Newton’s law modifications due to a Sol manifold extra dimensional space

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Abstract. The corrections to the gravitational potential due to a Sol extra dimensional compact manifold are studied. We compare the range of the corrections to the range of the 3-torus corrections. It is found that for small values of the radius of the extra dimensions the Sol manifold corrections are large compared to the 3-torus corrections. Also, Sol manifolds corrections can be larger, comparable or smaller compared to the 3-torus case, for larger compactification radii.

1. Introduction
In the last decade, the phenomenological implications of extra dimensional models [1] were extensively studied. Many of these studies are concentrated on the modifications caused on the gravitational potential due to extra dimensional manifolds [2, 3, 4, 5]. Generally the modifications are of Yukawa type (except in Randall-Sundrum warped extra dimensional models), differing only in their strength and range (for a very recent interesting application see [5]). We shall study the modification of the gravitational potential caused by a Sol extra dimensional manifold. Sol manifolds are compact and orientable three dimensional manifolds, denoted as $M^3_A$. We shall present the Sol manifolds geometric structure. Also we shall study the Laplace equation on such manifolds. Finally we compare the range of the corrections to the gravitational potential due to Sol manifolds, with the corresponding range caused by a $T^3$ extra dimensional manifold.

2. Sol Manifold Geometry and Topology
Sol manifolds are one of Thurston’s 8 three dimensional geometries [6] (geometric structures). The geometrization conjecture is an approach to the geometry of three dimensional manifolds through general topological arguments. One starts with the fact that three dimensional manifolds are composed by 2-spheres or torii surfaces. According to the properties and the details of the topological gluing map of the above with $R$ or $S^1$, the resulting manifolds acquire locally homogeneous metrics. Sol manifolds are described by one type of the 8 different homogeneous metrics.

Sol manifolds are obtained from $SL(2, Z)$ stiffennings of torus bundles over the circle. A theorem [7] states: given M a total bundle space of a $T^2$ bundle over $S^1$ with gluing map $\phi$ and let $A \in GL(2, Z)$ represent the automorphism of the fundamental group of the torus $T^2$ induced by the gluing map $\phi$. Then the total bundle space admits a Sol geometric structure.
if \( A \) is hyperbolic, an \( E^3 \) structure if \( A \) is periodic and finally a Nil structure otherwise. \( A \) is hyperbolic and gives rise to an orientable manifold if \( |\text{Tr}A| > 2 \) and we shall dwell on this choice. Thus Sol structure arises from the \( SL(2,\mathbb{Z}) \) stiffening of the mapping torus of a torus diffeomorphism \( \phi \) [6].

Two different hyperbolic gluings with \( A_1 \neq A_2 \), give rise to different geometric Sol structures. \( A \) will be represented by,

\[
A = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}
\]

so the different Sol structures are classified by the integer \( n \), with \( |n| = |\text{Tr}A| > 2 \). Also only the positive eigenvalues of \( A \) will be considered.

Let us analyze more the Sol manifold structure. Consider the manifold \( T^2 \times \mathbb{R} \) described by the periodic coordinates \( (x, y) \) for the torus, defined modulo \( \mathbb{R} \) (the radius of the compact dimensions) and \( z \in (-\infty, \infty) \) be a coordinate of \( \mathbb{R} \). The combined action of the torus mapping through the hyperbolic gluing map \( A \) diffeomorphism (which we denote as \( \Gamma \)) is:

\[
\Gamma : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} y \\ -x + ny \\ z + 2\pi R \end{pmatrix}
\]

The Sol manifold, which is denoted as \( M^3_A \) is the quotient of \( T^2 \times \mathbb{R} \) by the action of \( \Gamma \), that is \( M^3_A \equiv T^2 \times \mathbb{R}/\Gamma \). It is a total torus bundle over \( S^1 \), with \( T^2 \) the fiber, \( S^1 \) the base space and \( A \) the hyperbolic gluing map of the torus fibers. We consider Sol manifolds for which the eigenvalues \( \lambda \) of \( A \) are positive. For the case of (1) the characteristic polynomial of the matrix \( A \) reads:

\[
\lambda^2 - \text{Tr}A\lambda + 1 = 0
\]

or equivalently:

\[
\lambda^2 - n\lambda + 1 = 0
\]

The solutions to equation (4) are \( \lambda \) and \( \lambda^{-1} \) with \( \lambda + \lambda^{-1} = n = \text{Tr}A \). Also the discriminant of \( A \) is:

\[
D = (\lambda - \lambda^{-1})^2
\]

We use another coordinate system \( (u, v, z) \) on the Sol manifold. The \( (u, v) \) are linear coordinates of the torus fibres related to an eigenbasis of the hyperbolic map \( A \). These coordinates correspond to a rotated torus lattice. The action of \( \Gamma \) in the new coordinate system reads:

\[
\Gamma : \begin{pmatrix} u \\ v \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \lambda u \\ \lambda^{-1} v \\ z + 2\pi R \end{pmatrix}
\]

The original lattice was orthogonal while the new torus lattice is not. If \( e_u \) and \( e_v \) are the basis of the lattice after the action of \( \Gamma \) identification, then \( (e_u, e_v) = |e_u||e_v|\cos \theta \). Also the fiber coordinates \( T^2 \) are not periodic anymore. So in order two pairs \( (u_1, v_1) \) and \( (u_2, v_2) \) to define the same point on the torus lattice, the following must hold:

\[
(u_1 - u_2, v_1 - v_2) = ke_u + me_v
\]

with \( k, m \) integers and \( e_u, e_v \) defined previously.
2.1. Riemannian Sol group invariant metric on Sol manifolds
The Riemannian metric on Sol manifolds comes from the metric on the universal covering of $M^3_A$. The group invariant metric on the universal covering is a Sol group invariant metric. With the invariant metric on the universal covering of $M^3_A$ we can find the following class of metrics of Sol manifolds:

$$ds^2 = Ee^{2z\ln\lambda}du^2 + 2Fdu\,dv + Ge^{-2z\ln\lambda}dv^2 + dz^2 \quad (8)$$

We shall use the metric (8) in order to compute the Laplace-Beltrami operator on $M^3_A$.

Sol manifolds are hyperbolic manifolds with negative curvature. The hyperbolicity is a very interesting feature of Sol manifolds, regarding they are torus fibrations. Due to their hyperbolicity, the distribution of eigenvalues of the Laplacian is not Poisson. This might have effects on the gravitons mass splitting.

Newton’s law and extra dimensions
Our purpose is to examine the corrections to the gravitational potential caused by an extra dimensional Sol manifold. The spacetime manifold is of the form $M^4 \times M^n$. Let us review the general technique to obtain these corrections. The presentation is based on reference [2, 8].

Consider a spacetime of the form $M^4 \times M^n$, with $M^n$ an n-dimensional compact manifold and $M^4$ the four dimensional Minkowski spacetime and also a complete set of orthogonal harmonic functions on $M^4$, $\Psi_m$, satisfying the orthogonality condition:

$$\int_{M^n} \Psi_n(x)\Psi_m^*(x) = \delta_{n,m} \quad (9)$$

and the completeness relation:

$$\sum_m \Psi_m(x)\Psi_m^*(x') = \delta^{(n)}(x - x') \quad (10)$$

The functions $\Psi_m$ are eigenfunctions of the n-dimensional Laplace-Beltrami operator $\Delta_n$ of the manifold $M^n$, with eigenvalues $\mu_m^2$:

$$-\Delta_n \Psi_m = \mu_m^2 \Psi_m \quad (11)$$

The gravitational potential $V_{n+4}$ satisfies the Poisson equation in $n + 3$ spatial dimensions, when the Newtonian limit is taken:

$$\Delta_{n+3}V_{n+4} = (n + 1)\Omega_{n+2}G_{n+4}M\delta^{(n+3)}(x) \quad (12)$$

with $M$, the mass of the system, $G_{n+4}$ the Newton constant in $n + 4$ dimensions and

$$\Omega_{n+2} = \frac{2\pi^{\frac{n+3}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \quad (13)$$

Equation (12) corresponds to the case of large compact radius limit and has the solution:

$$V_{n+4} = -\frac{G_{n+4}M}{r_n^{n+1}} \quad (14)$$

with $r_n$ the radius of the n + 3-dimensional space in spherical coordinates. Denoting by $r$ the radius of the 3-dimensional space of our world, then $r_n = \sqrt{r^2 + x_1^2 + x_2^2 + ... + x_n^2}$. When the
compact dimensions have small lengths, we find the harmonic expansion of $V_{n+4}$ in terms of the eigenfunctions of the product space $M^4 \times M^n$, which reads:

$$V_{n+4} = \sum_m \Phi_m(r)\Psi_m(x)$$  \hspace{1cm} (15)

with $r$ denoting the coordinates of $M^4$ and $x$ denoting the coordinates of $M^n$. Consequently, the $\Phi_m$ obey:

$$\Delta_3 \Phi_m - \mu_m^2 \Phi_m = (n + 1)\Omega_n + 2 \Psi_m^*(0)G_{n+4}M\delta^3(x)$$  \hspace{1cm} (16)

with solution:

$$\Phi_m(r) = \frac{-\Omega_n G_{n+4}M\Psi_m^*(0) e^{-|\mu_m|r}}{2r}$$  \hspace{1cm} (17)

The gravitational potential is written as:

$$V_{n+4} = -\frac{\Omega_n G_{n+4}M}{2r} \sum_m \Psi_m^*(0)\Psi_m(x)e^{-|\mu_m|r}$$  \hspace{1cm} (18)

Since all point particles in the four dimensional spacetime have no dependence on the internal compact space $M^n$, we can take $x = 0$ in (18) to obtain the four dimensional gravitational potential:

$$V_4 = -\frac{G_4M}{r} \sum_m \Psi_m^*(0)\Psi_m(0)e^{-|\mu_m|r}$$  \hspace{1cm} (19)

which is valid for large values of $r$, compared to the lengths of the compact dimensions.

In the above general result of relation (19) we shall apply the eigenfunctions and eigenvalues of Sol manifolds.

3. Sol Manifold modification of Newton’s Law

In order to compute the corrections to the gravitational potential we must solve equation (11) for the case of Sol manifold $M^3_A$. A much more elaborate analysis of this section can be found in [8, 9]. Using the $(u, v, z)$ coordinates we introduced previously, the Laplace-Beltrami operator for the manifold $M^3_A$ is:

$$\Delta = Ee^{2z\ln \lambda} \frac{\partial^2}{\partial u^2} + 2F \frac{\partial^2}{\partial u \partial v} + Ge^{-2z\ln \lambda} \frac{\partial^2}{\partial v^2} + \frac{\partial^2}{\partial z^2}$$  \hspace{1cm} (20)

which stems from the Riemannian metric (8). As usual $E = |e_u|^2$, $F = |e_v|^2$ and $G = |(e_u, e_v)|$, where $e_u$ and $e_v$ the basis of the $T^2$ lattice. Thus we must solve

$$-\Delta \psi = E \psi$$  \hspace{1cm} (21)

A function $\Psi = e^{2\pi i(\gamma, w)}f(z)$ satisfies equation (21) if and only if $f(z)$ satisfies the modified Mathieu equation [8]:

$$\left(-\frac{d^2}{dz^2} + |\nu(\gamma)|\cosh 2\mu(z + \alpha(\gamma))\right)f(z) = \Lambda f(z)$$  \hspace{1cm} (22)

with $\mu = \ln \lambda$, $\nu(\gamma) = 8\pi^2CQ_A(\gamma)$ and $\alpha(\gamma) = \frac{\ln \left(\sqrt{\frac{\nu(\gamma, e_u)}{\nu(\gamma, e_v)}}\right)}{2 \ln \lambda}$. Also

$$C = \frac{1}{\sqrt{D\text{Vol}(M^3_A) \sin \theta}}$$  \hspace{1cm} (23)
The volume of the Sol manifold, \( \text{Vol}(M^3) \), is equal to the volume of the total bundle space \( T^2 \times S^1 \), and is equal to \( 2\pi R^3 \). \( D \) is the discriminant of the matrix \( A \) defined in (5) and \( Q_{A^*}(\gamma) \) is the quadratic form (see [9]) corresponding to \( A^* \) acting to the dual lattice of \( T^2 \), with \( Q_{A^*}(\gamma) = (\gamma, e_u)(\gamma, e_v)(\lambda - \lambda^{-1}) \). The eigenvalues \( E \) and \( \Lambda \) are related as follows:

\[
E = \Lambda + \nu(\gamma) \cos \theta
\]  

(24)

It is proved in reference [9] that the functions \( \Psi_\gamma = e^{2\pi i(\gamma, w)}f_\gamma(z) \) form a complete orthogonal basis on the Sol manifold. The spectrum of the Laplace-Beltrami operator consists of two parts:

- The trivial part, with eigenvalues \( E_k = \frac{4\pi^2k^2}{R^2} \)
- The non-trivial part with eigenvalues \( E_{k,\nu} = \Lambda_k(\nu[\gamma]) + \nu(\gamma) \cos \theta \) and eigenfunctions the solutions of (22).

with \( k = 0, 1, 2, \ldots \) and \( R \) the radius of the universal covering space \( T^2 \times S^1 \). Thus we must compute the first eigenvalues and eigenfunctions. We shall use the most interesting case which is when \( \text{Vol}(M^3) \sin \theta \) is large (or equivalently \( \nu \to 0 \)). This term contains the compactification radius of the extra dimensions and the deformation of the lattice in terms of \( \theta \). When \( \text{Vol}(M^3) \sin \theta \) becomes large, \( \nu(\gamma) \) becomes small [9]. The eigenvalues of the Laplace-Beltrami operator are \( E_k = \Lambda_k \), with \( \Lambda_k \) the eigenvalues of the modified Mathieu operator,

\[
M = -\frac{d^2}{dz^2} + |\nu(\gamma)|\cosh 2\mu z
\]  

(25)

Let the order of the eigenvalues be \( E_0 = 0 \leq E_1 \leq E_2 \leq \ldots \). We shall use the first two, since the corrections to the gravitational potential fall exponentially as the eigenvalues grow larger. Thus the first two are \( E_0 = 0 \) and \( E_1 \), which asymptotically read:

\[
E_1 \sim \frac{(\ln \lambda)^2 \pi^2}{(\ln C)^2}
\]  

(26)

with \( C \):

\[
C = \frac{1}{\sqrt{D\text{Vol}(M^3)\sin \theta}}
\]  

(27)

The eigenfunctions corresponding to the Mathieu operator (25) are,

\[
f_m(z) = \sum_{k=0}^{\infty} A_{2k}^{2m} \cosh[2kz]
\]  

(28)

with "\( m \)" counting the eigenvalues, \( m = 0 \) corresponds to \( E_0 \) e.t.c.

We substitute \( E_0 \) and \( E_1 \) in relation (19) with \( |\mu_0| = \sqrt{E_0} \) and \( |\mu_1| = \sqrt{E_1} \). Also we substitute the eigenfunctions of the Sol manifold, thus:

\[
\Psi_m(0) = \sum_{k=0}^{\infty} A_{2k}^{2m}
\]  

(29)

The asymptotic behavior of the coefficients \( A_{2k}^{2m} \) for \( \nu \to 0 \) is really simple [?]. The terms of the form \( A_m^{(m)} \) tend to 1, while terms of the form \( A_m^{(k)} \), with \( m \neq k \) tend to zero and the gravitational potential of relation (19) becomes:

\[
V_4 = -\frac{G_4M}{r} \sum_{m} \sum_{k=0}^{\infty} A_{2k}^{2m} A_{2k}^{2m} e^{-|\mu_m|r}
\]  

(30)
For the first two eigenvalues we have

\[ V_4 = -\frac{G_4 M}{r} \left( A_0^0 A_0^0 e^{-|E_0|r} + A_2^2 A_2^2 e^{-|E_1|r} \right) \tag{31} \]

or (using the asymptotic behavior for the coefficients \( A_m^{(m)} \)):

\[ V_4 = -\frac{G_4 M}{r} \left( e^{-\sqrt{E_0} r} + e^{-\sqrt{E_1} r} \right) \tag{32} \]

Since \( E_0 = 0 \) and using relations (26) and (27) we obtain finally:

\[ V_4 \sim -\frac{G_4 M}{r} \left( 1 + e^{-\left| \frac{(\ln \lambda)^2}{4|n| r} \right|} \right) \tag{33} \]

It is clear that the Sol manifold correction to the Newton’s law gravitational potential depends on the compactification radii of the extra dimensions \( R \), the angle \( \theta \) of the vectors \( e_u \) and \( e_v \) and on the eigenvalues of the hyperbolic gluing map \( A \). In the next section we study the parameter space of the corrections found and we compare it with the \( T^3 \) manifold corrections.

4. Analysis of the parameter space and comparison with the \( T^3 \) corrections

We examine first the dependence of the Sol correction range \( e^{-\mu_m r} \) on the parameters \( \theta \) and \( n \). In figure (1) we plot the dependence for the compactification radius value \( R = 0.05 \, mm \) (left) (smaller than the current experimental bound \( r = 0.2 \, mm \)) and for \( r = 0.1 \, mm \), the values of \( R \) are 0.1 mm (right) and 0.2 mm (bottom) respectively. The Sol structure gives very large corrections to gravity if the compactification radius is very small (of order \( R \sim 10^{-6} \, m \) and smaller). Also we shall check out the behavior of the range of the corrections around \( R \sim 10^{-8} \, m \).
which is the expected scale that three extra dimensional spaces should have. According to figure (1), we can see that the right and bottom figures are similar. Thus for large values of \( n > 200 \) and for \( \theta > 0.3 \), the corrections are very small. Let us compare the range of Sol manifolds with the range of the \( T^3 \) torus corrections.

Figure 2. \( r \)-dependence of the range \( e^{-\mu_m r} \) of Sol corrections with \( R = 0.05 \text{mm}, n = 250 \) and \( \theta = \frac{\pi}{3} \) (left), \( r \)-dependence of the range of \( T^3 \) corrections for \( R = 0.05 \text{mm} \) (right).

Figure 3. Comparison of the \( r \)-dependence of the range of \( T^3 \) and Sol corrections for \( R = 0.2 \text{ mm} \) for various \( n \), and \( \theta = \pi/3 \).

In figure (2) we plot the range of the \( T^3 \) corrections (left) and the corresponding dependence for the Sol manifold case (right) as a function of \( r \) with \( R = 0.05 \text{mm} \) and \( \theta = \pi/3 \) (left). As it can be seen from the figure (2), the term \( e^{-\mu_m r} \) is very big compared to \( e^{-r/R} \), for small \( n (\sim 5) \). As \( n \) grows the term becomes smaller and smaller. As \( n \) grows, the range of Sol-corrections becomes comparable and after a value of \( n \), smaller compared to the \( T^3 \) corrections.

The case with \( R = 0.2 \text{ mm} \) is very interesting. According to figure (3) the range of Sol corrections can vary significantly depending on the value that \( n \) takes. Compared to the corresponding 3-Torus range, Sol corrections can be much larger (for small \( n \)) and comparable (for \( n > 450 \)) and even smaller.

Also Sol corrections can exist for compactification radii and at distances for which the corresponding \( T^3 \) torus range values are very large (and consequently not experimentally preferable).
5. Conclusions
We studied the corrections to the gravitational potential caused by a Sol manifold extra dimensional compact space. After investigating the parameter space we found that the range of Sol manifolds corrections can be similar to the $T^3$ torus results and can be very different compared to the $T^3$ results depending on the values of the parameters. We left unanswered two issues, the graviton production and how do we distinguish Sol manifolds corrections from other corrections (the last due to the rich parameter space of Sol manifolds).

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