Large deviations, condensation and giant response in a statistical system

Federico Corberi

Dipartimento di Fisica ‘E. R. Caianiello’, and INFN, Gruppo Collegato di Salerno, and CNISM, Unità di Salerno, Università di Salerno, via Giovanni Paolo II 132, 84084 Fisciano (SA), Italy

E-mail: corberi@sa.infn.it

Received 5 July 2015, revised 21 September 2015
Accepted for publication 25 September 2015
Published 26 October 2015

Abstract

We study the probability distribution $P$ of the sum of a large number of non-identically distributed random variables $n_m$. Condensation of fluctuations, the phenomenon whereby one of such variables provides a macroscopic contribution to the global probability, is discussed and interpreted by analogy with phase-transitions in statistical mechanics. A general expression for $P$ is derived, and its sensitivity to the details of the distribution of a single $n_m$ is worked out. These general results are verified by the analytical and numerical solution of some specific examples.

Keywords: probability, condensation, response

1. Introduction

Condensation is the phenomenon whereby a finite fraction of some quantity, e.g. a particle density, concentrates into a small region of phase-space, as in the paradigmatic example of a vapor transforming into a liquid when crossing a phase-transition. Condensation is observed in a number of different models, related to magnetic properties [1], gravity [2–4], mass transport and other issues [5–20]. Despite the prominent role played by molecular interactions in most cases, condensation can also be observed in non-interacting systems as, for instance, in a quantum Bose–Einstein condensate [21] or in classical models such as the spherical model of a ferromagnet [5]. In the above-mentioned cases condensation occurs because the condensing quantity—the particle number in the former example or the spin variance in the latter—is conserved. This constraint acts like an effective interaction among the constituents bringing about the transition [22, 23]. Indeed, condensation does not occur in a non-interacting boson gas—as in the case of photons—which does not conserve the number of constituents.

A different manifestation of condensation is observed when probability distributions of a fluctuating collective variable $\mathcal{N}$, such as the number of particles in a thermodynamic system,
are considered. In this case, a fluctuation $\mathcal{N} = N$ well above the typical value can be associated with a condensed configuration of the system [22–32]. This phenomenon, referred to as condensation of fluctuations, is not restricted to the particle number $\mathcal{N}$ but was observed for quantities as diverse as energy, exchanged heats, particles currents etc [22–32]. It was shown [22, 23] that in some systems condensation of fluctuations may occur because, from the mathematical point of view, asking for a specific value $\mathcal{N} = N$ constrains the system in a similar way to a conservation law.

In this paper we study the probability distribution $P$ of the sum $\mathcal{N}$ of $M$ non-identically distributed random variables. We discuss how an interpretation can be provided, along the guidelines of statistical mechanics, in terms of a phase-transition between a normal phase with a vanishing order parameter $\ell_r$ and a condensed one with $\ell_r > 0$. A general expression for the probability $P$ is found and the radically different behavior of this quantity in the normal and in the condensed phase are discussed and illustrated by comparing the analytical and numerical solution of some specific models. In particular, in the condensed phase, the notable phenomenon of the giant response—a dramatic change of $P$ as the statistical properties of even a single random variable is modified—is pointed out.

This paper is organized as follows: in section 2 we introduce the statistical model that will be studied and set the notation. In sections 3 and 4 its behavior is discussed when condensation does and does not occur, providing also an example by means of an analytically tractable case for identically distributed variables (sections 3.1 and 4.1, respectively). In section 5 the case of non-identically distributed variables is addressed and the phenomenon of the giant response is discussed (section 5.1). Some examples are considered in section 5.2. Finally, in section 6 we briefly summarize and conclude the paper.

2. The statistical model

In order to set the stage, let us consider the independent variables $n_m (m = 1, M)$ subject to a probability $p_m(n_m; K)$, where $K$ is a set of parameters. This probabilistic setup is suited to describe at a simple level a variety of systems ranging from physics to chemistry, biology and social sciences. For instance, one can consider $M$ receptors where $n_m$ ligand particles, like those of a pollutant, can be adsorbed, or $n_m$ electrons populating $M$ atomic levels. One can also think of $M$ individuals, or agents, collecting $n_m$ resources with a certain probability $p_m$. In the former examples the temperature can be one of the control parameters $K$ but, in general, others can be present.

We fix the language by speaking of $M$ receptors hosting a total number

$$\mathcal{N} = \sum_{m=1}^{M} n_m$$

of particles, with an average value $\langle \mathcal{N} \rangle = \sum_{m=1}^{M} \langle n_m \rangle$, where $\langle n_m \rangle = \sum_{n} n p_m(n; K)$. For ease of notation the dependence on $K$ will often be dropped, and $M$ will be considered large.

We are interested in the probability of observing a total number $\mathcal{N} = N$ of particles

$$P(N, M) = \sum_{n_1, n_2, \ldots, n_M} p_1(n_1)p_2(n_2)\cdots p_M(n_M)\delta_{\mathcal{N}, N}$$

$$= \frac{1}{2\pi i} \oint dz e^{M[\ln Q(z,M) - \rho \ln z]}.$$

(2)
where, for discrete variables, we used the representation 

\[ Q(z, M) = \left[ \prod_{m=1}^{M} \sum_{n_m} p_m(n_m) z^{n_m} \right]^\frac{1}{2} \]

and \( \rho = \frac{N-1}{M} \approx \frac{N}{M} \) is the particles’ density.

As explained in [7, 22, 23] the probability distribution \( P \) of the fluctuations of the particle number corresponds also to the partition function of a dual model where the number of particles is conserved. For example, with some particular choices of the microscopic probabilities \( p \) that will be considered below, such a dual model corresponds to specific instances of the so-called urn model (or balls in boxes model) or of a zero-range process. This duality, which to the best of our knowledge has never been discussed in connection with the above-mentioned models, allows us to borrow a number of well established results in this research area to illustrate the behavior of \( P \)—whose properties are discussed here in a rather large generality—in some exemplifying cases.

The following relation holds [4, 6]

\[ P(N, M) = \sum_{n=0}^{N} \pi(n, N, M) \]

where

\[ \pi(n, N, M) = P(N-n, M-1)p_m(n) \]

is the probability that, when the \( M \)th receptor is added to the previous \( M-1 \), \( n \) particles are stored in it. Equation (4) is a recurrence relation allowing one to determine the probability distribution of \( M \) variables once the one for \( M-1 \) is known.

Let us discuss the basic mechanisms whereby the recurrence (4) works to build up the full probability \( P(N, M) \). Denoting with \( \overline{N} = M\overline{\pi} \) the value of \( N \) where \( P \) is maximum, the largest \( P(N-n, M-1) \) in equation (5) is the one with \( n = N - \overline{N} \). Note that this term is present in equation (4) only if \( N \geq \overline{N} \).

### 3. Gas phase

According to the previous discussion, for \( N \leq \overline{N} \), \( \pi \) has a maximum at a value \( n = n_e \) which is microscopic and does not scale with \( M \) (see the inset of figure 1 for a specific example with \( n_e = 0 \), to be discussed below). This is so because the quantities \( P(N-n, M-1) \) in equation (5) lower with \( n \) (i.e. moving away from the maximum) and the same is true for \( p_m(n) \), for sufficiently large \( n > n_e \) (being \( p_m(n) \) normalized). If the \( p_m \)'s are monotonously decreasing, it is \( n_e = 0 \).

A similar setting, with \( \pi \) peaked at a microscopic \( n_e \), is also found for \( N > \overline{N} \), when the largest probability \( P(\overline{N}, M-1) \) is contained in the sum on the rhs of equation (5), but its maximum is tamed by the microscopic probabilities, i.e. \( \lim_{M \to \infty} P(\overline{N}, M-1)p_m(N - \overline{N}) = 0 \).

The situation with \( \pi \) peaked in \( n = n_e \) is physically intuitive: It expresses the fact that, when \( M \) is large, the occupancy \( \pi \) of the new receptor (the \( M \)th) is microscopic. We will denote this situation, with a uniformly small occupation, the normal (or gas) phase.
3.1. An example

Let us illustrate these behaviors by considering a specific example with power-law distribution

\[ p_m(n) = c (n + 1)^{-K_m}, \]  

where \( K_m > 1 \) and \( c \) is a normalization.

We start with the simplest case where \( K_m = K \) does not depend on \( m \) [32]. In the inset of figure 1, \( \pi \) is plotted for \( K = 3/2 \) and different choices of \( N \). Here one observes a sharp peak in \( n = n_g \equiv 0 \), as expected.

In the gas phase, \( P \) can be determined in the large-\( M \) limit by a steepest-descent evaluation [2, 3, 6, 17–20, 22, 23, 32, 33] of the integral on the rhs of equation (2), leading to

\[ P(N, M) \simeq e^{-MR(\rho)}, \]  

with an \( M \)-independent rate function

\[ R(\rho) = -\ln \left[ z^*(\rho) \right] + \rho \ln z^*(\rho), \]  

where \( z^*(\rho) \) is given by the saddle point equation

\[ z^*(\rho) = \frac{Q\left[ z^*(\rho) \right]}{Q\left[ z^*(\rho) \right]} = \rho. \]  

For the specific example above, this equation can be cast as

\[ \frac{Li_{K-1}(z^*)}{Li_K(z^*)} = \rho + 1, \]  

where \( Li_K(z) \) is the polylogarithm (Jonquière’s function). It admits a solution with \( z < 1 \) for any finite value of \( \rho \) if \( K \lesssim K_c = 2 [2–4, 6, 13, 17–20] \). \( P \) is then expressed by equations (7), (8) with [2–4, 6, 17–20].
The rate function $R(\rho)$ obtained in this way is plotted with a black heavy-dashed line in figure 2. In the same picture the quantity $M \ln M$ obtained from equation (2) by exact enumeration, is shown for different choices of $M$. One observes that $M$ approaches the asymptotic $M$-independent form $R(\rho)$ as $M$ is increased. Note that the convergence is faster at small densities. It must be recalled that, for $K \leq K_c$, the average number $\langle N \rangle$ of particles is not finite, meaning that in the large-$M$ limit fluctuations with large $N$ are very likely, as it is reflected by the vanishing of $R(\rho)$ at large densities. However, for finite $M$ such large values of $N$ cannot be sustained and $R$, after reaching a minimum, raises again increasing $\rho$, thus determining the existence of a most probable value of the fluctuations. The position of such a value is pushed to larger densities by increasing $M$, providing in this way a gradual convergence, from smaller to larger values of $\rho$, of $R$ towards $R$.

4. Condensed phase

A radically different situation occurs for $N > \overline{N}$, when $P(\overline{N}, M-1)$ grows fast enough to give $\lim_{M \to \infty} P(\overline{N}, M-1) p_M (N - \overline{N}) \neq 0$. For the choice (6) with $K_m = K$, this happens when $K > K_c$.

In this case the sum in equation (4) does not only take contributions around the microscopic value $n_\pi$, since $\pi$ can be non-negligible up to a certain $n = n_f$ of order $M$. This is because $\pi$ is triggered by the maximum of $P(N-n, M-1)$ at a value $n = N - \overline{N}$ which, for given $\rho$, is of order $M$ itself. Usually $P$ is sharply peaked around the maximum as to give $n_f \simeq N - \overline{N}$ for $N$ sufficiently larger than $\overline{N}$. Figure 1 shows this for the model (6) with $K = 3$: $\pi$ develops a second peak at $n = n_f \simeq N - \overline{N}$ and only for $n > n_f$ does it drop down.

\[
Q(z^*) = L_K(z^*). \tag{11}
\]

The rate function $R(\rho)$ obtained in this way is plotted with a black heavy-dashed line in figure 2. In the same picture the quantity

\[
\mathcal{R}(\rho, M) = -\frac{1}{M} \ln P(M\rho, M) \tag{12}
\]

obtained from equation (2) by exact enumeration, is shown for different choices of $M$. One observes that $\mathcal{R}$ approaches the asymptotic $M$-independent form $R(\rho)$ as $M$ is increased. Note that the convergence is faster at small densities. It must be recalled that, for $K \leq K_c$, the average number $\langle N \rangle$ of particles is not finite, meaning that in the large-$M$ limit fluctuations with large $N$ are very likely, as it is reflected by the vanishing of $R(\rho)$ at large densities. However, for finite $M$ such large values of $N$ cannot be sustained and $\mathcal{R}$, after reaching a minimum, raises again increasing $\rho$, thus determining the existence of a most probable value of the fluctuations. The position of such a value is pushed to larger densities by increasing $M$, providing in this way a gradual convergence, from smaller to larger values of $\rho$, of $\mathcal{R}$ towards $R$. 

\[\text{Figure 2.}\] $R(\rho, M)$ is plotted against $\rho$ for the model with probabilities (6) with a uniform $K_m = K = 3/2 < K_c$, for different values of $M$ in the key. The heavy-dashed line is the asymptotic (for $M \to \infty$) expression $R(\rho)$ obtained from equations (8), (10) and (11).
to negligible values. The properties of this second peak can be described using extreme-value statistics [34].

The physical interpretation is that, when a value of \( N \) larger than the typical one \( \bar{N} \) is attempted, the occupancy of the new receptor \( M \) can be either microscopic or macroscopic. Then, together with the uniformly scarcely populated gas phase, a liquid, or condensed phase coexists characterized by a single \( n_{ul} \) hosting a finite fraction of the \( N \) particles.

From the previous considerations a close resemblance emerges with the problem of a gas–liquid transition, with \( N \) being a control parameter playing the role of the volume, \( \rho_t = n_t/M \) an order parameter and \( \mathcal{H} \sim -\ln \pi \) an energetic landscape. Note that the amount of condensed versus normal fluctuations, described by \( \rho_t \), depends on \( N \): the fraction of condensate is absent at \( N = \bar{N} \), and increases with \( N \).

When condensation occurs equation (10) has no solutions. Therefore the steepest-descent evaluation of the integral in (2) cannot be carried over straightforwardly as done for the gas phase in section 3. In [32] an upgraded saddle point technique based on a density functional approach was shown for a case with continuous variables related to the example (6). Another way of proceeding—still resorting to the saddle point technique—is illustrated in the appendix. Here we prefer to determine the form of \( P \) in a different way, which is discussed now. Starting for simplicity from equally distributed variables, \( p_m(n) \equiv p(n) \), we assume that particles condensed in a single receptor contribute a fraction \( C(N, M) = (1-a)P \) of the global probability while the others, scattered over the remaining locations, provide the remaining part \( G(N, M) = aP \). The parameter \( a \) depends on \( N \) in such a way that \( a = 1 \) (there is no condensate) at \( N = \bar{N} \), while \( a \to 0 \) (all the particles are condensed) for \( N \to \infty \). Casting equation (4) as

\[
P(N, M) \approx \sum_{n=0}^{n_{ul}} \pi(n, N, M) + \sum_{n=n_{ul}}^{N} \pi(n, N, M) = G(N, M) + C(N, M),
\]

where \( n_{ul} \) is the value of \( n \) where \( \pi \) is minimum (see figure 1), allows one to identify \( G \) and \( C \) as the first and second sum on the rhs, respectively.

For \( N \gg \bar{N} \) the peak around \( n_t \) becomes sharper and a Gaussian approximation for the evaluation of the second term gives

\[
C(N, M) = (1-a)P(N, M) \approx \sigma P(N_t, M-1)p(N - N_t),
\]

where \( N_t = N - n_t \approx \bar{N} \) and \( \sigma = \sqrt{2\pi \left[ 1 - (\partial^2 \ln \pi/\partial n^2)_{n=n_t} \right]} \). Assuming that in the large-\( N \) limit \( \sigma P(N_t, M) \) has only a weak dependence on \( N \), equation (14) has an approximate solution with

\[
\sigma P(N_t, M-1) \approx (1-a)b_M,
\]

and

\[
P(N, M) \approx b_M p(N - \bar{N}),
\]

where we have confused \( N_t \) with \( \bar{N} \) for large \( N \) as expressed below equation (14). Equation (16) is a general expression for the probability in the condensed phase for identically distributed variables. A straightforward generalization to the case of non-identically distributed variables will be discussed below.

The quantity \( b_M \) depends on the structure of the microscopic probabilities. If the \( p_s \) do not depend on \( M \), we can infer \( b_M \) by observing that \( \sum_{N=\bar{N}}^{\infty} C(N, M) \) —the number of particles in the condensed phase—is of order \( M \). Recalling equations (14)–(16) one can argue that
\[ b_M \sim M. \] (17)

The results (16), (17) have a very transparent physical meaning [6, 35]: when condensation occurs a single receptor hosts a number \( N - \overline{N} \) of particles with a probability \( p(N - \overline{N}) \). The factor \( M \) is the number of ways to choose such a receptor out of \( M \), which is true when the receptors are identical (indeed we will show in section 5 that equation (17) can be violated for non-identically distributed variables). Note that our derivation is of a general character and does not rely on any specific form of the microscopic probabilities \( p(n) \).

We emphasize the crucial role played by the \( \delta \)-function in equation (2) which, as mentioned in the introduction, effectively constrains the total particle number thus invalidating the central limit theorem which would otherwise apply for the problem at hand, making the condensation phenomenon possible.

Before moving to the more general case of a non-identical distribution of the \( p_n \)'s, we illustrate all the above with an example.

### 4.1. An example

Let us consider again the distribution (6) with \( K > K_c \). It is easy to show that the saddle point solution to equation (10) exists only for \( \rho \leq \overline{\rho} \) defined by

\[ \overline{\rho} = \frac{L_{K-1}(1)}{L_K(1)} - 1 \] (18)

(\( \overline{\rho} \approx 0.368433 \) for \( K = 3 \)). For \( \rho \leq \overline{\rho} \) only the normal phase exists, the steepest-descent evaluation of the integral in equation (2) is appropriate, and one arrives at equations (8) and (10). The rate function obtained in this way is shown in the upper panel of figure 3 together with the behavior of \( R(\rho, M) \) (equation (12)) which, for \( \rho \leq \overline{\rho} \), approaches \( R(\rho) \) for large \( M \).

As already discussed, for values of the density larger than \( \overline{\rho} \) a straightforward steepest-descent evaluation of the integral in (2) breaks down. In this case \( P \) is not exponentially small in \( M \), as required by equation (8), as it can clearly be understood by observing in figure 3 that the dependence on \( M \) does not cancel and \( \mathcal{R} \) keeps decreasing to zero for any value of \( M \). In this region condensation occurs and, instead of equations (8) and (10), the solution (16), (17) applies. In order to see this, in the inset of figure 3 we plot \( M^{-1}P(N, M) \), since according to equations (16), (17) this quantity ought to be independent of \( M \) and proportional to \( p(N - \overline{N}) \). As expected, for \( \rho \gg \overline{\rho} \) the form (16), (17) describes the probability with great accuracy for large \( N \). Note that in the condensed region the convergence to the asymptotic form (equations (16), (17)) is much faster than the one in the gas phase (to equations (8), (10)), being achieved already for \( M \lesssim 100 \), a value for which \( \mathcal{R} \) is still quite different from \( R \) in the region \( \rho < \overline{\rho} \).

### 5. Non-identically distributed variables

Now we turn to study the phenomenon of condensation when the microscopic variables are not identically distributed. Specific issues of this and related problems have been addressed in [5, 12, 36–42]. Here we are interested in the derivation of a general form for \( P \), generalizing equation (16), and discussing the related phenomenon of the giant response on broad grounds.

When the variables are non-identically distributed one can argue that condensation occurs on the most favorable receptor [5, 10], namely the one with the larger \( p_m(n_m) \). In the example (6), it is the one with the smaller \( K_m \). Denoting \( \overline{m} \) by this term (i.e. \( p_m(n_m) > p_{\overline{m}}(n_{\overline{m}}) \) \( \forall m \neq \overline{m} \)), recalling the physical meaning of \( \pi \) in equation (5), it is clear that a structure like
the one in figure 3, with a sharp peak around $n$, will be present if the recently added receptor is the one where condensation occurs, namely if $p_M \equiv p_m$ in equation (5). Then, in order to proceed as in section 4, we define

$$\pi(n, N, M) = P(N - n, M - 1)p_m(n),$$

(19)

which amounts only to the choice of a particular labeling of the receptors. Proceeding as in section 4 one obtains the following equation

Figure 3. Upper panel: the function $R(\rho, M)$ is plotted against $\rho$ for the model with probabilities (6) with a uniform $K_m = 3 > K_c$ for the values of $M$ in the key. The heavy-dotted black line is this quantity for $M \rightarrow \infty$, which coincides with $R(\rho)$ obtained from equations (8) and (10) for $\rho \leq p$ and is identically zero for $\rho > p$. In the inset $M^{-1}P$ is plotted against $N - \overline{N}$. The heavy-dotted turquoise line is the law $(N - \overline{N})^{-1}$, i.e. equations (16) and (17). Lower panel: $P$ is plotted for $M = 333$ and the three different choices (see text): (i) $K_m \equiv K = 3, \forall m$, continuous brown, (ii) $K_m \equiv K = 6, \forall m$, green dashed with a square and (iii) $K_1 = 3, K_m = K = 6, \forall m > 1$, dotted maroon with a circle.
instead of equation (14), thus arriving at
\[ P(N, M) \simeq b_M p_m(N - \overline{N}) \]  
(21)
in place of equation (16). This form of the probability generalizes equations (16), (17) to the case of non-identically distributed variables. Note that no assumptions on the form of the microscopic probabilities \( p_m(n) \) have been made also in this case and, therefore, equation (21) is expected to hold quite generally. Together with equations (16), (17), this equation represents the main result of this paper. Note that the dependence on \( M \) of \( b_M \) can be very different from the one (17) holding for identically distributed variables. Indeed, when the microscopic probabilities depend on \( M \), the number of particles in the condensed state may not be simply proportional to \( M \), since the \( M \)th receptors can promote condensation differently from the previous ones. An example showing this will be shown in section 5.2. A straightforward consequence of equation (21) is the phenomenon of the extreme sensitivity of the global probability \( P \) to specific details of the microscopic ones \( p_m \), that we discuss below.

5.1. Giant response

It must be stressed that the solutions (16), (21) are totally different from (7), in particular concerning the dependence on \( M \). Indeed, while (7) is exponentially small for large \( M \) and transforms into a \( \delta(N - \overline{N}) \), equations (16) and (17) show that in the presence of condensation the dependence can be as weak as linear in \( M \), signaling the occurrence of anomalously large fluctuations.

Related to that, an extreme sensitivity of the macroscopic probability \( P \) to the details of the microscopic ones \( p_m \) arises. In fact, equation (21) clearly shows that the distribution of the single variable \( m \) fully determines the global quantity \( P \). Introducing a susceptibility \( \chi \)—the shift of the macroscopic probability due to the variation of a microscopic one—from equation (21) one has
\[ \chi(N, m) = \lim_{M \to \infty} \frac{\Delta P(N, M)}{\Delta p_m(N)} = \begin{cases} 0, & \text{gas} \\ B_M \delta_{m, \overline{m}}, & \text{condens.} \end{cases} \ (N \gg \overline{N}), \]  
(22)
where \( B_M \) is a constant. This shows that in the gas phase a shift of one (or even of a finite number) of the microscopic probabilities cannot alter the global behavior of \( P \), since this is determined by the synergic contribution of a number \( M \to \infty \) of variables, the situation is profoundly different when condensation occurs. In this case \( P \) is fully determined by the statistical properties of the most favorable receptor \( \overline{m} \). Therefore \( P \) is independent of the form of all the other \( M - 1 \) variables, whereas a macroscopic effect can be determined by altering the statistical properties of the single \( p_m \). As we will show by means of some examples in section 5.2 this may have dramatic effects on the form of the probability \( P \) in the condensed phase.

This anomalous susceptibility is reminiscent of the large response induced by gapless modes \([43, 44]\), such as massless Goldstone modes in systems with a spontaneously broken continuous symmetry. In fact, starting from equally distributed variables the symmetry between the receptors is broken by changing the properties of one of them. However the phenomenon of the giant susceptibility discussed here is more general since it also occurs when the modification of the probability of a single variable occurs in a set of (already) non-identically distributed ones, as will be illustrated by the second example of section 5.2.
To the best of our knowledge, this remarkable property of the susceptibility (22) has never been pointed out before.

5.2. Examples

The occurrence of condensation in the case of non-identically distributed variables and the giant response phenomenon can be illustrated using again the probabilities (6) with \(m\)-dependent \(K_m\)'s. The simplest non-trivial choice is when all the \(K_m\)s are equal except one, namely \(K_m = K, \forall m > 1\) while \(K_1\) can be different from \(K\).
In the lower panel of figure 3 we compare $P$ for the three cases: (i) $K_1 = K = 3$, (ii) $K_1 = K = 6$ and (iii) $K_1 = 3, K = 6$. One sees that the curves relative to the choices (ii) and (iii) coincide for $\rho \lesssim \overline{\rho}$ (i.e. up to the maximum of $P$). This is because in this region there is no condensation and the macroscopic probability is insensitive to a single $p_m$, equation (22). However, for $\rho > \overline{\rho}$ the two curves become totally different and instead case (iii) behaves as (i), apart from a vertical displacement due to the different value of the constant $b_M$ in equation (21). This shows that a single variable cannot influence the collective behavior unless it is the one where condensation occurs, in which case a giant response is observed.

The examples considered so far were based on power-law probabilities (6). However, the features above are more general and not only restricted to this case. We show this by considering the exponential form

$$ p_m(n) = c \exp[-K_m \cdot n^\alpha], \quad (23) $$

where $K_m = \beta \left( \frac{m}{\overline{m}} \right)^\alpha$ ($\beta$ and $c$ are constants), and $\kappa_m$ is an $M$-independent exponent. This case is also interesting because the microscopic probabilities $p$ do not depend only on $m$, but also on $M$. In this case the scaling (17), which was expected quite generally for $M$-independent $p$s, can—in principle—be spoiled, and a general form of $b_M$ is not available.

In order to illustrate the behavior of $P$ with the exponentially distributed microscopic probabilities we have evaluated it for different choices of the parameters entering equation (23). Starting with a uniform exponent $\kappa = 1$, setting $\beta = 1$ and $\alpha = 2$, the upper panel of figure 4 shows a pattern of behavior similar to the case (6) with $K > K_c$: for $N \lesssim \overline{N}$, $R$ approaches the form (7) with a rate function given by equations (8), (9), whereas for $N \gg \overline{N}$ the determination (21) holds (with $m = 1$), implying $P(N, M) \sim e^{-K_c M^{-2}}$, as shown in the inset. The data collapse is obtained by plotting $M^3 P(N, M)$ against $(N - \overline{N})/M^2$, implying that $b_M = bM^{-2}$. Note that the approach to the asymptotic form is much faster than for the fat-tailed probabilities (6), since already for $M \approx 100$ one has a good representation of the large-$M$ form in the range of densities considered, at variance with that observed in figure 3.

The phenomenon of the giant susceptibility is illustrated by comparing the case above with the one where we change the distribution of $n_1$ to have $\kappa_1 = 0.95$ and all the remaining ones are left untouched ($\kappa_m = \kappa, \forall \kappa > 1$). The lower panel of figure 4 shows that, while in the normal phase $N \lesssim \overline{N}$ this does not alter $P$ (a residual difference between the two curves is due to the finite value of $M$), a dramatic change is produced in the condensed region $N > \overline{N}$ because, since $\kappa_1 < \kappa$, the statistical properties of the condensing variable have been changed.

6. Conclusions

In this paper we have discussed the general problem of evaluating the probability distribution $P(N, M)$ of the sum $N$ of a large number $M$ of micro-variables, not necessarily identically distributed.

We have done this by means of the recurrence relation (4), which provides an analogy with a thermodynamic system where a condensation transition occurs and the identification of an order parameter $\rho_i$. Equation (4) also allows the derivation of a rather general expression for $P$ (equation (21)) which is valid, when condensation occurs, for finite values of $M$. From this expression, computing the susceptibility (22), the extreme sensitivity of $P$ to the distribution of even a single variable was explicitly shown.

These properties of the probability $P$ have been discussed by means of specific examples amenable to analytical and numerical computations, including identically and differently
distributed variables, with or without fat-tails and also in the case of a specific dependence of the microscopic probabilities $p$ on the number $M$ of micro-variables.

The noteworthy features discussed in this paper are associated with the existence of a condensation phenomenon and, therefore, they are not expected to only be relevant to the large deviations of $\mathcal{N}$, but also to those of different macrovariables, and to application to a large class of problems in physics and other areas, making the issue considered in this paper a broad and general research topic.

Acknowledgments

FC acknowledges financial support by MIUR PRIN 2010HXAW77_005.

Appendix A. Saddle point evaluation of $P$ in the condensed phase

In the condensed phase the symmetry between the receptors is broken since, as discussed in section 4, a single variable provides a contribution comparable to all the remaining ones. Let us, without loss of generality, indicate this variable as being the first, namely $n_1$. In view of that, we re-write the first line of equation (2) as follows

$$ P(N, M) = \sum_{n_1} p(n_1) \Omega(n_1, N) $$

with

$$ \Omega(n_1, N, M) = \sum_{n_2, n_3, \ldots, n_M} p_2(n_2) \cdots p_M(n_M) \delta_{\mathcal{N}, N-n_1}, $$

where $\mathcal{N} = \sum_{m=2}^{M} n_m$. A solution can be found by making the ansatz that in the condensed phase the argument of the sum in equation (24) is sharply peaked around a certain value $n_\ell$, with a certain width $\tilde{b}$, so that it can be evaluated as

$$ P(N, M) = M \tilde{b} p(n_\ell) \Omega(n_\ell, N, M) $$

where $\tilde{b} = \sqrt{2\pi[-(\partial^2 \ln(p\Omega)/\partial n_\ell)]_{n_\ell=n_\ell}}^{-1}$ and the factor $M$ in front of the rhs of equation (26) is due to the $M$ possible ways of choosing the variable denoted by $n_1$ among $M$. Using equation (26) as a starting point, instead of equation (2), one arrives at

$$ P(N, M) = \tilde{b} M p(n_\ell) \frac{1}{2\pi i} \oint dz e^{M[\ln Q(z, M) - (\rho - \rho_i) \ln z]}, $$

for large $M$, where now

$$ Q(z, M) = \left[ \prod_{m=2}^{M} \sum_{n_m} p_m(n_m) z^{n_m} \right]^{1/M} $$

and $\rho_i = n_i/M$ is the condensed particles’ density. The steepest-descent evaluation of the integral leads to the saddle point equation

$$ z^*(\rho) \frac{Q[z^*(\rho)]}{Q[z^*(\rho)]} = \rho - \rho_i. $$

In the condensed phase there is always a solution with $z = 1$ and $\rho_i = \rho - \frac{Q'(1)}{Q(1)}$, and the evaluation of the integral in equation (27) gives
\[ P(N, M) = b_M p(n_t) \tag{30} \]

with \( b_M \sim \frac{\bar{M}}{M} e^{\ln(Q(1)/2\pi i)} = \frac{\bar{M}}{M} / (2\pi i) \) (in the last passage we have used \( Q(1) \equiv 1 \) because of the normalization of the \( \bar{p} \)s). Recalling that \( n_t \approx N - \overline{N} \) (see section 3) one recovers the result (16) that was obtained in a different way—by using the recurrence relation (4)—in section 3.

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