Scalar-Tensor-Vector Gravity, Galaxy Rotation Curves, and Quadrupole Gravitational Polarization

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Abstract
The standard cold dark matter model with a cosmological constant is the best fit to cosmological observations and to galaxy rotation curves. However, unless there is a direct detection of dark matter, either in the laboratory or in astronomical observations, one should allow for modified gravity theories such as MOND or Scalar-Tensor-Vector Gravity [STVG] as possible explanations for flatness of galaxy rotation curves. The STVG theory due to Moffat and collaborators modifies general relativity by the addition of a massive vector field, and the vector field coupling constant, its mass, and the gravitational constant, are dynamical scalar fields. The theory is shown to yield a modified acceleration law which has a repulsive Yukawa component added to the Newtonian law of gravitational acceleration, and which can explain the observed flatness of a large class of galaxy rotation curves by fixing the values of two free parameters [a mass scale and a length scale]. In the present paper we provide a possible insight into the success of the STVG theory, by staying within general relativity, and by considering the effect of polarization [quadrupole approximation] on the averaged gravitational field inside a galaxy, due to the pull of neighboring galaxies. This effect is analogous to the polarization induced modification of averaged electromagnetic fields in an electrically charged macroscopic medium, and was studied by Szekeres in the context of propagation of gravitational waves. The study was generalized to the case of a static weak-field approximation by Zalaletdinov and collaborators, who showed that the effect of quadrupole polarization is to modify Poisson's equation for the gravitational potential to a fourth order [biharmonic] equation. We show that, remarkably, the biharmonic equation implies a modification of Newtonian acceleration which is precisely of the same repulsive Yukawa form as in the STVG theory, and the corrections could in principle be large enough to explain flatness of the rotation curves.

1 Introduction
Dark matter is the most well accepted explanation for the flatness of galaxy rotation curves, and it also fits very well into the standard ΛCDM paradigm for cosmology. However, unless there is a direct detection of one or more popular dark matter candidates in the laboratory, or via astronomical observations, alternate possible explanations should not be excluded. These alternatives include modified theories of gravity such as MOND (Modified Newtonian Dynamics) and the Scalar-Tensor-Vector Gravity theory developed by Moffat and collaborators [1], [2], [3]. It is the latter which we will be concerned with, in the present paper. STVG is a generalization of general relativity which includes a massive vector field, and also three dynamical scalar fields G(x), μ(x) and ω(x), which are respectively the gravitational ‘constant’, the mass of the vector field, and its coupling to gravity. It has been shown that STVG possesses a modified law of acceleration which can explain flat rotation curves for a large class of galaxies after fixing two parameters, a mass scale and a length scale, to universal values. The choice of the values of these parameters is dictated by phenomenology, and not determined by theory.

Here we report a modified acceleration law which is essentially identical to that in STVG, but which comes from an entirely different line of reasoning. As motivation, we begin by stating an analogy with electrodynamics. If one considers a microscopic [corpuscular] medium of molecules in an external electromagnetic field, averaging of the Maxwell equations for the microscopic system results in the inclusion of polarization terms coming from the induction of moments on individual molecules. This is of course well known from electrodynamics. In an elegant analysis motivated by the study of propagation
of gravitational waves in a material medium, Szekeres [4] developed an analogous description for general
relativity, via averaging of Bianchi identities. A modification results from quadrupole moments induced
on ‘molecules’ which in the present context are galaxies, whose ‘atoms’ are stars. Zalaletdinov and
collaborators [5], [6] extended this analysis to the static weak-field Newtonian approximation, and showed
that the effect of quadrupole polarization is to modify Poisson’s equation to a fourth order biharmonic
equation for the gravitation potential. In the present paper we obtain the modified acceleration law
implied by this modification of Poisson’s equation, and point out its significance for the observed galaxy
rotation curves.

The fact that our present analysis gives the same acceleration law as STVG is all the more intriguing
because we were not looking for an explanation of STVG. Rather, it was a subsequent realization that
the two laws are the same. Nonetheless, our presentation here emphasizes also the similarity between
the two results - the commonality in spite of apparently different origins suggests the possibility that
this may be a genuine effect, where the quadrupole induced modification can be mimicked by a field
theoretic [fifth force] modification of general relativity. It should be emphasized that at this stage our
analysis is intended to be illustrative, so as to bring out the physics involved, rather than to dwell into
precise fitting of galaxy rotation curves.

The plan of the paper is as follows. In Sections 2 and 3, we recall the acceleration law of STVG,
and the derivation of the biharmonic equation for the gravitational potential. In Section 4 we obtain the
acceleration law which results from this biharmonic equation and in Section 5 we compare it with the
result from STVG. Section 6 briefly discusses the theoretical determination of the free parameters, ad
Conclusions are given in Section 7.

2 Galaxy Rotation Curves in the STVG Theory

Here we briefly recall the form of the acceleration law, as derived in STVG theory. Details can be found
in [1], [2] and [3].

In the STVG theory, the action is given by

\[ S = S_{\text{Grav}} + S_\phi + S_S + S_M, \]

where

\[ S_{\text{Grav}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \frac{1}{2} G(R + 2\Lambda) \right], \]

\[ S_\phi = -\int d^4x \sqrt{-g} \left[ \omega \left( \frac{1}{4} B_{\mu\nu} B_{\mu\nu} + V(\phi) \right) \right], \]

and

\[ S_S = \int d^4x \sqrt{-g} \left[ \frac{1}{2G} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu G \nabla_\nu G - V(G) \right) \right. \]

\[ \left. + \frac{1}{G} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu \omega \nabla_\nu \omega - V(\omega) \right) + \frac{\mu^2 G}{1} \left( \frac{1}{2} g^{\mu\nu} \nabla_\mu \mu \nabla_\nu \mu - V(\mu) \right) \right]. \]

Here \( V(\phi) \) is the potential for the massive vector field \( \phi^\mu \) having mass \( \mu \) and coupling \( \omega; \)
\( V(G), V(\omega) \) and \( V(\mu) \) are the potentials associated with the three scalar fields \( G(x), \omega(x) \) and \( \mu(x) \), respectively, and

\[ B_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu. \]

By assuming that \( \Lambda = 0, \ V(\phi) = 0, \ \omega \ \text{ad} \ \mu \ \text{to be constants, and by considering the motion of a test}
particle coupled to gravity and to the vector field, it was shown that the law of acceleration is given by

\[ a(r) = -\frac{G_M}{r^2} + K \frac{\exp(-r/\rho)}{r^2} \left( 1 + \frac{r}{\rho_0} \right), \]

where \( G_M \) is defined to be the effective gravitational constant at infinity

\[ G_M = G \left( 1 + \sqrt{\frac{M_0}{M}} \right) \]
and \( r_0 = 1/\mu \). Here, \( M_0 \) denotes a parameter that vanishes when \( \omega = 0 \) and \( G \) is Newton’s gravitational constant. The constant \( K \) is assumed to equal

\[
K = G\sqrt{M M_0}.
\]

The choice of \( K \), which determines the strength of the coupling of \( B_{\mu\nu} \) to matter and the magnitude of the Yukawa force modification of weak Newtonian gravity, is based on phenomenology and \( M_0 \) is a free parameter of the theory.

By using (7), one can rewrite the acceleration in the form

\[
a(r) = -\frac{GM}{r^2} \left\{ \frac{1}{1 + \frac{M_0}{M} \left[ 1 - \exp(-r/r_0) \left( 1 + \frac{r}{r_0} \right) \right]} \right\}.
\]

It is assumed [and justified by further considerations] that one can generalize this to the case of a mass distribution by replacing the factor \( GM/r^2 \) in (9) by \( GM(r)/r^2 \). The rotational velocity of a star \( v_c \) is obtained from

\[
v_c = \sqrt{GM(r)/r} \left\{ 1 + \frac{M_0}{M} \left[ 1 - \exp(-r/r_0) \left( 1 + \frac{r}{r_0} \right) \right] \right\}^{1/2}.
\]

A good fit to a large number of galaxies has been achieved with the parameters:

\[
M_0 = 9.60 \times 10^{11} M_\odot, \quad r_0 = 13.92 \text{kpc} = 4.30 \times 10^{22} \text{cm}.
\]

In the fitting of the galaxy rotation curves for both LSB and HSB galaxies, using photometric data to determine the mass distribution \( M(r) \), only the mass-to-light ratio \( \langle M/L \rangle \) is employed, once the values of \( M_0 \) and \( r_0 \) are fixed universally for all LSB and HSB galaxies. Dwarf galaxies are also fitted with the parameters

\[
M_0 = 2.40 \times 10^{11} M_\odot, \quad r_0 = 6.96 \text{kpc} = 2.15 \times 10^{22} \text{cm}.
\]

We will now see how a modified acceleration law of the form Eqn. (9) is obtained without modifying general relativity, as a result of considering polarization effects on the averaged gravitational field. As shown in the work of Szekeres and Zalaletdinov et al., which we recall in this next section, such a consideration leads to the Poisson equation being modified into a fourth order biharmonic equation. In the subsequent section, we derive the modified acceleration law by solving the biharmonic equation.

### 3 Quadrupole gravitational polarization and the biharmonic equation for the potential

This section summarizes the work of Szekeres [4] and Zalaletdinov et al. [5], [6] leading to the derivation of the biharmonic equation.

Thinking of galaxies as ‘molecules’ made of atoms [the stars], one would like to analyze how the averaged gravitational field inside a galaxy, is modified by the polarization of the molecules, due to the external pull of other galaxies. Indeed one has at the back of the mind the polarization induced modification of electromagnetic fields in an electrically charged medium.

In vacuum, if the Riemann tensor \( R_{\mu\nu\rho\sigma} \) is regarded as being the gravitational analog of the Maxwell tensor \( F_{\mu\nu} \), the Bianchi identities

\[
R_{\mu\nu\rho\sigma} \rightarrow 0 \leftrightarrow R_{\mu\nu\rho\sigma} = 0
\]

show striking resemblance to the “source-free” Maxwell equations

\[
F_{\mu\nu,\rho} = 0, \quad F_{\mu\nu} = 0.
\]

In the non-vacuum case, the Weyl tensor

\[
C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu\rho} R_{\nu\sigma} - R_{\mu[\rho} g_{\sigma]\nu} + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu}\]

(15)
is the gravitational analog of the Maxwell tensor. The Bianchi identities now take the form,
\[ C_{\mu\nu\rho\sigma}^\ast = \kappa J_{\mu\nu}, \]
where
\[ J_{\mu\nu} = J_{[\mu|\nu]} = -(T_{\rho[\mu;\nu]} - \frac{1}{3}g_{\rho\mu}T_{\nu}). \]
Eqn. (16) is comparable to the Maxwell equations
\[ F_{\mu\nu} = \mu_0 j_{\mu}, \]
if \( C_{\mu\nu\rho\sigma} \) is interpreted as representing the free gravitational field and \( J_{\mu\nu} \) is a kind of matter current. And the corresponding conservation equation is:
\[ J_{\mu\nu} = 0. \]

The gravitational field created by a number of particles represented by a microscopic energy-momentum tensor \( T^{a(micro)}_{\mu\nu} \) is given by Einstein’s equations
\[ g^{\alpha\epsilon}r_{\epsilon\beta} = -\frac{1}{2}\delta_\beta^\alpha \frac{g^{\mu\nu}r_{\mu\nu}}{-\kappa\epsilon_\beta^{(micro)}} \]
where \( \kappa = 8\pi G/c^4 \). The Einstein equations for the microscopic distribution of gravitational molecules
\[ f^{(molecule)}_{\mu\nu}(x) = e^{-1}\Sigma_a \sum_i m_a \frac{d\tau_a}{dt_i} \frac{dz_1}{dz_2} \delta^4(x - y_a(\tau))d\tau_a \]
are then :
\[ g^{\alpha\epsilon}r_{\epsilon\beta} = -\frac{1}{2}\delta_\beta^\alpha \frac{g^{\mu\nu}r_{\mu\nu}}{-\kappa\epsilon_\beta^{(molecule)}}. \]

Averaging the left-hand side of the Einstein equations using Kaufmann’s method of molecular moments and by taking into account the expression \( [4] \) for the tensor of gravitational quadrupole polarization \( Q_{\mu\nu\rho\sigma} \) in terms of the covariant gravitational quadrupole moment \( q_{\mu\nu\rho\sigma} \)
\[ Q_{\mu\nu\rho\sigma} = e^{-1}\Sigma_a \int_{\Sigma_a} q_{\mu\nu\rho\sigma} \delta^4(x - y_a)d\tau_a \]
brings the averaged Einstein equations in the form:
\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa(T_{\mu\nu}^{(free)} + T_{\mu\nu}^{(GW)} + \frac{1}{2}c^2 Q_{\mu\nu\rho\sigma}), \]
where \( T_{\mu\nu}^{(GW)} \) is Isaacson’s energy-momentum tensor of gravitational waves and \( T_{\mu\nu}^{(free)} \) is the energy-momentum tensor of molecules
\[ T_{\mu\nu}^{(free)}(x) = e^{-1}\Sigma_a \int_{\Sigma_a} m_a \frac{dy_\mu^a}{dt_i} \frac{dy_\nu^a}{dt_i} \delta^4(x - y_a(\tau))d\tau_a. \]

A model of a static weak-field averaged gravitating medium with quadrupole gravitational polarization is developed next. The model is based on the following three assumptions.

1. The energy-momentum tensor of the molecules \( T_{\mu\nu}^{(free)} \) satisfies the standard conditions for the Newtonian approximation to hold :
\[ T_{00}^{(free)} \gg T_{ij}^{(free)}, \quad T_{00}^{(free)} \gg T_{0i}^{(free)}, \quad T_{00}^{(free)} = T_{\mu\nu}^{(free)}u^\mu u^\nu = \mu c^2. \]
2. The macroscopic metric tensor \( g^{(0)}_{\mu\nu} \) is static,
\[ \frac{\partial g^{(0)}_{\mu\nu}}{\partial t} = 0, \]
3. The averaged macroscopic metric obeys the weak field approximation: tensor $g_{\mu \nu}^{(0)}$,

$$g_{\mu \nu}^{(0)} = \eta_{\mu \nu} + \epsilon h_{\mu \nu}, \quad (28)$$

In general relativity the conditions (26)-(28) for a microscopic energy-momentum tensor $\epsilon_{\beta}^{\alpha \text{(micro)}}$ and for the metric tensor $g_{00} = - \left(1 + \frac{2\phi}{c^2}\right)$, $g_{0i} = 0$, $g_{ij} = \delta_{ij}$ lead to the Newtonian limit of the Einstein equations (20) and to result in the Poisson equation for the Newtonian gravitational potential $\phi = \phi(x^a)$

$$\nabla^2 \phi = 4\pi G\mu. \quad (29)$$

We are interested in the Newtonian limit of the macroscopic gravity equations (24) under conditions (26)-(28) for the macroscopic tensor $g_{\mu \nu}^{(0)}$

$$g_{00}^{(0)} = - \left(1 + \frac{2\phi}{c^2}\right), \quad g_{11}^{(0)} = 1, \quad g_{22}^{(0)} = 1, \quad g_{33}^{(0)} = 1, \quad g_{\mu \nu}^{(0)} = 0, \quad \mu \neq \nu, \quad (30)$$

which leads to a generalization of the Poisson equation which incorporates the effect of gravitational quadrupole polarization.

For the case of the static weak-field macroscopic medium Isaacson's energy-momentum tensor of gravitational waves $T_{\mu \nu}^{(GW)}$ vanishes

$$T_{\mu \nu}^{(GW)} = 0, \quad (31)$$

The matter quadrupole polarization tensor is modelled by assuming, in analogy with the electrostatic case, that

$$Q_{i0j0} = \epsilon_g C_{i0j0} \quad (32)$$

and the gravitational dielectric constant $\epsilon_g$ is modelled by [4] as

$$\epsilon_g = \frac{1}{4} \frac{mA^2 c^2}{\omega_0^2 N}. \quad (33)$$

Here, $A$ is the typical linear dimension of a molecule, $m$ is the typical mass of a molecule, $\omega_0$ is a typical frequency of harmonically oscillating atoms in the molecule and $N$ is the number density of the molecules in the medium.

For the metric

$$g_{00}^{(0)} = - \left(1 + \frac{2\phi}{c^2}\right), \quad g_{11}^{(0)} = 1, \quad g_{22}^{(0)} = 1, \quad g_{33}^{(0)} = 1, \quad g_{\mu \nu}^{(0)} = 0 \quad (\text{for } \mu \neq \nu), \quad (34)$$

we get

$$R_{00} = \frac{1}{c^2} \nabla^2 \phi, \quad (35)$$

$$R_{ii} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial x_i^2} \quad (\text{i is not summed over here.}), \quad (36)$$

$$R = g^{\mu \nu} R_{\mu \nu} = g^{00} R_{00} + g^{ii} R_{ii} \quad (37)$$

$$= \left(-1 + 2\phi/c^2\right) \left(-\frac{1}{c^2} \nabla^2 \phi\right) + \left(\frac{1}{c^2} \frac{\partial^2 \phi}{\partial x_i^2}\right) \quad (i \text{ is summed over here.}) \quad (38)$$

$$= \frac{2}{c^2} \nabla^2 \phi. \quad (39)$$

Terms only up to first order in $\phi$ have been considered. The last term in equation (24) is $Q_{\mu \nu \rho \sigma}$. The only non-zero components of this tensor are:

$$Q_{i0j0} = \epsilon_g R_{i0j0} = -\epsilon_g \frac{\partial^2 \phi}{c^2 \partial x^i \partial x^j} \quad (41)$$

$$\Rightarrow Q_{i0j0}^{ij} = -\frac{1}{c^2} \nabla^4 \phi. \quad (42)$$
Substituting these values in the trace of Eqn. (24), we get

\[
g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R g^{\mu\nu} g_{\mu\nu} = -\kappa (g^{\mu\nu} T_{\mu\nu} + \frac{1}{2} c^2 g^{\mu\nu} Q_{\mu\nu\sigma\rho}^\sigma) \quad (43)
\]

\[-R = -\kappa (T + \frac{1}{2} c^2 g^{00} Q_{0000}^0) \quad (44)
\]

\[R = \kappa (\mu c^2 + \frac{\epsilon g}{2} \nabla^4 \phi) \quad (45)
\]

\[
\frac{2}{c^2} \frac{1}{c^2} \nabla^2 \phi = \frac{8\pi G}{c^2} (\mu c^2 + \frac{\epsilon g}{2} \nabla^4 \phi) \quad (46)
\]

\[\nabla^2 \phi = 4\pi G \mu + \frac{2\pi G \epsilon g}{c^2} \nabla^4 \phi \quad (47)
\]

\[\nabla^2 \phi - \frac{2\pi G \epsilon g}{c^2} \nabla^4 \phi = 4\pi G \mu \quad (48)
\]

This is the biharmonic equation for the macroscopic gravitational potential. The origin of the fourth order term lies in the fact that the averaging procedure leads to the second derivative of the quadrupole polarization term in the averaged Einstein equations, and furthermore the polarization tensor is modelled as being proportional to the electric part of the Weyl tensor.

4 Solution to the biharmonic equation for the macroscopic gravitational potential

As reviewed in the previous section, in the model [5], the equation for the macroscopic gravitational potential is the biharmonic equation

\[
\nabla^4 \phi - k^2 \nabla^2 \phi = -4\pi Gk^2 \mu(r) \quad (49)
\]

where \(k\) is a fundamental constant of the theory, having dimensions of inverse length, and defined by \(k^2 = c^2 / 2\pi G \epsilon g\). We are interested here in the modification of the radial dependence of the potential. Assuming a separation of variables, the radial part of this equation is given by :

\[
\phi'''' + \frac{4}{r} \phi'''' - k^2 \phi'' - \frac{2}{r} k^2 \phi' = -4\pi Gk^2 \mu(r) \quad (50)
\]

where a prime denotes a derivative with respect to \(r\).

We find the series solution of this equation using the standard Frobenius method around the regular singular point \(r = 0\).

**Case 1**: The vacuum solution \(\mu = 0\) [homogeneous equation] :

In this case we get the following solution for the acceleration \(a = -\nabla \phi\)

\[
a(r) = - \left( C_0 + \frac{C_1}{k} \right) \frac{e^{kr}}{2kr} + \left( C_0 - \frac{C_1}{k} \right) \frac{e^{-kr}}{2kr} + \left( C_0 + \frac{C_1}{2k^2r^2} \right) \frac{e^{kr}}{2k^2r^2} + \left( C_0 - \frac{C_1}{k} \right) \frac{e^{-kr}}{2k^2r^2} - C_2 \frac{1}{r^2} \quad (51)
\]

Since we know that \(k > 0\), this implies that in Eqn. \(51\), terms proportional to \(e^{kr} \to \infty\) as \(r \to \infty\) which is unphysical. So we set the coefficients of \(e^{kr}\) to zero. Thus

\[
\left( C_0 + \frac{C_1}{k} \right) = 0 \Rightarrow C_1 = -kC_0. \quad (52)
\]

Hence

\[
a = C_0 \frac{e^{-kr}}{kr} + C_0 \frac{e^{-kr}}{k^2r^2} - C_2 \frac{e^{-kr}}{r^2} \quad (53)
\]

\[
a = -C_2 \frac{1}{r^2} + C_0 \frac{e^{-kr}}{k^2r^2} (1 + kr) \quad (54)
\]
Eqn. (54) is the solution for acceleration for vacuum [homogeneous] case. The similarity with the solution in the STVG case is immediately apparent. The constants $C_0$ and $C_2$ can be related by the following reasoning: For $kr \ll 1$ we assume Newton’s law of gravitation to hold, so that $C_2 = GM + C_0/k^2 \equiv G\infty M$ where $G\infty = G[1 + C_0/k^2MG]$. For $kr \gg 1$ the exponential term can be ignored, and $G\infty$ represents the effective gravitational constant at large distances. In the concluding section we will give an argument for estimating the physical value for the correction term proportional to $C_0$.

**Case 2: $\mu(r) \neq 0$ [Inhomogeneous Solution]:**

Since we know the solution for the homogeneous case ($\mu = 0$), we can construct the solutions for the inhomogeneous case using the homogeneous solutions. As is well known, if $y = au + bv$ is a solution of

$$y'' + P(x)y' + Q(x)y = 0$$

(55)

where $a$ and $b$ are constants, we can find the solution of

$$y'' + P(x)y' + Q(x)y = R(x)$$

(56)

in the form

$$y = A(x)u + B(x)v$$

(57)

where

$$A(x) = -\int \frac{vR}{W} dx, B(x) = \int \frac{uR}{W} dx$$

(58)

where $W$ is the Wronskian

$$W = uv' - u'v.$$  

(59)

In our problem,

$$A(r) = 4\pi Gk \int r \sinh(kr) \mu(r) dr + a_1, B(r) = -4\pi Gk^2 \int r \cosh(kr) \mu dr + a_2$$

(60)

Hence inside a medium with a density profile $\mu(r)$ the acceleration is given by

$$a = -\phi' = -\frac{1}{r^2} \int r^2 \left[ \frac{4\pi Gk \cosh(kr)}{r} \int r \sinh(kr) \mu dr - \frac{4\pi Gk \sinh(kr)}{r} \int r \cosh(kr) \mu dr + a_1 \frac{\cosh(kr)}{r} + a_2 \frac{\sinh(kr)}{kr} \right] dr - \frac{a_3}{r^2}$$

(61)

The last two terms in the bracket in the above equation are same as in the homogeneous case. These two will be reduced to $a_1e^{-kr}/k^2r^2(1 + kr)$ for the same reasons as discussed before Eqn. (54). Hence Eqn. (41) becomes

$$a = -\phi' = -\frac{1}{r^2} \int r^2 \left[ \frac{4\pi Gk \cosh(kr)}{r} \int r \sinh(kr) \mu dr - \frac{4\pi Gk \sinh(kr)}{r} \int r \cosh(kr) \mu dr + a_1 \frac{e^{-kr}}{k^2r^2}(1 + kr) - \frac{a_3}{r^2} \right] dr + a_4$$

(62)

Eqn. (41) gives the acceleration for any given $\mu(r)$. One can easily read off special cases from here:

**Case 1: $\mu = 0$:**

$$a = -\phi' = a_1 \frac{e^{-kr}}{k^2r^2}(1 + kr) - \frac{a_3}{r^2}$$

(63)

which is same as equation[54], as expected.

**Case II: $\mu =$constant$=\mu_0$:**

$$a = -\phi' = \frac{4\pi G\mu_0}{r^2} \int \left[ \frac{r^2k \cosh(kr)}{r} \right] \left( \int r \sinh(kr) dr \right) dr + \frac{4\pi G\mu_0}{r} \int \left[ \frac{r^2k \sinh(kr)}{r} \right] \left( \int r \cosh(kr) dr \right) dr + a_1 \frac{e^{-kr}}{k^2r^2}(1 + kr) - \frac{a_3}{r^2}$$

or

(64)
Eqn. (65) gives acceleration for a medium of constant density.

5 Comparison with the solution in Scalar-Tensor-Vector Gravity theory

As noted earlier, in STVG the acceleration outside of a body is given by

\[ a(r) = -\frac{G_\infty M}{r^2} + \frac{K}{r^2} \exp\left(-\frac{r}{r_0}\right) \left(1 + \frac{r}{r_0}\right) \]  

(66)

This form matches exactly with that derived in (54) for the biharmonic equation. Comparing Eqn. (54) with Eqn. (66),

\[ C_2 = \frac{G_\infty}{\sqrt{M_0}} M, \]  

(67)

\[ \frac{C_0}{k^2} = K = G \sqrt{M M_0} \]  

(68)

\[ k = \frac{1}{r_0}. \]  

(69)

The free parameter \( r_0 \) of STVG theory is thus now determined from microscopic considerations which determine \( k \) from the gravitational dielectric constant. In the concluding section we will comment on the possible physical origin of \( K \).

Next, in STVG theory, the acceleration inside a spherical mass distribution takes the form [as can be shown with some justification]

\[ a(r) = -\frac{GM(r)}{r^2} \left\{ 1 + \frac{M_0}{M} \left[ 1 - \exp\left(-r/r_0\right) \left(1 + \frac{r}{r_0}\right) \right] \right\} \]  

(70)

This is the form used to fit galaxy rotation curves. We write the above equation in the following form:

\[ a(r) = \frac{M(r)}{M} A(r) \]  

(71)

where

\[ A(r) = -\frac{GM}{r^2} \left\{ 1 + \frac{M_0}{M} \left[ 1 - \exp\left(-r/r_0\right) \left(1 + \frac{r}{r_0}\right) \right] \right\} \]  

(72)

Now one would like to check whether the biharmonic equation also gives the same form of acceleration inside a spherical mass distribution. In general this would be difficult to check; below we describe a way which suffices for our purpose.

The solution of the biharmonic equation for any given mass distribution \( \mu(r) \) is given by Eqn. (4). Rewriting (4),

\[ a(r) = -\frac{4\pi G \mu_0}{r^2} + \frac{a_1 e^{-kr}}{k^2 r^2} \left(1 + kr\right) - \frac{a_3}{r^2} \]  

(65)

(73)

with

\[ a_1 = k^2 G \sqrt{M M_0}, \quad a_3 = G \sqrt{M M_0} + GM \]  

(74)
we see that there is double integral over $\mu(r)$ in above equation. We construct a function of the acceleration and its derivatives which is independent of integrals. Such a function is

$$\chi(r) = \beta''(r) - k^2 \beta(r)$$

where

$$\beta = \frac{1}{r} \frac{d}{dr}[r^2(a(r) - A(r))]$$

From the solution of the biharmonic equation it can be shown that (as expected)

$$\chi = 4\pi Gk^2 r \mu(r)$$

We also find $\chi(r)$ from STVG theory also using equation (71). Now

$a_{STVG}(r) = a_{bh}(r)$,

$$\Rightarrow \chi_{STVG}(r) = \chi_{bh}(r)$$

where “bh” stands for biharmonic equation. Equating $\chi_{STVG}$ and $\chi_{bh}$, we get an equation of the form

$$P(r) M(r) + Q(r) M'(r) + R(r) M''(r) + S(r) M'''(r) + T(r) = 0.$$  \hspace{8.7cm} (80)

where

$$P(r) = -\frac{e^{-r/r_0}G\sqrt{\frac{M}{M}}}{r_0^2},$$

$$Q(r) = G\left[ -\frac{2}{r^3} + \sqrt{\frac{M_0}{M}} \left( -\frac{2}{r^3} + \frac{2e^{-r/r_0}}{r^3} + \frac{3e^{-r/r_0}}{r_0^2} + \frac{e^{-r/r_0}}{r^2} \right) \right] - \frac{4\pi}{r r_0^2},$$

$$R(r) = G\left( \frac{2}{r^3} + \frac{2\sqrt{\frac{M_0}{M}}}{r^2} - \frac{2e^{-r/r_0}}{r^3} \sqrt{\frac{M_0}{M}} + \frac{3e^{-r/r_0}}{r_0^2} \sqrt{\frac{M_0}{M}} \right),$$

$$S(r) = G\left( -\frac{1}{r} - \frac{\sqrt{\frac{M_0}{M}}}{r} + \frac{e^{-r/r_0}}{r} \sqrt{\frac{M_0}{M}} \right)$$

and

$$T(r) = -\frac{e^{-r/r_0}G M \sqrt{\frac{M}{M}}}{r_0^2}.$$  \hspace{8.7cm} (82)

\hspace{8.7cm} (83)

\hspace{8.7cm} (84)

\hspace{8.7cm} (85)

\hspace{8.7cm} (86)

Eqn. (80) contains one integral over $\mu(r)$. Dividing equation (80) by $P(r)$ and then differentiating, we get a differential equation of the form,

$$U(r) \mu(r) + V(r) \mu'(r) + W(r) \mu''(r) + X(r) \mu'''(r) = 0$$  \hspace{8.7cm} (87)

where

$$U(r) = \left( 4e^{r/r_0} M \sqrt{\frac{M_0}{M}} \pi r_0 (r + r_0) + M_0 (r^2 - 6rr_0 + 5r_0^2) \right),$$

$$V(r) = r_0 \left[ 2e^{r/r_0} M \sqrt{\frac{M_0}{M}} r_0 (2\pi r + r_0) + M_0 \left( -3r^2 + 11r_0 r + 2(-1 + e^{r/r_0}) r_0^2 \right) \right],$$

$$W(r) = r_0^2 \left[ e^{r/r_0} M \sqrt{r_0} r_0 (r + 3r_0) + M_0 \left( 3r^2 + (4 + e^{r/r_0}) r_0^2 + 3(-1 + e^{r/r_0}) r_0^2 \right) \right],$$

and

$$X(r) = -r_0^3 \left[ M_0 (r - e^{r/r_0} r_0 + r_0) - e^{r/r_0} M \sqrt{\frac{M_0}{M}} r_0 \right].$$  \hspace{8.7cm} (90)

\hspace{8.7cm} (91)
We do not get any term independent of \( \mu(r) \) in the above equation. The solution of Eqn. will tell us for what value of \( \mu(r) \), STVG theory and the biharmonic equation give the same acceleration inside a mass distribution. But it is not necessary to solve . Instead we take the observed density profile \( \mu(r) \) for a specific galaxy type and see whether it satisfies .

We assume the following density profile \( \mu(r) \)

\[
\mu(r) = \frac{3}{4\pi r^3} \beta M(r) \left[ \frac{r_c}{r + r_c} \right]
\]

where

\[
M(r) = 4\pi \int_0^r dr' r'^2 \mu(r')
\]

and

\[
\beta = \begin{cases} 
1 \text{ for HSB galaxies,} \\
2 \text{ for LSB & Dwarf galaxies.}
\end{cases}
\]

We have taken the data set of ten LSB, HSB and dwarf galaxy masses from Table (3) in [2]. Then we plot the left hand side of Eqn. after making it dimensionless using parameters \( r_0 \) and \( M_0 \). We have made the assumption that \( k = 1/r_0 \).

We have taken data set of 10 LSB, HSB and dwarf galaxies and for each of them we have plotted \( STVG - BH \) vs. \( r \) for three ranges of \( r \)- (0.0001 - 1) kpc, (1 - 100) kpc and (100 - 500) kpc. The plots are shown in Fig. 1-10. We see that for the range 1 - 100 kpc \( STVG - BH \) is very small, of the order of \( 10^{-4} \) or \( 10^{-3} \). And for other ranges \( STVG - BH \) takes high values. Since the range of length scales of interest in a galaxy lie between 1 - 100 kpc and \( STVG - BH \) is very small in this range, it can be said that the solution coming from STVG theory matches with the solution of the biharmonic equation, inside the medium, for the observed density profile. In this sense, the modified acceleration law resulting from the biharmonic equation can also explain the flatness of rotation curves, if \( k \) and \( M_0 \) are assumed to take the values needed to explain observations.

![Figure 1: for galaxy of mass 0.13 \( \times \) 10^{10}M_\odot and \( r_c = 0.53 \) kpc.](image)

### 6 Determination of parameters from theory

In the considerations based on the quadrupole polarization effect on the galactic field, \( k = 1/r_0 \) is a fundamental parameter derivable from microscopic considerations. We can find the value of \( k \) using the form given in [6]

\[
\frac{1}{k^2} = \frac{2\pi G}{c^2} \epsilon_g
\]

where

\[
\epsilon_g = \frac{1}{4} \frac{mA^2 c^2}{\omega_0^2} N
\]
Figure 2: for galaxy of mass $0.42 \times 10^{10} M_{\odot}$ and $r_c = 0.66$ kpc.

Figure 3: for galaxy of mass $0.40 \times 10^{10} M_{\odot}$ and $r_c = 0.82$ kpc.

Figure 4: for galaxy of mass $0.38 \times 10^{10} M_{\odot}$ and $r_c = 0.57$ kpc.

Figure 5: for galaxy of mass $2.26 \times 10^{10} M_{\odot}$ and $r_c = 1.06$ kpc.
Figure 6: for galaxy of mass $3.08 \times 10^{10} M_\odot$ and $r_c = 1.58$ kpc.

Figure 7: for galaxy of mass $5.46 \times 10^{10} M_\odot$ and $r_c = 1.40$ kpc.

Figure 8: for galaxy of mass $7.95 \times 10^{10} M_\odot$ and $r_c = 1.36$ kpc.

Figure 9: for galaxy of mass $9.12 \times 10^{10} M_\odot$ and $r_c = 1.04$ kpc.
where $\omega_0$ = typical characteristic frequency of harmonically oscillating particles in the molecule, $A$ = average linear dimension of a typical molecule, $m$ = the average mass of a molecule, $D$ is the mean distance between molecules, and $N$ = average number of molecules per unit volume. $k$ can also be written in the following form,

$$\frac{1}{k^2} = \frac{3}{8}\theta \left(\frac{A^3}{D^3}\right) A^2.$$  

(98)

Here the dimensionless factor $\theta$,

$$\theta = \frac{\omega_0^2}{4\pi G\mu/3},$$

(99)

reflects the nature of the field responsible for bounding discrete matter constituents into molecules. Here $\mu$ is the mass density of the medium. If $\theta \approx 1$, the molecules of a self-gravitating macroscopic medium are considered to be gravitationally bound.

In our calculation, the macroscopic medium is a galaxy cluster which is made up of galaxies [‘molecules’] of which the stars are atoms. Since in galaxy clusters, galaxies are gravitationally bound, $\theta \approx 1$, and assuming $A \approx 100$ kpc, and $D \approx 1$ Mpc. So we get

$$\frac{1}{k^2} \approx \frac{3}{8} \left(\frac{10^5}{1 \times 10^9}\right)^3 \times (100 \text{kpc})^2$$

(100)

or $1/k \approx 2$ kpc. Thus it follows that that the estimated value of $1/k = r_0$ is of the same order as desired in STVG theory. Of course this is only a ballpark estimate intended to show that $k$ lies in the kpc range as opposed to being somewhere very far from the kpc range.

7 Conclusions

We found the general series solution to the biharmonic equation for the macroscopic gravitational potential. We saw that the form of acceleration is very similar to that in STVG [1] in vacuum. Then we investigated further if the form of acceleration matches inside the matter distribution also. What we find is that both the solutions match for the observed matter density profile of galaxies. Also, the value of the parameter $r_0$ in STVG [2] is of the order of $k$ in the theory of macroscopic media in general relativity [3].

While we have not offered an analogous estimate for $M_0$ (equivalently $K$ or $G_\infty$), it is clear that $M_0$ also has to follow from microscopic considerations, since $k$ is the only new parameter that appears in the biharmonic equation. A suggestive estimate may be obtained by rewriting the biharmonic equation by taking the fourth order term to the right :

$$\nabla^2 \phi = 4\pi \mu(r)G \left(1 + \frac{\nabla^4 \phi}{4\pi Gk^2\mu(r)}\right)$$

(101)

The correction to $G$ on the right hand is of the order $1/k^2 r^2$ which as we seen above is of order unity in the region of interest, thus suggesting that $M$ is comparable to $M_0$, as assumed for explaining observations.
A few words about comparison with MOND. In MOND, the acceleration of a test particle is assumed to be given by the relation

\[ a = a_N \mu \left( \frac{a_N}{a_0} \right) \]  

(102)

where \( a_N \) is acceleration in Newtonian mechanics and \( a_0 \) is a new fundamental constant of nature. For small accelerations [large distances] it is assumed

\[ \mu \left( \frac{a_N}{a_0} \right) = \frac{a_N}{a_0} \]  

(103)

whereas \( \mu \) approaches the value one for accelerations encountered in the solar neighborhood. In our work and in STVG theory as well, the acceleration is of the form

\[ a(r) = a_N \frac{M(r)}{M} A(r) \]  

(104)

which is also a form of modified Newtonian dynamics. But the extra factor

\[ \frac{M(r)}{M} A(r) \neq \frac{a_N}{a_0} \]  

(105)

The fact that the modified acceleration law discussed in this paper also arises from apparently independent considerations as in STVG, is intriguing and suggests that further investigation would be of interest. Especially, a better understanding of the physics involved, which goes beyond the mathematical treatment, is essential. Also, one would like to understand better the relevance and validity of the functional form for the gravitational dielectric constant used here.

Of course no claim is made that we have proposed a definitive alternative to dark matter in galaxies, especially considering that any such proposals must also be studied for their role on cosmological scales, where dark matter is crucial for understanding structure formation. Our principal motive is to highlight that the role of averaged gravitational effects may not yet be fully understood on galactic scales, and even a percentage effect would be of interest and worth establishing accurately.

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ADDENDUM: After the completion of this work two important points were brought to our attention by Viktor T. Toth [71]. Firstly, in a more recent treatment [8] than the one discussed by us above the parameters \( M_0 \) and \( r_0 \) can, in fact, be obtained from three universal constants, which are fixed by observations ranging from the solar system to cosmological scales. It will be interesting to see in future work as to how the fundamental parameter \( k \) of the biharmonic equation relates to these universal constants.

Secondly, in [8] it is shown in their Eqn. (65) that \( r_0 \) is proportional to \( \sqrt{M} \), where \( M \) is the source mass. Remarkably it turns out that in our case too this proportionality holds as is evident from our Eqs. (96) and (97). (This is because the gravitational dielectric constant is proportional to the mass of the molecule). This further strengthens the likelihood of a fundamental connection between STVG and our work.

We would like to thank Viktor Toth for bringing these important observations to our attention. In further work we plan to explore the parallel between the two approaches in greater detail.
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