ON THE HOROFUNCTION BOUNDARIES OF HOMOGENEOUS GROUPS

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Abstract. We give a complete analytic and geometric description of the horofunction boundary for sub-Finsler polygonal metrics—that is, those that arise as asymptotic cones of word metrics—on the Heisenberg group. We develop theory for the more general case of horofunction boundaries in homogeneous groups by connecting horofunctions to Pansu derivatives of the distance function.

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1. INTRODUCTION

1.1. Describing the horofunction boundary. The study of boundaries in metric spaces has a rich history and has been fundamental in building bridges between the fields of algebra, topology, geometry, and dynamical systems. Understanding the boundary was essential in the proof of Mostow’s rigidity theorem for closed hyperbolic manifolds, and boundaries have also been used to classify isometries of metric spaces, to understand algebraic splittings of groups, and to study the asymptotic behavior of random walks.

The simplest and most classical setting for horofunctions is in the study of isometries of the hyperbolic plane. There, the isometry group splits and induces a geodesic flow and a horocycle flow on the tangent bundle; horocycles, or orbits of the horocycle flow, are level sets of horofunctions. The notion has since been abstracted by Busemann, generalized by Gromov, and used by Rieffel, Karlsson–Ledrappier, and many others to derive results in various fields. The horofunction boundary is obtained by embedding a metric space \( X \) into the space of continuous real-valued functions on \( X \) via the metric, as we will define below.

In this paper, we develop tools to study the horofunction boundary of homogeneous groups, in particular the real Heisenberg group \( \mathbb{H} \). The horofunction boundary of the Heisenberg group has been the subject of study in several publications. Klein and Nicas described the boundary of \( \mathbb{H} \) for the Korányi and sub-Riemannian metrics [10, 11], while several others have studied the boundaries of discrete word metrics in the integer Heisenberg group [22, 1]. In this paper, we aim to understand the horofunction boundary of the real Heisenberg group \( \mathbb{H} \) for a family of polygonal sub-Finsler metrics which arise as the asymptotic cones of the integer Heisenberg group for different word metrics [19].

While horofunction boundaries are not (yet) used as widely as visual boundaries or Poisson boundaries, they admit a theory which is useful across several fields including geometry, analysis, and dynamical systems. Whether it is classifying Busemann function, giving explicit formulas for the horofunctions, describing the topology of the boundary, or studying the action of isometries on the boundary, what it means to understand or to describe a horofunction boundary varies significantly between works.

In this paper, as is done in for the \( \ell^\infty \) metric on \( \mathbb{R}^n \) in [4], we hope to combine these analytic, topological, and dynamical descriptions while also introducing a more geometric approach. In particular, we want to associate a “direction” to every horofunction as well as a geometric condition for a sequence of points to induce a horofunction. In some settings, the horofunction boundary is made up entirely of limit points induced by geodesic rays—or in other words, every horofunction is a Busemann function.
It is known that in CAT(0) spaces \cite{2} as well as polyhedral normed vector spaces \cite{9}, the horofunction boundary is composed only of Busemann functions. This connection between horofunctions and geodesic rays provides a natural notion of directionality to the horofunction boundary, which is not present in settings of mixed curvature, as described in \cite{14}. For the model we develop in homogeneous metrics, sequences converging to a horofunction can often be dilated back to a well-defined point on the unit sphere, which we can then regard as a direction. In these sub-Finsler metrics, there are many directions with no infinite geodesics at all, so this provides one of the motivating senses in which the horofunction boundary is a better choice to capture the geometry and dynamics in nilpotent groups.

1.2. Outline of paper. For any homogeneous group, we convert the problem of describing the horofunction boundary to a study of directional derivatives, i.e., Pansu derivatives. It suffices to understand Pansu derivatives on the unit sphere. Therefore, in any homogeneous group where the unit sphere is understood, our method allows a description of the horofunction boundary.

Pansu-differentiable points on the sphere (i.e., points \( p \) at which distance to the origin has a well defined Pansu derivative) can be thought of as *directions* of horofunctions. Not all horofunctions are directional; the rest are *blow-ups* of non-differentiable points. Background on homogeneous groups, Pansu derivatives, and horofunctions is provided in §2. We use Kuratowski limits—a notion of set convergence in a metric space—to define the blow-up of a function in §3.

In the remainder of the paper, we focus on the Heisenberg group \( \mathbb{H} \). Recall that Klein–Nicas identified the horofunction boundary for sub-Riemannian metrics on \( \mathbb{H} \) with a topological disk. In §4 we show that the corresponding disk belongs to the boundary for the larger class of sub-Finsler metrics, but is a proper subset in many cases.

Our main theorem (Theorem 5.1 in §5) describes the horoboundary of polygonal sub-Finsler metrics on \( \mathbb{H} \) in terms of blow-ups. From this, we are able to give explicit expressions for the horofunctions, to describe the topology of the boundary, and to identify Busemann points. We also show that for any sub-Finsler metric, there is a topological disk in the horofunction boundary.

This description is extremely explicit and allows us to visualize the horofunction boundary and to understand it geometrically. We get a correspondence between “directions” on the sphere and functions in the boundary, as reflected by the colors in Figure 1. This description allows us to realize the horofunction boundary as a kind of dual to the unit sphere, generalizing previous observations for normed vector spaces and for the sub-Riemannian metric on \( \mathbb{H} \) \cite{7, 4, 9, 21, 11}.

\[
\begin{array}{c|c}
\mathbb{R}^2, d_{\text{Eucl}} & \mathbb{R}^2, d_{\text{hex}} \\
\text{Sphere} & \text{Sphere} \\
\text{Boundary} & \text{Boundary}
\end{array}
\]

\[
\begin{array}{c|c}
\mathbb{H}, d_{\text{subRiem}} & \mathbb{H}, d_{\text{subFins}} \\
\text{Sphere} & \text{Sphere} \\
\text{Boundary} & \text{Boundary}
\end{array}
\]

*Figure 1.* Dualities between unit spheres and horofunction boundaries for various metric spaces, where colors connect boundary points to directions on the unit sphere.
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2. Preliminaries on homogeneous groups and horofunctions

We begin with a brief introduction to graded Lie groups, homogeneous metrics, Pansu derivatives, and horofunctions. For a survey on graded Lie groups and homogeneous metrics, we refer the interested reader to [12].

2.1. Graded Lie groups. Let $V$ be a real vector space with finite dimension and $[\cdot, \cdot] : V \times V \to V$ be the Lie bracket of a Lie algebra $\mathfrak{g} = (V, [\cdot, \cdot])$. We say that $\mathfrak{g}$ is graded if subspaces $V_1, \ldots, V_s$ are fixed so that $V = V_1 \oplus \cdots \oplus V_s$ and $[V_i, V_j] := \text{span}\{[v, w] : v \in V_i, w \in V_j\} \subset V_{i+j}$ for all $i, j \in \{1, \ldots, s\}$, where $V_k = \{0\}$ if $k > s$. Graded Lie algebras are nilpotent. A graded Lie algebra is stratified of step $s$ if equality $[V_1, V_j] = V_{j+1}$ holds and $V_s \neq \{0\}$. Our main object of study are stratified Lie algebras, but we will often work with subspaces that are only graded Lie algebras.

On the vector space $V$ we define a group operation via the Baker–Campbell–Hausdorff formula

$$pq := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{s_j + r_j > 0, j = 1 \ldots n} [p^{r_1} q^{s_1} p^{r_2} q^{s_2} \cdots p^{r_n} q^{s_n}] \prod_{j=1}^n r_j! s_j! = p + q + \frac{1}{2} \{p, q\} + \ldots,$$

where

$$[p^{r_1} q^{s_1} p^{r_2} q^{s_2} \cdots p^{r_n} q^{s_n}] = [p_1, p_2, \ldots, q_1, q_2, \ldots, [p_1, \ldots, [p_n, q_1, \ldots, q_s] \ldots]].$$

The sum in the formula above is finite because $\mathfrak{g}$ is nilpotent. The resulting Lie group, which we denote by $G$, is nilpotent and simply connected; we will call it graded group or stratified group, depending on the type of grading of the Lie algebra. The identification $\mathbb{G} = G = \mathfrak{g}$ corresponds to the identification between Lie algebra and Lie group via the exponential map $\exp : \mathfrak{g} \to G$. Notice that $p^{-1} = -p$ for every $p \in G$ and that $0$ is the neutral element of $G$.

If $\mathfrak{g}'$ is another graded Lie algebra with underlying vector space $V'$ and Lie group $G'$, then, with the same identifications as above, a map $V \to V'$ is a Lie algebra morphism if and only if it is a Lie group morphism, and all such maps are linear. In particular, we denote by $\text{Hom}_\mathbb{G}(\mathfrak{g}; \mathfrak{g}')$ the space of all homogeneous morphisms from $\mathfrak{g}$ to $\mathfrak{g}'$, that is, all linear maps $V \to V'$ that are Lie algebra morphisms (equivalently, Lie group morphisms) and that map $V_j$ to $V_j'$. If $\mathfrak{g}$ is stratified, then homogeneous morphisms are uniquely determined by their restriction to $V_1$.

For $\lambda > 0$, define the dilations as the maps $\delta_\lambda : V \to V$ such that $\delta_\lambda v = \lambda v$ for $v \in V_j$. Notice that $\delta_\lambda \delta_{\mu} = \delta_{\lambda \mu}$ and that $\delta_\lambda \in \text{Hom}_\mathbb{G}(\mathfrak{g}; \mathfrak{g})$, for all $\lambda, \mu > 0$. Notice also that a Lie group morphism $F : G \to G'$ is homogeneous if and only if $F \circ \delta_\lambda = \delta_\lambda' \circ F$ for all $\lambda > 0$, where $\delta_\lambda'$ denotes the dilations in $G'$. We say that a subset $M$ of $V$ is homogeneous if $\delta_\lambda(M) = M$ for all $\lambda > 0$.

A homogeneous distance on $G$ is a distance function $d$ that is left-invariant and 1-homogeneous with respect to dilations, i.e.,

(i) $d(gx, gy) = d(x, y)$ for all $g, x, y \in G$;

(ii) $d(\delta_\lambda x, \delta_\lambda y) = \lambda d(x, y)$ for all $x, y \in G$ and all $\lambda > 0$.

When a stratified group $G$ is endowed with a homogeneous distance $d$, we call the metric Lie group $(G, d)$ a Carnot group. Homogeneous distances induce the topology of $G$, see [15, Proposition 2.26], and are biLipschitz equivalent to each other. Every homogeneous distance defines a homogeneous norm $d_e(\cdot) : G \to [0, \infty)$, $d_e(p) = d(e, p)$, where $e$ is the neutral element of $G$. We denote by $| \cdot |$ the Euclidean norm in $\mathbb{R}^t$.

2.2. The Heisenberg group. The Heisenberg group $\mathbb{H}$ is the simply connected Lie group whose Lie algebra $\mathfrak{h}$ is generated by three vectors $X, Y$, and $Z$, with the only nonzero relation $[X, Y] = Z$. The stratification is given by $V_1 = \text{span}\{X, Y\}$ and $V_2 = \text{span}\{Z\}$.

Via the exponential map, the Heisenberg group can be coordinatized as $\mathbb{R}^3$ with the following group multiplication:

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).$$
Under this group operation, the generating vectors in the Lie algebra correspond to the left-invariant vector fields

\[ X = \partial_x - \frac{1}{2} y \partial_z, \quad Y = \partial_y + \frac{1}{2} x \partial_z, \quad Z = \partial_z. \]

It will sometimes be convenient to coordinatize \( H \) as \( \mathbb{R}^2 \times \mathbb{R} \), in which case the group operation can be written

\[ (v, t)(w, s) = (v + w, t + s + \frac{1}{2} \omega(v, w)), \]

where \( \omega \) is the standard symplectic form on the plane.

Denote by \( \Delta \) the horizontal distribution, the sub-bundle (or plane field) generated by the vector fields \( X \) and \( Y \). We define \( \pi : H \to \mathbb{H}/[\mathbb{H}, \mathbb{H}] \simeq \mathbb{R}^2 \) to be the projection of a point to its horizontal components. Note that we will generally think of the image of \( \pi \) as a point \( \mathbb{R}^2 \) rather than an equivalence class in \( \mathbb{H}/[\mathbb{H}, \mathbb{H}] \). That is, \( \pi(x, y, z) = (x, y) \).

### 2.3. Pansu derivatives

Let \( G \) and \( G' \) be two Carnot groups with homogeneous metrics \( d \) and \( d' \), respectively, and let \( \Omega \subset G \) open. A function \( f : \Omega \to G' \) is called Pansu differentiable at \( p \in \Omega \) if there is \( L \in \text{Hom}_h(G; G') \) such that

\[ \lim_{x \to p} \frac{d'(f(p)^{-1}f(x), L(p^{-1}x))}{d(p, x)} = 0. \]

The map \( L \) is called Pansu derivative of \( f \) at \( p \) and it is denoted by \( \mathcal{D}f(p) \) or \( \mathcal{D}f|_p \). A map \( f : \Omega \to G' \) is of class \( C^1_h \) if \( f \) is Pansu differentiable at all points of \( \Omega \) and the Pansu derivative \( p \mapsto \mathcal{D}f|_p \) is continuous. We denote by \( C^1_h(\Omega; G') \) the space of all maps from \( \Omega \) to \( G' \) of class \( C^1_h \).

A function \( f : \Omega \to G' \) is strictly Pansu differentiable at \( p \in \Omega \) if there is \( L \in \text{Hom}_h(G; G') \) such that

\[ \lim_{x \to p} \sup_{c \to 0} \left\{ \frac{d'(f(y)^{-1}f(x), L(y^{-1}x))}{d(x, y)} : x, y \in B_d(p, \epsilon), x \neq y \right\} = 0, \]

where \( B_d(p, \epsilon) \) is the open \( \epsilon \)-ball centered at \( p \). Clearly, in this case \( f \) is Pansu differentiable at \( p \) and \( L = \mathcal{D}f|_p \).

The next result is proven in [8, Proposition 2.4 and Lemma 2.5].

**Proposition 2.1.** A function \( f : \Omega \to G' \) is of class \( C^1_h \) on \( \Omega \) if and only if \( f \) is strictly Pansu differentiable at all points of \( \Omega \). If \( f \in C^1_h(\Omega; G') \), then \( f : (\Omega, d) \to (G', d') \) is locally Lipschitz.

### 2.4. Sub-Finsler metrics

Let \( G \) be a stratified group and \( \| \cdot \| \) a norm on the first layer \( V_1 \subset T_e G \) of the stratification. Using left-translations, we extend the norm \( \| \cdot \| \) to the sub-bundle \( \Delta \subset TG \) of left-translates of \( V_1 \). We call a curve in \( G \) admissible if it is tangent to \( \Delta \) almost everywhere, and using the norm \( \| \cdot \| \) we can measure the length of any admissible curve. A classical result tells us that since \( V_1 \) bracket-generates the whole Lie algebra, any two points in \( G \) are connected by an admissible curve. We then define a Carnot-Carathéodory length metric by defining

\[ d(p, q) = \inf_{\gamma} \left\{ \int_a^b \| \gamma'(t) \| \, dt \right\}, \]

where the infimum is taken over all admissible \( \gamma \) connecting \( p \) to \( q \).

**Proposition 2.2** (Eikonal equation). If \( d \) is a homogeneous distance on \( G \), then \( d_e \) is Pansu differentiable almost everywhere. Moreover, if \( d \) is a sub-Finsler with norm \( \| \cdot \| \), then

\[ (1) \quad \| \mathcal{D}d_e|_p \| = 1 \quad \text{for a.e. } p \in G. \]

**Proof.** Since \( d_e \) is 1-Lipschitz, then it is Pansu differentiable almost everywhere by the Pansu–Rademacher Theorem [20, Theorem 2] and, in the sub-Finsler case, \( \| \mathcal{D}d_e \| \leq 1 \). To prove (1), let \( p \in G \) be such that there is a length minimizing curve \( \gamma : [0, T + h] \to G \) parametrized by arclength such that \( \gamma(T) = p \), and \( \gamma'(T) \) exists. Then, one easily sees that

\[ 1 = \lim_{t \to T} \frac{|d_e(\gamma(t)) - d_e(\gamma(T))|}{|t - T|} = \| \mathcal{D}d_e|_p[\gamma'(T)] \|, \]

\[ \square \]
2.5. Horoboundary of a metric space. Let \((X, d)\) be a metric space and \(\mathcal{C}(X)\) the space of continuous functions \(X \rightarrow \mathbb{R}\) endowed with the topology of the uniform convergence on compact sets. The map \(\iota: X \hookrightarrow \mathcal{C}(X), (\iota(x))(y) := d(x, y)\), is an embedding, i.e., a homeomorphism onto its image.

Let \(\mathcal{C}(X)/\mathbb{R}\) be the topological quotient of \(\mathcal{C}(X)\) with kernel the constant functions, i.e., for every \(f, g \in \mathcal{C}(X)\) we set the equivalence relation \(f \sim g \Leftrightarrow f - g\) is constant.

Then the map \(\hat{\iota}: X \hookrightarrow \mathcal{C}(X)/\mathbb{R}\) is still an embedding. Indeed, since the map \(\mathcal{C}(X) \rightarrow \mathcal{C}(X)/\mathbb{R}\) is continuous and open, we only need to show that \(\hat{\iota}\) is injective: if \(x, x' \in X\) are such that \(\iota(x)(z) - \iota(x')(z)\) is constant for all \(z \in X\), then taking \(z = x\) and then \(z = x'\) in turn tells us that \(c = d(x, x') = -d(x', x)\). Hence \(c = 0\) and \(x = x'\).

Define the **horoboundary of** \((X, d)\) as

\[ \partial_h X := \{ \hat{\iota}(i(x)) \setminus i(X) \subset \mathcal{C}(X)/\mathbb{R}, \]

where \(cl(i(X))\) is the topological closure.

Another description of the horoboundary is possible. Fix \(x_0 \in X\) and set

\[ \mathcal{C}(X)_{x_0} := \{ f \in \mathcal{C}(X) : f(x_0) = 0 \}. \]

Then the restriction \(\mathcal{C}(X)_{x_0} \rightarrow \mathcal{C}(X)/\mathbb{R}\) of the quotient map is an isomorphism of topological vector spaces. Indeed, one easily checks that it is both injective and surjective, and that its inverse map is \([f] \mapsto f - f(x_0)\), where \([f]\) is the class of equivalence of \(f \in \mathcal{C}(X)\).

Hence, we identify \(\partial_h X\) with a subset of \(\mathcal{C}(X)_{x_0}\). More explicitly: \(f \in \mathcal{C}(X)_{x_0}\) belongs to \(\partial_h X\) if and only if there is a sequence \(p_n \in X\) such that \(p_n \rightarrow \infty\) (i.e., for every compact \(K \subset X\) there is \(N \in \mathbb{N}\) such that \(p_n \notin K\) for all \(n > N\)) and the sequence of functions \(f_n \in \mathcal{C}(X)_{x_0}\),

\[ f_n(x) := d(p_n, x) - d(p_n, x_0), \]

converge uniformly on compact sets to \(f\).

If \(\gamma : [0, \infty) \rightarrow X\) is a geodesic ray, it is a simple exercise to check that \(\lim_{t \rightarrow \infty} \hat{\iota}(\gamma(t))\) exists, and the geodesic ray converges to a horofunction. These horofunctions which are the limits of geodesic rays, **Busemann functions**, have been widely studied and inspired the definition of general horofunctions.

2.6. Horofunctions on vertical fibers. From the basic ingredients above, we can deduce that all horofunctions are constant on vertical fibers in \(\mathbb{H}\), which we record because it may be of independent interest.

**Proposition 2.3** (Vertical invariance of horofunctions). **Horofunctions of** \((\mathbb{H}, d)\) **are constant along the cosets of the center** \([\mathbb{H}, \mathbb{H}]\). In particular, for every \(f \in \partial_h(\mathbb{H}, d)\) there is \(\hat{f} \in C(\mathbb{H}/[\mathbb{H}, \mathbb{H}])\) such that \(f = \hat{f} \circ \pi\).

**Proof.** Let \(\rho\) be a left-invariant Riemannian metric on \(\mathbb{H}\). Recall that, by the Ball-Box Theorem [17, 16, 6, 18], if \(\zeta \in [\mathbb{H}, \mathbb{H}]\) then \(\lim_{t \rightarrow 0^+} \frac{\rho(e, \delta_{\xi})}{\varepsilon} = 0\). Moreover, by [13, Proposition 3.3], if \(d\) is a sub-Finsler distance, then there is \(L > 0\) such that

\[ |d_{\rho}(x) - d_{\rho}(y)| \leq L \rho(x, y) \quad \forall x, y \in \mathbb{H} \setminus B(e, 1/2). \]

Since all homogeneous distances are biLipschitz equivalent with each other, the Lipschitz property (3) extends to every homogeneous distance \(d\) on \(\mathbb{H}\).

Now, fix \(f \in \partial_h(\mathbb{H}, d)\), and let \(p_n \in \partial B\) and \(\varepsilon_n \rightarrow 0\) as in Lemma 2.4. Then, for every \(\zeta \in [\mathbb{H}, \mathbb{H}]\) and \(x \in \mathbb{H}\),

\[ f(x\zeta) - f(x) = \lim_{n \rightarrow \infty} \frac{d_{\rho}(p_n \delta_{\zeta}(x\zeta)) - d_{\rho}(p_n \delta_{\zeta}, x)}{\varepsilon_n} = \lim_{n \rightarrow \infty} \frac{d((p_n \delta_{\zeta}, x)^{-1}, \delta_{\zeta}, \xi) - d((p_n \delta_{\zeta}, x)^{-1}, e)}{\varepsilon_n} \leq L \lim sup_{n \rightarrow \infty} \frac{\rho(e, \delta_{\zeta}, \xi)}{\varepsilon_n} = 0. \]

2.7. Horofunctions and the Pansu derivative.

We close this preliminary section with a fundamental observation connecting horofunctions on homogeneous groups to Pansu derivatives.

Let \(d\) be a homogeneous metric on \(\mathbb{G}\) with unit ball \(B\) and unit sphere \(\partial B\). Again, we denote by \(e\) the neutral element of \(\mathbb{G}\) and by \(d_e\) the function \(x \mapsto d(e, x)\). There is a strict connection between horofunctions and Pansu derivatives of the function \(d_e : x \mapsto d(e, x)\).
Lemma 2.4. Let \( d \) be a homogeneous metric on \( G \). If \( f \in \partial_h(G,d) \), then there is a sequence \((p_n, \epsilon_n) \in \partial B \times (0, +\infty) \) such that \( p_n \to p \in \partial B \), \( \epsilon_n \to 0 \) and

\[
f(x) = \lim_{n \to \infty} \frac{d_e(p_n \delta_{\epsilon_n} x) - d_e(p_n)}{\epsilon_n}, \quad \text{locally uniformly in } x \in G.
\]

On the other hand, if \((p_n, \epsilon_n) \in \mathbb{G} \times (0, +\infty) \) such that \( p_n \to p \in \partial B \), \( \epsilon_n \to 0 \) and \( f : \mathbb{G} \to \mathbb{R} \) is the locally uniform limit (4), then \( f \in \partial_h(G,d) \).

In particular, if \( d_e \) is strictly Pansu differentiable at \( p \), then \( f = \mathbf{D}d_e|_p \); if \( p_n \to p \) and \( d_e \) is Pansu differentiable at \( p \), then \( f = \mathbf{D}d_e|_p \). Moreover, the horofunction \( f \) is limit of the sequence of points

\[
q_n = \delta_{1/\epsilon_n} p_n^{-1}.
\]

Proof. A simple computation shows that, if \( p_n, q_n \in \mathbb{G} \) and \( \epsilon_n \in (0, +\infty) \) satisfy (5), then

\[
d(q_n, x) - d(q_n, e) = \frac{d_e(p_n \delta_{\epsilon_n} x) - d_e(p_n)}{\epsilon_n}.
\]

Therefore, if \( q_n \to f \in \partial_h(G,d) \), then we take \( \epsilon_n := (\epsilon, q_n)^{-1} \), which converges to 0, and \( p_n = \delta_{\epsilon_n} q_n^{-1} \in \partial B \). Then (4) holds and, up to passing to a subsequence, \( p_n \) converges to a point \( p \in \partial B \).

The opposite direction is also clear. \(\square\)

3. Blow-ups of sets and functions in homogeneous groups

As we observed in Lemma 2.4, in homogeneous groups there is a connection between horofunctions in the boundary and directional derivatives along the unit sphere. Wherever the unit sphere is smooth, this directional derivative is the Pansu derivative. While the unit sphere is Pansu differentiable almost everywhere, the nonsmooth points must be studied using a different strategy. In this section, we overview Kuratowski convergence of closed sets, sometimes credited to Kuratowski–Painlevé, and we use it define the blow-up of function.

3.1. Kuratowski limits in metric spaces. Let \((X, d)\) be a locally compact metric space and let \( \mathcal{CL}(X) \) be the family of all closed subsets of \( X \). If \( x \in X \) and \( C \subseteq X \), we set \( d(x, C) := \inf \{d(x, y) : y \in C\} \).

The Kuratowski limit inferior of a sequence \( \{C_n\}_{n \in \mathbb{N}} \subseteq \mathcal{CL}(X) \) is defined to be

\[
\text{Li}_{n \to \infty} C_n := \left\{ q \in X : \limsup_{n \to \infty} d(q, C_n) = 0 \right\} = \left\{ q \in X : \forall n \in \mathbb{N} \exists x_n \in C_n \text{ s.t. } \lim_{n \to \infty} x_n = q \right\},
\]

while the Kuratowski limit superior is defined to be

\[
\text{Ls}_{n \to \infty} C_n := \left\{ q \in X : \liminf_{n \to \infty} d(q, C_n) = 0 \right\} = \left\{ q \in X : \exists N \subseteq \mathbb{N} \text{ infinite } \forall k \in N \exists x_k \in C_k \text{ s.t. } \lim_{k \to \infty} x_k = q \right\}.
\]

It is clear that \( \text{Li}_n C_n \subseteq \text{Ls}_n C_n \) and that they are both closed. If \( \text{Li}_n C_n = \text{Ls}_n C_n = C \), then we say that the \( C \) is the Kuratowski limit of \( \{C_n\}_n \) and we write

\[
C = \text{K-lim}_{n \to \infty} C_n.
\]

If, for all \( n \in \mathbb{N} \) \( \Omega_n \subseteq X \) are closed sets and \( f_n : \Omega_n \to \mathbb{R} \) continuous functions, then we say that, for some \( \Omega \subseteq X \) closed and \( f : \Omega \to \mathbb{R} \) continuous,

\[
\text{K-lim}_{n \to \infty}(\Omega_n, f_n) = (\Omega, f)
\]

if \( \Omega = \text{K-lim}_{n} \Omega_n \) and if, for every \( x \in \Omega \) and every sequence \( \{x_n\}_{n \in \mathbb{N}} \) with \( x_n \in \Omega_n \) and \( x_n \to x \), we have \( f(x) = \lim_n f_n(x_n) \). Notice that this is equivalent to say that

\[
\text{K-lim}_{n \to \infty}\{(x, f_n(x)) : x \in \Omega_n\} = \{(x, f(x)) : x \in \Omega\}.
\]

If \( C^1_n, \ldots, C^J_n \) are sequences of closed sets, then one easily checks that

\[
\text{Ls}_{n \to \infty} \bigcup_{j=1}^J C^j_n \subseteq \bigcup_{j=1}^J \text{Ls}_{n \to \infty} C^j_n, \quad \text{and} \quad \text{Li}_{n \to \infty} \bigcup_{j=1}^J C^j_n \subseteq \bigcup_{j=1}^J \text{Li}_{n \to \infty} C^j_n.
\]
Therefore, if the limit $\text{K-lim}_{n \to \infty} C_n^j$ exists for each $j$, then we have

\begin{equation}
\text{K-lim}_{n \to \infty} \bigcup_{j=1}^{J} C_n^j = \bigcup_{j=1}^{J} \text{K-lim}_{n \to \infty} C_n^j.
\end{equation}

It is a classical result of Zarakiewicz that under mild conditions, $\text{CL}(X)$ is sequentially compact with respect to Kuratowski convergence.

**Theorem 3.1 (Zarakiewicz [23]).** If $(X,d)$ is a separable metric space, then the family of closed sets is sequentially compact with respect to the Kuratowski convergence, that is, if $(C_n)_{n \in \mathbb{N}}$ is a sequence of closed sets, then there is $N \subset \mathbb{N}$ infinite and $C \subset X$ closed such that $\text{K-lim}_{N \to \infty} C_n = C$.

For $\epsilon \geq 0$ and $C \subset X$, let

\[ \mathcal{N}_\epsilon(C) := \{ x : d(x,C) \leq \epsilon \}, \]

and $\mathcal{N}_-\epsilon(C) := \{ x : d(x,X \setminus C) > \epsilon \}$.

Notice that $\mathcal{N}_\epsilon(C) = X \setminus \mathcal{N}_\epsilon(X \setminus C)$.

A set $C \subset X$ is a regular closed set if it is the closure of its interior. If $C$ is a closed set, then $\overline{X \setminus C}$ is regular closed. If $C$ is a regular closed set, then

\[ C = \bigcap_{\epsilon > 0} \mathcal{N}_\epsilon(C) = \bigcup_{\epsilon > 0} \mathcal{N}_-\epsilon(C). \]

**Lemma 3.2.** Assume $X$ to be locally compact. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions uniformly converging to $f_\infty : X \to \mathbb{R}$. Then

\begin{equation}
\{ f_n < 0 \} \subset \bigcup_{n \to \infty} \{ f_n \leq 0 \} \subset \bigcup_{n \to \infty} \{ f_n \leq 0 \} \subset \{ f_\infty \leq 0 \}.
\end{equation}

In particular, if $\{ f_\infty < 0 \} = \{ f_\infty \leq 0 \}$, then

\[ \text{K-lim}_{n \to \infty} \{ f_n \leq 0 \} = \{ f_\infty \leq 0 \}. \]

**Proof.** For the first inclusion in (7), let $p \in X$ with $f_\infty(p) < -\epsilon < 0$ for some $\epsilon \leq 0$. Then there is $r > 0$ such that $B(p,r)$ is compact and $f_\infty(x) < -\epsilon$ for all $x \in B(p,r)$. By the uniform convergence on compact sets, there exist $N \subset \mathbb{N}$ such that $f_n(p) < -\epsilon/2 < 0$ for all $n > N$. Therefore, $p \in \bigcup_{n \to \infty} \{ f_n \leq 0 \}$.

For the third inclusion in (7), consider a sequence $(p_n)_{n \in \mathbb{N}} \subset X$ with $p_n \to p$ and $f_n(p_n) \leq 0$. Then, by the uniform convergence on compact sets, we have $\lim_n f_n(p_n) = f(p)$ and thus $f(p) \leq 0$.

The last statement is a direct consequence of the fact that Kuratowski superior and inferior limits are both closed. \hfill \Box

A family $\mathcal{F} \subset \mathbb{R}^X$ is strictly monotone if for every $p \in X$ there exists $\gamma_p : [-1,1] \to X$ continuous with $\gamma_p(0) = p$ such that $t \mapsto f(\gamma_p(t))$ is strictly increasing for every $f \in \mathcal{F}$.

**Lemma 3.3.** If $\mathcal{F} \subset C(X)$ is strictly monotone and finite, then

\[ \{ \max \mathcal{F} < 0 \} = \{ \max \mathcal{F} \leq 0 \}. \]

**Proof.** Let $p \in \{ \max \mathcal{F} \leq 0 \}$ with $\max \mathcal{F}(p) = 0$. Let $\gamma_p : [-1,1] \to X$ be continuous with $\gamma_p(0) = p$ such that $t \mapsto f(\gamma_p(t))$ is strictly increasing for every $f \in \mathcal{F}$. It follows that, for every $f \in \mathcal{F}$ and $t < 0$, we have $f(\gamma(t)) < f(\gamma(0)) \leq \max \mathcal{F}(p) = 0$. Then $p_n = \gamma(-1/n)$ is a sequence of points converging to $p$ with $p \in \{ \max \mathcal{F} < 0 \}$. We conclude that $p \in \{ \max \mathcal{F} \leq 0 \}$.

**Lemma 3.4.** Assume that $X$ is locally compact. For each $j$ integer between 1 and $J \in \mathbb{N}$, let $(f_n^j)_{n \in \mathbb{N}}$ be a sequence of continuous functions $f_n^j : X \to \mathbb{R}$ converging uniformly on compact sets to $f_\infty^j : X \to \mathbb{R}$. Then the sequence of continuous functions $g_n := \max\{ f_n^j \}$ converges uniformly on compact sets to $g_\infty := \max\{ f_\infty^j \}$.

Moreover, if $(f_\infty^j)_{j=1}^J$ is strictly monotone, then

\[ \text{K-lim}_{n \to \infty} \bigcap_{j=1}^{J} \{ f_n^j \leq 0 \} = \bigcap_{j=1}^{J} \{ f_\infty^j \leq 0 \} = \{ g_\infty \leq 0 \}. \]

**Proof.** Proceeding by induction on $J$, we assume $J = 2$. Notice that this works also for the last statement, because $\bigcap_{j=1}^{J} \{ f_n^j \leq 0 \} = \{ g_\infty \leq 0 \}$.

So, we assume $J = 2$. Let $K \subset X$, $\epsilon > 0$, and let $(f_1 \vee f_2)(x) = \max\{ f_1(x), f_2(x) \}$. Then there is $N \in \mathbb{N}$ such that $|f_n^1(x) - f_n^2(x)| < \epsilon$ for all $x \in K$ and $j$. We claim that $|(f_n^1 \vee f_n^2)(x) - (f_\infty^1 \vee f_\infty^2)(x)| < 3\epsilon$ for all $x \in K$. To prove the claim we need to check four cases, which by symmetry reduce to the following two: In the first case, $(f_n^1 \vee f_n^2)(x) = f_n^1(x)$ and $(f_\infty^1 \vee f_\infty^2)(x) = f_\infty^1(x)$. Then clearly
Therefore,
\[
|\left( f_n^1 \lor f_n^2 \right)(x) - \left( f_\infty^1 \lor f_\infty^2 \right)(x)| < \epsilon. \quad \text{In the second case,} \quad \left( f_n^1 \lor f_n^2 \right)(x) = f_n^1(x) \text{ and} \quad \left( f_\infty^1 \lor f_\infty^2 \right)(x) = f_\infty^2(x).
\]
Notice that
\[
0 \leq f_\infty^2(x) - f_n^1(x) \\
\leq f_\infty^2(x) - f_n^2(x) + f_n^2(x) - f_n^1(x) + f_n^1(x) - f_n^1(x) \\
\leq f_\infty^2(x) - f_n^2(x) + f_n^1(x) - f_n^2(x) \leq 2\epsilon.
\]

Therefore,
\[
|\left( f_n^1 \lor f_n^2 \right)(x) - \left( f_\infty^1 \lor f_\infty^2 \right)(x)| = |f_n^1(x) - f_\infty^2(x)| \leq |f_n^1(x) - f_\infty^1(x)| + |f_\infty^2(x) - f_\infty^2(x)| \leq 3\epsilon.
\]
This proves the claim and the first part of the lemma.

By Lemma 3.2, we only need to show that \( \{g_\infty < 0\} = \{g_\infty \leq 0\} \). But this is a consequence of the strict monotonicity and Lemma 3.3.

\[\square\]

3.2. Blow-ups of sets in homogeneous groups. Let \( G \) be a homogeneous group with a homogeneous distance \( d \). If \( \Omega \subset G \) is closed, \( \{p_n\}_{n \in \mathbb{N}} \subset G \) and \( \{e_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) are sequences, we define the blow-up set
\[
BU(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) := \text{K-lim}_{n \to \infty} \delta_{1/e_n}(p_n^{-1}\Omega),
\]
if it exists. We sometimes use also the intermediate blow-up sets
\[
BU^{-}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) := \text{Li}_{n \to \infty} \delta_{1/e_n}(p_n^{-1}\Omega),
\]
\[
BU^{+}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) := \text{Li}_{n \to \infty} \delta_{1/e_n}(p_n^{-1}\Omega),
\]
which are always well defined and
\[
(8) \quad BU^{-}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \subset BU^{+}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}).
\]

Proposition 3.5. Let \( \Omega \subset G \) be a nonempty closed set, \( \{p_n\}_{n \in \mathbb{N}} \subset G \) and \( \{e_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) sequences with \( e_n \to 0 \).

1. \( BU^{+}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \neq \emptyset \), if and only if \( \limsup_{n \to \infty} d(p_{n\Omega}, \Omega) = \infty \).

2. If \( BU^{-}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \notin G \), then \( \limsup_{n \to \infty} d(p_{n\Omega}, \Omega) = \infty \).

In particular, in case \( p_n \to p \) then we have:

1. If \( p \notin \Omega \), then \( BU(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) = \emptyset \).

2. If \( p \in \Omega^c \), then \( BU(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) = G \).

Proof. (1) Let \( q \in BU^{+}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \). Then there exists \( N \in \mathbb{N} \) infinite and a sequence \( \{x_k\}_{k \in \mathbb{N}} \subset \Omega \) such that \( q = \lim_{k \to \infty} \delta_{1/e_k}(p_k^{-1}x_k) \). Therefore,
\[
\lim_{n \to \infty} d(p_n, \Omega)_{e_n} \leq \limsup_{N \to \infty} d(p_k, x_k)_{e_k} = \liminf_{N \to \infty} d(e, \delta_{1/e_n}(p_n^{-1}x_k)) = d(e, q).
\]

Let \( N \in \mathbb{N} \) infinite and a sequence \( \{x_k\}_{k \in \mathbb{N}} \subset \Omega \) such that \( \lim_{N \to \infty} d(p_{n\Omega}, \Omega) = \infty \). Since \( d(p_{n\Omega}, \Omega) \leq d(e, \delta_{1/e_n}(p_n^{-1}x_k)) \), we can assume, up to passing to a subsequence, that the limit \( \lim_{N \to \infty} d(p_{n\Omega}, \Omega) = \infty \).

Thus, \( BU^{+}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) = \emptyset \).

(2) Let \( q \notin G \setminus BU^{-}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \) and define \( x := p_n e_n q \). Since \( q \in G \setminus BU^{-}(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \), there is \( N \in \mathbb{N} \) infinite such that \( x_k \notin \Omega \) for all \( k \in \mathbb{N} \). Therefore,
\[
\limsup_{n \to \infty} d(p_n, G \setminus \Omega)_{e_n} \leq \limsup_{n \to \infty} d(p_n, p_n e_n q)_{e_n} = d(e, q).
\]

\[\square\]

Proposition 3.6. Let \( \Omega \subset G \) be a nonempty closed set and \( p \in \partial\Omega \). Suppose that there exists a neighborhood \( U \) of \( p \) and a finite family of continuous functions \( F_j : U \to \mathbb{R} \) with \( j \in J \) finite such that \( \Omega \cap U = \bigcap_{j \in J} \{F_j(x) = 0\} \) and \( F_j(p) = 0 \) for all \( j \). Suppose also that each \( F_j \) is strictly Pansu differentiable at \( p \) and that
\[
(9) \quad 0 \notin \text{cvx}\{DF_j|_p\}_{j \in J}.
\]

Let \( p_n \to p \) and \( e_n \to 0^+ \), and assume that \( BU(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) \) exists. Then
\[
BU(\Omega, \{p_n\}_{n}, \{e_n\}_{n}) = \{x \in G : DF_j|_p(x) \leq t_j, j \in J\}
\]
with \( t_j \in \mathbb{R} \cup \{-\infty, +\infty\} \) defined as follows:

1. if \( \lim_{n \to \infty} d(p_n, F_j(0))_{e_n} = +\infty \), then \( t_j = -\infty \);
(2) if \( \lim_n \frac{d(p_n, G \setminus \{F_j \leq 0\})}{\epsilon_n} = +\infty \), then \( t_j = +\infty \);

(3) otherwise, there are \( g^j_n \in \{F_j = 0\} \) such that, up to a subsequence, \( \lim_n \delta_t/\epsilon_n((g^j_n)^{-1}p_n) =: v_j \), and we set \( t_j = -\nabla F_j|_p(v_j) \).

Proof. Let \( p_n \to p \) and \( \epsilon_n \to 0^+ \), assume that \( BU(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) \) exists.

If there is any \( j \) such that \( \lim_n \frac{d(p_n, \{F_j \leq 0\})}{\epsilon_n} = +\infty \) then \( BU(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) = \emptyset \) by Proposition 3.5.

Again by Proposition 3.5, for any \( j \) such that \( \lim_n \frac{d(p_n, \{F_j \leq 0\})}{\epsilon_n} = +\infty \), we know \( BU(\{F_j \leq 0\}, \{p_n\}_n, \{\epsilon_n\}_n) = \mathbb{G} = \{\nabla F_j|_p \leq +\infty\} \).

Let \( \hat{J} \) be the set of indices which do not fall into the first two cases. For all \( j \in \hat{J} \), there are \( g^j_n \in \{F_j = 0\} \) with \( d(p_n, g^j_n) = d(p_n, \{F_j = 0\}) \) and \( \limsup_n \frac{d(p_n, g^j_n)}{\epsilon_n} < \infty \). Up to a subsequence, we can assume that the limit \( \lim_n \delta_t/\epsilon_n((g^j_n)^{-1}p_n) =: v_j \) exists. Define

\[
f^j_n(x) := \frac{F_j(p_n \delta_{\epsilon_n} x)}{\epsilon_n} = \frac{F_j(p_n \delta_{\epsilon_n} x) - F(p_n)}{\epsilon_n} + \frac{F(p_n) - F(q_n)}{\epsilon_n}.
\]

By the strict Pansu differentiability of \( F_j \) at \( p \), the functions \( f^j_n \) converge uniformly on compact sets to \( f^j_\infty(x) := \nabla F_j|_p(x) + \nabla F_j|_p(v_j) \).

Condition (9) implies that there is \( w \in V_1 \) such that \( \nabla F_j|_p(w) > 0 \) for all \( j \). Define \( \gamma(t) = p \exp(tw) \). Then

\[
\frac{d}{dt} f^j_\infty(\gamma(t)) = \nabla F_j|_p(w),
\]

which is strictly positive in a neighborhood of 0, for all \( j \in \hat{J} \). Therefore, the family of functions \( \{f^j_n\}_{j \in \hat{J}} \) is strictly monotone and we conclude by Lemma 3.4 that

\[
BU(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) = \bigcap_{j \in \hat{J}} \{f^j_\infty \leq 0\}.
\]

\( \square \)

**Proposition 3.7.** Let \( \Omega \subset \mathbb{G} \) be a nonempty closed set and \( p \in \partial \Omega \). Suppose that there exists a neighborhood \( U \) of \( p \) and a finite family of continuous functions \( F_j : U \to \mathbb{R} \) with \( j \in J \) finite such that \( \Omega \cap U = \bigcap_{j \in J} \{F_j \leq 0\} \) and \( F_j(p) = 0 \). Suppose also that each \( F_j \) is strictly Pansu differentiable at \( p \) and that \( \nabla F_j|_p \) are linearly independent.

Then, for every \( (t_j)_j \in (\mathbb{R} \setminus \{\pm \infty\})^J \) and every \( \epsilon_n \to 0^+ \), there are \( p_n \to p \) such that

\[
BU(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) = \{x \in \mathbb{G} : \nabla F_j|_p(x) \leq t_j, \ j \in J\}.
\]

**Proof.** Fix \( (t_j)_j \in (\mathbb{R} \cup \{\pm \infty\})^J \) and \( \epsilon_n \to 0^+ \). Let \( A \subset J \) be the set of indices such that \( t_j \) is finite, and \( \hat{A} \) the set of indices for which \( t_j = +\infty \). Since \( \nabla F_j|_p \) are linearly independent, there exists an affine subspace \( W \subset \mathbb{G} \) such that for each \( j \in A \) and each \( w \in W \), we have \( \nabla F_j|_p(w) = -t_j \). Note that the topological dimension of \( W \) is at least \( |A| \). For \( n \) sufficiently large, the intersection \( B_{\sqrt{\epsilon_n}}(p) \cap W \) is nonempty, so we choose \( w_n \in B_{\sqrt{\epsilon_n}}(p) \) such that \( \nabla F_j|_p(w_n) \) diverges to \( -\infty \) for each \( j \in \hat{A} \), as \( n \to \infty \).

Set \( p_n := p\delta_{\epsilon_n} w_n \), and observe that \( p_n \to p \) due to our choice of \( w_n \). Notice that for \( j \in A \), the functions

\[
f^j_n(x) := \frac{F_j(p_n \delta_{\epsilon_n} x)}{\epsilon_n} = \frac{F_j(p_n \delta_{\epsilon_n} w_n x) - F_j(p)}{\epsilon_n}
\]

converge uniformly on compact sets to \( x \to \nabla F_j|_p(x) - t_j \). For \( j \in \hat{A}, M < 0 \), and \( K \) compact, eventually the restrictions of \( f^j_n(x) \) to \( K \) are bounded above by \( M \). Thus for \( j \in \hat{A} \) and \( M \) sufficiently large, \( f^j_n(x) \) does not affect the max function \( g^j_n(x) = \max\{f^j_n\}_j \). Now, the fact that \( \nabla F_j|_p \) are linearly independent implies \( 0 \notin \text{cvx}\{\nabla F_j|_p\}_{j \in J} \) and thus, as in the proof of Proposition 3.6, \( \{f^j_n\}_{j \in J} \) is strictly monotone. We conclude by Lemma 3.4 that

\[
BU(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) = \{\nabla F_j|_p(x) \leq t_j, j \in J\} = \{\nabla F_j|_p(x) \leq t_j, j \in A\}.
\]

\( \square \)
3.3. Blow-ups of functions in homogeneous groups. For a continuous function \( f : \Omega \rightarrow \mathbb{R} \), we define
\[
\text{BU}(\Omega, f, \{p_n\}_n, \{\epsilon_n\}_n) := \text{K-lim}_{n \rightarrow \infty} \left( \delta_{1/\epsilon_n}(p_n^{-1}\Omega), \frac{f(p_n\delta_{\epsilon_n}) - f(p_n)}{\epsilon_n} \right).
\]

**Proposition 3.8.** Let \( \Omega \subseteq G \) be a nonempty closed set, \( \{p_n\}_{n \in \mathbb{N}} \subseteq G \) and \( \{\epsilon_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) sequences with \( p_n \rightarrow p \in \Omega \) and \( \epsilon_n \rightarrow 0 \). Suppose that \( \Omega_0 := \text{BU}(\Omega, \{p_n\}_n, \{\epsilon_n\}_n) \) exists. Let \( f : G \rightarrow \mathbb{R} \) be a continuous function that is strictly Panus differentiable at \( p \). Then
\[
\text{BU}(\Omega, f, \{p_n\}_n, \{\epsilon_n\}_n) = (\Omega_0, \mathcal{D}f(p)|_{\Omega_0}).
\]

**Proof.** Let \( f_n(x) := \frac{f(p_n \delta_{\epsilon_n})(x) - f(p_n)}{\epsilon_n} \). If \( x_n \in \delta_{1/\epsilon_n}(p_n^{-1}\Omega) \) are such that \( x_n \rightarrow x \in \Omega_0 \), then \( f_n(x_n) \rightarrow \mathcal{D}f(p)[x] \), by the strict Panus differentiability of \( f \) at \( p \). \( \square \)

If \( Q \) is a closed set, we say that a function \( f : Q \rightarrow \mathbb{R} \) is smooth if there exists a smooth extension of \( f \) in a neighborhood of \( Q \). In particular, the derivative of \( f \) at points \( p \in \partial Q \) is well defined.

**Theorem 3.9.** Let \( \Omega \subset G \) be a closed set and there is a family of regular closed sets \( Q \) with disjoint interiors such that \( \Omega = \bigcup_{Q \in Q} Q \). For each \( Q \in Q \), let \( f_Q : G \rightarrow \mathbb{R} \) smooth such that the function \( f : \Omega \rightarrow \mathbb{R} \) defined by
\[
f(x) := \chi(x) \sum_{Q \in Q} f_Q(x) \mathbb{1}_Q(x)
\]
is Lipschitz continuous, where \( \chi(x) := \left( \sum_{Q \in Q} \mathbb{1}_Q(x) \right)^{-1} \).

Let \( \{p_n\}_{n \in \mathbb{N}} \subseteq G \) and \( \{\epsilon_n\}_{n \in \mathbb{N}} \subset (0, +\infty) \) sequences with \( p_n \rightarrow p \in \Omega^\circ \) and \( \epsilon_n \rightarrow 0 \). Assume that \( R_Q := \text{BU}(Q, \{p_n\}_n, \{\epsilon_n\}_n) \) exists for every \( Q \in Q \).

Then
\[
G = \bigcup_{Q \in Q} R_Q
\]
and \( \text{BU}(\Omega, f, \{p_n\}_n, \{\epsilon_n\}_n) = (G, g) \) exists, where
\[
g(x) = \tilde{\chi}(x) \sum_{Q \in Q} (\mathcal{D}f_Q|x) + c_Q \mathbb{1}_{R_Q}(x),
\]
with \( \tilde{\chi}(x) := \left( \sum_{Q \in Q} \mathbb{1}_{R_Q}(x) \right)^{-1} \) and \( c_Q \in \mathbb{R} \).

Notice that, if \( R_Q = \emptyset \), then \( c_Q \) is not uniquely determined. Moreover, the constants \( c_Q \) are such that \( g \) is continuous and \( g(e) = 0 \).

**Proof.** The fact that \( G = \bigcup_{Q \in Q} R_Q \) follows from \( p \in \Omega^\circ \), and...

Notice that there is a unique choice of constants \( c_Q \in \mathbb{R} \) for \( Q \in Q \) with \( R_Q \neq \emptyset \) such that a function defined as in (10) is continuous and takes the value 0 at \( e \). For instance, if \( e \in R_Q \), then \( c_Q \), must be zero; if \( x \in R_Q \cap R_{Q'} \), then \( c_Q = \mathcal{D}f_{Q'}|_x - \mathcal{D}f_Q|_x \), and so on.

Next, set \( g_n : \delta_{1/\epsilon_n}(p_n^{-1}\Omega) \rightarrow \mathbb{R} \), \( g_n(x) := \frac{f(p_n \delta_{\epsilon_n})(x) - f(p_n)}{\epsilon_n} \). The family of functions \( \{g_n\}_{n \in \mathbb{N}} \) is uniformly Lipschitz and \( g_n(e) = 0 \) for all \( n \). Thus, the set
\[
\mathcal{N} := \{N \subset \mathbb{N} \text{ infinite : } \{g_n\}_n \text{ converge}\}
\]
is nonempty and for every \( N \subset \mathbb{N} \) infinite there is \( N' \subset \mathbb{N} \) with \( N' \subset N \). For every \( N \subset \mathcal{N} \), define \( g^N := \lim_{N \ni n \rightarrow \infty} g_n \). We aim to prove that \( g^N = g \) for all \( N \subset \mathcal{N} \).

Let \( x \in R_Q \) for some \( Q \in Q \). Then there exist \( y_n \in Q \) such that \( x_n := \delta_{1/\epsilon_n}(p_n^{-1}y_n) \rightarrow x \). Therefore, \( g_n(x_n) \rightarrow g(x) \), where
\[
g_n(x_n) = \frac{f(y_n) - f(p_n)}{\epsilon_n} = \frac{f_Q(y_n) - f_Q(p_n)}{\epsilon_n} + \frac{f_Q(p_n) - f(p_n)}{\epsilon_n}.
\]
Since \( f_Q \) is smooth at \( p \), we have \( \lim_{n} \frac{f_Q(y_n) - f_Q(p_n)}{\epsilon_n} = \mathcal{D}f_Q|_y \). Therefore, if \( N \in \mathcal{N} \), then the limit \( c^N_Q := \lim_{N \ni n \rightarrow \infty} \frac{f_Q(p_n) - f(p_n)}{\epsilon_n} \) exists and it is equal to \( g^N(x) - \mathcal{D}f_Q|x \). Moreover,
\[
g^N(x) = \tilde{\chi}(x) \sum_{Q \in Q} (\mathcal{D}f_Q|x) + c^N_Q \mathbb{1}_{R_Q}(x).
\]
Finally, \( g^N \) is continuous and \( g^N(e) = e \), then \( c^N_Q = c_Q \), for all \( Q \in Q \) with \( R_Q \neq \emptyset \).
We conclude that $g^N = g$ for all $N \in \mathbb{N}$ and thus $g = \lim_{n \to \infty} g_n$. 

This theorem will allow us to finish our description of the horofunction boundary. At non-smooth points, horofunctions do not necessarily correspond to Pansu derivatives, but instead are piecewise defined by Pansu derivatives in each blow-up region.

This theorem can be used to recover results about the horofunction boundaries of normed spaces as in [7, 21].

4. Vertical sequences in the Heisenberg group $\mathbb{H}$

In this section, we work through some details for the case of the Heisenberg group $\mathbb{H}$, and show that vertical sequences induce a topological disk in the boundary for any sub-Finsler metric. Let $d$ be the sub-Finsler metric on $\mathbb{H}$ induced by a norm $\| \cdot \|$ on $\mathbb{R}^2$ with unit disk $Q$. In the Heisenberg group, much of the geometry is encoded in the horizontal plane. Indeed, given a path $\lambda$ for $t \in [0, \ell]$ in $\mathbb{H}$, there exists a unique lift to an admissible path $\tilde{\gamma}$ in $\mathbb{H}$ such that $\tilde{\gamma}$ has height $z_0$ at time zero and $\pi(\tilde{\gamma}) = \gamma$. A well-known result is that geodesics in sub-Finsler metrics are lifts of solutions to the Dido problem with respect to $\| \cdot \|$; that is, geodesics are lifts of arcs which traverse the perimeter of the isoperimetric $I$ for the given norm. This result connects results about geodesics in $\mathbb{H}$ to convex geometry in the plane.

We start with a technical lemma concerning convex geometry. Fix $b > 0$ and an open bounded convex set $Q \subset \mathbb{R}^2$. Dilate $Q$ by $\lambda$ for $\lambda > 1$, and take two points $p, q \in \partial \lambda Q$ so that $|p - q| \leq b$. The line passing through $p$ and $q$ cuts $\lambda Q$ into two parts with volumes $\lambda Q$.

**Lemma 4.1.** Let $Q \subset \mathbb{R}^2$ be an open bounded convex set. Fix $b > 0$ and define for $\lambda > 0$ 

$$\bar{x}(\lambda) := \inf \{ x \in \mathbb{R} : L^1 \{ y : (x, y) \in \lambda Q \} \geq b \},$$

$$Q^*_\lambda := \{ (x, y) \in \lambda Q : x \leq \bar{x}(\lambda) \}.$$

Then 

$$\sup \{ L^2(Q^*_\lambda) : \lambda > 1 \} < \infty,$$

where $L^1$ and $L^2$ denote the 1- and 2-dimensional Lebesgue measures, respectively.

**Proof.** For small $\lambda$ we have $L^2(Q^*_\lambda) \leq \lambda^2 L^2(Q)$. We must show that $L^2(Q^*_\lambda)$ remains bounded for $\lambda$ large. For large $\lambda$, we can assume $\bar{x}(\lambda) \in \mathbb{R}$. Define $V_\lambda$ to measure vertical strips in $\lambda Q$,

$$V_\lambda(x) = L^1 \{ y : (x, y) \in \lambda Q \},$$

and note that $V_\lambda(x) = \lambda V_1(x/\lambda)$. Up to translating $Q$, we can assume $V_1(x) = 0$ for all $x \leq 0$ and $V_1(x) > 0$ for small $x > 0$. Moreover, since $Q$ is convex, $V_1$ is a concave function. If $\lim_{x \to 0^+} V_1(x) > 0$, then $\bar{x}(\lambda) = 0$ and thus $L^2(Q^*_\lambda) = 0$ for $\lambda$ large.

Now assuming that $\lim_{x \to 0^+} V_1(x) = 0$, we have that $\bar{x}(\lambda) > 0$ for all $\lambda$. By concavity, there are $\epsilon, m > 0$ such that $V_1(x) \geq mx$ for all $x \in (0, \epsilon]$. By the definition of $\bar{x}$, if $x < \bar{x}(\lambda)$ then $V_\lambda(x) \leq b$. For $\lambda$ large, $V_\lambda(\lambda \epsilon) = \lambda V_1(\epsilon) \geq \lambda m \epsilon \geq b$, and so $\bar{x}(\lambda) < \epsilon$. It follows that

$$b > V_\lambda(\lambda \bar{x}(\lambda)/2) \geq m \bar{x}(\lambda)/2,$$

that is, $\bar{x}(\lambda) \leq 2b/m$. We conclude that 

$$L^2(C^*_\lambda) = \int_0^{\bar{x}(\lambda)} V_\lambda(x) \, dx \leq \frac{2b^2}{m},$$

for $\lambda$ large enough. 

**Lemma 4.2.** For any sub-Finsler metric $d$ on $\mathbb{H}$ and any $v \in \mathbb{R}^2$, 

$$\lim_{t \to \infty} [d_{c}(v(t)) - d_{c}(0, t)] = 0.$$

Moreover, the convergence is uniform in $v$ on compact sets.

**Proof.** By the triangle inequality, we have 

$$d_{c}(v(t)) - d_{c}(0, t) + d_{c}(v(0)) \geq 0$$

for all $t$. Let $Q^* \subset \mathbb{R}^2$ be the convex set dual to the unit ball $Q$ of the norm $\| \cdot \|$ on $\mathbb{R}^2$. Let $I$ be the rotation by $\frac{\pi}{2}$ of $Q^*$.

Define $a = a(t) = d_{c}(v(t))$, $b = d_{c}(v(0))$ and $h = h(t) = d_{c}(0, t)$. For $t$ large enough, the projection $\gamma_1 : [0, 1] \to \mathbb{R}^2$ of a geodesic from $(0, 0)$ to $(v, t)$ is a portion of the boundary of $M$, for some $\lambda$, with
Let \( \gamma_1(0) = (0,0) \) and \( \gamma_1(1) = v \). Notice that \( a \) is the length of \( \gamma_1 \), that \( b = \|v\| \) is the length of a chord of \( \partial(\lambda I) \) and that \( t \) is the area one of the two parts of \( \lambda I \) separated by the line passing through 0 and \( v \). Let \( s \) be the area of the other part and \( c \) the length of \( \partial(\lambda I) \setminus \gamma_1 \). If \( A \) is the area of \( I \) and \( \ell \) is the length of \( \partial I \), we have \( a + c = \lambda \ell \) and \( t + s = \lambda^2 A \). See Figure 2.

\[ \text{Figure 2. Convex geometry and vertical sequences} \]

The projection \( \gamma_2 \) of a geodesic from \((0,0)\) to \((0,t)\) is the boundary of \( \mu C \), for some \( \mu \) so that \( t = L^2(\mu C) = \mu^2 A \). Then \( h = d_e((0,0)) \) is the length of the boundary of \( \mu C \). Therefore,

\begin{equation}
(13) \quad h = \frac{\mu}{\lambda} (a + c) = \sqrt{\frac{t}{t+s}} (a + c) \geq \sqrt{\frac{t}{t+s}} (a + b).
\end{equation}

By Lemma 4.1, there is \( M > 0 \) such that \( s < M \) for all \( t \) sufficiently large. Thus, by combining (12) and (13), we see that \( h(t) \) converges to \( a(t) + b \), completing the first part of the proof.

For the uniform convergence, if we define \( f_t(v) = d_e((v,t)) - d_e((0,t)) + d_e((v,0)) \), then by the reverse triangle inequality, \( f_t : (\mathbb{R}^2 \times \{0\}, d) \to \mathbb{R} \) is Lipschitz, i.e.,

\[ |f_t(v) - f_t(w)| \leq d((v,t),(w,t)) + d((v,0),(w,0)) = 2d((v,0),(w,0)), \]

and \( f_t(c) = 0 \). Therefore, the pointwise convergence is uniform on compact sets. \( \square \)

Recall from the introduction that Klein–Nicas proved that horofunction boundaries for sub-Riemannian metrics are topological disks. This is no longer true for general sub-Finsler metrics, but the following theorem shows that the corresponding disk is a subset of the boundary in the general case.

**Theorem 4.3.** Let \( d \) be the sub-Finsler distance on \( \mathbb{H} \) generated by norm \( \| \cdot \| \) on the horizontal plane. Let \( \{w_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) be a bounded sequence and \( \{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) with \( |s_n| \to \infty \), and set \( p_n = (w_n, s_n) \in \mathbb{H} \). Then for all \((v,t) \in \mathbb{H}\)

\[ \lim_{n \to \infty} d(p_n,(v,t)) - d(p_n,e) - (\|w_n\| - \|w_n - v\|) = 0. \]

There is, therefore, a topological disk \( \{f(v,t) = \|w\| - \|w - v\| : w \in \mathbb{R}^2\} \) in the horofunction boundary.

**Proof.** It suffices to consider the case when \( s_n \to +\infty \). Notice that

\[
\begin{aligned}
d(p_n,(v,t)) &- d(p_n,e) - (\|w_n\| - \|w_n - v\|) \\
&= d_e((w_n - v, s_n - t - \omega(v,w_n)/2)) - d_e((0, s_n - t - \omega(v,w_n)/2)) + d_e((w_n - v, 0)) \\
&\quad - d_e((w_n, s_n)) + d_e((0, s_n)) - d_e((w_n, 0)) \\
&\quad + d_e((0, s_n - t - \omega(v,w_n)/2)) - d_e((0, s_n)),
\end{aligned}
\]

where, \( \omega \) is the standard symplectic form on \( \mathbb{R}^2 \). Using Lemma 4.2 and the boundedness of \( w_n \),

\[ \lim_{n \to \infty} d_e((w_n - v, s_n - t - \omega(v,w_n)/2)) - d_e((0, s_n - t - \omega(v,w_n)/2)) + d_e((w_n - v, 0)) = 0, \]

and

\[ \lim_{n \to \infty} -d_e((w_n, s_n)) + d_e((0, s_n)) - d_e((w_n, 0)) = 0. \]

Finally,

\[ \lim_{n \to \infty} d_e((0, s_n - t - \omega(v,w_n)/2)) - d_e((0, s_n)) = d_e((0,1)) \lim_{n \to \infty} \left( \sqrt{s_n - t - \omega(v,w_n)/2} - \sqrt{s_n} \right) = 0. \]
For the last statement, fix \( w \in \mathbb{R}^2 \) and set \( p_n = (w, n) \in \mathbb{H} \). Then \( p_n \to f(v, t) = \|w\| - \|w - v\| \) in the horofunction boundary.

**Remark 4.4.** For general homogeneous distances, Lemma 4.2 is not true. As an example, consider the function \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(x) := (-1)^{k(x)} \left( 2|x| - 3^{1+k(x)} \right), \quad \text{where} \quad k(x) := \left\lceil \frac{\log(x)}{\log(3)} \right\rceil,
\]
which is piecewise linear with derivative \( \pm 2 \) and satisfies \(-|x| \leq f(x) \leq |x| \) for all \( x \).

Consider the function \( \phi(v) := f(|v|) \) on the disk in \( \mathbb{R}^2 \). Since \( \phi \) is Lipschitz, then, by [13, Proposition 6.3], there is \( M \) such that \( \phi + M \) is the profile of the unit ball of a homogeneous distance \( d \) in \( \mathbb{H} \). If \( v \in \mathbb{R}^2 \setminus \{0\} \), then there is a sequence \( \{t_n\}_{n \in \mathbb{N}} \) with \( t_n \to +\infty \) such that \( f\left( \frac{|v||M|}{\sqrt{n}} \right) = 0 \), i.e., \( d_c\left( \frac{\sqrt{n}M}{\sqrt{|v|}}, M \right) = 1 = d_c((0, M)) \). Therefore,
\[
d_c((v, t_n)) - d_c((0, t_n)) + d_c((v, 0)) = \sqrt{\frac{t_n}{M}} \left( d_c\left( \frac{\sqrt{n}M}{\sqrt{|v|}}, M \right) - d_c((0, M)) \right) + d_c((v, 0)) = d_c((v, 0)),
\]
for all \( n \in \mathbb{N} \). We conclude that (11) cannot hold for such \( d \).

5. Horofunctions in polygonal sub-Finsler metrics on \( \mathbb{H} \)

Let \( d \) be a polygonal sub-Finsler metric on the Heisenberg group \( \mathbb{H} \). The fundamental lemma identifying horofunctions with Pansu derivatives (Lemma 2.4) applies in this case, but we need to take care in describing all possible blow-ups of the distance function at points on the sphere.

These blow-ups take two forms. Wherever \( d_c \) is strictly Pansu differentiable, the blow-up of \( d_c \) is the Pansu derivative of \( d_c \). As we will explain below, the function \( d_c \) is \( C^\infty \) almost everywhere for polygonal sub-Finsler metrics, and hence \( d_c \) is strictly Pansu differentiable almost everywhere on the unit sphere. We’ll call the exceptional non-smooth parts of the unit sphere the *seams*. On the seams, the blow-up of \( d_c \) is defined piecewise as described in Theorem 3.9.

**Theorem 5.1** (Main theorem). Every horofunction can be realized as a blow-up of the distance function at a point \( p \) on the unit sphere \( \partial B \). For smooth points, these horofunctions are Pansu derivatives. For non-smooth points, the blow-ups have piecewise descriptions as Pansu derivatives.

The cases that make up the parts of the main theorem are found in Theorems 5.2, Proposition 5.5, and Theorem 5.6, which appear in this section.

From the main theorem, we deduce the topology of the boundary in Theorem 5.7.

### 5.1. Geometry of polygonal sub-Finsler metrics

On \( \mathbb{R}^2 \), we denote by \( \langle \cdot, \cdot \rangle \) the standard scalar product, and by \( J \) the “multiplication by \( i \)”, i.e., the anticlockwise rotation by \( \pi/2 \). Notice that \( \omega(\cdot, \cdot) = \langle J\cdot, \cdot \rangle \) is the standard symplectic form.

Let \( Q \) be a centrally-symmetric polygon in \( \mathbb{R}^2 \) with boundary \( L \), and let \( \|\cdot\| = \|\cdot\|_L \) be the norm on \( \mathbb{R}^2 \) with unit metric circle \( L \). Enumerate the vertices \( \{v_j\}_j \) of \( L \) with \( j \) modulo \( 2N \), in an anticlockwise order. Define the \( j \)-th edge to be the vector
\[
e_j := v_{j+1} - v_j.
\]

For each \( j \), let \( \alpha_j \in (\mathbb{R}^2)^* \) be the linear map such that \( \alpha_j(v_j + t e_j) = 1 \) for all \( t \in \mathbb{R} \). Direct computation yields that
\[
\alpha_j = \frac{-\langle Je_j, \cdot \rangle}{\langle Jv_j, v_{j+1} \rangle}.
\]
Let \( \| \cdot \|' \) be the norm on \( (\mathbb{R}^2)^* \simeq \mathbb{R}^2 \) dual to \( \| \cdot \| \), and let \( Q^* \) be its unit ball, that is, the polar dual of \( Q \). Note that \( Q^* \) is the polygon with vertices \( \alpha_j \). Define \( L^* \) to be the boundary of \( Q^* \), i.e., the unit metric circle of \( \| \cdot \|' \).

A result of Busemann [3] tells us that the isoperimetric set \( I \), or isoperimetrix, in \( (\mathbb{R}^2, \| \cdot \|) \) is the set \( Q^* \) rotated by \( \pi/2 \) (or \(-\pi/2\)), that is, \( J^* Q^* \), seen as a subset of \( \mathbb{R}^2 \) via the equivalence \( \mathbb{R}^2 \simeq (\mathbb{R}^2)^* \) given by the scalar product. The isoperimetrix \( I \) is the polygon with vertices \( \{ J^* \alpha_j \} \), and we enumerate the edges of \( I \),

\[
\sigma_j := J^*(\alpha_j - \alpha_{j-1}).
\]

We will think of \( \sigma_j \) as the vector corresponding to the functional above (via the scalar product) and observe that by our choice of indexing, \( \sigma_j \) is a scalar multiple of the norm vertex \( v_j \), as in Figure 3.

For the case of polygonal sub-Finsler metrics on \( \mathbb{H} \), Duchin–Mooney [5] classify geodesics and describe the shape of the unit sphere. Here, we introduce some of their notation and summarize some key results.

First, Duchin–Mooney break geodesics into two categories: beelines and trace paths. Beeline geodesics are lifts of \( L \)-norm geodesics in the plane to admissible paths in \( \mathbb{H} \). Trace path geodesics, on the other hand, are lifts of paths in the plane which trace some portion of the perimeter of \( I \).

Let \( I_1 \) a scaled copy of \( I \) such that \( I_1 \) has perimeter \( 1 \) in the \( L \)-norm. Fix \( t : \mathbb{R} \to \mathbb{R}^2 \) to be a 1-periodic anticlockwise parametrization by arc-length of \( I_1 \), starting at one of the vertices of \( I_1 \). Notice that, for every \( t \in [0, 1] \) and \( T \in (t, t+1) \), the curve

\[
\text{trace}_{t,T}(s) := \frac{t(t + s(T - t)) - t(t)}{T - t}
\]

lifts to a unit-length trace path geodesic in \( (\mathbb{H}, d) \).

As in Duchin–Mooney, we partition \( Q \) into quadrilateral regions which are reached by trace paths which have similar combinatorial data, i.e., which traverse the same edges of \( I \). That is, define \( Q_{ij} \subset \mathbb{R}^2 \) to be the set of all endpoints of unit-length, positively-oriented trace paths in the plane whose parametrizations start by traversing a portion of \( \sigma_i \), traverse all of \( \sigma_{i+1}, \ldots, \sigma_{j-1} \), and end by traversing a portion of \( \sigma_j \). That is,

\[
Q_{ij} := \left\{ \frac{\iota(T) - \iota(t)}{T - t} : \iota(t) \in \sigma_i, \iota(T) \in \sigma_j \right\}.
\]

where \( \mu = \mu(r, s) = r||\sigma_i|| + ||\sigma_{i+1}|| + \ldots + ||\sigma_{j-1}|| + s||\sigma_j|| \) normalizes the length of the path. From Theorem 7 of [5], we know that \( Q_{ii} \) and \( Q_{i,i+1} \) are degenerate, that is, they have empty interior, and that the quadrilateral regions \( Q_{ij} \) are disjoint except along their boundaries and cover all of \( Q \).

Duchin–Mooney also describe the unit sphere of a polygonal sub-Finsler metrics. The spheres are made up of vertical wall panels passing through the the edges of \( Q \) and ceiling (and basement) panels which are the graphs of quadratic functions above (and below) the quadrilateral regions \( Q_{ij} \). See Figure 4.

Denote by \( \text{Panel}_{i+1} \) the vertical wall panel which projects to edge \( e_i \) through vertices \( v_i \) and \( v_{i+1} \). We denote the ceiling panel above \( Q_{ij} \) by \( \text{Panel}_{ij} \), and recall that it is the set of endpoints of lifts of all unit-length, positively-oriented trace paths whose endpoints lie in \( Q_{ij} \).

We will similarly define \( \text{Panel}_{ij} \) to be the basement panel below the quadrilateral region \( Q_{ij} \). We can also consider \( Q_{ij} \) to be the endpoints of negatively-oriented trace paths, i.e., those which trace the perimeter of \( I \) with a clockwise orientation. For the sake of consistency, with negatively-oriented trace paths we relabel the edges of \( I \) so that the edge vector \( \sigma_j \) is still a scalar multiple of \( v_j \). Then the negatively-oriented trace paths with endpoints in \( Q_{ij} \) parametrizations which begin by traversing a
portion of edge $\sigma_j$, traverse all of $\sigma_j, \ldots, \sigma_i + 1$, and end by traversing a portion of $\sigma_i$. See the right-most part of Figure 5 for an example. The basement panel $\text{Panel}_{ij}$ is then the set of endpoints of lifts of unit-length, negatively-oriented trace paths whose endpoints lie in $Q_{ij}$.

Fix a quadrilateral region $Q_{ij}$. Observe that trace paths provide a new coordinate system on $Q_{ij} \subset \mathbb{R}^2$ via $$(r,s) \in [0,1]^2,$$

(14) \[ u(r,s) = \frac{1}{\mu} (r\sigma_i + \sigma_{i+1} + \ldots + \sigma_{j-1} + s\sigma_j), \]

where $\mu = r\|\sigma_i\| + \|\sigma_{i+1}\| + \ldots + \|\sigma_{j-1}\| + s\|\sigma_j\|$ normalizes the length of the path. Define $\phi_{ij}$ to be the quadratic function which gives the height of the unit sphere above a point in $Q_{ij}$. This ceiling function $\phi_{ij}$ can also be expressed in terms of $r$ and $s$. An application of Green’s theorem tells us that the ceiling height is simply the area enclosed by trace path, so by breaking the path into triangles, we can write $\phi_{ij}$ as

(15) \[ \phi_{ij}(r,s) = \frac{1}{2\mu^2} \left( \sum_{i<k<j} \omega(r\sigma_i, \sigma_k) + \omega(r\sigma_i, s\sigma_j) + \sum_{i<k_1<k_2<j} \omega(\sigma_{k_1}, \sigma_{k_2}) + \sum_{i<k<j} \omega(\sigma_k, s\sigma_j) \right), \]

where $\omega$ is the standard symplectic form on the plane.

5.2. Blow-ups of $d_e$ at smooth points. First we consider blow-ups of $d_e$ at smooth points on $\partial B$, such as in the interior of each ceiling, basement, or wall panel making up the unit sphere. Since $d_e$ is smooth in the interior of each of these panels, it is strictly Pansu differentiable, and hence Lemma 2.4 tells us that any blow up of $d_e$ at $p$ is equal to the Pansu derivative.

It turns out that all of the smooth points on the ceiling, basement, and vertical walls of $\partial B$ contribute a circle’s worth of functions to the horoboundary. Indeed, they all have Pansu derivatives which lie in $L^*$.
the boundary of the dual ball $Q^*$. This is analogous to results in the sub-Riemannian case; Klein–Nicas in [11] showed that the smooth points contribute a circle’s worth of functions to the boundary, while the rest of the boundary comes from vertical sequences, analogous to our Theorem 4.3.

While the following theorem preserves this analogy, it is still quite surprising. We know from above that ceiling and basement points are reached by geodesics which are lifts of trace paths. It turns out that the Pansu derivative of $d_c$ on the ceiling or basement depends only on where in the isoperimetric $I$ the trace path ends and is independent of the rest of the shape of the trace path.

**Theorem 5.2** (Ceiling and basement Pansu derivatives). For any ceiling point $p$ in the interior of Panel$_{ij}$, the trace geodesic with coordinates $(r, s)$, the Pansu derivative of $d_c$ exists at $p$, and

$$
\mathbf{D}d_c|_p(q) = \mathbf{D}d_c|_p(v, t) = ((1 - s)\alpha_{j-1} + so_1)(v).
$$

Similarly, if $p$ is a basement point in the interior of Panel$_{ij}$, the trace geodesic with coordinates $(r, s)$, then the Pansu derivative of $d_c$ exists at $p$, and

$$
\mathbf{D}d_c|_p(q) = \mathbf{D}d_c|_p(v, t) = ((1 - s)\alpha_i + s\alpha_{i-1})(v).
$$

**Remark 5.3.** Again we note that the Pansu derivative forgets most of the combinatorial data of the trace path, remembering only where on $I$ the trace path ends (on $\sigma_j$ in the ceiling and on $\sigma_i$ in the basement). Indeed, if a positively-oriented trace path ends by traversing $so_j$, then its endpoint on $I$ is

$$(1 - s)J^*\alpha_{j-1} + sJ^*\alpha_j.$$ The Pansu derivative of $d_c$ at the ceiling endpoint of the lift of this trace path is equal to

$$-J^*((1 - s)J^*\alpha_{j-1} + sJ^*\alpha_j) = (1 - s)\alpha_{j-1} + so_1.$$ 

**Proof.** Given that $p$ is in the interior of Panel$_{ij}$, $d_c$ is smooth at $p$, and hence the Pansu derivative exists. Pansu derivatives are linear and invariant on vertical fibers, so we are looking for coefficients $c = (c_1, c_2)$ such that $\mathbf{D}d_c(p)(v, t) = \langle v, c \rangle$. Let $\gamma_1 : [0, 1] \to \mathbb{H}$ be the unit-speed trace path geodesic from $e$ to $p$. Since $p$ is in the interior of Panel$_{ij}$, the derivative $\gamma_1'(1)$ is defined and is equal to $\frac{\sigma_j}{|\sigma_j|} = v_j$. For sufficiently small $h$, $\gamma_1(1 + h) = p = \gamma_1(v_j)$, and so

$$1 = \lim_{h \to 0} \frac{d_c(\gamma_1(1 + h)) - d_c(\gamma_1(1))}{h} = \lim_{h \to 0} \frac{d_c(p\delta_hv_j) - d_c(p)}{h} = \mathbf{D}d_c|_p(v_j).$$

Next, let $\gamma_2 : (-\epsilon, \epsilon) \to \mathbb{H}$ be a path along the unit sphere $\partial B$ which is horizontal at $p$, with $\gamma_2(0) = p$ and $\gamma_2'(0) = w \in \Delta_p$. Since $\gamma_2$ is on the unit sphere, $d_c(\gamma_2(t)) \equiv 1$. A consequence of the horizontality of $\gamma_2$ at $p$ is the existence of the limit $\lim_{h \to 0} \delta_{\gamma_2(h)}(p^{-1}\gamma_2(h)) = \gamma_2'(0) = w$, and so by the strict Pansu differentiability of $d_c$, we have

$$0 = \lim_{h \to 0} \frac{d_c(\gamma_2(h)) - d_c(\gamma_2(0))}{h} = \mathbf{D}d_c|_p(w).$$

Now seek to find coordinates for $w$. To make use the geometry of the trace path geodesics, we change our coordinate system from the standard $(x, y)$-coordinates, and consider

$$\text{Panel}_{ij} = \{(u(r, s), \phi(r, s)) : r, s \in [0, 1]\}.$$ Then the tangent bundle to the unit sphere has a frame given by $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial \phi}\right)$.

We take the partial derivatives of $u$ and $\phi$ as defined in (14) and (15),

$$\begin{align*}
\partial_u u(r, s) &= \frac{\|\sigma_i\|}{\mu}(v_i - u(r, s)), \\
\partial_u \phi(r, s) &= \frac{\|\sigma_i\|}{\mu}\left(\frac{1}{2}\omega(v_i, u(r, s) - r\sigma_i) - 2\phi(r, s)\right), \\
\partial_s u(r, s) &= \frac{\|\sigma_j\|}{\mu}(v_j - u(r, s)), \\
\partial_s \phi(r, s) &= \frac{\|\sigma_j\|}{\mu}\left(\frac{1}{2}\omega(u(r, s) - s\sigma_j) - 2\phi(r, s)\right),
\end{align*}$$

where again $\mu = r\|\sigma_i\| + \|\sigma_{i+1}\| + \ldots + \|\sigma_{j-1}\| + s\|\sigma_j\|$. If $p = (u, \phi(u))$ has trace coordinates $(r, s)$ in Panel$_{ij}$, then after rescaling and simplifying, we have

$$T_p \partial B = \text{Span}\left\{\frac{v_i - v(r, s)}{\frac{1}{2}\omega(v_i, \sum_{i < k < j} \sigma_k + s\sigma_k) - 2\phi(r, s)}, \frac{v_j - v(r, s)}{\frac{1}{2}\omega(v_j, \sum_{i < k < j} \sigma_k, v_j) - 2\phi(r, s)}\right\}.$$ Meanwhile, the horizontal subspace at $p$ is spanned by $(v_i^{(1)} - u_1)X_p + (v_i^{(2)} - u_2)Y_p$ and $(v_j^{(1)} - u_1)X_p + (v_j^{(2)} - u_2)Y_p$. This gives

$$\Delta_p = \text{Span}\left\{\frac{v_i - u(r, s)}{\frac{1}{2}\omega(u(r, s), v_i)} , \frac{v_j - u(r, s)}{\frac{1}{2}\omega(u(r, s), v_j)}\right\}.$$
These two bases for $T_p \partial B$ and $\Delta_p$ allow us to find $w$ in the intersection. Indeed,

$$w = \left( \begin{array}{c} 2\phi(r,s)(v_j - v_i) + \omega(u,v_i)(v_j - u) \\ \phi(r,s)\omega(u,v_j - v_i) + \frac{1}{2}\omega(u,v_i)\omega(u,v_j) \end{array} \right).$$

Recall that our goal here is to solve for the vector $w$ such that the Pansu derivative of $d_e$ at $p$ evaluated at $w$ is equal to 0. Given that Pansu derivatives are invariant on vertical fibers, if $w = (w_1, w_2, w_3)$, it suffices to set $w_3 = 0$ and solve for the first two coordinates. After rescaling (17) and a significant amount of simplifying, one can show that

$$(w_1, w_2) = \left( -(1 - s)\alpha_j^{(2)} - s\alpha_j^{(2)}, (1 - s)\alpha_j^{(1)} + \alpha_j^{(1)} \right).$$

Now we have a system of two linear equations since $\langle c, v_j \rangle = 1$ and $\langle c, (w_1, w_2) \rangle = 0$. Solving the system provides the result for ceiling points. We reiterate from Remark 5.3 a few observations about this Pansu derivative. Note that if the trace path ends by traversing $\sigma_f$, then the trace path’s terminal endpoint on $I$ is $(1 - s)J^*\alpha_{j-1} + sJ^*\alpha_j$, and the Pansu derivative of $d_i$ at $p$ is equal to $-J^*((1 - s)J^*\alpha_{j-1} + sJ^*\alpha_j)$. To show the result for basement points, we will use an involution of $H$ and what we know about the Pansu derivatives on the ceiling. Let $\iota : H \to H$ be the involution taking $(x, y, z)$ to $(x, -y, -z)$. If $d$ is our polygonal sub-Finsler metric on $H$ induced by $\| \cdot \|_L$, define $d_i$ to be the metric such that $d_i(p, q) = d(\iota(p), \iota(q))$. That is, $d_i$ is the sub-Finsler metric induced by the norm $\| \cdot \|_{L(i)}$. Abusing notation, we also use $\iota$ for the restriction of this involution to the horizontal plane, and so the norm polygon $Q_i$, dual polygon $Q^*_i$, and isoperimetrix $I_i$ associated to $d_i$ are the images of $Q$, $Q^*$, and $I$, respectively, under $\iota$.

We observe that

$$\mathcal{D}d_e|_p(q) = \lim_{t \to 0} \frac{d(e, p\delta_t q) - d(e, p)}{t} = \lim_{t \to 0} \frac{d_i(e, \iota(p)\delta_t \iota(q)) - d_i(e, \iota(p))}{t} = \mathcal{D}d_i|_p(\iota(q)).$$

Let $p$ be a basement point in the interior of $\text{Panel}_{ij}^*$ with trace path coordinates $(r, s)$. Then the $d$-geodesic $\gamma$ between the origin $e$ and $p$ is the lift of a negatively-oriented trace path which traverses $\sigma_j$, traverses all of $\sigma_{j-1}, \ldots, \sigma_{i+1}$, and traverses $\sigma_i$. The endpoint of this trace path on $I$ is $(1 - s)(-J^*\alpha_i) + s(-J^*\alpha_{i-1})$. Note that this differs from the endpoint given for positively-oriented trace paths in (16); this difference reflects the fact that we reorder the names of the edges of $I$ for negatively-oriented trace paths.

The $d_i$-geodesic $\iota(\gamma)$ between $e$ and the ceiling point $\iota(p)$ is the lift of a positively-oriented trace path which ends by traversing $s\iota\sigma_i$. Then the endpoint of this trace path on $I$ is

$$(1 - s)(-J^*\alpha_i) + s(-J^*\alpha_{i-1}) = (1 - s)J^*\iota^*\alpha_i + sJ^*\iota^*\alpha_{i-1}.$$ Finaly, as described in Remark 5.3, we apply $-J^*$ to this endpoint. Hence,

$$\mathcal{D}d_e|_p(v, t) = \mathcal{D}d_i|_p(\iota(v, t)) = -(1 - s)(1 - s)J^*\iota^*\alpha_i + sJ^*\iota^*\alpha_{i-1})(\iota(v)) = ((1 - s)\alpha_i + s\alpha_{i-1})(v).$$

Since the Pansu derivative of $d_e$ on the ceiling depends only on the endpoints of trace paths, there are families of ceiling points spanning different ceiling panels which have the same Pansu derivative, as shown in Figure 6. Indeed, the Pansu derivatives of $d_e$ vary smoothly on the ceiling of $\partial B$ until you arrive at a star-like set, in red in the figure, near the north pole of $\partial B$. The same can be said of the basement, and we could draw a similar figure, where the families would spiral in anticlockwise, instead of clockwise.

**Corollary 5.4.** Except for star-like sets near the north and south poles, every point in the interior of the ceiling or basement of $\partial B$ is intrinsically smooth, and the Pansu derivative of $d_e$ and any of these points lies in the boundary circle $L^*$ of $Q^*$.

**Proposition 5.5 (Wall Pansu derivatives).** If $p$ is in the interior of the side panel $\text{Panel}_{i,i+1}$, then

$$\mathcal{D}d_e|_p(q) = \mathcal{D}d_e|_p(v, t) = \alpha_i(v).$$

**Proof.** Let $p = (u, t')$ be in the interior of the side panel $\text{Panel}_{i,i+1}$, and let $q = (v, t) \in H$. For sufficiently small $\epsilon > 0$, the point $p\delta_\epsilon q$ is inside the dilation cone of $\text{Panel}_{i,i+1}$. In this dilation cone, $\alpha_i(\cdot) = \alpha_i \circ \pi(\cdot)$. Thus, by definition of the Pansu derivative and the linearity of $\alpha_i$,

$$\mathcal{D}d_e|_p(q) = \lim_{\epsilon \to 0} \frac{d_e(p\delta_\epsilon q) - d_e(p)}{\epsilon} = \lim_{\epsilon \to 0} \frac{\alpha_i(u + \epsilon v) - 1}{\epsilon} = \alpha_i(v).$$

5.3. Blow-ups of $d_e$ at non-smooth points. We now consider blow-ups of the function $d_e$ at points on the unit sphere which are not smooth, i.e., along the seams of the sphere. The seams come in four varieties: the north and south poles, the vertices of the unit sphere, the star-like seams near the north and south poles, and the seams between ceiling or basement and wall panels. See Figure 7 for the seams along a hexagonal unit sphere. Each type of seam point intersects different combinations of panel dilation cones and hence provides a different kind of blow-up function, as described below and proved in Appendix A.

**Theorem 5.6** (Seam blow-ups). Let $p$ be a seam point on the unit sphere. Recall that $\{v_i\}_{i=1}^{2N}$ is the set of vertices of the norm ball $Q$, and $\{\alpha_i\}_{i=1}^{2N}$ is the set of linear functionals dual to the edges of $Q$. Suppose $\{p_n\}_n$ and $\{e_n\}_n$ are such that $e_n \to 0$ and $f = BU((\mathbb{H}, d_e), p)_n \to e_n$ exists. Then there exists $w \in \mathbb{R}^2$ or $i \in \{1, \ldots, 2N\}$, $C \in \mathbb{R} \cup \{-\infty, +\infty\}$, $c_1, c_2 \in \mathbb{R}$, and $s \in (0, 1]$ such that $f$ is in one of the following families of functions.

1. **Norm-like functions**
   
   $f(v, t) = \|u\| - \|w - v\|$  
   
   Limits of norm-like functions
   
   $f(v, t) = \begin{cases}  
   \alpha_i(v) + c_1 & \omega(v_i, v) \leq C \\
   \alpha_{i-1}(v) + c_2 & \omega(v_i, v) > C 
   \end{cases}$

2. **Isomorphic to $\partial_1(\mathbb{R}^2, \| \cdot \|_Q)$**

   $f(v, t) = \begin{cases}  
   \alpha_i(v) + c_1 & \omega(v_i, v) \geq C \\
   \alpha_{i-1}(v) + c_2 & \omega(v_i, v) < C 
   \end{cases}$

3. **Ceiling**

   $f(v, t) = \begin{cases}  
   \alpha_{i-1}(v) + c_1 & \omega(v_i, v) \geq C \\
   ((1 - s)\alpha_{i-1} + s\alpha_i)(v) + c_2 & \omega(v_i, v) < C 
   \end{cases}$

4. **Basement**

   $f(v, t) = \begin{cases}  
   \alpha_i(v) + c_1 & \omega(v_i, v) \leq C \\
   ((1 - s)\alpha_i + s\alpha_{i-1})(v) + c_2 & \omega(v_i, v) > C 
   \end{cases}$

5.4. Relation to horofunctions and the topology of the boundary. We are now able to describe the topology of the boundary. Recall that due to the left-invariance and homogeneity of polygonal CC metrics, every horofunction can be realized as the Pansu derivative or blow-up of $d_e(\cdot) = d(e, \cdot)$ at a point on the unit sphere $\partial B$. Therefore, by describing all possible blow-ups of the metric on the sphere, we have described all horofunctions in the boundary of $\mathbb{H}$ for polygonal sub-Finsler metrics.

**Theorem 5.7.** Let $d$ be a polygonal sub-Finsler metric on $\mathbb{H}$. Then the horofunction boundary is homeomorphic to a disk with spheres glued along the boundary.

In particular, if $d$ is induced by the polygonal norm $\| \cdot \|_Q$, then $\partial_h(\mathbb{H}, d)$ is homeomorphic to the polar dual $Q^\ast$ with two spheres glued in a chain along semicircular arcs to each edge of $Q^\ast$, as in Figure 8.

**Corollary 5.8.** Let $d$ be a polygonal sub-Finsler metric on $\mathbb{H}$. Then the set of Busamann functions is homeomorphic to a circle.
Indeed, the set of Busemann functions comes from blow-ups of the vertices of the unit sphere and is isomorphic to the horofunction boundary of $([\mathbb{R}^2, \|\cdot\|_Q]) \cong S^1$.

In Figure 8, we introduce a sense of directionality to the horofunction boundary. Recall that to any sequence $\{q_n\} \subset \mathbb{H}$ converging to a horofunction, we can associate sequences $\{p_n\}_n \subset \partial B$ and $\{\epsilon_n\}_n \subset \mathbb{R}$, where $\delta_n q_n = p_n$. For each horofunction $h \in \partial_h \mathbb{H}$, there exist sequences $\{q_n\}_n \leftrightarrow (\{p_n\}_n, \{\epsilon_n\}_n)$ such that $q_n \to h$ and $p_n \to p \in \partial B$. This assigns directions to horofunctions in the boundary. This correspondence between the boundary and the unit sphere is far from bijective. There exist families of directions, such as each blue vertical wall panel, which collapse to single points in the boundary. On the other hand, there are directions, such as the purple north and south poles, which blow-up to 1- or 2-dimensional families in the boundary. In these cases, which boundary point you converge to will depend on how exactly $q_n$ goes off to infinity. The colors in the figures allow us to see which directions on the sphere converge to which families horofunctions.

Figure 8. The unit sphere coming from the hexagonal norm on the left, and the corresponding sub-Finsler boundary on the right.
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APPENDIX A. BLOW-UPS OF THE DISTANCE FUNCTION AT NON-SMOOTH POINTS ON THE SPHERE

In this appendix, we provide the proof of Theorem 5.6, which describes the families of functions which arise as blow-ups of the metric along the seams of the unit sphere and, thus, as horofunctions of a polygonal sub-Finsler metric on $\mathbb{H}$ induced by the polygonal norm $\| \cdot \| = \| \cdot \|_Q$ on $\mathbb{R}^2$. We recall that there are 4 types of seam points on the sphere: the north and south poles, the vertices of the unit sphere, the star-like seams near the north and south poles, and the seams between ceiling or basement and wall panels.

We consider blow-ups of the function $d_e(\cdot) = d(e, \cdot)$ at seam points $p$ on the unit sphere $\partial B$. That is for a neighborhood $\Omega$ of $p$, we analyze the Kuratowski limit

$$\text{BU}(\Omega, d_e), \{p_n\}_n, \{\epsilon_n\}_n := \text{K-lim}_{n \to \infty} \left( \delta_{1/\epsilon_n}(p_n^{-1}\Omega), \frac{d_e(p_n \delta_{\epsilon_n}) - d_e(p_n)}{\epsilon_n} \right).$$

We will move between the three coordinate systems on $\mathbb{R}^2$ coming from the norm ball, the dual ball, and the isoperimetric, as described above in Section 5.1.

A.1. Blow-ups at north and south poles. Recall that Theorem 4.3 describes horofunctions, and hence blow-up functions, that come from vertical sequences $q_n = (w_n, s_n)$, where $\{w_n\}_n \subset \mathbb{R}^2$ is bounded and $|s_n| \to \infty$. When we dilate these sequences back to the unit sphere, getting a sequence $\{p_n\}_n \subset \partial B$ and a sequence of dilates $\{\epsilon_n\}_n$, the boundedness of $\{w_n\}_n$ makes it clear that if $p_n$ converges, it converges to either the north or south pole of $\partial B$. In particular, Theorem 4.3 implies the following.

**Proposition A.1.** Let $p$ be the north or south pole of the unit sphere $\partial B$. Then for the constant sequence $\{p\}_n$ and any sequence of positive numbers $\{\epsilon_n\}_n$ such that $\epsilon_n \to 0$, the blow-up $f = \text{BU}(\Omega, d_e), \{p\}_n, \{\epsilon_n\}_n$ of $d_e$ at $p$ is

$$f(q) = f(v, t) = -\|v\|.$$

Define $C_{i,i+1}$ to be the convex cone in $\mathbb{R}^2$ positively spanned by the vertices $v_i$ and $v_{i+1}$. That is, $C_{i,i+1} := \{\lambda v_i + \lambda_{i+1} v_{i+1} : \lambda, \lambda_{i+1} \geq 0\}$. Then $\pi^{-1}(C_{i,i+1})$ is a wedge-like piece of $\mathbb{H}$, and

$$-\|v\| = -\alpha_i(v), \quad q \in \pi^{-1}(C_{i,i+1}).$$

Note that each wedge $\pi^{-1}(C_{i,i+1})$ is defined by two linear inequalities. Thus we let the sequence $\{p_n\}_n$ vary in the application of Proposition 3.7 and Theorem 3.9, we get a two-parameter family of blow-up functions, resulting in a closed topological disk in the horofunction boundary.

A.2. Blow-ups at star-like seams near poles. Recall that in [5], Duchin–Mooney break up the metric-inducing polynomial into quadrilateral regions defined by the shape of trace path geodesics. For each $i$, the quadrilateral region $Q_{i,i}$, i.e., the set of points reached by starting and ending on the same edge while traversing all sides of the isoperimetric, is degenerate. Indeed, it is a line segment from the origin pointing in the direction of $-v_i$. We define the collection of these line segments, minus the origin, to be the star of $Q$. See Figure 7 for reference.

Observe that the innermost quadrilateral regions which intersect the star of $Q$ are of the form $Q_{i,i-1}$. Indeed, $Q_{i,i-1}$ is contained in the cone $-C_{i-1,i}$, and in trace path coordinates if you fix either $r$ or $s$ to be equal to 1 and let the other vary, you find yourself on the boundary of this cone, arriving at the origin when $r = s = 1$.

Let $p$ be a point in the degenerate $Panel_{s}^{+}$ above the segment of star($Q$) which points from the origin to vertex $-v_i$. Trace geodesics ending at $p$ are in general not unique, and can be parametrized as the lifts of $\frac{1}{m}(\tau_1 \sigma_1 + \sigma_1 + \cdots + \sigma_1 + \tau_2 \sigma_2)$, where $\mu$ normalizes the path to be unit length, and $0 \leq \tau_1 + \tau_2 \leq 1$. Note that if $\tau_1 + \tau_2 = 1$, then $p$ is the north pole, which we have already examined.

Assuming that $0 < \tau = \tau_1 + \tau_2 < 1$, the point $p$ is also in the intersection of two nondegenerate panels. The point $p$ lies in $P_1 := Panel_{s}^{+}$ with $(r,s)$-coordinates $(\tau,1)$, and in $P_2 := Panel_{r}^{+}$ with $(r,s)$-coordinates $(1, \tau)$. Let $\Omega_1$ and $\Omega_2$ be the dilation cones of $P_1$ and $P_2$, respectively. Note that near $p = (u,\phi(u))$, the cone $\Omega_1$ is of the form $\{q = (v,t) : \omega(-v_i, u + v) \leq 0\} = \{(v,t) : \omega(v, v) \geq 0\}$, and similarly $\Omega_2 = \{(v,t) : \omega(v, v) \leq 0\}$. So near $p$, the boundary between $\Omega_1$ and $\Omega_2$ is a vertical plane in.
the direction of $v_i$. This vertical boundary is preserved by left-translation by $p^{-1}$ and by dilation. Thus for any sequence $\{\epsilon_n\}_n$, $\epsilon_n \to 0$, the blow-ups of the sets $\Omega_1$ and $\Omega_2$ are

$$\text{BU}(\Omega_1, \{p\}_n, \{\epsilon_n\}_n) = \{(v, t) : \omega(v_i, v) \geq 0\} \quad \text{and} \quad \text{BU}(\Omega_2, \{p\}_n, \{\epsilon_n\}_n) = \{(v, t) : \omega(v_i, v) \leq 0\}.$$  

Finally, we use the $(r, s)$-coordinates of $p$ in panels $P_1$ and $P_2$ and apply Theorems 5.2 and 3.9 to compute the blow-up of $d_e$ at $p$. This leads to the following proposition.

**Proposition A.2.** Let $p$ be a ceiling point above the star of $Q$ in the degenerate panel $\text{Panel}_{i+1}^n$ such that the unit trace geodesic to $p$ traverses a proportion of $\sigma_i$ given by $s \in (0, 1)$. For the constant sequence $\sigma_n = p$ and for any $\epsilon_n \to 0$, the blow-up $f = \text{BU}(\{\Omega, d_e\}, \{\epsilon\}_n)$ is

$$f(q) = f(v, t) = \begin{cases} \alpha_{i-1}(v) & \omega(v_i, v) \geq 0 \\ \frac{(1-s)\alpha_i + s\alpha_i(v)}{(1-s)\alpha_i(v)} & \omega(v_i, v) < 0 \end{cases}.$$  

A similar analysis of points below star line segments in the basement of the unit sphere yields the following proposition.

**Proposition A.3.** Let $p$ be a basement point below the star of $Q$ in the degenerate panel $\text{Panel}_{i+1}^n$ such that the unit trace geodesic to $p$ traverses a proportion of $\sigma_i$ given by $s \in (0, 1)$. For the constant sequence $\sigma_n = p$ and for any $\epsilon_n \to 0$, the blow-up $f = \text{BU}(\{\Omega, d_e\}, \{\epsilon\}_n)$ is

$$f(q) = f(v, t) = \begin{cases} \alpha_i(v) & \omega(v_i, v) \leq 0 \\ \frac{(1-s)\alpha_i(v) + s\alpha_i(v)}{(1-s)\alpha_i(v)} & \omega(v_i, v) > 0 \end{cases}.$$  

If $t = t_1 + t_2 = 0$, then $p$ lies at the end of the star line segment and in the intersection of a third panel $\text{Panel}_{i+1, i+1}$. Checking the $(r, s)$ coordinates of $p$ in the three panels, one sees that the three pieces of the blow-up function are all equal to $\alpha_{i-1}$. Thus the Pansu derivative at the point exists and $d_e$ intrinsically smooth at $p$.

### A.3. Blow-ups along wall seams.

Here consider the vertical side panel $\text{Panel}_{i+1, i+1}$, and let $\Omega = \cup_{j} \partial \sigma_i(\text{Panel}_{i+1, i+1})$ be the dilation cone of this side panel. The boundary of $\Omega$ is made up of a top and a bottom piece, each of which is smooth, which we denote by $\partial \Omega^+$ and $\partial \Omega^-$, respectively. If $C_{i+1}$ is the convex cone in $\mathbb{R}^2$ positively spanned by $v_i$ and $v_{i+1}$, then there exists a function $\hat{F} : C_{i+1, i+1} \to \mathbb{R}$ which gives the height of $\partial \Omega^+$ above a point in $C_{i+1, i+1}$. Indeed, $\partial \Omega$ is parametrized by

$$\partial \Omega^\pm = \{e((1-t)v_i + tv_{i+1}) : \pm \frac{e^2}{2}\omega(v_i, v_{i+1})(t - t^2) : e \in (0, \infty), t \in [0, 1]\}.$$  

Using the parametrization, we solve for the height function,

$$\hat{F}(v) = \frac{\omega(v_i, v)\omega(v, v_{i+1})}{2\omega(v_i, v_{i+1}).}$$  

Thus, $\Omega = \{F \leq 0\}$, where $F(v, t) = |t| - \hat{F}(v)$, which is smooth except where $\partial \Omega^+$ and $\partial \Omega^-$ intersect on the boundary of the cone $C_{i+1, i+1}$ in the $xy$-plane.

**Lemma A.4.** Let $\Omega$ be the dilation cone the wall panel $\text{Panel}_{i+1, i+1}$ between vertices $v_i$ and $v_{i+1}$ on $\partial B$. If $p$ is in the interior of $\partial \Omega^+$, and $\epsilon_n \to 0$ then

$$\text{BU}(\Omega, \{p\}_n, \{\epsilon_n\}_n) = \{q = (v, t) : \omega(v_i, v) \leq 0\}.$$  

If $p$ is in the interior of $\partial \Omega^-$, then

$$\text{BU}(\Omega, \{p\}_n, \{\epsilon_n\}_n) = \{q = (v, t) : \omega(v, v_i) \geq 0\}.$$  

**Proof.** First, suppose $p = (u, \phi(u))$ is in the interior of $\partial \Omega^+$. Then $u \in \text{int}(C_{i+1})$ and locally, $\Omega$ looks like $\{F(v, t) = t - \hat{F}(v) \leq 0\}$. Then Proposition 3.6 tells us that

$$\text{BU}(\Omega, \{p\}_n, \{\epsilon_n\}_n) = \{\text{DF}_{|p} \leq 0\} = \{v, t) : \frac{\omega(u, v_i)}{\omega(v, v_{i+1})} \omega(v, v_{i+1}) \leq 0\} = \{(v, t) : \omega(v_i, v) \leq 0\},$$  

where the last equality comes from the fact that $u \in \text{int}(C_{i+1})$ implies that $\frac{\omega(u, v_i)}{\omega(v, v_{i+1})}$ is negative.

Next, suppose $p = (u, -\phi(u))$ is in the interior of $\partial \Omega^-$. Then locally, $\Omega$ looks like $\{F(v, t) = -t - \hat{F}(v) \leq 0\}$, and hence

$$\text{BU}(\Omega, \{p\}_n, \{\epsilon_n\}_n) = \{v, t) : \frac{\omega(u, v_{i+1})}{\omega(v, v_i)} \omega(v, v_i) \leq 0\} = \{(v, t) : \omega(v_i, v) \geq 0\},$$  

since $\omega(u, v_{i+1}) > 0$ and $\omega(v, v_{i+1}) > 0$. \hfill $\square$
Suppose \( p \) is a point on \( \partial B \) which lies in the interior of the seam between the vertical side \( \text{Panel}_{i,i+1} \) and the ceiling. Then \( p \) belongs to one of two ceiling panels. Either \( p \) has coordinates \((0, s)\), \( s \in (0, 1] \) in \( \text{Panel}^+_{i-1,i+1} \), or \( p \) has coordinates \((r, 0)\), \( r \in (0, 1] \) in \( \text{Panel}^+_{i,i+2} \). Either way, the trace geodesic ending at \( p \) ends by traversing some nonzero portion of edge \( \sigma_{i+1} \). We now combine Theorems 5.2 and 3.9 to compute the blow-up of \( d_e \) at \( p \).

**Proposition A.5.** Let \( p \) be a point on \( \partial B \) which lies on the seam between the vertical side \( \text{Panel}_{i,i+1} \) and the ceiling such that the unit trace geodesic to \( p \) ends by traversing a portion of \( \sigma_{i+1} \) given by \( s \in (0, 1] \). Let \( p_n = p \) for all \( n \) and \( \epsilon_n \to 0 \). Then the blow-up function \( f = \text{BU}(d_e, \{p\}_{n}, \{\epsilon_n\}_{n}) \) is

\[
f(q) = f(v, t) = \begin{cases} \alpha_i & \omega(v_{i+1}, v) \leq 0 \\ (1 - s)\alpha_i + s\alpha_{i+1} & \omega(v_{i+1}, v) > 0 \end{cases}.
\]

Similarly, we now let \( p \) be a point on \( \partial B \) which lies in the interior of the seam between the vertical side \( \text{Panel}_{i,i+1} \) and the basement. Then either \( p \) has coordinates \((0, s)\), \( s \in (0, 1] \) in \( \text{Panel}^+_{i,i+2} \), or \( p \) has coordinates \((r, 0)\), \( r \in (0, 1] \) in \( \text{Panel}^-_{i-1,i+1} \). Either way, the trace geodesic ending at \( p \) ends by traversing some nonzero portion of edge \( \sigma_i \). Thus we have the following proposition.

**Proposition A.6.** Let \( p \) be a point on \( \partial B \) which lies on the seam between the vertical side \( \text{Panel}_{i,i+1} \) and the basement such that the unit trace geodesic to \( p \) ends by traversing a portion of \( \sigma_i \) given by \( s \in (0, 1] \). Let \( p_n = p \) for all \( n \) and \( \epsilon_n \to 0 \). Then the blow-up function \( f = \text{BU}(d_e, \{p\}_{n}, \{\epsilon_n\}_{n}) \) is

\[
f(q) = f(v, t) = \begin{cases} \alpha_i & \omega(v_i, v) \geq 0 \\ (1 - s)\alpha_i + s\alpha_{i-1} & \omega(v_i, v) < 0 \end{cases}.
\]

**A.4. Blow-ups at vertices.** Let \( p \) be the vertex \( v_i \) in the unit sphere. Then \( p \) is in the intersection of 4 panels (wall panels \( \text{Panel}_{i-1,i} \) and \( \text{Panel}_{i,i+1} \); ceiling panel \( \text{Panel}^+_{i-1,i+1} \); and basement panel \( \text{Panel}^-_{i-1,i+1} \)) and hence 4 dilation cones. Denote the dilation cones of the panels listed above by \( W_{i-1,i} \), \( W_{i,i+1} \), \( \Omega_C \), and \( \Omega_B \), respectively. Let \( B_i \) be the boundary between \( W_{i-1,i} \) and \( \Omega_C \), \( B_2 \) the boundary between \( \Omega_C \) and \( W_{i,i+1} \), \( B_3 \) the boundary between \( W_{i,i+1} \) and \( \Omega_B \), and \( B_4 \) the boundary between \( \Omega_B \) and \( W_{i-1,i} \). Let \( F_i \) \( i = 1, \ldots, 4 \) be the smooth functions so that locally at \( p \), \( B_i = \{ F_i = 0 \} \). The proof of Lemma A.4 shows us that \( B_2 \) and \( B_4 \) are horizontal at \( p = v_i \). Then checking the four pieces of the blow-up function coming wall, ceiling, and basement Pansu derivatives and applying Theorem 3.9, we get the following proposition.

**Proposition A.7.** Let \( p \) be the vertex \( v_i \) on the unit sphere. Let \( p_n = p \) for all \( n \) and \( \epsilon_n \to 0 \). Then the blow-up function \( f = \text{BU}(d_e, \{p\}_n, \{\epsilon_n\}_n) \) is

\[
f(q) = f(v, t) = \begin{cases} \alpha_i & \omega(v_i, v) \geq 0 \\ \alpha_{i-1} & \omega(v_i, v) < 0 \end{cases}.
\]

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