We consider the precession of isolated neutron stars in which superfluid is not pinned to the stellar crust perfectly. In the case of perfect pinning, Shaham showed that there are no slowly oscillatory, long-lived modes. When the assumption of perfect pinning is relaxed, new modes are found that can be long lived but are expected to be damped rather than oscillatory, unless the drag force on moving superfluid vortex lines has a substantial component perpendicular to the direction of relative motion. The response of a neutron star to external torques, such as the spin-down torque, is also treated. We find that when computing the response of a star to perturbations, assuming perfect coupling of superfluid to normal matter from the start can miss some effects.

Subject headings: dense matter — stars: neutron

1. INTRODUCTION

Radio pulsars can be exceptionally stable clocks. The predictability of pulse arrival times has made possible precision tests of general relativity (e.g., Taylor et al. 1992) and the discovery of the first extrasolar planetary systems (Wolszczan & Frail 1992). However, imperfections in the rotation of most pulsars have been monitored for some time, most notably glitches (e.g., Boynton et al. 1969; Radhakrishnan & Manchester 1969; Backus, Taylor, & Damashek 1982; Downs 1982; Demianski & Proszynski 1983; Manchester et al. 1983; Lyne 1987; Lyne & Pritchard 1987; Cordes, Downs, & Krause-Polstorff 1988; McKenna & Lyne 1990; Hamilton et al. 1989; McCulloch et al. 1990; Flanagan 1990, 1993; Lyne, Smith, & Pritchard 1992; Shemar & Lyne 1996) and timing noise (e.g., Boynton et al. 1972; Groth 1975; Cordes & Helfand 1980; Cordes & Downs 1985; D’Alessandro et al. 1995). These deviations from stable spin convey information about the internal structure and dynamics of neutron stars. For example, the long-time healing of the spin frequencies and spin-down rates of glitching pulsars may be explained if the convulsions are due to sudden unpinning of superfluid vortex lines that were glued to nuclei in the pulsar’s crust before the glitch, migrate as a consequence of the glitch, and repin to other nuclei in its aftermath (e.g., Anderson & Itoh 1975; Alpar et al. 1981, 1984a, 1984b, 1993; Link, Epstein, & Baym 1993).

A small number of neutron stars also exhibit long-term cyclical but not precisely oscillatory variations in their spin. A particularly well-known case is the Crab pulsar, whose phase residuals (after careful fitting that accounts for spin-down and glitches) vary systematically, with a peak-to-peak range of order \(\pm 10\) ms and a characteristic cycle duration of about 20 months. (Lyne, Pritchard, & Smith 1988). After its Christmas 1988 glitch, the Vela pulsar showed—after accounting for exponential recovery from the glitch—damped oscillatory phase residuals with a period of order 25 days; evidence for oscillations in the frequency derivative of the pulsar both before and after the glitch with a period of “a few tens of days” was also reported (McClurio et al. 1990). Evidence for long-term variations (correlation times \(\sim 100\) days) in the pulse shape of the Vela pulsar has also been found in data spanning approximately 4 yr (Blaskiewicz 1992; Cordes 1993). A principal-component analysis of the pulse shape of PSR 1642–03 (Blaskiewicz 1992; Blaskiewicz & Cordes 1997, in preparation) yields evidence for cyclical pulse shape variations with a period of about 1000 days; long-term variations on a similar characteristic timescale are also seen in the timing residuals for this pulsar (Cordes 1993). Finally, although Her X-1 is an accreting X-ray pulsar, not an isolated radio pulsar, it has a well-known 35 day cycle on which it appears and disappears; observed variations in pulse shape over the cycle suggest that it is related to periodic variations in the rotation of the neutron star (e.g., Trümper et al. 1986; Alpar & Ögelman 1987).

Soon after the discovery of radio pulsars, it was suggested that long-term variations in their spin could result from free precession (e.g., Davis & Goldstein 1970; Goldreich 1970; Ruderman 1970; Brecher 1972; Pines, Pethick, & Lamb 1973; Pines & Shaham 1972, 1974; Lamb et al. 1975; Jones 1976) and that pulsar arcanae such as drifting subpulses might be related to precessional effects. If a neutron star were a rigid or semirigid solid (see Pines & Shaham 1972, 1974), it would precess with a period of order \(P/\epsilon\), where \(P\) is the spin period of the star and \(\epsilon\) is the fractional difference between its principal moments of inertia; \(\epsilon \lesssim 10^{-7}\) would imply precession periods \(\gtrsim P(s)\) yr, where \(P(s)\) is the spin period in seconds. However, once it became apparent that superfluid vortex lines pin to the crust of a neutron star, Shaham (1977) demonstrated that slow, persistent precession is impossible. Pinned superfluid alters the effective asymmetry of the star to where is the moment of inertia; would imply precession periods yr, where

\[
\frac{I_p}{I} \sim 10^{-2}
\]
the pessimism of theorists, the data demand an explanation. Ruderman (1997) has mentioned long-term variations in pulsar spin rates as one of the outstanding unresolved problems of neutron star physics.

This paper is the first of two that attack this problem. Our purpose in this paper is threefold: First, we revisit the arguments put forward by Shaham (1977) with an eye toward identifying possible loopholes. Although it will become apparent that there are several different possibilities, we concentrate here on Shaham’s assumption that superfluid pins perfectly to crustal nuclei. (Some of the other loopholes will be considered in a subsequent paper.) We develop the formalism for doing so in § 2.2 and solve the equations governing the spin dynamics of a neutron star in succeeding sections to varying degrees of complexity and realism. As expected, we do find modes in addition to those found by Shaham (1977), but we also argue that none of these modes is likely to be a long-period, slowly damped oscillation.

Second, we examine what can happen in a multicomponent star, in which some regions contain pinned superfluid and others unpinned superfluid. One might think that if some parts of a star are capable of slow oscillations of the spin—either precession or long-period fluid modes, such as Tkachenko modes—then there could be an observable signature of these modes in the detectable pulsar spin rate. However, we demonstrate that this situation is highly unlikely, for even if such regions do exist inside actual pulsars, persistent long-period oscillations in those domains are only possible if the coupling to the crust, where superfluid is pinned effectively, is very weak; but under such conditions, the crust is almost unaffected by the slow oscillations, which hardly manifest themselves in the crustal spin rate. Third, we begin a general examination of the effects of external torques—such as the spin-down torque—on the spin dynamics. For this purpose, we derive explicit expressions for the response of the various components of a neutron star to rather general time-dependent torques. Our treatment of this problem shows that the limit of perfect coupling must be taken carefully when the response to external torques is needed, because the additional modes that appear when pinning is imperfect contribute to the response and cannot be ignored.

As will become apparent in the succeeding sections, we do not believe that the long-term cyclic variability detected in the spins of some pulsars can be accounted for by free precession and that it is not likely to be due to forced precession either. However, we do believe that this paper begins to elucidate the complexity of the behavior of neutron star spin and clarifies the conditions that must be met for precession to occur, even if those conditions are not likely to be realized.

2. OVERVIEW

2.1. Pinned Superfluid Suppresses Precession (Shaham 1977)

Shaham (1977) showed that pinned crustal superfluid dramatically alters the physics of precession. Let us review his argument briefly. Consider a three-component neutron star that consists of (1) a rigid crust rotating at angular velocity \( \Omega_{cr} \); (2) pinned crustal superfluid, whose angular momentum \( L_p \) is independent of time in the frame rotating with the crust; and (3) a core (super)fluid rotating at angular velocity \( \Omega_c \). As seen in the inertial frame,

\[
I_c \frac{d\Omega_c}{dt} + \frac{d(I_{cr} \cdot \Omega_{cr})}{dt} + \Omega_{cr} \times L_p = 0 ,
\]  

(1)

if there are no external torques, where \( I_c \) is the moment of inertia of the core fluid and \( I_{cr} \) is the moment of inertia tensor of the crust. We assume that the moment of inertia tensor \( I_c \) of the core fluid is always of the form (\( \delta \) is the unit tensor)

\[
I_c = (I_c - \delta I_c)\delta + \Delta I_c \left( \frac{3\Omega_c \times \Omega_c - \delta}{2} \right),
\]  

(2)

so that its angular momentum \( L_c = I_c \cdot \Omega_c = I_c \Omega_c \). This amounts to assuming that the core fluid adjusts its shape instantaneously to an oblate spheroid flattened along its direction of rotation. We shall discuss this assumption more fully in a subsequent publication.

Introducing a dissipative torque that seeks to enforce corotation between the crust and the core, we get the coupled equations

\[
\frac{d(I_{cr} \cdot \Omega_{cr})}{dt} + \Omega_{cr} \times L_p = -K(\Omega_{cr} - \Omega_c),
\]  

(3)

\[
I_c \frac{d\Omega_c}{dt} = K(\Omega_{cr} - \Omega_c),
\]  

(4)

where \( K \) is a constant. These equations have a rich set of fixed points (where time derivatives vanish) depending on the orientation of \( L_p \) relative to the principal axes of \( I_{cr} \). The full set of fixed points, and their possible observable significance, will be discussed completely elsewhere; here we focus on the particularly simple—but far from general—situation in which \( L_p \) is along one of the principal axes of \( I_{cr} \). In that circumstance, the fixed-point solution is \( \Omega_{cr} = \Omega_c = \Omega \), with \( \Omega \parallel L_p \).

Perturbations about this fixed point are studied most easily in the frame corotating with the crust, where \( I_{cr} \) is independent of time. For definiteness, let us suppose that the principal moments of inertia of the crust are \( I_2 < I_3 < I_1 \) and, to parallel Shaham (1977) as closely as possible, suppose that the fixed point corresponds to rotation about the 3-axis at angular velocity \( \Omega = \dot{\Omega}_3 \). Then the linearized equations are

\[
\dot{\Omega}_{cr,1} + \left[ \left( \frac{I_3 - I_2}{I_3} \right) + \frac{L_p}{I_3 \Omega} \right] \Omega_{cr,2} = - \frac{K}{I_1 \Omega} (\Omega_{cr,1} - \Omega_{c,1}),
\]  

(5)
\[ \Omega_{cr,2} - \left[ \left( I_3 - I_1 \right) + \frac{L_p}{I_2 \Omega} \right] \Omega_{cr,1} = - \frac{K}{I_2 \Omega} \left( \Omega_{cr,2} - \Omega_{c,2} \right), \]  
(6)

\[ \Omega_{cr,3} = - \frac{K}{I_3 \Omega} \left( \Omega_{cr,3} - \Omega_{c,3} \right), \]  
(7)

\[ \dot{\Omega}_{c,1} + \dot{\Omega}_{cr,2} - \dot{\Omega}_{c,2} = \frac{K}{I_c \Omega} \left( \Omega_{cr,1} - \Omega_{c,1} \right), \]  
(8)

\[ \dot{\Omega}_{c,2} + \dot{\Omega}_{cr,1} - \dot{\Omega}_{c,1} = \frac{K}{I_c \Omega} \left( \Omega_{cr,2} - \Omega_{c,2} \right), \]  
(9)

\[ \dot{\Omega}_{c,3} = \frac{K}{I_c \Omega} \left( \Omega_{cr,3} - \Omega_{c,3} \right). \]  
(10)

In these equations, \( \dot{F} = \Omega^{-1} d^* F/dt \) is the time derivative of any vector \( F \) as seen in the frame rotating with the crust. It is clear that the perturbations along the 3-axis decouple from those along other axes and decay exponentially with a characteristic rate

\[ (\Omega_t)^{-1} = \frac{K}{\Omega} \left( \frac{1}{I_3} + \frac{1}{I_c} \right). \]  
(11)

this coupling time has been estimated by Alpar & Sauls (1988), according to whom \( \Omega_t/2\pi \approx 400-10^4 \) (and the relaxation of the electron distribution function is due to the scattering off the neutron vortex magnetization), and by Sedrakian & Sedrakian (1995), who find that \( \Omega_t/2\pi \) is rather sensitive to the mass density \( \rho \) and spans the range \( \Omega_t/2\pi \sim 10^2-10^3 \) for \( \rho \sim (1.6-3) \times 10^{14} \text{ g cm}^{-3} \) (here the electron scattering is off the proton vortex clusters coupled to the neutron vortex lattice). In the latter case the relaxation time spans a wide density range with a large gradient near the crust-core interface because of an exponential dependence of the size of the cluster on the proton effective mass. [See also Sedrakian et al. 1995; Sedrakian & Cordes 1998; for the decay of precession the effective coupling rate \( \gamma \) is a weighted average of the range found by Sedrakian & Sedrakian 1995, implying an effective coupling time closer to the smallest values; we adopt \( \Omega_t/2\pi \sim 100 \) for numerical estimates below.]

The remaining four equations have normal modes proportional to \( \exp (p \Omega t) \equiv \exp (p \phi) \), where \( \phi = \Omega t \) is pulse phase. It is straightforward to solve for the modes of a triaxial star, but the basic result can be derived under the assumption of axisymmetry, \( I_2 = I_1 \), (We have solved the corresponding triaxial problem, and there are no qualitatively different modes for slowly rotating neutron stars.) If we define

\[ \sigma = \left( I_3 - I_1 \right) + \frac{L_p}{I_1 \Omega}, \quad \gamma = \frac{K}{\Omega} \left( \frac{1}{I_3} + \frac{1}{I_c} \right) = \frac{I_3(I_1 + I_c)}{I_c \Omega \epsilon I_1(I_3 + I_c)}, \]  
(12)

then

\[ \dot{\Omega}_{cr,1} + \sigma \dot{\Omega}_{cr,2} = - \frac{\gamma I_c}{I_1 + I_c} \left( \Omega_{cr,1} - \Omega_{c,1} \right), \]  
(13)

\[ \dot{\Omega}_{cr,2} - \sigma \dot{\Omega}_{cr,1} = - \frac{\gamma I_c}{I_1 + I_c} \left( \Omega_{cr,2} - \Omega_{c,2} \right), \]  
(14)

\[ \dot{\Omega}_{c,1} + \dot{\Omega}_{cr,2} - \dot{\Omega}_{c,2} = \frac{\gamma I_1}{I_1 + I_c} \left( \Omega_{cr,1} - \Omega_{c,1} \right), \]  
(15)

\[ \dot{\Omega}_{c,2} + \dot{\Omega}_{cr,1} - \dot{\Omega}_{c,1} = \frac{\gamma I_1}{I_1 + I_c} \left( \Omega_{cr,2} - \Omega_{c,2} \right). \]  
(16)

If \( \gamma = 0 \), so the crust and core are uncoupled, then there are modes with \( p = \pm i \sigma \), which correspond to independent precession of \( \Omega_{cr} \), but at a frequency that is much larger than the conventional Euler frequency for reasonable values of \( L_p/I_1, \Omega \equiv I_p/I_1 \) (where \( I_p \) is the moment of inertia of pinned superfluid). The remaining modes with \( p = \pm i \) are an artifact of working in the frame that corotates with the crust and corresponds to \( \Omega_c \) fixed in the inertial frame of reference.

When \( \gamma \neq 0 \), the modes are damped, as was discussed by Bondi & Gold (1954) in the context of the rotation of the Earth (without considering pinned superfluid, of course!). The characteristic equation is fourth order in \( p \), but we expect the roots to come in complex conjugate pairs, so we can reduce the characteristic equation to second order by introducing the complex angular velocities

\[ \Omega_{cr}^+ = \Omega_{cr,1} + i \Omega_{cr,2}, \quad \Omega_{c}^+ = \Omega_{c,1} + i \Omega_{c,2}, \]  
(17)

which satisfy the equations

\[ \dot{\Omega}_{cr}^+ - i \sigma \Omega_{cr}^+ = - \frac{\gamma I_c}{I_1 + I_c} \left[ \Omega_{cr}^+ - \Omega_{c}^+ \right], \]  
(18)
\[
\dot{\Omega}_c^{(+)} - i\Omega_c^{(+)} + i\Omega_c^{(+)} - \frac{\gamma I_1}{I_1 + I_c} [\Omega_c^{(+)} - \Omega_c^{(+)}].
\] (19)

Substituting \([\Omega_c^{(+)} - \Omega_c^{(+)}] \propto \exp(p\phi)\) we find
\[
p^2 + p[\gamma + i(1 - \sigma)] + \sigma\left(1 - \frac{i\gamma I_1}{I_1 + I_c}\right) = 0.
\] (20)

The normal modes of the fourth-order system are the two solutions to this quadratic equation and their complex conjugates.

Although we can solve the second-order characteristic equation exactly, it is more instructive to find approximate solutions valid for small and large crust-core coupling. For small values of \(\gamma\), we rewrite the characteristic equation as
\[
(p + \delta)(p - i\sigma) + \gamma\left(p - \frac{i\gamma I_1}{I_1 + I_c}\right) = 0;
\] (21)
this form separates terms of zeroth and first order in \(\gamma\) explicitly. To first order in \(\gamma\), the solutions are
\[
p_d = -i - \frac{\gamma[1 + \sigma I_1/(I_1 + I_c)]}{1 + \sigma},
\] (22)
\[
p_p = i\sigma - \frac{\gamma\sigma I_c}{(I_1 + I_c)(1 + \sigma)}.
\] (23)

For large values of \(\gamma\), we rewrite the characteristic equation as
\[
p - i\sigma\frac{I_1}{I_1 + I_c} + \gamma^{-1}[p^2 + ip(1 - \sigma) + \sigma] = 0;
\] (24)
this form is useful for expanding in powers of \(\gamma^{-1}\). In this case, the solutions to first order in \(\gamma^{-1}\) are
\[
p_d = -\gamma - \frac{i}{1 - \frac{\sigma I_1}{I_c + I_c}},
\] (25)
\[
p_p = \frac{i\sigma I_1}{I_1 + I_c} - \frac{\sigma I_c}{\gamma(I_1 + I_c)}\left(\frac{1 + \sigma I_1}{I_1 + I_c}\right).
\] (26)

In each case, \(p_d\) represents damping of the angular velocity difference between crust and core and \(p_p\) is the precessing mode.

For the coupling times estimated by, for example, Alpar & Sauls (1988) or Sedrakian & Sedrakian (1995), the small \(\gamma\) limit is the relevant one. Since \(I_p/I_1 \gg I_c/I_1 - 1\), the precession period is far smaller than for a rigid body, approximately \(I_1/I_p\) spin periods. Moreover, the wobble damps away, lasting \(\sim \gamma^{-1}\) precession periods: \(\gamma^{-1} \approx 400 - 10^9\) according to Alpar & Sauls (1988), and a reasonable estimate for the effective coupling is \(\gamma^{-1} \approx 100P\) for Sedrakian & Sedrakian (1995). Even if \(\gamma\) were large, the precession period would be short, although it would be lengthened by a factor of \(1 + I_c/I_1\) relative to the small \(\gamma\) limit, implying a cycle \((I_c + I_1)/I_p\) spin periods long. The precession would persist for approximately \(I_1/I_p\) \(\Omega_p\) precession periods in this limit. Since the crust-core coupling time must exceed the light travel time across the star, \(\tau > R/c \approx 0.03\) ms and the damping time for the precession must be \(\lesssim 5000(I_1/I_c)P\) precession periods, where \(P\) is the rotation period in seconds.

In neither limit is the precession either long period or persistent. From this pessimistic result, one concludes that free precession cannot account for the cyclical behavior seen in the long-time monitoring of some pulsars. Moreover, to explain the data, one must invoke an excitation mechanism that acts relatively continuously, since it must fight the tendency for neutron star wobbles to decay rapidly. The characteristic cycle timescales of order months to years observed for these pulsars must reflect the underlying processes responsible for the continuous excitations.

### 2.2. Imperfect Pinning

In demonstrating that persistent, long-period precession is impossible for neutron stars with pinned superfluid, Shaham (1977) assumed perfect pinning. In actuality, superfluid vortex lines will not pin to crustal nuclei absolutely. One purpose of this paper is to see whether there are new oscillatory modes that emerge when pinning is assumed to be strong but not perfect.

To study this problem, we adopt a somewhat idealized approach. In actuality, the pinning of crustal superfluid is a highly inhomogeneous process involving the interaction of individual vortex lines and crustal nuclei. This coupling is modeled by effective potentials highly localized around discrete pinning sites in the vortex creep picture (e.g., Anderson & Itoh 1975; Alpar et al. 1984a; Link & Epstein 1991; Link et al. 1993) and by scattering of particles by and Kelvin excitation of moving vortex lines not pinned to crustal nuclei (e.g., Epstein & Baym 1992; Jones 1991, 1992). In our calculations, we use smoothed hydrodynamical equations to describe the coupling between the superfluid and normal components of the crust macroscopically, using the formalism developed by Khalatnikov (1965, § 16). This formulation of the problem is linked most naturally to a picture in which superfluid vortex lines experience drag forces as they move through a smooth medium of normal fluid but also may be applied directly in the vortex creep picture in the linear approximation (i.e., when the difference between the angular velocities of the superfluid and normal fluid are sufficiently small).

Shaham's results are recovered in the limit of perfect coupling, that is, when the coefficients of mutual friction are infinite. We can explore whether qualitatively new modes appear when the mutual friction is strong but pinning is not perfect. As we shall see, no new slowly damped, long-period modes arise.
General formulae for the mutual friction force are given by Khalatnikov (1965, § 16). If the superfluid vorticity is defined to be $\mathbf{V} \times \mathbf{v}_s$, where $\mathbf{v}_s$ is the superfluid velocity, then the net force per unit volume acting on the superfluid is

$$f = -\omega \times (\mathbf{V} \times \lambda \mathbf{v}) - \beta \rho_s \omega \times \mathbf{u} - \beta \rho_s \mathbf{v} \times (\omega \times \mathbf{u}) + \gamma \rho_s \mathbf{v} \omega \cdot \mathbf{u},$$

(27)

where $\rho_s$ is the superfluid mass density, $\mathbf{v} = \omega / |\omega|$, and, for a normal fluid velocity $\mathbf{v}_n$,

$$\mathbf{u} = \mathbf{v}_s - \frac{1}{\rho_s} \mathbf{V} \times \lambda \mathbf{v};$$

(28)

$$\lambda = (\rho_s \kappa / 4\pi) \ln (d/\xi),$$

where $\kappa$ is the quantum of circulation per vortex line, $d$ is the effective intervortex separation, and $\xi$ is the coherence length. The mutual friction force is defined to be $f + \omega \times (\mathbf{V} \times \lambda \mathbf{v}) / \rho_s$. The parameters $\beta$ and $\gamma$ must be positive for the rate of energy dissipation resulting from $f$ to be greater than zero locally.

Qualitatively, the terms in $f$ involving $\mathbf{v}$ arise from the bending of vortex lines and will be neglected here. Of the remaining contributions to $f$, the two proportional to $\beta$ and $\beta'$ are perpendicular to $\omega$, whereas the one proportional to $\gamma'$ is along $\omega$; the latter is expected to be small, and we neglect it too. With these simplifications, the form for $f$ used in this paper is

$$f = -\beta' \rho_s \omega \times (\mathbf{v}_n - \mathbf{v}_s) - \beta \rho_s \mathbf{v} \times (\omega \times (\mathbf{v}_n - \mathbf{v}_s)).$$

(29)

For getting a qualitative feeling for the relative sizes of the phenomenological quantities $\beta$ and $\beta'$, we use a different parameterization for the strength of the mutual friction force, based on the idea of vortex drag. The equation for the superfluid velocity including mutual friction is

$$\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s = -\mathbf{V} (\mu + \phi) + \frac{\mathbf{f}}{\rho_s},$$

(30)

where $\mu$ is the chemical potential and $\phi$ the gravitational potential; taking the curl of this equation gives

$$\frac{\partial \omega}{\partial t} = \nabla \times \left( \mathbf{v}_s \times \omega + \frac{\mathbf{f}}{\rho_s} \right).$$

(31)

If $f = 0$, then the superfluid vortex lines commove with the superfluid, but, in general, the vortex lines have a different velocity, $\mathbf{v}_L \neq \mathbf{v}_s$, and

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{v}_L \times \omega);$$

(32)

from the form for $f$ given in equation (29), we can read off

$$\mathbf{v}_L = \mathbf{v}_s + \beta' (\mathbf{v}_n - \mathbf{v}_s) + \beta \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_s) = \mathbf{v}_n + (\beta' - 1)(\mathbf{v}_n - \mathbf{v}_s) + \beta \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_s).$$

(33)

Only the components of $\mathbf{v}_n - \mathbf{v}_s$ perpendicular to $\mathbf{v}$ contribute to $\mathbf{v}_L$, as can be seen from the original expression for $f$. Clearly, vortex lines commove with the superfluid if $|\beta'|$ and $\beta$ are both small and commove with the normal fluid if $|\beta' - 1|$ and $\beta$ are small. (If superfluid rotates faster than normal fluid, vortices move slowly outward relative to normal fluid for $\beta$ small.)

The motion of a vortex is found by balancing the Magnus force due to superfluid streaming past the line and any other forces it experiences; for our purposes, the latter are drag forces perpendicular to the line, so the equation of motion is

$$\rho_s \kappa \mathbf{v} \times (\mathbf{v}_L - \mathbf{v}_s) + \mathbf{F}_d = 0.$$  

(34)

If the drag force per length on a vortex is

$$\mathbf{F}_d = -\eta (\mathbf{v}_L - \mathbf{v}_n) - \eta' \mathbf{v} \times (\mathbf{v}_L - \mathbf{v}_s),$$

(35)

then the vortex line velocity is

$$\mathbf{v}_L = \mathbf{v}_s + \frac{[\eta_2 - \eta' (\rho_s \kappa - \eta)][(\mathbf{v}_n - \mathbf{v}_s)] (\rho_s \kappa \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_s)) + \eta \kappa \rho_s \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_s)}{(\rho_s \kappa - \eta)^2 + \eta^2},$$

(36)

from which we infer the relations

$$\beta = \frac{\eta \rho_s \kappa}{(\rho_s \kappa - \eta)^2 + \eta^2},$$

(37)

$$\beta' = 1 - \frac{\rho_s \kappa (\rho_s \kappa - \eta')}{(\rho_s \kappa - \eta')^2 + \eta'^2}. $$

(38)

These results relate the drag coefficients $\eta$ and $\eta'$ with the parameters $\beta$ and $\beta'$ appearing in the mutual friction force.

In microscopic models for mutual friction developed so far, the coefficient $\eta'$, which determines the magnitude of the drag force perpendicular to the motion of a vortex line through the normal fluid, is negligible. If $\eta' = 0$, then equations (37) and (38)
where, from equation (29),

\[
\beta = \frac{\eta \rho_s \kappa}{(\rho_s \kappa)^2 + \eta^2},
\]

\[
\beta' = 1 - \frac{(\rho_s \kappa)^2}{(\rho_s \kappa)^2 + \eta^2}.
\]

From these relationships, we find that when \( \eta \gg \rho_s \kappa \), vortex lines are dragged effectively and tend to follow the normal fluid closely; in that limit, \( \beta \approx \rho_s \kappa / \eta \) and \( 1 - \beta' \approx (\rho_s \kappa / \eta)^2 \approx \beta^2 \). When \( \eta \ll \rho_s \kappa \), the drag is weak, vortex lines tend to follow the superfluid and \( \beta \approx \eta / \rho_s \kappa \) and \( \beta' \approx (\eta / \rho_s \kappa)^2 \approx \beta^2 \). As we shall see below, this means that the dissipative torque arising from mutual friction is much larger than the nondissipative torque when the drag is either very strong or very weak; the two torques are only comparable when \( \eta / \rho_s \kappa \sim 1 \). When \( \eta' \neq 0 \), the situation becomes more complicated, as equations (37) and (38) involve two nondimensional parameters, \( \eta / \rho_s \kappa \) and \( \eta / \rho_s \kappa \). If we assume that \( \eta' \ll \eta \), then in the strong damping limit, \( \beta \approx \rho_s \kappa / \eta \) as before and \( 1 - \beta' \approx (\rho_s \kappa / \eta)^2 \) if \( \eta' \ll \rho_s \kappa \), but \( 1 - \beta' \approx -\eta' \rho_s \kappa / \eta \approx -(\eta' / \eta) \beta \). In the weakly coupled domain, \( \beta \approx \eta / \rho_s \kappa \) as before, but \( \beta' \approx \beta^2 \) only when \( \eta' / \eta \ll \eta / \rho_s \kappa \), and instead \( \beta' \approx -\eta' \rho_s \kappa \approx -(\eta' / \eta) \beta \) when \( \eta' / \eta \gg \rho_s \kappa \). As we shall see, these results will make the existence of long-term oscillatory modes problematic when \( \eta' \ll \eta \) provided that at least part of the crust is coupled strongly to the crustal superfluid. For \( \eta' \gg \eta \), the situation turns out to be more favorable for the survival of oscillatory modes. In that case, \( \beta \approx \eta / \rho_s \kappa / \eta' \) and \( 1 - \beta' \approx -\eta' / \rho_s \kappa \approx -(\eta' / \eta) \beta \) in the limit of strong coupling and \( \beta \approx \eta / \rho_s \kappa \) and \( \beta' \approx -\eta' / \rho_s \kappa \approx -(\eta' / \eta) \beta \) in the limit of weak coupling.

The torque that results from mutual friction is

\[
N = \int d^3 r \mathbf{r} \times f(\mathbf{r}) \equiv N_\beta + N_\beta',
\]

where, from equation (29),

\[
N_\beta = -\int d^3 r \beta \rho_s \mathbf{r} \times \{ \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_a) \},
\]

\[
N_\beta' = -\int d^3 r \beta' \rho_s \mathbf{r} \times \{ \mathbf{v} \times (\mathbf{v}_n - \mathbf{v}_a) \}.
\]

We restrict ourselves to a uniformly rotating normal fluid, but the analogous restriction to uniformly rotating superfluid is dynamically inconsistent unless \( \beta \) and \( \beta' \) are independent of position. Consequently, we imagine that the star can be divided into "shells" in which \( \beta \) and \( \beta' \) are independent of position and the superfluid rotates uniformly. In these shells, \( \mathbf{v}_s = \mathbf{\Omega}_s \times r \) and \( \mathbf{v}_n = \mathbf{\Omega}_n \times r \), with \( \mathbf{\Omega}_s \) and \( \mathbf{\Omega}_n \) independent of \( r \); this also implies that \( \omega = 2 \mathbf{\Omega}_s \). In succeeding sections, we consider stars with one and two superfluid shells. These examples suffice to illustrate the complex behavior that may arise in a real neutron star, where \( \beta \) and \( \beta' \) vary continuously.

For uniform rotation, equations (42) and (43) become

\[
N_\beta = \mathbf{\Omega}_s \times \mathbf{T}_\beta \cdot (\mathbf{\Omega}_s \times \mathbf{\Omega}_a) + \mathbf{\Omega}_s (\mathbf{\Omega}_s - \mathbf{\Omega}_a) \cdot [\mathbf{T}_\beta - \delta \text{Tr} (\mathbf{T}_\beta)] ;
\]

\[
N_\beta' = -(\mathbf{\Omega}_s - \mathbf{\Omega}_a) \times \mathbf{T}_\beta' \cdot \mathbf{\Omega}_s ,
\]

where

\[
\mathbf{T}_\beta \equiv 2\beta \int d^3 r \rho_s \mathbf{r} \mathbf{r} \mathbf{r},
\]

\[
\mathbf{T}_\beta' \equiv 2\beta' \int d^3 r \rho_s \mathbf{r} \mathbf{r} \mathbf{r},
\]

\( \delta \) is the unit tensor, and Tr(\mathbf{T}_\beta) is the trace of \( \mathbf{T}_\beta \). It is easy to show that \( (\mathbf{\Omega}_s - \mathbf{\Omega}_a) \cdot N_\beta = 0 \) and \( (\mathbf{\Omega}_s - \mathbf{\Omega}_a) \cdot N_\beta < 0 \), so that \( (\mathbf{\Omega}_s - \mathbf{\Omega}_a) \cdot N < 0 \), that is, mutual friction torques are ultimately dissipative.

The tensors \( \mathbf{T}_\beta \) and \( \mathbf{T}_\beta' \) can be rather complicated in general. Even in uniformly rotating superfluid shells, the superfluid density \( \rho_s (r) \) may be slightly anisotropic, principally as a result of rotational flattening perpendicular to \( \mathbf{\Omega}_s \), which is time varying and not aligned with any of the principal axes of the crust in general. However, we shall neglect these complications, since the magnitudes of the anisotropies in \( \mathbf{T}_\beta \) and \( \mathbf{T}_\beta' \) are expected to be small for slowly rotating neutron stars, which we focus on here. Accordingly, we approximate

\[
\mathbf{T}_\beta = I_s \beta_{\text{eff}} \delta ,
\]

\[
\mathbf{T}_\beta' = I_s \beta'_{\text{eff}} \delta ,
\]

where \( I_s \) is the moment of inertia of the superfluid, and \( \beta_{\text{eff}} \) and \( \beta'_{\text{eff}} \) are suitably averaged \( \beta \) and \( \beta' \); henceforth, we drop the subscript "eff." With these expressions for \( \mathbf{T}_\beta \) and \( \mathbf{T}_\beta' \), the mutual friction torques simplify to

\[
N_\beta = -I_s \beta_{\text{eff}} (\mathbf{\Omega}_s - \mathbf{\Omega}_a) \cdot (\delta + \mathbf{\Omega}_s \mathbf{\Omega}_a) ;
\]
\[ N_\beta = I_\beta (\Omega_n \times \Omega_n) . \]  
(51)

We shall devote much of the remainder of this paper to examining the consequences of torques of this form. Here we neglect other torques that could be important, such as gravitational torques (both Newtonian and post-Newtonian) or fluid torques arising from boundary conditions, and ignore the various complications in \( T_\rho \) and \( T_\phi \) alluded to above. Some of these issues will be discussed in a subsequent publication.

3. TWO-COMPONENT STAR

Implicit in the review of Shaham (1977) presented in § 2.1 was a treatment of the two-component system consisting of the rigid crust and pinned crustal superfluid. This was the case when \( \gamma = 0 \) limit in which the crust and core decouple entirely. In that case, we found that the crust precesses at a frequency \( \sigma \) (see eq. [12]) under the additional assumption of axisymmetry; for \( \gamma = 0 \), this mode is undamped and with \( \Omega_n \) arising from boundary conditions, and ignore the various complications in \( T_\rho \) and \( T_\phi \) alluded to above. Some of these issues will be discussed in a subsequent publication.

3.1. Free Precession Reexamined

The coupled equations for the angular momenta of the crust and crustal superfluid are

\[ \frac{d(I_{cr} \cdot \Omega_{cr})}{dt} = -N_\beta - N_\rho = I_\beta (\Omega_s - \Omega_{cr}) \cdot (\delta + \Omega_s \hat{\Omega}_{s}) + I_\beta (\Omega_s \times \Omega_{cr}) \]
(52)

\[ I_\beta \frac{d\Omega_{cr}}{dt} = N_\beta + N_\rho = -I_\beta (\Omega_s - \Omega_{cr}) \cdot (\delta + \Omega_s \hat{\Omega}_{s}) + I_\beta (\Omega_{cr} \times \Omega_s) , \]
(53)

where we have substituted \( \Omega_s \) for \( \Omega_n \) in equations (50) and (51) and assumed that the angular momentum of the superfluid is \( I_\beta \Omega_s \), which is tantamount to assuming that the moment of inertia tensor of the superfluid is of the form

\[ I_\beta = (I_\beta - \Delta I_\beta) \delta + \Delta I_\beta \left( \frac{3 \Omega_s \hat{\Omega}_{s}}{2} - \delta \right) . \]
(54)

Notice that if \( \beta' = 1 \) and \( \beta = 0 \), these equations reduce to

\[ \frac{d(I_{cr} \cdot \Omega_{cr})}{dt} + I_\beta (\Omega_{cr} \times \Omega_s) = 0 , \]
(55)

which is equivalent to equation (1) with the contribution from the core component omitted, and

\[ \frac{d\Omega_{cr}}{dt} = \Omega_{cr} \times \Omega_s , \]
(56)

which implies that \( \Omega_s \) is fixed in the reference frame that rotates with the superfluid. This is the limit of perfect pinning and results in undamped precession at the frequency \( \sigma \).

When \( \beta' \neq 1 \) and \( \beta \neq 0 \), there are additional modes. Let us work in the frame rotating with the crust, in which case (recall that \( d^*F/dt \) is the time derivative of \( f \) in this frame)

\[ I_{cr} \cdot \frac{d^*\Omega_{cr}}{dt} + \Omega_{cr} \times (I_{cr} \cdot \Omega_{cr}) = I_\beta (\Omega_s - \Omega_{cr}) \cdot (\delta + \Omega_s \hat{\Omega}_{s}) + I_\beta (\Omega_s \times \Omega_{cr}) , \]
(57)

\[ \frac{d^*\Omega_{cr}}{dt} + (1 - \beta')(\Omega_{cr} \times \Omega_s) = -I_\beta (\Omega_s - \Omega_{cr}) \cdot (\delta + \Omega_s \hat{\Omega}_{s}) . \]
(58)

If we project equations (57) and (58) along the principal axes of the crust, and linearize around the fixed point at which \( \Omega_{cr} = \Omega_s = \Omega \) and \( \Omega \parallel \hat{\Omega}_{s} \), we find

\[ \Omega_{cr,1} + \left[ \begin{pmatrix} I_1 & -I_2 \\ I_2 & I_1 \end{pmatrix} + I_\beta \beta' \right] \Omega_{cr,2} = -I_\beta \Omega_{cr,2} \]
(59)

\[ \Omega_{cr,2} - \left[ \begin{pmatrix} I_1 & -I_2 \\ I_2 & I_1 \end{pmatrix} + I_\beta \beta' \right] \Omega_{cr,1} = -I_\beta \Omega_{cr,1} \]
(60)

\[ \Omega_{cr,3} = -2 \frac{I_\beta}{I_3} (\Omega_{cr,3} - \Omega_{s,3}) , \]
(61)

\[ \Omega_{s,1} - (1 - \beta') (\Omega_{s,2} - \Omega_{cr,2}) = -\beta (\Omega_{s,1} - \Omega_{cr,1}) , \]
(62)

\[ \Omega_{s,2} + (1 - \beta') (\Omega_{s,1} - \Omega_{cr,1}) = -\beta (\Omega_{s,2} - \Omega_{cr,2}) . \]
(63)
\[ \dot{\Omega}_{s,3} = -2\beta(\Omega_{s,3} - \Omega_{cr,3}) . \] (64)

As before, the evolution of the perturbations along the 3-axis decouple from those along the other axes and decay exponentially; the rate of decay is \(2\beta(1 + I_s/I_3)\). The remaining equations imply a fourth-order characteristic equation if we search for modes \(\propto \exp(p\phi)\).

### 3.1.1. Axisymmetric Crust

When the crust is axisymmetric, \(I_1 = I_2\), and equations (59) and (60) simplify to

\[ \dot{\Omega}_{cr,1} + \left( \frac{I_3 - I_1}{I_1} \right) \Omega_{cr,2} - \frac{I_s\beta'}{I_1} \Omega_{a,1} = -\frac{I_s \beta}{I_1} (\Omega_{cr,1} - \Omega_{a,1}) , \] (65)

\[ \dot{\Omega}_{cr,2} - \left( \frac{I_3 - I_1}{I_1} \right) \Omega_{cr,1} + \frac{I_s\beta'}{I_1} \Omega_{a,1} = -\frac{I_s \beta}{I_1} (\Omega_{cr,2} - \Omega_{a,2}) ; \] (66)

these couple to equations (62) and (63), which are unchanged.

As we found in § 2.1, the fourth-order characteristic equation may be reduced to second order in this case. Define

\[ \sigma' = \left( \frac{I_3 - I_1}{I_1} \right) + \frac{I_s\beta'}{I_1} \equiv \epsilon + \frac{I_s\beta'}{I_1} \] (67)

and let

\[ \Omega_s^{(+)} = \Omega_{a,1} + i\Omega_{a,2} ; \] (68)

then we get the two coupled equations

\[ \dot{\Omega}_s^{(+)} - i\sigma' \Omega_s^{(+)} + i \frac{I_s\beta'}{I_1} \Omega_s^{(+)} = -\frac{I_s \beta}{I_1} [\Omega_s^{(+)} - \Omega_s^{(+)}] ; \] (69)

\[ \dot{\Omega}_s^{(+)} + i(1 - \beta')[\Omega_s^{(+)} - \Omega_s^{(+)}] = -\beta[\Omega_s^{(+)} - \Omega_s^{(+)}] . \] (70)

It turns out to be convenient to use

\[ \Delta^{(+)} \equiv \Omega_s^{(+)} - \Omega_s^{(+)} \] (71)

instead of \(\Omega_s^{(+)}\); doing so yields the coupled equations

\[ \dot{\Omega}_s^{(+)} - i\epsilon \dot{\Omega}_s^{(+)} + \frac{I_s}{I_1} (i\beta' - \beta) \Delta^{(+)} = 0 , \] (72)

\[ \Delta^{(+)} + \left\{ \left[ 1 - \beta' \left(1 + \frac{I_s}{I_1} \right) \right] + \beta \left(1 + \frac{I_s}{I_1}\right) \right\} \Delta^{(+)} + i\epsilon \dot{\Omega}_s^{(+)} = 0 . \] (73)

Assuming that \([\Omega_{cr}^{(+)} , \Delta^{(+)}] \propto \exp(p\phi)\) we find the relation

\[ \Delta^{(+)} = -\frac{p\dot{\Omega}_{cr}^{(+)}}{p + i(1 - \beta') + \beta} \] (74)

and the characteristic equation

\[ p^2 + p \left[ i(1 - \beta' - \sigma') + \beta \left(1 + \frac{I_s}{I_1}\right) \right] + \epsilon(1 - \beta' - i\beta) = 0 . \] (75)

Equation (74) is useful for finding the eigenvectors once equation (75) is solved; these are needed to determine the response of the two spin components to external torques.

Although we can solve equation (75) exactly, it is instructive to consider the two limiting cases of weak and strong vortex drag separately. When vortex drag is weak, \(\beta'\) and \(\beta\) are small in magnitude, so we rewrite equation (75) as

\[ (p - i\epsilon)(p + i) - (\beta' + i\beta) \left[ ip \left(1 + \frac{I_s}{I_1}\right) + \epsilon \right] = 0 . \] (76)

The solutions to this equation to first order in the small quantities \(\beta\) and \(\beta'\) are

\[ p_d = -i + \frac{(i\beta' - \beta)(1 + \epsilon + I_s/I_1)}{1 + \epsilon} \] (77)

and (the Shakh mode)

\[ p_p = i\epsilon + \frac{(i\beta' - \beta)I_s}{I_1(1 + \epsilon)} . \] (78)
Both of these solutions damp slowly, at rates proportional to \( \beta > 0 \). The second mode reduces to the conventional Euler precession when \( \beta = \beta' = 0 \). The first mode arises because the superfluid angular velocity would remain fixed in the inertial frame if \( \beta \) and \( \beta' \) were zero but wanders slowly when the coupling is small but nonzero. (We discuss this point more fully in the context of three-component models; see § 4.1.1 below and discussion following eq. [148].) For strong vortex drag, \( 1 - \beta' \) and \( \beta' \) are small, and we rewrite equation (75) in the form

\[
p(p - i\alpha) + (1 - \beta')(ip + \epsilon) + \beta\left[p\left(1 + \frac{I_s}{I_1}\right) - i\epsilon\right] = 0.
\]  

(79)

The solutions to this equation to first order in the small quantities \( \beta \) and \( 1 - \beta' \) are

\[
p_d = -\frac{\epsilon[\beta + i(1 - \beta')]}{\sigma'},
\]

(80)

\[
p_p = i\sigma' - \frac{I_s[i(1 - \beta') + \beta(1 + \sigma')]}{I_1 \sigma'}.
\]  

(81)

The first of these solutions represents a slowly damped mode with an oscillatory part that is negligible when \( \eta' \ll \eta \), implying \( 1 - \beta' \ll \beta' \) in this limit. (Recall discussion following eqs. [37] and [38] in § 2.2.) The second mode corresponds to precession at \( \sigma' \) with slow damping: For \( I_s/I_1 \gg \epsilon \), equation (67) implies that \( \sigma' \approx I_s/I_1 \) for \( 1 - \beta' \ll 1 \), so the damping rate is approximately \( \beta(1 + I_s/I_1) \); in the unlikely event that \( \epsilon \gg I_s/I_1 \), then \( \sigma' \approx \epsilon \) and the damping rate is approximately \( (I_s/I_1)\beta(1 + \epsilon) \).

3.1.2. Nonaxisymmetric Crust

Since the neutron star crust may not be axisymmetric, it is worth checking that there are no surprises when \( I_1 \neq I_2 \). Define

\[
\sigma_1' = \left(\frac{I_3 - I_2}{I_1}\right) + \frac{I_s \beta'}{I_1} \equiv \epsilon_1 + \frac{I_s \beta'}{I_1},
\]

(82)

\[
\sigma_2' = \left(\frac{I_3 - I_1}{I_2}\right) + \frac{I_s \beta'}{I_2} \equiv \epsilon_2 + \frac{I_s \beta'}{I_2};
\]  

(83)

then equations (59) and (60) become

\[
\Omega_{cr,1} + \sigma_1' \Omega_{cr,2} - \frac{I_s \beta'}{I_1} \Omega_{s,2} = -\frac{I_s \beta'}{I_1} (\Omega_{cr,1} - \Omega_{s,1}),
\]

(84)

\[
\Omega_{cr,2} - \sigma_2' \Omega_{cr,1} + \frac{I_s \beta'}{I_2} \Omega_{s,1} = -\frac{I_s \beta'}{I_2} (\Omega_{cr,2} - \Omega_{s,2}),
\]  

(85)

which must be solved along with equations (62) and (63). When we look for solutions \( \propto \exp (p\phi) \) we find the fourth-order characteristic equation

\[
0 = p^4 + p^3 \beta (2 + \frac{I_s}{I_1} + \frac{I_s}{I_2})
\]

\[
+ p^2 \left[\beta^2 + (1 - \beta')^2\right] \left(1 + \frac{I_s}{I_1}\right) \left(1 + \frac{I_s}{I_2}\right) - (1 - \beta')^2 \frac{I_s^2}{I_1 I_2} - (1 - \beta') \left(\frac{I_s}{I_1} + \frac{I_s}{I_2}\right) + \sigma_1' \sigma_2'
\]

\[
+ p \beta \left(2\epsilon_1 \epsilon_2 + \frac{I_s \epsilon_2}{I_1} + \frac{I_s \epsilon_1}{I_2}\right) + \epsilon_1 \epsilon_2 [\beta^2 + (1 - \beta')^2].
\]  

(86)

It is not possible to factorize the characteristic equation into the product of two second-order equations because there is no guarantee that all of the roots are simply complex conjugate pairs.

In the limit of weak coupling, expanding equation (86) up to first order in \( \beta \) and \( \beta' \) yields

\[
0 = (p^2 + 1)(p^2 + \epsilon_1 \epsilon_2) + \beta \left[p^2 \left(2 + \frac{I_s}{I_1} + \frac{I_s}{I_2}\right) + p \left(2\epsilon_1 \epsilon_2 + \frac{I_s \epsilon_2}{I_1} + \frac{I_s \epsilon_1}{I_2}\right)\right]
\]

\[
- \beta \left[p^2 \left(2 + \frac{I_s(1 - \epsilon_2)}{I_1} + \frac{I_s(1 - \epsilon_1)}{I_2}\right) + 2\epsilon_1 \epsilon_2\right].
\]  

(87)

The approximate solutions of this form for the characteristic equation are

\[
p_d = -i + \frac{2(1 - \epsilon_1 \epsilon_2)}{2(1 - \epsilon_1 \epsilon_2)},
\]

(88)
\[ p_\rho = i\sqrt{\epsilon_1 \epsilon_2} + \frac{i\beta\sqrt{\epsilon_1 \epsilon_2}(I_d / I_1)(1 - \epsilon_2) + (I_d / I_2)(1 - \epsilon_1)}{2(1 - \epsilon_1 \epsilon_2)} - \frac{\beta(I_d / I_1)\epsilon_2(1 - \epsilon_1) + (I_d / I_2)\epsilon_1(1 - \epsilon_2)}{2(1 - \epsilon_1 \epsilon_2)}, \]  

and their complex conjugates. To get limiting results in the strongly coupled domain, we rewrite equation (86) in the slightly modified form

\[ 0 = p^2(p^2 + \sigma_1^2 \sigma_2^2) + p^2\beta\left(2 + \frac{I_d}{I_1} + \frac{I_d}{I_2}\right) \]

\[ + p^2\left[\beta^2 + (1 - \beta^2)\left(1 + \frac{I_d}{I_1}\right)\left(1 + \frac{I_d}{I_2}\right) - (1 - \beta^2)^2 \frac{I_d^2}{I_1 I_2} - (1 - \beta^2)\left(\frac{I_d}{I_1} + \frac{I_d}{I_2}\right)\right] \]

\[ + p\beta\left(2\epsilon_1 \epsilon_2 + \frac{I_d \epsilon_1}{I_1} + \frac{I_d \epsilon_2}{I_2}\right) + \epsilon_1 \epsilon_2[\beta^2 + (1 - \beta^2)^2]. \]  

(90)

From equation (90), it is evident that the modes are near \( p^2 = 0 \) and \( p^2 = -\sigma_1^2 \sigma_2^2 \). To get the first-order approximation to the modes with \( p^2 = 0 \), we substitute all terms in equation (90) that are potentially second order in small quantities; this leads to the quadratic equation

\[ 0 = p^2\sigma_1^2 \sigma_2^2 + p\beta\left(2\epsilon_1 \epsilon_2 + \frac{I_d \epsilon_1}{I_1} + \frac{I_d \epsilon_2}{I_2}\right) + \epsilon_1 \epsilon_2[\beta^2 + (1 - \beta^2)^2], \]  

(91)

which has the pair of roots

\[ p_\rho^\pm = -\frac{\beta(\epsilon_1 \sigma_2^2 + \epsilon_2 \sigma_1^2)}{2\sigma_1 \sigma_2} \pm \frac{1}{\sigma_1 \sigma_2} \left[\frac{\beta^2(\epsilon_1 \sigma_2^2 - \epsilon_2 \sigma_1^2)^2}{2\sigma_1 \sigma_2^2} - \frac{\epsilon_1 \epsilon_2(1 - \beta^2)^2}{\sigma_1 \sigma_2^2}\right]^{1/2}. \]  

(92)

In the axisymmetric limit, these two roots reduce to the damped mode found in § 3.1.1 and its complex conjugate, but although \( p_\rho^\pm \) both imply damping in general, they could be purely real and different in magnitude, especially since we expect \( 1 - \beta^2 \) to be much smaller than \( \beta \) in the strongly coupled regime. (This is why we could not factor eq. [86] into two quadratic equations.) It is also straightforward to expand equation (90) around the approximate root \( p \approx i\sqrt{\sigma_1 \sigma_2} \) to find

\[ p_\rho = i\sqrt{\sigma_1 \sigma_2} - \frac{i(1 - \beta)}{2\sqrt{\sigma_1 \sigma_2}} \left(\frac{I_d}{I_1} + \frac{I_d}{I_2}\right) - \frac{\beta}{2\sqrt{\sigma_1 \sigma_2}} \left[\frac{I_d}{I_1} \sigma_2(1 + \sigma_1) + \frac{I_d}{I_2} \sigma_1(1 + \sigma_2)\right], \]  

(93)

to first order in the small quantities \( \beta \) and \( 1 - \beta^2 \).

From this brief foray into the modes of a triaxial star, we conclude that deviations from axisymmetry do not alter the behavior of the precession qualitatively in either limit. The character of the damped modes may be different in the strong coupling limit for nonaxisymmetric stars, but if so, they become purely damped, with no oscillation at all. Consequently, from here on we specialize to axisymmetric crusts, since we do not expect to miss any important oscillatory modes.

### 3.2. Response to External Torques

As we have seen, the modes of free precession for this two-component model are rapidly oscillating and/or damped. Here, we consider the response of the system to external torques. Our analysis will reveal that the limit of perfect coupling between the normal fluid and superfluid must be taken with care when external torques act.

If we suppose that the crust is subject to an arbitrary time-dependent torque, \( N_\text{cr}(\phi) \), then equations (72) and (73) are changed to

\[ \dot{\Omega}_\text{cr}^{(+)} - i\epsilon \Omega_\text{cr}^{(+)} + \frac{I_d}{I_1} (i\beta - \beta) \Delta^{(+)} = \tilde{N}_\text{cr}^{(+)}(\phi), \]

(94)

\[ \dot{\Lambda}^{(+)} + \left[i(1 - \beta)(1 + \frac{I_d}{I_1}) + \beta\left(\frac{I_d}{I_1}\right)\right] \Delta^{(+)} + i\epsilon \Omega_\text{cr}^{(+)} = -\tilde{N}_\text{cr}^{(+)}(\phi), \]  

(95)

where \( \tilde{N}_\text{cr}^{(+)} \equiv I_1^{-1}(N_{\text{cr},1} + iN_{\text{cr},2}) \). Apart from decaying transients, the solution to these equations is

\[ \Omega_\text{cr}^{(+)} = \sum_{a=p,d} A_a \int_{-\infty}^{\phi} d\phi' \tilde{N}_\text{cr}^{(+)}(\phi') \exp \left[p_a(\phi - \phi')\right], \]  

(96)

\[ \Delta^{(+)} = -\sum_{a=p,d} \frac{p_a A_a}{p_a + i(1 - \beta) + \beta} \int_{-\infty}^{\phi} d\phi' \tilde{N}_\text{cr}^{(+)}(\phi') \exp \left[p_a(\phi - \phi')\right], \]  

(97)

where the coefficients are

\[ A_p = \frac{[p_p + i(1 - \beta) + \beta]}{(p_p - p_d)}, \quad A_d = \frac{[p_d + i(1 - \beta) + \beta]}{(p_d - p_p)}. \]  

(98)
In the limit of perfect coupling between the crust and superfluid, we take \( i(1 - \beta') + \beta \to 0 \) with \( p_d/[i(1 - \beta') + \beta] = -\epsilon/\sigma' \) constant; taking this limit of equations (96) and (97) naively yields

\[
\Omega_{cr}^{(+)} = \int_{-\infty}^{\phi} d\phi' N_{cr}^{(+)} \exp \left[ p_p(\phi - \phi') \right],
\]

with \( \Delta^{(+)} \to -\Omega_{cr}^{(+)} \).

Actually, the perfect coupling limit is a bit more subtle than the manipulations leading to equation (99). To see why, consider the response to a time-independent torque on the crust. Then equation (96) may be integrated readily and we find

\[
\Omega_{cr}^{(+)} = \frac{iN_{cr}^{(+)} + d}{\epsilon};
\]

(100)

integrating equation (97) yields \( \Delta^{(+)} = 0 \). These results ought to hold for any \( i(1 - \beta') + \beta \). However, equation (99), which purports to describe the limit of perfect coupling, \( i(1 - \beta') + \beta \equiv 0 \), yields

\[
\Omega_{cr}^{(+)} = -\frac{N_{cr}^{(+)} + d}{p_p} \frac{iN_{cr}^{(+)} + d}{\sigma'}
\]

(101)

(assuming a small, negative real part to \( p_p \)). Which of these results is correct?

To resolve the conundrum, consider a torque that turns on (or can be regarded as constant since) some time in the past, \( \phi_0 \). Then equations (96) and (97) yield the response

\[
\Omega_{cr}^{(+)} = \left\{ i + \left[ 1 + \frac{i(1 - \beta') + \beta}{p_p} \right] \frac{\exp \left[ p_p(\phi - \phi_0) \right]}{p_p - p_d} - \left[ 1 + \frac{i(1 - \beta') + \beta}{p_d} \right] \frac{\exp \left[ p_d(\phi - \phi_0) \right]}{p_p - p_d} \right\} N_{cr}^{(+)}.
\]

(102)

\[
\Delta^{(+)} = \left\{ -\exp \left[ p_p(\phi - \phi_0) \right] + \exp \left[ p_d(\phi - \phi_0) \right] \right\} N_{cr}^{(+)}.
\]

(103)

We suppose that the damping constant associated with the precessing mode, \( p_p \), is large enough that \( \exp \left[ p_p(\phi - \phi_0) \right] \to 0 \); then what we find depends on \( p_d(\phi - \phi_0) \). (Recall that the real parts of \( p_p \) and \( p_d \) are negative.) If \( |\text{Re}[p_d(\phi - \phi_0)]| \gg 1 \), so that any transient response has plenty of time to damp away between \( \phi_0 \) and \( \phi \), then we recover equation (100) and also find that \( \Delta^{(+)} \to 0 \). On the other hand, if \( |\text{Re}[p_d(\phi - \phi_0)]| \ll 1 \), then we recover equation (101) and \( \Delta^{(+)} \to -\Omega_{cr}^{(+)} \) in the limit of perfect coupling, \( i(1 - \beta') + \beta \to 0 \), to zeroth order in \( p_d(\phi - \phi_0) \). To first order we also find a term that grows linearly with \( p_d(\phi - \phi_0) \); over a sufficiently long time span, this growth would change the tilt from equation (100) to equation (101).

The resolution of the apparent paradox is that there is none: equations (96) and (97) are always the correct ones to use. What one gets in the limit of strong vortex drag depends on how the timescale on which the external torque changes compares with the timescales inherent in the coupling of superfluid to the normal crust. Equations (100) and (101) both have domains of validity; equation (101) is a lower bound to the steady state tilt of the rotational angular velocity away from the 3-axis in the strong coupling limit. As long as the damping timescale associated with \( p_d \) is short compared with any timescale associated with changes in the external torque, however, equation (100) gives the right response. In practical terms, pulsar spin-down provides a nearly constant torque on the crust, which can give rise to \( N_{cr}^{(+)} \). Thus, if \( p_d \) implies decay on timescales smaller than the pulsar spin-down time, then equation (100) describes the response of the crust, even in the strong pinning limit.

We emphasize that this result could not be found from a consideration of normal modes alone; arriving at it requires examining the response of the star to a torque. There is therefore a subtle aspect to the limiting case considered by Shaham (1977): While the modal frequencies he derived are correct, and his conclusions about free precession warranted, blithely using equation (99), which would follow from the assumption of perfect pinning, instead of equations (96) and (97), is wrong even in the limit of strong pinning.

The response of the star to torques along the 3-axis is found by solving the equations

\[
\Omega_{cr,3} = \frac{2I_p}{I_3} \Delta_3 = \tilde{N}_{cr,3}(\phi),
\]

(104)

\[
\Delta_3 + 2\beta \left( 1 + \frac{I_p}{I_3} \right) \Delta_3 = -\tilde{N}_{cr,3}(\phi),
\]

(105)

where \( \tilde{N}_{cr,3} = N_{cr,3}/I_3 \); the result is

\[
\Delta_3 = -\int_{-\infty}^{\phi} d\phi' \tilde{N}_{cr,3}(\phi') \exp \left[ -2\beta \left( 1 + \frac{I_p}{I_3} \right) (\phi - \phi') \right],
\]

(106)

\[
\Omega_{cr,3} = \frac{I_3}{I_3 + I_s} \int_{\phi}^{0} d\phi' \tilde{N}_{cr,3}(\phi') + \frac{I_s}{I_3 + I_s} \int_{-\infty}^{\phi} d\phi' \tilde{N}_{cr,3}(\phi') \exp \left[ -2\beta \left( 1 + \frac{I_p}{I_3} \right) (\phi - \phi') \right].
\]

(107)
In particular, a time-independent torque results in a steady angular velocity difference

$$\Delta_3 = -\frac{\tilde{N}_{cr,3}}{2\beta(1 + I_s/I_3)},$$

while the crustal angular velocity changes linearly:

$$\dot{\Omega}_{cr,3} = \frac{\tilde{N}_{cr,3} I_3}{I_s + I_3}.$$  \hspace{1cm} (109)$$

The response to an impulsive torque $\tilde{N}_{cr,3} = N_{0,3} \delta(\phi - \phi_0)$ is

$$\Omega_{cr,3} = \frac{N_{0,3}(I_3 + I_s) \exp \left( -2\beta(1 + I_s/I_3)(\phi - \phi_0) \right)}{I_s + I_3},$$

$$\Delta_3 = -N_{0,3} \exp \left[ -2\beta(1 + I_s/I_3)(\phi - \phi_0) \right].$$

Spin-down torques, which only change on very long timescales for all observed pulsars, provide a physical realization of a "time-independent" torque. Let us take the vacuum magnetic dipole torque, which is proportional to $-\mu \times (\mathbf{\omega} \times \mu)$ (e.g., Davis & Goldstein 1970; Goldreich 1970; Michel 1991); assuming a magnetic dipole moment $\mu$ that is fixed in the frame of the crust, with components

$$\mu = \cos \alpha \hat{e}_3 + \sin \alpha \hat{e}_1,$$

the torque may be written (to a good first approximation) as

$$N_d = N_{sd} \sin \alpha \hat{e}_1 \cos \alpha - \hat{e}_3 \sin \alpha \equiv N_{sd} \hat{e}_1 \cot \alpha - \hat{e}_3,$$

implying $\tilde{N}^{(+)\alpha}_{cr} = N_{sd} \cot \alpha / I_1$ and $\tilde{N}_{cr,3} = -N_{sd}/I_3$. The steady state response to these torques is

$$\Omega_{cr,3}^{(+\alpha)} = \frac{iN_{sd} \cot \alpha}{\epsilon I_1}, \quad \Delta^{(+\alpha)} = 0,$$

$$\Delta_3 = \frac{N_{sd}}{2\beta(I_s + I_3)},$$

with $\dot{\Omega}_{cr,3} = -N_{sd}/I_3$. Two aspects of these results are especially noteworthy. First, as is well known, the superfluid rotates faster than the crust as a consequence of the spin-down torque. Second, but not so widely appreciated, the steady state "tilt" in the angular velocity of the crust is surprisingly large, as it is proportional to $\epsilon^{-1}$. (When $\epsilon = 0$, eq. [75] implies that $p_d = 0$, and $\Omega_{cr,3}^{(+\alpha)}$ grows linearly with time according to eq. [96].)

### 3.3. Free Precession Encore

#### 3.3.1. Different Crust and Superfluid Angular Velocities in Unperturbed State

The fact that time-independent spin-down of the crust implies a difference between $\Omega_{cr,3}$ and $\Omega_{s,3}$ in steady state suggests that we explore free precession once again, but with a different unperturbed state than we used in §3.1. There we assumed that the undisturbed star rotates with $\Omega_s = \Omega_{cr} = \Omega_{3a}$. Let us consider instead what happens when $\Omega_s = \Omega_{cr,3}$ and $\Omega_s = \xi \Omega_{cr,3}$ in the unperturbed state. We shall not assume that $|\xi - 1|$ must be small, although we expect this to be true.

The equations governing the precession of an axisymmetric star given this new unperturbed state are

$$\dot{\Omega}_{cr,3}^{(+\alpha)} - i\sigma'_\xi \Omega_{cr,3}^{(+\alpha)} + i \frac{L}{I_1} \beta \Omega_{cr,3}^{(+\alpha)} = -\frac{I_s}{I_1} \beta \left[ \Omega_{cr,3}^{(+\alpha)} - \Omega_{cr,3}^{(+\ell)} \right],$$

$$\dot{\Omega}_{s,3}^{(+\alpha)} + i(1 - \beta')\Omega_{s,3}^{(+\alpha)} - \xi \Omega_{cr,3}^{(+\alpha)} = -\beta \left[ \Omega_{s,3}^{(+\alpha)} - \Omega_{cr,3}^{(+\alpha)} \right],$$

where

$$\sigma'_\xi \equiv \epsilon + \frac{\beta L_1 \xi}{I_1}.$$  \hspace{1cm} (118)$$

These equations yield the characteristic equation

$$p^2 + p \left[ i(1 - \beta' - \sigma'_\xi) + \beta \left( 1 + \frac{I_s}{I_1} \right) \right] + \epsilon(1 - \beta' - i\beta) + i \frac{\beta L}{I_1} (1 - \xi) = 0;$$

in the weakly coupled limit, the solutions are

$$p_d = -i + \frac{(i\beta' - \beta)(1 + \epsilon + \xi I_s/I_1)}{1 + \epsilon},$$

$$p_p = i\epsilon + \frac{\{i\epsilon\beta' - \beta[\epsilon - (\xi - 1)]\} I_s}{I_1(1 + \epsilon)},$$

$$p_m = \frac{i\epsilon}{I_1} (1 + \epsilon).$$

and in the strong coupling limit,

\[
p_a = -\frac{i\epsilon(1 - \beta') + \beta[\epsilon + I_a(\xi - 1)/I_a]}{\epsilon + \xi I_a I_1},
\]

\[
p_p = \frac{I_a[\xi(1 - \beta') + \beta(1 + \sigma \xi)]}{I_1 \sigma \xi}.
\]

These equations are hardly different than their \(\xi = 1\) counterparts except for two noteworthy differences. First, in the weakly coupled limit, the real part of \(p_p\) can become positive for \(\xi - 1 > \epsilon\), implying linear instability. However, this is not worrisome since the growth time of the mode is of order the spin-down time of the star if \(\xi = 1 = N_{a_0}/2\beta(I_s + I_b)\).

Second, in the strongly coupled limit, the real part of \(p_a\) is enhanced for \(\xi - 1 > 0\), implying faster damping. Otherwise, the effect of \(\xi \neq 1\) is merely to renormalize the various coefficients appearing in the solutions to the characteristic equation without introducing any qualitatively new behavior. Consequently, we shall not consider the effects of differential rotation further in this paper.

3.3.2. Interaction of Two Regions with Simple Modes

In §§ 3.1.1 and 3.1.2, we explored the characteristic modes of precession for axisymmetric and nonaxisymmetric crusts in the limits of weak and strong coupling to the crustal superfluid. In general, we can imagine that there are regions of the crust to which the superfluid couples with different strengths. These regions are presumably linked to one another by elastic forces that seek to enforce corotation.

To get a crude idea of what might transpire in such a situation, let us imagine dividing the crust into two components, \(a\) and \(b\). To get a schematic feeling for the normal modes of the coupled system, suppose the angular velocities can be described by

\[
\Omega_a^{(\pm)} - p_a \Omega_a^{(\pm)} = -\frac{\gamma I_b}{I_a + I_b} [\Omega^{(\pm)}_a - \Omega^{(\pm)}_b],
\]

\[
\Omega_b^{(\pm)} - p_b \Omega_b^{(\pm)} = -\frac{\gamma I_a}{I_a + I_b} [\Omega^{(\pm)}_b - \Omega^{(\pm)}_a],
\]

where \(\Omega^{(\pm)}_a = \Omega_{a,1} + i\Omega_{a,2}\) and \(\Omega^{(\pm)}_b = \Omega_{b,1} + i\Omega_{b,2}\). This system of equations has the characteristic equation

\[0 = p^2 - p(p_a + p_b - \gamma) + p_a p_b - \frac{\gamma(p_a I_a + p_b I_b)}{I_a + I_b}.\]

When \(\gamma\) is small, we rewrite the above equation as

\[0 = (p - p_a)(p - p_b) + \gamma(p - p_a I_a + p_b I_b)/(I_a + I_b);\]

to first order in \(\gamma\), the roots are

\[p_1 = p_a - \frac{\gamma I_b}{I_a + I_b},\]

\[p_2 = p_b - \frac{\gamma I_a}{I_a + I_b}.
\]

The effect of weak coupling is additional damping of the modes of the individual components. When \(\gamma\) is large, we rewrite the characteristic equation as

\[0 = p - \frac{p_a I_a + p_b I_b}{I_a + I_b} + \gamma^{-1}(p - p_a)(p - p_b).
\]

In this limit the two roots are

\[p_1 = -\gamma + \frac{p_a I_b + p_b I_a}{I_a + I_b},\]

\[p_2 = \frac{p_a I_a + p_b I_b}{I_a + I_b} - \frac{(p_b - p_a)(I_a - I_b)}{\gamma(I_a + I_b)}.
\]

In this case, \(p_1\) represents almost pure damping at a rate close to \(\gamma\), whereas \(p_2\) represents damped precession at a rate that is approximately the sum of the precession frequencies of the individual components weighted by their moment of inertia fractions.

Consequently, if the characteristic coupling timescales among different components of the crust are short, we find a mean precession frequency weighted toward regions that comprise the bulk of the crustal moment of inertia. Only if the coupling times are long will the precession frequencies of individual crustal components be apparent. This indicates that even if there are small regions of unpinned superfluid in the crust, precession at the Euler rate appropriate to those zones may only be seen
if they do not couple to the rest of the crust efficiently, and that even if this is so, the precession will damp out eventually as a consequence of the dissipative interaction.

From time to time it has also been suggested that Tkachenko modes of the core superfluid could manifest themselves in the observed rotation of a pulsar, which is presumably the angular velocity of its crust. If the crustal superfluid is strongly pinned, we can regard one component, $a$, as the crust, with $p_a = \sigma = \epsilon + I_a/I_1$, and the other as the core, with $p_b$ the oscillation frequency of the core in the zero coupling limit. Equation (129) shows that if the interaction between crust and core is weak, there is indeed a mode of oscillation close to $p_b$. For small but nonzero $\gamma$ the oscillations decay at a rate $\gamma I_a/(I_a + I_b)$, which implies a damping time of $\sim \gamma^{-1} I_a/I_1$ rotation periods, if we assume $I_a = I_1 \ll I_b = I_c$; this would be of order $(400-10^5) I_a/I_1$ for the damping times estimated by Alpar & Sauls (1988), or approximately days to months for $I_a/I_1 \sim 10^2$, comparable to the estimated periods of oscillation if they occur. Moreover, for small values of $\gamma$, the effect of the core oscillations on the crustal angular velocity is small: equation (124) implies that

$$\Omega_a^{(+) +} = \frac{\gamma I_b}{(I_b + I_0)(p_b - p_a) + \gamma I_b} \Omega_b^{(+) +},$$

which means that the amplitude of the oscillations in the angular velocity of the crust is $\sim \gamma I_a/p_a(I_a + I_b)$ times the amplitude of the oscillations of the angular velocity of the core, assuming that $|p_a| \gg |p_b|$. Thus, even if the coupling is so weak that the damping time $\tau_g$ associated with core oscillations is very long, their observable manifestation in the angular velocity of the crust is suppressed by a factor of $\sim (\Omega_g)^{-1}$.

4. THREE-COMPONENT STAR

The calculations presented in § 3 are valid if the crust and crustal superfluid are either completely decoupled from the core of the star or coupled to it perfectly. In the limit of perfect pinning, Shaham (1977) found that precession was damped as a consequence of imperfect coupling to the core. Here we examine a model consisting of three components, two of which are superfluid components that couple directly to the rigid crust. We have in mind two possible applications, one in which the three components are rigid crust, crustal superfluid, and core (super)fluid, and another in which the three components are rigid crust and two different regions of crustal superfluid with different frictional couplings to the rigid crust.

4.1. Free Precession

4.1.1. Crust, Crustal Superfluid, and Core (Super)Fluid

We assume that the core couples directly only to the rigid component of the crust, via a torque of the form

$$N_{cc} = -I_c \xi (\Omega_c - \Omega_{cc}) + I_c \xi (\Omega_{cr} \times \Omega_c),$$

for small differences between the angular velocities of the crust and core, which are both nearly $\Omega_c^\circ$. This form of the torque is analogous to equations (50) and (51), except that the dissipative torque has been assumed to be isotropic (by contrast to $N_p$). We have also included a nondissipative torque in $N_{cc}$, unlike Shaham (1977; see § 2.1); this contribution can be ignored by setting $\xi = 0$. Below, we shall assume that $\xi = \zeta$, at least for keeping track of small quantities.

If we define

$$\Delta_{(+)}^{(c)} = \Omega_{cr,1} - \Omega_{cr,1} + i(\Omega_{cr,2} - \Omega_{cr,1}),$$

then the coupled equations for the three-component star may be written in the form

$$\dot{\Omega}_{cr}^{(+)} - i\nu \Omega_{cr}^{(+)} + \frac{I_x}{I_1}(i\nu' - \beta)\Delta_{(+)}^{(c)} + \frac{I_c}{I_1}(i\nu' - \zeta)\Delta_{(+)}^{(c)} = 0,$$

$$\dot{\Delta}_{(+)}^{(c)} + \left\{ \begin{array}{l} \frac{1}{c'} - \frac{1}{c} \left( 1 + \frac{I_x}{I_1} \right) \\ \frac{1}{c'} - \frac{1}{c} \left( 1 + \frac{I_c}{I_1} \right) \end{array} \right\} \Delta_{(+)}^{(c)} + i\nu \Omega_{cr}^{(+)} - \frac{I_c}{I_1}(i\nu' - \beta)\Delta_{(+)}^{(c)} = 0,$$

$$\dot{\Delta}_{(+)}^{(c)} + \left\{ \begin{array}{l} \frac{1}{c'} - \frac{1}{c} \left( 1 + \frac{I_x}{I_1} \right) \\ \frac{1}{c'} - \frac{1}{c} \left( 1 + \frac{I_c}{I_1} \right) \end{array} \right\} \Delta_{(+)}^{(c)} + i\nu \Omega_{cr}^{(+)} - \frac{I_c}{I_1}(i\nu' - \beta)\Delta_{(+)}^{(c)} = 0.$$

Assuming modes proportional to $\exp(p\theta)$, we find that

$$\Delta_{(+)}^{(c)} = -\frac{p\Omega_{cr}^{(+)} + \beta}{p + i(1 - \beta')} + \beta,$$

$$\Delta_{(+)}^{(c)} = -\frac{p\Omega_{cr}^{(+)} + \beta}{p + i(1 - \beta')} + \beta.$$
When $\zeta$ and $\zeta'$ are small, two of the solutions of equation (140) are close to the solutions $p_d$ and $p_p$ of the second-order equation (75); to first order in $\zeta$ and $\zeta'$, the corrections to $p_d$ and $p_p$ are

$$
\delta p_d = \frac{I_c (i \zeta' - \zeta) [p_d (\delta - \beta I_c / I_1) - \epsilon (1 - \delta - i \beta)]}{(i + p_c) (p_d - p_p)},
$$

$$
\delta p_p = \frac{I_c (i \zeta' - \zeta) [p_p (\delta - \beta I_c / I_1) - \epsilon (1 - \delta - i \beta)]}{(i + p_c) (p_p - p_d)},
$$

respectively. The third solution is a new mode near $p = -i$; to first order in small quantities it is

$$
p_d' = -i + \frac{i (\beta - \delta)(i \zeta' - \zeta) [1 + \epsilon + (I_s + I_c) / I_1]}{(i + p_d) (i + p_p)}.
$$

When $\beta$ and $\beta'$ are small, we use equations (77) and (78) in equations (141) and (142) to find

$$
\delta p_d = \frac{I_s (i \zeta' - \zeta) (1 + I_c / I_1)}{I_1 (1 + \epsilon)};
$$

$$
\delta p_p = \frac{I_c (i \zeta' - \zeta)}{I_1 (1 + \epsilon)};
$$

the new mode is

$$
p_d = -i + \left(\frac{I \zeta' - \zeta (1 + \epsilon + I_c / I_1)}{1 + \epsilon + I_c / I_1}\right).
$$

When $\beta$ and $1 - \beta'$ are small, on using equations (80) and (81) in equations (141) and (142) we find that $\delta p_d$ is higher order in $\beta$ and $1 - \beta'$, and hence very small, while

$$
\delta p_p = \frac{I_s (i \zeta' - \zeta) \sigma'}{I_1 (1 + \sigma')};
$$

$$
p_d = -i + \left(\frac{I \zeta' - \zeta (1 + \sigma' + I_c / I_1)}{1 + \sigma'}\right);
$$

these two results are equivalent to equations (23) and (22) of § 2.1 if we substitute $\sigma$ for $\sigma'$ and $-\gamma$ for $(i \zeta' - \zeta)(1 + I_c / I_1)$. The qualitative conclusion reached by Shaham (1977) for weak crust-core coupling is duplicated here: according to equations (145) and (147), precession damps out in $I_c / I_1$, $z$ precession periods irrespective of the effectiveness of vortex drag. We note, however, that the mode corresponding to $p_d'$ implies angular velocities that are nearly but not precisely fixed in the inertial frame of the observer; if $\zeta' > \zeta$, these could complete at least one period of oscillation before decaying away (although we do not expect this to be the case generally). Such modes are also found in studies of the rotation of the Earth, where they may arise from departures from rigid rotation of the fluid core confined by the overlying crust; for the Earth, the result is a retrograde motion of the pole (see Lambeck 1980, § 3.3 for a physical and historical review). Since the mode arises because the core angular velocity remains fixed in the inertial frame when $\zeta$ and $\zeta'$ are zero identically, we expect that for small but finite crust-core coupling, the mode corresponds principally to oscillations of the angular velocity of the core. From equation (139) we find

$$
\frac{\Omega_s^{(z)}}{\Lambda_s^{(z)}} \approx (\zeta' + i \zeta' \left(\sigma' + \frac{I_c}{I_1}\right)) \approx (\zeta + i \zeta) \left(\frac{I_s + I_c}{I_1}\right),
$$

for this mode, which decreases linearly with the frequency of oscillation, but may be substantial nevertheless if $I_s + I_c \gg I_1$.

For large values of $|i \zeta' - \zeta|$, one of the roots of equation (140) is

$$
p_d = (i \zeta' - \zeta) \left(1 + \frac{I_c}{I_1}\right) - \delta \left(1 - \frac{\sigma' I_c}{I_s + I_1}\right),
$$

which is equivalent to equation (25) if we substitute $\sigma$ for $\sigma'$ and $-\gamma$ for $(i \zeta' - \zeta)(1 + I_c / I_1)$; this root does not depend on the strength of the vortex drag to lowest order in $|i \zeta' - \zeta|^{-1}$. When vortex drag is weak, the other two roots of equation (140) are

$$
p_d = -i + \frac{i \epsilon (b - \beta)(1 + \epsilon + I_c / I_1)}{1 + \epsilon + I_c / I_1},
$$

$$
p_p = \frac{I \epsilon (i \beta - \beta)(1 + \epsilon + I_c / I_1)}{I_1 (1 + \epsilon + I_c / I_1)} - \frac{I \epsilon (i \zeta' + \zeta) (1 + \epsilon + I_c / I_1)}{I_s (i \zeta')^2 + \zeta^2 (1 + I_c / I_1)^3}.
$$
(Corrections to eq. [151] proportional to $|i\zeta' - \zeta|^{-1}$ are higher order in $\beta$ and $\beta'$ and have been dropped.) In the limit of strong vortex drag,

$$ p_d = -\frac{\epsilon(1 - \beta') + \beta}{\sigma'}, \tag{153} $$

$$ p_p = \frac{i\sigma'}{1 + I_c/I_1} - \frac{I_s}{I_1\sigma'} \left\{ i(1 - \beta') + \beta \left[ 1 + \sigma' - \frac{I_s I_c}{I_1(I_1 + I_c)} \right] - I_c \sigma'(i\zeta' + \zeta)(1 + \epsilon + I_c/I_1) \right\} - \frac{I_c \sigma'(i\zeta' + \zeta)(1 + \epsilon + I_c/I_1)}{I_1[I(\zeta')^2 + \zeta^2](1 + I_c/I_1)^3}. \tag{154} $$

(Corrections to eq. [153] proportional to $|i\zeta' - \zeta|^{-1}$ are higher order in $\beta$ and $1 - \beta'$ and have been dropped.) The correction for strong but imperfect crust-core coupling in equation (154) is equivalent to equation (26) if $-\gamma$ is substituted for $(i\zeta' - \zeta)$ $(1 + I_c/I_1)$.

Interaction with the core of the star enhances the damping of the precessing modes in all of the limiting cases explored above. For very small or very large coupling between the crust and crustal superfluid, the crust-core interactions are the principal cause of decay. In the strongly pinned regime, the characteristic timescales for decay are just what Shaham (1977) estimated. Imperfect pinning allows a new eigenvalue $p_p$, but the associated mode damps out quickly and so cannot be the explanation for observations of persistent cyclical variations in pulsar spin rates.

4.1.2. Crust Coupled to Two Different Regions of Crustal Superfluid

The equations governing the spin dynamics of this system are the same as equations (136), (137), and (138) if we identify $\Omega_c$ with the angular velocity of the second crustal superfluid component and $\zeta$ and $\zeta'$ with the coefficients coupling this component to the rigid crust. Then it is clear that the characteristic equation for the normal modes of this system is still equation (140), which we rewrite in the form

$$ 0 = p^3 - i p^2 \left( \epsilon + \frac{I_s + I_c}{I_1} \right) $$

$$ + (1 - \beta' - i\beta) \left[ i p^2 \left( 1 - \frac{I_s}{I_1} \right) + p \left( \epsilon + \frac{I_c}{I_1} \right) \right] $$

$$ + (1 - \zeta' - i\zeta) \left[ i p^2 \left( 1 - \frac{I_c}{I_1} \right) + p \left( \epsilon + \frac{I_s}{I_1} \right) \right] $$

$$ + (1 - \beta' - i\beta)(1 - \zeta' - i\zeta) \left[ i \epsilon - p \left( 1 + \frac{I_s + I_c}{I_1} \right) \right], \tag{155} $$

which exhibits symmetry under interchange of superfluid components explicitly.

This form of the characteristic equation is especially useful when both superfluid components couple strongly to the rigid crust. In that limit one of the roots is

$$ p_p = i\sigma'' \left( \frac{1 + \sigma''}{\sigma''} \right) \left[ i(1 - \beta') + \beta \frac{I_s}{I_1} + [i(1 - \zeta') + \zeta'] \frac{I_c}{I_1} \right], \tag{156} $$

and have been dropped.) In the limit of

$$ \sigma'' \equiv \epsilon + \frac{I_s + I_c}{I_1}, \tag{157} $$

the appearance of this root suggests a simple generalization to a multicomponent superfluid, with separate moments of inertia $I_{s,j}$ and coupling coefficients $\beta_j$, and $\beta'_j$:

$$ p_p = i\sigma'' \left( \frac{1 + \sigma''}{\sigma''} \right) \sum_j \left[ i(1 - \beta'_j) + \beta'_j \right] \frac{I_{s,j}}{I_1}, \tag{158} $$

where

$$ \sigma'' \equiv \epsilon + \sum_j \frac{I_{s,j}}{I_1}. \tag{159} $$

The other two roots are first order small to leading order; they are approximately equal to the two roots of the quadratic equation

$$ 0 = p^2 \sigma'' + i p \left( 1 - \beta' - i\beta' \left( \epsilon + \frac{I_s}{I_1} \right) + (1 - \zeta' - i\zeta) \left( \epsilon + \frac{I_c}{I_1} \right) \right) - \epsilon(1 - \beta' - i\beta')(1 - \zeta' - i\zeta). \tag{160} $$

When the crust is only slightly nonspherical, so $\epsilon \ll 1$, the two roots of this equation are approximately

$$ p_+ \approx -\frac{i(1 - \beta') + \beta'(I_c/I_1) + [i(1 - \zeta') + \zeta'](I_s/I_1)}{\sigma''}. \tag{161} $$
\[ p_+ \approx \frac{\epsilon[\iota(1 - \beta') + \beta][\iota(1 - \zeta') + \zeta]}{[\iota(1 - \beta') + \beta](I_c/I_1) + [\iota(1 - \zeta') + \zeta](I_s/I_1)}. \] (162)

Notice that when one of the superfluid components is coupled to the rigid crust more strongly than the other, \( p_+ \) is dominated by the tighter coupled component, but the slow mode \( p_- \) is dominated by the weaker coupled one and reduces to equation (80), with \( \sigma' \approx I_s/I_1 \) if \( I_s/|\iota(1 - \zeta') + \zeta| \gg I_c/|\iota(1 - \beta') + \beta| \) and \( \sigma \approx I_c/I_1 \) if \( I_c/|\iota(1 - \beta') + \beta| \gg I_s/|\iota(1 - \zeta') + \zeta| \).

The limit in which one component couples strongly to the crust while the other couples only weakly is also potentially of interest, particularly in the aftermath of a pulsar glitch, in which some parts of the crustal superfluid may decouple rapidly and recouple only slowly if at all (e.g., Sedrakian 1995). The modes for that situation are given by equation (81) with the correction given in equation (147), equation (80), and equation (148). If \( \zeta' > \zeta \), it is possible that the mode corresponding to \( p_d \) given by equation (148) yields slowly damped oscillations in the angular velocity, as was discussed in § 4.1.1.

4.2. Response to External Torques

This section is analogous to § 3.2, except that we need to consider three distinct external torques, acting on the crust, crustal superfluid and core, respectively, and the linear response of the three different components to each.

4.2.1. Response to External Torques on the Crust

When the crust is subject to an external torque \( N_c(\phi) \),

\[ \Omega_c^{(+) - \iota \epsilon \Omega_c^{(+)}} + \frac{I_s}{I_1} (\iota \epsilon - \beta) \Delta^{(+)} + \frac{I_s}{I_1} (\iota \epsilon - \zeta) \Delta_c^{(+)} = \tilde{N}_c^{(+)}(\phi), \] (163)

\[ \Delta^{(+)} + \left\{ \left[ 1 - \beta (1 + \frac{I_s}{I_1}) \right] + \beta \left( 1 + \frac{I_s}{I_1} \right) \right\} \Delta^{(+)} + \iota \epsilon \Omega_c^{(+) - \iota \epsilon \Omega_c^{(+)}} - \frac{I_s}{I_1} (\iota \epsilon - \zeta) \Delta_c^{(+)} = -\tilde{N}_c^{(+)}(\phi), \] (164)

\[ \Delta_c^{(+)} + \left\{ \left[ 1 - \zeta (1 + \frac{I_s}{I_1}) \right] + \zeta (1 + \frac{I_s}{I_1}) \right\} \Delta_c^{(+)} - \frac{I_s}{I_1} (\iota \epsilon - \beta) \Delta^{(+)} = -\tilde{N}_c^{(+)}(\phi). \] (165)

It is straightforward to show that these equations have the particular solution

\[ \Omega_c^{(+)} = \sum_{p, d} A_{p, d} \int_{-\infty}^{\phi} d\phi' \tilde{N}_c^{(+)}(\phi') \exp \left[ p_c(\phi - \phi') \right], \] (166)

\[ \Delta^{(+)} = -\sum_{p, d} \frac{p_c A_{p, d}}{p_c + \iota(1 - \beta') + \beta} \int_{-\infty}^{\phi} d\phi' \tilde{N}_c^{(+)}(\phi') \exp \left[ p_c(\phi - \phi') \right], \] (167)

\[ \Delta_c^{(+)} = -\sum_{p, d} \frac{p_c A_{p, d}}{p_c + \iota(1 - \zeta') + \zeta} \int_{-\infty}^{\phi} d\phi' \tilde{N}_c^{(+)}(\phi') \exp \left[ p_c(\phi - \phi') \right], \] (168)

where the coefficients are

\[ A_p = \frac{[p_p + \iota(1 - \beta') + \beta][p_p + \iota(1 - \zeta') + \zeta]}{(p_p - p_d)(p_p - p_d)}, \] (169)

\[ A_d = \frac{[p_d + \iota(1 - \beta') + \beta][p_d + \iota(1 - \zeta') + \zeta]}{(p_d - p_d)(p_d - p_d)}, \] (170)

\[ A_d = \frac{[p_d + \iota(1 - \beta') + \beta][p_d + \iota(1 - \zeta') + \zeta]}{(p_d - p_d)(p_d - p_d)}. \] (171)

Qualitatively, these are similar to what we found for the response of a two-component star, except that the response timescales are shorter as a consequence of crust-core coupling, which implies (perhaps significantly) enhanced decay of the modes. Notice that as the coupling between the rigid crust and crustal superfluid becomes perfect, \( \Delta^{(+)} \to -\Omega_c^{(+)} \), and the response simplifies to

\[ \Omega_c^{(+)} = (p_p - p_d)^{-1} \int_{-\infty}^{\phi} d\phi' \tilde{N}_c^{(+)}(\phi') \left[ p_p + \iota(1 - \zeta') + \zeta \right] \exp \left[ p_c(\phi - \phi') \right] \]

\[ - \left[ p_d + \iota(1 - \zeta') + \zeta \right] \exp \left[ p_c(\phi - \phi') \right], \] (172)

\[ \Delta_c^{(+)} = (p_p - p_d)^{-1} \int_{-\infty}^{\phi} d\phi' \tilde{N}_c^{(+)}(\phi') \left[ -p_p \exp \left[ p_c(\phi - \phi') \right] + p_d \exp \left[ p_c(\phi - \phi') \right] \right] \]; (173)

these results are identical with equations (96) and (97) for the response of a two-component star, with the substitution of \( \Delta_c^{(+)} \) for \( \Delta^{(+)} \), \( p_d \) for \( p_d \), and \( \iota(1 - \zeta') + \zeta \) for \( \iota(1 - \beta') + \beta \). However, their usefulness is restricted to \( \tilde{N}_c^{(+)}(\phi) \) that vary on timescales short compared with the characteristic damping time implied by \( p_d \), the slowest decaying mode, as was discussed in § 3.2.
The response to a time-independent torque on the crust is simply $\Omega_{ct}^{(+)} = i\bar{N}_c^{(+)}/\epsilon$, just as in the two-component case, with $\Lambda^{(+)} = \Delta^{(+)} = 0$. These are the approximate responses obtained for a constant torque originating at a finite time $\phi_0$ in the past provided that $p_d(\phi - \phi_0) \gg 1$.

### 4.2.2. Response to Torques on the Core (Super)Fluid

If the core is subject to a torque $N_i(\phi)$ then

$$\dot{\Omega}_{ct}^{(+)} - i\epsilon\Omega_{ct}^{(+)} + \frac{I_y}{I_1} (i\beta' - \beta) \Delta^{(+)} + \frac{I_c}{I_1} (i\zeta' - \zeta) \Delta_c^{(+)} = 0,$$

(174)

$$\dot{\Lambda}^{(+)} + \left\{ \left[ 1 - \beta' \left( 1 + \frac{I_y}{I_1} \right) \right] + \beta \left( 1 + \frac{I_y}{I_1} \right) \right\} \Delta^{(+)} + i\epsilon \Omega_{ct}^{(+)} - \frac{I_y}{I_1} (i\beta' - \beta) \Delta^{(+)} = 0,$$

(175)

$$\dot{\Delta}_c^{(+)} + \left\{ \left[ 1 - \zeta' \left( 1 + \frac{I_y}{I_1} \right) \right] + \zeta \left( 1 + \frac{I_y}{I_1} \right) \right\} \Delta_c^{(+)} + i\epsilon \Omega_{ct}^{(+)} - \frac{I_y}{I_1} (i\beta' - \beta) \Delta^{(+)} = \bar{N}_c^{(+)}(\phi),$$

(176)

where $\bar{N}_c^{(+)} \equiv I^{-1}_c(N_{c,1} + iN_{c,2})$. Apart from decaying transients, the solution to these equations is

$$\Omega_{ct}^{(+)} = \sum_{p,\epsilon,\epsilon'} B_p \int_{-\infty}^{\phi} d\phi' \bar{N}_c^{(+)}(\phi') \exp \left[ p_d(\phi - \phi') \right],$$

(177)

$$\Delta^{(+)} = - \sum_{p,\epsilon,\epsilon'} \frac{p_d B_p}{\bar{p}_d + i(1 - \beta') + \beta} \int_{-\infty}^{\phi} d\phi' \bar{N}_c^{(+)}(\phi') \exp \left[ p_d(\phi - \phi') \right],$$

(178)

$$\Delta_c^{(+)} = - \sum_{p, \epsilon, \epsilon'} \frac{p_d B_p}{\bar{p}_d + i(1 - \zeta') + \zeta} \int_{-\infty}^{\phi} d\phi' \bar{N}_c^{(+)}(\phi') \exp \left[ p_d(\phi - \phi') \right],$$

(179)

where the coefficients are

$$B_p = \frac{[p_p + i(1 - \zeta') + \zeta][p_d + i(1 - \zeta') + \zeta][p_d + i(1 - \beta') + \beta]}{[i(1 - \zeta') + \zeta][i(1 - \beta') + \beta - i(1 - \zeta') - \beta][p_d + \bar{p}_d]} ,$$

(180)

$$B_d = \frac{[p_p + i(1 - \zeta') + \zeta][p_d + i(1 - \zeta') + \zeta][p_d + i(1 - \beta') + \beta]}{[i(1 - \zeta') + \zeta][i(1 - \beta') + \beta - i(1 - \zeta') - \beta][p_d + \bar{p}_d]} ,$$

(181)

$$B_d = \frac{[p_p + i(1 - \zeta') + \zeta][p_d + i(1 - \zeta') + \zeta][p_d + i(1 - \beta') + \beta]}{[i(1 - \zeta') + \zeta][i(1 - \beta') + \beta - i(1 - \zeta') - \beta][p_d + \bar{p}_d]} .$$

(182)

When the crust and crustal superfluid are coupled to one another perfectly, $p_d[i(1 - \beta') + \beta] \to -\epsilon/\sigma'$ and $B_d \to 0$, so $\Delta^{(+)} \to -\Delta_c^{(+)}$ and

$$\Omega_{ct}^{(+)} = - \frac{[p_p + i(1 - \zeta') + \zeta][p_d + i(1 - \zeta') + \zeta]}{[i(1 - \zeta') + \zeta][p_d + \bar{p}_d]} \int_{-\infty}^{\phi} d\phi' \bar{N}_c^{(+)}(\phi') \left[ \exp \left[ p_d(\phi - \phi') \right] - \exp \left[ p_d(\phi - \phi') \right] \right],$$

(183)

$$\Delta_c^{(+)} = \int_{-\infty}^{\phi} d\phi' \bar{N}_c^{(+)}(\phi') \left\{ \frac{p_p[p_d + i(1 - \zeta') + \zeta] \exp \left[ p_d(\phi - \phi') \right]}{[i(1 - \zeta') + \zeta][p_d + \bar{p}_d]} - \frac{p_d[p_p + i(1 - \zeta') + \zeta] \exp \left[ p_d(\phi - \phi') \right]}{[i(1 - \zeta') + \zeta][p_d + \bar{p}_d]} \right\} .$$

(184)

### 5. CONCLUSIONS

Shaham (1977) demonstrated that when superfluid pins perfectly to crustal nuclei, the precession period of a neutron star is shortened immensely and, moreover, the precession damps quickly as a result of weak coupling to the stellar core. One of the principal goals of this paper has been to examine whether there are additional modes with long periods and long damping timescales when the assumption of perfect coupling between crustal nuclei and superfluid is relaxed. In fact, when the coupling is strong but imperfect, there are new modes that have very long characteristic timescales. One new mode is given by equation (80),

$$p_d = - \frac{\epsilon[i(1 - \beta')]}{\sigma'},$$

for an axisymmetric, two-component star, where $\epsilon$ is the oblateness of the star and $\sigma' = \epsilon + I_y/I_1$, where $I_y$ is the moment of inertia of the crustal superfluid and $I_1$ one of the principal moments of inertia of the crust. (Coupling of the crust to the stellar core hardly alters this result; see discussion in § 4.1.) Since $\beta \ll 1$, in the limit of strong vortex drag, this mode is extremely long lived; moreover, since $1 - \beta' \ll 1$ in this limit, the mode undergoes oscillations that are also extremely slow. The problem is...
that we expect that $|1 - \beta'| \sim \beta^2 \ll \beta$ in the strong coupling domain as long as the vortex drag coefficient $\eta'$, which governs the strength of the drag force perpendicular to the direction of motion of the vortex through the normal fluid, is small compared with $\eta$, the analogous coefficient for the strength of the drag force antiparallel to the direction of motion (e.g., eqs. 37 and 38 and ensuing discussion). Thus, this mode is not actually oscillatory at all, for it damps before it can complete a single cycle. In fact, for a nonaxisymmetric star, $p_d$ splits into two modes, with (see eq. [92])

$$p_d^\pm = -\frac{\beta(\epsilon_1 \sigma_2' + \epsilon_2 \sigma_1')}{2 \sigma_1' \sigma_2'} \pm \frac{\left[ \frac{\beta^2(\epsilon_1 \sigma_2' - \epsilon_2 \sigma_1')^2}{(2 \sigma_1' \sigma_2')^2} - \frac{\epsilon_1 \epsilon_2 (1 - \beta')^2}{\sigma_1' \sigma_2'} \right]^{1/2}}{\sigma_1' \sigma_2'},$$

both of which may be purely real and decaying.

Another new mode arises in three-component models when, for example, the crustal superfluid is coupled strongly to the rigid crust in some regions and weakly in others, or else the crustal superfluid is strongly coupled to the rigid crust but the superfluid core is coupled to it only weakly. Under such circumstances, one solution to the three-component characteristic equation is equation (148)

$$p_d' = -i + \frac{(i\zeta - \zeta)(1 + \sigma + I_c/I_1)}{1 + \sigma'},$$

where $\zeta$ and $\zeta'$ are the coupling parameters between the rigid crust and the component that is barely tied to it. As was discussed in § 4.1.1, this mode can lead to a slow wandering of the pole of the neutron star as seen in the inertial reference frame. However, the excitation amplitude is relatively small for the crustal angular velocity in this mode (e.g., eq. [149]); moreover, in the weak coupling domain, we expect $\zeta' \ll \zeta$ if $\eta' \ll \eta$ (e.g., § 2.2), so the mode decays before completing one oscillation.

Thus, it appears likely that although there are new, possibly long-lived modes for a neutron star with strong but imperfect coupling between superfluid and rigid crust, these modes are not principally oscillatory as long as $\eta' \ll \eta$. Only if there are regions in the star where this inequality is reversed somehow could damped oscillations occur.

There may be regions of weak coupling between crustal superfluid and nuclei interspersed among regions of strong coupling. If so these regions could, if tied to the strong coupling regions tenuously, undergo nearly independent oscillations with both long cycle times and insignificant damping. Moreover, there could be regions of the core that may undergo long-period oscillations if detached from the crust. However, in both cases, the effect of nearly independent, slow and persistent oscillations on the portion of the crust where superfluid vortex lines are pinned would be minimal, tending to zero in the limit of complete independence. Thus, if slow, persistent oscillations can occur somewhere in the star, the chances that one can know about them from observations of the rotation rate of that part of the crust where superfluid is pinned strongly are remote. In the opposite limit, in which all regions of the star are coupled to one another closely, the observed frequencies are averages weighted by moment of inertia and tend to be dominated by regions of high frequency and/or large moment of inertia.

Under the combined action of external and internal torques, the angular velocity of the crust tends to tilt away from alignment with its principal axes. If the external torque is time independent or only varies on a very long timescale then ultimately the tilt angle approaches a constant value $\theta \sim |N_{ex}|/I_{ex} \Omega^2$, where $N_{ex}$ is the value of the constant external torque, $I_{ex}$ is the typical moment of inertia of the crust, and $\epsilon$ is the crustal oblateness. If $N_{ex} = I \Omega$ is the spin-down torque, where $I$ is the moment of inertia of the star, then the steady state tilt angle is $-\langle I \Omega \rangle/|\Omega|\langle I \rangle$, where $I$ is the total moment of inertia of the star. Even though $-\Omega/\Omega^2 \sim (\Omega \mu)^{-1}$, where $\mu$ is the spin-down timescale, is very small $[\approx 5 \times 10^{-9}P(\mu)/t_{sd}(s)\mu]$ and $|\Omega|/\mu$ may be very large and $\theta$ could be nonnegligible. An amusing side effect of this tilt is that even an axisymmetric neutron star can be a source of gravitational radiation, with an amplitude that can be determined from observables (e.g., spin-down timescale, period), quantities that can be inferred observationally with varying degrees of confidence (e.g., distance) and theoretically determined parameters (e.g., total moment of inertia) but do not depend on the oblateness $\epsilon$. Unfortunately, the implied wave amplitudes (strain amplitude $h \approx 10^{-30}$) are well below the projected capabilities of the advanced LIGO.

One key assumption behind this estimate of the steady state tilt angle is contained in the italicized word ultimately in the previous paragraph. As was discussed in § 3.2, the asymptotic value of $\theta$ is only attained if the slowest damped mode of the star decays in a time short compared with the timescales on which the external torque varies. Practically speaking, this means that if the crustal nuclei and superfluid are closely pinned, then the steady state tilt is approached on the damping time of $P_d$ (given by eq. [80]); if this is short compared with the spin-down timescale, then the asymptotic value of $\theta$ is reached. Otherwise, the tilt could be smaller—somewhere between $-\langle I \Omega \rangle/|\Omega|\langle I \rangle$ and $-\langle I \Omega \rangle/|\Omega|\langle I \rangle$—and time variable. As was pointed out in § 3.2, the correct steady state tilt angle, and the evolution toward that angle, could not be found from Shaham (1977), where perfect coupling between crust and crustal superfluid was assumed. The crux of the solution is in the timescales implied by the imperfection of the pinning.

We began this paper by proposing to examine perturbations about a particular fixed point of the equations governing the rotational dynamics of a neutron star, the one corresponding to equal angular velocities of all components lined up along one of the principal axes of the crust (the one with largest moment of inertia eigenvalue). We have only wavered from this program briefly, in § 3.3.1, where we considered perturbations about a state in which the rigid crust and crustal superfluid have parallel angular velocities with slightly different magnitudes. However, in spite of the constancy of our approach, we have uncovered some hints that it may be unrealistic, for when external torques are taken into account, the correct fixed points may involve tilts away from principal axes. In a sequel to this paper, we shall investigate the implications of time-dependent and time-independent tilts due to external torques, as well as to internal torques we have neglected here.
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