On the Rate of Convergence for a Characteristic of Multidimensional Birth-Death Process

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Abstract

We consider a multidimensional inhomogeneous birth-death process (BDP) and obtain bounds on the rate of convergence for the corresponding one-dimensional processes.

1 Introduction

Multidimensional birth-death processes (BDP) were objects of a number of studies in queueing theory and other applied fields, see \cite{1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13}. The problem of the product form solutions for such models was considered, for instance, in \cite{14} (also, see the references therein). If the process is inhomogeneous and transition intensities have a more general form, then the problem of computation of any probabilistic characteristics of the queueing model is much more difficult.

Our approach is particularly based on the method of investigation of inhomogeneous BDP, see the detailed discussion and some preliminary results in \cite{6, 15, 16, 17}. Estimates for the state probabilities of one-dimensional projections of a multidimensional BDP were studied in \cite{18} and \cite{19}.

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Here we slightly modify the approach to estimation and obtain some explicit bounds on the rate of convergence for one-dimensional projection of a multidimensional BDP.

Let $X(t) = (X_1(t), ..., X_d(t))$ be a $d$-dimensional BDP such that in the interval $(t, t + h)$ the following transitions are possible with order $h$: birth of a particle of type $j$, death of a particle of type $j$.

Let $\lambda_{j,m}(t)$ be the corresponding birth rate (from the state $m = (m_1, ..., m_d) = \sum_{i=1}^d m_i e_i$ to the state $m + e_j$) and $\mu_{j,m}(t)$ be the corresponding death intensity (from the state $m = (m_1, ..., m_d) = \sum_{i=1}^d m_i e_i$ to the state $m - e_j$). Denote $p_m(t) = \Pr(X(t) = m)$.

Let now the (countable) state space of the vector process under consideration be arranged in a special order, say $0, 1, \ldots$. Denote by $p_i(t)$ the corresponding state probabilities, and by $p(t)$ the corresponding column vector of state probabilities. Applying our standard approach (see details in [6, 16, 17]) we suppose in addition, that all intensities are nonnegative functions locally integrable on $[0, \infty)$, and, moreover, in new enumeration,

$$
\Pr(X(t + h) = j/X(t) = i) = \begin{cases} q_{ij}(t)h + \alpha_{ij}(t, h), & j \neq i, \\
1 - \sum_{k \neq i} q_{ik}(t)h + \alpha_i(t, h), & j = i, \end{cases}
$$

(1)

where all $\alpha_i(t, h)$ are $o(h)$ uniformly in $i$, i.e. $\sup_i |\alpha_i(t, h)| = o(h)$.

We suppose that $\lambda_{j,m}(t) \leq L < \infty$, $\mu_{j,m}(t) \leq M < \infty$, for any $j$, $m$ and almost all $t \geq 0$.

The probabilistic dynamics of the process is represented by the forward Kolmogorov system:

$$
\frac{dp}{dt} = A(t)p(t),
$$

(2)

where $A(t)$ is the corresponding infinitesimal (intensity) matrix.

Throughout the paper we denote the $l_1$-norm by $\| \cdot \|$, i.e. $\|x\| = \sum |x_i|$, and $\|B\| = \sup_j \sum_i |b_{ij}|$ for $B = (b_{ij})_{j=0}^\infty$.

Let $\Omega$ be the set all stochastic vectors, i.e., $l_1$-vectors with nonnegative coordinates and unit norm. We have the inequality $\|A(t)\| \leq 2d(L + M) < \infty$, for any $j$, $m$ and almost all $t \geq 0$. Hence, the operator function $A(t)$ from $l_1$ into itself is bounded for almost all $t \geq 0$ and is locally integrable on $[0; \infty)$. Therefore we can consider (2) as a differential equation in the space $l_1$ with bounded operator.
It is well known, see [2], that the Cauchy problem for differential equation (2) has unique solution for an arbitrary initial condition, and $p(s) \in \Omega$ implies $p(t) \in \Omega$ for $t \geq s \geq 0$.

We recall that a Markov chain $X(t)$ is called null-ergodic, if all $p_i(t) \to 0$ $t \to \infty$ for any initial condition, and it is called weakly ergodic, if $\|p^*(t) - p^{**}(t)\| \to 0$ as $t \to \infty$ for any initial condition $p^*(0), p^{**}(0)$.

2 Bounds on the rate of convergence for a differential equation

Consider the general differential equation
\[
\frac{dy}{dt} = Hy(t),
\] in the space of sequences $l_1$ under the assumption of existence and uniqueness of solution for any initial condition $y(0)$.

Let $H = (h_{ij})$, where all $h_{ij}$ depend on some parameters.

We have
\[
\frac{dy_i}{dt} = h_{ii}y_i + \sum_{j \neq i} h_{ij}y_j.
\]

Now, if $y_i \geq 0$, then
\[
\frac{d|y_i|}{dt} = \frac{dy_i}{dt} = h_{ii}y_i + \sum_{j \neq i} h_{ij}y_j \leq h_{ii}|y_i| + \sum_{j \neq i} |h_{ij}||y_j|,
\]
and if $y_i < 0$, then we also have
\[
\frac{d|y_i|}{dt} = -\frac{dy_i}{dt} = -h_{ii}y_i - \sum_{j \neq i} h_{ij}y_j \leq h_{ii}|y_i| + \sum_{j \neq i} |h_{ij}||y_j|.
\]

Finally, we obtain the bound
\[
\frac{d\|y\|}{dt} = \sum_i \frac{d|y_i|}{dt} \leq \sum_i \left( h_{ii}|y_i| + \sum_{j \neq i} |h_{ij}||y_j| \right) \leq \beta^*\|y\|,
\] where
\[
\beta^* = \sup_i \left( h_{ii} + \sum_{j \neq i} |h_{ji}| \right).
\]
Remark 1. One can see that inequality (4) implies the bound
\[ \|y(t)\| \leq e^{\int_0^t \beta^* du} \|y(0)\|. \]
Moreover, if \( H \) is bounded for any \( t \) linear operator function from \( l_1 \) to itself, then \( \beta^*(t) = \gamma(H(t)) \) is the corresponding logarithmic norm of \( H(t) \), see [6, 17].

3 Bounds on the rate of convergence for a projection of multidimensional BDP

Again consider the forward Kolmogorov system (2). Then we have
\[
\frac{dp_m}{dt} = \sum_l \lambda_{l,m-e_1}(t)p_{m-e_1} + \sum_l \mu_{l,m+e_1}(t)p_{m+e_1} - \sum_l (\lambda_{l,m} + \mu_{l,m}) (t)p_m,
\]
for any \( m \).

Fix \( j \) and consider the one-dimensional process \( X_j(t) \). Denote \( x_k(t) = \Pr(X_j(t) = k) \). Then \( x_k(t) = \sum_{m,m_j=k} p_m(t) \). The process \( X_j(t) \) has nonzero jump rates only for unit jumps (\( \pm 1 \)), namely, if \( X_j(t) = k \), then for small positive \( h \) only the jumps \( X(t+h) = k \pm 1 \) are possible with positive intensities, say \( \lambda_k \) and \( \mu_k \) respectively. Moreover, (7) implies the equalities
\[
\tilde{\lambda}_k x_k(t) = \sum_{m,m_j=k} \lambda_{j,m}(t)p_m(t), \tag{8}
\]
\[
\tilde{\mu}_k x_k(t) = \sum_{m,m_j=k} \mu_{j,m}(t)p_m(t), \tag{9}
\]
and hence
\[
\tilde{\lambda}_k = \frac{\sum_{m,m_j=k} \lambda_{j,m}(t)p_m(t)}{\sum_{m,m_j=k} p_m(t)}, \tag{10}
\]
and death intensities
\[
\tilde{\mu}_k = \frac{\sum_{m,m_j=k} \mu_{j,m}(t)p_m(t)}{\sum_{m,m_j=k} p_m(t)}. \tag{11}
\]

Then \( X_j(t) \) is the (generally non-Markovian) birth and death process with birth and death intensities \( \tilde{\lambda}_k \) and \( \tilde{\mu}_k \) respectively.
For any fixed initial distribution \( p(0) \) and any \( t > 0 \) the probability distribution \( p(t) \) is unique. Hence, \( \lambda_k = \lambda_k(p(0), t) \) and \( \mu_k = \mu_k(p(0), t) \) uniquely define the system
\[
\frac{dx}{dt} = \tilde{A}x(t),
\]
for the vector \( x(t) \) of state probabilities of the projection \( X_j(t) \) under the given initial condition. Here \( \tilde{A} \) is the corresponding three-diagonal “birth-death” transposed intensity matrix with nonnegative for any \( t \) and any initial condition \( p(0) \) off-diagonal elements and zero column sums.

Let for all \( m \) and any \( t \geq 0 \)
\[
l_j \leq \lambda_{j,m}(t) \leq L_j, \quad m_j \leq \mu_{j,m}(t) \leq M_j.
\]
Then from (10) and (11) we obtain the two-sided bounds
\[
l_j \leq \tilde{\lambda}_k \leq L_j, \quad m_j \leq \tilde{\mu}_k \leq M_j,
\]
for any \( k \).

1. Let
\[
M_j < l_j.
\]

Put \( \sigma = \sqrt{M_j/l_j} < 1, \delta_n = \sigma^n, \quad n \geq 0, \quad \tilde{x}_n = \delta_n x_n, \) and \( \tilde{x} = (\tilde{x}_0, \tilde{x}_1, \ldots) \). Let \( \Lambda \) be a diagonal matrix, \( \Lambda = diag(\delta_0, \delta_1, \ldots) \).

Then
\[
\frac{d\tilde{x}}{dt} = \Lambda \tilde{A} \Lambda^{-1} \tilde{x}(t).
\]
Hence
\[
\tilde{\lambda}_k + \tilde{\mu}_k - \frac{\delta_{k+1}}{\delta_k} \tilde{\lambda}_k - \frac{\delta_{k-1}}{\delta_k} \tilde{\mu}_k \geq \tilde{\lambda}_k (1 - \sigma) - \tilde{\mu}_k (1/\sigma - 1) \geq
\]
\[
l_j (1 - \sigma) - M_j (1/\sigma - 1) = \left( \sqrt{l_j} - \sqrt{M_j} \right)^2 = \alpha^*,
\]
we obtain the estimate
\[
\frac{d\|\tilde{x}\|}{dt} \leq \sup_k \left( \frac{\delta_{k+1}}{\delta_k} \tilde{\lambda}_k + \frac{\delta_{k-1}}{\delta_k} \tilde{\mu}_k - \tilde{\lambda}_k - \tilde{\mu}_k \right) =
\]
\[
- \inf_k \left( \tilde{\lambda}_k + \tilde{\mu}_k - \frac{\delta_{k+1}}{\delta_k} \tilde{\lambda}_k - \frac{\delta_{k-1}}{\delta_k} \tilde{\mu}_k \right) \leq -\alpha^* \|\tilde{x}\|,
\]
and the following statement.
Theorem 1. Let (15) hold for some \( j \). Then \( X_j(t) \) is null-ergodic and the following bounds hold:

\[
\|\tilde{x}(t)\| \leq e^{-\alpha^* t}\|\tilde{x}(0)\|, \tag{19}
\]

and

\[
\Pr(X_j(t) \leq n/X_j(0) = k) \leq \sigma^n \cdot e^{-\alpha^* t}. \tag{20}
\]

2. Let \( L_j < m_j, \alpha^* = l_j + m_j - 2\sqrt{L_j M_j} > 0. \tag{21} \)

The property \( x(t) \in \Omega \) for any \( t \geq 0 \) allows to set \( x_0(t) = 1 - \sum_{i \geq 1} x_i(t) \). Then from (12) we obtain the system

\[
\frac{dz}{dt} = \tilde{B}z + \tilde{f}, \tag{22}
\]

where \( z = (x_1, x_2, \ldots)^T \), \( \tilde{f} = \left(\tilde{\lambda}_0, 0, 0, \ldots\right)^T \), and the corresponding matrix

\[
\tilde{B} = \left(\tilde{b}_{ij}\right)_{i,j=1}^\infty, \quad \text{and} \quad \tilde{b}_{ij} = \tilde{a}_{ij} - \tilde{a}_{i0} \text{ for the corresponding elements of the matrix } \tilde{A}.
\]

For the solutions of system (22) the rate of convergence is determined by the system

\[
\frac{dw}{dt} = \tilde{B}w. \tag{23}
\]

Now let \( \beta = \sqrt{M_j/L_j} > 1 \) in accordance with (21). Let \( d_{k+1} = \beta^k, k \geq 0 \). Denote by \( D \) the upper triangular matrix

\[
D = \begin{pmatrix}
    d_1 & d_1 & d_1 & \cdots \\
    0 & d_2 & d_2 & \cdots \\
    0 & 0 & d_3 & \cdots \\
    \cdots & \cdots & \cdots & \cdots
\end{pmatrix}. \tag{24}
\]

Let \( \tilde{w} = Dw \). Then the following bound holds:

\[
\frac{d\|\tilde{w}\|}{dt} \leq \sup_{i \geq 0} \left( \frac{d_{i+1}}{d_i} \tilde{\lambda}_{i+1} + \frac{d_{i-1}}{d_i} \tilde{\mu}_i - \left(\tilde{\lambda}_i + \tilde{\mu}_{i+1}\right) \right) = -\inf_{i \geq 0} \left( \left(\tilde{\lambda}_i + \tilde{\mu}_{i+1} - \beta \tilde{\lambda}_{i+1} - \tilde{\mu}_i / \beta \right) \right) \leq -\alpha_* \|\tilde{w}\|, \tag{25}\]

and we obtain the following statement.
Theorem 2. Let (21) hold for some $j$. Then $X_j(t)$ is weakly ergodic and the following bound holds:

$$\|Dw(t)\| \leq e^{-\alpha t} \|Dw(0)\|,$$  

(26)

for any $t \geq 0$ and any corresponding initial conditions.

Remark 2. Instead of $X_j(t)$ we can obtain the same results for the one-dimensional process $Z(t) = |X(t)|$, that is, the number of all particles at the moment $t$.

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