MAXIMAL $\tau_d$-RIGID PAIRS

KARIN M. JACOBSEN AND PETER JØRGENSEN

Abstract. Let $\mathcal{T}$ be a 2-Calabi–Yau triangulated category, $T$ a cluster tilting object with endomorphism algebra $\Gamma$. Consider the functor $\mathcal{T}(T, -) : \mathcal{T} \to \text{mod} \Gamma$. It induces a bijection from the isomorphism classes of cluster tilting objects to the isomorphism classes of support $\tau$-tilting pairs. This is due to Adachi, Iyama, and Reiten.

The notion of $(d + 2)$-angulated categories is a higher analogue of triangulated categories. We show a higher analogue of the above result, based on the notion of maximal $\tau_d$-rigid pairs.

0. Introduction

In triangulated categories, the notions of cluster tilting objects (introduced in [4, p. 583]) and maximal rigid objects have recently been extensively investigated. They frequently coincide, by [22, thm. 2.6], and they are closely linked to the notion of support $\tau$-tilting pairs in abelian categories (introduced in [1, def. 0.3]). Indeed, there is often a bijection between the cluster tilting objects in a triangulated category and the support $\tau$-tilting pairs in a suitable (abelian) module category, see [1, thm. 4.1].

This paper investigates the analogous theory in $(d + 2)$-angulated and $d$-abelian categories, which are the main objects of higher homological algebra, see [8, def. 2.1] and [15, def. 3.1]. Several key properties from the classic case do not carry over. For example, cluster tilting objects are maximal $d$-rigid, but the converse is rarely true. Moreover, the higher analogue of support $\tau$-rigid pairs permit a bijection to the maximal $d$-rigid objects, but not to the cluster tilting objects.

For further reading in higher homological algebra a number of references have been included in the bibliography, see [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21].

Let $k$ be an algebraically closed field, $d \geq 1$ an integer, $\mathcal{T}$ a $k$-linear Hom-finite $(d + 2)$-angulated category with split idempotents, see [8, def. 2.1]. Assume that $\mathcal{T}$ is 2-$d$-Calabi–Yau, see [21, def. 5.2], and let $\Sigma^d$ denote the $d$-suspension functor of $\mathcal{T}$.

Cluster tilting and maximal $d$-rigid objects. An object $X \in \mathcal{T}$ is $d$-rigid if $\text{Ext}^d_{\mathcal{T}}(X, X) = 0$. We recall three important definitions.

Definition 0.1 ([21, def. 5.3]). An object $X \in \mathcal{T}$ is Oppermann–Thomas cluster tilting in $\mathcal{T}$ if:

(i) $X$ is $d$-rigid.

(ii) For any $Y \in \mathcal{T}$ there exists a $(d + 2)$-angle

$$X_d \to \cdots \to X_0 \to Y \to \Sigma^d X_d$$

with $X_i \in \text{add} X$ for all $0 \leq i \leq d$.

Definition 0.2. An object $X \in \mathcal{T}$ is $d$-self-perpendicular in $\mathcal{T}$ if

$$\text{add} X = \{ Y \in \mathcal{T} \mid \text{Ext}^d_{\mathcal{T}}(X, Y) = 0 \}.$$
Definition 0.3. An object $X \in \mathcal{T}$ is maximal $d$-rigid in $\mathcal{T}$ if
\[ \text{add } X = \{ Y \in \mathcal{T} \mid \text{Ext}^{d}_{\mathcal{T}}(X \oplus Y, X \oplus Y) = 0 \}. \]

Our first main result is:

Theorem A. $X$ is Oppermann–Thomas cluster tilting $\Rightarrow$ $X$ is $d$-self-perpendicular $\Rightarrow$ $X$ is maximal $d$-rigid.

We prove this in Theorem 1.1. Of equal importance is that the implications cannot be reversed in general, see Remark 1.2. In particular, when $d \geq 2$, the class of maximal $d$-rigid objects is typically strictly larger than the class of Oppermann–Thomas cluster tilting objects, in contrast to the classic case $d = 1$ where the two classes usually coincide, see [22, thm. 2.6].

Maximal $\tau_{d}$-rigid pairs. Let $T \in \mathcal{T}$ be an Oppermann–Thomas cluster tilting object and let $\Gamma = \text{End}_{\mathcal{T}}(T)$. Recall the following result.

Theorem 0.4 ([14, thm. 0.6]). Consider the essential image $\mathcal{D}$ of the functor $\mathcal{T}(T, -) : \mathcal{T} \to \text{mod } \Gamma$. Then $\mathcal{D}$ is a $d$-cluster tilting subcategory of $\text{mod } \Gamma$. There is a commutative diagram, as shown below, where the vertical arrow is the quotient functor and the diagonal arrow is an equivalence of categories:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\mathcal{T}(T, -)} & \mathcal{D} \\
(\sim) & \downarrow & \nearrow \\
\mathcal{T}/\text{add } \Sigma^{d}T. & & \\
\end{array}
\]

The category $\mathcal{D}$ is a $d$-abelian category by [15, thm. 3.16]. It has a $d$-Auslander–Reiten translation $\tau_{d}$, which is a higher analogue of the classic Auslander–Reiten translation $\tau$, see [12, sec. 1.4.1]. A module $M \in \mathcal{D}$ is called $\tau_{d}$-rigid if $\text{Hom}_{\Gamma}(M, \tau_{d}M) = 0$.

Remark 0.5. The classic add-proj-correspondence holds, as $\mathcal{T}(T, -)$ restricts to an equivalence $\text{add } T \to \text{proj } \Gamma$. The functor also restricts to an equivalence $\text{add } ST \to \text{inj } \Gamma$. [14, lem. 2.1]

It is natural to ask if $\mathcal{D}$ permits a higher analogue of the $\tau$-tilting theory of [1]. We will not answer this question, but will instead introduce the following definitions inspired by it.

Definition 0.6. A pair $(M, P)$ with $M \in \mathcal{D}$ and $P \in \text{proj } \Gamma$ is called a $\tau_{d}$-rigid pair in $\mathcal{D}$ if $M$ is $\tau_{d}$-rigid and $\text{Hom}_{\Gamma}(P, M) = 0$.

Definition 0.7. A pair $(M, P)$ with $M \in \mathcal{D}$ and $P \in \text{proj } \Gamma$ is called a maximal $\tau_{d}$-rigid pair in $\mathcal{D}$ if it satisfies:

(i) If $N \in \mathcal{D}$ then

\[ N \in \text{add } M \Leftrightarrow \left\{ \begin{array}{l}
\text{Hom}_{\Gamma}(M, \tau_{d}N) = 0, \\
\text{Hom}_{\Gamma}(N, \tau_{d}M) = 0, \\
\text{Hom}_{\Gamma}(P, N) = 0.
\end{array} \right. \]

(ii) If $Q \in \text{proj } \Gamma$, then

\[ Q \in \text{add } P \Leftrightarrow \text{Hom}_{\Gamma}(Q, M) = 0. \]

A maximal $\tau_{d}$-rigid pair is a $\tau_{d}$-rigid pair.

Our second main result is:
Theorem B. If each indecomposable object of $\mathcal{T}$ is $d$-rigid, then there is a bijection
\[
\left\{ \text{isomorphism classes of maximal } d\text{-rigid objects in } \mathcal{T} \right\} \to \left\{ \text{isomorphism classes of maximal } \tau_d\text{-rigid pairs in } \mathcal{D} \right\}.
\]

We prove this in Section 3. If $d = 1$, then $(M, P)$ is a maximal $\tau_1$-rigid pair if and only if it is a support $\tau$-tilting pair in the sense of [1, def. 0.3(b)], see [1, def. 0.3, prop. 2.3, and cor. 2.13]. Hence Theorem B is a higher analogue of the bijection
\[
\left\{ \text{isomorphism classes of cluster tilting object in } \mathcal{T} \right\} \to \left\{ \text{isomorphism classes of support } \tau\text{-tilting pairs in } \text{mod } \Gamma \right\}
\]
which exists by [1, thm. 4.1] when $\mathcal{T}$ is triangulated, i.e. in the case $d = 1$. However, when $d \geq 2$, we do not think of maximal $\tau_d$-rigid pairs as support $\tau_d$-tilting pairs. The reason is that by Theorem B, maximal $\tau_d$-rigid pairs are linked to maximal $d$-rigid objects in higher angulated categories. As remarked above, this class is typically strictly larger than the class of Oppermann–Thomas cluster tilting objects when $d \geq 2$.

Note that [19] makes an approach to higher support tilting theory.

This paper is organised as follows: Section 1 proves Theorem A, Section 2 investigates the precise relation between Hom spaces in $\mathcal{T}$ and $\mathcal{D}$, Section 3 proves Theorem B, and Section 4 gives an example.

Setup 0.8. Throughout the paper we use the following notation:

- $k$: An algebraically closed field.
- $D$: The duality functor $\text{Hom}_k(\cdot, k)$.
- $\mathcal{T}$: A $k$-linear, Hom-finite, $(d + 2)$-angulated category with split idempotents. We assume that $\mathcal{T}$ is $2d$-Calabi–Yau, that is $\mathcal{T}(X, Y) \cong D\mathcal{T}(Y, \Sigma^{2d}X)$ naturally in $X, Y \in \mathcal{T}$.
- $\Sigma^d$: The $d$-suspension functor on $\mathcal{T}$.
- $\mathcal{T}$: An Oppermann–Thomas cluster tilting object in $\mathcal{T}$.
- $(\cdot)$: The canonical functor $\mathcal{T} \to \mathcal{T}/\text{add } \Sigma^dT$, whose target is the naive quotient category of $\mathcal{T}$ modulo the morphisms which factor through an object in $\text{add } \Sigma^dT$.
- $\Gamma$: The endomorphism ring $\text{End}_\mathcal{T}(T)$.
- $\nu_\Gamma$: The Nakayama functor on $\text{mod } \Gamma$.
- $\tau_d$: The $d$-Auslander–Reiten translation on $\text{mod } \Gamma$.
- $\mathcal{D}$: The essential image of the functor $\mathcal{T}(T, \cdot) : \mathcal{T} \to \text{mod } \Gamma$.

1. Proof of Theorem A

Theorem 1.1. Let $X \in \mathcal{T}$ be given.

(i) There are implications
\[
X \text{ is Oppermann–Thomas cluster tilting} \Downarrow
\]
\[
X \text{ is } d\text{-self-perpendicular} \Downarrow
\]
\[
X \text{ is maximal } d\text{-rigid} \Downarrow
\]
\[
X \text{ is } d\text{-rigid}.
\]

(ii) If each indecomposable object in $\mathcal{T}$ is $d$-rigid, then
\[
X \text{ is } d\text{-self-perpendicular } \Leftrightarrow X \text{ is maximal } d\text{-rigid}.
\]
Proof. (i), the first implication: Suppose $X$ is Oppermann–Thomas cluster tilting. We must prove the equality in Definition 0.2, and the inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, suppose $\text{Ext}^d_{\mathcal{T}}(X, Y) = 0$. Then each morphism $X_0 \to \Sigma^d Y$ with $X_0 \in \text{add } X$ is zero. This applies in particular to the $(d + 2)$-angle $X_d \to \cdots \to X_0 \to \Sigma^d Y \to \Sigma^d X_d$ with $X_i \in \text{add } X$, which exists since $X$ is Oppermann–Thomas cluster tilting. But then the morphism $\Sigma^d Y \to \Sigma^d X_d$ is a split monomorphism, and applying $\Sigma^{-d}$ gives a split monomorphism $Y \to X_d$ proving $Y \in \text{add } X$.

(i), the second implication: Suppose that $X$ is $d$-self-perpendicular. We must prove the equality in Definition 0.3, and the inclusion $\subseteq$ is clear. For the inclusion $\supseteq$, suppose $\text{Ext}^d_{\mathcal{T}}(X \oplus Y, X \oplus Y) = 0$. Then in particular, $\text{Ext}^d_{\mathcal{T}}(X, Y) = 0$, whence $Y \in \text{add } X$.

(i), the third implication: This is clear.

(ii): Suppose that each indecomposable object in $\mathcal{T}$ is $d$-rigid. Because of part (i), it is enough to prove the implication $\Leftarrow$ in (ii), so suppose that $X$ is maximal $d$-rigid. We must prove the equality in Definition 0.2, and $\subseteq$ is clear.

For the inclusion $\supseteq$, observe that $\{ Y \in \mathcal{T} \mid \text{Ext}^d_{\mathcal{T}}(X, Y) = 0 \}$ is closed under direct sums and summands by additivity of $\text{Ext}$. Hence it is enough to suppose that $Y$ is an indecomposable object in this set and prove $Y \in \text{add } X$. However, $\text{Ext}^d_{\mathcal{T}}(X, Y) = 0$ implies $\text{Ext}^d_{\mathcal{T}}(Y, X) = 0$ because $\mathcal{T}$ is $2d$-Calabi–Yau, and $\text{Ext}^d_{\mathcal{T}}(Y, Y) = 0$ by assumption. Finally, $X$ is $d$-rigid by part (i), so $\text{Ext}^d_{\mathcal{T}}(X, X) = 0$. Combining these equalities shows $\text{Ext}^d_{\mathcal{T}}(X \oplus Y, X \oplus Y) = 0$, and $Y \in \text{add } X$ follows. \[\square\]

Remark 1.2. The implications in Theorem 1.1(i) cannot be reversed in general:

- An example of a $d$-self-perpendicular object $X$ which is not Oppermann–Thomas cluster tilting is given in Section 4. In fact, the objects in the last three rows of Figure 4 are such examples. The example was originally given in [21, p. 1735].
- An example of a maximal $d$-rigid object which is not $d$-self-perpendicular can be obtained by combining proposition 2.6 and corollary 2.7 in [5]. These results give a maximal 1-rigid object which is not cluster tilting, but in the triangulated setting of [5], cluster tilting is equivalent to 1-self-perpendicular, see [5, bottom of p. 963].
- Finally, an example of a $d$-rigid object which is not maximal $d$-rigid is the zero object, as soon as $\mathcal{T}$ has a non-zero $d$-rigid object.

We end the section by observing that Theorem 1.1(ii) can be applied to an important class of categories.

Proposition 1.3. Let $\Lambda$ be a $d$-representation finite algebra, $\mathcal{O}_\Lambda$ the $(d + 2)$-angulated cluster category associated to $\Lambda$ in [21, thm. 5.2]. Then each $X \in \mathcal{O}_\Lambda$ satisfies

$$X \text{ is } d\text{-self-perpendicular } \iff X \text{ is maximal } d\text{-rigid}.$$ 

Proof. Each indecomposable in $\mathcal{O}_\Lambda$ is $d$-rigid by [21, Lemma 5.41], so the equivalence follows from Theorem 1.1(ii). \[\square\]

2. A dimension formula for $\text{Ext}^d_{\mathcal{T}}$

Recall from Setup 0.8 that $T$ is a fixed Oppermann–Thomas cluster tilting object in $\mathcal{T}$, and that $\mathcal{T}$ is $2d$-Calabi–Yau, that is, $\mathcal{T}(X, Y) \cong D\mathcal{T}(Y, \Sigma^{2d} X)$ naturally in $X, Y \in \mathcal{T}$.

Lemma 2.1. There is a natural isomorphism

$$\nu_T \mathcal{T}(T, T') \cong \mathcal{T}(T, \Sigma^{2d}(T'))$$

for $T' \in \text{add } T$. 
Proof. By the $2d$-Calabi-Yau property we have
\[ \mathcal{F}(T, \Sigma^{2d}(T')) \cong D\mathcal{F}(T',T). \]
By [14, Lemma 2.2(i)],
\[ D\mathcal{F}(T',T) \cong DHom_{\Gamma}(\mathcal{F}(T,T'), \mathcal{F}(T,T)) = DHom_{\Gamma}(\mathcal{F}(T,T'), \Gamma). \]
Finally, by definition we have
\[ DHom_{\Gamma}(\mathcal{F}(T,T'), \Gamma) = \kappa T(T,T'). \]

□

Lemma 2.2. If $X \in \mathcal{F}$ has no non-zero direct summands in $\text{add} \, \Sigma^d T$, then there exists a $(d+2)$-angle
\[ T_d \to \cdots \to T_0 \to X \to \Sigma^d T_d \]
in $\mathcal{F}$ with the following properties: Each $T_i$ is in $\text{add} \, T$, and applying the functor $\mathcal{F}(T,-)$ gives a complex
\[ \mathcal{F}(T,T_d) \to \cdots \to \mathcal{F}(T,T_0) \to \mathcal{F}(T,X) \to 0 \]
which is the start of the augmented minimal projective resolution of $\mathcal{F}(T,X)$.

Proof. Given $X$, there exists a $(d+2)$-angle
\[ \Sigma^{-d}X \to T_d \to \cdots \to T_0 \to X \]
with each $T_i$ in $\text{add} \, T$ by Definition 0.1. Since $X$ has no non-zero direct summands in $\text{add} \, \Sigma^d T$, the first morphism in the $(d+2)$-angle is in the radical of $\mathcal{F}$. By dropping trivial summands of the form $T' \cong T$, we can assume that so are the other morphisms except the last morphism.

By [8, prop. 2.5(a)], applying the functor $\mathcal{F}(T,-)$ gives an exact sequence
\[ \mathcal{F}(T,\Sigma^{-d}X) \to \mathcal{F}(T,T_d) \to \cdots \to \mathcal{F}(T,T_0) \to \mathcal{F}(T,X) \to 0. \]
By Theorem 0.4, applying the functor $\mathcal{F}(T,-)$ is, up to isomorphism, just to apply a quotient functor, and this preserves radical morphisms. So in the exact sequence each morphism, except possibly $\mathcal{F}(T,T_0) \to \mathcal{F}(T,X)$, is in the radical of $\text{mod} \, \Gamma$. This proves the claim of the lemma. □

Lemma 2.3. If $X \in \mathcal{F}$ has no non-zero direct summands in $\text{add} \, \Sigma^d T$, then there is a natural isomorphism
\[ \tau_d \mathcal{F}(T,X) \cong \mathcal{F}(T,\Sigma^d X). \]

Proof. As $X$ has no non-zero direct summands in $\text{add} \, \Sigma^d T$, we can consider the $(d+2)$-angle from Lemma 2.2. Apply $\mathcal{F}(T,-)$ to get the following part of an augmented minimal projective resolution in $\text{mod} \, \Gamma$:
\[ \mathcal{F}(T,T_d) \to \cdots \to \mathcal{F}(T,T_0) \to \mathcal{F}(T,X) \to 0. \]
Using the Nakayama functor and Lemma 2.1 we get the following commutative diagram.
The top sequence is exact by the definition of \( \tau_d \), see [12, sec. 1.4.1]. The bottom sequence is exact because it is obtained by applying \( \text{Hom}_{\mathcal{T}}(T, -) \) to a \((d + 2)\)-angle in \( \mathcal{T} \), see [8, prop. 2.5(a)]. The first term of the bottom sequence is actually \( \mathcal{T}(T, \Sigma^d T_0) \), but this is zero. Since we have \( d \geq 1 \), the diagram implies
\[
\tau_d \mathcal{T}(T, X) \cong \mathcal{T}(T, \Sigma^d X).
\]

We write \([\text{add } T](X, Y) = \{ f \in \mathcal{T}(X, Y) \mid f \text{ factors through an object of } \text{add } T \}\).

**Lemma 2.4.** There is a natural isomorphism
\[
\text{D}[\text{add } T](X, Y) \cong \text{Hom}_{\mathcal{T}/\text{add } \Sigma^d T}(T', \Sigma^d X)
\]
for \( X, Y \in \mathcal{T} \).

**Proof.** Pick a \((d + 2)\)-angle in \( \mathcal{T} \):
\[
T_d \to \cdots \to T_0 \to Y \to \Sigma^d T_d,
\]
with \( T_i \in \text{add } T \). Use \( \mathcal{T}(X, -) \) to obtain the morphism \( \Psi : \mathcal{T}(X, T_0) \to \mathcal{T}(X, Y) \). This is a homomorphism of \( k \)-vector spaces, hence we can talk about the image of \( \Psi \). We first note that any morphism \( f \) in the image of \( \Psi \) must factor through \( \text{add } T \). Now suppose \( f \in \mathcal{T}(X, Y) \) factors through \( T' \in \text{add } T \). We have the following commutative diagram, where the lower row is a part of the \((d + 2)\)-angle above:

![Diagram](attachment:image.png)

The dashed arrow exists by completing the commutative square to a morphism of \((d + 2)\)-angles. We conclude that \( f \in \text{Im } \Psi \). Hence
\[
\text{Im } \Psi = [\text{add } T](X, Y).
\]

We now return to the long exact sequence
\[
\cdots \to \mathcal{T}(X, T_0) \overset{\Psi}{\to} \mathcal{T}(X, Y) \to \mathcal{T}(X, \Sigma^d T_d) \to \cdots .
\]
Using the duality functor \( \text{D} \) and Serre duality we get the following diagram with exact rows:
Lemma 2.6. Suppose \( \mathcal{F}(X, \Sigma^d T) \). Then we have a short exact sequence
\[
0 \to \mathrm{DHom}_{\mathcal{F}}(\Sigma^d Y, X, Y) \to \mathcal{F}(X, Y) \to \mathrm{Hom}_{\mathcal{F}}(\Sigma^d T, Y, X) \to 0.
\]

Proof. By the definition of the quotient functor we have a short exact sequence
\[
0 \to [\mathrm{add} \Sigma^d T](X, Y) \to \mathcal{F}(X, Y) \to \mathrm{Hom}_{\mathcal{F}}(\Sigma^d T, Y, X) \to 0.
\]

We have \([\mathrm{add} \Sigma^d T](X, Y) \cong [\mathrm{add} T](\Sigma^{-d} X, Y)\). By Lemma 2.4 we have
\[
[\mathrm{add} T](\Sigma^{-d} X, Y) \cong \mathrm{DHom}_{\mathcal{F}}(\Sigma^d T, Y, X) \cong \mathrm{DHom}_{\mathcal{F}}(\Sigma^d T, Y, X).
\]

We also know that \( \mathcal{F}(X, Y) \cong \mathrm{Ext}_{\mathcal{F}}^d(X, Y) \), so the conclusion follows.

Lemma 2.7. For each \( X \in \mathcal{F} \), pick an isomorphism \( X \cong X' \oplus X'' \) such that \( X' \) has no non-zero direct summands in \( \mathrm{add} \Sigma^d T \) and \( X'' \in \mathrm{add} \Sigma^d T \). Let
\[
\Delta(X) = (\mathcal{F}(T, X'), \mathcal{F}(T, \Sigma^{-d} X'')).
\]

This is a pair of \( \Gamma \)-modules where \( \mathcal{F}(T, X') \) is in \( \mathcal{D} \) and \( \mathcal{F}(T, \Sigma^{-d} X'') \) is in \( \text{proj} \Gamma \).
Proposition 2.8. Given $X, Y \in \mathcal{T}$, set $(M, P) = \Delta(X)$ and $(N, Q) = \Delta(Y)$, where $\Delta$ is the map in Definition 2.7. Then

$$\dim_k \text{Ext}^d_{\mathcal{T}}(X, Y) = \dim_k \text{Hom}_{\mathcal{T}}(M, \tau_d N) + \dim_k \text{Hom}_{\mathcal{T}}(N, \tau_d M)$$

$$+ \dim_k \text{Hom}_{\mathcal{T}}(P, N) + \dim_k \text{Hom}_{\mathcal{T}}(Q, M).$$

Proof. By additivity of Ext we have

$$\text{Ext}^d_{\mathcal{T}}(X, Y) \cong \text{Ext}^d_{\mathcal{T}}(X' \oplus X'', Y' \oplus Y'').$$

$$\cong \text{Ext}^d_{\mathcal{T}}(X', Y') \oplus \text{Ext}^d_{\mathcal{T}}(X'', Y'') \oplus \text{Ext}^d_{\mathcal{T}}(X', Y'') \oplus \text{Ext}^d_{\mathcal{T}}(X'', Y'').$$

As $T$ is $d$-rigid, we see that $\text{Ext}^d_{\mathcal{T}}(X'', Y'') = 0$, and hence we have

$$\dim \text{Ext}^d_{\mathcal{T}}(X, Y) = \dim \text{Ext}^d_{\mathcal{T}}(X', Y') + \dim \text{Ext}^d_{\mathcal{T}}(X'', Y'') + \dim \text{Ext}^d_{\mathcal{T}}(X', Y'').$$

(2.1)

From Lemma 2.6 we have the short exact sequence:

$$0 \rightarrow \text{DHom}_T (\mathcal{T}(T, Y''), \tau_d T(T, X')) \rightarrow \text{Ext}^d_{\mathcal{T}}(X', Y') \rightarrow \text{Hom}_T (\mathcal{T}(T, X'), \tau_d T(T, Y')) \rightarrow 0,$$

which means that

$$\dim \text{Ext}^d_{\mathcal{T}}(X', Y') = \dim_k \text{Hom}_{\mathcal{T}} (\mathcal{T}(T, X'), \tau_d T(T, Y')) + \dim_k \text{Hom}_{\mathcal{T}} (\mathcal{T}(T, Y'), \tau_d T(T, X'))$$

$$= \dim_k \text{Hom}_{\mathcal{T}}(M, \tau_d N) + \dim_k \text{Hom}_{\mathcal{T}}(N, \tau_d M).$$

(2.2)

We see that

$$\text{Ext}^d_{\mathcal{T}}(X'', Y'') \cong \mathcal{T}(X'', \Sigma^d Y'') \cong \mathcal{T}(\Sigma^{-d} X'', Y'') \cong \text{Hom}_T (\mathcal{T}(T, \Sigma^{-d} X''), \mathcal{T}(T, Y')) \cong \text{Hom}_T (P, N).$$

The third isomorphism follows from [14, Lemma 2.2(i)] and the fact that $\Sigma^{-d} X'' \in \text{add} T$. Similarly,

$$\text{Ext}^d_{\mathcal{T}}(X', Y'') \cong \text{DExt}^d_{\mathcal{T}}(Y'', X') \cong \text{DHom}_T (Q, M).$$

Thus we have

$$\dim \text{Ext}^d_{\mathcal{T}}(X'', Y'') = \dim_k \text{Hom}_{\mathcal{T}}(P, N)$$

(2.3)

$$\dim \text{Ext}^d_{\mathcal{T}}(X', Y'') = \dim_k \text{Hom}_{\mathcal{T}}(Q, M).$$

(2.4)

Substituting (2.2), (2.3), and (2.4) into (2.1) gives the result. □

As a consequence we have:

Corollary 2.9. Given $X, Y \in \mathcal{T}$, set $(M, P) = \Delta(X)$ and $(N, Q) = \Delta(Y)$. Then

$$\text{Ext}^d_{\mathcal{T}}(X, Y) = 0 \Leftrightarrow \text{Hom}_{\mathcal{T}}(M, \tau_d N) = \text{Hom}_{\mathcal{T}}(N, \tau_d M) = \text{Hom}_{\mathcal{T}}(P, N) = \text{Hom}_{\mathcal{T}}(Q, M) = 0.$$

3. PROOF OF THEOREM B

The following results use the map $\Delta$ from Definition 2.7.

Lemma 3.1. Given $X, Y \in \mathcal{T}$, set $(M, P) = \Delta(X)$ and $(N, Q) = \Delta(Y)$. Then $Y \in \text{add} X$ if and only if $N \in \text{add} M$ and $Q \in \text{add} P$.

Proof. Let $X \cong X' \oplus X''$ be the decomposition from Definition 2.7, where $X'$ has no non-zero direct summands from $\text{add} \Sigma^d T$ while $X''$ is in $\text{add} \Sigma^d T$. We have $(M, P) = (\mathcal{T}(T, X'), \mathcal{T}(T, \Sigma^{-d} X''))$. Similarly, $(N, Q) = (\mathcal{T}(T, Y'), \mathcal{T}(T, \Sigma^{-d} Y''))$.

The condition $Q \in \text{add} P$ is equivalent to $Y'' \in \text{add} X''$ by the add-proj-correspondence, (see Remark 0.5). The condition $N \in \text{add} M$ is equivalent to $Y' \in \text{add} X'$ by Theorem 0.4 because $X', Y'$ have no non-zero direct summands in $\text{add} \Sigma^d T$. The result follows. □
Lemma 3.2. The category $\mathcal{T}$ is skeletally small. The map $\Delta$ induces a bijection
\[ \delta : \text{iso } \mathcal{T} \to \text{iso } \mathcal{D} \times \text{iso proj } \Gamma, \] (3.1)
where iso denotes the set of isomorphism classes of a skeletally small category.

Proof. Let Iso denote the class of isomorphisms of a category. For a skeletally small category $\mathcal{C}$ we have that Iso $\mathcal{C} = \text{iso } \mathcal{C}$. Note that since a module category over a ring is skeletally small, we have that $\mathcal{D}, \text{proj } \Gamma \subseteq \text{mod } \Gamma$ are skeletally small.

It is clear that $\Delta$ induces a well-defined map of the form
\[ \delta' : \text{Iso } \mathcal{T} \to \text{iso } \mathcal{D} \times \text{iso proj } \Gamma. \]

To see that $\delta'$ is injective, argue like the proof of Lemma 3.1, replacing membership of add with isomorphism.

It follows that $\mathcal{T}$ is skeletally small. We can thus replace $\delta'$ with the map $\delta$ from (3.1).

To see that $\delta$ is surjective, let $(M, P)$ be a pair with $M \in \mathcal{D}$ and $P \in \text{proj } \Gamma$. By Theorem 0.4 there is an object $X' \in \mathcal{T}$ with no non-zero direct summands in add $\Sigma^d \mathcal{T}$ such that $M \cong \mathcal{T}(T, X')$. By the add-proj correspondence, see Remark 0.5, there is an object $X'' \in \text{add } \Sigma^d \mathcal{T}$ such that $P \cong \mathcal{T}(T, \Sigma^{-d}X'')$. Setting $X = X' \oplus X''$ gives $(M, P) \cong \Delta(X)$.

Lemma 3.3. If $X \in \mathcal{T}$ is $d$-self-perpendicular, then $(M, P) = \Delta(X)$ is a maximal $\tau_d$-rigid pair.

Proof. Let $N \in \mathcal{D}$ and $Q \in \text{proj } \Gamma$ be given. By Lemma 3.2, there is an object $Y \in \mathcal{T}$ such that $(N, Q) \cong \Delta(Y)$. Then
\[ N \in \text{add } M \text{ and } Q \in \text{add } P \]
\[ \Rightarrow Y \in \text{add } X \]
\[ \Leftrightarrow \text{Ext}_\mathcal{T}^d(X, Y) = 0 \]
\[ \Leftrightarrow \text{Hom}_\Gamma(M, \tau_dN) = \text{Hom}_\Gamma(N, \tau_dM) = \text{Hom}_\Gamma(P, N) = \text{Hom}_\Gamma(Q, M) = 0, \]
where the equivalences, respectively, are by Lemma 3.1, Definition 0.2, and Corollary 2.9.

The conditions of Definition 0.7 are recovered by setting $Q = 0$ respectively $N = 0$.

Lemma 3.4. Let $X \in \mathcal{T}$ be given. If $(M, P) = \Delta(X)$ is a maximal $\tau_d$-rigid pair, then $X$ is $d$-self-perpendicular.

Proof. Let $Y \in \mathcal{T}$ be given and set $(N, Q) \cong \Delta(Y)$. Then
\[ \text{Ext}_\mathcal{T}^d(X, Y) = 0 \]
\[ \Leftrightarrow \text{Hom}_\Gamma(M, \tau_dN) = \text{Hom}_\Gamma(N, \tau_dM) = \text{Hom}_\Gamma(P, N) = \text{Hom}_\Gamma(Q, M) = 0 \]
\[ \Leftrightarrow N \in \text{add } M \text{ and } Q \in \text{add } P \]
\[ \Leftrightarrow Y \in \text{add } X, \]
where the equivalences, respectively, are by Corollary 2.9, Definition 0.7, and Lemma 3.1.

Theorem 3.5. Recall that the map $\Delta$ from Definition 2.7 induces the bijection $\delta : \text{iso } \mathcal{T} \to \text{iso } \mathcal{D} \times \text{iso proj } \Gamma$ from Lemma 3.2.

(i) $\delta$ restricts to a bijection
\[ \{ \text{isomorphism classes of } \text{d-rigid objects in } \mathcal{T} \} \to \{ \text{isomorphism classes of } \tau_d \text{-rigid pairs in } \mathcal{D} \}.\]
(ii) \( \delta \) restricts further to a bijection
\[
\left\{ \text{isomorphism classes of } d\text{-self-perpendicular objects in } \mathcal{T} \right\} \rightarrow \left\{ \text{isomorphism classes of maximal } \tau_d\text{-rigid pairs in } \mathcal{D} \right\}.
\]

Proof. (i): Consider \( X \in \mathcal{T} \) and set \( (M, P) = \Delta(X) \). Then
\[
\text{Ext}_d^d(\mathcal{T})(X, X) = 0 \Leftrightarrow \text{Hom}_\mathcal{T}(M, \tau_dM) = 0 \text{ and } \text{Hom}_\mathcal{T}(P, M) = 0
\]
by Corollary 2.9, so the result follows.

(ii): See Lemmas 3.3 and 3.4. \( \square \)

Proof (of Theorem B from the introduction). Combine Theorems 3.5(ii) and 1.1(ii). \( \square \)

4. An example

In this section we let \( d = 3 \) and \( \mathcal{T} = \mathcal{O}_{A_2}^3 \). This is the 5-angulated (higher) cluster category of type \( A_2 \), see [21, def. 5.2, sec. 6, and sec. 8]. The indecomposable objects can be identified with the elements of the set
\[
\mathcal{C}_0^{13} = \{ 1357, 1358, 1368, 1468, 2468, 2469, 2479, 2579, 3579 \},
\]
see [21, sec. 8]. The AR quiver of \( \mathcal{T} \) is shown in Figure 1. By [21, thm. 5.5 and sec. 8], the object
\[
T = 1357 \oplus 1358 \oplus 1368 \oplus 1468
\]
is Oppermann–Thomas cluster tilting.

If \( X, Y \in \mathcal{T} \) are indecomposable objects, then
\[
\mathcal{T}(X, Y) = \begin{cases} k & \text{if } Y \text{ is } X \text{ or its immediate successor in the AR quiver}, \\ 0 & \text{otherwise}, \end{cases}
\]
see [21, prop. 6.1 and def. 6.9]. It follows that \( \Gamma = \text{End}_\mathcal{T}(T) = kQ/I \), where
\[
Q = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\]
and \( I \) is the ideal generated by all compositions of two consecutive arrows. The action of the functor \( \mathcal{T}(T, -) : \mathcal{T} \rightarrow \text{mod } \Gamma \) on indecomposable objects is shown in Figure 2, where \( P(q) \) and \( I(q) \) denote the indecomposable projective and injective modules associated to the vertex \( q \in Q \). Note that the
The maximal $\tau_d$-rigid pairs

| $X$ | 1357 | 1358 | 1368 | 1468 | 2468 | 2469 | 2479 | 2579 | 3579 |
|-----|------|------|------|------|------|------|------|------|------|
| $\mathcal{T}(T, X)$ | $P(4)$ | $P(3)$ | $P(2)$ | $P(1)$ | $I(1)$ | 0 | 0 | 0 | 0 |

Figure 2. The action of the functor $\mathcal{T}(T, -) : \mathcal{T} \to \text{mod } \Gamma$.

\[ \xymatrix{ & & & \circ \ar@{=>}[ddr] & \circ \ar@{=>}[drr] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[dll] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] \cr \circ & & & \circ \ar@{=>}[ddr] & \circ \ar@{=>}[drr] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[dll] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] & \circ \ar@{=>}[d] & \circ \ar@{=>}[dl] \cr Y_1 & & & X & & & & & & & Y_2 & & & & & & & \\
}

Figure 3. The functor $\text{Ext}^3_{\mathcal{T}}(X, -)$ is non-zero on $Y_1$ and $Y_2$. It is zero on every other indecomposable object.

The essential image of $\mathcal{T}(T, -)$ is

$$\mathcal{D} = \text{add} \{ P(4), P(3), P(2), P(1), I(1) \}.$$  

This is a $3$-cluster tilting subcategory of $\text{mod } \Gamma$ and hence it is $3$-abelian.

The $3$-suspension functor $\Sigma^3$ acts on the AR quiver by moving four steps clockwise. Combined with our knowledge of Hom, this shows that if $X$ is a fixed indecomposable object in $\mathcal{T}$, then the indecomposable objects $Y$ with $\text{Ext}^3_{\mathcal{T}}(X, Y) \neq 0$ are precisely the two objects furthest from $X$ in the AR quiver, see Figure 3.

Based on this, we can compute all basic $3$-self-perpendicular objects in $\mathcal{T}$, and by Proposition 1.3 they coincide with the basic maximal $3$-rigid objects in $\mathcal{T}$. For each such object $X$, there is a maximal $\tau_3$-rigid pair $\Delta(X) = (\mathcal{T}(T, X'), \mathcal{T}(T, \Sigma^{-3}X''))$ by Theorem B. See Figure 4. Note that the first nine objects in Figure 4 are Oppermann–Thomas cluster tilting, but the three last objects are not.

Acknowledgement. This work was supported by EPSRC grant EP/P016014/1 “Higher Dimensional Homological Algebra”. Karin M. Jacobsen is grateful for the hospitality of Newcastle University during her visit in October 2018.

References

[1] T. Adachi, O. Iyama, and I. Reiten, $\tau$-tilting theory, Compositio Math. 150 (2014), 415–452.
[2] I. Assem, D. Simson, and A. Skowroński, “Elements of the representation theory of associative algebras, Vol. 1, Techniques of representation theory”, London Math. Soc. Stud. Texts, Vol. 65, Cambridge University Press, Cambridge, 2006.
[3] P. A. Bergh and M. Thaule, The axioms for n-angulated categories, Algebr. Geom. Topol. 13 (2013), 2405–2428.
[4] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. 204 (2006), 572–618.
[5] A. B. Buan, R. J. Marsh, and D. F. Vatne, Cluster structures from 2-Calabi–Yau categories with loops, Math. Z. 265 (2010), 951–970.
| Maximal 3-rigid object $X$ | Maximal $\tau_3$-rigid pair $\Delta(X)$ |
|--------------------------|---------------------------------------|
| $1357 \oplus 1358 \oplus 1368 \oplus 1468$ | $(\Gamma, 0)$ |
| $1358 \oplus 1368 \oplus 1468 \oplus 2468$ | $(D\Gamma, 0)$ |
| $1368 \oplus 1468 \oplus 2468 \oplus 2469$ | $(P(2) \oplus P(1) \oplus I(1), P(4))$ |
| $1468 \oplus 2468 \oplus 2469 \oplus 2479$ | $(P(1) \oplus I(1), P(4) \oplus P(3))$ |
| $2468 \oplus 2469 \oplus 2479 \oplus 2579$ | $(I(1), P(4) \oplus P(3) \oplus P(2))$ |
| $2469 \oplus 2479 \oplus 2579 \oplus 3579$ | $(0, \Gamma)$ |
| $2479 \oplus 2579 \oplus 3579 \oplus 1357$ | $(P(4), P(3) \oplus P(2) \oplus P(1))$ |
| $2579 \oplus 3579 \oplus 1357 \oplus 1358$ | $(P(4) \oplus P(3), P(2) \oplus P(1))$ |
| $3579 \oplus 1357 \oplus 1358 \oplus 1368$ | $(P(4) \oplus P(3) \oplus P(2), P(1))$ |
| $1357 \oplus 1468 \oplus 2479$ | $(P(4) \oplus P(1), P(3))$ |
| $1358 \oplus 2468 \oplus 2579$ | $(P(3) \oplus I(1), P(2))$ |
| $1368 \oplus 2469 \oplus 3579$ | $(P(2), P(4) \oplus P(1))$ |

**Figure 4.** These are all the basic maximal 3-rigid objects of $\mathcal{T}$ and their corresponding maximal $\tau_3$-rigid pairs in $\mathcal{D}$. 

[6] F. Fedele, *Auslander-Reiten $(d + 2)$-angles in subcategories and a $(d + 2)$-angulated generalisation of a theorem by Brüning*, to appear in J. Pure Appl. Algebra.

[7] F. Fedele, *d-Auslander–Reiten sequences in subcategories*, preprint (2018). arXiv:1808.02709.

[8] C. Geiss, B. Keller, and S. Oppermann, *n-angulated categories*, J. Reine Angew. Math. 675 (2013), 101–120.

[9] M. Herschend and O. Iyama, *n-representation-finite algebras and twisted fractionally Calabi–Yau algebras*, Bull. London Math. Soc. 43 (2011), 449–466.

[10] M. Herschend and O. Iyama, *Selfinjective quivers with potential and 2-representation-finite algebras*, Compositio Math. 147 (2011), 1885–1920.

[11] O. Iyama, *Cluster tilting for higher Auslander algebras*, Adv. Math. 226 (2011), 1–61.

[12] O. Iyama, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 (2007), 22–50.

[13] O. Iyama and S. Oppermann, *Stable categories of higher preprojective algebras*, Adv. Math. 244 (2013), 23–68.

[14] K. M. Jacobsen and P. Jørgensen, *d-abelian quotients of $(d + 2)$-angulated categories*, J. Algebra 521 (2019), 114–136.

[15] G. Jasso, *n-abelian and n-exact categories*, Math. Z. 283 (2016), 203–759.

[16] G. Jasso and J. Külshammer, *Higher Nakayama algebras I: Construction*, preprint (2016). arXiv:1604.03500.

[17] G. Jasso and J. Külshammer, *Nakayama-type phenomena in higher Auslander–Reiten theory*, pp. 79–98 in: “Representations of algebras”, Contemp. Math., Vol. 705, American Mathematical Society, Providence, RI, 2018.

[18] G. Jasso and S. Kvaamme, *An introduction to higher Auslander–Reiten theory*, to appear in Bull. London Math. Soc.

[19] J. McMahon, *Higher support tilting I: Higher Auslander algebras of linearly oriented type A*, preprint (2018). arXiv:1808.05184.

[20] Y. Mizuno, *A Gabriel-type theorem for cluster tilting*, Proc. London Math. Soc. (3) 108 (2014), 836–868.

[21] S. Oppermann and H. Thomas, *Higher-dimensional cluster combinatorics and representation theory*, J. Eur. Math. Soc. (JEMS) 14 (2012), 1679–1737.

[22] Y. Zhou and B. Zhu, *Maximal rigid subcategories in 2-Calabi–Yau triangulated categories*, J. Algebra 348 (2011), 49–60.
JACOBSEN: NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, DEPARTMENT OF MATHEMATICAL SCIENCES, SENTRALBYGG 2, GLOSHAUGEN, 7491 TRONDHEIM, NORWAY

E-mail address: kjacobsen@math.uni-bielefeld.de

Current address: Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany

JØRGENSEN: SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE NE1 7RU, UNITED KINGDOM

E-mail address: peter.jorgensen@ncl.ac.uk

URL: http://www.staff.ncl.ac.uk/peter.jorgensen