FRACTIONAL TYPE MULTILINEAR COMMUTATORS
GENERATED BY FRACTIONAL INTEGRAL WITH ROUGH VARIABLE KERNEL AND LOCAL CAMПANATO FUNCTIONS
ON GENERALIZED VANISHING LOCAL MORREY SPACES

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Abstract. In this paper, we consider the boundedness of fractional type multilinear commutators generated by fractional integral with rough variable kernel and local Campanato functions on both generalized local (central) Morrey spaces and generalized vanishing local Morrey spaces, under generic size conditions which are satisfied by most of the operators in harmonic analysis, respectively.

1. Introduction and main lemmas

The classical Morrey spaces $L_{p,\lambda}$ were introduced by Morrey [23] in 1938 to study the local behavior of solutions of second order elliptic partial differential equations (PDEs). Later, there were many applications of Morrey space to the Navier-Stokes equations (see [21]), the Schrödinger equations (see [29]) and the elliptic problems with discontinuous coefficients (see [4, 12, 25]). The study of the operators of harmonic analysis in vanishing Morrey space, in fact has been almost not touched. A version of the classical Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where it is possible to approximate by "nice" functions is the so called vanishing Morrey space $VL_{p,\lambda}(\mathbb{R}^n)$ has been introduced by Vitanza in [33] and has been applied there to obtain a regularity result for elliptic PDEs. This is a subspace of functions in $L_{p,\lambda}(\mathbb{R}^n)$, which satisfies the condition

$$\lim_{r \to 0} \sup_{0 < t < r} \|f\|_{L_p(B(x,t))} = 0.$$ 

Later in [34] Vitanza has proved an existence theorem for a Dirichlet problem, under weaker assumptions than in [20] and a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to vanishing Morrey spaces depending on the dimension. Also Ragusa has proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VL_{p,\lambda}(\mathbb{R}^n)$ ([27, 28]). For the properties and applications of vanishing Morrey spaces, see also [5].

First of all, we recall some basic definitions and notations used in the paper.

Throughout the paper we assume that $x \in \mathbb{R}^n$ and $r > 0$, and also let $B(x,r)$ denote the open ball centered at $x$ of radius $r$, $B^C(x,r)$ denote its complement and
Remark 1. Define \( C > \ell \) \( b \in \ell M \). For the properties and applications of generalized local (central) Morrey spaces we denote by \( \ell M_{p,\varphi} \equiv \ell M_{p,\varphi}^{(x_0)}(\mathbb{R}^n) \) the generalized local Morrey space, the space of all functions \( f \in \ell L_{\text{loc}}^{(\mathbb{R}^n)} \) with finite quasinorm

\[
\|f\|_{\ell M_{p,\varphi}^{(x_0)}} = \sup_{r > 0} \varphi(x_0, r)^{-1} |B(x_0, r)|^{-\frac{1}{p}} \|f\|_{\ell L_{\text{loc}}(B(x_0, r))} < \infty.
\]

According to this definition, we recover the local Morrey space \( \ell L_{p,\lambda}^{(x_0)} \) under the choice \( \varphi(x_0, r) = r^{\frac{\lambda - \mu}{n}} \):

\[
\ell L_{p,\lambda}^{(x_0)} = \ell M_{p,\varphi}^{(x_0)}|_{\varphi(x_0, r) = r^{\frac{\lambda - \mu}{n}}}.
\]

For the properties and applications of generalized local (central) Morrey spaces \( \ell M_{p,\varphi}^{(x_0)} \), see also [9, 13].

**Definition 1.** (Generalized local (central) Morrey space) Let \( \varphi(x, r) \) be a positive measurable function on \( \mathbb{R}^n \times (0, \infty) \) and \( 1 \leq p < \infty \). For any fixed \( x_0 \in \mathbb{R}^n \) we define \( \ell M_{p,\varphi}^{(x_0)}(\mathbb{R}^n) \). Then \( \|f\|_{\ell L_{\text{loc}}(B(x_0, r))} < \infty \).

Lemma 1. [9, 13] Let \( b \) be function in \( \ell L_{p,\lambda}^{(x_0)}(\mathbb{R}^n) \), \( 1 \leq p < \infty \), \( 0 \leq \lambda < \frac{1}{n} \) and \( r_1, r_2 > 0 \). Then

\[
\left( \frac{1}{|B(x_0, r)|^{1+\lambda p}} \int_{B(x_0, r)} |b(y) - b(y)|^p dy \right)^{\frac{1}{p}} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) \|b\|_{\ell L_{p,\lambda}^{(x_0)}},
\]

where \( C > 0 \) is independent of \( b, r_1 \) and \( r_2 \).
From this inequality \( (1.2) \), we have
\[
|b_{B(x_0, r_1)} - b_{B(x_0, r_2)}| \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) |B(x_0, r_1)|^\lambda \|b\|_{L^p_{\lambda}(x_0)} ,
\]
and it is easy to see that
\[
\|b - (b)_B\|_{L^p(B)} \leq C \left( 1 + \ln \frac{r_1}{r_2} \right) r_1^{n+\lambda} \|b\|_{L^p_{\lambda}(x_0)} .
\]

For brevity, in the sequel we use the following notation
\[
\mathcal{M}_{p, \varphi}(f; x_0, r) := \frac{|B(x_0, r)|^{-\frac{1}{p}} \|f\|_{L^p(B(x_0, r))}}{\varphi(x_0, r)}.
\]

Extending the definition of the vanishing Morrey spaces to the case of the generalized local (central) Morrey spaces, we introduce the following definition.

**Definition 3. (generalized vanishing local Morrey space)** The generalized vanishing local Morrey space \( VLM_{p, \varphi}(x_0, R^n) \) is defined as the spaces of functions \( f \in LM_{p, \varphi}(x_0, R^n) \) such that
\[
\lim_{r \to 0} \mathcal{M}_{p, \varphi}(f; x_0, r) = 0.
\]

Throughout the sequel we assume that
\[
\lim_{r \to 0} \frac{1}{\varphi(x_0, r)} = 0,
\]
and
\[
\sup_{0 < r < \infty} \frac{1}{\varphi(x_0, r)} < \infty,
\]
which make the spaces \( VLM_{p, \varphi}(x_0, R^n) \) non-trivial, since bounded functions with compact support belong to this space. The space \( VLM_{p, \varphi}(x_0, R^n) \) is a Banach space with respect to the norm
\[
\|f\|_{VLM_{p, \varphi}(x_0, R^n)} \equiv \|f\|_{LM_{p, \varphi}(x_0, R^n)} = \sup_{r > 0} \mathcal{M}_{p, \varphi}(f; x_0, r).
\]

The space \( VLM_{p, \varphi}(x_0, R^n) \) is a closed subspace of the Banach space \( LM_{p, \varphi}(x_0, R^n) \), which may be shown by standard means.

Suppose that \( S^{n-1} \) is the unit sphere on \( R^n (n \geq 2) \) equipped with the normalized Lebesgue measure \( d\sigma(x') \). A function \( \Omega(x, z) \) defined on \( R^n \times R^n \) is said to belong to \( L_s(S^{n-1}) \) for \( s > 1 \), if \( \Omega \) satisfies the following conditions:

for any \( x, z \in R^n \) and \( \lambda > 0 \),
\[
\Omega(x, \lambda z) = \Omega(x, z);
\]
\[
\|
\Omega\|_{L_s(S^{n-1})} := \sup_{x \in R^n} \left( \int_{S_{s-1}^{n-1}} |\Omega(x, z')|^s d\sigma(z') \right)^{1/s} < \infty,
\]
where \( z' = z/|z| \), for any \( z \in R^n \setminus \{0\} \).
Suppose that $T_{\Omega, \alpha}$, $\alpha \in (0, n)$ represents a linear or a sublinear operator with rough variable kernel, which satisfies that for any $f \in S(\mathbb{R}^n)$ with compact support and $x \notin \text{supp}f$

$$
|T_{\Omega, \alpha}f(x)| \leq c_0 \int_{\mathbb{R}^n} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} |f(y)| \, dy,
$$

where $c_0$ is independent of $f$ and $x$. We point out that the condition (1.7) in the case of $\Omega \equiv 1$, $\alpha = 0$ has been introduced by Soria and Weiss in [32]. The condition (1.7) is satisfied by many interesting operators in harmonic analysis, such as the fractional Marcinkiewicz operator, fractional maximal operator, fractional integral operator (Riesz potential) and so on (see [18], [32] for details).

Then, the fractional integral operator with rough variable kernel $\tilde{T}_{\Omega, \alpha}$ is defined by

$$
\tilde{T}_{\Omega, \alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} f(y) \, dy \quad 0 < \alpha < n,
$$

and a related fractional maximal operator with rough variable kernel $M_{\Omega, \alpha}$ is given by

$$
M_{\Omega, \alpha}f(x) = \sup_{t>0} |B(x, t)|^{-1 + \frac{\alpha}{n}} \int_{B(x, t)} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} |f(y)| \, dy \quad 0 < \alpha < n,
$$

where $\Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})$, $s > 1$, is homogeneous of degree zero with respect to the second variable $y$ on $\mathbb{R}^n$ and these operators also satisfy condition (1.7).

If $\alpha = 0$, then $\tilde{T}_{\Omega, \alpha}$ and $M_{\Omega, \alpha}$ becomes the singular integral operator and the Hardy-Littlewood maximal operator by with rough variable kernels, respectively. It is obvious that when $\Omega \equiv 1$, $\tilde{T}_{1, \alpha} = \tilde{T}_\alpha$ and $M_{1, \alpha} = M_\alpha$ are the fractional integral operator (Riesz potential) and the fractional maximal operator, respectively.

In recent years, the mapping properties of $\tilde{T}_{\Omega, \alpha}$ on some kinds of function spaces have been studied in many papers (see [1], [2], [7], [24], [35], [37] for details). In particular, the boundedness of $\tilde{T}_{\Omega, \alpha}$ for rough variable kernel in Lebesgue spaces has been obtained by Muckenhoupt and Wheeden [24] as follows:

**Lemma 2.** [24] Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})$, $s > \frac{n}{n-\alpha}$, for $s' \leq p$ or $q < s$, then we have

$$
\|\tilde{T}_{\Omega, \alpha}f\|_{L_q(\mathbb{R}^n)} \leq C\|\Omega\|_{L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})} \|f\|_{L_p(\mathbb{R}^n)}.
$$

**Corollary 1.** Under the assumptions of Lemma 2 the operator $M_{\Omega, \alpha}$ from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ is bounded. Moreover, we have

$$
\|M_{\Omega, \alpha}f\|_{L_q(\mathbb{R}^n)} \leq C\|\Omega\|_{L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})} \|f\|_{L_p(\mathbb{R}^n)}.
$$

**Proof.** Set

$$
\tilde{T}_{\{\Omega, \alpha\}}(\{f\})(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} |f(y)| \, dy, \quad 0 < \alpha < n,
$$

where $\Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})$ $(s > 1)$ is homogeneous of degree zero with respect to the second variable $y$ on $\mathbb{R}^n$. It is easy to see that, $\tilde{T}_{\{\Omega, \alpha\}}$ satisfies Lemma 2. On
Here, for any $t > 0$, we have

$$\tilde{T}_{\Omega, \alpha}(\{f\})(x) \geq \int_{B(x,t)} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} \left| f(y) \right|dy$$

and

$$\geq \frac{1}{t^{n-\alpha}} \int_{B(x,t)} |\Omega(x, x - y)| \left| f(y) \right|dy.$$

Then by taking supremum for $t > 0$, we get

$$M_{\Omega, \alpha}f(x) \leq C_{n, \alpha}^{-1} \tilde{T}_{\Omega, \alpha}(\{f\})(x) \quad \text{for} \quad C_{n, \alpha} = |B(0, 1)|^{\frac{n}{n-\alpha}}.$$

On the other hand, for any $t > 0$, we have

$$\tilde{T}_{\Omega, \alpha}(\{f\})(x) \geq \int_{B(x,t)} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} \left| f(y) \right|dy$$

and

$$\geq \frac{1}{t^{n-\alpha}} \int_{B(x,t)} |\Omega(x, x - y)| \left| f(y) \right|dy.$$

Then by taking supremum for $t > 0$, we get

$$M_{\Omega, \alpha}f(x) \leq C_{n, \alpha}^{-1} \tilde{T}_{\Omega, \alpha}(\{f\})(x) \quad \text{for} \quad C_{n, \alpha} = |B(0, 1)|^{\frac{n}{n-\alpha}}.$$

For $m = 1$, $[\tilde{b}, \mathbf{T}_{\Omega, \alpha}]$ and $M_{\Omega, \tilde{b}, \alpha}$ are obviously the commutator operators of $\mathbf{T}_{\Omega, \alpha}$ and $M_{\Omega, \alpha}$.

$$[b, \mathbf{T}_{\Omega, \alpha}]f(x) \equiv b(x)\mathbf{T}_{\Omega, \alpha}f(x) - \mathbf{T}_{\Omega, \alpha}(bf)(x)$$

and

$$M_{\Omega, b, \alpha}(f)(x) \equiv b(x)M_{\Omega, \alpha}f(x) - M_{\Omega, \alpha}(bf)(x)$$

where $0 < \alpha < n$ and $f$ is a suitable function.

In [22], the authors obtain the boundedness for the multilinear commutators generated by singular integral operators with rough variable kernel and local Campanato functions on generalized local Morrey spaces.

Inspired by [22], in this paper we give local Campanato space estimates for fractional type multilinear commutators with rough variable kernel on both generalized local Morrey spaces and generalized vanishing local Morrey spaces, respectively.
But, the techniques and non-trivial estimates which have been used in the proofs of our main results are quite different from [22]. For example, using inequality about the weighted Hardy operator \( H_w \) in [22], in this paper we will only use the following relationship between essential supremum and essential infimum

\[
(1.8) \quad \left( \text{essinf}_{x \in E} f(x) \right)^{-1} = \text{esssup}_{x \in E} \frac{1}{f(x)},
\]

where \( f \) is any real-valued nonnegative function and measurable on \( E \) (see [30], page 143).

We first need some lemmas (our main lemmas) which are used in the proof of the main results. These lemmas with their proofs can be formulated as follows, respectively:

**Lemma 3.** Suppose that \( x_0 \in \mathbb{R}^n, \Omega \in L_\infty(\mathbb{R}^n) \times L_\alpha(S^{n-1}), s > 1, \) is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}. \) Let \( T_{\Omega, \alpha} \) be a sublinear operator satisfying condition [17], bounded from \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \).

If \( p > 1 \) and \( s' \leq p \), then the inequality

\[
(1.9) \quad \|T_{\Omega, \alpha} f\|_{L_q(B(x_0, r))} \leq r^\frac{n}{q} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0, t))} dt
\]

holds for any ball \( B(x_0, r) \) and for all \( f \in L_p^{loc}(\mathbb{R}^n) \).

If \( p > 1 \) and \( q < s \), then the inequality

\[
\|T_{\Omega, \alpha} f\|_{L_q(B(x_0, r))} \leq r^\frac{n}{q} \int_{2r}^\infty t^{-\frac{n}{q}-1} \|f\|_{L_p(B(x_0, t))} dt
\]

holds for any ball \( B(x_0, r) \) and for all \( f \in L_p^{loc}(\mathbb{R}^n) \).

**Proof.** Let \( 0 < \alpha < n, 1 \leq s' < p < \frac{n}{\alpha} \) and \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Set \( B = B(x_0, r) \) for the ball centered at \( x_0 \) and of radius \( r \) and \( 2B = B(x_0, 2r) \). We represent \( f \) as

\[
(1.10) \quad f = f_1 + f_2, \quad f_1(y) = f(y) \chi_{2B}(y), \quad f_2(y) = f(y) \chi_{(2B)^c}(y), \quad r > 0
\]

and have

\[
\|T_{\Omega, \alpha} f\|_{L_q(B)} \leq \|T_{\Omega, \alpha} f_1\|_{L_q(B)} + \|T_{\Omega, \alpha} f_2\|_{L_q(B)}.
\]

Since \( f_1 \in L_p(\mathbb{R}^n), T_{\Omega, \alpha} f_1 \in L_q(\mathbb{R}^n) \) and by the boundedness of \( T_{\Omega, \alpha} \) from \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \) (see Lemma 2) it follows that:

\[
\|T_{\Omega, \alpha} f_1\|_{L_q(B)} \leq \|T_{\Omega, \alpha} f_1\|_{L_q(\mathbb{R}^n)} \leq C \|f_1\|_{L_p(\mathbb{R}^n)} = C \|f\|_{L_p(2B)},
\]

where constant \( C > 0 \) is independent of \( f \).

It is clear that \( x \in B, y \in (2B)^c \) implies \( \frac{1}{2} |x_0 - y| \leq |x - y| \leq \frac{3}{2} |x_0 - y| \). Then we get

\[
|T_{\Omega, \alpha} f_2(x)| \leq 2^{n-\alpha} C_1 \int_{(2B)^c} \frac{|f(y)| |\Omega(x, x - y)|}{|x_0 - y|^{n-\alpha}} dy.
\]
By Fubini’s theorem, we have
\[
\int_{(2B)^c} \frac{|f(y)||\Omega(x, x - y)|}{|x_0 - y|^{n-\alpha}} \, dy \approx \int_{(2B)^c} \frac{|f(y)||\Omega(x, x - y)|}{|x_0 - y|} \, dt \frac{dt}{t^{n+1-\alpha}}
\]
\[
\approx \int_2^\infty \int_2^{\lceil x_0 - y \rceil \leq t} \frac{|f(y)||\Omega(x, x - y)|}{|x_0 - y|} \, dy \, dt \frac{dt}{t^{n+1-\alpha}}
\]
\[
\lesssim \int_2^\infty \int_{B(x_0, t)} \frac{|f(y)||\Omega(x, x - y)|}{|x_0 - y|} \, dy \, dt \frac{dt}{t^{n+1-\alpha}}.
\]

Applying Hölder’s inequality, we get
\[
\int_{(2B)^c} \frac{|f(y)||\Omega(x, x - y)|}{|x_0 - y|^{n-\alpha}} \, dy \lesssim \int_2^\infty \|f\|_{L^p(B(x_0, t))} \|\Omega(x, x - \cdot)\|_{L^q(B(x_0, t))} |B(x_0, t)|^{1 - \frac{1}{p} - \frac{1}{q}} \, dt \frac{dt}{t^{n+1-\alpha}}.
\]

(1.11)

For \( x \in B(x_0, t) \), notice that \( \Omega \) is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \) and \( \Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1}), s > 1 \). Then, we obtain
\[
\left( \int_{B(x_0, t)} |\Omega(x, x - y)|^s \, dy \right)^{\frac{1}{s}} \leq \left( \int_{B(0, 1/2)} |\Omega(x, x - z)|^s \, dz \right)^{\frac{1}{s}}
\]
\[
\leq \left( \int_{B(0, 1/2)} |\Omega(x, z)|^s \, dz \right)^{\frac{1}{s}}
\]
\[
= \left( \int_{S^{n-1}} |\Omega(x, z')|^s \, d\sigma(z') r^{n-1} \, dr \right)^{\frac{1}{s}}
\]
\[
= C \|\Omega\|_{L_\infty(\mathbb{R}^n) \times L_s(S^{n-1})} |B(x_0, 2t)|^{\frac{1}{s}}.
\]

(1.12)

Thus, by (1.12), it follows that:
\[
|T_{\Omega, \alpha} f_2 (x)| \lesssim \int_2^\infty \|f\|_{L^p(B(x_0, t))} \, dt \frac{dt}{t^{n+1-\alpha}}.
\]

Moreover, for all \( p \in (1, \infty) \) the inequality
\[
\|T_{\Omega, \alpha} f_2\|_{L^q(B)} \lesssim r^{\frac{1}{q}} \int_2^\infty \|f\|_{L^p(B(x_0, t))} \, dt \frac{dt}{t^{n+1+1}}.
\]

(1.13)
is valid. Thus, we obtain
\[
\| T_{\Omega, \alpha} f \|_{L_q(B)} \lesssim \| f \|_{L_p(2B)} + \frac{\| f \|_{L_p(B(x_0, t))}}{t^{\frac{1}{q} - 1}} d t.
\]

On the other hand, we have
\[
\| f \|_{L_p(2B)} \approx r^{\frac{\phi}{\beta}} \| f \|_{L_p(2B)} \int_{2r}^{\infty} \| f \|_{L_p(B(x_0, t))} \frac{d t}{t^{\frac{1}{q}+1}}.
\]
(1.14)

By combining the above inequalities, we obtain
\[
\| T_{\Omega, \alpha} f \|_{L_q(B)} \lesssim r^{\frac{\phi}{\beta}} \int_{2r}^{\infty} t^{-\frac{1}{q} - 1} \| f \|_{L_p(B(x_0, t))} d t.
\]

For the case of \(1 < q < s\), we can also use the same method, so we omit the details. This completes the proof of Lemma 3. \(\square\)

**Lemma 4.** Suppose that \(x_0 \in \mathbb{R}^n\), \(\Omega \in L_{\infty}(\mathbb{R}^n) \times L_{\alpha}(S^{n-1})\), \(s > 1\), is homogeneous of degree zero with respect to the second variable \(y\) on \(\mathbb{R}^n\). Let \(T_{\Omega, \alpha}\) be a linear operator satisfying condition (1.7). Let also \(0 < \alpha < n\) and \(1 < q, p_1, \ldots, p < \frac{n}{\alpha}\) with \(\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p_1} = \frac{1}{q} - \frac{\alpha}{n}\) and \(\overrightarrow{b} \in L^{m}(x_0)\) for \(0 \leq \lambda_i < \frac{1}{m}, i = 1, \ldots, m\).

Then, for \(s' \leq q\) the inequality
\[
\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \|_{L_{q_1}(B(x_0, r))} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{L^{q_1}(p_i, \lambda_i)} r^{\frac{\alpha}{q_1} - 1} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} t^{(\frac{\alpha}{q_1} + \sum_{i=1}^{m} \frac{\lambda_i}{m}) - 1} \| f \|_{L_p(B(x_0, t))} d t
\]
holds for any ball \(B(x_0, r)\) and for all \(f \in L_{p_1}^{m}(\mathbb{R}^n)\). Also, for \(q_1 < s\) the inequality
\[
\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \|_{L_{q_1}(B(x_0, r))} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{L^{m}(p_i, \lambda_i)} r^{\frac{\alpha}{m} - 1} \int_{2r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{m} t^{(\frac{\alpha}{m} + \sum_{i=1}^{m} \frac{\lambda_i}{m}) - 1} \| f \|_{L_p(B(x_0, t))} d t
\]
holds for any ball \(B(x_0, r)\) and for all \(f \in L_{p_1}^{m}(\mathbb{R}^n)\).

**Proof.** Without loss of generality, it is sufficient to show that the conclusion holds for \([\overrightarrow{b}, T_{\Omega, \alpha}] f = [(b_1, b_2), T_{\Omega, \alpha}] f\). As in the proof of Lemma 3 we represent \(f\) in form (1.10) and thus have
\[
\| (b_1, b_2), T_{\Omega, \alpha} \|_{L_{q_1}(B)} = \| (b_1, b_2), T_{\Omega, \alpha} \|_{L_{q_1}(B)} + \| (b_1, b_2), T_{\Omega, \alpha} \|_{L_{q_1}(B)} =: F + G.
\]

Let us estimate \(F + G\), respectively.
For \([b_1, b_2], T_{\Omega, \alpha}\)\(|f_1(x)\), we have the following decomposition,

\[
\begin{aligned}
\left[(b_1, b_2), T_{\Omega, \alpha}\right]|f_1(x) & = (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) T_{\Omega, \alpha} f_1(x) \\
& - (b_1(\cdot) - (b_1)_B) T_{\Omega, \alpha} ((b_2(\cdot) - (b_2)_B) f_1)(x) \\
& + (b_2(x) - (b_2)_B) T_{\Omega, \alpha} ((b_1(x) - (b_1)_B) f_1)(x) - T_{\Omega, \alpha} ((b_1(\cdot) - (b_1)_B)(b_2(\cdot) - (b_2)_B) f_1)(x).
\end{aligned}
\]

Hence, we get

\[
F = \|[(b_1, b_2), T_{\Omega, \alpha}] f_1\|_{L^q_1(B)} \lesssim
\|b_1 - (b_1)_B\|_{L^p_3(B)} \|b_2 - (b_2)_B\|_{L^p_3(B)} \|f\|_{L^p_\Omega(B)}
\]

One observes that the estimate of (1.16) is analogous to that of \(F_2\). Thus, we will only estimate \(F_1\), \(F_2\) and \(F_3\).

To estimate \(F_1\), let \(1 < \gamma_1, \gamma_2 < \infty\), such that \(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma}\). Then, using H{"o}lder’s inequality and by the boundedness of \(T_{\Omega, \alpha}\) from \(L_p\) into \(L_\Omega\) (see Lemma 2) it follows that:

\[
\begin{aligned}
F_1 & = \|b_1 - (b_1)_B\|_{L^p_1(B)} \|b_2 - (b_2)_B\|_{L^p_2(B)} \|T_{\Omega, \alpha} f_1\|_{L^q_1(B)} \\
& \lesssim \|b_1 - (b_1)_B\|_{L^p_1(B)} \|b_2 - (b_2)_B\|_{L^p_2(B)} \|T_{\Omega, \alpha} f_1\|_{L^q_1(B)} \\
& \lesssim \|b_1 - (b_1)_B\|_{L^p_1(B)} \|b_2 - (b_2)_B\|_{L^p_2(B)} \|f\|_{L^p_\Omega(B)} \\
& \lesssim \|b_1 - (b_1)_B\|_{L^p_1(B)} \|b_2 - (b_2)_B\|_{L^p_2(B)} \|f\|_{L^p_\Omega(B)} \frac{n(\frac{1}{r} - \frac{1}{\gamma})}{2r} \int_{2r}^\infty \frac{dt}{t^{\frac{\gamma}{\gamma - 1} + 1 - \alpha}}.
\end{aligned}
\]

From Lemma 1 it is easy to see that

\[
\|b_i - (b_i)_B\|_{L^p_i(B)} \leq C r^{\frac{n}{p_i} + n\lambda_i} \|b_i\|_{LC_{p_i, \lambda_i}^{(\sigma_i)}},
\]

and

\[
\|b_i - (b_i)_B\|_{L^p_i(2B)} \leq \|b_i - (b_i)_B\|_{L^p_i(B)} + \|b_i - (b_i)_B\|_{L^p_i(2B)} \lesssim r^{\frac{n}{p_i} + n\lambda_i} \|b_i\|_{LC_{p_i, \lambda_i}^{(\sigma_i)}},
\]

for \(i = 1, 2\). Hence, by (1.17) we get

\[
\begin{aligned}
F_1 & \lesssim \|b_1\|_{LC_{p_1, \lambda_1}^{(\sigma_1)}} \|b_2\|_{LC_{p_2, \lambda_2}^{(\sigma_2)}} r^{\frac{n}{p_1} + \frac{n}{p_2} + \frac{1}{p}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{\gamma} + \frac{n}{r} + \frac{1}{p}} \|f\|_{L^p(\Omega(x_0,t))} \, dt \\
& \lesssim \|b_1\|_{LC_{p_1, \lambda_1}^{(\sigma_1)}} \|b_2\|_{LC_{p_2, \lambda_2}^{(\sigma_2)}} r^{\frac{n}{p_1} + \frac{n}{p_2} + \frac{1}{p}} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{\gamma} + \frac{n}{r} + \frac{1}{p}} \|f\|_{L^p(\Omega(x_0,t))} \, dt.
\end{aligned}
\]
To estimate $F_2$, let $1 < \tau < \infty$, such that $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{\tau}$. Then, similar to the estimates for $F_1$, we have

$$F_2 = \|(b_1 - (b_1)_B) T_{1,\alpha} ((b_2 - (b_2)_B) f_1)\|_{L_{q_1}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(B)} \|T_{1,\alpha} ((b_2 - (b_2)_B) f_1)\|_{L_{\tau}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(B)} \|(b_2 - (b_2)_B) f_1\|_{L_{\tau}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(B)} \|b_2 - (b_2)_B\|_{L_{p_2}(2B)} \|f\|_{L_p(2B)},$$

where $1 < k < \frac{2p}{q_1}$, such that $\frac{1}{k} = \frac{1}{p_2} + \frac{1}{p} = \frac{1}{\tau} + \frac{1}{n}$. By (1.17) and (1.18), we get

$$F_2 \lesssim \|b_1\|_{L_{p_1}^{(x_0)}} \|b_2\|_{L_{p_2}^{(x_0)}} \tau^\frac{\gamma}{4r} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{4r} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{4r} + \frac{1}{n}\right)} \|f\|_{L_p(B(x_0,t))} dt.$$

In a similar way, $F_3$ has the same estimate as above, so we omit the details. Then we have that

$$F_3 \lesssim \|b_1\|_{L_{p_1}^{(x_0)}} \|b_2\|_{L_{p_2}^{(x_0)}} \tau^\frac{\gamma}{4r} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{4r} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{4r} + \frac{1}{n}\right)} \|f\|_{L_p(B(x_0,t))} dt.$$

Now let us consider the term $F_4$. Let $1 < q, \tau < \frac{2p}{n}$, such that $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$, $\frac{1}{\tau} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q_1} = \frac{1}{q} - \frac{1}{n}$. Then by the boundedness of $T_{1,\alpha}$ from $L_q$ into $L_{q_1}$ (see Lemma 2), Hölder’s inequality and (1.18), we obtain

$$F_4 = \|T_{1,\alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_1)\|_{L_{q_1}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(B)} \|T_{1,\alpha} ((b_2 - (b_2)_B) f_1)\|_{L_{\tau}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(B)} \|(b_2 - (b_2)_B) f_1\|_{L_{\tau}(B)} \lesssim \|(b_1 - (b_1)_B)\|_{L_{p_1}(2B)} \|b_2 - (b_2)_B\|_{L_{p_2}(2B)} \|f\|_{L_p(2B)} \lesssim \|b_1\|_{L_{p_1}^{(x_0)}} \|b_2\|_{L_{p_2}^{(x_0)}} \tau^\frac{\gamma}{4r} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{4r} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{4r} + \frac{1}{n}\right)} \|f\|_{L_p(B(x_0,t))} dt.$$

Combining all the estimates of $F_1, F_2, F_3, F_4$; we get

$$F = \|(b_1, b_2) , T_{1,\alpha} f_1\|_{L_{q_1}(B)} \lesssim \|b_1\|_{L_{p_1}^{(x_0)}} \|b_2\|_{L_{p_2}^{(x_0)}} \tau^\frac{\gamma}{4r} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{4r} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{4r} + \frac{1}{n}\right)} \|f\|_{L_p(B(x_0,t))} dt.$$
Now, let us estimate $G = \|[(b_1, b_2), T_{\Omega, \alpha}]f_2\|_{L^q(B)}$. For $G$, it’s similar to (1.10) we also write

$$G = \|[(b_1, b_2), T_{\Omega, \alpha}]f_2\|_{L^q(B)} \lesssim$$

$$\|b_1 - (b_1)_B\| (b_2 (x) - (b_2)_B) T_{\Omega, \alpha} f_2\|_{L^q(B)}$$

$$+ \|b_1 - (b_1)_B\| T_{\Omega, \alpha} ((b_2 - (b_2)_B) f_2)\|_{L^q(B)}$$

$$+ \|b_2 - (b_2)_B\| T_{\Omega, \alpha} ((b_1 - (b_1)_B) f_2)\|_{L^q(B)}$$

$$+ T_{\Omega, \alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_2)\|_{L^q(B)}$$

$$\equiv G_1 + G_2 + G_3 + G_4.$$

To estimate $G_1$, let $1 < p_1, p_2 < \frac{2n}{\alpha}$, such that $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{p} - \frac{n}{n}$. Then, using Hölder’s inequality, noting that in (1.13) and by (1.17), we have

$$G_1 = \|[(b_1 - (b_1)_B) (b_2 (x) - (b_2)_B)] T_{\Omega, \alpha} f_2\|_{L^q(B)}$$

$$\lesssim \|b_1 - (b_1)_B\| (b_2 - (b_2)_B) T_{\Omega, \alpha} f_2\|_{L^q(B)}$$

$$\lesssim \|b_1 - (b_1)_B\| L^{p_1 (r_1)} ||b_2 - (b_2)_B||_{L^r p_2 (r_2)} \int_{2r}^\infty \|f\|_{L^p (B(x_0, t))} t^{-\frac{n}{p} - 1} dt$$

$$\lesssim \|b_1\| L^{p_1 (r_1)} \|b_2\|_{L^r p_2 (r_2)} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{p} + n(\lambda_1 + \lambda_2) - 1 + \alpha} \|f\|_{L^p (B(x_0, t))} dt$$

$$\lesssim \|b_1\| L^{p_1 (r_1)} \|b_2\|_{L^r p_2 (r_2)} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{p} + n(\lambda_1 + \lambda_2) + n(\frac{n}{r_1} + \frac{n}{r_2}) - 1} \|f\|_{L^p (B(x_0, t))} dt.$$

On the other hand, for the estimates used in $G_2, G_3$, we have to prove the below inequality:

(1.19)

$$|T_{\Omega, \alpha} ((b_2 - (b_2)_B) f_2) (x)| \lesssim \|b_2\| L^{p_1 (r_1)} \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{n}{p} + n\lambda_2 - 1 + \alpha} \|f\|_{L^p (B(x_0, t))} dt.$$

Indeed, when $s' \leq q$, for $x \in B$, by Fubini’s theorem and applying Hölder’s inequality and from (1.13), (1.14), (1.12) we have

$$|T_{\Omega, \alpha} ((b_2 - (b_2)_B) f_2) (x)| \lesssim \int_{(2B)^\circ} |b_2 (y) - (b_2)_B| ||\Omega(x, x - y)|| \frac{|f(y)|}{|y|^{n - \alpha}} dy$$

$$\lesssim \int_{(2B)^\circ} |b_2 (y) - (b_2)_B| ||\Omega(x, x - y)|| \frac{|f(y)|}{|x_0 - y|^{n - \alpha}} dy$$

$$\approx \int_{2r}^\infty \int_{|x_0 - y| < t} |b_2 (y) - (b_2)_B| ||\Omega(x, x - y)|| |f (y)| dy \frac{dt}{y^{n - \alpha}}.$$
\[
\begin{align*}
\lambda \int \int_{2r}^{\infty} & \left| b_{2}(y) - (b_{2})_{B(x_0, t)} \right| \left| \Omega(x, x - y) \right| \left| f(y) \right| dy \frac{dt}{\tau_{n-\alpha+1}} \\
+ \int_{2r}^{\infty} \left| (b_{2})_{B(x_0, r)} - (b_{2})_{B(x_0, t)} \right| \int_{B(x_0, t)} \left| \Omega(x, x - y) \right| \left| f(y) \right| dy \frac{dt}{\tau_{n-\alpha+1}} \\
\lambda \int \int_{2r}^{\infty} & \left\| b_{2}(\cdot) - (b_{2})_{B(x_0, t)} \right\|_{L_{p_2}(B(x_0, t))} \left\| \Omega(x, x - \cdot) \right\|_{L_{\alpha}(B(x_0, t))} \left\| f \right\|_{L_{p}(B(x_0, t))} \left| B(x_0, t) \right|^{1 - \frac{1}{p_2} - \frac{1}{\tau_{n-\alpha+1}}} \\
+ \int_{2r}^{\infty} & \left\| (b_{2})_{B(x_0, r)} - (b_{2})_{B(x_0, t)} \right\|_{L_{p}(B(x_0, t))} \left\| \Omega(x, x - \cdot) \right\|_{L_{\alpha}(B(x_0, t))} \left| B(x_0, t) \right|^{1 - \frac{1}{p_2} - \frac{1}{\tau_{n-\alpha+1}}} \\
\lesssim & \left\| b \right\|_{L_{p_2}(x_0)} \int_{2r}^{\infty} (1 + \ln \frac{t}{\tau_{n-\alpha+1}}) t^{-\frac{\alpha}{\tau_{n-\alpha+1}} + \frac{n}{2}} \left\| f \right\|_{L_{p}(B(x_0, t))} dt \\
\end{align*}
\]

This completes the proof of inequality (1.19).

Let $1 < \tau < \infty$, such that $\frac{1}{q_1} = \frac{1}{p_1} + \frac{1}{\tau}$ and $\frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{\tau} - \frac{2}{n}$. Then, using Hölder's inequality and from (1.19) and (1.24), we get

\[
\begin{align*}
G_2 & = \left\| (b_1 - (b_1)_B) T_{\Omega, \alpha} ((b_2 - (b_2)_B) f_2) \right\|_{L_{q_1}(B)} \\
& \leq \left\| b_1 - (b_1)_B \right\|_{L_{p_1}(B)} \left\| T_{\Omega, \alpha} ((b_2 - (b_2)_B) f_2) \right\|_{L_{\tau}(B)} \\
& \leq \left\| b_1 - (b_1)_B \right\|_{L_{p_1}(B)} \left\| b_2 \right\|_{L_{p_2}(x_0)} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{\tau_{n-\alpha+1}}\right) t^{-\frac{\alpha}{\tau_{n-\alpha+1}} + \frac{n}{2}} \left\| f \right\|_{L_{p}(B(x_0, t))} dt \\
& \lesssim \left\| b_1 \right\|_{L_{p_1}(x_1, \lambda_1)} \left\| b_2 \right\|_{L_{p_2}(x_0, \lambda_2)} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{\tau_{n-\alpha+1}}\right) t^{-\frac{\alpha}{\tau_{n-\alpha+1}} + \frac{n}{2}} + \left(\frac{1}{\tau_{n-\alpha+1}} + \frac{1}{\tau}\right)^{-1} \left\| f \right\|_{L_{p}(B(x_0, t))} dt. \\
\end{align*}
\]

Similarly, $G_3$ has the same estimate above, so here we omit the details. Then the inequality

\[
G_3 = \left\| (b_2 - (b_2)_B) T_{\Omega, \alpha} ((b_1 - (b_1)_B) f_2) \right\|_{L_{q_1}(B)} \\
\lesssim \left\| b_1 \right\|_{L_{p_1}(x_1, \lambda_1)} \left\| b_2 \right\|_{L_{p_2}(x_0, \lambda_2)} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{\tau_{n-\alpha+1}}\right) t^{-\frac{\alpha}{\tau_{n-\alpha+1}} + \frac{n}{2}} + \left(\frac{1}{\tau_{n-\alpha+1}} + \frac{1}{\tau}\right)^{-1} \left\| f \right\|_{L_{p}(B(x_0, t))} dt
\]
is valid.

Now, let us estimate $G_4 = \left\| T_{\Omega, \alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_2) \right\|_{L_{q_1}(B)}$. It’s similar to the estimate of (1.19), for any $x \in B$, we also write $|T_{\Omega, \alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_2) (x)|$
\[ G_4 \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt. \]

Secondly, to estimate \( G_{42} \) and \( G_{43} \), from (1.19), (1.23) and (1.24), it follows that

\[ G_{42} \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt, \]

and

\[ G_{43} \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt. \]

Finally, to estimate \( G_{44} \), similar to the estimate of (1.19) and from (1.23) and (1.24), we have

\[ G_{44} \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt. \]

By the estimates of \( G_j \) above, where \( j = 1, 2, 3 \), we know that

\[ |T_{\Omega, \alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_2) (x)| \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt. \]

Then, we have

\[ G_4 = \|T_{\Omega, \alpha} ((b_1 - (b_1)_B) (b_2 - (b_2)_B) f_2)\|_{L^{\infty}_{\lambda_1}} \lesssim \|b_1\|_{L^{\infty}_{\lambda_1}} \cdot \|b_2\|_{L^{\infty}_{\lambda_2}} \cdot \frac{\mu}{\lambda_1} \cdot \int_{2r}^\infty \left(1 + \ln \frac{t}{r}\right)^2 t^{-\frac{\alpha}{q_1} + n(\lambda_1 + \lambda_2) + n\left(\frac{1}{p_1} + \frac{1}{p_2}\right)} \|f\|_{L^p(B(x_0, t))} \, dt. \]
So, combining all the estimates for \( G_1, G_2, G_3, G_4 \), we get
\[
G = \left\| (b_1, b_2), T_{\Omega, \alpha} \right\|_{L^{q_1}(B)} \lesssim \left\| b_1 \right\|_{LC_p^{(x_0)}} \left\| b_2 \right\|_{LC_p^{(x_0)}} \cdot \frac{1}{r^{\frac{s}{q_1}}}
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{- \frac{m}{2} + n(\lambda_1 + \lambda_2) + n \left( \frac{1}{m} + \frac{1}{r} \right)} \|f\|_{L_p(B(x_0, t))} \, dt.
\]
Thus, putting estimates \( F \) and \( G \) together, we get the desired conclusion
\[
\left\| (b_1, b_2), T_{\Omega, \alpha} \right\|_{L^{q_1}(B(x_0, r))} \lesssim \left\| b_1 \right\|_{LC_p^{(x_0)}} \left\| b_2 \right\|_{LC_p^{(x_0)}} \cdot \frac{1}{r^{\frac{s}{q_1}}}
\times \int_2^\infty \left( 1 + \ln \frac{t}{r} \right)^2 t^{- \frac{m}{2} + n(\lambda_1 + \lambda_2) + n \left( \frac{1}{m} + \frac{1}{r} \right)} \|f\|_{L_p(B(x_0, t))} \, dt.
\]
For the case of \( q_1 < s \), we can also use the same method, so we omit the details.
This completes the proof of Lemma 4. \( \Box \)

At last, throughout the paper we use the letter \( C \) for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence. By \( A \lesssim B \) we mean that \( A \leq CB \) with some positive constant \( C \) independent of appropriate quantities. If \( A \lesssim B \) and \( B \lesssim A \), we write \( A \approx B \) and say that \( A \) and \( B \) are equivalent.

2. Main Results

Now we are ready to give the following main results with their proofs, respectively.

**Theorem 1.** Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_\infty(\mathbb{R}^n) \times L_s(S^{n-1}) \), \( s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( T_{\Omega, \alpha} \) be a linear operator satisfying condition \( \square \). Let also \( 0 < \alpha < n \) and \( 1 < q, q_1, p, \frac{p}{q} < \frac{n}{\alpha} \) with \( \frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{p} \), \( \frac{1}{q_1} = \frac{1}{q} - \frac{n}{\alpha} \), and \( b \in LC_p^{(x_0)}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{n}, i = 1, \ldots, m \).

Let also, for \( s' \leq q \) the pair \((\varphi_1, \varphi_2)\) satisfies the condition
\[
\left( \begin{array}{c}
\int_r^{\infty} \left( 1 + \ln \frac{t}{r} \right)^m \\
\left( \frac{m}{\alpha} \right)
\end{array} \right)^{\frac{1}{m}} \leq C \varphi_2(x_0, r),
\]
and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfies the condition
\[
\left( \begin{array}{c}
\int_r^{\infty} \left( 1 + \ln \frac{t}{r} \right)^m \\
\left( \frac{m}{\alpha} \right)
\end{array} \right)^{\frac{1}{m}} \leq C \varphi_2(x_0, r),
\]
where \( C \) does not depend on \( r \).

Then, the operator \( \left[ \widetilde{b} \right], T_{\Omega, \alpha} \) is bounded from \( LM_p^{(x_0)} \) to \( LM_q^{(x_0)} \). Moreover,
\[
\left\| \left[ \widetilde{b}, T_{\Omega, \alpha} \right] \right\|_{LM_q^{(x_0)}} \lesssim \prod_{i=1}^m \left\| \widetilde{b} \right\|_{LC_p^{(x_0)}} \left\| f \right\|_{LM_p^{(x_0)}}.
\]
Proof. Since \( f \in LM_{P_1, \varphi_1}^{x_0} \), by (1.3) and it is also non-decreasing, with respect to \( t \), of the norm \( \| f \|_{L_p(B(x_0, t))} \), we get

\[
\begin{align*}
\text{essinf}_{0 < t < \tau < \infty} & \varphi_1(x_0, \tau) \tau^{\frac{m}{p}} \leq \text{esssup}_{0 < t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}} \\
& \leq \text{esssup}_{0 < \tau < \infty} \| f \|_{L_p(B(x_0, \tau))} \leq \| f \|_{LM_{P_1, \varphi_1}^{x_0}} .
\end{align*}
\]

(2.3)

For \( s' \leq q < \infty \), since \((\varphi_1, \varphi_2)\) satisfies (2.1) and by (2.3), we have

\[
\begin{align*}
\int \left( 1 + \ln \frac{t}{r} \right)^m \left( -\frac{1}{m} + \left( \sum_{i=1}^m \lambda_i + \sum_{n=1}^n \frac{1}{n} \right) \right)^{-1} \| f \|_{L_p(B(x_0, t))} dt \\
& \leq \int \left( 1 + \ln \frac{t}{r} \right)^m \text{essinf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}} \frac{\text{esssup}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}}}{n \left( \frac{1}{m} - \left( \sum_{i=1}^m \lambda_i + \sum_{n=1}^n \frac{1}{n} \right) \right)^{+1}} dt \\
& \leq C \| f \|_{LM_{P_1, \varphi_1}^{x_0}} \int \left( 1 + \ln \frac{t}{r} \right)^m \text{essinf}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}} \frac{\text{esssup}_{t < \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}}}{n \left( \frac{1}{m} - \left( \sum_{i=1}^m \lambda_i + \sum_{n=1}^n \frac{1}{n} \right) \right)^{+1}} dt \\
& \leq C \| f \|_{LM_{P_1, \varphi_1}^{x_0}} \varphi_2(x_0, r).
\end{align*}
\]

(2.4)

Then by (1.10) and (2.4), we get

\[
\begin{align*}
\left\| \overrightarrow{b} T_{\Omega, \alpha} f \right\|_{LM_{P_1, \varphi_2}^{x_0}} = & \sup_{r > 0} \varphi_2(x_0, r)^{-1} \| B(x_0, r) \|^{-\frac{1}{q}} \left\| \overrightarrow{b} T_{\Omega, \alpha} f \right\|_{L_{q_1}(B(x_0, r))} \\
& \leq C \prod_{i=1}^m \| \overrightarrow{b} \|_{L^{q_1, \alpha}_{P_i, \lambda_i}} \sup_{r > 0} \varphi_2(x_0, r)^{-1} \\
& \times \int \left( 1 + \ln \frac{t}{r} \right)^m \left( -\frac{1}{m} + \left( \sum_{i=1}^m \lambda_i + \sum_{n=1}^n \frac{1}{n} \right) \right)^{-1} \| f \|_{L_p(B(x_0, t))} dt \\
& \leq C \prod_{i=1}^m \| \overrightarrow{b} \|_{L^{q_1, \alpha}_{P_i, \lambda_i}} \| f \|_{LM_{P_1, \varphi_1}^{x_0}} .
\end{align*}
\]

For the case of \( q_1 < s \), we can also use the same method, so we omit the details.

Thus, we finish the proof of Theorem [1].

\[ \square \]

Corollary 2. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L^\infty(\mathbb{R}^n) \times L^s(S^{n-1}) \), \( s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( 0 < \alpha < n \) and \( 1 < q, q_1, p_i, p < \frac{n}{\alpha} \) with \( \frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} + \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{n} \) and \( \overrightarrow{b} \in LC^{(x_0)}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{n} \), \( i = 1, \ldots, m \). Let also, for \( s' \leq q \) the pair \((\varphi_1, \varphi_2)\) satisfy condition (2.1) and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfy condition (2.2). Then, the operators \( M_{\Omega, \beta, \alpha} \) and \( \left[ \overrightarrow{b}, T_{\Omega, \alpha} \right] \) are bounded from \( LM_{P_1, \varphi_1}^{x_0} \) to \( LM_{q_1, \varphi_2}^{x_0} \).
Corollary 3. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_{\infty}(\mathbb{R}^n) \times L_s(S^{n-1}) \), \( s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( T_{\Omega,\alpha} \) be a linear operator satisfying condition (1.7) and bounded from \( L_p(\mathbb{R}^n) \) to \( L_q(\mathbb{R}^n) \).

Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha} \), \( b \in L_C^{(x_0)}(\mathbb{R}^n) \), \( 0 \leq \lambda < \frac{1}{n}, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \frac{1}{q_1} = \frac{1}{p_1} - \frac{n}{m} \).

Let also, for \( s' \leq p \) the pair \((\varphi_1, \varphi_2)\) satisfy the condition

\[
\int_{r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right) \frac{\text{essinf}_{t \leq \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}}}{t^{\frac{1}{p_1} + 1 - \alpha n}} dt \leq C \varphi_2(x_0, r),
\]

and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfy the condition

\[
\int_{r}^{\infty} \left( 1 + \frac{t}{r} \right) \frac{\text{essinf}_{t \leq \tau < \infty} \varphi_1(x_0, \tau) \tau^{\frac{m}{p}}}{t^{\frac{1}{q_1} + 1 - \alpha n}} dt \leq C \varphi_2(x_0, r)^{\frac{m}{q}} ,
\]

where \( C \) does not depend on \( r \).

Then, the operator \([b, T_{\Omega,\alpha}]\) is bounded from \( LM_{p_1, q_1}^{(x_0)} \) to \( LM_{q_2, p_2}^{(x_0)} \). Moreover,

\[
\|b, T_{\Omega,\alpha}\|_{LM_{q_2, p_2}^{(x_0)}} \leq \|b\|_{LC_{p_2, \lambda}^{(x_0)}} \|f\|_{LM_{p_1, q_1}^{(x_0)}} .
\]

Theorem 2. Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_{\infty}(\mathbb{R}^n) \times L_s(S^{n-1}) \), \( s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( T_{\Omega,\alpha} \) be a linear operator satisfying condition (1.7). Let also \( 0 < \alpha < n \) and \( 1 < q, q_1, p_1, p < \frac{n}{\alpha} \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q_1} = \frac{1}{q_2} - \frac{\alpha}{n} \) and \( b \in L_C^{(x_0)}(\mathbb{R}^n) \) for \( 0 \leq \lambda < \frac{1}{n}, i = 1, \ldots, m \).

Let for \( s' \leq q \) the pair \((\varphi_1, \varphi_2)\) satisfies conditions (1.3)-(1.6) and

\[
(2.5) \quad \int_{r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right) \varphi_1(x_0, t) \frac{t^{\frac{m}{p}}}{n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) + 1} \leq C_0 \varphi_2(x_0, r),
\]

where \( C_0 \) does not depend on \( r > 0 \),

\[
(2.6) \quad \lim_{r \to 0} \frac{\ln \left( \frac{1}{r} \right)}{\varphi_2(x_0, r)} = 0
\]

and

\[
(2.7) \quad c_\delta := \int_{\delta}^{\infty} \left( 1 + \ln |t| \right)^m \varphi_1(x_0, t) \frac{t^{\frac{m}{p}}}{n \left( \frac{1}{p_1} + \frac{1}{p_2} \right) + 1} dt < \infty
\]

for every \( \delta > 0 \), and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfies conditions (1.3)-(1.6) and also

\[
(2.8) \quad \int_{r}^{\infty} \left( 1 + \ln \left( \frac{t}{r} \right) \right) \varphi_1(x_0, t) \frac{t^{\frac{m}{p}}}{n \left( \frac{1}{q_1} + \frac{1}{q_2} \right) + 1} \leq C \varphi_2(x_0, r)^{\frac{m}{q}},
\]
where $C_0$ does not depend on $r > 0$,

$$
\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\varphi_2(x_0, r)} = 0
$$

and

$$
c_{\delta'} := \int_{\delta'}^{\infty} \left(1 + \ln |t|\right)^m \varphi_1(x_0, t) \frac{t^{\frac{p}{q}}}{t \left(\frac{1}{q} - \frac{m}{n} + \sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}\right)} \, dt < \infty
$$

for every $\delta' > 0$.

Then the operator $[\overrightarrow{b}, T_{\Omega, \alpha}]$ is bounded from $VLM_{p, \varphi_1}^{(x_0)}$ to $VLM_{p, \varphi_2}^{(x_0)}$. Moreover,

$$
\left\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \right\|_{VLM_{p, \varphi_2}^{(x_0)}} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{LC_{p_i, \lambda_i}} \| f \|_{VLM_{p, \varphi_1}^{(x_0)}}.
$$

Proof. Since the inequality (2.10) holds by Theorem 1 we only have to prove the implication

$$
\lim_{r \to 0} r^{-\frac{1}{q}} \| f \|_{L^p(B(x_0, r))} = 0 \quad \implies \quad \lim_{r \to 0} r^{-\frac{1}{q}} \left\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \right\|_{L^q(B(x_0, r))} = 0.
$$

To show that

$$
\frac{r^{-\frac{1}{q}} \left\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \right\|_{L^q(B(x_0, r))}}{\varphi_2(x_0, r)} < \epsilon
$$

for small $r$, we use the estimate (2.15):

$$
\frac{r^{-\frac{1}{q}} \left\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \right\|_{L^q(B(x_0, r))}}{\varphi_2(x_0, r)} \lesssim \prod_{i=1}^{m} \| \overrightarrow{b} \|_{LC_{p_i, \lambda_i}} \int_{\delta}^{\infty} \frac{1 + \ln \frac{t}{r}}{t} \left(1 + \ln \frac{t}{r}\right)^m \left[n - \frac{1}{q} + \left(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}\right)\right] \| f \|_{L^p(B(x_0, t))} \, dt.
$$

We take $r < \delta_0$, where $\delta_0$ is small enough and split the integration:

$$
\frac{r^{-\frac{1}{q}} \left\| [\overrightarrow{b}, T_{\Omega, \alpha}] f \right\|_{L^q(B(x_0, r))}}{\varphi_2(x_0, r)} \leq C \left[ I_{\delta_0}(x_0, r) + J_{\delta_0}(x_0, r) \right],
$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$
I_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_{\delta_0}^{\delta_0} \left(1 + \ln \frac{t}{r}\right)^m \left[n - \frac{1}{q} + \left(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}\right)\right] \| f \|_{L^p(B(x_0, t))} \, dt,
$$

and

$$
J_{\delta_0}(x_0, r) := \frac{1}{\varphi_2(x_0, r)} \int_{\delta_0}^{\infty} \left(1 + \ln \frac{t}{r}\right)^m \left[n - \frac{1}{q} + \left(\sum_{i=1}^{m} \lambda_i + \sum_{i=1}^{m} \frac{1}{p_i}\right)\right] \| f \|_{L^p(B(x_0, t))} \, dt.
$$
and \( r < \delta_0 \). Now we can choose any fixed \( \delta_0 > 0 \) such that
\[
\frac{t^{-\frac{q}{p}} \| f \|_{L_p(B(x_0, t))}}{\varphi_1(x_0, t)} < \frac{\epsilon}{2C_0}, \quad t \leq \delta_0,
\]
where \( C \) and \( C_0 \) are constants from (2.5) and (2.12). This allows to estimate the first term uniformly in \( r \in (0, \delta_0) \):
\[
C \delta_0(x_0, r) < \frac{\epsilon}{2}, \quad 0 < r < \delta_0.
\]
For the second term, writing \( 1 + \ln \frac{1}{r} \leq 1 + |\ln t| + \ln \frac{1}{r} \), we obtain
\[
J_{\delta_0}(x_0, r) \leq \frac{c_{\delta_0} + c_{\delta_0} \ln \frac{1}{r}}{\varphi_2(x_0, r)} \| f \|_{L_{p,s}(x_0)},
\]
where \( c_{\delta_0} \) is the constant from (2.7) with \( \delta = \delta_0 \) and \( c_{\delta_0} \) is a similar constant with omitted logarithmic factor in the integrand. Then, by (2.6) we can choose small enough \( r \) such that
\[
J_{\delta_0}(x_0, r) < \frac{\epsilon}{2},
\]
which completes the proof of (2.11).

For the case of \( q_1 < s \), we can also use the same method, to obtain the desired result. Therefore, the proof of Theorem 2 is completed. \( \square \)

**Remark 2.** Conditions (2.7) and (2.9) are not needed in the case when \( \varphi(x_0, r) \) does not depend on \( x_0 \), since (2.7) follows from (2.5) and similarly, (2.9) follows from (2.5) in this case.

**Corollary 4.** Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_{\infty}(\mathbb{R}^n) \times L_s(S^{n-1}), s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( 0 < \alpha < n \) and \( 1 < q, q_1, p, \gamma < \frac{n}{\alpha} \) with \( \frac{1}{q} = \sum_{i=1}^{m} \frac{1}{p_i} + \frac{1}{p}, \frac{1}{q_1} = \frac{1}{q} - \frac{\alpha}{n} \) and \( \bar{b} \in L_{p, s}(\mathbb{R}^n) \) for \( 0 \leq \lambda_i < \frac{1}{1}, i = 1, \ldots, m \). Let also, for \( s' \leq q \) the pair \((\varphi_1, \varphi_2)\) satisfies conditions (1.8), (1.10), (2.8) and (2.9) and for \( q_1 < s \) the pair \((\varphi_1, \varphi_2)\) satisfies conditions (1.3), (1.7), (2.8) and (2.9). Then, the operators \( M_{11, b, \alpha} \) and \( [\bar{b}, \mathcal{R}_{\alpha, \alpha}] \) are bounded from \( VLM_{p, \varphi_1} \) to \( VLM_{p, \varphi_2} \).

**Corollary 5.** Suppose that \( x_0 \in \mathbb{R}^n \), \( \Omega \in L_{\infty}(\mathbb{R}^n) \times L_s(S^{n-1}), s > 1 \), is homogeneous of degree zero with respect to the second variable \( y \) on \( \mathbb{R}^n \). Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, b \in L_{p_2, \lambda}(\mathbb{R}^n), 0 \leq \lambda < \frac{1}{n}, \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \frac{1}{q_1} = \frac{1}{p_1} - \frac{\alpha}{n}, \) and \( \mathcal{T}_{\alpha, \alpha} \) is a linear operator satisfying condition (1.7) and bounded from \( L_p(\mathbb{R}^n) \) to \( L_{\gamma}(\mathbb{R}^n) \). Let also, for \( s' \leq p \) the pair \((\varphi_1, \varphi_2)\) satisfies conditions (1.3), (1.7) and
\[
\int_{r}^{\infty} \left( 1 + \ln \frac{t}{r} \right)^{\varphi_1(x_0, t)} \frac{t^{\frac{n}{p_1}} \| f \|_{L_p(B(x_0, t))}}{t^{\frac{n}{p_1} + 1 - \lambda \alpha}} dt \leq C_0 \varphi_2(x_0, r),
\]
where \( C_0 \) does not depend on \( r > 0 \),
\[
\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\varphi_2(x_0, r)} = 0
\]
and
\[
c_{\delta} := \int_{\delta}^{\infty} (1 + \ln |t|)^{\varphi_1(x_0, t)} \frac{t^{\frac{n}{p_1}} \| f \|_{L_p(B(x_0, t))}}{t^{\frac{n}{p_1} + 1 - \lambda \alpha}} dt < \infty.
\]
for every $\delta > 0$, and for $q_1 < s$ the pair $(\varphi_1, \varphi_2)$ satisfies conditions (1.3)-(1.6) and also

$$\int_r^\infty \left( 1 + \ln \frac{t}{r} \right) \varphi_1(x_0, t) \frac{t^{\frac{m}{p_1}}}{t^{\frac{m}{q_1}} + 1 - n\lambda} dt \leq C_0 \varphi_2(x_0, r)r^{\frac{s}{n}},$$

where $C_0$ does not depend on $r > 0$,

$$\lim_{r \to 0} \frac{1}{\varphi_2(x_0, r)} = 0$$

and

$$c_{\delta'} := \int_{\delta'}^\infty \left( 1 + \ln |t| \right) \varphi_1(x_0, t) \frac{t^{\frac{m}{p_1}}}{t^{\frac{m}{q_1}} + 1 - n\lambda} dt < \infty$$

for every $\delta' > 0$.

Then the operator $[b, T_{\Omega,\alpha}]$ is bounded from $VLM_{p_1,\varphi_1}^{(x_0)}$ to $VLM_{q,\varphi_2}^{(x_0)}$. Furthermore, we have

$$\|[b, T_{\Omega,\alpha}]f\|_{VLM_{q,\varphi_2}^{(x_0)}} \lesssim \|b\|_{L_{C_2}^{(x_0)}} \|f\|_{VLM_{p_1,\varphi_1}^{(x_0)}}.$$

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