A NOTE ON FEEBLY COMPACT SEMITOPOLOGICAL SYMMETRIC INVERSE SEMIGROUPS OF A BOUNDED FINITE RANK

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Abstract. We study feebly compact shift-continuous $T_1$-topologies on the symmetric inverse semigroup $\mathcal{S}_\lambda^n$ of finite transformations of the rank $\leq n$. It is proved that such $T_1$-topology is sequentially precompact if and only if it is feebly compact. Also, we show that every shift-continuous feebly $\omega$-bounded $T_1$-topology on $\mathcal{S}_\lambda^n$ is compact.

1. Introduction and preliminaries

We follow the terminology of the monographs [4,6,10,29,32,33]. If $X$ is a topological space and $A \subseteq X$, then by $\text{cl}_X(A)$ and $\text{int}_X(A)$ we denote the topological closure and interior of $A$ in $X$, respectively. By $|A|$ we denote the cardinality of a set $A$, by $A \Delta B$ the symmetric difference of sets $A$ and $B$, by $\mathbb{N}$ the set of positive integers, and by $\omega$ the first infinite cardinal. By $\mathcal{D}(\omega)$ and $\mathbb{R}$ we denote an infinite countable discrete space and the real numbers with the usual topology, respectively.

A semigroup $S$ is called inverse if every $a$ in $S$ possesses an unique inverse $a^{-1}$, i.e. if there exists an unique element $a^{-1}$ in $S$ such that

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$ 

A map which associates to any element of an inverse semigroup its inverse is called the inversion.

If $S$ is a semigroup, then by $E(S)$ we denote the subset of all idempotents of $S$. On the set of idempotents $E(S)$ there exists a natural partial order: $e \leq f$ if and only if $ef = fe = e$. A semilattice is a commutative semigroup of idempotents. We observe that the set of idempotents of an inverse semigroup is a semilattice [34].

Every inverse semigroup $S$ admits a partial order:

$$a \preceq b \quad \text{if and only if there exists} \quad e \in E(S) \quad \text{such that} \quad a = eb.$$ 

We shall say that $\preceq$ is the natural partial order on $S$ (see [4,34]).

Let $\lambda$ be an arbitrary nonzero cardinal. A map $\alpha$ from a subset $D$ of $\lambda$ into $\lambda$ is called a partial transformation of $\lambda$. In this case the set $D$ is called the domain of $\alpha$ and is denoted by $\text{dom} \alpha$. The image of an element $x \in \text{dom} \alpha$ under $\alpha$ is denoted by $x \alpha$. Also, the set $\{x \in \lambda: y \alpha = x \text{ for some } y \in Y\}$ is called the range of $\alpha$ and is denoted by $\text{ran} \alpha$. For convenience we denote by $\emptyset$ the empty transformation, a partial mapping with $\text{dom} \emptyset = \text{ran} \emptyset = \emptyset$.

Let $\mathcal{I}_\lambda$ denote the set of all partial one-to-one transformations of $\lambda$ together with the following semigroup operation:

$$x(\alpha \beta) = (x \alpha) \beta \quad \text{if} \quad x \in \text{dom}(\alpha \beta) = \{y \in \text{dom} \alpha: y \alpha \in \text{dom} \beta\}, \quad \text{for} \quad \alpha, \beta \in \mathcal{I}_\lambda.$$ 

The semigroup $\mathcal{I}_\lambda$ is called the symmetric inverse semigroup over the cardinal $\lambda$ (see [6]). For any $\alpha \in \mathcal{I}_\lambda$ the cardinality of $\text{dom} \alpha$ is called the rank of $\alpha$ and it is denoted by $\text{rank} \alpha$. The symmetric inverse semigroup was introduced by V. V. Wagner [34] and it plays a major role in the theory of semigroups.

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Put $\mathcal{I}_\lambda^n = \{\alpha \in \mathcal{I}_\lambda : \text{rank} \alpha \leq n\},$ for $n = 1, 2, 3, \ldots$. Obviously, $\mathcal{I}_\lambda^n$ ($n = 1, 2, 3, \ldots$) are inverse semigroups, $\mathcal{I}_\lambda^n$ is an ideal of $\mathcal{I}_\lambda$, for each $n = 1, 2, 3, \ldots$. The semigroup $\mathcal{I}_\lambda^n$ is called the symmetric inverse semigroup of finite transformations of the rank $\leq n$ [21]. By

$$\left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{array} \right)$$

we denote a partial one-to-one transformation which maps $x_1$ onto $y_1$, $x_2$ onto $y_2$, $\ldots$, and $x_n$ onto $y_n$. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$ ($i, j = 1, 2, 3, \ldots, n$). The empty partial map $\varnothing : \lambda \rightarrow \lambda$ is denoted by $\mathbf{0}$. It is obvious that $\mathbf{0}$ is zero of the semigroup $\mathcal{I}_\lambda^n$.

Let $\lambda$ be a nonzero cardinal. On the set $B_\lambda = (\lambda \times \lambda) \cup \{0\}$, where $0 \notin \lambda \times \lambda$, we define the semigroup operation “·” as follows

$$(a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c, \end{cases}$$

and $(a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0$ for $a, b, c, d \in \lambda$. The semigroup $B_\lambda$ is called the semigroup of $\lambda \times \lambda$-matrix units (see [6]). Obviously, for any cardinal $\lambda > 0$, the semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ is isomorphic to $\mathcal{I}_\lambda^1$.

A subset $A$ of a topological space $X$ is called regular open if $\text{int}_X(\text{cl}_X(A)) = A$.

We recall that a topological space $X$ is said to be

- semiregular if $X$ has a base consisting of regular open subsets;
- compact if each open cover of $X$ has a finite subcover;
- sequentially compact if each sequence $\{x_i\}_{i \in \mathbb{N}}$ of $X$ has a convergent subsequence in $X$;
- countably compact if each open countable cover of $X$ has a finite subcover;
- $H$-closed if $X$ is a closed subspace of every Hausdorff topological space in which it is contained;
- $\omega$-bounded-pracompact if $X$ contains a dense subset $D$ such that each countable subset of $D$ has the compact closure in $X$ [20];
- infra $H$-closed provided that any continuous image of $X$ into any first countable Hausdorff space is closed (see [27]);
- totally countably pracompact if there exists a dense subset $D$ of the space $X$ such that each sequence of points of the set $D$ has a subsequence with the compact closure in $X$ [20];
- sequentially pracompact if there exists a dense subset $D$ of the space $X$ such that each sequence of points of the set $D$ has a convergent subsequence [20];
- countably compact at a subset $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point $x$ in $X$ [1];
- countably pracompact if there exists a dense subset $A$ in $X$ such that $X$ is countably compact at $A$ [1];
- feebly $\omega$-bounded if for each sequence $\{U_n\}_{n \in \mathbb{N}}$ of nonempty open subsets of $X$ there is a compact subset $K$ of $X$ such that $K \cap U_n \neq \emptyset$ for each $n$ [20];
- selectively sequentially feebly compact if for every family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of $X$, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence ( [8]);
- sequentially feebly compact if for every family $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of $X$, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood $W$ of $x$ (see [9]);
- selectively feebly compact for each sequence $\{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of $X$, one can choose a point $x \in X$ and a point $x_n \in U_n$ for each $n \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : x_n \in W\}$ is finite for every open neighborhood $W$ of $x$ ( [8]);
- feebly compact (or lightly compact) if each locally finite open cover of $X$ is finite [3];
- $d$-feebly compact (or DFCC) if every discrete family of open subsets in $X$ is finite (see [31]);
- pseudocompact if $X$ is Tychonoff and each continuous real-valued function on $X$ is bounded;
- $Y$-compact for some topological space $Y$, if $f(X)$ is compact, for any continuous map $f : X \rightarrow Y$. 

According to Theorem 3.10.22 of [10], a Tychonoff topological space $X$ is feebly compact if and only if $X$ is pseudocompact. Also, a Hausdorff topological space $X$ is feebly compact if and only if every locally finite family of nonempty open subsets of $X$ is finite. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, every countably pracompact space is feebly compact (see [1]), every $H$-closed space is feebly compact too (see [19]). Also, every space feebly compact is infra $H$-closed by Proposition 2 and Theorem 3 of [27]. Using results of other authors we get that the following diagram which describes relations between the above defined classes of topological spaces.

A topological (semitopological) semigroup is a topological space together with a continuous (separately continuous) semigroup operation. If $S$ is a semigroup and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a semitopological semigroup, then we shall call $\tau$ a shift-continuous topology on $S$. An inverse topological semigroup with the continuous inversion is called a topological inverse semigroup.

Topological properties of an infinite (semi)topological semigroup $\lambda \times \lambda$-matrix units were studied in [15, 17]. In [15] it was shown that on the infinite semitopological semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ there exists a unique compact shift-continuous Hausdorff topology $\tau_e$ and also it is shown that every pseudocompact Hausdorff shift-continuous topology $\tau$ on $B_\lambda$ is compact. Also, in [15] it is proved that
every nonzero element of a Hausdorff semitopological semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) is an isolated point in the topological space \( B_\lambda \). In [15] it is shown that the infinite semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) cannot be embedded into a compact Hausdorff topological semigroup, every Hausdorff topological inverse semigroup \( S \) that contains \( B_\lambda \) as a subsemigroup, contains \( B_\lambda \) as a closed subsemigroup, i.e., \( B_\lambda \) is algebraically complete in the class of Hausdorff topological inverse semigroups. This result in [14] is extended onto so called inverse semigroups with tight ideal series and, as a corollary, onto the semigroup \( \mathcal{S}_\lambda^s \). Also, in [21] it was proved that for every positive integer \( n \) the semigroup \( \mathcal{S}_\lambda^n \) is algebraically h-complete in the class of Hausdorff topological inverse semigroups, i.e., every homomorphic image of \( \mathcal{S}_\lambda^n \) is algebraically complete in the class of Hausdorff topological inverse semigroups. In the paper [22] this result is extended onto the class of Hausdorff semitopological inverse semigroups and it is shown therein that for an infinite cardinal \( \lambda \) the semigroup \( \mathcal{S}_\lambda^n \) admits a unique Hausdorff topology \( \tau_c \) such that \((\mathcal{S}_\lambda^n, \tau_c)\) is a compact semitopological semigroup. Also, it was proved in [22] that every countably compact Hausdorff shift-continuous topology \( \tau \) on \( B_\lambda \) is compact. In [17] it was shown that a topological semigroup of finite partial bijections \( \mathcal{S}_\lambda^n \) with a compact subsemigroup of idempotents is absolutely H-closed (i.e., every homomorphic image of \( \mathcal{S}_\lambda^n \) is algebraically complete in the class of Hausdorff topological semigroups) and any Hausdorff countably compact topological semigroup does not contain \( \mathcal{S}_\lambda^n \) as a subsemigroup for an arbitrary infinite cardinal \( \lambda \) and any positive integer \( n \). In [17] there were given sufficient conditions onto a topological semigroup \( \mathcal{S}_\lambda^n \) to be non-H-closed. Also in [11] it is proved that an infinite semitopological semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) is H-closed in the class of semitopological semigroups if and only if the space \( B_\lambda \) is compact. In the paper [12] we studied feebly compact shift-continuous \( T_1 \)-topologies on the semigroup \( \mathcal{S}_\lambda^n \). For any positive integer \( n \geq 2 \) and any infinite cardinal \( \lambda \) a Hausdorff countably pracoompact non-compact shift-continuous topology on \( \mathcal{S}_\lambda^n \) is constructed there. In [12] it is shown that for an arbitrary positive integer \( n \) and an arbitrary infinite cardinal \( \lambda \) for a shift–continuous \( T_1 \)-topology \( \tau \) on \( \mathcal{S}_\lambda^n \) the following conditions are equivalent: 

(i) \( \tau \) is countably pracoompact; 
(ii) \( \tau \) is feebly compact; 
(iii) \( \tau \) is d-feebly compact; 
(iv) \((\mathcal{S}_\lambda^n, \tau)\) is H-closed; 
(v) \((\mathcal{S}_\lambda^n, \tau)\) is \( \mathcal{D}(\omega) \)-compact; 
(vi) \((\mathcal{S}_\lambda^n, \tau)\) is \( \mathbb{R} \)-compact; 
(vii) \((\mathcal{S}_\lambda^n, \tau)\) is infra H-closed. 

Also in [12] we proved that for an arbitrary positive integer \( n \) and an arbitrary infinite cardinal \( \lambda \) every shift-continuous semiregular feebly compact \( T_1 \)-topology \( \tau \) on \( \mathcal{S}_\lambda^n \) is compact. Similar results were obtained for a semitopological semilattice \((\exp_\lambda \lambda, \cap)\) in [23–25]. Also, in [26, 30] it is proved that feeble compactness implies compactness for semitopological bicyclic extensions.

In this paper we study feebly compact shift-continuous \( T_1 \)-topologies on the symmetric inverse semigroup \( \mathcal{S}_\lambda^n \) of finite transformations of the rank \( \leq n \). It is proved that such \( T_1 \)-topology is sequentially pracoompact if and only if it is feebly compact. Also, we show that every shift-continuous feebly \( \omega \)-bounded \( T_1 \)-topology on \( \mathcal{S}_\lambda^n \) is compact. The results of this paper is announced in [13].

**2. ON FEEBLY COMPACT SHIFT CONTINUOUS TOPOLOGIES ON THE SEMIGROUP \( \mathcal{S}_\lambda^n \)**

Later we shall assume that \( n \) is an arbitrary positive integer.

For every element \( \alpha \) of the semigroup \( \mathcal{S}_\lambda^n \) we put 

\[ \uparrow_\alpha = \{ \beta \in \mathcal{S}_\lambda^n : \alpha^{-1} \beta = \alpha \} \quad \text{and} \quad \uparrow^\alpha = \{ \beta \in \mathcal{S}_\lambda^n : \beta \alpha^{-1} \alpha = \alpha \}. \]

Then Proposition 5 of [22] implies that \( \uparrow_\alpha = \uparrow^\alpha \) and by Lemma 6 of [29, Section 1.4] we have that \( \alpha \leq \beta \) if and only if \( \beta \in \uparrow_\alpha \) for \( \alpha, \beta \in \mathcal{S}_\lambda^n \). Hence we put \( \uparrow_\alpha = \uparrow_\alpha = \uparrow^\alpha \) for any \( \alpha \in \mathcal{S}_\lambda^n \).

**Remark 1.** Later we identify every element \( \alpha \) of the semigroup \( \mathcal{S}_\lambda^n \) with the graph \( \text{graph}(\alpha) \) of the partial map \( \alpha : \lambda \rightarrow \lambda \) (see [29]). Then according to this identification we have that \( \alpha \leq \beta \) if and only if \( \alpha \subseteq \beta \).

**Lemma 2.** Let \( n \) be an arbitrary positive integer and \( \lambda \) be any infinite cardinal. Let \( \alpha \) be any nonzero element of the semigroup \( \mathcal{S}_\lambda^n \) with rank \( \alpha = m \leq n \). Then the poset \((\uparrow_\alpha, \leq)\) is order isomorphic to the poset \((\mathcal{S}_\lambda^{n-m}, \leq)\).
Proof. Suppose that
\[
\alpha = \begin{pmatrix}
 x_1 & \cdots & x_m \\
 y_1 & \cdots & y_m
\end{pmatrix}
\]
for some \( x_1, \ldots, x_m, y_1, \ldots, y_m \in \lambda \). If \( m = n \) then the inequality \( \alpha \preceq \beta \) in \((\mathcal{N}_\lambda^n, \preceq)\) implies \( \alpha = \beta \), and hence later we assume that \( m < n \). Then for any \( \beta \in \mathcal{N}_\lambda^n \) such that \( \alpha \preceq \beta \) by Remark 1 we have that
\[
\beta = \begin{pmatrix}
 x_1 & \cdots & x_m & x_{m+1} & \cdots & x_n \\
 y_1 & \cdots & y_m & y_{m+1} & \cdots & y_n
\end{pmatrix}
\]
for some \( x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n \in \lambda \). Since \( \lambda \) is infinite, \( |\lambda| = |\lambda \setminus \{x_1, \ldots, x_m\}| = |\lambda \setminus \{y_1, \ldots, y_m\}| \), and hence there exist bijective maps \( u: \lambda \setminus \{x_1, \ldots, x_m\} \to \lambda \) and \( v: \lambda \setminus \{y_1, \ldots, y_m\} \to \lambda \). Simple verifications show that the map \( \mathcal{J}: (\uparrow_{<,\alpha}, \preceq) \to (\mathcal{N}_\lambda^{n-m}, \preceq) \) defined in the following way \( \alpha \mapsto 0 \) and
\[
\begin{pmatrix}
 x_1 & \cdots & x_m & x_{m+1} & \cdots & x_n \\
 y_1 & \cdots & y_m & y_{m+1} & \cdots & y_n
\end{pmatrix}
\]
is an order isomorphism. \( \square \)

Later we need the following technical lemma from [12].

Lemma 3 ([12, Lemma 3]). Let \( n \) be an arbitrary positive integer and \( \lambda \) be an arbitrary infinite cardinal. Let \( \tau \) be a feebly compact shift-continuous \( T_1 \)-topology on the semigroup \( \mathcal{N}_\lambda^n \). Then for every \( \alpha \in \mathcal{N}_\lambda^n \) and any open neighbourhood \( U(\alpha) \) of \( \alpha \) in \((\mathcal{N}_\lambda^n, \tau)\) there exist finitely many \( \alpha_1, \ldots, \alpha_k \in \uparrow_{<,\alpha} \{\alpha\} \) such that
\[
\mathcal{N}_\lambda^n \setminus \mathcal{N}_\lambda^{n-1} \cap \uparrow_{<,\alpha} \subseteq U(\alpha) \cup \uparrow_{<,\alpha_1} \cup \cdots \cup \uparrow_{<,\alpha_k}.
\]

Lemma 4. Let \( \tau \) be a feebly compact topology on \( \mathcal{N}_\lambda^1 \) such that \( \uparrow_{<,\alpha} \) is closed-and-open for any \( \alpha \in \mathcal{N}_\lambda^1 \). Then \( \tau \) is compact.

The statement of Lemma 4 follows from the fact that all nonzero elements of the semigroup \( \mathcal{N}_\lambda^1 \) are closed-and-open in \((\mathcal{N}_\lambda^1, \tau)\).

A family of non-empty sets \{\( A_i : i \in \mathcal{I} \)\} is called a \( \Delta \)-system (a sunflower or a \( \Delta \)-family) if the pairwise intersections of the members are the same, i.e., \( A_i \cap A_j = S \) for some set \( S \) (for \( i \neq j \) in \( \mathcal{I} \)) [28]. The following statement is well known as the Sunflower Lemma or the Lemma about a \( \Delta \)-system (see [28, p. 107]).

Lemma 5. Every infinite family of \( n \)-element sets \( (n < \omega) \) contains an infinite \( \Delta \)-subfamily.

Proposition 6. Let \( n \) be an arbitrary positive integer and \( \lambda \) be an arbitrary infinite cardinal. Then every feebly compact shift-continuous \( T_1 \)-topology \( \tau \) on \( \mathcal{N}_\lambda^n \) is sequentially praco m pact.

Proof. Suppose to the contrary that there exists a feebly compact shift-continuous \( T_1 \)-topology \( \tau \) on \( \mathcal{N}_\lambda^n \) which is not sequentially countably praco m pact. Then every dense subset \( D \) of \((\mathcal{N}_\lambda^n, \tau)\) contains a sequence of points from \( D \) which has no a convergent subsequence.

By Proposition 2 of [12] the subset \( \mathcal{N}_\lambda^n \setminus \mathcal{N}_\lambda^{n-1} \) is dense in \((\mathcal{N}_\lambda^n, \tau)\) and by Lemma 2 from [12] every point of the set \( \mathcal{N}_\lambda^n \setminus \mathcal{N}_\lambda^{n-1} \) is isolated in \((\mathcal{N}_\lambda^n, \tau)\). Then the set \( \mathcal{N}_\lambda^n \setminus \mathcal{N}_\lambda^{n-1} \) contains an infinite sequence of points \( \{\chi_p : p \in \mathbb{N}\} \) which has not a convergent subsequence. If we identify elements of the semigroups with their graphs then by Lemma 3 the sequence \( \{\chi_p : p \in \mathbb{N}\} \) contains an infinite \( \Delta \)-subfamily, that is an infinite subsequence \( \{\chi_{i,j} : i, j \in \mathbb{N}\} \) such that there exists \( \chi \in \mathcal{N}_\lambda^n \) such that \( \chi_{i,j} \cap \chi_{j,i} = \chi \) for any distinct \( i, j \in \mathbb{N} \).

Suppose that \( \chi = 0 \) is the zero of the semigroup \( \mathcal{N}_\lambda^n \). Since the sequence \( \{\chi_{i,j} : i \in \mathbb{N}\} \) is an infinite \( \Delta \)-subfamily, the intersection \( \{\chi_{i,j} : i \in \mathbb{N}\} \cap \uparrow_{<,\gamma} \gamma \) contains at most one set for every non-zero element \( \gamma \in \mathcal{N}_\lambda^n \). Thus \((\mathcal{N}_\lambda^n, \tau)\) contains an infinite locally finite family of open non-empty subsets which contradicts the feebly compactness of \((\mathcal{N}_\lambda^n, \tau)\).

If \( \chi \) is a non-zero element of the semigroup \( \mathcal{N}_\lambda^n \) then by Lemma 2 from [12], \( \uparrow_{<,\chi} \) is an open-and-closed subspace of \((\mathcal{N}_\lambda^n, \tau)\), and hence by Theorem 14 from [3] the space \( \uparrow_{<,\chi} \) is feebly compact. We observe
that the element $\chi$ is the minimum of the poset $\uparrow_\preceq \chi$. Since the sequence $\{\chi_p: p \in \mathbb{N}\}$ is an infinite $\Delta$-subfamily, the intersection $\{\chi_p: i \in \mathbb{N}\} \cap \uparrow_\preceq \gamma$ contains at most one set for every element $\gamma \in \uparrow_\preceq \chi \setminus \{\chi\}$. Thus the subspace $\uparrow_\preceq \chi$ of $(\mathcal{I}_\lambda^\mathcal{A}, \tau)$ contains an infinite locally finite family of open non-empty subsets which contradicts the feebly compactness of $(\mathcal{I}_\lambda^\mathcal{A}, \tau)$. \qed

**Proposition 7.** Let $n$ be an arbitrary positive integer and $\lambda$ be an arbitrary infinite cardinal. Then every feebly compact shift-continuous $T_1$-topology $\tau$ on $\mathcal{I}_\lambda^n$ is totally countably pracoapact.

**Proof.** By Proposition 2 of [12] the subset $\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}$ is dense in $(\mathcal{I}_\lambda^n, \tau)$ and by Lemma 2 from [12] every point of the set $\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}$ is isolated in $(\mathcal{I}_\lambda^n, \tau)$. We put $D = \mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}$. Fix an arbitrary sequence $\{\chi_p: p \in \mathbb{N}\}$ of points of $D$.

It is obvious that at least one of the following conditions holds:

1. for any $\eta \in \mathcal{I}_\lambda^n \setminus \{0\}$ the set $\uparrow_\preceq \eta \cap \{\chi_p: p \in \mathbb{N}\}$ is finite;
2. there exists $\eta \in \mathcal{I}_\lambda^n \setminus \{0\}$ such that the set $\uparrow_\preceq \eta \cap \{\chi_p: p \in \mathbb{N}\}$ is infinite.

Suppose case (1) holds. By Lemma 2 of [12] for every point $\alpha \in \mathcal{I}_\lambda^n \setminus \{0\}$ there exists an open neighbourhood $U(\alpha)$ of $\alpha$ in $(\mathcal{I}_\lambda^n, \tau)$ such that $U(\alpha) \subseteq \uparrow_\preceq \alpha$ and hence our assumption implies that zero $0$ is a unique accumulation point of the sequence $\{\chi_p: p \in \mathbb{N}\}$. By Lemma 3 for an arbitrary open neighbourhood $W(0)$ of zero $0$ in $(\mathcal{I}_\lambda^n, \tau)$ there exist finitely many nonzero elements $\eta_1, \ldots, \eta_k \in \mathcal{I}_\lambda^n$ such that

$$(\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}) \subseteq W(0) \cup \uparrow_\preceq \eta_1 \cup \cdots \cup \uparrow_\preceq \eta_k,$$

and hence we get that $\{0\} \cup \{\chi_p: p \in \mathbb{N}\}$ is a compact subset of $(\mathcal{I}_\lambda^n, \tau)$.

Suppose case (2) holds: there exists $\eta_1 \in \mathcal{I}_\lambda^n \setminus \{0\}$ such that the set $\uparrow_\preceq \eta_1 \cap \{\chi_p: p \in \mathbb{N}\}$ is infinite. Then by Lemma 2 of [12], $\uparrow_\preceq \eta_1$ is an open-and-closed subset of $(\mathcal{I}_\lambda^n, \tau)$ and hence by Theorem 14 from [3] the subspace $\uparrow_\preceq \eta_1$ of $(\mathcal{I}_\lambda^n, \tau)$ is feebly compact. By Lemma 2 the poset $(\uparrow_\preceq \eta_1, \preceq)$ is order isomorphic to the poset $(\mathcal{I}_\lambda^{m_1}, \preceq)$ for some positive integer $m_1 = 2, \ldots, n - 1$.

We put $\{\chi^1_p: p \in \mathbb{N}\}$ is a subsequence of $\{\chi_p: p \in \mathbb{N}\}$ such that $\{\chi^1_p: p \in \mathbb{N}\} = \uparrow_\preceq \eta_1 \cap \{\chi_p: p \in \mathbb{N}\}$. Then for the feebly compact poset $(\uparrow_\preceq \eta_1, \preceq)$ and the sequence $\{\chi^1_p: p \in \mathbb{N}\}$ at least one of the following conditions holds:

1. for any $\eta \in \uparrow_\preceq \eta_1 \setminus \{\eta_1\}$ the set $\uparrow_\preceq \eta \cap \{\chi^1_p: p \in \mathbb{N}\}$ is finite;
2. there exists $\eta \in \uparrow_\preceq \eta_1 \setminus \{\eta_1\}$ such that the set $\uparrow_\preceq \eta \cap \{\chi^1_p: p \in \mathbb{N}\}$ is infinite.

Since every chain in the poset $(\uparrow_\preceq \eta_1, \preceq)$ is finite, repeating finitely many times our above procedure we obtain two chains of the length $s \leq n$:

(i) the chain $0 \preceq \eta_1 \preceq \cdots \preceq \eta^s$ of distinct elements of the poset $(\uparrow_\preceq \eta_1, \preceq)$; and
(ii) the chain $\{\chi_p: p \in \mathbb{N}\} \supseteq \{\chi^1_p: p \in \mathbb{N}\} \supseteq \cdots \supseteq \{\chi^s_p: p \in \mathbb{N}\}$ of infinite subsequences of the sequence $\{\chi_p: p \in \mathbb{N}\}$, such that the following conditions hold:

(a) $\{\chi^j_p: p \in \mathbb{N}\} \subseteq \uparrow_\preceq \eta^j$ for every $j = 1, \ldots, s$;
(b) either $\{\chi^j_p: p \in \mathbb{N}\} \cup \{\eta^j\}$ is a compact subset of the poset $(\uparrow_\preceq \eta^1, \preceq)$ or the poset $(\uparrow_\preceq \eta^s, \preceq)$ is order isomorphic to the poset $(\mathcal{I}_\lambda^1, \preceq)$.

If $\{\chi^s_p: p \in \mathbb{N}\} \cup \{\eta^s\}$ is a compact subset of $(\mathcal{I}_\lambda^n, \tau)$ then our above part of the proof implies that the sequence $\{\chi_p: p \in \mathbb{N}\}$ has the subshece $\{\chi^s_p: p \in \mathbb{N}\}$ with the compact closure.

If the poset $(\uparrow_\preceq \eta^s, \preceq)$ is order isomorphic to the poset $(\mathcal{I}_\lambda^1, \preceq)$, then by Lemma 2 of [12] the subspace $\uparrow_\preceq \eta^s$ of $(\mathcal{I}_\lambda^n, \tau)$ is open-and-closed and hence by Lemmas 2 and 4 the poset $(\uparrow_\preceq \eta^s, \preceq)$ is compact. Then the inclusion $\{\chi^s_p: p \in \mathbb{N}\} \subseteq \uparrow_\preceq \eta^s$ implies that the sequence $\{\chi_p: p \in \mathbb{N}\}$ has the subshece $\{\chi^s_p: p \in \mathbb{N}\}$ with the compact closure. This completed the proof of the proposition. \qed

We summarise our results in the following theorem.

**Theorem 8.** Let $n$ be any positive integer and $\lambda$ be any infinite cardinal. Then for any $T_1$-semitopological semigroup $\mathcal{I}_\lambda^n$ the following conditions are equivalent:

(i) $\mathcal{I}_\lambda^n$ is sequentially pracoapact;
(ii) $\mathcal{I}_h^n$ is totally countably pracompact;
(iii) $\mathcal{I}_h^n$ is feebly compact.

Proof. Implications (i) ⇒ (iii) and (ii) ⇒ (iii) are trivial. The corresponding their converse implica-
tions (iii) ⇒ (i) and (iii) ⇒ (ii) follow from Propositions 6 and 7, respectively.

It is well known that the (Tychonoff) product of pseudocompact spaces is not necessarily pseudocom-
 pact (see [10, Section 3.10]). On the other hand Comfort and Ross in [7] proved that a Tychonoff product
of an arbitrary family of pseudocompact topological groups is a pseudocompact topological group. The
Comfort–Ross Theorem is generalized in [2] and it is proved that a Tychonoff product of an arbitrary
non-empty family of feebly compact paratopological groups is feebly compact. Also, a counterpart of
the Comfort–Ross Theorem for pseudocompact primitive topological inverse semigroups and primitive
inverse semiregular feebly compact semitopological semigroups with closed maximal subgroups were
proved in [16] and [18], respectively.

Since a Tychonoff product of H-closed spaces is H-closed (see [5, Theorem 3] or [10, 3.12.5 (d)])
Theorem 8 implies a counterpart of the Comfort–Ross Theorem for feebly compact semitopological
semigroups $\mathcal{I}_h^n$:

Corollary 9. Let $\{\mathcal{I}_{n_i}: i \in \mathcal{I}\}$ be a family of non-empty feebly compact $T_1$-semitopological
semigroups and $n_i \in \mathbb{N}$ for all $i \in \mathcal{I}$. Then the Tychonoff product $\prod \{\mathcal{I}_{n_i}: i \in \mathcal{I}\}$ is feebly compact.

Definition 10. If $\{X_i: i \in \mathcal{I}\}$ is an uncountable family of sets, $X = \prod \{X_i: i \in \mathcal{I}\}$ is their Cartesian
product and $p$ is a point in $X$, then the subset
$$\Sigma(p, X) = \{x \in X: |\{i \in \mathcal{I}: x(i) \neq p(i)\}| \leq \omega\}$$
of $X$ is called the $\Sigma$-product of $\{X_i: i \in \mathcal{I}\}$ with the basis point $p \in X$. In the case when $\{X_i: i \in \mathcal{I}\}$
is a family of topological spaces we assume that $\Sigma(p, X)$ is a subspace of the Tychonoff product $X = \prod \{X_i: i \in \mathcal{I}\}$.

It is obvious that if $\{X_i: i \in \mathcal{I}\}$ is a family of semigroups then $X = \prod \{X_i: i \in \mathcal{I}\}$ is a semigroup
as well. Moreover $\Sigma(p, X)$ is a subspace of $X$ for arbitrary idempotent $p \in X$. Theorem 8 and
Proposition 2.2 of [20] imply the following corollary.

Corollary 11. Let $\{\mathcal{I}_{n_i}: i \in \mathcal{I}\}$ be a family of non-empty feebly compact $T_1$-semitopological
semigroups and $n_i \in \mathbb{N}$ for all $i \in \mathcal{I}$. Then for every idempotent $p$ of the product $X = \prod \{\mathcal{I}_{n_i}: i \in \mathcal{I}\}$
the $\Sigma$-product $\Sigma(p, X)$ is feebly compact.

3. ON COMPACT SHIFT CONTINUOUS TOPOLOGIES ON THE SEMIGROUP $\mathcal{I}_h^n$

The following example implies that there exists a countable feebly compact Hausdorff semitopological
semigroup $(\mathcal{I}_h^\omega, \tau_e)$ which is not $\omega$-bounded-pracompact.

Example 12. The following family
$$\mathcal{B}_e = \{U_\alpha(\alpha_1, \ldots, \alpha_k) = \uparrow_\omega \alpha \setminus (\uparrow_\omega \alpha_1 \cup \cdots \cup \uparrow_\omega \alpha_k):$$
$$\alpha_i \in \uparrow_\omega \alpha \setminus \{\alpha\}, \alpha, \alpha_i \in \mathcal{I}_h^\omega, i = 1, \ldots, k\}$$
determines a base of the topology $\tau_e$ on $\mathcal{I}_h^\omega$. By Proposition 10 from [22], $(\mathcal{I}_h^\omega, \tau_e)$ is a Hausdorff
compact semitopological semigroup with continuous inversion.

We construct a stronger topology $\tau_{e_2}$ on $\mathcal{I}_h^\omega$ in the following way. For every nonzero element $x \in \mathcal{I}_h^\omega$
we assume that the base $\mathcal{B}_{e_2}(x)$ of the topology $\tau_{e_2}$ at the point $x$ coincides with the base of the topology $\tau_e^2$ at $x$, and
$$\mathcal{B}_{e_2}(0) = \{U_0(0) = U(0) \setminus (\mathcal{I}_h^\omega \setminus \{0\}): U(0) \in \mathcal{B}_{e_2}(0)\}$$
form a base of the topology $\tau_{e_2}$ at zero $0$ of the semigroup $\mathcal{I}_h^\omega$. Since $(\mathcal{I}_h^\omega, \tau_{e_2})$ is a variant of the
semitopological semigroup defined in Example 3 of [12], $\tau_{e_2}$ is a Hausdorff topology on $\mathcal{I}_h^\omega$. Moreover,
by Proposition 1 of [12], $(\mathcal{I}_h^\omega, \tau_{e_2}^2)$ is a countably pracompact semitopological semigroup with continuous
inversion.
Proposition 13. The space \((\mathcal{I}_n^2, \tau_{2c})\) is not \(\omega\)-bounded-pracompact.

Proof. Since the space \((\mathcal{I}_n^2, \tau_{2c})\) is feebly compact and Hausdorff, by Proposition 2 of [12] the subset \(\mathcal{I}_n^2 \setminus \mathcal{I}_n^1\) is dense in \((\mathcal{I}_n^2, \tau_{2c})\), and by Lemma 2 from [12] every point of the set \(\mathcal{I}_n^2 \setminus \mathcal{I}_n^1\) is isolated in \((\mathcal{I}_n^2, \tau_{2c})\). This implies that every dense subset \(D\) of \((\mathcal{I}_n^2, \tau_{2c})\) contains the set \(\mathcal{I}_n^2 \setminus \mathcal{I}_n^{n-1}\). Then
\[
\text{cl}(\mathcal{I}_n^2 \setminus \mathcal{I}_n^1)(D) = \text{cl}(\mathcal{I}_n^2 \setminus \mathcal{I}_n^1)(\mathcal{I}_n^2 \setminus \mathcal{I}_n^{n-1}) = \mathcal{I}_n^2
\]
for every dense subset \(D\) of \((\mathcal{I}_n^2, \tau_{2c})\). Since \(\mathcal{I}_n^2\) is countable, so is \(D\), and hence the space \((\mathcal{I}_n^2, \tau_{2c})\) is not \(\omega\)-bounded-pracompact, because \((\mathcal{I}_n^2, \tau_{2c})\) is not compact. \(\Box\)

Proposition 14. Let \(n\) be any positive integer and \(\lambda\) be any infinite cardinal. If \(\mathcal{I}_\lambda^A\) is a \(T_1\)-semitopological semigroup then the following statements hold:

1. \(\mathcal{I}_\lambda^A\) is a compact subsemigroup of \(\mathcal{I}_\lambda^n\) for any subset \(A \subseteq \lambda\);
2. the band \(E(\mathcal{I}_\lambda^A)\) is a closed subset of \(\mathcal{I}_\lambda^A\).

Proof. (1) Fix an arbitrary \(\gamma \in \mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^A\). Then \(\text{dom} \gamma \notin A\) or \(\text{ran} \gamma \notin A\). Since \(\eta \trianglelefteq \delta\) if and only if \(\text{graph}\(\eta\) \subseteq \text{graph}\(\delta\)\) for \(\eta, \delta \in \mathcal{I}_\lambda^n\), the above arguments imply that \(\uparrow_{\trianglelefteq} \gamma \cap \mathcal{I}_\lambda^A = \emptyset\). By Lemma 2 of [12] the set \(\uparrow_{\trianglelefteq} \gamma\) is open in \(\mathcal{I}_\lambda^n\), which implies statement (1).

(2) Fix an arbitrary \(\gamma \in \mathcal{I}_\lambda^n \setminus E(\mathcal{I}_\lambda^A)\). Since \(\mathcal{I}_\lambda^A\) is an inverse subsemigroup of the symmetric inverse monoid \(\mathcal{I}_\lambda\), all idempotents of \(\mathcal{I}_\lambda^n\) are partial identity maps of rank \(r \leq n\). Then similar arguments as in statement (1) imply that \(E(\mathcal{I}_\lambda^A)\) is a closed subset of \(\mathcal{I}_\lambda^n\).

Proposition 14 implies the following corollary.

Corollary 15. Let \(n\) be any positive integer, \(\lambda\) be any infinite cardinal and \(A\) be an arbitrary infinite subset of \(\lambda\). If \(\mathcal{I}_\lambda^n\) is a compact \(T_1\)-semitopological semigroup then \(\mathcal{I}_\lambda^A\) is a compact \(\omega\)-bounded-pracompact space.

Lemma 16. Let \(n\) be any positive integer, \(\lambda\) be any infinite cardinal and \(A\) be an arbitrary infinite countable subset of \(\lambda\). If \(\mathcal{I}_\lambda^n\) is a \(\omega\)-bounded-pracompact \(T_1\)-semitopological semigroup then \(\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}\) is a dense subset of \(\mathcal{I}_\lambda^A\), and hence \(\mathcal{I}_\lambda^A\) is compact.

Proof. For any \(\alpha \in \mathcal{I}_\lambda^A\) we denote \(\uparrow_{\trianglelefteq} \alpha = \uparrow_{\trianglelefteq} \alpha \cap \mathcal{I}_\lambda^n\).

By induction we shall show that the set \(\uparrow_{\trianglelefteq} \alpha \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})\) is dense in \(\uparrow_{\trianglelefteq} \alpha\) for any \(\alpha \in \mathcal{I}_\lambda^A\). In the case when \(\text{rank} \alpha = n - 1\) by Lemmas 2 and 4 we have that the set \(\uparrow_{\trianglelefteq} \alpha\) is compact, and hence by Proposition 14(1), \(\uparrow_{\trianglelefteq} \alpha\) is compact as well. Since all points of \(\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}\) are isolated in \(\mathcal{I}_\lambda^n\), the set \(\uparrow_{\trianglelefteq} \alpha \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})\) is dense in \(\uparrow_{\trianglelefteq} \alpha\).

Next we show that the statement \(\uparrow_{\trianglelefteq} \alpha \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})\) is dense in \(\uparrow_{\trianglelefteq} \alpha\) for any \(\alpha \in \mathcal{I}_\lambda^n\) with \(\text{rank} \alpha = n - k\), for all \(k < m\) implies that the same is true for any \(\beta \in \mathcal{I}_\lambda^n\) with \(\text{rank} \beta = m - n\), where \(m \leq n\). Fix an arbitrary \(\beta \in \mathcal{I}_\lambda^n\) with \(\text{rank} \beta = n - m\). Suppose to the contrary that the set \(\uparrow_{\trianglelefteq} \beta \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})\) is not dense in \(\uparrow_{\trianglelefteq} \beta\). The assumption of induction implies that \(\gamma \in \text{cl}_{\mathcal{I}_\lambda^n}(\uparrow_{\trianglelefteq} \beta \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}))\) for any \(\gamma \in \uparrow_{\trianglelefteq} \beta \setminus \{\beta\}\), and hence \(\beta \notin \text{cl}_{\mathcal{I}_\lambda^n}(\uparrow_{\trianglelefteq} \beta \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1}))\). Then there exists an open neighbourhood \(U(\beta)\) of \(\beta\) in \(\mathcal{I}_\lambda^n\) such that \(U(\beta) \cap (\uparrow_{\trianglelefteq} \beta \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})) = \emptyset\). By Lemma 2 from [12] for any \(\delta \in \mathcal{I}_\lambda^n\) the set \(\uparrow_{\trianglelefteq} \delta\) is open-and-closed in \(\mathcal{I}_\lambda^n, \tau\), and hence \(\uparrow_{\trianglelefteq} \delta\) is open-and-closed in \(\mathcal{I}_\lambda^n\) as well. Hence we get that
\[
\text{cl}_{\mathcal{I}_\lambda^n}(\uparrow_{\trianglelefteq} \beta \cap (\mathcal{I}_\lambda^n \setminus \mathcal{I}_\lambda^{n-1})) = \uparrow_{\trianglelefteq} \beta \setminus \{\beta\}
\]
but the family \(\mathcal{U} = \{\uparrow_{\trianglelefteq} \delta: \delta \in \uparrow_{\trianglelefteq} \beta \setminus \{\beta\}\}\) is an open cover of \(\uparrow_{\trianglelefteq} \beta\) which hasn’t a finite subcover. This contradicts the condition that \(\mathcal{I}_\lambda^n\) is a \(\omega\)-bounded-pracompact space, which completes the proof of the first statement of the lemma. The last statement immediately follows from the first statement and the definition of the \(\omega\)-bounded-pracompact space. \(\Box\)

Theorem 17 describes feebly \(\omega\)-bounded shift-continuous \(T_1\)-topologies on the semigroup \(\mathcal{I}_\lambda^n\).

Theorem 17. Let \(n\) be any positive integer and \(\lambda\) be any infinite cardinal. Then for any \(T_1\)-semitopological semigroup \(\mathcal{I}_\lambda^n\), the following conditions are equivalent:
(i) $\mathcal{I}_n^\lambda$ compact;
(ii) $\mathcal{I}_n^\lambda$ is $\omega$-bounded-pracompact;
(iii) $\mathcal{I}_n^\lambda$ is feebly $\omega$-bounded.

Proof. Implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (ii) Let $\mathcal{I}_n^\lambda$ be a feebly $\omega$-bounded $T_1$-semitopological semigroup. By Proposition 2 of [12] the set $\mathcal{I}_n^\lambda \setminus \mathcal{I}_n^{\lambda-1}$ is dense in $\mathcal{I}_n^\lambda$. Fix an arbitrary infinite countable subset $D = \{\alpha_i : i \in \mathbb{N}\}$ in $\mathcal{I}_n^\lambda \setminus \mathcal{I}_n^{\lambda-1}$. By Lemma 2 from [12] every point of $D$ is isolated in $\mathcal{I}_n^\lambda$, and hence by feebly $\omega$-boundedness of $\mathcal{I}_n^\lambda$ we get that there exists a compact subset $K \subseteq \mathcal{I}_n^\lambda$ such that $D \subseteq K$. Since the closure of a subset in compact space is compact, so is the closure of $D$. Hence the space $\mathcal{I}_n^\lambda$ is $\omega$-bounded-pracompact.

(ii) $\Rightarrow$ (i) Suppose the contrary: there exists a noncompact $\omega$-bounded-pracompact $T_1$-semitopological semigroup $\mathcal{I}_n^\lambda$. By Theorem 1 of [12] the space $\mathcal{I}_n^\lambda$ is not countably compact. Then by Theorem 3.10.3 of [10] the space $\mathcal{I}_n^\lambda$ has an infinite countable closed discrete subspace $D$. We put

$$A = \{x \in \lambda : x \in \text{dom} \alpha \cup \text{ran} \alpha \text{ for some } \alpha \in D\}.$$ 

Since the set $D$ is countable, $\bigcup_{\alpha \in D} (\text{dom} \alpha \cup \text{ran} \alpha)$ is countable, and hence $A$ is countable, too. Then $\mathcal{I}_n^\lambda$ contains $D$. By Proposition 14(1), $\mathcal{I}_n^\lambda$ is a closed subspace of $\mathcal{I}_n^\lambda$, which implies that $D$ is an infinite countable closed discrete subspace of $\mathcal{I}_n^\lambda$. This contradicts Lemma 16, and hence $\mathcal{I}_n^\lambda$ is compact. □

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