A note about zonal polynomials

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Abstract
In this paper, we study some properties of multivariate gamma function and zonal polynomials.

1 Introduction

Zonal polynomials are of undeniable importance, in both theory and practice. Following the algorithms proposed by Koev and Edelman [2006], zonal polynomials are increasingly being used in various areas of knowledge. Undoubtedly, the initial studies by James (1960), James (1961a), James (1961b), Constantine (1963) and James (1964), among others, laid the foundations in this field. Subsequently, books by Farrell (1985) and Takemura (1984), among others, compiled many of these early results and proposed new theoretical considerations and many practical applications. Nevertheless, the book by Muirhead (1982), above all others, marks a watershed in these studies and has had an undeniable impact on recent generations of mathematicians and statisticians working in the field of multivariate analysis, see Wijsman (1984). Virtually all recent studies that bear upon the question of zonal polynomials have cited Muirhead’s book (1982). In particular, Li (1997), calculated the expectation of zonal and invariate polynomials, making use of various results published in Muirhead (1982). Unfortunately, the conclusions drawn by Li (1997) are incorrect, and this is because both Lemma 7.2.12 and the proof of Theorem 7.2.13 in Muirhead (1982) are incorrect. Due to the undeniable importance and impact of Muirhead’s book, and its influence on current and future studies, the present text proposes corrections to the above-mentioned lemma and theorem.

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(see Section 3). Prior to this, in Section 2 we propose an expression and alternative proof of the multivariate gamma function.

## 2 Preliminary results

The Pochhammer symbol is defined as

$$(x)_q = x(x+1)\cdots(x+q-1) = \prod_{i=1}^{q}(x+i-1) = \prod_{i=1}^{q}(x+q-i) = \frac{\Gamma[x+q]}{\Gamma[x]},$$

where $\Gamma[\cdot]$ is the gamma function. Also, observe that

$$(-x)_q = (-1)^q(x-q+1)_q = \frac{(-1)^q\Gamma[x+1]}{\Gamma[x-q+1]}.$$ (2)

Similarly

$$(x)^q = x(x-1)\cdots(x-q+1) = \prod_{i=1}^{q}(x-i+1) = \prod_{i=1}^{q}(x-q+i).$$

Then for any function $g : \mathbb{R} \to \mathbb{R}$

$$\prod_{i=1}^{q} g(x+i-1) = \prod_{i=1}^{q} g(x+q-i),$$ (3)

and

$$\prod_{i=1}^{q} g(x-i+1) = \prod_{i=1}^{q} g(x-q+i).$$ (4)

**Lemma 2.1.** Let $X$ be a real $m \times m$ positive definite matrix and $\Re(a) > (m-1)/2$. The multivariate gamma function, denoted by $\Gamma_m[a]$, is defined to be

$$\Gamma_m[a] = \int_{A>0} \text{etr}(-A)(\det A)^{a-(m+1)/2}(dA),$$

where $\text{etr}(\cdot) \equiv \exp \text{tr}(\cdot)$ and the integral is over the space of positive definite (and hence symmetric) $m \times m$ matrices. Here, $A > 0$ means that $A$ is a positive definite matrix. Then

$$\Gamma_m[a] = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma[a-(i-1)/2],$$

$$= \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma[a-(m-i)/2].$$ (6)

**Proof.** (5) is given in [Muirhead (1982, Theorem 2.1.12, pp. 62-63), see also, Mathai (1997, Example 1.24, pp. 56-57)].

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For proof (6), let $T$ be a real upper-triangular matrix with $t_{ii} > 0$, $i = 1, \ldots, m$ and consider the decomposition $X = TT'$, then

$$\text{tr} \ A = \text{tr} \ TT' = \text{tr} \ T'T = \sum_{i \leq j} t_{ij}^2,$$

$$\det \ A = \det \ TT' = (\det T)^2 = \prod_{i=1}^{m} t_{ii}^2,$$

and from [Mathai 1997, Theorem 1.28, p. 56]

$$(dA) = 2^m \prod_{i=1}^{m} t_{ii} \int_{-\infty}^{\infty} \int_{i \leq j} dt_{ij} = \prod_{i=1}^{m} (t_{ii}^2)^{(i-1)/2} \int_{i \leq j} dt_{ii} \int_{i \leq j} dt_{ij}.$$

Hence,

$$\Gamma_m[a] = \prod_{i<j}^{m} \left( \int_{-\infty}^{\infty} \exp \left(-t_{ij}^2\right) dt_{ij} \right) \prod_{i=1}^{m} \left( \int_{0}^{\infty} \exp \left(-t_{ii}^2\right) \left(t_{ii}^2\right)^{a-(m-i)/2-1} dt_{ii}^2 \right).$$

But

$$\int_{-\infty}^{\infty} \exp \left(-t_{ij}^2\right) dt_{ij} = \sqrt{\pi}$$

and

$$\int_{0}^{\infty} \exp \left(-t_{ii}^2\right) \left(t_{ii}^2\right)^{a-(m-i)/2-1} dt_{ii}^2 = \Gamma[a - (m - i)/2].$$

From where (6) follows. Alternatively, (5) is an immediate consequence of (4). \(\square\)

In a similar way to expression (4) it is readily apparent that for $k_i$ non negative integers, $i = 1, \ldots, q$,

$$\prod_{i=1}^{q} g(x \pm k_{q+1-i} - i + 1) = \prod_{i=1}^{q} g(x \pm k_i - q + i). \quad (7)$$

### 3 Zonal polynomials

In this section we propose the correct version of Lemma 7.2.12, p. 256 and the correct proof of Theorem 7.2.13, pp. 256-258 in [Muirhead 1982].

**Lemma 3.1.** If $Z = \text{diag}(z_1, \ldots, z_m)$ and $Y = (y_{ij})$ is an $m \times m$ positive definite matrix then

$$C_\kappa \left(Y^{-1}Z\right) = d_\kappa z_1^{k_1} \cdots z_m^{k_m} y_{11}^{-(k_{m-1}-k_{m-2})} \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-k_{m-1}-k_{m-2}} \cdots \det (Y)^{-k_1} + \text{terms of lower weight in the } z's.$$

where $\kappa = (k_1, \ldots, k_m)$. 

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Proof. If $A$ is a symmetric matrix with latent roots $a_1, \ldots, a_m$ then $A^{-1}$ is also a symmetric matrix with latent roots $\alpha_1, \ldots, \alpha_m$, such that $\alpha_i = a_i^{-1}$ for $i = 1, \ldots, m$. Then by Constantine (1966, without proof) and Takemura (1984, Lemma 2, p.54, with proof),

$$
\left(\det A\right)^n \frac{C_\kappa(A^{-1})}{C_\kappa(I_m)} = \frac{C_{\kappa^*}(A)}{C_{\kappa^*}(I_m)}
$$

where $n$ is any integer $\geq k_1$ and $\kappa^* = (n - k_m, \ldots, n - k_1)$. Thus

$$
C_{\kappa}(A^{-1}) = d_\kappa a_1^{k_1} \cdots a_m^{k_m} + \text{terms of lower weight}
$$

$$
= (\det A)^{-n} s_{\kappa, \kappa^*} C_{\kappa^*}(A)
$$

$$
= (\det A)^{-n} s_{\kappa, \kappa^*} [d_\kappa a_1^{n-k_m} \cdots a_m^{n-k_1} + \text{terms of lower weight}]
$$

Denoting

$$
s_{\kappa, \kappa^*} = \frac{C_\kappa(I_m)}{C_{\kappa^*}(I_m)}
$$

we have

$$
C_{\kappa}(A^{-1}) = (\det A)^{-n} s_{\kappa, \kappa^*} d_\kappa a_1^{n-k_m-\cdots-(n-k_m-1)} (a_1 a_2)^{n-k_m-\cdots-(n-k_m-2)} \cdots (a_1 a_2 \cdots a_m)^{-k_1 + \cdots}
$$

$$
= s_{\kappa, \kappa^*} d_\kappa a_1^{-(k_m-k_{m-1})} (a_1 a_2)^{-(k_{m-1}-k_m-2)} \cdots (a_1 a_2 \cdots a_m)^{-k_1 + \cdots}
$$

$$
= s_{\kappa, \kappa^*} d_\kappa r_1^{-(k_m-k_{m-1})} r_2^{-(k_{m-1}-k_m-2)} \cdots r_m^{-(k_1 + \cdots)}
$$

from (39) and (40) $(r_j = \text{tr}_j(A))$ in Muirhead (1982, p. 247)

$$
C_{\kappa}(A^{-1}) = s_{\kappa, \kappa^*} d_\kappa \text{tr}_1(A)^{-(k_m-k_{m-1})} \text{tr}_2(A)^{-(k_{m-1}-k_m-2)} \cdots \text{tr}_m(A)^{-k_1 + \cdots}
$$

$$
= s_{\kappa, \kappa^*} d_\kappa a_{11}^{-k_{m-k_m-1}} \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-k_{m-1}-k_m-2} \cdots (\det A)^{-k_1 + \cdots}.
$$

Now let $A^{-1} = Y^{-1}Z$, then $A = Z^{-1}Y$ and thus $a_{ij} = z_{ij}^{-1} y_{ij}$. From where

$$
C_{\kappa}(Y^{-1}Z) = s_{\kappa, \kappa^*} d_\kappa (z_1^{-1} y_{11})^{-(k_m-k_{m-1})} \det \begin{bmatrix} z_1^{-1} y_{11} & z_1^{-1} y_{12} \\ z_2^{-1} y_{21} & z_2^{-1} y_{22} \end{bmatrix}^{-(k_{m-1}-k_m-2)} \cdots (\det Z^{-1}Y)^{-k_1 + \cdots}
$$

$$
= s_{\kappa, \kappa^*} d_\kappa z_1^{k_1} \cdots z_m^{k_m} (y_{11})^{-k_{m-k_m-1}} \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-(k_{m-1}-k_m-2)} \cdots (\det Y)^{-k_1 + \cdots}.
$$

Finally, the result is obtained observe that:
i) In reality, the expression (10) in Constantine [1963], see also Muirhead [1982, expression (i)(1), p. 228]

\[ C_\kappa(Y) = d_\kappa y_1^{k_1} \cdots y_m^{k_m} + \text{terms of lower weight} \]

is

\[ C_\kappa(Y) = d_\kappa (y_1^{k_1} \cdots y_m^{k_m} + \cdots + \text{symmetric terms}) + \text{terms of lower weight} \]

where “symmetric terms” denotes the generic term \( y_{i_1}^{k_1} \cdots y_{i_m}^{k_m} \), with \( (i_1, \ldots, i_m) \) is a permutation of the \( m \) integers \( 1, \ldots, m \). Then, alternatively, for a fixed permutation \( (i_1, \ldots, i_m) \),

\[ C_\kappa(Y) = d_\kappa y_{i_1}^{k_1} \cdots y_{i_m}^{k_m} + \text{terms of lower weight} \]

or in particular

\[ C_\kappa(Y) = d_\kappa y_1^k \cdots y_m^k + \text{terms of lower weight} \]

ii) Finally, denoting \( s_{\kappa,\kappa'} \cdot d_\kappa \) by \( d_\kappa \) and observing that it denotes the constant of “\( z_1^{k_1} \cdots z_m^{k_m} + \cdots + \text{symmetric terms} \)” of \( C_\kappa(A^{-1}) (C_\kappa(Y^{-1}Z)) \) in terms of the latent roots of \( A \ (YZ^{-1}) \).

**Remark 3.1.** Also, observe that the “Hint” in problem 7.5 in Muirhead [1982] is also incorrect.

An application of Lemma 3.1, but in its wrong version, is given by Muirhead [1982, Theorem 7.2.13], surprisingly, the correct result is obtained. The following results were proposed, without proof, by Constantine [1966] and simultaneously, with an alternative proof to that given below, by Khatri [1966] and Takehura [1984, Lemma 1, p. 53].

**Theorem 3.1.** Let \( Z \) be a complex symmetric \( m \times m \) matrix with \( \text{Re}(Z) > 0 \). Then

\[
\int_{X>0} \text{etr}(-XZ)(\det X)^{a-(m+1)/2}C_\kappa(X^{-1}) \, (dX) = \frac{(-1)^k \Gamma_m[a]}{(-a + (m + 1)/2)_\kappa} (\det Z)^{-a} C_\kappa(Z) \]

\[
= \frac{\Gamma_m[a]}{(-a + (m + 1)/2)_\kappa} (\det Z)^{-a} C_\kappa(-Z) \]

for \( \text{Re}(a) > k_1 + (m - 1)/2 \), where \( \kappa = (k_1, \ldots, k_m) \) and \( k = k_1 + \cdots + k_m \).

**Proof.** First suppose that \( Z > 0 \) is real. Let \( f(Z) \) denote the integral on the left side of (8) and make the change of variable \( X = Z^{-1/2}YZ^{-1/2} \), with Jacobian \( (dX) = (\det Z)^{-(m+1)/2}(dY) \), to give

\[
f(Z) = \int_{Y>0} \text{etr}(-Y)(\det Y)^{a-(m+1)/2}C_\kappa \ (Y^{-1}Z) \ (dY)(\det Z)^{-a}.\]

Then, exactly as in the proof of Theorem 7.2.7 in Muirhead [1982, p. 256-257]

\[
f(Z) = \frac{f(I_m)}{C_\kappa(I_m)} C_\kappa(Z)(\det Z)^{-a}.\]

Assuming without loss of generality that \( Z = \text{diag}(z_1, \ldots, z_m) \), it then follows, using (i) from Definition 7.2.1 in Muirhead [1982], that

\[
f(Z) = \frac{f(I_m)}{C_\kappa(I_m)} (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} + \text{terms of lower weight}.\]
On the other hand, using the result of Lemma 3.1 in (9) gives

\[ f(Z) = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \int_{Y>0} \text{etr}(-Y) (\det Y)^{a-(m+1)/2} \times f(Y) \]

\[ \times y_{11}^{-(k_m-k_{m-1})} \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}^{-(k_{m-1}-k_{m-2})} \cdots \det Y^{-k_1} (dY) \]

+ terms of lower weight.

To evaluate this last integral, set \( Y = T^T \) where \( T \) is upper-triangular with positive diagonal elements;

\[ \text{tr} Y = \sum_{i<j} t_{ij}^2, \quad y_{11} = t_{ii}^2 \det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = t_{11}^2 t_{22}^2 \quad \cdots \det Y = \prod_{i=1}^m t_{ii}^2, \]

and, from Theorem 2.1.9 in Muirhead (1982, p. 60)

\[ (dY) = 2^m \prod_{i=1}^m t_{ii}^{m+1-i} \wedge dt_{ij} = \prod_{i=1}^m (t_{ii}^2)^{(m-1)/2} \wedge dt_{ii}^2 \wedge dt_{ij}. \]

Hence

\[ f(Z) = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \prod_{i<j} \left( \int_{-\infty}^{\infty} \exp (-t_{ij}^2) dt_{ij} \right) \]

\[ \times \prod_{i=1}^m \left( \int_{0}^{\infty} \exp (-t_{ii}^2) (t_{ii}^2)^{-k_m+1-i-(i-1)/2} dt_{ii}^2 \right) + \cdots \]

\[ = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} m(m-1)/4 \prod_{i=1}^m \Gamma[a-k_{m+1-i}-(i-1)/2] + \cdots \]

by (7), we have

\[ f(Z) = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} m(m-1)/4 \prod_{i=1}^m \Gamma[a-k_i-(m-i)/2] + \cdots \]

which is the result obtained by Khatri (1966, eq. (12)), see also Takemura (1984, Lemma 1, p. 53). Finally, by (2) and (6)

\[ f(Z) = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} m(m-1)/4 \prod_{i=1}^m \frac{(-1)^{k_i} \Gamma[a-(m-i)/2]}{(-a+(m-i)/2+1)k_i} + \cdots \]

\[ = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^{k_i} \Gamma[a-(m-i)/2]}{\prod_{i=1}^m (-a+(m-i)/2+1)k_i} + \cdots \]

\[ = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^{k_i} \Gamma_m[a]}{(-a+(m+1)/2-(i-1)/2)k_i} + \cdots \]

\[ = (\det Z)^{-a} d_\kappa z_1^{k_1} \cdots z_m^{k_m} \frac{(-1)^{k_i} \Gamma_m[a]}{(-a+(m+1)/2)k_i} + \cdots \]

(11)
where $k = k_1 + \cdots + k_m$ and $(b)_\kappa = \prod_{i=1}^m (b - (i - 1)/2)_{k_i}$. Equating coefficients of $z_1^{k_1} \cdots z_m^{k_m}$ in (10) and (11) it follows that
\[
\frac{f(I_m)}{C_\kappa(I_m)} = \frac{(-1)^k \Gamma_m[a]}{(-a + (m + 1)/2)_\kappa}.
\]
Hence we obtain the desired result for real $Z > 0$, and it follows for complex $Z$ with Re($Z$) > 0 by analytic continuation and recalling that $(-1)^k C_\kappa(A) = C_\kappa(-A)$. \hfill \-box

Remark 3.2. Observe that Muirhead (1982, penultimate line, p.257) obtains
\[
\int X > 0 \etr(-XV) (\det X)^{(a-(m+1)/2)} C_\kappa (TX^{-1}) (dX) = \prod_{i=1}^m (-a + (i + 1)/2)_{k_i} \Gamma_m[a]
\]

Corollary 3.1. Let $V$ be a complex symmetric $m \times m$ matrix with Re($V$) > 0, and let $T$ be an arbitrary complex symmetric matrix. Then
\[
\int_{X > 0} \etr(-XV) (\det X)^{a-(m+1)/2} C_\kappa (TX^{-1}) (dX) = \frac{(-1)^k \Gamma_m[a]}{(-a + (m + 1)/2)_\kappa} (\det V)^{-a} C_\kappa(VT)
\]
\[
= \frac{\Gamma_m[a]}{(-a + (m + 1)/2)_\kappa} (\det V)^{-a} C_\kappa(-VT)
\]
for Re($a$) > $k_1 + (m - 1)/2$, where $\kappa = (k_1, \ldots, k_m)$ and $k = k_1 + \cdots + k_m$.

Proof. Observe that if $V = I_m$ in (12) we obtain (9). For the general case substitute $V^{1/2} X V^{1/2}$ for $X$ in (9) with the Jacobian of the transformation $|V|^{(m+1)/2}$.

Conclusions

Let us stress that the aim of the present study is not to disparage the importance of Muirhead’s book, but rather to correct the minimal deficiencies we believe to have identified, and thus help prevent, or minimize, erroneous conclusions being drawn on the basis of this text, in both current and future work.
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