Faithfulness of Galois representations associated to hyperbolic curves

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Abstract

Let $C \to \text{Spec}(k)$ be a hyperbolic curve over a number field. The outer Galois representation

$$\rho_C : G_k \to \text{Out}(\pi_1(C \times_k \bar{k}))$$

is the associated monodromy representation, where $G_k$ is the absolute Galois group of $k$. We prove that $\rho_C$ is faithful for all hyperbolic curves, thus extending a result of Matsumoto [20], who had proved the faithfulness in the affine case.

Let $\mathcal{M}_{g,n}$, for $2g - 2 + n > 0$, be the moduli stack of smooth $n$-pointed, genus $g$ curves and let $\mathcal{C} \to \mathcal{M}_{g,n}$ be the universal $n$-punctured, genus $g$ curve. The arithmetic universal monodromy representation is the associated representation:

$$\mu_{g,n} : \pi_1(\mathcal{M}_{g,n} \times \mathbb{Q}, \bar{\xi}) \to \text{Out}(\pi_1(\mathcal{C}_\bar{\xi})),$$

where $\bar{\xi} \in \mathcal{M}_{g,n} \times \overline{\mathbb{Q}}$. We prove that $\mu_{g,n}$ is faithful for $g \leq 2$. Otherwise, its kernel can be identified with the congruence kernel of the profinite Teichmüller group $\hat{\Gamma}_{g,n}$.

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1 Introduction

Let $C$ be a hyperbolic curve defined over a number field $k$. The structural morphism $C \to \text{Spec}(k)$ induces a short exact sequence of algebraic fundamental groups:

$$1 \to \pi_1(C \times_k \overline{\mathbb{Q}}, \bar{\xi}) \to \pi_1(C, \bar{\xi}) \to G_k \to 1,$$

where we have chosen an embedding $k \subset \overline{\mathbb{Q}}$, a $\overline{\mathbb{Q}}$-valued point $\bar{\xi} \in C$ and the absolute Galois group $G_k$ is identified with the algebraic fundamental group $\pi_1(\text{Spec}(k), \text{Spec}(\overline{\mathbb{Q}}))$. Associated to the above short exact sequence, is then the outer Galois representation:

$$\rho_C : G_k \to \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \bar{\xi})).$$
This representation plays a central role in the arithmetic geometry of curves.

The representation $\rho_C$ can be interpreted as the monodromy representation associated to the curve $C \to \text{Spec}(k)$ and the given choices of base points. Hence, it can be recovered from the arithmetic universal monodromy representation as follows.

Let us suppose that $C$ is an $n$-punctured, genus $g$ curve, for $2g - 2 + n > 0$. Let then $\mathcal{M}_{g,n}$ be the moduli stack of smooth $n$-pointed, genus $g$ curves defined over some number field. It is a smooth irreducible Deligne–Mumford stack of dimension $3g - 3 + n$ defined over $\text{Spec}(\mathbb{Q})$, endowed with a universal $n$-punctured, genus $g$ curve $\mathcal{C} \to \mathcal{M}_{g,n}$. Let $\xi \in \mathcal{M}_{g,n}$ be the point corresponding to the curve $C$ and $\xi$ the $\mathbb{Q}$-valued point lying over it. Let us identify the fiber $\mathcal{C}_\xi$ with the curve $C \times_k \overline{\mathbb{Q}}$. There is then a short exact sequence of algebraic fundamental groups:

$$1 \to \pi_1(C \times_k \overline{\mathbb{Q}}, \hat{\xi}) \to \pi_1(\mathcal{C}, \hat{\xi}) \to \pi_1(\mathcal{M}_{g,n}, \xi) \to 1.$$ 

The associated outer representation

$$\mu_{g,n}: \pi_1(\mathcal{M}_{g,n}, \xi) \to \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \hat{\xi}))$$

is the arithmetic universal monodromy representation.

The morphism of pointed stacks $\xi: (\text{Spec}(k), \text{Spec}(\overline{\mathbb{Q}})) \to (\mathcal{M}_{g,n}, \xi)$ induces a homomorphism $\xi_*: G_k \to \pi_1(\mathcal{M}_{g,n}, \xi)$ on algebraic fundamental groups. The representation $\rho_C$ is then obtained composing $\xi_*$ with the arithmetic universal monodromy representation $\mu_{g,n}$. Thus, it is not a surprise that the representation $\mu_{g,n}$ contains essential informations on the properties shared by all Galois outer representations associated to smooth $n$-punctured, genus $g$ arithmetic curves. The study of the representation $\mu_{g,n}$ is the principal purpose of this paper.

The structural morphism $\mathcal{M}_{g,n} \to \text{Spec}(\mathbb{Q})$ induces the short exact sequence:

$$1 \to \pi_1(\mathcal{M}_{g,n} \times \overline{\mathbb{Q}}, \hat{\xi}) \to \pi_1(\mathcal{M}_{g,n}, \xi) \to G_{\mathbb{Q}} \to 1.$$ 

The left term of this short exact sequence is called the geometric algebraic fundamental group of $\mathcal{M}_{g,n}$. The reason for this terminology is that it is naturally isomorphic to the profinite completion of the topological fundamental group of $\mathcal{M}_{g,n} \times \mathbb{C}$ with base point $\hat{\xi}$.

Let $S_{g,n}$ be an $n$-punctured, genus $g$ Riemann surface and let $\Gamma_{g,n}$ be the associated Teichmüller modular group. The choice of a homeomorphism $\phi: S_{g,n} \to (C \times_k \mathbb{C})^{\text{top}}$ determines an isomorphism between $\Gamma_{g,n}$ and $\pi_1^{\text{top}}(\mathcal{M}_{g,n} \times \mathbb{C}, \overline{\xi})$ (for more details, see Section 2). So that we get also an identification of the geometric algebraic fundamental group of $\mathcal{M}_{g,n}$ with the profinite completion of the Teichmüller group $\Gamma_{g,n}$, which we simply call the profinite Teichmüller group $\hat{\Gamma}_{g,n}$.

Let us denote by $\Pi_{g,n}$ the topological fundamental group $\pi_1(S_{g,n}, \phi^{-1}(\hat{\xi}))$ and by $\hat{\Pi}_{g,n}$ its profinite completion. The restriction of the arithmetic universal monodromy representation $\mu_{g,n}$ to the geometric algebraic fundamental group then induces a representation:

$$\hat{\rho}_{g,n}: \hat{\Gamma}_{g,n} \to \text{Out}(\hat{\Pi}_{g,n}),$$
which we call the profinite universal monodromy representation. This can be also described as the homomorphism of profinite groups induced by the canonical homotopy action of \( \Gamma_{g,n} \) on \( S_{g,n} \) (see Section 2).

It is clear that, in order to gain some insight on the representation \( \mu_{g,n} \), we need to study before the representation \( \hat{\rho}_{g,n} \). The study of the latter representation is the subject of Section 2 till Section 7.

The first question which arises about the representation \( \hat{\rho}_{g,n} \) is whether or not it is faithful. This is better known as the congruence subgroup problem for the Teichmüller group \( \Gamma_{g,n} \). An affirmative answer is known only in genus \( \leq 2 \) (see [3] and [6]).

The investigation underlying this paper was stimulated by the idea that, in order to advance in all the above issues, it was necessary a complete understanding of the group-theoretic properties of the image of the representation \( \hat{\rho}_{g,n} \), which we then denote by \( \hat{\Gamma}_{g,n} \) and call the geometric profinite Teichmüller group.

The combinatorial group theory of the Teichmüller group \( \Gamma_{g,n} \) begins with the study of the relations occurring between words in its standard set of generators given by Dehn twists. For the geometric profinite Teichmüller group \( \hat{\Gamma}_{g,n} \), we then define the set of profinite twists to be the closure of the image of the set of Dehn twists via the natural homomorphism \( \Gamma_{g,n} \to \hat{\Gamma}_{g,n} \). Words in sets of commuting profinite Dehn twists are called profinite multi-twists.

The key technical result of this paper is a complete characterization of profinite multi-twists of \( \hat{\Gamma}_{g,n} \) (Theorem 6.1). As an immediate consequence, we get a complete description of the centralizers of profinite multi-twists and of the normalizers of the closed subgroups they span in \( \hat{\Gamma}_{g,n} \), in perfect analogy with the classical results for multi-twists in \( \Gamma_{g,n} \) (Corollary 6.9).

Deeper and more involved relations occurring between Dehn twists in \( \Gamma_{g,n} \) are encoded in the various curve complexes which can be associated to \( S_{g,n} \). Here, we mention just the most important one: the complex of curves \( C(S_{g,n}) \). This is the simplicial complex whose simplices are given by sets of distinct, non-trivial, isotopy classes of simple closed curves (briefly s.c.c.) on \( S_{g,n} \), such that they admit a set of disjoint representatives none of them bounding a disc with a single puncture.

There is a natural simplicial action of the Teichmüller group \( \Gamma_{g,n} \) on \( C(S_{g,n}) \). Moreover, there is a fundamental Theorem by Harer (see [12] and [13]) which asserts that the geometric realization of \( C(S_{g,n}) \) has the homotopy type of a bouquet of spheres of dimension \( -\chi(S_{g,n}) - 1 \), for \( n \geq 1 \), and \( -\chi(S_g) \), for \( n = 0 \) (here, by \( \chi(S_{g,n}) \), we denote the Euler characteristic of the Riemann surface \( S_{g,n} \)). In particular, it is simply connected for \( 3g - 3 + n > 2 \).

Sections 4, 5 and 6 are devoted to construct a satisfactory profinite analogue of the curve complex \( C(S_{g,n}) \). As the final result of our efforts, we get a profinite simplicial complex \( L(\Pi_{g,n}) \) which we call the complex of profinite curves. Eventually (Theorem 6.15), this is characterized as the simplicial complex whose \( k \)-simplices are the closed abelian subgroups of rank \( k + 1 \) spanned by profinite Dehn twists in \( \hat{\Gamma}_{g,n} \).

In Section 7 we prove that the obstructions for the congruence subgroup property
to hold true lie exactly in the fundamental groups of the profinite simplicial complexes $L(\Pi_{g,n})$, for $3g - 3 + n > 2$. Another way to formulate the congruence subgroup conjecture is indeed the assertion that all relations occurring between profinite Dehn twists in $\Gamma_{g,n}$ are consequence of those occurring between Dehn twists in $\Gamma_{g,n}$.

This is still an open problem, even though the results presented here suggest possible new strategies of attack, beyond the evidence offered by the flawed proof of Theorem 5.4 in [5].

All the results mentioned above then form the basis for the study of Galois representations associated to hyperbolic curves carried out in Section 8. The main result is that the outer representation $\rho_C$ associated to a hyperbolic curve $C$ over a number field $k$ is faithful (Theorem 8.6). We also reduce the faithfulness of the arithmetic universal monodromy representation $\mu_{g,n}$ to the congruence subgroup problem for $\Gamma_{g,n}$. In particular, we prove that $\mu_{g,n}$ is faithful for $g \leq 2$ (Corollary 8.9). The proofs of all these statements are inspired by ideas from Grothendieck-Teichmüller theory in the spirit of [11] (see, in particular, Theorem 8.3).

2 More on level structures over moduli of curves

The study we are going to carry out from Section 2 to Section 7 is in its essence topological. Hence, in order to avoid cumbersome notations, for a complex Deligne-Mumford (briefly D–M) stack $X$, we will denote by $\pi_1(X)$ its topological fundamental group and by $\hat{\pi}_1(X)$ its algebraic fundamental group (when unnecessary, we omit to mention base-points). This notation is consistent with the fact that $\hat{\pi}_1(X)$ is isomorphic to the profinite completion of $\pi_1(X)$. For the same reasons, $\overline{\mathcal{M}}_{g,n}$, for $2g - 2 + n > 0$, will denote the stack of $n$–pointed, genus $g$, stable complex algebraic curves. It is a smooth irreducible proper complex D–M stack of dimension $3g - 3 + n$, and it contains, as an open substack, the stack $\mathcal{M}_{g,n}$ of $n$–pointed, genus $g$, smooth complex algebraic curves. We will keep the same notations to denote the respective underlying analytic and topological stacks.

By Proposition 1.1 in [7], the stack $\overline{\mathcal{M}}_{g,n}$ is simply connected. On the contrary, the stack $\mathcal{M}_{g,n}$ has plenty of non-trivial covers which we are briefly going to introduce in this section. Its universal cover, the Teichmüller space $T_{g,n}$, is the stack of $n$–pointed, genus $g$, smooth complex analytic curves $C \to \mathcal{U}$ endowed with a topological trivialization $\Phi : S_{g,n} \times \mathcal{U} \sim C$ over $\mathcal{U}$, where two such trivializations are considered equivalent when they are homotopic over $\mathcal{U}$. We then denote by $(C \to \mathcal{U}, \Phi)$ the corresponding object of $T_{g,n}$ or, when $\mathcal{U}$ is just one point, simply by $(C, \Phi)$.

From the existence of Kuranishi families, it follows that the complex analytic stack $T_{g,n}$ is representable by a complex manifold (see [2], for more details on this approach). Then, it is not hard to prove that the complex manifold $T_{g,n}$ iscontractible and that the natural map of complex analytic stacks $T_{g,n} \to \mathcal{M}_{g,n}$ is a universal cover. The deck transformations’ group of this cover is described as follows.

Let $\text{Hom}^+(S_{g,n})$ be the group of orientation preserving self-homeomorphisms of $S_{g,n}$ and by $\text{Hom}^0(S_{g,n})$ the subgroup consisting of homeomorphisms homotopic to the identity.
The mapping class group \( \Gamma_{g,[n]} \) is classically defined to be the group of homotopy classes of homeomorphisms of \( S_{g,n} \) which preserve the orientation:

\[
\Gamma_{g,[n]} := \text{Hom}^+(S_{g,n})/\text{Hom}^0(S_{g,n}),
\]

where \( \text{Hom}_0(S_{g,n}) \) is the connected component of the identity in the topological group of homeomorphisms \( \text{Hom}^+(S_{g,n}) \). This group then is the group of covering transformations of the étale cover \( T_{g,n} \rightarrow \mathcal{M}_{g,[n]} \), where the latter denotes the stack of genus \( g \), smooth complex curves with \( n \) unordered labels. Therefore, there is a short exact sequence:

\[
1 \rightarrow \Gamma_{g,n} \rightarrow \Gamma_{g,[n]} \rightarrow \Sigma_n \rightarrow 1,
\]

where \( \Sigma_n \) denotes the symmetric group on the set of punctures of \( S_{g,n} \) and \( \Gamma_{g,n} \) is the deck transformations’ group of the étale cover \( T_{g,n} \rightarrow \mathcal{M}_{g,n} \).

There is a natural way to define homotopy groups for topological \( D\text--M \) stacks, as done, for instance, by Noohi in [22] and [23]. Therefore, the choice of a point \( a = [C] \in \mathcal{M}_{g,n} \) and a homeomorphism \( \phi: S_{g,n} \rightarrow C \) identifies the topological fundamental group \( \pi_1(\mathcal{M}_{g,n},a) \) with the pure Teichmüller modular group \( \Gamma_{g,n} \).

The following notation will turn out to be useful in the sequel. For a given oriented Riemann surface \( S \) of negative Euler characteristic, we denote by \( \Gamma(S) \) the mapping class group of \( S \) and by \( \mathcal{M}(S) \) and \( \overline{\mathcal{M}}(S) \), respectively, the \( D\text--M \) stack of smooth complex curves homeomorphic to \( S \) and its \( D\text--M \) compactification. In particular, \( \Gamma_{g,[n]} := \Gamma(S_{g,n}) \), \( \mathcal{M}_{g,[n]} := \mathcal{M}(S_{g,n}) \) and \( \overline{\mathcal{M}}_{g,[n]} := \overline{\mathcal{M}}(S_{g,n}) \).

The morphism \( \mathcal{M}_{g,n+1} \rightarrow \mathcal{M}_{g,n} \), forgetting the last label, is naturally isomorphic to the universal \( n \)-punctured, genus \( g \) curve \( p: \mathcal{C} \rightarrow \mathcal{M}_{g,n} \). Since \( p \) is a Serre fibration and \( \pi_2(\mathcal{M}_{g,n}) = \pi_2(T_{g,n}) = 0 \), there is a short exact sequence on fundamental groups

\[
1 \rightarrow \pi_1(\mathcal{C}_a, \tilde{a}) \rightarrow \pi_1(\mathcal{C}, \tilde{a}) \rightarrow \pi_1(\mathcal{M}_{g,n}, a) \rightarrow 1,
\]

where \( \tilde{a} \) is a point in the fiber \( \mathcal{C}_a \). By a standard argument this defines a monodromy representation:

\[
\rho_{g,n}: \pi_1(\mathcal{M}_{g,n}, a) \rightarrow \text{Out}(\pi_1(\mathcal{C}_a, \tilde{a})),
\]

called the universal monodromy representation.

Let us then fix a homeomorphism \( \phi: S_{g,n} \rightarrow \mathcal{C}_a \) and let \( \Pi_{g,n} \) be the fundamental group of \( S_{g,n} \) based in \( \phi^{-1}(\tilde{a}) \). Then, the representation \( \rho_{g,n} \) is identified with the faithful representation \( \Gamma_{g,n} \rightarrow \text{Out}(\Pi_{g,n}) \), induced by the homotopy action of \( \Gamma_{g,n} \) on the Riemann surface \( S_{g,n} \).

Let us give \( \Pi_{g,n} \) the standard presentation:

\[
\Pi_{g,n} = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, u_1, \ldots, u_n \right| \prod_{i=1}^{g} [\alpha_i, \beta_i] \cdot u_n \cdots u_1 \right\rangle,
\]

where \( u_i \), for \( i = 1, \ldots, n \), is a simple loop around the puncture \( P_i \). For \( n \geq 1 \), let \( A(g, n) \) be the group of automorphisms of \( \Pi_{g,n} \) which fix the conjugacy classes of all \( u_i \). For \( n = 0 \),
let instead $A(g,0)$ be the image of $A(g,1)$ in the automorphism group of $\Pi_g := \Pi_{g,0}$. Finally, let $I(g,n)$ be the group of inner automorphisms of $\Pi_{g,n}$. With these notations, the Nielsen realization Theorem says that the representation $\rho_{g,n}$ induces an isomorphism $\Gamma_{g,n} \cong A(g,n)/I(g,n)$.

In this paper, a level structure $M_\lambda$ is a finite, connected, Galois, étale cover of the stack $M_{g,n}$ (by étale cover, we mean here an étale, surjective, representable morphism of algebraic stacks), therefore it is also represented by a regular D–M stack $M_\lambda$. This is in contrast with [5], where the cover was not required to be Galois. The level associated to $M_\lambda$ is the finite index normal subgroup $\Gamma_\lambda := \pi_1(M_\lambda, a')$ of the Teichmüller group $\Gamma_{g,n}$.

A level structure $M_\lambda$ dominates $M_{\lambda'}$, if there is a natural étale morphism $M_{\lambda'} \to M_\lambda$ or, equivalently, $\Gamma_\lambda' \leq \Gamma_\lambda$. To mark the fact that $M_\lambda$ is a level structure over $M_{g,n}$, we will sometimes denote it by $M_{\lambda g,n}$.

The most natural way to define levels is provided by the universal monodromy representation $\rho_{g,n}$. In general, for a subgroup $K \leq \Pi_{g,n}$, which is invariant under $A(g,n)$ (in such case, we simply say that $K$ is invariant), let us define the representation:

$$\rho_K : \Gamma_{g,n} \to \text{Out}(\Pi_{g,n}/K),$$

whose kernel we denote by $\Gamma_K$. When $K$ has finite index in $\Pi_{g,n}$, then $\Gamma_K$ has finite index in $\Gamma_{g,n}$ and is called the geometric level associated to $K$. The corresponding level structure is denoted by $M_K$.

A class of finite index invariant subgroups of the group $\Pi_{g,n}$ one can consider, in order to define geometric level structures, is that obtained from the descending central series, twisting by $l$-th powers (see [24]). The descending central series is defined by $\Pi[1] := \Pi_{g,n}$ and $\Pi[k] := [\Pi^{[k-1]}, \Pi]$. Let then $\Pi^l$ be the invariant subgroup of $\Pi_{g,n}$ spanned by $l$-th powers and define, for $k \geq 1$ and $l > 0$:

$$\Pi^{[k,l]} := \Pi^{[k]} \cdot \Pi^l.$$

Of particular interest are the levels defined by the kernels of the representations:

$$\rho(m) : \Gamma_{g,n} \to \text{Sp}(H_1(S_g, \mathbb{Z}/m)), \quad \text{for } m \geq 2.$$

They are denoted by $\Gamma(m)$ and called abelian levels of order $m$. The corresponding level structures are then denoted by $M^{(m)}$. A classical result of Serre says that an automorphism of a smooth curve acting trivially on its first homology group with $\mathbb{Z}/m$ coefficients, for $m \geq 3$, is trivial. In particular, this implies that any level structure $M^{\lambda}$ dominating an abelian level structure $M^{(m)}$, with $m \geq 3$, is representable in the category of algebraic varieties.

There is another way to define levels of $\Gamma_{g,n}$ which turn out to be more treatable in a series of applications. They were basically introduced by Looijenga in [18]. Here, we give an obvious generalization of his definition giving, moreover, a throughout geometric interpretation of the construction.

Let $K$ be a normal finite index subgroup of $\Pi_{g,n}$ and let $p_K : S_K \to S_{g,n}$ be the étale Galois cover with covering transformation group $G_K$, associated to such subgroup. There
is then a natural monomorphism $G_K \hookrightarrow \Gamma(S_K)$. From the theory of moduli of Riemann surfaces with symmetries (see for instance [2]), we know that the quotient of the normalizer $N_{G_K}(\Gamma(S_K))$ by $G_K$ identifies with a finite index subgroup of $\Gamma_{g,[n]}$. If we assume, moreover, that $K$ is invariant for the action of $\Gamma_{g,[n]}$, then any homeomorphism $f: S_{g,n} \to S_{g,n}$ lifts to a homeomorphism $\tilde{f}: S_K \to S_K$. In this case, therefore there is a natural short exact sequence:

$$1 \to G_K \to N_{G_K}(\Gamma(S_K)) \to \Gamma_{g,[n]} \to 1.$$ 

Let us consider the natural representation $\rho_{(m)}: \Gamma(S_K) \to \text{Sp}(H_1(\overline{S}_K, \mathbb{Z}/m))$, for $m \geq 0$, where $\overline{S}_K$ is the Riemann surface obtained from $S_K$ filling in the punctures. For $m \geq 3$ and $m = 0$, the restriction of this representation to $G_K$ is faithful and we get a natural representation:

$$\rho_{K,(m)}: \Gamma_{g,[n]} \to N_{G_K}(\text{Sp}(H_1(\overline{S}_K, \mathbb{Z}/m)))/G_K.$$ 

We denote the kernel of $\rho_{K,(m)}$ by $\Gamma_{K,(m)}$ and call it the Looijenga level associated to the finite index invariant subgroup $K$ of $\Pi_{g,n}$. The corresponding level is denoted by $\mathcal{M}_{K,(m)}$.

For a given finite index invariant subgroup $K \leq \Pi_{g,n}$, the finite index verbal subgroup $K^{[2],m}$ of $K$ is invariant in $\Pi_{g,n}$ and it is clear that the associated geometric level $\Gamma_{K^{[2],m}}$ is contained in the Looijenga level $\Gamma_{K,(m)}$. On the other hand, it holds:

**Theorem 2.1.** Let $K$, for $2g - 2 + n > 0$, be a finite index invariant subgroup of $\Pi_{g,n}$. Then, for all integers $m \geq 3$, between the associated Looijenga and geometric levels, there is an inclusion:

$$\Gamma_{K,(m)} \leq \Gamma_K.$$ 

**Proof.** The action of the group $G_K$ on the surface $S_K$, as the group of covering transformations of the étale cover $S_K \to S_{g,n}$, induces, for $m \geq 3$, a faithful representation $j: G_K \hookrightarrow \text{Sp}(H_1(\overline{S}_K, \mathbb{Z}/m))$.

The representation $j$ can also be recovered as follows. The outer representation, associated to the short exact sequence

$$1 \to H_1(K, \mathbb{Z}/m) \to \Pi_{g,n}/K^{[2],m} \to G_K \to 1,$$

defines an action of $G_K$ on $H_1(K, \mathbb{Z}/m)$ and the representation $j$ is then induced by this action via the natural epimorphism $H_1(K, \mathbb{Z}/m) \to H_1(\overline{S}_K, \mathbb{Z}/m)$.

Let $f \in \Gamma_{K,(m)}$ and let $\tilde{f}$ be an automorphism of $\Pi_{g,n}$ in the class $f$ such that $\tilde{f}$ induces the identity on $H_1(\overline{S}_K, \mathbb{Z}/m)$. For all $h \in G_K$, it then holds $j(\tilde{f}(h)) = \tilde{f}(j(h))\tilde{f}^{-1} = j(h)$. From the faithfulness of $j$, it follows that $\tilde{f}$ acts trivially on $G_K$, i.e. $f \in \Gamma_K$. 

**Corollary 2.2.** For any $m \geq 3$, the set of Looijenga levels $\{\Gamma_{K,(m)}\}_{K \leq \Pi_{g,n}}$ forms an inverse system of finite index normal subgroups of $\Gamma_{g,n}$ which defines the same profinite topology than the tower of all geometric levels $\{\Gamma_K\}_{K \leq \Pi_{g,n}}$. 


Let us remark that the image of the geometric level $\Gamma^K$ associated to $K$ via the representation $\rho_{K,(m)}$ is contained in the quotient of the centralizer $Z_{G_K}(\text{Sp}(H_1(S_K, \mathbb{Z}/m)))$ by $G_K$. Let us then observe that, by the definition of Looijenga level $s$ and since the abelian level $\Gamma(m)$ of $\Gamma(S_K)$, for all $m \geq 3$, has trivial intersection with $G_K$, the natural epimorphism $N_{G_K}(\Gamma(S_K)) \to \Gamma_{g,[n]}$ induces, for all $m \geq 3$, an isomorphism:

$$N_{G_K}(\Gamma(S_K)) \cap \Gamma(m) \cong \Gamma_{K,(m)}.$$ 

More explicitly, an $f \in \Gamma_{K,(m)}$ has a unique lift $\tilde{f} : S_K \to S_K$ which acts trivially on $H_1(S_K, \mathbb{Z}/m)$. In particular, for every $m \geq 3$, the Looijenga level $\Gamma_{K,(m)}$ has an associated Torelli representation:

$$t_{K,(m)} : \Gamma_{K,(m)} \to Z_{G_K}(\text{Sp}(H_1(S_K, \mathbb{Z}))).$$

A natural question is whether the image of $t_{K,(m)}$ has finite index in the centralizer of $G_K$ in $\text{Sp}(H_1(S_K, \mathbb{Z}))$ for all invariant finite index subgroup $K$ of $\Pi_{g,[n]}$ and $m \geq 3$. This question was addressed positively by Looijenga in [19], for $n = 0$ and the levels associated to the subgroup $\Pi^2$ (the so-called Prym levels).

Looijenga level structures can also be described by means of the following geometric construction.

A marking $\phi : S_{g,n} \overset{\sim}{\to} C$, for $[C] \in \mathcal{M}_{g,[n]}$, identifies the group of automorphims $\text{Aut}(C)$ of a smooth $n$-pointed, genus $g$ curve $C$ with a finite subgroup of $\Gamma_{g,[n]}$. The embedding $\text{Aut}(C) \hookrightarrow \Gamma_{g,[n]}$ is then uniquely determined by $[C]$, modulo inner automorphims. In this way, more generally, we can associate to any subgroup $H$ of $\text{Aut}(C)$ a conjugacy class of finite subgroups of $\Gamma_{g,[n]}$.

The theory of Riemann surfaces with symmetry tells us that the locus of $\mathcal{M}_{g,[n]}$, parametrizing curves which have a group of automorphisms conjugated to a fixed finite subgroup $H$ of $\Gamma_{g,[n]}$, is an irreducible closed substack $\mathcal{M}_H$ of $\mathcal{M}_{g,[n]}$.

Let us assume from now on that the group $H$ acts freely on $S_{g,n}$ and that $\Pi_{g,[n]}$ identifies with a subgroup of $\pi_1(S_{g,n}/H)$ invariant for the action of $\Gamma(S_{g,n}/H)$. In this case, the normalization $\mathcal{M}_H'$ of $\mathcal{M}_H$ is a smooth $H$-gerbe over the moduli stack $\mathcal{M}(S_{g,n}/H)$ of curves homeomorphic to $S_{g,n}/H$. So, there is a natural short exact sequence:

$$1 \to H \to \pi_1(\mathcal{M}_H') \to \Gamma(S_{g,n}/H) \to 1.$$

A connected and analytically irreducible component of the inverse image of $\mathcal{M}_H$ in the Teichmüller space $T_{g,n}$ is given by the fixed point set $T_H$ of the action of the subgroup $H < \Gamma_{g,[n]}$. From this description, it follows, in particular, the natural isomorphism

$$\pi_1(\mathcal{M}_H') \cong \mathcal{N}_H(\Gamma_{g,[n]})$$

mentioned above.

The submanifold $T_H$ is also described as the set $(C, \phi)$ of Teichmüller points of $T_{g,n}$ such that the group of automorphisms of $C$ contains a subgroup which is conjugated to $H$ by means of the homeomorphism $\phi : S_{g,n} \overset{\sim}{\to} C$ (see [9], Theorem A and B).
In general, the substack $\mathcal{M}_H$ of $\mathcal{M}_{g,[n]}$ is not normal. Fixed an embedding of $\text{Aut}(C)$ in $\Gamma_{g,[n]}$, the subgroup $\text{Aut}(C) < \Gamma_{g,[n]}$ may contain, besides $H$, a $\Gamma_{g,[n]}$-conjugate of $H$ distinct from $H$. Such a situation would clearly give rise to a self-intersection of $\mathcal{M}_H$ inside the moduli stack $\mathcal{M}_{g,[n]}$.

Observe, however, that, for $g \geq 1$, the image of $T_H$ in a level structure $\mathcal{M}^\lambda$ over $\mathcal{M}_{g,[n]}$, dominating an abelian level of order at least 3, is normal and therefore smooth. In genus 0, the same holds for all level structures $\mathcal{M}^\lambda$ over $\mathcal{M}_{0,n}$.

In order to prove the above claim, let $f$ be an element of $\Gamma(m)$, for $m \geq 3$, such that both $H$ and $fHf^{-1}$ are contained in the finite subgroup $\text{Aut}(C)$ of $\Gamma_{g,[n]}$. The natural epimorphism $\rho_{(m)}: \Gamma_{g,[n]} \to \text{Sp}_{2g}(\mathbb{Z}/m)$ is injective when restricted to finite subgroups of $\Gamma_{g,[n]}$, for $g \geq 1$ and $m \geq 3$, and it holds:

$$\rho_{(m)}(fHf^{-1}) = \rho_{(m)}(f)\rho_{(m)}(H)\rho_{(m)}(f^{-1}) = \rho_{(m)}(H).$$

Therefore, it holds $fHf^{-1} = H$.

The above argument can be easily adapted to genus 0 replacing the abelian level structure $\mathcal{M}^{(m)}$ by $\mathcal{M}_{0,n}$ and the representation $\rho_{(m)}$ by the natural representation in the symmetric group $\Gamma_{0,[n]} \to \Sigma_n$.

So we have proved:

**Proposition 2.3.** Let $2g - 2 + n > 0$ and $m \geq 3$. The Looijenga level structure $\mathcal{M}^{K,(m)}$ over $\mathcal{M}_{g,n}$ is then isomorphic to a connected component of the inverse image of the closed substack $\mathcal{M}_{G_K}$ of $\mathcal{M}(S_K)$ in the abelian level structure $\mathcal{M}(S_K)^{(m)}$.

The usual way to compactify a level structure $\mathcal{M}^\lambda$ over $\mathcal{M}_{g,n}$ is to take the normalization of $\overline{\mathcal{M}}_{g,n}$ in the function field of $\mathcal{M}^\lambda$. A more functorial definition can be given in the category of log regular schemes as done by Mochizuki in [21]. Let $\partial$ be the logarithmic structure on $\overline{\mathcal{M}}_{g,n}$ associated to the normal crossing divisor $\partial \mathcal{M} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. We define a level structure over $(\overline{\mathcal{M}}_{g,n}, \partial)$ to be a finite, connected, log étale cover

$$(\overline{\mathcal{M}}^\lambda, \partial^\lambda) \to (\overline{\mathcal{M}}_{g,n}, \partial).$$

Then, by log purity Theorem, any level structure $\mathcal{M}^\lambda$ over $\mathcal{M}_{g,n}$ admits a canonical compactification to a level structure $\overline{\mathcal{M}}^\lambda$ over $(\overline{\mathcal{M}}_{g,n}, \partial)$, where $\overline{\mathcal{M}}^\lambda$ is the normalization of $\overline{\mathcal{M}}_{g,n}$ in the function field of $\mathcal{M}^\lambda$ and $\partial^\lambda$ the logarithmic structure associated to the normal crossing divisor $\partial \mathcal{M}^\lambda := \overline{\mathcal{M}}^\lambda \setminus \mathcal{M}^\lambda$ (the Deligne–Mumford boundary of $\overline{\mathcal{M}}^\lambda$). On the other hand, it is also clear that any level structure over $(\overline{\mathcal{M}}_{g,n}, \partial)$ can be realized in this way. So, forgetting the logarithmic structure, one is back to the previous definition. A basic property of (compactified) level structures is the following:

**Proposition 2.4.** If a level $\Gamma^\lambda$ is contained in an abelian level of order $m$, for some $m \geq 3$, then the level structure $\overline{\mathcal{M}}^\lambda$ is represented by a projective variety.

**Proof.** Even though this result is well known, its proof is rather technical and we prefer to give a sketch from the point of view of Teichmüller theory whose ideas will be useful later.
Let \( \bar{T}_{g,n} \) be the Bers compactification of the Teichmüller space \( T_{g,n} \) (see §3, Ch. II in [1] for details on this construction). It is a real analytic space compactifying \( T_{g,n} \). The natural action of \( \Gamma_{g,n} \) on \( T_{g,n} \) extends to \( \bar{T}_{g,n} \). However, this action is not anymore proper and discontinuous, since a boundary stratum has for stabilizer the free abelian group generated by the Dehn twists along the simple closed curves of \( S_{g,n} \) which are contracted on such boundary stratum.

The quotient stack \( \bar{T}_{g,n}/\Gamma_{g,n} \) then admits a natural map, non-representable, because of the extra-inertia at infinity, to the D–M stack \( \bar{M}_{g,n} \).

Thus, for a given level \( \Gamma^\lambda \) of \( \Gamma_{g,n} \), the level structure \( M^\lambda \) is the relative moduli space of the morphism \( \bar{T}_{g,n}/\Gamma^\lambda \to \bar{M}_{g,n} \) (this follows from the universal property of the normalization). In order to prove that \( M^\lambda \) is representable, we have to show that for all \( x = (C, \phi) \in \partial T_{g,n} \), the stabilizer \( \Gamma^\lambda_x \) equals its normal subgroup \( \Gamma^\lambda_x \cap I_x \), where \( I_x \) is the free abelian subgroup of \( \Gamma_{g,n} \) spanned by the Dehn twists along the s.c.c.’s on \( S_{g,n} \) which are contracted by the map \( \phi \).

Now, it is easy to see that \( \Gamma_x/I_x \) is naturally isomorphic to the automorphism group of the complex stable curve \( C \) and that elements in \( \Gamma(m) \cap \Gamma_x \) induce automorphisms acting trivially on \( H_1(C, \mathbb{Z}/m) \). Therefore, the claim and then the proposition follows from the fact that, for \( m \geq 3 \), the only such automorphism is the identity (see the proof of the above proposition sketched by Brylinski in [3]).

Let us return to moduli of curves with symmetry. As above, we assume, for simplicity, that the group \( H \) acts freely on the Riemann surface \( S_{g,n} \) and that \( \Pi_{g,n} \) identifies with an invariant subgroup of \( \pi_1(S_{g,n}/H) \). We have seen that a connected component of the inverse image of the closed substack \( \mathcal{M}_H \) of \( \mathcal{M}_{g,[n]} \) in the abelian level \( \mathcal{M}^{(m)} \) over \( \mathcal{M}_{g,[n]} \) is normal and hence indeed smooth for \( m \geq 3 \). Let us then see what happens to the inverse image of the closure \( \mathcal{M}_H \subset \mathcal{M}_{g,[n]} \) in the compactified abelian level structure \( \bar{\mathcal{M}}^{(m)} \).

Let us observe that \( \mathcal{M}_H \) may have self-intersections just occurring in its D–M boundary. On the other hand, its normalization \( \bar{\mathcal{M}}_H \) is still a smooth \( H \)-gerbe over \( \mathcal{M}(S_{g,n}/H) \) and a connected and analytically irreducible component of the inverse image of \( \mathcal{M}_H \) in the Bers compactification \( \bar{T}_{g,n} \) is given by the fixed point set \( \bar{T}_H \) for the action of the subgroup \( H \) of \( \Gamma_{g,[n]} \).

Above, we have seen that, for \( [C] \in \mathcal{M}_{g,[n]} \), once identified \( \text{Aut}(C) \) with a finite subgroup of \( \Gamma_{g,[n]} \), a self-intersection occurs in the point \( [C] \in \mathcal{M}_H \) whenever \( \text{Aut}(C) \) contains two distinct \( \Gamma_{g,[n]} \)-conjugates of \( H \).

A degenerate marking \( \phi: S_{g,n} \to C \), for \( [C] \in \partial \mathcal{M}_{g,[n]} \), determines a point \( x \in \partial T_{g,n} \). The stabilizer \( \Gamma_x \) of \( x \), for the action of \( \Gamma_{g,[n]} \) on the Bers compactification \( \bar{T}_{g,n} \), is then described by the short exact sequence:

\[
1 \to I_x \to \Gamma_x \to \text{Aut}(C) \to 1,
\]

where \( I_x \) is the free abelian group spanned by the Dehn twists along the s.c.c.’s on \( S_{g,n} \) which are contracted by the marking \( \phi \).

\( \square \)
A self-intersection of \(\overline{M}_H\) occurs at the point \([C] \in \partial \overline{M}_H\), if and only if the subgroup \(\Gamma_x\) of \(\Gamma_{g,[n]}\) contains two conjugates of \(H\) which project to distinct subgroups of \(\text{Aut}(C)\).

In general, the image of \(T_H\) in the level structure \(\overline{M}^\lambda\) over \(\overline{M}_{g,[n]}\) has self-intersections in the points parameterizing the curve \(C\), if and only if \(\Gamma_x\) contains two \(\Gamma^\lambda\)-conjugates of \(H\) which project to distinct subgroups of \(\text{Aut}(C)\).

Now, the fact that the natural representation \(\text{Aut}(C) \to \text{GL}(H_1(C,\mathbb{Z}/m))\), as we remarked in the proof of Proposition \([2.3]\), is injective for \(m \geq 3\), implies that two finite subgroups of \(\Gamma_x\), which differ by conjugation by an element of \(\Gamma(m)\), for \(m \geq 3\), project to the same subgroup of \(\text{Aut}(C)\). This completes the proof of the following theorem (observe that for \(g = 0\) the abelian level structure \(\overline{M}^{(m)}_{g,n}\) is just \(\overline{M}_{0,n}\):

**Theorem 2.5.** i.) For \(2g - 2 + n > 0\), let \(H\) be a finite subgroup of \(\Gamma_{g,[n]}\) acting freely on \(S_{g,n}\) and such that \(\Gamma_{g,n}\) identifies with an invariant subgroup of \(\pi_1(S_{g,n}/H)\). A connected component of the inverse image of \(\overline{M}_H \subset \overline{M}_{g,[n]}\) in a level structure \(\overline{M}^\lambda_{g,n}\), dominating an abelian level structure \(\overline{M}^{(m)}_{g,n}\) for \(m \geq 3\), is normal.

ii.) For \(2g - 2 + n > 0\) and \(m \geq 3\), the Looijenga level structure \(\overline{M}^{K,(m)}_{g,n}\) over \(\overline{M}_{g,n}\) is isomorphic to any of the connected components of the inverse image of the closed substack \(\overline{M}_{G_K}\) of \(\overline{M}(S_K)\) in the abelian level structure \(\overline{M}(S_K)^{(m)}\).

**Remark 2.6.** From the above description of a Looijenga level structure \(\overline{M}^{K,(m)}_{g,n}\), for \(m \geq 3\), we can derive a simple criterion for its smoothness.

The normalization \(\overline{M}_{G_K}\) of the \(G_K\)-gerbe \(\overline{M}_{G_K}\) is smooth and \(\overline{M}^{K,(m)}_{G_K}\) is isomorphic to a connected component of the cartesian product \(\overline{M}_{G_K} \times \overline{M}(S_K)\overline{M}(S_K)^{(m)}\). Therefore, it is clear that \(\overline{M}^{K,(m)}_{G_K}\), for \(m \geq 3\), is smooth whenever the inverse image of the closed substack \(\overline{M}_{G_K}\) of \(\overline{M}(S_K)\) in the abelian level structure \(\overline{M}(S_K)^{(m)}\) avoids its singular locus.

There is a very elementary and effective method to describe the compactifications \(\overline{M}^\lambda\), locally in the analytic topology. A neighborhood of a point \(x \in \mathcal{M}^\lambda\) is just the base of the local universal deformation of the fiber \(\mathcal{E}_x^\lambda\) of the universal family \(\mathcal{E}^\lambda \to \mathcal{M}^\lambda\). Let us then see how an analytic neighborhood of a point \(x \in \partial \mathcal{M}^\lambda\) can be described.

Let \(\mathcal{B} \to \overline{M}_{g,n}\) be an analytic neighborhood of the image \(y\) of \(x\) in \(\overline{M}_{g,n}\) such that:

- local coordinates \(z_1, \ldots, z_{3g-3+n}\) embeds \(\mathcal{B}\) in \(\mathbb{C}^{3g-3+n}\) as an open ball;

- \(C := \pi^{-1}(y)\) is the most degenerate curve in the pull-back \(\mathcal{E} \xrightarrow{\pi} \mathcal{B}\) of the universal family over \(\mathcal{B}\);

- an étale groupoid representing \(\overline{M}_{g,n}\) trivializes over \(\mathcal{B}\) to \(\text{Aut}(C) \times \mathcal{B} \xrightarrow{\pi} \mathcal{B}\).

Let \(\{Q_1, \ldots, Q_s\}\) be the set of singular points of \(C\) and let \(z_i\), for \(i = 1, \ldots, s\), parametrize curves where the singularity \(Q_i\) subsists. The discriminant locus \(\partial \mathcal{B} \subset \mathcal{B}\) of \(\pi\) has then
Theorem 2.7. With the above notations, the kernel of $\psi_U(m)$ is given by:

$$m N_{\Sigma(C)} + P_{\Sigma(C)} + S_{\Sigma(C)}.$$ 

Therefore, the singular locus of the abelian level structure $\overline{\mathcal{M}}^{(m)}$, for $m \geq 3$, is contained in the strata parametrized by cut pairs on $S_{g,n}$.
It is easy to determine the boundary strata of $\overline{M}(S_K)^{(m)}$ which are met by $p^{-1}(\overline{M}_{G_K})$ in terms of the cover $p_K: S_K \to S_{g,n}$. Let $\sigma = \{\gamma_0, \ldots, \gamma_k\}$ be a set of disjoint and non-homotopic, non-trivial s.c.c.’s on $S_{g,n}$. The inverse image $p_K^{-1}(\sigma)$ is a set of s.c.c. on $S_K$ with the same properties and so determines a closed boundary stratum $B_{p_K^{-1}(\sigma)}^{(m)}$ of $\overline{M}(S_K)^{(m)}$ (see Section 3 for this notation). Now, it is clear that:

$$p^{-1}(\overline{M}_{G_K}) \cap \partial \overline{M}(S_K)^{(m)} \subset \bigcup_{\sigma \in S_{g,n}} B_{p_K^{-1}(\sigma)}^{(m)}.$$ 

Therefore, if the cover $p_K: S_K \to S_{g,n}$ is such that, for all sets $\sigma$ of s.c.c.’s as above, the inverse image $p_K^{-1}(\sigma)$ does not contain cut pairs, then, by Remark 2.6 and Theorem 2.7, the Looijenga level structure $\overline{M}(S_K)^{(m)}$ is smooth for all $m \geq 3$.

It is now possible both to clarify and improve substantially the main result of [18]. From our perspective, all we need of [18] is Proposition 2, which states that, for $g \geq 2$ and $K = \Pi^2_2$, $\Pi^2_2$, the cover $p_K: S_K \to S_g$ is such that, for any set $\sigma$ of non-trivial s.c.c.’s on $S_g$, as above, the inverse image $p_K^{-1}(\sigma)$ does not contain cut pairs. It is not difficult to extend this result to the invariant subgroup $\Pi_2^1 \cdot \Pi_2^1$, where $p$ is any prime $\geq 2$.

Moreover, we can generalize Looijenga’s results to all pairs $(g, n)$ such that $2g - 2 + n > 0$ and find, for any given invariant finite index subgroup $K$ of $\Pi_{g,n}$, an invariant finite index subgroup $H$ of $\Pi_{g,n}$, contained in $K$, such that the corresponding Looijenga level structure $\overline{M}^{K,(m)}$ over $\overline{M}_{g,n}$ is smooth for all $m \geq 3$.

Let, indeed, $K$ be an invariant finite index proper subgroup of $\Pi_{g,n}$ such that, if $p_K: S_K \to S_{g,n}$ is the associated cover, for any set $\sigma$ of s.c.c.’s on $S_{g,n}$ as above, $S_K \setminus p_K^{-1}(\sigma)$ does not contain punctured open discs or cylinders (by Hurwitz’s Theorem, such covers are easy to construct). Let $N_K$ be the kernel of the natural epimorphism $\pi_1(S_K) \to \pi_1(\overline{S}_K)$ (induced filling in the punctures of $S_K$). Then $K' := K^2 \cdot N_K$, for a prime $p \geq 2$. From Looijenga’s result, it then follows that the cover $p_{K'}: S_{K'} \to S_{g,n}$ is such that, for any set $\sigma$ of non-trivial, disjoint and non-homotopic s.c.c.’s on $S_{g,n}$, the inverse image $p_{K'}^{-1}(\sigma)$ does not contain cut pairs. So, we have:

**Theorem 2.8.** For $2g - 2 + n > 0$, let $K$ be an invariant finite index subgroup of $\Pi_{g,n}$ and let $\overline{M}^{K,(m)}$ be the associated Looijenga level structure over $\overline{M}_{g,n}$.

i.) If the cover $p_K: S_K \to S_{g,n}$ is such that, for any set $\sigma$ of disjoint and non-homotopic, non-trivial s.c.c.’s on $S_{g,n}$, the inverse image $p_K^{-1}(\sigma)$ does not contain cut pairs, then the Looijenga level structure $\overline{M}^{K,(m)}$ is smooth for all $m \geq 3$.

ii.) Let $K$ be an invariant finite index proper subgroup of $\Pi_{g,n}$ with the property that the associated cover $p_K: S_K \to S_{g,n}$ is such that, for any set $\sigma$ of s.c.c.’s on $S_{g,n}$ as above, $S_K \setminus p_K^{-1}(\sigma)$ does not contain punctured open discs or cylinders. Let $K'_p$, for a prime $p \geq 2$, be the subgroup of $K$ defined above. Then, the associated Looijenga level structure $\overline{M}^{K'_p,(m)}$ is smooth for all $m \geq 3$. 
In i.) of Theorem 2.8 let us assume, moreover, that the cover \( p_K : S_K \to S_{g,n} \) is such that \( p_K^{-1}(\gamma) \) consists of non-separating s.c.c. for all non-trivial s.c.c. \( \gamma \) on \( S_{g,n} \). It is not difficult then to determine explicitly the local monodromy coefficients for the levels \( \Gamma_K^{(m)} \).

For a given non-trivial s.c.c. \( \gamma \) on \( S_{g,n} \), let us denote by \( c_\gamma \) the order of an element of \( \Pi_{g,n} \) freely isotopic to \( \gamma \) in the quotient group \( G_K = \Pi_{g,n}/K \). By the invariance of \( K \), the positive integer \( c_\gamma \) depends only on the topological type of \( S_{g,n} \setminus \gamma \). Let also \( c \) be the order of the finite group \( G_K \).

Let us then denote by \( k_\gamma \) the order of the image of the Dehn twist \( \tau_\gamma \) in the quotient group \( \Gamma_{g,n}/\Gamma_K^{(m)} \).

Let us show first that \( c_\gamma \) divides \( k_\gamma \). This follows from the fact that any lift to \( \overline{S}_K \) of a power \( \tau_\gamma^k \), where \( c_\gamma \) does not divide \( k \), either switches the connected components of \( \overline{S}_K \setminus p_K^{-1}(\gamma) \), and does not act trivially on \( H_1(\overline{S}_K, \mathbb{Z}/m) \) for \( m \geq 2 \), or restricts to a non-trivial finite order homeomorphism of some connected component \( S' \) of \( \overline{S}_K \setminus p_K^{-1}(\gamma) \), which then acts non-trivially on the image of \( H_1(S', \mathbb{Z}/m) \) in \( H_1(S_K, \mathbb{Z}/m) \), for \( m \geq 3 \).

The power \( \tau_\gamma^{c_\gamma} \in \Gamma_{g,n} \) of the given Dehn twist lifts canonically to a product \( \prod_{\gamma \in L_\gamma} \tau_\gamma \in \Gamma(S_K) \) of Dehn twists along a set \( L_\gamma \) of disjoint s.c.c. on \( S_K \) of cardinality \( c/c_\gamma \). By hypothesis, the set \( L_\gamma \) consists of non-separating s.c.c. and does not contain cut pairs. By Proposition 1 in [18], this implies that the image of \( \prod_{\gamma \in L_\gamma} \tau_\gamma \) in \( \text{Sp}(H_1(S_K, \mathbb{Z}/m)) \) has order \( m \). Therefore, it holds \( k_\gamma = mc_\gamma \) and we have proved:

**Proposition 2.9.** In the same hypotheses of item i.) of Theorem 2.8 let us assume, moreover, that the cover \( p_K : S_K \to S_{g,n} \) is such that, for all non-trivial s.c.c. \( \gamma \) on \( S_{g,n} \), the inverse image \( p_K^{-1}(\gamma) \) consists of non-separating s.c.c.’s. For a given non-trivial s.c.c. \( \gamma \) on \( S_{g,n} \), let \( c_\gamma \) be the order of an element of \( \Pi_{g,n} \) freely isotopic to \( \gamma \) in the quotient group \( G_K = \Pi_{g,n}/K \). With the same notations of Theorem 2.7, the kernel of \( \psi_U^{(m)} \) is given by:

\[
\sum_{e \in \Sigma(C)} mc_\gamma e.
\]

3 The boundary of level structures and the complex of curves

Let us recall Knudsen’s description of the Deligne-Mumford boundary of \( \overline{M}_{g,n} \). Let \( H = \{h_1, h_2, ..., h_{n_1}\} \) and \( K = \{k_1, k_2, ..., k_{n_2}\} \) be complementary subsets of \( \{1, 2, ..., n\} \) of cardinality \( n_1 \) and \( n_2 \) respectively. Let \( g_1 \) and \( g_2 \) be non-negative integers with \( g = g_1 + g_2 \) and satisfying the condition that \( n_i \geq 2 \) when \( g_i = 0 \). There are finite morphisms

\[
\beta_0 : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}
\]

\[
\beta_{g_1,g_2,H,K} : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}.
\]

Let us describe them more explicitly. Let \( [C] \in \overline{M}_{g-1,n+2} \), then \( \beta_0([C]) \) is the class of the curve obtained identifying to a node the labeled points \( P_{n+1} \) and \( P_{n+2} \) of \( C \). Similarly,
if \(([C_1], [C_2]) \in \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}\), then \(\beta_{g_1,g_2,H,K}([C_1], [C_2])\) is the class of the curve obtained from \(C_1 \amalg C_2\) identifying to a node the two labels on \(C_1\) and \(C_2\) not included in \(H\) and \(K\), respectively.

These morphisms define closed substacks \(B^0_{g,n}\) and \(B_{g_1,g_2,H,K}\) of \(\overline{\mathcal{M}}_{g,n}\), which are irreducible components of the boundary, and all the irreducible components of the boundary are obtained in this way.

More generally, for a point \([C] \in \overline{\mathcal{M}}_{g,n}\), let \(C_{g_1,n_1} \amalg \ldots \amalg C_{g_h,n_h}\) be the normalization of \(C\), where \(C_{g_i,n_i}\), for \(i = 1, \ldots, h\), is a genus \(g_i\) smooth curve with \(n_i\) labels on it (the labels include also the inverse images of singularities in \(C\)). Then, there is a natural morphism, which we call boundary map,

\[
\beta_C : \overline{\mathcal{M}}_{g_1,n_1} \times \cdots \times \overline{\mathcal{M}}_{g_h,n_h} \rightarrow \overline{\mathcal{M}}_{g,n}.
\]

The image of \(\mathcal{M}_{g_1,n_1} \times \cdots \times \mathcal{M}_{g_h,n_h}\) by \(\beta_C\) parametrizes curves homeomorphic to \(C\) and is called a stratum. We denote the restriction of \(\beta_C\) to \(\mathcal{M}_{g_1,n_1} \times \cdots \times \mathcal{M}_{g_h,n_h}\) by \(\hat{\beta}_C\) and call it a stratum map. In general, none of these morphisms is an embedding.

An analogue of the Deligne-Mumford compactification \(\overline{\mathcal{M}}_{g,n}\) in Teichmüller theory is the construction of a boundary with corners for the Teichmüller space \(T_{g,n}\).

Let \(\Pi_{C/\mathbb{R}}\) be the functor restriction of the scalars from \(C\) to \(\mathbb{R}\). Let us take the blow-up of \(\Pi_{C/\mathbb{R}}(\overline{\mathcal{M}}_{g,n})\) in \(\Pi_{C/\mathbb{R}}(\partial \mathcal{M}_{g,n})\). Its real points have a natural structure of real analytic stack. Cutting along the exceptional divisor, we get a real analytic stack with corners, denoted by \(\hat{\mathcal{M}}_{g,n}\) and called the real oriented blow-up of \(\overline{\mathcal{M}}_{g,n}\) along the D–M boundary.

The boundary \(\partial \hat{\mathcal{M}}_{g,n} := \hat{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}\) is naturally homeomorphic to a deleted tubular neighborhood of the D–M boundary of \(\overline{\mathcal{M}}_{g,n}\) and the natural projection \(\hat{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}\) restricts over each codimension \(k\) stratum to a bundle in \(k\)-dimensional tori. The inclusion \(\mathcal{M}_{g,n} \hookrightarrow \hat{\mathcal{M}}_{g,n}\) is a homotopy equivalence and then induces an inclusion of the respective universal covers \(\hat{T}_{g,n} \hookrightarrow \hat{\mathcal{M}}_{g,n}\).

From Proposition 2.4, it follows that \(\hat{T}_{g,n}\) is representable and, therefore, is a real analytic manifold with corners containing \(T_{g,n}\) as an open dense submanifold. The ideal boundary of Teichmüller space is defined by

\[
\partial \hat{\mathcal{T}}_{g,n} := \hat{T}_{g,n} \setminus T_{g,n}.
\]

It is possible to give a complete description of the ideal boundary of Teichmüller space. By means of the results of Knudsen displayed above and Fenchel-Nielsen coordinates, one can prove that an irreducible component of \(\partial \hat{\mathcal{T}}_{g,n}\) lying above \(\beta_0\) or \(\beta_{g_1,g_2,H,K}\) is isomorphic respectively to

\[
\mathbb{R}^+ \times \hat{T}_{g-1,n+2} \quad \text{and} \quad \mathbb{R}^+ \times \hat{T}_{g_1,n_1+1} \times \hat{T}_{g_2,n_2+1}.
\]

Likewise, the intersection of \(k\) irreducible components, lying above the boundary map \(\beta_C\) is isomorphic to \((\mathbb{R}^+)^k \times \hat{T}_{g_1,n_1} \times \cdots \times \hat{T}_{g_h,n_h}\).

Thus both the irreducible components of \(\partial \hat{T}_{g,n}\) and their intersections are contractible. So \(\partial \hat{T}_{g,n}\) is homotopically equivalent to the geometric realization of the nerve of this covering. By means of Fenchel-Nielsen coordinates, such nerve can be realized as the simplicial
complex whose simplices are given by sets of distinct, non-trivial isotopy classes of s.c.c. on $S_{g,n}$, such that they admit a set of disjoint representatives, none of them bounding a disc with a single puncture. This is the complex of curves $C(S_{g,n})$ of $S_{g,n}$. It is easy to check that the dimension of $C(S_{g,n})$ is one less the dimension of the moduli space $\mathcal{M}_{g,n}$, i.e.: $n - 4$ for $g = 0$ and $3g - 4 + n$ for $g \geq 1$.

There is a natural simplicial action of $\Gamma_{g,n}$ on $C(S_{g,n})$ which is compatible with the action on $\partial \hat{T}_{g,n}$.

Let $\hat{\mathcal{M}}_{g,n}$ be the oriented real blow-up of $\mathcal{M}_{g,n}$ along $\partial \mathcal{M}$. It holds $\hat{\mathcal{M}}_{g,n} \cong [\hat{T}_{g,n}/\Gamma_{g,n}]$ and then also $\partial \hat{\mathcal{M}}_{g,n} \cong [(\partial \hat{T}_{g,n})/\Gamma_{g,n}]$. Therefore, the quotient $|C(S_{g,n})|/\Gamma$ is a cellular complex which realizes the nerve of the covering of $\partial \hat{\mathcal{M}}_{g,n}$ by irreducible components and, thus, the nerve of the covering of $\partial \mathcal{M}$ by irreducible components. Let then $C(\mathcal{M}_{g,n})$, be the semi-simplicial set associated to the set of cells of $|C(S_{g,n})|/\Gamma$ and their face maps.

In particular, to a simplex $\mu \in C(S_{g,n})_k$, for $k \geq 0$, is associated a boundary map $\beta_C$ to $\hat{\mathcal{M}}_{g,n}$ whose image is a stratum of codimension $k + 1$. Consider the fibred product

$$
\begin{array}{ccc}
X & \longrightarrow & \hat{\mathcal{M}}_{g_1,n_1} \times \cdots \times \hat{\mathcal{M}}_{g_h,n_h} \\
\downarrow \beta_C & \square & \downarrow \beta_C \\
\hat{\mathcal{M}}_{g,n} & \longrightarrow & \mathcal{M}_{g,n}.
\end{array}
$$

By log étale base change, one has that $X$ is a normal D–M stack. In particular, its connected components are irreducible and then in bijective correspondence with the simplices in $C(\mathcal{M}_{g,n})_k$ in the orbit of $\mu$ by the action of $\Gamma_{g,n}$. For a simplex $\sigma$ in the $\Gamma_{g,n}/\Gamma$-orbit of $\mu$, let us denote by $\beta_\sigma^C : \delta_\sigma^C \to \hat{\mathcal{M}}_{g,n}$ the restriction of $\beta_C$ to the corresponding connected component of $X$ and call it the boundary map of the level structure $\hat{\mathcal{M}}_{g,n}$ in $\sigma$. We then denote by $B_\sigma^\lambda$ the image of the morphism $\beta_\sigma^\lambda$ and call it the closed boundary stratum of the level structure $\hat{\mathcal{M}}_{g,n}$ in $\sigma$.

We call the restriction $\beta_\sigma^\lambda : \delta_\sigma^\lambda \to \hat{\mathcal{M}}_{g,n}$ of $\beta_\sigma^\lambda$ over $\mathcal{M}_{g_1,n_1} \times \cdots \times \mathcal{M}_{g_h,n_h}$ the stratum map of $\hat{\mathcal{M}}_{g,n}$ in $\sigma$ and its image $\hat{B}_\sigma^\lambda$ is the boundary stratum of $\hat{\mathcal{M}}_{g,n}$ in $\sigma$. By log étale base change, the natural morphism $\delta_\sigma^\lambda \to \mathcal{M}_{g_1,n_1} \times \cdots \times \mathcal{M}_{g_h,n_h}$ is étale.

Similar definitions can be given with $\hat{\mathcal{M}}_{g,n}$ in place of $\hat{\mathcal{M}}_{g,n}$. For a simplex $\sigma \in C(S_{g,n})$, we define the ideal boundary map $\hat{\beta}_\sigma^\lambda : \hat{\Delta}_\sigma \to \hat{\mathcal{M}}_{g,n}$, as the pull-back of $\beta_\sigma^\lambda$ via the blow-up map $\hat{\mathcal{M}}_{g,n} \to \hat{\mathcal{M}}_{g,n}$.

The fundamental group of $\hat{\Delta}_\sigma$ is described as follows. Let $\delta_\sigma^\lambda$ be the oriented real blow-up of $\delta_\sigma$ along the divisor $\delta_\sigma^2 \setminus \delta_\sigma^1$. The embedding $\delta_\sigma^2 \to \delta_\sigma^1$ is a homotopy equivalence and $\hat{\Delta}_\sigma$ is a bundle over $\delta_\sigma^1$ in $k$-dimensional tori. Hence, there is an isomorphism $\pi_1(\hat{\Delta}_\sigma) \cong \pi_1(\delta_\sigma^1)$ and a short exact sequence:

$$
1 \to \bigoplus_{\gamma \in \sigma} \mathbb{Z} \cdot \gamma \to \pi_1(\hat{\Delta}_\sigma) \to \pi_1(\delta_\sigma^1) \to 1.
$$

A connected component of the fibred product $\hat{\Delta}_\sigma \times _{\hat{\mathcal{M}}_{g,n}} \hat{T}_{g,n}$ is naturally isomorphic
to \((\mathbb{R}^+)^k \times \hat{T}_{g_1,n_1} \times \cdots \times \hat{T}_{g_n,n_h}\). Therefore, the fundamental group of \(\hat{\Delta}_\sigma\) is isomorphic to the subgroup of elements in \(\Gamma_{g,n}\) which stabilize one of these connected components, preserving, moreover, the order of its factors. So, if we let

\[
\Gamma_\sigma := \{ f \in \Gamma_{g,n} | f(\overline{\gamma}) = \overline{\gamma}, \forall \gamma \in \sigma \},
\]

where \(\overline{\gamma}\) is the s.c.c. \(\gamma\) with a fixed orientation, it holds \(\pi_1(\hat{\Delta}_\sigma) \cong \Gamma_\sigma\) and, for \(\lambda\) the trivial level, the above short exact sequence takes the more familiar form

\[
1 \rightarrow \bigoplus_{\gamma \in \sigma} \mathbb{Z} \gamma \rightarrow \Gamma_\sigma \rightarrow \Gamma_{g_1,n_1} \times \cdots \times \Gamma_{g_n,n_h} \rightarrow 1.
\]

By the same argument, more generally, it holds \(\pi_1(\hat{\Delta}_\sigma^\lambda) \cong \Gamma^\lambda \cap \Gamma_\sigma\).

It is also clear that the fundamental group of \(\beta_\sigma(\hat{\Delta}_\sigma)\) is isomorphic to the stabilizer \(\Gamma_\sigma\) of \(\sigma\), for the action of \(\Gamma_{g,n}\) on \(C(S_{g,n})\) and more generally, it holds \(\pi_1(\beta_\sigma(\hat{\Delta}_\sigma^\lambda)) \cong \Gamma^\lambda \cap \Gamma_\sigma\).

Let us observe that an \(f \in \Gamma_\sigma\) can switch the s.c.c. in \(\sigma\) as well as their orientations. So, denoting by \(\Sigma_\sigma\{\pm\}\) the group of signed permutations of the set \(\sigma\), there is an exact sequence

\[
1 \rightarrow \Gamma^\lambda \cap \Gamma_\sigma \rightarrow \Gamma^\lambda \cap \Gamma_\sigma \rightarrow \Sigma_\sigma\{\pm\} \rightarrow 1
\]

and the Galois group of the étale cover \(\hat{\delta}_\sigma^\lambda \rightarrow \text{Im} \beta_\sigma^\lambda\) is isomorphic to the image of \(\Gamma^\lambda \cap \Gamma_\sigma\) in \(\Sigma_\sigma\{\pm\}\). Hence, the stratum map \(\beta_\sigma^\lambda\) is injective if and only if \(\Gamma^\lambda \cap \Gamma_\sigma = \Gamma^\lambda \cap \Gamma_\sigma\).

**Proposition 3.1.** For \(2g - 2 + n > 0\), let \(\bar{\mathcal{M}}^\mu\) be a smooth level structure over \(\bar{\mathcal{M}}_{g,n}\) dominating an abelian level structure \(\mathcal{M}^{(m)}\), for \(m \geq 3\). Then, for any level structure \(\bar{\mathcal{M}}^\lambda\) dominating \(\bar{\mathcal{M}}^\mu\), all boundary maps \(\delta_\sigma^\lambda \rightarrow \bar{\mathcal{M}}^\lambda\) are embeddings.

**Proof.** Let \(\sigma\) be a simplex in \(C(S_{g,n})\). By the above criterion, a stratum map \(\hat{\delta}_\sigma^\lambda \rightarrow \bar{\mathcal{M}}^\lambda\) is an embedding if and only if \(\Gamma^\lambda \cap \Gamma_\sigma = \Gamma^\lambda \cap \Gamma_\sigma\), i.e. if and only if an element \(f \in \Gamma^\lambda\) which fixes the set \(\sigma\), keeps fixed every s.c.c. in the set, preserving its orientation. So, in order to prove that the boundary map \(\hat{\delta}_\sigma^\lambda \rightarrow \bar{\mathcal{M}}^\lambda\) is an embedding, it is enough to show that, for any two disjoint oriented s.c.c.’s \(\overline{\gamma}_0, \overline{\gamma}_1\) on \(S_{g,n}\), which differ either in the isotopy class or in orientation, there is no \(f \in \Gamma^\lambda\) such that \(f(\overline{\gamma}_0) = \overline{\gamma}_1\). That there is no such \(f \in \Gamma^\mu\) and then in \(\Gamma^\lambda\) follows from simple observations and the proof of Proposition 2.1 in [7].

An immediate application of Proposition 3.1 is that, for the levels \(\Gamma^\lambda\) which satisfy its hypotheses, there is a direct way to realize the nerve of the D–M boundary of the associated level structure in the category of simplicial sets. From the remark which precedes the proposition, it follows indeed that such levels \(\Gamma^\lambda\) of \(\Gamma_{g,n}\) operate without inversion on the curve complex \(C(S_{g,n})\). So, let \(C(S_{g,n})_\bullet\) be the simplicial set associated to \(C(S_{g,n})\) and an ordering of its vertices compatible with the action of \(\Gamma^\lambda\) and let us define the simplicial finite set \(C^\lambda(S_{g,n})_\bullet\) to be the quotient of \(C(S_{g,n})_\bullet\) for the simplicial action of \(\Gamma^\lambda\). This simplicial set clearly describes the nerve of the covering of \(\partial \mathcal{M}^\lambda\) by irreducible components.
4  Profinite curve complexes

Let $X_\bullet$ be a simplicial set and $G$ a group. A geometric action of $G$ on $X_\bullet$ is an action of $G$ on $X_n$, defined for all $n \geq 0$, satisfying the following conditions:

i.) For all $g \in G$ and $\sigma \in X_n$, one has $\{ \partial_i g \cdot \sigma | i = 0, \ldots, n \} = \{ g \cdot \partial_i \sigma | i = 0, \ldots, n \}$. In this way, it is defined a permutation representation $\rho_\sigma : G \to \Sigma_n$ such that we require to be constant on the $G$-orbits, i.e. $\rho_\sigma(g) = \rho_{g \cdot \sigma}(g)$, for all $g, g' \in G$.

ii.) By the above condition, an action of $G$ on $\coprod_{n \geq 0} X_n \times \Delta_n$ is defined letting, for $(\sigma, x) \in X_n \times \Delta_n$ and $g \in G$:

$$g \cdot (\sigma, x) := (g \cdot \sigma, \rho_\sigma(g)(x)).$$

We then require that this action be compatible with the equivalence relation $\sim$ which defines the geometric realization $|X_\bullet| := \coprod_{n \geq 0} X_n \times \Delta_n / \sim$.

If the actions $G \times X_n \to X_n$ commute with the face and degeneracy operators, for $n \geq 0$, then $X_\bullet$ is said to be a simplicial $G$-set. In this case the quotient sets $X_n/G$, for $n \geq 0$, together with the induced face and degeneracy operators, form a simplicial set $X_\bullet/G$. Moreover, since the representations $\rho_\sigma$ are trivial for all $\sigma \in X_\bullet$, there is a natural isomorphism $|X_\bullet|/G \cong |X_\bullet/G|$. Observe that, by means of a barycentric subdivision, we may replace a simplicial set $X_\bullet$ on which $G$ acts geometrically, by a simplicial $G$-set $X_\bullet'$, such that there is a natural isomorphism $|X_\bullet| \cong |X_\bullet'|$ of $G$-CW-complex.

Let $G$ be a profinite group and $X_\bullet$ a simplicial profinite set (i.e. a simplicial object in the category formed by profinite sets and continuous maps). We define a continuous geometric action of $G$ on $X_\bullet$ to be a geometric action such that each action $G \times X_n \to X_n$ is continuous and with open orbits, for $n \geq 0$. Moreover, we require that there exists an open subgroup $U \leq G$ such that $X_\bullet$, with the induced $U$-action, is a simplicial $U$-set.

Let $\{G^\lambda\}_{\lambda \in \Lambda}$ be the tower of open subgroups of $G$ contained in $U$. Then, for all $\lambda \in \Lambda$, the quotient $X_\bullet/G^\lambda$ is a simplicial discrete finite set endowed with a geometric continuous action of $G$.

Let now $G \to G'$ be a homomorphism of a discrete group in a profinite group, with dense image, and let $\{G^\lambda\}_{\lambda \in \Lambda}$ be the tower of subgroups of $G$ which are inverse images of open subgroups of $G'$. Let $X_\bullet$ be a simplicial set endowed with a geometric $G$-action such that the set of $G$-orbits in $X_n$ is finite for all $n \geq 0$. Assume, moreover, that there exists a $\mu \in \Lambda$ such that $X_\bullet$, with the induced $G^\mu$-action, is a simplicial $G^\mu$-set.

The $G'$-completion of $X_\bullet$ is defined to be the simplicial profinite set

$$X_\bullet' := \lim_{\lambda \in \Lambda} X_\bullet/G^\lambda,$$

which is then endowed with a natural continuous geometric action of $G'$. The simplicial profinite set $X_\bullet$ has the following universal property. Let $f : X_\bullet \to Y_\bullet$ be a simplicial $G$-equivariant map, where $Y_\bullet$ is a simplicial profinite set endowed with a continuous geometric
action of $G'$. Then $f$ factors uniquely through the natural map $X_\bullet \to X'_\bullet$ and a simplicial $G'$-equivariant continuous map $f' : X'_\bullet \to Y_\bullet$.

Let us apply the definition of profinite $G'$-completion of a simplicial set endowed with a geometric $G$-action to the the Teichmüller group $\Gamma_{g,n}$ and a profinite completion $\Gamma'_{g,n}$ defined by a tower $\{\Gamma^\lambda\}_{\lambda \in \Lambda}$ of levels of $\Gamma_{g,n}$ which contains a level $\Gamma^\mu$ satisfying the hypotheses of Proposition 3.1.

From the remarks following Proposition 3.1, it follows that there is an ordering of the vertex set of the simplicial complex $C(S_{g,n})$ such that the associated simplicial set $C(S_{g,n})_\bullet$ is endowed with a natural geometric $\Gamma_{g,n}$-action. For this ordering, the conditions prescribed in order to define the $\Gamma'_{g,n}$-completion of the simplicial set $C(S_{g,n})_\bullet$ are all satisfied. We let then $C'(S_{g,n})_\bullet$ denote the $\Gamma'_{g,n}$-completion of $C(S_{g,n})_\bullet$. It comes with a natural $\Gamma_{g,n}$-equivariant map of simplicial sets $i_* : C(S_{g,n})_\bullet \to C'(S_{g,n})_\bullet$.

Let us recall the following result from [5]:

**Proposition 4.1.** Let $\Gamma_{g,n}$, for $2g - 2 + n > 0$, be a profinite completion $\Gamma'_{g,n}$ defined by a tower $\{\Gamma^\lambda\}_{\lambda \in \Lambda}$ of levels of $\Gamma_{g,n}$ such that, for some $\mu \in \Lambda$, the level $\Gamma^\mu$ satisfies the hypotheses of Proposition 3.1. Then, the natural homomorphism $i : \Gamma_{g,n} \to \Gamma'_{g,n}$ is injective if and only if the induced map $i_* : C(S_{g,n})_\bullet \to C'(S_{g,n})_\bullet$ is injective.

**Proof.** Let us assume first $i : \Gamma_{g,n} \to \Gamma'_{g,n}$ is injective. Let then $\alpha, \beta \in C(S_{g,n})_0$ and let $\tau_\alpha, \tau_\beta$ be the corresponding Dehn twists in $\Gamma_{g,n}$. Suppose that $i_*(\alpha) = i_*(\beta)$. This means that for every $\lambda \in \Lambda$ there is an $f_\lambda \in \Gamma^\lambda$, such that $f_\lambda(\alpha) = \beta$. From the identity $f_\lambda : \tau_\alpha \cdot f_\lambda^{-1} = f_\lambda \tau_\alpha = \beta$, it follows that $\tau_\alpha \equiv \tau_\beta \mod \Gamma^\lambda$ for every $\lambda \in \Lambda$ and then $i(\tau_\alpha) = i(\tau_\beta)$. Since $i$ is injective, one has $\alpha = \beta$.

For the other direction, we need a remark by N. Ivanov (see [14]). There is a natural representation $r : \Gamma_{g,n} \to \text{Aut}(C(S_{g,n})_\bullet)$ whose kernel is exactly the center of $\Gamma_{g,n}$. It is well known that for $g \geq 3$, $g = 0$ and $n \geq 4$, $g = 2$ and $n \geq 1$ or $g = 1$, $n \geq 3$ the center of $\Gamma_{g,n}$ is trivial. For $g = 2$ and $n = 0$ or $g = 1$ and $n \leq 1$ instead, the center is isomorphic to $\mathbb{Z}/2$ and is generated by, respectively, the elliptic and the hyperelliptic involution.

Let then $\text{Aut}(C'(S_{g,n})_\bullet)$ be the group of continuous automorphisms of the simplicial profinite set $C'(S_{g,n})_\bullet$. There is a natural continuous representation $\varphi : \Gamma'_{g,n} \to \text{Aut}(C'(S_{g,n})_\bullet)$ induced by $r$. Since $i_* : C(S_{g,n})_\bullet \to C'(S_{g,n})_\bullet$ has dense image, $\ker r \leq \ker(\varphi \cdot i)$. Therefore, $i_*$ induces a homomorphism $i^\sharp : \text{Im} r \to \text{Aut}(C'(S_{g,n})_\bullet)$. If $i_*$ is injective, then also the homomorphism $i^\sharp$ is injective. From the commutative diagram

\[
\begin{array}{ccc}
\Gamma_{g,n} & \xrightarrow{i} & \Gamma'_{g,n} \\
\downarrow r & & \downarrow \varphi \\
\text{Im} r & \xrightarrow{i^\sharp} & \text{Aut}(C'(S_{g,n})_\bullet),
\end{array}
\]

it follows that $\ker i \leq \ker r$. It is a classical result that $\Gamma(m)$, for $m \geq 3$, does not contain torsion elements. Since, by the above remark, the group $\ker r$ is finite, it follows that $\ker r$ injects in $\Gamma_{g,n}/\Gamma(m)$ and then in $\Gamma'_{g,n}$. Therefore, $\ker i = \{1\}$. 

\[\square\]
Another fundamental general result from \[5\] is the following:

**Proposition 4.2.** Let \(\Gamma_{g,n}'\), for \(2g - 2 + n > 0\), be a profinite completion of \(\Gamma_{g,n}\) satisfying the hypotheses of Proposition 4.1. Let then \(\Gamma^\lambda\) be a level of \(\Gamma_{g,n}\) and \(\tilde{\Gamma}^\lambda\) be its closure in the profinite completion \(\Gamma_{g,n}'\). The stabilizer \(\tilde{\Gamma}^\lambda_{\sigma}\) of a simplex \(\sigma \in \text{Im} C(S_{g,n})_\bullet \subset C'(S_{g,n})_\bullet\) is the closure in \(\Gamma_{g,n}'\) of the image of the stabilizer \(\Gamma^\lambda_{\sigma}\) for the action of \(\Gamma^\lambda\) on \(C(S_{g,n})_\bullet\).

**Proof.** Let \(\{\Gamma^\lambda\}_{\lambda \in \Lambda}\) be the tower of Galois levels of \(\Gamma_{g,n}\) inverse image of the tower of open normal subgroups of \(\Gamma_{g,n}'\). For all \(\lambda' \in \Lambda\) dominating \(\lambda\) and the level \(\mu\) (such that \(\overline{M}_{\sigma}\) is smooth and representable), the stabilizer of the image of \(\sigma\) in \(C^{\lambda'}(S_{g,n})_\bullet\), for the action of \(\Gamma_{g,n}/\Gamma^{\lambda'}\), is the Galois group of the étale cover \(\Delta^{\lambda'}_{\sigma} \rightarrow \beta^\lambda_{\sigma}(\Delta^{\lambda}_{\sigma})\) (see Section 3).

On the other hand, the fundamental group of \(\beta^\lambda_{\sigma}(\Delta^{\lambda}_{\sigma})\) is isomorphic to the stabilizer \(\Gamma^\lambda_{\sigma}\). Therefore, there is a natural epimorphism \(\Gamma^\lambda_{\sigma} \rightarrow (\Gamma^{\lambda}/\Gamma^{\lambda')}_{\sigma^\nu}\), for all \(\lambda' \in \Lambda\) as above. Since \(\Gamma^\lambda_{\sigma} \cong \lim_{\lambda' \in \Lambda} (\Gamma^{\lambda}/\Gamma^{\lambda'})_{\sigma^\nu}\), it follows that the image of \(\Gamma^\lambda_{\sigma}\) is dense in \(\Gamma^\lambda_{\sigma}\).

Let us introduce the profinite completion of the Teichmüller group \(\Gamma_{g,n}\) which will be our main object of study in the following sections. For \(2g - 2 + n > 0\), there is a short exact sequence:

\[
1 \rightarrow \hat{\Pi}_{g,n} \rightarrow \hat{\Gamma}_{g,n+1} \rightarrow \hat{\Gamma}_{g,n} \rightarrow 1.
\]

This short exact sequence induces the representation \(\hat{\rho}_{g,n}: \hat{\Gamma}_{g,n+1} \rightarrow \text{Aut}(\hat{\Pi}_{g,n})\) and the profinite universal monodromy representation \(\hat{\rho}_{g,n}: \hat{\Gamma}_{g,n} \rightarrow \text{Out}(\hat{\Pi}_{g,n})\).

**Definition 4.3.** Let us define the profinite groups \(\hat{\Gamma}_{g,n+1}\) and \(\hat{\Gamma}_{g,n}\) for \(2g - 2 + n > 0\), to be, respectively, the image of \(\hat{\rho}_{g,n}\) in \(\text{Aut}(\hat{\Pi}_{g,n})\) and of \(\hat{\rho}_{g,n}\) in \(\text{Out}(\hat{\Pi}_{g,n})\). We call \(\hat{\Gamma}_{g,n}\) the geometric profinite completion of the Teichmüller group \(\Gamma_{g,n}\) or, more simply, the geometric profinite Teichmüller group.

By definition, there are natural maps with dense image \(\Gamma_{g,n} \rightarrow \hat{\Gamma}_{g,n}\) and \(\Gamma_{g,n+1} \rightarrow \hat{\Gamma}_{g,n+1}\), but it is a deeper result by Grossman \[10\] that these maps are also injective.

In Theorem 2.5 of \[6\], it is claimed that \(\Gamma_{g,n+1} \equiv \hat{\Gamma}_{g,n+1}\), for \(n \geq 1\). However, as a consequence of the theory developed in this paper, we are going to provide a more direct proof, which covers also the \(n = 0\) case, and therefore we will not assume that result.

As an immediate consequence of Definition 4.3, for \(2g - 2 + n > 0\), there is a natural short exact sequence:

\[
1 \rightarrow \hat{\Pi}_{g,n} \rightarrow \hat{\Gamma}_{g,n+1} \rightarrow \hat{\Gamma}_{g,n} \rightarrow 1.
\]

By Theorem 2.1 and Theorem 2.8 the tower of levels which defines the geometric profinite completion of \(\Gamma_{g,n}\) contains a level \(\Gamma^n\) which satisfies the hypotheses of Proposition 3.1 and then acts without inversions on the curve complex \(C(S_{g,n})_\bullet\). So, we can define:

**Definition 4.4.** For \(2g - 2 + n > 0\), the geometric profinite curve complex \(\hat{C}(S_{g,n})_\bullet\) is the \(\hat{\Gamma}_{g,n}\)-completion of the simplicial set \(C(S_{g,n})_\bullet\).

**Remark 4.5.** The hypotheses of Proposition 4.1 are satisfied. Therefore, the natural map \(C(S_{g,n})_\bullet \rightarrow \hat{C}(S_{g,n})_\bullet\) is injective for \(2g - 2 + n > 0\).
The terminology ”curve complex” is a little inappropriate, since $\tilde{C}(S_{g,n})_{\bullet}$ is a simplicial set and not a simplicial complex, but it will be vindicated at the end of Section 6, where we show that $\tilde{C}(S_{g,n})_{\bullet}$ is the simplicial set associated to a genuine simplicial complex.

5 The geometric profinite curve complex

The purpose of this section is to provide an alternative, more intrinsic, description of the geometric profinite curve complex $\tilde{C}(S_{g,n})_{\bullet}$, in terms of the profinite fundamental group $\hat{\Pi}_{g,n}$ of the Riemann surface $S_{g,n}$.

Let $\mathcal{L} \cong C(S_{g,n})_0$, for $2g - 2 + n > 0$, denote the set of isotopy classes of non-trivial simple closed curves on $S_{g,n}$. Let $\Pi_{g,n}/\sim$ be the set of conjugacy classes of elements of $\Pi_{g,n}$ and let us denote by $\mathcal{P}_f(\Pi_{g,n}/\sim)$ the set of its finite subsets.

For a given $\gamma \in \Pi_{g,n}$, let us denote by $\gamma^{\pm1}$ the set $\{\gamma, \gamma^{-1}\}$ and by $[\gamma^{\pm1}]$ its equivalence class in $\mathcal{P}_f(\Pi_{g,n}/\sim)$. There is then a natural embedding $\iota: \mathcal{L} \hookrightarrow \mathcal{P}_f(\Pi_{g,n}/\sim)$, defined choosing for an element $s \in \mathcal{L}$ a loop $\gamma \in \Pi_{g,n}$ freely isotopic to $s$ and letting $\iota(s) := [\gamma^{\pm1}]$.

Let $\Pi_{g,n}/\sim$ be the set of conjugacy classes of elements of $\Pi_{g,n}$ and let us denote by $\mathcal{P}_f(\Pi_{g,n}/\sim)$ the profinite set of its finite subsets. From combinatorial group theory (see [10]), we know that the set $\Pi_{g,n}/\sim$ embeds in the profinite set $\Pi_{g,n}/\sim$. So, it is natural to define the set of profinite (non-trivial) s.c.c. $\mathcal{L}$ on $S_{g,n}$ to be the closure of the set $\iota(\mathcal{L})$ inside the profinite set $\mathcal{P}_f(\Pi_{g,n}/\sim)$.

The profinite set $\mathcal{L}$ consists of equivalence classes $[\alpha^{\pm1}]$ of couples of elements of $\Pi_{g,n}$, such that $\alpha$ is the limit of a sequence in $\Pi_{g,n}$ of elements representable by simple closed loops.

An alternative group-theoretic description of $\mathcal{L}$ is the following. Let $\mathcal{P}(\Pi_{g,n})/\sim$ be the set of conjugacy classes of subgroups of $\Pi_{g,n}$. There is a natural embedding $\iota': \mathcal{L} \hookrightarrow \mathcal{P}(\Pi_{g,n})/\sim$, defined sending $s \in \mathcal{L}$ to the conjugacy class of the cyclic subgroup of $\Pi_{g,n}$ generated by a loop $\gamma \in \Pi_{g,n}$ freely isotopic to $s$. So, let $\mathcal{P}(\Pi_{g,n})/\sim$ be the profinite set of conjugacy classes of closed subgroups of $\Pi_{g,n}$. There is a natural embedding of $\mathcal{P}(\Pi_{g,n})/\sim$ in $\mathcal{P}(\Pi_{g,n})/\sim$ and we define $\mathcal{L}'$ to be the closure of $\iota'(\mathcal{L})$ inside $\mathcal{P}(\Pi_{g,n})/\sim$. There is a natural continuous surjective map $\mathcal{L} \to \mathcal{L}'$. However, for the moment, it is not clear whether this map is also injective.

The main advantage of the definition of the profinite set $\mathcal{L}$ over $\mathcal{L}'$ is that it allows to define an orientation for an element $[\alpha^{\pm1}] \in \mathcal{L}$, fixing an order on the set $\{\alpha, \alpha^{-1}\}$ (this is preserved by the conjugacy action).

There are natural embeddings of the sets $C(S_{g,n})_k$ of $k$-simplices of the simplicial curve complex $C(S_{g,n})_{\bullet}$ in the profinite sets $\prod_{k+1}^{\mathcal{L}}$ and $\prod_{k+1}^{\mathcal{L}'}$. Let us then define the following simplicial profinite sets:

**Definition 5.1.** Let $L(\Pi_{g,n})_{\bullet}$ and $L'(\Pi_{g,n})_{\bullet}$, for $2g - 2 + n > 0$, be the simplicial profinite sets whose set of $k$-simplices are the closures of $C(S_{g,n})_k$ inside, respectively, the profinite sets $\prod_{k+1}^{\mathcal{L}}$ and $\prod_{k+1}^{\mathcal{L}'}$, for $k \geq 0$, and whose face and degeneracy maps are the continuous maps induced by the face and degeneracy maps of the simplicial set $C(S_{g,n})_{\bullet}$.
By their definition, the simplicial profinite sets defined above admit a natural continuous action of the geometric profinite Teichmüller group $\hat{\Gamma}_{g,n}$ and are linked by a natural continuous $\hat{\Gamma}_{g,n}$-equivariant surjective map of simplicial profinite sets $L(\hat{\Pi}_{g,n})_\bullet \rightarrow L'(\hat{\Pi}_{g,n})_\bullet$.

The embeddings $C(S_{g,n})_\bullet \rightarrow L(\hat{\Pi}_{g,n})_\bullet$ and $C(S_{g,n})_\bullet \rightarrow L'(\hat{\Pi}_{g,n})_\bullet$ have dense images and are, clearly, $\Gamma_{g,n}$-equivariant. Therefore, by the universal property of the $\hat{\Gamma}_{g,n}$-completion, there is, for all $2g - 2 + n > 0$, a series of natural surjective continuous $\hat{\Gamma}_{g,n}$-equivariant maps of simplicial profinite sets:

$$\hat{C}(S_{g,n})_\bullet \rightarrow L(\hat{\Pi}_{g,n})_\bullet \rightarrow L'(\hat{\Pi}_{g,n})_\bullet.$$

We will prove that the above maps are actually isomorphisms of simplicial profinite sets. It is clearly enough to prove that the composition of the two above maps, which we will denote by $\Phi_\bullet$, is an isomorphism.

In order to prove the claim, we will show that the actions of $\hat{\Gamma}_{g,n}$ on the two simplicial profinite sets $\hat{C}(S_{g,n})_\bullet$ and $L'(\hat{\Pi}_{g,n})_\bullet$ have the same stabilizers. From the proof, it will also follow that $\Phi_\bullet$ induces a bijection between orbits and then is a bijection.

Each $\hat{\Gamma}_{g,n}$-orbit in $L'(\hat{\Pi}_{g,n})_k$, for $k \geq 0$, contains simplices in the image of $C(S_{g,n})_\bullet$. Thus, by Proposition 4.2, the above claim about the stabilizers is equivalent to say that, for a simplex $\sigma \in C(S_{g,n})_\bullet$, the stabilizer of its image in $L'(\hat{\Pi}_{g,n})_\bullet$ for the action of $\hat{\Gamma}_{g,n}$ is the closure of the discrete stabilizer $\Gamma_{\sigma}$. This statement will follow from its explicit description.

For simplicity, given a simplex $\sigma \in C(S_{g,n})_\bullet$, we denote its image in $L'(\hat{\Pi}_{g,n})_\bullet$ also by $\sigma$ and by $\hat{\Gamma}_{\sigma}$ the corresponding $\hat{\Gamma}_{g,n}$-stabilizer.

**Theorem 5.2.** Let $\sigma \in L'(\hat{\Pi}_{g,n})_\bullet$, for $2g - 2 + n > 0$, be the image of a simplex of $C(S_{g,n})_\bullet$ consisting of the set $\{\gamma_1, \ldots, \gamma_k\}$ of s.c.c. on the Riemann surface $S_{g,n}$. Suppose that $S_{g,n} \setminus \{\gamma_1, \ldots, \gamma_k\} \cong S_{g_1,n_1} \amalg \cdots \amalg S_{g_h,n_h}$.

Then, the stabilizer $\hat{\Gamma}_{\sigma}$ fits into the two exact sequences:

$$1 \rightarrow \hat{\Gamma}_{\sigma} \rightarrow \hat{\Gamma}_{\sigma} \rightarrow \Sigma_{\sigma}\{\pm\},$$

$$1 \rightarrow \bigoplus_{i=1}^k\hat{\mathbb{Z}} \cdot \tau_{\gamma_i} \rightarrow \hat{\Gamma}_{\sigma} \rightarrow \hat{\Gamma}_{g_1,n_1} \times \cdots \times \hat{\Gamma}_{g_h,n_h} \rightarrow 1,$$

where $\Sigma_{\sigma}\{\pm\}$ is the group of signed permutations on the set $\{\gamma_1, \ldots, \gamma_k\}$.

The proof of Theorem 5.2 will consist of several steps. The essential case of the theorem is for $\sigma$ consisting of a single s.c.c. $\gamma$ on $S_{g,n}$ and we will consider this case first. To proceed further, we need a geometric description of the profinite representations:

$$\hat{\rho}_{g,n} : \hat{\Gamma}_{g,n} \rightarrow \text{Out}(\hat{\Pi}_{g,n}) \quad \text{and} \quad \hat{\rho}_{g,n} : \hat{\Gamma}_{g,n+1} \rightarrow \text{Aut}(\hat{\Pi}_{g,n}).$$

Let then $\hat{\mathcal{M}}$ (respectively, $\hat{\mathcal{C}}$) be the inverse limit of all the representable, connected, Galois covers of $\mathcal{M}_{g,n}$ (respectively, of $\mathcal{M}_{g,n+1}$) which ramify at most along the D–M
boundary. Let also \( \overline{\mathcal{C}} := \overline{M}_{g,n+1} \times_{\overline{M}_{g,n}} \overline{M} \) be the pull-back of the universal proper curve. For a rigorous construction of these spaces, we refer the reader to [26]. They are connected by the natural commutative diagram:

\[
\begin{array}{ccc}
\overline{\mathcal{C}} & \to & \overline{M} \\
\downarrow & & \downarrow \\
\overline{M}_{g,n+1} & \to & \overline{M}_{g,n}.
\end{array}
\]

The choices of lifts \( a' \in \overline{M} \) and \( \tilde{a}' \in \tilde{\mathcal{C}} \) of the base points \( a \in M_{g,n} \) and \( \tilde{a} \in \mathcal{C} \) identify the profinite Teichmüller groups \( \hat{\Gamma}_{g,n} \) and \( \hat{\Gamma}_{g,n+1} \), respectively, with the logarithmic algebraic fundamental groups \( \hat{\pi}_1(M_{g,n}, a) \) and \( \hat{\pi}_1(M_{g,n+1}, \tilde{a}) \), where we consider on \( M_{g,n} \) and \( M_{g,n+1} \) the logarithmic structures associated to their D–M boundaries.

The profinite Teichmüller group \( \hat{\Gamma}_{g,n} \) identifies with the covering transformations group of the profinite Galois covers \( \overline{M} \to \overline{M}_{g,n} \) and \( \overline{C} \to \overline{M}_{g,n+1} \), while \( \hat{\Gamma}_{g,n+1} \) identifies with the covering transformations group of the profinite Galois cover \( \overline{C} \to \overline{M}_{g,n+1} \).

Moreover, the kernel \( \hat{\Pi}_{g,n} \) of the natural epimorphism \( \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \) consists of the covering transformations of the cover \( \overline{C} \to \overline{M}_{g,n+1} \) which preserve the fibers of the natural morphism \( \overline{C} \to \overline{M} \). In this way, there is a canonical identification, modulo inner automorphisms, of the profinite group \( \hat{\Pi}_{g,n} \) and the logarithmic algebraic fundamental groups of the fibers of the morphism \( \overline{C} \to \overline{M} \), with respect to the logarithmic structure associated to the divisor of marked and singular points.

The representation \( \hat{\rho}_{g,n} : \hat{\Gamma}_{g,n+1} \to \text{Aut}(\hat{\Pi}_{g,n}) \) can be described as follows. For \( x \in \overline{M} \), let \( \overline{C}_x \) and \( \overline{C}_x \) be, respectively, the fibers of the morphisms \( \overline{C} \to \overline{M} \) and \( \overline{C} \to \overline{M} \). For all \( x \in \overline{M} \), let us identify the group \( \hat{\Pi}_{g,n} \) with the covering transformation group of the profinite cover \( \overline{C}_x \to \overline{C}_x \). Let us then denote by \( f \) the image of \( f \in \hat{\Gamma}_{g,n+1} \) via the natural epimorphism \( \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \). The element \( f \in \hat{\Gamma}_{g,n+1} \) induces, by restriction, an isomorphism \( f : \overline{C}_x \to \overline{C}_x \). The automorphism \( \hat{\rho}_{g,n}(f) : \hat{\Pi}_{g,n} \to \hat{\Pi}_{g,n} \) is defined, for \( \alpha \in \hat{\Pi}_{g,n} \), by \( \hat{\rho}_{g,n}(f)(\alpha) = f^{-1}\alpha f \).

The profinite universal monodromy representation \( \hat{\rho}_{g,n} : \hat{\Gamma}_{g,n} \to \text{Out}(\hat{\Pi}_{g,n}) \) is the outer representation associated to \( \hat{\rho}_{g,n} \). However, in the present geometric context, there is a more direct way to describe \( \hat{\rho}_{g,n} \).

Above, we identified \( \hat{\Gamma}_{g,n} \) with the covering transformations group of the profinite cover \( \overline{C} \to \overline{M}_{g,n+1} \). Therefore, an element \( f \in \hat{\Gamma}_{g,n} \) determines by restriction an isomorphism of algebraic curves \( f : \overline{C}_x \to \overline{C}_x f(x) \). This isomorphism induces on logarithmic algebraic fundamental groups an isomorphism \( f_* : \hat{\pi}_1(\overline{C}_x) \to \hat{\pi}_1(\overline{C}_x f(x)) \). Since both these groups identify, modulo inner automorphisms, with \( \hat{\Pi}_{g,n} \), it follows that \( f_* \) determines a unique element of \( \text{Out}(\hat{\Pi}_{g,n}) \).

The discrete universal monodromy representation \( \rho : \Gamma_{g,n} \to \text{Out}(\Pi_{g,n}) \) can be described in a similar way, where the Bers compactifications \( T_{g,n} \) and \( T_{g,n+1} \) and the universal
Lemma 5.3. Let \( \gamma \) be a s.c.c. on \( S_{g,n} \), for \( 2g-2+n > 0 \). The stabilizers of the simplices associated to \( \gamma \) in \( L(\hat{\Pi}_{g,n}) \) and \( L'(\hat{\Pi}_{g,n}) \), for the action of the profinite Teichmüller group \( \hat{\Gamma}_{g,n} \), are naturally isomorphic. Let then \( \hat{\Gamma}_{\gamma} \) be the subgroup of the stabilizer \( \hat{\Gamma}_{\gamma} \) of \( \gamma \) in \( L(\hat{\Pi}_{g,n}) \) consisting of elements which preserve an orientation of \( \hat{\gamma} \). It holds:

i.) If \( \gamma \) is non-separating, the profinite monodromy representation \( \hat{\rho}_{g,n} \) induces a representation \( \hat{\rho}_{\gamma} : \hat{\Gamma}_{\gamma} \to \text{Out}(\hat{\Pi}_{g-1,n+2}) \), such that \( \hat{\rho}_{\gamma}(\Gamma_{\gamma}) \) is dense inside \( \text{Im} \hat{\rho}_{\gamma} \).

ii.) If \( \gamma \) is separating and \( S_{g,n} \sim S_{g_1,n_1+1} \prod S_{g_2,n_2+1} \), the profinite monodromy representation \( \hat{\rho}_{g,n} \) induces a representation \( \hat{\rho}_{\gamma} : \hat{\Gamma}_{\gamma} \to \text{Out}(\hat{\Pi}_{g_1,n_1+1}) \times \text{Out}(\hat{\Pi}_{g_2,n_2+1}) \), such that \( \hat{\rho}_{\gamma}(\Gamma_{\gamma}) \) is dense inside \( \text{Im} \hat{\rho}_{\gamma} \).

**Proof.** Let us prove the lemma for \( \gamma \) a non-separating curve. The other case can be dealt with in a similar fashion.

Let \( (C_{\gamma}, \phi) \in \partial T_{g,n} \) be such that \( C_{\gamma} \) is a stable \( n \)-pointed, genus \( g \) irreducible curve with a single node and the degenerate marking \( \phi : S_{g,n} \to C_{\gamma} \) sends the s.c.c. \( \gamma \) to the node. Let \( x \in \hat{\mathcal{M}} \) be the image of the point \( (C_{\gamma}, \phi) \) via the natural embedding \( T_{g,n} \hookrightarrow \hat{\mathcal{M}} \). Let us identify the stable curve \( C_{\gamma} \) with the fiber \( \mathcal{C}_x \).

Let \( \hat{\mathcal{C}}_{\gamma} \) be the inverse limit of all logarithmic étale covers of \( C_{\gamma} \), with respect to the logarithmic structure associated to the divisor of marked and singular points. Let us identify the pro-curve \( \hat{\mathcal{C}}_{\gamma} \) with the fiber \( \mathcal{C}_x \) of the pro-curve \( \mathcal{C} \to \mathcal{M} \). So, we have also identified the group of covering transformations of the cover \( \hat{\mathcal{C}}_{\gamma} \to C_{\gamma} \) with the profinite group \( \hat{\Pi}_{g,n} := \pi_1(S_{g,n}) \). Let us recall that we gave to the group \( \Pi_{g,n} \) the presentation:

\[
\Pi_{g,n} = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, u_1, \ldots, u_n \mid \prod_{i=1}^{g} [\alpha_i, \beta_i] \cdot u_n \cdots u_1 \rangle, \tag{♣}
\]

where \( u_i \), for \( i = 1, \ldots, n \), is a simple loop around the puncture \( P_i \).

It is not restrictive to assume that \( \beta_g \) is freely isotopic to \( \gamma \). Let us put \( u_{n+1} := \beta_g^{-1} \) and \( u_{n+2} := \alpha_g \beta_g \alpha_g^{-1} \). The fundamental group of \( S_{g,n} \sim \gamma \) is then identified with the subgroup \( \Pi_{g-1,n+2} \) of \( \Pi_{g,n} \) of presentation:

\[
\Pi_{g-1,n+2} = \langle \alpha_1, \ldots, \alpha_{g-1}, \beta_1, \ldots, \beta_{g-1}, u_1, \ldots, u_{n+2} \mid \prod_{i=1}^{g-1} [\alpha_i, \beta_i] \cdot u_{n+2} \cdots u_1 \rangle. \tag{♦}
\]
Elements of $\hat{\Pi}_{g,n}$ in the conjugacy class of $\beta_g$ are in natural bijective correspondence with the set $\mathcal{N}_\gamma$ of nodes of $\hat{C}_\gamma$. Let us then denote the set of conjugates of $\beta_g$ by $\{\gamma_s\}_{s \in \mathcal{N}_\gamma}$. Each element $\gamma_s$, for $s \in \mathcal{N}_\gamma$, topologically generates the inertia group $I_s$ of the node $s \in \hat{C}_\gamma$.

Let us observe that, since the stabilizer of a node coincides with its inertia subgroup, the inertia subgroups $\{I_s\}_{s \in \mathcal{N}_\gamma}$ are their own normalizers in $\hat{\Pi}_{g,n}$. The same argument clearly applies to the inertia groups of marked points of $\hat{C}_\gamma$. So, we get the following lemma which will be useful later:

**Lemma 5.4.** Let $1 \neq x \in \Pi_{g,n}$, for $2g - 2 + n > 0$, be freely homotopic to a s.c.c. on $S_{g,n}$. Then, the closed subgroup topologically generated by $x^m$ in $\hat{\Pi}_{g,n}$, for $m \in \mathbb{N}^+$, has for normalizer the closed subgroup topologically generated by $x$.

By the above remarks, inertia groups corresponding to distinct nodes have trivial intersections. Now, if $f$ stabilizes the simplex associated to $\gamma$ in $L' (\hat{\Pi}_{g,n})$, its image in $\text{Out} (\hat{\Pi}_{g,n})$ can be lifted to an $\tilde{f} \in \text{Aut} (\hat{\Pi}_{g,n})$ such that $\tilde{f} (I_s) = I_s$, for a given $s \in \mathcal{N}_\gamma$. Therefore, the action of $\tilde{f}$ on $\hat{C}_\gamma$ stabilizes the node $s$ and so it holds $\tilde{f} (\gamma_s^{\pm 1}) = \gamma_s^{\pm 1}$, i.e. $f$ stabilizes the simplex associated to $\gamma$ in $L (\hat{\Pi}_{g,n})$ as well. This proves the first part of the lemma.

Similarly, there is a bijective correspondence between irreducible components of $\hat{C}_\gamma$ and conjugates of the subgroup $\hat{\Pi}_{g-1,n+2}$ in $\hat{\Pi}_{g,n}$ in particular, we see also that the subgroup $\hat{\Pi}_{g-1,n+2}$ of $\hat{\Pi}_{g,n}$ is its own normalizer.

In this way, the profinite completion $\hat{\Pi}_{g-1,n+2}$ is identified with the covering transformation group of the restriction of the cover $\hat{C}_\gamma \to C_\gamma$ to one of the two irreducible components of $\hat{C}_\gamma$ which contain the node corresponding to $\beta_g$.

Let us now come to the definition of the representation $\hat{\rho}_\gamma$. In virtue of the short exact sequence:

$$1 \to \hat{\Pi}_{g,n} \to \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \to 1,$$

given an element $f \in \hat{\Gamma}_{g,n}$, there is a lift $\tilde{f} \in \hat{\Gamma}_{g,n+1}$ such that $\tilde{f} (\beta_g) = \beta_g$. By Lemma 5.3, the image $\hat{\rho}_{g,n} (\tilde{f}) \in \text{Aut} (\hat{\Pi}_{g,n})$ is uniquely determined by $f$, modulo an inner automorphism by a power of $\beta_g$.

By the above discussion, it is clear that the isomorphism of fibers $\tilde{f} : \tilde{C}_x \to \tilde{C}_{f(x)}$, induced by $\tilde{f}$, sends the irreducible component of $\tilde{C}_x$ stabilized by the subgroup $\hat{\Pi}_{g-1,n+2}$ of $\hat{\Pi}_{g,n}$ to the irreducible component of $\tilde{C}_{f(x)}$ stabilized by the same subgroup and hence induces an automorphism $\tilde{f} : \hat{\Pi}_{g-1,n+2} \to \hat{\Pi}_{g-1,n+2}$. We let then $\hat{\rho}_\gamma (f)$ be the image of $\tilde{f}$ in $\text{Out} (\hat{\Pi}_{g-1,n+2})$.

In order to show that the elements of $\hat{\rho}_\gamma (\Gamma_\gamma)$ are dense in the image of $\hat{\rho}_\gamma$, it is enough to show that, for any given $f \in \Gamma_\gamma$, the image $\hat{\rho}_\gamma (f)$ is congruent, modulo any given characteristic open subgroup $K$ of $\hat{\Pi}_{g-1,n+2}$, to some element $f_K \in \Gamma_{g-1,n+2} < \text{Out} (\hat{\Pi}_{g-1,n+2})$.

Let us define, for the logarithmic algebraic fundamental group of a stable $n$-pointed, genus $g$ complex curve $C$, a standard set of generators to be a set of elements of $\hat{\pi}_1 (C^{\log})$ in the $\hat{\pi}_{g,n}$-orbit of a standard set of generators for the topological fundamental group $\pi_1 (C^{\log})$, i.e. of the type given in the presentation (♣).
In the description of the universal profinite monodromy representation, we considered isomorphisms \( \hat{\Pi}_{g,n} \sim \hat{\pi}_1(\hat{\mathcal{C}}_{x,y}) \), for \( y \in \mathcal{M} \), uniquely determined modulo inner automorphisms. Basically by definition, all such isomorphisms send standard sets of generators for \( \hat{\Pi}_{g,n} \) to standard sets of generators for \( \hat{\pi}_1(\hat{\mathcal{C}}_{x,y}) \). The same then holds for the restrictions of the isomorphisms \( \hat{\Pi}_{g,n} \sim \hat{\pi}_1(\hat{\mathcal{C}}_{x,y}) \) and \( \hat{\Pi}_{g,n} \sim \hat{\pi}_1(\hat{\mathcal{C}}_{f(x)}) \) to the subgroup \( \hat{\Pi}_{g-1,n+2} \) of \( \hat{\Pi}_{g,n} \) and its images in \( \hat{\pi}_1(\hat{\mathcal{C}}_{x,y}) \) and \( \hat{\pi}_1(\hat{\mathcal{C}}_{f(x)}) \) respectively.

Therefore, it is clear that the automorphism \( \hat{f}^\gamma : \hat{\Pi}_{g-1,n+2} \sim \hat{\Pi}_{g-1,n+2} \) induced by \( \gamma \), defined above, sends a set of standard generators for \( \hat{\Pi}_{g,n} \) to another such set.

A standard set of generators is characterized by the property that, for any open normal subgroup \( K \) of \( \hat{\Pi}_{g,n+2} \), it is congruent modulo \( K \) to a standard set of generators for \( \hat{\Pi}_{g-1,n+2} \) to another such set.

By the Nielsen realization Theorem, the set of standard generators for \( \Pi_{g-1,n+2} \) given by (\( \bullet \)) and another one, which is congruent, modulo \( K \), to its image in \( \hat{\Pi}_{g-1,n+2} \) via the automorphism \( \hat{f}^\gamma \), define an element \( f_K \in \Gamma_{g-1,n+2} \) with the desired properties.

\[ \square \]

For \( \gamma \) a non-separating s.c.c. on \( S_{g,n} \), the representation \( \hat{\rho}_\gamma \) of Lemma 5.3 factors through the natural epimorphism \( \hat{\Gamma}_\gamma \to \hat{\Gamma}_\gamma \) and a representation \( \hat{\rho}_\gamma : \hat{\Gamma}_\gamma \to \text{Out}(\hat{\Pi}_{g-1,n+2}) \). A similar statement holds for a separating s.c.c.

So, in order to complete the proof of Theorem 5.2 for the case of a 0-simplex of \( L'(\hat{\Pi}_{g,n})_\bullet \), it is now enough to show that the kernel of \( \hat{\rho}_\gamma \) is generated by the Dehn twist \( \tau_\gamma \). As above, we just treat the case of a non-separating s.c.c. \( \gamma \) on \( S_{g,n} \).

Let us keep the notations of the proof of Lemma 5.3 and let moreover \( t := \alpha_g \). We then get the standard presentation of \( \Pi_{g,n} \) as an HNN extension of \( \Pi_{g-1,n+2} \):

\[ \Pi_{g,n} = \langle \Pi_{g-1,n+2}, t | tu_{n+1}^{-1}t^{-1} = u_{n+2} \rangle. \]

In terms of graphs of groups, we are saying that \( \Pi_{g,n} \) is naturally isomorphic to the fundamental loop of the group of loops having for vertex group \( \Pi_{g-1,n+2} \) and for edge group the free cyclic group spanned by \( t \).

By the theory of graphs of profinite groups (see [27]), we also know that \( \hat{\Pi}_{g,n} \) is naturally isomorphic to the profinite fundamental group of the loop of groups having for vertex group \( \hat{\Pi}_{g-1,n+2} \) and for edge group the profinite free cyclic group spanned by \( t \).

Given \( f \in \hat{\Gamma}_\gamma \), we have seen that it admits a lift \( \tilde{f} \) to an automorphism of \( \hat{\Pi}_{g,n} \) which preserves the closed subgroup \( \hat{\Pi}_{g-1,n+2} \) and \( \hat{\rho}_\gamma(f) \) is defined by the restriction of \( \tilde{f} \) to this subgroup.

Let \( f \in \hat{\Gamma}_\gamma \) be such that \( \hat{\rho}_\gamma(f) = 1 \). It then admits a lift \( \tilde{f} \) as above which restricts to the identity on \( \hat{\Pi}_{g-1,n+2} \). In particular, \( \tilde{f}(u_{n+1}) = u_{n+1} \) and \( \tilde{f}(u_{n+2}) = u_{n+2} \). Hence, the action of \( \tilde{f} \) on \( t = \alpha_g \) satisfies the condition:

\[ \tilde{f}(u_{n+2}) = \tilde{f}(t)u_{n+1}^{-1}\tilde{f}(t)^{-1} = u_{n+2} = tu_{n+1}^{-1}t^{-1}. \]
By Lemma 5.4, the above identity implies \( \tilde{f}(t) = tu_{n+1}^k \), for some \( k \in \hat{\mathbb{Z}} \). Therefore, we see that \( f \) equals the power \( \tau_\gamma^{-k} \in \hat{\Gamma}_{g,n} \). Indeed, the Dehn twist \( \tau_\gamma \) admits as well a lift to an automorphism \( \tilde{\tau}_\gamma \) of \( \hat{\Pi}_{g,n} \) such that \( \tilde{\tau}_\gamma^{-k}(t) = tu_{n+1}^k \) and \( \tilde{\tau}_\gamma^{-k} \) restricts to the identity on the subgroup \( \hat{\Pi}_{g-1,n+2} < \hat{\Pi}_{g,n} \).

This concludes the proof of Theorem 5.2 in case \( \sigma \in L'(\hat{\Gamma}_{g,n}) \) is a 0-simplex. However, it is easy to deal with the general case by induction on the dimension of \( \sigma \), thanks to the proposition which we are now going to prove:

**Proposition 5.5.** Let \( \gamma \) be a s.c.c. on \( S_{g,n} \), for \( 2g - 2 + n > 0 \).

- i.) If \( \gamma \) is non-separating, let us identify \( S_{g,n} \setminus \gamma \) with the Riemann surface \( S_{g-1,n+2} \). The induced monomorphism \( \hat{\Pi}_{g-1,n+2} \hookrightarrow \hat{\Pi}_{g,n} \) then induces an embedding of the corresponding simplicial profinite sets of profinite s.c.c. \( L'(\hat{\Pi}_{g-1,n+2}) \hookrightarrow L'(\hat{\Pi}_{g,n}) \).

- ii.) If \( \gamma \) is separating, let us identify \( S_{g,n} \setminus \gamma \) with \( S_{g_1,n_1+1} \coprod S_{g_2,n_2+1} \). The induced monomorphisms \( \hat{\Pi}_{g,n+1} \hookrightarrow \hat{\Pi}_{g,n} \), for \( i = 1, 2 \), induce embeddings of the corresponding simplicial profinite sets of profinite s.c.c. \( L'(\hat{\Pi}_{g,n+1}) \hookrightarrow L'(\hat{\Pi}_{g,n}) \), for \( i = 1, 2 \).

**Proof.** As usual we deal only with the non-separating case. At least, it is clear that the embedding \( S_{g,n} \setminus \gamma \subset S_{g,n} \) induces a continuous map of profinite sets:

\[
i_\gamma: L'(\hat{\Pi}_{g-1,n+2})_0 \to L'(\hat{\Pi}_{g,n})_0.
\]

In order to prove that \( i_\gamma \) is injective, we have to show that a conjugacy class of closed subgroups \( \sigma(\hat{\Xi}) \subset L'(\hat{\Pi}_{g,n}) \), contained in the image of \( i_\gamma \), intersects the set of all closed subgroups of \( \hat{\Pi}_{g-1,n+2} < \hat{\Pi}_{g,n} \) in a single \( \hat{\Pi}_{g-1,n+2} \)-conjugacy class.

As we already remarked, the \( \hat{\Gamma}_{g-1,n+2} \)-orbit of \( \sigma(\hat{\Xi}) \) contains elements in the image of the set \( \mathcal{L} \) of isotopy classes of s.c.c. on \( S_{g-1,n+2} \). Hence, we can assume that \( \sigma(\hat{\Xi}) \) is the conjugacy class in \( \hat{\Pi}_{g,n} \) of the closure of the cyclic subgroup of \( \hat{\Pi}_{g-1,n+2} \) corresponding to a s.c.c. \( \alpha \) on \( S_{g,n} \setminus \gamma \).

Let then \( C_{\gamma \alpha} \) be a stable \( n \)-pointed, genus \( g \) curve with two nodes, endowed with a degenerate marking \( \psi: S_{g,n} \to C_{\gamma \alpha} \) which sends the s.c.c.'s \( \gamma \) and \( \alpha \) to the two nodes. Let \( \tilde{C}_{\gamma \alpha} \) be the inverse limit of all logarithmic étale covers of \( C_{\gamma \alpha} \), with respect to the logarithmic structure associated to the divisor of marked and singular points, and let us identify the covering transformation group of \( \tilde{C}_{\gamma \alpha} \to C_{\gamma \alpha} \) with the profinite group \( \hat{\Pi}_{g,n} \).

Let \( \mathcal{N}_\gamma \) and \( \mathcal{N}_\alpha \) be the sets of nodes of \( \tilde{C}_{\gamma \alpha} \) lying, respectively, above the nodes \( \psi(\gamma) \) and \( \psi(\alpha) \) of \( C_{\gamma \alpha} \). As we saw in the proof of Lemma 5.3, the set of inertia groups \( \{ I_s \}_{s \in \mathcal{N}_\alpha} \) of the nodes in \( \mathcal{N}_\alpha \) is in bijective correspondence with the elements of the conjugacy class \( \sigma(\hat{\Xi}) \). Moreover, the conjugates of the subgroup \( \hat{\Pi}_{g-1,n+2} \) of \( \hat{\Pi}_{g,n} \) are in bijective correspondence with the connected components of \( \tilde{C}_{\gamma \alpha} \setminus \mathcal{N}_\gamma \), which they stabilize.

Let \( \tilde{C}'_{\gamma \alpha} \) be the connected component of \( \tilde{C}_{\gamma \alpha} \setminus \mathcal{N}_\gamma \) stabilized by the subgroup \( \hat{\Pi}_{g-1,n+2} \) of \( \hat{\Pi}_{g,n} \). The set of inertia groups \( \{ I_s \}_{s \in \mathcal{N}_\alpha} \) then intersects the set of all closed subgroups
of \( \hat{\Pi}_{g-1,n+2} \) precisely in the set of inertia groups of the nodes of \( \hat{C}_\alpha \), which forms a single \( \hat{\Pi}_{g-1,n+2} \)-conjugacy class. This concludes the proof that \( i_{\gamma} \) is injective.

By the definition of higher dimensional simplices of \( L'(\hat{\Pi}_{g-1,n+2})_\bullet \) and \( L'(\hat{\Pi}_{g,n})_\bullet \), this is enough to conclude the proof of the proposition.

\[ \begin{align*}
\text{Theorem 5.6.} & \quad \text{For } 2g-2+n > 0, \text{ there is a series of natural } \hat{\Gamma}_{g,n} \text{-equivariant isomorphisms of simplicial profinite sets:} \\
& \quad \hat{C}(S_{g,n})_\bullet \sim L(\hat{\Pi}_{g,n})_\bullet \sim L'(\hat{\Pi}_{g,n})_\bullet.
\end{align*} \]

Proof. Each simplex of \( L'(\hat{\Pi}_{g,n})_\bullet \) is in the \( \hat{\Gamma}_{g,n} \)-orbit of a simplex in the image of \( C(S_{g,n})_\bullet \). On the other hand, from the geometric description of the profinite monodromy representation \( \hat{\rho}_{g,n} \), it follows that two simplices \( \sigma, \sigma' \) in the image of \( C(S_{g,n})_\bullet \) are in the same \( \hat{\Gamma}_{g,n} \)-orbit if and only if \( S_{g,n} \setminus \sigma \) and \( S_{g,n} \setminus \sigma' \) have the same topological type. Therefore, \( \Phi_k \) induces a natural bijective correspondence between the orbit sets of the actions of \( \hat{\Gamma}_{g,n} \) on \( \hat{C}(S_{g,n})_k \) and \( L'(\hat{\Pi}_{g,n})_k \), respectively, for all \( k \geq 0 \).

Thus, in order to prove that \( \Phi_\bullet \) is an isomorphism, it is enough to prove that, for all \( \sigma \in \hat{C}(S_{g,n})_\bullet \) in the image of \( C(S_{g,n})_\bullet \), it holds \( \hat{\Gamma}_\sigma \equiv \hat{\Gamma}_{\Phi(\sigma)} \).

Again from the geometric description of the profinite monodromy representation \( \hat{\rho}_{g,n} \), it follows that the images of \( \hat{\Gamma}_\sigma \) and \( \hat{\Gamma}_{\Phi(\sigma)} \) in the signed permutation group \( \Sigma_{\sigma}(\{\pm\}) \) are the same. So, it is enough to prove that \( \hat{\Gamma}_\sigma \equiv \hat{\Gamma}_{\Phi(\sigma)} \), which is the content of Lemma 4.2 and Theorem 5.2.

\[ \square \]

\[ \begin{align*}
\text{Remarks 5.7.} & \quad \text{i.) By the proof of Theorem 5.6, we can define the topological type of a simplex } \sigma \text{ in the geometric profinite curve complex } \hat{C}(S_{g,n})_\bullet, \text{ for } 2g-2+n > 0, \\
& \quad \text{as the topological type of the surface } S_{g,n} \setminus \sigma', \text{ where } \sigma' \text{ is any simplex of } C(S_{g,n})_\bullet \text{ whose image in } \hat{C}(S_{g,n})_\bullet \text{ is in the } \hat{\Gamma}_{g,n} \text{-orbit of } \sigma.
\end{align*} \]

ii.) For \( 2g-2+n > 0 \), by definition, each simplex of \( L(\hat{\Pi}_{g,n})_\bullet \) is determined by its set of vertices. So, by Theorem 5.6, the same property is shared by \( \hat{C}(S_{g,n})_\bullet \).

iii.) In particular, by Theorem 5.6, the natural map \( \hat{\mathcal{L}} \to \hat{\mathcal{L}}' \) is a bijection. This has the following interesting consequence. Let \( h: \hat{\Pi}_{g,n} \to \mathcal{P}_f(\hat{\Pi}_{g,n} / \sim) \) be the map defined sending an element \( \gamma \) to the class of the set \( \{\gamma, \gamma^{-1}\} \). Then, for \( \gamma \in h^{-1}(\hat{\mathcal{L}}) \), it holds \( \hat{\gamma}^k \cap h^{-1}(\hat{\mathcal{L}}') = \{\gamma, \gamma^{-1}\} \), i.e. the elements \( \gamma \) and \( \gamma^k \) of \( \hat{\Pi}_{g,n} \), for \( k \in \hat{\mathbb{Z}} \), can both determine a profinite s.c.c. if and only if \( k = \pm 1 \).

6 Profinite Dehn twists in \( \hat{\Gamma}_{g,n} \) and their centralizers

A basic fact of classical Teichmüller theory is that the set \( \mathcal{L} \) of isotopy classes of non-trivial s.c.c. on \( S_{g,n} \) parametrizes the set of Dehn twists of \( \Gamma_{g,n} \), which is the standard set
of generators for this group. In other words, the assignment $\gamma \mapsto \tau_\gamma$, for $\gamma \in \mathcal{L}$, defines an embedding $d : \mathcal{L} \hookrightarrow \Gamma_{g,n}$, for $2g - 2 + n > 0$.

The set $\{\tau_\gamma\}_{\gamma \in \mathcal{L}}$ of all Dehn twists of $\Gamma_{g,n}$ is closed under conjugation and falls in a finite set of conjugacy classes which are in bijective correspondence with the possible topological types of the surface $S_{g,n} \setminus \gamma$. Thus, it is natural to define, for a given profinite completion $\Gamma'_{g,n}$ of the Teichmüller group $\Gamma_{g,n}$, the set of \emph{profinite Dehn twists} of $\Gamma'_{g,n}$ to be the closure of the image of the set $\{\tau_\gamma\}_{\gamma \in \mathcal{L}}$ inside $\Gamma'_{g,n}$, which is the same as the union of the conjugacy classes in $\Gamma'_{g,n}$ of the images of the Dehn twists of $\Gamma_{g,n}$.

By Theorem \ref{thm:dehn-twist-closure}, the map $d$ extends to a continuous $\hat{\Gamma}_{g,n}$-equivariant map $\hat{d} : \hat{\mathcal{L}} \to \hat{\Gamma}_{g,n}$. Therefore, each profinite s.c.c. $\gamma \in \hat{\mathcal{L}}$ determines a profinite Dehn twist, which we denote by $\tau_\gamma$, in the geometric profinite Teichmüller group $\hat{\Gamma}_{g,n}$.

In this section, we will show that the map $\hat{d}$ is injective, i.e. the set of profinite Dehn twists of $\hat{\Gamma}_{g,n}$, for $2g - 2 + n > 0$, is parametrized by the profinite set $\hat{\mathcal{L}}$. We are actually going to prove a much stronger result. Let us observe that, for $\sigma = \{\gamma_0, \ldots, \gamma_s\}$ a $k$-simplex of $L(\hat{\Pi}_{g,n})$, the set $\{\tau_{\gamma_0}, \ldots, \tau_{\gamma_s}\}$ is a set of commuting profinite Dehn twists.

\textbf{Theorem 6.1.} For $2g - 2 + n > 0$, let $\sigma = \{\gamma_0, \ldots, \gamma_s\}$ and $\sigma' = \{\delta_0, \ldots, \delta_t\}$ be two non-degenerate simplices of $L(\hat{\Pi}_{g,n})$. Suppose that, in $\hat{\Gamma}_{g,n}$, there is an identity:

$$\tau_{\gamma_0}^{h_{\gamma_0}} \tau_{\gamma_1}^{h_{\gamma_1}} \cdots \tau_{\gamma_s}^{h_{\gamma_s}} = \tau_{\delta_0}^{k_0} \tau_{\delta_1}^{k_1} \cdots \tau_{\delta_t}^{k_t},$$

for some $h_i, k_j \in \hat{\mathbb{Z}}$.

Then, it holds:

\begin{enumerate}
  \item $t = s$;
  \item there is a permutation $\phi \in \Sigma_s$ such that $\delta_i = \gamma_{\phi(i)}$ and $k_i = h_{\phi(i)}$, for $i = 1, \ldots, s$.
\end{enumerate}

Before we start with the proof of Theorem \ref{thm:dehn-twist-closure}, we need some preliminary results which are also of independent interest.

Let us recall some notations from Section 2. For $K$ an open normal subgroup of $\hat{\Pi}_{g,n}$, there is an associated Galois cover $S_K \to S_{g,n}$ with covering transformations’ group $\hat{G}_K := \hat{\Pi}_{g,n}/K$ and we denote by $\mathfrak{S}_K$ the smooth compactification of the punctured Riemann surface $S_K$. For every element $\alpha \in \hat{\Pi}_{g,n}$, there is a minimal positive integer $m_\alpha$ such that $\alpha^{m_\alpha} \in K$. Let us denote then by $\hat{\alpha}$ the image of $\alpha^{m_\alpha}$ in the first homology group of the surface $\mathfrak{S}_K$ with coefficients in $\hat{\mathbb{Z}}$.

\textbf{Definition 6.2.} Let $M$ be a finitely generated free $\hat{\mathbb{Z}}$-module. We say that a submodule $N \subseteq M$ is primitive if both $N$ and $M/N$ are free $\hat{\mathbb{Z}}$-modules. An element $v \in M$ is then primitive if the closed subgroup generated by $v$ is a primitive submodule of $M$.

Let $h$ be the map defined in iiii.) of Remarks \ref{rem:dehn-twist-closure}. Given $\gamma \in \hat{\mathcal{L}}$, let $\alpha \in h^{-1}(\gamma)$. Then, we can associate to $\alpha$ and any open normal subgroup $K$ of $\hat{\Pi}_{g,n}$ the $G_K$-orbit $\{ \pm f \cdot \hat{\alpha} \}_{f \in G_K}$ of lax primitive vectors of $H_1(\mathfrak{S}_K, \hat{\mathbb{Z}})$. This orbit depends only on $\gamma \in \hat{\mathcal{L}}$ and not on the particular representative $\alpha \in h^{-1}(\gamma)$ chosen.
Proposition 6.3. For $2g - 2 + n > 0$, let $\gamma_0$ and $\gamma_1$ be distinct elements of $\mathcal{L}$ and let $\alpha_i \in h^{-1}(\gamma_i)$, for $i = 0, 1$. Then, there is an open characteristic subgroup $K$ of $\hat{\Pi}_{g,n}$ such that the primitive, cyclic $\hat{\mathbb{Z}}$-submodules of $H_1(\mathcal{S}_K, \hat{\mathbb{Z}})$ generated by elements in the $G_K$-orbits of $\hat{\alpha}_0$ and $\hat{\alpha}_1$ have mutual trivial intersections.

The following corollary of the above proposition is possibly of independent interest:

Corollary 6.4. Let $\gamma_0, \gamma_1$ be two non-homotopic s.c.c. on a Riemann surface $S_{g,n}$ then there is an unramified cover $\pi: S' \to S_{g,n}$ such that each of the connected components of $\pi^{-1}(\gamma_0)$ is non-homologous, in the smooth compactification $\overline{S'}$ of $S'$, to each of the connected components of $\pi^{-1}(\gamma_1)$.

Proof. In order to prove Proposition 6.3 let us observe that the profinite group $\hat{\Pi}_{g,n}$ can be realized as the inverse limit of all virtual pro-$p$ completions of $\Pi_{g,n}$, for a fixed prime $p$. For a normal finite index subgroup $N$ of $\Pi_{g,n}$, let us then define $\Pi_{N}^{(p)}$ be the virtual pro-$p$ group which fits in the short exact sequence:

$$1 \to N^{(p)} \to \Pi_{N}^{(p)} \to \Pi_{g,n}/N \to 1,$$

where, for a given group $G$ and a prime $p$, we denote by $G^{(p)}$ its pro-$p$ completion. It then holds $\lim_{N \leq \Pi_{g,n}} \Pi_{N}^{(p)} = \hat{\Pi}_{g,n}$.

For each of the virtual pro-$p$ completion $\Pi_N^{(p)}$ of $\Pi_{g,n}$, let us define, in the same way as we did for the full profinite completion $\hat{\Pi}_{g,n}$ at the beginning of Section 5, the profinite set $\mathcal{L}_{N}^{(p)}$ to be the closure of $\mathcal{L}$ inside $\mathcal{P}_f(\Pi_N^{(p)}/\sim)$. It is clear that it holds as well $\lim_{N \leq \Pi_{g,n}} \mathcal{L}_{N}^{(p)} = \mathcal{L}$.

Let $h$ be the map considered above and define also $h_N: \Pi_{N}^{(p)} \to \mathcal{P}_f(\Pi_{N}^{(p)}/\sim)$ to be the map defined sending an element $\gamma$ to the conjugacy class of the set $\{\gamma, \gamma^{-1}\}$. Let us then define the profinite sets:

$$\mathcal{R} := \{x^u \in \hat{\Pi}_{g,n} | u \in \hat{\mathbb{Z}}^* \text{ and } h(x) \in \mathcal{L}\}$$

$$\mathcal{R}_{N}^{(p)} := \{x^u \in \Pi_{N}^{(p)} | u \in \hat{\mathbb{Z}}^* \text{ and } h_N(x) \in \mathcal{L}_{N}^{(p)}\}.$$

As above, it holds $\lim_{N \leq \Pi_{g,n}} \mathcal{R}_{N}^{(p)} = \mathcal{R}$.

For every open characteristic subgroup $K$ of $\hat{\Pi}_{g,n}$, there is a natural continuous $\hat{\Pi}_{g,n}$-equivariant map:

$$\psi_K: \hat{\Pi}_{g,n} \to H_1(\mathcal{S}_K, \hat{\mathbb{Z}})$$

$$\alpha \mapsto \hat{\alpha},$$

where $\hat{\Pi}_{g,n}$ acts by conjugation on the domain and by the action induced by the covering transformations’ group $G_K$ on $H_1(\mathcal{S}_K, \hat{\mathbb{Z}})$.

Observe that the image by $\psi_K$ of the closed cyclic subgroup of $\hat{\Pi}_{g,n}$ generated by an element $\alpha$ is the closed cyclic subgroup of $H_1(\mathcal{S}_K, \hat{\mathbb{Z}})$ spanned by $\hat{\alpha}$. Moreover, for $u \in \hat{\mathbb{Z}}^*$, it holds $\psi_K(\alpha^u) = u \cdot \psi_K(\alpha)$. 

Let us consider the product map
\[
\Psi := \prod_{K \leq \hat{\Pi}_{g,n}} \psi_K : \hat{\Pi}_{g,n} \to \prod_{K \leq \hat{\Pi}_{g,n}} H_1(S_K, \hat{\mathbb{Z}}),
\]

where \( K \) varies over all open characteristic subgroups of \( \hat{\Pi}_{g,n} \). The map \( \Psi \) then is \( \hat{\Pi}_{g,n} \)-equivariant and has the property that, for all \( \alpha \in \hat{\Pi}_{g,n} \) and \( u \in \hat{\mathbb{Z}}^* \), it holds \( \Psi(\alpha^u) = u \cdot \Psi(\alpha) \).

**Lemma 6.5.** The restriction of \( \Psi \) to the profinite subset \( \hat{\mathcal{R}} \subset \hat{\Pi}_{g,n} \) is injective.

Lemma 6.5 implies Proposition 6.3. To see this, let \( \alpha_i \in \hat{\Pi}_{g,n} \), for \( i = 0, 1 \), be like in the hypotheses of Proposition 6.3. Then, if Lemma 6.5 holds, the primitive cyclic submodules \( \Psi(\alpha_0^2) \) and \( \Psi(\alpha_1^2) \) of \( \prod_{K \leq \Pi_{g,n}} H_1(S_K, \hat{\mathbb{Z}}) \) have trivial intersection. Otherwise, by iii.) of Remarks 5.7 it would hold \( \gamma_0 = \gamma_1 \).

For the same reason, since \( \Psi \) is \( \hat{\Pi}_{g,n} \)-equivariant, all primitive submodules in the \( \hat{\Pi}_{g,n} \)-orbits of \( \Psi(\alpha_0^2) \) and \( \Psi(\alpha_1^2) \) have also mutual trivial intersections. This implies that there exists an open characteristic subgroup \( K \) of \( \hat{\Pi}_{g,n} \) such that the same holds for the \( G_K \)-orbits of \( \psi_K(\alpha_0^2) \) and \( \psi_K(\alpha_1^2) \) in \( H_1(S_K, \hat{\mathbb{Z}}) \).

Let us define an analogue of the map \( \Psi \) for the virtual pro-\( p \) completions \( \Pi_{N}^{(p)} \):
\[
\Psi_{N}^{(p)} := \prod_{K \leq \Pi_{N}^{(p)}} \psi_{K}^{(p)} : \Pi_{N}^{(p)} \to \prod_{K \leq \Pi_{N}^{(p)}} H_1(S_K, \mathbb{Z}_p),
\]

where now \( K \) varies over all open characteristic subgroups of \( \Pi_{N}^{(p)} \).

**Lemma 6.6.** The restriction of \( \Psi_{N}^{(p)} \) to the profinite subset \( \hat{\mathcal{R}}_{N}^{(p)} \subset \Pi_{N}^{(p)} \) is injective.

It is clear that Lemma 6.6 implies Lemma 6.5. Thanks to the following lemma, which follows from an argument similar to the one given for Lemma 5.4, we will be able to further reduce to the case when \( N = \Pi_{g,n} \).

**Lemma 6.7.** Let \( x \neq 1 \) be an element of \( \hat{\mathcal{R}}_{N}^{(p)} \) and let \( n_x \) be the smallest positive integer such that \( x^{n_x} \in N^{(p)} \). Then, the closed subgroup topologically generated by \( x^{n_x} \) in \( \Pi_{N}^{(p)} \) has for normalizer the closed subgroup topologically generated by \( x \).

Indeed, Lemma 6.7 implies the injectivity of the restriction to \( \hat{\mathcal{R}}_{N}^{(p)} \) of the continuous map \( \Pi_{N}^{(p)} \to N^{(p)} \), defined sending \( x \in \Pi_{N}^{(p)} \) to \( x^{n_x} \in N^{(p)} \).

Since \( N^{(p)} \) is isomorphic to the pro-\( p \) completion of a hyperbolic surface group, we are finally reduced to prove that, for all \( 2g - 2 + n > 0 \), the product map
\[
\Psi^{(p)} := \prod_{K \leq \Pi_{g,n}^{(p)}} \psi_{K}^{(p)} : \Pi_{g,n}^{(p)} \to \prod_{K \leq \Pi_{g,n}^{(p)}} H_1(S_K, \mathbb{Z}_p),
\]

where \( K \) varies between all open characteristic subgroups of \( \Pi_{g,n}^{(p)} \), is injective.
It is well known that the pro-$p$ group $\Pi_{g,n}^{(p)}$ admits an exhaustive and nilpotent (i.e., $[\Pi_{g,n}^{(p)}, K_{s-1}] \triangleleft K_s$) sequence of characteristic open subgroups:

$$\ldots \triangleleft K_{s+1} \triangleleft K_s \triangleleft K_{s-1} \triangleleft \ldots \triangleleft K_0 = \Pi_{g,n}^{(p)}.$$ 

Let $x, y \in \Pi_{g,n}^{(p)}$ be two distinct elements. There is an integer $m$ such that the images of $x, y$ are equal in $\Pi_{g,n}^{(p)}/K_{m-1}$ and distinct in $\Pi_{g,n}^{(p)}/K_m$. Let $H$ be the open subgroup of $\Pi_{g,n}^{(p)}$ generated by $x$ and $K_{m-1}$. By our assumption, $y \in H$ as well and the images $\overline{x}, \overline{y}$ of $x, y$ in $H_1(H, \mathbb{Z}_p)$ are distinct. In fact, the quotient $H/K_m$ is an abelian subgroup of $\Pi_{g,n}^{(p)}/K_m$ which contains the images of $x$ and $y$.

Let $N$ be an open normal subgroup of $\Pi_{g,n}^{(p)}$ contained in $H$. The transfer homomorphism $\text{tr}: H_1(H, \mathbb{Z}_p) \to H_1(N, \mathbb{Z}_p)$ is injective. Let $x^{*s}, y^{*s}$ be the least positive powers of $x, y$ contained in $N$ and denote by $x', y'$ the images of $x^{*s}, y^{*s}$ in $H_1(N, \mathbb{Z}_p)$. It then holds $\text{tr}(\overline{x}) = \sum_{h \in H/N} h \cdot x' \neq \text{tr}(\overline{y}) = \sum_{h \in H/N} h \cdot y'$. In particular, $x' \neq y'$.

If $n = 0$, then $H_1(N, \mathbb{Z}_p) \cong H_1(\overline{S}_N, \mathbb{Z}_p)$ and we are done. If $n > 0$, let us just observe that there is an open characteristic subgroup $K$ of $N$ such that the natural homomorphism $H_1(N, \mathbb{Z}_p) \to H_1(\overline{S}_K, \mathbb{Z}_p)$, obtained composing the transfer homomorphism with the epimorphism $H_1(\overline{S}_K, \mathbb{Z}_p) \to H_1(\overline{S}_K, \mathbb{Z}_p)$, is injective.

We can now prove Theorem 6.1.

**Proof.** Let $V$ be a $\hat{\mathbb{Z}}$-lattice endowed with a non-degenerate symplectic form $\langle -,- \rangle$ and let $\text{Sp}(V)$ be the group of symplectic automorphisms of $V$.

A symplectic transvection $t_v: V \to V$ is defined, for a given primitive element $v \in V$ and all $w \in V$, by the assignment $w \mapsto w + \langle w, v \rangle v$. There is then a bijective correspondence between lax primitive vectors $\pm v$ of $V$ and all symplectic transvections $t_v$ of $V$. Observe, that, for $u \neq \pm 1$ a unit of $\hat{\mathbb{Z}}$, we have the non-trivial identity $t_{uv} = t_v^u$.

**Definition 6.8.** A family $F = \{x_0, \ldots, x_s\}$ of primitive elements of $V$ is totally unimodular if they generate a primitive submodule $L$ of $V$ and, in a base for $L$ formed by elements of $F$, the matrix with columns $x_0, \ldots, x_s$ is totally unimodular, i.e. every minor of the matrix is either $\pm 1$ or $0$. We say that the family $F$ is reduced if, moreover, in this matrix, there are not proportional columns.

For an isotropic, totally unimodular reduced set of lax primitive vectors $\{\pm v_0, \ldots, \pm v_s\}$ and a multi-index $(h_0, \ldots, h_s) \in (\hat{\mathbb{Z}} \setminus \{0\})^4$, let us define the associated weighted symplectic multi-transvection of $V$ to be the product $t_{v_0}^{h_0} \cdots t_{v_s}^{h_s}$. From the result in the appendix of [S], it follows that the elements in $\text{sp}(V)$, associated to the transvections $t_{v_i}$, for $i = 1, \ldots, s$, are linearly independent. Therefore, an identity between weighted symplectic multi-transvections of $V$

$$t_{v_0}^{h_0} \cdots t_{v_s}^{h_s} = t_{w_0}^{h_0} \cdots t_{w_t}^{h_t} \quad (\ast)$$

can subsist only if $s = t$ and the two sets of vectors $\{\pm v_0, \ldots, \pm v_s\}$ and $\{\pm w_0, \ldots, \pm w_t\}$ have the same linear span $L$ in $V$. 

Then, it is clear that, if we let $L = \oplus_{i=1}^{q} L_i$ be a decomposition of $L$ such that each of the sub-lattices $L_i$ of $V$, for $i = 1, \ldots, q$, is generated by elements of the set $\{\pm v_0, \ldots, \pm v_s\}$, then the identity $(\star)$ holds only if the decomposition $L = \oplus_{i=1}^{q} L_i$ has the same property with respect to the set $\{\pm w_0, \ldots, \pm w_s\}$.

So, let $\sigma = \{\gamma_0, \ldots, \gamma_s\}$ and $\sigma' = \{\delta_0, \ldots, \delta_l\}$ be as in the hypothesis of Theorem 6.1. By Proposition 6.3 there is an open characteristic subgroup $K$ of $\Pi_{g,n}$ such that all submodules of $H_1(\overline{S}_K, \hat{Z})$, in the $G_K$-orbits of the closed cyclic subgroups generated by the lax primitive vectors associated to distinct elements of the set $\sigma \cup \sigma'$, have mutual trivial intersections.

Let us assume also that $K$ satisfies the hypotheses of item ii.) in Theorem 2.8. By Proposition 2 in [18], the open characteristic subgroup $K_2 := K^{[2]} \cdot \overline{\Lambda} \cdot K^2$ of $\Pi_{g,n}$ which is contained in $K$ (and so has the above property), is then such that the union $F_\mu$ of the $G_K$-orbits of lax primitive vectors in $H_1(\overline{S}_K, \hat{Z})$, associated to a non-degenerate simplex $\mu$ in $L(\overline{\Pi}_{g,n})$, is a totally unimodular and reduced set.

For every non-trivial s.c.c. $\gamma$ on $S_K$, the cover $p': S_{K_2} \to S_K$ has the property that the pre-image $p'^{-1}(\gamma)$ disconnects the surface $S_{K_2}$. Therefore, every relation between elements of $F_\mu$ is a linear combination of those between elements which lie in the same $\pi_1(S_K)/\pi_1(S_{K_2})$-orbit.

Let $L_\mu$, for $\mu$ a non-degenerate simplex of $L(\overline{\Pi}_{g,n})$, be the linear span of the elements of the set $F_\mu$. Thus, there is a decomposition $L_\mu = \oplus_{i=1}^{q} L_i$, where each of the $L_i$, for $i = 1, \ldots, q$, is generated by the elements of a $\pi_1(S_K)/\pi_1(S_{K_2})$-orbit in the set $F_\mu$.

Now, any two $\pi_1(S_K)/\pi_1(S_{K_2})$-orbits in $F_\sigma \cup F_{\sigma'}$ lying above distinct elements of the set $\sigma \cup \sigma'$ generate sub-lattices of $H_1(\overline{S}_K, \hat{Z})$ which have trivial intersection. Hence, even if $L_\sigma = L_{\sigma'}$, for $\sigma \neq \sigma'$, the decomposition $L_{\sigma'} = L_{\sigma} = \oplus_{i=1}^{q} L_i$ would not be compatible with a basis chosen from elements of the family $F_{\sigma'}$.

Let $m$ be the least positive integer such that for any s.c.c. $\gamma$ on $S_{g,n}$ the power $\tau_{\gamma}^m$ belongs to the Looijenga level $\Gamma_{K_2}^{(2)}$. Therefore, in the open subgroup $\Gamma_{K_2}^{(2)} \subset \Gamma_{g,n}$, it holds the identity:

$$\tau_{\gamma_0}^{mh_0} \tau_{\gamma_1}^{mh_1} \ldots \tau_{\gamma_s}^{mh_s} = \tau_{\delta_{i_1}}^{mk_{i_1}} \tau_{\delta_{i_2}}^{mk_{i_2}} \ldots \tau_{\delta_{i_s}}^{mk_{i_s}}.$$

By the results of Section 2 there is a natural representation $\hat{\Gamma}_{K_2}^{(2)} \to \text{Sp}(H_1(\overline{S}_{K_2}, \hat{Z}))$, which sends a Dehn twist $\tau_{\gamma}^m$ in the weighted symplectic multi-transvection associated to the multi-index $(l, l, \ldots, l)$, for some positive integer $l|m$, and the $G_{K_2}$-orbit of lax primitive vectors of $H_1(\overline{S}_{K_2}, \hat{Z})$ associated to $\gamma$. Therefore, by our preliminary considerations, the above identity is possible only if $t = s$ and there is a permutation $\phi \in \Sigma_s$ such that $\delta_i = \gamma_{\phi(i)}$, for $i = 1, \ldots, s$. That it holds as well $k_i = h_{\phi(i)}$, for $i = 1, \ldots, s$, it then follows immediately.

\[\square\]

**Corollary 6.9.**\(\star\) Let $\sigma \in L(\overline{\Pi}_{g,n})$, for $2g - 2 + n > 0$, be a non-degenerate simplex of
Proof. The corollary is an immediate consequence of Theorem 5.2 and Theorem 6.1.

Corollary 6.10. For $2g - 2 + n > 0$, there is a natural continuous injective $\hat{\Gamma}_{g,n}$-equivariant map $\hat{d}: \mathcal{L} \hookrightarrow \hat{\Gamma}_{g,n}$ which assigns to a profinite s.c.c. $\gamma$ the Dehn twist $\tau_\gamma$.

Proof. By the first of Remarks 5.7, we know that the orbits of $\hat{\Gamma}_{g,n}$ in $\mathcal{L}$ are the closures of the orbits of $\Gamma_{g,n}$ in $\mathcal{L}$. On the other hand, from the description of local monodromy for the Looijenga levels given in Proposition 2.9, it follows that the conjugacy classes of profinite Dehn twists in $\hat{\Gamma}_{g,n}$ are obtained taking the closures of the conjugacy classes of Dehn twists in $\Gamma_{g,n}$ and are in bijective correspondence with them.

Thus, the map $\hat{d}$ induces a bijection between $\hat{\Gamma}_{g,n}$-orbits of profinite s.c.c. and conjugacy classes of profinite Dehn twists. By Theorem 5.2 and Corollary 6.9, the map $\hat{d}$ induces also an isomorphism between the respective $\hat{\Gamma}_{g,n}$-stabilizers. Therefore, it is injective.

Corollary 6.11. Let $2g - 2 + n > 0$. Then, for every open subgroup $U$ of $\hat{\Gamma}_{g,n}$, it holds:

$$Z(\hat{\Gamma}_{g,n}) = Z_{\hat{\Gamma}_{g,n}}(U) = Z(\Gamma_{g,n}).$$

In particular, all these groups are trivial for $(g,n) \neq (1,1), (2,0)$ and, otherwise, they are generated by the hyperelliptic involution.
Proof. This follows from the description of the centralizer of a multi-twist given above and from the description of centers of low genus profinite Teichmüller groups given in Proposition 3.2 in [6].

We can now give a proof of Theorem 2.5 in [6] which also covers the $n = 0$ case.

**Corollary 6.12.** For $2g - 2 + n > 0$, the natural representation $\tilde{\rho}_{g,n}: \hat{\Gamma}_{g,n+1} \to \text{Aut}(\hat{\Pi}_{g,n})$ is faithful. In particular, there is a short exact sequence: $1 \to \hat{\Pi}_{g,n} \to \hat{\Gamma}_{g,n+1} \to \hat{\Gamma}_{g,n} \to 1$.

**Proof.** From the description of the representation $\tilde{\rho}_{g,n}$ given in §2 of [6], it follows that its kernel is contained in all the centralizers of the multitwists of the form $\tau_{\gamma_0}\tau^{-1}_{\gamma_1}$, where $\{\gamma_0, \gamma_1\}$ is a cut pair on $S_{g,n+1}$ bounding a cylinder punctured only by $P_{n+1}$. Now, the result follows from Corollary 6.9 and Corollary 6.11.

A close inspection to the proof of Theorem 3.5 immediately shows that the $n = 0$ case of the congruence subgroup property for the hyperelliptic modular group (see [6] for all definitions) is an immediate consequence of Corollary 6.12:

**Theorem 6.13.** Let $H_{g,n}$, for $2g - 2 + n > 0$ and $g \geq 1$, be the moduli stack of $n$-pointed, genus $g$ smooth hyperelliptic complex curves. The profinite universal algebraic monodromy representation $\hat{\rho}_{g,n}: \hat{\pi}_1(H_{g,n}) \to \text{Out}(\hat{\Pi}_{g,n})$, associated to the universal $n$-punctured, genus $g$ hyperelliptic curve $H_{g,n+1} \to H_{g,n}$, is faithful.

The above results show that the simplicial profinite set $L(\hat{\Pi}_{g,n})_\bullet$ parametrizes also the abelian subgroups of $\hat{\Gamma}_{g,n}$ spanned by powers of profinite Dehn twists. Let us state this more precisely.

Let $\mathscr{F}$ be the profinite set of all closed subgroups of $\hat{\Gamma}_{g,n}$. The profinite group $\hat{\Gamma}_{g,n}$ acts continuously by conjugation on $\mathscr{F}$. A weight function on $\mathscr{F}$ is a $\hat{\Gamma}_{g,n}$-equivariant function $w: \hat{\mathcal{L}} \to \mathbb{N}^+$, where $\hat{\Gamma}_{g,n}$ acts trivially on $\mathbb{N}^+$.

Let then $\iota(w)_k: C(S_{g,n})_k \hookrightarrow \mathscr{F}$ be the natural embedding defined sending a $k$-simplex $\{\gamma_0, \ldots, \gamma_k\}$ to the closed subgroup $\hat{\mathbb{Z}} \cdot \tau^{w(\gamma_0)}_\gamma \cdots \tau^{w(\gamma_k)}_\gamma$ of $\hat{\Gamma}_{g,n}$ and let $\overline{C}^w(S_{g,n})_k$ be the closure, in the profinite topology, of the image of $\iota(w)_k$ in $\mathscr{F}$. The face and degeneracy operators extend to these profinite sets. As above, in this way, it is defined a simplicial profinite set, endowed with a natural, continuous, geometric $\hat{\Gamma}_{g,n}$-action, which we denote by $\overline{C}^w(S_{g,n})$ and call the weighted group-theoretic $\hat{\Gamma}_{g,n}$-completion of the complex of curves.

So, by the universal property of the completion $L(\hat{\Pi}_{g,n})_\bullet$, for every weight function $w: \hat{\mathcal{L}} \to \mathbb{N}^+$, there is a natural continuous $\hat{\Gamma}_{g,n}$-equivariant surjective map of simplicial profinite sets:

$$\mathscr{F}^w_\bullet: L(\hat{\Pi}_{g,n})_\bullet \twoheadrightarrow \overline{C}^w(S_{g,n}).$$

**Proposition 6.14.** Let $2g - 2 + n > 0$. For every weight function $w: \hat{\mathcal{L}} \to \mathbb{N}^+$, there is a natural continuous $\hat{\Gamma}_{g,n}$-equivariant isomorphism of simplicial profinite sets:

$$\mathscr{F}^w_\bullet: L(\hat{\Pi}_{g,n})_\bullet \simeq \overline{C}^w(S_{g,n}).$$
7 HOMOTOPY TYPE OF THE COMPLEX OF PROFINITE CURVES

Proof. Again by Proposition 2.9, the images of two simplices $\sigma, \sigma' \in C(S_{g,n})_k$ in the profinite set $C\hat{\omega} (\hat{C}(S_{g,n}))_k$ are in the same $\hat{\Gamma}_{g,n}$-orbit if and only if $S_{g,n} \setminus \sigma$ and $S_{g,n} \setminus \sigma'$ have the same topological type.

For all $k \geq 0$, the $\hat{\Gamma}_{g,n}$-orbits of $C\hat{\omega} (\hat{C}(S_{g,n}))_k$ are then in bijective correspondence with the $\Gamma_{g,n}$-orbits of $C(S_{g,n})_k$. Therefore, by Remarks 5.7, the $\hat{\Gamma}_{g,n}$-orbits of $L(\hat{\Pi}_{g,n})_\bullet$ are in bijective correspondence with the $\hat{\Gamma}_{g,n}$-orbits of $C\hat{\omega} (\hat{C}(S_{g,n}))_k$, for all $k \geq 0$, and the proposition follows if we show that the two actions have also isomorphic stabilizers. But this is an immediate consequence of Corollary 6.9, since the stabilizer of a $k$-simplex $\{\gamma_0, \ldots, \gamma_k\}$ of $C\hat{\omega} (\hat{C}(S_{g,n}))_k$ is precisely the normalizer of the subgroup $\hat{\mathbb{Z}} \cdot r_{\gamma_0}^{\mu_0(\gamma_0)} \oplus \ldots \oplus \hat{\mathbb{Z}} \cdot r_{\gamma_k}^{\mu_k(\gamma_k)}$ of $\hat{\Gamma}_{g,n}$. □

A profinite simplicial complex is an abstract simplicial complex whose sets of $k$-simplices are profinite, for all $k \geq 0$. For such simplicial complexes, the procedure which associates to each abstract simplicial complex a simplicial set produces a simplicial profinite set. The above results then sum up in the following completely intrinsic description of the geometric profinite curve complex $L(\hat{\Pi}_{g,n})_\bullet$.

**Theorem 6.15.** Let $2g - 2 + n > 0$. For any given open normal subgroup $\hat{\Gamma}^\lambda$ of $\hat{\Gamma}_{g,n}$, the simplicial profinite set $L(\hat{\Pi}_{g,n})_\bullet$ can be described as the simplicial profinite set associated to the profinite simplicial complex $L(\hat{\Pi}_{g,n})_\bullet$, whose set of $k$-simplices, for $k \geq 0$, consists of closed, primitive, free abelian subgroups of rank $k + 1$ in $\hat{\Gamma}_{g,n}$ spanned by powers of profinite Dehn twists.

**Remark 6.16.** Observe that $L(\hat{\Pi}_{g,n})_\bullet$, which we call the complex of profinite curves on $S_{g,n}$, is a flag complex, i.e. sets of vertices which are pairwise joinable are joinable. The barycentric subdivision of $L(\hat{\Pi}_{g,n})_\bullet$ is then more simply described as the flag complex associated to the poset of closed abelian subgroups of $\hat{\Gamma}_{g,n}$ spanned by profinite Dehn twists.

7 Homotopy type of the complex of profinite curves and the congruence subgroup property

The purpose of this section is to show that the obstruction to the validity of the subgroup congruence property for the Teichmüller group is the fundamental group of the complex of profinite curves.

In [25], Quick developed a homotopy theory for simplicial profinite sets, where the profinite fundamental group is defined in terms of finite covers. This approach turns out to be particularly useful here.

Let us recall that, in virtue of Theorem 5.6, the simplicial profinite set $L(\hat{\Pi}_{g,n})_\bullet$ is naturally isomorphic to the quotient of the $\hat{\Gamma}_{g,n}$-completion of the complex of curves $\hat{C}(S_{g,n})_\bullet$ (see Section 4), by the action of the congruence kernel, i.e. the kernel of the natural epimorphism $\hat{\Gamma}_{g,n} \rightarrow \hat{\Gamma}_{g,n}$.

The following result was essentially proved in [5].
Theorem 7.1. For $2g - 2 + n > 0$ and $3g - 3 + n > 2$, the simplicial profinite set $\hat{C}(S_{g,n})_\bullet$ is simply connected.

Proof. Let $\{\Gamma^\lambda\}_{\lambda \in \Lambda}$ be the set of all levels $\Gamma^\lambda$ of $\Gamma_{g,n}$. By definition, the simplicial profinite set $\hat{C}(S_{g,n})_\bullet$ is the inverse limit, over $\Lambda$, of the simplicial finite sets $C^\lambda(S_{g,n})_\bullet$. By Proposition 2.1 in [25] and the discussion which precedes it, there is then a natural isomorphism:

$$\hat{\pi}_1(\hat{C}(S_{g,n})_\bullet) \cong \lim_{\lambda \in \Lambda} \hat{\pi}_1(C^\lambda(S_{g,n})_\bullet),$$

where $\hat{\pi}_1(C^\lambda(S_{g,n})_\bullet)$ is the profinite completion of the topological fundamental group of the simplicial set $C^\lambda(S_{g,n})$, and $\hat{\pi}_1(\hat{C}(S_{g,n})_\bullet)$ is the profinite fundamental group of the simplicial profinite set $\hat{C}(S_{g,n})_\bullet$.

For a level structure $\hat{L}^\lambda$ which satisfies the hypotheses of Proposition 3.1, the simplicial set $C^\lambda(S_{g,n})$ is the nerve of the covering of the D–M boundary $\partial \hat{L}^\lambda$ by irreducible components and then the nerve of the corresponding covering of the deleted tubular neighborhood $\partial \hat{L}^\lambda$ of $\partial L^\lambda$ in $\hat{L}^\lambda$.

From Proposition 4 in [17], it follows that, for $3g - 3 + n > 2$, there is a natural isomorphism $\pi_1(\partial \hat{L}^\lambda) \cong \Gamma^\lambda$. On the other hand, by elementary topology, there is a natural epimorphism $\pi_1(\partial \hat{L}^\lambda) \rightarrow \pi_1(C^\lambda(S_{g,n})_\bullet)$. Therefore, passing to profinite completions, we get a natural epimorphism $\hat{\Gamma}^\lambda \rightarrow \hat{\pi}_1(C^\lambda(S_{g,n})_\bullet)$ and, passing to inverse limits, an epimorphism:

$$\lim_{\lambda \in \Lambda} \hat{\Gamma}^\lambda \twoheadrightarrow \hat{\pi}_1(\hat{C}(S_{g,n})_\bullet).$$

But now $\lim_{\lambda \in \Lambda} \hat{\Gamma}^\lambda = \cap_{\lambda \in \Lambda} \hat{\Gamma}^\lambda = \{1\}$ and the theorem follows.

Corollary 7.2. Let $2g - 2 + n > 0$. The congruence subgroup property for the Teichmüller group $\Gamma_{g,n}$ holds for all $(g, n)$ such that $3g - 3 + n \leq k$ if and only if $\hat{\pi}_1(L(\hat{\Pi}_{g,n})_\bullet) = \{1\}$ for all $(g, n)$ such that $2 < 3g - 3 + n \leq k$.

Proof. One implication is obvious. To prove the other, let us proceed by induction on $k$. The congruence subgroup property holds in genus $\leq 2$. So, the statement of the corollary holds at least for $k = 1, 2$.

In order to complete the induction, we have to show that, for $k \geq 3$, the assumptions that the congruence subgroup property holds for all $(g, n)$ such that $3g - 3 + n < k$ and that $\hat{\pi}_1(L(\hat{\Pi}_{g,n})_\bullet) = \{1\}$, for $3g - 3 + n = k$, implies the congruence subgroup property for $3g - 3 + n = k$.

Our induction hypothesis, together with Proposition 4.2 and Theorem 5.2, implies that, for $3g - 3 + n = k$, the stabilizers of $\hat{\Gamma}_{g,n}$ acting on $\hat{C}(S_{g,n})_\bullet$ and the stabilizers of $\hat{\Gamma}_{g,n}$ acting on $L(\hat{\Pi}_{g,n})_\bullet$ are naturally isomorphic. In other words, if $\Psi_{g,n}: \hat{\Gamma}_{g,n} \rightarrow \hat{\Gamma}_{g,n}$ denotes the natural epimorphism, the profinite group $\ker \Psi_{g,n}$ acts freely on $\hat{C}(S_{g,n})_\bullet$ with quotient naturally isomorphic to $L(\hat{\Pi}_{g,n})_\bullet$. From Corollary 2.3 in [25], it then follows that $\ker \Psi_{g,n} \cong \hat{\pi}_1(L(\hat{\Pi}_{g,n})_\bullet) = \{1\}$. 

A natural expectation is that, like the discrete curve complex $C(S_{g,n})$ (see [12], [13] and [16]), the simplicial profinite set $L(\hat{\Pi}_{g,n})\bullet$ is spherical of dimension $-\chi(S_{g,n}) - 1$, for $n \geq 1$, and $-\chi(S_g)$, for $n = 0$, where $\chi(S_{g,n}) = 2 - 2g - n$ is the Euler characteristic of $S_{g,n}$.

8 Faithfulness of Galois representations

In this section, for $2g - 2 + n > 0$, we denote by $\mathcal{M}_{g,n}$ the stack of smooth algebraic curves defined over some number field and by $\overline{\mathcal{M}}_{g,n}$ its D–M compactification (in contrast with Sections 2–7, where we considered only complex algebraic curves). Both of them are then defined over $\text{Spec}(\mathbb{Q})$.

Let $C$ be a smooth $n$-punctured, genus $g$ curve, defined over a number field $k$. To give such a curve, it is equivalent to give a point $\xi: \text{Spec}(k) \to \mathcal{M}_{g,n}$. If $\mathcal{C} \to \mathcal{M}_{g,n}$ denotes the universal $n$-punctured, genus $g$ curve, the curve $C$ is isomorphic to the fiber $\mathcal{C}_\xi$. Let $\xi$ be the geometric point of $\mathcal{M}_{g,n}$ associated to $\xi$ and a given embedding $k \subset \overline{\mathbb{Q}}$, let $\tilde{\xi}$ be a closed point of $C \times_k \overline{\mathbb{Q}}$ (which we identify with $\mathcal{C}_\xi$) and let $G_\mathbb{Q}$ be the absolute Galois group. There is then a commutative exact diagram:

$$
\begin{array}{ccccccc}
1 & \rightarrow & \pi_1(C \times_k \overline{\mathbb{Q}}, \tilde{\xi}) & \rightarrow & \pi_1(\mathcal{M}_{g,n} \times \overline{\mathbb{Q}}, \xi) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \text{\textcircled{★}} \\
1 & \rightarrow & \pi_1(\mathcal{C} \times \overline{\mathbb{Q}}, \tilde{\xi}) & \rightarrow & \pi_1(\mathcal{C}, \tilde{\xi}) & \rightarrow & G_\mathbb{Q} \rightarrow 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(\mathcal{M}_{g,n} \times \overline{\mathbb{Q}}, \xi) & \rightarrow & \pi_1(\mathcal{M}_{g,n}, \xi) & \rightarrow & G_\mathbb{Q} \rightarrow 1 \\
\end{array}
$$

Let $G_k$ be the absolute Galois group of $k$. Then, the group $G_k$ identifies with an open subgroup of $G_\mathbb{Q}$ and the algebraic fundamental group $\pi_1(\mathcal{M}_{g,n} \times k, \overline{\xi})$ with $p^{-1}(G_k)$.

Moreover, the point $\xi \in \mathcal{M}_{g,n}$ induces a homomorphism on algebraic fundamental groups $s_\xi: G_k \to \pi_1(\mathcal{M}_{g,n} \times k, \overline{\xi})$ which is a canonical section of the natural epimorphism $p: \pi_1(\mathcal{M}_{g,n} \times k, \overline{\xi}) \rightarrow G_k$.

There are various representations associated to the above diagram. To the middle column, is associated the arithmetic universal monodromy representation:

$$
\mu_{g,n}: \pi_1(\mathcal{M}_{g,n}, \overline{\xi}) \rightarrow \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \tilde{\xi})), \tag{1}
$$

which encloses all the others. Composing with the section $s_\xi$, we get indeed the outer monodromy representation associated to the curve $C/k$:

$$
\rho_C: G_k \rightarrow \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \tilde{\xi})). \tag{2}
$$
Instead, composing with the natural monomorphism $\pi_1(M_{g,n} \times \overline{Q}, \xi) \hookrightarrow \pi_1(M_{g,n}, \overline{\xi})$, we get the geometric universal monodromy representation:

$$\overline{\mu}_{g,n}: \pi_1(M_{g,n} \times \overline{Q}, \xi) \rightarrow \text{Out}(\pi_1(C \times_k \overline{Q}, \xi)).$$

Let us fix an embedding $\overline{Q} \subset C$ and let us assume that, with the notations of Section 2, it holds $\xi = a$ and $\overline{\xi} = \overline{a}$. Then, there are isomorphisms $\Gamma_{g,n} \cong \pi_1(M_{g,n} \times \overline{Q}, \xi)$, $\tilde{\Gamma}_{g,n+1} \cong \pi_1(M_{g,n+1} \times \overline{Q}, \xi)$ and $\tilde{\Pi}_{g,n} \cong \pi_1(C \times_k \overline{Q}, \xi)$, and the geometric universal monodromy representation $\overline{\mu}_{g,n}$ identifies with the profinite representation $\hat{\rho}_{g,n}$ induced by the topological universal monodromy representation $\rho_{g,n}$.

In this section, we are going to prove that the representation (2) is faithful for all hyperbolic curves $C$, thus extending to the projective case a classical result by M. Matsumoto who, in [20], proved the same result for hyperbolic affine curves. Moreover, we will show that the representation (1) is faithful if and only the representation (3) is faithful, i.e. if the subgroup congruence property holds for $\Gamma_{g,n}$. In particular, by Theorem 6.13 it will follow that $\mu_{g,n}$ is faithful for $2g - 2 + n > 0$ and $g \leq 2$.

In order to proceed with the above program, we need to consider the representation:

$$\mathcal{G}_{g,n}: \pi_1(M_{g,n}, \overline{\xi}) \rightarrow \text{Aut}^*(\tilde{\Gamma}_{g,n}),$$

where we have identified $\tilde{\Gamma}_{g,n}$ with the normal subgroup $\pi_1(M_{g,n} \times \overline{Q}, \overline{\xi})$ of $\pi_1(M_{g,n}, \overline{\xi})$, and the action is given by restriction of inner automorphisms.

We denote by $\text{Aut}^*(\tilde{\Gamma}_{g,n})$ the subgroup of $\text{Aut}(\tilde{\Gamma}_{g,n})$ consisting of those automorphism which preserve the set of inertia subgroups associated to the irreducible component of the D–M boundary of $M_{g,n}$. More explicitly, let us give the following definition.

**Definition 8.1.** Let $\Gamma'_{g,n}$, for $2g - 2 + n > 0$, be a profinite completion of the Teichmüller group. We then denote by $\text{Aut}^*(\Gamma'_{g,n})$ the group of automorphisms of $\Gamma'_{g,n}$ which preserve the set of closed cyclic subgroups of $\Gamma'_{g,n}$ generated by profinite Dehn twists.

For $2g - 2 + n > 0$, the geometric profinite Teichmüller group $\tilde{\Gamma}_{g,n}$ is defined as the image of the geometric universal monodromy representation $\overline{\mu}_{g,n}$. Thus, it is a normal subgroup of $\text{Im} \mu_{g,n}$ and $\pi_1(M_{g,n}, \overline{\xi})$ acts on it composing the representation $\mu_{g,n}$ with the action of inner automorphisms. So, we get a natural representation:

$$\mathcal{G}_{g,n}: \pi_1(M_{g,n}, \overline{\xi}) \rightarrow \text{Aut}^*(\tilde{\Gamma}_{g,n}),$$

which is compatible with $\mathcal{G}_{g,n}$. We will see that, thanks to the results of Section 6, the representation $\mathcal{G}_{g,n}$, while containing all the arithmetic information of $\mathcal{G}_{g,n}$, is much more treatable. For the moment, let us observe that, by lifting elements of $G_\overline{Q}$ in $\pi_1(M_{g,n}, \overline{\xi})$, we get also a natural representation, for $2g - 2 + n > 0$:

$$\overline{b}_{g,n}: G_\overline{Q} \rightarrow \text{Out}^*(\tilde{\Gamma}_{g,n}).$$

By the characterization given in Theorem 6.13 of the complex of profinite curves $L(\tilde{\Pi}_{g,n})$, there is a natural continuous action of $\text{Aut}^*(\tilde{\Gamma}_{g,n})$ on $\text{Aut}(L(\tilde{\Pi}_{g,n}))$, the group of continuous automorphism of the profinite simplicial complex $L(\tilde{\Pi}_{g,n})$. We claim that this action is faithful:
Theorem 8.2. For $2g - 2 + n > 0$, there is a natural faithful representation:

$$\text{Aut}^\ast(\hat{\Gamma}_{g,n}) \hookrightarrow \text{Aut}(L(\hat{\Pi}_{g,n}))$$

Proof. Let $\phi: \hat{\Gamma}_{g,n} \to \hat{\Gamma}_{g,n}$ be an automorphism such that, for every profinite Dehn twist $\tau_\gamma \in \hat{\Gamma}_{g,n}$, it holds $\phi(\tau^2_\gamma) = \tau^2_\gamma$, i.e., for every $\gamma \in L(\hat{\Pi}_{g,n})_0$, it holds $\phi(\tau_\gamma) = \tau^{\alpha_\gamma}_{\gamma}$, for some $\alpha_\gamma \in \mathbb{Z}^\ast$. Let us then show that $\phi$ is the identity.

Let $\gamma_0$, for $g \geq 1$ (resp. $g = 0$), be any non-separating s.c.c. (resp. a s.c.c. bounding a disc with two punctures) on $S_{g,n}$. The associated Dehn twists generate topologically $\hat{\Gamma}_{g,n}$. So, if we prove that $\phi(\tau_{\gamma_0}) = \tau_{\gamma_0}$, we are done.

Let then $\gamma_1$, for $g \geq 1$ (resp. $g = 0$), be a s.c.c. of the same type than $\gamma_0$ and which intersects $\gamma_0$, geometrically, only once (resp. only twice). The corresponding Dehn twists then satisfy the braid relation $\tau_{\gamma_0} \tau_{\gamma_1} \tau_{\gamma_0}^{-1} = \tau_{\gamma_1} \tau_{\gamma_0} \tau_{\gamma_1}^{-1}$. Applying the automorphism $\phi$, we get the identity:

$$\tau^{\alpha_{\gamma_0}}_{\gamma_0} \tau^{\alpha_{\gamma_1}}_{\gamma_1} \tau^{-\alpha_{\gamma_0}}_{\gamma_0} = \tau^{\alpha_{\gamma_1}}_{\gamma_1} \tau^{-\alpha_{\gamma_0}}_{\gamma_0} \tau^{-\alpha_{\gamma_1}}_{\gamma_1},$$

or, equivalently:

$$\tau^{\alpha_{\gamma_1}}_{\gamma_1 \gamma_0(\gamma_1)} = \tau^{\alpha_{\gamma_0}}_{\gamma_0 \gamma_1(\gamma_0)}.$$

Theorem [6,1] then implies that $\alpha_{\gamma_0} = \alpha_{\gamma_1}$ (let us then call $u$ this element) and that the profinite s.c.c.’s $\tau^{u}_{\gamma_0}(\gamma_1)$ and $\tau^{u}_{\gamma_1}(\gamma_0)$ have the same class in $\hat{L}$. This means that $\gamma_0 \cdot \gamma_1^u$ is conjugated in $\hat{\Pi}_{g,n}$ either to $\gamma_1 \cdot \gamma_0^u$ or to $\gamma_0^{-u} \cdot \gamma_1^{-1}$. Since the homology classes of $\gamma_0$ and $\gamma_1$ are distinct in $H_1(S_{g,n}, \hat{L})$, this is possible only for $u = 1$.

The following result is an interesting consequence of the description of centralizers of profinite Dehn twists given in Section 6.

Theorem 8.3. Let $(g,n)$ and $(g',n')$ be such that $2g - 2 + n > 0$, $2g' - 2 + n' > 0$, $g \geq g'$ and $3g - 3 + n > 3g' - 3 + n'$. Then, there is a natural $G_{\mathbb{Q}}$-equivariant homomorphism:

$$\text{Out}^\ast(\hat{\Gamma}_{g,n}) \to \text{Out}^\ast(\hat{\Gamma}_{g',n'}).$$

Proof. We have seen that there is a natural faithful representation $\Phi: \text{Aut}^\ast(\hat{\Gamma}_{g,n}) \hookrightarrow \text{Aut}(L(\hat{\Pi}_{g,n}))$. The key observation then is that elements of $\text{Aut}^\ast(\hat{\Gamma}_{g,n})$ preserve the topological type of profinite s.c.c.’s (see the first of Remarks 5.1). We actually need and we will prove here a slightly weaker assertion:

Lemma 8.4. For $2g - 2 + n > 0$, the action of $\text{Aut}^\ast(\hat{\Gamma}_{g,n})$ on $\hat{L}$ preserves the topological type of non-separating profinite s.c.c.’s and of profinite s.c.c.’s bounding a disc with two punctures.

Proof. Let us denote by $\hat{L}_0$ the $\hat{\Gamma}_{g,n}$-orbit in $\hat{L}$ consisting of non-separating profinite s.c.c.’s and, for $n \geq 2$, by $\hat{L}_1$ the $\hat{\Gamma}_{g,n}$-orbit in $\hat{L}$ consisting of profinite s.c.c.’s bounding a disc with two punctures. Let us show that the action of $\text{Aut}^\ast(\hat{\Gamma}_{g,n})$ on $\hat{L}_0$ preserves the union $\hat{L}_0 \cup \hat{L}_1$. 


Let $\gamma \in \mathcal{L}$, then, for all $f \in \text{Aut}^*(\tilde{\Gamma}_{g,n})$, it holds:

$$Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma) \cong Z_{\tilde{\Gamma}_{g,n}}(\tau_{f(\gamma)}) .$$

By Corollary 6.9, for $\gamma \in \mathcal{L}$, any pair of commuting profinite Dehn twists in $Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$ is conjugated, inside this group, to a pair of commuting Dehn twists contained in the discrete subgroup $Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$.

Now, any profinite s.c.c. $\gamma$ is in the $\tilde{\Gamma}_{g,n}$-orbit of a discrete one. So, it is easy to see that the centralizer of a profinite Dehn twist $\tau_\gamma$, for $\gamma \in \mathcal{L}_0 \cup \mathcal{L}_1$, is characterized by the property that, for any pair $\tau_\alpha, \tau_\beta \in Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$ of commuting profinite Dehn twists distinct from $\tau_\gamma$, there is a chain of profinite Dehn twists $\tau_\alpha = \tau_{\delta_1}, \ldots, \tau_{\delta_k} = \tau_\beta$ in $Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$ such that $\tau_{\delta_i}$ does not commute with $\tau_{\delta_i+1}$, for $i = 1, \ldots, k-1$.

The above argument already proves the lemma for $n \leq 1$. So let us assume $n \geq 2$. We need only to prove that, for any $\gamma \in \mathcal{L}_0$ and all $f \in \text{Aut}^*(\tilde{\Gamma}_{g,n})$, it holds $f(\gamma) \notin \mathcal{L}_1$.

Let us observe that the set of non-separating profinite Dehn twists forms a single conjugacy class in $\tilde{\Gamma}_{g,n}$, containing at least two distinct commuting elements for $n \geq 2$. On the other hand, to each pair of punctures on $S_{g,n}$, corresponds a distinct conjugacy class in $\tilde{\Gamma}_{g,n}$ of profinite Dehn twists along profinite s.c.c.’s bounding a disc with two punctures and none of these conjugacy classes contains a pair of distinct commuting elements.

Every automorphism $f$ of $\tilde{\Gamma}_{g,n}$ sends conjugacy classes to conjugacy classes and pairs of commuting elements to pairs of commuting elements. Therefore, for $\gamma \in \mathcal{L}_0$, it certainly does not hold $f(\gamma) \in \mathcal{L}_1$.

In order to prove Theorem 8.3, let us split it in the three assertions:

i.) for $g \geq 1$, there is a natural homomorphism $\text{Out}^*(\tilde{\Gamma}_{g,n}) \rightarrow \text{Out}^*(\tilde{\Gamma}_{g-1,n+2})$;

ii.) for $n \geq 2$, there is a natural homomorphism $\text{Out}^*(\tilde{\Gamma}_{g,n}) \rightarrow \text{Out}^*(\tilde{\Gamma}_{g,n-1})$;

iii.) for $g \geq 2$, there is a natural homomorphism $\text{Out}^*(\tilde{\Gamma}_{g,1}) \rightarrow \text{Out}^*(\tilde{\Gamma}_g)$.

Galois equivariance then follows from the naturality of the above homomorphisms.

Item i.) and ii.) are proved similarly. So, let us just give a proof of i.).

Let us choose an orientation for every element of $\mathcal{L}$. Let $\gamma$ be a non-separating s.c.c. on $S_{g,n}$. By Lemma 8.4, for any given $\tilde{f} \in \text{Aut}^*(\tilde{\Gamma}_{g,n})$, there is an $h \in \tilde{\Gamma}_{g,n}$ such that $\tilde{f}(\tau_\gamma) = \text{inn}(h)(\tau_\gamma)$. Moreover, since, for a non-separating s.c.c. $\gamma$ on $S_{g,n}$, there is a homeomorphism preserving $\gamma$ but switching its orientations, we can choose $h$ such that $h(\tilde{\gamma})$ has the chosen orientation.

Therefore, given an $f \in \text{Out}^*(\tilde{\Gamma}_{g,n})$, there is a lift $\tilde{f} \in \text{Out}^*(\tilde{\Gamma}_{g,n})$ such that $\tilde{f}(\gamma) = \gamma$. In particular, by restriction, $\tilde{f}$ induces an automorphism of $Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$, which preserves the set of closed cyclic groups spanned by profinite Dehn twists.

From the description of $Z_{\tilde{\Gamma}_{g,n}}(\tau_\gamma)$ given in Corollary 6.9, it then follows that this automorphism can be further restricted to its closed normal subgroup $\tilde{\Gamma}_\gamma$. Since, by Corollary 6.9 and Corollary 6.11, it holds $\tilde{\Gamma}_\gamma/Z(\tilde{\Gamma}_\gamma) \cong \tilde{\Gamma}_{g-1,n+2}$, it follows that $\tilde{f}$ induces an automorphism $\tilde{f}$ of $\tilde{\Gamma}_{g-1,n+2}$. 

\[ \square \]
Let \( \hat{\gamma} \in \text{Aut}^*(\hat{\Gamma}_{g,n}) \) be another lift of \( \gamma \) with the same properties and let \( \bar{\gamma} \) be the automorphism it induces on \( \hat{\Gamma}_{g-1,n+2} \). We have to prove that \( \hat{\gamma} \) and \( \bar{\gamma} \) differ by an inner automorphism of \( \hat{\Gamma}_{g-1,n+2} \).

By definition, the product \( \hat{\gamma}^{-1} \bar{\gamma} \) is an inner automorphism \( \text{inn}(x) \) of \( \hat{\Gamma}_{g,n} \), such that \( \text{inn}(x)(\tau_0) = \tau_0 \) and, moreover, \( x \) preserves the orientation of \( \hat{\gamma} \). Therefore, by Corollary 6.9, we have that \( x \in \hat{\Gamma}_{g} \) and the claim follows.

Let us now prove item \( \text{iii}. \). We just need to show that any element of \( \text{Aut}^*(\hat{\Gamma}_{g,1}) \) preserves the normal subgroup \( \hat{\Pi}_g \) of \( \hat{\Gamma}_{g,1} \). As a subgroup of \( \hat{\Gamma}_{g,1} \), the group \( \hat{\Pi}_g \) is topologically generated by products of Dehn twists of the form \( \tau_{\gamma_0} \tau_{\gamma_1}^{-1} \), where \( \{\gamma_0, \gamma_1\} \) is a cut pair on \( S_{g,1} \) bounding a cylinder containing the puncture. Therefore, it is enough to prove that elements of \( \text{Aut}^*(\hat{\Gamma}_{g,1}) \) preserve the topological type of the 1-simplices of \( L(\hat{\Pi}_{g,1}) \) corresponding to cut pairs on \( S_{g,1} \) of this kind.

So, given such a cut pair \( \{\gamma_0, \gamma_1\} \) on \( S_{g,1} \) and any \( f \in \text{Aut}^*(\hat{\Gamma}_{g,1}) \), let us show that \( \{f(\gamma_0), f(\gamma_1)\} \) has the same topological type. In order to prove this assertion, we are allowed to modify the given \( f \) by an inner automorphism of \( \hat{\Gamma}_{g,1} \).

Since \( \gamma_0 \) and \( \gamma_1 \) are non-separating, by the previous part of the proof, we may then assume that \( f(\gamma_0) = \gamma_0 \) and that \( f \) restricts to an automorphism of \( \hat{\Gamma}_{g,0} \). By Lemma 5.4, we also know that \( f(\gamma_1) \) is non-separating.

The s.c.c. \( \gamma_1 \) determines on \( S_{g,1} \setminus \gamma_0 \) a separating s.c.c. bounding a disc with two punctures, one of which is the only puncture on \( S_{g,1} \) and the other one is one of the two determined by the cutting. Then, in virtue of Lemma 5.4 applied to the group \( \text{Aut}^*(\hat{\Gamma}_{g,0}) \), the profinite s.c.c. \( f(\gamma_1) \) bounds also a disc containing two punctures. But now one of these punctures must be the one which corresponds to the only puncture on \( S_{g,1} \), otherwise \( f(\gamma_1) \) would be a separating s.c.c. on \( S_{g,1} \). Therefore, \( \{f(\gamma_0), f(\gamma_1)\} \) is a cut pair of the same topological type of \( \{\gamma_0, \gamma_1\} \).

\[ \square \]

**Corollary 8.5.** For \( 2g-2+n > 0 \) and \( 3g-3+n > 0 \), there is natural faithful representation \( \hat{b}_{g,n} : G_Q \hookrightarrow \text{Out}^*(\hat{\Gamma}_{g,n}) \).

**Proof.** For \( (g, n) = (0, 4), (1, 1) \) the above statement is well known and a direct consequence of Belyi’s Theorem (see [4]). The general case then follows, since, by Theorem 8.3 for \( 3g-3+n > 1 \), there is a natural \( G_Q \)-equivariant homomorphism \( \text{Out}^*(\hat{\Gamma}_{g,n}) \to \text{Out}^*(\hat{\Gamma}_{0,4}) \).

\[ \square \]

As an almost immediate consequence, we get the main result of this section:

**Theorem 8.6.** Let \( C \) be a hyperbolic curve defined over a number field \( k \). Then, the associated outer Galois representation \( \rho_C : G_k \to \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \xi)) \) is faithful.

**Proof.** Let us assume that \( C \) is the curve which appears in the diagram (★) at the beginning of this section.

From Corollary 8.5, it follows that the homomorphism \( \hat{b}_{g,n} \circ \hat{s}_{\xi} : G_k \to \text{Aut}^*(\hat{\Gamma}_{g,n}) \) is injective. This homomorphism can be described as follows. For \( h \in G_k \), the automorphism
homophy, this does not imply, at least not formally, the faithfulness of the representation $G_{\xi}$, for instance, by Proposition 3.2 in [6], for $g \geq 2$, it holds $Z(\hat{G}_{g,n}) = Z(\hat{G}_{g,n}).$ Therefore, it follows that $\ker \rho \leq \ker(\hat{G}_{g,n} \circ s_{\xi}) = \{1\}$.

**Remark 8.7.** For suitable choices of tangential base point $\xi$, for instance, if $C_\xi$ is a nodal curve with an elliptic tail defined over $Q$, it is not difficult to infer directly from Belyi’s Theorem that the associated representation $\hat{G}_{g,n} \circ s_{\xi} : G_Q \to \text{Aut}^*(\hat{G}_{g,n})$ is faithful. However, this does not imply, at least not formally, the faithfulness of the representation $\hat{G}_{g,n} \circ s_{\xi}$, for any other choice of base point $\xi$.

A $Q$-rational point of the moduli stack $\bar{M}_{g,n}$, given for instance by a suitable graph of punctured projective lines, determines a $Q$-rational tangential base point of $M_{g,n}$ and then a $\pi_1(M_{g,n} \times \bar{Q}, \xi)$-conjugacy class of splittings of the short exact sequence:

$$1 \to \pi_1(M_{g,n} \times \bar{Q}, \xi) \to \pi_1(M_{g,n}, \xi) \to G_Q \to 1.$$ 

So, let $\sigma : G_Q \to \pi_1(M_{g,n}, \xi)$ be such a section of $p$. It determines an isomorphism:

$$\pi_1(M_{g,n}, \xi) \cong \hat{G}_{g,n} \rtimes_{\sigma} G_Q,$$

where we have identified $\pi_1(M_{g,n} \times \bar{Q}, \xi)$ with $\hat{G}_{g,n}$ and $G_Q$ acts on it by means of the representation $\tilde{b} := \hat{G}_{g,n} \circ s_{\xi} : G_Q \to \text{Aut}^* (\hat{G}_{g,n}).$

In terms of this isomorphism, the representation $\hat{G}_{g,n} : \pi_1(M_{g,n}, \xi) \to \text{Aut}^* (\hat{G}_{g,n})$ is then described by sending an element $(f, h) \in \hat{G}_{g,n} \rtimes_{\sigma} G_Q$ to the automorphism of $\hat{G}_{g,n}$ given by the product $\text{inn } f \cdot \tilde{b}(h)$.

From Corollary 8.5, it follows that $\tilde{b}$ is faithful and that $\text{Im}(\hat{G}_{g,n}) \cap \text{Im} \tilde{b} = \{1\}$. Moreover, by Proposition 3.2 in [6], for $g \leq 2$, it holds $Z(\hat{G}_{g,n}) = Z(\hat{G}_{g,n}).$ So, we have:

**Proposition 8.8.** i.) For $2g - 2 + n > 0$, the kernel of the natural representation $\hat{G}_{g,n} : \pi_1(M_{g,n}, \xi) \to \text{Aut}^* (\hat{G}_{g,n})$ can be identified with the center of $\hat{G}_{g,n}$.

In particular, for $g \leq 2$, the representation $\hat{G}_{g,n}$ is faithful for $(g, n) \neq (1, 1), (2, 0)$. Otherwise its kernel is spanned by the hyperelliptic involution.

ii.) For $g \geq 3$ and $n \geq 0$, the kernel of the representation $\hat{G}_{g,n} : \pi_1(M_{g,n}, \xi) \to \text{Aut}^* (\hat{G}_{g,n})$ can be identified with the congruence kernel of $\hat{G}_{g,n}$.

The same argument used in the proof of Theorem 8.6 now yields:

**Corollary 8.9.** Let $2g - 2 + n > 0$. The arithmetic universal monodromy representation $\mu_{g,n} : \pi_1(M_{g,n}, \xi) \to \text{Out}(\pi_1(C \times_k \bar{Q}, \xi))$ is faithful for $g \leq 2$. Otherwise its kernel can be identified with the congruence kernel of $\hat{G}_{g,n}$.
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