On fixity of arc-transitive graphs

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Received May 27, 2020; accepted December 29, 2020; published online April 15, 2021

Abstract  The relative fixity of a permutation group is the maximum proportion of the points fixed by a non-trivial element of the group, and the relative fixity of a graph is the relative fixity of its automorphism group, viewed as a permutation group on the vertex-set of the graph. We prove in this paper that the relative fixity of connected 2-arc-transitive graphs of a fixed valence tends to 0 as the number of vertices grows to infinity. We prove the same result for the class of arc-transitive graphs of a fixed prime valence, and more generally, for any class of arc-transitive locally-\(L\) graphs, where \(L\) is a fixed quasiprimitive graph-restrictive permutation group.

Keywords  permutation group, fixity, minimal degree, graph, automorphism group, vertex-transitive, arc-transitive, fixed points

MSC(2020)  20B25

Citation:  Lehner F, Potočnik P, Spiga P. On fixity of arc-transitive graphs. Sci China Math, 2021, 64: 2603–2610, https://doi.org/10.1007/s11425-020-1825-1

1 Introduction

For a permutation group \(G\) acting on a finite set \(\Omega\) and an element \(g \in G\), let \(\Fix_\Omega(g) = \{\omega \in \Omega : \omega^g = \omega\}\) be the set of fixed points of \(g\), let

\[
fpr_\Omega(g) := \frac{|\Fix_\Omega(g)|}{|\Omega|}
\]

be the fixed-point-ratio of \(g\), and let

\[
\fix_\Omega(G) := \max\{|\Fix_\Omega(g)| : g \in G, g \neq \id_\Omega\}
\]

be the fixity and the relative fixity of \(G\), respectively.

Bounding the fixity of permutation groups has a long history, going back to a classical result of Jordan who proved that for every constant \(c\) apart from a finite list of exceptions (depending on \(c\)), every primitive permutation group \(G \leqslant \Sym(\Omega)\) not containing \(\Alt(\Omega)\) satisfies \(\fix_\Omega(G) \leqslant |\Omega| - c\). This result was later
improved by several authors (see, for example, Babai [1], Liebeck and Saxl [15], Saxl and Shalev [22] and Guralnick and Magaard [11]). Their work, among other results, amounts to a complete understanding of all the primitive groups $G \leqslant \text{Sym}(\Omega)$ with $\text{rfx}_G(G) > 1/2$. There are several results giving bounds on the fixity of transitive actions of almost simple groups (see, for example, [5,14,15]), but in general, not much is known about fixity of imprimitive permutation groups.

A natural relaxation of the primitivity condition appears in the theory of groups acting on graphs. Let $G$ be a transitive permutation group acting on a finite set $\Omega$ and let $\omega \in \Omega$. An orbit of the stabiliser $G_\omega$ in the set $\Omega \setminus \{\omega\}$ is then called a suborbit of $G$. Given a suborbit $\Sigma$ one can construct a so-called directed orbital graph whose vertex-set is $\Omega$ and the set of directed edges is $\{(\omega^\sigma, \sigma^\omega) : g \in G, \sigma \in \Sigma\}$. If this set of directed edges is invariant under the operation of interchanging the points in each ordered pair, then the suborbit is called self-paired and the directed orbital graph can be viewed as an undirected graph (called simply an orbital graph) upon which the group $G$ acts as a group of automorphisms acting transitively on the arcs (ordered pairs of adjacent vertices).

A remarkable observation of Higman [12] asserts that $G$ acts primitively on $\Omega$ if and only if each of its suborbits yields a connected directed orbital graph. With the existing results on fixity of primitive permutation groups in mind, it is now natural to ask to what extent these results carry over to the case where at least one directed orbital graph is connected. The class of permutation groups having such a suborbit is still too wide for any meaningful upper bound on the fixity. For example, the imprimitive wreath product of $C_n \wr \text{Sym}(m)$ acting on the Cartesian product $\Omega$ of a set of size $n$ and a set of size $m$ has a (non-self paired) suborbit of size $m$ yielding a connected directed orbital graph; namely, the lexicographic product of a directed cycle of length $n$ with an edgeless graph on $m$ vertices, in which the fixity is $|\Omega| - 2$.

As we shall show in this paper, the situation changes if a suborbit $\Sigma$ corresponding to a stabiliser $G_\omega$ in a transitive permutation group $G \leqslant \text{Aut}(\Omega)$ is self-paired and satisfies additional conditions either on its length and/or on the permutation group $G_\omega^\Sigma$ induced by the action of $G_\omega$ on $\Sigma$. The main result of the paper is the following (recall that the socle of a group is the subgroup generated by all the minimal normal subgroups):

**Theorem 1.1.** For every positive real number $\alpha$ there exists a constant $c_\alpha$ with the following property. Let $G$ be a transitive permutation group acting on a finite set $\Omega$, $|\Omega| > c_\alpha$, admitting a self-paired suborbit $\Sigma$ yielding a connected orbital graph, such that at least one of the following holds:

1. the cardinality of $\Sigma$ is a prime number;
2. $G_\omega^\Sigma$ is doubly-transitive;
3. $G_\omega^\Sigma$ is primitive with its socle acting regularly (i.e., $G_\omega^\Sigma$ is primitive of affine or twisted wreath type).

Then $\text{rfx}_G(G) < \alpha$.

In fact, we prove a slightly more general result, namely Theorem 3.1, which is stated in the graph theoretical language in Section 3 (where the proof of Theorem 1.1 can also be found). The fixity of the automorphism group of a graph was, to the best of our knowledge, first studied by Babai [2,3] and was motivated by the famous graph isomorphism problem (see [4]). In [7], a relationship between the fixity of the automorphism group and the distinguishing number of a graph was considered. The fixity of the automorphism group of vertex-transitive graphs of valence at most 4 was recently investigated in [18], and the present paper can be seen as a strengthening of the results proved there under additional assumptions on the group of automorphisms. Finally, we would also like to mention a very interesting line of research (see [6,13]), where the subgraphs induced by the sets of fixed points of automorphisms of cubic arc-transitive graphs were studied.

In what follows, we mostly use standard graph- and group-theoretical notation. In particular, a graph $\Gamma$ is determined by its (finite) vertex-set $V(\Gamma)$ and edge-set $E(\Gamma)$ consisting of unordered pairs of (distinct) adjacent vertices. All the graphs in this paper are finite and simple. An automorphism of a graph $\Gamma$ is by definition a permutation of $V(\Gamma)$ which, in its action on unordered pairs of elements of $V(\Gamma)$, preserves $E(\Gamma)$. The group of all the automorphisms of $\Gamma$ is denoted by $\text{Aut}(\Gamma)$. For a vertex $v$ of $\Gamma$, let $\Gamma(v)$ denote the neighbourhood of $v$. The image of $v$ under $g \in \text{Aut}(\Gamma)$ is denoted by $v^g$. For every
\( G \leq Aut(\Gamma) \) there are obvious induced actions of \( G \) on \( E(\Gamma) \), on the \textit{arc-set} \( A(\Gamma) := \{ (u, v) : \{ u, v \} \in E(\Gamma) \} \) of \( \Gamma \) and on the set \( A_2(\Gamma) := \{ (u, v, w) : \{ u, v, w \} \in E(\Gamma), u \neq w \} \) of 2-arcs of \( \Gamma \). If \( G \) is transitive on \( V(\Gamma), E(\Gamma), A(\Gamma) \) or \( A_2(\Gamma) \), then \( \Gamma \) is said to be \( G \)-vertex-transitive, \( G \)-edge-transitive, \( G \)-arc-transitive or \( (G,2) \)-arc-transitive, respectively; with the reference to \( G \) omitted when \( G = Aut(\Gamma) \).

For a permutation group \( G \) on a set \( \Omega \), let \( \Omega/G \) denote the set of all the \( G \)-orbits on \( \Omega \) and let \( G^+ \) be the group generated by all the point-stabilisers \( G_\omega, \omega \in \Omega \). Observe that \( G^+ \) is normal in \( G \), implying that \( \Omega/G^+ \) is a \( G \)-invariant partition of \( G \). For a set \( B \subseteq \Omega \), we let \( G_B = \{ g \in G : B^g = B \} \) be the set-wise stabiliser of \( B \) in \( G \). The centre of a group \( G \) will be denoted by \( Z(G) \). If \( g, x \in G \) we let \( g^x = x^{-1}gx \), write \( g^G = \{ g^x : x \in G \} \) and let \( C_G(g) = \{ x \in G : g^x = g \} \). A permutation group \( G \) is said to be \textit{quasiprimitive} provided that all the non-trivial normal subgroups of \( G \) (including \( G \) itself) are transitive.

2 Auxiliary results

In this section we prove a series of lemmas that are needed in the proof of Theorem 1.1. Some of them are standard and have appeared elsewhere in a similar form (such as Lemma 2.1), while some are, to the best of our knowledge, new and can be found interesting on its own (see, for example, Lemma 2.3).

**Lemma 2.1.** If \( G \leq Sym(\Omega) \) and \( g \in G \), then

\[
|\text{Fix}_\Omega(g)| \leq |C_G(g)||\Omega/G|.
\] (2.1)

Furthermore, if \( G \) is normal in a group acting transitively on \( \Omega \) and \( \omega \in \Omega \), then

\[
\text{fpr}_\Omega(g) \leq \frac{|G_\omega||C_G(g)|}{|G|}.
\] (2.2)

**Proof.** We prove the first inequality by double counting the set

\[
S = \{ (\delta, x) \mid \delta \in \Omega, x \in G, g^x \in G_\delta \}.
\]

By choosing first \( x \) and then \( \delta \) we see that

\[
|S| = \sum_{x \in G} |\{ \delta \in \Omega \mid g^x \in G_\delta \}| = \sum_{x \in G} |\text{Fix}(g^x)| = |\text{Fix}(g)||G|.
\] (2.3)

On the other hand,

\[
|S| = \sum_{\delta \in \Omega} |\{ x \in G \mid g^x \in G_\delta \}|.
\] (2.4)

Now let \( A_5 := \{ x \in G \mid g^x \in G_\delta \} \) and let \( \varphi : A_5 \to g^G \cap G_\delta \) be a function defined by \( \varphi(x) := g^x \). Let \( h \) be an arbitrary element of \( g^G \cap G_\delta \), and let \( \eta \in G \) satisfy that \( h = g^\eta \). Then the preimage \( \varphi^{-1}(h) \) consists of elements \( x \) such that \( g^\eta = g^x \), or equivalently, that \( xy^{-1} \in C_G(\eta) \). Hence \( \varphi^{-1}(h) = C_G(\eta)g \) and thus

\[
|A_5| = |g^G \cap G_\delta||C_G(g)|.
\]

By using (2.3) and (2.4), it follows that

\[
|\text{Fix}(g)||G| = |S| = \sum_{\delta \in \Omega} |g^G \cap G_\delta||C_G(g)| \leq |C_G(g)| \sum_{\delta \in \Omega} |G_\delta| = |C_G(g)||\Omega/G||G|,
\]

proving the inequality (2.1). If \( G \) is normal in a group acting transitively on \( \Omega \), then all of its orbits are of equal length and hence

\[
|\Omega| = |\omega^G||\Omega/G| = |G||\Omega/G||G_\omega|.
\]

The inequality (2.2) then follows by dividing the inequality (2.1) by \( |\Omega| \).

For a group \( X \) and an element \( g \in X \), let \( g^X \) denote the conjugacy class of \( g \) in \( X \) and let \( \langle g^X \rangle \) be the subgroup of \( X \) generated by all the elements of \( g^X \).

**Lemma 2.2.** There exists a strictly decreasing function \( f : [1, \infty) \to \mathbb{R}^+ \), \( \lim_{x \to \infty} f(x) = 0 \) with the following property: If \( g \) is an element of a transitive permutation group \( X \leq Sym(\Omega) \), \( \omega \in \Omega \) and \( G = \langle g^X \rangle \), then

\[
\text{fpr}_\Omega(g) \leq |G_\omega||\Omega : G| f(|G : Z(G)|).
\]
Proof. Observe first that $G$ is a normal subgroup of $X$. Now consider the action of $G$ on the conjugacy class $g^X$ by conjugation. The stabliser of a point $g' \in g^X$ is then the centraliser $C_G(g')$ and the kernel $K$ of this action consists of all the elements of $G$ that centralise every element of $g^X$. Since $G = \langle g^X \rangle$, it follows that $K = Z(G)$, implying that $G/Z(G)$ acts faithfully on $g^X$ and thus $|G/Z(G)| \leq |g^X|!$. On the other hand, 

$$|g^X| = \frac{|X|}{|C_X(g)|} = \frac{|X : G||C_G(g)|}{|C_X(g)|} \leq |X : G||G : C_G(g)|,$$

showing that $|G : Z(G)| \leq (|X : G||G : C_G(g)|)!$. In particular, by letting $f$ be the function mapping $x \in [1, \infty)$ to $\frac{1}{\Gamma(x)}$, where $\Gamma$ is the inverse of the Gamma function restricted to the interval $[2, \infty)$, we see that $f$ is a strictly decreasing function satisfying 

$$\frac{|C_G(g)|}{|G|} \leq |X : G|f(|G : Z(G)|).$$

The claim now follows from the inequality (2.2) of Lemma 2.1. \hfill $\Box$

**Lemma 2.3.** If $G$ is a transitive permutation group acting on a finite set $\Omega$, then 

$$\exp(G) \text{ divides } |G : Z(G)||\Omega/G^+|.$$

Proof. Observe first that $G^+$ is a normal subgroup of $G$, implying that $\Omega/G^+$ is a $G$-invariant partition of $\Omega$. Let $\omega$ be an arbitrary element of $\Omega$ and let $B = \omega^{G^+}$ be its $G^+$-orbit. Note that $(G_B)_{\omega} = G_{\omega} \leq G^+$. Since $G^+$, in its action on $B$, is a transitive subgroup of $G_B$, this implies that $G_B = G^+(G_B)_{\omega} = G^+$. Since $G_B$ is the stabiliser of the element $B$ in the induced action of $G$ on $\Omega/G^+$, this implies that the kernel of this action is $G^+$, and that the induced faithful action of $G/G^+$ on $\Omega/G^+$ is semiregular. In particular, $|G/G^+|$ divides $|\Omega/G^+|$.

Since the stabiliser $Z(G)_{\omega}$ of a vertex $\omega \in \Omega$ is a normal subgroup of $G$ contained in $G_{\omega}$, it follows that $Z(G)_{\omega} = 1$. Let $t = |G : Z(G)|$ and let 

$$\tau : G \to Z(G), \ x \mapsto x^t.$$

Then $\tau$ is a well-defined group homomorphism (see, for example, [21, Corollary 7.48]). Let $K = \text{Ker}(\tau)$. Since $|Z(G)G_{\omega}/Z(G)| = |G_{\omega}|$, we see that the order of $G_{\omega}$ divides $t$, and hence $G_{\omega}$ is a subgroup of $K$, implying that $G^+ \leq K$. Hence $\exp(N)$ divides $t$. The result now follows by the fact that $\exp(G)$ divides $\exp(G^+)\exp(G/G^+)$ and that $|G : G^+|$ divides $|\Omega/G^+|$.

A group $G$ acts semiregularly on a set $\Omega$ provided that $G_{\omega} = 1$ for every $\omega \in \Omega$. We call the cardinality of a smallest generating set of a group $G$ the rank of $G$ and denote it by rank($G$). The following lemma can be proved in many ways and we choose to use the tools from the theory of graph covers as described in [17] (or see [16] for a more succinct explanation of the theory).

**Lemma 2.4.** If $\Gamma$ is a connected graph and $G$ is a group of automorphisms of $\Gamma$ acting semiregularly on $V(\Gamma)$, then $\text{rank}(G) \leq |E(\Gamma)/G| - |V(\Gamma)/G| + 1$.

Proof. For the purpose of this proof we shall use a more general notion of a graph, namely one that allows parallel edges, loops and even semiedges (see [17, Section 3] or [16, Subsection 2.1] for exact definitions). Let $\Gamma' := \Gamma/G$ be the quotient graph of $\Gamma$ with respect to $G$ as defined in [16, Subsection 2.2]. What follows mimics the classical approach of the theory of graph covers of topological spaces with a small modification which is needed due to the possible existence of semiedges in the $\Gamma'$ which arise from the edge-reversing elements in $G$.

Since $G$ acts semiregularly on $V(\Gamma)$, the corresponding quotient projection $\varphi_G : \Gamma \to \Gamma'$ is a regular covering projection. The group of covering transformations (which is defined as the group of automorphisms of $\Gamma'$ preserving each fibre $\varphi_G^{-1}(x)$ where $x$ is either a vertex or a dart of $\Gamma'$) then equals the group $G$. By the definition of the quotient graph, we have $V(\Gamma') = V(\Gamma)/G$ and $E(\Gamma') = E(\Gamma)/G$.

Let $\pi(\Gamma', b)$ be the fundamental group based at a vertex $b$ of $\Gamma$, as defined in [17, Section 3] (or [16, Subsection 2.1]). Then (see [17, Section 3]) $\pi(\Gamma', b)$ is isomorphic to the free product of $m$ copies of $\mathbb{Z}_2$.
(where \(m\) equals the number of semiedges in \(\Gamma'\)) and \(\ell\) copies of \(\mathbb{Z}\). Moreover, \(m + \ell\) equals the Betti number of \(\Gamma'\), which equals the number of cotree edges in \(\Gamma'\) with respect to an arbitrary spanning tree of \(\Gamma'\). Hence \(\text{rank}(\pi(\Gamma'; b)) \leq m + \ell = |E(\Gamma')| - |V(\Gamma')| + 1\).

Furthermore, by using the procedure described in [16, Subsection 2.3], a homomorphism \(\zeta: \pi(\Gamma', b) \to G\) (called the voltage assignment) can be found which allows one to reconstruct the graph \(\Gamma\) from \(\Gamma', G\) and \(\zeta\) as the derived covering graph with respect to the locally transitive Cayley voltage space \((N; \zeta)\). One can easily see that the derived covering graph is connected if and only if the corresponding homomorphism \(\zeta: \pi(\Gamma', b) \to G\) is surjective. Since \(\Gamma\) is assumed to be connected, this then implies that \(\text{rank}(G) \leq \text{rank}(\pi(\Gamma'; b))\) and the result follows.

Proof. Let \(\Omega\) be an orbit of \(G\) in its action on \(\text{Aut}(\Gamma)\). Suppose first that \(\Omega \neq V(\Gamma)\). Then \(\Gamma\) is bipartite, \(\Omega\) is a part of the bipartition of \(\Gamma\) and \(G = G^+\). In particular, \(\epsilon = 0\) and \(|\Omega/G^+| = 1\). By assumption, the action of \(G\) on \(\Omega\) is faithful and hence \(G\) can be viewed as a transitive permutation group of \(\Omega\). By Lemma 2.3, it follows that \(\text{exp}(G)\) divides \(|G : Z(G)|\).

Suppose now that \(\Omega = V(\Gamma)\). Then \(G\) is arc-transitive and \(G^+\) has at most 2 orbits on \(\Omega\). Lemma 2.3 then yields that \(\text{exp}(G)\) divides \(2|G : Z(G)|\). Moreover, if \(\epsilon = 0\), then \(\Gamma\) is not bipartite, and thus \(|\Omega/G^+| = 1\). But then \(\text{exp}(G)\) divides \(|G : Z(G)|\), as claimed.

If \(\Gamma\) is a connected graph and \(G \leq \text{Aut}(\Gamma)\) such that for every vertex \(v \in V(\Gamma)\) the group \(G_v^{\Gamma(v)}\) is quasiprimitive (and thus transitive), then we say that \(\Gamma\) is \(G\)-locally quasiprimitive. Note that such a graph is automatically \(G\)-locally-arc-transitive. The following lemma is folklore, but for the sake of completeness we provide the proof.

Lemma 2.6. Let \(\Gamma\) be a connected \(G\)-locally quasiprimitive graph. If \(G\) acts unfaithfully on one of its orbits on \(V(\Gamma)\), then \(\Gamma\) is a complete bipartite graph.

Proof. Let \(\Omega\) be an orbit of \(G\) in its action on \(V(\Gamma)\). If the action of \(G\) on \(\Omega\) is not faithful, then \(V(\Gamma) \neq \Omega\) and hence \(\Gamma\) is bipartite and \(\Omega\) is one of the two sets of the partition of \(\Gamma\) with the other set of the bipartition being the second orbit of \(G\). Let \(K\) be the kernel of the action of \(G\) on \(\Omega\). Since \(K \neq 1\), there is a vertex \(u\) such that \(u^K \neq \{u\}\). Let \(v\) be a neighbour of \(u\). Since \(K\) is a normal subgroup of \(G_v\) and since \(K\) acts non-trivially on \(\Gamma(v)\), it follows that \(K\) is transitive on \(\Gamma(v)\) and hence \(\Gamma(v) = u^K\). But then \(\Gamma(u') = \Gamma(u)\) for every \(u' \in \Gamma(v)\). Consequently, the neighbourhood \(\Gamma(u')\) of every vertex \(u' \in \Gamma(u)\) contains \(\Gamma(v)\) and since \(v\) and \(v'\) are in the same \(G\)-orbit, this implies that \(\Gamma(u) = \Gamma(v')\). This shows that every walk starting in \(v\) never leaves the set \(\Gamma(v) \cup \Gamma(u)\). Since \(\Gamma\) is connected, this implies that \(V(\Gamma) = \Gamma(v) \cup \Gamma(u)\) and thus \(\Gamma\) is complete bipartite.

Lemma 2.7. There exists an unbounded strictly increasing function \(F: \mathbb{R}^+ \to \mathbb{R}^+\) such that for every connected \(G\)-locally-quasiprimitive graph \(\Gamma\) not isomorphic to a complete bipartite graph the following inequality holds:

\[ |G : Z(G)| \geq F(|G|). \]

Proof. Let \(\Gamma\) be a connected \(G\)-locally-quasiprimitive graph not isomorphic to a complete bipartite graph and let \(Z = Z(G)\). Since

\[ |G| = |G : Z||Z|, \quad (2.5) \]

it suffices to bound \(|Z|\) above in terms of \(|G : Z|\). Since \(\Gamma\) is not a complete bipartite graph, it follows from Lemma 2.6 that \(G\) acts faithfully on each of its orbits. By Corollary 2.5, it follows that \(\text{exp}(G) \leq 2|G : Z|\),
and since \( \exp(Z) \leq \exp(G) \), we see that
\[
\exp(Z) \leq 2|G : Z|.
\]

We will now establish an upper bound on the rank of \( Z \). Since \( G \) acts faithfully on each of its orbits, the vertex-stabiliser \( Z_v \) is trivial for every \( v \in V(\Gamma) \) and thus Lemma 2.4 applies. In particular,
\[
\text{rank}(Z) \leq |E(\Gamma)/Z| - |V(\Gamma)/Z| + 1 \leq |E(\Gamma)/Z|.
\]
Furthermore, since \( G \) acts transitively on \( E(\Gamma) \), it follows that \( G/Z \) acts transitively on \( E(\Gamma)/Z \), implying that
\[
\text{rank}(Z) \leq |E(\Gamma)/Z| \leq |G : Z|.
\]

By combining (2.5)–(2.7), we thus obtain
\[
|G| = |G : Z||Z| \leq |G : Z|\exp(Z)^\text{rank}(Z) \leq |G : Z|(2|G : Z|)^{|G : Z|}.
\]

Let \( F \) be the inverse of the (strictly increasing and bijective) function \( \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto x(2x)^2 \). The result now follows by applying the function \( F \) on both sides of the inequality (2.8).

\[\Box\]

3 Application to arc-transitive graphs

In this section we formulate and prove the main result of this paper (from which Theorem 1.1 follows easily). The formulation of the theorem is rather technical and uses the notion of locally-quasiprimitive group actions on graphs (introduced by Praeger [20]), and the notion of graph-restrictive permutation groups (introduced by Verret [26]), which can be defined as follows.

Let \( \Gamma \) be a \( G \)-vertex-transitive graph. If the group \( G^{\Gamma(v)}_v \), induced by the action of the vertex-stabiliser \( G_v \) on \( \Gamma(v) \), is permutation isomorphic to some permutation group \( L \), then we say that \( G \) is locally-\( L \). Similarly, if \( G^{\Gamma(v)}_v \) is a quasiprimitive permutation group, then we say that \( G \) is locally quasiprimitive. Following [26], we say that a transitive permutation group \( L \) is graph-restrictive provided there exists a constant \( c = c(L) \) such that whenever \( G \) is an arc-transitive, locally \( L \) group of automorphisms of a graph \( \Gamma \), the order of the stabiliser \( G_v \) is at most \( c(L) \).

**Theorem 3.1.** For every quasiprimitive and graph-restrictive permutation group \( L \) and every positive constant \( \alpha \) there exists an integer \( N_{L,\alpha} \) with the following property: If \( \Gamma \) is a connected \( X \)-arc-transitive graph with \( |V(\Gamma)| > N_{L,\alpha} \) and if \( X^{\Gamma(v)}_v \) is permutation isomorphic to \( L \) for every vertex \( v \), then
\[
\text{fpr}_{V(\Gamma)}(g) < \alpha
\]
for every nontrivial element \( g \) of \( X \).

**Remark 3.2.** Determining which transitive permutation groups are graph-restrictive is a classical topic in algebraic graph theory, going back to Tutte, who showed in [25] that the symmetric group of degree 3 is graph-restrictive with the corresponding constant being 48. Similarly, it can be deduced from the work of Gardiner [8] that the alternating group \( A_4 \) and the symmetric group \( S_4 \) (both of degree 4) are graph restrictive with corresponding constants \( c(A_4) = 36 \) and \( c(S_4) = 2^43^6 \). In [28, Conjecture 3.12], Weiss conjectured that every primitive permutation group is graph-restrictive. Weiss’s conjecture was later strengthened by Praeger [20], conjecturing that every quasiprimitive permutation group is graph-restrictive. Even though both conjectures are still open, one can deduce from the work of Weiss and Trofimov that every doubly transitive group is graph-restrictive (see [30, Introduction] or [19, Theorem 6]). The proof of this fact can be found by putting together pieces from many papers, but a nice summary is given in the introduction to a later paper by Weiss [30]. Together with another result of Weiss [29], this also implies that every permutation group of prime degree is graph-restrictive. In [23, 24], Spiga proved that every primitive permutation group of affine type or of twisted wreath type is graph restrictive (recall that a primitive permutation group is of affine type provided that it
contains a non-trivial abelian normal subgroup and of twisted wreath type if its socle is non-abelian and acts regularly); in short, primitive permutation groups whose socle (group generated by all the minimal normal subgroups) acts regularly on the points are graph-restrictive.

Other examples of graph-restrictive groups can be found in [9,10,26,27], and a summary of all (at that time) the known graph-restrictive groups is given in [19]. However, to deduce Theorem 1.1 from Theorem 3.1, all that needs to be remembered is that doubly-transitive permutation groups, primitive permutation groups of affine or twisted wreath type and transitive permutation groups of prime degree are graph-restrictive.

The rest of the section is devoted to proving Theorem 3.1, and to a deduction of Theorem 1.1 from Theorem 3.1. Let $L$ be a quasiprimitive and graph-restrictive permutation group. If the degree of $L$ (and thus the valence of $\Gamma$) is 1 or 2, then the result clearly holds. We may thus assume that the degree of $L$ is at least 3.

Let $\alpha > 0$ and let $\Gamma$ be a connected $X$-arc-transitive graph with $X_v$ permutation isomorphic to $L$, satisfying

$$f_{\text{pr}_{V(\Gamma)}}(g) \geq \alpha$$  

for some $g \in X \setminus \{1_X\}$. Let $c := c_L$ be the constant associated with graph-restrictive group $L$. Then $|X_v| \leq c$.

We need to show that $|V(\Gamma)|$ is bounded above by some constant $N$ depending only on $L$ and $\alpha$. Without loss of generality we may assume that $\Gamma$ is not a complete bipartite graph and moreover that $f_{\text{pr}_{V(\Gamma)}}(g) \geq \alpha$ implies that $g$ fixes at least one vertex of $\Gamma$. Since $\Gamma$ is connected and $g$ is a nontrivial automorphism, it then follows that there exists a vertex $v \in V(\Gamma)$ fixed by $g$, such that $g$ acts nontrivially on the neighbourhood $\Gamma(v)$.

Let $G = (g^X)$. Then $G$ is normal in $X$, implying that $G_v$ is a normal subgroup of $X_v$. Since $g \in G_v$, it follows that $G_v$ acts non-trivially on $\Gamma(v)$. Since $X_v^{\Gamma(v)}$ is quasiprimitive, this implies that $G_v$ acts transitively on $\Gamma(v)$. Moreover, since $G$ is normal in a vertex-transitive group $X$, it follows that $G_u$ is transitive on $\Gamma(u)$ for every $u \in V(\Gamma)$, i.e., $\Gamma$ is $G$-locally arc-transitive. Since $G \leq X$, it follows that

$$|G_v| \leq c.$$  

(3.2)

Moreover, since $G$ has at most 2 orbits on $V(\Gamma)$, it follows that

$$|X : G| = \frac{|V(\Gamma)||X_v|}{|G_v||G_v|} \leq 2|X_v : G_v| \leq |X_v| \leq c.$$  

(3.3)

By Lemma 2.7 we also see that

$$|G : Z(G)| \geq F(|G|)$$  

(3.4)

for some fixed unbounded strictly increasing function $F : \mathbb{R}^+ \to \mathbb{R}^+$, and by Lemma 2.2 it follows that

$$f_{\text{pr}_{V(\Gamma)}}(g) \leq |G_w||X : G| f(|G : Z(G)|)$$  

(3.5)

for some fixed strictly decreasing function $f : [1, \infty) \to \mathbb{R}^+$ such that $\lim_{x \to \infty} f(x) = 0$. Combining (3.2)–(3.5), we see that

$$\alpha \leq f_{\text{pr}_{V(\Gamma)}}(g) \leq c^2 \varphi(|G|),$$

where $\varphi := f \circ F$ is a strictly decreasing function such that $\lim_{x \to \infty} \varphi(x) = 0$. By dividing by $c^2$ and applying the inverse of $\varphi$, one thus concludes that

$$|G| \leq \varphi^{-1}(\alpha/c^2).$$

Since $|V(\Gamma)| \leq 2|G|/|G_v| \leq |G|$, this yields an upper bound $N_{L,\alpha} := \varphi^{-1}(\alpha/c^2)$ for $|V(\Gamma)|$ which depends only on $\alpha$ and $L$. In particular, if $|V(\Gamma)| > N_{L,\alpha}$, then the assumption (3.1) must be false. Hence $|V(\Gamma)| \leq N_{L,\alpha}$. This finishes the proof of Theorem 3.1.
Theorem 1.1 now follows easily from Theorem 3.1 and Remark 3.2. Indeed, let $\alpha$ be a positive constant, let $G$ be a transitive permutation group acting on a finite set $\Omega$, let $\omega \in \Omega$ and let $\Sigma = \delta G_{\omega}$ be a self-paired suborbit yielding a connected orbital graph $\Gamma$. Then $V(\Gamma) = \Omega$, $E(\Gamma) = \{ \{ \omega^g, \delta g \} : g \in G \}$ and $\Gamma$ is a $G$-arc-transitive graph of the valence $|\Sigma|$. Suppose in addition that either $|\Sigma|$ is a prime number or that the permutation group $G_{\Sigma}$ induced by the action of $G_{\omega}$ on $\Sigma$ is doubly-transitive or primitive of affine type. Then $L := G_{\Sigma}$ is clearly a primitive permutation group and in view of Remark 3.2, it is also graph-restrictive. By Theorem 3.1, there exists a constant $c_\alpha := N_{L,\alpha}$ such that $f_{pr}(V(\Gamma))(g) < \alpha$ for every $g \in G \setminus \{1\}$. In particular, $\text{rfx}_\Omega(G) \leq \alpha$, thus proving Theorem 1.1.

Acknowledgements The first author was supported by the Austrian Science Fund (FWF) Project W1230-N13. The second author was supported by the Research Programme P1-0294 and the Research Project J1-1691, both funded by the Slovenian Research Agency (ARRS).

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