A quantum Johnson-Lindenstrauss lemma via unitary $t$-designs

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Abstract

The famous Johnson-Lindenstrauss lemma [JL84] states that for any set of $n$ vectors \( \{v_i\}_{i=1}^n \in \mathbb{C}^{d_1} \) and any $\epsilon > 0$, there is a linear transformation $T : \mathbb{C}^{d_1} \rightarrow \mathbb{C}^{d_2}$, $d_2 = O(\epsilon^{-2} \log n)$ such that $\|T(v_i)\|_2 \in (1 \pm \epsilon)\|v_i\|_2$ for all $i \in [n]$. In fact, a Haar random $d_1 \times d_1$ unitary transformation followed by projection onto the first $d_2$ coordinates followed by a scaling of $\sqrt{d_1/d_2}$ works as a valid transformation $T$ with high probability. In this work, we show that the Haar random $d_1 \times d_1$ unitary can be replaced by a uniformly random unitary chosen from a finite set called an approximate unitary $t$-design for $t = O(d_2)$. Choosing a unitary from such a design requires only $O(d_2 \log d_1)$ random bits as opposed to $2^{\Omega(d_2)}$ random bits required to choose a Haar random unitary with reasonable precision. Moreover, since such unitaries can be efficiently implemented in the superpositional setting, our result can be viewed as an efficient quantum Johnson-Lindenstrauss transform akin to efficient quantum Fourier transforms widely used in earlier work on quantum algorithms.

We prove our result by leveraging a method of Low [Low09] for showing concentration for approximate unitary $t$-designs. We discuss algorithmic advantages and limitations of our result and conclude with a toy application to private information retrieval.

1 Introduction

The Johnson Lindenstrauss lemma is one of the oldest dimensionality reduction results for the $\ell_2$-norm and has applications to many problems in computer science, signal processing, compressed sensing etc. Informally speaking, it says that any set of $n$ points in high dimensional Euclidean space (say of dimension $d_1$) can be embedded into $d_2 := O(\epsilon^{-2} \log n)$-dimensional Euclidean space preserving all the $\binom{n}{2}$ pairwise distances to within a multiplicative factor of $1 \pm \epsilon$. An equivalent description would be that the embedding approximately preserves all the pairwise angles or inner products. Moreover, with high probability this embedding can be achieved by taking a Haar random $d_1 \times d_1$ unitary $U$, applying $U$ to all the points in the set, projecting onto the first $d_2$ coordinates and scaling the result by $\sqrt{d_1/d_2}$. The advantages of such an embedding are manifold: the embedding is linear, oblivious to the actual set of points, with target dimension independent of the source dimension, and can be implemented by a randomised algorithm in $O(d_1^2 \polylog(d_1))$ time. Fast Johnson Lindenstrauss transforms, akin to fast Fourier transforms arising from the discrete Fourier transform, have also been discovered (see e.g. [AC09]). They typically run in $O(d_1 \polylog(d_1))$ time.

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In this paper, we work in the quantum superpositional setting. By this we mean that our source vectors are not provided explicitly, but rather are the state vectors of pure quantum states with Hilbert space \( \mathbb{C}^{d_1} \). Then, if we choose a Haar random \( d_1 \times d_1 \) unitary \( U \), applying it via a quantum circuit to a pure state, measure the name of a block, where the \( d_1 \) coordinates are divided into \( d_1/d_2 \) blocks of \( d_2 \) coordinates each, then conditioned on a certain block name ‘i’ appearing, all the pairwise inner products are approximately preserved. In other words, even the unitary \( U \) is applied only in the superpositional setting. One may now wonder if we can implement the unitary \( U \) via an efficient quantum circuit (i.e. of size \( \text{polylog}(d_1) \)). If so, this would give rise to an efficient quantum Johnson Lindenstrauss transform, akin to efficient quantum Fourier transforms arising from classical discrete Fourier transforms (e.g. [Cop94, MZ04]). The efficient quantum Fourier transform is at the heart of many famous quantum algorithms, including Shor’s algorithms for integer factoring and discrete logarithm [Sho97].

We show that with high probability, a uniformly random \( d_1 \times d_1 \) unitary from an approximate \( t \)-design, where \( t = \Theta(d_2) \), suffices for an efficient quantum Johnson Lindenstrauss transform. For this value of \( t \), both choosing a uniformly random unitary from the \( t \)-design as well as applying it to quantum states are efficient to implement by quantum algorithms. This follows from the fact that so-called local random quantum circuits of size \( s = t^{10}(\log d_1)^2 \log(1/\alpha) \) form an \( \alpha \)-approximate \( t \)-design of \( d_1 \times d_1 \) unitaries with high probability [BHH16]. The number of random bits required to describe such a local random circuit is at most \( O(s \log s \log \log d_1) \).

A limitation of our quantum Johnson Lindenstrauss transform is that the distribution over the block names is almost uniform. We thus have no ‘control’ over the block name, because of which we cannot apply our transform for most of the classical settings where the Johnson Lindenstrauss lemma was used (in the classical explicit setting, one can always force the block to be the first block without any trouble). Nevertheless, we do give a toy application of our transform to the important problem of private information retrieval. Finding more applications of our transform is an important open problem.

**Related work:** Our quantum Johnson Lindenstrauss transform approximately preserves the pairwise inner products for a block name with high probability over the choice of the unitary from the design. The block dimension is \( d_2 \). If one wants to approximately preserve the pairwise overlaps averaged over all the unitaries from a finite set, then there much smaller block sizes suffice. This variant is also known as quantum identification codes. Fawzi, Hayden and Sen [FHS13] constructed such codes with very small block size by efficiently quantising (in the sense of quantum Fourier transform versus classical discrete Fourier transform) low distortion embeddings of \( \ell_2 \) into \( \ell_1 \).

Harrow, Montanaro and Short [HMS11] have shown the impossibility of obtaining a Johnson Lindenstrauss style dimensionality reduction for mixed quantum states under the Frobenius norm (aka Schatten 2-norm). The impossibility proof uses a feature similar to the observation above that the block name is essentially uniform.

The Johnson-Lindenstrauss lemma has found several applications in quantum algorithms and protocols too e.g. quantum fingerprinting [BCWd01, GKd06], non-local games [CHTW01] etc.
2 Preliminaries

Let \( \|v\|_2 := \sqrt{\sum_{i=1}^{d}|v_i|^2} \) denote the \( \ell_2 \)-norm of a vector \( v \in \mathbb{C}^d \). Similarly, for a matrix \( M \in \mathbb{C}^{d_1 \times d_2} \), let \( \|M\|_2 \) denote the Frobenius norm or Hilbert-Schmidt norm or the Schatten 2-norm which is nothing but the \( \ell_2 \)-norm of the \((d_1 d_2)\)-tuple obtained by stretching \( M \) to a long vector.

2.1 Unitary \( t \)-designs

We recall the definition of a tensor product expander (TPE) first defined by Harrow and Hastings [HH09].

Definition 1 (Tensor product expander). A \((d, s, \lambda, t)\)-tensor product expander (TPE) is a set of \( d \times d \) unitaries \( \{V_i\}_{i=1}^{s} \) such that

\[
\left\| \text{Design}_{V} [V \otimes t M (V^\dagger) \otimes t] - \text{Haar}_{U} [U \otimes t M (U^\dagger) \otimes t] \right\|_2 \leq \lambda \|M\|_2,
\]

for all linear operators \( M : (\mathbb{C}^d)^{\otimes t} \rightarrow (\mathbb{C}^d)^{\otimes t} \). The notation

\[
\text{Design}_{V} [V \otimes t M (V^\dagger) \otimes t] := \frac{1}{s} \sum_{i=1}^{s} V_i \otimes t M (V_i^\dagger) \otimes t
\]

denotes the expectation under the choice of a uniformly random unitary from the design. The notation \( \text{Haar}_{U} [\cdot] \) denotes the expectation under the choice of a unitary \( U \) picked from the Haar measure.

We now recall the definition of an approximate unitary \( t \)-design according to Low [Low09].

Definition 2 (Unitary \( t \)-design). Consider \( d^2 \) formal variables \( \{u_{ij}\}_{i,j=1}^{d} \). A monomial \( M \) in these formal variables is said to be balanced of degree \( t \) if it is a product of exactly \( t \) of the formal variables and exactly \( t \) of complex conjugates of the formal variables (the sets of unconjugated and conjugates variables bear no relation amongst them). For a \( d \times d \) unitary matrix \( V \), let \( M(V) \) denote the value of the monomial \( M \) obtained by evaluating it at the entries \( V_{ij} \) of \( V \). A balanced polynomial of degree \( t \) is a linear combination of balanced monomials of degree \( t \).

A unitary \((d, s, \alpha, t)\)-design is a set of \( d \times d \) unitaries \( \{V_i\}_{i=1}^{s} \) such that

\[
\left| \text{Design}_{V} [M(V)] - \text{Haar}_{U} [M(U)] \right| \leq \frac{\alpha}{d^t},
\]

for all balanced monomials \( M \) of degree \( t \).

Sequentially iterating a TPE twice means applying the super operator corresponding to the TPE twice in succession. This gives us a \((d, s^2, \lambda^2, t)\)-TPE where the \( s^2 \) unitaries are of the form \( V_i V_j \), \( 1 \leq i, j \leq s \). It is now easy to see that a \((d, s, \lambda, t)\)-TPE can be sequentially iterated \( O\left(\frac{\log d + \log \lambda}{\log \lambda} \right)^t \) times to obtain an \( \alpha \)-approximate unitary \( t \)-design. For a proof of this statement, we refer to [Low09, Lemma 2.7].
2.2 Johnson-Lindenstrauss lemma

We first recall the following well known concentration property of the sum of squares of iid Gaussians (aka the chi-square distribution), which can be easily proved Chernoff style using the exponential moment generating function.

Fact 1. Let $G_1, \ldots, G_n$ be independent Gaussians of mean 0 and variance 1 each. Let $\epsilon > 0$. Then

$$\Pr \left[ \sum_{i=1}^{n} G_i^2 \notin (1 \pm \epsilon) n \right] \leq 2 (e^{-\epsilon/2} \sqrt{1 + \epsilon})^n.$$  

For $\epsilon \leq 1$, we can further upper bound the right hand side by $2 (e^{-\epsilon/2} \sqrt{1 + \epsilon})^n \leq 2e^{-2-\epsilon^2 n}$.

We now state the main technical lemma behind the proof of the Johnson Lindenstrauss lemma which gives a concentration result for the length of the projection of a unit vector onto a Haar random subspace. This lemma can be proved by appealing to Levy’s lemma about concentration of a Lipschitz function defined on the unitary group around its mean, combined with Fact 1 above.

Fact 2. Let $v$ be a fixed vector in $\mathbb{C}^{d_1}$, $\|v\|_2 = 1$. Let $d_2 < d_1$. Let $U$ be a Haar random $d_1 \times d_1$ unitary. Let $\Pi_i$, $1 \leq i \leq \frac{d_2}{d_1}$ be the orthogonal projection in $\mathbb{C}^{d_1}$ onto the $i$th block of $d_2$ coordinates. Let $\epsilon > 0$. Then for any fixed $i$,

$$\Pr_U \left[ \|\Pi_i U v\|_2 \notin (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}} \right] \leq 4 \exp(-2^{-4} \epsilon^2 d_2).$$

Proof. By symmetry of the Haar measure, the desired probability is nothing but the probability that a random unit vector in $\mathbb{C}^{d_1}$ does not have length $(1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}}$ when projected onto the first $d_2$ coordinates. Since a Haar random unit vector $v \in \mathbb{C}^{d_1}$ can be generated by taking $2d_1$ independent real Gaussian random variables $\{G_i\}_{i=1}^{2d_1}$ with mean 0 and variance 1, forming a complex $d_1$-tuple out of them and then dividing by the $\ell_2$-norm of the tuple, we can see that for $0 < \epsilon \leq 1$,

$$\Pr_U \left[ \|\Pi_i U v\|_2 \notin (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}} \right] \leq \Pr \left[ \sum_{i=1}^{2d_1} G_i^2 \notin 2(1 \pm \frac{\epsilon}{2})d_2 \right] + \Pr \left[ \sum_{i=1}^{2d_1} G_i^2 \notin 2(1 \pm \frac{\epsilon}{2})d_1 \right] \leq 2 \exp(-2^{-4} \epsilon^2 d_2) + 2 \exp(-2^{-4} \epsilon^2 d_1) \leq 4 \exp(-2^{-4} \epsilon^2 d_2),$$

where we used Fact 1 in the second to last inequality.

For $\epsilon > 1$, only the upper tail is relevant i.e.

$$\Pr \left[ \|\Pi_i U v\|_2 \notin (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}} \right] = \Pr \left[ \|\Pi_i U v\|_2 > (1 + \epsilon) \sqrt{\frac{d_2}{d_1}} \right].$$

Define a real valued function $f(U) := \|\Pi_i U v\|_2$. Then $f(U)$ is 1-Lipschitz with respect to the Frobenius norm on $d_1 \times d_1$ unitary matrices. By Levy’s lemma [AGZ09, Corollary 4.4.28],

$$\Pr[|f(U) - E[f]| > \delta] \leq 2 \exp(-2^{-2} d_1 \delta^2).$$
where the probability and expectation are taken over the Haar measure on $d_1 \times d_1$ unitaries. Now observe by symmetry that $E[f(U)^2] = \frac{d_2}{d_1}$. By convexity of the square function, $E[f(U)] < \sqrt{\frac{d_2}{d_1}}$. Thus,

$$\Pr[f(U) > (1 + \epsilon)\sqrt{\frac{d_2}{d_1}}] \leq 2\exp(-2^{-2}\epsilon^2 d_2).$$

This covers both the cases of $\epsilon \leq 1$ and $\epsilon > 1$ and so completes the proof. \hfill \Box

The Johnson-Lindenstrauss lemma now follows easily from the above fact.

**Fact 3 (Johnson-Lindenstrauss Lemma).** Consider a set of $n$ vectors $\{v_i\}_{i=1}^n \in \mathbb{C}^{d_1}$. Let $\epsilon > 0$. Then there is a linear transformation $T : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ where $d_2 = O(\epsilon^{-2}\log n)$ such that $\|Tv_i\|_2 \in (1 \pm \epsilon)\|v_i\|_2$ for all $i \in [n]$.

**Proof.** Choose a Haar random $d_1 \times d_1$ unitary $U$. For $v \in \mathbb{C}^{d_1}$, define $T(v) := \sqrt{\frac{d_2}{d_1}}\Pi_1 Uv$. Fact 2 and a union bound on probability now completes the proof. \hfill \Box

### 3 An efficient quantum Johnson Lindenstrauss transform

In this section, we show that choosing a $d_1 \times d_1$ unitary uniformly at random from an approximate unitary $t$-design, for $t = \Theta(d_2)$, achieves similar performance as the Haar random unitary in Fact 2. We prove this by using the method of Low [Low09], who in turn adapted the classical $t$-moment method of Bellare and Rompel [BR94] to the quantum setting. It is also possible to give a more direct proof by truncating the exponential moment generating function, used to show concentration for sums of squares of independent Gaussians in Fact 1, at an appropriately chosen $\Theta(d_2)^{th}$ power and proving that the truncation does not affect the value of the generating function by much. However the value of $t$ obtained by this method is larger than the value obtained by using Low’s method. Hence we will only give the proof using Low’s method. The proof is deferred to Section 5.

**Proposition 1.** Let $v$ be a fixed vector in $\mathbb{C}^{d_1}$, $\|v\|_2 = 1$. Let $d_2 < d_1$. Let $U$ be a unitary chosen uniformly at random from a $(d_1, s, \lambda, t)$-TPE, for $t = 2^{-9}\epsilon^2 d_2$, $\lambda = (\frac{4s^2d_1}{d_1^4})^{1/2}e^{-t/2}$, and $\log s = O(d_2 \log d_1)$. Let $\Pi_i$, $1 \leq i \leq \frac{d_1}{d_2}$ be the orthogonal projection in $\mathbb{C}^{d_1}$ onto the $i$th block of $d_2$ coordinates. Let $0 < \epsilon < 1$. Then for any fixed $i$,

$$\Pr_U\left[\|\Pi_i Uv\|_2 \not\in (1 \pm \epsilon)\sqrt{\frac{d_2}{d_1}}\right] \leq 2^6 \exp(-2^{-10}\epsilon^2 d_2).$$

We can now define the quantum Johnson-Lindenstrauss transform and prove its main property.

**Theorem 1.** Consider a set of $n$ pure states $\{|v_i\rangle\}_{i=1}^n \in \mathbb{C}^{d_1}$, whose classical descriptions are known a priori. Let $0 < \epsilon, \delta < 1/4$. Let $d_2 = O(\epsilon^{-2}\log \frac{nd_1}{\delta})$. Let $U$ be a $d_1 \times d_1$ unitary chosen uniformly at random from a $(d_1, s, \lambda, t)$-TPE, for $t = 2^{-9}\epsilon^2 d_2$, $\lambda = (\frac{4s^2d_1}{d_1^4})^{1/2}e^{-t/2}$, and $\log s = O(d_2 \log d_1)$. Suppose we apply $U$ to the given pure state and measure the name of a block of $d_2$ coordinates i.e. we project onto the range of $\Pi_j$, for some $j$. Let $|v_i(j, U)\rangle$ be the normalised state resulting from
\(|v_i\rangle\) if the name of the measured block is \(j\) i.e. |v_i(j, U)\rangle = \frac{\Pi_j U|v_i\rangle}{\|\Pi_j U|v_i\rangle\|_2}. Then, with probability at least \(1 - \delta\) over the choice of \(U\)

\[
\|\Pi_j U|v_i\rangle\|_2 \in (1 \pm \epsilon)\sqrt{\frac{d_2}{d_1}} \quad \forall i \in [n], j \in \left[\frac{d_1}{d_2}\right], \\
\langle v_i(j, U)|v_{i'}(j, U)\rangle \in \langle v_i|v_{i'}\rangle \pm 8\epsilon \quad \forall i, i' \in [n], j \in \left[\frac{d_1}{d_2}\right].
\]

**Proof.** From Proposition\([\text{I}]\) and the union bound on probability, we see that

\[
\|\Pi_j U|v_i\rangle\|_2 \in (1 \pm \epsilon)\sqrt{\frac{d_2}{d_1}} \quad \forall i \in [n], j \in \left[\frac{d_1}{d_2}\right], \\
\|\Pi_j U|v_i\rangle - \Pi_j U|v_{i'}\rangle\|_2 \in (1 \pm \epsilon)\|v_i - v_{i'}\|_2 \sqrt{\frac{d_2}{d_1}} \quad \forall i, i' \in [n], j \in \left[\frac{d_1}{d_2}\right], \\
\|\Pi_j U|v_i\rangle - \sqrt{-1}\Pi_j U|v_{i'}\rangle\|_2 \in (1 \pm \epsilon)\|v_i - \sqrt{-1}v_{i'}\|_2 \sqrt{\frac{d_2}{d_1}} \quad \forall i, i' \in [n], j \in \left[\frac{d_1}{d_2}\right],
\]

with probability at least \(1 - \delta\) over the choice of \(U\). Using the above constraints, we get

\[
\langle v_i(j, U)|v_{i'}(j, U)\rangle
\]

\[
= \left(\frac{1}{2} \left\| \frac{\Pi_j U|v_i\rangle}{\|\Pi_j U|v_i\rangle\|_2} - \frac{\Pi_j U|v_{i'}\rangle}{\|\Pi_j U|v_{i'}\rangle\|_2} \right\|^2 - 1\right)
\]

\[
- \sqrt{-1} \left(\frac{1}{2} \left\| \frac{\Pi_j U|v_i\rangle}{\|\Pi_j U|v_i\rangle\|_2} - \sqrt{-1} \frac{\Pi_j U|v_{i'}\rangle}{\|\Pi_j U|v_{i'}\rangle\|_2} \right\|^2 - 1\right)
\]

\[
\in \left(\frac{1}{2} \frac{d_1}{d_2} \left(\|\Pi_j U|v_i\rangle - \Pi_j U|v_{i'}\rangle\|_2^2 + (4\epsilon/3)(\|\Pi_j U|v_i\rangle\|_2 + \|\Pi_j U|v_{i'}\rangle\|_2)^2\right)^2 - 1\right)
\]

\[
- \sqrt{-1} \left(\frac{1}{2} \frac{d_1}{d_2} \left(\|\Pi_j U|v_i\rangle - \sqrt{-1}\Pi_j U|v_{i'}\rangle\|_2^2 + (4\epsilon/3)(\|\Pi_j U|v_i\rangle\|_2 + \|\Pi_j U|v_{i'}\rangle\|_2)^2\right)^2 - 1\right)
\]

\[
\in \left(\frac{1}{2} \frac{d_1}{d_2} \left(\|\Pi_j U|v_i\rangle - \Pi_j U|v_{i'}\rangle\|_2^2 - 1\right)
\]

\[
- \sqrt{-1} \left(\frac{1}{2} \frac{d_1}{d_2} \left(\|\Pi_j U|v_i\rangle - \sqrt{-1}\Pi_j U|v_{i'}\rangle\|_2^2 - 1\right) \pm 5\epsilon\right)
\]

\[
\in \left(\frac{1}{2} \left(\|v_i\rangle - |v_{i'}\rangle\|_2^2 - 1\right) - \sqrt{-1} \left(\frac{1}{2} \left(\|v_i\rangle - \sqrt{-1}|v_{i'}\rangle\|_2^2 - 1\right) \pm 8\epsilon\right)
\]

This completes the proof. \(\square\)

## 4 A toy application

In this section, we will see a toy application of our quantum Johnson Lindenstrauss transform to protocols for private information retrieval. In this problem there are two parties, Alice and Bob. Alice is given a subset \(S \subseteq [m]\), of size \(|S| \leq n\). We work in the regime where \(n\) is very small compared to \(m\) viz. \(n \ll \frac{\log m}{\log\log m}\). Bob is given an element \(x \in [m]\) and he wants to whether \(x\) lies in \(S\) or not. For this purpose, Bob and Alice follow a two message communication protocol where Bob first sends a message to Alice, Alice responds and then Bob makes his conclusion whether \(x\) lies in \(S\) or not. Bob’s conclusion should be correct with probability at least \(3/4\). The privacy requirement is that Bob’s message should reveal very little information about \(x\).
Ideally, we would like the messages to be short and the computing resources used by Alice and Bob to be polynomial in \( n \) and \( \log m \). Is this possible? Yes! There is always the trivial protocol where Bob says nothing and Alice sends Bob the entire subset \( S \) using \( O(n \log m) \) bits. The trivial protocol guarantees perfect privacy for Bob.

We now ask if there is a protocol guaranteeing at least approximate privacy for Bob where Alice communications significantly less. Indeed, when \( n \ll m \) there is such a protocol based on the following fact proved by Buhrman, Miltersen, Radhakrishnan and Venkatesh \([BMRV02]\).

**Fact 4.** There exists a collection \( \{T_1, \ldots, T_m\} \) of subsets of \([n \log m]\), \( |T_i| = O(\log m) \), and for every subset \( S \subseteq [m] \), \( |S| \leq n \), a scheme of colouring the set \([n \log m]\) with zero or one, such that for \( x \in S \), at least 0.9 fraction of elements of \( T_x \) are coloured one, and for \( x \not\in S \), at least 0.9 fraction of elements of \( T_x \) are coloured zero.

The above fact suggests the following protocol for private information retrieval. Bob says nothing. Hence perfect privacy holds for Bob. Alice sends \( \Theta(n) \) random elements of \([n \log m]\) coloured one. Her message length is \( O(n \log(n \log m)) \) bits. Bob checks if the intersection of Alice’s message with \( T_x \) is above a certain constant If so, he declares that \( x \in S \); if not, he declares \( x \not\in S \). A standard Chernoff bound shows that there is a constant gap in the probability of Bob declaring \( x \in S \) depending on whether \( x \) really lies in \( S \) or not. A constant number of parallel repetitions of the protocol suffices to boost the gap and give a success probability of at least 0.75 for Bob.

One may now wonder if Alice’s communication can be made even more succinct. Unfortunately, not by much because there is a \( \Omega(n) \) lower bound for Alice’s message irrespective of Bob’s message length under the condition of approximate privacy of Bob, which holds for the quantum setting too. This can be proved by restricting Alice’s subset \( S \) to satisfy \( S \subseteq [n] \), Bob’s element \( x \) to satisfy \( x \in [n] \) and then applying the privacy-privacy tradeoff of \([JRS09]\) for the set membership problem. Nevertheless, there is still a gap between the upper and lower bounds for Alice’s message size.

We now ask if we can achieve approximate privacy for Bob, short message for Alice and make Bob’s internal computation efficient. Unfortunately, the set guaranteed by Fact 4 is non-explicit. Near explicit constructions of similar set systems were later provided by Ta-Shma \([Ta-02]\) and Capalbo, Reingold, Vadhan and Wigderson \([CRVW02]\), but their parameters are worse and Bob’s internal computation is still not proved to be efficient.

We now give a quantum protocol achieving approximate privacy for Bob, short message for Alice and efficient internal computation for Bob. Our protocol uses the efficient quantum Johnson-Lindenstrauss transform. The idea behind the protocol is as follows. For a subset \( S \subseteq [m] \), define the following pure quantum state \( |S\rangle := |S\rangle^{-1/2} \sum_{y \in S} |y\rangle \) in \( \mathbb{C}^m \). If \( x \in S \) \( \langle x|S\rangle \geq n^{-1/2} \). If \( x \not\in S \), \( \langle x|S\rangle = 0 \). Now suppose we apply the quantum Johnson Lindenstrauss transform of Theorem \( \Pi \) with \( \epsilon := 0.01n^{-3} \) and measure the name of a block, say \( i \in \left[ \frac{d_1}{d_2} \right] \), where \( d_1 := m \), \( d_2 = O(\epsilon^{-2} \log nd_1) = O(n^6 \log m) \). The unitary \( U \) from the \( (d_1, s, \lambda, t) \)-TPE where \( t = O(\epsilon^2 d_2) \), \( \lambda = (\frac{4\epsilon^2 d_2}{d_1})^{1/2} \epsilon^{-1/2} \), that is chosen by the transform can be described using \( \log s = O(d_2 \log d_1) \) bits. Moreover, constructing and applying the quantum circuit to quantum states, given the name of the unitary, can be done in time \( \text{poly}(n, \log m) \). Let \( |x\rangle', |S'\rangle \) be the resulting normalised projections in the \( i \)th block of dimension \( d_2 = O(n^6 \log m) \). Then, if \( x \in S \), \( \langle x'|S'\rangle \geq 0.9n^{-1/2} \); if \( x \not\in S \), \( \langle x'|S'\rangle \leq 0.1n^{-1/2} \). The distribution on the block names is within \( \ell_1 \)-distance \( \epsilon \) from the uniform distribution irrespective of the element \( x \in [m] \).

This leads naturally to the following quantum protocol for private information retrieval, where Alice is given \( S \subseteq [m] \), \( |S| \leq n \) and Bob is given \( x \in [m] \).
1. At first, independently of $x$, Bob chooses a uniformly random unitary $U$ from the TPE. He then applies $U$ to $|x\rangle$ and measures the name of a block. He stores the collapsed pure state that lives in the residual $d_2$-dimensional spaces. He repeats this process (with the same $U$ and $|x\rangle$) independently $\Theta(n^2)$ times. He then sends Alice the description of $U$, which is like a public coin, followed by the $\Theta(n^2)$ block names that were measured (note that in general, they are all different);

2. Alice makes $\Theta(n^2)$ projections of $|S\rangle$ into $d_2$-dimensional space corresponding to the unitary $U$ and the block names received from Bob. She then sends these $\Theta(n^2)$ pure quantum states to Bob;

3. Bob performs $\Theta(n^2)$ SWAP tests between the pure states that Alice sent versus the pure states that he obtained in the first step above by collapsing. From the results of these tests, he checks whether the fraction of successes was larger than $\frac{1}{2} + \frac{0.2}{n}$ or not. If yes, he declares that $x$ lies in $S$. If not, he declares that $x$ does not lie in $S$.

Bob’s message is classical and consists of $\log s = O(n^6(\log m)^2)$ bits of public coin followed by $O(n^2\log m)$ bits for the block names. Bob’s internal computation is efficient i.e. takes time $\text{poly}(n, \log m)$. The public coin can be reduced to $O(n \log m)$ bits by a standard technique of Newman [New91], but then Bob’s internal computation is no longer guaranteed to be efficient. Bob’s message is almost private since the probability distribution on the block names is at most $O(\epsilon n^2) = O(1/n)$ in $\ell_1$-distance from uniform. Alice’s message is quantum and consists of $O(n^2(\log n + \log \log m) \log \log m)$ qubits. For $n \ll \frac{\log m}{\log \log m}$, this is less than $O(n \log m)$. By a standard Chernoff bound, Bob reaches the correct conclusion whether $x$ lies in $S$ or not with probability at least $3/4$.

Remark: The efficient quantum identification code of Fawzi, Hayden and Sen [FHS13, Theorem 4.3] can also be easily exploited for private information retrieval. In that protocol, Bob’s message is classical and consists of $O(n^2 \log m)$ bits. Bob’s internal computation is efficient. Bob’s message is within $O(1/n)$ in $\ell_1$-distance from the uniform distribution. Alice’s message is quantum. However, it consists of $O(n^2(\log n + \log \log m) \log \log m)$ qubits, which is more than Alice’s message length in the protocol based on the quantum Johnson Lindenstrauss transform. The quantum Johnson Lindenstrauss transform based protocol achieves small number of qubits for Alice by trading off a larger number of bits for Bob, keeping Bob’s internal computation efficient.

5 Proof of Proposition 1

We use Low’s method [Low09]. Define the real valued function $f(U) := \|\Pi_i U v\|_2 - \sqrt{\frac{4}{d_1}}$ where $U$ is a $d_1 \times d_1$ unitary matrix. From Fact 2 for any $\lambda > 0$,

$$\Pr_U[|f(U)| \geq \lambda] \leq 4 \exp(-2^{-4}\lambda^2d_1),$$

where the probability is taken under the Haar measure on $U$. Combining this with [Low09, Lemma 3.3], we get

$$\mathbb{E}_U[(f(U))^{2m}] \leq 4 \left(\frac{2^4m}{d_1}\right)^m,$$
where the expectation is taken over the Haar measure on $U$. Now define the real valued function $g(U) := \|\Pi_i U v\|_2^2 - \frac{d_2}{d_1^2}$. Under the Haar measure on $U$, we have

\[
E_U[(g(U))^{2m}] = E_U[(f(U))^{2m} \left(\|\Pi_i U v\|_2 + \sqrt{\frac{d_2}{d_1}}\right)^{2m}]
\]

\[
\leq \left(\frac{4d_2}{d_1}\right)^m \Pr_U[\|\Pi_i U v\|_2 \leq \sqrt{\frac{d_2}{d_1}}] E_U[(f(U))^{2m}]
\]

\[
+ \sum_{i=2}^{\frac{\sqrt{d_2}}{d_1}} \left(\frac{(i + 1)^2 - 1}{d_1}d_2\right)^{2m} \Pr_U[i \sqrt{\frac{d_2}{d_1}} < \|\Pi_i U v\|_2 \leq (i + 1)\sqrt{\frac{d_2}{d_1}}]
\]

\[
\leq \left(\frac{4d_2}{d_1}\right)^m E_U[(f(U))^{2m}] + 4 \sum_{i=2}^{\frac{\sqrt{d_2}}{d_1}} \left(\frac{(i + 1)^2 - 1}{d_1}d_2\right)^{2m} \exp(-2^{-4i^2d_2})
\]

\[
\leq 2^4 \left(\frac{9^6m^2d_2}{d_1^2}\right)^m + 2^4 \left(\frac{9^6d_2^2}{d_1^4}\right)^m \exp(-2^{-2d_2}),
\]

where we used Fact \#2 again in the second inequality.

Now suppose we choose $U$ from a $(d_1, s, \lambda, 2m)$ tensor product expander instead of the Haar measure. Since $(g(U))^{2m}$ is a balanced degree $2m$ polynomial in the entries of $U$, its expectation under a TPE must be close to its expectation under the Haar measure. More precisely,

\[
\begin{align*}
|E_U[(g(U))^{2m}] - E_{\text{Haar}}[(g(U))^{2m}]| &\leq \frac{\text{TPE}}{\text{Haar}}[(g(U))^{2m}] \\
&= \frac{\text{TPE}}{\text{U}}[(\text{Tr} [\Pi_j U(|v\rangle\langle v| - \frac{1}{d_1^2}U^\dagger\Pi_j^\dagger)])^{2m}] - \frac{\text{Haar}}{\text{U}}[(\text{Tr} [\Pi_j U(|v\rangle\langle v| - \frac{1}{d_1^2}U^\dagger\Pi_j^\dagger)])^{2m}] \\
&= \text{TPE}_U[\text{Tr} [\Pi_j^{\otimes(2m)}U^{\otimes(2m)}(|v\rangle\langle v| - \frac{1}{d_1^2})^{\otimes(2m)}(U^\dagger)^{\otimes(2m)}]] - \frac{\text{Haar}}{\text{U}}[\text{Tr} [\Pi_j^{\otimes(2m)}U^{\otimes(2m)}(|v\rangle\langle v| - \frac{1}{d_1^2})^{\otimes(2m)}(U^\dagger)^{\otimes(2m)}]] \\
&\leq \left\|\Pi_j^{\otimes(2m)}\right\|_2 \left\|\text{TPE}_U[U^{\otimes(2m)}(|v\rangle\langle v| - \frac{1}{d_1^2})^{\otimes(2m)}(U^\dagger)^{\otimes(2m)}] - \frac{\text{Haar}}{\text{U}}[U^{\otimes(2m)}(|v\rangle\langle v| - \frac{1}{d_1^2})^{\otimes(2m)}(U^\dagger)^{\otimes(2m)}]\right\|_2 \\
&\leq (d_2)^m \lambda.
\end{align*}
\]

Recall that $\lambda$ can be made small at an exponential rate by simply sequentially iterating the TPE.

Now observe that for any probability distribution on $U$, by Markov’s inequality,

\[
\Pr_U[\|\Pi_i U v\|_2 \not\in (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}}] \leq \Pr_U[|g(U)| \geq 2\epsilon \frac{d_2}{d_1}] \leq E_U[(g(U))^{2m}] \left(\frac{d_1}{2\epsilon d_2}\right)^{2m},
\]

9
where $m$ is any positive integer. Thus,

$$
\text{TPE}_{P_U} \left[ \|\Pi_i U v\|_2 \not\in (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}} \right] 
\leq \left( \frac{d_1}{2 \epsilon d_2} \right)^{2m} \left( 2^4 \left( \frac{2^6 m d_2}{d_1^2} \right)^m \right) \exp(-2^{-2} d_2) + d_2^m \lambda 
= 2^4 \left( \frac{2^4 m}{\epsilon^2 d_2} \right)^m + 2^4 \left( \frac{4^4}{\epsilon^2} \right)^m \exp(-2^{-2} d_2) + \left( \frac{d_1^2}{4 \epsilon^2 d_2} \right)^m \lambda.
$$

Choosing $m := 2^{-10} \epsilon^2 d_2$, we get

$$
\text{TPE}_{P_U} \left[ \|\Pi_i U v\|_2 \not\in (1 \pm \epsilon) \sqrt{\frac{d_2}{d_1}} \right] 
\leq 2^4 \exp(-2^{-10} \epsilon^2 d_2) + 2^4 \exp(2^{-7} \epsilon^2 \ln(1/\epsilon) d_2) \exp(-2^{-2} d_2) + \left( \frac{d_1^2}{4 \epsilon^2 d_2} \right)^m \lambda 
\leq 2^6 \exp(-2^{-10} \epsilon^2 d_2),
$$

by taking $\lambda < \left( \frac{d_1^2}{4 \epsilon^2 d_2} \right)^{-m} e^{-2^{-10} \epsilon^2 d_2}$.

Note that starting from a TPE with constant value of parameter $\lambda_0$ and a constant number of unitaries $s$ we can sequentially iterate it $k := 2m \log d_1 + 2m \log(1/\epsilon) + 2^{-10} \epsilon^2 d_2 \log(1/\lambda)$ times in order to get $\lambda$ as small as above. Existence of $(d, \text{poly}(1/\lambda_0), \lambda_0, t)$-TPEs for constant $\lambda_0$ and $d \geq \text{poly}(t)$ was shown by Harrow and Hastings [HH09] via a probabilistic argument. Efficient constructions of such TPEs for $t = \text{polylog}(d)$ was shown by Sen [Sen18] by combining the existence result of Harrow and Hastings together with the zigzag product for quantum expanders [BST10]. For many applications including the one to Johnson-Lindenstrauss, the above expression for $k$ is polynomial in the input parameters. Moreover, choosing a uniformly random unitary from such a design takes only $O(k)$ random bits as opposed to the $(1/\lambda)^{O(d^2)}$ random bits required to choose a Haar random unitary to within Frobenius distance of $\lambda$.

This completes the proof of Proposition 1.

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**References**

[AC09] Ailon, N. and Chazelle, B. The fast Johnson-Lindenstrauss transform and approximate nearest neighbors. *SIAM Journal on Computing*, 39(1):302–322, 2009.
Anderson, G., Guionnet, A., and Zeitouni, O. *An introduction to random matrices*. Cambridge University Press, 2009.

Buhrman, H., Cleve, R., Watrous, J., and de Wolf, R. Quantum fingerprinting. *Phys. Rev. Lett.*, 87(16):167902–1–167902–4, 2001.

Brandao, F., Harrow, A., and Horodecki, M. Local random quantum circuits are approximate polynomial-designs. *Communications in Mathematical Physics*, 346(2):397–434, 2016.

Buhrman, H., Miltersen, P., Radhakrishnan, J., and Venkatesh, S. Are bitvectors optimal? *SIAM Journal on Computing*, 31(6):1723–1744, 2002.

M. Bellare and J. Rompel. Randomness-efficient oblivious sampling. In *Proceedings of the 35th Annual IEEE Symp. on Foundations of Comp. Sc. (FOCS)*, pages 276–287, 1994.

Ben-Aroya, H., Schwartz, O., and Ta-Shma, A. Quantum expanders: Motivation and construction. *Theory of Computing*, 6:47–79, 2010.

Cleve, R., Høyer, P., Toner, B., and Watrous, J. Consequences and limits of nonlocal strategies. In *Proceedings of the 19th Annual IEEE Conf. on Computational Complexity (CCC)*, pages 236–249, 2004.

D. Coppersmith. An approximate Fourier transform useful in quantum factoring. IBM Research Report RC 19642. Also available at arXiv:quant-ph/0201067., 1994.

Capalbo, M., Reingold, O., Vadhan, S., and Wigderson, A. Randomness conductors and constant-degree lossless expanders. In *Proceedings of the 34th Annual ACM Symp. on the Theory of Computing*, pages 659–668, 2002.

Fawzi, O., Hayden, P., and Sen, P. From low-distortion norm embeddings to explicit uncertainty relations and efficient information locking. *Journal of the ACM*, 60(6):44:1–44:61, 2013.

Gavinsky, D., Kempe, J., and de Wolf, R. Strengths and weaknesses of quantum fingerprinting. In *Proceedings of the 21st Annual IEEE Conf. on Computational Complexity (CCC)*, pages 288–298, 2006.

Hastings, M. and Harrow, A. Classical and quantum tensor product expanders. *Quantum Information and Computation*, 9(3):336–360, 2009.

Harrow, A., Montanaro, A., and Short, A. Limitations on quantum dimensionality reduction. In *Proc. Int. Colloq. on Aut. Lang. and Prog. (ICALP)*, pages 86–97, 2011. Also arXiv:1012.2262.

Johnson, W. and Lindenstrauss, J. Extensions of Lipschitz mappings into a Hilbert space. *Contemporary Mathematics*, 26:189–206, 1984.
[JRS09] Jain, R., Radhakrishnan, J., and Sen, P. A property of quantum relative entropy with an application to privacy in quantum communication. *Journal of the ACM*, 56(6):33:1–33:32, 2009.

[Low09] R. Low. Large deviation bounds for $k$-designs. *Proceedings of the Royal Society A*, 465:3289–3308, 2009.

[MZ04] Mosca, M. and Zalka, C. Exact quantum Fourier transforms and discrete logarithm algorithms. *International Journal of Quantum Information*, 2(1):91–100, 2004.

[New91] Newman, I. Private vs. common random bits in communication complexity. *Information Processing Letters*, 39(2):67–71, 1991.

[Sen18] Sen, P. Near Ramanujan quantum tensor product expanders via the generalised zigzag product. In preparation, 2018.

[Sho97] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997.

[Ta-02] Ta-Shma, A. Storing information with extractors. *Information Processing Letters*, 83(5):267–274, 2002.