Observables
I : Stone Spectra

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Abstract

In this work we discuss the notion of observable - both quantum and classical - from a new point of view. In classical mechanics, an observable is represented as a function (measurable, continuous or smooth), whereas in (von Neumann’s approach to) quantum physics, an observable is represented as a bonded selfadjoint operator on Hilbert space. We will show in part II of this work that there is a common structure behind these two different concepts. If $\mathcal{R}$ is a von Neumann algebra, a selfadjoint element $A \in \mathcal{R}$ induces a continuous function $f_A : Q(\mathcal{P}(\mathcal{R})) \to \mathbb{R}$ defined on the Stone spectrum $Q(\mathcal{P}(\mathcal{R}))$ of the lattice $\mathcal{P}(\mathcal{R})$ of projections in $\mathcal{R}$. The Stone spectrum $Q(\mathcal{L})$ of a general lattice $\mathcal{L}$ is the set of maximal dual ideals in $\mathcal{L}$, equipped with a canonical topology. $Q(\mathcal{L})$ coincides with Stone’s construction if $\mathcal{L}$ is a Boolean algebra (thereby “Stone”) and is homeomorphic to the Gelfand spectrum of an abelian von Neumann algebra $\mathcal{R}$ in case of $\mathcal{L} = \mathcal{P}(\mathcal{R})$ (thereby “spectrum”). Moreover, $Q(\mathcal{L})$ appears quite naturally in the construction of the sheafification of presheaves on a lattice $\mathcal{L}$. On the other hand, measurable or continuous functions can be described by spectral families and, therefore, as functions on appropriate Stone spectra. In this first part of our work, we investigate general properties of Stone spectra and, in more detail, Stone spectra of two specific classes of lattices: $\sigma$-algebras and projection lattices $\mathcal{P}(\mathcal{R})$ of von Neumann algebras $\mathcal{R}$. 
Für Karin
Chapter 1

Introduction and Overview

*Man hat viel erreicht, wenn einen sein Leben an ein volles Fass erinnert und nicht an einen leeren Eimer.*

(Hildegunst von Mythenmetz [21])

In this work I shall develop an unusual view of the notion of observable, both in quantum and in classical physics. Following Araki [1], an observable is an equivalence class of measuring instruments, two measuring instruments being equivalent if in any “state” of the “physical system” they lead upon a “large number of measurements” to the same distribution of (relative frequencies of) results. From this concept one can “derive” that

(i) in (von Neumann’s axiomatic approach to) quantum physics, an observable is represented by a bounded selfadjoint operator $A$ acting on a Hilbert space $\mathcal{H}$, and

(ii) in classical mechanics, an observable is represented by a real valued (smooth or continuous or measurable) function on an appropriate phase space.

Here a natural question arises: is the structural difference between classical and quantum observables fundamental, or is there some background structure, showing that classical and quantum observables are on the same footing? Indeed, such a background structure exists, and I shall describe some of its features and consequences.

This work consists of three parts. In part I we introduce and study the *Stone spectrum of a lattice* $\mathbb{L}$. This is a zero dimensional Hausdorff space that is of twofold importance: it is the base space for the etale
space in the sheafification of a presheaf on \( \mathbb{L} \) and it is a generalization of Stone’s representation of Boolean algebras as Boolean algebras of sets. If, in particular, \( \mathbb{L} \) is the lattice of projections in an abelian von Neumann algebra \( \mathcal{A} \), then the Stone spectrum of \( \mathbb{L} \) is homeomorphic to the Gelfand spectrum of \( \mathcal{A} \). Furthermore, we review in part I some basic definitions and results from lattice theory and the theory of operator algebras.

In part II we show that the selfadjoint operators \( A \) in a von Neumann algebra \( \mathcal{R} \) can be represented by bounded continuous functions \( f_A : \mathcal{Q}(\mathcal{R}) \to \mathbb{R} \) on the Stone spectrum \( \mathcal{Q}(\mathcal{R}) \) of the projection lattice \( \mathcal{P}(\mathcal{R}) \) of \( \mathcal{R} \). The mapping \( A \mapsto f_A \) from \( \mathcal{R}_{sa} \) to \( C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R}) \) is injective, but it is surjective if and only if \( \mathcal{R} \) is abelian. In this case, \( f_A \) is the Gelfand transform of \( A \). The main result of part II is an abstract characterization of observable functions. In the second chapter of part II we show that continuous real valued functions on a Hausdorff space \( M \) ("classical observables") can be characterized by certain spectral families in the lattice of open subsets of \( M \). Similar results are proved for measurable functions.

In part III we come back to the presheaf perspective and use the abstract characterization of observable functions to define the restriction of selfadjoint elements of a von Neumann algebra \( \mathcal{R} \) to a von Neumann subalgebra \( \mathcal{M} \) of \( \mathcal{R} \). This leads to the notion of contextual observables as global sections of a presheaf on the semi-lattice of abelian von Neumann subalgebras of \( \mathcal{R} \).

Now we describe the content in more detail.

A continuous classical observable is a continuous function \( f : M \to \mathbb{R} \) on a (locally compact) Hausdorff space \( M \). Equivalently, \( f \) can be considered as a global section of the presheaf \( \mathcal{C}_M \) of all real valued continuous functions that are defined on some nonempty open subsets of \( M \). This situation leads to a natural generalization. The set \( \mathcal{T}(M) \) of all open subsets of \( M \) can be seen as a complete lattice \(^1\) (definition \( 2.1 \)), the lattice operations being defined by

\[
\bigvee_{k \in K} U_k := \bigcup_{k \in K} U_k, \quad \bigwedge_{k \in K} U_k := \text{int} \left( \bigcap_{k \in K} U_k \right).
\]

It is straightforward to define presheaves and complete presheaves on an

\(^1\)In the English language the word “lattice” has two different meanings. Either it is a subgroup of the additive group \( \mathbb{Z}^d \) for some \( d \in \mathbb{N} \) (this is called “Gitter” in German) or it means a partially ordered set with certain additional properties. This is called “Verband” in German. We always use “lattice” in this second meaning.

\(^2\)definition (n,k) refers to the k-th definition in chapter n. The same system of internal reference is used for propositions, theorems etc.
arbitrary complete lattice. (As is well known from topos theory \([19]\), the theory of presheaves can be built on an arbitrary category.) It turns out, however, that on some important lattices, like the lattice \(L(H)\) of all closed subspaces of a Hilbert space \(H\), there are no nontrivial complete presheaves.

It is well known that one can associate to each presheaf \(S_M\) on a topological space \(M\) a sheaf on \(M\) in the following way:

If \(S\) is a presheaf on a topological space \(M\), i.e. on the lattice \(\mathcal{T}(M)\), then the corresponding \textit{etale space} \(\mathcal{E}(S)\) of \(S\) is the disjoint union of the \textit{stalks} of \(S\) at points in \(M\):

\[
\mathcal{E}(S) = \bigsqcup_{x \in M} S_x
\]

where

\[
S_x = \lim_{\longrightarrow} S(U),
\]

the \textit{inductive limit} of the family \(\{S(U)\}_{U \in \mathcal{U}(x)}\) (here \(\mathcal{U}(x)\) denotes the set of all open neighbourhoods of \(x\)), is the stalk in \(x \in M\). The stalk \(S_x\) consists of the \textit{germs} in \(x\) of elements \(f \in S(U),\ U \in \mathcal{U}(x)\). Germs are defined quite analogously to the case of ordinary functions. Let \(\pi : \mathcal{E}(S) \to M\) be the mapping that sends a germ in \(x\) to its basepoint \(x\). \(\mathcal{E}(S)\) can be given a topology for which \(\pi\) is a local homeomorphism. It is easy to see that the local sections of \(\pi\) form a complete presheaf on \(M\). If \(S\) was already complete, then this presheaf of local sections of \(\pi\) is isomorphic to \(S\).

A first attempt to generalize this construction to the situation of a presheaf on a general lattice \(L\) is to define a suitable notion of “point in a lattice”. This can be done in a quite natural manner, and it turns out that, for \textit{regular} topological spaces \(M\), the points in \(\mathcal{T}(M)\) are of the form \(\mathcal{U}(x)\), hence correspond to the elements of \(M\). But it also turns out that some important lattices, like \(L(H)\), do not have points at all (proposition \(3.3)\)!

For the definition of an inductive limit, however, we do not need a point, like \(\mathcal{U}(x)\), but only a partially ordered set \(I\) with the property

\[
\forall \alpha, \beta \in I \ \exists \gamma \in I : \gamma \leq \alpha \ \text{and} \ \gamma \leq \beta.
\]

In other words: a \textit{filter base} \(B\) in a lattice \(L\) is sufficient. It is obvious how to define a filter base in an arbitrary lattice \(L\) (definition \(3.3)\). The set of all filter bases in \(L\) is of course a rather unstructured object. Therefore it is reasonable to consider \textit{maximal} filter bases in \(L\). (By Zorn’s lemma, every filter base is contained in a maximal filter base in \(L\).) This leads to the following
**Definition 1.1** A nonempty subset $\mathcal{B}$ of a lattice $\mathbb{L}$ is called a quasipoint in $\mathbb{L}$ if and only if it is a maximal subset of $\mathbb{L}$ with the properties

(i) $0 \notin \mathcal{B}$,

(ii) $\forall a, b \in \mathcal{B} \exists c \in \mathcal{B}: c \leq a, c \leq b$.

It is easy to see that a quasipoint is nothing else but a maximal dual ideal in $\mathbb{L}$. By the way, it is rather obvious how to generalize this definition to small categories.

In 1936 M.H. Stone (25) showed that the set $\mathcal{Q}(\mathcal{B})$ of quasipoints in a Boolean algebra $\mathcal{B}$ can be given a topology such that $\mathcal{Q}(\mathcal{B})$ is a compact zero dimensional Hausdorff space and that the Boolean algebra $\mathcal{B}$ is isomorphic to the Boolean algebra of all closed open subsets of $\mathcal{Q}(\mathcal{B})$. A basis for this topology is simply given by the sets

$\mathcal{Q}_a(\mathcal{B}) := \{ \mathcal{B} \in \mathcal{Q}(\mathcal{B}) \mid a \in \mathcal{B} \}$

where $a$ is an arbitrary element of $\mathcal{B}$.

Of course we can generalize this construction to an arbitrary lattice $\mathbb{L}$. For $a \in \mathbb{L}$ let

$\mathcal{Q}_a(\mathbb{L}) := \{ \mathcal{B} \in \mathcal{Q}(\mathbb{L}) \mid a \in \mathcal{B} \}$.

It is quite obvious from the definition of a quasipoint that

$\mathcal{Q}_{a \wedge b}(\mathbb{L}) = \mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_b(\mathbb{L})$,

$\mathcal{Q}_0(\mathbb{L}) = \emptyset$ and $\mathcal{Q}_I(\mathbb{L}) = \mathcal{Q}(\mathbb{L})$

hold. Hence $\{ \mathcal{Q}_a(\mathbb{L}) \mid a \in \mathbb{L} \}$ is a basis for a topology on $\mathcal{Q}(\mathbb{L})$. It is easy to see, using the maximality of quasipoints, that in this topology the sets $\mathcal{Q}_a(\mathbb{L})$ are open and closed. Moreover, this topology is Hausdorff, zero-dimensional, and therefore also completely regular.

**Definition 1.2** $\mathcal{Q}(\mathbb{L})$, together with the topology defined by the basis $\{ \mathcal{Q}_a(\mathbb{L}) \mid a \in \mathbb{L} \}$, is called the Stone spectrum of the lattice $\mathbb{L}$.

Then we can mimic the construction of the etale space of a presheaf on a topological space $M$ to obtain from a presheaf $\mathcal{S}$ on a lattice $\mathbb{L}$ an etale space $\mathcal{E}(\mathcal{S})$ over the Stone spectrum $\mathcal{Q}(\mathbb{L})$ and a local homeomorphism
\( \pi_S : \mathcal{E}(\mathcal{S}) \rightarrow \mathcal{Q}(\mathbb{L}) \). From the etale space \( \mathcal{E}(\mathcal{S}) \) over \( \mathcal{Q}(\mathbb{L}) \) we obtain a complete presheaf \( \mathcal{S}^\mathcal{Q} \) on the topological space \( \mathcal{Q}(\mathbb{L}) \) by

\[
\mathcal{S}^\mathcal{Q}(\mathcal{V}) := \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))
\]

where \( \mathcal{V} \subseteq \mathcal{Q}(\mathbb{L}) \) is an open set and \( \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S})) \) is the set of sections of \( \pi_S \) over \( \mathcal{V} \), i.e. of all (necessarily continuous) mappings \( s_\mathcal{V} : \mathcal{V} \rightarrow \mathcal{E}(\mathcal{S}) \) such that \( \pi_S \circ s_\mathcal{V} = id_\mathcal{V} \). If \( \mathcal{S} \) is a presheaf of modules, then \( \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S})) \) is a module, too.

**Definition 1.3** The complete presheaf \( \mathcal{S}^\mathcal{Q} \) on the Stone spectrum \( \mathcal{Q}(\mathbb{L}) \) is called the sheaf associated to the presheaf \( \mathcal{S} \) on \( \mathbb{L} \).

Of course, Stone had quite another motivation for introducing the space \( \mathcal{Q}(\mathcal{B}) \) of a Boolean algebra \( \mathcal{B} \), namely to represent \( \mathcal{B} \) as a Boolean algebra of sets. The remarkable fact is that we arrive at a generalization of Stone’s concept from a completely different point of view.

In chapter 3 we will study properties of Stone spectra in general and of some specific types of lattices. In particular, it is shown that the Stone spectrum of a \( \sigma \)-algebra \( \mathfrak{A} \) of subsets of a nonempty set \( M \) is homeomorphic to the Gelfand spectrum of the \( C^* \)-algebra \( \mathcal{F}_{\mathfrak{A}}(M, \mathbb{C}) \) of all bounded \( \mathfrak{A} \)-measurable functions \( M \rightarrow \mathbb{C} \). Quite analogously, the Stone spectrum of the projection lattice of an abelian von Neumann algebra \( \mathcal{A} \) is homeomorphic to its Gelfand spectrum. Therefore, the Stone spectrum of an arbitrary von Neumann algebra is a noncommutative generalization of the Gelfand spectrum of an abelian von Neumann algebra.

But the real meaning of the Stone spectrum \( \mathcal{Q}(\mathcal{R}) \) of a von Neumann algebra \( \mathcal{R} \) is, that any selfadjoint element \( A \in \mathcal{R} \) can be represented as a continuous function \( f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R} \). Because selfadjoint operators are the (mathematical description of) observables in Quantum Theory, we have coined the name **observable function of \( A \)** for \( f_A \). If \( A \in \mathcal{R}_{sa} \), and if \( (E_\lambda)_{\lambda \in \mathbb{R}} \) is the spectral resolution of \( A \), then \( f_A \) is defined by

\[
\forall \mathcal{B} \in \mathcal{Q}(\mathcal{R}) : f_A(\mathcal{B}) := \inf \{ \lambda \in \mathbb{R} \mid E_\lambda \in \mathcal{B} \}.
\]

As the Stone spectrum is a generalization of the Gelfand spectrum, the mapping \( A \mapsto f_A \) will be proved to be a generalization of the Gelfand transformation. We motivate the definition of \( f_A \) in Part II using the presheaf of bounded spectral families in the lattice \( \mathcal{P}(\mathcal{L}(\mathcal{H})) \) of projections in \( \mathcal{L}(\mathcal{H}) \).
In Part II we shall study the properties of observable functions for general von Neumann algebras $\mathcal{R}$. It will be shown that observable functions are continuous and that the range of $f_A$ is precisely the spectrum of the operator $A$. But the mapping $A \mapsto f_A$ from $\mathcal{R}_{sa}$, the real vectorspace of selfadjoint operators in $\mathcal{R}$, to $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, the real vectorspace of real valued bounded continuous functions on $\mathcal{Q}(\mathcal{R})$, is linear if and only if $\mathcal{R}$ is abelian. This may appear as a shortcoming of the theory, because linear structures are indispensable in the theory of operator algebras. From the physical point of view, however, the possibility of adding two given observables to obtain a new one, is merely a mathematical reflex: what is the meaning of the sum of the position and the momentum operator or the sum of two different spin operators? Perhaps a similar question appears with the completion $\mathbb{R}$ of the rationals $\mathbb{Q}$: it is indispensable for analysis, but in the light of quantum theory it is worth to debate whether the continuum is of physical significance or not ([12]).

For the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$, we give an abstract characterization of observable functions, considered as functions on projective Hilbert space $\mathcal{P}\mathcal{H}$. We generalize this characterization for arbitrary von Neumann algebras. In order to achieve this, we extend the domain of definition of $f_A$ from the Stone spectrum $\mathcal{Q}(\mathcal{R})$ to the space $\mathcal{D}(\mathcal{R})$ of all dual ideals in $\mathcal{P}(\mathcal{R})$ in an obvious manner:

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_A(\mathcal{J}) := \inf \{ \lambda \in \mathbb{R} \mid E_\lambda \in \mathcal{J} \}.$$ 

The space $\mathcal{D}(\mathcal{R})$ can be equipped with a topology in the very same way as $\mathcal{Q}(\mathcal{R})$. It is not difficult to show that observable functions $f_A$, considered as functions on $\mathcal{D}(\mathcal{R})$, have the following properties:

(i) Let $(\mathcal{J}_j)_{j \in J}$ be a family in $\mathcal{D}(\mathcal{R})$. Then

$$f_A(\bigcap_{j \in J} \mathcal{J}_j) = \sup_{j \in J} f_A(\mathcal{J}_j).$$

(ii) $f_A : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is upper semicontinuous.

Giving (i) and (ii) the status of defining properties, we get the notion of an abstract observable function.

**Definition 1.4** A function $f : \mathcal{D}(\mathcal{R}) \to \mathbb{R}$ is called an abstract observable function if it is upper semicontinuous and satisfies the intersection condition

$$f(\bigcap_{j \in J} \mathcal{J}_j) = \sup_{j \in J} f(\mathcal{J}_j).$$
for all families $(\mathcal{J}_j)_{j \in \mathcal{J}}$ in $\mathcal{D}(\mathcal{R})$.

The intersection condition implies that an abstract observable function is decreasing. Let

$$H_P := \{ Q \in \mathcal{P}(\mathcal{R}) \mid Q \geq P \}$$

be the principle dual ideal in $\mathcal{P}(\mathcal{R})$, defined by $P \in \mathcal{P}_0(\mathcal{R})$ ($\mathcal{P}_0(\mathcal{R}) := \mathcal{P}(\mathcal{R}) \setminus \{0\}$). Then the definition of abstract observable functions can be reformulated so that it does not refer to the topology of $\mathcal{D}(\mathcal{R})$:

**Remark 1.1** $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function if and only if the following two properties hold for $f$:

(i) $\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) = \inf \{ f(H_P) \mid P \in \mathcal{J} \}$,

(ii) $f(\bigcap_{j \in J} J_j) = \sup_{j \in J} f(J_j)$ for all families $(\mathcal{J}_j)_{j \in \mathcal{J}}$ in $\mathcal{D}(\mathcal{R})$.

The central result in Part II, is the following

**Theorem 1.1** Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an abstract observable function. Then there is a unique $A \in \mathcal{R}_{sa}$ such that $f = f_A$.

In fact, this is a theorem about an abstract characterization of spectral families, and an inspection of its proof shows that it also holds for spectral families in any complete orthomodular lattice.

The set of non-zero elements $a$ of a lattice $\mathbb{L}$ is in one-to-one correspondence to the set $\mathcal{D}_{pr}(\mathbb{L})$ of principal dual ideals $H_a$ in $\mathbb{L}$. Hence any bounded function $r : \mathbb{L} \setminus \{0\} \rightarrow \mathbb{R}$ induces by

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) := \inf \{ r(a) \mid a \in \mathcal{J} \}$$

a function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$. Of course $r$ must satisfy some condition so that $f$ becomes an (abstract) observable function. In a complete lattice we have

$$H_{\bigvee_{k \in K} a_k} = \bigcap_{k \in K} H_{a_k}.$$

A necessary condition for $f$ to be an observable function is therefore

$$f(H_{\bigvee_{k \in K} a_k}) = \sup_{k \in K} f(H_{a_k}).$$

If $\mathbb{L}$ is a complete lattice, this requirement leads to the condition that

$$r(\bigvee_{k \in K} a_k) = \sup_{k \in K} r(a_k).$$
must be satisfied for every family \((a_k)_{k \in \mathbb{K}}\) in \(L \setminus \{0\}\). In this case \(r\) is called completely increasing. If this latter condition is fulfilled, then
\[
f(H_a) = r(a)
\]
for all \(a \in L \setminus \{0\}\) and \(f\) is an observable function. Conversely, if \(f : D(L) \to \mathbb{R}\) is an observable function, then \(r_f(a) := f(H_a)\) defines a completely increasing function \(r_f : L \setminus \{0\} \to \mathbb{R}\). This gives a bijection \(f \mapsto r_f\) between observable functions and completely increasing functions.

If \(M\) is a nonempty set and \(\mathcal{A}\) is a \(\sigma\)-algebra of subsets of \(M\), then every \(\mathcal{A}\)-measurable function \(g : M \to \mathbb{R}\) defines a spectral family \(\sigma_g\) in \(\mathcal{A}\) by
\[
\forall \lambda \in \mathbb{R} : \sigma_g(\lambda) := \left[ \frac{1}{g}(\lambda), \infty \right].
\]
Conversely, any spectral family \(\sigma\) in \(\mathcal{A}\) induces a function \(g_\sigma : M \to \mathbb{R}\) by
\[
\forall x \in M : g_\sigma(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\}.
\]
\(g_\sigma\) is \(\mathcal{A}\)-measurable because
\[
\forall \lambda \in \mathbb{R} : \frac{1}{g_\sigma}(\lambda) = \sigma(\lambda).
\]
Moreover, we will show that these constructions are inverse to each other, i.e.
\[
g_{\sigma_g} = g \quad \text{and} \quad \sigma_{g_\sigma} = \sigma
\]
for all \(\mathcal{A}\)-measurable functions \(g : M \to \mathbb{R}\) and all spectral families \(\sigma\) in \(\mathcal{A}\). On the other hand, every \(\mathcal{A}\)-measurable function \(g : M \to \mathbb{R}\) induces a function \(f_g : Q(\mathcal{A}) \to \mathbb{R}\) on the Stone spectrum \(Q(\mathcal{A})\) of \(\mathcal{A}\), defined by
\[
f_g(\mathcal{B}) := \inf\{\lambda \in \mathbb{R} \mid \sigma_g(\lambda) \in \mathcal{B}\}.
\]
We will show that \(f_g\) is the Gelfand transformation of \(g\):

**Theorem 1.2** Let \(\mathcal{A}(M)\) be a \(\sigma\)-algebra of subsets of a nonempty set \(M\) and let \(\mathcal{F}_{\mathcal{A}(M)}(M, \mathbb{C})\) be the \(C^*\)-algebra of all bounded \(\mathcal{A}(M)\)-measurable functions \(g : M \to \mathbb{C}\). Then the Gelfand spectrum \(\Omega(\mathcal{F}_{\mathcal{A}(M)}(M, \mathbb{C}))\) of \(\mathcal{F}_{\mathcal{A}(M)}(M, \mathbb{C})\) is homeomorphic to the Stone spectrum \(Q(\mathcal{A}(M))\) of \(\mathcal{A}(M)\) and the restriction of the Gelfand transformation to \(\mathcal{F}_{\mathcal{A}(M)}(M, \mathbb{R})\) is given, up to the homeomorphism \(Q(\mathcal{A}(M)) \cong \Omega(\mathcal{F}_{\mathcal{A}(M)}(M, \mathbb{C}))\), by \(g \mapsto f_g\), where
\[
f_g(\mathcal{B}) = \inf\{\lambda \in \mathbb{R} \mid \frac{1}{g}(\lambda) \in \mathcal{B}\}
\]
for all \(\mathcal{B} \in Q(\mathcal{A}(M))\).
Moreover, we will generalize this theorem to $\sigma$-algebras of the form $\mathcal{A}(M)/\mathcal{I}$, where $\mathcal{I}$ is a $\sigma$-ideal in $\mathcal{A}(M)$, which is, by a theorem of Loomis and Sikorski (24), up to isomorphism the general form of $\sigma$-algebras.

If $M$ is a Hausdorff space, then the interplay between continuous functions $f : M \to \mathbb{R}$ and spectral families $\sigma$ in the lattice $\mathcal{T}(M)$ of all open subsets of $M$ is not as simple as in the measurable case. This is due to the fact that for a family $(U_k)_{k \in \mathbb{K}}$ the infimum $\bigwedge_{k \in \mathbb{K}} U_k$ may the empty set but $\bigcap_{k \in \mathbb{K}} U_k$ is not empty.

Every continuous function $f : M \to \mathbb{R}$ defines a spectral family $\sigma_f$ in $\mathcal{T}(M)$ by

$$\sigma_f(\lambda) := \text{int} f([-\infty, \lambda]).$$

Conversely, if a spectral family $\sigma$ in $\mathcal{T}(M)$ is given, then

$$f_{\sigma}(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\}$$

is not necessarily defined for all $x \in M$. This leads to the following

**Definition 1.5** Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family in $\mathcal{T}(M)$. Then

$$\mathcal{D}(\sigma) := \{x \in M \mid \exists \lambda \in \mathbb{R} : x \notin \sigma(\lambda)\}$$

is called the admissible domain of $\sigma$.

It is easy to see that $\mathcal{D}(\sigma)$ is dense in $M$.

**Definition 1.6** Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a spectral family with admissible domain $\mathcal{D}(\sigma)$. Then the function $f_{\sigma} : \mathcal{D}(\sigma) \to \mathbb{R}$, defined by

$$\forall x \in \mathcal{D}(\sigma) : f_{\sigma}(x) := \inf\{\lambda \in \mathbb{R} \mid x \in \sigma(\lambda)\},$$

is called the function induced by $\sigma$.

If $\sigma = \sigma_f$ for a continuous function $f : M \to \mathbb{R}$, then $\sigma$ is regular in the following sense:

**Definition 1.7** A spectral family $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is called regular if

$$\forall \lambda < \mu : \overline{\sigma(\lambda)} \subseteq \sigma(\mu)$$

holds.

If $\sigma$ is a regular spectral family in $\mathcal{T}(M)$, then each $\sigma(\lambda)$ is a regular open set, i.e. it is the interior of its closure. Thus a regular spectral family has values in the complete Boolean algebra $\mathcal{T}_r(M)$ of regular open subsets of $M$. 


**Theorem 1.3** Let $M$ be a Hausdorff space. Then every continuous function $f : M \to \mathbb{R}$ induces a regular spectral family $\sigma_f : \mathbb{R} \to \mathcal{T}(M)$ by

$$\forall \lambda \in \mathbb{R} : \sigma_f(\lambda) := \text{int}(f([-\infty, \lambda])).$$

The admissible domain $\mathcal{D}(\sigma_f)$ equals $M$ and the function $f_{\sigma_f} : M \to \mathbb{R}$ induced by $\sigma_f$ is $f$. Conversely, if $\sigma : \mathbb{R} \to \mathcal{T}(M)$ is a regular spectral family, then the admissible domain of $\sigma$ is open and dense in $M$, the function $f_{\sigma} : \mathcal{D}(\sigma) \to \mathbb{R}$ induced by $\sigma$ is continuous and the induced spectral family $\sigma_{f_{\sigma}}$ in $\mathcal{T}(\mathcal{D}(\sigma))$ is the restriction of $\sigma$ to the admissible domain $\mathcal{D}(\sigma)$:

$$\forall \lambda \in \mathbb{R} : \sigma_{f_{\sigma}}(\lambda) = \sigma(\lambda) \cap \mathcal{D}(\sigma).$$

One may wonder why we have defined the function, that is induced by a spectral family $\sigma$, on $M$ and not on the Stone spectrum $\mathcal{Q}(\mathcal{T}(M))$. A quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{T}(M))$ is called finite if $\bigcap_{U \in \mathfrak{B}} U \neq \emptyset$. If $\mathfrak{B}$ is finite, then this intersection consists of a single element $x_\mathfrak{B} \in M$, and we call $\mathfrak{B}$ a quasipoint over $x_\mathfrak{B}$. Note that for a compact space $M$, all quasipoints are finite. Moreover, one can show that for compact $M$, the mapping $pt : \mathfrak{B} \mapsto x_\mathfrak{B}$ from $\mathcal{Q}(\mathcal{T}(M))$ onto $M$ is continuous and identifying.

**Remark 1.2** Let $\sigma : \mathbb{R} \to \mathcal{T}(M)$ be a regular spectral family and let $x \in \mathcal{D}(\sigma)$. Then for all quasipoints $\mathfrak{B}_x \in \mathcal{Q}(\mathcal{T}(M))$ over $x$ we have

$$f_{\sigma}(\mathfrak{B}_x) = f_{\sigma}(x).$$

Therefore, if $M$ is compact, it makes no difference whether we define $f_{\sigma}$ in $M$ or in $\mathcal{Q}(\mathcal{T}(M))$.

In part III we come back to the presheaf perspective. Basic to that is the semilattice $\mathfrak{A}(\mathcal{R})$ of all abelian von Neumann subalgebras of a von Neumann algebra $\mathcal{R}$. It can be seen also as the set of objects of a (small) category $\mathcal{CON}(\mathcal{R})$ whose morphisms are simply the inclusion maps. It is called the context category of the von Neumann algebra $\mathcal{R}$. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{A}(\mathcal{R})$ such that $\mathcal{A} \subseteq \mathcal{B}$, we can define, using the results of part II, a restriction map

$$\varrho^\mathcal{B}_\mathcal{A} : \mathcal{B} \to \mathcal{A}$$

in the following way. Identify $B \in \mathcal{B}_{sa}$ with the corresponding completely increasing function $r_B : \mathcal{P}_0(\mathcal{B}) \to \mathbb{R}$. Then $r_{\varrho^\mathcal{B}_\mathcal{A}} := r_{B \mid \mathcal{P}_0(\mathcal{A})} : \mathcal{P}_0(\mathcal{A}) \to \mathbb{R}$
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is completely increasing and, therefore, corresponds to a unique $g^B_A B \in A_{sa}$. For an arbitrary $B \in B$ we define $g^B_A B := g^B_A B_1 + i g^B_A B_2$, where $B = B_1 + i B_2$ is the decomposition of $B$ into selfadjoint parts. Obviously, the abelian von Neumann subalgebras of $\mathcal{R}$, together with the restriction maps $g^B_A$, form a presheaf

$$\Theta_\mathcal{R} := (\Theta(A), g^B_A A \subseteq B) := (A, g^B_A A \subseteq B)$$

on $\mathcal{CON}(\mathcal{R})$. We call $\Theta_\mathcal{R}$ the tautological presheaf on $\mathcal{CON}(\mathcal{R})$. The presheaf

$$\Theta_{\mathcal{R}_{sa}} := (A_{sa}, g^B_A A \subseteq B)$$

on $\mathcal{CON}(\mathcal{R})$ is a sub-presheaf of $\Theta_\mathcal{R}$, which is called the real tautological presheaf on $\mathcal{CON}(\mathcal{R})$ because

$$\Theta_\mathcal{R} = \Theta_{\mathcal{R}_{sa}} \oplus i \Theta_{\mathcal{R}_{sa}}.$$

Of course the definition of restricting an operator to a von Neumann subalgebra works for any pair $(\mathcal{M}, \mathcal{N})$ of von Neumann algebras such that $\mathcal{M} \subseteq \mathcal{N}$. We will interpret the restriction $g^\mathcal{R}_\mathcal{M} A$ of $A \in \mathcal{R}$ to a von Neumann subalgebra $\mathcal{M}$ of $\mathcal{R}$ as a coarse graining of $A$. This can already be seen in the following example: if $P \in \mathcal{R}$ is a projection, then $g^\mathcal{R}_\mathcal{M} P = s_\mathcal{M}(P)$, where $s_\mathcal{M}(P) := \bigwedge \{ Q \in \mathcal{P}(\mathcal{M}) \mid Q \geq P \}$ is the $\mathcal{M}$-support of $P$.

If $A$ is an observable, i.e. $A \in \mathcal{R}_{sa}$, then the family $(A_A)_{A \in \mathcal{A}(\mathcal{R})}$, defined by $A_A := g^\mathcal{R}_\mathcal{M} A$, is a global section of the presheaf $\Theta_{\mathcal{R}_{sa}}$. This means that the family $(A_A)_{A \in \mathcal{A}(\mathcal{R})}$ satisfies the conditions

$$A_A = g^B_A A_B \quad \text{if} \quad A \subseteq B.$$

Here the question arises whether every global section of $\Theta_{\mathcal{R}_{sa}}$ is induced by an operator $A \in \mathcal{R}_{sa}$. This is trivially the case if $\mathcal{R}$ is abelian, but not for $\mathcal{R} = \mathcal{L}(\mathbb{C}^2)$. However, it is not only the notorious type $I_2$ exception: we will present a generalizable example for $\mathcal{L}(\mathbb{C}^3)$. The reason for that phenomenon lies in the following result:

**Proposition 1.1** Let $\mathcal{R}$ be a von Neumann algebra. There is a one-to-one correspondence between global sections of the real tautological presheaf $\Theta_{\mathcal{R}_{sa}}$ and functions $f : \mathcal{P}_0(\mathcal{R}) \to \mathbb{R}$ that satisfy

(i) $f(\bigvee_{k \in \mathbb{K}} P_k) = \sup_{k \in \mathbb{K}} f(P_k)$ for all commuting families $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$,

(ii) $f|_{\mathcal{P}_0(\mathcal{R}) \cap A}$ is bounded for all $A \in \mathcal{A}(\mathcal{R})$.

Therefore, if one takes contextuality in quantum physics serious, it is natural to generalize the notion of quantum observable:
Definition 1.8 Let $\mathcal{R}$ be a von Neumann algebra. The global sections of the real tautological presheaf $\Theta_{\mathcal{R}_{sa}}$ are called **contextual observables**.

Moreover, we will discuss in part III some applications of our theory to positive operator valued measures.

To finish this introduction, I like to stress that the development of the theory presented here was *motivated* by conceptual notions of physics, it has been *guided*, however, by mathematical naturalness. This is not only due to the fact that I am a mathematician (although with strong inclination to physics), but mainly to my belief that the ultimate theory of physics will be in good mathematical shape.
Chapter 2

Preliminaries

In this chapter, we present the basic definitions and results from lattice theory and the theory of operator algebras that we shall use throughout this work. We omit proofs for most of the presented results because they can be found in the standard literature. An exception is section 2.2 which contains complete proofs.

2.1 Lattices

Definition 2.1 A lattice is a partially ordered set \((\mathbb{L}, \leq)\) such that any two elements \(a, b \in \mathbb{L}\) possess a maximum \(a \vee b \in \mathbb{L}\) and a minimum \(a \wedge b \in \mathbb{L}\). Let \(m\) be an infinite cardinal number. The lattice \(\mathbb{L}\) is called \(m\)-complete, if every family \((a_i)_{i \in I}\) has a supremum \(\bigvee_{i \in I} a_i\) and an infimum \(\bigwedge_{i \in I} a_i\) in \(\mathbb{L}\), provided that \(\#I \leq m\) holds. A lattice \(\mathbb{L}\) is simply called complete, if every family \((a_i)_{i \in I}\) in \(\mathbb{L}\) (without any restriction of the cardinality of \(I\)) has a supremum and an infimum in \(\mathbb{L}\). \(\mathbb{L}\) is said to be boundedly complete if every bounded family in \(\mathbb{L}\) has a supremum and an infimum.

If a lattice has a zero element 0 (i.e. \(\forall a \in \mathbb{L}: 0 \leq a\)) and a unit element 1 (i.e. \(\forall a \in \mathbb{L}: a \leq 1\)), completeness and bounded completeness are the same. A lattice \(\mathbb{L}\) is called distributive if the two distributive laws

\[
\begin{align*}
a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c) \\
a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c)
\end{align*}
\]

hold for all elements \(a, b, c \in \mathbb{L}\).

In fact it is an easy exercise to show that if one of these distributive laws is satisfied for all \(a, b, c \in \mathbb{L}\), so is the other.
\( \bigvee_{i \in I} a_i \) is characterized by the following universal property:

(i) \( \forall j \in I : a_j \leq \bigvee_{i \in I} a_i \)

(ii) \( \forall c \in \mathbb{L} : ( ( \forall i \in I : a_i \leq c ) \Rightarrow \bigvee_i a_i \leq c ) \).

An analogous universal property characterizes the infimum \( \bigwedge_i a_i \).

Note that if \( \mathbb{L} \) is a distributive complete lattice, then in general

\[
 a \land \left( \bigvee_{i \in I} b_i \right) \neq \bigvee_{i \in I} (a \land b_i),
\]

so completeness and distributivity together do not imply complete distributivity!

Let us give some important examples.

**Example 2.1** Let \( M \) be a topological space and \( \mathcal{T}(M) \) the topology of \( M \), i.e. the set of all open subsets of \( M \). \( \mathcal{T}(M) \) is a completely distributive lattice. The supremum of a family \( (U_i)_{i \in I} \) of open subsets \( U_i \) of \( M \) is given by

\[
 \bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i,
\]

the infimum, however, is given by

\[
 \bigwedge_{i \in I} U_i = \text{int}(\bigcap_{i \in I} U_i),
\]

where \( \text{int}N \) denotes the interior of a subset \( N \) of \( M \).

**Example 2.2** If \( U \in \mathcal{T}(M) \), then always

\[
 U \subseteq \text{int}\bar{U},
\]

but \( U \neq \text{int}\bar{U} \) in general. \( U \) fails to be the interior of its adherence \( \bar{U} \), if for example \( U \) has a “crack” or is obtained from an open set \( V \) by deleting some points of \( V \).

We call \( U \) a **regular open set**, if \( U = \text{int}\bar{U} \). Each \( U \in \mathcal{T}(M) \) has a **pseudocomplement**, defined by

\[
 U^c := M \setminus \bar{U},
\]

and together with the operation of pseudocomplementation \( \mathcal{T}(M) \) is a **Heyting algebra**:

\[
 \forall U \in \mathcal{T}(M) : U^{ccc} = U^c.
\]
$U \in \mathcal{T}(M)$ is regular if and only if $U = U^{cc}$. Let $\mathcal{T}_r(M)$ be the set of regular open subsets of $M$. If $U, V \in \mathcal{T}_r(M)$, then also $U \cap V \in \mathcal{T}_r(M)$. The union of two regular open sets, however, is not regular in general. Therefore one is forced to define the maximum of two elements $U, V \in \mathcal{T}_r(M)$ as

$$U \vee V := (U \cup V)^{cc}.$$  

It is then easy to see that $\mathcal{T}_r(M)$ is a distributive complete lattice with the lattice operations

$$U \wedge V := U \cap V, \quad U \vee V := (U \cup V)^{cc}.$$  

The pseudocomplement on $\mathcal{T}(M)$, restricted to $\mathcal{T}_r(M)$, gives an orthocomplement $U \mapsto U^c$ on $\mathcal{T}_r(M)$:

$$U^{cc} = U, \quad U^c \vee U = M, \quad U^c \wedge U = \emptyset, \quad (U \wedge V)^c = U^c \vee V^c$$

for all $U, V \in \mathcal{T}_r(M)$. Thus $\mathcal{T}_r(M)$ is a complete Boolean lattice i.e. a complete Boolean algebra.

**Example 2.3** Let $M$ be a topological space and $\mathcal{B}(M)$ the set of Borel subsets of $M$. $\mathcal{B}(M)$ together with the usual set theoretic operations is a distributive $\aleph_0$-complete Boolean lattice, usually called the $\sigma$-algebra of Borel subsets of $M$.

**Example 2.4** Let $\mathcal{H}$ be a (complex) Hilbert space and $\mathcal{L}(\mathcal{H})$ the set of all closed subspaces of $\mathcal{H}$. $\mathcal{L}(\mathcal{H})$ is a complete lattice with lattice operations defined by

$$U \wedge V := U \cap V$$

$$U \vee V := (U + V)^\perp$$

$$U^\perp := \text{orthogonal complement of } U \text{ in } \mathcal{H}.$$  

Contrary to the foregoing examples, $\mathcal{L}(\mathcal{H})$ is highly non-distributive! Of course $\mathcal{L}(\mathcal{H})$ is isomorphic to the lattice $\mathcal{P}(\mathcal{L}(\mathcal{H})) := \{P_U \mid U \in \mathcal{L}(\mathcal{H})\}$ of all orthogonal projections in the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators of $\mathcal{H}$. The non-distributivity of $\mathcal{L}(\mathcal{H})$ is equivalent to the fact that two projections $P_U, P_V \in \mathcal{P}(\mathcal{L}(\mathcal{H}))$ do not commute in general. $\mathcal{L}(\mathcal{H})$ is the basic lattice of quantum mechanics (§23). It represents “quantum logic” in contrast to classical “Boolean logic”.
2.2 Orthomodular Lattices

The most prominent examples of orthomodular lattices are Boolean algebras and the lattice of projections in a von Neumann algebra. A less popular example is the lattice of causally closed subsets of a spacetime ([4]). For the sake of completeness we give here the necessary definitions and prove the results we will use. Of course neither the results nor, probably, the presented proofs are new. Our general references are [2] and [18].

Definition 2.2 Let $\mathbb{L}$ be a lattice with a minimal element 0 and a maximal element 1. An orthocomplement for $\mathbb{L}$ is a mapping $\perp : \mathbb{L} \to \mathbb{L}$, $a \mapsto a^\perp$ with the following properties:

(i) $a \land a^\perp = 0$, $a \lor a^\perp = 1$ for all $a \in \mathbb{L}$,

(ii) $(a \land b)^\perp = a^\perp \lor b^\perp$, $(a \lor b)^\perp = a^\perp \land b^\perp$ for all $a, b \in \mathbb{L}$,

(iii) $a^{\perp \perp} = a$ for all $a \in \mathbb{L}$.

$\mathbb{L}$ together with an orthocomplement $\perp$ is called an orthocomplemented lattice (or an ortholattice for short).

Immediate consequences of these definitions are

Remark 2.1

(1) $0^\perp = 1$,

(2) $1^\perp = 0$,

(3) $a \leq b \iff b^\perp \leq a^\perp$, and, if $\mathbb{L}$ is complete,

(4) $\left( \bigwedge_{k \in K} a_k \right)^\perp = \bigvee_{k \in K} a_k^\perp$ and $\left( \bigvee_{k \in K} a_k \right)^\perp = \bigwedge_{k \in K} a_k^\perp$ for all families $(a_k)_{k \in K}$ in $\mathbb{L}$.

Proof: (1) and (2) follow from $a = (a^\perp \lor 0)^\perp = a \land 0^\perp$ and $a = (a^\perp \land 1)^\perp = a \lor 1^\perp$, (3) follows from $a \leq b \iff a \land b = a \iff a \lor b = b$, and (4) from the universal property of meet and join:

$$\bigvee_{i} a_i^\perp \geq a_k^\perp \implies \left( \bigvee_{i} a_i^\perp \right)^\perp \leq a_k$$

$$\implies \left( \bigvee_{i} a_i^\perp \right)^\perp \leq \bigwedge_{k} a_k$$

$$\implies \bigvee_{i} a_i^\perp \geq \left( \bigwedge_{k} a_k \right)^\perp$$
and
\[
\bigwedge_i a_i \leq a_k \implies (\bigwedge_i a_i) \perp \geq a_k \perp
\implies (\bigwedge_i a_i) \perp \geq \bigvee_k a_k. \quad \square
\]

**Definition 2.3** A lattice \( \mathbb{L} \) is called modular if
\[
\forall a, b, c \in \mathbb{L} : (b \leq a \implies a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c))
\]
holds.

The projection lattice \( \mathcal{P}(\mathcal{R}) \) of a von Neumann algebra \( \mathcal{R} \) is modular if \( \mathcal{R} \) is finite (26). In general \( \mathcal{P}(\mathcal{R}) \) is only orthomodular:

**Definition 2.4** An ortholattice \( \mathbb{L} \) is called orthomodular if
\[
\forall a, b, c \in \mathbb{L} : (b \leq a \text{ and } c \leq a^\perp \implies a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c))
\]
holds.

It is easy to see that the orthomodular law is equivalent to
\[
\forall a, b \in \mathbb{L} : (b \leq a \implies b = a \wedge (a^\perp \vee b)).
\]

An important consequence of orthomodularity is that an element \( d \in \mathbb{L} \) has at most one decomposition \( d = b \vee c \) with \( b \leq a \) and \( c \leq a^\perp \).

One can define commutativity of an element \( a \) with an element \( b \) in an arbitrary ortholattice \( \mathbb{L} \) by the relation
\[
aCb :\iff a = (a \wedge b) \vee (a \wedge b^\perp).
\]

However, the relation \( \mathcal{C} \) is not symmetric in general. In fact the symmetry of \( \mathcal{C} \) is equivalent to the orthomodularity of \( \mathbb{L} \):

**Proposition 2.1** (Nakamura) The commutativity relation \( \mathcal{C} \) in \( \mathbb{L} \) is symmetric if and only if \( \mathbb{L} \) is orthomodular.

**Proof:** If \( \mathbb{L} \) is orthomodular and \( (a, b) \in \mathcal{C} \), i.e. \( a = (a \wedge b) \vee (a \wedge b^\perp) \), then \( a^\perp = (a \wedge b)^\perp \wedge (a^\perp \vee b) \), hence
\[
a^\perp \wedge b = (a \wedge b)^\perp \wedge (a^\perp \vee b) \wedge b = (a \wedge b)^\perp \wedge b,
\]
\[(a \land b) \lor (a^\perp \land b) = (a \land b) \lor (a \land b^\perp).\]
a \land b \leq b \text{ implies } b^\perp \leq (a \land b)^\perp. \text{ Then we obtain, using the orthomodularity of } \mathbb{L},
\[(a \land b)^\perp \land ((a \land b) \lor b^\perp) = b^\perp,
\]
hence
\[b = (a \land b) \lor (a^\perp \land b),\]
\[\text{i.e. } (b,a) \in \mathcal{C}.\]
Conversely, let \(\mathcal{C}\) be symmetric. If \(b \leq a\) then \((b,a) \in \mathcal{C}\) and therefore \((a,b) \in \mathcal{C}\). Hence \((a,b^\perp)\) and so \((b^\perp,a)\) and \((b^\perp,a^\perp)\) also belong to \(\mathcal{C}\). We obtain
\[(a \land (a^\perp \lor b))^\perp = a^\perp \lor (a \land b^\perp) = (a^\perp \land b^\perp) \lor (a \land b^\perp) = b^\perp.\]
This shows that \(\mathbb{L}\) is orthomodular. \(\square\)

\textbf{Definition 2.5} Let \(\mathbb{L}\) be an orthomodular lattice and let \(M\) be a (nonvoid) subset of \(\mathbb{L}\). Then
\[M^\mathcal{C} := \{b \in \mathbb{L} \mid \forall a \in M : a \mathcal{C} b\}\]
is called the commutant of \(M\) in \(\mathbb{L}\). If \(M = \{a\}\), we simply write \(a^\mathcal{C}\) instead of \(\{a\}^\mathcal{C}\).

\textbf{Proposition 2.2} Let \(M\) be a subset of an orthomodular lattice \(\mathbb{L}\). Then \(M^\mathcal{C}\) is a lattice. If \(\mathbb{L}\) is complete then \(M^\mathcal{C}\) is complete, too.

\textit{Proof:} Without loss of generality we can assume that \(M = \{a\}\). Obviously 0, 1 \(\in a^\mathcal{C}\) and \(b^\perp \in a^\mathcal{C}\) for \(b \in \mathcal{C}\). Let \((b_k)_{k \in \mathbb{K}}\) be an arbitrary family in \(a^\mathcal{C}\), where we assume that \(\mathbb{K}\) is finite if \(\mathbb{L}\) is not complete. Then
\[
\bigvee_k b_k = \bigvee_k ((b_k \land a) \lor (b_k \land a^\perp))
\]
\[
= \bigvee_k (b_k \land a) \lor (\bigvee_k (b_k \land a^\perp))
\]
\[
\leq (\bigvee_k b_k) \land a \lor (\bigvee_k b_k \land a^\perp)
\]
\[
\leq \bigvee_k b_k.
\]
Hence
\[
((\bigvee_k b_k) \land a) \lor ((\bigvee_k b_k) \land a^\perp) = \bigvee_k b_k
\]
and, by orthomodularity,
\[
(\bigvee_k b_k) \land a = \bigvee_k (b_k \land a).
\]
The case of the meet \(\bigwedge_k b_k\) reduces to that of the join because of \(\bigwedge_k b_k = (\bigvee_k b_k^\perp)^\perp\). □

Remark 2.2 In the course of the foregoing proof we have also shown that \(M\) distributes over \(M^c\), i.e.
\[
\forall a \in M, (b_k)_{k \in K} \text{ in } M^c : a \land (\bigvee_k b_k) = \bigvee_k (a \land b_k).
\]

Lemma 2.1 Let \(M\) and \(N\) be subsets of an orthomodular lattice \(\mathbb{L}\). Then
\[
(i) \ M \subseteq N \implies N^c \subseteq M^c,
(ii) \ M \subseteq M^{ccc},
(iii) \ M^c = M^{ccc}.
\]

Proof: Property \((i)\) is obvious from the definition of \(C\). Properties \((ii)\) and \((iii)\) are essentially consequences of the symmetry of \(C\): If \(a \in M\) and \(b \in M^c\), then \(bCa\), hence \(aCb\) and therefore \(a \in M^{cc}\). This proves \((ii)\). From \((i)\) and \((ii)\) we obtain \(M^{ccc} \subseteq M^c\) and \((ii)\) implies the opposite inclusion. □

Proposition 2.3 Let \(\mathbb{L}\) be a complete orthomodular lattice and \(M \subseteq \mathbb{L}\) a sublattice. Then
\[
(i) \ M \text{ is distributive if and only if } M \subseteq M^c,
(ii) \ M \text{ is maximal distributive if and only if } M = M^c,
(iii) \ A \text{ maximal distributive sublattice is complete.}
\]
Proof: (i) is obvious from the definition of $C$. Let $\mathcal{M}$ be maximal distributive and $a \in \mathcal{M}^C$. Then $\{a\} \cup \mathcal{M} \subseteq (\{a\} \cup \mathcal{M})^C$ and therefore, by proposition 2.2, $\langle \{a\} \cup \mathcal{M} \rangle \subseteq (\{a\} \cup \mathcal{M})^C$. (If $M$ is a subset of $L$ then $\langle M \rangle$ denotes the sublattice generated by $M$.) Hence $a \in \mathcal{M}$ since $\mathcal{M}$ is a maximal distributive sublattice. Conversely, let $\mathcal{M} = \mathcal{M}^C$ and let $\mathcal{M}_{\text{max}}$ be a maximal distributive sublattice of $L$ that contains $\mathcal{M}$. Then $\mathcal{M}_{\text{max}} = \mathcal{M}_{\text{max}}^C$ and therefore $\mathcal{M}_{\text{max}} = \mathcal{M}_{\text{max}}^C \subseteq \mathcal{M}^C = \mathcal{M}$, i.e. $\mathcal{M}$ is maximal. (iii) follows from (ii) and proposition 2.2. □

Definition 2.6 A maximal distributive sublattice of a complete orthomodular lattice $L$ is called a Boolean sector of $L$.

Usually a maximal distributive sublattice of a complete orthomodular lattice $L$ is called a block. It will become clear in the next chapter why we deviate from common use.

2.3 Operator Algebras

We do not intend to give a real introduction into the subject of operator algebras here. We only want to fix our notations and to present some of the basic definitions and results in order to make this work more self-contained. Moreover, we restrict our discussion to operator algebras that are contained in $L(H)$, the algebra of all bounded linear operators of some (complex) Hilbert space $H$. By the GNS-construction ([14]), this is no real loss of generality. Our standard references are [14, 15, 16, 17, 26].

In what follows, $H$ denotes an arbitrary Hilbert space.

Definition 2.7 An operator algebra is a subalgebra $\mathcal{R}$ of the algebra $L(H)$ of all bounded linear operators $H \to H$ such that $T^* \in \mathcal{R}$ whenever $T \in \mathcal{R}$. $\mathcal{R}$ is called a $C^*$-algebra if it closed in the norm topology, and a von Neumann algebra if it is closed in the weak operator topology of $L(H)$.

Remark 2.3 More generally, an involutory algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$ that possesses an involution, i.e. a conjugate-linear mapping $*: \mathcal{A} \to \mathcal{A}$, $a \mapsto a^*$ which satisfies

(i) $a^{**} = a$ and

(ii) $(ab)^* = b^* a^*$
for all $a, b \in \mathfrak{A}$. A Banach algebra $\mathfrak{A}$ with an involution $*$ is called an abstract $C^*$-algebra if the norm of $\mathfrak{A}$ satisfies
\[
\forall a \in \mathfrak{A} : \ |a^*a| = |a|^2.
\]
The involution $*$ of an abstract $C^*$-algebra is necessarily isometric. A homomorphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ between involutary algebras $\mathfrak{A}$ and $\mathfrak{B}$ is called a $*$-homomorphism if $\Phi(a^*) = \Phi(a)^*$ holds for every $a \in \mathfrak{A}$. A $*$-homomorphism between $C^*$-algebras is continuous with norm less or equal to 1 and a $*$-isomorphism is necessarily isometric. An abstract $C^*$-algebra is, by the Gelfand-Neumark theorem ([14]), $*$-isomorphic to a (concrete) $C^*$-algebra $\mathfrak{R}$ in $\mathcal{L}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$.

Also von Neumann algebras have an abstract description: A $W^*$-algebra is a $C^*$-algebra that is the topological dual of a Banach space. $\mathcal{L}(\mathcal{H})$, for example, is a $W^*$-algebra, since $\mathcal{L}(\mathcal{H})$ is isomorphic to the topological dual of $\mathcal{L}_1(\mathcal{H})$, the Banach space of trace-class operators in $\mathcal{L}(\mathcal{H})$. It can be shown ([13]) that any $W^*$-algebra $\mathfrak{C}$ is $*$-isomorphic to a von Neumann algebra $\mathfrak{R}$ in $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

In contrast to general $C^*$-algebras, a von Neumann algebra always contains enough projections to be generated by them. We denote by $\mathcal{P}(\mathfrak{R})$ the set of projections in $\mathfrak{R}$. $\mathcal{P}(\mathfrak{R})$ is a complete lattice, hence there is a unique maximal projection $P_I$ in $\mathfrak{R}$. It is an immediate consequence of the spectral theorem that
\[
AP_I = P_I A = A
\]holds for all $A \in \mathfrak{R}$. This means that $P_I$ is the unit element of $\mathfrak{R}$ and that the closed subspace $\mathcal{H}_I := P_I \mathcal{H}$ of $\mathcal{H}$ is $\mathfrak{R}$-invariant. Therefore, we can always assume that a von Neumann algebra $\mathfrak{R} \subseteq \mathcal{L}(\mathcal{H})$ contains the identity operator $I := \text{id}_\mathcal{H}$.

The spectral theorem is a fundamental result that is used ubiquitously in the theory of operator algebras. It generalizes the diagonalization of hermitean matrices $A \in \mathbb{M}_n(\mathbb{C})$ to selfadjoint operators defined in $\mathcal{H}$.

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of the hermitean matrix $A \in \mathbb{M}_n(\mathbb{C})$, numbered in ascending order: $\lambda_1 < \cdots < \lambda_m$. Moreover, let $P_{\lambda_k}$ be the orthogonal projection onto the eigenspace $\mathcal{E}_{\lambda_k}$ for the eigenvalue $\lambda_k$ of $A$. Since the distinct eigenspaces of $A$ are pairwise orthogonal, the family $E^A = (E^A_\lambda)_{\lambda \in \mathbb{R}}$, defined by
\[
E^A_\lambda := \sum \{P_{\lambda_k} \mid \lambda_k \leq \lambda\},
\]has the following properties:
(i) $E^A_\lambda \leq E^A_\mu$ for $\lambda \leq \mu$,

(ii) $E^A_\lambda = \bigwedge_{\mu > \lambda} E^A_\mu$,

(iii) $E^A_\lambda = 0$ for $\lambda < \lambda_1$ and $E^A_\lambda = I$ for $\lambda \geq \lambda_m$,

(iv) $A = \sum_{k=1}^m \lambda_k (E^A_{\lambda_k} - E^A_{\lambda_{k-1}})$, where $\lambda_0 < \lambda_1$.

Property (ii) expresses that $E^A$ is continuous from the right. Equally well we could define

$$F^A_\lambda := \sum \{ P_{\lambda_k} \mid \lambda_k < \lambda \}.$$  

Then the family $F^A = (F^A_\lambda)_{\lambda \in \mathbb{R}}$ is continuous from the left, i.e.

$$F^A_\lambda = \bigvee_{\mu < \lambda} F^A_\mu,$$

and $A$ can be represented as

$$A = \sum_{k=1}^{m+1} \lambda_{k-1} (F^A_{\lambda_k} - F^A_{\lambda_{k-1}}),$$

where $\lambda_{m+1} > \lambda_m$.

This situation is generalized to arbitrary selfadjoint operators $A \in \mathcal{L}(\mathcal{H})$ in the following couple of theorems.

**Theorem 2.1** ([14], Theorem 5.2.2) If $A \in \mathcal{L}(\mathcal{H})$ is a selfadjoint operator and $A$ is an abelian von Neumann algebra containing $A$, there is a family $(E^A_\lambda)_{\lambda \in \mathbb{R}}$ of projections in $A$, called the spectral resolution of $A$, such that

(i) $E^A_\lambda = 0$ if $\lambda < -|A|$, and $E^A_\lambda = I$ if $\lambda \geq |A|$;

(ii) $E^A_\lambda \leq E^A_\mu$ if $\lambda \leq \mu$;

(iii) $E^A_\lambda = \bigwedge_{\mu > \lambda} E^A_\mu$;

(iv) $AE^A_\lambda \leq \lambda E^A_\lambda$ and $\lambda (I - E^A_\lambda) \leq A (I - E^A_\lambda)$ for each $\lambda \in \mathbb{R}$;

(v) $A = \int_{-|A|}^{|A|} \lambda dE^A_\lambda$ in the sense of norm convergence of approximating Riemann sums; and $A$ is the norm limit of finite linear combinations with coefficients in $\text{sp}(A)$ of orthogonal projections $E^A_\mu - E^A_{\lambda}$. 

This theorem is proved using the Gelfand representation of $\mathcal{A}$. As is shown in [22], $E_\lambda^A$ can be described quite explicitly: it is the projection onto the kernel of $(A - \lambda I)^+$. The spectral resolution of $A \in \mathcal{L} (\mathcal{H})_{sa}$ is a (bounded) spectral family, a notion which we need throughout this work not only in the projection lattice of a von Neumann algebra, but also in other complete lattices.

**Definition 2.8** Let $\mathbb{L}$ be a complete lattice. A (right-continuous) spectral family is a family $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ in $\mathbb{L}$ satisfying

(i) $E_\lambda \leq E_\mu$ for $\lambda \leq \mu$,

(ii) $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu$,

(iii) $\bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0$ and $\bigvee_{\lambda \in \mathbb{R}} E_\lambda = 1$.

$E$ is called bounded if there are $a_0, a_1 \in \mathbb{R}$ such that $E_\lambda = 0$ for $\lambda < a_0$ and $E_\lambda = 1$ for $\lambda > a_1$.

The converse of theorem 2.1 are

**Theorem 2.2** ([14], Theorem 5.2.3) If $(E_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral family and $A \in \mathcal{L} (\mathcal{H})$ is a selfadjoint operator such that $AE_\lambda \leq \lambda E_\lambda$ and $\lambda (I - E_\lambda) \leq A (I - E_\lambda)$ for each $\lambda \in \mathbb{R}$, or if $A = \int_{-a}^a \lambda dE_\lambda$ for each $a$ exceeding some $b \in \mathbb{R}$, then $(E_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral resolution of $A$ in $\mathcal{A}_0$, the abelian von Neumann algebra generated by $A$ and $I$.

and

**Theorem 2.3** ([14], Theorem 5.2.4) If $(E_\lambda)_{\lambda \in \mathbb{R}}$ is a bounded spectral family in $\mathcal{P} (\mathcal{L} (\mathcal{H}))$, then $\int_{-a}^a \lambda dE_\lambda$ converges to a selfadjoint operator $A$ on $\mathcal{H}$ such that $|A| \leq a$ and for which $(E_\lambda)_{\lambda \in \mathbb{R}}$ is the spectral resolution, where $E_\lambda = 0$ if $\lambda \leq -a$ and $E_\lambda = I$ if $\lambda \geq a$.

If $\mathbb{L}$ is an orthomodular complete lattice, then the notion of a spectral family can be generalized to that of a spectral measure:

**Definition 2.9** A spectral measure in a complete orthomodular lattice $\mathbb{L}$ is a mapping $\mathcal{E} : \mathcal{B} (\mathbb{R}) \rightarrow \mathbb{L}$, where $\mathcal{B} (\mathbb{R})$ denotes the $\sigma$- algebra of all Borel subsets of $\mathbb{R}$, such that the following two properties

(i) $\mathcal{E} (\bigcup_{n \in \mathbb{N}} M_n) = \bigvee_{n \in \mathbb{N}} \mathcal{E} (M_n)$ for all pairwise disjoint sequences $(M_n)_{n \in \mathbb{N}}$ in $\mathcal{B} (\mathbb{R})$, where $\bigvee_{n \in \mathbb{N}}$ indicates that $(\mathcal{E} (M_n))_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal elements in $\mathbb{L}$. 

(ii) $E(\mathbb{R}) = 1$.

are satisfied.

These two properties imply that a spectral measure has the following properties, too:

(iii) $E(M \setminus N) = E(M) \wedge E(N^\perp)$ for all $M, N \in \mathcal{B}(\mathbb{R})$, $N \subseteq M$.

(iv) $E(\bigcup_{n \in \mathbb{N}} M_n) = \bigvee_{n \in \mathbb{N}} E(M_n)$ for all sequences $(M_n)_{n \in \mathbb{N}}$ in $\mathcal{B}(\mathbb{R})$.

(v) $E(\bigcap_{n \in \mathbb{N}} M_n) = \bigwedge_{n \in \mathbb{N}} E(M_n)$ for all sequences $(M_n)_{n \in \mathbb{N}}$ in $\mathcal{B}(\mathbb{R})$.

A spectral family $E$ in an orthomodular complete lattice $\mathbb{L}$ induces a spectral measure $E : \mathcal{B}(\mathbb{R}) \to \mathbb{L}$ by

$$\forall a, b \in \mathbb{R}, a < b : \ E([a, b]) := E_b \wedge E_a^\perp,$$

and, consistent with this definition,

$$\forall a \in \mathbb{R} : \ E([-\infty, a]) := E_a.$$

This can be seen using standard measure theoretic techniques, since the complete orthomodular lattice generated by $\{E_\lambda \mid \lambda \in \mathbb{R}\}$ is completely distributive (by remark 2.2).

Conversely, if we start from a spectral measure, i.e. from a map $E : \mathcal{B}(\mathbb{R}) \to \mathbb{L}$ with the properties (i) and (ii), we obtain two spectral families:

1. $E = (E_\lambda)_{\lambda \in \mathbb{R}}$, defined by $E_\lambda := E([-\infty, \lambda])$, which is right-continuous, i.e. $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu$ for all $\lambda \in \mathbb{R}$, and

2. $F = (F_\lambda)_{\lambda \in \mathbb{R}}$, defined by $F_\lambda := E([\lambda, \infty])$, which is left-continuous, i.e. $F_\lambda = \bigvee_{\mu < \lambda} F_\mu$ for all $\lambda \in \mathbb{R}$.

Of course there is an analogous formulation of the spectral theorem for left-continuous spectral families (see, for example, appendix C in [11]).

Continuity from one side is necessary in order to make the spectral resolution of a selfadjoint operator unique. Any choice between left and right continuity brings, in principle, an asymmetry into the theory. In operator theory, this asymmetry does not play any rôle. But we shall see in part III of this work that it becomes manifest in some important constructions. Our preferred choice are right-continuous spectral families.
Abelian von Neumann algebras will play a significant rôle in our work. The fundamental theorem for abelian operator algebras is the Gelfand representation theorem.

Let \( \mathcal{A} \) be an abelian \( C^* \)-algebra and assume for simplicity that \( \mathcal{A} \) has a unit element. Let \( \Omega(\mathcal{A}) \) be the set of all non-zero multiplicative linear functionals \( \varphi : \mathcal{A} \to \mathbb{C} \) that are positive in the following sense:

\[
\forall A \in \mathcal{A} : \varphi(A^*A) \geq 0.
\]

The elements of \( \Omega(\mathcal{A}) \) are called characters of \( \mathcal{A} \), and the set \( \Omega(\mathcal{A}) \) itself is called the Gelfand spectrum of \( \mathcal{A} \). The sets

\[
N_{A,\varepsilon}(\varphi_0) := \{ \varphi \in \Omega(\mathcal{A}) \mid |\varphi(A) - \varphi_0(A)| < \varepsilon \} \quad (A \in \mathcal{A}, \varepsilon > 0)
\]

form a subbasis of neighbourhoods of \( \varphi_0 \in \Omega(\mathcal{A}) \) in a topology for \( \Omega(\mathcal{A}) \). This topology on \( \Omega(\mathcal{A}) \) is induced by the weak*- topology on \( \mathcal{A}' \), the topological dual of the Banach space \( \mathcal{A} \). Since \( \Omega(\mathcal{A}) \) is closed in the unit ball \( \{ \psi \in \mathcal{A}' \mid |\psi| \leq 1 \} \) of \( \mathcal{A}' \) with respect to the weak*- topology, and the latter is compact, \( \Omega(\mathcal{A}) \) becomes a compact Hausdorff space. The space \( C(\Omega(\mathcal{A})) \) of all continuous functions \( f : \Omega(\mathcal{A}) \to \mathbb{C} \) with pointwise defined algebraic operations, norm \( |f|_\infty := \sup_{\varphi \in \Omega(\mathcal{A})} |f(\varphi)| \) and “adjoint” \( f^*(\varphi) := \overline{f(\varphi)} \) is an abelian \( C^* \)-algebra, whose characters are the evaluation functionals

\[
f \mapsto f(\varphi) \quad (\varphi \in \Omega(\mathcal{A})).
\]

The Gelfand spectrum of \( C(\Omega(\mathcal{A})) \) is therefore homeomorphic to \( \Omega(\mathcal{A}) \).

**Theorem 2.4** (Gelfand representation theorem; [14], theorem 4.4.3)

Let \( \mathcal{A} \) be an abelian \( C^* \)-algebra with unit element. Then the mapping

\[
\mathcal{A} \to C(\Omega(\mathcal{A})), \quad A \mapsto \hat{A}, \text{ defined by}
\]

\[
\forall \varphi \in \Omega(\mathcal{A}) : \hat{A}(\varphi) := \varphi(A),
\]

is an isometric \( * \)-isomorphism from \( \mathcal{A} \) onto \( C(\Omega(\mathcal{A})) \).

The mapping \( A \mapsto \hat{A} \) is called the Gelfand transformation, the function \( \hat{A} \in C(\Omega(\mathcal{A})) \) the Gelfand transform of \( A \).

If \( A \in \mathcal{L}(\mathcal{H}) \) is a normal operator, i.e. \( A^*A = AA^* \), then the Gelfand spectrum of the abelian \( C^* \)-algebra \( C^*(I, A, A^*) \), generated by \( I, A \) and \( A^* \), is homeomorphic to the spectrum \( sp(A) \) of \( A \) ([14], theorem 4.4.5). In contrast to that, the Gelfand spectrum of an infinite dimensional abelian von Neumann algebra is always of a monstrous size. We demonstrate this at a very simple example. Let \( l^\infty(\mathbb{N}) \) be the algebra of bounded sequences in \( \mathbb{C} \) with
norm $|a_n|_\infty := \sup_{n \in \mathbb{N}} |a_n|$. $l^\infty(\mathbb{N})$ is an abelian von Neumann algebra acting on the Hilbert space $l^2(\mathbb{N})$ by multiplication operators. It is not difficult to show that the Gelfand spectrum of $l^\infty(\mathbb{N})$ is homeomorphic to the Stone-Čech compactification $\check{\mathbb{N}}$ of $\mathbb{N}$. $\check{\mathbb{N}}$, although separable and compact, is a very large space: its cardinality is $2^{2^{\aleph_0}}$.

The Gelfand spectrum of an abelian infinite dimensional von Neumann algebra is not only very large, but its topology is also rather bizarre: every open set has open closure. Such topological spaces are called extremely disconnected. However, not every extremely disconnected compact Hausdorff space is the Gelfand spectrum of an abelian von Neumann algebra. For an extensive discussion of this question we refer to [26].

The double commutant theorem shows that the weak (and strong) closure of an operator algebra $\mathfrak{A}$ in $\mathcal{L}(\mathcal{H})$ can be expressed in purely algebraic terms. Let $\mathcal{F}$ be a subset of $\mathcal{L}(\mathcal{H})$. Then

$$\mathcal{F}^c := \{C \in \mathcal{L}(\mathcal{H}) \mid \forall A \in \mathcal{F} : AC = CA\}$$

is called the commutant of $\mathcal{F}$. It is easy to see that the commutant has the following properties:

(i) If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{L}(\mathcal{H})$, then $\mathcal{G}^c \subseteq \mathcal{F}^c$.

(ii) $\mathcal{F} \subseteq \mathcal{F}^{cc}$.

(iii) $\mathcal{F}^c$ is weakly closed. If $\mathcal{F}$ is selfadjoint, i.e. $A^* \in \mathcal{F}$ whenever $A \in \mathcal{F}$, then $\mathcal{F}^c$ and $\mathcal{F}^{cc}$ are von Neumann algebras.

**Theorem 2.5** (Double commutant theorem; [14], theorem 5.3.1)

If $\mathfrak{A}$ is an operator algebra in $\mathcal{L}(\mathcal{H})$ containing the identity operator, then the weak and the strong closure of $\mathfrak{A}$ coincide with $\mathfrak{A}^{cc}$.

A direct consequence of this theorem is the following

**Corollary 2.1** The double commutant $\mathcal{F}^{cc}$ of a selfadjoint subset $\mathcal{F}$ of $\mathcal{L}(\mathcal{H})$ containing the identity operator is the von Neumann algebra generated by $\mathcal{F}$.

In the sequel, we need some structure theory of von Neumann algebras. This rests on the notion of equivalence of projections. The equivalence theory of projections can be seen as an adaption of naive set theory to the projection lattice of a von Neumann algebra.
**Definition 2.10** Let $\mathcal{R}$ be a von Neumann algebra and let $P,Q \in \mathcal{P}(\mathcal{R})$. $P$ is called equivalent to $Q$ (in $\mathcal{R}$), written $P \sim Q$, if there is a partial isometry $\theta \in \mathcal{R}$ such that $\theta^*\theta = P$ and $\theta \theta^* = Q$.

Recall that a partial isometry $\theta$ is an operator in $L(H)$ that is isometric on the orthogonal complement of its kernel. So, if $\theta^*\theta = P$ and $\theta \theta^* = Q$, then $\theta$ is an isometry from $P H$ onto $Q H$ and $\theta^*$ is an isometry from $Q H$ onto $P H$. Note that the definition of equivalence requires that the partial isometry $\theta$, joining $P$ with $Q$, belongs to $\mathcal{R}$. Hence, if $\mathcal{R}$ is abelian, $P \sim Q$ if and only if $P = Q$.

**Definition 2.11** Let $P,Q \in \mathcal{P}(\mathcal{R})$. $P$ is called weaker than $Q$, written $P \preceq Q$, if there is a projection $Q_0 \in \mathcal{P}(\mathcal{R})$ such that $Q_0 \leq Q$ and $P \sim Q_0$.

One can prove that $\preceq$ is a partial ordering of the classes of equivalent projections (see [15], chapter 6). The fundamental result of the comparison theory of projections is

**Theorem 2.6** *(Comparison Theorem; [15], 6.2.7)*  
If $E$ and $F$ are projections in a von Neumann algebra $\mathcal{R}$, there are unique orthogonal central projections $P$ and $Q$ maximal with respect to the properties $QE \sim QF$, and, if $P_0$ is a non-zero central subprojection of $P$, then $P_0 E \preceq P_0 F$. If $R_0$ is a non-zero central subprojection of $I - P - Q$, then $R_0 F \preceq R_0 E$.

In set theory, two sets are called equivalent if they can be mapped bijectively onto each other. A set is defined to be finite, if it is not equivalent to any of its proper subsets. Thus the following definition is natural.

**Definition 2.12** *(15), 6.3.1)*  
A projection $E$ in a von Neumann algebra $\mathcal{R}$ is said to be infinite relative to $\mathcal{R}$ when $E \sim E_0 < E$ for some $E_0 \in \mathcal{P}(\mathcal{R})$. Otherwise, $E$ is said to be finite relative to $\mathcal{R}$. If $E$ is infinite and $PE$ is either $0$ or infinite for each central projection $P$, $E$ is said to be properly infinite. $\mathcal{R}$ is a finite or properly infinite von Neumann algebra when $I$ is, respectively, finite or properly infinite.

Also the following important result comes from set theory.

**Proposition 2.4** *(Halving Lemma; [15], 6.3.3)*  
If $E$ is a properly infinite projection in a von Neumann algebra $\mathcal{R}$, there is a projection $F$ in $\mathcal{R}$ such that $F \leq E$ and $F \sim E - F \sim E$.
Definition 2.13 A projection $E$ in a von Neumann algebra $\mathcal{R}$ is said to be abelian in $\mathcal{R}$ when $E\mathcal{R}E$ is abelian.

The basic properties of abelian projections are summarized in

Proposition 2.5 (LE, 6.4.2) Each subprojection of an abelian projection in a von Neumann algebra $\mathcal{R}$ is the product of the abelian projection and a central projection. A projection in $\mathcal{R}$ is abelian if and only if it is minimal in the set of projections in $\mathcal{R}$ with the same central carrier. Each abelian projection in $\mathcal{R}$ is finite. If $\mathcal{C}$ is the center of $\mathcal{R}$ and $E$ is an abelian projection in $\mathcal{R}$, then $E\mathcal{R}E = \mathcal{C}E$.

If $E$ is a non-zero projection in a von Neumann algebra $\mathcal{R}$, then

$$E \text{ is abelian} \quad - \quad E \text{ is finite} \quad - \quad E \text{ is infinite}$$

is a chain of properties with from left to right ascending complexity. This leads to the following definition:

Definition 2.14 (LE, 6.5.1) A von Neumann algebra $\mathcal{R}$ is said to be of type I if it has an abelian projection with central carrier $I$ - of type $I_n$ if $I$ is the sum of $n$ equivalent abelian projections. If $\mathcal{R}$ has no non-zero abelian projections but has a finite projection with central carrier $I$, then $\mathcal{R}$ is said to be of type II - of type $II_1$ if $I$ is finite - of type $II_\infty$ if $I$ is properly infinite. If $\mathcal{R}$ has no non-zero finite projections, $\mathcal{R}$ is said to be of type III.

A first insight into the structure of a von Neumann algebra $\mathcal{R}$ is given by the following theorem, which says that $\mathcal{R}$ can be decomposed into von Neumann subalgebras of different types.

Theorem 2.7 (Type Decomposition; LE, 6.5.2) If $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, there are mutually orthogonal central projections $P_n$, $n$ not exceeding $\dim \mathcal{H}$, $P_{c_1}$, $P_{c_\infty}$, and $P_\infty$, with sum $I$, maximal with respect to the properties that $\mathcal{R}P_n$ is of type $I_n$ or $P_n = 0$, $\mathcal{R}P_{c_1}$ is of type $II_1$ or $P_{c_1} = 0$, $\mathcal{R}P_{c_\infty}$ is of type $II_\infty$ or $P_{c_\infty} = 0$, and $\mathcal{R}P_\infty$ is of type III or $P_\infty = 0$. 
Chapter 3

The Stone Spectrum of a Lattice

3.1 Presheaves and their Sheafification

Traditionally, the notions of a presheaf and a complete presheaf (complete presheaves are usually called “sheaves”) are defined for the lattice $\mathcal{T}(M)$ of a topological space $M$. The very definition of presheaves and sheaves, however, can be formulated also for an arbitrary lattice:

**Definition 3.1** A presheaf of sets (R-modules) on a lattice $L$ assigns to every element $a \in L$ a set (R-module) $S(a)$ and to every pair $(a, b) \in L \times L$ with $a \leq b$ a mapping (R-module homomorphism)

$$\rho^b_a : S(b) \to S(a)$$

such that the following two properties hold:

1. $\rho^a_a = id_{S(a)}$ for all $a \in L$,
2. $\rho^b_a \circ \rho^c_b = \rho^c_a$ for all $a, b, c \in L$ such that $a \leq b \leq c$.

The presheaf $(S(a), \rho^b_a)_{a \leq b}$ is called a complete presheaf (or a sheaf for short) if it has the additional property

3. If $a = \bigvee_{i \in I} a_i$ in $L$ and if $f_i \in S(a_i)$ ($i \in I$) are given such that

$$\forall i, j \in I : (a_i \land a_j \neq 0 \implies \rho^{a_i \land a_j}_{a_i}(f_i) = \rho^{a_i \land a_j}_{a_j}(f_j),$$

then there is exactly one $f \in S(a)$ such that

$$\forall i \in I : \rho^a_{a_i}(f) = f_i.$$
The mappings $\rho^b_a : S(b) \to S(a)$ are called restriction maps.

One of the most elementary and at the same time instructive examples is the sheaf of locally defined continuous complex valued functions on a topological space $M$: $S(U)$ is the space of continuous functions on the open set $U \subseteq M$ and for $U, V \in \mathcal{T}(M)$ with $U \subseteq V$

$$\rho^V_U : S(V) \to S(U)$$

is the restriction map $f \mapsto f|_U$. Property (3) in definition 3.1 expresses the elementary fact that one can glue together a family of locally defined continuous functions $f_i : U_i \to \mathbb{C}$ which agree on the non-empty overlaps $U_i \cap U_j$ to a continuous function $f$ on $\bigcup_{i \in I} U_i$ which coincides with $f_i$ on $U_i$ for each $i \in I$.

Are there interesting new examples for sheaves on a lattice other than $\mathcal{T}(M)$, in particular on the quantum lattice $\mathbb{L}(H)$? The story begins with a disappointing answer:

**Proposition 3.1** Let $(S(U), \rho^V_U)_{U \subseteq V}$ be a complete presheaf of nonempty sets on the quantum lattice $\mathbb{L}(H)$. Then

$$\# S(U) = 1$$

for all $U \in \mathbb{L}(H)$.

Thus complete presheaves on $\mathbb{L}(H)$ are completely trivial!

**Proof:** Each $U \in \mathbb{L}(H)$ can be written as

$$U = \bigvee_{C_x \subseteq U} C_x.$$

Because of $C_x \cap C_y = 0$ for $C_x \neq C_y$, the family $(S(C_x))_{C_x \subseteq U}$ satisfies in a trivial manner the compatibility conditions. Therefore to each family $(s_{C_x})_{C_x \subseteq U}$ of elements $s_{C_x} \in S(C_x)$ there is a unique $s_U \in S(U)$ such that $\rho^U_{C_x}(s_U) = s_{C_x}$ for all $C_x \subseteq U$. Hence there is a bijection

$$S(U) \cong \prod_{C_x \subseteq U} S(C_x).$$

Consequently, it suffices to prove that each $S(C_x)$ $(x \neq 0)$ consists of a single element.

Let $C_{e_1}, C_{e_2}$ be different lines in $\mathcal{H}$, $U = C_{e_1} + C_{e_2}$ and $0 \neq C_x \subseteq U$ such that $C_x \notin \{C_{e_1}, C_{e_2}\}$. Then

$$U = C_{e_1} \lor C_{e_2} = C_x \lor C_{e_1} \lor C_{e_2}$$
\[ \mathcal{S}(U) \cong \mathcal{S}(\mathbb{C}e_1) \times \mathcal{S}(\mathbb{C}e_2) \cong \mathcal{S}(\mathbb{C}x) \times \mathcal{S}(\mathbb{C}e_1) \times \mathcal{S}(\mathbb{C}e_2). \]

Let \( s_x, t_x \in \mathcal{S}(\mathbb{C}x) \) and fix elements \( s_{e_k} \in \mathcal{S}(\mathbb{C}e_k), \) \( (k = 1, 2) \). Then there are unique \( s, t \in \mathcal{S}(U) \) such that

\[ \rho^U_{\mathbb{C}x}(s) = s_x, \quad \rho^U_{\mathbb{C}e_k}(s) = s_{e_k} \quad (k = 1, 2), \quad \rho^U_{\mathbb{C}x}(t) = t_x, \quad \rho^U_{\mathbb{C}e_k}(t) = s_{e_k} \quad (k = 1, 2). \]

\( U = \mathbb{C}e_1 \lor \mathbb{C}e_2 \) implies \( s = t \), hence \( s_x = t_x \). This shows \( \# \mathcal{S}(\mathbb{C}x) = 1 \) for all lines in \( \mathcal{H} \) and therefore \( \# \mathcal{S}(U) = 1 \) for all \( U \in \mathbb{L}(\mathcal{H}) \). □

There are, however, non-trivial presheaves on \( \mathbb{L}(\mathcal{H}) \) and one of them, which we shall study in part II, turns out to be quite fruitful for quantum mechanics and the theory of operator algebras.

Moreover, there is also another perspective of sheaves: the etale space of a presheaf. Classically, for a topological space \( M \), a presheaf \( \mathcal{S} \) on \( \mathcal{T}(M) \) induces a sheaf of local sections of the etale space of \( \mathcal{S} \). This sheaf on \( \mathcal{T}(M) \) is called the “sheafification of the presheaf \( \mathcal{S} \)”.

In what follows we shall show that to each presheaf on a (complete) lattice \( \mathbb{L} \) one can assign a sheaf on a certain topological space derived from the lattice \( \mathbb{L} \), the *Stone spectrum* \( \mathcal{Q}(\mathbb{L}) \) of \( \mathbb{L} \). The construction is quite similar to the well-known construction called “sheafification of a presheaf”. If \( \mathcal{S} \) is a presheaf, say, of modules on a topological space \( M \), i.e. on the lattice \( \mathcal{T}(M) \), then the corresponding etale space \( \mathcal{E}(\mathcal{S}) \) of \( \mathcal{S} \) is the disjoint union of the stalks of \( \mathcal{S} \) at points in \( M \):

\[ \mathcal{E}(\mathcal{S}) = \coprod_{x \in M} \mathcal{S}_x \]

where

\[ \mathcal{S}_x = \lim_{U \in \mathbb{U}(x)} \mathcal{S}(U), \]

the inductive limit of the family \( (\mathcal{S}(U))_{U \in \mathbb{U}} \), is the stalk in \( x \in M \).

A first attempt to generalize stalks to the situation of lattices is to develop a general notion of a “point in a lattice”. This can be done in a quite satisfactory manner. The essential hint comes from the topological context.
Let $M$ and $N$ be topological spaces. The elements of $N$ are in one-to-one correspondence to the constant mappings $f : M \to N$. These constant mappings correspond via the inverse image morphisms

\[ V \mapsto f^{-1}(V) \quad (V \in \mathcal{T}(N)) \]

to the left continuous lattice morphisms

\[ \Phi : \mathcal{T}(N) \to \mathcal{T}(M) \]

with the property

\[ \forall V \in \mathcal{T}(N) : \Phi(V) \in \{\emptyset, M\}. \]

Here a lattice morphism $\Phi : \mathbb{L}_1 \to \mathbb{L}_2$ is called left continuous if $\Phi(\bigvee_k a_k) = \bigvee_k \Phi(a_k)$ holds for all families $(a_k)_{k \in \mathbb{K}}$ in $\mathbb{L}_1$. Analogously $\Phi$ is called right continuous if $\Phi(\bigwedge_k a_k) = \bigwedge_k \Phi(a_k)$. $\Phi$ is called continuous if it is both left and right continuous. The inverse image morphism is not right continuous in general.

It is immediate that the set

\[ p := \{V \in \mathcal{T}(N) \mid \Phi(V) = M\} \]

has the following properties:

1. \( \emptyset \notin p \).
2. If $V, W \in p$, then $V \cap W \in p$.
3. If $V \in p$ and $W \supseteq V$ in $\mathcal{T}(N)$, then $W \in p$.
4. If $(V_i)_{i \in I}$ is a family in $\mathcal{T}(N)$ and $\bigcup_{i \in I} V_i \in p$, then there is at least one $i_0 \in I$ such that $V_{i_0} \in p$.

Now these properties make perfectly sense in an arbitrary $\mathfrak{m}$-complete lattice, so we can use them to define points in a lattice:

**Definition 3.2** Let $\mathbb{L}$ be an $\mathfrak{m}$-complete lattice. A non-empty subset $p \subseteq \mathbb{L}$ is called a point in $\mathbb{L}$ if the following properties hold:

1. \( 0 \notin p \).
2. $a, b \in p \Rightarrow a \wedge b \in p$.
3. $a \in p, b \in \mathbb{L}, a \leq b \Rightarrow b \in p$. 

(4) Let \((a_i)_{i \in I}\) be a family in \(L\) such that \(#I \leq m\) and \(\bigvee_{i \in I} a_i \in p\) then \(a_i \in p\) for at least one \(i \in I\).

Example 3.1 Let \(M\) be a non-empty set and \(L \subseteq \text{pot}(M)\) an \(m\)-complete lattice such that

\[
0_L = \emptyset \\
1_L = M \\
\bigvee_{i \in I} U_i = \bigcup_{i \in I} U_i \quad (\#I \leq m).
\]

Then for each \(x \in M\)

\[p_x := \{U \in L \mid x \in U\}\]

is a point in \(L\).

Conversely, if \(L\) is the lattice \(T(M)\) of open sets of a regular topological space \(M\), we have

Proposition 3.2 Let \(M\) be a regular topological space. A non-empty subset \(p \subseteq T(M)\) is a point in the lattice \(T(M)\) if and only if \(p\) is the set of open neighbourhoods of an element \(x \in M\). \(x\) is uniquely determined by \(p\).

Proof: Let \(p\) be a point in \(T(M)\). Then \(\bigcap_{U \in p} U \neq \emptyset\), for otherwise \(\bigcup_{U \in p} U' = M \in p\) and therefore \(U' \in p\) for some \(U \in p\), a contradiction. Assume that \(\bigcap_{U \in p} U\) contains two different elements \(x, y\). Since \(M\) is regular, there are open neighbourhoods \(V, W\) of \(x, y\), respectively, such that \(V \cap W = \emptyset\). Then \(V \in p\) or \(W \in p\), and we may assume that \(V \in p\). But then \(x \in V\), and therefore \(V \cap \overline{V} \neq \emptyset\), a contradiction. Hence there is a unique \(x_p \in M\) such that

\[\{x_p\} = \bigcap_{U \in p} U\]

Assume that there is an open neighbourhood \(V\) of \(x_p\) that does not belong to \(p\). Take an open neighbourhood \(W\) of \(x_p\) such that \(\overline{W} \subseteq V\). Then \(V \cup W = M \in p\), so \(\overline{W} \in p\), and from \(x_p \in \overline{W}\) we get the contradiction \(W \cap \overline{W} \neq \emptyset\). Therefore, all open neighbourhoods of \(x_p\) belong to \(p\).

Finally, let \(U \in p\), but assume that \(x_p \notin U\). Let

\[\Omega := \{V \in T(M) \mid \overline{V} \subseteq U\}\]

Since \(M\) is regular, we have \(U = \bigcup_{V \in \Omega} V\), so \(V \in p\) for some \(V \in \Omega\). Therefore, \(x_p \notin \overline{V}\) by assumption, but \(x_p \in \overline{V}\) by construction. \(\square\)
Unfortunately, there are important lattices that do not possess any points!

There are plenty of points in $\mathcal{T}(M)$ and $\mathcal{B}(M)$; $\mathcal{T}_r(M)$ (for suitable topological spaces $M$) and $\mathbb{L}(\mathcal{H})$ possess no points at all. We will show this here only for the lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of the Hilbert space $\mathcal{H}$.

**Proposition 3.3** If $\dim \mathcal{H} > 1$, there are no points in $\mathbb{L}(\mathcal{H})$.

**Proof:** Let $p \subseteq \mathbb{L}(\mathcal{H})$ be a point. If $(e_\alpha)_{\alpha \in A}$ is an orthonormal basis of $\mathcal{H}$ then

$$\bigvee_{\alpha \in A} \mathbb{C}e_\alpha = \mathcal{H} \in p,$$

so $\mathbb{C}e_{\alpha_0} \in p$ for some $\alpha_0 \in A$. It follows that each $U \in p$ must contain the line $\mathbb{C}e_{\alpha_0}$. Now choose $U \in \mathbb{L}(\mathcal{H})$ such that neither $U$ nor $U^\perp$ contains $\mathbb{C}e_{\alpha_0}$.

Then $U, U^\perp \notin p$ but $U \lor U^\perp = \mathcal{H} \in p$ which is a contradiction to property (4) in the definition of a point in a lattice. Therefore, there are no points in $\mathbb{L}(\mathcal{H})$. □

Let $\mathcal{S} = (\mathcal{S}(U), \rho^U_V)_{V \subseteq U}$ be a presheaf on the topological space $M$. The *stalk* of $\mathcal{S}$ at $x \in M$ is the direct limit

$$\mathcal{S}_x := \lim_{U \in \mathfrak{U}(x)} \mathcal{S}(U)$$

where $\mathfrak{U}(x)$ denotes the set of open neighbourhoods of $x$ in $M$, i.e. the point in $\mathcal{T}(M)$ corresponding to $x$.

For the definition of the direct limit (see below), however, we do not need the point $\mathfrak{U}(x)$, but only a partially ordered set $I$ with the property

$$\forall \alpha, \beta \in I \ \exists \gamma \in I : \gamma \leq \alpha \text{ and } \gamma \leq \beta.$$

In other words: a *filter base* $B$ in a lattice $\mathbb{L}$ is sufficient. It is obvious how to define a filter base in an arbitrary lattice $\mathbb{L}$:

**Definition 3.3** A filter base $B$ in a lattice $\mathbb{L}$ is a non-empty subset $B \subseteq \mathbb{L}$ such that

1. $0 \notin B$,
2. $\forall a, b \in B \ \exists c \in B : c \leq a \land b$. 

Let $S = (\mathcal{S}(U), \rho^U_V)_{V \subseteq U}$ be a presheaf on the topological space $M$. The *stalk* of $S$ at $x \in M$ is the direct limit

$$S_x := \lim_{U \in \mathfrak{U}(x)} S(U)$$

where $\mathfrak{U}(x)$ denotes the set of open neighbourhoods of $x$ in $M$, i.e. the point in $\mathcal{T}(M)$ corresponding to $x$.

For the definition of the direct limit (see below), however, we do not need the point $\mathfrak{U}(x)$, but only a partially ordered set $I$ with the property

$$\forall \alpha, \beta \in I \ \exists \gamma \in I : \gamma \leq \alpha \text{ and } \gamma \leq \beta.$$

In other words: a *filter base* $B$ in a lattice $\mathbb{L}$ is sufficient. It is obvious how to define a filter base in an arbitrary lattice $\mathbb{L}$:

**Definition 3.3** A filter base $B$ in a lattice $\mathbb{L}$ is a non-empty subset $B \subseteq \mathbb{L}$ such that

1. $0 \notin B$,
2. $\forall a, b \in B \ \exists c \in B : c \leq a \land b$.
The set of all filter bases in a lattice \(L\) is of course a vast object. So it is reasonable to consider maximal filter bases in \(L\). (By Zorn’s lemma, every filter base is contained in a maximal filter base in \(L\).) This leads to the following

**Definition 3.4** A nonempty subset \(\mathcal{B}\) of a lattice \(L\) is called a quasipoint in \(L\) if and only if

1. \(0 \notin \mathcal{B}\),
2. \(\forall a, b \in \mathcal{B} \exists c \in \mathcal{B} : c \leq a \land b,
3. \(\mathcal{B}\) is a maximal subset having the properties (1) and (2).

**Proposition 3.4** Let \(\mathcal{B}\) be a quasipoint in the lattice \(L\). Then

\[ \forall a \in \mathcal{B} \forall b \in L : (a \leq b \implies b \in \mathcal{B}). \]

In particular

\[ \forall a, b \in \mathcal{B} : a \land b \in \mathcal{B}. \]

**Proof:** Let \(c \in \mathcal{B}\). Then \(a \land c \leq b \land c\) and from \(a, c \in \mathcal{B}\) we obtain a \(d \in \mathcal{B}\) such that

\[ d \leq a \land c \leq b \land c. \]

Therefore \(\mathcal{B} \cup \{b\}\) is a filter base in \(L\) containing \(\mathcal{B}\). Hence \(\mathcal{B} = \mathcal{B} \cup \{b\}\) by the maximality of \(\mathcal{B}\), i.e. \(b \in \mathcal{B}\). \(\square\)

This proposition shows that a quasipoint in \(L\) is nothing else but a maximal dual ideal in the lattice \(L\) \((\mathbb{2})\). The set of quasipoints in \(L\) is denoted by \(Q(L)\).

In 1936, M.H.Stone \((\mathbb{25})\) showed that the set \(Q(B)\) of quasipoints in a Boolean algebra \(B\) can be given a topology such that \(Q(B)\) is a compact zero dimensional Hausdorff space and that the Boolean algebra \(B\) is isomorphic to the Boolean algebra of all closed open subsets of \(Q(B)\). A base for this topology is simply given by the sets

\[ Q_U(B) := \{ \mathcal{B} \in Q(B) \mid U \in \mathcal{B}\} \]

where \(U\) is an arbitrary element of \(B\). Of course we can generalize this construction to an arbitrary lattice \(L\).

For \(a \in L\) let

\[ Q_a(L) := \{ \mathcal{B} \in Q(L) \mid a \in \mathcal{B}\}. \]
It is quite obvious from the definition of a quasipoint that
\[ Q_{a \wedge b}(L) = Q_a(L) \cap Q_b(L), \]
\[ Q_0(L) = \emptyset \quad \text{and} \quad Q_1(L) = Q(L) \]
hold. Hence \( \{Q_a(L) \mid a \in L\} \) is a base for a topology on \( Q(L) \). It is easy to see, using the maximality of quasipoints, that in this topology the sets \( Q_a(L) \) are open and closed: By definition, \( Q_a(L) \) is an open set. Let \( B \in Q(L) \setminus Q_a(L) \). Then \( a \not\in B \), so there is some \( b \in B \) such that \( a \wedge b = 0 \) and this implies \( Q_a(L) \cap Q_b(L) = \emptyset \), hence \( Q_a(L) \) is also closed. Therefore, the topology defined by the basic sets \( Q_a(L) \) is zero dimensional and, using the same argument, we see that it is also Hausdorff. Moreover, as the basic sets \( Q_a(L) \) are open and closed, this topology is completely regular.

**Definition 3.5** \( Q(L) \), together with the topology defined by the base \( \{Q_a(L) \mid a \in L\} \), is called the Stone spectrum of the lattice \( L \).

We have chosen this terminology because we will see in section 3.4 that the Stone spectrum is a generalization of the Gelfand spectrum of an abelian von Neumann algebra.

We will prove some general properties of Stone spectra for certain classes of lattices in the next sections.

A general lattice has no points. Our most important example for this situation is the quantum lattice \( L(\mathcal{H}) \) of closed subspaces of the Hilbert space \( \mathcal{H} \). However, putting aside some very special examples, we always have plenty of quasipoints, and we can define the stalk of a presheaf \( P \) on a lattice \( L \) over a quasipoint \( B \in Q(L) \) in the very same manner as in the topological situation.

Let \( S = (S(U), \rho_{UV})_{U \leq V} \) be a presheaf on the (complete) lattice \( L \).

**Definition 3.6** \( f \in S(U) \) is called equivalent to \( g \in S(V) \) at the quasipoint \( B \in Q_U(L) \) if and only if
\[ \exists W \in \mathcal{B} : W \leq U \wedge V \quad \text{and} \quad \rho_W^U(f) = \rho_W^V(g). \]

If \( f \) and \( g \) are equivalent at the quasipoint \( B \) we write \( f \sim_B g \).

It is easy to see that \( \sim_B \) is an equivalence relation. The equivalence class of \( f \in S(U) \) at the quasipoint \( B \in Q(L) \) is denoted by \([f]_B \). It is called
the germ of \( f \) at \( \mathfrak{B} \). Note that this only makes sense if \( \mathfrak{B} \in \mathcal{Q}_U(\mathbb{L}) \). Let \( \mathfrak{B} \in \mathcal{Q}_U(\mathbb{L}) \). Then we obtain a canonical mapping
\[
\rho^U_\mathfrak{B} : \mathcal{S}(U) \to \mathcal{S}_\mathfrak{B}
\]
of \( \mathcal{S}(U) \) onto the set \( \mathcal{S}_\mathfrak{B} \) of germs at the quasipoint \( \mathfrak{B} \), defined by the composition
\[
\mathcal{S}(U) \xrightarrow{i_U} \prod_{V \in \mathfrak{B}} \mathcal{S}(V) \xrightarrow{\pi_\mathfrak{B}} \left( \prod_{V \in \mathfrak{B}} \mathcal{S}(V) \right) / \sim_\mathfrak{B}
\]
where \( i_U \) is the canonical injection and \( \pi_\mathfrak{B} \) the canonical projection of the equivalence relation \( \sim_\mathfrak{B} \). (\( \mathcal{S}_\mathfrak{B} := (\prod_{V \in \mathfrak{B}} \mathcal{S}(V)) / \sim_\mathfrak{B} \) is nothing else but the direct limit \( \lim_{\longrightarrow} V \in \mathfrak{B} \mathcal{S}(V) \) ([?]) and \( \rho^U_\mathfrak{B}(f) \) is just another notation for the germ \([f]_\mathfrak{B}\) of \( f \in \mathcal{S}(U) \).

Let \( \mathcal{S} \) be a presheaf on the lattice \( \mathbb{L} \) and
\[
\mathcal{E}(\mathcal{S}) := \bigsqcup_{\mathfrak{B} \in \mathcal{Q}(\mathbb{L})} \mathcal{S}_\mathfrak{B}.
\]
Moreover, let
\[
\pi_\mathcal{S} : \mathcal{E}(\mathcal{S}) \to \mathcal{Q}(\mathbb{L})
\]
be the projection defined by
\[
\pi_\mathcal{S}(\mathcal{S}_\mathfrak{B}) := \{ \mathfrak{B} \}.
\]
We will define a topology on \( \mathcal{E}(\mathcal{S}) \) such that \( \pi_\mathcal{S} \) is a local homeomorphism. For \( U \in \mathbb{L} \) and \( f \in \mathcal{S}(U) \) let
\[
\mathcal{O}_{f,U} := \{ \rho^U_\mathfrak{B}(f) \mid \mathfrak{B} \in \mathcal{Q}_U(\mathbb{L}) \}.
\]
It is quite easy to see that \( \{ \mathcal{O}_{f,U} \mid f \in \mathcal{S}(U), \ U \in \mathbb{L} \} \) is a base for a topology on \( \mathcal{E}(\mathcal{S}) \). Together with this topology, \( \mathcal{E}(\mathcal{S}) \) is called the etale space of \( \mathcal{S} \) over \( \mathcal{Q}(\mathbb{L}) \). By the very definition of this topology the projection \( \pi_\mathcal{S} \) is a local homeomorphism, for \( \mathcal{O}_{f,U} \) is mapped bijectively onto \( \mathcal{Q}_U(\mathbb{L}) \).

If \( \mathcal{S} \) is a presheaf of modules or algebras, the algebraic operations can be transferred fibrewise to the etale space \( \mathcal{E}(\mathcal{S}) \).

Addition, for example, gives a mapping from
\[
\mathcal{E}(\mathcal{S}) \circ \mathcal{E}(\mathcal{S}) := \{ (a, b) \in \mathcal{E}(\mathcal{S}) \times \mathcal{E}(\mathcal{S}) \mid \pi_\mathcal{S}(a) = \pi_\mathcal{S}(b) \}
\]
to \( \mathcal{E}(\mathcal{S}) \), defined as follows:
Let \( f \in \mathcal{S}(U), \ g \in \mathcal{S}(V) \) be such that
\[
a = \rho^{U}_{\pi_\mathcal{S}(a)}(f), \quad b = \rho^{V}_{\pi_\mathcal{S}(b)}(g)
\]
and let $W \in \pi_\mathcal{S}(a)$ be some element such that $W \leq U \wedge V$. Then

$$a + b := \rho^W_{\pi_\mathcal{S}(a)}(\rho^U_W(f) + \rho^V_W(g))$$

is a well defined element of $\mathcal{E}(\mathcal{S})$. By standard techniques one can prove that the algebraic operations

$$\mathcal{E}(\mathcal{S}) \circ \mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{S})$$

$$(a,b) \mapsto a - b$$

(and $(a,b) \mapsto ab$, if $\mathcal{S}$ is a presheaf of algebras) and

$$\mathcal{E}(\mathcal{S}) \to \mathcal{E}(\mathcal{S})$$

$$a \mapsto aa$$

are continuous.

From the etale space $\mathcal{E}(\mathcal{S})$ over $Q(\mathbb{L})$ we obtain - as in ordinary sheaf theory - a complete presheaf $\mathcal{S}^Q$ on the topological space $Q(\mathbb{L})$ by

$$\mathcal{S}^Q(\mathcal{V}) := \Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$$

where $\mathcal{V} \subseteq Q(\mathbb{L})$ is an open set and $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$ is the set of continuous sections of $\pi_\mathcal{S}$ over $\mathcal{V}$, i.e. of all continuous mappings $s_\mathcal{V} : \mathcal{V} \to \mathcal{E}(\mathcal{S})$ such that $\pi_\mathcal{S} \circ s_\mathcal{V} = id_\mathcal{V}$. If $\mathcal{S}$ is a presheaf of modules, then $\Gamma(\mathcal{V}, \mathcal{E}(\mathcal{S}))$ is a module, too.

**Definition 3.7** The complete presheaf $\mathcal{S}^Q$ on the Stone spectrum $Q(\mathbb{L})$ is called the sheaf associated to the presheaf $\mathcal{S}$ on $\mathbb{L}$.

### 3.2 General Properties of Stone Spectra

In the following, let $\mathbb{L}$ be a lattice (with minimal element $0$ and maximal element $1$) and $Q(\mathbb{L})$ the Stone spectrum of $\mathbb{L}$.

We have seen that $Q_a(\mathbb{L}) \cap Q_b(\mathbb{L}) = Q_{a \land b}(\mathbb{L})$ holds for all $a, b \in \mathbb{L}$. Clearly $a \leq b$ implies $Q_a(\mathbb{L}) \subseteq Q_b(\mathbb{L})$, so

$$Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}) \subseteq Q_{a \lor b}(\mathbb{L}).$$

In an arbitrary lattice however, this inclusion may be proper.
Remark 3.1 If $\mathbb{L}$ is a distributive lattice then
\[ Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}) = Q_{a \lor b}(\mathbb{L}) \] (3.1)
for all $a, b \in \mathbb{L}$.

Proof: Assume that there is some $\mathfrak{B} \in Q_{a \lor b}(\mathbb{L}) \setminus (Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}))$. Then, by the maximality of quasipoints, we can choose $d, e \in \mathfrak{B}$ such that $d \land a = e \land b = 0$. Because of $d \land e, a \lor b \in \mathfrak{B}$ we obtain the contradiction $0 \neq d \land e \land (a \lor b) = (d \land e \land a) \lor (d \land e \land b) = 0$. □

Conversely, assume that $Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}) = Q_{a \lor b}(\mathbb{L})$ holds for all $a, b \in \mathbb{L}$. Then we get for all $a, b, c \in \mathbb{L}:
\[ Q_{(a \land b) \lor (a \land c)}(\mathbb{L}) = Q_{a \land b}(\mathbb{L}) \cup Q_{a \land c}(\mathbb{L}) \]
\[ = (Q_a(\mathbb{L}) \cap Q_b(\mathbb{L})) \cup (Q_a(\mathbb{L}) \cap Q_c(\mathbb{L})) \]
\[ = Q_a(\mathbb{L}) \cap (Q_b(\mathbb{L}) \cup Q_c(\mathbb{L})) \]
\[ = Q_{a \land (b \lor c)}(\mathbb{L}), \]
i.e. if property 3.1 holds then also
\[ Q_{(a \land b) \lor (a \land c)}(\mathbb{L}) = Q_{a \land (b \lor c)}(\mathbb{L}). \] (3.2)

A lattice satisfying property 3.2 is called quasidistributive. Quasidistributivity does not imply distributivity:

Example 3.2 Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\mathbb{L}(\mathcal{H})_{cof} := \{ U \in \mathbb{L}(\mathcal{H}) \mid \dim U^\perp < \infty \} \cup \{0\}$. Then $\mathbb{L}(\mathcal{H})_{cof}$ is a non-distributive lattice. As the intersection of two subspaces of finite codimension is never zero, $\mathbb{L}(\mathcal{H})_{cof}$ contains only one quasipoint, namely $\mathbb{L}(\mathcal{H})_{cof} \setminus \{0\}$. Therefore, this lattice is trivially quasidistributive but not distributive.

Such messy situations cannot occur for orthomodular lattices:

Lemma 3.1 Let $\mathbb{L}$ be an orthomodular lattice. Then $Q_a(\mathbb{L}) = Q_b(\mathbb{L})$ implies $a = b$.

Proof: It suffices to prove that $Q_a(\mathbb{L}) \subseteq Q_b(\mathbb{L})$ implies $a \leq b$. (In fact this is equivalent to the assertion.) If $a \not\leq b$ then $a \land b < a$ and therefore, as $a$ commutes with $a \land b$, $a \land (a \land b)^\perp \neq 0$. Take a quasipoint $\mathfrak{B}$ that contains $a \land (a \land b)^\perp$. Then $a \in \mathfrak{B}$ and therefore $b \in \mathfrak{B}$. Hence we get the contradiction $a \land b, a \land (a \land b)^\perp \in \mathfrak{B}$. □

Distributivity of an orthomodular lattice $\mathbb{L}$ can now be characterized by properties of the topological space $Q(\mathbb{L})$:
Proposition 3.5 The following properties of an orthomodular lattice $\mathbb{L}$ are equivalent:

(i) $\mathbb{L}$ is distributive.

(ii) $Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}) = Q_{a \lor b}(\mathbb{L})$ for all $a, b \in \mathbb{L}$.

(iii) $Q_a(\mathbb{L}) \cup Q_{a^\perp}(\mathbb{L}) = Q(\mathbb{L})$ for all $a \in \mathbb{L}$.

(iv) The only open closed subsets of $Q(\mathbb{L})$ are the sets $Q_a(\mathbb{L})$ ($a \in \mathbb{L}$).

Proof: (ii) follows from (i) by remark 3.1 and (iii) is a special case of (ii). If (iii) holds and if there is a quasipoint $\mathcal{B}$ that contains $a \lor b$ but neither $a$ nor $b$, then $a^\perp, b^\perp \in \mathcal{B}$ and therefore $(a \lor b)^\perp = a^\perp \land b^\perp \in \mathcal{B}$, contradicting $a \lor b \in \mathcal{B}$. (ii) implies that $\mathbb{L}$ is quasidistributive, hence distributive by lemma 3.1. If (iv) holds, then for all $a, b \in \mathbb{L}$ there is some $c \in \mathbb{L}$ such that $Q_a(\mathbb{L}) \cup Q_b(\mathbb{L}) = Q_c(\mathbb{L})$ holds. Hence, by lemma 3.1 $a, b \leq c \leq a \lor b$, i.e. $c = a \lor b$. This shows that (iv) implies the distributivity of $\mathbb{L}$. Conversely, if $\mathbb{L}$ is distributive then, being orthomodular, it is a Boolean algebra. Therefore $Q(\mathbb{L})$ is compact by Stone’s theorem ([2]). Let $\mathcal{O} \subseteq Q(\mathbb{L})$ be open and closed. Then $\mathcal{O}$ can be be represented as a finite union of sets $Q_{a_i}(\mathbb{L})$ and therefore, by (ii), $\mathcal{O} = Q_\bigvee a_i(\mathbb{L})$. □

Definition 3.8 A quasipoint $\mathcal{B}$ in a lattice $\mathbb{L}$ is called atomic if $\mathcal{B}$ is isolated in $Q(\mathbb{L})$.

Proposition 3.6 Let $\mathbb{L}$ be an orthomodular lattice. Then $\mathcal{B} \in Q(\mathbb{L})$ is atomic if and only if there is a (necessarily unique) atom $a_0 \in \mathbb{L}$ such that

$$\mathcal{B} = \{ a \in \mathbb{L} | a_0 \leq a \}.$$  \hspace{1cm} (3.3)

Proof: If $a_0 \in \mathbb{L}$ is an atom such that (3.3) is satisfied, then $\{ \mathcal{B} \} = Q_{a_0}(\mathbb{L})$, so $\mathcal{B}$ is atomic. If, conversely, $\mathcal{B}$ is atomic then there is some $a \in \mathbb{L}$ such that $\{ \mathcal{B} \} = Q_a(\mathbb{L})$. Assume that $a$ is not an atom, i.e. there is some $b \in \mathbb{L}$ such that $0 < b < a$. Then $Q_{a \land b^\perp}(\mathbb{L})$ is a proper nonempty subset of $Q_a(\mathbb{L})$, a contradiction. If $a_0 \in \mathbb{L}$ is an atom then clearly $\{ a \in \mathbb{L} | a_0 \leq a \}$ is a quasipoint in $\mathbb{L}$. □

Proposition 3.6 is not valid for arbitrary lattices as is shown by example 3.2. On the other hand, orthomodularity is not a necessary assumption for 3.6 because the proposition is true also for the lattice $\mathcal{T}(M)$ of open subsets of
We will now show that the Stone spectrum $Q(\mathcal{L})$ of a completely distributive lattice $\mathcal{L}$ is extremely disconnected, i.e. that the closure of every open subset of $Q(\mathcal{L})$ is open again. In order to prove this we must characterize the closure of the union of an arbitrary family of basic sets $Q_a(\mathcal{L})$. It is useful to do this for an arbitrary lattice.

Let $(a_k)_{k \in K}$ be an arbitrary family in a lattice $\mathcal{L}$. Then $\bigcup_{k \in K} Q_{a_k}(\mathcal{L})$ can be characterized in the following way:

$$B \in \bigcup_{k \in K} Q_{a_k}(\mathcal{L}) \iff \forall a \in B : Q_a(\mathcal{L}) \cap \bigcup_{k \in K} Q_{a_k}(\mathcal{L}) \neq \emptyset$$

$$\iff \forall a \in B \exists k : Q_a(\mathcal{L}) \cap Q_{a_k}(\mathcal{L}) \neq \emptyset$$

$$\iff \forall a \in B \exists k : a \land a_k \neq 0.$$  

**Proposition 3.7** If $\mathcal{L}$ is a completely distributive lattice then its Stone spectrum $Q(\mathcal{L})$ is extremely disconnected.

**Proof:** We will prove that for an arbitrary family $(a_k)_{k \in K}$

$$\bigcup_{k \in K} Q_{a_k}(\mathcal{L}) = Q_{\bigvee_{k \in K} a_k}(\mathcal{L}) \quad (3.4)$$

holds. Obviously, the left hand side of equation (3.4) is contained in the right hand side. Conversely, let $B$ be a quasipoint that contains $\bigvee_k a_k$. Then we obtain for all $a \in B$

$$0 \neq a \land (\bigvee_{k \in K} a_k) = \bigvee_{k \in K} (a \land a_k).$$

Hence $a \land a_k \neq 0$ for some $k \in K$. This means, by the foregoing characterization of the closure, that $B \in \bigcup_{k \in K} Q_{a_k}(\mathcal{L})$. \hfill $\square$

More general than completely distributive lattices are lattices of finite type:

**Definition 3.9** A lattice $\mathcal{L}$ is called of finite type if

$$\bigcup_{k \in K} Q_{a_k}(\mathcal{L}) = Q_{\bigvee_{k \in K} a_k}(\mathcal{L})$$

holds for all increasing families $(a_k)_{k \in K}$ in $\mathcal{L}$. 
Lemma 3.2 An orthomodular lattice $\mathbb{L}$ is of finite type if and only if
\[ a \land (\bigvee_{k \in K} a_k) = \bigvee_{k \in K} (a \land a_k) \]
for all $a \in \mathbb{L}$ and all increasing families $(a_k)_{k \in K}$ in $\mathbb{L}$.

Proof: The proof rests on the following simple observation: Let $M$ be a topological space and let $A, B \subseteq M$ be subsets such that $A$ is closed and open. Then
\[ \overline{A \cap B} = A \cap \overline{B}. \]
Indeed, let $U$ be an open neighborhood of $x \in A \cap \overline{B}$. Then $U \cap A$ is an open neighborhood of $x$ and therefore $U \cap A \cap B \neq \emptyset$, i.e. $x \in A \cap \overline{B}$. The reverse inclusion is obvious.
If $\mathbb{L}$ is of finite type and if $a$ and $(a_k)_{k \in K}$ are as above then $(a \land a_k)_{k \in K}$ is increasing and therefore
\[
\mathcal{Q}_{\bigvee_{k \in K} (a \land a_k)}(\mathbb{L}) = \bigcup_{k \in K} \mathcal{Q}_{a \land a_k}(\mathbb{L})
= \bigcup_{k \in K} (\mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_{a_k}(\mathbb{L}))
= \mathcal{Q}_a(\mathbb{L}) \cap \bigcup_{k \in K} \mathcal{Q}_{a_k}(\mathbb{L})
= \mathcal{Q}_a(\mathbb{L}) \cap \mathcal{Q}_{\bigvee_{k \in K} a_k}(\mathbb{L})
= \mathcal{Q}_{a \land (\bigvee_{k \in K} a_k)}(\mathbb{L}).
\]
Hence, by orthomodularity, $a \land (\bigvee_{k \in K} a_k) = \bigvee_{k \in K} (a \land a_k)$. The converse is shown by the same argument as in the proof of proposition 3.7. \hfill \square

Corollary 3.1 A complete Boolean algebra is completely distributive if and only if it is of finite type.

Proof: In a distributive lattice $\mathbb{L}$ we have $\mathcal{Q}_{\bigvee_{k \in K} a_k}(\mathbb{L}) = \bigcup_{k \in K} \mathcal{Q}_{a_k}(\mathbb{L})$ for finite $K$ and the join of an arbitrary family in $\mathbb{L}$ can be written as the join of an increasing family of finite subfamilies. \hfill \square

The term “finite type” is chosen because of the following
Theorem 3.1  The projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra $\mathcal{R}$ is of finite type if and only if $\mathcal{R}$ is of finite type.

We need a simple lemma on tensor products:

Lemma 3.3  Let $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ be a von Neumann algebra acting on a Hilbert space $\mathcal{K}$ with unity $I_M = \text{id}_\mathcal{K}$ and let $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ be a von Neumann algebra. Then for all $A, B \in \mathcal{R}_{sa}$ and all $P, Q \in \mathcal{P}(\mathcal{R})$:

(i) $I_M \otimes A \leq I_M \otimes B$ if and only if $A \leq B$.

(ii) $I_M \otimes (P \wedge Q) = (I_M \otimes P) \wedge (I_M \otimes Q)$.

(iii) $I_M \otimes (P \vee Q) = (I_M \otimes P) \vee (I_M \otimes Q)$.

(iv) $I_M \otimes (\bigvee_{k \in K} P_k) = \bigvee_{k \in K} (I_M \otimes P_k)$ for all families $(P_k)_{k \in K}$ in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. An analogous property holds for arbitrary meets.

Proof: We use some results on tensor products that can be found in [14, 15, 26]. Let $(e_b)_{b \in \mathcal{B}}$ be an orthonormal basis of $\mathcal{K}$. Then

$$U : \sum_{b \in \mathcal{B}} x_b \mapsto \sum_{b \in \mathcal{B}} (e_b \otimes x_b)$$

is a surjective isometry from $\bigoplus_{b \in \mathcal{B}} \mathcal{H}_b$ (with $\mathcal{H}_b = \mathcal{H}$ for all $b \in \mathcal{B}$) onto $\mathcal{K} \otimes \mathcal{H}$. Let $A \in \mathcal{R}$. Then $U$ intertwines $I_M \otimes A$ and $A$:

$$U^{-1}(I_M \otimes A)U = \bigoplus_{b \in \mathcal{B}} A_b$$

with $A_b = A$ for all $b \in \mathcal{B}$. This immediately implies (i). Note that $I_M \otimes A$ is a projection if and only if $A$ is. Then (ii) and (iii) follow from (i) and the universal property of minimum and maximum. In order to prove (iv) we use the fact that the mapping $A \mapsto I_M \otimes A$ from $\mathcal{R}$ to $\mathcal{M} \otimes \mathcal{R}$ is strongly continuous on bounded subsets of $\mathcal{R}$:

$$\bigvee_{k \in K} (I_M \otimes P_k) = I_M \otimes (\bigvee_{k \in K} P_k).$$

Hence also (iv) follows. □
Proof of theorem: Due to lemma 3.2 we have to show that $\mathcal{R}$ is of finite type if and only if for all $P \in \mathcal{P}(\mathcal{R})$ and every increasing net $(P_k)_{k \in \mathcal{K}}$ in $\mathcal{P}(\mathcal{R})$

$$P \land (\bigvee_{k \in \mathcal{K}} P_k) = \bigvee_{k \in \mathcal{K}} (P \land P_k) \quad (3.5)$$

holds. Now the right hand side of 3.5 is the limit of the increasing net $(P \land P_k)_{k \in \mathcal{K}}$ in the strong operator topology. If $\mathcal{R}$ is of finite type, this limit is equal to the left hand side of 3.5 (see [17], p.412).

If $\mathcal{R}$ is not of finite type we present an example for which 3.5 does not hold. We use a construction which is quite similar to one already used in [6]. Assume that $\mathcal{R}$ is not finite. Then $\mathcal{R}$ contains a direct summand of the form $\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$, where $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ is a suitable von Neumann algebra and $\mathcal{H}_0$ a separable Hilbert space of infinite dimension (see e.g. [26], Ch. V.1, essentially prop. 1.22: if $\mathcal{R}$ is not finite then $\mathcal{R}$ has a direct summand with properly infinite unity $I_0$. Use the halving lemma to construct a countable orthogonal sequence of pairwise equivalent projections with sum $I_0$ (see the proof of theorem 6.3.4 in [15]).

Now let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_0$, $x := \sum_{k=1}^{\infty} \frac{1}{k} e_k$, $P$ the projection onto $\mathbb{C} x$ and $P_n$ the projection onto $U_n = \mathbb{C} e_1 + \ldots + \mathbb{C} e_n$.

Note that $x \notin U_n$ for all $n \in \mathbb{N}$, hence $P \land P_n = 0$ for all $n \in \mathbb{N}$ and therefore $\bigvee_{n \in \mathbb{N}} (P \land P_n) = 0$. On the other hand $\bigvee_{n \in \mathbb{N}} P_n = I$ and therefore $P \land (\bigvee_{n \in \mathbb{N}} P_n) = P > 0$. Using lemma 3.3 we obtain

$$(I_{\mathcal{M}} \otimes P) \land (\bigvee_{n \in \mathbb{N}} (I_{\mathcal{M}} \otimes P_n)) = (I_{\mathcal{M}} \otimes P) \land (I_{\mathcal{M}} \otimes (\bigvee_{n \in \mathbb{N}} P_n))$$

$$= I_{\mathcal{M}} \otimes P$$

$$> 0$$

$$= I_{\mathcal{M}} \otimes (\bigvee_{n \in \mathbb{N}} (P \land P_n))$$

$$= \bigvee_{n \in \mathbb{N}} ((I_{\mathcal{M}} \otimes P) \land (I_{\mathcal{M}} \otimes P_n))$$.

Thus property 3.5 is not satisfied in $\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$ and therefore also not in $\mathcal{R}$. □
3.3 Stone Spectra of Some Distributive Lattices

In this section we discuss the structure of Stone spectra of two classes of examples: σ-algebras, the lattice $\mathcal{T}(M)$ and the sublattice $\mathcal{T}_r(M)$ for topological spaces $M$.

We begin with the lattice $\mathcal{T}(M)$ of open subsets of a locally compact Hausdorff space $M$.

Let $\mathcal{B}$ be a quasipoint in $\mathbb{L}$. We distinguish two cases. In the first case we assume that $B$ has an element that is a relatively compact open subset of $M$. Let $U_0 \in \mathcal{B}$ be such an element. Then

$$\bigcap_{U \in \mathcal{B}} \overline{U} \neq \emptyset,$$

for otherwise $\bigcap_{U \in \mathcal{B}} \overline{U} \cap U_0 = \emptyset$ and from the compactness of $\overline{U}_0$ we see that there are $U_1, \ldots, U_n \in \mathcal{B}$ such that $\bigcap_{i=1}^n U_i \cap U_0 = \emptyset$.

But then $U_0 \cap U_1 \cap \ldots \cap U_n = \emptyset$, contrary to the defining properties of a filter base. The maximality of $\mathcal{B}$ implies that every open neighbourhood of $x \in \bigcap_{U \in \mathcal{B}} \overline{U}$ belongs to $\mathcal{B}$. Therefore, as $M$ is a Hausdorff space, $\bigcap_{U \in \mathcal{B}} \overline{U}$ consists of precisely one element of $M$. We will denote this element by $\text{pt}(\mathcal{B})$ and call $\mathcal{B}$ a quasipoint over $x = \text{pt}(\mathcal{B})$.

Now consider the other case in which no element of the quasipoint $\mathcal{B}$ is relatively compact. It can be easily shown, using the maximality of $\mathcal{B}$ again, that in this case $M \setminus K \in \mathcal{B}$ for every compact subset $K$ of $M$. (See lemma 3.5 for a more general statement.) We summarize these facts in the following

**Proposition 3.8** Let $M$ be a locally compact Hausdorff space and $\mathcal{B}$ a quasipoint in the lattice $\mathcal{T}(M)$ of open subsets of $M$. Then either $M \setminus K \in \mathcal{B}$ for all compact subsets $K$ of $M$ or there is a unique element $x \in M$ such that $\bigcap_{U \in \mathcal{B}} \overline{U} = \{x\}$.

In the first case $\mathcal{B}$ is called an unbounded quasipoint, in the second a bounded quasipoint over $x$.

For a non-compact space $M$ let $M_\infty := M \cup \{\infty\}$ be the one-point compactification of $M$. Then the unbounded quasipoints in $\mathcal{T}(M)$ can be considered as quasipoints over $\infty$ in $\mathcal{T}(M_\infty)$.

Next we consider the Boolean σ-algebra $\mathcal{B}(M)$ of all Borel subsets of a Hausdorff topological space $M$. The orthocomplement of $A \in \mathcal{B}(M)$ is the ordinary set theoretic complement which we denote by $A'$. For some of our results
the topology of $M$ must fulfill some countability conditions. We suppose here that

(i) $M$ satisfies the first axiom of countability, i.e. each $x \in M$ has a countable base of neighbourhoods, and that

(ii) $M$ satisfies the Lindelöf condition, i.e. every open covering of $M$ can be refined by an at most countable subcovering.

These conditions are satisfied if e.g. the topology of $M$ has a countable base.

**Proposition 3.9** $p \subseteq \mathcal{B}(M)$ is a point in the lattice $\mathcal{B}(M)$ if and only if $p$ is an atomic quasipoint in $\mathcal{B}(M)$.

**Proof:** An atomic quasipoint in $\mathcal{B}(M)$ has the form $\{A \in \mathcal{B}(M) \mid x \in A\}$ for some $x \in M$, so it is obviously a point in $\mathcal{B}(M)$. Conversely, assume that $p \subseteq \mathcal{B}(M)$ is a point and let $\mathcal{B}$ be a quasipoint in $\mathcal{B}(M)$ that contains $p$. Let $A \in \mathcal{B}$. Because of $A \cup A' = M \in p$ we have $A \in p$ or $A' \in p$. As $A' \not\in \mathcal{B}$, this implies $A \in p$. Hence $p$ is a quasipoint. Moreover $\bigcap_{n \in \mathbb{N}} A_n \in p$ for all sequences $(A_n)_{n \in \mathbb{N}}$ in $p$: $\bigcap_{n \in \mathbb{N}} A_n \not\in p$ would imply $\bigcup_n A_n' = (\bigcap_n A_n)' \in p$, hence $A_{n_0}' \in p$ for some $n_0$, contradicting $A_{n_0} \in p$.

Now assume that $\bigcap_{A \in p} \overline{A} = \emptyset$, i.e. $\bigcup_{A \in p} \overline{A} = M$. The Lindelöf property implies that there is a sequence $(A_n)_{n \in \mathbb{N}}$ in $p$ such that $\bigcup_{n \in \mathbb{N}} \overline{A_n} = M$. Hence $\overline{A_n} \in p$ for some $n$, a contradiction. The maximality of $p$ implies that $\{y\} \in p$ for every $y \in \bigcap_{A \in p} \overline{A}$. Hence there is a unique $x \in M$ such that $\bigcap_{A \in p} \overline{A} = \{x\}$. This means that $p$ is an atomic quasipoint. $\Box$

Using similar arguments, we easily obtain

**Proposition 3.10** A quasipoint $\mathcal{B}$ in a $\sigma$-algebra $\mathcal{B}$ is a point in $\mathcal{B}$ if and only if $\bigwedge_{n \in \mathbb{N}} a_n \in \mathcal{B}$ for every sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathcal{B}$.

We will now present a sufficient condition for a $\sigma$-algebra to have no points. For the convenience of the reader we repeat some well known notions.

**Definition 3.10** A nonempty subset $\mathcal{I}$ of an $m$-complete lattice $\mathbb{L}$ is called an $m$-ideal if it has the following properties:

(i) If $a \in \mathcal{I}$, then $a \wedge b \in \mathcal{I}$ for all $b \in \mathbb{L}$.

(ii) $\bigvee_{k \in \mathbb{K}} a_k \in \mathcal{I}$ for every family $(a_k)_{k \in \mathbb{K}}$ in $\mathcal{I}$ such that $\# \mathbb{K} \leq m$.

An $m$-ideal is called proper if $1 \not\in \mathcal{I}$. 
If $\mathcal{I}$ is an $m$-ideal in $\mathbb{L}$ then the quotient $\mathbb{L}/\mathcal{I}$ is defined in the following way. We define an equivalence relation $\sim \subseteq \mathbb{L} \times \mathbb{L}$ by

$$a \sim b :\iff \exists p \in \mathcal{I} : a \lor p = b \lor p,$$

and we denote by $[a]$ the equivalence class of $a \in \mathbb{L}$. We define

$$[a] \lor [b] := [a \lor b]$$

and

$$[a] \leq [b] :\iff [a \lor b] = [b].$$

A routine calculation shows that these are well defined binary relations on $\mathbb{L}/\mathcal{I}$ which turn the quotient into a $\lor$-semilattice. The natural definition of a meet,

$$[a] \land [b] := [a \land b],$$

however, is only well defined if the lattice $\mathbb{L}$ is distributive. If $\mathcal{B}$ is a Boolean algebra, $\mathcal{B}/\mathcal{I}$ is also orthocomplemented by

$$[a]^\perp := [a^\perp].$$

The well definedness of this operation is most easily proved by using the following characterization of the equivalence modulo $\mathcal{I}$:

$$a \sim b \iff (a \land b^\perp) \lor (b \land a^\perp) \in \mathcal{I}.$$

We skip the essentially computational proof.

If $\mathbb{L}$ is complete (with or without restrictions to the cardinal defining the degree of completeness) then it is natural to define

$$\bigvee_{k \in \mathcal{K}} [a_k] := [\bigvee_{k \in \mathcal{K}} a_k].$$

Obviously, this is well defined. If $\mathbb{L}$ is orthocomplemented, we define the infinite meet by

$$\bigwedge_{k \in \mathcal{K}} [a_k] := (\bigvee_{k \in \mathcal{K}} [a]^\perp)^\perp.$$

This definition avoids the assumption of complete distributivity of $\mathbb{L}$ which would be needed when defining the infinite meet by $[\bigwedge_{k \in \mathcal{K}} a_k]$. Collecting these facts we obtain

**Proposition 3.11** If $\mathcal{I}$ is an $m$-ideal in an $m$-complete Boolean algebra $\mathcal{B}$ then the quotient $\mathcal{B}/\mathcal{I}$ is an $m$-complete Boolean algebra.
An \( m \)-complete Boolean algebra will be simply called an \( m \)-algebra. For \( m = \aleph_0 \) we use the traditional notation “\( \sigma \)-algebra”.

The following result is merely a corollary to proposition 3.9.

**Proposition 3.12** Let \( M \) be a Hausdorff space that satisfies the first axiom of countability and the Lindelöf condition. If \( \mathcal{I} \) is a \( \sigma \)-ideal in \( \mathcal{B}(M) \) that contains all atoms of \( \mathcal{B}(M) \) then the \( \sigma \)-algebra \( \mathcal{B} := \mathcal{B}(M)/\mathcal{I} \) has no points.

**Proof:** Consider the canonical projection

\[
\pi : \mathcal{B}(M) \rightarrow \mathcal{B}(M)/\mathcal{I}
\]

\( \pi \) is a \( \sigma \)-morphism of \( \sigma \)-algebras. Assume that there exists a point \( p \) in \( \mathcal{B} \). Then

\[
\pi^{-1}(p) = \{ A \in \mathcal{B}(M) \mid [A] \in p \}
\]

is a point in \( \mathcal{B}(M) \). By proposition 3.9 there is a unique \( x \in M \) such that \( \pi^{-1}(p) = \mathcal{B}_x \) and therefore \([\{x\}] \in p\). But \( \{x\} \in \mathcal{I} \), hence \([\{x\}] = [\emptyset] = 0\), a contradiction. \( \Box \)

We recall that a subset \( N \) of a Hausdorff topological space \( M \) is said to be nowhere dense, if the interior of its closure is empty. \( N \) is called a set of first category (or meagre) if it is a countable union of nowhere dense subsets, otherwise a set of second category. \( M \) is called a Baire space if every nonvoid open subset is of second category. By a theorem of Baire ([3], p.193) every locally compact and every complete metric space is a Baire space.

For any topological space \( M \) we signify by \( \mathcal{I}_1 \) the \( \sigma \)-ideal of all meagre Borel subsets of \( M \). The following result, not difficult to prove, can be found in [24]:

**Proposition 3.13** Let \( M \) be a Baire space. Then every equivalence class \( a \in \mathcal{B}(M)/\mathcal{I}_1 \) contains a unique regular open set \( U_a \). The mapping

\[
\Phi : \mathcal{B}(M)/\mathcal{I}_1 \rightarrow \mathcal{T}_r(M)
\]

\( a \mapsto U_a \)

is a \( \sigma \)-isomorphism of \( \sigma \)-algebras.

Together with proposition 3.12 we obtain
Corollary 3.2 Let $M$ be a separable complete metric space. Then $\mathcal{T}_r(M)$ has no points.

Proposition 3.14 Let $\mathcal{I}$ be a $\sigma$-ideal in the $\sigma$-algebra $\mathcal{B}$ and let $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ be the canonical projection onto the quotient $\sigma$-algebra $\mathcal{B}/\mathcal{I}$. Then $\mathfrak{B} \subseteq \mathcal{B}/\mathcal{I}$ is a quasipoint if and only if $\pi^-(\mathfrak{B})$ is a quasipoint in $\mathcal{B}$ such that

$$\pi^-(\mathfrak{B}) \cap \mathcal{I} = \emptyset$$

Proof: Let $\mathfrak{B}$ be a quasipoint in $\mathcal{B}/\mathcal{I}$. Then $\pi^-(\mathfrak{B})$ is a filter base in $\mathcal{B}$. If $a \in \mathcal{B}$ then $\pi(a)$ or $\pi(a)^\perp$ belongs to $\mathfrak{B}$, i.e. $a$ or $a^\perp$ belongs to $\pi^-(\mathfrak{B})$. Thus $\pi^-(\mathfrak{B})$ is a quasipoint in $\mathcal{B}$. Clearly $\mathcal{I} \cap \pi^-(\mathfrak{B}) = \emptyset$ because $a \in \mathcal{I}$ if and only if $\pi(a) = 0$.

Conversely let $\mathfrak{B}$ be a nonempty subset of $\mathcal{B}/\mathcal{I}$ such that $\pi^-(\mathfrak{B})$ is a quasipoint with $\mathcal{I} \cap \pi^-(\mathfrak{B}) = \emptyset$. Then $0 \notin \mathfrak{B}$. If $a, b \in \mathfrak{B}$, $A \in \pi^-(a)$, $B \in \pi^-(b)$ then $A \land B \in \pi^-(\mathfrak{B})$ and therefore $a \land b = \pi(A \land B) \in \mathcal{B}$. If $a$ is an arbitrary element of $\mathcal{B}/\mathcal{I}$ and $A \in \pi^-(a)$ then $A$ or $A^\_perp$ belongs to $\pi^-(\mathfrak{B})$, hence $a \in \mathfrak{B}$ or $a^\perp \in \mathfrak{B}$. This shows that $\mathfrak{B}$ is a quasipoint. □

Proposition 3.15 Let $\mathcal{I}$ be a $\sigma$-ideal in a $\sigma$-algebra $\mathcal{B}$ and let $\mathfrak{B} \subseteq \mathcal{B}$ be a quasipoint such that $\mathcal{I} \cap \mathfrak{B} = \emptyset$. Then $\pi(\mathfrak{B}) \subseteq \mathcal{B}/\mathcal{I}$ is a quasipoint and $\pi^-(\pi(\mathfrak{B})) = \mathfrak{B}$.

Proof: The same arguments as in the foregoing proof show that $\pi(\mathfrak{B})$ is a quasipoint. Then $\pi^-(\pi(\mathfrak{B}))$ is a quasipoint in $\mathcal{B}$ that contains $\mathfrak{B}$ and therefore $\mathfrak{B} = \pi^-(\pi(\mathfrak{B}))$ by maximality. □

Corollary 3.3 Let $\mathcal{B}$ be a $\sigma$-algebra and $\mathcal{I}$ a $\sigma$-ideal in $\mathcal{B}$. Then the set

$$Q^2(\mathcal{B}) := \{ \mathfrak{B} \in Q(\mathcal{B}) \mid \mathfrak{B} \cap \mathcal{I} = \emptyset \}$$

is compact and homeomorphic to the Stone spectrum of $\mathcal{B}/\mathcal{I}$.

Proof: By proposition 3.15 the canonical projection $\pi : \mathcal{B} \to \mathcal{B}/\mathcal{I}$ induces a mapping

$$\pi_* : Q^2(\mathcal{B}) \to Q(\mathcal{B}/\mathcal{I})$$

$$\mathfrak{B} \mapsto \pi(\mathfrak{B})$$
which is, according to propositions 3.14 and 3.15, a bijection \( \pi^{-1} : \mathcal{C} \mapsto \pi^{-1}(\mathcal{C}) \). Let \( A \in B \setminus \mathcal{I} \). Then \( \pi^{-1}(Q_A(B) \cap Q^2(B)) \subseteq Q_{[A]}(B/\mathcal{I}) \). If \( \mathfrak{B} \in Q_{[A]}(B/\mathcal{I}) \), then \( \pi^{-1}(\mathfrak{B}) \in Q_B(B) \cap Q^2(B) \) for some \( B \in [A] \). This shows that

\[
\pi^{-1}(Q_{[A]}(B/\mathcal{I})) = \bigcup_{B \in [A]} (Q_B(B) \cap Q^2(B)).
\]

Now observe that \( A, B \in [A] \) if and only if \( A \lor N = B \lor N \) for some \( N \in \mathcal{I} \). Therefore \( A \in \mathfrak{B} \) if and only if \( B \in \mathfrak{B} \) for all \( \mathfrak{B} \in Q^2(B) \). Thus

\[
\pi^{-1}(Q_{[A]}(B/\mathcal{I})) = Q_B(B) \cap Q^2(B).
\]

for all \( B \in [A] \). Hence \( \pi \) is a homeomorphism.

Now let \( \mathfrak{B} \notin Q^2(B) \). This means that there is some \( A \notin \mathfrak{B} \) and therefore \( Q_A(B) \cap Q^2(B) = \emptyset \). So \( Q^2(B) \) is closed. As \( Q(B) \) is compact, \( Q^2(B) \) is compact too. \( \square \)

**Remark 3.2** Since \( B \) is a distributive ortholattice, the condition \( \mathfrak{B} \cap \mathcal{I} = \emptyset \) is equivalent to \( \mathcal{I}^+ \subseteq \mathfrak{B} \) where \( \mathcal{I}^+ := \{ A^+ \mid A \in \mathcal{I} \} \).

By a theorem of Loomis and Sikorski (24) every \( \sigma \)-algebra is the quotient of a \( \sigma \)-algebra of Borel sets of a compact space modulo a \( \sigma \)-ideal. The construction is roughly as follows. Let \( B \) be a \( \sigma \)-algebra and let

\[
\eta : B \rightarrow \mathfrak{A}(Q(B))
\]

\[
a \mapsto Q_a(B)
\]

be the Stone isomorphism between the Boolean algebras \( B \) and \( \mathfrak{A}(Q(B)) \). Note that \( \mathfrak{A}(Q(B)) \) is in not a \( \sigma \)-algebra, because \( \bigcup_{n \in \mathbb{N}} Q_{a_n}(Q(B)) \) is open but, in general, not closed. Let \( \mathfrak{A}_\sigma(Q(B)) \) be the \( \sigma \)-algebra generated by \( \mathfrak{A}(Q(B)) \) in \( B(Q(B)) \). Then \( \eta \) can be considered as a homomorphism from \( B \) to \( \mathfrak{A}_\sigma(Q(B)) \). It is not a \( \sigma \)-homomorphism, because

\[
\bigcup_{n \in \mathbb{N}} Q_{a_n}(Q(B)) = Q_{\bigvee_{n \in \mathbb{N}} a_n}(Q(B))
\]

for every sequence \( (a_n)_{n \in \mathbb{N}} \) in \( B \). Thus the obstacles for \( \eta \) to be a \( \sigma \)-homomorphism are the boundaries of open sets of the form \( \bigcup_{n \in \mathbb{N}} Q_{a_n}(Q(B)) \).

It is easy to see that cutting them away means to go over to the quotient \( \mathfrak{A}_\sigma(Q(B))/\mathcal{I}(\mathfrak{N}) \) where \( \mathcal{I}(\mathfrak{N}) \) is the \( \sigma \)-ideal in \( \mathfrak{A}_\sigma(Q(B)) \) generated by

\[
\mathfrak{N} := \{ \bigcap_{n \in \mathbb{N}} Q_{a_n}(Q(B)) \mid \bigwedge_{n \in \mathbb{N}} a_n = 0 \}.
\]
Note that the elements of $I(N)$ are meagre subsets of $Q(B)$. It is not difficult to prove that the composition $\eta_\varrho := \varrho \circ \eta$ of $\eta$ with the canonical projection $\varrho : \mathfrak{A}_\sigma(Q(B)) \to \mathfrak{A}_\sigma(Q(B))/I(N)$ is a surjective $\sigma$-homomorphism

$$\eta_\varrho : B \to \mathfrak{A}_\sigma(Q(B))/I(N).$$

The technically difficult part of Loomis’s construction is the proof that $\eta_\varrho$ is injective. For this part we refer to Sikorski ([24]).

The theorem of Loomis and Sikorski represents an abstract $\sigma$-algebra $B$ as a $\sigma$-algebra of subsets of $Q(B)$ (with the standard set theoretic operations) modulo a $\sigma$-ideal. This is often not a very economic representation because, as we mentioned already, the Stone spectrum is typically very large. We will prove in the next section a $C^*$-algebraic description of Stone spectra of $\sigma$-algebras of the form $\mathfrak{A}(M)/I$ where $\mathfrak{A}(M)$ is an arbitrary $\sigma$-algebra of sets\(^1\) and $I$ is a $\sigma$-ideal in $\mathfrak{A}(M)$.

If $M$ is a complete lattice, isomorphic to $L$ via a lattice isomorphism $\Phi : L \to M$, then it is easy to see that $\Phi$ induces a homeomorphism

$$\Phi_* : Q(L) \to Q(M)$$

of the corresponding Stone spectra:

$$\Phi_*(\mathfrak{B}) := \{ \Phi(a) \mid a \in \mathfrak{B} \}.$$

The opposite conclusion, however, is not true. In fact we can show that the Stone spectra $Q(T(M))$ and $Q(T_r(M))$ are homeomorphic for every topological space $M$. But in general the lattice $T(M)$ of open subsets of $M$ is not isomorphic to the lattice $T_r(M)$ of regular open subsets of $M$, because $T(M)$ possesses points whereas, in general, $T_r(M)$ does not.

In section [2.1] we have seen that $T_r(M)$ is a Boolean algebra with complement operation

$$U \mapsto U^c$$

where $U^c := M \setminus \bar{U}$. Now it is easy to see that

$$U \cap V = \emptyset \implies U^{cc} \cap V^{cc} = \emptyset$$

holds for all open sets $U, V \subseteq M$. From this fact we get

\(^1\)This means in particular that the Boolean operations are the usual set theoretic ones.
Lemma 3.4 Let $M$ be a topological space and let $\mathcal{B}$ be a quasipoint in $\mathcal{T}(M)$. Then
\[ \mathcal{B}^r := \{U^{cc} \mid U \in \mathcal{B}\} \]
is a quasipoint in $\mathcal{T}_r(M)$.

Proposition 3.16 The mapping
\[ \rho : Q(\mathcal{T}(M)) \to Q(\mathcal{T}_r(M)) \]
\[ \mathcal{B} \mapsto \mathcal{B}^r \]
is a homeomorphism of Stone spectra.

Sketch of proof: The first thing to show is that every quasipoint $\mathfrak{R}$ in $\mathcal{T}_r(M)$ is contained in exactly one quasipoint in $\mathcal{T}(M)$. Thus $\rho$ is a bijection. Moreover
\[ U \in \mathcal{B} \iff U^{cc} \in \mathcal{B}^r \]
for every quasipoint $\mathcal{B}$ in $\mathcal{T}(M)$. This implies
\[ \rho(Q_U(\mathcal{T}(M))) = Q_{U^{cc}}(\mathcal{T}_r(M)) \]
and
\[ \rho^{-1}(Q_W(\mathcal{T}_r(M))) = Q_W(\mathcal{T}(M)), \]
i.e. $\rho$ is a homeomorphism. □

Corollary 3.4 The Stone spectrum $Q(\mathcal{T}(M))$ is compact.

Corollary 3.5 Let $M$ be a compact Hausdorff space and let
\[ pt : Q(\mathcal{T}(M)) \to M \]
be the map that assigns to $\mathcal{B} \in Q(\mathcal{T}(M))$ the element $pt(\mathcal{B}) \in M$ determined by $\bigcap_{U \in \mathcal{B}} \bar{U}$. Then the quotient topology of $M$ induced by $pt$ coincides with the given topology of $M$.

This follows from the fact that $pt$ is a continuous mapping and therefore the quotient topology is finer than the given topology. It cannot be strictly finer because both topologies are compact and Hausdorff.

This result gives an extreme example for the fact that the projection onto the quotient by an equivalence relation need not be an open mapping: let $M$ be a connected compact Hausdorff space. The compactness of the Stonean space $Q(\mathcal{T}(M))$ implies that $pt$ is a closed mapping. If it was also
an open mapping the total disconnectedness of $Q(T(M))$ would imply that the image $M$ of $pt$ is totally disconnected, too. As $M$ is connected, this is only possible for the trivial case that $M$ consists of a single element.

We want to show up some relations between the Stone spectrum of $B(M)$ and the Stone spectrum of $T(M)$ for a Baire space $M$, in particular for a locally compact space $M$.

Let $M$ be a Baire space and $I_1 \subseteq B(M)$ the $\sigma$-ideal of meagre Borel sets. The assumption that $M$ is a Baire space is expressed by

$$T(M) \cap I_1 = \emptyset.$$

In what follows, $\sim$ means equivalence modulo $I_1$.

**Proposition 3.17** Let $M$ be a Baire space and let $B \in Q(T(M))$. Then

$$\mathcal{C}(B) := \{ A \in B(M) \mid \exists U \in B : A \sim U \}$$

is a quasipoint in $B(M)$ with $\mathcal{C}(B) \cap I_1 = \emptyset$.

**Proof:** Let $\pi : B(M) \to B(M)/I_1$ be the canonical projection. Then

$$\mathcal{C}(B) = \pi^{-1} (\pi(B)).$$

Since $\pi$ is a lattice homomorphism and $B \cap I_1 = \emptyset$, $\pi(B)$ is a filter base. Let $\hat{B}$ be a quasipoint that contains $\pi(B)$. Then, by proposition 3.14, $\pi^{-1} (\hat{B})$ is a quasipoint in $B(M)$ such that $\pi^{-1} (\hat{B}) \cap I_1 = \emptyset$. Clearly $\mathcal{C}(B) \subseteq \pi^{-1} (\hat{B})$.

Assume that this inclusion is proper and take an element $A \in \pi^{-1} (\hat{B}) \setminus \mathcal{C}(B)$. By proposition 3.13 there is a $U \in T(M)$ such that $A \sim U$. According to our assumption we have $U \notin B$. But then also $U \notin \pi^{-1} (\hat{B})$, because $B \subseteq \pi^{-1} (\hat{B})$ and $U \in \pi^{-1} (\hat{B})$ would imply $V \cap U \neq \emptyset$ for all $V \in B$ and therefore $U \in B$. Hence $M \setminus U \in \pi^{-1} (\hat{B})$, so $A \cap (M \setminus U) \in \pi^{-1} (\hat{B})$. But $A \sim U$ implies $A \cap (M \setminus U) \in I_1$ which contradicts $\pi^{-1} (\hat{B}) \cap I_1 = \emptyset$. Therefore $\pi^{-1} (\hat{B}) = \mathcal{C}(B)$. \qed

**Definition 3.11** A quasipoint $\mathcal{C}$ in $B(M)$ is called a quasipoint of second category if $\mathcal{C} \cap I_1 = \emptyset$. Otherwise it is called a quasipoint of first category. As in corollary 3.3 we denote the set of quasipoints of second category by $Q^2(B(M))$. 

Proposition 3.18 Let $M$ be a Baire space. Then the mapping
\[
\pi_M : Q(T(M)) \to Q^2(B(M))
\]
\[
\mathcal{B} \mapsto \mathcal{C}^{\mathcal{B}}
\]
is a homeomorphism.

Proof: From propositions 3.13, 3.3, 3.16 and 3.17 we have the following homeomorphisms:
\[
Q(T(M)) \to Q(T_r(M)) \to Q(B(M)/I_1) \to Q^2(B(M))
\]
\[
\mathcal{B} \mapsto \mathcal{B}^r \mapsto \pi(\mathcal{B}^r) \mapsto \pi^{-1}(\pi(\mathcal{B}^r)).
\]
\[\pi_M\] is just the composition of these. \(\square\)

Let $M$ be a Baire space. It is obvious that a quasipoint $\mathcal{C} \in Q(B(M))$ contains at most one quasipoint $\mathcal{B} \in Q(T(M))$. If $\mathcal{C}$ is of second category, then it is of the form $\mathcal{C}(\mathcal{B})$ for exactly one $\mathcal{B} \in Q(T(M))$ (proposition 3.17). If $\mathcal{C}$ is an atomic quasipoint then it is of second category if the defining atom is isolated in $M$. If $\mathcal{C} \in Q(B(M))$ is atomic and of first category then $\mathcal{C}$ does not contain a quasipoint $\mathcal{B} \in Q(T(M))$: $\mathcal{C} = \{A \in B(M) \mid x \in A\}$ and $\mathcal{B} \subseteq \mathcal{C}$ would imply $x \in U$ for all $U \in \mathcal{B}$. But this is a contradiction to the maximality of $\mathcal{B}$ since $\{x\}$ is not open. On the other hand, a quasipoint $\mathcal{C} \in Q(B(M))$ of first category can very well contain a quasipoint $\mathcal{B} \in Q(T(M))$. Let for example $M = \mathbb{R}$ and take any quasipoint $\mathcal{B}$ in $Q(T(\mathbb{R}))$. Then $\mathcal{B} \cup \{Q\}$ is contained in a quasipoint $\mathcal{C} \in Q(B(\mathbb{R}))$. $\mathcal{C}$ is of first category because $Q$ is meagre in $\mathbb{R}$.

Lemma 3.5 Let $M$ be a topological space and let $\mathcal{J}$ be a dual ideal in $T(M)$. Then $\mathcal{J}$ is a quasipoint if and only if the following alternative is satisfied:
\[
\forall U \in T(M) : U \in \mathcal{J} \text{ or } U^c \in \mathcal{J}.
\]

Proof: We recall that the pseudocomplement $U^c$ of $U \in T(M)$ is defined as $U^c := M \setminus \overline{U}$. If $\mathcal{B}$ is a quasipoint in $T(M)$ and $U \notin \mathcal{B}$ then there is some $V \in \mathcal{B}$ such that $V \cap U = \emptyset$. But then also $V \cap \overline{U} = \emptyset$, hence $V \subseteq U^c$ and therefore $U^c \in \mathcal{B}$.

Conversely, let $\mathcal{J}$ be a dual ideal in $T(M)$ that satisfies the alternative in the lemma. Let $\mathcal{B}$ be a quasipoint that contains $\mathcal{J}$. If $W \in \mathcal{B}$ then also $W \in \mathcal{J}$, for otherwise $W^c \in \mathcal{J}$, hence also $W^c \in \mathcal{B}$ and therefore $\emptyset = W \cap W^c \in \mathcal{B}$, a contradiction. \(\square\)

We denote by $\partial A$ the boundary of the subset $A$ of $M$, i.e. $\partial A = \overline{A} \cap \overline{M} \setminus A$. 

Proposition 3.19 A quasipoint \( \mathcal{C} \) in \( \mathcal{B}(M) \) contains a quasipoint \( \mathcal{B} \in \mathcal{Q}(\mathcal{T}(M)) \) if and only if \( \partial U \notin \mathcal{C} \) for all \( U \in \mathcal{T}(M) \).

Proof: Let \( U \in \mathcal{T}(M) \) such that \( \partial U \in \mathcal{C} \). Then \( M \setminus U, \overline{U} \in \mathcal{C} \) and therefore \( U, U^c \notin \mathcal{C} \). If \( \mathcal{C} \) would contain a quasipoint \( \mathcal{B} \in \mathcal{Q}(\mathcal{T}(M)) \) then, according to lemma 3.5, \( U \) or \( U^c \) would belong to \( \mathcal{B} \), a contradiction.

Conversely, if \( \partial U \notin \mathcal{C} \) for all \( U \in \mathcal{T}(M) \), then \( U \cup U^c \in \mathcal{C} \) and, therefore, \( U \in \mathcal{C} \) or \( U^c \in \mathcal{C} \). Hence \( \mathcal{C} \cap \mathcal{T}(M) \in \mathcal{Q}(\mathcal{T}(M)) \). \( \square \)

### 3.4 Stone Spectra of von Neumann algebras

Let \( \mathcal{R} \) be a von Neumann algebra, considered as a subalgebra of \( \mathcal{L}(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \). The Stone spectrum of the projection lattice \( \mathcal{P}(\mathcal{R}) \) is called the Stone spectrum of \( \mathcal{R} \) and we denote it by \( \mathcal{Q}(\mathcal{R}) \).

The projection lattice \( \mathcal{P}(\mathcal{R}) \) reflects the structure of the von Neumann algebra \( \mathcal{R} \). \( \mathcal{P}(\mathcal{R}) \) is an orthomodular lattice and therefore a projection \( P \in \mathcal{P}(\mathcal{R}) \) can be identified with the open closed subset \( \mathcal{Q}_P(\mathcal{R}) \) of \( \mathcal{Q}(\mathcal{R}) \). This shows that the Stone spectrum \( \mathcal{Q}(\mathcal{R}) \) of \( \mathcal{R} \) is in general a highly complicated object. So we cannot expect a simple characterization of \( \mathcal{Q}(\mathcal{R}) \) for a general von Neumann algebra \( \mathcal{R} \). Indeed only partial results are known.

Putting aside von Neumann algebras of finite dimension, the most simple case is the Stone spectrum of an abelian von Neumann algebra \( \mathcal{A} \). We will prove now that the Stone spectrum \( \mathcal{Q}(\mathcal{A}) \) of \( \mathcal{A} \) is homeomorphic to the Gelfand spectrum of \( \mathcal{A} \) in a canonical manner.

We begin with some preparations.

**Definition 3.12** Let \( A \in lin_{\mathbb{C}} \mathcal{P}_0(\mathcal{A}) \).

\[
A = \sum_{j=1}^{m} b_j P_j
\]

is called an orthogonal representation of \( A \) if the projections \( P_j \) are pairwise orthogonal.
In analogy to standard measure theoretical methods one can easily see that each 
\( A = \sum_k a_k E_k \in \text{lin}_C P_0(\mathcal{A}) \) has an orthogonal representation:
\[
\sum_{k=1}^n a_k E_k = (a_1 + \ldots + a_n)E_1 \ldots E_n \\
+ \sum_{i=1}^n (a_1 + \ldots + \hat{a}_i + \ldots + a_n)E_1 \ldots (I - E_i) \ldots E_n \\
+ \sum_{1\leq i<j\leq n} (a_1 + \ldots + \hat{a}_i + \ldots + \hat{a}_j + \ldots + a_n)E_1 \ldots (I - E_i) \ldots (I - E_j) \ldots E_n \\
+ \ldots + \sum_{i=1}^n a_i (I - E_1) \ldots E_i \ldots (I - E_n).
\]

We call this the **standard orthogonal representation of** \( A \).

**Lemma 3.6** Let \( \sum_{j=1}^m b_j P_j \) and \( \sum_{k=1}^n c_k Q_k \) be two orthogonal represenations of \( A \in \text{lin}_C P_0(\mathcal{A}) \). Then
\[
\sum_{j=1}^m b_j \chi_{P_j}(\mathcal{A}) = \sum_{k=1}^n c_k \chi_{Q_k}(\mathcal{A}).
\]

**Proof:** Without loss of generality we may assume that the occurring coefficients \( b_1, \ldots, b_m, c_1, \ldots, c_n \) are all different from zero. Then \( \sum_j b_j P_j = \sum_k c_k Q_k \) implies that \( P_1 + \ldots + P_m = Q_1 + \ldots + Q_n \) holds. Indeed let \( V_j := \text{im} P_j, W_k := \text{im} Q_k \ (j \leq m, k \leq n) \) and let \( v_{j_0} \in V_{j_0} \). Then
\[
b_{j_0} v_{j_0} = (\sum_{j=1}^m b_j P_j)v_{j_0} = (\sum_{k=1}^n c_k Q_k)v_{j_0} \in W_1 + \ldots + W_n,
\]

hence \( v_{j_0} \in W_1 + \ldots + W_n \). This shows \( V_1 + \ldots + V_m \subseteq W_1 + \ldots + W_n \) and by symmetry we obtain \( V_1 + \ldots + V_m = W_1 + \ldots + W_n \).

Let \( \beta \in Q(\mathcal{A}) \). If \( P_1 + \ldots + P_m \notin \beta \) (and consequently \( Q_1 + \ldots + Q_n \notin \beta \)) then both sides of the asserted equality are zero. If \( P_1 + \ldots + P_m \in \beta \) then, due to orthogonality, there are uniquely determined indices \( j_0, k_0 \) such that \( P_{j_0}, Q_{k_0} \in \beta \). But then \( P_{j_0} Q_{k_0} \in \beta \), in particular \( P_{j_0} Q_{k_0} \neq 0 \). This shows \( b_{j_0} = c_{k_0} \) and because of \( \sum_j b_j \chi_{P_j}(\mathcal{A})(\beta) = b_{j_0}, \sum_k c_k \chi_{Q_k}(\mathcal{A})(\beta) = c_{k_0} \) the assertion follows also in this case. \( \square \)

The **Gelfand spectrum of the abelian von Neumann algebra** \( \mathcal{A} \) is the set of non-zero multiplicative linear functionals on \( \mathcal{A} \). The weak*-topology turns it into a compact Hausdorff space \((\Omega(\mathcal{A}))\) which is denoted by \( \Omega(\mathcal{A}) \).
Theorem 3.2 Let \( \mathcal{A} \) be an abelian von Neumann algebra. Then the Gelfand spectrum \( \Omega(\mathcal{A}) \) is homeomorphic to the Stone spectrum \( \mathcal{Q}(\mathcal{A}) \) of \( \mathcal{A} \).

Proof: Let \( \tau \in \Omega(\mathcal{A}) \). Then

\[
\beta_\tau := \{ P \in \mathcal{P}_0(\mathcal{A}) \mid \tau(P) = 1 \}
\]

is a quasipoint of \( \mathcal{P}(\mathcal{A}) \):
For all \( P \in \mathcal{P}(\mathcal{A}) \) we have \( \tau(P) \in \{0, 1\} \) because \( \tau \) is multiplicative. By definition \( 0 \notin \beta(\tau) \) and \( P, Q \in \beta(\tau) \) implies \( \tau(PQ) = \tau(P)\tau(Q) = 1 \), hence \( PQ \in \beta(\tau) \). Moreover \( \tau(I - P) = 1 - \tau(P) \) and therefore

\[
\forall P \in \mathcal{P}(\mathcal{A}) : P \in \beta_\tau \quad \text{or} \quad I - P \in \beta_\tau.
\]

As \( \mathcal{P}(\mathcal{A}) \) is distributive, this means that \( \beta_\tau \) is a quasipoint.
Let \( \sigma, \tau \in \Omega(\mathcal{A}) \) such that \( \beta_\sigma = \beta_\tau \). Then for all \( P \in \mathcal{P}(\mathcal{A}) \): \( \sigma(P) = 1 \) if and only if \( \tau(P) = 1 \). Hence \( \sigma \) and \( \tau \) agree on \( \mathcal{P}(\mathcal{A}) \) and therefore also on \( \text{lin}_\mathbb{C}\mathcal{P}(\mathcal{A}) \). Using their continuity we conclude from the spectral theorem that \( \sigma = \tau \). The mapping

\[
\Omega(\mathcal{A}) \to \mathcal{Q}(\mathcal{A})
\]

\[
\tau \mapsto \beta_\tau
\]

is therefore injective.
Conversely, let \( \beta \in \mathcal{Q}(\mathcal{A}) \) be given. We define a mapping

\[
\tau_\beta : \mathcal{P}(\mathcal{A}) \to \{0, 1\}
\]

by

\[
\tau_\beta(P) := \begin{cases} 
1 & \text{if } P \in \beta \\
0 & \text{otherwise.}
\end{cases}
\]

Because of \( P, Q \in \beta \) if and only if \( PQ \in \beta \) we have

\[
\forall P, Q \in \mathcal{P}(\mathcal{A}) : \tau_\beta(PQ) = \tau_\beta(P)\tau_\beta(Q).
\]

In the next step we extend \( \tau_\beta \) to a linear functional on \( \text{lin}_\mathbb{C}\mathcal{P}(\mathcal{A}) \) (which will be denoted by \( \tau_\beta \) too). Let \( A = \sum_{j=1}^m b_j P_j \) be an orthogonal representation of \( A \in \text{lin}_\mathbb{C}\mathcal{P}(\mathcal{A}) \). We may assume that \( P_1 + \ldots + P_m = I \) and that \( b_1, \ldots, b_{m-1} \) are different from zero. We call such a representation a complete orthogonal representation. Because of

\[
\mathcal{Q}(\mathcal{A}) = \bigcup_{j \leq m} \mathcal{Q}_{P_j}(\mathcal{A})
\]
we have $\beta \in Q_{P_j}(A)$ for exactly one $j \leq m$. Denote this $j$ by $j_\beta$. Of course we are forced to define

$$\tau_\beta(\sum_{j=1}^m b_j P_j) := b_{j_\beta}.$$  

We have to show that $\tau_\beta$ is well defined. Let $\sum_{j=1}^m b_j P_j$ and $\sum_{k=1}^n c_k Q_k$ be two (complete) orthogonal representations of $A \in \text{lin}_C P(A)$. Then, by lemma 3.6 $b_{j_\beta} = c_{k_\beta}$. This proves that $\tau_\beta$ is well defined on $\text{lin}_C P(A)$. Now let $\sum_{k=1}^n a_k E_k$ be an arbitrary element of $\text{lin}_C P(A)$. Using the standard orthogonal representation we see that

$$\tau_\beta(\sum_{k=1}^n a_k E_k) = a_{j_1} + \ldots + a_{j_s}$$

where $j_1, \ldots, j_s$ are precisely those indices for which $E_{j_1}, \ldots, E_{j_s}$ are elements of $\beta$. This shows that $\tau_\beta$ is linear. Multiplicativity follows from linearity and the fact that $\tau_\beta$ is multiplicative on projections. $\tau_\beta$ is continuous in norm because for orthogonal representations we have

$$\left| \sum_{k=1}^n a_k P_k \right| = \max_{k \leq n} |a_k|.$$  

The spectral theorem assures that $\tau_\beta$ has a unique extension to a multiplicative linear functional on $A$ which we denote again by $\tau_\beta$. By construction we have

$$\beta_\tau_\beta = \beta$$

and therefore the mapping $\tau \mapsto \beta_\tau$ is a bijection from $\Omega(A)$ onto $Q(A)$. In order to prove that this is a homeomorphism we have only to show that it is continuous because $\Omega(A)$ and $Q(A)$ are compact Hausdorff spaces. Let $\tau_0 \in \Omega(A), 0 < \varepsilon < 1$ and let $P \in P(A)$ such that $\tau_0(P) = 1$. Then

$$N_w(\tau_0) := \{ \tau \in \Omega(A) | |\tau(P) - \tau_0(P)| < \varepsilon \}$$

is an open neighborhood of $\tau_0$ and from $\varepsilon < 1$ we conclude

$$\tau \in N_w(\tau_0) \iff \tau(P) = \tau_0(P) \iff P \in \beta_\tau \iff \beta_\tau \in Q_P(A).$$

This means that $N_w(\tau_0)$ is mapped bijectively onto the open neighborhood $Q_P(A)$ of $\beta_{\tau_0}$. The $Q_P(A)$ with $P \in \beta_{\tau_0}$ form a neighborhood base of $\beta_{\tau_0}$ in the Stone topology of $Q(A)$. Hence $\tau \mapsto \beta_\tau$ is continuous.  \qed
We can use the proof of theorem 3.2 (with some obvious changings) to show that the Stone spectrum of a \( \sigma \)-algebra of the form \( \mathfrak{A}(M)/\mathcal{I} \), where \( \mathfrak{A}(M) \) is a \( \sigma \)-algebra of subsets of a non-empty set \( M \) and \( \mathcal{I} \subseteq \mathfrak{A}(M) \) is a \( \sigma \)-ideal, is the Gelfand spectrum of an abelian \( C^* \)-algebra that is canonically associated to the \( \sigma \)-algebra \( \mathfrak{A}(M)/\mathcal{I} \).

Let \( \mathcal{F} := \mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C}) \) be the algebra of all bounded \( \mathfrak{A}(M) \)-measurable functions \( M \rightarrow \mathbb{C} \). \( \mathcal{F} \), equipped with the norm of uniform convergence, is an abelian \( C^* \)-algebra.

For \( f \in \mathcal{F} \), let

\[
P(f) := \{ x \in M \mid f(x) \neq 0 \}.
\]

Because of

\[
P(fg) = P(f) \cap P(g) \quad \text{and} \quad P(f + g) \subseteq P(f) \cup P(g)
\]

for all \( f, g \in \mathcal{F} \), the set

\[
\mathcal{F}(\mathcal{I}) := \{ h \in \mathcal{F} \mid P(h) \in \mathcal{I} \}
\]

is an ideal in \( \mathcal{F} \). It is closed in the norm topology of \( \mathcal{F} \): let \( (h_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{F}(\mathcal{I}) \) that converges uniformly to \( h \in \mathcal{F} \). If \( x \in P(h) \), there is \( n_x \in \mathbb{N} \) such that \( |h_{n_x}(x) - h(x)| < |h(x)|/2 \). Hence

\[
P(h) \subseteq \bigcup_{n \in \mathbb{N}} P(h_n) \in \mathcal{I}
\]

and, therefore, \( h \in \mathcal{F}(\mathcal{I}) \). The quotient

\[
\mathcal{A}_{\mathcal{B}} := \mathcal{F}/\mathcal{F}(\mathcal{I}),
\]

where \( \mathcal{B} \) is an abbreviation for \( \mathfrak{A}(M)/\mathcal{I} \), is an abelian \( C^* \)-algebra, which we call the \( C^* \)-algebra associated to the \( \sigma \)-algebra \( \mathcal{B} \). The norm of \( \mathcal{A}_{\mathcal{B}} \) is the quotient norm

\[
||[f]|| := \inf\{||f + h|| \mid h \in \mathcal{F}(\mathcal{I})\},
\]

and it is easy to see that this norm is an essential supremum norm:

\[
||[f]|| = \inf\{c > 0 \mid \{ x \in M \mid |f(x)| > c \} \in \mathcal{I}\}.
\]

In the following we will show that the Stone spectrum \( Q(\mathcal{B}) \) of \( \mathcal{B} \) is homeomorphic to the Gelfand spectrum \( \Omega(\mathcal{A}_{\mathcal{B}}) \) of \( \mathcal{A}_{\mathcal{B}} \).
The Stone spectrum $Q(\mathcal{B})$ is homeomorphic to

$$Q(\mathfrak{A}(M))_I := \{ \mathcal{B} \in Q(\mathfrak{A}(M)) \mid I^\perp \subseteq \mathcal{B} \}.$$ 

$h \in \mathcal{F}(\mathcal{I})$ can be uniformly approximated by step functions of the form

$$s_I = \sum_{k=1}^n a_k \chi_{A_k}$$

with $a_1, \ldots, a_n \in \mathbb{C}$ and mutually disjoint $A_1, \ldots, A_n \in \mathcal{I}$. Let $\tau_\mathcal{B} \in \Omega(\mathcal{F})$ be the character induced by $\mathcal{B} \in Q(\mathfrak{A}(M))_I$. It is quite analogously defined as in the case of abelian von Neumann algebras:

$$\tau_\mathcal{B}(\chi_A) := \begin{cases} 1 & \text{if } A \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases}$$

for all $A \in \mathfrak{A}(M)$. It follows from the definition of $\tau_\mathcal{B}$ that $\tau_\mathcal{B}(s_I) = 0$ for all step functions $s_I$ of the above form and therefore, by continuity, $\tau_\mathcal{B}$ vanishes on $\mathcal{F}(\mathcal{I})$. Hence $\tau_\mathcal{B}$ induces a character $\theta_\mathcal{B}$ of $\mathcal{A}_I$. Conversely, if a character $\theta$ of $\mathcal{A}_I$ is given, we obtain a character $\tau := \theta \circ \varrho$ of $\mathcal{F}$ by composition with the canonical projection $\varrho : \mathcal{F} \to \mathcal{F}/\mathcal{F}(\mathcal{I})$. As in the case of von Neumann algebras, $\tau$ gives rise to a quasipoint $B_\theta := \{ A \in \mathfrak{A}(M) \mid \tau(\chi_A) = 1 \}$ which, by construction, belongs to $Q(\mathfrak{A}(M))_I$. Thus we get a bijection $\mathcal{B} \mapsto \theta_\mathcal{B}$ from $Q(\mathfrak{A}(M))_I$ onto $\Omega(\mathcal{A}_I)$. The same argument as in the proof of theorem 3.2 shows that it is a homeomorphism. Therefore, we have proved the following theorem which is a generalization of 5.7.20 in [10]:

**Theorem 3.3** Let $M$ be a non-empty set, $\mathfrak{A}(M)$ a $\sigma$-algebra of subsets of $M$ and $\mathcal{I}$ a $\sigma$-ideal in $\mathfrak{A}(M)$. Furthermore, let $\mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})$ be the abelian algebra of all bounded $\mathfrak{A}(M)$-measurable functions $M \to \mathbb{C}$. $\mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})$ is a $C^*$-algebra with respect to the supremum-norm and the set $\mathcal{F}(\mathcal{I})$ of all $f \in \mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})$ that vanish outside some set $A \in \mathcal{I}$ is a closed ideal in $\mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})$. Then the Gelfand spectrum of the quotient $C^*$-algebra $\mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})/\mathcal{F}(\mathcal{I})$ is homeomorphic to the Stone spectrum of the $\sigma$-algebra $\mathfrak{A}(M)/\mathcal{I}$.

Note that $\mathcal{F}_{\mathfrak{A}(M)}(M, \mathbb{C})/\mathcal{F}(\mathcal{I})$ may fail to be a von Neumann algebra ([10], 5.7.21(iv)).
If the von Neumann algebra $\mathcal{R}$ is a factor of type $I$, i.e. if $\mathcal{R}$ is isomorphic to $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, we have the following simple results for the Stone spectrum $Q(\mathcal{R})$. We formulate and prove these results in the lattice $\mathbb{L}(\mathcal{H})$ rather than in the equivalent projection lattice $\mathcal{P}(\mathcal{L}(\mathcal{H}))$.

**Proposition 3.20** Let $\mathcal{B}$ be a quasipoint in $\mathbb{L}(\mathcal{H})$. $\mathcal{B}$ contains an element of finite dimension if and only if there is a unique line $C x_0$ in $\mathcal{H}$ such that

$$\mathcal{B} = \{ U \in \mathbb{L}(\mathcal{H}) \mid C x_0 \subseteq U \}.$$

$\mathcal{B}$ does not contain an element of finite dimension if and only if $W \in \mathcal{B}$ for all $W \in \mathbb{L}(\mathcal{H})$ of finite codimension.

**Proof:** Let $U_0 \in \mathcal{B}$ be finite dimensional. Then $U \cap U_0 \neq 0$ for all $U \in \mathcal{B}$ and therefore $\{ U \cap U_0 \mid U \in \mathcal{B} \}$ contains an element $V_0$ of minimal dimension. Hence $V_0 \subseteq U$ for all $U \in \mathcal{B}$ and by the maximality of $\mathcal{B}$ $V_0$ must have dimension one.

Assume that a quasipoint $\mathcal{B}$ in $\mathbb{L}(\mathcal{H})$ contains every $W \in \mathbb{L}(\mathcal{H})$ of finite codimension. Let $U$ be a finite dimensional subspace of $\mathcal{H}$. Then $U^\perp \in \mathcal{B}$ and therefore $U \notin \mathcal{B}$ because of $U \cap U^\perp = 0$.

Let $V \in \mathbb{L}(\mathcal{H})$ be of finite codimension and $V \notin \mathcal{B}$. Then there is some $U \in \mathcal{B}$ such that $U \cap V = 0$. Consider the orthogonal projection

$$P_{V^\perp} : \mathcal{H} \rightarrow V^\perp$$

onto $V^\perp$. $U \cap V = 0$ means that the restriction of $P_{V^\perp}$ to $U$ is injective. As $V^\perp$ is finite dimensional, $U$ must be finite dimensional too. $\square$

Quasipoints in $\mathbb{L}(\mathcal{H})$ that contain a line are **atomic**. The non-atomic quasipoints are called **continuous**.

Whereas the structure of atomic quasipoints is trivial, the set of continuous quasipoints mirrors the whole complexity of spectral theory of linear operators in $\mathcal{H}$. Therefore it is a real challenge to classify the continuous quasipoints in $\mathcal{L}(\mathcal{H})$.

In contrast to the case of Boolean algebras, Stone spectra are not compact in general. The situation can be even worse, as the following important example shows:

**Proposition 3.21** Let $\mathcal{H}$ be a Hilbert space of dimension greater than one. Then the Stone spectrum $Q(\mathcal{H}) := Q(\mathbb{L}(\mathcal{H}))$ is not compact and, if the dimension of $\mathcal{H}$ is infinite, $Q(\mathcal{H})$ is not even locally compact.
Proof: This is an easy consequence of Baire’s category theorem and the general fact that the Stonean space \( Q(L_U) \) of the principal ideal \( L_U := \{ V \in L \mid V \leq U \} \) of an arbitrary lattice \( L \) and \( U \in L \setminus \{0\} \) is homeomorphic to \( Q_U(L) \).

Indeed, assume that \( Q(H) \) is compact. Then there are \( U_1, \ldots, U_n \in L(H) \) such that
\[
\bigcup_{k=1}^n Q_{U_k}(H) = Q(H).
\]

Let \( x \) be an arbitrary nonzero element of \( H \). Then the atomic quasipoint \( B_{Cx} \) belongs to \( Q_{U_k}(H) \) for at least one \( k \) and hence \( H = \bigcup_{k=1}^n U_k \), a contradiction. By Baire’s category theorem, the same argument works not only for finite \( n \) but also for \( n = \aleph_0 \). This shows that \( Q(H) \) is not Lindelöf compact. □

The structure of Stone spectra of finite von Neumann algebras of type I has been clarified by A. Döring in [8].

The basis of Döring’s result is the observation that the Stone spectrum \( Q(R) \) of an arbitrary von Neumann algebra \( R \) with center \( C \) can be mapped canonically onto the Stone spectrum \( Q(C) \) of \( C \). Later on we need a stronger and more general result:

Proposition 3.22 Let \( R \) be a von Neumann algebra with center \( C \) and let \( A \) be a von Neumann subalgebra of \( C \). Then the mapping
\[
\zeta_A : B \mapsto B \cap A
\]
is an open continuous, and therefore identifying, mapping from \( Q(R) \) onto \( Q(A) \). Moreover
\[
\zeta_A(B) = \{ s_A(P) \mid P \in B \}
\]
for all \( B \in Q(R) \), where
\[
s_A(P) := \bigwedge \{ Q \in \mathcal{P}(A) \mid P \leq Q \}
\]
is the \( A \)-support of \( P \in \mathcal{P}(R) \).

Proof: \( B \cap A \) is clearly a filter base in \( \mathcal{P}(A) \). Let \( \beta \in Q(A) \) be a quasipoint that contains \( B \cap A \) and let \( C \in \beta \). If \( C \notin B \cap A \) then \( C \notin B \). Hence there is some \( P \in B \) such that \( P \cap C = 0 \). Because \( C \) is central this means \( PC = 0 \). But then \( P = PC + P(I-C) = P(I-C) \), i.e. \( P \leq I-C \). This implies \( I-C \in B \cap A \subseteq \beta \), a contradiction to \( C \in \beta \). Hence \( B \cap A \) is a
quasipoint in $\mathcal{A}$.

It follows immediately from the definition of the $\mathcal{A}$-support that

$$\forall P, Q \in \mathcal{P}(\mathcal{R}) : P \leq s_\mathcal{A}(P) \text{ and } s_\mathcal{A}(P \land Q) \leq s_\mathcal{A}(P) \land s_\mathcal{A}(Q)$$

holds. This implies that $\{s_\mathcal{A}(P) \mid P \in \mathcal{B}\}$ is a filter base contained in $\mathcal{B} \cap \mathcal{A}$. Because of $s_\mathcal{A}(P) = P$ for all $P \in \mathcal{P}(\mathcal{A})$, we must have equality.

Now we prove that

(i) $\forall P \in \mathcal{P}(\mathcal{R}) : \zeta_\mathcal{A}(Q_P(\mathcal{R})) = Q_{s_\mathcal{A}(P)}(\mathcal{A})$ and

(ii) $\forall Q \in \mathcal{P}(\mathcal{A}) : \zeta_\mathcal{A}(Q_Q(\mathcal{A})) = Q_Q(\mathcal{R})$

hold: It is obvious that $\zeta_\mathcal{A}(Q_P(\mathcal{R}))$ is contained in $Q_{s_\mathcal{A}(P)}(\mathcal{A})$. Let $\gamma \in Q_{s_\mathcal{A}(P)}(\mathcal{A})$. Then $P \in s_\mathcal{A}^{-1}(\gamma)$, and we shall show that this implies that $\{P\} \cup \gamma$ is a filter base in $\mathcal{P}(\mathcal{R})$. $P$ being a central projection, $\{P\} \cup \gamma$ is a filter base if and only if

$$\forall Q \in \gamma : PQ \neq 0.$$ 

Assume that $PQ = 0$ for some $Q \in \gamma$. Then $P \leq I - Q$, hence also $s_\mathcal{A}(P) \leq I - Q$, contradicting $s_\mathcal{A}(P) \in \gamma$. Let $\mathcal{B}$ be a quasipoint in $\mathcal{P}(\mathcal{R})$ that contains $\{P\} \cup \gamma$. Because of $s_\mathcal{A}(Q) = Q$ for all $Q \in \gamma$ we obtain

$$\gamma = s_\mathcal{A}(\{P\} \cup \gamma) \subseteq s_\mathcal{A}(\mathcal{B}) = \zeta_\mathcal{A}(\mathcal{B}).$$

Hence $\gamma = \zeta_\mathcal{A}(\mathcal{B})$ since $\zeta_\mathcal{A}(\mathcal{B})$ and $\gamma$ are quasipoints in $\mathcal{P}(\mathcal{A})$. This proves (i). (ii) follows from the fact that each quasipoint in $\mathcal{P}(\mathcal{A})$ is contained in a quasipoint in $\mathcal{P}(\mathcal{R})$. Properties (i) and (ii) imply that $\zeta_\mathcal{A}$ is open, continuous and surjective. \qed

In [8] a quasipoint $\mathcal{B} \in Q(\mathcal{R})$ is called abelian if it contains an abelian projection. The term “abelian quasipoint” is motivated by the following fact: If $E \in \mathcal{B}$ is an abelian projection then the “$E$-socle”

$$\mathcal{B}_E := \{P \in \mathcal{B} \mid P \leq E\},$$

which determines $\mathcal{B}$ uniquely, consists entirely of abelian projections. Moreover every subprojection of an abelian projection $E$ is of the form $CE$ with a suitable central projection $C$. Hence

$$\mathcal{B}_E = \{CE \mid C \in \mathcal{B} \cap \mathcal{C}\}$$

if $E$ is abelian.
Let $\theta \in \mathcal{R}$ be a partial isometry, i.e. $E := \theta^*\theta$ and $F := \theta\theta^*$ are projections. $\theta$ has kernel $E(\mathcal{H})^\perp$ and maps $E(\mathcal{H})$ isometrically onto $F(\mathcal{H})$. Now it is easy to see that for any projection $P_U \leq E$ we have

$$\theta P_U \theta^* = P_{\theta(U)}.$$  \hfill (3.6)

A consequence of this relation is

$$\forall P, Q \leq E : \theta(P \wedge Q)\theta^* = (\theta P\theta^*) \wedge (\theta Q\theta^*). \hfill (3.7)$$

If $B \in Q_F(\mathcal{R})$ then

$$\theta_*(B_E) := \{ \theta P\theta^* | P \in B_E \} \hfill (3.8)$$

is the $F$-socle of a (uniquely determined) quasipoint $\theta_*(B) \in Q_F(\mathcal{R})$: Equation 3.7 guarantees that $\theta_*(B_E)$ is a filter base. Let $B$ be a quasipoint that contains $\theta_*(B_E)$. Then $\theta_*(B_E) \subseteq B_F$. Assume that this inclusion is proper. If $Q \in B_F \setminus \theta_*(B_E)$ then $\theta^*Q\theta \notin B_E$ and therefore there is some $P \in B_E$ such that $P \wedge \theta^*Q\theta = 0$. But then $\theta P\theta^* \wedge Q = 0$, a contradiction. This shows that we obtain a mapping

$$\theta_* : Q_{\theta^*}(\mathcal{R}) \to Q_{\theta\theta^*}(\mathcal{R})$$

$$\mathcal{B} \mapsto \theta_*(\mathcal{B}).$$

It is easy to see that $\theta_*$ is a homeomorphism with inverse $(\theta^*)_*$. Note that $\theta_*$ is globally defined if $\theta$ is a unitary operator. The following result is fundamental:

**Proposition 3.23** ([8]) Let $\mathcal{R}$ be a von Neumann algebra with center $\mathcal{C}$ and let $\mathcal{B}_1, \mathcal{B}_2 \in Q(\mathcal{R})$. Then $\zeta_{\mathcal{C}}(\mathcal{B}_1) = \zeta_{\mathcal{C}}(\mathcal{B}_2)$ if and only if there is a partial isometry $\theta \in \mathcal{R}$ such that $\theta_*(\mathcal{B}_1) = \mathcal{B}_2$. If $\mathcal{R}$ is finite then $\theta$ can be chosen as a unitary operator in $\mathcal{R}$.

It is well known ([15]) that a von Neumann algebra $\mathcal{R}$ of type $I_n$, $n < \infty$, is isomorphic to the algebra $M_n(\mathcal{C})$ of all $(n, n)$-matrices with entries from the center $\mathcal{C}$ of $\mathcal{R}$. Hence $\mathcal{R}$ can be considered as the endomorphism algebra of the free $\mathcal{C}$-module $\mathcal{C}^n$. Using methods from the theory of $C^*$-modules one can prove

**Theorem 3.4** ([8]) Let $\mathcal{R}$ be a finite von Neumann algebra of type $I$. Then all quasipoints in $\mathcal{R}$ are abelian.

Applying proposition 3.23 we obtain the structure of Stone spectra of finite von Neumann algebras of type $I$:

**Theorem 3.5** ([8]) Let $\mathcal{R}$ be a finite von Neumann algebra of type $I$ and let $\mathcal{C}$ be the center of $\mathcal{R}$. Then the orbits of the action of the unitary group $U_\mathcal{R}$ of $\mathcal{R}$ on $Q(\mathcal{R})$ are the fibres of $\zeta_\mathcal{C}$. 


3.5 Boolean Quasipoints

Boolean filter bases in an orthomodular lattice $\mathbb{L}$ are defined as filter bases in $\mathbb{L}$ whose elements commute with each other. Analogous to (ordinary) quasipoints we define:

**Definition 3.13** Let $\mathbb{L}$ be an orthomodular lattice. A subset $\beta$ of $\mathbb{L}$ is called a Boolean quasipoint if it satisfies the following requirements:

(i) $0 \notin \beta$,
(ii) $\forall a, b \in \beta \exists c \in \beta : c \leq a \land b$,
(iii) $\forall a, b \in \beta : a \mathcal{C} b$,
(iv) $\beta$ is a maximal subset that satisfies (i), (ii), (iii).

The set of all Boolean quasipoints in $\mathbb{L}$ is denoted by $Q^b(\mathbb{L})$.

A Boolean quasipoint is therefore a maximal Boolean filter base. As for ordinary quasipoints it is easy to see that

(v) Let $\beta$ be a Boolean quasipoint in an orthomodular lattice $\mathbb{L}$. If $c \in \mathbb{L}$ has the property that $a \mathcal{C} c$ for all $a \in \beta$ and $b \leq c$ for some $b \in \beta$ then $c \in \beta$.

holds.

**Lemma 3.7** A Boolean sector $\mathbb{B}$ of an orthomodular lattice $\mathbb{L}$ is generated by every Boolean quasipoint contained in $\mathbb{B}$.

**Proof:** Let $\beta \in Q^b(\mathbb{L})$. Then $\beta$ is contained in some Boolean sector $\mathbb{B}$ because the lattice $\mathbb{L}(\beta)$ generated by $\beta$ is distributive. Note that $\beta$ is a quasipoint in $\mathbb{B}$. Let $a \in \mathbb{B}$. Then, by remark 3.11, $a \in \beta$ or $a^+ \in \beta$. Hence $a \in \mathbb{L}(\beta)$ in any case, i.e. $\mathbb{L}(\beta) = \mathbb{B}$. □

**Corollary 3.6** A Boolean quasipoint in an orthomodular lattice $\mathbb{L}$ is contained in exactly one Boolean sector of $\mathbb{L}$.

This means that the maximal distributive sublattices of $\mathbb{L}$ induce a partition of $Q^b(\mathbb{L})$. Therefore the name “Boolean sector”.

**Proposition 3.24** The Stone spectrum $Q(\mathbb{B})$ of a Boolean sector of $\mathbb{L}$ coincides with the set of Boolean quasipoints contained in $\mathbb{B}$.
**Proof:** A Boolean quasipoint contained in $\mathbb{B}$ is obviously a quasipoint of the lattice $\mathbb{B}$. Conversely, let $\mathcal{B} \in \mathcal{Q}(\mathbb{B})$ and assume that $\mathcal{B}$ is not a Boolean quasipoint. Then $\mathcal{B}$ is properly contained in a Boolean quasipoint $\beta$ in $\mathbb{L}$. Let $b \in \beta \setminus \mathcal{B}$. Then there is some $a \in \mathbb{B}$ not commuting with $b$. But $a$ or $a^\perp$ belongs to $\mathcal{B}$ hence also to $\beta$. This gives a contradiction. □

**Proposition 3.25** Let $\mathcal{R}$ be a von Neumann algebra. Then there is a canonical one to one correspondence between the maximal abelian subalgebras of $\mathcal{R}$ and the Boolean sectors of the projection lattice $\mathcal{P}(\mathcal{R})$.

**Proof:** Let $\mathbb{B} \subseteq \mathcal{P}(\mathcal{R})$ be a Boolean sector and denote by $\mathcal{M}(\mathbb{B})$ the von Neumann algebra generated by $\mathbb{B}$, i.e. the double commutant of $\mathbb{B}$:

$$\mathcal{M}(\mathbb{B}) = \mathbb{B}^\mathcal{C}.$$

$\mathcal{M}(\mathbb{B})$ is an abelian von Neumann subalgebra of $\mathcal{R}$. It is contained in a maximal abelian subalgebra $\mathcal{M}$ of $\mathcal{R}$. Let $A \in \mathcal{M}_{sa}$. By the spectral theorem the spectral projections of $A$ commute with every operator that commutes with $A$. Hence all spectral projections of $A$ commute with all elements of $\mathbb{B}$. It follows from the maximality of $\mathbb{B}$ that all spectral projections of $A$ belong to $\mathbb{B}$. Therefore $A \in \mathcal{M}(\mathbb{B})$, i.e. $\mathcal{M}(\mathbb{B})$ is maximal.

Conversely, let $\mathcal{M}$ is a maximal abelian subalgebra of $\mathcal{R}$ and let $P \in \mathcal{P}(\mathcal{R})$ commute with all elements of $\mathcal{P}(\mathcal{M})$. Then, again by the spectral theorem, $P$ commutes with all elements of $\mathcal{M}$ and therefore $P \in \mathcal{M}$ since $\mathcal{M}$ is maximal. This shows that $\mathcal{P}(\mathcal{M})$ is a Boolean sector of $\mathcal{P}(\mathcal{R})$ and that the mapping $\mathbb{B} \mapsto \mathbb{B}^{\mathcal{C}}$ is a bijection from the set of Boolean sectors of $\mathcal{P}(\mathcal{R})$ onto the set of maximal abelian subalgebras of $\mathcal{R}$. □

For a general von Neumann algebra it is an intricate problem to determine its maximal abelian subalgebras. In the case $\mathcal{R} = \mathcal{L}(\mathcal{H})$ however, they can be determined up to unitary equivalence. Moreover every abelian von Neumann algebra is isomorphic to a maximal abelian subalgebra of $\mathcal{L}(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$. These are well known results ([15, 26]). We like to reprove them here in order to demonstrate the use of Stone spectra.

We recall some well known notions and facts. A Hausdorff space $\Omega$ is called a **Stone space** if it is compact and *extremely disconnected*. The Stone spectrum of an abelian von Neumann algebra is a Stone space, but the converse is not true ([16], p.228). If $\Omega$ is a Stone space, then $C(\Omega)$ is a von Neumann algebra if and only if $\Omega$ carries a family of normal probability measures that separates the continuous functions on $\Omega$. In this case $\Omega$ is called a **hyperstonean space** ([26], III.1).
Let $\mathcal{A}$ be an abelian von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. By the Gelfand representation theorem $\mathcal{A}$ is isomorphic to $C(\Omega)$, where $\Omega$ is the Gelfand spectrum of $\mathcal{A}$, and by theorem 3.2 $\Omega$ is homeomorphic to the Stone spectrum $\mathcal{Q}(\mathcal{A})$ of $\mathcal{A}$.

Let $D$ be the set of isolated points of $\Omega$. Since $\mathcal{H}$ is separable, $D$ is at most countably infinite. $D$ is an open subset of the extremely disconnected space $\Omega$. So its closure

$$\Omega_d := \overline{D}$$

is open and closed. This leads to a partition

$$\Omega = \Omega_c \sqcup \Omega_d$$

of $\Omega$ into open and closed sets. By the way, if $D$ is infinite then $\Omega_d$ is a very large set: it can be shown that $\Omega_d$ is the Stone-Čech compactification of $D$.

Because $\Omega_c$ and $\Omega_d$ are open and closed, this partition induces an isometric isomorphism

$$C(\Omega) \rightarrow C(\Omega_c) \times C(\Omega_d)$$

$$f \mapsto (f|_{\Omega_c}, f|_{\Omega_d}).$$

Using the embedding $C(\Omega_c) \rightarrow C(\Omega)$, $f \mapsto f_c$, defined by

$$f_c := \begin{cases} f & \text{on } \Omega_c \\ 0 & \text{on } \Omega_d \end{cases}$$

and the similar defined embedding $C(\Omega_d) \rightarrow C(\Omega)$, we see that $C(\Omega)$ can be written as a direct sum

$$C(\Omega) = C(\Omega_c) \oplus C(\Omega_d).$$

From this representation it is immediate that $\Omega$ is hyperstonean if and only $\Omega_c$ and $\Omega_d$ are.

Since $\Omega_c$ has no isolated points, $C(\Omega_c)$ is isomorphic to $L^\infty([0,1[, \lambda)$ where $\lambda$ denotes the Lebesgue measure on $[0,1[$ ([20], III.1). It remains to find a canonical Hilbert space representation for $C(\Omega_d)$.

If $\Omega$ is a Stonean space we denote by $\mathcal{OC}(\Omega)$ the set of open and closed subsets of $\Omega$. Since the interior of a closed subset of $\Omega$ is open
and closed, $\mathcal{O}C(\Omega)$ is a complete distributive lattice with respect to the operations

\[
\bigvee_{k \in K} U_k := \bigcup_{k \in K} U_k, \\
\bigwedge_{k \in K} U_k := \text{int}(\bigcap_{k \in K} U_k).
\]

It is easy to see that $\mathcal{O}C(\Omega)$ is even completely distributive.

**Lemma 3.8** Let $\Omega$ be a Stonean space. Then the Stone spectrum of $\mathcal{O}C(\Omega)$ is canonically homeomorphic to $\Omega$.

**Proof:** Let $\omega \in \Omega$ and let $\mathfrak{B}$ be a quasipoint over $\omega$. Then $\omega \in \overline{U}$ for all $U \in \mathfrak{B}$. But $U$ is open and closed, so $\mathfrak{B}$ consists of all open closed neighborhoods of $\omega$. This means that there is exactly one quasipoint over $\omega$. Hence the mapping

\[
pt : \mathcal{Q}(\mathcal{O}C(\Omega)) \to \Omega,
\]

defined by $\{pt(\mathfrak{B})\} := \bigcap_{U \in \mathfrak{B}} U$, is a bijection. Obviously

\[
pt(\mathcal{Q}_U(\mathcal{O}C(\Omega))) = U
\]

for all $U \in \mathcal{O}C(\Omega)$, so $pt$ is a homeomorphism. \(\square\)

**Proposition 3.26** Let $\mathcal{H}$ be a Hilbert space (of arbitrary dimension) and let $(e_k)_{k \in \mathbb{K}}$ be an orthonormal basis for $\mathcal{H}$. Then there is a unique Boolean sector $\mathfrak{B}$ of $\mathcal{L}(\mathcal{H})$ that contains all $\mathcal{C}e_k$ $(k \in \mathbb{K})$. $\mathfrak{B}$ consists of all $U \in \mathcal{L}(\mathcal{H})$ with the property

\[
\forall k \in \mathbb{K} : e_k \in U \cup U^\perp.
\]

**Proof:** It is obvious that $\{\mathcal{C}e_k \mid k \in \mathbb{K}\}$ is contained in a Boolean sector $\mathfrak{B}$. Let $U \in \mathfrak{B}$. $\mathcal{C}e_kCU$ means

\[
\mathcal{C}e_k = (\mathcal{C}e_k \cap U) + (\mathcal{C}e_k \cap U^\perp)
\]

and therefore, according to the minimality of $\mathcal{C}e_k$,

\[
e_k \in U \cup U^\perp.
\]

Hence it suffices to show that the elements of $\{U \in \mathcal{L}(\mathcal{H}) \mid \forall k \in \mathbb{K} : e_k \in U \cup U^\perp\}$ commute with each other. But this follows easily from remark 2.2. \(\square\)
The Boolean sector $\mathbb{B}$ occurring in proposition 3.26 is called **maximal atomic** because it contains a maximal number of atoms of $\mathbb{L}(\mathcal{H})$.

We return to the study of $C(\Omega_d)$. Because $\Omega_d = \overline{D}$ and the elements of $D$ are isolated, each open and closed subset of $\Omega_d$ is the closure of its intersection with $D$:

$$\forall \, O \in \mathcal{OC}(\Omega_d) : \ O = \overline{O \cap D},$$

that is

$$\forall \, O \in \mathcal{OC}(\Omega_d) : \ O = \bigvee_{\delta \in O \cap D} \{\delta\}.$$

The projections of $C(\Omega_d)$ are the characteristic functions of the elements $O \in \mathcal{OC}(\Omega_d)$. Therefore they can be written as

$$\chi_O = \bigvee_{\delta \in O \cap D} \chi_\delta.$$

Let $\mathcal{H}_D$ be a Hilbert space of dimension $\# D$ and take an orthonormal basis $(e_\delta)_{\delta \in D}$, labeled with the elements of $D$. Moreover, let $\mathbb{B}$ be the Boolean sector of $\mathbb{L}(\mathcal{H}_D)$ that is defined according to proposition 3.26 by this orthonormal basis and let

$$\mathcal{P}(\mathbb{B}) := \{P_U \mid U \in \mathbb{B}\}.$$

Since each $U \in \mathbb{B}$ is the join of the lines $\mathbb{C}e_\delta$ that are contained in $U$, we obtain a mapping

$$\tau : \mathcal{P}(\mathbb{B}) \rightarrow \mathcal{P}(C(\Omega_d))$$

$$P_U \mapsto \bigvee\{\chi_\delta \mid e_\delta \in U\}.$$  

It is easy to see that $\tau$ is a lattice isomorphism, so it induces a homeomorphism from the Stone spectrum $\mathcal{Q}(\mathcal{P}(\mathbb{B}))$ of $\mathcal{P}(\mathbb{B})$ onto the Stone spectrum $\mathcal{Q}(\mathcal{P}(C(\Omega_d)))$. According to the foregoing considerations and to lemma 3.8 the latter is homeomorphic to $\Omega_d$. This shows that the von Neumann algebra $C(\Omega_d)$ is isomorphic to the maximal abelian subalgebra $\mathcal{M}(\mathcal{P}(\mathbb{B}))$ of $\mathcal{L}(\mathcal{H}_D)$. Altogether we have proved:

**Proposition 3.27** Let $\mathcal{A}$ be an abelian von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Then there is a separable Hilbert space $\mathcal{H}_D$ and a maximal atomic Boolean sector $\mathbb{B}$ of $\mathcal{P}(\mathcal{H}_D)$ such that $\mathcal{A}$ is isomorphic to the maximal abelian subalgebra $L^\infty([0,1[, \lambda) \oplus \mathcal{M}(\mathbb{B})$ of $\mathcal{L}(L^2([0,1[, \lambda) \oplus \mathcal{H}_D)$, where $\lambda$ denotes the Lebesque measure on $[0,1]$. The isomorphy class of $\mathcal{A}$ is determined by the cardinality of the set $D$ of isolated points of the Gelfand spectrum $\Omega$ of $\mathcal{A}$ and fact whether $\Omega \setminus \overline{D}$ is empty or not.
Applying this result we can describe the isomorphy classes of Boolean sectors of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ for a separable Hilbert space $\mathcal{H}$. If $\mathcal{M}_1, \mathcal{M}_2$ are maximal abelian von Neumann subalgebras of $\mathcal{L}(\mathcal{H})$ and if $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism (recall that “isomorphism between von Neumann algebras” always means “$\ast$-isomorphism between von Neumann algebras”) then there is a unitary $U \in \mathcal{L}(\mathcal{H})$ such that $\varphi(A) = UAU^*$ for all $A \in \mathcal{M}_1$ (I3, p.661). We know that $\mathcal{M}_k = \mathcal{M}(\mathcal{B}_k) = \mathcal{B}_k^{cc}$ ($k = 1, 2$) for suitable Boolean sectors $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathcal{P}(\mathcal{L}(\mathcal{H}))$. Therefore the isomorphy of maximal abelian subalgebras of $\mathcal{L}(\mathcal{H})$ is the same as the unitary equivalence of the corresponding Boolean sectors of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$.

To each maximal abelian subalgebra $\mathcal{M}(\mathcal{B})$ of $\mathcal{L}(\mathcal{H})$ we assign a pair of numbers $(p, n) \in \{0, 1\} \times \{n \mid 0 \leq n \leq \dim \mathcal{H}\}$ defined in the following way: Let $D_\mathcal{B}$ be the set of isolated points in the Stone spectrum $\mathcal{Q}(\mathcal{B})$ and let

$$n := \# D_\mathcal{B}$$

and

$$p := \begin{cases} 0 & \text{if } D_\mathcal{B} = \mathcal{Q}(\mathcal{B}) \\ 1 & \text{otherwise.} \end{cases}$$

Then we obtain from proposition 3.27.

**Corollary 3.7** Let $\mathcal{H}$ be a separable Hilbert space. Then the unitary equivalence classes of Boolean sectors of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ correspond to the elements of $\{0, 1\} \times \{n \mid 0 \leq n \leq \dim \mathcal{H}\}$. 
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