VOGAN DIAGRAMS OF AFFINE TWISTED LIE SUPeralgebras

Biswajit Ransingh

Department of Mathematics
National Institute of Technology
Rourkela (India)
email- bransingh@gmail.com

Abstract. A Vogan diagram is a Dynkin diagram with a Cartan involution of twisted affine superalgebras based on maximally compact Cartan subalgebras. This article construct the Vogan diagrams of twisted affine superalgebras. This article is a part of completion of classification of vogan diagrams to superalgebras cases.

2010 AMS Subject Classification : 17B05, 17B22, 17B40

1. Introduction

Recent study of denominator identity of Lie superalgebra by Kac and et.al [10] shows a number of application to number theory, Vacuum modules and W algebras. We loud denominator identiy because it has direct linked to real form of Lie superalgebra and the primary ingredient of Vogan diagram is classification of real forms. It follows the study of twisted affine Lie superalgebra for similar application. Hence it is essential to roam inside the depth on Vogan diagram of twisted affine Lie superalgebras.

The real form of Lie superalgebra have a wider application not only in mathematics but also in theoretical physics. Classification of real form is always an important aspect of Lie superalgebras. There are two methods to classify the real form one is Satake or Tits-Satake diagram other one is Vogan diagrams. The former is based on the technique of maximally non compact Cartan subalgebras and later is based on maximally compact Cartan subalgebras. The Vogan diagram first introduced by A W Knapp to classifies the real form of semisimple Lie algebras and it is named after David Vogan. Since then the classification of Vogan diagram by different authors for affine Kac-Moody algebras (untwisted and twised), hyperbolic Kac-Moody algebras, Lie superalgebras and affine untwisted Lie superalgebras already developed. In this article we will developed Vogan diagrams of the rest superalgebras, twisted affine Lie superalgebras.

The classification of symmetric spaces by Satake diagram had been done in [7]. Similar classification of symmetric spaces had achieved with double Vogan diagram by Chuah in [15]. Recently the study of Kac-Moody symmetric spaces obtained by Freyn [6]. We hope an analogus classification of Kac-Moody symmetric superspaces can be obtained by Vogan graph theoretical method. So the exploration of Vogan digram of twisted Lie superalgebra is a preliminary step towards Kac-Moody symmetric superspaces.
2. Generalities

The following preliminary section deals with basic structure of Lie superalgebras which is helpfull for twisted affine extension of Lie superalgebras in the subsequent section.

2.1. The general linear Lie superalgebras. Let $V = V_0 \oplus V_1$ be a vector superspace, so that $\text{End}(V)$ is an associative superalgebra. The $\text{End}(V)$ with the supercommutator forms a Lie superalgebra, called the general linear Lie superalgebra and is denoted by $\mathfrak{gl}(m|n)$, where $V = \mathbb{C}^{m|n}$. With respect to an suitable ordered basis of $\text{End}(V)$, $\mathfrak{gl}(m|n)$ can be realized as $(m+n) \times (m+n)$ complex matrices of the block form.

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
$$

where $a$, $b$, $c$ and $d$ are respectively $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices. The even subalgebra of $\mathfrak{gl}(m|n)$ is $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$, which consists of matrices of the form

$$
\begin{pmatrix}
    a & 0 \\
    0 & d
\end{pmatrix},
$$

While the odd subspace consists of

$$
\begin{pmatrix}
    0 & b \\
    c & 0
\end{pmatrix}.
$$

Definition 2.1. A Lie superalgebra $G$ is an algebra graded over $\mathbb{Z}_2$, i.e., $G$ is a direct sum of vector spaces $G = G_0 \oplus G_1$, and such that the bracket satisfies

1. $[G_i, G_j] \subset G_{i+j(\text{mod } 2)}$,
2. $[x, y] = -(1)^{|x||y|}[y, x]$, (Skew supersymmetry) $\forall$ homogenous $x$ and $y \in G$ (Super Jacobi identity)
3. $[x, [y, z]] = [[x, y], z] + (1)^{|x||y|}[y, [x, z]] \forall x, y, z \in G$

A bilinear form $(.,.) : G \times G \rightarrow \mathbb{C}$ on a Lie superalgebra is called invariant if $(x, [y, z]) = ([x, y], z)$, for all $x, y, z \in G$

The Lie superalgebra $G$ has a root space decomposition with respect to $\mathfrak{h}$

$$
G = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} G_{\alpha}
$$

A root $\alpha$ is even if $G_{\alpha} \subset G_0$ and it is odd if $G_{\alpha} \subset G_1$

A Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of $G$ is defined to be a Cartan subalgebra of the even subalgebra $G_0$. Since every inner automorphism of $G_0$ extends to one of Lie superalgebra $\mathfrak{g}$ and Cartan subalgebras of $G_0$ are conjugate under inner automorphisms. So the Cartan subalgebras of $G$ are conjugate under inner automorphism.

3. Realization of twisted Affine Lie superalgebras

Let $G$ be a basic simple Lie superalgebra with non degenerate invariant bilinear form $(.,.)$ and $\sigma$ an automorphism of finite order $m > 1$. The eigenvalues of $\sigma$ are of the form $e^{\frac{2\pi ik}{m}}$, $k \in \mathbb{Z}_m$ and hence admits the following $\mathbb{Z}_m$ grading:

1. $G = \bigoplus_{k=0}^{m-1} G_k$, $m \geq 2$

such that

2. $[G_i, G_j] \subset G_{i+j}$, $i + j = i + j(\text{mod } m)$
and

\[ G_k = (G_k)_0 \oplus (G_k)_1 \]

(3)

\[ G_k = \{ x \in G | \sigma(x) = e^{2\pi i x/m} \} \]

(4)

The twisted affine Lie superalgebra is defined to be

\[ G^{(m)} = \left( \bigoplus_{k \in \mathbb{Z}_m} \mathbb{C} t^k \otimes G_{k(\text{mod } m)} \right) \oplus \mathbb{C} c \oplus \mathbb{C} d \]

The Lie superalgebra structure on \( G^{(m)} \) is such that \( c \) is the canonical central element and

\[ [x \otimes t^m + \lambda d, y \otimes t^n + \lambda_1 d] = ([x, y] \otimes t^{m+n} + \lambda ny \otimes t^n - \lambda_1 mx \otimes t^m + m\delta_{m,-n}(x, y)c \]

where \( x, y \in G^{(m)} \) and \( \lambda, \lambda_1 \in \mathbb{C} \). The element \( d \) acts diagonally on \( G \) with integer eigenvalues and induces \( \mathbb{Z} \) gradation.

3.1. Cartan Involution. Let \( g \) is a compact Lie algebra if the group \( \text{Int}g \) is compact. An involution \( \theta \) of a real semisimple Lie algebra \( g_0 \) such that symmetric bilinear form

\[ B_\theta(X, Y) = -B(X, \theta Y) \]

is positive definite is called a Cartan involution.

3.1.1. Cartan Involution of Contragradient Lie superalgebras. \( B \) is the supersymmetric nondegenerate invariant bilinear form on \( G \) define

\[ B_\theta(X, Y) = B(X, \theta Y) \]

We say that a real form of \( G \) has Cartan automorphism \( \theta \in \text{aut}_{2,4}(G) \) if \( B \) restricts to the Killing form on \( G_0 \) and \( B_\theta \) is symmetric negative definite on \( G^{(m)} \).

The bilinear form \((,.)\) on \( G \) gives rise to a nondegenerate symmetric invariant form on \( G^{(m)} \) by

\[ B^{(m)}(\mathbb{C}[t, t^{-1}] \otimes G, \mathbb{C} K \oplus \mathbb{C} d) = 0 \]

(8)

\[ \implies B^{(m)}(\bigoplus_{j \in \mathbb{Z}} t^j \otimes G(\sigma)_{j(\text{mod } m)}, \mathbb{C} K \oplus \mathbb{C} d) = 0 \]

(9)

\[ B^{(m)}(t^j \otimes X, t^k \otimes Y) = \lambda \delta_{j+k,0} B(X, Y) \]

(10)

\[ B^{(m)}(t^j \otimes X, K) = B^{(m)}(t^j \otimes X, d) = B^{(m)}(c, c) = B^{(m)}(d, d) = 0 \]

(11)

\[ B^{(m)}(c, d) = 1 \]

(12)

**Proposition 3.1.** Let \( \theta \in \text{aut}_{2,4}(G^{(m)}) \). There exists a real form \( G^{(m)}_R \) such that \( \theta \) restricts to a Cartan automorphism on \( G^{(m)}_R \).
Proof. Since $\theta$ is an $G^{(m)}$ automorphism, it preserves $B$, namely
\[ B^{(m)}(X, Y) = B^{(m)}(\theta X, \theta Y) \]
\[ B^{(m)}_\theta(X, Y) = B^{(m)}_\theta(Y, X), \quad B^{(m)}_\theta(X, \theta X) = 0 \]
\[ B^{(m)}_\theta(X \otimes t^m, Y \otimes t^m) = B^{(m)}_\theta(Y \otimes t^n, X \otimes t^m) = t^{m+n}B(X, Y) \]
for all $X, Y \in G_0$
\[ B^{(m)}(c, X \otimes t^k) = B(d, X \otimes t^k) = B^{(m)}(d, d) = B^{(m)}(c, c) = 0 \]
For $z \in L(t, t^{-1}) \otimes G_0$ and $X, Y \in L(t, t^{-1}) \otimes G_1$
\[ B^{(m)}_\theta(X, [Z, Y]) = B^{(m)}(X, [\theta Z, \theta Y]) = -B^{(m)}_\theta(X, [\theta Z, \theta Y]) \]
\[ B^{(m)}_\theta(X, [Z, Y]) = 0 \]
\[ \forall X \in \mathbb{C}c \text{ or } \mathbb{C}d \]
\[ G^{(m)}_R \simeq G^{(m)}_{\frac{\mathbb{R}}{\mathbb{R}}} \simeq G^{(m)}_{\frac{\mathbb{R}}{\mathbb{S}}}. \]
The above three real forms are isomorphic. So the Cartan decomposition of $G^{(m)}_R$ are isomorphic
to $G^{(m)}_R$.
\[ G^{(m)}_R = \mathfrak{g}_0 \oplus \mathfrak{p}_0 \]
\[ B_\theta(X, [Z, Y]) = \begin{cases} 
-\theta(B([Z, X], Y) & \text{if } Z \in \mathfrak{g}_0 \\
B_\theta([Z, X], Y) & \text{if } Z \in \mathfrak{p}_0
\end{cases} \]

We say that a real form of $G$ has Cartan automorphism $\theta \in \text{aut}_{2,4}(G)$ if $B$ restricts to the Killing form on $G_0$ and $B_\theta$ is symmetric negative definite on $G^{(m)}_{\frac{\mathbb{R}}{\mathbb{R}}}$ and $B_\theta$ is symmetric bilinear form on $G_1 = \{1 \otimes X_1, 1 \otimes X_2, \ldots, c, d\}$. $B_\theta(1 \otimes X_i, 1 \otimes X_j) = \delta_{ij}$. It follows that $B_\theta$ negative definite on $G^{(m)}_{\frac{\mathbb{R}}{\mathbb{R}}}$. So it is concluded that $\theta$ is a Cartan automorphism on $G^{(m)}$.

4. VOGAN DIAGRAM

Let $g_0$ be a real semisimple Lie algebra, Let $g$ be its complexification, let $\theta$ be a Cartan involution, let $g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition $A$ maximally compact $\theta$ stable Cartan subalgebra $h_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $g_0$ with complexification $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and we let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Choose a positive system $\Delta^+$ for $\Delta$ that takes $i\alpha_0$ before $\alpha$. $\theta(\Delta^+) = \Delta^+$
$\theta(h_0) = \mathfrak{k}_0 \oplus (-1)\mathfrak{p}_0$. Therefore $\theta$ permutes the simple roots. It must fix the simple roots that are imaginary and permute in 2-cycles the simple roots that are complex. By the Vogan diagram of the triple $(g_0, h_0, \Delta^+)$, we mean the Dynkin diagram of $\Delta^+$ with the 2 element orbits under $\theta$ so labeled and with the 1-element orbits painted or not, according as the corresponding imaginary simple root is noncompact or compact.

5. TWISTED AFFINE LIE SUPERAFFECTIONS

A Dynkin diagram of $G^{(m)}$ is obtained by adding a lowest weight (root) to the Dynkin diagram of $G$. 

\[ \text{□} \]
5.1. **Root systems.** We have mentioned the lowest root because it has the relation with Kac-Dynkin label. We can get canonical nontrivial Kac-Dynkin labels by lowest root from the fundamental representation.

The root systems of twisted affine Lie superalgebra $OSp(2m|2n)^{(2)}$ is given by

$$\Delta = \left\{ \frac{k}{2} - \delta_1, \delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, \delta_n - e_1, e_1 - e_2, \cdots, e_{m-1} - e_m, e_m \right\}$$

The $G_0$ representation $G_1$ is the fundamental representation of $Osp(2m - 1|2n)$ whose lowest weight is $-\delta_1$. For root systems of twisted affine Lie superalgebra $OSp(2|2n)^{(2)}$, there exist an automorphism $\tau$ such that the invariant subsuperalgebra $G_0$ is $Osp(1|2n)$. The simple root system of $G_0$ is

$$\Delta = \{ \delta_1 - \delta_2, \cdots, \delta_{n-1} - \delta_n, \delta_n \}$$

The lowest weight of the $G_1$ representation of $G_0$ is $\delta_1$. Similarly for twisted affine Lie superalgebra $Sl(1|2n+1)^{(4)}$, we know the invariant subalgebra can be taken to as $O(2n+1)$ and the lowest weight is $-\delta_1$.

### 6. Vogan diagrams of affine Lie superalgebras

Let $c$ the circling of vertices, $d$ diagram involution, $a_s$ numerical labeling and $D$ Dynkin diagram of $G^{(m)}$. $S$ is defined to be the set of $d$ orbit vertices.[4]

**Definition 6.1.** A Vogan diagram $(c, d)$ on $D$ and one of the following holds:

1. $d$ fixes grey vertices
2. $\sum S a_\alpha$ is odd.

The $\gamma$, $\delta$ and $c$ are expressed in terms of the bases given as follows

$$\gamma = \sum_{i=1}^{n} a_i \alpha_i \ , \ \delta = \sum_{i=0}^{n} a_i \alpha_i$$

Fix a set $\pi$ of simple roots of $G$, we take $\hat{\pi} = \{ \alpha_0 = \delta - \gamma \} \cup \pi$ be the simple roots of $G^{(m)} (\gamma$ is the highest weight in $\Delta_1^{(1)} \cup \Delta_1^{(1)}$).

If $\theta$ extend to $aut_{2,4}$ (automorphism of order 2 or 4) then $\theta$ permutes the extreme weight spaces $G^{(m)}$. Since $\theta|_{G_0}$ is represented by $(c,d)$ on $D_0$ (even part (set of even roots) of the Dynkin diagram), it permutes the simple root spaces of $G_0$. Hence $\theta$ permutes the lowest weight spaces of $G^{(m)}$ and $d$ extend to $inv(G^{(m)})$ (where $inv$ is involution on $\cdot$).

**Proposition 6.2.** Let $G_{\mathbb{R}}$ be a real form, with Cartan involution $\theta \in inv(G_{\mathbb{R}})$ and Vogan diagram $(c, d)$ of $G_0$. The following are equivalent

1. $\theta$ extend to $aut_{2,4}(G^{(m)})$.
2. $(G_{\mathbb{R}})$ extend to a real form of $G^{(m)}$.
3. $(c, d)$ extend to a Vogan diagram on $D$

**Proof.**

$$S =$$

\{vertices painted by p\}

\{white and adjacent 2-element d-orbits\}
{grey and non adjacent 2-element d-orbits}

Let $D$ be the Dynkin diagram of $G^{(m)}$ of simple root system $\Phi \cup \phi(\Phi$ simple root system with $\phi$ lowest root) with $D = D_0 + D_1$, where $D_0$ and $D_1$ are respectively the white and grey vertices. The numerical label of the diagram shows $\sum_{\alpha \in D_1} = 2$ has either two grey vertices with label 1 or one grey vertex with label 2.

(i) $D_1 = \{\gamma, \delta\}$ so the labelling of the odd vertices are 1.
(ii) $D_1 = \{\gamma\}$ so labelling is 2 ($a_\alpha = 2$) on odd vertex.

$\theta \in \text{inv}(G_\mathbb{R})$; $\theta$ permutes the weightspaces $L(t, t^{-1}) \otimes \bar{G}_1$ The rest part of proof of the proposition is followed the proof of the proposition 2.2 of [3].

When there is a $\sigma$ stable compact Cartan subalgebra then the Vogan diagrams are the following.

The Vogan diagrams of $\mathfrak{sl}(2m|2n)^{(2)}$ are

![Vogan Diagrams of sl(2m|2n)(2)]

The Vogan diagrams of $\mathfrak{sl}(2m|2n)^{(2)}$ are

![Vogan Diagrams of sl(2m|2n)(2)]
The Vogan diagrams of $\mathfrak{sl}(2m + 1|2n)^2$ are

The Vogan diagrams of $\mathfrak{sl}(2m + 1|2n + 1)^2$ are

The Vogan diagrams of $\mathfrak{sl}(2|2n + 1)^{(2)}$ are
The Vogan diagrams of $\mathfrak{sl}(2|2n)^{(2)}$ are

![Diagram of $\mathfrak{sl}(2|2n)^{(2)}$]

The Vogan diagrams of $\mathfrak{osp}(2m|2n)^{(2)}$ are

![Diagram of $\mathfrak{osp}(2m|2n)^{(2)}$]

The lowest weight representation $G_1$ of $G_0$ is $-\delta_1$ and that makes the following Dynkin diagram for $\mathfrak{osp}(2|2n)^2$. The Vogan diagrams of $\mathfrak{osp}(2|2n)^{(2)}$ are

![Diagram of $\mathfrak{osp}(2|2n)^{(2)}$]

The lowest weight representation $G_1$ of $G_0$ is $-\delta_1$ and that makes the following Dynkin diagram for $\mathfrak{sl}(1|2n + 1)^{(4)}$. The Vogan diagrams of $\mathfrak{sl}(1|2n + 1)^4$ are

![Diagram of $\mathfrak{sl}(1|2n + 1)^4$]
REFERENCES

[1] Batra P, Vogan diagrams of affine Kac-Moody algebras, Journal of Algebra 251, 80-97 (2002).
[2] Batra P, Invariant of Real forms of Affine Kac-Moody Lie algebras, Journal of Algebra 223,208-236 (2000).
[3] Chuah Meng-Kiat, Cartan automorphisms and Vogan superdiagrams, Math.Z. DOI 10.1007/s00209-012-1030-z.
[4] Chuah Meng-Kiat, Finite order automorphism on contragredient Lie superalgebras, Journal of Algebra 351, 138-159 (2012).
[5] Frappat L, Sciarrino A and Sorba, Structure of basic Lie superalgebras and of their affine extensions, commun. Math. Phys. 121, 457-500 (1989).
[6] Welter Freyn, Kac-Moody symmetric spaces and universal twin buildings, thesis (2009).
[7] Helgason Sigurdur, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press (2010).
[8] Kac V.G., Infinite dimensional Lie algebras, third edition 2003.
[9] Kac V.G., Lie superalgebras, Adv. Math. 26, 8 (1977).
[10] Kac V.G., Frajria P.M. and Papi P., Denominator Identities for finite dimensional Lie superalgebras and Howe duality for compact dual pairs, arXiv:1102.3785v1 [math.RT] (2011).
[11] Knapp A.W., Lie groups beyond an Introduction, second edition, vol. 140, Birkhuser, Boston (2002).
[12] Parker M, Classification of real simple Lie superalgebras of classical type, J. Math.Phys. 21(4), April 1980.
[13] Ransingh B and Pati K C, Vogan diagrams of Basic Lie superalgebra, arXiv: 1205.1394v3 (2013).
[14] Ransingh B, Vogan diagrams of affine untwisted Kac-Moody superalgebras, arXiv:
[15] Chuah, M.-K., and J.-S. Huang. Double Vogan Diagrams and Semisimple Symmetric Spaces, Transactions of the American Mathematical Society 362, 04 1721–1750 (2009).