Singular Lagrangians and the dynamics of the ”Kill the Winner” model and its descendants

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Abstract

We consider certain analytical features of the ”Kill the Winner” model which is a stochastic model that can explain among other things competition among species and simultaneous predation on the competing species. The model equations are shown to admit a Jacobi Last Multiplier which in turns allows for the construction of a Lagrangian. The Lagrangian is of singular nature so that construction of the Hamiltonian via a Legendre transformation is not possible. A Hamiltonian description of the model therefore requires the introduction of Dirac brackets. Explicit results are presented for the model and its reductions.

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1 Introduction

Population dynamics is a well established field and can lay claim to having increased our understanding of several phenomena is diverse areas. However, most theoretical models of population dynamics do not offer a satisfactory explanation for the so-called diversity paradox in nature. Roughly speaking when several species compete for the same finite resource, a theory called competitive exclusion indicates that one species will outperform the others and drive them to extinction, thereby limiting biodiversity. However, nature does not quite evolve in this manner and shows up a greater degree of diversity then predicted. Goldenfeld and Xue developed a stochastic model [19] that accounts for multiple factors observed in ecosystems, including competition among species and simultaneous predation on the competing species. Using bacteria and their host-specific viruses as an example, they showed that as the bacteria evolve defenses against the virus, the virus population also evolves to combat the bacteria. This "arms race" leads to a diverse population of both and to boom-bust cycles when a particular species dominates the ecosystem then collapses- the so-called "Kill the Winner" (KtW) phenomenon. This coevolutionary arms race is sufficient to yield a possible solution to the diversity paradox.

It is of interest to look at such systems from a purely dynamic point of view as they provide interesting concrete examples of first-order differential systems which have physical relevance. Of late there has been a lot of interest in the study of systems with singular Lagrangians which have arisen in the field of quantitative biology. Several well known models such as the Lotka-Volterra prey-predator system, the host-parasite model, the Bailey model etc. are the examples of systems that are described by singular Lagrangians, i.e. by Lagrangians that do not depend quadratically on the velocity parameter. A major problem in describing the dynamics of such systems on the phase space is in defining the appropriate canonical momentum. In order to analyze the dynamics of such systems, Dirac formulated a method which basically makes the use of a constrained surface on which the Poisson bracket may be suitably defined. Roughly speaking this restricted Poisson bracket defined on a sub-manifold of the complete phase space of the system is usually referred to the Dirac bracket.

A Lagrangian is said to be singular if the corresponding Hessian matrix is singular. The singular nature of the Hessian matrix prevents us from solving for the velocities in terms of the conjugate momenta and the coordinates. This in turn prevents us from constructing the Hamiltonian by means of a Legendre transformation. Most textbooks, while mentioning this particular feature do not generally provide too many illustrative examples of singular Lagrangians. Two illustrations are provided below.

\[ L = \frac{1}{2m}(\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2\mu}q_3^2 + V(q_1, q_2, q_3) \]
\[ L = (q_1\dot{q}_1 + \dot{q}_1\dot{q}_2)^2 + (q_2^2 + q_2^2)^2 \]

In each of the above cases it will be observed that the Hessian matrix \[ \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| \] has a vanishing
determinant. In the first instance the conjugate momenta are given by

\[ p_1 = \frac{1}{m}(\dot{q}_1 + \dot{q}_2), \quad p_2 = \frac{1}{m}(\dot{q}_1 + \dot{q}_2), \quad p_3 = \frac{1}{\mu}\dot{q}_3. \]

As we can not solve for the \( \dot{q}_i \) in terms of the coordinates and momenta we are forced to treat the above relations as a set of constraints, the so called primary constraints. We denote them by

\[ \phi_1 = p_1 - \frac{1}{m}(\dot{q}_1 + \dot{q}_2) \approx 0 \]
\[ \phi_2 = p_2 - \frac{1}{m}(\dot{q}_1 + \dot{q}_2) \approx 0 \]
\[ \phi_3 = p_3 - \frac{1}{\mu}\dot{q}_3 \approx 0 \]

The Hamiltonian

\[ H = p_1 \dot{q}_1 + p_2 \dot{q}_2 + p_3 \dot{q}_3 - L \]

is therefore a function of the "mixed" set of variables \((q_i, \dot{q}_i, p_i)\). To develop a Hamiltonian description of the evolution of the systems we define the primary Hamiltonian as

\[ H_p = H + \sum_i \lambda_i \phi_i \]

Because the primary constraints must hold good at all times it is necessary that

\[ \dot{\phi}_i = \{\phi_i, H_p\} = 0, \quad \forall i = 1, 2, 3. \]

It is evident that if the Lagrangian is linear in the velocities then it is singular in character because the Hessian matrix is trivial. As already mentioned a particularly rich field in which the dynamics may be described by singular Lagrangians arises in models describing ecological systems, predator-prey models etc.

In this article we focus on an interesting stochastic model dealing with the problem of biological diversity. The so called "Kill the Winner" (KtW) hypothesis has been quite extensively studied by many authors. The present analysis is based on the model equations introduced in [19]. While most studies of such systems have relied on a numerical approach we have adopted a more analytical view expanding on our previous work with biological systems. Several well known systems such as the Lotka-Volterra, Host-parasite, Kermack-McKendrick models may be shown to be special cases of the general KtW model studied here. It is interesting to note that the stochastic model equations admit a Jacobi Last Multiplier (JLM) which in turn enables us to set up a Lagrangian description of the system. However, it is found that the Lagrangian obtained is singular in character so that a proper Hamiltonian description of the system in phase space requires the introduction of Dirac brackets.

Our main result is the following:
Proposition 1.1 The generalized “Kill the Winner” model is described by the following system of ODEs

\[ \dot{x}_i = b_i x_i - \sum_{j=1}^{m} e_{ij} x_i x_j - p_i x_i y_i, \quad \dot{y}_i = q_i x_i y_i - d_i y_i, \quad i = 1, ..., m \]

where, \( a_i, b_i, c_i \) are constants.

(A) The Jacobi Last Multiplier for the system is given by

\[ M = \prod_{i=1}^{m} M_i, \quad M_i = e^{\gamma_i t} y_i^{\sigma_i} x_i, \quad \gamma_i = e_i d_i / q_i \quad \text{and} \quad \sigma_i = -1 + e_i / q_i. \]

(B) The Lagrangian and Hamiltonian are singular, these are given by

\[
L = \sum_{k=1}^{m} e^{\gamma_k t} y_k \left[ \frac{q_k \dot{x}_k}{e_{kk} x_k} - \log x_k \frac{\dot{y}_k}{y_k} + \left( 2q_k x_k - d_k \log x_k + \frac{2q_k}{e_{kk}} \left( \sum_{j=1}^{m} e_{kj} x_j - b_k \right) \right) + 2p_k y_k \left( \frac{q_k}{e_{kk}} + 1 \right) \right] \]

and

\[
H = -\sum_{k=1}^{m} e^{\gamma_k t} y_k \left[ 2q_k x_k - d_k \log y_k + \frac{2q_k}{e_{kk}} \left( \sum_{j=1}^{m} e_{kj} x_j - b_k \right) \right] + 2p_k y_k \left( \frac{q_k}{e_{kk}} + 1 \right) \]

respectively.

By computing the Dirac brackets associated to this Hamiltonian we establish the required equations of motion of the phase space variables.

The paper is organized as follows: In section 2 we outline the procedure for deriving a Lagrangian linear in the velocities (and hence singular) assuming the existence of a JLM. A geometric description of such singular Lagrangians is then presented followed by an introduction to the notion of Dirac brackets. In section 3 we introduce the “Kill the Winner” (KtW) model and considering its simplest version derive the explicit expression for the JLM. This is done to motivate the subsequent general results. The lagrangian for the general KtW model is then derived and the ingredients required for constructing the Dirac brackets is presented. This is followed by an explicit calculation of the Dirac brackets for some specific cases such as the Lotka-Volterra model with and without competition and the Gierer-Meinhardt model of pattern formation.

2 Preliminaries

Let us briefly recall the procedure described in [12] for finding Lagrangians for a planar system of ODEs from a knowledge of the last multiplier. We assume that the system

\[ \frac{dx}{dt} = f(t, x, y) \]
\[
\frac{dy}{dt} = g(t, x, y) \tag{2.2}
\]

admits a Lagrangian which is linear in the velocities, so that
\[
L(t, x, y, \dot{x}, \dot{y}) = F(t, x, y)\dot{x} + G(t, x, y)\dot{y} - V(t, x, y). \tag{2.3}
\]

Then the Euler-Lagrange equations of motion
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}} \right) = \frac{\partial L}{\partial u}, \quad \text{with } u = x \text{ and } y
\]
yields
\[
\dot{y} = \left( \frac{F_t + V_x}{G_x - F_y} \right) = g(t, x, y), \tag{2.4}
\]
and
\[
\dot{x} = -\left( \frac{G_t + V_y}{G_x - F_y} \right) = f(t, x, y), \tag{2.5}
\]
where the subscripts on \(F, G\) and \(V\) denote the partial derivatives while the over-dots represent the derivative with respect to time. It is obvious that one must have \(G_x \neq F_y\). In order to introduce the notion of Jacobi’s last multiplier we assume that \(G_x = -F_y\) and assign a common value,
\[
\mu(t, x, y) := G_x = -F_y. \tag{2.6}
\]

From (2.4) and (2.5), we have
\[
2\mu f(t, x, y) = -(G_t + V_y) \tag{2.7}
\]
\[
2\mu g(t, x, y) = (F_t + V_x). \tag{2.8}
\]

It is clear that the construction
\[
\frac{\partial}{\partial x} (2\mu f) + \frac{\partial}{\partial y} (2\mu g)
\]
leads to the following equation,
\[
\frac{d}{dt} \log \mu + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{2.9}
\]

using the original system of ODEs (2.1)-(2.2). However, (2.9) is precisely the defining relation for JLM \([18]\). Thus we see that given the solution of this equation one can easily construct from (2.4) the coefficient functions \(F\) and \(G\) occurring in the expression for the Lagrangian since
\[
F(t, x, y) = -\int \mu(t, x, y)dy \quad \text{and} \quad G(t, x, y) = \int \mu(t, x, y)dx. \tag{2.10}
\]
Once these functions are determined one can obtain an expression for the partial derivatives of $V$ from (2.4) and (2.5) as follows

$$\frac{\partial V}{\partial x} = 2\mu(t, x, y)g(t, x, y) + \frac{\partial}{\partial t} \left( \int \mu dy \right),$$

(2.11)

$$\frac{\partial V}{\partial y} = -2\mu(t, x, y)f(t, x, y) - \frac{\partial}{\partial t} \left( \int \mu dx \right).$$

(2.12)

In view of (2.9) it is easy to check the equality of the mixed derivatives,

$$\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}.$$ 

2.1 Geometric description of singular Lagrangian

The time evolution associated to a time-dependent mechanical system is usually represented by the flow of a vector field defined in $\mathbb{R} \times TQ$, where $Q$ is a smooth differentiable manifold which represents the original configuration space of the system. This is a space of 1-jets of the trivial bundle $\pi : \mathbb{R} \times Q \to \mathbb{R}$, i.e., $J^1\pi = \mathbb{R} \times TQ$. In local coordinates $(t, q^i, v^i)$ in $J^1\pi$ the local 1-forms $\theta^i = dq^i - v^i dt$, $i = 1, \cdots, n$, constitute a local basis for the contact 1-forms.

The $k$-jet bundle of $\pi$ is $J^k\pi = \mathbb{R} \times T^kQ$ and there exists a natural projection $\pi_{k,l} : \mathbb{R} \times T^kQ \to \mathbb{R} \times T^lQ$ for each pair of indices $k, l$ and $k > l$. In the Lagrangian formalism, the dynamics takes place in the manifold $\mathbb{R} \times TQ$ (for example, see for details [3, 16]).

We briefly introduce the basic definition of vertical endomorphism this will is to define the Poincaré-Cartan form. Let $v = v^i \frac{\partial}{\partial q^i} \in T_qQ$ and $V = \dot{q}^i \frac{\partial}{\partial q^i} + \dot{v}^i \frac{\partial}{\partial v^i} \in T_v(TQ)$ such that $\tau_{TQ}(V) = v$. In local coordinates, $\tau_{TQ}(q^i, \dot{q}^i, v^i) = (q^i, v^i)$. If $\tau_Q : v \in TQ \mapsto q \in Q$ denotes the natural projection, then, given a tangent vector $V \in T_v(TQ)$, we have $\tau_{TQ}(V) = v$. Let $v \in T_qQ$ be a vector tangent to $Q$ at some point $q \in Q$. Then the vertical lift of $v$ at a point $w \in T_qQ$ is the tangent vector $v^V_w \in T_w(TQ)$ given by

$$u^V_w(g) = \frac{d}{dt} g(w + tv)|_{t=0} \quad \forall g \in C^\infty(T_qQ).$$

In local coordinates of $T^2Q$ is given by $v^V_w = v^i \frac{\partial}{\partial v^i}|_w$, which is the Liouville vector field over $TQ$. The vertical endomorphism is the linear map $S : T^2Q \to T^2Q$ for which any vector $V \in T^2Q$ yields $S(V) = ((T_v\tau_Q)(V))^V$, where $v = \tau_{TQ}(V) \in TQ$.

We can extend this theory of vertical endomorphism to the nonautonomous case. Then in fibred coordinates, $(t, q^i, v^i) \in \mathbb{R} \times TQ$, $S$ is given as $S = (dq^i - v^i dt) \otimes \partial / \partial v^i$. The most important objects associated to the time-dependent Lagrangian are the Poincaré-Cartan one and two forms defined by

$$\Theta_L = dL \circ S + L dt, \quad \Omega_L := -d\Theta_L.$$

(2.13)
The local expressions are

\[
\Theta_L = \frac{\partial L}{\partial v^i}(dq^i - v^i dt) + Ldt = (L - v^i \frac{\partial L}{\partial v^i}) dt + \frac{\partial L}{\partial v^i} dq^i
\]

(2.14)

\[
\Omega_L = -d\left(\frac{\partial L}{\partial v^i}\right) \wedge dq^i + d\left(v^i \frac{\partial L}{\partial v^i} - L\right) \wedge dt.
\]

(2.15)

2.2 Dirac bracket

Let \( M \) be a \( 2n \)-dimensional symplectic manifold, this can be manifested as a classical phase space of a system with \( n \)-degrees of freedom. Dirac bracket were introduced as a modification of Poisson bracket in presence of constraints. If we impose \( 2m \) independent constraints, this yields \( 2(n - m) \)-dimensional symplectic manifold \( M_c \subset M \). In the neighbourhood of a point \( p \in M \), choose coordinates \( x_1, \ldots x_{2n} \in M \), such that \( M_c \) is given by \( x_1 = 0, \ldots, x_{2m-1} = 0, x_{2m} = 0 \), constraints are forcing the system to lie on \( M_c \). Instead of functions, we can supply \( 2(n - m) \) vector fields whose kernel is \( M_c \). Thus \( x_{2m+1}, \ldots, x_{2n} \) provide local coordinates on \( M_c \).

Let us define the matrix

\[
C^{rs}(x) = \{x_r, x_s\}, \quad r, s = 1, \ldots 2m,
\]

and suppose \( f, g \in C^\infty(M) \) be the smooth functions on \( M \), and let \( f', g' \) be their restriction to \( M_c \).

**Definition 2.1** A function \( f \in C^\infty(M) \) is called first class if \( \{f, x_r\} = 0, \forall r \), where \( x_r(q, p) = 0 \), otherwise it is called second class. All constraints are called second class provided \( \{x_r, x_s\} = C_{rs} \), and all constraints are called first class if \( \{y_r, y_s\} = f_{rs}y_w \approx 0 \).

Then the Dirac bracket is defined by

\[
\{f', g'\}_D := \{f, g\} - \sum_{r,s=1}^{2m} \{f, x_s\}[C_{rs}]^{-1}\{x_s, g\},
\]

(2.16)

where the double sum is taken for all second class constraints. For any second class constraint

\[
\{f', x_k\}_D := \{f, x_k\} - \sum_{r,s=1}^{2m} \{f, x_r\}[C_{rs}]^{-1}\{x_s, x_k\}\{f, x_s\}
\]

\[
= \{f, x_k\} - \sum_{r,s=1}^{2m} \{f, x_r\}[C_{rs}]^{-1}[C_{sk}]
\]

\[
= \{f, x_k\} - \sum_{r=1}^{2m} \{f, x_r\}\delta_{rk} = \{f, x_k\} - \{f, x_k\} = 0.
\]
which shows that the flow generated by the second class constraint vanishes, hence it does not leave the constraint manifold. Thus Dirac bracket provides a modification of the Poisson bracket so as to ensure that the Hamiltonian flow generated by the constraints with respect to the new Poisson structure is tangent to the constraint manifold.

Using bivector formalism, the Dirac bivector is given as

$$\Pi_D := \Pi + \frac{1}{2}[C_{rs}]^{-1}\mathcal{X}_r \wedge \mathcal{X}_s,$$  \hspace{1cm} (2.17)

where $\Pi$ stands for Poisson bivector and

$$\mathcal{X}_r = \{\cdot, x_r\} = -\left(\partial_i x_r\right)\Pi^{ij} \partial_j.$$

Let $\{p_j, q^j\}, 1 \leq j \leq 2n$, denote a set of dynamical variables, $\{u^a\}, 1 \leq a \leq 2m$, set of Lagrange multipliers, and $\{x_a(p, q)\}$ a set of constraints. The total Hamiltonian is given by

$$H_{Tot} = H(p, q) + u_a x_a(p, q).$$ \hspace{1cm} (2.18)

Then the dynamics of a constrained system can be obtained from the action principle

$$S = \int [p_j \dot{q}^j - H(p, q) - u_a x_a(p, q)] dt.$$

The resultant equations that arise from the action read

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} + u_a \frac{\partial x_a}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i} - u_a \frac{\partial x_a}{\partial p_i}, \quad \dot{x}_a = 0.$$  \hspace{1cm} (2.19)

The (constrained) Hamiltonian equation for any arbitrary smooth function $g(p, q) \in C^\infty(M)$ is given by

$$\dot{g} = \{g, H(p, q) + u_a x_a\} \approx \{g, H, x_a\} + u_a \{g, x_a\}.$$ \hspace{1cm} (2.20)

where the $\approx$ symbol means that we should not substitute $x_a = 0$ before evaluating Poisson bracket. This is called the weak equation. For a variation of the weak equation $\dot{g} \approx \{g, H(p, q) + u_a x_a\}$, the variation of LHS is not equal to the variation of RHS.

Let us assume that all the constraints are second class, then the corresponding undetermined coefficients are given by

$$u_b = -\{H, x_a\}[C_{ab}]^{-1}.$$ \hspace{1cm} (2.21)

We can express the equation of motion with second class constraints in terms of Dirac bracket.

**Proposition 2.1** The equation of motion for a system with second class constraints can be expressed by

$$\dot{g} \approx \{g, H\}_D = \{g, H\} - \sum_{a,b} \{g, x_a\}[C_{ab}]^{-1}\{x_b, H\}.$$ \hspace{1cm} (2.22)

This can be proved easily using (2.21).
3 Singular Lagrangians of biological systems and Dirac bracket

3.1 “Kill the Winner” Model

The “kill the winner (KtW)” hypothesis which attempts to get a stochastic model for the biological diversity problem in nature where the host-specific predators control the prey population of each species by preventing a winner from emerging. This natural phenomenon maintains the equilibrium coexistence of all the existing species in the system. An individual-level stochastic model in which predator-prey coevolution promotes the high diversity of the ecosystem by generating a persistent population flux of species has been developed in [19]. For a single species the model is described by the following system of differential equations [19]:

\[
\begin{align*}
\dot{x} &= a_1 x - b_1 x^2 - c_1 xy \\
\dot{y} &= a_2 xy - b_2 y
\end{align*}
\]

where, \(a_1, a_2, b_1, b_2, c_1\) are the constants and \(x(y)\) represents the bacterial(viral strains) density of the system. The Jacobi last multiplier for this system can be written as \(M = e^{\gamma t} x^\alpha y^\beta\) with

\[
\begin{align*}
\alpha &= -1 \\
\beta &= \frac{b_1 - a_2}{a_2} \\
\gamma &= \frac{b_1 b_2}{a_2}
\end{align*}
\]

The generalized KTW model is given by the system of equations

\[
\begin{align*}
\dot{x}_i &= b_i x_i - \sum_{j=1}^{m} e_{ij} x_i x_j - p_i x_i y_i \\
\dot{y}_i &= q_i x_i y_i - d_i y_i, \quad i = 1, \ldots, m
\end{align*}
\]

The Jacobi last multiplier for the system is given by

\[
M = \prod_{i=1}^{m} M_i
\]

with \(M_i = e^{\gamma_i t} y_i^{\sigma_i}/x_i\). Here \(\gamma_i = e_{ii} d_i / q_i\) and \(\sigma_i = -1 + e_{ii} / q_i\). Now assume the singular Lagrangian as

\[
L = \sum_{k=1}^{m} \left[ F_k(x,y) \dot{x}_k + G_k(x,y) \dot{y} \right] - U(x,y)
\]

substituting into the Euler-Lagrange equation it gives

\[
\sum_{k=1}^{m} \left[ (F_{ix_k} - F_{kx_k}) \dot{x} + (F_{iy_k} - G_{kx_k}) \dot{y} \right] = - \left( \frac{\partial U}{\partial x_i} + \frac{\partial F_i}{\partial t} \right)
\]
\[
\sum_{k=1}^{m} [(G_{xk} - F_{ky}) \dot{x} + (G_{yk} - G_{ky}) \dot{y}] = - \left( \frac{\partial U}{\partial y_i} + \frac{\partial G_i}{\partial t} \right) (3.11)
\]

This can be written in a matrix form as
\[
\begin{pmatrix}
\dot{X} \\
\dot{Y}
\end{pmatrix} = -\begin{pmatrix}
A & B \\
-B^T & D
\end{pmatrix}^{-1} \begin{pmatrix}
P \\
Q
\end{pmatrix} = -\frac{1}{\Delta} \begin{pmatrix}
D & -A^{-1}BA \\
D^{-1}B^T & A
\end{pmatrix} \begin{pmatrix}
P \\
Q
\end{pmatrix}
\]

where
\[
X = \begin{pmatrix}
x_1 \\
\vdots \\
x_m
\end{pmatrix}, \quad Y = \begin{pmatrix}
y_1 \\
\vdots \\
y_m
\end{pmatrix}, \quad P = \begin{pmatrix}
\frac{\partial F_1}{\partial t} + \frac{\partial U}{\partial x_1} \\
\vdots \\
\frac{\partial F_m}{\partial t} + \frac{\partial U}{\partial x_m}
\end{pmatrix}, \quad Q = \begin{pmatrix}
\frac{\partial G_1}{\partial t} + \frac{\partial U}{\partial y_1} \\
\vdots \\
\frac{\partial G_m}{\partial t} + \frac{\partial U}{\partial y_m}
\end{pmatrix}
\]

Now we make the assumption that \( A = B = 0 \) and \( B \) is diagonal with \( B_{ii} = F_{iy_i} - G_{ix_i} \). Therefore we can write
\[
\dot{y}_i = -\frac{1}{2M_i} \left( \frac{\partial U}{\partial y_i} + \frac{\partial F_i}{\partial t} \right) (3.12)
\]
and
\[
\dot{x}_i = -\frac{1}{2M_i} \left( \frac{\partial U}{\partial y_i} + \frac{\partial G_i}{\partial t} \right) (3.13)
\]

Next let us assume,
\[
\frac{\partial F_i}{\partial y_i} = -\frac{\partial G_i}{\partial x_i} = M_i (3.14)
\]

Therefore,
\[
\dot{y}_i = -\frac{1}{2M_i} \left( \frac{\partial U}{\partial x_i} + \frac{\partial F_i}{\partial t} \right) (3.15)
\]

&
\[
\dot{x}_i = -\frac{1}{2M_i} \left( \frac{\partial U}{\partial y_i} + \frac{\partial G_i}{\partial t} \right) (3.16)
\]

It follows that
\[
F_i = \int M_i dy_i + f_i(t, x_i) = e^{\gamma_i t} \frac{y_i^{\sigma_i+1}}{x_i (\sigma_i + 1)} + f_i(t, x_i) (3.17)
\]

and
\[
G_i = -\int M_i dx_i + g_i(t, y_i) = -e^{\gamma_i t} y_i^{\sigma_i} \log x_i + g_i(t, y_i) (3.18)
\]

Now from the above relations we can see that
\[
\frac{\partial U}{\partial x_i} = -\frac{e^{\gamma_i t} y_i^{\sigma_i+1}}{x_i} \left[ 2 \left( q_i x_i - d_i \right) + \frac{\gamma_i}{(\sigma_i + 1)} \right] (3.19)
\]
The solution for the potential \( U \) can be obtained from the above set of equations as

\[
U = -\sum_{i=1}^{m} e^{\gamma_i t} \left[ 2q_i x_i - d_i \log x_i + \frac{2p_i y_i}{\sigma_i + 2} \right] + e^{\gamma_i t} y_i^{\sigma_i+1} \frac{2q_i}{e_{ii}} \left( \sum_{j=1, j \neq i}^{m} e_{ij} x_j - b_i \right)
\]

(3.21)

Hence the Lagrangian for the system can be written as

\[
L = \sum_{k=1}^{m} e^{\gamma_k t} y_k^{b_k} \left[ \frac{q_k}{e_{kk}} \dot{x}_k - \log x_k \frac{\dot{y}_k}{y_k} + \left( 2q_k x_k - d_k \log x_k + \frac{2q_k}{e_{kk}} \left( \sum_{j=1, j \neq k}^{m} e_{kj} x_j - b_k \right) + 2p_k y_k \left( \frac{q_k}{e_{kk}} + 1 \right) \right) \right]
\]

(3.22)

The expression for the Hamiltonian turns out to be

\[
H = \sum_{k=1}^{m} \left[ \frac{\partial L}{\partial \dot{x}_k} \dot{x}_k + \frac{\partial L}{\partial \dot{y}_k} \dot{y}_k \right] - L = U(x, y)
\]

and is independent of the velocities as is natural for such singular Lagrangians. This completes the proof of the proposition (1.1).

Let us now identify the primary constraints of the system: It is easily seen that the conjugate momenta are given by

\[
p_x^k = \frac{\partial L}{\partial \dot{x}_k} = F_k = e^{\gamma_k t} \frac{q_k y_k^{e_{kk}/q_k}}{e_{kk}/x_k}
\]

\[
p_y^k = \frac{\partial L}{\partial \dot{y}_k} = G_k = -e^{\gamma_k t} \frac{y_k^{e_{kk}/q_k-1} \log x_k}{x_k}
\]

These lead us to the primary constraints of the model which are defined as

\[
\phi_k := p_x^k - F_k \approx 0, \quad \psi_k := p_y^k - G_k \approx 0.
\]

(3.23)

The primary Hamiltonian is therefore

\[
H_p = H + \sum_{k=1}^{m} (\lambda_k \phi_k + \mu_k \psi_k), \quad k = 1, ..., m
\]

(3.24)
As the primary constraints must be satisfied at all times it is necessary that their time evolution vanish. This requirement leads us to the second class constraints of the system.

\[ \dot{\phi}_k = \{\phi_k, H\} + \sum_{k=1}^{m} \mu_k \{\phi_x, \psi_k\} \approx 0 \]

We therefore define the secondary constraints corresponding to the \( \phi \)-class of primary constraints by

\[ \phi_{k+m} = \{p_k^x, U\} + \sum_{k=1}^{m} \mu_k \{\phi_k, \psi_k\} \approx 0, \quad k = 1, \ldots, m \quad (3.25) \]

In a similar manner for the \( \psi \)-class of primary constraints one defines the corresponding secondary constraints as

\[ \psi_{k+m} = \{p_k^y, U\} + \sum_{k=1}^{m} \lambda_k \{\psi_k, \phi_k\} \approx 0, \quad k = 1, \ldots, m \quad (3.26) \]

Let us now define the matrix of the Poisson brackets between the all the primary and second class constraints as \( C \) which is obviously a skew symmetric \( 4m \times 4m \) matrix.

\[ C = \begin{bmatrix} \{\phi_1, \phi_2\}_{2m \times 2m} & \{\phi_1, \psi_2\}_{2m \times 2m} \\ \{\psi_1, \phi_2\}_{2m \times 2m} & \{\psi_1, \psi_2\}_{2m \times 2m} \end{bmatrix} \]

This turns out to be non-singular and in terms of its elements the time evolution of any variable \( f \) is given by

\[ \dot{f} = \{f, H\} - \{\{f, \phi_1\}, \ldots, \{f, \phi_{2m}\}, \{f, \psi_1\}, \ldots, \{f, \psi_{2m}\}\} C^{-1} \begin{bmatrix} \{\phi_1, H\} \\ \vdots \\ \{\phi_{2m}, H\} \\ \{\psi_1, H\} \\ \vdots \\ \{\psi_{2m}, H\} \end{bmatrix} \quad (3.27) \]

we now present the explicit nature of the calculations stated above by considering the case \( m = 1 \), i.e, for the system (3.1) and (3.2). The JLM it will be recalled is given by \( J = e^\gamma y^\sigma / x \) where \( \gamma = e_{11} d / q \) and \( \sigma + 1 = e_{11} / q \). It follows that the Lagrangian for the system is given by

\[ L = \frac{e^\gamma y^\sigma + 1}{x(\sigma + 1)} \dot{x} - (e^\gamma y^\sigma \log x) \dot{y} - U(x, y, t) \]

where

\[ U(x, y, t) = -e^\gamma y^\sigma + 1 \left[ \frac{2p}{\sigma + 2} y + 2qx - d \log x - \frac{2qb_1}{e_{11}} \right] \]
The Hamiltonian is then found to be $H = U$ and the primary constraint equations which follow from the usual definition of the conjugate momenta are

$$
\phi_1 = p_x - \frac{e^{\gamma t} y^\sigma+1}{x(\sigma + 1)} \approx 0
$$

$$
\phi_2 = p_y + e^{\gamma t} y^\sigma \log x \approx 0
$$

The primary Hamiltonian is therefore given by

$$
H_p = U(x, y, t) + \lambda_1 \phi_1 + \lambda_2 \phi_2
$$

and from the time evolution of the primary constraints we arrive at the following second-class constraints, namely:

$$
\phi_3 = -U_x - \lambda_2 \left( \frac{2e^{\gamma t} y^\sigma}{x} \right) \approx 0
$$

$$
\phi_4 = -U_y + \lambda_1 \left( \frac{2e^{\gamma t} y^\sigma}{x} \right) \approx 0
$$

It is now a straightforward matter to calculate the matrix of the Poisson brackets of the primary and second-class constraints and we find that

$$
C = \begin{pmatrix}
0 & -\frac{2e^{\gamma t} y^\sigma}{x} & -\phi_3 x & -\phi_4 x \\
\frac{2e^{\gamma t} y^\sigma}{x} & 0 & -\phi_3 y & -\phi_4 y \\
\phi_3 x & \phi_3 y & 0 & 0 \\
\phi_4 x & \phi_4 y & 0 & 0
\end{pmatrix}
$$

and its inverse is given by

$$
C^{-1} = \frac{1}{\Delta} \begin{pmatrix}
0 & 0 & \phi_4 y & -\phi_3 y \\
0 & 0 & -\phi_4 x & \phi_3 x \\
-\phi_4 y & -\phi_4 x & 0 & -\frac{2e^{\gamma t} y^\sigma}{x} \\
\phi_3 y & \phi_3 x & \frac{2e^{\gamma t} y^\sigma}{x} & 0
\end{pmatrix}
$$

where $\Delta = \phi_3 x \phi_4 y - \phi_4 x \phi_3 y$. Explicit calculation of the time evolution of the phase space variables $(x, y, p_x, p_y)$ using the Dirac brackets now yields

$$
\dot{x} = 0,
$$

$$
\dot{y} = 0,
$$

$$
\dot{p}_x = (-1 + \Delta) U_x,
$$

$$
\dot{p}_y = (-1 + \Delta) U_y.
$$

We now turn to certain simplified reductions of the KtW model presented above and deduce explicitly the equations of motion for the phase space variables.
3.1.1 Lotka-Volterra model with competition

This is a model similar to the original Lotka-Volterra model but which incorporates the competition between species which is modeled by a term proportional to the product of the populations of the prey and predator. The equations are given by

\[
\dot{x} = x(a_1 - b_1 x - c_1 y) \quad (3.32)
\]

\[
\dot{y} = y(a_2 - b_2 y - c_2 x) \quad (3.33)
\]

where \( a_i, b_i, c_i > 0 \) \( \forall i = 1, 2 \). It is evident by comparison with (3.1) and (3.2) that this model is a special case of the KtW model with an extra term proportional to \( y^2 \) in the second equation. The Jacobi Last Multiplier for this system of equations is given by

\[
\mu = e^{\gamma t} x^\alpha y^\beta
\]

with the exponents being

\[
\alpha = \frac{b_2 c_2 + c_1 c_2 - 2b_1 b_2}{b_1 b_2 - c_1 c_2}
\]

\[
\beta = \frac{b_1 c_1 + c_1 c_2 - 2b_1 b_2}{b_1 b_2 - c_1 c_2}
\]

\[
\gamma = \frac{a_1(b_1 b_2 - b_2 c_2) + a_2(b_1 b_2 - b_1 c_1)}{b_1 b_2 - c_1 c_2}
\]

It turns out that a singular Lagrangian for the above system is given by

\[
L = -e^{\gamma t} x^{\alpha + 1} y^{\beta + 1} \frac{\dot{x}}{\beta + 1} + e^{\gamma t} x^{\alpha + 1} y^{\beta} \frac{\dot{y}}{\alpha + 1} - V(x, y, t)
\]

where

\[
V(x, y, t) = e^{\gamma t} x^{\alpha + 1} y^{\beta + 1} \left[ \frac{2(a_2 - b_2 y)}{\alpha + 1} + \frac{\gamma}{(\alpha + 1)(\beta + 1)} - \frac{2c_2 x}{\alpha + 2} \right]
\]

or alternately

\[
= e^{\gamma t} x^{\alpha + 1} y^{\beta + 1} \left[ -\frac{2(a_1 - b_1 x)}{\beta + 1} - \frac{\gamma}{(\alpha + 1)(\beta + 1)} + \frac{2c_1 y}{\beta + 2} \right]
\]

As characteristic feature of singular Lagrangians is that the Hamiltonian is given by

\[
H = p_x \dot{x} + p_y \dot{y} - L = V(x, y, t)
\]

and is therefore independent of the velocities. The primary constraints are therefore

\[
\phi_1 = p_x - F(x, y, t) \approx 0
\]

\[
\phi_2 = p_y - G(x, y, t) \approx 0
\]

where

\[
F(x, y, t) = -e^{\gamma t} x^{\alpha} y^{\beta + 1} \frac{1}{\beta + 1}
\]
\[ G(x, y, t) = e^{\gamma t} \frac{x^{\alpha} y^{\beta}}{\alpha + 1} \]

respectively. Hence the primary Hamiltonian is

\[ H_p = V + \lambda_1 \phi_1 + \lambda_2 \phi_2. \]

On the other hand the second class constraints are given by

\[ \phi_3 = \dot{\phi}_1 = \{ \phi_1, H_p \} = \{ p_x, V(x, y, t) \} + \lambda_2 \{ \phi_1, \phi_2 \} \approx 0 \]

\[ \phi_4 = \dot{\phi}_2 = \{ \phi_2, H_p \} = \{ p_y, V(x, y, t) \} + \lambda_1 \{ \phi_2, \phi_1 \} \approx 0 \]

that is

\[ \phi_3 = -V_x + \lambda_2 (2e^{\gamma t} x^\alpha y^\beta) \approx 0 \]

\[ \phi_4 = -V_y - \lambda_1 (2e^{\gamma t} x^\alpha y^\beta) \approx 0 \]

The matrix of the Poisson brackets between the constraints is given by

\[
C = \begin{pmatrix}
0 & 2e^{\gamma t} x^\alpha y^\beta & V_{xx} - \alpha V_x/x & V_{xy} + \alpha V_x/x \\
-2e^{\gamma t} x^\alpha y^\beta & 0 & V_{xy} + \beta V_y/y & V_{yy} - \beta V_y/y \\
-V_{xx} + \alpha V_x/x & -V_{xy} - \beta V_y/y & 0 & 0 \\
-V_{xy} - \alpha V_x/x & -V_{yy} + \beta V_y/y & 0 & 0
\end{pmatrix}
\]

This is a non-singular matrix and its inverse is

\[
C^{-1} = \frac{1}{\Delta} \begin{pmatrix}
0 & 0 & -V_{yy} + \beta V_y/y & V_{xy} + \beta V_y/y \\
0 & 0 & V_{xy} + \alpha V_x/x & -V_{xx} + \alpha V_x/x \\
V_{yy} - \beta V_y/y & -V_{xy} - \alpha V_x/x & 0 & 2e^{\gamma t} x^\alpha y^\beta \\
-V_{xy} - \beta V_y/y & V_{xx} - \alpha V_x/x & -2e^{\gamma t} x^\alpha y^\beta & 0
\end{pmatrix}
\]

where \( \Delta = (V_{xx} V_{yy} - V_{xy}^2) - \alpha V_x(V_{xy} + V_{yy})/x - \beta V_y(V_{xy} + V_{xx})/y. \)

The time evolution of a dynamical variable \( f \) is in terms of the Dirac brackets

\[ \dot{f} = \{ f, H \}_{DB} = \{ f, H \} - \{ f, \phi_i \} [C^{-1}]^{ij} \{ \phi_j, H \} \]

and up on using the entries of the matrix \( C^{-1} \) as given above we find that

\[ \dot{x} = 0, \]

\[ \dot{y} = 0, \]

\[ \dot{p}_x = -V_x - \frac{1}{\Delta} \left[ (V_{xx} - \frac{\alpha V_x}{x}) \{ (-V_{yy} + \frac{\beta V_y}{y})V_x + (V_{xy} + \frac{\alpha V_x}{x})V_y \} \\
+ (V_{xy} - \frac{\alpha V_y}{x}) \{ (-V_{xy} + \frac{\beta V_y}{y})V_x + (-V_{xx} + \frac{\alpha V_x}{x})V_y \} \right] \]

\[ \dot{p}_y = -V_y - \frac{1}{\Delta} \left[ (V_{xy} - \frac{\beta V_x}{y}) \{ (-V_{yy} + \frac{\beta V_y}{y})V_x + (V_{xy} + \frac{\alpha V_x}{x})V_y \} \\
+ (V_{yy} - \frac{\beta V_y}{y}) \{ (V_{xy} + \frac{\beta V_x}{y})V_x + (-V_{xx} + \frac{\alpha V_x}{x})V_y \} \right] \]
3.1.2 Lotka-Volterra model without competition

In the absence of competition, the coefficients \((b_1, b_2)\) in the Lotka-Volterra model with competition vanish and a simplified version of the predator-prey system becomes

\[
\begin{align*}
\dot{x} &= ax - bxy \\
\dot{y} &= -cy + dxy
\end{align*}
\]

where \(x\) and \(y\) refer to two species which live in a limited area with individual of the species \(y\) (predator) feed only on the species \(x\) (prey). The parameters \(a, b, c\) and \(d\) are assumed to be positive. The overdots refer to derivatives with respect to time with \(\dot{x}\) representing the growth rate of the prey population and \(\dot{y}\) being the growth rate of the predator population. This model can also be visualized as the special case of the KtW model with the absence of the term proportional to \(x^2\) in the \(\dot{x}\) equation.

One can recast the system (3.34) and (3.35) in the form of Euler-Lagrange equation with the Lagrangian

\[
L = \left( -\frac{\log x}{y} \dot{x} + \frac{\log y}{x} \dot{y} \right) - c \log x + a \log y + dx - by,
\]

and the corresponding momenta are

\[
\begin{align*}
p_x &= \frac{\partial L}{\partial \dot{x}} = -\frac{\log y}{x}, \\
p_y &= \frac{\partial L}{\partial \dot{y}} = \frac{\log x}{y}.
\end{align*}
\]

The Hamiltonian is therefore given by

\[
H = -2c \log x - 2a \log y + 2dx + 2by.
\]

The Hamiltonian is clearly independent of the momentum (or velocity) and is therefore of singular character. In order to employ study the dynamics in the phase space \((x, y, p_x, p_y)\) we treat the momenta as given in (3.37) as the primary constraints and express these in the form

\[
\begin{align*}
\phi_1 &= p_x + \frac{\log y}{x} \approx 0, \\
\phi_2 &= p_y - \frac{\log x}{y} \approx 0
\end{align*}
\]

The primary Hamiltonian is then defined by

\[
H_p = H + \lambda_1 \phi_1 + \lambda_2 \phi_2
\]

where \(\lambda_1\) and \(\lambda_2\) are Lagrange multipliers. In order to ensure that the constraints hold at all times it is necessary that their time evolutions with respect to the Hamiltonian \(H_p\) vanish. In other words we require that

\[
\dot{\phi}_1 = \{\phi_1, H\} = \frac{2c}{x} - 2d + \frac{2\lambda_2}{xy} \approx 0
\]
\[ \dot{\phi}_2 = \{\phi_2, H\} = \frac{2a}{y} - 2b - 2\frac{\lambda_1}{xy} \approx 0 \]

These furnish us with two more constraints which are referred to as the secondary constraints or constraints of the second class namely

\[ \phi_3 = 2c - 2d + 2\frac{\lambda_2}{xy} \approx 0, \quad \phi_4 = 2a - 2b - 2\frac{\lambda_1}{xy} \approx 0 \quad (3.41) \]

Thus we have in all four constraints. Let \( C \) denote that matrix formed by the Poisson brackets between \( \phi_i \) and \( \phi_j \), i.e, \( C_{ij} = \{\phi_i, \phi_j\} \). The \( \approx \) sign here means we can only substitute the value of \( \lambda_i \) from the secondary constraint equations after working out the respective Poisson brackets. With this in mind it is found that

\[
C = \frac{2}{xy} \begin{pmatrix}
0 & 1 & dy & -(a - by) \\
-1 & 0 & -(c - dx) & bx \\
dy & (c - dx) & 0 & 0 \\
(a - by) & -bx & 0 & 0
\end{pmatrix} \quad (3.42)
\]

It may be verified that \( C \) is non-singular and its inverse is given by

\[
C^{-1} = \frac{2}{xy\xi} \begin{pmatrix}
0 & 0 & -bx & -(c - dx) \\
0 & 0 & -(a - by) & -dy \\
-bx & (a - by) & 0 & 1 \\
(c - dx) & dy & -1 & 0
\end{pmatrix} \quad (3.43)
\]

where \( \det C = \xi = adx + bcy - ac \). The time evolution of any dynamical quantity in terms of the Dirac bracket is defined as

\[
\dot{f} = \{f, H\}_{DB} = \{f, H\} - \{f, \phi_i\}[C^{-1}]^{ij}\{\phi_j, H\} \quad (3.44)
\]

Using this definition we find in the present case the following equations of motion.

\[
\dot{x} = 0, \quad \dot{p}_x = \frac{2(c - dx)}{x} \left(1 - \frac{4}{x^2y^2}\right) \quad (3.45)
\]

\[
\dot{y} = 0, \quad \dot{p}_y = \frac{2(a - by)}{y} \left(1 - \frac{4}{x^2y^2}\right) \quad (3.46)
\]

### 3.1.3 Kermack-McKendreck model

A simplified version of this model which is often cited in the literature is given by the following system of differential equations:

\[
\dot{x} = -k_1xy \quad (3.47)
\]

\[
\dot{y} = k_1xy - k_2y \quad (3.48)
\]
with $k_1$ and $k_2$ being positive constants. This model can also be derived from the KtW model by suppressing some coefficients. The system may be derived from the Euler-lagrange equations with the Lagrangian

$$L = \frac{1}{2} \left( \frac{\log y}{x} \dot{x} - \frac{\log x}{y} \dot{y} \right) + k_1(x + y) - k_2 \log x$$

(3.49)

The conjugate momenta are obtained from

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\log y}{2x},$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = -\frac{\log x}{2y},$$

and are obviously velocity independent as the Lagrangian is singular in character. This necessitates that we defined the primary constraints by

$$\phi_1 = p_x - \frac{\log y}{2x} \approx 0, \quad \phi_2 = p_y + \frac{\log x}{2y} \approx 0.$$  

(3.50)

The Hamiltonian therefore has the appearance

$$H = p_x \dot{x} + p_y \dot{y} - L = -k_1(x + y) + k_2 \log x.$$  

(3.51)

It is easy to verify that $H$ is a constant of motion. We define the primary Hamiltonian by

$$H_p = H + \lambda_1 \phi_1 + \lambda_2 \phi_2.$$  

(3.52)

In order to ensure that the primary constraints hold at all times it is necessary that $\dot{\phi}_i = \{\phi_i, H_p\} i = 1, 2$ vanish. This in turn leads us to the secondary constraints, namely

$$\dot{\phi}_1 = \{\phi_1, H_p\} = \{\phi_1, H\} + \lambda_2 \{\phi_1, \phi_2\}$$

whence we have

$$\phi_3 = k_1 - \frac{k_2}{x} - \frac{\lambda_2}{xy} \approx 0.$$  

(3.53)

Similarly considering the vanishing of the time evolution of $G_2$ we find that

$$\phi_4 = k_1 + \frac{\lambda_1}{xy} \approx 0.$$  

(3.54)

The matrix of the Poisson brackets of the primary and secondary constraints is then given by

$$C = \begin{pmatrix}
  0 & -\frac{1}{xy} & -\frac{k_2}{x} & -\frac{k_1}{x} \\
  \frac{1}{xy} & 0 & -\frac{k_1}{y} & \frac{k_2}{xy} \\
  \frac{k_1}{x} & \frac{k_1}{y} & 0 & 0 \\
  \frac{k_1}{x} & \frac{k_2}{xy} & 0 & 0
\end{pmatrix}$$  

(3.55)
This is a non-singular matrix and its inverse is given by

\[
C^{-1} = \frac{x}{k_1 k_2} \begin{pmatrix}
0 & 0 & k_1 x & -k_1 x + k_2 \\
0 & 0 & -k_1 y & k_1 y \\
-k_1 x & k_1 y & 0 & -1 \\
k_1 x - k_2 & -k_1 y & 1 & 0
\end{pmatrix}
\]  \tag{3.56}

If we go back to the definition of the Dirac bracket it will be realized that the equations of motion of the phase space variables are now obtained from

\[
\dot{f} = \{f, H\}_{DB} = \{f, H\} - \frac{1}{k_1 k_2} (\{f, \phi_1\}, \{f, \phi_2\}, \{f, \phi_3\}, \{f, \phi_4\}) \begin{pmatrix}
0 \\
0 \\
-k_1^2 (x - y) + k_1 k_2 x \\
(k_1 x - k_2)^2 - k_1^2 x y
\end{pmatrix}
\]

It may be verified that this implies

\[
\begin{align*}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= (k_1 k_2 - 1) \left( -k_1 + \frac{k_2}{x} \right) \\
\dot{p}_y &= k_1 - k_1^2 k_2
\end{align*}
\]

### 3.2 Gierer-Meinhardt Model

This is a reaction-diffusion model which deals with the formation of patterns. A simplified version of this model neglecting the effects of diffusion is given by

\[
\begin{align*}
\dot{x} &= x^2 y - ax \\
\dot{y} &= b - x^2 y - y.
\end{align*}
\]  \tag{3.57}
\tag{3.58}

In order to express this system in the form of Euler-Lagrangian equations, it is found that unless the parameter \( b = 0 \), we cannot find a Lagrangian. Simple calculations show that with \( b = 0 \) one can derive a Jacobi Last Multiplier for this system which is given by

\[
\mu = \frac{e^{-at}}{x^2 y}
\]

and the Lagrangian (singular) is of the form

\[
L = e^{-at} \left[ \frac{\log y}{x^2} \dot{x} + \frac{1}{xy} \dot{y} - \left( 2(x - \frac{1}{x}) + 2y - a \frac{\log y}{x} \right) \right]
\]

The Hamiltonian is then given by

\[
H = e^{-at} \left[ 2(x - \frac{1}{x}) + 2y - a \frac{\log y}{x} \right]
\]
The primary constraints are therefore given by

\[ \phi_1 = p_x - e^{-at} \frac{\log y}{x^2} \approx 0 \]

\[ \phi_2 = p_y - e^{-at} \frac{1}{xy} \approx 0 \]

Consequently we define the primary Hamiltonian as

\[ H_p = H + \lambda_1 \phi_1 + \lambda_2 \phi_2 \]

By demanding that the time evolution of the primary constraints vanish we are led to the following secondary constraints, as already explained earlier, namely

\[ \phi_3 = e^{-at} \left( 2 + \frac{2}{x^2} + \frac{a}{x^2} \log y + \frac{2\lambda_2}{x^2y} \right) \approx 0 \]

\[ \phi_4 = e^{-at} \left( \frac{a}{xy} - 2 + \frac{2\lambda_1}{x^2y} \right) \approx 0 \]

The matrix of the Poisson brackets of the primary and secondary constraints is given by

\[
C = \frac{2e^{-at}}{x^2y} \begin{pmatrix}
0 & -1 & -2xy & 2xy - \frac{a}{2} \\
1 & 0 & \epsilon & \frac{x^2}{2} \\
2xy & -\epsilon & 0 & 0 \\
-2xy + \frac{a}{2} & -x^2 & 0 & 0
\end{pmatrix}
\]

where

\[ \epsilon = -(x^2 + 1 + \frac{a}{2} + \frac{a}{2} \log y) \]

Straightforward calculations give the inverse of the matrix \( M \) to be

\[
C^{-1} = \frac{e^{at} x^2 y}{2\delta} \begin{pmatrix}
0 & 0 & -x^2 & \epsilon \\
0 & 0 & -2xy - \frac{a}{2} & 2xy \\
x^2 & -2xy + \frac{a}{2} & 0 & 1 - \epsilon \\
-\epsilon & -2xy & 0 & 1
\end{pmatrix}
\]

where

\[ \delta = 2x^3 y - \epsilon (2xy - \frac{a}{2}) \]

The calculation of the Dirac brackets and the equations of motion of the phase space variables follow in the usual manner as in the earlier examples.
4 Conclusion

In the diverse field of microbial systems where the mutation rates of different species are very high and also in the field of quantitative biology like prey-predator system, host-parasite model etc. where the prey predator relationships becomes much more complicated with respect to the time parameter exact analytical results are often rare. In this paper we have attempted to provide a Lagrange/Hamiltonian description of an important stochastic model, popularly referred to as the ”Kill the Winner” model. It is evident that such a description is facilitated by the existence of a Jacobi Last Multiplier which is not very well known outside the community of mathematicians working on systems of ordinary differential equations. Its existence leads quite naturally to a Lagrange description of the model equations, which in the present case turns out to be of a singular character. As a result the corresponding Hamiltonian description of the phase space variables requires the introduction of Dirac brackets. We have presented explicit results for a number of models such a predator-prey systems with and without competition, pattern formation equations and of course the KtW model which forms the cornerstone of the article.

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