Igusa-Todorov distances of Artin algebras

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Abstract

We introduce Igusa-Todorov distances of Artin algebra, prove its invariance under derived equivalence, present its application to exterior algebra, and establish the link between the dimension of the singularity category and this distance.

1 Introduction

In the representation theory of Artin algebras, the finitistic dimension conjecture is a very important problem\cite{2,3,13}, which is closely related to many homology conjectures\cite{24}. Igusa and Todorov introduced two functions, \( \Phi \) and \( \Psi \)\cite{14}, which are called Igusa-Todorov functions. Igusa-Todorov functions have become an important tool for the study of the finitistic dimension conjecture.

Using the properties of Igusa-Todorov functions, Wei defines a class of algebras called \( (n-) \) Igusa-Todorov algebras\cite{22}, which satisfy the finitistic dimension conjecture. A natural question arises, namely, are all Artin algebras Igusa-Todorov algebras\cite{22}? Conde illustrates that not all Artin algebras are Igusa-Todorov algebras\cite{8}. Zheng defines \((m,n)\)-Igusa-Todorov algebra, which is a generalization of \((n-)\) Igusa-Todorov algebra. For each Artin algebra \( \Lambda \), there exist non-negative integers \( m \) and \( n \) such that \( \Lambda \) is an \((m,n)\)-Igusa-Todorov algebra\cite{26}. Zheng gives the upper bound of the derived dimension of the \((m,n)\)-Igusa-Todorov algebra\cite{26}.

In this article, we will introduce the concept of the Igusa-Todorov distance of Artin algebras, which can reflect how far an algebra is from an Igusa-Todorov algebra. We will show that this distance is an invariant under derived equivalence. In fact, this work can be seen as a generalization of Wei’s conclusion in\cite{23}. As we will illustrate, the Igusa-Todorov distance can be arbitrarily large. We also will prove the relationship between the dimension of the singularity category and the Igusa-Todorov distance.

The main results of this paper are as follows.

**Theorem 1.1.** (Theorem 2.32) Let \( k \) be a field and \( n \) a positive integer. Then

\[
\text{IT.dist}_\Lambda(k^n) = n - 1.
\]
Theorem 1.2. (Theorem 3.6) If Artin algebra \( A \) and \( B \) are derived equivalent, then \( \text{IT.dist} \ A = \text{IT.dist} \ B \).

Theorem 1.3. (Theorem 4.3) Given an Artin algebra \( \Lambda \). We have \( \text{tri.dim} \ D_{sg}(\text{mod} \ \Lambda) \leq \text{IT.dist} \ \Lambda \).

2 Preliminaries

2.1 The extension dimension of module category

Let \( \Lambda \) be an Artin algebra. All subcategories of \( \text{mod} \ \Lambda \) are full, additive and closed under isomorphisms and all functors between categories are additive. For a subclass \( \mathcal{U} \) of \( \text{mod} \ \Lambda \), we use \( \text{add} \ \mathcal{U} \) to denote the subcategory of \( \text{mod} \ \Lambda \) consisting of direct summands of finite direct sums of objects in \( \mathcal{U} \). Let us recall some notions and basic facts (for example, see [4, 30]). Let \( \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n \) be subcategories of \( \text{mod} \ \Lambda \). Define

\[
\mathcal{U}_1 \bullet \mathcal{U}_2 := \text{add}\{ M \in \text{mod} \ \Lambda \mid \text{there exists an sequence } 0 \to U_1 \to M \to U_2 \to 0 \text{ in } \text{mod} \ \Lambda \text{ with } U_1 \in \mathcal{U}_1 \text{ and } U_2 \in \mathcal{U}_2 \}\.
\]

Inductively, define

\[
\mathcal{U}_1 \bullet \mathcal{U}_2 \bullet \cdots \bullet \mathcal{U}_n := \text{add}\{ M \in \text{mod} \ \Lambda \mid \text{there exists an sequence } 0 \to U \to M \to V \to 0 \text{ in } \text{mod} \ \Lambda \text{ with } U \in \mathcal{U}_1 \text{ and } V \in \mathcal{U}_2 \bullet \cdots \bullet \mathcal{U}_n \}\.
\]

For a subcategory \( \mathcal{U} \) of \( \text{mod} \ \Lambda \), set \( [\mathcal{U}]_0 = 0, [\mathcal{U}]_1 = \text{add} \ \mathcal{U}, [\mathcal{U}]_n = [\mathcal{U}]_1 \bullet [\mathcal{U}]_{n-1} \) for any \( n \geq 2 \). If \( T \in \text{mod} \ \Lambda \), we write \( [T]_n \) instead of \( \{[T]\}_n \).

Let \( X \in \text{mod} \ \Lambda \). Given an epimorphism \( f : P \to X \) in \( \text{mod} \ \Lambda \) such that \( P \) is a projective cover of \( X \) in \( \text{mod} \ \Lambda \), then we write \( \Omega^{1}_\Lambda(X) := \text{Ker} f \) (the subscript can also be omitted if there is no misunderstanding, that is, \( \Omega^{1}_\Lambda(X) = \Omega^{1}(X) \)). Inductively, for any \( n \geq 2 \), we write \( \Omega^{n}(X) := \Omega^{1}(\Omega^{n-1}(X)) \). In particular, we set \( \Omega^{0}(X) := X \). Dually, if \( g : X \to I \) is an injective envelope of \( X \) with \( I \) injective, then the cokernel of \( g \) is called a cosyzygy of \( X \), denoted by \( \Omega^{-1}(X) \). Inductively, for each \( n \geq 2 \), we set \( \Omega^{-n}(X) := \Omega^{-1}(\Omega^{-(n-1)}(X)) \).

Definition 2.1. ([4]) The extension dimension \( \text{ext.dim} \ \Lambda \) of \( \text{mod} \ \Lambda \) is defined to be

\[
\text{ext.dim} \ \Lambda := \inf\{ n \geq 0 \mid \text{mod} \ \Lambda = [T]_{n+1} \text{ with } T \in \text{mod} \ \Lambda \}.
\]

Lemma 2.2. ([30 Corollary 2.3(1)]) For each \( T_1, T_2 \in \text{mod} \ \Lambda \) and nonnegative integer, we have

\[
[T_1]_m \bullet [T_1]_n \subseteq [T_1 \oplus T_2]_{m+n}.
\]

Lemma 2.3. ([4 Example 1.6(i)]) Let \( \Lambda \) be an Artin algebra. Then \( \Lambda \) is representation finite if and only if \( \text{ext.dim} \ \Lambda = 0 \).

Let \( \Lambda \) be an Artin algebra. Recall that \( \Lambda \) is called \textit{n-Gorenstein} if its left and right self-injective dimensions are at most \( n \). Let \( \mathcal{P} \) be the subcategory of \( \text{mod} \ \Lambda \) consisting of projective
modules. A module $G \in \text{mod} \Lambda$ is called \textbf{Gorenstein projective} if there exists a $\text{Hom}_\Lambda(-, P)$-exact exact sequence 

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$$

in $\text{mod} \Lambda$ with all $P_i, P^k$ in $\mathcal{P}$ such that $G \cong \text{Im}(P_0 \rightarrow P^0)$. Recall from [5] that $\Lambda$ is said to be of \textbf{finite Cohen-Macaulay type} (\textbf{finite CM-type} for short) if there are only finitely many non-isomorphic indecomposable Gorenstein projective modules in $\text{mod} \Lambda$.

**Lemma 2.4.** ([30]) If $\Lambda$ is an $n$-Gorenstein Artin algebra of finite CM-type, then $\text{ext.dim} \Lambda \leq n$.

**Example 2.5.** Let $k$ be an algebraically closed field, and $n = 4$ or $5$. Then 

$$T_2(k[x]/\langle x^n \rangle) := \left( \begin{array}{cc} k[x]/\langle x^n \rangle & 0 \\ k[x]/\langle x^n \rangle & k[x]/\langle x^n \rangle \end{array} \right)$$

is a representation-infinite, CM-finite 1-Gorenstein of infinite global dimension (see [17]). By Corollary 2.4, we know that $\text{ext.dim} T_2(k[x]/\langle x^n \rangle) \leq 1$. By Lemma 2.3, we see that $\text{ext.dim} T_2(k[x]/\langle x^n \rangle) > 0$.

And then, we can get that $\text{ext.dim} T_2(k[x]/\langle x^n \rangle) = 1$.

**Definition 2.6.** ([16]) The weak resolution $w. \text{resol.dim} \Lambda$ of Artin algebra $\Lambda$ as the minimal number $n \geq 0$ which satisfies the following equivalent conditions.

(i) There exists $M \in \text{mod} \Lambda$ such that, for any $X \in \text{mod} \Lambda$, there exists an exact sequence 

$$0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow X \rightarrow 0$$

with $M_i \in \text{add} M$ and $X \in \text{add} Y$.

(ii) There exists $M \in \text{mod} \Lambda$ such that, for any $X \in \text{mod} \Lambda$, there exists an exact sequence 

$$0 \rightarrow Y \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n \rightarrow 0$$

with $M_i \in \text{add} M$ and $X \in \text{add} Y$.

**Lemma 2.7.** ([25, Lemma 2.9] [30]) For an Artin algebra $\Lambda$, we have $w. \text{resol.dim} \Lambda = \text{ext.dim} \Lambda$.

Now let us recall the Oppermann weak resolution dimension $O.w. \text{resol.dim} \Lambda$ which is defined by Oppermann.

**Definition 2.8.** ([18]) The Oppermann weak resolution $O.w. \text{resol.dim} \Lambda$ of Artin algebra $\Lambda$ as the minimal number $n \geq 0$ which satisfies the following condition:

there exists $M \in \text{mod} \Lambda$ such that, for any $X \in \text{mod} \Lambda$, there exists an exact sequence 

$$0 \rightarrow M_n \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_0 \rightarrow X \rightarrow 0$$

with $M_i \in \text{add} M$.

Compare Definition 2.6 with Definition 2.8, we have
Lemma 2.9. For Artin algebra, we have $O_{w.resol.dim} \Lambda \geq w.resol.dim \Lambda$.

Lemma 2.10. ([30 Lemma 3.5]) Let $\Lambda$ be an Artin algebra.

(1) If

$$0 \rightarrow M \rightarrow X_0 \rightarrow X_{-1} \rightarrow \cdots \rightarrow X_{-n} \rightarrow 0,$$

is an exact sequence in $\text{mod} \Lambda$ with $n \geq 0$, then

$$M \in [\Omega^n(X_{-n})]_1 \bullet [\Omega^{n-1}(X_{-(n-1)})]_1 \bullet \cdots \bullet [\Omega^1(X_{-1})]_1 \bullet [X_0]_1 \subseteq \bigoplus_{i=0}^{n} \Omega^i(X_{-i})_{n+1}.$$ 

(2) If

$$0 \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0,$$

is an exact sequence in $\text{mod} \Lambda$ with $n \geq 0$, then

$$M \in [X_0]_1 \bullet [\Omega^{-1}(X_1)]_1 \bullet \cdots \bullet [\Omega^{-n}(X_n)]_1 \subseteq \bigoplus_{i=0}^{n} \Omega^{-i}(X_i)_{n+1}.$$ 

Lemma 2.11. ([29 Lemma 3.6(2)]) Let $X, Y \in \text{mod} \Lambda$ satisfy $[X]_1 \subseteq [Y]_m$. Then for any $p \geq 0$, we have $[\Omega^{-p}(X)]_m \subseteq [\Omega^{-p}(Y)]_m$.

2.2 The dimension of triangulated category

We recall some notions from [18, 20, 21]. Let $\mathcal{T}$ be a triangulated category and $\mathcal{I} \subseteq \text{Ob} \mathcal{T}$. Let $(\mathcal{I})_1$ be the full subcategory consisting of $\mathcal{T}$ of all direct summands of finite direct sums of shifts of objects in $\mathcal{I}$. Given two subclasses $\mathcal{I}_1, \mathcal{I}_2 \subseteq \text{Ob} \mathcal{T}$, we denote $\mathcal{I}_1 \ast \mathcal{I}_2$ by the full subcategory of all extensions between them, that is,

$$\mathcal{I}_1 \ast \mathcal{I}_2 = \{X | X_1 \rightarrow X \rightarrow X_2 \rightarrow X_1[1] \text{ with } X_1 \in \mathcal{I}_1 \text{ and } X_2 \in \mathcal{I}_2\}.$$ 

Write $\mathcal{I}_1 \circ \mathcal{I}_2 := (\mathcal{I}_1 \ast \mathcal{I}_2)_1$. Then $(\mathcal{I}_1 \circ \mathcal{I}_2) \circ \mathcal{I}_3 = \mathcal{I}_1 \circ (\mathcal{I}_2 \circ \mathcal{I}_3)$ for any subclasses $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ of $\mathcal{T}$ by the octahedral axiom. Write

$$(\mathcal{I})_0 := 0, \ (\mathcal{I})_{n+1} := (\mathcal{I})_n \circ (\mathcal{I})_1 \text{ for any } n \geq 1.$$

Definition 2.12. ([20 Definiton 3.2]) The dimension $\text{tri.dim} \mathcal{T}$ of a triangulated category $\mathcal{T}$ is the minimal $d$ such that there exists an object $M \in \mathcal{T}$ with $\mathcal{T} = (M)_d+1$. If no such $M$ exists for any $d$, then we set $\text{tri.dim} \mathcal{T} = \infty$.

Lemma 2.13. ([19 Lemma 7.3]) Let $\mathcal{T}$ be a triangulated category and let $X, Y$ be two objects of $\mathcal{T}$. Then

$$\langle X \rangle_m \circ \langle Y \rangle_n \subseteq \langle X \oplus Y \rangle_{m+n}$$

for any $m, n \geq 0$. 
2.3 \((m, n)\)-Igusa-Todorov algebras

**Definition 2.14.** ([22 Definition 2.2]) For nonnegative integer \(n\). The Artin algebra \(\Lambda\) is said to be an \(n\)-Igusa-Todorov algebra if there is a module \(V \in \text{mod } \Lambda\) such that for any module \(M\) there exists an exact sequence

\[
0 \to V_1 \to V_0 \to \Omega^n(M) \to 0
\]

where \(V_i \in \text{add } V\) for each \(0 \leq i \leq 1\). Such a module \(V\) is said to be an \(n\)-Igusa-Todorov module.

The following definition is a generalization of Definition 2.14.

**Definition 2.15.** ([26, Definition 2.1]) For two nonnegative integers \(m, n\). The Artin algebra \(\Lambda\) is said to be an \((m, n)\)-Igusa-Todorov algebra if there is a module \(V \in \text{mod } \Lambda\) such that for any module \(M \in \text{mod } \Lambda\) there exists an exact sequence

\[
0 \to V_m \to V_{m-1} \to \cdots \to V_1 \to V_0 \to \Omega^n(M) \to 0
\]

where \(V_i \in \text{add } V\) for each \(0 \leq i \leq m\). Such a module \(V\) is said to be an \((m, n)\)-Igusa-Todorov module.

For each \(n \geq 1\), we denote

\[
\Omega^n(\text{mod } \Lambda) := \{X|X = \Omega^n(Y) \oplus P\text{ for some } Y \in \text{mod } \Lambda\text{ and projective module } P\in \text{mod } \Lambda\}
\]

\[
= \{X \mid \text{there exists an exact sequence } 0 \to X \to P_n \to P_{n-1} \to \cdots \to P_1 \text{ with projective module } P_i \in \text{mod } \Lambda\text{ for each } 1 \leq i \leq n\}.
\]

And \(\Omega^0(\text{mod } \Lambda) := \text{mod } \Lambda\). Recall that \(\Lambda\) is said to be \(n\)-syzygy-finite if \(\Omega^n(\text{mod } \Lambda) = \text{add } M\) for some \(M \in \text{mod } \Lambda\). And \(\Lambda\) is said to be syzygy finite if there exists an nonnegative integer \(n\) such that \(\Omega^n(\text{mod } \Lambda) = \text{add } M\) for some \(M \in \text{mod } \Lambda\). In particular, \(\Lambda\) is \(0\)-syzygy-finite if and only if \(\Lambda\) is representation finite type.

**Remark 2.16.** By Definition 2.15 and Definition 2.14, we have the following easy observations.

1. \((1, n)\)-Igusa-Todorov algebras are the same as \(n\)-Igusa-Todorov algebras.
2. \((0, n)\)-Igusa-Todorov algebras are the same as \(n\)-syzygy-finite algebras.

2.4 The weak representation dimension of Artin algebra

**Definition 2.17.** ([1]) The **representation dimension** \(\text{rep.dim } \Lambda\) of Artin algebra \(\Lambda\) is defined as

\[
\text{rep.dim } \Lambda := \inf\{\text{gl.dim } \text{End}_\Lambda(M) \mid M \text{ is a generator-cogenerator for } \text{mod } \Lambda\}
\]

if \(\Lambda\) is non-semisimple; and \(\text{rep.dim } \Lambda = 1\) if \(\Lambda\) is semisimple.

In [15], Iyama prove that \(\text{rep.dim } \Lambda\) is finite for each Artin algebra \(\Lambda\).
Definition 2.18. ([20, Definition 3.2]) The weak representation dimension of Artin algebra \( \Lambda \), denoted by \( \text{w.rep.dim} \Lambda \), is the smallest integer \( i \geq 2 \) such that there is an object \( M \in \text{mod} \Lambda \) with the property that given any \( L \in \text{mod} \Lambda \), there is a bounded complex

\[
C = 0 \to C_r \to C_{r-1} \to \cdots \to C_{s+1} \to C_s \to 0
\]

of \( \text{add}(M) \) with

1. \( L \) isomorphic to a direct summand of \( H_0(C) \)
2. \( H_d(C) = 0 \) for \( d \neq 0 \) and
3. \( r - s \leq i - 2 \).

Rouquier suggest studying the following two dimensions, left weak representation dimension and right weak representation dimension (see [20, Remark 3.3]).

Definition 2.19. ([20, Remark 3.3]) The right weak representation dimension of Artin algebra \( \Lambda \), denoted by \( \text{r.w.rep.dim} \Lambda \), is the smallest integer \( i \geq 2 \) such that there is an object \( M \in \text{mod} \Lambda \) with the property that given any \( L \in \text{mod} \Lambda \), there is a bounded complex

\[
C = 0 \to C_r \to C_{r-1} \to \cdots \to C_1 \to C_0 \to 0
\]

of \( \text{add}(M) \) with

1. \( L \) isomorphic to a direct summand of \( H_0(C) \) in degree zero
2. \( H_d(C) = 0 \) for \( d > 0 \) and
3. \( r \leq i - 2 \).

Definition 2.20. ([20, Remark 3.3]) The right weak representation dimension of Artin algebra \( \Lambda \), denoted by \( \text{r.w.rep.dim} \Lambda \), is the smallest integer \( i \geq 2 \) such that there is an object \( M \in \text{mod} \Lambda \) with the property that given any \( L \in \text{mod} \Lambda \), there is a bounded complex

\[
C = 0 \to C_0 \to C_{-1} \to \cdots \to C_s \to 0
\]

of \( \text{add}(M) \) with

1. \( L \) isomorphic to a direct summand of the homology \( H_0(C) \) in degree zero
2. \( H_d(C) = 0 \) for \( d < 0 \) and
3. \( -s \leq i - 2 \).

Given an Artin algebra \( \Lambda \), we set

\[
\text{l.w.rep.dim} \Lambda := \text{l.w.rep.dim mod} \Lambda, \text{r.w.rep.dim} \Lambda := \text{r.w.rep.dim mod} \Lambda,
\]

\[
\text{w.rep.dim} \Lambda := \text{w.rep.dim mod} \Lambda, \text{ext.dim} \Lambda := \text{ext.dim mod} \Lambda.
\]

Rouquier establish the following important theorem, which provides the first known examples of representation dimension more than 3.

Theorem 2.21. (by [20, Proposition 3.6 and Theorem 4.1]) Let \( n \geq 1 \) be an integer and \( \Lambda(k^n) \) exterior algebras. Then

\[
\text{rep.dim} \Lambda(k^n) = \text{w.rep.dim} \Lambda(k^n) = n + 1.
\]
We will need the following relations.

**Lemma 2.22.** ([25, Lemma 2.9][30]) For an Artin algebra $\Lambda$, we have $\text{w.resol.dim} \Lambda = \text{ext.dim} \Lambda$.

Iyama point out the following facts

**Lemma 2.23.** ([16, Page 31, Definition 4.5]) For an Artin algebra $\Lambda$, we have

\[ r \cdot \text{w.rep.dim} \Lambda = \text{w.resol.dim} \Lambda + 2. \]

**Theorem 2.24.** Let $\Lambda$ be an Artin algebra. Then

\[ l \cdot \text{w.rep.dim} \Lambda = r \cdot \text{w.rep.dim} \Lambda = \text{w.rep.dim} \Lambda = \text{ext.dim} \Lambda + 2 = \text{w.resol.dim} \Lambda + 2. \]

**Proof.** By Lemma 2.22 we have $\text{ext.dim} \Lambda + 2 = \text{w.resol.dim} \Lambda + 2$.

Let $\text{w.rep.dim} \Lambda = n$. By Definition 2.13, there is an object $M \in \text{mod} \Lambda$ with the property that given any $L \in \text{mod} \Lambda$, there is a bounded complex

\[ C = 0 \to C_r \xrightarrow{d_r} \cdots \to C_{s+1} \xrightarrow{d_{s+1}} C_s \to 0 \]

of $\text{add}(M)$ with

1. $L$ isomorphic to a direct summand of $H_0(C)$
2. $H_d(C) = 0$ for $d \neq 0$ and
3. $r - s \leq n - 2$.

Then we have the following three exact sequences

\[ 0 \to C_r \to \cdots \to C_1 \to \text{Im} d_1 \to 0 \quad (2.1) \]

\[ 0 \to \text{Im} d_1 \to \text{Ker} d_0 \to H_0(C) \to 0 \quad (2.2) \]

and

\[ 0 \to \text{Ker} d_0 \to C_0 \to C_{-1} \to \cdots \to C_s \to 0 \quad (2.3) \]

Then we have

\[
\begin{align*}
L & \in [H_0(C)]_1 \\
& \subseteq [\text{Ker} d_0]_1 \bullet [\Omega^{-1}(\text{Im} d_1)]_1 \quad \text{(by Lemma 2.10(2) and exact sequence (2.22))}
& \subseteq [\bigoplus_{i=s}^{s+1} \Omega^i(C_i)]_{-s+1} \bullet [\Omega^{-1}(\bigoplus_{i=0}^{s-r} \Omega^{-i}(C_{i+1}))]_r \quad \text{(by Lemma 2.10(2.11) and exact sequences (2.1)(2.3))}
& \subseteq [\bigoplus_{i=s+r+1}^{s+r+1} \Omega^i(M)]_{-s+r+1} \quad \text{(by Lemma 2.2)}
\end{align*}
\]

that is, $\text{mod} \Lambda = [\bigoplus_{i=s+r+1} \Omega^i(M)]_{-s+r+1}$. Then $\text{ext.dim} \Lambda \leq r - s = n - 2 = \text{w.rep.dim} \Lambda - 2$.

Now, let $\text{ext.dim} \Lambda = m < n - 2$. By Lemma 2.22 we know that $\text{w.resol.dim} \Lambda = m$. And by Definition 2.6 there exists a module $M$ such that for any $X$ we have the following exact sequence

\[ 0 \to M_m \xrightarrow{d_m} \cdots \to M_1 \xrightarrow{d_1} M_0 \to X \oplus Z \to 0 \]
for some $Z \in \text{mod } \Lambda$, and $M_i \in \text{add } M$ for all $i$. And then, we can get the following complex

$$ W : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow M_m \overset{d_m}{\longrightarrow} \cdots \longrightarrow M_1 \overset{d_1}{\longrightarrow} M_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots $$

where $H_i(W) = 0$ for all $i > 0$ and $H_0(W) = M_0 / \text{Im } d_1 \cong X \oplus Z$. By Definition 2.18 we have $\text{w.resol.dim } \Lambda \leq m < n$, contradiction! Then we get that $\text{ext.dim } \Lambda = \text{w.rep.dim } \Lambda - 2$

Similarly, we also can get $\text{l.w.rep.dim } \Lambda = \text{ext.dim } \Lambda + 2$ and $\text{r.w.rep.dim } \Lambda = \text{ext.dim } \Lambda + 2$.

**Remark 2.25.** By Theorem 2.24 and [30, Corollary 3.6], for non-semisimple Artin algebra $\Lambda$, we have

$$ \text{w.resol.dim } \Lambda = \text{ext.dim } \Lambda = \text{w.rep.dim } \Lambda - 2 \leq \text{O.w.resol.dim } \Lambda \leq \text{rep.dim } \Lambda - 2. $$

**Remark 2.26.** Let $n \geq 1$ be an integer and $\Lambda := \bigwedge (k^n)$ exterior algebras. By Theorem 2.24 and Remark 2.25, we have

$$ \text{w.resol.dim } \Lambda = \text{ext.dim } \Lambda = \text{w.rep.dim } \Lambda - 2 = \text{O.w.resol.dim } \Lambda = n - 1. $$

### 2.5 Igusa-Todorov distances

Now we introduce the notion of the Igusa-Todorov distance of an Artin algebra.

**Definition 2.27.** Let $\Lambda$ be an Artin algebra. We set the Igusa-Todorov distance of $\Lambda$ as follows

$$ \text{IT.dist } \Lambda := \inf \bigcup_{n=0}^{\infty} \{ m \mid \Lambda \text{ is an } (m,n)\text{-Igusa-Todorov algebra} \}. $$

Using mathematical induction and horseshoe lemma, we can get the following conclusion.

**Lemma 2.28.** Let $\Lambda$ be an Artin algebra. If we have the following exact sequence

$$ 0 \to M_n \to \cdots \to M_1 \to M_0 \to 0 $$

in $\text{mod } \Lambda$, where $n \geq 2$. Then for each $t \geq 0$, we can get the following exact sequences

$$ 0 \to \Omega^t(M_n) \to \Omega^t(M_{n-1}) \oplus P_{n-1} \to \cdots \to \Omega^t(M_1) \oplus P_1 \to \Omega^t(M_0) \to 0 $$

in $\text{mod } \Lambda$, where $P_i$ is projective for each $1 \leq i \leq n - 1$, and

$$ 0 \to \Omega^{-t}(M_n) \to \Omega^{-t}(M_{n-1}) \oplus E_{n-1} \to \cdots \to \Omega^{-t}(M_1) \oplus E_1 \to \Omega^{-t}(M_0) \to 0 $$

in $\text{mod } \Lambda$, where $E_i$ is injective for each $1 \leq i \leq n - 1$.

**Lemma 2.29.** Given the following exact sequence

$$ 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \oplus P \longrightarrow 0 \hspace{1cm} (2.4) $$

in $\text{mod } \Lambda$, where $P$ is projective. Then we can get the following exact sequence

$$ 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \oplus P \longrightarrow M_0 \longrightarrow M \longrightarrow 0. \hspace{1cm} (2.5) $$
Proof. By the above exact sequence (2.4), we can get the following exact sequences

\[ 0 \to M_n \to M_{n-1} \to \cdots \to M_2 \to K_2 \to 0, \quad (2.6) \]

\[ 0 \to K_2 \to M_1 \to K_1 \to 0 \quad (2.7) \]

and

\[ 0 \to K_1 \to M_0 \to M \oplus P \to 0. \quad (2.8) \]

By (2.8), we can get the following pullback,

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
K_1 & M_0' \\
\downarrow & \downarrow \\
K_1 & M_0 \\
\downarrow & \downarrow \\
P & P \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

and we get the following two exact sequences

\[ 0 \to K_1 \to M_0' \to M \to 0 \quad (2.9) \]

\[ 0 \to M_0' \to M_0 \to P \to 0. \quad (2.10) \]

By (2.10) and \( P \) projective, we have

\[ M_0 \cong M_0' \oplus P. \quad (2.11) \]

By (2.9), we have the following exact sequence

\[ 0 \to K_1 \oplus P \to M_0' \oplus P \to M \to 0. \quad (2.12) \]

By (2.12) and (2.11), we have following exact sequence

\[ 0 \to K_1 \oplus P \to M_0 \to M \to 0. \quad (2.13) \]

By (2.7), we have following exact sequence

\[ 0 \to K_2 \to M_1 \oplus P \to K_1 \oplus P \to 0. \quad (2.14) \]

By (2.6) (2.14) and (2.13), we can get the exact sequence (2.5).
Lemma 2.30. ([2, Proposition 3.6]) Let $\Lambda$ be a selfinjective Artin algebra. For integer $m \geq 0$, we have
\[ M \oplus Q_1 \cong \Omega^{-m}(\Omega^m(M)) \oplus Q_2 \]
for some projective modules $Q_1, Q_2$ in mod $\Lambda$.

Proposition 2.31. Let $\Lambda$ be an Artin algebra. Suppose that $\Lambda$ is selfinjective. Then
\[ \text{IT.dist } = \text{O.w.resol.dim } \Lambda. \]

Proof. Set $\text{O.w.resol.dim } = m$. Since $\text{O.w.resol.dim } = m$, we know that $\Lambda$ is $(m, 0)$-Igusa-Todorov algebra. By Definition 2.27, we have $\text{IT.dist } \leq m = \text{O.w.resol.dim } \Lambda$.

Suppose that $\text{IT.dist } = p$. By Definition 2.27, we can set $\Lambda$ be a $(p, n)$-Igusa-Todorov algebra. By Definition 2.15, there exists a module $V$ such that for each module $M \in \text{mod } \Lambda$, we have the following exact sequence
\[ 0 \rightarrow V_p \rightarrow V_{p-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 \rightarrow \Omega^n(M) \rightarrow 0 \]
where $V_i \in [V]_i$ for each $0 \leq i \leq p$. By Lemma 2.28, we have the following exact sequence
\[ 0 \rightarrow \Omega^{-m}(V_p) \rightarrow \Omega^{-m}(V_{p-1}) \oplus E_{p-1} \rightarrow \cdots \rightarrow \Omega^{-m}(V_0) \oplus E_0 \rightarrow \Omega^{-m}(\Omega^m(M)) \rightarrow 0 \quad (2.15) \]
where $\Omega^{-m}(V_p), \Omega^{-m}(V_i) \oplus E_i \in [\Omega^{-m}(V) \oplus \Lambda]_i$ for each $0 \leq i \leq p - 1$.

By Lemma 2.30, we have
\[ M \oplus Q_1 \cong \Omega^{-m}(\Omega^m(M)) \oplus Q_2. \quad (2.16) \]

By the above exact sequence (2.15) and isomorphism (2.16), we have the following exact sequence
\[ 0 \rightarrow \Omega^{-m}(V_p) \rightarrow \Omega^{-m}(V_{p-1}) \oplus E_{p-1} \rightarrow \cdots \rightarrow \Omega^{-m}(V_1) \oplus E_1 \rightarrow \Omega^{-m}(V_0) \oplus E_0 \oplus Q_2 \rightarrow M \oplus Q_1 \rightarrow 0 \quad (2.17) \]
where $\Omega^{-m}(V_p), \Omega^{-m}(V_i) \oplus E_i \in [\Omega^{-m}(V) \oplus \Lambda]_i$ for each $0 \leq i \leq p - 1$.

By Lemma 2.29 and the exact sequence (2.17), we get the following exact sequence
\[ 0 \rightarrow \Omega^{-m}(V_p) \rightarrow \cdots \rightarrow \Omega^{-m}(V_1) \oplus E_1 \oplus Q_1 \rightarrow \Omega^{-m}(V_0) \oplus E_0 \oplus Q_2 \rightarrow M \rightarrow 0. \quad (2.18) \]

By Definition 2.8 and the exact sequence (2.18), we know that $\text{O.w.resol.dim } \Lambda \leq p = \text{IT.dist } \Lambda$. Moreover, $\text{O.w.resol.dim } \Lambda = \text{IT.dist } \Lambda$.

By Remark 2.26 and Proposition 2.31, we have the following theorem, which tell us that the Igusa-Todorov distance may be very large.

Theorem 2.32. Let $k$ be a field and $n$ positive integer. Then $\text{IT.dist } \wedge(k^n) = n - 1$.

Corollary 2.33. ([8]) Let $k$ be a field and $n \geq 3$ positive integer. Then $\wedge(k^n)$ is not IT.
Let \( \Lambda \) be an Artin algebra. The stable category of \( \text{mod } \Lambda \), denoted by \( \text{mod } \Lambda \), is defined to be the additive quotient \( \text{mod } \Lambda / \text{add } \Lambda \), where the objects are the same as those in \( \text{mod } \Lambda \) and the morphism space \( \text{Hom}_{\text{mod } \Lambda}(X, Y) \) is the quotient space of \( \text{Hom}_{\text{mod } \Lambda}(X, Y) \) modulo all morphisms factorizing through projective modules. Two objects \( X \) and \( Y \) are isomorphic in \( \text{mod } \Lambda \) if and only if there are projective modules \( P \) and \( Q \) such that \( X \oplus Q \cong Y \oplus P \) in \( \text{mod } \Lambda \).

By [6, Theorem 3.2 and Theorem 3.3] and [20, Proposition 3.7], we have the following theorem. The complexity of \( \Lambda / \text{rad } \Lambda \) and the \( \text{Fg} \) conditions can be seen in [6].

**Theorem 2.34.** If \( \Lambda \) is a non-semisimple selfinjective algebra and \( \text{Fg} \) holds, then

\[
\text{cx}(\Lambda / \text{rad } \Lambda) + 1 \leq \text{tri.dim } \text{mod } \Lambda + 2 \leq \text{w.rep.dim } \Lambda \leq \text{rep.dim } \Lambda \leq \text{LL}(\Lambda).
\]

**Example 2.35.** Let \( k \) be a field, and let \( n \geq 1 \) be an integer, and let \( \Lambda \) be the quantum exterior algebra

\[
\Lambda = k \langle X_1, \cdots, X_n \rangle / (X_i^2, X_iX_j - q_{ij}X_jX_i)_{i<j},
\]

where \( 0 \neq q_{ij} \in k \) and all the \( q_{ij} \) are the roots of unity.

By [6, Page 398, Examples (i)], we know that \( \Lambda \) is selfinjective, and \( \text{LL}(\Lambda) = n + 1 \), and \( \text{rep.dim } \Lambda = n + 1 \), and \( \text{cx}(\Lambda / \text{rad } \Lambda) = n \). And by Theorem 2.34, we have \( \text{w.rep.dim } \Lambda = n + 1 \). And by Remark 2.25, we have \( \text{ext.dim } \Lambda = O \). \( \text{w.resol.dim } \Lambda = n - 1 \). And by Proposition 2.31, we have \( \text{IT.dist } \Lambda = n - 1 \). If \( n > 2 \), we have \( \text{IT.dist } \Lambda = n - 1 > 1 \), and hence \( \Lambda \) is not Igusa-Todorov in this case.

### 3 Igusa-Todorov distance is an invariant under derived equivalence

We will review some of the basic facts and conclusions, as detailed in reference [12].

Given two Artin algebras \( A \) and \( B \). Let \( F : D^b(\text{mod } A) \to D^b(\text{mod } B) \) be derived equivalence. We can define a functor \( \overline{F} : \text{mod } A \to \text{mod } B \), which is called the stable functor of \( F \).

**Lemma 3.1.** ([11, Proposition 4.1],[12, Example 4.7(b)]) Given Artin algebra \( A \). Let \( n \) be a nonnegative integer. Then \( n \)th syzygy functor \( \Omega^n_A : \text{mod } A \to \text{mod } A \) is a stable functor of the derived equivalence \( [-n] : D^b(\text{mod } A) \to D^b(\text{mod } A) \).

**Lemma 3.2.** ([12, Theorem 4.11]) Given two Artin algebras \( A \) and \( B \). Let \( F : D^b(\text{mod } A) \to D^b(\text{mod } B) \) and \( G : D^b(\text{mod } B) \to D^b(\text{mod } C) \) be two triangle functors. Then the functors \( \overline{G} \circ F \) and \( \overline{G} \circ F \) are isomorphic.

**Lemma 3.3.** ([12, Corollary 4.12]) Given two Artin algebras \( A \) and \( B \). Let \( F : D^b(\text{mod } A) \to D^b(\text{mod } B) \) be a triangle functor. Then \( \overline{F} \circ \Omega_A \simeq \Omega_B \circ \overline{F} \).

**Lemma 3.4.** ([12, Proposition 4.13]) Given two Artin algebras \( A \) and \( B \). Let \( F : D^b(\text{mod } A) \to D^b(\text{mod } B) \) be a triangle functor. Suppose that

\[
0 \to X \to Y \to Z \to 0
\]
is an exact sequence in mod $A$. Then there is an exact sequence

$$0 \rightarrow \overline{F}(X) \rightarrow \overline{F}(Y) \oplus P \rightarrow \overline{F}(Z) \rightarrow 0$$

in mod $B$ for some projective module $P$.

By Lemma 3.4, we can get

**Corollary 3.5.** Given two Artin algebras $A$ and $B$. Let $F: D^b(\text{mod } A) \rightarrow D^b(\text{mod } B)$ be a triangle functor. Suppose that

$$0 \rightarrow X_k \rightarrow X_{k-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$$

is an exact sequence in mod $A$. Then there is an exact sequence

$$0 \rightarrow \overline{F}(X_k) \rightarrow \overline{F}(X_{k-1}) \oplus P_{k-1} \rightarrow \cdots \rightarrow \overline{F}(X_1) \oplus P_1 \rightarrow \overline{F}(X_0) \rightarrow 0$$

in mod $B$ for some projective modules $P_i$, where $0 \leq i \leq k - 1$.

**Theorem 3.6.** If Artin algebra $A$ and $B$ are derived equivalent, then $\text{IT.dist } A = \text{IT.dist } B$.

**Proof.** Suppose that $\text{IT.dist } B = m$ and $B$ is $(m, n)$-Igusa Todrov.

Let $F: D^b(\text{mod } A) \rightarrow D^b(\text{mod } B)$ be a derived equivalence, and $G$ be a quasi-inverse of $F$. Without loss of generality, we can assume that $F$ is nonnegative and the tilting complex associated to $F$ has terms only in degrees $0, -1, \cdots, -p$. Then $G[-p]$ is also nonnegative. Let $X \in \text{mod } A$. Then $\overline{F}(X) \in \text{mod } B$. By assumption $\text{IT.dist } B = m$ and Definition 2.15, we know that there exists an $(m, n)$-Igusa Todrov module $V$, and the following exact sequence in mod $B$

$$0 \rightarrow V_m \rightarrow V_{m-1} \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 \rightarrow \Omega_B^n(\overline{F}(X)) \rightarrow 0.$$  \hspace{1cm} (3.1)

Applying the stable functor of $G[-p]$ to the above exact sequence (3.1) and by Corollary 3.5 we can get the following exact sequence

$$0 \rightarrow \overline{G}[-p](V_m) \rightarrow \overline{G}[-p](V_{m-1}) \oplus P_{m-1} \rightarrow \cdots \rightarrow \overline{G}[-p](V_0) \oplus P_0 \rightarrow \overline{G}[-p](\Omega_B^n(\overline{F}(X))) \rightarrow 0.$$  \hspace{1cm} (3.2)

in mod $A$. On the other hand, we have the following isomorphisms in mod $A$

$$\overline{G}[-p](\Omega_B^n(\overline{F}(X))) \cong (\overline{G}[-p] \circ [-n] \circ \overline{F})(X) \quad \text{(by Lemma 3.1)}$$

$$\cong (\overline{G}[-p] \circ [-n] \circ \overline{F})(X) \quad \text{(by Lemma 3.2)}$$

$$\cong [-p-n](X) \quad \text{(by Lemma 3.3)}$$

$$\cong \Omega_A^{p+n}(X) \quad \text{(by Lemma 3.1)}.$$  \hspace{1cm} (3.3)

Then

$$\overline{G}[-p](\Omega_B^n(\overline{F}(X))) \oplus P \cong \Omega_A^{p+n}(X) \oplus Q.$$  \hspace{1cm} (3.3)

By the above exact sequence (3.2) and isomorphism (3.3), we have the following exact sequence
0 \to \overline{G[-p]}(V_m) \to \overline{G[-p]}(V_{m-1}) \oplus P_{m-1} \to \cdots \to \overline{G[-p]}(V_0) \oplus P_0 \oplus P \to \Omega^{n+p}_A(X) \oplus Q \to 0.

(3.4)

By Lemma 2.29 and the exact sequence (3.4), we get the following exact sequence

\[ 0 \to \overline{G[-p]}(V_m) \to \overline{G[-p]}(V_{m-1}) \oplus P_{m-1} \to \cdots \to \overline{G[-p]}(V_1) \oplus P_1 \oplus Q \to \overline{G[-p]}(V_0) \oplus P_0 \oplus P \to \Omega^{n+p}_A(X) \to 0. \]

where \( \overline{G[-p]}(V_m), \overline{G[-p]}(V_i) \oplus P_i, \overline{G[-p]}(V_0) \oplus P_0 \oplus P \in \text{add}(\overline{G[-p]}(V) \oplus A) \) for \( 1 \leq i \leq m - 1 \). By Definition 2.27, we have \( \text{IT.dist} B \leq m = \text{IT.dist} A \). Similarly, we also have \( \text{IT.dist} B \geq \text{IT.dist} A \). That is, \( \text{IT.dist} A = \text{IT.dist} B \).

\[ \square \]

Corollary 3.7. (12, 23, Theorem 4.5)) If Artin algebra \( \Lambda \) and \( \Gamma \) are derived equivalent, then \( \Lambda \) is syzygy-finite if and only if \( \Gamma \) is syzygy-finite. In other words, \( \text{IT.dist} \Lambda = 0 \) if and only if \( \text{IT.dist} \Gamma = 0 \).

Proof. By Remark 2.16 and Theorem 3.6 \( \square \)

Corollary 3.8. (23, Theorem 5.4) If Artin algebra \( \Lambda \) and \( \Gamma \) are derived equivalent, and \( \Lambda \) is an Igusa-Todorov algebra, then \( \Gamma \) is also an Igusa-Todorov algebra.

Proof. By Theorem 3.6 and Definition 2.14 \( \square \)

Recall that an \( \Lambda \)-module \( T \) is said to be a tilting module if \( T \) satisfied the following three conditions:

\begin{enumerate}
\item pd\((T) \leq n,
\item Ext^i_{\Lambda}(T, T) = 0 \text{ for all } i > 0, \text{ and}
\item there exists an exact sequence \( 0 \to \Lambda \to T_0 \to \cdots \to T_n \to 0 \) in \( \mod \Lambda \) with each \( T_i \) in \( \text{add}(T) \).
\end{enumerate}

By Theorem 3.6, we have

Corollary 3.9. Given an Artin algebra \( \Lambda \). Let \( T \) be a tilting module in \( \mod \Lambda \) and \( \Gamma = \text{End}_{\Lambda}(T) \). Then \( \text{IT.dist} \Lambda = \text{IT.dist} \Gamma \).

4 The singularity category and the Igusa-Todorov distance

Recall that the quotient triangulated category

\[ D_{sg}(\mod \Lambda) := D^b(\mod \Lambda)/K^b(\text{proj} \Lambda) \]

is the singularity category of \( \Lambda \), where \( K^b(\text{proj} \Lambda) \) is the bounded homotopy category. We denote by \( q : D^b(\mod \Lambda) \to D_{sg}(\mod \Lambda) \) the quotient.

Lemma 4.1. (9, Lemma 2.4(2)(a), 27, Lemma 2.1) Let \( X \in D_{sg}(\mod \Lambda) \). Then there exists a module \( M \in \mod \Lambda \) and \( r \in \mathbb{Z} \) such that \( X \cong q(S^0(M))[r] \) in \( D_{sg}(\mod \Lambda) \).
Lemma 4.2. ([7] Lemma 2.2) For each module $M \in \text{mod } \Lambda$, we have
\[ q(S^0(M)) \cong q(S^0(\Omega^n(M)))[n] \]
in $D_{sg}(\text{mod } \Lambda)$ for each nonnegative integer $n$.

Now we can establish the main result in this section.

Theorem 4.3. Given an Artin algebra $\Lambda$. We have $\text{tri.dim} \ D_{sg}(\text{mod } \Lambda) \leq \text{IT.dist } \Lambda$.

Proof. Let $X \in D_{sg}(\text{mod } \Lambda)$. By Lemma 4.1, there exists a module $M \in \text{mod } \Lambda$ and $r \in \mathbb{Z}$ such that $X \cong q(S^0(M))[r]$ in $D_{sg}(\text{mod } \Lambda)$. And by Lemma 4.2, we can get $X \cong q(S^0(\Omega^n(M)))[n+r]$, that is, $X[-r] \cong q(S^0(\Omega^n(M)))[n]$ in $D_{sg}(\text{mod } \Lambda)$.

We can set $\Lambda$ is an $(m,n)$-IT algebra. That is, there is a module $V$ such that for any module $M$, we have the following exact sequence
\[ 0 \to V_m \to V_{m-1} \to \cdots \to V_1 \to V_0 \to \Omega^n(M) \to 0, \tag{4.1} \]
where $V_i \in \text{add } V$ for $0 \leq i \leq n$.

By the above exact sequence (4.1), we can get the following short exact sequences
\[
\begin{aligned}
0 & \to K_1 \to V_0 \to S^0(\Omega^n(M)) \to 0 \\
0 & \to K_2 \to V_1 \to K_1 \to 0 \\
0 & \to K_2 \to V_1 \to K_1 \to 0 \\
& \vdots \\
0 & \to K_{m-1} \to V_{m-2} \to K_{m-2} \to 0 \\
0 & \to V_m \to V_{m-1} \to K_{m-1} \to 0.
\end{aligned}
\]

Then we have the following triangles in $D^b(\text{mod } \Lambda)$
\[
\begin{aligned}
S^0(K_1) \to S^0(V_0) \to S^0(\Omega^n(M)) \to S^0(K_1)[1] \\
S^0(K_2) \to S^0(V_1) \to S^0(K_1) \to S^0(K_2)[1] \\
& \vdots \\
S^0(K_{m-1}) \to S^0(V_{m-2}) \to S^0(K_{m-2}) \to S^0(K_{m-1})[1] \\
S^0(V_m) \to S^0(V_{m-1}) \to S^0(K_{m-1}) \to S^0(V_m)[1].
\end{aligned}
\]

Then we have the following triangles in $D_{sg}(\text{mod } \Lambda)$
\[
\begin{aligned}
q(S^0(K_1)) \to q(S^0(V_0)) \to q(S^0(\Omega^n(M))) \to q(S^0(K_1))[1] \\
q(S^0(K_2)) \to q(S^0(V_1)) \to q(S^0(K_1)) \to q(S^0(K_2))[1] \\
& \vdots \\
q(S^0(K_{m-1})) \to q(S^0(V_{m-2})) \to q(S^0(K_{m-2})) \to q(S^0(K_{m-1})[1] \\
q(S^0(V_m)) \to q(S^0(V_{m-1})) \to q(S^0(K_{m-1})) \to q(S^0(V_m))[1].
\end{aligned}
\]
Moreover, we can get the following triangles in $D_{sg}(\mod \Lambda)$,
\[
\begin{align*}
q(S^0(V_0))[n] & \longrightarrow q(S^0(\Omega^n(M)))[n] \longrightarrow q(S^0(K_1))[n + 1] \longrightarrow q(S^0(V_0))[n + 1] \\
q(S^0(V_1))[n + 1] & \longrightarrow q(S^0(K_1))[n + 1] \longrightarrow q(S^0(K_2))[n + 2] \longrightarrow q(S^0(V_1))[n + 2] \\
& \vdots \\
q(S^0(V_{m-2})[n + m - 2] & \longrightarrow q(S^0(K_{m-2})[n + m - 1] \longrightarrow q(S^0(K_{m-1}))[1] \longrightarrow q(S^0(V_{m-2}))[n + m - 1] \\
q(S^0(V_{m-1})[n + m - 1] & \longrightarrow q(S^0(K_{m-1}))[n + m - 1] \longrightarrow q(S^0(V_m))[n + m] \longrightarrow q(S^0(V_{m-1}))[n + m].
\end{align*}
\]

So we have
\[
X[-r] \cong q(S^0(\Omega^n(M)))[n] \\
\in \langle q(S^0((V_0))[n])_1 \circ q(S^0(K_1))[n + 1] \rangle_1 \\
\subseteq \langle q(S^0((V_0))[n])_1 \circ q(S^0(V_1))[n + 1] \circ q(S^0(K_2))[n + 1] \rangle_1 \\
\vdots \\
\subseteq \langle q(S^0(V_0))[n]_1 \circ q(S^0(V_1))[n + 1]_1 \circ \cdots \circ q(S^0(V_{m-1}))[n + m - 1]_1 \circ q(S^0(V_m))[n + m] \rangle_1 \\
\subseteq \langle q(S^0(V))_1 \circ q(S^0(V))_1 \circ \cdots \circ q(S^0(V))_1 \circ q(S^0(V))_1 \rangle_{m + 1}
\]

Then $D_{sg}(\mod \Lambda) = \langle q(S^0(V)) \rangle_{m + 1}$. Moreover, we have tri.dim $D_{sg}(\mod \Lambda) \leq m = IT.dist \Lambda$.

For a module $M \in \mod \Lambda$, we use rad $M$ to denote the radical of $M$. Let $\mathcal{V}$ be a subset of all simple modules, and $\mathcal{V}'$ the set of all the others simple modules in $\mod \Lambda$. We write
\[
\mathfrak{F}(\mathcal{V}) := \{M \in \mod \Lambda \mid \text{there exists a chain } 0 \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M \}
\]
of submodules of $M$ such that each quotients $M_i/M_{i-1} \in \mathcal{V}$.

Note that $\mathfrak{F}(\mathcal{V})$ is closed under extensions, submodules and quotients modules. Then we have a torsion pair $(T, \mathfrak{F}(\mathcal{V}))$, and the corresponding torsion radical is denoted by $t_{\mathcal{V}}$. For a subclass $\mathcal{B}$ of $\mod \Lambda$, the projective dimension $pd \mathcal{B}$ of $\mathcal{B}$ is defined as
\[
pd \mathcal{B} = \begin{cases} 
\sup\{pd M \mid M \in \mathcal{B}\}, & \text{if } \mathcal{B} \neq \emptyset; \\
-1, & \text{if } \mathcal{B} = \emptyset.
\end{cases}
\]

**Definition 4.4.** (\cite{10}) The $t_{\mathcal{V}}$-radical layer length is a function $\ell t_{\mathcal{V}} : \mod \Lambda \longrightarrow \mathbb{N} \cup \{\infty\}$ via
\[
\ell t_{\mathcal{V}}(M) = \inf\{i \geq 0 \mid t_{\mathcal{V}} \circ F^i_{t_{\mathcal{V}}}(M) = 0, M \in \mod \Lambda\}
\]
where $F^i_{t_{\mathcal{V}}} = \text{rad} \circ t_{\mathcal{V}}$.

**Theorem 4.5.** Let $\Lambda$ be an Artin algebra. $\mathcal{V}$ is the set of some simple modules with finite projective dimension. Then $\Lambda$ is a $(\max\{\ell t_{\mathcal{V}}(\Lambda) - 2, 0\}, pd \mathcal{V} + 2)$-Igusa-Todorov algebra.
Proof. If $\ell^t V(\Lambda) \leq 2$, then $\Lambda$ is $(\text{pd} V + 2)$-Igusa-Todorov algebra (see [28]). That is, $\Lambda$ is a $(0, \text{pd} V + 2)$-Igusa-Todorov algebra (see Remark 2.16(2)).

If $\ell^t V(\Lambda) \geq 2$, then $\Lambda$ is $(\ell^t V(\Lambda) - 2, \text{pd} V + 2)$-Igusa-Todorov algebra (see [26, Theorem 4.7]).

**Proposition 4.6.** Let $\Lambda$ be an Artin algebra. $\mathcal{V}$ is the set of some simple modules with finite projective dimension. Then $\text{IT.dist} \Lambda \leq \max\{\ell^t V(\Lambda) - 2, 0\}$.

**Proof.** By Theorem 4.5 and Definition 2.27.

**Corollary 4.7.** Let $\Lambda$ be an Artin algebra. $\mathcal{V}$ is the set of some simple modules with finite projective dimension. Then $\text{IT.dist} \Lambda \leq \max\{\ell^t V(\Lambda) - 2, 0\}$.

**Corollary 4.8.** ([27, Theorem 3.14]) Let $\Lambda$ be an Artin algebra. $\mathcal{V}$ is the set of some simple modules with finite projective dimension. Then $\text{tri.dim} D_{sg}(\text{mod} \Lambda) \leq \max\{\ell^t V(\Lambda) - 2, 0\}$.

**Proof.** By Proposition 4.6 and Theorem 4.3.

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