THE UNIVERSE INSIDE HALL ALGEBRAS OF COHERENT SHEAVES ON TORIC RESOLUTIONS

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**Abstract.** Let \( g \neq so_8 \) be a simple Lie algebra of type \( A, D, E \) with \( \hat{g} \) the corresponding affine Kac-Moody algebra and \( n_- \subset \hat{g} \) a nilpotent subalgebra. Given \( n_- \) as above, we provide an infinite collection of cyclic finite abelian subgroups of \( SL_3(\mathbb{C}) \) with the following properties. Let \( G \) be any group in the collection, \( Y = G - \text{Hilb}(\mathbb{C}^3) \) and \( \Psi : \text{Db}_G(\text{Coh}(\mathbb{C}^3)) \to \text{Db}(\text{Coh}(Y)) \) the derived equivalence of Bridgeland, King and Reid. We present an (explicitly described) subset of objects in \( \text{Coh}_G(\mathbb{C}^3) \), s.t. the Hall algebra generated by their images under \( \Psi \) is isomorphic to \( U(n_-) \). In case the field \( k \) (in place of \( \mathbb{C} \)) is finite and \( \text{char}(k) \) is coprime with the order of \( G \), we conjecture the isomorphisms of the corresponding ‘counting’ Ringel-Hall algebras and the specializations of quantized universal enveloping algebras \( U_v(n_-) \) at \( v = \sqrt{|k|} \).

1. Introduction

We begin with describing different versions of McKay correspondence. Let \( G \subset GL_n(\mathbb{C}) \) be a finite subgroup and consider the categorical quotient \( X = \mathbb{C}^n/G := \text{Spec}(\mathbb{C}[x_1, x_2, \ldots, x_n]^G) \). In addition assume the following properties:

- \( X \) has an isolated singularity at \( 0 \);
- there exists a projective resolution \( \rho : Y \to X \).

One way to understand McKay correspondence is via the bijection

\[
\begin{array}{c}
\left\{ \text{nontrivial irreducible representations of } G \right\} \\
\left\{ \text{irreducible components of the central fiber } \rho^{-1}(0) \right\}
\end{array} \xleftrightarrow{1:1} \left\{ \text{of } \text{Coh}_G(\mathbb{C}^n) \right\}
\]

Let \( \text{Coh}_G(\mathbb{C}^n) \) be the category of \( G \)-equivariant coherent sheaves on \( \mathbb{C}^n \) and \( \text{Coh}(Y) \) be the category of coherent sheaves on \( Y \). In modern language the McKay correspondence is usually understood as an equivalence of triangulated categories \( \text{Db}_G(\text{Coh}(\mathbb{C}^n)) \) and \( \text{Db}(\text{Coh}(Y)) \). Such an equivalence is known to hold in the following cases:

- \( G \subset SL_2(\mathbb{C}) \), any \( G \) ([KV00]);
- \( G \subset SL_3(\mathbb{C}), \text{any } G, Y = G - \text{Hilb}(\mathbb{C}^3) \) ([BKR01]);
- \( G \subset SL_3(\mathbb{C}), \text{any abelian } G \) and any crepant resolution \( (Y, \rho) \) ([CI04]);
- \( G \subset SP_{2n}(\mathbb{C}), \text{any } G \) and crepant symplectic resolution \( (Y, \rho) \) ([BK04]);
- \( G \subset SL_n(\mathbb{C}), \text{any abelian } G \) and any crepant symplectic resolution \( (Y, \rho) \) ([Kaw16]).

Any finite-dimensional representation \( V \) of \( G \) gives rise to two equivariant sheaves on \( \mathbb{C}^n \):

the skyscraper sheaf \( V^! = V \otimes_\mathbb{C} \mathcal{O}_0 \), whose fiber at \( 0 \) is \( V \) and all the other fibers vanish;

- the locally free sheaf \( \check{V} = V \otimes_\mathbb{C} \mathcal{O}_{\mathbb{C}^n} \).

Suppose the map \( \Psi : D^b_b(\mathbb{C}^n) \to D^b(Y) \) is an exact equivalence of triangulated categories. A natural question emerges.

**Question.** What are the images of \( \check{\chi} \) and \( \chi^i \) (for nontrivial irreducible representations \( \chi \) of \( G \)) under the equivalence \( \Psi \)?

The former, \( \Psi(\check{\chi}) \), is a vector bundle of dimension \( \dim(\check{\chi}) \) and is called a tautological or \( \text{GS}p-V \) sheaf (after Gonzales-Sprinberg and Verdier). Relatively little is known about \( \Psi(\chi^i) \).

The following results are due to Kapranov, Vasserot (see [KV00]) and Cautis, Craw, Logvinenko (see [CCL17]).

1. Let \( G \subset \text{SL}_2(\mathbb{C}) \) be a finite subgroup and \( \chi \in \text{Irr}(G) \setminus \text{triv} \). Then \( \Psi(\chi^i) \simeq \mathcal{O}_{\mathbb{P}^1(-1)[1]} \).
2. Let \( G \subset \text{SL}_3(\mathbb{C}) \) be a finite abelian subgroup. Then for any \( \rho \in \text{Irr}(G) \setminus \text{triv} \), the object \( \Psi(\chi^i) \in D^b_b(\text{Coh}(Y)) \) is pure (here \( Y = G - \text{Hilb}(\mathbb{C}^3) \) and an object is called pure provided all cohomology, except in a single degree, vanish).

Let \( \mathcal{C} \) be a \( \mathbb{C} \)-linear finitary abelian category (the former means that for any two objects \( A, B \in \mathcal{C} \) and \( i \in \mathbb{Z}_{\geq 0} \) one has \( \dim(\text{Ext}^i(A, B)) < \infty \)). There is a way to associate an algebra \( \mathcal{H}(\mathcal{C}) \) to \( \mathcal{C} \), which is called the Hall algebra of \( \mathcal{C} \) (see Section 4 and references therein). In Section 3 of [KV00] it was observed that in case \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are \( \mathbb{C} \)-linear finitary abelian categories, there is a derived equivalence \( \Psi : D^b_b(\mathcal{C}_1) \to D^b_b(\mathcal{C}_2) \) and a collection of objects \( \{a_1, \ldots, a_n\} \) in \( \mathcal{C}_1 \), s.t. \( \Psi(a_i) \) are all pure and concentrated in the same degree, then the Hall algebra generated by the objects \( \{a_1, \ldots, a_n\} \) is isomorphic to the Hall algebra generated by their images \( \{\Psi(a_1), \ldots, \Psi(a_n)\} \).

Next suppose \( Q \) is a quiver with no oriented cycles and let \( \text{Rep}(Q) \) be the category of representations of \( Q \). Let \( \mathcal{P}_Q \) be the path algebra of \( Q \). It is well known that the categories \( \text{Rep}(Q) \) and \( \mathcal{P}_Q \)-mod are equivalent. The assumptions on \( Q \) imply that the category \( \text{Rep}(Q) \) is hereditary, i.e. \( \text{Ext}^i(A, B) = 0 \) for any \( i \geq 2 \). Furthermore, both \( \dim(\text{Hom}(A, B)) \) and \( \dim(\text{Ext}^1(A, B)) < \infty \), while the presence of equivalence of categories \( \text{Rep}(Q) \simeq \mathcal{P}_Q \)-mod with the latter category being abelian guarantees that \( \text{Rep}(Q) \) is an abelian category as well. There is a natural way to associate a Kac-Moody Lie algebra \( \mathfrak{g}_Q \) to \( Q \). Namely, the Cartan matrix for \( \mathfrak{g}_Q \) is \( C = 2 \cdot I - A_Q - A_Q^t \), where \( A_Q \) is the adjacency matrix of \( Q \).

There is an embedding \( U(n_{-}) \to \mathcal{H}(\text{Rep}(Q)) \), where \( \mathcal{H}(\text{Rep}(Q)) \) is the Hall algebra for the category of quiver representations. The image is isomorphic to the composition subalgebra of \( \mathcal{H}(\text{Rep}(Q)) \) generated by characteristic functions of simple representations, see Example 4.25 in [Joy07].

Let \( G \subset \text{GL}_n(\mathbb{C}) \) be a finite group. The McKay quiver \( Q(G, \mathbb{C}^n) \) is the quiver whose vertices are in bijection with irreducible representations of \( G \) joined by \( \dim(\text{Hom}_G(\chi_k \otimes \mathbb{C}^n, \chi_\ell)) \) edges (possibly none) from vertex \( k \) to vertex \( \ell \) (here \( \chi_k \) and \( \chi_\ell \) are irreducible representations of \( G \)). The category of representations of McKay quiver \( Q(G, \mathbb{C}^n) \) is equivalent to the category \( \mathcal{P}_{Q(G, \mathbb{C}^n)}/\mathcal{L} \)-mod, where \( \mathcal{L} \subset \mathcal{P}_{Q(G, \mathbb{C}^n)} \) is an ideal. For instance, this ideal admits a nice description, when \( G \) is abelian (see Section 3.2). The category of McKay quiver representations will be denoted by \( \text{Rep}(Q(G, \mathbb{C}^n), \mathcal{R}) \) (here \( \mathcal{R} \) stands for ‘relations’, see Section 3.2 for details). There is an equivalence of abelian categories \( \text{Rep}(Q(G, \mathbb{C}^n), \mathcal{R}) \simeq \text{Coh}_G(\mathbb{C}^n) \).
Suppose $G \subset \GL_n(\mathbb{C})$ satisfies the following assumptions:

\(1\) the McKay quiver $Q(G, \mathbb{C}^n)$ contains a subquiver $Q'$ (without oriented cycles) with $\mathcal{I} \cap \mathcal{P}_Q' = \emptyset$;

\(2\) there is a derived equivalence $\Psi : D^b_c(\Coh(\mathbb{C}^n)) \to D^b(\Coh(Y))$;

\(3\) $\Psi$ sends the skyscraper sheaves $\chi' \in \Coh(\mathbb{C}^n)$, corresponding to the simple representations in $\Rep(Q(G, \mathbb{C}^n), \mathbb{R})$ supported at the vertices of $Q'$, to pure sheaves concentrated in the same degree.

Let $\mathcal{H}(\langle \Psi(\chi'_i) \rangle)_{i \in \mathcal{Q}_G^+}$ be the Hall algebra generated by the images of sheaves corresponding to simple representations of $Q'$ under $\Psi$ and $n_- \subset \mathfrak{g}_{Q'}$ stand for the corresponding nilpotent subalgebra of $\mathfrak{g}_{Q'}$. It follows from the discussion above that one has an isomorphism of algebras (see also diagram 13):

$$\Theta : U(n) \to \mathcal{H}(\langle \Psi(\chi'_i) \rangle)_{i \in \mathcal{Q}_G^+}.$$  

In this paper we present an infinite collection of cyclic finite abelian subgroups of $\SL_3(\mathbb{C})$ satisfying conditions \(1\)–\(3\) above with $Q'$ any simply laced Dynkin diagram of affine type except $D_4$ or $A_3$, hence, produce isomorphisms $U(n) \cong \mathcal{H}(\langle \Psi(\chi'_i) \rangle)_{i \in \mathcal{Q}_G^+}$ for $n_- \subset \mathfrak{g}$ with $\mathfrak{g} \neq \mathfrak{so}_8$ a simple Lie algebra of type $A, D, E$ and $\mathfrak{g}$ the corresponding affine Kac-Moody algebra. Let $\epsilon = e^{2\pi i/r}$ be the primitive root of unity. The indicated family consists of cyclic abelian groups $\varphi : \mathbb{Z}/r\mathbb{Z} \to \SL_3(\mathbb{C})$ with $\varphi(1) = \text{diag}(\epsilon, \epsilon^k, \epsilon^s)$, where $r = 1 + k + s$ and

- $s \equiv 0 \pmod{k}$,
- $s \equiv 0 \pmod{k+1}$.

Interestingly, as $s$ goes to infinity, the proportion of nontrivial characters $\{\chi \mid H^0(\Psi(\chi')) \neq 0\}$ tends to $\frac{k-1}{k+1}$ (see Corollary 5.4). In particular, as both $s$ and $k$ tend to infinity, for a uniformly randomly chosen character $\chi$, one has that $\Psi(\chi')$ is concentrated in degree 0 with probability 1.

The exposition in the paper is organized as follows. In Sections 2—4 we recall the generalities on McKay correspondence, quiver representations and Hall algebras, respectively. Each section has references for a more detailed overview of the corresponding topic. While these sections contain essentially no new results, the example presented in Section 2 is important for a better understanding of the construction. Finally, Section 5 introduces the families of finite cyclic subgroups in $\SL_3(\mathbb{C})$ and establishes the isomorphisms of algebras announced above (Theorems 5.3 and 5.6).

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2. McKay correspondence

We start with a quick chronological overview of the subject. In [McK80] John McKay has observed that for a finite subgroup $G \subset \SL_2(\mathbb{C})$ there is a bijection

$$\begin{cases} \text{nontrivial irreducible representations of } G \end{cases} \leftrightarrow \begin{cases} \text{irreducible components of the central fiber } \rho^{-1}(0) \end{cases}$$
where \( \rho : Y \to \mathbb{C}^2 / G \) is the minimal resolution of singularities. Notice that \( \mathbb{C}[G] \) (the ring of representations of \( G \)) is naturally isomorphic to \( K^G(\mathbb{C}^2) \), the Grothendieck group of \( G \)-equivariant coherent sheaves on \( \mathbb{C}^2 \). Following this observation, in [GSV83], the McKay correspondence was realized geometrically as an isomorphism of Grothendieck groups \( K^G(\mathbb{C}^2) \to K(Y) \). Next in [KV00] this isomorphism was lifted to an equivalence of triangulated categories of coherent sheaves:

\[
D^G(\mathbb{C}^2) \to D(Y).
\]

In particular, under this equivalence, \( \chi^* := \chi \otimes \mathcal{O}_0 \), the skyscraper sheaf at 0 associated to a nontrivial irreducible \( G \)-representation \( \chi \), is mapped to the structure sheaf of the corresponding exceptional divisor (twisted by \( \mathcal{O}(-1) \)). Then Bridgeland, King and Reid constructed the equivalence \( D^G(\mathbb{C}^3) \to D(Y) \) for any finite subgroup \( G \subset SL_3(\mathbb{C}) \) and \( Y = G \cdot \text{Hilb}(\mathbb{C}^3) \) (see [BKR01]). They showed that \( G \cdot \text{Hilb}(\mathbb{C}^3) \) is a crepant resolution of \( \mathbb{C}^3 \). It was established that the images of \( \chi^* \)'s are concentrated in a single degree in case \( G \) is abelian and \( \mathbb{C}^3 / G \) has a single isolated singularity (see [CL09]). The result was then extended to any finite abelian subgroup \( G \) of \( SL_3(\mathbb{C}) \) in [CCL17]. We briefly recall the setup.

2.1. \( G \cdot \text{Hilb}(\mathbb{C}^3) \) as a toric variety. We refer the reader to Section 2 of [Cra05] and [CR02] for a more comprehensive and detailed exposition.

Let \( G \subset SL_3(\mathbb{C}) \) be a finite abelian subgroup of order \( r = |G| \), and \( \varepsilon = e^{2\pi i / r} \) a primitive root of unity. We diagonalize the action of \( G \) and denote the corresponding coordinates on \( \mathbb{C}^3 \) by \( x, y, \) and \( z \). The lattice of exponents of Laurent monomials in \( x, y, z \) will be denoted by \( L = \mathbb{Z}^3 \) and the dual lattice by \( L^\vee \). Associate a vector \( \nu_g = \frac{1}{r}(\gamma_1, \gamma_2, \gamma_3) \) to each group element \( g = \text{diag}(\varepsilon^{\gamma_1}, \varepsilon^{\gamma_2}, \varepsilon^{\gamma_3}) \), define the lattice \( N := L^\vee + \sum_{g \in G} \mathbb{Z} \cdot \nu_g \) (with \( N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R} \)) and use \( M := \text{Hom}(N, \mathbb{Z}) \) for the dual lattice of \( G \)-invariant Laurent monomials. The categorical quotient \( X = \text{Spec} \mathbb{C}[x, y, z]^G \) is the toric variety \( \text{Spec} \mathbb{C}[\sigma^\vee \cap M] \) with the cone \( \sigma \) being the positive octant \( \sigma = \mathbb{R}_{>0} \epsilon_1 \subset N_\mathbb{R} \).

**Definition 2.1.** The **junior simplex** \( \Delta \subset N_\mathbb{R} \) is the triangle with vertices \( e_x = (1, 0, 0), e_y = (0, 1, 0) \) and \( e_z = (0, 0, 1) \). It contains the lattice points \( \frac{1}{r}(\gamma_1, \gamma_2, \gamma_3) \) with \( \gamma_1 + \gamma_2 + \gamma_3 = 1, \gamma_i \geq 0 \).

A subdivision of the cone \( \sigma \) gives rise to a fan \( \Sigma \) and, hence, a toric variety \( X_\Sigma \) together with a birational map \( X_\Sigma \to X \). A triangle inside the junior simplex is called **basic** in case the pyramid with this triangle as the base and origin as the apex has volume 1. If a fan \( \Sigma \) gives rise to a partition of the junior simplex into basic triangles, then the corresponding map \( X_\Sigma \to X \) is a crepant resolution of singularities. Notice that such a fan \( \Sigma \) is uniquely determined by the associated triangulation of the junior simplex into basic triangles (slightly abusing notation we will refer to such a triangulation as \( \Sigma \) as well).

**Definition 2.2.** A **G-cluster** is a \( G \)-invariant zero-dimensional subscheme \( Z \subset \mathbb{C}^n \) for which \( H^0(\mathcal{O}_Z) \) is isomorphic to the regular representation of \( G \) as a \( \mathbb{C}[G] \)-module. The **G-Hilbert scheme** is the variety \( Y = G \cdot \text{Hilb}(\mathbb{C}^n) \), which is the fine moduli space parameterizing \( G \)-clusters.

The \( G \)-Hilbert scheme is a toric variety and for \( G \subset SL_2(\mathbb{C}) \) or \( SL_3(\mathbb{C}) \) the map \( Y \to X \) is a crepant resolution of singularities. The partition of the junior simplex into basic triangles
for finite abelian $G \subset \text{SL}_3(\mathbb{C})$, giving rise to the fan of $Y$, can be computed according to the 3-step procedure below (see [Nak01] and [Cra05]). Prior to giving the algorithm a definition is due.

**Definition 2.3.** Let $r$ and $a$ be coprime positive integers with $r > a$. The **Hirzebruch–Jung continued fraction** of $\frac{r}{a}$ is the expression

$$
\frac{r}{a} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}}.
$$

We will also refer to the collection of numbers $[a_1 : a_2 : \ldots : a_k]$ as the **Hirzebruch–Jung sequence** of $\frac{r}{a}$ and denote it by $\text{HJ}(r, a)$.

**Remark 2.4.** Each number $a_i$ in a Hirzebruch–Jung sequence $[a_1 : a_2 : \ldots : a_k]$ is greater than or equal to 2.

**Example 2.5.** Let $r \geq 2$ be an integer. The Hirzebruch-Jung continued fraction for $(r, r - 1)$ is

$$
\frac{r}{r - 1} = 2 - \frac{1}{2 - \frac{1}{\ldots - 1 - 2}}
$$

with the corresponding sequence $\text{HJ}(r, r - 1) = [2 : 2 : \ldots : 2]$ consisting of an $(r - 1)$-tuple of twos.

Next we recall an algorithm for producing the partition of $\triangle$ into basic triangles corresponding to $Y$.

**Step 1.** Draw line segments connecting the vertices of $\triangle$ to lattice points on the boundary of the convex hull of $\triangle \setminus \{e_x, e_y, e_z\}$ in such a way that the line segments do not cross the interior of $\text{Conv}(\triangle \setminus \{e_x, e_y, e_z\})$. Let the Hirzebruch-Jung sequence at a vertex $\zeta \in \{e_x, e_y, e_z\}$ be $\text{HJ}_\zeta = [k_1 : k_2 : \ldots : k_s]$. There will be $s + 2$ line segments $L_0^\zeta, L_1^\zeta, \ldots, L_s^\zeta, L_{s+1}^\zeta$ emanating from $\zeta$ with $L_0^\zeta$ and $L_{s+1}^\zeta$ being parts of edges between $\zeta$ and the remaining two vertices of $\triangle$. Moreover, $L_{i+1}^\zeta = k_iL_i^\zeta - L_{i-1}^\zeta$ and we label the $i$th segment with $k_i$ (the edges $L_0^\zeta$ and $L_{s+1}^\zeta$ on the boundary of $\triangle$ receive no label).

**Step 2.** Extend the lines until they are ‘defeated’ according to the following rule: when lines meet at a point, the line with greatest label extends and its label value is reduced by 1 for every ‘rival’ it defeats; lines meeting with equal labels all terminate at that point.

**Step 3.** Draw $k-1$ lines parallel to the sides of each regular triangle of side $k$ (lattice triangle with $k + 1$ lattice points on each edge) to produce its regular tessellation into $k^2$ basic triangles.

**Example 2.6.** Let $G = \mathbb{Z}/15\mathbb{Z} \subset \text{SL}_3(\mathbb{C})$ with $v_1 = \frac{1}{15}(1, 2, 12)$. The exceptional divisors are (the rays corresponding to) the vectors with endpoints $v_k = \frac{1}{15}(k_{15}, 2k_{15}, 12k_{15}) \in \triangle$, i.e. $k_{15} + 2k_{15} + 12k_{15} = 15$ (we use the notation $s_{15}$ for $s$ modulo 15):
\[ E_1 = \frac{1}{15}(1, 2, 12), \quad E_2 = \frac{1}{15}(2, 4, 9) \]
\[ E_3 = \frac{1}{15}(3, 6, 6), \quad E_4 = \frac{1}{15}(4, 8, 3) \]
\[ E_5 = \frac{1}{15}(5, 10, 0), \quad E_6 = \frac{1}{15}(8, 1, 6) \]
\[ E_7 = \frac{1}{15}(9, 3, 3), \quad E_8 = \frac{1}{15}(10, 5, 0). \]

**Step 1.** We compute the Hirzebruch-Jung sequences:

\[ \frac{1}{15}(2, 12) \sim \frac{1}{15}(1, 6) \sim \frac{1}{5}(1, 2), \quad \frac{5}{2} = 3 - \frac{1}{2}, \quad \text{HJ}_x = \text{HJ}(15, 2) = [3 : 2]; \]

\[ \frac{1}{15}(12, 1) \sim \frac{1}{5}(4, 1) \sim \frac{1}{5}(1, 4), \quad \frac{5}{4} = 2 - \frac{1}{2}, \quad \text{HJ}_y = \text{HJ}(15, 4) = [2 : 2 : 2 : 2]; \]

\[ \frac{15}{2} = 8 - \frac{1}{2}, \quad \text{HJ}_z = \text{HJ}(15, 2) = [8 : 2] \] and draw

![Diagram](image_url)

**Figure 1.** Step 1 of the triangulation algorithm for \( G = \frac{1}{15}(1, 2, 12) \)
Step 2. Extending the lines according to their labels.

Step 3. Subdividing regular triangles into basic.
2.2. **Reid’s recipe.** Reid’s recipe (see [Rei02] and [Cra05]) is an algorithm to construct the cohomological version of the McKay correspondence for abelian subgroups of $SL_3(\mathbb{C})$. It starts with marking internal edges and vertices of the triangulation $\Sigma$ corresponding to $G \cdot \text{Hilb}(\mathbb{C}^3)$ with characters of $G$. For the purposes of this paper only the marking of edges will be required, which we recall below and refer the reader to Section 3 of [Cra05] for details. An edge $(e, f)$ in $\Sigma$ is labeled by a character of $G$ according to the following rule. The one-dimensional ray in $M$ perpendicular to the hyperplane $\langle e, f \rangle$ in $M_\mathbb{R}$ has a primitive generator given by exponents of $\frac{m_1}{m_2}$, where $m_1$ and $m_2$ are coprime regular monomials. As $M$ is the lattice of $G$-invariant Laurent monomials, $m_1$ and $m_2$ have the same character $\chi$ with respect to $G$-action. The edge $(e, f)$ is marked with character $\chi$.

**Example 2.7.** We determine the character that marks the edge $(e_z, E_1)$:

\[
\begin{align*}
\begin{cases}
  c = 0 \\
  a + 2b + 12c = 0
\end{cases} \iff a = -2b, c = 0,
\end{align*}
\]

hence, $m_1 = x^2$ and $m_2 = y^2z$ with $\chi = \chi_2$.

The computation below shows that the edge $(E_6, E_7)$ is marked by $\chi = \chi_1$:

\[
\begin{align*}
\begin{cases}
  8a + b + 6c = 0 \\
  9a + 3b + 3c = 0
\end{cases} \iff \begin{cases}
  3a + b + c = 0 \\
  a + c = 0
\end{cases} \iff a = -c, b = 2c,
\end{align*}
\]

thus, $m_1 = x$ and $m_2 = y^2z$.

![Figure 4. Σ fan and character marking for $G = \frac{1}{15}(1, 2, 12)$](image)

2.3. **Results for abelian subgroups** $G \subset SL_3(\mathbb{C})$. Let $G \subset SL_n(\mathbb{C})$ be a finite subgroup. The following result appeared in the celebrated paper of Bridgeland, King and Reid (see [BKR01]).
Theorem 2.8. Let $G \subset \text{SL}_n \mathbb{C}$ be a finite subgroup with $n \leq 3$.

1. The variety $G\text{-Hilb}(\mathbb{C}^n)$ is irreducible and the resolution $Y \to X$ is crepant.
2. The map $\Psi : \text{D}^b_G(\mathbb{C}^n) \to \text{D}^b(Y)$ is an exact equivalence of triangulated categories.

Let $\chi$ be an irreducible representation of $G$. There are two natural $G$-equivariant sheaves on $\mathbb{C}^n$ associated to $\chi$:

- $\tilde{\chi} := \chi \otimes O_{\mathbb{C}^n}$
- $\chi' := \chi \otimes O_0$,

where $O_0 = O_{\mathbb{C}^n}/m_0 = \mathbb{C}[x_1, x_2, \ldots, x_n]/(x_1, x_2, \ldots, x_n)$ is the structure sheaf of the origin in $\mathbb{C}^n$.

The image $\Psi(\tilde{\chi} \otimes O_{\mathbb{C}^n})$ admits a straightforward description. It is isomorphic to $L^{-\chi}_X$, where $L_X$ is the corresponding tautological vector bundle. The tautological vector bundles on $Y$ are defined as direct summands in the decomposition

$$p_*(O_Z) = \bigoplus L_{\chi} \otimes \chi,$$

with respect to the trivial $G$-action on $Y$ (here $Z \subset Y \times \mathbb{C}^n$ is the universal subscheme and $p$ the projection on $Y$).

However, the images of skyscraper sheaves $\chi'$ are more complicated to describe. In case of abelian $G$ the main result of [CCL17] provides such a description.

Theorem 2.9. Let $G \subset \text{SL}_3(\mathbb{C})$ be a finite abelian subgroup and let $\chi$ be an irreducible representation of $G$. Then $H^i(\chi') = 0$ unless $i \in \{0, -1, -2\}$. Moreover, one of the following holds:

| Reid’s recipe | $H^{-2}(\Psi(\chi'))$ | $H^{-1}(\Psi(\chi'))$ | $H^0(\Psi(\chi'))$ |
|---------------|------------------------|------------------------|---------------------|
| $\chi$ marks a single divisor $E$ | 0 | 0 | $L^{-1}_X \otimes O_E$ |
| $\chi$ marks a single curve $C$ | 0 | 0 | $L^{-1}_X \otimes O_C$ |
| $\chi$ marks a chain of divisors starting at $E$ and terminating at $F$ | 0 | $L^{-1}_X(-E - F) \otimes O_Z$ | 0 |
| $\chi$ marks three chains of divisors, starting at $E_x$, $E_y$ and $E_z$ and meeting at a divisor $P$ | 0 | $L^{-1}_X(-E_x - E_y - E_z) \otimes O_V$ | 0 |
| $\chi = \chi_0$ | $w_{\mathbb{Z}F_2}$ | $w_{\mathbb{Z}F_1}(\mathbb{Z}F_2)$ | 0 |

Table 1. Observed values

Remark 2.10. Apriori, each object $\Psi(\chi')$ is an abstract complex in $\text{D}^b(Y)$. It follows from Theorem 2.9 that for every nontrivial character $\chi$ the object $\Psi(\chi')$ is a pure sheaf (i.e. some shift of a coherent sheaf).

3. Quivers

The next ingredient that we need is McKay quivers. A good reference for basic concepts of quivers and representations thereof is the book [DW17]. We also invite the reader to look in the paper [IN00] for the exposition on realizations of $G$-Hilbert schemes as moduli spaces of McKay quiver representations.
3.1. **Generalities.**

**Definition 3.1.** A quiver $Q = (Q_0, Q_1)$ is a finite directed graph with finitely many vertices enumerated by the set $Q_0$ and finitely many edges indexed by $Q_1$. Each edge is uniquely determined by the pair of vertices it connects, which we will denote by $t(a)$ and $h(a)$ standing for 'tail' and 'head', respectively.

**Definition 3.2.** A path $p$ in a quiver $Q = (Q_0, Q_1)$ is a sequence $a_\ell a_{\ell-1} \ldots a_1$ of arrows in $Q_1$ such that $t(a_{i+1}) = h(a_i)$ for $i = 1, 2, \ldots, \ell - 1$. In addition, for every vertex $x \in Q_0$ we introduce a path $e_x$.

The path algebra $P_Q$ is a $\mathbb{C}$-algebra with a basis labeled by all paths in $Q$. The multiplication in $P_Q$ is given by

$$p \cdot q := \begin{cases} pq, & \text{if } t(p) = h(q) \\ 0, & \text{otherwise} \end{cases}$$

where $pq$ stands for the concatenation of paths subject to the conventions that $pe_x = p$ if $t(p) = x$, and $e_x p = p$ if $h(p) = x$.

**Remark 3.3.** Notice that $P_Q$ is of finite dimension over $\mathbb{C}$ if and only if $Q$ has no oriented cycles. The path algebra has a natural grading by path length with the subring of grade zero spanned by the trivial paths $e_x$ for $x \in Q_0$. It is a semisimple ring, in which the elements $e_x$ are orthogonal idempotents.

**Definition 3.4.** A representation of a quiver $Q$ consists of a collection of vector spaces $\{V_i\}_{i \in Q_0}$ and linear homomorphisms $\alpha_a \in \text{Hom}_\mathbb{C}(V_{ta}, V_{ha})$ for each arrow $a \in Q_1$.

Such representations form a category with morphisms being collections of $\mathbb{C}$-linear maps $\psi_i : V_i \to W_i$ for all $i \in Q_0$ such that the diagrams

$$
\begin{array}{ccc}
V_{ta} & \xrightarrow{\alpha_a} & V_{ha} \\
\downarrow{\psi_{ta}} & & \downarrow{\psi_{ha}} \\
W_{ta} & \xrightarrow{\alpha'_a} & W_{ha}
\end{array}
$$

commute. This category will be denoted by $\text{Rep}_\mathbb{C}(Q)$.

**Theorem 3.5.** The category $\text{Rep}_\mathbb{C}(Q)$ is equivalent to the category of finitely-generated left $P_Q$-modules. In particular, $\text{Rep}_\mathbb{C}(Q)$ is an abelian category.

Often, the algebra of interest is not the path algebra of a quiver $Q$, its quotient by an ideal of relations. A relation in $P_Q$ is a $\mathbb{C}$-linear combination of paths of length at least two, each with the same head and the same tail. A quiver with relations is a quiver $Q$ together with a finite set of relations $\mathcal{R}$. A representation of such a quiver is a representation of $Q$ where any composition of maps indexed by subsequent edges in a relation vanishes (is a zero map). Any finite set of relations $\mathcal{R}$ in $Q$ determines a two-sided ideal $\mathcal{I}_\mathcal{R} \subset P_Q$. As before, finite-dimensional representations of $(Q, \mathcal{R})$ form a category $\text{Rep}_\mathbb{C}(Q, \mathcal{R})$. Moreover, the analogue of Theorem 3.5 holds true, i.e. there is an equivalence of categories

$$(1) \quad \text{Rep}_\mathbb{C}(Q, \mathcal{R}) \simeq (P_Q/\mathcal{I}_\mathcal{R}) \text{-mod}.$$
3.2. McKay Quivers. Let $G \subset \text{SL}(V)$ be a finite subgroup.

**Definition 3.6.** The *McKay quiver* $Q(G, \mathbb{C}^n)$ is the quiver whose vertices are enumerated by irreducible representations of $G$ with $\text{dim}((\text{Hom}_G(\chi_k \otimes \mathbb{C}^n, \chi_\ell)))$ arrows (possibly none) from vertex $k$ to vertex $\ell$. Henceforth we assume that $G$ is abelian. Then all irreducible representations of $G$ are one-dimensional and correspond to characters of $G$:

$$\text{char}(G) := \{\chi : G \to \mathbb{C}^*\}.$$ 

In particular, as a representation of $G$, we have $\mathbb{C}^n = \bigoplus_{i=1}^n \mathbb{C}\chi_i =: \bigoplus_{i=1}^n \mathbb{C}e_i$ and let $x_1, x_2, \ldots, x_n \in (\mathbb{C}^n)^*$ be the dual basis to $\{e_1, e_2, \ldots, e_n\}$ with $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ the coordinate ring of $\mathbb{C}^n$. The chain of isomorphisms $\text{Hom}_G(\chi_k \otimes \mathbb{C}^n, \chi_\ell) \simeq \text{Hom}_G(\chi_k \otimes \bigoplus_{i=1}^n \mathbb{C}e_i, \chi_\ell) \simeq \bigoplus_{i=1}^n \text{Hom}_G(\chi_k \otimes \mathbb{C}e_i, \chi_\ell)$ provides a natural identification of the maps assigned to the arrows in the McKay quiver $Q(G, \mathbb{C}^n)$ with multiplication by $x_i$'s and, hence, impose the relations corresponding to the commutation of the latter:

$$\mathcal{I} := \langle a_i^{\chi \otimes \chi_j} a_j^\chi - a_i^{\chi \otimes \chi_l} a_l^\chi | \chi \in \text{char}(G), 1 \leq i, j \leq n \rangle \subset \mathbb{C}Q(G, \mathbb{C}^n)$$

(for every vertex $\chi \in Q_0$ there are $n$ arrows with head at $\chi$ denoted by $a_k^\chi : \chi \otimes \chi_k \to \chi$ and label the arrow $a_k^\chi$ by the monomial $x_k$).

**Example 3.7.** Let $G = \mathbb{Z}/n\mathbb{Z}$ be embedded in $\text{SL}(\mathbb{C}^2)$ via mapping $1$ to $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$ with $\varepsilon = e^{2\pi i/n}$ the primitive $n^{th}$ root of unity. There are $n$ irreducible representations of $G$, one-dimensional, to be denoted by $X_0, X_1, \ldots, X_{n-1}$ and $\mathbb{C}^2 \simeq X_1 \oplus X_{n-1}$. We label $x_k := \alpha_{X_1}^{X_k}$ and $y_k := \alpha_{X_{n-1}}^{X_k}$, then the ideal of relations is $\mathcal{I} = \langle x_i y_i - x_{i+1} y_{i+1} \rangle$.

The following Proposition will play an essential role for the construction in Section 5 (see Lemma 7.5 and its proof in [Cra08]).

**Proposition 3.8.** Let $G \subset \text{GL}(\mathbb{C}^n)$ be a finite subgroup. There exists a set of relations $\mathcal{R}$ in the McKay quiver $Q(G, \mathbb{C}^n)$ such that the categories $\text{Rep}_\mathbb{C}(Q(G, \mathbb{C}^n), \mathcal{R})$ and $\text{Coh}_G(\mathbb{C}^n)$ are equivalent.
4. Hall algebras

We will use two versions of the Hall algebra construction. The first variant appeared in [KV00] and was described in detail in [Joy07]. A very good overview of Ringel-Hall algebras over finite fields can be found in [Sch12].

Let $C$ be a $\mathbb{C}$-linear abelian finitary category (the latter means that all extension groups between any two objects in $C$ are finite dimensional).

4.1. Euler characteristic Hall algebra. The set of isomorphism classes of objects in $C$ will be denoted by $C^{\text{iso}}$. The space of functions $\text{Fun}(C^{\text{iso}}, \mathbb{C})$ can be made into an associative algebra $\mathcal{H}(C)$, called the Hall algebra of $C$. Let $G_C$ be the stack formed by pairs $(A, B)$ of objects of $C$, where $A$ is a subobject of $B$, and morphisms of such pairs. There are three morphisms $p_1, p_2, p_3 : G_C \to C^{\text{iso}}$ associating to the pair $(A, B)$ the objects $A, B$ and $B/A$, respectively. The fibers of $p_2$ are algebraic varieties. The multiplication on $\mathcal{H}(C)$ is given by

$$f \ast g := p_2(p_1^*(f)p_3^*(g)).$$

Let $A \in C$ be an object and $[C] \in \mathcal{H}(C)$ the characteristic function of $A$. The multiplicity of $[C]$ in $[A] \ast [B]$ is $\chi(G_{AB}^C)$, where $G_{AB}^C = \{ A' \subseteq C \mid A' \simeq A, C/A' \simeq B \}$ and $\chi$ stands for Euler characteristic with compact support.

The following proposition appearing in [KV00] (see Section 3.1) is a consequence of the fact that the heart of a triangulated category is stable under extensions. It will be an essential ingredient of the main construction in this paper.

**Proposition 4.1.** Let $C_1, C_2$ be two finitary abelian categories, and $\varphi : D^b(C_1) \to D^b(C_2)$ an equivalence of triangulated categories. If $A, B, C \in C_1$ are such that $\varphi(A), \varphi(B) \in C_2$ with $G_{AB}^C \neq \emptyset$, then $\varphi(C) \in C_2$ and $\varphi$ is an isomorphism of complex varieties $G_{AB}^C \simeq G_{\varphi(A)\varphi(B)}^{\varphi(C)}$.

**Corollary 4.2.** Let $C_1, C_2$ be two finitary abelian categories, and $\varphi : D^b(C_1) \to D^b(C_2)$ an equivalence of triangulated categories. If $A_1, A_2, \ldots, A_n \in C_1$ are such that $\varphi(A_1), \varphi(A_2), \ldots, \varphi(A_n) \in D^b(C_2)$ have cohomology concentrated in a single degree and that degree is the same for all $A_i$, then $\varphi$ induces an isomorphism of Hall algebras

$$\mathcal{H}(A_1, A_2, \ldots, A_n) \simeq \mathcal{H}(\varphi(A_1), \varphi(A_2), \ldots, \varphi(A_n)).$$

4.2. Ringel-Hall algebras over finite fields. Assume that $C$ is a finitary abelian category, such that

- $\text{gldim}(C) < \infty$;
- $|\text{Ext}^i(A, B)| < \infty$ for any two objects $A, B \in \text{Ob}(C)$ and all $i \geq 0$.

**Definition 4.3.** The multiplicative Euler form $\langle \cdot, \cdot \rangle : K(C \times C) \to \mathbb{C}$ is the form given by

$$\langle A, B \rangle := (\prod_{i=0}^{\infty} |\text{Ext}^i(A, B)|^{-1})^{1/2}.$$

Let $p^C_{A,B}$ be the number of short exact sequences $0 \to B \to C \to A \to 0$ and

$$p^C_{A,B} := \frac{p^C_{A,B}}{|\text{End}(A)||\text{End}(B)|}.$$
Consider the vector space \( \mathcal{H}_{\text{fin}}(C) := \bigoplus_{A \in C^\text{iso}} \mathbb{C}[A] \). The following defines the structure of an associative algebra on \( \mathcal{H}_{\text{fin}}(C) \):

\[
[A] \ast_{\text{fin}} [B] := \langle A, B \rangle \sum C \mathbb{P}^C_{A,B}[C]
\]

**Remark 4.4.** The unit \( i : \mathbb{C} \to \mathcal{H}_{\text{fin}}(C) \) is given by \( i(\lambda) = \lambda[0] \), where 0 is the initial object of \( C \).

Let \( Q \) be a quiver without oriented cycles, \( \text{Rep}_k(Q) \) the category of representations of \( Q \) over a finite field \( \mathbb{k} \) and \( U_q(\mathfrak{g}) \) the quantized enveloping algebra. Here \( \mathfrak{g} \) is the Lie algebra associated to the Dynkin diagram formed by \( Q \). We denote the simple roots of \( \mathfrak{g} \) by \( E_i \) and simple representations of \( Q \) by \( \{S_i\}_{i \in Q_0} \).

The following result was obtained by Ringel and Green (see Theorem 3.15 in [Sch12]).

**Theorem 4.5.** Let \( \nu = \sqrt{|k|} \). There is an embedding of algebras \( \varphi : U_\nu(n_-) \hookrightarrow \mathcal{H}_{\text{fin}}(\text{Rep}_k(Q)) \) with \( \varphi(E_i) = [S_i] \).

---

### 5. Main Result

In this section we will combine the results reviewed in Sections 2 – 4 to formulate and establish the main theorem of this paper. A few examples prior to giving the general statement will be helpful.

**Example 5.1.** Consider the groups \( G = \mathbb{Z}/6\mathbb{Z} \) and \( \bar{G} = \mathbb{Z}/7\mathbb{Z} \) with \( \nu_1 = \frac{1}{6}(1, 1, 4) \) and \( \bar{\nu}_1 = \frac{1}{7}(1, 1, 5) \). The partitions of junior simplices into basic triangles corresponding to \( G - \text{Hilb}(\mathbb{C}^3) \) and \( \bar{G} - \text{Hilb}(\mathbb{C}^3) \) are presented on Figure 6.

![Figure 6](image-url)

**Figure 6.** \( \Sigma \) fans and character labels for \( G = \frac{1}{6}(1, 1, 4) \) and \( \bar{G} = \frac{1}{7}(1, 1, 5) \)

The data on the images of skyscraper sheaves under \( \Psi \) appearing in Table 2 is obtained via a direct application of Theorem 2.9. Notice that the McKay quiver \( Q(G, \mathbb{C}^3) \) contains a
$Q' = \tilde{A}_1$ subquiver supported on the vertices enumerated by characters $\chi_1$ and $\chi_2$:

![Diagram](image)

**Table 2.** Images of $\chi_i^j$'s under $\Psi$ for $G = \frac{1}{6}(1, 1, 4)$ and $\tilde{G} = \frac{1}{7}(1, 1, 5)$

**Example 5.2.** We continue with Example 2.6. Recall that $G = \mathbb{Z}/15\mathbb{Z}$ with $\nu_1 = \frac{1}{15}(1, 2, 12)$. See Figure 4 for the partition of junior simplex into basic triangles corresponding to $G$ - Hilb$(\mathbb{C}^3)$ and marking of edges with characters. This time the McKay quiver $Q(G, \mathbb{C}^3)$ contains a $Q' = \tilde{A}_2$ subquiver supported on the vertices enumerated by characters $\chi_1$, $\chi_2$, and $\chi_3$:

![Diagram](image)

**Table 3.** Images of $\chi_{1,2,3}^j$ under $\Psi$ for $G = \frac{1}{15}(1, 2, 12)$

**Theorem 5.3.** Let $r = k + s + 1$ with $s \equiv 0 \pmod{k}$ and $s \equiv 0 \pmod{k+1}$. Set $t := \frac{r}{k+1}$ and consider the sequence

$$
\gamma_{0,t} = 1 \\
\gamma_{n,t} := nt - n(n-1).
$$

(1) If a character $\chi_c$ appears marking an edge, then $c$ satisfies at least one of the following conditions:

- $1 \leq c \leq k + 1$
- $c$ is divisible by $k$
- $c$ is divisible by $k + 1$

(2) The images of the $k+1$ skyscraper sheaves corresponding to the first $k+1$ nontrivial characters of $G$ are as presented below.
\begin{table}
\centering
\begin{tabular}{|c|c|}
\hline
$X$ & $H^{-1}(\Psi(X'))$ \\
\hline
$X_1$ & $\mathcal{L}^{-1}_{X_1}(-E_{\gamma_k}) \otimes \mathcal{O}_{E_{\gamma_{k-1},1,t+2\gamma_{k-1},1,t+3...\gamma_{k-1},t+1}}$ \\
\hline
$X_2$ & $\mathcal{L}^{-1}_{X_2}(-E_{\gamma_{k-1},1,t}) \otimes \mathcal{O}_{E_{\gamma_{k-2},2,t+2\gamma_{k-2},2,t+3...\gamma_{k-2},t+1}}$ \\
\hline
$\cdots$ & $\cdots$ \\
\hline
$X_k$ & $\mathcal{L}^{-1}_{X_k}(-E_{\gamma_{k+1}}) \otimes \mathcal{O}_{E_{\gamma_{k+1},t+1}}$ \\
\hline
$X_{k+1}$ & $\mathcal{L}^{-1}_{X_{k+1}}(-E_{xy}) \otimes \mathcal{O}_{E_{\gamma_{k+1},t+1}}$ \\
\hline
\end{tabular}
\caption{Images of $\chi_{1,2,...,k+1}$ under $\Psi$ for $G = \frac{1}{r}(1, k, s)$}
\end{table}

**Proof.** We follow Reid’s recipe (see Section 2). Let $\ell = \frac{s}{k}$ and compute the Hirzebruch-Jung fractions and sequences at vertices of $\triangle$ (see Example 2.5):

\[
\frac{r}{k} = \ell + 2 - \frac{k-1}{k}, \quad HJ_z = [\ell + 2 : 2 : \cdots : 2];
\]

\[
\frac{1}{r}(s, 1) = \frac{1}{r/(k+1)} \left( \frac{s}{k+1}, 1 \right) = \frac{1}{t}(t-1, 1) = \frac{1}{t}(1, t-1), \quad HJ_y = [2 : 2 : \cdots : 2];
\]

\[
\frac{1}{r}(k, s) = \frac{1}{r} \left( 1, \frac{s}{k} \right) = \frac{1}{k+1} \left( 1, \frac{s}{k(k+1)} \right) = \frac{1}{t} \left( 1, \frac{t-1}{k} \right), \quad HJ_x = [k + 1 : 2 : 2 : \cdots : 2];
\]

(we have used that $\frac{t}{(t-1)/k} = k + 1 - \frac{(t-1)/k-1}{(t-1)/k}$ to evaluate $HJ_x$).

As $HJ_x = [k + 1 : 2 : 2 : \cdots : 2]$, the line segment connecting $e_x$ and $E_{\gamma_{k-1},t-1+1}$ (or $E_{13}$ on Figure 7) continues until $E_1$. As $\ell + 2 = s/k + 2 > 2 = k + 1 - (k-1)$ and $s/k + 2 - t/k - (t-1) = 3 > 2$, it follows that each of the line segments emerging from $e_y$ and each of the line segments emanating from $e_z$ except $L_{13}^{\ell_1}$ does not continue beyond the first lattice point on it (see Step 2 of Algorithm 2.1). On the other hand $L_{13}^{\ell_1}$ defeats all the line segments on its way to $E_{1,t-1}$, which belongs to the edge $[e_x, e_y]$ of $\triangle$. The remaining edges of triangulation of $\triangle$ (see Step 3 of the algorithm) come in three families of parallel lines with slopes equal to those of

- $[e_z, E_1]$
- $[e_x, e_y]$
- $[e_x, E_{(t+1)/2}]$.

Next we verify that the edges of triangulation of $\triangle$ are marked as in the statement of the theorem. Let $1 \leq i \leq k$ and consider the edge $(e_z, E_{\gamma_{k-i},t+1})$. Denote $\alpha_i := \left( 1 + \frac{(r-1)(i-1)}{k} \right)$, then
\[
\begin{cases}
    c = 0 \\
    \alpha_i a + (k - i + 1) b = 0,
\end{cases}
\]

hence, \(m_1 = x^{k-i}\) and \(m_2 = y^{\alpha_i}\) with \(\chi = \chi_{k-i+1}\) (the solution is \((a, b, c) = (k - i + 1, -\alpha_i, 0))\).

For each \(1 \leq i \leq k\) the vertices \(E_{y_{k-i+1}}, E_{y_{k-i+2}}, \ldots, E_{y_{k-i+1}}\) lie on a line \(\ell_i\), moreover, all these lines are parallel and have direction vector \(v = (1, k, r - k - 1)\) (see Figure 7). As the solution \((a, b, c) = (k - i + 1, -\alpha_i, 0)\) satisfies the equation \(v \cdot (a, b, c) \equiv 0 \mod r\), it follows that all intervals on the line \(\ell_i\) are labeled by the character \(\chi_{k-i+1}\).

Now we check that the edge \((e_y, E_{t-1})\) is marked with \(\chi_{k+1}\). As \(E_{t-1} = \frac{1}{r}(t - 1, r - k - t, k + 1)\), we get the system of equations
\[
\begin{cases}
    b = 0 \\
    (t - 1) a + (k + 1) c = 0,
\end{cases}
\]

hence, \(m_1 = x^{k+1}\) and \(m_2 = z^{-1}\) with \(\chi = \chi_{k+1}\) (the solution is \((a, b, c) = (k + 1, 0, 1 - t))\). The vertices \(E_{t-1}, E_{y_{2t-1}}, \ldots, E_{y_{k-1}}\) lie on the same line, with the vector connecting any two subsequent vertices being \((t, -t, 0)\). As \((k + 1)t = (k + 1)\frac{r}{k + 1} = r\), we (inductively) get that the solution \((a, b, c) = (k + 1, 0, 1 - t)\) satisfies the equations imposed by \(E_{y_{i-1}}\) and \(E_{y_{i+1}}\), so each edge \((E_{y_{i-1}}, E_{y_{i+1}})\) is marked by \(\chi_{k+1}\). A straightforward computation shows that the edge \((e_x, E_{y_{k+1}})\) is marked with \(\chi_{k+1}\) as well. The remaining 'labeling assertions' can be checked similarly using the parallelism of corresponding lines. In particular, notice that

- a move by one unit to the right in the family of lines with slope equal to the one of \([e_x, E_1]\) results in decrease of the corresponding label by \(k\);
- a move by one unit to the right in the family of lines with slope equal to the one of \([e_x, e_y]\) results in decrease of the corresponding label by \(k + 1\);
- a move by one unit to the right in the family of lines with slope equal to the one of \([e_x, E_{(t+1/2)}]\) results in decrease of the corresponding label by \(k\) to the left and by \(k + 1\) to the right of the line segment \([e_x, E_{(t+1/2)}]\).

\[\square\]

**Corollary 5.4.** Let \(k\) and \(s\) satisfy the assumptions of Theorem 5.3 with \(k\) fixed. Denote the set of characters \(\chi\) with \(H^0(\Psi(\chi')) \neq 0\) by \(\mathcal{S}_0\). Then \(\lim_{s \to \infty} \frac{|\mathcal{S}_0|}{r - 1} \geq \frac{k - 1}{k + 1}\). In particular,

\[\lim_{k \to \infty} \frac{|\mathcal{S}_0|}{r - 1} = 1.\]

**Proof.** It follows from (1) in Theorem 5.3 combined with Theorem 2.9 that

\[
|\mathcal{S}_0| \geq r - (k + 1) - \frac{r - 1}{k} - \frac{r - 1}{k + 1} + \frac{r - 1}{k(k + 1)} = \frac{s k (k + 1) - (k + 1)(s + k) - k(s + k) + s + k}{k(k + 1)} = \frac{(k^2 - k)s - 2k^2}{k(k + 1)} = \frac{(k - 1)s - 2k}{k + 1}.
\]
Hence, \( \lim_{s \to \infty} \frac{|G_0|}{r - 1} \geq \lim_{s \to \infty} \frac{(k - 1)s - 2k}{(k + 1)(k + s)} = \frac{k - 1}{k + 1}. \)  

**Remark 5.5.** For \( s \gg k \gg 0 \) the cohomology of \( \Psi(\chi^i) \) tend to concentrate in degree 0.

**Theorem 5.6.** Let \( r = k + s + 1 \) with \( k \) and \( s \) satisfying the assumptions of Theorem 5.3 and \( s \gg 0 \).

1. Let \( n_- \subset \widehat{\mathfrak{sl}}_{k+1}(\mathbb{C}) \) be a nilpotent subalgebra and \( Q' \subset Q(G, \mathbb{C}^3) \) the subquiver supported on vertices \( Q'_0 = \{1, 2, \ldots, k + 1\} \). There is an isomorphism of algebras

\[
\mathbb{U}(n_-) \cong \mathcal{H}\langle [\Psi(\chi^i)]_{i \in Q'_0} \rangle.
\]
(2) Assume, in addition, that $k = 2q + 1 \geq 5$ is odd, $5 \leq n \leq k + 1$ and $\alpha$ satisfies the properties:

- $\alpha \equiv q \ (mod \ k)$, $\alpha \equiv -2 \ (mod \ k + 1)$;
- $\alpha + k(n - 3) + k + 1 \leq r - 1 \Leftrightarrow \alpha \leq s - k^2 + 3k - 1$.

Let $n_\sim \subset \mathfrak{g}_{2n}(\mathbb{C})$ with $5 \leq n \leq k + 1$ be a nilpotent subalgebra and $Q' \subset Q(G, \mathbb{C}^3)$ the subquiver supported on vertices $Q'_0 = \{\alpha - k - 1, \alpha, \alpha + 1, \alpha + k, \alpha + 2k, \ldots, \alpha + k(n - 3), \alpha + k(n - 3) - 1, \alpha + k(n - 3) + k + 1\}$. There is an isomorphism of algebras

$$U(n_\sim) \xrightarrow{\Theta} H(\langle \Psi(\chi^i) \rangle_{i \in Q'_0})$$

(3) Assume that $k > 8$ and let $\mathfrak{g}$ be the Lie algebra of type $E_6$ with $n_\sim \subset \mathfrak{g}$ a nilpotent subalgebra and $Q' \subset Q(G, \mathbb{C}^3)$ the subquiver supported on vertices $Q'_0 = \{k + 6, 2k + 6, 3k + 4, 3k + 5, 3k + 6, 4k + 7, 5k + 8\}$. There is an isomorphism of algebras

$$U(n_\sim) \xrightarrow{\Theta} H(\langle \Psi(\chi^i) \rangle_{i \in Q'_0})$$

Figure 8. $Q' = \tilde{A}_k \subset Q(G, \mathbb{C}^3)$

Figure 9. $Q' = \tilde{D}_{n+1} \subset Q(G, \mathbb{C}^3)$

Figure 10. $Q' = \tilde{E}_6 \subset Q(G, \mathbb{C}^3)$
(4) Assume that \( k > 9 \) and let \( \mathfrak{g} \) be the Lie algebra of type \( E_7 \) with \( n_\mathfrak{g} \subset \hat{\mathfrak{g}} \) a nilpotent subalgebra and \( Q' \subset Q(G, \mathbb{C}^3) \) the subquiver supported on vertices \( Q'_0 = \{ k + 6, 2k + 3, 2k + 4, 2k + 5, 2k + 6, 3k + 7, 4k + 8, 5k + 9 \} \). There is an isomorphism of algebras

\[
U(n_\mathfrak{g}) \cong \mathcal{H}(\bigoplus_{i \in Q'_0} \Psi(\chi_i)),
\]

\[
\begin{array}{cccccccc}
2k+3 & \rightarrow & 2k+4 & \rightarrow & 2k+5 & \rightarrow & 2k+6 & \leftarrow & 3k+7 & \leftarrow & 4k+8 & \leftarrow & 5k+9 & \\
\uparrow & & & & & & & & & & & & & & \downarrow k+6
\end{array}
\]

**Figure 11.** \( Q' = \tilde{E}_7 \subset Q(G, \mathbb{C}^3) \)

(5) Assume that \( k > 10 \) and let \( \mathfrak{g} \) be the Lie algebra of type \( E_8 \) with \( n_\mathfrak{g} \subset \hat{\mathfrak{g}} \) a nilpotent subalgebra and \( Q' \subset Q(G, \mathbb{C}^3) \) the subquiver supported on vertices \( Q'_0 = \{ k + 5, 2k + 3, 2k + 4, 2k + 5, 3k + 6, 4k + 7, 5k + 8, 6k + 9, 7k + 10 \} \). There is an isomorphism of algebras

\[
U(n_\mathfrak{g}) \cong \mathcal{H}(\bigoplus_{i \in Q'_0} \Psi(\chi_i)),
\]

\[
\begin{array}{cccccccc}
2k+3 & \rightarrow & 2k+4 & \rightarrow & 2k+5 & \leftarrow & 3k+6 & \leftarrow & 4k+7 & \leftarrow & 5k+8 & \leftarrow & 6k+9 & \leftarrow & 7k+10 & \\
\uparrow & & & & & & & & & & & & & & \downarrow k+5
\end{array}
\]

**Figure 12.** \( Q' = \tilde{E}_8 \subset Q(G, \mathbb{C}^3) \)

**Proof.** We present an argument for (1) with the verification of (2) – (5) being completely analogous.

The McKay quiver \( Q(G, \mathbb{C}^3) \) contains an \( \tilde{A}_k \) subquiver \( Q' \) supported on the vertices \( 1, 2, \ldots, k + 1 \). Notice that \( Q' \) has no oriented cycles, hence, \( U(n_\mathfrak{g}) \) is isomorphic to the composition subalgebra of \( \mathcal{H}(\text{Rep}(Q')) \) (subalgebra generated by characteristic functions of simple representations), see Example 4.25 in [Joy07]. Then subsequent application of (2), Theorem 2.8 and Proposition 3.8 together with Theorem 5.3 and Corollary 4.2 gives rise to the proposed isomorphism, see diagram on Figure 13 with embeddings, isomorphisms and correspondences below for a schematic summary. 

□

**Remark 5.7.** Let \( G = \mathbb{Z}/r\mathbb{Z} \hookrightarrow \text{SL}_3(\mathbb{C}) \) be a finite abelian subgroup with \( \nu_1 = \frac{1}{r}(1, k, s) \) and \( 1 + k + s = r \). The McKay quiver \( Q(G, \mathbb{C}^3) \) can not contain a \( \tilde{D}_4 \) subquiver. This is due to the fact that \( \tilde{D}_4 \) has a vertex of valency 4, which implies the existence of an oriented 3-cycle, supported on these vertex and 2 of the vertices connected to it. Indeed, let the vertex of valency 4 correspond to the character (irreducible representation) \( \chi_i \). Then, inevitably, there are two vertices indexed by \( \chi_{i-a} \) and \( \chi_{i+b} \) with \( a \neq b \in \{1, k, s\} \) that are included in the subgraph as well. Therefore, the subgraph contains an oriented 3-cycle supported on the vertices enumerated by \( \chi_i, \chi_{i-a} \) and \( \chi_{i+b} \).
Conjecture 5.8. Let $k$ be a finite field and $\text{char}(k)$ coprime with the order of $G$. Under the assumptions (1) – (5) of Theorem 5.6 one has the corresponding isomorphisms

$$U_v(n_-) \xrightarrow{\Theta_{\text{fin}}} \mathcal{H}(\text{Rep}_{\mathbb{C}}(Q')) \xrightarrow{\sim} \mathcal{H}(\text{Coh}_{G,Q_0'}(\mathbb{C}^n)) \xrightarrow{\sim} \mathcal{H}(\{\Psi(x_i)\}_{i \in Q_0'})$$

with $v = \sqrt{|k|}$.

Remark 5.9. We provide some facts in support of the conjecture. The Bridgeland-King-Reid equivalence ((2) in Theorem 2.8) is known to hold in the above setup (see the comment after Conjecture 2.24 in [Rou06]). Another required modification to the proof of Theorem 5.6 is to use Theorem 4.5 in place of the isomorphism from Example 4.25 in [Joy07]. The analogue of Corollary 4.2 holds true as well.
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