Representation of the inverse of a multiplier as a multiplier

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February 2, 2013

Abstract

Certain mathematical objects appear in a lot of scientific disciplines, like physics, signal processing and certainly mathematics. In a general setting they can be described as frame multipliers, consisting of analysis, multiplication by a fixed sequence (called the symbol), and synthesis. They are not only interesting mathematical objects, but also important for applications, for example for the realization of time-varying filters. In this paper we show a surprising result about the inverse of such operators, if existing, as well as new results about a core concept of frame theory, dual frames. We show that for semi-normalized symbols, the inverse of any invertible frame multiplier can be represented as a frame multiplier with dual frames and reciprocal symbol. Furthermore, one of those dual frames is uniquely defined and the other one is an arbitrary dual frame. We investigate sufficient conditions for the special case, when those dual frames can be chosen to be the canonical duals. In connection to the above, we show that the set of dual frames determines a frame uniquely. Finally, we investigate invertible Gabor multipliers; we show that the inverse of every invertible lattice-invariant operator (in particular, every invertible Gabor frame multiplier with a constant symbol (1)) can be represented as a Gabor frame multiplier with a constant symbol (1).

Keywords: multiplier, invertibility, frame, Riesz basis, Bessel sequence
MSC 2000: 42C15, 47A05

1 Introduction, Notation, and Motivation

In many scientific disciplines, certain objects play an important role. Those systems are described by an analysis procedure followed by a multiplication, followed by a synthesis. Those operators are of utmost importance in

- Mathematics, where they are used for the diagonalization of operators [17].
Physics, where they are a link between classical and quantum mechanics, so called quantization operators [1].

- Signal processing, where they are a particular way to implement time-variant filters [15].
- Acoustics, where those time-frequency filters are used in several fields, for example in Computational Auditory Scene Analysis [22].

In this paper we show a surprising result about the shape of the inverse of such operators, if existing. This also lead us to new results concerning dual frames, a concept at the core of frame theory.

To be able to describe those operators in a general setting, as an extension of Gabor multipliers [12], multipliers for general Bessel sequences were introduced by one of the authors [3]. Further, multipliers for general sequences were investigated in [13 19 20 21]. These are operators defined by

\[
M_{(m_n),(\phi_n),(\psi_n)} h = \sum_{n=1}^{\infty} m_n(h, \psi_n)\phi_n,
\]

for given sequences \((\phi_n)\) and \((\psi_n)\) with elements from a Hilbert space \(\mathcal{H}\), and a given complex scalar sequence \((m_n)\) called the symbol. Such operators are also investigated for continuous transforms - in a general [4] (continuous frame multipliers), wavelet [16] (Calderon-Toeplitz operators) and short-time Fourier setting [9] (localization operators). Here we stick to the discrete version.

Multipliers are interesting not only from a theoretical point of view, but also for applications. They are applied for example in psychoacoustical modelling [5] and denoising [14]. Multipliers are a particular way to implement time-variant filters [15]. Therefore, for some applications it is important to find the inverse of a multiplier if it exists. The paper [18] is devoted to invertibility of multipliers, necessary conditions for invertibility, sufficient conditions, and representation for the inverse via Neumann series.

In the present paper our attention is on how to express the inverses of invertible multipliers as multipliers.

**Motivation**

By the spectral theorem it is known that any compact operator on an infinite dimensional Hilbert space can be represented as a multiplier with orthonormal sequences. If such an operator is invertible, the inverse can just be found by inverting the symbol and switching the role of the sequences. To find this representation for a given operator by the singular value decomposition, can be numerically inefficient. Now consider invertible operators which are known to be representable as multipliers not necessarily with orthonormal sequences. Does this help in finding the inverse in a more efficient way? In [3] it is proved that, if \(m\) is semi-normalized, then a Riesz multiplier \(M_{m,\Phi,\Psi}\) is automatically invertible and

\[
M_{m,\Phi,\Psi}^{-1} = M_{1/m,\tilde{\Phi},\tilde{\Psi}},
\]
where \( \tilde{\Phi} \) and \( \tilde{\Psi} \) denote the canonical duals of \( \Phi \) and \( \Psi \). This therefore generalizes the representations of the inverses of compact operators using orthonormal sequences.

The result on Riesz multipliers has opened the following questions:

\[ Q1 \] Are there other invertible multipliers whose inverses can be written as multipliers? Can we represent them by inverting the symbols?

\[ Q2 \] Are there other invertible frame multipliers \( M_{m,\Phi,\Psi} \) whose inverses can be written as \( M_{1/m,\tilde{\Phi},\tilde{\Psi}} \) using the canonical duals?

The paper is devoted to these two questions. Section \( \S \) gives an affirmative answer of Question \( Q1 \). Among the invertible multipliers, we determine several classes of multipliers whose inverses can be written as multipliers. We show that the inverse of every invertible multiplier with semi-normalized symbol can be represented as a multiplier with the reciprocal symbol and dual frames. One of the dual frames is uniquely defined, while the other one can be arbitrarily chosen. In connection to this result, we prove that if the dual frames of a frame \( (\phi_n) \) are also dual to another frame \( (\psi_n) \), then \( (\phi_n) \) and \( (\psi_n) \) coincide (Section 2).

In Section \( \S \) we give an affirmative answer of Question \( Q2 \). We determine frame multipliers \( M_{m,\Phi,\Psi} \) (not necessarily being Riesz multipliers) which are invertible and their inverses can be written as \( M_{1/m,\tilde{\Phi},\tilde{\Psi}} \). We note that not all the invertible frame multipliers have such a representation for the inverse.

Section \( \S \) is devoted to Gabor multipliers. We determine equivalent conditions for an invertible operator on \( L^2(\mathbb{R}^d) \) (and its inverse) to be represented as a Gabor frame multiplier with a constant symbol.

**Notation and definitions**

Throughout the paper, \( \mathcal{H} \) denotes a Hilbert space, \( \Phi = (\phi_n)_{n=1}^\infty \) and \( \Psi = (\psi_n)_{n=1}^\infty \) are sequences with elements from \( \mathcal{H} \). The sequence \( (e_n)_{n=1}^\infty \) denotes an orthonormal basis of \( \mathcal{H} \) and \( (\delta_n)_{n=1}^\infty \) denotes the canonical basis of \( \ell^2 \). When the index set is omitted, \( \mathbb{N} \) should be understood as the index set. The letter \( m \) is used to denote a complex valued scalar sequence \( (m_n) \). Furthermore, \( m = (\overline{m_n}) \) and \( 1/m = (1/m_n) \). The sequence \( m \) is called semi-normalized if \( 0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty \). For \( m \in \ell^\infty \), we will use the operator \( \mathcal{M}_m : \ell^2 \to \ell^2 \) given by \( \mathcal{M}_m(e_n) = (m_nc_n) \), which is bounded with \( ||\mathcal{M}_m|| = ||m||_{\ell^\infty} \). A linear mapping is called an operator. An operator \( M : \mathcal{H} \to \mathcal{H} \) is called invertible if it is a bounded bijection from \( \mathcal{H} \) onto \( \mathcal{H} \). The identity operator on \( \mathcal{H} \) is denoted by \( \text{Id}_{\mathcal{H}} \).

Recall that \( \Phi \) is called a frame for \( \mathcal{H} \) with bounds \( A_\Phi, B_\Phi \) if \( 0 < A_\Phi \leq B_\Phi < \infty \) and \( A_\Phi ||h||^2 \leq \sum |(h,\phi_n)|^2 \leq B_\Phi ||h||^2 \) for every \( h \) in \( \mathcal{H} \). For definitions of Bessel sequence, Riesz basis, analysis, synthesis and frame operator, the canonical and other duals of a frame, we refer to \[ 5 \]. For a given frame \( \Phi \) for \( \mathcal{H} \), the
analysis operator is denoted by $U_\Phi$, the synthesis operator by $T_\Phi$, the frame operator by $S_\Phi$, and the canonical dual by $\tilde{\Phi} = (\tilde{\phi}_n)$.

For given $m$, $\Phi$, and $\Psi$, the operator $M_{m, \Phi, \Psi}$ given by Equation 1.1 is called a multiplier. The operator $M_{m, \Phi, \Psi}$ is called unconditionally convergent if the series in (1.1) converges unconditionally for every $h \in \mathcal{H}$. When $\Phi$ and $\Psi$ are Bessel sequences, frames, Riesz bases for $\mathcal{H}$, then $M_{m, \Phi, \Psi}$ will be called a Bessel multiplier, frame multiplier, Riesz multiplier, respectively. When $m \in \ell^\infty$, then a Bessel multiplier is a well defined operator from $\mathcal{H}$ into $\mathcal{H}$.\[2\]

The set of dual frames

In order to prove one of the statements in Section 3, we need a result which is of independent interest for frame theory, showing new properties of the set of dual frames.

**Theorem 2.1.** Let $\Phi$ be a frame for $\mathcal{H}$. Then the following statements hold.

(i) The closure of the union of all sets $\mathcal{R}(U_{\Phi^d})$, where $\Phi^d$ runs through all dual frames of $\Phi$, is $\ell^2$.

(ii) Let $\Psi$ be a frame for $\mathcal{H}$. If every dual frame $\Phi^d$ of $\Phi$ is a dual frame of $\Psi$, then $\Psi = \Phi$.

**Proof:** (i) Let the sequence $c = (c_n) \in \ell^2$ fulfill $c \perp \mathcal{R}(U_{\Phi^d})$ for every dual frame $\Phi^d$ of $\Phi$. Then

$$\langle T_{\Phi^d} f, h \rangle_{\mathcal{H}} = 0, \forall f \in \mathcal{H}, \forall \text{ dual frame } \Phi^d \text{ of } \Phi,$$

which implies that

$$T_{\Phi^d} c = 0, \forall \text{ dual frame } \Phi^d \text{ of } \Phi. \quad (2.1)$$

The dual frames of $\Phi$ are precisely the sequences

$$\tilde{\phi}_n + h_n - \sum_{j=1}^{\infty} \langle \tilde{\phi}_n, \phi_j \rangle h_j |_{n=1}^{\infty},$$
where \((h_n)_{n=1}^{\infty}\) is a Bessel sequence in \(\mathcal{H}\) (see, e.g., \([8\), Theorem 5.6.5\]). Therefore,

\[
\sum_{n=1}^{\infty} c_n \left( \tilde{\phi}_n + h_n - \sum_{j=1}^{\infty} \langle \tilde{\phi}_n, \phi_j \rangle h_j \right) = 0
\]

for every Bessel sequence \(\{h_n\}_{n=1}^{\infty}\) in \(\mathcal{H}\). By \((2.1)\) we have \(T_{\Phi}c = 0\), which implies that

\[
\sum_{n=1}^{\infty} c_n \left( h_n - \sum_{j=1}^{\infty} \langle \tilde{\phi}_n, \phi_j \rangle h_j \right) = 0
\]

\((2.2)\) for every Bessel sequence \(\{h_n\}_{n=1}^{\infty}\) in \(\mathcal{H}\). Using \((2.2)\) with the Bessel sequence \((h_n)_{n=1}^{\infty} = (e_1, 0, 0, 0, \ldots)\), we obtain

\[
c_1 \left( e_1 - \langle \tilde{\phi}_1, \phi_1 \rangle e_1 \right) + \sum_{n=2}^{\infty} c_n \left( -\langle \tilde{\phi}_n, \phi_1 \rangle e_1 \right) = 0,
\]

which implies that

\[
c_1 = \left( \sum_{n=1}^{\infty} c_n \tilde{\phi}_n, \phi_1 \right) = 0.
\]

In a similar way, using \((2.2)\) with the Bessel sequence \((h_n)_{n=1}^{\infty} = (0, \ldots, 0, e_j, 0, 0, 0, \ldots)\), where \(e_j\) stands at the \(j\)-th position, we obtain \(c_j = 0\) for every \(j \geq 2\). Therefore, \(c = (0)\), which completes the proof.

(ii) Assume that all dual frames \(\Phi^d\) of \(\Phi\) are dual frames of \(\Psi\). Then \(T_{\Psi}U_{\Phi^d} = Id_{\mathcal{H}} = T_{\Psi}U_{\Phi^d}\), which by (i) implies that \(T_{\Phi} = T_{\Psi}\) and hence, \(\Phi = \Psi\).

By the above result, different frames have different sets of dual frames; if two frames \(\Phi\) and \(\Psi\) for \(\mathcal{H}\) have the same sets of dual frames, then \(\Phi = \Psi\).

In particular, two different frames cannot have sets of dual frames which are included into one another.

3 Inversion of Multipliers by Inverted Symbol [Q1] and Dual Frames

Here we give an affirmative answer of Question \([Q1]\). In the next propositions we determine three classes of invertible multipliers whose inverses can be written as multipliers:

- the invertible frame multipliers with semi-normalized weights (see Ex. \([4.1]\));

- the invertible multipliers \(M_{m, \Phi, \Psi}\) which are unconditionally convergent (see Ex. \([4.1]\));

\(^1\) This is particularly interesting, because arbitrary frames \(\Psi\) and \(\Phi\) with arbitrary duals \(\Psi^d\) and \(\Phi^d\), respectively, do not necessarily satisfy \(T_{\Psi}U_{\Phi^d}U_{\Psi^d} = \text{Identity}\).
- the invertible multipliers $M_{m, \Phi, \Psi}$ which are unconditionally convergent and $\Phi$ is minimal (for an example of such a multiplier consider $M_{(1), (e_1, e_2, e_3, \ldots)}$ whose inverse can be written as $M_{(1), (e_1, e_2, e_3, \ldots)}$);

- the invertible multipliers $M_{m, \Phi, \Psi}$ which are unconditionally convergent and $\inf_n |m_n| \Vert \phi_n \Vert \Vert \psi_n \Vert > 0$ (for an example of such a multiplier consider $M_{(1), \Phi, \Psi}$, where $\Phi = (e_1, e_2, e_3, \ldots)$).

**Theorem 3.1.** Let $\Phi$ and $\Psi$ be frames for $\mathcal{H}$, and let $m$ be semi-normalized. Assume that $M_{m, \Phi, \Psi}$ is invertible. Then there exist a dual frame $\Phi^\dagger$ of $\Phi$ and a dual frame $\Psi^\dagger$ of $\Psi$, so that for any dual frame $\Phi^d$ of $\Phi$ and any dual frame $\Psi^d$ of $\Psi$ we have

$$M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Psi^\dagger, \Phi^d} = M_{1/m, \Psi^d, \Phi^\dagger}. \quad (3.1)$$

If $F = (f_n)$ is a Bessel sequence in $\mathcal{H}$ which fulfills $M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Psi^\dagger, F}$ (resp. $M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Phi^\dagger, \psi}$), then $F$ must be a dual frame of $\Phi$ (resp. $\Psi$).

Furthermore, $\Psi^\dagger$ is the only Bessel sequence in $\mathcal{H}$ which satisfies $M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Psi^\dagger, \Phi^d}$ for all dual frames $\Phi^d$ of $\Phi$ and $\Psi^\dagger$ is the only Bessel sequence in $\mathcal{H}$ which satisfies $M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Psi^d, \Phi^\dagger}$ for all dual frames $\Psi^d$ of $\Psi$.

**Proof:** Denote $M := M_{m, \Phi, \Psi}$. First observe that the sequence $(M^{-1}(m_n \phi_n))$ is a dual frame of $\Psi$. Denote it by $\Psi^\dagger$. Therefore, $M^{-1}T_\Phi \delta_n = T_{\Psi^\dagger} M_{1/m} \delta_n$, $n \in \mathbb{N}$. Now the boundedness of the operators imply that $M^{-1}T_\Phi = T_{\Psi^\dagger} M_{1/m}$ on $\ell^2$. Using any dual frame $\Phi^d$ of $\Phi$ we get $M^{-1} = T_{\Psi^\dagger} M_{1/m} U_{\Phi^d} = M_{1/m, \Psi^\dagger, \Phi^d}$ on $\mathcal{H}$.

In a similar way as above, it follows that the sequence $((M^{-1})^\ast(m_n \psi_n))$ is a dual frame of $\Phi$ (denoted by $\Phi^\dagger$) and hence,

$$(M^{-1})^\ast T_\Psi = T_{\Phi^\dagger} M_{1/m} \text{ on } \ell^2. \quad (3.2)$$

Therefore, $M^{-1} = T_{\Psi^\dagger} M_{1/m} U_{\Phi^\dagger} = M_{1/m, \Psi^\dagger, \Phi^\dagger}$.

Now assume that $F = \{f_n\}$ is a Bessel sequence in $\mathcal{H}$ which satisfies $M_{m, \Phi, \Psi}^{-1} = M_{1/m, F, \Phi^\dagger}$. By $(3.2)$, it follows that $T_{\Psi^\dagger} = T_\Phi M_{1/m} U_{\Phi^\dagger} = M^\ast (M^{-1})^\ast = \text{Id}_H$, which implies that $F$ is a dual frame of $\Psi$. In a similar way, every Bessel sequence $F$ in $\mathcal{H}$ which satisfies $M_{m, \Phi, \Psi}^{-1} = M_{1/m, \Psi^\dagger, F}$ must be a dual frame of $\Phi$.

On the other hand, assume that $F$ is a Bessel sequence in $\mathcal{H}$ which satisfies $M_{1/m, F, \Phi^d} = M_{1/m, \Psi^\dagger, \Phi^d}$ for all dual frames $\Phi^d$ of $\Phi$. Then $T_{\Phi} M_{1/m} U_{\Phi^d} = T_{\Psi^\dagger} M_{1/m} U_{\Phi^d}$ for all dual frames $\Phi^d$ of $\Phi$, which by Theorem 2.1(i) implies that $T_{\Phi} M_{1/m} = T_{\Psi^\dagger} M_{1/m}$. Since $m$ is semi-normalized (so, $M_{1/m}$ is invertible on $\ell^2$), it follows that $T_{\Phi} = T_{\Psi^\dagger}$ and hence, $F = \Psi^\dagger$.

The statement for $\Phi^\dagger$ follows in a similar way. □

Concerning Theorem 3.1 it is natural to ask whether the frame $\Psi^\dagger$ (resp $\Phi^\dagger$) is the canonical dual of $\Psi$ (resp. $\Phi$). Observe that in this context we have
\[ \Psi^\dagger = \tilde{\Psi} \quad (\text{resp. } \Phi^\dagger = \tilde{\Phi}) \quad \text{if and only if } \Psi \text{ is equivalent to } \langle m_n \phi_n \rangle \quad (\text{resp. } \Phi \text{ is equivalent to } \langle m_n \psi_n \rangle). \]

Note that (3.1) is not a constructive approach leading to an implementation for the inversion of \( M \). For the dual frame \( \Psi^\dagger \) (resp. \( \Phi^\dagger \)) we already had to apply \( M^{-1} \). For more constructive approaches to the inversion of multipliers see Sections 4 and 5, and [18].

A sub-result of this theorem, the representation of the inverse for the particular case of finite-dimensional spaces and \( \Psi = \Phi^d \), has been independently found in the context of frame diagonalization of matrices [13].

**Remark 3.2.** In [21] the following question is posed: For an invertible, unconditionally convergent multiplier \( M_{m, \Phi, \Psi} \), do there exist sequences \( (c_n) \) and \( (d_n) \) so that \( M_{m, \Phi, \Psi} = M_{\langle 1 \rangle, (c_n \phi_n), (d_n \psi_n)} \) and the sequences \( (c_n \phi_n) \) and \( (d_n \psi_n) \) are frames for \( \mathcal{H} \)? If a conjecture in [21] is true, the answer to this question is always ‘Yes’, and so the inverse of every such multiplier can be written as a multiplier by Theorem 3.1. In [21], several sufficient conditions for this conjecture to be true are given, which leads to the following results.

**Proposition 3.3.** Let \( M_{m, \Phi, \Psi} \) be invertible and unconditionally convergent.

(i) If \( \Phi \) is minimal, then \( M^{-1}_{m, \Phi, \Psi} \) can be written as a multiplier.

(ii) If \( \Phi = \Psi \), then \( M^{-1}_{m, \Phi, \Psi} \) can be written as a multiplier.

(iii) If \( \inf_n |m_n| \cdot \|\phi_n\| \cdot \|\psi_n\| > 0 \), then \( M^{-1}_{m, \Phi, \Psi} \) can be written as a multiplier.

**Proof:** For all these conditions, by [21], there exist sequences \( (c_n) \) and \( (d_n) \) so that \( m = (c_n d_n) \) and the sequences \( (c_n \phi_n) \) and \( (d_n \psi_n) \) are frames for \( \mathcal{H} \). Thus, \( M_{m, \Phi, \Psi} = M_{\langle 1 \rangle, (c_n \phi_n), (d_n \psi_n)} \) and one can apply Theorem 3.1. \( \square \)

For the most useful frames in applications, the following holds:

**Corollary 3.4.** Let \( \Phi \) and \( \Psi \) be Gabor systems and \( m \) be semi-normalized. If \( M_{m, \Phi, \Psi} \) is invertible and unconditionally convergent, then \( M^{-1}_{m, \Phi, \Psi} \) can be written as a multiplier.

This corollary is true for any ‘coherent system’ [1], where the frame elements are created by applying the unitary representation of a group on one element (or even on finitely many elements). In particular it is also true for wavelet systems.

### 4 Inversion of Multipliers Using the Canonical Duals [Q2]

The following gives examples where Question [Q2] is answered affirmatively.

\(^2\)For the definition of equivalent frames see Remark 4.4.
Example 4.1. The frame operator for any frame $\Phi$ is invertible and its inverse is the frame operator for $\tilde{\Phi}$ (see, e.g., [8, Lemma 5.1.5]). Therefore, every frame $\Phi$ for $\mathcal{H}$ fulfills $M_{(1),\Phi,\tilde{\Phi}} = M_{(1),\tilde{\Phi},\Phi}^{-1}$.

Example 4.2 shows a case when $M_{m,\Phi,\Psi}$ is invertible but the inverse is not equal to $M_{1/m,\tilde{\Phi},\tilde{\Psi}}$.

Example 4.2. Let $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, \ldots)$ and $\Psi = (e_1, -e_1, e_2, -e_2, e_3, -e_3, \ldots)$. Then $M_{(1),\tilde{\Phi},\tilde{\Psi}} = \frac{1}{c} \operatorname{Id}_\mathcal{H} \neq M_{(1),\Phi,\Psi}^{-1}$.

The next theorem determines a class of multipliers which are invertible and whose inverses can be written as in (1.2). While Theorems 3.1 and 3.3 assume that the multiplier is invertible, in Theorem 4.3 we investigate the invertibility of multipliers - we give sufficient conditions for invertibility and sufficient conditions for non-invertibility of multipliers. For the rest of the section the letter $c$ means a non-zero constant.

Theorem 4.3. Let $\Phi$ and $\Psi$ be frames for $\mathcal{H}$ and $(m_n) = (c)$. Then the following assertions hold.

(i) If $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$, then $M_{(1/c),\tilde{\Phi},\tilde{\Psi}}$ is a bounded right inverse of $M_{(c),\Phi,\Psi}$.

(ii) If $\mathcal{R}(U_\Psi) \subseteq \mathcal{R}(U_\Phi)$, then $M_{(1/c),\tilde{\Phi},\tilde{\Psi}}$ is a bounded left inverse of $M_{(c),\Phi,\Psi}$.

(iii) If $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is invertible and $M_{(c),\Phi,\Psi}^{-1} = M_{(1/c),\tilde{\Phi},\tilde{\Psi}}$.

(iv) If $\mathcal{R}(U_\Phi) \not\subseteq \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.

(v) If $\mathcal{R}(U_\Psi) \not\subseteq \mathcal{R}(U_\Phi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.

Proof: (i) Assume that $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$. For every $h \in \mathcal{H}$, the element $U_\Phi S_\Psi h$ can be written as $U_\Psi g^h$ for some $g^h \in \mathcal{H}$ and

$$M_{(c),\Phi,\Psi} M_{(1/c),\tilde{\Phi},\tilde{\Psi}} h = T_\Phi U_\Phi S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} h = T_\Phi U_\Psi g^h = h.$$ 

(ii) can be proved in a similar way as (i).

(iii) follows from (i) and (ii).

(iv) Assume that $\mathcal{R}(U_\Phi) \not\subseteq \mathcal{R}(U_\Psi)$ with $\mathcal{R}(U_\Psi) \neq \mathcal{R}(U_\Phi)$. By (i), the operator $M_{(1/c),\tilde{\Phi},\tilde{\Psi}}$ is a bounded right inverse of $M_{(c),\Phi,\Psi}$. We will prove that $M_{(1/c),\tilde{\Phi},\tilde{\Psi}}$ is not a left inverse of $M_{(c),\Phi,\Psi}$, which will imply that $M_{(c),\Phi,\Psi}$ can not be invertible. Consider an arbitrary element $g \in \mathcal{R}(U_\Psi) \setminus \mathcal{R}(U_\Phi)$ and write $g = U_\Psi h$ for some $h \in \mathcal{H}$. Since $\ell^2 = \mathcal{R}(U_\Phi) \oplus \ker(T_\Phi)$, we can also write $g = U_\Phi f + d$ for some $f \in \mathcal{H}$ and some $d \in \ker(T_\Phi)$. Then

$$M_{(1/c),\tilde{\Phi},\tilde{\Psi}} M_{(c),\Phi,\Psi} h = S_\Psi^{-1} T_\Phi U_\Phi S_\Phi^{-1} T_\Psi U_\Phi h = S_\Psi^{-1} T_\Phi U_\Phi S_\Phi^{-1} T_\Psi (U_\Phi f + d) = S_\Psi^{-1} T_\Phi (U_\Psi h - d) = h - S_\Psi^{-1} T_\Phi d.$$
Observe that $S^{-1}T_{\psi}d \neq 0$, which implies that $M_{(1/c),\tilde{\psi},\tilde{\phi}}$ is not a left inverse of $M_{(c),\phi,\psi}$.

(v) Assume that $\mathcal{R}(U_{\psi}) \subset \mathcal{R}(U_{\phi})$ with $\mathcal{R}(U_{\psi}) \neq \mathcal{R}(U_{\phi})$. By (i), $M_{(1/c),\tilde{\psi},\tilde{\phi}}$ is a bounded left inverse of $M_{(c),\phi,\psi}$. In a similar way as in (iv), one can prove that $M_{(1/c),\tilde{\psi},\tilde{\phi}}$ is not a right inverse of $M_{(c),\phi,\psi}$, which implies that $M_{(c),\phi,\psi}$ can not be invertible. □

Remark 4.4. Let $\Phi$ and $\Psi$ be frames for $H$. The condition $\mathcal{R}(U_{\Phi}) = \mathcal{R}(U_{\Psi})$ corresponds to $\Phi$ and $\Psi$ being equivalent frames \cite[Corollary 4.5]{4}], i.e., to the existence of a bounded and invertible operator $G: H \rightarrow H$, such that $G\phi_k = \psi_k$, $\forall k \in \mathbb{N}$. The condition $\mathcal{R}(U_{\Phi}) \subseteq \mathcal{R}(U_{\Psi})$ is identical to $\Psi$ being partial equivalent to $\Phi$ \cite[2], i.e., to the existence of a bounded operator $Q: H \rightarrow H$, such that $\phi_k = Q\psi_k$, $\forall k \in \mathbb{N}$.

Note that if none of $\mathcal{R}(U_{\Phi})$ and $\mathcal{R}(U_{\Psi})$ is a subset of the other one, then both invertibility and non-invertibility of $M_{m,\phi,\psi}$ are possible, see Examples 4.5 and 4.6.

Example 4.5. (invertible multiplier) Consider the frames $\Phi = (e_1, e_1, e_2, e_3, e_4, e_4, \ldots)$ and $\Psi = (e_1, e_1, 3, e_2, e_3, e_3, e_4, e_4, \ldots)$. Then $M_{(1),\phi,\psi}$ is the identity operator on $H$.

Example 4.6. (non-invertible multiplier) Consider the frames $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, e_4, \ldots)$ and $\Psi = (e_1, e_1, e_2, e_2, e_4, \ldots)$. Then $M_{(1),\phi,\psi}$ is not invertible.

Corollary 4.7. If $\Phi$ and $\Psi$ are equivalent frames, then $M^{-1}_{(c),\phi,\psi}$ is invertible and $M^{-1}_{(c),\phi,\psi} = M_{(c),\phi,\psi}$.

5 Gabor Multipliers

By Corollary 4.3 an invertible Gabor frame multiplier with semi-normalized symbol has an inverse which can be written as a multiplier. Here we are interested in cases when the inverse can be written as a Gabor multiplier.

Theorem 5.1. Let $g \in L^2(\mathbb{R}^d)$ and let $(\pi(\lambda)g)_{\lambda \in \Lambda}$ be a Gabor frame for $L^2(\mathbb{R}^d)$. Let $V: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be a bounded bijective operator. Then the following statements are equivalent.

(A1) For every $\lambda \in \Lambda$, $V\pi(\lambda)g = \pi(\lambda)Vg$.

(A2) For every $\lambda \in \Lambda$ and every $f \in L^2(\mathbb{R}^d)$, $V\pi(\lambda)f = \pi(\lambda)Vf$ (i.e., $V$ commutes with $\pi(\lambda)$ for every $\lambda \in \Lambda$).

(A3) $V$ can be written as a Gabor frame multiplier with a constant symbol (1).

(A4) $V^{-1}$ can be written as a Gabor frame multiplier with a constant symbol (1).
which means that the proof uses similar technics as in [8, Lemma 9.3.1].

This statement extends the result that the frame operator commutes with \( \pi \) for \( \lambda = (\tau, \omega) \in \Lambda \) and \( \lambda' = (\tau', \omega') \in \Lambda \) we have

\[
V \pi(\lambda)f = \sum_{\lambda' \in \Lambda} \langle \pi(\lambda)f, \pi(\lambda')u \rangle \pi(\lambda')v
\]

\[
= \sum_{\lambda' \in \Lambda} \langle f, e^{2\pi i \tau'(\omega' - \omega)} \pi(\lambda' - \lambda)u \rangle \pi(\lambda')v
\]

\[
= \sum_{\lambda' \in \Lambda} \langle f, e^{2\pi i \omega'} \pi(\lambda')u \rangle \pi(\lambda' + \lambda)v
\]

\[
= \sum_{\lambda'' \in \Lambda} \langle f, e^{2\pi i \omega''} \pi(\lambda'')u \rangle e^{2\pi i \omega''} \pi(\lambda) \pi(\lambda'')v
\]

\[
= \pi(\lambda)V f.
\]

This statement extends the result that the frame operator commutes with \( \pi(\lambda) \); the proof uses similar technics as in [8, Lemma 9.3.1].

\((A_2) \implies (A_1)\) is obvious.

\((A_1) \implies (A_3)\) For every \( f \in L^2(\mathbb{R}^d) \),

\[
V f = V \left( \sum_{\lambda \in \Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda \right) = \sum_{\lambda \in \Lambda} \langle f, \tilde{g}_\lambda \rangle \pi(\lambda) V g,
\]

which means that \( V \) can be written as a Gabor frame multiplier with symbol (1).

\((A_3) \implies (A_4)\) For \( \lambda \in \Lambda \), denote \( h_\lambda = \pi(\lambda) V g \). By what is already proved, \( V \) can be written as the multiplier \( M_{(1), (h_\lambda), (\tilde{g}_\lambda)} \). Since \( (h_\lambda) \) and \( (\tilde{g}_\lambda) \) are equivalent frames, Corollary [17] implies that \( M_{(1), (h_\lambda), (\tilde{g}_\lambda)} = M_{(1), (\tilde{g}_\lambda), (h_\lambda)} \).

\((A_4) \implies (A_2)\) Having in mind the implication \((A_3) \implies (A_2)\) applied to \( V^{-1} \), it follows that \( V^{-1} \) commutes with \( \pi(\lambda), \forall \lambda \in \Lambda \). Therefore, \( V \) also commutes with \( \pi(\lambda), \forall \lambda \in \Lambda \). \( \square \)

**Remark 5.2.** This result gives a nice representation and criterion for TF-lattice invariant operators [11], which correspond to condition \((A_2)\). Motivated by [10], the condition \((A_1)\) can be considered to define 'locally TF-lattice invariant' operators. We have shown that this local property already implies the global one.

As a consequence of Theorem [10], the inverse of every invertible Gabor frame multiplier with constant symbol can be written as a Gabor frame multiplier:

**Corollary 5.3.** Let \( V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) be an invertible Gabor frame multiplier \( M_{(1), (\pi(\lambda) v)_{\lambda \in \Lambda}, (\pi(\lambda) u)_{\lambda \in \Lambda}} \). Let \( (g_\lambda)_{\lambda \in \Lambda} = (\pi(\lambda) g)_{\lambda \in \Lambda} \) be any Gabor frame for \( L^2(\mathbb{R}^d) \). Then \( V^{-1} \) can be written as the Gabor frame multiplier \( M_{(1), (g_\lambda), (\tilde{h}_\lambda)} \), where \( h_\lambda = \pi(\lambda) V g, \lambda \in \Lambda \).
Concerning Theorem 5.1, note that if weaker assumptions on $V$ are made, then a similar proof can be used to show the following statements.

**Lemma 5.4.** As in Theorem 5.1, let $g \in L^2(\mathbb{R}^d)$ and let $(\pi(\lambda)g)_{\lambda \in \Lambda}$ be a Gabor frame for $L^2(\mathbb{R}^d)$. Let $V : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be an operator. Consider the condition

$$(A'_3) \quad V \text{ can be written as a Gabor Bessel multiplier with a constant symbol (1).}$$

Then the following statements hold.

(i) If $V$ is bounded, then $(A_3) \Rightarrow (A'_3) \Leftrightarrow (A_2) \Leftrightarrow (A_1)$;

(ii) If $V$ is bounded and surjective, then $(A_3) \Leftrightarrow (A'_3) \Leftrightarrow (A_2) \Leftrightarrow (A_1)$.

So, when a Gabor Bessel multiplier $M_{(1),(\pi(\lambda)v)_{\lambda \in \Lambda},(\pi(\lambda)u)_{\lambda \in \Lambda}}$ is bounded and surjective, it can always be written as a Gabor frame multiplier for appropriate frames. Note that if $V$ is the Gabor Bessel multiplier $M_{(1),(\pi(\lambda)v)_{\lambda \in \Lambda},(\pi(\lambda)u)_{\lambda \in \Lambda}}$ for some $u,v \in L^2(\mathbb{R}^d)$ and $V$ is a bounded surjective (resp. invertible) operator, then $(\pi(\lambda)v)_{\lambda \in \Lambda}$ is a frame (resp. $(\pi(\lambda)u)_{\lambda \in \Lambda}$ and $(\pi(\lambda)v)_{\lambda \in \Lambda}$ are frames) for $L^2(\mathbb{R}^d)$ (see e.g. [8] (resp. see [13]).

**Acknowledgments** The work on this paper was supported by the WWTF project MULAC ('Frame Multipliers: Theory and Application in Acoustics; MA07-025) and partly supported by the Austrian Science Fund (FWF) START-project FLAME ('Frames and Linear Operators for Acoustical Modeling and Parameter Estimation'; Y 551-N13). The authors are thankful to H. Feichtinger and D. Bayer for their valuable comments. The second author is grateful for the hospitality of the Acoustics Research Institute and the support from the projects MULAC and FLAME.

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