LOW-LYING ZEROS OF CUBIC DIRICHLET $L$-FUNCTIONS AND THE RATIOS CONJECTURE

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Abstract. We compute the one-level density for the family of cubic Dirichlet $L$-functions when the support of the Fourier transform of a test function is in $(-1,1)$. We also establish the Ratios conjecture prediction for the one-level density for this family, and confirm that it agrees with the one-level density we obtain.

1. Introduction

After the monumental work of Montgomery [25], number theorists have tried to understand the zeros of automorphic $L$-functions via random matrix theory. Katz and Sarnak [18] proposed a conjecture, which claims that the distributions of low-lying zeros of the $L$-functions in a family $\mathfrak{F}$ is governed by its corresponding symmetry type $G(\mathfrak{F})$. We refer to [19] as a kind introduction to the conjecture. For various families, the conjecture has been tested and all the results have supported it. Since it is impossible to give a complete list, we name just a few of them [1, 2, 5, 16, 23, 27, 30]. Those who are interested in this problem may take a look at the references therein. However, it seems out of reach to prove the conjecture fully. In this sense, it is meaningful to investigate the distributions of low-lying zeros of $L$-functions in a new family even if the result is limited.

In this work, we study the low-lying zeros of cubic Dirichlet $L$-functions. Let $\phi$ be an even Schwartz function whose Fourier transform is compactly supported. For a cubic Dirichlet character $\chi$, let $\rho$ denote the nontrivial zeros of $L(\chi, s)$ in the critical strip. Define

$$D_X(\chi : \phi) = \sum_{\gamma} \phi\left(\frac{\gamma L}{2\pi}\right),$$

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where \( \gamma = -i(\rho - 1/2) \) and \( L = \log \left( \frac{X}{\pi e} \right) \). Let \( w(t) \) be an even Schwartz function that is nonnegative and is nonzero. The total weight is defined by

\[
W^*(X) = \sum_{\chi}^* w(q/X) = \sum_{\alpha} \sum_{\gamma = -i(\rho - 1/2)} w \left( \frac{N(\alpha)}{X} \right)
\]

where \( \sum_{\chi}^* = \sum_{\alpha}^* \) is a sum over primitive cubic characters \( \chi \) parameterized by \( \alpha \in \mathbb{Z}[\omega] \), \( \alpha \equiv 1 \mod{3} \), \( \alpha \) is square-free and has no rational prime divisor as in [3, Lemma 2.1], and \( q = q(\chi) = N(\alpha) \) is the conductor of \( \chi \). The 1-level density we are interested in is

\[
D^*(\phi; X) = \frac{1}{W^*(X)} \sum_{\chi} \sum_{\chi}^* \sum_{\gamma = -i(\rho - 1/2)} w \left( \frac{q}{X} \right) D_X(\chi; \phi).
\]

In this work we reserve \( C \) for \( \sup(\supp(\hat{\phi})) \), and \( \sum_{\alpha} \) will denote the sum over \( \alpha \in \mathbb{Z}[\omega] \), where \( \omega = \frac{-1+\sqrt{-3}}{2} \). \( m, n, \ell \) denote natural numbers, and \( p \) represents a rational prime. Then, we obtain

**Theorem 1.1.** Under GRH for Dirichlet \( L \)-functions, for \( C := \sup(\supp(\hat{\phi})) < 1 \), we have

\[
D^*(\phi; X) = \frac{\widehat{\phi}(0)}{LW^*(X)} \sum_{\chi} \sum_{\chi}^* \sum_{\gamma = -i(\rho - 1/2)} w \left( \frac{q}{X} \right) \log q - \frac{\widehat{\phi}(0)}{L} \left( \gamma + 3 \log 2 + \frac{\pi}{2} + \log \pi \right)
- \frac{2}{L} \sum_{\ell \geq 1} \sum_{p} \frac{a(p) \log p \cdot (3\ell \log p)^{\ell/2}}{p^{3/2}} \frac{\widehat{\phi}(0)}{L} \left( \frac{3\ell \log p}{L} \right) + \frac{4\pi}{L} \int_{0}^{\infty} \frac{e^{-\pi x}}{1 - e^{-4\pi x}} \left( \widehat{\phi}(0) - \frac{\pi}{2} \left( \frac{2\pi x}{L} \right) \right) dx
+ O \left( X^{-1/2 + C/2 + \epsilon} \right)
= \frac{\widehat{\phi}(0)}{LW^*(X)} \left( R_{w,1} X \log X + R'_{w,1} X \right) - \frac{\widehat{\phi}(0)}{L} \left( \gamma + 3 \log 2 + \frac{\pi}{2} + \log \pi \right)
- \sum_{m=0}^{\infty} \sum_{\ell \geq 1} \sum_{p} \frac{2a(p) \log p \cdot (3\ell \log p)^{\ell/2} \widehat{\phi}^{(m)}(0)}{m!p^{3/2}L^{m+1}} + \sum_{m=1}^{\infty} \frac{\Psi^{(m)}(1/4) \phi^{(m)}(0)}{m!2^m L^{m+1}}
+ O \left( X^{-1/2 + C/2 + \epsilon} \right),
\]

where \( R_{w,1}, R'_{w,1} \) are defined in Lemma 2.2, \( \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \) is the digamma function, and

\[
a(p) = \begin{cases}
\frac{p}{p+2} & \text{if } p \equiv 1 \mod{3}, \\
1 & \text{otherwise}.
\end{cases}
\]

Now we can see that the possible symmetry type for the family of cubic Dirichlet \( L \)-functions is \( U \).
Corollary 1.2. Under GRH for Dirichlet $L$-functions, for $C := \sup(\text{supp}(\hat{\phi})) < 1$, we have
$$
\lim_{X \to \infty} D^*(\phi; X) = \hat{\phi}(0).
$$

Proof. See Remark 2.7

On the other hand, Conrey, Farmer and Zimbauer developed the Ratios conjecture, which is a powerful recipe which predicts various statistics regarding $L$-functions. The Ratios conjecture is applied to compute one-level density for many different families of $L$-functions [8, 10, 14, 24]. The Ratios conjecture reveals lower order terms which the Katz-Sarnak’s conjecture is silent about.

In Sec. 3, we establish the Ratios conjecture prediction for one-level density for our family.

Theorem 1.3. Assuming Ratios conjecture, we have

$$
D^*(\phi; X) = \frac{\hat{\phi}(0)}{LW^*(X)} \sum_x w \left( \frac{q}{X} \right) \log q - \frac{\hat{\phi}(0)}{L} \left( \log \pi + \frac{\pi}{2} + 3 \log 2 + \gamma - 2C_1(0) \right)
$$

$$
+ \sum_{m=1}^{\infty} \frac{\hat{\phi}(m)(0)}{m!L^{m+1}} \left( 2C_1(m)(0) + \frac{1 + (-1)^m}{2} \right) + O \left( X^{C-1+\epsilon} + X^{-1/2+\epsilon} \right),
$$

where $C_1(r)$ is given in (3.3).

In Sec. 4, we show that the one-level density for our family agrees with the Ratios conjecture prediction up to an error $O \left( X^{-1/2+C/2+\epsilon} \right)$.

Theorem 1.4. If $C < 1$, Theorem 1.4 and Theorem 1.3 coincide with the error bounded by $O \left( X^{-1/2+C/2+\epsilon} \right)$.

Fiorilli and Miller considered a family of all Dirichlet $L$-functions. They found a term which the Ratio conjecture failed to predict when $\text{sup}(\text{supp}(\hat{\phi})) > 1$. It would be interesting if we are able to expand the support and find such a term.

This work is inspired by a recent result of Fiorilli, Parks and Sodergren, which studied one-level density for quadratic Dirichlet $L$-functions for $\text{supp}(\hat{\phi}) \subset (-2, 2)$. We tried to obtain one-level density for $\text{sup}(\text{supp}(\hat{\phi})) > 1$ but were not able to do. The family of cubic Dirichlet $L$-functions differs from that of quadratic Dirichlet $L$-functions in the following sense:

1. The parameterization of the characters requires two independent numbers instead of one.
2. The character is not self dual.
3. The phase distribution of the cubic Gauss sums (or the Kummer sum) is not known exactly.
Whereas (2), (3) do not affect our argument much, (1) is the major obstacle in expanding the support of \( \hat{\phi}(x) \). Cubic Dirichlet \( L \)-functions of conductor \( < X \), like quadratic ones, form a relatively thin subset of cardinality \( O(X) \) among Dirichlet \( L \)-functions of conductor \( < X \) (whose cardinality is \( \gg X^2 \)). From the analytic point of view, this sparsity is an obstacle in expanding the support of the Fourier transform of the test function. The family of quadratic \( L \)-functions are fortunately parameterized by one variable, which facilitates additional savings to consider wider supports. On the other hand, (1) implies that we do not have such an additional saving for cubic Dirichlet characters. It forces the estimation of \( S_1(X) = \sum_p \frac{\log p}{L} \frac{\gamma}{L} (\log p) \sum_\chi \omega \left( \frac{q}{X} \right) \chi(p) \) to be inferior to that for quadratic \( L \)-functions when the support of \( \hat{\phi} \) is expanded.

It also worths mentioning that the cubic Dirichlet characters are given as a restriction of cubic Hecke characters. This affects the strength of the large sieve estimates by Heath-Brown [12], since the summatory variables we take in the large sieve inequality become even thinner. In many cases, this is another difficulty in making estimates involving cubic Dirichlet characters as sharp as the ones for quadratic characters.

In Appendix, we collect some prequites which were useful for this work.

2. One-level Density

First, we introduce Weil’s Explicit Formula which is one of the main tool for one-level density problem.

**Lemma 2.1** (Weil’s Explicit Formula).

\[
D^*(\phi; X) = \frac{\hat{\phi}(0)}{LW^*(X)} \sum_\chi w \left( \frac{q}{X} \right) \log q - \frac{\hat{\phi}(0)}{L} \left( \gamma + 3 \log 2 + \frac{\pi}{2} + \log \pi \right) \\
- \frac{1}{LW^*(X)} \sum_{p, m > 0} \frac{\log p}{p^{m/2}} \frac{\gamma}{L} \left( \frac{m \log p}{L} \right) \sum_\chi w \left( \frac{q}{X} \right) (\chi(p^m) + \overline{\chi}(p^m)) \\
+ \frac{4\pi}{L} \int_0^\infty \frac{e^{-\pi x}}{1 - e^{-4\pi x}} \left( \hat{\phi}(0) - \hat{\phi} \left( \frac{2\pi x}{L} \right) \right) dx,
\]

where \( \gamma = 0.5772156649 \cdots \) is the Euler-Mascheroni constant.

**Proof.** The proof of [9, Lemma 2.1] works the same. The only difference is that \( \chi \) is always even in our case. The Euler-Mascheroni constant appears from \( \Gamma' \left( \frac{3}{2} \right) = -\gamma - \log 8 + \frac{\pi}{2} \). [26, Theorem 12.13]
To compute the one-level density for our family, we need to know the average of the first term and the third term of Lemma 2.1. These parts are carried out in Sec. 2.1 to Sec. 2.2. In Sec. 2.3, we find a different expression for the fourth term so that we can compare the one-level density with the Ratios conjecture prediction.

2.1. The first term in Lemma 2.1 \( \frac{\hat{\phi}(0)}{L^W(\chi)} \sum * w \left( \frac{q}{X} \right) \log q \). To estimate the first term in Lemma 2.1 we compute the sum \( \sum'' w \left( \frac{N(\alpha)}{X} \right) \log N(\alpha) \), for which we follow the proof of [8, Lemma 2.5 and 2.8]. Let \( w(s) = \int_0^{\infty} w(t) t^{s-1} dt \) be the Mellin transform of \( w(t) \). In this section we prove the following lemma.

**Lemma 2.2** (The first term of the explicit formula). Under GRH,

\[
\sum \chi \left( \frac{q}{X} \right) \log q = R_{w,1}X \log X + R'_{w,1}X + R_{w,1/3}X^{1/3} \log X + R'_{w,1/3}X^{1/3} + O \left( X^{1/4+\epsilon} \right),
\]

where

\[
R_{w,r} = -I'(s)(s-1)^2w(s) \bigg|_{s=r}, \quad R'_{w,r} = -\frac{\partial}{\partial s}(I'(s)(s-1)^2w(s)) \bigg|_{s=r}
\]

and \( I(s) \) is defined in (2.2).

We state a proposition before we prove the above lemma.

**Proposition 2.3.** For Dirichlet characters \( \chi_1, \ldots, \chi_r \), we have

\[
\prod_{p} \left( 1 + \sum_{i=1}^{r} \frac{\chi_i(p)}{p^s} \right) = \prod_i L(\chi_i, s) \cdot \prod_{i<j} L(\chi_i \chi_j, 2s)^{-1} \cdot \prod_{i<j<k} L(\chi_i \chi_j \chi_k, 3s)^2 \cdot \prod_{i \neq j} L(\chi_i \chi_j, 3s) \cdot \prod_{i} L(\chi_i^4, 4s) \cdot \prod_{i<j} L(\chi_i^2 \chi_j^2, 4s) \cdot H(s),
\]

where \( H(s) = \prod_p \left( 1 + O \left( \frac{1}{p^s} \right) \right) \) converges absolutely for \( \text{Re}(s) > 1/4 \).

**Proof.** Writing \( x_i = \frac{\chi_i(p)}{p^s} \), it suffices to compute

\[
\left( 1 + \sum_{i=1}^{r} x_i \right) \prod_i (1 - x_i) \cdot \prod_i \frac{1 - x_i^4}{1 - x_i^2} \cdot \prod_{i<j} \frac{1 - x_i^2 x_j^2}{1 - x_i x_j} \cdot \prod_{i<j<k} (1 - x_i x_j x_k)^2 \cdot \prod_{j<k} (1 - x_j^2 x_k)(1 - x_j x_k^2) = 1 + f(x_1, \ldots, x_r),
\]

where \( f(x_1, \ldots, x_r) \) is a polynomial such that every term has degree at least 4. \( \square \)
Proof of Lemma 2.2. Let $\nu : \mathbb{Z}[\omega] \to \{0, 1\}$ be defined by $\nu(\alpha) = 1$ if $\alpha \equiv 1 \mod 3$, $\alpha$ is square-free and has no rational prime divisor, and $\nu(\alpha) = 0$ otherwise. Then

$$\sum_{\alpha} \frac{1}{N(\alpha)^s} = \sum_{\alpha} \frac{\nu(\alpha)}{N(\alpha)^s} = \prod_{p=pp^\prime; \text{split}} \left(1 + \frac{1}{N(p)^s} + \frac{1}{N(p')^s}\right) = \prod_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right).$$

Let $\chi$ be the real Dirichlet character modulo 3, given by $\chi(p) = \left(\frac{p}{3}\right)$. For $p \equiv 1 \mod 3$, we observe that $1 + \frac{2}{p^s} = 1 + \frac{1}{p^s} + \frac{\chi(p)}{p^s}$, and for $p \equiv 2 \mod 3$, $1 = 1 + \frac{1}{p^s} + \frac{\chi(p)}{p^s}$. We thus write

$$\sum_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right) = \prod_{p \equiv 1 \mod 3} \left(1 + \frac{1}{p^s} + \frac{\chi(p)}{p^s}\right).$$

By Proposition 2.3, we can write

$$\prod_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right) = \zeta(s)L(\chi, s)\zeta(2s)^{-2}L(\chi, 2s)^{-1}\zeta(3s)L(\chi, 3s)H(s) =: I(s),$$

where $H(s)$ converges absolutely for $Re(s) > 1/4$. We thus consider $I(s)$ as the analytic continuation of $\prod_{p \equiv 1 \mod 3} \left(1 + \frac{2}{p^s}\right)$.

Using the Mellin transform identity (Mellin inversion), we write

$$\sum_{\alpha} w \left(\frac{N(\alpha)}{X}\right) \log N(\alpha) = \frac{1}{2\pi i} \int_{(2)} \sum_{\alpha} \frac{\nu(\alpha) \log N(\alpha)}{N(\alpha)^s} X^s w(s) ds = -\frac{1}{2\pi i} \int_{(2)} \frac{\partial}{\partial s} I(s) X^s w(s) ds.$$

Under GRH, we see that the singularities of $I(s)$ in the region $Re(s) > 1/4$ are at $s = 1$ and $s = 1/3$, which are both simple poles. The derivative $\frac{\partial}{\partial s} I(s)$ then has singularities of order 2 at $s = 1$ and $s = 1/3$, and we can use Cauchy’s differentiation formula for these points. Moving the contour of integration in (2.3) to $(1/4 + \epsilon)$, by the convexity bound for $\zeta(s)$ and [4, Theorem 2], the fast decaying property of $w$ ([3, Lemma 2.1]), and (5.1), we get Lemma 2.2. 

2.2. The third term of Lemma 2.4. Under GRH,

$$\sum_{p} \frac{\log p}{p^m/2} \phi \left(\frac{m \log p}{L}\right) \sum_{q} w \left(\frac{q}{X}\right) (\chi(p^m) + \overline{\chi}(p^m)) = W^*(X) \sum_{\ell \geq 1} \sum_{p} \frac{2a(p) \log p}{p^3/2} \phi \left(\frac{3\ell \log p}{L}\right) + O \left(X^{1/2+C/2+\epsilon}\right),$$
where
\[ a(p) = \begin{cases} \frac{p}{p+2} & \text{if } p \equiv 1 \mod 3, \\ 1 & \text{otherwise.} \end{cases} \]

Proof. This is an immediate corollary of Lemma 2.9.

Taking the Taylor series of \( \hat{\phi} \left( \frac{3\ell \log p}{L} \right) \), we also have the following expression.

Corollary 2.5.
\[
\frac{1}{LW^*(X)} \sum_{p,m > 0} \log p \left( \frac{m \log p}{L} \right) \sum_{\chi} w \left( \frac{q}{X} \right) (\chi(p^m) + \overline{\chi}(p^m))
\]
\[
= \sum_{m=0}^{\infty} \sum_{\ell \geq 1} \sum_p \frac{2a(p) \log p \cdot (3\ell \log p)^m \phi(m)(0)}{p^m/2m \log m + 1} + O \left( X^{-1/2+C/2+\epsilon} \right).
\]

Let
\[ S_m(X) = \sum_p \log p \left( \frac{m \log p}{L} \right) \sum_{\chi} w \left( \frac{q}{X} \right) \chi(p^m), \quad \kappa(n) = \prod_{p|n} p, \]
and \( \psi_n(\alpha) = \left( \frac{n}{\alpha} \right)_3 \) (\( n = 3 \) is possible). We start with following estimations.

Lemma 2.6. For \( n = 3 \),
\[
\sum_{\alpha} ^m w \left( \frac{N(\alpha)}{X} \right) \psi_n(\alpha) = a(n)T_{w,1}X + b(n)T_{w,1/3}X^{1/3} + O \left( X^{1/4+\epsilon} \right)
\]
\[ = a(n)W^*(X) + O \left( X^{1/3} \right), \]
where \( T_{w,r} = \text{Res}_{s=r} \left( I(s)\mathfrak{w}(s) \right) \) and \( a(n) = \prod_{p|n} \mod 3 \frac{p}{p+2}, b(n) = \prod_{p|n} \mod 3 p^{1/3} \frac{p^{1/3}}{p^{1/3}+2}. \)

Remark 2.7. We also note that, since \( I(s) \) has a simple pole at \( s = 1, R_{w,1} = T_{w,1} \). It follows that
\[
\frac{R_{w,1}X \log X}{LW^*(X)} = \frac{W^*(X) \log X + O(X^{1/3} \log X)}{W^*(X)(\log X - \log 2\pi)} = 1 + O \left( \frac{1}{\log X} \right).
\]

Proof. Recall that, if \( p\mathbb{Z}[\omega] = (\alpha)(\alpha') \) and \( \alpha \equiv \alpha' \equiv 1 \mod 3 \), then \( \alpha \alpha' \) and \( p \) are conjugates. This implies that, for any \( n, (n, \alpha \alpha') = 1 \), we shall have \( \left( \frac{n}{\alpha} \right)_3 \left( \frac{n}{\alpha'} \right)_3 = \left( \frac{n}{p} \right)_3 = 1 \) by the properties of the cubic residue symbol. Recall that, for a square-free integer \( n, \psi_n \) has conductor \( \asymp n^2 \) and \( L_K(\psi_n, s) \) is of degree 2. The conductor of \( \psi_n \) is therefore \( \ll \kappa(n)^2 \). As before, we write
\[
\sum_{\alpha} ^m w \left( \frac{N(\alpha)}{X} \right) \psi_n(\alpha) = \frac{1}{2\pi i} \int (2) \sum_{\alpha} \nu(\alpha) \psi_n(\alpha) \frac{N(\alpha)^s}{X^s} \mathfrak{w}(s) ds.
\]
In case \( n = \mathfrak{p} \), the sum over \( \alpha \) in the integral reduces to 
\[
\sum_{(\alpha, n)=1} \frac{\nu(\alpha)}{N(\alpha)^s} = I(s) \prod_{p \mid n} p^{s/p+2}
\]
and we obtain an analogue of Lemma 2.2. In this case, since the integrand does not involve a double pole, there are no derivative terms and hence no \( X \log X \) terms appear. The last expression is obvious, since the case \( n = 1 \) gives \( W^*(X) \).

**Lemma 2.8.** Under GRH, for \( n \neq \mathfrak{p} \),
\[
\sum_{\alpha} w \left( \frac{N(\alpha)}{X} \right) \psi_n(\alpha) \ll \kappa(n)^\epsilon X^{1/2+\epsilon}.
\]

**Proof.** Assume \( n \neq \mathfrak{p} \). The quadratic character \( \chi(p) = \left( \frac{n}{p} \right) \) defined by the Legendre symbol can be extended to a Hecke character over the ideals of \( \mathbb{Z}[\omega] \), by putting \( \chi(p) = \chi(p) \) where \( p \subset \mathbb{Z}[\omega] \) lies over \( p \neq 3 \). When \( p = (\alpha) \) with a primary \( \alpha \), we also write \( \psi_n(p) \) for \( \psi_n(\alpha) \). By the property of the cubic residue symbol, we note that for any \( p \equiv 1 \mod 3 \) splitting into \( pp' \),
\[
\psi_n(p') = \overline{\psi}_n(p) = \chi(p) \overline{\psi}_n(p),
\]
and for any \( p \equiv 2 \mod 3 \), \( \psi_n(p) = 1 \) and \( \chi(p) \psi_n(p) = -1 \). Then, generalizing Proposition 2.3 to the Dedekind zeta function of \( K = \mathbb{Q}(\omega) \) and Hecke characters for \( K \), we write
\[
\sum_{\alpha} \frac{\nu(\alpha) \psi_n(\alpha)}{N(\alpha)^s} = \prod_{p \equiv 1 \mod 3 \atop p = pp'} \left( 1 + \frac{\psi_n(p)}{p^s} + \frac{\psi_n(p')}{p^s} \right) = \prod_{p \equiv 3} \left( 1 + \frac{\psi_n(p)}{N(p)^s} + \frac{\chi(p) \overline{\psi}_n(p)}{N(p)^s} \right)^{1/2}
\]
\[
= L_K(\psi_n, s)^{1/2} L_K(\chi \overline{\psi}_n, s)^{1/2} L_K(\psi_n, 2s)^{-1/2} L_K(\psi_n, s)^{-1/2} L_K(\chi, 2s)^{-1/2} L_K(\chi \psi_n, 3s)^{1/2} H(s)
\]
for some \( H(s) = \prod_p (1 + O(N(p)^{-4s}))^{1/2} \) which converges absolutely for \( \text{Re}(s) > 1/4 \). Hence the square of \( \sum_{\alpha} \frac{\nu(\alpha) \psi_n(\alpha)}{N(\alpha)^s} \) is holomorphic for \( \text{Re}(s) > 1/4 \) and, under GRH, has no zero in \( \text{Re}(s) > 1/2 \). As in Lemma 2.2 by [15, Theorem 5.19, Corollary 5.20], we can move the contour of integration in (2.2) from (2) to (1/2 + \( \epsilon \)). By (5.2), (5.3), and the fast decaying property of \( w(s) \), we obtain the lemma. \( \Box \)

We now are about to prove the following lemma.

**Lemma 2.9.**
\[
\sum_{\ell \geq 1} S_{3\ell}(X) = W^*(X) \sum_{\ell \geq 1} \sum_p \frac{a(p) \log p}{p^{3\ell/2}} \phi \left( \frac{3\ell \log p}{L} \right) + O \left( X^{1/3} \right),
\]

[1] Under GRH, \( L_K(\psi_n, s)^{1/2} \) has branching points at \( \text{Re}(s) = 1/2 \).
\[ S_2(X) \ll X^{1/2+\epsilon}, \sum_{m \not\equiv 0 \pmod{3}, m > 3} S_m(X) \ll X^{1/2+\epsilon}, S_1(X) \ll X^{1/2+C/2+\epsilon}. \]

**Proof.** Note that \( \psi_p^\ell = \psi_p^3 \) for any \( \ell \geq 1 \). The first assertion follows directly from Lemma 2.6.

For \( m \not\equiv 0 \pmod{3} \), we observe that \( \psi_p^m = \psi_p \) or \( \overline{\psi_p} \), both of which have conductor \( \asymp p^2 \). By Lemma 2.8 we have

\[ S_m(X) \ll \sum_p \frac{\log p}{p^{m/2} \hat{\phi} \left( \frac{m \log p}{L} \right)} p^f X^{1/2+\epsilon}, \]

and hence

\[ \sum_{m \not\equiv 0 \pmod{3}, m > 3} S_m(X) \ll X^{1/2+\epsilon}. \]

We also have

\[ S_2(X) \ll \sum_p \frac{\log p}{p} \hat{\phi} \left( \frac{2 \log p}{L} \right) p^f X^{1/2+\epsilon} \ll X^{1/2+\epsilon}, \]

and

\[ S_1(X) \ll \sum_p \frac{\log p}{\sqrt{p}} \hat{\phi} \left( \frac{\log p}{L} \right) p^f X^{1/2+\epsilon} \ll X^{1/2+C/2+\epsilon}. \]

\[ \square \]

### 2.3. The Fourth term of Lemma 2.1

\[ \frac{4\pi}{L} \int_0^\infty \frac{e^{-\pi x}}{1 - e^{-\pi x}} \left( \hat{\phi}(0) - \hat{\phi} \left( \frac{2\pi x}{L} \right) \right) dx. \]

As in the proof of [22, Lemme I.2.1], we have

\[ \int_0^\infty \frac{e^{-\pi x}}{1 - e^{-\pi x}} \left( \hat{\phi}(0) - \hat{\phi} \left( \frac{2\pi x}{L} \right) \right) dx = \frac{1}{2\pi} \int_{-\infty}^\infty \phi \left( \frac{t}{2\pi} \right) (\Psi(\sigma + iat) - \Psi(\sigma)) \frac{dt}{t}. \]

Putting \( \sigma = 1/4 \), \( a = \frac{1}{2L} \), we write

\[ 4\pi \int_0^\infty \frac{e^{-4\pi x}}{1 - e^{-4\pi x}} \left( \hat{\phi}(0) - \hat{\phi} \left( \frac{2\pi x}{L} \right) \right) dx = \int_{-\infty}^\infty \phi(\tau) \left( \Psi \left( \frac{1}{4} + \frac{\pi i \tau}{L} \right) - \Psi \left( \frac{1}{4} \right) \right) d\tau. \]

But by (3.6),

\[ \int_{-\infty}^\infty \phi(\tau) \Psi \left( \frac{1}{4} + \frac{\pi i \tau}{L} \right) d\tau = \sum_{m=0}^\infty \frac{\Psi^{(m)}(1/4) \hat{\phi}^{(m)}(0)}{m! (2\pi i)^m} \left( \frac{\pi i}{L} \right)^m, \]

and the fourth term of Lemma 2.1 becomes

\[ \sum_{m=1}^\infty \frac{\Psi^{(m)}(1/4) \hat{\phi}^{(m)}(0)}{m! 2^m L^{m+1}}. \]
3. The Ratios Conjecture

In this section, following the exposition of Conrey and Snaith \[7\], we want to get a precise expectation of \(D^*(\phi, X)\) based on the Ratios conjecture for cubic Dirichlet \(L\)-functions.

Let \(f\) be an even Schwartz function. Following the convention of \[7\, Section 3\], with \(1/2 + 1/\log X < c < 3/4\), we let

\[
S(f) = \sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \sum_{\gamma} f(\gamma\alpha)
\]

\[
= \sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \frac{1}{2\pi i} \left( \int_{(c)} - \int_{(1-c)} \right) \frac{L'(s, \chi_\alpha)}{L(s, \chi_\alpha)} f\left( -i \left( s - \frac{1}{2} \right) \right) ds.
\]

The integral over \((c)\) is

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} f\left( t - i \left( c - \frac{1}{2} \right) \right) \sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \frac{L'(\frac{1}{2} + (c - \frac{1}{2} + it), \chi_\alpha)}{L\left(\frac{1}{2} + (c - \frac{1}{2} + it), \chi_\alpha\right)} dt,
\]

and assuming RH we have \(L'/L \ll \log^2(|s|N(\alpha))\). Put

\[
R_w(\nu'; \nu) = \sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \frac{L\left(\frac{1}{2} + \nu', \chi_\alpha\right)}{L\left(\frac{1}{2} + \nu, \chi_\alpha\right)},
\]

so that

\[
\sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \frac{L'(\frac{1}{2} + (c - \frac{1}{2} + it), \chi_\alpha)}{L\left(\frac{1}{2} + (c - \frac{1}{2} + it), \chi_\alpha\right)} = \frac{\partial}{\partial \nu'} R_w(\nu'; \nu) \bigg|_{\nu' = \nu - c - 1/2 + it}.
\]

As before we put \(a(n) = \prod_{p|n \text{ mod } 3} p^{\nu(n)}/p^{\nu(n)+2}\), and for a cube \(n \in \mathbb{Z}\), we have

\[
\sum_{\chi}^* w\left(\frac{q}{X}\right) \chi(n) = a(n) \cdot \sum_{\chi}^* w\left(\frac{q}{X}\right) + \text{error}.
\]

When \(n\) is not a cube we assume \(\sum_{\chi}^* w\left(\frac{q}{X}\right) \chi(n) = \text{small}\).

We expect the following conjecture to hold.

**Conjecture 3.1.**

\[
R_w(\nu', \nu) = \sum_{\alpha} w\left(\frac{N(\alpha)}{X}\right) \frac{L\left(\frac{1}{2} + \nu', \chi_\alpha\right)}{L\left(\frac{1}{2} + \nu, \chi_\alpha\right)}
\]

\[
= \sum_{\chi} w\left(\frac{q}{X}\right) \left( \frac{\zeta\left(\frac{3}{2} + 3\nu'/2\right)}{\zeta\left(\frac{1}{2} + 2\nu + \nu'\right)} A(\nu'; \nu) + \left(\frac{q}{\pi}\right)^{-\nu'} \Gamma\left(\frac{1}{4} - \nu'\right) \zeta\left(\frac{3}{2} - 3\nu'/2\right) \zeta\left(\frac{1}{2} - 2\nu + \nu'\right) A(-\nu'; \nu) \right) + O\left(X^{1/2+\epsilon}\right),
\]
where

\[ A(\nu'; \nu) = \frac{\zeta \left( \frac{3}{2} + 2\nu' + \nu \right)}{\zeta \left( \frac{3}{2} + 3\nu' \right)} \prod_p \left( 1 + a(p) \frac{1 - p^{3/2 + 3\nu' - \nu}}{p^{3/2 + 3\nu' - \nu} - 1} \right) \]

\[ = \prod_{p \equiv 1 \mod 3} \left( 1 + O \left( p^{5/2 - 2\nu' - \nu} + p^{-5/2 - 3\nu' + p^{-3 - 4\nu' - 2\nu} + p^{-4}} \right) \right). \]

Note that the only term in the parenthesis of \( R_w(\nu', \nu) \) that depends on \( \chi \) is \( q^{-\nu} \), and \( A(r; r) = 1. \)

**Derivation of Conjecture [7,1]** We only consider the main term of the conjecture in this recipe. For the numerator of \( R_w(\nu'; \nu) \), we use the approximate functional equation. For a primitive cubic Dirichlet character \( \chi \), [20, Theorem 1] becomes

\[ L \left( \frac{1}{2} + \nu, \chi \right) = \sum_{n \leq x} \frac{\chi(n)}{n^{1/2 + \nu}} + \frac{(q/\pi)^{-\nu} \Gamma \left( \frac{1}{4} - \nu \right)}{\Gamma \left( \frac{1}{4} + \frac{\nu}{2} \right)} \sum_{n \leq y} \frac{\chi(n)}{n^{1/2 - \nu}} + \text{Remainder}. \]

For the denominator, as in [7, Section 2.2] we write

\[ \frac{1}{L(s, \chi)} = \sum_{h=1}^{\infty} \frac{\mu(h) \chi(h)}{h^s}. \]

(This series does not converge absolutely for \( Re(s) \leq 1 \), but its partial sum approximates the infinite sum well enough for \( Re(s) > 1/2 \), so we write it as above.)

The first term in the approximate functional equation, multiplied by \( 1/L(s, \chi) \), results in

\[ \sum_{\chi} \sum_{h, m \equiv \chi \mod 3} \frac{\mu(h)a(hm)}{h^{1/2 + \nu} m^{1/2 + \nu'}} = \sum_{\chi} \sum_{h, m \equiv \chi \mod 3} \frac{\mu(h)a(hm)}{h^{1/2 + \nu} m^{1/2 + \nu'}}. \]

For \( h = 0 \), we have \( \sum_{m \equiv 0 \mod 3} \frac{a(p^{m})}{p^{m(1/2 + \nu')}} = 1 + \frac{a(p)}{p^{3/2 + 3\nu' - 1}} \), and for \( h = 1 \),

\[ \sum_{m \equiv 2 \mod 3} \frac{-a(p^{h+m})}{p^{1/2 + \nu + m(1/2 + \nu')}} = \frac{-a(p)p^{\nu' - \nu}}{p^{3/2 + 3\nu' - 1}}. \]

The Euler factor at \( p \) becomes

\[ 1 + a(p) \cdot \frac{1 - p^{3/2 - 2\nu' - \nu}}{p^{3/2 - 3\nu' - 1}}. \]

For \( p \equiv 2 \mod 3 \), this is

\[ 1 + \frac{1 - p^{3/2 - 2\nu' - \nu}}{p^{3/2 - 3\nu' - 1}} = \frac{1 - p^{-3/2 - 2\nu' - \nu}}{1 - p^{-3/2 - 3\nu'}}. \]
For \( p \equiv 1 \mod 3 \), using the expression \( \frac{1}{1-x} = 1 + \epsilon + \epsilon^2 + \cdots \) repeatedly, we can write the Euler factor as
\[
1 + \frac{p - p^{1+\nu'-\nu}}{(p + 2)(p^{3/2+3\nu'} - 1)} = 1 + \frac{p^{-3/2-3\nu'} - p^{-3/2-2\nu'-\nu}}{(1 + 2p^{-1})(1 - p^{-3/2-3\nu'})}
\]
\[
= \frac{(1 - 2p^{-1} + 4p^{-2} + \cdots)(1 + 2p^{-1} - p^{-3/2-2\nu'-\nu} - 2p^{-5/2-3\nu'})}{1 - p^{-3/2-3\nu'}}
\]
\[
= \frac{1 - p^{-3/2-2\nu'-\nu} + O(p^{-2} + p^{-5/2-3\nu'} + p^{-5/2-2\nu'-\nu})}{1 - p^{-3/2-3\nu'}}
\]
\[
= \frac{1 - p^{-3/2-2\nu'-\nu} + O(p^{-2} + p^{-5/2-3\nu'} + p^{-5/2-2\nu'-\nu})}{1 - p^{-3/2-3\nu'}}
\]
\[
\times (1 - p^{-3/2-2\nu'-\nu})(1 + p^{-3/2-2\nu'-\nu} + p^{-3-4\nu'-2\nu} + \cdots)
\]
\[
= \frac{1 - p^{-3/2-2\nu'-\nu}}{1 - p^{-3/2-3\nu'}} \left(1 + O(p^{-2} + p^{-3-4\nu'-2\nu} + p^{-5/2-3\nu'} + p^{-5/2-2\nu'-\nu})\right).
\]

Taking the product over all \( p \), we get
\[
\zeta \left(\frac{3}{2} + 3\nu'\right) \zeta \left(\frac{3}{2} + 2\nu' + \nu\right)^{-1} A(\nu', \nu),
\]
where
\[
A(\nu', \nu) = \prod_{p \equiv 1 \mod 3} \left(1 + O(p^{-5/2-2\nu'-\nu} + p^{-5/2-3\nu'} + p^{-3-4\nu'-2\nu} + p^{-2})\right)
\]
converges absolutely for \( \text{Re}(2\nu' + \nu) > -1 \) and \( \text{Re}(\nu') > -1/2 \). This gives the first term of Conjecture 3.1.

Recall that the second term in the approximate functional equation comes from the functional equation, namely,
\[
L \left(\frac{1}{2} + \nu', \chi\right) = \left(\frac{q}{\pi}\right)^{-\nu'} \frac{\Gamma \left(\frac{1}{4} - \frac{\nu'}{2}\right)}{\Gamma \left(\frac{1}{4} + \frac{\nu'}{2}\right)} L \left(\frac{1}{2} - \nu', \chi\right).
\]
We can deduce an analogous computation for the second term multiplied by \( L(1/2+\nu, \chi)^{-1} \) using this relation, and get Conjecture 3.1.

\[
\bullet
\]

To compute \((3.3)\), we define \( C_1(r), C_2(r), C_3(r) \) as follows:
\[
(3.3) \quad \frac{\partial}{\partial \nu'} \left( \zeta \left(\frac{3}{2} + 3\nu'\right) \zeta \left(\frac{3}{2} + 2\nu' + \nu\right)^{-1} A(\nu', \nu) \right) \bigg|_{\nu' = \nu = r} = \frac{\zeta'(3/2 + 3r)}{\zeta(3/2 + 3r)} + A'(r; r) =: C_1(r),
\]
\[
\frac{\partial}{\partial \nu'} \left( \left(\frac{q}{\pi}\right)^{-\nu'} \frac{\Gamma(1/4 - \nu'/2)\zeta(3/2 - 3\nu')}{\Gamma(1/4 + \nu'/2)\zeta(3/2 - 2\nu' + \nu)} A(-\nu'; \nu) \right) \bigg|_{\nu' = \nu = r} = C_2(r) \left(\frac{q}{\pi}\right)^{-r} - C_3(r) \left(\frac{q}{\pi}\right)^{-r} \log \frac{q}{\pi},
\]
where
\[
C_2(r) = \frac{\partial}{\partial \nu'} \left( \frac{\Gamma(1/4 - \nu'/2)\zeta(3/2 - 3\nu')}{\Gamma(1/4 + \nu'/2)\zeta(3/2 - 2\nu' + \nu)} A(-\nu'; \nu) \right)_{\nu' = \nu = r}
\]
and
\[
C_3(r) = \frac{\Gamma(1/4 - r/2)\zeta(3/2 - 3r)}{\Gamma(1/4 + r/2)\zeta(3/2 - r)} A(-r; r).
\]
Note that \( A(r; r) = 1 \). We then expect the following conjecture as well.

**Conjecture 3.2.**

\[
S(f) = \frac{1}{\pi} \sum_{\chi} w \left( \frac{q}{\chi} \right) \int_{-\infty}^{\infty} f(t) \left( C_1(it) + C_2(it) \left( \frac{q}{\pi} \right)^{-it} - C_3(it) \left( \frac{q}{\pi} \right)^{-it} \log \frac{q}{\pi} \right) dt
+ C_4 \sum_{\chi} w \left( \frac{q}{\chi} \right) \log \frac{q}{\pi} + C_5 \sum_{\chi} w \left( \frac{q}{\chi} \right) + O \left( X^{1/2+\epsilon} \right),
\]
where
\[
C_4 = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(-t) dt = \frac{1}{2\pi} \hat{f}(0), \quad C_5 = \frac{1}{4\pi} \int_{-\infty}^{\infty} f(-t) \left( \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} - \frac{t}{2}i \right) + \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} + \frac{t}{2}i \right) \right) dt.
\]

**Derivation of Conjecture 3.2.** We assume that, after taking the derivative, the error term \( O \left( X^{1/2+\epsilon} \right) \) in \( R_w(\nu'; \nu) \) contributes \( O \left( X^{1/2+\epsilon} \right) \). Writing \( S(f) = \int_{(c)} - \int_{(1-c)} \), we shall have

\[
\int_{(c)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f \left( t - i \left( c - \frac{1}{2} \right) \right) \sum_{\chi} w \left( \frac{q}{\chi} \right) \left( C_1 \left( c - \frac{1}{2} + it \right) + C_2 \left( c - \frac{1}{2} + it \right) \right) \left( \frac{q}{\pi} \right)^{-c+1/2-it}
- C_3 \left( c - \frac{1}{2} + it \right) \left( \frac{q}{\pi} \right)^{-c+1/2-it} \log \frac{q}{\pi} dt + O \left( X^{1/2+\epsilon} \right).
\]

For \( \int_{(1-c)} \) as in \([\mathcal{R}] (3.8), (3.9)\), we write

\[
\int_{(1-c)} \frac{L'(s, \chi)}{L(s, \chi)} f \left( -i \left( s - \frac{1}{2} \right) \right) ds = \int_{(c)} \frac{L'(1-s, \chi)}{L(1-s, \chi)} f \left( i \left( s - \frac{1}{2} \right) \right) ds
= \int_{(c)} f \left( i \left( s - \frac{1}{2} \right) \right) \left( -\log \frac{q}{\pi} - \frac{1}{2} \Gamma' \left( \frac{1}{2} - s \right) - \frac{1}{2} \Gamma \left( \frac{1}{2} \right) \right) ds.
\]
Since $f$ is even, we have

$$S(f) = \frac{1}{2\pi i} \sum_{\chi} w\left(\frac{q}{X}\right) \left\{ \int_{(c)} f\left(i \left(s - \frac{1}{2}\right)\right) \frac{L'(s, \chi)}{L(s, \chi)} ds \right. \\
+ \int_{(c)} f\left(i \left(s - \frac{1}{2}\right)\right) \left( \log \frac{q}{\pi} + \frac{1}{2\Gamma} \left( \frac{1-s}{2} \right) + \frac{1}{2\Gamma} \left( \frac{s}{2} \right) + \frac{L'(s, \chi)}{L(s, \chi)} \right) ds \right\}$$

$$= \frac{1}{\pi i} \sum_{\chi} w\left(\frac{q}{X}\right) \int_{(c)} f\left(-i \left(s - \frac{1}{2}\right)\right) \frac{L'(s, \chi)}{L(s, \chi)} ds + \frac{1}{2\pi i} \sum_{\chi} w\left(\frac{q}{X}\right) \log \frac{q}{\pi} \int_{(c)} f\left(i \left(s - \frac{1}{2}\right)\right) ds$$

$$+ \frac{1}{4\pi i} \sum_{\chi} w\left(\frac{q}{X}\right) \int_{(c)} f\left(i \left(s - \frac{1}{2}\right)\right) \left( \frac{\Gamma'}{\Gamma} \left( \frac{1-s}{2} \right) + \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} \right) \right) ds.$$ 

As for the first term here, by (3.3) we can push the contour of integration (c) to $c = 1/2$. The conjecture follows immediately. □

Proof of Theorem 1.3. Put $f(t) = \phi\left(\frac{Lt}{2\pi}\right)$. Then

$$D^\ast(\phi; X) = \frac{S(f)}{W^*(X)},$$

and letting $\tau = \frac{L}{2\pi} t$, we have $C_4 = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(\tau) \frac{2\pi}{L} d\tau = \frac{\hat{\phi}(0)}{L}$ which gives the leading term. As for $C_5$, we recall the series expansion for the digamma function $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$:

$$\Psi(\sigma + z) = \Psi(\sigma) + \Psi^{(1)}(\sigma) z + \frac{\Psi^{(2)}(\sigma)}{2!} z^2 + \frac{\Psi^{(3)}(\sigma)}{3!} z^3 + \cdots,$$

where $\Psi^{(m)}(s)$ is the polygamma function, given by $\Psi^{(m)}(s) = (-1)^m m! \sum_{k=0}^{\infty} \frac{1}{(s+k)^{m+1}}$ for $m > 0$ and any complex number $s$ that is not a negative integer. We thus have

$$\Psi\left(\frac{1}{4} - i \frac{\pi \tau}{L}\right) + \Psi\left(\frac{1}{4} + i \frac{\pi \tau}{L}\right) = 2 \sum_{m=0 \text{ even}}^{\infty} \frac{(-1)^{m/2} \Psi^{(m)}(1/4) \pi^m}{m!} \frac{\tau^m}{L^m},$$

and

$$C_5 = \sum_{m=0 \text{ even}}^{\infty} \frac{(-1)^{m/2} \Psi^{(m)}(1/4) \pi^m}{m! L^{m+1}} \int_{-\infty}^{\infty} \phi(\tau) \tau^m d\tau = \sum_{m=0 \text{ even}}^{\infty} \frac{\Psi^{(m)}(1/4) \hat{\phi}(m)(0)}{2^m m! L^{m+1}}.$$

For a smooth function $C(z)$, we have

$$(3.6) \int_{-\infty}^{\infty} \phi(\tau) C(H\tau) e(\Delta \tau) d\tau = \sum_{m=0}^{\infty} \frac{C^{(m)}(0)}{m!} \int_{-\infty}^{\infty} \phi(\tau) H^m \tau^m e(\Delta \tau) d\tau = \sum_{m=0}^{\infty} \frac{C^{(m)}(0) \hat{\phi}(m)(-\Delta)}{m! (2\pi i)^m} H^m,$$

and writing

$$\left(\frac{q}{\pi}\right)^{-2\pi i \tau} = \exp \left(-2\pi i \frac{\tau}{L} \log \frac{q}{\pi}\right) = \exp \left(-\frac{\log(q/\pi)}{L}\tau\right),$$
the integral in (3.4) now becomes
\[
\sum_{m=0}^{\infty} \frac{2\pi}{m!L^{m+1}} \left( C_1^{(m)}(0) \hat{\phi}(m)(0) + C_2^{(m)}(0) \hat{\phi}'(m) \left( \log \frac{q}{L} \right) - C_3^{(m)}(0) \hat{\phi}''(m) \left( \frac{\log \pi}{m} \right) \right).
\]

Observe that, because \(\hat{\phi}\) is even,
\[
\sum_{\chi} w\left( \frac{q}{X} \right) \hat{\phi}\left( -\log\left( \frac{q}{\pi L} \right) \right) \ll \sum_{q < X^C} X^{1/2 + \epsilon},
\]
and because \(w(x)\) is nonnegative and nonzero, \(\sum_{\chi} w\left( q/X \right) \gg X\). The contributions of \(C_2^{(m)}(0)\) and \(C_3^{(m)}(0)\) therefore are absorbed in an error term of order \(O\left( X^{1/2 + \epsilon} \right)\). We collect the dominant terms in (3.4) to get
\[
S(f) = \frac{\hat{\phi}(0)}{L} \sum_{\chi} w\left( \frac{q}{X} \right) \log \frac{q}{\pi} + \sum_{\chi} w\left( \frac{q}{X} \right) \sum_{m=0}^{\infty} \frac{\hat{\phi}(m)(0)}{m!} \left( 2C_1^{(m)}(0) + \frac{1 + (-1)^m \Psi^{(m)}\left( \frac{1}{4} \right)}{2m} \right) \frac{1}{L^{m+1}} + O\left( X^{1/2 + \epsilon} + X^{1/2 + \epsilon} \right).
\]

Recalling the special value of the digamma function, \(\Gamma\left( \frac{1}{4} \right) = -\frac{3}{2} - 3 \log 2 - \gamma\), we have Theorem \(1.3\).

\[\square\]

4. The Ratios conjecture vs Weil’s explicit formula

Assume \(C < 1\). Referring to Lemma 2.2, we see that Theorem 1.1 and Theorem 1.3 shall coincide if we have the following equality:

\[
\sum_{m=0}^{\infty} \sum_{\ell \geq 1} \frac{2a(p) \log p \cdot (3\ell \log p)^m \hat{\phi}(m)(0)}{m!p^{3\ell/2}L^{m+1}} + \sum_{m=1}^{\infty} \frac{\hat{\phi}(m)(0)\Psi^{(m)}\left( \frac{1}{4} \right)}{m!2^m L^{m+1}} = \sum_{m=0}^{\infty} \frac{2\hat{\phi}'(m)(0)C_1^{(m)}(0)}{m!L^{m+1}} + \sum_{m=1}^{\infty} \frac{\hat{\phi}''(m)(0)\Psi^{(m)}\left( \frac{1}{4} \right)}{m!2^m L^{m+1}} + O\left( X^{-1/2 + C/2 + \epsilon} + X^{-1 + C + \epsilon} \right).
\]

Indeed, this equality holds and we can eliminate the error term as well. Recall that \(\hat{\phi}(x)\) is an even function. The odd power terms of the Taylor expansion of \(\hat{\phi}(x)\) at \(x = 0\) are therefore all zero, and \(\sum_{m=1}^{\infty} \frac{\hat{\phi}''(m)(0)\Psi^{(m)}\left( \frac{1}{4} \right)}{m!2^m L^{m+1}}\) is the same with the subsum over even \(m\)’s only.
As for $C_1^{(m)}(0)$, we have

$$C_1(z) = \frac{\partial}{\partial \nu'} \left[ \prod_p \left( 1 + a(p) \frac{1 - p^{\nu'-\nu}}{p^{3/2+3\nu'} - 1} \right) \right] \bigg|_{\nu'=\nu=z}$$

$$= \prod_p \left( 1 + a(p) \frac{1 - 1}{p^{3/2+3\nu'} - 1} \right) \cdot \sum_p \left( 1 + a(p) \frac{1 - 1}{p^{3/2+3\nu'} - 1} \right)^{-1} A(p, z)$$

$$= \sum_p -a(p) \log p \frac{1}{p^{3/2+3\nu'} - 1},$$

where $A(p, z) = \left[ \frac{a(p) \log p}{p^{3/2+3\nu'} - 1} \right] \left( -p^{\nu'-\nu+3/2+3\nu'} + p^{\nu'-\nu} - 3p^{3/2+3\nu'} + 3p^{\nu'-\nu+3/2+3\nu'} \right) \bigg|_{\nu'=\nu=z}.$

Writing $C_1(z) = -\sum_p \sum_{\ell \geq 1} \frac{a(p) \log p}{p^{3/2+3\nu'} - 1} \cdot (-3\ell \log p)^m \frac{1}{p^{3/2+3\nu'}},$ we see that

$$C_1^{(m)}(z) = -\sum_p \sum_{\ell \geq 1} \frac{a(p) \log p}{p^{3/2+3\nu'} - 1} \cdot (-3\ell \log p)^m \frac{1}{p^{3/2+3\nu'}}.$$

Again, since $\hat{\phi}^{(m)}(0) = 0$ for all odd $m$, (4.1) holds with exact equality and the error term disappears. Hence, Theorem 1.4 follows.

5. APPENDIX

A trifling modification of [15, (5.28)] is as follows: Assume GRH for $L(f, s)$. For $-1/2 \leq \sigma \leq 2$, $\sigma \neq 1/2$, as $|t| \to \infty$ we have

$$L'(f, \sigma + it) \ll \log |t|. \tag{5.1}$$

[2]

[15] Theoreme 5.19 and Corollary 5.20] Assuming GRH and Ramanujan-Peterson conjecture for $L(f, s)$, for $\sigma \geq 1/2 + \epsilon$,

$$q(f, s)^{-\epsilon} \ll p_r(s)L(f, s) \ll q(f, s)\epsilon. \tag{5.2}$$

[2] For $-1/4 \leq \sigma \leq 5/4$, $\sigma \neq 1/2$, this is also a direct consequence of [15, Theorem 5.17], [15, (5.115)] and the functional equation of logarithmic derivatives

$$-\frac{L'(f, s)}{L(f, s)} = \log q(f) + \frac{\gamma'(f, s)}{\gamma(f, s)} + \frac{\gamma'(\overline{f}, 1 - s)}{\gamma(f, 1 - s)} + \frac{L'(\overline{f}, 1 - s)}{L(f, 1 - s)}.$$
The functional equation can be written

$$L(f, 1 - s) = \frac{q(\mathcal{F})^{1/2}}{q(f)^{1/2}} \frac{\gamma(\mathcal{F}, s)}{\gamma(f, 1 - s)} L(\mathcal{F}, s),$$

from which, for $\sigma = 1/4 + \delta$, $0 \leq \delta < 1/4$, $q(\mathcal{F}) \propto q(f)$, we have

$$(5.3) \quad L\left( f, \frac{1}{4} + \delta + it \right) \ll \delta q(f)^{1/4 - \delta} q(f, s)^{\ast}.$$  

An element $\alpha \in \mathbb{Z}[\omega]$ is primary if $(\alpha, 3) = 1$ and $\alpha \equiv r \mod (1 - \omega)^2$ for some $r \in \mathbb{Z}$. Equivalently, $\alpha$ is primary if $\alpha \equiv \pm 1 \mod 3$ in $\mathbb{Z}[\omega]$. If $(\alpha, 3) = 1$ then one of $\alpha, \omega \alpha, \omega^2 \alpha$ is primary. The product of primary numbers is again primary, and the complex conjugate of a primary number is also primary. Any number $\alpha \in \mathbb{Z}[\omega]$ has a unique factorization of the form $(-1)^s \omega^k (1 - \omega)^h \pi_1^{e_1} \pi_2^{e_2} \cdots$ where each $\pi_i$ is primary, $s, k, h, e_i \in \mathbb{Z}_{\geq 0}$.

Let $\pi \in \mathbb{Z}[\omega]$ be a prime element coprime to 3. Observe that $N(\pi) \equiv 1 \mod 3$. We collect well-known properties of the cubic residue symbol as follows.

**Proposition 5.1.**

1. $\left( \frac{a}{3} \right)_3 \equiv \alpha^{N(\alpha) - 1} \mod \pi$ for a unique $k \in \{0, 1, 2\}$.

2. If $\alpha \equiv \beta \mod \pi$ then $\left( \frac{a}{3} \right)_3 = \left( \frac{\beta}{3} \right)_3$.

3. $\left( \frac{\alpha \beta}{\pi} \right)_3 = \left( \frac{\alpha}{\pi} \right)_3 \left( \frac{\beta}{3} \right)_3$. (\alpha, \beta need not be coprime to each other.)

4. $\left( \frac{a}{\pi} \right)_3 = \left( \frac{a}{3} \right)_3$.

5. If $\pi$ and $\theta$ are associates, $\left( \frac{a}{\pi} \right)_3 = \left( \frac{a}{\theta} \right)_3$. (In particular, $\left( \frac{a}{3} \right)_3 = \left( \frac{\alpha}{\pi} \right)_3$.)

6. $x^3 \equiv \alpha \mod \pi$ has a solution $x \in \mathbb{Z}[\omega]$ if and only if $\left( \frac{a}{\pi} \right)_3 = 1$. In particular, $\left( \frac{1}{3} \right)_3 = 1$.

7. If $a, b \in \mathbb{Z}$ satisfy $(a, b) = (b, 3) = 1$ then $\left( \frac{a}{3} \right)_3 = 1$.

8. The cubic residue symbol extends (in the denominator) into composite numbers by

$$\left( \frac{\alpha}{\pi_1^{e_1} \pi_2^{e_2} \cdots} \right)_3 = \left( \frac{\alpha}{\pi_1} \right)_3^{e_1} \left( \frac{\alpha}{\pi_2} \right)_3^{e_2} \cdots.$$  

9. (Cubic Reciprocity) For $\alpha, \beta$ primary, $\left( \frac{\alpha \beta}{3} \right)_3 = \left( \frac{\beta}{\alpha} \right)_3$.

Assume $\pi = 1 + 3a + 3b\omega$ is primary with $a, b \in \mathbb{Z}$. (If $\pi \equiv -1 \mod 3$, replace $\pi$ by $-\pi$.)

Then

10. $\left( \frac{\omega}{\pi} \right)_3 = \omega^{2a + 2b}$.

11. $\left( \frac{1 - \omega}{\pi} \right)_3 = \omega^a$.

12. $\left( \frac{3}{\pi} \right)_3 = \omega^b$.

[3] There are 9 residue classes modulo 3.
References

[1] L. Alpoge and S. J. Miller, Low-lying zeros of Maass form L -functions, Int. Math. Res. Not. IMRN 2015, no. 10, 2678–2701.
[2] V. Chandee and Y. Lee, n-level density of the low-lying zeros of primitive Dirichlet L-functions, preprint, 2017.
[3] S. Baier and M.P. Young, Mean values with cubic characters, J. Number Theory 130 (2010), no. 4, 879–903.
[4] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the critical strip, J. Number Theory 131 (2011), no. 3, 363–384.
[5] P.J. Cho and H.H. Kim, n-level densities of Artin L-functions, Int. Math. Res. Not. IMRN 2015, no. 17, 7861–7883.
[6] B. Conrey, D.W. Farmer and M.R. Zirnbauer, Autocorrelation of ratios of L-functions, Commun. Number Theory Phys. 2 (2008), no. 3, 593–636.
[7] J.B.Conrey and N.C.Snaith, Applications of the L-functions ratios conjectures, Proc. London Math. Soc., 94 (2007) no. 3, 594–646.
[8] D. Fiorilli, J. Parks and A. Södergren, Low-lying zeros of elliptic curve L-functions: Beyond the Ratios Conjecture, Math. Proc. Cambridge Philos. Soc. 160 (2016), no. 2, 315–351.
[9] D. Fiorilli and S.J. Miller, Surpassing the ratios conjecture in the 1-level density of Dirichlet L-functions, Algebra Number Theory 9 (2015), no. 1, 13–52.
[10] I.S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Elsevier/Academic Press, Amsterdam, 2015. xlvi+1133 pp. ISBN: 978-0-12-384933-5
[11] D.R. Heath-Brown, Kummer’s conjecture for cubic Gauss sums, Israel J. Math. 120 (2000), part A, 97–124.
[12] B. Hough, Topics in analytic number theory, Lent 2013. Lecture 22: Approximate functional equation,
[13] D.K. Huynh, S.J. Miller and R. Morrison, An elliptic curve test of the L-functions ratios conjecture, J. Number Theory 131 (2011), no. 6, 1117–1147.
[14] H. Iwaniec and E. Kowalski, Analytic Number Theory, American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004.
[15] H. Iwaniec, W. Luo and P. Sarnak, Low lying zeros of families of L-functions, Inst. Hautes tudes Sci. Publ. Math. No. 91 (2000), 55–131 (2001).
[16] H. Iwaniec and P. Sarnak, Dirichlet L-functions at the central point, Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), 941–952, de Gruyter, Berlin, 1999.
[17] N. Katz and P. Sarnak, Random Matrices, Frobenius Eigenvalues and Monodromy, American Mathematical Colloquium Publications 45, 1999.
[18] W. Luo, On Hecke L-series associated with cubic characters, Compos. Math. 140 (2004), no. 5, 119–1196.
[22] J-F, Mestre, Formules explicites et minorations de conducteurs de variétés algébriques, Compositio Math. 58 (1986), no. 2, 209–232.
[23] S. J. Miller, One- and two-level densities for rational families of elliptic curves: evidence for the underlying group symmetries, Compos. Math. 140 (2004), no. 4, 952–992.
[24] ———, A symplectic test of the L-functions ratios conjecture, Int. Math. Res. Not. IMRN 2008, no. 3, Art. ID rnm146, 36 pp.
[25] H.L. Montgomery, The pair correlation of zeros of the zeta function, Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pp. 181–193. Amer. Math. Soc., Providence, R.I., 1973.
[26] H.L. Montgomery and R.C. Vaughan, Multiplicative Number Theory I: Classical Theory, Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.
[27] M.O. Rubinstein, Evidence for a spectral interpretation of the zeros of L-functions, Phd Thesis, Princeton University, 1998.
[28] K. Soundararajan, Nonvanishing of quadratic Dirichlet L-functions at s = 1/2, Ann. of Math. (2) 152 (2000), no. 2, 447–488.
[29] E.C. Titchmarsh, The theory of the Riemann zeta-function, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.
[30] M.P. Young, Low-lying zeros of families of elliptic curves, J. Amer. Math. Soc. 19 (2006), no. 1, 205250.

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