INVARIANT MANIFOLDS WITH BOUNDARY FOR JUMP-DIFFUSIONS

DAMIR FILIPOVIĆ, STEFAN TAPPE, AND JOSEF TEICHMANN

Abstract. We provide necessary and sufficient conditions for stochastic invariance of finite dimensional submanifolds with boundary in Hilbert spaces for stochastic partial differential equations driven by Wiener processes and Poisson random measures. Several examples illustrate our results.

1. Introduction

Consider a stochastic partial differential equation (SPDE) of the form

\[
\begin{align*}
    dr_t &= (Ar_t + \alpha(r_t))dt + \sigma(r_t)dW_t + \int_E \gamma(r_{t-}, x)(\mu(dt, dx) - F(dx) dt) \\
    r_0 &= h_0
\end{align*}
\]

on a separable Hilbert space \( H \) driven by some trace class Wiener process \( W \) on a separable Hilbert space \( \mathbb{H} \) and a compensated Poisson random measure \( \mu \) on some mark space \( E \) with \( dt \otimes F(dx) \) being its compensator. Throughout this paper, we assume that \( A \) is the generator of a \( C_0 \)-semigroup on \( H \) and that the mappings \( \alpha, \sigma = (\sigma^j)_{j \in \mathbb{N}} \) and \( \gamma \) satisfy appropriate regularity conditions.

Given a finite dimensional \( C^3 \)-submanifold \( M \) with boundary of \( H \), we study the stochastic viability and invariance problem related to the SPDE (1.1). In particular, we provide necessary and sufficient conditions such that for each \( h_0 \in M \) there is a (local) mild solution \( r \) to (1.1) with \( r_0 = h_0 \) which stays (locally) on the submanifold \( M \).

Any finite dimensional invariant submanifold \( M \) for the SPDE (1.1) gives rise to a finite dimensional Markovian realization of the respective particular solution processes \( r \) with initial values in \( M \), i.e. a deterministic \( C^3 \)-function \( G \) and a finite dimensional Markov process \( X \) such that \( r_t = G(X_t) \) up to some stopping time. This proves to be very useful in applications, since it renders the stochastic evolution model (1.1) analytically and numerically tractable for initial values in \( M \). An important example is the so called Heath-Jarrow-Morton (HJM) SPDE that describes the evolution of the interest rate curve. Stochastic invariance for the HJM SPDE has been discussed in detail in [3, 4, 5, 10, 11, 16, 17, 23] for the diffusion case. The present paper completes the results from [12, 16, 17] by providing explicit stochastic invariance conditions for the general case of an SPDE with jumps.

Stochastic invariance has been extensively studied also for other sets than manifolds. In finite dimension the general stochastic invariance problem for closed sets has been treated, e.g., in [17] in the diffusion case, and in [27] in the case of jump-diffusions. In infinite dimension we mention, e.g., the works of [22, 28].

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where stochastic invariance has been established by means of support theorems for diffusion-type SPDEs.

We shall now present and explain the invariance conditions which we derive in this paper. Let us first consider the situation where the jumps in (1.1) are of finite variation. Then, the conditions

\[ M \subset D(A), \] (1.2)

\[ \sigma^j(h) \in \begin{cases} T_hM, & h \in M \setminus \partial M, \\ T_{h_0}\partial M, & h \in \partial M, \end{cases} \quad \text{for all } j \in \mathbb{N} \] (1.3)

\[ h + \gamma(h, x) \in \overline{M} \] for F-almost all \( x \in E \), for all \( h \in M \) (1.4)

\[ Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) \] (1.5)

are necessary and sufficient for stochastic invariance of \( M \) for (1.1).

Condition (1.2) says that the submanifold \( M \) lies in the domain of the infinitesimal generator \( A \). This ensures that the mapping in (1.5) is well-defined. Condition (1.3) means that the volatilities \( h \mapsto \sigma^j(h) \) must be tangential to \( M \) in its interior and tangential to the boundary \( \partial M \) at boundary points. Condition (1.4) says that the functions \( h \mapsto h + \gamma(h, x) \) map the submanifold \( M \) into its closure \( \overline{M} \). Condition (1.5) means that the adjusted drift must be tangential to \( M \) in its interior and additionally inward pointing at boundary points.

In the general situation, where the jumps in (1.1) may be of infinite variation, condition (1.5) is replaced by the three conditions

\[ \int_E |⟨\eta_h, \gamma(h, x)⟩| F(dx) < \infty, \quad h \in \partial M \] (1.6)

\[ Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} ⟨\eta_h, D\sigma^j(h)\sigma^j(h)⟩ \]

\[ - \int_E \langle \eta_h, \gamma(h, x)⟩ F(dx) \in \left\{ T_hM, \quad h \in M \setminus \partial M \right\} \]

\[ - \int_E \langle \eta_h, \gamma(h, x)⟩ F(dx) \geq 0, \quad h \in \partial M \] (1.7)

where \( \eta_h \) denotes the inward pointing normal (tangent) vector to \( \partial M \) at boundary points \( h \in \partial M \).

Condition (1.6) concerns the small jumps of \( r \) at the boundary of the submanifold and means that the discontinuous part of the solution must be of finite variation, unless it is parallel to the boundary \( \partial M \). Denoting by \( \Pi_K \) the orthogonal projection on a closed subspace \( K \subset H \), we decompose

\[ \gamma(h, x) = \Pi_{T_hM} \gamma(h, x) + \Pi_{(T_hM)^\perp} \gamma(h, x). \]

As we will show, condition (1.4) implies

\[ \int_E ||\Pi_{(T_hM)^\perp} \gamma(h, x)|| F(dx) < \infty, \quad h \in M. \] (1.8)
The essential idea is to perform a second order Taylor expansion for a parametrization \( \phi \) around \( h \) to obtain
\[
\| \Pi_{(T_h,\mathcal{M})} \gamma(h, x) \| = \| \gamma(h, x) - \Pi_{T_h,\mathcal{M}} \gamma(h, x) \| \leq C \| \gamma(h, x) \|^2
\]
for some constant \( C \geq 0 \). By virtue of \( \text{(1.9)} \), the integral in \( \text{(1.7)} \) exists, and hence, conditions \( \text{(1.7)}, \text{(1.8)} \) correspond to \( \text{(1.5)} \).

As in previous papers on this subject we are dealing with mild solutions of SPDEs, i.e. stochastic processes taking values in a Hilbert space whose drift characteristic is quite irregular (e.g., not continuous with respect to the state variables). Therefore, the arguments to translate stochastic invariance into conditions on the characteristics are not straightforward. The arguments to prove our stochastic invariance results can be structured as follows: First, we show that we can (pre-)localize the problem by separating big and small jumps. Second, prelocal invariance of parametrized submanifolds can be pulled back to \( \mathbb{R}^m \) by a linear projection argument tracing back to [13]. Both steps require a careful analysis of jump structures, which leads to the involved invariance conditions.

The remainder of this paper is organized as follows: We state our main results in Section 2. In Section 3 we prove invariance results for submanifolds with one chart, and in Section 4 we prove the main results. In Section 5 we apply our results to the particular situation, where the Poisson random measure is generated by finitely many independent Lévy processes, and in Section 6 we present several examples illustrating our results. For the convenience of the reader, we provide the prerequisites on SPDEs in Appendix A and on finite dimensional submanifolds with boundary in Appendix B. For the sake of lucidity, we postpone the proofs of some auxiliary results to Appendix C.

2. Statement of the main results

In this section we introduce the necessary terminology and state our main results. We fix a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual conditions. In Appendix A below we review some basic facts about SPDEs of the type \( \text{(1.1)} \) and we recall the concepts of (local) strong, weak and mild solutions. In particular, in view of \( \text{(A.2)} \), equation \( \text{(1.1)} \) can be rewritten equivalently
\[
\begin{align*}
\begin{cases}
  dr_t &= (Ar_t + \alpha(r_t))dt + \sum_{j \in \mathbb{N}} \sigma_j(r_t)d\beta^j_t \\
r_0 &= h_0,
\end{cases}
\end{align*}
\]
where \( (\beta^j)_{j \in \mathbb{N}} \) is a sequence of real-valued independent standard Wiener processes.

We next formulate the concept of stochastic invariance.

2.1. Definition. A non-empty Borel set \( B \subset H \) is called prelocally (locally) invariant for \( \text{(2.1)} \), if for all \( h_0 \in B \) there exists a local mild solution \( r = r^{(h_0)} \) to \( \text{(2.1)} \) with lifetime \( \tau > 0 \) such that up to an evanescent set
\[
(r^\tau)_- \in B \text{ and } r^\tau \in B
\]

where \( r^\tau \in B \).

The following standing assumptions prevail throughout this paper:

- A generates a \( C_0 \)-semigroup \((S_t)_{t \geq 0}\) on \( H \).
- The mapping \( \alpha : H \to H \) is locally Lipschitz continuous, that is, for each \( n \in \mathbb{N} \) there is a constant \( L_n \geq 0 \) such that
\[
\| \alpha(h_1) - \alpha(h_2) \| \leq L_n \| h_1 - h_2 \|, \quad h_1, h_2 \in H \text{ with } \| h_1 \|, \| h_2 \| \leq n.
\]

\footnote{A random set \( A \subset \Omega \times \mathbb{R}_+ \) is called evanescent if the set \( \{ \omega : (\omega, t) \in A \text{ for some } t \in \mathbb{R}_+ \} \) is a \( \mathbb{P} \)-nullset, cf. [19] 1.1.10.}"
• For each $n \in \mathbb{N}$ there exists a sequence $(\kappa_n^j)_{j \in \mathbb{N}} \subset \mathbb{R}_+$ with $\sum_{j \in \mathbb{N}} (\kappa_n^j)^2 < \infty$ such that for all $j \in \mathbb{N}$ the mapping $\sigma^j : H \to H$ satisfies

\begin{align}
(2.3) & \quad \|\sigma^j(h_1) - \sigma^j(h_2)\| \leq \kappa_n^j \|h_1 - h_2\|, \quad h_1, h_2 \in H \text{ with } \|h_1\|, \|h_2\| \leq n, \\
(2.4) & \quad \|\sigma^j(h)\| \leq \kappa_n^j, \quad h \in H \text{ with } \|h\| \leq n.
\end{align}

Consequently, for each $j \in \mathbb{N}$ the mapping $\sigma^j$ is locally Lipschitz continuous.

• The mapping $\gamma : H \times E \to H$ is measurable, and for each $n \in \mathbb{N}$ there exists a measurable function $\rho_n : E \to \mathbb{R}_+$ with

\begin{align}
(2.5) & \quad \int_E \left(\rho_n(x)^2 \vee \rho_n(x)^4\right) F(dx) < \infty
\end{align}

such that for all $x \in E$ the mapping $\gamma(\bullet, x) : H \to H$ satisfies

\begin{align}
(2.6) & \quad \|\gamma(h_1, x) - \gamma(h_2, x)\| \leq \rho_n(x) \|h_1 - h_2\|, \quad h_1, h_2 \in H \text{ with } \|h_1\|, \|h_2\| \leq n, \\
(2.7) & \quad \|\gamma(h, x)\| \leq \rho_n(x), \quad h \in H \text{ with } \|h\| \leq n.
\end{align}

Consequently, for each $x \in E$ the mapping $\gamma(\bullet, x)$ is locally Lipschitz continuous.

• We assume that for each $j \in \mathbb{N}$ the mapping $\sigma^j : H \to H$ is continuously differentiable, that is

\begin{align}
(2.8) & \quad \sigma^j \in C^1(H) \quad \text{for all } j \in \mathbb{N}.
\end{align}

The first four conditions ensure that we may apply the results about SPDEs from Appendix A. We furthermore assume that:

- $\mathcal{M}$ is a finite-dimensional $C^2$-submanifold with boundary of $H$, see Appendix B.

Our first main result now reads as follows.

2.2. Theorem. The following statements are equivalent:

1. $\mathcal{M}$ is prelocally invariant for (2.1).
2. We have (1.2), (1.4) and (1.6)–(1.8).

In either case, $A$ and the mapping in (1.7) are continuous on $\mathcal{M}$, and for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1). Moreover, if instead of (1.4) we even have

\begin{align}
(2.9) & \quad h + \gamma(h, x) \in \mathcal{M} \quad \text{for } F - \text{almost all } x \in E, \quad \text{for all } h \in \mathcal{M},
\end{align}

then $\mathcal{M}$ is locally invariant for (2.1).

2.3. Remark. It follows from Theorem 2.2 that (pre-)local invariance of $\mathcal{M}$ is a property which only depends on the parameters $\{\alpha, \rho^j, \gamma, F\}$ – that is, on the law of the solution to (2.1). It does not depend on the actual stochastic basis $\{(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), W, \mu\}$.

Note that local invariance of $\mathcal{M}$ does not imply (2.9), as the following example illustrates:

2.4. Example. Let $H = \mathbb{R}$, $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\mathcal{M} = [0, 1)$ and consider the SDE

\begin{align}
(2.10) & \quad \begin{cases}
\, dr_t = dt + \int_{\mathbb{R}} \gamma(r_{t-}, x) \mu(dt, dx) \\
\, r_0 = h_0,
\end{cases}
\end{align}

where the compensator $dt \otimes F(dx)$ of $\mu$ is given by the Dirac measure $F = \delta_{\{1\}}$ and $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\gamma(h, x) = 1 - 2h$. 

Then, $\mathcal{M}$ is locally invariant for (2.10). Indeed, let $h_0 \in \mathcal{M}$ be arbitrary. There exists $\epsilon > 0$ with $h_0 + \epsilon < 1$. We define the stopping time $\tau > 0$ as

$$\tau := \inf\{t \geq 0 : r_t = h_0 + \epsilon\} \wedge \inf\{t \geq 0 : \mu([0,t] \times \mathbb{R}) = 1\}.$$ 

Then we have $(r^{(h_0)})^\tau \in \mathcal{M}$ up to an evanescent set, because $h + \gamma(h,x) = 1 - h \in \mathcal{M}$, $h \in (0,1)$ showing that $\mathcal{M}$ is locally invariant for (2.10). However, the jump condition (2.9) is not satisfied, because for $h = 0$ we have $h + \gamma(h,x) \not\in \mathcal{M}$.

Nevertheless, we see that condition (1.4) holds true, because $1 \in \mathcal{M}$.

If $\mathcal{M}$ is a closed subset of $H$ and global Lipschitz conditions are satisfied, then we obtain global invariance. This is the content of our second main result. Recall that the semigroup $(S_t)_{t \geq 0}$ is called pseudo-contractive if

$$\|S_t\| \leq e^{\omega t}, \quad t \geq 0$$

for some $\omega \in \mathbb{R}$.

2.5. **Theorem.** Assume that the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive and that conditions (2.2)–(2.7) hold globally, i.e. the coefficients $L_n, (\kappa_n^j)_{j \in \mathbb{N}}, \mu_n$ do not depend on $n \in \mathbb{N}$, and with the right-hand sides of (2.4), (2.7) multiplied by $(1 + \|h\|)$. If $\mathcal{M}$ is a closed subset of $H$, then (1.2)–(1.4) and (1.6)–(1.8) imply that for any $h_0 \in \mathcal{M}$ there exists a unique strong solution $r = r^{(h_0)}$ to (2.1) and $r \in \mathcal{M}$ up to an evanescent set.

The above two theorems simplify in the case of jumps with finite variation:

2.6. **Theorem.** Suppose for each $n \in \mathbb{N}$ there exists a measurable function $\theta_n : E \to \mathbb{R}^+$ with $\int_E \theta_n(x)F(dx) < \infty$ such that

$$\|\gamma(h,x)\| \leq \theta_n(x) \quad \text{for all } h \in \mathcal{M} \text{ with } \|h\| \leq n \text{ and all } x \in E.$$ 

Then, Theorems 2.2 and 2.5 remain true with (1.6)–(1.8) being replaced by (1.5) and the mapping in (1.7) being replaced by the mapping in (1.5).

2.7. **Remark.** We shall briefly comment on our assumptions:

- The assumption that the submanifold $\mathcal{M}$ is of class $C^3$ is a technical assumption, which we require for the proof of Lemma 3.3. Theorems 2.2, 2.5 and 2.6 also hold true for $C^2$-submanifolds, but in this case the proof of Proposition 3.13 below is more involved, because we only obtain the existence of martingale solutions to (3.20).

- In Theorem 2.5 the assumption that the semigroup $(S_t)_{t \geq 0}$ is pseudo-contractive is also a technical assumption, which allows us to apply Theorem A.6 ensuring existence of mild solutions to (2.1) with càdlàg paths. Theorem 2.5 holds true for every $C_0$-semigroup, but then the proof becomes more involved, because we only obtain the existence of mild solutions to (2.1) which might not have a càdlàg version.

The following results supplement Theorems 2.2, 2.5 and 2.6 by providing necessary conditions for (pre-)local stochastic invariance. If the mark space $E$ is a Banach space, then the support of the measure $F$ is defined as

$$\text{supp}(F) := \{x \in E : F(B_\epsilon(x)) > 0 \text{ for all } \epsilon > 0\},$$

where $B_\epsilon(x)$ denotes the open ball $B_\epsilon(x) := \{\xi \in E : \|\xi - x\| < \epsilon\}$. 


2.8. **Proposition.** Suppose condition (1.4) is satisfied and the mark space $E$ is a Banach space. Let $h \in \mathcal{M}$ and $x \in E$ be such that $\gamma(h, \bullet) : E \to H$ is continuous in a neighborhood of $x$ and differentiable in $x$ with $\gamma(h, x) = 0$. Then, for every direction $v \in E$, $v \neq 0$ the following statements are true:

1. Suppose there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \to 0$ such that
   \[ \{x + t_n v : n \in \mathbb{N}\} \subset \text{supp}(F) \quad \text{or} \quad \{x - t_n v : t \in \mathbb{N}\} \subset \text{supp}(F). \]
   Then, we have
   \[ D_x \gamma(h, x) v \in T_h \mathcal{M}. \]

2. Suppose $h \in \partial \mathcal{M}$ and there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \to 0$ such that
   \[ \{x + t_n v : n \in \mathbb{N}\} \subset \text{supp}(F). \]
   Then, we have
   \[ D_x \gamma(h, x) v \in (T_h \mathcal{M})_+. \]

3. Suppose $h \in \partial \mathcal{M}$ and there exist sequences $(t_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n, s_n \to 0$ such that
   \[ \{x + t_n v : n \in \mathbb{N}\} \subset \text{supp}(F) \quad \text{and} \quad \{x - s_n v : t \in \mathbb{N}\} \subset \text{supp}(F). \]
   Then, we have
   \[ D_x \gamma(h, x) v \in T_h \partial \mathcal{M}. \]

Further examples illustrating our main results from this section are provided in Sections 5 and 6 below.

### 3. Invariance results for submanifolds with one chart

In this section, we shall prove invariance results for submanifolds with one chart. First, we provide a stronger invariance property than Definition 2.1. Let $\tau_0$ be a bounded stopping time and consider the time-shifted version of (2.1)

\[
\begin{align*}
\begin{cases}
    dr_t &= (Ar_t + \alpha(r_t))dt + \sum_{j \in \mathbb{N}} \sigma^j(r_t) d\beta_t^{(\tau_0,j)} + \int_E \gamma(r_t - , x)(\mu^{(\tau_0)}(dx)dt - F(dx)dt) \\
    r_0 &= h_0
\end{cases}
\end{align*}
\]

for some $\mathcal{F}_{\tau_0}$-measurable random variable $h_0 : \Omega \to H$. In (3.1), the sequence $(\beta_t^{(\tau_0,j)})_{j \in \mathbb{N}}$ is a sequence of real-valued independent standard Wiener processes and $\mu^{(\tau_0)}$ is a time-homogeneous Poisson random measure, both relative to the filtration $(\mathcal{F}_{\tau_0+t})_{t \geq 0}$. We refer to Appendix A for further details.

3.1. **Definition.** A non-empty Borel set $B \subset H$ is called locally strong invariant for (3.1), if for each bounded stopping time $\tau_0$ and each bounded $\mathcal{F}_{\tau_0}$-measurable random variable $h_0 : \Omega \to H$ with $\mathbb{P}(h_0 \in B) = 1$ there exists a local mild solution $r = r^{(\tau_0, h_0)}$ to (3.1) with lifetime $\tau > 0$ such that $r^\tau \in B$ up to an evanescent set.

For technical reasons, we will also need the following concepts of prelocal (strong) invariance:

3.2. **Definition.** Let $B_1 \subset B_2 \subset H$ be two nonempty Borel sets.

1. $B_1$ is called prelocally invariant in $B_2$ for (2.1), if for all $h_0 \in B_1$ there exists a local mild solution $r = r^{(h_0)}$ to (2.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \in B_2$ up to an evanescent set.
(2) $B_1$ is called prelocally strong invariant in $B_2$ for (3.1), if for each bounded stopping time $\tau_0$ and each bounded $\mathcal{F}_{\tau_0}$-measurable random variable $h_0 : \Omega \to H$ with $\mathbb{P}(h_0 \in B_1) = 1$ there exists a local mild solution $r = r^{(\tau_0, h_0)}$ to (3.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \in B_2$ up to an evanescent set.

Note that any non-empty Borel set $B \subset H$ is prelocally (strong) invariant for (2.1) in the sense of Definition 2.1 (Definition 3.1) if and only if $B$ is prelocally (strong) invariant in $B$ for (2.1) in the sense of Definition 3.2.

Now, let $G$ be another separable Hilbert space. For any $k \in \mathbb{N}$ we denote by $C^k_b(G ; H)$ the linear space consisting of all $f \in C^k_b(G ; H)$ such that $D^i f$ is bounded for all $i = 1, \ldots, k$. In particular, for each $f \in C^k_b(X ; Y)$ the mappings $D^i f$, $i = 0, \ldots, k - 1$ are Lipschitz continuous. We do not demand that $f$ itself is bounded, as this would exclude continuous linear operators $f \in L(G, H)$.

3.3. Definition. Let $\alpha : H \to H$, $\sigma^j : H \to H$, $j \in \mathbb{N}$ and $\gamma : H \times E \to H$ be mappings satisfying

\begin{equation}
\sum_{j \in \mathbb{N}} \|\sigma^j(h)\|^2 < \infty, \quad h \in H
\end{equation}

\begin{equation}
\int_E \|\gamma(h, x)\|^2 F(dx) < \infty, \quad h \in H
\end{equation}

and let $f : G \to H$ and $g \in C^2_b(H;G)$ be mappings. We define the mappings $(f,g)^\star \alpha : G \to G$, $(f,g)^\star \sigma^j : G \to G$, $j \in \mathbb{N}$ and $(f,g)^\star \gamma : G \times E \to G$ as

\[ ((f,g)^\star \alpha)(x) := Dg(h)\alpha(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h)) \]

\[ + \int_E (g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x)) F(dx), \]

\[ ((f,g)^\star \sigma^j)(x) := Dg(h)\sigma^j(h), \]

\[ ((f,g)^\star \gamma)(z, x) := g(h + \gamma(h, x)) - g(h), \]

where $h = f(z)$.

3.4. Remark. Note that the mapping $(f,g)^\star \alpha$ is well-defined. Indeed, for any $h \in H$, by (3.3) we have

\[ \sum_{j \in \mathbb{N}} \|D^2 g(h)(\sigma^j(h), \sigma^j(h))\| \leq \|D^2 g(h)\| \sum_{j \in \mathbb{N}} \|\sigma^j(h)\|^2 < \infty, \]

and by (3.3) and Taylor’s theorem we have

\[ \int_E \|g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x)\|^2 F(dx) \]

\[ \leq \frac{1}{2} \|D^2 g\| \int_E \|\gamma(h, x)\|^2 F(dx) < \infty. \]

3.5. Lemma. Let $\alpha : H \to H$, $\sigma^j : H \to H$, $j \in \mathbb{N}$ and $\gamma : H \times E \to H$ be mappings satisfying the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). Then, the following statements are true:

1. The mappings $(f,g)^\star \alpha$, $(f,g)^\star \sigma^j$, $(f,g)^\star \gamma$ also fulfill the regularity conditions (2.2)–(2.4) and (2.6)–(2.8) with the mappings $\rho_n : E \to \mathbb{R}_+$, $n \in \mathbb{N}$ appearing in (2.6), (2.7) satisfying

\begin{equation}
\int_E \rho_n(x)^2 F(dx) < \infty.
\end{equation}
Proof. We define the mappings $\hat{a} : H \rightarrow G$, $\hat{b}^j : H \rightarrow G$, $j \in \mathbb{N}$ and $\hat{c} : H \times E \rightarrow G$ as

$$
\hat{a}(h) := \hat{a}_1(h) + \hat{a}_2(h) + \hat{a}_3(h),
$$

$$
\hat{b}^j(h) := Dg(h)\sigma^j(h),
$$

$$
\hat{c}(h, x) := g(h + \gamma(h, x)) - g(h),
$$

where $\hat{a}_1, \hat{a}_2, \hat{a}_3 : H \rightarrow G$ are given by

$$
\hat{a}_1(h) := Dg(h)\alpha(h),
$$

$$
\hat{a}_2(h) := \frac{1}{2} \sum_{j \in \mathbb{N}} D^2g(h)(\sigma^j(h), \sigma^j(h)),
$$

$$
\hat{a}_3(h) := \int_E (g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x))F(dx).
$$

Then we have $\hat{b}^j \in C^1(H; G)$ for all $j \in \mathbb{N}$. By Taylor’s theorem, we have the representations

$$
\hat{a}_3(h) = \int_E \int_0^1 (1-t)D^2g(h + t\gamma(h, x))(\gamma(h, x), \gamma(h, x))dtF(dx), \quad h \in H
$$

(3.6) $\hat{c}(h, x) = \int_0^1 Dg(h + t\gamma(h, x))\gamma(h, x)dt, \quad (h, x) \in H \times E.

Let $n \in \mathbb{N}$ be arbitrary. Furthermore, let $h \in H$ with $\|h\| \leq n$ be arbitrary. By (2.4), for all $j \in \mathbb{N}$ we have

$$
\|\hat{b}^j(h)\| \leq \|Dg\|_\infty \|\sigma^j(h)\| \leq \|Dg\|_\infty \kappa^j_n,
$$

and by (2.7) and the representation (3.6), for all $x \in E$ we have

$$
\|\hat{c}(h, x)\| \leq \int_0^1 \|Dg\|_\infty \|\gamma(h, x)\|dt \leq \|Dg\|_\infty \rho_n(x).
$$

Now, let $h_1, h_2 \in H$ with $\|h_1\|, \|h_2\| \leq n$ be arbitrary. Using estimate (2.2), we obtain

$$
\|\hat{a}_1(h_1) - \hat{a}_1(h_2)\| = \|Dg(h_1)\alpha(h_1) - Dg(h_2)\alpha(h_2)\|
\leq \|Dg(h_1)\alpha(h_1) - Dg(h_2)\alpha(h_2)\| + \|Dg(h_2)\alpha(h_1) - Dg(h_2)\alpha(h_2)\|
\leq (\|D^2g\|_\infty \{L_n \kappa_1 + \|\alpha(0)\|\} + \|Dg\|_\infty L_n)\|h_1 - h_2\|.
$$

Moreover, we have

$$
\|\hat{a}_2(h_1) - \hat{a}_2(h_2)\| \leq \sum_{j \in \mathbb{N}} \|D^2g(h_1)(\sigma^j(h_1), \sigma^j(h_1)) - D^2g(h_2)(\sigma^j(h_2), \sigma^j(h_2))\|
\leq \|D^2g(h_1)\| \sum_{j \in \mathbb{N}} \|\sigma^j(h_1)\| \|\sigma^j(h_1) - \sigma^j(h_2)\|
\leq \|D^2g(h_1)\| \sum_{j \in \mathbb{N}} \|\sigma^j(h_1)\| \|\sigma^j(h_2)\|
\leq \|D^2g(h_2)\| \sum_{j \in \mathbb{N}} \|\sigma^j(h_1) - \sigma^j(h_2)\| \|\sigma^j(h_2)\|.
$$

By estimates (2.3), (2.4) we obtain

$$
\|\hat{a}_2(h_1) - \hat{a}_2(h_2)\| \leq (2\|D^2g\|_\infty + \|D^3g\|_\infty \sum_{j \in \mathbb{N}} (\kappa_j^j)^2)\|h_1 - h_2\|.$$

Furthermore, we have
\[
\|\hat{a}_3(h_1) - \hat{a}_3(h_2)\| \leq \int_E \int_0^1 \|D^2g(h_1 + t\gamma(h_1,x))(\gamma(h_1,x), h_1) - D^2g(h_2 + t\gamma(h_2,x))(\gamma(h_2,x), h_2)\|dtF(dx)
\]
\[
\leq \int_E \int_0^1 \|D^2g(h_1 + t\gamma(h_1,x))\| \|\gamma(h_1,x)\| \|\gamma(h_1,x) - \gamma(h_2,x)\|dtF(dx)
\]
\[
+ \int_E \int_0^1 \|D^2g(h_1 + t\gamma(h_1,x)) - D^2g(h_2 + t\gamma(h_2,x))\|
\times \|\gamma(h_1,x)\| \|\gamma(h_2,x)\|dtF(dx)
\]
\[
+ \int_E \int_0^1 \|D^2g(h_2 + t\gamma(h_2,x))\| \|\gamma(h_1,x) - \gamma(h_2,x)\| \|\gamma(h_2,x)\|dtF(dx).
\]

Noting that, by (2.6), for all \((x,t) \in E \times [0,1]\) we have
\[
\|D^2g(h_1 + t\gamma(h_1,x)) - D^2g(h_2 + t\gamma(h_2,x))\|
\leq \|D^3g\| \|h_1 + t\gamma(h_1,x) - h_2 - t\gamma(h_2,x)\|
\leq \|D^3g\| \|h_1 - h_2\| \|\gamma(h_1,x) - \gamma(h_2,x)\|
\leq \|D^3g\| \|1 + \rho_n(x)\| \|h_1 - h_2\|,
\]
using estimates (2.6), (2.7) we get
\[
\|\hat{a}_3(h_1) - \hat{a}_3(h_2)\|
\leq \left(2\|D^2g\| \|D^3g\| \|\rho_n(x)^2 F(dx)\| + \|D^3g\| \|D^2g\| \|\rho_n(x)^3 F(dx)\|\right) \|h_1 - h_2\|.
\]

By estimates (2.3), (2.4), for all \(j \in \mathbb{N}\) we obtain
\[
\|\hat{b}_j(1) - \hat{b}_j(2)\| = \|Dg(h_1)\sigma_j(h_1) - Dg(h_2)\sigma_j(h_2)\|
\leq \|Dg(h_1) - Dg(h_2)\| \|\sigma_j(h_1)\| + \|Dg(h_2)\| \|\sigma_j(h_1) - \sigma_j(h_2)\|
\leq (\|D^2g\| \|D^3g\| \kappa_n^j \|h_1 - h_2\|)
\]
For all \(x \in E\) we obtain
\[
\|\hat{c}_j(h_1,x) - \hat{c}_j(h_2,x)\|
\leq \int_0^1 \|Dg(h_1 + t\gamma(h_1,x))\gamma(h_1,x) - Dg(h_2 + t\gamma(h_2,x))\gamma(h_2,x)\|dt
\leq \int_0^1 \|Dg(h_1 + t\gamma(h_1,x)) - Dg(h_2 + t\gamma(h_2,x))\| \|\gamma(h_1,x)\|dt
\]
\[
+ \int_0^1 \|Dg(h_2 + t\gamma(h_2,x))\| \|\gamma(h_1,x) - \gamma(h_2,x)\|dt.
\]

Arguing as in (3.7), for all \((x,t) \in E \times [0,1]\) we have
\[
\|Dg(h_1 + t\gamma(h_1,x)) - Dg(h_2 + t\gamma(h_2,x))\| \leq \|D^2g\| \|1 + \rho_n(x)\| \|h_1 - h_2\|.
\]

Using estimates (2.6), (2.7), we obtain
\[
\|\hat{c}_j(h_1,x) - \hat{c}_j(h_2,x)\| \leq \left(\|D^2g\| \|\rho_n(x)^2 + \rho_n(x)^3\| \|Dg\| \|\rho_n(x)^4\|\right) \|h_1 - h_2\|.
\]

Since \((f,g)^* \alpha = \hat{a} \circ f\), \((f,g)^* \sigma^j = \hat{b}_j \circ f\), \(j \in \mathbb{N}\) and \(((f,g)^* \gamma)(\bullet,x) = \hat{c}_j(\bullet,x) \circ f\), \(x \in E\) we deduce that conditions (2.2), (2.4) and (2.6), (2.8) are satisfied with the mappings \(\rho_n : E \rightarrow \mathbb{R}_+, n \in \mathbb{N}\) appearing in (2.6), (2.7) satisfying (3.4). This proves the first statement. If \(g \in L(H,G)\), then we have \(D^2g \equiv 0\), and hence, estimate (3.8) shows that the mappings \(\rho_n : E \rightarrow \mathbb{R}_+, n \in \mathbb{N}\) appearing in (2.6), (2.7) even satisfy (2.5), establishing the second statement. □
Now, let $\mathcal{N}$ be a $C^3$-submanifold with boundary of $G$. We assume there exist parametrizations $\phi : V \to \mathcal{M}$ and $\psi : V \to \mathcal{N}$. Let $f := \phi \circ \psi^{-1} : \mathcal{N} \to \mathcal{M}$ and $g := f^{-1} : \mathcal{M} \to \mathcal{N}$. This is illustrated by the following diagram:

\[
\begin{array}{ccc}
\mathcal{N} \subset G & \xrightarrow{f} & \mathcal{M} \subset H \\
V \subset \mathbb{R}^n & \xrightarrow{\phi} & \mathcal{N} \subset G
\end{array}
\]

We assume that $\phi$, $\psi$, $\Phi := \phi^{-1}$, $\Psi := \psi^{-1}$ have extensions $\phi \in C^3_0(\mathbb{R}^n; H)$, $\psi \in C^3_0(\mathbb{R}^n; G)$, $\Phi \in C^3_0(H; \mathbb{R}^n)$, $\Psi \in C^3_0(G; \mathbb{R}^n)$. Consequently, the mappings $f$, $g$ have extensions $f \in C^3_0(G; H)$, $g \in C^3_0(H; G)$.

Let $O_M \subset C_M \subset \mathcal{M}$ be subsets. We define the subsets $O_N \subset C_N \subset \mathcal{N}$ by $O_N := g(O_M)$ and $C_N := g(C_M)$.

3.6. Definition. Let $\beta : O_M \to H$, $\sigma^j : O_M \to H$, $j \in \mathbb{N}$ and $\gamma : O_M \times E \to H$ be mappings satisfying

\[
\sum_{j \in \mathbb{N}} \|\sigma^j(h)\|^2 < \infty, \quad h \in O_M
\]

\[
\int_E \|\gamma(h, x)\|^2 F(dx) < \infty, \quad h \in O_M.
\]

We define the mappings $f^* \beta : O_N \to G$, $f^* \sigma^j : O_N \to G$, $j \in \mathbb{N}$ and $f^* \gamma : O_N \times E \to G$ as

\[
(f^* \beta)(z) := Dg(h)\beta(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h)) + \int_E \left( g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x) \right) F(dx),
\]

\[
(f^* \sigma^j)(z) := Dg(h)\sigma^j(h),
\]

\[
(f^* \gamma)(z, x) := g(h + \gamma(h, x)) - g(h),
\]

where $h = f(z) \in O_M$.

Now, we consider the $G$-valued SDE

\[
\begin{cases}
\frac{dZ_t}{dt} = a(Z_t)dt + \sum_{j \in \mathbb{N}} b^j(Z_t)d\beta^j_t + \int_E \epsilon(Z_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\
Z_0 = z_0
\end{cases}
\]  

as well as the time-shifted version

\[
\begin{cases}
\frac{dZ_t}{dt} = a(Z_t)dt + \sum_{j \in \mathbb{N}} b^j(Z_t)d\beta^{(\tau_0)^{-1}, j}_t + \int_E \epsilon(Z_{t-}, x)(\mu(\tau_0)(dt, dx) - F(dx)dt) \\
Z_0 = z_0
\end{cases}
\]  

where $a : G \to G$, $b^j : G \to G$, $j \in \mathbb{N}$ and $\epsilon : G \times E \to G$ are mappings satisfying the regularity conditions 2.2, 2.4 and 2.6–2.8.

For our subsequent analysis, the following technical definitions will be useful.

3.7. Definition. The set $O_M$ is called prerelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$, if for all $h_0 \in O_M$ there exists a local strong solution $Z = Z^{g(h_0)}$ to (3.9) with lifetime $\tau > 0$ such that $(Z^\tau)_+ \in O_N$ and $Z^\tau \in C_N$ up to an evanescent set and $f(Z)$ is a local mild solution to (2.1) with initial condition $h_0$ and lifetime $\tau$. 
3.8. **Definition.** The set $O_M$ is called prelocally strong invariant in $C_M$ for (3.1) with solutions given by (3.10) and $f$, if for each bounded stopping time $\tau_0$ and each bounded $F_{\tau_0}$-measurable random variable $h_0 : \Omega \rightarrow H$ with $P(h_0 \in O_M) = 1$ there exists a local strong solution $Z = Z^{(g(h_0))}$ to (3.10) with lifetime $\tau > 0$ such that $(Z^\gamma)_- \in O_N$ and $Z^\gamma \in C_N$ up to an evanescent set and $f(Z)$ is a local mild solution to (3.1) with initial condition $h_0$ and lifetime $\tau$.

The following two auxiliary results are direct consequences of the previous definitions.

3.9. **Lemma.** Suppose $O_M$ is prelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$. Then, the following statements are true:

1. $O_M$ is prelocally invariant in $C_M$ for (2.1).
2. $O_N$ is prelocally invariant in $C_N$ for (3.9).

3.10. **Lemma.** Suppose $O_M$ is prelocally strong invariant in $C_M$ for (3.1) with solutions given by (3.10) and $f$. Then, the following statements are true:

1. $O_M$ is prelocally strong invariant in $C_M$ for (3.1).
2. $O_N$ is prelocally strong invariant in $C_N$ for (3.10).

If the generator $A$ is a continuous, i.e. (2.1) is rather an SDE than an SPDE, then the just introduced invariance concept transfers to the sets $O_N$ and $C_N$.

3.11. **Lemma.** Suppose $A \in L(H)$. Then, the following statements are equivalent:

1. $O_M$ is prelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$.
2. $O_N$ is prelocally invariant in $C_N$ for (3.9) with solutions given by (2.1) and $g$.

**Proof.** (1) $\Rightarrow$ (2): Let $z_0 \in O_N$ be arbitrary and set $h_0 := f(z_0) \in O_M$. There exists a local strong solution $Z = Z^{(g(h_0))} = Z^{(z_0)}$ to (3.9) with lifetime $\tau > 0$ such that $(Z^\gamma)_- \in O_N$ and $Z^\gamma \in C_N$ up to an evanescent set and, since $A \in L(H)$, the process $r = f(Z)$ is a local strong solution to (2.1) with initial condition $h_0 = f(z_0)$. Therefore, we have $(r^\gamma)_- \in O_M$ and $r^\gamma \in C_M$ up to an evanescent set and $g(r)$ is a local strong solution to (2.1) with initial condition $z_0$ and lifetime $\tau$.

(2) $\Rightarrow$ (1): This implication is proven analogously. $\square$

3.12. **Lemma.** Suppose $A \in L(H)$. Then, the following statements are equivalent:

1. $O_M$ is prelocally strong invariant in $C_M$ for (3.1) with solutions given by (3.10) and $f$.
2. $O_N$ is prelocally strong invariant in $C_N$ for (3.10) with solutions given by (3.1) and $g$.

**Proof.** The proof is analogous to that of Lemma 3.11. $\square$

3.13. **Proposition.** The following statements are equivalent:

1. $O_M$ is prelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$.
2. $O_N$ is prelocally invariant in $C_N$ for (3.9) and we have

   \begin{align*}
   (A + a)(h) &= (g^*a)(h) \quad \text{for all } h \in O_M \\
   \sigma^j(h) &= (g^*b^j)(h) \quad \text{for all } j \in \mathbb{N}, \text{ for all } h \in O_M \\
   \gamma(h, x) &= (g^*c)(h,x) \quad \text{for } F\text{-almost all } x \in E, \text{ for all } h \in O_M.
   \end{align*}

In either case, $A$ is continuous on $O_M$. 

Proof. (1) ⇒ (2): By Lemma 3.9, the set $O_N$ is prelocally invariant in $C_N$ for (3.9). Let $h \in O_M$ be arbitrary. Since $O_M$ is prelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$, there exists a local strong solution $Z = Z^{(g,h)}$ to (3.9) with lifetime $\tau > 0$ such that $(Z^\tau)_- \in O_N$ and $Z^\tau \in C_N$ up to an evanescent set and $r := f(Z)$ is a local mild solution to (2.1) with initial condition $h_0$ and lifetime $\tau$. By Itô’s formula (Theorem A.16) we obtain $P$–almost surely

$$r_{t \wedge \tau} = f(Z_{t \wedge \tau}) = h + \int_0^{t \wedge \tau} (g^* a)(r_s)ds + \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} (g^* b^j)(r_s) d\beta_s^j + \int_0^{t \wedge \tau} \int_E (g^* c)(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.$$  

Let $\zeta \in \mathcal{D}(A^*)$ be arbitrary. Since $r$ is also a local weak solution to (2.1) with lifetime $\tau$, we have $P$–almost surely

$$\langle \zeta, r_{t \wedge \tau} \rangle = \langle \zeta, h \rangle + \int_0^{t \wedge \tau} \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle ds + \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} \langle \zeta, \sigma^j(r_s) \rangle d\beta_s^j + \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds), \quad t \geq 0.$$  

Therefore, we get up to an evanescent set

$$B + M^c + M^d = 0,$$

where the processes $B$, $M^c$, $M^d$ are given by

$$B_t := \int_0^{t \wedge \tau} \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) - (g^* a)(r_s) \rangle ds,$$

$$M^c_t := \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau} \langle \zeta, \sigma^j(r_s) \rangle - (g^* b^j)(r_s) d\beta_s^j,$$

$$M^d_t := \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) - (g^* c)(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds).$$

The process $B$ is a finite variation process which is continuous, and hence predictable, $M^c$ is a continuous square-integrable martingale and $M^d$ is a purely discontinuous square-integrable martingale. Therefore $B + M^c + M^d$ is a special semimartingale. Since the decomposition $B + M$ of a special semimartingale into a finite variation process $B$ and a local martingale $M$ is unique (see [19, Cor. I.3.16]) and the decomposition of a local martingale $M = M^c + M^d$ into a continuous local martingale $M^c$ and a purely discontinuous local martingale $M^d$ is unique (see [19, Thm. I.4.18]), we deduce that $B = M^c = M^d = 0$ up to an evanescent set. By the Itô isometry, we obtain $P$–almost surely

$$\int_0^{t \wedge \tau} \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) - (g^* a)(r_s) \rangle ds = 0, \quad t \geq 0$$

$$\int_0^{t \wedge \tau} \sum_{j \in \mathbb{N}} \langle \zeta, \sigma^j(r_s) \rangle - (g^* b^j)(r_s) \rangle d\beta_s^j = 0, \quad t \geq 0$$

$$\int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) - (g^* c)(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds) = 0, \quad t \geq 0.$$
Since the process $r$ is càdlàg, by Lemma 3.5 and Lebesgue’s dominated convergence theorem (applied to the sum $\sum_{j \in \mathbb{N}}$ and to the integral $\int_{E}$) the integrands appearing in (3.18), (3.17) are continuous in $s = 0$, and hence, we get

\begin{align}
&A \sum_{j \in \mathbb{N}} |\langle \zeta, \sigma^j(h) - (g^*b^j)(h) \rangle|^2 = 0, \\
&\int_{E} |\langle \zeta, \gamma(h,x) - (g^*c)(h,x) \rangle|^2 F(dx) = 0.
\end{align}

Identity (3.18) shows that $\zeta \mapsto \langle A^* \zeta, h \rangle$ is continuous on $\mathcal{D}(A^*)$, proving $h \in \mathcal{D}(A^{**})$. Since $A = A^{**}$, see [23, Thm. 13.12], we obtain $h \in \mathcal{D}(A)$, which yields (3.11). Using the identity $\langle A^* \zeta, h \rangle = \langle \zeta, Ah \rangle$, we obtain

$$\langle \zeta, Ah + \alpha(h) - (g^*a)(h) \rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*),$$

and hence (3.12). For an arbitrary $j \in \mathbb{N}$ we obtain, by using (3.19),

$$\langle \zeta, \sigma^j(h) - (g^*b^j)(h) \rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*),$$

showing (3.13). By (3.20), for all $\zeta \in \mathcal{D}(A^*)$ we have

$$\langle \zeta, \gamma(h,x) - (g^*c)(h,x) \rangle = 0 \quad \text{for } F\text{–almost all } x \in E.$$

Using Lemma A.22 for $F\text{–almost all } x \in E$ we obtain

$$\langle \zeta, \gamma(h,x) - (g^*c)(h,x) \rangle = 0 \quad \text{for all } \zeta \in \mathcal{D}(A^*).$$

This proves (3.14).

(2) $\Rightarrow$ (1): Let $h_0 \in O_M$ be arbitrary. Since $O_N$ is prelocally invariant in $C_{\mathcal{N}}$ for (3.9), there exists a local strong solution $Z = Z^{(g(h_0))}$ to (3.9) with lifetime $\tau > 0$ such that $(Z^t)_- \in O_N$ and $Z^t \in C_{\mathcal{N}}$ up to an evanescent set. By Itô’s formula (Theorem A.16) and taking into account the Itô isometry, the process $r := f(Z)$ satisfies $F$–almost surely

$$\tau_{t \wedge \tau} = f(Z_{t \wedge \tau}) = h_0 + \int_{0}^{t \wedge \tau} (g^*a)(r_s)ds + \sum_{j \in \mathbb{N}} \int_{0}^{t \wedge \tau} (g^*b^j)(r_s)d\beta^j_s + \int_{0}^{t \wedge \tau} \int_{E} (g^*c)(r_s,x)(\mu(ds,dx) - F(dx)ds)$$

$$\quad \quad + \int_{0}^{t \wedge \tau} \int_{E} \gamma(r_s,x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0.$$

Therefore, the process $r$ is a local strong solution to (2.1) with initial condition $h_0$ and lifetime $\tau$, showing that $O_M$ is prelocally strong invariant in $C_M$ for (3.1) with solutions given by (3.9) and $f$.

If condition (3.11), (3.12) are satisfied, then the continuity of $A$ on $O_M$ follows from Lemma 3.5 proving the additional statement.

\[ \square \]

3.14. Proposition. The following statements are equivalent:
In either case, $A$ is continuous on $O_M$. 

**Proof.** The proof is analogous to that of Proposition 3.13. □

For the rest of this section, let $G = \mathbb{R}^m$, where $m \in \mathbb{N}$ denotes the dimension of the submanifold $M$. We assume there exist elements $\zeta_1, \ldots, \zeta_m \in D(A^*)$ such that the mapping $f : N \to M$ has the inverse

$$(3.21) \quad f^{-1} : M \to N, \quad f^{-1}(h) = (\langle \zeta_1, h \rangle, \ldots, \langle \zeta_m, h \rangle).$$

This is illustrated by the following diagram:

$$\begin{array}{ccc}
N \subset \mathbb{R}^m & \xrightarrow{f} & M \subset H \\
\psi & \downarrow & \phi \\
V \subset \mathbb{R}^m_+ & \xrightarrow{\Theta} & N \subset \mathbb{R}^m
\end{array}$$

We define the subsets $O_V \subset C_V \subset V$ by $O_V := \psi^{-1}(O_N)$ and $C_V := \psi^{-1}(C_N)$.

We assume that $O_M$ is open in $M$ and $C_M$ is compact. Since $f : N \to M$ is a homeomorphism, $O_N$ is open in $N$ and $C_N$ is compact. Furthermore, since $\psi : V \to N$ is a homeomorphism, $O_V$ is open in $V$ and $C_V$ is compact. We define the mappings for the $\mathbb{R}^m$-valued SDE (3.9) as

$$a := \langle A^* \zeta, f \rangle + (\langle f, (\zeta, \bullet) \rangle)^* \alpha : \mathbb{R}^m \to \mathbb{R}^m,$$

$$b^j := (f, (\zeta, \bullet))^* \sigma^j : \mathbb{R}^m \to \mathbb{R}^m \quad \text{for } j \in \mathbb{N},$$

$$c := (f, (\zeta, \bullet))^* \gamma : \mathbb{R}^m \times E \to \mathbb{R}^m,$$

where $\langle A^* \zeta, f \rangle := (\langle A^* \zeta_1, f \rangle, \ldots, \langle A^* \zeta_m, f \rangle)$. Then, for each $h \in O_M$ we have

$$a(z) = \langle A^* \zeta, h \rangle + \langle \zeta, \alpha(h) \rangle,$$

$$b^j(z) = \langle \zeta, \sigma^j(h) \rangle, \quad j \in \mathbb{N}$$

$$c(z, x) = \langle \zeta, (\gamma(h, x)) \rangle, \quad x \in E$$

where $z = \langle \zeta, h \rangle \in O_N$. Furthermore, we define the mappings

$$\Theta := (\psi, \Psi)^* a : \mathbb{R}^m \to \mathbb{R}^m,$$

$$\Sigma^j := (\psi, \Psi)^* b^j : \mathbb{R}^m \to \mathbb{R}^m \quad \text{for } j \in \mathbb{N},$$

$$\Gamma := (\psi, \Psi)^* c : \mathbb{R}^m \times E \to \mathbb{R}^m$$

and consider the $\mathbb{R}^m$-valued SDE

$$\begin{cases}
    dY_t = \Theta(Y_t)dt + \sum_{j \in \mathbb{N}} \Sigma^j(Y_t)d\beta_t^j + \int_E \Gamma(Y_t-x)(\mu(dt, dx) - F(dx)dt) \\
    Y_0 = y_0
\end{cases}$$

as well as the time-shifted version

$$\begin{cases}
    dY_t = \Theta(Y_t)dt + \sum_{j \in \mathbb{N}} \Sigma^j(Y_t)d\beta_t^{(\tau_0,j)} + \int_E \Gamma(Y_t-x)(\mu^{(\tau_0)}(dt, dx) - F(dx)dt) \\
    Y_0 = y_0
\end{cases}$$

(3.27)
According to Lemma 3.5, the mappings $a$, $(b^j)_{j \in \mathbb{N}}$, $c$ as well as $\Theta$, $(\Sigma^j)_{j \in \mathbb{N}}$, $\Gamma$ satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8). Note that

$$\Theta(y) = (\psi^* a)(y), \quad y \in O_V$$

(3.28)

$$\Sigma^j(y) = (\psi^* b^j)(y), \quad j \in \mathbb{N} \text{ and } y \in O_V$$

(3.29)

$$\Gamma(y,x) = (\psi^* c)(y,x), \quad x \in E \text{ and } y \in O_V.$$  

(3.30)

The following result provides necessary and sufficient conditions regarding prelocal invariance of $C_V$ in $O_V$ for (3.26). Note that $V$ is also a $C^3$-submanifold with boundary of $\mathbb{R}^m$, and that for $y \in \partial V$ the inward pointing normal (tangent) vector to $\partial V$ at $y$ is given by the first unit vector $e_1 = (1,0,\ldots,0)$.

3.15. **Proposition.** The following statements are equivalent:

1. $O_V$ is prelocally strong invariant in $C_V$ for (3.27).
2. $O_V$ is prelocally invariant in $C_V$ for (3.26).
3. We have

$$\Sigma^j(y) \in T_y \partial V, \quad y \in O_V \cap \partial V, \quad \text{for all } j \in \mathbb{N}$$

(3.31)

$$y + \Gamma(y,x) \in C_V \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } y \in O_V$$

(3.32)

$$\int_E \langle |e_1, \Gamma(y,x)| F(dx) \rangle < \infty, \quad y \in O_V \cap \partial V$$

(3.33)

$$\langle e_1, \Theta(y) \rangle - \int_E \langle e_1, \Gamma(y,x) \rangle F(dx) \geq 0, \quad y \in O_V \cap \partial V.$$

(3.34)

**Proof.** For the sake of simplicity, we agree to write $O := O_V$ and $C := C_V$ during this proof.

(1) $\Rightarrow$ (2): This implication is obvious.

(2) $\Rightarrow$ (3): Let $y \in O$ be arbitrary. Since $O$ is prelocally invariant in $C$ for (3.26), there exists a local strong solution $Y = Y^y(t)$ to (3.26) with lifetime $\tau > 0$ such that $(Y^r)_- \in O$ and $Y^r \in C$ up to an evanescent set. Thus, Proposition A.23 yields (3.32), and for every finite stopping time $\varrho \leq \tau$ we have

$$\mathbb{P}(\langle e_1, Y_{\varrho}^\varrho \rangle \geq 0) = 1.$$  

(3.35)

From now on, we assume that $y \in \partial O := O \cap \partial V$. Let $(\Phi^j)_{j \in \mathbb{N}} \subset \mathbb{R}$ be a sequence with $\Phi^j \neq 0$ for only finitely many $j \in \mathbb{N}$, and let $\Psi : E \to \mathbb{R}$ be a measurable function of the form $\Psi = \psi \circ B$ with $c > -1$ and $B \in \mathcal{E}$ satisfying $F(B) < \infty$. Let $Z$ be the Doléans-Dade exponential

$$Z_t = \mathcal{E} \left( \sum_{j \in \mathbb{N}} \Phi^j \beta^j + \int_0^t \int_E \Psi(x)(\mu(ds, dx) - F(dx)ds) \right)_t, \quad t \geq 0.$$  

By [19] Thm. I.4.61] the process $Z$ is a solution of

$$Z_t = 1 + \sum_{j \in \mathbb{N}} \Phi^j \int_0^t Z_s dB^j_s + \int_0^t \int_E Z_s \Psi(x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0$$

and, since $\Psi > -1$, the process $Z$ is a strictly positive local martingale. There exists a strictly positive stopping time $\tau_1$ such that $Z^{\tau_1}$ is a martingale. Integration by parts (see [19] Thm. I.4.52]) yields

$$\langle e_1, Y_t \rangle Z_t = \int_0^t \langle e_1, Y_s \rangle dZ_s + \int_0^t Z_s d\langle e_1, Y_s \rangle$$

(3.36)

$$+ \langle \langle e_1, Y^c \rangle, Z^c \rangle_t + \sum_{s \leq t} \langle e_1, \Delta Y_s \rangle \Delta Z_s, \quad t \geq 0.$$
Taking into account the dynamics (3.26), we have
\[
(3.37) \quad \langle e_1, Y^c \rangle_t = \sum_{j \in \mathbb{N}} \Phi^j t \int_0^t Z_s \langle e_1, \Sigma^j(Y_s) \rangle ds, \quad t \geq 0,
\]
\[
(3.38) \quad \sum_{s \leq t} \langle e_1, \Delta Y_s \rangle \Delta Z_s = \int_0^t \int_{E} Z_{s-} \Psi(x) \langle e_1, \Gamma(Y_{s-}, x) \rangle \mu(ds, dx), \quad t \geq 0.
\]

Incorporating (3.26), (3.37) and (3.38) into (3.36), we obtain
\[
(3.38)
\]
\[
(3.37)
\]
\[
(3.36)
\]
\[
\]
\[
\]
Furthermore, let \( \Psi := 0 \) and \( y \) yields
\[
(3.39)
\]
\[
\]
Let \( \Phi := 0 \) and \( \Psi := -E \) such that
\[
F
\]
where \( \Psi \) and \( \Sigma \) there exist \( \Theta > 0 \) such that
\[
(3.40)
\]
\[
\]
By (3.40), (3.33) and Lebesgue’s dominated convergence theorem, we
\[
\]
\[
\]
where \( \Theta \) and \( \Sigma \) there exist \( \Theta > 0 \) such that
\[
(3.41)
\]
\[
\]
Let \( (\Phi_k)_{k \in \mathbb{N}} \subset \mathbb{R} \) be the sequence given by
\[
\Phi^k = \begin{cases} -\text{sign}(\langle e_1, \Sigma^k(y) \rangle) \bar{\Theta} + 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}
\]

Furthermore, let \( \Psi := 0 \) and \( g := \tau \land \tau_1 \land \tau_2 \land \tau_3 \land \tau_4 \). Taking expectation in (3.39) yields
\[
E[\langle e_1, Y_\bar{\theta} \rangle Z_{\bar{\theta}}] < 0, \quad \text{implying } P(\langle e_1, Y_\bar{\theta} \rangle < 0) > 0, \quad \text{which contradicts } (3.35).
\]
This proves (3.31).

Now suppose \( \int_E |\langle e_1, \Gamma(y, x) \rangle| F(dx) = \infty \). By Lemma A.18, for all \( B \in \mathcal{E} \) with \( F(B) < \infty \) the map \( y \mapsto \int_B \Gamma(y, x) F(dx) \) is continuous. Using the \( \sigma \)-finiteness of \( F \), there exist \( B \in \mathcal{E} \) with \( F(B) < \infty \) and a strictly positive stopping time \( \tau_4 \leq 1 \) such that
\[
-\frac{1}{2} \int_B |\langle e_1, \Gamma(Y_{(\tau_4 - )}, x) \rangle| F(dx) \leq -\langle \bar{\Theta}, 1 \rangle, \quad t \geq 0.
\]

Let \( \Phi := 0 \), \( \Psi := -\frac{1}{2} \tilde{B} \) and \( g := \tau \land \tau_1 \land \tau_2 \land \tau_3 \land \tau_4 \). Taking expectation in (3.39) we obtain
\[
E[\langle e_1, Y_\bar{\theta} \rangle Z_{\bar{\theta}}] < 0, \quad \text{implying } P(\langle e_1, Y_\bar{\theta} \rangle < 0) > 0, \quad \text{which contradicts } (3.35).
\]
This yields (3.33).

Since \( F \) is \( \sigma \)-finite, there exists a sequence \( (B_n)_{n \in \mathbb{N}} \subset \mathcal{E} \) with \( B_n \uparrow E \) and \( F(B_n) < \infty, \quad n \in \mathbb{N} \). We shall show for all \( n \in \mathbb{N} \) the relation
\[
(3.40) \quad \langle e_1, \Theta(y) \rangle + \int_E \Psi_n(x) \langle e_1, \Gamma(y, x) \rangle F(dx) \geq 0,
\]
where \( \Psi_n := -(1 - \frac{1}{2}) \mathbb{1}_{B_n} \). Suppose, on the contrary, that (3.40) is not satisfied for some \( n \in \mathbb{N} \). Then, there exist \( \eta > 0 \) and a strictly positive stopping time \( \tau_4 \leq 1 \) such that
\[
(3.41) \quad \langle e_1, \Theta(Y_{(\tau_4 \land )} - ) \rangle + \int_E \Psi_n(x) \langle e_1, \Gamma(Y_{(\tau_4 \land )} - ), x \rangle F(dx) \leq -\eta, \quad t \geq 0.
\]

Let \( \Phi := 0 \) and \( g := \tau \land \tau_1 \land \tau_2 \land \tau_3 \land \tau_4 \). Taking expectation in (3.39) we obtain
\[
E[\langle e_1, Y_\bar{\theta} \rangle Z_{\bar{\theta}}] < 0, \quad \text{implying } P(\langle e_1, Y_\bar{\theta} \rangle < 0) > 0, \quad \text{which contradicts } (3.35).
\]
This yields (3.40). By (3.40), (3.33) and Lebesgue’s dominated convergence theorem, we
and therefore, it satisfies
\[(3.41) \quad \Pi(y^1, y^2, \ldots, y^m) = ((y^1)^+, (y^2)^+, \ldots, (y^m)^+),\]
and therefore, it satisfies
\[\|\Pi(y_1) - \Pi(y_2)\| \leq \|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in \mathbb{R}^m.\]
Consequently, the mappings \(\Theta_\Pi : \mathbb{R}^m \to \mathbb{R}^m, \Sigma^j_\Pi : \mathbb{R}^m \to \mathbb{R}^m, j \in \mathbb{N}\) and \(\Gamma_\Pi : \mathbb{R}^m \times E \to \mathbb{R}^m\) defined as
\[\Theta_\Pi := \Theta \circ \Pi, \quad \Sigma^j_\Pi := \Sigma^j \circ \Pi \quad \text{and} \quad \Gamma_\Pi(\bullet, x) := \Gamma(\bullet, x) \circ \Pi\]
also satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8), which, due to Proposition 3.3, ensures existence and uniqueness of strong solutions to the SDE
\[\begin{align*}
\frac{dY_t}{Y_t} &= \Theta_\Pi(Y_t)dt + \sum_{j \in \mathbb{N}} \Sigma^j_\Pi(Y_t)d\beta_t^{(\tau_0),j} \\
Y_0 &= y_0.
\end{align*}\]
We define the boundary \(\partial O := O \cap \partial V\) and the interior \(\text{Int } O := O \setminus \partial O\) of the set \(O\). The interior \(\text{Int } O\) is an open subset in \(\mathbb{R}^m\) and we have the decomposition
\[O = \text{Int } O \cup \partial O.
\]
For each \(y \in \text{Int } O\) there exists \(\epsilon_y > 0\) such that \(B_{\epsilon_y}(y) \subset \text{Int } O\). Therefore, we have
\[\text{Int } O = \bigcup_{y \in \text{Int } O} B_{\epsilon_y}(y) = \bigcup_{y \in \text{Int } O} B_{\epsilon_y/2}(y).
\]
By Lindelöf’s Lemma \cite[Lemma 1.1.6]{[1]} there exist sequences \((y^\text{Int}_k)_{k \in \mathbb{N}} \subset \text{Int } O\) and \((\epsilon_k)_{k \in \mathbb{N}} \subset (0, \infty)\) such that
\[\text{Int } O = \bigcup_{k \in \mathbb{N}} B_{\epsilon_k}(y^\text{Int}_k) = \bigcup_{k \in \mathbb{N}} B_{\epsilon_k/2}(y^\text{Int}_k).
\]
Since \(O\) is open in \(\mathbb{R}^m_+\), there exists an open subset \(\hat{O} \subset \mathbb{R}^m\) such that \(\hat{O} \cap \mathbb{R}^m_+ = O\).
For each \(y \in \partial O\) there exists \(\delta_y > 0\) such that \(B_{\delta_y}(y) \subset \hat{O}\). Therefore, we have
\[\partial O \subset \bigcup_{y \in \partial O} B_{\delta_y}(y) = \bigcup_{y \in \partial O} B_{\delta_y/2}(y) = \bigcup_{y \in \partial O} B_{\delta_y/2}(y) \subset \hat{O}.
\]
By Lindelöf’s Lemma \cite[Lemma 1.1.6]{[1]} there exist sequences \((y^\partial_k)_{k \in \mathbb{N}} \subset \partial O\) and \((\delta_k)_{k \in \mathbb{N}} \subset (0, \infty)\) such that
\[\partial O \subset \bigcup_{k \in \mathbb{N}} B_{\delta_k}(y^\partial_k) = \bigcup_{k \in \mathbb{N}} B_{\delta_k/2}(y^\partial_k) \subset \hat{O}.
\]
Setting
\[P := \bigcup_{k \in \mathbb{N}} B_{\delta_k}(y^\partial_k) \quad \text{and} \quad \mathbb{R}^m := \{y \in \mathbb{R}^m : y_1 \leq 0\},
\]
by taking into account that the metric projection \(\Pi\) on \(\mathbb{R}^m_+\) is given by \((3.41)\), we have
\[\Pi(y) \in \partial O, \quad y \in P \cap \mathbb{R}^m_-.
\]
Defining the sequence \(P^\text{Int}_k \subset B_{\epsilon_k}(y^\text{Int}_k), \ k \in \mathbb{N}\) of disjoint sets as
\[P^\text{Int}_k := B_{\epsilon_k}(y^\text{Int}_k) \setminus \bigcup_{j=1}^{k-1} B_{\epsilon_j}(y^\text{Int}_j), \quad k \in \mathbb{N},
\]
we have the identities
\[ \text{Int } O = \bigcup_{k \in \mathbb{N}} P_k^{\text{Int}} = \bigcup_{k \in \mathbb{N}} P_k^{\text{Int}}, \]
and defining the sequence \( P_k^0 \subset B_{\delta_k}(y_k^0), \ k \in \mathbb{N} \) of disjoint sets as
\[ P_k^0 := B_{\delta_k}(y_k^0) \setminus \bigcup_{j=1}^{k-1} B_{\delta_j}(y_j^0), \ k \in \mathbb{N}, \]
we have the inclusions
\[ \partial O \subset P = \bigcup_{k \in \mathbb{N}} P_k^0 = \bigcup_{k \in \mathbb{N}} P_k^0 \subset \partial O. \]

Now, let \( \tau_0 \) be a bounded stopping time and let \( y_0 : \Omega \to \mathbb{R}^m \) be a bounded \( \mathcal{F}_{\tau_0} \)-measurable random variable. Defining the \( \mathcal{F}_{\tau_0} \)-measurable sets
\[ \Omega_k^{\text{Int}} := \{ y_0 \in P_k^{\text{Int}} \} \quad \text{and} \quad \Omega_k^0 := \{ y_0 \in P_k^0 \cap \partial O \} \quad \text{for } k \in \mathbb{N}, \]
we have the decomposition \( \Omega = (\bigcup_{k \in \mathbb{N}} \Omega_k^{\text{Int}}) \cup (\bigcup_{k \in \mathbb{N}} \Omega_k^0). \) According to Proposition A.8 for each \( k \in \mathbb{N} \) there exist local strong solutions \( Y_{t \wedge k}^{\text{Int}, k}, Y_{t \wedge k}^{\partial, k} \) to the SDE (3.42) with initial conditions \( y_0 \mathbb{1}_{\Omega_k^{\text{Int}}}, y_0 \mathbb{1}_{\Omega_k^0} \) and lifetimes \( \tau_k^{\text{Int}}, \tau_k^0 > 0. \) According to Lemma A.2, the mappings
\[ \tau_k^{\text{Int}} := \inf \{ t \geq 0 : Y_{t \vee k}^{\text{Int}, k} \notin B_{\delta_k}(y_k^{\text{Int}}) \} \]
\[ \tau_k^0 := \inf \{ t \geq 0 : Y_{t \vee k}^{\partial, k} \notin B_{\delta_k}(y_k^0) \} \]
are strictly positive stopping times. By Proposition A.9 the mapping
\[ \tau := \sum_{k \in \mathbb{N}} \tau_k^{\text{Int}} + \sum_{k \in \mathbb{N}} \tau_k^0 \]
is a strictly positive stopping time and
\[ Y := \sum_{k \in \mathbb{N}} Y_{t \wedge k}^{\text{Int}, k} \mathbb{1}_{\Omega_k^{\text{Int}}} + \sum_{k \in \mathbb{N}} Y_{t \wedge k}^{\partial, k} \mathbb{1}_{\Omega_k^0} \]
is a local strong solution to (3.42) with initial condition \( y_0 \) and lifetime \( \tau. \) We obtain on \( \{ y_0 \in \text{Int } O \} \) up to an evanescent set
\[ (Y^{\tau})_{-} = \sum_{k \in \mathbb{N}} ((Y_{t \wedge k}^{\text{Int}, k})_{-} \mathbb{1}_{\Omega_k^{\text{Int}}} \in \bigcup_{k \in \mathbb{N}} B_{\delta_k}(y_k^{\text{Int}}) = \text{Int } O \subset O. \]

By (3.41) and (3.32), for all \( y \in P \cap \mathbb{R}^m_+ \) we have
\[ \langle e_1, y + \xi \Gamma_{\Pi}(y, x) \rangle = (1 - \xi)\langle e_1, y \rangle + \xi(\langle e_1, y \rangle + \langle e_1, \Gamma_{\Pi}(y, x) \rangle) \]
\[ = (1 - \xi)\langle e_1, y \rangle + \xi(\langle e_1, y \rangle + \Gamma_{\Pi}(y, x)) \geq 0 \quad \text{for all } \xi \in [0, 1], \]
for \( F \)-almost all \( x \in E. \)

Furthermore, by (3.31), (3.33), and (3.43), for all \( y \in P \cap \mathbb{R}^m_+ \) we have
\[ \langle e_1, \Sigma_{\Pi}(y) \rangle = \langle e_1, \Sigma'(\Pi(y)) \rangle = 0, \quad \text{for all } j \in \mathbb{N}, \]
\[ \langle e_1, \Gamma_{\Pi}(y, x) \rangle = \langle e_1, \Pi_{\Pi}(y) \rangle + \langle e_1, \Gamma_{\Pi}(y, x) \rangle \]
\[ = \langle e_1, \Pi_{\Pi}(y) \rangle + \langle e_1, \Gamma_{\Pi}(y, x) \rangle \geq 0, \quad \text{for } F \text{-almost all } x \in E, \]
\[ \int_E |\langle e_1, \Gamma_{\Pi}(y) \rangle| F(dx) = \int_E |\langle e_1, \Gamma_{\Pi}(y) \rangle| F(dx) < \infty, \]
\[ \langle e_1, \Theta_{\Pi}(y) \rangle - \int_E \langle e_1, \Gamma_{\Pi}(y, x) \rangle F(dx) \]
\[ = \langle e_1, \Theta_{\Pi}(y) \rangle - \int_E \langle e_1, \Gamma_{\Pi}(y, x) \rangle F(dx) \geq 0. \]
The function \( \phi : \mathbb{R} \rightarrow \mathbb{R} \), \( \phi(y) := (-y^2)^+ \) is of class \( C^2(\mathbb{R}) \) and we have \( \phi'(y) < 0 \) for \( y < 0 \) and \( \phi'(y) = \phi''(y) = 0 \) for \( y \geq 0 \). By (3.44)–(3.48) and Lemma A.22 we obtain

\[
\phi'(e_1, y) \left( \langle e_1, \Theta_N(y) \rangle - \int_E \langle e_1, \Gamma_N(y, x) \rangle F(dx) \right) \leq 0, \quad y \in P
\]

(3.49)

\[
\phi''(e_1, y) \left| \langle e_1, \Sigma_N^j(y) \rangle \right|^2 = 0, \quad y \in P, \quad \text{for all } j \in \mathbb{N}
\]

(3.50)

\[
\phi'(e_1, y) \langle e_1, \Sigma_N^j(y) \rangle = 0, \quad y \in P, \quad \text{for all } j \in \mathbb{N}
\]

(3.51)

\[
\int \phi'(e_1, y) \left( \langle e_1, \Theta_N(y) \rangle \right) \int e_1, \Gamma_N(y, x) \rangle F(dx) \leq 0 \quad \text{for all } y \in P,
\]

for \( F \)-almost all \( x \in E \).

Let \( k \in \mathbb{N} \) be arbitrary. Applying Itô’s formula (Theorem A.16) yields \( \mathbb{P} \)-almost surely on \( \Omega_k^0 \)

\[
\phi(e_1, Y^{\partial,k}_{t \wedge \tau^0_k}) = \phi(e_1, y_0)
\]

\[
+ \int_0^{t \wedge \tau^0_k} \left( \phi'(e_1, Y^{\partial,k}_s) \langle e_1, \Theta_N(Y^{\partial,k}_s) \rangle + \frac{1}{2} \sum_{j \in \mathbb{N}} \phi''(e_1, Y^{\partial,k}_s) \left| \langle e_1, \Sigma_N^j(Y^{\partial,k}_s) \rangle \right|^2 \right) ds
\]

\[
+ \int_E \phi'(e_1, Y^{\partial,k}_s + \Gamma_N(Y^{\partial,k}_s, x)) - \phi(e_1, Y^{\partial,k}_s) \int e_1, \Gamma_N(Y^{\partial,k}_s, x) \rangle F(dx) \right) ds
\]

\[
+ \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau^0_k} \phi'(e_1, Y^{\partial,k}_s) \langle e_1, \Sigma_N^j(Y^{\partial,k}_s) \rangle d\beta^j_s(t_0)
\]

\[
+ \int_E \left( \int_0^1 \phi'(e_1, Y^{\partial,k}_{s-} + \xi \Gamma_N(Y^{\partial,k}_{s-}, x)) d\xi \right) \langle e_1, \Gamma_N(Y^{\partial,k}_s, x) \rangle \mu(s) ds
\]

Since \( \Omega_k^0 \subset \{ y_0 \in \partial O \} \), by (3.47) and Taylor’s theorem we obtain \( \mathbb{P} \)-almost surely on \( \Omega_k^0 \)

\[
\phi(e_1, Y^{\partial,k}_{t \wedge \tau^0_k}) = \int_0^{t \wedge \tau^0_k} \left[ \phi'(e_1, Y^{\partial,k}_s) \langle e_1, \Theta_N(Y^{\partial,k}_s) \rangle - \int_E \langle e_1, \Gamma_N(Y^{\partial,k}_s, x) \rangle F(dx) \right] ds
\]

\[
+ \frac{1}{2} \sum_{j \in \mathbb{N}} \phi''(e_1, Y^{\partial,k}_s) \left| \langle e_1, \Sigma_N^j(Y^{\partial,k}_s) \rangle \right|^2 ds
\]

\[
+ \sum_{j \in \mathbb{N}} \int_0^{t \wedge \tau^0_k} \phi'(e_1, Y^{\partial,k}_s) \langle e_1, \Sigma_N^j(Y^{\partial,k}_s) \rangle d\beta^j_s(t_0)
\]

\[
+ \int_E \left( \int_0^1 \phi'(e_1, Y^{\partial,k}_{s-} + \xi \Gamma_N(Y^{\partial,k}_{s-}, x)) d\xi \right) \langle e_1, \Gamma_N(Y^{\partial,k}_s, x) \rangle \mu(s) ds,
\]

\( t \geq 0. \)

By (3.49)–(3.52) and Lemmas A.14–A.15 we deduce that \( \phi(e_1, (Y^{\partial,k})^0_{t \wedge \tau^0_k}) \leq 0 \) on \( \Omega_k^0 \) up to an evanescent set. Therefore, we obtain on \( \{ y_0 \in \partial O \} \) up to an evanescent
\[
(Y^\tau)^{-} = \sum_{k \in \mathbb{N}} ((Y^{\Omega,k})^0)^{-} \mathbbm{1}_{\Omega_{k}} \in \bigcup_{k \in \mathbb{N}} B_{\delta_k}(y_k^0) \cap \mathbb{R}^m_+ \subset \bar{O} \cap \mathbb{R}^m_+ = O.
\]

Consequently, we get up to an evanescent set
\[
(Y^\tau)^{-} = (Y^\tau)^{-} \mathbbm{1}_{\{\gamma_0 \in \text{Int} \ O\}} + (Y^\tau)^{-} \mathbbm{1}_{\{\gamma_0 \in \partial \ O\}} \in O.
\]

Using (3.32) and Corollary A.25 we obtain \( Y^\tau \in O \) up to an evanescent set. Since \( \Theta|_C = \Theta|_C \), \( \Sigma^j|_C = \Sigma^j|_C \) for all \( j \in \mathbb{N} \) and \( \Gamma(\bullet, x)|_C = \Gamma(\bullet, x)|_C \) for all \( x \in E \), the process \( Y \) is also a local strong solution to (3.27) with lifetime \( \tau \), proving that \( O \) is prelocally strong invariant in \( C \) for (3.26).

\[\Box\]

3.16. Lemma. Suppose that (3.31) is satisfied. Then for all \( j \in \mathbb{N} \) we have \[\langle e_1, DS\gamma(y)\Sigma^j(y) \rangle = 0, \quad y \in O_v \cap \partial V.\]

Proof. This is a direct consequence of Lemma C.3. \[\Box\]

3.17. Lemma. Suppose that \( O_M \) is prelocally invariant in \( C_M \) for (2.1). Then, the set \( O_M \) is prelocally invariant in \( C_M \) for (2.7) with solutions given by (3.9) and \( f \).

Proof. Let \( h_0 \in O_M \) be arbitrary. Since \( O_M \) is prelocally invariant in \( C_M \) for (3.9), there exists a local mild solution \( r = r(h_0) \) to (2.1) with lifetime \( \tau > 0 \) such that \( (r^\tau)^{-} \in O_M \) and \( r^\tau \in C_M \) up to an evanescent set. Since \( \zeta_1, \ldots, \zeta_m \in D(A^*) \) and \( r \) is also a local weak solution to (2.1), setting \( Z := \langle \zeta, r \rangle \) we have, by taking into account (3.23) (3.25), \( \mathbb{P} \)-almost surely

\[
Z_{t\wedge \tau} = \langle \zeta, r_{t\wedge \tau} \rangle = \langle \zeta, h_0 \rangle + \int_0^{t\wedge \tau} \langle A^* \zeta, r_s \rangle + \langle \zeta, \alpha(r_s) \rangle ds
+ \sum_{j \in \mathbb{N}} \int_0^{t\wedge \tau} \langle \zeta, \sigma^j(r_s) \rangle d\beta^j_s + \int_0^{t\wedge \tau} \int_E \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds)
+ \langle \zeta, h_0 \rangle + \int_0^{t\wedge \tau} a(Z_s)ds + \sum_{j \in \mathbb{N}} \int_0^{t\wedge \tau} b^j(Z_s)d\beta^j_s
+ \int_0^{t\wedge \tau} \int_E c(Z_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]

Therefore, the process \( Z \) is a local strong solution to (3.9) with initial condition \( \langle \zeta, h_0 \rangle \) and lifetime \( \tau \) such that \( (Z^\tau)^{-} \in O_M \) and \( Z^\tau \in C_M \) up to an evanescent set. By (3.21), we have \( f(Z^\tau) = r^\tau \), and hence, the process \( f(Z) \) is a local mild solution to (2.1) with initial condition \( h_0 \) and lifetime \( \tau \). \[\Box\]

3.18. Proposition. The following statements are equivalent:

1. \( O_M \) is prelocally strong invariant in \( C_M \) for (3.7) with solutions given by (3.10) and \( f \).
2. \( O_M \) is prelocally strong invariant in \( C_M \) for (3.1).
3. \( O_M \) is prelocally invariant in \( C_M \) for (2.1).
4. \( O_M \) is prelocally invariant in \( C_M \) for (2.1) with solutions given by (3.9) and \( f \).
(5) The following conditions are satisfied:

\begin{align}
(3.53) & \quad O_M \subset D(A), \\
(3.54) & \quad \sigma^j(h) \in \begin{cases} T_hM, & h \in O_M \cap (M \setminus \partial M) \\ T_h\partial M, & h \in O_M \cap \partial M, \end{cases} \quad \text{for all } j \in \mathbb{N} \\
(3.55) & \quad h + \gamma(h, x) \in C_M \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_M \\
(3.56) & \quad \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) < \infty, \quad h \in O_M \cap \partial M \\
(3.57) & \quad Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) \\
& \quad - \int_E \Pi_{(T_hM)^\perp} \gamma(h, x) F(dx) \in T_hM, \quad h \in O_M \\
(3.58) & \quad \langle \eta_h, Ah + \alpha(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \eta_h, D\sigma^j(h)\sigma^j(h) \rangle \\
& \quad - \int_E \langle \eta_h, \gamma(h, x) \rangle F(dx) \geq 0, \quad h \in O_M \cap \partial M. 
\end{align}

In either case, \( A \) and the mapping in (3.57) are continuous on \( O_M \).

Proof. (1) \( \Rightarrow \) (2): This implication is clear.

(2) \( \Rightarrow \) (3): See Lemma 3.10.

(3) \( \Rightarrow \) (4): See Lemma 3.17.

(4) \( \Rightarrow \) (5): If \( O_M \) is prelocally invariant in \( C_M \) for (2.1) with solutions given by (3.9) and \( f \), then we have two implications:

- Proposition 3.13 yields (3.53) and

\begin{align}
(3.59) & \quad (A + \alpha)(h) = (\zeta, \bullet)^* a(h), \quad h \in O_M \\
(3.60) & \quad \sigma^j(h) = (\zeta, \bullet)^* b^j(h) \quad \text{for all } j \in \mathbb{N}, \quad \text{for all } h \in O_M \\
(3.61) & \quad \gamma(h, x) = (\zeta, \bullet)^* c(h, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_M.
\end{align}

- By Lemma 3.9, the set \( O_N \) is prelocally invariant in \( C_N \) for (3.9). Hence, by (3.28)–(3.30) and Proposition 3.13, the set \( O_V \) is prelocally invariant in \( C_V \) for (3.26) with solutions given by (3.9) and \( \Psi \).

The latter statement has two further consequences:

- By Lemma 3.11, the set \( O_N \) is prelocally invariant in \( C_N \) for (3.9) with solutions given by (3.26) and \( \psi \). Thus, Proposition 3.13 yields

\begin{align}
(3.62) & \quad a(z) = (\Psi^* \Theta)(z), \quad z \in O_N \\
(3.63) & \quad b^j(z) = (\Psi^* \Sigma^j)(z), \quad j \in \mathbb{N} \text{ and } z \in O_N \\
(3.64) & \quad c(z, x) = (\Psi^* \Gamma)(z, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } z \in O_N.
\end{align}

- By Lemma 3.9, the set \( O_V \) is prelocally invariant in \( C_V \) for (3.26). Proposition 3.13 implies that conditions (3.31)–(3.34) are satisfied.
In view of (3.31)–(3.34), Lemma 3.16, identities (3.62)–(3.64) and Proposition C.12 we obtain
(3.65) $b_j(z) \in T_z\partial N$, $z \in \Omega_N \cap \partial N$, for all $j \in \mathbb{N}$
(3.66) $z + c(z, x) \in C_N$ for $F$–almost all $x \in E$, for all $z \in \Omega_N$
(3.67) $\int_E |\langle \xi_z, c(z, x) \rangle| F(dx) < \infty$, $z \in \Omega_N \cap \partial N$
(3.68) $\langle \xi_z, a(z) \rangle = \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi_z, Db_j(z)b_j(z) \rangle$
\[ - \int_E |\langle \xi_z, c(z, x) \rangle| F(dx) \geq 0, \quad z \in \Omega_N \cap \partial N \]
where $\xi_z$ denotes the inward pointing normal (tangent) vector to $\partial N$ at $z$. Taking into account (3.59)–(3.61), applying Proposition C.12 we arrive at (3.54)–(3.58).

(5) $\Rightarrow$ (1): Suppose that conditions (3.53)–(3.58) are satisfied. By (3.53) and (3.23), for all $z \in \Omega_N$ we obtain
\[ a(z) = \langle A^* \zeta, h \rangle + \langle \zeta, \alpha(h) \rangle = \langle \zeta, Ah + \alpha(h) \rangle = (f^*(A + \alpha))(z), \]
where $h = f(z) \in \Omega_M$. Thus, we have
(3.69) $a(z) = (f^*(A + \alpha))(z), \quad z \in \Omega_N$
(3.70) $b_j(z) = (f^*\sigma^j)(z), \quad j \in \mathbb{N}$ and $z \in \Omega_N$
(3.71) $c(z, x) = (f^*\gamma)(z, x), \quad x \in E$ and $z \in \Omega_N$,
which has two implications:
- By (3.54), (3.55), (3.57) and Proposition C.22 we obtain (3.59)–(3.61).
- By (3.54), (3.58) and Proposition C.12 we have (3.65)–(3.68).

In view of (3.28)–(3.30), we obtain the following consequences:
- By (3.66) and Proposition C.22 we obtain (3.62)–(3.64).
- By (3.65), (3.68), Proposition C.12 and Lemma 3.16 we have (3.31)–(3.34).

Therefore, by Proposition 3.15 the set $\Omega_V$ is prelocally strong invariant in $C_V$ for (3.27). By (3.62), (3.64) and Proposition 3.14 the set $\Omega_N$ is prelocally strong invariant in $C_N$ for (3.10) with solutions given by (3.27) and $\psi$. According to Lemma 3.10 the set $\Omega_N$ is prelocally strong invariant in $C_N$ for (3.10). By (3.53), (3.59)–(3.61) and Proposition 3.14 the set $\Omega_M$ is prelocally invariant in $C_M$ for (3.1) with solutions given by (3.10) and $f$.

If $\Omega_M$ is prelocally invariant in $C_M$ for (2.1) with solutions given by (3.9) and $f$, then Proposition 3.14 implies that $A$ is continuous on $\Omega_M$. Using Lemma A.17 and Corollary C.17 we obtain that the mapping in (3.57) is continuous on $\Omega_M$, proving the additional statement. □

4. Proof of the main results

In this section, we shall prove our main results by using Proposition 3.18. Let $B \in E$ be a set with $F(B^c) < \infty$ and define the mappings $\alpha^B : H \to H$ and $\gamma^B : H \times E \to H$ as
\[ \alpha^B(h) := \alpha(h) - \int_{B^c} \gamma(h, x)F(dx), \]
\[ \gamma^B(h, x) := \gamma(h, x)1_B(x). \]
By Lemma A.18 the mappings \( \alpha^B, (\sigma^j)_{j \in \mathbb{N}}, \gamma^B \) are well-defined and satisfy the regularity conditions (2.2)–(2.4) and (2.6)–(2.8) from Section 2. We shall consider the SPDE

\[
\begin{align*}
&\text{(4.1)} \\
&\begin{cases}
&dr^B_t = \left(A r^B_t + \alpha^B(r^B_t)\right)dt + \sum_{j \in \mathbb{N}} \sigma^j(r^B_t)d\beta^j_t \\
&r^B_0 = h_0
\end{cases}
\end{align*}
\]

as well as the time-shifted version

\[
\begin{align*}
&\text{(4.2)} \\
&\begin{cases}
&dr^B_t = \left(A r^B_t + \alpha^B(r^B_t)\right)dt + \sum_{j \in \mathbb{N}} \sigma^j(r^B_t)d\beta^j_t\big|_{r^B_t=0} \\
&r^B_0 = h_0.
\end{cases}
\end{align*}
\]

According to Lemma A.20 the mapping

\[ g^B = \inf \{ t \geq 0 : \mu([0, t] \times B^c) = 1 \} \]

is a strictly positive stopping time.

4.1. **Proposition.** Let \( O_M \subset M \) be a subset which is open in \( M \), and suppose that \( O_M \subset D(A), \)

\[ h + \gamma(h, x) \in \bar{M} \] for \( F \)-almost all \( x \in E \), for all \( h \in O_M \).

Then, the following statements are true:

1. We have (3.56)–(3.58) if and only if

\[ \int_E |\langle \eta_h, \gamma^B(h, x)\rangle|F(dx) < \infty, \ h \in O_M \cap \partial M \]

\[ \int_E \Pi(T_{h, M}) \gamma^B(h, x)F(dx) \in T_{h, M}, \ h \in O_M \]

\[ \langle \eta_h, Ah + \alpha^B(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \eta_h, D\sigma^j(h)\sigma^j(h) \rangle \\
- \int_E \langle \eta_h, \gamma^B(h, x) \rangle F(dx) \geq 0, \ h \in O_M \cap \partial M. \]

2. The mapping in (3.57) is continuous on \( O_M \) if and only if the mapping in (4.4) is continuous on \( O_M \).

**Proof.** This follows from Lemma C.16 and Proposition C.8 \( \square \)

4.2. **Lemma.** Suppose that condition (4.4) is satisfied. Then, for all \( h_0 \in M \) there exist

- a constant \( \epsilon > 0 \) such that \( B_\epsilon(h_0) \cap M \) is a submanifold with one chart as in Section 3, i.e. as the submanifold \( M \) in Diagram (3.22),
- subsets \( O_M \subset C_M \subset B_\epsilon(h_0) \cap M \) with \( h_0 \in O_M \) as in Section 3, i.e. \( O_M \)
  is open in \( B_\epsilon(h_0) \cap M \) and \( C_M \) is compact,
- and a set \( B \in E \) with \( F(B^c) < \infty \)

such that we have

\[ h + \gamma^B(h, x) \in C_M \] for \( F \)-almost all \( x \in E \), for all \( h \in O_M \).

**Proof.** This follows from Lemmas B.11 and C.8 \( \square \)
4.1. Proof of Theorems 2.2 and 2.5. First, we shall prove that prelocal invariance of $\mathcal{M}$ for (2.1) implies conditions (1.2), (1.4), (1.6)–(1.8), the continuity of $A$ and the mapping in (1.7) on $\mathcal{M}$, and that for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1). Indeed, according to Proposition A.23 we have (1.4). Let $h_0 \in \mathcal{M}$ be arbitrary. By Lemma 4.2 there exist

- a constant $\epsilon > 0$ such that $B_r(h_0) \cap \mathcal{M}$ is a submanifold with one chart in Section 3 i.e. as the submanifold $\mathcal{M}$ in Diagram (3.22),
- subsets $O_{\mathcal{M}} \subset C_{\mathcal{M}} \subset B_r(h_0) \cap \mathcal{M}$ with $h_0 \in O_{\mathcal{M}}$ as in Section 3 i.e. $O_{\mathcal{M}}$ is open in $B_r(h_0) \cap \mathcal{M}$ and $C_{\mathcal{M}}$ is compact,
- and a set $B \in E$ with $F(B^c) < \infty$ such that (4.6) is valid. We will show that $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (1.1). Indeed, let $g_0 \in O_{\mathcal{M}}$ be arbitrary. Since $O_{\mathcal{M}}$ is open in $B_r(h_0) \cap \mathcal{M}$, there exists $\delta > 0$ such that $B_\delta(g_0) \cap \mathcal{M} \subset O_{\mathcal{M}}$. Since $\mathcal{M}$ is prelocally invariant for (2.1), there exist a local mild solution $r = r^{(g_0)}$ to (2.1) with lifetime $0 < \tau \leq \rho^B$ such that $(r^\tau)_- \in \mathcal{M}$ up to an evanescent set. According to Proposition A.21 there exists a local mild solution $r^B = r^{B,(g_0)}$ to (1.1) with lifetime $\tau$ such that $(r^\tau)_- = ((r^B)^\tau)_-$. Taking into account Lemma A.2 the mapping $\rho := \inf\{t \geq 0 : r_t \notin B_\delta(g_0)\}$ is a strictly positive stopping time. We obtain up to an evanescent set

$$(r^B)^\rho = (r^\rho)_- \in B_\delta(g_0) \cap \mathcal{M} \subset O_{\mathcal{M}}.$$  

Furthermore, using (4.6) and Corollary A.25 we obtain $r^B$ in $C_{\mathcal{M}}$ up to an evanescent set. Hence, the set $O_{\mathcal{M}}$ is prelocally invariant in $C_{\mathcal{M}}$ for (1.1). Proposition 3.18 applies and yields (3.53), (3.54), (3.55) and (3.57) and that $A$ and the mapping in (4.4) are continuous on $O_{\mathcal{M}}$. Since (3.53) and (1.4) are satisfied, by Proposition 4.1 we also have (3.56), (3.58) and the mapping in (3.57) is continuous on $O_{\mathcal{M}}$. Since $h_0 \in \mathcal{M}$ was arbitrary, we deduce (1.2), (1.3), (1.6)–(1.8) and that $A$ and the mapping in (1.7) are continuous on $\mathcal{M}$. By Lemma A.11 for each $h_0 \in \mathcal{M}$ there is a local strong solution $r = r^{(h_0)}$ to (2.1).

Next, we shall prove that conditions (1.2), (1.4) and (1.6)–(1.8) imply prelocal strong invariance of $\mathcal{M}$ for (3.1) and the remaining statement of Theorem 2.2. By Lemma 1.2 for each $h \in \mathcal{M}$ there exist

- a constant $\epsilon_h > 0$ such that $B_{\epsilon_h}(h) \cap \mathcal{M}$ is a submanifold with one chart as in Section 3 i.e. as the submanifold $\mathcal{M}$ in Diagram (3.22),
- subsets $O_h \subset C_h \subset B_{\epsilon_h}(h) \cap \mathcal{M}$ with $h \in O_h$ as in Section 3 i.e. $O_h$ is open in $B_{\epsilon_h}(h) \cap \mathcal{M}$ and $C_h$ is compact,
- and a set $B_h \in E$ with $F(B_h^c) < \infty$ such that we have

$$h + \gamma^B(h,x) \in C_h \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_h.$$  

Since we have the coverings

$$\mathcal{M} = \bigcup_{h \in \mathcal{M}} O_h = \bigcup_{h \in \mathcal{M}} C_h = \bigcup_{h \in \mathcal{M}} (B_{\epsilon_h}(h) \cap \mathcal{M}),$$

by Lindelöf’s Lemma [11, Lemma 1.1.6] there exist

- a sequence $(\mathcal{M}_k)_{k \in \mathbb{N}}$ of submanifolds with one chart as in Section 3 i.e. as the submanifold $\mathcal{M}$ in Diagram (3.22), and $\mathcal{M}_k \subset \mathcal{M}$ for all $k \in \mathbb{N}$,
- subsets $O_{\mathcal{M}_k} \subset C_{\mathcal{M}_k} \subset \mathcal{M}_k$, $k \in \mathbb{N}$ as in Section 3 i.e. $O_{\mathcal{M}_k}$ is open in $\mathcal{M}_k$ and $C_{\mathcal{M}_k}$ is compact,
- and sets $(B_k)_{k \in \mathbb{N}} \subset E$ with $F(B_k^c) < \infty$ 

such that we have
such that for all \( k \in \mathbb{N} \) we have
\[
(4.7) \quad h + \gamma^B(h, x) \in C_{M_k} \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_{M_k}
\]
and the submanifold \( M \) has the countable coverings
\[
\mathcal{M} = \bigcup_{k \in \mathbb{N}} O_{M_k} = \bigcup_{k \in \mathbb{N}} C_{M_k} = \bigcup_{k \in \mathbb{N}} M_k.
\]
Let \( k \in \mathbb{N} \) be arbitrary. By (1.2), (1.3) and (1.6)–(1.8) we have (3.53), (3.54) and (3.55)–(3.58) with \( M = M_k. \) Since (4.53) and (1.4) are satisfied, by Proposition 4.1 we also have (4.3)–(4.5) with \( M = M_k \) and \( B = B_k. \) Consequently, by (3.53), (3.54), (4.7), (4.3)–(4.5) and Proposition 3.18 the set \( C_{M_k} \) is prelocally strong invariant in \( O_{M_k} \) for (4.2) with \( B = B_k. \)

Now, let \( \tau_0 \) be a bounded stopping time and let \( h_0 : \Omega \to H \) be a bounded \( \mathcal{F}_{\tau_0} \)-measurable random variable with \( \mathbb{P}(h_0 \in \mathcal{M}) = 1. \) Defining the sequence \( P_k \subset O_{M_k}, k \in \mathbb{N} \) of disjoint subsets as
\[
P_k := O_{M_k} \setminus \bigcup_{j=1}^{k-1} O_{M_j}, \quad k \in \mathbb{N}
\]
the submanifold \( M \) has the disjoint covering \( \mathcal{M} = \bigcup_{k \in \mathbb{N}} P_k, \) and defining the \( \mathcal{F}_{\tau_0} \)-measurable sets
\[
\Omega_k := \{ h_0 \in P_k \}, \quad k \in \mathbb{N}
\]
we have the decomposition \( \Omega = \bigcup_{k \in \mathbb{N}} \Omega_k. \) Let \( k \in \mathbb{N} \) be arbitrary. Since \( C_{M_k} \) is prelocally strong invariant in \( O_{M_k} \) for (4.2) with \( B = B_k, \) there exists a local strong solution \( r^{B_k} \) to (4.2) with \( B = B_k, \) initial condition \( h_0 \mathbb{1}_{\Omega_k} \) and lifetime \( 0 < \tau_k \leq \rho^B \) such that on \( \Omega_k \) up to an evanescent set
\[
((r^{B_k})^{\tau_k})_\cdot \in O_{M_k} \quad \text{and} \quad (r^{B_k})^{\tau_k} \in C_{M_k}.
\]
According to Proposition A.21, there exists a local mild solution \( r^k \) to (3.1) with initial condition \( h_0 \mathbb{1}_{\Omega_k} \) and lifetime \( \tau_0 \) such that \( (r^k)^{\tau_0} \mathbb{1}_{[0, \tau_0]} = (r^{B_k})^{\tau_k} \mathbb{1}_{[0, \tau_k]} \). By Proposition A.9, the mapping \( r := \sum_{k \in \mathbb{N}} r^k \mathbb{1}_{\Omega_k} \) is a strictly positive stopping time and \( r := \sum_{k \in \mathbb{N}} r^k \mathbb{1}_{\Omega_k} \) is a local mild solution to (3.1) with initial condition \( h_0 \). We obtain up to an evanescent set
\[
(r^\tau)_\cdot = \sum_{k \in \mathbb{N}} ((r^k)^{\tau_k})_\cdot \mathbb{1}_{\Omega_k} = \sum_{k \in \mathbb{N}} ((r^{B_k})^{\tau_k})_\cdot \mathbb{1}_{\Omega_k} \in \bigcup_{k \in \mathbb{N}} O_{M_k} = \mathcal{M}
\]
as well as
\[
r^\tau \mathbb{1}_{[0, \tau]} = \sum_{k \in \mathbb{N}} (r^k)^{\tau_k} \mathbb{1}_{[0, \tau_k]} \mathbb{1}_{\Omega_k} = \sum_{k \in \mathbb{N}} (r^{B_k})^{\tau_k} \mathbb{1}_{[0, \tau_k]} \mathbb{1}_{\Omega_k} \in \bigcup_{k \in \mathbb{N}} C_{M_k} = \mathcal{M}.
\]
Using Proposition A.24 by (1.4) we obtain \( r^\tau \in \overline{\mathcal{M}} \) up to an evanescent set, proving that \( \mathcal{M} \) is prelocally invariant for (3.1). If even condition (2.9) is satisfied, then by Proposition A.24 we obtain \( r^\tau \in \overline{\mathcal{M}} \) up to an evanescent set, and hence, \( \mathcal{M} \) is locally strong invariant for (3.1). This concludes the proof of Theorem 2.2.

We shall now prove Theorem 2.5. Let \( h_0 \in \mathcal{M} \) be arbitrary. According to Theorem A.6 there exists a unique mild and weak solution \( r = r^{(h_0)} \) to (2.1). By Lemma A.2 the mapping
\[
(4.8) \quad \tau := \inf\{ t \geq 0 : r_t \notin \mathcal{M} \}
\]
is a stopping time. We claim that
\[
(4.9) \quad \mathbb{P}(\tau = \infty) = 1.
\]
Suppose, on the contrary, that relation (4.9) is not satisfied. Then, there exists $N \in \mathbb{N}$ such that $\mathbb{P}(\tau \leq N) > 0$. We define the bounded stopping time $\tau_0 := \tau \wedge N$. By the closedness of $\mathcal{M}$ in $\mathcal{H}$, we have $(r_{\tau_0})_\cdot \in \mathcal{M}$ up to an evanescent set. Therefore, by relation (1.4) and Corollary A.25 we obtain $r_{\tau_0} \in \mathcal{M}$ up to an evanescent set. The process $r_{\tau_0 + \cdot}$ is a weak solution to the time-shifted SPDE (3.1) with initial condition $r_{\tau_0}$. Indeed, for each $x \in \mathcal{D}(A^*)$ we have $\mathbb{P}$–almost surely

$\langle \zeta, r_{\tau_0 + t} \rangle = \langle \zeta, r_{\tau_0} \rangle + \int_{\tau_0}^{\tau_0 + t} \langle (A^* \zeta, r_s) + \langle \zeta, \alpha(r_s) \rangle \rangle ds + \sum_{j \in \mathbb{N}} \int_{\tau_0}^{\tau_0 + t} \langle \zeta, \sigma^j(r_s) \rangle d\beta^j_s$

$+ \int_{\tau_0}^{\tau_0 + t} \int_\mathcal{E} \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds)$

$= \langle \zeta, r_{\tau_0} \rangle + \int_{\tau_0}^{t} \langle (A^* \zeta, r_s) + \langle \zeta, \alpha(r_{s+s}) \rangle \rangle ds + \sum_{j \in \mathbb{N}} \int_{0}^{t} \langle \zeta, \sigma(r_{s+s}) \rangle d\beta^j_{s+\tau_0}, \rangle$

$+ \int_{0}^{t} \int_\mathcal{E} \langle \zeta, \gamma(r_{(s+s)} - s, x) \rangle (\mu_{\tau_0}(ds, dx) - F(dx)ds), \ t \geq 0.$

There exists $K \in \mathbb{N}$ such that $\mathbb{P}(\Gamma) > 0$, where

$\Gamma := \{ \tau \leq N \} \cap \{ \| r_{\tau_0} \| \leq K \}$.

According to Theorem A.6 there exists a unique mild solution $r^K_{\cdot}$ to (3.1) with initial condition being the bounded $\mathcal{F}_0$-measurable random variable $r_{\tau_0} \mathbb{1}_{\{\| r_{\tau_0} \| \leq K \}}$. By the second part of the proof of Theorem 2.2 the submanifold $\mathcal{M}$ is prelocally strong invariant for (3.1), and hence, there exists an $(\mathcal{F}_{\tau_0+\cdot})$-stopping time $\vartheta > 0$ such that $(r^K)^\vartheta \in \mathcal{M}$ up to an evanescent set. Noting that $\{ \tau \leq N \} = \{ \tau = \tau_0 \}$, by Proposition A.10 we obtain up to an evanescent set

$(r_{\tau_0 + \cdot})^\vartheta \mathbb{1}_{\Gamma} = (r_{\tau_0 + \cdot})^\vartheta \mathbb{1}_{\Gamma} = (r^K)^\vartheta \mathbb{1}_{\Gamma} \in \mathcal{M},$

which contradicts the Definition (4.8) of $\tau$. Therefore, relation (4.9) is valid and we obtain $r \in \mathcal{M}$ up to an evanescent set. According to Theorem 2.2, the operator $A$ is continuous on $\mathcal{M}$. Hence, Lemma A.11 implies that $r$ is a strong solution to (2.1). This finishes the proof of Theorem 2.5.

4.2. Proof of Theorem 2.6 Relation (2.12) implies (1.6). Furthermore, presuming (1.2), we have (1.5) if and only if (1.7), (1.8) are satisfied. Indeed, noting that

$Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_\mathcal{E} \gamma(h, x)F(dx)$

$= Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_\mathcal{E} \Pi_{(T_h \mathcal{M})^+} \gamma(h, x)F(dx)$

$- \Pi_{T_h \mathcal{M}} \int_\mathcal{E} \gamma(h, x)F(dx), \ h \in \mathcal{M}$

we have (1.7) if and only if

$Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_\mathcal{E} \gamma(h, x)F(dx) \in T_h \mathcal{M}, \ h \in \mathcal{M}$

and, by identity (B.7) from Lemma B.7 we have (1.8) if and only if

$Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_\mathcal{E} \gamma(h, x)F(dx) \in (T_h \mathcal{M})^+, \ h \in \partial \mathcal{M}$

showing that condition (1.5) is equivalent to (1.7), (1.8).
Since, by Theorem 2.2, the mapping in (1.7) is continuous on $\mathcal{M}$, identity (4.10) together with relations (2.6) and (2.12) and Lebesgue’s dominated convergence theorem shows that the mapping in (1.5) is continuous on $\mathcal{M}$. This finishes the proof of Theorem 2.3.

4.3. Proof of Proposition 2.8. We start with the proof of the first statement. By Lemma B.11, there exists a parametrization $\phi : V \subset \mathbb{R}^n \to U \cap \mathcal{M}$ around $h$ such that $\Phi := \phi^{-1}$ has an extension $\Phi \in C^2_b(H; \mathbb{R}^m)$. By Lemma B.10 there exists $\epsilon > 0$ such that $B_r(h) \cap \mathcal{M}$ is compact and we have

$$B_r(h) \cap \mathcal{M} \subset \overline{B_r(h)} \cap \mathcal{M} \subset U \cap \mathcal{M}.$$ 

Since $\gamma(h, \bullet)$ is continuous in $x$, by (1.3) there exists $\delta > 0$ such that

$$h + \gamma(h, \xi) \in \overline{B_r(h)} \cap \mathcal{M} \quad \text{for } F-\text{almost all } \xi \in B_r(h).$$ 

Hence, there is an $F$–nullset $N \subset B_r(h)$ such that

$$h + \gamma(h, \xi) \notin \overline{B_r(h)} \cap \mathcal{M} \quad \text{for all } \xi \in B_r(h) \setminus N.$$

By hypothesis, we may assume, without loss of generality, that $\|v\| = 1$, the function $\gamma(h, \bullet)$ is continuous on $B_r(h)$, and there is a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \delta)$ with $t_n \to 0$ such that

$$(4.11) \quad \{x + t_n v : n \in \mathbb{N}\} \subset \text{supp}(F).$$

We claim that

$$(4.12) \quad h + \gamma(h, x + t_n v) \in \overline{B_r(h)} \cap \mathcal{M} \quad \text{for all } n \in \mathbb{N}.$$ 

Indeed, suppose, on the contrary, there exists $n \in \mathbb{N}$ such that

$$h + \gamma(h, \xi) \notin \overline{B_r(h)} \cap \mathcal{M},$$

where $\xi := x + t_n v$. Since $\xi \in B_r(h)$ and $\gamma(h, \bullet)$ is continuous on $B_r(h)$, by the closedness of $\overline{B_r(h)} \cap \mathcal{M}$ in $H$ we obtain

$$h + \gamma(h, B_r(\xi)) \subset H \setminus (\overline{B_r(h)} \cap \mathcal{M})$$

for some $\eta > 0$ with $B_{\eta}(\xi) \subset B_r(h)$. This shows $B_\eta(\xi) \subset N$, and hence $F(B_\eta(\xi)) = 0$, which implies $\xi \notin \text{supp}(F)$. This contradicts (4.11), establishing (4.12). We set $y := \phi^{-1}(h) \in V$. Since $\gamma(h, x) = 0$, taking into account (4.12), we obtain

$$D_x \gamma(h, x) v = D_x (h + \gamma(h, x)) v = \lim_{n \to \infty} \frac{h + \gamma(h, x + t_n v) - (h + \gamma(h, x))}{t_n}$$

$$= \lim_{n \to \infty} \frac{\phi(\Phi(h + \gamma(h, x + t_n v))) - \phi(\Phi(h + \gamma(h, x)))}{t_n}$$

$$= D(\phi \circ \Phi \circ (h + \gamma(h, \bullet)))(x) v = D\phi(y) D(\Phi \circ (h + \gamma(h, \bullet)))(x) v \in T_h \mathcal{M},$$

proving (2.13). Under the hypothesis of the second statement, we have $y \in \partial V$, and thus

$$\langle e_1, D(\Phi \circ (h + \gamma(h, \bullet)))(x) v \rangle = \left( e_1, \lim_{n \to \infty} \frac{\Phi(h + \gamma(h, x + t_n v)) - \Phi(h + \gamma(h, x))}{t_n} \right)$$

$$= \lim_{n \to \infty} \langle e_1, \phi^{-1}(h + \gamma(h, x + t_n v)) \rangle - \langle e_1, y \rangle$$

$$= \lim_{n \to \infty} \langle e_1, \phi^{-1}(h + \gamma(h, x + t_n v)) \rangle \geq 0,$$

showing (2.14), and under the hypothesis of the third statement, the second statement and identity (B.4) yield

$$D_x \gamma(h, x) v \in (T_h \mathcal{M})_+ \cap -(T_h \mathcal{M})_+ = T_h \partial \mathcal{M},$$
proving (2.15). This completes the proof of Proposition 2.8.

4.4. Remarks on the existence of invariant manifolds. The existence of (locally) invariant submanifolds (with boundary) for a given jump-diffusion is an (even more) involved question. As in [16], Frobenius-type theorems have to be applied to construct (locally) invariant submanifolds for given drift, volatility and jump mappings. Assuming

\begin{equation}
\int_E \|\gamma(h, x)\| F(dx) < \infty, \quad h \in H
\end{equation}

we first construct submanifolds \( \mathcal{M} \) (closed for simplicity) such that \( \sigma^j \) and the mapping in (1.5) are tangent. Then \( \mathcal{M} \) is left invariant by the evolution of (1.1) if and only if the maps \( h \mapsto h + \gamma(h, x) \) leave \( \mathcal{M} \) invariant for \( F \)-almost all \( x \in E \). Notice that in this case the manifolds \( \mathcal{M} \) constructed by Frobenius-type methods from \( \sigma^j \) the mapping in (1.5) have to satisfy an additional condition.

If (4.13) is not fulfilled, then for \( h \in H \) we have to determine the set

\[ \mathcal{N}_h := \left\{ v \in H : \int_E \langle v, \gamma(h, x) \rangle |F(dx)| < \infty \right\}. \]

Let \( \mathcal{M} \) be a submanifold such that (1.4) is satisfied. Then we have (1.9), and hence (T_h,M) \( \subset \mathcal{N}_h \), which implies \( \mathcal{N}_h^+ \subset T_h \mathcal{M} \). Therefore we have to construct via the methods of [16] submanifolds \( \mathcal{M} \) such that \( \sigma^j \) and \( \mathcal{N}_h^+ \) are tangent. Then we have to check among these submanifolds \( \mathcal{M} \) those, which are left invariant by the maps \( h \mapsto h + \gamma(h, x) \) for \( F \)-almost all \( x \in E \), and which are tangent to the mapping in (1.7). Notice that in this case the actual construction of \( \mathcal{M} \) already involved the jump structure via the distribution \( h \mapsto \mathcal{N}_h^+ \).

5. Invariant manifolds with boundary for Lévy driven jump-diffusions

In this section, we investigate the invariance problem for submanifolds with boundary for the particular situation, where the Poisson random measure \( \mu \) in the SPDE (2.1) is generated by finitely many independent Lévy processes. The following additional assumptions prevail throughout this section:

- The mark space is \((E, \mathcal{E}) = (\mathbb{R}^e, B(\mathbb{R}^e))\) for some \( e \in \mathbb{N} \).
- Concerning the compensator \( dt \otimes F(dx) \) of the Poisson random measure \( \mu \), we assume that \( F \) is given by

\begin{equation}
F(B) := \sum_{k=1}^e \int_{\mathbb{R}} 1_B(xe_k) F_k(dx), \quad B \in B(\mathbb{R}^e)
\end{equation}

where \( F_1, \ldots, F_e \) are Lévy measures on \((\mathbb{R}, B(\mathbb{R}))\) such that

\begin{equation}
F_k(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} |x|^2 F_k(dx) < \infty \quad \text{for } k = 1, \ldots, e
\end{equation}

and where the \( (e_k)_{k=1,\ldots,e} \) denote the unit vectors in \( \mathbb{R}^e \).
- We assume there are a measurable mapping \( \Delta : \mathcal{M} \times \mathbb{R}^e \rightarrow H \) and continuous functions \( \delta^k : \mathcal{M} \rightarrow H, \quad k = 1, \ldots, e \) such that

\begin{equation}
\gamma(h, x) = \Delta(h, x) + \sum_{k=1}^e \delta^k(h)x_k, \quad (h, x) \in \mathcal{M} \times \mathbb{R}^e.
\end{equation}

Consequently, by (2.6) for each \( x \in \mathbb{R}^e \) the mapping \( \Delta(\cdot, x) \) is continuous.
- For each \( h \in \mathcal{M} \) we have \( \gamma(h, \cdot) \in C^1(\mathbb{R}^e; H) \) with \( \gamma(h, 0) = 0 \) and

\begin{equation}
\frac{\partial}{\partial x_k} \gamma(h, x) \bigg|_{x=0} = \delta^k(h), \quad h \in \mathcal{M}, \quad k = 1, \ldots, e.
\end{equation}
infinite variation”. Let us further introduce finite activity”, $K$ of Lévy processes from [26, Def. 11.9]. In terms of Lévy processes, (5.10) constitutes a decomposition of $K$ (5.9) for any nonnegative measurable function $g$ with $K$ means “infinite activity” but “finite variation”, and $B$ means “finite activity”, $K_B$ means “infinite activity” but “finite variation”, and $K_C$ means “infinite variation”. Let us further introduce (5.10) $K_B^+ := \{ k \in K : F_k(\mathbb{R}) = \infty \text{ and } F_k(\mathbb{R}_+) < \infty \}$, (5.11) $K_B^- := \{ k \in K : F_k(\mathbb{R}_+) = \infty \text{ and } F_k(\mathbb{R}_-) = \infty \}$. By symmetry, we may assume, without loss of generality, that we can decompose the set $K_B$ into $K_B = K_B^+ \cup K_B^-$. 5.1. Lemma. The following statements are true:

(1) For each $k \in K_B \cup K_C$ there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \to 0$ such that (5.12) $\{t_n e_k : n \in \mathbb{N}\} \subset \text{supp}(F)$ or $\{-t_n e_k : n \in \mathbb{N}\} \subset \text{supp}(F)$.

(2) For each $k \in K_B^+$ there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \to 0$ such that (5.13) $\{t_n e_k : n \in \mathbb{N}\} \subset \text{supp}(F)$. (3) For each $k \in K_B^-$ there exist sequences $(t_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n, s_n \to 0$ such that (5.14) $\{t_n e_k : n \in \mathbb{N}\} \subset \text{supp}(F)$ and $\{-s_n e_k : n \in \mathbb{N}\} \subset \text{supp}(F)$.

Proof. By the Definition (5.1) of $F$ we have (5.15) $\text{supp}(F) = \bigcup_{k=1}^\infty \{ x e_k : x \in \text{supp}(F_k) \}$.

We start with the proof of the second statement. Let $k \in K_B^+$ be arbitrary. By the Definition (5.10) of $K_B^+$ we have $F_k(\mathbb{R}_+) = \infty$. Therefore, for all $n \in \mathbb{N}$ we have (0, $\frac{1}{n}$) ∩ sup(F_k) ≠ ∅. Indeed, suppose, on the contrary, that (0, $\frac{1}{n}$) ∩ sup(F_k) = ∅ for some $n \in \mathbb{N}$. Then, for all $x \in (0, \frac{1}{n})$ there exists $\epsilon_x > 0$ such that $F_k(B_{\epsilon_x}(x)) = 0$. Using Lindelöf’s Lemma [11] Lemma 1.1.6], for some countable subset $I \subset (0, \frac{1}{n})$ we obtain $F_k\left(0, \frac{1}{n}\right) \leq F_k\left(\bigcup_{x \in I} B_{\epsilon_x}(x)\right) = F_k\left(\bigcup_{x \in I} B_{\epsilon_x}(x)\right) \leq \sum_{x \in I} F_k(B_{\epsilon_x}(x)) = 0,$
which contradicts $F_k((0, \frac{1}{\epsilon})) = \infty$. Hence, there exists a sequence $(t_n)_{n \in \mathbb{N}} \subset (0, \infty)$ with $t_n \to 0$ such that $(t_n : n \in \mathbb{N}) \subset \text{supp}(F_k)$. In view of (5.15), we obtain (5.13), proving the second statement. Analogous arguments provide the first and the third statement. □

5.2. Theorem. The statements of Theorems 2.2 and 2.5 remain true with (1.6)–(1.8) being replaced by

$$\delta^k(h) \in T_h \partial M, \quad h \in \partial M \text{ and } k \in K_C$$

(5.16)

$$Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_{\mathbb{R}^e} \left( \Delta(h, x) + \sum_{k \in K_A \cup K_B} \delta^k(h)x_k \right) F(dx) \in \begin{cases} T_h M, & h \in M \setminus \partial M \\ (T_h M)^+, & h \in \partial M \end{cases}$$

(5.17)

and the mapping in (1.7) being replaced by the mapping in (5.17). Furthermore, in either case we have

$$\delta^k(h) \in \begin{cases} T_h M, & h \in M \setminus \partial M \text{ and } k \in K_C \\ (T_h M)^+, & h \in \partial M \text{ and } k \in K_B^+ \\ T_h \partial M, & h \in \partial M \text{ and } k \in K_B^- \end{cases}$$

(5.18)

Proof. Suppose that condition (1.4) is satisfied. Taking into account Lemma 5.1 and relation (5.4), applying Proposition 2.8 with $v = e_k$ yields (5.18) and

$$\delta^k(h) \in T_h M, \quad h \in M \text{ and } k \in K_C.$$ (5.19)

Furthermore, presuming (1.2), (1.4), conditions (1.6)–(1.8) are equivalent to (5.16), (5.17). Indeed, by the decomposition (5.3) and identity (5.6), for all $h \in \partial M$ we have

$$\int_{\mathbb{R}^e} \frac{1}{e} \left| \langle \eta_h, \gamma(h, x) \rangle \right| F(dx) \leq \int_{\mathbb{R}^e} \left| \langle \eta_h, \Delta(h, x) \rangle \right| F(dx) + \sum_{k=1}^e \left| \langle \eta_h, \delta^k(h) \rangle \right| \int_{\mathbb{R}^e} |x| F_k(dx)$$

as well as

$$\sum_{k=1}^e \left| \langle \eta_h, \delta^k(h) \rangle \right| \int_{\mathbb{R}^e} |x| F_k(dx) \leq \int_{\mathbb{R}^e} \left| \langle \eta_h, \Delta(h, x) \rangle \right| F(dx) + \int_{\mathbb{R}^e} \left| \langle \eta_h, \gamma(h, x) \rangle \right| F(dx).$$

By (5.5) and the Definitions (5.7)–(5.9) of the sets $K_A, K_B, K_C$, we have (1.6) if and only if

$$\langle \eta_h, \delta^k(h) \rangle = 0, \quad h \in \partial M \text{ and } k \in K_C,$$

which, by identity (B.6) from Lemma B.7, is equivalent to (5.16). This establishes the equivalence (1.6) ⇔ (5.16). By the decomposition (5.3) and relations (5.19),
Since, by Theorem 2.2, the mapping in (1.7) is continuous on \(\Delta(\cdot, x)\), we have
\[
Ah + \alpha(h) = \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_{\mathbb{R}^n} \left( \Delta(h, x) + \sum_{k \in K_A \cup K_B} \delta^k(h)x_k \right) F(dx)
\]
\[
= Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_{\mathbb{R}^n} \left( \gamma(h, x) - \sum_{k \in K_C} \delta^k(h)x_k \right) F(dx)
\]
\[
= Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_{\mathbb{R}^n} \Pi_{(T, \mathcal{M})}^- \gamma(h, x) F(dx)
\]
\[
- \int_{\mathbb{R}^n} \left( \Pi_{T, \mathcal{M}} \gamma(h, x) - \sum_{k \in K_C} \delta^k(h)x_k \right) F(dx)
\]
\[
= Ah + \alpha(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_{\mathbb{R}^n} \Pi_{(T, \mathcal{M})}^- \gamma(h, x) F(dx)
\]
\[
- \Pi_{T, \mathcal{M}} \left( \int_{\mathbb{R}^n} \Delta(h, x) F(dx) + \sum_{k \in K_A \cup K_B} \delta^k(h) \int_{\mathbb{R}} x F_k(dx) \right), \quad h \in \mathcal{M}.
\]
Thus, we have \((1.7)\) if and only if
\[
Ah + \alpha(h) = \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)
\]
\[
- \int_{\mathbb{R}^n} \left( \Delta(h, x) + \sum_{k \in K_A \cup K_B} \delta^k(h)x_k \right) F(dx) \in T_h \mathcal{M}, \quad h \in \mathcal{M}.
\]
By the decomposition \((5.3)\) and identity \((5.20)\) we have
\[
\langle \eta_h, Ah + \alpha(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \eta_h, D\sigma^j(h)\sigma^j(h) \rangle - \int_{E} \langle \eta_h, \gamma(h, x) \rangle F(dx)
\]
\[
= \langle \eta_h, Ah + \alpha(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)
\]
\[
- \int_{\mathbb{R}^n} \left( \Delta(h, x) + \sum_{k \in K_A \cup K_B} \delta^k(h)x_k \right) F(dx) \}, \quad h \in \partial \mathcal{M}.
\]
Therefore, by identity \((B.7)\) from Lemma \((B.7)\) we have \((1.8)\) if and only if
\[
Ah + \alpha(h) = \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)
\]
\[
- \int_{\mathbb{R}^n} \left( \Delta(h, x) + \sum_{k \in K_A \cup K_B} \delta^k(h)x_k \right) F(dx) \in (T_h \mathcal{M})_+, \quad h \in \partial \mathcal{M}.
\]
Consequently, conditions \((1.6) - (1.8)\) are equivalent to \((5.16), (5.17)\).

By our assumptions stated at the beginning of the section, for each \(x \in E\) the mapping \(\Delta(\cdot, x)\) is continuous on \(\mathcal{M}\) and for each \(k \in K_A \cup K_B\) the mapping \(\delta^k\) is continuous on \(\mathcal{M}\). Since, by Theorem 2.2, the mapping in \((1.7)\) is continuous on \(\mathcal{M}\), identity \((5.21)\) together with relation \((5.5)\) and Lebesgue’s dominated convergence theorem shows that the mapping in \((5.17)\) is continuous on \(\mathcal{M}\).

6. Examples of invariant manifolds with boundary

In order to demonstrate our results from the previous sections, we provide four examples in this section. In the first two examples, we treat the unit circle and the closed unit ball, both in the Euclidean plane \(\mathbb{R}^2\). In our third example, we
consider Ornstein-Uhlenbeck processes on closed, convex cones in separable Hilbert spaces, and in the fourth example, we deal with the existence of finite dimensional realizations for HJM interest rate models with jumps.

6.1. **Stochastic invariance of the unit circle.** Let the state space $H = \mathbb{R}^2$ be the Euclidean plane and let

$$\mathcal{M} = \{ h \in \mathbb{R}^2 : \| h \| = 1 \}$$

be the unit circle. Then $\mathcal{M}$ is a one-dimensional submanifold without boundary, i.e. $\partial \mathcal{M} = \emptyset$, which is a closed subset of $\mathbb{R}^2$. We choose $\mathbb{H} = \mathbb{R}$, i.e. the Wiener process $W$ is one-dimensional, and the mark space $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Concerning the compensator $dt \otimes F(dx)$ of the Poisson random measure $\mu$, we assume that $F$ is a Lévy measure satisfying

$$F(\{ 0 \}) = 0, \quad \int_{-\pi}^{\pi} x^2 F(dx) < \infty \quad \text{and} \quad F(\mathbb{R} \setminus [-\pi, \pi]) = 0.$$

Consider the two-dimensional SDE

$$\begin{align*}
\mathrm{d} r_t &= \alpha(r_t) \mathrm{d} t + \sigma(r_t) \mathrm{d} W_t + \int_{\mathbb{R}} \gamma(r_{t-}, x) (\mu(\mathrm{d} t, dx) - F(dx) \mathrm{d} t) \\
\mathrm{r}_0 &= h_0,
\end{align*}$$

where the mappings $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ are given by

$$\begin{align*}
\alpha(h) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h - \left( \frac{1}{2} - \int_{-\pi}^{\pi} (\cos x - 1) F(dx) \right) h, \\
\sigma(h) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h = \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix}, \\
\gamma(h, x) &= \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} h - h = \begin{pmatrix} \cos x - 1 & \sin x \\ -\sin x & \cos x - 1 \end{pmatrix} h.
\end{align*}$$

In order to apply Theorem 2.5, it suffices to show that

$$\begin{align*}
\sigma(h) \in T_h \mathcal{M}, & \quad h \in \mathcal{M} \\
h + \gamma(h, x) \in \mathcal{M}, & \quad (h, x) \in \mathcal{M} \times [-\pi, \pi] \\
\alpha(h) - \frac{1}{2} D\sigma(h) \sigma(h) - \int_{\mathbb{R}} \Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x) F(dx) \in T_h \mathcal{M}, & \quad h \in \mathcal{M}.
\end{align*}$$

The unit circle $\mathcal{M}$ has the tangent spaces

$$T_h \mathcal{M} = \text{span} \left\{ \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix} \right\} \quad \text{and} \quad (T_h \mathcal{M})^\perp = \text{span}\{ h \} \quad \text{for} \ h \in \mathcal{M}.$$

Therefore, condition (6.2) satisfied. Furthermore, for all $(h, x) \in \mathcal{M} \times [-\pi, \pi]$ we have, by noting that $\| h \| = 1$,

$$\| h + \gamma(h, x) \|^2 = \left\| \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} h \right\|^2 = h_1^2 \cos^2 x + 2h_1 h_2 \cos x \sin x + h_2^2 \sin^2 x + h_1^2 \sin^2 x - 2h_1 h_2 \cos x \sin x + h_2^2 \cos^2 x = 1,$$

showing (6.3). Since

$$D\sigma(h) \sigma(h) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma(h) = -h, \quad h \in \mathbb{R}^2$$

and for $(h, x) \in \mathcal{M} \times \mathbb{R}$ we have, by noting that $\| h \| = 1$,

$$\Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x) = \Pi_{\text{span}(h)} \gamma(h, x) = \langle \gamma(h, x), h \rangle h$$

$$= (h_2^2 \cos x + h_1 h_2 \sin x - h_1 h_2 \sin x + h_2^2 \cos x) h - h = (\cos x - 1) h,$$
for all $h \in \mathcal{M}$ we obtain
\[ \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_{\mathbb{R}} \Pi_{(T_h\mathcal{M})^\perp} \gamma(h, x) F(dx) = \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix} \in T_h\mathcal{M} \]
showing that (6.4) is fulfilled. Now, Theorem 2.6 applies and yields that for each $h_0 \in \mathcal{M}$ there exists a unique strong solution $r = r^{(h_0)}$ to (6.1) and $r \in \mathcal{M}$ up to an evanescent set.

6.1. Remark. Note that due to (6.6) we have
\[ \int_{\mathbb{R}} \|\Pi_{(T_h\mathcal{M})^\perp} \gamma(h, x)\| F(dx) < \infty, \quad h \in \mathcal{M} \]
showing that relation (1.9) is indeed true.

An alternative way to prove the previous invariance result is established by applying Theorem 5.2. We shall consider the case $\int_{-\pi}^{\pi} |x| F(dx) = \infty$, i.e. $K_A = K_B = 0$ and $K_C = \{1\}$. Note that we have the decomposition
\[ \gamma(h, x) = \Delta(h, x) + \delta(h)x, \quad (h, x) \in \mathcal{M} \times \mathbb{R} \]
where the mappings $\delta : \mathcal{M} \to \mathbb{R}^2$ and $\Delta : \mathcal{M} \times \mathbb{R} \to \mathbb{R}^2$ are given by
\[ \delta(h) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h = \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix}, \]
\[ \Delta(h, x) := \begin{pmatrix} \cos x - 1 & \sin x - x \\ -(\sin x - x) & \cos x - 1 \end{pmatrix} h, \]
In order to apply Theorem 5.2 it suffices to show that
\[ \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_{-\pi}^{\pi} \Delta(h, x) F(dx) \in T_h\mathcal{M}, \quad h \in \mathcal{M}. \]
Taking into account (6.5), we obtain
\[ \alpha(h) - \frac{1}{2} D\sigma(h)\sigma(h) - \int_{-\pi}^{\pi} \Delta(h, x) F(dx) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h + \int_{-\pi}^{\pi} \begin{pmatrix} 0 & -(\sin x - x) \\ \sin x - x & 0 \end{pmatrix} F(dx) \]
\[ = \begin{pmatrix} 1 - \int_{-\pi}^{\pi} (\sin x - x) F(dx) \end{pmatrix} \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix} \in T_h\mathcal{M}, \quad h \in \mathcal{M} \]
showing (6.9). Now, Theorem 5.2 applies and yields that for each $h_0 \in \mathcal{M}$ there exists a unique strong solution $r = r^{(h_0)}$ to (6.1) and $r \in \mathcal{M}$ up to an evanescent set.

6.2. Remark. Note that by the Definition (6.7) of $\delta$ we have
\[ \delta(h) \in T_h\mathcal{M}, \quad h \in \mathcal{M} \]
showing that relation (5.18) is indeed true.

6.2. Stochastic invariance of the closed unit ball. As in Section 6.1, let the state space $H = \mathbb{R}^2$ be the Euclidean plane. Let the submanifold be the closed unit ball
\[ \mathcal{M} = \{ h \in \mathbb{R}^2 : \|h\| \leq 1 \}. \]
Then $\mathcal{M}$ is a two-dimensional submanifold with boundary of $\mathbb{R}^2$, which is a closed subset of $\mathbb{R}^2$, and its boundary is the unit circle
\[ \partial\mathcal{M} = \{ h \in \mathbb{R}^2 : \|h\| = 1 \}. \]
As before, we choose $H = \mathbb{R}$, i.e. the Wiener process $W$ is one-dimensional, and the mark space $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Concerning the compensator $dt \otimes F(dx)$ of the Poisson random measure $\mu$, we assume that $F$ is a Lévy measure satisfying

$$F([0]) = 0, \quad \int_0^1 x^2 F(dx) < \infty \quad \text{and} \quad F(\mathbb{R} \setminus [0,1]) = 0.$$  

We define the mappings $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ and $\gamma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ of the two-dimensional SDE (6.1) as

$$\alpha(h) := -\left(a + \frac{1}{2} + 2 \int_0^1 x^2 F(dx)\right) h,$$

$$\sigma(h) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} h = \begin{pmatrix} h_2 \\ -h_1 \end{pmatrix},$$

$$\gamma(h, x) := -2xf(h),$$

where $a \geq 0$ is a constant and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function satisfying the boundary condition

(6.10) \hspace{1cm} f(h) = 2, \quad h \in \partial \mathcal{M}.

In order to apply Theorem [2.3], it suffices to show that

(6.11) \hspace{1cm} \langle \eta_h, \sigma(h) \rangle = 0, \quad h \in \partial \mathcal{M}

(6.12) \hspace{1cm} h + \gamma(h, x) \in \mathcal{M}, \quad (h, x) \in \mathcal{M} \times [0,1]

(6.13) \hspace{1cm} \int_\mathbb{R} |\langle \eta_h, \gamma(h, x) \rangle| F(dx) < \infty, \quad h \in \partial \mathcal{M}

(6.14) \hspace{1cm} \langle \eta_h, \alpha(h) \rangle - \frac{1}{2} \langle \eta_h, D\sigma(h)\sigma(h) \rangle - \int_\mathbb{R} \langle \eta_h, \gamma(h, x) \rangle F(dx) \geq 0, \quad h \in \partial \mathcal{M}.

The inward pointing normal (tangent) vectors to $\partial \mathcal{M}$ at boundary points are given by

$$\eta_h = -h, \quad h \in \partial \mathcal{M}.$$  

Therefore, condition (6.11) is satisfied. Moreover, for all $(h, x) \in \mathcal{M} \times [0,1]$ we have, by noting that $\|h\| \leq 1$,

$$\|h + \gamma(h, x)\| = \|h - 2xf(h)\| = \|(1 - 2xf(h))h\| = |1 - 2xf(h)| \|h\| \leq 1 \in \mathcal{M},$$

providing (6.12). By the boundary condition (6.10), for all $h \in \partial \mathcal{M}$ we have, by noting that $\|h\| = 1$,

$$\langle \eta_h, \gamma(h, x) \rangle = \langle h, 2xf(h)h \rangle = 2x^2, \quad x \in \mathbb{R}$$

which gives us

(6.15) \hspace{1cm} \int_\mathbb{R} |\langle \eta_h, \gamma(h, x) \rangle| F(dx) = \int_\mathbb{R} \langle \eta_h, \gamma(h, x) \rangle F(dx) = 2 \int_0^1 x^2 F(dx) < \infty,

showing (6.13). Moreover, by (6.5) and (6.15), for each boundary point $h \in \partial \mathcal{M}$ we have, by noting that $\|h\| = 1$,

$$\langle \eta_h, \alpha(h) \rangle - \frac{1}{2} \langle \eta_h, D\sigma(h)\sigma(h) \rangle - \int_\mathbb{R} \langle \eta_h, \gamma(h, x) \rangle F(dx)$$

$$= a + \frac{1}{2} + 2 \int_0^1 x^2 F(dx) - \frac{1}{2} - 2 \int_0^1 x^2 F(dx) = a \geq 0,$$

which yields (6.14). Now, Theorem [2.6] applies and yields that for each $h_0 \in \mathcal{M}$ there exists a unique strong solution $r = r^{(h_0)}$ to (6.1) and $r \in \mathcal{M}$ up to an evanescent set.
6.3. **Ornstein-Uhlenbeck processes on closed, convex cones.** In our third example, let $H$ be a separable Hilbert space and let $A$ be the generator of a pseudo-contractive semigroup $(S_t)_{t \geq 0}$ on $H$. We consider an SPDE of Ornstein-Uhlenbeck type

\begin{equation}
\begin{aligned}
    dr_t &= (Ar_t + \alpha) dt + \sum_{j \in \mathbb{N}} \sigma^j d\beta^j_t + \int_E \gamma(x) (\mu(dt, dx) - F(dx) dt) \\
    r_0 &= h_0
\end{aligned}
\end{equation}

with $\alpha \in H$, a sequence $(\sigma^j)_{j \in \mathbb{N}} \subset H$ satisfying $\sum_{j \in \mathbb{N}} \|\sigma^j\|^2 < \infty$ and a measurable mapping $\gamma : E \to H$ such that $\int_E \|\gamma(x)\|^2 F(dx) < \infty$. Note that for each $h_0 \in H$ there exists a unique mild solution $r = r^{(h_0)}$ to (6.16), which is given by

\begin{equation}
\begin{aligned}
    r_t &= S_t h_0 + \int_0^t S_{t-s} \alpha ds + \sum_{j \in \mathbb{N}} \int_0^t S_{t-s} \sigma^j d\beta^j_s \\
    &+ \int_0^t \int_E S_{t-s} \gamma(x)(\mu(ds, dx) - F(dx) ds), \quad t \geq 0.
\end{aligned}
\end{equation}

Let $C \subset H$ be a closed, convex cone, i.e. $C$ is a nonempty, closed subset of $H$ such that $h + g \in C$ for all $h, g \in C$ and $\lambda h \in C$ for all $\lambda \geq 0$ and $h \in C$.

6.3. **Proposition.** Suppose we have

\begin{align}
    \sigma^j &= 0 \quad \text{for all } j \in \mathbb{N}, \\
    \gamma(x) &\in C \quad \text{for } F\text{-almost all } x \in E, \\
    \int_E \|\gamma(x)\| F(dx) &< \infty, \\
    \alpha - \int_E \gamma(x) F(dx) &\in C, \\
    S_t C &\subset C, \quad t \geq 0.
\end{align}

Then, for each $h_0 \in C$ the mild solution $r = r^{(h_0)}$ to (6.16) satisfies $r \in C$ up to an evanescent set.

**Proof.** Let $h_0 \in C$ be arbitrary. The mild solution $r = r^{(h_0)}$ to (6.16) is given by (6.17), and hence, by (6.18) and (6.20) we can write it as

\begin{equation}
\begin{aligned}
    r_t &= S_t h_0 + \int_0^t S_{t-s} \alpha ds - \int_0^t \int_E S_{t-s} \gamma(x)(\mu(ds, dx) - F(dx) ds), \quad t \geq 0.
\end{aligned}
\end{equation}

Since (6.19), (6.21) and (6.22) are satisfied, using Lemmas A.14 and A.15 we deduce that $r \in C$ up to an evanescent set. \hfill \Box

Now, we deal with the necessity of conditions (6.18)–(6.22). For the sake of simplicity, we shall assume that $C$ is a polyhedral cone generated by linearly independent vectors, that is

\[ C := \text{cone}\{v_1, \ldots, v_m\} := \left\{ \sum_{i=1}^m \lambda_i v_i : \lambda_i \geq 0 \text{ for } i = 1, \ldots, m \right\}. \]

where $m \in \mathbb{N}$ denotes a positive integer and $v_1, \ldots, v_m \in H$ are linearly independent vectors. We also define the interior of the cone

\[ (\text{cone}\{v_1, \ldots, v_m\})^\circ := \left\{ \sum_{i=1}^m \lambda_i v_i : \lambda_i > 0 \text{ for } i = 1, \ldots, m \right\} \]

and the finite dimensional subspace

\[ V := \text{span}\{v_1, \ldots, v_m\}. \]
6.4. **Theorem.** The following statements are equivalent:

1. For each \( h_0 \in C \) there exists a unique strong solution \( r = r^{(h_0)} \) to \((6.16)\) and \( r \in C \) up to an evanescent set.
2. We have \((6.18)\)–\((6.21)\) and

\[
\begin{align*}
(6.23) & \quad V \subset \mathcal{D}(A), \\
(6.24) & \quad AV \subset V, \\
(6.25) & \quad e^{tA}C \subset C, \quad t \geq 0
\end{align*}
\]

where \((e^{tA})_{t \geq 0}\) denotes the norm-continuous semigroup on \(V\) generated by the linear operator \(A|_V : V \to V\).

**Proof.** \((2) \Rightarrow (1):\) Conditions \((6.18)\)–\((6.21)\) imply that \( \alpha \in V, \sigma^j \in V \) for all \( j \in \mathbb{N}\) and \( \gamma(x) \in V \) for \( F\)-almost all \( x \in E\). Thus, in view of \((6.23), (6.24)\), the SPDE \((6.16)\) is an \(V\)-valued SDE. By \((6.25)\) and Proposition 5.3 for each \( h_0 \in C \) there exists a unique strong solution \( r = r^{(h_0)} \) to \((6.16)\) and \( r \in C \) up to an evanescent set.

\((1) \Rightarrow (2):\) We define \( \partial M_i := (\text{cone}\{v_j : j \neq i\})^\circ \) for \( i = 1, \ldots, m \). Then we have \( M_i \cap M_j = \emptyset \) for \( i \neq j \) and the subset

\[
\mathcal{M} := C^\circ \cup \left( \bigcup_{i=1}^m \partial M_i \right)
\]

is an \(m\)-dimensional \(C^0\)-submanifold with boundary \( \partial \mathcal{M} = \bigcup_{i=1}^m \partial M_i \). The closure of \( \mathcal{M} \) as a subset of \( H \) is given by \( \overline{\mathcal{M}} = C \), and we obtain the tangent spaces

\[
\begin{align*}
(6.26) & \quad T_h \mathcal{M} = V, \quad h \in \mathcal{M} \\
(6.27) & \quad (T_h \mathcal{M})_+ = \text{cone}\{v_i\} + \text{span}\{v_j : j \neq i\}, \quad i = 1, \ldots, m \quad \text{and} \quad h \in \partial M_i \\
(6.28) & \quad T_h \partial M = \text{span}\{v_j : j \neq i\}, \quad i = 1, \ldots, m \quad \text{and} \quad h \in \partial M_i.
\end{align*}
\]

There exist unique \( w_1, \ldots, w_m \in V \) such that

\[
(6.29) \quad \eta_h = w_{ij}, \quad i = 1, \ldots, m \quad \text{and} \quad h \in \partial M_i,
\]

where \( \eta_h \) denotes the inward pointing normal (tangent) vector to \( \partial M_i \) at \( h \). The vectors \( w_1, \ldots, w_m \in V \) are linearly independent. Indeed, let \( c \in \mathbb{R}^m \) with \( \sum_{j=1}^m c_j w_j = 0 \) be arbitrary. Then we have \( Bc = 0 \), where \( B \in \mathbb{R}^{m \times m} \) denotes the matrix given by \( B_{ij} := \langle v_i, v_j \rangle \). By Lemma B.7 we have \( B_{ii} > 0 \) for \( i = 1, \ldots, n \) and \( B_{ij} = 0 \) for \( i \neq j \). Therefore, we obtain \( \det B > 0 \), which implies \( c = 0 \).

By hypothesis, the submanifold \( \mathcal{M} \) is prelocally invariant for \((6.16)\). Theorem 2.2 applies and yields \((1.2)\)–\((1.4)\) as well as \((1.6)\)–\((1.8)\). For each \( i = 1, \ldots, m \) relation \((1.2)\) gives us

\[
\sum_{j=1}^m v_j \in \mathcal{D}(A) \quad \text{and} \quad \sum_{j \neq i} v_j \in \mathcal{D}(A),
\]

which yields

\[
v_i = \sum_{j=1}^m v_j - \sum_{j \neq i} v_j \in \mathcal{D}(A),
\]

showing \((6.23)\). Taking into account \((6.28)\), condition \((1.3)\) yields

\[
\sigma^k \in \bigcap_{i=1}^m \text{span}\{v_j : j \neq i\} = \{0\} \quad \text{for all} \quad k \in \mathbb{N},
\]

and therefore \((6.18)\). Using \((1.4)\) we obtain

\[
\frac{1}{n} \sum_{i=1}^m v_i + \gamma(x) \in \overline{\mathcal{M}} = C \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{for} \ F\text{-almost all} \ x \in E
\]
which implies
\[ \gamma(x) = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{m} v_i + \gamma(x) \right) \in C \quad \text{for } F\text{-almost all } x \in E, \]
showing (6.19). For an arbitrary \( w = \sum_{i=1}^{m} c_i w_i \in V \), by (6.29), (1.6) we obtain
\[ \int_{E} |\langle w, \gamma(x) \rangle| F(dx) \leq \sum_{i=1}^{m} |c_i| \int_{E} |\langle w_i, \gamma(x) \rangle| F(dx) < \infty. \]
Therefore, denoting by \( (e_j)_{j=1}^{m} \subset V \) an orthonormal basis of \( V \) and taking into account (6.19), we deduce
\[ \int_{E} \|\gamma(x)\| F(dx) \leq \sum_{j=1}^{m} \int_{E} |\langle e_j, \gamma(x) \rangle| F(dx) < \infty, \]
proving (6.20). Condition (1.7) yields
\[ A h + \alpha - \int_{E} \Pi_{(T_h M)^+} \gamma(x) F(dx) \in T_h M, \quad h \in \mathcal{M}. \]
Hence, by (6.19), (6.26) we get
\[ A h + \alpha - \int_{E} \gamma(x) F(dx) \in V, \quad h \in C^\circ. \]
By (1.8) and identity (B.7) from Lemma B.7 we have
\[ A h + \alpha - \int_{E} \gamma(x) F(dx) \in (T_h M)^+, \quad h \in \partial \mathcal{M}. \]
Thus, by (6.27) for all \( i = 1, \ldots, m \) we obtain
\[ A h + \alpha - \int_{E} \gamma(x) F(dx) \in \text{cone} \{v_i\} + \text{span} \{v_j : j \neq i\}, \quad h \in \partial \mathcal{M}_i, \]
which gives us
\[ \alpha - \int_{E} \gamma(x) F(dx) = \lim_{n \to \infty} \left[ A \left( \frac{1}{n} \sum_{j \neq i} v_j \right) + \alpha - \int_{E} \gamma(x) F(dx) \right] \in \text{cone} \{v_i\} + \text{span} \{v_j : j \neq i\}. \]
Consequently, we have
\[ \alpha - \int_{E} \gamma(x) F(dx) \in \bigcap_{i=1}^{m} \left( \text{cone} \{v_i\} + \text{span} \{v_j : j \neq i\} \right) = C, \]
proving (6.21). By (6.30), (6.31), for \( i = 1, \ldots, m \) we get
\[ A v_i = A \left( \sum_{j=1}^{m} v_j \right) + \alpha - \int_{E} \gamma(x) F(dx) - \left[ A \left( \sum_{j \neq i} v_j \right) + \alpha - \int_{E} \gamma(x) F(dx) \right] \in V, \]
showing (6.24). Let \( t \geq 0 \) and \( h \in C \) be arbitrary. Since \( r^{(h_0)} \) up to an evanescent set for all \( h_0 \in C \), by (6.17) and the continuity of the mapping \( \lambda \mapsto \lambda e^{t A} h \) we have \( \mathbb{P} \)-almost surely
\[ r_t^{(0)} + \lambda e^{t A} h = e^{t A} \lambda h + r_t^{(0)} = r_t^{(\lambda h)} \in C \quad \text{for all } \lambda \in \mathbb{R}_+, \]
which yields \( e^{t A} h \in C \), showing (6.25).
We can apply Theorem 6.4 in order to derive a well-known invariance result about Lévy processes. Let $X = (X^{(x_0)})_{x_0 \in \mathbb{R}}$ be a family of one-dimensional Lévy processes with Lévy-Itô decomposition

$$X^{(x_0)}_t = x_0 + \alpha t + \sigma W_t + \int_0^t \int_{\mathbb{R}} x \{ \mu(ds,dx) - F(dx)ds \}, \quad t \geq 0 \quad (6.32)$$

where $\alpha \in \mathbb{R}$ denotes the drift, $\sigma \geq 0$ the diffusion part, $W$ a one-dimensional Wiener process and $F$ the Lévy measure, which we assume to be square-integrable, that is $\int_{\mathbb{R}} x^2 F(dx) < \infty$. Then, by Theorem 6.4, the half space $\mathbb{R}_+$ is invariant for (6.32) if and only if we have

$$\sigma = 0, \quad F(\mathbb{R}_-) = 0, \quad \int_{\mathbb{R}_+} x F(dx) < \infty \quad \text{and} \quad \alpha - \int_{\mathbb{R}_+} x F(dx) \geq 0.$$ 

Indeed, these conditions are also known to be necessary and sufficient for a Lévy processes to be a subordinator, i.e., to have non-decreasing sample paths, see [26, Thm. 21.5].

6.4. HJM interest rate models from finance. In this section, we shall deal with the existence of finite dimensional realizations for the HJMM (Heath–Jarrow–Morton–Musiela) equation

$$\begin{cases}
\frac{dr_t}{r_t} &= \left( \frac{d}{dt} r_t + \alpha_{\text{HJM}}(r_t) \right) dt + \sum_{j \in \mathbb{N}} \sigma^j(r_t) d\beta^j_t \\
&\quad + \int_E \gamma(r_{t-},x)(\mu(dt,dx) - F(dx)dt) \\
r_0 &= h_0 \quad (6.33)
\end{cases}$$

from interest rate theory. The HJMM equation (6.33) describes the evolution of interest rates in a bond market

$$P(t, T) = \exp \left( - \int_0^{T-t} r_t(\xi) d\xi \right). \quad (6.34)$$

In order to ensure the absence of arbitrage, we consider the HJMM equation (6.33) under a martingale measure $Q$. Then, the drift is given by

$$\alpha_{\text{HJM}}(h) := \sum_{j \in \mathbb{N}} \sigma^j(h) \Sigma^j(h) - \int_E \gamma(h,x) \left( e^{\Gamma(h,x)} - 1 \right) F(dx),$$

where we have set

$$\Sigma^j(h)(\xi) := \int_0^\xi \sigma^j(h)(\eta)d\eta,$$

$$\Gamma(h,x)(\xi) := - \int_0^\xi \gamma(h,x)(\eta)d\eta.$$ 

The state space $H = H_\beta$ for the forward curve evolution (6.33) is the space of all absolutely continuous functions $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|h\|_{\beta} := \left( |h(0)|^2 + \int_{\mathbb{R}_+} |h'(\xi)|^2 e^{\beta \xi} d\xi \right)^{1/2} < \infty.$$

The parameter $\beta > 0$ is an arbitrary positive constant, and $(S_t)_{t \geq 0}$ denotes the shift semigroup defined by $S_t h := h(t + \cdot)$ for $t \geq 0$, which is generated by the differential operator $d/d\xi$. We refer to [13] for further details on this topic.

Proceeding as in Section 4.4, we construct a finite dimensional submanifold $\mathcal{M}$, which is (locally) invariant for the HJMM equation (6.33) if and only if for $F$–almost all $x \in E$ the mapping $h \mapsto h + \gamma(h,x)$ leaves $\mathcal{M}$ invariant. Under appropriate
conditions on the mappings, the results from [16] yield that we have an affine term structure, that is, the submanifold $\mathcal{M}$ has an affine parametrization of form
\[
\phi(t, y) = \psi(t) + \sum_{i=1}^{m} y_i h_i
\]
with a smooth map $\psi : [0, \epsilon) \to H$ and $h_1, \ldots, h_m \in H$. We immediately see that
\[
\gamma(h, x) \in \text{span}\{h_1, \ldots, h_m\}, \quad \text{for } F^-\text{-almost all } x \in E, \quad \text{for all } h \in H
\]
implies the jump condition (1.4), and hence, it is a sufficient condition for invariance of the submanifold $\mathcal{M}$ with boundary for the HJMM equation (6.33).

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Appendix A. SPDEs driven by Wiener processes and Poisson measures

For convenience of the reader, we provide the crucial results on SPDEs driven by Wiener processes and Poisson measures in this appendix. References on this topic are, e.g., [2] [21] [14].

In the sequel, \((\Omega, F, (F_t)_{t \geq 0}, \mathbb{P})\) denotes a filtered probability space satisfying the usual conditions. Let \(H\) be a separable Hilbert space and let \((S_t)_{t \geq 0}\) be a \(C_0\)-semigroup on \(H\) with infinitesimal generator \(A : \mathcal{D}(A) \subset H \rightarrow H\). We denote by \(A^* : \mathcal{D}(A^*) \subset H \rightarrow H\) the adjoint operator of \(A\). Recall that the domains \(\mathcal{D}(A)\) and \(\mathcal{D}(A^*)\) are dense in \(H\), see, e.g., [25] Thm. 13.35.c, Thm. 13.12.

Let \(\mathbb{H}\) be another separable Hilbert space and let \(Q \in L(\mathbb{H})\) be a nuclear, self-adjoint, positive definite linear operator. Then, there exist an orthonormal basis \((e_j)_{j \in \mathbb{N}}\) of \(\mathbb{H}\) and a sequence \((\lambda_j)_{j \in \mathbb{N}} \subset (0, \infty)\) with \(\sum_{j \in \mathbb{N}} \lambda_j < \infty\) such that

\[
Qu = \sum_{j \in \mathbb{N}} \lambda_j (u, e_j)_{\mathbb{H}} e_j, \quad u \in \mathbb{H}
\]

namely, the \(\lambda_j\) are the eigenvalues of \(Q\), and each \(e_j\) is an eigenvector corresponding to \(\lambda_j\). The space \(\mathbb{H}_0 := Q^{1/2}(\mathbb{H})\), equipped with the inner product

\[
(u, v)_\mathbb{H}_0 := (Q^{-1/2}u, Q^{-1/2}v)_\mathbb{H},
\]

is another separable Hilbert space. According to [8] Prop. 4.1, the sequence of stochastic processes \((\beta^j)_{j \in \mathbb{N}}\) defined as

\[
\beta^j := \frac{1}{\sqrt{\lambda_j}} (W, e_j), \quad j \in \mathbb{N}
\]

is a sequence of real-valued independent standard Wiener processes and we have the expansion

\[
W = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \beta^j e_j.
\]

Note that \(L^0_2(H) \cong \mathbb{L}^2(H)\), because

\[
(A.1) \quad L^0_2(H) \rightarrow \mathbb{L}^2(H), \quad \Phi \mapsto (\Phi^j)_{j \in \mathbb{N}} \quad \text{with} \quad \Phi^j := \sqrt{\lambda_j} \Phi e_j, \quad j \in \mathbb{N}
\]

is an isometric isomorphism. According to [8] Thm. 4.3, for every predictable process \(\Phi : \Omega \times \mathbb{R}_+ \rightarrow L^0_2(H)\) satisfying

\[
P\left( \int_0^t \|\Phi_s\|_{L^2_2(H)}^2 ds < \infty \right) = 1 \quad \text{for all} \quad t \geq 0
\]

we have the identity

\[
(A.2) \quad \int_0^t \Phi_s dW_s = \sum_{j \in \mathbb{N}} \int_0^t \Phi^j_s d\beta^j_s, \quad t \geq 0.
\]

Let \((E, \mathcal{E})\) be a measurable space which we assume to be a Blackwell space (see [9] [13]). We remark that every Polish space with its Borel \(\sigma\)-field is a Blackwell space. Furthermore, let \(\mu\) be a time-homogeneous Poisson random measure on \(\mathbb{R}_+ \times E\), see [19] Def. II.1.20. Then its compensator is of the form \(dt \otimes F(dx)\), where \(F\) is a \(\sigma\)-finite measure on \((E, \mathcal{E})\).
A.1. Lemma. Let $\tau_0$ be a bounded stopping time. We define:

- The filtration $(\mathcal{F}^{(\tau_0)}_t)_{t \geq 0}$ by
  $$\mathcal{F}^{(\tau_0)}_t := \mathcal{F}_{\tau_0 + t}, \quad t \geq 0.$$ 

- The $\mathbb{H}$-valued process $W^{(\tau_0)}$ by
  $$W^{(\tau_0)}_t := W_{\tau_0 + t} - W_{\tau_0}, \quad t \geq 0.$$ 

- The sequence $(\beta^{(\tau_0)}_j)_{j \in \mathbb{N}}$ of real-valued processes by
  $$\beta^{(\tau_0)}_j t := \beta^j_{\tau_0 + t} - \beta^j_{\tau_0}, \quad t \geq 0.$$ 

- The new random measure $\mu^{(\tau_0)}$ on $\mathbb{R}_+ \times E$ by
  $$\mu^{(\tau_0)}(\omega; B) := \mu(\omega; B(\sigma(\omega))), \quad \omega \in \Omega \text{ and } B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E},$$
  where we use the notation $B_{\tau_0} := \{(t + \tau_0, x) \in \mathbb{R}_+ \times E : (t, x) \in B\}.$

Then, $W^{(\tau_0)}$ is a $\mathcal{Q}$-Wiener process adapted to $(\mathcal{F}^{(\tau_0)}_t)_{t \geq 0}$, the sequence $(\beta^{(\tau_0)}_j)_{j \in \mathbb{N}}$ is a sequence of real-valued independent standard Wiener processes, adapted to $(\mathcal{F}^{(\tau_0)}_t)_{t \geq 0}$, we have the expansion

$$W^{(\tau_0)} = \sum_{j \in \mathbb{N}} \sqrt{\lambda_j} \beta^{(\tau_0)}_j e_j$$

and $\mu^{(\tau_0)}$ is a time-homogeneous Poisson random measure relative to the filtration $(\mathcal{F}^{(\tau_0)}_t)_{t \geq 0}$ with compensator $dt \otimes F(dx)$.

Proof. See [15, Lemma 4.6].

A.2. Lemma. Let $r$ be an $\mathbb{H}$-valued càdlàg adapted process and let $B \in \mathcal{B}(\mathbb{H})$ be a Borel set. Then $\tau : \Omega \to \mathbb{R}_+$ defined as

$$\tau := \inf\{t \geq 0 : r_t \in B\}$$

is a stopping time.

Proof. We claim that the process $r$ is optional, that is, measurable with respect to the optional $\sigma$-algebra $\mathcal{O}$, which is generated by all real-valued càdlàg adapted processes. Indeed, according to [8, Prop. 1.3] we have

$$\mathcal{B}(\mathbb{H}) = \sigma(\{(\langle h, \cdot \rangle) \in C : h \in \mathbb{H} \text{ and } C \in \mathcal{B}(\mathbb{R})\}).$$

But for any $h \in \mathbb{H}$ and $C \in \mathcal{B}(\mathbb{R})$ we have

$$r^{-1}(\langle h, \cdot \rangle \in C) = \{(h, r) \in C\} \in \mathcal{O},$$

because $\langle h, \cdot \rangle$ is a real-valued càdlàg adapted process. Therefore, the set $A = \{r \in B\}$ is optional, and hence

$$\tau(\omega) = \inf\{t \geq 0 : (\omega, t) \in A\}$$

is a stopping time due to [19, Thm. I.1.27].

We shall now focus on SPDEs of the type (1.1). Let $\alpha : H \to H$, $\sigma : H \to L^2_0(H)$ and $\gamma : H \times E \to H$ be measurable mappings.
A.3. Definition. Let \( h_0 : \Omega \to H \) be an \( F_0 \)-measurable random variable. Furthermore, let \( r = r^{(h_0)} \) be an \( H \)-valued càdlàg adapted process and let \( \tau > 0 \) a stopping time such that for all \( t \geq 0 \) we have
\[
P\left( \int_0^\tau (\|r_s\| + \|\sigma(r_s)\|^2) + \int_E \|\gamma(r_s,x)\|^2 F(dx) \, ds \right) < \infty = 1. \]

Then, the process \( r \) is called

- a local strong solution to \((1.1)\), if

\begin{equation}
(A.3) \quad r_{t \wedge \tau} \in \mathcal{D}(A) \quad \text{for almost all } t \in \mathbb{R}_+, \quad P-\text{almost surely,}
\end{equation}

\begin{equation}
(A.4) \quad P\left( \int_0^{t \wedge \tau} \|Ar_s\| \, ds < \infty \right) = 1 \quad \text{for all } t \geq 0
\end{equation}

and we have \( P-\text{almost surely} \)
\[
r_{t \wedge \tau} = h_0 + \int_0^{t \wedge \tau} (Ar_s + \alpha(r_s)) \, ds + \int_0^{t \wedge \tau} \sigma(r_s) \, dW_s
\quad + \int_0^{t \wedge \tau} E \gamma(r_{s-},x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\]

- a local weak solution to \((1.1)\), if for all \( \zeta \in \mathcal{D}(A^*) \) we have \( P-\text{almost surely} \)
\[
\langle \zeta, r_{t \wedge \tau} \rangle = \langle \zeta, h_0 \rangle + \int_0^{t \wedge \tau} \langle A^* \zeta, r_s \rangle \, ds + \int_0^{t \wedge \tau} \langle \zeta, \sigma(r_s) \rangle dW_s
\quad + \int_0^{t \wedge \tau} E \langle \zeta, \gamma(r_{s-},x) \rangle (\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\]

- a local mild solution to \((1.1)\), if we have \( P-\text{almost surely} \)
\[
r_{t \wedge \tau} = S_{t \wedge \tau} h_0 + \int_0^{t \wedge \tau} S_{(t \wedge \tau)-s} \alpha(r_s) \, ds + \int_0^{t \wedge \tau} S_{(t \wedge \tau)-s} \sigma(r_s) \, dW_s
\quad + \int_0^{t \wedge \tau} E S_{(t \wedge \tau)-s} \gamma(r_{s-},x)(\mu(ds,dx) - F(dx)ds), \quad t \geq 0.
\]

We call \( \tau \) the lifetime of \( r \). If \( \tau = \infty \), then we call \( r \) a strong, weak or mild solution to \((1.1)\), respectively.

A.4. Remark. Since the process \( r \) is càdlàg, we have
\[
r_t = r_{t-} \quad \text{for almost all } t \in \mathbb{R}_+, \quad P-\text{almost surely,}
\]

and hence, relation \((A.3)\) implies
\[
r_{(t \wedge \tau)-} \in \mathcal{D}(A) \quad \text{for almost all } t \in \mathbb{R}_+, \quad P-\text{almost surely.}
\]

According to \cite[Lemma 2.4.2]{[13]}, the process \( f \) defined by
\[
(A.6) \quad f_t := \begin{cases} Ar_{t-}, & \text{if } r_{t-} \in \mathcal{D}(A) \\ 0, & \text{otherwise} \end{cases}
\]
is predictable. By slight abuse of notation, we have written \( Ar \) instead of \( f \) in \((A.3)\) and \((A.5)\).

A.5. Remark. The following results are well-known:

- Every (local) strong solution to \((1.1)\) is also a (local) weak solution to \((1.1)\).
- Every (local) weak solution to \((1.1)\) is also a (local) mild solution to \((1.1)\).
- If \( A \) is bounded, i.e. generates a norm-continuous semigroup \( (S_t)_{t \geq 0} \), then the concepts of (local) strong, weak and mild solutions to \((1.1)\) are equivalent.

For our upcoming results, we introduce the abbreviation \( L^2(F) := L^2(E, \mathcal{E}, F; H) \).
A.6. Theorem. We suppose that:
• The semigroup \((S_t)_{t \geq 0}\) is pseudo-contractive.
• \(\alpha : H \to H\) is Lipschitz continuous.
• \(\sigma : H \to L^0_2(H)\) is Lipschitz continuous.
• For each \(h \in H\) we have \(\gamma(h, \bullet) \in L^2(F)\) and \(\gamma : H \to L^2(F)\) is Lipschitz continuous.

Then, for each random variable \(h_0 \in L^2(F_0; H)\) there exists a mild and weak solution \(r = r(h_0)\) to \((\ref{1.1})\), which is unique up to indistinguishability.

Proof. This is a direct consequence of \cite{[14]} Cor. 10.9. \hfill \Box

A.7. Remark. Recall that the semigroup \((S_t)_{t \geq 0}\) is called pseudo-contractive if \((\ref{2.17})\) is satisfied for some \(\omega \in \mathbb{R}\). The pseudo-contractive property is needed to ensure that mild solutions to the SPDE \((\ref{1.1})\) have càdlàg sample paths, which we demand in Definition \ref{A.3}. If \(A\) is the generator of a general \(C_0\)-semigroup, then, under appropriate regularity conditions we also have existence and uniqueness of mild solutions, see \cite{[21]}, but the mild solution to \((\ref{1.1})\) might not have a càdlàg version, see the counterexample in \cite{[24]} Prop. 9.25.

A.8. Proposition. We suppose that:
• The semigroup \((S_t)_{t \geq 0}\) is pseudo-contractive.
• \(\alpha : H \to H\) is locally Lipschitz continuous.
• \(\sigma : H \to L^0_2(H)\) is locally Lipschitz continuous.
• For each \(h \in H\) we have \(\gamma(h, \bullet) \in L^2(F)\) and \(\gamma : H \to L^2(F)\) is locally Lipschitz continuous.

Then, for every bounded \(F_0\)-measurable random variable \(h_0 : \Omega \to H\) there exists a local mild and weak solution \(r = r(h_0)\) to \((\ref{1.1})\).

Proof. Let \(h_0 : \Omega \to H\) be a bounded \(F_0\)-measurable random variable. Then, there exists \(N \in \mathbb{N}\) such that \(\|h_0\| < N\). We define the retraction
\[
R : H \to C, \quad R(h) := \begin{cases} h, & \text{if } \|h\| \leq N, \\ N \cdot \frac{h}{\|h\|}, & \text{if } \|h\| > N, \end{cases}
\]
where \(C := \{h \in H : \|h\| \leq N\}\), and the mappings \(\alpha_R : H \to H\), \(\sigma_R : H \to L^0_2(H)\) and \(\gamma_R : H \times E \to H\) as
\[
\alpha_R := \alpha \circ R, \quad \sigma_R := \sigma \circ R \quad \text{and} \quad \gamma_R(\bullet, x) := \gamma(\bullet, x) \circ R.
\]
These mappings satisfy the Lipschitz conditions from Theorem \ref{A.6} and hence, there exists a unique mild and weak solution \(r = r(h_0)\) to the SPDE
\[
\begin{align*}
\{ & dr_t = (Ar_t + \alpha_R(r_t))dt + \sigma_R(r_t)dW_t + \int_E \gamma_R(r_{t-}, x)(\mu(dt, dx) - F(dx)dt) \\
r_0 & = h_0,
\}
\]
By Lemma \ref{A.2} the mapping
\[
\tau := \inf\{t \geq 0 : \|r_t\| \geq N\}
\]
is a strictly positive stopping time. Since \((r^*_\tau)_- \in C\) up to an evanescent set, \(\alpha_C = \alpha_R|_C\), \(\sigma_C = \sigma_R|_C\) and \(\gamma(\bullet, x)|_C = \gamma(\bullet, x)\circ R|_C\) for all \(x \in E\), we deduce that \(r\) is a local mild and weak solution to \((\ref{1.1})\) with lifetime \(\tau\). \hfill \Box

A.9. Proposition. For each \(k \in \mathbb{N}\) let \(h^k_0 : \Omega \to H\) be an \(F_0\)-measurable random variable and let \(r^k\) be a local mild solution to \((\ref{1.1})\) with initial condition \(h^k_0\) and lifetime \(\tau^k > 0\). Let \((\Omega_k)_{k \in \mathbb{N}} \subset F_0\) be a decomposition of \(\Omega\). Then, \(\tau := \sum_{k \in \mathbb{N}} \tau_k \|\Omega_k\|\) is a strictly positive stopping time and \(r := \sum_{k \in \mathbb{N}} r^k \|\Omega_k\|\) is a local mild solution to \((\ref{1.1})\) with initial condition \(h_0 := \sum_{k \in \mathbb{N}} h^k_0 \|\Omega_k\|\) and lifetime \(\tau\).
Proof. For each \( k \in \mathbb{N} \) the mapping \( \tau_k \mathbb{1}_{\Omega_k} \) is a stopping time, because, taking into account that \( \Omega_k \in \mathcal{F}_t \) we have
\[
\{ \tau_k \mathbb{1}_{\Omega_k} \leq t \} = (\Omega_k \cap \{ \tau_k \leq t \}) \cup \Omega_k^c \in \mathcal{F}_t \quad \text{for all } t \geq 0.
\]
Consequently, \( \tau = \bigvee_{k \in \mathbb{N}} \tau_k \mathbb{1}_{\Omega_k} \) is a strictly positive stopping time by [19, Thm. I.1.18]. For each \( k \in \mathbb{N} \), the process \( r^k \) be a local mild solution to \([\text{1.1}]\) with initial condition \( h_0^k \) and lifetime \( \tau_k \), and hence, we have \( \mathbb{P} \)-almost surely
\[
r_{t \wedge \tau}^k = \sum_{k \in \mathbb{N}} r_{t \wedge \tau}^k \mathbb{1}_{\Omega_k}
= \sum_{k \in \mathbb{N}} \mathbb{1}_{\Omega_k} \left( S_{t \wedge \tau_k} h_0^k + \int_0^{t \wedge \tau_k} S_{(t \wedge \tau_k) - s} \alpha(r_s^k) ds + \int_0^{t \wedge \tau_k} S_{(t \wedge \tau_k) - s} \sigma(r_s^k) dW_s \right)
+ \int_0^{t \wedge \tau_k} \int_E S_{(t \wedge \tau_k) - s} \gamma(r_s^k, x) (\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]
Since \( \Omega_k \in \mathcal{F}_0 \) for each \( k \in \mathbb{N} \), this implies \( \mathbb{P} \)-almost surely
\[
r_{t \wedge \tau} = \sum_{k \in \mathbb{N}} r_{t \wedge \tau}^k \mathbb{1}_{\Omega_k}
= \sum_{k \in \mathbb{N}} \mathbb{1}_{\Omega_k} \left( S_{t \wedge \tau_k} h_0^k + \int_0^{t \wedge \tau_k} S_{(t \wedge \tau_k) - s} \alpha(r_s^k) ds + \int_0^{t \wedge \tau_k} S_{(t \wedge \tau_k) - s} \sigma(r_s^k) dW_s \right)
+ \int_0^{t \wedge \tau_k} \int_E S_{(t \wedge \tau_k) - s} \gamma(r_s^k, x) (\mu(ds, dx) - F(dx)ds)
\]
showing that \( r \) is a local mild solution to \([\text{1.1}]\) with initial condition \( h_0 \) and lifetime \( \tau \).

A.10. Proposition. We suppose that:

- \( \alpha : H \to H \) is locally Lipschitz continuous.
- \( \sigma : H \to L^2_0(H) \) is locally Lipschitz continuous.
- For each \( h \in H \) we have \( \gamma(h, \bullet) \in L^2(F) \) and \( \gamma : H \to L^2(F) \) is locally Lipschitz continuous.

Then, for all \( \mathcal{F}_0 \)-measurable random variables \( h_0, g_0 : \Omega \to H \) and any two local mild solutions \( r^{(h_0)}, r^{(g_0)} \) to \([\text{1.1}]\) with initial conditions \( h_0, g_0 \) and lifetimes
For all \( t \) we have up to indistinguishability
\[
(r^{(h_0)})^\tau(h_0) \mathbb{1}_{\{h_0=g_0\}} = (r^{(g_0)})^\tau(g_0) \mathbb{1}_{\{h_0=g_0\}}.
\]

**Proof.** By Lemma A.2, the increasing sequence \((\tau_n)_{n \in \mathbb{N}}\) given by
\[
\tau_n := \inf \{ t \geq 0 : \|r_t^{(h_0)} \| \geq n \} \wedge \inf \{ t \geq 0 : \|r_t^{(g_0)} \| \geq n \} \wedge \tau^{(h_0)} \wedge \tau^{(g_0)}
\]
is a sequence of stopping times. Since the sample paths of \( r^{(h_0)} \) and \( r^{(g_0)} \) are càdlàg, we have \( \mathbb{P} \)–almost surely
\[
\sup_{t \in [0,N]} \|r_t^{(h_0)} \| < \infty \quad \text{and} \quad \sup_{t \in [0,N]} \|r_t^{(g_0)} \| < \infty \quad \text{for all } N \in \mathbb{N}.
\]
Therefore, we have \( \mathbb{P}(\tau_n \to \tau^{(h_0)} \wedge \tau^{(g_0)}) = 1 \). Set \( \Gamma := \{ h_0 = g_0 \} \in \mathcal{F}_0 \) and let \( n \in \mathbb{N} \) and \( T \geq 0 \) be arbitrary. We define the process \( X^n \) as
\[
X^n_t := \mathbb{1}_\Gamma (r_t^{(h_0)} - r_t^{(g_0)}), \quad t \in [0,T].
\]
For all \( t \in [0,T] \) we have \( \mathbb{P} \)–almost surely
\[
X^n_t = \mathbb{1}_\Gamma \left( \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} (\alpha(r_u^{(h_0)})) \, du + \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} (\sigma(r_u^{(h_0)})) \, dW_u \right.
\]
\[
+ \left. \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} \mathcal{L}(\gamma(r_u^{(h_0)}, x) - \gamma(r_u^{(g_0)}, x)) \right) (\mu(du, dx) - F(dx)du). \tag{A.7}
\]

Since \( \Gamma \in \mathcal{F}_0 \), for all \( t \in [0,T] \) we obtain \( \mathbb{P} \)–almost surely
\[
X^n_t = \mathbb{1}_\Gamma \left( \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} (\alpha(r_u^{(h_0)})) \, du + \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} (\sigma(r_u^{(h_0)})) \, dW_u \right.
\]
\[
+ \left. \int_0^{\tau^{(h_0)} \wedge \tau^{(g_0)}} S_{t \wedge \tau_n} S_{(t \wedge \tau_n) - u} \mathcal{L}(\gamma(r_u^{(h_0)}, x) - \gamma(r_u^{(g_0)}, x)) \right) (\mu(du, dx) - F(dx)du). \tag{A.7}
\]

There are constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that
\[
\|S_t\| \leq Me^{\omega t} \quad \text{for all } t \geq 0,
\]
see, e.g., [23] Thm. 13.35.a. Furthermore, by the assumed local Lipschitz continuity of the mappings, there exists a constant \( L_n \geq 0 \) such that
\[
\|\alpha(h_1) - \alpha(h_2)\| \leq L_n \|h_1 - h_2\|,
\]
\[
\|\sigma(h_1) - \sigma(h_2)\| \leq L_n \|h_1 - h_2\|,
\]
\[
\left( \int_E \|\gamma(h_1, x) - \gamma(h_2, x)\|^2 F(dx) \right)^{1/2} \leq L_n \|h_1 - h_2\|
\]
for all \( h_1, h_2 \in H \) with \( \|h_1\|, \|h_2\| \leq n \). By the Cauchy-Schwarz inequality and the Itô isometry, for all \( 0 \leq s \leq t \leq T \) we obtain
\[
\mathbb{E} \left[ \|X^n_t - X^n_s\|^2 \right] \leq C_n \int_s^t \mathbb{E} \left[ \|X^n_u\|^2 \right] du.
\]
where the constant \( C_n > 0 \) is given by
\[
C_n = 3(T + 2)(Me^{\omega T}L_n)^2.
\]
Note that for all \( h, g \in H \) we have
\[
\|h\|^2 - \|g\|^2 \leq \|h - g\|^2 + 2\|g\| \|h - g\|.
\]
Therefore, estimate (A.7) and the Cauchy-Schwarz inequality yield
\[
|E[\|X_t^n\|^2] - E[\|X_s^n\|^2]| \leq E[\|X_t^n - X_s^n\|^2] + 2E[(\|X_t^n\|^2)^{1/2}E[(\|X_s^n - X_t^n\|^2)^{1/2}]
\leq C_n(2n)^2(t - s) + 2\sqrt{C_n(2n)^2(t - s)}C_n(2n)^2(t - s) \leq C_n(2n)^2((t - s) + 2\sqrt{T} \sqrt{T - s}) \text{ for all } 0 \leq s \leq t \leq T.
\]
Hence, the nonnegative function
\[
[0, T] \to \mathbb{R}, \quad t \mapsto E[\|X_t^n\|^2]
\]
is continuous. In view of (A.7), the Gronwall lemma applies and yields
\[
E[\|X_t^n\|^2] = 0 \text{ for all } t \in [0, T].
\]
Since \( T \geq 0 \) was arbitrary and the sample paths of \( r^{(h_0)} \) and \( r^{(g_0)} \) are càdlàg, recalling that \( \mathbb{P}(\tau_n \to \tau^{(h_0)} \wedge \tau^{(g_0)}) = 1 \) we deduce that up to an evanescent set
\[
(r^{(h_0)})^{(h_0)} \wedge (g_0) \mathbb{1}_\Gamma = \lim_{n \to \infty} (r^{(h_0)})^{\tau_n} \mathbb{1}_\Gamma = \lim_{n \to \infty} (r^{(g_0)})^{\tau_n} \mathbb{1}_\Gamma = (r^{(g_0)})^{(h_0) \wedge (g_0)} \mathbb{1}_\Gamma,
\]
completing the proof.

**A.11. Lemma.** Let \( \mathcal{M} \subset D(A) \) be a subset such that \( A \) is continuous on \( \mathcal{M} \), and let \( r = r^{(h_0)} \) be a local weak solution to (1.1) with lifetime \( \tau > 0 \) for some \( \mathcal{F}_0 \)-measurable random variable \( h_0 : \Omega \to H \) such that \( (r^\tau)_- \in \mathcal{M} \) up to an evanescent set. Then, \( r \) is also a local strong solution to (1.1) with lifetime \( \tau \).

**Proof.** Condition (A.3) is satisfied, because \((r^\tau)_- \in \mathcal{M} \subset D(A)\) up to an evanescent set, and condition (A.4) is satisfied due to the continuity of \( A \) on \( \mathcal{M} \). Taking into account Remark A.4 we obtain for each \( \zeta \in D(A^*) \) that \( \mathbb{P} \)-almost surely
\[
\langle \zeta, r_{t, \wedge \tau} \rangle = \langle \zeta, h_0 \rangle + \int_0^{t \wedge \tau} (A^* \zeta, r_s) + \langle \zeta, \sigma(r_s) \rangle ds + \int_0^{t \wedge \tau} \langle \zeta, \sigma(r_s) \rangle dW_s
+ \int_0^{t \wedge \tau} \langle \zeta, \gamma(r_{s-}, x) \rangle (\mu(ds, dx) - F(dx)ds)
\leq \langle \zeta, h_0 \rangle + \int_0^{t \wedge \tau} (Ar_s + \alpha(r_s))ds + \int_0^{t \wedge \tau} \sigma(r_s)dW_s
+ \int_0^{t \wedge \tau} \int_E \gamma(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\]
Since \( D(A^*) \) is dense in \( H \), we get \( \mathbb{P} \)-almost surely
\[
r_{t, \wedge \tau} = h_0 + \int_0^{t \wedge \tau} (Ar_s + \alpha(r_s))ds + \int_0^{t \wedge \tau} \sigma(r_s)dW_s
+ \int_0^{t \wedge \tau} \int_E \gamma(r_{s-}, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0,
\]
showing that \( r \) is a local strong solution to (1.1) with lifetime \( \tau \).

**A.12. Remark.** According to [19] Prop. II.1.14], there exist a sequence \((\tau_n)_{n \in \mathbb{N}}\) of finite stopping times with \([\tau_n] \cap [\tau_m] = \emptyset \) for \( n \neq m \) and an \( E \)-valued optional process \( \xi \) such that for every optional process \( \gamma : \Omega \times \mathbb{R}_+ \times E \to H \) with
\[
\mathbb{P}\left( \int_0^t \int_E \|\gamma(s, x)\|\mu(ds, dx) < \infty \right) = 1 \quad \text{for all } t \geq 0
\]
we have
\[
\int_0^t \int_E \gamma(s, x)\mu(ds, dx) = \sum_{n \in \mathbb{N}} \gamma(\tau_n, \xi_{\tau_n}) \mathbb{1}_{\{\tau_n \leq t\}}, \quad t \geq 0.
\]
Furthermore, for every predictable process \( \gamma : \Omega \times \mathbb{R}_+ \times E \to H \) with
\[
\mathbb{P} \left( \int_0^t \int_E \| \gamma(s,x) \|^2 F(dx) ds < \infty \right) = 1 \quad \text{for all } t \geq 0
\]
the jumps of the integral process
\[
X_t := \int_0^t \int_E \gamma(s,x) (\mu(ds,dx) - F(dx) ds), \quad t \geq 0
\]
are given by
\[
(A.10) \quad \Delta X_t = \gamma(t, \zeta_t) \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau_n = t\}}, \quad t \geq 0,
\]
see [19, Sec. II.1.d].

A.13. **Lemma.** Let \( r = r^{(h_0)} \) be a local weak solution to \((1.1)\) with lifetime \( \tau > 0 \) for some \( \mathcal{F}_0 \)-measurable random variable \( h_0 : \Omega \to H \). Then, the following statements are true:

1. The jumps of the stopped process \( r^\tau \) are given by
\[
\Delta r_{t \wedge \tau} = \gamma(r(t \wedge \tau)^-, \xi_{t \wedge \tau}) \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau_n = t \wedge \tau\}}, \quad t \geq 0.
\]

2. For each \( n \in \mathbb{N} \) we have
\[
\Delta r_{\tau_n} \mathbb{1}_{\{\tau_n \leq \tau\}} = \gamma(r_{\tau_n}^-, \xi_{\tau_n}) \mathbb{1}_{\{\tau_n \leq \tau\}}.
\]

**Proof.** Let \( X \) be the process
\[
X_t := \int_0^{t \wedge \tau} \int_E \langle \zeta, \gamma(r_{t \wedge \tau}^-, x) \rangle (\mu(ds,dx) - F(dx) ds), \quad t \geq 0.
\]
Since \( r \) is a local weak solution to \((1.1)\), for every \( \zeta \in \mathcal{D}(A^*) \) we have, by using \((A.10)\),
\[
\langle \zeta, \Delta r_{t \wedge \tau} \rangle = \Delta \langle \zeta, r_{t \wedge \tau} \rangle = \Delta X_{t \wedge \tau} = \langle \zeta, \gamma(r(t \wedge \tau)^-, \xi_{t \wedge \tau}) \rangle \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau_n = t \wedge \tau\}}
\]
\[
= \langle \zeta, \gamma(r(t \wedge \tau)^-, \xi_{t \wedge \tau}) \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\tau_n = t \wedge \tau\}} \rangle, \quad t \geq 0.
\]
Taking into account that \( \mathcal{D}(A^*) \) is dense in \( H \), the first statement follows. Since \([\tau_n] \cap [\tau_m] = \emptyset \) for \( n \neq m \), we deduce that
\[
\Delta r_{\tau_n} \mathbb{1}_{\{\tau_n \leq \tau\}} = \Delta r_{\tau_n \wedge \tau} \mathbb{1}_{\{\tau_n \leq \tau\}} = \left( \gamma(r_{\tau_n \wedge \tau}^-, \xi_{\tau_n \wedge \tau}) \sum_{m \in \mathbb{N}} \mathbb{1}_{\{\tau_m = \tau_n \wedge \tau\}} \right) \mathbb{1}_{\{\tau_n \leq \tau\}}
\]
\[
= \left( \gamma(r_{\tau_n}^-, \xi_{\tau_n}) \sum_{m \in \mathbb{N}} \mathbb{1}_{\{\tau_m = \tau_n\}} \right) \mathbb{1}_{\{\tau_n \leq \tau\}} = \gamma(r_{\tau_n}^-, \xi_{\tau_n}) \mathbb{1}_{\{\tau_n \leq \tau\}}
\]
for each \( n \in \mathbb{N} \), establishing the second statement. \( \square \)

Recall that a closed, convex cone \( C \) is a nonempty, closed subset \( C \subset H \) such that \( h + g \in C \) for all \( h, g \in C \) and \( \lambda h \in C \) for all \( \lambda \geq 0 \) and \( h \in C \).

A.14. **Lemma.** Let \((G, G, \nu)\) be a \( \sigma \)-finite measure space, let \( C \subset H \) be a closed, convex cone and let \( f \in L^1(G; H) \) be such that \( f(x) \in C \) for \( \nu \)-almost all \( x \in G \). Then we have
\[
\int_G f \, d\nu \in C.
\]
Proof. First, we assume that $f \in L^1(G; H)$ is a simple function of the form

$$f = \sum_{k=1}^{m} c_k 1_{A_k}$$

with $c_k \in C$ and $A_k \in G$ satisfying $\nu(A_k) < \infty$ for $k = 1, \ldots, m$. Then we have

$$\int_G f d\nu = \sum_{k=1}^{m} c_k \nu(A_k) \in C.$$

Now, let $f \in L^1(G; H)$ be an arbitrary function such that $f(x) \in C$ for $\nu$–almost all $x \in G$. Arguing as in the proof of [8, Lemma 1.1], there exists a a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions of the form (A.11) such that $f_n \to f$ in $L^1(G; H)$. Therefore, we get

$$\int_G f d\nu = \lim_{n \to \infty} \int_G f_n d\nu \in C,$$

finishing the proof. \qed

A.15. Lemma. Let $C \subset H$ be a closed, convex cone and let $\gamma : \Omega \times \mathbb{R}_+ \times E \to H$ be an optional process satisfying (A.8) such that $\gamma(\bullet, x) \in C$ up to an evanescent set, for $F$–almost all $x \in E$. Then we have $X \in C$ up to an evanescent set, where $X$ denotes the integral process $X_t := \int_0^t \int_E \gamma(s, x) \mu(ds, dx)$, $t \geq 0$.

Proof. By assumption, there is an $F$–nullset $N$ such that $\gamma(\bullet, x) \in C$ up to an evanescent set, for all $x \in N^c$. Using identity (A.9) and [19, Thm. II.1.8] we obtain

$$\mathbb{E}\left[\sum_{n \in \mathbb{N}} 1_{\{\xi_n \in N\}}\right] = \mathbb{E}\left[\int_0^\infty \int_E 1_{\{x \in N\}} \mu(dx) ds\right] = \mathbb{E}\left[\int_0^\infty \int_E 1_{\{x \in N\}} F(dx) ds\right] = 0,$$

which gives us $\mathbb{P}(\xi_n \notin N$ for all $n \in \mathbb{N}) = 1$. Using (A.9) we obtain $\mathbb{P}$–almost surely

$$X_t = \sum_{n \in \mathbb{N}} \gamma(\tau_n, \xi_n) 1_{\{\tau_n \leq t\}} \in C \quad \text{for all } t \geq 0,$$

finishing the proof. \qed

In this text, we apply the following version of Itô’s formula.

A.16. Theorem. Let $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}^m$, $\sigma : \Omega \times \mathbb{R}_+ \to L_2^0(\mathbb{R}^m)$ and $\gamma : \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}^m$ be predictable processes such that for all $t \geq 0$ we have

$$\mathbb{P}\left(\int_0^t \left(\|\alpha_s\| + \|\sigma_s\|^2_{L_2^0(\mathbb{R}^m)} + \int_E \|\gamma(s, x)\|^2 F(dx) ds\right) ds < \infty\right) = 1.$$

Furthermore, let $Y_0 : \Omega \to \mathbb{R}^m$ be an $\mathcal{F}_0$-measurable random variable, let $Y$ be the $\mathbb{R}^m$-valued Itô process

$$Y_t = Y_0 + \int_0^t \alpha_s ds + \sum_{j \in \mathbb{N}} \int_0^t \sigma_{j,s} d\beta^j_s + \int_0^t \int_E \gamma(s, x) (\mu(ds, dx) - F(dx)) ds,$$
and let \( \phi \in C^2(\mathbb{R}^m; H) \) be arbitrary. Then we have \( \mathbb{P} \)-almost surely

\[
\phi(Y_t) = \phi(Y_0) + \int_0^t \left( D\phi(Y_s)\alpha_s + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2\phi(Y_s)(\sigma_j^s, \sigma_j^s) \right) ds + \int_E \left( \phi(Y_s + \gamma(s, x)) - \phi(Y_s) - D\phi(Y_s)\gamma(s, x) \right) F(dx) ds \\
+ \sum_{j \in \mathbb{N}} \int_0^t D\phi(Y_s)\sigma_j^s d\beta^j_s \\
+ \int_0^t \left( \phi(Y_{s-} + \gamma(s, x)) - \phi(Y_{s-}) \right) (\mu(ds, dx) - F(dx)ds), \quad t \geq 0
\]

where \( \sigma_j^s :\mathbb{N} \to \mathbb{R} \) for each \( j \in \mathbb{N} \).

**Proof.** This follows from applying Itô’s formula for finite dimensional semimartingales (see [13, Thm. A.4.57]) to \( (h, \phi(Y)) \) for each \( h \in H \). \( \square \)

From now on, we fix mappings \( \alpha : H \to H, \sigma^j : H \to H, j \in \mathbb{N}, \gamma : H \times E \to H \) satisfying the regularity conditions (2.2), (2.4) and (2.6), (2.8), where the respective functions \( \rho_n : E \to \mathbb{R}_+ \), only need to satisfy (3.1). Using the identification \( L^0_2(H) \cong \ell^2(H) \), which holds true by the isometric isomorphism defined in (A.1), we can identify the sequence \( (\sigma^j)_{j \in \mathbb{N}} \) of mappings \( \sigma^j : H \to H \) with a locally Lipschitz continuous mapping \( \sigma : H \to L^2_0(H) \).

**A.17. Lemma.** The following statements are true:

1. For each \( h \in H \) we have

\[
\sum_{j \in \mathbb{N}} \| D\sigma^j(h)\sigma^j(h) \| < \infty.
\]

2. The mapping

\[
H \to H, \quad h \mapsto \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)
\]

is continuous.

**Proof.** Let \( j \in \mathbb{N} \) be arbitrary. Furthermore, let \( h \in H \) be arbitrary. There exists \( n \in \mathbb{N} \) such that \( \| h \| \leq n \). By estimates (2.3), (2.4) we have

\[
\sum_{j \in \mathbb{N}} \| \sigma^j(h) \|^2 < \infty,
\]

showing that the first statement holds true. For each \( j \in \mathbb{N} \) the mapping

\[
H \to H, \quad D\sigma^j(h)\sigma^j(h)
\]

is continuous, because for all \( h_1, h_2 \in H \) we have

\[
\begin{align*}
\| D\sigma^j(h_1)\sigma^j(h_1) - D\sigma^j(h_2)\sigma^j(h_2) \| & \leq \| D\sigma^j(h_1)\sigma^j(h_1) - D\sigma^j(h_1)\sigma^j(h_2) \| + \| D\sigma^j(h_1)\sigma^j(h_2) - D\sigma^j(h_2)\sigma^j(h_2) \| \\
& \leq \| D\sigma^j(h_1) \| \| \sigma^j(h_1) - \sigma^j(h_2) \| + \| D\sigma^j(h_2) \| \| \sigma^j(h_2) \|.
\end{align*}
\]

Denoting by \( \nu \) the counting measure on \( (\mathbb{N}, \mathcal{B}(\mathbb{N})) \) given by \( \nu(\{ j \}) = 1 \) for all \( j \in \mathbb{N} \), we can express the mapping

\[
\sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) = \int_\mathbb{N} D\sigma^j(h)\sigma^j(h) \nu(\{ j \}).
\]

Taking into account estimate (A.14), Lebesgue’s dominated convergence theorem yields the continuity of the mapping (A.13). \( \square \)
A.18. Lemma. Let $B \in \mathcal{E}$ be a set with $F(B^c) < \infty$.

(1) For each $h \in H$ we have

\[(A.15) \quad \int_{B^c} \|\gamma(h,x)\| F(dx) < \infty.\]

(2) The mappings $\alpha^B : H \to H$ and $\gamma^B : H \times E \to H$ defined as

\[(A.16) \quad \alpha^B(h) := \alpha(h) - \int_{B^c} \gamma(h,x) F(dx),\]
\[(A.17) \quad \gamma^B(h,x) := \gamma(h,x) \mathbb{1}_B(x)\]

also satisfy the regularity conditions (2.2), (2.6), (2.7).

Proof. Let $h \in H$ be arbitrary. There exists $n \in \mathbb{N}$ with $\|h\| \leq n$. By the Cauchy-Schwarz inequality and (2.7), (3.4) we have

\[
\int_{B^c} \|\gamma(h,x)\| F(dx) \leq F(B^c)^{1/2} \left( \int_{E} \|\gamma(h,x)\|^2 F(dx) \right)^{1/2}
\]

\[
\leq F(B^c)^{1/2} \left( \int_{E} \rho_n(x)^2 F(dx) \right)^{1/2} < \infty,
\]

showing (A.15). Now, let $n \in \mathbb{N}$ and $h_1, h_2 \in H$ with $\|h_1\|, \|h_2\| \leq n$ be arbitrary. By the Cauchy-Schwarz inequality and (2.6) we obtain

\[
\left\| \int_{B^c} \gamma(h_1,x) F(dx) - \int_{B^c} \gamma(h_2,x) F(dx) \right\| \leq \int_{B^c} \|\gamma(h_1,x) - \gamma(h_2,x)\| F(dx)
\]

\[
\leq F(B^c)^{1/2} \left( \int_{E} \|\gamma(h_1,x) - \gamma(h_2,x)\|^2 F(dx) \right)^{1/2}
\]

\[
\leq F(B^c)^{1/2} \left( \int_{E} \rho_n(x)^2 F(dx) \right)^{1/2} \|h_1 - h_2\|,
\]

which, in view of (3.4), proves that $\alpha^B$ also satisfies (2.2). Furthermore, the mapping $\gamma^B$ also satisfies (2.6), (2.7), which directly follows from its Definition (A.17). \hfill \Box

A.19. Lemma. For every set $B \in \mathcal{E}$ with $F(B^c) < \infty$ the process

\[N_t := \mu([0,t] \times B^c), \quad t \geq 0\]

is a càdlàg, adapted process with $N_0 = 0$, $N \in \mathbb{N}_0$ and $\Delta N \in \{0,1\}$ up to an evanescent set, and we have the representation

\[(A.18) \quad N_t = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\xi_n \not\equiv B\}} \mathbb{1}_{\{\tau_n \leq t\}}, \quad t \geq 0.\]

Proof. We have $N_0 = 0$, because $\mu(\omega; \{0\} \times E) = 0$ for all $\omega \in \Omega$ by the definition of a random measure, see \cite{[10]} Def. II.1.3. By (A.9) we have

\[N_t = \mu([0,t] \times B^c) = \int_0^t \int_{E} \mathbb{1}_{\{x \not\equiv B\}} \mathbb{1}_{\{s \leq t\}} \mu(ds,dx)
\]

\[= \sum_{n \in \mathbb{N}} \mathbb{1}_{\{\xi_n \not\equiv B\}} \mathbb{1}_{\{\tau_n \leq t\}}, \quad t \geq 0\]

which provides the representation (A.18) and shows that $N \in \mathbb{N}_0$. Since

\[E[N_t] = E[\mu([0,t] \times B^c)] = t F(B^c) < \infty \quad \text{for all } t \geq 0,
\]

we deduce that $P(N_t < \infty) = 1$ for all $t \geq 0$. Therefore, the representation (A.18) shows that the process $N$ is càdlàg, adapted with $N \in \mathbb{N}_0$ up to an evanescent set. Since $\mu(\omega; \{t\} \times E) \leq 1$ for all $(\omega,t) \in \Omega \times \mathbb{R}_+$ by the definition of an integer-valued random measure, see \cite{[10]} Def. II.1.13, we obtain $\Delta N \in \{0,1\}$.
For any set $B \in \mathcal{E}$ we define the mapping $\varrho^B : \Omega \to \mathbb{R}_+$ as

$$
\varrho^B := \inf \{ t \geq 0 : \mu([0, t] \times B^c) = 1 \}.
$$

For the representation (A.20) below we recall that for any stopping time $\tau$ and any set $A \in \mathcal{F}_\tau$ the mapping $\tau^A : \Omega \to \mathbb{R}_+$ given by

$$
\tau^A(\omega) := \begin{cases} 
\tau(\omega), & \omega \in A \\
\infty, & \omega \notin A
\end{cases}
$$

is also a stopping time.

**A.20. Lemma.** For every set $B \in \mathcal{E}$ with $\mathcal{F}(B^c) < \infty$ the mapping $\varrho^B$ is a strictly positive stopping time and we have the representation

$$
(A.20) \quad \varrho^B = \min_{n \in \mathbb{N}} \tau_{\varrho^B_n}^{\{\xi_{\varrho^B_n} \notin B\}}.
$$

**Proof.** This is a direct consequence of Lemma [A.19]. $\square$

We shall now consider the SPDE

$$
(A.21) \quad \begin{cases} 
\frac{dr^B_t}{dt} = (Ar^B_t + \alpha^B(t^B) + \sigma(r^B_t))dt + \gamma^B(t^B, x)(\mu^{(\tau_0)})(dt, dx) - F(dx)dt \\
r^B_0 = h_0,
\end{cases}
$$

where $B \in \mathcal{E}$ is a set with $\mathcal{F}(B^c) < \infty$ and the mappings $\alpha^B : H \to H$ and $\gamma^B : H \times E \to H$ are given by (A.16), (A.17).

**A.21. Proposition.** Let $h_0 : \Omega \to H$ be an $\mathcal{F}_0$-measurable random variable, let $0 < \tau \leq \varrho^B$ be a stopping time, and let $B \in \mathcal{E}$ be a set with $\mathcal{F}(B^c) < \infty$. Then, the following statements are true:

1. If there exists a local mild solution $r$ to (1.1) with lifetime $\tau$, then there also exists a local mild solution $r^B$ to (A.21) with lifetime $\tau$ such that

$$
(A.22) \quad (r^B)_{[0, \tau]} = (r^B)_{[0, \tau]}^{\tau_{\varrho^B}}.
$$

2. If there exists a local mild solution $r^B$ to (A.21) with lifetime $\tau$, then there also exists a local mild solution $r$ to (1.1) with lifetime $\tau$ such that (A.22) is satisfied.

In particular, in either case we have $(r^\tau)_- = ((r^B)^\tau)_-.$

**Proof.** Let $r$ be a local mild solution to (1.1) with lifetime $\tau$. We define the process $r^B$ by

$$
r^B := r - \gamma(r_{\varrho^B -}, \xi_{\varrho^B})1_{[\varrho^B, \infty[}. 
$$

Then, $r^B$ is càdlàg and adapted, because $\gamma(r_{\varrho^B -}, \xi_{\varrho^B})$ is $\mathcal{F}_{\varrho^B}$-measurable, and, since $\tau \leq \varrho^B$, we have

$$
(r^B)^\tau = r^\tau - \gamma(r_{\varrho^B -}, \xi_{\varrho^B})1_{[\varrho^B, \infty[} = r^\tau - \gamma(r_{\tau -}, \xi_\tau)1_{[\tau, \infty[}.
$$
Therefore, we have (A.22), and hence \((r^\tau)_- = ((r^B)^\tau)_-\). Since \(r\) is a local mild solution to (1.1) with lifetime \(\tau\), we have \(\mathbb{P}\)-almost surely

\[
\begin{align*}
r^B_{t \wedge \tau} &= S_{t \wedge \tau} h_0 + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \alpha(r_s) ds + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \sigma(r_s) dW_s \\
&\quad + \int_0^{t \wedge \tau} \int_E S_{(t \wedge \tau) - s} \gamma(r_s, x)(\mu(ds, dx) - F(dx)ds) \\
&\quad - \gamma(r_{\tau^-}, \xi_\tau) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}} \\
&= S_{t \wedge \tau} h_0 + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \alpha(r^B_s) ds + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \sigma(r^B_s) dW_s \\
&\quad + \int_0^{t \wedge \tau} \int_E S_{(t \wedge \tau) - s} \gamma(r^B_s, x)(\mu(ds, dx) - F(dx)ds) \\
&\quad - \gamma(r_{\tau^-}, \xi_\tau) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0.
\end{align*}
\]

Hence, by the Definitions (A.16), (A.17) of \(\alpha^B, \gamma^B\) we get \(\mathbb{P}\)-almost surely

\[
\begin{align*}
r^B_{t \wedge \tau} &= S_{t \wedge \tau} h_0 + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \alpha^B(r^B_s) ds + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \sigma(r^B_s) dW_s \\
&\quad + \int_0^{t \wedge \tau} \int_E S_{(t \wedge \tau) - s} \gamma^B(r^B_s, x)(\mu(ds, dx) - F(dx)ds) \\
&\quad - \gamma(r_{\tau^-}, \xi_\tau) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}} + \int_0^{t \wedge \tau} \int_{B^c} S_{(t \wedge \tau) - s} \gamma(r^B_s, x) F(dx) ds \\
&\quad + \int_0^{t \wedge \tau} \int_{B^c} S_{(t \wedge \tau) - s} \gamma(r^B_s, x)(\mu(ds, dx) - F(dx)ds), \quad t \geq 0.
\end{align*}
\]

By (A.9) and the representation (A.20) from Lemma (A.20) we have \(\mathbb{P}\)-almost surely

\[
\int_0^{t \wedge \tau} \int_{B^c} S_{(t \wedge \tau) - s} \gamma(r^B_s, x) \mu(ds, dx)
= \sum_{n \in \mathbb{N}} S_{(t \wedge \tau) - \tau_n} \gamma(r^B_{\tau_n}, \xi_{\tau_n}) \mathbb{1}_{\{\tau_n \leq t \wedge \tau\}}
= S_{(t \wedge \tau) - \theta^n} \gamma(r^B_{\theta^n}, \xi_{\theta^n}) \mathbb{1}_{\{\theta^n \leq t \wedge \tau\}} = S_{(t \wedge \tau) - \theta^n} \gamma(r^B_{\theta^n}, \xi_{\theta^n}) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}}
= \gamma(r_{\tau^-}, \xi_\tau) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0.
\]

Therefore, we obtain \(\mathbb{P}\)-almost surely

\[
\begin{align*}
r^B_{t \wedge \tau} &= S_{t \wedge \tau} h_0 + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \alpha^B(r^B_s) ds + \int_0^{t \wedge \tau} S_{(t \wedge \tau) - s} \sigma(r^B_s) dW_s \\
&\quad + \int_0^{t \wedge \tau} \int_B S_{(t \wedge \tau) - s} \gamma^B(r^B_s, x)(\mu(ds, dx) - F(dx)) ds, \quad t \geq 0
\end{align*}
\]

showing that \(r^B\) is a local mild solution to (A.21) with lifetime \(\tau\). This proves the first statement.

Now, let \(r^B\) be a local mild solution to (A.21) with lifetime \(\tau\). We define the process \(r\) by

\[
r := r^B + \gamma(r^B_{\theta^n}, \xi_{\theta^n}) \mathbb{1}_{\{\theta^n \leq \tau\}}.
\]

Then, \(r\) is càdlàg and adapted, because \(\gamma(r^B_{\theta^n}, \xi_{\theta^n})\) is \(\mathcal{F}_{\theta^n}\)-measurable, and, since \(\tau \leq \theta^n\), we have

\[
r^\tau = (r^B)^\tau + \gamma(r^B_{\theta^n}, \xi_{\theta^n}) \mathbb{1}_{\{\theta^n = \tau\}} = (r^B)^\tau + \gamma(r_{\tau^-}, \xi_\tau) \mathbb{1}_{\{\tau = \theta^n\}} \mathbb{1}_{\{\tau \leq t\}}.
\]
Therefore, we have \( \tau^\ast \) and hence \((r^\ast)^\tau = ((r^B)^\tau)^\tau \). Since \( r^B \) is a local mild solution to (A.2.1) with lifetime \( \tau \), we have \( \mathbb{P} \)–almost surely
\[
\begin{align*}
    r_{t,T} &= S_{t,T}h_0 + \int_0^{t\wedge T} S_{(t,T)}\rho \alpha (r_s) ds + \int_0^{t\wedge T} S_{(t,T)}\rho \sigma (r_s) dW_s \\
    &\quad + \int_0^{t\wedge T} \int_E S_{(t,T)}\gamma (r_s,x) \mu (ds, dx) - F(dx) ds \\
    &\quad + \gamma (r^B_{t-}, \xi_T) \mathbb{1}_{\{\tau = \rho^n\}} \mathbb{I}_{\{\tau \leq \ell\}} + \int_0^{t\wedge T} \int_B S_{(t,T)}\gamma (r_s, x) (\mu (ds, dx) - F(dx) ds
\end{align*}
\]
Hence, by the Definitions (A.16) and (A.17) of \( \alpha^B, \gamma^B \) we get \( \mathbb{P} \)–almost surely
\[
\begin{align*}
    r_{t,T} &= S_{t,T}h_0 + \int_0^{t\wedge T} S_{(t,T)}\rho \alpha (r_s) ds + \int_0^{t\wedge T} S_{(t,T)}\rho \sigma (r_s) dW_s \\
    &\quad + \int_0^{t\wedge T} \int_B S_{(t,T)}\gamma (r_s,x) (\mu (ds, dx) - F(dx) ds \\
    &\quad + \gamma (r^B_{t-}, \xi_T) \mathbb{1}_{\{\tau = \rho^n\}} \mathbb{I}_{\{\tau \leq \ell\}} - \int_0^{t\wedge T} \int_B S_{(t,T)}\gamma (r_s, x) F(dx) ds, \quad t \geq 0.
\end{align*}
\]
Arguing as in (A.23), we have \( \mathbb{P} \)–almost surely
\[
\begin{align*}
    \int_0^{t\wedge T} \int_B S_{(t,T)}\gamma (r_s,x) \mu (ds, dx) &= \gamma (r^B_{t-}, \xi_T) \mathbb{1}_{\{\tau = \rho^n\}} \mathbb{I}_{\{\tau \leq \ell\}}, \quad t \geq 0.
\end{align*}
\]
Therefore, we obtain \( \mathbb{P} \)–almost surely
\[
\begin{align*}
    r_{t,T} &= S_{t,T}h_0 + \int_0^{t\wedge T} S_{(t,T)}\rho \alpha (r_s) ds + \int_0^{t\wedge T} S_{(t,T)}\rho \sigma (r_s) dW_s \\
    &\quad + \int_0^{t\wedge T} \int_E S_{(t,T)}\gamma (r_s,x) (\mu (ds, dx) - F(dx) ds, \quad t \geq 0
\end{align*}
\]
showing that \( r \) is a local mild solution to (1.1) with lifetime \( \tau \). This proves the second statement. \( \square \)

**A.22. Lemma.** Let \( G_1, G_2 \) be metric spaces such that \( G_1 \) is separable. Let \( B \subset G_1 \) be a Borel set, let \( C \subset G_2 \) be a closed set and let \( \delta : G_1 \times E \to G_2 \) be a measurable mapping such that \( \delta (\bullet, x) : G_1 \to G_2 \) is continuous for all \( x \in E \). Suppose that
\[
\delta (h, x) \in C \quad \text{for } F\text{–almost all } x \in E, \quad \text{for all } h \in B.
\]
Then, we even have
\[
\delta (h, x) \in C \quad \text{for all } h \in B, \quad \text{for } F\text{–almost all } x \in E.
\]

**Proof.** By separability of \( G_1 \) there exists a countable set \( D \), which is dense in \( B \). By (A.24), for each \( h \in D \) there exists an \( F\)-nullset \( N_h \) such that
\[
\delta (h, x) \in C \quad \text{for all } x \in N_h^c.
\]
The set \( N := \bigcup_{h \in D} N_h \) is also an \( F\)-nullset. Now, let \( h \in B \) be arbitrary. Then, there exists a sequence \( (h_n)_{n \in \mathbb{N}} \subset D \) with \( h_n \to h \), and hence
\[
\delta (h_n, x) \in C \quad \text{for all } n \in \mathbb{N} \text{ and } x \in N^c.
\]
Proof. We denote by \( \delta(h,x) = \lim_{n \to \infty} \delta(h_n, x) \) for all \( x \in N \), providing (A.25).

A.23. Proposition. Let \( B_1 \subseteq B_2 \subseteq H \) be two nonempty Borel sets such that \( B_1 \) is prelocally invariant in \( B_2 \) for (1.1), see Definition 3.2. Then we have

\[
\text{if } h + \gamma(h, x) \in B_2 \text{ for } F\text{-almost all } x \in E, \text{ for all } h \in B_1.
\]

Proof. We denote by

\[
d_{B_2} : H \to \mathbb{R}_+, \quad d_{B_2}(h) := \inf_{g \in B_2} \| h - g \|
\]

the distance function of the set \( B_2 \). Since

\[
|d_{B_2}(h_1) - d_{B_2}(h_2)| \leq \| h_1 - h_2 \| \quad \text{for all } h_1, h_2 \in H,
\]

by the linear growth condition (2.7), for all \( n \in \mathbb{N} \), all \( h \in B_2 \) with \( \| h \| \leq n \) and all \( x \in E \) we have

\[
(A.26) \quad |d_{B_2}(h + \gamma(h, x))| = |d_{B_2}(h + \gamma(h, x)) - d_{B_2}(h)| \leq \| \gamma(h, x) \| \leq \rho_n(x).
\]

By (2.6) and Lebesgue’s dominated convergence theorem, the mapping

\[
(A.27) \quad B_2 \to \mathbb{R}, \quad h \mapsto \int_E |d_{B_2}(h + \gamma(h, x))|^2 F(dx)
\]

is continuous. Now, let \( h \in B_1 \) be arbitrary. Since \( B_1 \) is prelocally invariant in \( B_2 \) for (1.1), there exists a local mild solution \( r = r^{(h)} \) to (1.1) with lifetime \( \tau > 0 \) such that \((\tau^-)_- \in B_1\) and \( r^- \in B_2\) up to an evanescent set. Taking into account [19, Thm. II.1.8], identity (A.9) and Lemma A.13 we obtain

\[
\mathbb{E} \left[ \int_0^\tau \int_E |d_{B_2}(r_{s^-} + \gamma(r_{s^-}, x))|^2 F(dx) ds \right] = \mathbb{E} \left[ \int_0^\tau \int_E |d_{B_2}(r_{s^-} + \gamma(r_{s^-}, x))|^2 \mu(ds, dx) \right] = \mathbb{E} \left[ \sum_{n \in \mathbb{N}} |d_{B_2}(r_{\tau_n^-} + \gamma(r_{\tau_n^-}, \xi_{\tau_n}))|^2 \mathbb{1}_{\{\tau_n \leq \tau\}} \right] = \mathbb{E} \left[ \sum_{n \in \mathbb{N}} |d_{B_2}(r_{\tau_n})|^2 \mathbb{1}_{\{\tau_n \leq \tau\}} \right] = 0.
\]

Therefore, we have \( \mathbb{P} \)-almost surely

\[
(A.28) \quad \int_0^\tau \left( \int_E |d_{B_2}(r_{s^-} + \gamma(r_{s^-}, x))|^2 F(dx) \right) ds = 0, \quad t \geq 0.
\]

Since the process \( r \) is càdlàg with \((\tau^-)_- \in B_1\) up to an evanescent set and the mapping (A.27) is continuous, the integrand appearing in (A.28) is continuous in \( s = 0 \). Thus, we deduce that

\[
\int_E |d_{B_2}(h + \gamma(h, x))|^2 F(dx) = 0.
\]

This provides

\[
d_{B_2}(h + \gamma(h, x)) = 0 \quad \text{for } F\text{-almost all } x \in E,
\]

and hence

\[
h + \gamma(h, x) \in \overline{B}_2 \quad \text{for } F\text{-almost all } x \in E,
\]

completing the proof. \( \square \)
A.24. Proposition. Let $B_1 \subset B_2 \subset H$ be two nonempty Borel sets such that
\[ \text{(A.29)} \quad h + \gamma(h, x) \in B_2 \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B_1. \]

Let $h_0 : \Omega \to H$ be an $\mathcal{F}_0$-measurable random variable and let $r = r^{(h_0)}$ be a local mild solution to (1.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \mathbb{1}_{[0,\tau]} \in B_2$ up to an evanescent set. Then we have $r^\tau \in B_2$ up to an evanescent set.

Proof. By the closedness of $H$, we have $r_0 = \mathbb{1}_{[0,\tau]} \in B_2$ up to an evanescent set.

(A.30) \[ \mathbb{P}(r^\tau \mathbb{1}_{(\tau < \infty)} \in B_2) = 1. \]

By (A.9), [19] Thm. II.1.8 and (A.29) we obtain
\[ \mathbb{E} \left[ \sum_{n \in \mathbb{N}} \mathbb{1}_{\{r_{\tau_n} + \gamma(r_{\tau_n} - \xi_{\tau_n}) \notin B_2\}} \right] = \mathbb{E} \left[ \int_0^\infty \int_E \mathbb{1}_{\{r_{\tau_n} + \gamma(r_{\tau_n} - x) \notin B_2\}} \mu(dx, dx) \right] = \mathbb{E} \left[ \int_0^\infty \int_E \mathbb{1}_{\{r_{\tau_n} + \gamma(r_{\tau_n} - x) \notin B_2\}} F(dx) ds \right] = 0, \]
which yields
\[ \mathbb{P}(r_{\tau_n} + \gamma(r_{\tau_n} - \xi_{\tau_n}) \in B_2 \text{ for all } n \in \mathbb{N}) = 1. \]

Therefore, by Lemma [A.13] we obtain $\mathbb{P}$-almost surely
\[ r^\tau \mathbb{1}_{(\tau < \infty)} = \left( r^\tau + \Delta r^\tau \right) \mathbb{1}_{(\tau < \infty)} = \left( r^\tau + \gamma(r^\tau, \xi^\tau) \sum_{n \in \mathbb{N}} \mathbb{1}_{(r_{\tau_n} = r)} \mathbb{1}_{(\tau < \infty)} \right) \mathbb{1}_{(\tau < \infty)} \in B_2, \]
proving (A.30).

A.25. Corollary. Let $B \subset C \subset H$ be two nonempty Borel sets such that $C$ is closed in $H$ and
\[ h + \gamma(h, x) \in C \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in B. \]

Let $h_0 : \Omega \to H$ be an $\mathcal{F}_0$-measurable random variable and let $r = r^{(h_0)}$ be a local mild solution to (1.1) with lifetime $\tau > 0$ such that $(r^\tau)_- \in B_1$ and $r^\tau \mathbb{1}_{[0,\tau]} \in B_2$ up to an evanescent set. Then we have $r^\tau \in C$ up to an evanescent set.

Proof. By the closedness of $C$ in $H$, we have $r^\tau \mathbb{1}_{[0,\tau]} \in C$ up to an evanescent set. Thus, the statement follows from Proposition A.24.

Appendix B. Finite dimensional submanifolds with boundary in Hilbert spaces

For convenience of the reader, we provide the crucial properties of finite dimensional submanifolds with boundary in Hilbert spaces. For more details, we refer to any textbook about manifolds, e.g., [1], [20] or [29].

Let $H$ be a Hilbert space and let $m \in \mathbb{N}$ be a positive integer. We denote by $\mathbb{R}^m_+$ the set of $m$-tuples $y \in \mathbb{R}^m$ with non-negative first coordinate $y_1 \geq 0$, that is
\[ \mathbb{R}^m_+ = \mathbb{R}^+_1 \times \mathbb{R}^{m-1} = \{ y \in \mathbb{R}^m : y_1 \geq 0 \}. \]
We consider the relative topology on $\mathbb{R}^m_+$. Let $V$ be an open subset in $\mathbb{R}^m_+$, i.e., there exists an open set $\tilde{V} \subset \mathbb{R}^m$ such that $\tilde{V} \cap \mathbb{R}^m_+ = V$. A boundary point of $V$ is by definition any point $y \in V$ with vanishing first coordinate $y_1 = 0$. The set of all boundary points of $V$ is denoted by $\partial V$, i.e.
\[ \partial V = \{ y \in V : y_1 = 0 \}. \]
Let $k \in \mathbb{N}$ be arbitrary.
B.1. Definition. A map \( \phi : V \subseteq \mathbb{R}_+^m \to H \) is called a \( C^k \)-map, if there is an open set \( \tilde{V} \subseteq \mathbb{R}^m \) together with a \( C^k \)-map \( \tilde{\phi} : \tilde{V} \to H \) such that \( \tilde{V} \cap \mathbb{R}_+^m = V \) and \( \tilde{\phi}|_V = \phi \).

For a \( C^k \)-map \( \phi : V \subseteq \mathbb{R}_+^m \to H \) and \( y \in V \) we define the derivative \( D\phi(y) := D\tilde{\phi}(y) \). Note that this definition does not depend on the choice of \( \tilde{\phi} \).

B.2. Definition. A map \( \phi : V \subseteq \mathbb{R}_+^m \to W \subseteq \mathbb{R}_+^l \) is called a \( C^k \)-diffeomorphism, if \( \phi \) is bijective and both, \( \phi \) and \( \phi^{-1} \), are \( C^k \)-maps.

The following lemma is a standard result, whence we omit the proof.

B.3. Lemma. Let \( \phi : V \subseteq \mathbb{R}_+^m \to W \subseteq \mathbb{R}_+^m \) be a \( C^k \)-diffeomorphism for some \( k \in \mathbb{N} \). Then the following statements are true:

1. We have \( \phi(\partial V) = \partial W \).
2. For each \( y \in \partial V \) we have \( D\phi(y)|_{\mathbb{R}_+^m} \subseteq \mathbb{R}_+^m \).

Hence, boundary points of \( V \) are mapped to boundary points of \( W \) under a \( C^k \)-diffeomorphism.

B.4. Definition. Let \( \mathcal{M} \subseteq H \) be a nonempty subset.

1. \( \mathcal{M} \) is an \( m \)-dimensional \( C^k \)-submanifold with boundary of \( H \), if for all \( h \in \mathcal{M} \) there exist an open neighborhood \( U \subseteq H \) of \( h \), an open set \( V \subseteq \mathbb{R}_+^m \) and a map \( \phi \in C^k(V; H) \) such that
   a. \( \phi : V \to U \cap \mathcal{M} \) is a homeomorphism;
   b. \( D\phi(y) \) is one to one for all \( y \in V \).

The map \( \phi \) is called a parametrization of \( \mathcal{M} \) around \( h \).

2. The boundary of \( \mathcal{M} \) is defined as the set of all points \( h \in \mathcal{M} \) such that \( \phi^{-1}(h) \in \partial V \) for some parametrization \( \phi : V \to H \) around \( h \). The set of all boundary points is denoted by \( \partial \mathcal{M} \) and is a submanifold without boundary of dimension \( m-1 \) of \( H \). Parametrizations of \( \partial \mathcal{M} \) are provided by restricting parametrizations \( \phi : V \to H \) of \( \mathcal{M} \) to the boundary \( \partial V \).

Notice that any submanifold is a submanifold with empty boundary. In what follows, let \( \mathcal{M} \) be an \( m \)-dimensional \( C^k \)-submanifold with boundary of \( H \).

B.5. Definition. Let \( h \in \mathcal{M} \) be arbitrary and let \( \phi : V \subseteq \mathbb{R}_+^m \to U \cap \mathcal{M} \) be a parametrization around \( h \).

1. The tangent space to \( \mathcal{M} \) at \( h \) is the subspace

\[
T_h\mathcal{M} := D\phi(y)\mathbb{R}_+^m, \quad y = \phi^{-1}(h) \in V.
\]

2. For \( h \in \partial \mathcal{M} \) we can distinguish a half space in \( T_h\mathcal{M} \), namely the set of all inward pointing directions in \( \mathcal{M} \), given by

\[
(T_h\mathcal{M})_+ := D\phi(y)\mathbb{R}_+^m, \quad y = \phi^{-1}(h) \in \partial V.
\]

B.6. Remark. By \cite[Lemma 6.1.1]{1993} and Lemma \ref{B.3} the Definitions \ref{B.1}, \ref{B.2} of the tangent spaces \( T_h\mathcal{M} \) and \( (T_h\mathcal{M})_+ \) are independent of the choice of the parametrization.

Since parametrizations of \( \partial \mathcal{M} \) are provided by restricting parametrizations \( \phi : V \subseteq \mathbb{R}_+^m \to U \cap \mathcal{M} \) of \( \mathcal{M} \) to the boundary \( \partial V \), for any \( h \in \partial \mathcal{M} \) we have

\[
T_h\partial \mathcal{M} = D\phi(y)\partial \mathbb{R}_+^m, \quad y = \phi^{-1}(h) \in \partial V.
\]

In particular, we see that

\[
T_h\partial \mathcal{M} = (T_h\mathcal{M})_+ \cap -(T_h\mathcal{M})_+ \subseteq (T_h\mathcal{M})_+
\]

\[
\subseteq (T_h\mathcal{M})_+ \cap -(T_h\mathcal{M})_+ = T_h\mathcal{M}, \quad h \in \partial \mathcal{M}.
\]
For a subset $A \subset H$ we define
\[ A^\perp := \{ h \in H : \langle h, g \rangle = 0 \text{ for all } g \in A \}, \]
\[ A^+ := \{ h \in H : \langle h, g \rangle \geq 0 \text{ for all } g \in A \}. \]

In order to introduce the inward pointing normal (tangent) vectors at boundary points of the submanifold $M$, we require the following auxiliary result. The proof is elementary and therefore omitted.

B.7. Lemma. For each $h \in \partial M$ there exists a unique vector $\eta_h \in (T_h M)^+ \cap (T_h \partial M)^\perp$ with $||\eta_h|| = 1$ such that
\[ (B.5) \quad T_h M = T_h \partial M \oplus \text{span}\{\eta_h\}. \]
Moreover, for each $h \in \partial M$ we have
\[ (B.6) \quad T_h \partial M = T_h M \cap \{\eta_h\}^\perp, \]
\[ (B.7) \quad (T_h M)^+ = T_h M \cap \{\eta_h\}^+. \]

B.8. Definition. For each $h \in \partial M$ we call $\eta_h$ the inward pointing normal (tangent) vector to $\partial M$ at $h$.

In the sequel, the vector $e_1 \in \mathbb{R}^m$ denotes the first unit vector $e_1 = (1, 0, \ldots, 0)$.

B.9. Lemma. Let $\phi : V \subset \mathbb{R}^m_+ \to U \cap M$ be a parametrization. Then, for every $h \in U \cap \partial M$ there exists a unique number $\lambda > 0$ such that
\[ (B.8) \quad \langle \eta_h, D\phi(y)v \rangle = \lambda \langle e_1, v \rangle \quad \text{for all } v \in \mathbb{R}^m, \]
where $y = \phi^{-1}(h)$.

Proof. Let $h \in U \cap \partial M$ be arbitrary. We define the continuous linear functional
\[ \ell : \mathbb{R}^m \to \mathbb{R}, \quad \ell(v) := \langle \eta_h, D\phi(y)v \rangle. \]

There is a unique $z \in \mathbb{R}^m$ such that
\[ (B.9) \quad \ell(v) = \langle z, v \rangle \quad \text{for all } v \in \mathbb{R}^m. \]

In order to complete the proof, we shall show that $z = \lambda e_1$ for some $\lambda > 0$. By identity \(B.6\) from Lemma \(B.7\), for any $v \in \mathbb{R}^m$ we have $\ell(v) = 0$ if and only if $D\phi(y)v \in T_h \partial M$, which, in view of \(B.3\), means that $v \in \partial \mathbb{R}^m_+$. This shows $\ker(\ell) = \partial \mathbb{R}^m_+$, and hence, there exists a unique $\lambda \in \mathbb{R}$ such that $z = \lambda e_1$. Consequently, identity \(B.8\) is valid. By \(B.2\), \(B.3\) we have $D\phi(y)e_1 \in (T_h M)^+ \setminus T_h \partial M$, and hence, inserting $v = e_1$ into \(B.8\), by \(B.6\), \(B.7\) we obtain
\[ \lambda = \lambda \langle e_1, e_1 \rangle = \langle \eta_h, D\phi(y)e_1 \rangle > 0, \]
finishing the proof. \qed

In the sequel, for $h_0 \in H$ and $\epsilon > 0$ we denote by $B_\epsilon(h_0)$ the open ball
\[ B_\epsilon(h_0) = \{ h \in H : ||h - h_0|| < \epsilon \}. \]

B.10. Lemma. For each $h_0 \in \mathcal{M}$ there exists $\epsilon_0 > 0$ such that for all $0 \leq \epsilon \leq \epsilon_0$ the following statements are true:

1. The set $B_\epsilon(h_0) \cap \mathcal{M}$ is compact.
2. We have $B_\epsilon(h_0) \cap \mathcal{M} \subset B_{\epsilon_0}(h_0) \cap \mathcal{M}.$

Proof. Let $h_0 \in \mathcal{M}$ be arbitrary, let $\phi : V \subset \mathbb{R}^m_+ \to U \cap \mathcal{M}$ be a parametrization around $h_0$ and set $y_0 := \phi^{-1}(h_0) \in V$. Since $V$ is open in $\mathbb{R}^m_+$, there exist $X \subset K \subset V$ such that $X$ is open in $\mathbb{R}^m_+$ and $K$ is compact. Since $\phi : V \to U \cap \mathcal{M}$ is
Taking into account \[13, \text{Prop. 6.1.2}\], there exist a homeomorphism, \(\phi(X)\) is open in \(U \cap \mathcal{M}\) and \(\phi(K)\) is compact. Therefore, and since \(U\) is an open neighborhood of \(h_0\), there exists \(\epsilon_0 > 0\) such that
\[\overline{B_\epsilon(h_0)} \subset U \quad \text{and} \quad \overline{B_\epsilon(h_0)} \cap (U \cap \mathcal{M}) \subset \phi(X)\].

Let \(0 \leq \epsilon \leq \epsilon_0\) be arbitrary. Since \(\phi(X) \subset \phi(K) \subset U \cap \mathcal{M}\), we have the identity
\[\overline{B_\epsilon(h_0) \cap \mathcal{M}} = \overline{B_\epsilon(h_0) \cap \phi(K)}\],
showing that \(\overline{B_\epsilon(h_0) \cap \mathcal{M}}\) is closed in \(\phi(K)\). Since \(\phi(K)\) is compact, we deduce that \(\overline{B_\epsilon(h_0) \cap \mathcal{M}}\) is compact, establishing the first statement.

For the proof of the second statement, let \(h \in B_\epsilon(h_0) \cap \overline{\mathcal{M}}\) be arbitrary. Since \(h \in \overline{\mathcal{M}}\), there exists a sequence \((h_n)_{n \in \mathbb{N}} \subset \mathcal{M}\) with \(h_n \to h\). Therefore, and since \(h \in B_\epsilon(h_0)\), there exists an index \(n_0 \in \mathbb{N}\) such that \(h_n \in B_\epsilon(h_0)\) for all \(n \geq n_0\). Consequently, we have \(h_n \in \overline{B_\epsilon(h_0) \cap \mathcal{M}}\) for all \(n \geq n_0\). By the closedness of \(\overline{B_\epsilon(h_0) \cap \mathcal{M}}\) we deduce that \(h \in \overline{B_\epsilon(h_0) \cap \mathcal{M}}\), completing the proof. \(\square\)

For two Banach spaces \(X, Y\) we denote by \(C^k_b(X; Y)\) the linear space consisting of all \(f \in C^k(X; Y)\) such that \(D^i f\) is bounded for all \(i = 1, \ldots, k\). In particular, for each \(f \in C^k_b(X; Y)\) the mappings \(D^i f, i = 0, \ldots, k - 1\) are Lipschitz continuous.

We do not demand that \(f\) itself is bounded, as this would exclude continuous linear operators \(f \in L(X, Y)\).

**B.11. Lemma.** Let \(\mathcal{M}_0 \subset H\) be an \(m\)-dimensional \(C^k\)-submanifold with boundary of \(H\), let \(h_0 \in \mathcal{M}\) be arbitrary and let \(D \subset H\) be a dense subset. Then there exist
- a constant \(\epsilon > 0\) such that \(\mathcal{M} := B_\epsilon(h_0) \cap \mathcal{M}_0\) is an \(m\)-dimensional \(C^k\)-submanifold with boundary of \(H\),
- an \(m\)-dimensional \(C^k\)-submanifold \(\mathcal{N}\) with boundary of \(\mathbb{R}^m\),
- parametrizations \(\phi : V \to \mathcal{M}\) and \(\psi : V \to \mathcal{N}\),
- and elements \(\zeta_1, \ldots, \zeta_m \in D\) such that the mapping \(f := \phi \circ \psi^{-1} : \mathcal{N} \to \mathcal{M}\) has the inverse
\[(B.10) \quad f^{-1} : \mathcal{M} \to \mathcal{N}, \quad f^{-1}(\zeta) = (\zeta_1, \ldots, \zeta_m, h)\]

Furthermore, the mappings \(\phi, \psi, \Phi := \phi^{-1}, \Psi := \psi^{-1}\) have extensions \(\phi \in C^k_b(\mathbb{R}^m; H), \psi \in C^k_b(\mathbb{R}^m), \Phi \in C^k_b(H; \mathbb{R}^m), \Psi \in C^k_b(\mathbb{R}^m)\).

**Proof.** Taking into account \cite{13} Prop. 6.1.2, there exist
- a constant \(\epsilon > 0\),
- an \(m\)-dimensional \(C^k\)-submanifold \(\tilde{\mathcal{M}}\) of \(H\) without boundary,
- a parametrization \(\tilde{\phi} : \tilde{V} \subset \mathbb{R}^m \to \tilde{\mathcal{M}}\) and such that \(\tilde{\phi}(\tilde{V}) = \mathcal{M}\), where \(\tilde{V} := \tilde{V} \cap \mathbb{R}^m\) and \(\tilde{\mathcal{M}} := B_\epsilon(h_0) \cap \mathcal{M}_0\),
- elements \(\zeta_1, \ldots, \zeta_m \in D\) and a parametrization \(\tilde{f} : \tilde{\mathcal{N}} \subset \mathbb{R}^m \to \tilde{\mathcal{M}}\) with inverse
\[\tilde{f}^{-1} : \tilde{\mathcal{M}} \to \tilde{\mathcal{N}}, \quad \tilde{f}^{-1}(\zeta) = (\zeta_1, \ldots, \zeta_m, h)\).

We set \(\phi := \tilde{\phi}|_{\tilde{V}}, \mathcal{N} := \tilde{f}^{-1}(\mathcal{M}), f := \tilde{f}|_{\mathcal{N}}\) and \(\psi := f^{-1} \circ \phi\). Then, \(\phi : \tilde{V} \subset \mathbb{R}^m \to \mathcal{M}\) is a parametrization, \(\mathcal{N}\) is an \(m\)-dimensional \(C^k\)-submanifold with boundary of \(\mathbb{R}^m\) and \(\psi : \tilde{V} \subset \mathbb{R}^m \to \mathcal{N}\) is a parametrization.

By the inverse mapping theorem, see \cite{hi} Thm. 2.5.2, the parametrization \(\psi\) is a local diffeomorphism. Hence, arguing as in \cite{13} Remark 6.1.1, we may assume that the mappings \(\phi, \psi, \Phi := \phi^{-1}, \Psi := \psi^{-1}\) (after restricting to smaller neighborhoods, if necessary) have the desired extensions. \(\square\)
The following diagram illustrates the situation provided by Lemma B.11:

\[ N \subset \mathbb{R}^m \xrightarrow{f} M \subset H \]

\[ V \subset \mathbb{R}^m \]

B.12. Lemma. Let \( h \in M \) be arbitrary and let \( \phi : V \subset \mathbb{R}^m_+ \rightarrow U \cap M \) be a parametrization around \( h \) such that \( \Phi := \phi^{-1} \) has an extension \( \Phi \in C^k(H; \mathbb{R}^m) \).
Then we have

\[ D\phi(y)^{-1}w = D\Phi(h)w \quad \text{for all } w \in T_hM, \]

where \( y = \phi^{-1}(h) \).

Proof. The identity \( D\Phi(h)D\phi(y) = D(\Phi \circ \phi)(y) = \text{Id}|_{\mathbb{R}^m} \) yields the assertion. □

Appendix C. Further auxiliary results

Let \( M \) be an \( m \)-dimensional \( C^3 \)-submanifold with boundary of \( H \). For \( h \in \partial M \) the vector \( \eta_h \) denotes the inward pointing normal (tangent) vector to \( \partial M \) at \( h \).

C.1. Lemma. Let \( \phi : V \subset \mathbb{R}^m_+ \rightarrow U \cap M \) be a parametrization and let \( \sigma \in C^1(H) \) be a mapping such that \( \sigma(h) \in T_hM, \ h \in U \cap M \).

We define the mapping \( \theta : V \rightarrow \mathbb{R}^m \), \( \theta(y) := D\phi(y)^{-1}\sigma(h) \), where \( h := \phi(y) \in U \cap M \).

1. For each \( h \in U \cap M \) we have the decomposition

\[ D\sigma(h)\sigma(h) = D\phi(y)(D\theta(y)\theta(y)) + D^2\phi(y)(\theta(y), \theta(y)), \]

where \( y = \phi^{-1}(h) \in V \).

2. If, moreover, we have

\[ \sigma(h) \in T_{h\partial M}, \ h \in U \cap \partial M \]

then for each \( h \in U \cap \partial M \) we have

\[ \langle \eta_h, D\sigma(h)\sigma(h) \rangle = \langle \eta_h, D^2\phi(y)(\theta(y), \theta(y)) \rangle, \]

where \( y = \phi^{-1}(h) \in \partial V \).

Proof. Let \( h \in U \cap M \) be arbitrary and set \( y := \phi^{-1}(h) \in V \). There exist \( \epsilon > 0 \) and \( \Lambda \in \{-1, 1\} \) such that

\[ y + \Lambda t\theta(y) \in V \quad \text{for all } t \in [0, \epsilon). \]

Consequently, the curve

\[ c : [0, \epsilon) \rightarrow U \cap M, \ c(t) := \phi(y + \Lambda t\theta(y)) \]

is well-defined and we have \( c \in C^1([0, \epsilon); H) \). Note that

\[ c(0) = h \quad \text{and} \quad \frac{d}{dt}c(t) \bigg|_{t=0} = \Lambda D\phi(y)\theta(y) = \Lambda\sigma(h) \]

by the Definition (C.2) of \( \theta \). Therefore, we have

\[ \frac{d}{dt}\sigma(c(t)) \bigg|_{t=0} = \Lambda D\sigma(h)\sigma(h). \]
On the other hand, by (C.2),
\[
\frac{d}{dt} \gamma(c(t)) \bigg|_{t=0} = \frac{d}{dt} D\phi(y + \Lambda t\theta(y))\theta(y + \Lambda t\theta(y)) \bigg|_{t=0}
= \Lambda \left( D\phi(y)(D\theta(y)\theta(y)) + D^2\phi(y)(\theta(y), \theta(y)) \right).
\]
Combining the latter two identities yields (C.3), proving the first statement.

Now, suppose that (C.4) is satisfied. Then, we have \( \theta(y) \in \partial \mathbb{R}_+^m \) for all \( y \in \partial V \), and therefore
\[
\langle e_1, \theta(y) \rangle = 0 \quad \text{for all} \quad y \in \partial V.
\]
Let \( h \in U \cap \partial M \) be arbitrary and set \( y := \phi^{-1}(h) \in \partial V \). There exist \( \epsilon > 0 \) and \( \Lambda \in \{-1, 1\} \) such that (C.6) is satisfied. Moreover, we have
\[
\langle e_1, y + \Lambda t\theta(y) \rangle = \langle e_1, y \rangle + \Lambda t(e_1, \theta(y)) = 0 \quad \text{for all} \quad t \in [0, \epsilon),
\]
which gives us
\[
y + \Lambda t\theta(y) \in \partial V \quad \text{for all} \quad t \in [0, \epsilon).
\]
Consequently, using Lemma B.9 and (C.7), for some \( \lambda > 0 \) we obtain
\[
\langle \eta_h, D\phi(y)(D\theta(y)\theta(y)) \rangle = \Lambda \langle e_1, D\theta(y)\theta(y) \rangle
= \lambda \lim_{t \downarrow 0} \frac{\langle e_1, \theta(y + \Lambda t\theta(y)) \rangle - \langle e_1, \theta(y) \rangle}{t} = 0.
\]
In view of (C.3), identity (C.5) follows, establishing the second statement. \( \square \)

Let \( \gamma : H \times E \to H \) be a mapping fulfilling conditions (2.6), (2.7) with the mappings \( \rho_n : E \to \mathbb{R}_+ \), \( n \in \mathbb{N} \) satisfying (3.4).

C.2. Definition. We introduce the following notions:

1. Let \( h_0 \in \mathcal{M} \) be arbitrary. We say that \( \gamma \) satisfies the \( \epsilon - \delta \)-jump condition in \( h_0 \), if there exists \( \epsilon_0 > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_0 \) the set \( \overline{B}_\epsilon(h_0) \cap \mathcal{M} \) is compact, and there are \( 0 < \delta < \epsilon \) and a set \( B \in \mathcal{E} \) with \( F(B^c) < \infty \) such that
\[
h + \gamma(h, x) \in \overline{B}_\epsilon(h_0) \cap \mathcal{M} \quad \text{for} \quad \text{F-almost all} \quad x \in B, \quad \text{for all} \quad h \in B_\delta(h_0) \cap \mathcal{M}.
\]

2. We say that \( \gamma \) satisfies the \( \epsilon - \delta \)-jump condition on \( \mathcal{M} \), if \( \gamma \) satisfies the \( \epsilon - \delta \)-jump condition in \( h_0 \) for each \( h_0 \in \mathcal{M} \).

C.3. Lemma. Let \( h_0 \in \mathcal{M} \) be such that for some neighborhood \( U \) of \( h_0 \) we have
\[
h_0 + \gamma(h_0, x) \in \overline{\mathcal{M}} \quad \text{for} \quad \text{F-almost all} \quad x \in E, \quad \text{for all} \quad h \in U \cap \mathcal{M}.
\]
Then \( \gamma \) satisfies the \( \epsilon - \delta \)-jump condition in \( h_0 \).

Proof. By Lemma B.10 there exists \( \epsilon_0 > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_0 \) the set \( \overline{B}_\epsilon(h_0) \cap \mathcal{M} \) is compact and we have \( B_\epsilon(h_0) \cap \mathcal{M} \subset \overline{B}_\epsilon(h_0) \cap \mathcal{M} \). Let \( 0 < \epsilon \leq \epsilon_0 \) be arbitrary. There exists \( 0 < \delta < \epsilon/2 \) such that \( B_\delta(h_0) \subset U \). Moreover, there is \( n \in \mathbb{N} \) such that \( ||h|| \leq n \) for all \( h \in B_\delta(h_0) \cap \mathcal{M} \). Setting \( B := \{ \rho_n < \delta \} \in \mathcal{E} \), by (3.4) and Chebyshev’s inequality we obtain
\[
F(B^c) \leq \left( \frac{2}{\delta} \right)^2 \int_E \rho_n(x)^2 F(dx) < \infty.
\]
Let \( h \in B_\delta(h_0) \cap \mathcal{M} \) be arbitrary. By (2.7) we have
\[
||\gamma(h, x)|| \leq \rho_n(x) < \delta \quad \text{for all} \quad x \in B.
\]
Taking into account (C.9), we deduce
\[
h + \gamma(h, x) \in B_\epsilon(h_0) \cap \overline{\mathcal{M}} \subset \overline{B}_\epsilon(h_0) \cap \mathcal{M} \quad \text{for} \quad \text{F-almost all} \quad x \in E,
\]
showing that $\gamma$ satisfies the $\epsilon$–$\delta$–jump condition in $h_0$. \hfill \blacksquare

**C.4. Lemma.** Let $h_0 \in \mathcal{M}$ be such that $\gamma$ satisfies the $\epsilon$–$\delta$–jump condition in $h_0$. Let $\phi : V \subset \mathbb{R}^n_+ \to U \cap \mathcal{M}$ be a parametrization around $h_0$ such that $\phi$ and $\Phi := \phi^{-1}$ have extensions $\phi \in C^2_0(\mathbb{R}^m; H)$ and $\Phi \in C^1_0(H; \mathbb{R}^m)$. Then, there exist $\delta > 0$, a set $B \in \mathcal{E}$ with $F(B^c) < \infty$ and a measurable mapping $\rho : E \to \mathbb{R}_+$ satisfying $\int_E \rho(x)^2 F(dx) < \infty$ such that

\begin{equation}
\|\gamma(h, x) - D\phi(y)(\Phi(h + \gamma(h, x)) - \Phi(h))\| \leq \rho(x)^2 \quad \text{for } F\text{–almost all } x \in B,
\end{equation}

for all $h \in B_3(h_0) \cap \mathcal{M}$, where $y = \phi^{-1}(h)$.

**Proof.** Since $\gamma$ satisfies the $\epsilon$–$\delta$–jump condition in $h_0$, there exist $\delta > 0$ and a set $B \in \mathcal{E}$ with $F(B^c) < \infty$ such that

$$h + \gamma(h, x) \in U \cap \mathcal{M} \quad \text{for } F\text{–almost all } x \in B, \quad \text{for all } h \in B_3(h_0) \cap \mathcal{M}.$$ 

Furthermore, there exists $n \in \mathbb{N}$ such that $\|h\| \leq n$ for all $h \in B_3(h_0) \cap \mathcal{M}$. Let $h \in B_3(h_0) \cap \mathcal{M}$ be arbitrary and set $y := \phi^{-1}(h)$. With $M := \|D^2\phi\|_\infty$ and $N := \|D\Phi\|_\infty$, by Taylor’s theorem and (2.7), for $F$–almost all $B \in \mathcal{E}$ we obtain

$$\|\gamma(h, x) - D\phi(y)(\Phi(h + \gamma(h, x)) - \Phi(h))\|
\leq \|\phi(\Phi(h + \gamma(h, x))) - \Phi(h)\|
\leq 1 \frac{M}{2} \|\Phi(h + \gamma(h, x)) - \Phi(h)\|^2
\leq \frac{1}{2} M N \|\gamma(h, x)\|^2
\leq \frac{1}{2} M N \rho(x)^2,$$

proving (C.10). \hfill \blacksquare

For a closed subspace $K \subset H$ we denote by $\Pi_K : H \to K$ the orthogonal projection on $K$, that is, for each $h \in H$ the vector $\Pi_K h$ is the unique element from $K$ such that

$$\|\Pi_K h - h\| = \inf_{g \in K} \|g - h\|.$$

**C.5. Lemma.** Suppose that $\gamma$ satisfies the $\epsilon$–$\delta$–jump condition on $\mathcal{M}$. Then, the following statements are true:

1. For each $h \in \mathcal{M}$ we have

\begin{equation}
\int_E \|\Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x)\| F(dx) < \infty.
\end{equation}

2. The mapping

\begin{equation}
\mathcal{M} \to H, \quad h \mapsto \int_E \Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x) F(dx)
\end{equation}

is continuous.

**Proof.** Let $h_0 \in \mathcal{M}$ be arbitrary. By Lemma B.11 there exists a parametrization $\phi : V \subset \mathbb{R}^n_+ \to U \cap \mathcal{M}$ around $h_0$ such that $\phi$ and $\Phi := \phi^{-1}$ have extensions $\phi \in C^2_0(\mathbb{R}^m; H)$ and $\Phi \in C^1_0(H; \mathbb{R}^m)$. According to Lemma C.4 there exist $\delta > 0$, a set $B \in \mathcal{E}$ with $F(B^c) < \infty$ and a measurable mapping $\rho : E \to \mathbb{R}_+$ satisfying $\int_E \rho(x)^2 F(dx) < \infty$ such that (C.10) is satisfied. Let $h \in B_3(h_0) \cap \mathcal{M}$ be arbitrary. Then, for $F$–almost all $x \in B$ we obtain

$$\|\Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x)\| = \|\gamma(h, x) - \Pi_{T_h \mathcal{M}} \gamma(h, x)\|
\leq \|\gamma(h, x) - D\phi(y)(\Phi(h + \gamma(h, x)) - \Phi(h))\|
\leq \rho(x)^2.$$  

Moreover, by (2.6), for each $x \in E$ the mapping

$$H \to H, \quad h \mapsto \Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x)$$
is continuous. Together with Lebesgue’s dominated convergence theorem and Lemma A.18 we deduce (1.11) and the continuity of the mapping (C.12).

C.6. Remark. Lemmas C.3 and C.5 show that the jump condition (1.4) implies condition (1.9), as pointed out in the introduction.

C.7. Lemma. Suppose that \( \gamma \) satisfies the \( \epsilon, \delta \)-jump condition on \( \mathcal{M} \) and let \( \phi : V \subset \mathbb{R}^m \to U \cap \mathcal{M} \) be a parametrization such that \( \phi \) and \( \Phi := \phi^{-1} \) have extensions \( \phi \in C^2_b(\mathbb{R}^m; H) \) and \( \Phi \in C^1_b(H; \mathbb{R}^m) \). Then, the following statements are equivalent:

1. We have
\[
\int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) < \infty, \quad h \in U \cap \partial \mathcal{M}.
\]

2. We have
\[
\int_E |\langle \eta_h, D\phi(y) (\Phi(h + \gamma(h, x)) - \Phi(h)) \rangle| F(dx) < \infty, \quad h \in U \cap \partial \mathcal{M}
\]
where \( y = \phi^{-1}(h) \).

Proof. Let \( h \in U \cap \partial \mathcal{M} \) be arbitrary and set \( y := \phi^{-1}(h) \). By Lemma C.4 there exists set \( B \in \mathcal{E} \) with \( F(B^c) < \infty \) such that
\[
\int_B |\langle \eta_h, \gamma(h, x) \rangle - D\phi(y) (\Phi(h + \gamma(h, x)) - \Phi(h)) \rangle| F(dx) < \infty.
\]
Setting \( M := \|D\phi\|_\infty \) and \( N := \|D\Phi\|_\infty \), by using Lemma A.18 we have
\[
\int_{B^c} |\langle \eta_h, \gamma(h, x) \rangle| F(dx) \leq \|\eta_h\| \int_B \|\gamma(h, x)\| F(dx) < \infty
\]
as well as
\[
\leq \|\eta_h\| MN \int_{B^c} \|\gamma(h, x)\| F(dx) < \infty.
\]
Therefore, the claimed equivalence follows.

Let \( \beta : H \to H \) and \( \gamma : H \times E \to H \) be mappings such that conditions (2.6), (2.7) are fulfilled with the mappings \( \rho_n : E \to \mathbb{R}_+ \), \( n \in \mathbb{N} \) satisfying (3.4). Let \( B \in \mathcal{E} \) be a set with \( F(B^c) < \infty \) and define the mappings \( \beta^B : H \to H \) and \( \gamma^B : H \times E \to H \) as
\[
\beta^B(h) := \beta(h) - \int_{B^c} \gamma(h, x) F(dx),
\]
\[
\gamma^B(h, x) := \gamma(h, x) 1_{B}(x).
\]
Note that \( \beta^B \) is well-defined according to Lemma A.18

C.8. Proposition. Suppose that \( \gamma \) satisfies the \( \epsilon, \delta \)-jump condition on \( \mathcal{M} \). Then, the following statements are true:

1. We have
\[
\int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx), \quad h \in \partial \mathcal{M}
\]
\[
\beta(h) - \int_E \Pi_{(T_h \mathcal{M})^\perp} \gamma(h, x) F(dx) \in T_h \mathcal{M}, \quad h \in \mathcal{M}
\]
\[
\langle \eta_h, \beta(h) \rangle - \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) \geq 0, \quad h \in \partial \mathcal{M}
\]
Lemma.

(C.9) \[ \begin{align*} &\text{If and only if if} \quad \\ &\int_E |\langle \eta_h, \gamma^B(h, x) \rangle| F(dx), \quad h \in \partial M \quad \text{(C.16)} \\ &\beta^B(h) - \int_E \Pi_{(T_h, M)}^+ \gamma^B(h, x) F(dx) \in T_h M, \quad h \in M \quad \text{(C.17)} \\ &\langle \eta_h, \beta^B(h) \rangle - \int_E |\langle \eta_h, \gamma^B(h, x) \rangle| F(dx) \geq 0, \quad h \in \partial M. \quad \text{(C.18)} \end{align*} \]

Proof. This is a consequence of Lemmas C.9 – C.11 below.

Proof. Let the following statements are true:

(C.14) The mapping in \[ (C.14) \] is continuous on \[ (M) \] if and only if the mapping in \[ (C.17) \] is continuous on \[ (M) \].

Proof. Let \( h \in \partial M \) be arbitrary. Then we have

\[ \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) = \int_{E^c} |\langle \eta_h, \gamma(h, x) \rangle| F(dx) + \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) \]

\[ = \int_{E^c} |\langle \eta_h, \gamma(h, x) \rangle| F(dx) + \int_E |\langle \eta_h, \gamma^B(h, x) \rangle| F(dx). \]

Taking into account Lemma A.18, the claimed equivalence \( (\text{C.13}) \Leftrightarrow (\text{C.16}) \) follows.

C.9. Lemma. Conditions \( \text{(C.13)} \) and \( \text{(C.16)} \) are equivalent.

Proof. Let \( h \in \partial M \) be arbitrary. Then we have

\[ \beta^B(h) - \int_E \Pi_{(T_h, M)}^+ \gamma^B(h, x) F(dx) \]

\[ = \beta(h) - \int_{E^c} \gamma(h, x) F(dx) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) \]

\[ = \beta(h) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) - \int_{E^c} \gamma(h, x) F(dx) \]

\[ - \int_{E^c} \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) + \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) \]

\[ = \beta(h) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) - \Pi_{T_h M} \int_{E^c} \gamma(h, x) F(dx), \]

together with Lemma A.18, proves the claimed equivalences.

C.10. Lemma. Suppose that \( \gamma \) satisfies the \( \epsilon - \delta \) jump condition on \( M \). Then, the following statements are true:

1. Conditions \( (\text{C.14}) \) and \( (\text{C.17}) \) are equivalent.
2. The mapping in \( (\text{C.14}) \) is continuous on \( M \) if and only if the mapping in \( (\text{C.17}) \) is continuous on \( M \).

Proof. Let \( h \in M \) be arbitrary. The calculation

\[ \beta^B(h) - \int_E \Pi_{(T_h, M)}^+ \gamma^B(h, x) F(dx) \]

\[ = \beta(h) - \int_{E^c} \gamma(h, x) F(dx) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) \]

\[ = \beta(h) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) - \int_{E^c} \gamma(h, x) F(dx) \]

\[ - \int_{E^c} \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) + \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) \]

\[ = \beta(h) - \int_E \Pi_{(T_h, M)}^+ \gamma(h, x) F(dx) - \Pi_{T_h M} \int_{E^c} \gamma(h, x) F(dx), \]

together with Lemma A.18, proves the claimed equivalences.

C.11. Lemma. Suppose that \( (\text{C.13}) \) is satisfied. Then, conditions \( (\text{C.15}) \) and \( (\text{C.18}) \) are equivalent.

Proof. According to Lemma C.9, condition \( (\text{C.16}) \) is satisfied, too. Let \( h \in \partial M \) be arbitrary. Then we have

\[ \langle \eta_h, \beta^B(h) \rangle - \int_E |\langle \eta_h, \gamma^B(h, x) \rangle| F(dx) \]

\[ = \langle \eta_h, \beta(h) - \int_E \gamma(h, x) F(dx) \rangle - \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx) \]

\[ = \langle \eta_h, \beta(h) \rangle - \int_E |\langle \eta_h, \gamma(h, x) \rangle| F(dx), \]
proving the claimed equivalence \( C.15 \Leftrightarrow C.18 \).

Let \( G \) be another separable Hilbert space and let \( \mathcal{N} \) an \( m \)-dimensional \( C^3 \)-submanifold with boundary of \( G \). We assume there exist parametrizations \( \phi : V \subset \mathbb{R}^m_+ \to M \) and \( \psi : V \subset \mathbb{R}^m_+ \to \mathcal{N} \). Defining \( f := \phi \circ \psi^{-1} : \mathcal{N} \to M \) and \( g := \psi \circ \phi^{-1} : M \to \mathcal{N} \), the situation is illustrated by the following diagram:

\[
\begin{array}{ccc}
\mathcal{N} \subset G & \xrightarrow{\phi} & M \subset H \\
\downarrow \psi & & \downarrow \phi \\
V \subset \mathbb{R}^m_+ & &
\end{array}
\]

We assume that \( \phi, \psi, \Phi := \phi^{-1}, \Psi := \psi^{-1} \) have extensions \( \phi \in C^3_b(\mathbb{R}^m; H), \psi \in C^3_b(\mathbb{R}^m; G), \Phi \in C^3_b(H; \mathbb{R}^m), \Psi \in C^3_b(G; \mathbb{R}^m) \). Consequently, the mappings \( f, g \) have extensions \( f \in C^3_b(G; H), g \in C^3_b(H; G) \). Let \( O_M \subset C_M \subset M \) be subsets such that \( O_M \) is open in \( M \). We define the subsets \( O_N \subset C_N \subset N \) by \( O_N := g(O_M), C_N := g(C_M) \) and the subsets \( O_V \subset C_V \subset V \) by \( O_V := \psi^{-1}(O_N) \), \( C_V := \psi^{-1}(C_N) \). Since \( f : \mathcal{N} \to M \) and \( \psi : V \to \mathcal{N} \) are homeomorphisms, \( O_N \) is open in \( \mathcal{N} \) and \( O_V \) is open in \( V \).

Let \( \beta : O_M \to H, \sigma^j : H \to H, \) \( j \in \mathbb{N}, \gamma : H \times E \to H \) and \( a : O_N \to G, b^j : G \to G, \) \( j \in \mathbb{N}, c : G \times E \to G \) be mappings satisfying the regularity conditions \( (2.3), (2.4) \) and \( (2.6) - (2.8) \).

In the sequel, for \( z \in \partial \mathcal{N} \) the vector \( \xi_z \) denotes the inward pointing normal (tangent) vector to \( \partial \mathcal{N} \) at \( z \).

C.12. Proposition. **Suppose that**

\[
\begin{align*}
(C.19) \quad & a(z) = (f^* \beta)(z), \quad z \in O_N \\
(C.20) \quad & b^j(z) = (f^* \sigma^j)(z), \quad j \in \mathbb{N} \text{ and } z \in O_N \\
(C.21) \quad & c(z, x) = (f^* \gamma)(z, x) \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } z \in O_N
\end{align*}
\]

and that the following conditions are satisfied:

\[
\begin{align*}
(C.22) \quad & \sigma^j(h) \in T_h M, \quad h \in O_M, \quad j \in \mathbb{N} \\
(C.23) \quad & \sigma^j(h) \in T_h \partial M, \quad h \in O_M \cap \partial M, \quad j \in \mathbb{N} \\
(C.24) \quad & h + \gamma(h, x) \in C_M \quad \text{for } F\text{-almost all } x \in E, \quad \text{for all } h \in O_M \\
(C.25) \quad & \int_E \langle \eta_h, \gamma(h, x) \rangle |F(dx)| < \infty, \quad h \in O_M \cap \partial M \\
(C.26) \quad & \beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D \sigma^j(h) \sigma^j(h) \\
& \quad - \int_E 1_{(T_h M)^*} \gamma(h, x) F(dx) \in T_h M, \quad h \in O_M \\
(C.27) \quad & \langle \eta_h, \beta(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \eta_h, D \sigma^j(h) \sigma^j(h) \rangle \\
& \quad - \int_E \langle \eta_h, \gamma(h, x) \rangle F(dx) \geq 0, \quad h \in O_M \cap \partial M.
\end{align*}
\]
Then, the following conditions also hold true:

(C.28) \( b^j(z) \in T_z \mathcal{N}, \quad z \in \mathcal{O_N}, \quad j \in \mathbb{N} \)

(C.29) \( b^j(z) \in T_z \partial \mathcal{N}, \quad z \in \mathcal{O_N} \cap \partial \mathcal{N}, \quad j \in \mathbb{N} \)

(C.30) \( z + c(z, x) \in \mathcal{C_N} \) for \( F \)-almost all \( x \in E \), for all \( z \in \mathcal{O_N} \)

(C.31) \[ \int_E |(\xi_z, c(z, x))| F(dx) < \infty, \quad z \in \mathcal{O_N} \cap \partial \mathcal{N} \]

(C.32) \[ a(z) - \frac{1}{2} \sum_{j \in \mathbb{N}} D^b^j(z) b^j(z) - \int_E \Pi_{(T_z \mathcal{N})^+} c(z, x) F(dx) \in T_z \mathcal{N}, \quad z \in \mathcal{O_N} \]

(C.33) \[ \langle \xi_z, a(z) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi_z, D^b^j(z) b^j(z) \rangle - \int_E \langle \xi_z, c(z, x) \rangle F(dx) \geq 0, \quad z \in \mathcal{O_N} \cap \partial \mathcal{N}. \]

For the proof of Proposition C.12 we prepare several auxiliary results. Note that, under conditions (C.19)–(C.21), for all \( z \in \mathcal{O_N} \) we have

(C.34) \[ a(z) = Dg(h) \beta(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h)) \]

\[ + \int_E (g(h + \gamma(h, x)) - g(h) - Dg(h) \gamma(h, x)) F(dx), \]

(C.35) \[ b^j(z) = Dg(h) \sigma^j(h) \quad \text{for all } j \in \mathbb{N}, \]

(C.36) \[ c(z, x) = g(h + \gamma(h, x)) - g(h) \quad \text{for } F\text{-almost all } x \in E, \]

where \( h = f(z) \in \mathcal{O_M} \).

C.13. Lemma. Let \( h \in \mathcal{M} \) be arbitrary and set \( z = g(h) \in \mathcal{N} \).

(1) For each \( w \in T_h \mathcal{M} \) we have \( Dg(h)w \in T_z \mathcal{N} \).

(2) For each \( w \in (T_h \mathcal{M})_+ \) we have \( Dg(h)w \in (T_z \mathcal{N})_+ \).

(3) For each \( w \in T_h \partial \mathcal{M} \) we have \( Dg(h)w \in T_z \partial \mathcal{N} \).

Proof. Let \( w \in T_h \mathcal{M} \) be arbitrary and set \( y := \phi^{-1}(h) \in V \). By Lemma B.12 we have

\[ Dg(h)w = D(\psi \circ \Phi)(h)w = D\psi(y)D\Phi(h)w = D\psi(y)(D\phi(y)^{-1}w), \]

proving the three assertions. \( \square \)

C.14. Lemma. Suppose that \( C.20 \) is satisfied. Then, the following statements hold true:

(1) Condition \( C.22 \) implies \( C.28 \).

(2) Condition \( C.23 \) implies \( C.29 \).

Proof. This follows from \( C.35 \) and Lemma C.13. \( \square \)

C.15. Lemma. Suppose that \( C.21 \) is satisfied. Then, condition \( C.24 \) implies \( C.30 \).

Proof. Let \( z \in \mathcal{O_N} \) be arbitrary and set \( h := f(z) \in \mathcal{O_M} \). Then, by \( C.36 \) and \( C.24 \), for \( F \)-almost all \( x \in E \) we obtain

\[ z + c(z, x) = z + g(h + \gamma(h, x)) - g(h) = g(h + \gamma(h, x)) \in \mathcal{C_N}, \]

showing \( C.30 \). \( \square \)
C.16. Lemma. Suppose that
\[ h + \gamma(h, x) \in \overline{M} \] for \( F \)-almost all \( x \in E \), for all \( h \in O_M \).
Then \( \gamma \) satisfies the \( \epsilon - \delta \)-jump condition on \( O_M \).

Proof. Let \( h_0 \in O_M \) be arbitrary. By Lemma C.3 and since \( O_M \) is open in \( M \),
there exist \( 0 < \delta < \epsilon \) and a set \( B \in \mathcal{E} \) with \( F(B^c) < \infty \) such that \( \overline{B}(h_0) \cap M \) is compact, \( \overline{B}(h_0) \cap M \subset O_M \) and (C.8) is satisfied. Noting that
\[ \overline{B}(h_0) \cap M = \overline{B}(h_0) \cap O_M, \]
we deduce that
\[ h + \gamma(h, x) \in \overline{B}(h_0) \cap O_M \] for \( F \)-almost all \( x \in B \), for all \( h \in \overline{B}(h_0) \cap O_M \),
finishing the proof. \( \square \)

C.17. Corollary. Suppose that condition (C.24) is satisfied. Then, the following statements are true:

1. For each \( h \in O_M \) we have
\[ \int_E \| \Pi_{(T_h M)^\perp} \gamma(h, x) \| F(dx) < \infty. \]
2. The mapping
\[ O_M \to H, \ h \mapsto \int_E \Pi_{(T_h M)^\perp} \gamma(h, x) F(dx) \]
is continuous.

Proof. This is a direct consequence of Lemmas C.16 and C.5. \( \square \)

C.18. Lemma. For every \( h \in \partial M \) there exists a unique number \( \lambda > 0 \) such that
\[ \langle \xi_z, D\psi(y)v \rangle = \lambda \langle \eta_{\gamma(h)} D\phi(y)v \rangle \] for all \( v \in \mathbb{R}^m \),
where \( y = \phi^{-1}(h) \) and \( z = \psi(y) \). Moreover, we have
\[ \langle \xi_z, Dg(h)w \rangle = \lambda \langle \eta_{\gamma(h)} w \rangle \] for all \( w \in T_h M \).

Proof. Identity (C.37) is a direct consequence of Lemma B.9. Using Lemma B.12 and (C.37), for all \( w \in T_h M \) we obtain
\[ \langle \xi_z, Dg(h)w \rangle = \langle \xi_z, D(\psi \circ \Phi)(h)w \rangle = \langle \xi_z, D\psi(y)D\Phi(h)w \rangle = \langle \xi_z, D\psi(y)D\phi^{-1}(y)w \rangle = \lambda \langle \eta_{\gamma(h)} w \rangle, \]
which proves (C.38). \( \square \)

C.19. Lemma. Suppose that (C.21), (C.24) are satisfied. Then, condition (C.25) implies (C.31).

Proof. According to Lemma C.15 condition (C.30) is satisfied, too. Let \( z \in O_M \cap \partial N \) be arbitrary. We set \( h := f(z) \in O_M \cap \partial M \) and \( y := \phi^{-1}(h) \in O_V \cap \partial V \). By Lemma C.7 we have
\[ \int_E |\langle \eta_{\gamma(h)}, D\phi(y)(\Phi(h + \gamma(h, x)) - \Phi(h)) \rangle| F(dx) < \infty. \]
Using (C.36) and Lemma C.18 for some \( \lambda > 0 \) we obtain
\[
\int_E |\langle \xi_z, D\psi(y)\rangle (\Psi(z + c(z, x)) - \Psi(z))| F(dx)
\]
\[
= \int_E |\langle \xi_z, D\psi(y)\rangle (\Phi(g + \gamma(h, x)) - \Phi(h))| F(dx)
\]
\[
= \int_E |\langle \xi_z, D\psi(y)\rangle (\Phi(h + \gamma(h, x)) - \Phi(h))| F(dx)
\]
\[
= \lambda \int_E |\langle \eta_h, D\psi(y)\rangle (\Phi(h + \gamma(h, x)) - \Phi(h))| F(dx) < \infty.
\]
Applying Lemma C.7 yields condition (C.31).

C.20. Lemma. Suppose that \((C.20), (C.22)\) are satisfied and let \( j \in \mathbb{N} \) be arbitrary. For each \( z \in O_N \) we have the decomposition
\[
D^j h(z) b^j(z) = Dg(h)(D\sigma^j(h)\sigma^j(h)) + D^2 g(h)(\sigma^j(h), \sigma^j(h)),
\]
where \( h = f(z) \in O_M \).

Proof. According to Lemma C.14 condition (C.28) is satisfied, too. Note that, by Lemma B.12 and (C.35), for all \( y \in O_V \) we have
\[
D\phi(y)^{-1} \sigma^j(h) = D\phi(h) \sigma^j(h) = D(\Psi \circ g)(h) \sigma^j(h) = D(\Psi(z) Dg(h) \sigma^j(h)
\]
\[
= D\Psi(z) b^j(z) = D\psi(y)^{-1} b^j(z),
\]
where \( h := \phi(y) \in O_M \) and \( z := \psi(y) \in O_N \). We define the mapping
\[
\theta^j : O_Y \to \mathbb{R}^m, \quad \theta^j : D\phi(y)^{-1} \sigma^j(h), \quad h := \phi(y).
\]
Let \( z \in O_N \) be arbitrary. We set \( h := f(z) \in O_M \) and \( y := \phi^{-1}(h) \in O_U \). Using Lemma C.9 we obtain the decompositions
\[
D\sigma^j(h) \sigma^j(h) = D\phi(y)(D\theta^j(y) \theta^j(y)) + D^2 \phi(y)(\theta^j(y), \theta^j(y)),
\]
\[
D h(z) b^j(z) = D\psi(y)(D\theta^j(y) \theta^j(y)) + D^2 \psi(y)(\theta^j(y), \theta^j(y)).
\]
Note that we have
\[
D\psi(y)(D\theta^j(y) \theta^j(y)) = (D \circ \phi(y))(D\theta^j(y) \theta^j(y)) = Dg(h) D\phi(y)(D\theta^j(y) \theta^j(y)).
\]
By the second order chain rule we obtain
\[
D^2 \psi(y)(\theta^j(y), \theta^j(y)) = D^2 (\circ \phi(y))(\theta^j(y), \theta^j(y))
\]
\[
= D^2 g(h)(D\phi(y)(D\theta^j(y) \theta^j(y)), D\phi(y)(D\theta^j(y) \theta^j(y))) + Dg(h)(D^2 \phi(y)(\theta^j(y), \theta^j(y)))
\]
\[
= D^2 g(h)(\sigma^j(h), \sigma^j(h)) + Dg(h)(D^2 \phi(y)(\theta^j(y), \theta^j(y))).
\]
Moreover, by (C.39) we have
\[
Dg(h)(D^2 \phi(y)(\theta^j(y), \theta^j(y)))
\]
\[
= Dg(h)(D\sigma^j(h) D\sigma^j(h)) - Dg(h) D\phi(y)(D\theta^j(y) \theta^j(y)).
\]
Inserting (C.41)–(C.43) into (C.40) we arrive at
\[
D h(z) b^j(z) = Dg(h) D\phi(y)(D\theta^j(y) \theta^j(y)) + D^2 g(h)(\sigma^j(h), \sigma^j(h))
\]
\[
+ Dg(h)(D^2 \phi(y)(\theta^j(y), \theta^j(y)))
\]
\[
= Dg(h) D\phi(y)(D\theta^j(y) \theta^j(y)) + D^2 g(h)(\sigma^j(h), \sigma^j(h))
\]
\[
+ Dg(h)(D^2 \phi(y)(\theta^j(y), \theta^j(y))) - Dg(h) D\phi(y)(D\theta^j(y) \theta^j(y))
\]
\[
= Dg(h)(D(\sigma^j(h) \sigma^j(h)) + D^2 g(h)(\sigma^j(h), \sigma^j(h)),
\]
completing the proof. □
C.21. Lemma. Suppose that (C.19)–(C.21) and (C.22), (C.24) are satisfied. Then, condition (C.26) implies (C.32).

Proof. According to Lemma C.15 condition (C.30) is satisfied, too. Let \( z \in O_\mathcal{N} \) be arbitrary and set \( h := f(z) \in O_M \). By (C.34), (C.36) we obtain

\[
a(z) - \frac{1}{2} \sum_{j \in \mathbb{N}} Db_j^i(z)b_j^i(z) - \int_E \Pi_{(T_z\mathcal{N})^\perp} c(z, x) F(dx)
= Dg(h) \beta(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h))
+ \int_E \left( g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x) \right) F(dx)
- \frac{1}{2} \sum_{j \in \mathbb{N}} Db_j^i(z)b_j^i(z) - \int_E \Pi_{(T_z\mathcal{N})^\perp} \left( g(h + \gamma(h, x)) - g(h) \right) F(dx).
\]

Thus, by Lemma C.20 relation (C.26) and Lemma C.13 we arrive at

\[
a(z) - \frac{1}{2} \sum_{j \in \mathbb{N}} Db_j^i(z)b_j^i(z) - \int_E \Pi_{(T_z\mathcal{N})^\perp} c(z, x) F(dx)
= Dg(h) \beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} Dg(h)(D\sigma^j(h)\sigma^j(h))
- \int_E \Pi_{(T_z\mathcal{N})^\perp} (g(h + \gamma(h, x)) - g(h)) F(dx)
= Dg(h) \left( \beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \Pi_{(T_h\mathcal{M})^\perp}\gamma(h, x) F(dx) \right)
- \int_E \Pi_{(T_z\mathcal{N})^\perp} (g(h + \gamma(h, x)) - g(h)) F(dx) \in T_z\mathcal{N},
\]

proving that (C.32) is fulfilled. \( \square \)

C.1. Proof of Proposition C.12. According to Lemmas C.14, C.15, C.19, C.21 conditions (C.28)–(C.32) are satisfied. Let \( z \in O_\mathcal{N} \) be arbitrary and set \( h := f(z) \in O_M \). By (C.34), (C.36) and Lemma C.20 we obtain

\[
\langle \xi, a(z) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi, Db_j^i(z)b_j^i(z) \rangle - \int_E \langle \xi, c(z, x) \rangle F(dx)
= \langle \xi, Dg(h) \beta(h) \rangle + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h))
+ \int_E \left( g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x) \right) F(dx)
- \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi, Db_j^i(z)b_j^i(z) \rangle - \int_E \langle \xi, g(h + \gamma(h, x)) - g(h) \rangle F(dx)
= \langle \xi, Dg(h) \beta(h) \rangle - \frac{1}{2} \sum_{j \in \mathbb{N}} \langle \xi, Dg(h)(D\sigma^j(h)\sigma^j(h)) \rangle
- \int_E \langle \xi, Dg(h)\gamma(h, x) \rangle F(dx).
\]
Taking into account Lemma C.18 by (C.26), (C.27) for some $\lambda > 70$ DAMIR FILIPOVIĆ, STEFAN TAPPE, AND JOSEF TEICHMANN

Then, the following conditions also hold true:

C.22. Proposition. showing that (C.33) is satisfied. This completes the proof of Proposition C.12.

C.23. Proposition. Suppose we have (C.19)–(C.21) and (C.22), (C.24), (C.26).

For the proof of Proposition C.22 we prepare some auxiliary results. Note that Conditions (C.20), (C.22) imply (C.45).

Proof. Let $h$ be arbitrary and set $z := g(h) \in \mathcal{N}$. By Lemma B.12 for all $w \in T_h \mathcal{M}$ we have

$$Df(z)Dg(h)w = D(\phi \circ \Psi)(z)D(\psi \circ \Phi)(h)w = D\phi(y)[D\Psi(z)D\psi(y)D\Phi(h)]w$$

$$= D\phi(y)D\psi(y)^{-1}D\psi(y)D\phi(y)^{-1}w = w,$$

which proves the claimed identity. □

C.24. Lemma. Conditions (C.26), (C.23) imply (C.45).

Proof. Let $h \in O_M$ be arbitrary and set $z := g(h) \in \mathcal{N}$. By (C.48), (C.35), Lemma C.23 and (C.22), for each $j \in \mathbb{N}$ we obtain

$$(g^* b^j)(h) = Df(z)b^j(z) = Df(z)Dg(h)\sigma^j(h) = \sigma^j(h),$$

proving that (C.45) is fulfilled. □
C.25. Lemma. Conditions (C.21), (C.24) imply (C.46).

Proof. Let \( h \in O_M \) be arbitrary and set \( z := g(h) \in O_N \). By (C.49), (C.36) and (C.46), for \( F \)-almost all \( x \in E \) we obtain

\[
(g^*(c))(h, x) = f(z + c(z, x)) - f(z) = f(z + g(h + \gamma(h, x)) - g(h)) - f(z) = f(g(h + \gamma(h, x))) - h = \gamma(h, x),
\]

showing that (C.46) is satisfied. □

C.2. Proof of Proposition C.22. By Lemmas C.24, C.25 conditions (C.45), (C.46) are satisfied. Let \( h \in O_M \) be arbitrary and set \( z := g(h) \in O_N \). By (C.47), (C.34) we obtain

\[
(g^*(a))(h) = Df(z)a(z) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 f(z)(b^j(z), b^j(z)) + \int_E (f(z + c(z, x)) - f(z) - Df(z)c(z, x)) F(dx)
\]

\[
= Df(z)\left(Dg(h)\beta(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 g(h)(\sigma^j(h), \sigma^j(h)) + \int_E (g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x)) F(dx)\right)
\]

\[
+ \frac{1}{2} \sum_{j \in \mathbb{N}} D^2 f(z)(b^j(z), b^j(z)) + \int_E (f(z + c(z, x)) - f(z) - Df(z)c(z, x)) F(dx).
\]

By (C.22) and Lemma C.13, condition (C.28) is satisfied, too. Hence, applying Lemma C.20 two times, by taking into account (C.20), (C.22) and (C.45), (C.28), and using (C.36) as well as (C.46), (C.49) we get

\[
(g^*(a))(h) = Df(z)\left(Dg(h)\beta(h) + \frac{1}{2} \sum_{j \in \mathbb{N}} \left(Db^j(z)b^j(z) - Dg(h)(D\sigma^j(h)\sigma^j(h))\right)\right)
\]

\[
+ \int_E (g(h + \gamma(h, x)) - g(h) - Dg(h)\gamma(h, x)) F(dx)
\]

\[
+ \frac{1}{2} \sum_{j \in \mathbb{N}} (D\sigma^j(h)\sigma^j(h) - Df(z)(Db^j(z)b^j(z)))
\]

\[
+ \int_E (\gamma(h, x) - Df(z)(g(h + \gamma(h, x)) - g(h))) F(dx)
\]

\[
= Df(z)Dg(h)\left(\beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)\right) + \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)
\]

\[
+ \int_E (\gamma(h, x) - Df(z)Dg(h)\gamma(h, x)) F(dx).
\]
Using (C.26), by Lemma C.23, we obtain

\[(g^*a)(h) = Df(z)Dg(h)\left(\beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)\right) + \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h)\]

\[+ \int_E (\Pi_{T_h,M}\gamma(h,x) - Df(z)Dg(h)\Pi_{T_h,M}\gamma(h,x)) F(dx) + \Pi_{(T_h,M)^{\perp}}\gamma(h,x) F(dx)\]

\[= Df(z)Dg(h)\left(\beta(h) - \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) - \int_E \Pi_{(T_h,M)^{\perp}}\gamma(h,x) F(dx)\right)\]

\[+ \frac{1}{2} \sum_{j \in \mathbb{N}} D\sigma^j(h)\sigma^j(h) + \int_E \Pi_{(T_h,M)^{\perp}}\gamma(h,x) F(dx) = \beta(h),\]

showing that (C.44) is fulfilled. This completes the proof of Proposition C.22.

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EPFL and Swiss Finance Institute, Quartier UNIL-Dorigny, Extranef 218, CH-1015 Lausanne, Switzerland

E-mail address: damir.filipovic@epfl.ch

Leibniz Universität Hannover, Institut für Mathematische Stochastik, Welfengarten 1, D-30167 Hannover, Germany

E-mail address: tappe@stochastik.uni-hannover.de

ETH Zürich, Department of Mathematics, Rämistrasse 101, CH-8092 Zürich, Switzerland

E-mail address: josef.teichmann@math.ethz.ch