The stable norm on the 2-torus at irrational directions

Stefan Klempnauer and Jan Philipp Schröder

Faculty of Mathematics, Ruhr University, 44780 Bochum, Germany
E-mail: stefan.klempnauer@rub.de and jan.schroeder-a57@rub.de

Received 18 December 2015, revised 26 October 2016
Accepted for publication 21 December 2016
Published 18 January 2017

Recommended by Professor Dmitry V Treschev

Abstract
We study properties of the stable norm on the first homology group of the 2-torus with respect to Riemannian or Finsler metrics, focusing on points with irrational slope. Our results show that the stable norm detects KAM-tori and hyperbolicity in the geodesic flow. Along the way, we shall prove new inequalities for the stable norm near rational directions. Moreover, we study the stable norm in some natural examples reflecting the new results in this paper.

Keywords: Finsler metric, stable norm, Mather’s action functional, minimal geodesic, KAM-torus, hyperbolicity
Mathematics Subject Classification numbers: 37E45 (primary), 37J99 (secondary)

(Some figures may appear in colour only in the online journal)

1. Introduction and main results

In this paper we study properties of Finsler metrics \( F : T\mathbb{T}^2 \to \mathbb{R} \) on the 2-torus \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \), see [BCS00] for information on Finsler metrics. The Finsler metrics are not assumed to be reversible, so our results apply to general Tonelli Lagrangians \( L : T\mathbb{T}^2 \to \mathbb{R} \), see [CIPP98]. Readers unfamiliar with Finsler metrics may think of the norm \( F(v) = \sqrt{g(v, v)} \) of a Riemannian metric \( g \) in \( \mathbb{T}^2 \).

We write

\[ l_F(c) = l_F(c; [a, b]) = \int_a^b F(\dot{c})\,dt \]

for the \( F \)-length structure. Identifying \( \mathbb{T}^2 \cong \mathbb{R}/\mathbb{Z}^2 \) the marked length spectrum is defined as
\[
\begin{cases}
\sigma_F : \mathbb{R}^2 \to \mathbb{R}, \\
\sigma_F(z) := \inf \{l_F(c) : c \text{ is a closed curve with homotopy class } [c] = z\}.
\end{cases}
\]

Thus, \(\sigma_F\) contains information on closed \(F\)-geodesics.

We extend \(\sigma_F\) to a norm on \(\mathbb{R}^2\). By the results of Hedlund [Hed32] (see also theorem 4.1 (i) below), \(\sigma_F\) is positively homogeneous:

\[
\sigma_F(a \cdot z) = a \cdot \sigma_F(z) \quad \forall z \in \mathbb{R}^2, \ a \in \mathbb{N}_0.
\]

Moreover, using the fact that lifts of closed curves in linearly independent homotopy classes intersect in the universal cover \(\mathbb{R}^2\), one infers

\[
z, w \in \mathbb{Z}^2 \text{ linearly independent} \implies \sigma_F(z + w) < \sigma_F(z) + \sigma_F(w).
\]

Extending \(\sigma_F\) first homogeneously along rays of rational slope and then continuously to \(\mathbb{R}^2\), we obtain a convex (in general non-reversible) norm

\[
\sigma_F : \mathbb{R}^2 \to \mathbb{R}
\]

called the \textit{stable norm} (see section 2 of [Ban89]). Note that \(\sigma_F\) is related to Mather’s \(\beta\)-function

\[
\beta_F := \frac{1}{2} \sigma_F^2
\]

of the ‘Tonelli’ Lagrangian \(L = \frac{1}{2} F^2\) (see section 1 of [Mas96]).

Recall that a \textit{minimal geodesic} is a geodesic \(c : \mathbb{R} \to \mathbb{T}^2\) with the property that the lifts \(\tilde{c} : \mathbb{R} \to \mathbb{R}^2\) to the universal cover minimize the length between any two of their points. Writing \(d_F\) for the (in general non-symmetric) distance induced by the length \(l_F\) on the universal cover \(\mathbb{R}^2\), this means for the lifts \(\tilde{c}\), that

\[
l_F(\tilde{c} ; [a,b]) = d_F(\tilde{c}(a), \tilde{c}(b)) \quad \forall a \leq b.
\]

Let us write

\[
ST^2 = \{F = 1\} \subset T\mathbb{T}^2
\]

for the unit tangent bundle and

\[
\phi_F^t : ST^2 \to ST^2
\]

for the geodesic flow of \(F\). For a point \(\xi \in S^1\) we define the \(\phi_F^t\)-invariant set

\[
\mathcal{M}(\xi) \subset S\mathbb{T}^2
\]

to be the set of initial conditions \(\tilde{c}(0)\) of arc-length minimal geodesics \(c : \mathbb{R} \to \mathbb{T}^2\) with asymptotic direction \(\delta^+(c) = \xi\), where

\[
\delta^+(c) := \lim_{t \to \infty} \frac{\tilde{c}(t)}{|\tilde{c}(t)|},
\]

\(\tilde{c} : \mathbb{R} \to \mathbb{R}^2\) being any lift of \(c\) and \(|\cdot|\) the euclidean norm on \(\mathbb{R}^2\). It is known that the above limit exists for all minimal geodesics. We shall write \(\mathcal{M}(a \cdot \xi) = \mathcal{M}(\xi)\) for \(a > 0\). It is furthermore known that the shortest closed geodesics in the definition of \(\sigma_F(z)\) lie in the set \(\mathcal{M}(z)\) for \(z \in \mathbb{Z}^2\). For these and more facts we refer to [Hed32] and [Ban88] in the Riemannian case and to [Zau62] and [Sch15a] for the general Finsler case.

Motivated by classical results due to Mather and Bangert (see theorem 2.1 below), we state the following problem.
Problem 1.1. Relate the properties of $\sigma_F$ at a given point $\xi \in \mathbb{R}^2 - \{0\}$ to the structure of the set $\mathcal{M}(\xi) \subset S^\mathbb{T}^2$.

A particularly nice structure of $\mathcal{M}(\xi)$ would be that it is a KAM-torus.

Definition 1.2. Let $\phi^t : X \to X$ be a $C^\infty$-flow on a $C^\infty$-manifold $X$. A $C^k$-KAM-torus (of dimension $n$) is a $C^k$-submanifold $T \subset X$, so that

(i) $T$ is invariant under $\phi^t$,
(ii) there exists a $C^k$-diffeomorphism $T \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ conjugating $\phi^t|_T$ to a linear flow $\psi^t$ on $\mathbb{T}^n$ of the form

$$\psi^t(x) = x + t\rho \mod \mathbb{Z}^n.$$ 

In our case $X = S\mathbb{T}^2$ and $\phi^t = \phi^t_F$, we consider only KAM-tori of dimension $n = 2$ (half the dimension of the symplectic manifold $T^*\mathbb{T}^2$). Moreover, the vector $\rho \in \mathbb{R}^2$ will have irrational slope $\rho \in \mathbb{Q}$. Another possible structure of $\mathcal{M}(\xi)$ would be hyperbolicity.

Definition 1.3. Let $\phi^t : X \to X$ be a $C^\infty$-flow on a $C^\infty$-manifold $X$. A subset $\Lambda \subset X$ is uniformly hyperbolic for $\phi^t$, if there exist constants $\lambda > 0$ and distributions $E^+(x), E^-(x) \subset T^x X$, (i.e. for each $x \in \Lambda$ vector subspaces $E^+(x), E^-(x) \subset T^x X$), such that

(i) $\Lambda$ is compact and $\phi^t$-invariant,
(ii) the distributions are $\phi^t$-invariant:

$$D\phi^t|_\Lambda E^+(x) = E^+(\phi^t(x)), \quad D\phi^t|_\Lambda E^-(x) = E^-(\phi^t(x)),$$
(iii) the distributions together with the flow direction span the tangent spaces:

$$T^x X = \left( \frac{d}{dt} \right)_{t=0} \phi^t(x) \mathbb{R} \oplus E^+(x) \oplus E^-(x),$$
(iv) with respect to some Riemannian metric on $X$ we have contraction:

$$\|D\phi^t|_\Lambda v\| \leq C \cdot \exp(-\lambda t) \cdot \|v\| \quad \forall t \geq 0, v \in E^+(x),$$
$$\|D\phi^{-t}|_\Lambda v\| \leq C \cdot \exp(-\lambda t) \cdot \|v\| \quad \forall t \geq 0, v \in E^-(x).$$

We can now state our main result concerning the structure of the stable norm at points of irrational slope.

Main Theorem 1.4. Let $F$ be any Finsler metric on $\mathbb{T}^2$ with stable norm $\sigma_F$ and let $\xi \in \mathbb{R}^2 - \{0\}$ have irrational slope $\xi/\xi \in \mathbb{R} - \mathbb{Q}$.

(i) If the set $\mathcal{M}(\xi) \subset S\mathbb{T}^2$ is a $C^3$-KAM-torus for the geodesic flow $\phi^t_F$, then the (square of the) stable norm $\sigma_F$ is strongly convex near $\xi$. More precisely, there exist constants $C, C' > 0$, so that Mather’s $\beta$-function

$$\beta_F = \frac{1}{2} \sigma_F^2$$

satisfies for all $v \in \mathbb{R}^2$ the estimate

$$C \cdot |v|^2 \leq \beta_F(\xi + v) - \beta_F(\xi) - D\beta_F(\xi)[v] \leq C' \cdot |v|^2.$$

(ii) Assume additionally that in each non-trivial free homotopy class of $\mathbb{T}^2$, there exists only one shortest closed $F$-geodesic.
If the set $\mathcal{M}(\xi) \subset S^2$ is uniformly hyperbolic for the geodesic flow $\phi^t_F$, then the stable norm $\sigma_F$ is exponentially flat near $\xi$. More precisely, there exist constants $C, \lambda > 0$, such that in all choices of rays $R \subset \mathbb{R}^2$ emanating from the origin there exist sequences $v_n \to 0$, $v_n \neq 0$, so that

$$0 \leq \sigma_F(\xi + v_n) - \sigma_F(\xi) - D\sigma_F(\xi)[v_n] \leq |v_n|^1/4 \cdot C \cdot \exp\left(-\lambda \cdot \frac{1}{||v_n||^{1/2}}\right).$$

Observe that the extra assumption on the uniqueness of shortest closed $F$-geodesics in item (ii) is true for generic Finsler and Riemannian metrics, see proposition 2.5 below. The hyperbolicity of $\mathcal{M}(\xi)$ in the irrational case occurs for generic directions and generic metrics; see proposition 2.6. On the other hand, the famous KAM-theorem tells us that the case where $\mathcal{M}(\xi)$ is a smooth KAM-torus, appears frequently for $\xi \in \mathbb{R}^2$ satisfying a Diophantine condition. Note moreover that the differentiability of $\sigma_F$ and $\beta_F$ at $\xi$ is stated in theorem 2.1 (i). The first inequality in part (ii) follows easily from the convexity of $\sigma_F$.

Main theorem 1.4 distinguishes two fundamentally opposite dynamical situations—integrability and hyperbolicity—via the variationally defined stable norm. We describe the shape of $\sigma_F$ in these cases more explicitly. Let us consider a point $\xi \in \mathbb{R}^2$ with $\sigma_F(\xi) = 1$ and irrational slope and let $v \in S^1$ be a direction tangent to the indicatrix $\{ v \cdot \xi \} \subset S^2$.

$$S_F := \{ \sigma_F = 1 \} \subset \mathbb{R}^2,$$

i.e. $D\sigma_F(\xi)[v] = 0$. If $\mathcal{M}(\xi)$ is a $C^3$-KAM-torus, then by main theorem 1.4 (i) we have for $|t| \leq 2/\sqrt{C}$

$$\sigma_F(\xi + tv) \geq \sqrt{1 + 2C \cdot t^2} \geq 1 + (C/2) \cdot t^2.$$

On the other hand, if $\mathcal{M}(\xi)$ is uniformly hyperbolic, then by main theorem 1.4 (ii) there exists a sequence $t_n \to 0$ (and a sequence $t_n \not\to 0$) with

$$\sigma_F(\xi + t_n v) \leq 1 + t_n^{1/4} \cdot C \cdot \exp\left(-\lambda \cdot \frac{1}{t_n^{1/2}}\right).$$

The latter function coincides with 1 in $t = 0$ to infinite order. Considering the distance of $S_F$ to its tangent line at $\xi$, it thus behaves like a quadratic function in the first case, i.e. $S_F$ looks like a parabola near $\xi$, and in the latter case the distance vanishes to infinite order, i.e. $S_F$ looks like a straight line near $\xi$. We will give some examples of Finsler metrics together with their stable norms in section 6 below. Intuitively, one sees that when perturbing the euclidean metric on $\mathbb{T}^2$ with $S_F = S^1$, the convexity of $S_F$ moves notably into vertices at rational directions, while at directions with irrational slope the unit circle $S_F$ looks more and more like a straight line. Indeed, in the case of a hyperbolic metric on the (punctured) torus $S_F$ looks polygonal even though it is strictly convex, see section 6.3. This shift in convexity corresponds to the break-up of more and more KAM-tori, until they all disappear.

For further discussions see section 2.

We will also prove a theorem on the structure of $\sigma_F$ in rational directions. Recall that generally the set $\mathcal{M}(\xi)$ is hyperbolic for all $\xi$ with rational or infinite slope, see proposition 2.6. In this case we can sharpen the classical results stated in theorem 2.1 (ii) below, obtaining an estimate analogous to main theorem 1.4 (ii). We write

$$D^+\sigma_F(\xi)[v] := \inf_{t > 0} \frac{\sigma_F(\xi + tv) - \sigma_F(\xi)}{t} = \lim_{t \downarrow 0} \frac{\sigma_F(\xi + tv) - \sigma_F(\xi)}{t}$$

for the forward directional derivative of $\sigma_F$. The second equality holds due to convexity.
Theorem 1.5. Let $\xi \in \mathbb{R}^2 - \{0\}$ with rational or infinite slope. Assume that the set of periodic minimal geodesics $\mathcal{M}^{\text{per}}(\xi) \subset S\mathbb{T}^2$ is uniformly hyperbolic for the geodesic flow $\phi^t_F$. Then there exist constants $C, \lambda, \varepsilon > 0$ so that for all $v \in \mathbb{R}^2$ with euclidean norm $|v| \leq \varepsilon$

$$0 \leq \sigma_F(\xi + v) - \sigma_F(\xi) - D^+ \sigma_F(\xi)[v] \leq |v| \cdot C \cdot \exp\left(-\lambda \cdot \frac{1}{|v|}\right)$$

See figure 2 below for the stable norm of the rotational torus, where theorem 1.5 applies to $\xi = \pm e_1$.

Structure of this paper. In section 2 we present classical and some related results as well as open questions. Main theorem 1.4 (i) is proved in section 3. The arguments for main theorem 1.4 (ii) and theorem 1.5 are contained in section 4. In section 5, we sketch the proof of proposition 2.6; in section 6 we study some natural examples of Finsler metrics and their stable norms.

2. Further discussion

Let us recall the classical result on $\sigma_F$ due to Mather [Mat90]. The theorem has a different proof due to Bangert [Ban94]. (These authors prove the theorem for reversible Finsler metrics, while for non-reversible Finsler metrics the same results hold, see [Sch15a].)

**Theorem 2.1 (Mather, Bangert).** Let $F$ be any Finsler metric on $\mathbb{T}^2$ with stable norm $\sigma_F$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 - \{0\}$.

(i) If $\xi$ has irrational slope, then the stable norm $\sigma_F$ is differentiable at $\xi$.

(ii) If $\xi$ has rational or infinite slope $\xi_1/\xi_2 \in \mathbb{Q} \cup \{\infty\}$, then the stable norm $\sigma_F$ is differentiable at $\xi$ if and only if there exists a foliation of $\mathbb{T}^2$ by shortest closed geodesics in the homotopy class $z$, where $z$ is the prime element in $\mathbb{Z} \cap \mathbb{R}_{>0} \xi_1$.

**Remark 2.2.** There is a well-known rigidity phenomenon: $\sigma_F : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ is $C^1$ if and only if the geodesic flow $\phi_F^t$ of $F$ is $C^0$-integrable in $S\mathbb{T}^2$, that is, $S\mathbb{T}^2$ is $C^0$-foliated by invariant graphs [MS11]. In the Riemannian case, the $C^0$-integrability of $C^0$-integrability of $\phi_F^t$ is equivalent to the flatness of the metric $F$ by a classical result of Hopf [Hop48]. In general, however, two Finsler or Riemannian metrics with the same stable norm need not be isometric; see the discussion in section 6 of [Ban94].

Theorem 2.1 (ii) can be rephrased as follows:

- $\sigma_F$ is differentiable at a point $\xi \in \mathbb{R}^2 - \{0\}$ of rational slope if and only if the set $\mathcal{M}(\xi)$ is a $C^0$-KAM-torus.

The following problem arises, making problem 1.1 more explicit.

**Problem 2.3.** Give a criterion on the stable norm $\sigma_F$ near a given irrational direction $\xi \in \mathbb{R}^2 - \{0\}$, which is equivalent to the case where the set $\mathcal{M}(\xi)$ is a $C^0$-KAM-torus for the geodesic flow $\phi_F^t$.

We saw in theorem 2.1 that the answer in the rational case was given in terms of the differentiability of $\sigma_F$, while for the irrational case this is not possible since $\sigma_F$ is always differentiable here. In this light, main theorem 1.4 partially answers problem 2.3 in terms of the flatness properties of $\sigma_F$ near $\xi$. However, there are several issues to discuss.
Let us start with the condition in main theorem 1.4 (ii), that each free homotopy class contains only one shortest closed geodesic. This is certainly not fulfilled for every Finsler metric.

**Definition 2.4.** A property of Finsler metrics is said to be **conformally generic** if, given an arbitrary Finsler metric \( F_0 \) on \( T^2 \), the property holds for all Finsler metrics \( F \) of the form

\[
F(x,v) = f(x) \cdot F_0(x,v)
\]

with \( f \) belonging to a residual subset of \( C^\infty(T^2, \mathbb{R}, >0) \) in the \( C^\infty \)-topology. Here, a residual set in a topological space is a countable intersection of open and dense subsets.

In the Lagrangian setting, the above notion of genericity is related via Maupertuis’ principle to R. Mañé’s way of perturbing a Tonelli Lagrangian \( L_0 \) by a potential into \( L(x,v) = L_0(x,v) + f(x) \), see [Mañ96].

The next proposition is proved in [Sch16].

**Proposition 2.5.** The property to admit only one shortest closed geodesic in each free homotopy class is conformally generic.

This shows that main theorem 1.4 applies to ‘most’ Finsler metrics on \( T^2 \) without the extra condition in item (ii).

Next, let us see, what alternatives there are for the structure of \( \xi_M \):

(A) \( \mathcal{M}(\xi) \) is a \( C^3 \)-KAM-torus for \( \phi_F^t \),

(B) \( \mathcal{M}(\xi) \) is uniformly hyperbolic for \( \phi_F^t \),

(C) none of the above two.

First we note that case (A) occurs frequently by KAM-theory, if the Finsler metric \( F \) is close to one with an integrable geodesic flow (e.g. the euclidean metric), see e.g. [Mos62], while we do not attempt to give a full overview on the literature on KAM-theory. The reason for us to use \( C^3 \)-regularity is given in remark 3.2.

We will prove the following proposition concerning case (B).

**Proposition 2.6.** The following property of Finsler metrics on \( T^2 \) is conformally generic:

- For an open and dense subset \( U \subset S^1 \), the sets \( \mathcal{M}(\xi) \) are uniformly hyperbolic for all \( \xi \in U \). The set \( U \) strictly contains all \( \xi \in S^1 \) with rational or infinite slope.

Put together, one expects that cases (A) and (B) occur quite frequently. Aiming at problem 2.3, for conformally generic Finsler metrics, main theorem 1.4 yields conditions on \( \sigma_F \) distinguishing the cases (A) and (B), while we are not able to distinguish case (C) via \( \sigma_F \).

Let us have a brief look at case (C). This case could be quite subtle and will be left for future research. See also the discussion in section 10 of [Mac92]. This case contains the following situations:

(CA) \( \mathcal{M}(\xi) \) is a \( C^0 \)-KAM-torus, but not a \( C^3 \)-KAM-torus,

(CB) \( \mathcal{M}(\xi) \) is not a \( C^0 \)-KAM-torus, but also not uniformly hyperbolic.

As cases (A) and (C) are excluded for rational \( \xi \), if the Finsler metric is chosen generically (proposition 2.6), let us assume that \( \xi \) has irrational slope. One might expect that case (CA) can be treated as a generalization of case (A) with some degeneracy to be expected. In case (CB) the set \( \pi(\mathcal{M}^\text{rec}(\xi)) \) of recurrent minimal geodesics projected to \( \mathbb{T}^2 \) is nowhere dense in \( \mathbb{T}^2 \) [Ban88]. (The case where \( \pi(\mathcal{M}^\text{rec}(\xi)) \) is nowhere dense occurs for generic Finsler metrics, fixing \( \xi \in S^1 \) with slope \( \xi_2/\xi_1 \) a Liouville number, see [Mat88].) Here, it is known that homoclinic
behavior of geodesics close to $\mathcal{M}^{\text{rec}}(\xi)$ occurs, i.e. one could expect some hyperbolicity. However, it is still possible that $\pi(\mathcal{M}(\xi)) = \mathbb{T}^2$; also, one can have vanishing or non-vanishing Lyapunov exponents (non-uniform hyperbolicity). All these topics will not be treated here. Let us in this connection refer to the work of Arnaud: [Arn11], [Arn13], [AB14].

Note that, intuitively, there are relations of theorem 1.5 and main theorem 1.4 (ii) to [BQ07] in the setting of ergodic optimization.

We close this discussion with a remark on the stable norm on higher genus surfaces.

**Remark 2.7.** We saw that in the torus case, the stable norm contains much information on the dynamics of $\phi'_F$. The natural question is, whether this is true also for higher genus surfaces. Here, there are results analogous to theorem 2.1 due to Massart [Mas03] (note, however, the erratum [Mas15]). In [Sch15b] the second author proves that a similar asymptotic object associated to $F$, namely the horofunction boundary is generically homeomorphic to that of a constant curvature metric. The following question should be an interesting topic for future research.

**Question.** Are the differentiability properties of the stable norm of a generic Finsler metric $F$ on a closed orientable surface $M$ of genus at least two the same as those of the stable norm of a Riemannian metric with constant curvature?

### 3. The case of a KAM-torus

We fix the Finsler metric $F$. The associated sets $\mathcal{M}(\xi)$ can be seen as remnants of KAM-tori (recall definition 1.2 for the definition of a KAM-torus). More precisely, if a KAM-torus $T \subset S\mathbb{T}^2$ is a Lipschitz graph over the base $\mathbb{T}^2$, then it is well-known (see theorem 17.4 in [MF94] or section 3 in [Sch13]) that $T \subset \mathcal{M}(\xi)$ for some $\xi$. In this section we fix $\xi \in \mathbb{R}^2 - \{0\}$ and in order to prove main theorem 1.4 (i) we assume that

- the set $T = \mathcal{M}(\xi)$ is a $C^k$-KAM-torus for the geodesic flow $\phi'_F$, while $\phi'_F|_T$ is conjugated via some diffeomorphism $\Phi : T \to \mathbb{T}^2$ to the linear flow $\psi = x + tp$.

We shall call $\rho$ the frequency vector of $\mathcal{M}(\xi)$.

The aim of this section is to study the stable norm $\sigma_F$ close to $\xi$. We shall follow the ideas of Siburg, see [Sib00] or chapter 4 in [Sib04]. Note, however, that in our setting we do not need symplectic coordinate changes and the condition for our KAM-torus to be positive definite is fulfilled automatically, see below.

**Lemma 3.1.** If the set $\mathcal{M}(\xi)$ is a $C^k$-KAM-torus for the geodesic flow $\phi'_F$, as assumed above, then there exists a $C^k$-diffeomorphism $\varphi : \mathbb{T}^2 \to \mathbb{T}^2$, such that the push-forward $C^{k-1}$-Finsler metric $\varphi^*F$ admits the straight lines

$$x + tp \mod \mathbb{Z}^2$$

as arc-length geodesics.

**Remark 3.2.** For the push-forward $\varphi F$ to be a Finsler metric, it should be at least $C^2$ away from the zero section, hence the KAM-torus in lemma 3.1 should be at least $C^1$.

**Proof.** First we claim that the canonical projection $\pi|_{\mathcal{M}(\xi)} : \mathcal{M}(\xi) \to \mathbb{T}^2$ is a bi-Lipschitz homeomorphism. Indeed, by assumption all orbits in $T = \mathcal{M}(\xi)$ are recurrent under $\phi'_F$, while it is known that $\pi$ restricted to the set of recurrent minimal geodesics $\mathcal{M}^{\text{rec}}(\xi) \subset \mathcal{M}(\xi)$ is a bi-Lipschitz homeomorphism onto its image in $\mathbb{T}^2$; for this let us refer to [Sch15a], in particular
in the non-reversible Finsler case, while the extensive literature on the subject starts already with [Hed32]. The claim follows.

As $\pi : T^2 \to T^2$ is smooth and the $C^k$-submanifold $M(\xi)$ is a Lipschitz-graph (the image of $\pi^{-1}_{M(\xi)} : T^2 \to T^2$), we find that $\pi^{-1}_{M(\xi)} : M(\xi) \to T^2$ is a $C^k$-diffeomorphism. Consider the $C^k$-diffeomorphism $\Phi : M(\xi) \to T^2$ conjugating $\phi^\xi_{M(\xi)}$ to the linear flow $\psi x = x + t\rho$. Then for the geodesic $c_v : R \to T^2$ corresponding to $v \in M(\xi)$ we find

$$c_v(t) = \pi^{-1}_{M(\xi)} \circ \phi^\xi_{M(\xi)}(v) = \pi^{-1}_{M(\xi)} \circ \psi t \circ \Phi(v) = \pi^{-1}_{M(\xi)} \circ \psi t \circ \Phi(v) + t\rho.$$  

This shows that the $C^k$-diffeomorphism

$$\varphi := \Phi \circ \pi^{-1}_{M(\xi)} : T^2 \to T^2,$$

sends the $F$-geodesics from $M(\xi)$ to the desired straight lines. □

We write

$$\tilde{F} := \varphi F : T^2 \to R$$

for the push-forward Finsler metric found in lemma 3.1. Let us see how the stable norm $\sigma_F$ transforms under $\varphi$.

**Lemma 3.3.** If $\varphi : T^2 \to T^2$ is the diffeomorphism from lemma 3.1, then there exists a linear isomorphism $L_\varphi : R^2 \to R^2$ with

$$\sigma_F = \sigma_F \circ L_\varphi^{-1}.$$  

Moreover, there exists $\lambda > 0$ with

$$\rho = \lambda \cdot L_\varphi \xi.$$  

**Proof.** We defined $\sigma_F$ as a function on $R^2$, while it could be equivalently defined on the first homology group $H_1(T^2, R) \cong R^2$ via

$$\sigma_F(h) = \inf \left\{ \sum_{i=1}^k r_i p(c_i) \mid r_i \in R, c_i \text{ closed curve in } T^2, h = \sum_{i=1}^k r_i c_i \right\},$$

see section 2 of [Ban89]. The diffeomorphism $\varphi : T^2 \to T^2$ induces a linear isomorphism

$$L_\varphi : H_1(T^2, R) \to H_1(T^2, R), \quad L_\varphi [c] = [\varphi \circ c].$$

see corollary 4.3 on p 176 in [Bre93]. Using that $l_F(c) = l_F(\varphi^{-1} \circ c)$ for the length of curves, we find with the above definition of $\sigma_F$, that

$$\sigma_F(L_\varphi h) = \inf \left\{ \sum_{i=1}^k r_i l_F(c_i) \mid L_\varphi h = \sum_{i=1}^k r_i \varphi^{-1} \circ c_i \right\}$$

$= \inf \left\{ \sum_{i=1}^k r_i l_F(\varphi^{-1} \circ c_i) \mid L_\varphi h = \sum_{i=1}^k r_i L_\varphi[c_i] \right\}$

$= \inf \left\{ \sum_{i=1}^k r_i l_F(\varphi^{-1} \circ c_i) \mid h = \sum_{i=1}^k r_i \varphi^{-1} \circ c_i \right\}$

$= \sigma_F(h).$
i.e. the first claim follows. Let now \( v \in M(\xi) \), then the \( F \)-geodesic \( c_v \) is recurrent and there exists a sequence \( T_n \to \infty \) with \( c_v(T_n) \to c_v(0) \). We close \( c_v|_{[0,T_n]} \) by a short segment \( \varepsilon_n \) and write \( \tilde{c}_v : \mathbb{R} \to \mathbb{R}^2 \) for some lift of \( c_v \). It follows for the homology class of \( c_v|_{[0,T_n]} \) seen as a point in \( \mathbb{Z}^2 \subset \mathbb{R}^2 \), that

\[
\lim_{n \to \infty} [c_v|_{[0,T_n]} \ast \varepsilon_n] - \tilde{c}_v(T_n) - \tilde{c}_v(0) = 0. 
\]

(1)

By Birkhoff’s ergodic theorem, the rotation vector

\[
\rho_0 := \lim_{T \to \infty} \frac{\tilde{c}_v(T)}{T}
\]

exists (at least for almost every \( v \in M(\xi) \)), such that by definition of \( \xi = \lim_{T \to \infty} \frac{\tilde{c}_v(T)}{|\tilde{c}_v(T)|} \) we find

\[
\rho_0 = \lim_{T \to \infty} \frac{\tilde{c}_v(T)}{T} = \lim_{T \to \infty} \frac{|\tilde{c}_v(T)|}{T} \cdot \tilde{c}_v(T) = \lambda \cdot \xi.
\]

Moreover, by (1) and the analogous statement for \( \phi \circ c_v(t) = x + t \rho \)

\[
L_{-\theta} \rho_0 = L_{-\theta} \left( \lim_{T \to \infty} \frac{\tilde{c}_v(T)}{T} \right) = L_{-\theta} \left( \lim_{n \to \infty} \frac{[c_v|_{[0,T_n]} \ast \varepsilon_n]}{T_n} \right)
\]

\[
= \lim_{n \to \infty} \frac{[\phi \circ c_v|_{[0,T_n]} \ast \varepsilon_n]}{T_n} = \lim_{n \to \infty} \frac{\rho \cdot T_n}{T_n} = \rho.
\]

This proves the second claim. \( \square \)

In the following, we shall use the standard coordinates \( T^2 \times \mathbb{R}^2 \) for \( T^2 \times \mathbb{R}^2 \).

**Lemma 3.4.** Let \( \rho \) be the frequency vector of \( M(\xi) \) and suppose that \( \xi \) has irrational slope \( \xi_2/\xi_1 \in \mathbb{R} - \mathbb{Q} \). Then

\[
\hat{F}(., \rho) : T^2 \to \mathbb{R}, \quad \frac{\partial \hat{F}}{\partial v}(., \rho) : T^2 \to (\mathbb{R}^2)^\prime
\]

are constant.

**Proof.** For the first claim note that the curves \( x + t \rho \) are arc-length \( \hat{F} \)-geodesics, i.e. \( \hat{F}(x, \rho) = 1 \) for all \( x \). Now consider the Euler–Lagrange equation for \( \hat{F} \), fulfilled by the \( \hat{F} \)-geodesics:

\[
\frac{d}{dt} \frac{\partial \hat{F}}{\partial v}(c, \dot{c}) = \frac{\partial \hat{F}}{\partial x}(c, \dot{c}).
\]

Then, for \( c(t) = t \rho \) mod \( \mathbb{Z}^2 \) we find by \( \hat{F}(., \rho) = \text{const.} \), that

\[
\frac{\partial \hat{F}}{\partial x}(c, \dot{c}) = \frac{\partial \hat{F}}{\partial x}(t \rho, \rho) = 0.
\]

Hence, by the Euler–Lagrange equation

920
\[
\frac{d}{dt} \frac{\partial \hat{F}}{\partial v}(t\rho, \rho) = 0,
\]
such that \(\frac{d}{dv}(t\rho, \rho)\) is independent of \(t\). By the irrationality of \(\xi\), the vector \(\rho\) is also irrational (otherwise, \(\mathcal{M}(\xi)\) would contain periodic orbits), such that the curve \(c(t) = t\rho\) is dense in \(\mathbb{T}^3\). The claim follows. \(\square\)

In the next lemma, we obtain the desired estimates in theorem 1.4 (i) for the Finsler metric \(\hat{F}\). Recall the notation

\[\beta_{\hat{F}} = \frac{1}{2} \sigma_{\hat{F}}^2.\]

Also note that \(\beta_{\hat{F}}\) is differentiable in \(\rho\) by theorem 2.1 (i).

**Lemma 3.5.** Let the regularity of the KAM-torus \(\mathcal{M}(\xi)\) be \(k \geq 3\) and let \(\xi\) have irrational slope. Then there exists a constant \(C \geq 1\), such that for all \(h \in \mathbb{R}^2\)

\[\frac{1}{C} |h - \rho|^2 \leq \beta_{\hat{F}}(h) = \beta_{\hat{F}}(\rho) - D\beta_{\hat{F}}(\rho)[h - \rho] \leq C|h - \rho|^2.\]

**Proof.** We consider the Lagrangian \(L = \frac{1}{2} \hat{F}^2\). Then by a Taylor expansion, for some \(t_\ast \in (0, 1)\)

\[L(x, v) = L(x, \rho) + \frac{\partial L}{\partial v}(x, \rho)[v - \rho] + \frac{1}{2} \frac{\partial^2 L}{\partial v^2}(x, \rho + t_\ast (v - \rho))[v - \rho].\]

As the Hessian \(\frac{\partial^2 L}{\partial x^2}(x, \rho)\) is positive definite by the definition of a Finsler metric, we find a constant \(C \geq 1\), such that

\[\forall (x, v) \in \mathcal{T}^3, w \in \mathbb{R}^2 : \quad \frac{1}{C} |w|^2 \leq \frac{1}{2} \frac{\partial^2 L}{\partial v^2}(x, v)[w, w] \leq C |w|^2.\]

Using lemma 3.4, the functions

\[L(x, \rho) = \frac{1}{2} \hat{F}(x, \rho)^2,\]

\[\frac{\partial L}{\partial v}(x, \rho)[w] = \frac{1}{2} \frac{d}{dt} \Bigg|_{t=0} \frac{\partial \hat{F}(x, \rho + tw)^2}{\partial v} = \hat{F}(x, \rho) \cdot \frac{\partial \hat{F}}{\partial v}(x, \rho)[w]\]

are independent of \(x \in \mathcal{T}^3\). Then consider the new Lagrangians

\[L_0^-(v) = L(x, \rho) + \frac{\partial L}{\partial v}(x, \rho)[v - \rho] + \frac{1}{C} |v - \rho|^2,\]

\[L_0^+(v) = L(x, \rho) + \frac{\partial L}{\partial v}(x, \rho)[v - \rho] + C |v - \rho|^2\]

satisfying
\[ L_0^-(v) \leq L(x, v) \leq L_0^+(v) \]  

(2)

for all \((x, v) \in T^{1,2}\). We shall use (2) to find an analogous estimate for \(\hat{\beta}_F\). For any curve \(c : [0, T] \rightarrow \mathbb{R}^2\) we obtain by the \(L^2\)-Cauchy–Schwarz inequality, that

\[ l_F(c; [0, T])^2 = (\frac{1}{T} \int_0^T \| \dot{c}(t) \|^2 dt)^2 \leq \frac{1}{T^2} \int_0^T (\dot{c}(t))^2 dt \leq \frac{1}{T} \int_0^T l(c, \dot{c}) dt, \]

(3)

with equality if and only if \(\dot{\hat{F}}(c, \dot{c}) \equiv \text{const.}\). Let \(z \in \mathbb{Z}^2\) and let \(c : [0, T] \rightarrow \mathbb{R}^2\) be given by \(c_0(t) = c(0) + t \hat{c}_T\). If \(L_0 : \mathbb{T}^2 \rightarrow \mathbb{R}\) is any Tonelli Lagrangian independent of the base variable \(x \in \mathbb{T}^2\), then fixing \(T > 0\) and the endpoints \(c(0), c(T) \in \mathbb{R}^2\), it is known that the action \(\int_0^T L_0(c, \dot{c}) dt\) is minimized by the curve \(c_0\) with constant velocity vector (by Tonelli’s theorem there exist minimizers for fixed endpoints and fixed connection time, which have to be solutions of the Euler–Lagrange equation; by \(L_0\) being independent of \(x\), such minimizers have to be straight lines). It follows by (2) and (3), that

\[ L_0^-(\frac{z}{T}) = \frac{1}{T} \int_0^T L_0^-(c_0, \dot{c}_0) dt \leq \frac{1}{T} \int_0^T L_0^-(c, \dot{c}) dt \leq \frac{1}{T} \int_0^T L(c, \dot{c}) dt \]

\[ = \frac{1}{2T^2} \int_0^T l_F(c; [0, T])^2 \leq \frac{1}{2T^2} \int_0^T l_F(c_0; [0, T])^2 \leq \frac{1}{T} \int_0^T L(c, \dot{c}) dt \]

\[ \leq \frac{1}{T} \int_0^T L_0^+(c_0, \dot{c}_0) dt = L_0^+(\frac{z}{T}). \]

By \(l_F(c; [0, T]) = \sigma_F(z)\) and homogeneity of \(\sigma_F\), we obtain

\[ L_0^-(\frac{z}{T}) \leq \beta_F(\frac{z}{T}) = \frac{1}{2} \| \hat{c}(\frac{z}{T}) \|^2 \leq L_0^+(\frac{z}{T}) \]

\(\forall z \in \mathbb{Z}^2, T > 0\).

Using continuity it follows that

\[ L_0^- \leq \beta_F \leq L_0^+ \]

everywhere. Using the definition of \(L_0^+\) one infers

\[ L(x, \rho) = \beta_F(\rho), \quad \frac{\partial L}{\partial v}(x, \rho) = D \beta_F(\rho) \]

and the claim follows. 

We now translate lemma 3.5 into a statement for the original Finsler metric \(F\) and obtain the main result in this section, which is stated as item (i) of main theorem 1.4 in the introduction.

**Theorem 3.6.** If \(\xi \in \mathbb{R}^2 - \{0\}\) has irrational slope and if the set \(\mathcal{M}(\xi)\) is a \(C^3\)-KAM-torus for the geodesic flow \(\phi^t\), then the square of the stable norm \(\sigma_F\) is strongly convex near \(\xi\). More precisely, there exists a constant \(C \geq 1\), such that for the function
\[ \beta_F = \frac{1}{2} \sigma_F^2 \]

we have for all \( h \in \mathbb{R}^2 \)

\[ \frac{1}{C} |h - \xi|^2 \leq \beta_F(h) - \beta_F(\xi) - D\beta_F(\xi)[h - \xi] \leq C |h - \xi|^2. \]

**Proof.** By lemma 3.5, we find

\[ \frac{1}{C} |h - \rho|^2 \leq \beta_F(h) - \beta_F(\rho) - D\beta_F(\rho)[h - \rho] \leq C |h - \rho|^2. \]

Using the first part of lemma 3.3,

\[ \beta_F = \frac{1}{2} \sigma_F^2 = \frac{1}{2} (\sigma_F \circ L^{-1}_\varphi)^2 = \beta_F \circ L^{-1}_\varphi, \]

\[ D\beta_F(\rho) = D\beta_F(L^{-1}_\varphi \rho) \circ L^{-1}_\varphi. \]

Also observe for the operator norm \( \| \| \), that

\[ |h - \rho|^2 \leq \| L_\varphi \|^2 \| L^{-1}_\varphi h - L^{-1}_\varphi \rho \|^2, \]

\[ |L^{-1}_\varphi h - L^{-1}_\varphi \rho|^2 \leq \| L^{-1}_\varphi \|^2 \| h - \rho \|^2. \]

Summarizing,

\[ \frac{1}{C\| L^{-1}_\varphi \|^2} \| L^{-1}_\varphi h - L^{-1}_\varphi \rho \|^2 \]

\[ \leq \beta_F(L^{-1}_\varphi h) - \beta_F(L^{-1}_\varphi \rho) - D\beta_F(L^{-1}_\varphi \rho)[L^{-1}_\varphi h - L^{-1}_\varphi \rho] \]

\[ \leq C\| L_\varphi \|^2 \| L^{-1}_\varphi h - L^{-1}_\varphi \rho \|^2. \]

The second part of lemma 3.3 showed \( L^{-1}_\varphi \rho = \lambda \cdot \xi \) for some \( \lambda > 0 \). Replacing \( C \) by a new constant and \( L^{-1}_\varphi \) by \( \lambda \cdot h \) (as \( h \) is arbitrary) yields

\[ \frac{1}{C} |\lambda \cdot h - \lambda \cdot \xi|^2 \leq \beta_F(\lambda \cdot h) - \beta_F(\lambda \cdot \xi) - D\beta_F(\lambda \cdot \xi)[\lambda \cdot h - \lambda \cdot \xi] \]

\[ \leq C |\lambda \cdot h - \lambda \cdot \xi|^2. \]

Finally, using \( \beta_F(\lambda \cdot h) = \lambda^2 \cdot \beta_F(h) \) and \( D\beta_F(\lambda \cdot \xi) = \lambda \cdot D\beta_F(\xi) \) due to homogeneity and dividing the estimates by \( \lambda^2 \), the claim follows. \( \square \)

**4. The hyperbolic case**

In this section we study the stable norm \( \sigma = \sigma_F \) of the Finsler metric \( F \) near a direction \( \xi \in S^1 \) under the assumption, that the set \( \mathcal{M}(\xi) \subset \mathbb{ST}^2 \) of minimal geodesics with asymptotic direction \( \xi \) carries hyperbolicity (see definition 1.3). We shall again write \( \mathcal{M}(a \cdot \xi) = \mathcal{M}(\xi) \) for all \( a > 0 \). The ideas in this section are motivated by the works of Mather, [Mat88] and [Mat90].
Let us first recall the structure of the set $\mathcal{M}(\xi)$ for the case where $\xi \in S^1$ has rational or infinite slope. For simplicity, we will in this section rescale $\xi$:

$$\mathbb{Z} \xi \cdot = \{z \in \mathbb{Z} : a \cdot z = \min \{t > 0 : t \cdot \xi \in \mathbb{Z} \} \}.$$ 

In particular $z$ is prime, i.e. there does not exist $w \in \mathbb{Z}$ and some $n \geq 2$ with $z = n \cdot w$.

The following theorem is due to H. M. Morse [Mor24] and G. A. Hedlund [Hed32]. The result has been generalized to non-reversible Finsler metrics by several authors, see e.g. [Zau62], [CR06] and [Sch15a]. Let us write

$$\mathcal{M}^{\text{per}}(z) := \{v \in \mathcal{M}(z) : c_v \text{ is } z\text{-periodic}\}.$$ 

Here, a $z$-periodic geodesic is a closed geodesic in the homotopy class $z \in \mathbb{Z} \cong \eta(\mathbb{T}^2)$; equivalently, the lifts $\tilde{c} : \mathbb{R} \rightarrow \mathbb{R}^2$ are invariant under the translation by $z$.

**Theorem 4.1 (Morse, Hedlund).** Let $\xi \in S^1$ have rational or infinite slope and $z \in \mathbb{Z}$ as defined above. Then we have the following:

(i) $\mathcal{M}^{\text{per}}(z) \neq \emptyset$ and $\mathcal{M}^{\text{per}}(z)$ determines a closed lamination of $\mathbb{T}^2$, i.e. no two $z$-periodic minimal geodesics intersect. Moreover, if $z = kw$ for some $w \in \mathbb{Z}$ and $k \geq 2$, then the $z$-periodic minimal geodesics are the $k$th iterates of the $w$-periodic minimal geodesics.

In particular, we can say that a pair $c_0, c_1 : \mathbb{R} \rightarrow \mathbb{R}^2$ of minimal geodesics lifted from $\mathcal{M}^{\text{per}}(z)$ is neighboring, if in the strip between $c_0(\mathbb{R})$, $c_1(\mathbb{R}) \subset \mathbb{R}^2$ there are no further geodesics lifted from $\mathcal{M}^{\text{per}}(z)$.

(ii) If $c_0, c_1 : \mathbb{R} \rightarrow \mathbb{R}^2$ is a pair of neighboring minimal geodesics lifted from $\mathcal{M}^{\text{per}}(z)$, then there exist two minimal geodesics $c^- : \mathbb{R} \rightarrow \mathbb{R}^2$ lifted from $\mathcal{M}(z) - \mathcal{M}^{\text{per}}(z)$, which are heteroclinic between $c_0, c_1$ with opposite asymptotic behavior (see figure 1).

**Remark 4.2.** It is known that $\mathcal{M}^{\text{per}}(z)$ consists precisely of the shortest closed geodesics in the homotopy class $z$. If the set $\mathcal{M}^{\text{per}}(z)$ is hyperbolic, then the heteroclinic connections between neighboring periodic minimal in theorem 4.1 (ii) are parts of the (un)stable manifolds of the periodic orbits.

For the rest of this section 4, whenever we speak of a minimal geodesic, we mean a lift to the universal cover $\mathbb{R}^2$. We introduce some further notation.

**Definition 4.3.** For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ we write $\xi^\pm := i \cdot \xi = (-\xi_2, \xi_1)$. We let $k(z) \in \mathbb{N} \cup \{\infty\}$ be the number of $z$-periodic minimal in $\mathbb{R}^2$ in the strip between a $z$-periodic minimal $c_0$ and its translate $c_0 + z^{-}$. 

![Figure 1. Some geodesics from $\mathcal{M}(\xi)$ for $z \in \mathbb{Z} - \{0\}$. There are some strips foliated by $z$-periodic geodesics and in between them there are gaps overstretched by heteroclinics. On the right one can see a geometric situation generating closed and heteroclinic minimal geodesics.](image-url)
Remark 4.4.

(i) In the case \( k(z) < \infty \), we can choose any \( z \)-periodic minimal \( c_0 : \mathbb{R} \to \mathbb{R}^2 \) and have the \( z \)-periodic minimal in the strip between \( c_0, c_0 + z^\perp \) ordered as

\[
c_0 < c_1 < ... < c_{k(z)} = c_0 + z^\perp,
\]

where we wrote \( c_i < c_{i+1} \), if the image \( c_{i+1}(\mathbb{R}) \subset \mathbb{R}^2 \) lies left of \( c_i(\mathbb{R}) \) with respect to the orientation given by the frame \( (c_i, (c_i)^\perp) \).

(ii) Assume that \( \mathcal{M}^\text{per}(z) \) consists of a single orbit (this is conformally generic by proposition 2.5) and let \( c_0 : \mathbb{R} \to \mathbb{R}^2 \) be a (lifted) \( z \)-periodic minimal. This means that in the strip between \( c_0, c_0 + z^\perp \), the \( z \)-periodic minimal correspond to the \( \mathbb{Z}^+ \)-points in that strip. The number \( k(z) \) counts only the images of such minimal, so that using that \( z \) is chosen to be prime one can easily show that

\[
k(z) = 1 + \text{card}(\mathbb{Z}^2 \cap Q(z)),
\]

where we wrote \( Q(z) \) for the interior of the square spanned by \( z, z^\perp \). Combining this with Pick’s theorem, we find

\[
k(z) = |z| \cdot |z^\perp| = |z|^2,
\]

where \( |.| \) is the euclidean norm on \( \mathbb{R}^2 \).

(iii) If \( \mathcal{M}^\text{per}(z) \) is assumed to be uniformly hyperbolic for the geodesic flow \( \phi^t \), then \( \mathcal{M}^\text{per}(z) \subset S^\mathbb{T}^2 \) is a finite union of closed orbits (by the lamination property in theorem 4.1). Letting \( n \) be the number of distinct orbits in \( \mathcal{M}^\text{per}(z) \) we find

\[
k(z) = n \cdot |z|^2 < \infty.
\]

The aim of this section is to prove theorem 1.5 from the introduction. For this we assume uniform hyperbolicity of the set \( \mathcal{M}^\text{per}(z) \). As we noted after theorem 4.1, the heteroclinic minimal \( c^- \) between pairs of neighboring periodic minimal \( c_0, c_1 \) belong to the (un)stable manifolds. The hyperbolicity of \( \mathcal{M}^\text{per}(z) \) will be used in the following form: There exist constants \( C, \lambda > 0 \) and parameters \( t \in S\mathbb{T}^2 \), such that for all \( t \geq 0 \)

\[
\begin{aligned}
&d_p(c_0(-t), c_0^-(S^- - t)) \leq C \exp(-\lambda t) \\
&d_p(c^-(T^- + t), c_0(t)) \leq C \exp(-\lambda t) \\
&d_p(c_0(-t), c^+(S^+ - t)) \leq C \exp(-\lambda t) \\
&d_p(c^+(T^+ + t), c_0(t)) \leq C \exp(-\lambda t)
\end{aligned}
\]

(4)

The notation will usually be that \( c_1 \) lies to the left of \( c_0 \) (using the orientation of the geodesics) and the choice of \( c^- \), \( c^+ \) is such that the above inequalities hold.

The following lemma is the key observation in order to recognize hyperbolicity in the stable norm \( \sigma = \sigma_p \) of \( F \). Recall the definition

\[
D^+\sigma(\xi)[v] := \inf_{t > 0} \frac{\sigma(\xi + tv) - \sigma(\xi)}{t} = \lim_{t \searrow 0} \frac{\sigma(\xi + tv) - \sigma(\xi)}{t}
\]

and note that, due to homogeneity of \( \sigma \), for \( a, b > 0 \)

\[
D^+\sigma(a \cdot \xi)[b \cdot v] = b \cdot D^+\sigma(\xi)[v].
\]

Lemma 4.5. Let \( n \in \mathbb{N} \), \( s \in \{-1, 1\} \) and \( z \in \mathbb{Z}^2 \setminus \{0\} \), such that \( \mathcal{M}^\text{per}(z) \) is a hyperbolic set for the geodesic flow with hyperbolicity constants \( C, \lambda \) (in the sense of (4)). Then for integers \( n \), with
\[ 0 = n_0 \leq n_1 \leq \ldots \leq n_{k-1} \leq n_k = n \]

there exists a continuous, piecewise \( C^1 \)-periodic curve \( \gamma \) with

\[
I_\sigma(\gamma) \leq \sigma(nz) + D^+ \sigma(z)[sz^+] + 2C\sum_{i=1}^{k(z)} \exp\left(-\frac{\lambda \sigma(z)}{2} (n_i - n_{i-1})\right).
\]

**Proof.** We first consider the case \( s = 1 \). Set \( k = k(z) \), let \( c_0, \ldots, c_{k-1} \) be the ordered sequence of periodic minimals between \( c_0 \) and \( c_k \). Consider heteroclinic minimals \( c_i^+ \) connecting \( c_{i-1} \) to \( c_i \) for \( i = 1, \ldots, k \). By hyperbolicity, we find \( \lambda > C, 0 \) and \( R \in T_{1, \ldots, k} \) with

\[
 \left\{ \begin{array}{l}
df(c_{i-1}(-t), c_i^+(-t)) \leq C \exp(-\lambda t) \\
df(c_i^+(T_i + t), c_i(t)) \leq C \exp(-\lambda t)
\end{array} \right.
\]

for all \( t \geq 0 \). Here we change the parameter of the \( c_i^+ \) if necessary to have \( T_i \) only in the second line. Set \( \theta = \sigma(z) \) and given the integers \( n_i \), define

\[
 S_i := \theta \cdot \frac{n_i - n_{i-1}}{2} \geq 0, \quad i = 1, \ldots, k, \quad S_0 := S_k.
\]

Consider minimal geodesic segments \( \delta_i, \varepsilon_i \) for \( i = 1, \ldots, k \), connecting

\[
 \delta_i : c_{i-1}(-S_{i-1}) \to c_i^+(-S_{i-1}), \quad \varepsilon_i : c_i^+(T_i + S_i) \to c_i(S_i),
\]

and set

\[
 \gamma := \left[ n_0 \cdot z + (\delta_1 \ast c_1^+(-S_0 - T_0 + S_1) \ast \varepsilon_1) \right] \\
 \ast \ldots \\
 \ast \left[ n_{i-1} \cdot z + (\delta_i \ast c_i^+(-S_{i-1} - T_i + S_{i+1}) \ast \varepsilon_i) \right] \\
 \ast \ldots \\
 \ast \left[ n_{k-1} \cdot z + (\delta_k \ast c_k^+(-S_{k-1} - T_k + S_k) \ast \varepsilon_k) \right].
\]

Here we wrote \( \ast \) for the concatenation of curves. By definition of \( S_i \) and \( \theta \) being the period of \( c_i \) we have for \( i = 1, \ldots, k - 1 \), that

\[
c_i(S_i) - c_i(-S_i) = (n_i - n_{i-1}) \cdot z,
\]

showing

\[
n_{i-1} \cdot z + c_i(S_i) = n_i \cdot z + c_i(-S_i).
\]

Hence, the curve \( \gamma \) is a continuous, piecewise \( C^1 \)-connection from \( c_0(-S_0) \) to \( n_{k-1} \cdot z + c_k(S_k) \). Moreover, by \( c_k = c_0 + z^+ \) and \( S_0 = S_k \) we have

\[
n_{k-1} \cdot z + c_k(S_k) = n_{k-1} \cdot z + z^+ + c_0(S_0) \\
= n_{k-1} \cdot z + z^+ + c_0(\theta \cdot (n_k - n_{k-1}) - S_0) \\
= n_k \cdot z + z^+ + c_0(-S_0),
\]
i.e. with \( n_k = n \) the curve \( \gamma \) is \((nz + z^\perp)\)-periodic. Using the hyperbolicity and \( S_k = S_0 \), we find

\[
\sigma(nz + z^\perp) \leq \lambda \int \sum_{i=1}^{k} [T_i + S_i + S_{i-1} + \lambda L(i) + L(\xi)]
\]
\[
\leq \sum_{i=1}^{k} T_i + 2 \sum_{i=1}^{k} S_i + C \sum_{i=1}^{k} (\exp(-\lambda S_{i-1}) + \exp(-\lambda S_i))
\]
\[
= \sum_{i=1}^{k} T_i + n\theta + 2C \sum_{i=1}^{k} \exp \left( -\lambda \theta \frac{n_i - n_{i-1}}{2} \right).
\]  

(6)

Note that \( n\theta = n\sigma(z) = \sigma(nz) \). To finish the proof we show

\[
\sum_{i=1}^{k} T_i \leq D^+ \sigma(z)[z^\perp].
\]

Let \( \gamma_n \) be a \((nz + z^\perp)\)-periodic minimal. We find intersections (the \( c_0, ..., c_{k-1}, c_k = c_0 + z^\perp \) as before)

\[
\gamma_n(S_{n,i}) \in c_i(\mathbb{R})
\]

and w.l.o.g. we have

\[
0 = S_{n,0} < S_{n,1} < ... < S_{n,k} = \sigma(nz + z^\perp)
\]

and

\[
\gamma_n(S_{n,i}) = c_i(0) \quad \forall i = 1, ..., k - 1, \quad \gamma_n(S_{n,k}) = c_k(n\theta).
\]

Define for \( i = 1, ..., k \)

\[
\gamma^+_n(t) := \begin{cases} 
  c_{i-1}(t) & : t \leq 0 \\
  \gamma_n(S_{n,i-1} + t) & : 0 \leq t \leq S_{n,i} - S_{n,i-1} \\
  c_i(t - [S_{n,i} - S_{n,i-1}]) & : t \geq S_{n,i} - S_{n,i-1} \quad \text{&} \quad i < k \\
  c_i(t - [S_{n,i} - S_{n,i-1}] + n\theta) & : t \geq S_{n,i} - S_{n,i-1} \quad \text{&} \quad i = k
  \end{cases}
\]

The \( \gamma^+_n \) are heteroclinic curves connecting \( c_{i-1} \) to \( c_i \). For large \( m \), the curve \( c^+_i \big|_{[-m\theta, m\theta + T_i]} \) connects by (5) approximately the point

\[
c_{i-1}(-m\theta) = \gamma^+_n(-m\theta)
\]

to the point

\[
c_i(m\theta) = \begin{cases} 
  \gamma^+_n(m\theta + [S_{n,i} - S_{n,i-1}]) & : i < k \\
  \gamma^+_n((m - n)\theta + [S_{n,i} - S_{n,i-1}]) & : i = k
  \end{cases}
\]

By the minimality of the curves \( c^+_i \) we find for all \( n \)
The claim follows, observing that

\[
D^+ \sigma(z)[z^+] = \lim_{n \to \infty} \frac{\sigma(z + \frac{1}{n} z^+) - \sigma(z)}{\frac{1}{n}} = \lim_{n \to \infty} \sigma(nz + z^+) - \sigma(nz) - \sum_{i=1}^{k} T_i
\]

The estimates for a \((nz - z^+)-periodic curve, i.e. the case \(s = -1\) follow by the same lines just using the heteroclinics \(\epsilon_i\).

**Remark 4.6.** In the proof of lemma 4.5 we showed \(\sum T_i \leq D^+ \sigma(z)[z^+]\). The reverse inequality also holds, which can be seen using the curve \(\gamma\) associated to the integers \(n_i = i \cdot \lfloor n/k \rfloor\) for \(i < k\) and then applying the estimate (6) to the formula (7).

We need to refine lemma 4.5.

**Lemma 4.7.** Let \(a \in [-1, 1]\) and \(z \in \mathbb{Z} - \{0\}\), such that \(\mathcal{M}_{\text{last}}(z)\) is a hyperbolic set with hyperbolicity constants \(C, \lambda\) in (4). Then with the Gaussian bracket \([x] = \max\{n \in \mathbb{Z} : n \leq x\}

\[
\sigma(z + az^+) \leq \sigma(z) + D^+ \sigma(z)[az^+] + 2C|a|k(z) \exp\left(-\frac{\lambda\sigma(z)}{2}\left|\frac{1}{|a|k(z)}\right|\right)
\]

**Proof.** Consider integers \(N \geq M \geq 1\) and write \(k = k(z)\). We choose integers \(n_i, n'_i\) defined as

\[
n_i := i \cdot \left\lfloor \frac{N}{Mk} \right\rfloor, \quad i = 0, \ldots, k,
\]

\[
n'_i := n_i, \quad i = 0, \ldots, k-1, \quad n'_{k} := N - (M-1) \cdot n_k.
\]

Observe that by \(N \geq M\)

\[
n'_{k} = N - (M-1) \cdot k \cdot \left\lfloor \frac{N}{Mk} \right\rfloor \geq k \cdot \left\lfloor \frac{N}{Mk} \right\rfloor = n'_{k-1} + \left\lfloor \frac{N}{Mk} \right\rfloor
\]
For \( s \in \{-1, 1\} \) let \( \gamma \) be the \((n_kz + sz^+)\)-periodic curve from Lemma 4.5 associated to the integers \( n_i \) and analogously \( \gamma^* \) the \((n_kz + sz^+)\)-periodic curve associated to the integers \( n_i^* \). Consider the new curve
\[
\Gamma := \gamma * (\gamma + [n_kz + sz^+]) * ... * (\gamma + (M - 2)[n_kz + sz^+]) * (\gamma^* + (M - 1)[n_kz + sz^+]).
\]
We find for the homotopy class
\[
[\Gamma] = (M - 1)(n_kz + sz^+) + (n_k^*z + sz^+) = Nz + sMz^+.
\]
Note that \( n_i - n_{i-1}, n_i^* - n_{i-1}^* = \left\lfloor \frac{N}{M} \right\rfloor \) so that with Lemma 4.5
\[
\sigma(Nz + sMz^+) \leq \int_\varphi(\Gamma) = (M - 1)\int_\varphi(\gamma) + \int_\varphi(\gamma^*)
\]
\[
\leq (M - 1)\sigma(n_kz) + \sigma(n_k^*z) + MD^+\sigma(z)[sz^+]
\]
\[
+ 2CMk \exp\left(\frac{-\lambda\sigma(z)}{2} \left\lfloor \frac{N}{Mk} \right\rfloor\right)
\]
\[
= \sigma(Nz) + MD^+\sigma(z)[sz^+] + 2CMk \exp\left(\frac{-\lambda\sigma(z)}{2} \left\lfloor \frac{N}{Mk} \right\rfloor\right)
\]
Now let \( a \in [-1, 1] - \{0\} \) be arbitrary (note that the lemma is trivial for \( a = 0 \)). Let \( s = \text{sign}(a) \) and choose sequences of integers \( N_n \geq M_n \geq 1 \) with \( M_n / N_n \to |a| \), so that
\[
(1, |a|) = \lim_{n \to \infty} \frac{1}{N_n} N_m, \quad \frac{N_m}{M_n} \geq \left\lfloor \frac{1}{|a|} \right\rfloor.
\]
We find by the continuity of \( \sigma \), monotonicity of \( |\cdot| \) and homogeneity of \( D^+\sigma(z)[\cdot] \)
\[
\sigma(z + az^+) = \lim_{n \to \infty} \frac{1}{N_n} \sigma(n_kz + sMz^+)
\]
\[
\leq \lim_{n \to \infty} \frac{1}{N_n} \left[ \sigma(n_kz) + M_n D^+\sigma(z)[sz^+] + 2CM_n k \exp\left(\frac{-\lambda\sigma(z)}{2} \left\lfloor \frac{1}{|a| k} \right\rfloor\right)\right]
\]
\[
= \sigma(z) + D^+\sigma(z)[s|a|z^+] + 2C|a|k \exp\left(\frac{-\lambda\sigma(z)}{2} \left\lfloor \frac{1}{|a| k} \right\rfloor\right)
\]
The lemma follows.

Let us prove a simple lemma, which will be useful again later.

**Lemma 4.8.** Fix \( \xi \in \mathbb{R}^2 - \{0\} \) and consider the maps
\[
\Pi_1, \Pi_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \Pi_1(\nu) := (|\xi| + \langle \nu, \frac{\xi}{|\xi|} \rangle) \frac{\xi}{|\xi|}, \quad \Pi_2(\nu) := \nu - \langle \nu, \frac{\xi}{|\xi|} \rangle \frac{\xi}{|\xi|}.
\]
Then

(i) \( \xi + v = \Pi_1(v) + \Pi_2(v) \) and \( \Pi_1(v) \perp \Pi_2(v) \).

(ii) for \( |v| \leq |\xi| \) we have

\[
|\Pi_1(v)| = |\xi| + \langle v, \frac{\xi}{|\xi|} \rangle, \quad |\Pi_2(v)| = \left| \langle v, \frac{\xi^\perp}{|\xi|} \rangle \right|.
\]

(iii) for \( |v| < |\xi| \) we have

\[
\sigma(\xi + v) - \sigma(\xi) = D^+ \sigma(\xi)[v]
\]

\[
= \sigma(\Pi_1(v) + \Pi_2(v)) - \sigma(\Pi_1(v)) - D^+ \sigma(\Pi_2(v))[\Pi_2(v)].
\]

**Proof.** Items (i) and (ii) follow directly from the definitions. Let us prove (iii). Using homogeneity of \( \sigma \) we find for \( a \in \mathbb{R} \)

\[
D^+ \sigma(\xi)[a\xi + v] = \lim_{t \searrow 0} \frac{\sigma((1 + ta)\xi + tv) - \sigma(\xi)}{t}
\]

\[
= \lim_{t \searrow 0} \frac{(1 + ta)\sigma(\xi + \frac{t}{1 + ta}v) - (1 + ta)\sigma(\xi) + ta\sigma(\xi)}{t}
\]

\[
= \lim_{t \searrow 0} \frac{\sigma(\xi + \frac{t}{1 + ta}v) - \sigma(\xi)}{t} + a\sigma(\xi)
\]

\[
= a\sigma(\xi) + D^+ \sigma(\xi)[v].
\]

Then using \( |v| < |\xi| \) and homogeneity it follows that

\[
\sigma(\xi + v) = \sigma(\xi) - D^+ \sigma(\xi)[v]
\]

\[
= \sigma(\Pi_1(v) + \Pi_2(v)) - \sigma(\Pi_1(v)) - \left( \langle v, \frac{\xi}{|\xi|} \rangle \right) \sigma(\xi) + D^+ \sigma(\Pi_2(v))[\Pi_2(v)]
\]

\[
= \sigma(\Pi_1(v) + \Pi_2(v)) - \sigma(\Pi_1(v)) - D^+ \sigma(\Pi_2(v))[\Pi_2(v)].
\]

We can now prove the main result of section 4.1.

**Theorem 4.9.** Let \( \xi, v \in \mathbb{R}^2 \) with \( \xi \neq 0 \) having rational or infinite slope, such that \( \mathcal{M}^{per}(\xi) \) is uniformly hyperbolic. Moreover, let \( z \) be the prime element in \( \mathbb{Z} \cap \mathbb{R}_0 \xi \) and \( C, \lambda \) the hyperbolicity constants from (4). Then for all \( v \in \mathbb{R}^2 \) with \( \left| \langle v, \frac{\xi}{|\xi|} \rangle \right| < |\xi| + \langle v, \frac{\xi}{|\xi|} \rangle \)

\[
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi)[v] \leq \left( \langle v, \frac{\xi}{|\xi|} \rangle \right) \cdot \frac{k(z)}{|z|} \cdot 2C \cdot \exp \left( -\frac{\lambda \sigma(\xi)}{2} \cdot \frac{|\xi| + \langle v, \frac{\xi}{|\xi|} \rangle}{\left| \langle v, \frac{\xi}{|\xi|} \rangle \cdot k(z) \right|} \right).
\]

**Proof.** We first prove the theorem for \( v \perp \xi \) and choose some \( a > 0 \) and \( b \in \mathbb{R} \) with

\[
\xi = a \cdot z, \quad v = b \cdot z^\perp.
\]

Then by lemma 4.7, \( D^+ \sigma(\xi)[v] = D^+ \sigma(\xi)[b] \) and \( a = |\xi|/|z| > |b| = |v|/|z| \).
\[
\sigma(\xi + v) = a \cdot \sigma(\frac{b}{a} + \xi) \leq a \left[ \sigma(\xi) + D^+ \sigma(\xi) \left( \frac{b}{a} \right) \right] \\
+ 2C\frac{|b|}{a} k(z) \cdot \exp \left( -\frac{\lambda \sigma(z)}{2} \left[ \frac{a}{|b|k(z)} \right] \right) \\
= \sigma(\xi) + D^+ \sigma(\xi) [v] + 2C|v| \frac{k(z)}{|z|} \cdot \exp \left( -\frac{\lambda \sigma(z)}{2} \frac{|\xi|}{|v|k(z)} \right).
\]

Let now \( v \in \mathbb{R}^2 \) with \( \langle v, \xi \rangle \leq |\Pi_2(v)| < |\xi| + \langle v, \frac{\xi}{|\xi|} \rangle = |\Pi_1(v)| \). We can reduce this case to the case \( v \perp \xi \) by using lemma 4.8. Namely,
\[
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi) [v] \\
= \sigma(\Pi_1(v) + \Pi_2(v)) - \sigma(\Pi_1(v)) - D^+ \sigma(\Pi_1(v)) \Pi_2(v) \\
\leq 2C |\Pi_2(v)| \frac{k(z)}{|z|} \cdot \exp \left( -\frac{\lambda \sigma(z)}{2} \frac{|\Pi_1(v)|}{|\Pi_2(v)|k(z)} \right).
\]
This proves the theorem using the formulae in lemma 4.8.

We can now easily prove theorem 1.5 from the introduction.

**Proof of theorem 1.5.** Recall that \( x/2 \leq \lfloor x \rfloor \forall x \geq 1 \).

Also note that \( \left| \langle v, \frac{\xi}{|\xi|} \rangle \right| \leq |v| \). With \( |\xi| + \langle v, \frac{\xi}{|\xi|} \rangle \geq \frac{|\xi|}{2} \) for \(|v| \leq |\xi|/2\) we find by the monotonicity of \([\cdot]\)
\[
\left| \frac{|\xi| + \langle v, \frac{\xi}{|\xi|} \rangle}{\langle v, \frac{\xi}{|\xi|} \rangle k(z)} \right| \geq \frac{|\xi|}{2|v| k(z)} \geq \frac{|\xi|}{4|v| k(z)}
\]
for \( \frac{|\xi|}{2|v| k(z)} \geq 1 \), or equivalently \(|v| \leq \frac{|\xi|}{2k(z)}\). We apply theorem 4.9 and find
\[
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi) [v] \leq |v| 2C \frac{k(z)}{|z|} \exp \left( -\frac{\lambda \sigma(z)|\xi|}{8k(z)} \frac{1}{|v|} \right).
\]
The claim follows.

**4.2. The irrational case**

In order to see hyperbolicity in irrational directions using the stable norm, we will approximate irrationals by rationals and then use the results of the previous section 4.1. We shall use the following well-known continuity property of the sets \( M(\xi) \), based on the continuity of the asymptotic direction \( \delta^+ : \cup_{\xi \in S} M(\xi) \to S^I \) and on the closedness of the minimality-condition. We omit the proof; see e.g. corollary 3.16 in [Ban88].

931
Lemma 4.10. Let $\xi_n \to \xi$ in $S^1$ and consider any sequence $\nu_n \in \mathcal{M}(\xi_n)$. Then any limit point of $\{\nu_n\}$ lies in $\mathcal{M}(\xi)$. Equivalently, if $\xi_n \to \xi$, then for any open neighborhood $U \subset S^2$ of $\mathcal{M}(\xi)$, there exists $n_0 \in \mathbb{N}$, such that $\mathcal{M}(\xi_n) \subset U$ for $n \geq n_0$.

The following theorem is item (ii) of theorem 1.4 from the introduction. In the proof, we recover theorem 4.9 in the irrational case.

Theorem 4.11. Let $\xi, \xi' \in \mathbb{R}^2 - \{0\}$ have irrational slope $\xi' \in \mathbb{R} - \mathbb{Q}$ and assume that the set $\mathcal{M}(\xi)$ is uniformly hyperbolic. Assume moreover that $k(z) = |z|^2$ for all $z \in \mathbb{R}^2 - \{0\}$ (see remark 4.4). Then there exist constants $C, \lambda > 0$, such that for all for all choices of rays $R \subset \mathbb{R}^2$ emanating from the origin (i.e. $R = \mathbb{R}_{>0} \cdot v$ for some $v \neq 0$) there exist sequences $\nu_n \in R$ with $\nu_n \to 0$, so that

$$\sigma(\xi + \nu_n) - \sigma(\xi) - D^+ \sigma(\xi) |\nu_n| \leq |\nu_n|^{1/4} C \cdot \exp\left(-\lambda \cdot \frac{1}{|\nu_n|^{1/4}}\right).$$

Proof. By proposition 6.4.6 on p 265 in [KH95], there exists an open neighborhood $U \supset \mathcal{M}(\xi)$, such that the set

$$\Lambda := \bigcap_{r \in \mathbb{R}} \phi'_r(U)$$

is uniformly hyperbolic for $\phi'_r$. If we approach $\xi$ by a sequence $\xi_n \in S^1$ with rational slopes, then for large $n$ the sets $\mathcal{M}(\xi_n)$ will lie in the neighborhood $U$ using the upper semi-continuity of $\xi \mapsto \mathcal{M}(\xi)$ in lemma 4.10. By the flow-invariance of these sets, all sets $\mathcal{M}(\xi_n)$ will lie in $\Lambda$ and hence be hyperbolic with the hyperbolicity constants $C, \lambda > 0$ of $\Lambda$ (for $n$ sufficiently large). We will use the hyperbolicity of such $\mathcal{M}(\xi_n)$ in the sense of (4).

We first assume $\xi \bot R$ in the euclidean sense. We approximate $\xi$ by a point $\eta \in \xi - R$ of rational slope $\eta \in \mathbb{Q}$. More precisely, using continued fractions one can find infinitely many approximations $r := p/q \in \mathbb{Q}$ of the slope $\omega := \xi_2/\xi_1 \in \mathbb{R} - \mathbb{Q}$ of $\xi$ satisfying

$$|\omega - r| \leq \frac{1}{q(q + 1)}, \quad \text{(8)}$$

such $r$’s lying on either side of $\omega$. Thus, the set

$$Q = Q(\xi, R) := \{q \in \mathbb{N} \mid \exists p \in \mathbb{Z} : \eta \in \xi - R \text{ has slope } r = p/q \text{ with (8)}\}$$

is unbounded.

We fix $q \in Q$ and estimate the distance $|\xi - \eta|$. Using that $\eta$ is the only point in the straight line $\xi + \text{span}(R)$ of slope $\eta_2/\eta_1 = r$, one verifies that

$$\eta = \xi + \frac{\xi_2 - \xi_1}{\xi_1 + r \xi_2} \xi_1 = \xi + \frac{r - \omega}{1 + r \omega} \xi_1.$$

This shows with $r \omega > 0$ by $r \approx \omega$ and (8)

$$\frac{|\xi - \eta|}{|\xi|} = \frac{|\omega - r|}{1 + r \omega} \leq \frac{1}{q(q + 1)} \cdot \frac{1}{1 + r \omega}, \quad \text{(9)}$$

932
Next, we let $v \in R$ with
\[
\frac{1}{2q^2} \leq \frac{|v|}{|\xi|} \leq \frac{1}{q^2}.
\] (10)
For the intuition observe moreover that using the orientation given by $R$, the points $\eta, \xi, \xi + v$ are ordered along the line $\xi + \text{span}(R)$ as
\[
\eta < \xi < \xi + v.
\] (11)

We shall prove the theorem in the case $\xi \perp R$ in two steps. In the first step, we estimate the Gaussian bracket appearing in our application of theorem 4.9 from below by 1. As the proof of Step 1 is quite technical and long, we delay it to the end of this proof of theorem 4.11.

**Step 1.** Assuming $\xi \perp R$, with $\eta, v$ satisfying (9), (10), $z$ being the prime element in $\mathbb{Z}^2 \cap [\mathbb{R}_{>0} \eta]$ and $t := \frac{|\xi - \eta|}{|v|}$ we have for sufficiently large $q \in Q$, that
\[
\frac{|\eta| + ((1 + t)v, \frac{\eta}{|\eta|})}{((1 + t)v, \frac{\eta}{|\eta|})} k(z) \geq 1.
\] (12)

**Step 2.** Assuming $\xi \perp R$, there exist constants $C', \lambda' > 0$ (depending only on $\xi/|\xi| \in S^1$), so that for sufficiently large $q \in Q$ and $v \in R$ satisfying (10) we have
\[
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi)[v] \leq |\xi|^{1/4} |v|^{1/4} \cdot C' \cdot \exp\left(-\lambda' \cdot \frac{|\xi|^{1/4} |v|^{1/4}}{|v|^{1/2}}\right).
\] (13)

**Proof of Step 2.** We let $t = \frac{|\xi - \eta|}{|v|} > 0$ as above, so that $\xi = \eta + tv$ (recall (11)). By definition of $D^+ \sigma$ one finds
\[
\sigma(\xi) = \sigma(\eta + tv) \geq \sigma(\eta) + t \cdot D^+ \sigma(\eta)[v].
\]
Moreover, by convexity of $\sigma$ and (11) we find
\[
D^+ \sigma(\eta)[v] = \inf_{s > 0} \frac{\sigma(\eta + sv) - \sigma(\eta)}{s} \leq \inf_{s > 0} \frac{\sigma(\xi + sv) - \sigma(\xi)}{s} = D^+ \sigma(\xi)[v].
\]
Hence,
\[
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi)[v] 
\leq \sigma(\xi + v) - \sigma(\eta) - t \cdot D^+ \sigma(\eta)[v] - D^+ \sigma(\eta)[v] 
= \sigma(\eta + (1 + t)v) - \sigma(\eta) - D^+ \sigma(\eta)[(1 + t)v].
\] (14)

Our aim is to apply theorem 4.9 to the rational direction $\eta$, the vector $(1 + t)v$ and $z$ the prime element in $\mathbb{Z}^2 \cap [\mathbb{R}_{>0} \eta]$. Choose some $\alpha > 0$ with $\alpha |.| \leq F$ and note that
\[
\sigma \geq \alpha |.|.
\]
Choosing $D > 1 + \omega^2$, we find for large $q$
Due to $t = \frac{\xi - \eta}{|\eta|}$, (9) and (10) we have

$$
\frac{1}{q} \leq 2 \frac{|v|^{1/4}}{|\xi|^{1/4}}.
$$

Also, (10) implies

$$
\frac{1}{q} \leq 2 \frac{|v|^{1/4}}{|\xi|^{1/4}}.
$$

We now apply theorem 4.9 to (14) obtaining by Step 1 and $k(z) = |z|^2$

$$
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi) [v]
\leq \frac{|\xi|}{q} \cdot 4C \sqrt{D} \cdot \exp\left(-\frac{\lambda\alpha}{2} \cdot q\right)
\leq |\xi|^{1/4} \cdot |v|^{1/4} \cdot 8C \sqrt{D} \cdot \exp\left(-\frac{\lambda\alpha}{4} \cdot \frac{|\xi|^{1/4}}{|v|^{1/4}}\right).
$$

Observe that the condition on the vector $(1 + t) v$ in theorem 4.9 is satisfied due to Step 1.

Next, we generalize our estimate from Step 2 to the general case, not assuming $\xi \perp \mathbb{R}$. Observe that the case $R \subset \mathbb{R} \xi$ is trivial, so we assume $R \not\subset \mathbb{R} \xi$. Using lemma 4.8 and Step 2, we find

$$
\sigma(\xi + v) - \sigma(\xi) - D^+ \sigma(\xi) [v] = \sigma(\Pi_1(v) + \Pi_2(v)) - \sigma(\Pi_1(v)) - D^+ \sigma(\Pi_1(v)) [\Pi_2(v)]
\leq |\Pi_1(v)|^{1/4} \cdot |\Pi_2(v)|^{1/4} \cdot C' \cdot \exp\left(-\lambda' \cdot \frac{|\Pi_1(v)|^{1/4}}{|\Pi_2(v)|^{1/4}}\right).
$$

For the above argument we need to meet the requirement (10) for Step 2, i.e. we need

$$
\frac{1}{2q^4} \leq \frac{|\Pi_2(v)|}{|\Pi_1(v)|} \leq \frac{1}{q^4}.
$$

Here, $q \in Q = Q(\Pi_1(\xi), \Pi_2(R))$ and $Q$ is unbounded. Choosing $v_0 \in R$ with $|v_0| = 1$, observe that by lemma 4.8

$$
|\Pi_1(v)| = |\xi| + |v| \cdot \langle v_0, \frac{\xi}{|\xi|} \rangle = : a + b \cdot |v|,
|\Pi_2(v)| = |v| \cdot \langle v_0, \frac{\xi}{|\xi|} \rangle = : c \cdot |v|.
$$
Hence, (15) translates for small $|v|$ (using $a, c > 0$) into the range
\[
\frac{a}{c \cdot 2q^2 - b} \leq |v| \leq \frac{a}{c \cdot q^2 - b}
\]
for $|v|$. The theorem follows, using for $|v| \leq |\xi|/2$ the estimates
\[
\frac{1}{2} |\xi| \leq |\xi| - |v| \leq |\Pi_t(v)| \leq 2|\xi|, \quad |\Pi_t(v)| \leq |v|.
\]

**Proof of Step 1.** With $v = a\xi \perp$ for some $a \in \mathbb{R}$ by the assumption $\xi \perp R$ and $t = \frac{|\xi - \eta|}{|v|}$ we find
\[
\left\langle (1 + t)v, \eta \right\rangle = |(1 + t)a| \cdot |(\xi + \eta)|
\]
\[
= \left|\frac{|\xi + \eta|}{|\xi|} \cdot |\xi, \eta| \right|
\]
\[
= \left|\frac{|\xi + \eta|}{|\xi|} \cdot |\xi, \eta - \xi + \xi| \right|
\]
\[
= \left|\frac{|\xi + \eta|}{|\xi|} |\xi, \eta - \xi + \xi| \right|.
\]
In the last equality we used $\eta - \xi \perp \xi$ due to $\xi \perp R$. Similarly, using $\xi = \eta + n\nu$ due to (11) we obtain
\[
|\eta|^2 + \left\langle (1 + t)v, \eta \right\rangle = \langle \xi + v, \eta \rangle
\]
\[
= \langle \xi + v, \xi - \nu \rangle
\]
\[
= \langle \xi + v, \xi \rangle - t\langle \xi + v, \nu \rangle
\]
\[
= |\xi|^2 - |\xi - \eta| |v|.
\]
Recalling the assumption $k(z) = |z|^2$, we find that (12) is equivalent to
\[
\frac{|\eta|^2}{1 - \frac{|\eta|^2}{|\xi|^2}} \leq \frac{1}{|\xi|^2}. \tag{16}
\]
Observe that in general, for $0 \leq x, y \leq 1/2$ one can verify that
\[
\frac{x + y}{1 - xy} \leq x + y + 2(x^2 + y^2).
\]
Hence, using (9) and (10) we find for the left hand side of (16)
\[
A := \frac{|\eta|^2}{1 - \frac{|\eta|^2}{|\xi|^2}} \leq \frac{|\eta|^2}{|\xi|^2} + \frac{|\xi|^2}{|\xi|^2} + 2\left(\frac{|\eta|}{|\xi|} + \frac{|\xi|}{|\xi|} \right)^2
\]
\[
\leq \frac{1}{q^4} + \frac{1}{q(q + 1)} \frac{1}{1 + r\omega} + 2\left(\frac{1}{q^4} + \frac{1}{q^4} \frac{1}{(1 + r\omega)^2}\right)
\]
On the other hand, by \( z = (q, p), r = p/q \) and (8), we have for the right hand side of (16)

\[
\frac{1}{|z|^2} = \frac{1}{q^2(1 + r^2)} = \frac{1}{q^2} \cdot \frac{1}{1 + r\omega + r(r - \omega)} \geq \frac{1}{q^2} \cdot \frac{1}{1 + r\omega + |r|\frac{1}{q(q + 1)}}.
\]

Put together, we obtain

\[
q^2(q + 1) \cdot \left( \frac{1}{|z|^2} - A \right) \geq \frac{q + 1}{1 + r\omega + |r|\frac{1}{q(q + 1)}} - \left[ \frac{g + 1}{q^2} + \frac{q}{1 + r\omega} + 2 \frac{q + 1}{q^2} \left( \frac{1}{q^2} + \frac{1}{1 + r\omega} \right) \right] \geq \frac{q + 1}{1 + r\omega + |r|\frac{1}{q(q + 1)}} - \left[ \frac{g + 1}{q^2} + \frac{q + 1}{1 + r\omega} \right] \geq \frac{(q + 1)(1 + r\omega) - q(1 + r\omega + |r|\frac{1}{q(q + 1)})}{(1 + r\omega + |r|\frac{1}{q(q + 1))(1 + r\omega)} - 5 \frac{q + 1}{q^2}} = \frac{1 + r\omega - |r|\frac{1}{q + 1}}{(1 + r\omega + |r|\frac{1}{q(q + 1))(1 + r\omega)} - 5 \frac{q + 1}{q^2}}.
\]

For \( q \to \infty \), the last term converges to \( (1 + \omega^2)^{-1} > 0 \), proving that (16) is fulfilled for sufficiently large \( q \).

This finishes the proof of theorem 4.11.

5. The proof of proposition 2.6

In this section we sketch the proof of proposition 2.6. It stated that for conformally generic Finsler metrics (see definition 2.4) there exists an open and dense subset \( U \subset S^1 \), containing the points with rational and infinite slope, such that the set \( \mathcal{M}(\xi) \) is uniformly hyperbolic for all \( \xi \in U \). The central arguments were given by LeCalvez in [LeC88].

First, recall that by proposition 2.5, the shortest closed geodesic in each homotopy class is unique, if \( F \) is conformally generic. The main property of our Finsler metric that we need for the proof will be that it is a ‘Kupka–Smale metric’. This is formulated in the following theorem, which is a variant of theorem D in [CI99] due to Contreras and Iturriaga.

Theorem 5.1. The following property is conformally generic for Finsler metrics on \( \mathbb{T}^2 \):

- in all non-trivial free homotopy classes of \( \mathbb{T}^2 \), the unique shortest closed geodesic is hyperbolic and its stable and unstable manifolds intersect transversely.
We will not prove theorem 5.1 here. The interested reader is referred to the proof of theorem D in [C199], noting that the perturbation \( L + \phi \) is replaced by \( \phi \cdot L \), where the Lagrangian \( L \) is given by \( L = \frac{1}{2} F^2 \).

Next, we can use the arguments in [LeC88] to show the following.

**Theorem 5.2.** Assuming the assertion in theorem 5.1, the set \( \mathcal{M}(\xi) \) is hyperbolic for all \( \xi \in S^1 \) with rational or infinite slope.

**Sketch of the proof.** We only sketch the proof, referring to [LeC88] for the details. Roughly, the argument is as follows. By assumption, the set of periodic minimal geodesics \( \mathcal{M}^{\text{per}}(\xi) \) is a single hyperbolic orbit with (un)stable manifolds intersecting transversely. Then one notes that \( A := \mathcal{M}(\xi) - \mathcal{M}^{\text{per}}(\xi) \) is a subset of the (un)stable manifolds of the hyperbolic periodic orbit (see theorem 4.1). An application of the \( \lambda \)-lemma shows that the closure of each orbit in \( A \) is uniformly hyperbolic. Then one uses the transversality of the intersection of (un)stable manifolds to infer that \( A \) consists of only finitely many orbits. Hence, \( \mathcal{M}(\xi) \) itself consists of only finitely many orbits, each of which has a uniformly hyperbolic closure. The hyperbolicity constants of these finitely many sets yield the hyperbolicity constants of the finite union \( \mathcal{M}(\xi) \). Hence, \( \mathcal{M}(\xi) \) is hyperbolic.

Using the well-known stability of hyperbolic sets, we can now prove proposition 2.6.

**Proof of proposition 2.6.** We show that for each \( \xi \in S^1 \) with rational or infinite slope, there exists a small neighborhood \( U_\xi \subset S^1 \) of \( \xi \), such that \( \mathcal{M}(\eta) \) is hyperbolic for all \( \eta \in U_\xi \). Assuming the contrary, fix \( \xi \) with rational slope and let \( \xi_n \to \xi \), such that \( \mathcal{M}(\xi_n) \) is not hyperbolic. By the upper semi-continuity of \( \xi \to \mathcal{M}(\xi) \) (lemma 4.10), we find by proposition 6.4.6 on p 265 in [KH95], that \( \mathcal{M}(\xi_n) \) has to be hyperbolic for large \( n \), which is a contradiction. \( \square \)

6. Examples

6.1. The flat torus

The simplest example is the flat torus. More generally, consider any Finsler metric \( F : TT^2 \to \mathbb{R} \), which does not depend on the base variable \( x \in T^2 \) (in standard coordinates \( TT^2 = T^2 \times \mathbb{R}^2 \)). Here we find that

\[ \sigma_F = F, \]

which is everywhere strongly convex by the definition of a Finsler metric. By main theorem 1.4 (i), this corresponds to KAM-tori in \( ST^2 \). Indeed, these are simply given by

\[ \mathcal{M}(\xi) = \{(x, v) : x \in T^2, v = \xi/F(\xi)\} \subset ST^2. \]

A special case is the euclidean norm \( F(v) = |v| \), where

\[ \{\sigma_F = 1\} = S^1. \]

6.2. The rotational torus

The next well-known example is a rotational torus in \( \mathbb{R}^3 \), obtained by rotating a circle in the \( x_1-x_3 \)-plane about the \( x_3 \)-axis. Here, one has the ‘inner’ closed geodesic as a hyperbolic closed geodesic, such that \( \mathcal{M}^{\text{per}}(\pm e_3) \) become hyperbolic sets and theorem 1.5 applies. In the following, we will find a formula for drawing the unit circle \( \{\sigma_F = 1\} \subset \mathbb{R}^2 \) with a computer.

Let \( c = (c_1, c_2, c_3) : \mathbb{R} \to \mathbb{R}^3 \) be a curve with \( c_1 > 0, c_2 = 0 \) and \( |c| = 1 \). We can parametrize a surface of revolution in \( \mathbb{R}^3 \) via
The level set \( \{ \sigma_g = 1 \} \) for the rotational metric \( g = f \cdot \langle \cdot \rangle \) with \( f(x) = \cos(2\pi x) + 2 \). Note that \( \{ \sigma_g = 1 \} \) has tangencies to infinite order to its one-sided tangent spaces at the points \( \pm e_1 \) (the parts of \( \{ \sigma_g = 1 \} \) look like a straight line). This corresponds via theorem 1.5 to the hyperbolicity of \( M^{\text{po}}( \pm e_1 ) \). On the other hand, \( \sigma_g \) is strongly convex in \( \mathbb{R}^2 - \mathbb{R} e_1 \), corresponding to \( \mathcal{C}^\infty \)-KAM-tori via main theorem 1.4 (i). See also the proof of theorem 6.1.

\[
\phi : \mathbb{R}^2 \to \mathbb{R}^3, \quad \phi(s, t) = \begin{pmatrix} \cos(s) - \sin(s) & 0 \\ \sin(s) & \cos(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1(t) \\ 0 \\ c_3(t) \end{pmatrix}.
\]

If \( h : \mathbb{R} \to \mathbb{R} \) is a solution of \( h' = c_1 \circ h \), then \( h \) is strictly increasing by \( c_1 > 0 \). We find with

\[
\phi^*(\langle \cdot \rangle_{\mathbb{R}^2})_{\mathbb{R}^2} = f(x_2) \cdot \langle \cdot \rangle, \quad f := (c_1 \circ h)^2.
\]

Moreover, if \( c \) is a periodic curve, then the function \( f \) will be periodic.

A Riemannian metric \( g \) on \( \mathbb{R}^2 \) of the form

\[
g_{c}(v, w) = f(x_2) \cdot \langle v, w \rangle, \quad x = (x_1, x_2)
\]

is called a rotational metric. One can draw the level set \( \{ \sigma_g = 1 \} \) using a computer and the formula in the following theorem, see figure 2.

**Theorem 6.1.** If \( g_{c} = f(x_2) \cdot \langle \cdot \rangle \) is a rotational metric on \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \), then the unit circle \( \{ \sigma_g = 1 \} \) of the stable norm is given by the union of the two curves

\[
t \mapsto \pm \frac{1}{\int_{0}^{1} f(x) \frac{dx}{\sqrt{f(x) - t^2}}} \cdot \left( \int_{0}^{t} \frac{t}{\sqrt{f(x_2) - t^2}} \frac{dx_2}{1} \right)
\]

with \( |t| \leq \sqrt{\min f} \).

**Proof.** First, we move to the Hamiltonian setting via the Legendre transform

\[
\mathcal{L} : T^*T^2 \to T^*\mathbb{T}^2, \quad \mathcal{L}(x, v) = g_{c}(v, \cdot) = f(x_2) \cdot \langle v, \cdot \rangle,
\]

where the geodesic flow of \( g \) is described in standard coordinates of \( T^*T^2 = T^2 \times \mathbb{R}^2 \) by the Hamiltonian

\[
\rightlink{938}
which is dual to the Lagrangian \( L(x,v) = \frac{1}{2} g(x,v) \). \( H \) admits the coordinate function \( p_1 \) as a first integral, such that for \((a,b) \in \mathbb{R}^2\) the sets

\[
\Sigma_{a,b} := \{ H = a, p_1 = b \} \subset T^* \mathbb{T}^2
\]

are invariant under the Hamiltonian flow of \( H \). We find

\[
H(x,p) = a \land p_1 = b \iff p = p(x) = \left( b, \pm \sqrt{2af(x) - b^2} \right).
\]

For the case \( 2a \min f > b^2 \), the above formula defines two smooth, invariant, Lagrangian graphs

\[
\Sigma_{a,b}^\pm := \{ (x,p(x)) : x \in \mathbb{T}^2 \}
\]

(which are in fact \( C^\infty \)-KAM-tori in the sense of definition 1.2). We can write \( p(x) \) seen as a closed 1-form on \( T^2 \) in the form \( \eta = \alpha \wedge 2a f \), where \( \eta \in \Omega^1(T^2, \mathbb{R}) \) is a constant 1-form

and \( \alpha : T^2 \rightarrow \mathbb{R} \) is some function (see e.g. lemma 3.4 in [Sch13]); \( \eta = [\alpha] \) is called the Liouville class of the Lagrangian graph. We thus find a find a formula for the Liouville class of the graph \( \Sigma_{a,b}^\pm \) given by

\[
[S_{a,b}^\pm] = \eta = \int_{\mathbb{T}^2} \rho(x)dx = \left\{ \int_0^1 \left[ b \int_0^1 \pm \sqrt{2af(x) - b^2} \, dx \right] \right\}.
\]

We shall use a bit of language from Mather theory. Namely, the convex dual of Mather’s \( \beta \)-function \( \beta = \frac{1}{2} \sigma_\theta^2 \) is Mather’s \( \alpha \)-function. Note that \( \alpha \) is a \( C^1 \)-function by \( \beta \) being strictly convex in the case of \( T^2 \), as we already noted in the introduction. Moreover, if \( \eta \) is the Liouville class of an invariant Lagrangian graph, then \( \alpha(\eta) \) equals the energy of the corresponding graph. Hence,

\[
\alpha([\Sigma_{a,b}^\pm]) = H_{\Sigma_{a,b}^\pm} = a.
\]

This shows

\[
\left\{ \left[ b \int_0^1 \pm \sqrt{2af(x) - b^2} \, dx \right] : b^2 < 2a \min f \right\} \subset \{ \alpha = a \}.
\]

Next, observe that by Fenchel duality and \( \alpha(\eta) = \beta(\nabla \alpha(\eta)) \) we have for the euclidean gradient \( \nabla \alpha \) of \( \alpha \), that

\[
\beta(\nabla \alpha(\eta)) = \langle \nabla \alpha(\eta), \eta \rangle - \alpha(\eta) = \alpha(\eta) \quad \forall \eta.
\]

Hence, a point \( \eta \in \{ \alpha = a \} \) yields a point

\[
\nabla \alpha(\eta) \in \{ \beta = a \} = \{ \sigma_\cdot = \sqrt{2a} \}.
\]

Hence, our aim is to compute \( \nabla \alpha(\eta) \) for \( \eta \in \{ \alpha = a \} \).
We fix the value \( a = 1/2 \) and consider the function

\[ g(t) := \int_{0}^{1} \frac{f(x_2)}{\sqrt{f(x_2) - t^2}} \, dx_2, \quad |t| < \sqrt{\min f}. \]

Then the curve

\[ \gamma(t) := (t, g(t)), \quad |t| < \sqrt{\min f} \]

parametrizes the upper half of the set found in \( \{ \alpha = 1/2 \} \) above. The velocity vector \( \dot{\gamma} = (1, g') \) is orthogonal to \( \nabla \alpha \circ \gamma \), i.e. we find some function \( \lambda(t) \) with

\[ \nabla \alpha(\gamma(t)) = \lambda(t) \cdot (-g'(t), 1). \]

Using \( \langle \nabla \alpha(\eta), \eta \rangle = 2\alpha(\eta) \) due to homogeneity of degree 2, we find

\[ 1 = 2\alpha(\gamma(t)) = \langle \nabla \alpha(\gamma(t)), \gamma(t) \rangle = \lambda(t) \cdot (g(t) - t \cdot g'(t)). \]

Hence, for \( |t| < \sqrt{\min f} \)

\[ \nabla \alpha(\gamma(t)) = \frac{1}{g(t) - t \cdot g'(t)} \cdot (-g'(t), 1) \in \{ \beta = 1/2 \}. \]

Next, observe that the second component of \( \nabla \alpha(\gamma(t)) \) vanishes as \( |t| \to \sqrt{\min f} \). Indeed, one computes

\[ g(t) - t \cdot g'(t) = \int_{0}^{1} \frac{f(x_2)}{\sqrt{f(x_2) - t^2}} \, dx_2 \geq \min f \cdot \int_{0}^{1} \frac{1}{\sqrt{f(x_2) - t^2}} \, dx_2. \]

On the other hand,

\[ \int_{0}^{1} \frac{1}{\sqrt{f(x_2) - t^2}} \, dx_2 \to \infty, \quad \text{as} \quad |t| / \sqrt{\min f}. \]

This shows that the two segments \( \pm \nabla \alpha \circ \gamma(t) \) with \( |t| \leq \sqrt{\min f} \) form a closed curve in \( \{ \beta = 1/2 \} \). The theorem follows. \( \Box \)

### 6.3. The punctured torus

Here we state a result due to McShane and Rivin, see [MR95a] and [MR95b]. These authors treat hyperbolic metrics \( g \) on the punctured torus \( T^2 \). While this case does not quite fit into our setting of a metric on the closed torus \( T^2 \), the same phenomena appear. Let us state theorem 2.1 from [MR95b].

**Theorem 6.2.** Let \( g \) be a hyperbolic metric on the once punctured torus \( T^2 \) with finite area. Then the stable norm \( \sigma_\xi \) of \( g \) on \( H_0(T^2, \mathbb{R}) \cong \mathbb{R}^2 \) is flat to infinite order at points \( \xi \in S^1 \) of irrational slope. At points \( \xi \in S^1 \) of rational or infinite slope, the stable norm is not differentiable and the analogous statement holds on each side of the line \( \mathbb{R}^+ \xi \).

See also the figure on page 5 of [MR00]: The level set \( \{ \sigma_\xi = 1 \} \) in theorem 6.2 for the modular torus looks like a polygon, even though it is in fact strictly convex. Of course, in this
case there are no KAM-tori and the geodesic flow is hyperbolic due to negative curvature. In this light, theorem 6.2 confirms main theorem 1.4 (ii) and theorem 1.5.

It was pointed out the authors that theorem 6.2 has no proof in the literature. The direction of the proof suggested in [MR95b], however, is quite different from the arguments in the present paper. Without going into any details, one should be able to prove theorem 6.2 using the arguments in this paper and the hyperbolicity of the geodesic flow on $T^2$.

References

[Arn11] Arnaud M-C 2011 The link between the shape of irrational Aubry-Mather sets and their Lyapunov exponents Ann. Math. 174 1571–601
[Arn13] Arnaud M-C 2013 Boundaries of instability zones for symplectic twist maps J. Inst. Math. Jussieu 13 19–41
[AB14] Arnaud M-C and Berger P 2016 The non-hyperbolicity of irrational invariant curves for twist maps and all that follows Rev. Mat. Iberoamericana 32 1295–310
[Ban88] Bangert V 1988 Mather sets for twist maps and geodesics on tori Dyn. Reported 1 1–56
[Ban89] Bangert V 1989 Minimal geodesics Ergod. Theor. Dyn. Syst. 10 263–86
[Ban94] Bangert V 1994 Geodesic rays, Busemann functions and monotone twist maps Calculus Var. PDE 2 49–63
[BCS00] Bao D D-W, Chern S S and Shen Z 2000 An Introduction to Riemann–Finsler Geometry (Graduate Texts in Mathematics vol 200) (Berlin: Springer)
[Bre93] Bredon G E 1993 Topology and Geometry (Graduate Texts in Mathematics vol 139) (Berlin: Springer)
[BQ07] Bressaud X and Quas A 2007 Rate of approximation of minimizing measures Nonlinearity 20 845–53
[CR06] Carneiro M J D and Ruggiero R O 2006 On Birkhoff theorems for Lagrangian invariant tori with closed orbits Manuscr. Math. 119 411–32
[CI99] Contreras G and Iturriaga R 1999 Convex Hamiltonians without conjugate points Ergod. Theor. Dyn. Syst. 19 901–52
[CIPP98] Contreras G, Iturriaga R, Paternain G P and Paternain M 1998 Lagrangian graphs, minimizing measures and Mañé’s critical values Geom. Funct. Anal. 8 788–809
[Hed32] Hedlund G A 1932 Geodesics on a two-dimensional Riemannian manifold with periodic coefficients Ann. Math. 33 719–39
[Hop48] Hopf E 1948 Closed surfaces without conjugate points Proc. Natl Acad. Sci. USA 34 47–51
[KH95] Katok A, Hasselblatt B 1995 Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications vol 54) (Cambridge: Cambridge University Press)
[LeC88] LeCalvez P 1988 Les ensembles d’Aubry-Mather d’un difféomorphisme conservatif de l’anneau d’évitant la verticale sont en général hyperboliques C. R. Acad. Sci. I 306 51–4
[Mac92] MacKay R S 1996 Greene’s residue criterion Nonlinearity 5 161–87
[Mañ96] Mañé R 1996 Generic properties and problems of minimizing measures of Lagrangian systems Nonlinearity 9 273–310
[Mass96] Massart D 1996 Normes stables des surfaces Thése Doctorat Ecole Normale Supérieure de Lyon
[Mas03] Massart D 2003 On Aubry sets and Mather’s action functional Isr. J. Math. 134 157–71
[Mass15] Massart D 2015 Erratum to ‘On Aubry sets and Mather’s action functional’ Isr. J. Math. 207 1001
[MS11] Massart D and Sorrentino A 2011 Differentiability of Mather’s average action and integrability on closed surfaces Nonlinearity 24 1777–93
[Mat88] Mather J N 1988 Destruction of invariant circles Ergod. Theor. Dynam. Syst. 8 199–214
[Mat90] Mather J N 1990 Differentiability of the minimal average action as a function of the rotation number Bol. Soc. Bras. Mat. 21 59–70
[MF94] Mather J N and Forni G 1994 Action minimizing orbits in Hamiltonian systems. *Transition to Chaos in Classical and Quantum Mechanics* (Berlin: Springer) pp 92–186

[MR95a] McShane G and Rivin I 1995 A norm on homology of surfaces and counting simple geodesics *Int. Math. Res. Not.* 2 61–9

[MR95b] McShane G and Rivin I 1995 Simple curves on hyperbolic tori *C. R. Acad. Sci. I* 320 1523–8

[MR00] McShane G and Rivin I 2000 Simple curves on hyperbolic tori (arXiv:math/0005220 [math.GT])

[Mor24] Morse H M 1924 A fundamental class of geodesics on any closed surface of genus greater than one *Trans. Am. Math. Soc.* 26 25–60

[Mos62] Moser J 1962 On invariant curves of area-preserving mappings of an annulus *Nachr. Akad. Wiss. Göttingen Math.—Phys. Klasse II* 1962 1–20

[Sch13] Schröder J P 2013 Tonelli Lagrangians on the 2-torus: global minimizers, invariant tori and topological entropy *PhD Thesis* Ruhr-Universität Bochum (available online at www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Lehrstuhl-X/jan.html)

[Sch15a] Schröder J P 2015 Global minimizers for Tonelli Lagrangians on the 2-torus *J. Topol. Anal.* 7 261–91

[Sch15b] Schröder J P 2015 Minimal rays on closed surfaces preprint to appear in *Isr. J. Math.*

[Sch16] Schröder J P 2016 Generic uniqueness of shortest closed geodesics *Calculus Var. PDE* 55 1–12

[Sib00] Siburg K F 2000 Symplectic invariants of elliptic fixed points *Commentarii Math. Helvetici* 75 681–700

[Sib04] Siburg K F 2004 *The Principle of Least Action in Geometry and Dynamics* (Lecture Notes in Mathematics vol 1844) (Berlin: Springer)

[Zau62] Zaustinsky E M 1962 Extremals on compact E-surfaces *Trans. Am. Math. Soc.* 102 433–45