Robust estimation with Lasso when
outputs are adversarially contaminated

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Abstract: We consider robust estimation when outputs are adversarially contaminated. Nguyen and Tran (2012) proposed an extended Lasso for robust parameter estimation and then they showed the convergence rate of the estimation error. Recently, Dalalyan and Thompson (2019) gave some useful inequalities and then they showed a faster convergence rate than Nguyen and Tran (2012). They focused on the fact that the minimization problem of the extended Lasso can become that of the penalized Huber loss function with $L_1$ penalty. The distinguishing point is that the Huber loss function includes an extra tuning parameter, which is different from the conventional method. We give the proof, which is different from Dalalyan and Thompson (2019) and then we give the same convergence rate as Dalalyan and Thompson (2019). The significance of our proof is to use some specific properties of the Huber function. Such techniques have not been used in the past proofs.

MSC 2010 subject classifications: 62G35, 62G05.
Keywords and phrases: Lasso, Robustness, convergence rate, Huber loss.

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1. Introduction

We consider a linear regression model in a high dimensional case, given by

\[ y_i = X_i^\top \beta^* + \xi_i, \quad i = 1, \cdots, n, \]

where \((X_1, y_1), \cdots, (X_n, y_n) \in \mathbb{R}^d \times \mathbb{R}\) are input-output pairs, \(\beta^*\) is the true regression coefficient vector, and \(\xi_1, \cdots, \xi_n \in \mathbb{R}\) are noises. The most popular sparse modeling is the least absolute shrinkage and selection operator (Lasso, Tibshirani (1996)) by virtue of its generality and convexity. Lasso estimates the true regression coefficient \(\beta^*\) by solving the following convex optimization problem with the tuning parameter \(\lambda_s\):

\[ \hat{\beta} \in \arg \min_{\beta \in \mathbb{R}^d} \left\{ \frac{1}{2n} \|Y - X\beta\|_2^2 + \lambda_s \|\beta\|_1 \right\}, \tag{1.1} \]

where \(Y = (y_1, \cdots, y_n)^\top\) and \(X = (X_1, \cdots, X_n)^\top\). Many parameter estimation methods with sparsity on \(\beta^*\) have been proposed by Tibshirani (1996), Fan and Li (2001), Zou and Hastie (2005), Yuan and Lin (2006), Candes et al. (2007), Bellec, Lecué and Tsybakov (2018), and so on.

Suppose that the outputs may be contaminated by an adversarial noise \(\sqrt{n} \theta^* \in \mathbb{R}^n\), where

- non-zero entries of \(\theta^*\) can take arbitrary values and \(\sqrt{n}\) is used for normalization. In this case, \(Y\) is replaced by \(Y + \sqrt{n} \theta^*\). Then, the optimization problem (1.1) may not give an appropriate estimate of \(\beta^*\) due to adversarial contamination. To weaken an adverse effect of adversarial contamination, Nguyen and
Tran (2012) proposed an extended Lasso, which estimates $\beta^*$ and $\theta^*$ simultaneously by solving the following convex optimization problem with two tuning parameters $\lambda_s$ and $\lambda_o$:

$$
(\hat{\beta}, \hat{\theta}) \in \arg\min_{(\beta, \theta) \in \mathbb{R}^d \times \mathbb{R}^n} \left\{ \frac{1}{2n} \| Y - X\beta - \sqrt{n}\theta \|_2^2 + \lambda_s \| \beta \|_1 + \lambda_o \| \theta \|_1 \right\}.
$$

(1.2)

Suppose that $X_1, \cdots, X_n$ and $\xi_1, \cdots, \xi_n$ are i.i.d. random samples from $\mathcal{N}(0, \Sigma)$ and $\mathcal{N}(0, \sigma^2)$, respectively, and $X_i$s and $\xi_i$s are mutually independent. In this paper, we assume $d \geq 3$. Let $\|a\|_0$ be the number of non-zero components of $a$. Assume that $\|\beta^*\|_0 \leq s$ and $\|\theta^*\|_0 \leq o$. Nguyen and Tran (2012) derived the convergence rate of $\|\beta^* - \hat{\beta}\|_2 + \|\theta^* - \hat{\theta}\|_2$, which implies the convergence rate of $\|\beta^* - \hat{\beta}\|_2$, given by

$$
O \left( \sqrt{\frac{s \log d}{n}} + \frac{o}{n \log n} \right).
$$

(1.3)

Recently, Dalalyan and Thompson (2019) insisted a faster convergence rate using different tuning parameters from Nguyen and Tran (2012), given by

$$
O \left( \sqrt{\frac{s \log d}{n}} + \frac{o}{n \sqrt{\log n \log n}} \right).
$$

(1.4)

In this paper, we give a correct proof of the convergence rate with a different technique from Dalalyan and Thompson (2019). On the other hand, Propositions 3 and 4 of Dalalyan and Thompson (2019) are very attractive, which also play important roles in the proofs of this paper. In the past proofs, the convexity and Lipschitz continuity of the Huber loss function were used, in other words, general properties of the loss function were used. In the proof of this paper, we use a specific property of the Huber loss function. (It changes the behavior at the threshold from a quadratic function to a linear function.) By such a careful analysis, we can give a sharper convergence rate than Nguyen and Tran (2012), even when the number of outliers is large.

The convergence rate of robust estimation has been examined rapidly in recent years. First, the robust estimation of the mean and scatter matrix was examined under the Huber’s contamination. Chen, Gao and Ren (2018) derived the minimax rate and proposed a method that achieves the minimax rate. However, the computation is of exponential time. Lai, Rao and Vempala (2016) proposed another method that is of polynomial time and achieves the optimal rate up to logarithmic factor. Lai, Rao and Vempala (2016) has been followed by Diakonikolas et al. (2017, 2018, 2019a,b), Cheng, Diakonikolas and Ge (2019), and so on. The robust and sparse estimation in linear regression has also been studied under the Huber’s contamination. Gao (2020) derived the minimax rate of the regression coefficient estimation, given by

$$
O \left( \sqrt{\frac{s \log(d/s)}{n}} + \frac{o}{n} \right).
$$

(1.5)
The adversarial contamination over the Huber’s contamination has also been discussed. Nguyen and Tran (2012) considered the case where the outputs were adversarially contaminated. The convergence rate is given by (1.3). It is slower than (1.5). This is because the adversarial contamination includes various types of contamination over the Huber’s contamination. Chen, Caramanis and Mannor (2013) treated the extended case where both outputs and inputs were adversarially contaminated, and then they derived the convergence rate, but it is not optimal and it depends on the true value $\beta^*$. The case with a fixed $d$ was also discussed. In this case, we can show a faster convergence rate than that by (1.3) and (1.4), because $d$ is fixed. Diakonikolas, Kong and Stewart (2019) considered the case that $\beta^*$ was not sparse and proposed a new method based on filtering, and showed that the convergence rate is $O(\frac{n}{\log(n/o)})$. Liu et al. (2018) proposed another new method based on iterative hard thresholding, and showed that the convergence rate is $O(\sqrt{o n})$ in which the order $\log n$ is omitted. For more information on recent developments in robust estimation, see the survey paper by Diakonikolas and Kane (2019).

This paper is organized as follows. In Section 2, we roughly give the main theorem and the outline of the proof. In Section 3, we prepare some key propositions to prove the main theorem, including Propositions 3 and 4 of Dalalyan and Thompson (2019), and arrange the necessary conditions. In Section 4, we give a simple relation between $\|\beta^* - \hat{\beta}\|_1$ and $\|\Sigma^{1/2}(\beta^* - \hat{\beta})\|_2$, which plays an important role to obtain the convergence rate of the estimation error of the Lasso. In Section 5, we investigate a behavior of $C_{cut}$, which has a close relation to a specific property of the Huber loss function. This is a distinguishing point of this paper. In Section 6, we give the outline of the proofs of the key propositions. In Section 7, we give a rigorous statement of the main theorem and prove the main theorem, combining the results prepared in the previous sections.

2. Main theorem and outline of proof

2.1. Main theorem

The main theorem is roughly given in the following. A rigorous statement of the main theorem, including detailed conditions, is given in Section 7. The main theorem is compared with the past theorem from the point of view of the conditions and convergence rate.

Here we prepare some notations related to $\Sigma$. Let $\rho^2 = \max_i (\Sigma_{ii})$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalues of $\Sigma$, respectively.

**Theorem 2.1.** Suppose that $\Sigma$ satisfies the restricted eigenvalue condition $\text{RE}(s, 5, \kappa)$ (cf. Definition 4.1). Assume that $\delta$ is sufficiently small and $n$ is sufficiently large. Let

\[
\lambda_o = C_{\lambda_o} \sqrt{\frac{2\sigma^2\log(n/\delta)}{n}}, \quad \lambda_s = \frac{4\sqrt{2}}{\sqrt{3}} C_{\lambda_s} \lambda_o,
\]

(2.1)
where $C_{\lambda_1}$ is an appropriate numerical constant,

$$C_{\lambda_1} = C_z + \sqrt{\frac{2}{s}} g(o), \quad C_z = \sqrt{\frac{3}{\lambda_1^2} \log \frac{d}{\delta}},$$

$$g(o) = \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right) + 1.2c_\alpha \sqrt{2n \log d} + 4.8c_\alpha \sqrt{\frac{\sigma}{n}} \sqrt{4 + \log \frac{n}{\delta}}.$$

Assume that $\lambda_0$ and $\lambda_s$ satisfy

$$8 \max \left( 3.6 \sqrt{\frac{2\sigma^2 \log d}{n}}, 2.4 \frac{\lambda_s}{\lambda_0} \sqrt{\frac{2\log n}{n}} \right) \sqrt{\frac{s}{\kappa^2} + \frac{6.25\alpha^2}{\lambda_s^2}} \leq C_{n,\delta}, \quad (2.2)$$

where $C_{n,\delta}$ is given later. Under some additional conditions, with probability at least $1 - 7\delta$, we have

$$\| \Sigma^{1/2} (\beta^* - \hat{\beta}) \|_2 \leq C_{\delta,\kappa,\rho,\sigma} r_{n,d,s,o}, \quad (2.3)$$

where $C_{\delta,\kappa,\rho,\sigma}$ is a constant depending on $\delta, \kappa, \rho, \sigma,$ and

$$r_{n,d,s,o} = \sqrt{\frac{s \log d}{n}} + \frac{o}{n} \sqrt{\log \frac{n}{\delta} \log n}.$$

**Remark 2.1.** In Section 7, first, we will obtain a more general estimation bound than (2.3) under a weaker condition than (2.1), given by

$$\lambda_0 \geq C_\lambda \sqrt{\frac{2\sigma^2 \log (n/\delta)}{n}}, \quad \lambda_s \geq \frac{4\sqrt{2}}{\sqrt{3}} C_\lambda \lambda_0. \quad (2.4)$$

Hereafter, we basically assume (2.4) instead of (2.1).

**Corollary 2.1.** Assume the conditions used in Theorem 2.1. Suppose that $\lambda_{\min}$ is positive. Then, the inequality (2.3) of Theorem 2.1 implies

$$\| \beta^* - \hat{\beta} \|_2 \leq C_{\delta,\kappa,\rho,\sigma} \frac{r_{n,d,s,o}}{\lambda_{\min}}. \quad (2.5)$$

Here we compare the above with the convergence rate of Nguyen and Tran (2012).

**Theorem 2.2** (Nguyen and Tran (2012)). Assume that $\lambda_{\min}$ is positive and $\lambda_{\max}$ satisfies $\lambda_{\max} \rho^2 = O(1)$. Suppose that $n$ is sufficiently large with $n \geq c_{\min}^2 \sigma^2 \log d$ and $o$ is sufficiently small with $o \leq \min \left( c_1 \frac{n}{\gamma \log n}, c_2 n \right)$, where $c, c_1, c_2$ are some constants and $\gamma \in (0, 1]$. Let

$$\lambda_0 = 2 \sqrt{\frac{2\sigma^2 \log n}{n}}, \quad \lambda_s = \frac{2}{\gamma} \sqrt{\frac{2\sigma^2 \rho^2 \log d}{n}} \left( 1 + \sqrt{\frac{2 \log d}{n}} \right).$$
With probability at least $1 - c_3 e^{-c_4 n}$, we have
\[
\|\beta^* - \hat{\beta}\|_2 + \|\theta^* - \hat{\theta}\|_2 \leq C_{\kappa, \rho, \sigma} \left( \sqrt{\frac{s \log d}{n}} + \sqrt{\frac{o \log n}{n}} \right),
\]
where $c_3$ and $c_4$ are some constants and $C_{\kappa, \rho, \sigma}$ is a constant depending on $\kappa, \rho, \sigma$.

A remarkable difference between the above two theorems is the convergence rate of the estimation error on the second term related to the number of outliers, $o$. The main theorem shows a faster convergence rate than Theorem 2.2. This arises from careful analysis with a different setting of tuning parameters in the main theorem.

A large difference of conditions between the above two theorems is that the parameter $\delta$ does not appear in the tuning parameters in Theorem 2.2, but the parameter $\delta$ is incorporated into the tuning parameters in the main theorem. The condition (2.2) may be complicated, however it is satisfied under some condition, including that $n$ is sufficiently large, as seen in Appendix C.

### 2.2. Outline of the proof of the main theorem

Let $L(\beta)$ be the loss function with the parameter $\beta$. When we obtain the convergence rate of the estimation error of Lasso, we usually start with the inequality that $L(\hat{\beta}) \leq L(\beta^*)$, where $\hat{\beta}$ is the minimizer of $L(\beta)$ and $\beta^*$ is the true value. In this paper, we employ this approach to obtain a simple but weak relation between $\|\beta^* - \hat{\beta}\|_1$ and $\|\Sigma^* (\beta^* - \hat{\beta})\|_2$ in Section 4.2, by virtue of the restricted eigenvalue condition. Although we partly use this approach, we mainly adopt a different approach to obtain a faster convergence rate. First, we focus on the fact that the derivative of the loss function is zero at the minimizer. Next, we divide the derivative of the loss function into three parts via a specific property of the Huber function. These are shown in this subsection. Some properties related to the three terms are given in the subsequent sections. Combining the results, we can show the main theorem.

As shown in She and Owen (2011), the optimization problem (1.2) becomes
\[
\hat{\beta} \in \arg\min_{\beta} L(\beta), \quad L(\beta) = \lambda_0^2 \sum_{i=1}^n H \left( \frac{y_i - X_i^T \beta}{\lambda_0 \sqrt{n}} \right) + \lambda_s \|\beta\|_1, \tag{2.6}
\]
where $H(t)$ is the Huber loss function, given by
\[
H(t) = \begin{cases} 
|t| - 1/2 & (|t| > 1) \\
\frac{t^2}{2} & (|t| \leq 1) 
\end{cases}.
\]

It should be noted that this is not a standard penalized Huber loss function, because the tuning parameter $\lambda_0$ is included in the Huber loss function. Let
\[
r_i(\beta) = \frac{y_i - X_i^T \beta}{\lambda_0 \sqrt{n}}.
\]
Since the derivative of $L(\beta)$ about $\beta$ is zero at $\beta = \hat{\beta}$, we have
\begin{equation}
\lambda_o \sum_{i=1}^{n} \frac{X_i}{\sqrt{n}} \hat{\psi}_i = \lambda_o \partial \|\hat{\beta}\|_1, \tag{2.7}
\end{equation}
where $\psi(t) = H'(t)$, $\hat{\psi}_i = \psi \left( r_i(\beta) \right)$, $\partial \| \cdot \|_1$ is a subdifferential of $\| \cdot \|_1$. By multiplying $(\beta^* - \hat{\beta})^T$ to the both side of the above equation, we have
\begin{equation}
\lambda_o \sum_{i=1}^{n} (\beta^* - \hat{\beta})^T \frac{X_i}{\sqrt{n}} \hat{\psi}_i = \lambda_o (\beta^* - \hat{\beta})^T \partial \| \hat{\beta}\|_1 \leq \lambda_o \|\beta^* - \hat{\beta}\|_1. \tag{2.8}
\end{equation}

Let $I_u$ and $I_o$ be the index sets for uncontaminated and contaminated outputs, respectively. Let
\begin{equation}
I_\succ = \left\{ i \in I_u : \left| r_i(\hat{\beta}) \right| > 1 \right\}, \quad C_{cut} = \#I_\succ. \tag{2.9}
\end{equation}
These play important roles in the proof, because the Huber loss function $H(r_i(\beta))$ changes the behavior according to whether $|r_i(\beta)|$ is larger than one or not. Let $I_\prec = \left\{ i \in I_u : \left| r_i(\hat{\beta}) \right| \leq 1 \right\} = I_u - I_\succ$. We see $\#I_u = n - o$, $\#I_o = o$ and $\#I_\prec = n - o - C_{cut}$. The left-hand side (L.H.S.) of (2.8) can be divided into three parts, given by
\begin{equation}
T_1 + T_2 + T_3 \leq \lambda_o \|\beta^* - \hat{\beta}\|_1, \tag{2.10}
\end{equation}
where $T_1$, $T_2$ and $T_3$ correspond to the index sets $I_\prec$, $I_\succ$ and $I_o$, respectively.

As described later, each term of (2.10) can be evaluated with a link to the estimation error
\begin{equation}
E = \| \Sigma^\frac{1}{2} (\beta^* - \hat{\beta}) \|_2. \end{equation}
The right-hand side (R.H.S.) of (2.10) is evaluated in Section 4. The value $C_{cut}$ is evaluated in Section 5. The L.H.S of (2.10), including $C_{cut}$, is evaluated in Section 6. Using these results, we can prove the main theorem.

3. Preliminary

First, we present some properties related to the Gaussian width and Gaussian random matrix. Next, we prepare some concentration inequalities. These are the results shown by Dalalyan and Thompson (2019) and others. Finally, we summarize the conditions used in this section.

First, we state the definition of the Gaussian width and show three properties of the Gaussian width.

**Definition 3.1 (Gaussian width).** For a subset $T \subset \mathbb{R}^d$, the Gaussian width is defined by
\begin{equation}
\mathcal{G}(T) := E \sup_{x \in T} (g, x),
\end{equation}
where $g \sim \mathcal{N}(0, I_d)$. 
Lemma 3.1 (Theorem 2.5 of Boucheron, Lugosi and Massart (2013)). We have

\[ G(\Sigma_{\Sigma}^{\frac{1}{2}}B_1^d) \leq \sqrt{2\rho^2 \log d}. \]

Let \( B_1^m = \{ u \in \mathbb{R}^m : \| u \|_1 \leq 1 \} \) and \( B_2^m = \{ u \in \mathbb{R}^m : \| u \|_2 \leq 1 \} \).

Lemma 3.2. For any vector \( u \in \mathbb{R}^n \), we have

\[ G(\| u \|_1 B_1^n \cap \| u \|_2 B_2^n) \leq \| u \|_1 \sqrt{2 \log n}. \]

(3.1)

Proof.

\[ G(\| u \|_1 B_1^n \cap \| u \|_2 B_2^n) \leq G(\| u \|_1 B_1^n) = \| u \|_1 G(B_1^n) \leq \| u \|_1 \sqrt{2 \log n}. \]

The last inequality follow from Lemma 3.1.

Lemma 3.3. For any \( m \)-sparse vector \( u \in \mathbb{R}^n \), we have

\[ G(\| u \|_1 B_1^n \cap \| u \|_2 B_2^n) \leq \frac{4}{\sqrt{e}} \sqrt{\frac{8}{4 + \log (8n/m)}} \frac{\| u \|_1}{\sqrt{n}}. \]

(3.2)

Proof. Because \( u \) is a \( m \)-sparse vector, \( \| u \|_1 \leq \sqrt{m} \| u \|_2 \) holds. Hence, setting \( o = m \) on (39) in Remark 4 of Dalalyan and Thompson (2019), we have

\[ G(\| u \|_1 B_1^n \cap \| u \|_2 B_2^n) \leq 4\sqrt{e} \| u \|_2 \sqrt{1 + \log (8n/m)}. \]

This inequality shows the inequality (3.2) from \( 1 + \log 8 < 4 \).

Here, we introduce two concentration inequalities of Gaussian random matrix.

Proposition 3.1 (Proposition 3 of Dalalyan and Thompson (2019)). Let \( Z \in \mathbb{R}^{n \times p} \) be a random matrix with i.i.d. \( \mathcal{N}(0, \Sigma) \) rows. For any \( \delta \in (0, 1/7] \) and \( n \geq 100 \), the following property holds with probability at least \( 1 - \delta \): for any \( v \in \mathbb{R}^d \),

\[ \left\| \frac{Z}{\sqrt{n}} \right\|_2 \geq a_1 \| \Sigma^{\frac{1}{2}} v \|_2 - \frac{1.2 G(\Sigma_{\Sigma}^{\frac{1}{2}} B_1^d)}{\sqrt{n}} \| v \|_1, \]

where

\[ a_1 = 1 - \frac{4.3 + \sqrt{2 \log (9/\delta)}}{\sqrt{n}}. \]

Proposition 3.2 (Proposition 4 of Dalalyan and Thompson (2019)). Let \( Z \in \mathbb{R}^{n \times p} \) be a random matrix with i.i.d. \( \mathcal{N}(0, \Sigma) \) rows. For any \( \delta \in (0, 1/7] \) and \( n \in \mathbb{N} \), the following property holds with probability at least \( 1 - \delta \): for any \( u \in \mathbb{R}^n \) and \( v \in \mathbb{R}^d \),

\[ \left| u^\top \frac{Z}{\sqrt{n}} v \right| \leq \| \Sigma^{\frac{1}{2}} v \|_2 \| u \|_2 \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\frac{81}{\delta}} \right) + 1.2 \| u \|_1 \| u \|_2 \frac{G(\Sigma_{\Sigma}^{\frac{1}{2}} B_1^d)}{n} + 1.2 \| \Sigma^{\frac{1}{2}} v \|_2 \frac{G(\| u \|_1 B_1^n \cap \| u \|_2 B_2^n)}{\sqrt{n}}. \]
By Lemmas 3.3 and 3.1 and Propositions 3.1 and 3.2, we can easily show the following corollaries.

**Corollary 3.1.** Let $Z \in \mathbb{R}^{n \times p}$ be a random matrix with i.i.d. $\mathcal{N}(0, \Sigma)$ rows. For any $\delta \in (0, 1/7]$ and $n \geq 100$, the following property holds with probability at least $(1 - \delta)$: for any $v \in \mathbb{R}^d$,
\[
\left\| \frac{Z}{\sqrt{n}} v \right\|_2 \geq a_1 \left\| \Sigma^{1/2} v \right\|_2 - 1.2 \frac{2\rho^2 \log d}{n} \left\| v \right\|_1.
\]

**Corollary 3.2.** Let $Z \in \mathbb{R}^{n \times p}$ be a random matrix with i.i.d. $\mathcal{N}(0, \Sigma)$ rows. For any $\delta \in (0, 1/7]$ and $n \in \mathbb{N}$, the following property holds with probability at least $(1 - \delta)$: for any $m$-sparse $u \in \mathbb{R}^n$ and any $v \in \mathbb{R}^d$,
\[
\left\| u^\top \frac{Z}{\sqrt{n}} v \right\| \leq \left\| \Sigma^{1/2} v \right\|_2 \left\| u \right\|_2 2 \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \left( \frac{81}{\delta} \right)} \right) + 1.2 \left\| v \right\|_1 \left\| u \right\|_2 \sqrt{\frac{2\rho^2 \log d}{n}}
\]
\[
+ 4.8 \sqrt{\log \left( \frac{2}{\delta} \right)} \left\| \Sigma^{1/2} v \right\|_2 \left\| u \right\|_2 \frac{m}{n} \sqrt{4 + \log \frac{n}{m}}.
\]

Next, we prepare three inequalities of concentration inequalities.

**Proposition 3.3** (Lemma 2 of Dalalyan and Thompson (2019)). Let $\{\xi_i\}_{i=1}^n$ be a sequence with i.i.d random variables drawn from $\mathcal{N}(0, \sigma^2)$ and $\{X_i\}_{i=1}^n$ drawn from $\mathcal{N}(0, \Sigma)$. Let $\xi = (\xi_1, \cdots, \xi_n)^\top$ and $X = [X_1, \cdots, X_n]^\top$. For any $\delta \in (0, 1)$ and $n \geq 2 \log(d/\delta)$, with probability at least $(1 - \delta)^3$, we have
\[
\left\| \frac{\xi}{\sqrt{n}} \right\|_\infty \leq \sqrt{\frac{2\sigma^2 \log(n/\delta)}{n}},
\]
\[
\left\| \frac{X^\top \xi}{n} \right\|_\infty \leq 2 \sqrt{\frac{2\sigma^2 \rho^2 \log(d/\delta)}{n}}.
\]

**Proposition 3.4.** Let $\{\xi_i\}_{i=1}^n$ be a sequence with i.i.d random variables drawn from $\mathcal{N}(0, \sigma^2)$ and $\{X_i\}_{i=1}^n$ drawn from $\mathcal{N}(0, \Sigma)$. Let $z_{ij} = X_{ij} \psi \left( \frac{z_{ij}}{\lambda_0 \sqrt{n}} \right)$ and $z = (\sum_{i=1}^n z_{i1}, \cdots, \sum_{i=1}^n z_{id})$. For any $\delta \in (0, 1)$ and $n$ such that $\sqrt{\log(d/\delta)} \leq \sqrt{3} - \sqrt{2}$, with probability at least $1 - \delta$, we have
\[
\left\| \frac{z}{\sqrt{n}} \right\|_\infty \leq \sqrt{\frac{3\rho^2 \sigma^2}{n\lambda_0^2} \log \frac{d}{\delta}} =: C_z.
\]

The proof of Proposition 3.4 is given in Appendix A.

**Proposition 3.5** (Lemma 1 of Laurent and Massart (2000)). Let $\{\xi_i\}_{i=1}^n$ be a sequence with i.i.d random variables drawn from $\mathcal{N}(0, \sigma^2)$. For any $\delta \in (0, 1)$ and $n$ such that $2\sqrt{n \log(1/\delta)} + 2 \log(1/\delta) \leq n$, with probability at least $1 - \delta$, we have
\[
\frac{1}{n} \sum_{i=1}^n \xi_i^2 \leq 2\sigma^2.
\]
Finally, we summarize the conditions used above:

(c1) \( \delta \in (0, 1/7] \), \( n \geq 100 \).

(c2) \( \sqrt{\frac{\log(d/\delta)}{n}} \leq \sqrt{3} - \sqrt{2} \). (This implies \( 2 \log(d/\delta) \leq n \) with \( n \geq 100 \)).

(c3) \( 2 \sqrt{n \log(1/\delta)} + 2 \log(1/\delta) \leq n \).

Based on the results prepared in this section, we will show many propositions and finally prove the main theorem. Hereafter, we will use the phrase “with a high probability” without explicit probability in the propositions. We give an explicit probability in Section 7.3.

4. Relation between \( \|\beta^* - \hat{\beta}\|_1 \) and \( \|\Sigma^{1/2}(\beta^* - \hat{\beta})\|_2 \)

As seen in Section 2, a relation between \( \|\beta^* - \hat{\beta}\|_1 \) and \( \|\Sigma^{1/2}(\beta^* - \hat{\beta})\|_2 \) plays an important role to obtain the convergence rate of the estimation error.

First, we introduce a restricted eigenvalue condition and a simple lemma, which are often used to obtain the convergence rate of the estimation error. Next, we obtain a relation between \( \|\beta^* - \hat{\beta}\|_1 \) and \( \|\Sigma^{1/2}(\beta^* - \hat{\beta})\|_2 \).

4.1. Restricted eigenvalue condition

For a set \( J \), let \( \#J \) represent the number of elements of \( J \). For a vector \( v = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d \) and a set \( J \subset \{1, \ldots, d\} \), let \( v_J \) be the vector whose \( j \)th component is \( v_j \) for \( j \in J \) and 0 for \( j \notin J \).

The restricted eigenvalue condition for \( \Sigma \) is defined in the following and this condition enable us to deal with the case where \( \Sigma \) is singular.

**Definition 4.1** (Restricted Eigenvalue Condition Dalalyan and Thompson (2019)). The matrix \( \Sigma \) is said to satisfy the restricted eigenvalue condition \( \text{RE}(s, c_0, \kappa) \) with a positive integer \( s \) and positive values \( c_0 \) and \( \kappa \), if

\[
\kappa \|v_J\|_2 \leq \|\Sigma^{1/2}v\|_2
\]

for any set \( J \subset \{1, \cdots, d\} \) and any vector \( v \in \mathbb{R}^d \) such that \( |J| \leq s \),

\[
\|v_{J^c}\|_1 \leq c_0 \|v_J\|_1. \tag{4.1}
\]

When the matrix \( \Sigma \) satisfies the restricted eigenvalue condition \( \text{RE}(s, c_0, \kappa) \), we immediately obtain the following lemma.

**Lemma 4.1**. Suppose that \( \Sigma \) satisfies \( \text{RE}(s, c_0, \kappa) \). Then, we have

\[
\|v\|_1 \leq \frac{c_0}{\kappa} \sqrt{s} \|\Sigma^{1/2}v\|_2, \tag{4.2}
\]

for any \( v \in \mathbb{R}^d \) satisfying (4.1) for every \( J \subset \{1, \cdots, d\} \) with \( |J| \leq s \).
Proof.

\[ \|v\|_1 = \|v_J\|_1 + \|v_{J^c}\|_1 \leq (c_0 + 1)\|v_J\|_1 = \frac{c_0 + 1}{\kappa} \sqrt{s_K}\|v_J\|_2 \leq \frac{c_0 + 1}{\kappa} \sqrt{s}\|\Sigma^\frac{1}{2} v\|_2. \]

\[\Box\]

4.2. Relation between \(\|\beta^* - \hat{\beta}\|_1\) and \(\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\)

The following proposition plays an important role to show a relation between \(\|\beta^* - \hat{\beta}\|_1\) and \(\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\). The proof is given in Section 4.4. Let the active set be denoted by \(S = \{i : \beta_i^* \neq 0\}\).

Proposition 4.1. Assume the conditions (c1) and (c2). Suppose that \(\lambda_s - C_{\lambda_s} \lambda_o = 0\) and \(\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2 \leq \frac{1}{\sqrt{s}}\|\beta^* - \hat{\beta}\|_1\). Then, with a high probability, we have

\[\|\beta^*_S - \hat{\beta}_S\|_1 \leq \frac{\lambda_s + C_{\lambda_s} \lambda_o}{\lambda_s - C_{\lambda_s} \lambda_o} \|\beta^*_S - \hat{\beta}_S\|_1,\]

where \(C_{\lambda_s}\) is defined in Theorem 2.1.

Combining Lemma 4.1 with Proposition 4.1, we can easily prove the following proposition, which shows a relation between \(\|\beta^* - \hat{\beta}\|_1\) and \(\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\).

Proposition 4.2. Assume the conditions (c1) and (c2). Suppose that \(\Sigma\) satisfies \(\text{RE}(s, c_0, \kappa)\), \(\lambda_s - C_{\lambda_s} \lambda_o > 0\) and

\[\frac{\lambda_s + C_{\lambda_s} \lambda_o}{\lambda_s - C_{\lambda_s} \lambda_o} \leq c_0.\] (4.3)

Then, with a high probability, we have

\[\|\beta^* - \hat{\beta}\|_1 \leq c_\kappa \sqrt{s}\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2,\] (4.4)

where \(c_\kappa := \frac{c_0 + 1}{\kappa} + 1\).

Proof. When \(\|\beta^* - \hat{\beta}\|_1 < \sqrt{s}\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\), we have (4.4) immediately since \(c_\kappa \geq 1\). Consider the case where \(\|\beta^* - \hat{\beta}\|_1 \geq \sqrt{s}\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\). Let \(J = S\) and then we have \(|J| = |S| \leq s\). From Proposition 4.1 and condition (4.3), \(v = \beta^* - \hat{\beta}\) satisfies \(\|v_J\|_1 \leq c_0\|v_J\|_1\), that is, the condition (4.1). Hence, since \(\Sigma\) satisfies \(\text{RE}(s, c_0, \kappa)\), we have the property (4.2) with \(v = \beta^* - \hat{\beta}\), so that we see \(\|v\|_1 \leq c_\kappa \sqrt{s}\|\Sigma^\frac{1}{2} v\|_2\) since \((c_0 + 1)/\kappa \leq c_\kappa\), and then the property (4.4) holds. \(\Box\)

4.3. Inequalities related to the estimation error

Using Corollaries 3.1 and 3.2 and Proposition 4.2, we can easily show the following two propositions related to the estimation error \(\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\).
Proposition 4.3. Assume the conditions used in Proposition 4.2. Then, with a high probability, we have

\[ \left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2 \geq C_\kappa \| \Sigma_1^{1/2} (\beta^* - \hat{\beta}) \|_2, \quad C_\kappa = a_1 - 1.2c_\kappa \sqrt{\frac{2\rho^2 s \log d}{n}}. \]

Proof. By letting \( v = \beta^* - \hat{\beta} \) in Corollary 3.1, we have

\[ \left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2 \geq a_1 \| \Sigma_1^{1/2} (\beta^* - \hat{\beta}) \|_2 - 1.2 \sqrt{\frac{2\rho^2 \log d}{n}} \| \beta^* - \hat{\beta} \|_1. \]

The proof is complete from (4.4) in Proposition 4.2 \( \square \)

Proposition 4.4. Assume the conditions used in Proposition 4.2. Then, the following property holds with a high probability: for any \( m \)-sparse vector \( u \in \mathbb{R}^n \),

\[ \left\| u^\top \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\| \leq \| \Sigma_1^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 g(m). \]

Proof. By letting \( v = \beta^* - \hat{\beta} \) in Corollary 3.2, we have

\[ \left\| u^\top \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\| \leq \| \Sigma_1^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 \sqrt{2} \left( 4.8 + \sqrt{\frac{81}{\delta}} \right) + 1.2 \| \beta^* - \hat{\beta} \|_1 \| u \|_2 \sqrt{\frac{2\rho^2 \log d}{n}} \]

\[ + 4.8 \sqrt{e} \| \Sigma_1^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 \sqrt{\frac{m}{n}} \sqrt{4 + \log \frac{n}{m}}. \]

The proof is complete from (4.4) in Proposition 4.2. \( \square \)

4.4. Proof of Proposition 4.1

Since \( \hat{\beta} \) is the minimizer of \( L(\beta) \), we have \( L(\hat{\beta}) \leq L(\beta^*) \), which implies

\[ \lambda_\alpha^2 \sum_{i=1}^n H(r_i(\hat{\beta})) - \lambda_\alpha^2 \sum_{i=1}^n H(r_i(\beta^*)) \leq \lambda_\alpha (\| \beta^* \|_1 - \| \hat{\beta} \|_1). \quad (4.5) \]

Since \( H(z) \) is convex, we have

\[ H(r_i(\hat{\beta})) - H(r_i(\beta^*)) \geq \psi(r_i(\beta^*)) \{ r_i(\hat{\beta}) - r_i(\beta^*) \} = \psi(r_i(\beta^*)) \frac{X_i^\top (\beta^* - \hat{\beta})}{\lambda_\alpha \sqrt{n}} \]

and then

\[ \frac{\lambda_\alpha}{\sqrt{n}} \sum_{i=1}^n \psi(r_i(\beta^*)) X_i^\top (\beta^* - \hat{\beta}) \leq \lambda_\alpha (\| \beta^* \|_1 - \| \hat{\beta} \|_1). \quad (4.7) \]
Since $Y_i = X_i^T \beta^* + \xi_i$ for $i \in I_u$ and $Y_i = X_i^T \beta^* + \sqrt{n}\theta_i + \xi_i$ for $i \in I_o$, we have

$$r_i(\beta^*) = \begin{cases} 
\frac{\xi_i}{\lambda_o \sqrt{n}} & (i \in I_u) \\
\sqrt{n}\theta_i + \xi_i & (i \in I_o) 
\end{cases}.$$  

We divide the summation in (4.7) into two parts:

$$\lambda_o(||\beta^*||_1 - ||\hat{\beta}||_1) \geq \frac{\lambda_o}{\sqrt{n}} \sum_{i \in I_u} \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) X_i^T (\beta^* - \hat{\beta})$$

$$+ \frac{\lambda_o}{\sqrt{n}} \sum_{i \in I_o} \psi \left( \frac{\sqrt{n}\theta_i + \xi_i}{\lambda_o \sqrt{n}} \right) X_i^T (\beta^* - \hat{\beta}) \quad (4.8)$$

$$= \frac{\lambda_o}{\sqrt{n}} \sum_{i=1}^n \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) X_i^T (\beta^* - \hat{\beta}) + \frac{\lambda_o}{\sqrt{n}} \sum_{i=1}^n u_i X_i^T (\beta^* - \hat{\beta}),$$

where

$$u_i = \begin{cases} 
0 & (i \in I_u) \\
\psi \left( \frac{\sqrt{n}\theta_i + \xi_i}{\lambda_o \sqrt{n}} \right) - \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) & (i \in I_o) 
\end{cases}.$$  

The first term can be evaluated by a standard technique, because it includes only standard quantities. The second term is difficult to be evaluated, because it includes the terms related to adversarial outliers, $\sqrt{n}\theta_i$'s. It can be evaluated by virtue of Corollary 3.2, because $\#I_o = o$ and $u = (u_1, \ldots, u_n)^T$ is an $o$-sparse vector. Remember the notations $z_{ij} = X_{ij} \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right)$ and $z = (\sum_{i=1}^n z_{i1}, \ldots, \sum_{i=1}^n z_{id})^T$. From (4.8), we see

$$0 \leq -\frac{\lambda_o}{\sqrt{n}} z^T (\beta^* - \hat{\beta}) - \lambda_o u^T \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) + \lambda_o (||\beta^*||_1 - ||\hat{\beta}||_1)$$

$$\leq \frac{\lambda_o}{\sqrt{n}} ||z||_\infty \|\beta^* - \hat{\beta}\|_1 + \lambda_o \left( u^T \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right) + \lambda_o (||\beta^*||_1 - ||\hat{\beta}||_1)$$

$$\leq \frac{\lambda_o}{\sqrt{n}} ||z||_\infty \|\beta^* - \hat{\beta}\|_1 + \lambda_o \left( u^T \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right) + \lambda_o (||\beta^*||_1 - ||\hat{\beta}||_1). \quad (4.9)$$

First, we consider the first term of (4.9), which can be evaluated from Proposition 3.4. Next, we consider the second term of (4.9). Since $|\psi(t)| \leq 1$, we have $|u_i| \leq 2$ for $i \in I_o$ and $\|u\| \leq \sqrt{2o}$. Since $u$ is an $o$-sparse vector, it holds from
Corollary 3.2 and the assumption \(|\Sigma^\dagger(\beta* - \hat{\beta})| \leq \frac{1}{\sqrt{n}}\|\beta* - \hat{\beta}\|_1\) that

\[
\left| u^\top X \sqrt{n}(\beta* - \hat{\beta}) \right| \leq \|u\|_2 \left\{ \|\Sigma^\dagger(\beta* - \hat{\beta})\|_2 \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right) + 1.2\|\beta* - \hat{\beta}\|_1 \sqrt{\frac{2\rho^2 \log d}{n}} \right. \\
+ 4.8\sqrt{\|\Sigma^\dagger(\beta* - \hat{\beta})\|_2} \sqrt{\frac{\alpha}{n}} \left( 4 + \log \frac{n}{\delta} \right) \right\} \\
\leq \|\beta* - \hat{\beta}\|_1 C_s, \quad (4.10)
\]

where \(C_s = \sqrt{20/\delta} \log(\alpha)\). Applying (4.10) and Proposition 3.4 to (4.9), we have

\[
0 \leq \lambda_s \|\beta* - \hat{\beta}\|_1 C_s + \lambda_o \|\beta* - \hat{\beta}\|_1 C_s + \lambda_s (\|\beta*\|_1 - \|\hat{\beta}\|_1) \\
= (C_s + C_s) \lambda_s (\|\beta_s - \hat{\beta}_s\|_1 + \|\beta_{S^c} - \hat{\beta}_{S^c}\|_1) + \lambda_s (\|\beta_s\|_1 - \|\hat{\beta}_s\|_1) + \lambda_s (\|\beta_{S^c}\|_1 - \|\hat{\beta}_{S^c}\|_1) \\
= C_s \lambda_s \|\beta_s - \hat{\beta}_s\|_1 + \lambda_s (\|\beta_{S^c}\|_1 - \|\hat{\beta}_{S^c}\|_1) \\
\leq C_s \lambda_s (\|\beta_s - \hat{\beta}_s\|_1 + \|\beta_{S^c}\|_1) + \lambda_s \|\beta_{S^c}\|_1 + \lambda_s \|\beta_{S^c}\|_1 - \|\hat{\beta}_{S^c}\|_1) \\
= (\lambda_s + C_s \lambda_s) \|\beta_{S^c}\|_1 - \|\hat{\beta}_{S^c}\|_1 (\lambda_s + C_s \lambda_s) \|\beta_{S^c}\|_1.
\]

The proof is complete.

5. Evaluation of \(C_{cut}\)

As seen in Section 2.2, the integer \(C_{cut}\) plays an important role in the proofs. In this section, we give an upper bound of \(C_{cut}\).

First, we give two lemmas.

**Lemma 5.1.** Assume the condition (c1) and (c2). Then, with a high probability, we have

\[
\frac{C_{cut}}{2} \leq \sum_{i \in I_>} H \left( r_i(\hat{\beta}) \right), \\
\sum_{i \in I_>} H \left( r_i(\beta*) \right) \leq \frac{C_{cut} \sigma^2 \log(n/\delta)}{\lambda^2 n}.
\]

**Proof.** For \(i \in I_>\), we have \(|r_i(\hat{\beta})| > 1\). The Huber function satisfies \(H(t) = |t| - 1/2\) for \(|t| > 1\), so that \(H(r_i(\hat{\beta})) > 1/2\) for \(i \in I_>\), which shows the first inequality since \(#I_> = C_{cut}\). The Huber function satisfies \(H(t) \leq t^2/2\). We have \(r_i(\beta*) = \xi_i / \lambda_o \sqrt{n}\) for \(i \in I_> \subseteq I_o\). Proposition 3.3 holds from the conditions (c1) and (c2), so that we have \(\xi^2_i / n < 2\sigma^2 \log(n/\delta)/n\). Combining these results, we see

\[
\sum_{i \in I_>} H \left( r_i(\beta*) \right) \leq \sum_{i \in I_>} \frac{r_i(\beta*)^2}{2} \leq \sum_{i \in I_>} \frac{\xi^2_i}{2\lambda^2 n} \leq \frac{C_{cut} \sigma^2 \log(n/\delta)}{\lambda^2 n},
\]
since \( \#I_o = C_{\text{cut}} \), which shows the second inequality. \(\square\)

**Lemma 5.2.** Assume (c3) and the conditions used in Proposition 4.2. Then, with a high probability, we have

\[
\left| \sum_{i \in I_o} \psi(r_i(\beta^*)) \frac{X_i(\beta^* - \beta)}{\lambda_o \sqrt{n}} \right| \leq \frac{\sqrt{\sigma}}{\lambda_o} g(o) \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2,
\]

\[
\left| \sum_{i \in I_\text{cut}} \psi(r_i(\beta^*)) \frac{X_i(\beta^* - \beta)}{\lambda_o \sqrt{n}} \right| \leq \frac{\sqrt{2\sigma^2}}{\lambda_o} g(n-o) \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2.
\]

**Proof.** Let

\[
u_i = \begin{cases} \psi(r_i(\beta^*)) & (i \in I_o) \\ 0 & (i \notin I_o) \end{cases}.
\]

We see that \( u \) is an \( o \)-sparse vector and \( \|u\|_2 \leq \sqrt{\sigma} \) since \( |\psi(t)| \leq 1 \) and \( \#I_o = o \). From Proposition 4.4,

\[
\left| \sum_{i \in I_o} \psi(r_i(\beta^*)) \frac{X_i(\beta^* - \beta)}{\lambda_o \sqrt{n}} \right| = \frac{1}{\lambda_o} \left| u^\top \frac{1}{\sqrt{n}} X(\beta^* - \hat{\beta}) \right| \leq \frac{1}{\lambda_o} \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 g(o)
\]

\[
\leq \frac{\sqrt{\sigma}}{\lambda_o} \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2 g(o),
\]

which shows the first inequality of the proposition. Let \( u = (u_1, \ldots, u_n)^\top \) be redefined by

\[
u_i = \begin{cases} \psi(r_i(\beta^*)) & (i \in I_\text{cut}) \\ 0 & (i \notin I_\text{cut}) \end{cases}.
\]

From a similar discussion to the above, we have

\[
\left| \sum_{i \in I_\text{cut}} \psi(r_i(\beta^*)) \frac{X_i(\beta^* - \beta)}{\lambda_o \sqrt{n}} \right| \leq \frac{1}{\lambda_o} \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 g(n - C_{\text{cut}} - o)
\]

\[
\leq \frac{1}{\lambda_o} \| \Sigma_{\omega}^{1/2} (\beta^* - \hat{\beta}) \|_2 \| u \|_2 g(n - o),
\]

using the monotonicity of \( g(m) \). Proposition 3.5 holds from the condition (c3). By \( \psi(u) \leq |u| \) and Proposition 3.5,

\[
\|u\|_2 = \sqrt{\sum_{i \in I_\text{cut}} \psi(r_i(\beta^*))^2} \leq \sqrt{\sum_{i \in I_\text{cut}} r_i(\beta^*)^2} = \sqrt{\sum_{i=1}^n \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right)^2} \leq \sqrt{\frac{2\sigma^2}{\lambda_o}}.
\]

The above two inequalities show the second inequality of the proposition. \(\square\)

Using the above two lemmas, we give an upper bound of \( C_{\text{cut}} \).
Proposition 5.1. Assume (c3) and the conditions used in Proposition 4.2. Suppose that $\Sigma$ satisfies $RE(s,c_0,\kappa)$ and $C_{\lambda s} > 1$. Then, with a high probability, we have

$$C_{\text{cut}} \leq \frac{2C_r}{\lambda_0^2} \left( \sqrt{2\sigma^2 g(n - o)} + \sqrt{o\lambda_0 g(o)} + \sqrt{s}\lambda_s \right) \| \Sigma^{\frac{1}{2}}(\beta^* - \hat{\beta}) \|_2.$$ 

where $C_r = 1/(1 - \frac{2\sigma^2 \log(n/\delta)}{\lambda_0^2 n}) > 0$.

Proof. From (4.5), we have

$$\lambda_0^2 \sum_{i=1}^n \left\{ H(r_i(\hat{\beta})) - H(r_i(\beta^*)) \right\} \leq \lambda_0 \left( \| \beta^* \|_1 - \| \hat{\beta} \|_1 \right) \leq \lambda_0 \| \beta - \beta^* \|_1.$$

From (4.6), we have

$$\lambda_0 \| \beta - \beta^* \|_1 \geq \lambda_0^2 \left\{ \sum_{i \in I_>^c} + \sum_{i \in I_<^c} + \sum_{i \in I_o} \right\} \left\{ H(r_i(\hat{\beta})) - H(r_i(\beta^*)) \right\}$$

$$\geq \lambda_0^2 \sum_{i \in I_>^c} \left\{ H(r_i(\hat{\beta})) - H(r_i(\beta^*)) \right\} + \lambda_0^2 \left\{ \sum_{i \in I_<^c} + \sum_{i \in I_o} \right\} \psi(r_i(\beta^*)) \frac{X_i^T (\beta^* - \hat{\beta})}{\lambda_0 \sqrt{n}}$$

and

$$\lambda_0^2 \left\{ \sum_{i \in I_<^c} + \sum_{i \in I_o} \right\} \psi(r_i(\beta^*)) \frac{X_i^T (\beta^* - \hat{\beta})}{\lambda_0 \sqrt{n}} + \lambda_0 \| \beta - \beta^* \|_1$$

$$\geq \lambda_0^2 \sum_{i \in I_>^c} H(r_i(\hat{\beta})) - H(r_i(\beta^*)) \right\}.$$

From Lemmas 5.1 and 5.2 and Proposition 4.2, we have

$$\| \Sigma^{\frac{1}{2}}(\beta^* - \hat{\beta}) \|_2 \left( \sqrt{2\sigma^2 g(n - o)} + \sqrt{o\lambda_0 g(o)} + \sqrt{s}\lambda_s \right)$$

$$\geq C_{\text{cut}} \left( \frac{\lambda_0^2}{2} - \frac{\sigma^2 \log(n/\delta)}{n} \right) = C_{\text{cut}} \frac{\lambda_0^2}{2C_r}.$$

From (2.4) with $C_{\lambda s} > 1$, we have $C_r > 0$ and then

$$C_{\text{cut}} \leq \frac{2C_r}{\lambda_0^2} \left( \sqrt{2\sigma^2 g(n - o)} + \sqrt{o\lambda_0 g(o)} + \sqrt{s}\lambda_s \right) \| \Sigma^{\frac{1}{2}}(\beta^* - \hat{\beta}) \|_2.$$
6. Outline of the proofs of the key propositions

In this section, we evaluate $T_1$, $T_2$ and $T_3$ in a rough manner. Detailed evaluations are given in Section 7.

Let $X_{I_c}$ be the $\#I_c \times d$ matrix whose row vectors consist of $X_i$s ($i \in I_c$). Let $X_{I_\leq}$ and $X_{I_\geq}$ be defined in a similar manner. Let $\xi_{I_c}$ be the $\#I_c$ dimensional vector whose components consist of $\xi_i$s ($i \in I_c$). Let $\xi_{I_\leq}$ and $\xi_{I_\geq}$ be defined in a similar manner.

We see

$$T_2 = \lambda_0 \sum_{i \in I_\leq} \psi \left( r_1(\hat{\beta}) \right) \frac{X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta}) = \lambda_0 \sum_{i \in I_\leq} \text{sgn}(r_1(\hat{\beta})) \frac{X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta}),$$

$$T_3 = \lambda_0 \sum_{i \in I_\geq} \psi \left( r_1(\hat{\beta}) \right) \frac{X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta}),$$

$$T_1 = \lambda_0 \sum_{i \in I_{\leq}} \psi \left( r_1(\hat{\beta}) \right) \frac{X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta}) = \lambda_0 \sum_{i \in I_{\leq}} \frac{y_i - X_i^T \hat{\beta}}{\lambda_0 \sqrt{n}} \frac{X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta}) = \lambda_0 \sum_{i \in I_{\leq}} \frac{X_i^T (\beta^* - \hat{\beta}) + \xi_i X_i^T}{\sqrt{n}} (\beta^* - \hat{\beta})$$

$$= T_a + T_b,$$

where

$$T_a = \left\| \frac{X_{I_{\leq}}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2, \quad T_b = \frac{1}{\sqrt{n}} \xi_{I_{\leq}} \frac{X_{I_{\leq}}}{\sqrt{n}} (\beta^* - \hat{\beta}).$$

First, we evaluate $T_2$ and $T_3$, because they can be easily evaluated. Next, we evaluate $T_a$ and $T_b$.

**Lemma 6.1.** Assume the conditions used in Proposition 4.2. Then, with a high probability, we have

$$|T_2| \leq C_2 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2, \quad C_2 = \lambda_0 \sqrt{C_{\text{cut}} g(C_{\text{cut}})}.$$

Proof. Let $u$ be the $n$-dimensional vector whose $i$th component is $\text{sgn}(r_i(\hat{\beta}))$ for $i \in I_\leq$ and other components are zero. We see that $u$ is a $C_{\text{cut}} = \#I_\leq$ sparse vector and $\|u\|_2 \leq \sqrt{C_{\text{cut}}}$. From Proposition 4.4, we have

$$|T_2| = \lambda_0 \left| u^T \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right| \leq \lambda_0 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2 \|u\|_2 g(C_{\text{cut}})$$

$$\leq \lambda_0 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2 \sqrt{C_{\text{cut}} g(C_{\text{cut}})}.$$ 

□

**Lemma 6.2.** Assume the conditions used in Proposition 4.2. Then, with a high probability, we have

$$|T_3| \leq C_3 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2, \quad C_3 = \lambda_0 \sqrt{g(o)}.$$

Proof. Let $u$ be the $n$-dimensional vector whose $i$th component is $\text{sgn}(r_i(\hat{\beta}))$ for $i \in I_\leq$ and other components are zero. We see that $u$ is a $C_{\text{cut}} = \#I_\leq$ sparse vector and $\|u\|_2 \leq \sqrt{C_{\text{cut}}}$. From Proposition 4.4, we have

$$|T_3| = \lambda_0 \left| u^T \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right| \leq \lambda_0 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2 \|u\|_2 g(C_{\text{cut}})$$

$$\leq \lambda_0 \| \Sigma^{\frac{1}{2}} (\beta^* - \hat{\beta}) \|_2 \sqrt{C_{\text{cut}} g(C_{\text{cut}})}.$$ 

□
Proof. Let $u$ be the $n$-dimensional vector whose $i$th component is $\psi(r_i(\hat{\beta}))$ for $i \in I_o$ and other components are zero. We see that $u$ is an $o = \# I_o$ sparse vector and $\|u\|_2 \leq \sqrt{o}$ since $|\psi(u)| \leq 1$. From Proposition 4.4, we have

$$|T_3| = \lambda_o \left\| u^\top \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\| \leq \lambda_o \| \Sigma \hat{\beta} - \beta \|_2 \|u\|_2 g(o)$$

$$\leq \| \Sigma \hat{\beta} - \beta \|_2 \lambda_o \sqrt{o} g(o).$$

\[\Box\]

Lemma 6.3. Assume the conditions used in Proposition 4.2. Then, with a high probability, we have

$$\sqrt{T_a} \geq (a_1 - C_{a_1} - C_{a_2}) \| \Sigma \hat{\beta} - \beta \|_2,$$

where

$$C_{a_1} = 1.2 \kappa \sqrt{\frac{2p^2 \log d}{n}}, \quad C_{a_2} = g(C_{\text{cut}} + o).$$

Proof. We see

$$T_a = \left\| \frac{X_{I_o}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2 = \left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2 - \left\| \frac{X_{I_o \cup I_o}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2 \geq 0$$

and

$$\sqrt{T_a} = \sqrt{\left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2 - \left\| \frac{X_{I_o \cup I_o}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2} \geq \left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2 - \left\| \frac{X_{I_o \cup I_o}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2,$$

because $\sqrt{A^2 - B^2} \geq A - B$ for $A \geq B \geq 0$. From Proposition 4.3, we have

$$\left\| \frac{X}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2 \geq C_\kappa \| \Sigma \hat{\beta} - \beta \|_2,$$

where $C_\kappa = a_1 - C_{a_1}$. Let $u$ be the $n$-dimensional vector whose $i$th component is $\frac{X_i}{\sqrt{n}} (\beta^* - \hat{\beta})$ for $i \in I_o \cup I_o$ and other components are zero. This is a $C_{\text{cut}} + o = \# I_o + \# I_o$ sparse vector. From Proposition 4.4, we see

$$\|u\|_2^2 = \left\| \frac{X_{I_o \cup I_o}}{\sqrt{n}} (\beta^* - \hat{\beta}) \right\|_2^2 \leq \| \Sigma \hat{\beta} - \beta \|_2 \|u\|_2 g(C_{\text{cut}} + o) \quad (6.1)$$

and then we have $\|u\|_2 \leq \| \Sigma \hat{\beta} - \beta \|_2 g(C_{\text{cut}} + o)$. Combining the results, the proof is complete. \[\Box\]
Lemma 6.4. Assume the conditions used in Proposition 4.2. Then, with a high probability, we have

$$|T_b| \leq C_b\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2,$$

where

$$C_b = 2c_n\sqrt{\frac{2\sigma^2\rho^2s\log(d/\delta)}{n}} + g(C_{cut} + o)\sqrt{\frac{2\sigma^2(C_{cut} + o)\log(n/\delta)}{n}}.$$

Proof. We see

$$|T_b| = \left| \xi_{I_\times}^\top \frac{X_{I_\times}}{n}(\beta^* - \hat{\beta}) \right|$$

$$= \frac{1}{\sqrt{n}} \left| \xi^\top \frac{X}{\sqrt{n}}(\beta^* - \hat{\beta}) - \xi_{I_\times \cup I_o}^\top \frac{X_{I_\times \cup I_o}}{\sqrt{n}}(\beta^* - \hat{\beta}) \right|$$

$$\leq \left| \xi^\top \frac{X}{n}(\beta^* - \hat{\beta}) \right| + \frac{1}{\sqrt{n}} \left| \xi_{I_\times \cup I_o}^\top \frac{X_{I_\times \cup I_o}}{\sqrt{n}}(\beta^* - \hat{\beta}) \right|.$$

From Propositions 3.3 and 4.2,

$$\left| \xi^\top \frac{X}{n}(\beta^* - \hat{\beta}) \right| \leq \|\beta^* - \hat{\beta}\|_1 \left| \xi^\top \frac{X}{n} \right|_\infty \leq \|\beta^* - \hat{\beta}\|_2 \sqrt{\frac{2\sigma^2\rho^2\log(3d/\delta)}{n}}$$

$$\leq \|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_22c_n\sqrt{\frac{2\sigma^2\rho^2s\log(3d/\delta)}{n}}.$$

Let $u$ be the $n$-dimensional vector whose $i$th component is $\xi_i$ for $i \in I_\times \cup I_o$ and other components are zero. This is a $C_{cut} + o = \#I_\times + \#I_o$ sparse vector. From Proposition 4.4,

$$\left| \xi_{I_\times \cup I_o}^\top \frac{X_{I_\times \cup I_o}}{\sqrt{n}}(\beta^* - \hat{\beta}) \right| = \left| u^\top \frac{X}{\sqrt{n}}(\beta^* - \hat{\beta}) \right| \leq \|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2\|u\|_2 g(C_{cut} + o).$$

From Proposition 3.3,

$$\|u\|_2^2 \leq (C_{cut} + o)\|\xi\|_\infty^2 \leq 2\sigma^2(C_{cut} + o)\log(n/\delta).$$

Combining the above results, the proof is complete. □

Combining the above results, we can easily show the following proposition.

Proposition 6.1. Assume the conditions used in Proposition 4.2. Suppose $a_1 > 0$. Then, with a high probability, we have

$$\frac{a_1^2}{2}\|\Sigma(\beta^* - \hat{\beta})\|_2 \leq 2(C_{a_1}^2 + C_{a_2}^2)\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2 + C_b + C_2 + C_3 + \lambda_2 c_n\sqrt{s}.$$

Proof. We have

$$\lambda_o \sum_{i=1}^n \psi\left(r_i(\hat{\beta})\right) \frac{X_i^\top}{\sqrt{n}}(\beta^* - \hat{\beta}) = T_1 + T_2 + T_3 \geq T_a - |T_b| - |T_2| - |T_3|.$$
Let $E = \| \Sigma^{1/2} (\beta^* - \hat{\beta}) \|_2$. From Proposition 4.2, we have $\| \beta^* - \hat{\beta} \|_1 \leq c_\kappa \sqrt{s} E$. Combining these two inequalities on (2.8), we have

$$T_a \leq |T_b| + |T_2| + |T_3| + \lambda s c_\kappa \sqrt{s} E.$$  

From $T_a \geq 0$, we can take a square root on both sides. From Lemmas 6.1, 6.2 and 6.4, we see

$$\sqrt{T_a} \leq \sqrt{|T_b| + |T_2| + |T_3| + \lambda s c_\kappa \sqrt{s} E}.$$  

From Lemma 6.3, we have

$$\sqrt{T_a} \geq (a_1 - C_{a_1} - C_{a_2}) E.$$  

Then,

$$a_1 E \leq (C_{a_1} + C_{a_2}) E + \sqrt{(C_b + C_2 + C_3 + \lambda s c_\kappa \sqrt{s}) E}$$  

and

$$a_2^2 E \leq \left( (C_{a_1} + C_{a_2}) \sqrt{E} + \sqrt{C_b + C_2 + C_3 + \lambda s c_\kappa \sqrt{s}} \right)^2$$  

$$\leq 2 \left( (C_{a_1} + C_{a_2})^2 E + C_b + C_2 + C_3 + \lambda s c_\kappa \sqrt{s} \right).$$  

The proof is completed from $(C_{a_1} + C_{a_2})^2 \leq 2(C_{a_1}^2 + C_{a_2}^2)$.

7. Proof of main theorem

7.1. Notation

Let

$$r_1 = \sqrt{\frac{s \log d}{n}}, \quad r_2 = \frac{o}{n} \sqrt{\frac{n}{\log o \log n}},$$  

$$r_{21} = \sqrt{\frac{o}{n} \log \frac{n}{o}}, \quad r_{22} = \frac{o}{n} \log n \quad (\geq r_{21}).$$  

Then,

$$r_{n,d,s,o} = r_1 + r_2, \quad r_2 = r_{21} r_{22}.$$  

Let

$$\eta_5 = \sqrt{\frac{\log (n/\delta)}{\log n}}, \quad \eta_4 = \sqrt{\frac{4 + \log (n/o)}{\log (n/o)}}.$$  

These are larger than 1 and bounded above by some constants, as shown in Lemma 7.4.

Let $A$ be an upper bound of $A$. In this section, we consider two cases; $C_{cut} \leq o$ and $C_{cut} > o$. The corresponding upper bounds are denoted by $A^<$ and $A^>$. 

7.2. Theorem

We present a general estimation error bound in Theorem 7.1. By selecting a special tuning parameter in Theorem 7.1, we can have the main theorem, which is given in Theorem 7.2. In this section, we prove Theorem 7.1 and the main theorem.

Theorem 7.1. Consider the optimization problem (1.2). Suppose that \( \Sigma \) satisfies \( \text{RE}(s, 5, \kappa) \). Assume that \( \delta \) and \( n \) satisfy

(c1) \( \delta \in (0, 1/7] \), \( n \geq 100 \),
(c2) \( \sqrt{\frac{\log(d/\delta)}{n}} \leq \sqrt{3} - \sqrt{2} \) (This implies \( 2 \log(d/\delta) \leq n \) from \( n \geq 100 \).),
(c3) \( 2\sqrt{n \log(1/\delta)} + 2 \log(1/\delta) \leq n \),
(c4) \( a_1 > 3/4 \),
(c5) \( b_1 < 1/4 \),

where

\[
a_1 = 1 - \frac{4.3 + \sqrt{2 \log(9/\delta)}}{\sqrt{n}}, \quad b_1 = \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\frac{81}{\delta}} \right).
\]

Suppose that \( \lambda_s \) and \( \lambda_o \) satisfy

\[
\lambda_o \geq C_{\lambda_o} \sqrt{\frac{2\rho^2 \log(n/\delta)}{n}}, \quad \lambda_s \geq \frac{4\sqrt{2}}{\sqrt{3}} C_{\lambda_o} \lambda_o, \tag{7.1}
\]

\[
8 \max \left( 3.6 \sqrt{\frac{2\rho^2 \log d}{n}}, 2.4 \sqrt{\frac{2 \log n}{n}} \right) \sqrt{\frac{s}{\kappa^2} + \frac{6.25 \alpha \lambda_o^2}{\lambda_s^2}} \leq C_{n, \delta}, \tag{7.2}
\]

where

\[
C_{\lambda_o} = C_z + \sqrt{2}\rho \varphi(g(o)), \tag{7.3}
\]

\[
C_{n, \delta} = \sqrt{a_1^2 + b_1 + 1/4 - \sqrt{2(b_1 + 1/4)}}. \tag{7.4}
\]

Assume that (4.3) is satisfied with \( c_0 = 5 \), \( C_{\lambda_o} \) is a sufficiently large constant such that \( C_{\lambda_o} \geq 2 \) and

\[
C_> = \frac{9}{32} - 2 \times 9.6^2 \tilde{\eta}_4 \frac{C_{\lambda_o}}{C_{\lambda_o}^2 - 1} > 0,
\]

where \( \tilde{\eta}_4 \) is a constant given in Lemma 7.4. Then, with probability at least \( 1 - 7\delta \), the optimal solution \( \hat{\beta} \) satisfies

\[
\|\Sigma^{1/2} (\beta^* - \hat{\beta})\|_2 \leq \frac{32}{9} \left( R + 2 C_{\rho 2} \nu E + \tilde{C}_{\rho 2} + \tilde{C}_{\rho 2}^c + \tilde{C}_{\rho 2}^c \right), \quad \text{if} \ C_{\text{cut}} \leq o, \tag{7.5}
\]

\[
\leq \frac{1}{C_>} \left( R + 2 C_{\rho 2} \nu E + \tilde{C}_{\rho 2}^c + \tilde{C}_{\rho 2}^c \right), \quad \text{if} \ C_{\text{cut}} > o, \tag{7.6}
\]
where $\tilde{C}s$ and $\nu_E$ are given in Lemmas 7.1, 7.8, 7.11, 7.13, Proposition 7.1 and (7.12),

$$R = 2\tilde{C}_1\nu_E + \tilde{C}_b + \tilde{C}_3 + \lambda_s c_\kappa \sqrt{s}.$$

**Theorem 7.2.** Consider the optimization problem (1.2). Assume the same conditions as in Theorem 7.1 except for

$$\lambda_o = \frac{C\lambda_o}{\sqrt{2}} \sigma^2 \log(n/\delta) n,$$

$$\lambda_s = \frac{4\sqrt{2}}{\sqrt{3}} C\lambda_o.$$

(7.7)

The, with probability at least $1 - 7\delta$, the optimal solution $\hat{\beta}$ satisfies

$$\|\Sigma^{1/2}(\beta^* - \hat{\beta})\|_2 \leq C_{\delta,\kappa,\rho,\sigma} r_{n,d,s,o},$$

where $C_{\delta,\kappa,\rho,\sigma}$ is a constant depending on $\delta, \kappa, \rho, \sigma$.

**7.3. Outline of The Proof of Theorem 7.1**

We restrict the sample space with probability at least $1 - \delta$ via Propositions 3.1, 3.2, 3.3, 3.4 and 3.5. Hence, the theorem hold with probability at least $1 - 7\delta$.

We prove Theorem 7.1, using the basic inequality given in Proposition 6.1. Here, we again write the basic inequality:

$$\frac{a^2}{2} E \leq 2(C_{01} + C_{02})E + C_b + C_2 + C_3 + \lambda_s c_\kappa \sqrt{s},$$

(7.8)

where

$$C_{01} = C_{a_1}^2 = (1.2c_\kappa)^2 \frac{2\rho^2 s \log d}{n}, \quad C_{02} = C_{a_2}^2 = g(C_{\text{cut}} + o)^2,$$

$$C_b = \tilde{C}_b + \tilde{C}_b', \quad C_{b_1} = 2c_\kappa \sqrt{\frac{2\sigma^2 \rho^2 s \log(d/\delta)}{n}},$$

$$C_{b_2} = g(C_{\text{cut}} + o) \sqrt{2\sigma^2(C_{\text{cut}}/o + 1)} C_{b_2'}, \quad C_{b_2'} = \frac{o \log(n/\delta)}{n},$$

$$C_2 = \lambda_o \sqrt{C_{\text{cut}} g(C_{\text{cut}})}, \quad C_3 = \lambda_o \sqrt{o g(o)}.$$
Lemma 7.1. We have

\[ C_{01} = (1.2c_\kappa)^2 \frac{2\rho^2 s \log d}{n} \leq (1.2c_\kappa)^2 2\rho^2 r_1^2 =: \bar{C}_{01}, \]

\[ C_{b1} = 2c_\kappa \sqrt{\frac{2\sigma^2 \rho^2 s \log(d/\delta)}{n}} \leq 2c_\kappa \sqrt{2\sigma^2 \rho^2} \sqrt{1 + \log(1/\delta)} r_1 =: \bar{C}_{b1}, \]

\[ C_{b2}' = \sqrt{\frac{o \log(n/\delta)}{n}} = \eta_6 r_{22}. \]

Proof. The first and third inequalities hold immediately from the definitions of \( r_1, r_{22} \) and \( \eta_6 \). The second inequality follows from \( \log(d/\delta) = \log(d) + \log(1/\delta) \leq \log(d) + \log(1/\delta) \log(d) \).

The remaining terms are related to the function \( g(m) \) and \( C_{\text{cut}} \). First, we show some simple properties of \( g(m) \). Next, we consider \( C_{\text{cut}} \) with two cases: \( C_{\text{cut}} \leq o \) and \( C_{\text{cut}} > o \). The case \( C_{\text{cut}} \leq o \) is easily treated. The case \( C_{\text{cut}} > o \) is treated later in detail.

7.4. Constant Bounds

In this subsection, we give some constant bounds.

Lemma 7.2. We have

\[ \underline{C} \leq C_{n,\delta} \leq \bar{C}, \tag{7.9} \]

where \( \underline{C} = \sqrt{17/16} - 1 > 0 \) and \( \bar{C} = (\sqrt{5} - \sqrt{2})/2 \).

Proof. We can easily see that \( C_{n,\delta} = \sqrt{a_1^2 + b_1 + 1/4} - \sqrt{2(b_1 + 1/4)} \) is monotonically increasing on \( a_1 \) and decreasing on \( b_1 \). Since \( 3/4 \leq a_1 \leq 1 \) and \( 0 \leq b_1 \leq 1/4 \), we have

\[ C_{n,\delta} \leq \sqrt{1 + 1/4} - \sqrt{2(1/4)} = \underline{C}, \]

\[ C_{n,\delta} \geq \sqrt{(3/4)^2 + 1/4 + 1/4 - \sqrt{2(1/4 + 1/4)}} = \bar{C}. \]

Lemma 7.3. We have

\[ r_1 \leq \bar{C}\kappa/28.8\sqrt{2}\rho, \quad r_{22} \leq \bar{C}/19.2\sqrt{12.5}. \]

Proof. From (7.2), we know

\[ C_{n,\delta} \geq 8 \max \left( 3.6 \sqrt{\frac{2\rho^2 \log d}{n}}, 2.4 \frac{\lambda_s}{\lambda_0} \sqrt{\frac{2\log n}{n}}, \sqrt{\frac{s}{n^2}} + \frac{6.25\alpha^2}{\lambda_s^2} \right). \]
From this inequality, we see
\[
C_{n, \delta} \geq 8 \times 3.6 \sqrt{\frac{2\rho^2 \log d}{n}} \sqrt{\frac{s}{\kappa^2}} \geq 28.8 \sqrt{2} \rho \sqrt{\frac{s \log d}{n}},
\]
\[
C_{n, \delta} \geq 8 \times 2.4 \frac{\lambda_o}{\lambda_o} \sqrt{\frac{2 \log n}{n}} \sqrt{\frac{6.25 \lambda_o^2}{\lambda_o^2}} \geq 19.2 \sqrt{12.5} \sqrt{\frac{o \log n}{n}}.
\]
The proof is complete from Lemma 7.9.

**Lemma 7.4.** We have
\[
\eta_o = \sqrt{\frac{\log(n/\delta)}{\log n}} \leq \sqrt{\frac{\log 100/\delta}{\log 100}}, \tag{7.10}
\]
\[
\eta_4 = \sqrt{\frac{4 + \log(n/o)}{\log(n/o)}} \leq \sqrt{\frac{4 + \log C_{on}}{\log C_{on}}}, \tag{7.11}
\]
where \(C_{on} = (19.2 \sqrt{12.5})^2 \log 100/C\).

**Proof.** Note that \((a + \log(x))/\log(x)\) with \(a > 0\) and \(x > 0\) is a monotone decreasing function of \(x\). The first inequality holds from \(n \geq 100\). Next, we consider the second inequality. From Lemma 7.4 and \(r_{22} = \sqrt{o \log n/n}\), we see
\[
o/n \leq \frac{(C/19.2 \sqrt{12.5})^2}{\log n} \leq \frac{(C/19.2 \sqrt{12.5})^2}{\log 100} =: 1/C_{on},
\]
which implies
\[
\eta_4 = \sqrt{\frac{4 + \log(n/o)}{\log(n/o)}} \leq \sqrt{\frac{4 + \log C_{on}}{\log C_{on}}},
\]
\]

**Lemma 7.5.** We have
\[
C_r = \frac{1}{1 - 2\sigma^2 \log(n/\delta)/\lambda_o^2 n} \leq \frac{C_{\lambda_o}^2}{C_{\lambda_o}^2 - 1}.
\]

**Proof.** From the condition (7.1), we know \(\lambda_o \geq \sqrt{2 \sigma^2 \log(n/\delta)/n}\), which implies
\[
\frac{1}{C_r} = 1 - \frac{2\sigma^2 \log(n/\delta)}{n \lambda_o^2} \geq 1 - \frac{1}{C_{\lambda_o}^2}.
\]
The proof is complete.
7.5. Evaluation of $g(\cdot)$

Let

$$g(m) = g_1 + g_2(m),$$

where

$$g_1 = \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right) + 1.2c_\kappa \sqrt{\frac{2\rho^2 s \log d}{n}},$$

$$g_2(m) = 4.8\sqrt{e} \sqrt{\frac{m}{n}} \sqrt{4 + \log \frac{n}{m}}.$$

We can easily see that $g_2(m)$ and $g(m)$ are monotone increasing functions of $m$.

**Lemma 7.6.** We have

$$g_1 \leq c_g r_1,$$

where $c_g = 4.8\sqrt{2} + \sqrt{2 \log(81/\delta) + 1.2c_\kappa \sqrt{2\rho^2}}$.

**Proof.** We have

$$\sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right) \leq \sqrt{2} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right) r_1.$$

and

$$1.2c_\kappa \sqrt{\frac{2\rho^2 s \log d}{n}} \leq 1.2c_\kappa \sqrt{2\rho^2 r_1}.$$

**Lemma 7.7.** We have

$$g_2(o) = 4.8\sqrt{e} \eta_1 r_{21}.$$

**Proof.** We have

$$g_2(o) = 4.8\sqrt{e} \sqrt{o \sqrt{4 + \log \frac{n}{o}}} = 4.8\sqrt{e} \eta_1 \sqrt{o \sqrt{\log \frac{n}{o}}}.$$

The following lemma holds immediately from Lemmas 7.6 and 7.7.

**Lemma 7.8.** We have

$$C_3 = \lambda_o \sqrt{o} g(o) \leq (c_gr_1 + 4.8\sqrt{e} \eta_1 r_{21}) \sqrt{o} \lambda_o =: \bar{C}_3.$$
7.6. Simple estimation bound of $E = \|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2$

First, we introduce the concept of augmented transfer principle.

**Definition 7.1** (Definition 1 of Dalalyan and Thompson (2019)). We say that $X$ satisfies the augmented transfer principle ATP$_\Sigma(c_1, c_2, c_3)$ for some positive numbers $c_1$, $c_2$ and $c_3$, when for any $v \in \mathbb{R}^d$ and $u \in \mathbb{R}^n$, we have

$$\|X \sqrt{n} v + u\|_2 \geq c_1 \left(\|\Sigma^\frac{1}{2} v\|_2 + \|u\|_2\right) - c_2\|v\|_1 - c_3\|u\|_1.$$

The following lemma is a slight modification of Lemma 7 of Dalalyan and Thompson (2019), because we suspect the correctness. The proof of the following lemma is given in Appendix B.

**Lemma 7.9** (Modification of Lemma 7 of Dalalyan and Thompson (2019)). Let $Z \in \mathbb{R}^{n \times d}$ be a random matrix satisfying

$$\|Z \sqrt{n} v\|_2 \geq a_1\|\Sigma^\frac{1}{2} v\|_2 - a_2\|v\|_1$$

and

$$\|u^T Z \sqrt{n} v\|_2 \leq b_1\|\Sigma^\frac{1}{2} v\|_2\|u\|_2 + b_2\|v\|_1\|u\|_2 + b_3\|\Sigma^\frac{1}{2} v\|_2\|u\|_1$$

for some positive constants $a_1 \in (0, 1)$, $a_2$, $b_1$, $b_2$, $b_3$. Then, for any $\alpha > 0$, $Z$ satisfies

$$\|Z \sqrt{n} v + u\|_2 \geq c_1 \left(\|\Sigma^\frac{1}{2} v\|_2 + \|u\|_2\right) - c_2\|v\|_1 - c_3\|u\|_1$$

with the constants $c_1 = \sqrt{a_1^2 + b_1 + \alpha^2 - \sqrt{2(b_1 + \alpha^2)}}$, $c_2 = a_2 + b_2/\alpha$, $c_3 = b_3/\alpha$. If $a_1^2 > b_1 + \alpha^2$, then we have $c_1 > 0$.

We can obtain a simple estimation bound of $E = \|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2$ from Proposition 1 of Dalalyan and Thompson (2019). As seen later in Lemma 7.18, this bound is roughly of order $r_1 + r_2^2$

**Proposition 7.1.** We have

$$\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2^2 + \|\theta^* - \hat{\theta}\|_2^2 \leq \nu_E^2,$$

where

$$\nu_E = \frac{6}{C_{n, \delta}^2} \sqrt{\frac{\lambda_2^2 s}{\kappa^2} + 6.25\lambda_2^2\alpha}.$$

In addition, we have

$$\|\Sigma^\frac{1}{2}(\beta^* - \hat{\beta})\|_2 \leq \nu_E.$$
Proof. The first result is the same as in Proposition 1 of Dalalyan and Thompson (2019). It is enough to verify the conditions assumed in Proposition 1 of Dalalyan and Thompson (2019). The same conditions are assumed in Theorem 7.1, except for a similar condition to (7.2) and the conditions that \( \lambda_o \geq \frac{2}{\sqrt{n}}\|\xi\|_{\infty} \), \( \lambda_s \geq \frac{2}{n}\|X^\top \xi\|_{\infty} \), and \( X \) satisfies ATP_{\Sigma}(c_1; c_2; c_3) with the constants \( c_1 = C_{n,\delta} > 0 \), \( c_2 = 3.6\sqrt{2\rho^2 \log d/n} \), \( c_3 = 2.4\sqrt{2\log n/n} \). The condition (7.2) is just a slight modification of the condition assumed in Theorem 7.1. As a result, the proof is complete by verifying that these three conditions hold from the conditions assumed in Theorem 7.1. The ATP condition is proved in Proposition 7.2. The condition \( \lambda_o \geq \frac{2}{\sqrt{n}}\|\xi\|_{\infty} \) can be easily proved from (7.1) and Proposition 3.3. The condition \( \lambda_s \geq \frac{2}{n}\|X^\top \xi\|_{\infty} \) is proved as follows. From (7.1),

\[
\lambda_s \geq \frac{4\sqrt{2}}{\sqrt{3}} C, \lambda_o \geq \frac{4\sqrt{2}}{\sqrt{3}} C, \lambda_o = \frac{4\sqrt{2}}{\sqrt{3}} \frac{\rho^2 \sigma^2}{n} \log \frac{d}{\delta} \delta \geq \frac{2}{n} \|X^\top \xi\|_{\infty},
\]

since the last inequality holds from Proposition 3.3 and (c2).

**Proposition 7.2.** \( X \) satisfies ATP_{\Sigma}(c_1; c_2; c_3) with the constants \( c_1 = C_{n,\delta} \), \( c_2 = 3.6\sqrt{2\rho^2 \log d/n} \), \( c_3 = 2.4\sqrt{2\log n/n} \).

**Proof.** From Lemma 3.1 and Proposition 3.1, we have

\[
\left\| \frac{Z}{\sqrt{n}} \right\|_2 \geq a_1 \|\Sigma^\top v\|_2 - a_2 \|v\|_1,
\]

where

\[
a_1 = 1 - \frac{4.3 + \sqrt{2\log(9/\delta)}}{\sqrt{n}}, \quad a_2 = 1.2 \sqrt{2\rho^2 \log d/n}.
\]

From Lemma 3.2 and Corollary 3.2, we have

\[
\left\| u^\top \frac{Z}{\sqrt{n}} v \right\| \leq b_1 \|\Sigma^\top v\|_2 \|u\|_2 + b_2 \|v\|_1 \|u\|_2 + b_3 \|\Sigma^\top v\|_2 \|u\|_1,
\]

where

\[
b_1 = \sqrt{\frac{2}{n}} \left( 4.8 + \sqrt{\log \frac{81}{\delta}} \right), \quad b_2 = 1.2 \sqrt{2\rho^2 \log d/n}, \quad b_3 = 1.2 \sqrt{2\log n/n}.
\]

Let

\[
c_1 = C_{n,\delta} = \sqrt{a_1^2 + b_1 + 1/4 - \sqrt{2(b_1 + 1/4)}},
\]

\[
c_2 = a_2 + 2b_2 = 3.6 \sqrt{2\rho^2 \log d/n}, \quad c_3 = 2b_3 = 2.4 \sqrt{2\log n/n}.
\]

The condition \( c_1 > 0 \) holds from Lemma 7.2. From Lemma 7.9 with \( \alpha = 1/2 \), \( X \) satisfies ATP_{\Sigma}(c_1; c_2; c_3).
7.7. Case of $C_{\text{cut}} \leq o$

Lemma 7.10. We have

\[ g(C_{\text{cut}}) \leq g(o) = g_1 + g_2(o) \leq c_g r_1 + 4.8\sqrt{e} \eta_4 r_{21}, \]
\[ g(C_{\text{cut}} + o) \leq g(2o) = g_1 + g_2(2o) \leq c_g r_1 + 4.8\sqrt{2e} \eta_4 r_{21}. \]

Proof. From Lemmas 7.6 and 7.7, we have $g_1 \leq g_2 r_1$, $g_2(o) = 4.8\sqrt{e} \eta_4 r_{21}$ and

\[ g_2(2o) = 4.8\sqrt{e} \sqrt{\frac{2o}{n}} \sqrt{4 + \log \frac{n}{2o}} \leq 4.8\sqrt{2e} \sqrt{\frac{o}{n}} \sqrt{4 + \log \frac{n}{o}}. \]

\[ = 4.8\sqrt{2e} \eta_4 \sqrt{\frac{o}{n}} \log \frac{n}{o}. \]

\[ \square \]

Lemma 7.11. We have

\[ C_{02} \leq 2(c_g)^2 r_1^2 + 2 \times 4.8^2 e \eta_4^2 r_2 =: \tilde{C}_{02}, \]
\[ C_{b2} \leq 2c_g \sqrt{\sigma^2 \eta_b r_1 r_{22}} + 9.6 \sqrt{2e} \sigma \eta_b r_2 =: \tilde{C}_{b2}, \]
\[ C_2 \leq \lambda_o \sqrt{c}(c_g r_1 + 4.8 \sqrt{e} \eta_4 r_{21}) =: \tilde{C}_2. \]

Proof. From Lemmas 7.10 and 7.1, we see

\[ C_{02} = g(C_{\text{cut}} + o)^2 \leq g(2o)^2 \leq (c_g r_1 + 4.8\sqrt{2e} \eta_4 r_{21})^2 \]
\[ \leq 2 \left\{ (c_g)^2 r_1^2 + 2 \times 4.8^2 e \eta_4^2 r_2^2 \right\} \leq 2(c_g)^2 r_1^2 + 2 \times 4.8^2 e \eta_4^2 r_2, \]
\[ C_{b2} = g(C_{\text{cut}} + o) \sqrt{2\sigma^2 (C_{\text{cut}}/o + 1) C_{b2}} \leq g(2o) \sqrt{4\sigma^2 C_{b2}} \]
\[ \leq (c_g r_1 + 4.8\sqrt{2e} \eta_4 r_{21}) \sqrt{4\sigma^2 \eta_b r_2 \lambda} = 2c_g \sqrt{\sigma^2 \eta_b r_1 r_{22}} + 9.6 \sqrt{2e} \sigma \eta_b r_2, \]
\[ C_2 = \lambda_o \sqrt{C_{\text{cut}} g(C_{\text{cut}})} \leq \lambda_o \sqrt{c} g(o) \leq \lambda_o \sqrt{c}(c_g r_1 + 4.8 \sqrt{e} \eta_4 r_{21}). \]

\[ \square \]

Using the basic inequality (7.8) with the upper bounds obtained in Lemmas 7.1, 7.8 and 7.11 and Proposition 7.1, we can easily obtain the following proposition, which shows the estimation error (7.5) of $E$ from (c4).

Proposition 7.3. We have

\[ \frac{\sigma_1^2}{2} E \leq 2(\tilde{C}_{01} + \tilde{C}_{02}) \nu_E + (\tilde{C}_{b1} + \tilde{C}_{b2}) + \tilde{C}_2 + \tilde{C}_3 + \lambda_s c \sqrt{s}, \]

where $\tilde{C}$s and $\nu_E$ are given in Lemmas 7.1, 7.8 and 7.11 and Proposition 7.1.
7.8. Case of $C_{\text{cut}} > o$

We can obtain an upper bound of $C_{\text{cut}}$ from Proposition 5.1.

**Lemma 7.12.** We have

$$C_{\text{cut}} \leq v_{\text{cut}} = v_{1}^{\text{cut}} + v_{2}^{\text{cut}},$$

where

$$v_{1}^{\text{cut}} = 2C_{r} \left( \frac{\sqrt{2} \sigma^{2}}{\lambda_{o}^{2}} c_{g} r_{1} + \frac{\sqrt{\delta}}{\lambda_{o}} c_{g} r_{1} + \frac{\sqrt{\delta}}{\lambda_{o}} 1.8 \sqrt{c_{\eta} r_{21}} + \sqrt{s c_{\kappa} \lambda_{s}} \right) E,$$

$$v_{2}^{\text{cut}} = C_{r} \frac{1}{\lambda_{o}^{2}} E; \quad C_{r} = 19.2 \sqrt{2} \sigma^{2} C_{r}.$$

**Proof.** We see $g(n - o) \leq g(n) = g_{1} + g_{2}(n) \leq c_{g} r_{1} + 9.6 \sqrt{e}$ and $g(o) \leq c_{g} r_{1} + 4.8 \sqrt{e_{\eta} r_{22}}$ from Lemmas 7.6 and 7.7. From Proposition 5.1, we have

$$\frac{C_{\text{cut}}}{2 C_{r} E} \leq \sqrt{\frac{2 \sigma^{2}}{\lambda_{o}^{2}}} g(n - o) + \frac{\sqrt{\delta}}{\lambda_{o}} g(o) + \sqrt{s c_{\kappa} \lambda_{s}}$$

$$\leq \sqrt{\frac{2 \sigma^{2}}{\lambda_{o}^{2}}} c_{g} r_{1} + \frac{\sqrt{2 \sigma^{2}}}{\lambda_{o}^{2}} 9.6 \sqrt{e} + \frac{\sqrt{\delta}}{\lambda_{o}} c_{g} r_{1} + \frac{\sqrt{\delta}}{\lambda_{o}} 4.8 \sqrt{e_{\eta} r_{22}} + \sqrt{s c_{\kappa} \lambda_{s}}.$$

The proof is complete. \(\square\)

Roughly speaking, $\lambda_{o}^{2} v_{2}^{\text{cut}} / E = C_{v2} = O(1)$, but $\lambda_{o}^{2} v_{1}^{\text{cut}} / E \asymp r_{1} + \sqrt{\delta o} (r_{1} + r_{21}) + \sqrt{s} \lambda_{s}$, which is of order $O(r_{n,d,s,o})$, as shown later. Taking into consideration the difference between these orders, we will evaluate various terms.

Using Lemma 7.12, we evaluate each term of the basic inequality, in a similar manner to the above.

**Lemma 7.13.** We have

$$C_{02} = g(C_{\text{cut}} + o)^{2}$$

$$\leq (c_{g})^{2} r_{1}^{2} + 9.6 c_{g} \sqrt{e_{\eta} r_{1} r_{21}} \left( \frac{v_{\text{cut}}}{o} + 1 + 4.8^{2} c_{\eta} r_{21}^{2} \left( \frac{v_{\text{cut}}}{o} + 1 \right) \right) =: \tilde{C}_{02},$$

$$C_{2} = \lambda_{o} \sqrt{C_{\text{cut}} g(C_{\text{cut}})} \leq \lambda_{o} \left( c_{g} r_{1} \sqrt{v_{\text{cut}}} + 4.8 \sqrt{e_{\eta} r_{21} v_{\text{cut}}^{2}} \right) =: \tilde{C}_{2},$$

$$C_{b2} = \sqrt{2 \sigma^{2} C_{b2}^{2}} \sqrt{\frac{C_{\text{cut}}}{o} + 1} g(C_{\text{cut}} + o)$$

$$\leq \sqrt{2 \sigma^{2} \eta_{d} r_{22}} \left( c_{g} r_{1} \sqrt{\frac{v_{\text{cut}}}{o}} + 1 + 4.8 \sqrt{e_{\eta} r_{21} \left( \frac{v_{\text{cut}}}{o} + 1 \right)} \right) =: \tilde{C}_{b2}.$$
Proof. In this proof, we often use $g_1 \leq c_g r_1$ from Lemma 7.6. We see

$$\sqrt{C_{cut}} g(C_{cut}) = \sqrt{C_{cut}} \{g_1 + g_2(C_{cut})\}$$

$$\leq c_g r_1 \sqrt{C_{cut}} + 4.8 \sqrt{e^{C_{cut}/\sqrt{n}}} \sqrt{4 + \log n}$$

$$\leq c_g r_1 \sqrt{C_{cut}} + 4.8 \sqrt{e^{C_{cut}/\sqrt{n}}} \sqrt{4 + \log \frac{n}{o}}$$

$$= c_g r_1 \sqrt{C_{cut}} + 4.8 \sqrt{e^{C_{cut}/\sqrt{o}}} \sqrt{\log \frac{n}{o}}$$

$$= c_g r_1 \sqrt{C_{cut}} + 4.8 \sqrt{\eta} r_21 \sqrt{C_{cut}/\sqrt{o}}$$

The final formula is a monotone increasing function of $C_{cut}$. We know $C_{cut} \leq \nu_{cut}$ from Lemma 7.12. By replacing $C_{cut}$ by the upper bound $\nu_{cut}$, the second inequality of the lemma is proved. We see

$$g(C_{cut} + o) \leq c_g r_1 + 4.8 \sqrt{e^{C_{cut}/\sqrt{n}}} \sqrt{4 + \log \frac{n}{o}}$$

$$\leq c_g r_1 + 4.8 \sqrt{e^{C_{cut}/\sqrt{o}}} \sqrt{4 + \log \frac{n}{o}}$$

$$= c_g r_1 + 4.8 \sqrt{\eta} r_21 \sqrt{C_{cut}/\sqrt{o}} + 1.$$

Hence,

$$g(C_{cut} + o)^2 \leq (c_g)^2 r_1^2 + 9.6 c_g \sqrt{\eta} r_1 r_21 \sqrt{C_{cut}/\sqrt{o}} + 1 + 4.8^2 \eta r_21^2 \left(\frac{C_{cut}}{\sqrt{o}} + 1\right),$$

$$\sqrt{\frac{C_{cut}}{\sqrt{o}} + 1} g(C_{cut} + o) \leq c_g r_1 \sqrt{\frac{C_{cut}}{\sqrt{o}} + 1} + 4.8 \sqrt{\eta} r_21 \left(\frac{C_{cut}}{\sqrt{o}} + 1\right).$$

Two final formulas are monotone increasing functions of $C_{cut}$. We know $C_{cut} \leq \nu_{cut}$ from Lemma 7.12. By replacing $C_{cut}$ by the upper bound $\nu_{cut}$, the first inequality of the lemma is proved and the third inequality is proved since $C_{b2} = \eta r_{22}$ from Lemma 7.1. \hfill \Box

Here, we focus on two terms related to $r_{21} \nu_{cut}$ in the upper bounds of $C_2$ and $C_{b2}$ in Lemma 7.13. These terms have slower convergence rates than others, as seen later, and hence they are evaluated in a different way from others. Let

$$\tilde{C}^2_2 = \tilde{C}^2_{21} + \tilde{C}^2_{22}, \quad \tilde{C}^2_{b2} = \tilde{C}^2_{b21} + \tilde{C}^2_{b22},$$

(7.12)
where
\[
\begin{align*}
\bar{C}_{21} &= \lambda_0 \left( c_g r_1 \sqrt{v_{cut}} + 4.8 \sqrt{e \eta_4 r_1 \frac{v_{cut}}{\sqrt{o}}} \right), \\
\bar{C}_{22} &= 4.8 \sqrt{e \eta_4 \lambda_0 r_1 \frac{v_{cut}}{\sqrt{o}}}, \\
\bar{C}_{b_{21}} &= \sqrt{2 \sigma^2 \eta_8 r_2} \left( c_g r_1 \frac{v_{cut}}{\sqrt{o}} + 1 + 4.8 \sqrt{e \eta_4 r_1 \frac{v_{cut}}{\sqrt{o}}} + 1 \right), \\
\bar{C}_{b_{22}} &= 4.8 \sqrt{2 e \sigma^2 \eta_8 \eta_4 r_2 \frac{v_{cut}}{\sqrt{o}}}. 
\end{align*}
\]

**Lemma 7.14.** We have
\[
\begin{align*}
\bar{C}_{22} &= 4.8 \sqrt{e \eta_4 \lambda_0 r_1 \frac{v_{cut}}{\sqrt{o}}} \leq 9.6^2 e \eta_4 \frac{C_{\lambda_0}}{C_{\lambda_0}^2 - 1} E, \\
\bar{C}_{b_{22}} &= 4.8 \sqrt{2 e \sigma^2 \eta_8 \eta_4 r_2 \frac{v_{cut}}{\sqrt{o}}} \leq 9.6^2 e \eta_4 \frac{C_{\lambda_0}}{C_{\lambda_0}^2 - 1} E. 
\end{align*}
\]

**Proof.** We see
\[
\begin{align*}
\lambda_0 &\geq C_{\lambda_0} \sqrt{\frac{2 \sigma^2 \log(n/\delta)}{n}} = C_{\lambda_0} \sqrt{2 \sigma^2 \eta_8 \sqrt{\frac{\log n}{n}}} = C_{\lambda_0} \sqrt{2 \sigma^2 \eta_8 \frac{1}{\sqrt{o}}} r_{22}.
\end{align*}
\]
We know \( v_{cut} = C_{v_2} E / \lambda_0^2 \) with \( C_{v_2} = 19.2 \sqrt{2 e \sigma^2 C_r} \) from Lemma 7.12. We also know \( r_{21} \leq r_{22} \) and \( \eta_8 \geq 1 \) from the definition, and \( C_r \leq C_{\lambda_0}^2 / (C_{\lambda_0}^2 - 1) \) from Lemma 7.5. We see
\[
\begin{align*}
\lambda_0 r_1 \frac{v_{cut}}{\sqrt{o}} &= \lambda_0 r_1 \frac{1}{\sqrt{o}} \frac{C_{v_2} E}{\lambda_0^2} = r_1 C_{v_2} \frac{1}{\sqrt{o}} \lambda_0 E \leq \frac{C_{v_2} r_1}{C_{\lambda_0}^2 \eta_8} E \leq \frac{19.2 \sqrt{\sigma} C_r}{C_{\lambda_0}^2 \eta_8} E \leq \frac{19.2 \sqrt{\sigma} C_{\lambda_0} E}{C_{\lambda_0}^2 - 1} E
\end{align*}
\]
and
\[
\begin{align*}
\sqrt{2 \sigma^2 \eta_8 r_2} \frac{v_{cut}}{\sqrt{o}} &= \sqrt{2 \sigma^2 \eta_8 r_2} \frac{1}{\sqrt{o}} \frac{C_{v_2} E}{\lambda_0^2} \leq \sqrt{2 \sigma^2 \eta_8 r_2 \frac{C_{v_2} E}{C_{\lambda_0}^2 (2 \sigma^2) \eta_8^2 r_2}} E \leq \frac{19.2 \sqrt{\sigma} C_r}{C_{\lambda_0}^2 \eta_8} E \leq \frac{19.2 \sqrt{\sigma} \lambda_0 e \sqrt{s}}{C_{\lambda_0}^2 - 1} E.
\end{align*}
\]

**Proposition 7.4.** We have
\[
C_{\geq} E \leq (\bar{C}_{01} + \bar{C}_{02}) \nu E + \bar{C}_{b_1} + \bar{C}_{b_{21}} + \bar{C}_{21} + \bar{C}_{3} + \lambda_0 c_n \sqrt{s}.
\]
Proof. We extract two terms $\bar{C}_{b_{22}}$ and $\bar{C}_{22}$, which have slower convergence rates, from the basic inequality (7.8). From (c4) and $C_{\lambda_o} > 1$, the corresponding L.H.S. of (7.8) is expressed as

$$
\text{LHS}^- = \frac{\alpha^2}{2} E - \bar{C}_{22} - \bar{C}_{b_{22}} \geq \left( \frac{9}{32} - 9.6^2 \eta_4 \frac{C_{\lambda_o}}{C_{\lambda_o}^2 - 1} - 9.6^2 \eta_4 \frac{1}{C_{\lambda_o}^2 - 1} \right) E
$$

$$
\geq \left( \frac{9}{32} - 2 \times 9.6^2 \eta_4 \frac{C_{\lambda_o}}{C_{\lambda_o}^2 - 1} \right) E = C_\geq E.
$$

From the assumption of Theorem 7.1, the coefficient of $E$ is positive. From Proposition 7.1, the corresponding R.H.S. of (7.8) is given by

$$
\text{RHS}^- = 2(\bar{C}_{01} + \bar{C}_{02}) + \bar{C}_b + \bar{C}_2 + \bar{C}_3 + \lambda_s c_\kappa \sqrt{s} - \bar{C}_{22} - \bar{C}_{b_{22}} \\
\leq 2(\bar{C}_{01} + \bar{C}_{01}) + \bar{C}_b + \bar{C}_{b_{21}} + \bar{C}_{21} + \bar{C}_3 + \lambda_s c_\kappa \sqrt{s}.
$$

From $\text{RHS}^- \geq \text{LHS}^-$, the proof is complete. \(\square\)

7.9. Proof of Theorem 7.1

The case $C_{\text{cut}} \leq o$ is proved by Proposition 7.3. The case $C_{\text{cut}} > o$ is proved by Proposition 7.4.

7.10. Proof of Theorem 7.2

First, we rewrite the upper bounds obtained above in the special case where

$$
\lambda_o = C_{\lambda_o} \sqrt{2 \sigma^2 \log(n/\delta) / n}, \quad \lambda_s = \frac{4 \sqrt{2}}{\sqrt{3}} C_{\lambda_o} \lambda_o.
$$

Lemma 7.15.

$$
\lambda_o = C_{\lambda_o} \sqrt{2 \sigma^2 \log(n/\delta) / n} = C_{\lambda_o} \sqrt{2 \sigma^2 \eta_\delta} \frac{1}{\sqrt{o}} r_{22}.
$$

Proof. We have

$$
\lambda_o = C_{\lambda_o} \sqrt{2 \sigma^2 \log(n/\delta) / n} = C_{\lambda_o} \sqrt{2 \sigma^2 \eta_\delta} \sqrt{\log \frac{n}{n}}.
$$

The second equality of the lemma holds from $r_{22} = \sqrt{o} \log n / n$. \(\square\)

Lemma 7.16. We have

$$
\lambda_s = \frac{1}{\sqrt{s}} O(r_{n,d,s,o}).
$$
Proof. We see

$$\lambda_0 C_z = \lambda_0 \sqrt{\frac{3p^2\sigma^2 \log(d/\delta)}{\lambda_0^2 n}} \leq \sqrt{\frac{3p^2\sigma^2 \log d}{n}} \sqrt{1 + \log(1/\delta)} = \frac{1}{\sqrt{s}} O(r_1)$$

We have $g(o) = O(r_1 + r_{21})$ from Lemmas 7.4, 7.6 and 7.7. We know $\lambda_0 \sqrt{o} = O(r_{22})$ from Lemma 7.15 and $r_{22} = O(1)$ from Lemma 7.3. Then we have

$$\lambda_0 \sqrt{o} g(o) = O(r_{22}) O(r_1 + r_{21}) = O(r_{22} r_1 + r_2) = O(r_1 + r_2) = O(r_{n,d,s,o}).$$

Hence,

$$\lambda_s = \frac{4\sqrt{2}}{\sqrt{3}} C_{\lambda_s} \lambda_0 = \frac{4\sqrt{2}}{\sqrt{3}} \lambda_0 \left( C_z + \sqrt{\frac{\sigma g(o)}{s}} \right) = \frac{1}{\sqrt{s}} \frac{1}{\sqrt{s}} O(r_1) + \frac{1}{\sqrt{s}} O(r_{n,d,s,o}) = \frac{1}{\sqrt{s}} O(r_{n,d,s,o}).$$

Lemma 7.17. We have

$$\bar{C}_{01} = O(r_1), \quad \bar{C}_{b1} = O(r_1), \quad \bar{C}_3 = O(r_{n,d,s,o}).$$

Proof. We know $r_1 = O(1)$ from Lemma 7.3. Then we have $\bar{C}_{01} = (1.2c_\kappa)^2 2\rho^2 r_1^2 = O(r_1)$. We see $\bar{C}_{b1} = 2c_\kappa \sqrt{2\sigma^2 \rho} \sqrt{1 + \log(1/\delta)} r_1 = O(r_1)$. From Lemmas 7.15, 7.4 and 7.3, we see

$$\bar{C}_3 = (c_\gamma r_1 + 4.8 \sqrt{c_\eta r_{21}}) \sqrt{o} \lambda_0 = O(r_1 + r_{21}) O(r_{22}) = O(r_1 r_{22} + r_2) = O(r_{n,d,s,o}).$$

Lemma 7.18. We have

$$\nu_E = \frac{6}{C_{n,d}^2} \sqrt{\frac{\lambda_s^2 \kappa}{\kappa^2} + 6.25 \lambda_o^2} = O(r_1 + r_{22}).$$

Proof. We know $1/C_{n,d} = O(1)$ from Lemma 7.2, $\sqrt{o} \lambda_0 = O(r_{22})$ from Lemma 7.15, $\sqrt{s} \lambda_s = O(r_1 + r_{21} r_{22})$ from Lemma 7.16, and $r_{21} = O(1)$ from Lemma 7.3. Hence, by $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$ for $A, B > 0$,

$$\nu_E \leq \frac{6}{C_{n,d}^2} \left( \frac{\lambda_s \sqrt{s}}{\kappa} + \sqrt{6.25 \lambda_o^2} \right) = O(r_1 + r_{22}).$$

Lemma 7.19. We have

$$\bar{C}_{02} = O(r_{n,d,s,o}), \quad \bar{C}_{b2} = O(r_{n,d,s,o}), \quad \bar{C}_2 < O(r_{n,d,s,o}).$$
Proof. We know $\eta_4 = O(1)$ and $\eta_5 = O(1)$ from Lemma 7.4 and $r_1 = O(1)$ and $r_{22} = O(1)$ from Lemma 7.3. We also know $\lambda_o \sqrt{o} = O(r_{22})$ from Lemma 7.15.

Hence,

$$C_{02}^\leq = 2(c_g)^2 r_1^2 + 2 \times 4.8^2 e \eta_4^2 r_2 = O(r_1 + r_2) = O(r_{n,d,s,o}),$$

$$C_{02}^\geq = 2c_g \sqrt{2 \sigma^2 \eta_5 r_{22}} = O(r_1 + r_2) = O(r_{n,d,s,o}),$$

$$C_2^\geq = \lambda_o \sqrt{o}(c_g r_1 + 4.8 \sqrt{e} \eta_4 r_{21}) = O(r_1 r_{22} + r_{21} r_{22}) = O(r_1 + r_2) = O(r_{n,d,s,o}).$$

\[\Box\]

**Proposition 7.5.** In the case $C_{\text{cut}} \leq o$, we have $E = O(r_{n,d,s,o})$.

Proof. Each term of the upper bound of $E$ in (7.3) is shown to be $O(r_{n,d,s,o})$ from Lemmas 7.17, 7.19, 7.18, 7.3 and 7.16. The proof is complete. \[\Box\]

**Lemma 7.20.** We have

$$\tilde{C}_{02}^\geq E = O(r_{n,d,s,o}), \quad \tilde{C}_{02}^\geq = O(r_{n,d,s,o}), \quad \tilde{C}_{21}^\geq = O(r_{n,d,s,o}).$$

Proof. From Lemma 7.14, we know

$$\lambda_o = C_{\lambda_o} \sqrt{2 \sigma^2 \eta_5} \frac{1}{\sqrt{o}} r_{22},$$

and then $\lambda_o = O(r_{22}/\sqrt{o})$ and $1/\lambda_o = O(1)\sqrt{o}/r_{22}$. Hence, from Lemmas 7.16, 7.18 and 7.3,

$$v_1^{\text{cut}} = 2C_r \left( \frac{2 \sigma^2}{\lambda_o^2} c_g r_1 + \frac{\sqrt{o}}{\lambda_o} c_g r_1 + \frac{\sqrt{o}}{\lambda_o} 4.8 \sqrt{e} \eta_4 r_{21} + \sqrt{e} c_o \frac{\lambda_o}{\lambda_0^2} \right) E$$

$$= \left( \frac{o}{r_{22}^2} O(r_1) + \frac{o}{r_{22}} O(r_1 + r_{21}) + O(r_{n,d,s,o}) \frac{o}{r_{22}^2} \right) O(r_1 + r_{22})$$

$$= \frac{o}{r_{22}} O(r_{n,d,s,o}),$$

$$v_2^{\text{cut}} = C_{\alpha_2} \frac{1}{\lambda_0^2} E = \frac{o}{r_{22}} O(r_1 + r_{22}).$$

Then, from $\sqrt{A + B} \leq \sqrt{A} + \sqrt{B}$ for $A, B > 0$,

$$\sqrt{\frac{v_1^{\text{cut}}}{o} + 1} \leq \sqrt{\frac{v_1^{\text{cut}}}{o}} + \sqrt{\frac{v_2^{\text{cut}}}{o}} + 1 = 1 + \frac{1}{r_{22}} O(\sqrt{r_1} + \sqrt{r_{22}}).$$
Hence, from Lemmas 7.4 and 7.3, and \( r_{21} \leq r_{22} \), we see

\[
\tilde{C}_{02} = (\ell_g)^2 \nu_1^2 + 9.6 \ell_g \sqrt{c \eta_4 r_1 r_{21}} \sqrt{\frac{\nu_{cut}}{o}} + 1 + 4.8 \ell_g \sqrt{c \eta_4 r_{21}} \left( \frac{\nu_{cut}}{o} + 1 \right)
\]

\[
= O(\ell_1^2) + O(r_1 r_{21}) + \frac{r_1 r_{21}}{r_{22}} O(\sqrt{\ell_1} + \sqrt{r_{22}}) + \frac{r_{21}^2}{r_{22}} O(r_1 + r_{22})
\]

\[
= O(r_1) + \frac{r_{21}^2}{r_{22}} O(1),
\]

\[
\tilde{C}_{02} \mathbb{E} = \left( O(r_1) + \frac{r_{21}^2}{r_{22}} O(1) \right) O(r_1 + r_{22})
\]

\[
= O(r_1) + O(r_{21}^2) = O(r_1) + O(r_{21} r_{22}) = O(r_{n,d,s,o}),
\]

\[
\tilde{C}_{021} = \sqrt{2 \sigma^2 \eta_4 r_{22} \nu_{cut}} \left( c_g r_1 \sqrt{\frac{\nu_{cut}}{o}} + 1 + 4.8 \sqrt{c \eta_4 r_{21}} \left( \frac{\nu_{cut}}{o} + 1 \right) \right)
\]

\[
= O(r_{22}) \left\{ r_1 \left( 1 + \frac{1}{r_{22}} O(\sqrt{\ell_1} + \sqrt{r_{22}}) \right) + O(r_{21}) \left( 1 + \frac{1}{r_{22}} O(r_{n,d,s,o}) \right) \right\}
\]

\[
= O(r_1 (r_{22} + \sqrt{\ell_1} + \sqrt{r_{22}})) + O(r_{22} r_{21}) + O(r_{n,d,s,o}) = O(r_{n,d,s,o}),
\]

\[
\tilde{C}_{21} = \lambda_0 \left( c_g r_1 \sqrt{\frac{\nu_{cut}}{o}} + 4.8 \sqrt{c \eta_4 r_{21}} \nu_{cut} \sqrt{\frac{1}{o}} \right)
\]

\[
= O \left( \frac{r_{22}}{\sqrt{o}} \right) \left\{ r_1 \frac{\sqrt{o}}{r_{22}} O(\sqrt{\ell_1} + \sqrt{r_{22}}) + r_{21} \frac{\sqrt{o}}{r_{22}} O(r_{n,d,s,o}) \right\}
\]

\[
= O(r_1) + O(r_{n,d,s,o}) = O(r_{n,d,s,o}).
\]

\[
\square
\]

Proposition 7.6. In the case \( C_{cut} > o \), we have \( E = O(r_{n,d,s,o}) \).

Proof. Each term of the upper bound of \( E \) in (7.6) is shown to be \( O(r_{n,d,s,o}) \) from Lemmas 7.17, 7.18, 7.3, 7.16 and 7.20. The proof is complete. \( \square \)

Appendix A: Proof of Proposition 3.4

Here we give the Bernstein concentration inequality.

Theorem A.1 (Bernstein concentration inequality). Let \( \{Z_i\}_{i=1}^n \) be a sequence with i.i.d random variables. We assume that

\[
\sum_{i=1}^n \text{E}[X_i^2] \leq \nu, \sum_{i=1}^n \text{E}[(X_i)^k] \leq \frac{k!}{2} \nu \nu^{k-2}
\]

for \( i = 1, \cdots n \) and for \( k \in \mathbb{N} \) such that \( k \geq 3 \). Then, we have

\[
P \left( \left| \sum_{i=1}^n (X_i - \text{E}[X_i]) \right| \leq \sqrt{2\nu t + ct} \right) \geq 1 - e^{-t}
\]

for any \( t > 0 \).
Using Theorem A.1, we can prove Proposition 3.4, which is given in the following.

**Proposition A.1.** Let \( \{\xi_i\}_{i=1}^n \) be a sequence with i.i.d random variables drawn from \( \mathcal{N}(0, \sigma^2) \) and \( \{X_i\}_{i=1}^n \) drawn from \( \mathcal{N}(0, \Sigma) \). Let \( z_{ij} = X_{ij} \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right) \) and \( z = (\sum_{i=1}^n z_{i1}, \ldots, \sum_{i=1}^n z_{id}) \). For any \( \delta \in (0, 1) \) and \( n \) such that \( \sqrt{\frac{\log(d/\delta)}{n}} \leq \sqrt{3} - \sqrt{2} \), with probability at least \( 1 - \delta \), we have

\[
\left\| \frac{z}{\sqrt{n}} \right\|_\infty \leq \sqrt{3} \frac{\rho^2 \sigma^2}{n \lambda_o^2} \log \frac{d}{\delta} = C_z.
\]

**Proof.** We have \( \mathbb{E}[z_{ij}] = 0 \). Since \( |\psi(t)| \leq |t| \), we see

\[
\sum_{i=1}^n \mathbb{E}[z_{ij}^2] = \sum_{i=1}^n \mathbb{E}[X_{ij}^2] \mathbb{E}\left[ \psi \left( \frac{\xi_i}{\lambda_o \sqrt{n}} \right)^2 \right] \leq \sum_{i=1}^n \mathbb{E}[X_{ij}^2] \mathbb{E}\left[ \frac{\xi_i^2}{\lambda_o^2 n} \right] \leq \frac{\rho^2 \sigma^2}{\lambda_o^2}.
\]

From Proposition 3.2. of Rivasplata (2012), we can show that the absolute \( k(\geq 3) \)th moment of \( z_{ij} \) is bounded above, as follows:

\[
\sum_{i=1}^n \mathbb{E}[|z_{ij}|^k] \leq \sum_{i=1}^n \mathbb{E}[|X_{ij}|^k] \mathbb{E}\left[ \frac{|\xi_i|^k}{\lambda_o^k n^k} \right] \leq k! \frac{\rho^2 \sigma^2}{\lambda_o^2} \left( \frac{\rho^2 \sigma^2}{\lambda_o^2 n} \right)^{\frac{k-2}{2}}.
\]

From Theorem A.1 with \( t = \log(d/\delta) \), \( v = \frac{\rho^2 \sigma^2}{\lambda_o^2} \) and \( c = \sqrt{\frac{\rho^2 \sigma^2}{\lambda_o^2 n}} \), we have

\[
\mathbb{P}\left[ \sum_{i=1}^n z_{ij} \right] \leq \sqrt{2} \frac{\rho^2 \sigma^2}{\lambda_o^2} \log(d/\delta) + \sqrt{\frac{\rho^2 \sigma^2}{\lambda_o^2 n} \log(d/\delta)} \geq 1 - \delta/d.
\]

By the condition \( \sqrt{\frac{\log(d/\delta)}{n}} \leq \sqrt{3} - \sqrt{2} \), the above inequality is

\[
\mathbb{P}\left[ \sum_{i=1}^n z_{ij} \leq \sqrt{nC_z} \right] \geq 1 - \delta/d.
\]

Hence,

\[
\mathbb{P}\left[ \|z\|_\infty \leq \sqrt{nC_z} \right] = \mathbb{P}\left[ \sup_j \left\{ \sum_{i=1}^n |z_{ij}| \leq \sqrt{nC_z} \right\} \right] = 1 - \mathbb{P}\left[ \sup_j \left\{ \sum_{i=1}^n |z_{ij}| > \sqrt{nC_z} \right\} \right]
\]

\[
= 1 - \mathbb{P}\left[ \bigcup_{j=1}^d \left\{ \sum_{i=1}^n |z_{ij}| > \sqrt{nC_z} \right\} \right] \geq 1 - \sum_{j=1}^d \mathbb{P}\left[ \sum_{i=1}^n |z_{ij}| > \sqrt{nC_z} \right]
\]

\[
\geq 1 - (\delta/d) d = 1 - \delta.
\]

\( \square \)
Appendix B: Proof of Lemma 7.9

We give Lemma 7.9 again in the following.

**Lemma B.1** (Modification of Lemma 7 of Dalalyan and Thompson (2019)). Let $Z \in \mathbb{R}^{n \times d}$ be a random matrix satisfying

\[ \left\| \frac{Z}{\sqrt{n}} v \right\|_2 \geq a_1 \| \Sigma^{1/2} v \|_2 - a_2 \| v \|_1 \]  

(B.1)

and

\[ \left| u^T \frac{Z}{\sqrt{n}} v \right| \leq b_1 \| \Sigma^{1/2} v \|_2 \| u \|_2 + b_2 \| c \|_1 \| u \|_2 + b_3 \| \Sigma^{1/2} v \|_2 \| u \|_1 \]  

(B.2)

for some positive constants $a_1 \in (0, 1)$, $a_2$, $b_1$, $b_2$, $b_3$. Then, for any $\alpha > 0$, $Z$ satisfies

\[ \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2 \geq c_1 \left( \| \Sigma^{1/2} v \|_2 + \| u \|_2 \right) - c_2 \| v \|_1 - c_3 \| u \|_1 \]

with the constants $c_1 = \sqrt{a_1^2 + b_1 + \alpha^2 - \sqrt{2(b_1 + \alpha^2)}}$, $c_2 = a_2 + b_2/\alpha$, $c_3 = b_3/\alpha$. If $a_1^2 > b_1 + \alpha^2$, then we have $c_1 > 0$.

**Proof.** From (B.1) and simple calculation,

\[
\sqrt{a_1^2 + b_1 + \alpha^2} \left\{ \| \Sigma^{1/2} v \|_2 + \| u \|_2 \right\}^{1/2} \\
= \left\{ a_1^2 \| \Sigma^{1/2} v \|_2^2 + a_2^2 \| u \|_2^2 + (b_1 + \alpha^2)(\| \Sigma^{1/2} v \|_2^2 + \| u \|_2^2) \right\}^{1/2} \\
\leq \left\{ \left( \left\| \frac{Z}{\sqrt{n}} v \right\|_2 + a_2 \| v \|_1 \right)^2 + a_1^2 \| u \|_2^2 + (b_1 + \alpha^2)(\| \Sigma^{1/2} v \|_2^2 + \| u \|_2^2) \right\}^{1/2} \\
\leq \left\{ \left\| \frac{Z}{\sqrt{n}} v \right\|_2^2 + \| u \|_2^2 + (b_1 + \alpha^2)(\| \Sigma^{1/2} v \|_2^2 + \| u \|_2^2) \right\}^{1/2} + a_2 \| v \|_1. \\
\]

From Young’s inequality, we know $uv \leq (\gamma/2)u^2 + (1/2\gamma)v^2$ for $\gamma > 0$. Using this inequality and B.2, we see

\[
\left\| \frac{Z}{\sqrt{n}} v \right\|_2^2 + \| u \|_2^2 = \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2^2 - 2 u^T \frac{Z}{\sqrt{n}} v \\
\leq \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2^2 + 2b_1 \| \Sigma^{1/2} v \|_2 \| u \|_2 + 2b_2 \| v \|_1 \| u \|_2 + 2b_3 \| \Sigma^{1/2} v \|_2 \| u \|_1 \\
\leq \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2^2 + (b_1 + \alpha^2) \left( \| \Sigma^{1/2} v \|_2^2 + \| u \|_2^2 \right) + \frac{b_2^2}{\alpha^2} \| v \|_1^2 + \frac{b_3^2}{\alpha^2} \| u \|_1^2.
\]
Combining the above two properties,

\[
\sqrt{a_1^2 + b_1 + \alpha^2} \left\{ \frac{\|\Sigma^{1/2}v\|^2}{2} + \|u\|^2 \right\}^{1/2} \\
\leq \left\{ \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2^2 + 2(b_1 + \alpha^2)(\|\Sigma^{1/2}v\|^2 + \|u\|^2) + \frac{b_2^2}{\alpha^2} \|v\|_1^2 + \frac{b_3^2}{\alpha^2} \|u\|_1^2 \right\}^{1/2} + a_2 \|v\|_1 \\
\leq \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2 + \sqrt{2(b_1 + \alpha^2)} \left\{ \frac{\|\Sigma^{1/2}v\|^2}{2} + \|u\|^2 \right\}^{1/2} + \frac{b_2}{\alpha} \|v\|_1 + \frac{b_3}{\alpha} \|u\|_1 + a_2 \|v\|_1
\]

Rearranging the terms,

\[
\left( \sqrt{a_1^2 + b_1 + \alpha^2} - \sqrt{2(b_1 + \alpha^2)} \right) \left\{ \frac{\|\Sigma^{1/2}v\|^2}{2} + \|u\|^2 \right\}^{1/2} \\
\leq \left\| \frac{Z}{\sqrt{n}} v + u \right\|_2 + \left( \frac{b_2}{\alpha} + a_2 \right) \|v\|_1 + \frac{b_3}{\alpha} \|u\|_1.
\]

The condition \(a_1^2 > b_1 + \alpha^2\) implies \(c_1 > 0\).

\[\Box\]

**Appendix C: Condition (7.2)**

We investigate the condition (7.2) in detail. We assume the conditions used in Theorem 7.2. As seen in Lemma 7.2, the R.H.S. of (7.2), \(C_{n, \delta}\), is bounded above 0. We will show that the L.H.S. of (7.2) can be sufficiently small under some conditions, so that the condition (7.2) is satisfied.

Let

\[
A_1 = \frac{\log d}{n}, \quad A_2 = \frac{\lambda^2}{\lambda_o} n, \quad B_1 = s, \quad B_2 = o\frac{\lambda^2}{\lambda_o}.
\]

The L.H.S. of (7.2) is bounded up to constant by the square root of \(A_1B_1 + A_1B_2 + A_2B_1 + A_2B_2\). Hereafter, we evaluate each term. We see

\[
A_1B_1 = s^2 \frac{\log d}{n} = v_1^2, \quad A_1B_2 = o\frac{\log n}{n} = v_2^2.
\]

From Lemmas 7.15 and 7.16 and \(\eta_o \geq 1\),

\[
A_2B_1 = s^2 \frac{\lambda^2}{\lambda_o} n = s \left\{ \frac{1}{\eta o \sqrt{22/\sqrt{s}}} \frac{O(r_{n,d,s,o})}{\sqrt{s}} \right\}^2 \frac{\log n}{n} = O(v_{n,d,s,o}^2).
\]

Here, we see

\[
\frac{\lambda}{\lambda_o} = \frac{4\sqrt{2}}{\sqrt{3}} C_{\lambda_o} \geq \frac{4\sqrt{2}}{\sqrt{3}} C_o = \frac{4\sqrt{2}}{\sqrt{3}} \frac{3\rho^2\sigma^2 \log(d/\delta)}{\lambda^2 n} \\
= \frac{4\sqrt{2}}{\sqrt{3}} C_{\lambda_o} \frac{3\rho^2\sigma^2 \log(d/\delta)}{\lambda^2} \geq \frac{2\sqrt{\sigma} \rho^2 \log(d/\delta)}{C_{\lambda_o} \sigma^2 \log(n/\delta)}.
\]
Hence,

\[
A_1 B_2 = \frac{\log d}{n} \frac{\lambda_2}{\lambda_3} = \frac{C_2^2}{2 \sqrt{6} \rho^2} \frac{n \log(n/\delta)}{\log(n/\delta)} \leq \frac{C_2^2}{2 \sqrt{6} \rho^2} \frac{n \log d}{\log(d/\delta)} = O(r_{22}^2),
\]

because \( \eta_0 \) is bounded above from Lemma 7.4 and \( \log d / \log(d/\delta) \leq 1 \) from \( \delta \in (0, 1/7) \). Therefore, if \( r_1 \) and \( r_{22} \) are sufficiently small, then the L.H.S. of (7.2) is sufficiently small, so that (7.2) is satisfied.

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