MULTIFRACTALITY AS A LINK BETWEEN LUMINOSITY AND SPACE DISTRIBUTION OF VISIBLE MATTER

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Abstract

We discuss how luminosity and space distribution of galaxies are naturally linked in view of their multifractal properties. In particular we show that the mass (luminosity) function corresponding to a multifractal distribution in a given observed volume, consists of a power law followed by an exponential cut-off. This implies that the amplitude \(\phi^*\) of the Schechter function scales with the sample depth, as confirmed by various observational data. This effect is analogous to the scaling of the space density due to the fractal nature of the space distribution. This has the important consequence that the luminosity function can be properly defined only in a volume limited sample. Also the so-called "luminosity segregation" and the concept of bias correspond to a natural consequence of multifractality. This implies however that they should be considered from a different perspective with respect to the usual one. Such a concept allows us to unify the space and luminosity distributions as being shaped by a single cause: multifractality which should therefore claim a central stage in theoretical investigations.

Subject headings: galaxies: clustering - galaxies: structure - luminosity function - large scale structure of the universe
1 Introduction

In a well-known review on the galaxy luminosity function (LF) Binggeli et al (1988) state that "as the distribution of galaxies is known to be inhomogeneous on all scales up to a least $100h^{-1}\text{Mpc}$, a rich cluster of galaxies is like a Matterhorn in a grand Alpine landscape of mountain ridges and valleys of length up to 100 Km." The aim of this paper is to consider this point of view in the light of the concept of multifractality of the mass distribution. We show how the main observational aspects of galaxy luminosity and space distributions are strongly related and can be naturally linked and explained as a multifractal (MF) distribution. The concept of MF is appropriate to discuss physical systems with local properties of self-similarity, in which the scaling properties are defined by a continuous distribution of exponents. Roughly speaking one can visualize this property as having different scaling properties for different regions of the system. The fundamental point of this paper is that not only the pure space distribution of the luminous matter is self-similar (fractal), at least up to a certain scale (Coleman and Pietronero 1992 - CP92, Guzzo et al 1992; Baryshev et al., 1994 - BSLMP94; Pietronero & Sylos Labini, 1995; Sylos Labini et al., 1996a - SLGMP96 Sylos Labini et al., 1996b Di Nella et al., 1996; Sylos Labini & Amendola, 1996), but the whole matter distribution, i.e. weighing each point by its mass, is also self-similar. This situation requires the generalization of the simple fractal scaling to a MF distribution in which a continuous set of exponents is necessary to describe the spatial scaling of peaks of different weight (mass or luminosity). In this respect the mass and space distributions become naturally entangled with each other.

In section 2 we will briefly discuss some characteristic features of the galaxy distribution such as the space correlation, the morphological segregation, the morphology-density relation, the so-called "luminosity segregation" and the dwarf and giant galaxies distributions.

The distribution of luminous matter has MF properties: this result has been derived from the analysis of various redshift catalogs (CP92, SLGMP96) containing information both on space and luminous distribution, i.e. position in space and intrinsic luminosity. Multifractality naturally links together space and luminous (mass) distributions and provides a mathematical framework to go beyond the standard approximation that galaxian luminosities are not correlated with spatial location (section 3). This fact has fundamental consequences on the method adopted to determine the galaxy LF. In particular we show that the LF has to be studied only in volume limited samples in order to avoid the scaling in LF shape and amplitude that arises with varying sample-size. However given the limited redshift depth of the available three dimensional samples,
the shape of the LF is not greatly distorted and it can fairly accurately rendered also by magnitude limited samples, which are generally used in literature.

After having introduced the main properties of fractal and multifractal distributions (section 4), in section 5 we show that from them one can derive the shape of the LF as well as the exponent of the two-point correlation function. Moreover we will relate various observational issues with the multifractal behavior of the matter distribution.

Finally in section 6 we discuss the theoretical implications of multifractality and in particular its consequences on the theory of galaxy formation.

2 Galaxy space and luminosity distributions

We briefly list here the main features of the luminosity and space distributions of galaxies together with the various morphological properties that will be naturally embedded in a MF scheme.

1. The main statistical tool to study the spatial distribution of galaxies is the two point correlation function (CF)

\[ G(r) = \langle n(\vec{r}_0)n(\vec{r} + \vec{r}_0) \rangle \sim r^{-\gamma} \]  

(1)

where the last equality holds in the case of fractal distribution, and \( D = 3 - \gamma \) is the fractal dimension. CP92 by analyzing the CfA1 redshift survey, find that the correlation function in Eq.1 is a power law with exponent \( \gamma \sim 1.5 \) up to the sample limit \( \sim 20h^{-1}Mpc \). According to CP92, the CF in Eq.1 is the appropriate statistical tool that should be used in the description of systems with long-range correlations, rather than the usual \( \xi(r) \) (Peebles 1980; Davis & Peebles 1983; see Sec.4 for a detailed discussion). Recently there have been available various determinations of the correlation properties in redshift surveys which seem to point to a somewhat higher value for the fractal dimension. For example Guzzo et al.(1992) found that in the Perseus-Pisces survey (Giovanelli & Haynes, 1993) \( D \approx 2 \) up to 30\( h^{-1}Mpc \), while they found an evidence towards homogenization at larger distances. This result is confirmed by the analysis of Sylos Labini et al.(1996a, 1996b), even though they found no evidence towards homogenization and on the contrary they extend the power law behavior for the density up to \( \sim 130h^{-1}Mpc \). The discrepancy between these results is probably due to the fact that Guzzo et al.(1992) have used weighing schemes for the treatment of boundary conditions in the correlation analysis, that produce an artificial homogenization (CP92). Moreover Di Nella et al.(1996) by analyzing the LEDA database found a similar value for the fractal dimension up to \( \sim 150h^{-1}Mpc \) (Fig.1). Another evidence for a higher
value of the fractal dimension is given by Park et al. (1994), who found \( \gamma \approx 1 \) analyzing the CfA2 redshift survey (see also Sylos Labini & Amendola 1996). This value of the correlation exponent is lower than that found in the CGCC angular catalog, that is \( \gamma \approx 1.7 \), (Zwicky et al., 1968). The exponent \( \gamma \approx 1.7 \) has been obtained also in the APM galaxy catalog by Collins et al. (1992). This difference is probably due to the subtle effects that occur in the determination of the angular projection of a fractal structure (CP92, SLGMP96). Finally we want to stress that in the case of fractal distributions the \( \xi(r) \) CF (or the angular CF \( \omega(\theta) \)) has a power law behaviour at small scales and then there is a break that in log-log scale corresponds to a steep cut-off. The curved shape of \( \xi(r) \), or \( \omega(\theta) \), makes a precise determination of the correlation exponent very difficult. In particular all the determinations of this exponent by the \( \xi(r) \) or \( \omega(\theta) \) are affected by a systematic trend towards a larger value of \( \gamma \) (see Sec.4 for a detailed discussion of such an effect).

2. One of the main characteristics of galaxy surveys is that one finds that groups of galaxies comprise at least 70% of all galaxies not being part of clusters, and truly isolated galaxies are very rare. Tully (1988) by analyzing the Nearby Atlas of Galaxies (Tully and Fisher 1987) finds that essentially all galaxies can be grouped into clouds and that roughly 70% of these can be assigned to groups.

3. Various studies (Eder et al. 1989, Binggeli et al. 1990, Ferguson & Binggeli, 1994) of the spatial distribution of dwarf galaxies show that these galaxies fall into the structures defined by the luminous ones and that there is no evidence of segregation of bright and faint galaxies on large scale: dwarf galaxies are not more uniformly distributed than giants and the dwarfs, as the giants, belong to clouds, groups or clusters and there is evidence that the dwarfs fall well into the large scale patterns suggested by the giants consisting of filaments, walls and arcs. In particular, there is no evidence for them to fill voids (Thuan 1987, Bothun et al. 1988).

Disney & Phillips (1987) pointed out that galaxies of very low surface brightness (LSB) are entirely missed for an observational selection effect, and that what one can see is the ”tip of the iceberg”. Bothun et al. (1988) concluded that the patterns of Large Scale Structures appears to be mostly independent from galaxy surface brightness. Binggeli et al. (1988) stressed that most of the galaxies that are entirely missed because of their low surface brightness seem to be also of low total brightness, so that this observational bias has the effect of a lower cut-off in brightness.

4. An observation that is particularly important from a theoretical point of view is the behavior of the giant-to-dwarf ratio as a function of the local density:
we ask whether dwarf galaxies exist in low density regions, where giants are rare, or if they are only found as satellites of giants so that the giants-to-dwarfs ratio does not depend on environmental density. There is a clear experimental indication that the dwarf-to-giant ratio depends on the local density (Binggeli et al. 1988, Binggeli et al. 1990, Ferguson & Binggeli, 1994). Iovino et al. (1993) found that bright galaxies are relatively scarce in low density regions, while faint spirals are poorly present in high-density regions.

5. Einasto and Einasto (1985) found that the brightest galaxies in groups and clusters are brighter than in the field by up to 1 magnitude: the brightest galaxies lie preferentially in dense environments. In particular Dressler (1984) pointed out that the most luminous elliptical galaxies usually reside in the clusters cores, at local density maxima, and are not present in low density fields, so that these objects seem to be the product of dense environments.

6. The fact that giant galaxies are "more clustered" than the dwarfs has been interpreted as corresponding to a larger value of the amplitude of the correlation function for the giants than for dwarfs: this is the so-called "luminosity segregation" phenomenon (Davis et al., 1988; Iovino et al., 1993; Park et al., 1994; Benoist et al. 1996). On the contrary we show here that the segregation of giant galaxies in clusters arises as a consequence of self-similarity of matter distribution, and that in this case the only relevant parameter is the exponent of the correlation function, while the amplitude is a spurious quantity that has no direct physical meaning and depends explicitly on the sample size. For a detailed discussion of the luminosity segregation phenomenon we refer the reader to Sylos Labini et al. (1996b). In Fig.2a we report the behaviour of the so-called "correlation length" \( r_0 \) (defined as \( \xi(r_0) = 1 \)) computed in volume limited samples of the Perseus-Pisces survey with increasing depth \( R_s \): the linear scaling is in agreement with the fractal behaviour (CP92). Moreover in Fig.2b we show \( r_0 \) computed in volume limited samples with the same cut in absolute magnitude, but with different depth. If the luminosity segregation paradigm were to hold one should find that \( r_0 \) is independent on sample depth for galaxies with the same absolute luminosity, while that is clearly not the case. On the contrary the fractal nature of the galaxy distribution naturally explains the scaling of \( r_0 \) with depth.

7. There is evidence that galaxies in different environments are morphologically different and may followed different evolutionary paths. There is in particular a predominance of early type of galaxies in rich clusters: high density regions are dominated by E and S0 galaxies which themselves are hard to find in the field. Numerous studies have analyzed the variation in the pop-
ulation fraction and its possible relationship with cluster morphology (Oemler 1974); the morphological segregation has been studied systematically by Dressler (1980), who examined the variations in the relative fractions of E, S0, and spiral galaxies as a function of the local density and hence quantified the so-called morphology-density relation. He discovered that the local density of galaxies governs the mixture of Hubble types in any local environment of a cluster, independently of cluster global parameters like richness or size. The correlation of the morphological mix with local density is continuous and monotonic. This behavior has been shown to extend continuously over 6 orders of magnitude in space density from rich clusters to low density groups (de Souza et al. 1982, Postman & Geller, 1984). The main features of the morphological segregation is the decrease in spiral population with increasing local density and the increase with density of the fraction of S0 and elliptical galaxies.

Several authors (Oelmer, 1974; Melnik and Sergent 1977; Dressler 1980; Haynes & Giovanelli 1988; Iovino et al.1993) found that the relative abundance of elliptical, lenticular and spiral galaxies in clusters and their peripheries is a function of the local density: 80% of field galaxies are spirals and 15% of galaxies in rich clusters show spiral structure. The morphology-density relation in rich clusters is continuous over six orders of magnitude in space density and, correspondingly, the galaxian density is a continuous parameter: the consequence is that the separation between the luminosity function and the space density is seriously questionable. The morphology-density relation is found to hold also for dwarf galaxies (Ferguson & Binggeli, 1994). Recently Iovino et al.(1993) found clear evidence that the morphology and, in a weaker way, luminosity, are two independent parameters that affect galaxy distribution as a function of the local density.

8. The characteristics of morphology segregation can also be described by a comparison of the angular correlation function for representative samples of different morphology. Davis et al. (1976) found that elliptical-elliptical angular correlation function can be described by a power law with a slope significantly steeper than that of the corresponding spiral sample. Moreover the slope of the angular correlation function that characterizes the S0-S0 clustering is intermediate to other classes. Giovanelli et al.(1986), by analyzing the Perseus-Pisces redshift survey found that the slopes of \( w(\theta) \sim A\theta^\beta \) are significantly steeper for early type of galaxies: for early galaxies \( \beta = -0.90 \), while for early spirals \( \beta = -0.65 \) and for late spirals \( \beta = -0.37 \).

Davis and Djorgovski (1985) stressed that this result implies that the luminous galaxy distribution may not be a fair tracer of the mass distribution on any scale. In fact they argued that if the distribution of light in the Universe is a good tracer of the mass distribution then the spatial correlation function \( \xi(r) \)
should be the same for giant and dwarf galaxies. We show in the following that if the galaxy distribution is multifractal this apparent paradox is resolved and one expects that galaxies of different masses correlate with different exponents of the correlation function.

In the following we will see that morphological segregation is naturally explained within the context of a multifractal description, providing in the process a quantitative mathematical description of the phenomena.

3 Standard analysis of the Luminosity Function

The differential luminosity function, $\phi(L)$, gives the probability of finding a galaxy with luminosity in the range $[L, L + dL]$ in the unit volume ($Mpc^{-3}$). In literature (see Binggeli et al. 1988 for a review) one finds several methods to determine the LF for field galaxies and cluster galaxies. Special emphasis is devoted to the systematic differences in the LF for the various Hubble types. Here we are interested in the determination of the general LF defined as the sum over all Hubble types for field galaxies.

Let $\nu(L, \vec{r})$ denote the number of galaxies lying in volume $dV$ at $\vec{r}$ that have intrinsic luminosity between $L$ and $L + dL$. The main assumption generally used (Binggeli et al., 1988) is that galaxian luminosities are not correlated with spatial location. Under such an hypothesis one can write

$$\nu(L, \vec{r})dLdV = \phi(L)D(\vec{r})dLdV$$

(2)

where $\phi(L)$ gives the fraction of galaxies per unit luminosity having intrinsic luminosity in the interval $(L, L + dL)$, and $D(\vec{r})$ gives the number of galaxies of all luminosities per unit volume at $\vec{r}$.

The so-called classical method to determine the LF, in addition to the assumptions (1), (2) and (3), is based on the hypothesis that the galaxy distribution in the samples under analysis has reached homogeneity so that the average density $n_0$ of galaxies in space is constant and well defined. This method is highly sensitive to the spatial inhomogeneities in the distributions of galaxies that should distort the shape of the LF. For this reason many authors in the past (Felten 1977) excluded a region of solid angle containing strong "inhomogeneities" in galaxy distribution as the Virgo cluster.

Given the highly irregular character of galaxy distribution in all the recent redshift surveys (Haynes & Giovanelli, 1988; Paturel et al., 1988; Da Costa 1994; Vettolani et al. 1994), the assumption of constant density and homogeneous distribution is questionable and, in fact, the amplitude of the LF, that
is the average galaxy number density, is a strongly fluctuating and not well-defined quantity in the available samples. For this reason all new methods to determine the LF aim at a separation between the shape and the amplitude. In particular the so-called inhomogeneity-independent methods have been developed with the intent to determine only the shape of the LF. The basic idea is to consider the ratio of galaxies having intrinsic luminosity between \( L \) and \( L + dL \) to the total number of galaxies brighter than \( L \). If Eq.\(\ref{eq:2} \) holds then

\[
\nu(L, \vec{r})dLdV = \frac{\phi(L)D(\vec{r})dLdV}{\nu(L', \vec{r})dL'dV} = \frac{\phi(L)dL}{\Phi(L)} \sim d\log \Phi(L). \tag{3}
\]

By differentiating the integrated LF \( \Phi(L) \) one obtains the differential LF \( \phi(L) \). This technique allows recovery of the shape for the LF undisturbed by space inhomogeneities.

Usually the LF is assumed to be described by an analytical function. The most popular is the one proposed by Schechter (1975):

\[
\phi(L)d(L/L^*) = \phi^*(L/L^*)^\alpha \exp(-L/L^*)d(L/L^*) \tag{4}
\]

where \( L^* \) is the cut-off, \( \phi^* \) is the normalization constant (amplitude) and \( \alpha \) is the exponent. The LF has been measured by several authors in different redshift surveys (De Lapparent et al., 1986; Loveday et al., 1992; Da Costa et al., 1994; Marzke et al., 1994; Vettolani et al., 1994) and the agreement between the various determinations in very different volumes is excellent. The best fit parameters are \( \alpha = -1.13 \) and \( M_{bj}^* = -18.70 \) (Vettolani et al., 1994). The amplitude \( \phi^* \) is the most uncertain parameter of the LF because of spatial inhomogeneity in the available samples (De Lapparent et al., 1986; Da Costa et al., 1994). We discuss this point in the following.

We will show that not only the homogeneity assumption is inappropriate for the determination of the LF, but also that the assumption in Eq.(2) is not satisfied by the actual distribution of visible matter. As the available samples show structures as large as the survey depth we will see that not only the amplitude \( \phi^* \) but also the cut-off \( L^* \) of the LF are dependent on the sample depth. Our essential points will be the following. The galaxian luminosities are strongly correlated with their positions in space. This clear observational fact can be studied quantitatively with the MF formalism. In particular in such a scheme one can determine analytically the shape and the amplitude of the LF, and unify the various observational issues in quantitative mathematical scheme.
4 Essential properties of fractal structures

In this section we mention the essential properties of fractal structures and in the following section we introduce the MF formalism. However in no way these properties are assumed or used in the analysis itself. A fractal consists of a system in which more and more structures appear at smaller and smaller scales and the structures at small scales are similar to the one at large scales. Starting from a point occupied by an object, we count how many objects are present within a volume characterized by a certain length scale in order to establish a generalized "mass-length" relation from which one can define the fractal dimension. We can then define a relation between \( N \) ("mass") and \( r \) ("length") of type (Mandelbrot, 1982)

\[
N(r) = B \cdot r^D
\]  

(5)

where the fractal dimension is \( D \). Fractal structures in physics usually develop for length scales limited by a lower and/or an upper cut-off. In particular there exists a lower scale \( r_0 \) up to which the self-similarity holds, and the prefactor \( B \) is related to the number of elementary objects \( N_0 \) (galaxies) that are present within \( r_0 \) (CP92), i.e.

\[
B = \frac{N_0}{r_0^D}
\]  

(6)

Fractal structures are systems intrinsically irregular at all scales, and the self-similarity that characterizes their properties, implies the absence of regularity or analyticity everywhere in the system. From a mathematical point of view the property of self-similarity is associated to power-law functions for which the relevant property is the exponent (fractal dimension); the amplitude provides a is connection with the lower cut-offs of the distribution (CP92, BSLMP94, SLGMP96).

From Eq.5 we can readily compute the average density \( < n > \) within a spherical volume of radius \( R_s \)

\[
< n > = \frac{N(R_s)}{V(R_s)} = \frac{3}{4\pi} B R_s^{(3-D)}
\]  

(7)

From Eq.6 we see that the average density is not a meaningful concept in a fractal because it depends explicitly on the sample size \( R_s \). We can also see that for \( R_s \to \infty \) the average density \( < n > \to 0 \); therefore a fractal structure is asymptotically dominated by voids. We can define the conditional average density, that is the density in a spherical shell of area \( S(r) \) from an occupied point

\[
\Gamma(r) = S(r)^{-1} \frac{dN(r)}{dr} = \frac{D}{4\pi} B r^{(3-D)}
\]  

(8)
Usually the exponent \((3 - D)\) that defines the decay of the conditional density is called the codimension and it is related to the two point correlation function exponent \(\gamma\) as shown in Eq.(1) (CP92). The conditional average density, as given by Eq.8, is well defined in terms of its exponent, the fractal dimension. Moreover it is easy to show that the standard two-point correlation function in a spherical sample containing a fractal is (CP92)

\[
\xi(r) = \left[ (3 - \gamma) / 3 \right] \left( r / R_s \right)^{-\gamma - 1}
\]  

(9)

so that the so-called ”correlation length” is simply a linear fraction of the sample size (Fig.2a)

\[
x_0 = \left[ (3 - \gamma) / 6 \right] ^ {1/\gamma} R_s
\]  

(10)

Finally we would like to stress that \(\xi(r)\) is a power law only for

\[
\frac{3 - \gamma}{3} \left( \frac{r}{R_s} \right)^{-\gamma} \gg 1
\]  

(11)

hence for \(r \ll x_0\); for larger distances there is a clear deviation from the power law behaviour due to the definition of \(\xi(r)\). This deviation, however, is just due to the size of the observational sample and does not correspond to any real change of the correlation properties. It is clear that if one estimates the exponent of \(\xi(r)\) at distances \(r \lesssim x_0\), one systematically obtains a higher value of the correlation exponent due to the break of \(\xi(r)\) in the log-log plot. The analysis performed by \(\xi(r)\) is therefore mathematically inconsistent, if a clear cut-off towards homogeneity has not been reached, because it gives an information that is not related to the real physical features of the distribution in the sample, but to the size of the sample itself.

5 Multifractal measure

We now briefly introduce the concept of multifractal measure. The multifractal picture is a refinement and generalization of the fractal properties (Paladin & Vulpiani, 1987; Benzi et al., 1984, CP92, BSLMP94, SLGMP96, Sylos Labini et al.1996b) that arises naturally in the case of self-similar distributions. If one does not consider the mass one has a simple set given by the galaxy positions (that we call the support of the measure distribution). Multifractality instead becomes interesting and a physically relevant property when one includes the galaxy masses and consider the entire matter distribution (Pietronero, 1987; CP92). In this case the measure distribution is defined by assigning to each galaxy a weight which is proportional to its mass. The question of the self-similarity versus homogeneity of this set can be exhaustively discussed in terms
of the single correlation exponent that corresponds to the fractal dimension of
the support of the measure distribution. Several authors (Martinez & Jones,
1990) instead considered the eventual multifractality of the support itself. How-
ever the physical implication of such an analysis is not clear, and it does not
add much to the question above.

In the more complex case of MF distributions the scaling properties can be
different for different regions of the system and one has to introduce a contin-
uous set of exponents to characterize the system (the multifractal spectrum).
The discussion presented in the previous section was meant to distinguish be-
tween homogeneity and scale invariant properties; it is appropriate also in the
case of a multifractal. In the latter case the correlation functions we have
considered would correspond to a single exponent of a multifractal spectrum
of exponents, but the issue of homogeneity versus scale invariance (fractal or
multifractal) remains exactly the same.

Suppose that the total volume of the sample consists of a 3-dimensional
cube of size $L$. The density distribution of visible matter is described by

$$\rho(\vec{r}) = \sum_{i=1}^{N} m_i \delta(\vec{r} - \vec{r}_i)$$

(12)

where $m_i$ is the mass of the $i$-th galaxy and $N$ is the number of points in the
sample and $\delta(\vec{r})$ is the Dirac delta function. We assume that this distribution
corresponds to a measure defined on the set of points which have the corre-
lation properties described by Eq.8. It is possible to define the dimensionless
normalized density function

$$\mu(\vec{r}) = \sum_{i=1}^{N} \mu_i \delta(\vec{r} - \vec{r}_i)$$

(13)

with $\mu_i = m_i/M_T$ and $M_T = \sum_{i=1}^{N} m_i$, the total mass in the sample. We divide
this volume into boxes of linear size $l$. We label each box by the index $i$ and
construct for each box the function

$$\mu_i(\epsilon) = \int_{i-th\ box} \mu(r) dr$$

(14)

where $\epsilon = l/L$ and $0 < \mu_i < 1$. The definition of the box-counting fractal
dimension is

$$\lim_{\epsilon \rightarrow 0} \mu_i(\epsilon) \sim \epsilon^{\alpha(\vec{x})}$$

(15)

where $\alpha(\vec{x})$ is constant and equal to $D$ in all the occupied boxes in the case
of a simple fractal. This exponent fluctuates widely with the position $\vec{x}$ in the
In general we will find several boxes with a measure that scales with the same exponent \( \alpha \). These boxes form a fractal subset with dimension \( f \) that depends on the exponent \( \alpha \), i.e. \( f = f(\alpha) \). Hence the number of boxes that have a measure \( \mu \) that scales with exponent in the range \([\alpha, \alpha + d\alpha]\) varies with \( \epsilon \) as

\[
N(\alpha, \epsilon)d\alpha \sim e^{-f(\alpha)}d\alpha.
\] (16)

The function \( f(\alpha) \) is usually (Paladin & Vulpiani 1987) a single humped function with the maximum at \( \max_\alpha f(\alpha) = D \), where \( D \) is the dimension of the support. In the case of a single fractal, the function \( f(\alpha) \) is reduced to a single point: \( f(\alpha) = \alpha = D \).

In order to analyze the mass distribution of galaxies, obviously one needs to know the density distribution \( \rho(\vec{r}) \). The mass of each galaxy may be related to its total luminosity in a simple way

\[
M = k(i)L^\beta
\] (17)

where \( k \) is the mass to light ratio and depends on the galaxy morphological type \( i \). With respect to the MF properties, \( k \) plays a little role because the important quantity is the range of galaxy mass, which can be as large as a factor \( 10^6 \) or more. Therefore a modification of \( k \) produces small effects on a logarithm scale. The exponent \( \beta \) is more important, and here we assume (Faber and Gallagher 1979) that \( \beta \approx 1 \). However a different value of \( \beta \) should not change the MF nature of the mass distribution, if it is present in the sample, but only the parameters of the spectrum.

From a practical point of view one does not determine directly the spectrum of exponents \([f(\alpha), \alpha]\); it is more convenient to compute its Legendre transformation \([\tau(q), q] \) given by

\[
\begin{cases}
\tau(q) = q \cdot \alpha(q) - f(q) \\
\frac{d\alpha(q)}{dq} = \alpha(q)
\end{cases}
\] (18)

In the case of a simple fractal one has \( \alpha = f(\alpha) = D \). In terms of the Legendre transformation this corresponds to

\[
\tau(q) = D(q - 1)
\] (19)

i.e. the behaviour of \( \tau(q) \) versus \( q \) is a straight line with coefficient given by the fractal dimension.

The analyses carried out on CfA1 (CP92) and Perseus-Pisces (Sylos Labini et al., 1996b) redshift surveys provide unambiguous evidence for a MF behavior as shown by the non linear behaviour of \( \tau(q) \) in Fig.3.
5.1 Multifractal measure distribution

We have seen in the previous section the basic formulae that describe the scaling properties of a multifractal measure (MF). Suppose now we have a MF sample in a well defined volume $V$, and we want to study the behavior of the number of boxes with measure in the range $\mu$ to $\mu + d\mu$, having fixed the partitioning of the measure with boxes of size $\epsilon$. By changing variables and using Eq.15, then the measure distribution (Eq.16) becomes

$$N_\epsilon(\mu)d\mu \sim \epsilon^{-f(\alpha(\mu))} \frac{1}{|\log(\epsilon)|} \frac{d\mu}{\mu}.$$  

(20)

From this equation we can see that the distribution of the measure, at fixed resolution $\epsilon$, does not scale as a power law in $\mu$, because the exponent $f(\alpha(\mu))$ is a complex function of $\mu$. The self-similarity of the distribution is recovered by looking at the measure distribution as a function of the scale $\epsilon$.

Suppose we fix the dimension of the box at the scale $\epsilon$: for example, we can suppose that this can be the galactic scale, or the cluster scale. The function $N_\epsilon(\mu)$ is bell-shaped and convex with a maximum corresponding to the point at which

$$\frac{\partial N_\epsilon(\alpha)}{\partial \mu} = \frac{\partial N_\epsilon(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \mu} = -(f'(\alpha) + 1)N_\epsilon(\alpha) \frac{1}{\mu} = 0$$  

(21)

this condition corresponds

$$\left( \frac{\partial f(\alpha)}{\partial \alpha} \right)_{\alpha_c} = -1.$$  

(22)

The maximum of $N_\epsilon(\mu)$ fixes the most probable value of $\mu$. Well beyond this maximum the function can be well fitted by a power law. In practice this is the only observable part of the measure distribution in the case of galaxies because the higher values of $\alpha$ correspond to the smallest galaxies that are not present in the sample (CP92). For still higher values of $\mu$ the function shows an exponential-like decay. The tail is fixed by the point at which the derivative Eq.22 has a maximum. This happens for $\alpha = \alpha_{\text{min}}$, namely at the value corresponding to the box that contains the maximum measure (i.e. the strongest singularity)

$$\mu^* \sim \epsilon^{\alpha_{\text{min}}}.$$  

(23)

In order to compute the exponent characterizing the leading power law behavior we study the derivative of $\log(N_\epsilon(\mu))$ with respect to $\log(\mu)$. By performing the logarithmic derivative of Eq.20 we obtain

$$\frac{\partial \log(N_\epsilon(\mu))}{\partial \log(\mu)} = -\left( \frac{\partial f(\alpha)}{\partial \alpha} + 1 \right).$$  

(24)
We can try to fit Eq.20 with a power law function of $\mu$, plus an exponential tail. From Eq.24 we can define an effective exponent $\delta$, which depends explicitly on $\mu$. This implies that the power law approximation can be considered as a local fit

$$\delta = - \left( \frac{\partial f(\alpha)}{\partial \alpha} + 1 \right).$$  

This leads to $\delta = 0$ for $\alpha = \alpha_c$ and $\delta = -1$ for $\alpha_0$ such that $f(\alpha_0) = D(0)$. Locally we can expand $f'(\alpha)$ in power series of $\mu$, so that the Measure Distribution (MD) of Eq.20 in a certain range of $\mu$, is well fitted by a power law function with a cut-off

$$N(\mu) \sim \mu^\delta e^{-\mu^*}.$$ 

The exponent $\delta$ depends on the shape of the derivative of $f(\alpha)$, as well as on the value of $\alpha$ around which one develops $f(\alpha)$. We shall see, with the help of computer simulations that the value of $\delta$ is usually in the range $[-2,-1]$ for a wide range of $f(\alpha)$ spectra.

5.2 Numerical simulations

In order to examine a more general case we now consider a random multiplicative measure generated by a random fragmentation process using $m$ normalized generators $\chi^{(\gamma)}$ and $\gamma = 1, \ldots, m$ in the $d$-dimensional Euclidean space (Fig.7). The analytical shape of the measure distribution can be computed from the knowledge of $f(\alpha)$, determined by the Legendre transformation of $\tau(q)$ (computed from the $\chi^{(\gamma)}$). It is interesting to see how a MF distribution naturally leads to the various morphological properties that we have discussed in Sec. 2.

(1) The two point number-number correlation function

We have seen that the characteristics of a MF distribution is the $f(\alpha)$ spectrum of exponents. The relation of the properties of this distribution with the usual correlations analysis is straightforward. The exponent that describes the power law behaviour of the number density corresponds to the support of the MF distribution and therefore it is related to the maximum of $f(\alpha)$ (see Sec. 5.1).

(2) Luminosity function

In Fig.(8) we show the behaviour of the tail of $P(\mu)$ computed for the random MF distribution shown in Fig.7. It is possible to fit this curve with a Schechter-like function (Eq.26): the best fit gives $\delta = -1$ in agreement with
Eq.41. We can also see that at small $\mu$ the power law behaviour is a little bit steeper than in the intermediate region.

Consider a portion of a MF distribution in a spherical volume of radius $R$. The number of singularities of type $\alpha$ in the range $[\alpha, \alpha + d\alpha]$, is given by Eq.16. The corresponding average space density scales therefore as $<\nu(\alpha, R)> \sim R^{(f(\alpha)-3)}$. This relation implies that the space density is a complex function of $\alpha$ and $R$. Although not strictly valid, a separation between a space and a luminosity distribution is useful in the analysis of real catalogs. We approximate such a separation as follows. Integrating Eq.16 in $d\alpha$ we obtain that the total number of boxes of size $\epsilon$, divided for the volume, will be

$$n(R) = \frac{N(\epsilon(R))}{V} = \frac{3}{4\pi}BR^{D(0)-3}$$

(27)

where the last equality follows from Eq.7.

The probability that a certain box (at a given scale $\epsilon$) has the measure in the range $(\mu, \mu + d\mu)$ is determined by Eq.20 (or Eq.24). Hence the average number of boxes for unit measure and unit volume can be written as

$$<\nu(R, \mu)> = \frac{3}{4\pi}BR^{D(0)-3}\mu^\delta e^{-\frac{\mu}{\mu^*}}$$

(28)

This equation can be read as the average probability of having a galaxy of a certain luminosity and in a certain volume, in a MF Universe.

The amplitude of $\nu(R, \mu)$ is related to the lower cut-offs of the distribution and the size of the sample volume, and therefore has no special physical meaning. We stress in the following (see (5)) that the shape of the LF is not completely independent on the size of the sample volume. This implies that the approximation of Eq.28 does not strictly hold, because it does not consider the correlation between space and luminosity distribution, that for MF we know to be an important feature (see (6)). Nevertheless we stress that Eq.28 can be used in practice in the analysis of real redshift surveys, with great accuracy. In fact, the result of Eq.28 has been obtained under the approximation of Eq.26, while in the more general case the cut-off $\mu^*$ depends explicitly on the sample size. However this dependence is very weak in the available samples.

In summary, in order to study a MF distribution the three dimensional volume must be well defined: only in such a kind of volume one may define the scaling properties of the MF. This implies that an understanding of the distortions of the LF shape requires that volume limited samples should be used, rather than magnitude limited ones. Several authors, by analyzing the mean density in redshift surveys (i.e. the amplitude of the LF), concluded that the samples are not large enough to be fair because the fluctuations are
too large (Da Costa et al., 1994; Marzke et al., 1994). Our conclusion is that in magnitude limited samples it is still possible to use the inhomogeneity-independent technique to determine, if the galaxy distribution is MF, the shape of the LF, but its is certainly not possible to recover the amplitude of the LF, that is related to the space distribution via the average density. In this respect one should consider a volume limited sample and normalize for the global luminosity selection that is related to such volume limited sample (SLGMP96, Sylos Labini et al.1996b).

(3) Morphological and luminosities properties: a new interpretation of "luminosity-segregation"

Massive galaxies are mostly found in rich clusters while field galaxies are usually spirals or gas rich dwarfs (see Sec.2). These observational properties are consistent with multifractality, i.e. with the self-similar behaviour of the whole matter distribution. In Fig.4 we show the case of a deterministic MF while Fig.7 corresponds to a random MF. The largest peaks are located in the largest clusters. For the self-similarity each point of the structure belongs to a cluster or to a group of galaxies, because a certain portion of the fractal distribution is always made of smaller and smaller structures. Moreover the observations that the dwarf and low-surface-brightness galaxies do not fill the voids, is consistent with the fact that the galaxy distribution continues to be fractal even for the lowest peaks of the MF. In this picture the giant-to-dwarf ratio depends on the environmental density. In fact, the dwarf galaxies can belong to the rich clusters where the giants lie, but they can also be in small groups. The morphological-segregation can be seen as the self-similar character of the matter distribution. Multifractality is a description that can be useful for the statistical characterization of the system, but, of course, it cannot explain in detail various evidences which require a more appropriate morphological analysis.

From our result we can conclude that the fractal nature of galaxy distribution in the available samples, can account for the scaling of $r_0$ with sample depth. In this sense the luminosity segregation, intended as a different clustering properties of brighter and fainter galaxies in terms of the amplitude of the standard correlation function (Davis et al., 1988; Park et al., 1994; Benoist et al., 1996) has no experimental support. The linear dependence of $r_0$ on the sample size can be completely explained by the fractal nature of galaxy distribution (see also CP92, BSLMP94, Sylos Labini et al., 1996; Di Nella et al., 1996; Sylos Labini & Amendola 1996). The correct perspective to describe the different clustering of brighter and fainter galaxies is the MF picture, for which we have given ample evidences, and that implies that massive are mostly
into large clusters as observed. The quantitative characterization of such a phenomenon is therefore in terms of the exponent of the correlation function rather than its amplitude. In particular the brighter galaxies should have a greater correlation exponent than the fainter ones (see (4)).

(4) Multifractal spectrum and multiscaling

In Sec.4 we have introduced the MF spectrum $f(\alpha)$ and now we clarify its basic properties. Multifractality implies that if we select only the largest peaks in the measure distribution, the set defined by these peaks may have different fractal dimension than the set defined by the entire distribution. One can define a cut-off in the measure and consider only those singularities that are above it. If the distribution is MF the fractal dimension decreases as the cut-off increases. We note that, strictly speaking, the presence of the cut-off can lead (for a certain well defined value of the cut-off itself) to the so-called multiscaling behavior of the MF measure (Jansen et al., 1991). In fact, the presence of a lower cut-off in the calculation of the generalized correlation function affects the single-scaling regime of $\chi(\epsilon,q)$ for a well determined value of the cut-off $\alpha_{\text{cut-off}}$ such that $\alpha_{\text{cut-off}} < \alpha_c$, and this function exhibits a slowing varying exponent proportional to the logarithm of the scale $\epsilon$. However some authors (Martinez et al., 1995) misinterpret the multiscaling of a MF distribution as the variation of the fractal dimension with the density of the sample.

The fractal dimension $D$ of the support corresponds to the peak of the $f(\alpha)$-spectrum and raising the cut-off implies a drift of $\alpha$ towards $\alpha_{\text{min}}$ so that $f(\alpha) < D$ (Fig.5). This behavior can be connected with the different correlation exponent found by the angular correlation function for the elliptical, lenticular and spiral galaxies (see Sec. 2). In particular the observational evidence is that the correlation exponent is higher for elliptical than for spiral galaxies: this trend is compatible with a lower fractal dimension for the more massive galaxies than for the smaller ones, in agreement with a MF behavior.

(5) Scaling of the maximum mass

As shown in CP92 an important feature of the $f(\alpha)$ spectrum is represented by the value at $\alpha_{\text{min}}$: $f(\alpha_{\text{min}})$. This exponent corresponds to the scaling of the maximum singularity

$$\mu_{\text{max}}(\epsilon) \sim \epsilon^{\alpha_{\text{min}}}$$

where $\mu_{\text{max}}(\epsilon)$ is the maximum measure among all the boxes corresponding to a gridding of size $\epsilon$. The corresponding maximum density $\rho_{\text{max}}$ is therefore

$$\rho_{\text{max}} \sim \mu_{\text{max}}/\epsilon^3$$

where $\epsilon \sim 1/R_S$ and $R_S$ is the total size of the system.
Under the physical assumption that the maximum mass $M_{\text{max}}$ is related to the maximum density one can conclude that:

$$M_{\text{max}} \sim R_s^{3-\alpha_{\text{min}}}$$

(30)

i.e. that the maximum galaxy mass we can observe in a certain sample is related to the size of the sample itself. To study such an effect the sample size should be varied over a large range of length scales. In practice the depth of the VL samples that can be extracted from the available redshift surveys does not allow one to detect the scaling implied by Eq.30 so that the inhomogeneity-independent method in magnitude limited samples remains the most suitable to study the LF.

(6) Correlation between space and luminosity distribution

In Sylos Labini et al. (1996b) we have performed a test to check the MF nature of the observed Perseus-Pisces catalog. We have randomized the absolute magnitudes of the galaxies, i.e. we have fixed the galaxy position and we have assigned to each galaxy an absolute magnitude chosen at random among all the other galaxies. By doing this one destroys the correlations between the spatial locations and magnitudes of galaxies. As shown in Fig.9, in this case the dependence of $\tau(q)$ versus $q$ is linear, as for a simple fractal, with slope $\approx 2$. Instead in Fig.3 we show that the original distribution as a curved shape for $\tau(q)$. This result shows that the locations of galaxies are intrinsically correlated with their luminosities, i.e. the existence of a luminosity-position correlation. The MF framework provides a mathematical tool to study such a distribution.

6 Discussion and Conclusions

In this paper we have shown that it is possible to frame the main properties of the galaxy space and luminosity distribution in a unified scheme, by using the concept of multifractality (MF). In fact, the continuous set of exponents $[\alpha, f(\alpha)]$ that describes a MF distribution can characterize completely the galaxy distribution when one considers the mass (or luminosity) of galaxies in the analysis. In this way many observational evidences are linked together and arise naturally from the self-similar properties of the distribution.

Considering a MF distribution, the usual power-law space correlation properties correspond just to a single exponent of the $f(\alpha)$ spectrum: such an exponent simply describes the space distribution of the support of the MF measure. Furthermore the shape of the luminosity function (LF), i.e. the probability of
finding a galaxy of a certain luminosity per unit volume, is related to the \( f(\alpha) \) spectrum of exponents of the MF. We have shown that, under MF conditions, the LF is well approximated by a power law function with an exponential tail. Such a function corresponds to the Schechter LF observed in real galaxy catalogs. In this case the shape of the LF is almost independent on the sample size. Indeed we have shown that a weak dependence on sample size is still present because the cut-off of the Schechter function for a MF distribution turns out to be related to the sample depth: \( L^* \) increases with sample depth. In practice as this quantity is a strongly fluctuating one, in order to study its dependence on the sample size one should have a very large sample and should vary the depth over a large range of length scales. Given this situation a sample size independent shape of the LF can be well defined using the inhomogeneity-independent method in magnitude limited samples. Indeed such a technique has been introduced to take into account the highly irregular nature of the large scale galaxy distribution. For example a fractal distribution is non-analytic in each point and it is not possible to define a meaningful average density. This is because the intrinsic fluctuations that characterize such a distribution can be large as the sample itself, and the extent of the largest structures is limited only by the boundaries of the available catalogs.

Moreover if the distribution is MF, the amplitude of the LF depends on the sample size as a power law function. To determine the amplitude of the LF, as well as the average density, one should have a well defined volume limited sample, extracted from a three dimensional survey. We refer the reader to SLGMP96 for a detailed discussion on the determination of the average density that is related to the determination of the lower cut-off of the fractal distribution and to the subtle problems related with the finite size effects.

These results have important consequences from a theoretical point of view. In fact, when one deals with self-similar structures the relevant physical phenomenon that leads to the scale-invariant structures is characterized by the exponent and not the amplitude of the physical quantities that characterizes such distributions. Indeed, the only relevant and meaningful quantity is the exponent of the power law correlation function (or of the space density), while the amplitude of the correlation function, or of the space density and of the LF, is just related to the sample size and to the lower cut-offs of the distribution.

The geometric self-similarity has deep implications for the non-analyticity of these structures. In fact, analyticity or regularity would imply that at some small scale the profile becomes smooth and one can define a unique tangent. Clearly this is impossible in a self-similar structure because at any small scale a new structure appears and the distribution is never smooth. Self-similar structures are therefore intrinsically irregular at all scales and correspondingly one
has to change the theoretical framework into one which is capable of dealing with non-analytical fluctuations. This means going from differential equations to something like the Renormalization Group to study the exponents. For example the so-called ”Biased theory of galaxy formation” (Kaiser, 1984) is implemented considering the evolution of density fluctuations within an analytic Gaussian framework, while the non-analyticity of fractal fluctuations implies a breakdown of the central limit theorem which is the cornerstone of Gaussian processes (Pietronero & Tosatti 1986, CP92, BSLMP94).

In this scheme the space correlations and the luminosity function are then two aspects of the same phenomenon, the MF distribution of visible matter. The more complete and direct way to study such a distribution, and hence at the same time the space and the luminosity properties, is represented by the computation of the MF spectrum of exponents. This is the natural objective of theoretical investigation in order to explain the formation and the distribution of galactic structures. In fact, from a theoretical point of view one would like to identify the dynamical processes that can lead to such a MF distribution. As a preliminary step in this direction we have developed a simple stochastic model (Sylos Labini & Pietronero, 1995a; Sylos Labini & Pietronero, 1995b) in order to study which are the fundamental physical effects that lead to such a MF structure in an aggregation process. This is a very complex problem but to study it correctly one has to use the appropriate concepts and statistical tools.

If a crossover towards homogeneity would eventually be detected, this would not change the above discussion but simply introduce a crossover into it. The (multi)fractal nature of the observed structures would in any case, require a change of theoretical perspective.

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Appendix: The case of the Binomial multifractal

We discuss here a simple example of self-similar multifractal distribution for which all the properties can be derived analytically. This will provide a useful playground for the discussion of the real matter distribution. The binomial measure is defined by the following process: we start with constant density \( \rho = 1 \) defined in the unit interval \([0, 1]\) (the so-called support of the distribution). The first iteration consists of dividing the unit interval into two equal pieces, one having weight \( \mu_1 \) and the other one with weight \( \mu_2 \) such that \( \mu_1 + \mu_2 = 1 \). During the next iteration we apply the same procedure for the two subintervals: after an infinite number of iterations we are left with a well defined binomial measure (Fig.4). After \( k \) iterations the probability of having a singularity with measure (Pietronero & Siebesma, 1986)

\[
\mu = \mu_1^i \mu_2^{k-i}
\]

is

\[
P(k, i) = \frac{1}{2^k} \binom{k}{i}
\]

with \( i = 1, ..., k \). Considering the continuous limit of the binomial for \( k \to \infty \) we have:

\[
P(k, i)di \sim \frac{1}{2\pi \sigma^2} e^{-\frac{(i-<i>)^2}{2\sigma^2}} di
\]

with:

\[
<i> = k/2
\]
\[
\sigma^2 = k/4
\]

Inverting Eq.31, we obtain

\[
i = a \log(\mu) + b
\]

where

\[
a = \frac{1}{\log(\mu_1) - \log(\mu_2)}
\]
\[
b = -ka \log(\mu_1)
\]

By changing the variables in Eq.32, using Eq.36, we obtain

\[
P(\mu)d\mu \sim \frac{1}{2\pi \sigma^2} e^{-\frac{(a \log(\mu) + c)^2}{2\sigma^2}} \frac{a}{\mu} d\mu
\]

\[
c = b - <n>
\]
From Eq.39 we can see that

\[ P(\mu) \sim \mu^{-1} \]  

(41)

if the following inequality is satisfied

\[
\frac{a^2(\log(\mu))^2}{2\sigma^2} < \left( 1 + \frac{|ac|}{\sigma^2} \right) |\log(\mu)| \theta
\]

(42)

or

\[
|\log(\mu)| < \frac{2\sigma^2}{a^2} \left( 1 + \frac{|ac|}{\sigma^2} \right) \theta
\]

(43)

where we have introduced the parameter \( \theta \) to define quantitatively the precision of the approximation. For example for \( \theta = 0.1 \) then Eq.41 holds to about 10% accuracy (Fig.5). Beyond this region the function has a decay that can be fitted by an exponential tail.
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Figure captions

- Fig.1 The average conditional density $\Gamma(r)$ for VL samples of several redshift surveys (normalized to the luminosity selection of the VL sample): Perseus-Pisces (Sylos Labini et al.1996b, SLGMP96), LEDA (Di Nella et al.1996), CfA1 (CP92). The reference line has a slope $-\gamma = -1 \ (D \approx 2)$.

- Fig.2a The ”characteristic length scale” $r_0 \ (\xi(r_0) \equiv 1)$ plotted as a function of the sample radius $R_s$ for various volume limited samples of the Perseus-Pisces redshift survey. If the catalog is homogenous we should find a constant value for $r_0$. The fitting line has a slope of $D/6$ in agreement with a fractal behaviour. 2b The ”characteristic length scale” $r_0$ plotted as function of the absolute magnitude $M_{lim}$ of the volume limited samples with different depth. To test the luminosity segregation hypothesis, one should find that $r_0$ is the same for sample with the same $M_{lim}$ and different depth. It is clear that there is no agreement between these values.

- Fig. 3 The scaling exponents $\tau(q)$ as a function of the moment $q$ for the Perseus-Pisces redshift survey (from Sylos Labini et al.1996b). The multifractal behaviour is shown by the change of slope. For negative momenta the data are erratic because they are dominated by the smallest galaxies not present in the sample.

- Fig. 4 (a) The first four iterations for the construction of a multifractal binomial measure with $\mu_1 = 1/5$ and $\mu_2 = 4/5$. (b) The same MF measure of (a) but after 15 iterations.

- Fig. 5 The $f(\alpha)$ spectrum for the binomial multifractal of Fig.4. The support of the measure distribution is compact in this case.

- Fig. 6 (a) The measure distribution $P(\mu)$ (Eq.39) for the binomial multifractal of Fig.4. (b) The tail of $P(\mu)$: the fitting curve is a power law with an exponential cut-off according to Eq.26. the exponent is $\delta = -1$ according to Eq.41.

- Fig. 7 (a) A random multifractal in the one-dimensional Euclidean space. (b) A random multifractal in the two-dimensional Euclidean space

- Fig. 8 The measure distribution $P(\mu)$ (Eq.39) for the binomial multifractal of Fig.7. The fitting curve is a power law with an exponential cut-off according to Eq.26. the exponent is $\delta = -1.2$. 

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• Fig.9 The scaling exponents $\tau(q)$ as a function of the moment $q$ for the Perseus-Pisces redshift survey (from Sylos Labini et al., 1996b). In this case we have randomized the absolute magnitudes, and hence we have broken the correlation between galaxy luminosities and positions. The multifractal behaviour is also broken and the reference line has slope 2 in agreement with the simple fractal properties.