On the Properties of the Synthetic Control Estimator with Many Periods and Many Controls*

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Abstract

We consider the asymptotic properties of the original Synthetic Control (SC) estimator when both the number of pre-treatment periods and control units are large. If potential outcomes follow a linear factor model, we provide conditions under which the factor loadings of the SC unit converge in probability to the factor loadings of the treated unit. This happens when there are weights diluted among many control units such that a weighted average of the factor loadings of the control units reconstructs the factor loadings of the treated unit. In this case, the SC estimator is asymptotically unbiased even when treatment assignment is correlated with time-varying unobservables. This result can be valid even when the number of control units is larger than the number of pre-treatment periods.

Keywords: counterfactual analysis, comparative studies, synthetic control, policy evaluation, panel data, factor models.

JEL Codes: C13; C21; C23

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1 Introduction

The Synthetic Control (SC) estimator, proposed in a series of influential papers by Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015), quickly became one of the most popular methods for policy evaluation (e.g., Athey and Imbens (2016)). An important advantage of the SC method is that it can potentially allow for correlation between treatment assignment and time-varying unobserved covariates. Assuming a perfect pre-treatment fit condition, Abadie et al. (2010) show that the bias of the SC estimator is bounded by a function that asymptotes to zero when the number of pre-treatment periods increases and the number of control units is fixed. However, when the perfect pre-treatment fit condition is relaxed and the number of control units is fixed, Ferman and Pinto (2019) show that the SC estimator is generally biased when treatment assignment is correlated with time-varying unobservables. In settings where the number of control units and pre-treatment periods are both large, there is a series of alternative methods, many of them based on the original SC estimator, that allow for selection on time-varying unobservables. However, the properties of the original SC estimator — which remains commonly used in empirical applications — when both the number of pre-treatment periods and control units go to infinity received less attention.

In this paper, we consider the asymptotic properties of the original SC estimator when both the number of pre-treatment periods and the number of control units increase. We consider a linear factor model structure for potential outcomes, and derive conditions under which, in this setting, the factor loadings of the SC unit — which is a weighted average of the factor loadings of the control units — converge in probability to the factor loadings of the treated unit. This will be the case when (i) there exist weighted averages of the factor

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1 We refer to perfect pre-treatment fit as the existence of weights such that a weighted average of the control units equal to outcome of the treated unit for all pre-treatment periods. Botosaru and Ferman (2019) and Powell (2019) also consider the properties of the SC and related estimators with a perfect pre-treatment fit condition.

2 See, for example, Arkhangelsky et al. (2018), Athey et al. (2017), Gobillon and Magnac (2016), Bai (2009), and Xu (2017).
loadings of the control units that recover the factor loadings of the treated unit, and (ii) when the number of control units goes to infinity, it is possible to construct such weighted averages with weights that are diluted among many control units. Under these conditions, the factor loadings of the SC unit are consistent estimators of the factor loadings of the treated unit even in settings in which the number of pre-treatment periods and the number of control units are roughly of the same size, or there are more control units than pre-treatment periods, which is common in SC applications (e.g., Doudchenko and Imbens (2016)).

The intuition is the following. Ferman and Pinto (2019) show that, in a setting with a fixed number of control units and imperfect pre-treatment fit, the SC weights converge to weights that, in general, do not converge to weights that recover the factor loadings of the treated unit when the number of pre-treatment periods increases. The reason is that the SC weights converge to weights that attempt to, at the same time, recover the factor loadings of the treated unit and minimize the variance of a linear combination of the transitory shocks. However, when the number of control units increases, the importance of this variance of the linear combination of the transitory shocks vanishes if it is possible to recover the factor loadings of the treated unit with weights that are diluted among an increasing number of control units. Therefore, the SC weights converge to weights that recover the factor loadings of the treated unit. As a consequence, the SC estimator is asymptotically unbiased even when treatment assignment is correlated with time-varying unobservables.3

While increasing the number of control units increases the number of parameters to be estimated, as shown by Chernozhukov et al. (2017), the non-negativity and adding-up constraints work as a regularization method. This is why it is possible to consistently estimate the counterfactual for the treated unit even when the number of control units grows at a faster rate than the number of pre-treatment periods. Our main result is to provide

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3Arkhangelsky et al. (2018) show that SC weights with an $L_2$ penalization, to ensure that in large samples there will be many units with positive weights, consistently estimates a low-rank matrix structure when the penalization constraint becomes tighter. While they also rely on a condition that there are weights diluted among an increasing number of control units that recover the factor loadings of the treated unit, we do not require an $L_2$ penalization in the estimation of the weights, so our results are valid for the original SC weights, which does not use such penalization.
conditions for the consistency of the factor loadings of the SC unit even when the linear factor model structure induces a non-zero correlation between the outcome of the control units and the error in a linear model that relates the outcomes of the treated and the control units using balancing weights.\textsuperscript{4} Importantly, we also show that such regularization implies that, asymptotically, there is no over-fitting. Asymptotically, the SC unit absorbs only the common factor structure, so that the pre-treatment fit will not be perfect due to the transitory shocks, even when the number of control units increases. This highlights that the asymptotic unbiasedness of the SC estimator we derive does not come from improvements in the pre-treatment fit due to an increased number of control units. Rather, it comes from the fact that, under the conditions we consider for the factor loadings, it is possible to construct balancing weights such that the linear combination of the transitory shocks of the control units using those weights converge to zero.

If we relax the non-negativity constraint, then the estimator for the factor loadings of the treated unit will still be asymptotically unbiased when both the number of pre-treatment periods and the number of controls increase.\textsuperscript{5} However, due to the lack of regularization, this estimator will not converge, unless the difference between the number of pre-treatment periods and the number of control units goes to infinity. As a consequence, while the estimator for the treatment effects based on those weights would be asymptotically unbiased, the variance of the estimator would diverge. This highlights the importance of using regularization methods when the number of pre-treatment periods is not much larger than the number of control units.

\textsuperscript{4}We refer to “balancing weights” as weights such that a linear combination of the factor loadings of the control units recover the factor loadings of the treated unit.

\textsuperscript{5}In this case, we need that the number of pre-treatment periods is greater or equal to the number of control units, so that the estimator is well defined.
2 Setting

We start with \( j = 0, 1, \ldots, J \) units, where unit 0 is treated and the other units are control. Potential outcomes are determined by a linear factor model,

\[
\begin{align*}
  y_{it}(0) &= \lambda_t \mu_i + \epsilon_{it} \\
  y_{it}(1) &= \alpha_{it} + y_{it}(0),
\end{align*}
\]

where \( \lambda_t \) is an \( 1 \times F \) vector of common factors, \( \mu_i \) is an \( F \times 1 \) vector of unknown factor loadings, and the error terms \( \epsilon_{it} \) are unobserved transitory shocks. We only observe \( y_{it} = d_{it}y_{it}(1) + (1 - d_{it})y_{it}(0) \), where \( d_{it} = 1 \) if unit \( i \) is treated at time \( t \). We treat the vector of unknown factor loads (\( \mu_i \)) as fixed and the common factors (\( \lambda_t \)) as random variables. Let \( \mu \) be a \( J \times F \) matrix that collects the information on the factor loadings of the control units (that is, each line \( j \) of \( \mu \) is equal to \( \mu_i' \)).

We consider that we observe \( (y_{0t}, \ldots, y_{Jt}) \) for periods \( t \in \{-T_0 + 1, \ldots, -1, 0, 1, \ldots, T_1\} \), where treatment is assigned to unit 0 after time 0. Therefore, we have \( T_0 \) pre-treatment periods and \( T_1 \) post-treatment periods. Let \( T_0 \) (\( T_1 \)) be the set of time indices in the pre-treatment (post-treatment) periods. The main goal of the SC method is to estimate the effect of the treatment for unit 1 for each post-treatment \( t \), that is \( \{\alpha_{01}, \ldots, \alpha_{0T_1}\} \).

In a sequence of papers, Abadie and Gardeazabal (2003), Abadie et al. (2010), and Abadie et al. (2015) proposed the SC method to estimate weights for the control units to construct a counter-factual for \( \{y_{01}(0), \ldots, y_{0T_1}(0)\} \). In a version of the method where all pre-treatment outcome lags are included as predictor variables, those weights are estimated by minimizing the pre-treatment sum of squared residuals subject to the constraint that weights must be non-negative and sum one. Abadie et al. (2010) show that, if there are weights that provide a perfect pre-treatment fit, then the bias of the SC estimator is bounded by a function that asymptotes to zero when \( T_0 \) increases, even when \( J \) is fixed. By perfect pre-treatment fit we mean that there is a \( (w_1, \ldots, w_J) \in \Delta^{J-1} \) such that \( y_{0t} = \sum_{j=1}^{J} w_j y_{jt} \) for all \( t \in T_0 \), where
\( \Delta^{J-1} \equiv \{(w_1, ..., w_J) \in \mathbb{R}^J \mid w_j \geq 0 \text{ and } \sum_{j=1}^J w_j = 1 \} \). However, Ferman and Pinto (2019) show that, if the pre-treatment fit is imperfect, then the SC weights will not generally recover the factor loadings of the treated unit, so the SC estimator will be biased if there is selection on unobservables. They show that this result is valid even when \( T_0 \to \infty \), as long as \( J \) is fixed. The main reason is that, for any \( w^* \in \mathbb{R}^J \) such that \( \mu_0 = \mu' w^* \), it is possible to write

\[
y_{0t}(0) = y'_t w^* + \epsilon_{0t} - \epsilon'_t w^*,
\]

where \( y_t = (y_{1t}, ..., y_{Jt}) \), and \( \epsilon_t = (\epsilon_{1t}, ..., \epsilon_{Jt}) \). Therefore, the outcomes of the control units serve as a proxy for the factor loadings of the treated unit. However, the linear factor model structure inherently generates a correlation between \( y_t \) and the error in this model due to the transitory shocks \( \epsilon_t \).

In this paper, we consider a setting in which both the number of control units and the number of treated units increase. This provides a better asymptotic approximation to settings in which the number of pre-treatment periods and the number of control observations are roughly of the same size, as is common in SC applications (e.g., Doudchenko and Imbens (2016)).

We consider a setting in which the number of control units increases by replicating the structure of the initial \( J \) control units \( P \) times. Therefore, for a given \( P \), we have \( \tilde{J} = P \times J \) control units, and the \( PJ \times F \) matrix \( \tilde{\mu} = [\mu' \cdots \mu']' \) collects information on the factor loadings of all \( \tilde{J} \) control units. Given this structure, \( \mu_j = \mu_{(p-1)J+j} \) for all \( j \in \{1, ..., J\} \) and \( p \in \{1, ..., P\} \). We assume that \( \epsilon_{jt} \) is uncorrelated across \( j \), so all possible spatial correlated shocks are captured by the linear factor model structure. However, we allow for serial correlation for both \( \epsilon_{jt} \) and \( \lambda_t \). Finally, let \( T_0(P) \) be the number of pre-treatment observations when we consider the case with \( P \) replications, so that the number of pre-treatment observations can increase with the number of control units. This sampling scheme mimics a setting in which there is a large number of control units, but this large group of
control units is relatively homogeneous, so that the correlation among units is based on a factor model with a fixed number of factors, $F$. Moreover, the correlation between units do not vanish when the number of control units increases. If there is a $w^* \in \mathbb{R}^J$ such that $\mu_0 = \mu' w^*$, then $\tilde{w}^* = (\frac{w'}{J}, ..., \frac{w'}{P})$ is such that $\mu_0 = \tilde{\mu}' \tilde{w}^*$, so there are weights that reconstruct $\mu_0$ that get diluted among many control units when $P \to \infty$. Note that this would not approximate well a setting in which new units become more uncorrelated with the treated unit as $\tilde{J}$ increases.

3 Asymptotic Behavior of the Original SC Estimator

Let $y^p_t$ be the $J \times 1$ vector with outcome values of the $p$-th replication of the controls, and the $PJ \times 1$ vector $\tilde{y}_t = [y^1_t, ..., y^P_t]'$. Let $\epsilon^p_t$ and $\tilde{\epsilon}_t$ be defined similarly. The SC weights are given by

$$\hat{w}_{SC} = \arg\min_{\tilde{w} \in \Delta_{PJ-1}} \left\{ \frac{1}{T_0(P)} \sum_{t \in T_0} (y_{0t} - \tilde{w}' \tilde{y}_t)^2 \right\}. \tag{3}$$

When $P \to \infty$, the dimension of $\hat{w}_{SC}$ increases. However, we are not inherently interested in $\hat{w}_{SC}$, but in the estimator of the factor loading of the treated unit that is generated from $\hat{w}_{SC}$, the $F \times 1$ vector $\hat{\mu}_0 = \tilde{\mu}' \hat{w}_{SC}$. We consider, therefore, the asymptotic behavior of $\hat{\mu}_0$. For a given $\tilde{w}$, let $\bar{\mu} \equiv \tilde{\mu}' \tilde{w}$. From the objective function in equation (3),

$$\frac{1}{T_0(P)} \sum_{t \in T_0} (y_{0t} - \tilde{w}' \tilde{y}_t)^2 = \frac{1}{T_0(P)} \sum_{t \in T_0} (\lambda_t(\mu_0 - \bar{\mu}) + \epsilon_{0t} - \tilde{w}' \tilde{\epsilon}_t)^2. \tag{4}$$

Therefore, if we define

$$\mathcal{H}_{T_0}(\bar{\mu}) = \min_{\tilde{w} \in \Delta_{PJ-1}} \left\{ \frac{1}{T_0(P)} \sum_{t \in T_0} (\lambda_t(\bar{\mu} - \tilde{\epsilon}_t)^2 \right\}, \tag{5}$$

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where \( \bar{\lambda}_t(\bar{\mu}) \equiv \lambda_t(\mu_0 - \bar{\mu}) + \epsilon_{0t} \), then \( \hat{\mu}_0 = \bar{\mu}' \hat{w}_{sc} = \arg\min_{\mu \in \mathcal{M}} \mathcal{H}_{T_0}(\mu) \), where \( \mathcal{M} = \{ \mu \in \mathbb{R}^F | \mu = \bar{\mu}'w \) for some \( w \in \Delta^{P-1} \} \).

Using this characterization of \( \hat{\mu}_0 \), we provide conditions under which \( \hat{\mu}_0 \xrightarrow{p} \mu_0 \) when \( P \to \infty \).

**Proposition 1** Let \( \hat{\mu}_0 \) be defined as \( \bar{\mu}' \hat{w}_{sc} \), where \( \hat{w}_{sc} \) is defined in equation (3). Assume that there exists weights \( w^* \in \Delta^{J-1} \) such that \( \mu'w^* = \mu_0 \). Also, assume the data is \( \beta \)-mixing with exponential speed, and satisfy other assumptions listed in the proof. Then, if \( \frac{P_J}{T_0(P)} \to c \in \mathbb{R}_+ \), \( \hat{\mu}_0 \xrightarrow{p} \mu_0 \) when \( P \to \infty \).

**Proof.** See details in Appendix A.1 ■

The intuition of the proof is that \( \mathcal{H}_{T_0}(\bar{\mu}) \xrightarrow{p} \sigma_\lambda^2(\bar{\mu}) \equiv \text{plim}_{T_0(P)} \frac{1}{T_0(P)} \sum_{t \in T_0} [\lambda_t(\mu_0 - \bar{\mu}) + \epsilon_{0t}]^2 \) uniformly in \( \bar{\mu} \in \mathcal{M} \) when \( P \to \infty \). Note that \( \sigma_\lambda^2(\bar{\mu}) \) is equal to the variance of the transitory shocks of the treated unit plus the second moment of the common shocks that affect the treated unit that remain after we subtract the common shocks that affect a weighted average of the control units. Therefore, \( \sigma_\lambda^2(\bar{\mu}) \) is uniquely minimized when \( \bar{\mu} = \mu_0 \).

Ferman and Pinto (2019) show that, when the number of control units is fixed, the SC weights converge to weights that do not, in general, recover \( \mu_0 \). This happens because, in a setting with a fixed number of control units, the SC weights converge to weights that simultaneously attempt to minimize both the second moments of this remaining common shocks, and the variance of a weighted average of the transitory shocks of the control units. Proposition 1 shows that, when both the number of pre-treatment periods and the number of controls increase, the importance of this variance of a weighted average of the transitory shocks of the control units vanishes, so that the asymptotic bias on the estimator of the SC weights disappears when both the number of pre-treatment periods and then number of controls increase.

A crucial condition for this result is that, as the number of control units increase, it is possible to recover \( \mu_0 \) with weights that are diluted among more control units. If we consider,
for example, a setting such that there is only a fixed number of control units that can be
used to recover $\mu_0$, and the additional control units are uncorrelated with $y_{0t}$, then the result
from Ferman and Pinto (2019) would still apply, and $\hat{\mu}_0$ would not converge to $\mu_0$. Note
that such setting would not be consistent with the sampling scheme described in Section 2.

The convergence in probability of $\hat{\mu}$ even when the number of pre-treatment periods is
smaller than the number of control units occurs because the non-negativity and adding-up
constraints work as a regularization method. Therefore, there is no over-fitting in the mini-
mimization presented in equation (5). The result that such restrictions work as a regularization
method is presented by Chernozhukov et al. (2017), who derive conditions under which the
original SC estimator converges in probability. While they consider the case in which the
outcome of the control units is uncorrelated with the error in a model similar to the one
presented in equation (2), our Proposition 1 provides conditions under which the original SC
estimator converges to weights that recover $\mu_0$ even when the linear factor model structure
induces such correlation. Increasing the number of control units is not sufficient to generate
this result. It is crucial that the number of control units that can be used to recover $\mu_0$
increases with the total number of control units.

Our results are also closely linked to Theorem 5 by Arkhangelsky et al. (2018), who con-
sider a penalized version of the SC weights. This penalized SC weights solve the minimization
problem presented in equation (3) subject to the additional constraint that $||\tilde{w}||_2 \leq a_w$. Since
$||\tilde{w}||_1 = 1 \Rightarrow ||\tilde{w}||_2 \leq 1$, note that the original SC weights are equivalent to the penalized
SC weights with $a_w = 1$. They show that the approximation error for their low-rank matrix
structure is bounded by the sum of three terms, where one of them is $a_w$. In contrast, we
show that, in our setting, the SC weights achieve such balancing even when the penalty term
$a_w$ does not go to zero, so it is not necessary to force weights to be positive for many control
units in large samples with an $L_2$ penalization term.\footnote{Another term in their bounds is the performance of an oracle estimator that balances on the factor
structure rather than on the outcomes (which they call $\delta_w$). Given our assumption that there exists weights
$w^* \in \Delta^{J-1}$ such that $\mu^* w^* = \mu_0$, and the structure presented in Section 2, it is possible to set $a_w \rightarrow 0$ such
that $\delta_w = 0$ for all $P$.}
Since the factor loadings of the treated unit converge in probability to $\mu_0$ and the SC weights get diluted among many control units, it follows that, for $t \in T_1$, $\hat{\alpha}_t^{sc} \overset{P}{\to} \alpha_t + \epsilon_{0t}$. Therefore, the SC estimator is asymptotically unbiased for $\alpha_t$ even when treatment assignment is correlated with time-varying unobserved variables. Moreover, asymptotically, the variance of the SC estimator depends only on the transitory shocks of the treated unit in period $t$.

Finally, note that, under the assumptions used in Proposition 1, $\frac{1}{T_0(P)} \sum_{t \in T_0} (y_{0t} - \tilde{w}'_{sc} \tilde{y}_t)^2$ converges in probability to $\sigma^2_\lambda(\mu_0)$, which is the variance of $\epsilon_{0t}$. Therefore, the SC unit will asymptotically absorb all variability of $y_{0t}$ that is related to the factor structure, but will not over-fit the idiosyncratic shocks of the treated unit, so we should not expect a perfect pre-treatment fit in this setting even when $\tilde{J}$ is larger than $T_0$. This highlights that the asymptotic unbiasedness result from Proposition 1 does not come from a better pre-treatment fit when we increase the number of control units. Rather, it comes from the fact that, in this sampling scheme, increasing the number of control units implies existence of balancing weights that are diluted among an increasing number of control units, implying that the problems highlighted by Ferman and Pinto (2019) become asymptotically irrelevant.

4 Relaxing the non-negativity constraint

We show that the convergence of the original SC estimator derived in Section 3 rely crucially on the non-negativity constraint. When the non-negativity constraint is relaxed, the estimator of $\mu_0$ would remain asymptotically unbiased, but it does not converge in probability, unless $\frac{P_J}{T_0(P)} \to c < 1$. We consider first the case with no constraint. The case with only the adding-up constraint is similar. In this case, the weights are estimated using the OLS regression

$$\hat{w}_{OLS} = \arg\min_{\tilde{w} \in \mathbb{R}^{P_J}} \left\{ \frac{1}{T_0(P)} \sum_{t \in T_0} (y_{0t} - \tilde{w}' \tilde{y}_t)^2 \right\}. \quad (6)$$
Following the same arguments presented in Section 3, \( \hat{\mu}_{OLS} \equiv \tilde{\mu}_{OLS}^T \tilde{w}_{OLS} \) is the solution to

\[
\mathcal{H}_{T_0}^{OLS}(\bar{\mu}) = \min_{\tilde{w} \in \mathbb{R}^{PJ}} \left\{ \frac{1}{T_0(P)} \sum_{t \in T_0} (\tilde{\lambda}_t(\bar{\mu}) - \tilde{w}^T \tilde{\epsilon}_t)^2 \right\},
\]

(7)

A crucial difference in this case is that, by not imposing any restriction on \( \tilde{w} \), this minimization problem will be subject to over-fitting when the number of degrees of freedom in (6) does not go to infinity when \( P \to \infty \). In the extreme example in which \( T_0(P) = PJ - 1 \), note that \( \mathcal{H}_{T_0}^{OLS}(\bar{\mu}) = 0 \) for all \( \bar{\mu} \) and for all \( P \), so \( \mathcal{H}_{T_0}^{OLS}(\bar{\mu}) \) does not converge to a function that is uniquely minimized at \( \mu_0 \). We can still show, however, that, under some conditions, \( \mathbb{E}[\hat{\mu}_{OLS} - \mu_0] \to 0 \). Let \( K(P) = T_0(P) - PJ \).

**Proposition 2** Suppose \( 0 \leq K(P) \leq \bar{K} \) for all \( P \). Assume that there exists weights \( w^* \in \mathbb{R}^J \) such that \( \mu^T w^* = \mu_0 \). Also, assume that \( \lambda_t \) is i.i.d. across \( t \) and jointly normally distributed with a positive definite covariance matrix \( \Omega \), \( \epsilon_{jt} \) is independent across \( j \) and independent of \( \lambda_{\tau} \) for all \( \tau \in T_0 \), and that \( J = F \) with \( \text{rank}(\mu) = J \). Then \( \mathbb{E}[\hat{\mu}_{OLS} - \mu_0] \to 0 \) when \( P \to \infty \), but \( \hat{\mu}_{OLS} \) converges to a non-degenerate distribution.

**Proof.** See details in Appendix A.2 ■

The intuition for \( \mathbb{E}[\hat{\mu}_{OLS} - \mu_0] \to 0 \) is the same as the intuition from Proposition 1. When the number of control increases, we are able to have a more diluted weighted average of the control units that recover \( \mu_0 \). This reduces the importance of the variance of the linear combination of the transitory shocks of the control units in the minimization problem (6) for the estimation of \( \tilde{w}_{OLS} \). We show in Appendix A.3 that this result remains valid when \( J > F \). However, differently from the original SC estimator, once we relax all constraints, the increase in the number of parameters estimated when \( P \to \infty \) implies a reduction of the same order in the number of degrees of freedom in the estimation. If \( 0 \leq K(P) \leq \bar{K} \), then there will only be at most \( K(P) \leq \bar{K} \) degrees of freedom to estimate \( \hat{\mu}_{OLS} \), regardless of \( P \), so this estimator will not converge in probability.7

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7While we derive these results under i.i.d. and normality assumptions on the factor loadings, the result
While $\hat{\mu}_{\text{OLS}}$ is asymptotically unbiased, it aggregates an increasing number of positive and negative components of $\hat{\mathbf{w}}_{\text{OLS}}$, where none of the components in $\hat{\mathbf{w}}_{\text{OLS}}$ converge in probability. Therefore, the variance of $\hat{\alpha}_{\text{OLS}} = y_{0t} - (\hat{\mathbf{w}}_{\text{OLS}})' \mathbf{y}_t = \lambda_t (\mu_0 - \hat{\mu}_{\text{OLS}}) + \epsilon_{0t} - (\hat{\mathbf{w}}_{\text{OLS}})' \mathbf{e}_t$, for $t \in T_1$, will diverge when $P \to \infty$.

When $\frac{P_J}{I_0(P)} \to c < 1$, the number of degrees of freedom increase with $P$, so $\hat{\mu}_{\text{OLS}} \overset{P}{\to} \mu_0$.

**Proposition 3** Suppose $\frac{P_J}{I_0(P)} \to c < 1$ when $P \to \infty$. Assume that there exists weights $\mathbf{w}^* \in \mathbb{R}^J$ such that $\mu' \mathbf{w}^* = \mu_0$. Also, assume that $\lambda_t$ and $\epsilon_{jt}$ are weakly dependent stationary processes, where $\epsilon_{jt}$ is independent across $j$ and independent of $\lambda_t$, and that $J = F$ with $\text{rank}(\mu) = J$. Then $\hat{\mu}_0 \overset{P}{\to} \mu_0$ when $P \to \infty$.

**Proof.** See details in Appendix A.4 ■

Therefore, using an OLS regression to estimate the weights without any regularization method can be a reasonable idea when the number of control units is large, but the number of pre-treatment periods is much larger than the number of controls units. An advantage relative to the original SC estimator is that the condition on the existence of $\mathbf{w}^* \in \mathbb{R}^J$ such that $\mu' \mathbf{w}^* = \mu_0$ is weaker than the condition on the existence of $\mathbf{w}^* \in \Delta^{J-1}$ such that $\mu' \mathbf{w}^* = \mu_0$. However, such procedure is problematic when the number of pre-treatment periods is not much larger than the number of control units. Including only the adding-up constraint (without the non-negativity constraint) only increases the number of degrees of freedom by one, so all results in this section remain valid in this case. When the number of pre-treatment periods is not much larger than the number of control units, other regularization methods could be used, as considered by, for example, Doudchenko and Imbens (2016), Arkhangelsky et al (2018), Carvalho et al. (2018), and Chernozhukov et al. (2017).

that the estimator of the factor loadings converges to a non-degenerate distribution should remain valid if we relax these assumptions.
5 Monte Carlo simulations

We present a simple Monte Carlo (MC) exercise to illustrate the main results presented in Sections 3 and 4. We consider a setting in which there are two common factors, \( \lambda_{1t} \) and \( \lambda_{2t} \). Potential outcomes for the treated unit and for half of the control units depend on the first common factor, so \( y_{jt} = \lambda_{1t} + \epsilon_{jt} \) for \( j = 0, 1, \ldots, \tilde{J}/2 \) and \( y_{jt} = \lambda_{2t} + \epsilon_{jt} \) for \( j = \tilde{J}/2 + 1, \ldots, \tilde{J} \). In this case, \( \mu_0 = (\mu_{1,0}, \mu_{2,0}) = (1, 0) \). Therefore, the goal of the SC method is to set positive weights only to units \( j = 1, \ldots, \tilde{J}/2 \), which would imply that the asymptotic distribution of \( \hat{\alpha}_t \) does not depend on the common factors. The common factors are normally distributed with a serial correlation equal to 0.5 and variance equal to 1; \( \lambda_{1t} \) and \( \lambda_{2t} \) are independent. The transitory shocks \( \epsilon_{jt} \) are i.i.d. normally distributed with variance equal to 1.

In Panel A of Table 1 we present results for the SC method when \( T_0(\tilde{J}) = \tilde{J} + 5 \), so the number of pre-treatment periods and the number of control units are roughly of the same size. When the number of control units is small (\( \tilde{J} = 4 \) or \( \tilde{J} = 10 \)), there is distortion in the proportion of weights allocated to the control units that follow the same common factor as the treated unit. For example, when there are 10 control units, around 82% of the weights are correctly allocated, while around 18% of the weights are misallocated. When \( \tilde{J} \) and \( T_0 \) increase, the proportion of misallocated weights goes to zero, which is consistent with Proposition 1. Interestingly, the standard error of \( \hat{\mu}_0 \) goes to zero when \( \tilde{J} \) increases, even when \( \tilde{J} \) and \( T_0 \) remains roughly at the same size. Moreover, the standard error of the treatment effect one period ahead, \( \hat{\alpha} \), converges to the standard error of the transitory shocks (\( \sigma = 1 \)).

We present in Table 2 results using OLS to estimate the weights. In this case, \( E[\hat{\mu}_{1,0}^{\text{OLS}}] < 1 \) when \( \tilde{J} \) is small, due to the endogeneity generated by the transitory shocks of the control units. When \( \tilde{J} \) increases, however, \( E[\hat{\mu}_{1,0}^{\text{OLS}}] \to 1 \), which is consistent with Propositions 2 and 3. However, differently from the SC weights, the standard error of \( \hat{\mu}_{1,0}^{\text{OLS}} \) does not go to zero, and remains roughly constant when \( \tilde{J} \) increases but \( \tilde{J} \) and \( T_0 \) remains roughly at the same size (Panel A). In contrast, when \( T_0 - \tilde{J} \) increases (Panel B), then the standard error of \( \hat{\mu}_{1,0}^{\text{OLS}} \)
goes to zero. The standard error of $\hat{\alpha}$ diverge with $\tilde{J}$ when $T_0 = \tilde{J} + 5$, but is decreasing with $\tilde{J}$ when $T_0 = 2 \times \tilde{J}$.

When weights are estimated with OLS using only the adding-up constraint, results are similar to the unrestricted OLS. The only difference is that $E[\hat{\mu}_{2,0}^{\text{OLS}}] = 0$ regardless of $\tilde{J}$ when we consider the unrestricted OLS. This happens because $\mu_{2,0} = 0$, so there is no endogeneity problem for this parameter when we consider the unrestricted OLS. In contrast, there is distortion in $\hat{\mu}_{2,0}$ when we include the restriction that weights should sum one (see Table 3).

6 Conclusion

We provide conditions under which that the SC estimator is asymptotically unbiased when both the number of pre-treatment periods and the number of control units increase. This will be the case when (i) there are weighted averages of the factor loadings of the control units that recover the factor loadings of the treated unit, and (ii) when the number of control units goes to infinity, it is possible to construct such weighted averages with weights that are diluted among many control units. Under these conditions, the SC estimator can be asymptotically unbiased even when there are more control units than pre-treatment periods, which is common SC applications.

We show that the non-negative and adding-up constraints are crucial for this result, as they provide regularization for cases in which the number of parameters to be estimated is larger than the number of pre-treatment periods. Without these constraints, the estimator for the treatment effect remains asymptotically unbiased, but its variance diverge, unless the number of pre-treatment periods is much larger than then number of control units.

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Table 1: Monte Carlo Simulations - Original Synthetic Control Estimator

|     | $J$ | 4   | 10  | 50  | 100 | (1) | (2) | (3) | (4) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Panel A: $T_0 = J + 5$ | $\mathbb{E}[^{\hat{\mu}}_{01}]$ | 0.752 | 0.820 | 0.907 | 0.933 | se[^{\hat{\mu}}_{01}] | 0.206 | 0.154 | 0.069 | 0.052 |
|     | $\mathbb{E}[^{\hat{\mu}}_{02}]$ | 0.248 | 0.180 | 0.093 | 0.067 | se[^{\hat{\mu}}_{02}] | 0.206 | 0.154 | 0.069 | 0.052 |
|     | $se(\hat{\alpha})$ | 1.313 | 1.236 | 1.098 | 1.029 | |
| Panel B $T_0 = 2 \times \tilde{J}$ | $\mathbb{E}[^{\hat{\mu}}_{01}]$ | 0.755 | 0.831 | 0.922 | 0.945 | se[^{\hat{\mu}}_{01}] | 0.215 | 0.135 | 0.055 | 0.039 |
|     | $\mathbb{E}[^{\hat{\mu}}_{02}]$ | 0.245 | 0.169 | 0.078 | 0.055 | se[^{\hat{\mu}}_{02}] | 0.215 | 0.135 | 0.055 | 0.039 |
|     | $se(\hat{\alpha})$ | 1.342 | 1.212 | 1.052 | 1.040 | |

Notes: this table presents the expected value and the standard error of the estimators for $\mu_0 = (\mu_{01}, \mu_{02})$ using the original SC method. It also presents the standard error of $\hat{\alpha}$ for this method. Since $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\tilde{\alpha}] = 0$, which is the true treatment effect. Panel A presents results with $T_0 = J + 5$, while Panel B presents results with $T_0 = 2 \times \tilde{J}$. The DGP is described in detail in Section 5.
Table 2: Monte Carlo Simulations - Unconstrained OLS

| J  | 4    | 10   | 50   | 100  |
|----|------|------|------|------|
|    | (1)  | (2)  | (3)  | (4)  |
| Panel A: $T_0 = \tilde{J} + 5$ |
| $\mathbb{E}[\hat{\mu}_{01}]$ | 0.649 | 0.813 | 0.944 | 1.004 |
| $se[\hat{\mu}_{01}]$       | 0.522 | 0.504 | 0.547 | 0.499 |
| $\mathbb{E}[\hat{\mu}_{02}]$ | 0.003 | -0.008 | 0.005 | -0.005 |
| $se[\hat{\mu}_{02}]$       | 0.496 | 0.490 | 0.509 | 0.505 |
| $se(\hat{\alpha})$         | 1.606 | 2.142 | 3.827 | 5.213 |

Panel B $T_0 = 2 \times \tilde{J}$

| $\mathbb{E}[\hat{\mu}_{01}]$ | 0.630 | 0.836 | 0.961 | 0.990 |
| $se[\hat{\mu}_{01}]$       | 0.550 | 0.336 | 0.141 | 0.099 |
| $\mathbb{E}[\hat{\mu}_{02}]$ | 0.009 | 0.000 | 0.004 | -0.005 |
| $se[\hat{\mu}_{02}]$       | 0.564 | 0.335 | 0.147 | 0.103 |
| $se(\hat{\alpha})$         | 1.826 | 1.513 | 1.446 | 1.441 |

Notes: This table presents the expected value and the standard error of the estimators for $\mu_0 = (\mu_{01}, \mu_{02})$ where weights are estimated using unrestricted OLS. It also presents the standard error of $\hat{\alpha}$ for this method. Since $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\hat{\alpha}] = 0$, which is the true treatment effect. Panel A presents results with $T_0 = \tilde{J} + 5$, while Panel B presents results with $T_0 = 2 \times \tilde{J}$. The DGP is described in detail in Section 5.
Table 3: Monte Carlo Simulations - OLS with Adding-up Constraint

|      | 4   | 10  | 50  | 100 |
|------|-----|-----|-----|-----|
|      | (1) | (2) | (3) | (4) |

Panel A: $T_0 = \tilde{J} + 5$

|      |      |      |      |      |
|------|------|------|------|------|
| $\mathbb{E}[\hat{\mu}_{01}]$ | 0.832 | 0.911 | 0.978 | 1.008 |
| $se[\hat{\mu}_{01}]$     | 0.317 | 0.326 | 0.310 | 0.310 |
| $\mathbb{E}[\hat{\mu}_{02}]$ | 0.168 | 0.089 | 0.022 | -0.008 |
| $se[\hat{\mu}_{02}]$     | 0.317 | 0.326 | 0.310 | 0.310 |
| $se(\hat{\alpha})$       | 1.477 | 1.946 | 3.394 | 4.701 |

Panel B $T_0 = 2 \times \tilde{J}$

|      |      |      |      |
|------|------|------|------|
| $\mathbb{E}[\hat{\mu}_{01}]$ | 0.812 | 0.918 | 0.979 | 0.998 |
| $se[\hat{\mu}_{01}]$     | 0.352 | 0.223 | 0.102 | 0.072 |
| $\mathbb{E}[\hat{\mu}_{02}]$ | 0.188 | 0.082 | 0.021 | 0.002 |
| $se[\hat{\mu}_{02}]$     | 0.352 | 0.223 | 0.102 | 0.072 |
| $se(\hat{\alpha})$       | 1.645 | 1.473 | 1.439 | 1.435 |

Notes: this table presents the expected value and the standard error of the estimators for $\mu_0 = (\mu_{01}, \mu_{02})$ where weights are estimated using OLS with the adding-up constraint. It also presents the standard error of $\hat{\alpha}$ for this method. Since $\mathbb{E}[\lambda_t]$, $\mathbb{E}[\hat{\alpha}] = 0$, which is the true treatment effect. Panel A presents results with $T_0 = \tilde{J} + 5$, while Panel B presents results with $T_0 = 2 \times \tilde{J}$. The DGP is described in detail in Section 5.
A Appendix

A.1 Proof of Proposition 1

Proof.

We show that \(H_{T_0}(\mu) \xrightarrow{p} H(\mu)\) uniformly in \(\mu\), where \(H(\mu)\) is continuous and uniquely minimized at \(\mu_0\).

First, let \(\bar{w}(\bar{\mu}) \in \mathbb{R}^J\) be such that \(\mu' \bar{w}(\bar{\mu}) = \bar{\mu}\). Then, if we consider \(\tilde{w}(\bar{\mu}) = [\frac{w(\bar{\mu})'}{P}, \ldots, \frac{w(\bar{\mu})'}{P}]'\), we have that \(\tilde{\mu}'\tilde{w}(\bar{\mu}) = \bar{\mu}\). Therefore, since \(\tilde{w}(\bar{\mu})\) is a candidate for solution in the minimization problem presented in equation 5,

\[
H_{T_0}(\bar{\mu}) \leq H_{T_0}^{UB} \equiv \frac{1}{T_0(P)} \sum_{t \in T_0} \left( \bar{\lambda}_t(\bar{\mu}) - \frac{1}{P} \sum_{p=1}^P \bar{w}(\bar{\mu})' \bar{e}_t^p \right)^2
\]

(8)

\[
= \frac{1}{T_0(P)} \sum_{t \in T_0} \bar{\lambda}_t(\bar{\mu})^2 - \frac{2}{T_0(P)} \sum_{t \in T_0} \frac{1}{P} \sum_{p=1}^P \bar{e}_t^p \bar{\lambda}_t(\bar{\mu})
\]

(9)

\[
+ \bar{w}(\bar{\mu})' \frac{1}{T_0(P)} \frac{1}{P^2} \sum_{t \in T_0} \left[ \left( \sum_{p=1}^P \bar{e}_t^p \right) \left( \sum_{p=1}^P \bar{e}_t^p \right)' \right] \bar{w}(\bar{\mu}),
\]

(10)

where \(\frac{1}{T_0(P)} \sum_{t \in T_0} \bar{\lambda}_t(\bar{\mu})^2 \xrightarrow{p} \sigma^2_\lambda(\bar{\mu})\equiv \plim_{T_0(P)} \sum_{t \in T_0} \bar{\lambda}_t(\bar{\mu})^2\). Since \(\bar{\mu}\) is bounded, it follows that this convergence is uniform. Similar calculations imply that the other two terms converge uniformly in probability to zero. Therefore, \(H_{T_0}^{UB} \xrightarrow{p} \sigma^2_\lambda(\bar{\mu})\) uniformly in \(\bar{\mu} \in \mathcal{M}\).

Now note that

\[
H_{T_0}(\bar{\mu}) \geq H_{T_0}^{LB} \equiv \min_{b \in \Delta^{P-1}} \left\{ \frac{1}{T_0(P)} \sum_{t \in T_0} (\bar{\lambda}_t(\bar{\mu}) - b' \bar{e}_t)^2 \right\},
\]

(11)

since \(H_{T_0}^{LB}\) considers the same minimization problem as in \(H_{T_0}(\bar{\mu})\), but without the restriction \(\bar{\mu}'\bar{w} = \bar{\mu}\). Let \(\tilde{b}\) be the solution to this minimization problem, which is the SC estimator of \(\bar{\lambda}_t(\bar{\mu})\) on \(\bar{e}_t\). Since \(\bar{\lambda}_t(\bar{\mu})\) and \(\bar{e}_t\) are independent, Lemma 2 from Chernozhukov et al. (2017) implies that \(\frac{1}{T_0(P)} \sum_{t \in T_0} \left( \tilde{b}' \bar{e}_t \right)^2 = o_p(1)\), which implies that \(H_{T_0}^{LB} \xrightarrow{p} \sigma^2_\lambda(\bar{\mu})\). This convergence is uniform because \(\sigma^2_\lambda(\bar{\mu})\) is bounded. Therefore, \(H_{T_0}(\bar{\mu})\) converges uniformly
to $\sigma^2_\lambda(\bar{\mu})$, which is minimized at $\mu_0$. We just need to check the conditions presented in the proof of Lemma 2 of Chernozhukov et al. (2017). Condition 1 is satisfied by the fact that $\bar{\lambda}_t(\bar{\mu})$ and $\tilde{\epsilon}_t$ are independent. For condition 2, we assume that there are constants $c_1$ and $c_2$ such that $\mathbb{E}[\bar{\lambda}_t(\bar{\mu})\epsilon^p_{jt}]^2 \geq c_1$ and $\mathbb{E}[\bar{\lambda}_t(\bar{\mu})\epsilon^p_{jt}]^3 \leq c_2$ for all $t \in T_0$, $j = 1, ..., J$, and $p = 1, ..., P$.

For condition 3, we assume that, for any $j = 1, ..., J$, and $p = 1, ..., P$, $\{\bar{\lambda}_t(\bar{\mu})\epsilon^p_{jt}\}_{t \in T_0}$ is $\beta$-mixing and the $\beta$-mixing coefficients satisfies that $\beta(t) \leq a_1 \exp(-a_2 t^\tau)$, where $a_1, a_2, \tau > 0$ are constants. For condition 4, we assume there exists a constant $c_3 > 0$ such that $\max_{j,p} \sum_{t \in T_0} \bar{\lambda}_t(\bar{\mu})^2(\epsilon^p_{jt})^2 \leq T_0(P)c_3$ with probability $1 - o(1)$. Condition 5 is satisfied by the assumption that $\frac{P J}{T_0(P)} \to c \in \mathbb{R}_+$. Combining these results, $\mathcal{H}^L_{T_0} \leq \mathcal{H}_{T_0}(\bar{\mu}) \leq \mathcal{H}^U_{T_0}$, where both $\mathcal{H}^L_{T_0}$ and $\mathcal{H}^U_{T_0}$ converge in probability uniformly to $\sigma^2_\lambda(\bar{\mu})$. Therefore, $\mathcal{H}_{T_0}(\bar{\mu})$ converges in probability uniformly.

Since the parameter space for $\mu$ ($\mathcal{M} = \{\mu \in \mathbb{R}^F | \mu = \tilde{\mu}' w \text{ for some } w \in \Delta_{P-1}^J\}$) is compact, and $\mathcal{H}_{T_0}(\bar{\mu})$ converges uniformly in probability to $\sigma^2_\lambda(\bar{\mu})$, which is continuous and uniquely minimized at $\mu_0$, from Theorem 2.1 from Newey and McFadden (1994) we have that $\hat{\mu} \overset{p}{\to} \mu_0$. □

### A.2 Proof of Proposition 2

**Proof.**

Let $w^* \in \mathbb{R}^J$ be such that $\mu'w^* = \mu_0$, and let $\tilde{w}^* = [\tilde{w}^{'0}, \ldots, \tilde{w}^{'P}]'$. If there is more than one $w$ that satisfy this condition, then let $w^*$ be the one that minimizes $\text{var}((w^*)'e^t)$. Then

$$y^t_0 = \bar{v}^t_0 \tilde{w}^* + \epsilon^t_0 - \bar{\epsilon}^t_0 \tilde{w}^*.$$  \hspace{1cm} (12)

Since we are only interested in $\sum_{p=1}^P \hat{w}^p$, where $\hat{w}^p$ is the OLS estimator for the $J$ control units in replication $p$, and not in the individual $\hat{w}^p$, consider the following change in variables
for equation (12)

\[ y_{0t} = \tilde{y}_t H^{-1} \tilde{w} + \epsilon_{0t} - \tilde{\epsilon}_t \tilde{w}, \]  

(13)

where

\[
H = \begin{bmatrix}
\mathbb{I}_J & \mathbb{I}_J & \mathbb{I}_J & \cdots & \mathbb{I}_J \\
0 & \mathbb{I}_J & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & \mathbb{I}_J
\end{bmatrix}
\quad \text{and} \quad
H^{-1} = \begin{bmatrix}
\mathbb{I}_J & -\mathbb{I}_J & -\mathbb{I}_J & \cdots & -\mathbb{I}_J \\
0 & \mathbb{I}_J & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & \mathbb{I}_J
\end{bmatrix}
\]  

(14)

Therefore,

\[
H \tilde{w} = \begin{bmatrix}
\sum_{p=1}^{P} \tilde{w}^p \\
\tilde{w}^2 \\
\vdots \\
\tilde{w}^P
\end{bmatrix}
\quad \text{and} \quad
\tilde{y}_t H^{-1} = \begin{bmatrix}
y_1^{1\prime} (y_2^{2} - y_1^{1})' & \ldots & (y_P^{P} - y_1^{1})'
\end{bmatrix}
\]  

(15)

Let \( Y^p \) be a \( T_0(P) \times J \) matrix with information on \( y^p_t \) for all pre-treatment periods, and \( y^0 \) an \( T_0(P) \times 1 \) vector with information on \( y_{0t} \) for all pre-treatment periods. Likewise, \( \epsilon^0 \) is an \( T_0(P) \times 1 \) vector with information on \( \epsilon_{0t} \) for all pre-treatment periods, \( \epsilon^p \) is an \( T_0(P) \times J \) matrix with information on \( \epsilon^p_t \) for all pre-treatment periods, and \( \tilde{\epsilon} \) is an \( T_0(P) \times PJ \) matrix with information \( \epsilon^p \) for all replications \( p = 1, \ldots, P \).

Using Frisch-Waugh-Lovell theorem, we have that

\[
\sum_{p=1}^{P} \tilde{w}^p = \left( Y_1^{1\prime} MY_1^{1} \right)^{-1} Y_1^{1\prime} My^0
\]  

(16)

\[
= w^* + \left( Y_1^{1\prime} MY_1^{1} \right)^{-1} Y_1^{1\prime} M \epsilon^0 + \left( Y_1^{1\prime} MY_1^{1} \right)^{-1} Y_1^{1\prime} M \tilde{\epsilon} w^*,
\]  

(17)

where \( M \) is the residual-maker matrix of an OLS regression with \( Y^2 - Y^1, \ldots, Y^P - Y^1 \) as
regressors. Note that \( \hat{\mu}_{\text{OLS}} = \mu' P \hat{\gamma} \).

We start showing that \( \mathbb{E}[\hat{\mu}_{\text{OLS}} - \mu_0] \rightarrow 0 \) when \( P \rightarrow \infty \). First, note that \( \mathbb{E}[(Y^1'MY^1)^{-1} Y^1'Me^0|\tilde{Y}] = (Y^1'MY^1)^{-1} Y^1'Me^0[\tilde{Y}] = 0 \), which implies that \( \mathbb{E}[(Y^1'MY^1)^{-1} Y^1'Me^0] = 0 \).

Now we show that \( (Y^1'MY^1)^{-1} Y^1'M\tilde{w} \rightarrow 0 \). Since \( M \) is idempotent, it follows that

\[
Y^1'M\tilde{w} = (MY^1)'(M\tilde{w}) = (MY^1)' \left( M \frac{1}{P} \sum_{p=1}^P \epsilon^p \tilde{w}^* \right) \tag{18}
\]

Let \( \epsilon^p_j \) be the \( j \)-th column of \( \epsilon^p \), and consider the \( T_0(P) \times (P - 1)J \) matrix \( V \equiv [Y^2 - Y^1 Y^3 - Y^1 \ldots Y^P - Y^1] \). Then, by definition of matrix \( M \),

\[
\left\| \frac{1}{P} \sum_{p=1}^P \epsilon^p ight\|^2_2 = \arg\min_{b \in \mathbb{R}^{(P-1)J}} \left\| \frac{1}{P} \sum_{p=1}^P \epsilon^p_j - Vb \right\|^2_2 \leq \sum_{t=1}^{T_0(P)-(P-1)J} \left( \frac{1}{P} \sum_{p=1}^P \epsilon^p_{j,t} \right)^2 \leq \sum_{t=1}^{K+J} \left( \frac{1}{P} \sum_{p=1}^P \epsilon^p_{j,t} \right)^2 \tag{19}
\]

where the inequality follows from the fact that we can always choose \( b \in \mathbb{R}^{(P-1)J} \) such that the first \( (P - 1)J \) coordinates of \( \frac{1}{P} \sum_{p=1}^P \epsilon^p_j - Vb \) are equal to zero. Since, for any given \( j \) and \( t \), \( \frac{1}{P} \sum_{p=1}^P \epsilon^p_{j,t} \rightarrow 0 \) when \( P \rightarrow \infty \), it follows that \( \left\| \frac{1}{P} \sum_{p=1}^P \epsilon^p \right\|^2_2 = o_p(1) \).

Likewise, \( \left\| MY^1 \right\|^2_2 \leq \sum_{t=1}^{K+J} (y^1_{j,t})^2 = O_p(1) \), which implies that \( \left\| Y^1'M\tilde{w}^* \right\| \leq \left\| MY^1 \right\| \times \left\| M\tilde{w}^* \right\| = O_p(1) o_p(1) = o_p(1) \). Note that this result does not depend on the normality and independence assumption on the common and idiosyncratic shocks, nor that \( \mu \) is an invertible square matrix.

Now we have to show that \( (Y^1'MY^1)^{-1} = O_p(1) \). Note that

\[
Y^1'MY^1 = \mu' \lambda'M\lambda'\mu' + \mu' \lambda'Me^1 + (\epsilon^1)'M(\epsilon^1) + (\epsilon^1)'\lambda'M\mu', \tag{20}
\]

where \( \lambda \) is an \( T_0(P) \times F \) matrix with information on \( \lambda_t \) for all pre-treatment periods. Following the same arguments as above, \( \mu' \lambda'Me^1 + (\epsilon^1)'M(\epsilon^1) + (\epsilon^1)'\lambda'M\mu' = o_p(1) \). Under the assumption that \( \mu \) is an \( J \times J \) invertible matrix,

\[
(\mu' \lambda'M\lambda'\mu')^{-1} = (\mu')^{-1}(\lambda'M\lambda')^{-1}(\mu)^{-1}, \tag{21}
\]
where $\mu$ is fixed, so we only need to show that $(\lambda' M \lambda)^{-1} = O_P(1)$. Under the normality and independence assumption, $(\lambda' M \lambda)$ follows a Wishart distribution with $\nu = K(P)$ degrees of freedom. Since $\nu > J$, $(\lambda' M \lambda)$ is invertible with probability one, and $(\lambda' M \lambda)^{-1}$ follows an inverse-Wishart distribution, so it is $O_P(1)$. Therefore, $(Y^t M Y)^{-1} Y^t \tilde{M} \tilde{w}^* \overset{p}{\to} 0$, implying that $E[\hat{\mu}_{OLS} - \mu_0] \to 0$ when $P \to \infty$.

However, $\hat{\mu}_{OLS}$ will not converge in probability to $\mu_0$ because $(Y^t MY^t)^{-1} = O_P(1)$ and, since $M$ and $e^0$ are independent, $Me^0 = O_P(1)$, but not $o_P(1)$. Therefore, $\hat{\mu}_{OLS}$ will have a non-degenerate distribution even when $P \to \infty$. ■

A.3 OLS with no restrictions: case with $J > F$

This case generates additional complications because, in general, the matrix $(\mu \lambda' M \lambda \mu')$ will not have full rank. It is still possible to show, however, that $E[\hat{\mu}_{OLS} - \mu_0] \to 0$.

Consider the OLS estimator of $\tilde{w}$, which is given by

$$\tilde{w}_{OLS} = (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' \tilde{y}^0 = \tilde{w}^* + (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' e^0 + (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' e \tilde{w}^*.$$  \hspace{1cm} (22)

Therefore,

$$\hat{\mu}_{OLS} - \mu_0 = \mu' [\tilde{w}_{OLS} - \tilde{w}^*] = \mu' (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' e^0 + \mu' (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' e \tilde{w}^*.$$ \hspace{1cm} (23)

Since $e^0$ is independent of $\tilde{Y}$, we have that $E[\hat{\mu}' (\tilde{Y}' \tilde{Y})^{-1} \tilde{Y}' e^0 | \tilde{Y}] = 0$. The problem with the second term is that $E[e | \tilde{Y}] \neq 0$. Under the normality and i.i.d. assumptions, we have that

$$E[\tilde{e}_l | \tilde{Y}] = E[\tilde{e}_l | \tilde{y}_l] = (E[\tilde{e}_r \tilde{y}_r']) (E[\tilde{y}_r \tilde{y}_r'])^{-1} \tilde{y}_l,$$ \hspace{1cm} (24)
where \( E[\tilde{Y}_\tau \tilde{Y}'_\tau] \) is the expected value for a generic time period \( \tau \). Now note that

\[
E[\tilde{\epsilon}|\tilde{Y}] \tilde{w}^* = \tilde{Y}(E[\tilde{Y}_\tau \tilde{Y}'_\tau])(E[\tilde{\epsilon}_\tau |\tilde{Y}]) \tilde{w}^* 
\]

(25)

where \( \tilde{\epsilon} \) is the expected value for a generic time period \( \tau \). Now note that

\[
E[\tilde{\epsilon}|\tilde{Y}] \tilde{w}^* = \tilde{Y}(j_P j_P^\prime \otimes \mu \Omega \mu' + \mathbb{I}_P \otimes \Sigma)^{-1}(\mathbb{I}_P \otimes \Sigma) \tilde{w}^*,
\]

(26)

where \( j_P \) is an \( P \times 1 \) vector of ones.

Therefore, plugging into equation (23) and with some matrix algebra we can show that

\[
E[\hat{\mu}_0 - \mu_0] = E[\tilde{\mu}'(\tilde{Y}'\tilde{Y})^{-1}\tilde{Y}\tilde{\epsilon}\tilde{w}'\tilde{Y}]
\]

(27)

\[
= \tilde{\mu}'(j_P j_P^\prime \otimes \mu \Omega \mu' + \mathbb{I}_P \otimes \Sigma)^{-1}(\mathbb{I}_P \otimes \Sigma) \tilde{w}^*
\]

(28)

\[
= \frac{1}{P} \mu' \left[ \frac{1}{P} \Sigma + \mu \Omega \mu' \right]^{-1} \Sigma \tilde{w}^*,
\]

(29)

where we used that the \( PJ \times F \) matrix \( \tilde{\mu} \) stacks \( P \) times the matrix \( \mu \), and the \( PJ \times 1 \) vector \( \tilde{w}^* \) stacks \( P \) times the vector \( w^* \).

If \( \mu \Omega \mu' \) is invertible, then \( \frac{1}{P} \Sigma + \mu \Omega \mu' \rightarrow \mu \Omega \mu' \), which implies that \( (\frac{1}{P} \Sigma + \mu \Omega \mu')^{-1} \rightarrow (\mu \Omega \mu')^{-1} \). Therefore, in this case, it is easy to show that \( E[\hat{\mu}_{OL} - \mu_0] \rightarrow 0 \) when \( P \rightarrow \infty \).

Now we show that this result is valid even when \( \mu \Omega \mu' \) is singular. We can assume that \( rank(\mu \Omega \mu') = F \) (otherwise, we would have redundant common factors). Therefore, there are \( E_1, ..., E_F \) positive definite \( F \times F \) matrices such that \( P \mu \Omega \mu' = PE_1 + ... + E_F \). Define \( C_{k+1}(P) = \Sigma + PE_1 + ... + E_k \). Then, from Miller (1981),

\[
C_2(P)^{-1} = \Sigma^{-1} - \frac{P}{1 + \text{Tr}(PE_1 \Sigma^{-1})} \Sigma^{-1}E_1 \Sigma^{-1} \rightarrow \Sigma^{-1} - \frac{1}{\text{Tr}(E_1 \Sigma^{-1})} \Sigma^{-1}E_1 \Sigma^{-1} \equiv A_2
\]

(30)

when \( P \rightarrow \infty \). Now suppose \( C_k(P)^{-1} \rightarrow A_k \). Then we can follow the same steps as above to show that \( C_{k+1}(P)^{-1} \) converge to a matrix \( A_{k+1} \). Therefore, by induction, it follows that \( C_k(P)^{-1} \) converge to a matrix for any \( k \). Setting \( k + 1 = F \), we have

\[
[\Sigma + P \mu \Omega \mu']^{-1} = \frac{1}{P} \left[ \frac{1}{P} \Sigma + \mu \Omega \mu' \right]^{-1} \rightarrow A.
\]

(31)
Now let $U_P = \frac{1}{P} \Sigma + \mu \Omega \mu'$. Then for any $x \in \mathbb{R}^j$, $\frac{1}{P} U_P^{-1} \rightarrow Ax$ when $P \rightarrow \infty$. Since $U_P \rightarrow \mu \Omega \mu'$, then

$$\frac{1}{P} x = \frac{1}{P} U_P U_P^{-1} x \rightarrow \mu \Omega \mu' A x, \quad (32)$$

which implies that $\mu \Omega \mu' A x = 0$. Therefore, $\mu \Omega \mu' A x = 0$, which implies that $(\mu' A x)' \Omega (\mu' A x) = 0$. Since $\Omega$ is positive definite and this equality is valid for any $x \in \mathbb{R}^j$, it follows that $\mu' A = 0$. Therefore, $\mu' [\Sigma + P \mu \Omega \mu']^{-1} \Sigma w^* \rightarrow 0$.

\[A.4\text{ Proof of Proposition 3}\]

Proof. Following the same steps as in the proof of Proposition 2,

$$\sum_{p=1}^{P} \hat{w}^p - w^* = \left( \frac{1}{T_0(P)} Y l'MY^1 \right)^{-1} \frac{1}{T_0(P)} Y l'Me^0$$

$$+ \left( \frac{1}{T_0(P)} Y l'MY^1 \right)^{-1} \frac{1}{T_0(P)} Y l'M \left( \frac{1}{P} \sum_{p=1}^{P} e^p w^* \right). \quad (34)$$

For a given $j$,

$$\left\| M \frac{1}{P} \sum_{p=1}^{P} e^p_j \right\|^2 = \text{argmin}_{b \in \mathbb{R}^{K(P)-J}} \left\| \frac{1}{P} \sum_{p=1}^{P} (e^p_j - Vb) \right\|^2 \leq \sum_{t=1}^{K(P)+J} \left( \frac{1}{P} \sum_{p=1}^{P} e^p_{j,t} \right)^2 \quad (35)$$

$$= \frac{\sigma^2}{P} \sum_{t=1}^{K(P)+J} \left( \frac{1}{\sqrt{P}} \sum_{p=1}^{P} e^p_{j,t} / \sigma_t \right)^2, \quad (36)$$

where $\| . \|$ denoted the Frobenius norm.

We consider the case in which $\frac{P l}{T_0(P)} \rightarrow c \in (0,1)$.\(^8\) Note that $\left( \frac{1}{\sqrt{P}} \sum_{p=1}^{P} e^p_{j,t} / \sigma_t \right)^2$ converges

\(^8\)The results we derive here are valid as well for the extreme case in which $\frac{P l}{T_0(P)} \rightarrow 0$.\]
in distribution to a $\chi^2_1$, which implies

$$\frac{\sigma^2}{P} \sum_{t=1}^{K(P)+J} \left( \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \frac{\epsilon_{j,t}^p}{\sigma} \right)^2 \overset{P}{\rightarrow} \frac{\sigma^2(2K(P) + J)}{P(K(P) + J)} \sum_{t=1}^{K(P)+J} \left( \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \frac{\epsilon_{j,t}^p}{\sigma} \right)^2$$

(37)

$$\overset{P}{\rightarrow} \lim \left\{ \frac{\sigma^2}{P} \left( \sum_{t=1}^{K(P)+J} \frac{1}{\sqrt{P}} \sum_{p=1}^{P} \epsilon_{j,t}^p \right)^2 \right\} = \sigma^2 \left( \frac{1-c}{c} \right) J.$$  

(38)

Therefore, $\frac{1}{\sqrt{T_0(P)}} \frac{1}{P} \sum_{p=1}^{P} \epsilon_{j}^p \overset{P}{\rightarrow} 0$. Likewise, $\frac{1}{\sqrt{T_0(P)}} M Y_j^1 = \frac{1}{\sqrt{T_0(P)}} M \lambda \mu' + \frac{1}{\sqrt{T_0(P)}} M \epsilon^1$, where $\frac{1}{\sqrt{T_0(P)}} M \epsilon^1 = o_P(1)$. By definition of $M$,

$$\left\| \frac{1}{\sqrt{T_0(P)}} M \lambda \right\|^2 \leq \sum_{f=1}^{F} \left( \frac{1}{\sqrt{T_0(P)}} \sum_{t=1}^{K(P)-J} \lambda_{j_t}^2 \right) \overset{P}{\rightarrow} \left( \frac{1-c}{c} \right) J \sum_{f=1}^{F} \mathbb{E} \left[ \lambda_{j_t}^2 \right],$$

(39)

which implies that $\frac{1}{T_0(P)} Y^1 M \tilde{\epsilon} \tilde{\omega} \overset{P}{\rightarrow} 0$.

Now using again the fact that $M$ and $\lambda$ are independent, $\frac{1}{T_0(P)} (Y^1 M Y^1) \overset{P}{\rightarrow} \left( \frac{1-c}{c} \right) J \mathbb{E} [\lambda^t \lambda^t'] \mu'$, which is positive definite. Therefore, $(Y^1 M Y^1)^{-1} Y^1 M \tilde{\epsilon} \tilde{\omega} \overset{P}{\rightarrow} 0$. Similar calculations imply that $(Y^1 M Y^1)^{-1} Y^1 M \epsilon \overset{P}{\rightarrow} 0$. $\blacksquare$