DOUBLING THE EQUATORIAL FOR THE PRESCRIBED SCALAR CURVATURE PROBLEM ON $S^N$

LIPENG DUAN, MONICA MUSSO, AND SUTING WEI

Abstract. We consider the prescribed scalar curvature problem on $S^N$

$$\Delta_{S^N} v - \frac{N(N-2)}{2} v + \tilde{K}(y)v^\frac{N+2}{N-2} = 0 \quad \text{on } S^N, \quad v > 0 \quad \text{in } S^N,$$

under the assumptions that the scalar curvature $\tilde{K}$ is rotationally symmetric, and has a positive local maximum point between the poles. We prove the existence of infinitely many non-radial positive solutions, whose energy can be made arbitrarily large. These solutions are invariant under some non-trivial sub-group of $O(3)$ obtained doubling the equatorial. We use the finite dimensional Lyapunov-Schmidt reduction method.

Keyword: Prescribed scalar curvature problem, Finite dimensional Lyapunov-Schmidt reduction

AMS Subject Classification: 35A01, 35B09, 35B38.

1. INTRODUCTION

Given the $N$-th sphere $(S^N, g)$ equipped with the standard metric $g$ and a fixed smooth function $\tilde{K}$, the prescribed scalar curvature problem on $S^N$ consists in understanding whether it is possible to find another metric $\tilde{g}$ in the conformal class of $g$, such that the scalar curvature of $\tilde{g}$ is $\tilde{K}$. For a conformal change of the metric

$$\tilde{g} = v^{\frac{4}{N-2}} g,$$

for some positive function $v : S^N \to \mathbb{R}$, the scalar curvature with respect to $\tilde{g}$ is given by

$$v^\frac{N+2}{N-2} \left( \Delta_{S^N} v - \frac{N(N-2)}{2} v \right),$$

where $\Delta_{S^N}$ is the Laplace-Beltrami operator on $S^N$. Thus the prescribed scalar curvature problem on $S^N$ can be addressed by studying the solvability of the problem

$$\Delta_{S^N} v - \frac{N(N-2)}{2} v + \tilde{K}(y)v^\frac{N+2}{N-2} = 0 \quad \text{on } S^N, \quad v > 0 \quad \text{in } S^N. \quad (1.1)$$

Testing the equation against $v$ and integrating on $S^N$, we get that a necessary condition for the solvability of this problem is that $\tilde{K}(y)$ must be positive somewhere. There are other obstructions for the existence of solutions in the energy space, which are said to be of topological type. For instance, a solution $v$ must satisfy the following Kazdan-Warner type condition (see [15]):

$$\int_{S^N} \nabla \tilde{K}(y) \cdot \nabla v v^{\frac{N}{N-2}} d\sigma = 0. \quad (1.2)$$

This condition is a direct consequence of Theorem 5.17 in [16]: Kazdan and Warner proved that given a positive solution $v$ to

$$\Delta_{S^N} v - \frac{N(N-2)}{2} v + H(y)v^a = 0$$

Date: June 7, 2022.
on the standard sphere $\mathbb{S}^N$, $N \geq 3$, then
\[
\int_{\mathbb{S}^N} v^{a+1} \nabla H \cdot \nabla F = \frac{1}{2} (N-2) \left( \frac{N+2}{N-2} - a \right) \int_{\mathbb{S}^N} v^{a+1} H F
\]
for any spherical harmonics $F$ of degree 1. Taking $a = \frac{N+2}{N-2}$, $H = \tilde{K}$ and $F = y$, we obtain (1.2).

The problem of determining which $\tilde{K}(y)$ admits a solution has been the object of several studies in the past years. We refer the readers to [2] [3] [4] [6] [7] [8] [10] [11] [13] [15] [30], and the references therein.

The prescribed scalar curvature problem on $\mathbb{S}^N$ (1.1) can be transformed into a semi-linear elliptic equation in the flat space $\mathbb{R}^N$ via the stereographic projection. Letting $D^{1,2}(\mathbb{R}^N)$ denote the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\int_{\mathbb{R}^N} |\nabla v|^2$, solving (1.1) in the energy space is equivalent to solving
\[
\Delta v + K(y)v^{2^*-1} = 0, \quad v > 0, \quad \text{in } \mathbb{R}^N, \quad v \in D^{1,2}(\mathbb{R}^N).
\]

Because of its geometry roots, it is of interest to establish under what kind of assumptions on $K$ problem (1.3) admits one or multiple solutions.

For $N = 3$, Y.Y. Li [17] showed problem (1.3) has infinitely many solutions provided that $K(y)$ is bounded below, and periodic in one of its variables, and the set $\{x \mid K(x) = \max_{y \in \mathbb{R}^3} K(y)\}$ is not empty and contains at least one bounded connected component.

If $K$ has the form $K(y) = 1 + \epsilon h(y)$, namely it is a perturbation of the constant 1, D. Cao, E. Noussair and S. Yan [5] proved the existence of multiple solutions.

If $K(y)$ has a sequence of strictly local maximum points moving to infinity, S. Yan [32] constructed infinitely many solutions.

In [31], J. Wei and S. Yan showed that problem (1.3) has infinitely many solutions provided $K$ is radially symmetric $K(y) = K(r)$, $r = |y|$, and has a local maximum around a given $r_0 > 0$. More precisely, they ask that there are $r_0$, $c_0 > 0$ and $m \in [2, N-2)$ such that
\[
K(s) = K(r_0) - c_0 |s - r_0|^m + O(|s - r_0|^{m+\sigma}), \quad s \in (r_0 - \delta, r_0 + \delta),
\]
for some $\sigma, \delta$ small positive constants. These solutions are obtained by gluing together a large number of Aubin-Talenti bubbles (see [29])
\[
U_{x, \Lambda}(y) = c_N \left( \frac{\Lambda}{1 + \Lambda^2 |x - x_0|^2} \right)^{\frac{N+2}{2}}, \quad c_N = [N(N-2)]^{\frac{N-2}{4}}.
\]
For any $x \in \mathbb{R}^N$ and $\Lambda \in \mathbb{R}^+$, these functions solve
\[
\Delta u + u^{N+2 \over 2} = 0 \quad \text{in } \mathbb{R}^N.
\]
In fact they are the only positive solutions to (1.4). At main order their solution looks like
\[
\tilde{u}_k \sim \sum_{j=1}^k U_{x_j, \Lambda},
\]
where $\Lambda$ is a positive constant and the points $x_j$ are distributed along the vertices of a regular polygon of $k$ edges in the $(y_1, y_2)$-plane, with $|x_j| \to r_0$ as $k \to \infty:
\[
x_j = \left( \tilde{r} \cos \frac{2(j-1)\pi}{k}, \tilde{r} \sin \frac{2(j-1)\pi}{k}, 0, \cdots, 0 \right), \quad j = 1, \cdots, k,
\]
with $\tilde{r} \to r_0$ as $k \to \infty$.

Under the weaker symmetry conditions for $K(y) = K(|y'|, y'')$ with $y = (|y'|, y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, S. Peng, C. Wang and S. Wei [27] constructed infinitely many bubbling solutions, which concentrate at the saddle points of the potential $K(y)$. Their method uses local Pohozaev identities adapted to problem (1.3). Y. Guo and B. Li [11] generalized these results for problems (1.3) with polyharmonic operators [31]. For the fractional scalar field equation, we refer to [13] [23].
DOUBLING THE EQUATORIAL FOR THE PRESCRIBED SCALAR CURVATURE PROBLEM ON $S^N$

The study of other aspects of problem (1.3), such as radial symmetry of their solutions, uniqueness of solutions, Liouville type theorem, a priori estimates, and bubbling analysis, have been the object of investigation of several researchers. We refer the readers to the papers [11, 9, 18, 20, 21, 22, 25, 26, 32] and the references therein.

Recently, Y. Guo, M. Musso, S. Peng and S. Yan [12] investigated the spectral property of the linearized problem associated to (1.3) around the solution $\tilde{u}_k$ found in [31]. They proved a non-degeneracy result for such operator using a refined version of local Pohozaev identities. As an application of this non-degeneracy result, they built new type of solutions by gluing another large number of bubbles, whose centers lie near the circle $|y| = r_0$ in the $(y_3, y_4)$-plane.

All these results concern solutions made by gluing together Aubin-Talenti bubbles with centres distributed along the vertices of one or more planar polygons, thus of two-dimensional nature.

The purpose of this paper is to present a different type of solution to (1.3) with a more complex concentration structure, which cannot be reduced to a two-dimensional one.

To present our result, we made the following assumptions on $K$: it is radially symmetric and $H$: There are $r_0$ and $c_0 > 0$ such that

$$K(s) = K(r_0) - c_0|s - r_0|^m + O(|s - r_0|^{m+\sigma}), \quad s \in (r_0 - \delta, r_0 + \delta),$$

where $\sigma, \delta > 0$ are small constants,

$$m \in \begin{cases} [2, N-2) & \text{if } N = 5 \text{ or } 6, \\ \left(\frac{N-2}{2}, N-2 \right) & \text{if } N \geq 7. \end{cases}$$

There is a slight difference between our assumptions on $K(s)$ and the ones in [31]. We will comment on this issue later in the introduction.

Without loss of generality, we assume $r_0 = 1, K(1) = 1$. For any integer $k$, we define

$$r = k^{\frac{N-2}{2}}.$$

We scale $u(y) = r^{-\frac{N-2}{2}}u\left(\frac{|y|}{r}\right)$, so that problem (1.3) becomes

$$-\Delta u = K\left(\frac{|y|}{r}\right)u^{2^*-1}, \quad u > 0, \quad \text{in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N),$$

where

$$2^* = \frac{2N}{N-2}.$$

Consider the points

$$x_j = r\left(\sqrt{1 - h^2 \cos \frac{2(j-1)\pi}{k}}, \sqrt{1 - h^2 \sin \frac{2(j-1)\pi}{k}}, h, 0\right), \quad j = 1, \cdots, k,$$

$$z_j = r\left(\sqrt{1 - h^2 \cos \frac{2(j-1)\pi}{k}}, \sqrt{1 - h^2 \sin \frac{2(j-1)\pi}{k}}, -h, 0\right), \quad j = 1, \cdots, k,$$

where $0$ is the zero vector in $\mathbb{R}^{N-3}$ and $h, r$ are positive parameters.

We define

$$W_{r,h,\Lambda}(y) = \sum_{j=1}^k U_{x_j,\Lambda}(y) + \sum_{j=1}^k U_{z_j,\Lambda}(y), \quad y \in \mathbb{R}^N.$$  

(1.7)

We will produce a family of solutions to (1.6) which are small perturbations of $W_{r,h,\Lambda}$, for any integer $k$ sufficiently large. The Aubin-Talenti bubbles are now centred at points lying on the top and the bottom circles of a cylinder and this configuration is now invariant under a non-trivial sub-group of $O(3)$ rather than $O(2)$.
Throughout of the present paper, we assume $N \geq 5$ and $(r, h, \Lambda) \in \mathcal{S}_k$, where

$$\mathcal{S}_k = \left\{(r, h, \Lambda) \mid r \in \left[k^{N-2-m} - \hat{\sigma}, k^{N-2-m} + \hat{\sigma}\right], \quad \Lambda \in \left[\Lambda_0 - \hat{\sigma}, \Lambda_0 + \hat{\sigma}\right], \right\},$$

with $\Lambda_0, B'$ are the constants in (3.7), (3.10) and $\hat{\sigma}$ is a small fixed number, independent of $k$. For convenience, we denote

$$r = k^{N-2-m}, \quad h = \frac{B'}{k^{N-1}}.$$

Since $h \to 0$ as $k \to \infty$, the two circles where the points $x_j$ and $x'_{j'}$ are distributed become closer to each other as $k$ increases.

In this paper, we prove that for any $k$ large enough problem (1.6) has a family of solutions $u_k$ with the approximate form

$$u_k(y) \sim W_{r,h,\Lambda}. \tag{1.9}$$

These solutions have polygonal symmetry in the $(y_1, y_2)$-plane, are even in the $y_3$ direction and radially symmetric in the variables $y_4, \ldots, y_N$. Our solutions are thus different from the ones obtained in [31] and have strong analogies with the doubling construction of the entire finite energy sign-changing solutions for the Yamabe equation in [24].

Define the symmetric Sobolev space:

$$H_s = \left\{ u : u \in H^1(\mathbb{R}^N), \ u \text{ is even in } y_\ell, \ell = 2, 3, 4, \ldots, N, \right\},$$

$$\begin{align*}
&u\left(\sqrt{y_1^2 + y_2^2 \cos \theta}, \sqrt{y_1^2 + y_2^2 \sin \theta}, y_3, y''\right) \\
&= u\left(\sqrt{y_1^2 + y_2^2 \cos \left(\theta + \frac{2j\pi}{k}\right)}, \sqrt{y_1^2 + y_2^2 \sin \left(\theta + \frac{2j\pi}{k}\right)}, y_3, y''\right)\right\},
\end{align*}$$

where $\theta = \arctan \frac{y_2}{y_1}$. Let us define the following norms which capture the decay property of functions

$$\|u\|_* = \sup_{y \in \mathbb{R}^N} \left(\sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - x_j|)^{N-2+m+\tau}} + \frac{1}{(1 + |y - x'_{j'}|)^{N-2+m+\tau}} \right]^{-1} |u(y)| \right), \tag{1.10}$$

and

$$\|f\|_* = \sup_{y \in \mathbb{R}^N} \left(\sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - x_j|)^{N-2+m+\tau}} + \frac{1}{(1 + |y - x'_{j'}|)^{N-2+m+\tau}} \right]^{-1} |f(y)| \right), \tag{1.11}$$

where

$$\tau = \left(\frac{N-2-m}{N-2}, \frac{N-2-m}{N-2} + \epsilon_1\right), \tag{1.12}$$

for some $\epsilon_1$ small. The main results of this paper are the following:
Theorem 1.1. Suppose that $K(|y|)$ satisfies $H$ and $N \geq 5$. Then there exists a large integer $k_0$, such that for each integer $k \geq k_0$, problem (1.6) has a solution $u_k$ of the form

$$u_k(y) = W_{r_k,h_k,A_k}(y) + \phi_k(y),$$

(1.13)

where $\phi_k \in H_s$, $(r_k,s_k,A_k) \in \mathcal{S}_k$ and $\phi_k$ satisfies

$$\|\phi_k\|_s = o_k(1), \text{ as } k \to \infty.$$  

Equivalently, problem (1.3) has solution $v_k(y)$ of the form

$$v_k(y) = r^{\frac{N-k}{2}} \left[ W_{r_k,h_k,\Lambda_k}(ry) + \phi_k(ry) \right].$$

Let us outline the main ideas in the proof of Theorem 1.1. The first step in our argument is to find $\phi$ so that $u = W_{r,h,\Lambda} + \phi$ solves the auxiliary problem

$$\left\{ \begin{array}{l}
-\Delta (W_{r,h,\Lambda} + \phi) = K\left(\frac{|y|}{r}\right) (W_{r,h,\Lambda} + \phi)^{2^*-1} \\
+ \sum_{\ell=1}^{3} \sum_{j=1}^{k} c_{\ell} \left( U_{r,h,\Lambda}^{2^*-2} \overline{\mu_{\ell j}} + U_{r,h,\Lambda}^{2^*-2} \overline{v_{\ell j}} \right) \text{ in } \mathbb{R}^N, \\
\phi \in \mathcal{E},
\end{array} \right.$$  

(1.14)

for some scalars $c_{\ell}$ for $\ell = 1, 2, 3$. In (1.14), the functions $\overline{\mu_{\ell j}}$ and $\overline{v_{\ell j}}$ are given by

$$\overline{\mu_{1j}} = \frac{\partial U_{r,h,\Lambda}}{\partial r}, \quad \overline{\mu_{2j}} = \frac{\partial U_{r,h,\Lambda}}{\partial h}, \quad \overline{\mu_{3j}} = \frac{\partial U_{r,h,\Lambda}}{\partial \Lambda},$$

$$\overline{v_{1j}} = \frac{\partial U_{r,h,\Lambda}}{\partial r}, \quad \overline{v_{2j}} = \frac{\partial U_{r,h,\Lambda}}{\partial h}, \quad \overline{v_{3j}} = \frac{\partial U_{r,h,\Lambda}}{\partial \Lambda},$$

for $j = 1, \ldots, k$. Moreover the function $\phi$ belongs to the set $\mathcal{E}$ given by

$$\mathcal{E} = \left\{ v : v \in H_s, \int_{\mathbb{R}^N} U_{r,h,\Lambda}^{2^*-2} \overline{\mu_{\ell j}} v \, dy = 0 \quad \text{and} \quad \int_{\mathbb{R}^N} U_{r,h,\Lambda}^{2^*-2} \overline{v_{\ell j}} v \, dy = 0, \quad j = 1, \ldots, k, \quad \ell = 1, 2, 3 \right\}.\quad (1.15)$$

From the linear theory developed in Section 2 (1.14) can be solved by means of the contraction mapping theorem. More precisely, we prove that, for any $(r,h,\Lambda) \in \mathcal{S}_k$ there exist $\phi = \phi_{r,h,\Lambda} \in \mathcal{E}$ and constants $c_{\ell}, \ell = 1, 2, 3$ which solve the auxiliary problem (1.14).

The results in Theorem 1.1 can be then proved if there exists a choice of $(r,h,\Lambda) \in \mathcal{S}_k$ so that the multipliers $c_{\ell} (\ell = 1, 2, 3)$ in (1.14) can be made equal zero. We now make the following observation: take the functional corresponding to the problem (1.6)

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dy - \frac{1}{2} \int_{\mathbb{R}^N} K\left(\frac{|y|}{r}\right) |u|^{2^*} \, dy.$$

(1.16)

If we can choose $(r,h,\Lambda)$ to be a critical point of function

$$F(r,h,\Lambda) := I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}) \quad \text{for } \phi \in \mathcal{E},$$

then the constants $c_{\ell}, \ell = 1, 2, 3$ would be zero. Thus finding solutions of problem (1.6) would be reduced to find a critical point of $F(r,h,\Lambda)$. This is the result in Proposition 3.1.

An important work of this paper is to give an accurate expression of $F(r,h,\Lambda)$ (see Proposition 3.2). Under the assumptions $r \sim k^{\frac{N-2}{N-2-m}}, h \to 0, \frac{1}{hk} \to 0$ as $k \to \infty$, we first get the expansion of energy functional $I(W_{r,h,\Lambda})$

$$F_1(r,h,\Lambda) := I(W_{r,h,\Lambda}) = k A_1 - \frac{k}{\Lambda^{N-2}} \left[ \frac{B_1 k^{N-2}}{(r \sqrt{1-h^2})^{N-2}} + \frac{B_3 k}{r^{N-2} h^{N-3} \sqrt{1-h^2}} \right].$$
\[ + k \left[ \frac{A_2}{\Lambda^{m} k^{(N-2)m} N^{-2m}} + \frac{A_3}{\Lambda^{m-2} k^{(N-2)m} N^{-2m}} (r - r)^2 \right] + kO \left( \frac{1}{k^{(N-2)m + \sigma}} \right), \]

where \( A_i \) for \( i = 1, 2, 3 \) and \( B_j \) for \( j = 4, 5 \) are constants. Denote
\[
\mathcal{G}(h) := \frac{B_4 k^{N-2}}{(\sqrt{1 - h^2})^{N-2}} + \frac{B_5 k}{h^{N-3} \sqrt{1 - h^2}}.
\]

Let \( h \) be the solution of \( \partial_h \mathcal{G}(h) = 0 \), then
\[
h = \frac{B'}{k^{N-1}} (1 + o(1)), \quad \text{as} \quad k \to \infty
\]
for some \( B' > 0 \). If \( r \sim k^{\frac{N-2}{4-2m}}, h \sim \frac{B'}{k^{N-1}}, \) then
\[
\frac{B_5 k}{r^{N-2} h^{N-3} \sqrt{1 - h^2}} = \frac{\tilde{B}}{k^{\frac{(N-2)m}{2} + \frac{2(N-3)}{N-1}}} (1 + o(1)), \quad \text{as} \quad k \to \infty
\]
for some constant \( \tilde{B} \).

However, we now find that the term \( O \left( \frac{1}{k^{(N-2)m + \sigma}} \right) \) in the expansion of \( F_1(r, h, \Lambda) \) competes with the term \( \frac{B_5 k}{r^{N-2} h^{N-3} \sqrt{1 - h^2}} \), making it impossible to identify a critical point for \( F_1(r, h, \Lambda) \). In reality though the remainder \( O \left( \frac{1}{k^{(N-2)m + \sigma}} \right) \) can be estimated in a more accurate way (see Proposition A.4) under our assumptions \( \mathcal{H} \).

We need to expand the full energy \( F(r, h, \Lambda) = I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}) \). We need a strong control on the size of \( \phi_{r,h,\Lambda} \) in order not to destroy the critical point structure of \( F_1(r, h, \Lambda) \) and to ensure the qualitative properties of the solutions as stated in Theorem 1.1. This is another delicate step of our construction, where we make full use of the assumption \( \mathcal{H} \) on \( K \).

**Structure of the paper.** The remaining part of this paper is devoted to the proof of Theorem 1.1 which will be organized as follows:

1. In Section 2 we will establish the linearized theory for the linearized projected problem. We will give estimates for the error terms in this Section.
2. In Section 3 we shall proof Theorem 1.1 by showing there exists a critical point of reduction function \( F(r, h, \Lambda) \).
3. Some tedious computations and some useful Lemmas will be given in Appendices A-B.

**Notation and preliminary results.** For the readers’ convenience, we will provide a collection of notation. Throughout this paper, we employ \( C, C_j \) to denote certain constants and \( \sigma, \tau, \sigma_j \) to denote some small constants or functions. We also note that \( \delta_{ij} \) is Kronecker delta function:
\[
\delta_{ij} = \begin{cases} 
1, & \text{if} \ i = j, \\
0, & \text{if} \ i \neq j.
\end{cases}
\]

Furthermore, we also employ the common notation by writing \( O(f(r, h)), o(f(r, h)) \) for the functions which satisfy
\[
\text{if} \quad g(r, h) \in O(f(r, h)) \quad \text{then} \quad \lim_{k \to +\infty} \left| \frac{g(r, h)}{f(r, h)} \right| \leq C < +\infty,
\]
and
\[
\text{if} \quad g(r, h) \in o(f(r, h)) \quad \text{then} \quad \lim_{k \to +\infty} \frac{g(r, h)}{f(r, h)} = 0.
\]
2. Finite dimensional reduction

For $j = 1, \cdots, k$, we divide $\mathbb{R}^N$ into $k$ parts:

$$\Omega_j := \left\{ y = (y_1, y_2, y_3, y'') \in \mathbb{R}^3 \times \mathbb{R}^{N-3} : \right\}$$

$$\left\langle \left( \frac{y_1}{\|y_1, y_2\|}, \left( \cos \frac{2(j-1)\pi}{k}, \sin \frac{2(j-1)\pi}{k} \right) \right) \right\rangle_{\mathbb{R}^2} \geq \cos \frac{\pi}{k} \right\}.$$  

where $\langle, \rangle_{\mathbb{R}^2}$ denote the dot product in $\mathbb{R}^2$. For $\Omega_j$, we further divide it into two parts:

$$\Omega_j^+ = \left\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 \geq 0 \right\},$$

$$\Omega_j^- = \left\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 < 0 \right\}.$$  

We can know that

$$\mathbb{R}^N = \bigcup_{j=1}^k \Omega_j, \quad \Omega_j = \Omega_j^+ \cup \Omega_j^-$$  

and

$$\Omega_j \cap \Omega_i = \emptyset, \quad \Omega_j^+ \cap \Omega_j^- = \emptyset, \quad \text{if} \quad i \neq j.$$  

We consider the following linearized problem

$$\left\{ \begin{array}{ll}
-\Delta \phi - (2^* - 1)K \left( \frac{|y|}{r} \right) W_{r, h, \Lambda}^{2^*-2} \phi &= f + \sum_{i=1}^k \sum_{\ell=1}^3 \left( c_{i \ell} U_{z_i, \Lambda}^{2^*-2} \nabla z_{i \ell} + c_{i \ell} U_{z_i, \Lambda}^{2^*-2} z_{i \ell} \right) \\
\phi &\in \mathcal{E},
\end{array} \right. \quad \text{in} \ \mathbb{R}^N,$$  

(2.1)

for some constants $c_{i \ell}$.

Coming back to equation (1.4), we recall that the functions

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \ldots, N, \quad Z_{N+1}(y) := \frac{N-2}{2} U(y) + y \cdot \nabla U(y).$$  

(2.2)

belong to the null space of the linearized problem associated to (1.4) around an Aubin-Talenti bubble, namely they solve

$$\Delta \phi + (2^* - 1)U^{2^*-2} \phi = 0, \quad \text{in} \ \mathbb{R}^N, \quad \phi \in D^{1,2}(\mathbb{R}^N).$$  

(2.3)

It is known [28] that these functions span the set of the solutions to (2.3). This fact will be used in the following crucial lemma which concerns the linearized problem (2.1).

**Lemma 2.1.** Suppose that $\phi_k$ solves (2.1) for $f = f_k$. If $\|f_k\|_{**} \to 0$ as $k \to \infty$, so does $\|\phi_k\|_{*}$.

The norms $\| \cdot \|_{*}$ and $\| \cdot \|_{**}$ are defined respectively in (1.10) and (1.11).

**Proof.** We prove the Lemma by contradiction. Suppose that there exists a sequence of $(r_k, h_k, \Lambda_k) \in \mathcal{J}_k$, and for $\phi_k$ satisfies (2.1) with $f = f_k, r = r_k, h = h_k, \Lambda = \Lambda_k$, with $\|f_k\|_{**} \to 0$, and $\|\phi_k\|_{*} \geq c' > 0$. Without loss of generality, we can assume that $\|\phi_k\|_{*} = 1$. For convenience, we drop the subscript $k$. 

From (2.1), we know that
\[
\phi(y) = (2^* - 1) \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} K\left(\frac{|z|}{r}\right) W_{r,h,\Lambda}^{2^* - 2} \phi(z) \, dz + \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} f(z) \, dz
\]
\[
+ \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \sum_{j=1}^{k} \sum_{\ell=1}^{3} \left( c_{\ell} U_{r,\Lambda}^{2^* - 2} \otimes_{\Lambda} Z_{\ell, j} + c_{\ell} U_{r,\Lambda}^{2^* - 2} \otimes_{\Lambda} Z_{\ell, j} \right) \, dz
\]
\[
:= M_1 + M_2 + M_3.
\]
For the first term $M_1$, we make use of Lemma [B.5] so that
\[
M_1 \leq C\|\phi\|_{*} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} W_{r,h,\Lambda}^{2^* - 2} \left( \sum_{j=1}^{k} \left[ \frac{1}{(1 + |z - x_j|)^{N - 2 + \tau}} + \frac{1}{(1 + |z - x_j|)^{N - 2 + \tau}} \right] \right) \, dz
\]
\[
\leq C\|\phi\|_{*} \sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - x_j|)^{N - 2 + \tau}} + \frac{1}{(1 + |y - x_j|)^{N - 2 + \tau}} \right].
\]
For the second term $M_2$, we make use of Lemma [B.4] so that
\[
M_2 \leq C\|f\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \sum_{j=1}^{k} \left[ \frac{1}{(1 + |z - x_j|)^{N - 2 + \tau}} + \frac{1}{(1 + |z - x_j|)^{N - 2 + \tau}} \right] \, dz
\]
\[
\leq C\|f\|_{**} \sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - x_j|)^{N - 2 + \tau}} + \frac{1}{(1 + |y - x_j|)^{N - 2 + \tau}} \right].
\]
In order to estimate the term $M_3$, we will first give the estimates of $Z_{1j}$ and $Z_{1j}$
\[
|Z_{1j}| \leq \frac{C}{(1 + |y - x_j|)^{N - 2}}, \quad |Z_{2j}| \leq \frac{C r}{(1 + |y - x_j|)^{N - 2}}, \quad |Z_{3j}| \leq \frac{C}{(1 + |y - x_j|)^{N - 2}},
\]
\[
|Z_{1j}| \leq \frac{C}{(1 + |y - x_j|)^{N - 2}}, \quad |Z_{2j}| \leq \frac{C r}{(1 + |y - x_j|)^{N - 2}}, \quad |Z_{3j}| \leq \frac{C}{(1 + |y - x_j|)^{N - 2}}. \tag{2.4}
\]
Combining estimates (2.4) and Lemma [B.4] we have
\[
\sum_{j=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} U_{r,\Lambda}^{2^* - 2} \otimes_{\Lambda} Z_{\ell, j} \, dz \leq C \sum_{j=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} \frac{(1 + r \delta_{\ell 2})}{(1 + |z - x_j|)^{N + 2}} \, dz
\]
\[
\leq C \sum_{j=1}^{k} \frac{(1 + r \delta_{\ell 2})}{(1 + |y - x_j|)^{N - 2 + \tau}}, \quad \text{for } \ell = 1, 2, 3,
\]
where $\delta_{\ell 2} = 0$ if $\ell \neq 2$, $\delta_{\ell 2} = 1$ if $\ell = 2$. Similarly, we have
\[
\sum_{j=1}^{k} \int_{\mathbb{R}^N} \frac{1}{|z - y|^{N-2}} U_{r,\Lambda}^{2^* - 2} \otimes_{\Lambda} Z_{\ell, j} \, dz \leq C \sum_{j=1}^{k} \frac{(1 + r \delta_{\ell 2})}{(1 + |y - x_j|)^{N - 2 + \tau}}, \quad \text{for } \ell = 1, 2, 3.
\]
Next, we will give the estimates of $c_{\ell}, \ell = 1, 2, 3$. Multiply both sides of (2.11) by $\mathcal{Z}_q, q = 1, 2, 3$, then we obtain that

$$
\int_{\mathbb{R}^N} \left[ -\Delta \phi - (2^* - 1) K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^*-2} \phi \right] \mathcal{Z}_q
$$

$$
= \int_{\mathbb{R}^N} f \mathcal{Z}_q + \sum_{j=1}^k \sum_{\ell=1}^3 \int_{\mathbb{R}^N} \left( c_{\ell} U_{q,j,\Lambda}^{2^*-2} \mathcal{Z}_j + c_{\ell} U_{q,j,\Lambda}^{2^*-2} \mathcal{Z}_j \right) \mathcal{Z}_q,
$$

(2.5)

Using Lemma [3.3], we can get

$$
\int_{\mathbb{R}^N} f \mathcal{Z}_q \leq C \|f\|_{\ast\ast} \sum_{j=1}^k \int_{\mathbb{R}^N} \frac{1 + r \delta_{q,j}}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \left[ \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} + \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \right]
$$

$$
\leq C(1 + r \delta_{q,j}) \|f\|_{\ast\ast}.
$$

The discussion on the left side of (2.5) may be more tricky, in fact, we have

$$
\int_{\mathbb{R}^N} \left[ -\Delta \phi - (2^* - 1) K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^*-2} \phi \right] \mathcal{Z}_q
$$

$$
= \int_{\mathbb{R}^N} \left[ -\Delta \mathcal{Z}_q - (2^* - 1) K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^*-2} \mathcal{Z}_q \phi \right]
$$

$$
= (2^* - 1) \int_{\mathbb{R}^N} \left[ 1 - K \left( \frac{|y|}{r} \right) \right] W_{r,h,\Lambda}^{2^*-2} \mathcal{Z}_q \phi + \left( U_{1,\Lambda}^{2^*-2} - W_{r,h,\Lambda}^{2^*-2} \right) \mathcal{Z}_q \phi
$$

$$
:= J_1 + J_2.
$$

Using the property of $K(s)$, similar to the proof of Lemma [3.5], we can get

$$
J_1 \leq C \|\phi\|_{\ast} \int_{\mathbb{R}^N} \left[ 1 - K \left( \frac{|y|}{r} \right) \right] W_{r,h,\Lambda}^{2^*-2} \mathcal{Z}_q \sum_{j=1}^k \left[ \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} + \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \right]
$$

$$
= C \|\phi\|_{\ast} \int_{||y|-r| \leq \sqrt{r}} \left[ 1 - K \left( \frac{|y|}{r} \right) \right] W_{r,h,\Lambda}^{2^*-2} \mathcal{Z}_q \sum_{j=1}^k \left[ \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} + \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \right]
$$

$$
+ C \|\phi\|_{\ast} \int_{||y|-r| \geq \sqrt{r}} \left[ 1 - K \left( \frac{|y|}{r} \right) \right] W_{r,h,\Lambda}^{2^*-2} \mathcal{Z}_q \sum_{j=1}^k \left[ \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} + \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \right]
$$

$$
\leq \frac{C}{\sqrt{r}} \int_{\mathbb{R}^N} W_{r,h,\Lambda}^{2^*-2} (y) \frac{1 + r \delta_{q,j}}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \sum_{j=1}^k \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}}
$$

$$
+ \frac{C}{r^2} \int_{\mathbb{R}^N} W_{r,h,\Lambda}^{2^*-2} (y) \frac{1 + r \delta_{q,j}}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \sum_{j=1}^k \frac{1}{(1 + |y - \mathcal{Z}_j|)^{N-2}} \leq C(1 + r \delta_{q,j}).
$$

For $J_2$, it is easy to derive that

$$
J_2 \leq \int_{\mathbb{R}^N} \left| U_{1,\Lambda}^{2^*-2} - W_{r,h,\Lambda}^{2^*-2} \right| \frac{1 + r \delta_{q,j}}{(1 + |y - \mathcal{Z}_j|)^{N-2}}
$$
we can get
\[
\frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau} + \frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau}
\]
\[
\leq \frac{C}{r^\sigma} (1 + r \delta_\ell).
\]
Then, we get
\[
\int_{\mathbb{R}^N} \left[-\Delta \phi - (2^* - 1) K \left(\frac{|y|}{r}\right) W_{r,h,\Lambda}^{2^* - 2} \phi\right] \mathbb{Z}_{q1} \leq \frac{C}{r^\sigma} (1 + r \delta_\ell) \|\phi\|_L.
\]
On the other hand, there holds
\[
\sum_{j=1}^k \int_{\mathbb{R}^N} (U_{\tau_j,\Lambda}^{2^* - 2} \mathbb{Z}_{\ell j} + U_{\tau_j,\Lambda}^{2^* - 2} \mathbb{Z}_{\ell j}) \mathbb{Z}_{q1} = c_\ell \delta_{\ell q} (1 + \delta_\ell r^2) + o(1), \quad \text{as} \quad k \to \infty.
\]
Note that
\[
\int_{\mathbb{R}^N} U_{\tau,\Lambda}^{2^* - 2} \mathbb{Z}_{\ell 1} \mathbb{Z}_{q1} = \left\{ \begin{array}{ll}
0, & \text{if } \ell \neq q, \\
c_\ell (1 + \delta_\ell r^2), & \text{if } \ell = q,
\end{array} \right.
\]
for some constant $c_\ell > 0$. Then we can get
\[
c_\ell = \frac{1 + r \delta_\ell}{1 + \delta_\ell r^2} O \left(\frac{1}{r^\sigma} \|\phi\|_L + \|f\|_L\right) = o(1), \quad \text{as} \quad k \to \infty.
\]
Then we have
\[
|\phi| \leq \left(\|f\|_L \sum_{j=1}^k \left[ \frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau} + \frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau}\right]
\right.
\]
\[
\left. + \sum_{j=1}^k \left[ \frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau + \sigma} + \frac{1}{1 + |y - \tau_j|} \frac{N - 2}{2 + \tau + \sigma}\right]\right).
\]
Combining this fact and $\|\phi\|_L = 1$, we have the following claim:

\textbf{Claim 1}: There exist some positive constants $R$, $\delta_1$ such that
\[
\|\phi\|_{L^\infty(B_R(\tau_1))} \geq \delta_1 > 0,
\]
for some $l \in \{1, 2, \ldots, k\}$.

Since $\phi \in H_s$, we assume that $l = 1$. By using local elliptic estimates and (2.7), we can get, up to subsequence, $\tilde{\phi}(y) = \phi(y - \tau_1)$ converge uniformly in any compact set to a solution
\[
-\Delta \tilde{u} - (2^* - 1) U_{0,\Lambda}^{2^* - 2} \tilde{u} = 0, \quad \text{in } \mathbb{R}^N,
\]
for some $\Lambda \in [L_1, L_2]$. Since $\phi$ is even in $y_d, d = 2, 4, \ldots, N$, we know that $\tilde{u}$ is also even in $y_d, d = 2, 4, \ldots, N$. Then we know that $\tilde{u}$ must be a linear combination of the functions
\[
\frac{\partial U_{0,\Lambda}}{\partial y_1}, \quad \frac{\partial U_{0,\Lambda}}{\partial y_3}, \quad y \cdot \nabla U_{0,\Lambda} + (N - 2) U_{0,\Lambda}.
\]
From the assumptions
\[
\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \mathbb{Z}_{\ell 1} \tilde{\phi} = 0 \quad \text{for } \ell = 1, 2, 3,
\]
we can get
\[
\sqrt{1 - h^2} \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_1} \tilde{\phi} + h \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_3} \tilde{\phi} = 0,
\]
\[
\sqrt{1-h^2} \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_1} \tilde{\phi} + h \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_3} \tilde{\phi} = 0,
\]
and
\[
\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \left[ y \cdot \nabla U_{0,\Lambda} + (N - 2)U_{0,\Lambda} \right] \tilde{\phi} = 0.
\]
By taking limit, we have
\[
\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_1} u = \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \frac{\partial U_{0,\Lambda}}{\partial y_3} u = \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^* - 2} \left[ y \cdot \nabla U_{0,\Lambda} + (N - 2)U_{0,\Lambda} \right] u = 0.
\]
So we have \( u = 0 \). This is a contradiction to (2.8). \( \square \)

For the linearized problem (2.1), we have the following existence, uniqueness results. Furthermore, we can give the estimates of \( \phi \) and \( c_\ell, \ell = 1, 2, 3 \).

**Proposition 2.2.** There exist \( k_0 > 0 \) and a constant \( C > 0 \) such that for all \( k \geq k_0 \) and all \( f \in L^\infty(\mathbb{R}^N) \), problem (2.1) has a unique solution \( \phi \equiv \mathbf{L}_k(f) \). Besides,
\[
\|\phi\|_* \leq C\|f\|_{**}, \quad |c_\ell| \leq \frac{C}{1 + \delta_{2r}}\|f\|_{**}, \quad \ell = 1, 2, 3.
\]

**Proof.** Recall the definition of \( E \) as in (1.15), we can rewrite problem (2.1) in the form
\[
-\Delta \phi = f + (2^* - 1)K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^* - 2} \phi \quad \text{for all } \phi \in E,
\]
in the sense of distribution. Furthermore, by using Riesz’s representation theorem, equation (2.10) can be rewritten in the operational form
\[
(\mathbb{I} - T_k)\phi = \tilde{f}, \quad \text{in } E,
\]
where \( \mathbb{I} \) is identity operator and \( T_k \) is a compact operator. Fredholm’s alternative yields that problem (2.11) is uniquely solvable for any \( \tilde{f} \) when the homogeneous equation
\[
(\mathbb{I} - T_k)\phi = 0, \quad \text{in } E,
\]
has only the trivial solution. Moreover, problem (2.12) can be rewritten as following
\[
\begin{align*}
-\Delta \phi - (2^* - 1)K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^* - 2} \phi &= \sum_{i=1}^{3} \sum_{\ell=1}^{3} \left( c_i U_{x_i,\Lambda}^{2^* - 2} \bar{\zeta}_{i,\ell} + c_\ell U_{x_i,\Lambda}^{2^* - 2} \bar{\zeta}_{i,\ell} \right) \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
Suppose that (2.13) has nontrivial solution \( \phi_k \) and satisfies \( \|\phi_k\|_* = 1 \). From Lemma (2.1) we know \( \|\phi_k\|_* \) tends to zero as \( k \to +\infty \), which is a contradiction. Thus problem (2.12) (or (2.13)) only has trivial solution. So we can get unique solvability for problem (2.1). Using Lemma (2.1) the estimates (2.9) can be proved by a standard method. \( \square \)

We can rewrite problem (1.14) as following
\[
\begin{align*}
-\Delta \phi - (2^* - 1)K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^* - 2} \phi &= \mathbf{N}(\phi) + \mathbf{I}_k \\
&+ \sum_{j=1}^{k} \sum_{\ell=1}^{3} \left( c_i U_{x_j,\Lambda}^{2^* - 2} \bar{\zeta}_{j,\ell} + c_\ell U_{x_j,\Lambda}^{2^* - 2} \bar{\zeta}_{j,\ell} \right) \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
where
\[
\mathbf{N}(\phi) = K \left( \frac{|y|}{r} \right) \left( W_{r,h,\Lambda} + \phi \right)^{2^* - 1} - W_{r,h,\Lambda}^{2^* - 1} - (2^* - 1)W_{r,h,\Lambda}^{2^* - 2} \phi.
\]
We can rewrite
\[ l_k = K \left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^*-1} - \sum_{j=1}^{k} \left( U_{x_j,\Lambda}^{2^*-1} + U_{x_j,\Lambda}^{2^*-1} \right). \]

Next, we will use the Contraction Mapping Principle to show that problem (2.14) has a unique solution in the set that \( \|\phi\|_s \) is small enough. Before that, we will give the estimate of \( N(\phi) \) and \( l_k \).

**Lemma 2.3.** Suppose \( N \geq 5 \). There exists \( C > 0 \) such that
\[ \| N(\phi) \|_{**} \leq C \| \phi \|_{\min(2^*-1,2)}, \]
for all \( \phi \in E \).

**Proof.** The proof is similar to that of Lemma 2.4 in [31]. Here we omit it. \( \square \)

We next give the estimate of \( l_k \).

**Lemma 2.4.** Suppose \( K(|y|) \) satisfies \( H \) and \( N \geq 5 \), \( (r,h,\Lambda) \in \mathcal{S}_k \). There exists \( k_0 \) and \( C > 0 \) such that for all \( k \geq k_0 \)
\[ \| l_k \|_{**} \leq C \max \left\{ \frac{1}{k^{(N-2)/2}}, \frac{1}{k^{(N-2)/2}} \right\}, \]
where \( \epsilon_1 \) is small constant given in (1.12).

**Proof.** We can rewrite \( l_k \) as
\[ l_k = K \left( \frac{|y|}{r} \right) \left[ W_{r,h,\Lambda}^{2^*-1} - \sum_{j=1}^{k} \left( U_{x_j,\Lambda}^{2^*-1} + U_{x_j,\Lambda}^{2^*-1} \right) \right], \]
\[ + \sum_{j=1}^{k} \left[ K \left( \frac{|y|}{r} \right) - 1 \right] \left( U_{x_j,\Lambda}^{2^*-1} + U_{x_j,\Lambda}^{2^*-1} \right) := S_1 + S_2. \]

Assume that \( y \in \Omega^+, \) then we get
\[ S_1 = K \left( \frac{|y|}{r} \right) \left[ \left( \sum_{j=1}^{k} U_{x_j,\Lambda} + U_{x_j,\Lambda} \right)^{2^*-1} - \sum_{j=1}^{k} \left( U_{x_j,\Lambda}^{2^*-1} + U_{x_j,\Lambda}^{2^*-1} \right) \right] \leq CK \left( \frac{|y|}{r} \right) \left[ \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right] + \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)^{2^*-1}. \]

Thus, we have
\[ S_1 \leq C \sum_{j=2}^{k} \left( \frac{1}{1 + |y_1|} \right)^{2^*-1} \sum_{j=2}^{k} \left( \frac{1}{1 + |y_j|} \right)^{N-2} + C \sum_{j=1}^{k} \left( \frac{1}{1 + |y_1|} \right)^{N-2} \]
\[ + C \sum_{j=2}^{k} \left( \frac{1}{1 + |y_j|} \right)^{N-2} \left( \sum_{j=2}^{k} \frac{1}{|x_j - x_1|} \right)^{2^*-1} := S_{11} + S_{12} + S_{13}. \]

We first consider the case \( N = 5 \). It is easy to get that
\[ S_{11} \big|_{N=5} \leq C \sum_{j=2}^{k} \left( \frac{1}{|x_j - x_1|} \right)^{2^*-1} \sum_{j=2}^{k} \left( \frac{1}{|x_j - x_1|} \right)^{3} \]
\[
\leq C \frac{1}{(1 + |y - x_1|)^\frac{3}{2} + \tau \left(\frac{k}{r}\right)^\alpha_1}. \tag{2.16}
\]

When \( N \geq 6 \), similar to the proof of Lemma \([3.1]\), for any \( 1 < \alpha_1 < N - 2 \), we have

\[
\sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \leq C \frac{1}{(1 + |y - x_1|)^{N-2-\alpha_1}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\alpha_1}}.
\]

Since \( \tau \in \left(\frac{N-2-m}{N-2}, \frac{N-2-m}{N-2} + \epsilon_1\right) \), we can choose \( \alpha_1 \) satisfies

\[
\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1 < \alpha_1 < \frac{N+2}{2} - \tau < N-2.
\]

Then

\[
S_{11}\big|_{N\geq6} \leq \frac{C}{(1 + |y - x_1|)^{N+2-\alpha_1}} \sum_{j=2}^{k} \frac{1}{|x_j - x_1|^{\alpha_1}} \leq \frac{C}{(1 + |y - x_1|)^{N+2-\alpha_1}} \left(\frac{k}{r}\right)^{\alpha_1} \leq C \frac{k^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}}{(1 + |y - x_1|)^{N+2-\tau} \left(\frac{k}{r}\right)^\alpha_1}. \tag{2.17}
\]

Then combining (2.16) and (2.17), we can get

\[
\|S_{11}\|_{**} \leq \begin{cases} 
C \left(\frac{k}{r}\right)^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}, & \text{if } N \geq 6, \\
C \left(\frac{k}{r}\right)^3, & \text{if } N = 5.
\end{cases} \tag{2.18}
\]

For \( S_{12} \), we can rewrite it as following

\[
S_{12} = C \frac{1}{(1 + |y - x_1|)^2} \left[ \frac{1}{(1 + |y - x_1|)^{N-2}} + \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right]
\leq C \frac{1}{(1 + |y - x_1|)^2} \left[ \frac{1}{(1 + |y - x_1|)^{N-2}} + \sum_{j=2}^{k} \frac{1}{(1 + |y - x_j|)^{N-2}} \right].
\]

Similarly to (2.16), we can obtain

\[
\left|S_{12}\right|_{N=5} \leq C \frac{1}{(1 + |y - x_1|)^\frac{3}{2} + \tau \left(\frac{k}{r}\right)^\alpha_1}. \tag{2.16}_1
\]

For \( N \geq 6 \) and the same \( \alpha_1 \) as in (2.18), it is easy to derive that

\[
\frac{1}{(1 + |y - x_1|)^4} \left(\frac{1}{(1 + |y - x_1|)^{N-2}} \right)
\leq \left[ \frac{1}{(1 + |y - x_1|)^{N+2-\alpha_1}} + \frac{1}{(1 + |y - x_1|)^{N+2-\alpha_1}} \right] \frac{1}{|x_1 - x_1|^{\alpha_1}} \leq \frac{C}{(1 + |y - x_1|)^{N+2-\alpha_1}} \left(\frac{k}{r}\right)^\alpha_1.
\]
\[ \leq C \frac{1}{(1 + |y - \mathbf{r}|)^{\frac{N+2}{2} + \tau}} \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}, \]

where we have used the fact \( hr > C \frac{k}{r} \). Thus, we can obtain that

\[
\|S_{12}\|_{**} \leq \begin{cases} 
C \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}, & \text{if } N \geq 6, \\
C \left( \frac{k}{r} \right)^{3}, & \text{if } N = 5.
\end{cases} (2.19)
\]

Next, we consider \( S_{13} \). For \( y \in \Omega^+_1 \),

\[
\sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{r}_j|)^{N-2}} \leq \sum_{j=2}^k \frac{1}{(1 + |y - \mathbf{r}_1|)^{\frac{N+2}{2}}} \frac{1}{(1 + |y - \mathbf{r}_j|)^{\frac{N-2}{2}}}
\]

\[
\leq \sum_{j=2}^k \frac{C}{|\mathbf{r}_j - \mathbf{r}_1|^{\frac{N+2}{2} - \frac{N-2-m}{N-2} \tau}} \frac{1}{(1 + |y - \mathbf{r}_1|)^{\frac{N-2}{2} + \frac{N-2}{N-2} \tau}} 
\leq C \left( \frac{k}{r} \right)^{-\frac{N+2}{2} + \tau} \frac{C}{(1 + |y - \mathbf{r}_1|)^{\frac{N+2}{2} + \tau}} \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}.
\]

Thus we have

\[
S_{13} \leq \left( \frac{k}{r} \right)^{-\frac{N+2}{2} + \tau} \frac{C}{(1 + |y - \mathbf{r}_1|)^{\frac{N+2}{2} + \tau}} \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}.
\]

Since \( (\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1) \big|_{N=5} > 3 \) for \( m \in \{2, 3\} \), then we have

\[
\|S_{13}\|_{**} \leq \begin{cases} 
C \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}, & \text{if } N \geq 6, \\
C \left( \frac{k}{r} \right)^{3}, & \text{if } N = 5.
\end{cases} (2.20)
\]

Combining (2.18), (2.19), (2.20), we obtain

\[
\|S_1\|_{**} \leq \begin{cases} 
C \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{N-2} - \epsilon_1}, & \text{if } N \geq 6, \\
C \left( \frac{k}{r} \right)^{3}, & \text{if } N = 5.
\end{cases} (2.21)
\]

We now consider the estimate of \( S_2 \). For \( y \in \Omega^+_1 \), we have

\[
S_2 \leq 2 \sum_{j=1}^k \left[ K \left( \frac{|y|}{r} \right) - 1 \right] U_{\mathbf{r}_j, \Lambda}^{2^* - 1} 
= 2 U_{\mathbf{r}_1, \Lambda}^{2^* - 1} \left[ K \left( \frac{|y|}{r} \right) - 1 \right] + 2 \sum_{j=2}^k U_{\mathbf{r}_j, \Lambda}^{2^* - 1} \left[ K \left( \frac{|y|}{r} \right) - 1 \right]
\]
Thus, we can get, if \( m > N \) since \( \delta > \delta_1 > 0 \), then
\[
|y - \bar{x}_1| \geq |y| - r - |r - \bar{x}_1| \geq \frac{1}{2} \delta_1 r.
\]
As a result, we get
\[
U^{2^* - 1}_{\bar{x}_1, \Lambda} \left[ K(\frac{|y|}{r}) - 1 \right] \leq \frac{C}{(1 + |y - \bar{x}_1|)^{\frac{N+2}{2} + \tau}} \frac{1}{r^{\frac{N+2}{2} + \tau}}
\]
\[
\leq \frac{C}{(1 + |y - \bar{x}_1|)^{\frac{N+2}{2} + \tau}} \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N - 2 - m}{2} - \epsilon_1}.
\]
• If \( |\frac{|y|}{r} - 1| \leq \delta_1 \), then
\[
\left[ K(\frac{|y|}{r}) - 1 \right] \leq C \left| \frac{|y|}{r} - 1 \right|^m = \frac{C}{r^m} |y| - r |y|^m
\]
\[
\leq \frac{C}{r^m} \left[ ||y| - |\bar{x}_1||^m + |\bar{x}_1| - r |\bar{x}_1| \right]
\]
\[
\leq \frac{C}{r^m} \left[ ||y| - |\bar{x}_1||^m + \frac{1}{k^{\delta m}} \right].
\]
Thus, we can get, if \( m > 3 \),
\[
U^{2^* - 1}_{\bar{x}_1, \Lambda} \left[ K(\frac{|y|}{r}) - 1 \right] \leq \frac{C}{r^m} \left[ ||y| - |\bar{x}_1||^m + \frac{1}{k^{\delta m}} \right] \frac{C}{(1 + |y - \bar{x}_1|)^{N+2}}
\]
\[
\leq \frac{C}{r^{m+3}} \left[ \left( \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right)^\frac{m+3}{2} \right] + \frac{1}{r^{\frac{m-2}{2}} k^{\delta m}} \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}}
\]
\[
\leq \frac{C}{r^{m+3}} \left[ \left( \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right)^\frac{N+2 - \tau - m + 3}{2} + \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right]
\]
\[
\leq \frac{1}{r^{\frac{m+3}{2}}} \frac{C}{(1 + |y - \bar{x}_1|)^{\frac{N+2}{2} + \tau}},
\]
the last inequality holds due to \( \frac{N+2}{2} - \tau - \frac{m+3}{2} > 0 \).

On the other hand, if \( m \leq 3 \), we have
\[
U^{2^* - 1}_{\bar{x}_1, \Lambda} \left[ K(\frac{|y|}{r}) - 1 \right] \leq \frac{C}{r^m} \left[ ||y| - |\bar{x}_1||^m + \frac{1}{k^{\delta m}} \right] \frac{C}{(1 + |y - \bar{x}_1|)^{N+2}}
\]
\[
\leq \frac{C}{r^m} \left[ \left( \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right)^\frac{N+2 - \tau - m}{2} + \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right]
\]
\[
\leq \frac{C}{r^m} \left[ \left( \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right)^\frac{N+2 - \tau}{2} + \frac{1}{(1 + |y - \bar{x}_1|)^{N+2}} \right]
\]
\[
\leq \frac{C}{r^m} \left( \frac{1}{(1 + |y - \bar{x}_1|)^{\frac{N+2}{2} + \tau}} \right),
\]
since \( \frac{N+2}{2} - \tau - m > 0 \). Thus we have
\[
U^{2^* - 1}_{\bar{x}_1, \Lambda} \left[ K(\frac{|y|}{r}) - 1 \right] \leq \frac{C}{r^{\min\{m, \frac{m+3}{2}\}}} \frac{1}{(1 + |y - \bar{x}_1|)^{\frac{N+2}{2} + \tau}}.
\]
As a result,
\[ S_{21} \leq C \max \left\{ \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}, \frac{1}{r^{\min(m, \frac{m+3}{2})}} \right\} \left( \frac{1}{1 + \frac{|y - \omega_1|}{N+2}} \right)^{\frac{N+2}{2} + r}. \tag{2.22} \]

Since \( y \in \Omega_1^+ \), then for \( j = 2, \cdots, k \), there holds
\[ |\omega_1 - \omega_j| \leq |y - \omega_1| + |y - \omega_j| \leq 2|y - \omega_j|. \]
Therefore, it is easy to derive that
\[ S_{22} \leq C \left( \frac{1}{1 + \frac{|y - \omega_1|}{N+2}} \right)^{\frac{N+2}{2} + r} \sum_{j=2}^{k} \frac{1}{(1 + \frac{|y - \omega_j|}{N+2})^{\frac{N+2}{2}}} \leq C \left( \frac{1}{1 + \frac{|y - \omega_1|}{N+2}} \right)^{\frac{N+2}{2} + r} \sum_{j=2}^{k} \frac{1}{|\omega_1 - \omega_j|^{\frac{N+2}{2}}} \leq C \left( \frac{1}{1 + \frac{|y - \omega_1|}{N+2}} \right)^{\frac{N+2}{2} + r} \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}. \tag{2.23} \]
Combining (2.22) with (2.23), we obtain
\[ \|S_2\|_{**} \leq C \max \left\{ \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}, \frac{1}{r^{\min(m, \frac{m+3}{2})}} \right\}. \]

If \( N = 5 \), we can check that \( \frac{1}{r^{m}} = \left( \frac{k}{r} \right)^{3} \). Thus, we can rewrite (2.21) as
\[ \|S_1\|_{**} \leq C \max \left\{ \left( \frac{k}{r} \right)^{\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1}, \frac{1}{r^{\min(m, \frac{m+3}{2})}} \right\}. \]
Therefore, we showed (2.15). \[ \square \]

The solvability theory for the projected problem (2.14) can be provided in the following:

**Proposition 2.5.** Suppose that \( K(|y|) \) satisfies \( H \) and \( N \geq 5 \), \( (r, h, \Lambda) \in S_k \). There exists an integer \( k_0 \) large enough, such that for all \( k \geq k_0 \) problem (2.14) has a unique solution \( \phi_k \) which satisfies
\[ \|\phi_k\|_* \leq C \max \left\{ \frac{1}{k^{\frac{m}{N-2-m}}(\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1)}, \frac{1}{k^{(\frac{N+2}{2} - \frac{N-2-m}{2}) \min\{m, \frac{m+3}{2}\}}} \right\}, \tag{2.24} \]
and
\[ |c_\ell| \leq \frac{C}{(1 + \delta \epsilon_2 r)} \max \left\{ \frac{1}{k^{\frac{m}{N-2-m}}(\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1)}, \frac{1}{k^{(\frac{N+2}{2} - \frac{N-2-m}{2}) \min\{m, \frac{m+3}{2}\}}} \right\}, \quad \text{for } \ell = 1, 2, 3. \tag{2.25} \]

**Proof.** We first denote
\[ B := \left\{ v : v \in E, \|v\|_* \leq C \max \left\{ \frac{1}{k^{\frac{m}{N-2-m}}(\frac{N+2}{2} - \frac{N-2-m}{2} - \epsilon_1)}, \frac{1}{k^{(\frac{N+2}{2} - \frac{N-2-m}{2}) \min\{m, \frac{m+3}{2}\}}} \right\} \right\}. \]
From Proposition 2.2 we know that problem (2.14) is equivalent to the following fixed point problem
\[ \phi = L_k (N(\phi) + 1_k) =: A(\phi), \]
where \( L_k \) is the linear bounded operator defined in Proposition 2.2.

From Lemma 2.3 and Lemma 2.4, we know, for \( \phi \in B \)
\[ \|A(\phi)\|_* \leq C \left( \|N(\phi)\|_{**} + \|1_k\|_{**} \right) \]
is a critical point of 

denote functions which are independent of 

Proposition 3.1. 

So the operator \( \mathbf{A} \) maps from \( \mathcal{B} \) to \( \mathcal{B} \). Furthermore, we can show that \( \mathbf{A} \) is a contraction mapping. In fact, for any \( \phi_1, \phi_2 \in \mathcal{B} \), we have 

\[
\| \mathbf{A}(\phi_1) - \mathbf{A}(\phi_2) \| \leq C \| \mathbf{N}(\phi_1) - \mathbf{N}(\phi_2) \|.
\]

Since \( \mathbf{N}(\phi) \) has a power-like behavior with power greater than one, then we can easily get 

\[
\| \mathbf{A}(\phi_1) - \mathbf{A}(\phi_2) \| \leq o(1) \| \phi_1 - \phi_2 \|.
\]

A direct application of the contraction mapping principle yields that problem (2.14) has a unique solution \( \phi \in \mathcal{B} \). The estimates for \( c_k, \ell = 1, 2, 3 \) can be got easily from (2.6). 

\\[ \square \]

3. Proof of Theorem 1.1

**Proposition 3.1.** Let \( \phi_{r,h,\Lambda} \) be a function obtained in Proposition 2.5 and 

\[ F(r, h, \Lambda) := I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}). \]

If \( (r, h, \Lambda) \) is a critical point of \( F(r, h, \Lambda) \), then 

\[ u = W_{r,h,\Lambda} + \phi_{r,h,\Lambda} \]

is a critical point of \( I(u) \) in \( H^1(\mathbb{R}^N) \). 

We will give the expression of \( F(r, h, \Lambda) \). We first note that we employ the notation \( C(r, \Lambda) \) to denote functions which are independent of \( h \) and uniformly bounded.

**Proposition 3.2.** Suppose that \( K(|y|) \) satisfies \( \mathbf{H} \) and \( N \geq 5 \), \( (r, h, \Lambda) \in \mathcal{I}_k \). We have the following expansion as \( k \to \infty \)

\[
F(r, h, \Lambda) = I(W_{r,h,\Lambda}) + kO\left( \frac{1}{k^{\left( \frac{m(N-2)}{N-2-m} + \frac{2(N-3)}{N-1} + \sigma \right)}} \right)
\]

\[ = kA_1 - \frac{k}{\Lambda^{N-2}} \left[ \frac{B_3k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5k}{r^N h^{N-3} \sqrt{1-h^2}} \right] + k\frac{A_2}{\Lambda^{m(N-2)m}} + \frac{A_3}{\Lambda^{m-2} k^{N-2}} (r - r)^2 + k \frac{C(r, \Lambda)}{k^{m(N-2)}} (r - r)^{2+\sigma} + k \frac{C(r, \Lambda)}{k^{m(N-2)}} (r - r)^{2+\sigma} + kO\left( \frac{1}{k^{\left( \frac{m(N-2)}{N-2-m} + \frac{2(N-3)}{N-1} + \sigma \right)}} \right), \]

where \( A_1, A_2, A_3, B_4, B_5 \) are positive constants.

**Proof.** The proof of Proposition 3.2 is similar to that of Proposition 3.1 in [31]. We omit it here. 

Next, we will give the expansions of \( \frac{\partial F(r,h,\Lambda)}{\partial \Lambda} \) and \( \frac{\partial F(r,h,\Lambda)}{\partial h} \). 

**Proposition 3.3.** Suppose that \( K(|y|) \) satisfies \( \mathbf{H} \) and \( N \geq 5 \), \( (r, h, \Lambda) \in \mathcal{I}_k \). We have the following expansion for \( k \to \infty \)

\[
\frac{\partial F(r, h, \Lambda)}{\partial \Lambda} = \frac{k(N-2)}{\Lambda^{N-1}} \left[ \frac{B_3k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5k}{r^N h^{N-3} \sqrt{1-h^2}} \right]
\]
\[ -k \left[ \frac{m A_2}{\Lambda^{m+1} k^{N-2-m}} + \frac{(m-2) A_3}{\Lambda^{m-1} k^{N-2-m}} (r - r)^2 \right] + k O \left( \frac{1}{k^{N-2-m+\sigma}} \right), \quad (3.1) \]

where \( A_2, A_3, B_4, B_5 \) are positive constants.

**Proof.** The proof of this proposition can be found in [31]. We omit it here. \( \square \)

**Proposition 3.4.** Suppose that \( K(|y|) \) satisfies \( H \) and \( N \geq 5 \), \( (r, h, \Lambda) \in \mathcal{S}_k \). We have the following expansion

\[
\frac{\partial F(r, h, \Lambda)}{\partial h} = -\frac{k}{\Lambda^{N-2}} \left[ (N-2) \frac{B_4 k^{N-2}}{r^{N-2} (\sqrt{1 - h^2})^N} - (N-3) \frac{B_5 k}{r^{N-2} h^{N-2} \sqrt{1 - h^2}} \right] + k O \left( \frac{1}{k^{N-2-m+\sigma}} \right), \quad (3.2)
\]

where \( B_4, B_5 \) are positive constants.

**Proof.** Notice that \( F(r, h, \Lambda) = I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}) \), there holds

\[
\frac{\partial F(r, h, \Lambda)}{\partial h} = \left< I'(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}), \frac{\partial (W_{r,h,\Lambda} + \phi_{r,h,\Lambda})}{\partial h} \right>
\]

\[
= \left< I'(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h} \right> + \left< I'(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}), \frac{\partial \phi_{r,h,\Lambda}}{\partial h} \right>
\]

\[
= \left< I'(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h} \right> + \left< \sum_{j=1}^{k} \sum_{\ell=1}^{3} \left( c_{\ell} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} + c_{\ell} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} \right), \frac{\partial \phi_{r,h,\Lambda}}{\partial h} \right>. \quad (3.3)
\]

Since \( \int_{\mathbb{R}^N} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} \phi_{r,h,\Lambda} = \int_{\mathbb{R}^N} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} \phi_{r,h,\Lambda} = 0 \), we can get easily

\[
\left< U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j}, \frac{\partial \phi_{r,h,\Lambda}}{\partial h} \right> = -\left< \frac{\partial (U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j})}{\partial h}, \phi_{r,h,\Lambda} \right>,
\]

\[
\left< U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j}, \frac{\partial \phi_{r,h,\Lambda}}{\partial h} \right> = -\left< \frac{\partial (U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j})}{\partial h}, \phi_{r,h,\Lambda} \right>.
\]

Then

\[
\left< \sum_{j=1}^{k} \left( c_{\ell} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} + c_{\ell} U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j} \right), \frac{\partial \phi_{r,h,\Lambda}}{\partial h} \right>
\]

\[
\leq C \|c\| \|\phi_{r,h,\Lambda}\| \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{\partial (U_{\ell_j,\Lambda}^{2,2} \bar{Z}_{\ell_j})}{\partial h} \left( \sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - \ell_j|)^{N/2+\tau}} + \frac{1}{(1 + |y - \ell_j|)^{N/2+\tau}} \right] \right)
\]

\[
\leq C \|c\| \|\phi_{r,h,\Lambda}\| \times \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{r (1 + \delta r)}{(1 + |y - \ell_i|)^{N+3}} \left( \sum_{j=1}^{k} \left[ \frac{1}{(1 + |y - \ell_j|)^{N/2+\tau}} + \frac{1}{(1 + |y - \ell_j|)^{N/2+\tau}} \right] \right)
\]
\[
\leq C r \max \left\{ \frac{1}{k^{N-2m}(N+2-2N-2m-2\epsilon)} \right\}, \quad \frac{1}{k^{N-2m}(N+2m-3)(2m+3)} + C \frac{1}{k^{N-2m}(N+2m-3)(2m+3)} \min\{2m,m+3\},
\]

where we used the estimates (2.24)-(2.25) and the inequalities
\[
\left| \frac{\partial \left( U^{2^*\alpha} - 2 \Phi_\alpha \right) \right| \leq C r (1 + \delta_{2r}) \frac{1}{(1 + |y - \frac{r}{2}|)^{N+3}} \text{ for } i = 1, \ldots, k, \ell = 1, 2, 3.
\]

On the other hand, we have
\[
\left\langle I'(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h} \right\rangle
\]
\[
= \int_{R^N} \nabla (W_{r,h,\Lambda} + \phi_{r,h,\Lambda}) \nabla W_{r,h,\Lambda} - \int_{R^N} K \left( \frac{|y|}{r} \right) (W_{r,h,\Lambda} + \phi_{r,h,\Lambda})^{2^* - 1} \frac{\partial W_{r,h,\Lambda}}{\partial h}
\]
\[
= \int_{R^N} \nabla W_{r,h,\Lambda} \nabla \frac{\partial W_{r,h,\Lambda}}{\partial h} - \int_{R^N} K \left( \frac{|y|}{r} \right) (W_{r,h,\Lambda} + \phi_{r,h,\Lambda})^{2^* - 1} \frac{\partial W_{r,h,\Lambda}}{\partial h}
\]
\[
= \frac{\partial I(W_{r,h,\Lambda})}{\partial h} + (2^* - 1) \int_{R^N} K \left( \frac{|y|}{r} \right) W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} \phi_{r,h,\Lambda} + O \left( \int_{R^N} \phi_{r,h,\Lambda}^2 \right). \tag{3.5}
\]

For the second term in (3.5), using the decay property of \( K(|y|) \) and orthogonality of \( \phi_{r,h,\Lambda} \), we can show this term is small. In fact, we have
\[
\int_{R^N} K \left( \frac{|y|}{r} \right) W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} \phi_{r,h,\Lambda}
\]
\[
= \int_{R^N} K \left( \frac{|y|}{r} \right) \left[ W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \left( U^{2^* - 2 \Phi_{2i}} + U^{2^* - 2 \Phi_{2i}} \right) \right] \phi_{r,h,\Lambda}
\]
\[
= \sum_{i=1}^{k} \int_{R^N} K \left( \frac{|y|}{r} \right) \left[ W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \left( U^{2^* - 2 \Phi_{2i}} + U^{2^* - 2 \Phi_{2i}} \right) \right] \phi_{r,h,\Lambda}
\]
\[
= 2k \int_{\Omega^+_{1}} K \left( \frac{|y|}{r} \right) \left[ W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \left( U^{2^* - 2 \Phi_{2i}} + U^{2^* - 2 \Phi_{2i}} \right) \right] \phi_{r,h,\Lambda}
\]
\[
+ 2k \int_{R^N} K \left( \frac{|y|}{r} \right) \left[ W_r^{2^* - 2 \Phi_{2i}} \right] \phi_{r,h,\Lambda}.
\]

According to the expression of \( W_{r,h,\Lambda} \), we can obtain that
\[
\int_{\Omega^+_{1}} K \left( \frac{|y|}{r} \right) \left[ W_r^{2^* - 2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \left( U^{2^* - 2 \Phi_{2i}} + U^{2^* - 2 \Phi_{2i}} \right) \right] \phi_{r,h,\Lambda}
\]
\[
\leq C \int_{\Omega^+_{1}} \left[ \sum_{j=2}^{k} \sum_{i=1}^{k} \left( U^{2^* - 2 \Phi_{2i}} + U^{2^* - 2 \Phi_{2i}} \right) \phi_{r,h,\Lambda}
\]
\[
\leq C \left( \frac{n}{r} \right)^{2^* - 2^*} \int_{\Omega^+_{1}} \frac{r}{(1 + |y - \frac{r}{2}|)^{N+2^*}} \phi_{r,h,\Lambda}
\]
Combining all above, we can get

\[ \leq C \left( \frac{k}{r} \right)^{N + \frac{2}{N - 2} - \tau} \| \phi_{r, h, \Lambda} \|_* \int_{\Omega_1} \frac{r}{(1 + |y - x|)^{N + \frac{2}{N - 2} + \tau}} \left( \sum_{j=1}^k \left[ \frac{1}{(1 + |y - x_j|)^{N + \frac{2}{N - 2} + \tau}} + \frac{1}{(1 + |y - x_j|)^{N + \frac{2}{N - 2} + \tau}} \right] \right) \]

\[ \leq C r \left( \frac{k}{r} \right)^{N + \frac{2}{N - 2} - \tau} \| \phi_{r, h, \Lambda} \|_* \leq C r \max \left\{ \frac{1}{k^{(N + \frac{2}{N - 2})/(N + 2 - 2N - 2m - 2e_1)}}, \frac{1}{k^{(N - 2 - m) \min\{2m,m+3\}}} \right\} \]

And it’s easy to show that

\[ \int_{\mathbb{R}^N} K \left( \frac{|y|}{r} \right) - 1 \right] U_{2r_1 \Lambda}^2 \frac{1}{\Omega_2(\phi_{r, h, \Lambda}} \]

\[ \leq C r \max \left\{ \frac{1}{k^{(N + \frac{2}{N - 2})/(N + 2 - 2N - 2m - 2e_1)}}, \frac{1}{k^{(N - 2 - m) \min\{2m,m+3\}}} \right\} \].

Combining all above, we can get

\[ \frac{\partial F(r, h, \Lambda)}{\partial h} = \frac{\partial I(W_{r,h,\Lambda})}{\partial h} + kO(\max \left\{ \frac{1}{k^{(N + \frac{2}{N - 2})/(N + 2 - 2N - 2m - 2e_1)}}, \frac{1}{k^{(N - 2 - m) \min\{2m,m+3\}}} \right\}). \quad (3.6) \]

Combining (3.6), Proposition A.6 and Lemma B.6 we can get (3.2) \hfill \square

**Remark 3.5.** The expansions of \( \frac{\partial F(r, h, \Lambda)}{\partial h} \) and \( \frac{\partial F(r, h, \Lambda)}{\partial \Lambda} \) would be applied in the proof of Proposition 3.6, which is essential for proving the existence critical point of \( F(r, h, \Lambda) \). In order to get a proper expansion of \( \frac{\partial F(r, h, \Lambda)}{\partial h} \), we need accurate estimates for \( \phi_{r, h, \Lambda} \). \hfill \square

**Rewritten the expansion of the energy functional.**

Let \( \Lambda_0 \) be

\[ \Lambda_0 = \left[ \frac{(N - 2)B_4}{A_2m} \right]^{1 \over N - 2} \]. \quad (3.7)

Then it solves

\[ \frac{B_4(N - 2)}{\Lambda^{N - 1}} - \frac{A_2m}{\Lambda^{m + 1}} = 0. \]

Denote

\[ G(h) := \frac{B_4k^{N-2}}{(\sqrt{1 - h^2})^{N-2}} + \frac{B_5k}{h^{N-3}\sqrt{1 - h^2}} \]

then

\[ G'(h) = (N - 2) \frac{B_4k^{N-2}h}{(\sqrt{1 - h^2})^{N-2}} - (N - 3) \frac{B_5k}{h^{N-2}\sqrt{1 - h^2}} + h \frac{B_5k}{h^{N-4}(1 - h^2)\sqrt{1 - h^2}} \]

\[ = (N - 2)B_4k^{N-2}h[1 + O(h^2)] \]

\[ - (N - 3) \frac{B_5k}{h^{N-2}}[1 + O(h^2)] + \frac{B_5k}{h^{N-4}}[1 + O(h^2)] \]

\[ = \left[ (N - 2)B_4k^{N-2}h - (N - 3) \frac{B_5k}{h^{N-2}} \right] + O\left( \frac{k}{h^{N-4}} \right) \]
\[ G''(h) = (N - 2) \frac{B_4 k^{N-2}}{(\sqrt{1 - h^2})^N} + (N - 2)N \frac{B_4 k^{N-2} h^2}{(\sqrt{1 - h^2})^{N+2}} \]
\[ \quad + (N - 3)(N - 2) \frac{B_5 k}{h^{N-1} \sqrt{1 - h^2}} - (N - 3) \frac{B_5 k}{h^{N-3} (1 - h^2) \frac{3}{2}} \]
\[ \quad - (N - 4) \frac{B_5 k}{h^{N-3} (1 - h^2) \frac{3}{2}} + \frac{3B_5 k}{h^{N-5} (1 - h^2) \frac{5}{2}} \]
\[ = (N - 2)B_4 k^{N-2} + (N - 3)(N - 2) \frac{B_5 k}{h^{N-1}} + O(h^2 k^{N-2}) + O\left(\frac{k}{h^{N-3}}\right), \quad (3.8) \]

and
\[ G'''(h) = O\left(\frac{k}{h^N}\right). \quad (3.9) \]

Let \( h \) be a solution of
\[ \left[(N - 2)B_4 k^{N-2} h - (N - 3) \frac{B_5 k}{h^{N-2}} \right] = 0, \]
then
\[ h = \frac{B'}{k^{N-1}}, \quad \text{with} \quad B' = \left[\frac{(N - 3)B_5}{(N - 2)B_4}\right] \frac{1}{\theta^2}. \quad (3.10) \]

Define
\[ S_k = \left\{(r, h, \Lambda) \mid r \in [k^{N-2-m} - \frac{1}{k^\theta}, k^{N-2-m} + \frac{1}{k^\theta}], \quad \Lambda \in [\Lambda_0 - \frac{1}{k^m}, \Lambda_0 + \frac{1}{k^m}] \right\}, \]
\[ h \in \left[\frac{B'}{k^{N-1}} \left(1 - \frac{1}{k^\theta}\right), \frac{B'}{k^{N-1}} \left(1 + \frac{1}{k^\theta}\right)\right], \]
for \( \bar{\theta} \) is a small constant such that \( \bar{\theta} \leq \frac{\sigma}{100} \). In fact, \( S_k \) is a subset of \( \mathcal{X}_k \). We will find a critical point of \( F(r, h, \Lambda) \) in \( S_k \).

A direct Taylor expansion gives that
\[ G(h) = G(h) + G'(h)(h - h) + \frac{1}{2}G''(h)(h - h)^2 + O\left(G'''(h + (1 - \iota)h\right))(h - h)^3, \quad (3.11) \]
where
\[ G(h) = B_4 k^{N-2} \left[ 1 + \frac{N - 2}{2} h^2 + O(h^4) \right] + \frac{B_5 k}{h^{N-3}} \left[ 1 + \frac{1}{2} h^2 + O(h^4) \right], \]
\[ G'(h) = O\left(\frac{k}{h^{N-1}}\right) \]
and
\[ G''(h) = \frac{(N - 2)}{2} \left[ B_4 k^{N-2} + (N - 3) \frac{B_5 k}{h^{N-1}} \right] + O(h^2 k^{N-2}). \]

Since \( G(h), G''(h) \) are independent of \( h, r, \Lambda \), for simplicity, in the following, we will denote
\[ G(h) = B_4 k^{N-2} + \frac{(N - 2)B_4 k^{N-2}}{2} h^2 + \frac{B_5 k}{h^{N-3}}, \quad (3.12) \]
\[ G''(h) = \frac{(N - 2)}{2} \left[ B_4 k^{N-2} + (N - 3) \frac{B_5 k}{h^{N-1}} \right]. \quad (3.13) \]
Therefore, we get
$$G(h) = B_4 k^{N-2} + \frac{(N-2)B_4}{2} k^{N-2} h^2 + \frac{B_5 k}{h^{N-3}} + O\left(\frac{k}{h^{N-4}}\right)(h - h)$$
$$+ \frac{(N-2)}{2} \left[ B_4 k^{N-2} + (N-3)B_5 \right] (h - h) + O\left(\frac{k}{h^{N}}\right)(h - h)^3.$$ 

Therefore, we get
$$G(h) = B_4 k^{N-2} + \frac{(N-2)B_4 B'^2}{2} + \frac{B_5}{B'^{N-3}} \frac{k^{N-2}}{k^{2(N-3)}}$$
$$+ \frac{(N-2)}{2} \left[ B_4 B'^2 + \frac{(N-3)B_5}{B'^{N-3}} \right] \frac{k^{N-2}}{k^{2(N-3)}} (1 - h^{-1}h)^2 + O\left(\frac{k^{N-2}}{k^{2(N-3)}}\right)(1 - h^{-1}h)^3$$
$$= B_4 k^{N-2} + B_6 \frac{k^{N-2}}{k^{N-3}} + B_7 \frac{k^{N-2}}{k^{N-3}} (1 - h^{-1}h)^2 + O\left(\frac{k^{N-2}}{k^{N-3}}\right)(1 - h^{-1}h)^3, \quad (3.14)$$

where
$$B_6 = \frac{(N-2)B_4 B'^2}{2} + \frac{B_5}{B'^{N-3}}, \quad B_7 = \frac{(N-2)}{2} \left[ B_4 B'^2 + \frac{(N-3)B_5}{B'^{N-3}} \right].$$

Since
$$r \in \left[ k^{\frac{N-2}{2-m}} - \frac{1}{k^\theta}, \quad k^{\frac{N-2}{2-m}} + \frac{1}{k^\theta} \right],$$
then
$$r^{N-2} = k^{\frac{(N-2)^2}{2-m}} \left(1 + \frac{C(r, \Lambda)}{k^{\frac{(N-2)}{2-m} + \theta}}\right).$$

We now rewrite
$$\frac{B_4 k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}$$
$$= \frac{B_4}{k^{N-2} k^{N-3} + 2(N-3)} + \frac{B_6}{k^{(N-2)m} + 2(N-3)} + \frac{C(r, \Lambda)}{k^{(N-2)m} + \sigma}$$
$$+ \frac{B_7}{k^{(N-2)m} + \frac{1}{2(N-3)}} (1 - h^{-1}h)^2 + O\left(\frac{k^{(N-2)m}}{k^{(N-3)m} + \frac{2(N-3)}{N-1}}\right)(1 - h^{-1}h)^3.$$

Then we can express $F(r, h, \Lambda)$ as
$$F(r, h, \Lambda) = kA_1 - k\left[ \frac{B_4}{\Lambda^{N-2} k^{N-2} m} + \frac{B_6}{\Lambda^{N-2} k^{N-2} m + \frac{2(N-3)}{N-1}} \right]$$
$$+ \frac{B_7}{\Lambda^{N-2} k^{N-2} m + \frac{1}{2(N-3)}} (1 - h^{-1}h)^2$$
$$+ k\left[ \frac{A_2}{\Lambda^{N-2} k^{N-2} m} + \frac{A_3}{\Lambda^{N-2} k^{N-2} m} (r - r)^2 \right] + k\frac{C(r, \Lambda)}{\Lambda^{N-2} k^{N-2} m} (r - r)^{2+\sigma}$$
$$+ kO\left(\frac{1}{k^{\frac{m(N-2)}{N-2} m + \frac{2(N-3)}{N-1} + \sigma}}\right)(1 - h^{-1}h)^3 + kO\left(\frac{1}{k^{\frac{m(N-2)}{N-2} m + \frac{2(N-3)}{N-1} + \sigma}}\right). \quad (3.15)$$
And similarly, we have
\[ \frac{\partial F(r, h, \Lambda)}{\partial \Lambda} = k \left[ \frac{(N-2)B_1}{\Lambda^{N-1} k^{\frac{(N-2)m}{N-2-m}}} - \frac{mA_2}{\Lambda^{m+1} k^{\frac{(N-2)m}{N-2-m}}} \right] + \frac{(m-2)A_3}{\Lambda^{m-1} k^{\frac{(N-2)m}{N-2-m}}} (r - r)^2 + kO \left( \frac{1}{k^{\frac{(N-2)m}{N-2-m}}} (r - r)^2 + \sigma \right); \]
and from (3.2), by using some calculations, we have
\[ \frac{\partial F(r, h, \Lambda)}{\partial h} = \frac{k}{\Lambda^{N-2}} \left[ \frac{2B_7}{\Lambda^{N-2} k^{\frac{(N-2)m}{N-2-m} + \frac{(N-3)}{N-1}}} (1 - h^{-1} h) \right] + kO \left( \frac{1}{k^{\frac{(N-2)m}{N-2-m} + \frac{(N-3)}{N-1}}} (1 - h^{-1} h)^2 \right) + kO \left( \frac{1}{k^{\frac{(N-2)m}{N-2-m} + \frac{(N-3)}{N-1} + \sigma}} \right), \quad (3.16) \]
Now define
\[ \bar{F}(r, h, \Lambda) = -F(r, h, \Lambda), \quad (3.17) \]
and
\[ t_2 = k(-A_1 + \eta_1), \quad t_1 = k \left( -A_1 - \frac{A_2}{\Lambda_0^2} - \frac{B_4}{A_0^{N-2}} - \frac{1}{k^{\frac{(N-2)m}{N-2-m} + \frac{(N-3)}{N-1}}} \right), \]
where \( \eta_1 > 0 \) small. We also define the energy level set
\[ \bar{F}^t = \left\{ (r, h, \Lambda) \mid (r, h, \Lambda) \in S_k, \bar{F}(r, h, \Lambda) \leq t \right\}. \]
We consider the following gradient flow system
\[
\begin{align*}
\frac{dr}{dt} &= -\bar{F}_r, \quad t > 0; \\
\frac{dh}{dt} &= -\bar{F}_h, \quad t > 0; \\
\frac{d\Lambda}{dt} &= -\bar{F}_\Lambda, \quad t > 0; \\
(r, h, \Lambda)|_{t=0} &\in \bar{F}^{t_2}.
\end{align*}
\]
The next proposition would play an important role in the proof of Theorem 1.1

**Proposition 3.6.** The flow would not leave \( S_k \) before it reaches \( \bar{F}^{t_1} \).

**Proof.** There are three positions that the flow tends to leave \( S_k \):

**position 1.** \( |r - r| = \frac{1}{k^\sigma} \) and \( |1 - h^{-1} h| \leq \frac{1}{k^\sigma}, \quad |\Lambda - \Lambda_0| \leq \frac{1}{k^{\frac{(N-2)m}{N-2-m}}}; \)

**position 2.** \( |1 - h^{-1} h| = \frac{1}{k^\sigma} \) when \( |r - r| \leq \frac{1}{k^\sigma}, \quad |\Lambda - \Lambda_0| \leq \frac{1}{k^{\frac{(N-2)m}{N-2-m}}}; \)

**position 3.** \( |\Lambda - \Lambda_0| = \frac{1}{k^{\frac{(N-2)m}{N-2-m}}} \) when \( |r - r| \leq \frac{1}{k^\sigma}, \quad |1 - h^{-1} h| \leq \frac{1}{k^\sigma}. \)

\[ \blacklozenge \] We now consider **position 1.** Since \( |\Lambda - \Lambda_0| \leq \frac{1}{k^{\frac{(N-2)m}{N-2-m}}}, \) it is easy to derive that
\[
\left( \frac{B_4}{A_0^{N-2}} - \frac{A_2}{A_0^m} \right) = \left( \frac{B_4}{A_0^{N-2}} - \frac{A_2}{A_0^m} \right) + O\left( |\Lambda - \Lambda_0|^2 \right) + O\left( \frac{1}{k^{3\sigma}} \right). \quad (3.18)
\]
Combining (3.15), (3.17), (3.18), we can obtain that, if \((r, h, \Lambda)\) lies in position 1,
\[
\bar{F}(r, h, \Lambda) = -kA_1 + k\left(\frac{B_4}{\Lambda_0^{N-2}k^{N-2-m}} - \frac{A_k}{\Lambda_0 \Lambda_0^{m-2}k^{N-2-m}}\right)
\]
\[
- k\frac{A_3}{\Lambda_0^{m-2}k^{N-2-m} + 2\theta} + O\left(\frac{1}{k^{N-2-m} + 2\theta}\right) < t_1.
\]

\(\blacklozenge\) On the other hand, we claim that it’s impossible for the flow \((r(t), h(t), \Lambda(t))\) leaves \(S_k\) when it lies in position 2. If \(1 - h^{-1}h = \frac{1}{k^{2\theta}}\), then from (3.16) and (3.17), we have
\[
\frac{\partial \bar{F}(r, h, \Lambda)}{\partial h} = -k\frac{2B_7}{\Lambda^{N-2}}\left(\frac{2B_7}{\Lambda^{N-2}k^{N-2-m} + 2\theta}\right) + O\left(\frac{1}{k^{N-2-m} + 2\theta}\right) < 0. \tag{3.19}
\]
On the other hand, if \(1 - h^{-1}h = -\frac{1}{k^{2\theta}}\),
\[
\frac{\partial \bar{F}(r, h, \Lambda)}{\partial h} = k\frac{2B_7}{\Lambda^{N-2}}\left(\frac{2B_7}{\Lambda^{N-2}k^{N-2-m} + 2\theta}\right) + O\left(\frac{1}{k^{N-2-m} + 2\theta}\right) > 0. \tag{3.20}
\]
So it’s impossible for the flow leaves \(S_k\) when it lies in position 2.

\(\blacklozenge\) Finally, we consider position 3. If \(\Lambda = \Lambda_0 + \frac{1}{k^{2\theta}}\), from (3.1) and (3.17), there exists a constant \(C_1\) such that
\[
\frac{\partial \bar{F}(r, h, \Lambda)}{\partial \Lambda} = k\left[C_1\left(\frac{1}{k^{N-2-m} + 2\theta}\right) + O\left(\frac{1}{k^{N-2-m} + 2\theta}\right)\right] > 0.
\]
On the other hand, if \(\Lambda = \Lambda_0 - \frac{1}{k^{2\theta}}\), there exists a constant \(C_2\) such that
\[
\frac{\partial \bar{F}(r, h, \Lambda)}{\partial \Lambda} = k\left[-C_2\left(\frac{1}{k^{N-2-m} + 2\theta}\right) + O\left(\frac{1}{k^{N-2-m} + 2\theta}\right)\right] < 0.
\]
Hence the flow \((r(t), h(t), \Lambda(t))\) does not leave \(S_k\) when \(|\Lambda - \Lambda_0| = \frac{1}{k^{2\theta}}\).

Combining above results, we conclude that the flow would not leave \(S_k\) before it reach \(\bar{F}^{t_1}\). \(\square\)

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1** According to Proposition 3.1, in order to show Theorem 1.1, we only need to show that function \(\bar{F}(r, h, \Lambda)\), and thus \(F(r, h, \Lambda)\), has a critical point in \(S_k\).

Define
\[
\Gamma = \left\{ \gamma : \gamma(r, h, \Lambda) = (\gamma_1(r, h, \Lambda), \gamma_2(r, h, \Lambda), \gamma_3(r, h, \Lambda)) \in S_k, (r, h, \Lambda) \in S_k; \gamma(r, h, \Lambda) = (r, h, \Lambda), \, \text{if} \, |r - r| = \frac{1}{k^{2\theta}} \right\}.
\]
Let
\[
c = \inf_{\gamma \in \Gamma} \max_{(r, h, \Lambda) \in S_k} \bar{F}(\gamma(r, h, \Lambda)).
\]
We claim that \(c\) is a critical value of \(\bar{F}(r, h, \Lambda)\) and can be achieved by some \((r, h, \Lambda) \in S_k\). By the minmax theory, we need to show that
\[
(i) \quad t_1 < c < t_2;
\]
\[
(ii) \quad \sup_{|r - r'| = \frac{1}{k^{2\theta}}} \bar{F}(\gamma(r, h, \Lambda)) < t_1, \, \forall \gamma \in \Gamma.
\]
Using the results in Proposition 3.6 we can prove (i) and (ii) easily.

Finally, for every \(k\) large enough, we get the critical point \((r_k, h_k, \Lambda_k)\) of \(F(r, h, \Lambda)\). \(\square\)
Appendix A. Expansions for the Energy Functional

This section is devoted to the computation of the expansion for the energy functional $I(W_r,h,\Lambda)$. We first give the following Lemma.

**Lemma A.1.** $N \geq 5$ and $(r,h,\Lambda) \in \mathcal{I}_k$. We have the following expansions for $k \to \infty$:

\[
\sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^{N-2}} = \frac{k^{N-2}}{(r\sqrt{1 - h^2})^{N-2}} (B_1 + \sigma_1(k)),
\]

\[
\sum_{i=1}^{k} \frac{1}{|x_1 - x_i|^{N-2}} = \frac{B_2 k}{r^{N-2}h^{N-3}\sqrt{1 - h^2}} \left(1 + \sigma_2(k)\right) + \frac{\sigma_1(k)k^{N-2}}{(r\sqrt{1 - h^2})^{N-2}},
\]

where

\[
B_1 = \frac{2}{(2\pi)^{N-2}} \sum_{i=1}^{\infty} \frac{1}{i^{N-2}}, \quad B_2 = \frac{1}{2^{N-3}\pi} \int_{0}^{+\infty} \frac{1}{(s^2 + 1)^{N/2}} ds,
\]

and

\[
\sigma_1(k) = \begin{cases} O\left(\frac{1}{k^2}\right), & N \geq 6, \\ O\left(\ln \frac{k}{h}\right), & N = 5, \end{cases} \quad \sigma_2(k) = O((hk)^{-1}).
\]

**Proof.** In fact, for $\frac{1}{2} < c_3 \leq c_4 \leq 1$, we have

\[
c_3 \frac{i\pi}{k} \leq \sin \frac{i\pi}{k} \leq c_4 \frac{i\pi}{k}, \quad \text{for} \ i \in \{1, \ldots, \frac{k}{2}\}.
\]

Without loss of generality, we can assume $k$ is even. It is easy to derive that

\[
\sum_{i=2}^{k} \frac{1}{|x_1 - x_i|^{N-2}} = \sum_{i=1}^{k} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2}
\]

\[
= \sum_{i=1}^{k} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2} + \sum_{i=\frac{k}{2}+1}^{k} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2}.
\]

Direct computations show that

\[
\sum_{i=1}^{\frac{k}{2}} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2} \]

\[
= \sum_{i=1}^{\frac{k}{2}} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2} + \sum_{i=\frac{k}{2}+1}^{\frac{k}{2}} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2}
\]

\[
= \sum_{i=1}^{\frac{k}{2}} \left(\frac{1}{2r\sqrt{1 - h^2}\sin \frac{i\pi}{k}}\right)^{N-2} \left(1 + O\left(\frac{k^2}{k^2}\right)\right) + O\left(\frac{k}{(2r\sqrt{1 - h^2})^{N-2}}\right)
\]

\[
= \left(\frac{k}{r\sqrt{1 - h^2}}\right)^{N-2} \left(D_1 + \sigma_1(k)\right),
\]

(A.5)
where $D_1 = \frac{1}{2\pi r^2} \sum_{i=1}^{\infty} \frac{1}{i^{\gamma/2}}$ and $\sigma_1(k)$ is defined in (A.4). Using symmetry of function $\sin x$, we can easily show
\[
\sum_{i=\frac{b}{2}+1}^{k} \left(\frac{1}{2r\sqrt{1-h^2}\sin \frac{\pi i}{k}}\right)^{N-2} = \left(\frac{k}{r\sqrt{1-h^2}}\right)^{N-2} \left(D_1 + \sigma_1(k)\right).
\]
Thus we proved (A.1).

Similarly, we can obtain
\[
\sum_{i=1}^{k} \frac{1}{|x_1 - x_2|^{N-2}} = \sum_{i=1}^{k} \frac{1}{\left(2r\left[1-h^2\right]\sin^2 \left(\frac{(i-1)\pi}{k}\right) + h^2\right)^{N/2}}.
\]
Consider $O((hk)^{-1}) = o(1)$ as $k \to \infty$. Since
\[
\sum_{j=1}^{k} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{N/2}} \geq \int_{0}^{\frac{1}{2}} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{N/2}} \, dx
\]
\[
\geq \int_{0}^{2} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{N/2}} \, dx + \sum_{j=4}^{k+1} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{N/2}},
\]
then we have
\[
\sum_{j=1}^{k} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{N/2}} = \int_{0}^{\frac{1}{2}} \frac{1}{\left(\frac{(1-h^2)(j-1)^2\pi^2}{k^2} + 1\right)^{\frac{N-2}{2}}} \, dx + 1 + o(1)
\]
\[
= \frac{hk}{\sqrt{1-h^2\pi}} \int_{0}^{(1-h^2)\pi} \frac{1}{(s^2 + 1)^{\frac{N-2}{2}}} \, ds + 1 + o(1)
\]
\[
= \frac{hk}{\sqrt{1-h^2\pi}} \int_{0}^{\infty} \frac{1}{(s^2 + 1)^{\frac{N-2}{2}}} \, ds \left(1 + O((hk)^{-1})\right).
\]
Combining above calculations, we can obtain that
\[
\sum_{i=1}^{k} \frac{1}{|x_1 - x_2|^{N-2}} = \frac{1}{(rh)^{N-2}} \sqrt{1-h^2} \left(B_2 h k \left(1 + \sigma_2(k)\right) + O\left(\frac{\sigma_1(k)k^{N-2}}{(r\sqrt{1-h^2})^{N-2}}\right)\right),
\]
where $B_2$ and $\sigma_2$ are defined in (A.3), (A.4).
**Lemma A.2.** We have the expansion, for \( k \to \infty \)

\[
\int_{\mathbb{R}^N} U_{x_i, \Lambda}^{2^*} U_{x_j, \Lambda} = \frac{B_0}{\Lambda^{N-2}|x_1 - x_j|^{N-2}} + O\left(\frac{1}{|x_1 - x_j|^{N-\epsilon_0}}\right),
\]

and

\[
\int_{\mathbb{R}^N} U_{x_i, \Lambda}^{2^*} U_{x_j, \Lambda} = \frac{B_0}{\Lambda^{N-2}|x_1 - x_j|^{N-2}} + O\left(\frac{1}{|x_1 - x_j|^{N-\epsilon_0}}\right),
\]

where \( B_0 = \int_{\mathbb{R}^N} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \) and \( \epsilon_0 \) is constant small enough.

**Proof.** Let \( d_j = |x_1 - x_j|, d_j = |x_1 - x_j| \) for \( j = 1, \ldots, k \). We consider

\[
\int_{\mathbb{R}^N} U_{x_i, \Lambda}^{2^*} U_{x_j, \Lambda} = \int_{\mathbb{R}^N} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{N-2}}{(1 + \Lambda^2|y - x_i|^2)^{\frac{N-2}{2}}}
\]

\[
\left\{ \begin{array}{c}
\int_{B_{\frac{d_j}{\Lambda}}(x_i)} + \int_{\mathbb{R}^N \setminus B_{\frac{d_j}{\Lambda}}(x_i)} \end{array} \right\} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{N-2}}{(1 + \Lambda^2|y - x_i|^2)^{\frac{N-2}{2}}},
\]

(A.6)

First, we have

\[
\int_{B_{\frac{d_j}{\Lambda}}(x_i)} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{N-2}}{(1 + \Lambda^2|y - x_i|^2)^{\frac{N-2}{2}}}
\]

\[
= \int_{B_{\frac{d_j}{\Lambda}}(x)} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \frac{1}{(1 + z^2 + 2\Lambda(z, x_1 - x_i) + \Lambda^2|x_1 - x_i|^2)^{\frac{N-2}{2}}}
\]

\[
= \frac{1}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} \int_{B_{\frac{d_j}{\Lambda}}(0)} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \left(1 - \frac{N - 2 + z^2 + 2\Lambda(z, x_1 - x_i)}{2 \Lambda^2|x_1 - x_i|^2}\right)
\]

\[+ O\left(\frac{(1 + z^2 + 2\Lambda(z, x_1 - x_i))^2}{\Lambda^2|x_1 - x_i|^2}\right).\]

It is easy to check that

\[
\frac{1}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} O\left(\int_{B_{\frac{d_j}{\Lambda}}(0)} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \left(1 + z^2 + 2\Lambda(z, x_1 - x_i)\right)^2\right) = O\left(\frac{1}{|x_1 - x_i|^N}\right),
\]

(A.8)

and

\[
\frac{1}{\Lambda^N|x_1 - x_i|^N} \int_{B_{\frac{d_j}{\Lambda}}(0)} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \left(1 + z^2 + 2\Lambda(z, x_1 - x_i)\right) = O\left(\frac{1}{|x_1 - x_i|^{N-\epsilon_0}}\right).
\]

(A.9)

Standard calculation implies that

\[
\frac{1}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} \int_{B_{\frac{d_j}{\Lambda}}(0)} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} = \frac{B_0}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} + O\left(\frac{1}{|x_1 - x_i|^N}\right),
\]

(A.10)

where \( B_0 = \int_{\mathbb{R}^N} \frac{1}{(1 + z^2)^{\frac{N+2}{2}}} \).
From (A.7)–(A.10), we get
\[
\int_{B_{\frac{3}{4}}(x_1)} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_i|^2)^{\frac{N+2}{2}}}
= \frac{B_0}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} + O\left(\frac{1}{|x_1 - x_i|^{N-\epsilon_0}}\right).
\] (A.11)

When \(y \in \mathbb{R}^N \setminus B_{\frac{3}{4}}(x_1)\), there holds
\[|y - x_1| \geq \frac{1}{4}|x_1 - x_i|.
\]

It’s easy to get
\[
\int_{\mathbb{R}^N \setminus B_{\frac{3}{4}}(x_1)} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{N+2}}{(1 + \Lambda^2|y - x_i|^2)^{\frac{N+2}{2}}} = O\left(\frac{1}{|x_1 - x_i|^{N-\epsilon_0}}\right).
\] (A.12)

Combining (A.6), (A.11) and (A.12), we can get
\[
\int_{\mathbb{R}^N} U_{x_i,\Lambda}^{2^* - 1} U_{x_i,\Lambda} = \frac{B_0}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} + O\left(\frac{1}{|x_1 - x_i|^{N-\epsilon_0}}\right).
\] (A.13)

Similarly, we can get
\[
\int_{\mathbb{R}^N} U_{x_j,\Lambda}^{2^* - 1} U_{x_j,\Lambda} = \frac{B_0}{\Lambda^{N-2}|x_1 - x_i|^{N-2}} + O\left(\frac{1}{|x_1 - x_i|^{N-\epsilon_0}}\right),
\]
for \(i = 1, \ldots, k\).

Lemma A.3. Suppose that \(K(|y|)\) satisfies H and \(N \geq 5\), \((r, h, \Lambda) \in \mathcal{S}_k\). We have the expansion for \(k \to \infty\)
\[
I(W_{r, h, \Lambda}) = kA_1 - k \int_{\mathbb{R}^N} U_{x_1,\Lambda}^{2^* - 1}\left(\sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda}\right)
+ k\left[\frac{A_2}{\Lambda^{m_{x_1}}} + \frac{A_3}{\Lambda^{m_{x_2}}} (r - r)^2\right] + k\frac{C(r, \Lambda)}{r^m (r - r)^{2+\sigma}}
+ k\frac{C(r, \Lambda)}{r^{m+\sigma}} + kO\left(\frac{k}{r}\right)^{2+\sigma} + kO\left(\frac{1}{r}\right)^{N-2},
\]
where \(C(r, \Lambda)\) denotes function independent of \(h\) and should be order of \(O(1)\),
\[
A_1 = \left(1 - \frac{2}{2^*}\right) \int_{\mathbb{R}^N} |U_{0,1}|^{2^*}, \quad A_2 = \frac{2c_0}{2^*} \int_{\mathbb{R}^N} |y_1|^{m} U_{0,1}^{2^*},
\] (A.14)
\[
A_3 = \frac{c_0(m - 1)}{2^*} \int_{\mathbb{R}^N} |y_1|^{m - 2} U_{0,1}^{2^*},
\] (A.15)
and \(\epsilon_0\) is constant can be chosen small enough.

Proof. Recalling the definition of \(I(u)\) as in (1.16), then we obtain that
\[
I(W_{r, h, \Lambda}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla W_{r, h, \Lambda}|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} K\left(\frac{|y|}{r}\right) W_{r, h, \Lambda}^{2^*}
:= I_1 - I_2.
\] (A.16)
According to the expression of $W_{r,h,:}$, we have

$$I_1 = \frac{1}{2} \sum_{j=1}^{k} \sum_{i=1}^{k} \int_{\mathbb{R}^N} -\Delta (U_{\tau_j,:} + U_{\xi_i,:}) (U_{\tau_i,:} + U_{\xi_i,:})$$

$$= k \sum_{j=1}^{k} \int_{\mathbb{R}^N} \left( U_{2^*_{\tau_j,:} - 1} U_{\tau_j,:} + U_{2^*_{\xi_i,:} - 1} U_{\xi_i,:} \right)$$

$$= k \int_{\mathbb{R}^N} U_{2^*_{\tau,:}} + k \int_{\mathbb{R}^N} \sum_{j=2}^{k} \sum_{j=1}^{k} U_{2^*_{\tau,:}} + \sum_{j=1}^{k} U_{\xi,:},$$

$$\leq \int_{\mathbb{R}^N} \left( U_{2^*_{\tau,:}} + \sum_{j=2}^{k} \sum_{j=1}^{k} U_{\tau,:} + \sum_{j=1}^{k} U_{\xi,:} \right) \equiv O \left( \int_{\mathbb{R}^N} \right) \equiv O \left( \left( \frac{k}{r} \right)^{N-2} \right) \equiv 0 \left( \left( \frac{k}{r} \right)^{N-2} \right).$$

For $y \in \Omega_1^+$, from Lemma [B.1] we have

$$\left( \sum_{j=2}^{k} \sum_{j=1}^{k} U_{\tau,:} + \sum_{j=1}^{k} U_{\xi,:} \right) \leq \frac{C}{\left( 1 + |y - \tau| \right)} \left( \frac{k}{r} \right)^{N-2},$$

with $\epsilon_0 > 0$ can be chosen small enough. Then we can get

$$I_{23} = O \left( \int_{\Omega_1^+} \right) \equiv O \left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right). \quad (A.19)$$

For $I_{21}$, we can rewrite it as following

$$I_{21} = \int_{\Omega_1^+} U_{2^*_{\tau,:} - 1} - \left[ K \left( \frac{|y|}{r} \right) - 1 \right] U_{2^*_{\tau,:}}$$

$$\equiv \int_{\mathbb{R}^N} U_{2^*_{\tau,:}} + \int_{\Omega_1^+} \left[ K \left( \frac{|y|}{r} \right) - 1 \right] U_{2^*_{\tau,:}} + O \left( \left( \frac{k}{r} \right)^{N} \right).$$

Furthermore, we obtain

$$\int_{\Omega_1^+} \left[ K \left( \frac{|y|}{r} \right) - 1 \right] U_{2^*_{\tau,:}} = \int_{\Omega_1^+ \cap \{ y : |y - \tau| \geq \delta \}} \left[ K \left( \frac{|y|}{r} \right) - 1 \right] U_{2^*_{\tau,:}}$$
\[
\int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} \left[ K\left(\frac{|y|}{r}\right) - 1 \right] U_\Omega^{2*}_{1,A}.
\]

When \(|\frac{|y|}{r} - 1| \geq \delta\), there holds
\[
|y - \overline{x}_1| \geq |y| - |r| - |r - \overline{x}_1| \geq \frac{1}{2} \delta r.
\]

Thus we can easily get
\[
\int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} \left[ K\left(\frac{|y|}{r}\right) - 1 \right] U_\Omega^{2*}_{1,A} \leq \frac{C}{r^{N-\epsilon_0}}.
\]

If \(|\frac{|y|}{r} - 1| \leq \delta\), recalling the decay property of \(K\), we can obtain that
\[
\int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} \left[ K\left(\frac{|y|}{r}\right) - 1 \right] U_\Omega^{2*}_{1,A}
= -c_0 \frac{1}{r} \int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} |y| - |r|^m U_\Omega^{2*}_{1,A}
+ O\left(\frac{1}{r^{m+\sigma}} \int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} |y| - |r|^{m+\sigma} U_\Omega^{2*}_{1,A}\right)
= -c_0 \frac{1}{r} \int_{\mathbb{R}^N} |y - r|^m U\Omega_{1,A}^{2*}
+ O\left(\frac{1}{r^{m+\sigma}} \int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} |y| - |r|^{m+\sigma} U_\Omega^{2*}_{1,A}\right)
= -c_0 \frac{1}{r} \int_{\mathbb{R}^N} |y + \overline{x}_1 - r|^m U\Omega_{1,A}^{2*}
+ O\left(\frac{1}{r^{m+\sigma}} \int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} |y| - |r|^{m+\sigma} U_\Omega^{2*}_{1,A}\right) + O\left(\left(\frac{k}{r}\right)^{N-\epsilon_0}\right).
\]

Furthermore, recalling \(|\overline{x}_1| = r\) and using the symmetry property, we have
\[
\int_{\mathbb{R}^N} |y + \overline{x}_1 - r|^m U\Omega_{0,A}^{2*} = \int_{\mathbb{R}^N} |y + e_1 r - r|^m U\Omega_{0,A}^{2*},
\]
where \(e_1 = (1, 0, \cdot, 0)\).

We get
\[
\int_{\mathbb{R}^N} |y + \overline{x}_1 - r|^m U\Omega_{0,A}^{2*}
= \int_{\mathbb{R}^N} |y|^m U\Omega_{0,A}^{2*} + \frac{1}{2} m (m - 1) \int_{\mathbb{R}^N} |y|^{m-2} U\Omega_{0,A}^{2*} (r - r)^2 + \mathcal{C}(r, \Lambda)(r - r)^{2+\sigma},
\]
here \(\mathcal{C}(r, \Lambda)\) denote functions which are independent of \(h\) and can be absorbed in \(O(1)\).

Similarly, we can also have the following expression
\[
O\left(\frac{1}{r^{m+\sigma}} \int_{\Omega_1^+ \cap \{y: |\frac{y}{r} - 1| \leq \delta\}} |y| - |r|^{m+\sigma} U_\Omega^{2*}_{1,A}\right)
= O\left(\frac{1}{r^{m+\sigma}} \int_{\mathbb{R}^N} |y| - |r|^{m+\sigma} U_\Omega^{2*}_{1,A}\right) + O\left(\left(\frac{k}{r}\right)^{N-\epsilon_0}\right)
\]
\[ C(r, \Lambda) \frac{r^{m+\sigma}}{m} + O\left( \frac{k}{r} \right)^{N-\epsilon_0}. \]

Then, we can obtain that

\[ I_{21} = \int_{\mathbb{R}^N} |U_{0,1}|^{2^*} - \frac{c_0}{\Lambda^m r^m} \int_{\mathbb{R}^N} |y_1|^m U_{0,1}^{2^*} \]
\[ - \frac{1}{2} m (m-1) \frac{c_0}{\Lambda^{m-2} r^m} \int_{\mathbb{R}^N} |y_1|^{m-2} U_{0,1}^{2^*} (r-r)^2 \]
\[ + C(r, \Lambda) (r-r)^{2+\sigma} + C(r, \Lambda) + O\left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right). \tag{A.20} \]

Finally, we consider \( I_{22} \)

\[ I_{22} = 2^* \int_{\Omega^+_1} U_{\xi, \Lambda}^{2^*-1} \left( \sum_{j=2}^k U_{\xi, \Lambda} + \sum_{j=1}^k U_{\xi, \Lambda} \right) \]
\[ + 2^* \int_{\Omega^+_1} \left[ K \left( \frac{|y_1|}{r} \right) - 1 \right] U_{\xi, \Lambda}^{2^*-1} \left( \sum_{j=2}^k U_{\xi, \Lambda} + \sum_{j=1}^k U_{\xi, \Lambda} \right) \]
\[ = 2^* \int_{\mathbb{R}^N} U_{\xi, \Lambda}^{2^*-1} \left( \sum_{j=2}^k U_{\xi, \Lambda} + \sum_{j=1}^k U_{\xi, \Lambda} \right) - 2^* \int_{\mathbb{R}^N \setminus \Omega^+_1} U_{\xi, \Lambda}^{2^*-1} \left( \sum_{j=2}^k U_{\xi, \Lambda} + \sum_{j=1}^k U_{\xi, \Lambda} \right) \]
\[ + 2^* \int_{\Omega^+_1} \left[ K \left( \frac{|y_1|}{r} \right) - 1 \right] U_{\xi, \Lambda}^{2^*-1} \left( \sum_{j=2}^k U_{\xi, \Lambda} + \sum_{j=1}^k U_{\xi, \Lambda} \right) \]
\[ =: I_{221} + I_{222} + I_{223}. \]

For \( I_{222} \), it is easy to derive that

\[ \sum_{j=1}^k \int_{\mathbb{R}^N \setminus \Omega^+_1} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} = O\left( \frac{k^N}{r^N} \right). \]

Moreover, we know that

\[ \sum_{j=2}^k \int_{\mathbb{R}^N \setminus \Omega^+_1} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} \]
\[ = \sum_{j=2}^k \int_{(\mathbb{R}^N \setminus \Omega^+_1) \cap B_{\xi, 1/2}(\xi_1)} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} + \sum_{j=2}^k \int_{(\mathbb{R}^N \setminus \Omega^+_1) \setminus B_{\xi, 1/2}(\xi_1)} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} \]
\[ = \sum_{j=2}^k \int_{(\mathbb{R}^N \setminus \Omega^+_1) \cap B_{\xi, 1/2}(\xi_1)} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} + O\left( \sum_{j=2}^k \frac{1}{|\xi_1 - \xi_j|^{N-\epsilon_0}} \right) \]
\[ \leq C \sum_{j=2}^k \int_{B_{\xi, 1/2}(\xi_1) \setminus B_{\xi, 1/2}(\xi_1)} U_{\xi, \Lambda}^{2^*-1} U_{\xi, \Lambda} + O\left( \sum_{j=2}^k \frac{1}{|\xi_1 - \xi_j|^{N-\epsilon_0}} \right) \]
\[ = C \sum_{j=2}^k \frac{1}{|\xi_1 - \xi_j|^{N-2}} \int_{B_{\xi, 1/2}(\xi_1) \setminus B_{\xi, 1/2}(\xi_1)} \left( 1 + z^2 \right)^{N+2} + O\left( \sum_{j=2}^k \frac{1}{|\xi_1 - \xi_j|^{N-\epsilon_0}} \right) \]
\[
\leq C \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N-2}} O\left( \frac{1}{d_2^2} \right) + O\left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N-\epsilon_0}} \right) \\
= O \left( \frac{k^2}{r^2} \right) \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N-2}} + O\left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N-\epsilon_0}} \right) \\
= O \left( \frac{k^N}{r^N} \right) + O\left( \sum_{j=2}^{k} \frac{1}{|x_1 - x_j|^{N-\epsilon_0}} \right),
\]

where \( d_j = |x_1 - x_j| \) for \( j = 2, \ldots, k \) and \( d_2 = |x_1 - x_2| = 2r\sqrt{1-h^2}\sin \frac{\theta}{4} = O\left( \frac{r}{k} \right) \). Then we get
\[
I_{222} = O \left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right). \tag{A.21}
\]

Next, we consider the term \( I_{223} \). In fact, we have
\[
I_{223} = \int_{\Omega_1^+ \cap \{|y|/r - 1| \geq \delta\}} \left[ K\left( \frac{|y|}{r} \right) - 1 \right] U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)
+ \int_{\Omega_1^+ \cap \{|y|/r - 1| \leq \delta\}} \left[ K\left( \frac{|y|}{r} \right) - 1 \right] U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right).
\]

When \(|y/r - 1| \geq \delta\), there hold
\[
|y - x_1| \geq |y| - r - |r - x_1| \geq \frac{1}{2} \delta r.
\]

And for \( y \in \Omega_1^+ \) and \(|y/r - 1| \geq \delta\), we have
\[
\left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right) \leq C \left( \frac{k}{r} \right)^{\alpha} \frac{1}{(1 + |y - x_1|)^{N-2-\alpha}}, \tag{A.22}
\]

with \( \alpha = (\frac{N-2-m}{N-2}, \frac{N-2}{2}) \). Then we can get easily
\[
\int_{\Omega_1^+ \cap \{|y|/r - 1| \geq \delta\}} \left[ K\left( \frac{|y|}{r} \right) - 1 \right] U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)
\leq C \frac{C}{r^{N-\alpha-\epsilon_0}} \left( \frac{k}{r} \right)^{\alpha} \leq C \left( \frac{k}{r} \right)^{N}.
\]

If \(|y/r - 1| \leq \delta\), then
\[
\int_{\Omega_1^+ \cap \{|y|/r - 1| \leq \delta\}} \left[ K\left( \frac{|y|}{r} \right) - 1 \right] U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)
\leq C \frac{C}{r^{m}} \int_{\Omega_1^+ \cap \{|y|/r - 1| \leq \delta\}} |y| - r|^m U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)
\leq C \frac{C}{r^{m}} \int_{\Omega_1^+ \cap \{|y|/r - 1| \leq \delta\} \cap \{|y| - r \leq \frac{m}{2} r\}} |y| - r|^m U_{x_1,\Lambda}^{2^*-1} \left( \sum_{j=2}^{k} U_{x_j,\Lambda} + \sum_{j=1}^{k} U_{x_j,\Lambda} \right)
\]
\[ \frac{C}{r^m} \int_{\Omega_1 \cap \{ \frac{|y|}{r} - 1 \leq \delta \}} |y| - r |^m \left( \sum_{j=2}^{k} U_{\varphi_{j}, \Lambda} + \sum_{j=1}^{k} U_{\varphi_{j}, \Lambda} \right), \]

where \( \delta_1 \) is a small constant. If \( |y - \varphi_1| \leq \frac{\delta_1 r}{k} \), it is easy to derive

\[ |y| - r | \leq |y - \varphi_1| + |\varphi_1| - r | \leq \frac{\delta_2 r}{k}, \]

for some small \( \delta_2 \). Therefore,

\[ \frac{C}{r^m} |y| - r |^m \leq \frac{C}{k^m}. \]

Hence

\[ \frac{C}{r^m} \int_{\Omega_1 \cap \{ \frac{|y|}{r} - 1 \leq \delta \}} |y| - r |^m \left( \sum_{j=2}^{k} U_{\varphi_{j}, \Lambda} + \sum_{j=1}^{k} U_{\varphi_{j}, \Lambda} \right) \]

\[ \leq \frac{C}{k^m} \int_{\Omega_1} U_{\varphi_{j}, \Lambda} \left( \sum_{j=2}^{k} U_{\varphi_{j}, \Lambda} + \sum_{j=1}^{k} U_{\varphi_{j}, \Lambda} \right) \]

\[ \leq \frac{C}{k^m} \left( \frac{k}{r} \right)^{N-2}. \]

When \( |y - \varphi_1| \geq \frac{\delta_1 r}{k} \), combining (A.22), we can get easily,

\[ \frac{C}{r^m} \int_{\Omega_1 \cap \{ \frac{|y|}{r} - 1 \leq \delta \}} |y| - r |^m \left( \sum_{j=2}^{k} U_{\varphi_{j}, \Lambda} + \sum_{j=1}^{k} U_{\varphi_{j}, \Lambda} \right) \]

\[ \leq C \left( \frac{k}{r} \right)^{N-\epsilon_0}. \]

Thus we can get

\[ I_{223} = O \left( \left( \frac{k}{r} \right)^{N-\epsilon_0} + O \left( \frac{1}{k^m} \left( \frac{k}{r} \right)^{N-2} \right) \right). \quad (A.23) \]

Combining (A.17), (A.18), (A.20), (A.19), (A.21) and (A.23), we can get

\[ I(W_{r,h,\Lambda}) = k \left( 1 - \frac{2}{2^*} \right) \int \limits_{\mathbb{R}^N} |U_{0,1}|^{2^*} - k \int \limits_{\mathbb{R}^N} U_{\varphi_{j}, \Lambda}^{2^*-2} \left( \sum_{j=2}^{k} U_{\varphi_{j}, \Lambda} + \sum_{j=1}^{k} U_{\varphi_{j}, \Lambda} \right) \]

\[ + \frac{2k}{2^*} \left( \frac{c_0 m r^m}{\Lambda^m} \right) \int \limits_{\mathbb{R}^N} |y_1|^m U_{0,1}^{2^*} + \frac{c_0 m (m - 1)}{2^* \Lambda^{m-2} r^m} \int \limits_{\mathbb{R}^N} |y_1|^{m-2} U_{0,1}^{2^*} (r - r)^2 \]

\[ + k \frac{C(r, \Lambda)}{m^{N-2}} (r - r)^{2+\sigma} + k \frac{C(r, \Lambda)}{m^{N-2} r^{m+\sigma}} + k O \left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right) + k O \left( \frac{1}{k^m} \left( \frac{k}{r} \right)^{N-2} \right). \]

Combining Lemma A.1-A.3, we can get the following Proposition which gives the expression of \( I(W_{r,h,\Lambda}) \).

**Proposition A.4.** Suppose that \( K(|y|) \) satisfies H and \( N \geq 5 \), \((r, h, \Lambda) \in \mathcal{S}_k \). Then we have

\[ I(W_{r,h,\Lambda}) = kA_1 - \frac{k}{\Lambda N-2} \left[ \frac{B_4 k^{N-2}}{(r \sqrt{1 - h^2})^{N-2}} + \frac{B_5 k}{r^{N-2} h^{N-3} \sqrt{1 - h^2}} \right] \]
The asymptotic expression of \( h, r \) can be absorbed in \( O(h, r) \), where the constants \( \sigma_i, i = 1, 2, 3 \), are defined in Proposition A.4.

**Proof.** A direct result of Lemma 4.1 (4.3) is

\[
I(W_{r,h}, \Lambda) = kA_1 - \frac{k}{N-2} \left[ \frac{B_4 k^{N-2}}{(r \sqrt{1 - h^2}) N - 2} + \frac{B_5 k}{r N - 2 h N^3 \sqrt{1 - h^2}} \right] + k \left[ \frac{A_2}{\Lambda^{m+1}} + \frac{A_3}{\Lambda^{m+2-m}} (r - r)^2 \right] + k \frac{C(r, \Lambda)}{k^{N-2-m} + \sigma} + kO\left( \frac{1}{k^{m(N-2-m) + \sigma}} \right),
\]

as \( k \to \infty \), where \( A_i, i = 1, 2, 3 \), \( B_4, B_5 \) are positive constants.

Proof. A direct result of Lemma 4.1 (4.3) is

\[
I(W_{r,h}, \Lambda) = kA_1 - \frac{k}{N-2} \left[ \frac{B_4 k^{N-2}}{(r \sqrt{1 - h^2}) N - 2} + \frac{B_5 k}{r N - 2 h N^3 \sqrt{1 - h^2}} \right] + k \left[ \frac{A_2}{\Lambda^{m+1}} + \frac{A_3}{\Lambda^{m+2-m}} (r - r)^2 \right] + k \frac{C(r, \Lambda)}{k^{N-2-m} + \sigma} + kO\left( \frac{1}{k^{m(N-2-m) + \sigma}} \right),
\]

with \( B_4 = B_0 B_1, B_5 = B_0 B_2 \) are positive constants. From the expressions of \( \sigma_1(k), \sigma_2(k) \) and asymptotic expression of \( h, r \) as in (4.4), (4.8), we can show that

\[
\frac{\sigma_1(k) k^{N-2}}{(r \sqrt{1 - h^2}) N^2}, \quad \frac{\sigma_2(k) k}{r N - 2 h N^3 \sqrt{1 - h^2}},
\]

can be absorbed in \( O\left( \frac{1}{k^{m(N-2-m) + \sigma}} \right) \).

Noting that \( m > \frac{N-2}{2} \) implies

\[
\frac{N - 3}{N - 1} < \frac{m}{N - 2 - m},
\]

thus provided with \( \epsilon_0, \sigma \) small enough, we can get

\[
\left( \frac{k}{r} \right)^{N - \epsilon_0} = \frac{1}{k^{N-2-m}} \frac{1}{k^{N-2-m}} \leq C \frac{1}{k^{(m(N-2-m) + \sigma)}}.
\]

Since \( m \geq 2 \), we can check that

\[
\frac{1}{k^{m}} \left( \frac{k}{r} \right)^{N-2} \leq C \frac{1}{k^{(m(N-2-m) + \sigma)}}.
\]

Thus we can get (4.24). \( \square \)

To get the expansions of \( \frac{F(r,h,\Lambda)}{\partial \Lambda}, \frac{F(r,h,\Lambda)}{\partial h} \), we need the following expansions for \( \frac{\partial I(W_{r,h}, \Lambda)}{\partial \Lambda}, \frac{\partial I(W_{r,h}, \Lambda)}{\partial h} \).

**Proposition A.5.** Suppose that \( K(|y|) \) satisfies H and \( N \geq 5 \), \( (r, h, \Lambda) \in \mathcal{S}_k \). We have

\[
\frac{\partial I(W_{r,h}, \Lambda)}{\partial \Lambda} = \frac{k}{N-1} \left[ \frac{B_4 k^{N-2}}{(r \sqrt{1 - h^2}) N^2} + \frac{B_5 k}{r N - 2 h N^3 \sqrt{1 - h^2}} \right] - k \left[ \frac{m A_2}{\Lambda^{m+1}} + \frac{(m - 2) A_3}{\Lambda^{m+2-m}} (r - r)^2 \right] + kO\left( \frac{1}{k^{m(N-2-m) + \sigma}} \right),
\]

as \( k \to \infty \), where the constants \( B_i, i = 4, 5 \) and \( A_i, i = 2, 3 \) are defined in Proposition 4.4.
Proof. The proof of this proposition is standard and the reader can refer to [31] for details. □

Proposition A.6. Suppose that \( K(|y|) \) satisfies \( \mathbf{H} \) and \( N \geq 5 \), \((r, h, \Lambda) \in \mathcal{S}_k\). Then we have

\[
\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = -\frac{k}{A^{N-2}} \left[ (N-2) \frac{B_4 k^{N-2}}{r^{N-2} (\sqrt{1 - h^2})^N} h - (N-3) \frac{B_5 k}{r^{N-2} h^{N-2} (\sqrt{1 - h^2})} \right] + kO\left( \frac{1}{k^{(m(N-2)/m + (N-3)/N) + \sigma}} \right)
\]

as \( k \to \infty \).

Proof. Recall

\[
\mathbb{Z}_{2j} \leq C \frac{r}{(1 + |y - x_j|)^{N-1}}, \quad \mathbb{Z}_{2j} \leq C \frac{r}{(1 + |y - x_j|)^{N-1}}.
\]

We know that

\[
\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = \frac{1}{2} \frac{\partial}{\partial h} \int_{\mathbb{R}^N} |\nabla W_{r,h,\Lambda}|^2 - \frac{1}{2} \frac{\partial}{\partial h} \int_{\mathbb{R}^N} K\left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^2
\]

\[
= k \frac{\partial}{\partial h} \int_{\mathbb{R}^N} U_{\mathbb{R}_{1,\Lambda}}^{2^{*}-1} \left( \sum_{j=2}^{k} U_{\mathbb{R}_{j,\Lambda}} + \sum_{i=1}^{k} U_{\mathbb{Z}_{j,\Lambda}} \right)
\]

\[
- \int_{\mathbb{R}^N} K\left( \frac{|y|}{r} \right) W_{r,h,\Lambda}^{2^{*}-1} \left( \mathbb{Z}_{21} + \sum_{j=2}^{k} \mathbb{Z}_{2j} + \sum_{j=1}^{k} \mathbb{Z}_{2j} \right).
\]

From (A.27), similar to the calculations in the proof of Proposition A.3, we can get

\[
\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = -k \frac{\partial}{\partial h} \int_{\mathbb{R}^N} U_{\mathbb{R}_{1,\Lambda}}^{2^{*}-1} \left( \sum_{j=2}^{k} U_{\mathbb{R}_{j,\Lambda}} + \sum_{i=1}^{k} U_{\mathbb{Z}_{j,\Lambda}} \right) + k^2 O\left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right).
\]

Then by some tedious but straightforward analysis, we can get

\[
\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = -\frac{k}{A^{N-2}} \left[ (N-2) \frac{B_4 k^{N-2}}{r^{N-2} (\sqrt{1 - h^2})^N} h - (N-3) \frac{B_5 k}{r^{N-2} h^{N-2} (\sqrt{1 - h^2})} \right]
\]

\[
+ h \frac{B_5 k}{r^{N-2} h^{N-3} (1 - h^2)^{3/2}} + k^2 O\left( \left( \frac{k}{r} \right)^{N-\epsilon_0} \right),
\]

for some \( \epsilon_0 \) small enough. In fact, we know that \( k \left( \frac{k}{r} \right)^{N-\epsilon_0} \) and \( h \frac{B_5 k}{r^{N-2} h^{N-3} (1 - h^2)^{3/2}} \) can be absorbed in \( O\left( \frac{1}{k^{(m(N-2)/m + (N-3)/N) + \sigma}} \right) \) provided with \( m \) satisfying (1.5) and \( \epsilon_0, \sigma \) small enough. Then we can get (A.25) directly.

□

APPENDIX B. SOME BASIC ESTIMATES AND LEMMAS

Lemma B.1. Under the condition \((r, h, \Lambda) \in \mathcal{S}_k\), for \( y \in \Omega_1^+ \) there exists a constant \( C \) such that

\[
\left( \sum_{j=2}^{k} U_{\mathbb{R}_{j,\Lambda}} + \sum_{j=1}^{k} U_{\mathbb{Z}_{j,\Lambda}} \right) \leq C \left( \frac{k}{r} \right)^{\alpha} \frac{1}{(1 + |y - \mathbb{P}|)^{N-2-\alpha}},
\]

with \( \alpha = (1, N-2) \).
Lemma B.3. For any constants 0 < \gamma_1, \gamma_2 \leq \min\{\gamma_1, \gamma_2\}, there is a constant C > 0, such that

\[ g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\gamma_1} \left( \frac{1}{(1 + |y - x_i|)^{\gamma_1 + \gamma_2 - \gamma_1}} + \frac{1}{(1 + |y - x_j|)^{\gamma_1 + \gamma_2 - \gamma_2}} \right). \]

Lemma B.4. For any constant 0 < \beta < N - 2, there is a constant C > 0, such that

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{(1 + |z|)^{2+\beta}} \, dz \leq \frac{C}{(1 + |y|)^\beta}. \]

Lemma B.5. Suppose that N \geq 5 and \tau \in (0, 2), y = (y_1, \cdots, y_N). Then there is a small \sigma > 0, such that when y_3 \geq 0,

\[ \int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \frac{1}{W_{r,h}\Lambda(z)} \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2 + \tau}} \, dz \leq C \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{N-2 + \tau + \sigma}}, \]
Lemma B.6. Suppose that $N \geq 5$ and $m$ satisfies (1.5). We have

\[ \frac{r}{k(\frac{m}{N-2-m})(N+2-\frac{2N-2-m}{N-2}-2\epsilon_1)} \leq \frac{C}{k(\frac{m(N-2)}{N-2-m} + \frac{(N-3)}{(N-1)+\sigma})}, \]

provided with $\sigma, \epsilon_1$ small enough.

Proof. It’s easy to show that

\[ \frac{r}{k(\frac{m}{N-2-m})(N+2-\frac{2N-2-m}{N-2}-2\epsilon_1)} \leq \frac{C}{k(\frac{m(N-2)}{N-2-m} + \frac{(N-3)}{(N-1)+\sigma})}, \]

for $m \geq 2$. In order to get (B.1), we just need to show

\[ \frac{r}{k(\frac{m}{N-2-m})(N+2-\frac{2N-2-m}{N-2}-2\epsilon_1)} = \frac{k^{\frac{N-2}{N-2-m}}}{k^{\frac{m}{N-2-m})(N+2-\frac{2N-2-m}{N-2}-2\epsilon_1)} \leq \frac{C}{k(\frac{m(N-2)}{N-2-m} + \frac{(N-3)}{(N-1)+\sigma})}, \]

for some $\sigma, \epsilon_1$ small. The problem to show (B.2) can be reduced to show that $6 + \frac{(N-3)}{(N-1)} < 3(\frac{N-2}{N-2-m}) + 2\frac{2N-2-m}{N-2}$, for $m$ satisfying (1.5). This inequality follows by simple computations. This fact concludes the proof. \(\Box\)

Acknowledgements: L. Duan was supported by the China Scholarship Council and NSFC grant (No.11771167) and Technology Foundation of Guizhou Province (2001)ZK008). M. Musso was supported by EPSRC research Grant EP/T008458/1. S. Wei was supported by the NSFC grant (No.12001203) and Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515110622). Some part of the work was done during the visit of L. Duan to Prof. M. Musso at the University of Bath. L. Duan would like to thank the Department of Mathematical Sciences for its warm hospitality and supports.

References

[1] A. Ambrosetti, G. Azorero, J. Peral, Perturbation of $-\Delta u - u^{\frac{N+2}{N-2}} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topics, J. Funct. Anal. 165(1999), 117–149.

[2] A. Bahri, J. Coron, The scalar-curvature problem on the standard three-dimensional sphere, J. Funct. Anal. 95(1991), 106-172.

[3] G. Bianchi, Non-existence and symmetry of solutions to the scalar curvature equation, Comm. Partial Differential Equations 21(1996), 229-234.

[4] H. Brezis, L.A. Peletier, Elliptic equations with critical exponent on spherical caps of $S^3$, J. Anal. Math. 98(2006), 279-316.

[5] D. Cao, E. Noussair, S. Yan, On the scalar curvature equation $-\Delta u = (1 + \epsilon K) u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 15(2002), 403-419.

[6] S.-Y. A. Chang, P. C. Yang, A perturbation result in prescribing scalar curvature on $S^N$, Duke Math. J. 64(1991), 27-69.

[7] C.-C. Chen, C.-S. Lin, Estimate of the conformal scalar curvature equation via the method of moving planes. II, J. Differential Geom. 49(1998), 115-178.

[8] W.-X. Chen, W.-Y. Ding, Scalar curvature on $S^2$, Trans. Amer. Math. Soc. 303(1987), 369-382.
[9] Y. Deng, C.-S. Lin, S. Yan, On the prescribed scalar curvature problem in $\mathbb{R}^N$, local uniqueness and periodicity, J. Math. Pures Appl. 104(2015), 1013-1044.

[10] O. Druet, From one bubble to several bubbles: the low-dimensional case, J. Differential Geom. 63(2003), 399-473.

[11] Y. Guo, B. Li, Infinitely many solutions for the prescribed curvature problem of polyharmonic operator, Calc. Var. Partial Differential Equations 46(2013), 809-836.

[12] Y. Guo, M. Musso, S. Peng, S. Yan, Non-degeneracy of multi-bubbling solutions for the prescribed scalar curvature equations and applications, J. Funct. Anal. 279(2020), 108553, 22pp.

[13] Y. Guo, J. Nie, Infinitely many non-radial solutions for the prescribed curvature problem of fractional operator, Discrete Contin. Dyn. Syst. 36(2016), 6873-6898.

[14] Z.-C. Han, Prescribing Gaussian curvature on $S^2$, Duke Math. J. 61(1990), 679-703.

[15] J. Kazdan, F. Warner, Existence and conformal deformation of metrics with prescribed Gaussian and Scalar curvature, Ann. Math. 101(1975), 317-331.

[16] J. Kazdan, F. Warner, Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geom. 10(1), (1975), 113-134.

[17] Y. Y. Li, On $-\Delta u = K(x)u^\frac{N+2}{N-2}$ in $\mathbb{R}^N$, Comm. Pure Appl. Math. 46(1993), 303-340.

[18] Y. Y. Li, Prescribing scalar curvature on $S^3$, $S^4$ and related problems, J. Funct. Anal. 118(1993), 43-118.

[19] Y. Y. Li, Prescribing scalar curvature on $S^n$ and related problems. I, J. Differential Equations 120(1995), 319-410.

[20] Y. Y. Li, Prescribing scalar curvature on $S^n$ and related problems. II, Existence and compactness. Comm. Pure Appl. Math. 49(1996), 541-597.

[21] Y. Y. Li, J. Wei, H. Xu, Multi-bump solutions of $-\Delta u = K(x)u^{\frac{n+2}{N-2}}$ on lattices in $\mathbb{R}^n$, J. Reine Angew. Math. 743(2018), 163-211.

[22] C. S. Lin, S.-S. Lin, Positive radial solutions for $\Delta u + K(r)u^{\frac{n+2}{N-2}} = 0$ in $\mathbb{R}^n$ and related topics, Appl. Anal. 38(1990), 121-159.

[23] W. Long, S. Peng, J. Yang, Infinitely many positive and sign-changing scalar field equations, Discrete Contin. Dyn. Syst. 36(2016), 917-939.

[24] M. Medina, M. Musso, Doubling nodal solution to the Yamabe equation in $\mathbb{R}^n$ with maximal rank, J. Math. Pures Appl. (9) 152 (2021), 145-188.

[25] W. M. Ni, On the elliptic equation $\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$ its generalizations and applications in geometry, Indiana Univ. Math. J. 31(1982), 493-529.

[26] E. Noussair, S. Yan, The scalar curvature equation on $\mathbb{R}^N$, Nonlinear Anal. 45(2001), 483-514.

[27] S. Peng, C. Wang, S. Wei, Construction of solutions for the prescribed scalar curvature problem via local Pohozaev identities, J. Differential Equations. 267(2019), 2503-2530.

[28] O. Rey, The role of the Green’s function in a nonlinear elliptic problem involving the critical Sobolev exponent, J. Funct. Anal. 89(1990), 1-52.

[29] G. Talenti, Best constants in Sobolev inequality, Ann. Mat. Pura Appl. 110(1976), 353-372.

[30] R. Schoen, D. Zhang, Prescribed scalar curvature on the $n$-sphere, Calc. Var. Partial Differential Equations 4(1996), 1-25.

[31] J. Wei, S. Yan, Infinitely many solutions for the prescribed scalar curvature problem on $\mathbb{R}^N$, J. Funct. Anal. 258(2010), 3048-3081.

[32] S. Yan, Concentration of solutions for the scalar curvature equation on $\mathbb{R}^N$, J. Differential Equations 163(2000), 239-264.

Lipeng Duan,
School of Mathematics and Information Science, Guangzhou University,
Guangzhou 510006, P. R. China.
Email address: lpduan777@sina.com

Monica Musso,
Department of Mathematical Sciences, University of Bath,
Bath BA2 7AY, United Kingdom
Email address: mm2683@bath.ac.uk

Suting Wei,
Department of Mathematics, South China Agricultural University,
Guangzhou, 510642, P. R. China.
Email address: stwei@scau.edu.cn