Systematic calculation of fine structure in the $\alpha$ decay of deformed nuclei

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Abstract. We perform an extensive investigation on the $\alpha$ decay of well-deformed even-even nuclei by using the coupled-channel Schrödinger equation with outgoing wave boundary conditions. The internal effect of daughter states is taken into account in dealing with the interaction matrix and the $\alpha$-cluster formation. In contrast to the traditional $\alpha$-decay theories, the five-channel microscopic calculations of this work well reproduce the available experimental data concerning $\alpha$-decay half-lives and fine structures. Some predictions on the fine structure are made for superheavy nuclei. Moreover, the sensitivity of the results to the model quantities is discussed in detail.

1. Introduction

Alpha decay was observed by Rutherford about a century ago and interpreted as a quantum tunneling effect of preformed $\alpha$ particles independently by Gamow [1] and by Condon and Gurney [2] in 1928. It was the first successful application of quantum mechanics to a nuclear physics problem. More importantly, it proved the validity of quantum mechanics for nuclei and ushered in a new era in nuclear physics. Experimentally, an experimental combination of $\alpha$-decay and $\gamma$ emission can greatly help the spectroscopic study of neutron-deficient nuclei [3, 4]. Identification and knowledge of new synthesized elements and nuclides wholly or mainly resort to observing $\alpha$-decay chains [5, 6, 7].

In the spherical system, favored $\alpha$ transitions are always preferred during the decay. Since the pioneering work of Gamow, lots of effort have been devoted to pursue a quantitative interpretation of such a transition [8, 9, 10, 11, 12, 13, 14, 15]. The exact quantum mechanics treatment of the decay width is also made [16, 17], where one only needs to solve a single radial Schrödinger equation. In the deformed system, the ground and low-lying excited states in the daughter nucleus are closely distributed so that they are all accessible to $\alpha$ transitions. This is called fine structure in $\alpha$ decay. For even-even nuclei, the decay generally ends up in various members of the ground-state rotational band in the daughter nucleus. Furthermore, there is significant mixing of these decay channels during the tunneling. This poses a tough test of traditional $\alpha$-decay theories and a stringent challenge of experimental studies. To evaluate the decay width, some semiclassical methods are extended to a deformed case in a phenomenological manner, such as the density-dependent cluster model (DDCM) [18] and the unified model for $\alpha$ decay and $\alpha$ capture (UMADAC) [19]. Additionally, by introducing the non-zero angular
momentum, the attempt to describe the fine structure in $\alpha$-decay has been achieved for even-even nuclei recently [20, 21, 22]. Nevertheless, considering the nature of $\alpha$ decay, i.e., the three-dimensional quantum tunneling effect, the more consistent analysis is the coupled-channel approach [23, 24, 25, 26]. Such a study is less documented than the semiclassical calculation. The objective of this article is to extend the previous presentation of our coupled-channels study made in [27] with a comprehensive analysis and perspective. The weak dependence of the results on the model parameters is shown, confirming the good reliability of our theoretical approach. As shown below, the available experimental data concerning total $\alpha$-decay half-lives and branching ratios (b.r.) for various daughter states can be well reproduced in a straightforward and consistent manner.

2. Theoretical framework for deformed $\alpha$ decays

The picture we consider here is that of a system consisting of a spherical $\alpha$ cluster coupled to an axial-symmetric and well-deformed daughter nucleus. The total wave function of the system of the channel and radial components:

$$\Psi_{JM} = \phi(\alpha)r^{-1}\sum_{\ell I} u_{n\ell I}(r) Y_{\ell I}(\mathbf{r}) \otimes \Phi_I]_{JM}. \quad (1)$$

where $\phi(\alpha)$ is the internal wave function of the $\alpha$ particle, $u_\alpha^{\ell I} [\alpha \equiv (n\ell I) labels the channel quantum numbers] is the cluster radial function, $Y_{\ell I}(\mathbf{r})$ is the orbital wave function of the $\alpha$ particle, and $\Phi_I$ is the wave function of the daughter nucleus characterizing the rotation of the core with excitation energies, $H_\ell \Phi_I = E_I \Phi_I$.

After inserting (1) into the Schrödinger equation and projecting the equation onto the angular part, one obtains the coupled-channel equations for the radial components

$$\left[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - \frac{\ell_\alpha(\ell_\alpha + 1)}{r^2} \right) - (Q_0 - E_I) \right] u_\alpha(r) + \sum_{\alpha'} V_{\alpha,\alpha'}(r) u_{\alpha'}(r) = 0. \quad (2)$$

In this equation, $Q_0$ is the $Q_\alpha$ value for the decay to the ground state, and the quantity $V_{\alpha,\alpha'}(r)$ is the matrix element of the interaction $V$ taken between channels $\alpha$ and $\alpha'$. The deformed potential $V$ consists of the attractive nuclear and repulsive Coulomb potentials. The nuclear potential between the $\alpha$ cluster and the deformed core nucleus has a simple axially deformed Woods-Saxon (WS) form [27]. The Coulomb potential of the $\alpha$ cluster with the core is approximated to the first order in $\sum_\ell \beta_\lambda Y_{\lambda 0}(\theta)$ [26, 28].

In order to evaluate the interaction matrix elements, we need to make a multipole expansion of the potential $V$ [23, 24, 25], $V = \sum_{\lambda=0}^{\lambda_{\text{max}}} v_\lambda(r)(\Omega_\lambda \otimes Y_\lambda)_{00}$, where only even values of $\lambda$ appear in the summation owing to the axial symmetry. Taking into account the dynamics of the rotational core nuclei, one evaluates the coupling potential between channels $\alpha$ and $\alpha'$ [27, 29, 30]:

$$V_{\alpha,\alpha'}(r) = \sum_\lambda v_\lambda(r) \frac{(-1)^\lambda}{4\pi} (2\lambda + 1) \sqrt{(2\ell + 1)(2\ell' + 1)} \langle I' \lambda K 0 | I K \rangle W(\ell' \lambda J; \ell I), \quad (3)$$

where $\langle abc| \beta \gamma \rangle$ is the Clebsch-Gordan coefficient and $W(ab; cd)\langle ef \rangle$ the Racah coefficient.

The coupled equations (2) are solved with the following boundary conditions. Each solution should behave as outgoing Coulomb-Hankel waves at large distances and should be regular at origin. Moreover, the eigencharacteristic of each solution should follow the Wildermuth condition [31], $G = 2n + \ell = \sum_{i=1}^{4} g_i$. After solving the equations (2), one can express the partial width of the channel $\ell I$ as [25, 26, 27]

$$\Gamma_{\ell I}(R) = \frac{\hbar^2 k_I}{\mu} \frac{|u_{n\ell I}(R)|^2}{G_{\ell}(k_I R)^2 + F_{\ell}(k_I R)^2}. \quad (4)$$
Table 1. Branching ratios (in units of 1%) for the $^{244}$Cm ground-state decay to the $0^+$, $2^+$, $4^+$, $6^+$, and $8^+$ states of $^{240}$Pu, together with total $\alpha$-decay half-lives (in units of second). The calculations are performed with different expansion order $\lambda_{\text{max}}$.

| $\lambda_{\text{max}}$ | b.r.$(0^+)$ | b.r.$(2^+)$ | b.r.$(4^+)$ | b.r.$(6^+)$ | b.r.$(8^+)$ | $T_{1/2}^{\text{cal}}$ |
|-----------------|----------|----------|----------|----------|----------|---------------|
| 6               | 69.15    | 30.82    | 0.019    | 0.0062   | 1.98$\times 10^{-5}$ | 5.27$\times 10^8$ |
| 8               | 70.17    | 29.78    | 0.041    | 0.0071   | 2.64$\times 10^{-5}$ | 5.63$\times 10^8$ |
| 10              | 71.43    | 28.51    | 0.049    | 0.0078   | 2.55$\times 10^{-5}$ | 5.69$\times 10^8$ |
| 12              | 71.34    | 28.60    | 0.048    | 0.0073   | 2.75$\times 10^{-5}$ | 5.68$\times 10^8$ |
| 14              | 71.27    | 28.68    | 0.046    | 0.0072   | 2.74$\times 10^{-5}$ | 5.64$\times 10^8$ |
| 16              | 71.31    | 28.63    | 0.048    | 0.0073   | 2.75$\times 10^{-5}$ | 5.66$\times 10^8$ |

where $R$ denotes large distances beyond the range of the nuclear potential and beyond the distance where the Coulomb potential can be regarded as spherically symmetric. It should be particularly noted that the expression of $\Gamma_{\text{II}}(R)$ is rather insensitive to the choice of $R$. This provides a stringent test of the reliability of the exact formalism presented here. If $\Gamma_{\text{II}}(R)$ has a clear dependence on $R$, then the theory is incorrect.

Ultimately, the structure part of $\alpha$ decay can be evaluated by a constant $\alpha$-preformation factor $P_\alpha$, together with the hypothesis of the Boltzmann distribution (BD) for daughter states, $\rho(E_f) = \exp(-cE_f)$. The motivation of this is clearly shown in [17, 25, 26, 27]. For additional details on this subject, see [32, 33, 34, 35, 36, 37, 38]. It has been shown in [27] that the experimental $\alpha$-decay rates can be well reproduced with the same factor $P_\alpha = 0.36$ and the same BD $c = 2.38$ MeV$^{-1}$.

3. Numerical results and discussion

3.1. Sensitivity of results to the model quantities used in the calculations

It is interesting to discern the sensitivity of the results to the model quantities such as the expansion order $\lambda_{\text{max}}$ of the interaction potential. We present in table 1 the detailed results for the $\alpha$ decay of $^{244}$Cm. The calculations are performed by expanding the potential $V$ in spherical multipoles to different orders, and the results are listed according to the $\lambda_{\text{max}}$ values. In each case, the excitation spectrum and deformation parameters of daughter nuclei, together with the decay $Q_0$ value, remain the same. As can be seen, the expansion order $\lambda_{\text{max}}$ has a value of more than 8, yielding the approximately same results. So it is sufficient to take $\lambda_{\text{max}} = 10$ for proper convergence. Nevertheless, for the sake of good stability, the potential $V$ is expanded in multipoles to order 12 in all the calculations.

Another effects on the results arise from the details of the WS potential such as the radius $r_0$ and diffuseness $a$. Taking the $\alpha$ decay of $^{244}$Cm for example, as the radius $r_0$ is changed from the used value 1.24 fm to the large value of 1.34 fm, the half-life is decreased by about one order of magnitude. In a similar way, as the diffuseness $a$ is changed from the used value 0.62 fm to 0.75 fm, the half-life is decreased by a factor of about 4 to 5. This dependence of $\alpha$-decay half-lives on the potential parameters is different from that of proton-emission half-lives, where the effect of the potential parameters is less than a factor of about 3 [39]. This can be easily understood in terms of the semiclassical Wentzel-Kramers-Brillouin (WKB) approximation. In the case of proton emission, the centrifugal barrier plays an important role owing to the small proton-daughter reduced mass and also because in most cases the emitted proton carries a non-zero angular momentum. And the decay energy of proton emission is generally small. These cause an extensive integrating range in the WKB exponent so that the main contribution to the exponent comes from the Coulomb-plus-centrifugal potential rather than the short-range...
nuclear potential. It is therefore not so surprising to see the weak dependence of the half-life on the nuclear potential parameters. On the contrary, the role of the centrifugal barrier is rather weaker in the $\alpha$ decay of even-even nuclei and the $\alpha$-decay energy is relatively large especially for deformed emitters. So the effect of the nuclear potential on the WKB exponent is evident to some extent, leading to the slightly strong dependence of the half-life on the nuclear potential. For the b.r., reasonable variations in the potential parameters result in minor changes.

3.2. Systematic study of well-deformed even-even emitters
Systematic five-channel calculations have been performed for 35 even-even emitters with $92 \leq Z \leq 106$. First, we restrict our attention to the calculations of $\alpha$-decay half-lives. Figure 1 shows the deviations of the calculated $\alpha$-decay half-lives from the experimental data as a function of the mass number $A_p$ of the parent nuclei. As can be seen, all the values of $\log_{10}(T_{cal}/T_{exp})$ are within the range of $-0.4$ to $0.4$, which corresponds to the values of the ratio $T_{cal}/T_{exp}$ within the range of $0.4$ to $2.5$. In spite of large deformation involved, the calculated $\alpha$-decay half-lives are in excellent agreement with the experimental data.

![Figure 1](image.png)

**Figure 1.** Deviations of the calculated half-lives from the experimental values versus the mass number $A_p$ of the parent nuclei with $Z = 92–106$.

Next, we transfer our attention from the total $\alpha$-decay half-lives to the b.r.. For the b.r. to $0^+$ and $2^+$ states, there exists satisfactory agreement in both the systematic behavior of the various isotopic chains and the magnitude of b.r.. The present calculations well reproduce the available experimental b.r. to $0^+$ and $2^+$ states with mean factors of 1.1 and 1.3, respectively.

In contrast to the semiclassical WKB calculations which generally substantially overestimate the b.r. to excited $4^+$ states by more than one order of magnitude [20, 21, 22], the present calculations can give a precise description of the b.r. to excited $4^+$ states. The comparison of the calculated b.r. to excited $4^+$ states with the available experimental data is shown in figure 2, where each isotopic chain is sorted by the mass number $A_p$ of the parent nuclei, smallest to largest. To aid the eye, consecutive isotopes of a given element are connected with a line segment. Empty symbols denote the experimental data and solid points stand for the theoretical results. Note that the x-axis of figure 2 is the serial number of the decay event, from 1 to 35. Although the b.r. to excited $4^+$ states show a high sensitivity to the nuclear structure properties, one can see that, the theoretical b.r. follow the experimental ones well. In a similar fashion, figure 3 illustrates the b.r. to excited $6^+$ states, and the good description of the b.r. to excited $6^+$ states is also seen. The largest deviation of our analysis emerges at the emitter $^{240}\text{Cm}$, for which the b.r. to the excited $4^+$ and $6^+$ states are predicted as 0.437% and 0.0012% but the experimental values are known as 0.052% and 0.014%. New measurements of these b.r. with improved accuracy would be a good way to test the validity of the present study. For the transitions to excited $8^+$ states, there are quite little data in experiments. Among 35 $\alpha$-decay
Table 2. The predicted branching ratios (b.r.) for various daughter states (in units of 1%) and theoretical $\alpha$-decay half-lives (in seconds).

| Emitter | b.r.$(0^+)$ | b.r.$(2^+)$ | b.r.$(4^+)$ | b.r.$(6^+)$ | b.r.$(8^+)$ | $T_{\text{exp}}^{1/2}$ | $T_{\text{cal}}^{1/2}$ |
|---------|------------|------------|------------|------------|------------|----------------|----------------|
| $^{252}$No | 74.19 | 23.95 | 1.81 | 0.0465 | 8.98$\times$10$^{-4}$ | 3.91$\times$10$^0$ | 3.22$\times$10$^0$ |
| $^{254}$No | 76.53 | 21.60 | 1.84 | 0.0258 | 8.21$\times$10$^{-4}$ | 5.67$\times$10$^1$ | 3.08$\times$10$^1$ |
| $^{256}$Rf | 75.22 | 23.46 | 1.30 | 0.0116 | 1.43$\times$10$^{-3}$ | 2.00$\times$10$^0$ | 8.56$\times$10$^{-1}$ |
| $^{258}$Rf | 70.51 | 27.64 | 1.83 | 0.0206 | 4.29$\times$10$^{-5}$ | 2.00$\times$10$^0$ | 8.56$\times$10$^{-1}$ |

On the whole, there exists good agreement between experiment and theory for the fine structure. This gives us some guaranty for reliable predictions of $\alpha$-decay properties. In table 2, predictions on the fine structure are made for the $\alpha$ decay of $^{252,254}$No and $^{256,258}$Rf. It would be of particular interest to compare the theoretical predictions with future measurements with these emitters.

4. Summary
In summary, we have presented in this paper an extensive study of the fine structure in the $\alpha$ decay of well-deformed even-even nuclei. The coupled-channel Schrödinger equation with outgoing wave boundary conditions is employed to calculate $\alpha$-decay half-lives and branching ratios. The internal effect of nuclear states in the daughter nucleus is taken into account by including the dynamics of the core in the interaction matrix and by introducing the BD hypothesis of daughter states for the $\alpha$-cluster formation. We also perform a detailed analysis of the dependence of the results on the model parameters such as the expansion order and the WS parameters, showing the good stability of the present calculations. In contrast to the traditional semiclassical approximation which usually overrates the branching ratio to highly excited states by one order of magnitude, the present calculations can give a robust description concerning both the $\alpha$-decay half-life and the fine structure observed in $\alpha$ decay. Some predictions are made
for some superheavy nuclei where the branching ratios to various daughter states are still not measured.

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