Coalgebra Encoding for Efficient Minimization

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Abstract

Recently, we have developed an efficient generic partition refinement algorithm, which computes behavioural equivalence on a state-based system given as an encoded coalgebra, and implemented it in the tool CoPaR. Here we extend this to a fully fledged minimization algorithm and tool by integrating two new aspects: (1) the computation of the transition structure on the minimized state set, and (2) the computation of the reachable part of the given system. In our generic coalgebraic setting these two aspects turn out to be surprisingly non-trivial requiring us to extend the previous theory. In particular, we identify a sufficient condition on encodings of coalgebras, and we show how to augment the existing interface, which encapsulates computations that are specific for the coalgebraic type functor, to make the above extensions possible. Both extensions have linear run time.

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1 Introduction

The task of minimizing a given finite state-based system has arisen in different contexts throughout computer science and for various types of systems, such as standard deterministic automata, tree automata, transition systems, Markov chains, probabilistic or other weighted systems. In addition to the obvious goal of reducing the mere memory consumption of the state space, minimization often appears as a subtask of a more complex problem. For instance, probabilistic model checkers benefit from minimizing the input system before performing the actual model checking algorithm, as e.g. demonstrated in benchmarking by Katoen et al. [32].

Another example is the graph isomorphism problem. A considerable portion of input instances can already be decided correctly by performing a step called colour refinement [9], which amounts to the minimization of a weighted transition system wrt. weighted bisimilarity.

Minimization algorithms typically perform two steps: first a reachable subset of the state set of the given system is computed by a standard graph search, and second, in the resulting reachable system all behaviourally equivalent states are identified. For the latter step one uses partition refinement or lumping algorithms that start by identifying all states and then iteratively refine the resulting partition of the state set by looking one step into the transition structure of the given system. There has been a lot of research on efficient partition refinement procedures, and the most efficient algorithms for various concrete system types have a run
time in $O(m \log n)$, for a system with $n$ states and $m$ transitions, e.g. Hopcroft’s algorithm for
deterministic automata [30] and the algorithm by Paige and Tarjan [36] for transition systems,
even if the number of action labels is not fixed [43]. Partition refinement of probabilistic
systems also underwent a dynamic development [18,53], and the best algorithms for Markov
chain lumping now match the complexity of the relational Paige-Tarjan algorithm [22,31,44].
For the minimization of more complex system types such as Segala systems [6,26] (combining
probabilities and non-determinism) or weighted tree automata [29], partition refinement
algorithms with a similar quasilinear run time have been designed over the years.

Recently, we have developed a generic partition refinement algorithm [23,49] and im-
plemented it in the tool CoPaR [19,52]. This generic algorithm computes the partition
of the state set modulo behavioural equivalence for a wide variety of stated-based system
types, including all the above. This genericity in the system type is achieved by working
with coalgebras for a functor which encapsulates the specific types of transitions of the input
system. More precisely, the algorithm takes as input a syntactic description of a set functor
and an encoding of a coalgebra for that functor and then computes the simple quotient,
i.e. the quotient of the state set modulo behavioural equivalence. The algorithm works
correctly for every zippable set functor (Definition 2.8). It matches, and in some cases even
improves on, the run-time complexity of the best known partition refinement algorithms for
many concrete system types [52, Table 1].

The reasons why this run-time complexity can be stated and proven generically are: first,
the encoding allows us to talk about the number of states and, in particular, the number
of transitions of an input coalgebra. But more importantly, every iterative step of partition
refinement requires only very few system-type specific computations. These computations are
encapsulated in the refinement interface [49], which is then used by the generic algorithm.

An important feature of our coalgebraic algorithm is its modularity: in the tool the user
can freely combine functors with already implemented refinement interfaces by products,
coproducts and functor composition. A refinement interface for the combined functor is
then automatically derived. In this way more structured systems types such as (simple and
general) Segala systems and weighted tree automata can be handled.

In the present paper, we extend our algorithm to a fully fledged minimizer. In previous
work [3] it has been shown that for set functors preserving intersections, every coalgebra
equipped with a point, modelling initial states, has a minimization called the well-pointed
modification. Well-pointedness means that the coalgebra does not have any proper quotients
(i.e. it is simple) nor proper pointed subcoalgebras (i.e. it is reachable), in analogy to minimal
deterministic automata being reachable and observable (see e.g. [5, p. 256]). The well-pointed
modification is obtained by taking the reachable part of the simple quotient of a given
pointed coalgebra [3] (and the more usual reversed order, simple quotient of the reachable
part, is correct for functors preserving inverse images [51, Sec. 7.2]). Our previous work on
coalgebraic minimization algorithms has focused on computing the simple quotient. Here we
extend our algorithm by two missing aspects of minimization and provide their correctness
proofs: the computation of (1) the transition structure of the minimized system, and (2) the
reachable states of an input coalgebra.

One may wonder why (1) is a step worth mentioning at all because for many concrete
system types this is trivial, e.g. for deterministic automata where the transitions between
equivalence classes are simply defined by choosing representatives and copying their transitions
from the input automaton. However, for other system types this step is not that obvious,
e.g. for weighted automata where transition weights need to be summed up and transitions
might actually disappear in the minimized system because weights cancel out. We found
that in the generic coalgebraic setting enabling the computation of the (encoding of) the transition structure of the minimized coalgebra is surprisingly non-trivial, requiring us to extend the theory behind our algorithm.

In order to be able to perform this computation generically we work with uniform encodings, which are encodings that satisfy a coherence property (Definition 3.10). We prove that all encodings used in our previous work are uniform, and that the constructions enabling modularity of our algorithm preserve uniformity (Prop. 3.12). We also prove that uniform encodings are subnatural transformations, but the converse does not hold in general. In addition, we introduce the minimization interface containing the new function \texttt{merge} (to be implemented together with the refinement interface for each new system type) which takes care of transitions that change as a result of minimization. We provide \texttt{merge} operations for all functors with explicitly implemented refinement interfaces (Example 4.4), and show that for combined system types minimization interfaces can be automatically derived (Prop. 4.11); similarly as for refinement interfaces. Our main result is that the (encoded) transition structure of the minimized coalgebra can be correctly computed in linear time (Thm. 4.9).

Concerning extension (2), the computation of reachable states, it is well-known that every pointed coalgebra has a reachable part (being the smallest subcoalgebra) \cite{3, 50}. Moreover, for a set functor preserving intersections it coincides with the reachable part of the canonical graph of the coalgebra \cite[Lem. 3.16]{3}. Recently, it was shown that the reachable part of a pointed coalgebra can be constructed iteratively \cite[Thm. 5.20]{50} and that this corresponds to performing a standard breadth-first search on the canonical graph. The missing ingredient to turn our previous partition refinement algorithm into a minimizer is to relate the canonical graph with the encoding of the input coalgebra. We prove that for a functor with a subnatural encoding, the encoding (considered as a graph) of every coalgebra coincides with its canonical graph (Theorem 5.6).

Putting everything together, we obtain an algorithm that computes the well-pointed modification of a given pointed coalgebra. Both additions can be implemented with linear run time in the size of the input coalgebra and hence do not add to the run-time complexity of the previous partition refinement algorithm. We have provided such an implementation with the new version of our tool CoPaR.

All proofs and additional details can be found in the appendix.

**Reachability in Coalgebraic Minimization** There are several works on coalgebraic minimization, ranging from abstract constructions to concrete and implemented algorithms \cite{1, 34, 35, 49, 52}, that compute the simple quotient \cite{27} of a given coalgebra. These are not concerned with reachability since coalgebras are not equipped with initial states in general.

In Brzozowski’s automata minimization algorithm \cite{16}, reachability is one of the main ingredients. This is due to the duality of reachability and observability described by Arbib and Manes \cite{4}, and this duality is used twice in the algorithm. Consequently, reachability also appears as a subtask in the categorical generalizations of Brzozowski’s algorithm \cite{10, 14, 15, 35, 38}. These generalizations concern automata processing input words and so do not cover minimization of (weighted) tree automata. Segala systems are not treated either. Due to the dualization, Brzozowski’s classical algorithm for deterministic automata has doubly exponential time complexity in the worst case (although it performs well on certain types of non-deterministic automata, compared to determination followed by minimization \cite{41}).
2 Background

Our algorithmic framework [49] is defined on the level of coalgebras for set functors, following the paradigm of universal coalgebra [39]. Coalgebras can model a wide variety of systems.

In the following we recall standard notation for sets and functions as well as basic notions from the theory of coalgebras. We fix a singleton set 1 = {*}; for each set \( X \), we have a unique map \( !: X \to 1 \). We denote the disjoint union (coproduct) of sets \( A,B \) by \( A + B \) and use\( \text{inl},\text{inr} \) for the canonical injections into the coproduct, as well as \( \text{pr}_1,\text{pr}_2 \) for the projections out of the product. We use the notation \( \langle \cdot,\cdot \rangle \), respectively \( [\cdot,\cdot] \), for the unique map induced by the universal property of a product, respectively coproduct. We also fix two sets \( S = \{0,1\} \) and \( T = \{0,1,2\} \) and use the former as a set of boolean values with 0 and 1 denoting false and true, respectively. For each subset \( S \) of a set \( X \), the characteristic function \( \chi_S : X \to S \) assigns 1 to elements of \( S \) and 0 to elements of \( X \setminus S \). We denote by Set the category of all sets and maps. We shall indicate injective and surjective maps by \( \rightarrow \) and \( \rightarrow\rightarrow \), respectively.

Recall that an endofunctor \( F : \text{Set} \to \text{Set} \) assigns to each set \( X \) a set \( FX \), and to each map \( f : X \to Y \) a map \( FFf : FX \to FY \), preserving identities and composition, that is we have \( F(\text{id}_X) = \text{id}_{FX} \) and \( F(g \circ f) = Fg \circ FFf \). We denote the composition of maps by \( \cdot \), written infix, as usual. An \( F \)-coalgebra is a pair \( (X,c) \) that consists of a set \( X \) of states and a map \( c : X \to FX \) called (transition) structure. A morphism \( h : (X,c) \to (Y,d) \) of \( F \)-coalgebras is a map \( h : X \to Y \) preserving the transition structure, i.e. \( Fh \cdot c = d \cdot h \). Two states \( x,y \in X \) of a coalgebra \( (X,c) \) are behaviourally equivalent if there exists a coalgebra morphism \( h \) with \( h(x) = h(y) \).

Example 2.1. Coalgebras and the generic notion for behavioural equivalence instantiate to a variety of well-known system types and their equivalences:

1. The finite powerset functor \( \mathcal{P}_f \) maps a set to the set of all its finite subsets and functions \( f : X \to Y \) to \( \mathcal{P}_fX \to \mathcal{P}_fY \) taking direct images. Its coalgebras are finitely branching (unlabelled) transition systems and coalgebraic behavioural equivalence coincides with Milner and Park’s (strong) bisimilarity.

2. Given a commutative monoid \( (M,+,0) \), the monoid-valued functor \( M(-) \) maps a set \( X \) to the set of finitely supported functions from \( X \) to \( M \). These are the maps \( f : X \to M \), such that \( f(x) = 0 \) for all except finitely many \( x \in X \). Given a map \( h : X \to Y \) and a finitely supported function \( f : X \to M \), \( M^h(f) : M(X) \to M(Y) \) is defined as \( M^h(f)(y) = \sum_{x \in X, h(x) = y} f(x) \). Coalgebras for \( M(-) \) correspond to finitely branching weighted transition systems with weights from \( M \). If a coalgebra morphism \( h : (X,c) \to (Y,d) \) merges two states \( s_1,s_2 \), then for all transitions \( x \xrightarrow{m_1} s_1, x \xrightarrow{m_2} s_2 \) in \( (X,c) \) there must be a transition \( h(x) \xrightarrow{m_1+m_2} h(s_1) = h(s_2) \) in \( (Y,d) \) and similarly if more than two states are merged. Coalgebraic behavioural equivalence captures weighted bisimilarity [33, Prop. 2].

Note that the monoid may have inverses: if \( s_2 = -s_1 \), then the transitions in the above example cancel each other out, leading to a transition \( h(x) \xrightarrow{m_1} h(s_1) \) with weight 0, which in fact represents the absence of a transition. This happens for example for the monoid \( (\mathbb{R},+,0) \) of real numbers. A simple minimization algorithm for real weighted transition (i.e. \( \mathbb{R}(-)\)-coalgebras) systems is given by Valmari and Franceschinis [44]. These systems subsume Markov chains which are precisely the coalgebras for the finite probability distribution functor \( \mathcal{D} \), a subfunctor of \( \mathbb{R}(-) \).

3. Given a signature \( \Sigma \) consisting of operation symbols \( \sigma \), each with a prescribed natural number, its arity \( \text{ar}(\sigma) \), the polynomial functor \( F\Sigma \) sends each set \( X \) to the set of (shallow)
Another example are behavioural equivalence coincides with \( \text{ras} [27] \). That is, every quotient coalgebra morphism, for which we write \( \text{e.g.} [2, \text{Sec. 9}] \).

Coalgebraic concepts. For a more detailed and well-motivated discussion with examples, see means to compute its well-pointed modification. We now briefly recall the corresponding further, \( \text{Segala systems} \) (or \( \text{probabilistic LTSs} [26] \)) are coalgebras for \( FX = \mathcal{P}_1(\Sigma \times X) \), for which coalgebraic behavioural equivalence instantiates to probabilistic bisimilarity [7].

Example 2.2 (Modularity). New system types can be constructed from existing ones by functor composition. For example, labelled transition systems (LTSs) are coalgebras for the functor \( FX = \mathcal{P}_1(\mathcal{A} \times X) \), which is the composite of \( \mathcal{P}_1 \) and \( \mathcal{A} \times - \) for a label alphabet \( \mathcal{A} \), and precisely the bisimilar states in an \( F \)-coalgebra are behaviourally equivalent. Composing further, Segala systems (or probabilistic LTSs [26]) are coalgebras for \( FX = \mathcal{P}_1(\mathcal{A} \times \mathcal{D} X) \), for which coalgebraic behavioural equivalence instantiates to probabilistic bisimilarity [7].

Another example are weighted tree automata [29] with weights in a commutative monoid \( \mathcal{M} \) and input signature \( \Sigma \); they are coalgebras for the composed functor \( FX = \mathcal{M}^{\Sigma X} \), for which behavioural equivalence coincides with backwards bisimilarity [20].

Simple, Reachable, and Well-Pointed Coalgebras Minimizing a given pointed coalgebra means to compute its well-pointed modification. We now briefly recall the corresponding coalgebraic concepts. For a more detailed and well-motivated discussion with examples, see e.g. [2, Sec. 9].

First, a quotient coalgebra of an \( F \)-coalgebra \( (X, c) \) is represented by a surjective \( F \)-coalgebra morphism, for which we write \( q: (X, c) \rightarrow (Y, d) \), and a subcoalgebra of \( (X, c) \) is represented by an injective \( F \)-coalgebra morphism \( m: (S, s) \rightarrow (X, c) \).

A coalgebra \( (X, c) \) is called simple if it does not have any proper quotient coalgebras [27]. That is, every quotient \( q: (X, c) \rightarrow (Y, d) \) is an isomorphism. Equivalently, distinct
states \(x, y \in X\) are never behaviourally equivalent. Every coalgebra has an (up to isomorphism) unique simple quotient (see e.g. [2, Prop. 9.1.5]).

▶ Example 2.3. 1. A deterministic automaton regarded as a coalgebra for \(FX = 2 \times X^A\) is simple iff it is observable [5, p. 256], that is, no distinct states accept the same formal language.

2. A finitely branching transition system considered as a \(\mathcal{P}_I\)-coalgebra is simple, if it has no pairs of strongly bisimilar but distinct states; in other words if two states \(x, y\) are strongly bisimilar, then \(x = y\).

3. A similar characterization holds for monoid-valued functors (such as the bag functor) wrt. weighted bisimilarity.

A pointed coalgebra is a coalgebra \((X, c)\) equipped with a point \(i: 1 \to X\), equivalently a distinguished element \(i \in X\), modelling an initial state. Morphisms of pointed coalgebras are the point-preserving coalgebra morphisms, i.e. morphisms \(h: (X, c, i) \to (Y, d, j)\) satisfying \(h \cdot i = j\). Quotients and subcoalgebras of pointed coalgebras are defined wrt. these morphisms.

A pointed coalgebra \((X, c, i)\) is called \emph{reachable} if it has no proper subcoalgebra, that is, every subcoalgebra \(m: (S, s, j) \hookrightarrow (X, c, i)\) is an isomorphism. Every pointed coalgebra has a unique reachable subcoalgebra (see e.g. [2, Prop. 9.2.6]). The notion of reachable coalgebras corresponds well with graph theoretic reachability in concrete examples. We elaborate on this a bit more in Section 5.

▶ Example 2.4. 1. A deterministic automaton considered as a pointed coalgebra for \(FX = 2 \times X^A\) (with the point given by the initial state) is reachable if all of its states are reachable from the initial state.

2. A pointed \(\mathcal{P}_I\)-coalgebra is a finitely branching directed graph with a root node. It is reachable precisely when every node is reachable from the root node.

3. Similarly, for monoid-valued functors such as the bag functor, reachability is precisely graph theoretic reachability, where a transition weight of 0 means ‘no edge’.

Finally, a pointed coalgebra \((X, c, i)\) is \emph{well-pointed} if it is reachable and simple. Every pointed coalgebra has a \emph{well-pointed modification}, which is obtained by taking the reachable part of its simple quotient (see [2, Not. 9.3.4]).

▶ Remark 2.5. For a functor preserving inverse images, one may reverse the two constructions: the well-pointed modification is the simple quotient of the reachable part of a given pointed coalgebra [51, Sec. 7.2]. This is the usual order in which minimization of systems is performed algorithmically. However, for a functor that does not preserve inverse images, quotients of reachable coalgebras need not be reachable again [51, Ex. 5.3.27], possibly rendering the usual order incorrect.

Our present paper is concerned with the \emph{minimization problem} for coalgebras, i.e. the problem to compute the well-pointed modification of a given pointed coalgebra in terms of its encoding.

▶ Remark 2.6. Recall that a (sub)natural transformation \(\sigma\) from a functor \(F\) to a functor \(G\) is a set-indexed family of maps \(\sigma_X: FX \to GX\) such that for every (injective) function \(m: X \to Y\) the square on the right below commutes; we also say that \(\sigma\) is (sub)natural in \(X\).

From previous results (see [49, Prop. 2.13] and [50, Thm. 4.6]) one obtains the following sufficient condition for reductions of reachability and simplicity. Given a family of maps \(\sigma_X: FX \to GX\), then every \(F\)-coalgebra \((X, c)\) yields a \(G\)-coalgebra \((X, \sigma_X \cdot c)\) and we can reduce minimization tasks from \(F\)-coalgebras to \(G\)-coalgebras as follows:
1. Suppose that $\sigma : F \to G$ is sub-cartesian, that is the squares on the right are pullbacks for every injective map $m : X \to Y$. Then the reachable part of a pointed $F$-coalgebra $(X, c, i)$ is obtained from the reachable part of the $G$-coalgebra $(X, \sigma_X \cdot c, i)$.

2. Suppose that $F$ is a subfunctor of $G$, i.e. we have a natural transformation $\sigma$ with injective components $\sigma_X : FX \to GX$. Then the problem of computing the simple quotient for $F$-coalgebras reduces to that for $G$-coalgebras: the simple quotient of $(X, \sigma_X \cdot c)$ yields that of $(X, c)$.

Consequently, if $F$ is a subfunctor of $G$ via a subcartesian $\sigma$, the minimization problem for $F$-coalgebras reduces to that for $G$-coalgebras. For example, the distribution functor $\mathcal{D}$ is a subcartesian subfunctor of $\mathbb{R}^\leftarrow$.

**Preliminaries on Bags** The bag functor defined in Example 2.1 plays an important role in our minimization algorithm, not only as one of many possible system types, but bags are also used as a data structure. To this end, we use a couple of additional properties of this functor.

**Remark 2.7.** 1. Since $\mathcal{B}$ can also be regarded as a monoid-valued functor for $(\mathbb{N}, +, 0)$, every bag $b = \{x_1, \ldots, x_n\} \in \mathcal{B}X$ may be identified with a finitely supported function $X \to \mathbb{N}$, assigning to each $x \in X$ its multiplicity in $b$. We shall often make use of this fact and represent bags as functions.

2. The set $\mathcal{B}X$ itself is a commutative monoid with bag-union as the operation and the empty bag $\emptyset$ as the identity element. In fact, this is the free commutative monoid over $X$. It therefore makes sense to consider the monoid-valued functor $(\mathcal{B}X)^\leftarrow$ for a monoid of bags. Note that for every pair of sets $A, X$, the set $(\mathcal{B}A)^{(X)}$ of finitely supported functions from $X$ to $\mathcal{B}A$ is isomorphic to $\mathcal{B}(A \times X)$ as witnessed by the following isomorphism (where swap, curry and uncurry are the evident canonical bijections):

$$
\text{group} = (\mathcal{B}(A \times X) \xrightarrow{B(\text{swap})} \mathcal{B}(X \times A) \xrightarrow{\text{curry}} (\mathcal{B}A)^{(X)}), \quad \text{and}
$$

$$
\text{ungroup} = ((\mathcal{B}A)^{(X)} \xrightarrow{\text{uncurry}} \mathcal{B}(X \times A) \xrightarrow{B(\text{swap})} \mathcal{B}(A \times X)).
$$

Note that since swap is self-inverse and curry, uncurry are mutually inverse, group and ungroup are mutually inverse, too. In symbols:

$$
group \cdot \text{ungroup} = \text{id}_{(\mathcal{B}A)^{(X)}}, \quad \text{ungroup} \cdot \text{group} = \text{id}_{\mathcal{B}(A \times X)}. \quad (1)
$$

We often need to filter a bag of tuples $\mathcal{B}(A \times X)$ by a subset $S \subseteq X$. To this end we define the maps $\text{fil}_S : \mathcal{B}(A \times X) \to \mathcal{B}(A)$ for sets $S \subseteq X$ and $A$ by

$$
\text{fil}_S(f) = (a \mapsto \sum_{x \in S} f(a, x)) = \{a | (a, x) \in f, x \in S\},
$$

where the multiset comprehension is given for intuition.

**Zippable Functors** One crucial ingredient for the efficiency of the generic partition refinement algorithm [49] is that the coalgebraic type functor is zippable:

**Definition 2.8 [49, Def. 5.1].** A set functor $F$ is called zippable if the following maps are injective for every pair $A, B$ of sets:

$$
F(A + B) \xrightarrow{(F(A + 1), F(1 + B))_F} F(A + 1) \times F(1 + B).
$$
Zippability of a functor allows that partitions are refined incrementally by the algorithm [49, Prop. 5.18], which in turn is the key for allowing a low run time complexity of the implementation. For additional visual explanations of zippability, see [49, Fig. 2]. We shall need this notion in the proof of Proposition 3.3, and later proofs use this result.

It was shown in [49] that all functors in Example 2.1 are zippable. In addition, zippable functors are closed under products, coproducts and subfunctors. However, they are not closed under functor composition, e.g. \( P_f P_l \) is not zippable [49, Ex. 5.10].

### The Trnková Hull

For purposes of universal coalgebra, we may assume without loss of generality that set functors preserve injections. Indeed, every set functor preserves nonempty injections (being the split monomorphisms in \( \text{Set} \)). As shown by Trnková [42, Prop. II.4 and III.5], for every set functor \( F \) there exists an essentially unique set functor \( \overline{F} \) which coincides with \( F \) on nonempty sets and functions, and preserves finite intersections (whence injections). The functor \( \overline{F} \) is called the **Trnková hull** of \( F \). Since \( F \) and \( \overline{F} \) coincide on nonempty sets and maps, the categories of coalgebras for \( F \) and \( \overline{F} \) are isomorphic.

### 3 Coalgebra Encodings

In order to make abstract coalgebras tractable for computers and to have a notion of the size of a coalgebra structure in terms of nodes and edges as for standard transition systems, our algorithmic framework encodes coalgebras using a graph-like data structure. To this end, we require functors to be equipped with an encoding as follows.

- **Definition 3.1.** An encoding of a set functor \( F \) consists of a set \( A \) of labels and a family of maps \( \flat_X : FX \to B(A \times X) \), one for every set \( X \), such that the following map is injective:

\[
FX \xrightarrow{(F!, \flat_X)} F1 \times B(A \times X).
\]

An encoding of a coalgebra \( c : X \to FX \) is given by \( \langle F!, \flat_X \rangle \cdot c : X \to F1 \times B(A \times X) \).

Intuitively, the encoding \( \flat_X \) of a functor \( F \) specifies how an \( F \)-coalgebra should be represented as a directed graph, and the required injectivity models that different coalgebras have different representations.

- **Remark 3.2.** Previously [49, Def. 6.1], the map \( \langle F!, \flat_X \rangle \) was not explicitly required to be injective. Instead, a family of maps \( \flat_X : FX \to B(A \times X) \) and a refinement interface for \( F \) was assumed. The definition of a refinement interface for \( F \) is tailored towards the computation of behaviourally equivalent states and its details are therefore not relevant for the present work. All we need here is that the existence of a refinement interface implies the injectivity condition of Definition 3.1 and consequently, we inherit all examples of encodings from the previous work.

- **Proposition 3.3.** For every zippable set functor \( F \) with a family of maps \( \flat_X : FX \to B(A \times X) \) and a refinement interface, the family \( \flat_X \) is an encoding for \( F \).

- **Example 3.4.** We recall a number of encodings from [49]; the injectivity is clear, and in fact implied by Proposition 3.3:

1. Our encoding for the finite powerset functor \( \mathcal{P}_l \) resembles unlabelled transition systems by taking the singleton set \( A = 1 \) as labels. The map \( \flat_X : \mathcal{P}_l(X) \to B(1 \times X) \cong B(X) \) is the obvious inclusion, i.e. \( \flat_X(t)(*, x) = 1 \) if \( x \in t \) and 0 otherwise.

2. The monoid-valued functor \( M(-) \) has labels from \( A = M \) and \( \flat_X : M(X) \to B(M \times X) \) is given by \( \flat_X(t)(m, x) = 1 \) if \( t(x) = m \neq 0 \) and 0 otherwise.
3. For a polynomial functor $F_{\Sigma}$, we use $A = \mathbb{N}$ as the label set and define the maps $\flat_X : F_{\Sigma}X \to B(\mathbb{N} \times X)$ by $\flat_X((x_1, \ldots, x_n)) = \{(1, x_1), \ldots, (n, x_n)\}$. Note that $\flat_X$ itself is not injective if $\Sigma$ has at least two operation symbols with the same arity. E.g. for DFAs ($F_{\Sigma}X = 2 \times X^A$), $\flat_X$ only retrieves information about successor states but disregards the ‘finality’ of states. However, pairing $\flat_X$ with $F! : FX \to F1$ yields an injective map.

4. The bag functor $B$ itself also has $A = \mathbb{N}$ as labels and $\flat_X(t)(n, x) = 1$ if $t(x) = n$ and 0 otherwise. This is just the special case of the encoding for a monoid-valued functor for the monoid $(\mathbb{N}, +, 0)$.

The encoding does by no means imply a reduction of the problem of minimizing $F$-coalgebras to that of coalgebras for $B(A \times -)$ (cf. Remark 2.6). In fact, the notions of behavioural equivalence for $F$-coalgebras and coalgebras for $B(A \times -)$, can be radically different. If $\flat_X$ is natural in $X$, then behavioural equivalence wrt. $F$ implies that for $B(A \times -)$, but not necessarily conversely. However, we do not assume naturality of $\flat_X$, and in fact it fails in all of our examples except one:

▶ Proposition 3.5. The encoding $\flat_X : F_{\Sigma}X \to B(A \times X)$ for the polynomial functor $F_{\Sigma}$ is a natural transformation.

▶ Example 3.6. The encoding $\flat_X : P_1(X) \to B(1 \times X) \cong B(X)$ in Example 3.4 item 1 is not natural. Indeed, consider the map $! : 2 \to 1$, for which we have

$$B(! \cdot \flat_2((0, 1))) = B(!)[0, 1] = \{*, *\} \neq \{*\} = \flat_1((*) \cdot P_1(!)((0, 1))).$$

Similar examples show that the encodings in Example 3.4 item 2 (for all non-trivial monoids) and item 4 are not natural.

An important feature of our algorithm and tool is that all implemented functors can be combined by products, coproducts and functor composition. That is, the functors from Example 3.4 are implemented directly, but the algorithm also automatically handles coalgebras for more complicated combined functors, like those in Example 2.2, e.g. $P_1(A \times -)$. The mechanism that underpins this feature is detailed in previous work [20, 49] and depends crucially on the ability to form coproducts and products of encodings:

▶ Construction 3.7 [20, 49]. Given a family of functors $(F_i)_{i \in I}$ with encodings $(\flat_{X_i}, i \in I)$ and $(A_i, i \in I)$, we obtain the following encodings with labels $A = \prod_{i \in I} A_i$:

1. for the coproduct functor $F = \coprod_{i \in I} F_i$ we take

$$\flat_X : \prod_{i \in I} F_i \mathrel{\coprod_{i \in I}} \flat_{X_i} \mathrel{\coprod_{i \in I}} B(A_i \times X) \mathrel{\coprod_{i \in I}} B(\prod_{i \in I} A_i \times X).$$

2. for the product functor $F = \prod_{i \in I} F_i$ we take

$$\flat_X : \prod_{i \in I} F_i \mathrel{\prod_{i \in I}} B(\prod_{i \in I} A_i \times X) \quad \flat_X(t)(\operatorname{in}_i(a), x) = \flat_i(\operatorname{pr}_i(t))(a, x),$$

where $\operatorname{in}_i : A_i \to \prod_{j \neq i} A_j$ and $\operatorname{pr}_i : \prod_{j} F_j \to F_i X$ denote the canonical coproduct injections and product projections, respectively.

▶ Proposition 3.8. The families $\flat_X$ defined in Construction 3.7 yield encodings for the functors $\Pi_{i \in I} F_i$ and $\coprod_{i \in I} F_i$, respectively.
Remark 3.9. Since zippable functors are not closed under composition, modularity cannot be achieved by simply providing a construction of an encoding for a composed functor (at least not without giving up on the efficient run-time complexity). Functor composition is reduced to coproducts making a detour via many-sorted sets. Here is a rough explanation of how this works. Suppose that $F$ is a finitary set functor, which means that for every $x \in FX$ there exists a finite subset $Y \subseteq X$ and $x' \in FY$ such that $x = Fm(x')$ for the inclusion map $m: Y \hookrightarrow X$. Given a finite coalgebra $c: X \rightarrow FGX$, it can be turned into a 2-sorted coalgebra $(c', d'): (X, Y) \rightarrow (FY, GX)$ as follows: since $F$ is finitary one picks a finite subset $Y$ of $GX$ such that there exists a map $c': X \rightarrow FY$ with $c =Fd' \cdot c'$, where $d': Y \hookrightarrow GX$ is the inclusion map. Then $c'$ and $d'$ are combined into one coalgebra on the disjoint union $X + Y$ as shown below:

$$\begin{align*}
X + Y &\xrightarrow{c'+d'} FY + GX \\
&\xrightarrow{[F \text{inr}, G \text{inl}]} (F + G)(X + Y)
\end{align*}$$

for the coproduct of the functors $F$ and $G$, where $\text{inl}: X \rightarrow X + Y$ and $\text{inr}: Y \rightarrow X + Y$ are the two coproduct injections. Full details may be found in [49, Sec. 8].

For the sake of computing the coalgebra structure of the minimized coalgebra, we require that, intuitively, the labels used for encoding $FX$ are independent of the cardinality of $X$:

Definition 3.10. An encoding $\flat_X$ for a set functor $F$ is called uniform if it fulfils the following property for every $x \in X$:

$$\begin{align*}
FX &\xrightarrow{\flat_X} B(A \times X) \\
F\{x\} &\xrightarrow{\text{fil}_{\{x\}}} B(A)
\end{align*}$$

(2)

Intuitively, the condition in Definition 3.10 expresses that in an encoded coalgebra, the edges (and their labels) to a state $x$ do not change if other states $y, z \in X \setminus \{x\}$ are identified by a possible partition on the state space. Diagram (2) expresses the extreme case of such a partition, particularly the one where all elements of $X$ except for $x$ are identified in a block, with $x$ being in a separate singleton block.

Fortunately, requiring uniformity does not exclude any of the existing encodings that we recalled above.

Proposition 3.11. All encodings from Example 3.4 are uniform.

Uniform encodings interact nicely with the modularity constructions:

Proposition 3.12. Uniform encodings are closed under product and coproduct.

That is, given functors $(Fi)_{i \in I}$ with uniform encodings $(\flat_i)_{i \in I}$, then the encodings for the functors $\prod_{i \in I} F_i$ and $\bigvee_{i \in I} F_i$, as defined in Construction 3.7, are uniform.

Admittedly, the condition in Definition 3.10 is slightly technical. However, we will now prove that it sits strictly between two standard properties, naturality and subnaturality.

Proposition 3.13. 1. Every natural encoding is uniform.

2. Every uniform encoding is a subnatural transformation.

The converses of both of the above implications fail in general. For the converse of 1 we saw a counterexample in Example 3.6, and for the converse of 2 we have the following counterexample.
Example 3.14. Consider the following encoding for the functor $FX = X \times X \times X$ given by $A = 3 + 3$ and

$$\sharp_X : FX \rightarrow \mathcal{B}(A \times X)$$

$$\sharp_X(x, y, z) = \begin{cases} \{(\text{inl } 0, x), (\text{inl } 1, y), (\text{inl } 2, z)\} & \text{if } y = z, \\ \{(\text{inr } 0, x), (\text{inr } 1, y), (\text{inr } 2, z)\} & \text{if } y \neq z. \end{cases}$$

This encoding is subnatural, since the value of $y = z$ is preserved by injections under $F$. But it is not uniform, for if $x \neq y \neq z$, then we have

$$\text{fil}_{\{1\}}(\sharp(\text{fil}_1(\sharp(x, y, z)))) = \text{fil}_{\{1\}}(\sharp(1, 0, 0)) = \{\text{inl } 0\} \neq \{\text{inr } 0\} = \text{fil}_{\{2\}}(\sharp(x, y, z)).$$

4 Computing the Simple Quotient

The previous coalgebraic partition refinement algorithm and its tool implementation in CoPaR compute for a given encoding of a coalgebra $(X, c)$ the state set of its simple quotient $q : (X, c) \rightarrow (Y, d)$, that is the partition $Y$ of the set $X$ corresponding to behavioural equivalence. But the algorithm does not compute the coalgebra structure $d$ of the simple quotient (and note that it is not given the structure $c$ explicitly, to begin with). Here we will fill this gap. We are interested in computing the encoding $FY \xrightarrow{\sharp} \mathcal{B}(A \times Y)$ given the encoding $X \xrightarrow{\sharp} FX \xrightarrow{\text{fil}_1} \mathcal{B}(A \times X)$ of the input coalgebra and the quotient map $q : X \rightarrow Y$.

The edge labels in the encoding of the quotient coalgebra relate to the labels in the encoded input coalgebra in a functor specific way. For example, for weighted transition systems, the labels are the transition weights, which are added whenever states are identified. In contrast, for deterministic automata (or when $F$ is a polynomial functor), the labels (i.e. input symbols) on the transitions remain the same even when states are identified.

Thus, when computing the encoding of the simple quotient, the modification of edge labels is functor specific. Algorithmically, this is reflected by specifying a new interface containing one function $\text{merge}$, which is intended to be implemented together with the refinement interface (Section 3) for every functor of interest. The abstract function $\text{merge}$ is then used in the generic Construction 4.8 in order to compute the encoding of the simple quotient.

Definition 4.1. A minimization interface for a set functor $F$ equipped with a functor encoding $\sharp_X : FX \rightarrow \mathcal{B}(A \times X)$ is a function $\text{merge} : \mathcal{B}(A) \rightarrow \mathcal{B}(A)$ such that the following diagram commutes for all $S \subseteq X$:

\[
\begin{array}{c}
FX \xrightarrow{\sharp_X} \mathcal{B}(A \times X) \xrightarrow{\text{fil}_1} \mathcal{B}(A) \\
F \mathcal{S} \xrightarrow{\sharp_2} \mathcal{B}(A \times 2) \xrightarrow{\text{fil}_{\{1\}}} \mathcal{B}(A) \\
\end{array}
\]

Intuitively, $\text{merge}$ expresses what happens on the labels of edges from one state to one block. It receives the bag of all labels of edges from a particular source state $x$ to a set of states $S$ that the minimization procedure identified as equivalent. It then computes the edge labels from $x$ to the merged state $S$ of the minimized coalgebra in a functor specific way. Figure 1 depicts this process for a monoid-valued functor (cf. Example 2.1, item 2). In this example, $\text{merge}$ sums up the labels (which are monoid elements), resulting in a correct transition label to the new merged state.
Before we give formal definitions of \( \text{merge} \) for the functors of interest, let us show that there is a close connection between properties of \( \text{merge} \) and the encoding; this will simplify the definition of \( \text{merge} \) later (Example 4.4).

First, if \( \text{merge} \) receives the bag of labels from a source state to a single target state, then there is nothing to be merged and thus \( \text{merge} \) should simply return its input bag. Moreover, we can even characterize uniform encodings by this property:

\[
\begin{align*}
\text{Lemma 4.2.} & \quad \text{Given a minimization interface, the following are equivalent:} \\
1. & \quad \text{merge}(\text{fil}_{x}(\mathcal{P}_X(t))) = \text{fil}_{x}(\mathcal{P}_X(t)) \quad \text{for all } t \in FX. \\
2. & \quad \mathcal{P}_X \text{ is uniform.}
\end{align*}
\]

Similarly, the property that \( \text{merge} \) is always the identity characterizes natural encodings:

\[
\begin{align*}
\text{Lemma 4.3.} & \quad \text{For every encoding } \mathcal{P}_X : FX \to \mathcal{B}(A \times X), \text{ the following are equivalent:} \\
1. & \quad \text{The identity on } \mathcal{B}A \text{ is a minimization interface.} \\
2. & \quad \mathcal{P}_X \text{ is a natural transformation.}
\end{align*}
\]

\[
\text{Example 4.4.} \quad 1. \quad \text{For the finite powerset functor } \mathcal{P}_f(\mathcal{X}), \text{ with labels } A = 1, \text{ we define } \text{merge} : \mathcal{B}1 \to \mathcal{B}1 \text{ by } \text{merge}(\ell)(\ast) = \min(1, \ell(\ast)).
\]

\[
2. \quad \text{For monoid-valued functors } M(\mathcal{X}) \text{ with } A = M, \text{ merge is defined as}
\]

\[
\text{merge}(\ell) = \begin{cases} 
\{ \Sigma \ell \} & \Sigma \ell \neq 0 \\
\{ \} & \text{otherwise,}
\end{cases}
\]

where \( \Sigma : \mathcal{B}(M) \to M \) is defined by \( \Sigma(\{ m_1, \ldots, m_n \}) = m_1 + \cdots + m_n. \)

\[
3. \quad \text{The encoding for the polynomial functor } F_{\Sigma} \text{ for a signature } \Sigma \text{ is a natural transformation and hence its minimization interface is given by } \text{merge} = \text{id (see Lemma 4.3).}
\]

\[
\text{Proposition 4.5.} \quad \text{All } \text{merge} \text{ maps in Example 4.4 are minimization interfaces and run in linear time in the size of their input bag.}
\]

Having \( \text{merge} \) defined for the functors of interest, we can now use it to compute the encoding of the simple quotient.

\[
\text{Assumption 4.6.} \quad \text{For the remainder of this section we assume that } F1 \neq \emptyset.
\]

This is w.l.o.g. since \( F1 = \emptyset \) if and only if \( FX = \emptyset \) for all sets \( X \), for which there is only one coalgebra (which is therefore its own simple quotient already).

\[
\text{Proposition 4.7.} \quad \text{Suppose that the set functor } F \text{ is equipped with a uniform encoding } \mathcal{P}_X : FX \to \mathcal{B}(A \times X) \text{ and a minimization interface } \text{merge}. \text{ Then the diagram below commutes for every map } q : X \to Y,
\]

\[
\begin{array}{cccccccc}
FX & \xrightarrow{\mathcal{P}_X} & \mathcal{B}(A \times X) & \xrightarrow{\mathcal{B}(A \times q)} & \mathcal{B}(A \times Y) & \xrightarrow{\text{group}} & \mathcal{B}(A)(Y) \\
\downarrow Fq & & \downarrow \xrightarrow{\text{group}} & & \downarrow \xrightarrow{\text{merge}(Y)} \\
FY & \xrightarrow{\mathcal{P}_Y} & \mathcal{B}(A \times Y) & \xrightarrow{\text{ungroup}} & \mathcal{B}(A)(Y)
\end{array}
\]
Note that the dashed arrow is not simply the identity map because \( \mathcal{b}_X \) fails to be natural for most functors of interest (Example 3.6).

**Proof (Sketch).** One first proves that \( \text{merge} \) preserves empty bags: \( \text{merge}(\emptyset) = \emptyset \). The commutativity of the desired diagram (4) is proven by extending it by every evaluation map \( \text{ev}(y) : B(A)^{(Y)} \to B(A) \), \( y \in Y \), which form a jointly injective family. The extended diagram for \( y \in Y \) is then proven commutative using (2) for \( y \). (3) for \( S = q^{-1}[y] \), which is also used in the form \( \chi|_S \cdot q = \chi_{SY} \) in addition to two easy properties of \( \text{ev} \) and \( \text{fil} \): \( \text{fil}_{(y)} = \text{ev}(y) \cdot \text{group} \) and \( \text{fil}_{(y)} \cdot B(A \times q) = \text{fil}_S \). ▶

**Construction 4.8.** Given the encoded \( F \)-coalgebra \( (X, \mathcal{b}_X \cdot c) \), the quotient \( q : X \to Y \), and a minimization interface for \( F \), we define the map \( e : Y \to B(A \times Y) \) as follows: given an element \( y \in Y \), choose any \( x \in X \) with \( q(x) = y \) and put

\[
e(y) := (\text{ungroup} \cdot \text{merge}^{(Y)}) \cdot \text{group} \cdot B(A \times q) \cdot \mathcal{b}_X \cdot c)(x),
\]

where the involved types are as follows:

\[
\begin{array}{cccc}
X & \xrightarrow{c} & FX & \xrightarrow{\mathcal{b}_X} B(A \times X) & \xrightarrow{B(A \times q)} B(A \times Y) & \xrightarrow{\text{group}} B(A)^{(Y)} \\
q \downarrow & & \downarrow & & & \\
Y & \xrightarrow{e} & B(A \times Y) & \leftarrow \text{ungroup} & B(A)^{(Y)}
\end{array}
\] (5)

For the well-definedness and the correctness of Construction 4.8, we need to prove that (5) commutes. Moreover, observe that \( c \) is not directly given as input, and that the structure \( d : Y \to FY \) of the simple quotient is not computed; only their encodings \( \mathcal{b}_X \cdot c \) and \( e = \mathcal{b}_Y \cdot d \) are.

**Theorem 4.9.** Suppose that \( q : (X, c) \to (Y, d) \) represents a quotient coalgebra. Then Construction 4.8 correctly yields the encoding \( e = \mathcal{b}_Y \cdot d \) given the encoding \( \mathcal{b}_X \cdot c \) and the partition of \( X \) associated to \( q \).

If \( \text{merge} \) runs in linear time (in its parameter), then Construction 4.8 can be implemented with linear time (in the size of the input coalgebra \( \mathcal{b}_X \cdot c \)).

In the run time analysis, a bit of care is needed so that the implementation of \( \text{group} \) has linear run time; see the appendix for details. From Proposition 4.5 we see that for every functor from Example 2.1, Construction 4.8 can be implemented with linear run time.

### 4.1 Modularity of Minimization Interfaces

Modularity in the system type is gained by reducing functor composition to products and coproducts (Remark 3.9). Since we want the construction of the minimized coalgebra structure to benefit from the same modularity, we need to verify closure under product and coproduct for the notions required in Proposition 4.7. We have already done so for uniform encodings (Proposition 3.12); hence it remains to show that minimization interfaces can also be combined by product and coproduct:

**Construction 4.10.** Given a family of functors \( (F_i)_{i \in I} \) together with uniform encodings \( \mathcal{b}_i : F_i X \to B(A_i \times X) \) and minimization interfaces \( \text{merge}_i : B(A_i) \to B(A_i) \), we define \( \text{merge} \) for the (co)product functors \( \coprod_{i \in I} F_i \) and \( \prod_{i \in I} F_i \) as follows:

\[
\text{merge} : B(\coprod_{i \in I} A_i) \to B(\coprod_{i \in I} A_i) \quad \text{merge}(t)(\text{in}_i a) = \text{merge}_i(\text{filter}_i(t))(a),
\]

where \( \text{filter}_i : B(\prod_{j \in I} A_j) \to B(A_j) \) is given by \( \text{filter}_i(f)(a) = f(\text{in}_i(a)) \).
Curiously, the definition of `merge` is the same for products and coproducts, e.g. because the label sets are the same (see Construction 3.7). However, the correctness proofs turns out to be quite different. Note that for coproducts, all labels in the image of `fil_S ⋆ X` are in the same coproduct component. Thus, `filter`, never removes elements and acts as a mere type-cast when the above `merge` is used in accordance with its specification.

\[\text{Proposition 4.11.} \text{ The } \text{merge} \text{ function defined in Construction 4.10 yields a minimization interface for the functors } \prod_{i \in I} F_i \text{ and } \coprod_{i \in I} F_i. \text{ It can be implemented with linear run-time if each } \text{merge}, \text{ is linear in its input.}\]

\[\text{Corollary 4.12.} \text{ The class of set functors having a minimization interface contains all polynomial and all monoid-valued functors and is closed under product and coproduct.}\]

Consequently, Construction 4.8 correctly yields encoded quotient coalgebras for those functors. Note that all functors from Example 4.4 are contained in this class. Furthermore, functor composition can be dealt with by using coproducts as explained in Remark 3.9.

5 Reachability

Having quotiented an encoded coalgebra by behavioural equivalence, the remaining task is to restrict the coalgebra to the states that are actually reachable from a distinguished initial state. For an intersection preserving set functor, the reachable part of a pointed coalgebra can be constructed iteratively, and this reduces to standard graph search on the canonical graph of the coalgebra [50, Cor. 5.26f], which we now recall. Throughout, \(\mathcal{P}\) denotes the (full) powerset functor. The following is inspired by Gumm [28, Def. 7.2]:

\[\text{Definition 5.1.} \text{ Given a functor } F: \text{Set} \to \text{Set}, \text{ we define a family of maps } \tau^F_X: FX \to \mathcal{P}X \text{ by } \tau^F_X(t) = \{ x \in X \mid 1 \to FX \text{ does not factorize through } F(X \setminus \{ x \}) \xrightarrow{\text{fix}} FX \}, \text{ where } i: X \setminus \{ x \} \hookrightarrow X \text{ denotes the inclusion map.}\]

The canonical graph of a coalgebra \((X, \tau^F_X, c, i)\) is the directed graph \(X \xrightarrow{\tau^F_X} FX \xrightarrow{\text{fix}} \mathcal{P}X.\) The nodes are the states of \((X, c)\) and one has an edge from \(x\) to \(y\) whenever \(y \in \tau^F_X(c(x))\). Note that for a pointed coalgebra \((X, c, i)\) its canonical graph is equipped with the same point \(i: 1 \to X\), that is, the canonical graph is equipped with a root node \(i(*) \in X\). As we pointed out in Section 2, reachability of the pointed \(\mathcal{P}\)-coalgebra \((X, \tau^F_X \cdot c, i)\) precisely means that every \(x \in X\) is reachable from the root node in the canonical graph.

\[\text{Example 5.2.} \text{ 1. For a deterministic automaton considered as a coalgebra for } FX = 2 \times X^A, \text{ the canonical graph is precisely its usual underlying state transition graph.}\]

\[\text{2. For the finite powerset functor } \mathcal{P}_i, \text{ it is easy to see that } \tau^F_{\mathcal{P}_i}: \mathcal{P}_i X \hookrightarrow \mathcal{P}X \text{ is the inclusion map. Thus, the canonical graph of a } \mathcal{P}_i\text{-coalgebra (a finitely branching graph) is itself.}\]

\[\text{3. For the functor } \mathcal{B}(A \times -) \text{ the maps } \tau^F_{\mathcal{B}(A \times -)}: \mathcal{B}(A \times X) \to \mathcal{P}X \text{ act as follows}\]

\[\left\langle (a_1, x_1), \ldots, (a_n, x_n) \right\rangle \mapsto \{x_1, \ldots, x_n\}.\]

Hence, if we view a coalgebra \(X \to \mathcal{B}(A \times X)\) as a finitely-branching graph whose edges are labelled by pairs of elements of \(A\) and \(\mathbb{N}\), then the canonical graph is that same graph but without the edge labels. This holds similarly also for other monoid-valued functors.

To perform reachability analysis on encoded coalgebras, we would like that the canonical graph of a coalgebra and its encoding coincide. This clearly follows when, given a set functor \(F\) with encoding \(\gamma_X: FX \to \mathcal{B}(A \times X)\), the following equation holds for every set \(X\):

\[\tau^F_X = (FX \xrightarrow{\gamma_X} \mathcal{B}(A \times X) \xrightarrow{\tau^F_{\mathcal{B}(A \times -)}} \mathcal{P}X).\]
Assumption 5.3. For the rest of this section we assume that $F$ is an intersection preserving set functor equipped with a subnatural encoding $\flat_X : FX \to \mathcal{B}(A \times X)$.

Remark 5.4. That $F$ preserves intersections is an extremely mild condition for set functors. All the functors in Example 3.4 preserve intersections. Furthermore, the collection of intersection preserving set functors is closed under products, coproducts, and functor composition. A subfunctor $\sigma : F \to G$ of an intersection preserving functor $G$ preserves intersections if $\sigma$ is a cartesian natural transformation, that is all naturality squares are pullbacks (cf. Remark 2.6).

Let us note that for every finitary set functor (cf. Remark 3.9) the Trnková hull $\bar{F}$ (see p. 8) preserves intersections [2, Cor. 8.1.17].

We are now ready to show the desired equality (6) by point-wise inclusion in either direction. Under the running Assumption 5.3 it follows that the encoding of a coalgebra can only mention states that are in the coalgebra’s canonical graph:

Proposition 5.5. For every $t \in FX$ we have that $\tau^{\mathcal{B}(A \times -)}_X (\flat_X (t)) \subseteq \tau^F_X (t)$.

Proof (Sketch). This is shown by contraposition. If $x$ is not in $\tau^F_X (t)$, then we know that the map $t : 1 \to FX$ factorizes through $F(X \setminus \{x\} ) \xrightarrow{F\cdot i} FX$ (cf. Definition 5.1). Using the subnaturality square of $\flat$ for the map $i$ then yields $x \notin \tau^{\mathcal{B}(A \times -)}_X (\flat_X (t))$.

For the converse inclusion, we additionally require that $F$ meets the assumptions of the partition refinement algorithm:

Theorem 5.6. The canonical graph of a finite coalgebra coincides with that of its encoding.

For every finite set $X$ one proves the equation (6): $\tau^F_X = \tau^{\mathcal{B}(A \times -)}_X \cdot \flat_X$. It suffices to prove the reverse of the inclusion in Proposition 5.5 – again by contraposition. This time the argument is more involved using that the map $\langle F!, \flat_X \rangle$ is injective (Definition 3.1), and that $F$ preserves intersections. (For details see the appendix.)

As a consequence of Theorem 5.6, the states in the reachable part of a pointed coalgebra $(X, c, i)$ are precisely the states reachable from the node $i(*) \in X$ in the (underlying graph of the) encoding $\flat_X \cdot c : X \to \mathcal{B}(A \times X)$, cf. Example 5.23. Thus, given (the encoding of) a pointed coalgebra $(X, c, i)$, its reachable part can be computed in linear time by a standard breadth-first search on the encoding viewed as a graph (ignoring the labels).

This holds for all the functors in Example 3.4 and every functor obtained from them by forming products, coproducts and functor composition.

6 Conclusions and Future Work

We have shown how to extend a generic coalgebraic partition refinement algorithm to a fully fledged minimization algorithm. Conceptually, this is the step from computing the simple quotient of a coalgebra to computing the well-pointed modification of a pointed coalgebra. To achieve this, our extension includes two new aspects: (1) the computation of the transition structure of the simple quotient given an encoding of the input coalgebra and the partition of its state space modulo behavioural equivalence, and (2) the computation of the encoding of the reachable part from the encoding of a given pointed coalgebra. Both of these new steps have also been implemented in the Coalgebraic Partition Refiner CoPaR, together with a new pretty-printing module that prints out the resulting encoded coalgebra in a functor-specific human-readable syntax.

There are a number of questions for further work. This mainly concerns broadening the scope of generic coalgebraic partition refinement algorithms. First, we will further broaden
the range of system types that our algorithm and tool can accommodate, and provide support for base categories beside the sets as studied in the present work, e.g. nominal sets, which underlie nominal automata [13, 40].

Concerning genericity, there is an orthogonal approach by Ranzato and Tapparo [37], which is variable in the choice of the notion of process equivalence — however within the realm of standard labelled transition systems (see also [25]). Similarly, Blom and Orzan [11,12] use a technique called signature refinement, which handles strong and branching bisimulation as well as Markov chain lumping (see also [46]).

To overcome the bottleneck on memory consumption that is inherent in partition refinement [43,44], symbolic and distributed methods have been employed for many concrete system types [8,11,12,24,46,48]. We will explore in future work whether these methods, possibly generic in the equivalence notion, can be extended to the coalgebraic generality.

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A  Additional Notation in Omitted Proofs

Recall that the quotients of a set $X$, represented by surjective maps $X \twoheadrightarrow P$ are in one-to-one correspondence with partitions on $X$. More generally, every map $f: X \to Y$ induces an equivalence relation

$$\ker(f) = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\}$$

called the kernel of $f$. If $f: X \twoheadrightarrow Y$ is surjective, then $\ker(f)$ is the equivalence relation corresponding to the partition $Y$ on $X$.

B  Omitted Proofs

B.1  Proofs for Section 2

Details for Remark 2.6

Recall that a (sub)natural transformation $\sigma$ from a functor $F$ to a functor $G$ is a set-indexed family of maps $\sigma_X: FX \to GX$ such that for every (injective) function $f: X \to Y$ we have

$$\begin{array}{ccc}
FX & \xrightarrow{\sigma_X} & GX \\
Ff & & Gf \\
FY & \xrightarrow{\sigma_Y} & GY
\end{array}$$

As usual, we shall also say that the family $\sigma$ is (sub)natural in $X$.

A subnatural transformation $\sigma: F \to G$ is called subcartesian if the above “naturality squares” are pullbacks for every injective map $f$.

Given a natural transformation $\sigma: F \to G$ every $F$-coalgebra $(X, c)$ yields a $G$-coalgebra $(X, \sigma_X \cdot c)$.

Recall that $F$ is a subfunctor of $G$ if there is a natural transformation $\sigma: F \to G$ all of whose components $\sigma_X: FX \hookrightarrow GX$ are injective maps.

In the following proposition point 1 is standard, for point 2 see [49, Prop. 2.13], and point 3 can be gleaned from [50, Thm. 4.6]. We provide a full proof for the convenience of the reader.

▶ Proposition B.1. Let $\sigma: F \to G$ be natural transformation.

1. Behavioural equivalence wrt. $F$ implies that for $G$.

2. If $F$ is a subfunctor of $G$ via $\sigma$, then the problem of computing the simple quotient for $F$-coalgebras reduces to that for $G$-coalgebras.

3. If $\sigma$ is subcartesian, then the problem of computing the reachable part for pointed $F$-coalgebras reduces to that for pointed $G$-coalgebras.

Consequently, if $F$ is a subfunctor of $G$ via a subcartesian $\sigma$ the minimization problem for $F$-coalgebras reduces to that for $G$-coalgebras.

Proof. 1. This follows from the fact that for every morphism $h: (X, c) \to (Y, d)$ of $F$-coalgebras we have the following commutative diagram due to the naturality of $\sigma$:

$$\begin{array}{ccc}
X & \xrightarrow{c} & FX \\
\downarrow{h} & & \downarrow{Fh} \\
Y & \xrightarrow{d} & FY \\
\end{array} \xrightarrow{\sigma_X} \begin{array}{ccc}
FX & \xrightarrow{\sigma_X} & GX \\
GFh & & GY
\end{array}$$
This actually shows that the object assignment \((X, c) \mapsto (X, \sigma_X \cdot c)\) is a functor from the category \(\text{Coalg}_F\) of all \(F\)-coalgebras to the category \(\text{Coalg}_G\) of all \(G\)-coalgebras, which acts as the identity on morphisms.

2. We first prove that the above functor \(\text{Coalg}_F \to \text{Coalg}_G\) preserves and reflects quotient coalgebras if \(\sigma_X\) is injective.

For preservation, note that every quotient \(q: (Y, d) \xrightarrow{\sim} (X, c)\) yields the quotient \(q: (Y, \sigma_Y \cdot d) \xrightarrow{\sim} (X, \sigma_X \cdot c)\) wrt. the functor \(G\), cf. Diagram (7).

For reflection, let \((X, c)\) be an \(F\)-coalgebra and let \(q: (X, \sigma_X \cdot c) \xrightarrow{\sim} (Y, d')\) be any quotient of \(G\)-coalgebras. Since \(q\) is surjective and \(\sigma_Y\) injective we obtain a unique coalgebra structure \(d: Y \to FY\) such that \(q\) is a morphism of \(F\)-coalgebras:

\[
\begin{array}{ccc}
X & \xrightarrow{c} & FX \\
\downarrow q & & \downarrow Gq \\
Y & \xrightarrow{\sigma_Y} & FY
\end{array}
\]

The desired reduction is now obvious since the simple quotient of an \(F\)-coalgebra \((X, c)\) coincides with that of the \(G\)-coalgebra \((X, \sigma_X \cdot c)\).

3. We first prove that the functor \(\text{Coalg}_F \to \text{Coalg}_G\) induced by \(\sigma\) preserves and reflects pointed subcoalgebras.

Preservation is clear by using Diagram (7) and the fact that a morphism \(h: (X, c, i) \to (Y, d, j)\) of pointed coalgebras preserves the point: \(h \cdot i = j\).

For reflection, let \((X, c, i)\) be any \(F\)-coalgebra and let \(m: (S, s', j) \xrightarrow{\sim} (X, \sigma_X \cdot c, i)\) be a subcoalgebra. Then from the fact that \(\sigma\) is subcartesian we obtain a unique coalgebra structure \(s: S \to FS\) such that \(m: (S, s, j) \xrightarrow{\sim} (X, c, i)\) is a pointed subcoalgebra wrt. \(F\):

\[
\begin{array}{ccc}
S & \xrightarrow{s'} & FS \\
\downarrow m & & \downarrow Gm \\
X & \xrightarrow{c} & FX
\end{array}
\]

This implies that the reachable parts of \((X, c, i)\) wrt. \(F\) and \((X, \sigma_X \cdot c, i)\) wrt. \(G\) coincide, which clearly establishes the desired reduction. ▶

For further use we collect a few properties of the filter function \(\text{fil}_S\).

\[\textbf{Lemma B.2.}\]
1. The maps \(\text{fil}_S: B(A \times X) \to B(A)\) are natural in \(A\).

2. For every \(x \in X\), we have

\[\text{fil}_{\{x\}} = \text{ev}(x) \cdot \text{group},\]

where \(\text{ev}: X \to Y^X \to Y\) is the evaluation of the exponential \(Y^X\) (in curried form).

3. The function \(\langle \text{fil}_{\{x\}} \rangle_{x \in X}: B(A \times X) \to (BA)^X\) has a codomain restriction to \((BA)^X\), and this is equal to the function \(\text{group}\).

4. For every function \(f: X \to Y\) and \(S \subseteq X\),

\[\text{fil}_S = \text{fil}_{f[S]}: B(A \times f).\]

\[\text{Proof.}\] 1. This was proved in previous work [49, Rem. 6.5].
2. Given \( x \in X, a \in A \) and \( f \in \mathcal{B}(A \times X) \),
\[
\text{fil}_{\{x\}}(f)(a) = \sum_{y \in \{x\}} f(a, y) = f(a, x) (\lambda b. f(b, x))(a) = \text{ev}(x)(\lambda y. \lambda b. f(b, y))(a) = \text{ev}(x)(\text{group}(f))(a).
\]

3. The first part of the statement is clear, and the second part follows from item 2.

4. Given \( t \in \mathcal{B}(A \times X), f : X \rightarrow Y, S \subseteq X \) and \( a \in A \):
\[
\text{fil}_{f[S]}(\mathcal{B}(A \times f))(t)(a) = \sum_{y \in f[S]} \mathcal{B}(A \times f)(t)(a, y) = \sum_{y \in f[S]} \sum_{x \in X \atop f(x) = y} t(a, x) = \sum_{x \in S \atop f(x) \in f[S]} t(a, x) = \text{fil}_S(t)(a).
\]

\[\blacktriangle\]

### B.2 Proofs for Section 3

#### Proof of Proposition 3.3

We recall the definition of a refinement interface of a functor \( F \) here for the convenience of the reader. For detailed treatment, please refer to our previous work [49,52].

**Definition B.3.** Given sets \( S \subseteq C \subseteq X \), the map \( \chi^C_S : X \rightarrow 3 \) is defined by
\[
\chi^C_S(x) = \begin{cases} 
2 & \text{if } x \in S, \\
1 & \text{if } x \in C \setminus S, \\
0 & \text{if } x \in X \setminus C.
\end{cases}
\]

Intuitively, this is a ‘three-valued characteristic function’ and is equivalent to the map \( \langle \chi_S, \chi_C \rangle : X \rightarrow 2 \times 2 \) without the impossible case \((1,0) \ (x \in S, x \notin C, S \subseteq C)\).

**Definition B.4** [49]. Given a set \( A \) and a family of maps \( \triangleright_X : FX \rightarrow \mathcal{B}(A \times X) \) (one for every set \( X \)), a **refinement interface** for a functor \( F \) is formed by a set \( W \) of **weights** and functions
\[
\text{init} : F1 \times BA \rightarrow W, \quad \text{update} : BA \times W \rightarrow W \times F3 \times W
\]
such that there exists a family of **weight maps** \( w : \mathcal{P}X \rightarrow (FX \rightarrow W) \) such that for all \( S \subseteq C \subseteq X \), the diagrams
\[
\begin{array}{ccc}
F1 \times BA & \xrightarrow{\text{init}} & W \\
\downarrow^{(F1, \text{fil}_X \triangleright)} & & \downarrow^{w(X)} \\
FX & & FX
\end{array}
\quad
\begin{array}{ccc}
BA \times W & \xrightarrow{\text{update}} & W \times F3 \times W \\
\downarrow^{(\text{fil}_S, \triangleright_X \triangleright w(C))} & & \downarrow^{(w(S), F\chi^C_S, w(C \setminus S))} \\
FX & & FX
\end{array}
\]

commute.

**Remark B.5** [49]. The crucial part of forming a refinement interface for a functor \( F \) is finding an appropriate set \( W \) and maps \( w : \mathcal{P}X \rightarrow (FX \rightarrow W) \). However, only the maps \( \text{init} \) and \( \text{update} \) are implemented, whereas \( w \) just ensures correctness (and is not thus implemented). In most instances, we have
\[
W := F2 \quad \text{and} \quad w(C) := F\chi_C : FX \rightarrow F2 \quad \text{for } C \subseteq X.
\]
The only exception is the monoid valued functor $FX = M^{(X)}$ for a non-cancellative monoid $(M, +, 0)$ [52] (so in particular also for $FX = \mathcal{P}_1 X$ [49]), where we have:

$$W := M \times B(M \setminus \{0\}) \quad \text{and}$$

$$w(C): M^{(X)} \to M \times B(M \setminus \{0\})$$

$$w(C)(\mu) = \left( \sum_{x \in X \setminus C} \mu(x), m \mapsto |\{x \in X \mid \mu(x) = m\}| \right)$$

where we define the bag $B(M \setminus \{0\})$ by a map $(M \setminus \{0\}) \to \mathbb{N}$. For $FX = \mathcal{P}_1 X$, we have $M = 2$ and $B(2 \setminus \{0\}) = B\{1\} \cong \mathbb{N}$, hence $w(C): \mathcal{P}_1 X \to 2 \times \mathbb{N}$ sends a successor structure $t \in \mathcal{P}_1 X$ to $w(C)(t) \in 2 \times \mathbb{N}$ which provides (a) the information whether $t$ contains a successor outside of $C$ and (b) the number of successors in $C$. Keeping track of the number of successors in the blocks $C$ of the partition is one of the main ideas of the $O(m \log n)$ algorithm by Paige and Tarjan [36].

The functions $\text{init}, \text{update}$, which are to be implemented for every functor $F$ of interest, incrementally compute these weights (in $W$) and the three valued characteristic function $F\chi_C^S: FX \to F3$.

**Lemma B.6.** Given a functor $F$ with a refinement interface, we have for all sets $S \subseteq C \subseteq X$ a map $r_S^C: F1 \times B(A \times X) \to F3$ with $r_S^C \cdot (F!, \flat_X) = F\chi_S^C$.

![Diagram](image)

**Proof.** First, we define maps $v_X$ and $v_C$ by the commutativity of the left-hand parts of the diagrams below, respectively. We also observe that precomposing these maps with $(F!, \flat_X)$ yields $w(X)$ and $w(C)$, respectively, using the axioms of the refinement interface (note that the left-hand triangle in the right-hand diagram commutes by the left-hand diagram):

$$v_X := \begin{array}{c} F1 \times B(A \times X) \xrightarrow{(F!, \flat_X)} FX \\ F1 \times BA \xrightarrow{\text{init}} W \end{array}$$

$$v_C := \begin{array}{c} F1 \times B(A \times X) \xrightarrow{(F!, \flat_X)} FX \\ W \times BA \xrightarrow{\text{update}} W \end{array}$$

Now we can define $r_S^C$ by the commutativity of the left-hand part in the diagram below and show that $r_S^C \cdot (v_C, \flat_S \cdot pr_2) = (w(C), \flat_C \cdot \flat_X)$:

$$r_S^C := \begin{array}{c} W \times BA \xrightarrow{\text{update}} F3 \\ W \times BA \xrightarrow{(v_C, \flat_S \cdot pr_2)} FX \xrightarrow{F\chi_S^C} F3 \end{array}$$

$\blacksquare$
Proof of Proposition 3.3. Let $X = \{x_0, \ldots, x_{n-1}\}$. We define the following family of subsets of $X$:

$$S_i = \{x_i\}, \quad C_i = \{x_i, \ldots, x_{n-1}\} \quad \text{for } 0 \leq i < n.$$

For every $k$, $0 \leq k \leq n$, we define the map

$$q_k := \langle \chi_{S_i}^C \rangle_{0 \leq i < k} : X \to \prod_{0 \leq i < k} 3 \cong 3^k.$$

The partitions corresponding to $\chi_{S_i}^C$ and $q_k$ are illustrated in Figure 2. Note that the partition for $q_{k+1}$ is formed by the (nonempty) intersections of the blocks from the partitions for $q_k$ and $\chi_{S_i}^C$. Clearly, the union of the equivalence relations $\ker(q_k)$ and $\ker(\chi_{S_i}^C)$ is again an equivalence relation, for every $0 \leq k < n$. Thus, we can apply [49, Prop. 5.18] to obtain

$$\ker(Fq_k, \chi_{S_i}^C) = \ker(Fq_k, \chi_{S_i}^C) \quad \text{for all } 0 \leq k < n.$$

Combining these $n$-many equalities, we obtain

$$\ker Fq_n = \ker \langle F\chi_{S_i}^C \rangle_{0 \leq k < n}. \quad (10)$$

Here, the map $q_n : X \to 3^n$ is injective, because for every $x_i \in X$ the element $q_n(x_i) \in 3^n$ is clearly the only element in the image of $q_n$ that has 2 in the $i$th component:

$$\text{pr}_i(q_n(x_i)) = \chi_{S_i}^C(x_i) = \chi_{S_i}^C(x_i) \overset{\text{Def.}}{=} S_i \overset{\text{Def.}}{=} \chi \overset{\text{Def.}}{=} 2.$$

Since $F$ preserves injective maps, $Fq_n : FX \to F3^n$ is injective, too. Thus, so is $\langle F\chi_{S_i}^C \rangle_{0 \leq k < n}$ by equation (10) of kernels (note that a map $f : A \to B$ is injective iff the relation $\ker(f)$ is the identity relation on $A$). Finally, we pair all the maps $r_{S_i}^C, 0 \leq k < n$, that we have derived from the refinement interface in Lemma B.6. We obtain the following commutative diagram:

\[
\begin{array}{cccc}
FX & \langle F\chi_{S_i}^C \rangle_{0 \leq k < n} & \prod_{0 \leq k < n} F3 \\
\downarrow & & \downarrow \\
(F!, b_X) \times B(A \times X) & \langle F!, b_X \rangle & \langle F!, b_X \rangle_{0 \leq k < n}
\end{array}
\]

We know that the map at the top is injective. It follows from the standard laws for injective maps that $\langle F!, b_X \rangle$ is injective, as desired.
Proof of Proposition 3.5

Proof. Given \( f : X \to Y \) and \( \sigma(x_1, \ldots, x_n) \in F_X X \), we calculate

\[
B(A \times f)(\nu_X(\sigma(x_1, \ldots, x_n))) = B(A \times f)((1, x_1), \ldots, (n, x_n))
\]

\[
= [(1, f(x_1)), \ldots, (n, f(x_n))]
\]

\[
= \nu_Y(\sigma(f(x_1), \ldots, f(x_n)))
\]

\[
= \nu_Y(F_X f(\sigma(x_1, \ldots, x_n))).
\]

Proof of Proposition 3.8

Proof. By the assumption on the individual functors \( F_i \), we know that \( \langle F_i!, \nu_X,i \rangle \) is injective for every \( i \in I \). We now prove the required injectivity for the coproduct and product functors respectively:

1. For the coproduct \( \coprod_{i \in I} F_i \), assume \( t_1, t_2 \in \coprod_{i \in I} F_i X \) such that \( \langle F!, \nu_X \rangle(t_1) = \langle F!, \nu_X \rangle(t_2) \). Since this implies that \( \nu_X(F t_1) = \nu_X(F t_2) \), we know that \( j = k \). We now expand the definition of \( \nu_X \) and calculate for \( i = 1, 2 \):

\[
\langle F!, \nu_X \rangle(t_i) = \langle F!, [B(\text{id} \times \text{id})]_{k \in I} \cdot \coprod_{k \in I} \nu_{X,k} \rangle(t_i)
\]

\[
= \text{id} \times [B(\text{id} \times \text{id})]_{k \in I} \cdot \langle F!, \coprod_{k \in I} \nu_{X,k} \rangle(t_i)
\]

\[
= \nu_X(\nu_X(F t_i))(t_i).
\]

Since \( \nu_X \) is an injective map, \( B \) preserves injections, and injections are stable under product, we see that \( \nu_X \times [B (\text{id} \times \text{id})]_{k \in I} \) is injective, whence so is its composite with \( \langle F!, \nu_{X,j} \rangle \). Since this composite merges \( t_1 \) and \( t_2 \) by assumption, we conclude \( t_1 = t_2 \).

2. For the product \( \prod_{i \in I} F_i \), assume \( t_1, t_2 \in \prod_{i \in I} F_i X \) such that \( \langle F!, \nu_X \rangle(t_1) = \langle F!, \nu_X \rangle(t_2) \). We will show that \( t_1 = t_2 \). The assumption implies that \( F!(t_1) = F!(t_2) \) as well as \( \nu_X(t_1) = \nu_X(t_2) \). The former implies that for all \( i \in I \) we have \( F_i!(\nu_X(i)) = F_i!(\nu_X(i)) \). From the latter and the definition of \( \nu_X \) we obtain for every \( i \in I \), \( a \in A_i \), and \( x \in X \) that

\[
\nu_X(i)(a, x) = \nu_X(t_i)(a, x) = \nu_X(t_2)(a, x) = \nu_X(i)(a, x).
\]

Consequently, for all \( i \in I \) we have \( \langle F_i!, \nu_{X,i} \rangle(\nu_X(i)) = \langle F_i!, \nu_{X,i} \rangle(\nu_X(i)) \) which implies \( \nu_X(i) = \nu_X(i) \) by the assumption. Hence, \( t_1 = t_2 \) since the projections \( \nu_{X,i}, i \in I \), form a jointly monic family.

Proof of Proposition 3.11

Proof. First, note that for every bag \( b \in B(A \times X) \) and every pair \((a, x) \in A \times X\) we have \( \text{fil}_{(a)}(b)(a) = b(a, x) \).

1. For the finite powerset functor \( \mathcal{P}_f(-) \), we have \( A = 1 \) and \( b : \mathcal{P}_f X \to B(1 \times X) \cong B(X) \) given by

\[
b(t)(*, x) = \begin{cases} 1 & x \in t, \\ 0 & \text{otherwise}. \end{cases}
\]
Observe that $1 \in P_{\chi_x}(t) \iff x \in t$. We then have

$$\text{fil}_{\chi_x}(b(t))(*) = b(t)(*, x) = \begin{cases} 1 & x \in t \\ 0 & \text{otherwise}, \end{cases}$$

$$\text{fil}_{\chi_x}(b(P_{\chi_x}(t)))(*) = b(P_{\chi_x}(t))(*, 1) = \begin{cases} 1 & 1 \in P_{\chi_x}(t) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{fil}_{\chi_x}(1)(*) = 1 \in P_{\chi_x}(t) \iff x \in t.$$  

2. The monoid-valued functor $M^{(-)}$ for a given monoid $M$ has labels $A = M$ and $\flat : M^X \to B(M \times X)$ given by

$$\flat(t)(m, x) = \begin{cases} 1 & t(x) = m \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Observe that $M^{X(\sigma)}(t)(1) = t(x)$. We then have

$$\text{fil}_{\chi_x}(b(t))(m) = b(t)(m, x) = \begin{cases} 1 & t(x) = m \neq 0 \\ 0 & \text{otherwise}, \end{cases}$$

$$\text{fil}_{\chi_x}(b(M^{X(\sigma)}(t)))(m) = b(M^{X(\sigma)}(t))(m, 1) = \begin{cases} 1 & M^{X(\sigma)}(t)(1) = m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{fil}_{\chi_x}(1)(m) = 1 \in M^{X(\sigma)}(t) \iff t(x) = m \neq 0.$$  

3. The polynomial functor $F\Sigma$ for a signature $\Sigma$ has labels $A = \mathbb{N}$, and the map $\flat : F\Sigma X \to B(\mathbb{N} \times X)$ is given by $\flat(\sigma(x_1, \ldots, x_n)) = \{(1, x_1), \ldots, (n, x_n)\}$. Since this $\flat$ is natural by Proposition 3.5, the desired result follows from Proposition 3.13(1).

Proof of Proposition 3.12

Proof. 1. For the coproduct of $(F_i)_{i \in I}$, $\flat_X : \coprod_{i \in I} F_i X \to B(\coprod_{i \in I} A_i \times X)$ is defined in Construction 3.7 as

$$\flat_X : \coprod_{i \in I} F_i X \xrightarrow{\coprod_{i \in I} \text{fil}_{X,i}} \coprod_{i \in I} B(A_i \times X) \xrightarrow{[B(m_i \times X)]_{i \in I}} B(\coprod_{i \in I} A_i \times X).$$
We evaluate both sides of the condition for uniform encodings:

\[
\begin{align*}
\text{fil}_{\{x\}}(b(\text{in}_i t))(\text{in}_j a) &= b(\text{in}_i t)(\text{in}_j a, x) \\
&= (B(\text{in}_i X))_{k \in I}((\prod_{k \in I} b_k)(\text{in}_i t))(\text{in}_j a, x) \\
&= (B(\text{in}_i X))_{k \in I} (\text{in}_i (\text{in}_k t))(\text{in}_j a, x) \\
&= \begin{cases} 
0 & i \neq j \\
\gamma_i(a, x) & \text{otherwise}, 
\end{cases}
\end{align*}
\]

and \(\gamma_i(\prod_{k \in I} F_k \chi_{\{x\}})(\text{in}_i t))(\text{in}_j a, 1) = \gamma_i(\text{in}_i F_i \chi_{\{x\}}(t))(\text{in}_j a, 1)
\]

\[
= \begin{cases} 
0 & i \neq j \\
\gamma_i(F_i \chi_{\{x\}}(t))(a, 1) & \text{otherwise}, 
\end{cases}
\]

Therefore, both sides agree.

2. For the product \(\prod_{i \in I} F_i\) we define \(\gamma_X : \prod_{i \in I} F_i X \to B(\prod_{i \in I} A_i \times X)\) in Construction 3.7 as

\[
\gamma_X(t)(\text{in}_i(a), x) = \gamma_i(\text{pr}_i(t))(a, x).
\]

We evaluate both sides again:

\[
\begin{align*}
\text{fil}_{\{x\}}(b(t))(\text{in}_i(a)) &= b(t)(\text{in}_i a, x) = \gamma_i(\text{pr}_i(t))(a, x) \\
\text{fil}_{\{1\}}(b((\prod_{k \in I} F_k \chi_{\{x\}}(t))))(\text{in}_i a) &= \\
&= b((\prod_{k \in I} F_k \chi_{\{x\}})(t))(\text{in}_i a, 1) \\
&= \gamma_i(\text{pr}_i((\prod_{k \in I} F_k \chi_{\{x\}}(t)))(a, 1) \\
&= \gamma_i(F_i \chi_{\{x\}}(\text{pr}_i(t)))(a, 1) \\
&= \gamma_i(\text{pr}_i(t)(a, x),
\end{align*}
\]

where the last line uses the fact that \(\gamma_i\) is uniform.

\[\blacktriangleleft\]

**Proof of Proposition 3.13**

In order to prove that uniform encodings are subnatural we use the following lemma:

**Lemma B.7.** The following diagram commutes for all uniform encodings:

\[
\begin{array}{ccc}
F1 & \xrightarrow{1} & 1 \\
F0 \downarrow & & \downarrow 0 \\
F2 \xrightarrow{b_2} B(A \times 2) & \xrightarrow{\text{fil}_{\{\cdot\}}(1)} & BA,
\end{array}
\]

where \(0 : 1 = \{0\} \hookrightarrow \{0, 1\} = 2\) is the obvious inclusion map.
Proof. The following diagram commutes for all \( n \in \mathbb{N} \):

\[
\begin{array}{c}
F^1 \xrightarrow{F \text{inl}} F(1 + \mathbb{N}) \xrightarrow{\beta_{1+\mathbb{N}}} B(A \times (1 + \mathbb{N})) \\
\downarrow F_{\text{inl}(n)} \quad \text{(Def. 3.10)} \\
F^2 \xrightarrow{\beta_2} B(A \times 2) \xrightarrow{\text{fil}(n)} B(A).
\end{array}
\]

Let \( t \in F^1 \) and, for the sake of contradiction, suppose that \( \text{fil}(1)(\beta_2(F^0(t))) \) is nonempty and contains the element \( a \). Then, by the above diagram we have \( a \in \text{fil}(n)(\beta_{1+\mathbb{N}}(\text{Finl}(t))) \) and therefore

\[(a, \text{inr} n) \in \beta_{1+\mathbb{N}}(\text{Finl}(t)) \quad \text{for all } n \in \mathbb{N}.
\]

However, this contradicts the finiteness of the bag \( \beta_{1+\mathbb{N}}(\text{Finl}(t)) \).

\[\blacklozenge\]

We are now ready to prove the main proposition:

Proof of Proposition 3.13. 1. Given an encoding \( \beta_X : FX \to B(A \times X) \) which is natural in \( X \), we have the following commutative diagram:

\[
\begin{array}{c}
FX \xrightarrow{\beta_X} B(A \times X) \\
\downarrow F_{\text{inl}(x)} \quad \text{(Def. 3.10)} \\
FY \xrightarrow{\beta_Y} B(A \times Y) \xrightarrow{\text{fil}(y)} BA
\end{array}
\]

Indeed, the left-hand square commutes due to the naturality of \( \beta \) and the right-hand triangle commutes by Lemma B.2(4).

2. Let \( \beta_X : FX \to B(A \times X) \) be a uniform encoding. First we show that the family \( \{ B(A \times Y) \xrightarrow{\text{fil}(y)} B(A) \}_{y \in Y} \) is jointly monic. Indeed, recall from Lemma B.2 that the morphism \( \langle \text{fil}(y) \rangle_{y \in Y} \) is equal to \( \text{group} : B(A \times Y) \to (BA)^Y \), which is an isomorphism, whence a split mono. It therefore suffices to prove that the following diagram commutes for all \( y \in Y \) and all monomorphisms \( m : X \hookrightarrow Y \):

\[
\begin{array}{c}
FX \xrightarrow{\beta_X} B(A \times X) \xrightarrow{B(A \times m)} B(A \times Y) \\
\downarrow F_{\text{inl}(x)} \quad \text{(Def. 3.10)} \\
FY \xrightarrow{\beta_Y} B(A \times Y) \xrightarrow{\text{fil}(y)} BA
\end{array}
\]

We distinguish two cases:

a. If \( y \in m[X] \), equivalently, \( y = m(x) \) for an \( x \in X \), the following diagram commutes:

\[
\begin{array}{c}
FX \xrightarrow{\beta_X} B(A \times X) \xrightarrow{B(A \times m)} B(A \times Y) \\
\downarrow F_{\text{inl}(x)} \quad \text{(Def. 3.10)} \\
FY \xrightarrow{\beta_Y} B(A \times Y) \xrightarrow{\text{fil}(m(x))}
\end{array}
\]

Therefore, (11) commutes for \( y = m(x) \in m[X] \).
b. If \( y \in (Y \setminus m[X]) \), equivalently, \( \chi_{(y)} \cdot m = 0! \), then the following diagram commutes:

\[
\begin{array}{ccccccccc}
FX & \xrightarrow{b_X} & F(A \times X) & \xrightarrow{B(A \times m)} & F(B(A \times Y)) \\
\downarrow{Fm} & & \downarrow{F1} & \downarrow{F0} & \downarrow{\mathfrak{b}_m} \\
FY & \xrightarrow{F\chi_{(y)}} & F2 & \xrightarrow{b_2} & B(A \times Y) & \xrightarrow{\mathfrak{b}_2} & B(A) \\
\end{array}
\]

Therefore, (11) also commutes for \( y \not\in m[X] \).

\[\blacksquare\]

### B.3 Proofs for Section 4

#### Proof of Lemma 4.2

**Proof.** To see this, instantiate (3) for \( S = \{x\} \) and compare it with the diagram (2) of Definition 3.10:

\[
\begin{array}{ccccccccc}
FX & \xrightarrow{\mathfrak{b}_x \cdot b_X} & F(A) & \xrightarrow{B(A \times x)} & B(A \times X) \\
\downarrow{F\chi_{(x)}} & & \downarrow{\text{merge}} & \downarrow{\mathfrak{b}_2} & \downarrow{\mathfrak{b}_1} \\
F2 & & & & B(A) \\
\end{array}
\]

We see that the upper inner part of the diagram commutes if and only if the outside does. This establishes the desired equivalence. \[\blacksquare\]

#### Proof of Lemma 4.3

We first establish the following easy lemma:

➤ **Lemma B.8.** For every map \( f: X \to Y \) and subset \( S \subseteq Y \) we have

\[
\begin{array}{cccc}
X & \xrightarrow{f} & Y \\
\chi_{f^{-1}[S]} & & \chi_S \\
\end{array}
\]

**Proof.** Indeed, we have

\[\chi_S(f(x)) = 1 \iff f(x) \in S \iff x \in f^{-1}[S] \iff \chi_{f^{-1}[S]}(x) = 1.\]

**Proof of Lemma 4.3.** \( 2 \Rightarrow 1 \): The required axiom (3) for \( \text{merge} = \text{id} \) follows by combining the given naturality of \( b_X \) with Equation (9):

\[
\begin{array}{ccccccccc}
FX & \xrightarrow{b_X} & F(A \times X) & \xrightarrow{B(A \times \chi_S)} & B(A) \\
\downarrow{F\chi_{S}} & & \downarrow{\mathfrak{b}_S} & \downarrow{\mathfrak{b}_1} \\
F2 & \xrightarrow{b_2} & B(A \times 2) & \xrightarrow{\mathfrak{b}_2} & B(A) \\
\end{array}
\]
1 ⇒ 2: Suppose \( \text{merge} = \text{id} \) is a minimization interface. Then the axiom (3) simplifies as follows:

\[
\begin{align*}
FX & \xrightarrow{b_X} B(A \times X) \\
& \xrightarrow{\text{fil}_0} B(A). \\
F2 & \xrightarrow{b_2} B(A \times 2) \\
& \xrightarrow{\text{fil}_{(1)}} B(A).
\end{align*}
\] (12)

In order to show that \( b \) is a natural transformation we use that the family \((B(A \times Y) \xrightarrow{\text{fil}_{(2)}} B(A))_{y \in Y}\) is jointly monic. Hence, it suffices to prove that the following diagram commutes for all functions \( f: X \to Y \) and \( y \in Y \):

\[
\begin{align*}
FX & \xrightarrow{b_X} B(A \times X) \\
& \xrightarrow{B(A \times f)} B(A \times Y) \\
FY & \xrightarrow{b_y} B(A \times Y) \\
& \xrightarrow{\text{fil}_{(y)}} B(A)
\end{align*}
\] (13)

Indeed, let \( y \in Y \) and \( S \subseteq X \) be the inverse image of \( y \) under \( f \): \( S = f^{-1}[y] \). Then the following diagram commutes:

\[
\begin{align*}
FX & \xrightarrow{b_X} B(A \times X) \\
& \xrightarrow{B(A \times f)} B(A \times Y) \\
FY & \xrightarrow{b_y} B(A \times Y) \\
& \xrightarrow{\text{fil}_{(y)}} B(A)
\end{align*}
\] (12) for \( S = \{y\} \)

Therefore, (13) commutes for all \( f: X \to Y \) and \( y \in Y \) as desired. ▸

**Proof of Proposition 4.5**

**Proof.** 1. For the finite powerset functor \( \mathcal{P}_f(-) \), with \( A = 1 \), we define \text{merge} by

\[
\text{merge}(\ell)(*) = \min(1, \ell(*)).
\]

To show that the axiom holds, we calculate both sides:

\[
\text{fil}_{(1)}(b(\mathcal{P}_f\chi_S(t)))(*) = b(\mathcal{P}_f\chi_S(t))(*, 1)
\]

\[
= \begin{cases} 
 1 & 1 \in \mathcal{P}_f\chi_S(t) \\
 0 & \text{otherwise}
\end{cases}
= \begin{cases} 
 1 & S \cap t \neq \emptyset \\
 0 & \text{otherwise}
\end{cases}
\]

\[
\text{merge}(\text{fil}_{S}(b(t)))(*) = \min(1, \text{fil}_{S}(b(t))(*))
\]

\[
= \min(1, \sum_{x \in S} b(t)(*, x))
\]

\[
= \begin{cases} 
 1 & S \cap t \neq \emptyset \\
 0 & \text{otherwise}
\end{cases}
\]

This \text{merge} can be implemented in constant time, since it just needs to check if its input bag is empty and return one of two possible constants, depending on that result.
2. For monoid-valued functors $M(-)$ with $A = M$, $\text{merge}$ is defined as

$$\text{merge}(\ell) = \begin{cases} \{ \Sigma \ell \} & \Sigma \ell \neq 0 \\ \{ \} & \text{otherwise.} \end{cases}$$

To show that this fulfils the required property, we first need the following facts:

$$(M^{\chi_S}t)(1) = \sum_{x \in X} t(x) = \sum_{x \in S} t(x),$$

and

$$\Sigma(\text{fil}_S(\flat t)) = \sum_{m \in M} m \cdot (\text{fil}_S(\flat t))(m)$$

$$= \sum_{m \in M} m \cdot \left( \sum_{x \in S} \flat t(m, x) \right)$$

$$= \sum_{x \in S} \sum_{m \in M} m \cdot \frac{\flat t(m, x)}{1 \text{ if } t(x) = m \neq 0 \text{ else } 0}$$

$$= \sum_{x \in S} t(x).$$

Now we have that

$$\text{fil}_{\{1\}}(\flat (M^{\chi_S}t))(m) = (M^{\chi_S}t)(m, 1)$$

$$= \begin{cases} 1 & \sum_{x \in S} t(x) = m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and also

$$\text{merge}(\text{fil}_S(\flat t))(m) = \begin{cases} 1 & \Sigma(\text{fil}_S(\flat t)) = m \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \sum_{x \in S} t(x) = m \\ 0 & \text{otherwise.} \end{cases}$$

Since this $\text{merge}$ has to sum up the monoid elements in its input bag, it runs in linear time in the size of that bag, provided that addition of monoid elements is a constant-time operation.

3. For the polynomial functor $F_{\Sigma}$, the encoding $\flat : F_{\Sigma}X \to B(\mathbb{N} \times X)$ is already natural (see Proposition 3.5). Thus, $\text{merge} = \text{id}$ is a minimization interface by Lemma 4.3 with constant run-time.

**Proof of Proposition 4.7**

We first prove the following technical proposition about merge:

**Proposition B.9.** Suppose $F$ is not the constant empty set functor ($CX = \emptyset$, $F \neq C$) and is equipped with a subnatural encoding and a minimization interface $\text{merge}$. Then we have $\text{merge}(\{\}) = \{\}$. 
Proof. Consider the diagram for the injective \( \chi_\emptyset : 1 \mapsto 2 \):

\[
\begin{align*}
F1 & \xrightarrow{\chi_1} \mathcal{B}(A \times 1) \xrightarrow{\text{fil}_0} \mathcal{B} A \\
F\chi_1 & \xrightarrow{y} \mathcal{B}(A \times \chi_0) \xrightarrow{\text{merge}} \mathcal{B} A \\
F2 & \xrightarrow{\chi_2} \mathcal{B}(A \times 2) \xrightarrow{\text{fil}_1} \mathcal{B} A
\end{align*}
\]

Note that \( F1 \) is nonempty for all \( \text{Set} \)-functors except for the constant empty set functor, which is excluded by assumption. Hence, there is some \( x \in F1 \) and we have:

\[
\text{merge}(\emptyset) = \text{merge}(\text{fil}_0(\chi_1(x))) = \text{fil}_1(\chi_1(y(\chi_0(x)))) = \text{fil}_1(\mathcal{B}(A \times \chi_0)(\chi_1(x))) = \emptyset
\]

as desired. \( \blacktriangleleft \)

We now proceed to prove Proposition 4.7:

Proof. First, observe that for every function \( f : X \to Y \) the following squares commute:

\[
\begin{array}{ccc}
X^Z & \xrightarrow{\text{ev}(z)} & X \\
\downarrow{f^Z} & \downarrow{f} & \\
Y^Z & \xrightarrow{\text{ev}(z)} & Y
\end{array}
\]

for every \( z \in Z \). (14)

We verify that the following diagram commutes for every \( y \in Y \), where we define \( S = q^{-1}[y] \):

\[
\begin{array}{ccc}
FX & \xrightarrow{\chi_x} & \mathcal{B}(A \times X) \\
\downarrow{F\chi_x} & \downarrow{\text{fil}_0} & \\
B(A \times Y) & \xrightarrow{\text{group}} & \mathcal{B} A
\end{array}
\]

Instantiating (14) for \( f = \text{merge} \) yields \( \text{ev}(y) \cdot \text{merge}^Y = \text{merge} \cdot \text{ev}(y) \). By virtue of Proposition B.9 and Assumption 4.6, \( \text{merge} \) preserves empty bags. Hence, the function \( \text{merge}^Y : \mathcal{B}(A)^Y \to \mathcal{B}(A)^Y \) defined as \( f \mapsto \text{merge} \cdot f \) preserves finite support and therefore restricts to the monoid-valued functor as \( \text{merge}^Y : \mathcal{B}(A)^Y \to \mathcal{B}(A)^Y \). Therefore, the outside of the diagram together with the fact that \( (\text{ev}(y))_{y \in Y} \) is a jointly injective family implies

\[
\begin{array}{ccc}
FX & \xrightarrow{\chi_x} & \mathcal{B}(A \times X) \\
\downarrow{F\chi_x} & \downarrow{\text{fil}_0} & \\
\mathcal{B}(A \times Y) & \xrightarrow{\text{group}} & \mathcal{B}(A)^Y
\end{array}
\]

Post-composition with \( \text{ungroup} \) and the application of equation (1) now yields the desired result. \( \blacktriangleleft \)
Proof of Theorem 4.9

Proof. (1) Correctness. Combining that \( q \) is a coalgebra homomorphism with Proposition 4.7 yields the following diagram, whose commutativity we discuss next:

\[
\begin{array}{cccccc}
X & \overset{c}{\rightarrow} & FX & \overset{b_X}{\rightarrow} & B(A \times X) & \overset{B(A \times q)}{\rightarrow} & B(A \times Y) & \overset{\text{group}}{\rightarrow} & B(A)^Y \\
\downarrow{q} & \downarrow{\text{coalgebra}} & \downarrow{Fq} & & & & & & \\
Y & \overset{d}{\rightarrow} & FY & \overset{b_Y}{\rightarrow} & B(A \times Y) & \overset{\text{merge}^Y}{\leftarrow} & B(A)^Y & \overset{\text{ungroup}}{\leftarrow} & B(A)^Y
\end{array}
\]

The two rectangles commute, and the outside of the diagram commutes by Construction 4.8. Hence, \( e \cdot q = b_Y \cdot d \cdot q \). Since \( q \) is surjective, we have \( e = b_Y \cdot d \) as desired.

(2) Runtime. For the implementation of Construction 4.8, assume that the encoded input coalgebra \( b_X \cdot c \colon X \rightarrow B(A \times X) \) is given as adjacency lists and that the quotient map \( q \colon X \rightarrow Y \) is given as a partition on \( X \). Such a partition is represented as an assignment \( q' \colon X \rightarrow \{0, \ldots, |Y| - 1\} \) which sends an element of \( X \) to the number of its block and which can be evaluated in \( \mathcal{O}(1) \) (e.g. the refinable partition structure \([45]\) represents partitions in such a way and is in fact used by the coalgebraic algorithm \([20]\)); in other words, we implicitly use the bijection \( Y = \{0, \ldots, |Y| - 1\} \). We now compute the composition

\[
\text{ungroup} \cdot \text{merge}^Y \cdot \text{group} \cdot B(A \times q) \cdot b_X \cdot c 
\]

from Construction 4.8 step by step:

1. \( s_0 := b_X \cdot c \colon X \rightarrow B(A \times X) \) is the given input, encoded using adjacency lists, i.e. as an array of size \(|X|\) whose entries are lists of elements from \( A \times X \). We denote its size by

\[
m := |X| + \sum_{x \in X} |b_X(c(x))|.
\]

2. For \( s_1 := B(A \times q) \cdot s_0 \colon X \rightarrow B(A \times Y) \), we iterate over all edges in the adjacency lists and replace every right-hand side \( x \in X \) of an edge by \( q(x) \in Y \). This takes \( \mathcal{O}(m) \) time (\( \mathcal{O}(1) \) time for each of the \( m \) entries).

3. For \( s_2 := \text{group} \cdot s_1 \colon X \rightarrow B(A)^Y \), we represent a map \( t \in B(A)^Y \) as a list of pairs \((y, t(y)) \in Y \times B(A)\) with \( t(y) \) non-empty and compute this list for all \( x \in X \) as follows. Allocate an array \( \text{idx} \colon Y \rightarrow \mathbb{Z} \) (initially \(-1\) everywhere) and then do the following for every \( x \in X \):

   a. Allocate an array \( p \colon \mathbb{N} \rightarrow Y \times B(A) \) of size \(|s_1(x)|\) and initialize an integer \( i := 0 \) (intuitively, \( i \) is the index of the first unused cell in \( p \)).

   b. For every \((a, y) \in s_1(x)\), we distinguish whether we have seen \( y \) in \( s_1(x) \) before:

      = If \( \text{idx}(y) < 0 \), then it is the first time we see \( y \) in \( s_1(x) \). Thus put \( \text{idx}(y) := i \), increment \( i := i + 1 \), and define \( p(\text{idx}(y)) := (y, \{a\}) \).
If $\text{idx}(y) \geq 0$, then we have seen $y$ before and simply append $a$ to the second component of $p(\text{idx}(y))$.

c. For every $(y, \ell)$ in the first $i$ entries of $p$, put $\text{idx}(y) = -1$. (Thus, $\text{idx}$ is again $-1$ everywhere.)

d. Let $s_2(x)$ be the first $i$ entries of $p$

For $x \in X$ each of the above steps runs in $O(|s_1(x)|)$, thus doing these for all $x \in X$ runs in $O(m)$ in total.

4. For $s_3 := \text{merge}^{(Y)} \cdot s_2 : X \to \mathcal{B}(A)^{(Y)}$, apply $\text{merge} : \mathcal{B}(A) \to \mathcal{B}(A)$ to every bag in the list $s_2(x) \in \mathcal{B}(A)^{(Y)}$ (we have represented $s_2(x)$ as a list of elements from $Y \times \mathcal{B}(A)$ in the definition of $s_2$). Since by assumption, $\text{merge}$ runs in linear time, the present step runs in $O(m)$ time and moreover the size of the resulting $s_3$ is still of size $O(m)$.

5. For $s_4 := \text{ungroup} \cdot s_3 : X \to \mathcal{B}(A \times Y)$, first note that for every $x \in X$, the bag $s_3(x) \in \mathcal{B}(A)^{(Y)}$ is represented by a list of elements of $Y \times \mathcal{B}(A)$, i.e. every $\ell \in s_3(x)$ is of type $\ell \in Y \times \mathcal{B}(A)$, thus we define $s_4$ as the following multiset-comprehension:

$$s_4(x) := \{ (a, y) \mid (y, \ell) \in s_3(x), a \in \ell \} \in \mathcal{B}(A \times Y).$$

This is computed in time $|s_3(x)|$ for every $x \in X$ and thus $s_4$ can be computed in time $O(m)$.

Finally, for the definition of $e : Y \to \mathcal{B}(A \times Y)$, we allocate $|Y|$ new adjacency lists, all of them empty initially. Then, for every $x \in X$, we put $e(q(x)) := s_4(x)$ if $e(q(x))$ is empty (and skip otherwise). By the well-definedness of Construction 4.8 it does not matter which $x \in X$ defines the outgoing edges of $q(x) \in Y$. This takes $|X| \leq m$ time. Thus, all steps $s_1, \ldots, s_4$ and the final definition of $e$ take $O(m)$ time in total.

### Proof of Proposition 4.11

We first note a few technicalities before proceeding to the proof of Proposition 4.11.

- Remark B.10. 1. We observe that for every $i \in I$, we have

$$\text{filter}_i(\text{merge}(t))(a) = \text{merge}(t)(\text{in}_i a) = \text{merge}_i(\text{filter}_i(t))(a).$$

(15)

2. In order to show that $\text{merge}$ in Construction 4.10 indeed constitutes a lawful minimization interface, we use a different, but equivalent, definition of $\triangleright$ for $\prod_{i \in I} F_i$:

$$y' = \prod_{i \in I} F_i X \xrightarrow{\prod_{i \in I} \triangleright} \prod_{i \in I} \mathcal{B}(A_i \times X) \xrightarrow{\text{concat}} \mathcal{B}(\prod_{i \in I} \mathcal{A}_i \times X),$$

(16)

with $\text{concat}$ given by $\text{concat}(t)(\text{in}_i a, x) = \text{pr}_i(t)(a, x)$. This is indeed equivalent to the original definition:

$$\triangleright(t)(\text{in}_i a, x) = \text{concat}(\prod_{j \in I} \triangleright_j(t))(\text{in}_i a, x)
= \text{pr}_i(\prod_{j \in I} \triangleright_j(t))(a, x) = \triangleright_i(\text{pr}_i(t))(a, x)
= \triangleright(t)(\text{in}_i a, x).$$

We also need another auxiliary definition similar to $\text{concat}$

$$\text{concat}' : \prod_{i \in I} \mathcal{B}(A_i) \to \mathcal{B}(\prod_{i \in I} A_i)
\text{concat}'(t)(\text{in}_i a) = \text{pr}_i(t)(a),$$

(17)
for which we observe the following properties:

\[
\prod_{j \in I} B(A_j) \xrightarrow{\text{concat}'} B(\prod_{j \in I} A_j)
\]

and

\[
\prod_{i \in I} B(A_i \times X) \xrightarrow{\text{concat}} B(\prod_{i \in I} A_i \times X)
\]

Indeed, we have

\[
\text{fil}_S(\text{concat}(t))(\text{in}_i a) = \sum_{x \in S} \text{concat}(t)(\text{in}_i a, x)
\]

\[
= \sum_{x \in S} \text{pr}_i(t)(a, x) = \text{fil}_S(\text{pr}_i(t))(a)
\]

\[
= \text{pr}_i(\prod_{j \in I} \text{fil}_S(t))(a)
\]

\[
= \text{concat}'(\prod_{j \in I} \text{fil}_S(t))(\text{in}_i a)
\]

and

\[
\text{filter}_i(\text{concat}'(t))(a) = \text{concat}'(t)(\text{in}_i a) = \text{pr}_i(t)(a).
\]

3. The function \(\text{filter}_i\) behaves as expected when injecting all elements of a bag into a coproduct and then immediately filtering this bag. Specifically, we have that

\[
\text{filter}_i • B(\text{in}_j) = \begin{cases} 
\text{id} & \text{if } i = j, \\
\{\} & \text{if } i \neq j.
\end{cases}
\]

We are now ready to prove the main proposition:

**Proof of Proposition 4.11.** 1. For the product functor \(\prod_{i \in I} F_i\), the following diagram com-
Observe that for any two \( f, g \in \mathcal{B}(\bigsqcup_{i \in I} A_i) \) with \( f \neq g \), there exists a \( j \in I \) such that \( \text{filter}_j(f) \neq \text{filter}_j(g) \): Let w.l.o.g. be \( x = \text{in}_i a \in \bigsqcup_{j \in I} A_j \) such that \( f(x) \neq g(x) \). Then we have

\[
\text{filter}_i(f)(a) = f(\text{in}_i a) = f(x) \neq g(x) = g(\text{in}_i a) = \text{filter}_i(g)(a).
\]

Hence, the family \((\text{filter}_i)_{i \in I}\) is a point-separating source and therefore jointly monic.

The desired equation \( \text{merge} \cdot \text{fil}_S \cdot \delta' = \text{fil}_{(1)} \cdot \delta' \cdot \prod_{j \in I} F_j \chi_S \) thus follows from the diagram above.

2. For the coproduct functor \( \bigsqcup_{i \in I} F_i \), we assume without loss of generality that \( F_i 1 \neq \emptyset \) for all \( i \in I \) because summands which are constantly \( \emptyset \) may be omitted from the coproduct without changing it.

We need to show

\[
\text{merge}(\text{fil}_S(b(\text{in}_i t)))(\text{in}_j a) = \text{fil}_{(1)}(b((\bigsqcup_{k \in I} F_k \chi_S)(\text{in}_i t)))(\text{in}_j a)
\]
for every \( n_i t \in \prod_{k \in I} F_k X \) and \( n_j a \in \prod_{k \in I} A_k \).

We calculate as follows:

\[
\begin{align*}
\text{merge}_j \text{fil}_S(\psi(n_i t))(n_j a) \\
= \text{merge}_j(\text{filter}_j(\text{fil}_S(\psi(n_i t)))(a)) & \text{ Def. of merge} \\
= \text{merge}_j(\text{filter}_j(\text{fil}_S([B(\psi(n_k X)]_{k \in I}((\prod_{k \in I} \hat{b}_k)(n_i t)))))(a)) & \text{ Def. of } \psi \\
= \text{merge}_j(\text{filter}_j(\text{fil}_S([B(\psi n_k X)]_{k \in I}(\prod_{k \in I} b_k))(n_i t)))(a) & \prod_{k \in I} b_k \cdot n_i = \psi \cdot n_i \\
= \text{merge}_j(\text{filter}_j(\text{fil}_S(B(\psi n_k X))(\hat{b}_i(t))))(a) & [f_k] \cdot n_i = f_i \\
= \text{merge}_j(\text{filter}_j(B(\psi n_k X))(\hat{b}_i(t)))(a) & \text{Lemma B.2.1}
\end{align*}
\]

From here we proceed by case distinction. If \( i = j \), we have

\[
\begin{align*}
\text{merge}_i(\text{filter}_i(B(\psi n_k X))(\hat{b}_i(t)))(a) \\
= \text{merge}_i(\text{fil}_S(\psi(n_i t)))(a) & \text{Remark B.10(3)} \\
= \text{fil}_S([\hat{b}_i(F_i X S(t))](a)) & \text{Axiom of merge}_i \\
= \text{filter}(B(\psi n_k X))(\text{fil}_S([\hat{b}_i(F_i X S(t))]))(a) & \text{Remark B.10(3)} \\
= B(\psi n_k X)(\text{fil}_S([\hat{b}_i(F_i X S(t))]))(n_i a) & \text{Def. of filter}_i \\
= \text{fil}_S([\hat{b}_i(F_i X S(t))])(n_i a) & \text{Lemma B.2.1} \\
= \text{fil}_S([\prod_{k \in I} F_k X S(t)])(n_i a) & \text{UMP of } \prod \\
= \text{fil}_S([\prod_{k \in I} F_k X S(t)])(n_i a) & \text{Def. of } \hat{b}_i
\end{align*}
\]

In the second case, \( i \neq j \), we have

\[
\begin{align*}
\text{merge}_j(\text{filter}_j(B(\psi n_k X))(\hat{b}_i(t)))(a) \\
= \text{merge}_j(\text{fil}_S(\psi(n_i t)))(a) & \text{Remark B.10} \\
= \text{fil}_S([\hat{b}_i(F_i X S(t))](a)) & \text{Proposition B.9 & Assumption 4.6} \\
= \text{filter}(B(\psi n_k X))(\text{fil}_S([\hat{b}_i(F_i X S(t))]))(a) & \text{Remark B.10}
\end{align*}
\]

The remainder of the calculation is completely analogous to the first case.

3. Since both the product and coproduct of functors share the same definition of \text{merge}, its linear run-time complexity only needs to be verified once. To this end, we represent bags \( B(A) \) as (linked) lists of elements from \( A \) (in lieu of maps \( A \rightarrow \mathbb{N} \)) and rewrite the definition of \text{merge} such that it uses \text{concat'} from (17):

\[
\begin{align*}
\text{merge}(t)(n_j a) = \text{merge}_j(\text{filter}_j(t))(a) & \text{Construction 4.10} \\
= \text{pr}_j(\langle \text{merge}_i \cdot \text{filter}_i \rangle_{i \in I}(t))(a) & \text{Def. of } \langle \cdots \rangle \\
= \text{concat'}(\langle \text{merge}_i \cdot \text{filter}_i \rangle_{i \in I}(t))(n_j a) & (17)
\end{align*}
\]

Hence, \text{merge} is the composition

\[
\begin{align*}
B(\prod_{i \in I} A_i) \xrightarrow{(\text{filter}_i)_{i \in I}} \prod_{i \in I} B(A_i) \xrightarrow{\prod_{i \in I} \text{merge}_i} \prod_{i \in I} B(A_i) \xrightarrow{\text{concat'}} B(\prod_{i \in I} A_i).
\end{align*}
\]

This composition can be readily implemented by the following algorithm. Given a bag \( t \in B(\prod_{i \in I} A_i) \), let \( n \) be the number of elements in \( t \) and do:
a. Allocate an array of length $|I|$ initially containing an empty bag of type $B(A_i)$ in the $i$th component for all $i \in I$ (this array represents an element of $\prod_{i \in I} B(A_i)$).

b. Insert each label $a_i$ from $t$ into the $i$th bag; this implements $\langle \text{filter}_i \rangle_{i \in I}$ above.

c. For each $i \in I$, apply $\text{merge}_i$ on the $i$th bag.

d. Concatenate the resulting $|I|$ lists (encoding bags of type $B(A_i)$) stored in our array to one list encoding the result bag of type $B(\textstyle\bigcup_{i \in I} A_i)$.

Each of those steps runs in $O(|I| + n)$ time if $\text{merge}_i$ has linear run-time for every $i \in I$. Since $|I|$ is constant, this amounts to $O(n)$ overall. $\blacktriangleleft$

### B.4 Proofs for Section 5

Gumm [28, Def. 7.2] defined the maps $\tau^F_X : FX \to PX$ differently. We show that his definition is equivalent to ours.

**Lemma B.11.** The definition of $\tau^F_X$ in Definition 5.1 is equivalent to Gumm’s definition in op.cit.

**Proof.** Before showing the equivalence, we need to recall other definitions that are used by Gumm [28]. Recall that a filter $G$ on a set $X$ is a nonempty family $G \subseteq PX$ that is closed under binary intersection and supersets. The filter functor $F$ is the Set-functor that sends a set $X$ to the set of all filters on $X$ (its definition on maps is not relevant to this proof). For a given Set-functor $F : \text{Set} \to \text{Set}$ and sets $U \subseteq X$, the set

$$[F^X_U] := F(i : U \hookrightarrow X)[FU] \subseteq FX$$

denotes the image of $Fi : FU \to FX$, where $i : U \hookrightarrow X$ is the inclusion map. The notation $[F^X_U]$ is monotone in $U$ [28, Lemma 1], that is, $V \subseteq U$ implies $[F^X_V] \subseteq [F^X_U]$. Moreover, one can easily prove for $t \in FX$ that

$$t \in [F^X_U] \iff t : 1 \to FX \text{ factorizes through } Fi. \quad (21)$$

Using this notation, we define the following family of maps:

$$\mu_X : FX \to FX \quad \mu_X(t) := \{U \subseteq X \mid t \in [F^X_U] \}.$$  

For $t \in FX$, the intersection of all elements in $\mu_X(t)$ yields a subset of $X$:

$$\bigcap \mu_X(t) = \{ x \in X \mid \forall U \in \mu_X(t) : x \in U \} \in PX.$$  

This is the definition of $\tau^F_X$ in op.cit. In order to prove that this definition is equivalent to ours in Definition 5.1, we will prove that

$$\tau^F_X(t) = \bigcap \mu_X(t) \quad \text{for all } t \in FX.$$  

We have the following chain of equal sets, whose equality is established by performing
equivalent rewrites in the comprehension formula:

\[ \bigcap \mu_X(t) = \{ x \in X \mid \forall U \in \mu_X(t) : x \in U \} \quad \text{(Def. \( \bigcap \))} \]

\[ = \{ x \in X \mid \forall U \subseteq X : t \in [F_U^X] \rightarrow x \in U \} \quad \text{(Def. \( \mu_X \))} \]

\[ = \{ x \in X \mid \forall U \subseteq X : x \notin U \rightarrow t \notin [F_U^X] \} \quad \text{(Contraposition)} \]

\[ = \{ x \in X \mid \forall U \subseteq X \text{ with } x \notin U : t \notin [F_U^X] \} \]

This completes the proof.

**Proof of Proposition 5.5**

*Proof.* We prove \( \tau_X^{B(A \times -)}(\bar{\sigma}_X(t)) \subseteq \tau_X^Z(t) \) by contraposition. If \( x \in X \) is not in \( \tau_X^Z(t) \), then we show that it is not in \( \tau_X^{B(A \times -)}(\bar{\sigma}_X(t)) \) by proving that the following diagram commutes:

\[
\begin{array}{ccc}
1 & \rightarrow & FX \\
\downarrow & & \downarrow F_i \\
F(X \setminus \{x\}) & \Rightarrow & B(A \times X) \\
\end{array}
\]

First, observe that \( x \notin \tau_X^Z(t) \) implies that the map \( t : 1 \rightarrow FX \) factorizes through \( F(X \setminus \{x\}) \rightarrow FX \) (cf. Definition 5.1), and we therefore obtain \( t' \) as shown in the left triangle. The right rectangle commutes by the subnaturality of \( \bar{\sigma} \). Therefore, \( 1 \xrightarrow{t} FX \xrightarrow{\tau_X^Z} B(A \times X) \) factors through \( B(A \times X \setminus \{x\}) \xrightarrow{B(A \times i)} B(A \times X) \) and thus \( x \) can not be in \( \tau_X^{B(A \times -)}(\bar{\sigma}(t)) \). We have shown that \( x \notin \tau_X^Z(t) \) implies \( x \notin \tau_X^{B(A \times -)}(\bar{\sigma}(t)) \); equivalently, we have \( \tau_X^{B(A \times -)}(\bar{\sigma}(t)) \subseteq \tau_X^Z(t) \) as required.

**Proof of Theorem 5.6**

*Proof.* Having established one inclusion in Proposition 5.5 already, we prove the remaining inclusion \( \tau_X^Z(t) \subseteq \tau_X^{B(A \times -)}(\bar{\sigma}(t)) \) by contraposition:

\[ x \notin \tau_X^{B(A \times -)}(\bar{\sigma}_X(t)) \implies x \notin \tau_X^Z(t) \quad \text{for all } x \in X. \]

To this end, suppose that \( x \in X \) satisfies \( x \notin \tau_X^{B(A \times -)}(\bar{\sigma}_X(t)) \). This implies that there exists some \( t' \in B(A \times (X \setminus \{x\})) \) such that the diagram below commutes:

\[
\begin{array}{ccc}
1 & \rightarrow & FX \\
\downarrow & & \downarrow F_i \\
B(A \times (X \setminus \{x\})) & \Rightarrow & B(A \times X) \\
\end{array}
\]

\[ X \quad \text{with} \quad \begin{cases} b & \text{if } y = x \\ y & \text{otherwise.} \end{cases} \]

For a fixed \( b \in 2 = \{0, 1\} \), we define the injective auxiliary map \( (x \rightsquigarrow b) \) by
We will now prove the following equality

\[ F(x \rightsquigarrow 0)(t) = F(x \rightsquigarrow 1)(t). \] (22)

We have the following commutative diagrams (for \( b = 0, 1 \)):

\[
\begin{array}{ccc}
F(X \setminus \{x\} + 2) & \xrightarrow{(F!,b_{X \setminus \{x\} + 2})} & F1 \times \mathcal{B}(A \times (X \setminus \{x\} + 2)) \\
F(x \rightsquigarrow b) & \downarrow \text{(trivial, subnaturality)} & \downarrow F1 \times \mathcal{B}(A \times (x \rightsquigarrow b)) \\
FX & \xrightarrow{(F!,b_X)} & F1 \times \mathcal{B}(A \times X) \\
\end{array}
\]

\[
\begin{array}{ccc}
t & \uparrow t' & \uparrow F1 \times \mathcal{B}(A \times \text{inl}) \\
1 & \xrightarrow{(F!,t,t')} & F1 \times \mathcal{B}(A \times (X \setminus \{x\})) \\
\end{array}
\]

Note that the lower right outside path via \( t' \) does not mention \( b \) at all. Hence, we have:

\[
\langle F!,b_{X \setminus \{x\} + 2}\rangle(F(x \rightsquigarrow 0)(t)) = (F1 \times \mathcal{B}(A \times \text{inl}))(F!(t),t') = \langle F!,b_{X \setminus \{x\} + 2}\rangle(F(x \rightsquigarrow 1)(t)).
\]

By assumption \( F \) has a refinement interface, and so we know that the map \( \langle F!,b_{X \setminus \{x\} + 2}\rangle \) is injective by Proposition 3.3. Thus, we obtain the desired equation (22). Rephrased as a diagram, we see that the outside of the following diagram commutes:

\[
\begin{array}{ccc}
FX & \xrightarrow{F(x \rightsquigarrow 0)} & F(X \setminus \{x\} + 2) \\
\end{array}
\]

\[
\begin{array}{ccc}
t & \uparrow F1 & \uparrow F(x \rightsquigarrow 1) \\
F(X \setminus \{x\}) & \xrightarrow{Fi} & FX \\
\end{array}
\]

Regarding the square, note that \( X \setminus \{x\} \) is the intersection of the injective maps \( (x \rightsquigarrow 0): X \hookrightarrow X \setminus \{x\} + 2 \) and \( (x \rightsquigarrow 1): X \hookrightarrow X \setminus \{x\} + 2 \). Since \( F \) preserves intersections, we thus see that the above square is a pullback. Since the outside commutes, we obtain the above dashed map \( t'': 1 \to F(X \setminus \{x\}) \) with \( Fi \cdot t'' = t \). By the definition of \( \tau_X^F \) (see Definition 5.1), this implies that \( x \not\in \tau_X^F(t) \), as desired. \( \blacktriangleleft \)