PROJECTIVITY OF THE MODULI SPACE OF STABLE LOG-VARIETIES
AND SUBADDITIVITY OF LOG-KODAIRA DIMENSION

SÁNDOR J KOVÁCS AND ZSOLT PATAKFALVI

Abstract. We prove a strengthening of Kollár’s Ampleness Lemma and use it to prove that any proper coarse moduli space of stable log-varieties of general type is projective. We also prove subadditivity of log-Kodaira dimension for fiber spaces whose general fiber is of log general type.

1. Introduction

Since Mumford’s seminal work on the subject, \( \mathcal{M}_g \), the moduli space of smooth projective curves of genus \( g \geq 2 \), has occupied a central place in algebraic geometry and the study of \( \mathcal{M}_g \) has yielded numerous applications. An important aspect of the applicability of the theory is that these moduli spaces are naturally contained as open sets in \( \mathcal{M}_g \) the moduli space of stable curves of genus \( g \) and the fact that this later space admits a projective coarse moduli scheme.

Even more applications stem from the generalization of this moduli space, \( \mathcal{M}_{g,n} \), the moduli space of \( n \)-pointed smooth projective curves of genus \( g \) and its projective compactification, \( \overline{\mathcal{M}}_{g,n} \), the moduli space of \( n \)-pointed stable curves of genus \( g \).

It is no surprise that after the success of the moduli theory of curves huge efforts were devoted to develop a similar theory for higher dimensional varieties. However, the methods used in the curve case, most notably GIT, proved inadequate for the higher dimensional case. Gieseker [Gie77] proved that the moduli space of smooth projective surfaces of general type is quasi-projective, but the proof did not provide a modular projective compactification. In fact, Wang and Xu showed recently that such GIT compactification using asymptotic Chow stability is impossible [WX14]. The right definition of stable surfaces only emerged after the development of the minimal model program allowed bypassing the GIT approach [KSB88] and the existence and projectivity of the moduli space of stable surfaces and higher dimensional varieties have only been proved very recently as the combined result of the effort of several people over several years [KSB88, Kol90, Ale94, Vie95, HK04, AH11, Kol08, Kol13a, Kol13b, Fuj12, HMX14, Kol14].
Naturally, one would also like a higher dimensional analogue of \( n \)-pointed curves and extend the existing results to that case [Ale96]. The obvious analogue of an \( n \)-pointed smooth projective curve is a smooth projective log-variety, that is, a pair \((X, D)\) consisting of a smooth projective variety \(X\) and a simple normal crossing divisor \(D \subseteq X\). For reasons originating in the minimal model theory of higher dimensional varieties, one would also like to allow some mild singularities of \(X\) and \(D\) and fractional coefficients in \(D\), but we will defer the discussion of the precise definition to a later point in the paper (see Definition 2.9). In any case, one should mention that the introduction of fractional coefficients for higher dimensional pairs led Hassett to go back to the case of \( n \)-pointed curves and study a weighted version in [Has03]. These moduli spaces are more numerous and have greater flexibility than the traditional ones. In fact, they admit natural birational transformations and demonstrate the workings of the minimal model program in concrete highly non-trivial examples. Furthermore, the log canonical models of these moduli spaces of weighted stable curves may be considered to approximate the canonical model of \( \overline{\mathcal{M}}_{g,n} \) [HH09, HH13].

It turns out that the theory of *moduli of stable log-varieties*, also known as *moduli of semi-log canonical models* or *KSBA stable pairs*, which may be regarded as the higher dimensional analogues of Hassett’s moduli spaces above, is still very much in the making. It is clear what a stable log-variety should be: the correct class (for surfaces) was identified in [KSB88] and further developed in [Ale96]. This notion, is easy to generalize to arbitrary dimension cf. [Kol13a]. On the other hand, at the time of the writing of this article it is not entirely obvious what the right definition of the corresponding moduli functor is over non reduced bases. For a discussion of this issue we refer to [Kol13a, §6]. A major difficulty is that in higher dimensions when the coefficients of \(D\) are not all greater than \(1/2\) a deformation of a log-variety cannot be simplified to studying deformations of the ambient variety \(X\) and then deformations of the divisor \(D\). An example of this phenomenon, due to Hassett, is presented in Section 1.2, where a family \((X, D) \to \mathbb{P}^1\) of stable log varieties is given such that \(D \to \mathbb{P}^1\) does not form a flat family of pure codimension one subvarieties. In fact, the flat limit \(D_0\) acquires an embedded point, or equivalently, the scheme theoretic restriction of \(D\) onto a fiber is not equal to the divisorial restriction. Therefore, in the moduli functor of stable log-varieties one should allow both deformations that acquire and also ones that do not acquire embedded points on the boundary divisors. This is easy to phrase over nice (e.g., normal) bases see Definition 5.2 for details. However, at this point it is not completely clarified how it should be presented in more intricate cases, such as for instance over a non-reduced base. Loosely speaking the infinitesimal structure of the moduli space is not determined yet (see Remark 5.15 for a discussion on this), although there are also issues about the implementation of labels or markings on the components of the boundary divisor (cf. Remark 6.17).

By the above reasons, several functors have been suggested, but none of them yet emerged as the obvious “best”. However, our results apply to any moduli functor for which the objects are the stable log-varieties (see Definition 5.2 for the precise condition on the functors). In particular, our results apply to any moduli space that is sometimes called a *KSBA compactification* of the moduli space of log-canonical models.

Our main result is the following. Throughout the article we are working over an algebraically closed base field \(k\) of characteristic zero.

**Theorem 1.1** (=Corollary 6.3). *Any algebraic space that is the coarse moduli space of a moduli functor of stable log-varieties with fixed volume, dimension and coefficient set (as defined in Definition 5.2) is a projective variety over \(k\).*

For auxiliary use, in Section 5.1 we also present one particular functor as above, based on a functor suggested by Kollár [Kol13a, §6]. In particular, the above result is not vacuous.

As mentioned Mumford’s GIT method used in the case of moduli of stable curves does not work in higher dimensions and so we study the question of projectivity in a different manner. The properness of any algebraic space as in Theorem 1.1 is shown in [Kol14]. For the precise
statement see Proposition 5.4. Hence, to prove projectivity over $k$ one only has to exhibit an ample line bundle on any such algebraic space. Variants of this approach have been already used in [Kum83, Kol90, Has03]. Generalizing Kollár’s method to our setting [Kol90], we use the polarizing line bundle det $f_*\mathcal{O}_X(r(K_{X/Y} + D))$, where $f : (X, D) \to Y$ is a stable family and $r > 0$ is a divisible enough integer. Following Kollár’s idea and using the Nakai-Moishezon criterion it is enough to prove that this line bundle is big for a maximal variation family over a normal base. However, Kollár’s Ampleness Lemma [Kol90, 3.9.3.13] is unfortunately not strong enough for our purposes and hence we prove a stronger version in Theorem 4.1. There, we also manage to drop an inconvenient condition on the stabilizers from [Kol90, 3.9.3.13], which is not necessary for the current application, but we hope will be useful in the future. Applying Theorem 4.1 and some other arguments outlined in Section 1.1 we prove that the above line bundle is big in Theorem 6.1.

A side benefit of this approach is that proving a positivity property of $K_{X/Y} + D$ opens the door to other applications. For example, a related problem in the classification theory of algebraic varieties is the subadditivity of log-Kodaira dimension. We prove this assuming the general fiber is of log general type in Theorem 8.4. This generalizes to the logarithmic case the celebrated results on the subadditivity of Kodaira dimension [Kaw81, Kaw85, Vie83a, Vie83b, Kol87], also known as Itaka’s conjecture $C_{n,m}$ and its strengthening by Viehweg $C_{n,m}^+$. For Theorem 8.4 we refer to Section 8, here we only state two corollaries that need less preparation.

**Theorem 1.2.** (= Theorem 8.5 and Corollary 8.6)

1. If $f : (X, D) \to (Y, E)$ is a surjective map of log-smooth projective pairs with coefficients at most 1, such that $D \geq f^*E$ and $K_{X_\eta} + D_\eta$ is big, where $\eta$ is the generic point of $Y$, then
   \[ \kappa(K_X + D) \geq \kappa(K_{X_\eta} + D_\eta) + \kappa(K_Y + E). \]

2. Let $f : X \to Y$ be a dominant map of (not necessarily proper) algebraic varieties such that the generic fiber has maximal Kodaira dimension. Then
   \[ \kappa(X) \geq \kappa(X_\eta) + \kappa(Y). \]

In the logarithmic case Fujino obtained results similar to the above in the case of maximal Kodaira dimensional base [Fuj14a, Thm 1.7] and relative one dimensional families [Fuj15]. Another related result of Fujino is subadditivity of the numerical log-Kodaira dimension [Fuj14b]. A version of the latter, under some additional assumptions, was also proved by Nakayama [Nak04, V.4.1]. The numerical log-Kodaira dimension is expected to be equal to the usual log-Kodaira dimension by the Abundance Conjecture. However, the latter is usually considered the most difficult open problem in birational geometry currently. Our proof does not use either the Abundance Conjecture or the notion of numerical log-Kodaira dimension.

Further note that our proof of Theorem 1.2 is primarily algebraic. That is, we obtain our positivity results, from which Theorem 1.2 is deduced, algebraically, starting from the semi-positivity results of Fujino [Fuj12, Fuj14a]. Hence, our approach has a good chance to be portable to positive characteristic when the appropriate semi-positivity results (and other ingredients such as the mmp) become available in that setting. See [Pat12b] for the currently available semi-positivity results in positive characteristic, and [CZ13, Pat13] for results on subadditivity of Kodaira-dimension.

**Theorem 1.2** is based on the following theorem stating that the sheaves $f_*\mathcal{O}_X(r(K_{X/Y} + \Delta))$ have more positivity properties than just that their determinants are ample. This is a generalization of [Kol87] and [EV90, Thm 3.1] to the logarithmic case.

**Theorem 1.3.** (=Theorem 7.1) If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal, projective variety $Y$ with klt general fiber, then $f_*\mathcal{O}_X(r(K_{X/Y} + D))$ is big for every divisible enough integer $r > 0$.

Note that Theorem 1.3 fails without the klt assumption. Also, Theorem 1.3 allows for numerous applications, such as, the already mentioned Theorem 1.2, as well as upcoming work in progress.
on a log-version of [Abr97] in [AT15] and on the ampleness of the CM line bundle on the moduli space of stable varieties in [PX15]. We also state Theorem 1.3 and our other positivity results over almost projective bases in Section 9, that is, over bases that are big open sets in projective varieties. We hope this will be helpful for some applications.

1.1. Outline of the proof

As mentioned above, using the Nakai-Moishezon criterion for ampleness, Theorem 1.1 reduces to the following statement (= Proposition 6.16): given a family of stable log-varieties \( f : (X, D) \rightarrow Y \) with maximal variation over a smooth, projective variety, \( \det f_* \mathcal{O}_X(r(K_{X/Y} + D)) \) is big for every divisible enough integer \( q > 0 \). This follows relatively easily from the bigness of \( K_{X/Y} + D \). To be precise it also follows from the bigness of the log canonical divisor \( K_{X(r)/Y} + D_{X(r)} \) of some large enough fiber power for some integer \( r > 0 \) (see Notation 2.12 and the proof of Proposition 6.16).

In fact, one cannot expect to do better for higher dimensional bases, see Remark 6.2 for details. Here we review the proof of the bigness of these relative canonical divisors, going from the simpler cases to the harder ones.

1.1.1. The case of \( \dim Y = 1 \) and \( \dim X = 2 \). In this situation, roughly speaking, we have a family of weighted stable curves as defined by Hassett [Has03]. The only difference is that in our notion of a family of stable varieties there is no marking (that is, the points are not ordered). This means that the marked points are allowed to form not only sections but multisectons as well. However, over a finite cover of \( Y \) these multisectons become unions of sections, and hence we may indeed assume that we have a family of weighted stable curves. Denote by \( s_i : Y \rightarrow X \) \((1, \ldots, m)\) the sections given by the marking and let \( D_i \) be the images of these sections. Hassett proved projectivity [Has03, Thm 2.1, Prop 3.9] by showing that the following line bundle is ample:

\[
\det f_* \mathcal{O}_X(r(K_{X/Y} + D)) \otimes \left( \bigotimes_{i=1}^m s_i^* \mathcal{O}_X(r(K_{X/Y} + D)) \right).
\]

Unfortunately, this approach does not work for higher dimensional fibers, because according to the example of Section 1.2, the sheaves corresponding to \( s_i^* \mathcal{O}_X(r(K_{X/Y} + D)) \) are not functorial in higher dimensions. In fact, the function \( y \mapsto h^0 \left( \left( D_i \right)_y, \mathcal{O}_{D_i}(r(K_{X/Y} + D)) \right) \) jumps down in the limit in the case of example of Section 1.2, which means that there is no possibility to collect the corresponding space of sections on the fibers into a pushforward sheaf. Note that here it is important that \( (D_i)_y \) means the divisorial restriction of \( D_i \) onto \( X_y \). Indeed, with the scheme theoretic restriction there would be no jumping down, since \( D_i \) is flat as a scheme over \( Y \). However, the scheme theoretic restriction of \( D_i \) onto \( X_y \) contains an embedded point and therefore the space of sections on the divisorial restriction is one less dimensional than on the scheme theoretic restriction.

So, the idea is to try to prove the ampleness of \( \det f_* \mathcal{O}_X(r(K_{X/Y} + D)) \) in the setup of the previous paragraph, hoping that that argument would generalize to higher dimensions. Assume that \( \det f_* \mathcal{O}_X(r(K_{X/Y} + D)) \) is not ample. Then by the ampleness of (1.3.1), for some \( 1 \leq i \leq m \), \( s_i^* \mathcal{O}_X(r(K_{X/Y} + D)) \) must be ample. Therefore, for this value of \( i \), \( D_i \cdot (K_{X/Y} + D) > 0 \). Furthermore, by decreasing the coefficients slightly, the family is still a family of weighted stable curves. Hence \( K_{X/Y} + D - \varepsilon D_i \) is nef for every \( 0 \leq \varepsilon \ll 1 \) (see Lemma 6.7, although this has been known by other methods for curves). Putting these two facts together yields that

\[
(K_{X/Y} + D)^2 \geq (K_{X/Y} + D) \cdot (K_{X/Y} + D - \varepsilon D_i) + (K_{X/Y} + D) \cdot \varepsilon D_i > 0.
\]

This proves the bigness of \( K_{X/Y} + D \), and the argument indeed generalizes to higher dimensions as explained below.
1.1.2. The case of $\dim Y = 1$ and arbitrary $\dim X$. Let $f : (X, D) \to Y$ be an arbitrary family of non-isotrivial stable log-varieties over a smooth projective curve. Let $D_i$ ($i = 1, \ldots, m$) be the union of the divisors (with reduced structure) of the same coefficient (cf. Definition 6.4). The argument in the previous case suggests that the key is to obtain an inequality of the form

$$(1.3.2) \quad \left( (K_{X/Y} + D)\big|_{D_i} \right)^{\dim D_i} > 0.$$ 

Note that it is considerably harder to reach the same conclusion from this inequality, than in the previous case, because the $D_i$ are not necessarily $\mathbb{Q}$-factorial and then $(X, D - \varepsilon D_i)$ might not be a stable family. To remedy this issue we pass to a $\mathbb{Q}$-factorial dlt-blowup. For details see Lemma 6.13.

Let us now turn to how one might obtain (1.3.2). First, we prove using our generalization (see Theorem 4.1) of the Ampleness Lemma a higher dimensional analogue of (1.3.1) in Proposition 6.8, namely, that the following line bundle is ample:

$$(1.3.3) \quad \det f_* \mathcal{O}_X(r(K_{X/Y} + D)) \otimes \left( \bigotimes_{i=1}^m \det (f|_{D_i})_* \mathcal{O}_{D_i}(r(K_{X/Y} + D)|_{D_i}) \right).$$

The main difference compared to (1.3.1) is that $f|_{D_i}$ is no longer an isomorphism between $D_i$ and $Y$ as it was in the previous case where the $D_i$ were sections. In fact, $D_i \to Y$ has positive dimensional fibers and hence $\mathcal{E}_i := (f|_{D_i})_* \mathcal{O}_{D_i}(r(K_{X/Y} + D)|_{D_i})$ is a vector bundle of higher rank. As before, if $\det f_* \mathcal{O}_X(r(K_{X/Y} + D))$ is not ample, then for some $i$, $\det \mathcal{E}_i$ has to be. However, since $\mathcal{E}_i$ is higher rank now, it is not as easy to obtain intersection theoretic information as earlier.

As a result one has to utilize a classic trick of Viehweg which leads to working with fibered powers. Viehweg’s trick is using the fact that there is an inclusion

$$(1.3.4) \quad \det \mathcal{E}_i \hookrightarrow \bigotimes_{j=1}^d \mathcal{E}_j,$$

where $d := \text{rk} \mathcal{E}_i$, and where the latter sheaf can be identified with a pushforward from the fiber product space $D_i^{(d)} \to Y$ (see Notation 2.12). This way one obtains that

$$\left( (K_{X^{(d)}/Y} + D_{X^{(d)}})\big|_{D_i^{(d)}} \right)^{\dim D_i^{(d)}} > 0,$$

from which it is an easy computation to prove (1.3.2)

1.1.3. The case of both $\dim Y$ and $\dim X$ arbitrary. We only mention briefly what goes wrong here compared to the previous case, and what the solution is. The argument is very similar to the previous case until we show that (1.3.3) is big. However, it is no longer true that if $\det f_* \mathcal{O}_X(r(K_{X/Y} + D))$ is not big, then one of the $\det \mathcal{E}_i$ is big. So, the solution is to treat all the sheaves at once via an embedding as in (1.3.4) of the whole sheaf from (1.3.3) into a tensor-product sheaf that can be identified with a pushforward from an appropriate fiber product (see (6.12.1)). The downside of this approach is that one then has to work on $X^{(l)}$ for some big $l$, but we still obtain an equation of the type (1.3.2), although with $D_i$ replaced with a somewhat cumbersome subvariety of fiber product type.

After that an enhanced version of the previous arguments yields that $K_{X^{(l)}/Y} + D_{X^{(l)}}$ is big on at least one component, which is enough for our purposes. In fact, in this case we cannot expect that $K_{X/Y} + D$ would be big on any particular component, cf. Remark 6.2. However, the bigness of $K_{X^{(l)}/Y} + D^{(l)}$ on a component already implies the bigness of $\det f_* \mathcal{O}_X(r(K_{X/Y} + D))$ (see Proposition 6.16). This argument is worked out in Section 6.
1.1.4. Subadditivity of log-Kodaira dimension. First we prove Theorem 1.3 in Section 7 using ideas originating in the works of Viehweg. This implies that although in Section 6 we were not able to prove the bigness of $K_{X/Y} + D$ (only of $K_{X(Y)} + D(0)$), it actually does hold for stable families of maximal variation with klt general fibers (cf. Corollary 7.3). Then with a comparison process (see the proof of Proposition 8.7) of an arbitrary log-fiber space $f' : (X', D') \to Y'$ and of the image in moduli of the log-canonical model of its generic fiber, we are able to obtain enough positivity of $K_{X'/Y'} + D'$ to deduce subadditivity of log-Kodaira dimension if the log-canonical divisor of the general fiber is big.

1.2. An important example

The following example is due to Hassett (cf. [Kol13a, Example 42]), and has been referenced at a couple of places in the introduction.

Let $\mathcal{X}$ be the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with polarization $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)$ and let $\mathcal{D}$ be the conic divisor $\frac{1}{2}p_2^*P + \frac{1}{2}p_2^*Q$, where $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection to the second factor, and $P$ and $Q$ are general points. Let $H_0$ be a cone over a hyperplane section $C$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with the given polarization, and $H_\infty$ a general hyperplane section of $\mathcal{X}$ (which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$). Note that since deg $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)|_C = 4$, $H_0$ is a cone over a rational normal curve of degree 4. Let $f : \mathcal{X} \to \mathbb{P}^1$ be the pencil of $H_0$ and $H_\infty$. It is naturally a subscheme of the blowup $\mathcal{X}'$ of $\mathcal{X}$ along $H_0 \cap H_\infty$. Furthermore, the pullback of $\mathcal{D}$ to $\mathcal{X}'$ induces a divisor $\mathcal{D}'$ on $\mathcal{X}'$, such that

1. its reduced fiber over 0 is a cone over the intersection of $\frac{1}{2}p_2^*P + \frac{1}{2}p_2^*Q$ with $C$, that is, over 4 distinct points on $\mathbb{P}^1$ with coefficients $\frac{1}{2}$, and
2. its fiber over $\infty$ is two members of one of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ with coefficients $\frac{1}{2}$. In the limit both of these lines degenerate to a singular conic, and they are glued together at their singular points.

In case the reader is wondering how this is relevant to stable log-varieties of general type, we note that this is actually a local model of a degeneration of stable log-varieties, but one can globalize it by taking a cyclic cover branched over a large enough degree general hyperplane section of $\mathcal{X}$. For us only the local behaviour matters, so we will stick to the above setup. Note that since $\chi(\mathcal{O}_{\mathcal{X}'}) = 2$, the above described reduced structure cannot agree with the scheme theoretic restriction $\mathcal{D}'_{0,\text{sch}}$ of $\mathcal{D}'$ over 0, since then $\chi(\mathcal{O}_{\mathcal{D}',0,\text{sch}}) = 1$ would hold. Therefore $\mathcal{D}'_{0,\text{sch}}$ is non-reduced at the cone point. Furthermore, note that the log canonical divisor of $(\mathcal{X}, \mathcal{D})$ is the cone over a divisor corresponding to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2 + 2, -2 + 1 + \frac{1}{2} + \frac{1}{2}) \simeq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$. In particular, this log canonical class is $\mathbb{Q}$-Cartier, and hence $(\mathcal{H}, \mathcal{D}')$ does yield a local model of a degeneration of stable log-varieties.

1.3. Organization

We introduce the basic notions on general and on almost proper varieties in Section 2 and Section 3. In Section 4 we state our version of the Ampleness Lemma. In Section 5 we define moduli functors of stable log-varieties and we also give an example of a concrete moduli functor for auxiliary use. Section 6 contains the proof of Theorem 1.1 as well as of the necessary positivity of $\det f_*\mathcal{O}_X(r(K_{X/Y} + D))$. Section 7 is devoted to the proof of Theorem 1.3. Section 8 contains the statements and the proofs of the subadditivity statements including Theorem 1.2. Finally, in
Section 9 we shortly deduce almost projective base versions of the previously proven positivity statements.

**Acknowledgement.** The authors are thankful to János Kollár for many insightful conversations on the topic; to Maksym Fedorchuk for the detailed answers on their questions about the curve case; to James McKernan and Chenyang Xu for the information on the results in the article [HMX14].

## 2. Basic tools and definitions

We will be working over an algebraically closed base field $k$ characteristic zero in the entire article. In this section we give those definitions and auxiliary statements that are used in multiple sections of the article. Most importantly we define stable log-varieties and their families here.

**Definition 2.1.** A *variety* will mean a reduced but possibly reducible separated scheme of finite type over $k$. A *vector bundle* $W$ on a variety $Z$ in this article will mean a locally free sheaf. Its dual is denoted by $W^*$.

**Remark 2.2.** It will always be assumed that the support of a divisor does not contain any irreducible component of the conductor subscheme. Obviously this is only relevant on non-normal schemes. The theory of Weil, Cartier, and Q-Cartier divisors work essentially the same on *demi-normal* schemes, i.e., on schemes that satisfy Serre’s condition $S_2$ and are semi-normal and Gorenstein in codimension 1. For more details on demi-normal schemes and their properties, including the definition and basic properties of divisors on demi-normal schemes see [Kol13b, §5.1].

**Definition 2.3.** Let $Z$ be a scheme. A *big open subset* $U$ of $Z$ is an open subset $U \subseteq Z$ such that depth$_{Z\setminus U} \mathcal{O}_Z \geq 2$. If $Z$ is $S_2$, e.g., if it is normal, then this is equivalent to the condition that codim$_Z(Z \setminus U) \geq 2$.

**Definition 2.4.** The dual of a coherent sheaf $\mathcal{F}$ on a scheme $Z$ will be denoted by $\mathcal{F}^*$ and the sheaf $\mathcal{F}^{**}$ is called the *reflexive hull* of $\mathcal{F}$. If the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism, then $\mathcal{F}$ is called reflexive. For the basic properties of reflexive sheaves see [Har80, §1].

Let $Z$ be an $S_2$ scheme and $\mathcal{F}$ a coherent sheaf on $Z$. Then the reflexive powers of $\mathcal{F}$ are the reflexive hulls of tensor powers of $\mathcal{F}$ and are denoted the following way:

$$\mathcal{F}^m := (\mathcal{F}^\otimes m)^{**}$$

Obviously, $\mathcal{F}$ is reflexive if and only if $\mathcal{F} \simeq \mathcal{F}^{[1]}$. Let $\mathcal{I}$ be coherent sheaf on $Z$. Then the reflexive product of $\mathcal{F}$ and $\mathcal{I}$ (resp. reflexive symmetric power of $\mathcal{F}$) is the reflexive hull of their tensor product (resp. of the symmetric power of $\mathcal{F}$) and is denoted the following way:

$$\mathcal{F} \otimes \mathcal{I} := (\mathcal{F} \otimes \mathcal{I})^{**} \quad \text{Sym}^a(\mathcal{F}) := (\text{Sym}^a(\mathcal{F}))^{**}$$

**Notation 2.5.** Let $f : X \to Y$ and $Z \to Y$ be morphisms of schemes. Then the base change to $Z$ will be denoted by

$$f_Z : X_Z \to Z,$$

where $X_Z := X \times_Y Z$ and $f_Z := f \times_Y \text{id}_Z$. If $Z = \{y\}$ for a point $y \in Y$, then we will use $X_y$ and $f_y$ to denote $X_{\{y\}}$ and $f_{\{y\}}$.

**Lemma 2.6.** Let $f : X \to Y$ and $g : Z \to Y$ be surjective morphisms such that $Y$ is normal and let $\mathcal{L}$ and $\mathcal{N}$ be line bundles on $X$ and $Z$ respectively. Assume that there is a big open set of $Y$ over which $X$ and $Z$ are flat and $f_* \mathcal{L}$ and $g_* \mathcal{N}$ are locally free. Then

$$((fZ)_*(p_X^* \mathcal{L} \otimes p_Z^* \mathcal{N}))^{**} \simeq f_* \mathcal{L} \otimes g_* \mathcal{N}.$$

Furthermore, if $X$ and $Z$ are flat and $f_* \mathcal{L}$ and $g_* \mathcal{N}$ are locally free over the entire $Y$, then the above isomorphism is true without taking reflexive hulls.
Proof. Since the statement is about reflexive sheaves, we may freely pass to big open sets. In particular, we may assume that \( f \) and \( g \) are flat and \( f_* \mathcal{L} \) and \( g_* \mathcal{N} \) are locally free. Then

\[
(f \circ g)_*(p_X^* \mathcal{L} \otimes p_Z^* \mathcal{N}) \simeq g_* \left( (p_Z)_*(p_X^* \mathcal{L} \otimes \mathcal{N}) \right) \simeq g_* (g^* (g_* \mathcal{N} \otimes \mathcal{L})) \simeq f_* \mathcal{L} \otimes g_* \mathcal{N}.
\]

\[\square\]

**Notation 2.7.** Let \( f : X \to Y \) be a flat equidimensional morphism of demi-normal schemes, and \( Z \to Y \) a morphism between normal varieties. Then for a \( \mathbb{Q} \)-divisor \( D \) on \( X \) that avoids the generic and codimension 1 singular points of the fibers of \( f \), we will denote by \( D_Z \) the *divisorial pull-back* of \( D \) to \( X_Z \), which is defined as follows: As \( D \) avoids the singular codimension 1 points of the fibers, there is a big open set \( U \subseteq X \) such that \( D|_U \) is \( \mathbb{Q} \)-Cartier. Clearly, \( U_Z \) is also a big open set in \( X_Z \) and we define \( D_Z \) to be the unique divisor on \( X_Z \) whose restrict to \( U_Z \) is \( (D|_U)_Z \).

**Remark 2.8.** Note that this construction agrees with the usual pullback if \( D \) itself is \( \mathbb{Q} \)-Cartier, because the two divisors agree on \( U_Z \).

Also note that \( D_Z \) is not necessarily the (scheme theoretic) base change of \( D \) as a subscheme of \( X \). In particular, for a point \( y \in Y \), \( D_y \) is not necessarily equal to the scheme theoretic fiber of \( D \) over \( y \). The latter may contain smaller dimensional embedded components, but we restrict our attention to the divisorial part of this scheme theoretic fiber. This issue has already come up multiple times in Section 1, in particular in the example of Section 1.2.

Finally, note that if \( q(K_{X/Y} + D) \) is Cartier, then using this definition the line bundle \( \mathcal{O}_X(q(K_{X/Y} + D)) \) is compatible with base-change, that is, for a morphism \( Z \to Y \),

\[
\left( \mathcal{O}_X(q(K_{X/Y} + D)) \right)_Z \simeq \mathcal{O}_Z(q(K_{X/Z} + D_Z)).
\]

To see this, recall that this holds over \( U_Z \) by definition and both sheaves are reflexive on \( Z \). (See Definition 2.10 for the precise definition of \( K_{X/Z} \).)

**Definition 2.9.** A pair \((Z, \Gamma)\) consist of an equidimensional demi-normal variety \( Z \) and an effective \( \mathbb{Q} \)-divisor \( \Gamma \subseteq Z \). A *stable log-variety* \((Z, \Gamma)\) is a pair such that

1. \( Z \) is proper,
2. \((Z, \Gamma)\) has slc singularities, and
3. the \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( K_Z + \Gamma \) is ample.

For the definition of slc singularities the reader is referred to [Kol13b, 5.10]

**Definition 2.10.** If \( f : X \to Y \) is either

1. a flat projective family of equidimensional demi-normal varieties, or
2. a surjective morphism between normal projective varieties,

then \( \omega_{X/Y} \) is defined to be \( f_* \mathcal{O}_Y \). In particular, if \( Y \) is Gorenstein (e.g., \( Y \) is smooth), then

\[
\omega_{X/Y} \simeq \omega_X \otimes f^* \omega_Y^{-1}.
\]

In any case, \( \omega_{X/Y} \) is a reflexive sheaf (c.f., [PS14, Lemma 4.9]) of rank 1. Furthermore, if either in the first case \( Y \) is also normal or in the second case \( Y \) is smooth, then \( \omega_{X/Y} \) is trivial at the codimension one points, and hence it corresponds to a Weil divisor that avoids the singular codimension one points [Kol13b, 5.6]. This divisor can be obtained by fixing a big open set \( U \subseteq X \) over which \( \omega_{X/Y} \) is a line bundle, and hence over which it corresponds to a Cartier divisor, and then extending this Cartier divisor to the unique Weil-divisor extension on \( X \). Note that in the first case \( U \) can be chosen to be the relative Gorenstein locus of \( f \), and in the second case the regular locus of \( X \). Furthermore, in the first case, we have \( K_{X_Y/V} \mid V \sim K_{X^V/Y} \) for any \( V \to Y \) base-change from a normal variety (here restriction is taken in the sense of Notation 2.7).

**Definition 2.11.** A family of stable log-varieties, \( f : (X, D) \to Y \) over a normal variety consists of a pair \((X, D)\) and a flat proper surjective morphism \( f : X \to Y \) such that

1. \( D \) avoids the generic and codimension 1 singular points of every fiber,
(2) $K_{X/Y} + D$ is $\mathbb{Q}$-Cartier, and
(3) $(X_y, D_y)$ is a connected stable log-variety for all $y \in Y$.

**Notation 2.12.** For a morphism $f : X \to Y$ of schemes and $m \in \mathbb{N}_+$, define

$$X_Y^{(m)} := \prod_{i=1}^{m} X_Y = X \times_Y X \times_Y \cdots \times_Y X \tag{m \text{ times}}$$

and let $f_Y^{(m)} : X_Y^{(m)} \to Y$ be the induced natural map. For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ define

$$\mathcal{F}_Y^{(m)} := \bigotimes_{i=1}^{m} p_i^* \mathcal{F},$$

where $p_i$ is the $i$-th projection $X_Y^{(m)} \to X$. Similarly, if $f$ is flat, equidimensional with demi-normal fibers, then for a divisor $\Gamma$ on $X$ define

$$\Gamma_X^{(m)} := \sum_{i=1}^{m} p_i^* \Gamma,$$

a divisor on $X_Y^{(m)}$.

Finally, for a subscheme $Z \subseteq X$, $Z_Y^{(m)}$ is naturally a subscheme of $X_Y^{(m)}$. Notice however that if $m > 1$ and $Z$ has positive codimension in $X$, then $Z_Y^{(m)}$ is never a divisor in $X_Y^{(m)}$. In particular, if $Y$ is normal, $f$ is flat, equidimensional and has demi-normal fibers, and $\Gamma$ is an effective divisor that does not contain any generic or singular codimension 1 points of the fibers of $f$, then

$$\left( \Gamma_Y^{(m)} \right)_{\text{red}} = \left( \bigcap_{i=1}^{m} p_i^* \Gamma \right)_{\text{red}} \tag{2.12.1}$$

Notice the difference between $\Gamma_X^{(m)}$ and $\Gamma_Y^{(m)}$. The former corresponds to taking the $(m)$th box-power of a divisor as a sheaf, while the latter to taking fiber power as a subscheme. In particular,

$$\mathcal{O}_{X_Y^{(m)}}(\Gamma_X^{(m)}) \simeq (\mathcal{O}_X(\Gamma))_Y^{(m)},$$

while $\Gamma_Y^{(m)}$ is not even a divisor if $m > 1$.

In most cases, we omit $Y$ from the notation. I.e., we use $X^{(m)}$, $\Gamma_X^{(m)}$, $\Gamma^{(m)}$, $f^{(m)}$ and $\mathcal{F}^{(m)}$ instead of $X_Y^{(m)}$, $\Gamma_X^{(m)}$, $\Gamma_Y^{(m)}$, $f_Y^{(m)}$ and $\mathcal{F}_Y^{(m)}$, respectively.

### 3. Almost Proper Varieties and Big Line Bundles

**Definition 3.1.** An **almost proper** variety is a variety $Y$ that admits an embedding as a big open set into a proper variety $Y \hookrightarrow Y$. If $Y$ is almost proper, then a **proper closure** will mean a proper variety with such an embedding. The proper closure is not unique, but also, obviously, an almost proper variety is not necessarily an open set for an arbitrary embedding into a proper (or other) variety. An almost proper variety $Y$ is called **almost projective** when it has a proper closure $\overline{Y}$ which is projective. Such a proper closure will be called a **projective closure**.

**Lemma 3.2.** Let $Y$ be an almost projective variety of dimension $n$ and $B$ a Cartier divisor on $Y$. Then there exists a constant $c > 0$ such that for all $m > 0$

$$h^0(Y, \mathcal{O}_Y(mB)) \leq c \cdot m^n$$
Proof. Let \( \iota : Y \hookrightarrow \overline{Y} \) be a projective closure of \( Y \) and set \( \mathcal{B}_m = \iota_* \mathcal{O}_Y(mB) \). Let \( \mathcal{H} \) be a very ample invertible sheaf on \( \overline{Y} \) such that \( H^0(\overline{Y}, \mathcal{H} \otimes (\mathcal{B}_1)^*) \neq 0 \) where \((\mathcal{B}_1)^* \) is the dual of \( \mathcal{B}_1 \). It follows that there exists an embedding \( \mathcal{O}_Y(B) \hookrightarrow \mathcal{H}|_Y \) and hence for all \( m > 0 \) another embedding \( \mathcal{O}_Y(mB) \hookrightarrow \mathcal{H}^m|_Y \). Pushing this forward to \( Y \) one obtains that \( \mathcal{B}_m \subseteq \iota_* \mathcal{H}^m|_Y \simeq \mathcal{H}^m \). Note that the last isomorphism follows by the condition of \( Y \) being almost projective/proper, that is, because \( \text{depth}_{\overline{Y}} \mathcal{O}_{\overline{Y}} \mathcal{O}_{\overline{Y}} \leq 2 \). Finally this implies that
\[
H^0(Y, \mathcal{O}_Y(mB)) = H^0(\overline{Y}, \mathcal{B}_m) \leq H^0(\overline{Y}, \mathcal{H}^m) \sim c \cdot m^n,
\]
where the last inequality follows from [Har77, I.7.5]. \( \square \)

**Definition 3.3.** Let \( Y \) be an almost proper variety of dimension \( n \). A Cartier divisor \( B \) on \( Y \) is called **big** if \( H^0(Y, \mathcal{O}_Y(mB)) > c \cdot m^n \) for some \( c > 0 \) constant and \( m \gg 1 \) integer. A line bundle \( \mathcal{L} \) is called **big** if the associated Cartier divisor is big.

**Lemma 3.4.** Let \( Y \) be an almost proper variety of dimension \( n \) and \( \iota : Y \hookrightarrow \overline{Y} \) a projective closure of \( Y \). Let \( \mathcal{B} \) be a Cartier divisor on \( \overline{Y} \) and denote its restriction to \( Y \) by \( B = \mathcal{B}|_Y \). Then \( B \) is big if and only if \( \mathcal{B} \) is big.

**Proof.** Clear from the definition and the fact that \( \iota_* \mathcal{O}_Y(mB) \simeq \mathcal{O}_{\overline{Y}}(m\mathcal{B}) \) for every \( m \in \mathbb{Z} \). \( \square \)

**Remark 3.5.** Note that it is generally not assumed that \( B \) extends to \( \overline{Y} \) as a Cartier divisor.

**Lemma 3.6.** Let \( Y \) be an almost projective variety of dimension \( n \) and \( B \) a Cartier divisor on \( Y \). Then the following are equivalent:

1. \( mB \sim A + E \) where \( A \) is ample and \( E \) is effective for some \( m > 0 \),
2. the rational map \( \phi_{|mB|} \) associated to the linear system \( |mB| \) is birational for some \( m > 0 \),
3. the projective closure of the image \( \phi_{|mB|} \) has dimension \( n \) for some \( m > 0 \), and
4. \( B \) is big.

**Proof.** The proof included in [KM98, 2.60] works almost verbatim. We include it for the benefit of the reader since we are applying it in a somewhat unusual setup.

Clearly the implications (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are obvious. To prove (3) \( \Rightarrow \) (4), let \( T = \overline{\phi_{|mB|}(Y)} \subseteq \mathbb{P}^N \). By assumption \( \dim T = n \), so by [Har77, I.7.5] the Hilbert polynomial of \( T \) is \( h^0(T, \mathcal{O}_T(l)) = (\deg T/n!) \cdot l^n + \text{(lower order terms)} \). By definition of the associated rational map \( \phi_{|mB|} \) induces an injection \( H^0(T, \mathcal{O}_T(l)) \subseteq H^0(Y, \mathcal{O}_Y(mB)) \), which proves (3) \( \Rightarrow \) (4).

To prove (4) \( \Rightarrow \) (1), let \( B \) be a Cartier divisor on \( Y \) and let \( \iota : Y \hookrightarrow \overline{Y} \) be a projective closure of \( Y \). Further let \( \mathcal{A} \) be a general member of a very ample linear system on \( Y \). Then \( A := \mathcal{A} \cap Y \) is an almost projective variety by [Fle77, 5.2]. It follows by Lemma 3.2 that \( h^0(A, \mathcal{O}_A(mB|_A)) \leq c \cdot m^{n-1} \), which, combined with the exact sequence
\[
0 \to H^0(Y, \mathcal{O}_Y(mB - A)) \to H^0(Y, \mathcal{O}_Y(mB)) \to H^0(A, \mathcal{O}_A(mB|_A)),
\]
shows that if \( B \) is big, then \( H^0(Y, \mathcal{O}_Y(mB - A)) \neq 0 \) for \( m \gg 0 \) which implies (1) as desired. \( \square \)

The notion of weak-positivity used in this article is somewhat weaker than that of [Vie95]. The main difference is that we do not require being global generated on a fixed open set for every \( b > 0 \) in the next definition. This is a minor technical issue and proofs of the basic properties works just as for the definitions of [Vie95], after disregarding the fixed open set. The reason why this weaker form is enough for us is that we use it only as a tool to prove bigness, where there is no difference between our definition and that of [Vie95].

**Definition 3.7.** Let \( X \) be a normal, almost projective variety and \( \mathcal{H} \) an ample line bundle on \( X \).

1. A coherent sheaf \( \mathcal{F} \) on \( X \) is weakly-positive, if for every integer \( a > 0 \) there is an integer \( b > 0 \), such that \( \text{Sym}^{[a b]}(\mathcal{F}) \otimes \mathcal{H}^b \) is generically globally generated. Note that this does not depend on the choice of \( \mathcal{H} \) [Vie95, Lem 2.14.a].
(2) A coherent sheaf $\mathcal{F}$ on $X$ is big if there is an integer $a > 0$ such that $\text{Sym}^a(\mathcal{F}) \otimes \mathcal{H}^{-1}$ is generically globally generated. This definition also does not depend on the choice of $\mathcal{H}$ by a similar argument as for the previous point. Further, this definition is compatible with the above definition of bigness for divisors and the correspondence between divisors and rank one reflexive sheaves.

**Lemma 3.8.** Let $X$ be a normal, almost projective variety, $\mathcal{F}$ a weakly-positive and $\mathcal{I}$ a big coherent sheaf. Then

1. $\bigoplus \mathcal{F}, \text{Sym}^a(\mathcal{F}), \bigotimes \mathcal{F}, \det \mathcal{F}$ are weakly-positive,
2. generically surjective images of $\mathcal{F}$ are weakly-positive, and those of $\mathcal{I}$ are big,
3. if $\mathcal{I}$ is an ample line bundle, then $\mathcal{F} \otimes \mathcal{I}$ is big, and
4. if $\mathcal{I}$ is of rank 1, then $\mathcal{F} \otimes \mathcal{I}$ is big.

**Proof.** Let us fix an ample line bundle $\mathcal{H}$. (1) follows verbatim from [Vie95, 2.16(b) and 2.20], and (2) follows immediately from the definition. Indeed, given generically surjective morphisms $\mathcal{F} \rightarrow \mathcal{F}'$ and $\mathcal{I} \rightarrow \mathcal{I}'$, there are generically surjective morphisms $\text{Sym}^{ab}(\mathcal{F}) \otimes \mathcal{H}^b \rightarrow \text{Sym}^{ab}(\mathcal{F}') \otimes \mathcal{H}^b$ and $\text{Sym}^a(\mathcal{I}) \otimes \mathcal{H}^{-1} \rightarrow \text{Sym}^a(\mathcal{I}') \otimes \mathcal{H}^{-1}$ proving the required generic global generation.

To prove (3), take an $a > 0$, such that $\mathcal{H}^a \otimes \mathcal{H}^{-1}$ is effective and $\mathcal{H}^c$ is very ample for $c > a$. Then for a $b > a$ such that $\text{Sym}^{ab}(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated, the embedding $\text{Sym}^{[3b]}(\mathcal{F}) \otimes \mathcal{I}^b \hookrightarrow \text{Sym}^{[3b]}(\mathcal{F} \otimes \mathcal{I}) \otimes \mathcal{I}^{3b-a} \simeq \text{Sym}^{[3b]}(\mathcal{F} \otimes \mathcal{I}) \otimes \mathcal{I}^{-a} \hookrightarrow \text{Sym}^{[3b]}(\mathcal{F} \otimes \mathcal{I}) \otimes \mathcal{H}^{-1}$ is generically surjective which implies the statement.

To prove (4) take an $a$, such that $\mathcal{H}^{-1} \otimes \mathcal{H}^a$ is generically globally generated. This corresponds to a generically surjective embedding $\mathcal{H} \rightarrow \mathcal{I}^a$. According to (1) and (3), $\left(\text{Sym}^{a}(\mathcal{F}) \otimes \mathcal{H}\right)$ is big. Hence, by (2), $\text{Sym}^{a}(\mathcal{F}) \otimes \mathcal{H}^a \simeq \text{Sym}^{a}(\mathcal{F} \otimes \mathcal{I})$ is also big. Therefore, for some $b > a$, $\text{Sym}^{b}(\text{Sym}^{a}(\mathcal{F} \otimes \mathcal{I})) \otimes \mathcal{H}^{-1}$ is generically globally generated and then the surjection $\text{Sym}^{ab}(\text{Sym}^{a}(\mathcal{F} \otimes \mathcal{I})) \rightarrow \text{Sym}^{ab}(\mathcal{F} \otimes \mathcal{I})$ concludes the proof. \qed

### 4. Ampleness Lemma

**Theorem 4.1.** Let $W$ be a weakly-positive vector bundle of rank $w$ on a normal almost projective variety $Y$ with a reductive structure group $G \leq \text{GL}(k, w)$ the closure of the image of which in the projectivization $\mathbb{P} \left( \text{Mat}(k, w) \right)$ of the space of $w \times w$ matrices is normal and let $Q_i$ be vector bundles of rank $q_i$ on $Y$ admitting generically surjective homomorphisms $\alpha_i : W \rightarrow Q_i$ for $i = 0, \ldots, n$. Let $Y(k) \rightarrow \bigotimes_{i=0}^n \text{Gr}(w, q_i)(k)/G(k)$ be the induced classifying map of sets. Assume that this map has finite fibers on a dense open set of $Y$. Then $\bigotimes_{i=0}^n \det Q_i$ is big.

**Remark 4.2.** One way to define the above classifying map is to choose a basis on every fiber of $W$ over every closed point up to the action of $G(k)$. For this it is enough to fix a basis on one fiber of $W$ over a closed point, and transport it around using the $G$-structure. In fact, a little less is enough. Given a basis, multiplying every basis vector by an element of $k^\times$ does not change the corresponding rank $q$ quotient space, and hence the classifying map, so we only need to fix a basis up to scaling by an element of $k^\times$. To make it easier to talk about these in the sequel we will call a basis which is determined up to scaling by an element of $k^\times$ a homogenous basis.

**Remark 4.3.** The normality assumption in Theorem 4.1 is satisfied if $W = \text{Sym}^d V$ with $v := \text{rk} V$ and $G := \text{GL}(k, v)$ acting via the representation $\text{Sym}^d$. Indeed, in this case the closure of the image of $G$ in $\mathbb{P} \left( \text{Mat}(k, w) \right)$ agrees with the image of the embedding $\text{Sym}^d : \mathbb{P} \left( \text{Mat}(k, v) \right) \rightarrow \mathbb{P} \left( \text{Mat}(k, w) \right)$. In particular, it is isomorphic to $\mathbb{P} \left( \text{Mat}(k, v) \right)$, which is smooth.

For more results regarding when this normality assumption is satisfied in more general situations see [Tim03, DC04, BGRM11] and other references in those papers.
Remark 4.4. Theorem 4.1 is a direct generalization of the core statement [Kol90, 3.13] of Kollár’s Ampleness Lemma [Kol90, 3.9]. This statement is more general in several ways:

- The finiteness assumption on the classifying map is weaker (no assumption on the stabilizers).
- The ambient variety $Y$ is only assumed to be almost projective instead of projective.

Our proof is based on Kollár’s original idea with some modifications to allow for weakening the finiteness assumptions.

Note that if $Y$ is projective and $W$ is nef on $Y$, then it is also weakly positive [Vie95, Prop. 2.9.e].

We will start by making a number of reduction steps to simplify the statement. The goal of this reduction is to show that it is enough to prove the following theorem which contains the essential statement.

**Theorem 4.5.** Let $W$ be a weakly-positive vector bundle of rank $w$ on a normal almost projective variety $Y$ with a reductive structure group $G \subseteq \text{GL}(k, w)$ the closure of the image of which in the projectivization $\mathbb{P}(\text{Mat}(k, w))$ of the space of $w \times w$ matrices is normal and let $\alpha : W \to Q$ be a surjective morphism onto a vector bundle of rank $q$. Let $Y(k) \to \text{Gr}(w, q)(k)/G(k)$ be the induced classifying map. If this map has finite fibers on a dense open set of $Y$, then the line bundle $\det Q$ is big.

**Lemma 4.6.** Theorem 4.5 implies Theorem 4.1.

**Proof.** Step 1. We may assume that the $\alpha_i$ are surjective. Let $Q_i^- = \text{im} \alpha_i \subseteq Q_i$. Then there exists a big open subset $U : Y \hookrightarrow Y$ such that $Q_i^- |_U$ is locally free of rank $q_i$. If $\bigotimes_{i=1}^n \det Q_i^- |_U$ is big, then so is $\bigotimes_{i=1}^n \det Q_i^- = \imath_* \left( \bigotimes_{i=1}^n \det (Q_i^- |_U) \right)$ and hence so is $\bigotimes_{i=1}^n \det Q_i$. Therefore we may replace $Y$ with $U$ and $Q_i$ with $Q_i^- |_U$.

Step 2. It is enough to prove the statement for one quotient bundle. Indeed, let $W' = \bigoplus_{i=0}^n W$ with the diagonal $G$-action, $Q' = \bigoplus_{i=0}^n Q_i$, and $\alpha := \bigoplus_{i=0}^n \alpha_i : W' \to Q'$ the induced morphism. If all the $\alpha_i$ are surjective, then so is $\alpha$.

Furthermore, there is a natural injective $G$-invariant morphism

$$
\bigotimes_{i=0}^n \text{Gr}(w, q_i) \hookrightarrow \text{Gr} \left( \sum_{i=0}^n q_i \right) \quad \text{via} \quad (L_1, \ldots, L_r) \mapsto \bigoplus_{i=0}^n L_i.
$$

Since the $G$-action on $\bigotimes_{i=0}^n \text{Gr}(w, q_i)$ is the restriction of the $G$-action on $\text{Gr} \left( \sum_{i=0}^n q_i \right)$ via this embedding it follows that the induced map on the quotients remain injective:

$$
\bigotimes_{i=0}^n \text{Gr}(w, q_i) / G \hookrightarrow \text{Gr} \left( \sum_{i=0}^n q_i \right) / G.
$$

It follows that the classifying map of $\alpha' : W' \to Q'$ also has finite fibers and then the statement follows because $\det Q \simeq \bigotimes_{i=0}^n \det Q_i$.

**Lemma 4.7.** If $V \subseteq W$ is a $G$-invariant sub-vector bundle of the $G$-vector bundle $W$ on a normal almost projective variety $X$, and $W$ is weakly positive, then so is $V$.

**Proof.** $V$ corresponds to a subrepresentation of $G$, and by the characteristic zero and reductivity assumptions it follows that $V$ is a direct summand of $W$, so $V$ is also weakly positive.

**Remark 4.8.** The above lemma, which is used in the last paragraph of the proof, is the only place where the characteristic zero assumption is used in the proof of Theorem 4.1. In particular, the
statement holds in positive characteristic for a given $W$ if the $G$-subbundles of $W$ are weakly-positive whenever $W$ is. According to [Kol90, Prop 3.5] this holds for example if $Y$ is projective and $W$ is nef satisfying the assumption ($\Delta$) of [Kol90, Prop 3.6]. The latter is satisfied for example if $W = \text{Sym}^d(W')$ for a nef vector bundle $W'$ of rank $w'$ and $G = \text{GL}(k, w')$.

**Proof of Theorem 4.5.** We start with the same setup as in [Kol90, 3.13]. Let $\pi : \mathbb{P} = \mathbb{P}(\oplus_{i=1}^{w} W^*) \to Y$, which can be viewed as the space of matrices with columns in $W$, and consider the universal basis map

$$
\varsigma : \bigoplus_{j=1}^{w} \mathcal{O}_\mathbb{P}(-1) \to \pi^* W,
$$

formally given via the identification $H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(1) \otimes \pi^* W) \simeq H^0(Y, \bigoplus_{j=1}^{w} W^* \otimes W)$ by the identity sections of the different summands of the form $W^* \otimes W$. Informally, the closed points of $\mathbb{P}$ over $y \in Y$ can be thought of as $w$-tuples $(x_1, \ldots, x_w) \in W_y$ and hence a dense open subset of $\mathbb{P}_y$ corresponds to the choice of a basis of $W_y$ up to scaling by an element of $k^\times$, i.e., to a homogenous basis. Similarly, the map $\varsigma$ gives $w$ local sections of $\pi^* W$ which over $(x_1, \ldots, x_w)$ take the values $x_1, \ldots, x_w$, up to scaling by an element of $k^\times$ where this scaling corresponds to the transition functions of $\mathcal{O}_\mathbb{P}(-1)$.

As explained in Remark 4.2, to define the classifying map we need to fix a homogenous basis of a fiber over a fixed closed point. Let us fix such a point $y_0 \in Y$ and a homogenous basis on $W_{y_0}$ and keep these fixed throughout the proof. This choice yields an identification of $\mathbb{P}_{y_0}$ with $\mathbb{P}(\text{Mat}(k, w))$. Notice that the dense open set of $\mathbb{P}_{y_0}$ corresponding to the different choices of a homogenous basis of $W_{y_0}$ is identified with the image of $\text{GL}(k, w)$ in $\mathbb{P}(\text{Mat}(k, w))$ and the point in $\mathbb{P}_{y_0}$ representing the fixed homogenous basis above is identified with the image of the identity matrix in $\mathbb{P}(\text{Mat}(k, w))$. Now we want to restrict to a $G$ orbit inside all the choices of homogenous bases. Let $\tilde{G}$ denote the closure of the image of $G \subseteq \text{GL}(k, w)$ in $\mathbb{P}(\text{Mat}(k, w))$. Via the identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\text{Mat}(k, w))$, $\tilde{G}$ corresponds to a $G$-invariant closed subscheme of $\mathbb{P}_{y_0}$, which carried over by the $G$-action defines a $G$-invariant closed subscheme $\mathbb{P} \subseteq \mathbb{P}$. Note that since $\tilde{G}$ is assumed to be normal, so is $\mathbb{P}$ by [EGA-IV, II 6.5.4]. To simplify notation let us denote the restriction $\pi|_{\mathbb{P}}$ also by $\pi$. Restricting the universal basis map to $\mathbb{P}$ and twisting by $\mathcal{O}_\mathbb{P}(1)$ gives

$$
\beta := \varsigma|_{\mathbb{P}} \otimes \text{id}_{\mathcal{O}_\mathbb{P}(1)} : \bigoplus_{j=1}^{w} \mathcal{O}_\mathbb{P} \to \pi^* W \otimes \mathcal{O}_\mathbb{P}(1).
$$

Let $\Upsilon \subset \mathbb{P}$ be the divisor where this map is not surjective, i.e., those points that correspond to non-invertible matrices via the above identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\text{Mat}(k, w))$. By construction, $\beta$ gives a trivialization of $\pi^* \mathcal{O}_\mathbb{P}(1)$ over $\mathbb{P} \setminus \Upsilon$. It is important to note the following fact about this trivialization: let $p \in \mathbb{P}_{y_0}$ be the closed point that via the above identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\text{Mat}(k, w))$ corresponds to the image of the identity matrix in $\mathbb{P}(\text{Mat}(k, w))$. Then the trivialization of $\pi^* W \otimes \mathcal{O}_\mathbb{P}(1)$ given by $\beta$ gives a basis on $(\pi^* W_{y_0})_p$, which is compatible with our fixed homogenous basis on $W_{y_0}$.

Furthermore, for any $p' \in (\mathbb{P} \setminus \Upsilon)_{y_0}$ the basis on $(\pi^* W_{y_0})_{p'}$ given by $\beta$ corresponds to the fixed homogenous basis of $W_{y_0}$ twisted by the matrix (which is only given up to scaling by an element of $k^\times$) corresponding to the point $p' \in \mathbb{P}_{y_0}$ via the identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\text{Mat}(k, w))$. Note that as $G$ is reductive, it is closed in $\text{GL}(k, w)$ and hence $G(k)$ is transitive on $(\mathbb{P} \setminus \Upsilon)_{y_0}$. It follows that then the choices of homogenous bases of $W_{y_0}$ given by $\beta$ on $(\pi^* W_{y_0})_{p'}$ for $p' \in (\mathbb{P} \setminus \Upsilon)_{y_0}$ form a $G(k)$-orbit, and this orbit may be identified with $(\mathbb{P} \setminus \Upsilon)_{y_0}$.

Transporting this identification around $Y$ using the $G$-action we obtain: For every $y \in Y(k)$,

$$
(4.5.1) \quad (\mathbb{P} \setminus \Upsilon)_y \text{ may be identified with the } G(k) \text{-orbit of homogenous bases of } W_y \text{ containing the homogenous basis obtained from the fixed homogenous basis of } W_{y_0} \text{ via the } G \text{-structure.}$$
Next consider the composition of \( \alpha = \pi^*\alpha \otimes \text{id}_{O_P(1)} \) and \( \beta \):

\[
\gamma : \bigoplus_{j=1}^w O_P \xrightarrow{\beta} \pi^* W \otimes O_P(1) \xrightarrow{\alpha} \pi^* Q \otimes O_P(1)
\]

which is surjective on \( P \setminus Y \). Taking \( q \)-th wedge products yields

\[
\gamma^q : \bigoplus_{j=1}^w O_P \xrightarrow{\beta^q} \pi^* (\wedge^q W) \otimes O_P(q) \xrightarrow{\alpha^q} \pi^* \det Q \otimes O_P(q)
\]

which is still surjective outside \( Y \) and hence gives a morphism

\[
\nu : P \setminus Y \to \text{Gr}(w, q) \subseteq P \left( \bigwedge^q (k^{\oplus w}) \right) =: P_{Gr},
\]

such that

\[
\circ \quad \text{(4.5.1)}, \quad \text{on the } k \text{-points } \nu \text{ is a lift of the classifying map } Y \to \text{Gr} / G, \quad \text{where}
\]

\[
\text{Gr} := \text{Gr}(w, q) \text{ is the Grassmannian of rank } q \text{ quotients of a rank } w \text{ vectorspace, and}
\]

\[
\circ \quad \nu^* O_{Gr}(1) \simeq (\pi^* \det Q \otimes O_P(q)) |_{P \setminus Y}, \quad \text{where } O_{Gr}(1) \text{ is the restriction of } O_{P_{Gr}}(1) \text{ via the Plücker embedding.}
\]

We will also view \( \nu \) as a rational map \( \nu : P \to \text{Gr} \). Let \( \sigma : \tilde{P} \to P \) be the blow up of \( (\text{im } \gamma^q) \otimes (\pi^* \det Q \otimes O_P(q))^{-1} \subseteq O_P \) and set \( \tilde{\nu} := \pi \circ \sigma \). It follows that \( \tilde{\nu} = \nu \circ \sigma : \tilde{P} \to \text{Gr} \) is well-defined everywhere on \( P \) and there exists an effective Cartier divisor \( E \) on \( P \) such that

\[
(4.5.2) \quad \sigma^* (\pi^* \det Q \otimes O_P(q)) \simeq \tilde{\nu}^* O_{Gr}(1) \otimes O_P(E).
\]

Let \( Y^0 \subseteq Y \) be the dense open set where the classifying map has finite fibers and let \( P^0 := \tilde{\pi}^{-1}(Y^0) \setminus \sigma^{-1}(Y) \subseteq \tilde{P} \). Observe that \( P^0 \simeq \pi^{-1}(Y^0) \setminus Y \) via \( \sigma \).

Next let \( T \) be the image of the product map \( (\tilde{\pi} \times \tilde{\nu}) : \tilde{P} \to Y \times \text{Gr} \):

\[
T := \text{im } [(\tilde{\pi} \times \tilde{\nu}) : \tilde{P} \to Y \times \text{Gr}],
\]

and let \( \tau : T \to \text{Gr} \) and \( \phi : T \to Y \) be the projection. Furthermore, let \( \vartheta : \tilde{P} \to T \) denote the induced morphism. We summarize our notation in the following diagram. Note that although \( Y \) is only almost proper, every scheme in the diagram (except \( \text{Gr} \) which is proper over \( k \)) is proper over \( Y \).

![Diagram](image)

**Claim 3.** The map \( \tau |_{\vartheta(P^0)} \) has finite fibers.

**Proof.** Since \( k \) is assumed to be algebraically closed, it is enough to show that for every \( k \)-point \( x \) of \( \text{Gr} \) there are finitely many \( k \)-points of \( \vartheta(P^0) \) mapping onto \( x \). Let \( (y, x) \) be such a \( k \)-point, where \( y \in Y(k) \). Choose then \( z \in P^0(k) \) such that \( \vartheta(z) = (y, x) \). Then \( \pi(z) = y \) and \( \nu(z) = x \). Furthermore, if \( \psi \) is the classifying map and \( \xi \) is the quotient map \( \text{Gr}(k) \to \text{Gr}(k)/G(k) \), then

\[
\psi(y) = \psi(\pi(z)) = \xi(\nu(z)) = \xi(x).
\]

Therefore, \( y \in \psi^{-1}(\xi(x)) \). However, by the finiteness of the classifying map there are only finitely many such \( y \).

\( \Box \)
By construction \( \partial (P^o) \) is dense in \( T \) and it is constructible by Chevalley’s Theorem. Then the dimension of the generic fiber of \( \tau \) equals the dimension of the generic fiber of \( \tau_{|\partial (P^o)} \) and hence \( \tau \) is generically finite.

Next consider a projective closure \( Y \hookrightarrow \overline{Y} \) of \( Y \) and let \( \overline{T} \subseteq \overline{Y} \times \text{Gr} \) denote the closure of \( T \) in \( \overline{Y} \times \text{Gr} \). Let \( \overline{\varphi} : \overline{T} \rightarrow \overline{Y} \) and \( \varphi : \overline{T} \rightarrow \text{Gr} \) denote the projections. Clearly, \( \overline{\varphi}|_T = \varphi, \overline{\varphi}|_{T'} = \tau, \) and \( \overline{\varphi} \) is also generically finite. Let \( \overline{H} \) be an ample Cartier divisor on \( \overline{Y} \). Since \( \overline{\varphi} \circ \varphi = 1 \) is big, there is an \( m \), such that \( \overline{\varphi}^* \varphi^* (\overline{\varphi}^{-1} (\overline{H})) \) has a non-zero section. Let \( H = \overline{H}|_{Y} \) and restrict this section to \( T \). It follows that the line bundle

\[
(4.5.4) \quad \varphi^* (\varphi^* \varphi^* (\overline{\varphi}^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \simeq \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))
\]

also has a non-zero section, and then by (4.5.2) and (4.5.4) there is also a non-zero section of

\[
(4.5.5) \quad (\varphi^* \varphi^* \varphi^* (\overline{\varphi}^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \simeq (\varphi^* (\overline{\varphi}^{-1} (\overline{H})) \otimes \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H})))
\]

Pushing this section down via \( \varphi \) and using the projection formula we obtain a section of

\[
\sigma^* (\varphi^* (\overline{\varphi}^{-1} (\overline{H})) \otimes \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H})) \otimes \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}) \otimes \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H})))
\]

Pushing this section down via \( \varphi \) and rearranging the sheaves on the two sides of the arrow we obtain a non-zero morphism

\[
(5.2.1) \quad (\varphi^* \varphi^* \varphi^* (\overline{\varphi}^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \simeq (\varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \otimes (\varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \otimes (\varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H})))
\]

Now observe, that by construction

\[
(\varphi^* \varphi^* \varphi^* (\overline{\varphi}^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \simeq (\text{Sym}^m (w W))^\star \simeq \text{Sym}^m (w W)
\]

is weakly-positive and \( (\varphi^* \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \) is a \( G \)-invariant subbundle of \( (\varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \) for \( m \gg 0 \). In particular, by Lemma 4.7, \( (\varphi^* \varphi^* \varphi^* (\overline{\varphi}^{-1} (\overline{H}))) \) is weakly positive as well. Then by (4.5.5) and Lemma 3.8 it follows that \( \varphi^* \overline{\varphi}^{-1} (\overline{H}) \) is big.

\[ \square \]

5. **Moduli spaces of stable log-varieties**

**Definition 5.1.** A set \( I \subseteq [0,1] \) of coefficients is said to be closed under addition, if for every integer \( s > 0 \) and every \( x_1, \ldots, x_s \in I \) such that \( \sum_{i=1}^s x_i \leq 1 \) it holds that \( \sum_{i=1}^s x_i = I \).

**Definition 5.2.** Fix \( 0 < v \in \mathbb{Q}, 0 < n \in \mathbb{Z} \) and a finite set of coefficients \( I \subseteq [0,1] \) closed under addition. A functor \( \mathcal{M} : \mathcal{G}ch_k \rightarrow \mathcal{G}et_s \) (or to groupoids) is a moduli (pseudo-)functor of stable log-varieties of dimension \( n \), volume \( v \) and coefficient set \( I \), if for each normal \( Y \),

\[
(5.2.1) \quad \mathcal{M}(Y) = \begin{cases} (X,D) \rightarrow f \in \text{Fl} & (1) \ f \text{ is a flat morphism}, \\ Y \end{cases}
\]

\[
(2) \ D \text{ is a Weil-divisor on } X \text{ avoiding the generic and the codimension 1 singular points of } X_y \text{ for all } y \in Y, \\
(3) \text{ for each } y \in Y, (X_y,D_y) \text{ is a stable log-variety of dimension } n, \text{ such that the coefficients of } D_y \text{ are in } I, \text{ and } (K_{X_y} + D_y)^n = v, \text{ and } \\
(4) \ K_{X/Y} + D \text{ is } \mathbb{Q}-\text{Cartier.}
\]

and the line bundle \( Y \rightarrow \text{det } f_* E_X (r(K_{X/Y} + D)) \) associated to every family as above extends to a functorial line bundle on the entire (pseudo-)functor for every divisible enough integer \( r > 0 \).

Also note that if \( \mathcal{M} \) is regarded as a functor in groupoids, then in (5.2.1) instead of equality only equivalence of categories should be required.
Remark 5.3. (1) The condition “\(D\) is a Weil-divisor on \(X\) avoiding the generic and the codimension 1 singular points of \(X\) for all \(y \in Y\)” guarantees that \(D_y\) can be defined sensibly. Indeed, according to this condition, there is a big open set of \(X_y\), over which \(D\) is \(\mathbb{Q}\)-Cartier.

(2) The condition ”\(K_{X/Y} + D\) is \(\mathbb{Q}\)-Cartier“ is superfluous according to a recent, yet unpublished result of Kollár, which states that for a flat family with stable stable fibers if \(y \mapsto (K_{X_y} + D_y)\) is constant, then \(K_{X/Y} + D\) is automatically \(\mathbb{Q}\)-Cartier.

(3) \(I\) has to be closed under addition, to guarantee properness. Indeed, divisors with coefficients \(c_1, c_2, \ldots, c_n\), respectively, can come together in the limit to form a divisor with a coefficient \(\sum_{i=1}^{s} c_i\).

(4) According to [HMX14, Thm 1.1], after fixing \(n, v\) and a DCC set \(I \subseteq [0, 1]\), there exist automatically

(a) a finite set \(I_0 \subseteq I\) containing all the possible coefficients of stable log-varieties of dimension \(n\), volume \(v\) and coefficient set \(I\), and

(b) a uniform \(m\) such that \(m(K_X + D)\) is Cartier for all stable log-varieties \((X, D)\) of dimension \(n\), volume \(v\) and coefficient set \(I\).

In particular, \(m\) may also be fixed in the above definition if it is chosen to be divisible enough after fixing the other three numerical invariants.

Proposition 5.4. Let \(n > 0\) be an integer, \(v > 0\) a rational number and \(I \subseteq [0, 1]\) a finite coefficient set closed under addition. Then any moduli (pseudo-)functor of stable log-varieties of dimension \(n\), volume \(v\) and coefficient set \(I\) is proper. That is, if it admits a coarse moduli space which is an algebraic space then that coarse moduli space is proper over \(k\). If in addition the pseudo-functor itself is a DM-stack then it is a proper DM-stack over \(k\) (from which the existence of the coarse moduli space as above follows [KeM97, Con05]).

Proof. This is shown in [Kol14, Thm 12.11]. \(\square\)

Proposition 5.5. If \((X, D)\) is a stable log-variety then \(\text{Aut}(X, D)\) is finite.

Proof. Let \(\pi : \overline{X} \to X\) be the normalization of \(X\) and \(\overline{D}\) is defined via

\[
K_{\overline{X}} + \overline{D} = \pi^*(K_X + D)
\]

where \(K_X\) and \(K_{\overline{X}}\) are chosen compatibly such that \(K_X\) avoids the singular codimension one points of \(X\). Note that \(\overline{D} \geq 0\) by [Kol13b, (5.75)]. Any automorphism of \((X, D)\) extends to an automorphism of \((\overline{X}, \overline{D})\), hence we may assume that \((X, D)\) is normal. Furthermore, since \(X\) has finitely many irreducible components, the automorphisms fixing each component form a finite index subgroup. Therefore, we may also assume that \(X\) is irreducible. Let \(U \subseteq X\) be the regular locus of \(X \setminus \text{Supp} D\). Note that \(U\) is \(\text{Aut}(X, D)\)-invariant, hence there is an embedding \(\text{Aut}(X, D) \hookrightarrow \text{Aut}(U)\). In particular, it is enough to show that \(\text{Aut}(U)\) is finite. Next let \(g : (Y, E) \to (X, D)\) be a log-crepant resolution that is an isomorphism over \(U\) and for which \(g^{-1}(X \setminus U)\) is a normal-crossing divisor. Let \(F\) be the reduced divisor with support equal to \(g^{-1}(X \setminus U)\). Then \((Y, E)\) is log-canonical, and \(E \leq F\). Therefore, \(g^*(K_X + D) = K_Y + E \leq K_Y + F\) and hence \((Y, F)\) is of log general type. However, \(U = Y \setminus \text{Supp} F\), and hence \(U\) itself is of general type. Then by [lit82, Thm 11.12] a group (which is called \(\text{SBir}(U)\) there) containing \(\text{Aut}(U)\) is finite. \(\square\)

5.1. A particular functor of stable log-varieties

In what follows we describe a particular functor of stable log-varieties introduced by János Kollár [Kol13a, (3) of page 21]. The main reason we do so is to be able to give Definition 5.16 and prove Corollary 5.18 and Corollary 5.19. These are used in the following sections.

In fact, our method will be somewhat non-standard: we define a pseudo-functor \(\mathcal{M}_{n,m,h}\) which is larger than needed in Definition 5.6. We show that \(\mathcal{M}_{n,m,h}\) is a DM-stack (Proposition 5.11) and if \(m\) is divisible enough (after fixing \(n\) and \(v\)), the locus of stable log-varieties of dimension \(n\), volume
The issue in general about functors of stable log-varieties is that, as Definition 5.2 suggests, it is not clear what their values should be over non-reduced schemes. The main problem is to understand the nature and behavior of $D$ in those situations. Kollár’s solution to this is that instead of trying to figure out how $D$ should be defined over non-reduced schemes, let us replace $D$ as part of the data with some other data equivalent to (5.2.1) that has an obvious extension to non-reduced schemes. This “other” data is as follows: instead of remembering $D$ the nature and behavior of $m$ should be identified eventually.

There are two things we note before proceeding to the precise definition.

1. A global choice of $m$ as above is possible according to Remark 5.3.

2. Fixing $(X, \phi : \omega_X^{\otimes m} \to \mathcal{L})$ is slightly more than just fixing $(X, D)$, since composing $\phi$ with an automorphism $\xi$ of $\mathcal{L}$ is formally different, but yields the same $D$. In particular, we have to remember that different pairs $(X, \phi)$ that only differ by an automorphism $\xi$ of $\mathcal{L}$ should be identified eventually.

We define our auxiliary functor $\mathcal{M}_{n,m,h}$ according to the above considerations.

**Definition 5.6.** Fix an integer $n > 0$, a polynomial $h : \mathbb{Z} \to \mathbb{Z}$ and an integer $m > 0$ divisible enough (after fixing $n$ and $h$). We define the auxiliary pseudo-functor $\mathcal{M}_{n,m,h}$ as

\[
\mathcal{M}_{n,m,h}(Y) = \left\{ \left( \begin{array}{c} X \\ \phi : \omega_{X/Y}^{\otimes m} \to \mathcal{L} \\ Y \end{array} \right) \right| \begin{array}{l}
(1) f \text{ is a flat morphism of pure relative dimension } n, \\
(2) \mathcal{L} \text{ is a relatively very ample line bundle on } X \text{ such that } R^i f_* (\mathcal{L}^r) = 0 \text{ for every } r > 0, \text{ and }\\
(3) \text{ for all } y \in Y: \\
\text{i. } \phi \text{ is an isomorphism at the generic points and at the codimension 1 singular points of } X_y, \\
\text{and hence it determines a divisor } D_y, \text{ such that } \mathcal{L}_y \cong \omega_y (m(K_{X_y} + D_y)), \\
\text{ii. } (X_y, D_y) \text{ is slc, and } \\
\text{iii. } h(r) = \chi(X_y, \mathcal{L}_y^r) \text{ for every integer } r > 0. \\
\end{array} \right\} 
\]

where

(a) as indicated earlier, if $Y$ is normal, $\phi$ corresponds to an actual divisor $D$ such that $\omega_X (m(K_{X/Y} + D)) \cong \mathcal{L}$. Explicitly, $D$ is the closure of $E/m$, where $E$ is the divisor determined by $\phi$ on the relatively Gorenstein locus $U$.

(b) The arrows in $\mathcal{M}_{n,m,h}$ between

\[
\left( X \to S, \phi : \omega_{X/S}^{\otimes m} \to \mathcal{L} \right) \in \mathcal{M}_{n,m,h}(S),
\]

and

\[
\left( X' \to T, \phi' : \omega_{X'/T}^{\otimes m} \to \mathcal{L}' \right) \in \mathcal{M}_{n,m,h}(T),
\]

over a fixed $T \to S$ are of the form $(\alpha : X' \to X, \xi : \alpha^* \mathcal{L} \to \mathcal{L}')$, such that the square

\[
\begin{array}{c}
X' \\
\downarrow^{\alpha} \\
T \\
\downarrow \\
S
\end{array}
\]
is Cartesian, and \( \xi \) is an isomorphism such that the following diagram is commutative.

\[
\begin{array}{ccc}
\alpha^* \omega_{X/S}^m & \xrightarrow{\alpha^* \phi} & (\alpha^* \omega_{X/S}^m)^{**} \\
\omega_{X/T}^m & \xrightarrow{m} & \omega_{X'/T}^m
\end{array}
\]

(5.6.2)

In other words, \( \phi' \) corresponds to \( \xi \circ \alpha^* \phi \) via the natural identification

\[
\text{Hom} \left( \alpha^* \omega_{X/S}^m, \mathcal{L}' \right) = \text{Hom} \left( \omega_{X'/T}^m, \mathcal{L}' \right).
\]

(c) An arrow as above is an isomorphism if \( T \to S \) is the identity and \( \alpha \) is an isomorphisms.

(d) We fix the following pullback construction. It features subtleties similar to that of (5.6.2) stemming from the fact that only the hull \( \omega_{X/Y}^m \) of \( \omega_{X/Y}^m \) is compatible with base-change.

So, let us consider \((X, \phi) := (X \to S, \phi : \omega_{X/S}^m \to \mathcal{L}) \in \mathcal{M}_{n,m,h}(S)\) and a \( k \)-morphism \( T \to S \). Then \((X, \phi)_T := (X_T \to T, \phi[T] : \omega_{X/T}^m \to \mathcal{L}_T)\), where \( \phi[T] \) is defined via the following commutative diagram.

\[
\begin{array}{ccc}
\omega_{X_T/T}^m & \xrightarrow{\phi_T} & \mathcal{L}_T \\
\omega_{X/T}^m & \xrightarrow{m} & \mathcal{L}_T
\end{array}
\]

In other words, via the natural identification \( \text{Hom} \left( \left( \omega_{X/Y}^m \right)_T, \mathcal{L}_T \right) = \text{Hom} \left( \omega_{X'/T}^m, \mathcal{L}_T \right) \), \( \phi_T \) corresponds to \( \phi[T] \).

We leave the proof of the following statement to the reader. We only note that the main reason it holds is that the presence of the very ample line bundle \( \mathcal{L} \) makes descent work.

**Proposition 5.7.** When viewed as a pseudo-functor (or equivalently as a category fibered in groupoids) \( \mathcal{M}_{n,m,h} \) is an étale (or even fppf) stack.

**Proposition 5.8.** Let \((f : X \to Y, \phi : \omega_{X/Y}^m \to \mathcal{L})\) and \((f' : X' \to Y, \phi' : \omega_{X'/Y}^m \to \mathcal{L}')\) be two objects in \( \mathcal{M}_{n,m,h}(Y) \). Then the isomorphism functor of these two families \( \text{Isom}_Y((X, \phi), (X', \phi')) \) is representable by a quasi-projective scheme over \( Y \), which is denoted by \( \text{Isom}_Y((X, \phi), (X', \phi')) \). Furthermore, this isomorphism scheme, \( \text{Isom}_Y((X, \phi), (X', \phi')) \), is unramified over \( Y \).

**Remark 5.9.** Recall that, by definition, \( \text{Isom}_Y((X, \phi), (X', \phi'))(T) \) is the set of \( T \)-isomorphisms between \((X, \phi)_T\) and \((X', \phi')_T\) for any scheme \( T \) over \( k \).

**Proof.** First, we show the representability part of the statement. Let \( I := \text{Isom}_Y^1(X, X') \to Y \) be the connected components of the Isom scheme \( \text{Isom}_Y(X, X') \) parametrizing isomorphisms...
γ : X_T → X'_T such that γ^*L_T'' ≅ T L_T [Kol96, Exercise 1.10.2]. It comes equipped with a universal isomorphism α : X_I → X'_I. Now, let J := Isom_Y(α^*L'_I, L_I) [Kol08, 33] parametrizing isomorphisms. This space also comes equipped with a universal isomorphism ξ : α^*L'_I → L_J. This space J, with the universal family α_I : X_I → X'_I and ξ : α^*L'_I → L_J is a fine moduli space for the functor

\[ T \mapsto \{ (\beta, \zeta) | \beta : X_T → X'_T \text{ and } \zeta : \beta^*L_T'' → L_T \text{ are isomorphisms} \}. \]

This is almost the functor Isom_Y((X, φ), (X', φ')), except in the latter there is an extra condition that the following diagram commutes:

\[
\begin{array}{ccc}
\beta^*ω_{X_T/T}' & \xrightarrow{\beta^*φ_{[T]}} & \beta^*L_T'' \\
\downarrow{\simeq} & & \downarrow{\zeta} \\
ω_{X_T/T}' & \xrightarrow{φ_{[T]}} & L_T''
\end{array}
\]

(5.9.1)

Note that here we do not have to take hulls. Indeed, \( β^*ω_{X'_T/T}'' \) itself is isomorphic to \( ω_{X_T/T}' \) via the \( m \)-th tensor power of the unique extension of the canonical map of [Con00, Thm 3.6.1] from the relative Gorenstein locus, since \( β \) is an isomorphism and hence \( β^*ω_{X'_T/T}'' \) is reflexive.

Hence we are left to show that the condition of the commutativity of (5.9.1) is a closed condition. That is, there is a closed subscheme \( S \subseteq J \), such that the condition of (5.9.1) holds if and only if the induced map \( T → J \) factors through \( S \).

Set \( ψ := φ_{[J]} \) and let \( \psi' \) be the composition

\[
ω_{X_J/J}' \simeq α^*ω_{X'_J/J}' → α^*φ_{[J]} → α^*L'_J → ξ : L_J.
\]

Consider \( M := \text{Hom}\left(ω_{X'_J/J}' L_J\right) \) [Kol08, 33]. The homomorphisms \( ψ \) and \( ψ' \) give two sections \( s, s' : J → M \). Let \( S := s'^{-1}(s(J)) \).

In the remainder of the proof we show the above claimed universal property of \( S \). Take a scheme \( T \) over \( k \) and a pair of isomorphisms \( (β, ζ) \), where \( β \) is a morphism \( X_T → X'_T \) and \( ζ \) is a homomorphism \( β^*L_T'' → L_T \). Let \( μ : T → J \) be the moduli map, that is via this map \( β = α_T \) and \( ζ = ξ_T \). We have to show that the commutativity of (5.9.1) holds if and only if \( μ \) factors through the closed subscheme \( S \subseteq J \).

First, by the natural identification \( \text{Hom}(ω_{X_T/T}', L_T) = \text{Hom}\left(ω_{X_J/J}' T, L_T\right) \) the commutativity of (5.9.1) is equivalent to \( ψ_T = ψ'_T \). Second, by functoriality of Mor, the latter condition is equivalent to \( s_T = s'_T \) (as sections of \( MT → T \)). However, the latter is equivalent to the factorization of \( T → J \) through \( S \), which shows that indeed \( \text{Isom}_Y((X, φ), (X', φ')) := S \) represents the functor \( \text{Isom}_Y((X, φ), (X', φ')) \).

For the addendum, note that \( \text{Isom}_Y((X, φ), (X', φ')) \) is a group scheme over \( Y \). Since \( \text{char} k = 0 \), the characteristics of all the geometric points is 0 and hence all the geometric fibers are smooth. This implies that \( \text{Isom}_Y((X, φ), (X', φ')) \) is unramified over \( Y \) [StacksProject, Tag 02G8], since its geometric fibers are finite by Proposition 5.5.

**Lemma 5.10.** Let \( f : X → Y, ω_{X/Y}' → L \) satisfy conditions (1), (2), (3i) and (3iii) in (5.6.1), i.e., do not assume that \( (X_y, D_y) \) is slc. Further assume that \( X_y \) is demi-normal for all \( y ∈ Y \) and \( Y \) is essentially of finite type over \( k \). Then the subset \( Y^c := \{ y ∈ Y | (X_y, D_y) \text{ is slc} \} \) is constructible. For that it is enough to show that there is a non-empty
open set $U$ of $Y$ such that either $(X_y, D_y)$ is slc for all $y \in U$ or $(X_y, D_y)$ is not slc for all $y \in U$ and conclude by noetherian induction. To prove the existence of such a $U$, we may assume that $Y$ is irreducible. Let $\rho : X' \to X$ be a semi-smooth log-resolution and $U \subseteq Y$ an open set for which

- $\rho^{-1} f^{-1} U \to U$ is flat,
- $X'_y \to X_y$ is a semi-smooth log-resolution for all $y \in U$, and
- for any exceptional divisor $E$ of $\rho$ that does not dominate $Y$ (i.e., which is $f$-vertical) $f(\rho(E)) \cap U = \emptyset$.

It follows that for $y \in U$, the discrepancies of $(X_y, D_y)$ are independent of $y$. Hence, either every such $(X_y, D_y)$ is slc or all of them are not slc.

Next, we prove that the locus \{ $y \in Y \mid (X_y, D_y)$ is slc \} is closed under generalization, which will conclude our proof by [Har77, Exc I.3.18.c]. For, this we should prove that if $Y$ is a DVR, essentially of finite type over $k$, and $(X_\xi, D_\xi)$ is slc for the closed point $\xi \in Y$, then so is $(X_\eta, D_\eta)$ for the generic point $\eta \in Y$. However, this follows immediatley by inversion of adjunction for slc varieties [Pat12a, Cor 2.11], since that implies that $(X, D + X_\xi)$ is slc and then by localizing at $\eta$ we obtain that $(X_\eta, D_\eta)$ is slc.

\textbf{Proposition 5.11.} $\mathcal{M}_{n,m,h}$ is a DM-stack of finite type over $k$.

\textbf{Proof.} For simplicity let us denote $\mathcal{M}_{n,m,h}$ by $\mathcal{M}$. According to [DM69, 4.21] we have to show that $\mathcal{M}$ has representable and unramified diagonal, and there is a smooth surjection onto $\mathcal{M}$ from a scheme of finite type over $k$. For any stack $\mathcal{X}$ and a morphism from a scheme $T \to \mathcal{X} \times_k \mathcal{X}$ corresponding to $s, t \in \mathcal{X}(T)$, the fiber product $\mathcal{X} \times_{\mathcal{X} \times_k \mathcal{X}} T$ can be identified with $\text{Isom}_T(s, t)$. Hence the first condition follows from Proposition 5.8. For the second condition we are to construct a cover $S$ of $\mathcal{M}$ by a scheme such that $S \to \mathcal{M}$ is formally smooth. The rest of the proof is devoted to this.

Set $N := p(1) - 1$. Then, $\mathfrak{Sil}_{\mathcal{X},N}^h$ contains every $(X, \phi : \omega_X^{\otimes m} \to \mathcal{L}) \in \mathcal{M}(k)$, where $X$ is embedded into $\mathbb{P}^N_k$ using $H^0(X, \mathcal{L})$. Let $\mathcal{H}_1 := \mathfrak{Sil}_{\mathcal{X},N}^h$ be the open subscheme corresponding to $X \subseteq \mathbb{P}^N_k$, such that $H^i(X, \mathcal{O}_X(r)) = 0$ for all integers $i > 0$ and $r > 0$. According to [EGA-IV, III.12.2.1], there is an open subscheme $\mathcal{H}_2 \subseteq \mathcal{H}_1$ parametrizing the reduced equidimensional and $S_2$ varieties. Since small deformations of nodes are either nodes or regular points, we see that there is an open subscheme $\mathcal{H}_3 \subseteq \mathcal{H}_2$ parametrizing the demi-normal varieties (where reducedness and equidimensionality is included in demi-normality). Let $\mathcal{U}_3$ be the universal family over $\mathcal{H}_3$. According to [Kol08, 33] there is a fine moduli scheme $M_4 := \text{Hom}_{\mathcal{H}_3} \left( \omega_{\mathcal{U}_3/\mathcal{H}_3}^{\otimes m}, \mathcal{O}_{\mathcal{U}_3}(1) \right)$. Define $\mathcal{U}_4$ and $\mathcal{O}_{\mathcal{U}_4}(1)$ to be the pullback of $\mathcal{U}_3$ and of $\mathcal{O}_{\mathcal{U}_3}(1)$ over $M_4$. Then there is a universal homomorphism $\gamma : \omega_{\mathcal{U}_4/M_4}^{\otimes m} \to \mathcal{O}_{\mathcal{U}_4}(1)$. Let $M_5 \subseteq M_4$ be the open locus where $\gamma$ is an isomorphism at every generic point and singular codimension one point of each fiber is open. Let $\mathcal{U}_5$ and $\mathcal{O}_{\mathcal{U}_5}(1)$ the restrictions of $\mathcal{U}_4$ and $\mathcal{O}_{\mathcal{U}_4}(1)$ over $M_5$. According to Lemma 5.10, there is an even smaller open locus $M_6 \subseteq M_5$ defined by

$$M_6 := \left\{ t \in M_5 \left| \omega_{(\mathcal{U}_5)_t}^{\otimes m} \to \mathcal{O}_{(\mathcal{U}_5)_t}(1) \text{ corresponds to an slc pair} \right. \right\}.$$ 

Then define $S := M_6$ and $g : U \to S$ and $\phi : \omega_{U/S}^{\otimes m} \to \mathcal{O}_U(1)$ to be respectively the restrictions of $\mathcal{U}_5$ to $M_5$ and of $\gamma$ over $M_6$. From Definition 5.6 and by cohomology and base-change it follows that for each $(h : X_T \to T, \phi' : \omega_{X_T/T}^{\otimes m} \to \mathcal{L}_T) \in \mathcal{M}(T)$ such that $T$ is Noetherian,

1. the sheaf $h_*\mathcal{L}_T$ is locally free, and
2. giving a map $\nu : T \to S$ and an isomorphism $(\alpha, \xi)$ between $(h : X_T \to T, \phi' : \omega_{X_T/T}^{\otimes m} \to \mathcal{L}_T)$ and $(U_T \to T, \phi[T] : \omega_{U_T/T}^{\otimes m} \to \mathcal{O}_{U_T}(1))$ is equivalent to fixing a set of free generators $s_0, \ldots, s_n \in h_*\mathcal{L}_T$. 

Indeed, for the second statement, fixing such a generator set is equivalent to giving a closed embedding \( \iota : X_T \to \mathbb{P}^N_T \) with Hilbert polynomial \( h \) together with an isomorphism \( \zeta : \mathcal{L}_T \to \iota^* \mathcal{O}_{\mathbb{P}^N}(1) \). Furthermore, the latter is equivalent to a map \( \nu_{\text{pre}} : T \to \mathcal{H}_3 \) together with isomorphisms \( \alpha : X_T \to (\mathcal{Y}_3)_{T} \) and \( \xi : \mathcal{L}_T \to \alpha^* \mathcal{O}_{(\mathcal{Y}_3)_T}(1) \). Then the composition
\[
\alpha^* \omega^{\otimes m}_{(\mathcal{Y}_3)_{T}/T} \overset{\sim}{\longrightarrow} \omega^{\otimes m}_{X_T/T} \overset{\phi'}{\longrightarrow} \mathcal{L}_T \overset{\xi}{\longrightarrow} \iota^* \mathcal{O}_{(\mathcal{Y}_3)_T}(1)
\]
yields a lifting of \( \nu_{\text{pre}} \) to a morphism \( \nu : T \to S \), such that \( (\alpha, \xi^{-1}) \) is an isomorphism between \((X_T, \phi')\) and \((U_T, \phi_{|T})\).

Now, we show that the map \( S \to \mathcal{M} \) induced by the universal family over \( S \) is smooth. It is of finite type by construction, so we have to show that it is formally smooth. Let \( \delta : (A', \mathfrak{m}') \to (A, \mathfrak{m}) \) be a surjection of Artinian local rings over \( k \) such that \( \mathfrak{m}(\ker \delta) = 0 \). Set \( T := \text{Spec} A \) and \( T' := \text{Spec} A' \). According to [EGA-IV, IV.17.14.2], we need to show that if there is a 2-commutative diagram of solid arrows as follows, then one can find a dashed arrow keeping the diagram 2-commutative.

\[
\begin{array}{ccc}
S & \searrow & T \\
\downarrow & & \downarrow \\
\mathcal{M} & \longleftarrow & T'
\end{array}
\]

In other words, given a family \( \left(h : X_{T'} \to T', \phi' : \omega^{\otimes m}_{X_{T'}/T'} \to \mathcal{L}\right) \in \mathcal{M}(T') \), with an isomorphism \((\beta, \zeta)\) between \((X_T, \phi_T)\) and \((U_T, \phi_{|T})\). We are supposed to prove that \((\beta, \zeta)\) extends over \( T' \). However, as explained above, \((\beta, \zeta)\) corresponds to free generators of \((h_T)^* \mathcal{L}_T \), which can be lifted over \( T' \) since \( T \to T' \) is an infinitesimal extension of Artinian local schemes. \(\square\)

**Lemma 5.12.** Let \( \left(f : X \to Y, \omega^{\otimes m}_{X/Y} \to \mathcal{L}\right) \in \mathcal{M}_{n,m,h}(T) \) for some \( T \) essentially of finite type over \( k \) and \( I \subseteq [0, 1] \) a finite coefficient set closed under addition. Then the locus
\[
\{t \in T | (X_t, D_t) \text{ has coefficients in } I\}
\]
is closed (here \( D_t \) is the divisor corresponding to \( \phi_t \)). Furthermore, if \( m \) is divisible enough (after fixing \( n, v \) and \( I \)), then the above locus is proper over \( k \).

**Proof.** For the first statement, according to [Har77, Exc II.3.18.c] we are supposed to prove that the above locus is constructible and closed under specialization. Both of these follow from the fact that if \( T \) is normal, and \( D_t \) is the divisor corresponding to \( \phi_T \), then there is a dense open set \( U \subseteq T \) such that the coefficients of \( D_T \) and of \( D_t \) agree for all \( t \in U \). For the “closed under specialization” part one should also add that if \( T \) is a DVR with generic point \( \eta \) and special point \( \varepsilon \), then the coefficient set of \( D_\eta \) agrees with the coefficient set of \( D \), and the coefficients of \( D_\varepsilon \) are sums formed from coefficients of \( D \). Since \( I \) is closed under addition, if \( D_\eta \) has coefficients in \( I \), so does \( D_\varepsilon \).

The properness statement follows from [Kol14, Thm 12.11] and [HMX14, Thm 1.1]. \(\square\)

**Notation 5.13.** Fix an integer \( n > 0 \), a rational number \( v > 0 \) and a finite coefficient set \( I \subseteq [0, 1] \) closed under addition. After this choose an \( m \) that is divisible enough. For stable log-varieties \((X, D)\) over \( k \) for which \( \dim X = n \), \( (K_X + D)^n = v \) and the coefficient set is in \( I \), there are finitely many possibilities for the Hilbert polynomial \( h(r) = \chi(X, rm(K_X + D)) \) by [HMX14, Thm 1.1]. Let \( h_1, \ldots, h_s \) be these values. For each integer \( 1 \leq i \leq s \), let \( \mathcal{M}_i \) denote the reduced structure on the locus (5.12.1) of \( \mathcal{M}_{n,m,h_i} \) and let \( \mathcal{M}_{n,v,I} := \coprod_{i=1}^s \mathcal{M}_i \) (where \( \coprod \) denotes disjoint union).

**Proposition 5.14.** \( \mathcal{M}_{n,v,I} \) is a pseudo-functor for stable log-varieties of dimension \( n \), volume \( v \) and coefficient set \( I \).
Proof. Given a normal variety $T$, $\mathcal{M}_{n,v,I}(T) = \Pi_{i=1}^s \mathcal{M}_i(T)$. Since in Notation 5.13, $\mathcal{M}_i$ were defined by taking reduced structures, for reduced schemes $T$, there are no infinitesimal conditions on $\mathcal{M}_i(T)$. That is it is equivalent to the sub-groupoid of $\mathcal{M}_{n,m,h_i}(T)$ consisting of $(X \to T, \phi : \omega_X^{\otimes m} \to \mathcal{L})$, such that the coefficients of $(X_t, D_t)$ is in $I$. Then it follows by construction that the disjoint union of these is equivalent to the groupoid given in (5.2.1) and that the line bundle $\det f_* \mathcal{L}^j$ associated to $(X \to T, \phi : \omega_X^{\otimes m} \to \mathcal{L}) \in \mathcal{M}_{n,v,I}(T)$ yields a polarization for every integer $j > 0$.

Remark 5.15. $\mathcal{M}_{n,v,I}$ a-priori depends on the choice of $m$, which will not matter for our applications. However, one can show by exhibiting isomorphic groupoid representations that in fact the normalization of any DM-stack $\mathcal{M}$ which is a pseudo-functor of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$ is isomorphic to the normalization of $\mathcal{M}_{n,v,I}$.

Definition 5.16. Given a family $f : (X, D) \to Y$ of stable log-varieties over an irreducible normal variety, such that the dimension $\dim X_y = n$ and the volume $(K_{X_y} + D_y)^n$ of the fibers are fixed. Let $I$ be the set of all possible sums, at most 1, formed from the coefficients of $D$. Then, there is an associated moduli map $\mu : Y \to \mathcal{M}_{n,v,I}$. The variation $\text{Var} f$ of $f$ is defined as the dimension of the image of $\mu$.

Note that this does not depend on the choice of $m$ or $I$ (see Remark 5.15), since it is $\dim Y - d$, where $d$ is the general dimension of the isomorphism equivalence classes of the fibers $(X_y, D_y)$. This general dimension exists, because it can also be expressed as the general fiber dimension of $\text{Isom}_Y((X, \phi), (X, \phi))$, where $(X, \phi) \in \mathcal{M}_{n,m,h}(Y)$ corresponds to $(X, D)$.

Further note that it follows from the above discussion that using any pseudo-functor of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$ instead of $\mathcal{M}_{n,v,I}$ leads to the same definition of variation.

Remark 5.17. Corollary 5.20 gives another alternative definition of variation: it is the smallest number $d$ such that there exists a diagram as in Corollary 5.20 with $d = \dim Y'$.

Corollary 5.18. Given $f : (X, D) \to Y$ a family of stable log-varieties over a normal variety $Y$, and a compactification $\overline{Y} \supseteq Y$, there is a generically finite proper morphism $\tau : \overline{Y} \to \overline{Y}$ from a normal variety, and a family $f : (\overline{X}, \overline{D}) \to \overline{Y}$ of stable log-varieties, such that $(\overline{X}_{Y'}, \overline{D}_{Y'}) \simeq (X_{Y'}, D_{Y'})$, where $Y' := \tau^{-1}Y$.

Proof. Let $n$ be the dimension and $v$ the volume of the fibers of $f$. Let $I \subseteq [0, 1]$ be a finite coefficient set closed under addition that contains the coefficients of $D$. Denote for simplicity $\mathcal{M}_{n,v,I}$ by $\mathcal{M}$. According to [LMB00, Thm 16.6], there is a finite, generically étale surjective map $S \to \mathcal{M}$, and $f : (X, D) \to Y$ induces another one $Y \to \mathcal{M}$. Let $Y'$ be a component of the normalization of $Y \times \mathcal{M} S$ dominating $Y$. Note that since $\mathcal{M}$ is a DM-stack, $Y$ is a scheme and $Y' \to Y$ is finite and surjective. Hence, we may compactify $Y'$ to obtain a normal projective variety $\overline{Y'}$, such that the maps $Y' \to S$ and $Y' \to Y$ extend to morphisms $\overline{Y'} \to S$ and $\overline{Y'} \to \overline{Y}$ (note that both $S$ and $\overline{Y'}$ are proper over $k$). Hence, we have a 2-commutative diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\tau} & \overline{Y'} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\mu} & S
\end{array}
\]

which shows that the induced family on $\overline{Y'}$ has the property as required, that is, by pulling back to $Y'$ it becomes isomorphic to the pullback of $(X, D)$ to $Y'$.

\qed
Corollary 5.19. If \( \mathcal{M} \) is a moduli (pseudo-)functor of stable log-varieties of dimension \( n \), volume \( v \) and coefficient set \( I \) admitting a coarse moduli space \( \mathcal{M} \) which is an algebraic space, then there is a finite cover \( S \rightarrow \mathcal{M} \) from a normal scheme \( S \) induced by a family \( f \in \mathcal{M}(S) \).

Proof. Since for every moduli (pseudo-)functor \( \mathcal{M} \) of stable log-varieties of dimension \( n \), volume \( v \) and coefficient set \( I \), \( \mathcal{M}(k) \) is the same (as a set or as a groupoid), and furthermore \( \mathcal{M} \) is proper over \( k \) according to Proposition 5.4, it is enough to show that there is a proper \( k \)-scheme \( S \), such that \( S \) supports a family \( f \in \mathcal{M}(S) \) for which

1. the isomorphism equivalence classes of the fibers of \( f \) are finite, and
2. every isomorphism class in \( \mathcal{M}(k) \) appears as a fiber of \( f \).

However, the existence of this follows by [LMB00, Thm 16.6] and Proposition 5.11. □

Corollary 5.20. Given a family \( f : (X,D) \rightarrow Y \) of stable log-varieties over a normal variety, there is diagram

\[
\begin{array}{ccc}
(X', D') & \xrightarrow{(X'', D'')} & (X, D) \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{Y''} & Y
\end{array}
\]

with Cartesian squares, such that

1. \( Y' \) and \( Y'' \) are normal,
2. \( \text{Var } f = \dim Y' \),
3. \( Y'' \rightarrow Y \) is finite, surjective, and
4. \( f' : (X', D') \rightarrow Y' \) is a family of stable log-varieties for which the induced moduli map is finite. In particular, the fiber isomorphism classes of \( f' : (X', D') \rightarrow Y' \) are finite.

Proof. Set \( n := \dim X_y \) and \( v := (K_{X_y} + D_y)^n \). Let \( I \) be the set of all possible sums, at most 1, formed from the coefficients of \( D \). Then there is an induced moduli map \( \nu : Y \rightarrow \mathcal{M}_{n,v,I} \). Let \( S \rightarrow \mathcal{M}_{n,v,I} \) be the finite cover given by Corollary 5.19. The map \( Y \times_{\mathcal{M}_{n,v,I}} S \rightarrow Y \) is finite and surjective. Define \( Y'' \) to be the normalization of an irreducible component of \( Y \times_{\mathcal{M}_{n,v,I}} S \) that dominates \( Y \) and define \( Y' \) to be the normalization of the image of \( Y'' \) in \( S \). That is, we obtain a 2-commutative diagram

\[
\begin{array}{ccc}
Y'' & \xrightarrow{Y'} & . \\
\downarrow{\nu} & & \downarrow{\mathcal{M}_{n,v,I}} \\
Y & & 
\end{array}
\]

This yields families over \( Y' \) and \( Y'' \) as required by the statement. □

6. Determinants of Pushforwards

The main results of this section are the following theorem and its corollary. For the definition of stable families see Definition 2.11 and for the definition of variation see Definition 5.16 and Remark 5.17. We also use Notation 2.12 in the next statement.

Theorem 6.1. If \( f : (X,D) \rightarrow Y \) is a family of stable log-varieties of maximal variation over a smooth projective variety, then

1. there exists an \( r > 0 \) such that \( K_{X^{(r)}/Y} + D_{X^{(r)}} \) is big on at least one component of \( X^{(r)} \),

or equivalently

\[
\left( K_{X^{(r)}/Y} + D_{X^{(r)}} \right)^{\dim X^{(r)}} > 0,
\]

and
(2) for every divisible enough \( q > 0 \), det \( f_*\Theta_X(q(K_{X/Y} + \Delta)) \) is big.

**Remark 6.2.** The \( r \)-th fiber power in point (1) of Theorem 6.1 cannot be dropped. This is because there exist families \( f : X \to Y \) of maximal variation that are not varying maximally on any of the components of \( X \). Note the following about such a family:

1. \( K_{X/Y} \) cannot be big on any component \( X_i \) of \( X \). Indeed, since the variation of \( f|_{X_i} \) is not maximal, after passing to a generically finite cover of \( X_i \), \( K_{X/Y}|_{X_i} \) is a pull back from a lower dimensional variety.
2. On the other hand, \( X^{(r)} \to Y \) will have a component of maximal variation for \( r \gg 0 \). In particular, \( K_{X^{(r)}/Y} \) does have a chance to be big on at least one component.

To construct a family as above, start with two non-isotrivial smooth families \( g_i : Z_i \to C_i \) of curves of different genera, both at least two [BPVdV84, Sec V.14]. Take a multisection on each of these. By taking a base-change via the multisections, we may assume that in fact each \( g_i \) is endowed with a section \( s_i : C_i \to Z_i \). Now define \( f_1 := g_1 \times \text{id}_{C_2} : X_1 := Z_1 \times C_2 \to Y := C_1 \times C_2 \) and \( f_2 := \text{id}_{C_1} \times g_2 : X_2 := C_1 \times Z_2 \to Y \). The section \( s_i \) of \( g_i \) induce sections of \( f_i \) as well. Let \( D_i \) be the images of these. Then, according to [Kol13b, Thm 5.13], \( (X_1, D_1) \) and \( (X_2, D_2) \) glues along \( D_1 \) and \( D_2 \) to form a stable family \( f : X \to Y \) as desired. Also notice that in this example \( f^{(2)} : X^{(2)} \to Y \) has a component of maximal variation.

**Corollary 6.3.** Any algebraic space that is the coarse moduli space of a functor of stable log-varieties with fixed volume, dimension and coefficient set (as in Definition 5.2) is a projective variety over \( k \).

The rest of the section contains the proofs of Theorem 6.1 and Corollary 6.3. The first major step is Proposition 6.8, which needs a significant amount of notation to be introduced.

**Definition 6.4.** For a \( \mathbb{Q} \)-Weil divisor \( D \) on a demi-normal variety and for a \( c \in \mathbb{Q} \) we define the \( c \)-coefficient part of \( D \) to be the reduced effective divisor

\[
D_c := \sum_{\text{coeff } E = c} E,
\]

where the sum runs over all prime divisors. Clearly

\[
D = \sum_{c \in \mathbb{Q}} cD_c.
\]

Notice that \( D_c \) is invariant under any automorphism of the pair \( (X, D) \), that is, under any automorphism of \( X \) that leaves \( D \) invariant. In fact, an automorphism of \( X \) is an automorphism of the pair \( (X, D) \) if and only if it leaves \( D_c \) invariant for every \( c \in \mathbb{Q} \).

**Definition 6.5.** Let \( f : (X, D) \to Y \) be a family of stable log-varieties. We will say that the coefficients of \( D \) are compatible with base-change if for each \( c \in \mathbb{Q} \) and \( y \in Y \),

\[
D_{c}|_{X_y} = (D_y)_{c}.
\]

Note that this condition is automatically satisfied if all the coefficients are greater than \( \frac{1}{2} \).

**Notation 6.6.** Let \( f : (X, D) \to Y \) be a family of stable log-varieties over a smooth projective variety. For a fixed \( m \in \mathbb{Z} \) that is divisible by the Cartier index of \( K_{X/Y} + D \), and an arbitrary \( d \in \mathbb{Z} \) set \( \mathcal{L}_d := \mathcal{O}_X(dm(K_{X/Y} + D)) \).

Observe that there exists a dense big open subset \( U \subset Y \) over which all the possible unions of the components of \( D \) (with the reduced structure) are flat. Our goal is to apply Theorem 4.1 for \( f_U : X_U \to U \) (we allow shrinking \( U \) after fixing \( d \) and \( m \), while keeping \( U \) a big open set).

Next we will group the components of \( D \) according to their coefficients. Recall the definition of \( D_c \) from Definition 6.4 where \( c \in \mathbb{Q} \) and observe that there is an open set \( V \subset U \) over which
(A) $D_c$ is compatible with base-change as in Definition 6.5 for all $c \in \mathbb{Q}$, and

(B) the scheme theoretic fiber of $D_c$ over $v \in V$ is reduced and therefore is equal to its divisorial restriction (see the definition of the latter in Notation 2.7).

To simplify notation we will make the following definitions: Let $\{c_1, \ldots, c_n\} := \{c \in \mathbb{Q} \mid D_c \neq \emptyset\}$ be the set of coefficients appearing in $D$ and let $D_i := D_{c_i}$, for $i = 1, \ldots, n$.

Next we choose an $m \in \mathbb{Z}$ satisfying the following conditions for every integer $i, j, d > 0$:

(C) $m(K_{X/Y} + D)$ is Cartier,

(D) $L_d = \mathcal{O}_X(dm(K_{X/Y} + D))$ is $f$-very ample,

(E) $R^1f_*L_d = 0$,

(F) $(R^1(f|_{D_i})_*\mathcal{L}_d|_{D_i})|_V = 0$, and

(G) $(f_*\mathcal{L}_1)|_V \rightarrow ((f|_{D_i})_*\mathcal{L}_1|_{D_i})|_V$ is surjective.

These conditions imply that

(H) $\mathbb{N} \ni N := h^0(\mathcal{L}_1|_{x_y}) - 1$ is independent of $y \in Y$, and in fact

(I) $f_*\mathcal{L}_d$ and $((f|_{D_i})_*\mathcal{L}_d|_{D_i})|_V$ are locally free and compatible with base-change.

By possibly increasing $m$ we may also assume that

(J) the multiplication maps

$$\text{Sym}^d(f_*\mathcal{L}_1) \rightarrow (f_*\mathcal{L}_d) \quad \text{and} \quad \text{Sym}^d(f_*\mathcal{L}_1)|_V \rightarrow ((f|_{D_i})_*\mathcal{L}_d|_{D_i})|_V$$

are surjective.

For the surjectivity of the map $\text{Sym}^d(f_*\mathcal{L}_1)|_V \rightarrow ((f|_{D_i})_*\mathcal{L}_d|_{D_i})|_V$ we write it as the composition of the restriction map $\text{Sym}^d(f_*\mathcal{L}_1)|_V \rightarrow \text{Sym}^d((f|_{D_i})_*\mathcal{L}_1|_{D_i})|_V$ and the multiplication map $\text{Sym}^d((f|_{D_i})_*\mathcal{L}_1|_{D_i})|_V \rightarrow ((f|_{D_i})_*\mathcal{L}_d|_{D_i})|_V$. The former is surjective by the choice of $m$ and condition (G) while the surjectivity of the latter follows by the finite generation of the relative section ring, after an adequate increase of $m$.

We fix an $m$ satisfying the above requirements for the rest of the section and use the global sections of $\mathcal{L}_1|_{x_y}$ to embed $X_y$ (and hence $D_i|_{x_y}$ as well) into the fixed projective space $\mathbb{P}_k^N$ for every closed point $y \in V$. The ideal sheaves corresponding to these embeddings will be denoted by $\mathcal{I}_{x_y}$ and $\mathcal{I}_{D_i|x_y}$ respectively. As the embedding of $X_y$ is well-defined only up to the action of $\text{GL}(N+1, k)$, the corresponding ideal sheaf is also well-defined only up to this action. Furthermore, in what follows we deal with only such properties of $X_y$, $D_i|_{x_y}$, $\mathcal{I}_{x_y}$ and $\mathcal{I}_{D_i|x_y}$ that are invariant under the $\text{GL}(N+1, k)$ action.

So, finally, we choose a $d > 0$ such that

(K) for all $y \in V$, $X_y$ as well as $D_i|_{x_y}$ are defined by degree $d$ equations.

From now on we keep $d$ fixed with the above chosen value and we suppress it from the notation. We make the following definitions:

(L) $W := \text{Sym}^d(f_*\mathcal{L}_1)|_U$, and

(M) $Q_0 := (f_*\mathcal{L}_d)|_U$.

Further note that $(f|_{D_i})_*\mathcal{L}_d|_{D_i}$ is torsion-free, since $f|_{D_i}$ is surjective on all components and $D_i$ is reduced. Hence by possibly shrinking $U$, but keeping it still a big open set, we may assume that

(N) $Q_i := ((f|_{D_i})_*\mathcal{L}_d|_{D_i})|_U$ is locally free for all for $i > 0$.

Our setup ensures that we have natural homomorphisms $\alpha_i : W \rightarrow Q_i$ which are surjective over $V$ and we may make the following identifications for all closed points $y \in V$ up to the above explained...
GL(N + 1, k) action:

\[
\begin{align*}
    Q_0 \otimes k(y) & \xrightarrow{} H^0 \left( \mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \right) \\
    \ker \left[ W \otimes k(y) \rightarrow Q_0 \otimes k(y) \right] & \xrightarrow{} H^0 \left( \mathbb{P}^N, \mathcal{I}_{\mathbb{P}^y}(d) \right) \\
    \ker \left[ W \otimes k(y) \rightarrow Q_i \otimes k(y) \right] & \xrightarrow{} H^0 \left( \mathbb{P}^N, \mathcal{I}_{\mathbb{D}_{i\mid\mathbb{P}}} \right), \text{ for } i > 0.
\end{align*}
\]

We will use this setup and notation for the rest of the present section.

**Lemma 6.7.** Let \( f : (X, D) \rightarrow Y \) be a family of stable log-varieties over a normal proper variety \( Y \), and let \( m \geq 0 \) be an integer such that

1. \( m(K_{X/Y} + D) \) is Cartier,
2. \( m(K_{X/Y} + D) \) is relatively basepoint-free with respect to \( f \), and
3. \( R^i f_* \mathcal{O}_X(m(K_{X/Y} + D)) = 0 \) for all \( i > 0 \).

Then \( f_* \mathcal{O}_X(m(K_{X/Y} + D)) \) is a nef locally free sheaf. Further note, that the above conditions and hence the statement hold for every divisible enough \( m \). In particular, it applies for the \( m \) chosen in Notation 6.6, and hence \( f_* \mathcal{L}_d \) is weakly positive for all \( d > 0 \).

**Proof.** The assumptions guarantee that \( f_* \mathcal{O}_X(m(K_{X/Y} + D)) \) is compatible with base-change. As being nef is decided on curves, we may assume that \( Y \) is a smooth curve. Note that then by the slc version of inversion of adjunction (e.g., [Pat12a, Cor 2.11]) \( (X, D) \) itself is slc. Hence, [Fuj12, Theorem 1.13] applies and yields the statement.

**Proposition 6.8.** In the situation of Notation 6.6, assume that the variation is maximal. Then for all \( d \gg 0 \),

\[
\det f_* \mathcal{L}_d \otimes \left( \otimes_{i=1}^n \det \left( (f|_{D_i})_* \mathcal{L}_d|_{D_i} \right) \right)
\]

is big.

**Proof.** Note that \( f_* \mathcal{L}_1 \) is weakly positive by Lemma 6.7 and hence so is \( W = \text{Sym}^d f_* \mathcal{L}_1 \). This will allow us to use Theorem 4.1 in the situation of Notation 6.6 by setting \( G := \text{GL}(N + 1, k) \) (see Remark 4.3) with the natural action on \( W \) if we prove that the restriction over \( V \) of the classifying map of the morphisms \( \alpha_i \) for \( i = 0, \ldots, n \) have finite fibers.

Translating this required finiteness to geometric terms means that fixing a general \( y \in V(k) \) and the fiber \( X_y \), there are only finitely many other general \( z \in V(k) \), such that for the fiber \( X_z \) the degree \( d \) forms in the ideals of \( X_y \) and \( D_i|_{X_y} \) can be taken by a single \( \phi \in \text{GL}(N + 1, k) \) to the degree \( d \) forms in the ideals of \( X_z \) and \( D_i|_{X_z} \). However, if such a \( \phi \) exists, then \( (X_y, D_y) \simeq (X_z, D_z) \) meaning that \( y \) and \( z \) lie in the same fiber of the associated moduli map \( \mu : Y \rightarrow \mathcal{M}_{m, c, 1} \) (see Section 5.1). The maximal variation assumption implies that \( \mu \) is generically finite, so there is an open \( Y^0 \subseteq Y \), over which \( \mu \) has finite fibers, which is exactly what we need. By shrinking \( V \), we may assume that \( V \subseteq Y^0 \) and applying Theorem 4.1 yields the statement.

**Lemma 6.9.** Let \( f : (X, D) \rightarrow Y \) be a family of stable log-varieties over a smooth variety. Then \( D_c|_T \) is flat for all \( c \in \mathbb{Q} \), where \( T \) is the locus over which \( D_c \) is Cartier. Note that \( T|_{X_y} \) is a big open set for every \( y \in Y \).

**Proof.** As \( \mathcal{O}_{D_c|_T} \) is the cokernel of \( \epsilon : \mathcal{O}_T(-D_c) \rightarrow \mathcal{O}_T \), it is enough to prove that \( \epsilon_y : \mathcal{O}_T(-D_c) \otimes \mathcal{O}_{T_y} \rightarrow \mathcal{O}_{T_y} \) is injective for every \( y \in Y \) [StacksProject, Tag 00MD]. However, as \( \mathcal{O}_T(-D_c) \otimes \mathcal{O}_{T_y} \) is a line bundle on \( T_y \), and hence \( S_2 \), and the map \( \epsilon_y \) is an isomorphism, in particular injective, at every generic point of \( T_y \), it is in fact injective everywhere.

**Lemma 6.10.** Let \( f : (X, D) \rightarrow Y \) be a family of stable log-varieties over a smooth variety. Then \( D_c \rightarrow Y \) is an equidimensional morphism for all \( c \in \mathbb{Q} \).
Proof. By assumption $D_c$ has codimension 1 in $X$ and it does not contain any irreducible components of any fiber. It follows that the general fiber of $D_c$ over $Y$ has codimension 1 in the corresponding fiber of $X$ and that this is the maximum dimension any of its fibers may achieve. Since the dimension of the fibers is semi-continuous this implies that all fibers of $D_c$ have the same dimension.

\begin{lemma}
Let $f : (X, D) \to Y$ be a family of stable log-varieties over a smooth variety. Let $Z$ be the fiber product over $Y$ of some copies of $X$ and of the $D_i = D_{c_i}$’s. Then
1. every irreducible component of $Z$ dominates $Y$,
2. there is a big open set of $Y$ over which $Z$ is flat and reduced,
3. $Z$ is equidimensional over $X$, and
4. $X$ is regular at every generic point of $Z$.
\end{lemma}

\begin{proof}
First notice that (3) follows directly from Lemma 6.10.

Next recall that we have already noted in Notation 6.6 that there exists a big open set $U \subseteq Y$, over which $X$ and all the possible unions of the components of $D$ are flat, and hence so is $Z$. It follows that all the embedded points of $Z$ over $U$ map to the generic point $\eta$ of $Y$. However $Z_\eta$ is reduced, so $Z$ is not only flat, but also reduced over $U$. This proves (2).

On the other hand, $Z$ can definitely have multiple irreducible or even connected components. Assume that there exists an irreducible component $S$ that does not dominate $Y$. Then $S$ is contained in the non-flat locus of $Z$. However, according to Lemma 6.9, the non-flat locus of $D_i$ has codimension at least one in each fiber of $D_i \to Y$ for all $i$’s. Therefore, the non-flat locus of $Z$ also has codimension at least one in each fiber. Hence, the existence of $S$ would contradict (3) (and ultimately Lemma 6.10). This proves (1).

By (1) the generic points of $Z$ are dominating the generic points of $D_i$. At these points the corresponding fibers of $X$ are regular and so (4) follows.
\end{proof}

Notation 6.12 is used in the proof of Theorem 6.1.1, which is presented right after it.

\begin{notation}
Assume that we are in the situation of Notation 6.6, in particular, recall the definition $D_i = D_{c_i}$. To simplify the notation we also set $D_0 := X$. For a fixed positive natural number $r \in \mathbb{N}_+$ consider a partition of $r$: i.e., a set of natural numbers $r_i \in \mathbb{N}$ for $i = 0, \ldots, n$ such that $\sum_{i=0}^{n} r_i = r$. We will denote a partition by $[r_0, r_1, \ldots, r_n]$. For $[r_0, r_1, \ldots, r_n]$ we define the following mixed product (we omit $Y$ from the notation for sanity):

$$D_{(r_0, r_1, \ldots, r_n)} := \left( \prod_{i=0}^{n} D_{r_i} \right)_{\text{red}} = \left( D_{0}^{(r_0)} \times \cdots \times Y D_{n}^{(r_n)} \right)_{\text{red}}.$$

Observe that $D_{(r_0, r_1, \ldots, r_n)}$ is naturally a closed subscheme of $X_Y^{(r)}$.

Let us assume now that $r_j > 0$ for some $j$. Then $[r_0 + 1, r_1, \ldots, r_j - 1, \ldots, r_n]$ is another partition of the same $r$ and

$$D_{(r_0, r_1, \ldots, r_n)} \subseteq D_{(r_0 + 1, r_1, \ldots, r_j - 1, \ldots, r_n)}$$

is a reduced effective Weil divisor no component of which is contained in the singular locus of $D_{(r_0 + 1, r_1, \ldots, r_j - 1, \ldots, r_n)}$ according to Lemma 6.11. In particular, for a sequence of partitions,

$[r_0, r_1, 2, \ldots, r_n], [r_0 + 1, r_1 - 1, r_2, \ldots, r_n], \ldots, [r_0 + r_1, 0, r_2, \ldots, r_n], [r_0 + r_1 + 1, 0, r_2 - 1, \ldots, r_n], \ldots, [r_0 + r_1 + r_2, 0, 0, \ldots, r_n], \ldots, [r_0 + \cdots + r_{n-1}, 0, 0, \ldots, r_n], [r_0 + \cdots + r_{n-1} + 1, 0, 0, \ldots, 0, r_n - 1], \ldots, [r, 0, 0, \ldots, 0]$,

we obtain a filtration of $X_Y^{(r)}$ where each consecutive embedding is a reduced effective Weil divisor in the subsequent member of the filtration and furthermore no component of the former is contained
in the singular locus of the latter:

\[ D^\prime_{\tau_0} \subset D^\prime_{\tau_0+1} \subset \cdots \subset D^\prime_{\tau_0+r_1-1} \subset D^\prime_{\tau_0+r_1} \subset \cdots \subset D^\prime_{\tau_0+r_1+r_2-1} \subset \cdots \subset D^\prime_{\tau_0+r_1+r_2} \subset \cdots \subset D^\prime_{\tau_0} = X^\prime. \]

In fact, using Lemma 6.11, one can see that for every (not necessarily subsequent) pair \( D' \subseteq D'' \) of schemes appearing in the above filtration, \( D'' \) is regular at the generic points of \( D' \). Indeed, according to Lemma 6.11 every generic point \( \xi \) of \( D' \) is over the generic point \( \eta \) of \( Y \). Hence it is enough to see that \( D''_\eta \) is regular at \( \xi \). Observe, that \( D''_\eta \) is a product over \( \text{Spec} k(\eta) \), and not over a positive dimensional scheme as \( D'' \) is. Hence it is enough to see that all the components of \( D''_\eta \) are regular at the appropriate projection of \( \xi \). However, this follows immediately from our definition of stable families (Definition 2.11), that is, by the assumption that \( D_i \) avoid the codimension one singular points of the fibers and hence in particular of \( X_\eta \).

**Proof of Theorem 6.1.1.** We will use the setup established in Notation 6.6 and 6.12. As before, \( f_*L_d \) is a nef vector bundle by Lemma 6.7. Therefore, by the surjective natural map \( f^*f_*L_d \to L_d \), \( K X/Y + D \) is nef as well. Clearly the same holds for \( K_{X/Y} + \sum_j D_j \) for any integer \( j > 0 \).

Now, let \( r_0 := \text{rk} f_*L_d \) and for \( i > 0 \) let \( r_i := \text{rk} (f|_{D_i})_*L_d|_{D_i} \). Furthermore, set \( r := \sum r_i \), \( Z := D^{(r_0,r_1,\ldots,r_n)} \) and \( \eta : Z \to Z \) the normalization of \( Z \). Note that \( Z \) can be reducible and a priori even non-reduced, but it is a closed subscheme of \( X^\prime \), its irreducible components dominate \( Y \) and non-reducedness on \( Z \) may happen only in large codimension by Lemma 6.11.

Consider the natural injection below, which can be defined first over the big open set \( U \subseteq Y \) of Notation 6.6, and then extended reflexively to \( Y \),

\[
\iota_d : \mathcal{A}_d := \det (f_*L_d) \otimes \left( \bigotimes_{i=1}^n \det ((f|_{D_i})_*L_d|_{D_i}) \right) \to f_*L_d \otimes \left( \bigotimes_{i=1}^n \left( \bigotimes_{j=1}^{r_i} (f|_{D_i})_*L_d|_{D_i} \right) \right) \simeq \left( \left( \left( f^\prime \right)_*L_d \right)_Z \right)_Z \text{ iterated use of Lemma 2.6}.
\]

By a slight abuse of notation we will denote the composition of restriction from \( X^\prime \) to \( Z \) and the pull-back via the normalization morphism \( \eta : Z \to Z \) by restriction to \( Z \). In other words we make the following definition:

\[
(f^\prime)_Z := \eta^* \circ (f)_Z.
\]

So, for instance, \( (f^\prime)_Z \) denotes the pulling back by the composition \( \xymatrix{ Z \ar[r]^-{\eta} & Z \ar[r]^-{f} & X^\prime \ar[r]^-{f} & Y } \).

As in its definition above, if we restrict \( \iota_d \) to \( U \), then the reflexive hulls are unnecessary on the right hand side of (6.12.1). Then by adjointness we obtain a non-zero homomorphism

\[
\left( (f^\prime)_Z \right)_U^* \mathcal{A}_d \to L_d \left( (f^\prime)_Z \right)_U^* \mathcal{A}_d.
\]

Pulling this further back over \( Z \) yields a non-zero homomorphism

\[
(6.12.2) \quad \left( (f^\prime)_Z \right)_Z^* \mathcal{A}_d \to L_d \left( (f^\prime)_Z \right)_Z^* \mathcal{A}_d.
\]
Since $Z \to Y$ and hence also $\tilde{Z} \to Y$ is an equidimensional morphism, $(f^{(r)}|_{\tilde{Z}})^{-1} U$ is also a big open set in $\tilde{Z}$ and hence (6.12.2) induces a non-zero homomorphism

$$\left( f^{(r)} \right)^* \mathcal{A}_d \to \mathcal{L}_d^{(r)}\big|_{\tilde{Z}}. \tag{6.12.3}$$

The non-zero map (6.12.3) induces another non-zero map

$$\mathcal{L}_d^{(r)}\big|_{\tilde{Z}} \otimes \left( f^{(r)} \right)^* \mathcal{A}_d \to \left( \mathcal{L}_d^{(r)} \right)^{\otimes 2}\big|_{\tilde{Z}},$$

where on the left hand side we have a relatively ample and nef line bundle tensored with the pullback of a big line bundle. Hence the line bundle on the left hand side is big on every component of $\tilde{Z}$. Therefore the line bundle on the right hand side is big on at least one component. Let $L^{(r)}$ denote a Cartier divisor corresponding to $\mathcal{L}_d^{(r)}$. Then by the nefness of $L^{(r)}$ it follows that

$$0 < L^{(r)}\big|_{\tilde{Z}}^{\dim \tilde{Z}},$$

and then also

$$0 < L^{(r)}\big|_{Z}^{\dim Z}. \tag{6.12.4}$$

Next we will define a filtration starting with $X^{(r)}$ and ending with $Z$ where each consecutive member is a reduced divisor in the previous member. Recall that $r = \sum_{i=0}^{n} r_i$ and observe that for any integer $r_0 \leq t < r$ there is a unique $0 \leq j < n$ such that

$$\sum_{i=0}^{j} r_i \leq t < \sum_{i=0}^{j+1} r_i,$$

and hence

$$0 \leq t_{j+1} := t - \sum_{i=0}^{j} r_i < r_{j+1}.$$ 

Now recall Notation 6.12 and let us define $Z_r := X^{(r)}$ and for any $t, r_0 \leq t < r$,

$$Z_t := D^{\sum_{i=0}^{j} r_i + t_{j+1}, 0, \ldots, 0, r_{j+1} - t_{j+1}, r_{j+2}, \ldots, r_n},$$

where each consecutive $r_i$ is the intersection of $Z_{t+1}$ with $p_t^* D_{j+1}$. We claim that this is in fact true also divisorially. Indeed, $Z_t$ is reduced and by Lemma 6.11 it is equidimensional. So, it is enough to check that $Z_t$ and the divisorial restriction $p_t^* D_{j+1}$ agrees at all codimension one points $\xi$ of $Z_{t+1}$. If $p_t^* D_{j+1}$ contains $\xi$ in its support, then $D_{j+1}$ contains $p_t(\xi)$, hence $p_t(\xi)$ has to be a codimension 1 regular point of $X$ lying over the generic point $\eta$ of $Y$. Note that $\text{mult}_\xi p_t^* D_{j+1} = \text{mult}_{p_t(\xi)} D_{j+1} = 1$, and that $Z_{j+1}$ contains exactly the same codimension one points of $Z_{t+1}$, which concludes our claim that

$$Z_t = p_t^* D_{j+1}|_{Z_{t+1}}. \tag{6.12.5}$$

Our goal is to show that

$$0 < \left( L^{(r)} \right)^{\dim X^{(r)}} \left( = \left( L^{(r)} \right)^{\dim Z_r} \right).$$
For any rational number $1 \gg \varepsilon > 0$ we have
\[
(L^{(r)})^{\dim Z_r} = (L^{(r)})^{\dim Z_r} + \sum_{t=r_0}^{r-1} \varepsilon^{r-j} \left( L^{(r)} \right|_{Z_t}^{\dim Z_t} - L^{(r)} \right|_{Z_t}^{\dim Z_t}) = \\
= L^{(r)} \left|_Z \right. + \sum_{t=r_0}^{r-1} \varepsilon^{r-j-1} \left( L^{(r)} \right|_{Z_{t+1}}^{\dim Z_{t+1}} - \varepsilon L^{(r)} \right|_{Z_t}^{\dim Z_t}).
\]

Thus, according to (6.12.4), it is enough to prove that for each integer $r_0 \leq t < r$,

\[0 \leq L^{(r)} \left|_{Z_{t+1}}^{\dim Z_{t+1}} - \varepsilon L^{(r)} \right|_{Z_t}^{\dim Z_t}.\]

(6.12.6)

In the rest of the proof we fix an integer $r_0 \leq t < r$, and prove (6.12.6) for that value of $t$. Let $\tilde{Z}_{t+1}$ be the normalization of $Z_{t+1}$, and let $S$ be the strict transform of $Z_t$ in $\tilde{Z}_{t+1}$. Denote by $\rho$ the composition $\tilde{Z}_{t+1} \to Z_{t+1} \to X^{(r)}$. According to the discussion in Notation 6.12, $\tilde{Z}_{t+1} \to Z_{t+1}$ is an isomorphism at the generic point of $Z_t$. Hence it is enough to prove that

\[0 \leq (\rho^* L^{(r)}) \left|_{\tilde{Z}_{t+1}}^{\dim \tilde{Z}_{t+1}} - \varepsilon (\rho^* L^{(r)}) \right|_{S}^{\dim S} = (\rho^* L^{(r)}) \left|_{\tilde{Z}_{t+1}}^{\dim \tilde{Z}_{t+1} - 1} \right. \cdot (\rho^* L^{(r)} - \varepsilon S).\]

Note that the right most expression is the intersection of several Cartier divisors with a Weil $\mathbb{Q}$-divisor, and hence it is well-defined. Furthermore, since $\rho^* L^{(r)}$ is nef, to prove the above inequality it is enough to prove that the $\mathbb{Q}$-divisor $(\rho^* L^{(r)} - \varepsilon S)$ is pseudo-effective on every component of $\tilde{Z}_{t+1}$. This follows if we apply Lemma 6.13 by setting $Z := Z_{t+1}$, $\tilde{Z} := \tilde{Z}_{t+1}$, $E := p_t^* D_{j+1}$ and by using (6.12.5) (and its implication that $S = p_t^* D_{j+1}|_{\tilde{Z}_{t+1}}$).

Recall that a $\mathbb{Q}$-Weil divisor $D$ is called $\mathbb{Q}$-effective if $mD$ is linearly equivalent to an effective divisor for some integer $m > 0$.

**Lemma 6.13.**

1. Let $f : (X,D) \to Y$ be an equidimensional, surjective, projective morphism from a semi-log canonical pair onto a smooth projective variety, such that $K_{X/Y} + D$ is $f$-ample and all irreducible components of $X$ dominate $Y$.
2. Let $Z$ be a closed subscheme of $X$, which is equidimensional over $Y$, reduced, and all its irreducible components dominate $Y$.
3. Let $E$ be a reduced effective divisor on $X$ with support in $\text{Supp} \, D$, in particular, no component of $E$ is contained in the singular locus of $X$. Assume that $E$ does not contain any component of $Z$ and that both $Z$ and $X$ are regular at the generic points of $Z$ and at the codimension one points of $Z$ that are contained in $E$.
4. Let $\tilde{Z} \to Z$ be the normalization.

Then $(K_{X/Y} + D - \varepsilon E)|_{\tilde{Z}}$ is pseudo-effective for every $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon \ll 1$, meaning that for any fixed ample divisor $A$ on $Z$, $(K_{X/Y} + D - \varepsilon E)|_{\tilde{Z}} + \delta A$ is $\mathbb{Q}$-effective on every component of $\tilde{Z}$ for every $\delta \in \mathbb{Q}$, $0 < \delta \ll 1$.

**Remark 6.14.** In the above statement $E|_{\tilde{Z}}$ is defined by considering the (big) open locus in $Z$, where $E$ is Cartier, pulling back to $\tilde{Z}$ and taking the closure there using that the complement has codimension at least 2.

**Proof.** Reduction step: Let $\pi : (\overline{X}, \overline{D}) \to (X,D)$ be the normalization and $\overline{Z}$ and $\overline{E}$ the strict transforms (by the regularity assumptions $\pi$ is an isomorphism at all generic points of $\overline{Z}$ and $\overline{E}$ so these strict transforms are meaningful). Since $\tilde{Z} \to Z$ factors through $\overline{Z} \to Z$, this setup shows that we may assume that $(X,D)$ is log canonical.

Summary of assumptions after the reduction step:
(1) $f : (X, D) \to Y$ is an equidimensional, surjective, projective morphism from a log canonical pair onto a smooth projective variety, such that $K_{X/Y} + D$ is $f$-ample,
(2) $Z$ is equidimensional over $Y$, reduced, and all its irreducible components dominate $Y$,
(3) $\text{Supp } E \subseteq \text{Supp } D$,
(4) no irreducible component of $Z$ is contained in the support of $E$, and
(5) regularity assumptions: $X$ is regular at the generic points of $Z$ and both $E$ and $Z$ are regular at the codimension one points of $Z$ that are contained in $E$.

The argument. Set $L := K_{X/Y} + D$, $\mathcal{L} := \mathcal{O}_X(L)$ and $S := E|_{\tilde{Z}}$ and let $\rho$ be the composition $\tilde{Z} \to Z \to X$. Note that to establish that $\rho^* L - \varepsilon S$ is pseudo-effective one may use an arbitrary Cartier divisor $A$ on $\tilde{Z}$, and show that $\rho^* L - \varepsilon S + \delta A$ is $\mathbb{Q}$-effective on every component for every $0 < \delta \ll 1$. Indeed, choosing an ample $A'$, it follows that $t A' - A$ is effective on every component for some $t > 0$, and hence then

$$\rho^* L - \varepsilon S + \delta t A' = \rho^* L - \varepsilon S + \delta A + \delta (t A' - A)$$

is also $\mathbb{Q}$-effective on every component as well. Here we will choose $A$ to be the pullback of an appropriate ample line bundle on $Y$.

Let us take a $\mathbb{Q}$-factorial dlt model $\tau : (T, \Theta) \to (X, D)$ such that $K_T + \Theta = \tau^*(K_X + D)$ (cf. [KK10, 3.1]) and define $g := f \circ \tau$. Note that $\tau$ is an isomorphism both at the generic points of $Z$ and at the codimension one points of $Z$ that are contained in $E$, since $X$ is regular at all these points. Set $\Gamma := \tau^{-1}_* E$. Consider

$$q \tau^* L - \Gamma = q \left( K_{T/Y} + \Theta - \frac{1}{q} \Gamma \right).$$

for a divisible enough integer $q > 0$. There are two important facts about the above divisor. On one hand,

$$(6.13.1) \quad \tau_* \mathcal{O}_T(q \tau^* L - \Gamma) \subseteq \mathcal{O}_X(q L - E),$$

on the other hand, the above divisor is the $q$th multiple of the relative log-canonical divisor of a dlt pair. Hence according to [Fuj14a, Thm 1.1], for every divisible enough $q$,

$$g_* \mathcal{O}_T(q \tau^* L - \Gamma)$$

is weakly positive. Therefore after fixing an ample line bundle $H$ on $Y$, for each $a > 0$, there is a $b > 0$, such that

$$\text{Sym}^{ab}(g_* \mathcal{O}_T(q \tau^* L - \Gamma)) \otimes H^b$$

is generically globally generated.

Let $U$ be the open set where both $g_* \mathcal{O}_T(q \tau^* L - \Gamma)$ and $f_* \mathcal{O}_X(q L - E)$ are locally free. Over $U$ consider the composition of the following homomorphisms, where the left most one is the push-forward of the embedding in (6.13.1):

$$(6.13.2) \quad f^* \text{Sym}^{ab}(g_* \mathcal{O}_T(q \tau^* L - \Gamma)) \to f^* \text{Sym}^{ab}(f_* \mathcal{O}_X(q L - E)) \to f^* f_* \mathcal{O}_X(ab(q L - E)) \to \mathcal{O}_X(ab(q L - E)).$$

Let us pause for a moment and recall that $q L - E$ is not necessarily Cartier in general. However, it is Cartier over a big open set of $f^* U$, so the natural map $\text{Sym}^{ab}(f_* \mathcal{O}_X(q L - E)) \to f_* \mathcal{O}_X(ab(q L - E))$, which yields the middle arrow above, can still be constructed over that big open set and then extended uniquely, since $X$ is normal.

Setting $h := f \circ \rho$, still over $U$, we obtain the following natural morphism by pulling back the composition of (6.13.2) via $\rho$.

$$h^* \text{Sym}^{ab}(g_* \mathcal{O}_T(q \tau^* L - \Gamma)) \to \mathcal{O}_{\tilde{Z}}(ab(q \rho^* L - S)).$$
Again, note that $qL - E$ is not necessarily Cartier over $Z$. However, by our regularity assumption it is Cartier over a big open set $U_Z$ of $Z$. So the above map is constructed first over $\rho^{-1}(U_Z \cap f^{-1}U)$ and then extended uniquely using that $\tilde{Z}$ is normal.

So, since $\tilde{Z} \to Y$ is equidimensional, $h^{-1}U$ is a big open set of $\tilde{Z}$. In particular, we obtain a homomorphism

\[(6.13.3) \quad h^[a]\text{Sym}^{[ab]}(g_*\mathcal{O}_T(q\tau^*L - \Gamma)) \otimes h^*H^b \to \mathcal{O}_{\tilde{Z}}(ab(q\rho^*L - S)) \otimes h^*H^b.\]

Now choose $q$ divisible enough so that $\tau_*\mathcal{O}_T(q\tau^*L - \Gamma) \simeq \mathcal{O}_X(qL) \otimes \tau_*\mathcal{O}_T(-\Gamma)$ is $f$-globally generated (recall that $L$ is $f$-ample). Note that the ideal $\tau_*\mathcal{O}_T(-\Gamma)$ is supported on $\text{Supp } E$ and $\text{Supp } E$ does not contain any component of $Z$ by assumption. Hence, it follows that the natural map

\[h^*g_*\mathcal{O}_T(q\tau^*L - \Gamma)) \to \mathcal{O}_{\tilde{Z}}(q\rho^*L - S)\]

is surjective at all generic points of $\tilde{Z}$ and then the same holds for the map in $(6.13.3)$. Furthermore, the sheaf on the left hand side in $(6.13.3)$ is globally generated at every generic point of $\tilde{Z}$. This gives us the desired sections of $\mathcal{O}_{\tilde{Z}}(ab(q\rho^*L - S)) \otimes h^*H^b$ and concludes the proof. \hfill \Box

We will need the following analog of Lemma 3.6 for reducible schemes.

**Lemma 6.15.** If $X$ is a projective scheme of pure dimension $n$ over $k$ and $L$ a nef Cartier divisor which is big on at least one component (that is, $L^n > 0$), then for every Cartier divisor $D$ that does not contain any component of $X$, $L - \varepsilon D$ is $Q$-effective for every rational number $0 < \varepsilon \ll 1$ (however the corresponding effective divisor may be zero on every irreducible component but one).

**Proof.** Let $\mathcal{L} := \mathcal{O}_X(L)$. Consider the exact sequence,

\[
0 \longrightarrow \mathcal{L}^a(-D) \longrightarrow \mathcal{L}^a \longrightarrow \mathcal{L}^a|_D \longrightarrow 0
\]

Since $L$ is nef, by the asymptotic Riemann-Roch Theorem [Laz04a, Corollary 1.4.41], $h^0(L^a) = a^n L^n + O(a^{n-1})$. Furthermore, $h^0(\mathcal{L}^a|_D) = O(a^{n-1})$. Hence, for every $a \gg 0$, $H^0(\mathcal{L}^a|_D) \neq 0$. \hfill \Box

**Theorem 6.1.2** is an immediate consequence of the following statement.

**Proposition 6.16.** If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal proper variety, then there exists an integer $q > 0$ and a proper closed subvariety $S \subseteq Y$, such that for every integer $a > 0$, and closed irreducible subvariety $T \subseteq Y$ not contained in $S$, $\text{det } f_*\mathcal{O}_X(aq(K_{X/Y} + \Delta))|_T$ is big, where $\bar{T}$ is the normalization of $T$.

**Proof.** First, note that since $q$ can be chosen to be divisible enough, $f_*\mathcal{O}_X(aq(K_{X/Y} + \Delta))$ commutes with base-change, and hence we may replace $Y$ by any of its resolution. That is, we may assume that $Y$ is smooth and projective. We may also replace $\bar{T}$ by a resolution of $T$ in the statement.

Let $H$ be any ample effective Cartier divisor on $Y$, and let $\mathcal{H} := \mathcal{O}_Y(H)$ be the associated line bundle. Let $r > 0$ be the integer given by Theorem 6.1.1. Since every component of $X^{(r)}$ dominates $Y$, according to Lemma 6.15, $q(K_{X^{(r)}/Y} + D_{X^{(r)}}) - (f^{(r)})^*H$ is linearly equivalent to an effective divisor for some multiple $q$ of $dm$. Equivalently, there is a non-zero map

\[(6.16.1) \quad (f^{(r)})^*\mathcal{H} \to \mathcal{O}_{X^{(r)}}(q(K_{X^{(r)}/Y} + D_{X^{(r)}})) \cdot\]

Let $S \subseteq Y$ be the (proper) closed set over which $(6.16.1)$ is zero. For any integer $a > 0$ consider the following non-zero map induced by the $a$th tensor power of $(6.16.1)$.

\[(6.16.2) \quad \mathcal{H}^a \simeq f_*^{(r)}(f^{(r)})^*\mathcal{H}^a \to f_*^{(r)}\mathcal{O}_{X^{(r)}}(aq(K_{X^{(r)}/Y} + D_{X^{(r)}})) \simeq \bigotimes f_*\mathcal{O}_X(aq(K_{X^{(r)}/Y} + D_{X^{(r)}}))\]

\text{Lemma 2.6}
This is necessarily an embedding, because $Y$ is integral. Let $\sigma : \tilde{T} \to Y$ be the resolution of an irreducible closed subset $T$ of $Y$ that is not contained in $S$. Then, the induced map
\[
\sigma^* \mathcal{H}^a \to \bigotimes^r \sigma^* f_* \mathcal{O}_X (aq (K_{X/Y} + D)) \simeq \bigotimes^r (f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} \left( aq \left( K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}} \right) \right)
\]
is not zero and therefore it is actually an embedding. Let $\mathcal{B}$ denote the saturation of $\sigma^* \mathcal{H}^a$ in $\bigotimes^r (f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} (aq(K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}}))$. Then $\mathcal{B}$ is big since $\mathcal{H}$ is ample and it induces another exact sequence
\[
0 \longrightarrow \mathcal{B} \longrightarrow \bigotimes^r (f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} \left( aq \left( K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}} \right) \right) \longrightarrow \mathcal{G} \longrightarrow 0,
\]
where $\mathcal{G}$ is locally free in codimension one. Since according to Lemma 6.7, $(f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} (aq(K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}}))$ is nef, $\mathcal{G}$ is weakly-positive according to [Vie95, prop 2.9.e] and point (2) of Lemma 3.8. Note that we cannot infer that $\mathcal{G}$ is nef, since $\mathcal{G}$ does not have to be locally free. However, we can infer that $det \mathcal{G}$ is weakly-positive as well by (1) of Lemma 3.8 and then for some $N > 0$,
\[
det \left( \bigotimes^r (f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} \left( aq \left( K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}} \right) \right) \right) \simeq \left( \det (f_{\tilde{T}})_* \mathcal{O}_{\tilde{T}} \left( aq \left( K_{X_{\tilde{T}}/\tilde{T}} + D_{\tilde{T}} \right) \right) \right)^N \simeq \mathcal{B} \otimes \det \mathcal{G}
\]
is big by (4) of Lemma 3.8. This concludes the proof. 

\[\square\]

**Proof of Corollary 6.3.** Let $M$ be the algebraic space in the statement, and $\mathcal{M}$ the (pseudo-)functor that it coarsely represents. First note that by finiteness of the automorphism groups (Proposition 5.5), an appropriate power of the functorial polarization required in Definition 5.2 descends to $M$. Since $M$ is proper by Proposition 5.4, according to the Nakai-Moishezon criterion we only need to show that the highest self-intersection of this polarization on every proper irreducible subspace of $M$ is positive. However, by Corollary 5.19 it is enough to show this, instead of $M$, for a proper, normal scheme $Z$, that supports a family $f : (X_Z, D_Z) \to Z$ with the property that each fiber of $f$ is isomorphic to only finitely many others.

Let us state our goal precisely at this point: we are supposed to exhibit an $r > 0$ such that for any closed irreducible subvariety $V \subseteq Z$,
\[
c_1 \left( \det (f_V)_* \mathcal{O}_V \left( r \left( K_{X_V/V} + D_V \right) \right) \right)^{\dim V} > 0.
\]
In fact we, are proving something slightly stronger. We claim that there exist an integer $q > 0$, such that for every integer $a > 0$ and closed irreducible subvariety $V \subseteq Z$,
\[
c_1 \left( \det (f_V)_* \mathcal{O}_V \left( aq \left( K_{X_V/V} + D_V \right) \right) \right)^{\dim V} > 0.
\]
We prove this statement by induction. For $\dim Z = 0$ it is vacuous, so we may assume that $\dim Z > 0$. By Proposition 6.16 there exist a $q_Z > 0$ and a closed subset $S \subseteq Z$ that does not contain any component of $Z$, such that for every $a > 0$ and every irreducible closed subset $T \subseteq Z$ not contained in $S$, if we set $\mathcal{M}_{aq} := \det f_* \mathcal{O}_X \left( aq \left( K_{X/Z} + D_Z \right) \right)$, then $c_1 (\mathcal{M}_{aqZ}|_T)^{\dim T} > 0$. Let $\tilde{S}$ denote the normalization of $S$. Then by induction, since $\dim S < \dim Z$, there exists a $q_{\tilde{S}} > 0$, such that for every $a > 0$ and every irreducible closed subset $V \subseteq \tilde{S}$, $c_1 (\mathcal{M}_{aq_{\tilde{S}}} | V)^{\dim V} > 0$. Taking $q := \max\{q_Z, q_{\tilde{S}}\}$ concludes the proofs of the claim and of Corollary 6.3. 

\[\square\]

**Remark 6.17.** If one allows labeling of the components as well, which was excluded up to this point from Definition 5.2 for simplicity, then Theorem 6.1 still yields projectivity as in Corollary 6.3 for the unlabeled case. This follows from the fact that each stable log-variety admits at most finitely many labelings. Hence, forgetting the labeling of a labeled family with finite isomorphism equivalence classes yields a non-labeled family with finite isomorphism equivalence classes. In particular, the proof of Corollary 6.3 implies that the polarization by $\det f_* \mathcal{O}_X (dm(K_{X/Y} + D))$ yields an ample line bundle on the base of the labeled family as well.
7. **Pushforwards without determinants**

The main goal of this section is to prove the following result.

**Theorem 7.1.** If \( f : (X, D) \to Y \) is a family of stable log-varieties of maximal variation over a normal, projective variety \( Y \) with klt general fiber, then \( f_*\mathcal{O}_X(q(K_{X/Y} + D)) \) is big for every divisible enough integer \( q > 0 \).

**Remark 7.2.** One might wonder if this could be true without assuming that the general fiber is klt. We will show below that that assumption is in fact necessary.

**Corollary 7.3.** If \( f : (X, D) \to Y \) is a family of stable log-varieties of maximal variation over a normal, projective variety \( Y \) with klt general fiber, then \( K_{X/Y} + D \) is big.

This corollary follows from Theorem 7.1 by a rather general argument which we present in the following lemma.

**Lemma 7.4.** Let \( f : X \to Y \) be a surjective morphism between normal proper varieties and assume also that \( Y \) is projective. Let \( \mathcal{L} \) be an \( f \)-big line bundle on \( X \) such that \( f_*\mathcal{L} \) is a big vector bundle. Then \( \mathcal{L} \) itself is big.

**Proof.** Choose an ample line bundle \( \mathcal{A} \) on \( Y \) such that \( f^*\mathcal{A} \otimes \mathcal{L} \) is big. Then by Definition 3.7 there is a generically isomorphic inclusion for some integer \( a > 0 \):

\[
\bigoplus \mathcal{A} \hookrightarrow \text{Sym}^a(f_*\mathcal{L})
\]

This induces the following non-zero composition of homomorphisms, which concludes the proof:

\[
\bigoplus f^*\mathcal{A} \otimes \mathcal{L} \twoheadrightarrow f^*\text{Sym}^a(f_*\mathcal{L}) \otimes \mathcal{L} \to f^*f_*\mathcal{L}^a \otimes \mathcal{L} \to \mathcal{L}^{a+1}.
\]

**Proof of Corollary 7.3.** Take \( \mathcal{L} = \mathcal{O}_X(q(K_{X/Y} + D)) \) for a divisible enough \( q > 0 \).

Next we show that the klt assumption in Theorem 7.1 is necessary.

**Example 7.5.** Let \( f : X \to Y \) be an arbitrary non-isotrivial smooth projective family of curves over a smooth projective curve. Assume that it admits a section \( \sigma : Y \to X \) (this can be easily achieved after a base change) and let \( D = \text{im} \sigma \subset X \). This is one of the simplest examples of a family of stable log-varieties. Notice that the fibers are log canonical, but not klt. By adjunction \( K_P = (K_X + D)|_D \) and as \( f|_D : D \to Y \) is an isomorphism, it follows that \( \mathcal{O}_X(K_{X/Y} + D)|_D \simeq \mathcal{O}_D \). The following claim implies that \( f_*\mathcal{O}_X(r(K_{X/Y} + D)) \) cannot be big for any integer \( r > 0 \).

**Claim 1.** Let \( f : X \to Y \) be a flat morphism, \( \mathcal{E} \) a torsion-free sheaf on \( X \), and \( \mathcal{E} \) a locally free sheaf on \( Y \). Further let \( D \subset X \) be the image of a section \( \sigma : Y \to X \) and assume that \( Y \) is irreducible, that \( \mathcal{E}|_D \subseteq \mathcal{O}_D \), and that there exists a homomorphism \( \varrho : f^*\mathcal{E} \to \mathcal{L} \) such that \( \varrho|_D \not\equiv 0 \). Then \( \mathcal{E} \) cannot be big.

**Proof.** Since \( f|_D \) is an isomorphism, if \( \mathcal{E} \) were big, so would be \( (f^*\mathcal{E})|_D \) and then \( \varrho|_D \) would imply that \( \mathcal{O}_D \) is big. This is a contradiction which proves the statement.

A variant of Example 7.5 shows that even assuming that \( D = 0 \) would not be enough to get the statement of Theorem 7.1 without the klt assumption:

**Example 7.6.** Let \( f : X \to Y \) be an arbitrary non-isotrivial smooth projective family of curves over a smooth projective curve. Assume that it admits two disjoint sections \( \sigma_i : Y \to X \) for \( i = 1, 2 \) and let \( D_i = \text{im} \sigma \subset X \). Next glue \( X \) to itself by identifying \( D_1 \) and \( D_2 \) via the isomorphism \( \sigma_1 \circ \sigma_2^{-1} \) and call the resulting variety \( X' \). Then the induced \( f' : X' \to Y \) is a family of stable varieties. The same computation as above shows that \( f'_*\mathcal{O}_{X'}(rK_{X'/Y}) \) cannot be big for any \( r > 0 \) for this example as well. For computing the canonical class of non-normal varieties see [Kol13b, 5.7].
A variant of the above examples can be found in [Kee99, Thm. 3.0], for which not only $K_X/Y + D$ is numerically trivial on a curve $C$ contained in $D$ (and hence other ones can be constructed where the same happens over the double locus), but $K_X/Y + D|_C$ is not even semi-ample.

One might complain that in Example 7.6 the fibers are not normal. One can construct a similar example of a family of stable varieties where the general fiber is log canonical (and hence normal) that shows that the klt assumption is necessary, but this is a little bit more complicated.

**Example 7.7.** Let $Z$ be a projective cone over a genus 1 curve $C$. Assume that $Z \subseteq \mathbb{P}^3$ is embedded compatibly with this cone structure, that is, via this embedding, $Z \cap \mathbb{P}^2 = C$ for some fixed $\mathbb{P}^2 \subseteq \mathbb{P}^3$. Fix also coordinates $x_0, \ldots, x_3$ such that $x_1, x_2, x_3$ are coordinates for $\mathbb{P}^2$ and the cone point is $P := [1,0,0,0]$. Choose two general polynomials $f(x_1, x_2, x_3)$ and $g(x_0, x_1, x_2, x_3)$. Consider the pencil of hypersurfaces in $Z$ defined by these two equations. This yields a hypersurface $\mathcal{D} \subseteq Z \times \mathbb{P}^1$ with $\mathcal{D}_0 = V(f) \cap Z$ a general conic hypersurface section of $Z$ and $\mathcal{D}_\infty = V(g) \cap Z$ a general hypersurface section of $Z$. Since $g$ was chosen generally, $P \notin \mathcal{D}_\infty$. On the other hand, $P \in \mathcal{D}_0$, and hence $P \notin \mathcal{D}_t$ for $t \neq 0$. Furthermore, since in codimension 1 hypersurface sections of $Z$ disjoint from $P$ acquire only nodes $\mathcal{D}_t$ is either smooth or has only nodes for $t \neq 0$. Hence, for $d \gg 0$ the family $(Z \times \mathbb{P}^1, \mathcal{D}) \to \mathbb{P}^1$ is a family of stable log-varieties outside $t = 0$. For $t = 0$ we run stable reduction. Since the stable limit is unique, we may figure out the stable limit without going through the meticulous process by hand: it is enough to exhibit one family that is isomorphic in a neighborhood of 0 to the original family after a base-change and which does have a stable limit. The pencil $\mathcal{D}$ around $t = 0$ is described by the equation $f(x_1, x_2, x_3) + tg(x_0, x_1, x_2, x_3)$. Extract a $d$-th root from $t$ and denote the new family also by $(Z \times \text{Spec} k[t], \mathcal{D})$ (i.e., we keep the same notation for the boundary). Then $\mathcal{D}$ around $t = 0$ is described by the equation

$$F_1(t, x_0, x_1, x_2, x_3) := f(x_1, x_2, x_3) + t^d g(x_0, x_1, x_2, x_3).$$

Now set

$$F_2(t, x_0, x_1, x_2, x_3) := f(x_1, x_2, x_3) + t^d g(x_0/t, x_1, x_2, x_3),$$

and let $\mathcal{D}'$ be the hypersurface of $Z \times \text{Spec} k[t]$ defined by $F_2$. Then in a punctured neighborhood of $t = 0$, $(Z \times \text{Spec} k[t], \mathcal{D})$ is isomorphic to $(Z \times \text{Spec} k[t], \mathcal{D}')$, via the map

$$x_i \mapsto x_i (i \neq 0) \quad t \mapsto t \quad x_0 \mapsto t \cdot x_0.$$

Here the key is that $Z$, being a cone, is invariant under scaling by $x_0$. Note that since $g$ is general, $x_0^d$ has a non-zero coefficient, say $c$. Then it is easy to see that $F_2(0, x_1, x_2, x_3) = f(x_1, x_2, x_3) + cx_0^d$. That is, $\mathcal{D}_0'$ is a $d$-th cyclic cover of $V(f) \cap C \subseteq \mathbb{P}^2$ in $Z$. Since $f$ is general, $V(f) \cap C$ is smooth (i.e., a union of reduced points), and hence $\mathcal{D}_0'$ is also smooth. Furthermore, $\mathcal{D}_0'$ avoids $P$. It follows that $(Z, \mathcal{D}_0')$ is log canonical, whence stable and therefore it has to be the central fiber of the stable reduction.

Summarizing, after the stable reduction, we obtain a family $(\mathcal{Z}, \mathcal{D}) \to Y$ of stable log-pairs over a smooth projective curve (we denote the divisor by $\mathcal{D}$ here as well for simplicity), such that $\mathcal{Z}_y \simeq Z$ and $\mathcal{D}_y$ avoids the cone point in $\mathcal{Z}_y$ for each $y \in Y$. Note that $\mathcal{Z}$ cannot be isomorphic to $Y \times Z$ anymore (not even after a proper base-change), since then $\mathcal{D}_y$ would give a proper family of moving divisors in $Z$ that does not contain $P$. This is impossible, since a proper family covers a proper image, which would have to be the entire $Z$.

In any case, after possibly a finite base-change, we are able to take the cyclic cover of $\mathcal{Z}$ of degree $d$ ramified along $\mathcal{D}$. For $d \gg 0$ the obtained family $X \to Y$ is stable of maximal variation over the projective curve $Y$. It has elliptic singularities along a curve $B$ that covers $d$ times the singularity locus of $\mathcal{Z} \to Y$. Hence, $B \to Y$ is proper and has $d$ preimages over each point. In particular it is étale (though $B$ might be reducible). If we blow-up $B$, and resolve the other singular points as well (which are necessarily disjoint from $B$, since they originate from the nodal fibers of $\mathcal{D} \to Y$), we obtain a resolution $\pi : V \to X$. Let $E$ be the (reduced) preimage of $B$. Then we have that
$K_{V/Y} + E + F \equiv \pi^*K_{X/Y}$, where $F$ is exceptional and disjoint from $E$. In particular then

$$K_{E/Y} \equiv (K_{V/Y} + E)|_E \equiv (K_{V/Y} + E + F)|_E \equiv \pi^*K_{X/Y}|_E \equiv (\pi|_E)^* (K_{X/Y}|_B).$$

Hence it is enough to show that $K_{E/Y} \equiv 0$ (since then we have found a horizontal curve over which $K_{X/Y}$ is numerically trivial). Since $B \to Y$ is étale, it is enough to show that $K_{E/B} \equiv 0$. However $E \to B$ is a smooth family of isomorphic genus one curves. In particular, after a finite base-change we may also assume that it has a section, in which case we do know that its relative canonical sheaf is numerically trivial. However, then it is numerically trivial even without the base-change. It follows that $K_{X/Y}|_B$ is numerically trivial and the same argument as above shows that then it cannot be big.

Recall that if $(X, D)$ is a klt pair and $\Gamma$ a $\mathbb{Q}$-Cartier divisor, then the log canonical threshold is defined as

$$\sup \{ t | (X, D + t\Gamma) \text{ is log canonical} \}.$$

**Lemma 7.8.** The log canonical threshold is lower semi-continuous in projective, flat families with $\mathbb{Q}$-Cartier relative log canonical bundle. That is, if $f : (X, D) \to S$ is a projective, flat morphism with $S$ normal and essentially of finite type over $k$ such that $K_{X/S} + D$ is $\mathbb{Q}$-Cartier, $(X_s, D_s)$ is klt for all $s \in S$ and $\Gamma \geq 0$ is a $\mathbb{Q}$-Cartier divisor on $X$ not containing any fibers, then $\lct(\Gamma_s; X_s, D_\Gamma)$ is lower semi-continuous.

Furthermore, if $S$ is regular, then for every $s \in S$ there is a neighborhood $U$ of $s$, such that

$$\lct(\Gamma|_{f^{-1}U}; f^{-1}U, D|_{f^{-1}U}) \geq \lct(\Gamma_s; X_s, D_s).$$

**Proof.** Let us first show the second statement, which is an application of inversion of adjunction. Let $A = f^{-1}H$ for some very ample reduced effective divisor $H$. Then

$$(A, D|_A + t\Gamma|_A) \text{ is lc } \Rightarrow (X, D + t\Gamma + A) \text{ is lc in a neighborhood of } A \Rightarrow$$

$$\Rightarrow (X, D + t\Gamma) \text{ is lc in a neighborhood of } A.$$

Applying this inductively gives the second statement, since for regular schemes every point can be (locally) displayed as the intersection of hyperplanes.

Next, let us prove that $s \mapsto \lct(\Gamma_s; X_s, D_s)$ is constant on a dense open set $U$ and that $U$ can be chosen such that $\lct(\Gamma|_{f^{-1}U}; f^{-1}U, D|_{f^{-1}U})$ agrees with this constant value. For this we may assume that $S$ is smooth. Take a resolution $\pi : Y \to X$ of $(X, D + \Gamma)$. By replacing $X$ with a dense Zariski open set we may assume that all exceptional divisors of $\pi$ are horizontal and that $f \circ \pi$ is smooth. However, then the discrepancies of $(X_s, D_s + t\Gamma_s)$ agree for all $s \in S$ and $t \in \mathbb{Q}$ and furthermore, this is the same set as the discrepancies of $(X, D + t\Gamma)$. This concludes our claim.

The above two claims show that we have semi-continuity over smooth curves, and also that the function is constructible. These together show that the function is semi-continuous in general. $\square$

**Definition 7.9.** We define the log canonical threshold of a line bundle $\mathcal{L}$ on a projective pair $(X, \Delta)$ as the minimum of the log canonical thresholds of the effective divisors in $\mathbb{P}(H^0(X, \mathcal{L})^*)$, the complete linear system of $\mathcal{L}$:

$$\lct(\mathcal{L}; X, \Delta) := \min \{ \lct(\Gamma; X, \Delta) \mid \Gamma \in \mathbb{P}(H^0(X, \mathcal{L})^*) \}.$$

By the above lemma this minimum exists.

**Lemma 7.10.** The log canonical threshold of a line bundle is bounded in projective, flat families. That is, let $f : (X, D) \to T$ be a projective flat morphism with $T$ normal and essentially of finite type over $k$ and $\mathcal{L}$ a line bundle on $X$. Assume that $(X_t, D_t)$ is klt for all $t \in T$ and $K_{X/T} + D$ is $\mathbb{Q}$-Cartier. Then there exists a real number $c$, such that $\lct(\mathcal{L}_t; X_t, D_t) \geq c$ for all $t \in T$. 

**Proof.** First assume that $f_*\mathcal{L}$ commutes with base-change (and it is consequently locally free) and let $\mathbb{P} := \text{Proj}_T((f_*\mathcal{L})^*)$. Notice that the points of $\mathbb{P}$ for $t \in T$ may be identified with elements of the linear systems $\mathbb{P}(H^0(X,\mathcal{L})^*)$. Further let $\Gamma$ be the universal divisor on $X \times_T \mathbb{P}$ corresponding to $\mathcal{L}$, that is, $(x,[D]) \in \Gamma$ iff $x \in D$. Now, applying Lemma 7.8 to $X \times_T \mathbb{P} \to \mathbb{P}$ and $\Gamma$ yields the statement.

In the general case, we work by induction on the dimension of $T$. We can find a dense open set over which $f_*\mathcal{L}$ commutes with base change. So, there is a lower bound as above over this open set, and there is another lower bound on the complement. Combining the two gives a lower bound over the entire $T$.

The essence of the argument of the proof of the following proposition was taken from [Vie95, Lemma 5.18], though the context is slightly different.

**Proposition 7.11.** Let $f : (X,D) \to Y$ be a flat morphism such that $D$ does not contain any fibers, $(X_y,D_y)$ is klt for a fixed $y \in Y$, $K_X + D$ is $\mathbb{Q}$-Cartier and $Y$ is smooth. Let $\Gamma$ be a $\mathbb{Q}$-Cartier effective divisor on $X$ that contains no fibers, let $\Delta$ be a normal crossing divisor (with arbitrary $\mathbb{Q}$-coefficients) on $Y$, let $\tau : Z \to X$ be a log-resolution of singularities of $(X,D + \Gamma + f^*\Delta)$ and finally let $t$ be a real number such that $t < \text{lct}(\Gamma|_{X_y};X_y,D_y)$. Then in a neighborhood of $X_y$

$$\tau_*\mathcal{O}_Z\left([K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta)]\right) \cong \mathcal{O}_X([-f^*\Delta])$$

**Proof.** Since the statement is local near $y \in Y$, we may replace $Y$ with any arbitrarily small neighborhood of $y$, which we will do multiple times as needed without explicitly saying so.

In order to prove the desired isomorphism it is enough to find a map

$$\tau_*\mathcal{O}_Z\left([K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta)]\right) \xrightarrow{\zeta} \mathcal{O}_X([-f^*\Delta])$$

which is surjective in a neighborhood of $X_y$. Indeed, $\tau_*\mathcal{O}_Z\left([K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta)]\right)$ is torsion-free of rank 1, so if $\zeta$ is surjective, then it is generically an isomorphism and hence $\ker \zeta$ would be a torsion sheaf and hence zero. Therefore, it is enough to prove the existence of a map as in (7.11.1).

Next we will prove that such a map exists. By Lemma 7.8 and the klt assumption

$$K_Z + \tau_*^{-1}(D + t\Gamma) = \tau^*(K_X + D + t\Gamma) + \sum a_i E_i$$

where $a_i > -1$ and $E_i$ are pairwise distinct irreducible $\tau$-exceptional divisors. (In order to get equality we choose canonical divisors on $X$ and $Z$ in a coherent manner). Let us write $\tau^*f^*\Delta$ in a similar fashion:

$$\tau^*f^*\Delta = \tau_*^{-1}f^*\Delta + \sum b_i E_i$$

with some appropriate $b_i \in \mathbb{Q}$. Putting these together we get

$$K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta) = -\tau_*^{-1}(D + t\Gamma + f^*\Delta) + \sum (a_i - b_i) E_i,$$

which, using the facts that $D$ and $\Gamma$ has no $f$-vertical components and that $|D + t\Gamma| = 0$ by the klt property, yields that

$$\begin{align*}
[K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta)] &= [-\tau_*^{-1}(D + t\Gamma + f^*\Delta) + \sum (a_i - b_i) E_i] \\
&= [-f^*\Delta] + \sum [a_i - b_i] E_i.
\end{align*}$$

Since the $E_i$ are $\tau$-exceptional, after pushing forward via $\tau$, the components with non-negative coefficient $[a_i - b_i]$ disappear and hence we obtain a map in a neighborhood of $X_y$ as requested in (7.11.1):

$$\tau_*\mathcal{O}_Z\left([K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta)]\right) \xrightarrow{\zeta} \mathcal{O}_X([-f^*\Delta]).$$
In the rest of the proof we will show that this map is surjective in a neighborhood of $X_y$.

Notice that the integral part of $\Delta$ makes no difference by the projection formula, so we may replace $\Delta$ with $\{\Delta\}$, that is, we may assume that $|\Delta| = 0$. Furthermore, by the klt assumption on $(X_y, D_y)$ it follows that $X_y$ is reduced and hence we may assume that the pre-image of any component of $\Delta$ is reduced, which implies that any estimate or rounding of the coefficients of $\Delta$ remain true for $f^*\Delta$. In particular, we may assume that $[-f^*\Delta] = 0$

We will use induction on the number of components of $\Delta$. If $\Delta = 0$, then the statement follows from (7.11.2), since in this case $b_i = 0$. Next, let $E$ be an arbitrary component of $\Delta$. Define $H := f^*E$, and let $\tilde{H}$ be the strict transform of $H$ in $Z$. Note that $\tau|_{\tilde{H}} : \tilde{H} \to H$ is a log resolution of $(H, D|H + \Gamma|_H + f^*((\Delta - (\text{coeff}_E \Delta)|E)))$.

Observe, that in order to prove that $\varsigma$ of (7.11.1) is surjective, using Nakayama’s lemma, it is enough to prove that it is surjective after composing with the natural surjective map $\mathcal{O}_X \to \mathcal{O}_H$. We will denote this composition by $\delta$.

Consider the following commutative diagram. After the diagram we explain why the indicated maps exists and why $\alpha$ and $\gamma$ are surjective.

\[
\begin{array}{c}
q_\tau \mathcal{O}_Z \left( \left[ K_{Z/X} - \tau^*(D + t\Gamma + H + f^*(\Delta - (\text{coeff}_E \Delta)|E)) + \tilde{H} \right] \right) \\
\downarrow \alpha & \beta & \downarrow \tau_\sigma \mathcal{O}_Z \left( \left[ K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta) \right] \right) \\
\tau_\sigma \mathcal{O}_{\tilde{H}} \left( \left[ K_{\tilde{H}/H} - \tau^*(D + t\Gamma + f^*(\Delta - (\text{coeff}_E \Delta)|E)) \right] \right) & \delta & \tau_\sigma \mathcal{O}_H
\end{array}
\]

(7.11.3)

Using adjunction on $X$ and $Z$ respectively we have that $K_H = (K_X + H)|_H$ and $K_{\tilde{H}} = (K_Z + \tilde{H})|_{\tilde{H}}$ and hence $K_{\tilde{H}/H} = (K_{Z/X} + \tilde{H} - \tau^*H)|_{\tilde{H}}$, so $\alpha$ is simply the $\tau_\sigma$ of the restriction map from $Z$ to $\tilde{H}$. By [Laz04b, Theorem 9.4.15],

\[
R^1\tau_\sigma \mathcal{O}_Z \left( \left[ K_{Z/X} - \tau^*(D + t\Gamma + H + f^*(\Delta - (\text{coeff}_E \Delta)|E)) \right] \right) = 0,
\]

which implies that $\alpha$ is surjective. The map $\gamma$ is the equivalent of $\varsigma$ for $H$ and hence it is surjective by the inductive hypothesis. To construct $\beta$ we will show that

\[
(7.11.4) \quad \left[ K_{Z/X} - \tau^*(D + t\Gamma + H + f^*(\Delta - (\text{coeff}_E \Delta)|E)) + \tilde{H} \right] \leq \left[ K_{Z/X} - \tau^*(D + t\Gamma + f^*\Delta) \right].
\]

This also proves that $\beta$ is injective, but we do not need that fact.

To show (7.11.4) first note that the coefficient of $\tilde{H}$ in the divisor on the left is zero (even before the round-up) and it is zero after the round-up on the right side, because of our assumption that $|\Delta| = 0$. To compare the coefficients of the other prime divisors, note that the difference in (7.11.4) between the divisor on the left and on the right side (before the round-up) is

\[
F := \tilde{H} - \tau^*H + \tau^*f^*((\text{coeff}_E \Delta)|E) = \tilde{H} - \tau^*(H - f^*((\text{coeff}_E \Delta)|E)) = \tilde{H} - \tau^*(1 - \text{coeff}_E \Delta)H.
\]

Since $0 \leq \text{coeff}_E \Delta < 1$ by our initial simplification, (7.11.5) implies that $\text{coeff}_G F \leq 0$ for every prime divisor $\tilde{H} \neq G \subseteq \text{Supp} \tau^*H$. This shows that (7.11.4) is satisfied over each divisor, except
possibly over \(\tilde{H}\). To see what happens over \(\tilde{H}\) let us compute the coefficients over it on the two sides of (7.11.4) (before the round-up): On the left hand side the terms containing \(H\) are \(\tilde{H} - \tau^*(H + f^*(\Delta - (\text{coeff}_E \Delta)E))\), however the coefficients of these over \(\tilde{H}\) cancel out. On the right hand side the coefficient of \(\tilde{H}\) is \(-\text{coeff}_H \tau^* f^* \Delta = -\text{coeff}_E \Delta\), which is at most 0 but greater than \(-1\). Hence, after rounding up the coefficients of \(\tilde{H}\) on both sides end up being 0. This proves the existence (and injectivity) of \(\beta\) and then (7.11.3) shows that \(\delta\) is surjective and we had already observed that this implies the surjectivity of \(\zeta\) by Nakayama’s lemma completing the proof. \(\square\)

**Proposition 7.12.** Let \((V, D_V)\) and \((Y, D_Y)\) be two klt pairs and \(\mathcal{L}\) and \(\mathcal{N}\) two line bundles on \(V\) and \(Y\) respectively. Then

\[
\text{lt}(p_V^* \mathcal{L} \otimes p_Y^* \mathcal{N}; V \times Y, p_V^* D_V + p_Y^* D_Y) = \min\{\text{lt}(\mathcal{L}; V, D_V), \text{lt}(\mathcal{N}; Y, D_Y)\}
\]

**Proof.** It is obvious that

\[
\text{lt}(p_V^* \mathcal{L} \otimes p_Y^* \mathcal{N}; V \times Y, p_V^* D_V + p_Y^* D_Y) \leq \min\{\text{lt}(\mathcal{L}; V, D_V), \text{lt}(\mathcal{N}; Y, D_Y)\}.
\]

We have to prove the opposite inequality. To do that, choose \(\Gamma \in |p_V^* \mathcal{L} \otimes p_Y^* \mathcal{N}|, x \in V \times Y\) and \(t < \min\{\text{lt}(\mathcal{L}; V, D_V), \text{lt}(\mathcal{N}; Y, D_Y)\}\). Let \(\rho : \tilde{Y} \to (Y, D_Y)\) be a log-resolution, \(D_{\tilde{Y}} := \tau^* D_Y\), \(\pi : X := V \times \tilde{Y} \to V \times Y\) the natural morphism, \(\tilde{f} : X \to \tilde{Y}\) the projection and \(\tau : Z \to X\) a log resolution of \(\left(X, \pi^* \Gamma + p_V^* D_V + \tilde{f}^* D_{\tilde{Y}}\right)\). Note that, according to [Vie95, Claim 5.20], \(\tilde{Y}\) can be chosen such that \(\pi^* \Gamma = \Gamma' + \tilde{f}^* \Delta\) where \(\Delta\) is simple normal crossing on \(\tilde{Y}\) and \(\Gamma'\) contains no fibers.

According to Proposition 7.11, there is an isomorphism

\[
\pi_* \tau_* \mathcal{O}_Z \left(\left[K_{Z/V \times Y} - \tau^* \pi^* (p_Y^* D_V + p_Y^* D_Y + t \Gamma)\right]\right)
\]

\[
\simeq \pi_* \tau_* \mathcal{O}_Z \left(\left[K_{Z/X} - \pi^* \left(p_Y^* D_V + t \Gamma' - \tilde{f}^* \left(K_{\tilde{Y}/Y} + D_{\tilde{Y}} - t \Delta\right)\right)\right]\right)
\]

\[
\simeq \pi_* \mathcal{O}_X \left(\left[\tilde{f}^* \left(K_{\tilde{Y}/Y} + D_{\tilde{Y}} - t \Delta\right)\right]\right).
\]

Note that by the choice of \(t\), \(\left[\tilde{f}^* \left(K_{\tilde{Y}/Y} + D_{\tilde{Y}} - t \Delta\right)\right] \geq 0\) and then

\[
\pi_* \mathcal{O}_X \left(\left[\tilde{f}^* \left(K_{\tilde{Y}/Y} + D_{\tilde{Y}} - t \Delta\right)\right]\right) \simeq \mathcal{O}_{V \times Y}.
\]

This finishes the proof. \(\square\)

For the next statement recall Notation 2.12.

**Corollary 7.13.** If \((X, D)\) is a projective klt pair, \(\mathcal{L}\) a line bundle on \(X\), then for all integers \(m > 0\),

\[
\text{lt} \left(\mathcal{L}^{(m)}; X^{(m)}, D_{X^{(m)}}\right) = \text{lt}(L; X, D).
\]

In the next statement multiplier ideals are used. Recall that the multiplier ideal of a pair \((X, D)\) of a normal variety and an effective \(\mathbb{Q}\)-divisor is \(\mathcal{J}(X, D) := \tau_* \mathcal{O}_Z \left(\left[K_{X/X} - \tau^* D\right]\right) \subseteq \mathcal{O}_X\).

**Proposition 7.14.** Let \(f : X \to Y\) be a surjective morphism between projective, normal varieties with equidimensional, reduced \(S_2\) fibers, \(L\) a Cartier divisor and \(\Delta \geq 0\) an effective divisor on \(X\) such that \(\Delta\) contains no general fibers, \((X_y, \Delta_y)\) is klt for general \(y \in Y\) and \(L - K_{X/Y} - \Delta\) is a nef and \(f\)-ample \(\mathbb{Q}\)-Cartier divisor. Assume further that \(K_Y\) is Cartier. Then \(f_* \mathcal{O}_X(L)\) is weakly-positive (in the weak sense).

**Proof.** Set \(\mathcal{L} := \mathcal{O}_X(L)\). Let \(A\) be a general very ample effective divisor on \(Y\) and \(m > 0\) an integer. In this proof a subscript of \(A\) will denote a base change to \(A\).
Claim 1. For any nef Cartier divisor $N$ on $Y$ the natural restriction map,

$$H^0 \left( X^{(m)}; \mathcal{L} \left( X^{(m)}, \Delta_{X^{(m)}} \right) \otimes \mathcal{L}^{(m)} \left( f^{(m)} \right) \left( K_Y + 2A + N \right) \right) \rightarrow H^0 \left( X^{(m)}_A; \mathcal{L} \left( X^{(m)}_A, \Delta_{X^{(m)}_A} \right) \otimes \mathcal{L}^{(m)} \left( f^{(m)} \right) \left( K_Y + 2A + N \right) \right)$$

is surjective.

Proof. Note that in the statement we are already using the fact that $\mathcal{L} \left( X^{(m)}_A, \Delta_{X^{(m)}_A} \right) \cong \mathcal{L} \left( X^{(m)}_A, \Delta_{X^{(m)}_A} \right)$, which follows from the general choice of $A$. For the above homomorphism to be surjective, it is proved that the corresponding equation holds:

$$(7.14.2) \quad H^1 \left( X^{(m)}; \mathcal{L} \left( X^{(m)}, \Delta_{X^{(m)}} \right) \otimes \mathcal{L}^{(m)} \left( f^{(m)} \right) \left( K_Y + A + N \right) \right) = 0$$

However,

$$L_{X^{(m)}} + \left( f^{(m)} \right)^* \left( K_Y + A + N \right) - \left( K_{X^{(m)}} + \Delta_{X^{(m)}} \right) = L - K_{X/Y} - \Delta + \left( f^{(m)} \right)^* \left( A + N \right)$$

is ample, hence (7.14.2) holds by Nadel-vanishing. This proves the claim. □

Note that the assumptions of the proposition remain valid for $f \mid_{X_A} : X_A \to A$ and $\Delta \mid_{X_A}$. Hence we may use Claim 1 iteratively. By the klt assumption on the general fiber, we may further leave out the multiplier ideal in the last term. Thus we obtain a surjective homomorphism

$$H^0 \left( X^{(m)}; \mathcal{L} \left( X^{(m)}, \Delta_{X^{(m)}} \right) \otimes \mathcal{L}^{(m)} \left( f^{(m)} \right) \left( K_Y + A + \sum_{i=1}^n A_i \right) \right) \rightarrow H^0 \left( Y, \mathcal{L} \left( Y, \Delta_{Y} \right) \otimes f^*_s \mathcal{L} \left( Y \right) \right)$$

where $A_1, \ldots, A_n \in |A|$ are general, $y \in \bigcap_{i=1}^n A_i$ is arbitrary and $n := \dim Y$. Since the left hand side of this homomorphism is a subspace of

$$H^0 \left( Y, \mathcal{L} \left( Y, \Delta_{Y} \right) \otimes f^*_s \mathcal{L} \left( Y \right) \right)$$

and the right hand side can be identified with $f^*_s \mathcal{L} \left( Y \right) \otimes k(y)$ (recall $y \in Y$ is general), we obtain that

$$H^0 \left( Y, \mathcal{L} \left( Y, \Delta_{Y} \right) \otimes f^*_s \mathcal{L} \left( Y \right) \right) \rightarrow f^*_s \mathcal{L} \left( Y \right) \otimes k(y)$$

is surjective. Therefore,

$$\mathcal{L} \left( Y, \Delta_{Y} \right) \otimes f^*_s \mathcal{L} \left( Y \right) \cong \mathcal{L} \left( Y, \Delta_{Y} \right) \otimes \left[ \otimes_{j=1}^m \right] f_s \mathcal{L}$$

is generically globally generated for all $m > 0$. However, then so is $\mathcal{L} \left( Y, \Delta_{Y} \right) \otimes \text{Sym}^n \left[ f_s \mathcal{L} \right]$, since there is a generically surjective homomorphism from the former to the latter. This yields weak positivity (in the weak sense). □

Proof of Theorem 7.1. By resolving $Y$ and then pulling back $X$ to the resolution we may assume that $Y$ is smooth. According to Theorem 6.1, for all divisible enough $q > 0$, det $f_s \mathcal{L} \left( q(K_Y + D) \right)$ is big. Fix such a $q$. According to Lemma 7.10 there is a real number $c > 0$ such that

$$c < \text{let} \left( \mathcal{O}_{X_y} \left( q \left( K_{X_y} + D_y \right) \right); X_y, D_y \right)$$

for every $y \in U$, where $U$ is the open locus over which the fibers $(X_y, D_y)$ are klt. Fix also such a $c$ and let $l := \left\lceil \frac{1}{c} \right\rceil$. By replacing $Y$ with a finite cover, we may assume that det $f_s \mathcal{L} \left( q(K_Y + D) \right) =
Lemma 6.7 has a local splitting, Proposition 7.14 we may obtain that (7.14.5)

\[ \text{Lemma 2.6} \]

which implies that

(7.14.4)

\[ (f^{(m)})^* IA + \Gamma \sim q \left( K_{X^{(m)}/Y} + D_{X^{(m)}} \right) \]

for some appropriate effective divisor $\Gamma$ on $X^{(m)}$. Note that since (7.14.3) has a local splitting, $\Gamma_y \neq 0$ for any $y \in Y$. In particular, $\Gamma$ does not contain any $X_y^{(m)}$ for any $y \in U$, since fibers over $U$ are irreducible.

By (7.14.4) we obtain that

(7.14.5)

\[ \frac{1}{l} \Gamma + \frac{q(2l-1)}{l} \left( K_{X^{(m)}/Y} + D_{X^{(m)}} \right) \sim q 2q \left( K_{X^{(m)}/Y} + D_{X^{(m)}} \right) - (f^{(m)})^* A. \]

Note that for each $y \in U$,

\[ \text{lct} \left( \frac{1}{l} \Gamma_y; X_y^{(m)}, (D_{X^{(m)})}_y \right) \leq \]

\[ \leq l \cdot \text{lct} \left( \mathcal{O}_{X_y^{(m)}} \left( q \left( K_{X_y^{(m)}} + (D_{X^{(m)})}_y \right) \right); X_y^{(m)}, (D_{X^{(m)})}_y \right) = \]

\[ = l \cdot \text{lct} \left( \mathcal{O}_{X_y^{(m)}} \left( q \left( K_{X_y^{(m)}} + D_y \right) \right); X_y, D_y \right) > \left\lceil \frac{1}{c} \right\rceil c \geq 1 \]

Corollary 7.13

Therefore, \( \left( X_y^{(m)}, \frac{1}{l} \Gamma_y + (D_{X^{(m)})}_y \right) \) is klt for all $y \in U$. Then by (7.14.5) and Lemma 6.7 we may apply Proposition 7.14 to show that

\[ f^{(m)}_* \mathcal{O}_{X^{(m)}} \left( 2q \left( K_{X^{(m)}/Y} + D_{X^{(m)}} \right) - (f^{(m)})^* A \right) \]

\[ \simeq f^{(m)}_* \mathcal{O}_{X^{(m)}} \left( 2q \left( K_{X^{(m)}/Y} + D_{X^{(m)}} \right) \right) \otimes \mathcal{O}_Y(-A) \]

\[ \simeq \mathcal{O}_Y(-A) \otimes \bigotimes_{i=1}^{m} f_* \mathcal{O}_X \left( 2q \left( K_{X/Y} + D \right) \right) \]

Lemma 2.6

is weakly-positive. Therefore there exists an integer $b > 0$ such that

\[ \mathcal{O}_Y(bA) \otimes \text{Sym}^{2b} \left( \mathcal{O}_Y(-A) \otimes \bigotimes_{i=1}^{m} f_* \mathcal{O}_X \left( 2q \left( K_{X/Y} + D \right) \right) \right) \]

\[ \simeq \mathcal{O}_Y(-bA) \otimes \text{Sym}^{2bm} \left( \bigotimes_{i=1}^{m} f_* \mathcal{O}_X \left( 2q \left( K_{X/Y} + D \right) \right) \right) \]

\[ \longrightarrow \mathcal{O}_Y(-bA) \otimes \text{Sym}^{2bm} \left( f_* \mathcal{O}_X \left( 2q \left( K_{X/Y} + D \right) \right) \right) \]

is generically globally generated. Hence $f_* \mathcal{O}_X \left( 2q \left( K_{X/Y} + D \right) \right)$ is big. \( \square \)
8. SUBADDITIONIVITY OF LOG-KODAIRA DIMENSION

In this section we are considering the question of subadditivity of log-Kodaira dimension. Since, at this point, there are multiple non-equivalent statements of this conjecture in the literature, we state a couple of them. All of these are straightforward consequences of Proposition 8.7.

**Definition 8.1.** A log canonical fiber space is a surjective morphism \( f : (X, D) \rightarrow Y \) such that

1. both \( X \) and \( Y \) are irreducible, normal and projective,
2. \( K_X + D \) is \( \mathbb{Q} \)-Cartier and
3. \((X_\eta, D_\eta)\) has log canonical singularities.

Next we define the notion of variation for log canonical fiber spaces. Unfortunately, at this time we have to put a restriction on the log canonical fiber spaces on which the definition works. The main issue is that in Definition 5.16, variation is defined only for families of stable log-varieties.

For general log canonical fiber spaces as in Definition 8.1 the reasonable expectation is that we would define variation as the variation of the relative log canonical model of \((X, D)\) (restricted to the open locus where it is a stable family). However, for log canonical singularities, the existence of a log canonical model is not known even in the log-general type case. Hence, in order to make this definition, we assume that a relative log canonical model exists. This is known for example if the general fiber is klt.

**Definition 8.2.** Let \( f : (X, D) \rightarrow Y \) be a log canonical fiber space such that \( K_{X_\eta} + D_\eta \) is big and \((X_\eta, D_\eta)\) admits a log canonical model, where \( \eta \) is the generic point of \( Y \). Then let \( \text{Var} \ f \) to be the variation of the log canonical model of \((X_\eta, D_\eta)\) as defined in Definition 5.16.

**Remark 8.3.** If \((X_\eta, D_\eta)\) is klt and \( K_{X_\eta} + D_\eta \) is big, then \((X_\eta, D_\eta)\) admits a log canonical model by [BCHM10, Thm 1.2]) and hence in this case \( \text{Var} \ f \) is defined.

**Theorem 8.4.** If \( f : (X, D) \rightarrow Y \) is a log canonical fiber space with \( K_{X_\eta} + D_\eta \) big, where \( \eta \) is the generic point of \( Y \), then subadditivity of log-Kodaira dimension holds. That is,

\[
\kappa(K_X + D) \geq \kappa(Y) + \kappa(K_{X_\eta} + D_\eta).
\]

Furthermore, if \((X_\eta, D_\eta)\) is klt, then

\[
\kappa(K_X + D) \geq \max\{\kappa(Y), \text{Var} \ f\} + \kappa(K_{X_\eta} + D_\eta).
\]

**Theorem 8.5.** If \( f : (X, D) \rightarrow (Y, E) \) is a surjective map of log-smooth and log canonical projective pairs, such that \( D \geq f^*E \) and \( K_{X_\eta} + D_\eta \) is big, where \( \eta \) is the generic point of \( Y \), then

\[
\kappa(K_X + D) \geq \kappa(K_{X_\eta} + D_\eta) + \kappa(K_Y + E).
\]

In the next corollary we use the notion of Kodaira dimension of an arbitrary algebraic variety \( X \). It is defined via finding a resolution \( X'_0 \) of \( X \) with a projective compactification \( X' \) such that \( D' := (X' \setminus X'_0)_{\text{red}} \) is simple normal crossing, and then setting

\[
\kappa(X) := \kappa(K_{X'} + D').
\]

The following statement follows immediately from Theorem 8.5.

**Corollary 8.6.** Let \( f : X \rightarrow Y \) be a dominant map of (not necessarily proper) algebraic varieties such that the generic fiber has maximal Kodaira dimension. Then

\[
\kappa(X) \geq \kappa(X_\eta) + \kappa(Y).
\]

**Proposition 8.7.** Let \( f : (X, D) \rightarrow Y \) be a log canonical fiber space such that

1. \( K_{X_\eta} + D_\eta \) is big, where \( \eta \) is the generic point of \( Y \),
2. \((X, D)\) is log-smooth, and
3. \( Y \) is smooth.
Further let \( M \) be a \( \mathbb{Q} \)-Cartier divisor on \( Y \) with \( \kappa(M) \geq 0 \). Then
\[
\kappa(K_{X/Y} + D + f^*M) \geq \kappa(M) + \kappa(K_{X/Y} + D).
\]
Furthermore, if \((X_\eta, D_\eta)\) is klt, then
\[
\kappa(K_{X/Y} + D + f^*M) \geq \max\{\kappa(M), \mathrm{Var} \ f\} + \kappa(K_{X/Y} + D_\eta).
\]

**Proof of Theorem 8.4 using Proposition 8.7.** Let \( \tau : Y' \to Y \) be a resolution of \( Y \), and let \( X' \) be a resolution of the component of \( X \times_Y Y' \) that dominates \( X \) such that \( \tau^*D \) is a simple normal crossing divisor, where \( \pi : X' \to X \) is the induced map. Choose canonical divisors \( K_X \) and \( K_{X'} \) such that they agree on the locus where \( \pi \) is an isomorphism. Then define \( D' \) and \( E \) via
\[
K_{X'} + D' = \pi^*(K_X + D) + E
\]
such that \( E, D' \geq 0 \) and have no common components. Then we have \( \kappa(K_X + D) = \kappa(K_{X'} + D') \), \( \kappa(K_{X/Y} + D_\eta) = \kappa\left(K_{X/Y} + D'_\eta\right) \) and if \((X_\eta, D_\eta)\) is klt then also \( \mathrm{Var} \ f = \mathrm{Var} \ f' \), where \( f' : (X', D') \to Y' \) is the induced morphism. Proposition 8.7 applied to \( f' \) with \( M = K_{Y'} \) completes the proof. \( \square \)

**Proof of Theorem 8.5 using Proposition 8.7.** Let \( D' := D - f^*E \), set \( M := K_Y + E \), and apply Proposition 8.7 to \( f : (X, D') \to Y \) and \( M \). Notice that we may assume that \( \kappa(K_Y + E) \geq 0 \), since otherwise the statement is trivial. This yields
\[
\kappa(K_X + D) = \kappa(K_{X/Y} + D' + f^*(K_Y + E)) \geq \kappa(K_Y + E) + \kappa(K_{X/Y} + D'_\eta) = \kappa(K_Y + E) + \kappa(K_{X/Y} + D_\eta). \quad \square
\]

The rest of the section concerns proving Proposition 8.7.

**Lemma 8.8.** Consider the following commutative diagram of normal varieties, where \( f \) is flat and Gorenstein, \( \tau \) is surjective, \( \overline{X} := X \times_Y Y' \) and \( X_\eta \) is the normalization of the component of \( X \times_Y Y' \) dominating \( Y' \).

\[
\begin{array}{ccc}
X & \xleftarrow{\alpha} & \overline{X} \\
f & & \beta \\
Y & \xleftarrow{\tau} & Y'
\end{array}
\]

Then, there is a natural embedding \( \omega_{X_\eta/Y'} \hookrightarrow \beta^* \alpha^* \omega_{X/Y} \).

**Proof.** Since \( f \) is flat and Gorenstein, \( \omega_{\overline{X}/Y'} \simeq \alpha^* \omega_{X/Y} \) according to [Con00, Thm 3.6.1]. In particular, \( \omega_{\overline{X}/Y'} \) is a line bundle. Consider then the Gorthendieck trace of \( \beta, \beta_* \omega_{X_\eta/Y'} \to \omega_{\overline{X}/Y'} \). Pulling this back we obtain \( \phi : \beta^* \beta_* \omega_{X_\eta/Y'} \to \beta^* \omega_{\overline{X}/Y'} \). We claim that \( \phi \) factors through the natural map \( \xi : \beta^* \beta_* \omega_{X_\eta/Y'} \to \omega_{\overline{X}/Y'} \). For this note first that since \( \beta^* \omega_{\overline{X}/Y'} \) is a line bundle, it is torsion-free. Hence if \( \mathcal{T} \) is the torsion part of \( \beta^* \beta_* \omega_{X_\eta/Y'} \), \( \phi \) factors through the natural map \( \beta^* \beta_* \omega_{X_\eta/Y'} \to \beta^* \beta_* \omega_{\overline{X}/Y'}/\mathcal{T} \). Therefore, it is enough to show that the latter map is isomorphic to \( \xi \), that is, that \( \ker \xi = \mathcal{T} \) and that \( \xi \) is surjective. The surjectivity follows immediately, since \( \beta \) is affine and for any ring map \( A \to B \) and \( B \)-module \( M \), the natural map \( M \otimes_A B \to M \) is surjective. To show that \( \ker \xi \subseteq \mathcal{T} \) we just note that \( \beta \) is generically an isomorphism, and hence \( \xi \) is generically an isomorphism. The opposite containment, that is, that \( \ker \xi \supseteq \mathcal{T} \), follows from the fact that \( \omega_{X_\eta/Y'} \) is torsion-free. This concludes our claim. Hence we obtain an embedding \( \omega_{X_\eta/Y'} \hookrightarrow \beta^* \omega_{\overline{X}/Y'} \simeq \beta^* \alpha^* \omega_{X/Y} \). \( \square \)

**Proof of Proposition 8.7.**

**Step 0:** Assuming klt. If \((X_\eta, D_\eta)\) is not klt, then by decreasing the coefficients of \( D \) a little all our assumptions remain true, and so we may assume that \((X_\eta, D_\eta)\) is klt.
Step 1: Allowing an extra divisor avoiding a big open set of the base. According to [Vie83a, Lemma 7.3], there is a birational morphism $\tilde{Y} \to Y$ from a smooth projective variety, and another one from $\tilde{X}$ onto the component of $X \times_Y \tilde{Y}$ dominating $\tilde{Y}$, such that for the induced map $\tilde{f} : \tilde{X} \to \tilde{Y}$ and for every prime divisor $E \subseteq \tilde{X}$, if $\text{codim}_{\tilde{Y}} \tilde{f}(E) \geq 2$, then $E$ is $\tilde{X} \to X$ exceptional. Furthermore, it follows from the proof of [Vie83a, Lemma 7.3] that we may choose $\tilde{X} \to X$ and $\tilde{Y} \to Y$ to be isomorphisms over the smooth locus of $f$ on $Y$. Let $\rho : \tilde{X} \to X$ and $\tau : \tilde{Y} \to Y$ be the induced maps and set $\tilde{D} := \rho^*D$ and $\tilde{M} := \tau^*M$.

Claim 4. It is enough to prove that for some divisor $0 \leq B$ on $\tilde{X}$, for which $\text{codim}_{\tilde{Y}} \tilde{f}(B) \geq 2$ the following holds:

$$\kappa(K_{\tilde{X}/\tilde{Y}} + \tilde{D} + \tilde{f}^*\tilde{M} + B) \geq \max\{\kappa(\tilde{M}), \text{Var} \tilde{f}\} + \kappa(K_{\tilde{X}/\tilde{Y}} + \tilde{D}_\eta)$$

Proof of Claim 4. We have that $X_\eta = \tilde{X}_\eta$, $\kappa(K_{X_\eta} + D_\eta) = \kappa(K_{\tilde{X}_\eta} + \tilde{D}_\eta)$, $\text{Var} f = \text{Var} \tilde{f}$ and $\kappa(M) = \kappa(\tilde{M})$. Furthermore, note that since both $Y$ and $\tilde{Y}$ are smooth, there is an effective divisor $E$ on $\tilde{Y}$ such that $K_{\tilde{Y}} = \tau^*KY + E$. In particular, the following holds.

$$(8.7.5) \quad K_{\tilde{X}/\tilde{Y}} = K_{\tilde{X}} - \tilde{f}^*K_{\tilde{Y}} = K_{\tilde{X}} - \tilde{f}^*(\tau^*KY + E) = K_{\tilde{X}} - \rho^*f^*KY - \tilde{f}^*E$$

Since $B$ is $\rho$ exceptional and $\rho$ is birational, we obtain using (8.7.5) that for every $m > 0$ integer

$$\rho_*\mathcal{E}_{\tilde{X}}(m(K_{\tilde{X}/\tilde{Y}} + B)) \to \rho_*\mathcal{E}_{\tilde{X}}(m(K_{\tilde{X}} - \rho^*f^*KY + B)).$$

Furthermore, by construction, $\tilde{f}^*\tilde{M} + \tilde{D} = \rho^*(f^*M + D)$ and hence for every divisible enough $m > 0$ integer there is an injection

$$\rho_*\mathcal{E}_{\tilde{X}}(m(K_{\tilde{X}/\tilde{Y}} + \tilde{f}^*\tilde{M} + \tilde{D} + B)) \to \mathcal{E}_X(m(K_{X/Y} + D + f^*M))$$

This shows that $\kappa(K_{\tilde{X}/\tilde{Y}} + \tilde{D} + \tilde{f}^*\tilde{M} + B) \leq \kappa(K_{X/Y} + D + f^*M)$, which implies the claim.

From now on our goal is to prove that for some $0 \leq B$ for which $\text{codim}_{\tilde{Y}} \tilde{f}(B) \geq 2$,

$$(8.7.6) \quad \kappa(K_{X/Y} + D + f^*M + B) \geq \max\{\kappa(M), \text{Var} f\} + \kappa(K_{X_\eta} + D_\eta)$$

Step 2: Disallowing vertical components of $D$. If $D$ contains a vertical component, after deleting that component from $D$ our assumptions are still satisfied. In other words, we may assume that $D$ contains no vertical components.

Step 3: Replacing $\text{Var} f$ by $\text{Var} f_{\text{can}}$. Let $f_{\text{can}} : (X_{\text{can}}, D_{\text{can}}) \to Y$ be the log canonical model of $(X, D)$ over some dense open set $U \subseteq Y$ over which $(X, D)$ is klt. By shrinking $U$ we may assume that $f_{\text{can}}$ is a stable family. Note that if $(X, D)$ was klt to start with, then $\text{Var} f = \text{Var} f_{\text{can}}$ (where the latter is taken as the variation as a stable family of log-varieties). Hence, in order to obtain (8.7.6) it is enough to prove the following inequality:

$$(8.7.7) \quad \kappa(K_{X/Y} + D + f^*M + B) \geq \max\{\kappa(M), \text{Var} f_{\text{can}}\} + \kappa(K_{X_\eta} + D_\eta).$$

Step 4: An auxiliary base change. Set $n := \dim X - \dim Y$, $v := \text{vol}(K_{X_\eta} + D_\eta)$, where $\eta$ is the generic point of $Y$. Let $I \subseteq [0,1]$ be a finite coefficient set closed under addition (Definition 5.1) that contains the coefficients of $D$. Let $\mu : U \to \mathcal{M}_{n,v,I}$ be the moduli map associated to $(V_{\text{can}}, D_{\text{can}} \mid V_{\text{can}})$, where $V_{\text{can}} := f_{\text{can}}^{-1}(U)$ and let $S \to \mathcal{M}_{n,v,I}$ be the finite cover granted by Corollary 5.19. Define then $Y_{\text{aux}}$ to be the resolution of a compactification of a component of $U \times \mathbb{A}_{n,v,I}$ that dominates $U$. We may further assume that the birational map $\delta : Y_{\text{aux}} \to Y$ is a morphism. Let $Y''$ be the normalization of the image of $Y_{\text{aux}}$ in $S$ and $f'' : (X'', D'') \to Y''$ the family over $Y''$ induced by $f \in \mathcal{M}(S)$ given in Corollary 5.19. Then the pullback of this family over $\delta^{-1}(U)$ is isomorphic to the pullback of $(X_{\text{can}}, D_{\text{can}})$ and hence $\dim Y'' = \text{Var} f_{\text{can}}$. 

\[\quad\]
Step 5: Local stable reduction. That is, we construct a generically finite map $Y' \to Y^\text{aux}$ and a normal pair $(X', D')$, such that

- $Y'$ is smooth
- $X'$ maps birationally onto the component of $X \times_Y Y'$ dominating $Y'$,
- $X' \to X \times_Y Y'$ is an isomorphism over the generic point of $Y'$,
- if $\tau : Y' \to Y$ and $\rho : X' \to X$ are the induced maps, then $D' \leq \rho^* D$, and
- $(X', D')$ is a locally stable family at every $y \in Y'$ for which $\text{codim} \tau(y) = 1$, that is, at every such $y \in Y'$ the following two equivalent conditions hold:
  - $(X', \overline{X'_y} + D')$ is lc around $X'_y$, where $\overline{X'_y}$ is the closure of $X'_y$, or equivalently
  - $(X'_y, D'_y)$ is slc and $K_{X'} + D'$ is $\mathbb{Q}$-Cartier around $X'_y$.

To obtain the above, we apply the process described in [Kol14, first 6 paragraphs in the proof of Thm 12.11] to the fibers of $X \times_Y Y^\text{aux} \to Y^\text{aux}$ over $y \in Y^\text{aux}$ mapping to codimension one points of $Y$. That is, first we resolve the main component of $X \times_Y Y^\text{aux}$ to obtain $X^\text{nc}$ such that $X^\text{nc}$ is smooth, and if $D^\text{nc}$ is the horizontal part of the pullback of $D$ to $X^\text{nc}$, then $(X^\text{nc}, D^\text{nc} + X^\text{nc}_y)$ is a normal crossing pair around $X^\text{nc}_y$ for each $y \in Y^\text{aux}$ mapping to a codimension one point of $Y$. Then a generically finite cover of the base with prescribed ramifications at finitely many codimension one points (allowing further ramifications at unprescribed points) yields the required $Y'$ as in [Kol14, proof of Thm 12.11]. $X'$ is defined to be the normalization of the main component of $X^\text{nc} \times_Y Y^\text{aux} Y'$.

Step 6: Choosing nice big open sets. Let $f' : X' \to Y'$ be the natural morphism and let $Y_0 \subseteq Y$ be the big open set over which

1. $f'$ is flat over $\tau^{-1} Y_0$,
2. $(X'_0, D|_{X'_0})$ is klt and forms a flat locally stable family of log-varieties, where $X'_0 := f'^{-1} \tau^{-1} Y_0$.

Let $Y'_0 := \tau^{-1} Y_0$ and let $f'_\text{can} : (X'_\text{can}, D'_\text{can}) \to Y'_0$ be the log canonical model of $(X'_0, D|_{X'_0})$ over $Y'_0$. By shrinking $Y_0$ (and $Y'_0$ and $X'_0$ compatibly, keeping $Y_0$ big), we may further assume that

3. $f'_\text{can}$ is flat (and hence it is a family of stable log-varieties).

Let $\eta'$ be the generic point of $Y'$. Then we have $(X'_{\eta'}, D'_{\eta'}) \simeq (X_{\eta}, D_{\eta}|_{\eta'})$, since over the locus (in $Y$) over which $f$ is smooth and $(X, D)$ is a relative normal crossing divisor, $(X', D')$ is isomorphic to $(X, D) \times_Y Y'$. Therefore

$$(X'_{\text{can}}, D'_{\text{can}})_{\eta'} \simeq (X_{\text{can}}, D_{\text{can}})_{\eta'} \simeq (X'', D'')_{\eta'}.$$ 

That is, $(X'', D'') \times_Y Y'_0$ is isomorphic to $(X'_\text{can}, D'_\text{can})$ over $\eta'$. Equivalently, their Isom scheme has a rational point over $\eta'$. The closure of this rational point yields a section of the Isom scheme over a big open set of $Y'_0$. Hence, by further restricting $Y_0$, we may assume that

4. $(X'', D'') \times_Y Y'_0$ is isomorphic to $(X'_\text{can}, D'_\text{can})$.

Step 7: Bounding $\kappa \left( K_{X'_\text{can}/Y'_0} + D'_{\text{can}} + f'_{\text{can}}^* \pi^* M \right)$. According to Corollary 7.3, $K_{X''/Y''} + D''$ is big. In particular, there is an ample divisor $H$ and an effective divisor $E$ on $X''$, such that $H + E \sim q(K_{X''/Y''} + D'')$ for some $q > 0$ divisible enough. Let $\pi : X'_\text{can} \to X''$ be the induced map and let $V \subseteq |\pi^* H|$ be a linear system inducing $\pi$. Further let $W \subseteq |q f'_{\text{can}}^* \pi^* M|$ be the linear
system that identifies with $|qM|$ via the natural embedding $|qM| \hookrightarrow |qf_{\text{can}}^*\tau^*M|$.

We compute the dimension of a general fiber of $\phi_{V+W}$. For that choose an open set $U' \subseteq X'_{\text{can}}$, such that $\phi_{V+W}$ is a morphism over $U'$ and $\phi|_{qM}$ is a morphism over $\tau(f_{\text{can}}^*U')$. In the next few sentences, when computing fibers of $\phi_{V+W}$, $\phi_W$ and $\phi|_{qM}$, we take $U'$ and $\tau_s(f_{\text{can}}'(x))$ as the domain. So, choose $z \in Z$ and $x \in X''$ general. We have $\phi_{V+W}^{-1}(x, z) = \phi_W^{-1}(z) \cap \phi_{V}^{-1}(x)$. Furthermore, $\phi_W^{-1}(z)$ is of the form $f_{\text{can}}^{-1}(Z')$ for a variety $Z'$ of dimension $\dim Y - \kappa(M)$. On the other hand, $\phi_{V}^{-1}(x)$ intersects each fiber of $f_{\text{can}}'$ in at most one point and has dimension $\dim Y - \text{Var} f_{\text{can}}$. Therefore,

\begin{equation}
(8.7.8) \quad \dim \phi_{V+W}^{-1}(x, z) \leq \min\{\dim Y - \text{Var} f_{\text{can}}, \dim Y - \kappa(M)\}.
\end{equation}

Hence,

\begin{equation}
(8.7.9) \quad \kappa\left(\left(K_{X'_{\text{can}}/Y_0} + D_{\text{can}} + f_{\text{can}}^*\tau^*M\right) \geq \kappa\left(\pi^*H + qf_{\text{can}}^*\tau^*M\right) \geq \dim \phi_{V+W} \geq n + \dim Y - \min\{\dim Y - \text{Var} f_{\text{can}}, \dim Y - \kappa(M)\} = n + \max\{\text{Var} f_{\text{can}}, \kappa(M)\}.
\end{equation}

\begin{equation}
(8.7.8)
\end{equation}

**Step 8: Conclusion.** We use here the notations introduced in Lemma 8.8. First, note that that

\[ H^0\left(X'_0, q\left(K_{X'/Y_0} + D' + f_*^*\tau^*M\right) |_{X'_0}\right) = \kappa\left(X'_{\text{can}}, q\left(K_{X'_{\text{can}}/Y_0} + D_{\text{can}} + f_{\text{can}}^*\tau^*M\right)\right), \]

Hence, by (8.7.9) there is an effective divisor $B'$ on $X'$ supported in $X' \setminus X'_0$, such that

\begin{equation}
(8.7.10) \quad \kappa(K_{X'/Y'} + D' + f_*^*\tau^*M + B') \geq n + \max\{\text{Var} f_{\text{can}}, \kappa(M)\} : q
\end{equation}

Note that $K_{X'/Y'} + D' + f_*^*\tau^*M + B'$ might not be a $\mathbb{Q}$-Cartier divisor, so what we mean by the above statement is that for some $q > 0$ divisible enough $q$-times multiple of this divisor defines a rational map with image of dimension at least $n + \max\{\text{Var} f_{\text{can}}, \kappa(M)\}$.

Let $\gamma : X' \to X_n$ be the induced map and let $\xymatrix{X_n \ar[r]^-{\xi} & X'_n \ar[r]^-{\xi} & X}$ be the Stein-factorization of $\alpha \circ \beta$. Then

\begin{equation}
(8.7.11) \quad \kappa\left(\zeta^*(K_X + D + f^*M) + \xi_*\gamma_*B'\right) \geq \kappa\left(\beta^*\alpha^*(K_X + D + f^*M) + \gamma_*B'\right) \geq\n \end{equation}

\[ \geq \kappa\left(\left(K_{X_n/Y_0} + \beta^*\alpha^*D + \beta^*\alpha^*f^*M + \gamma_*B'\right) = \kappa\left(\left(\zeta_*\left(\left(K_{X'/Y'} + D' + f_*^*\tau^*M + B'\right)\right) \geq n + \max\{\text{Var} f_{\text{can}}, \kappa(M)\}\right)\right).\]

Choose now an effective divisor $B$ with support in $X \setminus f^{-1}Y_0$, such that $\zeta^*B \geq \xi_*\gamma_*B'$ (recall, $\xi_*\gamma_*B'$ is disjoint from $\zeta^{-1}f^{-1}Y_0$). By (8.7.11), for this choice of $B$

\[ \kappa(\zeta^*(K_X + D + f^*M + B)) \geq n + \max\{\text{Var} f_{\text{can}}, \kappa(M)\} = \kappa(K_{X_0} + D_{\eta}) + \max\{\text{Var} f_{\text{can}}, \kappa(M)\} \].

Hence, since Kodaira-dimension of a line bundle is invariant under finite pullback, (8.7.7) holds. \qed
9. **Almost Proper Bases**

**Lemma 9.1.** Consider the following commutative diagram of normal, irreducible varieties, where

1. \( \overline{Y} \) and \( \overline{Y}' \) are projective over \( k \)
2. \( \tau \) is generically finite,
3. \( Y' = \tau^{-1}Y \),
4. \( Y \) is a big open set of \( \overline{Y} \),
5. there are vector bundles \( \mathcal{F} \) and \( \mathcal{G} \) given on \( Y \) and \( \overline{Y}' \) respectively, such that \( \mathcal{G} \) is big and \( \tau^* \mathcal{F} \cong \mathcal{G}|_{Y'} \).

\[
\begin{array}{ccc}
Y' & \xrightarrow{\tau} & \overline{Y}' \\
\downarrow{\tau} & & \downarrow{\tau} \\
Y & \xrightarrow{\nu} & Z \\
\end{array}
\]

Then \( \mathcal{F} \) is big as well.

**Proof.** Choose ample line bundles \( \mathcal{K} \) and \( \mathcal{A} \) on \( \overline{Y} \) and \( \overline{Y}' \), respectively. Let \( b > 0 \) be an integers such that there is an injection \( \mathcal{K} \otimes \mathcal{A}^b \hookrightarrow \mathcal{A} \). Since \( \mathcal{G} \) is big, there is an integer \( a > 0 \) such that \( \text{Sym}^a(\mathcal{G}) \otimes \mathcal{A}^{-1} \) is generically globally generated. Hence, so is \( \text{Sym}^a(\mathcal{G}) \otimes \mathcal{A}^{-b} \). So, by the embedding \( \text{Sym}^a(\mathcal{G}) \otimes \mathcal{A}^{-b} \hookrightarrow \text{Sym}^a(\mathcal{G}) \otimes \mathcal{K}^{-1} \), the latter sheaf is generically globally as well. In particular, so is

\[
\text{Sym}^a(\mathcal{G}) \otimes \mathcal{K}^{-1}|_{Y'} \cong \text{Sym}^a(\tau^* \mathcal{F}) \otimes \tau^* \mathcal{K}|_{Y'}^{-1}
\]

Let

\[
\begin{array}{ccc}
Y' & \xrightarrow{\nu} & Z \\
\downarrow{\tau} & & \downarrow{\rho} \\
Y & &
\end{array}
\]

be the Stein factorization of \( \tau \). Then since \( \nu \) is birational,

\[
\nu_* \left( \text{Sym}^a(\tau^* \mathcal{F}) \otimes \tau^* \mathcal{K}|_{Y'}^{-1} \right) \cong \text{Sym}^a(\rho^* \mathcal{F}) \otimes \rho^* \mathcal{K}|_{Y}^{-1}
\]

is also generically globally generated. Then [VZ02, Lem 1.3] shows that \( \text{Sym}^a(\mathcal{F}) \otimes \mathcal{K}^{-1}|_{Y} \) is generically globally generated, and hence \( \mathcal{F} \) is big indeed.

Using Lemma 9.1 and Corollary 5.18 immediately follow versions of point (2) of Theorem 6.1 and of Theorem 7.1 for the almost projective base case.

**Corollary 9.2.** If \( f : (X,D) \to Y \) is a family of stable log-varieties of maximal variation over a normal almost projective variety, then

1. for every divisible enough \( q > 0 \), \( \det f_* \mathcal{O}_X(q(K_X + \Delta)) \) is big.
2. \( f_* \mathcal{O}_X(q(K_X/Y + D)) \) is big for every divisible enough integer \( q > 0 \), provided that \( (X,D) \) has klt general fibers over \( Y \).

**References**

[Abr97] D. Abramovich: A high fibered power of a family of varieties of general type dominates a variety of general type, Invent. Math. 128 (1997), no. 3, 481–494.

[AH11] D. Abramovich and B. Hassett: Stable varieties with a twist, Classification of algebraic varieties, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 1–38.

[Ale94] V. Alexeev: Boundedness and \( K^2 \) for log surfaces, Internat. J. Math. 5 (1994), no. 6, 779–810.

[Ale96] V. Alexeev: Moduli spaces \( M_{g,n}(W) \) for surfaces, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22.

[AT15] K. Ascher and A. Turchet: Work in progress, in preparation.
Log canonical singularities are Du Bois, J. Amer. Math. Soc. 23 (2010), no. 3, 791–813.

Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

Champs algébriques, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 39, Springer-Verlag, Berlin, 2000.

Positivity in algebraic geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48, Springer-Verlag, Berlin, 2004.

Positivity in algebraic geometry. II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 49, Springer-Verlag, Berlin, 2004.

Zariski-decomposition and abundance, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.

Fibered stable varieties, arXiv:1208.1787, to appear in the Transactions of the American Mathematical Society (2012).

Semi-positivity in positive characteristics, arXiv:1208.5391, to appear in Annales Scientifiques de l’École Normale Supérieure (2012).

On subadditivity of Kodaira dimension in positive characteristic, arXiv:1308.5371 (2013).

Depth of $F$-singularities and base change of relative canonical sheaves, J. Inst. Math. Jussieu 13 (2014), no. 1, 43–63.

Projectivity of CM line bundle on moduli space of canonically polarized varieties, in preparation.

Equivariant compactifications of reductive groups, Mat. Sb. 194 (2003), 119–146.

Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, Algebraic varieties and analytic varieties (Tokyo, 1981), Adv. Stud. Pure Math., vol. 1, North-Holland, Amsterdam, 1983, pp. 329–353.

Weak positivity and the additivity of the Kodaira dimension. II. The local Torelli map, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 567–589.

Quasi-projective moduli for polarized manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 30, Springer-Verlag, Berlin, 1995.

Base spaces of non-isotrivial families of smooth minimal models, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328.

Nonexistence of asymptotic GIT compactification, Duke Math. J. 163 (2014), no. 12, 2217–2241.

University of Washington, Department of Mathematics, Box 354350 Seattle, WA 98195-4350, USA

E-mail address: skovacs@uw.edu

Department of Mathematics, Princeton University, Fine Hall, Washington Road, NJ 08544-1000, USA

E-mail address: pzs@math.princeton.edu