EQUIVARIANT POINCARÉ DUALITY FOR QUANTUM GROUP ACTIONS

RYSZARD NEST AND CHRISTIAN VOIGT

Abstract. We extend the notion of Poincaré duality in $KK$-theory to the setting of quantum group actions. An important ingredient in our approach is the replacement of ordinary tensor products by braided tensor products. Along the way we discuss general properties of equivariant $KK$-theory for locally compact quantum groups, including the construction of exterior products. As an example, we prove that the standard Podleś sphere is equivariantly Poincaré dual to itself.

1. Introduction

The notion of Poincaré duality in $K$-theory plays an important rôle in noncommutative geometry. In particular, it is a fundamental ingredient in the theory of noncommutative manifolds due to Connes [11]. A noncommutative manifold is given by a spectral triple $(\mathcal{A}, H, D)$ where $\mathcal{A}$ is a $\ast$-algebra represented on a Hilbert space $H$ and $D$ is an unbounded self-adjoint operator on $H$. The basic requirements on this data are that $D$ has compact resolvent and that the commutators $[D, a]$ are bounded for all $a \in \mathcal{A}$. There are further ingredients in the definition of a noncommutative manifold, in particular a grading and the concept of a real structure [12], [13]. An important recent result due to Connes is the reconstruction theorem [14], which asserts that in the commutative case, under some natural conditions, the algebra $\mathcal{A}$ is isomorphic to $C^\infty(M)$ for a unique smooth manifold $M$. The real structure produces a version of $KO$-Poincaré duality, which is a necessary ingredient for the existence of a smooth structure.

Quantum groups and their homogeneous spaces give natural and interesting examples of noncommutative spaces, and several cases of associated spectral triples have been constructed [15], [10], [16], [17], [40]. An important guiding principle in all these constructions is equivariance with respect to the action of a quantum group. In [46], [18] a general framework for equivariant spectral triples is formulated, including an equivariance condition for real structures. However, in some examples the original axioms in [12] are only satisfied up to infinitesimals in this setup [16], [17]. The $K$-theoretic interpretation of a real structure up to infinitesimals is unclear.

In this paper we introduce a notion of $K$-theoretic Poincaré duality which is particularly adapted to the symmetry of quantum group actions. More precisely, we generalize the definition of Poincaré duality in $KK$-theory given by Connes [11] to $C^\ast$-algebras with a coaction of a quantum group using braided tensor products. Braided tensor products are well-known in the algebraic approach to quantum groups [30], in our context they are constructed using coactions of the Drinfeld double of a locally compact quantum group.

The example we study in detail is the standard Podleś sphere, and we prove that it is equivariantly Poincaré dual to itself with respect to the natural action of $SU_q(2)$. The Drinfeld double of $SU_q(2)$, appearing as the symmetry group in this case, is

2000 Mathematics Subject Classification. 46L80, 19K35, 20G42, 46L65.
the quantum Lorentz group \cite{42}, a noncompact quantum group built up out of a compact and a discrete part. We remark that the additional symmetry of the Podleś sphere which is encoded in the discrete part of the quantum Lorentz group is not visible classically.

The spectral triple corresponding to the Dirac operator on the standard Podleś sphere \cite{15} can be equipped with a real structure, and, due to \cite{52}, it satisfies Poincaré duality in the sense of \cite{12}. From this point of view the standard Podleś sphere is very well-behaved. However, already in this example the formulation of equivariant Poincaré duality requires the setup proposed in this paper.

Usually, the symmetry of an equivariant spectral triple is implemented by the action of a quantized universal enveloping algebra. In our approach we have to work with coactions of the quantized algebra of functions instead. Both descriptions are essentially equivalent, but an advantage of coactions is that the correct definition of equivariant $K$-theory and $K$-homology in this setting is already contained in \cite{1}.

In particular, we do not need to consider constructions of equivariant $K$-theory as in \cite{39}, \cite{52} which do not extend to general quantum groups.

Let us now describe how the paper is organized. In the first part of the paper we discuss some results related to locally compact quantum groups and $KK$-theory. Section 2 contains an introduction to locally compact quantum groups, their coactions and associated crossed products. In particular, we review parts of the foundational work of Vaes on induced coactions \cite{48} which are relevant to this paper.

In section 3 we introduce Yetter-Drinfeld-$C^*$-algebras and braided tensor products and discuss their basic properties, including compatibility with induction and restriction. Then, in section 4 we review the definition of equivariant $KK$-theory for quantum groups following Baaj and Skandalis \cite{1}. In particular, we show that $KK^G$ for a regular locally compact quantum group $G$ satisfies a universal property as in the group case. A new feature in the quantum setting is the construction of exterior products for $KK^G$. The non-triviality of it is related to the fact that a tensor product of two algebras with a coaction of a quantum group does not inherit a natural coaction in general, in distinction to the case of a group action. We deal with this problem using braided tensor products.

Basic facts concerning $SU_q(2)$ and the standard Podleś sphere $SU_q(2)/T$ are reviewed in section 5. The main definition and results are contained in section 6 where we introduce the concept of equivariant Poincaré duality with respect to quantum group actions and show that $SU_q(2)/T$ is equivariantly Poincaré dual to itself. As an immediate consequence we determine the equivariant $K$-homology of the Podleś sphere.

Let us make some remarks on notation. We write $\mathbb{L}(\mathcal{E}, \mathcal{F})$ for the space of adjointable operators between Hilbert $A$-modules $\mathcal{E}$ and $\mathcal{F}$. Moreover $\mathbb{K}(\mathcal{E}, \mathcal{F})$ denotes the space of compact operators. If $\mathcal{E} = \mathcal{F}$ we write simply $\mathbb{L}(\mathcal{E})$ and $\mathbb{K}(\mathcal{E})$, respectively. The closed linear span of a subset $X$ of a Banach space is denoted by $[X]$. Depending on the context, the symbol $\otimes$ denotes either the tensor product of Hilbert spaces, the minimal tensor product of $C^*$-algebras, or the tensor product of von Neumann algebras. For operators on multiple tensor products we use the leg numbering notation.

It is a pleasure to thank Uli Krähmer for interesting discussions on the subject of this paper. The second author is indebted to Stefaan Vaes for helpful explanations concerning induced coactions and braided tensor products. A part of this work was done during stays of the authors in Warsaw supported by EU-grant MKTD-CT-2004-509794. We are grateful to Piotr Hajac for his kind hospitality.
2. Locally compact quantum groups and their coactions

In this section we recall basic definitions and results from the theory of locally compact quantum groups and fix our notation. For more detailed information we refer to the literature [28], [29], [48].

Let \( \phi \) be a normal, semifinite and faithful weight on a von Neumann algebra \( M \). We use the standard notation

\[
M^+_\phi = \{ x \in M_+ | \phi(x) < \infty \}, \quad N_\phi = \{ x \in M | \phi(x^*x) < \infty \}
\]

and write \( M^+_\phi \) for the space of positive normal linear functionals on \( M \). Assume that \( \Delta : M \to M \otimes M \) is a normal unital \(*\)-homomorphism. The weight \( \phi \) is called left invariant with respect to \( \Delta \) if

\[
\phi((\omega \otimes \text{id})\Delta(x)) = \phi(x)\omega(1)
\]

for all \( x \in M^+_\phi \) and \( \omega \in M^+_\phi \). Similarly one defines the notion of a right invariant weight.

**Definition 2.1.** A locally compact quantum group \( G \) is given by a von Neumann algebra \( L^\infty(G) \) together with a normal unital \(*\)-homomorphism \( \Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \) satisfying the coassociativity relation

\[
(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta
\]

and normal semifinite faithful weights \( \phi \) and \( \psi \) on \( L^\infty(G) \) which are left and right invariant, respectively.

Our notation for locally compact quantum groups is intended to make clear how ordinary locally compact groups can be viewed as quantum groups. Indeed, if \( G \) is a locally compact group, then the algebra \( L^\infty(G) \) of essentially bounded measurable functions on \( G \) together with the comultiplication \( \Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G) \) given by

\[
\Delta(f)(s,t) = f(st)
\]

defines a locally compact quantum group. The weights \( \phi \) and \( \psi \) are given in this case by left and right Haar measures, respectively.

Of course, for a general locally compact quantum group \( G \) the notation \( L^\infty(G) \) is purely formal. Similar remarks apply to the \( C^* \)-algebras \( C^*_\text{r}(G) \), \( C^*_{\text{r}}(G) \) and \( C^*_\text{l}(G), C^*_{\text{l}}(G) \) associated to \( G \) that we discuss below. It is convenient to view all of them as different appearances of the quantum group \( G \).

Let \( G \) be a locally compact quantum group and let \( \Lambda : N_\phi \to \mathbb{H}_G \) be a GNS-construction for the weight \( \phi \). Throughout the paper we will only consider quantum groups for which \( \mathbb{H}_G \) is a separable Hilbert space. One obtains a unitary \( W_G = W \) on \( \mathbb{H}_G \otimes \mathbb{H}_G \) by

\[
W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))
\]

for all \( x, y \in N_\phi \). This unitary is multiplicative, which means that \( W \) satisfies the pentagonal equation

\[
W_{12}W_{13}W_{23} = W_{23}W_{12}.
\]

From \( W \) one can recover the von Neumann algebra \( L^\infty(G) \) as the strong closure of the algebra \( (\text{id} \otimes L(\mathbb{H}_G),)(W') \) where \( L(\mathbb{H}_G) \) denotes the space of normal linear functionals on \( L(\mathbb{H}_G) \). Moreover one has

\[
\Delta(x) = W^*(1 \otimes x)W
\]

for all \( x \in M \). The algebra \( L^\infty(G) \) has an antipode which is an unbounded, \( \sigma \)-strong* closed linear map \( S \) given by \( S(\text{id} \otimes \omega)(W) = (\text{id} \otimes \omega)(W^*) \) for \( \omega \in L(\mathbb{H}_G)_\text{r} \).
Moreover there is a polar decomposition $S = R\tau_{-i/2}$ where $R$ is an antiautomorphism of $L^\infty(G)$ called the unitary antipode and $(\tau_t)$ is a strongly continuous one-parameter group of automorphisms of $L^\infty(G)$ called the scaling group. The unitary antipode satisfies $\sigma(R \otimes R)\Delta = \Delta R$.

The group-von Neumann algebra $L(G)$ of the quantum group $G$ is the strong closure of the algebra $(L(H_G) \otimes \text{id})(W)$ with the comultiplication $\Delta : L(G) \to L(G) \otimes L(G)$ given by

$$\hat{\Delta}(y) = \hat{W}^*(1 \otimes y)\hat{W}$$

where $\hat{W} = \Sigma W^*\Sigma$ and $\Sigma \in L(H_G \otimes H_G)$ is the flip map. It defines a locally compact quantum group $\hat{G}$ which is called the dual of $G$. The left invariant weight $\hat{\phi}$ for the dual quantum group has a GNS-construction $\Lambda : \mathcal{N}_\phi \to H_G$, and according to our conventions we have $L(G) = L^\infty(\hat{G})$.

The modular conjugations of the weights $\phi$ and $\hat{\phi}$ are denoted by $J$ and $\hat{J}$, respectively. These operators implement the unitary antipodes in the sense that

$$R(x) = \hat{J}x^*\hat{J}, \quad \hat{R}(y) = Jy^*J$$

for $x \in L^\infty(G)$ and $y \in L(G)$. Note that $L^\infty(G)' = JL^\infty(G)J$ and $L(G)' = J\hat{L}(G)\hat{J}$ for the commutants of $L^\infty(G)$ and $L(G)$. Using $J$ and $\hat{J}$ one obtains multiplicative unitaries

$$V = (J \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J}), \quad \hat{V} = (J \otimes J)W(J \otimes J)$$

which satisfy $V \in \hat{L}(G)' \otimes L^\infty(G)$ and $\hat{V} \in L^\infty(G)' \otimes L(G)$, respectively.

We will mainly work with the $C^*$-algebras associated to the locally compact quantum group $G$. The algebra $[(\text{id} \otimes \hat{L}(H_G))(W)]$ is a strongly dense $C^*$-subalgebra of $L^\infty(G)$ which we denote by $C^*_0(G)$. Dually, the algebra $[(\hat{L}(H_G) \otimes \text{id})(W)]$ is a strongly dense $C^*$-subalgebra of $L(G)$ which we denote by $C^*\tau(G)$. These algebras are the reduced algebra of continuous functions vanishing at infinity on $G$ and the reduced group $C^*$-algebra of $G$, respectively. One has $W \in M(C^*_0(G) \otimes C^*\tau(G))$.

Restriction of the comultiplications on $L^\infty(G)$ and $L(G)$ turns $C^*_0(G)$ and $C^*\tau(G)$ into Hopf-$C^*$-algebras in the following sense.

**Definition 2.2.** A Hopf $C^*$-algebra is a $C^*$-algebra $S$ together with an injective nondegenerate $*$-homomorphism $\Delta : S \to M(S \otimes S)$ such that the diagram

$$\begin{array}{ccc}
S & \xrightarrow{\Delta} & M(S \otimes S) \\
\downarrow & & \downarrow \text{id} \otimes \Delta \\
M(S \otimes S) & \xrightarrow{\Delta \otimes \text{id}} & M(S \otimes S \otimes S)
\end{array}$$

is commutative and $[\Delta(S)(1 \otimes S)] = S \otimes S = [(S \otimes 1)\Delta(S)]$.

A morphism between Hopf-$C^*$-algebras $(S, \Delta_S)$ and $(T, \Delta_T)$ is a nondegenerate $*$-homomorphism $\pi : S \to M(T)$ such that $\Delta_T \pi = (\pi \otimes \pi)\Delta_S$.

If $S$ is a Hopf-$C^*$-algebra we write $S^{\text{cop}}$ for the Hopf-$C^*$-algebra obtained by equipping $S$ with the opposite comultiplication $\Delta^{\text{cop}} = \sigma \Delta$.

A unitary corepresentation of a Hopf-$C^*$-algebra $S$ on a Hilbert $B$-module $E$ is a unitary $X \in L(S \otimes E)$ satisfying

$$(\Delta \otimes \text{id})(X) = X_{13}X_{23}.$$ 

A universal dual of $S$ is a Hopf-$C^*$-algebra $\hat{S}$ together with a unitary corepresentation $X \in M(S \otimes \hat{S})$ satisfying the following universal property. For every Hilbert $B$-module $E$ and every unitary corepresentations $X \in L(S \otimes E)$ there exists a unique nondegenerate $*$-homomorphism $\pi_X : \hat{S} \to L(E)$ such that $(\text{id} \otimes \pi_X)(X) = X$.

For every locally compact quantum group $G$ there exists a universal dual $C^*_\tau(G)$
of $C_b^0(G)$ and a universal dual $C_b^0(G)$ of $C^*_r(G)$, respectively [27]. We call $C^*_r(G)$ the maximal group $C^*$-algebra of $G$ and $C^b(G)$ the maximal algebra of continuous functions on $G$ vanishing at infinity. Since $\mathbb{H}_G$ is assumed to be separable the $C^*$-algebras $C_b^0(G), C_b^0(G)$ and $C^*_r(G), C^*_r(G)$ are separable. The quantum group $G$ is called compact if $C^*_r(G)$ is unital, and it is called discrete if $C^*_r(G)$ is unital. In the compact case we also write $C^0(G)$ and $C^0(G)$ instead of $C^*_r(G)$ and $C^*_r(G)$, respectively.

In general, we have a surjective morphism $\hat{\pi} : C^*_r(G) \to C^*_r(G)$ of Hopf-$C^*$-algebras associated to the left regular corepresentation $W \in M(C_0(G) \otimes C^*_r(G))$. Similarly, there is a surjective morphism $\pi : C^b_0(G) \to C^b_0(G)$. We will call the quantum group $G$ amenable if $\hat{\pi} : C^*_r(G) \to C^*_r(G)$ is an isomorphism and coamenable if $\pi : C_b^0(G) \to C^b_0(G)$ is an isomorphism. If $G$ is amenable or coamenable, respectively, we also write $C^*(G)$ and $C_b(G)$ for the corresponding $C^*$-algebras. For more information on amenability for locally compact quantum groups see [6].

Let $S$ be a $C^*$-algebra. The $S$-relative multiplier algebra $M_S(S \otimes A) \subset M(S \otimes A)$ of a $C^*$-algebra $A$ consists of all $x \in M(S \otimes A)$ such that the relations

$$x(S \otimes 1) \subset S \otimes A,$$

$$(S \otimes 1)x \subset S \otimes A$$

hold. In the sequel we tacitly use basic properties of relative multiplier algebras which can be found in [20].

**Definition 2.3.** A (left) coaction of a Hopf $C^*$-algebra $S$ on a $C^*$-algebra $A$ is an injective nondegenerate $*$-homomorphism $\alpha : A \to M(S \otimes A)$ such that the diagram

$${\begin{array}{ccc} A & \xrightarrow{\alpha} & M(S \otimes A) \\ \downarrow{\alpha} & & \downarrow{\Delta \otimes \text{id}} \\ M(S \otimes A) & \xrightarrow{\text{id} \otimes \alpha} & M(S \otimes S \otimes A) \end{array}}$$

is commutative and $\alpha(A) \subset M_S(S \otimes A)$. The coaction is called continuous if $[\alpha(A)(S \otimes 1)] = S \otimes A$.

If $(A, \alpha)$ and $(B, \beta)$ are $C^*$-algebras with coactions of $S$, then a $*$-homomorphism $f : A \to M(B)$ is called $S$-colinear if $\beta f = (\text{id} \otimes f) \alpha$.

We remark that some authors do not require a coaction to be injective. For a discussion of the continuity condition see [3].

A $C^*$-algebra $A$ equipped with a continuous coaction of the Hopf-$C^*$-algebra $S$ will be called an $S$-$C^*$-algebra. If $S = C_b^0(G)$ for a locally compact quantum group $G$ we also say that $A$ is $G$-$C^*$-algebra. Moreover, in this case $S$-colinear $*$-homomorphisms will be called $G$-equivariant or simply equivariant. We write $G\text{-Alg}$ for the category of separable $G$-$C^*$-algebras and equivariant $*$-homomorphisms.

A (nondegenerate) covariant representation of a $G$-$C^*$-algebra $A$ on a Hilbert-B-module $E$ consists of a (nondegenerate) $*$-homomorphism $f : A \to \mathbb{L}(E)$ and a unitary corepresentation $X \in \mathbb{L}(C_b^0(G) \otimes E)$ such that

$$(\text{id} \otimes f)\alpha(a) = X^*(1 \otimes f(a))X$$

for all $a \in A$. There exists a $C^*$-algebra $C^*_r(G)_{\text{op}} \ltimes_l A$, called the full crossed product, together with a nondegenerate covariant representation $(j_A, X_A)$ of $A$ on $C^*_r(G)_{\text{op}} \ltimes_l A$ which satisfies the following universal property. For every nondegenerate covariant representation $(f, X)$ of $A$ on a Hilbert-$B$-module $E$ there exists a unique nondegenerate $*$-homomorphism $F : C^*_r(G)_{\text{op}} \ltimes_l A \to \mathbb{L}(E)$, called the integrated form of $(f, X)$, such that

$$X = (\text{id} \otimes F)(X_A), \quad f = F j_A.$$
Remark that the corepresentation $X_A$ corresponds to a unique nondegenerate $*$-
-homomorphism $g_A : C_r^*(G)^{\text{cop}} \to M(C_r^*(G)^{\text{cop}} \rtimes_f A)$.

On the Hilbert $A$-module $\mathbb{H} \otimes A$ we have a covariant representation of $A$ given
by the coaction $\alpha : A \to \mathbb{L}(\mathbb{H} \otimes A)$ and $W \otimes 1$. The reduced crossed product $C_r^*(G)^{\text{cop}} \rtimes_f A$ is the image of $C_r^*(G)^{\text{cop}} \rtimes_f A$ under the corresponding integrated form. Explicitly, we have

$$C_r^*(G)^{\text{cop}} \rtimes_f A = [(C_r^*(G) \otimes 1)\alpha(A)]$$

inside $M(\mathbb{K}_G \otimes A) = \mathbb{L}(\mathbb{H}_G \otimes A)$ using the notation $\mathbb{K}_G = \mathbb{K}(\mathbb{H}_G)$. There is a nondegenerate $*$-homomorphism $j_A : A \to M(C_r^*(G)^{\text{cop}} \rtimes_f A)$ induced by $\alpha$.

Similarly, we have a canonical nondegenerate $*$-homomorphism $g_A : C_r^*(G)^{\text{cop}} \to M(C_r^*(G)^{\text{cop}} \rtimes_f A)$.

The full and the reduced crossed products admit continuous dual coactions of $C_r^*(G)^{\text{cop}}$ and $C_r^*(G)^{\text{cop}}$, respectively. In both cases the dual coaction leaves the copy of $A$ inside the crossed product invariant and acts by the (opposite) comultiplication on the group $C^*$-algebra. If $G$ is amenable then the canonical map $C_r^*(G)^{\text{cop}} \rtimes_f A \to C_r^*(G)^{\text{cop}} \rtimes_f A$ is an isomorphism for all $G$-$C^*$-algebras $A$, and we will also write $C^*(G)^{\text{cop}} \rtimes_f A$ for the crossed product in this case.

The comultiplication $\Delta : C_r^0(G) \to M(C_r^0(G) \otimes C_r^0(G))$ defines a coaction of $C_r^0(G)$ on itself. On the Hilbert space $\mathbb{H}_G$ we have a covariant representation of $C_r^0(G)$ given by the identical representation of $C_r^0(G)$ and $W \in M(C_r^0(G) \otimes \mathbb{K}_G)$. The quantum group $G$ is called strongly regular if the associated integrated form induces an isomorphism $C_r^*(G)^{\text{cop}} \rtimes_f C_r^*(G) \cong \mathbb{K}_G$. Similarly, $G$ is called regular if the corresponding homomorphism on the reduced level gives an isomorphism $C_r^*(G)^{\text{cop}} \rtimes_f C_r^*(G) \cong \mathbb{K}_G$. Every strongly regular quantum group is regular, it is not known whether there exist regular quantum groups which are not strongly regular. If $G$ is regular then the dual $\hat{G}$ is regular as well.

Let $\mathcal{E}_B$ be a right Hilbert module. The multiplier module $M(\mathcal{E})$ of $\mathcal{E}$ is the right Hilbert-$M(\mathcal{B})$-module $M(\mathcal{E}) = \mathbb{L}(B, \mathcal{E})$. There is a natural embedding $\mathcal{E} \cong \mathbb{K}(B, \mathcal{E}) \to \mathbb{L}(B, \mathcal{E}) = M(\mathcal{E})$. If $\mathcal{E}_A$ and $\mathcal{F}_B$ are Hilbert modules, then a morphism from $\mathcal{E}$ to $\mathcal{F}$ is a linear map $\Phi : \mathcal{E} \to M(\mathcal{F})$ together with a $*$-homomorphism $\phi : A \to M(B)$ such that

$$(\Phi(\xi), \Phi(\eta)) = \phi((\xi, \eta))$$

for all $\xi, \eta \in \mathcal{E}$. In this case $\Phi$ is automatically norm-decreasing and satisfies $\Phi(\lambda(a) \xi) = \Phi(\xi) \phi(a)$ for all $\xi \in \mathcal{E}$ and $a \in A$. The morphism $\Phi$ is called nondegenerate if $\phi$ is nondegenerate and $[\Phi(\mathcal{E})B] = \mathcal{F}$.

Let $S$ be a $C^*$-algebra and let $\mathcal{E}_A$ be a Hilbert module. The $S$-relative multiplier module $M_S(S \otimes \mathcal{E})$ is the submodule of $M(S \otimes \mathcal{E})$ consisting of all multipliers $x$ satisfying $x(S \otimes 1) \subset S \otimes \mathcal{E}$ and $(S \otimes 1)x \subset S \otimes \mathcal{E}$. For further information we refer again to [20].

**Definition 2.4.** Let $S$ be a Hopf-$C^*$-algebra and let $\beta : B \to M(S \otimes B)$ be a
coaction of $S$ on the $C^*$-algebra $B$. A coaction of $S$ on a Hilbert module $\mathcal{E}_B$ is a nondegenerate morphism $\lambda : \mathcal{E} \to M(S \otimes \mathcal{E})$ such that the diagram

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\lambda} & M(S \otimes \mathcal{E}) \\
\downarrow{\lambda} & & \downarrow{\Delta \otimes \text{id}} \\
M(S \otimes \mathcal{E}) & \xrightarrow{\text{id} \otimes \lambda} & M(S \otimes S \otimes \mathcal{E})
\end{array}$$

is commutative and $\lambda(\mathcal{E}) \subset M_S(S \otimes \mathcal{E})$. The coaction $\lambda$ is called continuous if $[(S \otimes 1) \lambda(\mathcal{E})] = S \otimes \mathcal{E} = [\lambda(\mathcal{E})](S \otimes 1)]$.

A morphism $\Phi : \mathcal{E} \to M(\mathcal{F})$ of Hilbert $B$-modules with coactions $\lambda_\mathcal{E}$ and $\lambda_\mathcal{F}$, respectively, is called $S$-colinear if $\lambda_\mathcal{F} \Phi = (\text{id} \otimes \Phi) \lambda_\mathcal{E}$. 
If \( \lambda : \mathcal{E} \to M(S \otimes \mathcal{E}) \) is a coaction on the Hilbert-\( B \)-module \( \mathcal{E} \) then the map \( \lambda \) is automatically isometric and hence injective.

Let \( G \) be a locally compact quantum group and let \( B \) be a \( G \)-\( C^* \)-algebra. A \( G \)-Hilbert \( B \)-module is a Hilbert module \( \mathcal{E}_B \) with a continuous coaction \( \lambda : \mathcal{E} \to M(S \otimes \mathcal{E}) \) for \( S = C_0^*(G) \). If \( G \) is regular then continuity of the coaction \( \lambda \) is in fact automatic. Instead of \( S \)-colinear morphisms we also speak of equivariant morphisms between \( G \)-Hilbert \( B \)-modules.

Let \( B \) be a \( C^* \)-algebra equipped with a coaction of the Hopf-\( C^* \)-algebra \( S \). Given a Hilbert module \( \mathcal{E}_B \) with coaction \( \lambda : \mathcal{E} \to M(S \otimes \mathcal{E}) \) one obtains a unitary operator

\[
V_\lambda : \mathcal{E} \otimes_B (S \otimes B) \to S \otimes \mathcal{E}
\]

by

\[
V_\lambda(\xi \otimes x) = \lambda(\xi)x
\]

for \( \xi \in \mathcal{E} \) and \( x \in S \otimes B \). Here the tensor product over \( B \) is formed with respect to the coaction \( \beta : B \to M(S \otimes B) \). This unitary satisfies the relation

\[
(id \otimes_C V_\lambda)(V_\lambda \otimes (id \otimes_B) id) = V_\lambda \otimes (\Delta \otimes id) id
\]

in \( L(\mathcal{E} \otimes (\Delta \otimes id) \beta (S \otimes S \otimes B), S \otimes S \otimes \mathcal{E}) \). Moreover, the equation

\[
\text{ad}_\lambda(T) = V_\lambda(T \otimes id)V^*_\lambda
\]

determines a coaction \( \text{ad}_\lambda : \mathbb{K}(\mathcal{E}) \to M(S \otimes \mathbb{K}(\mathcal{E})) = L(S \otimes \mathcal{E}) \). If the coaction \( \lambda \) is continuous then \( \text{ad}_\lambda \) is continuous as well. In particular, if \( \mathcal{E} \) is a \( G \)-Hilbert \( B \)-module with coaction \( \lambda \), then the associated coaction \( \text{ad}_\lambda \) turns \( \mathbb{K}(\mathcal{E}) \) into a \( G \)-\( C^* \)-algebra.

Let \( B \) be a \( C^* \)-algebra equipped with the trivial coaction of the Hopf-\( C^* \)-algebra \( S \) and let \( \lambda : \mathcal{E} \to M(S \otimes \mathcal{E}) \) be a coaction on the Hilbert module \( \mathcal{E}_B \). Then using the natural identification \( \mathcal{E} \otimes_B (S \otimes B) \cong \mathcal{E} \otimes S \cong S \otimes \mathcal{E} \) the associated unitary \( V_\lambda \)

determines a unitary corepresentation \( V^*_\lambda \) in \( L(S \otimes \mathcal{E}) \). Conversely, if \( V \in L(S \otimes \mathcal{E}) \) is a unitary corepresentation then \( \lambda_V : \mathcal{E} \to M(S \otimes \mathcal{E}) \) given by \( \lambda_V(\xi) = V^*(1 \otimes \xi) \)

is a nondegenerate morphism of Hilbert modules satisfying the coaction identity. If \( S = C_0^*(G) \) for a regular quantum group \( G \), then \( \lambda_V \) defines a continuous coaction on \( \mathcal{E} \). As a consequence, for a regular quantum group \( G \) and a trivial \( G \)-\( C^* \)-algebra \( B \), continuous coactions on a Hilbert \( B \)-module \( \mathcal{E} \) correspond uniquely to unitary corepresentations of \( C_0^*(G) \) on \( \mathcal{E} \).

Let \( G \) be a regular quantum group and let \( \mathcal{E}_B \) be a \( G \)-Hilbert module with coaction \( \lambda_V : \mathcal{E} \to M(C_0^*(G) \otimes \mathcal{E}) \). Then \( \mathbb{H}_G \otimes \mathcal{E} \) becomes a \( G \)-Hilbert \( B \)-module with the coaction \( \lambda_{\mathbb{H}_G \otimes \mathcal{E}}(x \otimes \xi) = X_0 \Sigma_{i=1}^d (id \otimes \lambda_V)(x \otimes \xi) \)

where \( X = \Sigma \mathcal{V} \Sigma \in L(C_0^*(G) \otimes \mathbb{K}G) \). In particular, for \( \mathcal{E} = B \) the algebra \( \mathbb{K}G \otimes B = \mathbb{K}(\mathbb{H}_G \otimes B) \) can be viewed as a \( G \)-\( C^* \)-algebra. We now state the following version of the Takesaki-Takai duality theorem [2].

**Theorem 2.5.** Let \( G \) be a regular locally compact quantum group and let \( A \) be a \( G \)-\( C^* \)-algebra. Then there is a natural isomorphism

\[
C_0^*(G) \ltimes_T C_r^*(G)^{\text{cop}} \ltimes_T A \cong \mathbb{K}_G \otimes A
\]

of \( G \)-\( C^* \)-algebras.

An equivariant Morita equivalence between \( G \)-\( C^* \)-algebras \( A \) and \( B \) is given by an equivariant \( A \)-\( B \)-imprimitivity bimodule, that is, a full \( G \)-Hilbert \( B \)-module \( \mathcal{E} \) together with an isomorphism \( \mathcal{E} \cong \mathbb{K}(\mathcal{E}) \) of \( G \)-\( C^* \)-algebras. Theorem 2.5 shows that the double crossed product \( C_0^*(G) \ltimes_T C_r^*(G)^{\text{cop}} \ltimes_T A \) is equivariantly Morita equivalent to \( A \) for every \( G \)-\( C^* \)-algebra \( A \) provided \( G \) is regular.

A morphism \( H \to G \) of locally compact quantum groups is a nondegenerate *-homomorphism \( \pi : C_0^*(G) \to M(C_0^*(H)) \) which is compatible with the comultiplications in the sense that \( (\pi \otimes \pi)\Delta_G = \Delta_H \pi \). Every such morphism induces
canonically a dual morphism \( \hat{\pi} : \cop{C^*_r(H)} \to M(\cop{C^*_r(G)}) \). A closed quantum subgroup \( H \subset G \) is a morphism \( H \to G \) for which the latter map is accompanied by a faithful normal \(*\)-homomorphism \( \mathcal{L}(H) \to \mathcal{L}(G) \) of the group-von Neumann algebras, see [39, 48]. In the classical case this notion recovers precisely the closed subgroups of a locally compact group \( G \). Observe that there is in general no associated homomorphism \( L^\infty(G) \to L^\infty(H) \) for a quantum subgroup, this fails already in the group case.

Let \( H \to G \) be a morphism of quantum groups and let \( B \) be a \( G\)-\( C^* \)-algebra with coaction \( \beta : B \to M(\cop{C^*_0(G)} \otimes B) \). Identifying \( \beta \) with a normal coaction of the full \( C^* \)-algebra \( \cop{C^*_0(G)} \), the map \( \pi : \cop{C^*_0(G)} \to M(\cop{C^*_0(H)}) \) induces on \( B \) a continuous coaction \( \text{res}(\beta) : B \to M(\cop{C^*_0(H)} \otimes B) \). We write \( \text{res}^B_H \) (\( B \)) for the resulting \( H\)-\( C^* \)-algebra. In this way we obtain a functor \( \text{res}^B_H : \text{G-Alg} \to \text{H-Alg} \).

Conversely, let \( G \) be a strongly regular quantum group and let \( H \subset G \) be a closed quantum subgroup. Given an \( H\)-\( C^* \)-algebra \( B \), there exists an induced \( G\)-\( C^* \)-algebra \( \text{ind}_B^G \) such that the following version of Green’s imprimitivity theorem holds [48].

**Theorem 2.6.** Let \( G \) be a strongly regular quantum group and let \( H \subset G \) be a closed quantum subgroup. Then there is a natural \( \cop{C^*_r(G)} \)-\( \text{colinear Morita equivalence} \)

\[
\cop{C^*_r(G)} \circ \text{ind}_H^G(B) \sim_M \cop{C^*_r(H)} \circ \beta, \quad B
\]

for all \( H\)-\( C^* \)-algebras \( B \).

In fact, the induced \( C^* \)-algebra \( \text{ind}_H^G(B) \) is defined by Vaes in [48] using a generalized Landstad theorem after construction of its reduced crossed product. A description of \( \cop{C^*_r(G)} \circ \beta, \text{ind}_H^G(B) \) can be given as follows. From the quantum subgroup \( H \subset G \) one first obtains a right coaction \( L^\infty(G) \to L^\infty(G) \otimes L^\infty(H) \) on the level of von Neumann algebras. The von-Neumann algebraic homogeneous space \( L^\infty(G/H) \subset L^\infty(G) \) is defined as the subalgebra of invariants under this coaction. If \( \hat{\pi} : \mathcal{L}(H)' \to \mathcal{L}(G)' \) is the homomorphism \( \hat{\pi}(x) = J_G \hat{\pi}(j_H x j_H) j_G \) induced by \( \pi : \mathcal{L}(H) \to \mathcal{L}(G) \), then

\[
I = \{ v \in \mathcal{L}(H,H) \mid vx = \hat{\pi}(x)v \text{ for all } x \in \mathcal{L}(H)' \}
\]

defines a von-Neumann algebraic imprimitivity bimodule between the von Neumann algebraic crossed product \( \mathcal{L}(G) \circ \hat{\pi}, \text{ind}_H^G(B) \) and \( \mathcal{L}(H) \circ \beta, \text{ind}_H^G(B) \). There is a \( C^* \)-algebraic homogeneous space \( \cop{C^*_0(G/H)} \subset L^\infty(G/H) \) and a \( C^* \)-algebraic imprimitivity bimodule \( \mathcal{T} \subset I \) which implements a Morita equivalence between \( \cop{C^*_r(G)} \circ \beta, \text{ind}_H^G(B) \) and \( \cop{C^*_r(H)} \circ \beta, \text{ind}_H^G(B) \). Explicitly, we have

\[
\mathcal{T} \circ \mathbb{H}_G = [V_G(I \otimes 1)(\text{id} \otimes \hat{\pi})(\hat{V}_H^*)(\cop{C^*_r(H)} \circ \mathbb{H}_G)].
\]

The crossed product of the induced \( C^* \)-algebra \( \text{ind}_H^G(B) \) is then given by

\[
\cop{C^*_r(G)} \circ \beta, \text{ind}_H^G(B) = [(\mathcal{T} \otimes 1)\beta(B)(\mathcal{T}^* \otimes 1)]
\]

where \( \beta : B \to M(\cop{C^*_0(H)} \otimes B) \) is the coaction on \( B \).

At several points of the paper we will rely on techniques developed in [48]. Firstly, as indicated in [48], let us note that we have induction in stages.

**Proposition 2.7.** Let \( H \subset K \subset G \) be strongly regular quantum groups. Then there is a natural \( G \)-equivariant isomorphism

\[
\text{ind}_H^K(B) \cong \text{ind}_H^K(B)
\]

for every \( H\)-\( C^* \)-algebra \( B \).
Proof. Let \( \hat{\pi}_K^G : \mathcal{L}(H) \to \mathcal{L}(G) \) be the normal \( * \)-homomorphism corresponding to the inclusion \( H \subset G \), and denote by \( I_K^G \subset I_H^G \subset \mathcal{L}(\mathbb{H}_K, \mathbb{H}_G) \) the associated imprimitivity bimodules. For the inclusions \( H \subset K \) and \( K \subset G \) we use analogous notation. By assumption we have \( \hat{\pi}_K^G \hat{\pi}_K^H = \hat{\pi}_H^G \), and we observe that \( I_K^H I_K^G \subset I_K^G \) is strongly dense.

Since the \( * \)-homomorphism \( \hat{\pi}_K^G \) is normal and injective we obtain
\[
I_K^G \cong \mathbb{H}_G = [(\text{id} \otimes \hat{\pi}_K^G)(\hat{\Theta}_K)(I_K^G \otimes 1)(\text{id} \otimes \hat{\pi}_H^G)(\hat{\Theta}_H)(C^*_r(H) \otimes \mathbb{H}_G)]
\]
which yields
\[
[I_K^G I_K^G] \otimes \mathbb{H}_G = [(\text{id} \otimes \hat{\pi}_K^G)(\hat{\Theta}_K)(I_K^G \otimes 1)(\text{id} \otimes \hat{\pi}_H^G)(\hat{\Theta}_H)(C^*_r(H) \otimes \mathbb{H}_G)].
\]

Using the normality of \( \hat{\pi}_K^G \) we see that if \( (v_i) \in I \) is a bounded net in \( I^G_K \) converging strongly to zero then \( \hat{V}_G(v) \otimes 1 \) converges to zero in norm for all \( x \in C^*_r(H) \) and \( \xi \in \mathbb{H}_G \). As a consequence we obtain \( I^G_K = [I^G_K I^G_K] \) for the \( C^*_r \)-algebraic imprimitivity bimodules.

Now let \( B \) be an \( H-C^*_r \)-algebra with coaction \( \beta \). Then we have
\[
C^*_r(G) \otimes K = \text{ind}^{G}_{K} \text{ind}^{K}_{H}(B) = [(I^{G}_{K} \otimes \text{id} \otimes \text{id})(\Delta_K \otimes \text{id})(\text{ind}^{K}_{H}(B))(I^{G}_{K})^* \otimes \text{id} \otimes \text{id})]
\]
\[
\cong [(I^{G}_{K} \otimes \text{id} \otimes \text{id})(\Delta_K \otimes \text{id})(\text{ind}^{K}_{H}(B))(\text{ind}^{K}_{H}(B))\Delta_{K} \otimes \text{id} \otimes \text{id}]
\]
\[
\cong [(I^{G}_{K} \otimes \text{id} \otimes \text{id})(\text{ind}^{K}_{H}(B))(I^{K}_{H})^* \otimes \text{id} \otimes \text{id})]
\]
using conjugation with the unitary \((\hat{\pi}_K^G \otimes \text{id})(V_K^*)\) in the second step. The resulting isomorphism between the crossed products \( C^*_r(G) \otimes K \) and \( C^*_r(G) \otimes H \) is \( C^*_r(G) \)-colinear and identifies the natural corepresentations of \( C^*_r(G) \) on both sides. Hence theorem 6.7 in [15] yields the assertion. \( \square \)

Let \( H \subset G \) be a quantum subgroup of a strongly regular quantum group \( G \) and let \( B \) be an \( H-C^*_r \)-algebra with coaction \( \beta \). If \( E \) denotes the trivial group, then due to proposition 2.4 we have
\[
\text{ind}^G_{H}(C^*_r(H) \otimes B) = \text{ind}^G_{H} \text{ind}^H_{E}(B) \cong \text{ind}^{G}_{K} \text{res}^{H}_{E}(B) = C^*_r(G) \otimes B
\]
where \( C^*_r(H) \otimes B \) is viewed as an \( H-C^*_r \)-algebra via comultiplication on the first tensor factor. The \( * \)-homomorphism \( \beta : B \to M(C^*_r(H) \otimes B) \) induces an injective \( G \)-equivariant \( * \)-homomorphism \( \text{ind}^G_{H}(\beta) : \text{ind}^G_{H}(B) \to M(\text{ind}^G_{H}(C^*_r(H) \otimes B)) \), and it follows that \( \text{ind}^G_{H}(B) \) is contained in \( M(C^*_r(G) \otimes B) \). Using that the coaction \( \beta \) is continuous we see that \( \text{ind}^G_{H}(B) \) is in fact contained in the \( C^*_r(G) \)-relative multiplier algebra of \( C^*_r(H) \). \( \Box \)

Now let \( A \) and \( B \) be \( H-C^*_r \)-algebras. According to the previous observations every \( H \)-equivariant \( * \)-homomorphism \( f : A \to B \) induces a \( G \)-equivariant \( * \)-homomorphism \( \text{ind}^G_{H}(f) : \text{ind}^G_{H}(A) \to \text{ind}^G_{H}(B) \) in a natural way. We conclude that induction defines a functor \( \text{ind}^G_{H} : H-\text{Alg} \to G-\text{Alg} \).

3. YETTER-DRINFELD ALGEBRAS AND BRAIDED TENSOR PRODUCTS

In this section we study Yetter-Drinfeld-\( C^*_r \)-algebras and braided tensor products. We remark that these concepts are well-known in the algebraic approach to quantum groups [30]. Yetter-Drinfeld modules for compact quantum groups are discussed in [12].

Let us begin with the definition of a Yetter-Drinfeld \( C^*_r \)-algebra.

**Definition 3.1.** Let \( G \) be a locally compact quantum group and let \( S = C^*_r(H) \) and \( \hat{S} = C^*_r(G) \) be the associated reduced Hopf-\( C^*_r \)-algebras. A \( G \)-Yetter-Drinfeld
$C^*$-algebra is a $C^*$-algebra $A$ equipped with continuous coactions $\alpha$ of $S$ and $\lambda$ of $\hat{S}$ such that the diagram

$$
\begin{array}{ccc}
A & \overset{\lambda}{\longrightarrow} & M(\hat{S} \otimes A) \\
\downarrow{\alpha} & & \downarrow{id \otimes \alpha} \\
M(S \otimes A) & \overset{id \otimes \lambda}{\longrightarrow} & M(S \otimes \hat{S} \otimes A) \\
\end{array}
$$

is commutative. Here $\text{ad}(W)(x) = WxW^*$ denotes the adjoint action of the fundamental unitary $W \in M(S \otimes \hat{S})$.

In order to compare definition 3.1 with the notion of a Yetter-Drinfeld module in the algebraic setting, one should keep in mind that we work with the opposite comultiplication in the sequel. We will also refer to $G$-Yetter-Drinfeld $C^*$-algebras as $G$-YD-algebras. A homomorphism of $G$-YD-algebras $f : A \to B$ is a $*$-homomorphism which is both $G$-equivariant and $\hat{G}$-equivariant. We remark that the concept of a Yetter-Drinfeld-$C^*$-algebra is self-dual, that is, $G$-YD-algebras are the same thing as $\hat{G}$-YD-algebras.

Let us discuss some basic examples of Yetter-Drinfeld-$C^*$-algebras. Consider first the case that $G$ is an ordinary locally compact group. Since $C_0(G)$ is commutative, every $G$-$C^*$-algebra becomes a $G$-YD-algebra with the trivial coaction of $C_0^*(G)$. If $G = \hat{G}$ is discrete then such coactions correspond to Fell bundles over $G$. In this case a Yetter-Drinfeld structure is determined by an action of $G$ on the bundle which is compatible with the adjoint action on the base $G$.

Let $G$ be a locally compact quantum group and consider the $G$-$C^*$-algebra $C_0^*(G)$ with coaction $\Delta$. If $G$ is regular the map $\lambda : C_0^*(G) \to M(C^*_r(G) \otimes C_0^*(G))$ given by $\lambda(f) = \hat{W}^*(1 \otimes f)\hat{W}$ defines a continuous coaction. Moreover

$$(\text{ad}(W) \otimes \text{id})(\text{id} \otimes \lambda)\Delta(f) = W_{12}\hat{W}_{23}^*(W_{13}^*W_{12}^*(1 \otimes 1 \otimes f)W_{13}\hat{W}_{23}W_{12})$$

shows that $C_0^*(G)$ together with $\Delta$ and $\lambda$ is a $G$-YD-algebra. More generally, we can consider a crossed product $C_0^*(G) \rtimes_r A$ for a regular quantum group $G$. The dual coaction together with conjugation by $\hat{W}^*$ as above yield a $G$-YD-algebra structure on $C_0^*(G) \rtimes_r A$.

There is another way to obtain a Yetter-Drinfeld-$C^*$-algebra structure on a crossed product. Let again $G$ be a regular quantum group and let $A$ be a $G$-YD-algebra. We obtain a continuous coaction $\hat{\lambda} : C_0^*(G)^{\text{cop}} \rtimes_r A \to M(C_r^*(G) \otimes (C^*_r(G)^{\text{cop}} \rtimes_r A))$ by

$$\hat{\lambda}(x) = \hat{W}_{12}^* (\text{id} \otimes \lambda)(x)_{213} \hat{W}_{12}$$

for $x \in C_0^*(G)^{\text{cop}} \rtimes_r A \subset \text{L}(\mathbb{H}_G \otimes A)$. On the copy of $A$ in the multiplier algebra of the crossed product this coaction implements $\lambda$, and on the copy of $C_0^*(G)^{\text{cop}} = C^*_r(G)$ it is given by the comultiplication $\Delta$ of $C^*_r(G)$. In addition we have a continuous coaction $\hat{\alpha} : C_0^*(G)^{\text{cop}} \rtimes_r A \to M(C_0^*(G) \otimes (C^*_r(G)^{\text{cop}} \rtimes_r A))$ given by

$$\hat{\alpha}(x) = W_{12}^*(1 \otimes x)W_{12}.$$

Remark that on the copy of $A$ in the multiplier algebra this coaction implements $\alpha$, and on the copy of $C_r^*(G)^{\text{cop}} = C_r^*(G)$ it implements the adjoint coaction. It is
We have by restriction. These coactions are determined by the conditions where the maps \( ∗ \mathcal{a} \) nondegenerate associated continuous coactions \( α \mathcal{G} \). Conversely, assume that \( G \) and since the coaction \( γ \mathcal{G} \) and \( \mathcal{G} \) as closed quantum subgroups. If \( G \) is regular then \( \mathcal{D}(G) \) is again regular.

**Proposition 3.2.** Let \( G \) be a locally compact quantum group and let \( \mathcal{D}(G) \) be its Drinfeld double. Then a \( G \)-Yetter-Drinfeld \( C^* \)-algebra is the same thing as a \( \mathcal{D}(G) \)-\( C^* \)-algebra.

**Proof.** Let us first assume that \( A \) is a \( \mathcal{D}(G) \)-\( C^* \)-algebra with coaction \( γ : A \to M(\mathcal{C}_0^0(\mathcal{D}(G)) \otimes A) \). Since \( G \) and \( \hat{\mathcal{G}} \) are quantum subgroups of \( \mathcal{D}(G) \) we obtain associated continuous coactions \( α : A \to M(\mathcal{C}_0^0(\mathcal{G}) \otimes A) \) and \( λ : A \to M(\mathcal{C}_0^0(\mathcal{G}) \otimes A) \) by restriction. These coactions are determined by the conditions

\[
(δ \otimes \text{id})γ = (\text{id} \otimes α)γ, \quad (\hat{δ} \otimes \text{id})γ = (\text{id} \otimes λ)γ
\]

where the maps \( δ : \mathcal{C}_0^0(\mathcal{D}(G)) \to M(\mathcal{C}_0^0(\mathcal{D}(G)) \otimes \mathcal{C}_0^0(\mathcal{G})) \) and \( \hat{δ} : \mathcal{C}_0^0(\mathcal{D}(G)) \to M(\mathcal{C}_0^0(\mathcal{D}(G)) \otimes \mathcal{C}_0^0(\mathcal{G})) \) are given by

\[
δ = (\text{id} \otimes σ)\text{ad}(W_{23})(\Delta \otimes \text{id}), \quad \hat{δ} = \text{id} \otimes \hat{Δ}.
\]

We have

\[
\text{ad}(W_{23})(\text{id} \otimes \text{id} \otimes λ)(\text{id} \otimes α)γ = \text{ad}(W_{23})(\text{id} \otimes \text{id} \otimes λ)(δ \otimes \text{id})γ
\]

\[
= \text{ad}(W_{23})(δ \otimes \text{id} \otimes \text{id})(\hat{δ} \otimes \text{id})γ
\]

\[
= \text{ad}(W_{23})(δ \otimes \text{id} \otimes \text{id} \otimes \text{id})\text{ad}(W_{23})(\Delta \otimes \hat{Δ} \otimes \text{id})γ
\]

\[
= (\text{id} \otimes σ \otimes \text{id} \otimes \text{id})\text{ad}(W_{23})W_{23}(\Delta \otimes \hat{Δ} \otimes \text{id})γ
\]

\[
= (\text{id} \otimes σ \otimes \text{id} \otimes \text{id})(\text{id} \otimes \Delta)\text{ad}(W_{23})(\Delta \otimes \hat{Δ} \otimes \text{id})γ
\]

\[
= (\text{id} \otimes σ \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{id})δ \otimes \text{id})γ
\]

\[
= (\text{id} \otimes σ \otimes \text{id})(\text{id} \otimes \text{id} \otimes α)(\text{id} \otimes λ)γ,
\]

and since the coaction \( γ \) is continuous this implies

\[
\text{ad}(W_{12})(\text{id} \otimes λ)α = (σ \otimes \text{id})(\text{id} \otimes α)λ.
\]

It follows that we have obtained a \( G \)-YD-algebra structure on \( A \).

Conversely, assume that \( A \) is equipped with a \( G \)-YD-algebra structure. We define a nondegenerate \( * \)-homomorphism \( γ : A \to M(\mathcal{C}_0^0(\mathcal{D}(G)) \otimes A) \) by

\[
γ = (\text{id} \otimes λ)α
\]
and compute
\[
(id \otimes \gamma) \gamma = (id \otimes id \otimes id \otimes \lambda)(id \otimes id \otimes \alpha)(id \otimes \lambda) \alpha \\
= (id \otimes id \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(id \otimes \alpha)(id \otimes id \otimes \lambda)(id \otimes \lambda) \alpha \\
= (id \otimes \sigma \otimes id \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(\Delta \otimes \Delta \otimes id) \alpha \\
= (id \otimes \sigma \otimes id \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(id \otimes \sigma \otimes id \otimes \lambda)(\Delta \otimes \Delta \otimes id)(id \otimes \lambda) \alpha \\
= (\Delta_{\mathcal{D}(G)} \otimes id)(\Delta \otimes \Delta \otimes id)(\Delta \otimes id) \gamma \\
= (\Delta_{\mathcal{D}(G)} \otimes id) \gamma.
\]

It follows that $\gamma$ is a continuous coaction which turns $A$ into a $\mathcal{D}(G)$-$C^*$-algebra.

One checks easily that the two operations above are inverse to each other. □

We shall now define the braided tensor product $A \boxtimes_B$ of a $G$-$C^*$-algebra $A$ with a $G$-$C^*$-algebra $B$. Observe first that the $C^*$-algebra $B$ acts on the Hilbert module $\mathbb{H} \otimes B$ by $(\pi \otimes id)\beta$ where $\pi : C^*_{\mathcal{D}}(G) \to L(\mathbb{H})$ denotes the defining representation on $\mathbb{H} = H_G$. Similarly, the $C^*$-algebra $A$ acts on $\mathbb{H} \otimes A$ by $(\hat{\pi} \otimes id)\lambda$ where $\hat{\pi} : C^*_{\mathcal{D}}(G) \to L(\mathbb{H})$ is the defining representation. From this we obtain two $*$-homomorphisms $\iota_A : A \to L(\mathbb{H} \otimes A \otimes B)$ and $\iota_B : B \to L(\mathbb{H} \otimes A \otimes B)$ by acting with the identity on the factor $B$ and $A$, respectively.

**Definition 3.3.** Let $G$ be a locally compact quantum group, let $A$ be a $G$-$C^*$-algebra and $B$ a $G$-$C^*$-algebra. With the notation as above, the braided tensor product $A \boxtimes_B$ is the $C^*$-subalgebra of $L(\mathbb{H} \otimes A \otimes B)$ generated by all elements $\iota_A(a)\iota_B(b)$ for $a \in A$ and $b \in B$.

We will also write $A \boxtimes B$ instead of $A \boxtimes_B$ if the quantum group $G$ is clear from the context. The braided tensor product $A \boxtimes B$ is in fact equal to the closed linear span $[\iota_A(A)\iota_B(B)]$. This follows from proposition 8.3 in [BG], we reproduce the argument for the convenience of the reader. Clearly it suffices to prove $[\iota_A(A)\iota_B(B)] = [\iota_B(B)\iota_A(A)]$. Using continuity of the coaction $\lambda$ and $\tilde{V} = (\tilde{J} \otimes 1)W^*(\tilde{J} \otimes 1)$ we get

\[
\lambda(A) = [(L(\mathbb{H}_G) *, \otimes id \otimes id)(\Delta \otimes id)\lambda(A)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id)((V_{12} \lambda(A) \lambda_1 \lambda_2)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id)((W_{12}^* \mu(A) \lambda_2 \lambda_1)]
\]

where $\mu(x) = (\tilde{J} \otimes 1)x(\tilde{J} \otimes 1)$ for $x \in A$. Since $\beta : B \to M(C^*_{\mathcal{D}}(G) \otimes B)$ is a continuous coaction we obtain

\[
[\lambda(A)_{12}\beta(B)_{13}] = [(L(\mathbb{H}_G) *, \otimes id \otimes id \otimes id)(W_{12}^* \mu(A) \lambda_2 \lambda_1 \lambda_2 \lambda_1)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id \otimes id)(W_{12}^* \mu(A) \lambda_2 \lambda_1 \lambda_2 \lambda_1)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id \otimes id)(W_{12}^* \mu(A) \lambda_2 \lambda_1 \lambda_2 \lambda_1)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id \otimes id)(\Delta \otimes id)\beta(B)_{12} \mu(A)_{13} \lambda_1 \lambda_2)] \\
= [(L(\mathbb{H}_G) *, \otimes id \otimes id \otimes id)\beta(B)_{12} \mu(A)_{13} \lambda_1 \lambda_2)] \\
= [(\lambda_1 \lambda_2) * \lambda(A)]
\]

which yields the claim. It follows in particular that we have natural nondegenerate $*$-homomorphisms $\iota_A : A \to M(A \boxtimes B)$ and $\iota_B : B \to M(A \boxtimes B)$.

The braided tensor product $A \boxtimes B$ becomes a $G$-$C^*$-algebra in a canonical way. In fact, we have a nondegenerate $*$-homomorphism $\alpha \boxtimes \beta : A \boxtimes B \to M(C^*_{\mathcal{D}}(G) \otimes (A \boxtimes B))$ given by

\[
(\alpha \boxtimes \beta)(\lambda(a)_{12}\beta(b)_{13}) = W_{12}^* \sigma \otimes id)((id \otimes \alpha) \lambda(a))_{123} \beta(b)_{24}W_{12} \\
= (id \otimes \lambda) \alpha(a)_{123} \beta(b)_{24}W_{12},
\]
and it is straightforward to check that $\alpha \boxtimes \beta$ defines a continuous coaction of $C^*_\gamma(G)$ such that the $*$-homomorphisms $\iota_A$ and $\iota_B$ are $G$-equivariant.

If $B$ is a $G$-YD-algebra with coaction $\gamma : B \to M(C^*_\gamma(G) \otimes B)$ then we obtain a nondegenerate $*$-homomorphism $\lambda \boxtimes \gamma : A \boxtimes B \to M(C^*_\gamma(G) \otimes (A \boxtimes B))$ by the formula

$$(\lambda \boxtimes \gamma)(\lambda(a)_{12}\beta(b)_{13}) = \hat{W}_{12}\lambda(a)_{23}(\sigma \otimes \id)((\id \otimes \gamma)\beta(b))_{124}\hat{W}_{12} = (\id \otimes \lambda)\lambda(a)_{123}(\id \otimes \beta)\gamma(b)_{124}.$$  

In the same way as above one finds that $\lambda \boxtimes \gamma$ yields a continuous coaction of $C^*_\gamma(G)$ such that $\iota_A$ and $\iota_B$ are $G$-equivariant. From the equivariance of $\iota_A$ and $\iota_B$ it follows that $A \boxtimes B$ together with the coactions $\alpha \boxtimes \beta$ and $\lambda \boxtimes \gamma$ becomes a $G$-YD-algebra.

If $A$ is a $G$-YD-algebra and $f : B \to C$ a possibly degenerate equivariant $*$-homomorphism of $G$-$C^*$-algebras, then we obtain an induced $*$-homomorphism $M_K(K \otimes A \otimes B) \to M_K(K \otimes A \otimes C)$ between the relative multiplier algebras. Since $A \boxtimes B \subset M(K \otimes A \otimes B)$ is in fact contained in $M_K(K \otimes A \otimes B)$, this map restricts to an equivariant $*$-homomorphism $\id \otimes f : A \boxtimes B \to A \boxtimes C$. It follows that the braided tensor product defines a functor $A \boxtimes -$ from $G$-$\Alg$ to $G$-$\Alg$. Similarly, if $f : A \to B$ is a homomorphism of $G$-YD-algebras we obtain for every $G$-algebra $C$ an equivariant $*$-homomorphism $f \boxtimes \id : A \boxtimes C \to B \boxtimes C$ and a functor $- \boxtimes C$ from $D(G)$-$\Alg$ to $G$-$\Alg$. There are analogous functors $A \boxtimes -$ and $- \boxtimes C$ from $D(G)$-$\Alg$ to $D(G)$-$\Alg$ if we consider $G$-YD-algebras in the second variable.

Assume now that $A$ and $B$ are $G$-YD-algebras and that $C$ is a $G$-$C^*$-algebra. According to our previous observations we can form the braided tensor products $(A \boxtimes B) \boxtimes C$ and $A \boxtimes (B \boxtimes C)$, respectively. We have

$$(A \boxtimes B) \boxtimes C = [(\id \otimes \lambda^A)\lambda^A(A)_{123}(\id \otimes \beta)\lambda^B(B)_{124}\gamma(C)_{15}]$$

$= [\hat{W}_{12}\lambda^A(A)_{23}\hat{W}_{12}(\id \otimes \beta)\lambda^B(B)_{124}\hat{W}_{12}\Sigma_{12}\hat{W}_{12}\Sigma_{12}\gamma(C)_{15}]$

$= [\hat{W}_{12}\lambda^A(A)_{23}\Sigma_{12}(\id \otimes \lambda^B)\beta(B)_{124}\Sigma_{12}\gamma(C)_{15}\Sigma_{12}\hat{W}_{12}]$

$= [\hat{W}_{12}\Sigma_{12}\lambda^A(A)_{13}(\id \otimes \lambda^B)\beta(B)_{124}(\id \otimes \gamma)\gamma(C)_{12}\Sigma_{12}\hat{W}_{12}]$

$\cong [\lambda^A(A)_{13}(\beta \boxtimes \gamma)(B \boxtimes C)_{124}] \cong A \boxtimes (B \boxtimes C),$  

and the resulting isomorphism $(A \boxtimes B) \boxtimes C \cong A \boxtimes (B \boxtimes C)$ is $G$-equivariant. If $C$ is a $G$-YD-algebra then this isomorphism is in addition $G$-equivariant. We conclude that the braided tensor product is associative in a natural way. If $B$ is a trivial $G$-algebra then the braided tensor product $A \boxtimes B$ is isomorphic to $A \otimes B$ with the coaction induced from $A$. Similarly, if the coaction of $C^*_\gamma(G)$ on the $G$-YD-algebra $A$ is trivial then $A \boxtimes B$ is isomorphic to $A \otimes B$. Recall that if $G$ is a locally compact group we may view all $G$-algebras as $G$-YD-algebras with the trivial coaction of the group $C^*$-algebra. In this case the braided tensor product reduces to the ordinary tensor product of $G$-$C^*$-algebras with the diagonal $G$-action. For general quantum groups the braided tensor product should be viewed as a substitute for the latter construction.

Following an idea of Vaes, we shall now discuss the compatibility of the braided tensor product with induction and restriction. Let $G$ be a strongly regular quantum group and let $H \subset G$ be a closed quantum subgroup determined by the faithful normal $*$-homomorphism $\hat{\pi} : \mathcal{L}(H) \to \mathcal{L}(G)$. Keeping our notation from section 2, we denote by $I$ the corresponding von-Neumann algebraic imprimitivity bimodule for $\mathcal{L}(G)^\text{cop} \ltimes \mathcal{L}(G/H)$ and $\mathcal{L}(H)^\text{cop}$, and by $I \subset I$ the $C^*$-algebraic imprimitivity bimodule for $C^*_\gamma(G)^\text{cop} \ltimes, C^*_\gamma(G/H) and C^*_\gamma(H)^\text{cop}$. 
Proposition 3.4. Let $G$ be a strongly regular quantum group and let $H \subset G$ be a closed quantum subgroup. If $A$ is an $H$-YD-algebra then the induced $C^*$-algebra $\text{ind}_H^G(A)$ is a $G$-YD-algebra in a natural way.

Proof. Let $\alpha : A \to M(C_0^r(H) \otimes A)$ be the coaction of $C_0^r(H)$ on $A$. From the construction of $\text{ind}_H^G(A)$ in [48] we have the induced coaction $\text{ind}(\alpha) : \text{ind}_H^G(A) \to M(C_0^r(G) \otimes A)$ given by $\text{ind}(\alpha)(x) = (W_G^r)_{12}(1 \otimes x)(W_G^r)_{12}$ for $x \in \text{ind}_H^G(A) \subset L(H \otimes G)$. Our task is to define a continuous coaction of $C^*_r(G) \otimes \text{ind}_H^G(A)$ satisfying the YD-condition.

Denote by $\lambda : A \to M(C^*_r(H) \otimes A)$ the coaction which determines the $H$-YD-algebra structure on $A$. This coaction induces a coaction $\text{res}(\hat{\lambda}) : A \to M(C^*_r(G) \otimes A)$ because $H \subset G$ is a closed quantum subgroup. Since $A$ is an $H$-YD-algebra we have in addition the coaction $\hat{\lambda}$ on the crossed product $C^*_r(H)^{\text{cop}} \ltimes \alpha$, and a corresponding coaction $\text{res}(\hat{\lambda})$ of $C^*_r(G)$.

We abbreviate $B = C^*_r(H)^{\text{cop}} \ltimes \alpha$ and consider the Hilbert $B$-module $\mathcal{E} = B$ with the corepresentation $X = W_H \otimes \text{id} \in M(C_0^r(H) \otimes \mathbb{K}(\mathcal{E})) = M(C_0^r(H) \otimes B)$. The corresponding induced Hilbert $B$-module $\text{ind}_H^G(\mathcal{E})$ is constructed in [48] such that

$$H \otimes \text{ind}_H^G(\mathcal{E}) \cong I \otimes_{\pi_H} \mathcal{F}$$

where $\mathcal{F} = \mathbb{H}_G \otimes \mathcal{E}$ and the strict $\ast$-homomorphism $\pi_H : L(H) \to \mathbb{L}(\mathcal{F})$ is determined by $(\text{id} \otimes \pi_H)(W_H) = (\text{id} \otimes \pi)(W_H)_{12}X_{13}$.

Let us define a coaction on $I \otimes_{\pi_H} \mathcal{F}$ as follows. On $I$ we have the adjoint action $\eta : I \to L(G) \otimes I$ given by

$$\eta(v) = \tilde{W}_G^*(1 \otimes v)(\hat{\pi} \otimes \text{id})(W_H)$$

which is compatible with the coaction $(\hat{\pi} \otimes \text{id})\hat{\Delta}_H : L(H) \to L(G) \otimes L(H)$. In addition consider the coaction $\beta_{\mathcal{F}}$ of $C^*_r(G)$ on $\mathcal{F}$ given by $\beta_{\mathcal{F}} = (\sigma \otimes \text{id})((\hat{\pi} \otimes \text{id})\text{res}(\hat{\lambda}))$.

By construction, $\beta_{\mathcal{F}}$ is compatible with the coaction $\text{res}(\hat{\lambda})$ on $B$. Moreover, the $\ast$-homomorphism $\pi_H : L(H) \to L(\mathcal{F})$ is covariant in the sense that

$$(\text{id} \otimes \pi_H)((\hat{\pi} \otimes \text{id})\hat{\Delta}_H(x) = \text{ad}_{\beta_{\mathcal{F}}} (\pi_H(x))$$

in $L(\mathbb{H}_G \otimes \mathcal{F})$ for all $x \in L(H)$. According to proposition 12.13 in [48] we obtain a product coaction of $C^*_r(G)$ on $I \otimes_{\pi_H} \mathcal{F}$.

Under the above isomorphism, this product coaction leaves invariant the natural representations of $L^\infty(G)'$ and $L(G)'$ on the first tensor factor of the left hand side. Hence there is an induced coaction $\gamma : \text{ind}_H^G(\mathcal{E}) \to M(C^*_r(G) \otimes \text{ind}_H^G(\mathcal{E}))$ on $\text{ind}_H^G(\mathcal{E})$. Using the identification $\text{ind}_H^G(\mathcal{E}) \cong [(I \otimes 1)\alpha(A)]$ we see that $\gamma$ is given by

$$\gamma((v \otimes 1)\alpha(a)) = (\eta(v) \otimes 1)(\text{id} \otimes \alpha)\text{res}(\hat{\lambda})(a)$$

for $v \in I$ and $a \in A$. Since $\mathbb{K}(\text{ind}_H^G(\mathcal{E})) = C^*_r(G)^{\text{cop}} \ltimes \alpha$, $\text{ind}_H^G(A)$ we obtain a coaction $\text{ad}_\gamma$ on $C^*_r(G)^{\text{cop}} \ltimes \alpha$, $\text{ind}_H^G(A)$. By construction, the coaction $\text{ad}_\gamma$ commutes with the dual coaction and is given by $\hat{\Delta}$ on the copy of $C^*_r(G)^{\text{cop}}$. It follows that $\text{ad}_\gamma$ induces a continuous coaction $\delta : \text{ind}_H^G(A) \to M(C^*_r(G) \otimes \text{ind}_H^G(A))$. Explicitly, this coaction is given by

$$\delta(x) = (W_G^r)_{12}(\sigma \otimes \text{id})(\text{id} \otimes \text{res}(\hat{\lambda}))(x)(W_G^r)_{12}$$
for $x \in \text{ind}_{H}^{G}(A) \subset L(H) \otimes A)$. Writing $W_G = W$ and $\hat{W}_G = \hat{W}$ we calculate

$$\text{ad}(W_{12})(\text{id} \otimes \delta) \text{ind}(\alpha)(x)$$

$$= W_{12} \hat{W}_{23} \Sigma_{23} (\text{id} \otimes \text{id} \otimes \text{res}(\lambda))(W_{12}^* (1 \otimes x) W_{12}) \hat{W}_{23} \Sigma_{23} \hat{W}_{12}$$

$$= \Sigma_{23} W_{13} W_{23} W_{12}^* (\text{id} \otimes \text{id} \otimes \text{res}(\lambda))(1 \otimes x) W_{12} W_{23}^* \Sigma_{23} \hat{W}_{12}$$

$$= \Sigma_{23} W_{12}^* W_{23} (\text{id} \otimes \text{id} \otimes \text{res}(\lambda))(1 \otimes x) W_{23} \Sigma_{23} \hat{W}_{23} W_{13}$$

$$= (x \otimes \text{id})(\text{id} \otimes \text{ind}(\alpha))\delta(x)$$

which shows that $\text{ind}(\alpha)$ and $\delta$ combine to turn $\text{ind}_{H}^{G}(A)$ into a $G$-YD-algebra. □

Let $G$ be a regular locally compact quantum group and let $A$ be a $G$-YD-algebra with coactions $\alpha$ and $\lambda$. As explained above, the crossed product $C_r^*(G)^{\text{cop}} \rtimes_r A$ is again a $G$-YD-algebra in a natural way. Moreover let $B$ be a $G$-algebra with coaction $\beta$ and observe

$$\text{ind}(\beta)(r^*(A \boxtimes_G B)) = ((C^*_r(G)^{\text{cop}} \rtimes_r (A \boxtimes_G B) \cong [\beta(C^*_r(G)^{\text{cop}} \rtimes_r (A \boxtimes_G B)] = \text{ind}(\beta)(B))_{13}$$

Under this isomorphism the dual coaction on the left hand side corresponds to the coaction determined by the dual coaction on $C_r^*(G)^{\text{cop}} \rtimes_r A$ and the trivial coaction on $B$ on the right hand side. As a consequence we obtain the following lemma.

**Lemma 3.5.** Let $G$ be a regular locally compact quantum group, let $A$ be a $G$-YD-algebra and let $B$ be a $G$-algebra. Then there is a natural $C_r^*(G)^{\text{cop}}$-colinear isomorphism

$$C_r^*(G)^{\text{cop}} \rtimes_r (A \boxtimes_G B) \cong (C_r^*(G)^{\text{cop}} \rtimes_r A) \boxtimes_G B.$$ 

After these preparations we shall now describe the compatibility of restriction, induction and braided tensor products.

**Theorem 3.6.** Let $G$ be a strongly regular quantum group and let $H \subset G$ be a closed quantum subgroup. Moreover let $A$ be an $H$-YD-algebra and let $B$ be a $G$-algebra. Then there is a natural $G$-equivariant isomorphism

$$\text{ind}_{H}^{G}(A \boxtimes_{H} \text{res}_{H}^{G}(B)) \cong \text{ind}_{H}^{G}(A) \boxtimes_{G} B.$$ 

Proof. Note that the case $A = \mathbb{C}$ with the trivial action is treated in [15]. We denote by $\text{res}(\beta)$ the restriction to $C_r^*(H)$ of the coaction $\beta : B \to M(C_r^*(H) \otimes B)$. Moreover let $\text{res}(\lambda)$ be the push-forward of the coaction $\lambda : A \to M(C_r^*(H) \otimes A)$ to $C_r^*(G)$. Then

$$[(\text{id} \otimes \text{id} \otimes \beta)(\lambda(A)_{12} \text{res}(\beta)(B))_{13}] = [\lambda(A)_{12} (\text{id} \otimes \text{id} \otimes \beta) \text{res}(\beta)(B)]_{134}$$

$$= [\lambda(A)]_{12} (\text{id} \otimes \hat{\beta})(W_H^*)_{13} \beta(B)_{34} (\text{id} \otimes \hat{\beta})(W_H)_{13}$$

$$= [(\hat{\beta} \otimes \text{id}) (\text{id} \otimes \lambda)(\lambda(A))]_{312} \beta(B)_{34} (\text{id} \otimes \hat{\beta})(W_H)_{13}$$

$$= [(\hat{\beta} \otimes \text{id}) (\text{id} \otimes \lambda)(\lambda(A))]_{312} \beta(B)_{34}$$

and hence

$$A \boxtimes H \text{res}_{H}^{G}(B) \cong [\text{res}(\lambda)(A)_{12} \beta(B)]_{13}.$$
Writing $W_G = W$ we conclude

$$C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A \boxtimes_H r \text{res}_{H}^{G}(B))$$

$$\cong [(I \otimes \text{id} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \text{res}(\lambda))(\text{id} \otimes \text{res}(\beta))(\text{id} \otimes \beta)]_{123}(I^* \otimes \text{id} \otimes \text{id} \otimes \text{id})$$

$$= [(\text{id} \otimes \text{res}(\lambda))(I \otimes \text{id} \otimes \text{id})(I^* \otimes \text{id})]_{123}(\Delta_G \otimes \text{id})\beta(B)_{124}$$

$$= [W^*_{12}((W \otimes \text{id})(\text{id} \otimes \text{res}(\lambda))(\text{id} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \beta))(W^* \otimes 1)]_{123}\beta(B)_{24}W_{12}$$

using that $[(I \otimes \text{id})\text{res}(\beta)(B)] = [\beta(B)(I \otimes \text{id})]$ for the restricted coaction $\text{res}(\beta)$. Moreover

$$[W^*_{12}((W \otimes \text{id})(\text{id} \otimes \text{res}(\lambda))(\text{id} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \beta))(W^* \otimes 1)]_{123}\beta(B)_{24}W_{12}$$

$$\cong [((W \otimes \text{id})(I \otimes \text{id} \otimes \text{id})(I^* \otimes \text{id}))(\text{id} \otimes \text{id} \otimes \text{id})(\text{id} \otimes \beta)]_{123,3}\beta(B)_{24}$$

$$\cong (C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A)) \boxtimes_{G} B$$

where $\delta : C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A) \rightarrow M(C^*_r(G) \otimes (C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A)))$ is the natural coaction on the cross product of the G-YD-algebra $\text{ind}_{H}^{G}(A)$. Under these identifications, the dual coaction on $C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A \boxtimes_H r \text{res}_{H}^{G}(B))$ corresponds on $(C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A)) \boxtimes_{G} B$ to the dual coaction on the crossed product and the trivial coaction on $B$. As a consequence, using lemma 6.3 we obtain a $C^*_r(G)^{\text{cov}}$-equivariant isomorphism

$$C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A \boxtimes_H r \text{res}_{H}^{G}(B)) \cong C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A) \boxtimes_{G} B)$$

Moreover, the element $W \otimes \text{id} \in M(C^*_r(G) \otimes C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A \boxtimes_H r \text{res}_{H}^{G}(B)))$ is mapped to $W \otimes \text{id} \in M(C^*_r(G) \otimes C^*_r(G)^{\text{cov}} \ltimes r \text{ind}_{H}^{G}(A) \boxtimes_{G} B))$ under this isomorphism. Due to theorem 6.7 in [13] this shows that there is a $G$-equivariant isomorphism

$$\text{ind}_{H}^{G}(A \boxtimes_H r \text{res}_{H}^{G}(B)) \cong \text{ind}_{H}^{G}(A) \boxtimes_{G} B)$$

as desired.

We also need braided tensor products of Hilbert modules. Since the constructions and arguments are similar to the algebra case treated above our discussion will be rather brief. Assume that $A$ is a $G$-YD-algebra and that $B$ is a $G$-algebra. Moreover let $E_A$ be a $D(G)$-Hilbert module and let $F_B$ be a $G$-Hilbert module. As in the algebra case, a $D(G)$-Hilbert module $E$ is the same thing as a Hilbert module equipped with continuous coactions $\alpha_E$ of $C^*_0(G)$ and $\lambda_E$ of $C^*_r(G)$ satisfying the Yetter-Drinfeld compatibility condition in the sense that

$$(\sigma \otimes \text{id})(\text{id} \otimes \alpha_E)\lambda_E = (\text{ad}(W) \otimes \text{id})(\text{id} \otimes \lambda_E)\alpha_E$$

where $\text{ad}(W)$ is the adjoint action.

The braided tensor product of $E$ and $F$ is defined as

$$E \boxtimes_{G} F = [\lambda_E(E)_{12} \beta_{\text{F}}(F)_{13}] \subset M_{K}(K \otimes E \otimes F)$$

where $\lambda_E$ denotes the coaction of $C^*_r(G)$ on $E$ and $\beta_{\text{F}}$ is the coaction of $C^*_0(G)$ on $F$. One has $[\lambda_E(E)_{12} \beta_{\text{F}}(F)_{13}] = [\beta_{\text{F}}(F)_{13} \lambda_E(E)_{12}]$, and $E \boxtimes_{G} F$ is closed under right multiplication by elements from $A \boxtimes_{G} B \subset M_{K}(K \otimes A \otimes B)$. Moreover the restriction to $E \boxtimes_{G} F$ of the scalar product of $M_{K}(K \otimes E \otimes F)$ takes values in $A \boxtimes_{G} B$. It follows that $E \boxtimes_{G} F$ is a Hilbert-$A \boxtimes_{G} B$-module.

As in the algebra case there is a continuous coaction of $C^*_0(G)$ on $E \boxtimes_{G} F$ given by

$$\text{ad}(W_{12})(\sigma \otimes \text{id})(\text{id} \otimes \alpha_E \otimes \text{id}).$$

Similarly, if $B$ is a $G$-YD-algebra and $F$ is a $D(G)$-Hilbert module we have a continuous $C^*_r(G)$-coaction. The braided tensor product becomes a $D(G)$-Hilbert module in this case.
There are canonical nondegenerate $\ast$-homomorphisms $\mathbb{K}(\mathcal{E}) \to \mathbb{L}(\mathcal{E} \boxtimes G \mathcal{F})$ and $\mathbb{K}(\mathcal{F}) \to \mathbb{L}(\mathcal{E} \boxtimes G \mathcal{F})$. Combining these homomorphisms yields an identification $\mathbb{K}(\mathcal{E}) \boxtimes G \mathbb{K}(\mathcal{F}) \cong \mathbb{K}(\mathcal{E} \boxtimes G \mathcal{F})$.

We conclude this section with a discussion of stability properties.

**Proposition 3.7.** Let $G$ be a regular locally compact quantum group and let $A$ be a $G$-YD-algebra.

a) For every $G$-C*-algebra $B$ there is a natural $G$-equivariant Morita equivalence 
\[ (\mathbb{K}_{\mathcal{D}(G)} \otimes A) \boxtimes G B \sim_M A \boxtimes G B. \]

If $B$ is a $G$-YD-algebra this Morita equivalence is $\mathbb{D}(G)$-equivariant.

b) For every $G$-C*-algebra $B$ there is a natural $G$-equivariant Morita equivalence 
\[ A \boxtimes G (\mathbb{K}_{\mathcal{D}(G)} \otimes B) \sim_M A \boxtimes G B. \]

If $B$ is a $G$-YD-algebra there is a natural $\mathbb{D}(G)$-equivariant Morita equivalence 
\[ A \boxtimes G (\mathbb{K}_{\mathcal{D}(G)} \otimes B) \sim_M A \boxtimes G B. \]

**Proof.** We consider the coaction of $\mathbb{D}(G)$ on $\mathbb{H}_{\mathcal{D}(G)}$ coming from the regular representation. From $\mathcal{H}$ we know that the corresponding corepresentation of $C^*_r(G)$ on $\mathbb{H}_{\mathcal{D}(G)} = \mathbb{H}_G \boxtimes \mathbb{H}_G$ is $W_{12} \in M(C^*_r(G) \otimes \mathbb{K}_{\mathcal{D}(G)})$. The corresponding corepresentation of $C^*_r(G)$ on $\mathbb{H}_{\mathcal{D}(G)}$ is given by $Z_{23}^* \hat{W}_{13} Z_{23}$ where $Z = W(J \otimes J)^* W(J \otimes J)$.

To prove (a) we observe that $Z_{23}^* \hat{W}_{13} Z_{23}$ implements a $G$-equivariant isomorphism 
\[ (\mathbb{H}_{\mathcal{D}(G)} \otimes A) \boxtimes G B = [Z_{23}^* \hat{W}_{13} Z_{23} \sigma_{12} (\text{id} \otimes \lambda)(\mathbb{H}_{\mathcal{D}(G)} \otimes A)_{123} \beta(B)_{14}] \cong \mathbb{H}_{\mathcal{D}(G)} \otimes [\lambda(A)_{12} \beta(B)_{14}] = \mathbb{H}_{\mathcal{D}(G)} \otimes (A \boxtimes G B) \]

of Hilbert modules. This yields 
\[ (\mathbb{K}_{\mathcal{D}(G)} \otimes A) \boxtimes G B \cong \mathbb{K}((\mathbb{H}_{\mathcal{D}(G)} \otimes A) \boxtimes G B) \cong \mathbb{K}(\mathbb{H}_{\mathcal{D}(G)} \otimes (A \boxtimes G B)) \sim_M \mathbb{K}(A \boxtimes G B) = A \boxtimes G B \]
in a way compatible with the coaction of $C^*_r(G)$. If $B$ is a $G$-YD-algebra the above isomorphisms and the Morita equivalence are $\mathbb{D}(G)$-equivariant. The assertions in (b) are proved in a similar fashion. \(\square\)

4. The equivariant Kasparov category

In this section we first review the definition of equivariant Kasparov theory given by Baaj and Skandalis $\mathcal{H}$. Then we explain how to extend several standard results from the case of locally compact groups to the setting of regular locally compact quantum groups. In particular, we adapt the Cuntz picture of $KK$-theory $\mathcal{H}$ to show that equivariant $KK$-classes can be described by homotopy classes of equivariant homomorphisms. As a consequence, we obtain the universal property of equivariant Kasparov theory. We describe its structure as a triangulated category and discuss the restriction and induction functors. Finally, based on the construction of the braided tensor product in the previous section we construct exterior products in equivariant $KK$-theory.

Let us recall the definition of equivariant Kasparov theory $\mathcal{H}$. For simplicity we will assume that all C*-algebras are separable. Let $S$ be a Hopf-C*-algebra and let $A$ and $B$ be graded $S$-C*-algebras. An $S$-equivariant Kasparov $A-B$-module is a countably generated graded $S$-equivariant Hilbert $B$-module $\mathcal{E}$ together with an $S$-colinear graded $\ast$-homomorphism $\phi : A \rightarrow \mathbb{L}(\mathcal{E})$ and an odd operator $F \in \mathbb{L}(\mathcal{E})$ such that 
\[ [F, \phi(a)], \quad (F^2 - 1)\phi(a), \quad (F - F^\ast)\phi(a) \]
are contained in \( \mathbb{K}(\mathcal{E}) \) for all \( a \in A \) and \( F \) is almost invariant in the sense that
\[
(\text{id} \otimes \phi)(x)(1 \otimes F - \text{ad}_\lambda(F)) \subseteq S \otimes \mathbb{K}(\mathcal{E})
\]
for all \( x \in S \otimes A \). Here \( S \otimes \mathbb{K}(\mathcal{E}) = \mathbb{K}(S \otimes \mathcal{E}) \) is viewed as a subset of \( \mathbb{L}(S \otimes \mathcal{E}) \) and \( \text{ad}_\lambda \) is the adjoint coaction associated to the given coaction \( \lambda : \mathcal{E} \to M(S \otimes \mathcal{E}) \) on \( \mathcal{E} \).

Two \( S \)-equivariant Kasparov \( A \)-\( B \)-modules \((E_0, \phi_0, F_0)\) and \((E_1, \phi_1, F_1)\) are called unitarily equivalent if there is an \( S \)-colinear unitary \( U \in \mathbb{L}(E_0, E_1) \) of degree zero such that \( U \phi_0(a) = \phi_1(a)U \) for all \( a \in A \) and \( F_1U = UF_0 \). We write \((E_0, \phi_0, F_0) \cong (E_1, \phi_1, F_1)\) in this case. Let \( E_S(A, B) \) be the set of unitary equivalence classes of \( S \)-equivariant Kasparov \( A \)-\( B \)-modules. This set is functorial for graded \( S \)-colinear \(*\)-homomorphisms in both variables. If \( f : B_1 \to B_2 \) is a graded \( S \)-colinear \(*\)-homomorphism and \((\mathcal{E}, \phi, F)\) is an \( S \)-equivariant Kasparov \( A \)-\( B_1 \)-module, then
\[
f_*((\mathcal{E}, \phi, F)) = (\mathcal{E} \otimes_f B_2, \phi \otimes \text{id}, F \otimes 1)
\]
is the corresponding Kasparov \( A \)-\( B_2 \)-module. A homotopy between \( S \)-equivariant Kasparov \( A \)-\( B \)-modules \((E_0, \phi_0, F_0)\) and \((E_1, \phi_1, F_1)\) is an \( S \)-equivariant Kasparov \( A \)-\( B \)-\([0,1]\)-module \((\mathcal{E}, \phi, F)\) such that \((\text{ev}_t)_*((\mathcal{E}, \phi, F)) \cong (E_t, \phi_t, F_t)\) for \( t = 0, 1 \). Here \( B[0,1] = B \otimes C[0,1] \) where \( C[0,1] \) is equipped with the trivial action and grading and \( \text{ev}_t : B[0,1] \to B \) is evaluation at \( t \).

**Definition 4.1.** Let \( S \) be a Hopf-\( C^* \)-algebra and let \( A \) and \( B \) be graded \( S \)-\( C^* \)-algebras. The \( S \)-equivariant Kasparov group \( KK^S(A, B) \) is the set of homotopy classes of \( S \)-equivariant Kasparov \( A \)-\( B \)-modules.

In the definition of \( KK^S(A, B) \) one can restrict to Kasparov triples \((\mathcal{E}, \phi, F)\) which are essential in the sense that \( [\phi(A)\mathcal{E}] = \mathcal{E} \), compare [33]. We note that \( KK^S(A, B) \) becomes an abelian group with addition given by the direct sum of Kasparov modules. Many properties of ordinary \( KK \)-theory carry over to the \( S \)-equivariant situation, in particular the construction of the Kasparov composition product and Bott periodicity [1]. As usual we write \( KK^S_0(A, B) = KK^S(A, B) \) and let \( KK^S_1(A, B) \) be the odd \( KK \)-group obtained by suspension in either variable. In the case \( S = C_0(G) \) for a locally compact group \( G \) one reobtains the definition of \( G \)-equivariant \( KK \)-theory [25].

Our first aim is to establish the Cuntz picture of equivariant \( KK \)-theory in the setting of regular locally compact quantum groups. This can be done parallel to the account in the group case given by Meyer [33]. For convenience we restrict ourselves to trivially graded \( C^* \)-algebras and present a short argument using Baaj-Skandalis duality.

Let \( S \) be a Hopf-\( C^* \)-algebra and let \( A_1 \) and \( A_2 \) be \( S \)-\( C^* \)-algebras. Consider the free product \( A_1 * A_2 \) together with the canonical \(*\)-homomorphisms \( \iota_j : A_j \to A_1 * A_2 \) for \( j = 1, 2 \). We compose the coaction \( \alpha_j : A_j \to MS(S \otimes A_j) \) with the \(*\)-homomorphism \( MS(S \otimes A_j) \to MS(S \otimes (A_1 * A_2)) \) induced by \( \iota_j \) and combine these maps to obtain a \(*\)-homomorphism \( \alpha : A_1 * A_2 \to MS(S \otimes (A_1 * A_2)) \). This map satisfies all properties of a continuous coaction in the sense of definition [23] except that it is not obvious whether \( \alpha \) is always injective. If necessary, this technicality can be overcome by passing to a quotient of \( A_1 * A_2 \). More precisely, on \( A_1 * S^j A_2 = (A_1 * A_2) / \ker(\alpha) \) the map \( \alpha \) induces the structure of an \( S \)-\( C^* \)-algebra, and we have canonical \( S \)-colinear \(*\)-homomorphisms \( A_j \to A_1 * S^j A_2 \) for \( j = 1, 2 \) again denoted by \( \iota_j \). The resulting \( S \)-\( C^* \)-algebra is universal for pairs of \( S \)-colinear \(*\)-homomorphisms \( f_1 : A_1 \to C \) and \( f_2 : A_2 \to C \) into \( S \)-\( C^* \)-algebras \( C \). That is, for any such pair of \(*\)-homomorphisms there exists a unique \( S \)-colinear \(*\)-homomorphism \( f : A_1 * S^j A_2 \to C \) such that \( f \iota_j = f_j \) for \( j = 1, 2 \). By abuse of notation, we will still write \( A_1 * A_2 \) instead of \( A_1 * S^j A_2 \) in the sequel. We point out that in the arguments below we could equally well work with the ordinary free
product together with its possibly noninjective coaction.
Let $A$ be an $S$-$C^*$-algebra and consider $QA = A \ast A$. The algebra $\mathbb{K} \otimes QA$ is $S$-collinearly homotopy equivalent to $\mathbb{K} \otimes (A \otimes A)$ where $\mathbb{K}$ denotes the algebra of compact operators on a separable Hilbert space $\mathbb{H}$. Moreover there is an extension
\[
\begin{array}{c}
0 \longrightarrow QA \xrightarrow{\pi} A \longrightarrow 0
\end{array}
\]
of $S$-$C^*$-algebras with $S$-colinear splitting, here $\pi$ is the homomorphism associated to the pair $f_1 = f_2 = \text{id}_A$ and $qA$ its kernel.

We shall now restrict attention from general Hopf-$C^*$-algebras to regular locally compact quantum groups and state the Baaj-Skandalis duality theorem \([11], [2]\).

**Theorem 4.2.** Let $G$ be a regular locally compact quantum group and let $S = C_G^0(G)$ and $\hat{S} = C^*_r(G)^\text{cop}$. For all $S$-$C^*$-algebras $A$ and $B$ there is a canonical isomorphism
\[
J_S : KK^S(A, B) \rightarrow KK^S(\hat{S} \ltimes A, \hat{S} \ltimes B)
\]
which is multiplicative with respect to the composition product.

For our purposes it is important that under this isomorphism the class of an $S$-equivariant Kasparov $A$-$B$-module $([E, \phi, F])$ is mapped to the class of an $\hat{S}$-equivariant Kasparov module $(J_S([E]), J_S(\phi), J_S(F))$ with an operator $J_S(F)$ which is exactly invariant under the coaction of $\hat{S}$.

Let $G$ be a regular locally compact quantum group and let $E$ and $F$ be $G$-Hilbert $B$-modules which are isomorphic as Hilbert $B$-modules. Then we have a $G$-equivariant isomorphism
\[
H_G \otimes E \cong H_G \otimes F
\]
of $G$-Hilbert $B$-modules where $H_G$ is viewed as a $G$-Hilbert space using the left regular corepresentation, see \([51]\). Using the Kasparov stabilization theorem we deduce that there is a $G$-equivariant Hilbert $B$-module isomorphism
\[
(H_G \otimes E) \oplus (H_G \otimes H \otimes B) \cong H_G \otimes H \otimes B
\]
for every countably generated $G$-Hilbert $B$-module $E$. This result will be referred to as the equivariant stabilization theorem.

In the sequel we will frequently write $KK^G$ instead of $KK^S$ for $S = C_G^0(G)$ and call the defining cycles of this group $G$-equivariant Kasparov modules. It follows from Baaj-Skandalis duality that $KK^G(A, B)$ can be represented by homotopy classes of $G$-equivariant Kasparov $(\mathbb{K}_G \otimes A)$-$(\mathbb{K}_G \otimes B)$-modules $(\mathcal{E}, \phi, F)$ with $G$-invariant operator $F$. Taking Kasparov product with the $\mathbb{K}_G \otimes B$-bimodule $(\mathbb{H}_G \otimes B, \text{id}, 0)$ we see that $KK^G(A, B)$ can be represented by homotopy classes of equivariant Kasparov $(\mathbb{K}_G \otimes A)$-$B$ modules of the form $(\mathbb{H}_G \otimes \mathcal{E}, \phi, F)$ with invariant $F$. Using the equivariant stabilization theorem we can furthermore assume that $(\mathbb{H}_G \otimes \mathcal{E})_A = \mathbb{H}_G \otimes H \otimes B$ is the standard $G$-Hilbert $B$-module.

From this point on we follow the arguments in \([33]\). Writing $[A, B]_G$ for the set of equivariant homotopy classes of $G$-equivariant $*$-homomorphisms between $G$-$C^*$-algebras $A$ and $B$, we arrive at the following description of the equivariant $KK$-groups.

**Theorem 4.3.** Let $G$ be a regular locally compact quantum group. Then there is a natural isomorphism
\[
KK^G(A, B) \cong [q(\mathbb{K}_G \otimes A), \mathbb{K}_G \otimes \mathbb{K} \otimes B]_G
\]
for all separable $G$-$C^*$-algebras $A$ and $B$. We also have a natural isomorphism
\[
KK^G(A, B) \cong [\mathbb{K}_G \otimes \mathbb{K} \otimes q(\mathbb{K}_G \otimes A), \mathbb{K}_G \otimes \mathbb{K} \otimes q(\mathbb{K}_G \otimes \mathbb{K} \otimes B)]_G
\]
under which the Kasparov product corresponds to the composition of homomorphisms.
Consider the category $G\text{-Alg}$ of separable $G$-$C^*$-algebras for a regular quantum group $G$. A functor $F$ from $G\text{-Alg}$ to an additive category $\mathcal{C}$ is called a homotopy functor if $F(f_0) = F(f_1)$ whenever $f_0$ and $f_1$ are $G$-equivariant $*$-homomorphisms. It is called stable if for all pairs of separable $G$-Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ the maps $F(\mathbb{K}(\mathcal{H}_j) \otimes A) \to F(\mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes A)$ induced by the canonical inclusions $\mathcal{H}_j \to \mathcal{H}_1 \oplus \mathcal{H}_2$ for $j = 1, 2$ are isomorphisms. As in the group case, a homotopy functor $F$ is stable if there exists a natural isomorphism $F(A) \cong F(\mathbb{K}_G \otimes \mathbb{K} \otimes A)$ for all $A$. Finally, $F$ is called split exact if for every extension $0 \to K \to E \to Q \to 0$ of $G$-$C^*$-algebras that splits by an equivariant $*$-homomorphism $\sigma : Q \to E$ the induced sequence $0 \to F(K) \to F(E) \to F(Q) \to 0$ in $\mathcal{C}$ is split exact.

Equivariant $KK$-theory can be viewed as an additive category $KK^G$ with separable $G$-$C^*$-algebras as objects and $KK^G(A, B)$ as the set of morphisms between two objects $A$ and $B$. Composition of morphisms is given by the Kasparov product. There is a canonical functor $\iota : G\text{-Alg} \to KK^G$ which is the identity on objects and sends equivariant $*$-homomorphisms to the corresponding $KK$-elements. This functor is a split exact stable homotopy functor.

As a consequence of theorem 4.3, we obtain the following universal property of $KK^G$, see again [33]. We remark that a related assertion is stated in [43], however, some of the arguments in [43] are incorrect.

**Theorem 4.4.** Let $G$ be a regular locally compact quantum group. The functor $\iota : G\text{-Alg} \to KK^G$ is the universal split exact stable homotopy functor on the category $G\text{-Alg}$. More precisely, if $F : G\text{-Alg} \to \mathcal{C}$ is any split exact stable homotopy functor with values in an additive category $\mathcal{C}$ then there exists a unique functor $f : KK^G \to \mathcal{C}$ such that $F = f\iota$.

Let us explain how $KK^G$ becomes a triangulated category. We follow the discussion in [34], for the definition of a triangulated category see [35]. Let $\Sigma A$ denote the suspension $C_0(\mathbb{R}) \otimes A$ of a $G$-$C^*$-algebra $A$. Here $C_0(\mathbb{R})$ is equipped with the trivial coaction. The corresponding functor $\Sigma : KK^G \to KK^G$ determines the translation automorphism. If $f : A \to B$ is a $G$-equivariant $*$-homomorphism then the mapping cone

$$C_f = \{(a, b) \in A \times C_0((0, 1], B) | b(1) = f(a)\}$$

is a $G$-$C^*$-algebra in a natural way, and there is a canonical diagram

$$\Sigma B \longrightarrow C_f \longrightarrow A \xrightarrow{f} B$$

of $G$-equivariant $*$-homomorphisms. Diagrams of this form are called mapping cone triangles. By definition, an exact triangle is a diagram $\Sigma Q \to K \to E \to Q$ in $KK^G$ which is isomorphic to a mapping cone triangle.

The proof of the following proposition is carried out in the same way as for locally compact groups [34].

**Proposition 4.5.** Let $G$ be a regular locally compact quantum group. Then the category $KK^G$ together with the translation functor and the exact triangles described above is triangulated.

Several results about the equivariant $KK$-groups for ordinary groups extend in a straightforward way to the setting of quantum groups. As an example, let us state the Green-Julg theorem for compact quantum groups and its dual version for discrete quantum groups. If $G$ is a locally compact quantum group and $A$ is a $C^*$-algebra we write $\text{res}^G_A$ for the $G$-$C^*$-algebra $A$ with the trivial coaction. A detailed proof of the following result is contained in [51].
Theorem 4.6. Let $G$ be a compact quantum group. Then there is a natural isomorphism
\[ KK^G(\text{res}^E_G(A), B) \cong KK(A, \text{C}^*(G)^{\text{cop}} \rtimes B) \]
for all $\text{C}^*$-algebras $A$ and all $G$-$\text{C}^*$-algebras $B$.

Dually, let $G$ be a discrete quantum group. Then there is a natural isomorphism
\[ KK^G(A, \text{res}^E_G(B)) \cong KK(\text{C}^*_t(G)^{\text{cop}} \rtimes_\epsilon A, B) \]
for all $G$-$\text{C}^*$-algebras $A$ and all $C$-$\text{C}^*$-algebras $B$.

Let $G$ be a strongly regular quantum group and let $H \subset G$ be a regular closed quantum subgroup. It is easy to check that restriction from $G$ to $H$ induces a triangulated functor $\text{res}^G_H$ : $KK^G \to KK^H$. This functor associates to a $G$-$\text{C}^*$-algebra $A$ the $G$-$\text{C}^*$-algebra $\text{res}^G_H(A) = A$ obtained by restricting the action. Similarly, using the universal property of theorem 4.4 we obtain a triangulated functor $\text{ind}^G_H$ : $KK^H \to KK^G$ which maps an $H$-$\text{C}^*$-algebra $A$ to the induced $G$-$\text{C}^*$-algebra $\text{ind}^G_H(A)$. Note that the compatibility of induction with stabilizations follows from Vaes’ imprimitivity theorem stated above as theorem 2.3.

A closed quantum subgroup $H \subset G$ is called cocompact if the $\text{C}^*$-algebraic quantum homogeneous space $C_0^H(G/H)$ is a unital $\text{C}^*$-algebra. It is easy to check that restriction from $G$ to $G/H$ induces a $\text{C}^*$-algebra $\text{res}^{G/H}_H$ which maps an $H$-$\text{C}^*$-algebra $A$ to the induced $G$-$\text{C}^*$-algebra $\text{res}^{G/H}_H(A)$. Note that, although not being a $\text{C}^*$-algebra, the map $\text{res}^{G/H}_H(A) \to C_0^H(G/H)$ is a $\text{C}^*$-algebra map. We obtain $\text{C}^*_r(G/H)$ instead of $C_0^H(G/H)$. Recall that a locally compact quantum group $G$ is coamenable if the natural map $C_0^r(G) \to C_0^r(G)$ is an isomorphism. Strong regularity is equivalent to regularity in this case.

Proposition 4.7. Let $H \subset G$ be a cocompact regular quantum subgroup of a strongly regular quantum group $G$. If $G$ is coamenable there is a natural isomorphism
\[ KK^H(\text{res}^H_G(A), B) \cong KK^G(A, \text{ind}^G_H(B)) \]
for all $G$-$\text{C}^*$-algebras $A$ and all $H$-$\text{C}^*$-algebras $B$.

Proof. We describe the unit $\eta$ and the counit $\kappa$ of this adjunction. For a $G$-$\text{C}^*$-algebra $A$ let $\eta_A : A \to \text{ind}^G_H \text{res}^H_G(A) \cong \text{C}^*(G/H) \boxtimes_G A$ be the $G$-equivariant $\text{C}^*$-homomorphism obtained from the embedding of $A$ in the braided tensor product. Here we use theorem 2.3 and the assumption that $H \subset G$ is cocompact. In order to define the counit $\kappa$ recall that the induced $\text{C}^*$-algebra $\text{ind}^G_H(B)$ of an $H$-$\text{C}^*$-algebra $B$ is contained in the $C_0^r(G)$-relative multiplier algebra of $C_0^r(G) \otimes B$. We obtain an $H$-equivariant $\text{C}^*$-homomorphism $\kappa_B : \text{res}^G_H \text{ind}^G_H(B) \to B$ as the restriction of $\epsilon \otimes \text{id} : M(C_0^r(G) \otimes B) \to M(B)$ where $\epsilon : C_0^r(G) \to \mathbb{C}$ is the counit. Here we use coamenability of $G$.

Let $A$ be a $G$-$\text{C}^*$-algebra with coaction $\alpha$ and let $\text{res}(\alpha) : A \to M(C_0^r(H) \otimes A)$ be the restriction of $\alpha$ to $H$. Using the relation $[(\mathbb{I} \otimes 1) \text{res}(\alpha)(A)] = [\alpha(A)(\mathbb{I} \otimes 1)]$ established in [15] we see that $\kappa_{\text{res}(A)}$ is given by $\epsilon \boxtimes \text{id} : \text{C}^*(G/H) \boxtimes_G A \to \mathbb{C} \boxtimes_G A \cong A$. Note that, although not being $G$-equivariant, the map $\epsilon$ is $C_0^r(G)$-colinear and hence induces a $\text{C}^*$-homomorphism between the braided tensor products as desired. It follows that the composition
\[ \text{res}^G_H(A) \xrightarrow{\text{res}(\eta_A)} \text{res}^G_H \text{ind}^G_H \text{res}^H_G(A) \xrightarrow{\text{res}(\kappa_A)} \text{res}^G_H(A) \]

is the identity in $KK^H(\text{res}^H_G(A), \text{res}^G_H(A))$ for every $G$-$\text{C}^*$-algebra $A$.

Identifying the isomorphism $\text{ind}^G_H \text{res}^G_H \text{ind}^G_H(B) \cong \text{C}^*(G/H) \boxtimes_G \text{ind}^G_H(B)$ and using the counit identity $(\text{id} \otimes \epsilon) \Delta = \text{id}$ for $C_0^r(G)$ we see that
\[ \text{ind}^G_H(B) \xrightarrow{\text{ind}(\kappa_B)} \text{ind}^G_H \text{res}^G_H \text{ind}^G_H(B) \xrightarrow{\text{ind}(\kappa_B)} \text{ind}^G_H(B) \]

is the identity in $KK^G(\text{ind}^G_H(B), \text{ind}^G_H(B))$ for every $G$-$\text{C}^*$-algebra $B$. This yields the assertion. \qed
Based on the braided tensor product we introduce exterior products in equivariant Kasparov theory.

**Proposition 4.8.** Let $G$ be a regular locally compact quantum group, let $A$ and $B$ be $G$-$C^*$-algebras and let $D$ be a $G$-$YD$-algebra. Then there exists a natural homomorphism

$$\lambda_D : KK^G(A, B) \to KK^G(D \boxtimes_G A, D \boxtimes_G B)$$

defining a triangulated functor $\lambda_D : KK^G \to KK^G$.

If $A$ and $B$ are $G$-$YD$-algebras then there is an analogous homomorphism

$$\lambda_D : KK^{D(G)}(A, B) \to KK^{D(G)}(D \boxtimes_G A, D \boxtimes_G B)$$

defining a triangulated functor $\lambda_D : KK^{D(G)} \to KK^{D(G)}$.

**Proof.** We shall only discuss the first assertion, the case of $G$-$YD$-algebras is treated analogously. Taking the braided tensor product with $G$-Alg to $KK^G$. According to proposition 5.7 this functor is stable. Hence the existence of $\lambda_D$ is a consequence of the universal property of $KK^G$ established in theorem 4.4 and the resulting functor is easily seen to be triangulated.

The same arguments yield the following right-handed version of proposition 4.8.

**Proposition 4.9.** Let $G$ be a regular locally compact quantum group, let $C$ and $D$ be $G$-$YD$-algebras and let $B$ be a $G$-$C^*$-algebra. Then there exists a natural homomorphism

$$\rho_B : KK^{D(G)}(C, D) \to KK^G(C \boxtimes_G B, D \boxtimes_G B)$$

defining a triangulated functor $\rho_B : KK^{D(G)} \to KK^G$.

If $B$ is a $G$-$YD$-algebra we obtain a natural homomorphism

$$\rho_B : KK^{D(G)}(C, D) \to KK^{D(G)}(C \boxtimes_G B, D \boxtimes_G B)$$

defining a triangulated functor $\rho_B : KK^{D(G)} \to KK^{D(G)}$.

By construction, the class of a $G$-equivariant $\ast$-homomorphism $f : A \to B$ is mapped to the class of $f \boxtimes \text{id}$ under $\lambda_D : KK^G \to KK^G$, and similar remarks apply to the other functors obtained above.

Of course one can also give direct definitions on the level of Kasparov modules for the constructions in propositions 4.8 and 4.9. For instance, let $(E, \phi, F)$ be a $G$-equivariant Kasparov $A$-$B$-module. Then $D \boxtimes_G E$ is a $D \boxtimes_G B$-Hilbert module, and the map $\phi : A \to L(E) = M(\mathcal{K}(E))$ induces a $G$-equivariant $\ast$-homomorphism $id \boxtimes_G \phi : D \boxtimes_G A \to M(D \boxtimes_G \mathcal{K}(E)) \cong L(D \boxtimes_G E)$. Moreover, we obtain $id \boxtimes_G F \in L(D \boxtimes_G E)$ by applying the canonical map $\mathbb{L}(E) \to L(D \boxtimes_G E)$. It is readily checked that this yields a $G$-equivariant Kasparov module. The construction is compatible with homotopies and induces $\lambda_D : KK^G(A, B) \to KK^G(D \boxtimes_G A, D \boxtimes_G B)$.

Let $A_1, B_1$ and $D$ be $G$-$YD$ algebras and let $A_2, B_2$ be $G$-$C^*$-algebras. We define the exterior Kasparov product

$$KK^{D(G)}(A_1, B_1 \boxtimes_G D) \times KK^G(D \boxtimes_G A_2, B_2) \to KK^G(A_1 \boxtimes_G A_2, B_1 \boxtimes_G B_2)$$

as the map which sends $(x, y)$ to $\rho_{A_2}(x) \circ \lambda_{B_1}(y)$. Here $\circ$ denotes the Kasparov composition product, and we use $(B_1 \boxtimes_G D) \boxtimes_G A_2 \cong B_1 \boxtimes_G (D \boxtimes_G A_2)$.

If $A_2, B_2$ are $G$-$YD$-algebras we obtain an exterior product

$$KK^{D(G)}(A_1, B_1 \boxtimes_G D) \times KK^{D(G)}(D \boxtimes_G A_2, B_2) \to KK^{D(G)}(A_1 \boxtimes_G A_2, B_1 \boxtimes_G B_2)$$

in the same way.

We summarize the main properties of the above exterior Kasparov products in analogy with the ordinary exterior Kasparov product, see [7].
Theorem 4.10. Let $G$ be a regular locally compact quantum group. Moreover let $A_1, B_1$ and $D$ be $G$-YD algebras and let $A_2, B_2$ be $G$-$C^*$-algebras. The exterior Kasparov product

$$KK^{D(G)}(A_1, B_1 \boxtimes_G D) \times KK^{D(G)}(D \boxtimes_G A_2, B_2) \to KK^{D(G)}(A_1 \boxtimes_G A_2, B_1 \boxtimes_G B_2)$$

is associative and functorial in all possible senses. An analogous statement holds provided $A_2, B_2$ are $G$-YD algebras.

Recall that every $G$-$C^*$-algebra for a locally compact group $G$ can be viewed as a $G$-YD-algebra with the trivial coaction of $C^*_\text{r}(G)$. In this case our constructions reduce to the classical exterior product in equivariant $KK$-theory. Still, even for classical groups the products defined above are more general since we may consider $G$-YD-algebras that are equipped with a nontrivial coaction of the group $C^*$-algebra.

5. THE QUANTUM GROUP $SU_q(2)$

In this section we recall some definitions and constructions related to the compact quantum group $SU_q(2)$. For more information on the algebraic aspects of compact quantum groups we refer to [23].

Let us fix a number $q \in (0,1]$ and describe the $C^*$-algebra of continuous functions on $SU_q(2)$. Since $SU_q(2)$ is coamenable [30], [3] there is no need to distinguish between the full and reduced $C^*$-algebras. By definition, $C(SU_q(2))$ is the universal $C^*$-algebra generated by two elements $\alpha$ and $\gamma$ satisfying the relations

$$\alpha \gamma = q \gamma \alpha, \quad \alpha^* \gamma = q \gamma^* \alpha, \quad \gamma^* \gamma = \gamma \gamma^* = 1, \quad \alpha^* \alpha + q^2 \gamma^* \gamma = 1.$$ 

The comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is given on the generators by

$$\Delta(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$ 

From a conceptual point of view, it is useful to interpret these formulas in terms of the fundamental matrix

$$u = \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$ 

In fact, the defining relations for $C(SU_q(2))$ are equivalent to saying that the fundamental matrix is unitary, and the comultiplication of $C(SU_q(2))$ can be written in a concise way as

$$\Delta \left( \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) = \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$ 

We will also work with the dense *-subalgebra $\mathbb{C}[SU_q(2)] \subset C(SU_q(2))$ generated by $\alpha$ and $\gamma$. Together with the counit $\epsilon : \mathbb{C}[SU_q(2)] \to \mathbb{C}$ and the antipode $S : \mathbb{C}[SU_q(2)] \to \mathbb{C}[SU_q(2)]$ determined by

$$\epsilon \left( \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \left( \begin{pmatrix} \alpha & -q \gamma^* \\ \gamma & \alpha^* \end{pmatrix} \right) = \begin{pmatrix} \alpha^* & \gamma^* \\ -q \gamma & \alpha \end{pmatrix},$$

the algebra $\mathbb{C}[SU_q(2)]$ becomes a Hopf-*-algebra. We use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for the comultiplication and write

$$f \to x = x_{(1)} f(x_{(2)}), \quad x \leftarrow f = f(x_{(1)}) x_{(2)}$$

for elements $x \in \mathbb{C}[SU_q(2)]$ and linear functionals $f : \mathbb{C}[SU_q(2)] \to \mathbb{C}$.

The antipode is an algebra antihomomorphism satisfying $S(S(x)^*) = x$ for all
$x \in \mathbb{C}[SU_q(2)]$, in particular the map $S$ is invertible. The inverse of $S$ can be written as

$$S^{-1}(x) = \delta \rightarrow S(x) \leftarrow \delta^{-1}$$

where $\delta : \mathbb{C}[SU_q(2)] \rightarrow \mathbb{C}$ is the modular character determined by

$$\delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

Apart from its role in connection with the antipode, the character $\delta$ describes the modular properties of the Haar state $\phi$ of $C(SU_q(2))$ in the sense that

$$\phi(xy) = \phi(y(\delta \rightarrow x \leftarrow \delta))$$

for all $x, y \in \mathbb{C}[SU_q(2)]$. The Hilbert space $\mathbb{H}_{SU_q(2)}$ associated to $SU_q(2)$ is the GNS-construction of $\phi$ and will be denoted by $L^2(SU_q(2))$ in the sequel. The irreducible corepresentations $\psi_l$ of $C(SU_q(2))$ are parametrized by $l \in \frac{1}{2}\mathbb{N}$, and the dimension of $\psi_l$ is $2l + 1$ as for the classical group $SU(2)$. According to the Peter-Weyl theorem, the Hilbert space $L^2(SU_q(2))$ has an orthonormal basis $e_{i,j}^{(l)}$ with $l \in \frac{1}{2}\mathbb{N}$ and $i, j \in \{-l, -l+1, \ldots, l\}$ corresponding to the decomposition of the regular representation. In this picture, the GNS-representation of $C(SU_q(2))$ is given by

$$\alpha e_{i,j}^{(l)} = a_+(l, i, j) e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(l+1)} + a_-(l, i, j) e_{i-\frac{1}{2}, j+\frac{1}{2}}^{(l+1)}$$

$$\gamma e_{i,j}^{(l)} = c_+(l, i, j) e_{i+\frac{1}{2}, j-\frac{1}{2}}^{(l+1)} + c_-(l, i, j) e_{i-\frac{1}{2}, j+\frac{1}{2}}^{(l+1)}$$

where the explicit form of $a_\pm$ and $c_\pm$ for $q \in (0, 1)$ is

$$a_+(l, i, j) = q^{2l+i+j+1} \frac{(1 - q^{2l+2i+2})^{1/2}(1 - q^{2l+2i+2})^{1/2}}{(1 - q^{4l+2})^{1/2}(1 - q^{4l+2})^{1/2}}$$

$$a_-(l, i, j) = \frac{(1 - q^{2l+2})^{1/2}(1 - q^{2l+2})^{1/2}}{(1 - q^{4l+2})^{1/2}(1 - q^{4l+2})^{1/2}}$$

and

$$c_+(l, i, j) = -q^{l+j} \frac{(1 - q^{2l+2})^{1/2}(1 - q^{2l+2})^{1/2}}{(1 - q^{4l+2})^{1/2}(1 - q^{4l+2})^{1/2}}$$

$$c_-(l, i, j) = q^{l+i} \frac{(1 - q^{2l+2})^{1/2}(1 - q^{2l+2})^{1/2}}{(1 - q^{4l+2})^{1/2}(1 - q^{4l+2})^{1/2}}$$

In the above formulas the vectors $e_{i,j}^{(l)}$ are declared to be zero if one of the indices $i, j$ is not contained in $\{-l, -l+1, \ldots, l\}$.

We will frequently use the fact that the classical torus $T = S^1$ is a closed quantum subgroup of $SU_q(2)$. The inclusion $T \subset SU_q(2)$ is determined by the $*$-homomorphism $\pi : \mathbb{C}[SU_q(2)] \rightarrow \mathbb{C}[T] = \mathbb{C}[z, z^{-1}]$ given by

$$\pi \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$
is a noncommutative analogue of the space of sections of the homogeneous vector bundle $SU(2) \times_T V$ over $SU(2)/T$. Clearly $\Gamma(SU_q(2) \times_T V)$ is a $\mathbb{C}[SU_q(2)/T]$-bimodule in a natural way. In accordance with the Serre-Swan theorem, the space of sections $\Gamma(SU_q(2) \times_T V)$ is finitely generated and projective both as a left and right $\mathbb{C}[[SU_q(2)/T]]$-module. This follows from the fact that $\mathbb{C}[[SU_q(2)/T]] \subset \mathbb{C}[SU_q(2)]$ is a faithfully flat Hopf-Galois extension, see \cite{35}, \cite{45}. If $V = \mathbb{C}_k$ is the irreducible representation of $T$ of weight $k \in \mathbb{Z}$ we write $L^2(SU_q(2) \times_T \mathbb{C}_k)$ for the $SU_q(2)$-Hilbert space obtained by taking the closure of $\Gamma(SU_q(2) \times_T \mathbb{C}_k)$ inside $L^2(SU_q(2))$.

We also note the Frobenius reciprocity isomorphism

$$\text{Hom}_T(\text{res}^{SU_q(2)}_T(V), \mathbb{C}_k) \cong \text{Hom}_{SU_q(2)}(V, L^2(SU_q(2) \times_T \mathbb{C}_k))$$

for all finite dimensional corepresentations $V$ of $C(SU_q(2))$.

6. EQUIVARIANT POINCARÉ DUALITY FOR THE PODOLEŠ SPHERE

Poincaré duality in Kasparov theory plays an important rôle in noncommutative geometry, for instance in connection with the Dirac-dual Dirac method for proving the Novikov conjecture \cite{25}. In this section we extend this concept to the setting of quantum group actions and show that the standard Podleš sphere is equivariantly Poincaré dual to itself.

Let us begin with the following terminology, generalizing the definition given by Connes in \cite{11}. Recall that we write $D(G)$ for the Drinfeld double of a locally compact quantum group $G$.

**Definition 6.1.** Let $G$ be a regular locally compact quantum group. Two $G$-YD-algebras $P$ and $Q$ are called $G$-equivariantly Poincaré dual to each other if there exists a natural isomorphism

$$KK_*^{D(G)}(P \boxtimes_G A, B) \cong KK_*^{D(G)}(A, Q \boxtimes_G B)$$

for all $G$-YD-algebras $A$ and $B$.

Using the notation introduced in proposition \ref{prop11} we may rephrase this by saying that the $G$-YD-algebras $P$ and $Q$ are $G$-equivariantly Poincaré dual to each other iff $\lambda_P$ and $\lambda_Q$ are adjoint functors. In particular, the unit and counit of the adjunction determine elements

$$\alpha \in KK_*^{D(G)}(P \boxtimes_G Q, \mathbb{C}), \quad \beta \in KK_*^{D(G)}(\mathbb{C}, Q \boxtimes_G P)$$

if $P$ and $Q$ are Poincaré dual. In this case one also has a duality on the level of $G$-equivariant Kasparov theory in the sense that there is a natural isomorphism

$$KK^G_*(P \boxtimes_G A, B) \cong KK^G_*(A, Q \boxtimes_G B)$$

for all $G$-$*$-algebras $A$ and $B$.

In the sequel we restrict attention to $G_q = SU_q(2)$. Our aim is to show that the standard Podleš sphere is $SU_q(2)$-equivariantly Poincaré dual to itself in the sense of definition \ref{def11}. As a first ingredient we need the $K$-homology class of the Dirac operator on $G_q/T$ for $q \in (0, 1)$. We review briefly the construction in \cite{15}, however, instead of working with the action of the quantized universal enveloping algebra we consider the corresponding coaction of $C(G_q)$. Using the notation from section \ref{sec5} the underlying graded $G_q$-Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of the spectral triple is given by

$$\mathcal{H}_\pm = L^2(G_q \times_T \mathbb{C}_{\pm 1})$$

with its natural coaction of $C(G_q)$. The covariant representation $\phi = \phi_+ \oplus \phi_-$ of the $C(G_q/T)$ is given by left multiplication. Finally, the Dirac operator $D$ on $\mathcal{H}$ is
the odd operator
\[ D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \]
where
\[ D^\pm |l, m\rangle_\pm = [l + 1/2]_q |l, m\rangle_\pm \]
and \(|l, m\rangle_\pm\) are the standard basis vectors in \(V_l \subset \mathcal{H}_\pm\) and
\[ [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}} \]
for a nonzero number \(a \in \mathbb{C}\). Note that \(\mathcal{H}_+\) and \(\mathcal{H}_-\) are isomorphic corepresentations of \(C(G_q)\) according to Frobenius reciprocity. It follows that the phase \(F\) of \(D\) can be written as
\[ F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
and the triple \((\mathcal{H}, \phi, F)\) is a \(G_q\)-equivariant Fredholm module. In this way \(D\) determines an element in \(KK^{G_q}_0(C(G_q/T), \mathbb{C})\).

According to proposition 3.4 the \(C^*\)-algebra \(C(G_q/T) = \text{ind}^{G_q}_T(\mathbb{C})\) is a \(G_q\)-YD-algebra. For our purposes the following fact is important.

**Proposition 6.2.** The Dirac operator on the standard Podleś sphere defines an element in \(KK^{D(G_q)}(C(G_q/T), \mathbb{C})\) of \(C(G_q/T), \mathbb{C}\) in a natural way.

**Proof.** With the notation as above, we consider the operator \(D\) on the Hilbert space \(\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-\). Using \(\text{ind}^{G_q}_T \text{res}^{G_q}_G(C(G_q/T)) \cong C(G_q/T) \boxtimes_{G_q} C(G_q/T)\) we obtain a graded \(G_q\)-equivariant \(*\)-homomorphism \(\psi : C(G_q/T) \boxtimes_{G_q} C(G_q/T) \rightarrow \mathbb{L}(\mathcal{H})\) by applying the induction functor to the counit \(\epsilon : C(G_q/T) \rightarrow \mathbb{C}\) and composing the resulting map with the natural representation of \(C(G_q/T)\) on \(\mathcal{H}\). On both copies of \(C(G_q/T)\) the map \(\psi\) is given by the homomorphism \(\phi\) from above. In particular, the commutators of \(F\) with elements from \(C(G_q/T) \boxtimes_{G_q} C(G_q/T)\) are compact.

The coaction \(\lambda : \mathcal{H} \rightarrow M(C^*(G_q) \otimes \mathcal{H})\) which turns \(\mathcal{H}\) into a \(D(G_q)\)-Hilbert space is obtained from the action of \(C[G_q]\) on \(\Gamma(G_q \times_T \mathbb{C})\), given by
\[ f \cdot h = f_{(1)} h \delta \rightarrow S(f_{(2)}) \]
where \(\delta\) is the modular character. The homomorphism \(\psi\) is \(C^*(G_q)\)-colinear with respect to this coaction, and in order to show
\[ (C^*(G_q) \otimes 1) (1 \otimes F - \text{ad}_\lambda(F)) \subset C^*(G_q) \otimes \mathbb{K}(\mathcal{H}) \]
it suffices to check that \(F\) commutes with the above action of \(C[G_q]\) up to compact operators. This in turn is a lengthy but straightforward calculation based on the explicit formulas for the GNS-representation of \(C(G_q)\) in section 5. It follows that \((\mathcal{H}, \psi, F)\) is a \(D(G_q)\)-equivariant Kasparov module as desired. \(\square\)

Note that in the construction of the Dirac cycle in proposition 6.2 we use two identical representations of \(C(G_q/T)\) as in the case of a classical spin manifold. The difference to the classical situation lies in the replacement of the ordinary tensor product with the braided tensor product.

Let us formally write \(\mathcal{E}_k = G_q \times_T C_k\) for the induced vector bundle associated to the representation of weight \(k\), and denote by \(C(\mathcal{E}_k)\) the closure of \(\Gamma(\mathcal{E}_k)\) inside \(C(G_q)\).

The space \(C(\mathcal{E}_k)\) is a \(G_q\)-equivariant Hilbert \(C(G_q/T)\)-module with the coaction induced by comultiplication, and the coaction \(\lambda : C(\mathcal{E}_k) \rightarrow M(C^*(G_q) \otimes C(\mathcal{E}_k))\) given by \(\lambda(f) = \hat{W}^*(1 \otimes f)\hat{W}\) turns it into a \(D(G_q)\)-equivariant Hilbert module. Left multiplication yields a \(D(G_q)\)-equivariant \(*\)-homomorphism \(\mu : C(G_q/T) \rightarrow \mathbb{K}(C(\mathcal{E}_k))\). Hence \((C(\mathcal{E}_k), \mu, 0)\) defines a class \([\mathcal{E}_k]\) in \(KK^D_0(C(G_q/T), C(G_q/T))\).

Next observe that the unit homomorphism \(u : \mathbb{C} \rightarrow C(G_q/T)\) induces an element
$[u] \in KK_0^{D(G_q)}(\mathbb{C}, C(G_q/T))$. We obtain a class $[\mathcal{E}_k]$ in $KK_0^{D(G_q)}(\mathbb{C}, C(G_q/T))$ by restricting $[[\mathcal{E}_k]]$ along $u$, or equivalently, by taking the product

$$[\mathcal{E}_k] = [u] \circ [[\mathcal{E}_k]].$$

Under the forgetful map from $KK_0^{D(G_q)}$ to $KK^{G_q}$, this class is mapped to the $K$-theory class in $KK^{G_q}(\mathbb{C}, C(G_q/T))$ corresponding to $\mathcal{C}_k$ in $R(T)$ under Frobenius reciprocity.

In addition we define elements $[D \otimes \mathcal{E}_k] \in KK_0^{G_q}(C(G_q/T), \mathbb{C})$ by

$$[D \otimes \mathcal{E}_k] = [[\mathcal{E}_k]] \circ [D]$$

where $[D] \in KK_0^{G_q}(C(G_q/T), \mathbb{C})$ is the class of the Dirac operator. We remark that these elements correspond to twisted Dirac operators on $G_q/T$ as studied by Sitarz in [27].

Let us determine the equivariant indices of these twisted Dirac operators.

**Proposition 6.3.** Consider the classes $[\mathcal{E}_k] \in KK^{G_q}(\mathbb{C}, C(G_q/T))$ and $[D \otimes \mathcal{E}_k] \in KK^{G_q}(C(G_q/T), \mathbb{C})$ introduced above. The Kasparov product $[\mathcal{E}_k] \circ [D \otimes \mathcal{E}_k] \in KK_0^{G_q}(\mathbb{C}, \mathbb{C}) = R(G_q)$ is given by

$$[\mathcal{E}_k] \circ [D \otimes \mathcal{E}_l] = \begin{cases} -[V_{(k+l-1)/2}] & \text{for } k + l > 0 \\ 0 & \text{for } k + l = 0 \\ [V_{-(k+l+1)/2}] & \text{for } k + l < 0 \end{cases}$$

for $k, l \in \mathbb{Z}$.

**Proof.** This is analogous to calculating the index of a homogenous differential operator [9]. Since we have $[[\mathcal{E}_m]] \circ [[\mathcal{E}_n]] = [[\mathcal{E}_{m+n}]]$ for all $m, n \in \mathbb{Z}$ it suffices to consider the case $k = 0$. The product $[\mathcal{E}_0] \circ [D \otimes \mathcal{E}_l]$ is given by the equivariant index of the $G_q$-equivariant Fredholm operator representing $[D \otimes \mathcal{E}_l]$. This operator can be viewed as an odd operator on $L^2(\mathcal{E}_{l+1}) \oplus L^2(\mathcal{E}_{l-1})$. By equivariance, the claim follows from Frobenius reciprocity; we only have to subtract the classes of $L^2(\mathcal{E}_{l+1})$ and $L^2(\mathcal{E}_{l-1})$ in the formal representation ring of $G_q$. $\Box$

We note that for the above computation there is no need to pass to cyclic cohomology or twisted cyclic cohomology.

In order to proceed we need a generalization of the Drinfeld double. The relative Drinfeld double $D(T, \hat{G}_q)$ is defined as the double crossed product [11] of $C(T)$ and $C^*(G_q)$ using the matching $m(x) = \bar{x}x\bar{x}^*$ where $\mathcal{Z} = (\pi \otimes \text{id})(W_{G_q})$ and $\pi : C(G_q) \to C(T)$ is the quotient map. That is, we have $C_0^*(D(T, \hat{G}_q)) = C(T) \otimes C^*(G_q)$ with the comultiplication

$$\Delta_{D(T, \hat{G}_q)} = (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes m \otimes \text{id})(\Delta \otimes \hat{\Delta}).$$

The relative Drinfeld double $D(T, \hat{G}_q)$ is a cocompact closed quantum subgroup of $D(G_q)$, and the quantum homogeneous space $C^*(D(G_q)/D(T, \hat{G}_q))$ is isomorphic to $C(G_q/T)$. Under this identification, the natural $D(G_q)$-algebra structure on the homogeneous space corresponds to the $\mathcal{YD}$-algebra structure on the induced algebra $C(G_q/T) = \text{ind}_{G_q}^{G_q}(\mathbb{C})$ obtained from proposition 5.2.

Every continuous coaction of $C(T)$ on a $C^*$-algebra $B$ restricts to a continuous coaction of $C_0^*(D_q) = C_0^*(D(T, \hat{G}_q))$ in a natural way, and we write $\text{res}_{D_q}(B)$ for the resulting $D(T, \hat{G}_q)$-$C^*$-algebra. Indeed, since $C(T)$ is commutative, the canonical $*$-homomorphism $C(T) \to M(C_0^*(D(T, \hat{G}_q)))$ is compatible with the comultiplications. The following result is a variant of the dual Green-Julg theorem, see theorem 4.6.
Lemma 6.4. Let $D_q = \mathcal{D}(T, \hat{G}_q)$ be the relative Drinfeld double of $G_q$. Then there is a natural isomorphism

$$KK^{D_q}(A, \text{res}^T_{q}(B)) \cong KK^T(C(G_q)^{\text{cop}} \times A, B)$$

for all $D_q$-$C^*$-algebras $A$ and all $T$-$C^*$-algebras $B$.

Proof. If $A$ is a $D_q$-$C^*$-algebra then the crossed product $C(G_q)^{\text{cop}} \times A$ becomes a $T$-$C^*$-algebra using the adjoint action on $C(G_q)$ and the restriction of the given coaction on $A$. The natural map $\psi_A : A \to C(G_q)^{\text{cop}} \times A$ is $T$-equivariant and $C^*(G_q)$-colinear with respect to the coaction on the crossed product induced by the corepresentation $\hat{W}_G$.

Assume that $(\mathcal{E}, \phi, F)$ is a $D_q$-equivariant Kasparov $A$-$\text{res}^T_q(B)$-module which is essential in the sense that the $*$-homomorphism $\phi : A \to \mathbb{L}(\mathcal{E})$ is nondegenerate. The coaction of $C^*_0(D_q)$ on $\mathcal{E}$ is determined by a coaction of $C(T)$ and a unitary corepresentation of $C^*(G_q)$. Together with $\phi$, this corepresentation corresponds to a nondegenerate $*$-homomorphism $\psi : C(G_q)^{\text{cop}} \times A \to \mathbb{L}(\mathcal{E})$ which yields a $T$-equivariant Kasparov $C(G_q)^{\text{cop}} \times A$-$B$-module $(\mathcal{E}, \psi, F)$. Conversely, assume that $(\mathcal{E}, \psi, F)$ is an essential $T$-equivariant Kasparov $C(G_q)^{\text{cop}} \times A$-$B$-module. Then $\psi$ is determined by a covariant pair consisting of a nondegenerate $*$-homomorphism $\phi : A \to \mathbb{L}(\mathcal{E})$ and a unitary corepresentation of $C^*(G_q)$ on $\mathcal{E}$. In combination with the given $C(T)$-coaction, this corepresentation determines a coaction of $C^*_0(D_q)$ on $\mathcal{E}$ such that $(\mathcal{E}, \phi, F)$ is a $D_q$-equivariant Kasparov module. The assertion follows easily from these observations.

Before we proceed we need some further facts about the structure of $q$-deformations. Note that $C(G_q)$ can be viewed as a $T \times T$-$C^*$-algebra with the action given by left and right translations. The $C^*$-algebras $C(G_q)$ assemble into a $T \times T$-$C^*$-equivariant continuous field $G = (C(G_q))_{q \in [0,1]}$ of $C^*$-algebras, compare [8], [37]. In particular, the algebra $C^*_0(G)$ of $C_0$-sections of the field is a $T \times T$-$C^*$-algebra in a natural way.

We can also associate equivariant continuous fields to certain braided tensor products. For instance, the braided tensor products $C(G_q) \boxtimes_{G_q} C(G_q)$ yield a continuous field of $C^*$-algebras over $(0, 1]$ whose section algebra we denote by $C^*_0(G) \boxtimes_{G} C^*_0(G)$. This is easily seen using that $C(G_q) \boxtimes_{G_q} C(G_q) \cong C(G_q) \otimes C(G_q)$ as $C^*$-algebras and the fact that $C(G_q)$ is nuclear for all $q \in [0,1]$. A similar argument works for the quantum flag manifolds $C(G_q/T)$ instead of $C(G_q)$.

As a consequence of lemma 6.4, we obtain in particular that the Dirac operator on $G_q/T$ determines an element in $KK^T(C(G_q/T) \boxtimes_{G_q} C(G_q/T) \boxtimes_{G_q} C(G_q/T))$ since we have

$$C(G_q)^{\text{cop}} \times A \cong A \boxtimes_{G_q} C(G_q)$$

for every $G_q$-$YD$-algebra $A$ by definition of the braided tensor product.

In fact, these elements depend in a continuous way on the deformation parameter. More precisely, if we fix $q \in (0,1]$ then the proof of proposition 6.2 shows that the Dirac operators on $G_t/T$ for different values of $t \in [q,1]$ yield an element

$$[D] \in KK^T(C(G/T) \boxtimes_{G} C(G/T) \boxtimes_{G} C(G/q, 1])$$

where $C(G/T) \boxtimes_{G} C(G/T) \boxtimes_{G} C(G)$ denotes the algebra of sections of the continuous field over $[q, 1]$ with fibers $C(G_t/T) \boxtimes_{G_t} C(G_t/T) \boxtimes_{G_t} C(G_t)$. Similarly, writing $C(G/T)$ for the algebra of sections of the continuous field over $[q, 1]$ given by the Podleś spheres, the induced vector bundle $\mathcal{E}_k$ determines a class in $KK^T(C(G/T), C(G/T))$. Composition of this class with the canonical homomorphism $C[q,1] \to C(G/T)$ yields an element in $KK^T(C[q,1], C(G/T))$. After these preparations we prove the following main result.
Theorem 6.5. The Podleś sphere $C(G_q/T)$ is $G_q$-equivariantly Poincaré dual to itself. That is, there is a natural isomorphism

$$KK^0(\mathbb{C}, C(G_q/T) \boxtimes_{G_q} A, B) \cong KK^0(\mathbb{C}, C(G_q/T) \boxtimes_{G_q} B)$$

for all $G_q$-YD-algebras $A$ and $B$.

Proof. According to proposition 6.2 the Dirac operator on $G_q/T$ yields an element $[D_q] \in KK^0(\mathbb{C}, C(G_q/T) \boxtimes_{G_q} C(G_q/T), \mathbb{C})$. Let us define a dual element $\eta_q$ in $KK^0(\mathbb{C}, C(G_q/T) \boxtimes_{G_q} C(G_q/T))$ by $\eta_q = [\mathcal{E}_0] \boxtimes [\mathcal{E}_0] - [\mathcal{E}_0] \boxtimes [\mathcal{E}_1]$ where we write $\boxtimes$ for the exterior product obtained in theorem 4.10. In order to show that $\eta_q$ and $[D_q]$ are the unit and counit of the desired adjunction we have to study the endomorphisms $(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id})$ and $(\eta_q \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q])$ of $C(G_q/T)$ in $KK^0(\mathbb{C}, C(G_q/T))$.

First we consider the classical case $q = 1$. Since all $C^*_r(G_1)$-coactions in the construction of $[D_1]$ and $\eta_1$ are trivial it suffices to work with the above morphisms at the level of $KK^0(G_1)$. Due to proposition 4.7 the counit $\epsilon : C(G_q/T) \to \mathbb{C}$ induces an isomorphism $KK^0(T, C(G_q/T), C(G_q/T)) \cong KK^0(T, C(G_q/T), \mathbb{C})$. Hence, according to the universal coefficient theorem for $T$-equivariant $KK$-theory 14, in order to identify $(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id})$ we only have to compute the action of $(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q])$ on $KK^0(T, C(G_q/T))$. Using proposition 6.3 we obtain

$$(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q]) = [\mathcal{E}_0] \circ (\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q]) = [\mathcal{E}_0] \circ (\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) = [\mathcal{E}_0] \circ (\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q]) = \mathcal{E}_0 \otimes \mathcal{E}_0$$

This yields an element $\mathcal{E}_0 \otimes \mathcal{E}_0$ in $KK^0(T, C(G_q/T), \mathbb{C})$. As already indicated above, we conclude that these identities hold at the level of $KK^0(\mathbb{C}, C(G_q/T))$ as well.

For general $q \in (0, 1]$ we observe that the Drinfeld double $D(G_q)$ is cocommutative and recall that $D(T, \hat{G}_q) \subset D(G_q)$ is a cocompact quantum subgroup. According to proposition 4.7 this implies

$$KK^0(T, C(G_q/T), C(G_q/T)) \cong KK^0(T, C(G_q/T), \mathbb{C})$$

since $C(G_q/T) \cong \text{ind}_{D_q}(G_q/T, C(G_q/T))$. Moreover, due to lemma 6.4 we have

$$KK^0(T, C(G_q/T), \mathbb{C}) = KK^0(T, C(G_q/T) \boxtimes_{G_q} C(G_q), \mathbb{C})$$

using $C(G_q)^{\text{cop}} \times C(G_q/T) \cong C(G_q/T) \boxtimes_{G_q} C(G_q)$. Recall that $T$ acts by conjugation on the copy of $C(G_q)$. The element in $KK^0(T, C(G_q/T) \boxtimes_{G_q} C(G_q), \mathbb{C})$ corresponding to $(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q]) \circ (\text{id} \boxtimes [D_q])$ is given by

$$(\text{id} \boxtimes \eta_q) \circ ([D_q] \boxtimes \text{id}) \circ (\text{id} \boxtimes [D_q]) \circ (\text{id} \boxtimes [D_q]) = \mathcal{E}_0 \otimes \mathcal{E}_0$$

We observe that the individual elements in this composition assemble into $KK^0$-classes for the corresponding continuous fields over $[q, 1]$. Let us denote by $c_q \in E^0(T, C(G_q/T) \boxtimes_{G_q} C(G_q))$ the $T$-equivariant comparison element for the field $C(G/T) \boxtimes_{G_q} C(G_q)$ over $[q, 1]$.

Using again the universal coefficient theorem for $T$-equivariant $KK$-theory we obtain a commutative diagram

$$\begin{array}{c}
C(G/T) \boxtimes_{G_q} C(G_q) \\
\downarrow \delta_1 \\
\mathbb{C}
\end{array} \quad \begin{array}{c}
C(G/T) \boxtimes_{G_q} C(G_q) \\
\downarrow \delta_2 \\
\mathbb{C}
\end{array}$$
in $KK^T$ where $c_q$ is an isomorphism. Moreover, due to our previous considerations in the case $q = 1$ we have $\delta_1 = (\varepsilon \boxtimes \id) \circ \varepsilon$. This implies $\delta_q = (\varepsilon \boxtimes \id) \circ \varepsilon$ and hence $(\id \boxtimes D_q) \circ ([D_q] \boxtimes \id) \equiv \id$ in $KK^{D(G_q)}$. In a similar way one obtains $(\eta_q \boxtimes \id) \circ (\id \boxtimes [D_q]) \equiv \id$ in $KK^{D(G_q)}$. According to the characterization of adjoint functors in terms of unit and counit this yields the assertion. □

As a corollary we determine the equivariant $K$-homology of the Podleś sphere.

**Corollary 6.6.** For the standard Podleś sphere $C(G_q/T)$ we have

$$KK^{G_q}_0(C(G_q/T), \mathbb{C}) \cong R(G_q) \oplus R(G_q), \quad KK^{G_q}_1(C(G_q/T), \mathbb{C}) = 0.$$ 

Let us also discuss the following result which is closely related to theorem 6.5.

**Theorem 6.7.** The standard Podleś sphere $C(G_q/T)$ is a direct summand of $\mathbb{C} \oplus \mathbb{C}$ in $KK^{D(G_q)}$.

**Proof.** Let us consider the elements $\alpha_q \in KK^{D(G_q)}_0(C(G_q/T), \mathbb{C} \oplus \mathbb{C})$ and $\beta_q \in KK^{D(G_q)}_0(\mathbb{C} \oplus \mathbb{C}, C(G_q/T))$ given by

$$\alpha_q = [D] \oplus [D \otimes \mathcal{E}_{-1}], \quad \beta_q = (-[\mathcal{E}_1]) \oplus \mathcal{E}_0,$$

respectively. Following the proof of theorem 6.5 we shall show $\alpha_q \circ \beta_q = \id$.

Consider first the case $q = 1$. All $C^*(G_1)$-coactions in the construction of $\alpha_1$ and $\beta_1$ are trivial, and it suffices to check $\alpha_1 \circ \beta_1 = \id$ in $KK^{G_1}_0(C(G_1/T), C(G_1/T))$. Using $KK^{G_1}_0(C(G_1/T), C(G_1/T)) \cong KK^{T}_0(C(G_1/T), \mathbb{C})$ and the universal coefficient theorem for $KK^T$ we only have to compare the corresponding actions on $K^*_T(C(G_1/T))$. One obtains

$$[\mathcal{E}_0] \circ [\alpha_1] \circ [\beta_1] \circ \varepsilon = \mathcal{E}_0 \circ [D \otimes \mathcal{E}_{-1}] \circ \mathcal{E}_0 \circ \varepsilon \equiv \mathcal{E}_0 \circ (D \otimes \mathcal{E}_{-1}) = \mathcal{E}_0$$

in $R(T)$ due to proposition 6.3 and similarly $[\mathcal{E}_1] \circ [\alpha_1] \circ [\beta_1] \circ \varepsilon = \mathcal{C}_1$. Taking into account McLeod’s theorem 6.3 this yields the assertion for $q = 1$.

For general $q \in (0,1]$ we recall

$$KK^{D(G_q)}_0(C(G_q/T), C(G_q/T)) \cong KK^{T}_0(C(G_q/T) \boxtimes G_q C(G_q), \mathbb{C})$$

and notice that the elements corresponding to $\alpha_t \circ \beta_t$ for $t \in [q,1]$ assemble into a class in $KK^T(C(G/T) \boxtimes G(C,G), \mathbb{C}, [q,1])$. The comparison argument in the proof of theorem 6.5 carries over and yields $\alpha_q \circ \beta_q = \id$ in $KK^{D(G_q)}_0$. □

On the level of $G_q$-equivariant Kasparov theory one can strengthen the assertion of theorem 6.7 as follows.

**Proposition 6.8.** The standard Podleś sphere $C(G_q/T)$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ in $KK^{G_q}$.

**Proof.** We have already seen that the elements $\alpha_q$ and $\beta_q$ defined in theorem 6.7 satisfy $\alpha_q \circ \beta_q = \id$ in $KK^{D(G_q)}$, hence this relation holds in $KK^{G_q}$ as well. Using proposition 6.3 one immediately calculates $\beta_q \circ \alpha_q = \id$ in $KK^{G_q}$. □

**REFERENCES**

[1] Baaj, S., Skandalis, G., $C^*$-algèbres de Hopf et théorie de Kasparov équivariante, K-theory 2 (1988), 683 - 721
[2] Baaj, S., Skandalis, G., Unitaires multiplicatifs et dualité pour les produits croisés des $C^*$-algèbres, Ann. Sci. École Norm. Sup. 26 (1993), 425 - 488
[3] Baaj, S., Skandalis, G., Vaes, S., Non-semi-regular quantum groups coming from number theory, Commun. Math. Phys. 235 (2003), 139 - 167
[4] Baaj, S., Vaes, S., Double crossed products of locally compact quantum groups, Journal Inst. Math. Jussieu 4 (2005), 135 - 173
[5] Banica, T., Fusion rules for representations of compact quantum groups, Exposition. Math. 17 (1999), 313 - 337
[6] Bédos, E., Tuset, L., Amenability and co-amenability for locally compact quantum groups, Internat. J. Math. 14 (2003), 865 - 884
[7] Blackadar, B., K-theory for operator algebras. Second edition. Cambridge University Press, Cambridge, 1998
[8] Blanchard, E., Déformations de C*-algèbres de Hopf, Bull. Soc. Math. France 124 (1996), 141 - 215
[9] Bott, R., The index theorem for homogenous differential operators, in: Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), 167 - 186, Princeton Univ. Press, 1965
[10] Chakraborty, P. S., Pal, A., Equivariant spectral triples on the quantum SU(2) group, K-Theory 28 (2003), 107 - 126
[11] Connes, A., Noncommutative Geometry, Academic Press, 1994
[12] Connes, A., Noncommutative geometry and reality, J. Math. Phys. 36 (1995), 6194 - 6231
[13] Connes, A., Gravity coupled with matter and the foundation of non-commutative geometry, Comm. Math. Phys. 182 (1996), 155 - 176
[14] Connes, A. On the spectral characterization of manifolds, arXiv:math.OA/0810.2088 (2008)
[15] Dabrowski, L., Sitarz, A., Dirac operator on the standard Podleś quantum sphere, in: Non-commutative geometry and quantum groups (Warsaw, 2001), 49 - 58, Banach Center Publ. 61, Polish Acad. Sci., Warsaw, 2003
[16] Dabrowski, L., Landi, G., Sitarz, A., van Suijlekom, W., Várilly, J., The Dirac operator on SU_q(2), Comm. Math. Phys. 259 (2005), 729 - 759
[17] Dabrowski, L., D’Andrea, F., Landi, G., Wagner, E., Dirac operators on all Podleś quantum spheres, J. Noncommut. Geom. 1 (2007), 213 - 239
[18] Dabrowski, L., Geometry of quantum spheres, J. Geom. Phys. 56 (2006), 86 - 107
[19] Echterhoff, S., Kaliszewski, S., Quigg, J., Raeburn, I., A categorical approach to imprimitivity theorems for C*-dynamical systems, Mem. Amer. Math. Soc. 180 (2006)
[20] Fischer, R., Volle verschrankte Produkte für Quantengruppen und äquivariante KK-Theorie, PhD Thesis, Münster, 2003
[21] Gover, A. R., Zhang, R. B., Geometry of quantum homogeneous vector bundles and representation theory of quantum Groups I, Rev. Math. Phys. 11 (1999), 533 - 552
[22] Hajac, P. M., Bundles over quantum sphere and noncommutative index theorem, K-Theory 21 (2000), 141 - 150
[23] Kasparov, G. G., The operator K-functor and extensions of C*-algebras, Izv. Akad. Nauk SSSR Ser. Mat. 44 (1980), 571 - 636
[24] Kasparov, G. G., Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1988), 147 - 201
[25] Klimyk, A., Schmuedgen, K., Quantum groups and their representations, Texts and Monographs in Physics, Springer, Berlin, 1997
[26] Kustermans, J., Locally compact quantum groups in the universal setting, Internat. J. Math. 12 (2001), 289 - 338
[27] Kustermans, J., Vaes, S., Locally compact quantum groups, Ann. Sci. École Norm. Sup. 33 (2000), 837 - 934
[28] Kustermans, J., Vaes, S., Locally compact quantum groups in the von Neumann algebraic setting, Math. Scand. 92 (2003), 68 - 92
[29] Majid, S., Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995
[30] Masuda, T., Nakagami, Y., Watanabe, J., Noncommutative differential geometry on the quantum SU(2), II: An algebraic viewpoint, K-theory 4 (1990), 157 - 180
[31] McLeod, J., The Küneth formula in equivariant K-theory, Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978), 316 - 333, LNM 741
[32] Meyer, R., Equivariant Kasparov theory and generalized homomorphisms, K-Theory 21 (2000), 201 - 228
[33] Meyer, R., Nest, R., The Baum-Connes conjecture via localisation of categories, Topology 45 (2006), 209 - 259
[34] Müller, E. F., Schneider, H.-J., Quantum homogeneous spaces with faithfully flat module structures, Israel J. Math. 111 (1999), 157-190
[35] Nagy, G., On the Haar measure of the quantum SU(N) group, Comm. Math. Phys. 153 (1993), 217 - 228
[36] Nagy, G., Deformation quantization and K-theory, Perspectives on quantization (South Hadley, MA, 1996), 111 - 134, Contemp. Math. 214, 1998

POINCARÉ DUALITY
[38] Neeman, A., Triangulated categories, Annals of Mathematics Studies 148, Princeton University Press, 2001

[39] Neshveyev, S., Tuset, L., Hopf algebra equivariant cyclic cohomology, K-theory and index formulas, K-Theory 31 (2004), 357 - 378

[40] Neshveyev, S., Tuset, L., The Dirac operator on compact quantum groups, arXiv:math.OA/0703161 (2007)

[41] Podleś, P., Quantum spheres, Lett. Math. Phys. 14 (1987), 193 - 202

[42] Podleś, P., Woronowicz, S. L., Quantum deformation of Lorentz group, Comm. Math. Phys. 130 (1990), 381 - 431

[43] Popescu, R., Coactions of Hopf-C*-algebras and equivariant E-theory, arXiv:math.KT/0410023 (2004)

[44] Rosenberg, J., Schochet, C., The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory, Mem. Amer. Math. Soc. 62 (1986)

[45] Schauenburg, P., Hopf-Galois and bi-Galois extensions, in: Galois theory, Hopf algebras, and semiabelian categories, Fields Inst. Commun. 43, 469 - 515, 2004

[46] Sitarz, A., Equivariant spectral triples, in: Noncommutative geometry and quantum groups (Warsaw, 2001), 231 - 263, Banach Center Publ. 61, Polish Acad. Sci., Warsaw, 2003

[47] Sitarz, A., Twisted Dirac operators over quantum spheres, J. Math. Phys. 49 (2008), 033509

[48] Vaes, S., A new approach to induction and imprimitivity results, J. Funct. Anal. 229 (2005), 317 - 374

[49] Vaes, S., Vainerman, L., Extensions of locally compact quantum groups and the bicrossed product construction, Adv. Math. 175 (2003), 1 - 101

[50] Vaes, S., Vainerman, L., On low-dimensional locally compact quantum groups, in: Locally compact quantum groups and groupoids (Strasbourg, 2002), 127 - 187, IRMA Lect. Math. Theor. Phys. 2, de Gruyter, Berlin, 2003

[51] Vergnioux, R., KK-théorie équivariante et opérateur de Julg-Valette pour les groupes quantiques, PhD thesis, Paris, 2002

[52] Wagner, E., On the noncommutative spin geometry of the standard Podles sphere and index computations, arXiv:math/QA/0707.3403 v2

[53] Woronowicz, S. L., Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. RIMS Kyoto 23 (1987), 117 - 181

RYSZARD NEST, INSTITUT FOR MATEMATISKE FAG, UNIVERSITET KØBENHAVN, UNIVERSITETSPARKEN 5, 2100 KØBENHAVN, DENMARK

E-mail address: rnest@math.ku.dk

CHRISTIAN VOIGT, MATHEMATISCHES INSTITUT, GEORG-AUGUST-UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 5-5, 37073 GÖTTINGEN, GERMANY

E-mail address: cvoigt@uni-math.gwdg.de