On the Solution of the Van der Pol Equation

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Abstract

We linearize and solve the Van der Pol equation (with additional nonlinear terms) by the application of a generalized form of Cole-Hopf transformation. We classify also Lienard equations with low order polynomial coefficients which can be linearized by this transformation.
1 Background

The Van der Pol equation [1] was derived originally to model electrical circuits in vacuum tubes. However since then it has been used in both the physical [2-6] and biological sciences. For instance, in biology, Fitzhugh and Nagumo [7] considered a two dimensional version of this equation as a model for the action potentials of neurons.

While there is no existing general theory for integrating nonlinear ordinary differential equations (ODEs), the derivation of exact (closed form) solutions for these equations is a non-trivial and important problem. Some methods to this end are Lie, and related symmetry methods [8]. Many other ad-hoc methods for solving nonlinear ODES and partial differential equations were independently suggested during the last decades. Among these methods Cole-Hopf transformation [9,10,11,12] has been used originally to linearize the Burger’s equation [11] which has various applications in applied mathematics and theoretical physics. Recently however, we introduced [13] a generalization of Cole-Hopf transformation and used it to linearize and solve various nonlinear ordinary differential equations e.g Duffing equation. We showed also that many of the special functions of mathematical physics are exact solutions for a class of nonlinear equations.

In this paper we apply in Sec. 2 the same generalization of Cole-Hopf transformation to linearize the Van der Pol equation with additional quadratic and cubic nonlinear terms. These terms can be considered as perturbations to the original equation. Next in Sec. 3 we present explicit solution (in closed analytic form) to this equation with no external forcing. Solutions to this equation when external forcing term is included can be obtained also by this method. In Sec. 4 we consider the linearization of Lienard equations with low order polynomial coefficients (in the dependent function). We end up in Sec. 5 with some conclusions.

2 The Perturbed Van der Pol Equation

The original form of Van der Pol equation with no forcing can be written as

\[ \psi''(x) = \mu(\beta - \psi(x)^2)\psi'(x) - \alpha \psi(x) \] (2.1)
where $\alpha$ and $\beta$ and $\mu$ are arbitrary constants. In this paper however we consider a more general form of this equation with additional nonlinear and forcing terms

$$\psi(x)'' = \mu(\beta - \psi(x)^2)\psi(x)' - \alpha \psi(x) + v(x)\psi(x)^2 + h(x)\psi(x)^3 + g(x)\psi(x)^4 + f(x). \quad (2.2)$$

We shall say that the solutions of (2.2) and

$$\phi(x)'' = U(x)\phi(x) \quad (2.3)$$

are related by a generalized Cole-Hopf transformation if we can find a function $P(x)$ so that

$$\psi(x) = P(x) + \frac{\phi(x)'}{\phi(x)}. \quad (2.4)$$

To classify those nonlinear equations of the form (2.2) whose solutions can be obtained from those of the linear equation (2.3) we differentiate (2.4) twice and in each step replace the second order derivative of $\phi(x)$ using (2.3). Substituting these results in (2.2) leads to the following equation

$$a_4(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^4 + a_3(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^3 + a_2(x) \left( \frac{\phi(x)'}{\phi(x)} \right)^2 + a_1(x) \frac{\phi(x)'}{\phi(x)} + a_0(x) = 0 \quad (2.5)$$

where

$$a_4(x) = -\mu - g(x) \quad (2.6)$$
$$a_3(x) = -[2\mu + 4g(x)]P(x) - h(x) + 2 \quad (2.7)$$
$$a_2(x) = \mu P(x)' - (6g(x) + \mu)P(x)^2 - 3h(x)P(x) - v(x) + \mu U(x) + \mu \beta \quad (2.8)$$
$$a_1(x) = -4g(x)P(x)^3 - 3h(x)P(x)^2 + [2\mu P(x)' - 2v(x) + 2\mu U(x)]P(x) - 2U(x) + \alpha \quad (2.9)$$
$$a_0(x) = P(x)'' + \mu [P(x)^2 - \beta]P(x)' + U(x)' - g(x)P(x)^4 - h(x)P(x)^3 + [\mu U(x) - v(x)]P(x)^2 + \alpha P(x) - \mu \beta U(x) - f(x) \quad (2.10)$$

To satisfy (2.5) it is sufficient to let $a_i(x) = 0$, $i = 1, 2, 3, 4$. From $a_4 = 0$ we obtain

$$g(x) = -\mu. \quad (2.11)$$

Using this result and $a_3(x) = 0$ we solve for $h(x)$ (in terms of $P(x)$)

$$h(x) = 2(\mu P(x) + 1). \quad (2.12)$$
The condition $a_2(x) = 0$ can be solved then for $v(x)$

$$v(x) = \mu P(x)' + \mu U(x) - [\mu P(x) + 6]P(x) + \mu \beta.$$  \hspace{1cm} (2.13)

Substituting (2.11)-(2.13) in $a_1 = 0$ we solve for $U(x)$ in terms of $P(x)$

$$U(x) = 3P(x)^2 - \mu \beta P(x) + \frac{\alpha}{2}.$$ \hspace{1cm} (2.14)

Finally using the expressions derived in (2.11)-(2.14) in $a_0 = 0$ we have

$$f(x) = P(x)'' - 2\mu \beta P(x)' + [6P(x)' + \alpha + \mu^2 \beta^2]P(x) + 4[P(x) - \mu \beta]P(x)^2 - \frac{\mu \beta \alpha}{2}.$$ \hspace{1cm} (2.15)

This equation can be solved (in principle) for $P(x)$ for a given $f(x)$ or it can be used to determine $f(x)$ for apriori choice of $P(x)$. Eqs. (2.12)-(2.14) can be solved then (in reverse order) to compute the functions $U(x), v(x)$ and $h(x)$.

Using this algorithm we provide in the next section explicit solutions to (2.2) under various conditions.

### 3 Solutions to the Van der Pol Equation

#### 3.1 Van der Pol Equation with No External Forcing

When $f(x) = 0$ the general solution of (2.15) for $P(x)$ is

$$P(x) = \frac{2C_1 \mu \beta e^{\frac{\mu \beta x}{2}} + C_2 (\mu \beta + k) e^{\frac{k x}{2}} + (\mu \beta - k) e^{-\frac{k x}{2}}}{4[C_1 e^{\frac{\mu \beta x}{2}} + C_2 e^{\frac{k x}{2}} + e^{-\frac{k x}{2}}]}.$$ \hspace{1cm} (3.1)

where $k^2 = \mu^2 \beta^2 - 4\alpha$ and $C_1, C_2$ are arbitrary constants. Similar (but more cumbersome) expressions can be obtained for $P(x)$ when $f(x) = c$ where $c$ is a non zero constant.

It is now only a matter of simple algebra to compute the general form of the functions $U(x), v(x)$ and $h(x)$ and then derive solutions to (2.2) by solving the linear equation (2.3).

We consider some simple cases.

**Case 1:** $C_1 = C_2 = 0$. In this case (3.1) reduces to

$$P(x) = \frac{\mu \beta - k}{4}.$$ \hspace{1cm} (3.2)
The resulting expressions for \( U(x) \), \( v(x) \) and \( h(x) \) are

\[
U(x) = \alpha/2 + \frac{(3k + \mu\beta)(k - \mu\beta)}{16} \tag{3.3}
\]

\[
v(x) = -\frac{\mu^2\beta^2}{8} + \frac{\mu}{2} \left( \alpha - \beta + \frac{k^2}{4} \right) + \frac{3k}{2} \tag{3.4}
\]

\[
h(x) = \frac{\mu}{2}(\mu\beta - k) + 2. \tag{3.5}
\]

With these settings the general solution of \( (2.3) \) is

\[
\phi(x) = C_3 \cos(\omega x) + C_4 \sin(\omega x) \tag{3.6}
\]

where \( \omega = \frac{1}{4}\sqrt{\mu^2\beta^2 + 2\mu\beta k - 3k^2 - 8\alpha} \). These solutions are related to the solutions of \((2.2)\) by the transformation \((2.4)\).

**Case 2:** Assume \( k = 0 \) and \( C_1 = 0 \).

In this case

\[
P(x) = \frac{\mu\beta}{4}, \quad U(x) = \alpha/2 - \frac{\mu^2\beta^2}{16}, \quad h(x) = \frac{\mu^2\beta}{2} + 2, \quad v(x) = \frac{\mu}{2} \left( \alpha - \beta - \frac{\mu^2\beta^2}{4} \right) \tag{3.7}
\]

The solution for \( \phi(x) \) is in the same form as in \((3.6)\) with \( \omega = \frac{1}{4}\sqrt{\mu^2\beta^2 - 8\alpha} \).

**Case 3:** Assume \( \alpha = 0 \).

When \( \alpha = 0 \), \( k = \pm \mu\beta \). (In the following we consider only the plus sign). Under this constraint \((3.1)\) reduces to

\[
P(x) = \frac{(C_1 + C_2)e^{\mu\beta x}}{(C_1 + C_2 + e^{-\mu\beta x})} \tag{3.8}
\]

Using \((2.14)\) yields

\[
U(x) = -\frac{c\mu^2\beta^2(e^{-\mu\beta x} - c)}{e^{-\mu\beta x} + c} \tag{3.9}
\]

where \( c = C_1 + C_2 \) Similarly \((2.13)\) and \((2.14)\) lead to

\[
v(x) = \frac{\mu\beta(e^{-\mu\beta x} - 2c)}{e^{-\mu\beta x} + c}, \quad h(x) = 2 + \frac{\mu^2\beta c}{e^{-\mu\beta x} + c} \tag{3.10}
\]

With this data the general solution for \((2.3)\) is

\[
\phi(x) = \frac{C_3 + C_4(c e^{\mu\beta x} + \mu\beta x)}{\sqrt{1 + ce^{\mu\beta x}}} \tag{3.11}
\]
3.2 Van der Pol Equation with External forcing

When the forcing function $f(x)$ is not zero or a constant \( (2.15) \) cannot be solved in closed form for $P(x)$. It is expedient in this case to reverse the process and classify those equations of the form \( (2.2) \) which can be linearized for a given $P(x)$.

To make a proper choice of $P(x)$ we observe that in order to obtain closed form solutions for $\psi(x)$ the function $U(x)$ has to be in a form which enables the solution of \( (2.3) \) to be expressed in "simple form".

To carry this out it is easy to see from \( (2.14) \) that when

\[
U(x) = 3g(x)^2 - \mu \beta g(x) + \frac{\alpha}{2},
\]

(where $g(x)$ is an arbitrary smooth function) then

\[
P(x) = g(x), \quad \text{or} \quad P(x) = -g(x) + \frac{\mu \beta}{3}
\]

The computation of the functions $f(x), v(x)$ and $h(x)$ can be carried out then by substitution. We present some examples.

**Example 1:** Let $g(x) = ax$ where $a$ is a constant. We then have

\[
U(x) = 3a^2x^2 + a\mu \beta x - \frac{\alpha}{2},
\]

\[
f(x) = -4a^3x^3 + a \left( 6a - \alpha + \frac{\mu^2 \beta^2}{3} \right) x - \frac{\mu \beta \alpha}{6} + \frac{\mu^3 \beta^3}{27}
\]

The general solution for $\phi(x)$ can be expressed then in terms of Hypergeometric functions.

**Example 2:** let $g(x) = tan(x)$. In this case we obtain an oscillatory forcing function and the solutions of \( (2.3) \) for $\phi(x)$ can be expressed again in terms of Hypergeometric functions.

4 Polynomial Lienard Equations

The general form of Lienard equations is

\[
\frac{d^2 \psi}{dt^2} + f(\psi) \frac{dx}{dt} + g(\psi) = 0 \quad (4.1)
\]

Since Van der Pol equation with no external forcing belongs to this class of equation it is appropriate to ask if the same method used to solve \( (2.2) \) can be applied to this more general
class of equations. In this section we explore this application to (4.1) when \( f(x) \) and \( g(x) \) are respectively a second and fourth order polynomials

\[
f(x) = \sum_{i=0}^{2} c_i \psi(x)^i, \quad g(\psi) = \sum_{i=0}^{4} b_i \psi(x)^i
\]

Applying the transformation (2.4) with \( \phi(x) \) satisfying (2.3) to (4.1) leads to an equation similar to (ref3.3). which can be satisfied if we set \( a_i = 0 \) for \( i = 0, \ldots, 4 \). We solve this set of equations for \( b_i \) in terms of \( a_i, P(x) \) and \( U(x) \). We obtain the following relations:

\[
\begin{align*}
b_4(x) &= c_2(x), \quad b_3(x) = c_1(x) - 2c_2(x)P(x) + 2, \\
b_2(x) &= c_2(x)P(x)^2 - 2(3 - c_1(x))P(x) - c_2(x)(P(x)' - U(x)) + c_0(x) \\
b_1(x) &= (6 + c_1(x))P(x)^2 - 2c_0(x)P(x) - c_1(x)P(x)' - (c_1(x) + 2)U(x), \\
b_0(x) &= P(x)'' - c_0(x)P(x)' + U(x)' - 2P(x)^3 + 2P(x)U(x) + c_0(x)(P(x)^2 - U(x))
\end{align*}
\]

For apriori choice of the functions \( c_i(x) \) and \( U(x) \), \( P(x) \) these equations can be solved for the \( b_i(x) \). When these relations hold we obtain an equation of the form (4.1) whose solutions are related to those of (2.3) by the transformation (2.6).

An interesting case arises when \( b_0 = 0 \). Under this condition (4.3) will be satisfied if

\[
U(x) = P(x)^2 - P(x)'
\]

which is a Riccati equation and \( a_0(x) \) remains a free parameter.

5 Conclusions

We demonstrated in this paper that a generalized form of Cole-Hopf transformation (2.6) can be used to find solutions of the perturbed Van der Pol equation without forcing. The method can be used also to find solutions of this equation with forcing. However in this case one can not specify apriori the forcing term.
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