ANALYSIS OF A DYNAMIC PREMIUM STRATEGY: FROM THEORETICAL AND MARKETING PERSPECTIVES

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(Communicated by Ken Siu)

Abstract. Premium rate for an insurance policy is often reviewed and updated periodically according to past claim experience in real-life. In this paper, a dynamic premium strategy that depends on the past claim experience is proposed under the discrete-time risk model. The Gerber-Shiu function is analyzed under this model. The marketing implications of the dynamic premium strategy will also be discussed.

1. Introduction. Determining an optimal premium strategy is always an important issue to insurance companies. A good premium strategy should not just serve the purpose of keeping the risk within the tolerance level but at the same time be attractive to customers. In the literature, constant premium rate is a common assumption. This is a fair assumption for large portfolios of insurance policies. However, a flat premium rate for all policyholders maybe unfair, and hence unattractive, to individual ones because the severity of an individual risk varies and risks from the pool are not identical to each other. With the more flexible premium strategy studied in this paper, the price of the insurance product can be set to be more fair and competitive. We will compare the constant premium strategy and the dynamic premium strategy from the marketing point of view, which is seldom discussed in detail in the literature.

In the risk theory literature, there has been accrued interest in dynamic premium strategy. Some recent examples that considered dynamic premium strategy in the continuous-time risk model are as follows. [2] updated the premium according to Buhlmann’s credibility model and calculated the ruin probabilities numerically. [12] studied the analytical form of Gerber-Shiu function when the premium rate depends only on the previous claim size. [5] (Chapter VIII) considered a risk process where the premium rate depends on the current surplus level, while [10] reviewed the premium rate whenever the surplus level reaches a new minimum. Later, [11] and

2010 Mathematics Subject Classification. Primary: 60E99, 60D05.
Key words and phrases. Dynamic premium, discrete-time risk model, Gerber-Shiu function, geometric claims, marketing implications.

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studied the case when premium rate varies according to the surplus increments between consecutive review times.

There are few results on the discrete-time risk model with varying premium when compared to the continuous-time risk model. However, business is usually reviewed periodically in real-life. [14] studied a discrete-time risk model with credibility premium. The ruin probabilities under the assumption of exponential claims with Poisson arrival was calculated by Monte-Carlo simulation. [13] investigated the bounds and asymptotic results for ruin probabilities in the cases of light-tailed claims and heavy-tailed claims respectively.

In this paper, we will study a dynamic premium strategy under the discrete-time risk model. The premium rate will be updated according to the amount of the most recent claim. Take health insurance as an example. If a large claim is reported, it will be taken as a warning sign that the policyholder’s current health condition turns “bad” and may report large claims later on. A higher premium rate will be charged to the policyholder. On the contrary, should the reported claim amount be small or even zero, it is reasonable to believe that the policyholder keeps or regains the “good” health condition and will be charged a lower premium rate in the next period. In short, the most recent claim experience best reflects the current health condition of the policyholder and can be used as the basis for insurance company to determine the amount of premium charged in the next period.

From practical point of view, the dynamic premium strategy proposed in this paper is an experience rating strategy and has a close relation to the credibility theory in insurance mathematics. Studies on credibility-based premium risk models can be found in [4], [3] and [1] and references therein. More recently, [13] provides a study on the effect of the horizon of credibility on ruin probability by assuming Buhlmann credibility premium. It turns out that the ultimate ruin probability may not be necessarily reduced by choosing a longer period of past claim experiences. This gives us a motivation to consider the dynamic premium strategy in this paper that only depends on the past one period. The advantages are two-fold. First, the calculation is mathematically tractable. Second, the premium rate is set in advance in this strategy. This benefits insurance companies from the marketing perspective, which will be discussed in detail later.

In next section, the discrete-time risk model with varying premium rate will be introduced. In Section 3, structural properties of the Gerber-Shiu function will be studied. The Gerber-Shiu functions given different starting premium rates will be shown to follow a recurrence relation. In Section 4, explicit solution for the Gerber-Shiu functions is derived under the assumption of geometric claims. There are two reasons to study the special case of geometric claims. First, the result gives us insight on the form of the Gerber-Shiu functions under the dynamic premium strategy. The Gerber-Shiu functions are found to have the same form irrespective of the starting premium rate. Second, since the result is tractable, it is valuable and useful for further analysis from the marketing perspective in Section 5.

2. The model. Consider the following discrete-time insurer’s surplus process

\[
U(t) = u + \sum_{j=1}^{t} \eta_j - \sum_{j=1}^{N(t)} X_j, \quad t \in \mathbb{N}
\]  

(1)

where \( u \in \mathbb{N} \) is the insurer’s initial surplus level, \( \eta_j \) is the premium charged for the \( j \)th period, and \( \{X_j\}_{j=1}^{\infty} \) are independent and identically distributed (iid) claim size
random variables with probability mass function (pmf) \( p(x) \) for \( x = 1, 2, \ldots \). The claim number process \( \{N(t), t \in \mathbb{Z}^+\} \) is defined through a sequence of iid interclaim time random variables \( \{W_j\}_{j=1}^{\infty} \). Denote the pmf of \( W_j \) by \( k(t) \) for \( t = 1, 2, \ldots \). Assume that \( \{W_j\}_{j=1}^{\infty} \) and \( \{X_j\}_{j=1}^{\infty} \) are all mutually independent. Since time value of money is not considered in (1), it is reasonably to assume that the insurer will receive premium and pay the indemnity for claims, if any, at the end of each period.

Define the time to ruin as

\[
T = \inf\{t \in \mathbb{N}^+: U(t) < 0\},
\]

where \( T = \infty \) if \( U(t) > 0 \) for all \( t \in \mathbb{N}^+ \), by convention. Let \( U(T - 1) \) be the surplus prior to ruin and \( |U(T)| \) be the deficit at ruin.

Also, define the surplus process immediately after the \( n \)th claim \( \{R(n), n \in \mathbb{N}\} \) as follows

\[
R(0) = u \text{ and } R(n) = u + \eta_1 + \eta_2 + \cdots + \eta \sum_{j=1}^{n} W_j - \sum_{j=1}^{n} X_j, \quad (2)
\]

where \( \sum_{j=1}^{n} W_j \) is the time to the \( n \)th claim for \( n \in \mathbb{N}^+ \). Thus, \( R(N(T) - 1) \) represents the surplus immediately after the second last claim before ruin for \( N(T) > 1 \), and \( R(N(T) - 1) = u \) for \( N(T) = 1 \). The surplus immediately after the second last claim before ruin, \( R(N(T) - 1) \), is introduced to the Gerber-Shiu analysis in [6]. As mentioned therein, through the analysis of \( R(N(T) - 1) \), the pair of interclaim time and claim size just before ruin is considered. This information is important for risk management purpose. Interested readers can refer to e.g. [7] and [8] for further analysis involving \( R(N(T) - 1) \).

The premium process \( \{\eta_j\}_{j=1}^{\infty} \) in (1) has a dynamic setting. There are two levels of premium rate, \( c_1 \) and \( c_2 \), which are set in advance. Assume without loss of generality that \( c_1 \leq c_2 \). If the policyholder is classified as “good”, then the premium rate charged to the policyholder is \( c_1 \). On the other hand, if the policyholder is classified as “bad”, then the premium rate charged to the policyholder is \( c_2 \). Such classification is updated at the beginning of each time period according to the most recent claim. If the claim amount is greater than or equal to the current premium charged, then it is a warning to the insurer that the current premium rate may be too low. The policyholder will be classified as “bad” and be charged \( c_2 \) for the next time period. On the contrary, if the claim amount is strictly less than the premium charged or there is no claim, the policyholder will be classified as “good” and be charged \( c_1 \) for the next time period. In short, the premium schedule can be expressed as

\[
\eta_{j+1} = \begin{cases} 
  c_2, & \text{if the claim at time } j \text{ is greater than or equal to } \eta_j, \\
  c_1, & \text{otherwise,}
\end{cases} \quad (3)
\]

with the starting premium \( \eta_1 \), where \( c_1 \leq c_2 \) and \( c_1, c_2 \in \mathbb{N}^+ \). The starting premium \( \eta_1 \) will be determined by the insurance company based on its marketing strategy and basic information of the policyholder. Let \( \delta = c_2 - c_1 \), then \( \delta \in \mathbb{N} \) and may be interpreted as an extra risk loading. Furthermore, the positive security loading condition, \( c_1 \mathbb{E}[W_1] > \mathbb{E}[X_1] \), is assumed to be satisfied.

Given a discount factor \( v \in [0, 1] \) and a non-negative penalty function \( w(x, y, r) \), the Gerber-Shiu functions for the surplus process (1) with premium schedule (3)
starting from \( \eta_1 = c_i, \ i = 1, 2 \), are in the form
\[
m_{c_i,v}(u) = \mathbb{E} \left[ v^T w \left( U(T-1) \right) \right] \mathbb{I}(T < \infty) \left| U(0) = u, \eta_1 = c_i \right., \tag{4}
\]
where \( \mathbb{I}(A) \) is an indicator function that equals to 1 if event \( A \) occurs and 0 otherwise.

The concept of Gerber-Shiu function was first introduced in [9], and it becomes a powerful tool for analyzing surplus process soon after that. In particular, [16] analyzed the type of Gerber-Shiu function shown in (4) in discrete-time risk model with \( c_1 = c_2 = 1 \). A special case of (4), when \( w(x,y,r) = w^*(y) \), is
\[
m^{*}_{c_i,v}(u) = \mathbb{E} \left[ v^T w^* \left( (U(T)) \right) \mathbb{I}(T < \infty) \right| U(0) = u, \eta_1 = c_i \right. \tag{5}
\]
Throughout this paper, denote the generating function of an arbitrary function \( a(\cdot) \), which has support on all non-negative integers, by \( \hat{a}(s) = \sum_{u=0}^{\infty} s^u a(u) \). Also, the notational convention from which empty sum equals to zero is adopted.

3. Structural analysis of Gerber-Shiu function. In this section, we analyze the structural properties of the Gerber-Shiu functions defined in (4). The Gerber-Shiu functions will be shown to satisfy a recurrence relation.

3.1. Discounted probability functions. To start with, consider the case when ruin occurs on the first claim. If the insured is initially classified to be “good”, i.e. \( \eta_1 = c_1 \), then the premium rate will remain to be \( c_1 \) until ruin (since there are no claims until ruin). The surplus prior to ruin is the initial surplus accumulated with \( T-1 \) periods of premium, i.e. \( U(T-1) = u + c_1(T-1) \), and therefore \( T = \frac{U(T-1) - u}{c_1} + 1 \). Meanwhile, \( R(N(T) - 1) = R(0) = u \) since \( N(T) = 1 \). The joint defective probability function (pf) of the surplus prior to ruin \( (x) \) and the deficit at ruin \( (y) \) when \( \eta_1 = c_1 \) is given by
\[
h_{c_1,1}(x,y|u) = k \left( \frac{x-u}{c_1} + 1 \right) p(x+y+c_1),
\]
for \( x \in \mathbb{N} \) and \( y \in \mathbb{N}^+ \). It should be noted that \( \frac{x-u}{c_1} + 1 \) may not be an integer, while non-integer cases would not occur under our model assumption. Therefore, it is reasonable to assume that \( k(x) = 0 \) when \( x \) is not a positive integer.

If the insured is initially classified to be “bad”, i.e. \( \eta_1 = c_2 \), then the premium rate will be \( c_2 \) in the first period and will reduce to \( c_1 \) in the subsequent periods (since there are no claims until ruin). The joint defective pf of the surplus prior to ruin \( (x) \) and the deficit at ruin \( (y) \) when \( \eta_1 = c_2 \) is given by
\[
h_{c_2,1}(x,y|u) = \begin{cases} 
    k(1)p(x+y+c_2), & \text{if } x = u, \\
    k \left( \frac{x-u-c_2}{c_1} + 2 \right) p(x+y+c_1), & \text{if } x > u
\end{cases}
\]
\[
= k \left( \frac{x-u}{c_1} + 1 - \frac{\delta}{c_1} \mathbb{I} \{ x > u \} \right) p \left( x+y+c_1+\delta \mathbb{I} \{ x = u \} \right),
\]
for \( x \in \mathbb{N} \) and \( y \in \mathbb{N}^+ \). Also, \( R(N(T) - 1) = u \).

Define the discounted joint pf corresponding to \( h_{c_1,1} \) and \( h_{c_2,1} \) respectively as
\[
g_{c_1,1,u}(x,y|u) = v^{\frac{x-u}{c_1}+1} h_{c_1,1}(x,y|u), \tag{6}
\]
and
\[
g_{c_2,1,u}(x,y|u) = v^{\frac{x-u}{c_1}+1-\frac{\delta}{c_1} \mathbb{I} \{ x > u \}} h_{c_2,1}(x,y|u). \tag{7}
\]
Next, consider the case when ruin occurs on a claim subsequent to the first. For any surplus process with initial premium rate \( \eta_1 = c_i, \ i = 1, 2 \), the premium rate
just after the second last claim could be either $c_1$ or $c_2$. When $\eta_1 = c_i$, $i = 1, 2$, define the joint defective pf of $(T, U(T - 1), |U(T)|, R(N(T) - 1))$ at $(t, x, y, r)$ such that $t = 2, 3, \ldots; x \in \mathbb{N}; y \in \mathbb{N}^r; r = 0, 1, \ldots, x$, by
\begin{equation}
    h_{c_i,1}(t, x, y, r | u) = h_{c_i,1,2}(t, x, y, r | u) + h_{c_i,2,2}(t, x, y, r | u),
\end{equation}
where $h_{c_i,1,2}(t, x, y, r | u)$ is the joint defective pf when $\eta_1 = c_i$ and the premium rate just after the second last claim is $c_j$, $j = 1, 2$. Furthermore, it can be argued probabilistically that, for $i = 1, 2$,
\begin{equation}
    h_{c_i,1,2}(t, x, y, r | u) = \left\{ \sum_{a=1}^{\infty} h_{c_i,1,2}(t, x, a, r | u) \right\} \frac{h_{c_i,1}(x, y | r)}{\sum_{z=1}^{\infty} h_{c_i,1}(x, z | r)}
\end{equation}
and
\begin{equation}
    h_{c_i,2,2}(t, x, y, r | u) = \left\{ \sum_{a=1}^{\infty} h_{c_i,2,2}(t, x, a, r | u) \right\} \frac{h_{c_i,2}(x, y | r)}{\sum_{z=1}^{\infty} h_{c_i,2}(x, z | r)}
\end{equation}

The discounted joint pf corresponding to $h_{c_i,2}$, for $i = 1, 2$, are defined as
\begin{equation}
    g_{c_i,2,v}(x, y, r | u) = \sum_{t=2}^{\infty} v^t h_{c_i,2}(t, x, y, r | u).
\end{equation}

### 3.2. Recurrence relation.

Now, we will study the Gerber-Shiu function by conditioning on the first drop below initial surplus $u$ (referred to as first drop below). The following proposition shows that $m_{c_1,v}(u)$ and $m_{c_2,v}(u)$ satisfy a recurrence relation.

**Proposition 1.** The Gerber-Shiu functions $m_{c_1,v}(u)$ and $m_{c_2,v}(u)$ satisfy the following recurrence relation.

\begin{equation}
    m_{c_1,v}(u) = \phi_{c_1,v} \sum_{y=1}^{u} m_{c_2,v}(u-y) f_{c_1,v}(y) + \xi_{c_1,v}(u)
\end{equation}

and
\begin{equation}
    m_{c_2,v}(u) = \phi_{c_2,v} \sum_{y=1}^{u} m_{c_2,v}(u-y) f_{c_2,v}(y) + \xi_{c_2,v}(u),
\end{equation}

where
\begin{equation}
    \xi_{c_i,v}(u) = \sum_{y=u+1}^{\infty} \sum_{x=0}^{\infty} \left\{ w(x + u, y - u, u) g_{c_i,1,v}(x, y | 0) \right. \\
    \left. + \sum_{r=0}^{x} w(x + u, y - u, r + u) g_{c_i,2,v}(x, y, r | 0) \right\},
\end{equation}
Also,
\[
\phi_{c_i,v}(u) = \sum_{y=1}^{\infty} \left\{ \sum_{x=0}^{\infty} g_{c_i,1,v}(x,y|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{x} g_{c_i,2,v}(x,y,r|0) \right\},
\]
and
\[
f_{c_i,v}(y) = \frac{1}{\phi_{c_i,v}} \left\{ \sum_{x=0}^{\infty} g_{c_i,1,v}(x,y|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{x} g_{c_i,2,v}(x,y,r|0) \right\},
\]
for \( y \in \mathbb{N}^+ \), which is the ladder height pf.

Proof. Suppose that the insured starts with premium rate \( \eta_1 = c_i \), where \( i = 1, 2 \). If the first drop occurs on the first claim, then \( h_{c_i,1}(x,y|0) \) gives the joint pf of the amount of surplus above \( u \) one period before the drop \( (x) \) and the amount of drop below \( u \) \( (y) \). If the first drop occurs on claim subsequent to the first, then \( h_{c_i,2}(t,x,y,r|0) \) gives the joint pf the time to the first drop \( (t) \), the amount of surplus above \( u \) one period before the drop \( (x) \), the amount of drop below \( u \) \( (y) \), and the amount of surplus above \( u \) immediately after the second last claim before the drop \( (r) \). In case of \( y \leq u \) \( (\text{the first drop does not cause ruin}) \), the surplus process starts again with initial capital \( u - y \) and premium rate \( c_2 \). It should be pointed out here that, the fact of dropping below the initial capital implies that the claim causes the first drop must be larger than the premium charged just before the claim. Therefore, immediately after the first drop, the premium rate should be \( c_2 \). In case of \( y > u \), the first drop causes ruin. If this first drop occurs on the first claim, then \( U(T - 1) = x + u, |U(T)| = y - u, \) and \( R(N(T) - 1) = u \). If this first drop occurs on claim subsequent to the first, then \( U(T - 1) = x + u, |U(T)| = y - u, \) and \( R(N(T) - 1) = r + u \). Therefore, for \( i = 1, 2 \) and \( u \in \mathbb{N} \),
\[
m_{c_i,v}(u) = \sum_{y=1}^{u} m_{c_2,v}(u-y) \left\{ \sum_{x=0}^{\infty} g_{c_i,1,v}(x,y|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{x} g_{c_i,2,v}(x,y,r|0) \right\},
\]
where \( \xi_{c_i,v}(u) \) is given by (14). With definitions (15) and (16), it follows that (17) can be rewritten as (12) and (13).

Equations (12) and (13) show that the form of \( m_{c_1,v}(u) \) depends on the form of \( m_{c_2,v}(u) \). Therefore, in next section, we are going to first determine the form of \( m_{c_2,v}(u) \), which will then be used to find out the form of \( m_{c_1,v}(u) \).

4. Explicit solution for geometric claim sizes. In this section, explicit form of Gerber-Shiu function will be derived by assuming that the claim sizes \( \{X_j\}_{j=1}^{\infty} \) follow geometric distribution. The pf of claim sizes is given by
\[
p(x) = (1-q)x^{-1}q, \quad \forall x = 1, 2, \ldots,
\]
where \( 0 < q < 1 \). The analytical results for the Gerber-Shiu function \( m_{c_i,v}(u) \) shown in (5), in which the penalty function includes only the deficit at ruin, are derived in the following.

Lemma 4.1. Suppose that (18) holds. Then the ladder height probability functions \( f_{c_i,v}(y), i = 1, 2 \), given in (16) follow the same geometric distribution as \( X_i \).
Proof. Using (18), the discounted joint pf (6) becomes
\[ g_{c,1,v}(x, y|u) = g_{c,1,v}(x|u)(1 - q)^{y-1}q \] (19)
where
\[ g_{c,1,v}(x|u) \triangleq v \frac{x-u}{c_1} + 1 \frac{(x-u)^{(c_1)} + 1}{1 - q^{x+c_1}}, \]
and (7) becomes
\[ g_{c,2,v}(x, y|u) = g_{c,2,v}(x|u)(1 - q)^{y-1}q \] (20)
where
\[ g_{c,2,v}(x|u) \triangleq v \frac{x-u+1-\delta}{c_1(x>u)}k \left( \frac{x-u}{c_1} + 1 - \delta \right) \delta \{x > u\} (1 - q)^{x+c_1+\delta(x=u)} \]
respectively.
Moreover, under (18), the joint defective pf (9) can be simplified as
\[ h_{c_i,v}(t, x, y, r|u) = \left( \sum_{a=1}^{\infty} h_{c_i,v}(t, x, a, r|u) \right) \frac{(1 - q)^{x+y+c_1-1}q}{(1 - q)^{x+c_1}} \]
for \( i = 1, 2 \). Also, the joint defective pf (10) becomes
\[ h_{c_i,v}(t, x, y, r|u) \triangleq \left( \sum_{a=1}^{\infty} h_{c_i,v}(t, x, a, r|u) \right) (1 - q)^{y-1}q \] (21)
for \( i = 1, 2 \). Using (21), (22) and (8), we have
\[ h_{c_i,v}(t, x, y, r|u) = \left( \sum_{a=1}^{\infty} \left\{ h_{c_i,v}(t, x, a, r|u) + h_{c_i,v}(t, x, a, r|u) \right\} \right) (1 - q)^{y-1}q, \]
(23)
Then, substituting (23) into (11) yields
\[ g_{c_i,v}(x, y, r|u) = g_{c_i,v}(x, r|u)(1 - q)^{y-1}q, \] (24)
where
\[ g_{c_i,v}(x, r|u) \triangleq \sum_{t=2}^{\infty} \sum_{y=1}^{\infty} v^t h_{c_i,v}(t, x, y, r|u). \]
Next, let us consider the ladder height pf given in (16). By using (19), (20) and (24), it yields that
\[ f_{c_i,v}(y) = \frac{1}{\phi_{c_i,v}} \left\{ \sum_{x=0}^{\infty} g_{c_i,1,v}(x|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{\infty} g_{c_i,2,v}(x, r|0) \right\} (1 - q)^{y-1}q \]
\[ = \theta_{c_i,v}(1 - q)^{y-1}q, \]
where

\[
\theta_{c_i,v} \triangleq \frac{1}{\phi_{c_i,v}} \left\{ \sum_{x=0}^{\infty} g_{c_i,1,v}(x|0) + \sum_{x=0}^{\infty} \sum_{r=0}^{x} g_{c_i,2,v}(x,r|0) \right\}.
\]

Note that

\[
1 = \sum_{y=1}^{\infty} f_{c_i,v}(y) = \theta_{c_i,v} \sum_{y=1}^{\infty} (1 - q)^{y-1} q = \theta_{c_i,v}.
\]

Thus,

\[
f_{c_i,v}(y) = (1 - q)^{y-1} q
\]

for \(i = 1, 2\), which is the same geometric distribution as \(X_i\).

Lemma 4.1 shows that the ladder height probability functions follow geometric distribution, irrespective of whether \(\eta_1 = c_1\) or \(\eta_1 = c_2\).

From the proof of Lemma 4.1, it shows that it is necessary to incorporate the surplus prior to ruin and the second last claim before ruin into the analysis even when the penalty function depends on deficit at ruin only. The surplus prior to ruin is used to determine the claim size causing ruin. The second last claim before ruin is important as the analysis depends on the premium rate and the surplus immediately after the second last claim before ruin. The surplus prior to ruin and the second last claim before ruin are useful tools to make the analysis mathematically tractable.

Next, we determine the explicit form of Gerber-Shiu functions \(m_{c_i,v}^*(u)\), \(i = 1, 2\).

**Lemma 4.2.** Suppose that (18) holds and assume that \(\phi_{c_2,v}\) is non-zero. Then, the Gerber-Shiu functions defined in (5) have the forms

\[
m_{c_1,v}^*(u) = Q_{c_1,1}(1 - R)^u + Q_{c_1,2}(1 - q)^u,
\]

\[
m_{c_2,v}^*(u) = Q_{c_2}(1 - R)^u,
\]

for some unknown constants \(Q_{c_1,1}, Q_{c_1,2}, Q_{c_2}\) and \(R\).

**Proof.** The proof will be separated into two cases according to the initial premium level.

1. **Suppose that \(\eta_1 = c_2\).** From (13), it follows that

\[
m_{c_2,v}^*(u) = \phi_{c_2,v} \sum_{y=1}^{u} m_{c_2,v}^*(u - y)f_{c_2,v}(y) + \xi_{c_2,v}^*(u)
\]

with

\[
\xi_{c_2,v}^*(u) = \sum_{y=u+1}^{\infty} w^*(y-u) \sum_{x=0}^{\infty} \left\{ g_{c_2,1,v}(x,y|0) + \sum_{r=0}^{x} g_{c_2,2,v}(x,y,r|0) \right\}
\]

\[
= \sum_{y=u+1}^{\infty} w^*(y-u)\phi_{c_2,v}f_{c_2,v}(y) = \sum_{y=u+1}^{\infty} w^*(y-u)\phi_{c_2,v}(1 - q)^{y-1} q
\]

\[
= \alpha_{c_2,v}(1 - q)^u q,
\]

and

\[
\alpha_{c_2,v} = \phi_{c_2,v} \sum_{y=u+1}^{\infty} w^*(y-u)(1 - q)^{y-u-1} = \phi_{c_2,v} \sum_{y=1}^{\infty} w^*(y)(1 - q)^{y-1}.
\]
Take the generating functions on both sides of (28) yields
\[ \hat{m}_{c_2,v}(z) = \phi_{c_2,v}\hat{m}_{c_2,v}(z)\hat{f}_{c_2,v}(z) + \hat{\xi}_{c_2,v}(z). \] (30)

Applying (25) and (29) to (30) leads to
\[ \hat{m}_{c_2,v}(z) = \phi_{c_2,v}\hat{m}_{c_2,v}(z)\frac{zq}{1-z(1-q)} + \alpha_{c_2,v}\frac{q}{1-z(1-q)}, \]
which gives
\[ \hat{m}_{c_2,v}(z) = \frac{\alpha_{c_2,v}q}{1-z(1-q)} + \phi_{c_2,v}zq. \] (31)

Invert (31) with respect to \( z \), we obtain (27), where \( Q_{c_2} = \alpha_{c_2,v}q \) and \( R = q(1-\phi_{c_2,v}) \).

2. Suppose that \( \eta_1 = c_1 \). From (12), we have
\[ m_{c_1,v}^*(u) = \phi_{c_1,v}\sum_{y=1}^{u} m_{c_2,v}(u-y)f_{c_1,v}(y) + \xi_{c_1,v}(u), \] (32)
where
\[ \xi_{c_1,v}(u) = \sum_{y=u+1}^{\infty} w^*(y-u)\sum_{x=0}^{\infty} \left\{ g_{c_1,1,v}(x,y,0) + \sum_{r=0}^{x} g_{c_1,2,v}(x,y,r) \right\}, \]
\[ = \sum_{y=u+1}^{\infty} w^*(y-u)\phi_{c_1,v} f_{c_1,v}(y) = \sum_{y=u+1}^{\infty} w^*(y-u)\phi_{c_1,v}(1-q)^{y-1}q \]
\[ = \alpha_{c_1,v}(1-q)^u q \] (33)
with
\[ \alpha_{c_1,v} = \phi_{c_1,v}\sum_{y=1}^{\infty} w^*(y)(1-q)^{y-1}. \]

Consider the generating function of (32), which gives
\[ \hat{m}_{c_1,v}(z) = \phi_{c_1,v}\hat{m}_{c_2,v}(z)\hat{f}_{c_1,v}(z) + \hat{\xi}_{c_1,v}(z). \]

By using (25), (33) and (31), we get
\[ \hat{m}_{c_1,v}(z) = \phi_{c_1,v}\hat{m}_{c_2,v}(z)\frac{zq}{1-z(1-q)} + \frac{\alpha_{c_1,v}q}{1-z(1-q)} \]
\[ = \frac{\alpha_{c_2,v}q}{1-z(1-q)} + \frac{\alpha_{c_1,v}q}{1-z(1-q)}. \] (34)

By partial fraction expansion, we get
\[ \hat{m}_{c_1,v}(z) = \frac{Q_{c_1,1}}{1-z(1-q)} + \frac{Q_{c_1,2}}{1-z(1-q)}, \]
where \( Q_{c_1,1} = \alpha_{c_2,v}\phi_{c_1,v}q/\phi_{c_2,v} \) and \( Q_{c_1,2} = \alpha_{c_1,v}\phi_{c_2,v} - \alpha_{c_2,v}\phi_{c_1,v} \).

Finally, (26) is obtained by inverting (34) with respect to \( z \).

\[ \square \]

The unknown constants in Lemma 4.2 will be determined in the following theorem by conditioning on the time and size of the first claim. It will be shown that (26), the expression for \( m_{c_1,v}^*(u) \), can be further simplified.
Theorem 4.3. Suppose the conditions of Lemma 4.2 holds. Consider (26) and (27). \( R \in (0, 1) \) is a real number that satisfies

\[
(1 + \beta) \left[ \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1} - \alpha^{-1} \right] + (1 - R)^{c_2 - c_1} = 1,
\]

where

\[
\alpha \equiv \left( \frac{1 - q}{1 - R} \right)^{c_1 - 1} \frac{q}{q - R} k(\nu(1 - R)^{c_1}), \\
\beta \equiv \frac{q}{q - R} \nu k(1) \left[ (1 - R)^{c_2 - c_1 + 1}(1 - q)^{c_1 - 1} - (1 - R)(1 - q)^{c_1 - 1} \right].
\]

The coefficients satisfy the following equations

\[
Q_{c_2} = \left( 1 - \frac{R}{q} \right) \text{E}[\nu^*(G)],
\]

where \( G \) represents a geometric random variable with pf given by (18); and

\[
Q_{c_1, 1} = \frac{Q_{c_2}}{1 + \alpha^{-1} - \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1}}, \\
Q_{c_1, 2} = 0.
\]

Thus, the Gerber-Shiu functions (26) and (27) are reduced to

\[
m_{c_1, v}^*(u) = Q_{c_1, 1}(1 - R)^u
\]

and

\[
m_{c_2, v}^*(u) = Q_{c_2}(1 - R)^u.
\]

Proof. First consider \( \eta_1 = c_1 \). By conditioning on the time and the amount of the first claim, it yields that

\[
m_{c_1, v}^*(u) = \sum_{t=1}^{\infty} \sum_{y=u+c_1(t+1)}^{\infty} v^t k(t)p(y)w^*(y - u - c_1 t) \\
+ \sum_{t=1}^{\infty} \sum_{y=c_1}^{\infty} v^t k(t)p(y)m_{c_2, v}^*(u + c_1 t - y) \\
+ \sum_{t=1}^{\infty} \sum_{y=1}^{c_1 - 1} v^t k(t)p(y)m_{c_1, v}^*(u + c_1 t - y).
\]

Substitute (18), (26) and (27) into (39), which can be rearranged to obtain

\[
C_{1, R}(1 - R)^u = C_{1, q}(1 - q)^u
\]

with \( C_{1, R} \) given by (50) and \( C_{1, q} \) given by (52). The coefficients \( C_{1, R} \) and \( C_{1, q} \) do not depend on \( u \) and detailed calculation is shown in Appendix A.

Next, consider \( \eta_1 = c_2 \). Conditioning on the time and the amount of the first claim gives

\[
m_{c_2, v}^*(u) = \sum_{t=1}^{\infty} \sum_{y=u+c_2+c_1(t+1)+1}^{\infty} v^t k(t)p(y)w^*(y - u - c_2 - c_1(t - 1))
\]
On the other hand, from (35), it follows that

\[ C = \text{the discounted ruin probability when the initial surplus is zero and the starting } \]

with \( C \) given by (52) and \( q \) given by (53). Again, the coefficients \( C, q \) do not depend on \( u \) and detailed calculation is shown in Appendix A.

By substituting (18), (26) and (27) into (41), it follows after simplification that

\[ C_2, R(1 - R)^u = C_2, q(1 - q)^u \]

with \( C_2, R \) given by (52) and \( C_2, q \) given by (53). Again, the coefficients \( C_2, R \) and \( C_2, q \) do not depend on \( u \) and detailed calculation is shown in Appendix A.

Since (40) and (42) hold for all \( u = 0, 1, 2, \ldots \), it can be concluded that

\[ C_1, R = C_1, q = C_2, R = C_2, q = 0. \]

From \( C_1, R = C_1, q = 0 \), it gives that

\[ Q_{c_2} = \left[ 1 + \alpha^{-1} - \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1} \right] Q_{c_1, 1} \]

(43)

\[ \frac{q}{q - R} Q_{c_2} + \frac{1}{k (v(1 - q)^{c_2})} \left( \frac{q(c_1 - 1)}{1 - q} \right) Q_{c_1, 2} = E \left[ w^*(G) \right]. \]

(44)

On the other hand, from \( C_2, R = C_2, q = 0 \), it yields that

\[ [1 + \beta - \alpha(1 - R)^{c_2 - c_1}] Q_{c_2} = \left[ (1 - R)^{c_2 - c_1} \right] \alpha \left( \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1} - 1 \right) + \beta \] \[ Q_{c_1, 1, 1}, \]

(45)

\[ \frac{q}{q - R} Q_{c_2} + \frac{-vk(1)q(1 - q)^{c_1 - 1}(c_2 - c_1)}{k (v(1 - q)^{c_2})} - \frac{q(c_1 - 1)}{1 - q} \] \[ Q_{c_1, 2} = E \left[ w^*(G) \right]. \]

(46)

By combining (44) and (46), one gets two results of either \( vk(1)q(1 - q)^{c_1 - 1}(c_2 - c_1) = -1 \) or \( Q_{c_1, 2} = 0 \). However, the result \( vk(1)q(1 - q)^{c_1 - 1}(c_2 - c_1) = -1 \) is rejected since \( c_2 \geq c_1 \geq 1 \) guarantees that the LHS is non-negative. Therefore, it can be conclude that \( Q_{c_1, 2} = 0 \). \( Q_{c_1, 1} \) and \( Q_{c_2} \) can be solved through (43) and (44).

Finally, since \( Q_{c_1, 1} \) and \( Q_{c_2} \) are non-zero, combining (43) and (45) yields

\[ 1 + \alpha^{-1} - \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1} = \frac{(1 - R)^{c_2 - c_1} \alpha \left( \left( \frac{1 - R}{1 - q} \right)^{c_1 - 1} - 1 \right) + \beta}{1 + \beta - \alpha(1 - R)^{c_2 - c_1}}, \]

which can be simplified to be (35). Furthermore, we know from the proof of Lemma 4.2 that \( R = q(1 - \phi_{c_2, u}) \). By definition, \( 0 < q < 1 \) and

\[ \phi_{c_2, u} = E \left[ v^T I(T < \infty) \right] U(0) = 0, \eta_t = c_2 \]

is the discounted ruin probability when the initial surplus is zero and the starting premium is \( c_2 \). Therefore, it can be concluded \( R \) is a real number between 0 and 1 that satisfies (35).

\[ \square \]
It is interesting to note that, since $Q_{c_1,2} = 0$, the Gerber-Shiu functions have the same form given in (37) and (38) with different coefficients $Q_{c_1,1}$ and $Q_{c_2}$ which depends on $c_1$ and $c_2$. Moreover, the Gerber-Shiu functions move proportionally:

$$\frac{m_{c_2,v}(u)}{m_{c_1,v}(u)} = \frac{Q_{c_2}}{Q_{c_1,1}} = 1 + \alpha^{-1} - \left(\frac{1 - R}{1 - q}\right) c_1^{-1}. $$

**Corollary 1.** In the degenerated case $c_1 = c_2 = c$, the coefficients are $Q_{c_1,2} = 0$ and $Q_{c_1,1} = Q_{c_2} = \left(1 - \frac{R}{q}\right) \mathbb{E}[w^*(G)]$. Moreover, $R$ can be solved by

$$\frac{q}{q - R} \hat{k}(v(1 - R)^c) = 1. \tag{47}$$

**Proof.** In case of $c_1 = c_2 = c$, (35) and (36) both reduce to (47).

For $c = 1$, corollary 1 is consistent with the result in [17].

### 5. Analysis from a marketing perspective.

In this section, we study the marketing implications of the dynamic premium strategy by two numerical examples.

**Example 5.1.** Let the claim size distribution be $p(x) = 0.8x^{-1.02}$ for $x = 1, 2, \ldots$, and interclaim time distribution be $k(t) = 0.4t^{-1.6}$ for $t = 1, 2, \ldots$.

Suppose that an insurance company is considering which premium strategy to use for its new line of insurance product. Strategy 1 is to adopt the dynamic premium strategy studied in this paper with varying premium rates $c_1$ and $c_2$. Strategy 2 is to adopt a constant premium strategy with constant premium rate $c$.

For Strategy 1, the insurance company will set $c_1 = 4$ and $c_2 = 6$. As for Strategy 2, $c = 4$ or $c = 5$ are two possible options that the insurance company may consider. The insurance product will be competitive in terms of price if the constant premium rate is set at 4, which is a relatively low level. However, this may greatly increase the risk. Therefore, another alternative that the insurance company may consider to offer is a constant premium rate in between of $c_1$ and $c_2$. We adopt the premium rate 5 in our numerical calculation, which is an average of $c_1$ and $c_2$.

First, let us compare Strategy 1 and Strategy 2 with $c = 4$. Figure 1 below shows the probability of ruin for each strategy. Note that for Strategy 1, the starting premium may be $c_1$ or $c_2$, depends on whether the policyholder is classified as "good" or "bad" initially.

Strategy 2 aims to attract customers with a low constant premium. This would make the insurance product price competitive in the market. However, it can be seen from Figure 1 that this strategy has a high risk in which the ruin probability is high. This strategy basically assumes that all policyholders are "good" and will be "good". But this assumption threatens the insurance company as the policyholders may be or become "bad" such that the actual claims turn out to be much more severe than expected.

If Strategy 1 is adopted, the ruin probability would be greatly reduced as shown in Figure 1, no matter which initial classification the policyholder is in. If the insurance company is aggressive, it can offer a low starting premium ($\eta_1 = 4$) to all customers as an attraction. This assumes that all policyholders are "good" initially. However, Strategy 1 can protect the insurance company by preserving the right to raise the premium when there are warning signs in annual reviews. If the insurance company is conservative, it can offer a relatively high premium rate ($\eta_1 = 6$) to
all policyholders initially. The possibility of premium discount in the future can be
used as a selling point to attract customers.

Next, Figure 2 shows the comparison between Strategy 1 and Strategy 2 with
\( c = 5 \). As mentioned before, \( c = 5 \) is another constant premium rate that the
insurance company may choose to offer. This may be viewed as a moderate premium
rate to balance the risks between “good” and “bad” customers.

If a constant premium rate of 5 is offered, the product is less price competitive in
the market. “Good” customers may be attracted to other insurance companies which
can offer a lower price. From Figure 2, the ruin probability from Strategy 1 with
\( \eta_1 = 6 \) is close to but still lower than the ruin probability from Strategy 2 with \( \eta_2 = 5 \). This result suggest that, to strike a balance between the level of risk exposure
and price competitiveness, the insurance company can adopt Strategy 1 and offer an
initial premium according to the policyholder’s historical record. The beneficiaries
due to the strategy can be argued from two ways. From one hand, the dynamic
premium strategy gives a policyholder certain level of flexibility to control his or her
premiums in the future. The higher premium level 6 can be viewed as a penalty to
careless behaviours while the lower premium level 4 stimulates cautious behaviours
of the policyholder. The policyholders with high initial premium level have the
possibilities to obtain the premium discount which cost them less than Strategy
2 in the future. As a result, the dynamic premium strategy helps the insurance
company to well control its total risk exposure by establishing a incentive system.
among its policyholders. On the other hand, [15] pointed out an interesting result in the cumulative perspective theory that people show risk-seeking tendency when they must choose between a sure loss and a probability of a larger loss. Therefore, compared to Strategy 2 which means a certain loss in premium, Strategy 1 with \( \eta_1 = 4 \) accords with people’s behaviour preference and is expected to be more prevalent in the market, and the risk exposure can be well controlled by charging high premium to costumers who have bad historical record.

Example 5.2. Let the claim size distribution be \( p(x) = \left( \frac{7}{9} \right)^{x-1} \frac{2}{9} \) for \( x = 1, 2, \ldots \), and interclaim time distribution be \( k(t) = \frac{e^{-\frac{t}{\tau}}}{\tau(1-e^{-\frac{t}{\tau}})} \) for \( t = 1, 2, \ldots \).

Let us compare the two strategies discussed in Example 5.1 again in this case. The ruin probabilities under each strategy are given in Figure 3 and Figure 4.

We can observe that results are similar to that in Example 5.1. In addition to the advantages mentioned in Example 5.1, dynamic premium strategy also helps to improve customer satisfaction. For constant premium strategy, the constant premium rate can be viewed as a balancing rate between “good” and “bad” customers. “Good” customers are actually penalized by paying more for “bad” customers. As for dynamic premium strategy, “good” customers will be rewarded by paying a low rate if they continue to be “good”. They will only be penalized when severe claims occur. Similarly as in Example 5.1, this may also lower the customers’ incentive to make a claim.
6. **Conclusion.** In this paper, we consider a discrete-time risk model with a two-stage dynamic premium strategy. Policyholders are classified into two categories with different premium rates, and the classification is updated periodically according to the realized claim amount in the previous time period. The Gerber-Shiu functions are shown to have a recurrence relation. In particular, if the claim size is geometric distributed, the explicit formulae of the Gerber-Shiu functions are obtained. By using the results obtained, advantages of the dynamic premium strategy are further discussed from the marketing perspective as compared to the constant premium strategy.

**Acknowledgments.** The authors would like to thank the anonymous referees for giving useful suggestions. The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (UGC/FDS14/P03/16) for Wing Yan Lee, and by the grants from the National Natural Science Foundation of China (Grant No. 11601540), 111 Project (B17050) and Program for Innovation Research in Central University of Finance and Economics for Fangda Liu.

**Appendix A. Appendix.**

*Detailed calculation for Theorem 4.3.* Following from (39), \( Q_{c_1,1}(1-R)^u + Q_{c_1,2}(1-q)^u = S_1 + S_2 + S_3 \), where
Further calculation of each term $S_i$, $i = 1, 2, 3$, are given as follows:

$$S_1 = \sum_{t=1}^{\infty} \sum_{y=u+c_1 t+1}^{\infty} v^t k(t)(1-q)^{y-1} qw(y-u-c_1 t)$$

(48)

$$S_2 = \sum_{t=1}^{\infty} \sum_{y=c_1}^{\infty} v^t k(t)(1-q)^{y-1} qQ_{c_2}(1-R)^{u+c_1 t-y}$$

$$S_3 = \sum_{t=1}^{\infty} \sum_{y=1}^{u+c_1 t-1} v^t k(t)(1-q)^{y-1} q \{ Q_{c_1,1}(1-R)^{u+c_1 t-y} + Q_{c_1,2}(1-q)^{u+c_1 t-y} \}$$

(49)

Figure 4. Strategy 1 ($c_1 = 4$ and $c_2 = 6$) vs Strategy 2 ($c = 5$)
Therefore, (48) can be simplified as (40) with

\[ (1 - R)^u \frac{q}{1 - q} Q_{c_2} \sum_{t=1}^{\infty} v^t k(t)(1 - R)^c t \sum_{y=0}^{u+c_1} \left( \frac{1-q}{1-R} \right)^y \]

= \( (1 - R)^u \frac{q}{q - R} Q_{c_2} \hat{k}(v(1 - R)^c_1) \left( \frac{1-q}{1-R} \right)^{c_1-1} \)

and

Next, following from (41), we have

\[ Q_{c_1} T = \frac{q}{q - R} Q_{c_2} \hat{k}(v(1 - R)^c_1) \]

Therefore, (48) can be simplified as (40) with

\[ C_{1,R} = Q_{c_1,1} - \alpha Q_{c_2} - \frac{q}{q - R} Q_{c_1,1} \hat{k}(v(1 - R)^c_1) + \alpha Q_{c_1,1} \]

and

Next, following from (41), we have \( Q_{c_2}(1 - R)^u = T_1 + T_2 + T_3 + T_4 + T_5 \), where

\[ T_1 = \sum_{t=1}^{\infty} \sum_{y=0}^{u+c_2+c_1(t-1)+1} v^t k(t)(1 - q)^{y-1} q w(y - u - c_2 - c_1(t - 1)) \]

\[ T_2 = v \sum_{y=1}^{u+c_2} k(1)(1 - q)^{y-1} q [Q_{c_1,1}(1 - R)^u+c_2+y + Q_{c_1,2}(1 - q)^u+c_2+y] \]

\[ T_3 = v k(1) \sum_{y=c_2}^{u+c_2} (1 - q)^{y-1} q Q_{c_2}(1 - R)^u+c_2+y \]

\[ T_4 = \sum_{t=2}^{\infty} \sum_{y=1}^{c_1-1} v^t k(t)(1 - q)^{y-1} q \]

\[ \times [Q_{c_1,1}(1 - R)^u+c_2+c_1(t-1)-y + Q_{c_1,2}(1 - q)^u+c_2+c_1(t-1)-y] \]

\[ T_5 = \sum_{t=2}^{\infty} \sum_{y=c_1}^{u+c_2+c_1(t-1)-y} v^t k(t)(1 - q)^{y-1} q Q_{c_2}(1 - R)^u+c_2+c_1(t-1)-y. \]
Calculate each summand in (51):

\[
T_1 = \sum_{t=1}^{\infty} \sum_{y=u+c_2+c_1(t-1)+1}^{\infty} v^t k(t) (1-q)^{y-1} qw(y - u - c_2 - c_1(t-1))
\]

\[
= \sum_{t=1}^{\infty} \sum_{y=1}^{\infty} v^t k(t) (1-q)^{y+u+c_2+c_1(t-1)-1} qw(y)
\]

\[
= (1-q)^u (1-q)^{c_2-c_1} k(v(1-q)^{c_1}) \mathbb{E}[w(G)].
\]

The second term is

\[
T_2 = vk(1) \sum_{y=1}^{c_2-1} (1-q)^{y-1} \left[ Q_{c_1,1}(1-R)^{u+c_2-y} + Q_{c_1,2}(1-q)^{u+c_2-y} \right]
\]

\[
= (1-R)^u vk(1) Q_{c_1,1} q (1-R)^{c_2-1} \frac{1-R}{q-R} \left[ 1 - \left( \frac{1-q}{1-R} \right)^{c_2-1} \right]
\]

\[
+ vk(1) Q_{c_1,2} q (1-q)^{c_2-1} (c_2-1)
\]

\[
= (1-R)^u Q_{c_1,2} v k(1) \frac{q}{q-R} (1-R)^{c_2-1} - (1-q)^u Q_{c_1,2} v k(1) \frac{q}{q-R} (1-q)^{c_2}.
\]

The third term is

\[
T_3 = vk(1) \sum_{y=c_2}^{u+c_2} (1-q)^{y-1} q Q_{c_2} (1-R)^{u+c_2-y}
\]

\[
= vk(1) Q_{c_2} \frac{q}{1-q} (1-R)^{u+c_2} \left( \frac{1-q}{1-R} \right)^{c_2} \frac{1-R}{q-R} \left[ 1 - \left( \frac{1-q}{1-R} \right)^{u+1} \right]
\]

\[
= (1-R)^u Q_{c_2} v k(1) \frac{q}{q-R} (1-R)^{c_2-1} - (1-q)^u Q_{c_2} v k(1) \frac{q}{q-R} (1-q)^{c_2}.
\]

The fourth term is

\[
T_4 = \sum_{t=2}^{\infty} \sum_{y=1}^{c_1-1} v^t k(t) (1-q)^{u+c_2+c_1(t-1)-1} \left[ Q_{c_1,1} \left( \frac{1-R}{1-q} \right)^{u+c_2+c_1(t-1)-y} + Q_{c_1,2} \right]
\]

\[
= (1-R)^u Q_{c_1,1} q (1-R)^{c_2-c_1-1} \sum_{t=2}^{\infty} v^t k(t) (1-R)^{c_1 t} \sum_{y=1}^{c_1-1} \left( \frac{1-q}{1-R} \right)^{y-1}
\]

\[
+ (1-q)^u Q_{c_1,2} q \sum_{t=2}^{\infty} \sum_{y=1}^{c_1-1} v^t k(t) (1-q)^{c_2+c_1(t-1)-1}
\]

\[
= (1-R)^u Q_{c_1,1} \frac{q}{q-R} (1-R)^{c_2-c_1} \left( \hat{k} v (1-R)^{c_1} - v k(1)(1-R)^{c_1} \right)
\]

\[
\times \left( 1 - \left( \frac{1-q}{1-R} \right)^{c_1-1} \right)
\]

\[
+ (1-q)^u Q_{c_1,2} q (1-q)^{c_2-c_1-1} (c_1-1) \left( \hat{k} v (1-q)^{c_1} - v k(1)(1-q)^{c_1} \right).
\]
The last term is
\[
T_5 = \sum_{t=2}^{\infty} \sum_{y=c_1}^{u+c_2+c_1(t-1)} v^t k(t) (1-q)^{y-1} q Q c_2 (1-R)^u c_2 + c_1 (t-1) - y
\]

\[
= (1-R)^u Q c_2 \frac{q}{q-R} (1-R)^{c_2-c_1} \left( \frac{1-q}{1-R} \right)^{c_1-1} \sum_{t=2}^{\infty} v^t k(t) (1-R)^{c_1 t}
\]

\[
- (1-q)^u Q c_2 \frac{q}{q-R} \sum_{t=2}^{\infty} v^t k(t) (1-q)^{c_2+c_1(t-1)}
\]

\[
= (1-R)^u Q c_2 \frac{q}{q-R} (1-R)^{c_2-c_1} \left( \frac{1-q}{1-R} \right)^{c_1-1} \left( \hat{k}(v(1-R)^{c_1}) - v k(1)(1-R)^{c_1} \right)
\]

\[
- (1-q)^u Q c_2 \frac{q}{q-R} (1-q)^{c_2-c_1} \left( \hat{k}(v(1-q)^{c_1}) - v k(1)(1-q)^{c_1} \right).
\]

Therefore, (51) can be simplified as (42) with
\[
C_{2,R} = \mathcal{E}_1 Q_{c_1,1} + \mathcal{E} Q c_2,
\]

where
\[
\mathcal{E}_1 = \frac{q}{q-R} (1-R)^{c_2-c_1} \hat{k}(v(1-R)^{c_1}) \left( 1 - \left( \frac{1-q}{1-R} \right)^{c_1-1} \right)
\]

\[
+ \frac{q}{q-R} v k(1) \left[ (1-R)^{c_2} - (1-R)(1-q)^{c_2-1} - (1-R)^{c_2} \left( 1 - \left( \frac{1-q}{1-R} \right)^{c_1-1} \right) \right]
\]

\[
= (1-R)^{c_2-c_1} \alpha \left[ \left( \frac{1-R}{1-q} \right)^{c_1-1} - 1 \right] + \beta,
\]

\[
\mathcal{E} = \frac{q}{q-R} (1-R)^{c_2-c_1} \left( \frac{1-q}{1-R} \right)^{c_1-1} \left( \hat{k}(v(1-R)^{c_1}) - v k(1)(1-R)^{c_1} \right)
\]

\[
+ v k(1) \frac{q}{q-R} (1-R)(1-q)^{c_2-1} - 1
\]

\[
= \alpha (1-R)^{c_2-c_1} - 1 - \beta;
\]

and
\[
C_{2,q} = - \mathcal{E}_2 Q_{c_1,2} + \mathcal{E} Q c_2 - (1-q)^{c_2-c_1} \hat{k}(v(1-q)^{c_1}) \mathbb{E}[w(G)],
\]

where
\[
\mathcal{E}_2 = q(1-q)^{c_2-c_1-1}(c_1-1) \hat{k}(v(1-q)^{c_1}) + v k(1) q(1-q)^{c_2-1}(c_2 - c_1)
\]

and
\[
\mathcal{E}_q = - \frac{q}{q-R} (1-q)^{c_2-c_1} \hat{k}(v(1-q)^{c_1}).
\]

\[\square\]

REFERENCES

[1] L. B. Afonso, A. D. Egidio dos Reis and H. R. Waters, Calculating continuous time ruin probabilities for a large portfolio with varying premium, *ASTIN Bulletin*, **39** (2009), 117–136.

[2] L. B. Afonso, A. D. Egidio dos Reis and H. R. Waters, Numerical evaluation of continuous time ruin probabilities for a portfolio with credibility updated premiums, *ASTIN Bulletin*, **40** (2010), 399–414.
[3] L. B. Afonso, Evaluation of ruin probabilities for surplus process with credibility and surplus dependent premium, Ph.D. Thesis, 2008.
[4] S. Asmussen, On the ruin problem for some adapted premium rules, MaPhySto Research Report No. 5 University of Aarhus, Denmark., 1999.
[5] S. Asmussen and H. Albrecher, Ruin Probabilities, World Scientific, 2010.
[6] E. C. K. Cheung, D. Landriault, G. E. Willmot and J.-K. Woo, Gerber-Shiu analysis with a generalized penalty function, Scandinavian Actuarial Journal, 2010 (2010), 185–199.
[7] E. C. K. Cheung, D. Landriault, G. E. Willmot and J.-K. Woo, Structural properties of Gerber-Shiu function in dependent Sparre Andersen models, Insurance: Mathematics and Economics, 46 (2010), 117–126.
[8] E. C. K. Cheung, D. Landriault, G. E. Willmot and J.-K. Woo, On orderings and bounds in a generalized sparre andersen risk model, Applied Stochastic Models in Business and Industry, 27 (2011), 51–60.
[9] H. U. Gerber and E. S. W. Shiu, On the time value of ruin, North American Actuarial Journal, 2 (1998), 48–78.
[10] D. Landriault, C. Lemieux and G. E. Willmot, An adaptive premium policy with a Bayesian motivation in the classical risk model, Insurance: Mathematics and Economics, 51 (2012), 370–378.
[11] S. Li, D. Landriault and C. Lemieux, A risk model with varying premiums: Its risk management implications, Scandinavian Actuarial Journal, 2012 (2012), 201–224.
[12] Z. Li and K. P. Sendova, On a ruin model with both interclaim times and premiums depending on claim sizes, Scandinavian Actuarial Journal, 2015 (2015), 245–265.
[13] S. Loisel and J. Trufin, Ultimate ruin probability in discrete time with Buhlmann credibility premium adjustments, Bulletin Francais d’Actuariat, 13 (2013), 73–102.
[14] C. C.-L. Tsai and G. Parker, Ruin probabilities: Classical versus credibility, NTU International Conference on Finance, 2004.
[15] A. Tversky and D. Kahneman, Advances in prospect theory: Cumulative representation of uncertainty, Journal of Risk and Uncertainty, 5 (1992), 297–323.
[16] J.-K. Woo, A generalized penalty function for a class of discrete renewal processes, Scandinavian Actuarial Journal, 2012 (2012), 130–152.
[17] X. Wu and S. Li, On the discounted penalty function in a discrete time renewal risk model with general interclaim times, Scandinavian Actuarial Journal, 2009 (2009), 281–294.
[18] Z. Zhang, Y. Yang and C. Liu, On a perturbed compound Poisson model with varying premium rates, Journal of Industrial and Management Optimization, 13 (2017), 721–736.

Received February 2017; revised June 2017.

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