Gaussian intrinsic entanglement for states with partial minimum uncertainty

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We develop a theory of a quantifier of bipartite Gaussian entanglement called Gaussian intrinsic entanglement (GIE) which was proposed recently in [L. Mišta et al., Phys. Rev. Lett. 117, 240505 (2016)]. The GIE provides a compromise between computable and physically meaningful entanglement quantifiers and so far it was calculated for two-mode Gaussian states including all symmetric partial minimum-uncertainty states, weakly mixed asymmetric squeezed thermal states with partial minimum uncertainty, and weakly mixed symmetric squeezed thermal states. We improve the method of derivation of GIE and we show, that all previously derived formulas for GIE of weakly mixed states in fact hold for states with higher mixedness. In addition, we derive analytical formulas for GIE for several new classes of two-mode Gaussian states with partial minimum uncertainty. Finally, it is shown, that like for all previously known states, also for all new states the GIE is equal to Gaussian Rényi-2 entanglement of formation. This finding strengthens a conjecture about equivalence of GIE and Gaussian Rényi-2 entanglement of formation for all bipartite Gaussian states.

I. INTRODUCTION

Since quantum entanglement saw the light of day \cite{1, 2}, it metamorphosed from a puzzling ingredient of quantum mechanics to a unique concept opening new paradigms in communication and computing. The development of theory and experiment exploring entanglement unveiled, that it is imperative not only to be able to certify its presence \cite{2, 3}, but also to quantify it properly. For example, entanglement exhibits various monogamy properties \cite{4, 5} which are quantitative, and therefore they can be captured only with the help of entanglement measures. Likewise, entanglement measures are indispensable in proofs of impossibility \cite{6, 7} or limitation \cite{8} of some quantum-information protocols, and they provide useful bounds on several important hardly computable quantities \cite{9, 10}. As far as the experiment is concerned, entanglement measures are needed to assess the quality of experimentally prepared entangled states \cite{11} and entangling gates \cite{12}, and what is more, they are vital for verification of successful demonstration of some crucial concepts of quantum communication such as entanglement distillation \cite{13} or the existence of a gap between secret key content and distillable entanglement \cite{14}.

Demand for entanglement quantifiers which would stay on solid grounds triggered the development of the axiomatic theory of entanglement measures \cite{15, 16}. Primarily, any good entanglement measure should be a non-negative function, which vanishes on separable (disentangled) states, and which does not increase under local operations and classical communication. Additionally, a good entanglement measure should be also equal to marginal von Neumann entropy on pure states, it should be convex, additive on tensor product and asymptotically continuous.

At present, there is a number of different entanglement measures which quantify entanglement in different ways. The most widely used measure is undoubtedly a relatively easily computable logarithmic negativity \cite{17, 18, 19}, which quantifies entanglement of a given quantum state according to how much a partial transpose of the state deviates from a physical state. Other means of entanglement quantification provide the so called operational measures known as entanglement of formation and distillable entanglement \cite{20}, which quantify how much maximal pure-state entanglement one needs to create a shared quantum state and how much maximal pure-state entanglement one can distill from a shared quantum state, respectively. A conceptually different way of entanglement quantification provide geometric measures, which quantify the amount of entanglement in a quantum state via a distance of the state from the set of separable states \cite{21}. Yet another way of entanglement quantification exists, which utilizes information-theoretical measures of correlations and the measure in question is the so called squashed entanglement \cite{22} defined as a quantum conditional mutual information of an extension of the investigated quantum state, which is minimized over all the extensions.

Each of the measures listed above has its advantages as well as weaknesses. First, for most of the measures mentioned above, some of the axioms are relaxed. Second, the measures either possess a good operational meaning or are computable but not both. One exception to the first rule is the squashed entanglement, for which all axioms imposed on a proper entanglement measure are satisfied \cite{23}, but up to exceptions \cite{24}, it is hard to compute. The other candidate for a good entanglement measure is the entanglement of formation, which was so far computed for two qubits \cite{25} and symmetric Gaussian states \cite{26}, but which has operational meaning beyond some practically usable task.

Recently, an attempt has been made \cite{27, 28, 29} to extend the family of candidates for a good entanglement measure, aiming at the same time at probing the gap between computable and physically meaningful entanglement measures. This resulted in the proposal of a new quantifier of entanglement called intrinsic entanglement (IE) \cite{27, 28}. The introduction of the IE closely follows the idea of Gisin and Wolf \cite{30} to quantify entan-
In this paper we further develop the theory of GIE and investigate its relation to GR2EoF. First, we show that analytical formulas for GIE of symmetric two-mode squeezed thermal states and asymmetric squeezed thermal states with a three-mode purification derived in Ref. 29 hold for a larger set of the states. Next, we derive an analytical formula for GIE for several new classes of two-mode Gaussian states with a three-mode purification. Finally, we discuss the relation of GIE to other entanglement measures encompassing logarithmic negativity and GR2EoF. It is shown, in particular, that the GIE for the new classes of states is again equal to the GR2EoF, which further strengthens the conjectured equivalence of the two quantities. As a byproduct of derivation of new formulas for GIE, we also obtain an explicit form of a symplectic matrix which symplectically diagonalizes an arbitrary two-mode covariance matrix in standard form the off-diagonal block of which has a negative determinant.

The paper is organized as follows. In Section II we explain basics of a new quantifier of Gaussian intrinsic entanglement called intrinsic entanglement. In Section III we give a brief introduction into the formalism of bipartite entanglement. Section IV is dedicated to the explanation of the concept of Gaussian intrinsic entanglement. In Section V and Section VI we describe in detail a generic method of derivation of the Gaussian intrinsic entanglement. In Section VII we derive Gaussian intrinsic entanglement for several new classes of two-mode Gaussian states with a three-mode purification. Section VIII deals with relation of Gaussian intrinsic entanglement to logarithmic negativity and Gaussian Rényi-2 entanglement of formation. Finally, Section IX contains conclusions.

II. INTRINSIC ENTANGLEMENT

The definition of IE is based on the classical measure of entanglement [30] which utilizes mapping of quantum states onto probability distributions. First, for the state of interest \( \rho_{AB} \) a purification \( |\Psi_{ABE}\rangle \) is constructed, where \( \text{Tr}_E|\Psi_{ABE}\rangle = \rho_{AB} \). Next, the purification is mapped by pure-state measurements \( \{\Pi_A\} \) and \( \{\Pi_B\} \), and a generic measurement \( \{\Pi_E\} \), on subsystems \( A, B \) and \( E \), onto a probability distribution

\[
P(A, B, E) = \text{Tr}(\Pi_A \otimes \Pi_B \otimes \Pi_E |\Psi_{ABE}\rangle \langle \Psi_{ABE}|).
\]

The key quantity in the definition of the classical measure of entanglement, which also stays behind introduction of the squashed entanglement [23], is the so-called intrinsic information [34] of the distribution,

\[
I(A; B \downarrow E) = \inf_{E \to \tilde{E}} [I(A; B|\tilde{E})].
\]

Here the infimum is taken over all conditional probability distributions \( P(\tilde{E}|E) \) defining a new random variable \( \tilde{E} \), and

\[
I(A; B|E) = H(A|E) - H(A|B, E),
\]

is the mutual information between \( A \) and \( B \) conditioned on \( E \). Here, \( H(X|Y) \) is the conditional Shannon entropy given by \( H(X|Y) = H(X, Y) - H(Y) \), where \( H(X, Y) \) and \( H(Y) \) are joint and marginal Shannon entropies [35], respectively.

From Eqs. (2) and (3) it follows that the intrinsic information quantifies how much reduces Bob’s uncertainty
about Alice’s variable \( A \) if he looks at his variable \( B \) after Eve announces her variable \( E \) (or a function of her variable) \( 38 \). The intrinsic information also provides an upper bound \( 24 \) (not always tight \( 27 \)) on the rate at which a secret key can be generated from the probability distribution \( P \) in the secret key agreement protocol \( 31 \), and more importantly, it is conjectured to be equal to a secret key rate in the modification of the protocol called public Eve scenario \( 38, 39 \).

Because Alice and Bob may perform unsuitable measurements on an entangled state such that intrinsic information vanishes, and on the other hand, a bad measurement on Eve’s side may allow Alice and Bob to get a strictly positive intrinsic information even for a separable state \( 30 \), some optimization is needed to get a quantity which faithfully maps entanglement onto secret correlations. For this reason, Gisin and Wolf defined the classical measure of entanglement as the following optimized intrinsic information \( 30 \):

\[
\mu(\rho_{AB}) = \inf_{\{\Pi_A, \Pi_B\}} \left\{ \sup_{\{\Pi_{AB}, \Pi_E\}} \left[ I(A; B \downarrow E) \right] \right\},
\]

where the minimization is carried out also over all purifications of the studied state \( \rho_{AB} \). The IE is then obtained \( 28, 29 \) by reversing the order of optimization in the previous formula, i.e.,

\[
E_{\infty}(\rho_{AB}) = \sup_{\{\Pi_{AB}, \Pi_E\}} \left\{ \inf_{\{\Pi_{AB}, \Pi_E\}} \left[ I(A; B \downarrow E) \right] \right\}.
\]

In the rest of the present paper we investigate the IE for the case, when all states \( \rho_{AB} \), purifications \( |\Psi\rangle_{ABE} \), measurements \( \Pi_A, \Pi_B \) and \( \Pi_E \), as well as the conditional probability distributions \( P(E|E) \), are Gaussian. Therefore, in the following section we give a brief introduction into the theory of Gaussian quantum states.

### III. GAUSSIAN STATES

In this paper we work with quantum states of systems with infinite-dimensional Hilbert space state, which we shall call modes in what follows. A system of \( N \) modes is described by a vector of quadratures \( \xi = (x_1, p_1, \ldots, x_N, p_N)^T \) with components satisfying the canonical commutation rules \( [\xi_j, \xi_k] = i(\Omega_N)_{jk} \), where

\[
\Omega_N = \bigoplus_{i=1}^N J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

is the so called symplectic matrix. We restrict our attention to Gaussian states of modes, which are defined as states with a Gaussian Wigner function. Any \( N \)-mode Gaussian state is thus fully characterized by a \( 2N \times 2N \) covariance matrix (CM) \( \gamma \) with entries \( \gamma_{jk} = \langle \xi_j \xi_k + \xi_k \xi_j \rangle - 2\langle \xi_j \rangle \langle \xi_k \rangle \) and by a \( 2N \times 1 \) vector of first moments \( \langle \xi \rangle \). Since the first moments can be set to zero by local displacements which do not change entanglement of the state, they are irrelevant as far as the entanglement is concerned and therefore they are assumed to be zero in the rest of the paper.

Apart from Gaussian states we will also utilize Gaussian unitary operations defined as unitary operations which preserve Gaussian character of states. On the CM level an \( N \)-mode Gaussian unitary is represented by a real \( 2N \times 2N \) symplectic matrix \( S \) satisfying condition

\[
S\Omega_N S^T = \Omega_N
\]

and it transforms a given CM \( \gamma \) to \( \gamma' = S\gamma S^T \).

In this paper we focus on Gaussian states of two-modes \( A \) and \( B \), and therefore we denote CMs of the states as \( \gamma_{AB} \). The quantifiers of Gaussian entanglement including GIE \( 28 \) are always invariant with respect to local Gaussian unitary operations on modes \( A \) and \( B \), and thus we can work without loosing any generality only with a canonical form of the CM with respect to the operations, the so called standard form \( 40 \),

\[
\gamma_{AB} = \begin{pmatrix} a & 0 & c_x & 0 \\ 0 & a & 0 & c_p \\ c_x & 0 & b & 0 \\ 0 & c_p & 0 & b \end{pmatrix},
\]

where we can assume \( c_x \geq |c_p| \geq 0 \). Since states with \( c_x c_p \geq 0 \) are separable \( 41 \) and thus possess zero GIE \( 28 \), in calculations we can restrict ourself only to CMs satisfying \( c_x c_p < 0 \). Introducing new more convenient parameters \( k_x \equiv c_x \) and \( k_p \equiv |c_p| = -c_p \), we arrive at the following standard-form CM which we shall consider in what follows \( 27 \):

\[
\gamma_{AB} = \begin{pmatrix} a & 0 & k_x & 0 \\ 0 & a & 0 & -k_p \\ k_x & 0 & b & 0 \\ 0 & -k_p & 0 & b \end{pmatrix},
\]

where \( k_x \geq k_p > 0 \). Construction of explicit examples of CMs of entangled Gaussian states requires to know when the matrix \( 11 \) corresponds to a physical quantum state and when the state is entangled. For this purpose, we can use a necessary and sufficient condition for a strictly positive matrix \( 11 \) to be a CM of a physical quantum state, which is given by the following inequalities \( 11 \):

\[
(ab - k_x^2)(ab - k_p^2) + 1 \geq a^2 + b^2 - 2k_x k_p, \\
abk_x^2 \geq 1.
\]

Additionally, CM \( 11 \) describes an entangled state if and only if \( 41 \)

\[
(ab - k_x^2)(ab - k_p^2) + 1 < a^2 + b^2 + 2k_x k_p.
\]

Further, we also need to find a Gaussian purification of the state with CM \( 11 \), i.e., a pure Gaussian state \( |\Psi\rangle_{ABE} \) satisfying condition \( \text{Tr}_E |\Psi\rangle_{ABE} \langle \Psi | = \rho_{AB} \). This can be done easily with the help of Williamson theorem.
which says, that for any two-mode CM $\gamma_{AB}$ there is always a symplectic transformation $S$ which brings the CM to the normal form,

$$S\gamma_{AB}S^T = \text{diag}(\nu_1, \nu_1, \nu_2, \nu_2) \equiv \gamma_{AB}^{(0)},$$

where $\nu_1 \geq \nu_2 \geq 1$ are symplectic eigenvalues of $\gamma_{AB}$. In physical terms Williamson theorem tells us that for any two-mode Gaussian state there always exists a global Gaussian unitary which brings the state into a tensor product of two thermal states with CMs $\nu_j \mathbb{I}$, $j = 1, 2$, where $\mathbb{I}$ is the $2 \times 2$ identity matrix.

The symplectic eigenvalues of CM $\gamma_{AB}$ can be calculated conveniently from the eigenvalues of the matrix $i\Omega_{AB}$ which are of the form $\{\pm \nu_1, \pm \nu_2\}$. In terms of parameters $a, b, k_x$ and $k_p$ they read explicitly as

$$\nu_{1,2} = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4D}}{2}},$$

where

$$\Delta = a^2 + b^2 - 2k_xk_p,$$

$$D = \Delta^2 - 4\det\gamma_{AB} = (a^2 - b^2)^2 + 4M\tilde{M},$$

with

$$M \equiv ak_x - bk_p, \quad \tilde{M} \equiv bk_x - ak_p.$$  

Similarly, we can express the symplectic matrix $S$ which brings the CM to Williamson normal form in terms of parameters $a, b, k_x$ and $k_p$. This can be done using either a method of Ref. [12] or a method of Ref. [14]. The derivation of the matrix for an arbitrary CM by means of the method of Ref. [13] is rather technical and it is placed into the Appendix A.

Having now both symplectic eigenvalues and symplectic matrix from Eq. (12) in hands, we can proceed to the construction of a CM ($\gamma_{AB}$) of a purification of the state with CM $\gamma_{AB}$. Obviously, the structure of the purification will depend on the so called symplectic rank $R$ of the CM, which is defined as the number of its symplectic eigenvalues different from 1 [12].

In the most simple case of $R = 0$ CM $\gamma_{AB}$ describes a pure state $|\psi\rangle_{AB}$ and the purifying subsystem $E$ is completely independent of modes $A$ and $B$. Consequently, $|\psi\rangle_{AB}E = |\psi\rangle_{AB}|\varphi\rangle_E$, where $|\varphi\rangle_E$ is the state of a purifying system, and thus $\gamma_E = \gamma_{AB} \oplus \gamma_E$, where $\gamma_E$ is a CM of the state $|\varphi\rangle_E$.

For $R > 0$ the construction of the purification relies on replacement of each of the $R$ modes with symplectic eigenvalue $\nu_i > 1, i = 1, \ldots, R$, in the Williamson normal form $\gamma_{AB}^{(0)}$.

$$\gamma_{AB}^{(0)} = \bigoplus_{i=1}^{R} \nu_i \mathbb{I} \oplus \mathbb{I}_{2(2-R)},$$

where $\mathbb{I}_K$ is the $K \times K$ identity matrix, with one mode of the two-mode squeezed vacuum state with CM

$$\gamma_{TMSV}(\nu_i) = \left( \begin{array}{cc} \nu_i \mathbb{I} & \sqrt{\nu_i^2 - 1} \sigma_z \\ \sqrt{\nu_i^2 - 1} \sigma_z & \nu_i \mathbb{I} \end{array} \right).$$

Hence, we get the following $(2 + R)$-mode CM

$$\gamma_{AB}^{(0)} = \left( \begin{array}{cc} \gamma_{AB}^{(0)} & \gamma_{ABE}^{(0)} \\ (\gamma_{ABE}^{(0)})^T & \gamma_E^{(0)} \end{array} \right),$$

with

$$\gamma_{ABE}^{(0)} = \bigoplus_{i=1}^{R} \nu_i \mathbb{I},$$

where $\sigma_z = \text{diag}(1, -1)$ is the diagonal Pauli-$z$ matrix and $\mathbb{I}_{2R}$ is the $2R \times 2R$ identity matrix, to CM $\gamma_{AB}$ which gives the sought CM of the purification

$$\gamma_{AB} = S^{-1}\gamma_{ABE}(S^T)^{-1}.$$  

### IV. GAUSSIAN INTRINSIC ENTANGLEMENT

Having established all needed ingredients we are now in a position to provide a definition of GIE and give analytical formulas for it, which have been obtained so far. In the case of a two-mode Gaussian state $\rho_{AB}$ the GIE is defined as

$$E_G^{G}(\rho_{AB}) = \sup_{\Gamma_A, \Gamma_B} \inf_{E} [\mathcal{I}(A; B|E)],$$

where

$$\mathcal{I}(A; B|E) = \frac{1}{2} \ln \left( \frac{\det \sigma_A \det \sigma_B}{\det \sigma_{AB}} \right).$$

Here

$$\sigma_{AB} = \gamma_{AB}|E| + \Gamma_A \oplus \Gamma_B,$$

where $\sigma_{AB}$ are local submatrices of $\sigma_{AB}$ and $\Gamma_A$ and $\Gamma_B$ are single-mode CMs of pure-state Gaussian measurements on modes $A$ and $B$, respectively. Further,

$$\gamma_{AB}|E| = \gamma_{AB} - \gamma_{ABE} (\gamma_E + \Gamma_E)^{-1} \gamma_{ABE}.$$
is a CM of a conditional state $\rho_{AB}\mid E$ of modes $A$ and $B$ obtained by a Gaussian measurement with CM $\Gamma_E$ on purifying subsystem $E$ of the purification with CM $\rho_{AB}^{(1)}$. The use of Eq. (21) on the right-hand side (RHS) of Eq. (26) further yields for the CM $\gamma_{AB}\mid E$ the expression

$$\gamma_{AB}\mid E = S^{-1}\gamma_{AB}\mid E(S^{-1})^T$$

(27)

with

$$\gamma_{AB}\mid E = \gamma_{AB} - \gamma_{AB}\mid E(\gamma_{E} + \Gamma_E)^{-1}(\gamma_{AB}\mid E)^T,$$  

(28)

where matrices $\gamma_{AB}$, $\gamma_{AB}\mid E$ and $\gamma_{E}$ are given in Eqs. (16) and (19).

Before summarizing currently known formulas for GIE, let us make a brief remark on uniqueness of the definition of GIE. Namely, it is obvious that the symplectic transformation $S$ that brings CM $\gamma_{AB}$ to Williamson normal form, Eq. (12), is not determined uniquely. More precisely, if CM $\gamma_{AB}$ has non-degenerate (degenerate) symplectic eigenvalues, $\nu_1 \neq \nu_2$ ($\nu_1 = \nu_2$), $S$ is determined uniquely up to local orthogonal symplectic transformations $O_A$ and $O_B$ (global orthogonal symplectic transformation $O_{AB}$) on modes $A$ and $B$. Nevertheless, despite the ambiguity in determination of matrix $S$, the GIE is determined uniquely. To show this, imagine that instead of using symplectic matrix $S$, we would use in CM of purification $\rho_{AB}\mid E$ a symplectic matrix $\bar{S} = OS$, where $O = O_A \oplus O_B$ ($O = O_{AB}$) in the non-degenerate (degenerate) case. The change $S \rightarrow \bar{S}$ entails, that the correlation matrix (25) changes to

$$\bar{S}_{AB} = \tilde{\gamma}_{AB}\mid E + \Gamma_A \oplus \Gamma_B,$$  

(29)

where

$$\tilde{\gamma}_{AB}\mid E = S^{-1}O^T\gamma_{AB}\mid E O(S^{-1})^T$$

(30)

and $\gamma_{AB}\mid E$ is given in Eq. (28). Further, making use in the latter formula Eqs. (12) and (19), and utilizing orthogonality of matrix $O$, one finds that CM (30) boils down to

$$\tilde{\gamma}_{AB}\mid E = S^{-1}\gamma_{AB}\mid E(S^{-1})^T,$$  

(31)

where

$$\tilde{\gamma}_{AB}\mid E \equiv \gamma_{AB} - \gamma_{AB}\mid E(\gamma_{E} + \Gamma_E)^{-1}(\gamma_{AB}\mid E)^T.$$  

(32)

Here $\Gamma_E$ is a CM of a new Gaussian measurement with

$$\bar{\sigma} = \left\{ \begin{array}{ll} \sigma_O \sigma_{z}, & \text{if } R = 1; \\ (\sigma_z \oplus \sigma_z)O(\sigma_z \oplus \sigma_z), & \text{if } R = 2, \end{array} \right.$$  

(33)

being an orthogonal symplectic matrix with $O = O_A \oplus O_B$ if $\nu_1 \neq \nu_2$ and $O = O_{AB}$ if $\nu_1 = \nu_2$. Hence we see, that if we use for calculation of GIE symplectic matrix $\bar{S}$ instead of symplectic matrix $S$, the conditional mutual information (24) that is to be optimized is obtained by replacing correlation matrix $\sigma_{AB}$ on the RHS of Eq. (24) with correlation matrix (29), which is further equivalent to calculation with the original correlation matrix $\sigma_{AB}$, Eq. (25), in which CM $\Gamma_E$ is replaced with CM $\bar{\Gamma}_E$. Since in the definition of GIE (23) we carry out minimization over all CMs $\Gamma_E$, also the new CM $\bar{\Gamma}_E$ runs over all CMs in the course of the minimization. Consequently, minimization with respect to all CMs $\Gamma_E$ of the conditional mutual information calculated from correlation matrix $\sigma_{AB}$ can be replaced with minimization over all CMs $\bar{\Gamma}_E$ and thus we get the same value of GIE irrespective of whether we work with symplectic matrix $S$ or $\bar{S}$, as we set out to prove.

Up to now, GIE was calculated for the following three classes of states with CM (9) [28, 29].

1) **Symmetric GLEMS.** The GLEMS are Gaussian states with least negativity for given global and local purities [41, 42]. Entangled GLEMS satisfy equality $\nu_2 = 1$ if $a$ and $b$ are symmetric they also fulfill condition $a = b$. For all symmetric entangled GLEMS ($\equiv \rho_{AB}^{(1)}$) GIE reads as

$$E_+^G(\rho_{AB}^{(1)}) = \ln \left( \frac{a}{\sqrt{a^2 - b^2}} \right).$$  

(34)

Further, if $k_x = k_p \equiv k$, symmetric GLEMS satisfy $a^2 - k^2 = 1$, they reduce to pure states ($\equiv \rho_{AB}^{(p)}$), and the GIE is given by [28]

$$E_+^G(\rho_{AB}^{(p)}) = \ln(a).$$  

(35)

2) **Symmetric squeezed thermal states** [48]. The squeezed thermal states are characterized by the condition $k_x = k_p \equiv k$. The symmetric squeezed thermal states ($\equiv \rho_{AB}^{(2)}$) further fulfill condition $a = b$, and they are entangled iff $a - k < 1$ [27, 40]. For all entangled symmetric squeezed thermal states satisfying inequality $a \leq 2.41$ GIE is equal to

$$E_+^G(\rho_{AB}^{(2)}) = \ln \left( \frac{(a - k)^2 + 1}{2(a - k)} \right).$$  

(36)

3) **Asymmetric squeezed thermal GLEMS.** The states ($\equiv \rho_{AB}^{(3)}$) satisfy conditions $k_x = k_p \equiv k$ and $\nu_2 = 1$, whereas $a$ and $b$ generally differ. For all the states for which $\sqrt{ab} \leq 2.41$ GIE is given by

$$E_+^G(\rho_{AB}^{(3)}) = \ln \left( \frac{a + b}{|a - b| + 2} \right).$$  

(37)

Previous analytical expressions for GIE can be derived in two steps. The first step consists of calculation of an easier computable upper bound on GIE. In the second step it is shown, that for some fixed measurements on modes $A$ and $B$, the minimum of the
conditional mutual information \( \rho \) over all measurements on subsystem \( E \) saturates the bound. As a consequence, the upper bound is tight and it gives the sought value of the GIE. The derivation of the upper bound for symmetric squeezed thermal states and asymmetric squeezed thermal GLEMS proved to be tractable only for “weakly mixed” states satisfying inequalities \( a \leq 2.41 \) and \( \sqrt{ab} \leq 2.41 \), respectively, which is the cause why formulas \( 36 \) and \( 37 \) are currently known to hold only for states fulfilling the latter inequalities.

In the next section we improve the method of derivation of GIE, which is later used for calculation of GIE for new classes of two-mode Gaussian states. As a byproduct, we get a stronger condition under which formulas \( 36 \) and \( 37 \) are valid thus extending the set of states for which GIE is known.

V. UPPER BOUND ON GIE

A key role in derivation of analytical formulas for GIE given in Eqs. (34), (35), (36) and (37) plays the so called Gaussian classical mutual information (GCMI) of a bipartite Gaussian quantum state. This quantity has been introduced in Ref. [41] by restricting the classical mutual information of a quantum state \( \rho_{AB} \) to Gaussian states and measurements. Here \( I(A;B) = H(A) + H(B) - H(A,B) \) is the classical mutual information of the probability distribution \( P(A,B) = \text{Tr}[(\Pi_A \otimes \Pi_B)\rho_{AB}] \) of outcomes of local measurements \( \Pi_A \) and \( \Pi_B \) on state \( \rho_{AB} \). In this way, one gets for a Gaussian state \( \rho_{AB} \) with CM \( \gamma_{AB} \) the GCMI in the form \( 40 \):

\[
\mathcal{T}_c^G(\rho_{AB}) = \sup_{\Gamma_{A,B}} [I(A;B)],
\]

with

\[
I(A;B) = \frac{1}{2} \ln \left[ \frac{\det(\gamma_A + \Gamma_A)\det(\gamma_B + \Gamma_B)}{\det(\gamma_{AB} + \Gamma_A + \Gamma_B)} \right],
\]

where \( \gamma_{AB} \) are local CMs of \( \gamma_{AB} \).

For a generic two-mode Gaussian state with the standard form CM \( 42 \) optimization in Eq. (39) requires finding of roots of a 12th-order polynomial \( 43 \), which can be done generally only numerically. Nevertheless, for a certain region of parameters \( a, b, c_x \) and \( c_p \) of the CM, the optimization can be performed analytically. Specifically, if the parameters satisfy inequality \( 29 \)

\[
G = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + 1 - \sqrt{ab - c_x^2} \geq 0,
\]

the GCMI reads as

\[
\mathcal{T}_{c,h}^G(\rho_{AB}) = \frac{1}{2} \ln \left( \frac{ab}{ab - c_x^2} \right),
\]

and it is reached by double homodyne detection of quadratures \( x_A \) and \( x_B \) on modes \( A \) and \( B \). Needless to say for completeness, that if an opposite inequality \( G < 0 \) holds, then homodyning is not optimal anymore, and a larger value of GCMI can be obtained, e.g., by double heterodyne detection on modes \( A \) and \( B \), i.e., projection of the modes onto coherent states.

Previous findings about GCMI represent a backbone of the method used in Refs. [28, 29] to evaluate formulas \( 34 \)–\( 37 \) for GIE for various classes of two-mode Gaussian states. The method consists of calculation of an easier computable quantity,

\[
U(\rho_{AB}) = \inf_{\Gamma_E} \left[ \mathcal{T}_c^G(\rho_{AB|E}) \right],
\]

where

\[
\mathcal{T}_c^G(\rho_{AB|E}) = \sup_{\Gamma_{A,B}} [I(A;B|E)]
\]

is the GMI of the conditional state \( \rho_{AB|E} \) with CM \( 24 \), which is an upper bound on GIE as follows from the max-min inequality \( 51 \), \( E_0^G(\rho_{AB}) \leq U(\rho_{AB}) \). In the next step, for some fixed CMs \( \Gamma_A \) and \( \Gamma_B \) we find \( \inf_{\Gamma_E} [I(A;B|E)] \), which saturates the bound, and thus the bound gives us the value of GIE we are looking for,

\[
E_0^G(\rho_{AB}) = U(\rho_{AB}).
\]

A specific feature of all states for which GIE was calculated so far, i.e., states for which Eqs. (34)–(37) hold, is that for any CM \( \Gamma_E \) the optimal measurement on modes \( A \) and \( B \) of the conditional state \( \rho_{AB|E} \) in the standard form, which reaches GMI \( 44 \), is always homodyne detection of quadratures \( x_A \) and \( x_B \). This property in fact makes evaluation of GIE possible and for this reason, in the present paper we will also restrict ourself to states equipped with this property.

To find the condition under which for a given Gaussian state the GMI \( 44 \) is attained by double homodyning for any choice of CM \( \Gamma_E \), one can use inequality \( 45 \). Since the GMI is invariant with respect to local symplectic transformations, the CM of the conditional state \( \rho_{AB|E} \) can be taken in the standard form

\[
\gamma_{AB|E} = \begin{pmatrix} \tilde{a} & 0 & \tilde{c}_x & 0 \\ 0 & \tilde{a} & 0 & \tilde{c}_p \\ \tilde{c}_x & 0 & \tilde{b} & 0 \\ 0 & \tilde{c}_p & \tilde{b} & 0 \end{pmatrix},
\]

where \( \tilde{c}_x \geq |\tilde{c}_p| \geq 0 \). In terms of the parameters of CM \( 46 \) condition \( 47 \) reads as

\[
\tilde{G} = \sqrt{\frac{\tilde{a}}{\tilde{b}}} + \sqrt{\frac{\tilde{b}}{\tilde{a}}} + 1 - \sqrt{\tilde{a}\tilde{b} - \tilde{c}_x^2} \geq 0.
\]

In order for a given Gaussian state \( \rho_{AB} \) the GMI \( 33 \) to be reached by double homodyning, i.e., to be of the form

\[
\mathcal{T}_{c,h}^G(\rho_{AB|E}) = \frac{1}{2} \ln \left( \frac{\tilde{a}\tilde{b}}{\tilde{a}\tilde{b} - \tilde{c}_x^2} \right),
\]
for any CM $\Gamma_E$, inequality (47) has to be fulfilled for any CM $\Gamma_E$. In Ref. [29] it was shown, that if a symmetric squeezed thermal state satisfies inequality $a \leq 2.41$, then inequality (47) holds for any CM $\Gamma_E$. Similarly, in Ref. [29] the validity of inequality (47) for any CM $\Gamma_E$ was also shown for asymmetric squeezed thermal GLEMS fulfilling condition $\sqrt{ab} \leq 2.41$. In what follows, we derive a stronger condition under which inequality (47) is satisfied for any CM $\Gamma_E$ thereby extending the set of states for which GIE is known.

We start with an observation [33, 52], that the CM $\gamma_{AB}$ of the investigated state $\rho_{AB}$, and the CM $\gamma_{AB|E}$ of the conditional state $\rho_{AB|E}$, satisfy inequality $\gamma_{AB} \geq \gamma_{AB|E}$. Further, both matrices appearing in the latter inequality are physical CMs which are positive definite [40, 53], which together with the latter inequality implies that $\det \gamma_{AB} \geq \det \gamma_{AB|E}$ [52]. As a consequence, the following inequality holds:

$$\nu_1^2 \nu_2^2 \geq (\tilde{a} \tilde{b})^2 (\tilde{a} \tilde{b} - c_p^2)^2 \geq (\tilde{a} \tilde{b} - c_p^2)^2.$$  \hspace{1cm} (49)

Here, to get the first inequality we used Eqs. (12) and (46), whereas the second inequality follows from inequality $\tilde{c}_p \geq |c_p|$. By taking finally the fourth root of inequality $\nu_1^2 \nu_2^2 \geq (\tilde{a} \tilde{b} - c_p^2)^2$, we obtain

$$\nu_1 \nu_2 \geq \sqrt{\tilde{a} \tilde{b} - c_p^2}. \hspace{1cm} (50)$$

Matrix inequality $\gamma_{AB} \geq \gamma_{AB|E}$ also imposes a restriction on local symplectic eigenvalues $\tilde{a}$ and $\tilde{b}$ appearing in the standard form of CM $\gamma_{AB|E}$, Eq. (46). Consider now the CM $\gamma_{AB|E}$ of the conditional state $\rho_{AB|E}$ after a Gaussian measurement with a generic CM $\Gamma_E$ on a purifying subsystem $E$ of the state $\rho_{AB}$, expressed with respect to $A|B$ splitting,

$$\gamma_{AB|E} = \left( \begin{array}{cc} A & C^T \\ \phantom{A} & B \end{array} \right), \hspace{1cm} (51)$$

which will not be generally in the standard form [46]. Inequality $\gamma_{AB} \geq \gamma_{AB|E}$ then implies the following inequalities for the local CMs of modes $A$ and $B$, $a \not\geq \tilde{A}$, and $b \not\geq \tilde{B}$ [52], respectively, where $a$ and $b$ are local symplectic eigenvalues of CM $\gamma_{AB}$, Eq. (46). By exactly the same argument which leads to inequality (50) we then have $a^2 \geq \det \tilde{A}$ and $b^2 \geq \det \tilde{B}$, which finally implies the following inequalities

$$a \geq \tilde{a}, \hspace{0.5cm} b \geq \tilde{b}, \hspace{1cm} (52)$$

where we have used equalities $\tilde{a} = \sqrt{\det \tilde{A}}$ and $\tilde{b} = \sqrt{\det \tilde{B}}$.

If we now combine inequalities (50) and (52) with inequality

$$\sqrt{\frac{\tilde{a}}{b}} + \sqrt{\frac{\tilde{b}}{a}} = \frac{\tilde{a} + \tilde{b}}{\sqrt{\tilde{a} \tilde{b}}} \geq 2,$$  \hspace{1cm} (53)

which follows from the inequality of arithmetic and geometric means, we arrive at a new lower bound $\bar{G}_{\text{min}}$ on $\bar{G}$, $\bar{G} \geq \bar{G}_{\text{min}}$, of the form:

$$\bar{G}_{\text{min}} = 2 + \frac{1}{\sqrt{ab}} - \nu_1 \nu_2, \hspace{1cm} (54)$$

where $\nu_{1,2}$ are symplectic eigenvalues (13) of the investigated state $\rho_{AB}$. Hence, if for a two-mode Gaussian state with standard-form CM (9) inequality

$$2 + \frac{1}{\sqrt{ab}} \geq \nu_1 \nu_2 \hspace{1cm} (55)$$

is obeyed, $\bar{G} \geq 0$ for any CM $\Gamma_E$, and homodyne detection of quadratures $x_A$ and $x_B$ on modes $A$ and $B$ is optimal for any $\Gamma_E$. The GCMI then always reads as in Eq. (48), and for symmetric squeezed thermal states as well as asymmetric squeezed thermal GLEMS the GIE is given by Eqs. (50) and (57), respectively.

Note first, that in contrast with the derivation of original inequalities $a \leq 2.41$ and $\sqrt{ab} \leq 2.41$, which utilized a specific structure of states for which they were derived, no similar restrictive assumptions have been made when deriving inequality (54), and thus it holds for any two-mode Gaussian state. Needles to say further, that inequality $\bar{G}_{\text{min}} \geq 0$ provides a strictly stronger condition, i.e., it is satisfied by a strictly larger set of states, than original inequalities. This is a consequence of inequality

$$\sqrt{\nu_1 \nu_2} = \sqrt[4]{\det \gamma_{AB}} = \sqrt[4]{(ab - k_x^2)(ab - k_p^2)} < \sqrt{ab}, \hspace{1cm} (56)$$

where the strict inequality follows from inequality $k_x \geq k_p > 0$ given below Eq. (9). Now, if we combine Eq. (54) with inequality (56) we get

$$\bar{G}_{\text{min}} > 2 \left( 1 - \frac{\sqrt{ab} - \sqrt{ab}}{2} \right). \hspace{1cm} (57)$$

If now $\sqrt{ab} \leq 2.41$, or $a \leq 2.41$ for the case $a = b$, the RHS of inequality (57) is nonnegative, which implies $\bar{G}_{\text{min}} > 0$, and thus condition (55) is satisfied for all states for which the original inequalities hold. Consider now an entangled symmetric two-mode squeezed thermal state with parameters $a = \sqrt{6} \approx 2.45$ and $k = 2$. For this state inequality $a \leq 2.41$ is clearly not satisfied, whereas $\bar{G}_{\text{min}} \geq 0.99 > 0$ still holds, and thus condition (55) is indeed stronger than the original one. Finally, the bound (54) is tight for some classes of states but it is not tight always. For instance, for the class of symmetric two-mode squeezed thermal states $\gamma_{AB}^{(2)}$, the bound boils down to

$$\bar{G}_{\text{min}}^{(2)} = 2 + \frac{1}{a} - \nu, \hspace{1cm} (58)$$

which is tight, because it is reached by dropping the purifying subsystem $E$, or equivalently, by projecting the subsystem onto a product of two infinitely hot thermal
states with CM \( \Gamma^{(n)}_{E} \), where \( \Gamma^{(n)}_{E} = (2(n) + 1)\mathbb{I} \). On the other hand, in the case of symmetric GLEMS one can minimize analytically \( \tilde{G} \), Eq. (17), over all CMs \( \Gamma_{E} \), which yields another lower bound (\( \equiv \tilde{G}_{\text{opt}} \)) of the form \( \tilde{G}_{\text{opt}} \equiv 2 + \frac{1}{a} - \sqrt{a^{2} - k_{x}^{2}} \). (59)

Consider now a mixed symmetric GLEMS, which has to fulfill inequality \( k_{x} > k_{p} \), because equality \( k_{x} = k_{p} \) implies purity of the state. Then, according to the latter inequality and the left-hand side of inequality \( (56) \), we have

\[
\sqrt{\nu_{1}^{2} + \nu_{2}^{2}} = \sqrt{(a^{2} - k_{x}^{2})(a^{2} - k_{p}^{2})} > \sqrt{a^{2} - k_{x}^{2}},
\]  

(60)

where equality \( \nu_{2} = 1 \) was used. Hence, \( \tilde{G}_{\text{min}} < \tilde{G}_{\text{opt}} \), and the lower bound \( (52) \) is not tight.

In this section we have derived a sufficient condition for the GCMI \( (14) \) for a generic two-mode Gaussian state with CM \( (9) \) to be always reached by double homodyne detection. The condition attains a particularly simple form for symmetric two-mode squeezed thermal states, when it simplifies to

\[

\nu \leq 2 + \frac{1}{a}.
\]  

(61)

Because the condition is stronger than the original condition \( a \leq 2.41 \), our finding extends the formula for GIE, Eq. (35), to all symmetric two-mode squeezed thermal states satisfying inequality \( (61) \).

A distinctive feature of condition \( (55) \) is that it is valid for any two-mode Gaussian state. This gives us a prospect that we will be able to calculate GIE even for more generic two-mode Gaussian states, including those states with \( a \neq b \) and simultaneously \( k_{x} \neq k_{p} \). To achieve this goal, we have to be able to perform minimization on the RHS of Eq. (43). By rewriting Eq. (43) as

\[

U(\rho_{AB}) = -\ln \sqrt{1 - h_{\text{min}}},
\]  

(62)

where

\[

h_{\text{min}} = \inf_{I_{E}} \left( \frac{c_{x}^{2}}{ab} \right),
\]  

(63)

we see, that minimization in Eq. (43) is equivalent with minimization on the RHS of Eq. (62). Our ability to carry out the minimization on the RHS of the last formula strongly depends on the structure of the investigated state \( \rho_{AB} \). Previously \( (28, 29) \), this approach proved to be successful in derivation of the upper bound \( (48) \) for symmetric GLEMS and asymmetric two-mode squeezed thermal GLEMS. Later in this paper we show, that the same method can be also used for evaluation of the upper bound on GIE for several new classes of two-mode GLEMS. Before doing that, however, we first briefly explain the last step of the method of calculation of GIE, that is, the saturation of the upper bound.

VI. SATURATION OF THE UPPER BOUND

We now move to the description of a method, which allows us to show for all states investigated here as well as in Ref. \( (24) \), that the conditional mutual information \( (24) \) for homodyne detection of quadratures \( x_{A} \) and \( x_{B} \) on modes \( A \) and \( B \), respectively, which is minimized with respect to all CMs \( \Gamma_{E} \), saturates the upper bound \( (43) \).

For this purpose, it is convenient to express blocks \( \gamma_{AB} \) and \( \gamma_{ABE} \) of CM \( \gamma_{E} \), Eq. (20), as

\[

\gamma_{AB} = \left( \begin{array}{cc} \gamma_{A} & \omega_{AB} \\ \omega_{AB}^{T} & \gamma_{B} \end{array} \right), \quad \gamma_{ABE} = \left( \begin{array}{c} \gamma_{AE} \\ \gamma_{BE} \end{array} \right). \]  

(64)

Next, we apply to the matrix \( \Sigma_{ABE} \equiv \gamma_{E} + \Gamma_{A} \oplus \Gamma_{B} \oplus \Gamma_{E} \) the determinant formula \( (52) \):

\[

\det(M) = \det(\mathfrak{D})\det(\mathfrak{A} - \mathfrak{B} \mathfrak{D}^{-1} \mathfrak{C}),
\]  

(65)

which is valid for any \( (n + m) \times (n + m) \) matrix

\[

M = \left( \begin{array}{ccc} \mathfrak{A} & \mathfrak{B} & \mathfrak{C} \\ \mathfrak{C} \mathfrak{D} & \mathfrak{D} \end{array} \right),
\]  

(66)

where \( \mathfrak{A}, \mathfrak{B} \) and \( \mathfrak{C} \) are respectively \( n \times n, n \times m \) and \( m \times n \) matrices and \( \mathfrak{D} \) is an \( m \times m \) invertible matrix. This allows us to express the determinant of the correlation matrix \( \Sigma_{AB} \) as

\[

\det\sigma_{AB} = \frac{\det(\Gamma_{A} \oplus \Gamma_{B} + \gamma_{AB})\det(\Gamma_{E} + X_{AB})}{\det(\Gamma_{E} + \gamma_{E})},
\]  

(67)

where

\[

X_{AB} = \gamma_{E} - \gamma_{ABE}^{T}(\Gamma_{A} \oplus \Gamma_{B} + \gamma_{AB})^{-1}\gamma_{ABE}.
\]  

(68)

Likewise, application of the determinant formula \( (65) \) to the reduced matrices \( \Sigma_{jE}, j = A, B, \) of subsystem \( (jE) \), yields

\[

\det\sigma_{j} = \frac{\det(\Gamma_{j} + \gamma_{j})\det(\Gamma_{E} + X_{j})}{\det(\Gamma_{E} + \gamma_{E})},
\]  

(69)

where

\[

X_{j} = \gamma_{E} - \gamma_{jE}^{T}(\Gamma_{j} + \gamma_{j})^{-1}\gamma_{jE}.
\]  

(70)

Hence, the conditional mutual information \( (24) \) can be rewritten into the form

\[

I(A; B|E) = I(A; B) + K(E|A; B),
\]  

(71)

where \( I(A; B) \) is given in Eq. (40) and

\[

K(E|A; B) = \frac{1}{2} \ln \mathcal{K},
\]  

(72)

with

\[

\mathcal{K} = \frac{\det(\Gamma_{E} + X_{A})\det(\Gamma_{E} + X_{B})}{\det(\Gamma_{E} + X_{AB})\det(\Gamma_{E} + \gamma_{E})}.
\]  

(73)
The expression of the conditional mutual information given on the RHS of Eq. (71) simplifies its minimization over all CMs \( \Gamma_E \). Consider now homodyne detection of quadratures \( x_A \) and \( x_B \) on modes \( A \) and \( B \), which is described by CMs \( \Gamma_A^t \equiv \text{diag}(e^{-2t}, e^{2t}) \) and \( \Gamma_B^t \equiv \text{diag}(e^{-2t}, e^{2t}) \) in the limit \( t \rightarrow +\infty \). In this case, Eq. (71) boils down to

\[
I_h(A; B|E) = \mathcal{I}_h^c(\rho_{AB}) + K_h(E|A; B),
\]

where \( \mathcal{I}_h^c(\rho_{AB}) \) is obtained from Eq. (42) by replacing \( c_x \) with \( k_x \), and

\[
K_h(E|A; B) = \frac{1}{2} \ln \mathcal{K}_h,
\]

where \( \mathcal{K}_h \) is obtained from the RHS of Eq. (73), by putting \( \Gamma_A = \Gamma_A^t \) and \( \Gamma_B = \Gamma_B^t \) and taking the limit \( t \rightarrow +\infty \).

The remaining step is the minimization of the conditional mutual information (74) over all CMs \( \Gamma_E \), which boils down to finding of the quantity

\[
L(\rho_{AB}) = \inf_{\Gamma_E} [I_h(A; B|E)] = \frac{1}{2} \ln \left( \frac{ab}{ab - k_x^2} \right) + \frac{1}{2} \ln \mathcal{K}_{\min},
\]

where

\[
\mathcal{K}_{\min} = \inf_{\Gamma_E} \mathcal{K}_h,
\]

and where we used Eq. (42) and monotonicity of the logarithmic function. Now, if for some state \( \rho_{AB} \) the quantity (76) is equal to the upper bound (13), we found for fixed measurements on modes \( A \) and \( B \) the minimal conditional mutual information (24) with respect to all CMs \( \Gamma_E \), which cannot be improved, and thus the upper bound coincides with GIE. In what follows we illustrate the utility of this approach for derivation of GIE for several new classes of GLEMS.

### VII. GIE FOR GLEMS

As we have already mentioned, GLEMS are Gaussian states with minimal negativity for fixed global and local purities \( [46, 47] \), and they naturally appear in a cryptographic setting involving two-mode squeezed vacuum \([17]\) with one mode transmitted through a purely lossy channel. Here, we restrict ourself to a subset of GLEMS with CM \([9]\), which is characterized by condition \( a + b - 1 > \sqrt{\det \gamma_{AB}} \) \([40]\), and which is relevant for calculation of GIE because it contains all entangled GLEMS. The GLEMS from the subset, which we call from now simply as GLEMS for brevity, saturate the first of inequalities \([10]\) which express the Heisenberg uncertainty principle, and thus they are states with partial minimum uncertainty. From Eqs. (13) and (14) it follows, that saturation of the first of inequalities \([10]\) is equivalent with equality \( \nu_2 = 1 \), whereas the other symplectic eigenvalue is equal to

\[
\nu \equiv \nu_1 = \sqrt{\det \gamma_{AB}}.
\]

The symplectic matrix which brings CM \([9]\) for GLEMS to Williamson normal form \([12]\) reads as

\[
S = \begin{pmatrix}
x_1 & 0 & x_2 & 0 & 0 & x_3 & 0 & x_4 \\
0 & x_5 & 0 & x_6 & 0 & 0 & x_7 & 0 & x_8
\end{pmatrix},
\]

where the explicit expression of matrix elements \( x_1, x_2, \ldots, x_8 \) in terms of parameters \( a, b, k_x \) and \( k_p \) is given in the Appendix A.

Since we already know GIE for GLEMS with \( \nu = 1 \), which coincide with pure states, symmetric GLEMS with \( a = b \) and \( \nu > 1 \), as well as asymmetric GLEMS with \( a \neq b \) and \( k_x = k_p \), here we focus on derivation of GIE for several other classes of GLEMS with \( \nu > 1 \). Derivation of the symplectic matrix which brings CM \([9]\) to the Williamson normal form performed in Appendix A unveils, that depending on the relation among parameters \( a, b, k_x \) and \( k_p \) we have to distinguish another four sets of GLEMS. This includes GLEMS \((\equiv \rho_{AB}^{(4)})\) with \( a > b \) and \( bk_x = ak_p, \ (\equiv \rho_{AB}^{(5)}) \) with \( a < b \) and \( ak_x = bk_p, \ (\equiv \rho_{AB}^{(6)}) \) with \( a > b \) and \( bk_x \neq ak_p, \) and \((\equiv \rho_{AB}^{(7)}) \) with \( a < b \) and \( ak_x \neq bk_p \). In what follows, we compute an analytical formula for GIE of all states \( \rho_{AB}^{(4)} \) and \( \rho_{AB}^{(5)} \) satisfying condition (55). Moreover, we also outline how to calculate GIE for states \( \rho_{AB}^{(6)} \) and \( \rho_{AB}^{(7)} \) obeying inequality (65), by calculating it explicitly for a particular example of a state \( \rho_{AB}^{(6)} \).

#### A. Upper bound for GLEMS

For evaluation of the upper bound on GIE, Eq. (13), we need to calculate the quantity (55), which requires to express parameters of the standard-form CM \([40]\) as functions of parameters of CM \( \Gamma_E \). Owing to condition \( \nu_2 = 1 \), GLEMS possess unit symplectic rank, \( R = 1 \), and thus their Williamson normal form \([12]\) reads as \( \gamma_{AB}^{(0)} = (\nu I) \oplus I \). In addition, from equation (19) it follows that

\[
\gamma_{AB}^{(0)} = \begin{pmatrix}
\nu^2 - 1 & 0 \\
0 & \frac{1}{\nu} I
\end{pmatrix}, \quad \gamma_E^{(0)} = \nu I,
\]

which reveals that the purifying subsystem \( E \) is single-mode. This allows us to take CM \( \Gamma_E \) appearing in Eq. (28) in the form:

\[
\Gamma_E = P(\varphi) \text{diag}(V_x, V_y) P^T(\varphi),
\]

where

\[
P(\varphi) = \begin{pmatrix}
\cos \varphi & -\sin \varphi \\
\sin \varphi & \cos \varphi
\end{pmatrix}
\]
with \( \varphi \in [0, \pi) \), \( V_x = \tau e^{2i t} \) and \( V_p = \tau e^{-2i t} \), where \( \tau \geq 1 \) and \( t \geq 0 \). By calculating the inverse matrix on the RHS of Eq. (23), and making use of Eqs. (53), together with relation \( PT(\varphi) = \sigma_1 P(\varphi) \sigma_3 \), we get after some algebra CM (27) in the form (29):

\[
\gamma_{AB|E} = S^{-1} (\gamma_{A|E} \otimes I_B) (S^{-1})^T. \tag{83}
\]

Here,

\[
\gamma_{A|E} = \begin{pmatrix} \rho & & \\
0 & V_x & V_y \\
0 & V_y^* & V_x^* \end{pmatrix} P(\varphi) = \begin{pmatrix} V_+ & V_- \cos(2\varphi) & V_- \sin(2\varphi) \\
V_- \cos(2\varphi) & -V_- & V_+ \sin(2\varphi) \\
V_- \sin(2\varphi) & V_+ & -V_- \cos(2\varphi) \end{pmatrix}, \tag{84}
\]

with \( V_{\pm} = (V_x \pm V_p)/2 \), where

\[
V_x = \frac{\nu V_x + 1}{\nu + V_x}, \quad V_p = \frac{\nu V_p + 1}{\nu + V_p}, \tag{85}
\]

\( \nu \geq V_x \geq V_p \geq 1/\nu \), are eigenvalues of CM (83), and

\[
S^{-1} = \begin{pmatrix} x_3 & 0 & x_7 \\
x_0 & x_1 & 0 \\
x_4 & x_5 & 0 \\
x_0 & x_2 & x_6 \end{pmatrix} \tag{86}
\]

is the inverse of symplectic matrix (79), which can be calculated with the help of formula \( S^{-1} = \Omega_2 S^T \Omega_2 \). If we now substitute matrix (56) into the RHS of Eq. (83) and we express the obtained matrix in the block form (51), we can calculate all parameters needed for calculation of the quantity \( h_{min} \), Eq. (63). Below we will see, that all we need are parameters \( \tilde{a}^2, \tilde{b}^2 \) and \( \tilde{c}_x \tilde{c}_p \), which can be obtained from the formulas \( \tilde{a}^2 = \det \tilde{A}, \tilde{b}^2 = \det \tilde{B} \) and \( \tilde{c}_x \tilde{c}_p = \det \tilde{C} \) in the form

\[
\tilde{a}^2 = x_1^2 x_2^2 V_x V_p + x_4^2 x_5^2 \gamma_{11} + x_1^2 x_7^2 \gamma_{22} + x_2^2 x_7^2, \\
\tilde{b}^2 = x_2^2 x_4^2 V_x V_p + x_4^2 x_5^2 \gamma_{11} + x_2^2 x_5^2 \gamma_{22} + x_2^2 x_5^2, \\
\tilde{c}_x \tilde{c}_p = x_1 x_2 x_3 x_4 V_x V_p + x_3 x_4 x_5 x_6 \gamma_{11} + x_1 x_2 x_7 x_8 \gamma_{22} + x_5 x_6 x_7 x_8, \tag{87}
\]

where we have set \( \gamma_{i i} = (\gamma_{A|E})_{i i}, \quad i = 1, 2, \) for the sake of simplicity. To proceed further with calculation of the upper bound (13) we have to express parameters \( x_1, x_2, \ldots, x_8 \) appearing in Eqs. (87) via parameters \( a, b, k_x \) and \( k_p \). This requires to distinguish the following cases:

1. **GLEMS with \( a > b \) and \( bk_x = ak_p \)**

Let us consider first GLEMS \( \rho_{AB}^{(4)} \) with \( a > b \) and \( bk_x = ak_p \). This class of states is relevant from the point of view of calculation of the upper bound on GIE based on formula (48), because there exist physical entangled GLEMS satisfying equation \( bk_x = ak_p \), for which inequality (55) is obeyed. Indeed, consider a matrix (29) with \( a = 2\sqrt{2}, b = k_x = \sqrt{2} \) and \( k_p = 1/\sqrt{2} \), which clearly satisfies equality \( bk_x = ak_p \). The matrix also describes a physical entangled state, because both state conditions (10) as well as entanglement condition (11) are fulfilled. Additionally, one has \( v_2 = \sqrt{\nu - k_x k_p} = 1 \), Eq. (A11) of Appendix A and thus the state is GLEMS. Finally, for the lower bound (54) one gets \( \tilde{C}_{min} = 2.5 - \sqrt{7} \approx 0.873 > 0 \), which implies that double homodyning on modes \( A \) and \( B \) is optimal for the present state, and therefore the upper bound (13) can be calculated by carrying out minimization on the RHS of Eq. (63).

For this purpose, we calculate parameters (51), which attain a particularly simple form. Indeed, from Eq. (A12) of Appendix A one finds that

\[
x_1 = \frac{\sqrt{\nu}}{a}, \quad x_2 = 0, \quad x_3 = \frac{\sqrt{a}}{\nu}, \quad x_4 = \frac{k_x}{\sqrt{\nu a}}, \quad x_5 = \frac{k_p}{\sqrt{b}}, \quad x_6 = \frac{\sqrt{b}}{\sqrt{b}}, \quad x_7 = 0, \quad x_8 = \frac{1}{\sqrt{b}} \tag{88}
\]

where

\[
\nu = \sqrt{a^2 - k_x k_p}, \tag{89}
\]

and hence Eqs. (57) yield

\[
\tilde{a}^2 = V_x V_p + \frac{k_x k_p}{\nu} [V_+ + V_- \cos(2\varphi)], \\
\tilde{b}^2 = 1 + \frac{k_p}{\nu} [V_+ + V_- \cos(2\varphi)], \\
\tilde{c}_x \tilde{c}_p = -\frac{k_x k_p}{\nu} [V_+ + V_- \cos(2\varphi)], \tag{90}
\]

where we used matrix (51) and condition \( bk_x = ak_p \).

Moving to minimization on the RHS of Eq. (63) one can see, that it can be carried out using the following chain of inequalities:

\[
\frac{c_x^2}{ab} \geq -\frac{c_x \tilde{c}_p}{a^2} \geq -\frac{c_x \tilde{c}_p}{\tilde{a}^2} = 1 - \frac{1}{1 + \frac{k_x k_p}{\nu} [V_+ + V_- \cos(2\varphi)]} \geq 1 - \frac{1}{1 + \frac{k_x k_p}{\nu} [V_+ + V_- \cos(2\varphi)]} = \frac{k_x k_p}{a^2}. \tag{91}
\]

Here, the first inequality is a consequence of inequality \( \tilde{c}_x \tilde{c}_p < 0 \), which follows from the third of Eqs. (90), and inequality \( \tilde{c}_x \geq |\tilde{c}_p| \geq 0 \) given below Eq. (40), whereas the second inequality stems from inequality \( \tilde{a} \geq \tilde{b} \) resulting from the first two of Eqs. (90) and the fact that \( V_x V_p \geq 1 \). Further, the third inequality is obtained if we notice that since \( V_x \geq V_p \) one has \( V_- \geq 0 \), and thus the expression in the square brackets is minimized for \( \varphi = \pi/2 \). Finally, as \( V_x \leq \nu \) the last inequality is satisfied, while Eq. (59) has been used to obtain the last equality.

Importantly, the lower bound (51) is tight, because it is reached for CM (51), e.g., with \( \varphi = \pi/2, V_x V_p = 1 \) and
in the limit \( t \to +\infty \), which corresponds to homodyne detection of quadrature \( x_E \) on mode \( E \). As a result, one gets \( \hbar_{\text{min}} = k_x k_p / a^2 \), Eq. (63), which gives after the substitution into the RHS of Eq. (62) the upper bound we are looking for,

\[
U \left( \rho^{(4)}_{AB} \right) = \ln \left( \frac{a}{b} \right).
\]

(92)

2. GLEMS with \( a < b \) and \( ak_x = bk_p \)

Derivation of the upper bound (43) for GLEMS \( \rho^{(5)}_{AB} \) with \( a < b \) and \( ak_x = bk_p \) closely follows derivation performed in previous case. First, it is straightforward to find an example of a state from the considered class of states. Namely, owing to symmetry of conditions (10) and (11) with respect to exchange \( a \leftrightarrow b \) it is obvious, that both the conditions are satisfied also by a CM obtained from CM of previous example by exchanging the values of parameters \( a \) and \( b \), i.e., by a CM (9) with \( b = 2 \sqrt{2} \), \( a = k_x = \sqrt{2} \) and \( k_p = 1/\sqrt{2} \). It is also clear that the new CM fulfills both other conditions \( a < b \) and \( ak_x = bk_p \). Moreover, since states satisfying the latter two conditions possess symplectic eigenvalues

\[
\hat{\nu} \equiv \nu_1 = -\sqrt{k_x k_p}
\]

(93)

and \( \nu_2 = \sqrt{a^2 - k_x k_p} \), Eq. (117) of Appendix A the new CM has the same symplectic eigenvalues as the original CM and thus it describes a GLEMS with a strictly positive lower bound (54). As a result, the new CM is again the sought example of a state from the investigated class of states, for which the upper bound (43) can be obtained by calculating the quantity (63).

Further, making use of equation (A18) of Appendix A one finds the elements of symplectic matrix (80) to be

\[
\begin{align*}
x_1 &= 0, & x_2 &= \sqrt{\frac{\nu}{b}}, & x_3 &= \frac{k_x}{\sqrt{b}}, & x_4 &= \sqrt{\frac{1}{p}}, \\
x_5 &= \sqrt{a}, & x_6 &= -\frac{k_p}{\sqrt{a}}, & x_7 &= \frac{1}{a}, & x_8 &= 0,
\end{align*}
\]

(94)

which gives after substitution into RHS of Eq. (87)

\[
\begin{align*}
\hat{a}^2 &= 1 + \frac{k_x k_p}{\nu} \nu_+ + \nu_- \cos(2\varphi), \\
\hat{b}^2 &= \nu_+ \nu_+ + \frac{k_x k_p}{\nu} \nu_+ \nu_- \cos(2\varphi), \\
\hat{c}_x \hat{c}_p &= -\frac{k_x k_p}{\nu} \nu_+ \nu_- \cos(2\varphi).
\end{align*}
\]

(95)

Comparison of the latter parameters with parameters (90) unveils that the former can be obtained from the latter by replacing \( \nu \) with \( \hat{\nu} \) and exchanging the righthand sides of equations for \( \hat{a}^2 \) and \( \hat{b}^2 \). Repeating the same procedure as that of leading to the lower bound (91) we get the same chain of inequalities as in (91) just with \( \hat{a}^2 \) replaced with \( \hat{b}^2 \) on the RHS of the second inequality, \( \nu \) replaced with \( \hat{\nu} \) in the remaining inequalities, and \( a \) replaced with \( b \) in the final lower bound. Thus one finds the lower bound in the following form:

\[
\frac{\hat{c}_x^2}{\hat{c}_p} \geq \frac{k_x k_p}{\hat{b}^2},
\]

(96)

which is again saturated by homodyne detection of quadrature \( x_E \). Hence, from Eq. (62) we immediately arrive at the upper bound

\[
U \left( \rho^{(5)}_{AB} \right) = \ln \left( \frac{b}{\hat{b}} \right).
\]

(97)

3. Generic GLEMS

Previous method of derivation of the upper bound (43) can be extended to more generic GLEMS the parameters of which do not satisfy any additional condition except for the defining equality \( \nu_2 = 1 \). To illustrate this, we calculate the bound for one example of a state \( \rho^{(6)}_{AB} \) which also gives us a recipe of how to evaluate the bound for some other states \( \rho^{(6)}_{AB} \) and \( \rho^{(7)}_{AB} \).

The example state (\( \equiv \rho^{(6)}_{AB} \)) has a CM (9) with parameters \( a = 2 \sqrt{2} \), \( b = 6 \) and \( k_{x,p} = (\sqrt{97} \pm 1)/8 \). As the first of inequalities (10) is satisfied whereas the second one is fulfilled owing to inequality \( ab - k_x^2 = (79 - \sqrt{97})/32 \approx 2.161 > 1 \), the considered CM describes a physical GLEMS. Further, because \( a > b \) and \( \hat{M} = bk_x - ak_p = (3 - \sqrt{97})/(4\sqrt{2}) \approx -1.211 < 0 \), the state belongs to the class of states \( \rho^{(6)}_{AB} \). Finally, since inequality (11) boils down to inequality \( 7 < 13 \), the GLEMS is entangled, and as \( \hat{C}_{\text{min}} = 5/2 - \sqrt{6} \approx 0.935 > 0 \), Eq. (34), the upper bound (43) can be calculated using formulas (62) and (63).

To calculate the bound we can proceed analogously as in previous two cases. First, we use conditions

\[
\begin{align*}
x_1 x_3 + x_2 x_4 &= 1, & x_1 x_7 + x_2 x_8 &= 0, \\
x_5 x_7 + x_6 x_8 &= 1, & x_3 x_5 + x_4 x_6 &= 0,
\end{align*}
\]

(98)

being a consequence of the symplectic condition (7) with \( N = 2 \), and conditions

\[
\begin{align*}
x_1 x_3 + x_5 x_7 &= 1, & x_2 x_4 + x_6 x_8 &= 1,
\end{align*}
\]

(99)

which follow from equation \( S^{-1} S = 1_4 \). Next, with the help of the conditions we can express the product \( \hat{c}_x \hat{c}_p \), Eq. (87), as

\[
\hat{c}_x \hat{c}_p = x_1 x_3 x_5 x_7 (\nu_+ \nu_+ + 1) - x_2 x_4 x_6 x_8 (\nu_+ \nu_- \cos(2\varphi))
\]

\[
= x_1 x_3 \nu_+ \nu_+ + x_5 x_7 - \hat{a}^2,
\]

(100)

where \( \hat{a}^2 \) is given in Eq. (87). From the explicit form of parameters \( x_1, x_2, \ldots, x_8 \) given in Eq. (A15) of the
where quantities $D, M$ and $	ilde{M}$ are defined in Eqs. (14) and (15), respectively. Consequently, first of Eqs. (100) reveals that $c_x > 0$ and if we take into account condition $c_x > |\tilde{c}_p| \geq 0$, equality $\tilde{a}^2 - \tilde{b}^2 = (x_1x_3 - x_3x_7)(\nu \frac{V_p}{\tilde{c}_p} - 1)$, and the second equation in Eq. (100), we find after some algebra for the quantity $c_x^2/\tilde{a}b$ to be minimized the following lower bound:

$$\frac{c_x^2}{\tilde{a}b} \geq -\frac{\tilde{c}_x\tilde{c}_p}{\tilde{a}b} \left( \left[ 1 + \frac{x_1x_3\nu \frac{V_p}{\tilde{c}_p} + x_5x_7}{\tilde{a}^2 - (x_1x_3\nu \frac{V_p}{\tilde{c}_p} + x_5x_7)} \right] - \frac{1}{\tilde{a}^2 - (x_1x_3\nu \frac{V_p}{\tilde{c}_p} + x_5x_7)} \right) \right) \equiv h. \quad (102)$$

Our goal is now to minimize function $h$ over $\varphi \in [0, \pi)$ and eigenvalues $\nu \frac{V_p}{\tilde{c}_p}$, Eq. (85), such that if $\nu \frac{V_p}{\tilde{c}_p} \in [1, \nu/\kappa]$, then $\nu \frac{V_p}{\tilde{c}_p} \in [1, \nu/\kappa]$, whereas if $\nu \frac{V_p}{\tilde{c}_p} \in [1, \nu/\kappa]$, then $\nu \frac{V_p}{\tilde{c}_p} \in [1, \nu/\kappa]$, and we minimize the quantity $\tilde{a}^2$ with respect to $\nu \frac{V_p}{\tilde{c}_p}$ and $\kappa$. Now, making use of the first of conditions (99), relation $x_1x_3 - x_3x_7 = (a^2 - b^2)/\sqrt{D}$, and inequality $a > b$, one finds that $x_1x_3 > 0$ and hence utilizing Eq. (101) it also follows that $x_5x_7 < 0$. This implies using inequality $\kappa \geq 1$ that $x_1x_3\kappa^2 + x_5x_7 \geq x_1x_3 + x_5x_7 = 1 > 0$. Likewise, as $\kappa < \nu$ one gets for the present state $x_5x_7\kappa^2 + x_1x_3 \geq x_5x_7\nu^2 + x_1x_3 = 1/2 > 0$ and both second terms in square brackets on the RHS of Eq. (102) are positive. Obviously, both the terms depend on variables $\varphi$ and $z$ only through the parameter $\tilde{a}^2 = \tilde{a}^2(\varphi, z, \kappa)$ and because they are both positive, minimization of function $h = h(\varphi, z, \kappa)$, Eq. (102), with respect to the variables can be performed by minimization of the parameter $\tilde{a}^2$. By substituting for elements $\gamma_{11}$ and $\gamma_{22}$ from Eq. (84) into the RHS of expression for $\tilde{a}^2$, Eq. (57), we get

$$\tilde{a}^2(\varphi, z, \kappa) = x_1^2x_3^2\kappa^2 + x_5^2x_7^2 + (x_1^2x_3^2 + x_5^2x_7^2)\nu_+ + (x_1^2x_3^2 - x_5^2x_7^2)\nu_- \cos(2\rho). \quad (103)$$

where parameters $\nu_\pm$ are defined below Eq. (84). The minimization of $\tilde{a}^2$ with respect to $\nu$ is now straightforward. Namely, if we note that $x_1^2x_3^2 - x_5^2x_7^2 = ab(k_x^2 - k_z^2)/\sqrt{\det\gamma_{AB}} > 0$ and $\nu_- \geq 0$, we immediately see, that the minimum is reached for $\varphi = \pi/2$ and it reads as

$$\tilde{a}^2(\pi/2, z, \kappa) = x_1^2x_3^2\kappa^2 + x_5^2x_7^2 + \kappa\left(x_1^2x_3^2z + x_5^2x_7^2z\right). \quad (104)$$

It is also easy to minimize the latter quantity with respect to variable $z$. By solving extremal equation

$$\partial^2 \left( \frac{\pi}{2}, z, \kappa \right) / \partial z = 0,$$
which is strictly larger than the minimum \( h^{(1)}_{\text{min}} \), Eq. (108). As a result, the minimum \( h^{(1)}_{\text{min}} \) coincides with the sought minimal value \( h_{\text{min}} \), Eq. (63), which gives after the substitution into the formula (62) the upper bound

\[
U \left( \rho_{AB}^{(6)} \right) = \ln \left( \frac{6}{5} \right).
\]

(111)

The results of the present subsection show, that it is possible to calculate the upper bound (43) even for a generic GLEMS which does not possess any further symmetry. Although we have derived the bound for a particular state, in the course of the derivation we just used inequalities \( a > b, M < 0, z_1 < \nu \) and \( x_3 x_7 b^2 + x_1 x_3 > 0 \). Therefore, for all states \( \rho_{AB}^{(6)} \) which are entangled GLEMS satisfying condition (55) and the latter inequalities, the quantity \( h_{\text{min}} \), Eq. (63), is equal to \( h_{\text{min}} = -4 M M/(a^2 - b^2)^2 \), Eq. (108), and the upper bound (43) then reads as follows:

\[
U \left( \rho_{AB}^{(6)} \right) = \ln \left( \frac{a^2 - b^2}{\sqrt{D}} \right).
\]

(112)

Note finally, that one can expect that a straightforward modification of previous procedure would allow us to derive the upper bound (43) also for a subclass of states \( \rho_{AB}^{(6)} \), which in addition to condition \( a > b \), satisfy inequality \( M < 0 \) as well as respective analogies of other inequalities needed for derivation of formula (112). While this programme is deferred for further research, in the following section we show, that for all states investigated in the present section the formulas for the upper bound (43) in fact coincide with the GIE.

B. Saturation of the upper bound for GLEMS

Let us start with observation, that from Eqs. (21), (80) and (86) it follows that \( \gamma_E = \nu \mathbb{1} \), whereas for blocks \( \gamma_{AE} \) and \( \gamma_{BE} \) of matrix \( \gamma_{AB} \), Eq. (63), one gets

\[
\gamma_{AE} = \sqrt{\nu^2 - 1} \begin{pmatrix} x_3 & 0 \\ 0 & -x_1 \end{pmatrix}, \\
\gamma_{BE} = \sqrt{\nu^2 - 1} \begin{pmatrix} x_4 & 0 \\ 0 & -x_2 \end{pmatrix}.
\]

(113)

Substituting the latter matrices into Eqs. (63) and (70), setting \( \Gamma_A = \Gamma_A^\nu = \text{diag}(e^{-2t}, e^{2t}) \) and \( \Gamma_B = \Gamma_B^\nu = \text{diag}(e^{-2t}, e^{2t}) \), and performing the limit \( t \to +\infty \), one finds after some calculations that the matrices \( X_{AB}, X_A \) and \( X_B \) attain the same form

\[
X_k = \nu \mathbb{1} - \alpha_k |0\rangle \langle 0|,
\]

(114)

\( k = A, B, AB \), where

\[
\alpha_A = \left( \frac{\nu^2 - 1}{a} \right) x_3^2, \quad \alpha_B = \left( \frac{\nu^2 - 1}{b} \right) x_4^2, \\
\alpha_{AB} = \left( \frac{\nu^2 - 1}{ab - k_2^2} \right) (a x_3^2 + b x_4^2 - 2k_2 x_3 x_4).
\]

(115)

and \( |0\rangle = (1,0)^T \). By substituting from Eq. (114) for matrices \( X_A, X_B \) and \( X_{AB} \) into the RHS of Eq. (73), and using the formula (54)

\[
\det(\mathcal{X} + |c\rangle \langle r|) = \left( 1 + \langle r|\mathcal{X}^{-1}|c\rangle \right) \det \mathcal{X},
\]

(116)

which is valid for any invertible matrix \( \mathcal{X} \), we arrive at the following simple expression for quantity (73),

\[
\mathcal{X}_h = \frac{(1 - \alpha_A Q)(1 - \alpha_B Q)}{(1 - \alpha_{AB} Q)},
\]

(117)

where \( Q = \langle 0| (\Gamma_E + \nu \mathbb{1})^{-1} |0\rangle \). By calculating the inverse matrix \( (\Gamma_E + \nu \mathbb{1})^{-1} \), we further get

\[
Q = \frac{(\Gamma_E)_{22} + \nu}{\det(\Gamma_E + \nu \mathbb{1})}.
\]

(118)

Let us now express CM \( \Gamma_E \) as in Eq. (34), which can be further rewritten in analogy with Eq. (31) as

\[
\Gamma_E = \begin{pmatrix} V_+ + V_- \cos(2\varphi) & V_- \sin(2\varphi) \\
V_- \sin(2\varphi) & V_+ - V_- \cos(2\varphi) \end{pmatrix}
\]

(119)

with

\[
V_+ = \frac{V_x + V_y}{2} = \tau \cosh(2t), \\
V_- = \frac{V_x - V_y}{2} = \tau \sinh(2t).
\]

(120)

Inserting from here for \((\Gamma_E)_{22}\) into the RHS of Eq. (118) and taking into account that

\[
\det(\Gamma_E + \nu \mathbb{1}) = \tau^2 + 2 \tau \cosh(2t) \nu + \nu^2,
\]

(121)

one finds the variable \( Q \) appearing on the RHS of Eq. (117) is equal to

\[
Q = \frac{\tau\cosh(2t) - \sinh(2t) \cos(2\varphi) + \nu}{\tau^2 + 2 \tau \cosh(2t) \nu + \nu^2}.
\]

(122)

For evaluation of the quantity (76) it remains to perform minimization on the RHS of Eq. (77). This can be done by minimization of the quantity (117), where \( Q \) is given in Eq. (122) over \( \varphi \in [0, \pi] \), \( \tau \geq 1 \) and \( t \geq 0 \). In fact, the minimization can be greatly simplified. Namely, if we look on the RHS of Eq. (117) we see, that it is a function of just a single variable \( Q \). Thus, if we find the interval of values in which the variable \( Q \) may vary, it is sufficient to minimize the quantity (117) with respect to the single variable \( Q \) on the found interval. Provided that the minimum lies at some point \( Q_{\text{min}} \) which can be attained for some admissible values \( \varphi_{\text{min}} \in [0, \pi] \), \( \tau_{\text{min}} \in [1, +\infty) \) and \( t_{\text{min}} \in [0, +\infty) \), we get the sought optimized quantity (77).

The latter interval is easy to find with the help of the following inequalities:

\[
0 < \frac{1}{V_x + \nu} \leq Q \leq \frac{1}{V_p + \nu} < \frac{1}{\nu}.
\]

(123)
Here, the inner inequalities follow from the fact that $Q$ lies between the least eigenvalue $1/(V_x + \nu)$ and the largest eigenvalue $1/(V_p + \nu)$ of matrix $(\Gamma_E + \nu \mathbb{1})^{-1}$, which are easy to find using the expression of CM $\Gamma_E$ given in Eq. (51). The outer inequalities represent the lower and the upper bound on the least and largest eigenvalue, which is reached in the limit $V_x \to +\infty$ and for $V_p = 0$, respectively. Instead of carrying out minimization in Eq. (77), we thus calculate the quantity

$$K_{\text{min}} = \inf_{Q \in (0, \nu)} \mathcal{K}_h, \quad (124)$$

which requires to compare the values of $\mathcal{K}_h$ at stationary points lying in the interval $(0, 1/\nu)$ as well as at the boundary points 0 and $1/\nu$ of the interval and find the least value.

To proceed further with evaluation of the quantity (124), we need to know the expression of parameters $\alpha_A, \alpha_B$ and $\alpha_{AB}$, Eq. (115), in terms of parameters $a, b, k_x$ and $k_p$. In analogy with previous section, we again analyze each of the considered types of GLEMS separately.

1. **GLEMS with $a > b$ and $bk_x = ak_p$**

Let us now move to calculation of the quantity $L$, Eq. (76), for states $\rho_{(4)}^{AB}$. By taking from Eq. (58) explicit expressions for parameters $x_3$ and $x_4$ and substituting them into Eq. (115), we arrive after some algebra at

$$\alpha_A = \alpha_{AB} = \frac{\nu^2 - 1}{\nu}, \quad \alpha_B = \left(\frac{\nu^2 - 1}{\nu}\right) \frac{k_x^2}{ab}, \quad (125)$$

where $\nu$ is the symplectic eigenvalue defined in Eq. (59).

Owing to equality $\alpha_A = \alpha_{AB}$ the quantity to be minimized, Eq. (117), reduces to the following simple form:

$$\mathcal{K}_h = 1 - \alpha_B Q. \quad (126)$$

Since $\alpha_B > 0$ in the present case, previous function is monotonically decreasing function of $Q$ attaining minimum of

$$K_{\text{min}} = 1 - \left(\frac{\nu^2 - 1}{\nu}\right) \frac{k_x^2}{ab}, \quad (127)$$

at the boundary point $1/\nu$. The point is reached, e.g., for the measurement with CM (51), where $\varphi = \pi/2, \tau = 1$ and in the limit for $t \to +\infty$, which is homodyne detection of quadrature $x_E$ on purifying mode $E$. As a consequence, the quantity (127) coincides with the quantity $\mathcal{K}_{\text{min}}$, Eq. (177), and it gives after substitution into Eq. (76) and some algebra

$$L\left(\rho_{(4)}^{AB}\right) = \frac{1}{2} \ln \left(\frac{a^2}{a^2 - k_x k_p}\right) = \ln \left(\frac{a}{\nu}\right). \quad (128)$$

Comparison of the latter quantity with the upper bound (92) reveals, that they are equal. Thus we have shown, that for the class of states considered here the upper bound on GIE is reached by conditional mutual information (41) for distribution of outcomes of homodyne detections of quadratures $x_A$ and $x_B$ of modes $A$ and $B$, which is minimized over all measurements on mode $E$.

This implies that for GLEMS with $a > b$ and $bk_x = ak_p$, which satisfy condition (55), GIE is given by

$$E^{G}_\psi\left(\rho_{AB}^{(4)}\right) = \ln \left(\frac{a}{\nu}\right). \quad (129)$$

In particular, for the state with $a = 2\sqrt{2}, b = k_x = \sqrt{2}$ and $k_p = 1/\sqrt{2}$, formula (129) yields $E^{G}_\psi(\rho_{AB}^{(4)}) = \ln(2\sqrt{2/7}) = 0.067$.

2. **GLEMS with $a < b$ and $ak_x = bk_p$**

Let us now investigate states $\rho_{AB}^{(5)}$, i.e., GLEMS satisfying conditions $a < b$ and $ak_x = bk_p$. From Eq. (91) one finds easily that the parameters (115) read as

$$\alpha_A = \left(\frac{\nu^2 - 1}{\nu}\right) \frac{k_x^2}{ab}, \quad \alpha_B = \alpha_{AB} = \frac{\nu^2 - 1}{\nu}, \quad (130)$$

and hence the quantity (117) boils down to

$$\mathcal{K}_h = 1 - \alpha_A Q. \quad (131)$$

By minimizing the RHS with respect to $Q$ on the interval $(0, 1/\nu)$ and taking into account inequality $\alpha_A > 0$ one finds immediately the optimized quantity (177) to be

$$K_{\text{min}} = 1 - \left(\frac{\nu^2 - 1}{\nu^2}\right) \frac{k_x^2}{ab}, \quad (132)$$

and it is again reached if Eve carries out homodyne detection of quadrature $x_E$ on her mode $E$. If we now insert the latter quantity into the RHS of Eq. (76), we get

$$L\left(\rho_{AB}^{(5)}\right) = \frac{1}{2} \ln \left(\frac{b^2}{b^2 - k_x k_p}\right) = \ln \left(\frac{b}{\nu}\right). \quad (133)$$

Hence we see again, that the conditional mutual information (41) for fixed homodyne detections of quadratures $x_A$ and $x_B$, which is minimized over all CMs $\Gamma_E$, Eq. (76), saturates the upper bound (97) and thus the GIE for states $\rho_{AB}^{(5)}$ satisfying condition (55) is given by

$$E^{G}_\psi\left(\rho_{AB}^{(5)}\right) = \ln \left(\frac{b}{\nu}\right). \quad (134)$$

Since the example of a state $\rho_{AB}^{(5)}$ with $b = 2\sqrt{2}, a = k_x = \sqrt{2}$ and $k_p = 1/\sqrt{2}$ differs from previous example just by an exchange of the values of $a$ and $b$ and the same holds also for formulas (129) and (134), we get again $E^{G}_\psi(\rho_{AB}^{(5)}) = 0.067$. 
3. Generic GLEMS

Like in the case of upper bound (13) derived in Subsec. VII A, also the method of derivation of the quantity (70) presented in previous two subsections can be extended to more generic GLEMS. Although we again illustrate this on one concrete state investigated in Subsubsec. VII A 3, we first carry out the derivation in full generality and the concrete values of the parameters \(a, b, k_x\) and \(k_p\) are substituted into the final formulas only at the end of our calculations. This implies, that most of the results presented here do not hold only for the considered state but they can be straightforwardly used to derive the quantity (70) also for other generic GLEMS.

In the general case GLEMS with \(a > b\) (a \(b\) satisfy \(bk_x \neq ak_p\) (\(ak_x \neq bk_p\)), and the function (117) has two generally different stationary points. They can be obtained as solutions of the extremal equation \(d\mathcal{K}_h/d\mathcal{Q} = 0\), which is equivalent with the following quadratic equation:

\[
\alpha_A\alpha_B\alpha_{AB}Q^2 - 2\alpha_A\alpha_B Q + \alpha_A + \alpha_B - \alpha_{AB} = 0, \quad (135)
\]

and which possesses the following two solutions:

\[
Q_{1,2} = \frac{1}{\alpha_{AB}} \left[ 1 \pm \sqrt{\left( \frac{\alpha_{AB}}{\alpha_A} - 1 \right) \left( \frac{\alpha_{AB}}{\alpha_B} - 1 \right)} \right]. \quad (136)
\]

Using formulas (115) we can now write the ratios in the round brackets on the RHS of Eq. (136) as

\[
\frac{\alpha_{AB}}{\alpha_A} = 1 + x^2, \quad \frac{\alpha_{AB}}{\alpha_B} = 1 + y^2, \quad (137)
\]

where we introduced

\[
x \equiv \frac{a}{\sqrt{ab-b^2}} \left( \frac{x_4}{x_3} - \frac{k_x}{a} \right), \quad y \equiv \frac{b}{\sqrt{ab-b^2}} \left( \frac{x_3}{x_4} - \frac{k_x}{b} \right). \quad (138)
\]

For states with \(a > b\) and \(bk_x \neq ak_p\) (\(ak_x \neq bk_p\)) we can further substitute here from relation \(x_4/x_3 = -M/L_1\), Eq. (A15) of Appendix A \((x_4/x_3 = M/L_2\), Eq. (A19) of Appendix A), where \(L_1\) and \(L_2\) are quantities defined in Eq. (A3) of Appendix A. This allows us to express parameters \(a, b, k_x\) and \(k_p\), in terms of parameters \(a, b, k_x\) and \(k_p\).

By introducing parameters \(x, y\) we simplified stationary points (136) to \(Q_{1,2} = (1 + |xy|)/\alpha_{AB}\). With the help of the latter formulas together with Eq. (137) we finally express the value of quantity (117) in the stationary points as

\[
\mathcal{K}_h(Q_{1,2}) = \frac{(|x| + |y|)^2}{(1 + x^2)(1 + y^2)}. \quad (139)
\]

To get the optimized quantity (124) we now have to identify which of the stationary points \(Q_{1,2}\) lies in the interval \((0, 1/\nu)\). By comparing the values (139) for the stationary points lying in the interval with the values of the quantity (117) in the boundary points, \(\mathcal{K}_h(0) = 1\) and

\[
\mathcal{K}_h\left(\frac{1}{\nu}\right) = \frac{(\nu - \alpha_A)(\nu - \alpha_B)}{\nu(\nu - \alpha_{AB})}, \quad (140)
\]

and selecting the least value, we get the quantity (124). If one can further find parameters \(\varphi, \tau\) and \(t\) of CM \(\Gamma_E\) giving the value of \(Q\) at which the least value is reached, the quantity (124) coincides with the quantity (77) we are looking for.

By applying previous algorithm to the state \(\tilde{\rho}_A^{(6)}\) of Subsubsec. VII A 3 with \(a = 2\sqrt{2}, b = \sqrt{2}\) and \(k_{x,p} = (\sqrt{97} \pm 1)/8\) we get

\[
Q_1 = 0.578 > \frac{1}{\nu} = 0.408 > Q_2 = 0.402 \quad (141)
\]

and

\[
\mathcal{K}_h(Q_2) = \frac{9}{800}(79 - \sqrt{97}) \approx 0.7780, \quad \mathcal{K}_h\left(\frac{1}{\nu}\right) = \frac{3169 - 79\sqrt{97}}{3072} \approx 0.7783, \quad (142)
\]

which implies that \(\mathcal{K}_{\min} = \mathcal{K}_h(Q_2)\). Further, the value of \(Q_2\) can be reached by a CM \(\Gamma_E\), Eq. (81), with \(\varphi = \pi/2, \tau = 1\) and \(t = 1.613\), which corresponds to a projection onto a pure state squeezed in quadrature \(x_E\) with finite squeezing. Consequently, it holds that \(\mathcal{K}_{\min} = \mathcal{K}_h(\tilde{\rho}_A^{(6)})\), the quantity (129) is equal to \(L(\tilde{\rho}_A^{(6)}) = \ln(6/5)\) and it coincides with the upper bound (111). Hence, GIE for the state \(\tilde{\rho}_A^{(6)}\) is given by

\[
E_G(\tilde{\rho}_A^{(6)}) = \ln \left( \frac{6}{5} \right). \quad (143)
\]

Needles to say finally, that the quantity \(L(\tilde{\rho}_A^{(6)})\), Eq. (76), represents at fixed homodyne detections of quadratures \(x_A\) and \(x_B\) on modes \(A\) and \(B\) the least conditional mutual information over all Gaussian measurements on mode \(E\), and thus it gives a lower bound on GIE. In the course of derivation of GIE for symmetric GLEMS and asymmetric squeezed thermal GLEMS performed in Ref. [29], equality of the upper bound (145) given by the RHS of Eqs. (54) and (57) to the latter minimal conditional mutual information was utilized, without proving it explicitly. In Appendix B we prove the equality by showing equality of the quantity \(L\) to the RHS of formulas (54) and (57), thereby complementing derivation of GIE for symmetric GLEMS and asymmetric squeezed thermal GLEMS.

VIII. RELATION TO OTHER ENTANGLEMENT MEASURES

An important question which has to be addressed for any entanglement quantifier is its relation to other know...
entanglement quantifiers. Here, we first study whether there is a connection between GIE and the most popular easily computable logarithmic negativity [18–20]. Next, we move to analysis of relation of GIE and the GR2EoF [32, 33].

A. Relation to logarithmic negativity

The obtained formulas for GIE of symmetric states can be compactly written as [20]

$$E^G_{\frac{1}{2}}(\rho_{AB}) = \left\{ \begin{array}{l} \ln \left[ \frac{\tilde{\nu} - (\tilde{\nu}^{-1})^{-1}}{2} \right], \text{ if } \tilde{\nu} < 1; \\ 0, \text{ if } \tilde{\nu} \geq 1, \end{array} \right. \quad (144)$$

where $\tilde{\nu} = \sqrt{(a - k_x)(a - k_y)}$ is the lower symplectic eigenvalue of the partial transpose of the investigated state $\rho_{AB}$. Hence we see, that for considered symmetric states the GIE is a monotonically decreasing function of the symplectic eigenvalue $\tilde{\nu}$ and thus in this respect it stays in line with other frequently used Gaussian entanglement measure called logarithmic negativity [18–20] which is defined as $E_N(\rho_{AB}) = \max[0, -\ln(\tilde{\nu})]$. However, for asymmetric states this rule is violated as it is witnessed by the formula (144), which cannot be rewritten as a function solely of the symplectic eigenvalue $\nu = \sqrt{a + b - \sqrt{(a + b)^2 - 4\nu^2}}/2$, where $\nu = 1 + |a - b|$.

Further comparison of GIE with logarithmic negativity unveils that $E_N(\rho_{AB}) \geq E^G_{\frac{1}{2}}(\rho_{AB})$ in the case of symmetric states and thus one might be tempted to think, that there is a fixed hierarchy between the two quantifiers which holds for all bipartite Gaussian states. While this conjecture might be true for two-mode Gaussian states, it is false in general because $E_N(\rho_{PPT}^{AB}) = 0$ for Gaussian entangled states with partial transpose ($\equiv \rho_{PPT}^{AB}$) [55], whereas $E^G_{\frac{1}{2}}(\rho_{PPT}^{AB}) > 0$ due to the faithfulness property.

B. Relation to Gaussian Rényi-2 entanglement of formation

A systematic investigation of GIE carried out in Refs. [28, 29] revealed a remarkable relation of the quantity to GR2EoF [32, 33]. Concretely, it was shown, that formulas for GIE for symmetric GLEMS, Eq. (94), symmetric squeezed thermal states, Eq. (93), as well as asymmetric squeezed thermal GLEMS, Eq. (97), coincide with the formulas for GR2EoF. Based on this observation, in Ref. [29] a conjecture has been expressed that GIE and GR2EoF are equal on all bipartite Gaussian states.

In this section we further strengthen the conjecture by showing, that also expressions (129), (134) and (143) for GIE of GLEMS $\rho_{AB}^{(4)}$, $\rho_{AB}^{(5)}$, as well as an example state $\rho_{AB}^{(6)}$, are again equal to GR2EoF.

Recall, that GR2EoF ($\equiv E^G_{\frac{1}{2}}$) for a two-mode reduction of a pure state of three modes $A_1, A_2$ and $A_3$ can be calculated from the following standard-form CM of the state [56]:

$$\gamma_{A_1, A_2, A_3} = \left( \begin{array}{ccc} a_1 & 0 & c_2^- \\ 0 & a_1 & 0 \\ c_2^+ & 0 & 0 \end{array} \right), \quad (145)$$

with

$$c_2^\pm = \frac{\sqrt{\sqrt{a_2} - a_2^2 + \pm \sqrt{a_2} - a_2^2}}{4\sqrt{a_2}}. \quad (146)$$

where

$$a_\pm = (a_1 + 1)^2 - (a_j - a_k)^2, \quad (147)$$

and $|a_j - a_k| + 1 \leq a_j \leq a_j + a_k - 1$, where $\{i, j, k\}$ run over all possible permutations of $\{1, 2, 3\}$. The GR2EoF of a reduced state $\rho_{A_iA_j}$ of modes $A_i$ and $A_j$ with CM $\gamma_{A_i, A_j}$ is given by [32]

$$E^G_{\frac{1}{2}}(\rho_{A_iA_j}) = \frac{1}{2} \ln g_k, \quad (148)$$

where [57]

$$g_k = \left\{ \begin{array}{ll} 1, & \text{if } a_k \geq \sqrt{a_i^2 + a_j^2 - 1}; \\ \frac{a_i}{a_i^2}, & \text{if } a_k < \sqrt{a_i^2 + a_j^2 - 1}; \\ \left(\frac{a_i^2 - a_j^2}{a_i^2 - a_k^2}\right)^\frac{1}{2}, & \text{if } a_k \leq a_k. \end{array} \right. \quad (149)$$

Here,

$$\alpha_k = \left\{ \begin{array}{l} 1, \quad \text{if } a_k \geq \sqrt{a_i^2 + a_j^2 - 1}; \\ \frac{a_i^2 - a_j^2}{a_i^2 + a_k^2}, \quad \text{if } a_k < \sqrt{a_i^2 + a_j^2 - 1}; \\ \frac{a_i^2 - a_j^2}{a_i^2 - a_k^2}, \quad \text{if } a_k \leq a_k. \end{array} \right. \quad (150)$$

At the outset, we calculate GR2EoF for states $\rho_{AB}^{(4)}$. In this case, from Eqs. (94), (143) and the fact that $\gamma_E = \nu I$, we get immediately the parameters of the pure-state CM (145), which are needed for evaluation of GR2EoF, in the form $a_1 = a, a_2 = b$ and $a_3 = \nu$. According to Eq. (148) the GR2EoF is equal to $E^G_{\frac{1}{2}}(\rho_{AB}^{(4)}) = \ln(g_3)/2$, where $g_3$ is obtained from Eq. (149) for $k = 3$, and where we identified modes as $A = A_1, B = A_2$ and $E = A_3$. Since $\sqrt{a_i^2 + a_j^2 - 1} = \sqrt{a_i^2 + k_xk_p} > \sqrt{a_i^2 - k_xk_p} = \nu = a_3$, where the first equality follows from condition
\[ \nu_2 = \sqrt{b^2 - k_x k_y} = 1 \] and the inequality is a consequence of the assumption \( k_x k_y > 0 \), the first branch of Eq. (149) never applies. The latter two conditions further imply, that \( b > 1 \) and thus \( (a^2 + b^2)(b^2 - 1) > 0 \) which is equivalent with inequality

\[ \frac{a^2 - b^2 + \sqrt{(a^2 - b^2)^2 + 8(a^2 + b^2)}}{2(a^2 + b^2)} < 1 \quad (151) \]

being further equivalent with inequality \( \alpha_3 < \alpha_3 \), and thus only second branch in Eq. (149) applies. Next, after some algebra one finds that \( \sqrt{\delta} = 4(a^2 - b^2)(b^2 - 1) \) and consequently \( \zeta = 8a^2 \), where the condition \( a > b \) has been used. As a result, the GR2EoF for states \( \rho_{AB}^{(4)} \) is equal to

\[ E_{F,2}^G \left( \rho_{AB}^{(4)} \right) = \ln \left( \frac{a}{p} \right) \quad (152) \]

By comparing the latter formula with GIE for the same class of states, Eq. (129), one has seen that like in many other cases the two quantities coincide.

Derivation of GR2EoF for the class of states \( \rho_{AB}^{(5)} \) can be performed exactly in the same way as in the previous case. As for parameters of CM (135) one has \( a_1 = a, a_2 = b \) and \( \alpha_3 = \tilde{\nu} \) and condition \( \nu_2 = \sqrt{a^2 - k_x k_y} = 1 \) is obeyed, it is obvious that \( \sqrt{a_1^2 + a_2^2 - 1} = \sqrt{b^2 + k_x k_y} > \sqrt{b^2 - k_x k_y} = \tilde{\nu} = \alpha_3 \) and the first branch of Eq. (149) never occurs. Additionally, since inequality obtained by exchanging \( a \) and \( b \) in inequality (151) is fulfilled for states \( \rho_{AB}^{(5)} \), we have \( \alpha_3 < \alpha_3 \) for the states and thus one always has to take the second branch of Eq. (149).

Finally, because \( \sqrt{\delta} = 4(b^2 - a^2)(a^2 - 1) \) and \( \zeta = 8b^2 \), the GR2EoF is equal to

\[ E_{F,2}^G \left( \rho_{AB}^{(5)} \right) = \ln \left( \frac{b}{p} \right) \quad (153) \]

according to Eq. (148). A quick look at formula (134) uncovers again that equality of GIE and GR2EoF holds also for the class of states \( \rho_{AB}^{(5)} \).

We conclude this section by calculating the GR2EoF for the state \( \tilde{\rho}_{AB}^{(6)} \) from Subsubsection VII A 3. In this case, one has \( a = 2\sqrt{2}, b = \sqrt{2} \) and \( k_x, k_y = (\sqrt{97} \pm 1)/8 \), whence the relevant parameters of CM (145) are given by \( a_1 = a, a_2 = b \) and \( \alpha_3 = \nu = \sqrt{5} = 2.45 \). Further, one finds out that \( \alpha_3 < \alpha_3 = \sqrt{(14 + 3\sqrt{20})/5} = 2.456 \) and thus the third branch of Eq. (149) has to be taken. Finally, according to Eq. (148) the GR2EoF is equal to

\[ E_{F,2}^G \left( \tilde{\rho}_{AB}^{(6)} \right) = \ln \left( \frac{6}{5} \right) \quad (154) \]

Comparing the result with Eq. (148) one can see that GIE and GR2EoF are equal also for this state.

The results presented in this subsection reveal, that also in the case of states \( \rho_{AB}^{(4)}, \rho_{AB}^{(5)} \) as well as \( \tilde{\rho}_{AB}^{(6)} \) GIE coincides with GR2EoF. This even strengthens already a strongly supported conjecture about equivalence of the two quantities.

\section*{IX. Conclusions}

Entanglement quantification based on tripartite extensions of quantified states as embodied by squashed entanglement [23] proves to be currently most successful. This is not only because this approach allows to fulfill all axioms imposed on a good entanglement measure [24], but also due to the fact that the cryptographic nature of the underlying scenario may give the quantifier a cryptographic meaning [58].

In this paper, we developed a theory of another representative of such quantifiers called GIE, which was initiated in Refs. 28 29. The GIE is in fact a Gaussian relative of squashed entanglement because it is defined as optimized intrinsic information being the mother of squashed entanglement.

First, we have shown, that the analytic formulas for GIE derived in Ref. 29 hold for larger classes of states than previously thought. Second, we have derived analytical expressions for GIE for two new classes of two-mode Gaussian states with partial minimum uncertainty, which have certain symmetry. Moreover, by deriving GIE for one particular state we have demonstrated, that it can be calculated also for generic partial minimum-uncertainty states which possess no further symmetry. Finally, we have proved, that like for all states studied in Ref. 29, also for the new states investigated here the GIE is always equal to the GR2EoF. In view of our results we think, that equality of GIE and GR2EoF conjectured in Ref. 29 is very likely. The proof of this conjecture, which would equip Gaussian entanglement theory with a unique entanglement measure, is left for further research.

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\section*{Appendix A: Symplectic diagonalization}

In this section we derive for any CM [9] an explicit expression for a symplectic matrix \( S \) which brings the CM to the Williamson normal form [12]. This can be done either using a method of Ref. 13 or a method of Ref. 14. In the first method [13] we seek matrix \( S \) in the form of a product \( S = (\oplus_{i=1}^N U_i^*) V^T \), where

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ i & i \end{pmatrix} \quad (A1) \]

and \( V \) contains in its columns the eigenvectors of the matrix \( \Omega_{\gamma AB} \), which are chosen such that \( S \) is real and it satisfies the symplectic condition (17) with \( N = 2, \Omega_{\gamma} S^T = \Omega_{\gamma} \). The matrix \( S \) can be found using the aforementioned method with an additional constraint that
it does not mix position and momentum quadratures. Hence we get
\[ S = \begin{pmatrix} x_1 & 0 & x_2 & 0 \\ 0 & x_3 & 0 & x_4 \\ x_5 & 0 & x_6 & 0 \\ 0 & x_7 & 0 & x_8 \end{pmatrix}, \] (A2)

where the real parameters \( x_1, x_2, \ldots, x_8 \) are related to the elements of eigenvectors \( u_{\nu_1} \) and \( w_{\nu_2} \) of the matrix \( i\Omega_{AB} \) corresponding to the eigenvalues \( \nu_1 \) and \( \nu_2 \) as 
\[ u_{\nu_1} = (ix_1, x_3, ix_2, x_4)^T \] and 
\[ w_{\nu_2} = (ix_5, x_7, ix_6, x_8)^T \]
and thus satisfy the set of equations
\[
Mx_3 + L_1x_4 = 0, \quad Mx_7 + L_2x_8 = 0, \\
Mx_4 - L_2x_3 = 0, \quad Mx_8 - L_1x_7 = 0, \\
x_1 = \frac{ax_3 - kp_4}{\nu_1}, \quad x_2 = \frac{bx_4 - kp_3}{\nu_1}, \\
x_5 = \frac{ax_7 - kp_8}{\nu_2}, \quad x_6 = \frac{bx_8 - kp_7}{\nu_2}, \] (A3)

where we defined
\[ M \equiv ak_x - bk_p, \quad \tilde{M} \equiv bk_x - ak_p, \]
\[ L_{1,2} \equiv b^2 - k_x k_p - \nu_1^2 = \frac{b^2 - a^2 + \sqrt{D}}{2}, \] (A4)

where quantity \( D \) is defined in Eq. (14). On the top of that, parameters \( x_1, \ldots, x_8 \) also have to satisfy further constraints
\[
x_1x_3 + x_2x_4 = 1, \quad x_1x_7 + x_2x_8 = 0, \\
x_3x_5 + x_6x_8 = 1, \quad x_3x_5 + x_4x_6 = 0 \] (A5)
imposed by condition \( S\Omega_2 S^T = \Omega_2 \).

Several cases must be distinguished when solving sets of equations (A3) and (A5) depending on the relations between the parameters \( a, b, k_x \), and \( k_p \).

1. If \( a = b \) and \( k_x \geq k_p > 0 \), Eq. (13) gives
\[
\nu_1 = \sqrt{(a + k_x)(a - k_p)}, \\
\nu_2 = \sqrt{(a - k_x)(a + k_p)}, \] (A6)

and CM (6) is brought to the Williamson normal form (12) by symplectic matrix
\[ S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} z_A^{-1} & 0 & z_A^{-1} & 0 \\ 0 & z_A & 0 & z_A \\ -z_B & 0 & z_B & 0 \\ 0 & -z_B^{-1} & 0 & z_B^{-1} \end{pmatrix}, \] (A7)

where \( z_A = \sqrt{\frac{a + k_x}{k_p}} > 1 \) and \( z_B = \sqrt{\frac{a + k_p}{k_x}} > 1 \). A closer look at matrix (A7) reveals that it can be expressed as the following product
\[ S = (S_A \oplus S_B)U_{BS}, \] (A8)

which describes a composition of a balanced beam splitter described by symplectic matrix
\[ U_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -\nu_1 & -\nu_2 & 0 \\ \nu_1 & 0 & 0 & \nu_2 \\ 0 & \nu_1 & \nu_2 & 0 \end{pmatrix}, \] (A9)

followed by local squeezing transformations of quadratures \( x_A \) and \( p_B \), described by diagonal symplectic matrices
\[ S_A = \begin{pmatrix} z_A^{-1} & 0 \\ 0 & z_A \end{pmatrix}, \quad S_B = \begin{pmatrix} z_B & 0 \\ 0 & z_B^{-1} \end{pmatrix}. \] (A10)

2. If \( a > b \), then \( M > 0 \) and \( L_1 < 0 \).

a) If \( bk_x = ak_p \), we get \( \tilde{M} = 0 \) and
\[ \nu_1 = \sqrt{a^2 - k_x k_p}, \quad \nu_2 = \sqrt{b^2 - k_x k_p} \] (A11)
by Eq. (13). Further, \( L_2 = 0 \) and by solving Eqs. (A3) and (A5) we arrive at matrix (A2) of the form
\[ S_{2a} = \begin{pmatrix} \sqrt{\frac{a^2}{\nu_1}} & 0 & 0 & 0 \\ 0 & \sqrt{\nu_1} & 0 & \frac{k_x}{\sqrt{a^2}} \\ \sqrt{\frac{k_x}{\nu_2}} & 0 & \sqrt{\nu_2} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{k_x}{\nu_2}} \end{pmatrix}. \] (A12)

Similar to case 1, we can decompose the symplectic matrix in terms of more simple symplectic matrices. Here, the decomposition attains the following form:
\[ S_{2a} = (S_A \oplus S_B)S_{QND}, \] (A13)

where the matrices \( S_A \) and \( S_B \) are given in Eq. (A10) with \( z_A = \sqrt{\frac{k_x}{\nu_1}} > 1 \) and \( z_B = \sqrt{\frac{k_x}{\nu_2}} > 1 \), and they describe squeezing in quadratures \( x_A \) and \( p_B \) of modes \( A \) and \( B \), respectively. The matrix \( S_{QND} \) is a symplectic matrix of the quantum non-demolishing interaction
\[ S_{QND} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ -q & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (A14)

with interaction constant \( q = \frac{k_p}{a} = \frac{k_x}{b} \).

Note, that the present set of states is non-empty. For instance, a CM (11) with \( a = 3, b = 2, k_x = 2 \) and \( k_p = 4/3 \) satisfies both inequalities (10) and thus represents a physical state with \( a > b \) and \( ak_p = bk_x \).

b) If \( bk_x > ak_p \) we have \( L_2 > 0 \), whereas if \( bk_x < ak_p \) we have \( L_2 < 0 \). Thus, if \( bk_x \neq ak_p \) then \( L_2 \neq 0 \). By solving Eqs. (A3) and (A5) we get a matrix \( S_{2b} \) of the
form (A2) with
\[
x_3 = -\frac{L_1}{M} x_4, \quad x_7 = \frac{L_2}{M} x_8, \\
x_1 = -\frac{a L_1 + k_p M}{\nu_1 M} x_4, \quad x_2 = \frac{k_p L_1 + b M}{\nu_1 M} x_4, \\
x_5 = -\frac{a L_2 + k_p M}{\nu_2 M} x_8, \quad x_6 = \frac{k_p L_2 + b M}{\nu_2 M} x_8, \\
x_4 = M \sqrt{\frac{\nu_1}{a L_1^2 + 2 k_p L_1 M + b M^2}}, \\
x_8 = M \sqrt{\frac{\nu_2}{a L_2^2 + 2 k_p L_2 M + b M^2}},
\]  
(A15)

3. If \(a < b\), we can find the symplectic matrix (A2) by transforming the present case to case 2 with the help of symplectic matrix
\[
T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]  
(A16)

which exchanges modes \(A\) and \(B\). The matrix \(T\) transforms CM \(\gamma_{AB}\), Eq. (9), to CM \(\tilde{\gamma}_{AB} \equiv T \gamma_{AB} T^T\), which is again of the form (9), but with parameters \(a\) and \(b\) exchanged. Next, we calculate a symplectic matrix \((\equiv \tilde{S})\) which brings CM \(\tilde{\gamma}_{AB}\) to the Williamson normal form by solving a set of equations obtained from Eq. (A3) by exchanging parameters \(a\) and \(b\), and a set of equations (A5). The symplectic matrix which brings the original CM (9) to the Williamson normal form is then given by \(S = ST\).

Let us now apply previous algorithm do derive an explicit form of the symplectic matrix \(S\), Eq. (A2), which brings CM (9) with \(a < b\) to the Williamson normal form. We have already mentioned, that one set of equations to be solved to get \(S\) is obtained from Eqs. (A3) by exchanging parameters \(a\) and \(b\). This entails the following exchanges \(M \leftrightarrow M\) and \(L_{1,2} \leftrightarrow -L_{2,1}\), whereas symplectic eigenvalues \((\equiv 1)\) remain unchanged. In analogy with previous case 2 we see, that if \(a \neq b\) then \(\tilde{M} > 0\), \(L_2 > 0\), and in dependence on whether the quantity \(M\) vanishes or not, we distinguish the following two cases:

a) If \(a k_x = b k_p\), we get \(M = 0\) and hence
\[
\nu_1 = \sqrt{b^2 - k_x k_p}, \quad \nu_2 = \sqrt{a^2 - k_x k_p}, \tag{A17}
\]
as well as \(L_1 = 0\). The symplectic matrix \(\tilde{S}_{3a}\) is obtained from the symplectic matrix (A12) by exchanging \(a\) and \(b\). By multiplying the latter matrix by the matrix (A16) from the right, we finally get
\[
\tilde{S}_{3a} = \begin{pmatrix} 0 & 0 & \sqrt{\frac{k_x}{\nu_2}} & 0 \\ 0 & \sqrt{\frac{\nu_2}{k_x}} & 0 & \sqrt{\frac{k_x}{\nu_2}} \\ \sqrt{\frac{k_x}{\nu_2}} & 0 & -\frac{k_x}{\nu_2} & 0 \\ 0 & 0 & \sqrt{\frac{\nu_2}{k_x}} & 0 \end{pmatrix} \tag{A18}
\]
b) If \(a k_x > b k_p\) we have \(L_1 < 0\), while if \(a k_x < b k_p\) we have \(L_1 > 0\). Thus, if \(a k_x \neq b k_p\) then \(L_1 \neq 0\). By solving Eqs. (A3) and (A5) with \(a\) and \(b\) exchanged we get a matrix \(\tilde{S}_{3b}\) of the form (A2) with
\[
x_3 = \frac{L_2}{M} x_4, \quad x_7 = \frac{L_1}{M} x_8, \\
x_1 = b L_2 - k_p M x_4, \quad x_2 = a M - k_p L_2 x_4, \\
x_5 = b L_1 - k_p M x_8, \quad x_6 = a M - k_p L_1 x_8, \\
x_4 = M \sqrt{\frac{\nu_1}{a L_2^2 - 2 k_p L_2 M + a M^2}}, \\
x_8 = M \sqrt{\frac{\nu_2}{b L_1^2 - 2 k_p L_1 M + b M^2}}.
\]  
(A19)

Hence, the sought matrix \((\equiv S_{3b})\), which brings the original CM (9) with \(a < b\) and \(a k_x \neq b k_p\) to the Williamson normal form reads as
\[
S_{3b} = \tilde{S}_{3b} T = \begin{pmatrix} x_2 & 0 & 0 & x_1 \\ 0 & x_1 & 0 & x_3 \\ x_6 & 0 & x_5 & 0 \\ 0 & x_8 & 0 & x_7 \end{pmatrix}, \tag{A20}
\]
where the elements \(x_1, x_2, \ldots, x_8\) are given in Eq. (A19).

Needless to say finally, that sets of Eqs. (A3) and (A5), which we used to derive symplectic matrix \(S\) bringing CM (9) to the Williamson normal form (A2), do not determine the matrix uniquely. The structure of the equations reveals, that they remain unchanged under the following transformation:
\[
x_1 \rightarrow -x_1, \quad x_2 \rightarrow -x_2, \quad x_3 \rightarrow -x_3, \quad x_4 \rightarrow -x_4,
\]  
(A21)
as well as under the transformation
\[
x_5 \rightarrow -x_5, \quad x_6 \rightarrow -x_6, \quad x_7 \rightarrow -x_7, \quad x_8 \rightarrow -x_8.
\]  
(A22)

Thus, by solving Eqs. (A3) and (A5) we not only get the matrix \(S_j\), \(j = 1, 2a, 2b, 3a, 3b\), but also matrices
\[
[1 \oplus (-1)] S_j, \quad [(-1) \oplus 1] S_j, \quad [(-1) \oplus (-1)] S_j. \tag{A23}
\]
Let us stress here, that the ambiguity in determination of the matrix \(S\) does not cause any nonuniqueness in determination of GIE. This is because as we have shown in Sec. IV GIE is invariant under the transformation \(S \rightarrow (O_A \oplus O_B) S\), where \(O_A\) and \(O_B\) are local orthogonal symplectic matrices and thus any of the matrices (A23) yields the same value of GIE as the matrix \(S_j\). Therefore, for evaluation of GIE we can take either the matrix \(S_j\) or any of the matrices (A23) and for the sake of simplicity we work with the most simple matrix \(S_j\) in the main text.

Appendix B: Saturation of the upper bound for states \(\rho_{AB}^{(1)}\) and \(\rho_{AB}^{(3)}\)

In this section we prove that the quantity \(L\), Eq. (76), for symmetric GLEMS \(\rho_{AB}^{(1)}\) is equal to the RHS of
Eq. (34) and for asymmetric squeezed thermal GLEMS \( \rho_{AB}^{(3)} \) it is equal to the RHS of Eq. (37).

We start with symmetric GLEMS that are described by CM \( (\bar{\rho}) \) with \( a = b \) and \( \nu_{2} = \sqrt{(a - k_{x})(a + k_{p})} = 1 \), Eq. (A6). From Eq. (A7) of Appendix A it follows that \( x_{3} = x_{4} = z_{A}/\sqrt{2} \), where the parameter \( z_{A} \) is defined below the equation. Inserting from here for \( x_{3} \) and \( x_{4} \) into the RHS of equation (115) further yields

\[
\alpha_{A} = \alpha_{B} = \left( \frac{\nu^{2} - 1}{a} \right) \frac{a^{2}}{2}, \quad \alpha_{AB} = \frac{\nu^{2} - 1}{\nu},
\]

and the quantity (117) which is to be minimized on the interval \( Q \in (0, 1/\nu) \) to get the quantity (124) attains the form:

\[
\mathcal{K}_{h} = \left( 1 - \frac{\alpha_{A}Q^{2}}{\alpha_{AB}Q} \right).
\]

Solution of the extremal equation \( d\mathcal{K}_{h}/dQ = 0 \) gives two stationary points \( Q_{1} = 1/\alpha_{A} \) and \( Q_{2} = 2/\alpha_{AB} - 1/\alpha_{A} \). It is easy to show with the help of equations \( \nu = \sqrt{(a + k_{x})(a - k_{p})} \) and \( (a - k_{x})(a + k_{p}) = 1 \), and inequalities \( k_{x} \leq k_{p} \) and \( a > k_{x} \), where the second inequality follows from the second of inequalities (10), that \( Q_{1} = 1/\alpha_{A} > 1/\nu \) and thus \( Q_{1} \notin (0, 1/\nu) \). Likewise, using inequalities \( a > k_{p} \) and \( k_{x} < k_{p} \) we also find that \( Q_{2} = k_{x}/(\nu(k_{x} - k_{p})) > 1/\nu \), whence \( Q_{2} \notin (0, 1/\nu) \). At the boundary points 0 and 1/\( \nu \) the quantity (122) then satisfies inequality \( \mathcal{K}_{h}(0) = 1 \geq (a^{2} - k_{x}^{2})/(a^{2} - k_{p}^{2}) = \mathcal{K}_{h}(1/\nu) \). As a consequence, the quantity \( \mathcal{K}_{\text{min}} \), Eq. (124), is equal to \( \mathcal{K}_{\text{min}} = (a^{2} - k_{x}^{2})/(a^{2} - k_{p}^{2}) \). Because the optimum is reached at point 1/\( \nu \) which has been shown below Eq. (127) to be reached by homodyne detection of quadrature \( x_{E} \), quantities \( \mathcal{K}_{\text{min}} \) and \( \mathcal{K}_{h} \), Eq. (177), are equal, and one finds the quantity (176) to be

\[
L\left( \rho_{AB}^{(1)} \right) = \ln \left( \frac{a}{a^{2} - k_{p}^{2}} \right),
\]

which clearly coincides with the RHS of Eq. (34) as we set out to prove.

It is also possible to prove equality of the quantity \( L \), Eq. (76), to GIE for asymmetric two-mode squeezed thermal GLEMS \( \rho_{AB}^{(3)} \), Eq. (37). The latter states are defined by conditions \( k_{x} = k_{p} \equiv k \) and \( \nu_{2} = \sqrt{(a + b)^{2} - 4k^{2} - (a - b)^{2}}/2 = 1 \), which lead to the following CM (29):

\[
\gamma_{AB}^{(3)} = \begin{pmatrix} a & k_{x} & k_{x} \\ k_{x} & b & \end{pmatrix}
\]

with

\[
k = \begin{cases} \sqrt{(a + b)(b - 1)}, & \text{if } a \geq b; \\ \sqrt{(a - b)(a - 1)}, & \text{if } a < b. \end{cases}
\]

First, we investigate states \( (\rho_{AB}^{(3)}) \) with \( a \geq b \). For \( a = b \) asymmetric squeezed thermal GLEMS reduce to pure states \( \rho_{AB}^{(\rho)} \) with \( k = \sqrt{a^{2} - 1} \), and for them the purifying subsystem \( E \) is completely decoupled from modes \( A \) and \( B \). This implies, that matrices (68) and (70) read as \( A_{X} = X_{B} = X_{AB} = \gamma_{E} \) and thus \( \gamma_{i} = \mathcal{K}_{1} \) by Eq. (73).

As a result, the quantity \( L \) is given only by the first term on the RHS of Eq. (70), i.e., \( L(\rho_{AB}^{(\rho)}) = \ln(a) \), which is equal to GIE given in Eq. (33). For the case \( a > b \) the symplectic matrix which symplectically diagonalizes CM (194) is of the form (A2), where parameters \( x_{1}, x_{2}, \ldots, x_{8} \), are given in Eq. (A15). Making use of the first branch of Eq. (B5), the explicit form of the larger symplectic eigenvalue \( \nu \equiv \nu_{1} = 1 + a - b \), and condition \( \nu_{2} = 1 \), we arrive at the parameters \( x_{3} \) and \( x_{4} \) appearing in Eq. (115).

\[
x_{3} = \sqrt{\frac{a + 1}{b - a + 2}}, \quad x_{4} = \sqrt{\frac{b - 1}{b - a + 2}},
\]

which further give

\[
\alpha_{A} = \frac{(a - b)(a + 1)}{a}, \quad \alpha_{B} = \frac{(a - b)(b - 1)}{b},
\]

\[
\alpha_{AB} = \frac{(a - b)(a + b - 2)}{a - b + 1}.
\]

In this case, quantity (117) has two stationary points \( \nu_{2} = 1/\nu \), which can be brought using Eq. (177) into the form \( Q_{1,2} = 1/(\nu \mp 1) \). As \( Q_{1} > 1/\nu \) we have \( Q_{1} \notin (0, 1/\nu) \), whereas it is obvious, that \( Q_{2} \in (0, 1/\nu) \). From Eqs. (B7) and expression for the larger symplectic eigenvalue \( \nu \) given above Eq. (B6), it further follows that \( \mathcal{K}_{h}(1/\nu) = 1 \), which is equal to the value at the other boundary point, \( \mathcal{K}_{h}(0) = 1 \). Finally, substitution of \( Q_{2} = 1/(\nu + 1) \) into the RHS of Eq. (117) and utilization of formulas (B7) gives

\[
\mathcal{K}_{h}(Q_{2}) = \frac{(a + b)^{2}(a - b + 1)}{ab(a - b + 2)^{2}} < 1.
\]

This implies, that \( \mathcal{K}_{\text{min}} = \mathcal{K}_{h}(Q_{2}) \) and because according to Eq. (122) the stationary point \( Q_{2} = 1/(\nu + 1) \) is reached for \( \varphi = 0, \tau = 1 \) and \( t = 0 \), which corresponds to heterodyne detection, i.e., projection onto a coherent state with CM \( \Gamma_{E} = 1 \), \( \mathcal{K}_{\text{min}} \) is again equal to \( \mathcal{K}_{h} \). Hence, one gets using Eq. (76) the expression

\[
L\left( \rho_{AB}^{(3)} \right) = \ln \left( \frac{a + b}{a - b + 2} \right).
\]

The remaining part is a proof of the equality of GIE to the quantity \( L \) for the asymmetric squeezed thermal GLEMS with \( a < b \). The proof goes along the same lines as the proof for the case with \( a > b \). Like previously, the matrix which symplectically diagonalizes CM (B3) with \( k \) given by the second branch in Eq. (B5) is of the form (A20), where parameters \( x_{1}, x_{2}, \ldots, x_{8}, \) are given in Eq. (A19). This gives us for the parameters \( x_{3} \) and \( x_{4} \) appearing in Eqs. (115) expressions

\[
x_{3} = \sqrt{\frac{a - 1}{b - a + 2}}, \quad x_{4} = \sqrt{\frac{b + 1}{b - a + 2}}.
\]
and thus
\[
\alpha_A = \frac{(b-a)(a-1)}{a}, \quad \alpha_B = \frac{(b-a)(b+1)}{b}, \\
\alpha_{AB} = \frac{(b-a)(b-a+2)}{b-a+1}.
\]  
(B11)

By calculating ratios \(\alpha_{AB}/\alpha_A\) and \(\alpha_{AB}/\alpha_B\) and substituting the obtained expressions into Eq. (136), we get the following equations:

\[
\mathcal{X}_h(0) = \frac{1}{\nu}, \quad \mathcal{X}_h(1/\nu) = 1, \quad \mathcal{X}_h(Q_2) = \frac{(a+b)^2(b-a+1)}{ab(b-a+2)^2} < 1, \quad \text{(B12)}
\]

whence \(\mathcal{K}_{\text{min}} = \mathcal{X}_h(Q_2)\). Finally, since \(Q_2\) is reached by heterodyne detection on mode \(E\) we have that \(\mathcal{K}_{\text{min}} = \mathcal{K}_{\text{min}}\) and thus

\[
L\left(\rho_{AB}^{(3)}\right) = \ln \left(\frac{a+b}{b-a+2}\right), \quad \text{(B13)}
\]

Combining Eqs. (159) and (113) we arrive at a single formula for quantity \(\mathcal{J}_0\) for all states \(\rho_{AB}^{(3)}\):

\[
L\left(\rho_{AB}^{(3)}\right) = \ln \left(\frac{a+b}{|a-b|+2}\right). \quad \text{(B14)}
\]

If we now compare the latter equation with Eq. (89) it is clear, that also in the case of states \(\rho_{AB}^{(3)}\) the quantity \(L\) coincides with the GIE which accomplishes our proof.

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