Generalized Gauged Thirring Model on Curved Space-Times

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Abstract: We analyse the interacting theory of charged fermions, scalars, pseudo-scalars and photons propagating in 2-dimensional curved spacetime in detail. For certain values of the coupling constants the theory reduces to the gauged Thirring model and for others the Schwinger model in curved spacetime. It is shown that the interaction of the fermions with the pseudo-scalars shields the electromagnetic interaction, and that the non-minimal coupling of the scalars to the gravitational field amplifies the Hawking radiation. We solve the finite temperature and density model by using functional techniques and in particular derive the exact equation of state. The explicit temperature and curvature dependence of the chiral condensate is found. When the electromagnetic field is switched off the model reduces to a conformal field theory. We determine the physically relevant expectation values and conformal weights of the fundamental fields in the theory.

1. Introduction

The response of physical systems to a change of external conditions is of eminent importance in physics. In particular the dependence of expectation values on temperature, the particle density, the space region, the imposed boundary conditions or external fields has been widely studied [1]. Despite
all these efforts we are still unable to understand, for example, the mechanism leading to the spontaneous symmetry breaking of the $SU_A(N)$ in low temperature QCD [2]. Clearly such subtle effects require a better understanding of the nonperturbative effects and in particular nonperturbative vacuum sector of gauge theories. From our experience with 2-dimensional gauge theories [3] which we suppose to mimic one-flavour QCD [4], we are lead to believe that gauge fields with windings are responsible for the non-vanishing chiral condensate and in particular its temperature dependence. A related problem is how quantum systems behave in a hot and dense environment as it exists or existed in heavy ion collision, neutron stars or the early epochs of the universe [2].

On another front there has been much effort to quantize self-interacting field theories in a background gravitational field [5]. For example, one is interested whether a black hole still emits thermal radiation when self-interaction is included. Due to general arguments by Gibbons and Perry [6] this question is intimately connected with universality of the second law of thermodynamics.

Rather than seeking new partial results for more general and realistic 4-dimensional systems we have chosen an idealized 2-dimensional model with self-interaction to investigate the questions mentioned and others. It is a theory containing photons\(^1\), charged massless fermions, scalars and pseudoscalars in interaction with themselves and a gravitational background field. The model has the action

\[
S = \int \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^{\mu} (\nabla_{\mu} - ig_2 \eta_{\mu\nu} \partial_{\nu} \phi) \psi \\
+ g^{\mu\nu} (\partial_{\mu} \phi \partial_{\nu} \phi + \partial_{\mu} \lambda \partial_{\nu} \lambda) - g_3 R \lambda \right],
\]

where $F_{\mu\nu}$ is the electromagnetic field strength, the gamma-matrices in curved space are related to the flat ones as $\gamma^{\mu} = e_\mu^a \tilde{\gamma}^a$, $\nabla_{\mu} = \partial_{\mu} + i \omega_{\mu} - i e A_{\mu}$

\(^1\) Althouth photons in 1+1 dimensions possess no transversal degrees of freedom we still call them photons. However, thru their interaction with charged fermions they may become dynamical fields as exemplified by the Schwinger mechanism.
is the generally and gauge covariant derivative containing the $U(1)$ gauge potential and spin connection, $\eta_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu}$ denotes the totally antisymmetric tensor and $\mathcal{R}$ the Ricci scalar. The gravitational field $g_{\mu\nu}$ (or rather the 2-bein $e^a_\mu$, since the theory contains fermions) is treated as classical background field, whereas the 'photons' $A_\mu$, 'electrons' $\psi$, scalars $\lambda$ and pseudoscalars $\phi$ are fully quantized. In two dimensions the electric charge $e$ has the dimension of a mass. The other 3 couplings are dimensionless. The classical theory is invariant under $U(1)$ gauge- and axial transformations and correspondingly possesses conserved vector and axial-vector currents.

We have chosen this model since it allows to address the above raised questions and since it relates to known soluble models for certain values of the coupling constants. For example, it contains the **gauged Thirring model**, the *Schwinger model in curved space time* and the *minimal models* in conformal field theory as particular limits. Its vacuum structure is non-trivial, i.e. it has $\theta$-vacua like more realistic 4-dimensional gauge theories [7]. The 'photon' acquires a mass $m_\gamma^2 = e^2/(\pi + \frac{1}{2}g_2^2)$ via the Schinger mechanism. For finite volumes the theory possesses instantons which minimize the euclidean action. These instantons lead to chirality violating vacuum expectation values. For example, a non-zero chiral condensate develops which only for high temperature and large curvature vanish exponentially.

For $e = 0$ all coupling constants are dimensionless and the theory becomes conformally invariant. In this limit the vacuum structure becomes trivial. Despite its complexity the general model (1.1) is solvable for arbitrary classical backgrounds $g_{\mu\nu}$ and allows for an analytical treatment. This in turn enables the entire stress tensor in *any* curved space, the induced currents, their correlators and the equation of state to be constructed.

The physical role of the coupling constants is the following: The coupling of $\phi$ to the transversal current decreases the effective electromagnetic interaction between fermions. For example, the electric charge becomes renormalized to $e_R = e/\sqrt{1 + g_2^2/2\pi}$, the chiral condensate decreases as $\sim (2\pi + g_2^2)^{-\frac{1}{2}}$. 
The mass in the bosonised theory depends on $g_2$. In the ungauged sector the Kac-Moody central extension, conformal weights and $U(1)$ charges depend on $g_2$. The coupling constant $g_3$ amplifies the Hawking radiation which remains thermal for the interacting model. It is $(3 + 24\pi g_3^2)$ times as strong as that of a free massless scalar field. The central charge and conformal weights depend on $g_3$. Actually, the weights of the fermionic fields become complex for $g_3 \neq 0$. However, $g_3$ does not enter in the finite size effects. The coupling constant $g_1$ to the longitudinal current weakens the long range gauge invariant electron-electron correlators in the one-instanton sector (see 6.27). In the ungauged sector it enters in expectation values of local operators and in particular in the short distance expansions of the fermionic fields and energy momentum tensor. It does not influence the thermodynamics of the model.

Since for particular choices of the coupling constants the model reduces to wellknown and wellstudied exactly soluble models there are many earlier works which are related to ours. Some of them concentrated more on the gauge sector and investigated the renormalization of the electric charge in the gauged Thirring model by the four-fermi interaction [8] or the non-trivial vacuum structure in the Schwinger model [3,9]. Others concentrated on the ungauged conformal sector. Freedman and Pilch calculated the partition function of the ungauged Thirring model on arbitrary Riemann surfaces [10]. We do not agree with their result and in particular show that there is no holomorphic factorization for general fermionic boundary conditions. Also we deviate from Destri and deVega [11] which investigated the ungauged model on the cylinder with twisted boundary conditions. We shall comment on the discrepancies in sections 3 and 7. Other papers which are relevant and are dealing with different aspects of certain limiting cases of (1.1) are [12], where the thermodynamics of the Thirring model has been studied or [5] in which the Hawking radiation has been derived.
The paper is organized as follows: In section 2 we analyse the classical model to prepare the ground for the quantization. In particular we derive the general solution of the field equations, discuss the conservation laws and investigate the limiting theories. By employing the graded structure we derive the classical Poisson (anti) commutators of the fundamental fields with the energy momentum tensor. In the following section we quantize the finite temperature model. To avoid infrared problems we assume space to be finite. Together with the finite temperature boundary conditions we are lead to considering the theory on the 2-dimensional euclidean torus. Due to the twists in the fermionic boundary conditions, the non-trivial vacuum structure and the associated instantons and fermionic zero-modes the quantization is rather subtle. Actually we show that some of the results in the literature are incorrect. In section 4 the general results are applied to derive the partition function of the gauged model. Its dependence on the spatial size, temperature and gravitational field is explicitly found. In section 5 we show that for equal couplings the gauged model on curved spacetime can be bosonized. It turns out that only the non-constant parts of the currents can be bosonized and that for this part the wellknown flat spacetime rules need just be covariatized. In the following section the chiral symmetry breaking is studied. The exact form of the chiral condensate is found. On the flat torus the formula simplifies to (6.13). Various limits, e.g. \( L \to \infty \), \( T \to 0 \), \( T \to \infty \) or \( g_2 \to \infty \) are investigated. By comparing the temperature and curvature dependence of the condensate we derive an effective curvature induced temperature. In section 7 the thermodynamics of the ungauged model is studied. We derive the ground state energy and its dependence on the coupling constants, size of the system and boundary conditions. We compute the equation of state and our result does not agree with [12]. In the last section we investigate the conformal sector of (1.1), that is the ungauged model in flat spacetime. Besides the Virasoro algebra the model contains an \( U(1) \) Kac-Moody algebra. We calculate the important commu-
tactors and in particular determine the conformal weights and $U(1)$-charges of the fundamental fields from first principles. Also we show that the finite size effects are in general not proportional to the central charge as has been conjectured by Cardy [13]. The appendix A contains our conventions and scaling formulae for the various geometrical objects. In appendix B we collected some useful variational formulae which we have used in this work. In appendix C we derive the partition function within the canonical approach.

2. Classical theory

2.1 Equations of Motion

The field equations of the model (1.1) are

$$i\gamma^{\mu}(\nabla_{\mu} - ig_{1}\partial_{\mu}\lambda + ig_{2}\eta_{\mu\nu}\partial^{\nu}\phi) \; \psi \equiv i\gamma^{\mu}D_{\mu} = 0$$

$$2 \nabla^{2}\lambda = -g_{3}\mathcal{R} - g_{1}\nabla_{\mu}j^{\mu}$$

$$2 \nabla^{2}\phi = -g_{2}\nabla_{\mu}j^{5\mu}$$

$$\nabla_{\nu}F^{\mu\nu} = e^{j^{\mu}}$$

which are the Dirac equation for massless charged fermions propagating in a curved space-time and interacting with the scalar and pseudoscalar-fields, Klein Gordon type of equations and Maxwell equation. Here $j^{5\mu}$ is the axial vector current which is defined by

$$j^{5\mu} = \bar{\psi}\gamma^{\mu}\gamma_{5}\psi = \eta^{\mu}_{\nu}j^{\nu}.$$  \hspace{1cm} (2.2)

When one decomposes the gauge field as

$$A_{\mu} = \partial_{\mu}\alpha - \eta_{\mu\rho}\partial^{\rho}\varphi \quad \text{so that} \quad F_{01} = \sqrt{-g}\nabla^{2}\varphi,$$  \hspace{1cm} (2.3)

and choses isothermal coordinates for which $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$, then the generalized Dirac operator reads

$$\mathcal{D} = e^{iF - i\gamma_{5}G - \xi^{2}\sigma} \; \mathcal{D} e^{-iF - i\gamma_{5}G + \xi^{2}\sigma}, \quad \text{where}$$

$$F = g_{1}\lambda + e\alpha \quad , \quad G = g_{2}\phi + e\varphi.$$  \hspace{1cm} (2.4)
Hence, if $\psi_0(x)$ solves the free Dirac equation in flat Minkowski space-time, then

$$\psi(x) \equiv e^{iF + i\gamma_5 G - \frac{1}{2}\sigma} \psi_0$$  \hspace{1cm} (2.5)

solves the Dirac equation of the interacting theory on curved spacetime. The vector currents are related as

$$j^\mu = \bar{\psi} \gamma^\mu \psi = \bar{\psi}_0 \tilde{\gamma}^\mu \psi_0 e^{-2\sigma} \equiv \frac{1}{\sqrt{-g}} j^\mu_0.$$  \hspace{1cm} (2.6)

The same relation holds for the axial vector current. From $\sqrt{-g} \nabla_\mu j^\mu = \partial_\mu \sqrt{-g} j^\mu$ the conservation of the vector and axial currents follow at once,

$$\nabla_\mu j^\mu = \nabla_\mu \tilde{j}^{\mu} = 0,$$  \hspace{1cm} (2.7)

expressing the classical $U(1) \times U_A(1)$ invariance of the model. Using (2.7) in (2.1) we conclude that

$$2\nabla^2 \lambda = -g_\lambda \mathcal{R} \quad \text{and} \quad \nabla^2 \phi = 0$$ \hspace{1cm} (2.8)

or that there is no backreaction from fermions onto scalars. Finally the conservation laws imply that the currents are free fields

$$\nabla^2 j^\mu = \nabla^2 \tilde{j}^{\mu} = 0,$$  \hspace{1cm} (2.9)

which is the reason which accounts for the solubility of the model [14], even in the presence of photons and an external gravitational field. As is well-known, for any gauge invariant regularization the axial current possesses an anomalous divergence in the quantized model and (2.9) is modified. Thus the normal $U_A(1)$ Ward identities in the ungauged Thirring model [8] become anomalous when the fermions couple to a gauge field.

In isothermal coordinates the general solution of the field equations can be expressed in terms of 6 chiral functions as follows: Introducing lightcone coordinates $x^\pm = x^0 \pm x^1$ so that $ds^2 = e^{2\sigma} dx^+ dx^-$, the solutions of (2.8) read

$$\lambda = g_\lambda \sigma + \lambda_+(x^+) + \lambda_-(x^-), \quad \text{and} \quad \phi = \phi_+(x^+) + \phi_-(x^-)$$  \hspace{1cm} (2.10)
and depend on 4 chiral functions which are fixed by the initial data on some spacelike hypersurface. The solutions of the free Dirac equations depend on 2 chiral functions as
\[
\psi_0 = \begin{pmatrix} \psi_-(x^-) \\ \psi_+(x^+) \end{pmatrix}.
\]
In these coordinate system the Maxwell equations (2.1) can easily be integrated and one finds
\[
\partial_+ \partial_- \phi = F_{01} = 2e^{2\alpha} \left[ \int \psi_+^\dagger(\xi)\psi_-(\xi) d\xi - \int \psi_-^\dagger(\xi)\psi_+(\xi) d\xi \right].
\]
To go further we must fix the gauge. Conveniently one chooses the Lorentz gauge such that \( \alpha = 0 \) in (2.3) and thus \( \phi \) in (2.12) determines \( A_\mu \). We see that in isothermal coordinates and this gauge the general solution of (2.1) is given by (2.10), (2.12) and (2.5), that is in terms of 6 chiral functions.

Besides the currents the symmetric energy-momentum tensor of the matter fields
\[
T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}
\]
plays an important role in any theory in curved space time. Applying the variational identities in Appendix B one obtains after a lengthy but straightforward computation
\[
T^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\sigma\rho} F_{\sigma\rho} - F^{\sigma\nu} F_{\sigma}^\mu + \frac{i}{2} \left[ \bar{\psi} \gamma^{(\mu} D^{\nu)} \psi - (D^{(\mu} \bar{\psi}) \gamma^{\nu)} \psi \right]
+ 2 \nabla^\mu \phi \nabla^\nu \phi - g^{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi + (\phi \leftrightarrow \lambda)
- 2g_3 \left( g^{\mu\nu} \nabla^2 - \nabla^\mu \nabla^\nu \right) \lambda
+ \frac{1}{2} j^\mu \left( g_1 \nabla^\nu \lambda - g_2 \eta^{\nu\alpha} \nabla_\alpha \phi \right) + (\mu \leftrightarrow \nu)
+ g_2 g^{\mu\nu} j^\alpha \eta_{\alpha\beta} \nabla^\beta \phi - 2g_2 j^\alpha \eta_{\alpha}^{(\mu} \nabla^{\nu)} \phi,
\]
where we have introduced the symmetrization \( A^{(\mu B^\nu)} = \frac{1}{2} (A^\mu B^\nu + A^\nu B^\mu) \).

The first two lines are just the energy momentum of the electromagnetic field, charged fermions and free neutral (pseudo-) scalars. The terms containing second derivatives of \( \lambda \) are the improvement terms [15] which are
always present when one couples scalars non-minimally to a background curvature. The remaining terms reflect the interaction between the fermionic and auxiliary fields.

On shell $T^{\mu\nu}$ is conserved as required by general covariance. Using the field equations for $\psi$ and $\lambda$ its trace reads

$$T^{\mu}_{\mu} = g_3^2 R - \frac{1}{2} F^{\sigma\rho} F_{\sigma\rho}. \quad (2.15)$$

In particular for $g_3=0$ and $A_\mu=0$ it vanishes, and the theory becomes Weyl-invariant. As a consequence it reduces to a conformal field theory in the flat spacetime limit [16]. It is remarkable that it can be made Weyl invariant even when $g_3 \neq 0$. Indeed, without changing the flat spacetime limit we may add a nonlocal Wess-Zumino-type term to the action, namely

$$S' = S - \frac{g_3^2}{4} S_p \quad \text{where} \quad S_p = \int \sqrt{-g} R \frac{1}{\nabla^2} R \quad (2.16)$$

the variation of which is

$$\delta S_p = \int \left\{ 4[g^{\mu\nu} R - \nabla^{\mu} \nabla^{\nu} \frac{1}{\nabla^2} R] + 2\nabla^{\mu} \left( \frac{1}{\nabla^2} R \right) \nabla^{\nu} \left( \frac{1}{\nabla^2} R \right) \right. \nabla^{\alpha} \left( \frac{1}{\nabla^2} R \right) \left( \frac{1}{\nabla^2} R \right) \left\} \sqrt{-g} \delta g_{\mu\nu} \quad (2.17)$$

The trace of the modified energy momentum tensor is now zero, and for $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ the Lagrangian corresponds to a conformal field theory in Minkowski spacetime.

Choosing the coupling constants appropriately, the model reduces to various well known exactly solvable models:

- For the special choice $g_1 = g_2 = e = 0$ and for vanishing gauge field the $\lambda$-dependent part of (1.1) is just the Lagrangian of scalar fields coupled to a background charge and and for imaginary $g_3$ describes the minimal models of conformal field theory [17].
- For \( g_3 = 0 \) and \( g_1^2 = -g_2^2 = g^2 \) the fermionic sector reduces to the gauged version of the Thirring model \[18\] in curved space time. To see that we solve the Klein Gordon equations in (2.1) for the \( U(1) \) current which yields

\[
\mathcal{L}[\bar{\psi}, \psi] = \bar{\psi} \gamma^\mu \nabla_\mu \psi - \frac{g^2}{2} j^\mu \gamma_\mu \psi = 0 ,
\]

which is the field equation of the gauged Thirring model in curved spacetime with Lagrangean

\[
\mathcal{L}[\bar{\psi}, \psi] = \bar{\psi} \gamma^\mu \nabla_\mu \psi - \frac{g^2}{4} j^\mu j_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} .
\]

- If we further specialize to \( g = 0 \) we recover the Schwinger model in curved spacetime \[9\].

In the following sections we are lead to consider the euclidean version of the model. Then one must replace the lorentzian \( \gamma^\mu, g_{\mu\nu} \) and \( \omega_\mu \) by there euclidean counterparts. For example, with our conventions (see appendix A) the relation (2.2) becomes

\[
\bar{\psi} \gamma^\mu \nabla_\mu \psi = 0 ,
\]

and as a consequence the generalized Dirac operator in euclidean spacetime becomes

\[
\mathcal{D} = e^{iF + \gamma_5 G - \frac{i}{2} \sigma} \mathcal{D} e^{-iF + \gamma_5 G + \frac{i}{2} \sigma}
\]

instead of (2.4). Also, to recover the euclidean Thirring model as particular limit of (1.1) we must set \( g_3 = 0 \) and \( g_1^2 = g_2^2 = g^2 \).

2.2 Hamiltonian formalism and classical conformal structure
In this subsection we investigate the Hamiltonian structure of the model (1.1) in the conformal limit, i.e. in flat Minkowski space and for vanishing gauge field. In the presence of both fermions and bosons it is convenient to exploit the graded Poisson structure [19]. We recall, that the equal time Poisson brackets are

\[
\{A(x), B(y)\} \equiv \sum_O \int dz^1 \left( \frac{A(x)}{\delta \pi_O(z)} \frac{\delta B(y)}{\delta O(z)} + \frac{A(x)}{\delta O(z)} \frac{\delta B(y)}{\delta \pi_O(z)} \right) \bigg|_{x^0 = y^0}.
\]

The sum is over all fundamental fields \(O(x)\) in the theory. The sign is minus if one or both of the fields \(A\) and \(B\) are bosonic (even) and it is plus if both are fermionic (odd). The momentum densities \(\pi_O(x)\) conjugate to the \(O\)-fields are given by functional left-derivatives

\[
\pi_O(x) = \frac{\delta S}{\delta \partial_0 O(x)}.
\]

A simple calculation yields the following momenta

\[
\pi_\psi = -i\psi^\dagger, \quad \pi_\phi = g_2 j_0^5 + 2\partial_0 \phi \quad \text{and} \quad \pi_\lambda = g_1 j_0 + 2\partial_0 \lambda
\]

which form the fundamental Poisson brackets with the fields

\[
\{\psi^\dagger_\alpha(x), \psi_\beta(y)\} = i\delta_{\alpha\beta}\delta(x^1 - y^1),
\]

\[
\{\pi_\lambda(x), \lambda(y)\} = \delta(x^1 - y^1), \quad \{\pi_\phi(x), \phi(y)\} = \delta(x^1 - y^1).
\]

For the Hamiltonian we obtain

\[
H = \int dx^1 \left[ \partial_0 \psi \pi_\psi + \partial_0 \lambda \pi_\lambda + \partial_0 \phi \pi_\phi - \mathcal{L} \right]
\]

\[
= \int dx^1 \left[ \pi_\psi \gamma_5 \partial_1 \psi - ig_1 \partial_1 \lambda \pi_\psi \gamma_5 \psi - ig_2 \partial_1 \phi \pi_\psi \psi + (\partial_1 \lambda)^2 \right.
\]

\[
\left. + (\partial_1 \phi)^2 + \frac{1}{4}(\pi_\lambda - ig_1 \pi_\psi \psi)^2 + \frac{1}{4}(\pi_\phi - ig_2 \pi_\psi \gamma_5 \psi)^2 \right].
\]

It can be checked that the corresponding Hamiltonian equations are just the field equations (2.1) with flat metric and vanishing gauge potential, as required. Since \(T^\nu_\mu = 0\) (see 2.15) the only non-zero components of \(T^{\mu\nu}\) are the lightcone components \(T_{++}\) and \(T_{--}\). To continue it is convenient to introduce adapted light cone coordinates
\[ x^\pm = x^0 \pm x^1 \] so that \[ \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1), \] (2.27)

and the chiral components of the Dirac spinor \( \psi_\pm = \frac{1}{2}(1 \pm \gamma_5)\psi \). Then \( T_{--} \) in (2.14) simplifies to

\[
T_{--} = -\frac{1}{2}(\pi_\psi \partial_- \psi_+ - \partial_- \pi_\psi \psi_+) + 2(\partial_- \lambda)^2 + 2(\partial_- \phi)^2 + 2g_3 \partial_-^2 \lambda + i\partial_- (g_1 \lambda + g_2 \phi)\pi_\psi \psi_+. \] (2.28)

Using the equations of motion one shows explicitly that it is a chiral field, i.e. depends only on \( x^- \). With (2.28) we can now find the conformal weights of the fundamental fields which determine their transformations under infinitesimal conformal symmetry transformations. For that we must calculate the commutator of the symmetry generators \( T_f = \int dx^- f(x^-)T_{--} \) with the fields. The result is

\[
\{ T_f, \phi \} = f \partial_- \phi \\
\{ T_f, \lambda \} = f \partial_- \lambda - \frac{g_3}{2} \partial_- f \\
\{ T_f, \psi_+ \} = f \partial_- \psi_+ + \frac{1}{2}(1 - ig_1 g_3)\psi_+ \partial_- f \\
\{ T_f, \psi_+^\dagger \} = f \partial_- \psi_+^\dagger + \frac{1}{2}(1 + ig_1 g_3)\psi_+^\dagger \partial_- f. \] (2.29)

Whereas \( \phi \) and \( \psi_+ \) are primary fields, \( \lambda \) is not. Actually, the non-primary character of \( \lambda \) is very much linked with the \( g_3 \)-dependent term in the transformation of the Dirac field. To see that more clearly we note that under an infinitesimal left conformal transformation generated by \( \bar{T}_f = \int dx^+ f(x^+)T_{++} \) the scalar and fermi field transform as

\[
\{ \bar{T}_f, \lambda \} = f \partial_- \lambda - \frac{g_3}{2} \partial_- f \quad \text{and} \quad \{ \bar{T}_f, \psi_+ \} = f \partial_- \psi_+ - ig_1 g_3 \psi_+ \partial_- f. \] (2.30)

Since \( \psi_+ \) is not any longer a scalar under left transformation the term

\[
\int dx^+ dx^- \left( 2i\psi_+^\dagger (\partial_+ - ig_1 \partial_+ \lambda)\psi_+ \right)
\]
appearing in the action is only conformally invariant because $\lambda$ transforms inhomogenously like a spin connection. It may be surprising that the symmetry transformations depend on the coupling constant $g_3$ which is not present in the flat space time Lagrangean. However, the same happens for example in 4 dimensions if one couples a scalar field conformally, that is non-minimally, to gravity. Although the Lagrangeans for the minimally and conformally coupled particles are the same on Minkowski spacetime, their energy momentum tensors are not. The same happens for the conformally invariant nonabelian Toda theories which admit several energy momentum tensors and hence several conformal structures [20].

The current transforms as

$$\{T_f, j_\cdot\} = f \partial_- j_\cdot + j_\cdot \partial_- f$$

(2.31)

and the energy momentum tensor as

$$\{T_f, T_{\cdot\cdot}\} = f \partial_- T_{\cdot\cdot} + 2T_{\cdot\cdot} \partial_- f - g_3^2 \partial_3^3 f.$$  (2.32)

Recalling that a primary field $O$ with weight $h$ transforms as

$$\{T_f, O\} = f \partial_- O + hO \partial_- f$$

and comparing with the above results we have found the following structure:

- The pseudoscalar field $\phi$ is primary with $h_\phi = 0$. The scalar field $\lambda$ is only primary for $g_3 = 0$ in which case $h_\lambda = 0$.
- The Dirac field $\psi_\cdot$ is primary with $h_{\psi_\cdot} = \frac{1}{2}(1 - ig_1g_3)$. The conformal weight is real for imaginary $g_3$.
- The current is primary with weight 1.
- Already on the classical level the energy momentum tensor is only quasi-primary. The corresponding Virasoro algebra (2.32) has central charge $c = 24\pi g_3^2$. 
3. Quantization of the generalized gauged Thirring model

In this section we quantize the general model (1.1) in curved space-times. The results are then applied in the following sections, where we calculate the partition function, ground state energy, equation of state and certain correlators of interest and their dependence on the chemical potential, volume of space, temperature and background metric. To do that we couple the conserved U(1)-charge to a chemical potential $\mu$. We enclose the system in a box with length $L$ to avoid infrared divergences. To investigate the temperature dependence the time is taken to be purely imaginary in the functional approach [21]. The imaginary time $x^0$ varies then from zero to the inverse temperature $\beta$ and we must impose periodic- and antiperiodic boundary conditions for the bosonic- and fermionic fields, respectively. Thus to study the finite temperature model we must assume that space-time is an euclidean torus $[0, \beta] \times [0, L]$.

To see how the partition function and correlators depend on the gravitational field we assume that the torus is equipped with an arbitrary metric with euclidean signature or equivalently with a 2-bein $e_{\mu a}$. The curved gamma matrices are $\gamma_\mu = e_{\mu a} \hat{\gamma}^a$ and in particular $\gamma_5 = -\frac{i}{2} \eta_{\mu \nu} \gamma^\mu \gamma^\nu = \sigma_3$ is constant (see appendix A for our conventions). We can always choose (quasi) isothermal coordinates and a Lorentz frame such that

\[ e_{\mu a} = e^\sigma \hat{e}_{\mu a} \equiv e^\sigma \begin{pmatrix} \tau_0 & \tau_1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad g_{\mu \nu} = e^{2\sigma} \hat{g}_{\mu \nu} \equiv e^{2\sigma} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \]

(3.1)

where $\tau = \tau_1 + i\tau_0$ is the Teichmueller parameter and $\sigma$ the gravitational Liouville field. Space-time is then a square of length $L$ and has volume $V = \int_0^L d^2 x \sqrt{g}$. We allow for the general twisted boundary conditions for the fermions

\[ \psi(x^0 + L, x^1) = -e^{2\pi i (\alpha_0 + \beta_0 \gamma_5)} \psi(x^0, x^1) \]
\[ \psi(x^0, x^1 + L) = -e^{2\pi i (\alpha_1 + \beta_1 \gamma_5)} \psi(x^0, x^1). \]

(3.2)
The parameters $\alpha_i$ and $\beta_i$ represent vectorial and chiral twists, respectively. We could allow for twisted boundary conditions for the (pseudo) scalars as well, e.g. $\phi(x^0 + nL, x^1 + mL) = \phi(x^1, x^0) + 2\pi(m+n)$. However, to recover the Thirring model for certain values of the couplings we assume that these fields are periodic. For $\sigma = 0$, $\tau = i\beta/L$ and $\alpha_0 = \beta_0 = 0$ the partition function has then the usual thermodynamical interpretation. Its logarithm is proportional to the free energy at temperature $T = 1/\beta$.

### 3.1 Fermionic path integral

Twisted boundary conditions as in (3.2) require some care in the fermionic path integral. Indeed the fermionic determinant is not uniquely defined when one allows for such twists. The ambiguities are not related to the unavoidable ultra-violet divergences but to the transition from Minkowski- to Euclidean space-time. To see that more clearly let $S^\pm$ denote the set of fermionic fields in Minkowski space-time with chirality $\pm 1$. Since both the commutation relations and the action do not connect $S^+$ and $S^-$ we can consistently impose different boundary conditions on $S^+$ and $S^-$. On the other hand, in the euclidean path integral for the generating functional

$$Z_F[\eta, \bar{\eta}] = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \ e^{\int \sqrt{g} \psi^\dagger i\mathcal{D}\psi + \int \sqrt{g} (\bar{\eta}\psi + \psi^\dagger \eta)}, \quad (3.3)$$

the Dirac operator

$$\mathcal{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

exchanges the two chiral components of $\psi$, i.e. $\mathcal{D} : S^\pm \rightarrow S^\mp$. Thus, in contrast to the situation in Minkowski space the two chiral sectors are related in the action. Of course, the eigenvalue problem for $i\mathcal{D}$ is then not well defined. This is the origin of the ambiguity in the definition of the determinant. It is related to the ambiguities one encounters when one quantizes chiral fermions [22]. To solve this problem we shall analytically continue the well-defined determinants in the untwisted sector $\beta = 0$ to $\beta \neq 0$. The
resulting determinants do not factorize into (anti-) holomorphic pieces and differ from previous ones in the literature [10]. In appendix C we give further arguments in favour of our result by calculating the determinants in a different way.

Let us now study the generating functional for fermions in an external gravitational and gauge field and coupled to the auxiliary fields. For that we observe that on the torus the decomposition (2.2) of the gauge potential generalises to

\[ A_\mu = A_\mu^I + \frac{2\pi}{L} t_\mu + \partial_\mu \alpha - \eta_{\mu\nu} \partial^\nu \varphi, \]

where the last 3 terms are recognized as Hodge decomposition of the single valued part of \( A \) in a given topological sector, that is the harmonic-, exact- and coexact pieces. In arbitrary coordinates the toron field \( t_\mu \) obeys the harmonic conditions \( \nabla_\mu t^\mu = t_{[\mu,\nu]} = 0 \). It follows then that in isothermal coordinates \( t_\mu \) must be constant. The role of the toron fields has recently been emphasized within the canonical approach [23]. In the Hamiltonian formulation they are quantum mechanical degrees of freedom which are needed for an understanding of the infrared sector in gauge theories. Also, in [24] it has been demonstrated that the \( Z_N \)-phases of hot pure Yang-Mills theories [25] should correspond to the same physical state if one takes care of the toron fields.

The first term in (3.4a) is an \textit{instanton potential} which gives rise to a nonvanishing quantized flux \( \Phi \) or integer-valued instanton number \( k \):

\[ \Phi = e \int F_{01} \equiv e \int E = e \int E^I = 2\pi k. \]

As representative in the \( k \)-instanton sector we choose the, up to gauge transformations, \textit{unique absolute minimum} of the Maxwell action in (1.1). It has field strength \( e E^I = \sqrt{g} \Phi/V \). As instanton potential we choose

\[ eA^I_\mu = e\hat{A}^I_\mu - \Phi \eta_{\mu}^{\nu} \partial_\nu \chi, \quad \text{where} \quad e\hat{A}^I = -\frac{\sqrt{g}}{V}(x^1, 0) \]
is the instanton potential on the flat torus with the same flux but field strength $\sqrt{g} \Phi/\hat{V}$. The function $\chi$ is then determined (up to a constant) by

$$\sqrt{g} \frac{\Phi}{\hat{V}} - \sqrt{\hat{g}} \frac{\Phi}{\hat{V}} = \sqrt{g} \Delta \chi.$$  \hspace{1cm} (3.4c)

The solution of this equation is given by

$$\chi(x) = -\frac{1}{V} \left( \frac{1}{\Delta} e^{-2\sigma} \right)(x) = \frac{1}{V} \int d^2y \sqrt{g(y)} G_0(x,y) e^{-2\sigma(y)},$$  \hspace{1cm} (3.4d)

where

$$G_0(x,y) = \langle x | \frac{1}{-\Delta} | y \rangle = \sum_{\lambda_n > 0} \frac{\phi_n(x) \phi_n^*(y)}{\lambda_n}$$  \hspace{1cm} (3.6a)

is the Greenfunction belonging to $-\Delta$. In deriving (3.4d) we have used that $\frac{1}{\Delta}(\Phi/\hat{V}) = 0$ which follows from the spectral resolution (3.6a) for the Green function in which the constant zero mode $\phi_0 = 1/\sqrt{V}$ of $\Delta$ is missing.

Note that 2-dimensional gauge theories are not scale or Weyl invariant as 4-dimensional ones are. For that reason the instantons on conformally flat spacetimes are not just the 'flat' instantons.

To be more explicit we relate $G_0$ to the Greenfunction $\hat{G}_0$ on the flat torus with the hatted metric [26]

$$\hat{G}_0(x,y) = -\frac{1}{4\pi} \log \left| \frac{1}{\eta(\tau)} \left[ \frac{1}{2} + \frac{\xi_0^0}{L} \right] (0,\tau) \right|^2,$$  \hspace{1cm} (3.6b)

For that we note that due to the missing zero-mode in (3.6a) the usual flat spacetime equations for the Greenfunctions are modified to

$$-\Delta_x G_0(x,y) = \frac{\delta(x-y)}{\sqrt{g}} - \frac{1}{V}, \quad -\hat{\Delta}_x \hat{G}_0(x,y) = \frac{\delta(x-y)}{\sqrt{\hat{g}}} - \frac{1}{\hat{V}}.$$  \hspace{1cm} (3.7a)

Furthermore one sees at once that both Green functions annihilate the corresponding constant zeromodes

$$\int d^2y \sqrt{g(y)} G_0(x,y) = \int d^2y \sqrt{\hat{g}} \hat{G}_0(x,y) = 0.$$  \hspace{1cm} (3.7b)

From these two equations one concludes that Greenfunction on the curved torus is related to the flat one (3.6b) as
\[ G_0(x,y) = \hat{G}_0(x,y) + \frac{1}{\sqrt{V}} \int d^2u d^2v \sqrt{g(u)g(v)} \hat{G}_0(u,v) \]
\[ - \frac{1}{\sqrt{V}} \int \hat{G}_0(x,u) \sqrt{g(u)} du \]
\[ - \frac{1}{\sqrt{V}} \int d^2u \sqrt{g(u)} \hat{G}_0(u,y) \]

(3.8)

and this replaces the infinite space relations \( G_0 = \hat{G}_0 \) [27].

Our choice for the instanton potential (3.4) corresponds to a particular trivialization of the \( U(1) \)-bundle over the torus [3]. In other words, the gauge potentials and fermion fields at \( (x^0, x^1) \) and \( (x^0, x^1 + L) \) are necessarily related by a nontrivial gauge transformation with windings
\[ A_\mu(x^0, x^1 + L) - A_\mu(x^0, x^1) = \partial_\mu \alpha(x) \]
\[ \psi(x^0, x^1 + L) = -e^{ie\alpha(x)} e^{2\pi i(\alpha_1 + \beta_1 \gamma_3)} \psi(x^0, x^1). \]

(3.9a)

For the choice (3.4b) we find
\[ e^{\alpha(x)} = -\frac{\Phi}{L} x^0. \]

(3.9b)

Note that \( A \) is still periodic in \( x^0 \) with period \( L \) and \( \psi \) still obeys the first boundary condition in (3.2). Our trivialization differs from the one chosen in [28] and so do our instantons and fermionic zero modes.

Similarly as for the gauge potential we must add a harmonic piece to the auxiliary vector field \( B_\mu \) to which the fermions couple in (1.1), so that
\[ B_\mu = \frac{2\pi}{L} g_0 h_\mu + g_1 \partial_\mu \lambda - g_2 \eta_{\mu\nu} \partial^\nu \phi \]

(3.10)

appears in the Dirac operator in (1.1) on the torus. \( \lambda \) and \( \phi \) couple to the divergence of the vector and axial vector currents. The harmonic fields \( h_\mu \) couple to the harmonic part of the current and are needed to recover the Thirring model in the limit \( g_0^2 = g_1^2 = g_2^2 \). Also, we shall see that \( t_\mu \) and \( h_\mu \) are essential to obtain the correct answer for the thermodynamic potential. Note that \( B_\mu \) contains no instanton part since it couples to the gauge invariant fermionic current.

Finally we introduce a chemical potential for the conserved \( U(1) \) charge. In the euclidean functional approach this is equivalent to coupling the fermions to a constant imaginary gauge potential \( A_0 \) [29].
Inserting the above decompositions and the chemical potential into the Dirac operator finally yields in isothermal coordinates

\[ \gamma^\nu D_\nu = e^{iF + \gamma_5(G + \Phi\chi) - \frac{2i}{\sigma}} \mathcal{D} e^{-iF + \gamma_5(G + \Phi\chi) + \frac{2i}{\sigma}}, \]

where

\[ \mathcal{D} = \gamma^\mu \left( \partial_\mu + i\hat{\omega}_\mu - ie\hat{A}_I^\mu \right) - \frac{2\pi i}{L} [H_\mu + \mu_\mu], \] (3.11)

\[ H_\mu = e t_\mu + g_0 h_\mu \quad \text{and} \quad \mu_\mu = -i\frac{\tau_0 L}{2\pi} \delta_{\mu0}. \]

Here \( \hat{\omega} \) is the spin connection belonging to \( \hat{e}_\mu a \). It vanishes for our choice of the reference zweibein. \( \hat{A}_I^\mu \) is the instanton potential (3.4b) on the flat torus. The scalar and pseudoscalar functions \( F, G \) and \( \chi \) have been introduced in (2.4) and (3.4d). In the chosen coordinates \( t \) and \( h \) and hence \( H \) are all constant. In [3] it has been shown that \( \mathcal{D} \) possesses \(|k|\) zero-modes of definite chirality and their chirality is given by the sign of \( k \). They are crucial in any correct quantization. For example, if one would leave out instanton sectors in which \( i\mathcal{D} \) has zero-modes then the cluster property would be violated.

In a first step we quantize the fermions in the flat instanton and harmonic background and reference metric \( \hat{g}_{\mu\nu} \), that is we assume \( \mathcal{D} \rightarrow \mathcal{D} \) in (3.3). The dependence on the remaining fields \( F, G, \chi \) and \( \sigma \), that is the relation between \( Z_F \) and \( \hat{Z}_F \), is then found by integrating the chiral and trace anomalies [30] and exploiting the relation (3.11) between \( \mathcal{D} \) and \( \mathcal{D} \).

We expand the fermionic field in a orthonormal basis of the Hilbert space

\[ \psi(x) = \sum_n a_n \psi_{n+}(x) + \sum_n b_n \psi_{n-}(x) \]
\[ \psi^\dagger(x) = \sum_n \bar{a}_n \chi_{n+}(x) + \sum_n \bar{b}_n \chi_{n-}(x), \] (3.12)

where \( a_n, b_n, \bar{a}_n, \bar{b}_n \) are independent Grassmann variables.

**Topologically trivial sector**

For \( k = 0 \) or vanishing instanton potential we can immediately write down a basis

\[ \psi_{n\pm}(x) = \frac{1}{\sqrt{V}} e^{i(p_n^\pm x)} e_\pm, \quad \text{where} \quad (p_n^\pm)_i = \frac{2\pi}{L} \left( \frac{1}{2} + \alpha_i \pm \beta_i + n_i \right), \] (3.13)
and $e_{\pm}$ are the eigenvectors of $\gamma_5$. The $\psi_{n+}$ and $\psi_{n-}$ must obey the $S^+$ and $S^-$ boundary conditions, respectively. These boundary conditions fix the admissable momenta $p^{\pm}_n$ in (3.13). Since the Dirac operator maps $S^\pm$ into $S^{\mp}$ the $\chi_{n\pm}$ must then obey the same boundary conditions as the $\psi_{n\mp}$. Thus $\chi_{n\pm}(x)$ is obtained from $\psi_{n\pm}(x)$ by exchanging $p^{+}_n$ and $p^{-}_n$. It follows then that

$$i\hat{D}\psi_{n\pm} = \lambda^{\pm}_n \chi_{n\mp}$$

(3.14a)

with

$$\lambda^{+}_n = \frac{2\pi}{\tau_0 L} [\tau(\frac{1}{2} + a_1 + \beta_1 + n_1) - (\frac{1}{2} + a_0 + \beta_0 + n_0)]$$

$$\lambda^{-}_n = \frac{2\pi}{\tau_0 L} [\tau(\frac{1}{2} + a_1 - \beta_1 + n_1) - (\frac{1}{2} + a_0 - \beta_0 + n_0)].$$

(3.14b)

Here we have introduced $a_\mu \equiv \alpha_\mu - H_\mu - \mu_\mu$. Substituting (3.12,14) into the generating functional (3.3) and applying the standard Grassmann integration rules we arrive at

$$\hat{Z}_F[\eta, \bar{\eta}] = \det i\hat{D} e^{\int \bar{\eta}(x)\hat{S}(x,y)\eta(y)}, \quad \det i\hat{D} = \prod_n \lambda^{+}_n \lambda^{-}_n,$$

$$\hat{S}(x,y) = \sum_n \left( \frac{\psi_{n+}(x)\chi^{\dagger}_{n-}(y)}{\lambda^{+}_n} + \frac{\psi_{n-}(x)\chi^{\dagger}_{n+}(y)}{\lambda^{-}_n} \right).$$

(3.15)

$\hat{S}$ is the fermionic Green function in the 0-instanton sector. Note that both the 'eigenvalues' and the Green function depend on the Teichmueller parameter, harmonic potentials, twists and chemical potential.

We proceed to calculate the infinite product or generalized determinant in (3.15). This is one of the central points of our article and for non-zero chiral twists and chemical potential our result deviates from previous ones [10]. Actually the twists and chemical potential are related as one can see from (3.14). One may be tempted so identify

$$\det(D_+D_-) \sim \prod \lambda^{+}_n \lambda^{-}_n \quad \text{and} \quad \det D_+ \det D_- \sim \prod \lambda^{+}_n \prod \lambda^{-}_m$$

(3.16)
and thus conclude that the determinant is a product, \( f(\tau)\bar{f}(\tau) \), that is factorizes into holomorphic and anti-holomorphic pieces (the overall factor \( \sim 1/\tau_0 L \) in the eigenvalues (3.14b) drops in the infinite product, since the torus has vanishing Euler number). However, the infinite product in (3.15) must be regularized and the two expressions in (3.16) may differ. In conformal field theory [26] one is naturally lead to consider the individual chiral sectors and thus finds holomorphic factorization. For Dirac fermions one uses \( \mathcal{D}^2 \) to regularize the product and this leads to the determinant of the product \( D_+ D_- \).

To continue we recast the infinite product in the form
\[
\prod_{n=1}^{\infty} \lambda_n^+ \lambda_n^- = \prod_{n \in \mathbb{Z}^2} \left( \frac{2\pi}{L} \right)^2 \hat{g}^{\mu\nu} (\frac{1}{2} + c_\mu + n_\mu)(\frac{1}{2} + c_\nu + n_\nu)
\]
where \( \hat{g}^{\mu\nu} \) is the inverse of the reference metric (3.1) and
\[
c_\mu = a_\mu + \hat{\eta}_{\mu\nu} \beta_\nu, \quad \text{where} \quad (\hat{\eta}_{\mu\nu}) = -\frac{1}{\tau_0} \begin{pmatrix} \tau_1 & -|\tau|^2 \\ 1 & -\tau_1 \end{pmatrix}.
\]
The point is that for real \( c_\mu \), that is for vanishing chiral twists \( \beta_\mu \) and chemical potential (see the definitions of \( a_\mu \) below (3.14b) and \( \mu_\mu \) in (3.11)) the zeta function defined by
\[
\zeta(s) = \sum_n (\lambda_n^+ \lambda_n^-)^{-s}
\]
has a well defined analytic continuation to \( s < 1 \) via a Poisson resummation. An explicit calculation yields [3,31,38]
\[
\det(i\mathcal{D}) \equiv \left( \prod_n \lambda_n^+ \lambda_n^- \right)_{\text{reg}} = e^{-\zeta'(s)|_{s=0}}, \quad \text{where}
\]
\[
\zeta'(s)|_{s=0} = -\log \left[ \frac{1}{|\eta(\tau)|^2} \Theta\left[ \frac{-c_1}{c_0} \right](0, \tau) \Theta\left[ \frac{-c_1}{c_0} \right](0, \tau) \right].
\]
However, for complex \( c_\mu \) the Poisson resummation is not applicable and \( \zeta'(0) \) cannot be calculated by direct means. To circumvent these difficulties we note that the infinite product (3.17c) defining the \( \zeta \)-function for \( s > 1 \) is a meromorphic function in \( c \). Thus we may first continue to \( s < 1 \) for real \( c_\mu \) and then continue the result to complex values. Using the transformation properties of theta functions the resulting determinant can be written as
\[
\det(i\hat{D}) = e^{2\pi(\sqrt{\hat{g}}\hat{\gamma}_\mu \beta_\mu \beta_\nu - 2i\beta_1 a_0)} \cdot \frac{1}{|\eta(\tau)|^2} \Theta \left[ \frac{-a_1 + \beta_1}{a_0 - \beta_0} \right](0, \tau) \bar{\Theta} \left[ \frac{-\bar{a}_1 - \beta_1}{\bar{a}_0 + \beta_0} \right](0, \tau). (3.19)
\]

It can be shown that this determinant is \textit{gauge invariant}, i.e. invariant under \(\alpha_\mu \rightarrow \alpha_\mu + 1\), but not invariant under chiral transformations, \(\beta_\mu \rightarrow \beta_\mu + 1\), as expected. Furthermore it transforms covariantly under modular transformations \(\tau \rightarrow \tau + 1\) and \(\tau \rightarrow -1/\tau\). In other words, \(\det i\hat{D}\) is invariant under modular transformations if at the same time the boundary conditions are transformed accordingly. The exponential prefactor is needed for modular covariance and is not present in the literature \[10\]. It correlates the two chiral sectors and will have important consequences. In the appendix C we confirm (3.19) with operator methods.

**Topologically nontrivial sectors**

For definiteness we assume \(k > 0\). Then \(i\hat{D}\) possesses \(k\) zero-modes \(\hat{\psi}_{0+}^p, p = 1, \ldots, k\) with positive chirality and \(S^+\) boundary conditions. Together with the excited modes \(\psi_{n+}\) they form a basis of \(S^+\). Thus we must add the zero-mode contribution \(\sum c_n \hat{\psi}_{0+}^p\) to \(\psi\) in (3.12). Similarly we must add \(\bar{c}_n \bar{\chi}_{0+}^{p\dagger}\) to \(\psi^\dagger\) in (3.12). The zero modes \(\hat{\chi}_{0+}^p\) in \(\psi\) must obey \(S_-\) boundary conditions (see the discussion below (3.13)). Thus the zero-modes in the expansions of \(\psi\) and \(\psi^\dagger\) have the same chirality but obey different boundary conditions. This is required for the zero- and excited modes to form a complete basis and is consistent since \(i\hat{D}\) does not relate the zero-mode sector of \(S^+\) with \(S^-\). The Grassmann integral over the variables belonging to the excited modes is performed as in the trivial sector. Also, the integration over the \(c_n\) and \(\bar{c}_n\) can easily be done and one obtains
\[
\hat{Z}_F[\eta, \bar{\eta}] = \prod_{p=1}^{[k]} (\bar{\eta}, \Sigma_{0+}^p, \psi_{0+}) \det' i\hat{\mathcal{D}} e^{-\int \bar{\eta}(x) \hat{S}_e(x, y) \eta(y)},
\]

\[
\det' i\hat{\mathcal{D}} = \prod_{\lambda_n \neq 0} \lambda_n^+ \lambda_n^-,
\]

\[
\hat{S}_e(x, y) = \sum_{\lambda_n \neq 0} \left( \frac{\psi_{n^+}(x) \chi^+_n(y)}{\lambda_n^+} + (+ \leftrightarrow -) \right).
\]

Note that the excited Green function \(S_e\) anticommutes with \(\gamma_5\).

To calculate the determinant we observe that

\[
\hat{\mathcal{D}}^2 = \left( \begin{array}{cc}
D_+ D_- & 0 \\
0 & D_+ D_-
\end{array} \right) = \frac{1}{\sqrt{g}} D_\mu \sqrt{g} g^\mu\nu D_\nu - \frac{1}{4} \mathcal{R} + \frac{e}{2} \eta^{\mu \nu} F_{\mu \nu} \gamma_5
\]

(3.21)

simplifies in the instanton background \(\hat{A}^I\) and on the flat torus to

\[
-\hat{\mathcal{D}}^2 = -\hat{g}^{\mu \nu} \hat{D}_\mu \hat{D}_\nu - \frac{\Phi}{V} \gamma_5.
\]

(3.22)

In other words, it is the same in the two chiral sectors, up to the constant \(2\Phi/\hat{V}\). This observation allows one to reconstruct the spectrum of \(-\hat{\mathcal{D}}^2\) as follows:

First note that we can define two sets of normalizable zero-modes in the positive chirality \((\gamma_5 = 1)\) sector. One containing \(k\) modes obeying the \(S^+\) boundary conditions and the other consisting of \(k\) modes with the \(S^-\) boundary conditions. The first set is admissible and are just the \(k\) zero modes \(\psi_{0+}^p\) appearing in (3.20). The other \(k\) zeromodes \(\chi_{0+}^p\) appear also in (3.20) but the Dirac operator does not act on them. But because of (3.22) they are at the same time eigenmodes of \(-\hat{\mathcal{D}}^2\) in the negative chirality sector with the correct \(S^-\) boundary conditions and eigenvalues \(2\Phi/\hat{V}\). Of course, the \(\psi_{0+}^p\) are also eigenmodes in the \(\gamma_5 = -1\) sector but with the wrong boundary conditions. However, applying \(\hat{D}_-\) to them produces \(k\) eigenmodes in the positive chirality sector with the correct boundary conditions and eigenvalues \(2\Phi/\hat{V}\). This procedure may now be iterated and one ends up with the following spectrum of \(-\hat{\mathcal{D}}^2:\)

\[
\lambda_n^2 \begin{cases} 
0 & \text{degeneracy } = k \\
2n\Phi/\hat{V} & \text{degeneracy } = 2k.
\end{cases}
\]

(3.23)
With the explicit spectrum at hand we can compute the zero-mode truncated determinant with zeta-function methods and find [3]

\[ \det'(i\hat{D}) = \left( \frac{\pi V}{\Phi} \right)^{\phi/4\pi}. \]  

(3.24)

We proceed with computing the zero modes of \( \hat{D}^2 \). For that we note that the operator commutes with the time translations which leads to the ansatz

\[ \tilde{\chi}_p = e^{2\pi i c_p x^0/L} e^{2\pi i H_1 x^1/L} \xi_p(x^1) e_+, \quad c_p = p + (\frac{1}{2} + \alpha_0 + \beta_0), \]  

(3.25a)

where we have assumed \( k > 0 \). The choice of \( c_p \) is dictated by the time-like boundary conditions in (3.2). Inserting this ansatz into the zero mode equation \( \hat{D}^2 \tilde{\chi}_p = 0 \) yields

\[ (|\tau|^2 \frac{d^2}{dy^2} - \frac{\Phi^2}{L^4} y^2 - 2i\tau \frac{\Phi}{L^2} \frac{d}{dy} - i\tau \frac{\Phi}{L^2}) \xi_p = 0, \]

where \( y = x^1 + \frac{L}{k}(c_p - H_0 - \mu_0) \).

This is just the differential equation for the ground state of a generalized harmonic oscillator to which it reduces for \( \tau = i\tau_0 \). The solution is given by

\[ \xi_p = \exp \left[ - \frac{\Phi}{2i\tau L^2} \left\{ x^1 + \frac{L}{k}(c_p - H_0 - \mu_0) \right\}^2 \right]. \]

These functions do not obey the boundary condition (3.9), but the correct eigenmodes can be constructed as superpositions of them. For that we observe that

\[ \tilde{\chi}_p(x^0, x^1 + L) = e^{-i\Phi x^0/\beta} e^{2i\pi H_1} \tilde{\chi}_{p+k}(x^0, x^1) \]

so that the sums

\[ \tilde{\psi}_{0+}^p = \left( \frac{2k\tau_0}{\sqrt{|\tau|V}} \right)^{\frac{1}{4}} e^{\pi \nu_0^2} e^{2i\pi i(H_0 - \alpha_0 - \frac{1}{2})\beta_1} \sum_{n \in \mathbb{Z}} e^{-2i\pi (n+p/k)(\frac{1}{2} + \alpha_1 + \beta_1 - H_1)} \tilde{\chi}_{p+nk} e_+, \]  

(3.25b)

where \( p = 1, \ldots, k \), obey the boundary conditions and thus are the \( k \) required zero-modes. We have chosen the phase such that the accompanying zero-modes in (3.20) are just
Actually the product $\hat{\chi}^p_0+\hat{\psi}^p_0$ is only determined up to a (possibly $\alpha$ and $\beta$-dependent) phase. But gauge invariance requires that

$$\chi^p_0+(x)\psi^p_0+(x) = \chi^p_0+(x,-\beta,\ldots)\exp[i\int_y^x A]$$

are both invariant under $\alpha_\mu \to \alpha_\mu + n_\mu$ and $et_\mu \to et_\mu + n_\mu$, where the $n_\mu$ are integers. This almost fixes the relative phases of the zero-modes in (3.25b) and (3.25c). Also, the overall factor normalizes these functions to one. Modes with different $p$ are orthogonal to each other, so that the system (3.25b) forms an orthonormal basis of the zero-mode subspace. For $k < 0$ the zero-modes are the same if one replaces $\beta_\mu$ by $-\beta_\mu$ and $e_+$ by $e_-$. 

### Integrating the chiral and trace anomalies

To relate the determinants of $i\hat{D}$ and $i\hat{\nabla}$ we introduce the one-parameter family of Dirac operators

$$\hat{D}_\tau = e^{\tau iF+\gamma_5(G+\Phi\chi)-\frac{i}{2}\sigma} \hat{D} e^{\tau [-iF+\gamma_5(G+\Phi\chi)+\frac{1}{2}\sigma]}$$

which interpolates between $\hat{D}$ and $\hat{\nabla}$ [30]. The $\tau$-derivative of the corresponding determinants is determined by the chiral and trace anomaly. An explicit calculation yields

$$\log \frac{\det' i\hat{D}}{\det' i\hat{\nabla}} = \frac{1}{4\pi} \int_0^1 d\tau \int \sqrt{g}\, tr a_1^\tau \left(2\gamma_5(G+\Phi\chi) - \sigma\right) + \log \det \frac{\hat{N}_\psi}{\hat{N}_{\hat{\psi}}}.$$ (3.27)

Here $g^{\tau}$ is the determinant of the deformed metric $g^{\tau}_{\mu\nu} = e^{2\tau\sigma} \hat{g}_{\mu\nu}$, and

$$a_1^\tau = -\frac{1}{12} R^\tau + \gamma_5 \tau \Delta^\tau G + \frac{1}{\sqrt{g}} \left[ (1 - \tau) \sqrt{g} \frac{\Phi}{V} + \tau \sqrt{g} \frac{\Phi}{V} \right] \gamma_5$$

is the relevant Seeley-deWitt coefficient of $\hat{D}^2\tau$. Furthermore, $\hat{N}_\psi$ is the norm-matrix of the zero-modes $\hat{\psi}^p_0$ in (3.25b). Since those are orthonormal it is just the $k$-dimensional identity matrix. $\hat{N}_\psi$ is the norm-matrix of the zero-modes of $i\hat{D}$ which are related to the $\hat{\psi}^p_0$ as
\[ \psi_{0+}^p = e^{iF - \gamma_5(G + \Phi \chi) - \frac{1}{2} \sigma} \psi_{0+} \] (3.29)

as follows from (3.11). Inserting (3.28) into (3.27) one finds the following formula for the determinant in arbitrary background gravitational and gauge fields:

\[
\det' i \Psi = \det \frac{N_\psi}{N_\psi} \det'(i \psi) \exp \left( \frac{S_L}{24\pi} \right) \cdot \exp \left( \frac{1}{2\pi} \int \sqrt{g} G \Delta G + \frac{2k}{V} \int \sqrt{g} G + \frac{\Phi^2}{2\pi V} \int \sqrt{g} \chi \right),
\]

where

\[ S_L = \int \sqrt{g} [\tilde{R} \sigma - \sigma \tilde{\Delta} \sigma] \] (3.30b)

is the Liouville action. In deriving this result we used that \( \int \sqrt{g} \chi = 0 \). Actually, for our reference metric the Ricci scalar \( \tilde{\mathcal{R}} \) vanishes and the Liouville action simplifies to \(- \int \sqrt{g} \sigma \tilde{\Delta} \sigma \). However, as it stands the formula (3.30) holds for arbitrary reference metrics and arbitrary Riemannian surfaces.

As expected for a gauge-invariant regularisation, the function \( F \) and thus the pure gauge part of the vector potential does not appear in the determinant.

For later use we also give the analogous formula for the zero-mode truncated scalar determinant [32]

\[
\det' \frac{1}{2}(-\Delta) = \det' \frac{1}{2}(-\Delta) \left( \frac{V}{V} \right) \frac{1}{2} \exp \left( - \frac{1}{24\pi} S_L \right). \] (3.31)

This completes the computations of the determinants.

The generating functional for the full theory is then obtained as follows: First one notes that the formulae (3.15) and (3.20) for the fermionic functionals still hold without hats. Thus to calculate the functionals in arbitrary gauge-, auxiliary- and gauge fields we need to know the Greenfunctions, determinants and zero-modes in these backgrounds.

To relate the fermionic Greenfunctions \( S \) in the different topological sectors to the hatted ones we define

\[
S_1(x, y) = e^{-g(x)} \hat{S}(x, y) e^{-\bar{g}(y)}, \quad g = -iF + \gamma_5(G + \Phi \chi) + \frac{1}{2} \sigma. \] (3.32a)
On the infinite space we would have $S = S_1$ [27]. However, if the Dirac operator possesses zero modes this simple relation is modified to to

$$S(x, y) = S_1(x, y) + \int P_0(x, u)S_1(u, v)P_0(v, y)\sqrt{g(u)g(v)}d^2u d^2v$$

$$- \int S_1(x, u)P_0(u, y)\sqrt{g(u)}d^2u - \int P_0(x, u)S_1(u, y)\sqrt{g(u)}d^2u,$$

and this formula should be compared with the analogous one for scalars (3.8). Here $P_0$ is the orthonormal projector onto the zero modes. For gauge fields with vanishing flux $S = S_1$. Together with the relation (3.30) between the full and hatted determinant and the explicit form (3.18, 19) for $\det i\hat{D}$ this yields the fermionic generating functional in the various topological sectors.

In the trivial sector one finds explicitly

$$Z_F[\eta, \bar{\eta}] = \frac{1}{|\eta(\tau)|^2} \Theta \left[ \frac{-c_1}{c_0} \right] (0, \tau) \Theta \left[ \frac{-\bar{c}_1}{\bar{c}_0} \right] (0, \tau)$$

$$e^{-\int \bar{\eta}(x)S(x, y)\eta(y) + \frac{1}{24\pi}S_L + \frac{1}{2\pi} \int \sqrt{g}G \Delta G},$$

By using the scaling properties of the Ricci-scalar and Laplacian (see appendix B) the exponent can be rewritten as

$$-\frac{1}{96\pi} \int \sqrt{g}R \frac{1}{\Delta}R + \frac{1}{2\pi} \int \sqrt{g}G \Delta G,$$

which makes clear that the resulting functional is diffeomorphism invariant. Here we used that $\mathcal{R}$ integrates to zero or that the Euler number of the torus vanishes. On the sphere or higher genus surfaces the last formula is modified.

To relate the hatted and full functionals in the non-trivial sectors one recalls that in the formula (3.20) for the full partition function (without hats) one must use orthonormal zero-modes. These can be expanded in terms of the un-normalized modes $\psi_{0+}^P$ and $\chi_{0+}^P$ defined in (3.29). Inserting these expansions into (3.20) yields the inverse square roots of the determinants of the corresponding normmatrices $\mathcal{N}_\psi$ and $\mathcal{N}_\chi$ which partly chancel $\det \mathcal{N}_\psi$ in (3.20). Thus one ends up with
\[Z_F[\eta, \bar{\eta}] = \frac{\det \frac{1}{2} \mathcal{N}_\psi}{\det \frac{1}{2} \mathcal{N}_\chi} \left( \frac{\pi V}{\Phi} \right)^{\frac{3}{2}} e^{\Phi^2/2\pi V} \cdot \int \sqrt{\bar{g}} \prod_{p=1}^{|k|} (\bar{\eta}, \psi^p_0)(\chi^p_0, \eta) \cdot e^{\int \bar{\eta}(x)S_e(x,y)\eta(y)} \exp \left( \frac{S_L}{24\pi} + \frac{1}{2\pi} \int \sqrt{g} \bar{G} \Delta G + \frac{2k}{V} \int \sqrt{g} G \right), \] (3.34)

where the \( \psi^p_0 \) and \( \chi^p_0 \) are the un-normalized zero-modes (3.25).

### 3.2 Bosonic path integral

To arrive at the generating functional for the complete theory we must finally quantize the photon and auxiliary fields \( A_\mu \) and \( B_\mu \) (see (3.10)). For that we insert the decomposition (3.4a) into the bosonic part of the (euclidean) action (1.1). This results in

\[S_B = \frac{\bar{g}^2}{2e^2V} + (2\pi)^2 \sqrt{g} \bar{g}^{\mu\nu} h_\mu h_\nu + \int \sqrt{g} \left( \frac{1}{2} \varphi \Delta^2 \varphi - \lambda \Delta \lambda - \phi \Delta \phi - g_3 R \lambda \right).\] (3.35)

The term quadratic in the \( h \) field is not present in the action (1.1) on Minkowski space-time. But on the torus \( h \) is part of the Hodge decomposition of \( B_\mu \) and thus on the same footing as \( \partial \lambda \) and \( \eta \partial \phi \). Since \( S_B \) and the fermionic determinants are both gauge invariant and thus independent of the pure gauge mode \( \alpha \) in (3.4a), it is natural to change variables from \( A_\mu \) to \( (\varphi, \alpha, t_\mu, \Phi) \) in each topological sector. One can show [3] that this transformation is one to one, provided

\[\int \sqrt{g} \varphi = \int \sqrt{g} \alpha = 0 \quad \text{and} \quad et_\mu \in [0, 1].\] (3.36)

The measures are related as

\[\mathcal{D}A_\mu = J \sum_k dt_0 dt_1 \mathcal{D} \varphi \mathcal{D} \alpha, \quad \text{where} \quad J = (2\pi)^2 \det'(-\Delta).\] (3.37)

The Jacobian \( J \) is independent of the dynamical fields. In expectation values of gauge invariant and thus -independent operators the \( \alpha \)-integration cancels against the normalization. This is of course related to the fact that in \( QED \) the ghosts decouple in the Lorentz gauge.
Finally observe that via the derivative couplings to the fermionic current \cite{33} we introduced artificial degrees of freedom. The relation between $B_\mu$ in (3.10) and the fields $(\phi, \lambda, h_\mu)$ is only one to one if we impose the conditions similar to (3.36), namely

$$\bar{\phi} \equiv \frac{1}{V} \int \sqrt{g} \phi = 0, \quad \bar{\lambda} = 0 \quad \text{and} \quad h_\mu \in [-\infty, \infty]. \quad (3.38)$$

There is no restriction on the harmonic part of the auxiliary field, since $B_\mu$ is not a gauge field. The constraints are imposed in the functional integral as

$$\int dh_0 dh_1 D\phi D\lambda \delta(\bar{\phi}) \delta(\bar{\lambda}) \cdots. \quad (3.39)$$

The normalization by the volume in (3.38) is needed such that the constraints and hence the partition function are both dimensionless. For example, expanding $\phi$ in eigenmodes of the Laplacian as

$$\phi = a_0 \phi_0 + \sum_{n>0} a_n \phi_n, \quad \text{where} \quad \phi_0 = \frac{1}{\sqrt{V}}$$

is the zeromode, one finds the dimensionless partition function

$$\int D\phi \delta(\bar{\phi}) e^{\phi \Delta \phi} = \sqrt{V} \frac{1}{\text{det}'^2 (-\Delta)} \quad (3.40)$$

for free bosons.

Constraining the mean field to zero as in (3.40) is equivalent to fixing the field at an arbitrary point $\xi$ on the torus to zero \cite{34}

$$\int D\phi \delta(\bar{\phi}) \cdots = \int D\phi \delta(\phi(\xi)) \cdots. \quad (3.41)$$

This can be seen as follows:

$$\int D\phi \delta(\phi(\xi)) \cdots = \int du \delta(\bar{\phi} - u) D\phi \delta(\phi(\xi)) \cdots. \quad (3.42a)$$

Now one shifts the field as $\phi \rightarrow \phi + u$. Using that the action is left invariant by this shift, the measure becomes

$$\int du D\phi \delta(\bar{\phi}) \delta(\phi(\xi) + u) \cdots = \int D\phi \delta(\bar{\phi}) \cdots \quad (3.42b)$$
which shows that the two constraints are the same. When integrating over the auxiliary fields it is always understood that the divergent zeromodes are suppressed as in (3.39).

4. Partition function of the generalized Thirring model

As a first application of our general results we calculate the partition function of the theory (1.1). To compute it we must put the sources $\eta$ and $\bar{\eta}$ in (3.3) to zero. Then it is evident from (3.34) that the non-trivial sectors do not contribute and hence we may assume $\Phi=0$. Thus the partition function is given by

$$Z_0 = J \int d^2t d^2\varphi d\phi d\lambda \ Z_F[0,0] \ e^{-S_B[\Phi=0]},$$

(4.1)

where $J$ is the Jacobian of the transformation (3.37). $Z_F$ is the fermionic partition function (3.33) in the trivial sector and the integration is over fields obeying the conditions (3.36,38). Now we perform the various integrals in turn.

*Integration over the harmonics:*

By using the series representation of the theta functions one computes

$$\int_0^1 d^2(\tau) \Theta[-c_1,0,\tau] \ \bar{\Theta}[-c_1,0,\tau] = \frac{1}{\sqrt{2\tau_0}}.$$ 

(4.2)

Since the result appears always together with the $\eta$-function factor in (3.33) it is convenient to introduce

$$\kappa := \frac{1}{\sqrt{2\tau_0}} \frac{1}{|\eta(\tau)|^2}$$

in the following expressions. The result (4.2) does not depend on the $h$-field and hence the $h$-integration in (4.1) becomes Gaussian. It yields a factor $1/4\pi$ so that
\[ Z_0 = \pi \kappa \det'(-\triangle) e^{S_L/24\pi} \int \mathcal{D}_\delta(\varphi \phi \lambda) e^{\frac{1}{2} \gamma \int \sqrt{g} \varphi \chi \delta G - S_B[h=0]}, \]  

where \( G \) has been defined in (2.4). We inserted the explicit expression (3.37) for the Jacobian. It is interesting to note that already the toron-integration in (4.2) washes out the dependence on the boundary conditions and chemical potential. We shall comment on this point later on.

**Integration over \( \lambda \) and \( \phi \):**

The integral over \( \lambda \), subject to the condition (3.38), modifies the Liouville factor and yields one inverse square-root of the determinant of \(-2\triangle\) in (4.3).

To continue we recall the scaling formula for the determinant of \( \triangle \) [35]:

\[
\log \frac{\det'(-a\triangle)}{\det'(-\triangle)} = \log a \cdot \zeta(0) = \log a \cdot \left[ \frac{1}{4\pi} \int a_1 - p \right],
\]

where \( p \) is the number of zero modes of the operator. On the torus \( \int a_1 = 0 \) and we find

\[
\det'(-a\triangle) = \frac{1}{a} \det'(-\triangle). \tag{4.4b}
\]

Using this scaling property the \( \lambda \)-integral together with (3.40) we obtain

\[ Z_0 = \sqrt{2V} \kappa \pi \det'^{\frac{1}{2}}(-\triangle) e^{(g_2^2+1/24\pi)S_L} \int \mathcal{D}_\delta(\varphi \phi) e^{\frac{1}{2} \gamma \int \sqrt{g} \varphi \chi \delta G - S_B[h=\lambda=0]}, \]

To quantize the \( \phi \) field we need to recall that \( G = e\varphi + g_2\phi \). Since \( \varphi \triangle \varphi \sim (A_T, A_T) \), the anomalous term \( \sim \int G \triangle G \) in the exponent contains an explicit photon mass term with bare-mass \( e/\sqrt{\pi} \). However, when quantizing the \( \phi \) field this mass is renormalized. This can be seen explicitly in the resulting expression for the partition function after the \( \phi \)-integration has been performed

\[ Z_0 = \frac{2\sqrt{\pi} \kappa eV}{m_\gamma} e^{(g_2^2+1/24\pi)S_L} \int \mathcal{D}\varphi e^{-\frac{1}{2} \gamma \int \sqrt{g} \varphi (\triangle^2 - m_\gamma^2 \triangle) \varphi}, \]

where the renormalized photon mass is

\[ m_\gamma^2 = \frac{e^2}{\pi} \frac{2\pi}{2\pi + g_2^2}. \tag{4.7} \]

**Integration over \( \varphi \):**
The zeta-function regulated determinant which one obtains when performing the integral (4.6) factorizes

\[ \det'(\triangle^2 - m_\gamma^2 \triangle) = \det'(-\triangle) \cdot \det'(-\triangle + m_\gamma^2). \]

This factorization property is not obvious since all determinants must be regulated. But it holds for commuting operators and in the zeta-function scheme. Then the partition function simplifies to

\[ Z_0 = \frac{2\sqrt{\pi k e V}}{m_\gamma} (\det'(-\triangle)\det'(-\triangle + m_\gamma^2))^{\frac{1}{2}} \exp\left((g_3^2 + \frac{1}{24\pi}) S_L\right). \quad (4.8) \]

We can go further by using (3.31) and the known result for the determinant of \( \hat{\triangle} \) [26] which together yield

\[ \det'\frac{1}{2}(-\triangle) = \tau_0 L |\eta(\tau)|^2 \sqrt{\frac{V}{V}} \exp\left(-\frac{1}{24\pi} S_L\right) \quad (4.9) \]

which finally leads to

\[ Z_0 = \sqrt{2\pi V} \frac{e}{m_\gamma \tau_0 |\eta(\tau)|^4} \frac{1}{\det'\frac{1}{2}(-\triangle + m_\gamma^2)} \exp\left(\frac{1}{12\pi} + g_3^2\right) S_L \quad (4.10) \]

for the partition function of the general model (1.1) on curved spaces. It shows explicitly that in the topologically trivial sector the theory should be equivalent to a theory of free massless and massive bosons with mass \( m_\gamma \).

It is interesting to follow the various contributions to the explicit dependence on the gravitational field since they contribute to the Hawking radiation. For that we recall that when one quantizes a conformal field theory with central charge \( c \) in an external gravitational fields one ends up with the Liouville term, \( Z \sim \exp[c S_L/24\pi] \) [32]. Thus the fermions contribute with \( c = 1 \), as expected. The \( \phi \) and \( \lambda \) field contribute with \( 1 + 24\pi g_3^2 \), respectively.

However, the Jacobian combined with the conformal part of the gauge sector contribute with \( c = -1 \) and we are left with a total central charge \( c = 2 + 24\pi g_3^2 \). Of course, the gauged model is not conformally invariant and the breaking is manifest in the massive determinant in (4.10). The partition function of the ungauged theory is (4.6) multiplied by an inverse determinant.
(the missing Jacobian) and without $\varphi$-integration. In this limit one obtains a conformal theory with central charge $c = 3 + 24\pi g_2^2$.

By using an elegant result of Christensen and Fulling [36], that relates the conformal anomaly to the asymptotic Hawking flux, one concludes that the Hawking radiation of the ungauged model is $3 + 24\pi g_2^2$ times that of free massless scalars. For the gauged model the Hawking radiation is still thermal and consists of massless and massive particles.

The appearance of $m_\gamma$ in (4.6) should be interpreted as renormalization of the electric charge induced by the interaction of the auxiliary fields with the fermions. After summing over all fermion-loops this leads to an effective coupling between the photons and the $\phi$-field and in turn to a modified effective mass for the photons in (4.6). In the limit $g_2 \to 0$ this mass tends to the well-known Schwinger model result, $m_\gamma \to e/\sqrt{\pi}$ [37].

We have already mentioned that the chemical potential coupled to the electric charge has completely disappeared from the partition function. This does not come as a surprise since the only particle in the gauged Thirring model is a massive neutral boson. This has no charge which may couple to the chemical potential. If the partition function would depend on $\mu$ then the expectation value of the charge would not vanish, in contradiction to the integrated Gauss law.

Also note that the final result is independent of the chiral and non-chiral twists. The normal twists have been wiped out by the toron integration. Since the chiral twists are equivalent to chemical potentials the partition function should not depend on them in order not to violate gauge invariance. If one would assume holomorphic factorization for the fermionic determinant [11] then the partition function would depend on the chiral twists.

We conclude this subsection with deriving an explicit formula for the partition function on the flat torus. Applying the results in [38] one obtains for the massive determinant
det′(-\Delta + m_2^2)^{1/2} = \frac{1}{m_\gamma} e^{-\frac{1}{2} \zeta'(0)}, \quad (4.11a)

with

\zeta'(0) = \sum_{n \neq 0} \frac{1}{\pi L} \sqrt{(n,n)} K_1 (m_\gamma L \sqrt{(n,n)}) - \frac{\hat{V} m_\gamma^2}{4\pi}, \quad (4.11b)

where \((n,n) = \hat{g}_{ij} n^i n^j\) is the inner product taken with the reference metric, and the sum is over all \((n^i) \in \mathbb{Z}^2\) with the origin excluded. For \(g_{\mu\nu} = \delta_{\mu\nu}\), in which case the partition function has the usual thermodynamical interpretation, the result reduces to one derived previously by Ambjorn [39]. In addition, if \(L\) approaches infinity we recover a result in [29]. The free energy for \(\tau_1 = 0\) and on flat space simplifies then to

\[F = -\frac{1}{\beta} \log Z = \frac{1}{2\beta} \zeta'(0).\]  (4.12)

with \(\zeta'(0)\) from (4.11b) and the particular choice for the parameters.

5. Bosonization of the gauged Thirring model

In the classical analysis we have already seen that in the limiting case \(g_3 = 0\) and \(g_1 = g_2 = g\) the general model reduces to the gauged Thirring model. In this section we show that the same is true for the quantized theory on the torus if in addition we set \(g_0 = g\). More precisely, the Hubbard-Stratonovich transform [40] of the Thirring model is just the derivative coupling model (1.1) with identical couplings. In the process of showing that we shall arrive at the bosonization formulae for the gauged Thirring model on the curved torus. We shall see that only the non-harmonic part of the fermion current can naively be bosonized and that for this part the rules of the ungauged model on flat space time [41] need just be covariantized.

For that we calculate the partition function (4.1a) in a different order. First we integrate out the auxiliary fields. In order to understand the role of \(\lambda\) and \(\phi\) we introduce sources for them. Thus we study the generating functional for the correlators of the auxiliary fields
\[ Z[\xi, \zeta] = \int \mathcal{D}(\lambda \phi h \psi A_\mu) e^{-S + \int \sqrt{g} [\xi A + \zeta \phi]} . \] 

(5.1a)

Here
\[ S = -i \int \sqrt{g} \psi'^\dagger \gamma \psi + S_B[g_3 = 0] \] 

(5.1b)

is the action of the full theory. \( \mathcal{D} \) is the Dirac operator in (3.11) with all couplings set equal and \( S_B \) the bosonic action (3.35). Since \( \lambda \) and \( \phi \) integrate to zero (see (3.38)) we may assume the same property to hold for the sources. Also, since there are no fermionic sources only configurations in the trivial sector contribute, so that there is not instanton potential in (3.11) and hence \( \Phi = 0 \) in (3.35). The integration over the auxiliary fields is gaussian and yields
\[ Z = N_0 \int \mathcal{D}(\psi A_\mu) e^{-S_T} \exp \int \sqrt{g} \left[ -\frac{1}{4} (\xi A \xi + \zeta A \zeta) + \frac{g}{2} (\xi A j_\mu^A + \zeta A j_\mu^B) \right] \] 

(5.2)

where
\[ S_T = -\frac{1}{4} \int \sqrt{g} \left( F_{\mu\nu} F^{\mu\nu} - i \psi'^\dagger \gamma \psi - \frac{g^2}{4} j_\mu^A j_\mu^A \right) \] 

(5.3a)

is the action of the gauged Thirring model on curved space-time and
\[ N_0 = \frac{V}{2\pi \text{det}'(-\Delta)} \] 

(5.3b)

comes from the integration over the auxiliary fields.

Let us first consider the partition function, that is set the sources to zero. Comparing (5.2) with (4.6) and using (4.9) we easily find
\[ \int \mathcal{D}(\psi t) e^{-S_T} = \sqrt{\frac{1}{2} + \frac{g^2}{4\pi}} e^{-\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \int \mathcal{D}(A) \delta(\gamma) e^{-S_A}} \] 

(5.4a)

where \( \gamma \) is the mean field (see 3.38) and we used (3.37) and (4.4b). The action for the neutral scalar field \( \gamma \) is found to be
\[ S_\gamma = \frac{1}{2} \int \sqrt{g} \partial_\mu \gamma \partial_\mu \gamma - \frac{ie}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \int \sqrt{g} \Delta \varphi . \] 

(5.4b)

Since (5.4) holds for any \( \varphi \) (and thus for the non-harmonic part of any \( A_\mu \), because of gauge-invariance) we read off the following bosonization rules:
\[ j^\mu \rightarrow \frac{i}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \eta^{\mu\nu} \partial_\nu \gamma \quad \text{and} \quad j_5^\mu \rightarrow -\frac{i}{\sqrt{\pi}} \frac{1}{\sqrt{1 + g^2/2\pi}} \partial^\mu \gamma, \]

(5.5)

where prime denotes the non-harmonic part of the currents. Thus, only the non-harmonic parts of the currents can be bosonized in terms of a single valued scalar field. To bosonize their harmonic parts one would have to allow for a scalar field \( \gamma \) with windings as \( \phi \) below (3.2). On the infinite plane the harmonic part is not present and we may leave out the primes in (5.5). If we further assume space time to be flat we recover the well-known bosonization rules in [41]. What we have shown then, is that for the gauged model on curved space time the bosonization rules are just the flat ones properly covariantized and with the omission of the zero-modes.

Since (5.4a) holds for any gauge field the current correlators in the Thirring model are correctly reproduced by the bosonization rules (5.4,5). To see that more clearly we calculate the two-point functions of the auxiliary fields in the Thirring model (5.2,3). For that we differentiate (5.2) (\( \varphi \) is treated as external field) with respect to the sources and find

\[
\langle \lambda(x)\lambda(y) \rangle = \frac{1}{2} G_0(x, y) + \frac{g^2}{4} \int \langle G_0(x, u)j^\mu_{\lambda\mu}(u)G_0(y, v)j^\nu_{\nu\nu}(v) \rangle_T \tag{5.6}
\]

\[
\langle \phi(x)\phi(y) \rangle = \frac{1}{2} G_0(x, y) + \frac{g^2}{4} \int \langle G_0(x, u)j^\mu_{5\mu}(u)G_0(y, v)j^\nu_{5\nu}(v) \rangle_T,
\]

where \( G_0 \) is the free massless Greenfunction (3.6a,3.8) in curved space-time and the integrations are over the variables \( u \) and \( v \) with the invariant measure on the curved torus. Here \( \langle \ldots \rangle_T \) are vacuum expectation values of the Thirring model (5.3a). Alternatively we can calculate these expectation values from (4.3) and (4.5), where the fermionic integration has been performed and find

\[
\langle \lambda(x)\lambda(y) \rangle = \frac{1}{2} G_0(x, y)
\]

\[
\langle \phi(x)\phi(y) \rangle = \frac{\pi m}{2e^2} G_0(x, y) + \frac{m^2}{2} \left( 1 - \frac{\pi m^2}{e^2} \right) \varphi(x)\varphi(y). \tag{5.7}
\]

Comparing the results (5.6) and (5.7) we see at once that
The correlators (5.8) express the gauge invariance and the axial anomaly 
\langle j^\mu_{5} \rangle = -m_\gamma \Delta \varphi$ in the gauged Thirring model. They can be correctly reproduced with the bosonisation rules (5.5).

6. Chiral condensate

We have seen that the partition function of the gauged model does not depend on the chiral twists $\beta$. As we pointed out this property is very much linked to gauge invariance. The same should then hold in the topological non-trivial sectors. For that reason we shall set $\beta = 0$ in this section. This assumption will simplify the manipulations below considerably. For example, $\mathcal{N}_\psi = \mathcal{N}_\chi$ in (3.34).

Recalling that $S_e$ in (3.34) anti-commutes with $\gamma_5$ one sees at once that only configuration supporting one fermionic zero-mode with positive chirality contribute to the chiral condensate

$$\langle \psi^\dagger P_+ \psi \rangle = -\frac{J}{Z_0} \frac{\delta^2}{\delta \eta_+(x) \delta \bar{\eta}_+(x)} \int \mathcal{D}(\ldots) Z_F[\eta, \bar{\eta}]|_{\eta=\bar{\eta}=0} e^{-S_B},$$

(6.1)

where $\eta_+ = P_+ \eta$. Earlier we have seen that these are the gauge fields with flux $\Phi = 2\pi$ or instanton number $k = 1$. Thus (6.1) reads

$$\langle \psi^\dagger P_+ \psi \rangle = -\frac{J}{Z_0} \sqrt{\frac{V}{2}} \int \mathcal{D}(\ldots) \psi_{0+}^\dagger(x) \psi_{0+}(x) \exp(\ldots) e^{-S_B[k=1]},$$

(6.2)

where $\exp(\ldots)$ is the last exponential factor in (3.34). Here we have used that the chiral twists vanish, such that $\chi_{0+} = \psi_{0+}$ (see (3.25)). First we integrate over the toron field $t$. The $t$ dependence enters only through the zero mode and more specifically $\hat{\psi}_{0+}$ in (3.29) and (3.25) with $p = 1$. Using the series representation for the theta functions one finds

$$\int \langle G_0(x, u) j^\mu_{\mu}(u) G_0(y, v) j^\nu_{\nu}(v) \rangle_T = 0$$

$$\int \langle G_0(x, u) j^\mu_{5\mu}(u) G_0(y, v) j^\nu_{5\nu}(v) \rangle_T = \frac{m_\gamma^2}{e^2} (m_\gamma \varphi(x) \varphi(y) - G_0(x, y)).$$

(5.8)
\[
\int d^2 t \hat{\psi}_{0+}^\dagger (x) \hat{\psi}_{0+}(x) = \frac{1}{\hat{V}}. \tag{6.3}
\]

Note that the result does not depend on the chemical potential similarly to our calculation of the partition function.

To continue we observe that the term \( \int \sqrt{g} G \) in \( \exp(\ldots) \) vanishes because of our conditions (3.36) and (3.38) on the fields \( \varphi \) and \( \phi \). Also note, that the fermionic Green function does not enter in the expression for the chiral condensate. It follows that the fermionic functional (3.34) in the trivial sector and (3.34) in the one-instanton sector are the same, up to the factors in the fist lines. From (6.3) and (4.3) we see that the toron integral of the first line in (3.34) is \( |\eta|^2 \sqrt{\tau_0/\hat{V}} \exp(2\pi \int \sqrt{g} \chi / \hat{V}) \) times the toron integral over the factor in (3.33). Also, since

\[ S_B[k = 1] = S_B[k = 0] + \frac{2\pi^2}{e^2\hat{V}} \]

the functional integral and normalizing partition function in (6.2) are the same, up to these factor and the field-dependent factors which relate the hatted and unhatted zero-modes in (3.29). Finally note that the \( \lambda \) integrals in (6.2) and in the normalizing partition function cancel so that we end up with the following formula for the condensate

\[
\langle \psi^\dagger P_+ \psi \rangle = \sqrt{\frac{\tau_0}{\hat{V}}} |\eta(\tau)|^2 e^{-2\pi^2/e^2\hat{V} + 2\pi \int \sqrt{g} \chi / \hat{V}} \int \sqrt{\delta x} \left< e^{-2(g\phi + e\varphi)(x) - \sigma(x)} \right> \phi \varphi. \tag{6.4a}
\]

The expectation value is evaluated with

\[
S_{\text{eff}} = \int \sqrt{g} \left[ \frac{1}{2} \varphi(\Delta^2 - \frac{e^2}{\pi} \Delta) \varphi - \frac{e^2}{\pi m_\gamma^2} \phi \Delta \phi - \frac{eg_2}{\pi} \phi \Delta \varphi \right]. \tag{6.4b}
\]

A formal calculation of the resulting gaussian integrals yield

\[
\langle \psi^\dagger P_+ \psi \rangle = \sqrt{\frac{\tau_0}{\hat{V}}} |\eta(\tau)|^2 e^{-2\pi^2/e^2\hat{V} + 2\pi \int \sqrt{g} \chi / \hat{V}} e^{-\sigma(x) - 2g\chi(x)} \cdot \exp \left[ \frac{2\pi^2 m_\gamma^4}{e^2} K(x, x) \right] \exp \left[ \frac{2\pi g_2^2}{2\pi + g_2^2} G_0(x, x) \right], \tag{6.5}
\]

where
\[ K(x, y) = \langle x| \frac{1}{\Delta^2 - m^2} \Delta |y \rangle = \frac{1}{m^2} \left( G_0(x, y) - G_m(x, y) \right) \] (6.6)

and \( G_m, G_0 \) are the massive and massless Green functions.

Here we encounter ultraviolet divergences since \( G_0(x, y) \) is logarithmically divergent when \( x \) tends to \( y \). To extract a finite answer we need to renormalize the operator \( \exp(\alpha \phi) \). This wave function renormalization is equivalent to the renormalization of the fermion field in the Thirring model and thus is very much expected [33,41]. In order to do that we first determine the short distance behaviour of the massless Green function (3.6b).

Using the identity
\[
|\Theta \left[ \frac{1}{2} + \frac{\xi_0^2}{L} \right] (0, \tau) |^2 = |e^{i\pi \tau (\xi_0^2/L)^2} \Theta_1 \left( \frac{\tau \xi_0^2 + \xi_1^2}{L} , \tau \right) |^2
\]
and the small \( z \) expansion
\[
\Theta_1(z, \tau) = 2\pi \eta(\tau)^3 z + O(z^2),
\]
we see that \( \hat{G}_0 \) possesses the expected logarithmic short distance singularity
\[
\hat{G}_0(x, y) = -\frac{1}{4\pi} \log \frac{\hat{g}_{\mu\nu}\xi_0^2\xi_\nu}{V} - \frac{1}{4\pi} \log \left( 4\pi^2 \tau_0 |\eta(\tau)|^4 \right) + O(\xi). \quad (6.7)
\]

From the relation (3.8) between the full and hatted Green function and the definition of \( \chi \) in (3.4c) it follows that \( G_0 \) has the short distance expansion
\[
G_0(x, y) \sim \hat{G}_0(x, y) + 2\chi(x) - \frac{1}{V} \int \sqrt{g} \chi + O(\xi) \quad (6.8)
\]

To continue we need to regularize the composite operator \( \exp(\alpha \phi) \) appearing in (6.4a). The normal ordering prescription
\[
: e^{\alpha \phi(x)} := \frac{e^{\alpha \phi(x)}}{\langle e^{\alpha \phi(x)} \rangle}. \quad (6.9)
\]
works well on the whole plane [33,41]. On the curved torus we must be more careful when renormalizing this operator. The required wave function renormalization is not unique but it is very much restricted by the following requirements: First we take as reference system (the denominator in 6.9) one with a minimal number of dynamical degrees of freedom since we do not want to lose information by our regularization. Second, the renormalized operator should have a well-defined infinite volume limit and its expectation values should cluster. Finally, the regularization should respect general covariance. These requirements then force us to take as reference system the infinite plane with metric $g_{\mu\nu}$. The flat metric $\delta_{\mu\nu}$ is not permitted since it leads to a ill-defined expression for $\langle \exp(\alpha \phi) \rangle$. With these choice the normal ordering in (6.9) is equivalent to replacing the massless Greenfunction in (6.5) by

$$G_0^{reg}(x,y) := G_0(x,y) + \frac{1}{4\pi} \log \left[ \mu^2 s^2(x,y) \right]. \tag{6.10}$$

Here $s(x,y)$ denotes the geodesic distance between $x$ and $y$. The occurrence of the arbitrary mass scale $\mu$ comes from the ambiguities in the required ultraviolett regularization. On the flat torus $\hat{G}_0^{reg}$ has now the finite coincidence limit

$$\hat{G}_0^{reg}(x,x) = -\frac{1}{4\pi} \log \left( \frac{4\pi^2 \tau_0 \eta(\tau)^4}{\mu^2 \hat{V}} \right). \tag{6.11}$$

To determine the chiral condensate we also need to determine $K(x,y)$ on the diagonal. In a first step we shall obtain it for the flat torus. Its $\sigma$-dependence is then determined in a second step. For $\sigma=0$ and $\tau=i\tau_0$ the Greenfunction $\hat{K}$ has been computed in [3]. The generalization to arbitrary $\tau$ is found to be

$$m_\gamma^2 \hat{K}(x,x) = -\frac{1}{2m_\gamma L \tau_0} \coth \left( \frac{\pi \tau_0 a}{|\tau|^2} \right) + \frac{1}{m_\gamma^2 \hat{V}}$$

$$+ \frac{1}{2\pi} \left( -\log |\eta(-1/\tau)|^2 + F(L,\tau) - H(L,\tau) \right), \tag{6.12a}$$

where we introduced the dimensionless constant $a = L m_\gamma |\tau|/2\pi$ and the functions
\[ F(L, \tau) = \sum_{n>0} \left[ \frac{1}{n} - \frac{1}{\sqrt{n^2 + a^2}} \right] \]
\[ H(L, \tau) = \sum_{n>0} \frac{1}{\sqrt{n^2 + a^2}} \left[ \frac{1}{e^{-2\pi i z_+(n)} - 1} + \frac{1}{e^{2\pi i z_-(n)} - 1} \right]. \]

\[ z_\pm = \frac{1}{|\tau|^2} (n\tau_1 \pm i\tau_0 \sqrt{n^2 + a^2}). \]

We used the abbreviations

\[ z_\pm = \frac{1}{|\tau|^2} (n\tau_1 \pm i\tau_0 \sqrt{n^2 + a^2}). \]

Substituting (6.12) and (6.11) into (6.5) with \( \sigma = 0 \) we obtain the following exact formula for the chiral condensate on the torus with flat metric \( \hat{g}_{\mu\nu} \):

\[ \langle \psi^+ P_+ \psi \rangle_{\hat{g}} = \frac{1}{L|\tau|} \left( \frac{m_\gamma L|\tau|}{2\pi} \right)^{\frac{a_2^2}{2a+g_2^2}} \exp \left( \frac{\pi^2 m_\gamma}{e^2 L\tau_0} \coth \frac{Lm_\gamma \tau_0}{2|\tau|} \right) \]
\[ \cdot \exp \left[ \frac{\pi m_\gamma^2}{e^2} \left( F(L, \tau) - H(L, \tau) \right) \right], \]

where we used that on the flat torus \( \chi = 0 \) and \( V = \hat{V} \). Furthermore, we identified \( \mu \) with the natural mass scale \( m_\gamma \) of the theory.

To study the finite temperature behaviour of the chiral condensate we must assume that \( \tau = i\beta/L \) and then \( \beta = 1/T \) is just the inverse temperature. Furthermore we perform the thermodynamic limit \( L \to \infty \). Then \( \coth(...) \to 1 \), \( H \to 0 \) and the expression for the chiral condensate simplifies to

\[ \langle \psi^+ P_+ \psi \rangle_{\beta} = -T \left( \frac{m_\gamma}{2\pi} \right)^{\frac{a_2^2}{2a+g_2^2}} \exp \left[ -\frac{\pi^2 m_\gamma^2}{e^2} T + \frac{2\pi}{2\pi + g_2^2} F \right]. \]

Let us now investigate the low and high temperature limits in turn. To study the low temperature limit we use that

\[ F(\beta) \to \gamma + \log a \frac{1}{2} + \frac{1}{2a} \quad \text{for} \quad a \to \infty, \]

where \( \gamma = 0.57721\ldots \) is the Euler number. Inserting this expansion into (6.14) yields

\[ \langle \psi^+ P_+ \psi \rangle = -\frac{m_\gamma}{4\pi} 2\gamma^2/(2\pi + g_2^2) \exp \left( \frac{2\pi}{2\pi + g_2^2} \gamma \right) \quad \text{for} \quad T \to 0. \]
For temperatures large compared to the induced photon mass $F$ vanishes. Thus we obtain the high temperature behaviour

$$\langle \psi^+ P_+ \psi \rangle_T = -T \left( \frac{m_\gamma}{2\pi T} \right)^{\frac{g_2^2}{2\pi^2}} \exp \left( -\frac{\pi^2 m_\gamma T}{e^2} \right) \quad \text{for } T \to \infty \quad (6.16)$$

It is instructive to discuss the various limiting cases. For all $g_i = 0$, i.e. the Schwinger model limit, the exact result (6.14) simplifies to

$$\langle \psi^+ P_+ \psi \rangle_T = -Te^{-\frac{\pi m_\gamma}{4\pi} + F(\beta)} \rightarrow \begin{cases} -\frac{m_\gamma}{4\pi} e^{\gamma} & \text{for } T \to 0 \\ -Te^{-\pi T/m_\gamma} & \text{for } T \to \infty , \end{cases} \quad (6.17)$$

where now $m_\gamma^2 = e^2/\pi$ is the induced photon mass in the Schwinger model. This formula for the temperature dependence of the chiral condensate in $QED_2$ agrees with the earlier results in [3].

Next we wish to investigate how the selfinteraction of the fermions affect the breaking. For large coupling $g_2$ and fixed temperature the exponent in (6.14) vanishes so that

$$\langle \psi^+ P_+ \psi \rangle_T \sim \frac{1}{\sqrt{2\pi + g_2}} \quad \text{for } T \text{ fixed, } g_2 \to \infty . \quad (6.18)$$

Hence, for very large current-current coupling the chiral condensate vanishes. Or in other words, the electromagnetic interaction which is responsible for the chiral condensate, is shielded by the pseudoscalar-fermion interaction.

For intermediate temperature and coupling $g_2$ we must retreat to numerical evaluations of the sums defining the chiral condensate in (6.14). The results of the numerical calculations are depicted in Figure 1.

The study of the influence of the gravitational field is complicated by the presence of the massive Green function $G_{m_\gamma}$ in (6.5,6). This Green function is known only for very particular curved spaces. Fortunately we only need the coincidence limit for which we can use its short distance expansion [42]. For simplicity we assume infinite volume and zero temperature. Then [43]

$$G_m(x, y) \sim \frac{1}{4i} \sum_{j=0}^{\infty} a_j(x, y) \left( -\frac{\partial}{\partial m^2} \right)^j H^{(2)}_0 (ms) , \quad (6.19)$$
for small geodesic distances \( s = s(x, y) \). Here \( H^{(2)}_0 \) denotes the Hankel function of the second kind and order zero. In particular

\[
H^{(2)}_0(z) \to \frac{2}{i\pi} \left[ \log \frac{z}{2} + \gamma \right] \quad \text{for} \quad z \to 0.
\] (6.20)

Inserting that into (6.19) we find with \( G_0 = \hat{G}_0 \) from (6.7) the following short distance expansion

\[
G_0(x, y) - G_m(x, y) \sim -\frac{1}{2\pi} \left[ \log \left( \frac{2\pi|\eta(\tau)|^2}{m_\gamma L e^{\sigma(x)}} \right) - \gamma \right] + \frac{1}{4\pi} \sum_{j=1}^{\infty} a_j(x) \left( -\frac{\partial}{\partial m^2} \right)^j \log(m^2).
\] (6.21)

We have used that \( a_0(x) = 1 \) and \( s \sim e^{\sigma(x)} \hat{s} \), where \( \hat{s} \) is the geodesic distance on the flat spacetime with hatted metric, \( \hat{s}^2 = \hat{g}_{\mu\nu}(x-y)^\mu(x-y)^\nu \). Finally, substituting (6.21) into (6.5) we end up with

\[
\langle \psi^+ P_+ \psi \rangle_{\sigma} = \langle \psi^+ P_+ \psi \rangle_{\sigma=0} \cdot \exp \left[ -\frac{1}{2} \left( \frac{\pi m_{\gamma}}{e} \right)^2 \sum_{j=1}^{\infty} a_j(x) \frac{(j-1)!}{m^{2j}} \right].
\] (6.22)

The Seeley-deWitt coefficients \( a_j \) have been computed up to \( j = 5 \) \([44]\). They are of order \( j \) in the curvature and its derivatives. The first two are

\[
a_0(x) = 1 \quad \text{and} \quad a_1(x) = \frac{1}{6} \mathcal{R}.
\]

For \( \mathcal{R} \ll m^2 \) and slowly varying \( \mathcal{R} \) we conclude that the chiral condensate decreases with increasing curvature as

\[
\langle \psi^+ P_+ \psi \rangle \sim \exp \left[ -\frac{\pi^2}{12e^2} \mathcal{R} \right].
\] (6.23)

If we compare this with the temperature dependence (6.16) we are lead to define a curvature induced effective temperature

\[
T_{eff} = \frac{\mathcal{R}}{12m_{\gamma}}.
\] (6.24)

For this formal identification of curvature with temperature no horizon is needed as in black hole physics where the temperature is related to the surface gravity at the horizon. Note that contrary to the temperature the curvature may become negative. Then the condensate is amplified.
Finally we consider the *chiral two point function* for non-coinciding points. The gauge invariant form reads

\[ S_+(x, y) \equiv \langle \psi(x) e^{ie \int_x^y A_\mu dx^\mu} P_+ \psi(y) \rangle. \]

It is related to a bound state between a static external charge and a dynamical fermion [45].

The integration over the various fields is similar to (6.3-12). The result takes a simple form in the infinite volume and zero temperature limit:

\[
S_+(x, y) = S_+(x) \hat{S}_+(y) \exp \left[ \frac{1}{2} \left( \frac{\pi m_e}{e} \right)^2 (K(x, y) + K(y, x)) \right] 
\times \exp \left[ (\frac{\pi g_2^2}{2\pi} + \frac{g_1^2}{2}) G_0(x, y) - \frac{g_1^2}{4} (G_0(x, x) + G_0(y, y)) \right],
\]

where \( S_+(x) \equiv S_+(x, x) = \langle \psi(x) \psi(x) \rangle \) denotes the chiral condensate.

Again the massless propagator must be regularized. We do this using the prescription (6.10). Then

\[
S_+(x, y) = S_+(x) \hat{S}_+(y) \exp \left[ \frac{1}{2} \left( \frac{\pi m_e}{e} \right)^2 (G_{m_\gamma}(x, y) + G_{m_\gamma}(y, x)) \right]
\times \sqrt{\frac{\pi g_2^2}{2\pi}} (g(x) \hat{s} g(y) \hat{s})^{\frac{1}{2}} \left( \frac{g_1^2}{2\pi} \right)^{\frac{1}{2}} (1 + \frac{g_1^2}{2\pi})
\]

Note that the coupling strength \( g_1 \) to the longitudinal current enters the scaling exponent. On flat space \( G_m \) reduces to \( \frac{1}{2\pi} K_0(m\hat{s}) \) which decays exponentially for large separations. Hence we find

\[
\hat{S}_+(x, y) \sim \frac{\hat{S}_+(x) \frac{1}{2}}{\sqrt{2\pi \hat{s} (m_\gamma \hat{s})^{\frac{1}{4}}}}
\]

for large separations of \( x \) and \( y \). We have used that the chiral condensate \( \hat{S}_+(x) \) in (6.15) is constant, due to translational invariance. For \( g_1 = 0 \) this simplifies to the Schwinger model result [3]

\[
\hat{S}_+(x, y) \sim \sqrt{\frac{m_\gamma e^\gamma}{2\pi}} \frac{1}{2\pi \sqrt{|x - y|}}.
\]
Unlike the correlators of fields which in the bosonized version are local in the massive boson field, this two-point function does not decay exponentially. However the long range correlations are suppressed by the coupling to the longitudinal current.

7. Thermodynamics of the ungauged Model

In this chapter we derive the grand canonical potential, equation of state and ground state energy for $A_\mu = 0$. For the ungauged model there is no Gauss constraint and the charge of the vacuum need not vanish. Indeed, for $A_\mu = 0$ the partition function depends on the chemical potential and on the fermionic boundary conditions. Technically this is due to the absence of the toron integration which for the gauged model wiped out any dependence on $\mu, \alpha$ and $\beta$.

The partition function of the ungauged model is given by

$$ Z = \int d^2 h D\phi D\lambda \ Z_F[\eta = \bar{\eta} = A = 0] \ e^{-S_B[A=0]}, \quad (7.1) $$

where $Z_F$ is the fermionic generating functional (3.33) and $S_B$ the bosonic action (3.35). The integration over the harmonic fields is Gaussian and yields

$$ \int_{-\infty}^{\infty} d^2 h \Theta[-c_1/c_0] \bar{\Theta}[-\bar{c}_1/\bar{c}_0] e^{-(2\pi)^2 \sqrt{g_0} \ h_\mu h_\nu} = \frac{\Theta[u/w](\Lambda)}{4\pi \sqrt{1 + g_0^2/2\pi}} \quad (7.2) $$

where

$$ \Theta[u/w](\Lambda) = \sum_{n \in \mathbb{Z}^2} e^{i\pi(n+u)\Lambda(n+u) + 2\pi i(n+u)w} \quad (7.3a) $$

is the thetafunction with characteristics

$$ u = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\alpha_1 + i\eta_1^\nu \beta_\nu) \quad \text{and} \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}(\alpha_0 + i\eta_0^\nu \beta_\nu - \mu_0) \quad (7.3b) $$

and covariance

$$ \Lambda = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix} + \frac{i \pi g_0^2 \tau_0}{2\pi + g_0^2} \begin{pmatrix} g_0^2 & -4\pi - g_0^2 \\ -4\pi - g_0^2 & g_0^2 \end{pmatrix}. \quad (7.3c) $$
The remaining functional integrals in (7.1) are performed as in chapter 4. To obtain the partition function of the Thirring model in the limit \( g_i = g \) we divide \( Z \) by the corresponding partition function \( N_0 \) of the free bosons (see 5.3b)). Using (4.4b) and (3.33) we obtain

\[
\frac{Z}{N_0} = \frac{1}{|\eta(\tau)|^2} \sqrt{\frac{2\pi + g \frac{2}{6}}{2\pi + g_0^2}} \Theta \left[ \frac{u}{w} \right](\Lambda) e^{(1/24\pi + g^2)S_L}. \tag{7.4}
\]

In the Thirring model limit \( g_2 = g_0 \) and the square-root in this formula disappears.

### 7.1 Zero-temperature limit

To investigate the thermodynamics of the model we assume spacetime to be flat and that \( \tau = i\beta/L \). Then

\[
\Omega = -\frac{1}{\beta} \log \frac{Z}{N_0} \tag{7.5}
\]

is the grand canonical potential. Let us now investigate the low temperature limit of \( \Omega \). For \( \mu = 0 \) this yields the ground state energy.

To study this limit we observe that for \( \tau = i\beta/L \) the covariance matrix \( \Lambda \) in (7.3c) simplifies to

\[
i \pi \Lambda = -\frac{\pi \beta}{L} \left[ \text{Id} + \frac{g_0^2}{4\pi} \frac{1}{2\pi + g_0^2} \begin{pmatrix}
\frac{g_0^2}{2} & -4\pi - g_0^2 \\
-4\pi - g_0^2 & g_0^2
\end{pmatrix}\right] \tag{7.6a}
\]

and has eigenvalues

\[
\lambda_1 = -\frac{\pi \beta}{L} \frac{2\pi + g_0^2}{2\pi} \quad \text{and} \quad \lambda_2 = -\frac{\pi \beta}{L} \frac{2\pi}{2\pi + g_0^2} \tag{7.6b}
\]

with corresponding eigenvectors

\[
v_1 = (-1, 1) \quad \text{and} \quad v_2 = (1, 1). \tag{7.6c}
\]

Also the \( \eta \) tensor (see 3.17b) and \( \mu_0 \) (see 3.11) in (7.3b) simplify to

\[
\eta^\nu_{\mu} = \begin{pmatrix}
0 & \beta/L \\
-L/\beta & 0
\end{pmatrix} \quad \text{and} \quad \mu_0 = -\frac{\beta}{2\pi} \mu.
\]

Now we can determine the low temperature limit of the grand potential from (7.4) (with $S_L = 0$) and (7.6). For that we note that the saddle point approximation to the Gaussian sum (7.3a) defining the theta-function becomes exact when $\beta \to \infty$. Also, using that
\[
\log |\eta(\tau)|^2 \longrightarrow -\frac{\pi \beta}{6L} \quad \text{for} \quad \beta \to \infty
\]
we end up with
\[
\Omega(\beta \to \infty) = -\frac{\pi}{6L} - \frac{4\pi}{2\pi + g_0^2 L} \left(\beta_1 + \frac{\mu L}{2\pi}\right)^2
\]
\[
+ \frac{\pi}{2L} \min_{n \in \mathbb{Z}^2} \left\{ \frac{2\pi + g_0^2}{2\pi} \left\{ n_2 - n_1 - \frac{4\pi}{2\pi + g_0^2} (\beta_1 + \frac{\mu L}{2\pi}) \right\}^2 \right. \\
+ \left. \frac{2\pi}{2\pi + g_0^2} \left\{ n_1 + n_2 - 2\alpha_1 \right\}^2 \right\}
\]
for the zero-temperature grand potential of the ungauged model. The chemical potential and chiral twist enter only through the combination $\beta_1 + \mu L / 2\pi$.
Up to the second term the potential is invariant under
\[
\alpha_1 \longrightarrow \alpha_1 + 1 \quad \text{and} \quad \beta_1 + \frac{\mu L}{2\pi} \longrightarrow \beta_1 + \frac{\mu L}{2\pi} + 1 + g_0^2 / 2\pi.
\]
Let us now discuss the potential in the various limiting cases.
First assume that there is no chiral twist, $\beta_1 = 0$, and that the chemical potential vanishes. Then $\Omega(\beta \to \infty)$ coincides with the groundstate energy. The minimum in (7.8) is attained for $n_1 = n_2 = \lfloor \frac{1}{2} + \alpha_1 \rfloor$ and we find
\[
E_0(L, \alpha_1, \beta_1 = 0) = -\frac{\pi}{6L} + \frac{2\pi}{L} \frac{2\pi}{2\pi + g_0^2} (\alpha_1 - \left\lfloor \frac{1}{2} + \alpha_1 \right\rfloor)^2.
\]
Only for anti-periodic boundary conditions, that is for $\alpha_1 = 0$, does this Casimir energy coincide with the corresponding result for free fermions. For $g_0^2 \geq 4\pi$ the Casimir force is always attractive whereas for $g_0^2 < 4\pi$ it can be attractive or repulsive, depending on the value of $\alpha_1$. The result (7.10) is in agreement with the literature [11]. For example, it coincides with De Vegas and Destri’s result if we make the identification $\omega_{DD} = 2\pi \alpha_1$ and $1/\beta_{DD} = 1 + g_0^2 / 2\pi$ in formula (42) of that paper.
For small twists and chemical potential the minimum is assumed for $n_i = 0$ and the potential simplifies to

$$\Omega(\beta \to \infty) = -\frac{\pi}{6L} + \frac{2\pi}{L} \frac{2\pi}{2\pi + g_0^2} \alpha_1^2$$

(7.11)

and does not depend on the chemical potential.

For vanishing $g_0$, that is for free fermions, the minimum of (7.8) is attained for

$$n_1 = \left[ \frac{1}{2} + \alpha_1 - \beta_1 - \frac{\mu L}{2\pi} \right] \quad \text{and} \quad n_2 = \left[ \frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi} \right],$$

where $[x]$ denotes the biggest integer which is smaller or equal to $x$. This then leads to the following zero temperature potential

$$\Omega = -\frac{\pi}{6L} - \frac{2\pi}{L} (\beta_1 + \frac{\mu L}{2\pi})^2$$

$$+ \frac{\pi}{L} \left( \frac{\alpha_1 - \beta_1 - \mu L}{2\pi} - \left[ \frac{1}{2} + \alpha_1 - \frac{\mu L}{2\pi} \right] \right)^2$$

$$+ \frac{\pi}{L} \left( \frac{\alpha_1 + \beta_1 + \mu L}{2\pi} - \left[ \frac{1}{2} + \alpha_1 + \beta_1 + \frac{\mu L}{2\pi} \right] \right)^2.$$  

(7.12)

For $\mu = \beta_1 = 0$ this reduces to the Casimir energy for free fermions with left-right symmetric twists and agrees with the results in [46].

Note, however, that for $\beta_1 \neq 0$ we disagree with [11]. The difference is due to the second term on the right in (7.8). Let us give two arguments in favour of our result:

The discrepancy arises from the prefactor appearing in the fermionic determinant (3.19). As discussed earlier this prefactor implies the breakdown of holomorphic factorization, a property which has been presupposed in [11].

In appendix C we show that our results can be reproduced by starting with massive fermions and taking the limit $m \to 0$.

The second argument goes as follows: Suppose that $\beta_1 = \alpha_1 = 0$. Then (7.12) simplifies to

$$\Omega(\beta \to \infty) = -\frac{\pi}{6L} - \frac{2\pi}{L} \left( \frac{\mu L}{2\pi} \right)^2$$

$$+ \frac{2\pi}{L} \left( \frac{\mu L}{2\pi} - \left[ \frac{1}{2} + \frac{\mu L}{2\pi} \right] \right)^2.$$  

(7.13)
For massless fermions the Fermi energy is just $\mu$ and at $T=0$ all electron states with energies less then $\mu$ and all positron states with energies less then $-\mu$ are filled. The other states are empty. Since $d\Omega/d\mu$ is the expectation value of the electric charge in the presence of $\mu$ we see that it must jump if $\mu$ crosses an eigenvalue of the first quantized Dirac Hamiltonian $h$. For vanishing twists the eigenvalues of $h$ are just $E_n=(n-\frac{1}{2})\pi/L$. Indeed, from (7.13) one finds that the electric charge
\begin{equation}
\langle Q \rangle = \frac{d\Omega}{d\mu} = 2\left[\frac{1}{2} + \frac{\mu L}{2\pi}\right] = 2n \quad \text{for} \quad E_n \leq \mu < E_{n+1}
\end{equation}
jumps at these values for $\mu$. Further observe, that in the thermodynamic limit $L \to \infty$ the density
\begin{equation}
\frac{\Omega}{L} \to -\frac{2\pi}{2\pi + g_0^2} \frac{\mu^2}{\pi},
\end{equation}
reduces for $g_0=0$ to the standard result for free electrons.

### 7.2 Equation of state

We wish to derive the equation of state for finite $T$ in the infinite volume limit $L \to \infty$. This may be achieved by interchanging the roles played by $L$ and $\beta$. More precisely, using that
\begin{equation}
\Theta[^u_w](\Lambda) = \sqrt{\det(i\Lambda^{-1})} e^{2\pi i w \cdot u} \Theta[^-w_u](i\Lambda^{-1})
\end{equation}
we find in analogy with the low temperature limit that for $L \to \infty$ the pressure is given by
\begin{equation}
\beta p = \lim_{L \to \infty} \frac{1}{L} \log \frac{Z}{N_0} = \frac{\pi}{6\beta} + \frac{2\pi}{\beta} \frac{2\pi + g_0^2}{2\pi} \beta_0^2
\end{equation}
\begin{equation}
- \frac{\pi}{2\beta} \min_{n \in \mathbb{Z}} \left[ \frac{2\pi + g_0^2}{2\pi} \left\{ n_1 + n_2 + 2\beta_0 \right\}^2 \right] + \frac{2\pi}{2\pi + g_0^2} \left\{ n_2 - n_1 + 2\alpha_0 + 2i\frac{\beta \mu}{2\pi} \right\}^2.
\end{equation}
Here the minimum of the real part has to be taken. Again the minimization arises from the saddle point approximation to the theta function which becomes exact when $L \to \infty$. For small twists the minimum is assumed for $n_i = 0$ and then

$$
\beta p = \frac{\pi}{6\beta} - \frac{2\pi}{\beta} \frac{2\pi}{2\pi + g_0^2} \left(\alpha_0 + i\frac{\beta \mu}{2\pi}\right)^2
$$

(7.18)

becomes independent on the chiral twist $\beta_0$. As we have interchanged the roles of the temporal and spatial twists this is consistent with the earlier result that for small twists $\Omega$ is independent of $\beta_1$. In particular, for $\alpha_0 = 0$, we are lead to the following equation of state

$$
p(\beta, \mu, \alpha_0 = 0) = \frac{\pi}{6\beta^2} + \frac{\mu^2}{2\pi} \frac{2\pi}{2\pi + g_0^2},
$$

(7.19)

which for small $\beta_0$ relates the pressure to the chemical potential and temperature. This result is consistent with the renormalization of the electric charge which is conjugate to the chemical potential. It shows in particular that the thermodynamic behaviour of the Thirring model is not just the one of free fermions as has been claimed in [12]. Indeed, the zero point pressure is multiplied by a factor $2\pi/(2\pi + g_0^2)$. This modification arises from the coupling of the current to the harmonic fields. It can not be seen if only the local part of the auxilliary field is considered, which is the case if one quantizes the model on the infinite Euclidean space. Furthermore, we see that the 'pressure' $p$ is real only for $\alpha_0 = 0$. This phenomenon occurs also in the Hamiltonian formalism [47]. However, finite temperature physics dictates anti-periodic boundary conditions, i.e $\alpha_0 = 0$, and then $p$ becomes real.

8. Conformal structure of the ungauged model

When we discussed the properties of the classical model (1.1) in chapter 2 we have noticed that for $A_\mu = 0$ it reduces to a conformal field theory on flat Minkowski spacetime. We have found the results listed at the end of section 2.
We determine the quantum corrections to these classical results. As in the previous chapters we do that within the Euclidean functional approach. Thus we start from first principles and need not postulate the emerging \textit{Kac-Moody} and \textit{Virasoro algebras} in advance [8,16]. When comparing the classical with the quantum results one should keep in mind that roles of $\psi^\dagger_0$ and $\psi^\dagger_1$ are interchanged when one switches from Minkowski to Euclidean spacetime. For further changes the reader is referred to appendix A.

In what follows it is convenient to exploit the holomorphic structure of the model. On the torus with flat metric $\hat{g}_{\mu\nu}$ the Cauchy-Riemann equations read

\[ (\eta^\nu_{\mu} \partial_{\nu} - i \partial_{\mu}) f = 0. \]  

(8.1)

Then one chooses coordinates $x'^a = e^a_{\mu} x^\mu$ and the corresponding complex coordinates $x = x'^0 + i x'^1$ such that (8.1) takes the standard form. More explicitly we chose

\[ x = i \bar{\tau} x^0 + i x^1 \quad \text{so that} \quad \partial_x = \frac{1}{2\tau_0} (\partial_{x^0} - \tau \partial_{x^1}). \]  

(8.2)

In this section $x$ and $\bar{x}$ always denote the complex coordinates belonging to $x^\mu$. In these coordinates the free Dirac operator and the corresponding Greensfunction are simple

\[ i\hat{\partial} = 2i \begin{pmatrix} 0 & \partial_{\bar{x}} \\ \partial_{x} & 0 \end{pmatrix} \quad \text{and} \quad S(x^\alpha, y^\beta) = \frac{1}{2\pi i} \begin{pmatrix} 0 & 1/\xi \\ 1/\bar{\xi} & 0 \end{pmatrix} + O(1), \]  

(8.3)

where $\xi = x - y$. The chiral components of the energy momentum tensor and current are then given by

\[ T_{xx} = \frac{\tau_0}{2i}(\tau T^{00} + T^{01}) = \frac{\tau_0}{2i} \frac{d\hat{g}_{\mu\nu}}{d\tau} T^{\mu\nu} \quad \text{and} \quad j_x = \frac{1}{2i}(\tau j^0 - j^1). \]  

(8.4)

Using that the energy momentum tensor is conserved and traceless and that the vector and axial-vector currents are conserved it is easy to check that these chiral components only depend on $x$ and not on $\bar{x}$.
8.1 Virasoro and Kac-Moody algebras

First we determine the central charge from the short distance expansion of the $T_{xx}$ correlators. As in the classical theory (see (2.13)) the symmetric energy momentum tensor measures the change of the effective action $\Gamma = - \log Z$ under arbitrary variations of the metric. For the torus there are two independent contributions. One being due to variations of the modular parameter $\tau$ and its conjugate $\bar{\tau}$ which depend implicitly on the metric. The other is due to the variations of terms which depend explicitly on the metric. Since the chiral component $T_{xx}$ is gotten by contracting $T^{\mu\nu}$ with $\delta g_{\mu\nu}/\delta \bar{\tau}$ it follows that

$$\langle T_{xx} \rangle = \frac{i\tau_0}{\sqrt{g(x^\alpha)}} \left( \frac{1}{L^2} \frac{\partial}{\partial \bar{\tau}} + \frac{\delta \mu_\nu}{\delta \gamma_{\mu\nu}(x^\alpha)} \right) \Gamma[g, \tau, \bar{\tau}] \equiv \delta_x \Gamma[g, \tau, \bar{\tau}] \text{.}$$

It is always understood when doing metric variations, that we take the flat spacetime limit afterwards. The $\bar{\tau}$ variation is constant and may be skipped in the short distance expansion.

Taking several metric variations of the curvature dependent part of $\log Z$ with $Z$ from (7.4), (5.3b) and (4.9) we find the following short distance expansions for the three point correlation function

$$\langle T_{uu} T_{vv} T_{zz} \rangle \sim - \frac{3 + 24\pi g_3^2}{(2\pi)^3} \frac{1}{(u-v)^2(u-z)^2(v-z)^2} \text{.}$$

Comparing with the general expression [16] we read off the central charge and the conformal weight of the energy momentum tensor

$$c = 3 + 24g_3^2\pi \quad \text{and} \quad h_{T_{xx}} = 2 \text{.}$$

The first contribution is that of three free fields. The $g_3$–dependent term already appeared in the classical analysis and is related to the coupling to the background curvature. It is well known from the minimal conformal series. Note that the couplings $g_1$ and $g_2$ do not affect the central charge. In particular, if we subtract the central charge of the auxiliary fields and set $g_3=0$ then the value of $c$ is the same as the one for the Thirring model, namely $c=1$ [16].
Next we determine the Kac-Moody algebra of the $U(1)$ currents. To derive the correlation functions with current insertions we couple the fermions to a gauge field, that is consider the ‘gauged’ model without Maxwell term. For example,

$$<j^\mu(x^\alpha) j^\nu(y^\beta)> = \frac{1}{e^2 \sqrt{g(x^\alpha)g(y^\beta)}} \frac{\delta^2 \Gamma[g, A]}{\delta A_\nu(x^\alpha) \delta A_\mu(y^\beta)} |_{A=0}.$$  \hspace{1cm} (8.8)

Using (4.6) on flat spacetime and without Maxwell term, together with

$$\partial_\mu \phi = \eta_\mu^\nu A_\nu^T,$$

where $A_\mu^T = A_\mu - \frac{2\pi}{L} t_\mu - \nabla_\mu \frac{1}{\Delta} \nabla^\nu A_\nu$ \hspace{1cm} (8.9)

is the transversal part of $A_\mu$, one obtains the following short distance expansion

$$\langle j_x j_y \rangle \sim -\frac{1}{2\pi} \frac{1}{2\pi + g^2} \frac{1}{(x-y)^2}.$$ \hspace{1cm} (8.10)

We read off the value $k$ of the central extension in the $U(1)$-Kac-Moody algebra to be

$$k = \frac{2\pi}{2\pi + g^2}.$$ \hspace{1cm} (8.11)

Finally we need to determine the conformal weight of the current. From

$$\langle j_x j_y T_{zz} \rangle \sim -\frac{1}{4\pi^2} \frac{1}{2\pi + g^2} \frac{1}{(x-z)^2(y-z)^2}$$ \hspace{1cm} (8.12)

we obtain $h_j = 1$. To summarize, the symmetry algebra is the semidirect product of a Virasoro algebra with central charge (8.7) and a $U(1)$ Kac-Moody current algebra with central extension (8.11).

### 8.2 Conformal weights

To unravel the possible representations of the Virasoro algebra realized in the model we must determine the conformal weights of the fundamental fields. The short distance expansions of the fermionic two-point function with $T_{zz}$ follows from the metric variation of the Greensfunction

$$\langle \psi_0(x) \psi_1^\dagger(y) \rangle = S_{ij}(x,y) \cdot e^{i g_1 g_3 \sigma(x) + \alpha G_R(x,x) - [x \rightarrow y] - 2\alpha G(x,y)}$$ \hspace{1cm} (8.13a)
where
\[
\alpha = \frac{1}{4} \left( g_1^2 - \frac{2\pi g_2^2}{2\pi + g_2^2} \right). \tag{8.13b}
\]

and \( S_{ij} \) is the fermionic Green function in the external gravitational field and harmonic gauge field but with \( \phi \) and \( \lambda \) set to zero. More precisely,
\[
\langle \psi_0(x) \psi_1^*(y) T_{zz} \rangle = \frac{1}{Z} \delta_z \left( Z \langle \psi_0(x) \psi_1^*(y) \rangle \right).
\]

However, since \( Z \sim \exp[F(R^2)] \), its metric variation vanishes after the flat spacetime limit has been taken. We refer to appendix B for the variation of \( S_{ij} \) and \( G(x,y) \). Collecting the most singular terms, we arrive at
\[
\langle \psi_0(x) \psi_1^*(y) T_{zz} \rangle \sim \frac{1}{2\pi i} \frac{1}{4\pi} \left[ \frac{1}{(z-x) (z-y)} \left( \frac{1}{z-x} - \frac{1}{z-y} \right) - \frac{ig_1 g_3}{x-y} \left( \frac{1}{(z-x)^2} - \frac{1}{(z-y)^2} \right) + \frac{\alpha}{2\pi} \left( \frac{1}{z-x} - \frac{1}{z-y} \right)^2 \right] e^{2\alpha G(x,y)}. \tag{8.14}
\]

Using that
\[
\partial_x e^{2\alpha G(x,y)} = -\partial_y e^{2\alpha G(x,y)} = -\frac{\alpha}{2\pi} \frac{1}{x-y} e^{2\alpha G(x,y)}, \tag{8.15}
\]
we find that the 2-point function varies under a infinitesimal conformal transformation, paramatrized by \( f(z) \), as
\[
\frac{1}{i} \oint dz f(z) \langle \psi_0(x) \psi_1^*(y) T_{zz} \rangle = \left\{ f(x) \partial_x + f(y) \partial_y + \frac{1}{2} \left( 1 + \frac{\alpha}{2\pi} \right) \left[ f'(x) + f'(y) \right] - \frac{ig_1 g_3}{2} \left[ f'(x) - f'(y) \right] \right\} \langle \psi_0(x) \psi_1^*(y) \rangle. \tag{8.16}
\]

Note that the exponential factor has been absorbed to recover the correlation function \( \langle \psi_0(x) \psi_1^*(y) \rangle \). The short distance expansion with \( T_{zzz} \) is calculated similarly. Then one reads off the conformal weights
\[
h_{\psi_0} = \frac{1}{2} + \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi} \frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{ig_1 g_3}{2},
\]
\[
h_{\psi_1^\dagger} = (h_{\psi_0})^\dagger
\]
\[
\bar{h}_{\psi_0} = \frac{1}{16\pi} g_1^2 - \frac{1}{16\pi} \frac{2\pi g_2^2}{2\pi + g_2^2} - \frac{ig_1 g_3}{2}. \tag{8.17}
\]
Thus we have reproduced the classical results supplemented by additional $g_1$ and $g_2$ dependent quantum corrections. In the Thirring model limit $g_3 = 0$ and $g_1 = g_2 = g$, these terms add up to give the known anomalous dimension appearing in the Thirring model [16]. The last classical term is a peculiar feature of the solution. For the conformal weight to be real we are obliged to choose $g_3$ imaginary.

Let us now turn to the auxiliary fields. It is straightforward to compute the correlators

$$
\langle \lambda_x \lambda_y T_{zz} \rangle \sim -\frac{1}{32\pi^2} \frac{1}{(x-z)(y-z)} \tag{8.18}
$$

$$
\langle \phi_x \phi_y T_{zz} \rangle \sim -\frac{1}{16\pi} \frac{1}{(x-z)(y-z)}.
$$

We see that the classical results are unchanged, that is for $g_3 \neq 0$ the scalar field $\lambda$ is not primary and for $g_3 = 0$ we find the conformal weights $h_\lambda = h_\phi = 0$.

Finally we turn to vertex operators or exponentials of the auxiliary fields. In contrast to $\lambda$ and $\phi$ those are well defined even on the extended plane. Recalling the regularization prescription (6.10) we find

$$
\langle :e^{\alpha_1 \phi(x)} :e^{\alpha_2 \phi(y)} :T_{zz} \rangle \sim -\frac{1}{16\pi} \left[ \frac{\alpha_1}{2\pi + g_2^2} \frac{1}{z-x} + \frac{\alpha_2}{z-y} \right] \tag{8.19}
$$

and hence

$$
\frac{1}{i} \int_C f(z) \langle :e^{\alpha_1 \phi(x)} :e^{\alpha_2 \phi(y)} :T_{zz} \rangle \sim \left[ f(x) \partial_x + f(y) \partial_y 
- \frac{1}{8(2\pi + g_2^2)} \left( \alpha_1^2 f'(x) + \alpha_2^2 f'(y) \right) \right] \langle :e^{\alpha_1 \phi(x)} :e^{\alpha_2 \phi(y)} :\rangle \tag{8.20}
$$

From this we read off the conformal weights of the vertex operators

$$
h_i = \tilde{h}_i = -\frac{\alpha_i^2}{8(2\pi + g_2^2)} \tag{8.21}
$$

Note that $\alpha_i$ must be imaginary to get a positive weight. A similar analysis for the $\lambda$-field yields
\[ \frac{1}{i} \int_C f(z) \langle e^{\alpha_1 \lambda(x)} : : e^{\alpha_2 \lambda(y)} : T_{zz} \rangle \sim \left[ f(x)\partial_x + f(y)\partial_y - \frac{\alpha_1}{2} (\frac{\alpha_1}{8\pi} + g_3) f'(x) - \frac{\alpha_2}{2} (\frac{\alpha_2}{8\pi} + g_3) f'(y) \right] \langle e^{\alpha_1 \lambda(x)} : : e^{\alpha_2 \lambda(y)} : \rangle \]

(8.22)

and hence

\[ h_i = -\frac{1}{2} \alpha_i \left( \frac{\alpha_i}{8\pi} + g_3 \right). \]

(8.23)

Here both \( \alpha_i \) and \( g_3 \) must be imaginary for the weights to be positive. Note that contrary to \( \lambda \) the fields \( : e^{\alpha \lambda} : \) remain primary when the \( \lambda R \) coupling is switched on. This coupling results only in a shift of the conformal weights.

### 8.3 \( U(1) \)-charges

To see how the left and right Kac Moody currents act on the fermionic fields we notice that after the integration over the auxiliary fields the A-dependence of the fermionic Greenfunction factorizes as

\[ \langle \psi_0(x) \psi_1(y) \rangle_A = e^{\frac{i}{4} m_+ \int \phi \Delta \phi} \cdot e^{-eg(x)} \langle \psi_0(x) \psi_1(y) \rangle_{A=0} e^{-e\bar{g}(y)}, \]

where

\[ g(x) = -i\alpha(x) + \gamma_5\beta \varphi(x), \quad \beta = \frac{2\pi}{2\pi + g_2^2}. \]

Also, using that on flat spacetime

\[ \phi(x) = -i \int \partial_z G(x, z) A^z + i \int \partial_{\bar{z}} G(x, z) A^{\bar{z}}, \]

\[ \alpha(x) = \int \partial_z G(x, z) A^z + \int \partial_{\bar{z}} G(x, z) A^{\bar{z}}, \]

(8.24)

one ends up with

\[ \langle \psi_0(x) \psi_1(y) \rangle_{j_z} = \frac{1}{4\pi i} \left[ \frac{4\pi + g_2^2}{2\pi + g_2^2} \frac{1}{z-x} + \frac{g_2^2}{2\pi + g_2^2} \frac{1}{z-y} \right] \langle \psi_0(x) \psi_1(y) \rangle \]

\[ \langle \psi_0(x) \psi_1(y) \rangle_{j_{\bar{z}}} = \frac{1}{4\pi i} \left[ \frac{g_2^2}{2\pi + g_2^2} \frac{1}{z-x} + \frac{4\pi + g_2^2}{2\pi + g_2^2} \frac{1}{z-y} \right] \langle \psi_0(x) \psi_1(y) \rangle \]

(8.25)

and thus obtains the following the \( U(1) \) charges

\[ q_{\psi_0} = \frac{1}{2} \left( 1 + \frac{2\pi}{2\pi + g_2^2} \right), \quad \bar{q}_{\psi_0} = \frac{1}{2} \left( 1 - \frac{2\pi}{2\pi + g_2^2} \right). \]

(8.26)
We have used the convention where the electric charge $q + \bar{q}$ is unity. In the Thirring model limit we can compare (8.26) with the results obtained in [16]. For that we need to rescale the currents such that the central extension (8.11) of the Kac-Moody algebra becomes unity

$$j_z \to \sqrt{1 + g_z^2/2\pi} j_z.$$  \hfill (8.27)

Now it is easy to see that we agree with [16] if we make the identification

$$\bar{g}_F u = \frac{g_z^2}{4\pi} \frac{1}{\sqrt{1 + g_z^2/2\pi}}.$$  \hfill (8.28)

Let us summarize our results. The classical conformal and axial transformations of all fields besides $\phi$ and $\lambda$ are deformed. The longitudinal part of the current-current interaction in (1.1) changes the conformal weights of the fermion field only. The transversal part affects all weights and $U(1)$-charges. The background charge changes the conformal weight of the vertex operators belonging to the scalar field.

Of course, the same structure is found in the other chiral sector.

### 8.4 Finite size effects

When quantizing a conformal field theory on a spacetime with finite volume one introduces a length scale. The presence of this length scale in turn breaks the conformal invariance and gives rise to finite size effects. It has been conjectured [13] that the finite size effects are proportional to the central charge. For example when one stretches space time, $x^\alpha \to ax^\alpha$, then the change of the effective action is proportional to $c$:

$$\Gamma_{ax} - \Gamma_x = -\frac{c}{6} \log a \cdot \chi,$$  \hfill (8.29)

where $\chi$ is the Euler number of the euclidean space time. In [48] this conjecture has been proven for a class of conformal field theories on spaces with boundaries. The only important assumption has been that the regularization respects general covariance. In this subsection we shall show that the conjecture does not hold for the model (1.1) on Riemannian surfaces.
Unfortunately, the only global conformal transformations on the torus are translations which do not give rise to finite size effects. Also, the Euler number vanishes and according to (8.29) the finite size effects are insensitive to the value of $c$. For that reason we quantize the ungauged model (1.1) on the sphere where the global conformal group is the Moebius group.

An effective method to compute finite size effects has been developed in [48]. It is based on the following observation: Any conformal transformation $z \rightarrow w(z)$ is a composition of a diffeomorphism (defined by the same $w$) and a compensating Weyl transformation $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$ with

$$e^{2\sigma} = \frac{dw(z)}{dz} \frac{d\bar{w}(\bar{z})}{d\bar{z}}, \quad z = x^0 + ix^1. \quad (8.30)$$

Therefore, choosing a diffeomorphism invariant regularization one has

$$0 = \delta\Gamma_{Diff} = \delta\Gamma_{Conf} - \delta\Gamma_{Weyl}. \quad (8.31)$$

Now we apply the techniques of the previous sections to derive the change $\delta\Gamma_{Weyl}$ of the effective action on the sphere under Weyl transformations. This change is given by the trace anomaly.

The change of the effective action under Weyl rescalings is

$$\delta\Gamma_{Weyl} = - \log \frac{\int D(\lambda\phi) \det(i\not{D}) \exp(-S_B[A = 0, g])}{\int D(\lambda\phi) \det(i\not{D}) \exp(-S_B[A = 0, \hat{g}])}, \quad (8.32)$$

where $S_B$ is the bosonic action (3.35) with vanishing gauge field. Also, since on the sphere there are no harmonic vector fields the term $\sim h^2$ in $S_B$ is not present. Thus the calculation on the sphere is actually simpler as on the torus (see 7.1) since there is no integration over the harmonic fields. As on the torus we must impose the conditions (3.38) in order to eliminate the additional degrees of freedom we introduced in the derivative coupling representation. Thus we obtain

$$\delta\Gamma_{Weyl} = \log \frac{\hat{V}}{V} - \frac{S_L}{24\pi} + \frac{g_3^2}{4} \int R \frac{1}{\triangle} R + \log \frac{\det'\triangle}{\det'\hat{\triangle}}. \quad (8.33)$$
Here we used that (3.30a) in the trivial sector still holds on the sphere. Also we used the scaling law (4.4b). $S_L$ is the Liouville action (3.30b) in which we can not put $\hat{\mathcal{R}}$ to zero, since

$$\int \sqrt{g} \mathcal{R} = 8\pi = 4\pi \chi$$

(8.34)

for any curvature and thus in particular for $\hat{\mathcal{R}}$. As for the fermions (see 3.26) one introduces the 1-parametric family of Laplacians

$$\Delta_\tau = e^{-2\tau \sigma} \hat{\Delta}$$

(8.35)

interpolating between $\hat{\Delta}$ and $\Delta$. The $\tau$ derivative of the corresponding determinant is given by the trace anomaly [32,48]. The explicit calculation yields

$$\log \det' \frac{\Delta_\tau}{\hat{\Delta}} = 2 \int_0^1 d\tau \int \sqrt{g^\tau} \left( -\frac{1}{4\pi} a^\tau_1 - P^\tau \right) \sigma,$$

(8.36)

Again $g^\tau$ is the determinant and $a^\tau_1 = \frac{1}{6} \mathcal{R}^\tau$ the relevant Seeley-deWitt coefficient of the deformed metric $g^\tau_{\mu\nu} = e^{2\tau \sigma} \hat{g}_{\mu\nu}$. $P^\tau$ is the projection onto the zero-mode of $\Delta^\tau$. Using that the normalized zero-mode is constant and $\sim 1/\sqrt{V^\tau}$, one finds

$$\log \det' \frac{\Delta_\tau}{\hat{\Delta}} = \log \frac{V}{V} + \frac{1}{12\pi} S_L.$$

(8.37)

The $\sim \log V$ term cancels against the same term in (8.33) and we end up with

$$\delta \Gamma = \frac{g_5^2}{4} \int \sqrt{g} \mathcal{R} \frac{1}{\Delta} (\mathcal{R} - \frac{8\pi}{V}) - \frac{3}{24\pi} \int \sqrt{\hat{g}} \hat{\mathcal{R}} \sigma + \frac{3}{24\pi} \int \sqrt{\hat{g}} \sigma \hat{\Delta} \sigma.$$  

(8.38)

Now we can see why the finite size conjecture generally fails to be true, although it holds for theories without background charge on domains with boundaries [48]. Take the simple case of a dilatation $w(z) = az$. Then, the conformal angle is a constant $\sigma = \log a$ and $(\mathcal{R} - 8\pi/V) = 0$. Thus the first term in (8.38) vanishes and the finite size effect does not depend on $g_5^2$. It is given by

$$\delta \Gamma = -\frac{3}{24\pi} \log a \int \sqrt{\hat{g}} \hat{\mathcal{R}} = -\log a$$  

(8.39)
and does not agrees with (8.29) since $c$ in (8.7) depends on $g_3$. Thus we have disproved the conjecture. On other Riemannian surfaces one would find the same result: the effective action scales as in (8.7) where $c$ is the central charge of the model without background charge. It is evident that the finite size scaling comes from the middle term $\sim \log a \int \sqrt{g} \hat{R}$ in (8.38).

It is interesting to compare the finite size scaling on Riemannian surfaces with the one on domains with boundaries. In the presence of boundaries (8.36) is modified to

$$\log \det \frac{\triangle}{\hat{\triangle}} = -\frac{1}{2\pi} \int_0^1 d\tau \left( \int \sqrt{g} a_1^\tau \sigma + \oint \sqrt{\tilde{g}} b_1^\tau \sigma \right), \quad (8.40)$$

where the second integral is over the boundary of spacetime and $\tilde{g}_{\mu\nu}$ the induced metric on this boundary. On a domain we can always put $\hat{R}$ to zero and the middle term in (8.40) does not contribute to the scaling. The scaling comes from the surface term in (8.40). Diffeomorphism invariance implies that the bulk term determines the surface term (up to diffeomorphism invariant surface terms). This is how the central charge, defined by the short distance expansion of the $T_{zz}$-correlators and thus by the bulk term, reemerges in the scaling law (8.7), which is determined by the surface term.
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Appendix A: Conventions

In this appendix we set up our notation and give a list of useful formulae. Let \( g_{\mu\nu} \) be the metric of spacetime. The sign convention for the curvature tensors is such that

\[
R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\delta\gamma,\beta} - \Gamma^\alpha_{\delta\beta,\gamma} + \Gamma^\alpha_{\gamma\beta,\delta} - \Gamma^\sigma_{\gamma\beta} \Gamma^\alpha_{\delta\sigma} \quad \text{and} \quad R_{\beta\delta} = R^\alpha_{\beta\alpha\delta}. \quad (A.1)
\]

In 2 dimensions the only independent component is \( R_{0101} \). In order to couple fermions to gravity we must introduce a local Lorentz frame (or tetrad), \( e_{\mu a} \), relating the Lorentz and spacetime indices:

\[
e_{\alpha a} e_{\beta b} = g_{\alpha\beta}, \quad e_{\alpha a} e^{a\alpha} = \eta_{ab}, \quad \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.2)
\]

The latin and greek indices are Lorentz and spacetime indices, respectively. All physical laws should be general- and Lorentz covariant. If \( g_{\alpha\beta} \) has Euclidean signature then \( \eta_{ab} \) in (A.2) is changed to \( \delta_{ab} \).

The ‘curved’ gamma matrices are related to the flat ones as

\[
\gamma^\mu = e^\mu_a \tilde{\gamma}^a. \quad (A.3)
\]

We use the following chiral representation for the flat \( \gamma \)'s:

\[
\hat{\gamma}^0_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\gamma}^1_M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (A.4)
\]

and in Euclidean spacetime we may choose
\[ \hat{\gamma}_E^0 = \hat{\gamma}_M^0, \quad \hat{\gamma}_E^1 = i \hat{\gamma}_M^1. \]  

We may also define

\[ \tilde{\gamma}_5 = \hat{\gamma}_M^0 \hat{\gamma}_M^1 = -i \hat{\gamma}_E^0 \hat{\gamma}_E^1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]  

\[(A.5)\]

The relations

\[ \tilde{\gamma}_a^a \tilde{\gamma}_5 = \epsilon_{ab} \tilde{\gamma}_b^b, \quad \tilde{\gamma}_E^a \tilde{\gamma}_5 = -i \epsilon_{ab} \tilde{\gamma}_b^b, \quad \text{where} \quad \epsilon_{ab} = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right). \]  

\[(A.6a)\]

are particular to 2 dimensions and play an important role in the body of the paper. Note that depending whether one is in Minkowskian or Euclidian spacetime the Lorentzindex \( a \) is raised with \( \eta^{ab} \) or \( \delta^{ab} \). The curved space analogue of \((A.6a)\) reads

\[ \gamma_5 = \frac{1}{2} \eta_{\mu\nu} \gamma_M^\mu \gamma_M^\nu = \frac{1}{2i} \eta_{\mu\nu} \gamma_E^\mu \gamma_E^\nu = \tilde{\gamma}_5, \]  

\[(A.7)\]

where \( \eta_{\mu\nu} = \sqrt{|g|} \epsilon_{\mu\nu} \) is the antisymmetric tensor (whereas the flat metric has Lorentz- indices, the antisymmetric tensor has space-time indices). To implement local Lorentz invariance one needs to introduce a connection \( \omega_{\mu ab} \).

For example, in the Lagrangean the Lorentz-covariant derivative acting on the spinors read

\[ D_\mu = \partial_\mu + i \omega_\mu, \]  

\[(A.8)\]

where the spin connection \( \omega_\mu \) is defined by

\[ D_\mu \equiv \partial_\mu e_a^\mu - \Gamma_{\mu\nu}^\lambda e_\lambda^a + \omega_{\mu ab} e_b^\mu = 0, \quad \omega_\mu = \frac{1}{2} \omega_{\mu ab} \Sigma^{ab}, \quad \Sigma^{ab} = \frac{1}{4i} [\tilde{\gamma}_a, \tilde{\gamma}_b]. \]  

\[(A.9)\]

In 2 dimensions this reduces to

\[ \omega_\mu^M = \frac{1}{2i} \omega_{\mu 01} \gamma_5 \quad \text{or} \quad \omega_\mu^E = \frac{1}{2} \omega_{\mu 01} \gamma_5. \]  

\[(A.10)\]

Finally we list some useful scaling relations. If the 2-bein scales as \( e_\mu^a = e^\sigma \hat{e}_\mu^a \) then the above introduced quantities scale as
$g_{\mu \nu} = e^{2\sigma} \hat{g}_{\mu \nu}$, $\sqrt{g} = e^{2\sigma} \sqrt{\hat{g}}$, $\mathcal{R} = e^{-2\sigma} (\hat{R} - 2\hat{\Delta}\sigma)$

\begin{equation}
\omega_{\mu ab} = \hat{\omega}_{\mu ab} - \partial_a \sigma \hat{e}_{\mu b} + \partial_b \sigma \hat{e}_{\mu a},
\end{equation}

\begin{equation}
\Gamma_{\mu \nu}^{\alpha} = \hat{\Gamma}_{\mu \nu}^{\alpha} + \left( \partial_\mu \sigma \delta_\nu^{\alpha} + \partial_\nu \sigma \delta_\mu^{\alpha} - \partial_\beta \sigma \delta_\mu^{\alpha} \hat{g}_{\beta \mu} \right),
\end{equation}

\begin{equation}
\triangle = e^{-2\sigma} \hat{\Delta}, \quad \phi + i\psi = e^{-\frac{\chi}{2}\sigma}(\hat{\phi} + i\hat{\psi})e^{\frac{\chi}{2}\sigma}.
\end{equation}

**Appendix B: Variational formulae**

In the following $D_\mu$ denotes the spacetime and Lorentz covariant derivative. How it acts on spacetime and Lorentz tensors follows from the first formula in (A.9).

Using the definition of the Christoffel symbol and (A.2) it is straightforward to show that

\begin{equation}
\delta g_{\mu \nu} = \delta e_\mu^{\alpha} e_{\nu \alpha} + e_\mu^{\alpha} \delta e_{\nu \alpha}, \quad \delta \sqrt{g} = \frac{1}{2} \sqrt{\hat{g}} \delta g^{\mu \nu} \delta g_{\mu \nu}
\end{equation}

\begin{equation}
\delta \gamma^{\mu} = -\gamma^{\nu} e^{\mu}_a \delta e_{\nu \alpha}, \quad \delta \eta_\mu^{\nu} = \frac{1}{2} (\eta^{\alpha \nu} \delta g_{\mu \alpha} - \eta^{\mu \sigma} g^{\nu \rho} \delta g_{\sigma \rho})
\end{equation}

\begin{equation}
\delta \Gamma_{\mu \nu}^{\alpha} = \frac{1}{2} g^{\alpha \beta} (D_\nu \delta g_{\beta \mu} + D_\mu \delta g_{\beta \nu} - D_\beta \delta g_{\mu \nu}).
\end{equation}

For some formulae related to the variation of the tetrad let us refer to [49]

\begin{equation}
\delta e_\mu^a = \frac{1}{2} e_{\nu a} \delta g^{\mu \nu} - t_\alpha^b e_\mu^b, \quad \delta e_\mu^a = \frac{1}{2} e^{\nu a} \delta g_{\mu \nu} - t_\beta^b \epsilon_\mu^b
\end{equation}

where $t_\alpha^a = \frac{1}{2} (e^{\nu a} \delta e_{\nu b} - e^{\nu b} \delta e_{\nu a})$. Then using (A.9) it is easy to see that

\begin{equation}
\delta \omega_{\mu ab} = D_\mu t_{ab} - \alpha_{\mu ab} \quad \alpha_{\mu ab} = \frac{1}{2} e_\alpha^a e_\beta^b (D_\alpha \delta g_{\beta \mu} - D_\beta \delta g_{\alpha \mu}).
\end{equation}

When performing the variation of curvature dependent expressions we have used the identities

\begin{equation}
\delta g^{\mu \nu} \delta \mathcal{R}_{\mu \nu} = \omega_\alpha^{\mu \nu}, \quad \text{where} \quad \omega_\alpha^{\mu \nu} = g^{\mu \nu} \delta \Gamma_{\mu \nu}^{\alpha} - g^{\alpha \nu} \delta \Gamma_{\mu \nu}^{\mu}
\end{equation}

\begin{equation}
\text{and} \quad \int \sqrt{g} \omega_\alpha A_\alpha = \int \sqrt{\hat{g}} \{g^{\alpha \beta} \nabla_\mu A_\mu - \nabla_\alpha A^\beta\} \delta g_{\alpha \beta}.
\end{equation}
Depending on the topology of spacetime, the induced curvature $\hat{R}$ appearing in (A.11) may be different from zero. In this case it is not possible to express the conformal angle $\sigma$ in terms of the curvature scalar. Nevertheless, to perform variations of $\sigma$-dependent expressions, the identity

$$\delta(\sqrt{g}R) = -2\delta(\sqrt{g}\triangle\sigma) \quad (B.5)$$

proves to be useful.

Taking the variations of the equations

$$\sqrt{g}\Delta G(x, y) = -\delta(x - y) \quad \text{and} \quad \sqrt{g}i\partial S(x, y) = \delta^2(x - y) \quad (B.6)$$

for the scalar and fermionic Greensfunctions we may derive (up to contact terms) the following variational formulae

$$\delta G = \int \left( -\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} + g^{\alpha\mu}g^{\beta\nu} \right) \partial_\alpha G(x, u) \partial_\beta G(u, y) \sqrt{g}\delta g_{\mu\nu} \quad \delta S = \frac{i}{4} \int \left( 2S(x, u)\gamma^\mu D^\nu S(u, y) - D_\alpha[S(x, u)\gamma^\mu \eta_\delta^\nu \delta^\alpha S(u, y)] \right) \sqrt{g}\delta g_{\mu\nu}, \quad (B.9)$$

where all arguments and derivatives which are not made explicit in the integral refer to the coordinate $u$ over which is integrated. Finally, we need the following formula for the variation of the inverse Laplacian

$$\delta \left( \frac{1}{\triangle} f \right) = \frac{1}{\triangle} \left( \delta f - \delta(\triangle) \frac{1}{\triangle} f \right) - \frac{1}{2V} \int \sqrt{g}\delta g^{\mu\nu}\delta g_{\mu\nu} \frac{1}{\triangle} f, \quad (B.10)$$

where $V$ is the volume of spacetime and $f$ an arbitrary function. To prove this identity we note that for $f \in (\text{Kern}\triangle)^\perp$ we have

$$\triangle \frac{1}{\triangle} f = f.$$ 

Varying this equation yields

$$\triangle(\delta \frac{1}{\triangle} f) = \delta f - (\delta\triangle) \frac{1}{\triangle} f$$

which may be inverted to give

$$\delta \left( \frac{1}{\triangle} f \right) = \frac{1}{\triangle} \left( \delta f - \delta(\triangle) \frac{1}{\triangle} f \right) + \frac{1}{V} \int \sqrt{g}\delta \left( \frac{1}{\triangle} f \right). \quad (B.11)$$
Varying the identity
\[ \frac{1}{V} \int \sqrt{g} \frac{1}{\Delta} f = 0 \]
allows to replace the last term of (B.11) to obtain the required result (B.10).

**Appendix C: Canonical approach to the partition function**

In this appendix we compute the partition function for massive Dirac fermions in the canonical formalism. In the limit \( m \to 0 \) we confirm some of the results in sections 3 and 7. For massive fermions one cannot consistently impose chirally twisted boundary conditions. However, from the explicit eigenvalues (3.17) one sees at once that the chiral twist \( \beta_1 \) and the chemical potential are equivalent. One can easily verify that this equivalence holds also for massless fermions in the canonical approach and that \( \beta_1 \sim \mu L/2\pi \). Let us therefore compute the partition function

\[ Z(\beta) = Tr[e^{-\beta(H-\mu Q)}] \quad (C.1) \]

for massive Dirac fermions with chemical potential \( \mu \) on a cylinder with (nonchiral) twisted boundary conditions

\[ \psi(x + L, t) = e^{-2i\pi \alpha_1} \psi(x, t). \quad (C.2) \]

For massive particles it is more convenient to use the Dirac representation

\[ \gamma^0 = \sigma_3, \quad \gamma^1 = -i\sigma_2, \quad \gamma^5 = \gamma^0\gamma^1 = -\sigma_1. \quad (C.3) \]

The Dirac field is expanded in terms of the eigenmodes of the first quantized Hamiltonian

\[ h = \begin{pmatrix} m & i\partial_x \\ i\partial_x & -m \end{pmatrix} \quad (C.4) \]

as

\[ \Psi(x, t) = \sum_n \psi_{n,\uparrow} b_n + \sum_n \psi_{n,\downarrow} d_n^*, \quad (C.5a) \]
where the $\psi_{n,+}$ and $\psi_{n,-}$ are the positive and negative energy modes,

$$\psi_{n,+} = e^{-i\omega_n t - i\lambda_n x} c_n, \quad \psi_{n,-} = e^{i\omega_n t - i\lambda_n x} \gamma_1 c_n,$$

$$c_n = \frac{1}{\sqrt{2\omega_n(\omega_n + m)L}} \begin{pmatrix} \omega_n + m \\ \lambda_n \end{pmatrix}. \quad (C.5b)$$

The momenta $\lambda_n$ and energies $\omega_n$ are determined by the boundary condition (C.2) to be

$$\lambda_n = \frac{2\pi}{L} \left( n - \frac{1}{2} - \alpha_1 \right) \quad \text{and} \quad \omega_n = \sqrt{m^2 + \lambda_n^2}. \quad (C.5c)$$

After normal ordering the 'positron' operators with respect to the Fock vacuum defined by $H$ we find

$$(H - \mu Q) = \sum_n (\omega_n - \mu) b_n^\dagger b_n + \sum_n (\omega_n + \mu) d_n^\dagger d_n - \sum_n (\omega_n + \mu), \quad (C.6)$$

where the last $c$-number term represents the infinite vacuum contribution which must be regularized. To do that we employ the zeta function regularization. That is we define the zeta-function for $s > 1$ by the sum

$$\zeta(s) = \sum_n (\omega_n + \mu)^{-s}$$

which in turn defines an analytic function on the whole complex $s$-plane up to a simple pole at $s = 1$. The analytic continuation is constructed by a Poisson resummation

$$\sum_n (\omega_n + \mu)^{-s} = \frac{L^s}{2\pi} \sum_n F(n) \quad (C.7a)$$

where

$$F(\xi) = e^{2\pi i \xi (\frac{1}{2} - \alpha_1)} \int dy e^{i\xi y} \left[ \sqrt{\tilde{m}^2 + y^2} + \tilde{\mu} \right]^{-s}, \quad (C.7b)$$

and $\tilde{m} = Lm$, $\tilde{\mu} = L\mu$. Taking the Mellin transform of (C.7b) we find

$$F(\xi) = e^{2\pi i \xi (\frac{1}{2} - \alpha_1)} \frac{1}{\Gamma(s)} \int dt \int dt^{-1} e^{-t\sqrt{\tilde{m}^2 + y^2} - t\tilde{\mu}}$$

$$= -\frac{2}{\Gamma(s)} e^{2\pi i \xi (\frac{1}{2} - \alpha_1)} \int dt \int dt^{-1} e^{-t\tilde{\mu}} \frac{d}{dt} K_0(\tilde{\mu} \sqrt{\xi^2 + t^2}) \quad (C.7c)$$

$$= \frac{2\tilde{m}}{\Gamma(s)} e^{2\pi i \xi (\frac{1}{2} - \alpha_1)} \int dt \int t^s e^{-t\tilde{\mu}} \frac{K_1(\tilde{\mu} \sqrt{\xi^2 + t^2})}{\sqrt{\xi^2 + t^2}}.$$
F diverges at 0 since the Kelvin function $K_1(z) \sim 1/z$ for small $z$. It follows that the $n = 0$ term in (C.7a) diverges. This divergence is regularized by subtracting the ground state energy of the infinite volume system. Indeed, because of the exponential decay of the Bessel function for large arguments, only the $n = 0$ term contributes for infinite volume. So we find for the regularized sum

$$
\sum_n (\omega_n + \mu)^{-s} = \frac{\tilde{m} s L}{\Gamma(s) \pi} \sum_{n \neq 0} \int dt \ e^{2\pi i n (\frac{1}{2} - \alpha_1)} t^s e^{-t \tilde{\mu}} K_1(n \sqrt{n^2 + t^2}) \frac{1}{\sqrt{n^2 + t^2}}. \quad (C.8)
$$

Noe we perform the limit $m \to 0$. Only the most singular term in the expansion of the Bessel function contributes, hence

$$
\sum_n (\omega_n + \mu)^{-s} = \frac{L^s}{\Gamma(s) \pi} \sum_{n \neq 0} \int dt \ e^{2\pi i n (\frac{1}{2} - \alpha_1)} t^s e^{-t \tilde{\mu}} \frac{1}{(n^2 + t^2)} = -s \frac{L^s}{\pi} \sum_{n \neq 0} e^{2\pi i n (\frac{1}{2} - \alpha_1)} \sqrt{\tilde{\mu}} n^{s - \frac{1}{2}} S_{-s - \frac{1}{2}}(\tilde{\mu} n), \quad (C.9)
$$

where $S_{a;b}(z)$ is the Lommel function [50]. In particular for $s = -1$ this function is $S = 1/z$ so that finally

$$
\sum_n (\omega_n + \mu)^{reg} = -\frac{1}{\pi L} \sum_{n \neq 0} \frac{(-)^n}{n^2} e^{-2\pi i n \alpha_1} = \frac{\pi}{6L} - \frac{2\pi}{L} (\alpha_1 - [\alpha_1 + \frac{1}{2}])^2. \quad (C.10)
$$

Inserting this into (C.6) then yields the regularized expression

$$
: H - \mu Q : = \sum_n (\omega_n - \mu) b_n^\dagger b_n + \sum_n (\omega_n + \mu) b_n^\dagger d_n - \frac{\pi}{6L} + \frac{2\pi}{L} (\alpha_1 - [\alpha_1 + \frac{1}{2}])^2.
$$

For small $\mu$ the normal ordering is $\mu$-independent so that

$$
\langle 0 | : H - \mu Q : | 0 \rangle = -\frac{\pi}{6L} + \frac{2\pi}{L} (\alpha_1 - [\alpha_1 + \frac{1}{2}])^2 = \langle 0 : H : 0 \rangle. \quad (C.12)
$$

is independent of $\mu$ and coincides with the Casimir energy [46].

Finally we compute the partition function. Using (C.12) we easily find
\[ Z(\beta) = \text{tr} \left[ e^{-\beta(H - \mu Q)} \right] = q^{\left[ \alpha_1^2 - \frac{1}{4} \right]} \]
\[ = \prod_{n > \left\lceil \frac{1}{2} + \alpha_1 \right\rceil} (1 + q^{(n - \frac{1}{2} + \alpha_1)} e^{\beta\mu}) \prod_{n > -\left\lfloor \frac{1}{2} - \alpha_1 \right\rfloor} (1 + q^{(n - \frac{1}{2} + \alpha_1)} e^{-\beta\mu}) \]
\[ = \frac{1}{|\eta(\tau)|^2} \Theta \left[ -\alpha_1 \right] (0, \tau) \Theta \left[ -\alpha_1 - i\mu \frac{\beta}{2\pi} \right] (0, \tau), \]

where we have used the product representation of the theta functions in the last identity and that \( q = e^{2\pi i \tau} = e^{-2\pi \beta/L} \). A non-vanishing chiral twist \( \beta_1 \) can now be included by shifting the chemical potential. Thus we have confirmed the formula (3.19) in our paper.

Note that for \( \mu \neq 0 \) the zero-temperature limit of the grand potential is not equal to the vacuum expectation value of \( H - \mu Q \). For \( \mu \neq 0 \) all states up to the \( \mu \)-dependent fermi energy are filled. For example, for \( \omega_1 < \mu < \omega_2 \) the limit \( \lim_{\beta \to \infty} \Omega \) reduces to the expectation value of \( H - \mu Q \) in the one-electron state.

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