MORI DREAM SPACES AND GIT

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Dedicated to Bill Fulton

An important advance in algebraic geometry in the last ten years is the theory of variation of geometric invariant theory quotient (VGIT), see [BP], [H], [DH], [T]. Several authors, have observed that VGIT has implications for birational geometry, e.g. it gives natural examples of Mori flips and contractions, [R2], [DH], [T]. In this paper we observe that the connection is quite fundamental – Mori theory is, at an almost tautological level, an instance of VGIT, see (2.14). Here are more details:

Given a projective variety $X$ a natural problem is to understand the collection of all morphisms (with connected fibres) from $X$ to other projective varieties. Ideally one would like to decompose each map into simple steps, and parameterize the possibilities, both for the maps, and for the factorizations of each map. An important insight, principally of Reid and Mori, is that the picture is often simplified if one allows in addition to morphisms, small modifications, i.e. rational maps that are isomorphisms in codimension one. With this extension a natural framework is the category of rational contractions. In many cases there is a nice combinatorial parameterization, given by a decomposition of a convex polyhedral cone, the cone of effective divisors $\overline{NE}^1(X)$, into convex polyhedral chambers, which we call Mori chambers. Instances of this structure have been studied in various circumstances: The existence of such a parameterizing decomposition for Calabi-Yau manifolds was conjectured by Morrison, motivated by ideas in mirror symmetry [M]. The conjecture was proven in dimension three by Kawamata, [Ka]. Oda and Park study the decomposition for toric varieties, motivated by questions in combinatorics, [OP]. Shokurov studies such a decomposition for parameterizing log minimal models, [Sh]. In Geometric Invariant Theory there is a similar combinatorial structure, a decomposition of the $G$-ample cone into GIT chambers parameterizing GIT quotients, [DH]. The main observation of this paper is that whenever a good Mori chamber decomposition exists, it is in a natural way a GIT decomposition.
The main goal of this paper is to study varieties $X$ with a good Mori chamber decomposition (see §1 for the meaning of good). We call such varieties Mori Dream Spaces. There turn out to be many examples, including quasi-smooth projective toric (or more generally, spherical) varieties, many GIT quotients, and log Fano 3-folds. We will show that a Mori dream space is in a natural way a GIT quotients of affine variety by a torus in a manner generalizing Cox’s construction of toric varieties as quotients of affine space, [C]. Via the quotient description, the chamber decomposition of the cone of divisors is naturally identified with the decomposition of the $G$-ample cone from VGIT, See (2.9). In particular every rational contraction of a Mori dream space comes from GIT, and all possible factorizations of a rational contraction (into other contractions) can be read off from the chamber decomposition. See (2.3), (2.9) and (2.11).

Content overview: In §1 we define Mori chambers and Mori dream spaces. The main theorems are proven in §2. In §3 we note connections with a question of Fulton about $\overline{M}_{0,n}$.

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§1 Mori equivalence. Throughout the paper $N^1(X)$ indicates the Neron-Severi group of divisors, with rational coefficients.

We begin with a few definitions:

1.0 Definitions-Lemma. Let $f : X \dashrightarrow Y$ be a rational map between normal projective varieties. Let $(p, q) : W \rightarrow X \times Y$ be a resolution of $f$ with, $W$ projective, and $p$ birational. We say $f$ has connected fibres if $q$ does. If $f$ is birational, we call it a birational contraction if every $p$ exceptional divisor is $q$ exceptional. For a $\mathbb{Q}$-Cartier divisor $D \subset Y$, $f^*(D)$ is defined to be $p_*(q^*(D))$. All of these are independent of the resolution.

Warning: $f^*$ for rational maps is not in general functorial.

It is useful to generalize the notion of rational contraction to the non-birational case. Intuitively this should be a composition of a small modification (see (1.8) below) and a morphism. Our definition is different – we do not want to assume at the outset the existence of small modifications but in the cases we consider it will be equivalent, see (1.11) below.

1.1 Definition. Notation as in (1.0). An effective divisor $E$ on $W$ is called $q$-fixed if no effective Cartier divisor whose support is contained in the support of $E$ is $q$-moving
(see [Ka]), i.e. for every such divisor $D$ the natural map

$$\mathcal{O}_Y \to q_*(\mathcal{O}(D))$$

is an isomorphism. $f$ is called a contraction if every $p$ exceptional divisor is $q$-fixed. An effective divisor $E \subset X$ is called $f$-fixed if any effective divisor of $W$ supported on the union of the strict transform of $E$ with the exceptional divisor of $p$ is $q$-fixed.

One checks easily that for birational maps a divisor is fixed iff it is exceptional.

1.2 Definition. For a line bundle $L$ on a scheme $X$ the section ring is the graded ring

$$R(X, L) := \bigoplus_{n \in \mathbb{N}} H^0(X, L^\otimes n).$$

We will often mix the notation of divisors and line bundles, e.g. writing $H^0(X, D)$ for $H^0(X, \mathcal{O}(D))$ for a divisor $D$. We recall that the moving cone $\text{Mov}(X) \subset \overline{\text{NE}}^1(X)$ is the collection of (numerical classes of) divisors with no stable base components.

If $R(X, D)$ is finitely generated, and $D$ is effective, then there is an induced rational map

$$f_D : X \dasharrow \text{Proj}(R(X, D))$$

which is regular outside the stable base locus of $|nD|$.

1.3 Definition-Lemma: Mori Equivalence. Let $D_1$ and $D_2$ be two $\mathbb{Q}$-Cartier divisors on $X$ with finitely generated section rings. Then we say $D_1$ and $D_2$ are Mori equivalent if the rational maps $f_{D_i}$ have the same Stein factorization i.e. there is an isomorphism between their images which makes the obvious triangular diagram commutative. This occurs iff the rational maps $f_{mD_i}$ are the same for some $m > 0$.

1.4. Definition. Let $X$ be a projective variety such that $R(X, L)$ is finitely generated for all line bundles $L$ and $\text{Pic}(X)_\mathbb{Q} = N^1(X)$. By a Mori chamber of $N^1(X)$ we mean the closure of an equivalence class whose interior is open in $N^1(X)$.

Contractions and finite generation turn out to be closely related.

1.5 Lemma. Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be an $\mathbb{N}$-graded ring, finitely generated as an algebra over $R_0$. Then for some $m > 0$ the natural map

$$\text{sym}_k(R_m) \to R_{km}$$

is surjective for all $k > 0$.

Proof. Let $Y := \text{Proj}(R)$. Then for some $m > 0$, $H^0(Y, \mathcal{O}_Y(km)) = R_{km}$ for all $k > 0$. The result follows. □
1.6 Lemma. If a divisor $D$ has a finitely generated section ring, then, after replacing $D$ by a positive multiple, $f_D$ is a contracting rational map and $D = f_D^*(\mathcal{O}(1)) + E$, for some $f_D$ fixed effective divisor $E$. Conversely, if $f : X \dashrightarrow Y$ is a contracting rational map and $D = f_D^*(A) + E$ for $A$ ample on $Y$, and $E$ fixed by $f$, then $D$ has a finitely generated section ring, and $f = f_{mD}$ for some $m > 0$.

Proof. Suppose $R(X, D)$ is finitely generated. Let $D = M + F$ be the canonical decomposition of $D$ into its moving and fixed components. By (1.5) after replacing $D$ by a multiple

$$\text{sym}_k(H^0(X, M)) \rightarrow H^0(X, kM)$$

is surjective and

$$(1.6.1) \quad H^0(X, kM) \rightarrow H^0(X, kM + rF)$$

is an isomorphism for any $k, r > 0$. By passing to the blowup of the scheme-theoretic base locus of $|M|$ we may assume $f$ is regular, and $M = f^*(A)$ for some ample $A$ on $Y$. Now $F$ is $f$-fixed by (1.6.1).

Now consider the converse, with notation as in the statement. Let $p, q, W$ be a resolution as in (1.0). By negativity of contraction, [Kolláretal,2.19] $p^*f^*(A) = q^*(A) + E'$ where $E'$ is $p$ exceptional and effective. Thus $p^*(D) = q^*(A) + E''$ where $E''$ is effective and $q$-fixed. Thus $R(X, D) = R(W, p^*(D)) = R(Y, A)$ is finitely generated. We can check $f = f_{mD}$ after throwing away the base locus of $D$, where the equality is familiar. □

1.7 Lemma. Let $f : X \dashrightarrow Y$ and $g : X \dashrightarrow Z$ be birational contractions. Suppose $f^*(A) + E = g^*(B) + F$ for $A$ ample, $B$ nef, $E$ $f$-exceptional and $F$ $g$-exceptional. then $f \circ g^{-1} : Z \rightarrow Y$ is regular.

Proof. By negativity of contractions we can pass to a resolution and assume $f$ and $g$ are regular. Now by negativity of contraction $E = F$, and so $f^*(A) = g^*(B)$. The result follows from the rigidity lemma, see e.g. [Ke,1.0]. □

1.8 Definition. By a small $\mathbb{Q}$-factorial modification (SQM) of a projective variety $X$ we mean a contracting birational map $f : X \dashrightarrow X'$ with $X'$ projective and $\mathbb{Q}$-factorial, such that $f$ is an isomorphism in codimension one.

The most important examples of SQMs are flips.
1.9 Definition. Let $q : X \rightarrow Y$ a small birational morphism, and let $D$ be a $\mathbb{Q}$-Cartier divisor such that $-D$ is $q$-ample. By a $D$-flip of $q$ we mean a small birational morphism $q' : X' \rightarrow Y$ such that the strict transform of $D$ on $X'$ is $\mathbb{Q}$-Cartier and $q'$ ample. We say the flip is of relative Picard number one if $q$ and $q'$ are of relative Picard number one.

The $D$-flip if it exists is unique. In the relative Picard number one case it is independent of $D$. See e.g. [KM].

1.10 Definition: Mori Dream Space. We will call a normal projective variety $X$ a Mori Dream Space provided the following hold:

1. $X$ is $\mathbb{Q}$-factorial and $\text{Pic}(X)_\mathbb{Q} = N^1(X)$
2. $\text{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.
3. There is a finite collection of SQMs $f_i : X \rightarrow X_i$ such that each $X_i$ satisfies (2) and $\text{Mov}(X)$ is the union of the $f_i^*(\text{Nef}(X_i))$.

1.11 Proposition. Let $X$ be a Mori dream space. The following hold:

1. Mori’s program can be carried out for any divisor on $X$, i.e. the necessary contractions and flips exist, any sequence terminates, and if at some point the divisor becomes nef then at that point it becomes semi-ample.
2. The $f_i$ of (1.10) are the only SQMs of $X$. $X_i, X_j$ in adjacent chambers are related by a flip. $\overline{\text{NE}}^1(X)$ is the affine hull of finitely many effective divisors. There are finitely many birational contractions $g_i : X \rightarrow Y_i$ with $Y_i$ Mori dream spaces, such that

$$\overline{\text{NE}}^1(X) = \bigcup_i g_i^*(\text{Nef}(Y_i)) \times \text{ex}(g_i)$$

is a decomposition of $\overline{\text{NE}}^1(X)$ into closed convex chambers, with disjoint interiors. The cones $g_i^*(\text{Nef}(Y_i)) \times \text{ex}(g_i)$ are precisely the Mori chambers of $\overline{\text{NE}}^1(X)$. They are in one to one correspondence with birational contractions of $X$ with $\mathbb{Q}$-factorial image.
3. The chambers $f_i^*(\text{Nef}(X_i))$ together with their faces gives a fan, with support $\text{Mov}(X)$. The cones in the fan are in one to one correspondence with contracting rational maps $g : X \rightarrow Y$ with $Y$ normal and projective via

$$[g : X \rightarrow Y] \rightarrow [g^*(\text{Nef}(Y)) \subset \text{Mov}(X)].$$
Let $D$ be an effective divisor on $X$

(4) $R(X, D)$ is finitely generated.

(5) After replacing $D$ by a multiple, the canonical decomposition $D = M + F$ into moving and fixed part has the following properties: There is a Mori chamber containing $D$, so that if $g_i : X \to Y_i$ is the corresponding birational contraction of (2), then $F$ has support the exceptional locus of $g_i$ and $M$ is the pullback of a semi-ample line bundle on $Y_i$.

Proof. These all follow from the definition and purely formal properties of Mori’s program. Here is a sketch of the proof.

Note if $f : X \to Y$ is a small birational morphism, then $f^*(A)$ for $A$ ample is in the interior of $\text{Mov}(X)$. Thus from (1.10.3) all the small contractions of any $X_i$ have a flip, given by another $X_j$. Now let $D$ be a divisor. If it is nef it is semi-ample by assumption, and Mori’s program for $D$ terminates. So we can assume it is not nef. Choose a general ample divisor $A \in \text{Ample}(X)$ and look at the intersection point of the line segment $\overline{AD}$ with the boundary of $\text{Nef}(X)$. This defines a $D$-negative contraction. We can assume (by taking a bigger boundary wall) that it is of relative Picard number one. If it’s small, we can flip it. If it’s not birational the program stops. So we can assume it’s a divisorial contraction of relative Picard number one $f : X \to Y$. Thus $Y$ is $\mathbb{Q}$-factorial. $D - f^*(f_*(D))$ is effective (since $-D$ is $f$-ample), so we can replace $D$ by $f^*(f_*(D))$ and assume $D$ is pulled back. Now we can work in $f^*(\text{Pic}(Y)_\mathbb{Q}) \subset \text{Pic}(X)_\mathbb{Q}$, and induct on the Picard number of $Y$. Eventually we reduce to the case when $Y$ has Picard number one, and $D$ is either the pullback of ample, trivial, or anti-ample. This proves (1).

(4) follows from (1).

Given an effective divisor $D$, running Mori’s program for $D$ yields a birational contraction (indeed a composition of birational morphisms and flips each of relative Picard number one) $g : X \to Y$, with $Y$, $\mathbb{Q}$-factorial, such that $D = g^*(A) + E$ with $A$ semi-ample and $E$ effective with support the full $g$-exceptional locus. Clearly $g^*(A)$ and $E$ are the moving and fixed part of $D$. $g^*(\text{Nef}(Y)) \times \text{ex}(g)$ is a Mori chamber by (1.7). This proves (5).

The contracting morphisms with domain $X_i$ are in one to one correspondence with the faces of $\text{Nef}(X_i)$. Let $g : X \to X'$ be a contracting rational map. Choose $X_i$ so $g^*(A) \subset f'_i(\text{Nef}(X_i))$ It follows that $X_i \to X'$ is regular. This proves (2). (3) can be similarly proved. □
1.12 Remark. (1.11.4) is a natural condition, especially in view of (1.6). Unfortunately by itself it does not imply Mori dream space, or even that nef divisors are semi-ample. For example let \( p : S \to C \) be the projectivization of the non-split extension of \( \mathcal{O}_C \) by itself, for \( C \) an elliptic curve in characteristic zero. Then the cone of effective divisors is two dimensional, with edges \( F \), the fibre of \( p \), and \( C \), the section with trivial normal bundle. Every effective divisor is nef, and the only non-semi-ample effective divisor is (a multiple of) \( C \). \( R(S,C) \) is a polynomial ring. Thus all the section rings are finitely generated. However a natural strengthening of (1.11.4) is indeed an equivalent characterization of a Mori dream space, see (2.9).

\[ \S 2 \] Mori Theory and GIT.

We refer to [DH] for basic notions from VGIT. We recall in particular that two \( G \)-ample line bundles are called GIT-equivalent if they have the same semi-stable locus (and thus in particular give the same GIT quotients). The equivalence classes are always locally polyhedral (and in the cases we consider, will always be polyhedral). We note one difference from the notation of [DH]: Here by a GIT chamber we simply mean a top dimensional GIT equivalence class (in [DH] the term is reserved for equivalence classes for which the stable and semi-stable loci are the same).

2.0 Notation. Let \( V \) be an affine variety over \( k \). Let \( G \) be a reductive group acting on \( V \). Let \( L \) be the trivial line bundle with the trivial induced action (i.e. the action is only on the \( V \) component). For each character \( \chi \in \chi(G) \), let \( U_{\chi} = V^{ss}(L_{\chi}) \), with quotient \( q_{\chi} : U_{\chi} \to U_{\chi}/G := Q_{\chi} \). Let \( C := C^G(V) \cap \ker(f) \) where \( f \) is the forgetful map \( f : C^G(V) \to NS^1(V) \). We denote the complement of the semi-stable locus (i.e. the non-semi-stable locus) by \( V^{nss}(L_{\chi}) \).

2.0.1 Lemma. Notation as above. \( C \) is the affine hull of finitely many characters.

Proof. This is well known. See e.g., [DH, 1.1.5] or [T,2.3]. \( \square \)

2.1 Lemma. Let \( f : U \to Q \) be a geometric quotient by a reductive group \( G \) acting with finite stabilizers. If \( U \) is \( \mathbb{Q} \)-factorial, and for each \( G \)-invariant Cartier divisor \( D \subset U \), \( \mathcal{O}_U(mD) \) has a linearization for some \( m > 0 \), then \( Q \) is \( \mathbb{Q} \)-factorial.

If \( G \) is connected then the converse holds.

Proof. First we consider the forward implication. Let \( D' \subset Q \) be an effective Weil divisor. Replacing \( D' \) by a multiple we may assume the inverse image, \( D \) is Cartier and \( \mathcal{O}_U(D) \) has a linearization. Then \( D \) is the zero locus of a section, \( \sigma \), on which \( G \) acts
by a character, $\chi$. Thus if we adjust the linearization, $\sigma$ is an invariant section. The line bundle and the section descend, by Kempf’s descent lemma, after taking multiples.

For the reverse direction assume $G$ is connected. By [V, Theorem 1], since $Q$ is $\mathbb{Q}$-factorial, the composition

$$
\text{Pic}(Q)_\mathbb{Q} \xrightarrow{f^*} \text{Pic}^G(U)_\mathbb{Q} \to \text{Pic}(U)_\mathbb{Q} \to A^1(U)_\mathbb{Q}
$$

is surjective. The first map is an isomorphism by the descent lemma. The result follows. □

2.2 Lemma. Notation as in (2.0). Let $x$ be a character such that the quotient $Q_x$ is projective. Consider the following conditions

(1) $V^{ss}(L_x) = V^s(L_x)$ and the complement $V^{nss}(L_x) \subset V$ has codimension at least two.

(2) $V$ has torsion class group.

(3) $Q_x$ is $\mathbb{Q}$-factorial.

(4) Both of the maps

$$\chi(G)_\mathbb{Q} \xrightarrow{\chi \cdot L_x|_{U_x}} \text{Pic}^G(U_x)_\mathbb{Q} \xleftarrow{g^*_x} \text{Pic}(Q_x)_\mathbb{Q}$$

are isomorphisms.

(1-2) imply (3-4). If $G$ is connected, then (1), (3) and (4) together imply (2).

Proof. Assume (1-2). The second map in (4) is an isomorphism by Kempf’s descent lemma, and the first map is injective by the codimension condition of (1). As any two linearizations of a $\mathbb{Q}$-line bundle differ by a character, (2) implies the first map is surjective.

Assume (1), (3) and (4) and that $G$ is connected. $V$ and $U_x$ have the same class group by the codimension assumption of (1). $U_x$ is $\mathbb{Q}$-factorial by (2.1), and so has torsion class group by the first map in (4). □

2.3 Theorem. Notation as in (2.0). Let $x$ be a character such that $Q_x$ is projective. If conditions (2.2.1-2.2.4) hold (e.g. by the Lemma, if either (2.1.1-2.1.2) hold, or $G$ is connected and (2.2.1),(2.2.3), and (2.2.4) hold) then $Q_x$ is a Mori dream space. Moreover: The isomorphism $\psi : \chi(G)_\mathbb{Q} \to N^1(Q_x)$ (induced by (2.1.4)) identifies $\overline{NE}^1(Q_x)$ with $C$, and under this identification, Mori chambers are identified with GIT
chambers. Every contraction $f : Q_x \rightarrow Y$ with ($Y$ normal and projective) is induced by GIT, i.e. $Y = Q_y$ for some character $y$, and $f$ is the induced map.

2.3.1 Remark. (2.3) has an obvious analog for quotients of a projective variety where we vary the linearization on powers of a fixed ample divisor. For the proof one passes to the cone over the variety, and applies (2.3). We leave the details to the reader. We expect one could further generalize the proposition, to show that GIT quotients of Mori dream spaces are again Mori dream spaces.

Proof of 2.3. Pic ($Q_x$) is finitely generated by (2.2.4) and thus we have (1.10.1). Every line bundle on $Q_x$ is of form $\psi(L_y)$ and $L_y|_{U_x} = q^*_x(\psi(L_y))$. By descent and the codimension condition we have canonical identifications

\begin{equation}
H^0(V, L_y)^G = H^0(U_x, L_y)^G = H^0(Q_x, \psi(L_y)).
\end{equation}

Thus $\psi$ identifies $C$ with $\overline{\text{NE}}^1(Q_x)$.

By the GIT construction, $L_y|_{U_y} = q^*_y(L_y')$ for an ample line bundle $L_y'$ on $Q_y$, and there are canonical identifications

\begin{equation}
H^0(V, L_y)^G = H^0(U_y, L_y)^G = H^0(Q_y, L_y')
\end{equation}

By the codimension condition we have further identifications

\begin{equation}
H^0(U_y, L_y)^G = H^0(U_y \cap U_x, L_y)^G = H^0(q_x(U_y \cap U_x), \psi(L_y)).
\end{equation}

(Note since $q_x$ is a geometric quotient, $q_x(U_y \cap U_x)$ is open and its inverse image under $q_x$ is $U_x \cap U_y$).

Every section ring on $Q_x$ is finitely generated by Nagata’s theorem. So Mori equivalence is well defined on the cone of divisors. Let

$$f_y : Q_x \rightarrow Q_y$$

be the induced rational map. By (2.3.3) $f_y = f_{\psi(L_y)}$ and in particular by (1.6) a contraction. Further by (1.6)

\begin{equation}
\psi(L_y) = f_y^*(L_y') + E_y
\end{equation}

for some effective $f_y$ exceptional divisor $E_y$. Via $\psi$ we have both Mori and GIT equivalence on $\overline{\text{NE}}^1(Q_x)$. Clearly GIT equivalence is finer: if the semi-stable loci are the
same, the associated contractions of $Q_x$ are the same. By the theory of VGIT, the GIT chambers are finite polyhedral, the affine hulls of finitely many effective divisors. Thus $\overline{NE}^1(Q_x)$ is a union of finitely many Mori chambers, each finite polyhedral.

Now suppose $y$ and $z$ are general members of the same Mori chamber. We will show they are in the same GIT chamber (thus showing GIT and Mori chambers are the same). By assumption $f_z$ and $f_y$ are the same, they are birational as the corresponding divisors are big. By dimension considerations (since the Mori equivalence class is maximal dimensional), $E_y$ and $E_z$ have the same support, the full divisorial exceptional locus of $f_z = f_y$, and the number of components of either is the relative Picard number and $Q_z = Q_y$ is $\mathbb{Q}$-factorial. We argue now that $U_z = U_y$.

Of course it is enough to show $U_z \subseteq U_y$.

Let $z$ be a point of $U_z$. Then by the construction of GIT quotients, there is a section $\sigma \in H^0(V, L_y)^G$ such that $\sigma|_{U_y} = q_y^*(\sigma')$ for a section $\sigma' \in H^0(Q_y, L'_y)$ which does not vanish at $q_z(z) \in Q_z = Q_y$. We claim that

(2.3.7) \quad $L_y|_{U_z} = q_z^*(L'_y)$ and $\sigma|_{U_z} = q_z^*(\sigma')$.

This will of course imply $\sigma(z) \neq 0$ and $z \in U_y$. We can check (2.3.7) after removing any codimension two subset from $U_z$. By (2.3.3) and (2.3.6) $U_x \cap U_y$ and $U_x \cap U_z$ are equal in codimension one: the complement of either is, up to codimension one, the inverse image under $q_x$ of the divisorial exceptional locus of $f_z = f_y$. Thus $U_z$ and $U_y$ are equal in codimension one, and we can check (2.3.7) after restricting to $U_z \cap U_y$ where it obviously holds.

Thus the Mori and GIT chambers have the same interiors, and (up to closure) each chamber is of form $f_z^\ast (\text{Ample}(Q_z)) \times \text{ex}(f_z)$ for linearizations $z$ such that $Q_z$ is $\mathbb{Q}$-factorial. In particular (up to closure) the moving cone will be the union of the (finitely many) chambers with $f_z$ small. To finish the proof we need only show that on these $Q_z$ the nef cones are generated by finitely many semi-ample line bundles. Let $z$ be such a character. Note since $f_z$ is small, $\overline{NE}^1(Q_z)$ and $\overline{NE}^1(Q_x)$ are canonically identified by $f_z^\ast$. Let $C_z \subset C$ be the closure of the GIT chamber of $z$ (which we know is the closure of the ample cone of $Q_z$). Choose $y \in \partial C_z$. By the VGIT theory there is an inclusion $U_z \subset U_y$. It follows that the rational map

$f_{zy} = f_y \circ f_z^{-1} : Q_z \to Q_y$
is regular. By negativity of contraction, since $\psi(L_y)$ (being on the boundary of the ample cone) is nef on $Q_z$ the term $E_y$ in (2.3.6) is empty, and $\phi(L_y) = f_y^*(L'_y)$. Since $L'_y$ is ample, and $f_{zy}$ is regular, $\psi(L_y)$ is semi-ample on $Q_z$. □

**2.4 Corollary.** Let $X$ be a projective geometric GIT quotient for the action of an algebraic torus on an affine variety with torsion class group. If the non-stable locus has codimension at least two, then $X$ is a Mori dream space satisfying the conclusions of (2.3). Moreover GIT quotients from linearizations in the interior of Mori chambers are geometric quotients (i.e. the Mori chambers are chambers in the sense of [DH].)

Suppose furthermore that $V$ is smooth. Then any rational contraction of $X$ with $\mathbb{Q}$-factorial image is a composition of weighted flips, weighted blowdowns, and étale locally trivial (on the image) fibrations of relative Picard number one with fibre a quotient of weighted projective space by a finite abelian group (in particular the image of any such a contraction has cyclic quotient singularities). Indeed the factorization is obtained by the series of (necessarily codimension one) wall crossings connecting a general member of the ample cone of $X$ with a general member of the chamber corresponding to the contraction.

The smooth case of (2.4) is obviously an optimal situation –the contractions are parameterized in a nice combinatorial way, and each contraction is naturally factored into simple parts. We note that in general such a factorization is only possible once one allows small modifications –there will be no such factorization if one restricts themselves to morphisms. e.g. there are birational morphisms $f : X \to Y$ of relative Picard number two between smooth projective toric varieties which do not factor through any morphism $X \to Y'$ of relative Picard number one with $Y'$ $\mathbb{Q}$-factorial.

**Proof of (2.4).** Except for the final claim of the first paragraph, everything is immediate from (2.3) and the theory of VGIT [DH,0.2.5] or [T,5.6]. We follow the notation of the proof of (2.3). Consider a linearization $y$ in the interior of a Mori chamber. It’s enough to show $q_y^*$ is an isomorphism: Then for any character $\nu$, $L_{mv}|U_y$ is pulled back from $Q_y$ (for some $m > 0$), thus the stabilizer of any point of $U_y$ is in the kernel of $mv$ for all $\nu$, and so the stabilizer is finite. We can check $q_y^*$ is an isomorphism after removing codimension two subsets from $Q_y$ and $U_y$. Thus we can restrict to $U_y \cap U_x$ and to the locus where $f_y^{-1}$ is an isomorphism. Here the quotient is geometric, so $q_y^*$ is an isomorphism by Kempf’s descent lemma. □

(2.4) applies to any quasi-smooth projective toric variety $X$ by Cox’s construction which gives an essentially canonical way of writing $X = X(\Delta)$ (for the fan $\Delta$ with
support the lattice \( N = \mathbb{N}^n \) as a GIT quotient of \( \mathbb{A}^r \), \( r = \#(\Delta(1)) \) (where \( \Delta(k) \) is the collection of \( k \)-dimensional cones in the fan). by \( T = \text{Hom}(A_{n-1}, \mathbb{G}_m) \), satisfying (2.1.1-2). See [C].

For a \( \rho \) dimensional torus \( T \) acting on affine space the GIT chambers are particularly simple: An action of \( T \) on \( \mathbb{A}^r \) is given by \( r \) characters \( \chi_i \in \chi(T) \). The \( T \)-ample cone is the affine hull of the characters. The GIT chambers are the affine hulls of all subsets of \( \rho \) independent characters. See e.g., [DH]. Combining this with Cox’s construction and (3.3) gives a simple algorithm for describing the Mori chambers of any quasi-smooth projective toric variety. This description was obtained by Oda and Park, [OP], using Reid’s Toric Mori’s program [R1]. The factorization in (2.4) gives a cheap form of Morelli’s factorization theorem, [Mo], cheap in that even in factoring a birational map between smooth spaces we allow cyclic quotient singularities. The factorization does however have an important advantage over Morelli’s: Morelli factors birational maps, but even to factor a birational morphism he may have to blowup an indeterminate number of times. In particular there could á prior be infinitely many such factorizations. On the other hand all possible factorizations into contractions are encoded in the chamber decomposition of (2.4).

We note that by [BK] quasi-smooth projective spherical varieties give further examples of Mori dream spaces.

Notation: For a collection of \( r \) line bundles \( L_1, \ldots, L_r \) and a vector of integers \( v = (n_1, \ldots, n_r) \in \mathbb{Z}^r \) we use the notion

\[
L^v := L_1^{\otimes n_1} \otimes L_2^{\otimes n_2} \cdots \otimes L_r^{\otimes n_r}.
\]

**2.5 Definition.** For line bundles \( L_1, \ldots, L_r \) on \( X \) let

\[
\text{R}(X, L_1, \ldots, L_r) := \bigoplus_{v \in \mathbb{N}^r} H^0(X, L^v).
\]

**2.6 Definition.** Let \( X \) be a projective variety such that \( \text{Pic}(X)_\mathbb{Q} = N^1(X) \). By a Cox ring for \( X \) we mean the ring

\[
\text{Cox}(X) := \text{R}(X, L_1, \ldots, L_r)
\]

for a choice of line bundles \( L_1, \ldots, L_r \) which are a basis of \( \text{Pic}(X)_\mathbb{Q} \), and whose affine hull contains \( \overline{NE}^1(X) \).
Remark. Rather than have choices as in (2.6) we would prefer to use
\[ \bigoplus_{L \in \text{Pic}(X)} H^0(X, L) \]
but this does not have a well defined ring structure (for an isomorphism class \(L\) the vector space \(H^0(X, L)\) is only determined up to a scalar). Of course \(\text{Cox}(X)\) as we’ve defined it depends on the choice of basis. If we choose two \(\mathbb{Z}\)-basis of the torsion free part of \(\text{Pic}(X)\) then the two rings are isomorphic. If we replace the line bundles by positive powers, the original Cox ring is a finite extension of the new Cox ring. Thus finite generation of the Cox ring, which for our purposes will be the main issue, is independent of choice. For any toric variety \(\text{Cox}(X)\) is a polynomial ring, Cox’s coordinate ring, \([C]\), whence the name.

2.7 Lemma. Let \(\sigma_1, \sigma_2 \in H^0(X, L)\) be two sections of a non-torsion line bundle whose zero divisors have no common component. Then \((\sigma_1, \sigma_2) \subset \text{Cox}(X)\) is a regular sequence.

Proof. Suppose \(a \cdot \sigma_1 = b \cdot \sigma_2\). We can assume \(a\) and \(b\) are homogeneous. Thus \(a\) and \(b\) are sections of the same line bundle \(M\) and \(a \otimes \sigma_1 = b \otimes \sigma_2\). Let \(A, B, D_1, D_2\) be the zero divisors of \(a, b, \sigma_1, \sigma_2\). We have an equality of Weil divisors
\[ A + D_1 = B + D_2 \]
It follows that \(A - D_2 = B - D_1\) is effective and Cartier. Thus \(d = a/\sigma_2 = b/\sigma_1\) is a regular section of \(M \otimes L^*\), and \(a = \sigma_2 \cdot d, b = \sigma_1 \cdot d\). \(\square\)

2.8 Lemma (Zariski). Let \(L_1, \ldots, L_d\) be semi-ample line bundles on a projective variety \(Y\). Then \(\text{R}(Y, L_1, \ldots, L_d)\) is finitely generated, and there exists an integer \(m > 0\), such that for any \(k > 0\), after replacing \(L_i\) by \(L_i^\otimes km\) the canonical map
\[ H^0(Y, L_1)^{\otimes n_1} \otimes \ldots \otimes H^0(Y, L_d)^{\otimes n_d} \to H^0(Y, L^{(n_1, \ldots, n_d)}) \]
is surjective for all \(n_i \geq 0\).

Proof. If \(\mathbb{P} = \mathbb{P}(L_1 \oplus \cdots \oplus L_r)\) then \(\text{R}(Y, L_1, \ldots, L_r) = \text{R}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))\), so we we reduce to a single semi-ample line bundle, where finite generation is a familiar result due to Zariski. For some \(m > 0\),
\[ \text{sym}_k(H^0(\mathbb{P}, \mathcal{O}(m))) \to H^0(\mathbb{P}, \mathcal{O}(km)) \]
is surjective for all \(k\). The second statement follows by considering the appropriate graded piece. \(\square\)
2.9 Proposition. Let $X$ be a $\mathbb{Q}$-factorial projective variety such that $\text{Pic}(X)_{\mathbb{Q}} = N^1(X)$. $X$ is a Mori dream space iff $\text{Cox}(X)$ is finitely generated.

If $X$ is a Mori dream space then $X$ is a GIT quotient of $V = \text{spec}(\text{Cox}(X))$ by the torus $G = \text{Hom}(\mathbb{N}^r, \mathbb{G}_m)$, where $r$ is the Picard number of $X$, satisfying the conditions of (2.3). Moreover we may choose the Cox ring so that $G$ acts freely on the semi-stable loci of any linearization in the interior of a Mori chamber.

Proof. Let $R = \text{Cox}(X) = \bigoplus_{v \in \mathbb{N}^r} R_v$.

Assume $X$ is a Mori dream space. For each (closed) Mori chamber $C \subset \overline{\text{NE}}(X)$ let $R_C = \bigoplus_{v \in C} R_v$. As there are only finitely many chambers, and any homogenous element of $R$ lies in some $R_C$, to show $R$ is finitely generated it is enough to show $R_C$ is finitely generated for each $C$. Choose a chamber $C$ and line bundles $J_1, \ldots, J_d \in C$ which generate $C$ (as a semi-group). Expressing the $J_i$ as tensor products of the $L_j$ induces a surjection

$$R(X, J_1, \ldots, J_d) \rightarrow R_C$$

so we need only show $R(X, J_1, \ldots, J_d)$ is finitely generated. By (1.11.2) there is a contracting rational map $f : X \dashrightarrow Y$ to a projective $\mathbb{Q}$-factorial normal variety $Y$ such that each $J_i = f^*(A_i)(E_i)$ for a $A_i$ semi-ample, and $E_i$ effective and $f$ exceptional. Thus by the projection formula there is a natural identification

$$R(X, J_1, \ldots, J_d) = R(Y, A_1, \ldots, A_d)$$

The latter is finitely generated by (2.8).

Now suppose $R$ is finitely generated. Note $G$ acts naturally on $R$ so that

$$R = \bigoplus_{v \in \chi(T) = \mathbb{N}^r} R_v$$

is the eigenspace decomposition for the action. Thus

$$H^0(V, L_v)^G = R_v$$

and for $v = L \in \text{Pic}(X)$ the ring of invariants is

$$R(V, L_v)^G = R(X, L).$$

Thus $X$ is the GIT quotient for any linearization $v \in \text{Ample}(X) \subset \chi(G)_{\mathbb{Q}}$, and for any linearization $v = L$ the induced rational map $X \dashrightarrow Q_v$ is $f_L$.  

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Let \( h : R \to \mathbb{C} \) be a point of \( V \), and \( v \in \overline{NE}^1(X) \) a linearization. By our description of the invariants \( h \) is \( L_v \) semi-stable iff \( h(R_{nv}) \neq 0 \) for some \( n > 0 \). Suppose for some \( m > 0 \) and \( v_1, \ldots, v_d \) with \( \sum v_i = mv \) that

\[
R_{nv_1} \otimes \cdots \otimes R_{nv_d} \to R_{nmv}
\]

is surjective for all \( n > 0 \). Then

\[
V^{ss}(L_v) = \bigcap_{i=1}^{i=d} V^{ss}(L_{v_i}).
\]

It follows in particular from (2.8) that any ample \( v \) has the same semi-stable locus, say \( U \). Furthermore, \( \lambda \in G_h \) (the stablizer of \( h \)) iff \( \lambda^v = 1 \) for all \( v \) such that \( h(R_v) \neq 0 \). In particular \( \lambda^v \) is torsion if \( h \) is \( L_v \) semi-stable. The ample cone generates \( N^e = \chi(T) \) (as a group) thus any \( h \) semi-stable for an ample \( v \) has finite stabilizer. Thus \( X \) is a geometric quotient of \( V \). Choose two sections \( \sigma_1, \sigma_2 \) of some ample line bundle \( L \), whose zero divisors have no common component. Let \( I \) be the ideal of the non semi-stable locus \( U^c \) (with reduced structure). \( \sigma_1, \sigma_2 \in I \), so by (2.7), \( U^c \) has codimension at least two.

Thus the quotient \( X \) satisfies the conditions of (2.3), so \( X \) is a Mori dream space.

Now choose a Mori chamber \( C \) and generating line bundles \( J_1, \ldots, J_d \), with associated contracting birational map \( f \) as above. After replacing \( X \) by a SQM (which is again a Mori dream space with the same Cox ring) we may assume \( f \) is a morphism. One sees that \( V^{ss}(L_v) \) is constant for \( v \) in the interior of \( C \), and that any point in this open set has finite stabilizers using (2.8) exactly as in the case of \( C = \text{Nef}(X) \) argued above. The same argument shows that after replacing the \( L_i \) by powers, the stabilizers are trivial. \( \square \)

**2.10 Corollary.** Let \( X \) be a smooth projective variety with \( \text{Pic}(X)_\mathbb{Q} = N^1(X) \). \( X \) is a toric variety iff it has a Cox ring which is a polynomial ring.

**Proof.** In the smooth toric case \( \text{Cox}(X) \) is Cox’s homogeneous coordinate ring. By (2.9) if \( \text{Cox}(X) \) is finitely generated, then \( X \) is a geometric GIT quotient of \( \text{spec}(\text{Cox}(X)) \) by a torus, and and the quotient of affine space by a torus is a toric variety. \( \square \)

The next proposition indicates that the birational contractions of a Mori dream space are induced from toric geometry.
2.11 Proposition. Let $X$ be a Mori dream space. Then there is embedding $X \subset W$ into a quasi-smooth projective toric variety such that

1. The restriction $\text{Pic}(W)_\mathbb{Q} \to \text{Pic}(X)_\mathbb{Q}$ is an isomorphism.
2. The isomorphism of (1) induces an isomorphism $\overline{NE}^1(W) \to \overline{NE}^1(X)$.
3. Every Mori chamber of $X$ is a union of finitely many Mori chambers of $W$.
4. For every rational contraction $f : X \to X'$ there is toric rational contraction $\tilde{f} : W \to X'$, regular at the generic point of $X$, such that $f = \tilde{f}|_X$.

Proof. Let $R = \text{Cox}(X) = \bigoplus_{v \in \mathbb{N}} R_v$. By (2.9) $R$ is finitely generated over $R_0 = k$. Choose homogenous generators whose degrees (in the grading) are non-trivial effective divisors. This defines a $k$-algebra surjection $A \to R$ from a polynomial ring $A$, and a compatible action of $T = \text{Hom}(\mathbb{N}^r, \mathbb{G}_m)$ on $A$ such that $A^T = k$. Let $\mathbb{A} = \text{spec}(A)$. We have an equivariant embedding $V = \text{spec}(R) \subset \mathbb{A}$. Let $M_v$ (resp. $L_v$) be twistings by the character $v \in \chi(T)$ of the trivial line bundle on $\mathbb{A}$ (resp. $V$). We follow the notation of (2.0). $\mathbb{A}^{ss}(M_v) \cap V = V^{ss}(L_v)$ for any $v$. Thus GIT equivalence on $V$ is finer than GIT equivalence on $\mathbb{A}$. Choose $v$ a general member of a Mori chamber of $\text{Mov}(X)$. We claim that the quotient $W_v := \mathbb{A}^{ss}(M_v)//T$ satisfies the conditions of (2.3). As remarked in the proof of (2.4) we need only check the codimension condition of (2.2.1). Suppose $\mathbb{A}^{ss}(M_v)$ has a divisorial component. By (2.9) $Q_v$ satisfies the conditions of (2.3), so there is a non-constant function $g \in \mathcal{O}(\mathbb{A})$ on which $T$ acts by some character, $\chi$, whose restriction to $V$ is a unit. But then $L_\chi \in \text{Pic}^G(U_v)$ is trivial. So $\chi$ is trivial. But then $f$ is a non-constant invariant function, a contradiction. Thus (2.3) applies to $Q_v$ and $W_v$. The result follows. □

There is a natural local (in the cone of divisors) generalization of (1.10).

2.12 Definition. Let $C \subset \overline{NE}^1(X)$ be the affine hull of finitely many effective divisors. We say that $C$ is a Mori Dream Region provided the following hold:

1. There exist a finite collection of birational contractions $f_i : X \to Y_i$ such that $C_i := C \cap f_i^*(\text{Nef}(Y_i)) \times \text{ex}(f_i)$ is the affine hull of finitely many effective divisors.
2. $C$ is the union of the $C_i$.
3. Any line bundle in $(f_i)_*(C_i) \cap \text{Nef}(Y_i)$ is semi-ample.

(2.9) has the following analog

2.13 Theorem. Let $X$ be a normal projective variety and let $C \subset N^1(X)$ be a rational polyhedral cone (i.e. the affine hull of the classes of finitely many line bundles). $C \cap$
$\overline{NE}^1(X)$ is a Mori dream region iff there are generators $L_1, \ldots, L_r$ of $C$ such that $R(X, L_1, \ldots, L_r)$ is finitely generated.

Proof. Analogous to that of (2.12). □

It is natural to expect that the region of the cone of divisors studied by Mori theory is itself (at least locally) a Mori dream region. This leads to the following conjecture, which by the ideas of the proof of (2.9) contains all the main conjectures/theorems (e.g. cone and contraction theorems, existence of log flips, and log abundance) of Mori’s program:

2.14 Conjecture. Let $\Delta_1, \ldots, \Delta_r$ be a collection of boundaries such that $K_X + \Delta_i$ is Kawamata log terminal. Choose an integer $m$ so that $L_i = m(K_X + \Delta_i)$ are all Cartier. Then $R(X, L_1, \ldots, L_r)$ is finitely generated.

2.15 Corollary. The conjecture holds in dimension three or less.

Proof. By [Sh,6.20] the intersection of the affine hull of the $L_i$ with $\overline{NE}^1(X)$ is a Mori dream region. □

2.16 Corollary. Let $X$ be a log Fano $n$-fold, with $n \leq 3$. Then $X$ is a Mori dream space.

Proof. Let $K_X + \Delta$ be KLT and anti-ample. Choose a basis $L_1, \ldots, L_r$ of $\text{Pic}(X)$ whose affine hull contains $\overline{NE}^1(X)$. Choose $n > 0$ so that $A_i = L_i - n(K_X + \Delta)$ is ample for all $i$. Let $\Delta_i = 1/nmD_i + \Delta$ for $D_i$ a general member of $|mA_i|$. Note $L_i = n(K_X + \Delta_i)$ (in $\text{Pic}(X)_\mathbb{Q}$), and $\Delta_i$ is KLT for sufficiently large $m$. Now apply (2.13). □

§3 Connections with $\overline{M}_{0,n}$.

The original motivation for this paper was to try to understand the geometric meaning of the cone of effective divisors, in connection with questions of Fulton on $\overline{M}_{0,n}$, the moduli space of stable $n$-pointed rational curves.

3.1 Question (Fulton). Is $\overline{NE}_1(\overline{M}_{0,n})$ (resp $\overline{NE}^1(\overline{M}_{0,n})$) the affine hull of the one dimensional (resp. codimension one) strata?

See [KeM] for definitions and partial results, and an indication of the wide range of contexts in which $\overline{M}_{0,n}$ naturally appears.

The connection with GIT is as follows:

Consider the diagonal action of $G = \text{SL}_2$ on the $n$-fold product $(\mathbb{P}^1)^n$. By the Gelfand-Macpherson correspondence, the VGIT theory for this action is identified with
that of the torus \( T = \mathbb{G}_m^n \) on the Grassmannian \( G(2, m) \), e.g. the \( G \)-ample cones and their chamber decompositions are naturally identified, and the corresponding GIT quotients are the same (in the first case we vary the line bundle, the linearization on each is canonical, in the second the line bundle is fixed and we vary the linearization by characters). (2.4) (see (2.3.1)) now applies. The \( G \)-ample cone and chamber decomposition are easy to describe, see [DH], and one obtains a complete description of the rational contractions on any of the GIT quotients. By [K], \( \overline{M}_{0,n} \) is the inverse limit of all the GIT quotients.

### 3.2 Question. Is \( \overline{M}_{0,n} \) a Mori dream space?

One result of [KeM] is that any extremal ray of \( \overline{NE}_1(\overline{M}_{0,n}) \) which can be contracted by a map of relative Picard number one is generated by a stratum, so long as the exceptional locus of the map has dimension at least two (any stratum can be contracted, and the exceptional locus of the contraction satisfies the dimension condition for any \( n \geq 9 \)). By (1.11) if \( \overline{M}_{0,n} \) is a Mori dream space, then any extremal ray of the Mori cone is contracted by a map of relative Picard number one. Thus a positive answer to (3.2) would nearly answer Fulton’s question for \( \overline{NE}_1(\overline{M}_{0,n}) \).

There is a natural action of the symmetric group \( S_n \) on \( \overline{M}_{0,n} \) and it is natural to consider the \( S_n \) equivariant geometry, or equivalently the geometry of the quotient \( \tilde{M}_{0,n} \). This quotient is itself an important moduli space, e.g. \( \tilde{M}_{0,2g+2} \subset \overline{M}_g \) is the hyperelliptic locus.

Let \( k = \lfloor n/2 \rfloor \). Let \( B_i \subset \overline{M}_{0,n}, k \geq i \geq 2 \), be the union of codimension one strata whose generic point corresponds to a curve with two components, and exactly \( i \) marked points on one of the components. The analog of (3.1) for \( \overline{NE}_1(\tilde{M}_{0,n}) \) is proven in [KeM]. \( \overline{NE}_1(\tilde{M}_{0,n}) \) is in fact simplicial, generated by the (images of the) \( B_i \). Furthermore, every moving divisor is big (thus every rational contraction of \( \tilde{M}_{0,n} \) is birational).

In particular by (2.9), \( \tilde{M}_{0,n} \) is a Mori dream space iff ring

\[
\bigoplus_{(d_2, \ldots, d_k) \in \mathbb{N}^{k-2}} H^0(\overline{M}_{0,n}, \sum d_i B_i)
\]

is finitely generated.

Furthermore, by (1.10), a positive answer to (3.2) would imply the following:

### 3.4 Implication. For each \( k \geq i \geq 2 \) there exists a unique birational contraction \( f_i : \tilde{M}_{0,n} \rightarrow Q_i \), where \( Q_i \) is \( \mathbb{Q} \)-factorial of Picard number one, and the exceptional
divisors of $f_i$ are exactly the $B_j$ with $j \neq i$. The moving cone of $\widetilde{M}_{0,n}$ is simplicial, generated by pullbacks of ample classes from the $Q_j$.

$f_2$ of (3.4) exists, it is the (regular) contraction to the GIT quotient of $\mathrm{SL}_2$ for the action on the $n^{th}$ symmetric product of the standard representation (i.e. the GIT quotient for $n$ unmarked points on $\mathbb{P}^1$).

We finish by giving a result which gives another connection between $\overline{M}_{0,n}$ and GIT. Though rather unrelated to the rest of the paper, we hope the reader will find it of interest.

In [FM] Fulton and MacPherson construct a functorial compactification $X[n]$ of the locus of distinct points in a smooth variety $X$. As we now indicate, $\overline{M}_{0,n}$ occurs as a GIT quotient of $\mathbb{P}^1[n]$ by the natural action of $G = \mathrm{SL}_2$.

There is a proper birational morphism $f : \mathbb{P}^1[n] \to (\mathbb{P}^1)^{\times n}$. Let $E$ be an effective divisor, with support the full exceptional locus of $f$, such that $-E$ is $f$-ample (such an $E$ exists for any proper birational morphism between $\mathbb{Q}$-factorial varieties).

3.5 Theorem. For each linearization $L \in \mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$ such that $((\mathbb{P}^1)^{\times n})^{ss}(L) = ((\mathbb{P}^1)^{\times n})^s(L) \neq \emptyset$, and each sufficiently small $\epsilon > 0$ the line bundle $L' = f^*(L)(-\epsilon E)$ is ample and

$$(\mathbb{P}^1[n])^{ss}(L') = (\mathbb{P}^1[n])^s(L') = f^{-1}((\mathbb{P}^1)^{\times n})^{ss}(L).$$

There is a canonical identification

$$(\mathbb{P}^1[n])^{ss}(L')/G = \overline{M}_{0,n}$$

and a commutative diagram

$$
\begin{array}{ccc}
(\mathbb{P}^1[n])^{ss}(L') & \xrightarrow{f} & ((\mathbb{P}^1)^{\times n})^{ss}(L) \\
q \downarrow & & q \downarrow \\
\overline{M}_{0,n} & \xrightarrow{g} & ((\mathbb{P}^1)^{\times n})^{ss}(L)/G
\end{array}
$$

where $q$ indicates the geometric quotient, and the various $g$ are Kapranov’s blowup expressions for $\overline{M}_{0,n}$, realizing it as the inverse limit of all the GIT quotients of $(\mathbb{P}^1)^{\times n}$.

Proof. We follow the notation of [FM] for divisors on $X[n]$ and [H1] for the chamber decomposition for $\mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$. For a subset $S \subset \{1,2,\ldots,n\}$ let $l_S$ be the linear functional on $\mathrm{Pic}^G((\mathbb{P}^1)^{\times n})$

$$l_S(x_1,\ldots,x_n) = \sum_{i \in S} x_i - \sum_{i \notin S} x_i.$$
For the first statement see [H2]. We let $U$ be the semi-stable locus for a linearization on $(\mathbb{P}^1)^n$ corresponding to a chamber, and $U' = f^{-1}(U)$. Let the corresponding quotients be $Q$ and $Q'$. By [FM, pg. 195, pg. 212], there is a natural $G$-equivariant surjection $\mathbb{P}^1[n] \to \overline{M}_{0,n}$, where $G$ acts trivially on $\overline{M}_{0,n}$. Thus there is an induced proper birational morphism $Q' \to \overline{M}_{0,n}$. To prove this is an isomorphism, both sides being $\mathbb{Q}$-factorial, it is enough to show that both sides have the same Picard number.

$$\rho(Q') = \rho(U') = \rho(U) + e_U = \rho(Q) + e_U$$

where $e_U$ is the number of $f$-exceptional divisors that meet $U'$, or equivalently, the number of diagonals $\Delta_S$ which meet $U$ and have $|S| > 2$. We show first that $\rho(Q')$ is constant (i.e. independent of the chamber). It’s enough to check two chambers sharing the codimension one wall $W_S$. Let the two open sets be $U_1, U_2$ where we assume $U_1$ meets $\Delta_S$ and $|S| \leq |S_c|$. Note the $U_i'$ meet the same divisors $D(T)$ except that $U_1'$ meets $D(S)$ (and not $D(S_c)$) while $U_2'$ meets $D(S_c)$ (and not $D(S)$). If $|S| \geq 2$ then $Q_1 \to Q_2$ is a small modification, so $\rho(Q_1) = \rho(Q_2)$, and $e_{U_1} = e_{U_2}$. Suppose $|S| = 2$. Then $Q_1 \to Q_2$ is a birational contraction with exceptional divisor (the image of) $\Delta_S$. Thus $\rho(Q_1) = \rho(Q_2) + 1$. On the other hand $e_{U_1} = e_{U_2} - 1$, since $D(S)$ is not exceptional (its image is divisorial), while $D(S_c)$ is exceptional.

Now we compute $\rho(Q')$ for the case of the chamber given by inequalities $l_S < 0$ for all $1 \notin S$. In this case $Q = \mathbb{P}^{n-3}$, while the $f$ exceptional divisors that meet $U'$ are precisely the $D(S)$ with $1 \notin S$, and $n - 2 \geq |S| \geq 3$. Thus

$$\rho(Q') = 2^{n-1} - \binom{n}{2} - 1 = \rho(\overline{M}_{0,n}). \quad \square$$

[BK] M. Brion and F. Knop, *Constructions and flips for varieties with group action of small complexity*, J. Math. Sci. Univ. Tokyo 1 (1994), 641–655.

[BP] M. Brion and C. Procesi, *Action d’un tore dans une variété projective*, Progress in Math. 192, 509–539.

[C] D. Cox, *The homogeneous coordinate ring of a toric variety*, Jour. Algebraic Geom. 4, 17–50.

[DN] J. Drezet and M. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semistable sur les courbes algébriques*, Inven. Math. 97 (1989), 53–94.

[DH] I. Dolgachev and Y. Hu, *Variation of geometric invariant theory quotients*, Inst. Hautes Études Sci. Publ. Math. 87 (1998), 5–56.
[FM] W. Fulton and R. MacPherson,, *A compactification of configurations spaces*, Ann. of Math. 139, 183–225.

[H] Y. Hu, *The Geometry and Topology of Quotient Varieties of Torus Actions*, Duke Math. Jour. 68, 151-183.

[H1] Y. Hu, *Moduli of stable polygons and symplectic structures on $\overline{M}_{0,n}$*, Compositio Mathematicae 118, 159-187.

[H2] Y. Hu, *Relative Geometric Invariant Theory and Universal Moduli Spaces*, International Journal of Mathematics 7, 151 – 181.

[K] M. M. Kapranov, *Chow Quotients of Grassmannians. I.*, I.M. Gelfand Seminar, S. Gelfand, S. Gindikin eds., Advances in Soviet Mathematics vol. 16, part 2., A.M.S., 1993, pp. 29-110.

[Ka] Y. Kawamata, *On the cone of divisors of Calabi-Yau fibre spaces*, Internal J. Math. 8 (1997), 665-687.

[Ke] S. Keel, *Basepoint freeness for nef and big linebundles in positive characteristic*, Annals of Math. (1999), 253–286.

[KeM] S. Keel and J. McKernan, *Contractible extremal rays of $\overline{M}_{0,n}$*, preprint alg-geom/9707016.

[Kolláretal] J. Kollár (with 14 coauthors), *Flips and Abundance for Algebraic Threefolds*, Astélique 211 (1992).

[KM] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, vol. 134, Cambridge Univ. Press, 1998.

[Mo] R. Morelli, *The birational geometry of toric varieties*, J. Algebraic Geom. 5 (1996), 751–782.

[M] D. Morrison, *Compactifications of moduli spaces inspired by mirror symmetry*, Asterique 218, 243–271.

[OP] T. Oda and H. Park, *Linear Gale transforms and Gelfand-Kapranov-Zelevinskiy decompositions*, Tohoku Math. Jour. 43, 375–379.

[R1] M. Reid, *Decomposition of toric morphisms*, Progr. Math. 36, 395–418.

[R2] M. Reid, *What is a flip?*, preprint (1992).

[T] M. Thaddeus, *Geometric Invariant Theory and Flips*, Journal of the A. M. S. 9 (1996), 691–723.

[V] A. Vistoli, *Chow groups of quotient varieties*, Jour. of Algebra 107, 410–424.