On completely faithful Selmer groups of elliptic curves and Hida
deformations

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Abstract

In this paper, we study completely faithful torsion \( \mathbb{Z}_p[G] \)-modules with applications to the study of Selmer
groups. Namely, if \( G \) is a nonabelian group belonging to certain classes of polycyclic pro-\( p \) group, we establish
the abundance of faithful torsion \( \mathbb{Z}_p[G] \)-modules, i.e., non-trivial torsion modules whose global annihilator
ideal is zero. We then show that such \( \mathbb{Z}_p[G] \)-modules occur naturally in arithmetic, namely in the form of
Selmer groups of elliptic curves and Selmer groups of Hida deformations. It is interesting to note that faithful
Selmer groups of Hida deformations do not seem to appear in literature before. We will also show that faithful
Selmer groups have various arithmetic properties. Namely, we show that faithfulness is an isogeny invariant,
and we will prove “control theorem” results on the faithfulness of Selmer groups over a general admissible
\( p \)-adic Lie extension.

Keywords and Phrases: Completely faithful modules, Selmer groups, elliptic curves, Hida deformations.

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1 Introduction

Throughout the paper, \( p \) will always denote an odd prime. Let \( E \) be an elliptic curve defined over
\( \mathbb{Q} \) which has good ordinary reduction at the prime \( p \). The Iwasawa main conjecture predicts that the
Mazur-Swinnerton-Dyer \( p \)-adic \( L \)-function \( L_p(E) \) associated to \( E \) can be interpreted as an element of the
Iwasawa-algebra \( \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})]] \) of the cyclotomic \( \mathbb{Z}_p \)-extension \( \mathbb{Q}^{\text{cyc}} \) of \( \mathbb{Q} \) and is a generator of the
characteristic ideal of the Pontryagin dual \( X(E/\mathbb{Q}^{\text{cyc}}) \) of the Selmer group of \( E \) over \( \mathbb{Q}^{\text{cyc}} \) (see [MSw]).
Furthermore, if \( X(E/\mathbb{Q}^{\text{cyc}}) \) does not have any nonzero pseudo-null submodule, it will follow from the
main conjecture that the \( p \)-adic \( L \)-function \( L_p(E) \) annihilates \( X(E/\mathbb{Q}^{\text{cyc}}) \). We should mention that the
Iwasawa main conjecture in this context has been well understood and largely proven (see [K1, R, SU]).

It is natural to consider generalization of the above by considering field extensions \( F_{\infty} \) of some number
field \( F \) whose Galois group \( G = \text{Gal}(F_{\infty}/F) \) is a nonabelian \( p \)-adic Lie group, and this has been the
central theme in noncommutative Iwasawa theory. One of the earliest approach towards understanding
and formulating this theory is to investigate the global annihilator ideal of \( X(E/F_{\infty}) \) (for instance,

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see [CSS, Ha]. Inspired by the cyclotomic situation, it was then hoped that such an investigation might give some insight to the noncommutative $p$-adic $L$-function which is, even today, still largely conjectural in most situations (although one now has a slightly better understanding of the shape of the $p$-adic $L$-functions and the form of the main conjecture via an algebraic $K$-theoretical approach; see [BV, CFKSV, FK, K2]). As it turns out, such an approach via global annihilators had been shown to be not feasible in general. In fact, Venjakob was able to establish the existence of a class of modules over the Iwasawa algebra of the nonabelian group $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ which cannot be annihilated by a single global element in the Iwasawa algebra (see [V2]). Building on this work, he and Hachimori were able to give examples of Selmer groups of elliptic curves over a noncommutative $p$-adic Lie extension whose Galois group is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ which do not have a global annihilator (see [HV]).

In this paper, following the footsteps of Venjakob, we will establish the nonexistence of global annihilators for a class of modules over the Iwasawa algebra of a nonabelian group which is an extension of a polycyclic pro-$p$ group by $\mathbb{Z}_p \rtimes \mathbb{Z}_p$ (see Theorem 3.3). The technique we used in establishing this result derives essentially from [V2]. In fact, to prove our main result, we will also require the case of $\mathbb{Z}_p \times \mathbb{Z}_p$ which was first established by Venjakob in [V2]. Then as in the paper of Hachimori and Venjakob [HV], we apply our result to obtain examples of completely faithful Selmer groups of elliptic curves and Hida deformations over noncommutative $p$-adic Lie extensions (see Theorems 4.1 and 4.4). To the best of the author’s knowledge, completely faithful Selmer groups of Hida deformations do not seem to be observed in literature before. We mention that we can also find examples of Selmer groups of elliptic curves which are faithful but not completely faithful. We also mention that it should be quite evident that our results can be applied to obtain completely faithful Selmer groups for $p$-adic representations defined over coefficient rings $\mathbb{Z}_p[[X_1, X_2, \ldots, X_n]]$. However, in this paper, we will content ourselves mainly with Hida deformations and a short remark in the general aspect.

For the remainder of the paper, we discuss further properties on the faithfulness of Selmer groups. In particular, we show that faithfulness is an isogeny invariant (see Proposition 5.1). However, on the other hand, completely faithfulness is not an isogeny invariant and we will give an example to show this. In the final section of the paper, we will prove some “control theorem” type results for the faithfulness of Selmer groups (see Propositions 6.1 and 6.5). It would seem that “control theorem” type results for faithfulness of Selmer group have not been observed in literature before.

We should also mention that completely faithful modules and Selmer groups of elliptic curves over Iwasawa algebras of compact $p$-adic Lie group other than the ones considered in this paper have also been studied in [A, BZ]. Our results here may therefore be viewed as complement to the results there.

We end the introductory section discussing some (negative) consequences and significance of our results. Let $E$ be an elliptic curve defined over $\mathbb{Q}$ which has good ordinary reduction at the prime $p$ and set $F = \mathbb{Q}(\mu_p)$. Let $F_\infty$ be a false Tate extension of $F$ in the sense of [HV]. As shown loc. cit., there are cases of $X(E/F_\infty)$ being completely faithful. One may then naively consider adjoining multiple $\mathbb{Z}_p$-extensions of $F$ to $F_\infty$ and perhaps hope to obtain nontrivial global annihilator of the Selmer groups which is now defined over a larger $p$-adic Lie extension. The rationale (which now seems irrational)
behind this thought is that our Selmer group is now a module over an Iwasawa algebra of the group \(\mathbb{Z}_p \times (\mathbb{Z}_p \rtimes \mathbb{Z}_p)\) which has a large “commutative” component and, therefore, one might naively hope that having large “commutative” component may somehow force the existence of nontrivial global annihilator for our Selmer group. However, as our results (both algebraic and arithmetic) will show, such an idea is not feasible in general.

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2 Algebraic Preliminaries

In this section, we establish some algebraic preliminaries and notation. Throughout the paper, we will always work with left modules over a ring. Let \(\Lambda\) be a (not necessarily commutative) Noetherian ring which has no zero divisors. Then it admits a skew field of fractions \(K(\Lambda)\) which is flat over \(\Lambda\) (see [GW, Chapters 6 and 10] or [Lam, Chapter 4, §9 and §10]). If \(M\) is a finitely generated \(\Lambda\)-module, we define the \(\Lambda\)-rank of \(M\) to be

\[
\operatorname{rank}_\Lambda M = \dim_{K(\Lambda)} K(\Lambda) \otimes_\Lambda M.
\]

Clearly, one has \(\operatorname{rank}_\Lambda M = 0\) if and only if \(K(\Lambda) \otimes_\Lambda M = 0\). We say that \(M\) is a torsion-module if \(\operatorname{rank}_\Lambda M = 0\). We shall record a simple lemma which is a special case of [Lim1, Lemma 4.1].

Lemma 2.1. Let \(x\) be a central element of \(\Lambda\) with the property that \(\Omega := \Lambda / x\Lambda\) also has no zero divisors. Let \(M\) be a finitely generated \(\Lambda\)-module. Then

\[
\operatorname{rank}_\Omega M/xM = \operatorname{rank}_\Omega M[x] + \operatorname{rank}_\Lambda M,
\]

where \(M[x]\) is the submodule of \(M\) killed by \(x\).

For a nonzero \(\Lambda\)-module \(M\), we define the global annihilator “ideal”

\[
\operatorname{Ann}_\Lambda(M) = \{\lambda \in \Lambda : \lambda m = 0 \text{ for all } m \in M\}.
\]

Note that the above “ideal” is naturally an additive subgroup of \(\Lambda\). However, if \(\Lambda\) is not commutative, it may not be an ideal. Despite this, we will still keep the terminology “ideal”. We will say that \(M\) is a faithful \(\Lambda\)-module if \(\operatorname{Ann}_\Lambda(M) = 0\).

Now if \(x \in \Lambda\), we denote \(M[x]\) to be the set consisting of elements of \(M\) annihilated by \(x\). If \(x\) is not central, \(M[x]\) is at most an additive subgroup of \(M\). However, if we assume further that \(x\Lambda = \Lambda x\), then it is easy to see that \(M[x]\) is a \(\Lambda\)-submodule of \(M\). In fact, for given \(\lambda \in \Lambda\) and \(m \in M\), it follows from the hypothesis \(x\Lambda = \Lambda x\) that \(\lambda'x = x\lambda\) for some \(\lambda' \in \Lambda\), and therefore, one has \(x\lambda m = \lambda'xm = 0\).

Continuing to assume that \(x\Lambda = \Lambda x\), one can verify that

\[
xM = \{xm : m \in M\}
\]
is a $\Lambda$-submodule of $M$, and that $M/xM$ is a $\Lambda/x\Lambda$-module.

The following lemma is a natural generalization of [V2, Lemma 4.5] and will be a crucial ingredient in proving our main result in Section 3. It will be of interest to have an analogous statement for completely faithful modules (see Remark 6.2) but we are not able to establish such a statement at the point of writing.

**Lemma 2.2.** Let $x$ be a nonzero element of $\Lambda$ with the property that $x\Lambda = \Lambda x$ and suppose that the ring $\Omega := \Lambda/x\Lambda$ has no zero divisors. Write $I = x\Lambda (= \Lambda x)$. Let $M$ be a finitely generated $\Lambda$-module. Suppose that $M[x] = 0$, and suppose that $\cap_{i \geq 1} I^i = 0$.

If $M/xM$ is a faithful $\Omega$-module, then $M$ is a faithful $\Lambda$-module.

**Proof.** We will prove the contrapositive statement. Suppose that $\text{Ann}_\Lambda(M)$ contains a nonzero element $\lambda$. Since $\cap_{i \geq 1} I^i = 0$, we can find $n$ such that $\lambda \in I^n$ but $\lambda \notin I^{n+1}$. This in turn implies that $\lambda = x^n\lambda_0$ for some $\lambda_0 \notin I$ (note that such a representation is possible by the assumption that $x\Lambda = \Lambda x$). But since $M[x] = 0$, we actually have $\lambda_0 \in \text{Ann}_\Lambda(M)$. Since $\lambda_0 \notin I$, the image of $\lambda_0$ in $\Omega$ is nonzero and lies in $\text{Ann}_\Omega(M/xM)$.

We record two more lemmas. The first has an easy proof which is left to the reader.

**Lemma 2.3.** Suppose that we are given an exact sequence

$$0 \longrightarrow B' \longrightarrow M' \longrightarrow M \longrightarrow B \longrightarrow 0$$

of $\Lambda$-modules, where $B'$ and $B$ are both annihilated by a central element $\lambda$ of $\Lambda$. Then $M'$ is faithful over $\Lambda$ if and only if $M$ is faithful over $\Lambda$.

**Lemma 2.4.** Suppose that we are given an exact sequence

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of $\Lambda$-modules, where $M'$ is nonzero and $M''$ is finite. Then if $M$ is faithful over $\Lambda$, so is $M'$.

**Proof.** If the ring $\Lambda$ has characteristic zero, one can probably give a proof along the line of the proof of Lemma 2.3. In fact, for much of the discussion in the paper, this case suffices. However, we thought that it may be of interest to give a general proof that works in general which we do now. We will prove the contrapositive statement. Suppose that $\text{Ann}_\Lambda(M')$ contains a nonzero element $\lambda$. Since $M''$ is finite, it follows that for each $z \in M''$, there exists $n_z < m_z$ such that $\lambda^{n_z} z = \lambda^{m_z} z$. This in turns implies that $\lambda^{n_z}(\lambda^{m_z-n_z} - 1)z = 0$. Set $n = 1 + \max_{z \in M'' \setminus \{0\}} n_z$ and $m = \prod_{z \in M'' \setminus \{0\}} (m_z - n_z)$. Clearly, one has $\lambda^n(\lambda^m - 1)z = 0$ for every $z \in M''$. Also, since $n > 0$ by our choice, we have that $\lambda^n(\lambda^m - 1)$ lies in $\text{Ann}_\Lambda(M')$. Therefore, $\lambda^n(\lambda^m - 1)$ lies in $\text{Ann}_\Lambda(M)$. It remains to show that $\lambda^n(\lambda^m - 1)$ is a nonzero element of $\Lambda$. Let $w$ be a nonzero element of $M'$ (such an element exists by our hypothesis that $M' \neq 0$). Then $(\lambda^m - 1)w = -w \neq 0$, and this in turn implies that $\lambda^m - 1 \neq 0$. Since $\Lambda$ has no zero divisors and $\lambda \neq 0$, it follows that $\lambda^n(\lambda^m - 1)$ is also nonzero. This completes the proof of the lemma.  

□
Let $\Lambda$ be an Auslander regular ring (see [V1, Definition 3.3]) with no zero divisors. Let $M$ be a finitely generated $\Lambda$-module. Then $M$ is a torsion $\Lambda$-module if and only if $\text{Hom}_\Lambda(M, \Lambda) = 0$ (cf. [Lim1, Lemma 4.2]). If $M$ is a torsion $\Lambda$-module, we say that $M$ is a pseudo-null $\Lambda$-module if $\text{Ext}^1_\Lambda(M, \Lambda) = 0$. Let $\mathcal{M}$ denote the category of all finitely generated $\Lambda$-modules, let $\mathcal{C}$ denote the full subcategory of all pseudo-null modules in $\mathcal{M}$ and let $q: \mathcal{M} \rightarrow \mathcal{M}/\mathcal{C}$ denote the quotient functor. For a finitely generated $\Lambda$-module $M$, we say that $M$ is completely faithful if $\text{Ann}_\Lambda(N) = 0$ for any $N \in \mathcal{M}$ such that $q(N)$ is isomorphic to a non-zero subquotient of $q(M)$.

**Lemma 2.5.** Let $\Lambda$ be an Auslander regular ring with no zero divisors. Then we have the following statements.

(a) If $M$ is completely faithful over $\Lambda$, so is every non pseudo-null subquotient of $M$.

(b) An extension of completely faithful $\Lambda$-modules is also completely faithful.

**Proof.** This is straightforward from the definition. $\square$

3 Completely faithful modules over completed group algebras

In this section, we will prove our main theorem which is an extension of [V2, Theorem 6.3]. As before, $p$ will denote a fixed odd prime. Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. It is well known that $\mathbb{Z}_p[G]$ is an Auslander regular ring (cf. [V1, Theorems 3.26]). Furthermore, the ring $\mathbb{Z}_p[G]$ has no zero divisors (cf. [Neu]), and therefore, as seen in the previous section, there is a well-defined notion of $\mathbb{Z}_p[G]$-rank and torsion $\mathbb{Z}_p[G]$-module. We record the following well-known and important result of Venjakob (cf. [V2, Example 2.3 and Proposition 5.4]).

**Theorem 3.1 (Venjakob).** Suppose that $H$ is a closed normal subgroup of $G$ with $G/H \cong \mathbb{Z}_p$. Let $M$ be a compact $\mathbb{Z}_p[G]$-module which is finitely generated over $\mathbb{Z}_p[H]$. Then $M$ is a pseudo-null $\mathbb{Z}_p[G]$-module if and only if $M$ is a torsion $\mathbb{Z}_p[H]$-module.

We record another useful lemma whose proof is left to the reader (or see [Lim1, Lemma 4.5]).

**Lemma 3.2.** Let $H$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $N$ be a closed normal subgroup of $H$ such that $N \cong \mathbb{Z}_p$ and such that $H/N$ is also a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $M$ be a finitely generated $\mathbb{Z}_p[H]$-module. Then $H_1(N, M)$ is finitely generated over $\mathbb{Z}_p[H/N]$ for each $i$ and $H_i(N, M) = 0$ for $i \geq 2$. Furthermore, we have an equality

$$\text{rank}_{\mathbb{Z}_p[H]} M = \text{rank}_{\mathbb{Z}_p[H/N]} M_N - \text{rank}_{\mathbb{Z}_p[H/N]} H_1(N, M).$$

We can now state and prove the main theorem of this section which generalizes [V2, Theorem 6.3].

**Theorem 3.3.** Suppose that $N$ and $H$ are two closed normal subgroups of $G$, and suppose that the following statements are satisfied.
Let $M$ be a completely faithful $\mathbb{Z}$-module. Since $M$ is a $\mathbb{Z}$-module, it is finitely generated over $\mathbb{Z}$, and so it is not faithful over $\mathbb{Z}$. Hence, $M$ is a faithful $\mathbb{Z}$-module.

**Proof.** Since $M$ is finitely generated over $\mathbb{Z}$, it follows from the result of Venjakob mentioned above that every subquotient of $M$ which is not pseudo-null has positive $\mathbb{Z}_p[H]$-rank. Therefore, it suffices to show that every $\mathbb{Z}_p[G]$-module that is finitely generated over $\mathbb{Z}_p$ with positive $\mathbb{Z}_p[H]$-rank is faithful over $\mathbb{Z}_p[G]$. Let $M$ be such a module. We will proceed by induction on $r$. When $r = 0$, this is precisely [V2 Corollary 4.3]. Now suppose that $r \geq 1$. Then choose a topological generator $\gamma_1$ of $N_1$. The ideal generated by $\gamma_1 - 1$ is precisely the augmentation kernel $I_{N_1}$ of the canonical quotient map $\mathbb{Z}_p[G] \rightarrow \mathbb{Z}_p[G/N_1]$, and one has $I_{N_1} = (\gamma_1 - 1)\mathbb{Z}_p[G] = \mathbb{Z}_p[G]((\gamma_1 - 1)$. Therefore, $M[\gamma_1 - 1]$ is a $\mathbb{Z}_p[G]$-submodule of $M$. Set $M'' = M/M[\gamma_1 - 1]$. Since $M[\gamma_1 - 1]$ is a torsion $\mathbb{Z}_p[H]$-module, it follows that $M''$ has positive $\mathbb{Z}_p[H]$-rank and $M''[\gamma_0 - 1] = 0$. On the other hand, one clearly has $\text{Ann}_{\mathbb{Z}_p[G]}(M) \subseteq \text{Ann}_{\mathbb{Z}_p[G]}(M'')$, and therefore, we may assume that the module $M$ we start with satisfies the property that $M[\gamma_1 - 1] = 0$. Now, note that $M_{N_1} = M/(\gamma_1 - 1)M$ is finitely generated over $\mathbb{Z}_p[G]/(\gamma_1 - 1)\mathbb{Z}_p[G] \cong \mathbb{Z}_p[G/N_1]$ and it follows from Lemma 3.2 that

$$\text{rank}_{\mathbb{Z}_p[G/N_1]} M_{N_1} = \text{rank}_{\mathbb{Z}_p[G]} M + \text{rank}_{\mathbb{Z}_p[G/N_1]} H_1(N_1, M) = \text{rank}_{\mathbb{Z}_p[G]} M > 0.$$ 

Here the second equality follows from the facts that $H_1(N_1, M) = M[\gamma_1 - 1]$ and that $M[\gamma_1 - 1] = 0$. Hence, $M_{N_1}$ is a $\mathbb{Z}_p[G/N_1]$-module which is finitely generated over $\mathbb{Z}_p[H/N_1]$ with positive $\mathbb{Z}_p[H/N_1]$-rank. By our induction hypothesis, we have that $M_{N_1}$ is (completely) faithful over $\mathbb{Z}_p[G/N_1]$. Finally, $I_{N_1}$ is closed in $\mathbb{Z}_p[G]$ and so $\bigcap_{i \geq 1} I_{N_i} = 0$. Thus, we may now apply Lemma 2.2 to conclude that $M$ is faithful over $\mathbb{Z}_p[G]$. This completes the proof of the theorem. 

**Remark 3.4.** One can of course prove analogous result as in Theorem 3.3 replacing $\mathbb{Z}_p$ by $\mathbb{F}_p$. This is achieved by combining the argument in Theorem 3.3 with [V2 Proposition 4.2(i)]. In fact, it is not difficult (though not immediate) to see that one can also prove analogous results as in the paper [V2] and Theorem 3.3 for a ring of integer of a finite extension of $\mathbb{Q}_p$ and its residue field.

We should mention here that the conclusion in Theorem 3.3 does not hold for a general $G$. We give a class of counterexamples. For the remainder of this paragraph, we assume that $G = H \times G/H$. Let $M = \mathbb{Z}_p[H]$ be the $\mathbb{Z}_p[G]$-module where the action of $H$ is the natural one, and the action of $G/H$ is the trivial one. Denote $\gamma$ to be a topological generator of $G/H$. Clearly, $M$ is clearly annihilated by $\gamma - 1$, and so it is not faithful over $\mathbb{Z}_p[G]$.

Despite the counterexamples, it is still of interest to ask if there exists other torsionfree pro-$p$ $p$-adic Lie group $G$ where a similar conclusion in Theorem 3.3 holds. Alternatively, one may ask if $G$ is a torsionfree pro-$p$ $p$-adic Lie group with a normal subgroup $H$ such that $G/H \cong \mathbb{Z}_p$, and has the property
that every finitely generated \( \mathbb{Z}_p[H] \)-module of positive \( \mathbb{Z}_p[H] \)-rank is completely faithful over \( \mathbb{Z}_p[G] \), what can one say about the structure of \( G \)? The author does not have an answer to these questions at this point of writing.

Finally, we end the section mentioning how the results in this section can be extended to a certain class of finitely generated torsion \( \mathbb{Z}_p[G] \) which was first introduced in [CFKSV]. In particular, this class of modules is a source of examples of \textbf{faithful modules that are not completely faithful}. As before, \( G \) is a compact pro-\( p \) Lie group without \( p \)-torsion and \( H \) is a closed normal subgroup of \( G \) such that \( G/H \cong \mathbb{Z}_p^r \). Therefore, an algebraic \( \mathbb{Z}_p[G] \)-module which contains a submodule which is not pseudo-null and not faithful, and therefore, is not completely faithful.

**Proposition 3.5.** Suppose that \( N \) and \( H \) are two closed normal subgroups of \( G \), and suppose that the following statements are satisfied.

(i) \( N \subseteq H \), \( G/H \cong \mathbb{Z}_p \) and \( G/N \) is a non-abelian group isomorphic to \( \mathbb{Z}_p \times \mathbb{Z}_p \).

(ii) There is a finite family of closed normal subgroups \( N_i \) (0 \leq i \leq r) of \( G \) such that \( 1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N \) and such that \( N_i/N_{i-1} \cong \mathbb{Z}_p^t \) for \( 1 \leq i \leq r \).

Let \( M \) be a \( \mathbb{Z}_p[G] \)-module which belongs to \( \mathfrak{M}_H(G) \) and has the property that \( M/M(p) \) has a positive \( \mathbb{Z}_p[H] \)-rank. Then \( M \) is a faithful \( \mathbb{Z}_p[G] \)-module.

Furthermore, \( M \) is a completely faithful \( \mathbb{Z}_p[G] \)-module if and only if \( M(p) \) is a pseudo-null \( \mathbb{Z}_p[G] \)-module.

**Proof.** Since \( M \) is finitely generated over \( \mathbb{Z}_p[G] \), the module \( M(p) \) is annihilated by a power of \( p \). The first assertion is now an immediate consequence from Lemma 2.3 and Theorem 3.3.

To prove the second assertion, we first recall that \( q : \mathcal{M} \rightarrow \mathcal{M}/C \) is the quotient functor, where \( \mathcal{M} \) denotes the category of all finitely generated \( \Lambda \)-modules and \( C \) denotes the full subcategory of all pseudo-null modules in \( \mathcal{M} \). Now suppose that \( M(p) \) is a pseudo-null \( \mathbb{Z}_p[G] \)-module, then we have \( q(M) = q(M/M(p)) \). Since \( M/M(p) \) is completely faithful by Theorem 3.3, it follows that \( M \) is also completely faithful. On the other hand, if \( M(p) \) is not a pseudo-null \( \mathbb{Z}_p[G] \)-module and is annihilated by some power of \( p \), then \( M \) contains a submodule which is not pseudo-null and not faithful, and therefore, is not completely faithful. \( \square \)

**Remark 3.6.** Note that \( M(p) \) is a pseudo-null \( \mathbb{Z}_p[G] \)-module if and only if its \( \mu_G \)-invariant (see [VI, Definition 3.32] for definition) vanishes (cf. [VI, Remark 3.33]).

### 4 Completely faithful Selmer groups

Let \( F \) be a number field. Fix once and for all an algebraic closure \( \bar{F} \) of \( F \). Therefore, an algebraic (possibly infinite) extension of \( F \) will mean a subfield of \( \bar{F} \) which contains \( F \). Let \( E \) be an elliptic
curve over $F$. Assume that for every prime $v$ of $F$ above $p$, our elliptic curve $E$ has either good ordinary reduction or multiplicative reduction at $v$.

Let $v$ be a prime of $F$. For every finite extension $L$ of $F$, we define

$$J_v(E/L) = \bigoplus_{w|v} H^1(L_w, E)_{p\infty},$$

where $w$ runs over the (finite) set of primes of $L$ above $v$. If $\mathcal{L}$ is an infinite extension of $F$, we define

$$J_v(E/\mathcal{L}) = \lim_{\leftarrow L} J_v(E/L),$$

where the direct limit is taken over all finite extensions $L$ of $F$ contained in $\mathcal{L}$. For any algebraic (possibly infinite) extension $\mathcal{L}$ of $F$, the Selmer group of $E$ over $\mathcal{L}$ is defined to be

$$S(E/\mathcal{L}) = \ker \left( H^1(\mathcal{L}, E_{p\infty}) \to \bigoplus_v J_v(E/\mathcal{L}) \right),$$

where $\mathcal{L}$ runs through all the primes of $F$.

We say that $F_\infty$ is an admissible $p$-adic Lie extension of $F$ if (i) $\text{Gal}(F_\infty/F)$ is a compact $p$-adic Lie group, (ii) $F_\infty$ contains the cyclotomic $\mathbb{Z}_p$-extension $F^\text{cyc}$ of $F$ and (iii) $F_\infty$ is unramified outside a finite set of primes of $F$. Furthermore, an admissible $p$-adic Lie extension $F_\infty$ of $F$ will be said to be strongly admissible if $\text{Gal}(F_\infty/F)$ is a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Write $G = \text{Gal}(F_\infty/F)$, $H = \text{Gal}(F_\infty/F^\text{cyc})$ and $\Gamma = \text{Gal}(F^\text{cyc}/F)$. Let $S$ be a finite set of primes of $F$ which contains the primes above $p$, the infinite primes, the primes at which $E$ has bad reduction and the primes that are ramified in $F_\infty/F$. Denote $F_S$ to be the maximal algebraic extension of $F$ unramified outside $S$. For each algebraic (possibly infinite) extension $\mathcal{L}$ of $F$ contained in $F_S$, we write $G_S(\mathcal{L}) = \text{Gal}(F_S/\mathcal{L})$. The following alternative equivalent description of the Selmer group of $E$ over $F_\infty$

$$S(E/F_\infty) = \ker \left( H^1(G_S(F_\infty), E_{p\infty}) \xrightarrow{\lambda_{E/F_\infty}} \bigoplus_{v \in S} J_v(E/F_\infty) \right)$$

is well-known (for instance, see [CS2 Lemma 2.2]). We will denote $X(E/F_\infty)$ to be the Pontryagin dual of $S(E/F_\infty)$. The following is then an immediate consequence of Theorem 3.3.

**Theorem 4.1.** Let $E$ be an elliptic curve over $F$ which has either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$. Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that $G$ contains a closed normal subgroup $N$ which is contained in $H$ and suppose that the following conditions are satisfied.

(i) There is a finite family of closed normal subgroups $N_i$ ($0 \leq i \leq r$) of $G$ such that $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$ and such that $N_i/N_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$.

(ii) $G/N$ is a non-abelian group isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_p$.

If $X(E/F_\infty)$ is finitely generated over $\mathbb{Z}_p[H]$ with positive $\mathbb{Z}_p[H]$-rank, then $X(E/F_\infty)$ is completely faithful over $\mathbb{Z}_p[G]$. 

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We record the following corollary of the theorem which is useful in obtaining examples of completely faithfully Selmer groups.

**Corollary 4.2.** Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that $G$ contains a closed normal subgroup $N$ which is contained in $H$ and suppose that conditions (i) and (ii) of Theorem 4.1 are satisfied. Assume that $X(E/F_{\text{cyc}})$ is finitely generated over $\mathbb{Z}_p$. Furthermore, suppose that either of the following conditions is satisfied.

(a) $X(E/F_{\text{cyc}})$ has positive $\mathbb{Z}_p$-rank.

(b) The field $F$ is not totally real, the elliptic curve $E$ has good ordinary reduction at every prime of $F$ above $p$ and $X(E/F_{\infty}) \neq 0$.

Then $X(E/F_{\infty})$ is completely faithful over $\mathbb{Z}_p[G]$.

**Proof.** It suffices to show that $X(E/F_{\infty})$ has positive $\mathbb{Z}_p[H]$-rank under the assumption of either conditions. If (a) holds, a standard argument in the spirit of [HM] will allow us establish the positivity of $\mathbb{Z}_p[H]$-rank (alternatively, one can make use of [HSh] Theorem 5.4] directly). Now if (b) holds, we may apply the main result in [Mat] to conclude that $X(E/F_{\infty})$ has positive $\mathbb{Z}_p[H]$-rank (for instance, see [Lim2, Lemma 5.8]).

**Remark 4.3.** Of course, one can apply Theorem 3.3 to obtain completely faithful Selmer groups of $p$-ordinary modular forms, or even more general $p$-adic representations.

We now discuss the complete faithfulness of Selmer groups of Hida deformations. Let $E$ be an elliptic curve over $\mathbb{Q}$ with ordinary reduction at $p$ and assume that $E[p]$ is an absolutely irreducible Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)-representation. By Hida theory (for instance, see [Hill, Hida]), there exists a commutative complete Noetherian local domain $R$ which is flat over the power series ring $\mathbb{Z}_p[X]$ in one variable, and a free $R$-module $T$ of rank 2 with $T/P \cong T_pE$ for some prime ideal $P$ of $R$. We will further assume that $R = \mathbb{Z}_p[X]$ in all our discussion. For more detailed description of fundamental and important arithmetic properties of the Hida deformations, we refer readers to [CS2, Hida, Hida3, Hill, SS]. We will just mention two properties of $T$ which we require to define an appropriate Selmer group of the Hida deformation. The first is that $T$ is unramified outside the set $S$, where $S$ is any finite set of primes of $F$ which contains the primes above $p$, the infinite primes, the primes at which $E$ has bad reduction and the primes that are ramified in $F_{\infty}/F$. The second property we will mention is that there exists a $R$-submodule $T^+$ of $T$ which is invariant under the action of Gal($\bar{\mathbb{Q}}_p/\mathbb{Q}_p$) and such that both $T^+$ and $T/T^+$ are free $R$-modules of rank one.

Set $A = T \otimes_R \text{Hom}_{cts}(R, \mathbb{Q}_p/\mathbb{Z}_p)$ and $A^+ = T^+ \otimes_R \text{Hom}_{cts}(R, \mathbb{Q}_p/\mathbb{Z}_p)$. We note that one has $E_p^\infty = A[P]$. Then following [CS2, Section 4] or [SS, Section 6], we define the Selmer group of the Hida deformation over an admissible $p$-adic Lie extension $F_{\infty}$ of $\mathbb{Q}$ by

$$S(A/F_{\infty}) = \ker \left( H^1(G_{\infty}(F_{\infty}), A) \longrightarrow \bigoplus_{v \in S} J_v(A, F_{\infty}) \right),$$

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is the same as the $Z$-R3.3. For the second part, it suffices to show that $X$ implies that $R$ over $Z$.

Proof. By identifying $X$ that contains a closed normal subgroup $N$ which is finitely generated over $Z$ in $[SS, \text{Theorem 7.4}]$, one can show that $S$ is completely faithful over $Z$.

Remark 4.5. One can also obtain completely faithful Selmer groups of $p$-adic representations defined over coefficient rings $Z_p[[X_1, X_2, ..., X_n]]$ over strongly admissible $p$-adic Lie extensions of the form considered in this section.

We now give some examples to illustrate the results in this section.

(a) Let $E$ be the elliptic curve $11a2$ of Cremona’s table which is given by

$$y^2 + y = x^3 - x.$$ 

Take $p = 5$, $F = Q(\mu_5)$ and $L_\infty = Q(\mu_{5\infty}, 11^{5\infty})$. By [HV] Theorem 6.2, $X(E/L_\infty)$ is a free $Z_5[[\text{Gal}(L_\infty/F^{\text{cyc}})]]$-module of rank four. Let $F_\infty$ be a strongly admissible $5$-adic Lie extension of $F$. 

We will denote by $X(A/F_\infty)$ the Pontryagin dual of this Selmer group. We will consider this dual Selmer group as a (compact) $\text{Gal}(F_\infty/F)$-module for some finite extension $F$ of $Q$ in $F_\infty$, where $F_\infty$ is a strongly admissible $p$-adic Lie extension of $F$.

**Theorem 4.4.** Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with Galois group $G$. Suppose that $G$ contains a closed normal subgroup $N$ which is contained in $H$, and suppose that the following statements are satisfied.

(i) There is a finite family of closed normal subgroups $N_i$ ($0 \leq i \leq r$) of $G$ such that $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$ and such that $N_i/N_{i-1} \cong \mathbb{Z}_p$ for $1 \leq i \leq r$.

(ii) $G/N$ is a non-abelian group isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

If $X(A/F_\infty)$ is finitely generated over $R[[H]]$ with positive $R[[H]]$-rank, then $X(A/F_\infty)$ is completely faithful over $R[[G]]$. In particular, if $X(E/F_\infty)$ is finitely generated over $Z_p[[H]]$ with positive $Z_p[[H]]$-rank, then $X(A/F_\infty)$ is completely faithful over $R[[G]]$.

Proof. By identifying $R[G] \cong Z_p[Z_p \times G]$, the first part of the theorem is immediate from Theorem 3.3. For the second part, it suffices to show that $X(A/F_\infty)$ is finitely generated over $R[[H]]$ with positive $R[[H]]$-rank. By [CS2, Theorem 4.2], the map

$$X(A/F_\infty)/P \rightarrow X(E/F_\infty)$$

has cokernel which is finitely generated over $Z_p$. Since $X(E/F_\infty)$ has positive $Z_p[[H]]$-rank, this in turn implies that $X(A/F_\infty)/P$ has positive $Z_p[[H]]$-rank. By an application of an argument in the spirit to that in [SS, Theorem 7.4], one can show that $S(A/F_\infty)/P = 0$. Equivalently, this is the same as saying that $X(A/F_\infty)[P] = 0$. Therefore, we may apply Lemma 2.4 to conclude that $R[[H]]$-rank of $X(A/F_\infty)$ is the same as the $Z_p[[H]]$-rank of $X(E/F_\infty)$ and, in particular, is positive which is as required. \qed

Remark 4.5. One can also obtain completely faithful Selmer groups of $p$-adic representations defined over coefficient rings $Z_p[[X_1, X_2, ..., X_n]]$ over strongly admissible $p$-adic Lie extensions of the form considered in this section.
contains $L_{\infty}$ and that the group $N = \text{Gal}(F_{\infty}/L_{\infty})$ satisfies the conditions in Theorem 4.1. A similar argument as in Corollary 4.2 shows that $X(E/F_{\infty})$ is a finitely generated $\mathbb{Z}_p[H]$-module of positive rank. Then we may apply Theorem 4.1 to conclude that $X(E/F_{\infty})$ is completely faithful over $\mathbb{Z}_p[G]$. Now if $E'$ is either 11a1 or 11a3, then it follows from Proposition 5.1 that $X(E'/F_{\infty})$ is faithful over $\mathbb{Z}_p[G]$. We claim that in either cases, $X(E'/F_{\infty})$ is not completely faithful. To see this, it suffices, by Proposition 3.3, to show that $X(E'/F_{\infty})(5)$ is not pseudo-null, or equivalently, the $\mu_{G}$-invariant of $X(E'/F_{\infty})$ is positive. By an application of [CFKSV, Lemma 5.6], we have that $X(A/F_{\infty})$ belongs to $\mathfrak{M}_{\infty}(G)$. This in turn allows us to apply [Lim2, Theorem 3.1] to conclude that $\mu_{G}(X(E'/F_{\infty})) = \mu_{T}(X(E'/F_{\infty}))$. But this latter quantity is well-known to be nonzero (cf. [CS1, Theorem 5.28]) and hence our claim is established.

(b) The next example is taken from [Jh]. Let $E$ be the elliptic curve 79a1 of Cremona’s table which is given by

$$y^2 + xy + y = x^3 + x^2 - 2x.$$ 

Take $p = 3$ and $F = \mathbb{Q}(\mu_3)$. As noted in [Jh], $X(E/F^c)$ is isomorphic to $\mathbb{Z}_3$. Let $F_{\infty}$ be a strongly admissible 5-adic Lie extension of $F$ that satisfies the conditions in Theorem 4.1. Write $G = \text{Gal}(F_{\infty}/F)$. By Corollary 4.2, $X(E/F_{\infty})$ is a completely faithful $\mathbb{Z}_3[G]$-module. Let $A$ be the Galois module obtained from the Hida family associated to $E$ as above. Therefore, one may apply Theorem 4.1 to conclude that $X(A/F_{\infty})$ is a completely faithful $\mathbb{Z}_3[G]$-module. Examples of strongly admissible 3-adic extensions of $F$ that one can take are:

$$\mathbb{Q}(\mu_3, 2^{3^{-\infty}}), \quad \mathbb{Q}(\mu_3, 2^{3^{-\infty}}, 3^{3^{-\infty}}), \quad \mathbb{Q}(\mu_3, 3^{3^{-\infty}}, 5^{3^{-\infty}}), \quad \mathbb{Q}(\mu_3, 2^{3^{-\infty}}, 3^{3^{-\infty}}, 5^{3^{-\infty}}), \quad \text{etc.}$$

5 Isogeny invariance of faithful Selmer groups

In this short section, we will show that the property of faithfulness is an isogeny invariant. Namely, we will prove the following statement.

**Proposition 5.1.** Let $E_1$ and $E_2$ be two elliptic curves over $F$ with either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$ which are isogenous to each other. Let $F_{\infty}$ be a strongly admissible noncommutative $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_{\infty}/F)$. Assume that both $X(E_1/F_{\infty})$ and $X(E_2/F_{\infty})$ are torsion over $\mathbb{Z}_p[G]$, and that the localization maps $\lambda_{E_1/F_{\infty}}$ and $\lambda_{E_2/F_{\infty}}$ are surjective. Then $X(E_1/F_{\infty})$ is a faithful $\mathbb{Z}_p[G]$-module if and only if $X(E_2/F_{\infty})$ is a faithful $\mathbb{Z}_p[G]$-module.

**Proof.** Let $\varphi : E_1 \longrightarrow E_2$ be an isogeny defined over $F$. By a standard argument to that in the proof of [HV] Theorem 5.1 or [Ho] Theorem 3.1, we can show that $\varphi$ induces a $\mathbb{Z}_p[G]$-homomorphism

$$X(E_2/F_{\infty}) \longrightarrow X(E_1/F_{\infty}),$$

whose kernel and cokernel are killed by $p^n$ for some large enough $n$. The required conclusion is now immediate from an application of Lemma 2.3.
6 Control theorems for faithfulness of Selmer groups

In this section, we will prove two control theorems on faithfulness of Selmer groups which can be applied to a \( p \)-adic Lie extension whose Galois group is not of the form in Theorem 3.3. We retain the notation of Section 4. In particular, our result will also show that one cannot obtain nonfaithful Selmer groups from a faithful Selmer group by adjoining \( \mathbb{Z}_p \)-extension or moving into the Hida deformation in general.

The first control theorem concerns extensions of admissible \( p \)-adic Lie extensions. Recall that \( S \) is a finite set of primes of \( F \) which contains the primes above \( p \), the infinite primes, the primes at which \( E \) has bad reduction and the primes that are ramified in \( F_\infty /F \).

**Proposition 6.1.** Let \( E \) be an elliptic curve over \( F \) with either good ordinary reduction or multiplicative reduction at every prime of \( F \) above \( p \). Let \( F_\infty \) be a strongly admissible \( p \)-adic Lie extension of \( F \) with \( G = \text{Gal}(F_\infty /F) \). Suppose that the following statements hold.

(i) \( N \) is a closed normal subgroup of \( G \) which is contained in \( H \), and there is a finite family of closed normal subgroups \( N_i \) \((0 \leq i \leq r)\) of \( G \) such that \( 1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N \) and such that \( N_i/N_{i-1} \cong \mathbb{Z}_p \) for \( 1 \leq i \leq r \).

(ii) \( G/N \) is a non-abelian pro-\( p \) \( p \)-adic Lie group without \( p \)-torsion. (In particular, the dimension of the \( p \)-adic Lie group \( G/N \) is necessarily \( \geq 2 \).)

(iii) Set \( L_\infty := F_\infty^N \). Suppose that \( X(E/L_\infty) \) is torsion over \( \mathbb{Z}_p[G/N] \).

(iv) Suppose that either (a) or (b) holds.

(a) \( E_{F_\infty} \) is not rational over \( F_\infty^N \) and \( X(E/L_\infty) \) is a faithful \( \mathbb{Z}_p[G/N]\)-module.

(b) For every \( v \in S \), the decomposition group of \( \text{Gal}(L_\infty /F) \) at \( v \) has dimension \( \geq 2 \), \( X(E/L_\infty) \) is a completely faithful \( \mathbb{Z}_p[G/N]\)-module and \( r = 1 \).

Then \( X(E/F_\infty) \) is a faithful \( \mathbb{Z}_p[G]\)-module.

**Proof.** We first note that it follows from condition (i) and (iii) that \( X(E/F_\infty) \) is torsion over \( \mathbb{Z}_p[G] \). This can be seen by applying either a similar argument to that in [SS] Theorem 7.4] iteratively or an argument to that in [HO] Theorem 2.3]. Now suppose that condition (iv)(a) holds. Clearly, it suffices to prove the proposition in this case assuming \( r = 1 \). By an application of an argument in the spirit to that in [SS] Theorem 7.4], one can show that \( H_1(N, X(A/F_\infty)) = 0 \). This in turn implies that \( X(E/F_\infty)[\gamma_N - 1] = 0 \), where \( \gamma_N \) is a topological generator of \( N \). As observed in the proof of Theorem 3.3, \( I_N \) is closed in \( \mathbb{Z}_p[G] \) and so \( \cap_{i \geq 1} I_{N_i} = 0 \). Therefore, by Lemma 2.2, we are reduced to proving that \( X(E/F_\infty)_N \) is a faithful
\[ Z_p[[G/N]] \text{-module. Now, applying an argument similar to that in } [\text{CS2}] \text{ Lemma 2.4}, \text{ we have that the dual of the cokernel of the map} \]
\[ \alpha : X(E/F_\infty)_N \rightarrow X(E/L_\infty) \]
\[ \text{is contained in } H^1(N, E(F_\infty)_{p^\infty}). \text{ By the first assumption in condition (iv)(a) and } [\mathbb{Z}] \text{ Proposition 10}, \text{ we have that } H^0(N, E(F_\infty)_{p^\infty}) = E(L_\infty)_{p^\infty} \text{ is finite. On the other hand, it follows from Lemma } [\text{SS2}] \text{ that} \]
\[ \text{corank}_{Z_p} E(L_\infty)_{p^\infty} = \text{corank}_{Z_p} H^1(N, E(F_\infty)_{p^\infty}) + \text{corank}_{Z_p} [N] E(F_\infty)_{p^\infty}. \]
\[ \text{Therefore, it follows that } H^1(N, E(F_\infty)_{p^\infty}), \text{ and hence the cokernel of } \alpha, \text{ is finite. We may now combine Lemma } [\text{2.4}] \text{ with the second assumption of condition (iv)(a) to conclude that } X(E/F_\infty)_N \text{ is a faithful } Z_p[[G/N]] \text{-module.} \]

We now consider the case when condition (iv)(b) holds. As above, it suffices to show that \( X(E/F_\infty)_N \) is a faithful \( Z_p[[G/N]] \)-module. Again, by the argument of \([\text{CS2}] \text{ Lemma 2.4}\), one can show that the map
\[ \alpha : X(E/F_\infty)_N \rightarrow X(E/L_\infty) \]
has kernel which is cofinitely generated over \( Z_p[[H/N]] \), and cokernel which is cofinitely generated over \( Z_p \). In particular, by condition (ii) and Theorem \([\text{2.1}]\) the dual of the cokernel of \( \alpha \) is pseudo-null over \( Z_p[[G/N]] \). Furthermore, in view of the first assumption of condition (iv)(b), one can apply a similar argument in the spirit of the proof of \([\text{SS}] \text{ Lemma 8.7}\) to show that the dual of the cokernel of \( \alpha \) is a finitely generated torsion \( Z_p[[H/N]] \)-module, and therefore, is pseudo-null over \( Z_p[[G/N]] \). Hence we have
\[ q(X(E/F_\infty)_N) = q(X(E/L_\infty)), \]
where \( q \) is the quotient functor from the category of finitely generated \( Z_p[[G/N]] \)-modules to the category of finitely generated \( Z_p[[G/N]] \)-modules modulo pseudo-null \( Z_p[[G/N]] \)-modules. Since \( X(E/L_\infty) \) is completely faithful over \( Z_p[[G/N]] \), it follows that \( X(E/F_\infty)_N \) is a faithful \( Z_p[[G/N]] \)-module, as required. \( \square \)

**Remark 6.2.** It is clear from the proof that under condition (iv)(b), one actually shows that \( X(E/F_\infty)_N \) is a completely faithful \( Z_p[[G/N]] \)-module. However, due to the lack of an analogous result for completely faithful modules in the direction of Lemma \([\text{2.2}]\), we are not able to deduce complete faithfulness of \( X(E/F_\infty) \) from the complete faithfulness of \( X(E/F_\infty)_N \). This is also precisely the reason why we require the extra assumption that \( r = 1 \) in condition (iv)(b).

In the next proposition, we mention the best we can do when we do not assume \( r = 1 \) in condition (iv)(b) which might be of interest.

**Proposition 6.3.** Retaining the assumptions of Proposition \([\text{6.7}]\) Furthermore, we assume that the action of \( G \) on \( \{N_i/N_{i-1}\} \) by inner automorphism is given by a homomorphism \( \chi_i : G/N \rightarrow Z_p^\times \) for every \( i \).
Suppose that for every \( v \in S \), the decomposition group of \( \text{Gal}(L_\infty/F) \) at \( v \) has dimension \( \geq 2 \), and suppose that \( X(E/L_\infty) \) is a completely faithful \( Z_p[[G/N]] \)-module. Then \( X(E/F_\infty)_N \) is a completely faithful \( Z_p[[G/N]] \)-module, and for \( i \geq 1 \), \( H_i(N, X(E/F_\infty)) \) is either a pseudo-null \( Z_p[[G/N]] \)-module or a completely faithful \( Z_p[[G/N]] \)-module.
Proof. The proof of Proposition 6.1 carries over to show that $X(E/F_\infty)_N$ is a completely faithful $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module. By [K3, Proposition 4.2], $H_i(N, X(E/F_\infty))$ is a successive extension of twists of a $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-subquotient $T$ of $X(E/F_\infty)_N$ by a one dimensional character. Therefore, if $H_i(N, X(E/F_\infty))$ is not a pseudo-null $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module, then $T$ cannot be a pseudo-null $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module. Since $X(E/F_\infty)_N$ is completely faithful over $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$, so is $T$. It is not difficult to verify that every twist of $T$ by a one dimensional character is also completely faithful over $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$. Hence we may apply Lemma 2.5 to conclude that $H_i(N, X(E/F_\infty))$ is completely faithful over $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$.

We also mention that it is clear from the proof of Proposition 6.1 that we can prove the following proposition for a general $N$.

**Proposition 6.4.** Let $E$ be an elliptic curve over $F$ with either good ordinary reduction or multiplicative reduction at every prime of $F$ above $p$. Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with $G = \text{Gal}(F_\infty/F)$. Suppose that the following statements hold.

(i) $N$ is a closed normal subgroup of $G$ which is contained in $H$.

(ii) $G/N$ is a non-abelian pro-$p$ $p$-adic Lie group without $p$-torsion. (In particular, the dimension of the $p$-adic Lie group $G/N$ is necessarily $\geq 2$.)

(iii) Set $L_\infty := F_{\infty}^N$. Suppose that either (a) or (b) holds.

(a) $E_{p^\infty}$ is not rational over $F_\infty$ (note the slight difference here) and $X(E/L_\infty)$ is a faithful $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module.

(b) For every $v \in S$, the decomposition group of $\text{Gal}(L_\infty/F)$ at $v$ has dimension $\geq 2$, and $X(E/L_\infty)$ is a completely faithful $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module.

Then $X(E/F_\infty)_N$ is a faithful $\mathbb{Z}_p[\mathbb{G}/\mathbb{N}]$-module.

The next control theorem is in the direction of a Hida deformation. We recall that $A$ is the $R$-cofree Galois module attached to the Hida deformation as defined at the end of Section 4, where $R = \mathbb{Z}_p[[X]]$, and has the property that $A[P] = E_{p^\infty}$ for some prime ideal $P$ of $R$. As before, we denote by $X(A/F_\infty)$ the dual Selmer group of the Hida deformation.

**Proposition 6.5.** Let $F_\infty$ be a strongly admissible $p$-adic Lie extension of $F$ with Galois group $G$. Suppose that the following statements hold.

(i) $G$ is non-abelian and has dimension $\geq 2$.

(ii) $X(E/F_\infty)$ is torsion over $\mathbb{Z}_p[G]$.

(iii) Either (a) or (b) holds.

(a) $E_{p^\infty}$ is not rational over $F_\infty$ and $X(E/F_\infty)$ is a faithful $\mathbb{Z}_p[G]$-module.
(b) For every \( v \in S \), the decomposition group of \( G \) at \( v \) has dimension \( \geq 2 \), and \( X(E/F_\infty) \) is a completely faithful \( \mathbb{Z}_p[G] \)-module.

Then \( X(A/F_\infty) \) is faithful over \( \mathcal{R}[G] \).

Proof. The proof is essentially similar to that in Proposition 6.1. The only thing which perhaps requires additional attention is to show that the cokernel of the map
\[
\beta : X(A/F_\infty)/P \longrightarrow X(E/F_\infty)
\]
is finite under the assumption of condition (iii)(a). Note that the dual of its cokernel is contained in \( A(F_\infty)/P \), where we write \( A(F_\infty) = A^{\text{Gal}(F/F_\infty)} \). Noting that \( A[P] = E_p^\infty \), it then follows from (the dual of) Lemma 2.1 that
\[
\text{corank}_{\mathbb{Z}_p} E(F_\infty)^{\text{p}} = \text{corank}_R A(F_\infty) + \text{corank}_{\mathbb{Z}_p} A(F_\infty)/P.
\]
Again, by the first assumption of condition (iii)(a) and \([Z, \text{Proposition 10}]\), we have that \( E(F_\infty)^{\text{p}} \) is finite. Combining this observation with the above equation, we have that the cokernel of \( \beta \) is finite. The remainder of the proof proceeds as in Proposition 6.1.

Remark 6.6. Of course, one can have a control theorem result for the faithfulness of the Selmer group of the Hida deformation for other specializations. In particular, one can also generalize the above control theorem to (appropriate) Selmer groups of more general deformations over \( \mathbb{Z}_p[X_1,\ldots,X_r] \) and their various intermediate specializations as considered in \([G]\).

We end the paper discussing an example to illustrate our control theorem results. Let \( p = 5 \). Let \( E \) be the elliptic curve 21a4 of Cremona’s tables given by
\[
y^2 + xy = x^3 + x
\]
and let \( A \) be an elliptic curve 1950y1 of Cremona’s tables
\[
A : y^2 + xy = x^3 - 355303x - 89334583.
\]
Let \( p = 5 \), \( F = \mathbb{Q}(\mu_5) \) and \( L_\infty = F(A_5^\infty) \). As discussed in \([Z, \text{Section 7}]\), if \( X(E/F) \) is finite (as suggested by its \( p \)-adic \( L \)-function), then \( X(E/L_\infty) \) is a completely faithful \( \mathbb{Z}_5[\text{Gal}(L_\infty/F)] \)-module. We will assume this latter property throughout our discussion here. Let \( F_\infty \) be any strongly admissible 5-adic Lie extension of \( F \) which contains \( L_\infty \) and such that \( N = \text{Gal}(F_\infty/L_\infty) \) satisfies the condition in Proposition 6.1. It is not difficult to see that \( E_5^\infty \) is not rational over \( F_\infty \). Hence we can apply Proposition 6.1 to conclude that \( X(E/F_\infty) \) is a faithful \( \mathbb{Z}_5[\text{Gal}(F_\infty/F)] \)-module. Examples of strongly admissible 5-adic extensions \( F_\infty \) that we may take are:
\[
\mathbb{Q}(A[5^\infty], 2^5\infty), \quad \mathbb{Q}(A[5^\infty], 2^5\infty, 3^5\infty), \quad \mathbb{Q}(A[5^\infty], 3^5\infty, 5^5\infty), \quad \mathbb{Q}(A[5^\infty], 2^5\infty, 3^5\infty, 5^5\infty)
\]
\[ Q(A[5^\infty], 5^{-\infty}, 3^{-\infty}, 5^{-\infty}, 11^{-\infty}), \ldots \text{etc}, \]
\[ M_\infty(A[5^\infty], 2^{-\infty}, 3^{-\infty}), \]
\[ M_\infty(A[5^\infty], 2^{-\infty}, 3^{-\infty}, 5^{-\infty}), \]
\[ M_\infty(A[5^\infty], 2^{-\infty}, 3^{-\infty}, 5^{-\infty}, 7^{-\infty}), \ldots \text{etc}, \]

where \( M_\infty \) is any \( \mathbb{Z}_p \)-extension of \( F \) disjoint from \( F_{\text{cyc}} \) for \( 1 \leq r \leq 2 \). However, at present, we are not able to determine whether or not \( X(E/F_\infty) \) is completely faithful for any of such \( F_\infty \).

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