THE REGULAR REPRESENTATION OF $U_{\nu}(\mathfrak{gl}_{m|n})$

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Abstract. Using quantum differential operators, we construct a super representation of $U_{\nu}(\mathfrak{gl}_{m|n})$ on a certain polynomial superalgebra. We then extend the representation to its formal power series algebra which contains a $U_{\nu}(\mathfrak{gl}_{m|n})$-submodule isomorphic to the regular representation of $U_{\nu}(\mathfrak{gl}_{m|n})$. In this way, we obtain a presentation of $U_{\nu}(\mathfrak{gl}_{m|n})$ by a basis together with explicit multiplication formulas of the basis elements by generators.

1. Introduction

Arising from the natural representation $V$ of the quantum supergroup $U_{\nu}(\mathfrak{gl}_{m|n})$, the investigation on the tensor products $V^{\otimes r}$ for all $r \geq 0$ has recently produced interesting outcomes. For example, the root-of-unity theory resulted in a new proof for the quantum Mullineux conjecture (see [9]). On the other hand, the generic theory on $\nu$-Schur superalgebras, which are homomorphic images of the representations $U_{\nu}(\mathfrak{gl}_{m|n}) \to \text{End}(V^{\otimes r})$, gives rise to a new construction for $U_{\nu}(\mathfrak{gl}_{m|n})$ itself (see [7]). This latter work extends the geometric realisation of quantum $\mathfrak{gl}_n$, given by Beilinson–Lusztig–MacPherson (BLM) in [1], to the super case. The BLM work has also been generalised to the quantum affine case [3, 5] and the case for the other classical types [2, 10].

Furthermore, in the nonsuper case, there are other representations of $U_{\nu}(\mathfrak{gl}_n)$ arising from the symmetric and exterior algebras $S(V)$ and $\Lambda(V)$ of the natural representation $V$; see, e.g., [11] and [12, §§5A.6-7], where the module actions are defined by using certain quantum differential operators. Can these representations be used to determine the structure of a quantum supergroup? We will provide an affirmative answer in this paper.

We will start with the natural super representations $V = V_0 \oplus V_1$ of $U_{\nu}(\mathfrak{gl}_{m|n})$. We first introduce two types of symmetric superalgebras $S_{0|1}(V) = S(V_0) \otimes \Lambda(V_1)$ and $S_{1|0}(V) = \Lambda(V_0) \otimes S(V_1)$ and their mixed tensor product $S^{m|n}(V)$. The supermodule structure on each of them is defined via quantum differential (super) operators. We then extend the supermodule structure to the formal power series algebra $\tilde{S}^{m|n}(V)$.
which is naturally a $U_{\nu}(\mathfrak{gl}_{m|n})$-module. We will extract a submodule from $\tilde{S}^{|\rho|}_{m|n}(V)$ which naturally possesses a supermodule structure. We prove in the main theorem (Theorem 5.3) that this supermodule is isomorphic to the regular representation of $U_{\nu}(\mathfrak{gl}_{m|n})$. Thus, we obtain a new presentation for $U_{\nu}(\mathfrak{gl}_{m|n})$ (cf. Lemma 5.1).

Surprising enough, this presentation from the regular representation of $U_{\nu}(\mathfrak{gl}_{m|n})$ coincides with the one from [1, Lemma 5.3] when $n = 0$ and with the one as given in [7, Thm 8.4] in general, both of which were obtained either by a geometric method involving quantum Schur algebras or by an algebraic method involving quantum Schur superalgebras.

2. The quantum supergroup $U_{\nu}(\mathfrak{gl}_{m|n})$ and differential operators

For fixed non-negative integers $m, n$ with $m+n > 0$, let $[1, m+n] := \{1, 2, \ldots, m+n\}$, and define the parity function $\hat{\iota}: [1, m+n] \to \mathbb{Z}_2$, $i \mapsto \hat{\iota}$ by

$$\hat{\iota} = \begin{cases} 0, & \text{if } 1 \leq i \leq m; \\ 1, & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

We will always regard $\mathbb{Z}_2 = \{0, 1\}$ as a subset of $\mathbb{N}$ unless it is used for the grading of a super structure. For any superspace $V$ and a homogeneous element $\nu \in V$, we often use $\hat{\nu}$ to denote its parity.

Let $\{e_1, e_2, \ldots, e_{m+n}\}$ be the standard basis for $\mathbb{Z}^{m+n}$, and define the “super dot product” on $\mathbb{Z}^{m+n}$ by

$$e_i \ast e_j = (e_i, e_j)_s = (-1)^{\hat{\iota}_i \hat{\iota}_j} \delta_{ij}. \quad (2.0.1)$$

Let $\mathbb{Q}(\nu)$ be the field of rational functions in indeterminate $\nu$ and let

$$\nu_h = \nu^{(-1)^h} \quad (h \in [1, m+n]), \quad [a]! = [1][2] \cdots [a], \quad \hat{\nu} = \frac{\nu^i - \nu^{-i}}{\nu - \nu^{-1}} \quad (a \in \mathbb{N}).$$

Let $[\hat{\nu}]_q$ denote the value at $q$.

Define the super (or graded) commutator on the homogeneous elements $X, Y$ for an (associative) superalgebra by

$$[X, Y] = [X, Y]_s = XY - (-1)^{\hat{\iota}X \hat{\iota}Y} YX.$$

The following quantum enveloping superalgebra $U_{\nu}(\mathfrak{gl}_{m|n})$ is defined in [13].

**Definition 2.1.** (1) The quantum enveloping superalgebra $U_{\nu}(\mathfrak{gl}_{m|n})$ over $\mathbb{Q}(\nu)$ is generated by

$$\begin{cases} \text{even generators:} & E_h, F_h, K_j^{\pm 1}, \, 1 \leq h, j \leq m+n, h \neq m, m+n; \\ \text{odd generators:} & E_m, F_m. \end{cases}$$

These elements are subject to the following relations:

(QG1) $K_a K_b = K_b K_a$, $K_a K_a^{-1} = K_a^{-1} K_a = 1$;

(QG2) $K_a E_h = \nu^{(\epsilon_a, \alpha_h)} E_h K_a$, $K_a F_b = \nu^{(\epsilon_a, -\alpha_b)} F_b K_a$;

(QG3) $[E_a, F_b] = \delta_{a,b} \frac{K_a K_{a+1}^{-1} K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{\nu_a - \nu_a^{-1}}$. 

where we may similarly define the quantum group

Following [12, 5A.6], we define

representations of its hyperalgebra at roots of unity). We will need two special

Example 2.2. Let

Note that, if \( n = 0 \), then (QG1)–(QG5) form a presentation for the quantum group

A Hopf algebra structure on \( U_v(\mathfrak{gl}_{m|n}) \) is defined (see [13, Section II]) by:

\[
\Delta(K_i) = K_i \otimes K_i, \\
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \\
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \\
\varepsilon(K_i) = 1, \\
\varepsilon(E_i) = \varepsilon(F_i) = 0, \\
S(K_i) = K_i^{-1}, \\
S(E_i) = -E_i K_i^{-1}, \\
S(F_i) = -K_i F_i,
\]

where \( K_i = K_i K_i^{-1} \).

Representations of \( U_v(\mathfrak{gl}_{m|n}) \) have been investigated in [13] (see also [9] for representations of its hyperalgebra at roots of unity). We will need two special \( U_v(\mathfrak{gl}_{m|n}) \)supermodules in the next section for our construction. They are built on the following two \( U_v(\mathfrak{gl}_N) \)-modules defined by quantum differential operators.

Example 2.2. Let \( V \) be a vector space over a field \( \mathbb{k} \) of dimension \( N \) and let \( \mathbb{k}[x_1, x_2, \ldots, x_N] \) be the polynomial algebra over \( \mathbb{k} \) in indeterminates \( x_1, \ldots, x_N \).

(1) Let \( S(V) \) be the symmetric algebra on \( V \), identified as \( S(V) = \mathbb{k}[x_1, x_2, \ldots, x_N] \).

Following [12, 5A.6], we define quantum differential operators \( \mathcal{D}_i : S(V) \to S(V) \) by

\[
\mathcal{D}_i(x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}) = \begin{cases} 
[a_i] q^{a_1} x_2^{a_2} \cdots x_i^{a_i-1} \cdots x_N^{a_N}, & \text{if } a_i \geq 1; \\
0, & \text{otherwise.}
\end{cases}
\]

We also introduce algebra automorphism \( \mathcal{K}_i : S(V) \to S(V) \) by setting

\[
\mathcal{K}_i(x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}) = q^{a_1} x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}.
\]

Let \( \mathcal{E}_i = x_i \circ \mathcal{D}_{i+1} \) and \( \mathcal{F}_i = x_{i+1} \circ \mathcal{D}_i \). Then, by [12, Prop. 5A.6], the following map

\[
E_i \mapsto \mathcal{E}_i, \\
F_i \mapsto \mathcal{F}_i, \\
K_j \mapsto \mathcal{K}_j
\]

for all \( 1 \leq i, j \leq N \) (\( i \neq N \)) defines an algebra homomorphism from \( U_q(\mathfrak{gl}_N) \) to the endomorphism algebra of \( S(V) \). Hence, \( S(V) \) becomes a \( U_q(\mathfrak{gl}_N) \)-module (cf. [11, Thm 4.1(A)]).
(2) Let $\Lambda(V)$ be the exterior algebra on $V$. In this case, we may identify $\Lambda(V)$ with the Grassman superalgebra $\Lambda(d_1, \ldots, d_N)$ with odd generators $d_1, \ldots, d_N$ and relations
\[ d_i^2 = 0 \quad (1 \leq i \leq N), \quad d_id_j = -d_jd_i \quad (1 \leq i \neq j \leq N). \]

Thus, $\Lambda(V)$ has a basis $d^a := d_1^{a_1} \cdots d_N^{a_N}$, $a \in \mathbb{Z}_2^N$. Define a $U_q(\mathfrak{gl}_N)$-action on $\Lambda(V)$ by
\[ K_id^a = q^{\alpha_i}d^a, \quad Ehd^a = \begin{cases} d^{a+\alpha_h}, & \text{if } a_{h+1} > 0; \\ 0, & \text{otherwise}, \end{cases} \quad Fhd^a = \begin{cases} d^{a-\alpha_h}, & \text{if } a_h > 0; \\ 0, & \text{otherwise}, \end{cases} \]
for all $1 \leq h, i \leq N$, $h \neq N$. It is direct to check that all relations (QG1-5) are satisfied. Hence, $\Lambda(V)$ becomes a $U_q(\mathfrak{gl}_N)$-module (cf. [11, §§2.4]).

3. The polynomial superalgebra $S_{m/n}(V)$ as a $U_\nu(\mathfrak{gl}_{m/n})$-supermodule

We generalize the constructions of the module structures on symmetric and exterior algebras to the supergroup $U_\nu(\mathfrak{gl}_{m/n})$.

Consider the natural representation on the superspace $V = V_0 \oplus V_1$ of $\mathfrak{gl}_{m/n}(k)$ where $\dim V_0 = m$ and $\dim V_1 = n$. We will consider two superalgebras in the notation of Example 2.2:
\[ S(V_0) \otimes \Lambda(V_1) = \mathbb{k}[x_1, \ldots, x_m, d_1, \ldots, d_n], \]
\[ \Lambda(V_0) \otimes S(V_1) = \mathbb{k}[d_1, \ldots, d_m, x_1, \ldots, x_n]. \]
These are known as polynomial superalgebras with even generators $x_i$ and odd generators $d_j$. By Example 2.2, both algebras are also $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$-modules.

We now assume $\mathbb{k} = \mathbb{Q}(\nu)$. In order to introduce supermodule structure for $U_\nu(\mathfrak{gl}_{m/n})$, we set
\[ S_{0|1} = S_{0|1}(V) := \mathbb{Q}(\nu)[X_1, X_2, \ldots, X_{m+n}] \quad \text{with } X_i = x_i, X_{m+j} = d_j, \]
\[ S_{1|0} = S_{1|0}(V) := \mathbb{Q}(\nu)[X_1, X_2, \ldots, X_{m+n}] \quad \text{with } X_i = d_i, X_{m+j} = x_j, \]
(3.0.1)

where $1 \leq i \leq m, 1 \leq j \leq n$. We use divided powers to denote their monomial bases:
\[ X^{(a)} = X_1^{(a_1)} X_2^{(a_2)} \cdots X_{m+n}^{(a_{m+n})}, \]
where $a = (a_1, \ldots, a_{m+n}) \in \mathbb{N}^m \times \mathbb{Z}_2^n$ for $S_{0|1}$, $a = (a_1, \ldots, a_{m+n}) \in \mathbb{Z}_2^n \times \mathbb{N}^n$ for $S_{1|0}$, and $X_i^{(a_i)} = \frac{X_i^{a_i}}{a_i!}$.

For the superspace structure, we have, for $i \in \mathbb{Z}_2$, $X^{(a)} \in (S_{0|1})$, if and only if $\hat{a} := \sum_{j=1}^n a_{m+j} \equiv i (\text{mod } 2)$, while $X^{(a)} \in (S_{1|0})$, if and only if $\hat{a} := \sum_{j=1}^n a_j \equiv i (\text{mod } 2)$.

As algebras, both $S_{0|1}$ and $S_{1|0}$ have a graded structure $S_{0|1}(r)$ and $S_{1|0}(r)$ for all $r \geq 0$, where $S_{0|1}(r)$ (resp., $S_{1|0}(r)$) is the $r$-th homogeneous component spanned by all $X^{(a)}$ with $\deg(X^{(a)}) = r$. Here $\deg(X^{(a)}) := \sum_{i=1}^{m+n} a_i$. 
Lemma 3.1. Both $S_{0|1}$ and $S_{1|0}$ are $U_q(gl_{m|n})$-supermodules under the actions above. In particular, their homogeneous components $S_{0|1}(r), S_{1|0}(r), r \geq 0$ are all subsupermodules.

Proof. We only need to verify the defining relations that involve the odd generators. We only prove the case for $S_{0|1}$. It is easy to verify the relations (QG2) and (QG4). Note that the actions of $E_m, F_m$ is consistent with those for even generators, so (QG5) holds. It remains to check (QG3) and (QG6).

The relations $[E_m, E_b] = 0 = [E_b, F_m]$ with $m \neq b$ in (QG3) are clear. Assume now $a = b = m$. Let $a = (a_1, \ldots, a_{m+n}) \in \mathbb{N}^m \times \mathbb{Z}_2^n$. If $a_{m+1} = 1$ then

$$E_{m}F_{m} + F_{m}E_{m} \cdot X^{(a)} = E_{m}F_{m} \cdot X^{(a)} = [a_{m} + 1]F_{m}X^{(a + \alpha_{m})} = [a_{m} + 1]X^{(a)}.$$ 

$$K_{m}K_{m+1}^{-1} - K_{m}^{-1}K_{m+1}X^{(a)} = \frac{\nu^{a_{m} + 1} - \nu^{-a_{m} - 1}}{\nu - \nu^{-1}} \cdot X^{(a)} = [a_{m} + 1]X^{(a)}.$$ 

If $a_{m+1} = 0, a_{m} > 0$ then

$$(E_{m}F_{m} + F_{m}E_{m}) \cdot X^{(a)} = E_{m}F_{m} \cdot X^{(a)} = E_{m}X^{(a - \alpha_{m})} = [a_{m}]X^{(a)}.$$ 

$$= K_{m}K_{m+1}^{-1} - K_{m}^{-1}K_{m+1} \cdot X^{(a)}.$$ 

If $a_{m+1} = 0, a_{m} = 0$ then

$$(E_{m}F_{m} + F_{m}E_{m}) \cdot X^{(a)} = 0 = \frac{K_{m}K_{m+1}^{-1} - K_{m}^{-1}K_{m+1}}{\nu - \nu^{-1}} \cdot X^{(a)}.$$ 

So, in all three cases, we obtain

$$(E_{m}F_{m} + F_{m}E_{m}) \cdot X^{(a)} = \frac{K_{m}K_{m+1}^{-1} - K_{m}^{-1}K_{m+1}}{\nu - \nu^{-1}} \cdot X^{(a)},$$ 

for all $a \in \mathbb{N}^m \times \mathbb{Z}_2^n$, proving (QG3).
Finally, we prove the four relations in (QG6). As $a_{m+1} \leq 1$, we have $E_m^2 \cdot X(a) = 0 = F_m^2 \cdot X(a)$ for all $a$. For the other two relations, if $a_{m+1} = 1, a_{m+2} = 1$ then

\[
E_m E_{m-1, m+2} \cdot X(a) = (E_{m-1} E_{m} E_{m+1} + \nu E_{m-1} E_{m+1} E_{m} - \nu^{-1} E_{m} E_{m+1} E_{m} - E_{m+1} E_{m} E_{m+1} E_{m-1})X(a)
\]

\[
= (-\nu E_{m-1} E_{m+1} E_{m} + E_{m} E_{m+1} E_{m-1} E_{m})X(a)
\]

\[
= (-\nu [\alpha + 1] [\alpha - 1] [\alpha + 1] + [\alpha - 1] [\alpha + 1] [\alpha + 1]) X(a'),
\]

where $a' = a + \alpha - 1 + 2 \alpha + \alpha + 1 = a + e_{m-1} + e_m - e_{m+1} - e_{m+2}$. On the other hand,

\[
E_{m-1, m+2} E_m \cdot X(a) = (E_{m-1} E_{m} E_{m+1} + \nu E_{m-1} E_{m+1} E_{m} - \nu^{-1} E_{m} E_{m+1} E_{m} - E_{m+1} E_{m} E_{m+1} E_{m-1})X(a)
\]

\[
= (E_{m-1} E_{m} E_{m+1} - \nu^{-1} E_{m} E_{m+1} E_{m})X(a)
\]

\[
= ([\alpha + 1] [\alpha + 2] + [\alpha - 1] [\alpha + 1] [\alpha + 1]) X(a') = 0.
\]

Since $[\alpha] + [\alpha + 2] - (\nu + \nu^{-1})[\alpha + 1] = 0$, it follows that

\[
(E_{m-1, m+2} E_m + E_{m} E_{m-1, m+2}) \cdot X(a)
\]

\[
= [\alpha - 1] [\alpha + 1] [\alpha + 2] + (\nu + \nu^{-1})[\alpha + 1]) X(a') = 0.
\]

If $a_{m+1} = 0$ or $a_{m+2} = 0$ then $E_{m-1, m+2} E_m \cdot X(a) = 0 = E_m E_{m-1, m+2} \cdot X(a)$ by the definition of the actions. The last case can be proved similarly. This proves (QG6).

The following result is a super analog of a result stated at the end of [12, 5A.7] (see also [11, Thms. 4.1(A), 4.2]). Recall from, say, [9] that irreducible weight $U_{\nu}(\mathfrak{g}l_{m+n})$-modules are indexed by

\[
N_{++}^{m+n} = \{ \lambda \in \mathbb{N}^{m+n} | \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n} \}.
\]

**Corollary 3.2.** Let $\Delta(\rho e_1)$ (resp., $\nabla(\rho e_{m+n})$) be the irreducible weight $U_{\nu}(\mathfrak{g}l_{m+n})$-module of highest (resp., lowest) weight $\rho e_1$ (resp., $\rho e_{m+n}$). Then, there are $U_{\nu}(\mathfrak{g}l_{m+n})$-module isomorphisms:

\[
S_{0|1}(r) \cong \Delta(\rho e_1), \quad S_{1|0}(r) \cong \nabla(\rho e_{m+n}).
\]

**Proof.** Let $\lambda = \rho e_1 \in N_{++}^{m+n}$. Then $X(\lambda) \in S_{0|1}(r)$ is a highest weight vector, since, for any $a = (a_1, \cdots, a_{m+n}) \in \mathbb{N}^m \times \mathbb{Z}_2$ with $|a| = r$, $r e_1 - a = a_2 (e_1 - e_2) + a_3 (e_1 - e_3) + \cdots + a_{m+n} (e_1 - e_{m+n})$ and

\[
(F_1^{(a_2)} F_2^{(a_3)} F_1^{(a_3)} \cdots F_{m+n-1}^{(a_m)} F_{m+n}^{(a_{m+n})}) \cdot X(\lambda) = X(a).
\]

Hence, $S_{0|1}(r)$ is generated by an highest weight vector. On the other hand, a reversed sequence in the $F_i^{(a_i)}$’s send $X(a)$ back to $X(\lambda)$. Thus, $S_{0|1}(r)$ is irreducible. The proof for $S_{1|0}(r)$ is similar. \qed
Consider the tensor product
\[ S^{m|n} = S^{m|n}(V) = (S_{0|1})^m \otimes (S_{1|0})^n \cong \mathbb{Q}(v)[X_{i,j}]_{1 \leq i, j \leq m+n}, \] (3.2.1)
where \( X_{i,j} \) denotes the \( i \)-th tensor factor. Thus, we may regard \( S^{m|n} \) as the polynomial superalgebra as indicated by the right hand side of (3.2.1), which has even generators \( X_{i,j} \) for all \( i, j \) with \( \hat{i} + \hat{j} = 0 \) and odd generators \( X_{i,j} \) for all \( i, j \) with \( \hat{i} + \hat{j} = 1 \). In particular, we may describe the monomial basis for \( S^{m|n} \) in terms of the following matrix set:
\[ M(m|n) = \left\{ \left( \begin{array}{ll} X & Q \\ Q' & Y \end{array} \right) \mid X \in M_m(\mathbb{N}), Q \in M_{m \times n}(\mathbb{Z}_2), Q' \in M_{n \times m}(\mathbb{Z}_2), Y \in M_n(\mathbb{N}) \right\}. \] (3.2.2)
For \( A = (a_{i,j}) \in M(m|n) \), let
\[ c_i = c_i(A) = (a_{1,i}, a_{2,i}, \ldots, a_{m+n,i}) \]
be the \( i \)-th column of \( A \) and let
\[ X^{[A]} := X^{(e_1)} \otimes X^{(e_2)} \cdots \otimes X^{(e_{m+n})}. \]
The parity of \( X^{[A]} \) is given by \( \hat{A} := \sum_{i+j=1} a_{i,j} \).

Via the coalgebra structure (2.1.1) of \( U_v (\mathfrak{gl}_{m|n}) \), \( S^{m|n} \) becomes a \( U_v (\mathfrak{gl}_{m|n}) \)-module (see the lemma below). Recall also the sign rule: for supermodules \( V_1, V_2 \) over a superalgebra \( U \), if \( u_1, u_2 \subset U, v_1 \in V_i \) with \( u_2, v_1 \) homogeneous, then
\[ (u_1 \otimes u_2)(v_1 \otimes v_2) = (-1)^{|u_2|} u_1 v_1 \otimes u_2 v_2. \]

For \( A \in M(m|n), i \in [1, m+n] \), let
\[ \sigma(i, A) = \begin{cases} \sum_{s > m, t < i} a_{s,t}, & \text{if } 1 \leq i \leq m; \\ \sum_{s > m, t \leq m} a_{s,t} + \sum_{s \leq m, m < t < i} a_{s,t}, & \text{if } m + 1 \leq i \leq m+n, \end{cases} \] (3.2.3)
and
\[ f(i, A) = \sum_{j > i} a_{h,j} - (-1)^{\delta_{h,m}} \sum_{j > i} a_{h+1,j}, \]
\[ g(i, A) = \sum_{j < i} a_{h+1,j} - (-1)^{\delta_{h,m}} \sum_{j < i} a_{h,j}. \] (3.2.4)

**Lemma 3.3.** The set \( \{ X^{[A]} \mid A \in M(m|n) \} \) forms a \( \mathbb{Q}(v) \)-basis for the \( U_v (\mathfrak{gl}_{m|n}) \)-supermodule \( S^{m|n} \) which has the following actions:

1. \( K_i X^{[A]} = v_i \sum_{1 \leq j \leq m} a_{i,j} X^{[A]} \),
2. \( E_h X^{[A]} = \sum_{1 \leq i \leq m+n} (-1)^{\sigma_h,m(i,A)} v_h^f(i,A)[a_{h,i} + 1] X^{[A+E_h,i-E_{h+1},i]} \),
3. \( F_h X^{[A]} = \sum_{1 \leq i \leq m+n} (-1)^{\sigma_h,m(i,A)} v_h^{g(i,A)}[a_{h+1,i} + 1] X^{[A-E_h,i+E_{h+1},i]} \),
where
\[
\sigma_{h,m}(i, A) = \begin{cases} 
\delta_{h,m}\sigma(i, A), & \text{if } h = m; \\
0, & \text{if } h \neq m.
\end{cases}
\] (3.3.1)

Proof. Let \(\Delta^{(N)} = (\Delta \otimes 1 \otimes \cdots 1) \circ \cdots \circ (\Delta \otimes 1) \circ \Delta\). Then, for \(N = m + n - 1\),
\[
\begin{align*}
\Delta^{(N)}(K_i) &= K_i \otimes \cdots \otimes K_i, \\
\Delta^{(N)}(E_h) &= \sum_{i=1}^{m+n} 1 \otimes 1 \cdots \otimes 1 \otimes E_h \otimes \tilde{K}_h \cdots \otimes \tilde{K}_h, \\
\Delta^{(N)}(F_h) &= \sum_{i=1}^{m+n} \tilde{K}_h^{-1} \cdots \otimes \tilde{K}_h^{-1} \otimes F_h \otimes 1 \otimes 1 \cdots \otimes 1. 
\end{align*}
\] (3.3.2)

Thus, by (3.0.2) and the sign rule (and, for \(h = m\), noting \(\nu_m = \nu, \nu_{m+1} = \nu^{-1}\)),
\[
E_h.X^{[A]} = \Delta^{(N)}(E_h).X^{[A]}
= \sum_{i=1}^{m+n} (-1)^{\delta_{h,m}(\sum_{j<i} c_j)} (X^{(c_1)} \otimes \cdots \otimes X^{(c_{i-1})} \otimes E_h.X^{(c_i)})
\otimes \tilde{K}_h.X^{(c_{i+1})} \cdots \otimes \tilde{K}_h.X^{(c_{m+n})})
= \sum_{1 \leq a_{h,i} \leq 1} \sum_{1 \leq j \leq m+n} \nu_h^{f(i,A)} [a_{h,i} + 1] X^{[A+E_h,i-E_{h+1,j}]}.
\]

The actions of \(F_h, K_i\) can be proved similarly. \(\square\)

Remark 3.4. We remark that these module formulas are easily obtained, but are the key to the determination of the regular representation of \(U_\nu(\mathfrak{gl}_{m|n})\). As a comparison, analogous formulas for quantum Schur superalgebras are certain multiplication formulas (see [7, Props 4.4-5]) which are obtained by rather lengthy calculations.

4. The formal power series algebra \(\tilde{S}^{m|n}(V)\)

We now extend the module structure on \(S^{m|n}\) to its formal power series algebra and then focus on a submodule which has a \(U_\nu(\mathfrak{gl}_{m|n})\)-supermodule structure. We will displayed explicitly the actions on a basis.

Recall from (3.2.1) the polynomial superalgebra \(S^{m|n}\) and its basis \(\{X^{[A]}\}_{A \in M(m|n)}\). By turning the direct sum of all \(Q(\nu).X^{[A]}\) into a direct product, we obtain the formal power series algebra:
\[
\tilde{S}^{m|n} = \tilde{S}^{m|n}(V) := \prod_{A \in M(m|n)} Q(\nu).X^{[A]} \cong Q(\nu)[[X_{i,j}]]_{1 \leq i,j \leq m+n}.
\] (4.0.1)

For clarity of the \(U_\nu(\mathfrak{gl}_{m|n})\)-actions below, we continue to write the elements in \(\tilde{S}^{m|n}\) by infinite series in \(X^{[A]}\)'s. Naturally, the \(U_\nu(\mathfrak{gl}_{m|n})\)-action on \(S^{m|n}\) extends to \(\tilde{S}^{m|n}\) so that \(\tilde{S}^{m|n}\) becomes a \(U_\nu(\mathfrak{gl}_{m|n})\)-module. We now construct a submodule on which a natural super structure can be built.
Let

\[ M(m|n)^\mp = \{ A = (a_{i,j}) \in M(m|n) \mid a_{i,i} = 0 \ \forall i \}. \]

For \( \lambda \in \mathbb{N}^{m+n} \), \( A \in M(m|n)^\mp \), let \( A + \lambda = A + \text{diag}(\lambda) \) and, for \( j \in \mathbb{Z}^{m+n} \), define

\[ A(j) = \sum_{\lambda \in \mathbb{N}^{m+n}} \nu^{\lambda j} X^{[A+\lambda]} \in \tilde{S}^{m|n}. \] (4.0.2)

Let \( U(m|n) \) be the subspace of \( \tilde{S}^{m|n} \) spanned by \( A(j) \) for all \( A \in M(m|n)^\mp, j \in \mathbb{Z}^{m+n} \). Since every \( X^{[A+\lambda]} \) in \( A(j) \) has parity, \( A(j) \) is a supermodule. In the rest of the section, we will prove that \( U(m|n) \) is a \( U_v(\mathfrak{gl}_{m|n}) \)-supermodule.

Let \( \alpha_h = e_h - e_{h+1}, \beta_h = e_h + e_{h+1}, \sigma_{h,m}(i) = \sigma_{h,m}(i, A), f(i) = f(i, A) \), and \( g(i) = g(i, A) \) (see (3.2.4) and (3.3.1)).

**Theorem 4.1.** The superspace \( U(m|n) \) is a \( U_v(\mathfrak{gl}_{m|n}) \)-submodule of \( \tilde{S}^{m|n} \) with basis \( \{ A(j) \mid A \in M(m|n)^\mp, j \in \mathbb{Z}^{m+n} \} \) and the following explicit actions of \( E_h, F_h, K_i \): for \( A = (a_{s,t}), j = (j_h), 1 \leq i \leq m + n, \) and \( 1 \leq h < m + n, \)

\[ K_i A(j) = \nu^i \sum_{1 \leq j \leq m+n} a_{i,j} A(j + e_i), \]

\[ E_h A(j) = \sum_{i > h, a_{h,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \nu^{h \uparrow i} \left[ a_{h,i} + 1 \right] (A + E_{h,i} - E_{h+1,i})(j) \]
\[ + \sum_{i < h, a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \nu^{h \uparrow i} \left[ a_{h,i} + 1 \right] (A + E_{h,i} - E_{h+1,i})(j + \alpha_h) \]
\[ + \nu^{h \downarrow i} (\nu_h - \nu_{h+1}) \left[ a_{h,h+1} + 1 \right] (A + E_{h,h+1})(j - \beta_h), \] (4.1.1)

\[ F_h A(j) = \sum_{i < h, a_{h,i} \geq 1} (-1)^{\sigma_{h,m}(i,A)} \nu^{h \uparrow i} \left[ a_{h+1,i} + 1 \right] (A - E_{h,i} - E_{h+1,i})(j) \]
\[ + \sum_{i > h, a_{h,i} \geq 1} (-1)^{\sigma_{h,m}(i,A)} \nu^{h \uparrow i} \left[ a_{h+1,i} + 1 \right] (A - E_{h,i} + E_{h+1,i})(j - \alpha_h) \]
\[ + \nu^{h \downarrow i} (\nu_h - \nu_{h+1}) \left[ a_{h,h+1} + 1 \right] (A + E_{h,h+1})(j) \]
\[ + \nu^{h \downarrow i} (\nu_h - \nu_{h+1}) \left[ a_{h,h+1} + 1 \right] (A - E_{h,h+1})(j - \beta_h) \] (4.1.2)

where \( \uparrow, \downarrow \) is 0 if \( a_{h+1,h} = 0 \) (resp., \( a_{h,h+1} = 0 \)), and is 1 otherwise.

Moreover, it is a \( U_v(\mathfrak{gl}_{m|n}) \)-supermodule.

**Proof.** The proof of linear independence is similar to that of [6, Prop. 4.1(2)].
By Lemma 3.3(1),
\[ K_{i,A}(j) = \sum_{\lambda \in \mathbb{N}^{m+n}} \mathbf{v}^{\lambda j} K_{i,A}^\lambda \mathbf{x}^{[A+\lambda]} = \sum_{\lambda \in \mathbb{N}^{m+n}} \mathbf{v}^{\lambda j} \mathbf{v}^\lambda \sum_{j=1}^{m+n} a_{i,j} \mathbf{x}_{A+\lambda}^{[A+\lambda]} \]
\[ = \sum_{j=1}^{m+n} a_{i,j} \sum_{\lambda \in \mathbb{N}^{m+n}} \mathbf{v}^{\lambda j} \mathbf{x}_{A+\lambda}^{[A+\lambda]} = \sum_{j=1}^{m+n} a_{i,j} A(j + e_i). \]

Similarly, by Lemma 3.3(2), and noting \( \sigma(i, A) = \sigma(i, A + \lambda) \),
\[ E_{h,A}(j) = \sum_{\lambda \in \mathbb{N}^{m+n}} \mathbf{v}^{\lambda j} E_{e_i A}^\lambda \mathbf{x}^{[A+\lambda]} \]
\[ = \sum_{\lambda \in \mathbb{N}^{m+n}} \sum_{1 \leq i \leq m+n, a_{h+1,i} \geq 1} \mathbf{v}^{\lambda j} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{f(i,A+\lambda)}_{h}(a_{h,i}+1) \mathbf{x}_{A+\lambda+e_{h,i}-E_{h+1,i}}^{[A+\lambda+e_{h,i}-E_{h+1,i}]} \]
\[ = \sum_{1 \leq i \leq m+n, \lambda \in \mathbb{N}^{m+n}, a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{v}^{f(i,A+\lambda)}_{h}(a_{h,i}+1) \mathbf{x}_{A+\lambda+e_{h,i}-E_{h+1,i}}^{[A+\lambda+e_{h,i}-E_{h+1,i}]} \]
\[ = \sum_{1 \leq i \leq m+n, \lambda \in \mathbb{N}^{m+n}, a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+\lambda+e_{h,i}-E_{h+1,i}]} \]
\[ = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \]

where
\[ \Sigma_1 = \sum_{i=h+1; a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h,i}-E_{h+1,i}]} \]
\[ \Sigma_2 = \sum_{i<h; a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h,i}-E_{h+1,i}]} \]
\[ \Sigma_3 = \sum_{\lambda \in \mathbb{N}^{m+n}, a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(h+1)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A-E_{h+1,i}+E_{h+1,i}]} \]
\[ \Sigma_4 = \sum_{\lambda \in \mathbb{N}^{m+n}, a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(h+1)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h+1,i}]} \]

By (3.2.4), for \( i \geq h+1 \), we have \( f(i, A + \lambda) = f(i, A) \). Thus,
\[ \Sigma_1 = \sum_{i=h+1; a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h,i}-E_{h+1,i}]} \]
\[ = \sum_{i=h+1; a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h,i}-E_{h+1,i}]}(j), \]

and
\[ \Sigma_4 = (-1)^{\sigma_{h,m}(h+1)} \mathbf{v}^{f(h+1)+(-1)^{h,m} e_{h+1}} h j a_{h,h+1}^{h+1} \mathbf{x}^{[A+E_{h+1,i}+\lambda-E_{h+1,i}]} \]
\[ = \sum_{\lambda \in \mathbb{N}^{m+n}, \lambda_{h+1} > 0} (-1)^{\sigma_{h,m}(h+1)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h+1,i}+\lambda-E_{h+1,i}]}(j). \]

Similarly, for \( i < h \), \( f(i, A + \lambda) = f(i, A) + \lambda_h - (-1)^{h,m} \lambda_{h+1} \). So, by (2.0.1),
\[ \Sigma_2 = \sum_{i<h; a_{h+1,i} \geq 1} (-1)^{\sigma_{h,m}(i)} \mathbf{v}^{\lambda j} \mathbf{x}^{[A+E_{h,i}-E_{h+1,i}]}(j + e_h - e_{h+1}). \]
Finally, for $\Sigma_3$ when $a_{h+1,h} > 0$, since $f(h, A + \lambda) = f(h, A) - (-1)^{h,m} \lambda_{h+1}$ and

$$\nu^{\lambda_1} v^n f_{h}^{-1}[-(-1)^{h,m} \lambda_{h+1}] = \nu^n f_{h}^{-1}[-(-1)^{h,m} \lambda_{h+1}] = v^n f_{h}^{-1}[-(-1)^{h,m} \lambda_{h+1}]$$

it follows that

$$\Sigma_3 = \sum_{\lambda \in \mathbb{N}^{m+n}} (-1)^{r_{h,m}(h)} \nu^{\lambda_1} v^n f_{h}^{-1}[-(-1)^{h,m} \lambda_{h+1}] X^{[A-E_{h+1,h}+\lambda+E_{h,h}]}$$

$$= (-1)^{r_{h,m}(h)} \nu^n f_{h}^{-1} [-(-1)^{h,m} \lambda_{h+1}] X^{[A-E_{h+1,h}+\lambda+E_{h,h}]}$$

$$= (-1)^{r_{h,m}(h)} \nu^n f_{h}^{-1} [-(-1)^{h,m} \lambda_{h+1}] (A-E_{h+1,h}) (j+e_h-e_{h+1}) - (A-E_{h+1,h}) (j-e_h-e_{h+1})$$

proving (4.1.1). (Notice a cancellation for the terms associated to those $\lambda$ with $\lambda_h = 0$ when expanding the numerator of the last expression.)

The proof for the action of $F_h$ is similar. Finally, the supermodule assertion follows easily from the action formulas.

\[\square\]

5. The main result

We are now ready to prove the main result of the paper by the following.

**Lemma 5.1.** Let $U$ be an algebra over a field $\mathbb{k}$ with generators $g_i$, $1 \leq i \leq N$. Suppose $Uv$ is a cyclic $U$-module with basis $b_j = u_{j,v}$, $j \in J$ ($u_j \in U$), and trivial annihilator $\text{ann}_U(v) = 0$. Then the matrix representations $g_i b_j = \sum_{k \in J} \lambda_{i,j,k} b_k$ of the generators give rise to a presentation of $U$ by basis $\{u_j \mid j \in J\}$ and the multiplication formulas:

$$g_i u_j = \sum_{k \in J} \lambda_{i,j,k} u_k, \quad \text{for all } 1 \leq i \leq N, j \in J.$$ 

**Proof.** Since the $U$-module homomorphism $\phi : U \rightarrow Uv, u \mapsto u.v$ is an isomorphism, the basis claim is clear and so are the multiplication formulas. \[\square\]

For $\lambda = (a_{i,j}) \in M(m|n)$, let

$$A_{s,t}^+ = \sum_{i \leq s, j \geq t} a_{i,j} \text{ if } s < t, \quad A_{s,t}^- = \sum_{i \geq s, j \leq t} a_{i,j} \text{ if } s > t.$$
Following [1, §3.5] or [7, (8.0.1)], define a preorder relation on $M(m|n)$:

$$A \preceq B \iff \begin{cases} A_{s,t}^k \leq B_{s,t}^k, & \text{for all } s < t; \\ A_{s,t}^- \leq B_{s,t}^-, & \text{for all } s > t. \end{cases}$$

Note that this is a partial order relation on $M(m|n)$. The $U_v(^{\mathfrak{gl}}_m|_n)$-actions in Theorem 4.1 satisfy certain “triangular relations” relative to $\preceq$. The “lower terms” below means a linear combination of $B(j')$ with $B < \text{the leading matrix}$.

**Lemma 5.2.** Let $A = (a_{i,j}) \in M(m|n)$, $j \in \mathbb{Z}^{m+n}$, and $h, k \in [1, m + n]$. 

1. If $h < k$, $A_{h+1,k+1} = 0$, $a_{h,k} = 0$, and $a_{h+1,k} \geq a > 0$, then, for some $b \in \mathbb{Z}$,
   $$E_h^{(a)}(A) = \pm \nu^b(A + a E_{h,k} - a E_{h+1,k})(j) + \text{(lower terms)}.$$ 

2. If $h + 1 > k$, $A_{h-1,k} = 0$, $a_{h,k} = 0$, and $a_{h,k} \geq a > 0$, then, for some $c \in \mathbb{Z}$,
   $$F_h^{(a)}(A) = \pm \nu^c(A - a E_{h,k} + a E_{h+1,k})(j) + \text{(lower terms)}.$$ 

**Proof.** This follows easily from repeatedly applying the actions in Theorem 4.1. For example, the first summation in $E_h(A(0))$ contains only the terms $(A - E_{h,i} + E_{h+1,i})(0)$, for some $h + 1 < i \leq k$, and $A + E_{h,k} - E_{h+1,k} \geq A + E_{h,i} - E_{h+1,i}$ if all $i < h$ or $h + 1 < i < k$ if it occurs in the first two summations. One sees also $A + E_{h,k} - E_{h+1,k} \geq A - E_{h+1,k}, A + E_{h,h+1}$. Hence, $E_h(A(0)) = \pm \nu^b(A + E_{h,k} - E_{h+1,k})(0) + \text{(lower terms)}$. Inductively, $E_h^n(A(0)) = \pm \nu^b[a]^n(A + a E_{h,k} - a E_{h+1,k})(0) + \text{(lower terms)}$. Hence, the desired formulas follow. 

**Theorem 5.3.** The $U_v(^{\mathfrak{gl}}_m|_n)$-supermodule $\mathcal{U}(m|n)$ is a cyclic module generated by $O(0)$, where $O \in M(m|n)$ and $0 \in \mathbb{N}^{m+n}$ are the zero elements, and the module homomorphism

$$f : U_v(^{\mathfrak{gl}}_m|_n) \rightarrow \mathcal{U}(m|n), u \mapsto u.O(0). \quad (5.3.1)$$

is an isomorphism.

**Proof.** By Lemma 5.2, we may use an argument similar to that for [1, Proposition 3.9]). Consider reduced expressions of the longest elements in the symmetric groups $S_{\{1,2,\ldots,j\}}$ for $j = 2, 3, \ldots, m + n$ and $S_{\{k,k+1,\ldots,m+n\}}$ for $k = 1, 2, \ldots, m + n - 1$:

$s_{j-1}(s_{j-2}s_{j-1}) \cdots (s_1s_2 \cdots s_{j-1}), \quad s_k(s_{k+1}s_k) \cdots (s_{m+n-1} \cdots s_{k+1}s_k).$ 

For any $A = (a_{i,j}) \in M(m|n)$ and $j \in \mathbb{Z}^{m+n}$, let

$$m_j^+ = m_j^+ (A) = E_{j-1}^{(a_{j-1,j})} E_{j-2}^{(a_{j-2,j})} \cdots E_1^{(a_{1,j})} E_j^{(a_{j,j})} \cdots E_j^{(a_{j,j})},$$

$$m_k^- = m_k^- (A) = F_{k+1}^{(a_{k+1,k})} F_{k+2}^{(a_{k+2,k})} \cdots F_{m+n-1}^{(a_{m+n-1,k})} F_{m+n-1}^{(a_{m+n-1,k})} F_k^{(a_{m+n,k})} F_k^{(a_{m+n,k})},$$

and let $m_{AJ} = m_{J}^+ K^J m_j^-$, where $m_{J}^- = m_{J}^- m_2^- \cdots m_{m+n-1}^-$, $K^J = K_1^J \cdots K_{m+n}^J$, and $m_{J}^+ = m_{J}^+ \cdots m_3^+$. For example, if $m = 2, n = 2$, $A \in M(2|2)^\pm, j \in \mathbb{Z}^4$, then

$$m_{AJ} = (E_1^{(a_{21})} E_2^{(a_{31})} E_3^{(a_{41})} E_4^{(a_{41})} F_1^{(a_{11})} F_2^{(a_{11})} F_3^{(a_{11})} F_4^{(a_{11})}) (F_2^{(a_{22})} F_3^{(a_{22})} F_4^{(a_{22})} F_5^{(a_{22})}) (E_2^{(a_{22})} E_3^{(a_{22})} E_4^{(a_{22})} E_5^{(a_{22})}) (E_2^{(a_{32})} E_3^{(a_{32})} E_4^{(a_{32})} E_5^{(a_{32})}) (E_2^{(a_{42})} E_3^{(a_{42})} E_4^{(a_{42})} E_5^{(a_{42})}) (E_2^{(a_{52})} E_3^{(a_{52})} E_4^{(a_{52})} E_5^{(a_{52})}).$$
Repeatedly applying Lemma 5.2, we obtain
\[ m^{Aj}.O(0) = \pm \alpha^c A(j) + \text{(lower terms)} \quad (c \in \mathbb{Z}). \]
In fact, \( m^+_A. O(0) \) has the leading term \((a_{1,2}E_{1,2})(0), m^+_3 m^+_A. O(0) \) has the leading term \((a_{1,2}E_{1,2} + a_{1,3}E_{1,3} + a_{2,3}E_{2,3})(0), \ldots, m^+_3 m^+_A O(0) \) has the leading term \( A^+(0) \), where \( A^+ \) is the upper triangular part of \( A \). Similarly, \( K^j m^+_A. O(0) \) has the leading term \( A^+=a_{m+n, m+n-1}E_{m+n,m+n-1}(j) \), and so on.

Since \( \{A(j) \mid A \in M(m|n)^\mp, j \in \mathbb{N}^{m+n} \} \) forms a basis for \( \mathcal{U}(m|n) \) by Theorem 4.1, the triangular relation above implies that \( \{m^{Aj}.O(0) \mid A \in M(m|n)^\mp, j \in \mathbb{N}^{m+n} \} \) are linearly independent. Hence, the module homomorphism (5.3.1) must be an isomorphism.

The theorem above gives immediately a presentation for \( U_v(\mathfrak{gl}_{m|n}) \).

**Corollary 5.4.** The supergroup \( U_v(\mathfrak{gl}_{m|n}) \) contains a basis
\[ \{A(j) \mid A \in M(m|n)^\mp, j \in \mathbb{Z}^{m+n} \} \]
such that \( E_h = E_{h,h+1}(0), F_h = E_{h+1,h}(0), \) and \( K_i = O(e_i) \), and the \( U_v(\mathfrak{gl}_{m|n}) \)-action formulas given in Theorem 4.1 become the multiplication formulas of the basis elements \( A(j) \) by the generators.

**Proof.** By the module isomorphism (5.3.1) (and by abuse of notation), let \( A(j) := f^{-1} A(j) \). Since \( E_h.O(0) = E_{h,h+1}(0), F_h.O(0) = E_{h+1,h}(0), \) and \( K_i.O(0) = O(e_i) \), we have \( E_h = E_{h,h+1}(0), F_h = E_{h+1,h}(0), \) and \( K_i = O(e_i) \). The assertion now follows from Lemma 5.1. \( \square \)

**Remark 5.5.** The presentation above for \( U_v(\mathfrak{gl}_{m|n}) \) coincides with the one from [1, Lemma 5.3] (or [4, Theorem 14.8]) in the quantum \( \mathfrak{gl}_n \) case and with the one in [7, Thm 8.4] in general after a sign modification given below.

For any \( A = (a_{i,j}) \in M(m|n) \), let \( ^1 \)
\[ \overline{A} = \sum_{\substack{1 \leq i, k \leq m \\atop m < j < l \leq m+n}} a_{i,j}a_{k,l}. \quad (5.5.1) \]

**Lemma 5.6.** For \( \lambda \in \mathbb{N}^{m+n}, A = (a_{i,j}) \in M(m|n)^\mp \) and \( 1 \leq h, k \leq m + n \) with \( h < m + n \), then
\[
(1) \quad \overline{A + \lambda} = \overline{A};
\]
\[
(2) \quad \overline{A + \delta_h, m \sigma(k, A)} = \overline{A + E_h, k - E_{h+1,k} + \delta_{h,m}} \left( \sum_{\substack{i > m \\atop j \leq \min(k-1,m)}} a_{i,j} - \delta_{k,m} \sum_{\substack{i < m \\atop j > k}} a_{i,j} \right). \]

Here, \( \delta_{k,m} = 1 \) if \( k > m \) and 0 otherwise.

\( ^1 \)This number \( \overline{A} \) is different from the number \( \overline{A} \) defined in [7, (5.0.1)], where the super grading structure on the tensor space is under consideration.
Proof. If we write \( A = \left( \begin{array}{c|c} X & Q \\ \hline Q' & Y \end{array} \right) \) in blocks as in (3.2.2), then the entries involved in \( \overline{A} \) are all in \( Q \). Thus, (1) and (2) for \( h \neq m \) or \( h = m, k \leq m \) are all clear. Assume now \( h = m, k > m \). Then, by definition,
\[
\overline{A} + E_{m,k} - E_{m+1,k} = \overline{A} + \sum_{i \leq m, m < j < k} a_{i,j} + \sum_{i \leq m, j > k} a_{i,j} - (\sum_{i > m, j \leq m} a_{i,j} - \sum_{i \leq m, j > k} a_{i,j})
\]
\[
= \overline{A} + \sigma(k, A) - (\sum_{i > m, j \leq m} a_{i,j} - \sum_{i \leq m, j > k} a_{i,j}),
\]
as desired. \(\square\)

Let \( \overline{A}(j) = (-1)^A A(j) \) for all \( A \in M(m|n)^\perp, j \in \mathbb{Z}^{m+n} \).

**Theorem 5.7.** Modifying the multiplication formulas in Theorem 4.1 by using the basis \( \{ A(j) \mid A \in M(m|n)^\perp, j \in \mathbb{Z}^{m+n} \} \) for the supergroup \( U_{\nu}(gl_{m|n}) \) yields exactly the same formulas as given in [7, Thm 8.4].

**Proof.** We first observe that the generators \( E_h = E_{h,h+1}(0) = E_{h,h+1}(0) \), etc. are part of the new basis. After multiplying both sides of the multiplication formulas in Theorem 4.1 by \((-1)A\) and applying Lemma 5.6, the sign term becomes \((-1)^s(h,i)\) with
\[
s(h, i) = \delta_{h,m} \left( \sum_{s > m, t \leq \min\{i-1, m\}} a_{s,t} + \delta_{i,m} \sum_{s \leq m, t > i} a_{s,t} \right)
\]
This number \( s(h, i) \) is exactly the same number \( \varepsilon_{h,h+1}\sigma(i) \) defined in [7, (5.5.1-2)] and used in the multiplication formulas in [7, Thm 8.4]. \(\square\)

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