A basket half full: sparse portfolios

EKATERINA SEREGINA

Department of Economics, Colby College, Waterville, ME 04901, USA

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The existing approaches to sparse wealth allocations (1) are limited to low-dimensional setup when the number of assets is less than the sample size; (2) lack theoretical analysis of sparse wealth allocations and their impact on portfolio exposure; (3) are suboptimal due to the bias induced by an $\ell_1$-penalty. We address these shortcomings and develop an approach to construct sparse portfolios in high dimensions. Our contribution is twofold: from the theoretical perspective, we establish the oracle bounds of sparse weight estimators and provide guidance regarding their distribution. From the empirical perspective, we examine the merit of sparse portfolios during different market scenarios. We find that in contrast to non-sparse counterparts, our strategy is robust to recessions and can be used as a hedging vehicle during such times.

Keywords: High dimensionality; Portfolio optimization; Factor investing; De-biasing; Post-Lasso; Approximate factor model

JEL Classifications: C13, C55, C58, G11, G17

1. Introduction

The search for the optimal portfolio weights reduces to the questions (i) which stocks to buy and (ii) how much to invest in these stocks. Depending on the strategy used to address the first question, the existing allocation approaches can be further broken down into the ones that invest in all available stocks, and the ones that select a subset out of the stock universe. The latter is referred to as a sparse portfolio, since some assets will be excluded and get a zero weight leading to sparse wealth allocations. Asset allocation for non-sparse portfolios has been the main focus of the existing research on asset management for a long time. Standard portfolio optimization problems require the inverse covariance matrix, or precision matrix, of excess stock returns as an input. The sample estimators are only suitable when the number of assets $p$ is fixed and smaller than the sample size $T$ (number of periods), i.e. $\lim_{p,T \to \infty} p/T = 0$, which is known as the standard asymptotics in statistics (Le Cam and Yang 2000). Under this asymptotics, the traditional sample covariance is consistent and produces well-behaved estimates for portfolio weights.

What happens when $p$ and $T$ are comparable in size, or when $p$ is allowed to grow with $T$, i.e. $\lim_{p,T \to \infty} p/T = c \in (0, \infty)$? These cases fall under the umbrella of a high-dimensional scenario and have received increased attention in the literature that focuses on asset allocation for large-dimensional portfolios. Sample estimators lose favorable properties which leads to unstable portfolio weights (Brodie et al. 2009). High-dimensional scenarios are prevalent nowadays: according to World Federation of Exchanges (WFE) statistics portal, at the end of Q1 2022 the total number of listed companies stood at 58,200, which represented a 0.2% uptick on Q4 2021 and a 2.5% increase on Q1 2021. The World Bank estimates show that the total number of listed domestic companies worldwide has almost doubled from 1990 to 2020. This raises high dimensionality concerns even when using higher frequency data, such as daily. Furthermore, even if $p < T$, it is unrealistic to assume that $p$ is fixed as in the standard asymptotics.

There are two mainstream solutions to handle high-dimensional portfolio allocation. The first focuses on finding an improved estimator for covariance or precision matrix and the primary target are non-sparse portfolios. The second applies techniques such as reducing the number of assets to avoid dealing with high dimensionality. The aforementioned approaches illustrate a bias-variance trade-off: including more assets introduces higher estimation error, whereas removing some assets (e.g. illiquid assets) improves efficiency at the cost of higher bias. In this paper, we develop a statistical approach to reduce the number of assets and construct sparse portfolios that are guaranteed to provide consistent estimates of portfolio weights. We obtain the oracle bounds of sparse weight estimators and provide guidance regarding their distribution. From the empirical perspective, we examine the merit of sparse portfolios during the periods of economic growth, moderate market decline and severe economic downturns. We find that...
in contrast to non-sparse counterparts, our strategy is robust to recessions and can be used as a hedging vehicle during such times. Our approach is relevant to practitioners since it does not require an ad hoc pre-selection of assets and helps efficiently control transaction costs.

As pointed out above, estimating high-dimensional covariance or precision matrix to improve portfolio performance of non-sparse strategies has received a lot of attention in the existing literature. Ledoit and Wolf (2004, 2017) developed linear and non-linear shrinkage estimators of a covariance matrix, Fan et al. (2013, 2018) introduced a covariance matrix estimator when stock returns are driven by common factors under the assumption of a spiked covariance model. Once the covariance estimator is obtained, it is then inverted to get a precision matrix, the main input to any portfolio optimization problem. A parallel stream of literature has focused on estimating precision matrix directly, that is, avoiding the inversion step that leads to additional estimation errors, especially in high dimensions. Friedman et al. (2008) developed an iterative algorithm that estimates the entries of precision matrix column-wise using penalized Gaussian log-likelihood (Graphical Lasso); Meinshausen and Bühlmann (2006) used the relationship between regression coefficients and the entries of precision matrix to estimate the elements of the latter column by column (nodewise regression). Cai et al. (2011) use constrained $\ell_1$-minimization for inverse matrix estimation (CLIME). Callot et al. (2019) examined the performance of high-dimensional portfolios constructed using covariance and precision estimators and found that precision-based models outperform covariance-based counterparts in terms of the out-of-sample (OOS) Sharpe Ratio and portfolio return.

In order to create a sparse portfolio, that is, a portfolio with many zero entries in the weight vector, we can use an $\ell_1$-penalty (Lasso) on the portfolio weights which shrinks some of them to zero (see Brodie et al. 2009, Li 2015, Ao et al. 2019, Fan et al. 2019 among others). Caccioli et al. (2016) proved the mathematical equivalence of adding an $\ell_1$-penalty and controlling transaction costs associated with the bid-ask spread impact of single and sequential trades executed in a very short time. This indicates another advantage of sparse portfolios: market liquidity dries up during economic downturns which increases bid-ask spreads, a measure of liquidity costs. Henceforth, regularizing portfolio positions accounts for the increased liquidity risk associated with acquiring and liquidating positions. The existing literature on sparse wealth allocations is scarce and has several drawbacks: (1) it is limited to low-dimensional setup when $p < T$, whereas sparsity becomes especially important in high-dimensional scenarios; (2) it lacks theoretical analysis of sparse wealth allocations and their impact on portfolio exposure; (3) the use of an $\ell_1$-penalty produces biased estimates (see Javanmard and Montanari 2014a, 2014b, van de Geer et al. 2014, Zhang and Zhang 2014, Belloni et al. 2015, Javanmard and Montanari 2018 among others), however, this issue has been overlooked in the context of portfolio allocation. This paper addresses the aforementioned drawbacks and develops an approach to construct sparse portfolios in high dimensions. Our contribution is twofold: from the theoretical perspective, we establish consistency of high-dimensional sparse weight estimators, provide guidance regarding their distribution, and prove consistency of a new high-dimensional precision matrix estimator that bridges factor models and graphical models.

From the empirical perspective, we examine the merit of sparse portfolios during different market scenarios. We find that in contrast to non-sparse counterparts, our strategy is robust to recessions and can be used as a hedging vehicle during such times. To illustrate, the last two rows of table 1 show the performance of two sparse strategies proposed in this paper: both approaches outperform non-sparse counterparts in terms of total OOS Sharpe Ratio, and they produce positive cumulative excess return (CER) during the pandemic, as well as in the period preceding it. Figure 1 shows the stocks selected by post-Lasso in August 2019 and in May 2020: the colors serve as a visual guide to identify groups of closely related stocks (stocks of the same color do not necessarily correspond to the same sector). Our framework makes use of the tool from the network theory called nodewise regression which not only satisfies desirable statistical properties but also allows us to study whether certain industries could serve as safe havens during recessions. We find that such non-cyclical industries as consumer staples, healthcare, retail and food were driving the returns of the sparse portfolios during both the Global Financial Crisis (GFC) in 2007–2009 and the COVID-19 outbreak, whereas the insurance sector was the least attractive investment in both periods.

This paper is organized as follows: section 2 introduces sparse de-biased portfolio and sparse portfolio using post-Lasso. Section 3 develops a new high-dimensional precision

Table 1. Performance of non-sparse and sparse portfolios using daily data: return ($\times 100$), risk ($\times 100$) and Sharpe Ratio over the training period (left), CER ($\times 100$) and risk ($\times 100$) over two sub-periods (right).

| Portfolio          | Total OOS Performance 10/24/19–09/24/20 | Before the Pandemic 01/02/19–12/31/19 | During the Pandemic 01/02/20–06/30/20 |
|--------------------|------------------------------------------|----------------------------------------|----------------------------------------|
|                    | Return ($\times 100$) | Risk ($\times 100$) | Sharpe Ratio | CER ($\times 100$) | Risk ($\times 100$) | CER ($\times 100$) | Risk ($\times 100$) |
| EW                 | 0.0108 | 1.8781 | 0.0058 | 28.5420 | 0.8010 | −19.7207 | 3.3169 |
| Index              | 0.0351 | 1.7064 | 0.0206 | 27.8629 | 0.7868 | −9.0802 | 2.9272 |
| Nodewise Regr’     | 0.0322 | 1.6384 | 0.0196 | 29.6292 | 0.6856 | −11.7431 | 2.8939 |
| CLIME              | 0.0793 | 3.1279 | 0.0373 | 31.5294 | 1.0215 | −25.3004 | 3.8972 |
| LW                 | 0.0317 | 1.7190 | 0.0184 | 29.5513 | 0.7924 | −14.9328 | 3.0115 |
| Our Post-Lasso-based | 0.1428 | 1.6427 | 0.0869 | 41.6218 | 1.0193 | 2.3813 |
| Our De-biased Estimator | 0.0275 | 0.5231 | 0.0526 | 23.7629 | 0.4972 | 6.5813 | 0.5572 |

Weights are estimated using the Global Minimum-Variance formula. In-sample: 25 May 2018–23 October 2018 (105 obs.), Out-of-sample (OOS): 24 October 2018–24 September 2020 (483 obs.)
Section 7 concludes.

1.1. Notation

For the convenience of the reader, we summarize the notation to be used throughout the paper. A basket half full

estimator called Factor Nodewise regression. Section 4 contains theoretical results and section 5 validates these results using simulations. Section 6 provides empirical application. Section 7 concludes.

1.1. Notation

For the convenience of the reader, we summarize the notation to be used throughout the paper. Let $S_p$ denote the set of all $p \times p$ symmetric matrices. For any matrix $C$, its $(i,j)$th element is denoted as $c_{ij}$. Given a vector $y \in \mathbb{R}^d$ and parameter $a \in [1, \infty)$, let $\|y\|_a$ denote $\ell_a$-norm. Given a matrix $U \in S_p$, let $\Lambda_{\text{max}}(U) = \Lambda_1(U) \geq \Lambda_2(U) \geq \ldots \Lambda_{\text{min}}(U) = \Lambda_p(U)$ be the eigenvalues of $U$, and $\text{diag}(U) \in \mathbb{R}^{p \times p}$ denote the first $K \leq p$ normalized eigenvectors corresponding to $\Lambda_1(U), \ldots, \Lambda_K(U)$. Given parameters $a, b \in [1, \infty)$, let $\|U\|_{a,b} = \max_{1 \leq i \leq p} \|U_y\|_b$ denote the induced matrix-operator norm. The special cases are $\|U\|_1 = \max_{i \leq p} \sum_{j=1}^p |u_{ij}|$ for the $\ell_1/\ell_1$-operator norm; the operator norm ($\ell_2$-matrix norm) $\|U\|_2^2 = \Lambda_{\text{max}}(U^T U)$ is equal to the maximal singular value of $U$; $\|U\|_\infty = \max_{i \leq p} \sum_{j=1}^p |u_{ij}|$ for the $\ell_\infty/\ell_\infty$-operator norm. Finally, $\|U\|_{\text{max}} = \max_{i \leq p} |u_{ij}|$ denotes the element-wise maximum, and $\|U\|_F^2 = \sum_{i,j} u_{ij}^2$ denotes the Frobenius matrix norm, $\|\cdot\|_*$ denotes the trace norm, which is defined as the sum of the singular values of a matrix. We also use the following notations: $a \lor b = \max\{a, b\}$, and $a \land b = \min\{a, b\}$. For an event $A$, we say that $A \Rightarrow 1$ when $A$ occurs with probability approaching 1 as $T$ increases.

2. Sparse portfolios

There exist several widely used portfolio weight formulations depending on the type of optimization problem solved by an investor. Suppose we observe $p$ assets (indexed by $i$) over a period of time (indexed by $t$). Let $r_t = (r_{1t}, r_{2t}, \ldots, r_{pt})' \sim D(m, \Sigma)$ be a $p \times 1$ vector of excess returns drawn from a distribution $D$, where $m$ and $\Sigma$ are unconditional mean and covariance of excess returns, and $D$ belongs to either sub-Gaussian or elliptical families. When $D = \mathcal{N}$, the precision matrix $\Sigma^{-1} = \Theta$ contains information about conditional dependence between the variables. For instance, if $\theta_{ij}$, which is the $(i,j)$th element of the precision matrix, is zero, then the variables $i$ and $j$ are conditionally independent, given the other variables. The goal of the Markowitz theory is to choose assets weights in a portfolio optimally. We will study two criteria of optimality: the first is a well-known Markowitz weight-constrained optimization problem, and the second formulation relaxes constraints on portfolio weights.

The first optimization problem, which will be referred to as Markowitz weight-constrained problem (MWC), searches for asset weights such that the portfolio achieves a desired expected rate of return with minimum risk, under the restriction that all weights sum up to one. The aforementioned goal can be formulated as the following quadratic optimization problem:

$$\min \theta w' \Sigma w, \quad \text{s.t. } w' 1 = 1 \text{ and } m' w \geq \mu,$$

where $w$ is a $p \times 1$ vector of asset weights in the portfolio, $\iota$ is a $p \times 1$ vector of ones, and $\mu$ is a desired expected rate of portfolio return. The constraint in (1) requires portfolio weights to sum up to one—this assumption can be easily relaxed and we will demonstrate the implications of this constraint on portfolio weights.

If $m' w > \mu$, then the solution to (1) yields the global minimum-variance (GMV) portfolio weights $w_{GMV}$:

$$w_{GMV} = (\iota' \Theta)^{-1} \Theta \iota.$$
If \( \mathbf{m}' \mathbf{w} = \mu \), the solution to (1) is

\[
\mathbf{w}_{\text{MWC}} = (1 - a_t) \mathbf{w}_{\text{GMV}} + a_t \mathbf{w}_M, \tag{3}
\]

\[
\mathbf{w}_M = (\mathbf{\Theta} \mathbf{m})^{-1} \mathbf{\Theta} \mathbf{m}, \tag{4}
\]

\[
a_t = \frac{\mu (\mathbf{m}' \mathbf{\Theta}) (\mathbf{\Theta}' \mathbf{m}) - (\mathbf{m}' \mathbf{\Theta})^2}{(\mathbf{m}' \mathbf{\Theta}) (\mathbf{\Theta}' \mathbf{m}) - (\mathbf{m}' \mathbf{\Theta})^2}, \tag{5}
\]

where \( \mathbf{w}_{\text{MWC}} \) denotes the portfolio allocation with the constraint that the weights need to sum up to one and \( \mathbf{w}_M \) captures all mean-related market information.

The second optimization problem, which will be referred to as the Markowitz risk-constrained (MRC) problem, has the same objective as in (1), but portfolio weights are not required to sum up to one:

\[
\min_{\mathbf{w}} \frac{1}{2} \mathbf{w}' \mathbf{\Sigma} \mathbf{w} \quad \text{s.t.} \quad \mathbf{m}' \mathbf{w} = \mu. \tag{6}
\]

It can be easily shown that the solution to (6) is:

\[
\mathbf{w}_1^* = \frac{\mu \mathbf{\Theta} \mathbf{m}}{\mathbf{m}' \mathbf{\Theta} \mathbf{m}}. \tag{7}
\]

Alternatively, instead of searching for a portfolio with a specified desired expected rate of return and minimum risk, one can maximize expected portfolio return given a maximum risk-tolerance level:

\[
\max_{\mathbf{w}} \mathbf{w}' \mathbf{m} \quad \text{s.t.} \quad \mathbf{w}' \mathbf{\Sigma} \mathbf{w} \leq \sigma^2. \tag{8}
\]

In this case, the solution to (8) yields:

\[
\mathbf{w}_2^* = \frac{\sigma^2}{\mathbf{w}' \mathbf{m}} \mathbf{\Theta} \mathbf{m} = \frac{\sigma^2}{\mu} \mathbf{\Theta} \mathbf{m}. \tag{9}
\]

To get the second equality in (9) we used the definition of \( \mu \) from (1) and (6). It follows that if \( \mu = \sigma \sqrt{\theta} \), where \( \theta \equiv \mathbf{m}' \mathbf{\Theta} \mathbf{m} \) is the squared Sharpe Ratio, then the solution to (6) and (8) admits the following expression:

\[
\mathbf{w}_{\text{MRC}} = \frac{\sigma}{\sqrt{\mathbf{m}' \mathbf{\Theta} \mathbf{m}}} \mathbf{\Theta} \mathbf{m} = \frac{\sigma}{\sqrt{\theta}} \mathbf{m}. \tag{10}
\]

where \( \mathbf{\alpha} = \mathbf{\Theta} \mathbf{m} \). Equation (10) tells us that once an investor specifies the desired return, \( \mu \), and maximum risk-tolerance level, \( \sigma \), this pins down the Sharpe Ratio of the portfolio which makes the optimization problems of minimizing risk and maximizing expected return of the portfolio in (6) and (8) identical.

This brings us to three alternative portfolio allocations commonly used in the existing literature: GMV in (2), MWC in (3) and MRC in (10). Below we summarize the aforementioned portfolio weight expressions:

\[
\text{GMV : } \mathbf{w}_{\text{GMV}} = (\mathbf{\Theta}' \mathbf{\Theta})^{-1} \mathbf{\Theta} \mathbf{t}, \tag{11}
\]

\[
\text{MWC : } \mathbf{w}_{\text{MWC}} = (1 - a_t) \mathbf{w}_{\text{GMV}} + a_t \mathbf{w}_M, \tag{12}
\]

\[
\text{where } \mathbf{w}_M = (\mathbf{\Theta} \mathbf{m})^{-1} \mathbf{\Theta} \mathbf{m}, \quad a_t = \frac{\mu (\mathbf{m}' \mathbf{\Theta}) (\mathbf{\Theta}' \mathbf{m}) - (\mathbf{m}' \mathbf{\Theta})^2}{(\mathbf{m}' \mathbf{\Theta}) (\mathbf{\Theta}' \mathbf{m}) - (\mathbf{m}' \mathbf{\Theta})^2}, \tag{13}
\]

where \( \mathbf{\alpha} = \mathbf{\Theta} \mathbf{m} \). The second optimization problem, which will be referred to as the Markowitz risk-constrained (MRC) problem, has the same objective as in (1), but portfolio weights are not required to sum up to one:

so far we have considered allocation strategies that put nonzero weights to all assets in the financial portfolio. As an implication, an investor needs to buy a certain amount of each security even if there are a lot of small weights. However, oftentimes investors are interested in managing a few assets which significantly reduces monitoring and transaction costs and was shown to outperform equal-weighted and index portfolios in terms of the Sharpe Ratio and cumulative return (see Brodie et al. 2009, Li 2015, Ao et al. 2019, Fan et al. 2019 among others). This strategy is based on holding a sparse portfolio, that is, a portfolio with many zero entries in the weight vector.

2.1. Sparse de-biased portfolio

Let us motivate the use of a sparse portfolio by examining correlations between assets over time. We use a sample of monthly returns for 30 country equity indices† from 10/2002 to 03/2023. Figure 2 plots three-year average rolling correlations between equities: we observe a spike at the onset of the financial crisis in late 2008, and a spike at the beginning of 2020 at the onset of COVID-19 pandemic. This suggests reduced benefits of diversification during downturns and points us towards developing a more selective investment strategy that leverages holding fewer assets. This stylized fact is consistent with previous empirical studies: Tidmore et al. (2019) used active US equity funds’ quarterly data from January 2000 to December 2017 from Morningstar, Inc. to study the impact of concentration (measured by the number of holdings) on fund excess returns and found that the effect was significant and fluctuated considerably over time. Notably, the relationship became negative in the period preceding and including the global financial crisis.

To further support this hypothesis, we compare the performance of sparse vs non-sparse strategies in terms of utility gain to investors. Consider the following mean-variance utility problem: \( \min_{\mathbf{w}} -U \equiv \frac{1}{2} \mathbf{w}' \mathbf{\Sigma} \mathbf{w} - \mathbf{w}' \mathbf{m}, \) s.t. \( \mathbf{w}' \mathbf{1} = 1, \) \( \supp(\mathbf{w}) \leq \bar{p}, \) \( \bar{p} \leq p, \) where \( \mathbf{w} \) is a \( p \times 1 \) vector of portfolio weights, \( \supp(\mathbf{w}) = \{ i : w_i > 0 \} \) is the cardinality constraint that controls sparsity, and \( \gamma \) determines the risk of an investor under the assumption of a normal distribution. When \( \bar{p} = p \) the portfolio is non-sparse and the respective utility is denoted as \( U^{\text{Non-Sparse}} \), while when \( \bar{p} < p \) the utility of such sparse portfolio is denoted as \( U^{\text{Sparse}} \). Figure 3 reports the ratio of utilities using monthly data from 2003:04 to 2009:12 on the constituents of the S&P100 as a function of \( \bar{p} ; \) we

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† MSCI ITALY, MSCI SPAIN, MSCI PORTUGAL, MSCI FRANCE, MSCI GERMANY, MSCI AUSTRIA, MSCI DENMARK, MSCI FINLAND, MSCI NETHERLANDS, MSCI SWEDEN, MSCI SWITZERLAND, MSCI TURKEY, MSCI CANADA, MSCI BRAZIL, MSCI MEXICO, MSCI COLOMBIA, MSCI ARGENTINA, MSCI PERU, MSCI CHILE, MSCI CHINA, MSCI INDIA, MSCI INDONESIA, MSCI RUSSIA, MSCI JAPAN, MSCI MALAYSIA, MSCI SINGAPORE, MSCI TAIWAN, MSCI SOUTH AFRICA, MSCI AUSTRALIA, MSCI KOREA, MSCI US.
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Figure 2. Average 3-year rolling correlations using a sample of monthly returns for 30 country equity indices.

Figure 3. The ratio of non-sparse and sparse portfolio utilities averaged over the test window.

set \( \gamma = 3 \) and vary \( \bar{p} = \{5, 10, 15, 20, 30, \ldots, 90\} \). Our test sample includes two periods of particular interest: before the global financial crisis (GFC) (2004: 01–2006:12) and during the GFC (2007: 01–2009:12). As evidenced from figure 3: (1) for both time periods there exists a lower-dimensional subset of stocks which brings greater utility compared to non-sparse portfolios; (2) the number of stocks minimizing the ratio of utilities is smaller during the GFC compared to the period preceding it. Both findings are consistent with the stylized fact that including more stocks does not guarantee better performance and suggesting that holding a ‘basket half full’ instead can help achieve superior performance in stressed market scenarios.

Proceeding to constructing a sparse portfolio, we introduce some notations. The sample mean and sample covariance matrix have standard formulas: \( \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t \) and \( \hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} (r_t - \hat{\mu})(r_t - \hat{\mu})' \). Our empirical application shows that risk-constrained Markowitz allocation in (13) outperforms GMV and MWC portfolios in (11)–(12). Therefore, we

\[ \hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} r_t \]  
\[ \hat{\Sigma} = \frac{1}{T} \sum_{t=1}^{T} (r_t - \hat{\mu})(r_t - \hat{\mu})' \]

Since the optimization problem with a cardinality constraint is not convex, we find a solution using the Lagrangian relaxation procedure of Shaw et al. (2008)

† Since the optimization problem with a cardinality constraint is not convex, we find a solution using the Lagrangian relaxation procedure of Shaw et al. (2008)
first study sparse MRC portfolios. Our goal is to construct a sparse vector of portfolio weights given by (13). To achieve this, we use the following equivalent and unconstrained regression representation of the mean-variance optimization in (6) and (8):

$$\mathbf{w}_{\text{MRC}} = \arg\min_{\mathbf{w}} \mathbb{E} [\mathbf{y} - \mathbf{w} \mathbf{r}], \quad \text{where } \mathbf{y} = \frac{1 + \theta}{\theta} \mathbf{\mu} = \sigma \frac{1 + \theta}{\sqrt{\theta}}. \quad (14)$$

The sample counterpart of (14) is written as:

$$\mathbf{w}_{\text{MRC}} = \arg\min_{\mathbf{w}} \frac{1}{T} \sum_{t=1}^{T} (\mathbf{y} - \mathbf{w} \mathbf{r}_t)^2. \quad (15)$$

Ao et al. (2019) prove that the weight allocation from (14) is equivalent to (13). The sparsity is introduced through Lasso techniques (see Javanmard and Montanari 2014a, 2014b, van de Geer 2014, Zhang and Zhang 2014, Belloni et al. 2015, Javanmard and Montanari 2018 among others).

Now we propose two extensions to the setup (16). First, the estimator \( w_{\text{MRC,SPARSE}} \) is infeasible since \( \theta \) used for constructing \( y \) is unknown. Ao et al. (2019) construct an estimator of \( \theta \) under normally distributed excess returns, assuming \( p/T \to \rho \in (0, 1) \) and the sample size \( T \) is required to be larger than the number of assets \( p \). Their paper uses an unbiased estimator proposed in Kan and Zhou (2007): \( \hat{\theta} = ((T - p - 2) \hat{\mathbf{m}} \hat{\mathbf{Σ}}^{-1} \hat{\mathbf{m}} - p)/T \), where \( \hat{\mathbf{m}} \) and \( \hat{\mathbf{Σ}}^{-1} \) are sample mean and inverse of the sample covariance matrix respectively. One of the limitations of the model studied by Ao et al. (2019) is that it cannot handle high dimensions. In both simulations and empirical applications, the maximum number of stocks used by the authors is limited to 100. Another limitation of Ao et al. (2019) approach is that they do not correct the bias introduced by imposing \( \lambda \)-constraint in (16). However, it is well-known that the estimator in (16) is biased and the existing literature proposes several de-biasing techniques (see Javanmard and Montanari 2014a, 2014b, van de Geer et al. 2014, Zhang and Zhang 2014, Belloni et al. 2015, Javanmard and Montanari 2018 among others).

To address the first aforementioned limitation, we propose to use an estimator of a high-dimensional precision matrix appropriate for high-dimensional settings, it can handle cases when the sample size is less than the number of assets, and it is always non-negative by construction. Consequently, the estimator of \( y \) is

$$\tilde{\mathbf{y}} = \frac{1 + \hat{\theta}}{\hat{\theta}} \mathbf{\mu} = \sigma \frac{1 + \hat{\theta}}{\sqrt{\hat{\theta}}}. \quad (17)$$

To approach the second limitation, motivated by van de Geer et al. (2014), we propose the de-biasing technique that uses the node wise regression estimator of the precision matrix. First, let \( \mathbf{R} \) be a \( T \times p \) matrix of excess returns stacked over time and \( \hat{\mathbf{y}} \) be a \( T \times 1 \) constant vector. Consider a high-dimensional linear model

$$\hat{\mathbf{y}} = \mathbf{R}\mathbf{w} + \mathbf{e}, \quad \text{where } \mathbf{e} \sim \mathcal{D}(0, \sigma^2 I). \quad (18)$$

We study high-dimensional framework \( p \geq T \) and in the asymptotic results, we require \( \log p/T = o(1) \). Let us rewrite (16):

$$\mathbf{w}_{\text{MRC,SPARSE}} = \arg\min_{\mathbf{w} \in \mathbb{R}^p} \frac{1}{T} \|\hat{\mathbf{y}} - \mathbf{R}\mathbf{w}\|^2 + 2\lambda \|\mathbf{w}\|. \quad (19)$$

The estimator in (16) satisfies the following KKT conditions:

$$\mathbf{R}'(\tilde{\mathbf{y}} - \mathbf{R}\mathbf{\hat{w}}) + \lambda \hat{\mathbf{g}} = 0, \quad (20)$$

$$\|\hat{\mathbf{g}}\|_{\infty} \leq 1 \quad \text{and} \quad \hat{g}_i = \text{sign}(\hat{w}_i) \text{ if } \hat{w}_i \neq 0, \quad (21)$$

where \( \hat{\mathbf{g}} \) is a \( p \times 1 \) vector arising from the subgradient of \( \|\mathbf{w}\|_1 \). Let \( \hat{\mathbf{Σ}} = \mathbf{R}'(\mathbf{Σ}^{-1} \mathbf{R})/T \), then we can rewrite the KKT conditions:

$$\hat{\mathbf{Σ}}(\mathbf{\tilde{w}} - \mathbf{w}) + \lambda \hat{\mathbf{g}} = \mathbf{R}'\mathbf{e}/T. \quad (22)$$

Multiply both sides of (22) by \( \hat{\Theta} \) obtained from algorithm 2, add and subtract \( (\mathbf{\tilde{w}} - \mathbf{w}) \), and rearrange the terms:

$$\mathbf{\tilde{w}} - \mathbf{w} + \hat{\Theta}\hat{\mathbf{g}} = \hat{\Theta}\mathbf{R}'\mathbf{e}/T - \sqrt{T(\hat{\Theta}\hat{\mathbf{Σ}} - \mathbf{I})}(\mathbf{\tilde{w}} - \mathbf{w})/\sqrt{T}. \quad \Delta \quad (23)$$

In the section with the theoretical results, we show that \( \Delta \) is asymptotically negligible under certain sparsity assumptions. Combining (20) and (23) brings us to the de-biased estimator of portfolio weights:

$$\hat{\mathbf{w}}_{\text{MRC,DEBIASED}} = \mathbf{\tilde{w}} + \hat{\Theta}\hat{\mathbf{g}} = \hat{\mathbf{w}} + \hat{\Theta}\mathbf{R}'(\tilde{\mathbf{y}} - \mathbf{R}\mathbf{\hat{w}})/T. \quad (24)$$

The properties of the proposed de-biased estimator are examined in section 5.

2.2. Sparse portfolio using post-Lasso

One of the drawbacks of the de-biased portfolio weights in (24) is that the weight formula is tailored to a specific portfolio choice that maximizes an unconstrained Sharpe Ratio (i.e. MRC in (13)). However, it is desirable to accommodate preferences of different types of investors who might be interested in weight allocations corresponding to GMV (11) or MWC (12) portfolios. To overcome this difficulty we split the search for the optimal portfolio weights into two stages. First, we solve for the optimal subset of assets to be included in the portfolio. Second, a desired financial portfolio is constructed using the assets from the first stage. We now elaborate on both

† Our empirical results suggest that the unbiased estimator \( \hat{\theta} = ((T - p - 2) \hat{\mathbf{m}} \hat{\mathbf{Σ}}^{-1} \hat{\mathbf{m}} - p)/T \) is oftentimes negative even after using the adjusted estimator defined in Kan and Zhou (2007, p. 2906).

‡ Note that we cannot directly apply Theorem 2.2 of van de Geer et al. (2014) since \( \mathbf{y} \) needs to be estimated and we first need to show consistency of the respective estimator.
stages. Step 1 solves the following optimization problem:
\[
\hat{w}_t = \arg\min_{w \in \mathbb{R}^p} \frac{1}{2T} \| \bar{y}_t - Rw \|_2^2 + P_\zeta (w), \quad \text{s.t. } C_\zeta (w)
\] (25)
where \( \zeta \in \{ \text{MRC, MWC, GMV} \} \) determines the portfolio allocation that an investor is interested in, \( y_t \) is the dependent variable that corresponds to each allocation, \( P_\zeta (w) \) is the penalty that imposes sparsity on portfolio weights, and \( C_\zeta (w) \) is the set of constraints. As the choices for the dependent variable we use:
\[
\tilde{y}_{\zeta} (w) = \begin{cases} 
\frac{1 + \tilde{\theta}}{\tilde{\theta}} \mu \equiv \frac{1 + \tilde{\theta}}{\sqrt{\tilde{\theta}}} \zeta \in \{ \text{MRC, MWC} \}, \\
\rho_{\text{Baseline Asset}, j} \zeta \in \{ \text{GMV} \}
\end{cases}
\] (26)
where MRC and MWC formulations are the same as in (17), and the GMV uses the return of an (arbitrary selected) baseline asset, \( \rho_{\text{Baseline Asset}, j} \), which is motivated by the regression formulation studied in Belloni and Chernozhukov (2013). We use the returns of S&P500 composite index as a baseline asset for our empirical application. As the choice for the penalty function, we use:
\[
P_\zeta (w) = \begin{cases} 
2\lambda \| w \|_1, & \text{standard Lasso as in (19)} \\
2\lambda \left[ \gamma (R'R)^{1/2} + (1 - \gamma) (R'R + \mu I)^{-1} \right] & \text{precisionLasso}
\end{cases}
\]
The precision Lasso penalty was introduced by Wang et al. (2019) to account for correlations and linear dependencies in high-dimensional data. Following their paper, we set \( \gamma = 1/2 \) and \( \mu \) is a small positive parameter to make \( R'R \) invertible when \( p > T \). As the choices for the set of constraints, we have \( C_\zeta (w) : w_{j} = 1 \) for \( \zeta \in \{ \text{MWC, GMV} \} \).

Let \( \hat{\zeta} = \text{support}(\hat{w}) \) denote the model selected in step 1. Our step 2 is motivated by the literature on model selection in high-dimensional regression models: we apply additional thresholding to remove stocks with small estimated weights
\[
\hat{w}_t (j) = (\hat{w}_{\zeta(t)} \mathbb{I} [\hat{w}_{\zeta(t)} j > \gamma], j = 1, \ldots, p), \quad \text{where } t \geq 0 \text{ is the thresholding level.}
\]
The corresponding selected model is denoted as \( \hat{\zeta}_{t} (w) : \text{support}(\hat{w}(t)) \). When \( t = 0 \), \( \hat{\zeta}_{t} (w) = \hat{\zeta} \).

3. Factor nodewise regression

In this section, we introduce a high-dimensional estimator of precision matrix that bridges the gap between factor models and graphical models.

The arbitrage pricing theory (APT), developed by Ross (1976), postulates that expected returns on securities should be related to their covariance with the common components or factors only. The goal of the APT is to model the tendency of asset returns to move together via factor decomposition. Assume that the return-generating process \( (r_t) \) follows a \( K \)-factor model:
\[
r_t = B f_t + \epsilon_t, \quad t = 1, \ldots, T
\] (30)
where \( f = (f_1, \ldots, f_K)' \) are the factors, \( B \) is a \( p \times K \) matrix of factor loadings, and \( \epsilon_t \) is the idiosyncratic component that cannot be explained by the common factors. Factors
in (30) can be either observable, such as in Fama and French (1993, 2015), or can be estimated using statistical factor models.

In this subsection, we examine how to approach the portfolio allocation problems in (11)–(13) using a factor structure in the returns. Our approach, called Factor Nodewise Regression, uses the estimated common factors to obtain a sparse precision matrix of the idiosyncratic component. The resulting estimator is used to obtain the precision of the asset returns necessary to form portfolio weights.

As in Fan et al. (2013), we consider a spiked covariance model when the first $K$ principal eigenvalues of $\Sigma$ are growing with $p$, while the remaining $p-K$ eigenvalues are bounded and grow slower than $p$.

Rewrite equation (30) in matrix form:

$$
R_{p \times T} = \mathbb{B} F + \mathbb{E},
$$

(31)

Let $\Sigma = T^{-1}RR'$, $\Sigma_x = T^{-1}EE'$ and $\Sigma_f = T^{-1}FF'$ be covariance matrices of stock returns, idiosyncratic components and factors, and let $\Theta = \Sigma^{-1}$, $\Theta_x = \Sigma_x^{-1}$ and $\Theta_f = \Sigma_f^{-1}$ be their inverses. The factors and loadings in (31) are estimated by solving $(\widehat{\mathbb{B}}, \widehat{\mathbb{F}}) = \text{argmin}_F \|R - \mathbb{BF}\|_F^2$ s.t. $\approx\mathbb{FF}' = 1_k$, $\mathbb{BB}$ is diagonal. The constraints are needed to identify the factors (Fan et al. 2018). It was shown (Stock and Watson 2002) that $\widehat{\mathbb{F}} = \sqrt{T} \text{eig}(R'R)$ and $\widehat{\mathbb{B}} = T^{-1}R\widehat{\mathbb{F}}$. Given $\widehat{\mathbb{F}}, \widehat{\mathbb{B}}$, define $\widehat{\mathbb{E}} = R - \mathbb{BF}$.

Since our interest is in constructing portfolio weights, our goal is to estimate a precision matrix of the excess returns $\Theta_x$, which is used for constructing de-biased portfolios in (24) and post-Lasso portfolios in (25) (i.e. for constructing $\widehat{\gamma}$).

Most previous studies that focus on direct estimation of precision matrix impose sparsity on $\Theta$ shrinking many entries to zero. However, as pointed out by Koike (2020), when common factors are present across the excess returns, the precision matrix cannot be sparse because all pairs of the returns are partially correlated given other excess returns through the common factors. Therefore, we impose a sparsity assumption on the precision matrix of the idiosyncratic errors, $\Theta_f$, which is obtained using the estimated residuals after removing the co-movements induced by the factors (see Barigozzi et al. 2018, Brownlees et al. 2018, Koike 2020).

We use the nodewise regression as a shrinkage technique to estimate the precision matrix of residuals. In the spirit of Meinshausen and Bühlmann (2006), we solve for $\Theta_x$ one column at a time via linear regressions, replacing population moments by their sample counterparts. When we repeat this procedure for each variable $j = 1, \ldots, p$, we will estimate the elements of $\Theta_x$ column by column using $\{\hat{\delta}_{ij}\}_{j=1}^p$ via $p$ linear regressions. This method is known as the ‘nodewise’ regression.

Let $\hat{\gamma}_j$ be a $T \times 1$ vector of estimated residuals for the $j$-th regressor, the remaining covariates are collected in a $T \times (p-1)$ matrix $\hat{\gamma}_{-j}$. For each $j = 1, \ldots, p$ we run the following Lasso regressions:

$$
\hat{\gamma}_j = \arg\min_{\gamma \in \mathbb{R}^{p-1}} \left( \| \gamma - \hat{\gamma}_{-j} \|_2^2 / T + 2 \lambda_j \| \gamma \|_1 \right),
$$

(32)

where $\hat{\gamma}_j = \{\hat{\gamma}_{ij} : j = 1, \ldots, p, k \neq j\}$ is a $(p-1) \times 1$ vector of the estimated regression coefficients that will be used to construct the estimate of the precision matrix, $\Theta_x$. Define

$$
\widehat{C} = \begin{pmatrix}
1 & -\hat{\gamma}_{1,2} & \cdots & -\hat{\gamma}_{1,p} \\
-\hat{\gamma}_{2,1} & 1 & \cdots & -\hat{\gamma}_{2,p} \\
\vdots & \vdots & \ddots & \vdots \\
-\hat{\gamma}_{p,1} & -\hat{\gamma}_{p,2} & \cdots & 1
\end{pmatrix}.
$$

(33)

For $j = 1, \ldots, p$, define

$$
\hat{\varepsilon}_j^2 = \| \hat{\varepsilon}_j - \hat{\gamma}_{-j} \|_2^2 / T + \lambda_j \| \hat{\gamma}_j \|_1
$$

and write

$$
\tilde{T}^2 = \text{diag}(\hat{\varepsilon}_1^2, \ldots, \hat{\varepsilon}_p^2).
$$

(35)

The approximate inverse is defined as

$$
\hat{\Theta}_x = \tilde{T}^{-2} \hat{C}.
$$

(36)

One of the caveats to keep in mind when using the node-wise regression method is that the estimator in (36) is not self-adjoint. To resolve this issue we use the matrix symmetrization procedure as in Fan et al. (2018) and then use eigenvalue cleaning as in Hautsch et al. (2012). First, we symmetric matrix is constructed as

$$
\hat{\Theta}_x = \hat{\Theta}_x^T \hat{C}.
$$

(39)

where $\hat{\Theta}_{x,ij}$ is the $(i,j)$th element of the estimated precision matrix from (36). Second, we use eigenvalue cleaning to make $\hat{\Theta}_x$ positive definite: write the spectral decomposition $\hat{\Theta}_x = \hat{\Lambda} \hat{V} \hat{V}^T$, where $\hat{\Lambda}$ is a matrix of eigenvalues and $\hat{V}$ is a diagonal matrix with $p$ eigenvalues $\hat{\Lambda}_{x,ii}$ on its diagonal. Let $\hat{\Lambda}_{x,ii} = \min(\hat{\Lambda}_{x,ii}, \hat{\Lambda}_{x,ii} > 0)$. We replace all $\hat{\Lambda}_{x,ii} < \hat{\Lambda}_{x,ii}$ with $\hat{\Lambda}_{x,ii}$ and define the diagonal matrix with cleaned eigenvalues as $\hat{\Lambda}_x$. We use $\hat{\Theta}_x = \hat{V}^T \hat{\Lambda}_x \hat{V}$ which is symmetric and positive definite.

Having estimated all necessary components, we use the Sherman–Morrison–Woodbury formula to estimate the precision of excess returns:

$$
\hat{\Theta}_x = \hat{\Theta}_x \hat{B} \hat{\Theta}_x + \hat{\Theta}_x \hat{B} \hat{\Theta}_x \hat{B}^{-1} \hat{B} \hat{\Theta}_x.
$$

(38)

To obtain $\hat{\Theta}_x = \hat{\Sigma}^{-1}_f$, we use the inverse of the sample covariance of the estimated factors $\hat{\Sigma}_f = T^{-1} \hat{\mathbb{F}} \hat{\mathbb{F}}'$. The proposed procedure is called Factor Nodewise Regression and it is summarized in algorithm 2.

Algorithm 2 involves a tuning parameter $\lambda_j$ in (39): we choose shrinkage intensity by minimizing the generalized information criterion (GIC). Let $[\hat{S}_j(\lambda_j)]$ denote the estimated number of nonzero parameters in the vector $\hat{\gamma}_j$.

$$
\text{GIC}(\lambda_j) = \log(\| \hat{\gamma}_j - \hat{\gamma}_{-j} \|_2^2 / T) + |\hat{S}_j(\lambda_j)| \frac{\log(p)}{T} \log(\log(T)).
$$

We can use $\hat{\Theta}_x$ obtained in (40) to estimate $y$ in equation (17) and obtain sparse portfolio weights in (24) and algorithm 1.
Algorithm 2 Factor nodewise regression (FMB)

1: Estimate factors, \( \hat{\mathbf{F}} \), and factor loadings, \( \hat{\mathbf{B}} \), using PCA.
   Obtain \( \hat{\mathbf{\Sigma}}_j = T^{-1} \hat{\mathbf{F}} \hat{\mathbf{F}}' \hat{\mathbf{\Sigma}}_j^{-1} \) and \( \hat{\mathbf{\epsilon}}_j = \mathbf{r}_j - \hat{\mathbf{B}} \hat{\mathbf{\epsilon}}_j \).
2: Estimate a sparse \( \hat{\mathbf{\Theta}}_j \) using nodewise regression: run Lasso regressions in (32) for \( \hat{\mathbf{\epsilon}}_j \),
   
   \[
   \hat{\mathbf{y}}_j = \arg \min_{\mathbf{y} \in \mathbb{R}^p} \left( \| \mathbf{\epsilon}_j - \hat{\mathbf{\Theta}}_j \mathbf{y} \|_2^2 / T + 2\lambda_j \| \mathbf{y} \|_1 \right),
   \]

   to get \( \hat{\mathbf{\Theta}}_j \).
3: Use \( \hat{\mathbf{\Theta}}_j \) from step 1 and \( \hat{\mathbf{\Theta}}_j \) from step 2 to estimate \( \hat{\mathbf{\Theta}} \)
   using the sample counterpart of the Sherman–Morrison–Woodbury formula in (38):
   
   \[
   \hat{\mathbf{\Theta}} = \hat{\mathbf{\Theta}}_j - \hat{\mathbf{\Theta}} \hat{\mathbf{B}} \left[ \hat{\mathbf{\Theta}}_j + \hat{\mathbf{B}}' \hat{\mathbf{\Theta}}_j \hat{\mathbf{B}} \right]^{-1} \hat{\mathbf{B}}' \hat{\mathbf{\Theta}}_j. \tag{40}
   \]

4.1. Assumptions

We now list the assumptions on the model (41):

(A.1) (Spiked covariance model) As \( p \to \infty \), \( \Lambda_j(\mathbf{\Sigma}) \geq \Lambda_{j+1}(\mathbf{\Sigma}) \geq \cdots \geq \Lambda_K(\mathbf{\Sigma}) \geq 0 \),
   where \( \Lambda_j(\mathbf{\Sigma}) \sim \mathcal{O}(p) \) for \( j \leq K \), while the non-
   spiked eigenvalues are bounded, \( \Lambda_j(\mathbf{\Sigma}) = o(1) \) for \( j > K \).

(A.2) (Pervasive factors) There exists a positive definite \( K \times K \) matrix \( \hat{\mathbf{B}} \) such that
   
   \[
   \left\| p^{1/2} \mathbf{B} - \hat{\mathbf{B}} \right\|_2 \to 0 \text{ and } \Lambda_{\text{min}}(\mathbf{B})^{-1} = \mathcal{O}(1) \text{ as } p \to \infty.
   \]

(A.3) (Beta mixing) Let \( \mathcal{F}^{F_j} \) and \( \mathcal{F}^{F_j^c} \) denote the \( \sigma \)-algebras that are generated by \( \{ \mathbf{e}_u : u \leq t \} \) and \( \{ \mathbf{e}_u : u > t \} \)
   respectively. Then \( \{ \mathbf{e}_u \}_u \) is \( \beta \)-mixing in the sense that
   
   \[
   \beta_k = \sup \mathbb{E} \left[ \sup_{\mathbf{B} \in \mathcal{F}^{F_j}} \left| \text{Pr} \left( \mathbf{B} \in \mathcal{F}^{F_j^c} \right) - \text{Pr} \left( \hat{\mathbf{B}} \in \mathcal{F}^{F_j^c} \right) \right| \right]. \tag{42}
   \]

Some comments regarding the aforementioned assumptions are in order. Assumptions (A.1)–(A.2) are the same as in Fan et al. (2018), and assumption (A.3) is required to consistently estimate the precision matrix for de-biasing portfolio weights. Assumption (A.1) divides the eigenvalues into the diverging and bounded ones. Without loss of generality, we assume that \( K \) largest eigenvalues have a multiplicity of 1.

The assumption of a spiked covariance model is common in the literature on approximate factor models, however, we note that the model studied in this paper can be characterized as a ‘very spiked model’. In other words, the gap between the first \( K \) eigenvalues and the rest is increasing with \( p \). As pointed out by Fan et al. (2018), Assumption (A.1) is typically satisfied by the factor model with pervasive factors, which brings us to the assumption (A.2): the factors impact a non-vanishing proportion of individual time-series. Assumption (A.3) allows for weak dependence in the residuals of the factor model in (41): causal ARMA processes, certain stationary Markov chains and stationary GARCH models with finite second moments satisfy this assumption. We note that our assumption (A.3) is much weaker than in Callot et al. (2019), the latter requires weak dependence of the returns series, whereas we only restrict dependence of the idiosyncratic components.

Let \( \Sigma = \mathbf{\Gamma} \mathbf{A} \mathbf{\Gamma}' \), where \( \Sigma \) is the covariance matrix of returns that follow the factor structure described in equation (41). Define \( \hat{\mathbf{\Sigma}}, \hat{\mathbf{\Lambda}}, \hat{\mathbf{\Gamma}} \) to be the estimators of \( \mathbf{\Sigma}, \mathbf{A}, \mathbf{\Gamma} \). We further let \( \hat{\mathbf{\Lambda}}_k = \text{diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_k) \) and \( \hat{\mathbf{\Gamma}} = (\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_K) \) be constructed by the first \( K \) leading empirical eigenvalues and the corresponding eigenvectors of \( \hat{\mathbf{\Sigma}} \) and \( \hat{\mathbf{B}} \hat{\mathbf{B}}' = \hat{\mathbf{\Lambda}}_K \hat{\mathbf{\Gamma}} \hat{\mathbf{\Sigma}} \hat{\mathbf{\Gamma}}'. \) Similarly to Fan et al. (2018), we require the following bounds on the componentwise maximums of the estimators:

(B.1) \( \| \hat{\mathbf{\Sigma}} - \mathbf{\Sigma} \|_{\text{max}} = \mathcal{O}(\sqrt{\log p / T}) \),

(B.2) \( \| (\hat{\mathbf{\Lambda}}_K - \mathbf{\Lambda})^{-1} \|_{\text{max}} = \mathcal{O}(\sqrt{\log p / T}) \),

(B.3) \( \| \hat{\mathbf{\Gamma}} - \mathbf{\Gamma} \|_{\text{max}} = \mathcal{O}(\sqrt{\log p / (Tp)}) \).
Let \( \hat{\Sigma}^{SG} \) be the sample covariance matrix, with \( \hat{A}_K^{SG} \) and \( \hat{I}_K^{SG} \) constructed with the first \( K \) leading empirical eigenvalues and eigenvectors of \( \hat{\Sigma}^{SG} \) respectively. Also, let \( \hat{\Sigma}^{ELI} = \text{DR}_1 \text{D} \), where \( \text{DR}_1 \) is obtained using the Kendall’s tau correlation coefficients and \( \text{D} \) is a robust estimator of variances constructed using the Huber loss. Furthermore, let \( \hat{\Sigma}^{EL2} = \text{DR}_2 \text{D} \), where \( \text{DR}_2 \) is obtained using the spatial Kendall’s tau estimator. Define \( \hat{A}_K \) to be the matrix of the first \( K \) leading empirical eigenvalues of \( \hat{\Sigma}^{ELI} \), and \( \hat{I}_K \) is the matrix of the first \( K \) leading empirical eigenvectors of \( \hat{\Sigma}^{EL2} \). For more details regarding constructing \( \hat{\Sigma}^{SG} \), \( \hat{\Sigma}^{ELI} \) and \( \hat{\Sigma}^{EL2} \) see Fan et al. (2018, Sections 3 and 4).

**Theorem 1** (Fan et al. 2018) For sub-Gaussian distributions, \( \hat{\Sigma}^{SG}, \hat{A}_K^{SG} \) and \( \hat{I}_K^{SG} \) satisfy (B.1)–(B.3).

For elliptical distributions, \( \hat{\Sigma}^{ELI}, \hat{A}_K \) and \( \hat{I}_K \) satisfy (B.1)–(B.3).

Theorem 1 is essentially a rephrasing of the results obtained in Fan et al. (2018, Sections 3 and 4). Since there is no separate statement of these results in their paper (it is rather a summary of several theorems), we separated it as a Theorem for the convenience of the reader. As evidenced from the above theorem, \( \hat{\Sigma}^{ELI} \) is only used for estimating the eigenvectors. This is necessary due to the fact that, in contrast with \( \hat{\Sigma}^{SG} \), the theoretical properties of the eigenvectors of \( \hat{\Sigma}^{EL} \) are mathematically involved because of the sin function.

In addition, the following structural assumption on the model is imposed:

\[
(C.1) \quad \| \Sigma \|_{\text{max}} = O(1) \quad \text{and} \quad \| B \|_{\text{max}} = O(1),
\]

which is a natural assumption on the population quantities.

In contrast to Fan et al. (2018), instead of estimating and inverting the covariance matrix, we focus on obtaining the precision matrix directly since it is the ultimate input to any portfolio optimization problem.

### 4.2. Asymptotic properties of non-sparse portfolio weights

Recall that we used equation (38) to estimate \( \Theta \). Therefore, in order to establish consistency of the estimator in (38), we first show consistency of \( \hat{\Theta} \). Proofs of all the theorems are in appendix.

**Theorem 2** Suppose that assumptions (A.1)–(A.3), (B.1)–(B.3) and (C.1) hold. Let \( \omega_T \equiv \sqrt{\log p/T} + 1/\sqrt{p} \). Then

\[
\max_{1 \leq j \leq p}(1/T) \sum_{t=1}^{T} | \hat{\epsilon}_{it} - \epsilon_{it} | = O_p(\omega_T^2) \quad \text{and} \quad \max_{1 \leq j \leq p} | \hat{\epsilon}_{ij} - \epsilon_{ij} | = O_p(\omega_T) = o_p(1).
\]

Under the sparsity assumption \( \hat{\omega}_T^2 = o(1) \), with \( \hat{\omega}_T \equiv \sqrt{\sum_{1 \leq j \leq p} \hat{\omega}_j^2} \), we have

\[
\max_{1 \leq j \leq p} \| \hat{\Theta}_{ij} - \Theta_{ij} \|_1 = O_p(\hat{\omega}_T),
\]

\[
\max_{1 \leq j \leq p} \| \hat{\Theta}_{ij} - \Theta_{ij} \|_2^2 = O_p(\hat{\omega}_T^3).
\]

Some comments are in order. First, the sparsity assumption \( \hat{\omega}_T^2 = o(1) \) is stronger than that required for convergence of \( \hat{\Theta} \); this is necessary to ensure consistency for \( \hat{\Theta} \) established in theorem 3, so we impose a stronger assumption at the beginning. We also note that at the first glance, our sparsity assumption in theorem 3 is stronger than that required by van de Geer et al. (2014) and Callot et al. (2019), however, recall that we impose sparsity on \( \Theta \), not \( \Theta \) as opposed to the two aforementioned papers. Hence, this assumption can be easily satisfied once the common factors have been accounted for and the precision of the idiosyncratic components is expected to be sparse. The bounds derived in theorem 2 help us establish the convergence properties of the precision matrix of stock returns in equation (38).

**Theorem 3** Under the assumptions of theorem 2 and, in addition, assuming \( \| \Theta_{0,j} \|_2 = O(1) \), we have

\[
\max_{1 \leq j \leq p} \| \hat{\Theta}_{ij} - \Theta_{ij} \|_1 = O_p(\sqrt{\omega_T}),
\]

\[
\max_{1 \leq j \leq p} \| \hat{\Theta}_{ij} - \Theta_{ij} \|_2^2 = O_p(\hat{\omega}_T^2).
\]

Using theorem 3 we can then establish the consistency of the non-sparse counterpart of the estimated MRC portfolio weight in (19).

**Theorem 4** Under the assumptions of theorem 3, algorithm 2 consistently estimates non-sparse MRC portfolio weights such that \( \| \hat{\Theta}_{MRC} - \Theta_{MRC} \|_2 = O_p(\sqrt{\omega_T}) \).

Note that the rate in theorem 4 depends on the sparsity of \( \Theta \). If, instead, sparsity on \( \Theta \) is imposed, the rate becomes similar to the one derived by Callot et al. (2019): \( \hat{\omega}(\Theta)^{3/2} = o_p(1) \), where \( \hat{\omega}(\Theta) \) is the maximum vertex degree of \( \Theta \). In their case, if the precision matrix of stock returns is not sparse, consistent estimation of portfolio weights is possible if \( (p - 1)^{3/2}(\log p/\sqrt{T}) + 1/\sqrt{p} = o(1) \). However, this excludes high-dimensional cases since \( p \) is required to be less than \( T^{1/6} \).

### 4.3. Asymptotic properties of de-biased portfolio weights

We now proceed to examining the properties of sparse MRC portfolio weights for de-biased portfolio, as summarized by the following theorem:

**Theorem 5** Let \( \hat{\Sigma} \) be an estimator of covariance matrix satisfying (B.1), and \( \hat{\Theta} \) be the estimator of precision obtained using FMB in algorithm 2. Under the assumptions of theorem 3, consider the linear model (18) with \( e \sim \mathcal{N}(0, \sigma_e^2 I) \), where \( \sigma_e^2 = O(1) \). Consider a suitable choice of the regularization parameters \( \lambda \approx \omega_T \) for the Lasso regression in (19) and \( \lambda_j \approx \omega_T \) uniformly in \( j \) for the Lasso for nodewise regression in (32). Assume \( (s_0 \sqrt{\hat{\omega}_T^2})(\log p/\sqrt{T} + \sqrt{T}/p) = o(1) \).

Then

\[
\sqrt{T} (\hat{\Theta}_{\text{DEBIASED}} - \Theta) = W + \Delta,
\]

\[
W = \hat{\Theta} R e/\sqrt{T},
\]

\[
\| \Delta \|_\infty = O_p\left((s_0 \sqrt{\hat{\omega}_T^2})(\log p/\sqrt{T} + \sqrt{T}/p)\right) = o_p(1).
\]

Furthermore, if \( e \sim \mathcal{N}(0, \sigma_e^2 I) \), let \( \hat{\Theta} = \hat{\Xi} \hat{\Theta} = \hat{\Xi} \hat{\Sigma} \hat{\Theta} \). Then \( W|R \sim \mathcal{N}(0, \sigma_e^2 \hat{\Xi} \hat{\Theta}) \) and \( \| \hat{\Xi} - \Theta \|_\infty = o_p(1) \).

Some comments are in order. Our theorem 5 is an extension of Theorem 2.4 of van de Geer et al. (2014) for non-iid
case, where the latter is achieved with a help of Chang et al. (2018). Furthermore, there are several fundamental differences between theorem 5 and Theorem 2.4 of van de Geer et al. (2014): first, we apply nodewise regression to estimate sparse precision matrix of factor-adjusted returns, which explains the difference in convergence rates. Concretely, van de Geer et al. (2014) have \( \omega_T = \sqrt{\log p/T} \), whereas we have \( \omega_T = \sqrt{\log p/T + 1/\sqrt{p}} \), where \( 1/\sqrt{p} \) arises due to the fact that factors need to be estimated. However, we note that since we deal with high-dimensional regime \( p \geq T \), this additional term is asymptotically negligible, we only keep it for identification purposes. Second, in contrast with van de Geer et al. (2014), the dependent variable in the Lasso regression in (19) is unknown and needs to be estimated. Lemma A.2 shows that \( \hat{\gamma} \) constructed using the precision matrix estimator from theorem 3 is consistent and shares the same rate as \( \omega_t \).

Finally, let us compare the rates of non-sparse MRC portfolio weights in theorem 4, de-biased weights in theorem 5, and post-Lasso weights in theorem 6: de-biased estimator exhibits fastest convergence, followed by post-Lasso and non-sparse weights. This result is further supported by our simulations presented in the next section.

5. Monte Carlo

We study the consistency for estimating portfolio weights in (10) of (i) sparse portfolios that use the standard Lasso without de-biasing in (19), (ii) Lasso with de-biasing in (24), (iii) post-Lasso in algorithm 1, and (iv) non-sparse portfolios that use FMB from algorithm 2. Our simulation results are divided into two parts: the first part examines the performance of models (i)–(iv) under the Gaussian setting, and the second part examines the robustness of performance under the elliptical distributions (to be described later). Each part is further subdivided into two cases: with \( p < T \) (case 1) and with \( p > T \) (case 2), in both cases we allow the number of stocks to increase with the sample size, i.e. \( p = p_T \to \infty \) as \( T \to \infty \).

In case 1 we let \( p = T^{1/2}, \delta = 0.85 \) and \( T = \lfloor 2^{h} \rfloor \), for \( h = 7, 7.5, 8, \ldots, 9.5 \), in case 2 we let \( p = 3 \cdot T^{1/2}, \delta = 0.85 \), all else equal.

First, consider the following data-generating process for stock returns:

\[
\begin{align*}
\mathbf{r}_t & = \mathbf{m} + \mathbf{B} \cdot \epsilon_t, & t = 1, \ldots, T
\end{align*}
\]  

(43)

where \( \mathbf{m}_t \sim \mathcal{N}(1, 1) \) independently for each \( i = 1, \ldots, p \), \( \epsilon_t \) is a \( p \times 1 \) random vector of idiosyncratic errors following \( \mathcal{N}(\mathbf{0}, \mathbf{I}_p) \), with a Toeplitz matrix \( \mathbf{I}_p \) parameterized by \( \rho \); that is, \( \mathbf{I}_p = \mathcal{E}(\rho) \), where \( (\mathcal{E}(\rho))_{ij} = \rho^{j-i}, i, j \in 1, \ldots, p \) which leads to sparse \( \mathbf{X}_t, \mathbf{t} \) is a \( K \times 1 \) vector of factors drawn from \( \mathcal{N}(\mathbf{0}, \mathbf{I}_K) \), \( \mathbf{B} \) is a \( p \times K \) matrix of factor loadings drawn from \( \mathcal{N}(\mathbf{0}, \mathbf{I}_K/100) \). We set \( \rho = 0.5 \) and fix the number of factors \( K = 3 \).

Let \( \mathbf{X} = \mathbf{X}_t' \mathbf{S}_t \). To create sparse MRC portfolio weights we use the following procedure: first, we threshold the vector \( \mathbf{S}_t^{-1} \mathbf{m} \) to keep the top \( p/2 \) entries with the largest absolute values. This yields sparse vector \( \mathbf{a} = \mathbf{S}_t^{-1} \mathbf{m} \).
defined in (13). We use $\Sigma\alpha$ and $\Sigma$ as the values for the mean and covariance matrix parameters to generate multivariate Gaussian returns in (43). Note that the low-rank plus sparse structure of the covariance matrix is preserved under this transformation.

Figure 4 shows the averaged (over Monte Carlo simulations) errors of the estimators of the weight $w_{\text{MRC}}$ versus the sample size $T$ in the logarithmic scale (base 2). As evidenced by figure 4, (1) sparse estimators outperform non-sparse counterparts; (2) using de-biasing or post-Lasso improves the performance compared to the standard Lasso estimator. As expected from theorems 5–6, the Lasso, de-biased Lasso and post-Lasso exhibit similar rates, but the two latter estimators enjoy lower estimation error. The ranking remains similar for case 2, however, as illustrated in figure 4, the performance of all estimators slightly deteriorates.

The Gaussian-tail assumption is too restrictive for modeling the behavior of financial returns. Hence, as a second exercise, we check the robustness of our sparse portfolio allocation estimators under the elliptical distributions, which we briefly review based on Fan et al. (2018). The elliptical distribution family generalizes the multivariate normal distribution and multivariate $t$-distribution. Let $\mathbf{m} \in \mathbb{R}^p$ and $\Sigma \in \mathbb{R}^{p \times p}$. A $p$-dimensional random vector $\mathbf{r}$ has an elliptical distribution, denoted by $\mathbf{r} \sim ED_p(\mathbf{m}, \Sigma, \zeta)$, if it has a stochastic representation

$$\mathbf{r} \overset{d}{=} \mathbf{m} + \zeta \mathbf{A} \mathbf{U},$$

where $\mathbf{U}$ is a random vector uniformly distributed on the unit sphere $S^{p-1}$ in $\mathbb{R}^p$, $\zeta \geq 0$ is a scalar random variable independent of $\mathbf{U}$, $\mathbf{A} \in \mathbb{R}^{p \times q}$ is a deterministic matrix satisfying $\mathbf{A}^t \mathbf{A} = \Sigma$. As pointed out by Fan et al. (2018), the representation in (44) is not identifiable, hence, we require $\mathbb{E}[\zeta^2] = q$, such that $\text{Cov}(\mathbf{r}) = \Sigma$. We only consider continuous elliptical distributions with $\mathbb{P}[\zeta = 0] = 0$. The advantage of the elliptical distribution for the financial returns is its ability to model heavy-tailed data and the tail dependence between variables.

Having reviewed the elliptical distribution, we proceed to the second part of simulation results. The data-generating process is similar to Fan et al. (2018): let $(\epsilon_t, \eta_t)$ from (43) jointly follow the multivariate $t$-distribution with zero mean and covariance matrix $\Sigma = \text{diag}(\Sigma_f, \Sigma_e)$, where $\Sigma_f = \mathbf{I}_K$. To construct $\Sigma_e$ we use a Toeplitz structure parameterized by $\rho = 0.5$, which leads to the sparse $\Theta_e = \Sigma_e^{-1}$. The rows of $\mathbf{B}$ are drawn from $\mathcal{N}(\mathbf{0}, \mathbf{1}_K / 100)$. Figure 5 reports the results for $\nu = 4.2$.† The performance of the standard Lasso estimator significantly deteriorates, which is further amplified in the high-dimensional case where it exhibits the worst performance. Noticeably, post-Lasso still achieves the lowest estimation error, followed by a de-biased estimator.

6. Empirical application

This section is divided into three main parts. First, we examine the performance of several non-sparse portfolios, including the equal-weighted and Index portfolios (reported as the composite S&P500 index listed as “GSPC”). Second, we study the performance of sparse portfolios that are based on de-biasing and post-Lasso. Third, we consider several interesting periods that include different states of the economy: we examine.

† The results for larger degrees of freedom do not provide any additional insight, hence we do not report them here. However, they are available upon request.
the merit of sparse vs non-sparse portfolios during the periods of economic growth, moderate market decline and severe economic downturns.

6.1. Data

We use monthly returns of the components of the S&P500 index.† The data on historical S&P500 constituents and stock returns is fetched from CRSP and Compustat using the SAS interface. The full sample has 480 observations on 355 stocks from 1 January 1980–1 December 2019. We use 1 January 1980–1 December 1994 (180 obs) as a training period and 1 January 1995–1 December 2019 (300 obs) as the out-of-sample test period. We roll the estimation window over the test sample to rebalance the portfolios monthly. At the end of each month, prior to portfolio construction, we remove stocks with less than 15 years of historical stock return data. For a sparse portfolio, we employ the following strategy to choose the tuning parameter λ in (16): we use the first two thirds of the training data (which we call the training window) to estimate weights and tune the shrinkage intensity λ in the remaining one third of the training sample to yield the highest Sharpe Ratio which serves as a validation window. We estimate factors and factor loadings in the training window and validation window combined. The risk-free rate and Fama–French factors are taken from Kenneth R. French’s data library.

6.2. Performance measures

We consider four metrics commonly reported in finance literature: the Sharpe Ratio, the portfolio turnover, the average return and risk of a portfolio. We consider two scenarios: with and without transaction costs. Let $T$ denote the total number of observations, the training sample consists of $m$ observations, and the test sample is $n = T - m$. When transaction costs are not taken into account, the out-of-sample average portfolio return, risk and Sharpe Ratio are

$$
\hat{\mu}_{test} = \frac{1}{n} \sum_{t=m}^{T-1} \hat{\mathbf{w}} \mathbf{r}_{t+1},
$$

$$
\hat{\sigma}_{test} = \sqrt{\frac{1}{n - 1} \sum_{t=m}^{T-1} (\hat{\mathbf{w}} \mathbf{r}_{t+1} - \hat{\mu}_{test})^2},
$$

$$
SR = \frac{\hat{\mu}_{test}}{\hat{\sigma}_{test}}.
$$

We follow DeMiguel et al. (2009), Li (2015), and Ban et al. (2018) to account for transaction costs (tc). In line with the aforementioned papers, we set $c = 50$bps. Define the excess portfolio at time $t + 1$ with transaction costs as

$$
r_{t+1,\text{portfolio}} = \hat{\mathbf{w}} \mathbf{r}_{t+1} - c (1 + \hat{\mathbf{w}} \mathbf{r}_{t+1})
$$

$$
\times \sum_{j=1}^{p} [\hat{w}_{t+1,j} - \hat{w}_{r_{t+1,j}}],
$$

where

$$
\hat{w}_{r_{t+1,j}} = \frac{1 + r_{t+1,j} + r'_{t+1,j}}{1 + r_{t+1,\text{portfolio}} + r'_{t+1}},
$$

where $r_{t+1,j} + r'_{t+1}$ is the sum of the excess return of the $j$th asset and risk-free rate, and $r_{t+1,\text{portfolio}} + r'_{t+1}$ is the sum of the excess return of the portfolio and risk-free rate. The out-of-sample average portfolio return, risk, Sharpe Ratio and turnover are defined accordingly:

$$
\hat{\mu}_{test,tc} = \frac{1}{n} \sum_{t=m}^{T-1} r_{t,\text{portfolio}},
$$
Accounting for the factor structure in stock returns improves portfolio performance in terms of the OOS Sharpe Ratio. Specifically, EW, Index, MB and CLIME which ignore factor structure perform worse than FMB and LW. (2) The models that use an improved estimator of covariance or precision matrix outperform EW and Index on the test sample. As a downside, such models have higher Turnover. This implies that superior performance is achieved at the cost of larger variability of portfolio positions over time and, as a consequence, increased risk associated with it.

The second set of results studies the performance of sparse portfolios: we include our proposed methods based on de-biasing and post-Lasso, as well as the approach studied in Ao et al. (2019) (Lasso) without factor investing. For post-Lasso we use algorithm 1. The tuning procedure for the threshold $t$ is the same as described in section 2.2. Standard Lasso is used as a penalty in the first stage (see (27)) since we did not find a significant difference in asset subset selection when using precision Lasso. As in equation (26), the post-Lasso GMV portfolio uses the return of S&P500 as the baseline asset. Therefore, since there is no estimation of a high-dimensional $\Theta$ in the first stage of post-Lasso GMV, the GMV results in table 3 are only reported for the $\Theta$ estimated using statistical factors in the second stage.

Let us comment on the results presented in table 3: (1) column one demonstrates that de-biasing leads to significant performance improvement in terms of the return and the OOS Sharpe Ratio. Note that even though the risk of de-biased portfolio is also higher, it still satisfies the risk-constraint. This result emphasizes the importance of correcting for the bias introduced by the $\ell_1$-regularization. (2) Comparing two

| Portfolio Type | Return | Risk | SR | Turnover |
|----------------|--------|------|----|----------|
| EW             | 0.0081 | 0.0520 | 0.1553 | – |
| Index          | 0.0063 | 0.0458 | 0.1389 | – |
| MB             | 0.0539 | 0.2522 | 0.2138 | – |
| FMB (PC)       | 0.0287 | 0.1049 | 0.2743 | – |
| CLIME          | 0.0372 | 0.2337 | 0.1593 | – |
| LW             | 0.0296 | 0.1049 | 0.2817 | – |
| FMB (FF1)      | 0.0497 | 0.2200 | 0.2258 | – |
| FMB (FF3)      | 0.0384 | 0.1319 | 0.2908 | – |
| FMB (FF5)      | 0.0373 | 0.1277 | 0.2921 | – |
| EW             | 0.0080 | 0.0520 | 0.1538 | 0.0630 |
| MB             | 0.0512 | 0.0637 | 0.2027 | 2.9458 |
| FMB (PC)       | 0.0248 | 0.1049 | 0.2368 | 3.7190 |
| CLIME          | 0.0334 | 0.2334 | 0.1429 | 4.9174 |
| LW             | 0.0237 | 0.1052 | 0.2257 | 5.5889 |
| FMB (FF1)      | 0.0470 | 0.2202 | 0.2136 | 2.7245 |
| FMB (FF3)      | 0.0356 | 0.1319 | 0.2694 | 2.4670 |
| FMB (FF5)      | 0.0345 | 0.1277 | 0.2699 | 2.4853 |

Table 2. Monthly portfolio returns, risk, Sharpe Ratio and turnover.
Table 3. Sparse portfolio (FMB is used for de-biasing): monthly portfolio returns, risk, Sharpe Ratio and turnover.

| Without TC | De-Biasing | Post-Lasso |
|------------|------------|------------|
|            | Markowitz (RC) | Markowitz (RC) | Markowitz (WC) | GMV |
|            | Return | Risk | SR | Turnover | Return | Risk | SR | Turnover | Return | Risk | SR | Turnover | Return | Risk | SR | Turnover |
| Lasso (PC0) | 0.0007 | 0.0048 | 0.1406 | – | 0.0020 | 0.0719 | 0.3055 | – | 0.0053 | 0.0338 | 0.1569 | – | – | – | – |
| De-biased Lasso (PC0) | 0.0023 | 0.0100 | 0.2266 | – | 0.0224 | 0.0715 | 0.3126 | – | 0.0055 | 0.0350 | 0.1584 | – | 0.0070 | 0.0324 | 0.2171 | – |
| Lasso (PC) | 0.0006 | 0.0052 | 0.1122 | – | 0.0209 | 0.0720 | 0.2903 | – | 0.0050 | 0.0337 | 0.1498 | – | – | – | – |
| De-biased Lasso (PC) | 0.0067 | 0.0265 | 0.2542 | – | 0.0199 | 0.0719 | 0.2770 | – | 0.0049 | 0.0336 | 0.1456 | – | – | – | – |
| Lasso (FF1) | 0.0007 | 0.0039 | 0.1902 | – | 0.0214 | 0.0740 | 0.2894 | – | 0.0055 | 0.0341 | 0.1615 | – | – | – | – |
| De-biased Lasso (FF1) | 0.0109 | 0.0346 | 0.3213 | – | 0.0140 | 0.0716 | 0.1958 | 0.0026 | 0.0337 | 0.0780 | 2.5592 | 1.5373 | – | – | – | – |
| Lasso (FF3) | 0.0004 | 0.0040 | 0.1113 | – | 0.0154 | 0.0714 | 0.2152 | 0.0028 | 0.0348 | 0.0802 | 2.6788 | 1.5777 | 0.0040 | 0.0323 | 0.1244 | 2.7955 |
| De-biased Lasso (FF3) | 0.0072 | 0.0265 | 0.2721 | – | 0.0129 | 0.0717 | 0.1801 | 0.0024 | 0.0336 | 0.0701 | 2.5787 | 1.7374 | – | – | – | – |
| Lasso (FF5) | 0.0002 | 0.0042 | 0.0577 | – | 0.0117 | 0.0715 | 0.1636 | 0.0022 | 0.0336 | 0.0670 | 2.5686 | 1.6208 | – | – | – | – |
| De-biased Lasso (FF5) | 0.0073 | 0.0300 | 0.2467 | – | 0.0133 | 0.0737 | 0.1804 | 0.0027 | 0.0339 | 0.0802 | 2.7007 | 1.6033 | – | – | – | – |

Transaction costs are set to 50 basis points, targeted risk is set at $\sigma = 0.05$ (which is the standard deviation of the monthly excess returns on S&P 500 index from 1980 to 1995, the first training period), monthly targeted return is 0.7974% which is equivalent to 10% yearly return when compounded. Factor Nodewise-regression estimator of precision matrix is used for de-biasing. The tuning procedure for the threshold $t$ for Post-Lasso is the same as described in section 2.2. In-sample: 1 January 1980–31 December 1995 (180 obs), Out-of-sample: 1 January 1995–31 December 2019 (300 obs).
bias-correction methods, de-biasing and post-Lasso, we find that the latter is characterized by higher return, higher risk, and higher Sharpe Ratio for MRC portfolios. (3) Sparse portfolios have lower return, risk and turnover compared to non-sparse counterparts in table 2, however, the OOS Sharpe Ratio is comparable, i.e. we do not see uniform superiority of either method. Therefore, incorporating sparsity allows investors to reduce portfolio risk at the cost of lower return while maintaining the Sharpe Ratio comparable to holding a non-sparse portfolio.

Tables 4–5 compare the performance of non-sparse and sparse (de-biased, ‘DL’, and post-Lasso, ‘PL’) portfolios for different time periods in terms of the cumulative excess return (CER) over the period of interest and risk. The first period of interest (1997–1998), which will be referred to as ‘Period I’, corresponds to economic growth since the Index exhibited positive CER during this time. The second period of interest, ‘Period II’, corresponds to moderate market decline since EW and Index had relatively small negative CER. Finally, ‘Period III’, corresponds to a severe economic downturn and a significant drop in the performance of EW and Index. We note that the references to the specific crises in tables 4–5 do not intend to limit these economic periods to these time spans. They merely provide the context for the time intervals of interest. Since the performance of MWC portfolios is similar to GMV, we only report MRC and GMV for the ease of presentation.

Let us summarize the findings from tables 4–5: (1) In Period I non-sparse portfolios that rely on the estimation of covariance or precision matrix outperformed EW and Index in terms of CER for both MRC and GMV. However, in Period II GMV portfolios exhibited slightly negative CER, whereas MRC portfolios had higher risk but positive CER (albeit being lower compared to Period I). Note that in Period III none of the non-sparse portfolios generated positive CER and portfolio risk increased rapidly. Examining the performance of sparse portfolios in table 5, we see that (2) our proposed sparse portfolios produce positive CER during all three periods of interest. Furthermore, the return generated by PL is higher than that by non-sparse portfolios even during Periods I and II. Interestingly, DL produces positive CER without having high-risk exposure. This suggests that our de-biased estimator of portfolio weights exhibits minimax properties. We leave the formal theoretical treatment of the latter for the future research.
As a final empirical exercise, we examine the robustness of sparse portfolios to shorter training periods and different return frequencies. We construct non-sparse GMV portfolios and explore their performance during the recent COVID-19 outbreak. Using daily returns of 495 constituents of the S&P500 from 25 May 2018–24 September 2020 (588 obs.) with 105 training observations, table 1 reports the performance of the selected strategies: we included EW and Index portfolios, as well as precision-based nodewise regression estimator by Meinshausen and Bühlmann (2006) (MB), LW, and CLIME. We use 25 May 2018–23 October 2018 (105 obs.) as a training period and 24 October 2018–24 September 2020 (483 obs.) as the out-of-sample test period. We roll the estimation window over the test sample to rebalance the portfolios monthly. The left panel of table 1 shows the return, risk and Sharpe Ratio of portfolios over the training period, and the right panel reports cumulative excess return (CER) and risk over two sub-periods of interest: before the pandemic (2 January 2019–31 December 2019) and during the first wave of the COVID-19 outbreak in the US (2 January 2020–30 June 2020). As evidenced by table 1, none of the non-sparse portfolios was robust to the downturn brought on by the pandemic and yielded negative CER.

7. Conclusion
This paper develops an approach to construct sparse portfolios in high dimensions that addresses the shortcomings of the existing sparse portfolio allocation techniques. We establish consistency of high-dimensional sparse weight estimators, provide guidance regarding their distribution, and prove consistency of a new high-dimensional precision matrix estimator that bridges factor models and graphical models. From the empirical perspective, we examine the merit of sparse portfolios during different market scenarios. We find that in contrast to non-sparse counterparts, our strategy is robust to recessions and can be used as a hedging vehicle during such times. Our framework makes use of the tool from the network theory called nodewise regression which not only satisfies desirable statistical properties but also allows us to study whether certain industries could serve as safe havens during recessions. We find that such non-cyclical industries as consumer staples, healthcare, retail and food were driving the returns of the sparse portfolios during both the global financial crisis of 2007–2009 and the COVID-19 outbreak, whereas the insurance sector was the least attractive investment in both periods. Finally, we develop a simple framework that provides clear guidelines on how to implement factor investing using the methodology developed in this paper.

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ORCID
Ekaterina Seregina http://orcid.org/0000-0003-4591-4239

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Appendix

In this appendix, we collected the proofs of theorems 2–5.

A.1. Proof of theorem 2

The first part of theorem 2 was proved by Fan et al. (2018) (see their proof of Theorem 2.1) under the assumptions (A.1)–(A.3), (B.1)–(B.3) and log p = o(T). To prove the convergence rates for the precision matrix of the factor-adjusted returns, we follow Chang et al. (2018), Caner and Kock (2018) and Callot et al. (2019). Using the facts that max_{j∈P}(1/T) \sum_{t=1}^{T} |\hat{e}_{jt} - e_{jt}| = O_p(\omega_T^2) and max_{j∈P} |\hat{e}_{jt} - e_{jt}| = O_p(\omega_T^2) = o_p(1), we get

max_{j∈P} ||\hat{y}_j - y_j||_1 = O_p(\sqrt{\omega_T}),

(A1)

where \hat{y}_j was defined in (32). The proof of (A1) is similar to the proof of the equation (23) of Chang et al. (2018), with \omega_T = \sqrt{\log p/T} for their case. Similarly to Callot et al. (2019), consider the following linear model:

\hat{e}_j = \tilde{E}_{-j} y_j + \eta_j, \quad \text{for } j = 1, \ldots, p,

(2)

and max_{j∈P} ||\hat{y}_j - y_j||_1/T = O_p(\omega_T).

(A2)

van de Geer et al. (2014) and Chang et al. (2018) showed that

\max_{j∈P} ||\hat{y}_j - y_j||_1/T = O_p(\omega_T).

(A3)

Note that the rate in (A4) is the same as in Lemma 1 of Chang et al. (2018) with \omega_T = \sqrt{\log p/T} for their case. However, the rate in (A4) is different from the one derived in van de Geer et al. (2014) since we allow time-dependence between factor-adjusted returns.

Recall that τ^2 = ||\hat{y} - \tilde{E}_j \hat{y}||_2^2/T + \lambda_j ||\hat{y}||_1. Using triangle inequality, we have:

\max_{j∈P} ||\hat{y}_j - y_j||_1/T \leq \max_{j∈P} ||\hat{y}_j - y_j||_1 + \max_{j∈P} ||\tilde{E}_{-j} (\hat{y}_j - y_j)||_1/T

+ \max_{j∈P} \max_{j∈P} ||\tilde{E}_{-j} (\hat{y}_j - y_j)||_1/T.

(A4)

The first term was bounded in (A4), we now bound the remaining terms:

\max_{j∈P} \tilde{E}_{-j} (\hat{y}_j - y_j)/T \leq \max_{j∈P} \tilde{E}_{-j} (\hat{y}_j - y_j)/T = O_p(\sqrt{\omega_T}),

where we used (A1) and (A3). For slowroman textbookwe have

\max_{j∈P} ||\tilde{E}_{-j} (\hat{y}_j - y_j)||_1 \leq \sqrt{\omega_T} \max_{j∈P} ||y_j||_1 = O_p(\sqrt{\omega_T}).

To bound the last term, we use KKT conditions in nodewise regression:

\max_{j∈P} ||\tilde{E}_{-j} (\hat{y}_j - y_j)||_1/T \leq \max_{j∈P} ||\tilde{E}_{-j} \eta_j||_1/T

+ \max_{j∈P} \lambda_j \max_{j∈P} ||\tilde{E}_{-j} \eta_j||_1/T = O_p(\omega_T).

(A4)
where we used (A3) and $\lambda_j \propto \omega_j$. It follows that

$$ IV = O_P(\omega) \max_{1 \leq j \leq p} \| y_j \|_1 = O_P(\sqrt{\omega} \cdot \tau). $$

Therefore, we now have shown that

$$ \max_{1 \leq j \leq p} \| \tilde{y}_j^2 - \hat{y}_j^2 \| = O_P(\sqrt{\omega} \cdot \tau). $$ (A5)

Using the fact that $1/\tilde{y}_j^2 = O(1)$, we also have

$$ 1/\hat{y}_j^2 - 1/\tilde{y}_j^2 = O_P(\sqrt{\omega} \cdot \tau). $$ (A6)

Finally, using the analysis in (B.51) – (B.53) of Caner and Kock (2018), we get

$$ \max_{1 \leq j \leq p} \| \hat{\theta}_j - \Theta_j \|_1 = O_P(\omega \cdot \tau). $$ (A7)

To prove the second rate for the precision of the factor-adjusted returns, we note that

$$ \max_{1 \leq j \leq p} \| \tilde{y}_j - y_j \|_2 = O_P(\sqrt{\omega} \cdot \tau), $$ (A8)

which was obtained by Chang et al. (2018) (see their Lemma 2). We can write

$$ \max_{1 \leq j \leq p} \| \hat{\theta}_j - \Theta_j \|_2 \leq \max_{1 \leq j \leq p} \| \tilde{y}_j - y_j \|_2/\tilde{y}_j^2 + \| y_j \|_2/\hat{y}_j^2 - 1/\tilde{y}_j^2 = O_P(\sqrt{\omega} \cdot \tau). $$ (A9)

A.2. Proof of theorem 3

Let $\hat{J} = \Lambda^{1/2} \hat{\Theta} \hat{\Gamma} \hat{\Lambda}^{1/2}$ and $\tilde{J} = \Lambda^{1/2} \tilde{\Theta} \tilde{\Gamma} \tilde{\Lambda}^{1/2}$. Also, define

$$ \Delta_{\text{inv}} = \hat{\Theta} \Lambda^{1/2} (\hat{I}_K + \hat{J})^{-1} \Lambda^{1/2} \hat{\Gamma} \hat{\Theta} - \Theta \hat{\Gamma} \hat{\Lambda}^{1/2} \hat{I}_K + \hat{J})^{-1} \hat{\Lambda}^{1/2} \hat{\Gamma} \Theta. $$

Using Sherman–Morrison–Woodbury formulas in (38), we have

$$ \| \hat{\Theta} - \Theta \|_1 \leq \| \hat{\Theta} - \Theta \|_1 + \| \Delta_{\text{inv}} \|_1. $$ (A10)

As pointed out by Fan et al. (2018), $\| \Delta_{\text{inv}} \|_1$ can be bounded by the following three terms:

$$ \| \hat{\Theta}_j - \Theta_j \|_1 \Lambda^{1/2} (\hat{I}_K + \hat{J})^{-1} \Lambda^{1/2} \hat{\Gamma} \hat{\Theta} \|_1 = O_P(\sqrt{\omega} \cdot \tau \cdot \bar{p} \cdot 1/\sqrt{\hat{\tau}}), $$
\[
\| \Theta_j (\tilde{\Theta}^{1/2} - \hat{\Theta}^{1/2})(\hat{I}_K + \hat{J})^{-1} \Lambda^{1/2} \hat{\Gamma} \hat{\Theta} \|_1 = O_P(\sqrt{\tau} \cdot \bar{p} \cdot \omega \cdot \tau \cdot 1/\sqrt{\hat{\tau}}), $$
\[
\| \Theta_j \Lambda^{1/2} \tilde{\Gamma} ((\hat{I}_K + \hat{J})^{-1} - (\hat{I}_K + \hat{J})^{-1}) \tilde{\Gamma} \hat{\Theta} \|_1 = O_P(\sqrt{\tau} \cdot \bar{p} \cdot \omega \cdot \tau \cdot 1/\sqrt{\hat{\tau}}). $$

To derive the above rates we used (B.1)–(B.3), theorem 2 and the fact that $\| \hat{\Gamma} \hat{\Lambda}^{1/2} - \tilde{\Gamma} \hat{\Lambda}^{1/2} \| = O_P(\omega \cdot \tau \cdot 1/\sqrt{\hat{\tau}}).$ The second rate in theorem 3 can be easily obtained using the technique described above for the $l_2$-norm.

A.3. Lemmas for theorems 4–thm5

Lemma A.1 Under the assumptions of theorem 3, we have

(a) $\| \tilde{m} - m \|_{\text{max}} = O_P(\sqrt{\log p/T})$, where $m$ is the unconditional mean of stock returns defined in section 3.2, and $\tilde{m}$ is the sample mean.

(b) $\| \Theta \|_1 = O(1)$.

Proof (a) The proof of Part (a) is provided in Chang et al. (2018, Lemma 1).

(b) To prove Part (b) we use Sherman–Morrison–Woodbury formula in (38):

$$ \| \Theta \|_1 \leq \| \Theta \|_1 + \| \Theta B (I_K + B' \Theta B)^{-1} B' \Theta \|_1 $$

$$ = O(\sqrt{\tau}) + O(\sqrt{\tau} \cdot p \cdot 1/\sqrt{\hat{\tau}}) = O(\hat{\tau}). $$ (A11)

The last equality in (A11) is obtained under the assumptions of theorem 5. This result is important in several aspects: it shows that the sparsity of the precision matrix of stock returns is controlled by the sparsity in the precision of the idiosyncratic returns. Hence, one does not need to impose an unrealistic sparsity assumption on the precision of returns a priori when the latter follow a factor structure - sparsity of the precision once the common movements have been taken into account would suffice.

Lemma A.2 Define $\theta = m' \Theta m / p$ and $\bar{g} = \sqrt{m' \Theta m / p}$. Also, let $\tilde{\theta} = \hat{m} / \sqrt{\hat{m} \Theta \hat{m} / p}$ and $\tilde{g} = \sqrt{\hat{m} \Theta \hat{m} / p}$. Under the assumptions of theorem 3:

(a) $\theta = O(1)$.

(b) $| \tilde{\theta} - \theta | = O_P(\sqrt{\omega} \cdot \tau) = O(1)$.

(c) $| \tilde{\theta} - \theta | = O_P(\sqrt{\omega} \cdot \tau) = O(1)$, where $\omega$ was defined in (17).

(d) $| \tilde{\theta} - \theta | = O_P(\sqrt{\omega} \cdot \tau) = O(1)$.

Proof (a) Part (a) is trivial and follows directly from $\| \Theta \|_2 = O(1)$.

(b) First, rewrite the expression of interest:

$$ | \tilde{\theta} - \theta | = \sqrt{(\tilde{m} - m)' (\tilde{\theta} - \Theta)(\tilde{m} - m) / p}, $$

$$ \leq | \tilde{m} - m |_{\text{max}} \| \tilde{\theta} - \Theta \|_1 + | \tilde{m} | (\tilde{\theta} - \Theta)(\tilde{m} - m) / p + | \tilde{m} | (\tilde{\theta} - \Theta)(\tilde{m} - m) / p. $$ (A12)

We now bound each of the terms in (A12) using the expressions derived in Callot et al. (2019) (see their Proof of lemma A.3, lemma A.1 and the fact that $\log p / T = o(1)$).

$$ | \tilde{m} - m |' (\tilde{\theta} - \Theta)(\tilde{m} - m) / p $$

$$ \leq | \tilde{m} - m |_{\text{max}} \| \tilde{\theta} - \Theta \|_1 = O_P \left( \sqrt{\log p / T} \cdot 2 \cdot \omega \cdot \tau \right). $$ (A13)

$$ | \tilde{m} - m |' (\tilde{\theta} - \Theta)(\tilde{m} - m) / p $$

$$ \leq | \tilde{m} - m |_{\text{max}} \| \tilde{\theta} - \Theta \|_1 = O_P \left( \sqrt{\log p / T} \cdot \hat{\tau} \right). $$ (A14)

$$ | \tilde{m} - m |' (\tilde{\theta} - \Theta)(\tilde{m} - m) / p $$

$$ \leq | \tilde{m} - m |_{\text{max}} \| \tilde{\theta} - \Theta \|_1 $$

$$ = O_P \left( \sqrt{\log p / T} \cdot \hat{\tau} \right). $$ (A15)

$$ | \tilde{m} |' (\tilde{\theta} - \Theta)(\tilde{m} - m) / p $$

$$ \leq | \tilde{m} - m |_{\text{max}} \| \tilde{\theta} - \Theta \|_1 $$

$$ = O_P \left( \sqrt{\log p / T} \cdot \hat{\tau} \right). $$ (A16)
where we used lemmas A.1–A.2 to obtain the rates.

(c) Part (c) trivially follows from Part (b).
(d) This is a direct consequence of Part (b) and the fact that \( \sqrt{\theta - \hat{\theta}} \geq \sqrt{\theta - \bar{\theta}} \).

\[ \] 

A.4. Proof of theorem 4

Using the definition of MRC weight in (13), we can rewrite

\[
\| \hat{w}_{\text{MRC}} - w_{\text{MRC}} \|_1 \leq \frac{\frac{p}{\bar{g}} \left[ \| (\hat{\Theta} - \Theta) \|_1 \right. + \left. \| (\Theta - \hat{\Theta}) \|_1 \right] + p \| \Theta \|_1 \| \| (\hat{\Theta} - \Theta) \|_1 \| | \| \Theta \|_1 | | m \|_{\text{max}}}{\frac{p}{\bar{g}}}
\]

\[
= O_{p} \left( \hat{s}^2 \Omega_1 + \frac{\log p}{\sqrt{T}} \right) + O_{p} \left( \hat{s}^2 \Omega_1 + \frac{\log p}{\sqrt{T}} \right)
\]

where we used lemmas A.1–A.2 to obtain the rates.

A.5. Proof of theorem 5

The KKT conditions for the nodewise Lasso in (32) imply that

\[
\hat{e}_{j} = (\hat{e}_{j} - \hat{e}_{j}) \hat{\theta}_{j}/T. \quad \text{hence,} \quad \hat{e}_{j} \hat{\Theta}_{j} \|_{\text{infty}} / T = 1.
\]

As shown in van de Geer et al. (2014), these KKT conditions also imply that

\[
\| \hat{\Theta}_{j} - \Theta_{j} \|_{\text{infty}} / T \leq \hat{\lambda}_{j}/\hat{\tau}_{j}^2.
\]

Therefore, the estimator of precision matrix needs to satisfy the following "extended KKT" condition:

\[
\| \hat{\Sigma} \hat{\Theta}_{j} - e_{j} \|_{\text{infty}} \leq \hat{\lambda}_{j}/\hat{\tau}_{j}^2,
\]

where \( e_{j} \) is the \( j \)th unit column vector. Combining the rate in \( \ell_1 \) norm in theorem 3 and (A19), we have:

\[
\| \hat{\Sigma} \hat{\Theta}_{j} - e_{j} \|_{\text{infty}} \leq \hat{\lambda}_{j}/\hat{\tau}_{j}^2.
\]

Using the definition of \( \Delta \) in (23), it is straightforward to see that

\[
\| \Delta \|_{\text{infty}} / \sqrt{T} = \| (\hat{\Theta} \hat{\Sigma} - I_{p})(\hat{w} - w) \|_{\text{infty}} \leq \| \hat{\Theta} \hat{\Sigma} - I_{p} \|_{\text{infty}} \| \hat{w} - w \|_{1}.
\]

Therefore, combining (A20) and (A21), we have

\[
\| \Delta \|_{\text{infty}} \leq \sqrt{T} \| \hat{w} - w \|_{1} \max_{j} \hat{\lambda}_{j}/\hat{\tau}_{j}^2 = O_{p} \left( \sqrt{T} \cdot (s_0 \vee \hat{s}) \omega_{T} \cdot \omega_{T} \right)
\]

Finally, we show that \( \| \hat{\Theta} - \Theta \|_{\infty} = o_{p}(1) \). Using theorem 3 and lemma A.1 we have \( \| \hat{\Theta} \|_{1} = O_{p}(\delta) \). Also, \( \hat{\Theta} = \hat{\Theta} \hat{\Sigma} \hat{\Theta} = (\hat{\Theta} \hat{\Sigma} - I_{p}) \hat{\Theta} + \hat{\Theta}' \).

And using (A20) and (A21) together with \( \max_{j} \hat{\lambda}_{j}/\hat{\tau}_{j}^2 = o_{p}(1) \):

\[
\| (\hat{\Theta} \hat{\Sigma} - I_{p}) \hat{\Theta} \|_{\infty} \leq \max_{j} \hat{\lambda}_{j}/\hat{\Theta}_{j} / \hat{\tau}_{j}^2 = o_{p}(1).
\]

It follows that

\[
\| \hat{\Theta} - \Theta \|_{\infty} \leq \max_{j} \| \hat{\Theta}_{j} - \Theta \|_{2} \leq \max_{j} \hat{\lambda}_{j} \sqrt{T} = o_{p}(1).
\]

Combining (A24)–(A26) completes the proof.