On the history of ring geometry
(with a thematical overview of literature)

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Abstract

In this survey paper we give an historical and at the same time thematical overview of the development of “ring geometry” from its origin to the current state of the art. A comprehensive up-to-date list of literature is added with articles that treat ring geometry within the scope of incidence geometry.

In questo documento di ricerca forniamo una panoramica storica e allo stesso tempo tematica dello sviluppo della “geometria sopra un anello” dalla sua origine allo stato attuale. È aggiunto una lista di letteratura aggiornata completa di articoli che trattano la geometria degli anelli nel contesto della geometria dell’incidenza.

In diesem Forschungsartikel geben wir einen historischen und gleichzeitig thematischen Überblick über die Entwicklung der “Ringgeometrie” von ihrem Ursprung bis zum aktuellen Stand der Technik, mit einer Liste aktualisierter Literatur einschließlich Artikeln zur Ringgeometrie im Kontext der Inzidenzgeometrie.

Dans ce document de recherche, nous fournissons un aperçu historique et à la fois thématique du développement de la “géométrie sur un anneau”, de son origine à l’état actuel des connaissances. Nous ajoutons une liste de la littérature actualisée comprenant des articles traitant la géométrie sur un anneau dans le contexte de la géométrie de l’incidence.

Keywords: Ring geometry, projective ring plane, Hjelmslev geometry, Klingenberg geometry, Barbilian plane, neighbor relation, bibliography

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1 Introduction

The current version of the Mathematics Subject Classification (MSC), a classification scheme used by the two major mathematical reviewing databases, Mathematical Reviews (MathSciNet) and Zentralblatt MATH (zbMATH), provides code 51C05 for all research papers dealing with Ring Geometry. This rather small category contains articles about geometries that are not only provided with an incidence relation but also with a neighbor relation (or its negation, a non–neighbor or remoteness relation). Included are all geometries obtained from rings that are not division rings. Ring geometry, i.e. the theory of geometries equipped with a neighbor relation in general and of geometries over rings in particular, is a rather young discipline. Its origin lies in the beginning of the 20th century and its importance has grown steadily. For that reason it has also got a full–fledged place as a chapter in the “Handbook of Incidence Geometry” [A24]. In the past decades several mathematicians have contributed to ring geometry. Newly discovered connections with coding theory, with the theory of Tits–buildings and with quantum information theory, have opened new horizons. In this survey paper we present an historical and thematical overview with attention to many aspects. An out-of-date list with articles on the subject, up to 1989, was composed by Törner and Veldkamp in [A23] as an addition and completion of two even older lists [A02] and [A12] written by respectively Artmann, Drake et al. and Jungnickel. Up to now such list of literature is not available for the period after 1990, except for a survey paper by the author [A13], dealing exclusively with plane projective geometries over finite rings. In the present work we fill that gap and we add a new updated list of existing literature, ordered thematically (and containing the relevant material from the preceding lists). Articles on algebraic geometry and on differential geometry over rings are not included. We also do not aim for completeness when it concerns metric aspects, neither for geometries on modular lattices nor for geometric algebra over rings. We think that the bibliography might be useful for researchers who want to attack the future challenges of ring geometry.

2 The first traces of ring geometry: dual numbers and Johannes Hjelmslev

The first traces of ring geometry date back to the beginning of the twentieth century. The Danish geometer Johannes Trolle Hjelmslev (1873–1950) who was born as Johannes Petersen but who changed his name in 1904, graduated in mathematics from the University of Copenhagen and received his PhD degree in 1897. Hjelmslev can be viewed as one of the early founders of ring geometry. In a series of four lectures, held at the University of Hamburg in July 1922 and published one year later in the Abhandlungen aus dem Mathematischen Seminar [A10], he presented an axiomatic framework for geometry that better reflected the properties observed in the real world. The basic observation made by Hjelmslev was that, if one draws lines “close” to each other (meaning that the
sharp angle they define is very small), then it is hard to identify the intersection point, and it looks as if the lines have a little segment in common. Dually, if two points are close to each other, they belong to a line segment that can be part of several joining lines. Hjelmslev called this Die natürliche Geometrie, the “natural geometry”. In fact, Hjelmslev put forward this idea already some years earlier, in 1916, in an article [A09] on what he called Die Geometrie der Wirklichkeit, the “geometry of reality”.

In order to obtain a model for his geometry, Hjelmslev made use of the ring of dual numbers of the form $a + b\varepsilon$ with $a$ and $b$ both real and $\varepsilon^2 = 0$. Dual numbers were already well-known before Hjelmslev. William Clifford defined them for the first time in 1873 [A06] and they were used as a convenient tool in mechanics by Aleksander Kotelnikov [A18] and by Eduard Study in his famous work “Geometrie der Dynamen” [A22].

According to Benz [A05] the first traces of ring geometry can be observed already in Study’s work and in that of some of his contemporaries like Josef GRÜNWALD, Pilo PREDELLA and Corrado SEGRE. In their papers dual numbers are treated from a geometrical viewpoint, by connecting the geometry of oriented lines (spears) in the real euclidean space with a spherical geometry over the ring of dual numbers [A08, A19, A21].

3 The pioneers of ring geometry: Baribilian and Klingenberg

Twenty years after Hjelmslev, Dan BARBILIAN (1895–1961), besides a mathematician at the University of Bucharest, also one of the greatest Romanian poets (with pseudonym Ion Barbu), took up the line again. In two papers [A03, A04] (extended versions of a lecture hold in Baden–Baden in 1940) he gave the first axiomatic foundation for plane projective ring geometry. He started by investigating the conditions which must be imposed upon an arbitrary associative ring in order that the corresponding geometry may have “useful” properties. It turns out that the rings must have a unit element and that all left singular elements must be two-sided singular or equivalently $ab = 1$ implies $ba = 1$. Baribilian called these rings “Zweiseitig singuläre Ringe” or Z–rings. Today such rings are also known as Dedekind–finite rings. They include of course all commutative rings but also all finite rings (even non–commutative) and many other classes of rings (e.g. matrix rings over a field). Starting with C–rings (“Kategorische Ringe”), being Z–rings with some additional property, Baribilian defined a kind of projective plane. Conversely, he formulated a set of axioms for a plane geometry with incidence and non–neighborship (“klare Lage”) and proved that this leads to a C–ring. Baribilian’s work showed some shortcomings, but nevertheless it was of great importance for the development of geometry over arbitrary rings as we will discuss further (see also section 11).

Another monument among the ring geometers was the German mathematician Wilhelm KLINGENBERG (1924–2010). He is best known for his work on diffe-
ential geometry, but in a series of papers [A14 A15 A16 A17], published in the mid-fifties, he also laid the foundation for affine, metric (euclidean) and projective geometries with neighbor relation ("mit Nachbarelementen"). His central idea is the existence of a natural map from the geometry under consideration onto an “underlying” ordinary geometry, a consequence of the assumption that the neighbor relation is transitive. Therefore Klingenberg called such geometries “Geometrien mit Homomorphismus”. He not only defined them axiomatically, but he also constructed explicit examples using rings. The assumption of a transitive neighbor relation is interlaced with the fact that the rings must be local ones. A not necessarily commutative local ring $R$ is a ring with a unique maximal left (or equivalently a unique maximal right) ideal $R_0$. The quotient ring $R/R_0$ is a (skew)field, coordinatizing an ordinary geometry which is the natural epimorphic image of the ring geometry over $R$.

The work of Klingenberg was the source of two mainstreams in ring geometry in the decades afterwards: Klingenberg geometry and Hjelmslev geometry. Geometries with a transitive neighbor relation or equivalently with a canonical map onto an ordinary geometry are now named Klingenberg geometries. Their epimorphic image is obtained by considering the equivalence classes of neighboring elements as “thick” elements. Non–neighboring elements behave like distinct elements in the epimorphic image (e.g. two non–neighboring points are incident with exactly one line in a projective Klingenberg plane since two distinct points in its epimorphic projective plane are incident with a unique line). No further assumptions are made about the neighboring elements.

Klingenberg geometries comprise the smaller, but more intensively studied class of Hjelmslev geometries for which additional axioms must hold when it comes to neighboring elements (e.g. two neighboring points can be connected by at least two distinct lines in a projective Hjelmslev plane). This idea relies on the natural geometry of Hjelmslev (see section 2). In the case of a Klingenberg plane over a local ring $R$, the Hjelmslev condition means that the ring must be a two–sided chain ring (a ring is a right chain ring if for any $a$ and $b$ in $R$ either $a \in bR$ or $b \in aR$, similar for left chain ring) with the additional property that its maximal ideal contains only zero divisors. Such rings are called PH–rings (projective Hjelmslev rings or $H$–rings). Finite chain rings are always $H$–rings and, equivalently, local principal ideal rings. Klingenberg himself considered both Klingenberg planes (over local rings) and Hjelmslev planes (over Hjelmslev rings) in [A16].

To be complete, we must mention also two isolated, rather unnoticed, contributions to early geometry over rings. J.W. ARCHBOLD [A01] wrote on projective geometry over group algebras $K[G]$ and worked out a small finite example taking $K = GF(2)$ and $G$ a group of order 2, while Rimhak REE in [A20] considered projective spaces as the modular lattices $L(R, M)$ of all sub–modules of a module $M$ over a ring $R$, in particular over a full matrix ring.
4 The Belgian contribution to early ring geometry: projective geometry over rings of matrices

The ideas of Barbilian concerning projective geometries over rings instead of fields, got some followers in the sixties and early seventies. Among them we find a lot of Belgian contributors who studied projective geometries over rings of matrices. Since such rings are not local, the geometries are not Klingenberg nor Hjelmslev geometries.

Julien DEPUNT studied in [B01] the projective line and in [B03] [B04] projective planes over the ring of ternions. This ring contains the elements $a_1\varepsilon_1 + a_2\varepsilon_2 + a_3\varepsilon_3$ with $a_i \in \mathbb{C}$ and $\varepsilon_i^2 = \varepsilon_i$ ($i = 1, 2$), $\varepsilon_1\varepsilon_2 = \varepsilon_2\varepsilon_3 = \varepsilon_3\varepsilon_1$ and all other products equal to 0. The ring of ternions is isomorphic to the ring of upper triangular matrices with elements in the complex field. The main result of Depunt concerns the embedding of the plane over the ternions into the 5–dimensional complex projective space.

Inspired by this work, Cléry VANHELLEPUTTE could generalize the results for planes over the full matrix ring of $2 \times 2$–matrices with elements in an arbitrary commutative field. His doctoral thesis was published in [B14] and fitted in the research program under Julien BİLO (1914–2006) who was head of the Geometry Department at the University of Ghent at that time.

A few years later, in 1969, Joseph THAS wrote his PhD thesis under the supervision of BİLO about the projective line over the ring of $3 \times 3$–matrices with elements in an algebraically closed field. His doctoral thesis was published in [B11], see also [B10]. Soon afterwards Thas published [B12] [B13] on projective ring geometry over matrix rings, in particular over finite rings $M_n(GF(q))$ of $n \times n$–matrices with entries in a Galois field of order $q$. Especially [B12] was important as it contains some concepts (ovals, arcs, caps, etc.) and combinatorial theorems which extend known results from classical Galois geometry over finite fields and which appear for the first time for ring geometries. Later Thas grew out to an authority in finite geometry, but his interest shifted from finite ring geometry to Galois geometry and other topics, in particular finite generalized quadrangles.

Thas was a colleague of Willem MIELANTS during many years. It was he who guided Hendrik VAN MALDEGHEM (see section 14) and the author towards ring geometry as PhD supervisor for both of us.

Paul DE WINNE was the last student of BİLO studying ring geometry, see [B05]. His doctoral thesis was published in [B06].

Around the same time, Xavier HUBAUT, another Belgian mathematician from the Université Libre de Bruxelles, introduced the projective line over a general associative algebra and investigated the structure of its group of projectivities in [B08] [B09]. His colleague Franz BINGEN could generalize the results to $n$–dimensional projective spaces over semi–primary rings in [B02].

Mind that the geometries over matrix rings studied by all these people, may not be confused with the “geometries of matrices” initiated by the Chinese mathematician Loo–Keng HUA in the mid–forties [B07]. Hua’s geometries of matrices are not ring geometries in the narrow sense, despite the suggestive
name (although a connection is possible with projective lines over rings, see section 13).

5 The foundations of plane affine ring geometry: from Benz to Leissner and beyond

Barbilian, as well as all members of the Belgian School, dealt exclusively with projective ring geometry. The first traces of affine geometries over rings can be found in a paper from 1942 by Cornelius Everett, concerning affine planes over rings without zero divisors \([C16]\).

Walter Benz (1931–2017), a renowned German geometer, especially for his work on circle geometries, considered in \([C04, C05]\) affine planes over a special kind of commutative rings and in \([C06]\) plane affine (and metric–affine) geometries over arbitrary rings with unit. A big part of this interesting paper discusses the relation between algebraic properties of the ring and geometric properties of the plane.

Peter Ashley Lawrence obtained a PhD under the supervision of Benz. His thesis “Affine mappings in the geometries of algebras” considers ring geometries associated with modules and algebras over a commutative ring and was published in \([C21]\). Two other publications \([C02, C03]\) by Hans–Joachim Arnold can be classified within this section. They contain a simultaneous generalization of affine geometry over rings and generalized affine spaces in the sense of Emanuel Sperner \([C41]\). Arnold starts with the axiomatic definition of an affine–line geometry which can be coordinatized by a vectorial groupoid and he adds some extra axioms to turn it into a module over a unitary ring. Also \([C18, C34]\) both fit into this approach.

Affine planes over local rings, today also known as desarguesian affine Klingenberg planes, appear as particular examples in the papers of Benz, but this was not for the first time. Klingenberg already considered them on the side of projective ring planes and in the more general context of “Affine Ebenen mit Nachbarelementen” in the articles \([A14, A15, A16, A17]\).

A reasonable part of the research concerning affine ring geometry focuses on affine Hjelmslev planes (AH–planes). An important paper \([C28]\) in that respect was written by Heinz Lüneburg (1935–2009) in 1962. It contains the main results of his doctoral thesis entitled “Affine Hjelmslev–Ebenen mit transitiver Translationsgruppe”, dealing with translation AH–planes as generalizations of ordinary translation planes. Werner Seier investigated in more detail desarguesian and translation AH–planes in \([C36, C37, C38, C39, C40]\). The central theme in his work is a characterization of desarguesian AH–planes by an affine variant of Desargues’ theorem and the transitivity of the set of translations in the plane. Some other papers on affine Hjelmslev planes are not mentioned here and we postpone them to sections 7 and 8 where they will be discussed in connection with projective Hjelmslev planes.

Without minimizing the importance of the papers cited above, we can state
that the real breakthrough of affine ring geometry in full generality was not 
achieved yet at this point. Therefore one has to wait for Werner Leissner, 
a student of Benz. Leissner wrote two papers in the mid–seventies on, what 
he called, “affine Barbilian planes” (today the name “Leissner planes” 
would be more convenient). In \( C^{22} \) such planes are defined axiomatically as affine 
structures in which two parallelodromy axioms hold (suitable substitutes for the 
affine Desargues theorem) and they are coordinatized by an arbitrary \( \mathbb{Z} \)–ring 
\( R \) together with a special subset \( B \) of \( R \times R \) (a Barbilian domain). In \( C^{23} \) 
the converse is proved: any affine ring plane over a \( \mathbb{Z} \)–ring is an affine Barbilian 
plane. Leissner considered even more general affine Barbilian structures in \( C^{24} \). 
The papers of Leissner were a source of inspiration for further generalizations by 
several people. Victoria Groze and Angela Vasiu considered affine structures 
over arbitrary rings in \( C^{17} \), Francisc Radó defined affine Barbilian structures 
in \( C^{33} \) by weakening Leissner’s axioms, Armentrout et al. investigated in 
\( C^{31} \) generalized affine planes which can be coordinatized by near-rings and 
Pickert studied tactical configurations over quasirings in \( C^{32} \). 

Other contributors to the theory of plane affine geometries with neighbor 
relation and parallelism were Gernot Dorn \( C^{13} \), Kvetoslav Burian \( C^{08} \) 
\( C^{09} \) \( C^{10} \), Frantisek Machala \( C^{29} \) \( C^{30} \) \( C^{31} \), Stefan Schmidt and Ralph 
Steinitz \( C^{35} \), Franco Eugeni \( C^{14} \) \( C^{15} \) and Angela Vasiu \( C^{42} \) \( C^{43} \) \( C^{44} \). 
The Barbilian domains defined by Leissner also formed a study object in them- 
selves. Several authors have contributed to this, see \( C^{07} \) \( C^{20} \) \( C^{25} \) \( C^{26} \) \( C^{27} \). 

For a recent and fairly complete overview of affine planes over finite 
rings we refer to the survey paper \( C^{19} \) of the author. We also mention here two papers 
\( C^{11} \) \( C^{12} \) by Basra Çelik on finite hyperbolic Klingenberg planes as they are 
not far away from finite affine Klingenberg planes.

6 Metric geometry over rings and the school of 
Bachmann

The goal of this section is just to give a glimpse of metric ring geometry. We 
do not aim for completeness when it concerns metric aspects, because they are 
only indirectly related to incidence geometry. The literature about this subject 
is extensive and we have selected only a few representative papers in which 
much more references can be found. The basis of “modern” metric geometry is 
provided by the work of several German mathematicians, including Bachmann, 
Lingenberg and Schröder.

The pivotal figure is Friedrich Bachmann (1909–1982). In the second edition 
of his famous book “Aufbau der Geometrie aus dem Spiegelungsbegriff” \( D^{01} \) 
he considers plane metric geometries in which points, lines, incidence and 
orthogonality are defined by means of a group of reflections. A group \( G \) with a 
subset \( S \) of involutory elements which are invariant under inner automorphisms 
of \( G \) and which generate \( G \), determines a group plane \( E = E(G, S) \) as follows: 
the elements of \( S \) are the lines of \( E \) and the involutory elements of \( S^2 \) are the
points. Two lines that commute are perpendicular. A point and a line that commute are incident. If the group plane $E$ satisfies the conditions (1) a point and a line determine a unique perpendicular, and (2) the product of three lines that are either concurrent or have a common perpendicular is again a line, then the pair $(G, S)$ is called a Hjelmslev group and the associated group plane $E$ is called a metric (non–elliptic) Hjelmslev plane.

If the uniqueness of the line incident with two distinct points is not required, then metric Hjelmslev planes stripped of their metric structure, reduce to incidence Hjelmslev planes. In section 2 we have already discussed the role of Hjelmslev as founder of (incidence) ring geometry. In some later work, the “Algemeine Kongruenzlehre” $[D05]$, Hjelmslev used special transformations (reflections) to define orthogonality. Hence, the rudiments of metric ring geometry also go back to Hjelmslev and have been worked out in more detail by Bachmann. There is also a minor contribution by Klingenberg who treats metric aspects in ring geometries in $[A15]$.

Rolf Lingenberg (1929–1978), a student of Bachmann, continues his work in $[D12]$. The main result is an algebraic characterization of classical metric group planes $E(G, S)$ with an additional axiom, as planes associated with a metric vectorspace $(V, q)$ with $q$ a quadratic form.

Eberhard Schröder in $[D18] [D19]$ considers metric planes of different kind starting from a pappian affine plane over a field and using two-dimensional algebras for the introduction of the metric notions of angle, distance and orthogonality. His work is strongly related with circle geometries studied intensively by Benz (see section 13). A good overview of classical metric geometry (but not including ring geometries) can be found in Chapter 17 written by Schröder in the Handbook of Incidence Geometry $[D20]$.

There are numerous contributions by Bachmann and his school to the theory of Hjelmslev groups, connecting geometric and algebraic properties of groups and planes. From Bachmann himself we mention the important additions $[D02] [D03] [D04]$ to his book $[D01]$. In the seventies and eighties his successors, most of them from the University of Kiel (Germany), have extended the knowledge. Finite Hjelmslev groups were characterized by Rolf Stöltig in his PhD thesis “Endliche Hjelmslev-Gruppen und Erweiterungen von Hjelmslev-Gruppen”, also published in $[D21]$. In $[D22]$ Hjelmslev groups are constructed from a module $M$ over a commutative ring $R$ endowed with a bilinear form.

Edzard Salow introduced singular Hjelmslev groups (in which the product of three points is always a point) in his doctoral thesis “Beiträge zur Theorie der Hjelmslev-Gruppen: Homomorphismen und Singuläre Hjelmslev-Gruppen” published in $[D15]$. The main result is the construction of a coordinate ring $R$ for the group plane of a singular Hjelmslev group, proving that these metric planes are indeed ring geometries. The process of algebraization of metric Hjelmslev planes is investigated also in $[D16]$. It is proved there that a Hjelmslev group with some additional axioms can be embedded in the orthogonal group of a metric module $(R^3, f)$ with $R$ a commutative ring with unit for which 2 and any non–zero–divisor is invertible and with $f$ a symmetric bilinear form of the free module $R^3$. 


In \[D17\] Salow studies another class of metric ring planes using a commutative algebra over a ring and the concept of an angle. In an early paper of Benz \[C04\] the metric notion of angle also played an important role and in \[C06\] he devotes a paragraph to metric geometry, using an elliptic form over an arbitrary commutative ring.

Similar work can be found in \[D13, D14\] in which Wolfgang Nolte\[D13\] proves that a class of metric planes \(E(G, S)\) with additional axioms can be embedded into a projective Hjelmslev plane over a local ring and that \(G\) is isomorphic to a subgroup of an orthogonal group. Other results in this direction were obtained by Frieder Knüppel in \[D07, D08, D09, D10, D27\], Gerald Fischbach \[D05\], Michael Kunze \[D11\] and Rolf and Horst Struve \[D23, D24, D25, D26\].

The influence of Bachmann is also clear from the large amount of doctoral theses produced in Germany on Hjelmslev groups and metric geometry, e.g. R. Schnabel (1974), M. Kunze (1975), H. Struve (1979), R. Struve (1979), M. Gegenwart (1987), W. Vonck (1988) and A. Bach (1998).

7 The florescence of Hjelmslev geometry in the era of Drake, Artmann and Törner

The first period of florescence of ring geometry (especially Hjelmslev geometry) regarded as incidence geometry, started in the late sixties and reached its culmi-
nation point in the seventies. This is reflected in a large number of publications. Two mathematicians who were very productive in this area and left their mark, were David Allyn Drake (1937–) from the University of Florida, Gainesville (USA) and Benno Artmann (1933–2010) from the University of Giessen (Germany).

Drake obtained his PhD in 1967 with a thesis entitled “Neighborhood collineations and affine projective extensions of Hjelmslev planes” under the supervision of Erwin Kleinfeld. Kleinfeld himself, an authority in algebra, with a lot of publications on non–associative alternative rings, published only one, though interesting paper \[E25\] about ring geometry. It was the first publication concerning finite Hjelmslev planes. Among other things he introduced a two parameter set \((s, t)\) of non–zero integers such that for each flag \((P, \ell)\) in a finite projective Hjelmslev plane, there are exactly \(t\) points on \(\ell\) neighboring with \(P\) and exactly \(s\) points on \(\ell\) not neighboring with \(P\). It was proved that \(s \leq t^2\) or \(t = 1\). If \(s = t^2\), the plane is called uniform. In that case all point neighborhoods have the structure of ordinary affine planes. Robert Craig proved in \[E11\] that any finite projective plane can be extended to a uniform projective Hjelmslev plane. The notion of uniformity (and its generalization to \(n\)–uniformity) has played a crucial role in the work of Drake. It was also related to another issue: the extension of an affine Hjelmslev plane to a projective Hjelmslev plane. An ordinary affine plane can always be extended to a projective plane, but for Hjelmslev planes the situation is much more complicated. Drake has written several papers about this problem in the period from 1968 to 1975. In \[E14\] he proves that any
uniform affine Hjelmslev plane has at least one (uniform) projective extension and he gives an example of a desarguesian uniform affine Hjelmslev plane with a non–desarguesian projective extension. In \[E15\] uniformity is generalized to \(n\)-uniformity inductively (a PH–plane is \(n\)-uniform if the point neighborhoods are \((n - 1)\)-uniform AH–planes). Strongly \(n\)-uniform planes are characterized by a local property, which leads to the theorem: an \(n\)-uniform PH–plane is strongly \(n\)-uniform if and only if its dual is \(n\)-uniform. Drake also proved that every finite desarguesian PH–plane is strongly \(n\)-uniform.

A further study of \(n\)-uniform Hjelmslev planes (projective and affine) was made in \[E16, E17, E18, E19, E20, E21, E22\] where even more general affine geometries with neighbor relation appear. Drake could also prove that there do exist affine Hjelmslev planes which cannot be extended to projective Hjelmslev planes.

Artmann was a contemporary of Drake and he wrote his doctoral thesis “Automorphismen und Koordinaten bei ebenen Verbänden” in 1965 under the supervision of Günther Pickert. In his early work on Hjelmslev geometry we can observe a strong relation with the theory of modular lattices. In \[E01\] he gives a sufficient condition for a modular lattice to define a projective Hjelmslev plane and in \[E04\] he proves that any uniform PH–plane can be derived from a modular lattice.

Like Drake, Artmann also studies refinements of the neighbor relation (“verfeinerten Nachbarschaftsrelationen”) and the affine–projective extension question. In \[E04\] he proves that a uniform affine Hjelmslev plane can be extended to at least two non–isomorphic projective Hjelmslev planes. A new concept introduced by him in \[E02\] is that of a projective Hjelmslev plane of level \(n\) (“\(n\)–ter Stufe”) based on the refinement of the neighbor relation. This definition was extended by Drake to the affine case in \[E20\]. Artmann proves that desarguesian PH–planes over a Hjelmslev ring \(R\) are of level \(n\) if and only if the maximal ideal of \(R\) is nilpotent of index \(n\), see \[E05, E06\].

Another theorem proved by Artmann \[E07\] states that for any projective plane \(P\) and any integer \(n > 0\) there exists a PH–plane of level \(n\) with \(P\) as epimorphic image. Moreover, given a sequence of PH–planes \(\ldots \rightarrow H_{i-1} \rightarrow H_i \rightarrow \ldots \rightarrow H_1 = P\) with \(H_i\) of level \(i\), the inverse limit is a projective plane. Arno CRONHEIM constructed in \[E12\] in a purely algebraic way, using formal power series over a cartesian group, a chain of Hjelmslev planes whose inverse limit is a projective plane. Cronheim also obtained a complete classification of all finite uniform desarguesian projective Hjelmslev planes in \[E13\]. They are either planes over a ring of twisted dual numbers over GF\((q)\) (a non–commutative generalization of the classical dual numbers) or over a truncated Witt ring \(W_2(q)\) of length 2.

Both Drake and Artmann had a great influence on the mathematical research in the domain of Hjelmslev geometry (even when Artmann’s interest shifted to other subjects after a few years). One of Drake’s students was Phyrne BACON. She wrote both her Master’s thesis “On Hjelmslev planes with small invariants” and her PhD thesis “Coordinatized Hjelmslev planes” on Hjelmslev geometry, resulting in two papers \[E08, E09\]. She proved that a finite Hjelmslev plane is strongly \(n\)–uniform if and only if it is of level \(n\) which unifies the two notions
introduced by Drake and Artmann respectively. Later, Bacon’s attention shifted to the more general Klingenberg geometries (see section 9).

Artmann was the supervisor of Manfred Dugas and Günther Törner who both made important contributions to Hjelmslev geometry. The PhD thesis of Dugas “Charakterisierungen endlicher desarguescher uniformer Hjelmslev-Ebenen” contains many new ideas, including a coordinatization method (see section 9). In [E23] Dugas proves that a finite translation AH–plane can be extended to a PH–plane if it is or a desarguesian PH–plane or an ordinary translation projective plane, while in [E24] he gives a necessary and sufficient condition for a projective Hjelmslev plane to be derivable from a lattice. In particular the PH–planes of level $n$ are always lattice–derivable.

Törner wrote his Master’s thesis on “Hjelmslev–Ringe und die Geometrie der Nachbarschaftsbereiche in der zugehörigen Ebenen” and obtained his PhD with “Eine Klassifizierung von Hjelmslev-Ringen und Hjelmslev-Ebenen” in the same year as Dugas, under the supervision of Artmann and Pickert. His research in the domain of ring geometry focusses on two main themes: the structure of (finite) Hjelmslev planes and the ideal structure of chain rings. Among his publications we mention here [E36] which contains the main results from his thesis: a classification of PH–planes based on congruence relations. He proves that the set of all congruence relations of a finite PH–plane is linearly ordered under inclusion and consequently, the canonical epimorphism onto the associated projective plane admits an essentially unique factorization into indecomposable epimorphisms. The plane $\mathcal{H}$ is of “type $n$” or “height $n$” if the canonical epimorphism $\varphi$ from $\mathcal{H}$ onto the projective plane $\mathcal{P}$ has a maximal factorization $\mathcal{H} = \mathcal{H}_n \rightarrow \mathcal{H}_{n-1} \rightarrow \ldots \rightarrow \mathcal{H}_1 = \mathcal{P}$. In [E40] Törner investigates the equivalence of finite $n$–uniform planes or planes of level $n$ as defined by Drake and Artmann to planes of type $n$. In [E39] he proves that $n$–uniform projective Hjelmslev planes are strongly $n$–uniform. Some of the results were later extended to the infinite case in [E43]. In [E40] much attention goes also to affine Hjelmslev planes, in particular translation AH–planes over near–rings. In Törner’s work homomorphisms play an important role as can also be seen from [E37] [E38].

In the desarguesian case (Hjelmslev planes over a chain ring) the structure of the plane is intrinsically connected with the ideal structure of the ring. The structure of chain rings and valuation rings was investigated by Törner partly in collaboration with Hans–Heinrich Brungs. One of their papers [E10] concerns the embedding of right chain rings into chain rings (related with the problem of embedding desarguesian affine Hjelmslev planes into projective ones), see also [E32] [E33] [E35] [E42].

With a postdoctoral scholarship Artmann stayed for a short time at the McMaster University in Ontario (Canada). There he inspired J.W. (Mike) Lorimer who would later become one of the leading figures in topological Hjelmslev geometry (see section 10). In [E31] Lorimer and Lane study desarguesian Hjelmslev planes. They prove that an affine Hjelmslev plane is desarguesian if and only if it can be coordinatized by an AH–ring and that not every desarguesian AH–plane can be extended to a desarguesian PH–plane. Morphisms between affine
Hjelmslev planes are the main subject in E26, E27, E30 while E28, E29 deal with the structure of Hjelmslev rings.

8 The continuation of the Hjelmslev epoch under Drake, Jungnickel and Sane

In his publications on $n$–uniform planes, we can observe already that Drake had a particular interest in finite Hjelmslev planes. This is continued when his attention goes more and more to the problem of existence and non–existence of finite Hjelmslev planes with given parameters. In a series of papers F05, F06, F07, F08, F13, F16, F20, some of them with co–author, he attacked this problem and he linked finite PH–planes to nets in F10, F19. Meanwhile, also finite Klingenberg planes came to the attention F09. Drake and Lenz considered a parameter set for finite PK–planes in F15 together with new examples of finite PH–planes. Structure theorems for finite chain rings (needed for finite desarguesian Klingenberg planes) were proved by Edwin Clark et al. in F03, F04 and independently by Arnold Neumaier in F33 and Al–Khamees F01. The classification of all chain rings is still an open problem but partial results are known. Galois rings GR($q^n, p^n$) with $q^n$ elements and characteristic $p^n$, with $q = p^r$, play a crucial role.

Beside Drake another player came to the forefront, Dieter Jungnickel, who was active at the Universities of Giessen and Augsburg (Germany). He was a student of Hanfried Lenz and became an expert in the theory of designs. In 1976 he wrote his Diplomarbeit at the University of Berlin (Germany) on “Klingenberge and Hjelmslev planes” and with the dissertation “Konstruktion transitiver Inzidenzstrukturen mit Differenzenverfahren” he obtained his doctoral degree. His most important contribution to the theory of Hjelmslev planes (and the more general class of Klingenberg planes) concerns “regularity” F14, F23, F24, F26, F29. A PK– or PH–plane is regular if it has an abelian automorphism group $G = Z \oplus N$, where $G$ acts regularly (sharply transitively) on the point set and on the line set and $N$ acts regularly on each neighborhood. It is proved in F21 that any finite PK–plane over a commutative local ring is regular. Regularity is also interpreted in terms of difference sets and auxiliary matrices, leading to new families of finite Hjelmslev and Klingenberg planes. An interesting result, connecting PH–planes with designs, is: the projective uniform Hjelmslev planes of order $q$ (with $q > 2$) are precisely the symmetric divisible partial designs on two classes with parameters $v = b = q^2(q^2 + q + 1), k = r = q(q + 1), s = q^2, t = q^2 + q + 1, \lambda_1 = q, \lambda_2 = 1$. For $q = 2$ counterexamples exist, see F27. The concepts of regularity and uniformity were also considered in $K$–structures, a further generalization of Klingenberg planes (see F09, F11, F12, F22, F25, F28, F30, F31). Nino Civolani F02 considers free extensions of partial Klingenberg planes.

Jungnickel’s work was continued by Sharad Sane. Sane studied at the Indian
Institute of Technology Bombay, Mumbai (India) and obtained his PhD with the dissertation “Studies in Partial Designs and Projective Hjelmslev Planes” under the supervision of Balmohan Vishnu Limaye. In [32], Sane and Limaye demonstrate that \( n \)-uniform PH–planes are a kind of divisible partial designs and by taking advantage of this property, they can give an alternative proof for the fact that \( n \)-uniform planes are strongly \( n \)-uniform, as was proved before in another way by Törner [39]. In some other papers [17, 18, 34, 35, 36, 37, 38] Sane contributes to the theory of finite Hjelmslev and Klingenber

### 9 The coordinatization of Hjelmslev and Klingenberg planes: a versatile story

The coordinatization of affine and projective planes is one of the most powerful tools in the study of such geometries. It permits to reformulate geometric properties into algebraic ones (and vice versa), leading to a better insight, including the construction of many non–desarguesian examples. This coordinatization goes back to Marshall Hall Jr. (1910–1990). His important paper [37] published in 1943 is still one of the most cited. The basic concept is a Hall ternary ring, also called PTR (planar ternary ring), an algebraic structure \((R, T)\) with \( R \) a non–empty set containing two distinct elements 0 and 1 and with \( T \) a ternary operation on \( R \) such that \( y = T(x, m, k) \) means that the point with coordinates \((x, y)\) lies on the line with coordinates \([m, k]\). With \((R, T)\) one can associate two loops \((R, +)\) and \((R, \circ)\) when defining \( a + b := T(a, 1, b) \) and \( a \circ b := T(a, b, 0) \). The properties of the plane (formulated in terms of the validity of Desargues’ configuration or in terms of transitivity of the automorphism group) are reflected in the richness of the coordinatizing algebraic structure. If the theorem of Desargues is always valid or equivalently if the plane is \((P, \ell)\)–transitive for any choice of the point \(P\) and the line \(\ell\), then it turns out that \((R, +)\) and \((R \setminus \{0\}, \circ)\) are both groups, hence \((R, +, \circ)\) is a division ring or skewfield. Conversely, any skewfield gives rise to a desarguesian projective plane. Slight variations on Hall’s coordinatization method were made by Daniel Hughes [38, 39] and by Günter Pickert [55]. Independently the russian mathematician Lev Anatolevich Skornyakov described in 1949 a similar coordinatization method in [58]. His work [59] was important for the distribution of the knowledge on projective planes in the russian speaking mathematical community.

In the seventies and the eighties several attempts were made to coordinatize in a similar way affine and projective Hjelmslev and Klingenberg planes. In 1967 the russian geometer V.K. Cyganova worked out a first successful coordinatization for the more restrictive class of affine Hjelmslev planes [21]. She used the concept of an \( H \)–ternar, an algebraic structure with two ternary operations, generalizing a Hall ternary ring (one of the main differences being the existence of zero divisors). Since her paper was written in russian, it remained unfortunately unaccessible for many people.
Independently from Cyganova, J.W. Lorimer introduced in 1971 in his PhD thesis “Hjelmslev Planes and Topological Hjelmslev Planes” generalized ternary rings (very similar to $H$–ternars) as the coordinatizing structures of affine Hjelmslev planes. Three years later, Phyrne Bacon streamlined the work of Cyganova and Lorimer in her thesis “Coordinatized Hjelmslev planes”, and she introduced the name biternary ring (in appendix A of her thesis she gives a comprehensive list of annotations including some mistakes and imperfections in the work of her predecessors). The interaction between the geometric properties of an affine Hjelmslev plane and the algebraic properties of its coordinatizing biternary ring was examined in more detail by Lorimer in $G46$, by Cyganova in $G20$, $G22$, $G23$, $G24$, $G25$, $G26$, $G27$, by Emelchenkov in $G33$, $G35$, $G36$, and by Shatokhin in $G50$, $G57$.

To be complete, we also have to mention a paper by Drake $G28$ in which he obtains a kind of coordinatization for a special class of affine Hjelmslev planes (radial planes) by means of a module.

After the coordinatization of affine Hjelmslev planes, a similar theory for the more general class of affine Klingenberg planes has been worked out by several authors. Bacon generalized her biternary rings. Drake, her supervisor, encouraged her to publish in mathematical journals, but she was stubborn and, apart from one single publication $G05$, she refused. Her voluminous work, totaling about 1000 pages, is contained in four books $G04$, edited in own management. For that reason it was often overlooked and seldom recognized as an acceptable reference. In $G05$ the “triangle theorem” is proved: a PK–plane $P$ possessing a nondegenerate triangle with sides $\ell_1$, $\ell_2$ and $\ell_3$ such that each derived AK–plane $A_i = P \setminus g_i$ is desarguesian, is itself a desarguesian PK–plane.

The Czech mathematician Frantisek Machala from the University of Olomouc introduced in $G49$ affine local ternary rings as an alternative for the coordinatization of affine Klingenberg planes. Much later he could prove the equivalence between his coordinatization method and the one of Bacon. He also proved that any “incomplete” biternary ring (with one ternary and one partial ternary operator) can be extended to a biternary ring with two (full) ternary operators $G52$. In $G11$ it is shown that this biternary ring extension is unique.

In the case of ordinary planes each (desarguesian) affine plane can be extended to a (desarguesian) projective plane. This does not hold any longer for Hjelmslev planes (see section 7). This observation has also a serious impact on the coordinatization of projective Hjelmslev planes: it does not follow immediately from the affine coordinatization. The projective case was first attacked by the Russian mathematician E.P. Emelchenkov in 1972 in his PhD thesis ”Ternars and automorphisms of Hjelmslev planes” (in Russian) and in $G34$.

Due to the language barrier, his work was not accessible for many researchers and for that reason, like Cyganova’s work, it was somewhat overlooked. Coordinatization methods for the more general case of projective Klingenberg planes were worked out by both Machala and Bacon. Their approach is totally different. Machala’s method is based on the concept of an extended local ternary ring, an algebraic structure $(R, R', T)$ with two disjoint sets of coordinates $R$ and $R'$ and one ternary operation $T$ on $R \cup R'$ (see $G47$, $G48$, $G50$).
This coordinatization was not very successful because it is not obvious to see any interaction between properties of the extended local ternary ring and geometric properties of the coordinatized plane.

The approach of Bacon is based on the fact that a PK–plane can be covered by three AK–planes corresponding to three biternary rings. This yields a coordinatizing structure for a projective Klingenberg plane as a triplet of biternary rings, called sexternary ring \[G04\]. In Bacon’s voluminous work, the interaction between geometric properties and the algebraic structure is examined in depth. Unfortunately, a big part of these results remained hidden for the reason mentioned above. In \[G31\] Manfred Dugas used a similar coordinatizing structure, with six ternary operations.

Independently from the people mentioned above, the author introduced in 1987 in his PhD thesis “Klingenberg incidence structures, a contribution to ring geometry” (in Dutch) (see also \[G41\] \[G42\] \[G43\]) planar sexternary rings (PSR’s) with one full and five partial ternary operators. His coordinatization method for PK–planes was inspired by the Hughes variant of the Hall ternary ring. A small deficiency in his method was detected later (as pointed out by Baker and Lorimer in \[G11\]). As a consequence of this shortcoming, the coordinatization of a PK–plane by a PSR wasn’t fully compatible with the coordinatization of a derived AK–plane by the biternary ring obtained from the PSR. To overcome this anomaly, Baker and Lorimer (op.cit.) developed a new coordinate ring, called incomplete sexternary ring (in the spirit of Dugas) as a substitute for the planar sexternary ring. They even proved that such a structure can be extended (in a unique way) to a sexternary ring with six full ternary operators.

The coordinatization of projective planes is a handy instrument for the construction of non–desarguesian examples in an algebraic manner. A lot of new planes were found using quasifields, nearfields or alternative division rings. Because of the bigger complexity of sexternary rings it seems that much less examples of non–desarguesian PK–planes were obtained in this manner. Nevertheless examples of non–desarguesian AK– and PK–planes obtained from algebraic structures, are known. The oldest examples are the Moult on affine Hjelmslev planes, given by Baker in \[G06\]. A projective version is constricted in \[G43\] by the author. Klingenberg planes over local nearrings and Hjelmslev planes over Hjelmslev–nearrings were studied by Emanuel Kolb in \[G44\] \[G45\]. Much attention has gone to Moufang planes which can be coordinatized by local alternative rings. Moufang–Hjelmslev planes first appear in a paper of Dugas \[G29\]. He proves that all finite uniform Moufang PH–planes are desarguesian. A stronger version of that theorem was first proved in \[G30\] (the uniformity condition could be dropped if the order of the plane is bigger than 2). Similar results were found by Baker, Lane and Lorimer for Moufang Klingenberg planes, see \[G07\] \[G09\] \[G10\]. They prove that the class of Moufang PK–planes coincides with the class of planes over local alternative rings and that a finite Moufang PK–plane in which any two points have at least one joining line, is a desarguesian projective Hjelmslev plane. Also a stronger version of Bacon’s triangle theorem was proved in \[G08\]: a PK–plane with a non–degenerate triangle for which the three derived AK–planes are translation AK–planes (and with
epimorphic image distinct from PG(2,2)), is Moufang. More recently, a group of mathematicians around Basri Çeliş and Sıleymen Çiftçi, from the University of Uludag, Bursa (Turkey), published several papers concerning a particular class of Moufang–Klingenberg planes $G_{01}, G_{02}, G_{14}, G_{13}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}$. Their results, all variations on the same theme, overlap with work of Andrea Blunck $G_{12}, G_{13}$. The role of Pappus’ theorem (its validity in a desarguesian plane implies the commutativity of the coordinatizing ring) was investigated by Nolte and Maurer in $G_{53}, G_{54}$.

10 Order and topology in Hjelmslev geometry: Machala and Lorimer

The theory of ordered incidence structures can be traced back mainly to Pasch, Vorlesungen über neuere Geometrie from 1882. An excellent survey paper on the axiomatics of ordered incidence geometry is $H_{27}$. Ordered affine and projective Hjelmslev planes were studied by a group of Canadian mathematicians, starting in the seventies. At least three dissertations were written in that period at the McMaster University of Hamilton, Ontario (Canada) under supervision of Norman Lane who was also a world-class canoeist, competing in two Olympic games (bronze medal in 1948 in London), before he started his academic career. In 1975, Lynda Ann Thomas, wrote her Master’s thesis on “Ordered desarguesian affine Hjelmslev planes” in which she proved that any ordered AH–ring gives rise to an ordered desarguesian affine Hjelmslev plane and vice versa. This result was published a few years later in $H_{30}$. James Laxton, another student of Lane, treated in his Master’s thesis “Ordered non–desarguesian affine Hjelmslev planes”. Catherine Baker, also a student of Lane, wrote her Master’s thesis on “Affine Hjelmslev and generalized affine Hjelmslev planes” and her doctoral thesis in 1978 on “Ordered Hjelmslev planes”. In that thesis she investigates in detail the relationship between ordered AH–planes and the coordinatizing ordered biternary rings, extending results of Laxton and Thomas. Baker published several papers about ordered (affine and projective) Hjelmslev planes (also with co–author): $H_{01}, H_{02}, H_{03}, H_{04}, H_{05}, H_{06}$. It would be disrespectful if we should attribute all the results on ordered ring geometries to the “Canadian School” only. Independently, a theory of orderings for Klingenberg planes was worked out by Machala. He published many papers on ordered Klingenberg planes $H_{18}, H_{19}, H_{20}, H_{21}, H_{22}, H_{23}, H_{24}, H_{25}$ and one overview work $H_{26}$. The work of Baker et al. resembles in many aspects Machala’s, but there are some differences. For a comparison between both approaches, one may consult $H_{06}$.

The study of topological Klingenberg and Hjelmslev planes remained an exclusive Canadian affair. The most prominent student of Lane, was undoubtedly J.W. (Mike) Lorimer. In his doctoral thesis on “Hjelmslev planes and topo-
logical Hjelmslev planes”, he not only introduced a coordinatization (see the previous section) but he also laid the foundation for topological Hjelmslev geometry. His work generalizes that of Salzmann [H28] and Skornyakov [H29] on topological projective planes. In a series of publications [H07, H08, H09, H10, H11, H12, H13, H14, H15, H16, H17] he further developed the theory in close connection with the coordinatization problem. Among the most important theorems proved by Lorimer, we mention the following characterization theorem: the only locally compact connected pappian projective Hjelmslev planes are the ones over the rings $\mathbb{K}[x]/\langle x^n \rangle$ with $\mathbb{K}$ the field of real or complex numbers.

11 The revival of ring geometry in the eighties and nineties under Veldkamp and Faulkner

In the seventies ring geometry was restricted almost exclusively to Hjelmslev and Klingenberg geometry (in the desarguesian case to geometries over local rings and Hjelmslev rings). The Dutch mathematician Ferdinand Douwe VELDKAMP (1931-1999), who is well-known for his work on geometries associated with exceptional Lie groups and in particular polar spaces, reverted back to the pioneering work of Barbilian where geometries over the broader class of $\mathbb{Z}$–rings were considered. It was Veldkamp’s aim to give an axiom system for projective planes (and higherdimensional spaces) over arbitrary rings with unit (without the imperfection in Barbilian’s attempt [A03, A04]). From conversations with his colleague van der Kallen at the University of Utrecht (an expert in $K$–theory), it became clear that the best setting for this project is provided by rings of stable rank two. A ring $R$ has stable rank two if the following property holds: if $a, b \in R$ and $Ra + Rb = R$ then there exists a $r$ in $R$ such that $a + rb$ is invertible in $R$. The class of stable rank two rings comprises all local rings. Hence, the projective ring planes introduced by Veldkamp include the desarguesian Klingenberg and Hjelmslev planes. A ring of stable rank two is always a $\mathbb{Z}$–ring.

Veldkamp first worked out the theory for planes in [I23] with some special cases in [I24] and later for spaces of higher dimension (see section 12). The ring planes defined by Veldkamp are also known today as desarguesian Veldkamp planes.

John Robert FAULKNER, an authority in the domain of non–associative algebra and geometry, further extended the theory of Veldkamp planes in the non–desarguesian direction by introducing alternative (non–associative) rings of stable rank two in [I01]. He then proved in [I02] that a Veldkamp plane has the Moufang property (i.e. $(P, \ell)$–transitivity holds for all $P, \ell$ with $P$ incident with $\ell$) if and only if it is a plane $P(A)$ over an alternative ring $A$ of stable rank two. Inspired by the work of his predecessors, Faulkner introduced in [I03] Faulkner planes as a very general class of plane incidence structures with neighbor (or remoteness) relation. They comprise the Veldkamp planes and the planes introduced by Barbilian. A (connected) Faulkner plane for which the group of $(P, \ell)$–transvections (automorphisms fixing all objects incident with
is transitive on the set of points not neighboring with \( \ell \), is called a transvection plane. The coordinatization of transvection Faulkner planes by a not necessarily associative alternative ring with the property that \( ab = 1 \) implies \( ba = 1 \) involves a rather technical procedure based on group theory and a lot of new concepts such as covering planes and tangent bundle planes. A transvection Faulkner plane for which the tangent bundle plane is also a transvection plane is called a Lie transvection Faulkner plane. To every such plane an alternative two-sided units ring can be attached and conversely with an alternative two-sided units ring \( R \) a corresponding Lie transvection Faulkner plane can be constructed. However, this plane is not determined unambiguously when the ring does not have stable rank 2. This high price has to be paid for the generalization from Veldkamp planes to Faulkner planes. In [I04] Faulkner gives a geometric construction of Barbilian planes coordinatized by composition algebras (including the Moufang plane) using Jordan algebras. His book [I07] is completely devoted to the role of such algebras in projective geometry.

Faulkner was surrounded by students at the University of Virginia, Charlottesville (USA), who all were involved with the study of ring geometries. Teresa Deltz Magnus obtained her PhD in 1991 on “Geometries over non–division rings” and she could generalize Faulkner’s axioms and results for geometries of higher dimension (see section 12). Eve Torrence also graduated under Faulkner’s supervision with “The coordinatization of a hexagonal–Barbilian plane by a quadratic Jordan algebra”, a generalization of the classical notion of generalized hexagon. Another student was Catherine Moore d’Ortona who studied homomorphisms between projective ring planes in her PhD thesis “Homomorphisms of remotely projective planes”, published in [I16]. Finally Karen Klintworth wrote her PhD thesis on “Affine remoteness planes”.

Faulkner himself considered a slightly more general axiomatization of Faulkner planes in [I06]. There he chooses for the remoteness relation, the negation of the neighbor relation. Much of the results obtained in [I03] are extended but the coordinate rings that appear are no longer always alternative. It is proved that \( P \) is a transvection plane if and only if \( P \) is isomorphic to \( P(G, N) \), the plane associated with a group \( G \) of Steinberg type parametrized by the ring \( R \) and with \( N \) a certain subgroup of \( G \). Necessary and sufficient conditions are given for \( R \) to be alternative, associative or commutative. In [I06] also projective remoteness planes with reflections (hence metric planes) have been considered.

The content of this paper is closely related to some work of Knüppel and Salow in [D10] (see section 6). It also contains a part on affine ring planes and elementary basis sets which are closely related to Barbilian domains as introduced by Leissner (see section 5).

In the slipstream of Veldkamp’s paper on projective ring planes, several slightly modified axiom systems have been described, leading to other classes of projective ring geometries. We have already mentioned Frieder Knüppel in the section on metric ring geometry, but some of his papers rather join the spirit of this section. In [I14] Knüppel considers ring geometries over associative rings based on four axioms adapted from Veldkamp (based on remoteness rather than neighborship). A coordinatization theorem is stated without proof. In [I15] he
studies homomorphisms between such planes. Renata Spanicciati defines near–Barbilian planes and strong near–Barbilian planes in [22] by adapting some of the axioms of Veldkamp. The neighbour relation between points turns out to be the identity, and the neighbour relation between lines becomes an equivalence relation. In [11] Guy Hanssens and Hendrik Van Maldeghem prove that any near–Barbilian plane is strong near–Barbilian. Kálman Pénalk gives a necessary and sufficient condition for a Veldkamp plane to be a direct product of a finite number of Veldkamp planes in [17], [18], [19].

The study of homomorphisms between ring geometries was also a central theme in several papers of Veldkamp (some of them in joint work with Joseph Ferrar) [10], [10], [10], [25], [26]. The geometric homomorphisms of distinct kind (incidence preserving, neighbor–preserving, distant–preserving) are characterized in terms of algebraic morphisms between the underlying rings. Veldkamp’s results were generalized by the author in the non–desarguesian case for homomorphisms between projective Klingenberg planes using the coordinatizing planar sexternary rings (see section 9) in [12], [13]. Thorsten Pfeiffer could generalize a well–known theorem for planes over fields to planes over rings: a desarguesian ring plane \( \mathcal{P}(R) \) is pappian (Pappus’ theorem is valid) if and only if \( R \) is commutative.

12 Projective and affine Hjelmslev spaces and spaces over arbitrary rings

Hitherto we only discussed plane ring geometries. The theory of higher dimensional projective spaces over rings has been developed by different people. Projective spaces over local rings appear for the first time in the work of Klingenberg [A17]. Today we call them Klingenberg spaces (PK–spaces). The first study of PK–spaces after Klingenberg, is due to Hans–Heinrich Lück, a student of Lüneburg. His PhD thesis, published as an article [J34] in 1970 under the somewhat misleading title “Projektive Hjelmslevräume”, contains an axiomatic characterization of a class of incidence structures which permit a coordinatization by local rings. Hence, the paper deals with projective Klingenberg spaces rather than with Hjelmslev spaces. Lück proves that in an axiomatically defined PK–space of dimension at least three, the theorem of Desargues holds and that it must be isomorphic to a space derived from a module over a local ring. Hence all projective Klingenberg spaces of dimension \( \geq 3 \) are desarguesian, a situation analogous to the case of classical projective spaces.

Independently from Lück, Machala defined and studied projective Klingenberg spaces (Projektive Räume mit Homomorphismus) of finite or infinite dimension in [J35], [J36], [J37], [J38], [J39]. He proved that the planes in a PK–space are PK–planes and that PK–spaces of dimension at least three come from modules over a local ring (cf. Lück). Machala also investigated homomorphisms between PK–spaces and the fundamental theorem (isomorphisms between spaces can be
represented by semilinear mappings between the underlying modules). A paper by Jukl \cite{13} is in conformity with this.

An axiomatic approach for the more restricted class of projective Hjelmslev spaces (PH–spaces) was initiated by John LAMB Jr. from the University of Texas at Austin (USA) in his PhD thesis, entitled “The Structure of Hjelmslev space, a generalization of projective space” (1969) but its content (related to lattice theory) was not published.

Much more widespread is the work of Karl Mathiak, who was very productive in this field. He defined a class of special projective Hjelmslev spaces starting from a vectorspace over a (skew)field endowed with a valuation (“Bewertete Vektorräume”). The structure of the ideals in the corresponding valuation ring plays a central role in his approach. The theory is thoroughly worked out in a series of papers, published over a period of twenty years between 1967 and 1987, see \cite{41,42,43,44,45,46,47,48,49,50}.

His compatriot Alexander KREUZER introduced an axiom system for arbitrary PH–spaces in his doctoral thesis “Projektive Hjelmslevräume” which was published afterwards as an article in \cite{18} with some preliminary work in \cite{17}. This study is continued in \cite{19,20}.

For the affine case we have to go to Canada again. In four papers, Tibor BISZTRICZKY together with J.W. LORIMER, worked out two axiom systems for affine Klingenberg spaces \cite{02,03,04,05}. Neither of their axiom systems assumes the existence of an overlying projective Klingenberg space or the existence of an underlying ordinary affine space. Machala in \cite{38} also defined affine Klingenberg spaces, but not separate from PK–spaces (similar to ordinary affine spaces obtained from projective spaces by deleting a hyperplane).

The most general study of projective ring spaces (over not necessarily local rings) was undertaken by Ferdinand VELDKAMP. We have already indicated his interest in section 11. In \cite{62,63}, Veldkamp gives a self–dual axiom system for projective “Barbilian spaces” of finite dimension using the basic concepts of points, hyperplanes, incidence and neighbor relation. We call them now Veldkamp spaces. The main result is that Veldkamp spaces of dimension $\geq 3$ are spaces over rings of stable rank 2.

Theresa MAGNUS defines Faulkner spaces as even more general geometries by extending the theory of Faulkner planes \cite{40}. She proves that any Faulkner space of dimension $n \geq 3$ is coordinatized by a unique associative two–sided units ring $R$ and that the group generated by all transvections is a group of Steinberg type over $R$. A Faulkner space over the ring $Z$ of integers is constructed, providing an example of a Faulkner geometry which is not a Veldkamp space, since $Z$ has stable rank 3. Under the Veldkamp spaces we also find the projective spaces over matrix rings over GF($q$), studied by Thas \cite{312} in the early days of ring geometry (see section 4). Other contributions to finite ring spaces are due to KAPRALOVA who considered projective spaces over the ring of dual numbers over a Galois field in \cite{14} and to Ivan LANDJEV and Peter VANDENDRIESE \cite{24,25}.

A well-developed theory for affine spaces over rings is also due to VELDKAMP. In \cite{64} he defines Barbilian domains in free modules of rank $n$ and
introduces \( n \)-dimensional affine ring geometries. A geometrical interpretation of Barbilian domains is given by Sprenger in [J61]. Other attempts for setting up a theory of higher dimensional affine ring geometries (incidence structures with parallelism) are scattered in the literature. The definitions and the methods used are very diverse. Contributions in this field are due Permutti and Pizzarello [J54, J55], Miron [J51], Leissner, Severin and Wolf [J31, J32], Ostrowski [J53], Schmidt and Weller [J58], Kreis [J15, J16], Seier [J59, J60], Bach [J01] and others.

One of the problems for higher dimensional geometries over rings that has got much attention is that of morphisms and the fundamental theorem. For classical projective geometries \( P(V) \) induced by a vectorspace over a field this theorem states that any bijective incidence preserving map (projectivity) between projective spaces \( P(V) \) and \( P(W) \) can be algebraically characterized by a semilinear map from \( V \) to \( W \). The first generalization of the fundamental theorem to ring geometries was obtained by the Indian mathematicians Ojanguren and Sridharan who could prove it in case of module–induced geometries \( P(M) \) with \( M \) a free module of finite rank \( \geq 3 \) over a commutative ring [J52]. Generalizations to other classes of rings were proved later by Sarath and Varadarajan in [J57] and by James in [J12] and Faure in [J06].

For the sake of completeness we also refer to module–induced geometries as defined by Marcus Greferath and Stefan Schmidt [J08, J09, J10, J11]. Instead of the usual definition of the pointset as the set of all submodules generated by a unimodular element in a free module (cfr. Veldkamp), they take all submodules of rank one, leading to the bizarre situation of points properly contained in bigger points. In [J07] an extension of the fundamental theorem is proved for such module–induced geometries. Close relative to this, there is an abundance of articles by the school of the Georgian mathematician Alexander Lashkhi. They all contain variations on the same theme: an extension of the fundamental theorem for affine and projective geometries related with modules over rings, from the lattice–theoretic point of view. We have not included the whole collection of papers by Lashki and his students. Some of them have been published multiple times in different journals (in Russian and in English). The literature list only mentions a few representative ones: [J21, J22, J23, J26, J27, J28, J29, J30, J31].

13 Projective lines and circle geometries over rings: Blunck, Havlicek and Keppens

The “smallest” projective geometries that can be considered, are the projective lines. From the viewpoint of incidence geometry not much can be said about these rather poor structures. But combining the study of projective lines with those of their automorphisms (the general linear group) puts them into a new light. The theory has common ground with what is usually called geometric algebra. Indeed, the projective line \( P(R) \) over any ring \( R \) can be defined in terms of the free left \( R \)-module \( R^2 \) as follows: it is the orbit of a starter point \( R(1,0) \).
under the action of the general linear group $GL_2(R)$ on $R^2$. Since geometric algebra over rings only fits sideways in the section of incidence geometry, we did not make an effort to be complete in the literature list for this item. Nevertheless we mention a number of relevant references in which more information can be found, e.g. [K46]. Central themes that keep returning are the notions of cross–ratio and harmonic quadruples and the fundamental theorem, also known as Von Staudt’s theorem. In the case of classical projective lines over a field or a skewfield, this theorem characterizes mappings of the projective line which preserve harmonicity as projectivities.

Among the first publications on projective ring lines (and we do not consider here papers only dealing with linear groups over rings) belong some articles by the Indian mathematicians Nirmala and Balmohan Limaye. They prove a generalization of Von Staudt’s theorem for some special classes of commutative and non–commutative rings in [K38]–[K42]. A little bit earlier, in 1968, Melchior, a student of Benz, wrote his PhD thesis, entitled “Die projektive Gerade über einem lokalen Ring: ihre lineare Gruppe und ihre Geometrie”. Other contributions to this item appeared in [K02]–[K10], [K16], [K22]–[K47]. Some papers by BiIo and Depunt [B01], Hubaut [B08, B09] and Thas [B10, B11] also deal with projective lines over rings (see section 4). They were followed by Havlicek et al. in [K26]–[K28].

Projective lines over rings are also intrinsically related with circle geometries. This relation was established for the first time by Benz in his famous book “Vorlesungen über Geometrie der Algebren” [K06] from 1973. He presented a unified treatment of plane circle geometries, now called Benz planes, using the projective line over a commutative ring which is a two–dimensional $K$–algebra over a field $K$. His definition was extended by Andrea Blunck, a student of Benz, and Armin Herzer who introduced more general chain geometries $\Sigma(K, R)$ with $R$ a (not necessarily two–dimensional) $K$–algebra. A chain geometry is an incidence structure whose point set is the set of points of the projective line over $R$ and the $GL_2(R)$ orbit of $P(K)$ is the set of chains. The plane circle geometries of Möbius, Laguerre and Minkowski type are particular chain geometries for $R = L$ (a quadratic field extension of $K$), $R = K + K\varepsilon$ with $\varepsilon^2 = 0$ (dual numbers) or $R = K + Kt$ with $t^2 = t$ (double numbers) respectively. For an overview of chain geometry we refer to [K19]. Chain geometries were also treated by Schaeffer, a student of Benz, in his doctoral thesis entitled “Zum Automorphismenproblem in affinen Geometrien und Kettengometrien über Ringen. The study of chain geometries is continued by Blunck who introduced generalized chain geometries by considering projective lines over non–associative, alternative rings. In her PhD thesis “Doppelverhältnisse und lokale Alternativringe” (1990) and in [K07] she extended the notion of cross–ratio and investigated chain geometries (in relation to projective lines over non–associative rings) in [K08, K11, K12].

Around the same time the author in [K32, K33] defined Klingen–Benz planes axiomatically, i.e. plane circle geometries with neighbor relation which admit a natural epimorphism onto a classical Benz plane. Using the projective line over three kinds of quadratic ring extensions of a local ring (instead
of a field), he was able to construct algebraic models of such geometries. Also Konrad Lang in [K36] studied independently of us a class of Hjelmslev–Möbius planes. Blunck and Stroppel extended our definition of Klingenberg–Benz planes to Klingenberg chain spaces in [K21] and Blunck also proved that a Klingenberg chain space can be embedded into a projective Klingenberg space, such that the points are identified with points of a quadric and the chains with plane sections [K09]. In [K50] Seier constructed analogously a class of chain geometries \( \Sigma(H, L) \) with \( H \) a Hjelmslev–ring and \( L \) a ring extension of \( H \) and in [K51] he defined a Möbius plane with neighbor relation of a different kind than the one defined by us.

A basic notion concerning the projective line over a ring \( R \) is its distant relation: two points are called distant if they can be represented by the elements of a two-element basis of \( R^2 \). The distant graph has as vertices the points of the projective line and as edges the pairs of distant points. The distant graph is connected precisely when \( GL_2(R) \) is generated by the elementary linear group \( E_2(R) \). This aspect of projective ring lines was studied in more detail by Blunck, Havlicek, Matraš and some others in [K01, K13, K14, K15, K29, K43, K44]. In [K17, K18] the interaction between ring geometry and the geometry of matrices in the sense of Hua (see [B07]) is investigated in more detail.

14 Ring geometries and buildings: Van Maldeghem and co.

The theory of buildings was invented by the Belgium-born French mathematician Jacques Tits. Roughly speaking, buildings are incidence geometries on which groups act. Tits received many awards for his fundamental and path-breaking mathematical ideas, including the Abel Prize in 2008. One of his achievements is the classification of affine buildings of rank at least 4. They are known to be “classical”, i.e. they arise from algebraic groups over a local field. In the rank three case (where affine buildings are of three possible types \( \tilde{A}_2 \), \( \tilde{C}_2 \) or \( \tilde{G}_2 \)) many non–classical counterexamples are known.

In his PhD thesis “Niet–klassieke driehoeksegebouwen” on triangle buildings (affine buildings of type \( \tilde{A}_2 \)) Hendrik Van Maldeghem observed that a special kind of ring geometry is present as the suitably defined geometry at distance \( n \) from any given vertex of the building (the so-called \( n–\)th floor). This was described among other things in [L18, L19]. A little bit later Hanssens and Van Maldeghem could prove that those ring geometries are in fact projective Hjelmslev planes of level \( n \), see [L07]. In [L08] they give a universal construction for level \( n \) Hjelmslev planes (see also [L06] for the 2–uniform case) and as a corollary any level \( n \) projective Hjelmslev plane is isomorphic to the \( n–\)th floor of a triangle building. This result links the theory of PH–planes to that of triangle buildings, a rather unexpected but fascinating fact. In the same spirit Van Maldeghem investigated another class of rank three affine buildings, of type \( \tilde{C}_2 \), and proved that the \( n–\)th floor turns out to be another type of ring geometry.
which can be seen as a generalization of an ordinary generalized quadrangle. He called it “Hjelmslev–quadrangle” of level \( n \) (see [L20, L21]). In joint work with Hanssens a complete characterization of \( C_2 \)-buildings by Hjelmslev quadrangles was obtained [L09, L10].

The author defined “Klingenberg–quadrangles” as another generalization of ordinary generalized quadrangles in [L12]. The connection between Klingenberg–quadrangles and Hjelmslev–quadrangles is explained in [L20].

The relation of polar spaces of higher rank to generalized quadrangles is comparable with the relation of projective spaces to projective planes. Generalized quadrangles are polar spaces of rank two. Projective ring spaces of dimension at least three have got as much attention in the literature as projective ring planes. This is not the case so far for general “Klingenberg–polar spaces” or “polar spaces over rings” if the rank is bigger than two. Only one paper by James [L11] is known to us. Certainly this topic offers perspectives for future research.

The discovery by Van Maldeghem of the connection between buildings and ring geometry has a precedent. Twenty years earlier, in 1968, Veldkamp in joint work with Tonny Springer considered a geometry over the split octonions (over the complex number field) in [L16]. This geometry is a kind of analogue of the non–desarguesian projective plane over the alternative division ring of (non–split) octonions \( O \) but in which two distinct lines may have more than one point in common and dually. It was called Moufang–Hjelmslev plane, but this name is misleading since it is not a projective Hjelmslev plane (the neighbor relation is not transitive) and hence it is completely distinct from the Moufang PH–plane (over an alternative local ring) studied elsewhere (see section 9). In two subsequent papers [L25, L26] more results on Hjelmslev–Moufang planes are obtained, concerning projective groups. The geometry of Veldkamp and Springer is the same as the one constructed by Tits [L17] starting from the split algebraic group of type \( E_6 \). In this geometry each line has the structure of a polar space and two lines can meet in more than one point (namely, in a maximal singular subspace of the corresponding polar spaces).

John Faulkner considered Hjelmslev–Moufang planes by allowing an arbitrary ground field instead of \( \mathbb{C} \) in [L04, L05] and also Robert Bix studied generalized Moufang planes in [L02, L03].

The relationship between Hjelmslev planes and buildings was further exploited by Van Maldeghem and Van Steen to give a characterization of some rank three buildings by automorphism groups [L22, L23, L24].

In the margins of the study of buildings some other questions have emerged. One of them concerns embeddings. Embeddings of point–line geometries into projective spaces are well–known in the literature. The embedding question for ring geometries, in particular for projective Hjelmslev planes, is first attacked by Artmann [L01] who shows that the PH–plane over the ring of plural numbers \( F[t]/t^n \) (\( F \) a field), can be embedded in the \((3n–1)\)-dimensional projective space over \( F \). In [L13] the author and Van Maldeghem prove a nice characterization theorem for embeddable Klingenberg planes: if \( P \) is a projective Klingenberg plane that is fully embedded in the projective space \( PG(5,K) \) for some skewfield.
If $\mathbb{K}$, then $\mathcal{P}$ is either a desarguesian Klingenberg plane over a ring of twisted dual numbers or a subgeometry of an ordinary projective plane. The embedding of the projective plane over a matrix ring with entries in GF$(q)$ into a projective space over GF$(q)$ was also observed by Thas in [B12, B13]. Veronesean sets are closely connected with embeddings. In [L14] Schillewaert and Van Maldeghem define geometries with an additional axiom by which the Hjelmslev–Moufang plane (in the sense of Springer–Veldkamp) and its relatives fit into the framework using the modern notion of parapolar spaces. In [L15] they provide a common characterisation of projective planes over two-dimensional quadratic algebras (over an arbitrary field) in terms of associated Veronesean sets. Anneleen De Schepper and Van Maldeghem [L27] have considered Veronese representations of Hjelmslev planes over quadratic alternative algebras as part of a more general study of Veronese varieties and Mazzocca–Melone sets.

15 Ring geometries in coding theory: Honold, Kiermaier and Landjev

One of the fastest growing disciplines in mathematics is coding theory. Since its introduction by Claude Shannon in 1948, the number of publications about codes has exploded, in particular due to its importance in cryptography, data transmission and data storage. Initially mostly linear codes over finite fields were studied, but after the publication in 1994 of the paper [M07] by Hammons et al., a new era has begun. In that paper it is proved that all (non–linear) binary Kerdock-, Preparata-, Goethals- and Delsarte-Goethals-codes are images of $\mathbb{Z}_4$–linear codes under the Gray map. This discovery was quite peculiar and the paper got in 1995 the Information Theory Paper Award from the IEEE Information Theory Society. It was the start of a search for new codes by considering linear codes over the ring $\mathbb{Z}_4$ and over more general finite rings (see e.g. [M01]). Also some papers by Aleksandr Nechaev [M47, M48] have led the research in that direction.

We will not give a survey of all results obtained up to now for codes over finite rings, because even this niche has become too wide. We restrict ourselves here (and in the literature list) to the publications in which the direct relation between codes over rings and ring geometries is exhibited. Indeed, linear codes over finite chain rings can be associated with finite projective Hjelmslev geometries, the hyperplanes corresponding to the codewords. This correspondence offers opportunities for investigating the structure and the construction of ring–linear codes by pure geometrical methods.

The technique was first applied in [M14] by Thomas Honold, now working at the Zhejiang Universität in Hangzhou (China), and Ivan Landjev from the New Bulgarian University of Sofia (Bulgaria). They prove that certain MacDonald codes can be represented by linear codes over the ring of twisted dual numbers on a finite field, using multisets of points in Hjelmslev spaces. In [M15] they prove that all Reed–Muller codes are linearly representable over the ring of dual
numbers over $\mathbb{Z}_2$. In [M16] a general theory of linear codes over finite chain rings has been developed as a natural generalization of the theory of linear codes over finite fields and the correspondence with Hjelmslev spaces is investigated. In [M17] and [M18] an update of that paper is given. Geometric arguments are also used explicitly in [M19] for the construction of particular linear codes over chain rings of order four, generalizing a result obtained by Michael Kiermaier and Johannes Zwanzger in [M34], [M35]. Keisuke Shiromoto and Leo Storme defined in [M49] a Griesmer type bound for linear codes over finite quasi-Frobenius rings and they give a geometrical characterization of linear codes meeting the bound, viz. a one-to-one correspondence between these codes and minihypers in projective Hjelmslev spaces.

Kiermaier was a student of Honold and wrote his Master’s thesis on “Arcs und Codes über endlichen Kettenringen” in 2006 and obtained his PhD in 2012 at the University of Bayreuth (Germany) with the thesis “Geometrische Konstruktionen linearer Codes über Galois-Ringen der Charakteristik 4 von hoher homogener Minimaldistanz”.

The study of codes over chain rings from the viewpoint of Hjelmslev geometry has also led to the generalization of several special point sets (arcs, ovals, blocking sets, caps), already well–known in classical geometry over fields. The thus far most studied sets in Hjelmslev planes are arcs. A $(k, n)$-arc refers to a set of $k$ points which meet every line in at most $n$ points. This definition was given by Honold and Kiermaier in [M43]. General upper bounds on the cardinality of such arcs were found as well as the maximum possible size. For a chain ring $R$ of length 2 with Jacobson radical $R_0$, such that $|R/R_0| = q$, the maximum size of a $(k, 2)$-arc in the projective Hjelmslev plane over $R$ is $q^2$ if $q$ is odd and $q^2 + q + 1$ if $q$ is even (see [M20]). In [M21] the existence of maximal $(q^2+q+1, 2)$-arcs (i.e. hyperovals) is proved for $q$ even and in [M12] the existence of maximal $(q^2, 2)$-arcs for $q$ odd is proved. The results on maximal arcs are also used to construct interesting codes with a linear representation over a chain ring. Examples, non–existence results and upper bounds for the length of arcs are also present in [M02], [M08], [M10], [M11], [M13], [M23], [M32], [M33], [M36], [M50].

Caps in finite projective Hjelmslev spaces over chain rings of nilpotency index 2 are defined by Honold and Landjev in [M22]. A geometric construction for caps in the three-dimensional space is given, using ovoids in the epimorphic space PG(3, $q$) as well as an algebraic construction using Teichmüller sets. Blocking sets in Hjelmslev planes and their relation with codes is the subject of [M03], [M40], [M42] while [M39] and [M54] deal with spreads.

Another aspect that appears in the literature is the correspondence between two–weight codes and strongly regular graphs. In [M04] regular projective two–weight codes over finite Frobenius rings are introduced and it is shown that such codes give rise to a strongly regular graph. In [M04], [M41] two-weight codes are constructed using ring geometries and they yield infinite families of strongly regular graphs with non-trivial parameters.

Ovals in an ordinary projective plane of order $q$ are just $q + 1$ arcs and every conic is an oval. By a celebrated theorem of Segre every $(q + 1)$–arc in PG(2, $q$) with $q$ odd, is a conic. The study of ovals, conics and unitals in finite projective
Klingenberg and Hjelmslev planes was initiated by the author in his PhD thesis but only a part of it was published, e.g. in [M31]. The author also defined polarities in projective Klingenberg planes and spaces and he investigated their sets of absolute points which give rise to ovals or ovoids in some cases. A comprehensive study of polarities in \( n \)-uniform Hjelmslev planes and spaces over the ring \( \text{GF}(q)[t]/t^n \) appeared in [M28]–[M30]. Also the papers [M37]–[M38] enlighten some aspects of ovals and conics in ring planes.

A combinatorial study of conics in finite desarguesian Hjelmslev planes was made by Ratislav Jurga and Viliam Chvál. Their papers contain formulas for the number of interior and exterior points, tangents and secants of a conic [M05]–[M27]. A related problem is the projective equivalence of quadrics in projective Klingenberg spaces. This was first formulated in [M05] and analyzed in detail by O.A. Starikova et al. in [M24]–[M53].

Recently, a special class of codes (toric codes) is found to be related to affine ring geometries (Leissner planes) by Little in [M45]–[M46], revealing still another correspondence between ring geometry and coding theory.

16 Ring geometry in quantum information theory: Saniga and Planat

Besides the fact that ring geometries play a substantial role in coding theory, there is another domain in which they arise unexpectedly, namely in quantum information theory. In this important branch of quantum physics it is studied how information can be stored and retrieved from a quantum mechanical system. In 2006 Metod Saniga from the Astronomical Institute at the Slovak Academy of Sciences and Michel Planat from the Université de Franche–Comté (France) discovered a connection between finite ring geometry and quantum information theory [N09]. It is not clear yet whether or not the correspondence goes further than just equality in numbers of objects, but anyhow it is remarkable that ring geometries seem to play a role in quantum physics.

The notion of mutually unbiased bases (MUBs) has turned into a cornerstone of the modern theory. Saniga and Planat observed that the basic combinatorial properties of a complete set of MUBs of a \( q \)-dimensional Hilbert space \( \mathcal{H}_q \) with \( q = p^r \), \( p \) being a prime and \( r \) a positive integer, are qualitatively mimicked by the configuration of points lying on a proper conic in a projective Hjelmslev plane defined over a Galois ring of characteristic \( p^2 \) and rank \( r \). The \( q \) vectors of a basis of \( \mathcal{H}_q \) correspond to the \( q \) points in a neighbour class and the \( q + 1 \) MUBs answer to the total number of pairwise disjoint neighbour classes on the conic. In a series of subsequently published papers, other combinatorial correspondences between concepts from quantum theory and ring geometries over finite rings (in particular projective ring lines) are observed. One of these similarities concerns the structure of the generalized Pauli group associated with a specific single \( d \)-dimensional qudit (qudits are generalizations to \( d \)-level quantum systems of qubits which are the basic units in quantum information systems of level two).
See \[ N01, N02, N03, N04, N06, N08, N11, N12, N13, N14, N15, N16, N17, N18 \] and the survey papers \[ N05, N10 \] for an overview on this topic. The study of the relation with finite geometry (not only restricted to ring geometry but also in connection with small generalized polygons and polar spaces) is still ongoing, see e.g. \[ N07 \].
17 Literature on ring geometry and geometry over rings

In the separate bibliographic list we only mention publications in scientific journals (or books). Other references, e.g. Master’s and PhD theses which are quoted explicitly in the foregoing text, are not repeated here, except in case they were also published. Articles belonging to conference proceedings are also omitted if there exist copies with almost identical content, published in other journals. We have grouped the references in themes, corresponding to the sections in the paper. Some papers can be classified in multiple sections. In that case they are listed in the section in which they are referred to for the first time. We do not claim that the bibliography is complete but we have tried to compose an accurate list which complements the existing obsolete lists, especially for the period after 1990. The author will be grateful for additions, completions and corrections to this list.

A First traces and pioneers of ring geometry

[A01] Archbold J.W., Projective geometry over an algebra, *Mathematika* 2, 105–115 (1955)

[A02] Artmann B., Dorn D., Drake D. and Törner G., Hjelmslev’sche Inzidenzgeometrie und verwandte Gebiete. Literaturverzeichnis, *J. Geom.* 7, 175–191 (1976)

[A03] Barbilian D., Zur Axiomatik der projektiven ebenen Ringgeometrien I, *Jahresber. Deutsch. Math.-Verein.* 50, 179–229 (1940)

[A04] Barbilian D., Zur Axiomatik der projektiven ebenen Ringgeometrien II, *Jahresber. Deutsch. Math.-Verein.* 51, 34–76 (1941)

[A05] Benz W., On Study’s Übertragungsprinzip, *J. Geom.* 64, 1–7 (1999)

[A06] Clifford W. K., Preliminary Sketch of Bi-Quaternions, *Proc. Lond. Math. Soc.* 4, 381–395 (1873)

[A07] Dembowski P., Finite Geometries, reprint of the 1968 Edition, Springer-Verlag, Berlin Heidelberg, 379 pp. (1997)

[A08] Grünwald J., Über duale Zahlen und ihre Anwendung in der Geometrie, *Monatsh. Math.* 17, 81–136 (1906)

[A09] Hjelmslev J., Die Geometrie der Wirklichkeit, *Acta Math.* 40, 35–66 (1916)

[A10] Hjelmslev J., Die natürliche Geometrie, *Abh. Math. Sem. Univ. Hamburg* 2, 1–36 (1923)
[A11] Iordănescu R., The geometrical Barbilian’s work from a modern point of view, *Balkan J. Geom. Appl.* 1, 31–36 (1996)

[A12] Jungnickel D., Hochelmslev’sche Inzidenzgeometrie und verwandte Gebiete. Literatureverzeichnis II, *J. Geom.* 16, 138–147 (1981)

[A13] Keppens D., 50 years of finite geometry, the “geometries over finite rings” part, *Innov. Incidence Geom.* 15, 123–143 (2017)

[A14] Klingenberg W., Projektive und affine Ebenen mit Nachbarelementen, *Math. Z.* 60, 384–406 (1954)

[A15] Klingenberg W., Euklidische Ebenen mit Nachbarelementen, *Math. Z.* 61, 1–25 (1954)

[A16] Klingenberg W., Desarguesche Ebenen mit Nachbarelementen, *Abh. Math. Sem. Univ. Hamburg* 20, 97–111 (1955)

[A17] Klingenberg W., Projektive Geometrien mit Homomorphismus, *Math. Ann.* 132, 180–200 (1956)

[A18] Kotelnikov A. P., Screw Calculus and Some Applications to Geometry and Mechanics (in Russian), *Annals of the Imperial University of Kazan*, Russia (1895)

[A19] Predella P., Saggio di Geometria non-Archimedea (Nota II), *Batt. G.* 3, 161–171 (1912)

[A20] Ree R., On projective geometry over full matrix rings, *Trans. Amer. Math. Soc.* 6, 144–150 (1955)

[A21] Segre C., Le geometrie proiettive nei campi di numeri duali, Nota I e II, *Atti R. Acc. Scienze Torino* 47, 114–133 and 164–185 (1911–1912) (in: Corrado Segre, Opere, a cura della Unione Matematica Italiana, Volume II, Edizione Cremonese, Roma, 396–431 (1958))

[A22] Study E., Geometrie der Dynamen, Leipzig, Germany, 603 pp. (1903)

[A23] Törner G. and Veldkamp F.D., Literature on geometry over rings, *J. Geom.* 42, 180–200 (1991)

[A24] Veldkamp F.D., Geometry over rings, chapter 19 in *Handbook of Incidence Geometry: Buildings and Foundations*. Edited by F. Buekenhout, Elsevier, Amsterdam, 1420 pp. (1995)

B The Belgian contribution to early ring geometry

[B01] Bilo J. and Depunt J., Over de lineaire transformaties van de ternionenrechte (in Dutch), *Med. Koninkl. Vlaamse Acad. Wet., Lett. schone Kunsten België, Kl. Wet.* 24 (8), 1–36 (1962)
[B02] Bingen F., Géométrie projective sur un anneau semi–primaire, *Acad. Roy. Belg. Bull. Cl. Sci.* **52**, 13–24 (1966)

[B03] Depunt J., Sur la géométrie ternionienne dans le plan, *Bull. Soc. Math. Belg.* **11**, 123–133 (1959)

[B04] Depunt J., Grondslagen van de analytische projectieve ternionen-meetkunde van het platte vlak (in Dutch), *Verh. Kon. Vl. Ac. voor Wet., Lett. en Sch. K. van België, Kl. der Wet.* **63**, 1–99 (1960)

[B05] De Winne P., An extension of the notion “cross-ratio of an ordered 4-tuple of points of the projective line” to an ordered \((n + 3)\)-tuple of points (resp. hyperplanes) of the \(n\)-dimensional space over an arbitrary ring with identity, part I: the \(n\)-dimensional projective space \(S_n\) over an arbitrary ring with identity, *Simon Stevin*, **47**, 139–159 (1974)

[B06] De Winne P., Een studie betreffende het projectieve vlak over de totale matrixalgebra \(M_2(K)\) der \(2 \times 2\)-matrices met elementen in een algebraïsch afgesloten veld (in Dutch), *Verh. Kon. Vl. Ac. voor Wet., Lett. en Sch. K. van België, Kl. der Wet.*, **144**, 1–80 (1978)

[B07] Hua L.-K., Geometries of Matrices I. Generalizations of Von Staudt’s theorem, *Trans. Amer. Math. Soc.* **57**, 441–481 (1945)

[B08] Hubaut X., Construction d’une droite projective sur une algèbre associative, *Acad. Roy. Belg. Bull. Cl. Sci.*, **50**, 618–623 (1964).

[B09] Hubaut X., Algèbres projectives, *Bull. Soc. Math. Belg.* **17**, 495–502 (1965)

[B10] Thas J.A., Dubbelverhouding van een geordend puntenviertal op de projectieve rechte over een associatieve algebra met éénelement (in Dutch), *Simon Stevin*, **42** (3), 97–111 (1969)

[B11] Thas J.A., Een studie betreffende de projectieve rechte over de totale matrix algebra \(M_3(K)\) der \(3 \times 3\)-matrices met elementen in een algebraïsch afgesloten veld \(K\) (in Dutch), *Verh. Kon. Vl. Ac. voor Wet., Lett. en Sch. K. van België, Kl. der Wet.*, **112**, 1–151 (1969)

[B12] Thas J.A., The \(m\)-dimensional projective space \(S_m(M_n(GF(q)))\) over the total matrix algebra \(M_n(GF(q))\) of the \(n \times n\)-matrices with elements in the Galois field \(GF(q)\), *Rend. Mat.* **4**, 459–532 (1971)

[B13] Thas J.A., Deduction of properties, valid in the projective space \(S_{3n-1}(K)\), using the projective plane over the total matrix algebra \(M_n(K)\), *Simon Stevin*, **46**, 3–16 (1972)

[B14] Vanhelleputte C., Een studie betreffende de projectieve meetkunde over de ring der \((2 \times 2)\)-matrices met elementen in een commutatief lichaam, *Verhdl. Vlaamse Acad. Wet., Lett. schone Kunsten België, Kl. Wet.* (in Dutch) **92**, 1–93 (1966)
C The foundations of plane affine ring geometry

[C01] Armengnout N., Hardy F. and Maxson C., On generalized affine planes, *J. Geom.* 4, 143–159 (1974)

[C02] Arnold H-J., Die Geometrie der Ringe im Rahmen allgemeiner affiner Strukturen, *Hamburger Mathematische Einzelschriften, Göttingen* 4, 86 pp. (1971)

[C03] Arnold H-J., A way to the geometry of rings, *J. Geom.* 1, 155–167 (1971)

[C04] Benz W., Süssche Gruppen in affinen Ebenen mit Nachbarelementen und allgemeineren Strukturen, *Abh. Math. Sem. Univ. Hamburg* 26, 83–101 (1963)

[C05] Benz W., Ω–Geometrie und Geometrie von Hjelmslev, *Math. Ann.* 164, 118–123 (1966)

[C06] Benz W., Eben Geometrie über einem Ring, *Math. Nachr.* 59, 163–193 (1974)

[C07] Benz W., On Barbilian domains over commutative rings, *J. Geom.* 12, 146–151 (1979)

[C08] Burian K., Affine H-structures, *Comment. Math. Univ. Carolin.* 13, 629–635 (1972)

[C09] Burian K., Affine parallel H–structures (in Czech), *Sb. Prací Ped. Fak. v Ostravě Ser. A* 9, 3–5 (1974)

[C10] Burian K., Translation H–structures, *Sb. Prací Ped. Fak. v Ostravě Ser. A Mat. Fyz.* 21, 15–25 (1986)

[C11] Çelik B., On the hyperbolic Klingenberg plane classes constructed by deleting subplanes, *J. Inequal. Appl.* 357, 1–6 (2013)

[C12] Çelik B., A hyperbolic characterization of projective Klingenberg planes, *Int. J. Math. Sci.* 2, 10–14 (2008)

[C13] Dorn G., Affine Geometrie über Matrizenringen, *Mitt. Math. Sem. Giessen* 109, 120 pp. (1974)

[C14] Eugeni F. and Galiè E., Sui piani costruiti su anelli, *Dipartimento M.E.T., Università di Teramo, Italy*, 143–162 (1991)

[C15] Eugeni F. and Maturo A., Generalized affine planes, *J. Inform. Optim. Sci.* 12, 431–439 (1991)

[C16] Everett C.J., Affine geometries of vector spaces over rings, *Duke Math. J.* 9, 873–878 (1942).
[C17] Groze V. and Vasiu A., Affine structures over an arbitrary ring, Studia Univ. Babes-Bolyai Math. 25, 28–31 (1980)

[C18] Kaerlein G., Zur Isomorphie vektorieller Gruppen und affiner Liniengeometrien, J. Geom. 16, 1–4 (1981)

[C19] Keppens D., Affine planes over finite rings, a summary, Aequationes Math. 91, 979–993 (2017)

[C20] Lantz D., Uniqueness of Barbilian domains, J. Geom. 15, 21–27 (1981)

[C21] Lawrence P.A., Affine mappings in the geometries of algebras, J. Geom. 2, 115–143 (1972)

[C22] Leissner W., Affine Barbilian-Ebenen. I, J. Geom. 6, 31–57 (1975)

[C23] Leissner W., Affine Barbilian-Ebenen. II, J. Geom. 6, 105–129 (1975)

[C24] Leissner W., Parallelogromie–Ebenen, J. Geom. 8, 117–135 (1976)

[C25] Leissner W., Barbilianbereiche, in Beiträge zur Geometrischen Algebra, Birkhäuser, Basel–Stuttgart, 219–224 (1977)

[C26] Leissner W., Rings of stable rank 2 are Barbilian rings, Result. Math. 20, 530–537 (1991)

[C27] Lorimer J. W., What is a collineation of the integer plane?, Amer. Math. Mon. 103, 687–691 (1996)

[C28] Lüneburg H., Affine Hjelmslev–Ebenen mit transitiver Translationsgruppe, Math. Z. 79, 260–288 (1962)

[C29] Machala F., Affine Klingenbergsche strukturen, J. Geom. 11, 16–34 (1978)

[C30] Machala F., Desarguesche affine Ebenen mit Homomorphismus, Geom. Dedicata 3, 493–509 (1975)

[C31] Machala F., Affine planes with homomorphism (in Czech), Knižnice, Odb. Věd. Spisu Vys. Uč. Tech. Brně 56, 79–84 (1975)

[C32] Pickert G., Taktische Konfigurationen über Quasiringen, Aequationes Math. 58, 31–40 (1999)

[C33] Radó F., Affine Barbilian structures, J. Geom. 14, 75–102 (1980)

[C34] Schleicher R., Die Eindeutigkeit der Koordinatisierung von affinen Liniengeometrien durch freie Moduln, J. Geom. 22, 143–148 (1984)

[C35] Schmidt S. and Steinitz R., The coordinatization of affine planes by rings, Geom. Dedicata 62, 299–317 (1996)
[C36] Seier W., Der kleine Satz von Desargues in affinen Hjelmslev-Ebenen, Geom. Dedicata 3, 215–219 (1974)

[C37] Seier W., Über Translationen in affinen Hjelmslev–Ebenen, Abh. Math. Sem. Univ. Hamburg 43, 224–228 (1975)

[C38] Seier W., Die Quasitranslationen desarguesscher affiner Hjelmslev-Ebenen, Math. Z. 177, 181–186 (1981)

[C39] Seier W., Streckungstransitive affine Hjelmslev–Ebenen, Geom. Dedicata 11, 329–336 (1981)

[C40] Seier W., Eine Bemerkung zum grossen Satz von Desargues in affinen Hjelmslev–Ebenen, J. Geom. 20, 181–191 (1983)

[C41] Sperner E., Affine Räume mit schwacher Inzidenz und zugehörige algebraische Strukturen, J. Reine und angewandte Math. 204, 205–215 (1960)

[C42] Vasiu A., Hjelmslev–Barbilian structures, Math., Rev. Anal. Numér. Théor. Approximation, Math. 27, 73–77 (1985)

[C43] Vasiu A., The coordinatisation of a class of \( B \)–structures (in Romanian), Stud. Univ. Babes-Bolyai, Math. 31, 35–40 (1986)

[C44] Vasiu A., On a class of planes with neighbouring elements, Prepr., Babes-Bolyai Univ., Fac. Math., Res. Semin. 10, 59–70 (1986)

D Metric geometry over rings

[D01] Bachmann F., Aufbau der Geometrie aus dem Spiegelungsbegriff, 2nd ed. Die Grundlehren der mathematischen Wissenschaften, Band 96. Springer-Verlag, Berlin-New York, 374 pp. (1973)

[D02] Bachmann F., Hjelmslev–Gruppen, Neudruck. Mit einem Kapitel und einem Anhang von R. Stölting. Arbeitsgemeinschaft über geometrische Fragen, Universität Kiel, Kiel, 175 pp. (1974)

[D03] Bachmann F., Eine neuere Entwicklung in der ebenen metrischen Geometrie, Ber. Math.-Statist. Sekt. Forsch. Graz 92–95, 22 pp. (1978)

[D04] Bachmann F., Ebene Spiegelungsgeometrie. Eine Vorlesung über Hjelmslev-Gruppen, Bibliographisches Institut, Mannheim, 340 pp. (1989)

[D05] Fischbach G., Ein Darstellungssatz für Translations-Hjelmslev-Ebenen mit Spiegelungen, J. Geom. 46, 45–54 (1993)
[D06] Hjelmslev J., Einleitung in die allgemeine Kongruenzlehre, 1. Mitt. Danske Vid. Selsk. mat.-fys. Medd. 8 (1929); 2. Mitt. 10 (1929); 3. Mitt. 19 (1942); 4. und 5. Mitt. 22 (1945); 6. Mitt. 25 (1949).

[D07] Knüppel F., Äquiforme Ebenen über kommutativen Ringen und singuläre Prä-Hjelmslev-Gruppen, Abh. Math. Semin. Univ. Hamburg 53, 229–257 (1983)

[D08] Knüppel F. and Kunze M., Neighbor relation and neighbor homomorphism of Hjelmslev groups, Canad. J. Math. 31, 680–699 (1979)

[D09] Knüppel F. and Kunze M., Reguläre Hjelmslev-Homomorphismen, Geom. Dedicata 11, 195–225 (1981)

[D10] Knüppel F. and Salow E., Plane elliptic geometry over rings, Pacific J. Math. 123, 337–384 (1986)

[D11] Kunze M., Angeordnete Hjelmslevsche Geometrie, ein Ergebnisbericht, Geom. Dedicata 10, 91–111 (1981)

[D12] Lingenberg R., Metric planes and metric vector spaces. Pure and Applied Mathematics, John Wiley & Sons, New York-Chichester-Brisbane, 209 pp. (1979)

[D13] Nolte W., Minkowskische Hjelmslevgruppen über lokalen Ringen, Result. Math. 12, 376–385 (1987)

[D14] Nolte W., Hjelmslevgruppen mit Nachbar-Homomorphismus, J. Geom. 38, 78–94 (1990)

[D15] Salow E., Singuläre Hjelmslev-Gruppen, Geom. Dedicata 1, 447–467 (1973)

[D16] Salow E., Einbettung von Hjelmslev-Gruppen in orthogonale Gruppen über kommutativen Ringen, Math. Z. 134, 143–170 (1973)

[D17] Salow E., Verallgemeinerte Halbdrehungsebenen, Geom. Dedicata 13, 67–85 (1982)

[D18] Schröder E., Gemeinsame Eigenschaften euklidischer, galileischer und minkowskischer Ebenen, Mitt. Math. Ges. Hamburg 10, 185–217 (1974)

[D19] Schröder E., Modelle ebener metrischer Ringgeometrien, Abh. Math. Sem. Univ. Hamburg 48, 139–170 (1979)

[D20] Schröder E., Metric geometry, chapter 17 in Handbook of Incidence Geometry: Buildings and Foundations. Edited by F. Buekenhout, Elsevier, Amsterdam, 1420 pp. (1995)

[D21] Stölting R., Über endliche Hjelmslev-Gruppen, Math. Z. 135, 249–255 (1973/74)
[D22] **Stölting R.**, Ebene metrische Geometrien über projektiven Moduln, *Abh. Math. Sem. Univ. Hamburg* **50**, 166–177 (1980)

[D23] **Struve H. and Struve R.**, Hjelmslevgruppen, in denen sich die Punkte gegen Geraden austauschen lassen, *Geom. Dedicata* **13**, 399–417 (1983)

[D24] **Struve H. and Struve R.**, Ein spiegelungsgeometrischer Aufbau der cominkowskischen Geometrie, *Abh. Math. Semin. Univ. Hamburg* **54**, 111–118 (1984)

[D25] **Struve H. and Struve R.**, Coeuklidische Hjelmslevgruppen, *J. Geom.* **34**, 181–186 (1989)

[D26] **Struve R.**, Algebraisierung singulärer Hjelmslevgruppen, *Geom. Dedicata* **13**, 309–323 (1982)

[D27] **von Benda H. and Knüppel F.**, Hjelmslev-Gruppen über lokalen Ringen, *Geom. Dedicata* **5**, 195–206 (1976)

**E The florescence of Hjelmslev geometry**

[E01] **Artmann B.**, Hjelmslev planes derived from modular lattices, *Canad. J. Math.* **21**, 76–83 (1969)

[E02] **Artmann B.**, Hjelmslev–Ebenen mit verfeinerten Nachbarschaftsrelationen, *Math. Z.* **112**, 163–180 (1969)

[E03] **Artmann B.**, Uniforme Hjelmslev–Ebenen und modulare Verbände, *Math. Z.* **111**, 15–45 (1969)

[E04] **Artmann B.**, Über die Einbettung uniformer affiner Hjelmslev–Ebenen in projektive Hjelmslev–Ebenen, *Abh. Math. Semin. Univ. Hamburg* **34**, 127–134 (1970)

[E05] **Artmann B.**, Existenz und projektive Limiten von Hjelmslev–Ebenen n-ter Stufe, *Atti Convegno Geom. combinator. Appl. Perugia*, 27–41 (1971)

[E06] **Artmann B.**, Desarguessche Hjelmslev-Ebenen n–ter Stufe, *Mitt. Math. Sem. Giessen* **91**, 1–19 (1971)

[E07] **Artmann B.**, Geometric aspects of primary lattices, *Pacific J. Math.* **43**, 15–25 (1972)

[E08] **Bacon P.**, Strongly n–uniform and level n Hjelmslev planes, *Math. Z.* **127**, 1–9 (1972)

[E09] **Bacon P.**, On the extension of projectively uniform affine Hjelmslev planes, *Abh. Math. Semin. Univ. Hamburg* **41**, 185–189 (1974)
[E10] **Brungs H. and Törner G.**, Embedding right chain rings in chain rings, *Canad. J. Math.* **30**, 1079–1086 (1978)

[E11] **Craig R.**, Extensions of finite projective planes. I. Uniform Hjelmslev planes, *Canad. J. Math.* **16**, 261–266 (1964)

[E12] **Cronheim A.**, Cartesian groups, formal power series and Hjelmslev planes, *Arch. Math.* **27**, 209–220 (1976)

[E13] **Cronheim A.**, Dual numbers, Witt vectors, and Hjelmslev planes, *Geom. Dedicata* **7**, 287–302 (1978)

[E14] **Drake D.A.**, Projective extensions of uniform affine Hjelmslev planes, *Math. Z.* **105**, 196–207 (1968)

[E15] **Drake D.A.**, On $n$–uniform Hjelmslev planes, *J. Comb. Theory Ser. A* **9**, 267–288 (1970)

[E16] **Drake D.A.**, The translation groups of $n$–uniform translation Hjelmslev planes, *Pacific J. Math.* **38**, 365–375 (1971)

[E17] **Drake D.A.**, The structure of $n$–uniform translation Hjelmslev planes, *Trans. Amer. Math. Soc.* **175**, 249–282 (1973)

[E18] **Drake D.A.**, Near affine Hjelmslev planes, *J. Comb. Theory Ser. A* **16**, 34–50 (1974)

[E19] **Drake D.A.**, Existence of parallelisms and projective extensions for strongly $n$–uniform near affine Hjelmslev planes, *Geom. Dedicata* **3**, 191–214 (1974)

[E20] **Drake D.A.**, Affine Hjelmslev-Ebenen mit verfeinerten Nachbarschaftsrelationen, *Math. Z.* **143**, 15–25 (1975)

[E21] **Drake D.A.**, All $n$–uniform quasitranslation Hjelmslev planes are strongly $n$–uniform, *Proc. Amer. Math. Soc.* **51**, 494–498 (1975)

[E22] **Drake D.A.**, Squeezing the accordion in a strongly $n$–uniform Hjelmslev plane, *Math. Z.* **185**, 151–166 (1984)

[E23] **Dugas M.**, Eine Kennzeichnung der endlichen desarguesschen Hjelmslev-Ebenen, *Geom. Dedicata* **3**, 295–324 (1974)

[E24] **Dugas M.**, Der Zusammenhang zwischen Hjelmslev-Ebenen und H-Verbänden, *Geom. Dedicata* **3**, 295–324 (1974)

[E25] **Kleinfeld E.**, Finite Hjelmslev planes, *Illinois J. Math.* **3**, 403–407 (1959)

[E26] **Lorimer J.W.**, The fundamental theorem of desarguesian affine Hjelmslev planes, *Mitt. Math. Sem. Giessen* **119**, 6–14 (1975)
[E27] Lorimer J.W., Morphisms and the fundamental theorem of affine Hjelmslev planes, Mathematical Report 64, McMaster University, Hamilton, Ontario, Canada (1973)

[E28] Lorimer J. W., Structure theorems for commutative Hjelmslev rings with nilpotent radicals, C. R. Math. Rep. Acad. Sci. Canada 6, 123–127 (1984)

[E29] Lorimer J. W., Affine Hjelmslev rings and planes, Annals of Discr. Math. 37, 265–276 (1988)

[E30] Lorimer J.W. and Lane N.D., Morphisms of affine Hjelmslev planes, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 56, 880–885 (1974)

[E31] Lorimer J.W. and Lane N.D., Desarguesian affine Hjelmslev planes, J. für die reine und angew. Math. 1, 336–352 (1975)

[E32] Machala F., Über projektive Erweiterung affiner Klingenberg’scher Ebenen, Czech. Math. J. 29, 116–129 (1979)

[E33] Machala F., Uber eine Klasse affiner Klingenburg’scher Ebenen, die projektiv erweiterbar sind, Acta Univ. Palacki. Olomuc., Fac. Rerum Nat. 19, 65–74 (1980)

[E34] Seier W., Zentrale und axiale Kollineationen in projektiven Hjelmslev–Ebenen, J. Geom. 17, 35–45 (1981)

[E35] Skornjakov, L. A., Rings chain–like from the left (in Russian) Izv. Vyssh. Uchebn. Zaved. Matematika 4, 114–117 (1966)

[E36] Törner G., Eine klassifizierung von Hjelmslev–ringen und Hjelmslev–Ebenen, Mitt. Math. Sem. Giessen, 107, 1–77 (1974)

[E37] Törner G., Über Homomorphismen projektiver Hjelmslev–Ebenen, J. Geom. 5, 1–13 (1974)

[E38] Törner G., Homomorphismen von affinen Hjelmslev–Ebenen, Math. Z. 141, 159–167 (1975)

[E39] Törner G., n-uniforme projektive Hjelmslev-Ebenen sind stark n-uniform, Geom. Dedicata 6, 291–295 (1977)

[E40] Törner G., Über den Stufenaufbau von Hjelmslev–Ebenen, Mitt. Math. Sem. Giessen, 126, 1–43 (1977)

[E41] Törner G., $(r^{n-1}, r)$ Hjelmslev–Ebenen des Typs $n$, Math. Z. 154, 189–201 (1977)

[E42] Törner G., Über ein Problem von Klingenberg, Arch. Math. 28, 253–254 (1977)

[E43] Törner G., Faktorisierungen von Epimorphismen projektiver Ebenen, Geom. Dedicata 18, 281–291 (1985)

38
The continuation of the Hjelmslev epoch

[F01] Al-Khamees Y., The enumeration of finite principal completely primary rings, *Abh. Math. Sem. Univ. Hamburg* **51**, 226–231 (1981)

[F02] Civolani N., Hyperfree extensions of partial Klingenberg planes, *Geom. Dedicata* **9**, 467–475 (1980)

[F03] Clark W.E. and Liang J., Enumeration of finite commutative chain rings, *J. Algebra* **27**, 445–453 (1973)

[F04] Clark W.E. and Drake D.A., Finite chain rings, *Abh. Math. Sem. Univ. Hamburg* **39**, 147–153 (1973)

[F05] Drake D.A., Nonexistence results for finite Hjelmslev planes, *Abh. Math. Sem. Univ. Hamburg* **40**, 100–110 (1974)

[F06] Drake D.A., More new integer pairs for finite Hjelmslev planes, *Illinois J. Math.* **19**, 618–627 (1975)

[F07] Drake D.A., Charakterisierungen der Hjelmslev-Ebenen mit Invarianten (4,2), *Arch. Math.* **27**, 436–440 (1976)

[F08] Drake D.A., Constructions of Hjelmslev planes, *J. Geom.* **10**, 179–193 (1977)

[F09] Drake D.A., The use of auxiliary sets of matrices in the construction of Hjelmslev and Klingenberg structures, *Lecture Notes in Pure and Appl. Math.* **82**, 129–153 (1983)

[F10] Drake D.A. and Hale M., Group constructible \((t, k)\)-nets and Hjelmslev planes, *J. Algebra* **48**, 301–331 (1977)

[F11] Drake D.A. and Jungnickel D., Klingenberg structures and partial designs I: Congruence relations and solutions, *J. Stat. Plann. Inference* **1**, 265–287 (1977)

[F12] Drake D.A. and Jungnickel D., Klingenberg structures and partial designs. II: Regularity and uniformity, *Pacific J. Math.* **76**, 389–415 (1978)

[F13] Drake D.A. and Jungnickel D., Das Existenzproblem für projektive \((8,5)\)-Hjelmslevebenen, *Abh. Math. Sem. Univ. Hamburg* **50**, 118–126 (1980)

[F14] Drake D.A. and Jungnickel D., Finite Hjelmslev planes and Klingenberg epimorphisms, in: *Rings and geometry*, Proc. NATO Adv. Study Inst., Istanbul/Turkey 1984, NATO ASI Ser., Ser. C 160, 153–231 (1985)

[F15] Drake D.A. and Lenz H., Finite Klingenberg planes, *Abh. Math. Sem. Univ. Hamburg* **44**, 70–83 (1975)
[F16] Drake D.A. and Lenz H., Finite Hjelmslev planes with new integer invariants, Bull. Amer. Math. Soc. 82, 265–267 (1976)

[F17] Drake D.A. and Sane S., Maximal intersecting families of finite sets and $n$–uniform Hjelmslev planes, Proc. Amer. Math. Soc. 86, 358–362 (1982)

[F18] Drake D.A. and Sane S., Auxiliary sets of matrices with new step parameter sequences, Linear Algebra Appl. 46, 131–153 (1982)

[F19] Drake D.A. and Shult E., Construction of Hjelmslev planes from $(t,k)$–nets, Geom. Dedicata 5, 377–392 (1976)

[F20] Drake D.A. and Törner G., Die Invarianten einer Klasse projektiver Hjelmslev–Ebenen, J. Geom. 7, 157–174 (1976)

[F21] Hale M. and Jungnickel D., A generalization of Singer’s theorem, Proc. Amer. Math. Soc. 71, 280–284 (1978)

[F22] Jungnickel D., Verallgemeinerte Klingenberg–Ebenen, Mitt. Math. Sem. Giessen 120, 1–10 (1976)

[F23] Jungnickel D., Hjelmslevebenen mit regulärer abelscher Kollineationsgruppe, Beiträge zur geometrischen Algebra (Proc. Sympos. Duisburg 1977), 157–165 (1977)

[F24] Jungnickel D., Regular Hjelmslev planes. II, Trans. Amer. Math. Soc. 241, 321–330 (1978)

[F25] Jungnickel D., On the congruence relations of regular Klingenberg structures, J. Combin. Inform. System Sci. 3, 49–57 (1978)

[F26] Jungnickel D., Regular Hjelmslev planes, J. Comb. Theory Ser. A 26, 20–37 (1979)

[F27] Jungnickel D., On an assertion of Dembowski, J. Geom. 12, 168–174 (1979)

[F28] Jungnickel D., Construction of regular proper CK–planes, J. Combin. Inform. System Sci. 4, 14–18 (1979)

[F29] Jungnickel D., On balanced regular Hjelmslev planes, Geom. Dedicata 8, 445–462 (1979)

[F30] Jungnickel D., Some new combinatorial results on finite Klingenberg structures, Utilitas Math. 16, 249–269 (1979)

[F31] Jungnickel D., A class of uniform Klingenberg matrices, Ars Combin. 10, 91–94 (1980)

[F32] Limaye B.V. and Sane S., On partial designs and $n$–uniform projective Hjelmslev planes, J. Combin. Inform. System Sci. 3, 223–227 (1978)
[F33] Neumaier A., Nichtkommutative Hjelmslev-Ringe, *Festband für H. Lenz, Freie Universität Berlin*, 200–213 (1976)

[F34] Sane S., Some new invariant pairs $(t,3)$ for projective Hjelmslev planes, *J. Geom.* 15, 64–73 (1981)

[F35] Sane S., New integer pairs for Hjelmslev planes, *Geom. Dedicata* 10, 35–48 (1981)

[F36] Sane S., On class-regular projective Hjelmslev planes, in: *Finite geometries and designs, London Math. Soc. Lecture Note Ser.* 49, 332–336 (1981)

[F37] Sane S., On the theorems of Drake and Lenz, *Aequationes Math.* 23, 223–232 (1981)

[F38] Sane S. and Singhi N., On the structure of a finite projective Klingenberg plane, *Congr. Numer.* 33, 285–292 (1981)

[F39] Sedlar V., Incidence matrices of finite projective uniform Hjelmslev planes (in Czech) *Sb. Praci Ped. Fak. v Ostrave Ser. A* 17, 25–38 (1982)

G The coordinatization of Klingenberg and Hjelmslev planes

[G01] Akpinar A., Çelik B. and Çiftçi S., Cross-ratios and 6-figures in some Moufang-Klingenberg planes, *Bull. Belg. Math. Soc. - Simon Stevin* 15, 49–64 (2008)

[G02] Akpinar A., Çelik B. and Çiftçi S., Cross-ratios of points and lines in some Moufang-Klingenberg planes, *Hacet. J. Math. Stat.* 40, 1–13 (2011)

[G03] Akpinar A., Dayıoğlu A., Doğan I., Boztemür B., Aslan D. and Gürel Z.S., A note on projective Klingenberg planes over rings of plural numbers, *Int. J. of New Technology and Research* 4, 103–105 (2018)

[G04] Bacon P., An introduction to Klingenberg planes. Vols. 1–4 (1976, 1977, 1979, 1983), published by the author, 3101 NW 2nd Av, Gainesville, Florida 32607, U.S.A..

[G05] Bacon P., Desarguesian Klingenberg planes, *Trans. Amer. Math. Soc.* 241, 343–355 (1978)

[G06] Baker, C., Moultton affine Hjelmslev planes *Canad. Math. Bull.* 21, 135–142 (1978)

[G07] Baker, C., Lane N. and Lorimer J.W., Local alternative rings and finite alternative right chain rings, *C. R. Math. Rep. Acad. Sci. Canada* 12, 53–58 (1990)
[G08] Baker, C., Lane N. and Lorimer J.W., An affine characterization of Moufang projective Klingenberg planes, Results Math. 17, 27–36 (1990)

[G09] Baker, C., Lane N. and Lorimer J.W., The Artin-Zorn theorem for finite punctually cohesive projective Klingenberg planes, Ars Comb. Ser. B 29, 143–149 (1990)

[G10] Baker, C., Lane N. and Lorimer J.W., A coordinatization for Moufang Klingenberg planes, Simon Stevin 65, 3–22 (1991)

[G11] Baker C.A. and Lorimer J.W., Coordinate rings of topological Klingenberg planes. II: The algebraic foundation for a projective theory, J. Geom. 73, 49–92 (2002)

[G12] Blunck A., Projectivities in Moufang-Klingenberg planes, Geom. Dedicata 40, 341–359 (1991)

[G13] Blunck A., Cross-ratios in Moufang-Klingenberg planes. Geom. Dedicata 43, 93–107 (1992)

[G14] Çelik B., Akpinar A. and Çiftçi S., 4-transitivity and 6-figures in some Moufang-Klingenberg planes, Monatsh. Math. 152, 283–294 (2007)

[G15] Çelik B. and Çiftçi S., Cross-ratios over the geometric structures which are coordinatized with alternative or local alternative rings, Commun. Fac. Sci. Univ. Ankara, Ser. A1 43, 105–117 (1994)

[G16] Çelik B. and Erdoğan F., On addition and multiplication of points in a certain class of projective Klingenberg planes, J. Inequal. Appl. 230, 1–9 (2013)

[G17] Çelik B. and Dayıoglu A., The collineations which act as addition and multiplication on points in a certain class of projective Klingenberg planes, J. Inequal. Appl. 193, 1–9 (2013)

[G18] Çelik N., Çiftçi S. and Akpinar A., Some properties of 6-figures and cross-ratios in some Moufang-Klingenberg planes, J. Algebra Appl. 9, 173–184 (2010)

[G19] Çelik B., Akpinar A. and Çiftçi S., On harmonicity in some Moufang-Klingenberg planes, Turk. J. Math. 34, 249–260 (2010)

[G20] Cyganova V. K., The impossibility of introducing universally comprehensible configurational propositions into a projective plane with neighboring elements (in Russian), Smolensk. Gos. Ped. Inst. Uchen. Zap. 18, 35–43 (1967)

[G21] Cyganova V. K., Ternary rings of affine Hjelmslev planes (in Russian), Smolensk. Gos. Ped. Inst. Uchen. Zap. 18, 44–69 (1967)
[G22] Cyganova V. K., H-Ternaries and configuration theorems with their algebraic equivalents in affine Hjelmslev planes (in Russian), Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk 3, 125–126 (1973)

[G23] Cyganova V. K., Dependence of some configurational theorems in affine Hjelmslev planes (in Russian), Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk 3, 127 (1973)

[G24] Cyganova V. K., Affine specialization of the configuration proposition $D_1(8,11,14)$ in an AH-plane, and its algebraic equivalent (in Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 5, 104–105 (1975)

[G25] Cyganova V. K., The configuration postulate $D_H(9,13,17)$, and its algebraic equivalent (in Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 1, 120–121 (1978)

[G26] Cyganova V. K., A geometric interpretation of the distributivity of an H-ternary (in Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 2, 31–35 (1980)

[G27] Cyganova V. K., The second minor Pappos theorem and its algebraic equivalent in Hjelmslev affine planes (in Russian), Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk 1, 107–109 (1984)

[G28] Drake D.A., Coordinatization of H-planes by H-modules, Math. Z. 115, 79–103 (1970)

[G29] Dugas M., Charakterisierungen endlicher Desarguesscher uniformer Hjelmslev-Ebenen, Geom. Dedicata 3, 295–324 (1974)

[G30] Dugas M., Moufang-Hjelmslev-Ebenen, Geom. Dedicata 3, 295–324 (1974)

[G31] Dugas M., Verallgemeinerte André–Ebenen mit Epimorphismen auf Hjelmslev–Ebenen, Geom. Dedicata 8, 105–123 (1979)

[G32] Eliseev, E.M., Desarguesian theorems and collineations of projective Hjelmslev planes, (in Russian) Geometry of incidence structures and differential equations, Collect. sci. Artic., Smolensk, 30–40 (1981)

[G33] Emel’chenkov E.P., Translation AH–planes and H–ternaries (in Russian), Smolenk. Gos. Ped. Inst. Uchen. Zap. 4, 74–83 (1973)

[G34] Emel’chenkov E.P., The PH–ternary of a projective Hjelmslev plane (in Russian), Smolenk. Gos. Ped. Inst. Uchen. Zap. 4, 93–101 (1973)

[G35] Emel’chenkov E.P., Homotheties of AH–planes and H–ternaries (in Russian), Gos. Ped. Inst., Leningrad, Geom. Topol. 2, 89–93 (1974)

[G36] Emel’chenkov E.P., On (II, l)–collineations of AH–planes (in Russian), in Modern Geometry, Gos. Ped. Inst., Leningrad, 58–60 (1978)
[G37] Hall M., Projective planes, Trans. Amer. Math. Soc. 54, 229–277 (1943)

[G38] Hughes D.R., Planar division neo-rings, Trans. Amer. Math. Soc. 80, 502–527 (1955)

[G39] Hughes D.R. and Piper F.C, Projective planes, Springer–Verlag, Berlin, 291 pp. (1973)

[G40] Jukl M., Desargues theorem for Klingenberg projective plane over certain local ring, Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. 36, 33–39 (1997)

[G41] Keppens D., Coordinatization of projective Klingenberg planes. I: Introduction of coordinates and planar sexternary rings, Simon Stevin 62, 63–90 (1988)

[G42] Keppens D., Coordinatization of projective Klingenberg planes. II: Connections between geometric properties of a PK-plane and algebraic properties of a coordinatizing PSR, Simon Stevin 62, 163–188 (1988)

[G43] Keppens D., Coordinatization of projective Klingenberg planes. III: Construction of planar sexternary rings and examples, Simon Stevin 63, 117–140 (1989)

[G44] Kolb E., Projective Klingenberg planes over nearrings, J. Geom. 46, 82–91 (1993)

[G45] Kolb E., Hjelmslev planes over nearrings, Discrete Math. 155, 147–155 (1996)

[G46] Lorimer J.W., Coordinate theorems for affine Hjelmslev planes, Ann. Mat. Pura Appl. 105, 171–190 (1975)

[G47] Machala F., Erweiterte lokale Ternärringe, Czech. Math. J. 27, 560–572 (1977)

[G48] Machala F., Koordinatisation projektiver Ebenen mit Homomorphismus, Czech. Math. J. 27, 573–590 (1977)

[G49] Machala F., Koordinatisation affiner Ebenen mit Homomorphismus, Math. Slovaca 27, 181–193 (1977)

[G50] Machala F., Projektive Ebenen mit Homomorphismus und erweiterte lokale Ternärringe, Math. Slovaca 29, 227–237 (1979)

[G51] Machala F., Epimorphismen von lokalen Ternärringen, Czech. Math. J. 33, 70–75 (1983)

[G52] Machala F., Biternärringe und affine lokale Ternärringe, Acta. Univ. Palack. Olomuc. Fac. Rerum. Natur. Math. 27, 25–37 (1988)
G53] Mäurer H. and Nolte W., A characterization of Pappian affine Hjelmslev planes, *Combinatorics ’86 (Trento, 1986)*, Ann. Discrete Math. **37**, 281–291 (1988)

G54] Nolte W., Pappussche affine Klingenbergebenen, *J. Geom.* **52**, 152–158 (1995)

G55] Pickert G., Projektiye Ebenen, Springer–Verlag, Berlin, second edition, 388 pp. (1975)

G56] Shatokhin N. L., Homomorphisms of H-planes and PH-ternars (in Russian), *Geometry of incidence structures and differential equations*, Collect. sci. Artic., Smolensk, 81–91 (1981)

G57] Shatokhin N. L., Frame isomorphisms of affine Hjelmslev planes and ω–isotopies of AH ternaries (in Russian), *Vladikavkaz. Mat. Zh.* **9**, 48–54 (2007)

G58] Skornyakov L.A., Natural domains of Veblen-Wedderburn projective planes (in Russian), *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **13**, 447–472 (1949); english translation in *Amer. Math. Soc. Translation* **58**, 37 pp. (1951)

H01] Baker C., Ordered affine Hjelmslev planes, *J. Geom.* **23**, 1–13 (1984)

H02] Baker C., Preordered uniform Hjelmslev planes, *J. Geom.* **24**, 14–17 (1985)

H03] Baker C. and Lorimer J.W., Coordinate rings of topological Klingenberg planes. I: The affine perspective, *Geom. Dedicata* **58**, 101–116 (1995)

H04] Baker C., Lane N.D. and Lorimer J.W., Order and topology in projective Hjelmslev planes, *J. Geom.* **19**, 8–42 (1982)

H05] Baker C., Lane N.D. and Lorimer J.W., A construction for topological non–desarguesian affine Hjelmslev planes, *Arch. Math.* **50**, 83–92 (1988)

H06] Baker C., Lane N.D., Laxton J.A. and Lorimer J.W., Preordered affine Hjelmslev planes, *J. Geom.* **23**, 14–44 (1984)

H07] Lorimer J.W., Topological Hjelmslev planes, *Geom. Dedicata* **7**, 185–207 (1978)
[H08] Lorimer J.W., Connectedness in topological Hjelmslev planes, *Ann. Mat. Pura Appl.* 118, 199–216 (1978)

[H09] Lorimer J.W., Locally compact Hjelmslev planes, *C. R. Math. Rep. Acad. Sci. Canada* 1, 309–314 (1978/79)

[H10] Lorimer J.W., Locally compact Desarguesian Hjelmslev planes of level n, *C. R. Math. Rep. Acad. Sci. Canada* 2, 141–145 (1980)

[H11] Lorimer J.W., Locally compact Hjelmslev planes and rings, *Canad. J. Math.* 33, 988–1021 (1981)

[H12] Lorimer J.W., Dual numbers and topological Hjelmslev planes, *Canad. Math. Bull.* 26, 297–302 (1983)

[H13] Lorimer J.W., Compactness in topological Hjelmslev planes, *Canad. Math. Bull.* 27, 423–429 (1984)

[H14] Lorimer J.W., A topological characterization of Hjelmslev’s classical geometries. in: Rings and geometry (Istanbul, 1984), 81–151, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 160, Reidel, Dordrecht, (1985)

[H15] Lorimer J.W., The classification of compact punctually cohesive Desarguesian projective Klingenberg planes, *Geom. Dedicata* 36, 347–358 (1990)

[H16] Lorimer J.W., Topological characterizations of finite desarguesian projective Hjelmslev planes, *Ars Comb. Ser. A* 29, 247–254 (1990)

[H17] Lorimer J.W., The classification of compact right chain rings, *Forum Math.* 4, 335–347 (1992)

[H18] Machala F., Angeordnete affine Klingenbergische Ebenen, *Czech. Math. J.* 30, 341—356 (1980)

[H19] Machala F., Angeordnete affine lokale Ternärringe und angeordnete affine Klingenbergischen Ebenen, *Czech. Math. J.* 30, 556–568 (1980)

[H20] Machala F., Fastgeordnete und geordnete affine Klingenbergische Ebenen, *Čas. Pěst. Mat.* 106, 138–155 (1981)

[H21] Machala F., Fastgeordnete und geordnete lokale Ringe und ihre geometrische Anwendung, *Čas. Pěst. Mat.* 106, 269–278 (1981)

[H22] Machala F., Über angeordnete affine Klingenbergische Ebenen, die sich in projektive Klingenbergische Ebenen einbetten lassen, *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.* 21, 9–31 (1982)

[H23] Machala F., Über die Fortsetzung einer Anordnung der affinen Klingenbergischen Ebene in einer Anordnung der projektiven Klingenbergischen Ebene, *Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math.* 22, 19–36 (1983)
I The revival of ring geometry (Veldkamp, Faulkner)

[Faulkner J.R.], Stable range and linear groups for alternative rings, *Geom. Dedicata* **14**, 177–188 (1983)

[Faulkner J.R.], Coordinatization of Moufang–Veldkamp planes, *Geom. Dedicata* **14**, 189–201 (1983)

[Faulkner J.R.], Barbilian planes, *Geom. Dedicata* **30**, 125–181 (1989)

[Faulkner J.R.], A geometric construction of Moufang planes, *Geom. Dedicata* **29**, 133–140 (1989)

[Faulkner J.R.], Current results on Barbilian planes, *Sci. Bull., Politech. Univ. Buchar., Ser. A* **55**, 146–152 (1993)

[Faulkner J.R.], Projective remoteness planes, *Geom. Dedicata* **60**, 237–275 (1996)

[Faulkner J.R.], The role of nonassociative algebra in projective geometry, *Graduate Studies in Mathematics*, 159. *American Mathematical Society*, Providence, RI, 229 pp. (2014)

[Ferrar J], Homomorphisms of Moufang-Veldkamp planes, *Geom. Dedicata* **46**, 299–311 (1993)
[109] Ferrar J. and Veldkamp F.D., Neighbor-preserving homomorphisms between projective ring planes, *Geom. Dedicata* **18**, 11–33 (1985)

[110] Ferrar J. and Veldkamp F.D., Admissible subrings revisited, *Geom. Dedicata* **23**, 229–236 (1987)

[111] Hanssens G. and Van Maldeghem H., A note on near–Barbilian planes, *Geom. Dedicata* **29**, 233–235 (1989)

[112] Keppens D., Neighbor-preserving epimorphisms between projective Klingenberg planes, *Geom. Dedicata* **29**, 209–219 (1989)

[113] Keppens D., Distant-preserving epimorphisms between projective Klingenberg planes, *Geom. Dedicata* **29**, 237–247 (1989)

[114] Knüppel F., Projective planes over rings, *Results Math.* **12**, 348–356 (1987)

[115] Knüppel F., Regular homomorphisms of generalized projective planes, *J. Geom.*, **29**, 170–181 (1987)

[116] Moore D’Ortona C., Homomorphisms of projective remoteness planes, *Geom. Dedicata* **72**, 111–122 (1998)

[117] Pén te k K., The minimal projective ring planes (in Hungarian), *Berzsenyi Dániel Tanárk. Főisk. Tud. Közl.*, Termtud. **8**, 33–70 (1992)

[118] Pén te k K., A generalization of the projective Klingenberg plane, *Berzsenyi Dániel Tanárk. Főisk. Tud. Közl.*, Termtud. **9**, 19–42 (1994)

[119] Pén te k K., The $\Delta^2$–configuration and the direct decomposition of projective Veldkamp planes (in Hungarian), *Berzsenyi Dániel Tanárk. Főisk. Tud. Közl.*, Termtud. **10**, 27–37 (1996)

[120] Pén te k K., On the direct decomposition of pappian projective Veldkamp planes, *Publ. Math.* **53**, 347–365 (1998)

[121] Pfeiffer T., Pappus’ theorem for ring-geometries, *Beitr. Algebra Geom.* **39**, 461–466 (1998)

[122] Spanicciati R., Near–Barbilian planes, *Geom. Dedicata* **24**, 311–318 (1987)

[123] Veldkamp F.D., Projective planes over rings of stable rank 2, *Geom. Dedicata* **11**, 285–308 (1981)

[124] Veldkamp F.D., Projective ring planes: some special cases, in *Atti Conv. Geometria combinatoria e di incidenza*, La Mendola, 1982, *Rend. Sem. Mat. Brescia* **7**, 609–615 (1984)
[I25] Veldkamp F.D., Distant–preserving homomorphisms between projective ring planes, *Nederl. Akad. Wetensch. Indag. Math.* **47**, 443–453 (1985)

[I26] Veldkamp F.D., Incidence–preserving mappings between ring planes, *Nederl. Akad. Wetensch. Indag. Math.* **47**, 455–459 (1985)

**J** Projective and affine Hjelmslev spaces and spaces over rings

[J01] Bach A., Teilverhältnisse in affinen Räumen über Moduln, *Beitr. Algebra Geom.* **38**, 385–398 (1997)

[J02] Bisztriczky T. and Lorimer J. W., Axiom systems for affine Klingenberg spaces, *Research and Lecture Notes in Mathematics* Combinatorics ’88, vol I, 185–200 (1991)

[J03] Bisztriczky T. and Lorimer J. W., On hyperplanes and free subspaces of affine Klingenberg spaces, *Aequationes Math.* **48**, 121–136 (1994)

[J04] Bisztriczky T. and Lorimer J. W., Subspace operations in affine Klingenberg spaces, *Bull. Belg. Math. Soc.–Simon Stevin* **2**, 99–108 (1995)

[J05] Bisztriczky T. and Lorimer J. W., Translations in affine Klingenberg spaces, *J. Geom.* **99**, 15–42 (2010)

[J06] Faure C.-A., Morphisms of projective spaces over rings, *Adv. Geom.* **4**, 19–31 (2004)

[J07] Greferath M., Global-affine morphisms of projective lattice geometries, *Results Math.* **24**, 76–83 (1993)

[J08] Greferath M. and Schmidt S.E., A unified approach to projective lattice geometries, *Geom. Dedicata* **43**, 243–264 (1992)

[J09] Greferath M. and Schmidt S.E., On Barbilian spaces in projective lattices geometries, *Geom. Dedicata* **43**, 337–349 (1992)

[J10] Greferath M. and Schmidt S.E., On point–irreducible projective lattice geometries, *J. Geom.* **50**, 73–83 (1994)

[J11] Greferath M. and Schmidt S.E., On stable geometries, *Geom. Dedicata* **51**, 181–199 (1994)

[J12] James D.G., Projective geometry over rings with stable range condition, *Linear and Multilinear Algebra* **23**, 299–304 (1988)
[J13] Jukl M., On homologies of Klingenberg projective spaces over special commutative local rings, *Publ. Math.* **55**, 113–121 (1999)

[J14] Kapralova, S. B., Dual Galois spaces (in Russian), *Trudy Geom. Sem. Kazan. Univ.* **12**, 38–44 (1980)

[J15] Kreis E., Koordinatisierung von verallgemeinerten affinen Räumen, *Result. Math.* **32**, 304–317 (1997)

[J16] Kreis E. and Schmidt S.E., Darstellung von Hyperebenen in verallgemeinerten affinen Räumen durch Moduln, *Results Math.* **26**, 39–50 (1994)

[J17] Kreuzer A., Hjelmslev-Räume, *Result. Math.* **12**, 148–156 (1987)

[J18] Kreuzer A., A system of axioms for projective Hjelmslev spaces, *J. Geom.* **40**, 125–147 (1991)

[J19] Kreuzer A., Fundamental theorem of projective Hjelmslev spaces, *Mitt. Math. Ges. Hamb.* **12**, 809–817 (1991)

[J20] Kreuzer A., Free modules over Hjelmslev rings in which not every maximal linearly independent subset is a basis, *J. Geom.* **45**, 105–113 (1992)

[J21] Kvirikashvili T.G., The fundamental theorem for affine geometries over rings (in russian), *Sooobshch. Akad. Nauk Gruz.* **148**, 196–197 (1993)

[J22] Kvirikashvili T.G., Projective geometries over rings and modular lattices. Algebra and geometry, *J. Math. Sci.* **153**, 495–505 (2008)

[J23] Kvirikashvili T.G. and Lashkhi A., Geometrical maps in ring affine geometries, *J. Math. Sci.* **186**, 759–765 (2012) [translated from Sovrem. Mat. Prilozhi. 74 (2011)]

[J24] Landjev I. and Vandendriesche P., On the point-by-subspace incidence matrices of projective Hjelmslev spaces, *C. R. Acad. Bulg. Sci.* **67**, 1485–1490 (2014)

[J25] Landjev I. and Vandendriesche P., On the rank of incidence matrices in projective Hjelmslev spaces, Des. Codes Cryptography **73**, 615–623 (2014)

[J26] Lashkhi A., The fundamental theorem of projective geometry for modules and Lie algebras, *J. Soviet Math.* **42**, 1991–2007 (1988) [translated from VINITI Itogi Nauki i tekni. Sovrem. Mat. i Prilozh., Geometria 18, 167–187 (1986)]

[J27] Lashkhi A., General geometric lattices and projective geometry of modules. *J. Math. Sci.* **74**, 1044–1077 (1995)
[J28] Lashkhi A., Ring geometries and their related lattices, *J. Math. Sci.* 144, 3960–3967 (2007) [translated from Fundam. Prikl. Mat. 11, 127–137 (2005)]

[J29] Lashkhi A. and Chkhatarashvili D., On the fundamental theorem of affine geometry over ring, *Bull. Georgian Acad. Sci.* 159, 17–19 (1999)

[J30] Lashkhi A. and Kvirikashvili T. G., Affine geometry of modules over a ring with an invariant basis number, *Math. Notes* 82, 756–765 (2007) [translated from Mat. Zametki 82, 838–849 (2007)].

[J31] Leissner W., On classifying affine Barbilian spaces, *Result. Math.* 12, 157–171 (1987)

[J32] Leissner W., Severin R. and Wolf K., Affine geometry over free unitary modules, *J. Geom.* 25, 101–120 (1985)

[J33] Limaye N. B., A generalization of Fano’s postulate, *Math. Stud.* 49, 125–127 (1981)

[J34] Lück H.-H., Projektive Hjelmslevräume, *J. Reine Angew. Math.* 243, 121–158 (1970)

[J35] Machala F., Projektive Abbildungen projektiver Räume mit Homomorphismus, *Czech. Math. J.* 25(100), 214–226 (1975)

[J36] Machala F., Homomorphismen projektiver Räume mit Homomorphismus, *Czech. Math. J.* 25(100), 454–474 (1975)

[J37] Machala F., Homomorphismen von projektiven Räumen und verallgemeinerte semilineare Abbildungen, *Čas. Pěst. Mat.* 100, 142–154 (1975)

[J38] Machala F., Eine Ebene im projektiven Raum mit Homomorphismus, *Acta Univ. Palack. Olomuc., Fac. Rer. Natur. Math.* 15, 5–21 (1976)

[J39] Machala F., Fundamentalsätze der projektiven Geometrie mit Homomorphismus, *Rozpr. Česk. Akad. Ved, Rada Mat. Prir. Ved* 90, 81 pp. (1980)

[J40] Magnus T. D., Faulkner geometry, *Geom. Dedicata* 59, 1–28 (1996)

[J41] Mathiak K., Eine geometrische Kennzeichnung von Homomorphismen desarguesscher projektiver Ebenen, *Math. Z.* 98, 259–267 (1967)

[J42] Mathiak K., Homomorphismen projektiver Räume und Hjelmslevsche Geometrie, *J. Reine Angew. Math.* 254, 41–73 (1972)

[J43] Mathiak K., Ein Beweis der Dimensionsformel in projektiven Hjelmslevschen Räumen, *J. Reine Angew. Math.* 256, 215–220 (1972)

[J44] Mathiak K., Bewertete Vektorräume, *J. Reine Angew. Math.* 257, 80–90 (1972)
[J45] Mathiak K., Kennzeichnende Eigenschaften bewerteter Vektorräume, *J. Reine Angew. Math.* 260, 127–132 (1973)

[J46] Mathiak K., *I*-Hülle und *I*-Kern von Moduln über Bewertungsringen, *J. Reine Angew. Math.* 283/284, 1–8 (1976)

[J47] Mathiak K., Schmitt und Verbindung in projektiven Hjelmslevschen Räumen, *J. Reine Angew. Math.* 283/284, 9–20 (1976)

[J48] Mathiak K., Projective Hjelmslevsche Räume im nicht invarianten Fall, *J. Reine Angew. Math.* 291, 182–188 (1977)

[J49] Mathiak K., Valuations of skew fields and projective Hjelmslev spaces, *Lecture Notes in Mathematics* 1175, Springer-Verlag, Berlin, 116 pp. (1986)

[J50] Mathiak K., Dualität in projektiven Hjelmslevschen Räumen, *Result. Math.* 12, 166–171 (1987)

[J51] Miron R., The minimality of Weyl’s system of axioms for the affine geometry over an unitary ring, *An. Stiint. Univ. “Al. I. Cuza” Iasi Sect. I a Mat. (N.S.)* 24, 15–19 (1978)

[J52] Ojanguren M. and Sridharan R., A note on the fundamental theorem of projective geometry, *Comment. Math. Helv.* 44, 310–315 (1969)

[J53] Ostrowski T. and Dunajewski K., An affine space over a module, *Int. Math. Forum* 4, 1457–1463 (2009)

[J54] Permutti R., Geometria affine su di un anello, *Atti Accad. Naz. Lincei, Mem., Cl. Sci. Fis. Mat. Nat., Sez. I, VIII.* 8, 259–287 (1967)

[J55] Pizzarello G., Sugli spazi affini sopra un anello, *Rend. Ist. Mat. Univ. Trieste* 1, 98–111 (1969)

[J56] Rostomashvili Z., Remark to the projective geometry over rings and corresponding lattices, *Bull. Georgian Acad. Sci.* 160, 211–212 (1999)

[J57] Sarath B. and Varadarajan K., Fundamental theory of projective geometry, *Comm. Algebra* 12, 937-952 (1984)

[J58] Schmidt S.E. and Weller S., Fundamentalsatz für affine Räume über Moduln, *Results Math.* 30, 151–159 (1996)

[J59] Seier W., Über Hjelmslev-Strukturen. I, *Abh. Math. Sem. Univ. Hamburg* 42, 107–133 (1974)

[J60] Seier W., Über Hjelmslev-Strukturen. II, *Abh. Math. Sem. Univ. Hamburg* 42, 236–254 (1974)
[J61] Sprenger N., Ein geometrischer Zugang zu Barbilianbereichen Berlin: Logos Verlag. Mainz: Univ. Mainz, Fachbereich Mathematik, 146 pp. (1999)

[J62] Veldkamp F.D., Projective Barbilian spaces. I., Results Math. 12, 222–240 (1987)

[J63] Veldkamp F.D., Projective Barbilian spaces. II., Results Math. 12, 434–449 (1987)

[J64] Veldkamp F.D., n–Barbilian domains, Results Math. 23, 177–200 (1993)

K Projective lines and circle geometries over rings

[K01] Bartnicka E. and Matraš A., The distant graph of the projective line over a finite ring with unity, Results Math. 72, 1943–1958 (2017)

[K02] Bartnicka E. and Matraš A., Free cyclic submodules in the context of the projective line, Results Math. 70, 567–580 (2016)

[K03] Bartolone C., Jordan homomorphisms, chain geometries and the fundamental theorem, Abh. Math. Sem. Univ. Hamburg 59, 93–99 (1989)

[K04] Bartolone C. and Bartolozzi F., Topics in geometric algebra over rings, in: Rings and geometry, Proc. NATO Adv. Study Inst., Istanbul/Turkey 1984, NATO ASI Ser., Ser. C 160, 353–389 (1985)

[K05] Bartolone C. and Di Franco F., A remark on the projectivities of the projective line over a commutative ring, Math. Z. 169, 23–29 (1979)

[K06] Benz W., Vorlesungen über Geometrie der Algebren, Springer, Berlin, 368 pp. (1973)

[K07] Blunck A., Cross–ratios over local alternative rings, Result. Math. 19, 246–256 (1991)

[K08] Blunck A., Chain geometries over local alternative algebras, J. Geom. 44, 33–44 (1992)

[K09] Blunck A., A quadric model for Klingenberg chain spaces, Geom. Dedicata 55, 237–246 (1995)

[K10] Blunck A. and Havlicek H., Projective representations. I: Projective lines over rings, Abh. Math. Sem. Univ. Hamburg 70, 287–299 (2000)

[K11] Blunck A. and Havlicek H., Projective representations. II: Generalized chain geometries, Abh. Math. Sem. Univ. Hamburg 70, 301–313 (2000)
[K12] **Blunck A. and Havlicek H.**, Extending the concept of chain geometry, *Geom. Dedicata* **83**, 119–130 (2000)

[K13] **Blunck A. and Havlicek H.**, The connected components on the projective line over a ring, *Adv. Geom.* **1**, 107–117 (2001)

[K14] **Blunck A. and Havlicek H.**, Radical parallelism on projective lines and non-linear models of affine spaces, *Math. Pannonica* **14**, 113–127 (2003)

[K15] **Blunck A. and Havlicek H.**, On distant-isomorphisms of projective lines, *Aequationes Math.* **69**, 146–163 (2005)

[K16] **Blunck A. and Havlicek H.**, Jordan homomorphisms and harmonic mappings, *Monatsh. Math.* **139**, 111–127 (2003)

[K17] **Blunck A. and Havlicek H.**, Projective lines over Jordan systems and geometry of Hermitian matrices, *Linear Algebra Appl.* **433**, 672–680 (2010)

[K18] **Blunck A. and Havlicek H.**, Geometries on $\sigma$–Hermitian matrices, *J. Math. Sci.* **186**, 715–719 (2012) [translated from Sovrem. Mat. Prilozh. 74 (2011)]

[K19] **Blunck A. and Herzer A.**, Kettengeometrien. Eine Einführung, *Berichte aus der Mathematik*, Shaker Verlag, Aachen, 337 pp. (2005)

[K20] **Blunck A. and Pianta S.**, Lines in 3–space, *Mitt. Math. Ges. Hamb.* **27**, 189–202 (2008)

[K21] **Blunck A. and Stroppel M.**, Klingenberg chain spaces, *Abh. Math. Sem. Univ. Hamburg* **65**, 225–238 (1995)

[K22] Chkhatarashvili D., K. von Staudt’s theorem over Ore domains, *Bull. Georgian Acad. Sci.* **158**, 18–20 (1998)

[K23] Cirlincione L. and Enea M.R., Una generalizzazione del birapporto sopra un anello, *Rend. Circ. Mat. Palermo (II)* **39**, 271–280 (1990)

[K24] **Havlicek H.**, Projective ring lines and their generalizations, *Electronic Notes in Discrete Mathematics* **40**, 151–155 (2013)

[K25] **Havlicek H.**, Von Staudt’s theorem revisited, *Aequationes Math.* **89**, 459–472 (2015)

[K26] **Havlicek H., Kosiorek J. and Odehnal B.**, A point model for the free cyclic submodules over ternions, *Result. Math.* **63**, 1071–1078 (2013)

[K27] **Havlicek H., Matraš A. and Pankov M.**, Geometry of free cyclic submodules over ternions, *Abh. Math. Sem. Univ. Hamburg* **81**, 237–249 (2011)
[K28] Havlicek H. and Saniga M., Vectors, cyclic submodules, and projective spaces linked with ternions, *J. Geom.* 92, 79–90 (2009)

[K29] Havlicek H. and Zanella C., Linear sets in the projective line over the endomorphism ring of a finite field, *J. Algebr. Comb.* 46, 297–312 (2017)

[K30] Herzer A. and Ramroth H., Die projektive Gerade über einem Ring, der direktes Produkt kommutativer Körper ist, *J. Algebra* 176, 1–11 (1995)

[K31] Jurga R., The cross-ratio in Hjelmslev planes, *Math. Bohem.* 122, 243–247 (1997)

[K32] Keppens D., Möbius planes with neighbor relation, *Simon Stevin* 61, 157–170 (1987)

[K33] Keppens D., Laguerre and Minkowski planes with neighbor relation, *J. Geom.* 30, 12–27 (1987)

[K34] Kulkarni M., Fundamental theorem of projective geometry over a commutative ring, *Indian J. Pure Appl. Math.* 11 (1980), 1561–1565 (1980)

[K35] Kvirikashvili T. and Lashkhi A., Harmonic maps and Von Staudt’s theorem over rings, *J. Math. Sci.* 195, 496–504 (2013)

[K36] Lang K., Spiegelungen in Hjelmslev’schen Kreisgeometrien, *Geom. Dedicata* 21, 107–121 (1986)

[K37] Lashkhi A., Harmonic maps over rings, *Georgian Math. J.* 4, 41–64 (1997)

[K38] Limaye N., Projectivities over local rings, *Math. Z.* 121, 175–180 (1971)

[K39] Limaye N., Cross ratios and projectivities of the line, *Math. Z.* 129, 49–53 (1972)

[K40] Limaye B and Limaye N., Fundamental theorem for the projective line over non-commutative local rings, *Arch. Math.* 28, 102–109 (1977)

[K41] Limaye B and Limaye N., The fundamental theorem for the projective line over commutative rings, *Aequationes Math.* 16, 275–281 (1977)

[K42] Limaye B. and Limaye N., Correction to “Fundamental theorem for the projective line over non-commutative local rings”, *Arch. Math.* 29, 672 (1977)

[K43] Matraś A. and Siemaszko A., The shortest path problem for the distant graph of the projective line over the ring of integers, *Bull. Malays. Math. Sci. Soc.* 41, 231–248 (2018)
[K44] Matraś A. and Siemaszko A., The Cayley property of some distant graphs and relationship with the Stern–Brocot tree, Result. Math. 73, No. 141, 14 pp. (2018)

[K45] McDonald, B. R., Projectivities over rings with many units, Comm. Algebra 9, 195–204 (1981)

[K46] McDonald, B. R., Geometric algebra over local rings, Pure and Applied Mathematics, No. 36. Marcel Dekker, Inc., New York-Basel, 421 pp. (1976)

[K47] Saniga M., Planat M., Kibler, M. and Pracna P., A classification of the projective lines over small rings, Chaos Solitons Fractals 33, 1095–1102 (2007)

[K47] Saniga M., Planat M. and Pracna P., A Classification of the projective lines over small rings II. Non-Commutative Case, arXiv:math/0606500 (2006)

[K49] Schaeffer H., Das von Staudtsche Theorem in der Geometrie der Algebren, J. Reine Angew. Math. 267, 133–142 (1974)

[K50] Seier W., Kettengeometrie über Hjelmslevringen, Beitr. Geom. Algebra, Proc. Symp. Duisburg 1976, 299–303 (1977)

[K51] Seier W., n–affine Ebenen mit Nachbarelementen, Math. Sem. Univ. Hamburg 50, 20–31 (1980)

L Ring geometries and buildings

[L01] Artmann B., Hjelmslev–Ebenen in projektiven Räumen, Arch. Math. 21, 304–307 (1970)

[L02] Bix R., Octonion planes over local rings, Trans. Amer. Math. Soc. 261, 417–438 (1980)

[L03] Bix R., Isomorphism theorems for octonion planes over local rings, Trans. Amer. Math. Soc. 266, 423–439 (1981)

[L04] Faulkner J.R., Octonion planes defined by quadratic Jordan algebras, Mem. Amer. Math. Soc. 104, 71 pp. (1970)

[L05] Faulkner J.R. and Ferrar J., Generalizing the Moufang plane, Rings and geometry (Istanbul, 1984), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Reidel, Dordrecht 160, 235–288 (1985)

[L06] Hall J. and Rao A., An algorithm for constructing Hjelmslev planes, in: Algebraic design theory and Hadamard matrices, Springer Proc. Math. Stat. 133, 137–147 (2015)
[L07] Hanssens G. and Van Maldeghem H., On projective Hjelmslev Planes of level $n$, *Glasgow Math. J.* **31**, 257–261 (1989)

[L08] Hanssens G. and Van Maldeghem H., A universal construction for projective Hjelmslev planes of level $n$, *Comp. Math.* **71**, 285–294 (1989)

[L09] Hanssens G. and Van Maldeghem H., Hjelmslev-Quadrangles of level $n$, *J. Combin. Theory Ser. A* **55**, 256–291 (1990)

[L10] Hanssens G. and Van Maldeghem H., A Characterization of $\tilde{C}_2$–buildings by floors, *Simon Stevin* **65**, 217–265 (1991)

[L11] James D.G., Polar spaces over rings with absolute stable range condition, *Linear Multilinear Algebra* **38**, 373–378 (1995)

[L12] Keppens D., Classical Klingenberg generalized quadrangles, *Arch. Math.* **55**, 619–624 (1990)

[L13] Keppens D. and Van Maldeghem H., Embeddings of projective Klingenberg planes in the projective space $\text{PG}(5,\mathbb{K})$, *Beiträge Algebra Geom.* **50**, 483–493 (2009)

[L14] Schillewaert J. and Van Maldeghem H., Imbrex geometries, *J. Comb. Theory, Ser. A* **127**, 286–302 (2014)

[L15] Schillewaert J. and Van Maldeghem H., Projective planes over 2–dimensional quadratic algebras, *Adv. Math.* **262**, 784–822 (2014)

[L16] Springer T. and Veldkamp F.D., On Hjelmslev–Moufang planes, *Math. Z.* **107**, 249–263 (1968)

[L17] Tits J., Sur la géométrie des $R$–espaces, *J. Math. Pure Appl.* **36**, 17–38 (1957)

[L18] Van Maldeghem H., Non–classical triangle buildings, *Geom. Dedicata* **24**, 123–206 (1987)

[L19] Van Maldeghem H., Valuations on PTR’s induced by triangle buildings, *Geom. Dedicata* **26**, 29–84 (1988)

[L20] Van Maldeghem H., Quadratic Quaternary Rings with Valuation and affine buildings of type $\tilde{C}_2$, *Mitt. Mathem. Sem. Giessen* **189**, 1–159 (1989)

[L21] Van Maldeghem H., An algebraic characterization of affine buildings of type $\tilde{C}_2$, *Mitt. Mathem. Sem. Giessen* **198**, 1–42 (1990)

[L22] Van Maldeghem H. and Van Steen K., Characterizations by automorphism groups of some rank 3 buildings I: Some properties of half strongly-transitive triangle buildings, *Geom. Dedicata* **73**, 119–142 (1998)
[L23] Van Maldeghem H. and Van Steen K., Characterizations by automorphism groups of some rank 3 buildings II: A half strongly-transitive locally finite triangle building is a Bruhat–Tits building, *Geom. Dedicata* 74, 113–133 (1999)

[L24] Van Steen K., Characterizations by automorphism groups of some rank 3 buildings III: Moufang-like conditions, *Geom. Dedicata* 74, 225–240 (1999)

[L25] Veldkamp F.D., Collineation groups in Hjelmslev–Moufang planes, *Math. Z.* 108, 37–52 (1968)

[L26] Veldkamp F.D., Unitary groups in Hjelmslev–Moufang planes, *Math. Z.* 108, 288–312 (1969)

[L27] De Schepper A. and Van Maldeghem H., Veronese representation of projective Hjelmslev planes over some quadratic alternative algebras, *Results Math.* 75, 1–51 (2020)

M Ring geometries and coding theory

[M01] Bini G. and Flamini F., Finite commutative rings and their applications, Kluwer, Boston–Dordrecht–London, 176 pp. (2002)

[M02] Boev S., Honold T. and Landjev I., Optimal arcs in Hjelmslev spaces of higher dimension, *C. R. Acad. Bulg. Sci.* 64, 625–632 (2011)

[M03] Boev S. and Landjev I., On blocking sets in affine Hjelmslev planes, *Serdica J. Comput.* 6, 175–184 (2012)

[M04] Byrne E., Greferath M. and Honold T., Ring geometries, two-weight codes, and strongly regular graphs, *Des. Codes Cryptography* 48, 1–16 (2008)

[M05] Chvál V. and Jurga R., Tangents of conics in Hjelmslev planes over a local ring of even characteristic, *Math. Slovaca* 48, 69–78 (1998)

[M06] Egorychev G. and Zima E., Simple formulae for the number of quadrics and symmetric forms of modules over local rings, *Commun. Algebra* 36, 1426–1436 (2008)

[M07] Hammons A. R., Kumar, P.V., Calderbank A.R., Sloane N.J.A. and Solé P., The Z4–linearity of Kerdock, Preparata, Goethals, and related codes, *IEEE Transactions on Information Theory* 40, 301–319 (1994)

[M08] Hemme L., Honold T. and Landjev I., Arcs in projective Hjelmslev spaces obtained from Teichmüller sets, in *Proceedings of the Seventh International Workshop on Algebraic and Combinatorial Coding Theory (ACCT 2000)*, Bansko, Bulgaria, 177–182 (2000)
[M09] Honold T., Modern Hjelmslev geometry (in Danish), Normat 33, 166–167 (1985)

[M10] Honold T. and Kiermaier M., The maximal size of 6- and 7-arcs in projective Hjelmslev planes over chain rings of order 9, Sci. China, Math. 55, 73–92 (2012)

[M11] Honold T. and Kiermaier M., Classification of maximal arcs in small projective Hjelmslev geometries, in Proceedings of the Tenth International Workshop on Algebraic and Combinatorial Coding Theory 2006, [arXiv:1503.02937], 112–117 (2006)

[M12] Honold T. and Kiermaier M., The existence of maximal \((q^2,2)\)-arcs in projective Hjelmslev planes over chain rings of length 2 and odd prime characteristic, Des. Codes Cryptography 68, 105–126 (2013)

[M13] Honold T., Kiermaier M. and Landjev I., New arcs of maximal size in projective Hjelmslev planes of order nine, C. R. Acad. Bulg. Sci. 63, 171–180 (2010)

[M14] Honold T. and Landjev I., Linearly representable codes over chain rings, Abh. Math. Sem. Univ. Hamburg 69, 187–203 (1999)

[M15] Honold T. and Landjev I., All Reed–Muller codes are linearly representable over the ring of dual numbers over \(Z_2\), IEEE Trans. Inf. Theory 45, 700–701 (1999)

[M16] Honold T. and Landjev I., Linear codes over finite chain rings, Electron. J. Comb. 7, Research paper R 11, 22 p. (2000); printed version J. Comb. 7 (2000)

[M17] Honold T. and Landjev I., Linear codes over finite chain rings and projective Hjelmslev geometries, in Codes over rings, Ser. Coding Theory Cryptol., 6, World Sci. Publ., Hackensack, NJ, 60–123 (2009)

[M18] Honold T. and Landjev I., Codes over rings and ring geometries, in Storme, Leo (ed.) et al., Current research topics in Galois geometry. New York, NY: Nova Science Publishers/Novinka, Mathematics Research Developments, 161–186 (2014)

[M19] Honold T. and Landjev I., Non–free extensions of the simplex codes over a chain ring with four elements, Des. Codes Cryptography 66, 27–38 (2013)

[M20] Honold T. and Landjev I., On arcs in projective Hjelmslev planes, Discrete Math. 231, 265–278 (2001)

[M21] Honold T. and Landjev I., On maximal arcs in projective Hjelmslev planes over chain rings of even characteristic, Finite Fields Appl. 11, 292–304 (2005)
[M22] Honold T. and Landjev I., Caps in projective Hjelmslev spaces over finite chain rings of nilpotency index 2, *Innov. Incidence Geom.* **4**, 13–25 (2006)

[M23] Honold T. and Landjev I., The dual construction for arcs in projective Hjelmslev spaces, *Adv. Math. Commun.* **5**, 11–21 (2011)

[M24] Jukl M. and Snášel V., Projective equivalence of quadrics in Klingenberg projective spaces over a special local ring, *Int. Electron. J. Geom.* **2**, 34–38 (2009)

[M25] Jurga R., Some combinatorial properties of conics in the Hjelmslev plane, *Math. Slovaca* **45**, 219–226 (1995)

[M26] Jurga R., Some problems of classification of points in the Desarguesian Hjelmslev plane, *Math. Slovaca* **47**, 563–574 (1997)

[M27] Jurga R., Some combinatorial results on the classification of lines in Desarguesian Hjelmslev planes, *Math. Slovaca* **48**, 79–85 (1998)

[M28] Keppens D., Polarities in finite 2-uniform projective Hjelmslev planes, *Geom. Dedicata* **24**, 51–76 (1987)

[M29] Keppens D., On polarities in the k-uniform n-dimensional projective Hjelmslev space PH(n,GF(q)[t]/t^k), q odd, *Results Math.* **12**, 297–324 (1987)

[M30] Keppens D., Polarities in n-uniform projective Hjelmslev planes, *Geom. Dedicata* **26**, 185–214 (1988)

[M31] Keppens D. and Mielants W., On the number of points on a plane algebraic curve over GF(q)[t]/t^n, *Ars Combin.* **40**, 121–128 (1995)

[M32] Kiermaier M., Koch M. and Kurz S., 2–arcs of maximal size in the affine and the projective Hjelmslev plane over Z_{25}, *Adv. Math. Commun.* **5**, 287–301 (2011)

[M33] Kiermaier M. and Kohrert A., New arcs in projective Hjelmslev planes over Galois rings, in *Proceedings of the Fifth International Workshop on Optimal Codes and Related Topics 2007*, [arXiv:1503.02932](https://arxiv.org/abs/1503.02932), 112–119 (2007)

[M34] Kiermaier M. and Zwanzger J., A Z_4–linear code of high minimum Lee distance derived from a hyperoval, *Adv. Math. Commun.* **5**, 275–286 (2011)

[M35] Kiermaier M. and Zwanzger J., New ring–linear codes from dualization in projective Hjelmslev geometries, *Des. Codes Cryptography* **66**, 39–55 (2013)
1. **Kohnert A.** Sets of type \((d_1, d_2)\) in projective Hjelmslev planes over Galois rings, in *Klin, Mikhail (ed.) et al., Algorithmic algebraic combinatorics and Gröbner bases. Dordrecht: Springer*, 269–278 (2009)

2. **Kossel M.** Symmetrische Ovale in Klingenberg–Ebenen, Aachen: Verlag Shaker. Darmstadt: TH Darmstadt, FB Math., 93 pp. (1996)

3. **Kulkarni M.** A generalisation of Pascal’s theorem to commutative rings, *Arch. Math.* **33**, 426–429 (1980)

4. **Landjev I.** Spreads in projective Hjelmslev spaces over finite chain rings, *Sci. Res.* **5**, 1–8 (2007)

5. **Landjev I.** On blocking sets in projective Hjelmslev planes, *Adv. Math. Commun.* **1**, 65–81 (2007)

6. **Landjev I. and Boev S.** A family of two-weight ring codes and strongly regular graphs, *C. R. Acad. Bulg. Sci.* **62**, 297–302 (2009)

7. **Landjev I. and Boev S.** Blocking sets of Rêdei type in projective Hjelmslev planes, *Discrete Math.* **310**, 2061–2068 (2010)

8. **Landjev I. and Honold T.** Arcs in projective Hjelmslev planes, *Discrete Math. Appl.* **11**, 53–70 (2001) [translated from Diskretnaya Mat. 13(1), 90–109 (2001) (in Russian)]

9. **Levchuk, V. M. and Starikova, O. A.** Quadratic forms of projective spaces over rings, *Sb. Math.* **197**, 887–899 (2006) [translated from Mat. Sb. 197, 97-110 (2006) (in Russian)].

10. **Little J.** Toric codes and finite geometries, *Finite Fields Appl.* **45**, 203–216 (2017)

11. **Little J.** Corrigendum to: “Toric codes and finite geometries”, *Finite Fields Appl.* **48**, 447–448 (2017)

12. **Nechaev, A.** Kerdock’s code in cyclic form, *Discrete Math. Appl.* **1**, 365–384 (1991) [translated from Diskret. Mat. 1, 123–139 (1989) (in Russian)]

13. **Nechaev A., Kuzmin A. and Markov, V.** Linear codes over finite rings and modules (in Russian), *Fundam. Prikl. Mat.* **3**, 195–254 (1997)

14. **Shiromoto K. and Storme L.** A Griesmer bound for codes over finite quasi–Frobenius rings, *International Workshop on Coding and Cryptography (Paris, 2001)*, 9 pp., *Electron. Notes Discrete Math.*, 6, Elsevier Sci. B. V., *Amsterdam* (2001)

15. **Stepień Z. and Szemaszkiewicz L.** Arcs in \(\mathbb{Z}_2^2\), *J. Comb. Optim.* **35**, 341–349 (2018)
[M51] Starikova, O. A. and Svistunova, A. V., Enumeration of quadrics of projective spaces over local rings, *Russ. Math.* 55, 48–51 (2011) [translated from Izv. Vyssh. Uchebn. Zaved., Mat. 2011, No. 12, 59–63 (2011) (in Russian)].

[M52] Starikova, O. A., Quadratic forms and quadrics of space over local rings, *J. Math. Sci.* 187, 177–186 (2012) [translated from Fundam. Prikl. Mat. 17, 97–110 (2012) (in Russian)].

[M53] Starikova, O. A., Classes of projectively equivalent quadrics over local rings, *Discrete Math. Appl.* 23 (2013), 385–398. [translated from Diskretn. Mat. 25, No. 2, 91–103 (2013) (in Russian)].

[M54] Landjev, I. and Georgieva, N., Conditions for the existence of spreads in projective Hjelmslev spaces, *Designs, Codes and Cryptography* 87 (2019), 785–794.

### N Ring geometries in quantum information theory

[N01] Havlicek H. and Saniga M., Projective ring line of a specific qudit, *J. Phys. A* 40, 943–952 (2007)

[N02] Havlicek H. and Saniga M., Projective ring line of an arbitrary single qudit, *J. Phys. A* 41, 12 pp. (2008)

[N03] Planat M. and Baboin A.-C., Qudits of composite dimension, mutually unbiased bases and projective ring geometry, *J. Phys. A, Math. Theor.* 40, 1005-1012 (2007)

[N04] Planat M., Baboin A.-C. and Saniga M., Multi-line geometry of qubit-qutrit and higher-order Pauli operators, *Int. J. Theor. Phys.* 47, 1127–1135 (2008)

[N05] Planat M., Rosu H. and Perrine S., A survey of finite algebraic geometrical structures underlying mutually unbiased quantum measurements, *Found. Phys.* 36, 1662–1680 (2006)

[N06] Planat M., Saniga M. and Kibler M., Quantum entanglement and projective ring geometry, *SIGMA Symmetry Integrability Geom. Methods Appl.* 2, 14 pp. (2006)

[N07] Saniga M. and Bartnicka E., Doily as subgeometry of a set of nonunimodular free cyclic submodules, *preprint arXiv:1812.01916* 5 december 2018], 1–5 (2018)

[N08] Saniga M., Havlicek H., Planat M. and Pracna P., Twin "Fano-snowflakes" over the smallest ring of ternions, *SIGMA Symmetry Integrability Geom. Methods Appl.* 4, Paper 050, 7 pp. (2008)
[N09] Saniga M. and Planat M., Hjelmslev geometry of mutually unbiased bases, J. Phys. A 39, 435–440 (2006)

[N10] Saniga M. and Planat M., Finite geometries in quantum theory: from Galois (fields) to Hjelmslev (rings), Internat. J. Modern Phys. B 20, 1885–1892 (2006)

[N11] Saniga M. and Planat M., A projective line over the finite quotient ring $\mathbb{GF}(2)[x]/(x^3 - x)$ and quantum entanglement: theoretical bases, Theoret. and Math. Phys. 151, 474–481 (2007) [translated from Teoret. Mat. Fiz. 151, 44–53 (2007) (in Russian)]

[N12] Saniga M. and Planat M., Projective planes over “Galois” double numbers and a geometrical principle of complementarity, Chaos Solitons Fractals 36, 374–381 (2008)

[N13] Saniga M. and Planat M., On the fine structure of the projective line over $\mathbb{GF}(2) \otimes \mathbb{GF}(2) \otimes \mathbb{GF}(2)$, Chaos Solitons Fractals 37, 337–345 (2008)

[N14] Saniga M., Planat M. and Minarovetch M., A projective line over the finite quotient ring $\mathbb{GF}(2)[x]/(x^3 - x)$ and quantum entanglement: the Mermin ”magic” square and pentagram, Theoret. and Math. Phys. 151, 625–631 (2007) [translated from Teoret. Mat. Fiz. 151, 219–227 (2007) (in Russian)]

[N15] Saniga M., Planat M. and Pracna P., Projective curves over a ring that includes two-qubits Theoret. and Math. Phys. 155, 905–913 (2008) [translated from Teoret. Mat. Fiz. 155, 463–473 (2008) (in Russian)].

[N16] Saniga M., Planat, M. and Pracna, P., Projective ring line encompassing two-qubits, Theor. Math. Phys. 155, 905–913 (2008) [translated from Teor. Mat. Fiz. 155, No. 3, 436–473 (2008) (in Russian)]

[N17] Saniga M. and Pracna P., A Jacobson radical decomposition of the Fano-Snowflake configuration, SIGMA Symmetry Integrability Geom. Methods Appl. 4, Paper 072, 7 pp. (2008)

[N18] Saniga M. and Pracna P., Space versus time: unimodular versus non-unimodular projective ring geometries?, Journal of Cosmology 4, 719–735 (2010)