Swapping and entangling qubits of single photons and atoms

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Abstract

A scheme is proposed here to achieve swapping and entangling of photonic and atomic qubits with high fidelity. The mechanism is based on the scattering of a single photon from a Λ-type three-level atom. The evolution of the coupled system is analyzed by projecting the quantum state onto a ‘bright’ and a ‘dark’ state. Quantum interference of these two states, which is determined by a frequency-dependent phase angle, can be exploited to perform various two-qubit transformations. It is remarkable that the probability of success of such transformations can approach unity in the strong coupling cavity QED regime.

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Quantum communication relies on the ability of transmitting and retrieving quantum information at specified locations. Particularly for a quantum network, communication between spatially separated nodes requires inter-conversions between flying qubits and stationary qubits with high fidelity. Therefore it is important to investigate mechanisms that can accomplish the task efficiently. Single photons and long-lived trapped atoms are considered as fundamental hardware in distributed quantum computing, with the former serving ideally as data bus, while the latter playing the role of local quantum memory [1]. Indeed, various authors have proposed protocols based on strong atom-field coupling in high-$Q$ cavities [2, 3]. These investigations suggest that a complete transfer of a qubit from an atom to a quantized electromagnetic field can in principle be feasible. Recently, there are also studies addressing methods of quantum communication using ensembles of atoms via their interactions with light in free space [4, 5].

In addition to the task of quantum state transfer, it is often desirable to exchange quantum information among carriers of distinct nature [6]. A basic operation of this kind is defined by the two-qubit transformation:

\[
(\alpha_1 |1\rangle_A + \alpha_0 |0\rangle_A) \otimes (c_1 |1\rangle_F + c_0 |0\rangle_F) \\
\rightarrow (c_1 |1\rangle_A + c_0 |0\rangle_A) \otimes (\alpha_1 |1\rangle_F + \alpha_0 |0\rangle_F),
\]

where $\alpha_1 |1\rangle_A + \alpha_0 |0\rangle_A$ represents an atomic qubit and $c_1 |1\rangle_F + c_0 |0\rangle_F$ denotes a photonic qubit. Through the transformation (1), the atomic qubit is deposited to the photon, and at the same time the photonic qubit is mapped onto the atom. Such a swapping process is more general than the typical one-way quantum state transfer previously discussed in cavity QED systems, because the latter corresponds to the special cases with either $c_1 = 0$ (atom to photon) or $\alpha_1 = 0$ (photon to atom) [2, 3]. To our knowledge, physical examples of swapping effects have only been investigated in atomic ensembles with collective spin variables [6] and NMR systems [7], a generic protocol of qubit swapping among single photons and atoms in leaky optical cavities has not yet been found.

In this Letter we present a mechanism to achieve qubit swapping based on scattering of a single photon by an atom in a high-$Q$ cavity. The two ground states of a trapped $\Lambda$-type three-level atom play the role of the two states of a stationary qubit, and the two polarizations of a single photon constitute a flying qubit. Quantum interference between the ‘bright’ state and the ‘dark’ state of the $\Lambda$-system holds the key to realization of qubit
swapping. We derive explicitly an exact analytic form of the scattering matrix, and determine a frequency-dependent phase shift of the bright state. Such a phase shift controls the quantum interference, and can be large in the strong cavity coupling regime. Therefore the frequency degree of freedom of photon would enable various kinds of two-qubit transformations. In addition to the swapping operation (1), we will also show how a maximally entangled photon-atom pair can be generated at suitable input photon frequencies.

To begin with, we consider a one-dimensional cavity with length $l$ and bounded by two mirrors. The left mirror at $x = 0$ is perfectly reflecting, while the other at $x = l$ is partially transparent. The normal modes of the electromagnetic field are characterized by a continuous wave number $k$. Specifically, the spatial mode functions $u_k(x)$ are given by [8]

$$u_k(x) = \begin{cases} 
  I(k) \sin kx & 0 < x < l \\
  e^{-ikx} + R(k)e^{ikx} & l < x < \infty
\end{cases},$$

(2)

where $I(k) = -2it/(1 + re^{2ikl})$ and $R(k) = (-r - t + re^{-2ikl})/(1 + re^{2ikl})$, with $r$ and $t$ being the reflection and transmission coefficients of the right mirror, respectively [9]. These continuous field modes provide a basis for the field quantization. A Λ-type three-level atom is located near the center of the cavity. The atom has two degenerate ground states $|L\rangle$, $|R\rangle$ and an excited state $|e\rangle$. The ground state $|L\rangle$ ($|R\rangle$) can be excited to $|e\rangle$ by absorbing a $|k_L\rangle$ ($|k_R\rangle$) mode photon as shown in Fig. 1, where the subscripts $L$, $R$ denote the polarization of the photon. In our scheme, there is one and only one photon involved in the scattering process, and no external classical pump fields are required. To facilitate our discussion, we first neglect the spontaneous emission from the atomic excited state into side modes of the cavity. The loss due to spontaneous decay will be discussed later in the paper.

The Hamiltonian of our model (in units of $\hbar = c = 1$) is given by

$$\hat{H} = \omega_e |e\rangle\langle e| + \int_0^\infty dk \sum_{\mu=L,R} k \hat{a}^{\dagger}_{k\mu} \hat{a}_{k\mu} + \int_0^\infty dk \sum_{\mu=L,R} g_{\mu}(k) \hat{a}_{k\mu} |e\rangle\langle \mu| + \text{h.c.}.$$  

(3)

Here, $\hat{a}_{k\mu}$ and $\hat{a}^{\dagger}_{k\mu}$ are the annihilation and creation operators associated with the field mode $u_k(x)$ with the polarization $\mu$ ($\mu = L, R$). The dipole coupling strength $g_{\mu}(k)$ is given by

$$g_{\mu}(k) = \frac{\lambda_{\mu} \sqrt{\kappa/\pi e^{i\theta_{\mu}}}}{k - k_c + ik},$$

(4)
which is proportional to the mode strength at the location of the atom, i.e., \(u_k(x = l/2)\). The \(\kappa = -\ln |r|/2l\) is the leakage rate of the cavity, \(\lambda^2_k = \int_{-\infty}^{\infty} |g_\mu(k)|^2 dk\), and \(\theta_\mu\) is a phase angle associated with dipole transition matrix elements. In writing Eq. (4), we have assumed that only one of the cavity quasi-modes (with a resonance frequency \(k_c\) defined by the real part of the pole of \(R(k)\)) interacts with the atom. Such an approximation is valid when \(\omega_e \approx k_c\), and all other quasi-modes are far off resonance. This requires a small cavity with high finesse in typical cavity QED experiments [10]. Note also that in the optical regime, the approximation allows us to extend the lower bound of the frequency of the photon from 0 to \(-\infty\).

Initially, the atom is prepared in a coherent superposition of the two ground states and a single photon wave packet of arbitrary spectrum and polarizations is injected into the cavity. Our task is to determine the analytic solution of the final state of the system. First, we notice that the scattering process is energy conserving. Hence for a given \(k\), the initial and final states dwell in the same \(k\)-subspace spanned by the four basis vectors: \(|L; k_L\rangle\), \(|L; k_R\rangle\), \(|R; k_L\rangle\), and \(|R; k_R\rangle\). It is obvious that \(|L; k_R\rangle\) and \(|R; k_L\rangle\) are eigenvectors of \(\hat{H}\). In addition, the coherent superposition \(g_R(k)|L; k_L\rangle - g_L(k)|R; k_R\rangle\) is a dark state well known in \(\Lambda\)-systems. The excited state \(|e\rangle\) couples only with the ‘bright’ state

\[
|\psi(k)\rangle \equiv \frac{1}{V(k)} [g_L^*(k)|L; k_L\rangle + g_R^*(k)|R; k_R\rangle].
\]  

where \(V(k) = \sqrt{|g_L(k)|^2 + |g_R(k)|^2}\). Therefore we only need to solve the evolution of \(|\psi(k)\rangle\), which is governed by the Hamiltonian (3) in the corresponding subspace:

\[
\hat{H}' = \omega_e|e; \phi\rangle\langle e; \phi| + \int_{-\infty}^{\infty} dk \, k |\psi(k)\rangle\langle \psi(k)|
\]

\[
+ \int_{-\infty}^{\infty} dk \, V(k) |e; \phi\rangle \langle e; \phi| + \text{h.c.},
\]

with \(\phi\) being the vacuum of the field.

For the scattering process considered here, we are interested in the transition matrix element \(U_{kk'} = \langle \psi(k)|\hat{U}(t)|\psi(k')\rangle\) in the asymptotic long time limit, where \(\hat{U}(t)\) is the evolution operator. With the help of the resolvent formalism [11], we obtain the relation:

\[
U_{kk'} = \delta(k - k') \left[1 - 2\pi iV(k)^2 \langle e; \phi|\hat{G}(k)|e; \phi\rangle \right] e^{-ikt},
\]

where \(\hat{G}(k) \equiv (k - \hat{H}')^{-1}\) is the resolvent. The element \(\langle e; \phi|\hat{G}(k)|e; \phi\rangle\) can be determined
by the projection operator method, which gives
\[
\langle e; \phi | \hat{G}(k) | e; \phi \rangle = \frac{\Delta k + i\kappa}{(\Delta k - \omega_+)(\Delta k - \omega_-)}.
\]
(8)

Here $\Delta k = k - k_c$, $\omega_\pm = (\delta_c - i\kappa)/2 \pm \sqrt{(\delta_c + i\kappa)^2/4 + \lambda_L^2 + \lambda_R^2}$ is related to vacuum Rabi-splitting $[12]$, and $\delta_c = \omega_c - k_c$. Together with Eq. (7), we have $\hat{U}(t, 0) |\psi(k)\rangle = e^{-i\kappa t} e^{i\delta_s(k)} |\psi(k)\rangle$. The exact expression of the phase shift $\delta_s(k)$ is given by
\[
e^{i\delta_s(k)} = \frac{(\Delta k - \delta_c)(\Delta k^2 + k^2) - (\Delta k + i\kappa)(\lambda_L^2 + \lambda_R^2)}{(\Delta k - \delta_c)(\Delta k^2 + k^2) - (\Delta k - i\kappa)(\lambda_L^2 + \lambda_R^2)}.
\]
(9)

Let us now denote a general input state (at a given $k$) by $|\text{in; } k\rangle = \alpha_1(k) |L; k_L\rangle + \alpha_2(k) |R; k_R\rangle + \alpha_3(k) |L; k_R\rangle + \alpha_4(k) |R; k_L\rangle$, and the corresponding output state by $|\text{out; } k\rangle = \beta_1(k) |L; k_L\rangle + \beta_2(k) |R; k_R\rangle + \beta_3(k) |L; k_R\rangle + \beta_4(k) |R; k_L\rangle$. With the help of Eq. (9) and taking account of the (non-interacting) dark state, the input-output relation can be conveniently expressed in a matrix equation,
\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{pmatrix} =
\begin{pmatrix}
T_{LL} & T_{LR} & 0 & 0 \\
T_{RL} & T_{RR} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}.
\]
(10)

Here $T_{LR}(k) = T_{RL}(k) e^{2i(\theta_R - \theta_L)} = g_L^*(k) g_R(k) (e^{i\delta_s(k)} - 1)/V(k)^2$, $T_{LL}(k) = e^{i\delta_s(k)}(|g_L(k)|^2 + |g_R(k)|^2)/V(k)^2$, $T_{RR}(k) = e^{i\delta_s(k)}(|g_R(k)|^2 + |g_L(k)|^2)/V(k)^2$, and we have omitted the trivial free evolution phase factor $e^{-i\kappa t}$.

Equation (10) describes a general transformation of photon-atom states before and after the scattering process. The matrix elements $T_{ij}(k)$ ($i, j = L, R$) are the consequences of interference between amplitudes associated with the dark state and the bright state $|\psi(k)\rangle$ that acquires a phase shift $\delta_s(k)$. Notice that the elements $T_{ij}(k)$ form a sub-matrix that is unitary. By tuning the photon frequency and cavity parameters, we are able to perform qubit transformations between the atom and the photon. In the following, we consider swapping and entangling of qubits.

**Swapping qubits**— If we treat the labels $L(k_L)$ and $R(k_R)$ as logical 1(0) and 0(1) for the atom(photon), respectively, the qubit swapping operation $[\Pi]$ corresponds to the conditions: $T_{LL} = T_{RR} = 0$ and $T_{LR} = T_{RL} = 1$. Such conditions are satisfied if $g_L(k) = -g_R(k)$ and
\[
\delta_s(k) = \pm \pi.
\]
(11)
The condition \( g_L(k) = -g_R(k) \) can be satisfied by choosing suitable atomic transition schemes. For example, the D1 line of sodium with hyperfine ground states \( |L\rangle = |F = 1, m_F = -1\rangle \) and \( |R\rangle = |F = 1, m_F = 1\rangle \), and the excited state \( |e\rangle = |F = 1, m_F = 0\rangle \) is a candidate of \( \Lambda \)-systems with equal but opposite dipole matrix elements.

If the cavity is tuned at the resonance \( k_c = \omega_e \), the requirement of a \( \pi \) phase shift \( \delta_s(k) \) is satisfied for the photon frequencies:

\[
k = k_c, k_c \pm \sqrt{2\lambda^2 - \kappa^2}, \quad (12)
\]

where \( \lambda \equiv \lambda_L = \lambda_R \). Hence, if the frequency of the incident photon is given by Eq. (12), perfect qubit swapping can be achieved as long as the loss due to spontaneous decay is ignorable.

Furthermore, we can extend our analysis to incorporate a non-zero spontaneous decay rate \( \gamma \) of the excited state. This can be done by adding a negative imaginary part \( -i\gamma \) to the atomic frequency \( \omega_e \). Accordingly, Eq. (9) is modified but the general form of input-output transformation given by Eqs. (10) remains unchanged. We find that among the three characteristic frequencies given in Eq. (12), \( k = k_c \) provides a robust performance against spontaneous emission loss in the strong coupling regime, where \( \lambda^2 \gg \kappa\gamma \) and consequently \( e^{i\delta_s(k_c)} \approx -1 + \kappa\gamma/\lambda^2 \). Therefore the loss due to spontaneous decay is of order \( \kappa\gamma/\lambda^2 \), and becomes insignificant in this regime.

To complete our analysis we need to consider realistic photons in forms of wavepackets, instead of being purely monochromatic light. We consider an initial state \( |\Phi_{\text{in}}\rangle = (A_L|L\rangle + A_R|R\rangle) \otimes \int_{-\infty}^{\infty} dk f_S(k) [C_L|k_L\rangle + C_R|k_R\rangle] \). To apply our solution (7)-(10), the spectral function \( f_S(k) \) of the photon packet should be taken such that the atom does not experience the field of the injected photon for \( t \leq 0 \). For concreteness, we consider a Gaussian photon packet which is traveling towards the cavity from a far distance \( x_0 \) at \( t = 0 \) with a peak frequency \( k_c \) and spectral width \( \kappa_{\text{in}} \):

\[
f_S(k') = \frac{1}{\pi^{1/4} \kappa_{\text{in}}} \exp \left[ -\frac{(k' - k_c)^2}{2\kappa_{\text{in}}^2} + i k' x_0 \right]. \quad (13)
\]

In the asymptotic long time limit, the output state \( |\Phi_{\text{out}}\rangle \) is determined by the transformation (10) for each \( k \). The swapping fidelity is defined by the overlap \( F = |\langle \Phi_{\text{swap}} | \Phi_{\text{out}} \rangle|^2 \), where \( |\Phi_{\text{swap}}\rangle = (C_R|L\rangle + C_L|R\rangle) \otimes \int_{-\infty}^{\infty} dk f_S(k) [A_R|k_L\rangle + A_L|k_R\rangle] \) is the ideal swapping of the input state.
After some straightforward calculations, we have

\[ F = 1 - 2\text{Re}(\xi)\eta + |\xi|^2\eta^2, \]

where \( \eta = |A_L C_L - A_R C_R|^2 \) and \( \xi = \int_{-\infty}^{\infty} T_{LL}(k)|f_S(k)|^2dk \). Therefore the fidelity depends on two parameters \( \eta \) and \( \xi \), which in turn depend on the qubit states and the photon spectrum. In essence \( \eta \) measures the overlap of the initial state with the interacting state \( |\psi(k)\rangle \) defined in Eq. (5), and \( \xi \) measures the deviations of \( T_{ij} \) from the ideal swapping matrix elements \( T_{LL} = 0 \) and \( T_{LR} = 1 \) averaged over the incident photon spectrum.

Further calculations show that \( F \) attains minimum when \( \eta = 1 \) for any \( \xi \), therefore \( F \) obeys the inequality,

\[ F \geq 1 - 2\text{Re}(\xi) + |\xi|^2 = F_{\text{min}}. \quad (14) \]

To provide numerical examples, the circles in Fig. 2 show the dependence of \( F_{\text{min}} \) on the normalized coupling strength \( \lambda/\kappa \). In the case \( \lambda/\kappa = 10, \gamma/\kappa = 0.5 \), we have \( F_{\text{min}} \approx 97\% \) for a Gaussian packet \( \kappa_{in} = 0.1\kappa \). Higher \( F_{\text{min}} \) can be achieved by decreasing the spectral width of the input photon.

**Entangling qubits** — If \( T_{RL}(k) = T_{LL}(k)e^{i\theta} \) (where \( \theta \) is a real phase angle) is satisfied at certain frequencies, then an initial product state \( |L, k_L\rangle \) at those frequencies will evolve into a Bell’s state: \( |\Phi_E\rangle = (|L, k_L\rangle + e^{i\theta}|R, k_R\rangle)/\sqrt{2} \). In other words, the scattered photon and the final state of the atom can become maximally entangled.

We find that solutions of \( k \) satisfying \( T_{RL}(k) = T_{LL}(k)e^{i\theta} \) do exist. Under the condition \( \delta_e = 0 \), we have \( \Delta k = \pm \kappa \) (or \( k = k_c \pm \kappa \)) as approximate solutions in the strong coupling regime \( \lambda \gg \gamma, \kappa \), and the corresponding \( \theta \) are \( \mp \pi/2 \). Such photon frequencies do not depend on the coupling strengths, hence the entangling operation is not sensitive to the position of the atom as long as the strong coupling requirement is satisfied.

To discuss the success probability, let us consider an initial state \( |\Phi_{in}\rangle = |L\rangle \otimes \int_{-\infty}^{\infty} dk f_E(k)|k_L\rangle \), where \( f_E(k) \) peaks at \( k = k_c + \kappa \) with a spectral width \( \kappa_{in} \). The success probability is defined by \( P = |\langle \Phi_E|\Phi_{out}\rangle|^2 \), where \( |\Phi_E\rangle = \int_{-\infty}^{\infty} dk f_E(k)(|L, k_L\rangle - i|R, k_R\rangle)/\sqrt{2} \) is the target Bell’s state. It can be shown that: \( P = \frac{1}{2} + |\xi|^2 - \text{Re}(\xi) + \text{Im}(\xi) \), where \( \xi \) is similarly defined by the previous expression, with \( f_S(k) \) replaced by \( f_E(k) \). It is useful to remark that \( P \) depends on the photon spectrum \( |f_E(k)|^2 \). The phase of \( f_E(k) \), which defines the spatial details of the photon wavefunction, does not affect the success probability. The same is also true for the swapping process above. We find that the probability of generating the Bell’s state is quite high in the strong coupling regime as long as the spectral width
of the incident photon is sufficiently narrow. Figure 2 illustrates how $P$ depends on the normalized coupling strength $\lambda/\kappa$. For example at $\lambda/\kappa = 10$, $P \approx 99.2\%$ can be achieved by using $\gamma = 0.5\kappa$ and a Gaussian amplitudes $f_E(k)$ with $\kappa_{in} = 0.1\kappa$.

To conclude, we have demonstrated how a single photon and an atom can effectively ‘communicate’ with each other through scattering in a strong coupling cavity QED environment. An interesting aspect of our approach is that the frequency of the photon serves as a control parameter. Various qubit transformations can be achieved in a tunable way according to the scattering matrix (10). The continuous dependence of frequencies in such matrix not only allows swapping and entangling in the same setup, but also enables the preparation of a wide range of photon-atom states when combining local transformations of the atom and the photon individually. Although we have used a one-dimensional cavity to illustrate the process, the Hamiltonian (11) itself is quite general. Our method is equally applicable to systems in higher dimensions as long as a single quasi-mode and a dominant input-output channel is involved in the process. In addition, since our protocol concerns only asymptotic states, no precise timing of interaction period is required as compared with typical state control strategies [14]. Finally, we note that the realization of our scheme relies on deterministic single photon sources [15, 16] and trapping of single atoms inside optical cavities [17, 18]. Our work in fact addresses a novel application of single photons scattering once these technologies become available.

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FIG. 1: A sketch of our cavity QED system.

FIG. 2: Dependence of minimum fidelity of swapping (filled circles) and success probability of Bell’s state generation (squares) on the normalized coupling strength $\lambda/\kappa$ for $\gamma = 0.5\kappa$ and $\kappa_{\text{in}} = 0.1\kappa$. 