Abstract

Let $G$ be a graph and $a, b$ vertices of $G$. A minimal $a, b$-separator of $G$ is an inclusion-wise minimal vertex set of $G$ that separates $a$ and $b$. We consider the problem of enumerating the minimal $a, b$-separators of $G$ that contain at most $k$ vertices, given some integer $k$. We give an algorithm which enumerates such minimal separators, outputting the first $R$ minimal separators in at most $\text{poly}(n)R \cdot \min(4^k, R)$ time for all $R$. Therefore, our algorithm can be classified as fixed-parameter-delay and incremental-polynomial time. To the best of our knowledge, no algorithms with non-trivial time complexity have been published for this problem before. We also discuss barriers for obtaining a polynomial-delay algorithm.

1 Introduction

Recent state-of-the-art algorithm implementations for determining the treewidth [14] and the treedepth [2, 10, 15] of a graph employ a subroutine that enumerates the minimal separators of the graph that contain at most $k$ vertices, for some bound $k$. This enumeration is in fact reported as the bottleneck of these implementations.

The problem of enumerating size bounded minimal separators is also a natural refinement of two well-known enumeration problems: the enumeration of (not necessarily minimal) $a, b$-separators of a graph with size at most $k$ and the enumeration of minimal $a, b$- separators without size bound. Both
of them have received significant attention [1, 6, 7, 8, 12, 13] and admit polynomial-delay algorithms [6, 13].

In this paper we give the following enumeration algorithm.

**Theorem 1.** There is an algorithm that given a graph $G$, a pair of vertices $a, b \in V(G)$, and an integer $k$, enumerates the minimal $a, b$-separators of $G$ that contain at most $k$ vertices, outputting the first $R$ minimal separators in $O^*(R \cdot \min(4^k, R))$ time for all $R$.

Our technique for obtaining this algorithm is to combine the algorithm of Takata [13] for enumerating minimal separators with the important separators technique developed in [3, 11] and exposed in [4]. We obtain our algorithm by using important separators to solve the following decision problem.

**Problem 1.** Given a graph $G$, a pair of vertices $a, b \in V(G)$, integer $k$, and vertex sets $C \subseteq V(G)$ and $X \subseteq N(C)$ such that $a \in C$ and $G[C]$ is connected, decide if there is a minimal $a, b$-separator $S \subseteq V(G) \setminus C$ with $|S| \leq k$ and $X \subseteq S$.

In particular, Problem 1 is the problem of determining if a given subtree of the search tree of Takata’s algorithm contains a minimal separator of size at most $k$. By the $O(n)$ depth of the search tree and standard techniques in enumeration algorithms, solving Problem 1 in time $f(G, k, R)$ implies an $f(G, k, R)$-delay algorithm for enumerating minimal $a, b$-separators of size at most $k$.

The following theorem gives evidence why this approach cannot be directly applied to obtain a polynomial-delay algorithm.

**Theorem 2.** Problem 1 is NP-complete, even when the graph $G$ is bipartite with a bipartition $\{\{a\} \cup N(b), \{b\} \cup N(a)\}$.

In this bipartite case of Theorem 2, size bounded minimal $a, b$-separator enumeration corresponds to size bounded minimal vertex cover enumeration in the graph $G \setminus \{a, b\}$, which to the best of our knowledge is a similarly open problem. We note that a recent paper gives an approximate enumeration algorithm for size bounded minimal vertex cover enumeration [9].

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1The $O^*(\cdot)$ notation suppresses factors polynomial in the input size.
2 Notation

We use standard graph notation. A graph $G$ has a vertex set $V(G)$ and edge set $E(G)$. The subgraph $G[X]$ induced by $X \subseteq V(G)$ has $V(G[X]) = X$ and $E(G[X]) = \{\{u, v\} \in E(G) \mid u, v \in X\}$. We denote induced subgraphs also by notation $G \setminus X = G[V(G) \setminus X]$. We denote the vertex sets of connected components of $G$ by $C(G)$. The set of neighbors of a vertex $v$ is denoted by $N(v)$ and the neighborhood of a vertex set $X$ is $N(X) = \bigcup_{v \in X} N(v) \setminus X$. The set of closed neighbors of $v$ is $N[v] = N(v) \cup \{v\}$ and the closed neighborhood of $X$ is $N[X] = N(X) \cup X$.

For a pair of vertices $a, b \in V(G)$, a minimal $a, b$-separator of $G$ is a vertex set $S \subseteq V(G)$ such that $a$ and $b$ are in different connected components of $G \setminus S$ and $S$ is inclusion-wise minimal with respect to this. A full component of a set $X \subseteq V(G)$ is a component $C \in C(G \setminus X)$ with $N(C) = X$. It is well-known that $S$ is a minimal $a, b$-separator if and only if $S$ has distinct full components containing $a$ and $b$.

An enumeration algorithm with input $I$ has delay $f(I, R)$ if for all $R$ it outputs the first $R$ solutions in at most $R \cdot f(I, R)$ time. A polynomial-delay enumeration algorithm has delay $f(I, R) = \text{poly}(|I|)$ for some polynomial $\text{poly}(|I|)$. An incremental-polynomial enumeration algorithm has delay $f(I, R) = \text{poly}(|I| + R)$ for some polynomial $\text{poly}(|I| + R)$.

3 The Algorithm

We first discuss Takata’s algorithm, then important separators, and then show how these can be combined to obtain our algorithm. In this section we always consider minimal $a, b$-separators of a graph $G$, so we will not spell this out in our definitions.

3.1 Takata’s Recurrence

We overview the Takata’s recurrence for enumerating all minimal $a, b$-separators with polynomial delay [13]. We give short proofs for completeness and because our presentation is different from [13].

Definition 1. Let $C$ and $X$ be vertex sets $C \subseteq V(G)$ and $X \subseteq N(C)$ so that $a \in C$ and $G[C]$ is connected. We denote by $\Delta(G, C, X)$ the set of minimal $a, b$-separators $S$ of $G$ such that $S \subseteq V(G) \setminus C$ and $X \subseteq S$. 

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The root of Takata’s recurrence is $\Delta(G) = \Delta(G, \{a\}, \emptyset)$. The leaves of the recurrence have $X = N(C)$, in which case $\Delta(G, C, N(C)) = \{N(C)\}$ if $N(C)$ is a minimal $a, b$-separator and $\emptyset$ otherwise. The internal nodes are defined by the following proposition.

**Proposition 1 ([13]).** Let $C$ and $X$ be as in Definition 1 and $v$ any vertex in $N(C) \setminus X$. The sets $\Delta(G, C \cup \{v\}, X)$ and $\Delta(G, C, X \cup \{v\})$ are disjoint, and their union is equal to $\Delta(G, C, X)$.

**Proof.** The first case corresponds to the minimal separators that do not contain $v$ and the second case to the minimal separators that contain $v$. \hfill $\Box$

Takata’s algorithm uses the search tree constructed by Proposition 1. To guarantee that the search in this tree finds minimal separators with polynomial delay, it is sufficient to observe that its depth is at most $n$, and to design a polynomial time algorithm for determining if the currently explored subtree is empty, i.e., if $\Delta(G, C, X) = \emptyset$. The following proposition provides this emptiness check.

**Proposition 2 ([13]).** The set $\Delta(G, C, X)$ is not empty if and only if there is a component $C_b \in C(G \setminus N[C])$ such that $b \in C_b$ and $X \subseteq N(C_b)$.

**Proof.** If $b \in N[C]$ there is no minimal $a, b$-separator that does not contain vertices in $C$. Otherwise $N(C_b)$ is a minimal $a, b$-separator in $\Delta(G, C, \emptyset)$ because it has a full component $C_b$ containing $b$ and $C_a \supseteq C$ containing $a$ because $N(C_b) \subseteq N(C)$. Now it suffices to show that if $N(C_b)$ does not subsume $X$ then no minimal separator in $\Delta(G, C, \emptyset)$ subsumes $X$. This follows from the fact that for any such minimal separator $S'$ the full component $C_b'$ of $S'$ containing $b$ is a subset of $C_b$ and thus $N(C_b') \cap N(C) \subseteq N(C_b) \cap N(C)$. \hfill $\Box$

Our algorithm is the same as Takata’s algorithm, except that we do not output minimal separators with size $> k$, and instead of determining if $\Delta(G, C, X)$ is empty we determine if it contains minimal separators of size at most $k$. For this we use important separators.

### 3.2 Important Separators

We overview the results on important separators [3, 11] that we use. This overview is based on the exposition of this technique in [4].
Definition 2. Let $A, B \subseteq V(G)$ be vertex sets of $G$. A set $S$ is a minimal $A, B$-separator if there exist components $C_A, C_B \in C(G \setminus S)$ with $A \subseteq C_A$, $B \subseteq C_B$, and $S = N(C_A) = N(C_B)$. A minimal $A, B$-separator is an important $A, B$-separator if there is no minimal $A, B$-separator $S'$ such that $|S'| \leq |S|$ and $C_A \subseteq C_A'$, where $C_A'$ is the component of $G \setminus S'$ containing $A$.

Important separators are exploited by using an enumeration algorithm that given vertex sets $A, B$ and an integer $k$ enumerates important $A, B$-separators of size at most $k$. In particular, we use the following proposition.

Proposition 3 ([4]). Let $A, B \subseteq V(G)$ be vertex sets of $G$ and $k$ an integer. There are at most $4k$ important $A, B$-separators of $G$ of size at most $k$ and they can be enumerated with polynomial delay.

Proposition 3 will be our tool to check if a subtree of the search tree in the enumeration algorithm is empty.

3.3 Proof of Theorem 1

Now we are ready to give our algorithm. We modify Definition 1 for the purpose of our algorithm.

Definition 3. Let $C, X$ be vertex sets $C \subseteq V(G)$ and $X \subseteq N(C)$ so that $a \in C$ and $G|C$ is connected. We denote by $\Delta(G, k, C, X)$ the set of minimal $a, b$-separators $S$ of $G$ such that $S \subseteq V(G) \setminus C$, $X \subseteq S$, and $|S| \leq k$.

This definition is analogous to Definition 1, except that it also includes a size bound $k$. By the same arguments as given for Takata’s algorithm, we can enumerate minimal $a, b$-separators of size at most $k$ with $f(G, k, R)$-delay if we have an $f(G, k, R)$ time algorithm for checking if $\Delta(G, k, C, X)$ is empty. The following lemma shows that we can use important separators to obtain this algorithm.

Lemma 1. The set $\Delta(G, k, C, X)$ is not empty if and only if there is an important $\{b\}, C$-separator $S$ such that $X \subseteq S$ and $|S| \leq k$.

Proof. For the if direction we observe that such $S$ satisfies $S \in \Delta(G, k, C, X)$. For the only if direction, let $S' \in \Delta(G, k, C, X)$ and denote by $C'_b$ the component of $G \setminus S'$ containing $b$. If $S'$ is not an important $\{b\}, C$-separator then there is an important $\{b\}, C$-separator $S$ with $|S| \leq k$ and a component $C_b \in C(G \setminus S)$ with $C'_b \subseteq C_b$. Because neither $C_b$ nor $C'_b$ intersects $N(C)$ we have that $N(C'_b) \cap N(C) \subseteq N(C_b) \cap N(C)$, and therefore $X \subseteq S$. □
Lemma 1 asserts that we can check if $\Delta(G, k, C, X)$ is empty by enumerating important $\{b\}, C$-separators of size at most $k$. By Proposition 3 this can be done in $O^*(4^k)$ time. To make the time complexity into $O^*(\min(4^k, R))$, where $R$ is the number of minimal separators already outputted, we note that the algorithm of Proposition 3 works in polynomial delay and all important separators outputted by it are also minimal $a, b$-separators of size at most $k$. Therefore we simply keep a set of already outputted minimal separators, and if an important separator given by Proposition 3 is not in this set we output it. Note that this will cause us to “miss” some outputs later, but this does not matter because the outputting of them can be seen just as moved forward. This completes the proof of Theorem 1.

4 Hardness

We show that the problem of testing if $\Delta(G, k, C, X)$ is empty is NP-hard even in graphs with bipartition $\{\{a\} \cup N(b), \{b\} \cup N(a)\}$, i.e., we prove Theorem 2. We reduce from set cover, which is NP-hard [5].

Let $U$ be a set and $F$ a family of subsets of $U$. Given $U$, $F$, and an integer $k$, the set cover problem is to determine if there is a subset $F' \subseteq F$ with $|F'| \leq k$ and $U = \bigcup_{T \in F'} T$.

We construct a graph $G(U, F)$ that has four layers, $\{a\}$, $N(a)$, $N(b)$, and $\{b\}$. The vertices of $N(b)$ corresponds to sets in $F$, i.e., for each set $T \in F$ there is a vertex $v_T \in N(b)$. For each vertex $v_T \in N(b)$ there are two vertices $u_T, w_T \in N(a)$ that are connected only to $v_T$ and $a$. The other vertices in $N(a)$ are the elements of $U$. We add an edge from $z \in U$ to $v_T \in N(b)$ if $z \in T$.

We first show that given a solution to the set cover problem we can construct a minimal $a, b$-separator in $\Delta(G(U, F), |U| + |F| + k, \{a\}, U)$.

**Lemma 2.** If there is a subset $F' \subseteq F$ with $|F'| \leq k$ and $U = \bigcup_{T \in F'} T$ then $\Delta(G(U, F), |U| + |F| + k, \{a\}, U)$ is not empty.

**Proof.** We construct a minimal $a, b$-separator that consists of vertices $z \in U$, vertices $v_T \in N(b)$ with $T \notin F'$, and vertices $u_T, w_T \in N(a)$ with $T \in F'$. The size of this separator is $|U| + |F| - k + 2k$. For any path $a, u_T, v_T, b$ or $a, w_T, v_T, b$ exactly one of the vertices is in the separator, so this indeed separates $a$ from $b$. This separator is minimal because each $z \in U$ is connected
to a vertex \( v_T \in N(b) \) with \( z \in T \in F' \) implying that \( v_T \) is not in the separator. \( \square \)

We complete the NP-completeness proof by showing that for a minimal \( a, b \)-separator in \( \Delta(G(U, F), |U| + |F| + k, \{a\}, U) \) we can construct a solution to the set cover problem.

**Lemma 3.** If there is \( S \in \Delta(G(U, F), |U| + |F| + k, \{a\}, U) \) then there is a subset \( F' \subseteq F \) with \( |F'| \leq k \) and \( U = \bigcup_{T \in F'} T \).

**Proof.** We construct the subset \( F' \) by including the sets \( T \) with \( v_T \notin S \). This is a set cover because for each \( z \in U \) there must be a vertex \( v_T \notin S \) with \( z \in T \) because otherwise \( S\setminus \{z\} \) would be an \( a, b \)-separator. We note that the vertices \( S \cap N(b) \) determine \( S \) uniquely, in particular forcing \( u_T, w_T \) to \( S \) if and only if \( v_T \notin S \), so we can compute that \( |U| + 2|F'| + |F| - |F'| = |S| \). \( \square \)

This completes the proof of Theorem 2.

5 Conclusion

We gave a fixed-parameter-delay and incremental-polynomial enumeration algorithm for enumerating minimal \( a, b \)-separators of size at most \( k \). To the best of our knowledge, this is the first algorithm for this problem with non-trivial time complexity. While our algorithm seems not completely impractical, ideally we would prefer a polynomial-delay algorithm to optimally implement the enumeration as a subroutine in various applications. We gave an NP-completeness proof that illustrates why our approach falls short in obtaining a polynomial-delay algorithm. Informally, our proof shows that any algorithm based on Takata’s recurrence must have a more “global” view on the search space than just the current subtree. The NP-completeness result also illustrates that enumerating minimal vertex covers of size at most \( k \) in a bipartite graph is an important open special case of this problem.

References

[1] Anne Berry, Jean Paul Bordat, and Olivier Cogis. Generating all the minimal separators of a graph. *Int. J. Found. Comput. Sci.*, 11(3):397–403, 2000.
[2] Ruben Brokkelkamp, Raymond van Venetië, Mees de Vries, and Jan Westerdiep. PACE Solver Description: tdULL. In 15th International Symposium on Parameterized and Exact Computation (IPEC 2020), volume 180 of LIPIcs, pages 29:1–29:4, 2020.

[3] Jianer Chen, Yang Liu, and Songjian Lu. An improved parameterized algorithm for the minimum node multiway cut problem. Algorithmica, 55(1):1–13, 2009.

[4] Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.

[5] M. R. Garey and David S. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

[6] Horst W. Hamacher. An O(k·n^4) algorithm for finding the k best cuts in a network. Oper. Res. Lett., 1(5):186–189, 1982.

[7] Arkady Kanevsky. On the number of minimum size separating vertex sets in a graph and how to find all of them. In David S. Johnson, editor, Proceedings of the First Annual ACM-SIAM Symposium on Discrete Algorithms, 22-24 January 1990, San Francisco, California, USA, pages 411–421. SIAM, 1990.

[8] Ton Kloks and Dieter Kratsch. Listing all minimal separators of a graph. SIAM J. Comput., 27(3):605–613, 1998.

[9] Yasuaki Kobayashi, Kazuhiro Kurita, and Kunihiro Wasa. Efficient constant-factor approximate enumeration of minimal subsets for monotone properties with cardinality constraints. CoRR, abs/2009.08830, 2020.

[10] Tuukka Korhonen. PACE Solver Description: SMS. In 15th International Symposium on Parameterized and Exact Computation (IPEC 2020), volume 180 of LIPIcs, pages 30:1–30:4, 2020.

[11] Dániel Marx. Parameterized graph separation problems. Theor. Comput. Sci., 351(3):394–406, 2006.
[12] Hong Shen and Weifa Liang. Efficient enumeration of all minimal separators in a graph. *Theor. Comput. Sci.*, 180(1-2):169–180, 1997.

[13] Ken Takata. Space-optimal, backtracking algorithms to list the minimal vertex separators of a graph. *Discret. Appl. Math.*, 158(15):1660–1667, 2010.

[14] Hisao Tamaki. Computing treewidth via exact and heuristic lists of minimal separators. In *Analysis of Experimental Algorithms - Special Event, SEA² 2019, Kalamata, Greece, June 24-29, 2019, Revised Selected Papers*, volume 11544 of *LNCS*, pages 219–236. Springer, 2019.

[15] Zijian Xu, Dejun Mao, and Vorapong Suppakitpaisarn. PACE Solver Description: Computing Exact Treedepth via Minimal Separators. In *15th International Symposium on Parameterized and Exact Computation (IPEC 2020)*, volume 180 of *LIPIcs*, pages 31:1–31:4, 2020.