{-\tau_k}) = G^h_{G} \cdot \text{Exp}_{e_0}(i(C^0_+ - C^0_-))$$

is mapped by $\kappa_h$ to

$$G^h_{G} \cdot \text{Exp}_{e_0}(C^0_+ + C^0_-) = W_M(h)_{e_0}.$$ 

For $g = (-1, -1, 1, \ldots, 1)$ we have $g.e_0 = -e_0$, $\text{Ad}(g)h = -h$ and $\text{Ad}(g)C^c = C^c$. Now

$$(T_M^{+})_{-e_0} = G^h_{G}g_0 \cdot \text{Exp}_{e_0}(iV_+ (e_0)^{-\tau_k})$$

is mapped by $\kappa_h$ to

$$G^h_{G} \kappa_h(g_0 \cdot \text{Exp}_{e_0}(iV_+ (e_0)^{-\tau_k})) = G^h_{G}g_0 \text{Exp}_{e_0}(\tau_k) = G^h_{G}g_0 \text{Exp}_{e_0}( -C^0_+ - C^0_-) = W_M(h)_{-e_0},$$

where the last equality follows from Lemma A.2. This completes the proof. \hfill \QED

## A Some facts on symmetric Lie groups

In this section we collect some basic facts on symmetric subgroups of a Lie group $G$ and on Olshanski semigroups.

**Lemma A.1.** Let $(G, \tau)$ be a symmetric Lie group. Then $G^\dagger := \{ g \in G : g^\dagger = g \}$ is a submanifold of $G$ containing $e$. Its identity component is

$$G^\dagger_e = \{gg^\dagger : g \in G_e\} \cong G_e/(G_e)^\tau.$$ 

For $g_0 \in G^\dagger$ and the involutive automorphism $\sigma(g) := g_0^{-1}g_0 \tau(g)g_0^{-1}$, the connected component of $G^\dagger$ containing $g_0$ is

$$G^\dagger_{g_0} = \{gg_0g^\dagger : g \in G_e\} = \{g \sigma(g)^{-1}g_0 : g \in G_e\}.$$ 

**Proof.** In the group $G_e := G \times \{ \text{id}_G, \tau \}$ the involutions form a submanifold $\text{Inv}(G_e)$ and $G_e$ acts transitively on its connected components by conjugation ([GN Ex. 4.6.12]). From

$$\text{Inv}(G_e) \cap (G \times \{ \tau \}) = G^\dagger \times \{ \tau \},$$

we thus derive that $G^\dagger$ is a submanifold of $G$ and that

$$G_e \tau = \{gg^\dagger : g \in G_e\} \times \{ \tau \}$$

is a connected component of $\text{Inv}(G_e)$. Therefore $G^\dagger_e$ is the $G_e$-orbit of $e$ with respect to the action given by $g.x := gxg^\dagger$. The same argument yields the description of $G^\dagger_{g_0}$. \hfill \QED

The following lemma could also be drawn from [KNÖ97 §IV], but for the sake of completeness we provide the complete arguments.
Lemma A.2. Let $(G, \sigma)$ be a connected symmetric Lie group, let $L \subseteq G^\circ$ be an open subgroup, and let $C \subseteq G^\circ$ be an open Ad$(L)$-invariant hyperbolic convex cone. Suppose further that $\exp_G$ is injective on $\mathfrak{g}(g) \cap \mathfrak{q}$, so that we can form the real Olshanski semigroup

$$S := L\exp(C) \subseteq G$$

for which the polar map $L \times C \to S, (g, x) \mapsto g \exp x$ is a homeomorphism. Let $\tau$ be an involutive automorphism of $G$ commuting with $\sigma$ such that $-\tau(C) = C$ and put $g^\tau := \tau(g)^{-1}$. Then the following assertions hold:

1. $S$ is $\tau$-invariant and the subset $S^\tau := \{ s \in S : s^\tau = s \}$ is invariant under the action of $S$ on itself by $s.t := sts^\tau$.

2. The projection $p : S \to L, p(g \exp x) := g$ satisfies

$$p(s^\tau) = p(s)^\tau \quad \text{and} \quad p(g, s) = gp(s)g^\tau \quad \text{for} \quad s \in S, g \in L. \quad (124)$$

3. The connected component $S^\tau_0$ of $S^\tau$ containing $\exp(C^-\tau)$ coincides with

$$L_e.\exp(C^-\tau) = \{ g \exp(x)g^\tau : g \in L_e, x \in C^-\tau \}$$

and the map

$$L_e \times (L_e)^\tau C^-\tau \to S^\tau_0, \quad [g, x] \mapsto g \exp(x)g^\tau \quad (125)$$

is a diffeomorphism.

4. $S^\tau_0 = \{ ss^\tau : s \in S_+ \}$.

5. For each connected component $C$ of $S^\tau$ there exists an element $\ell_0 \in L^\circ \cap C$ such that the involution $\gamma(g) := \ell_0^{-1}\tau(g)\ell_0$ of $G$ satisfies

(a) $\gamma$ commutes with $\sigma$.

(b) $-\gamma(C) = C$.

(c) The connected component $C$ is of the form

$$S^\tau_{\ell_0} = \ell_0\{ s\gamma(s)^{-1} : s \in S_+ \} = \bigcup_{\ell_1 \in L_e} \ell_0\ell_1\exp(C^-\tau)\ell_0\ell_1^\tau\ell_0,$$

which is diffeomorphic to $L_e \times (L_e)^\gamma C^-\tau$.

If $\mathfrak{g}$ is semisimple, then $\mathfrak{g}(g) = \{0\}$, so that the above condition on the exponential function is automatically satisfied. It is also satisfied if $G$ is simply connected.

Proof. For the existence of real Olshanski semigroups, we refer to [La94] Thm. 3.1]. To see that the assumptions of this theorem are satisfied, note that, as $Z(G)$ is a $\sigma$-invariant closed subgroup, so is $Z(G)^{\circ\tau} = \exp(\mathfrak{g}(g) \cap \mathfrak{q})$. Therefore the injectivity of $\exp$ on $\mathfrak{g}(g) \cap \mathfrak{q}$ implies in particular that, for every $x \in \mathfrak{q} \cap \mathfrak{g}$, the one-parameter subgroup $\exp(\mathfrak{R}L)$ is closed and isomorphic to $\mathfrak{R}$.

(1) follows immediately from $L^\circ = L$ and $(\exp x)^\tau = \exp(\tau(x)) \in \exp(C)$ for $x \in C$.

(2) From the bijective polar decomposition (cf. Lawson’s Theorem [Ne00] XI.1.7 and [La94]), we obtain the map $p : S \to L$ and (124) is easily verified.

(3) By (2), the subset $S^\tau_0$ is $L_e$-invariant and contains $\exp(C^-\tau)$. From (124) we get $p(S^\tau_0) \subseteq L^\circ$, and by continuity $p(S^\tau_0) \subseteq L^\circ$. As

$$L^\circ = \{ gg^\tau : g \in L_e \} \cong L_e/(L_e)^\tau$$

by Lemma [A.1] the group $L_e$ acts transitively on $L^\circ$. With (124) we conclude that

$$S^\tau_0 = L_e.p^{-1}(e), \quad \text{where} \quad p^{-1}(e) = \exp(C^-\tau).$$

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This proves the first part of (3). For the second part we observe that \( g_1 \exp(x_1) g_1^* = g_2 \exp(x_2) g_2^* \) is equivalent to \( g_2^{-1} g_1 \in L^t \) (which implies \( (g_2^{-1} g_1)^t = (g_2^{-1} g_1)^{-1} \)), and \( \text{Ad}(g_2^{-1} g_1)x_1 = x_2 \). Therefore the map in (3) is a bijection and the invertibility of its differential implies that it is a diffeomorphism.

(4) Clearly \( ss^t \in S^t \) for every \( s \in S_e \). Hence it suffices to show that every \( g \in (S^t)_e \) is of the form \( ss^t \) for some \( s \in S_e \). Since both sides are \( L_e \)-invariant, we may assume that \( g = \exp(2x) \) for some \( x \in C^{-t} \). Then the assertion follows with \( s := \exp x \).

(5) Let \( S^t_c \subseteq S^t \) be a connected component and \( \ell_0 \in p(S^t_c) \). As \( \ell_0 \in L \subseteq G^s \), the automorphism \( \gamma \) of \( G \) is an involution which commutes with \( \sigma \). Further \( -\gamma(C) = C \) follows from the fact that \( C \) is \( \text{Ad}(\ell_0) \)-invariant. Therefore \( \gamma \) satisfies both assumptions made for \( \tau \). Next we observe that \( \ell_0 = \ell_0^t \) leads to
\[
(\ell_0 s)^t = s^t \ell_0^t = \ell_0 (\ell_0^{-1} s^t) \ell_0 \]
implies that
\[
(\ell_0 s)^t = \ell_0 \{ s \in S : \gamma(s)^{-1} = s \}.
\]
The remaining assertions now follow from (3), applied to the involution \( \gamma \) instead of \( \tau \). □

**Remark A.3.** The preceding lemma extends to connected real Olshanski semigroups
\[
\Gamma_L(C) = L \exp(C)
\]
not contained in a group \( G \) (cf. [Ne00, Def. XI.1.11]). Here \( (g, \sigma) \) is a symmetric Lie algebra, \( L \) is a connected Lie group with Lie algebra \( l = g^\sigma \) to which the adjoint action of \( l \) on \( g^{-\sigma} \) integrates, and \( C \subseteq g^{-\sigma} \) is a pointed generating \( \text{Ad}(L) \)-invariant (weakly) hyperbolic cone.

If \( G \) is the simply connected Lie group with Lie algebra \( g \), these semigroups are constructed by first obtaining \( \Gamma_L(C) \), where \( q_L : \tilde{L} \to L \) is the universal covering, as the simply connected covering of \( \Gamma_G(C) = G^s \exp(C) \subseteq G \), and then factoring the discrete central subgroup \( \ker(q_l) \cong \pi_1(L) \subseteq Z(\tilde{L}) \) (see Definition 4.1 and [Ne00, §XI.1] for more details).

**B The topology on \( \mathcal{H}^\infty \) and the space \( \mathcal{H}^{-\infty} \)**

Let \( (U, \mathcal{H}) \) be a unitary representation of \( G \). A **smooth vector** is an element \( \eta \in \mathcal{H} \) for which the orbit map \( U^0 : G \to \mathcal{H}, g \mapsto U(g)\eta \) is smooth. We write \( \mathcal{H}^\infty = \mathcal{H}^\infty(U) \) for the space of smooth vectors. It carries the **derived representation** \( dU \) of the Lie algebra \( g \) given by
\[
dU(x)\eta = \lim_{t \to 0} \frac{U(\exp tx)\eta - \eta}{t}.
\]
We extend this representation to a homomorphism \( dU : U(g) \to \text{End}(\mathcal{H}^\infty) \), where \( U(g) \) is the complex enveloping algebra of \( g \). This algebra carries an involution \( D \to D^* \) determined uniquely by \( x^\star = -x \) for \( x \in g \). For \( D \in U(g) \), we obtain a seminorm on \( \mathcal{H}^\infty \) by
\[
p_D(\eta) = \|dU(D)\eta\| \quad \text{for} \quad \eta \in \mathcal{H}^\infty.
\]
These seminorms define a topology on \( \mathcal{H}^\infty \) which turn the injection
\[
\eta : \mathcal{H}^\infty \to \mathcal{H}^{\text{dU}(\mathcal{H})}, \quad \xi \mapsto (dU(D)\xi)_{D \in U(g)}
\]
into a topological embedding, where the right hand side carries the product topology (cf. [Mag92, 3.19]). Then \( \mathcal{H}^\infty \) becomes a complete locally convex space for which the linear operators \( dU(D), D \in U(g) \), are continuous. Since \( U(g) \) has a countable basis, countably many such seminorms already determine the topology, so that \( \mathcal{H}^\infty \) is metrizable, hence a Fréchet space. We also observe that the inclusion \( \mathcal{H}^\infty \hookrightarrow \mathcal{H} \) is continuous.
The space $H^\infty$ of smooth vectors is $G$-invariant and we denote the corresponding representation by $U^\infty$. We have the intertwining relation

$$dU(Ad(g)x) = U^\infty(g)dU(x)U^\infty(g)^{-1} \quad \text{for} \quad g \in G, x \in \mathfrak{g}.$$ 

If $\varphi \in C^\infty_c(G)$ and $\xi \in H$ then $U(\varphi)\xi = \int_G \varphi(g) d\mu(g) \xi \in H$. A sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $C^\infty_c(G)$ is called a $\delta$-sequence if $\varphi_n \geq 0$ and $\int_G \varphi_n(g) d\mu(g) = 1$ for all $n \in \mathbb{N}$ and for every $\varepsilon$-neighborhood $V \subseteq G$, we have $\text{supp}(\varphi_n) \subseteq V$ if $n$ is sufficiently large. If $(\varphi_n)_{n \in \mathbb{N}}$ is a $\delta$-sequence, then $U(\varphi_n) \xi \to \xi$, so that $H^\infty$ is dense in $H$.

We write $H^{-\infty}$ for the space of continuous anti-linear functionals on $H^\infty$. Its elements are called distribution vectors. The group $G$, $U(g)$ and $C^\infty_c(G)$ act on $\eta \in H^{-\infty}$ by

- $(U^{-\infty}(g)\eta)(\xi) := \eta(U(g^{-1})\xi)$, $g \in G, \xi \in H^\infty$.
- If $U : G \to \mathcal{A}U(H)$ is an antiunitary representation and $U(g)$ is antiunitary, then we have to modify this definition slightly by $(U^{-\infty}(g)\eta)(\xi) := \eta(U(g^{-1})\xi)$.
- $(dU^{-\infty}(D)\eta)(\xi) := \eta(dU(D^*\eta))$, $D \in \mathcal{U}(g), \xi \in H^\infty$.
- $U^{-\infty}(\varphi)\eta = \eta \circ U^{-\infty}(\varphi^*), \varphi \in C^\infty_c(G)$.

We have natural $G$-equivariant linear embeddings

$$H^\infty \hookrightarrow H \xrightarrow{\xi \mapsto \xi(-\xi)} H^{-\infty}. \quad (128)$$

For each $\varphi \in C^\infty_c(G)$, the map $U(\varphi) : H \to H^\infty$ is continuous, so that its adjoint defines a weak* continuous map $U^{-\infty}(\varphi^*) : H^{-\infty} \to H$. We actually have $U^{-\infty}(\varphi)H^{-\infty} \subseteq H^\infty$ as a consequence of the Dixmier–Malliavin Theorem [DM78, Thm. 3.1], which asserts that every $\varphi \in C^\infty_c(G)$ is a sum of functions of the form $\varphi_1 \ast \varphi_2$ with $\varphi_j \in C^\infty_c(G)$.

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