Monte Carlo Tree Search guided by Symbolic Advice for MDPs

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Abstract. In this paper, we consider the online computation of a strategy that aims at optimizing the expected average reward in a Markov decision process. The strategy is computed with a receding horizon and using Monte Carlo tree search (MCTS). We augment the MCTS algorithm with the notion of symbolic advice, and show that its classical theoretical guarantees are maintained. Symbolic advice are used to bias the selection and simulation strategies of MCTS. We describe how to use QBF and SAT solvers to implement symbolic advice in an efficient way. We illustrate our new algorithm using the popular game Pac-Man and show that the performances of our algorithm exceed those of plain MCTS as well as the performances of human players.

1 Introduction

Markov Decision processes (MDP) are an important mathematical formalism for modeling and solving sequential decision problems in stochastic environments [19]. The importance of this model has triggered a large number of works in different research communities within computer science, most notably in formal verification, and in artificial intelligence and machine learning. The works done in these research communities have respective weaknesses and complementary strengths. On the one hand, algorithms developed in formal verification are complete and provides strong guarantees on the optimality of computed solutions but they tend to be applicable to models of moderate sizes only. On the other hand, algorithms developed in artificial intelligence and machine learning usually scale to larger models but only provide weaker guarantees. Instead of opposing the two sets of algorithms, there have been recent works, see e.g. [5,10,9,16,1], that try to combine the strengths of the two approaches in order to offer new hybrid algorithms that scale better and provide stronger guarantees. The contributions described in this paper are part of this research agenda: we show how to integrate symbolic advice defined by formal specifications into Monte-Carlo Tree Search algorithms [6] using techniques such as SAT [17] and QBF [20].

When a MDP is too large to be analyzed offline using verification algorithms, receding horizon analysis combined with simulation techniques are used online. Receding horizon techniques work as follows. In the current state $s$ of the MDP, for a fixed horizon $H$, the receding horizon algorithm searches for an action $a$ that is the first action of a plan to act (close too) optimally on the finite horizon $H$. When such an action is identified, then it is played from $s$ and the state evolves stochastically to a new state $s'$ according to the dynamics specified by the MDP. The same process is repeated from $s'$. The optimization criteria over the $H$ next step depends on the long run measure that needs to be optimized. The tree unfolding from $s$ that needs to be analyzed is often very large (e.g. it may be exponential in $H$). As a consequence, receding horizon techniques are often coupled with sampling techniques that avoid the systematic exploration of the entire tree unfolding at the expense of approximation. The Monte Carlo Tree Search (MCTS) algorithm [6] is an increasingly popular tree search algorithm that implements those ideas, e.g. it is one of the core building block of the AlphaGo algorithm [21].

* Computational resources have been provided by the Consortium des Équipements de Calcul Intensif (CECI), funded by the Fonds de la Recherche Scientifique de Belgique (F.R.S.-FNRS) under Grant No. 2.5020.11.
While MCTS techniques may offer reasonable performances out of the shelf, they usually need substantial adjustments that depend on the application to really perform well. One way to adapt MCTS to a particular application is to bias the search towards promising subspaces taking into account properties of the application domain \[13,22\]. This is usually done by coding directly handcrafted search and sampling strategies. We show in this paper how to use techniques from formal verification to offer a flexible and rigorous framework to bias the search performed by MCTS using symbolic advice. A symbolic advice is a formal specification, that can be expressed for example in your favorite linear temporal logic, and which constrain the search and the sampling phases of the MCTS algorithm using QBF and SAT solvers. Our framework offers in principle the ability to easily experiment with precisely formulated bias expressed declaratively using logic.

**Contributions.** On the theoretical side, we study the impact of using symbolic advice on the guarantees offered by MCTS. We identify sufficient conditions for the symbolic advice to preserve the convergence guarantees of the MCTS algorithm (Theorem 2). Those results are partly based on an analysis of the incidence of sampling on those guarantees (Theorem 1) which can be of independent interest.

On a more practical side, we show how symbolic advice can be implemented using SAT and QBF techniques. More precisely, we use QBF to force \[13\] that all the prefixes explored by the MCTS algorithm in the partial tree unfolding have the property suggested by the selection advice (whenever possible) and we use SAT-based sampling techniques \[8\] to achieve uniform sampling among paths of the MDP that satisfy the sampling advice. The use of this symbolic exploration techniques is important as the underlying state space that we need to analyze is usually huge (e.g. exponential in the receding horizon \(H\)).

![Fig. 1: We used two grids of size 9 × 21 and 27 × 28 for our experiments. Pac-Man loses if he makes contact with a ghost, and wins if he eats all food pills (in white). The agents can travel in four directions unless they are blocked by the walls in the grid, and Ghosts cannot reverse their direction. The score decreases by 1 at each step, and increases by 10 whenever Pac-Man eats a food pill. A win (resp. loss), increases (resp. decreases) the score by 500. The game can be seen as an infinite duration game by saying that whenever Pac-Man wins or loses, the positions of the agents and of the food pills are reset.](image)

To demonstrate the practical interest of our techniques, we have applied our new MCTS with symbolic advice algorithm to play PAC-MAN. Fig. 1 shows a grid of the PAC-MAN game. In this version of the classical game, the agent Pac-Man has to eat food pills as fast as possible while avoiding being pinched by ghost. We have chosen this benchmark to evaluate our algorithm for several reasons. First, the state space of the underlying MDP is way too large for the state of the art implementations of complete algorithms. Indeed, the reachable state space of the small grid shown here has approximately \(10^{16}\) states, while the classical grid has approximately \(10^{23}\) states. Our algorithm can handle both grids. Second, this application allows for comparison between performances obtained from several versions of the MCTS algorithm but also with the performances that humans can attain in this game. In the PAC-MAN benchmark, we show that advice that
instructs Pac-Man on the one hand to avoid ghost at all cost during the selection phase of the
MCTS algorithm (enforced whenever possible by QBF) and on the other hand to bias the search
to path in which ghosts are avoided (using uniform sampling based on SAT) allow to attain or
surpass human level performances while the standard MCTS algorithm performs much worse.

Related works. Our analysis of the convergence of the MCTS algorithm with appropriate sym-
bolic advice is based on extensions of analysis results based on bias defined using UCT (bandit
algorithms) [15][2]. Those results are also related to sampling techniques for finite horizon objectives
in MDP [14].

Our concept of selection phase advice is related to the MCTS MinMax hybrid algorithm pro-
posed in [3]. There the selection phase advice is not specified declaratively using logic but encoded
directly in the code of the search strategy. No use of QBF nor SAT is advocated there and no use
of sampling advice either. In [1], the authors provide a general framework to add safety proper-
ties to reinforcement learning algorithm via shielding. These techniques analyse statically the full
state space of the game in order to compute a set of unsafe actions to avoid. This fits our advice
framework, so that such a shield could be used as an online selection advice in order to combine
their safety guarantees with our formal results for MCTS. Note that in general multiple ghosts
may prevent the existence of a strategy to enforce safety, i.e. always avoid pincer moves.

Our practical handling of symbolic sampling advice relies on symbolic sampling techniques
introduced in [7], while our handling of symbolic selection advice relies on natural encodings via
QBF that are similar to those defined in [15].

2 Preliminaries

A probability distribution on a finite set $S$ is a function $d : S \rightarrow [0, 1]$ such that $\sum_{s \in S} d(s) = 1$. We denote the set of all probability distributions on set $S$ by $\mathcal{D}(S)$. The support of a distribution $d \in \mathcal{D}(S)$ is $\text{Supp}(d) = \{ s \in S \mid d(s) > 0 \}$.

2.1 Markov decision process

Definition 1 (MDP). A Markov decision process is a tuple $M = (S, A, P, R, R_T)$, where $S$ is
a finite set of states, $A$ is a finite set of actions, $P$ is a mapping from $S \times A$ to $\mathcal{D}(S)$ such that
$P(s, a)(s')$ denotes the probability that action $a$ in state $s$ leads to state $s'$; $R : S \times A \rightarrow \mathbb{R}$ defines
the reward obtained for taking a given action at a given state, and $R_T : S \rightarrow \mathbb{R}$ assigns a terminal
reward to each state in $S$.

For a Markov Decision Process $M$, a path of length $i > 0$ is a sequence of $i$ consecutive states
and actions followed by a last state. We say that $p = s_0a_0s_1 \ldots s_i$ is an $i$-length path in the
MDP $M$ if for all $t \in \{0, i - 1\}$, $a_t \in A$ and $s_{t+1} \in \text{Supp}(P(s_t, a_t))$, and we denote $\text{last}(p) = s_i$ and
first$(p) = s_0$. We also consider states to be paths of length 0. An infinite path is an infinite sequence
$p = s_0a_0s_1 \ldots$ of states and actions such that for all $t \in \mathbb{N}$, $a_t \in A$ and $s_{t+1} \in \text{Supp}(P(s_t, a_t))$. We
denote the finite prefix of length $t$ of a finite or infinite path $p = s_0a_0s_1 \ldots$ by $p_t = s_0a_0 \ldots s_t$.
Let $p = s_0a_0s_1 \ldots s_i$ and $p' = s'_0a'_0s'_1 \ldots s'_j$ be two paths such that $s_i = s'_0$, let $a$ be an action and
$s$ be state of $M$. Then, $p \cdot p'$ denotes $s_0a_0s_1 \ldots s_i a'_0s'_1 \ldots s'_j$ and $p \cdot as$ denotes $s_0a_0s_1 \ldots s_i as$.

For a MDP $M$, the set of all finite paths of length $i$ is denoted by $\text{Paths}^i_M$. Let $\text{Paths}^i_M(s)$
denote the set of paths $p$ in $\text{Paths}^i_M$ such that $\text{first}(p) = s$. Similarly, if $p \in \text{Paths}^i_M$ and $i \leq j$,
then let $\text{Paths}^j_M(p)$ denote the set of paths $p'$ in $\text{Paths}^j_M$ such that there exists $p'' \in \text{Paths}^j_M$ with
$p' = p \cdot p''$. We denote the set of all finite paths in $M$ by $\text{Paths}_M$ and the set of finite paths of
length at most $H$ by $\text{Paths}^H_M$.

Definition 2. The total reward of a finite path $p = s_0a_0 \ldots s_n$ in $M$ is defined as

$$\text{Reward}_M(p) = \sum_{t=0}^{n-1} R(s_t, a_t) + R_T(s_n).$$
A (probabilistic) strategy is a function \( \sigma : \text{Paths}_M \rightarrow \mathcal{D}(A) \) that maps a path \( p \) to a probability distribution in \( \mathcal{D}(A) \). A strategy \( \sigma \) is deterministic if the support of the probability distributions \( \sigma(p) \) has size 1, it is memoryless if \( \sigma(p) \) depends only on \( \text{last}(p) \), i.e., if \( \sigma \) satisfies that for all \( p, p' \in \text{Paths}_M \), last\( (p) = \text{last}(p') \Rightarrow \sigma(p) = \sigma(p') \). For a probabilistic strategy \( \sigma \) and \( i \in \mathbb{N} \), let \( \text{Paths}^i_M(\sigma) \) denote the paths \( p = s_0 a_0 \ldots s_i \in \text{Paths}_M \) such that for all \( t \in [0, i-1] \), \( a_t \in \text{Supp}(\sigma(p_t)) \). For a finite path \( p \) of length \( i \in \mathbb{N} \) and some \( j \geq i \), let \( \text{Paths}^i_M(p, \sigma) \) denote \( \text{Paths}^i_M(\sigma) \cap \text{Paths}^j_M(p) \).

For a strategy \( \sigma \) and \( p \in \text{Paths}^i_M(\sigma) \), let the probability of \( \sigma = p = s_0 a_0 \ldots s_n \in M \) according to \( \sigma \) be defined as \( P^i_{M, \sigma}(p) = \prod_{t=0}^{n-1} \sigma(p_t)(a_t)P(s_t, a_t)(s_{t+1}) \). The mapping \( P^i_{M, \sigma} \) defines a probability distribution over \( \text{Paths}^i_M(\sigma) \).

**Definition 3.** The expected average reward of a deterministic strategy \( \sigma \) in a MDP \( M \), starting from state \( s \), is defined as

\[
\text{Val}_M(s, \sigma) = \liminf_{n \to \infty} \frac{1}{n} \mathbb{E}[\text{Reward}_M(p)],
\]

where \( p \) is a random variable over \( \text{Paths}^n_M(\sigma) \) following the distribution \( P^n_{M, \sigma} \).

**Definition 4.** The optimal expected average reward starting from a state \( s \) in a MDP \( M \) is defined over all strategies \( \sigma \) in \( M \) as \( \text{Val}_M(s) = \sup_\sigma \text{Val}_M(s, \sigma) \).

One can restrict the supremum to deterministic memoryless strategies \cite[Proposition 6.2.1]{19}. A strategy \( \sigma \) is called \( \epsilon \)-optimal for the expected average reward if \( \text{Val}_M(s, \sigma) \geq \text{Val}_M(s) - \epsilon \) for all \( s \).

**Definition 5.** The expected total reward of a deterministic strategy \( \sigma \) in a MDP \( M \), starting from state \( s \) and for a finite horizon \( i \), is defined as \( \text{Val}^i_M(s, \sigma) = \mathbb{E}[\text{Reward}_M(p)] \), where \( p \) is a random variable over \( \text{Paths}^i_M(\sigma) \) following the distribution \( P^i_{M, \sigma} \).

**Definition 6.** The optimal expected total reward starting from a state \( s \) in a MDP \( M \), with horizon \( i \in \mathbb{N} \), is defined over all strategies \( \sigma \) in \( M \) as \( \text{Val}^i_M(s) = \sup_\sigma \text{Val}^i_M(s, \sigma) \).

One can restrict the supremum to deterministic strategies \cite[Theorem 4.4.1.b]{19}.

Let \( \sigma^i_{M,s} \) denote a deterministic strategy that maximizes \( \text{Val}^i_M(s, \sigma) \), and refer to it as an optimal strategy for the expected total reward of horizon \( i \) at state \( s \). For \( i \in \mathbb{N} \), let \( \sigma^i_M \) refer to a deterministic memoryless strategy that maps every state \( s \) in \( M \) to the first action of a corresponding optimal strategy for the expected total reward of horizon \( i \), so that \( \sigma^i_M(s) = \sigma^i_{M,s}(s) \). This strategy can be obtained by the value iteration algorithm:

**Proposition 1** (Value iteration \cite[Section 4.5.]{19}). For an state \( s \) in MDP \( M \), for all \( i \in \mathbb{N} \),

\[
\begin{align*}
\text{Val}^i_M(s) &= \max_{a \in A} \mathbb{E}[\text{Reward}_M(s, a)] + \sum_{s' \in M} P(s, a)(s')\text{Val}^i_M(s')
\end{align*}
\]

Moreover, for a large class of MDPs and a large enough \( n \), the strategy \( \sigma^n_M \) is \( \epsilon \)-optimal for the expected average reward:

**Proposition 2.** \cite[Theorem 9.4.5]{19} For a strongly aperiodic Markov decision process \( M \), it holds that \( \text{Val}_M(s) = \lim_{n \to \infty} [\text{Val}^n_M(s) - \text{Val}^0_M(s)] \). Moreover, for any \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), \( \text{Val}_M(s, \sigma^n_M) \geq \text{Val}_M(s) - \epsilon \) for all \( s \).

A simple transformation can be used to make a MDP strongly aperiodic without changing the optimal expected average reward and the associated optimal strategies. Therefore, one can use an algorithm computing the strategy \( \sigma^n_M \) in order to optimise for the expected average reward, and obtain theoretical guarantees for an horizon \( H \) big enough. This is known as the receding horizon approach.

Finally, we will use the notation \( T(M, s_0, H) \) to refer to an MDP obtained as a tree-shaped unfolding of \( M \) from state \( s_0 \) and for a depth of \( H \). Then, it holds that:

**Lemma 1.** \( \text{Val}^H_M(s_0) \) is equal to \( \text{Val}^H_T(M, s_0, H)(s_0) \), and \( \sigma^H_M(s_0) \) is equal to \( \sigma^H_T(M, s_0, H)(s_0) \).

The aperiodicity and unfolding transformations are detailed in Appendix A.

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1 A Markov decision process is strongly aperiodic if \( P(s, a)(s) > 0 \) for all \( s \in S \) and \( a \in A(s) \).
2.2 Bandit problems and UCB

In this section, we present bandit problems, whose study forms the basis of a theoretical analysis of Monte Carlo tree search algorithms.

Let $A$ denote a finite set of actions. For each $a \in A$, let $(x_{a,t})_{t \geq 1}$ be a sequence of random payoffs associated to $a$. They correspond to successive plays of action $a$, and for every action $a$ and every $t \geq 1$, let $x_{a,t}$ be drawn in a probability distribution $D_{a,t}$ over $[0,1]$. We denote $X_{a,t}$ the random variable associated to this drawing. In a fixed distributions setting (the classical bandit problem), every action is associated to a fixed probability distribution $D_a$, so that $D_{a,t} = D_a$ for all $t \geq 1$.

The bandit problem consists of a succession of steps where the player selects an action and observes the associated payoff, while trying to maximise the cumulated gains. For example, selecting action $a$, then $b$ and then $a$ again would yield the respective payoffs $x_{a,1}$, $x_{b,1}$ and $x_{a,2}$ for the first three steps, drawn from their respective distributions. Let the regret $R_n$ denote the difference, after $n$ steps, between the optimal expected payoff $\max_{a \in A} E[\sum_{t=1}^{n} X_{a,t}]$ and the expected payoff associated to our action selection. The goal is to minimise the long-term regret when the number of steps $n$ diverges.

The algorithm UCB1 of [2] offers a practical solution to this problem, and offers theoretical guarantees. For an action $a$ and $n \geq 1$, let $\overline{X}_{a,n} = \frac{1}{n} \sum_{t=1}^{n} x_{a,t}$ denote the average payoff obtained from the first $n$ plays of $a$. Moreover, for a given step number $t$, let $t_a$ denote how many times action $a$ was selected in the first $t$ steps. The algorithm UCB1 chooses, at step $t + 1$, the action $a$ that maximises $\overline{X}_{a,t_a} + c_t \sqrt{\frac{\log t}{t_a}}$, where $c_t$ is defined as $\sqrt{\frac{2 \ln 2}{t}}$. This procedure enjoys optimality guarantees detailed in [2], as it keeps the regret $R_n$ below $O(\log n)$.

We will make use of an extension of these results to the general setting of non-stationary bandit problems, where the distributions $D_{a,t}$ are no longer fixed with respect to $t$. This problem has been studied in [15], and results were obtained for a class of distributions $D_{a,t}$ that respect assumptions referred to as drift conditions.

For a fixed $n \geq 1$, let $\overline{X}_{a,n}$ denote the random variable obtained as the average of the random variables associated with the first $n$ plays of $a$. Let $\mu_{a,n} = E[\overline{X}_{a,n}]$. We assume that these expected means eventually converge, and let $\mu_a = \lim_{n \to \infty} \mu_{a,n}$.

Definition 7 (Drift conditions). For all $a \in A$, the sequence $(\mu_{a,n})_{n \geq 1}$ converges to some value $\mu_a$. Moreover, there exists a constant $C_p > 0$ and an integer $N_p$, such that for $n \geq N_p$ and any $\delta > 0$, if $\Delta_n(\delta) = C_p \sqrt{n \ln(1/\delta)}$ then the tail inequalities $\mathbb{P}[n \overline{X}_{a,n} \geq n \mu_{a,n} + \Delta_n(\delta)] \leq \delta$ and $\mathbb{P}[n \overline{X}_{a,n} \leq n \mu_{a,n} - \Delta_n(\delta)] \leq \delta$ hold.

We recall in Appendix [3] the results of [15], and provide an informal description of those results here. Consider using the algorithm UCB1 on a non-stationary bandit problem satisfying the drift conditions, with $c_{t,t_a} = 2 C_p \sqrt{\frac{\log t}{t_a}}$. First, one can bound logarithmically the number of times a suboptimal action is played. This is used to bound the difference between $\mu_a$ and $E[\overline{X}]$ by $O(\ln n)$, where $a$ is an optimal action and where $\overline{X}_n$ denote the global average of payoffs received over the first $n$ steps. This is the main theoretical guarantee obtained for the optimality of UCB1. Also for any action $a$, the authors state a lower bound for number of times the action is played. The authors also prove a tail inequality similar to the one described in the drift conditions, but on the random variable $\overline{X}_n$ instead of $\overline{X}_{a,n}$. This will be useful for inductive proofs later on, when the usage of UCB1 is nested so that the global sequence $\overline{X}_n$ corresponds to a sequence $\overline{X}_{b,n}$ of an action $b$ of higher order. Finally, it is shown that the probability of making the wrong decision (choosing a suboptimal action) converges to 0 as the number of plays $n$ grows large enough.

3 Monte Carlo tree search with simulation

In a receding horizon approach, the objective is to compute $V^H_M(s_0)$ and $\sigma^H_M$ for some state $s_0$ and some horizon $H$. Exact procedures such as the recursive computation of Proposition [1] can
Backpropagation Phase.

One can compute $\text{Val}^H_M$ and $\sigma^H_M(s_0)$ on the unfolding $T(M, s_0, H)$. Note that rewards in the MDP $M$ are bounded. For the sake of simplicity we assume without loss of generality that for all paths $p$ of length at most $H$ the total reward $\text{Reward}_M(p)$ belongs to $[0, 1]$.

Given an initial state $s_0$, MCTS is an iterative process that incrementally constructs a search tree rooted at $s_0$ describing paths of $M$ and their associated values. This process goes on until a specified budget (of number of iterations or time) is exhausted. An iteration constructs a path in $M$ by following a decision strategy to select a sequence of nodes in the search tree. When a node that is not part of the current search tree is reached, the tree is expanded with this new node, whose expected reward is approximated by simulation. This value is then used to update the knowledge of all selected nodes, in a backpropagation.

In the search tree, each node represents a path. For a node $p$ and an action $a \in A$, let $\text{children}(p, a)$ be a list of nodes representing paths of the form $p \cdot as'$ where $s' \in \text{Supp}(P(\text{last}(p), a))$. For each node (resp. node-action pair) we store a value $\text{value}(p)$ (resp. $\text{value}(p, a)$) computed for node $p$ (resp. for playing $a$ from node $p$), meant to approximate $\text{Val}^H_M(p)$ (resp. $R(\text{last}(p), a) + \sum_{a'} P(\text{last}(p), a)(s')\text{Val}^{|p|-1}(s')$, and a counter $\text{count}(p)$ (resp. $\text{count}(p, a)$), that keeps track of the number of iterations that selected node $p$ (resp. that selected the action $a$ from $p$). We add subscripts $i \geq 1$ to these notations to denote the number of previous iterations, so that $\text{value}_i(p)$ is the value of $p$ obtained after $i$ iterations of MCTS, among which $p$ was selected $\text{count}_i(p)$ times. We also define $\text{total}_i(p)$ and $\text{total}_i(p, a)$ as shorthand for respectively $\text{value}_i(p) \times \text{count}_i(p)$ and $\text{value}_i(p, a) \times \text{count}_i(p, a)$. Each iteration consists of three phases. Let us describe these phases at iteration number $i$.

**Selection Phase.** Starting from the root node, MCTS descends through the existing search tree by choosing actions based on the current values and counters and by selecting next states stochastically according to the MDP. This continues until reaching a node $q$, either outside of the search tree or at depth $H$. In the former case, the simulation phase is called to obtain a value $\text{value}_i(q)$ that will be backpropagated along the path $q$. In the latter case, we use the exact value $\text{value}_i(q) = R_T(\text{last}(q))$ instead.

The action selection process needs to balance between the exploration of new paths and the exploitation of known, promising paths. A popular way to balance both is the Upper Confidence Bound for Trees (UCT) algorithm [15], that interprets the action selection problem of each node of the MCTS tree as a bandit problem, and selects an action $a^*$ at random in

$$\arg \max_{a \in A} \left[ \text{value}_{i-1}(p, a) + C \sqrt{\frac{\ln(\text{count}_{i-1}(p))}{\text{count}_{i-1}(p, a)}} \right],$$

for some constant $C$.

**Simulation Phase.** In the simulation phase, the goal is to get an initial approximation for the value of a node $p$, that will be refined in future iterations of MCTS. Classically, a sampling-based approach can be used, where one computes a fixed number $c \in \mathbb{N}$ of paths $p \cdot p'$ in $\text{Paths}^H_M(p)$. Then, one can compute $\text{value}_i(p) = \frac{1}{c} \sum_{p'} \text{Reward}_M(p')$, and fix $\text{count}_i(p)$ to 1. Usually, the samples are derived by selecting actions uniformly at random in the MDP.

In our theoretical analysis of MCTS, we take a more general approach to the simulation phase, defined by a finite domain $I \subseteq [0, 1]$ and a function $f : \text{Paths}^H_M \rightarrow \mathcal{D}(I)$ that maps every path $p$ to a probability distribution on $I$. In this approach, the simulation phase simply draws a value $\text{value}_i(p)$ at random according to the distribution $f(p)$, and sets $\text{count}_i(p) = 1$.

**Backpropagation Phase.** From the value $\text{value}_i(p)$ obtained at a leaf node $p = s_0a_0s_1\ldots s_h$ at depth $h$ in the search tree, let $\text{reward}_i(p_k) = \sum_{l=k}^{h-1} R(s_l, a_l) + \text{value}_i(p)$ denote the reward associated with the path from node $p_k$ to $p$ in the search tree. For $k$ from $0$ to $h - 1$ we update

\[2\text{ There are finitely many paths of length at most } H, \text{ with rewards in } \mathbb{R}.\]
the values according to $\text{value}_i(p_k) = \frac{\text{total}_{i-1}(p_k, a_k) + \text{reward}(p_k)}{\text{count}(p_k)}$. The value $\text{value}_i(p_k, a_k)$ is updated based on $\text{total}_{i-1}(p_k, a_k)$, $\text{reward}(p_k)$ and $\text{count}(p_k, a_k)$ with the same formula.

**Theoretical analysis.** In the remainder of this section, we prove Theorem 1 that provides theoretical properties of the MCTS algorithm with a general simulation phase (defined by some fixed $I$ and $f$). This theorem was proven in [15, Theorem 6] for a version of the algorithm that called MCTS recursively until leaves were reached, as opposed to the more practical sampling-based approach that has become standard in practice. Note that sampling-based approaches are captured by our general description of the simulation phase: For a number of samples $c = 1$, as one can pick $I$ to be the set of rewards associated with paths of $\text{Paths}_M$, while $f(p)$ is a probability distribution over $I$ such that a reward $\text{Reward}_x(p') \in I$ is given the probability of path $p'$ being selected with an uniform action selections in $T(M, s_0, H)$, starting from the node $p$. For $c > 1$ one simply needs to extend $I$ to be the set of average rewards over $c$ paths, while $f(p)$ becomes a distribution over average rewards.

**Theorem 1.** Consider a MDP $M$, an horizon $H$ and a state $s_0$. Let $V_n(s_0)$ be a random variable that represents the value $\text{value}_n(s_0)$ at the root of the search tree after $n$ iterations of the MCTS algorithm on $M$. Then, $\mathbb{E}[V_n(s_0)] - \text{Val}^H_I(s_0) = \mathcal{O}((\ln n)/n)$. Moreover, the failure probability $\mathbb{P}[\arg\max_a \text{value}_n(s_0, a) \neq \sigma^Y_I(s_0)]$ converges to zero.

Following the proof scheme of [15, Theorem 6], this theorem is obtained from the results mentioned in Section 2.2. To this end, every node $p$ of the search tree is considered to be an instance of a bandit problem with non-stationary distributions. Every time a node is selected, a step is processed in the corresponding bandit problem.

Let $(I_t(p))_{t \geq 1}$ be a sequence of iteration numbers for the MCTS algorithm that describes when the node $p$ is selected, so that the simulation phase was used on $p$ at iteration number $I_t(p)$, and so that the $i$-th selection of node $p$ happened on the iteration number $I_i(p)$. We define sequences $(I_i(p, a))_{i \in \mathbb{N}}$ similarly for node-action pairs.

For all paths $p$ and actions $a$, a payoff sequence $(x_{a,t})_{t \geq 1}$ of associated random variables $(X_{a,t})_{t \geq 1}$ is defined by $x_{a,t} = \text{reward}_{I_t(p,a)}(p)$.

According to the notations of Section 2.2, we have $(x_{a,t})_{t \geq 1}$ for all $t \geq 1$ we have $\text{count}_{I_i(p)}(p) = t$, $\text{count}_{I_i(p)}(p, a) = t_a$ and $\text{value}_{I_i(p)}(p, a) = \mathcal{E}_{a,t}$. Then, one can obtain Theorem 1 by applying inductively the UCB1 results recalled in Appendix B on the search tree in a bottom-up fashion. Indeed, as the root $s_0$ is selected at every iteration, $I_n(s_0) = n$ and $\text{value}_n(s_0) = \mathcal{E}_n$, while $\text{Val}^H_I(s_0)$ corresponds to recursively selecting optimal actions by Proposition 1.

The main difficulty, and the difference our simulation phase brings compared with the proof of [15, Theorem 6], lies in showing that our payoff sequences $(x_{a,t})_{t \geq 1}$, defined with an initial simulation step, still satisfy the drift conditions of Definition 4. We argue that this is true for all simulation phases defined by any $I$ and $f$.

**Lemma 2.** For any MDP $M$, horizon $H$ and state $s_0$, after $n$ iterations of MCTS, for any $p$, the sequences $(x_{a,t})_{t \geq 1}$ associated with $(\text{reward}_{I_t(p,a)}(p))_{t \geq 1}$ satisfy the drift conditions.

Although the long-term guarantees of Theorem 1 hold for any simulation phase independently of the MDP, in practice one would expect better results from a good simulation, that gives a value close to the real value of the current node. Domain-specific knowledge can be used to obtain such simulations, and also to guide the selection phase based on heuristics. Our goal will be to preserve the theoretical guarantees of MCTS in the process.

### 4 Symbolic advice for MCTS

In this section, we introduce a notion of advice meant to guide the construction of the Monte Carlo search tree. We argue that a symbolic approach is needed in order to handle large MDPs in

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3 If $p$ is selected at iteration $I_t(p, a)$ then $p$ must be a prefix of the leaf node reached in this iteration, so that $\text{reward}_{I_t(p, a)}(p)$ is defined in the backpropagation phase.
practice. Let a symbolic advice \( \mathcal{A} \) be a logical formula over finite paths whose truth value can be tested with an operator \( \models \).

Example 1. A number of standard notions can fit this framework. For example, reachability and safety properties, LTL formulae over finite traces or regular expressions could be used. We will use a safety property for Pac-Man as an example (see Fig. 1), by assuming that the losing states of the MDP should be avoided. This advice is thus satisfied by every path such that Pac-Man does not make contact with a ghost.

We denote \( \text{Paths}^H_M(\mathcal{A}) \) the set of paths \( p \in \text{Paths}^H_M \) such that \( p \models \mathcal{A} \). For a path \( p \in \text{Paths}^{<H}_M \), we denote \( \text{Paths}^H_M(p, \mathcal{A}) \) the set of paths \( p' \in \text{Paths}^H_M(p) \) such that \( p' \models \mathcal{A} \).

A nondeterministic strategy is a function \( \sigma : \text{Paths}_M \to 2^A \) that maps a finite path \( p \) to a subset of \( A \). For a strategy \( \sigma' \) and a nondeterministic strategy \( \sigma, \sigma' \subseteq \sigma \) if for all \( p, \text{Supp}(\sigma'(p)) \subseteq \sigma(p) \). Similarly, a nondeterministic strategy for the environment is a function \( \tau : \text{Paths}_M \times A \to 2^S \) that maps a finite path \( p \) and an action \( a \) to a subset of \( \text{Supp}(P(\text{last}(p), a)) \). We extend the notations used for probabilistic strategies to nondeterministic strategies in a natural way, so that \( \text{Paths}^H_M(\mathcal{A}) \) and \( \text{Paths}^H_M(\tau) \) denote the paths of length \( H \) compatible with the strategy \( \mathcal{A} \) or \( \tau \), respectively.

For a symbolic advice \( \mathcal{A} \) and an horizon \( H \), we define a nondeterministic strategy \( \sigma^H_M(\mathcal{A}) \) and a nondeterministic strategy \( \tau^H_M(\mathcal{A}) \) for the environment such that for all paths \( p \) with \( |p| < H \),

\[
\sigma^H_M(p) = \{ a \in A \mid \exists s \in S, \exists p' \in \text{Paths}^{H-|p|-1}_M(s), p \cdot a \cdot p' \models \mathcal{A} \},
\]

\[
\tau^H_M(p, a) = \{ s \in S \mid \exists p' \in \text{Paths}^{H-|p|-1}_M(s), p \cdot a \cdot p' \models \mathcal{A} \}.
\]

The strategies \( \sigma^H_M \) and \( \tau^H_M \) can be defined arbitrarily on paths \( p \) of length at least \( H \), for example with \( \sigma^H_M(p) = A \) and \( \tau^H_M(p, a) = \text{Supp}(P(\text{last}(p), a)) \) for all actions \( a \). Note that by definition, \( \text{Paths}^H_M(s, \mathcal{A}) = \text{Paths}^H_M(s, \sigma^H_M(\mathcal{A})) \cap \text{Paths}^H_M(s, \tau^H_M(\mathcal{A})) \) for all states \( s \).

Let \( \top \) (resp. \( \bot \)) denote the universal advice (resp. the empty advice) satisfied by every finite path (resp. never satisfied), and let \( \sigma_{\top} \) and \( \sigma_{\bot} \) (resp. \( \sigma_{\top} \) and \( \sigma_{\bot} \)) be the associated nondeterministic strategies. We define a class of advice that can be enforced against an adversarial environment by following a nondeterministic strategy, and that are minimal in the sense that paths that are not compatible with this strategy are not allowed.

Definition 8 (Strongly enforceable advice). A symbolic advice \( \mathcal{A} \) is called a strongly enforceable advice from a state \( s_0 \) and for an horizon \( H \) if there exists a nondeterministic strategy \( \sigma \) such that \( \text{Paths}^H_M(s_0, \sigma) = \text{Paths}^H_M(s_0, \mathcal{A}) \), and such that \( \sigma(p) \neq \emptyset \) for all paths \( p \) of length \( i < H \) in \( \text{Paths}^i_M(s_0, \sigma) \).

Note that Definition 8 ensures that paths that follow \( \sigma \) can always be extended into longer paths that follow \( \sigma \). This is a reasonable assumption to make for a nondeterministic strategy meant to enforce a property. In particular, \( s_0 \) is a path of length 0 in \( \text{Paths}^0_M(s_0, \sigma) \), so that \( \sigma(s_0) \neq \emptyset \) and so that by induction \( \text{Paths}^i_M(s_0, \sigma) \neq \emptyset \) for all \( i \in [0, H] \).

Lemma 3. Let \( \mathcal{A} \) be a strongly enforceable advice from \( s_0 \) with horizon \( H \). It holds that \( \text{Paths}^H_M(s_0, \sigma^H_M(\mathcal{A})) = \text{Paths}^H_M(s_0, \mathcal{A}) \). Moreover, for all paths \( p \in \text{Paths}^{<H-1}_M(s_0) \) and all actions \( a \), either \( \tau^H_M(p, a) = \tau_{\top}(p, a) \) or \( \tau^H_M(p, a) = \tau_{\bot}(p, a) \).

An strongly enforceable advice is encoding a notion of guarantee, as \( \sigma^H_M \) is a winning strategy for the reachability objective on \( T(M, s_0, H) \) defined by the set \( \text{Paths}^H_M(\mathcal{A}) \).

We say that the strongly enforceable advice \( \mathcal{A}' \) is extracted from a symbolic advice \( \mathcal{A} \) for an horizon \( H \) and a state \( s_0 \) if \( \mathcal{A}' \) is the greatest part of \( \mathcal{A} \) that can be guaranteed for the horizon \( H \) starting from \( s_0 \), i.e. if \( \text{Paths}^H_M(s_0, \mathcal{A}') \) is the greatest subset of \( \text{Paths}^H_M(s_0, \mathcal{A}) \) such that \( \sigma^H_M \), is a winning strategy for the reachability objective \( \text{Paths}^H_M(s, \mathcal{A}') \) on \( T(M, s_0, H) \). This greatest
subset always exists because if $\mathcal{A}'_1$ and $\mathcal{A}'_2$ are strongly enforceable advice in $\mathcal{A}'$, then $\mathcal{A}'_1 \cup \mathcal{A}'_2$ is strongly enforceable by union of the nondeterministic strategies associated with $\mathcal{A}'_1$ and $\mathcal{A}'_2$. However, this greatest subset may be empty, and as $\bot$ is not a strongly enforceable advice we say that in this case $\mathcal{A}'$ cannot be enforced from $s_0$ with horizon $H$.

**Example 2.** Consider a symbolic advice $\mathcal{A}$ described by the safety property for Pac-Man of Example 1. For a fixed horizon $H$, the associated nondeterministic strategies $\sigma^H_{M,s}$ and $\tau^H_{M,s}$ describe action choices and stochastic transitions compatible with this property. Notably, $\mathcal{A}$ may not be a strongly enforceable advice, as there may be situations $(p,a)$ where some stochastic transitions lead to bad states and some do not. In the small grid of Fig. 1 the path of length 1 that corresponds to Pac-Man going left and the red ghost going up is allowed by the advice $\mathcal{A}$, but not by any safe strategy for Pac-Man as there is a possibility of losing by playing left.

If a strongly enforceable advice $\mathcal{A}'$ can be extracted from $\mathcal{A}$, it is a more restrictive safety property, where the set of bad states is obtained by an attractor set computation. In this setting, $\mathcal{A}'$ corresponds to playing according to a strategy for Pac-Man that ensures not being eaten by adversarial ghosts for the next $H$ steps.

**Definition 9 (Pruned MDP).** For a MDP $M = (S,A,P,R)$ a horizon $H \in \mathbb{N}$, a state $s_0$ and an advice $\mathcal{A}$, let the pruned unfolding $T(M,s_0,H,\mathcal{A})$ be defined as a sub-MDP of $T(M,s_0,H)$ that contains exactly all paths in $\text{Paths}^H_{M,s_0}$ satisfying $\mathcal{A}$. It can be obtained by removing all action transitions that are not compatible with $\sigma^H_{M,s}$, and all stochastic transitions that are not compatible with $\tau^H_{M,s}$. The distributions $P(p,a)$ are then normalised over the stochastic transitions that are left.

Note that if $\mathcal{A}$ is a strongly enforceable advice, the pruning process only prunes stochastic transitions when the entire sub-tree is removed by the pruning, so that the normalisation step is not needed, and for all nodes $p$ in $T(M,s_0,H,\mathcal{A})$ and all actions $a$, the distributions $P(p,a)$ in $T(M,s_0,H,\mathcal{A})$ are the same as in $T(M,s_0,H)$.

**Definition 10 (Optimality assumption).** An advice $\mathcal{A}$ satisfies the optimality assumption for horizon $H$ if $\sigma^H_{M,s} \subseteq \sigma^H_{p,a}$ for all $s \in S$, where $\sigma^H_{M,s}$ is the optimal strategy for the expected total reward of horizon $H$ at state $s$.

**Lemma 4.** Let $\mathcal{A}$ be a strongly enforceable advice that satisfies the optimality assumption. Then, $\text{Val}^H_{T}(s_0) = \text{Val}^H_{T(M,s_0,H,\mathcal{A})}(s_0)$. Moreover, $\sigma^H_{M,s} = \sigma^H_{T(M,s_0,H,\mathcal{A})}(s_0)$.

**Example 3.** Let $\mathcal{A}'$ be a strongly enforceable safety advice for Pac-Man as described in Example 2. Assume that visiting a bad state leads to an irrecoverably bad reward, so that taking an unsafe action (i.e. an action such that there is a non-zero probability of losing associated with all Pac-Man strategies) is always worse (on expectation) than taking a safe action. Then, the optimality assumption holds for the advice $\mathcal{A}'$. This can be achieved by giving a score penalty for losing that is low enough.

### 4.1 MCTS under symbolic advice

We will augment the MCTS algorithm using two advice: a selection advice $\varphi$ to guide the MCTS tree construction, and a simulation advice $\psi$ to prune the sampling domain. We assume that the selection advice is a strongly enforceable advice that satisfies the optimality assumption. Notably, we make no such assumption for the simulation advice, so that any symbolic advice can be used.

**Selection Phase under advice.** We use the advice $\varphi$ to prune the tree according to $\sigma^H_{\varphi}$. Therefore, from any node $p$ our version of UCT selects the action

$$a^* = \arg\max_{a \in \sigma^H_{\varphi}(p)} \text{value}(p,a) + C \sqrt{\frac{\ln(\text{count}(p))}{\text{count}(p,a)}}.$$  

**Simulation Phase under advice.** For the simulation phase, we use a sampling-based approach
biased by the simulation advice: paths are sampled by picking actions uniformly at random in the pruned MDP $T(M, s_0, H, \psi)$, with a fixed prefix $p$ defined by the current node in the search tree. This can be interpreted as a probability distribution over $\text{paths}^H_T(p, \psi)$. If $p \notin T(M, s_0, H, \psi)$, the simulation phase outputs a value of 0 as it is not possible to satisfy $\psi$ from $p$. Another approach that does not require computing the pruned MDP repeats the following steps for a bounded number of time before returning 0 if no valid sample is found:

1. Pick a path $p \cdot p' \in \text{paths}^H_T(p)$ using an uniform sampling method;
2. If $p \cdot p' \not\models \psi$, reject and try again, otherwise output $p'$ as a sample.

We compute $\text{value}_n(p)$ by averaging the rewards of these samples.

**Theoretical analysis.** We show that the theoretical guarantees of the MCTS algorithm developed in Section 3 are maintained on the MCTS algorithm under symbolic advice.

**Theorem 2.** Consider a MDP $M$, an horizon $H$ and a state $s_0$. Let $V_n(s_0)$ be a random variable that represents the value $\text{value}_n(s_0)$ at the root of the search tree after $n$ iterations of the MCTS algorithm under a strongly enforceable advice $\varphi$ satisfying the optimality assumption and a simulation advice $\psi$. Then, $\mathbb{E}[V_n(s_0)] - \mathbb{E}[\text{value}^H_M(s_0)] = O((\ln n)/n)$. Moreover, the failure probability $\mathbb{P}[\arg \max_a \text{value}_n(s_0, a) \neq \sigma^H_M(s_0)]$ converges to zero.

In order to prove Theorem 2, we argue that running MCTS under a selection advice $\varphi$ and a simulation advice $\psi$ is equivalent to running the MCTS algorithm of Section 3 on the pruned MDP $T(M, s_0, H, \psi)$, with a simulation phase defined using the advice $\psi$.

The simulation phase biased by $\psi$ can be described in the formalism of Section 3 with a domain $I = \{\frac{1}{c} \sum_{i=1}^{c} \text{Reward}_M(p_i) \mid p_1, \ldots, p_c \in \text{paths}^H_{T(M, s_0, H, \psi)}\}$, and a mapping $f_\psi$ from paths $p$ in $\text{paths}^H_{T(M, s_0, H, \psi)}$ to a probability distribution on $I$ describing the outcome of a sampling phase launched from the node $p$. Formally, the weight of $\frac{1}{c} \sum_{i=1}^{c} \text{Reward}_M(p_i) \in I$ in $f(p)$ is the probability of sampling the sequence of paths $p_1, \ldots, p_c$ in the simulation phase under advice launched from $p$. Then, from Theorem 1 we obtain convergence properties of MCTS under symbolic advice towards the value and optimal strategy in the pruned MDP, and Lemma 4 lets us conclude the proof of Theorem 2 as those values and strategies are maintained in $M$ by the optimality assumption.

### 4.2 Using satisfiability solvers

We will now discuss the use of general-purpose solvers to implement symbolic advice according to the needs of MCTS.

A symbolic advice $\mathcal{A}$ describes a finite set of paths in $\text{paths}^H_M$, and as such can be encoded as a Boolean formula over a set of variables $V$, such that satisfying assignments $v : V \rightarrow \{\text{true, false}\}$ are in bijection with paths in $\text{paths}^H_M(\mathcal{A})$.

If a symbolic advice is described in Linear Temporal Logic, and a symbolic model of the MDP $M$ is available, one can encode $\mathcal{A}$ as a Boolean formula of size linear in the size of the LTL formula and $H$.

**On-the-fly computation of a strongly enforceable advice.** A direct encoding of a strongly enforceable advice may prove impractically large. We argue for an on-the-fly computation of $\sigma^H_{\mathcal{A}}$ instead, in the particular case where the strongly enforceable advice is extracted from a symbolic advice $\mathcal{A}$ with respect to the initial state $s_0$ and with horizon $H$.

**Lemma 5.** Let $\mathcal{A}'$ be a strongly enforceable advice extracted from $\mathcal{A}$ for horizon $H$. Consider a node $p$ at depth $i$ in $T(M, s_0, H, \mathcal{A}')$, for all $a_0 \in A$, $a_0 \in \sigma^H_{\mathcal{A}'}(p)$ if and only if

$$\forall s_1 \exists a_1 \forall s_2 \ldots \forall s_{H-i+1}, \ p \cdot a_0 s_1 a_1 s_2 \ldots s_{H-i+1} \models \mathcal{A}',$$

where actions are quantified over $A$ and every $s_k$ is quantified over $\text{Supp}(P(s_{k-1}, a_{k-1}))$. 

Therefore, given a Boolean formula encoding $A$, one can use a QBF solver to compute $\sigma^A$, the strongly enforceable advice extracted from $A$: this computation can be used whenever MCTS performs an action selection step under the advice $A'$, as described in Section 4.1.

The performance of this approach will crucially depend on the number of alternating quantifiers, and in practice one may limit themselves to a smaller depth $h < H - i$ in this step, so that safety is only guaranteed for the next $h$ steps.

Some properties can be inductively guaranteed, so that satisfying the QBF formula of Lemma 5 with a depth $H - i = 1$ is enough to guarantee the property globally. For example, if there always exists an action leading to states that are not bad, it is enough to check for safety locally with a depth of 1. This is the case in Pac-Man for a deadlock-free layout when there is only one ghost.

**Weighted sampling under a symbolic advice.** Given a symbolic advice $A$ as a Boolean formula, and a probability distribution $w \in D(\text{Paths}_M^H)$, our goal is to sample paths of $M$ that satisfy $A$ with respect to $w$. Let $\omega$ denote a weight function over boolean assignments that matches $w$. This reduces our problem to the weighted sampling of satisfying assignments in a Boolean formula. An exact solver for this problem may not be efficient, but one can use the techniques of [7] to perform approximate sampling in polynomial time:

**Proposition 3 ([7]).** Given a CNF formula $A$, a tolerance $\epsilon > 0$ and a weight function $\omega$, we can construct a probabilistic algorithm which outputs a satisfying assignment $z$ such that for all $y$ that satisfies $A$:

$$\frac{\omega(y)}{(1+\epsilon) \sum_{\psi} \omega(\psi)} \leq \Pr[z = y] \leq \frac{(1+\epsilon) \omega(y)}{\sum_{\psi} \omega(\psi)}$$

The above algorithm occasionally ‘fails’ (outputs no assignment even though there are satisfying assignments) but its failure probability can be bounded by any given $\delta$. Given an oracle for SAT, the above algorithm runs in time polynomial in $\ln(\frac{1}{\delta}), |\psi|, \frac{1}{\epsilon}$ and $r$ where $r$ is the ratio between highest and lowest weight according to $\omega$.

In particular, this algorithm uses $\omega$ as a black-box, and thus does not require precomputing the probabilities of all paths satisfying $A$. In our particular application of Proposition 3, the value $r$ can be bounded by $\left(\frac{p_{\text{max}} |A|}{p_{\text{min}}} \right)^H$ where $p_{\text{min}}$ and $p_{\text{max}}$ are the smallest and greatest probabilities for stochastic transitions in $M$.

Note that if we wish to sample from a given node $p$ of the search tree, we can force $p$ as a mandatory prefix of satisfying assignments by fixing the truth value of relevant variables in the Boolean formula.

### 5 A Pac-Man case study

We performed our experiments on the multi-agents game Pac-Man, using the code of [11]. The ghosts can have different strategies where they take actions based on their own position as well as position of Pac-Man. In our experiments, we used two different types of ghosts, the *random ghosts* (in green) always choose an action uniformly at random from the legal actions available, while the *Directional ghosts* (in red) take the legal action that minimizes the Manhattan distance to Pac-Man with probability 0.9, and move randomly otherwise.

The game can be seen as a Markov decision process, where states encode a position for each agent and for the food pills in the grid, where actions encode individual Pac-Man moves, and where stochastic transitions encode the moves of ghosts according to their probabilistic models. For each state and action pair, we define a reward based on the score gained or lost by this move, as explained in the caption of Figure 1. We also assign a terminal reward to each state, so as to allow MCTS to compare paths of length $H$ which would otherwise obtain the same score. Intuitively,

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5. The probability of a path $p$ being sampled should be equal to $w(p)/ \sum_{p' | p' \models A} w(p')$.

6. The last action played by ghosts should be stored as well, as they are not able to reverse their direction.
better terminal rewards are given to states where Pac-Man is closer to the food pills and further away from the ghosts, so that terminal rewards play the role of a static evaluation of positions.

**Experiments.** We used a receding horizon $H = 10$. The baseline is given by a standard implementation of the algorithm described in Section 3. A search tree is constructed with a maximum depth $H$, for 100 iterations, so that the search tree constructed by the MCTS algorithm contains up to 100 nodes. At the first selection of every node, 100 samples are obtained by using an uniform policy. Overall, this represents a tiny portion of the tree unfolding of depth 10, which underlines the importance of properly guiding the search to the most interesting neighborhoods. As a point of reference, we also had human players take control of Pac-Man, and computed the same statistics. The players had the ability to slow down the game as they saw fit, as we aimed for a comparison between the quality of the strategical decisions made by these approaches, and not of their reaction speeds.

We compare these baselines with the algorithm of Section 4.1, using the following advice. The *simulation advice* $\psi$ that we consider is defined as a safety property satisfied by every path such that Pac-Man does not make contact with a ghost, as in Example 1. We provide a Boolean formula encoding $\psi$, so that one can use a SAT solver to obtain samples, or by extension sampling tools as described in Proposition 3, such as WeightGen. We use UniGen to sample almost uniformly over the satisfying assignments of $\psi$.

From this simulation advice, we extract whenever possible a strongly enforceable *selection advice* $\phi$ that guarantees that Pac-Man will not make contact with a ghost, as described in Example 2. If safety cannot be enforced, $\top$ is used as a selection advice, so that no pruning is performed. This is implemented by using the Boolean formula $\psi$ in a QBF solver according to Lemma 5. For performance reason we guarantee safety for a smaller horizon $h < 10$, that we fixed at 3 in our experiments.

Several techniques were used to reduce the state-space of the MDP in order to obtain smaller formulæ. For example, a ghost that is too far away with respect to $H$ or $h$ can be safely ignored, and the current positions of the food pills is not relevant for safety.

**Results.** For each experiments, we ran 100 games in a high-end cluster using AMD Opteron Processors 6272 at 2.1 GHz. A summary of our results is displayed in Table 1. In the small grid

| Grid | Ghosts     | Algorithm          | win | loss | draw | food | score     |
|------|------------|--------------------|-----|------|------|------|-----------|
| 9 x 21 | 4 x Random | MCTS               | 17  | 59   | 24   | 16.65| -215.32  |
|       |            | MCTS+Selection advice | 25  | 54   | 21   | 17.84| -146.44  |
|       |            | MCTS+Simulation advice | 71  | 29   | 0    | 22.11| 291.80   |
|       |            | MCTS+both advice    | 85  | 15   | 0    | 23.42| 468.74   |
|       |            | Human               | 44  | 56   | 0    | 18.87| 57.76    |
| 1 x Directional + 3 x Random | MCTS             | 11  | 85   | 4    | 14.86| -339.99 |
|       |            | MCTS+Selection advice | 16  | 82   | 2    | 15.25| -290.6   |
|       |            | MCTS+Simulation advice | 27  | 70   | 3    | 17.14| -146.79  |
|       |            | MCTS+both advice    | 33  | 66   | 1    | 17.84| -92.47   |
|       |            | Human               | 24  | 76   | 0    | 15.50| 166.28   |
| 27 x 28 | 4 x Random | MCTS               | 1   | 10   | 89   | 14.85| -182.77  |
|       |            | MCTS+Selection advice | 1   | 10   | 89   | 14.85| -182.77  |
|       |            | MCTS+Simulation advice | 3   | 70   | 2    | 17.84| -92.47   |
|       |            | MCTS+both advice    | 95  | 5    | 5    | 24.10| 517.04   |

Table 1: Summary of experiments with different ghost models, algorithms and grid size. The win, loss and draw columns denote win/loss/draw rates in percents (the game ends in a draw after 300 game steps). The food eaten column refers to the number of food pills eaten on average, out of 25 food pills in total. Score refers to the average score obtained over all runs.

with four random ghosts, the baseline MCTS algorithm wins 17% of games. Adding the selection advice $\psi$ to MCTS implies an increase of 8 percentage points. This is reflected in a smaller score, which is consistent with the expected policies of the algorithm.

The distribution over path is slightly different than when sampling uniformly over actions in the pruned MDP $T(M, s_0, H, \psi)$, but UniGen enjoys better performances than WeightGen.
advice results in a slight increase of the win rate to 25%. The average score is improved as expected, but even if one ignores the win or loss score penalties we observe that more food pills were eaten on average as well. The simulation advice provides in turn a sizeable increase in both win rate (achieving 71%) and average score. Using both advice at the same time gave the best results overall, with a win rate of 85%. The same observations can be made in other settings as well, either with a directional ghost model or on a large grid. Moreover the simulation advice significantly reduces the number of game turns Pac-Man needs to win, resulting in less game draws, most notably on the large grid. This experiment was not designed to optimise or compare the performance of these approaches in term of computing time, but we note that the algorithm with both advice is about three times slower than the baseline. If we make use of the same time budget in the standard MCTS algorithm (roughly increasing the number of nodes in the MCTS tree threefold), the win rate climbs to 26%, which is still significantly under the 85% win rate achieved with advice. Supplementary material is available at https://debrajrc.github.io/pacman/.

6 Conclusion and future works

In this paper, we have introduced the notion of symbolic advice to guide the selection and the simulation phases of the MCTS algorithm. We have identified sufficient conditions to preserve the convergence guarantees offered by the MCTS algorithm while using symbolic advice. We have also explained how to implement them using SAT and QBF solvers in order to apply symbolic advice to large MDP defined symbolically rather than explicitly. We believe that the generality, the flexibility and the precision offered by logical formalism to express symbolic advice in MCTS can be used as the basis of a methodology to systematically inject domain knowledge in MCTS. We have shown that domain knowledge expressed as simple symbolic advice (safety properties) improves greatly the efficiency of the MCTS algorithm in the Pac-Man application. This application is challenging as the underlying MDPs have huge state spaces, i.e. up to $10^{23}$ states. In this application, symbolic advice allows the MCTS algorithm to reach or even surpass human level in playing.

As further works, we plan to offer a compiler from LTL to symbolic advice to automate their integration in the MCTS algorithm for diverse application domains. We also plan to work on the efficiency of the implementation. So far, we have developed a prototype implementation written in Python (an interpreted language). This implementation cannot be used to evaluate performances in absolute term but it was useful to show that if the same amount of resource is allocated to the two algorithms the one with advice performs much better. We believe that by using a well-optimised code base and by exploiting parallelism, we should be able to apply our algorithm in real-time and preserve the level of quality reported in the experimental section. Finally, we plan to study how learning can be incorporated in our framework. One natural option is to replace the static reward function used after $H$ steps by a function learned from previous runs of the algorithm and implemented using a neural network (as it is done in AlphaGo [21] for example).

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A Markov Decision Processes

One can make a MDP strongly aperiodic without changing the optimal expected average reward and its optimal strategies with the following transition:

Definition 11 (Aperiodic transformation). \[\text{Section 8.5.4}\] For a MDP \(M = (S, A, P, R)\), we define a new MDP \(M_\alpha = (S, A, P_\alpha, R_\alpha)\) for \(0 < \alpha < 1\)

\[
\begin{align*}
R_\alpha(s, a) &= R(s, a) \\
P_\alpha(s, a)(s') &= \alpha + (1 - \alpha)P(s, a)(s) \\
P_\alpha(s, a)(s') &= (1 - \alpha)P(s, a)(s')
\end{align*}
\]

Notice that \(M_\alpha\) is strongly aperiodic.

Every finite path in \(M\) is also in \(M_\alpha\). Thus for a strategy \(\tilde{\sigma}\) in \(M_\alpha\), there is a \(\sigma\) in \(M\) whose domain is restricted to the paths in \(M\).

Proposition 4. \[\text{Section 8.5.4}\] Let \(M\) be a MDP. \(M_\alpha\) is a new MDP generated by applying the aperiodic transformation mentioned above. Then the set of memoryless strategies that optimizes the expected average reward in \(M_\alpha\) is the same as the set of memoryless strategies the optimizes the expected average reward in \(M\). Also from any \(s\), \(\text{Val}_M(s) = \text{Val}_{M_\alpha}(s)\).

Notice that if we are interested in the best action to take from a certain state after a certain number of value iteration, it is not necessary to calculate values for all states in the MDP. Alternatively we can take an on-the-fly approach where we do a finite-horizon optimization each time we are at a state of the MDP. We fix a horizon \(H\). From the current state \(s\) of the MDP \(M\), we calculate \(\sigma_{M,s}^H\). Then only the first step according to the calculated strategy is taken, i.e. the action \(\sigma_{M,s}^H(s)\) is taken from state \(s\). Then we go to a new current state according to the MDP. We repeat the process from this current state with same horizon \(H\). This procedure is called Receding Horizon Control. It follows from Proposition 2 that

Corollary 1. For a strongly aperiodic Markov decision process, for any \(\epsilon\), there exists an \(H\) such that, for all \(n \geq H\), receding horizon control with horizon \(n\) gives an \(\epsilon\)-optimal strategy for the expected average reward.

This way, at each step, we only need to work on the finite horizon unfolding of a MDP from the current state.

Definition 12 (Finite horizon unfolding of a MDP). For a MDP \(M = (S, A, P, R, R_T)\), a horizon depth \(H \in \mathbb{N}\) and a state \(s_0\), the unfolding of \(M\) from \(s_0\) and with horizon \(H\) is a tree-shaped MDP defined as \(T(M, s_0, H) = (S' = S_0 \cup \ldots \cup S_H, A, P', R', R'_T)\), where for all \(i \in [0, H]\), \(S_i = \text{Paths}^i(s_0)\). The mappings \(P', R'\) and \(R'_T\) are inherited from \(P\), \(R\) and \(R_T\) in a natural way with additional self-loops at the leaves of the unfolding, so that for all \(i \in [0, H], p \in S_i, a \in A \) and \(p' \in S'\),

\[
\begin{align*}
P'(p, a)(p') &= \begin{cases} 
            P(\text{last}(p), a)(\text{last}(p')) & \text{if } i < H \text{ and } \exists s' \in S, p' = p \cdot a s' \\
            1 & \text{if } i = H \text{ and } p' = p \\
            0 & \text{otherwise,} 
          \end{cases} \\
R'(p, a) &= \begin{cases} 
            R(\text{last}(p), a) & \text{if } i < H \\
            0 & \text{otherwise.} 
          \end{cases} \\
R'_T(p) &= R_T(\text{last}(p))
\end{align*}
\]

Lemma 4 is obtained as a corollary of:

Lemma 6. For all \(i \in [0, H]\), for all \(p \in S_i\)
Proof. We prove the first statement by induction on $H - i$. For $H - i = 0$, for $p \in S_i$.

\[
\text{Val}^H_{M,i}(\text{last}(p)) = \text{Val}^H_{T(M,s_0,H)}(p) = R_T(\text{last}(p))
\]

Assume the statement is true for $H - i = k$. So we have for $p \in S_{H-k}$

\[
\text{Val}^k_{M,i}(\text{last}(p)) = \text{Val}^k_{T(M,s_0,H)}(p)
\]

Thus for $p \in S_{H-k-1}$, we have for all $a \in A$ and $s \in \text{Supp}(P(\text{last}(p), a))$,

\[
\text{Val}^k_{M,i}(s) = \text{Val}^k_{T(M,s_0,H)}(p \cdot as)
\]

So

\[
\text{Val}^{k+1}_{M,i}(\text{last}(p)) = \max_{a \in A}(R(\text{last}(p), a) + \sum_s P(\text{last}(p), a)\text{Val}^k_{M,i}(s))
\]

\[
= \max_{a \in A}(R(\text{last}(p), a) + \sum_s P(\text{last}(p), a)\text{Val}^k_{T(M,s_0,H)}(p \cdot as))
\]

\[
= \text{Val}^{k+1}_{T(M,s_0,H)}(p)
\]

Therefore

\[
\sigma^{H}_{M,i}(\text{last}(p)) = \arg\max_{a \in A}(R(\text{last}(p), a) + \sum_s P(\text{last}(p), a)\text{Val}^{H-1}_{M,i}(s))
\]

\[
= \arg\max_{a \in A}(R(\text{last}(p), a) + \sum_s P(\text{last}(p), a)\text{Val}^{H-1}_{T(M,s_0,H)}(p))
\]

\[
= \sigma^{H}_{T(M,s_0,H)}(p) \square
\]

B UCB

Let $X_{a,n} = \frac{1}{n} \sum_{i=1}^{n} X_{a,i}$ denote the average of the first $n$ plays of action $a$. Let $\mu_{a,n} = \mathbb{E}[X_{a,n}]$. We assume that these expected means eventually converge, and let $\mu_a = \lim_{n \to \infty} \mu_{a,n}$.

Definition 13 (Drift conditions).

For all $a \in A$, the sequence $(\mu_{a,n})_{n \geq 1}$ converges to some value $\mu_a$.

There exists a constant $C_p > 0$ and an integer $N_p$ such that for $n \geq N_p$ and any $\delta > 0$, $\Delta_n(\delta) = C_p \sqrt{n \ln(1/\delta)}$, the following bounds hold:

\[
\mathbb{P}[nX_{a,n} \geq n\mu_{a,n} + \Delta_n(\delta)] \leq \delta ,
\]

\[
\mathbb{P}[nX_{a,n} \leq n\mu_{a,n} - \Delta_n(\delta)] \leq \delta .
\]

We define $\delta_{a,n} = \mu_{a,n} - \mu_a$. Then, $\mu^*, \mu^*_n, \delta^*_n$ are defined as $\mu_j$, $\mu_{j,n}$, $\delta_{j,n}$ where $j$ is the optimal action. Moreover, let $\Delta_n = \mu^* - \mu_a$.

As $\delta_{a,n}$ converges to 0 by assumption, for all $\epsilon > 0$ there exists $N_0(\epsilon) \in \mathbb{N}$, such that for $t > N_0(\epsilon)$, then $2|\delta_{a,t}| \leq \epsilon \Delta_n$ and $2|\delta^*_{a,t}| \leq \epsilon \Delta_n$ for all suboptimal actions $a \in A$.

The authors start by bounding the number of time a suboptimal action is played:

\[\text{It is assumed, for simplicity, that a single action is optimal, i.e. a single } a \text{ maximises } \mathbb{E}[X_{a,n}] \text{ for } n \text{ large enough.}\]
\textbf{Theorem 3 (I5 Theorem 1).} Consider UCB1 applied to a non-stationary bandit problem with $C_{l,s} = 2C_p \sqrt{\frac{\ln t}{n}}$. Fix $\epsilon > 0$. Let $T_a(n)$ denote number of times action $a$ has been played at time $n$. Then under the drift conditions, there exists $N_p$ such that for all suboptimal actions $a \in A$,
\[
E[T_a(n)] \leq \frac{16C_p^2 \ln n}{(1 - \epsilon)^2 \Delta_a^2} + N_p + 1 + \frac{n^2}{3}.
\]

Let $X_n = \sum_{a \in A} \frac{T_a(n)}{n} X_{a,n} T_a(n)$ denote the global average of payoffs received up to time $n$. Then, one can bound the difference between $\mu^*$ and $X_n$:

\textbf{Theorem 4 (I5 Theorem 2).} Under the drift conditions of Definition 13, it holds that
\[
|E[X_n] - \mu^*| \leq |\delta_n| + O\left(\frac{|A|(C_p^2 \ln n + N_0(1/2))}{n}\right).
\]

The following theorem shows that the number of times an action is played can be lower bounded:

\textbf{Theorem 5 (I5 Theorem 3).} Under the drift conditions of Definition 13, there exists some positive constant $\rho$ such that after $n$ iterations for all action $a$, $T_a(n) \geq \lceil \rho \ln(n) \rceil$.

Then, the authors also prove a tail inequality similar to the one described in the drift conditions, but on the random variable $X_n$ instead of $X_{a,n}$:

\textbf{Theorem 6 (I5 Theorem 4).} Fix an arbitrary $\delta > 0$ and let $\Delta_n = 9 \sqrt{2n \ln(2/\delta)}$. Let $n_0$ be such that $\sqrt{n_0} \geq O(|A|(C_p^2 \ln n_0 + N_0(1/2)))$. Then under the drift conditions, for any $n \geq n_0$, the following holds true:
\[
P[nX_n \geq n E[X_n] + \Delta_n(\delta)] \leq \delta
\]
\[
P[nX_n \leq n E[X_n] - \Delta_n(\delta)] \leq \delta
\]

Finally, the authors argue that the probability of making the wrong decision (choosing a suboptimal action) converges to 0 as the number of plays grows:

\textbf{Theorem 7 (I5 Theorem 5).} Let $I_t$ be the action chosen at time $t$, and let $a^*$ be the optimal action. Then \( \lim_{t \to \infty} \Pr(I_t \neq a^*) = 0 \).

\section{MCTS with Simulation}

After $n$ iterations of MCTS, we have \( \text{total}_n(p) = \sum_{i|I_t(p) \leq n} \text{reward}_{I_t(p)}(p) \) and
\[
\text{total}(p, a) = \sum_{i|I_t(p,a) \leq n} \text{reward}_{I_t(p,a)}(p, a).
\]

We use the following observations, derived from the structure of the MCTS algorithm. For all nodes $p$ in the search tree, after $n$ iterations, we have:
\[
\text{total}_n(p) = \text{reward}_{T_t(p)}(p) + \sum_{a \in A} \text{total}_n(p, a)
\]
\[
\text{total}_n(p, a) = \sum_{s \in \text{Supp}(P\text{last}(p, a))} \text{total}_n(p \cdot as) + R(\text{last}(p, a)) \cdot \text{count}_n(p, a)
\]
\[
\text{value}_n(p) = \frac{\text{total}_n(p)}{\text{count}_n(p)}
\]
\[
\text{count}_n(p, a) = 1 + \sum_{a} \text{count}_n(p, a)
\]
\[
\text{count}_n(p, a) = \sum_{s} \text{count}_n(p \cdot as)
\]
Proof (of Lemma 2). We use following inequality (Chernoff-Hoeffding inequality) [12, Theorem 2] throughout the proof:

Let $X_1, X_2, \ldots, X_n$ be independent random variables in $[0, 1]$. Let $S_n = \sum_i X_i$. Then for all $a > 0$, $\mathbb{P}[S_n \geq \mathbb{E}[S_n] + t] \leq \exp \left(-\frac{2t^2}{n} \right)$ and $\mathbb{P}[S_n \leq \mathbb{E}[S_n] - t] \leq \exp \left(-\frac{2t^2}{n} \right)$.

We need to show the following conditions hold:

1. $\lim_{\text{count}_n(p) \to \infty} \mathbb{E}[\text{value}_n(p, a)]$ exists for all $a$.
2. There exists a constant $C_p > 0$ such that for $\text{count}_n(p, a)$ big enough and any $\delta > 0$, $\Delta_{\text{count}_n(p, a)}(\delta) = C_p \sqrt{\text{count}_n(p, a) \ln(1/\delta)}$, the following bounds hold:

$$
\mathbb{P}[\text{total}_n(p, a) \geq \mathbb{E}[\text{total}_n(p, a)] + \Delta_{\text{count}_n(p, a)}(\delta)] \leq \delta
$$

$$
\mathbb{P}[\text{total}_n(p, a) \leq \mathbb{E}[\text{total}_n(p, a)] - \Delta_{\text{count}_n(p, a)}(\delta)] \leq \delta
$$

We show it by induction on $H - |p|$. For $|P| = H - 1$: $\text{reward}_i(p, a)$ follows a stationary distribution: $\text{reward}_i(p, a) = R(\text{last}(p), a) + R_T(s)$ with probability $P(\text{last}(p), a)(s)$. Thus

$$
\mathbb{E}[\text{total}_n(p, a)] = \mathbb{E} \left[ \sum_{i | I_i(p, a) \leq n} \text{reward}_{I_i(p, a)}(p, a) \right] = \text{count}_n(p, a) \left( \sum_s R_T(s)P(\text{last}(p), a)(s) + R(\text{last}(p), a) \right)
$$

Thus $\mathbb{E}[\text{value}_n(p, a)] = \sum_s R_T(s)P(\text{last}(p), a)(s) + R(\text{last}(p), a)$.

Also from Chernoff-Hoeffding inequality,

$$
\mathbb{P} \left[ \sum_{i | I_i(p, a) \leq n} \text{reward}_{I_i(p, a)}(p, a) \geq \mathbb{E} \left[ \sum_{i | I_i(p, a) \leq n} \text{reward}_{I_i(p, a)}(p, a) \right] + \sqrt{\frac{\text{count}_n(p, a)}{2} \ln \frac{1}{\delta}} \right] \leq \delta
$$

and

$$
\mathbb{P} \left[ \sum_{i | I_i(p, a) \leq n} \text{reward}_{I_i(p, a)}(p, a) \leq \mathbb{E} \left[ \sum_{i | I_i(p, a) \leq n} \text{reward}_{I_i(p, a)}(p, a) \right] - \sqrt{\frac{\text{count}_n(p, a)}{2} \ln \frac{1}{\delta}} \right] \leq \delta
$$

So condition 2 also holds with $C_p = \frac{1}{\sqrt{2}}$.

Assume that the conditions are true for all $p \cdot a$s. So from Theorem 4 we get:

$$
\left| \mathbb{E} \left[ \frac{1}{\text{count}_n(p, a)} \sum_{a'} \text{value}_n(p, a') \right] - \lim_{\text{count}_n(p, a) \to \infty} \mathbb{E}(\text{value}_n(p \cdot a, a^*)) \right| \leq \left| \mathbb{E}(\text{value}_n(p \cdot a, a^*)) - \lim_{\text{count}_n(p, a) \to \infty} \mathbb{E}(\text{value}_n(p \cdot a, a^*)) \right| + O\left( \frac{\ln(\text{count}_n(p \cdot a) - 1)}{\text{count}_n(p \cdot a) - 1} \right)
$$

where $a^*$ is the optimal action from $p \cdot a$. Now,

$$
\lim_{\text{count}_n(p) \to \infty} \mathbb{E}(\text{value}_n(p \cdot a)) = \lim_{\text{count}_n(p) \to \infty} \mathbb{E} \left( \frac{\text{total}_n(p \cdot a)}{\text{count}_n(p \cdot a)} \right)
$$

$$
= \lim_{\text{count}_n(p) \to \infty} \mathbb{E} \left( \frac{\text{total}_n(p \cdot a) - \text{reward}_{I_i(p)(p \cdot a)}}{\text{count}_n(p \cdot a) - 1} \right)
$$

$$
= \lim_{\text{count}_n(p) \to \infty} \mathbb{E} \left( \frac{\sum_{a'} \text{total}_n(p \cdot a, a')}{\sum_{a'} \text{count}_n(p \cdot a, a')} \right)
$$
Let \( \lim_{\text{count}_n(p-as) \to \infty} \mathbb{E}(\text{value}_n(p-as, a^*)) = \mu_{p-as} \) (from induction hypothesis, we know that this limit exists). When \( \text{count}_n(p) \to \infty \), from Theorem 5 \( \text{count}_n(p, a) \to \infty \) for all \( a \). And as for all states \( s \), state \( s \) is chosen according to distribution \( P(p, a)(s) \), \( \text{count}_n(p-as) \to \infty \) with probability 1. Then,

\[
\lim_{\text{count}_n(p) \to \infty} \mathbb{E}(\text{value}_n(p \cdot a)) = \lim_{\text{count}_n(p) \to \infty} \mathbb{E} \left( \sum_s \text{value}_n(p \cdot a) \frac{\text{count}_n(p \cdot a)}{\text{count}_n(p)} + R(last(p), a) \right)
\]

\[
= R(last(p), a) + \sum_s \mu_{p-as} \cdot P(last(p), a)(s)
\]

So \( \lim_{\text{count}_n(p) \to \infty} \mathbb{E}(\text{value}_n(p \cdot a)) \) exists.

From Theorem 5 when \( \text{count}_n(p-as) \) is big enough, for all \( \delta > 0 \), \( P[\sum_{a} \text{total}_n(pas, a') \geq \mathbb{E}(\sum_{a} \text{total}_n(pas, a')) + \Delta_1^*(\delta)] \leq \frac{\delta}{2|S|} \), where \( \Delta_1^*(\delta) = 9 \left( \sqrt{\text{count}_n(p \cdot a) \ln \left( \frac{4|S|}{\delta} \right)} \right) \).

Therefore \( P[\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as) \geq \mathbb{E}(\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as)) + \Delta_1^*(\delta)] \leq \frac{\delta}{|S|} \).

Also the random variable associated to \( \text{reward}_{I_1}(p-as) \) is in \([0, 1]\) following a fixed stationary distribution \( f(p) \). So from Chernoff-Hoeffding inequality, \( P[\text{reward}_{I_1}(p-as) \geq \mathbb{E}(\text{reward}_{I_1}(p-as)) + \Delta_2^*(\delta)] \leq \frac{\delta}{|S|} \).

Now using the fact that for random variables \( n \) random variables \( \{A_i\}_{i \leq n} \) and \( n \) random variables \( \{B_i\}_{i \leq n} \), we get:

\[
P[\text{total}_n(p \cdot a) \geq \mathbb{E}(\text{total}_n(p \cdot a)) + \Delta_1^*(\delta) + \Delta_2^*(\delta)]
\]

\[
\leq P[\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as) \geq \mathbb{E}(\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as)) + \Delta_1^*(\delta)]
\]

\[
+ P[\text{reward}_{I_1}(p-as) \geq \mathbb{E}(\text{reward}_{I_1}(p-as)) + \Delta_2^*(\delta)] \leq \frac{\delta}{|S|}
\]

Also \( \text{count}_n(p \cdot a) \mathbb{E}(R(last(p), a)) = \mathbb{E}(\text{count}_n(p \cdot a) \cdot R(last(p), a)) \). Hence:

\[
P \left[ \text{total}_n(p, a) \geq \mathbb{E}(\text{total}_n(p, a)) + \sum_s (\Delta_1^*(\delta) + \Delta_2^*(\delta)) \right]
\]

\[
\leq \sum_s P[\text{total}_n(p \cdot a) \geq \mathbb{E}(\text{total}_n(p \cdot a)) + (\Delta_1^*(\delta) + \Delta_2^*(\delta))] \leq \delta
\]

Similarly, when \( \text{count}_n(p \cdot a) \) is big enough, for all \( \delta > 0 \) it holds that \( P[\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as) \leq \mathbb{E}(\text{total}_n(p \cdot a) - \text{reward}_{I_1}(p-as)) - \Delta_1^*(\delta)] \leq \frac{\delta}{|S|} \) and \( P[\text{reward}_{I_1}(p-as) \leq \mathbb{E}(\text{reward}_{I_1}(p-as)) + \Delta_2^*(\delta)] \leq \frac{\delta}{|S|} \).

Thus \( P[\text{total}_n(p, a) \leq \mathbb{E}(\text{total}_n(p, a)) - \sum_s (\Delta_1^*(\delta) + \Delta_2^*(\delta))] \leq \delta \).

As \( \text{count}_n(p \cdot a) \leq \text{count}_n(p, a) \), there exists \( C \in \mathbb{N} \) such that for \( \text{count}_n(p \cdot a) \) big enough and for all \( \delta > 0 \):

\[
\sum_s (\Delta_1^*(\delta) + \Delta_2^*(\delta)) \leq C \sum_s \sqrt{\text{count}_n(p \cdot a) \ln \left( \frac{1}{\delta} \right)}
\]

\[
\leq C \sum_s \sqrt{\text{count}_n(p, a) \ln \left( \frac{1}{\delta} \right)}
\]

\[
\leq C|S| \sqrt{\text{count}_n(p, a) \ln \left( \frac{1}{\delta} \right)}
\]

So, there is a constant \( C_p \) such that for \( \text{count}_n(p, a) \) big enough and any \( \delta > 0 \), it holds that \( \Delta_{\text{count}_n(p, a)}(\delta) = C_p \sqrt{\text{count}_n(p, a) \ln(1/\delta)} \geq \sum_s (\Delta_1^*(\delta) + \Delta_2^*(\delta)) \). Therefore following bounds hold:
\[ P \left[ \text{total}_n(p,a) \geq \mathbb{E}[\text{total}_n(p,a)] + \Delta_{\text{count}_n(p,a)}(\delta) \right] \text{ is upper bounded by} \]
\[ P \left[ \text{total}_n(p,a) \geq \mathbb{E}[\text{total}_n(p,a)] + \sum_{s}(\Delta_s^1(\delta) + \Delta_2(\delta)) \right] \leq \delta. \]

It follows that \( P[\text{total}_n(p,a) \geq \mathbb{E}[\text{total}_n(p,a)] + \Delta_{\text{count}_n(p,a)}(\delta)] \leq \delta. \)

Similarly, the following bounds hold: \( P \left[ \text{total}_n(p,a) \leq \mathbb{E}[\text{total}_n(p,a)] - \Delta_{\text{count}_n(p,a)}(\delta) \right] \leq \delta \).

This proves that for any \( p \), the sequences \( (x_{a,t})_{t \geq 1} \) associated with \( \text{reward}_{I_t(p,a)}(p) \) satisfy the drift conditions.

\[ \Box \]

D MCTS under Advice

Proof (of Lemma 2). We have \( \text{Paths}^H_T(\sigma) = \text{Paths}^H_T(\sigma) \cap \text{Paths}^H_T(\tau^H) \) for any advice \( \sigma \). Let us prove that \( \text{Paths}^H_T(\sigma) \subseteq \text{Paths}^H_T(\sigma) \) for a strongly enforceable advice \( \sigma \). Let \( p = p' \) be a path in \( \text{Paths}^H_T(\sigma) \). By definition of \( \sigma \), there exists \( s \) such that \( p' \cdot s \in \sigma \), so that \( p \cdot a \cdot s' \in \text{Paths}^H_T(\sigma) \subseteq \text{Paths}^H_T(\sigma) \). Since \( s \in \text{Supp}(P(p_{\text{last}}(p),a)) \), \( p = p' \cdot a \cdot s \) must also belong to \( \text{Paths}^H_T(\sigma) \).

Consider a path \( p \) and an action \( a \) such that \( |p| < H \). We want to prove that either all stochastic transitions starting from \( (p,a) \) are allowed by \( \sigma \), or none of them are. By contradiction, let us assume that there exists \( s_0 \) and \( s_1 \) in \( S \) such that for all \( p' \in \text{Paths}^H_T(\sigma) \), \( \cdot p \cdot a \cdot p' \in \sigma \), and such that there exists \( p' \in \text{Paths}^H_T(\sigma) \) with \( p \cdot a \cdot p' \in \sigma \).

From \( p \cdot p' \cdot p' \in \sigma \), we obtain \( p \cdot p' \cdot p' \in \text{Paths}^H_T(\sigma) \), so that \( p \cdot p' \cdot s \) is a path that follows \( \sigma \). Then, \( p \cdot a \cdot p' \) is a path that follows \( \sigma \) as well. It follows that \( \sigma(p \cdot a \cdot s) \neq 0 \), and \( p \cdot a \cdot s \) can be extended into a path \( p \cdot a \cdot s \cdot p' \in \text{Paths}^H_T(\sigma) \). This implies the contradiction \( p \cdot a \cdot s \in \sigma \).

Notice that for a strongly enforceable advice \( \sigma \), a state \( s_0 \) and a horizon \( H \), for any strategy \( \sigma \) in \( T(M,s_0,H,\sigma) \), \( \sigma(0) \subseteq \text{Paths}^H_T(\sigma) \). Also for any strategy \( \sigma \) in \( T(M,s_0,H,\sigma) \), \( \text{Paths}^H_T(M,s_0,H,\sigma) \subseteq \text{Paths}^H_T(M,s_0,H,\sigma) \).

Therefore for a strategy \( \sigma \) in \( T(M,s_0,H,\sigma) \), from lemma 1:

\[ \text{Val}^H_T(M,s_0,H,\sigma) = \text{Val}^H_T(M,s_0,H,\sigma) = \text{Val}^H_T(M,s_0,\sigma). \]

Proof (of Lemma 3). From optimality assumption, we get \( \sigma^H_{M,s} \subseteq \sigma^H_{\sigma} \). So:

\[ \sigma^H_{T(M,s_0,H,\sigma)}(s_0) = \arg \max_{\sigma \in \text{Paths}^H_T(M,s_0,H,\sigma)} \text{Val}^H_T(M,s_0,H,\sigma)(s_0,\sigma) \]
\[ = \arg \max_{\sigma \in \text{Paths}^H_T(M,s_0,H,\sigma)} \text{Val}^H_T(s_0,\sigma) \]
\[ = \sigma^H_{M,s_0}(s_0) \]

Therefore \( \sigma^H_{T(M,s_0,H,\sigma)}(s_0) = \sigma^H_{T(M,s_0,H,\sigma)}(s_0) = \sigma^H_{M,s_0}(s_0) \). Also \( \text{Val}^H_T(s_0,\sigma) = \text{Val}^H_T(s_0,\sigma) = \text{Val}^H_T(s_0,\sigma) \).

Proof (of Lemma 4). The proof is a reverse induction on the depth \( i \) of \( p \). For the initialisation step, with \( i = H \), let us prove that \( \forall s_1 \cdot p \cdot a \cdot s_1 \in \sigma \) if and only if \( a_0 \in \sigma^H_{M,s_0}(p) \). On the one hand, \( \sigma \) is guaranteed by playing \( a_0 \) from \( p \), then \( a_0 \) must be allowed by the greatest strongly enforceable subset of \( \sigma \). On the other hand, \( a_0 \in \sigma^H_{M,s_0}(p) \) implies \( \forall s_1, p \cdot a_0 \cdot s_1 \in \sigma \) as \( \sigma \) is strongly enforceable, and finally \( \sigma \Rightarrow \sigma \). We now assume the property holds for \( 1 \leq i \leq H \), and prove it for \( i - 1 \). If \( a_0 \in \sigma^H_{M,s_0}(p) \), then for all \( s_1 \) we have \( s_1 \in \tau^H_{M,s_0}(p,a_0) \), so that there exists \( a_1 \) with \( a_1 \in \sigma^H_{M,s_0}(p,a_0) \). As \( a \cdot a_0 \) is at depth \( i \) we can conclude that \( \forall s_1 \exists a_1 \exists s_2 \ldots \exists s_{H-1}, p \cdot a_0 \cdot a_1 \cdot s_2 \ldots s_{H-1} \in \sigma \) by assumption. For the converse direction, the alternation of quantifiers states that \( \sigma \) can be guaranteed from \( p \) by some deterministic strategy that starts by playing \( a_0 \), and therefore \( a_0 \) must be allowed by the strongly enforceable advice extracted from \( \sigma \).