PALEY-WIENER PROPERTIES FOR SPACES OF ENTIRE FUNCTIONS

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Abstract. We deduce Paley-Wiener results in the Bargmann setting. At the same time we deduce characterisations of Pilipović spaces of low orders. In particular we improve the characterisation of the Gröchenig test function space $H_♭ = \mathcal{S}_C$, deduced in [33].

0. Introduction

Paley-Wiener theorems characterize functions and distributions with certain restricted supports in terms of estimates of their Fourier-Laplace transforms. For example, let $f$ be a distribution on $\mathbb{R}^d$ and let $B_{r_0}(0) \subseteq \mathbb{R}^d$ be the ball with center at origin and radius $r_0$. Then $f$ is supported in $B_{r_0}(0)$ if and only if

$$|\hat{f}(\zeta)| \lesssim \langle \zeta \rangle^N e^{r_0 |\text{Im}(\zeta)|}, \quad \zeta \in \mathbb{C}^d,$$

for some $N \geq 0$. Furthermore, $f$ is supported in $B_{r_0}(0)$ and smooth, if and only if

$$|\hat{f}(\zeta)| \lesssim \langle \zeta \rangle^{-N} e^{r_0 |\text{Im}(\zeta)|}, \quad \zeta \in \mathbb{C}^d,$$

for every $N \geq 0$.

A similar approach for ultra-regular functions of Gevrey types and corresponding ultra-distribution spaces can be done. In fact, let $s > 1$, $\mathcal{D}'_s(\mathbb{R}^d)$ be the set of all Gevrey distributions of order $s$ and let $\mathcal{E}_s(\mathbb{R}^d)$ be the set of all smooth functions with Gevrey regularity $s$. Then it can be proved that $f \in \mathcal{D}'_s(\mathbb{R}^d)$ is supported in $B_{r_0}(0)$, if and only if

$$|\hat{f}(\zeta)| \lesssim e^{r_0 |\text{Im}(\zeta)|+r|\zeta|^s}, \quad \zeta \in \mathbb{C}^d,$$

for every $r > 0$. Furthermore $f \in \mathcal{E}_s(\mathbb{R}^d)$ is supported in $B_{r_0}(0)$, if and only if

$$|\hat{f}(\zeta)| \lesssim e^{r_0 |\text{Im}(\zeta)|-r|\zeta|^s}, \quad \zeta \in \mathbb{C}^d,$$

for some $r > 0$.

We observe that $s$ in the latter result can not be pushed to be smaller, because if $s \leq 1$, it does not make any sense to discuss compact support properties of $\mathcal{D}'_s(\mathbb{R}^d)$ and $\mathcal{E}_s(\mathbb{R}^d)$.

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In the paper we replace the Fourier-Laplace transform above with an analogous transform, designed by the reproducing kernel $\Pi_A$ attached to the Bargmann transform. In contrast to earlier contributions, our results concern situations involving Gevrey regularity of orders less than one.

We recall that if $(z, w)$ is the scalar product of $z, w \in \mathbb{C}^d$, then
\[
(\Pi_A F)(z) = \pi^{-d} \langle F, e^{(z, \cdot) - |\cdot|^2} \rangle
\]
when $F \in \mathcal{S}'(\mathbb{C}^d)$. We notice that this still makes sense when $F$ belongs to the Fourier-invariant Gelfand-Shilov distribution space $\mathcal{S}_b^r(\mathbb{C}^d)$, for any $s > 0$, or if
\[
z \mapsto F(z) e^{R|z|^2} \in L^1(\mathbb{C}^d), \quad R > 0.
\]
If (0.1) holds, then
\[
(\Pi_A F)(z) = \pi^{-d} \int_{\mathbb{C}^d} F(w) e^{(z,w) - |w|^2} d\lambda(w).
\]
Here $d\lambda(w)$ is the Lebesgue measure on $\mathbb{C}^d$.

In [33] it is proved that for the sets
\[
\mathcal{A}_\sigma(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|^2} \} \quad \text{for some } R > 0, \quad \sigma > 0,
\]
the mappings
\[
\Pi_A : \mathcal{S}'(\mathbb{C}^d) \to \mathcal{A}_{b_1}(\mathbb{C}^d),
\]
\[
\Pi_A : L^\infty(\mathbb{C}^d) \to \mathcal{A}_{b_1}(\mathbb{C}^d)
\]
are surjective. Here $L^\infty(\mathbb{C}^d) = \mathcal{S}'(\mathbb{C}^d) \cap L^\infty(\mathbb{C}^d)$.

In Section 2 we improve these mapping properties as well as deduce various kind of related mapping results. Firstly, let $A(K)$ be the set of all functions which are analytic in a neighbourhood of $K \subseteq \mathbb{C}^d$, and let $L_{\chi_K}^\infty(\mathbb{C}^d)$ be the set of functions of the form $F \cdot e^{(1/2) e^{s|\cdot|^2}}$, where $K \subseteq \mathbb{C}^d$ is compact, $\chi_K$ is the characteristic function of $K$ and $F_0 \in A(K)$. Then it is clear that $L_{\chi_K}^\infty(\mathbb{C}^d) \subseteq L^\infty(\mathbb{C}^d)$, and in Section 2 we improve the surjectivity of the mappings in (0.2), and prove that indeed
\[
\Pi_A : L_{\chi_K}^\infty(\mathbb{C}^d) \to \mathcal{A}_{b_1}(\mathbb{C}^d)
\]
is surjective (cf. Theorems 2.7, 2.9 and 2.10).

In Section 2 we also consider the slightly smaller subset
\[
\mathcal{A}_{b_0}^0(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d) ; |F(z)| \lesssim e^{R|z|^2} \} \quad \text{for every } R > 0, \quad \sigma > 0
\]
of $\mathcal{A}_{b_0}(\mathbb{C}^d)$, and deduce that $\Pi_A$ maps suitable subsets of $L_{\chi_K}^\infty(\mathbb{C}^d)$ surjectively on $\mathcal{A}_{b_0}^0(\mathbb{C}^d)$ when $\sigma \leq 1$ (cf. Theorems 2.7 and 2.11).

Finally, let $s \in [0, \frac{1}{2})$, let $\mathcal{A}_s(\mathbb{C}^d) (\mathcal{A}_s^0(\mathbb{C}^d))$ be the set of all $F \in A(\mathbb{C}^d)$ such that $|F(z)| \lesssim e^{R|\log(z)|^{1/2}}$ for some $R > 0$ (for every $R > 0$), and let $\chi_r$ be the characteristic function of the polydisc with
radii \( r \in \mathbb{R}^d_+ \) and center at origin. Then we prove in Section 2 that for any \( r \in \mathbb{R}^d_+ \), the map

\[
F \mapsto \Pi_A(F \cdot e^{\frac{|.|^2}{2}} \chi_r)
\]

is surjective from \( A_s(\mathbb{C}^d) \) to \( A_s(\mathbb{C}^d) \), and from \( A_s^0(\mathbb{C}^d) \) to \( A_s^0(\mathbb{C}^d) \) (cf. Theorem 2.14).

1. Preliminaries

In this section we recall some basic facts. We start by discussing Pilipović spaces and some of their properties. Then we recall some facts on modulation spaces. Finally we recall the Bargmann transform and some of its mapping properties, and introduce suitable classes of entire functions on \( \mathbb{C}^d \).

1.1. The Pilipović spaces. Next we make a review of Pilipović spaces. These spaces can be defined in terms of Hermite series expansions. We recall that the Hermite function of order \( \alpha \in \mathbb{N}^d \) is defined by

\[
h_{\alpha}(x) = \pi^{-\frac{d}{4}}((-1)^{|\alpha|}\alpha!\alpha! - \frac{1}{2} \sum_{\alpha \in \mathbb{N}^d} \left| \frac{\alpha!}{\alpha!} \right|^2 (\partial^\alpha e^{-|x|^2}).
\]

It follows that

\[
h_{\alpha}(x) = ((2\pi)^{\frac{d}{2}}\alpha!)^{-1} e^{-\frac{|x|^2}{2}} p_{\alpha}(x),
\]

for some polynomial \( p_{\alpha} \) on \( \mathbb{R}^d \), which is called the Hermite polynomial of order \( \alpha \). The Hermite functions are eigenfunctions to the Fourier transform, and to the Harmonic oscillator \( H_d \equiv |x|^2 - \Delta \) which acts on functions and (ultra-)distributions defined on \( \mathbb{R}^d \). More precisely, we have

\[
H_d h_{\alpha} = (2|\alpha| + d) h_{\alpha}, \quad H_d \equiv |x|^2 - \Delta.
\]

It is well-known that the set of Hermite functions is a basis for \( \mathcal{S}(\mathbb{R}^d) \) and an orthonormal basis for \( L^2(\mathbb{R}^d) \) (cf. [27]). In particular, if \( f \in L^2(\mathbb{R}^d) \), then

\[
\|f\|_{L^2(\mathbb{R}^d)}^2 = \sum_{\alpha \in \mathbb{N}^d} |c_h(f, \alpha)|^2,
\]

where

\[
f(x) = \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha) h_{\alpha}, \quad (1.1)
\]

is the Hermite series expansion of \( f \), and

\[
c_h(f, \alpha) = (f, h_{\alpha})_{L^2(\mathbb{R}^d)} \quad (1.2)
\]

is the Hermite coefficient of \( f \) of order \( \alpha \in \mathbb{R}^d \).

In order to define the full scale of Pilipović spaces, their order \( s \) should belong to the extended set

\[
\mathbb{R}_s = \mathbb{R}_+ \bigcup \{ b_\sigma ; \sigma \in \mathbb{R}_+ \},
\]

where
of \( \mathbb{R}_+ \), with extended inequality relations as
\[
s_1 < b_\sigma < s_2 \quad \text{and} \quad b_{\sigma_1} < b_{\sigma_2}
\]
when \( s_1 < \frac{1}{2} \leq s_2 \) and \( \sigma_1 < \sigma_2 \). (Cf. [33].)

For such \( s \) we set
\[
\vartheta_{r,s}(\alpha) \equiv \begin{cases}
eq e^{-\left(\frac{1}{r_1} \alpha_1^2 + \cdots + \frac{1}{r_d} \alpha_d^2\right)}, & s \in \mathbb{R}_+ \setminus \{\frac{1}{2}\}, \\
\alpha^a \frac{a!}{\alpha}, & s = b_\sigma, \\
\alpha^a, & s = \frac{1}{2}, \quad \alpha \in \mathbb{N}^d
\end{cases}
\]

and
\[
\vartheta'_{r,s}(\alpha) \equiv \begin{cases}
eq e^{\left(\frac{1}{r_1} \alpha_1^2 + \cdots + \frac{1}{r_d} \alpha_d^2\right)}, & s \in \mathbb{R}_+ \setminus \{\frac{1}{2}\}, \\
\alpha^a \frac{a!}{\alpha}, & s = b_\sigma, \\
\alpha^a, & s = \frac{1}{2}, \quad \alpha \in \mathbb{N}^d.
\end{cases}
\]

**Definition 1.1.** Let \( s \in \mathbb{R}_\delta = \mathbb{R}_\delta \cup \{0\} \), and let \( \vartheta_{r,s} \) and \( \vartheta'_{r,s} \) be as in (1.3) and (1.4).

1. \( H_0(\mathbb{R}^d) \) consists of all Hermite polynomials, and \( H_0^0(\mathbb{R}^d) \) consists of all formal Hermite series expansions in (1.1);
2. if \( s \in \mathbb{R}_\delta \), then \( H_s(\mathbb{R}^d) \) (\( H_0^0(\mathbb{R}^d) \)) consists of all \( f \in L^2(\mathbb{R}^d) \) such that
\[
|c_h(f, h_\alpha)| \lesssim \vartheta_{r,s}(\alpha)
\]
holds true for some \( r \in \mathbb{R}_+^d \) (for every \( r \in \mathbb{R}_+^d \));
3. if \( s \in \mathbb{R}_\delta \), then \( H_s'(\mathbb{R}^d) \) (\( (H_0^0)'(\mathbb{R}^d) \)) consists of all formal Hermite series expansions in (1.1) such that
\[
|c_h(f, h_\alpha)| \lesssim \vartheta'_{r,s}(\alpha)
\]
holds true for every \( r \in \mathbb{R}_+^d \) (for some \( r \in \mathbb{R}_+^d \)).

The spaces \( H_s(\mathbb{R}^d) \) and \( H_0^0(\mathbb{R}^d) \) are called **Pilipović spaces of Roumieu respectively Beurling types** of order \( s \), and \( H_s'(\mathbb{R}^d) \) and \( (H_0^0)'(\mathbb{R}^d) \) are called **Pilipović distribution spaces of Roumieu respectively Beurling types** of order \( s \).

**Remark 1.2.** Let \( \mathcal{S}_s(\mathbb{R}^d) \) and \( \Sigma_s(\mathbb{R}^d) \) be the Fourier invariant Gelfand-Shilov spaces of order \( s \in \mathbb{R}_+ \) and of Roumieu and Beurling types respectively (see [33] for notations). Then it is proved in [25, 26] that
\[
H_s^0(\mathbb{R}^d) = \Sigma_s(\mathbb{R}^d) \neq \{0\}, \quad s > \frac{1}{2},
\]
\[
H_s^0(\mathbb{R}^d) \neq \Sigma_s(\mathbb{R}^d) = \{0\}, \quad s \leq \frac{1}{2},
\]
\[
H_s(\mathbb{R}^d) = \mathcal{S}_s(\mathbb{R}^d) \neq \{0\}, \quad s \geq \frac{1}{2}.
\]
and

\[ \mathcal{H}_s(\mathbb{R}^d) \neq \mathcal{S}_s(\mathbb{R}^d) = \{0\}, \quad s < \frac{1}{2}. \]

In Proposition 1.3 below we give further characterisations of Pilipović spaces.

Next we recall the topologies for Pilipović spaces. Let \( s, r > 0 \), and let \( \|f\|_{H_{s,r}} \) and \( \|f\|_{H'_{s,r}} \) be given by

\[ \|f\|_{H_{s,r}} \equiv \sup_{\alpha \in \mathbb{N}^d} |c_h(f, \alpha)\vartheta_{r,s}(\alpha)|, \quad s \in \mathbb{R}, \tag{1.5} \]

and

\[ \|f\|_{H'_{s,r}} \equiv \sup_{\alpha \in \mathbb{N}^d} |c_h(f, \alpha)\vartheta_{r,s}(\alpha)|, \quad s \in \mathbb{R}, \tag{1.6} \]

when \( f \) is a formal expansion in (1.1). Then \( \mathcal{H}_{s,r}(\mathbb{R}^d) \) consists of all expansions (1.1) such that \( \|f\|_{H_{s,r}} \) is finite, and \( \mathcal{H}'_{s,r}(\mathbb{R}^d) \) consists of all expansions (1.1) such that \( \|f\|_{H'_{s,r}} \) is finite. It follows that both \( \mathcal{H}_{s,r}(\mathbb{R}^d) \) and \( \mathcal{H}'_{s,r}(\mathbb{R}^d) \) are Banach spaces under the norms \( f \mapsto \|f\|_{H_{s,r}} \) and \( f \mapsto \|f\|_{H'_{s,r}} \), respectively.

We let the topologies of \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{H}_s^0(\mathbb{R}^d) \) be the inductive respectively projective limit topology of \( \mathcal{H}_{s,r}(\mathbb{R}^d) \) with respect to \( r > 0 \). In the same way, the topologies of \( \mathcal{H}'_s(\mathbb{R}^d) \) and \( \mathcal{H}_s^0(\mathbb{R}^d) \) are the projective respectively inductive limit topology of \( \mathcal{H}'_{s,r}(\mathbb{R}^d) \) with respect to \( r > 0 \). It follows that all the spaces in Definition 1.1 are complete, and that \( \mathcal{H}_s^0(\mathbb{R}^d) \) and \( \mathcal{H}'_s(\mathbb{R}^d) \) are Fréchet space with semi-norms \( f \mapsto \|f\|_{H_{s,r}} \) and \( f \mapsto \|f\|_{H'_{s,r}} \), respectively.

The following characterisations for Pilipović spaces can be found in [33]. The proof is therefore omitted.

**Proposition 1.3.** Let \( s \in \mathbb{R}_+ \cup \{0\} \) and let \( f \in \mathcal{H}_s^0(\mathbb{R}^d) \). Then \( f \in \mathcal{H}_s^0(\mathbb{R}^d) \) (\( f \in \mathcal{H}_s(\mathbb{R}^d) \)), if and only if \( f \in C^\infty(\mathbb{R}^d) \) and satisfies \( |H^d_\alpha f(x)| \lesssim h^\alpha N!2^s \) for every \( h > 0 \) (for some \( h > 0 \)).

From now on we let

\[ \phi(x) = \pi^{-\frac{d}{4}} e^{-\frac{|x|^2}{2}}. \tag{1.7} \]

### 1.2. Spaces of entire functions and the Bargmann transform.

If \( \Omega \subseteq \mathbb{C}^d \) is open, the \( A(\Omega) \) is the set of all analytic functions in \( \Omega \). If instead \( \Omega \subseteq \mathbb{C}^d \) is closed, then \( A(\Omega) \) is the set of all functions which are analytic in an open neighbourhood of \( \Omega \).

We shall now consider the Bargmann transform which is defined by the formula

\[ (\mathfrak{M}_d f)(z) = \pi^{-\frac{d}{4}} \int_{\mathbb{R}^d} \exp \left( -\frac{1}{2} \langle z, z \rangle + |y|^2 + 2^{\frac{1}{2}} \langle z, y \rangle \right) f(y) dy, \]
when \( f \in L^2(\mathbb{R}^d) \) (cf. [1]). We note that if \( f \in L^2(\mathbb{R}^d) \), then the Bargmann transform \( \mathfrak{B}_d f \) of \( f \) is the entire function on \( \mathbb{C}^d \), given by

\[
(\mathfrak{B}_d f)(z) = \int_{\mathbb{R}^d} \mathfrak{A}_d(z, y) f(y) \, dy,
\]

or

\[
(\mathfrak{B}_d f)(z) = \langle f, \mathfrak{A}_d(z, \cdot) \rangle,
\]

where the Bargmann kernel \( \mathfrak{A}_d \) is given by

\[
\mathfrak{A}_d(z, y) = \pi^{-\frac{d}{4}} \exp \left( -\frac{1}{2} \langle z, z \rangle + 2 \frac{1}{2} \langle z, y \rangle \right).
\]

Here

\[
\langle z, w \rangle = \sum_{j=1}^{d} z_j w_j, \quad \text{when } z = (z_1, \ldots, z_d) \in \mathbb{C}^d
\]

and \( w = (w_1, \ldots, w_d) \in \mathbb{C}^d \), and otherwise \( \langle \cdot, \cdot \rangle \) denotes the duality between test function spaces and their corresponding duals. We note that the right-hand side in (1.8) makes sense when \( f \in S'_{2}^{1}(\mathbb{R}^d) \) and defines an element in \( A(\mathbb{C}^d) \), since \( y \mapsto \mathfrak{A}_d(z, y) \) can be interpreted as an element in \( S_{2}^{1}(\mathbb{R}^d) \) with values in \( A(\mathbb{C}^d) \). Here and in what follows, \( A(\Omega) \) denotes the set of analytic functions on the open set \( \Omega \subseteq \mathbb{C}^d \).

It was proved in [1] that \( f \mapsto \mathfrak{B}_d f \) is a bijective and isometric map from \( L^2(\mathbb{R}^d) \) to the Hilbert space \( A^2(\mathbb{C}^d) \equiv B^2(\mathbb{C}^d) \cap A(\mathbb{C}^d) \), where \( B^2(\mathbb{C}^d) \) consists of all measurable functions \( F \) on \( \mathbb{C}^d \) such that

\[
\| F \|_{B^2} \equiv \left( \int_{\mathbb{C}^d} |F(z)|^2 \, d\mu(z) \right)^{\frac{1}{2}} < \infty.
\]

Here \( d\mu(z) = \pi^{-d} e^{-|z|^2} \, d\lambda(z) \), where \( d\lambda(z) \) is the Lebesgue measure on \( \mathbb{C}^d \). We recall that \( A^2(\mathbb{C}^d) \) and \( B^2(\mathbb{C}^d) \) are Hilbert spaces, where the scalar product are given by

\[
(F, G)_{B^2} \equiv \int_{\mathbb{C}^d} F(z)\overline{G(z)} \, d\mu(z), \quad F, G \in B^2(\mathbb{C}^d).
\]

If \( F, G \in A^2(\mathbb{C}^d) \), then we set \( \| F \|_{A^2} = \| F \|_{B^2} \) and \( (F, G)_{A^2} = (F, G)_{B^2} \).

Furthermore, Bargmann proved that there is a convenient reproducing formula on \( A^2(\mathbb{C}^d) \). More precisely, let

\[
(\Pi F)(z) \equiv \int_{\mathbb{C}^d} F(w) e^{i\langle z, w \rangle} \, d\mu(w),
\]

(1.11)
when $z \mapsto F(z)e^{R|z|^2}$ belongs to $L^1(\mathbb{C}^d)$ for every $R \geq 0$. Here

$$(z, w) = \sum_{j=1}^{d} z_j \overline{w_j}, \quad \text{when} \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^d$$

and $w = (w_1, \ldots, w_d) \in \mathbb{C}^d$,
is the scalar product of $z \in \mathbb{C}^d$ and $w \in \mathbb{C}^d$. Then it is proved in [12] that $\Pi_A F = F$ when $F \in A^2(\mathbb{C}^d)$.

In [1] it is also proved that $V_d h_\alpha = e_\alpha$, where $e_\alpha(z) \equiv \frac{z^\alpha}{\sqrt{\alpha!}}$, $z \in \mathbb{C}^d$. (1.12)

In particular, the Bargmann transform maps the orthonormal basis \{h_\alpha\}_{\alpha \in \mathbb{N}^d} in $L^2(\mathbb{R}^d)$ bijectively into the orthonormal basis \{e_\alpha\}_{\alpha \in \mathbb{N}^d} of monomials in $A^2(\mathbb{C}^d)$. Hence, there is a natural way to identify formal Hermite series expansion by formal power series expansions

$$F(z) = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha) e_\alpha(z), \quad (1.13)$$

by letting the series (1.1) be mapped into

$$\sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha) e_\alpha(z). \quad (1.14)$$

It follows that if $f, g \in L^2(\mathbb{R}^d)$ and $F, G \in A^2(\mathbb{C}^d)$, then

$$(f, g)_{L^2(\mathbb{R}^d)} = \sum_{\alpha \in \mathbb{N}^d} c_h(f, \alpha) \overline{c_h(g, \alpha)}, \quad (1.15)$$

$$(F, G)_{A^2(\mathbb{C}^d)} = \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha) \overline{c(G, \alpha)}. \quad (1.15)$$

Here and in what follows, $(\cdot, \cdot)_{L^2(\mathbb{R}^d)}$ and $(\cdot, \cdot)_{A^2(\mathbb{C}^d)}$ denote the scalar products in $L^2(\mathbb{R}^d)$ and $A^2(\mathbb{C}^d)$, respectively. Furthermore,

$$c_h(f, \alpha) = c(F, \alpha) \quad \text{when} \quad F = \mathfrak{U}_d f, \quad G = \mathfrak{U}_d g. \quad (1.16)$$

We now recall the following spaces of power series expansions

**Definition 1.4.** Let $s \in \mathbb{R}_0 = \mathbb{R}_0 \cup \{0\}$, and let $\vartheta_{r,s}$ and $\vartheta'_{r,s}$ be as in (1.3) and (1.4).

1. $A_0(\mathbb{C}^d)$ consists of all analytic polynomials on $\mathbb{C}^d$, and $A'_0(\mathbb{C}^d)$ consists of all formal power series expansions on $\mathbb{C}^d$ in (1.13);
2. if $s \in \mathbb{R}_0$, then $A_s(\mathbb{C}^d)$ ($A'_0(\mathbb{C}^d)$) consists of all $F \in L^2(\mathbb{C}^d)$ such that

$$|c(F, h_\alpha)| \lesssim \vartheta_{r,s}(\alpha)$$

holds true for some $r > 0$ (for every $r > 0$);
(3) if \( s \in \mathbb{R}_\flat \), then \( \mathcal{A}'_s(\mathbb{C}^d) \) ((\( \mathcal{A}'_0(\mathbb{C}^d) \)) consists of all formal power series expansions in (1.13) such that

\[
|c(F, h_\alpha)| \lesssim \psi'_{r,s}(\alpha)
\]

holds true for every \( r > 0 \) (for some \( r > 0 \)).

Let \( f \in \mathcal{H}'_0(\mathbb{R}^d) \) with formal Hermite series expansion (1.1). Then the Bargmann transform \( \mathcal{U}_d f \) of \( f \) is defined to be the formal power series expansion (1.14). It follows that \( \mathcal{U}_d \) agrees with the earlier definition when acting on \( L^2(\mathbb{R}^d) \), that \( \mathcal{U}_d \) is linear and bijective from \( \mathcal{H}'_0(\mathbb{R}^d) \) to \( \mathcal{A}'_0(\mathbb{C}^d) \), and restricts to bijections from the spaces

\[
\mathcal{H}'_s(\mathbb{R}^d), \quad \mathcal{H}'_s(\mathbb{R}^d), \quad \mathcal{H}'_s(\mathbb{R}^d) \quad \text{and} \quad (\mathcal{H}'_s)^0(\mathbb{R}^d) \quad (1.17)
\]

to

\[
\mathcal{A}'_s(\mathbb{C}^d), \quad \mathcal{A}'_s(\mathbb{C}^d), \quad \mathcal{A}'_s(\mathbb{C}^d) \quad \text{and} \quad (\mathcal{A}'_s)^0(\mathbb{C}^d) \quad (1.18)
\]

respectively, when \( s \in \mathbb{R}_\flat \). We also let the topologies of the spaces in (1.18) be inherited from the spaces in (1.17).

If \( s \in \mathbb{R}_\flat \), \( f \in \mathcal{H}_s(\mathbb{R}^d) \), \( g \in \mathcal{H}_s(\mathbb{R}^d) \), \( F \in \mathcal{A}_s(\mathbb{C}^d) \) and \( G \in \mathcal{A}_s(\mathbb{C}^d) \), then \( (f, g)_{L^2(\mathbb{R}^d)} \) and \( (F, G)_{\mathcal{A}^2(\mathbb{C}^d)} \) are defined by the formula (1.15). It follows that (1.16) holds for such choices of \( f \) and \( g \). Furthermore, the duals of \( \mathcal{H}_s(\mathbb{R}^d) \) and \( \mathcal{A}_s(\mathbb{C}^d) \) can be identified with \( \mathcal{H}'_s(\mathbb{R}^d) \) and \( \mathcal{A}'_s(\mathbb{C}^d) \), respectively, through the forms in (1.15). The same holds true with

\[
\mathcal{H}'_s, \quad (\mathcal{H}'_s)^0, \quad \mathcal{A}'_s, \quad \text{and} \quad (\mathcal{A}'_s)^0
\]

in place of

\[
\mathcal{H}_s, \quad \mathcal{H}'_s, \quad \mathcal{A}_s, \quad \text{and} \quad \mathcal{A}'_s
\]

respectively, at each occurrence.
Proposition 1.5. Let \( r \) spaces of analytic functions, we let when

\[
M_{1,r,s}(z) = \begin{cases} 
    r_1 (\log(z_1))^\frac{1}{r_1} + \cdots + r_d (\log(z_d))^\frac{1}{r_d}, & s < \frac{1}{2}, \\
    r_1 |z_1|^\frac{2}{r_1} + \cdots + r_d |z_d|^\frac{2}{r_d}, & s = b_\sigma, \sigma > 0, \\
    \frac{|z|^2}{2} - (r_1 |z_1|^\frac{1}{r_1} + \cdots + r_d |z_d|^\frac{1}{r_d}), & s \geq \frac{1}{2},
\end{cases}
\]

when \( r \in \mathbb{R}_+^d \) and \( z \in \mathbb{C}^d \). For convenience we set \( M_r = M_{1,b_1,r} \).

By \([33]\) we have the following. The proof is therefore omitted.

Proposition 1.5. Let \( M_{1,r,s}, M^0_{1,r,s}, M_{2,r,s} \) and \( M^0_{2,r,s} \) be as in (1.19) when \( s \in \mathbb{R}_+ \) and \( r \in \mathbb{R}_+^d \). Then

\[
A_s^0(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d); Fe^{-M^0_{1,r,s}} \in L^\infty(\mathbb{C}^d) \text{ for every } r \in \mathbb{R}_+^d \}, \quad s > b_1,
\]

and

\[
A_s(\mathbb{C}^d) = \{ F \in A(\mathbb{C}^d); Fe^{-M_{1,r,s}} \in L^\infty(\mathbb{C}^d) \text{ for some } r \in \mathbb{R}_+^d \}, \quad s > b_1,
\]

Next we recall the link between the Bargmann transform and the short-time Fourier transform \( f \mapsto V_\phi f \) with window function \( \phi \) given by (1.7), defined by

\[
V_\phi f(x, \xi) \equiv \langle f, \phi(\cdot - x)e^{-i(\cdot, \xi)} \rangle.
\]

Let \( S \) be the dilation operator given by

\[
(SF)(x, \xi) = F(2^{-\frac{1}{2}}x, -2^{-\frac{1}{2}}\xi),
\]

when \( F \in L^1_{loc}(\mathbb{R}^{2d}) \). Then it follows by straightforward computations that

\[
(\mathcal{M}_d f)(z) = (\mathcal{M}_d f)(x + i\xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{4}(|z|^2 + |\xi|^2)} e^{-i(x, \xi)} V_\phi f(2^{\frac{1}{2}}x, -2^{\frac{1}{2}}\xi)
\]

\[
= (2\pi)^{\frac{d}{2}} e^{\frac{1}{4}(|z|^2 + |\xi|^2)} e^{-i(x, \xi)} (S^{-1}(V_\phi f))(x, \xi),
\]

(1.21)
or equivalently,

\[ V_0 f(x, \xi) = (2\pi)^{-\frac{d}{2}} e^{-\frac{1}{2}(|x|^2 + |\xi|^2)} e^{-i(x, \xi)/2}(\mathfrak{M}_d f)(2\frac{i}{d} x, -2\frac{i}{d} \xi). \]

\[ = (2\pi)^{-\frac{d}{2}} e^{-i(x, \xi)/2} S(e^{-\frac{1}{2} \frac{i}{d}}(\mathfrak{M}_d f))(x, \xi). \quad (1.22) \]

We observe that (1.21) and (1.22) can be formulated into

\[ \mathfrak{M}_d = U_{\varphi} \circ V_0, \quad \text{and} \quad U_{\varphi}^{-1} \circ \mathfrak{M}_d = V_0, \]

where \( U_{\varphi} \) is the linear, continuous and bijective operator on \( \mathcal{Q}'(\mathbb{R}^{2d}) \cong \mathcal{Q}'(\mathbb{C}^d) \), given by

\[ (U_{\varphi} f)(x, \xi) = (2\pi)^{\frac{d}{2}} e^{\frac{1}{2}(|x|^2 + |\xi|^2)} e^{-i(x, \xi)} f(2\frac{i}{d} x, -2\frac{i}{d} \xi). \quad (1.23) \]

Let \( D_{d,r}(z_0) \) be the polydisc

\[ \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d ; |z_j - z_{0,j}| \leq r_j, j = 1, \ldots, d \}, \]

with respect to

\[ z_0 = (z_{0,1}, \ldots, z_{0,d}) \in \mathbb{C}^d, \quad \text{and} \quad r = (r_1, \ldots, r_d) \in [0, \infty)^d, \]

and let \( A_d(D_{d,0}(z_0)) \) be the set of all functions which are defined and analytic near \( z_0 \). Then

\[ A(\mathbb{C}^d) = \bigcap_{r \in [0, \infty)^d} A(D_{d,r}(z_0)), \quad A_d(D_{d,0}(z_0)) = \bigcup_{r \in [0, \infty)^d} A(D_{d,r}(z_0)), \]

We also set \( D_{d,r} = D_{d,r}(0) \).

1.3. Hilbert spaces of power series expansions and analytic functions. The spaces in Definition 1.4 can also be described by related unions and intersections of Hilbert spaces of analytic functions and power series expansions as follows. (See also \[ \text{[33]} \].)

Let \( \vartheta \) be a weight on \( \mathbb{N}^d \), \( \omega \) be a weight on \( \mathbb{C}^d \), and let

\[ \| F \|_{A_0^d(\mathbb{C}^d)} \equiv \left( \sum_{\alpha \in \mathbb{N}^d} |c(F, \alpha)|^2 \vartheta(\alpha) \right)^{\frac{1}{2}} \quad (1.24) \]

when \( F \in A_0^d(\mathbb{C}^d) \) is given by (1.13), and

\[ \| F \|_{A_{\omega}^2(\mathbb{C}^d)} \equiv \left( \int_{\mathbb{C}^d} |F(z)\omega(2\frac{i}{d} z)|^2 \, d\mu(z) \right)^{\frac{1}{2}} \quad (1.25) \]

when \( F \in A(\mathbb{C}^d) \). We let \( A_0^d(\mathbb{C}^d) \) be the set of all \( F \in A_0^d(\mathbb{C}^d) \) such that \( \| F \|_{A_0^d} \) is finite, and \( A_{\omega}^2(\mathbb{C}^d) \) be the set of all \( F \in A(\mathbb{C}^d) \) such that \( \| F \|_{A_{\omega}^2} \) is finite. It follows that these spaces are Hilbert spaces under these norms.
If \( \vartheta \) and \( \omega \) are related to each other as

\[
\vartheta(\alpha) = \left( \frac{1}{\alpha!} \int_{\mathbb{R}^d_+} \omega_0(r)^2 r^{\alpha} \, dr \right)^{\frac{1}{2}}
\]  

(1.26)

and

\[
\omega(z) = e^{\frac{|z|^2}{2}} \omega_0(|z_1|^2, \ldots, |z_d|^2),
\]

(1.27)

for some suitable weight \( \omega_0 \) on \( \mathbb{R}^d_+ \), then the following result shows that \( A^2_{\vartheta}(\mathbb{C}^d) = A^2_\omega(\mathbb{C}^d) \) with equal norms. Here we identify entire functions with their power series expansions at origin. Consequently, the Bargmann transform is bijective and isometric from \( \mathcal{H}_2^{\vartheta}(\mathbb{R}^d) \) to \( \mathcal{A}^2_{\omega}(\mathbb{C}^d) \) for such choices of \( \vartheta \) and \( \omega \). See also [22, Theorem (4.1)] for related results in one dimension. We omit the proof since the result is an immediate consequence of [33, Theorem 3.5].

**Theorem 1.6.** Let \( e_\alpha \) be as in (1.12), \( \alpha \in \mathbb{N}^d \), and let \( \omega_0 \) be a positive measurable function on \( \mathbb{R}^d_+ \). Also let \( \vartheta \) and \( \omega \) be weights on \( \mathbb{N}^d \) and \( \mathbb{C}^d \), respectively, related to each other by (1.26) and (1.27), and such that

\[
\frac{r^{\alpha}}{(\alpha!)^\frac{1}{2}} \lesssim \vartheta(\alpha), \quad \alpha \in \mathbb{N}^d,
\]

(1.28)

holds for every \( r > 0 \). Then \( A^2_{\vartheta}(\mathbb{C}^d) = A^2_\omega(\mathbb{C}^d) \) with equal norms.

In our situation, the involved weights satisfy a split condition. In one dimension, (1.26), (1.27) and (1.28) take the forms

\[
\vartheta_j(\alpha_j) = \left( \frac{1}{\alpha_j!} \int_{\mathbb{R}^d_+} \omega_{0,j}(r)^2 r^{\alpha_j} \, dr \right)^{\frac{1}{2}}, \quad \alpha_j \in \mathbb{N},
\]

(1.26)'

\[
\omega_j(z_j) = e^{\frac{|z_j|^2}{2}} \omega_{0,j}(|z_j|^2), \quad z_j \in \mathbb{C}
\]

(1.27)'

and

\[
\frac{r^{\alpha_j}}{(\alpha_j!)^\frac{1}{2}} \lesssim \vartheta_j(\alpha_j), \quad r > 0, \quad \alpha_j \in \mathbb{N}.
\]

(1.28)'

**Lemma 1.7.** Let \( \omega_{0,j} \) be weights on \( \mathbb{R}^d_+ \), \( \omega_j \) be weights on \( \mathbb{C} \) and \( \vartheta_j \) be weights on \( \mathbb{N} \) such that (1.26) – (1.28) hold, \( j = 1, \ldots, d \), and set \( \omega_0(z) \equiv \prod_{j=1}^d \omega_{0,j}(z_j), \quad z \equiv (z_1, \ldots, z_d) \in \mathbb{C}^d \). If \( \vartheta \) and \( \omega \) are given by (1.26), and (1.27), then

\[
\vartheta(\alpha) = \prod_{j=1}^d \vartheta_j(\alpha_j), \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d
\]

(1.29)
\[ \omega(z) = \prod_{j=1}^{d} \omega_j(z_j), \quad z = (z_1, \ldots, z_d) \in \mathbb{C}^{d}. \quad (1.30) \]

**Proof.** By \([33, \text{Theorem 3.5}]\) and its proof, it follows that \(\omega_{0,j} \cdot r^{\alpha_j} \in L^1(\mathbb{R}_+)\) for all \(j \in \{1, \ldots, d\}\) and \(\alpha_j \in \mathbb{N}\). Hence, Fubini’s theorem gives

\[
\vartheta(\alpha) = \left( \frac{1}{\alpha!} \int_{\mathbb{R}^d_+} \omega_0(r^2 r^\alpha \, dr) \right)^{\frac{1}{2}} = \left( \frac{1}{\alpha!} \prod_{j=1}^{d} \int_{0}^{\infty} \omega_{0,j}(r_j^2 r_j^{\alpha_j} \, dr_j) \right)^{\frac{1}{2}} = \prod_{j=1}^{d} \vartheta_j(\alpha_j),
\]

and \((1.29)\) follows. The assertion \((1.30)\) follows from the definitions. \(\square\)

1.4. **A test function space introduced by Gröchenig.** In this section we recall some comparison results deduced in \([33]\), between a test function space, \(\mathcal{S}_C(\mathbb{R}^d)\), introduced by Gröchenig in \([18]\) to handle modulation spaces with ultra-distributions and Pilipović spaces.

The definition of \(\mathcal{S}_C(\mathbb{R}^d)\) is given as follows.

**Definition 1.8.** Let \(\phi(x) = \pi^{-\frac{d}{2}} e^{-\frac{|x|^2}{2}}\). Then \(\mathcal{S}_C(\mathbb{R}^d)\) and \(\mathcal{S}_G(\mathbb{R}^d)\) consist of all \(f \in \mathcal{S}'(\mathbb{R}^d)\) such that \(f = V_\phi F\), for some \(F \in L^\infty(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d)\) and \(F \in \mathcal{E}'(\mathbb{R}^d)\), respectively.

It follows that \(f \in \mathcal{S}_C(\mathbb{R}^d)\), if and only if

\[
f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^{2d}} F(y, \eta) e^{-\frac{1}{2} |x-y|^2} e^{i(x, \eta)} \, dyd\eta,
\]

for some \(F \in L^\infty(\mathbb{R}^d) \cap \mathcal{E}'(\mathbb{R}^d)\).

**Remark 1.9.** By the identity \((V_\phi h, F) = (h, V_\phi^* F)\) and the fact that the map \((f, \phi) \mapsto V_\phi f\) is continuous from \(\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)\) to \(\mathcal{S}'(\mathbb{R}^{2d})\), it follows that \(f = V_\phi^* F\) is uniquely defined as an element in \(\mathcal{S}'(\mathbb{R}^{2d})\) when \(F \in \mathcal{S}'(\mathbb{R}^{2d})\) (cf. \([31]\)). In particular, the space \(\mathcal{S}_G(\mathbb{R}^d)\) in Definition \(1.8\) is well-defined, and it is evident that \(\mathcal{S}_C(\mathbb{R}^d) \subseteq \mathcal{S}_G(\mathbb{R}^d)\).

The following is a restatement of \([33, \text{Lemma 4.9}]\).

**Lemma 1.10.** Let \(F \in L^\infty(\mathbb{C}^d) \cup \mathcal{E}'(\mathbb{C}^d)\). Then the Bargmann transform of \(f = V_\phi^* F\) is given by \(\Pi_A F_0\), where

\[
F_0(x + i\xi) = (2\pi^2)^\frac{d}{4} F(\sqrt{2}x, -\sqrt{2}\xi) e^{\frac{1}{4} |x|^2 + |\xi|^2} e^{-i(x, \xi)}.
\]
Moreover, the images of $S_C(\mathbb{R}^d)$ and $S_G(\mathbb{R}^d)$ under the Bargmann transform are given by

$$\{ \Pi_A F ; F \in L^\infty(\mathbb{C}^d) \bigcap \mathcal{O}(\mathbb{C}^d) \} \quad \text{and} \quad \{ \Pi_A F ; F \in \mathcal{O}'(\mathbb{C}^d) \},$$

(1.32) respectively.

Here recall that the map $\Pi_A$ is defined by (1.11).

The next result follows from [33, Theorem 4.10]. The proof is therefore omitted.

**Proposition 1.11.** It holds $S_C(\mathbb{R}^d) = S_G(\mathbb{R}^d) = H^\flat_1(\mathbb{R}^d)$.

Due to the image properties the spaces in Proposition 1.11 under the Bargmann transform, the next result is equivalent with the previous one.

**Proposition 1.12.** The sets in (1.32) are equal to $A^\flat_1(\mathbb{C}^d)$.

In the next section we shall extend Propositions 1.11 and 1.12 by proving that the conclusions in Proposition 1.12 holds for smaller sets than those in (1.32). We also deduce similar identifications for other Pilipović spaces and their Bargmann images.

2. Paley-Wiener properties for Bargmann-Pilipović spaces

In this section we consider spaces of compactly supported functions with interiors in $A_s(\mathbb{C}^d)$ or in $A'_s(\mathbb{C}^d)$. We show that the images of such functions under the reproducing kernel $\Pi_A$ are equal to $A_s(\mathbb{C}^d)$, for some other choice of $s \leq \flat_1$ (see Theorems 2.7 and 2.14). An essential part of these investigations concerns Proposition 2.3 where we make a detailed description between estimates on analytic functions, $F$, given in Proposition 1.5 and conditions on their coefficients, $c(F, \alpha)$.

First we note that if $A^2_{(\omega)} = A^2_{(\vartheta)}$, then a split of the variables in the weight $\omega$ in $A^2_{(\omega)}$ induce a split of the variables in $\vartheta$ in $A^2_{(\vartheta)}$.

By [31, Theorem 3.2], we have the following.

**Lemma 2.1.** Suppose $s \in \overline{\mathbb{R}}_0$, $r, r_0 \in \mathbb{R}_+^d$ such that $r_0 < r$ and let $F \in A(\mathbb{C}^d)$. Then

$$\| F \cdot e^{-M_{r,s}} \|_{L^2(\mathbb{C}^d)} \lesssim \| F \cdot e^{-M_{r_0,s}} \|_{L^\infty(\mathbb{C}^d)}$$

and

$$\| F \cdot e^{-M_{r,s}} \|_{L^\infty(\mathbb{C}^d)} \lesssim \| F \cdot e^{-M_{r_0,s}} \|_{L^2(\mathbb{C}^d)}.$$

**Lemma 2.2.** Let $\omega(z) = e^{\frac{1}{2}|z|^2 - M_{r,s}(z)}$, $z \in \mathbb{C}^d$, and $\alpha_0 = (1, \ldots, 1) \in \mathbb{N}^d$. Then the following is true:
If $F \in A(C^d)$ is given by (1.13), then

$$\|F \cdot e^{-M_{σ,r}}\|_{L^2(C^d)} = \left( \frac{σ + 1}{σ} \sum_{α \in \mathbb{N}^d} \left| c(F, α) (2r)^{-(α+α_0)\frac{σ+1}{σ}} \left( \frac{Γ((α + α_0)\frac{σ+1}{σ})}{α!} \right)^{\frac{1}{2}} \right|^2 \right)^{\frac{1}{2}};$$

(2) $A^2_{[θ]}(C^d) = A^2_{(ω)}(C^d)$ with equality in norms.

Proof. Since

$$e^{-M_{σ,r}(z)} = \prod_{j=1}^{d} e^{-r_j |z_j|^{\frac{2r}{σ+1}}},$$

Lemma [1.7] shows that we may assume that $d = 1$, giving that $r = r_1$ and $α_0 = 1$.

Since

$$θ(α) = \left( \frac{1}{α!} \int_{0}^{∞} e^{-2rt^{\frac{2r}{σ+1}}} t^α dt \right)^{\frac{1}{2}},$$

we obtain letting $u = 2rt^{\frac{2r}{σ+1}}$

$$θ(α) = \left( \frac{σ + 1}{σ} (2r)^{-(α+1)\frac{σ+1}{σ}} \frac{1}{α!} \int_{0}^{∞} e^{-u} u^{α(\frac{σ+1}{σ}+\frac{1}{2})} du \right)^{\frac{1}{2}},$$

$$= \left( \frac{σ + 1}{σ} (2r)^{-(α+1)\frac{σ+1}{σ}} \left( \frac{Γ((α + 1)\frac{σ+1}{σ})}{α!} \right)^{\frac{1}{2}} \right),$$

and the result follows from Theorem [1.6].

The next result links the sets in the following definition to each other.

Definition 2.3. Let $s \in \mathbb{R}$ and let $M_{1,r,s}^0$ and $θ_{r,s}$ be the same as in (1.3) and (1.19). Then

$$Ω_{1,r}(C^d) = \{ \Pi_A F ; F \in L^∞(C^d), \supp F \subseteq D_{d,r} \},$$

$$Ω_{2,r}(C^d) = \{ \Pi_A F ; F \in E'(C^d), \supp F \subseteq D_{d,r} \},$$

$$Ω_{3,r,s}(C^d) = \{ F \in A(C^d) ; |c(F, α)| \lessapprox θ_{r,s}(α), α \in \mathbb{N}^d \},$$

$$Ω_{4,r,s}(C^d) = \left\{ F \in A(C^d) ; |F(z)| \lessapprox e^{M_{r,σ}^0(z)} \right\},$$

$$Ω_{3,r}(C^d) = Ω_{3,r,1}(C^d) \text{ and } Ω_{4,r}(C^d) = Ω_{4,r,1}(C^d).$$
By (4.18) in [33] we have
\[
\bigcup_{r \in \mathbb{R}^d_+} \Omega_{1,r}(C^d) = \mathfrak{U}_d(S_C(\mathbb{R}^d)),
\]
\[
\bigcup_{r \in \mathbb{R}^d_+} \Omega_{2,r}(C^d) = \mathfrak{U}_d(S_G(\mathbb{R}^d)) \quad \text{and} \quad (2.6)
\]
\[
\bigcup_{r \in \mathbb{R}^d_+} \Omega_{3,\sigma}(C^d) = \bigcup_{r \in \mathbb{R}^d_+} \Omega_{4,r,\sigma}(C^d) = \mathfrak{U}_d(H_{\sigma}(\mathbb{R}^d)).
\]

We have now the following extension of [33, Theorem 4.10].

**Proposition 2.4.** Let \( \sigma \in (0, 1], r, r_0 \in \mathbb{R}^d_+ \) be such that \( r_0 < r \), and let \( \Omega_{j,r}(C^d), j = 1, 2 \) and \( \Omega_{j,r,\sigma}(C^d), j = 3, 4, \) be the same as in Definition 2.3. Then
\[
\Omega_{1,r}(C^d) \subseteq \Omega_{2,r}(C^d) \cap \Omega_{4,r}(C^d), \quad \Omega_{2,r_0}(C^d) \subseteq \Omega_{4,r}(C^d)
\]
\[
\Omega_{3,r_0}(C^d) \subseteq \Omega_{1,r}(C^d),
\]
\[
\Omega_{3,\sigma}(C^d) \subseteq \Omega_{4,\sigma}(C^d) \quad \text{when} \quad r_0 < \left( \frac{2\sigma}{\sigma + 1} \right)^{\frac{\sigma + 1}{2\sigma}}, \quad (2.7)
\]
and
\[
\Omega_{4,\sigma}(C^d) \subseteq \Omega_{3,\sigma}(C^d) \quad \text{when} \quad r_0 < \frac{r^2(\sigma + 1)}{2\sigma}. \quad (2.8)
\]

We need some preparations for the proof, and begin with the following lemma.

**Lemma 2.5.** Let \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d_+ \), \( \chi_r \) be the characteristic function of \( D_{d,r} \subseteq C^d \), and let
\[
F_{\alpha,r}(z) = \sqrt{\alpha!} \left( \prod_{j=1}^d r_j^{-2}(\alpha_j + 1) \right) r^{-2\alpha} z^{\alpha} e^{\|z\|^2} \chi_r(z), \quad z \in C^d, \ \alpha \in \mathbb{N}^d.
\]

Then
\[
\Pi_A F_{\alpha,r} = e_{\alpha} \quad (2.10)
\]

**Proof.** By Lemma 4.11 in [33], the result is true when \( D_{d,r} \) is the unit polydisc. For general \( r \in \mathbb{R}^d_+ \), the result follows by straight-forward computations by combining the latter case and the formula
\[
(\Pi_A F)(z) = (\Pi_A F_r)(r_1^{-1}z_1, \ldots, r_d^{-1}z_d), \quad (2.11)
\]
where
\[
F_r(z) = \left( \prod_{j=1}^d r_j^{-2} \right) e^{Q_r(z)} F(r_1^{-1}z_1, \ldots, r_d^{-1}z_d)
\]
and

\[ Q_r(z) = \sum_{j=1}^{d} (1 - r_j^{-2}) z_j^2 \]

for admissible \( F \), which also follows by straight-forward computations.

\[ \square \]

Lemma 2.6. Let \( \alpha \geq 1 \) be an integer. Then

\[
\frac{\Gamma \left( (\alpha + 1) \frac{\sigma + 1}{\sigma} \right)}{\alpha!} \asymp \frac{\left( \frac{\sigma + 1}{\sigma} + \frac{1}{\sigma} \right)^{\frac{\sigma + 1}{\sigma} + \frac{1}{\sigma} + \frac{1}{2}}}{e^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}}}. \]

Lemma 2.6 follows by repeated applications of Stirling’s formula and the standard limit

\[
\lim_{t \to \infty} \left( 1 + \frac{x}{t} \right)^t = e^x
\]

for every \( x \in \mathbb{R} \). In order to be self-contained we present the arguments.

Proof. We have

\[
\frac{\Gamma \left( (\alpha + 1) \frac{\sigma + 1}{\sigma} \right)}{\alpha!} \asymp \frac{\left( \frac{\sigma + 1}{\sigma} + \frac{1}{\sigma} \right)^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}}}{e^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}}} \cdot \frac{e^\alpha}{\alpha^{\frac{1}{2}}}.
\]

\[
= \frac{\left( \frac{\sigma + 1}{\sigma} + \frac{1}{\sigma} \right)^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}}}{e^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}}} \cdot \frac{\left( \frac{\alpha + 1}{\sigma} \right)^{\alpha + \frac{1}{2}}}{e^{\alpha + \frac{1}{2}}}.
\]

\[
\times \left( \frac{\sigma + 1}{\sigma} \right)^{\frac{\sigma + 1}{\sigma} + \frac{1}{2}} \times \frac{\alpha + 1}{\sigma} \times \frac{1}{e^{\alpha + \frac{1}{2}}} \times \left( \frac{\alpha + 1}{\sigma} \right)^{\alpha + \frac{1}{2}}.
\]

and the result follows.

\[ \square \]

Proof of Proposition 2.4. By (2.6), the result follows if we prove (2.7)–(2.9). Evidently, \( \Omega_{1,r} \subseteq \Omega_{2,r} \). Assume that \( F \in \Omega_{1,r} \). Then

\[
|F(w)e^{(z,w)}| \lesssim e^{M_r(z)}.
\]

This gives

\[
|\mathcal{M}_d f(z)| \lesssim e^{M_d(z)} \int_{C_d} d\mu(w) \asymp e^{M_d(z)},
\]

which implies \( \Omega_{1,r} \subseteq \Omega_{4,r} \).
Next assume that $F \in \Omega_{2,r_0}$. Then $F = \Pi_A F_0$ for some $F_0 \in \mathcal{E}'(\mathbb{C}^d)$ with supp $F_0 \subseteq D_{d,r_0}$. This implies that for some $N \geq 0$ we have

$$|F(z)| = \pi^{-d} \left| \langle F_0, e^{(z \cdot \cdot \cdot e^{-|\cdot|^2})} \rangle \right| \lesssim \sum_{|\alpha| \leq N} \| D^\alpha (e^{(z \cdot \cdot \cdot e^{-|\cdot|^2})} \|_{L^\infty(D_{d,r_0})} \lesssim \langle z \rangle^N e^{M\alpha_0(z)} \lesssim e^{M\alpha_0(z)}.$$  

Hence $F \in \Omega_{4,r}$, and it follows that $\Omega_{2,r_0} \subseteq \Omega_{4,r}$.

Next we prove $\Omega_{3,r_0} \subseteq \Omega_{1,r}$. By (2.11) we reduce to the case when $r_1 = \cdots = r_d = 1$.

Let $F \in \Omega_{3,r_0}$, and let $F_{\alpha,r}$ and $\chi_r$ be as in Lemma 2.5. Then (1.13) holds with

$$|c(F, \alpha)| \lesssim r_\alpha^0 \alpha!^{1/2}.$$  

By the Lemma 2.5 we get $F = \Pi_A G$, where

$$G(z) = \left( \sum_{\alpha \in \mathbb{N}^d} c(F, \alpha) \alpha!^{1/2} \left( \prod_{j=1}^d (\alpha_j + 1) \right) z^\alpha \right) e^{z|z|^2} \chi_1(z),$$  

Evidently, supp $G \subseteq D_{1,1}$. We also have

$$\|G\|_{L^\infty} \lesssim \sum_{\alpha \in \mathbb{N}^d} r_0^\alpha \left( \prod_{j=1}^d (\alpha_j + 1) \right) < \infty.$$  

Hence, $F \in \Omega_{1,r}$.

The last relation in (2.7) is a special case of (2.8) below, and is therefore omitted here.

Next we prove (2.8), and therefore suppose $F \in \Omega_{3,r_0,\sigma}(\mathbb{C})$, and let $r_1, r_2 \in \mathbb{R}_0^d$ be such that $r_0 < r_1 < r_2 < r$, and let $\alpha_0 = (1, \ldots, 1) \in \mathbb{N}^d$. 

\[17\]
Then Lemma 2.1 and 2.2 give
\[
\| F \cdot e^{-M_{1,r_0}} \|_{L^\infty(\mathbb{C}^d)} \lesssim \| F \cdot e^{-M_{1,r_2}} \|_{L^2(\mathbb{C}^d)}
\]
\[
= \left( \frac{\sigma + 1}{\sigma} \sum_{\alpha \in \mathbb{N}^d} |c(F, \alpha)(2r_2)^{-(\alpha+1)\frac{\sigma + 1}{2\sigma}} \left( \frac{\Gamma((\alpha + 1)\frac{\sigma + 1}{\alpha})}{\alpha!} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
= \left( \frac{\sigma + 1}{\sigma} \sum_{\alpha \in \mathbb{N}^d} |c(F, \alpha)(2r_1)^{-(\alpha+1)\frac{\sigma + 1}{2\sigma}} \left( \frac{\Gamma((\alpha + 1)\frac{\sigma + 1}{\alpha})}{\alpha!} \right)^{\frac{1}{2}} \left( \frac{r_1}{r_2} \right)^{2((\alpha+1)\frac{\sigma + 1}{2\sigma})} \right)^{\frac{1}{2}}
\]
\[
\lesssim \sup_{\alpha \in \mathbb{N}^d} \left| c(F, \alpha)(2r_1)^{-(\alpha+1)\frac{\sigma + 1}{2\sigma}} \left( \frac{\sigma + 1}{\sigma} \right)^{\frac{\alpha + 1}{2\sigma}} \alpha! \frac{1}{\alpha!} \right|
\]
\[
\times \sup_{\alpha \in \mathbb{N}^d} \left| c(F, \alpha) \right| \left( \frac{\sigma + 1}{2\sigma r_0} \right)^{\frac{\alpha + 1}{2\sigma}} \alpha! \frac{1}{\alpha!},
\]
where the second inequality follows from the fact that
\[
\sum_{\alpha \in \mathbb{N}^d} \left( \frac{r_1}{r_2} \right)^{2(\alpha + \alpha_0)}
\]
is convergent since \( r_1 < r_2 \), and the fifth relation follows from Lemma 2.6. This implies that \( F \in \Omega_{4,r_0,\rho}(\mathbb{C}^d) \), and we have proved (2.8).

Next assume that \( F \in \Omega_{4,r_0,\rho}(\mathbb{C}^d) \). By Lemma 2.1 we get
\[
\| F \cdot e^{-M_{1,r_0}} \|_{L^\infty(\mathbb{C}^d)} \gtrsim \| F \cdot e^{-M_{1,r_2}} \|_{L^2(\mathbb{C}^d)}
\]
\[
= \left( \frac{\sigma + 1}{\sigma} \sum_{\alpha \in \mathbb{N}^d} |c(F, \alpha)(2r_2)^{-(\alpha+1)\frac{\sigma + 1}{2\sigma}} \left( \frac{\Gamma((\alpha + 1)\frac{\sigma + 1}{\alpha})}{\alpha!} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
\geq \left( \frac{\sigma + 1}{\sigma} \right)^{\frac{1}{2}} \sup_{\alpha \in \mathbb{N}^d} \left| c(F, \alpha) \right| (2r_2)^{-(\alpha+1)\frac{\sigma + 1}{2\sigma}} \left( \frac{\Gamma((\alpha + 1)\frac{\sigma + 1}{\alpha})}{\alpha!} \right)^{\frac{1}{2}}
\]
\[
\gtrsim \sup_{\alpha \in \mathbb{N}^d} \left| c(F, \alpha) \right| \left( \frac{\sigma + 1}{2\sigma r} \right)^{\frac{\alpha + 1}{2\sigma}} (\alpha + \alpha_0)^{\frac{1}{\alpha!}} \alpha! \frac{1}{\alpha!}
\]
\[
\geq \sup_{\alpha \in \mathbb{N}^d} \left| c(F, \alpha) \right| \left( \frac{\sigma + 1}{2\sigma r} \right)^{\frac{\alpha + 1}{2\sigma}} \alpha! \frac{1}{\alpha!},
\]
where the third inequality follows from Lemma 2.6. This implies that $F \in \Omega_{3,m(r),\delta}(C^d)$ with

$$m(r) = \left( \frac{2\sigma r}{\sigma + 1} \right)^{\frac{1}{\frac{1}{2}\delta}},$$

i.e. $\Omega_{4,r_1,\sigma}(C^d) \subseteq \Omega_{3,m(r),\sigma}(C^d)$. This is the same as (2.9).

Next we show that if $F \in \Omega_{k,r,\delta}(C^d)$, $k = 3, 4$, then

$$F = \Pi_A(F_0 e^{|^2\chi_{r_0}})$$

for suitable function $F_0$ which is analytic near origin. We let

$$\Omega'_{k,r,s}(C^d) = \{ F \in A(C^d) ; |c(F, \alpha)| \lesssim \vartheta'_{r,s}(\alpha), \alpha \in \mathbb{N}^d \},$$

where $\vartheta'_{r,s}$ is given by (1.4).

**Theorem 2.7.** Let $\sigma \in (0, 1]$, $r_0, r_1, r_2 \in \mathbb{R}_+$ be such that $r_1 < r_2$, $s \in \mathbb{R}_+$, and let $\Omega_{k,r,s}(C^d)$ and $\Omega'_{k,r,s}(C^d)$ be the same as in (2.3), (2.4) and (2.14). Also let $\chi_r$ be the characteristic function of $D_{d,r}$. Then the following is true:

1. if $r_0 = r_2$ and $F \in \Omega_{k,r_1,\delta_1}(C^d)$, $k \in \{3, 4\}$, then (2.13) holds for some $F_0 \in A(D_{d,r_2})$. If instead $r_0 = r_1$ and $F_0 \in A(D_{d,r_1})$ and $F$ is given by (2.13), then $F \in \Omega_{3,r_1,\delta_1}(C^d) \subseteq \Omega_{4,r_2,\delta}(C^d)$;

2. if $\frac{1}{2} < \sigma < 1$, $\sigma_0 = \frac{\sigma}{2\sigma - 1}$ and $F \in \Omega_{3,r_1,\delta}(C^d)$,

then (2.13) holds for some $F_0 \in \Omega'_{3,r_0,2r_1,\sigma_0}(C^d)$. If instead $F_0 \in \Omega'_{3,r_0,2r_1,\sigma_0}(C^d)$ and $F$ is given by (2.13), then $F \in \Omega_{3,r_1,\delta}(C^d)$;

3. if $F \in \Omega_{3,r_1,\delta_1/2}(C^d)$, then (2.13) holds for some $F_0 \in \Omega_{3,r_0,2r_1,\delta_1/2}(C^d)$. If instead $F_0 \in \Omega_{3,r_0,2r_1,\delta_1/2}(C^d)$ and $F$ is given by (2.13), then $F \in \Omega_{3,r_1,\delta_1/2}(C^d)$;

4. if $0 < \sigma < \frac{1}{2}$, $\sigma_0 = \frac{\sigma}{1 - 2\sigma}$ and $F \in \Omega_{3,r_1,\delta}(C^d)$,

then (2.13) holds for some $F_0 \in \Omega'_{3,r_0,2r_2,\delta_0}(C^d)$. If instead $F_0 \in \Omega'_{3,r_0,2r_2,\delta_0}(C^d)$ and $F$ is given by (2.13), then $F \in \Omega_{3,r_1,\delta}(C^d)$

**Proof.** Let $F \in \Omega_{3,r_1,\delta_1}(C^d)$. Then there is a constant $C > 0$ such that

$$F = \sum_{\alpha} c(F, \alpha)e_\alpha, \quad |c(F, \alpha)| \leq Cr_1^\alpha \alpha!^{-\frac{1}{2\sigma}}.$$  (2.15)
By Lemma 2.5 we formally obtain
\[ F = \sum_{\alpha} c(F, \alpha) e_{\alpha} = \sum_{\alpha} c(F, \alpha) \Pi_A F_{\alpha, r_0} \]
\[ = \Pi_A \left( \sum_{\alpha} c(F, \alpha) F_{\alpha, r_0} \right) = \Pi_A (F_0 e^{1/2} \chi_{r_0}), \]
with
\[ F_0(z) = \sum_{\alpha} \left( c(F, \alpha) \alpha! \frac{1}{2} r_0^{-2\alpha} \rho(\alpha) z^\alpha \right), \quad \rho(\alpha) = \prod_{j=1}^d (r_{0,j}^{-2}(\alpha_j + 1)), \] (2.16)
that is,
\[ c(F_0, \alpha) = c(F, \alpha)! r_0^{-2\alpha} \rho(\alpha), \] (2.17)
provided \( F_0 \) makes sense, which is the case if we can show that the series in (2.16) is uniformly convergent in a neighbourhood of \( D_{d,r_2} \).

Let
\[ u_{\alpha}(z) = c(F, \alpha) \alpha! \frac{1}{2} r_0^{-2\alpha} \rho(\alpha) z^\alpha, \quad z \in D_{d,r_2}. \]
Since \( r_0 = r_2 \) we get for \( z \in D_{d,r_2} \) that
\[ |u_{\alpha}(z)| \leq |c(F, \alpha) \alpha! \frac{1}{2} r_0^{-2\alpha} \rho(\alpha) z^\alpha| \lesssim r_1^{-2\alpha} r_2^{-2\alpha} \rho(\alpha) r_2^\alpha = \rho(\alpha) \left( \frac{r_1}{r_2} \right)^\alpha. \]

By the fact \( r_1 < r_2 \), it follows that
\[ \sum_{\alpha \in \mathbb{N}^d} \rho(\alpha) \left( \frac{r_1}{r_2} \right)^\alpha \]
is convergent. Hence, Weierstrass theorem gives that
\[ F_0(z) = \sum_{\alpha \in \mathbb{N}^d} u_{\alpha}(z), \quad z \in D_{d,r_2}, \]
is uniformly convergent, and the first part of (1) follows.

In order to prove the second part of (1), let \( F_0 \in A(D_{d,r_1}) \). Then
\[ |c(F_0, \alpha)| \lesssim r_1^{-\alpha} \alpha! \frac{1}{2}, \]
and a combination of this estimate with (2.17) gives
\[ |c(F, \alpha)| \lesssim r_1^{-\alpha} \alpha! \frac{1}{2} r_0^{-2\alpha} \rho(\alpha)^{-1} \leq r_1^{-\alpha} \alpha! \frac{1}{2}, \]
which is the same as \( F \in \Omega_{3, r_1, \rho_1} \). This gives the second part of (1).

Suppose instead \( \frac{1}{2} < \sigma < 1 \). Then
\[ |c(F_0, \alpha)| = |c(F, \alpha) \alpha! r_0^{-2\alpha} \rho(\alpha)| \lesssim (r_0^{-2} r_1)^\alpha \rho(\alpha) \alpha!^{1-\frac{1}{2}} \]
\[ \lesssim (r_0^{-2} r_2)^\alpha \alpha!^{\sigma}, \] (2.18)
giving that \( F_0 \in \Omega_{3, r_0^{-2} r_2, \rho_0} \mathbb{C}^d \), which gives the first part of (2).
Conversely, if $F_0 \in \Omega_{3,r_0^2 r_1 \sigma_0} (C^d)$, then
\[
|c(F, \alpha)| \lesssim (r_0^{-2} r_1)^\alpha! \frac{1}{\sigma_0} \alpha!^{-1} r_0^{2\alpha} \rho(\alpha)^{-1} \leq r_1^\alpha \alpha!^{-\frac{1}{2}},
\]
which is the same as $F \in \Omega_{3,r_1 \sigma_0}$. This gives the second part of (2).

Next we consider the case when $\sigma = \frac{1}{2}$. Then
\[
|c(F_0, \alpha)| = |c(F, \alpha) r_0^{-2\alpha} \rho(\alpha)| \lesssim (r_0^{-2} r_1)^\alpha \rho(\alpha) \lesssim (r_0^{-2} r_2)^\alpha,
\]
which is the same as $F_0 \in \Omega_{3,r_0^2 r_2,1/2} (C^d)$.

If instead $F_0 \in \Omega_{3,r_0^{-2}, r_1,1/2} (C^d)$ and $F$ is given by (2.13), then
\[
|c(F, \alpha)| \lesssim (r_0^{-2} r_1)^\alpha! \frac{1}{\sigma_0} \alpha!^{-1} r_0^{2\alpha} \rho(\alpha)^{-1} \leq r_1^\alpha \alpha!^{-\frac{1}{2}},
\]
which is the same as $F \in \Omega_{3,r_1 \sigma_0}$, and the result follows in the case $\sigma = \frac{1}{2}$. This gives (3).

Finally, suppose $0 < \sigma < \frac{1}{2}$. Then (2.20) gives
\[
|c(F_0, \alpha)| \lesssim (r_0^{-2} r_2)^\alpha! \frac{1}{\sigma_0} \alpha!^{-\frac{1}{2}} = (r_0^{-2} r_2)^\alpha \alpha!^{-\frac{1}{2}},
\]
and the first part of (4) follows.

Conversely, let $F_0 \in \Omega_{3,r_0^{-2} r_1 \sigma_0} (C^d)$ and $F$ is given by (2.13). Then
\[
|c(F, \alpha)| \lesssim (r_0^{-2} r_1)^\alpha! \frac{1}{\sigma_0} \alpha!^{-1} r_0^{2\alpha} \rho(\alpha)^{-1} \leq r_1^\alpha \alpha!^{-\frac{1}{2}},
\]
which is the same as $F \in \Omega_{3,r_1 \sigma_0}$. This gives the second part of (4) and the proof is complete.

By similar arguments and using the fact that
\[
\bigcap_{r>0} \Omega_{4,r,\sigma_0} (C^d) = \mathcal{A}_{\sigma_0}^0 (C^d),
\]
which follows by straight-forward computations, we get the following. The details are left for the reader.

**Lemma 2.8.** Let $\sigma \in (0,1]$, $r \in \mathbb{R}^d$, $\chi_r$ be the characteristic function of $D_{d,r}$, and let $F \in \mathcal{A}_{\sigma_0}^0 (C^d)$. Then the following is true:

1. if $\sigma = 1$, then (2.13) holds for some $F_0 \in A(C^d)$;
2. if $\frac{1}{2} < \sigma < 1$, and $\sigma_0 = \frac{\sigma}{2\sigma - 1}$ then (2.13) holds for some $F_0 \in (A_{\sigma_0}^0)'(C^d)$;
3. if $\sigma = \frac{1}{2}$, then (2.13) holds for some $F_0 \in \mathcal{A}_{\sigma_0}^0 (C^d)$;
4. if $0 < \sigma < \frac{1}{2}$, and $\sigma_0 = \frac{\sigma}{1-2\sigma}$ then (2.13) holds for some $F_0 \in \mathcal{A}_{\sigma_0}^0 (C^d)$.

As an immediate consequence of the previous lemma and Proposition 2.4 we have the following.
Theorem 2.9. Let \( r, r_0 \in \mathbb{R}_+^d \) be such that \( r_0 < r \), \( \Omega_{k,r} \) be as in (2.11) for \( k = 1, \ldots, 4 \), and let
\[
\Omega_{5,r} = \{ \Pi_A F; \ F = F_0 e^{\cdot 1/2} \chi_r, \ F_0 \in A(D_{d,r}) \}.
\]
Then \( \Omega_{k,r_0} \subseteq \Omega_{5,r} \) for every \( k \).

Theorem 2.10. Let \( F \in A(\mathbb{C}^d) \). Then the following is true:

1. \( F \in A_{\theta_0}(\mathbb{C}^d) \), if and only if there is an \( r \in \mathbb{R}_+^d \) and \( F_0 \in A(D_{d,r}) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \);
2. \( F \in A^{\theta_0}(\mathbb{C}^d) \), if and only if for some \( r \in \mathbb{R}_+^d \) there is an \( F_0 \in A(\mathbb{C}^d) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \);
3. \( F \in A^{\theta_0}(\mathbb{C}^d) \), if and only if for every \( r \in \mathbb{R}_+^d \) there is an \( F_0 \in A(\mathbb{C}^d) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \).

Theorem 2.11. Let \( F \in A(\mathbb{C}^d) \), \( \sigma \in (0,1) \) and \( \sigma_0 = \sigma/(1-\sigma) \). Then the following is true:

1. \( F \in A_{\theta_0}^0(\mathbb{C}^d) \), if and only if there is an \( r \in \mathbb{R}_+^d \) and \( F_0 \in A_{\sigma_0}^{\theta_0}(\mathbb{C}^d) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \);
2. \( F \in A_{\sigma_0}^{\theta_0}(\mathbb{C}^d) \), if and only if for some \( r \in \mathbb{R}_+^d \) there is an \( F_0 \in A_{\sigma_0}^{\theta_0}(\mathbb{C}^d) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \);
3. \( F \in A_{\sigma_0}^{\theta_0}(\mathbb{C}^d) \), if and only if for every \( r \in \mathbb{R}_+^d \) there is an \( F_0 \in A_{\sigma_0}^{\theta_0}(\mathbb{C}^d) \) such that \( F = \Pi_A (F_0 e^{\cdot 1/2} \chi_r) \).

Proposition 2.12. Let \( s \in (0,1/2) \), \( r_0, r \in \mathbb{R}_+^d \) be such that \( r_0 < r \), and let
\[
\kappa_{d,s}(r_1, \ldots, r_d) = (1-2s)\frac{2^{1-d}}{2^d} (2s)^{-1} (r_1^{1-2s}, \ldots, r_d^{1-2s}), \quad (r_1, \ldots, r_d) \in \mathbb{R}_+^d.
\]
Then
\[
\Omega_{\kappa_d, s}(r_0, s) \subseteq \Omega_{4,r,s}(\mathbb{C}^d) \quad \text{and} \quad \Omega_{4,r_0,s}(\mathbb{C}^d) \subseteq \Omega_{\kappa_d, s}(r_0, s) \subseteq \Omega_{4,r_0,s}(\mathbb{C}^d).
\]
The preceding proposition is an immediate consequence of the following lemma.

Lemma 2.13. Let \( r_0, r_1, r_2 \in \mathbb{R}_+^d \) be such that \( r_1 < r_0 < r_2 \), \( p_1, p_2 \in [1, \infty] \), \( s \in (0,1/2) \), \( \kappa_{d,s} \) be the same as in Proposition 2.12 and let \( F \in A(\mathbb{C}^d) \). Then the following is true:

1. if \( \|F e^{-M_1 r_0 s}\|_{L^p_1} < \infty \), then
   \[
   \{ c(F,\alpha)/\partial_{\kappa_d, s}(r_2, s)(\alpha) \}_{s \in \mathbb{N}^d} \subseteq \ell^p(\mathbb{N}^d);
   \]
2. if
   \[
   \{ c(F,\alpha)/\partial_{\kappa_d, s}(r_1, s)(\alpha) \}_{s \in \mathbb{N}^d} \subseteq \ell^p(\mathbb{N}^d)
   \]
then \( \|F e^{-M_1 r_0 s}\|_{L^p_1} < \infty \).
Proof. By replacing \( r_1 \) with a slightly larger element in \( \mathbb{R}_+^d \), \( r_2 \) with a slightly smaller element in \( \mathbb{R}_+^d \) and using the fact that \( \{ e^{-M_{1,r_0,s}} \}_{r_0 \in \mathbb{R}_+^d} \) is an admissible family of weights in combination with Theorem 3.2 in [31], we reduce ourselves to the case when \( p_1 = p_2 = 2 \). Furthermore, by Lemma 1.7 we may assume that \( d = 1 \).

By Theorem 1.6 we have
\[
\| F \cdot e^{-M_{1,r_0,s}} \|_{L^2}^2 = \sum_{\alpha \in \mathbb{N}} |c(F, \alpha) \vartheta_{r_0}(\alpha)|^2,
\]
where
\[
\vartheta_{r_0}(\alpha) = \left( \frac{\pi}{2\alpha!} \int_0^\infty e^{-r_0(\log(t))^{\frac{1}{2}} t^\alpha} \, dt \right)^{\frac{1}{2}}.
\]

By the first inequality in (15) in [12] it follows that
\[
\vartheta_{r_0}(\alpha) \gtrsim e^{\kappa_{1,s}(r_2) \alpha^{\frac{1}{2}}}.
\]

In order to prove the opposite inequality
\[
\vartheta_{r_0}(\alpha) \lesssim e^{\kappa_{1,s}(r_1) \alpha^{\frac{1}{2}}},
\] (2.19)
we shall modify the proof of the second inequality in (15) in [12]. Let
\[
\theta = \frac{1}{1 - 2s}
\]
and
\[
g_\alpha(t) = e^{-r_1(\log t)^{\theta}} t^\alpha.
\]

Then
\[
\vartheta_{r_0}(\alpha)^2 \lesssim \int_e^\infty e^{(r_1-r_0)(\log t)\theta} g_\alpha(t) \, dt
\]
\[
\lesssim \sup_{t \geq e} (g_\alpha(t)) \int_e^\infty e^{(r_1-r_0)(\log t)\theta} \, dt \approx \sup_{t \geq e} (g_\alpha(t)).
\]

By straight-forward computations it follows that \( g_\alpha(t) \) attains its global maximum for
\[
t_\alpha = \exp \left( \frac{\alpha}{\theta r_1} \right),
\]
and that
\[
g(t_\alpha) = \exp \left( \frac{\theta - \frac{1}{2}\theta(\theta - 1)\alpha^{\frac{1}{2}}}{r_1^{\frac{1}{2} - 1}} \right) = e^{\kappa_{1,s}(r_1) \alpha^{\frac{1}{2}}},
\]
and (2.19) follows. \( \square \)

**Theorem 2.14.** Let \( r_0, r_1, r_2 \in \mathbb{R}_+^d \) be such that \( r_1 < r_2, s < \frac{1}{2}, \) and let \( \Omega_{3,r,s}(\mathbb{C}^d) \) be the same as in (2.3) and (2.4). Also let \( \chi_r \) be the characteristic function of \( D_{d,r} \). Then the following is true:

1. if \( k \in \{3,4\} \) and \( F \in \Omega_{k,r_1,s}(\mathbb{C}^d) \), then (2.13) holds for some \( F_0 \in \Omega_{k,r_2,s}(\mathbb{C}^d) \);
(2) If \( F_0 \in \Omega_{3,r_1,s}(\mathbb{C}^d) \) and \( F \) is given by (2.13), then \( F \in \Omega_{3,r_1,s}(\mathbb{C}^d) \).

(3) If \( F_0 \in \Omega_{4,r_1,s}(\mathbb{C}^d) \) and \( F \) is given by (2.13), then \( F \in \Omega_{4,r_2,s}(\mathbb{C}^d) \).

Proof. By Proposition 2.12 it suffices to prove (1) in the case \( k = 3 \), and (2).

Let \( \kappa_{d,s} \) be the same as in Proposition 2.12 and suppose \( F \in \Omega_{3,r_1,s}(\mathbb{C}^d) \). Then

\[
|c(F_0, \alpha)| = |c(F, \alpha)\alpha!r_0^{-2\alpha}\rho(\alpha)| \lesssim \alpha!r_0^{-2\alpha}\rho(\alpha)\vartheta_{s,\kappa_{d,s}}(r_1)(\alpha) 
\]

\[
\lesssim \vartheta_{s,\kappa_{d,s}}(r_2)(\alpha) \quad (2.20)
\]

giving that \( F_0 \in \Omega_{3,\kappa_{d,s}(r_2),s}(\mathbb{C}^d) \), which gives (1).

Assume instead \( F_0 \in \Omega_{3,\kappa_{d,s}(r_1),s}(\mathbb{C}^d) \). Then

\[
|c(F, \alpha)| = (\alpha!^{-1}r_0^{-2\alpha}\rho(\alpha)^{-1})|c(F_0, \alpha)| \lesssim |c(F_0, \alpha)| \lesssim \vartheta_{s,\kappa_{d,s}}(r_1)(\alpha)
\]

which is the same as \( F \in \Omega_{3,\kappa_{d,s}(r_1),s} \), and (2) and thereby the result follows. \( \square \)

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