Hamiltonian analysis of Lagrange multiplier modified gravity

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Abstract

We develop a Hamiltonian formalism for Lagrange multiplier modified gravity. We further calculate the Poisson brackets between constraints and we show that they coincide with the algebra of constraints in the Hamiltonian formulation of general relativity.

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1. Introduction

One of the most important problems in present cosmology is the understanding of the origin of late-time cosmic acceleration (the so-called dark energy (DE) epoch). Recently, a new interesting DE model was proposed in [1, 2]. This model consists of two scalar fields where one of scalars represents the Lagrange multiplier. The multiplier puts constraint on the second scalar field and as a result the theory contains single degrees of freedom. It was shown that the energy of the system flows along a time-like geodesic that is similar to dust; however, the theory contains non-zero energy. The behavior of this system suggests that it can be a natural candidate for the unification of dark energy and dark matter. The cosmological implications of these models were then analyzed in [4–6]. The role of Lagrange multipliers in the context of \( f(R) \) gravities was studied in [3]. Moreover, the Lagrange multipliers in the context of modified gravity may improve the ultraviolet properties of the covariant Hořava–Lifshitz gravity [9] leading to its renormalizability conjecture [7, 8].

As was shown in all these papers, the presence of the Lagrange multipliers in the action has a strong impact on the form of the resulting equations of motions. Then it is natural to ask the question, how the presence of Lagrange multipliers modifies the Hamiltonian structure of the given theory. Moreover, we would like to see whether the Hamiltonian of these systems is again given as a linear combination of constraints and whether these constraints are the first class and their Poisson algebra respects the basic principles of geometrodynamics [10–12]. It turns out that the Hamiltonian structure of the given theory is very interesting. We show that
the presence of the first scalar field that plays the role of the Lagrange multiplier implies an existence of the second-class constraints. Then after solving them, we find the Hamiltonian equations of motion for the second scalar field that are autonomous in the sense that the time evolution of the scalar field does not depend on its conjugate momenta. Such systems were studied in the past especially in the context of the ’t Hooft deterministic approach to quantum mechanics [13–16]. We also find that the resulting theory is a fully constrained system with the algebra of constraints that has the same form as in general relativity.

As the second example of the Lagrange multiplier modified theory, we consider the gravity action introduced in [3]. This action is the Lagrange modification of $F(R)$ gravity theories\(^1\). We show that the resulting Hamiltonian is given as a linear combination of constraints and has a similar structure as the Hamiltonian of $F(R)$ gravities [21, 22]\(^2\). However, there is an important difference that follows the fact that the presence of the Lagrange multiplier implies that the original auxiliary fields become dynamical in Hamiltonian formulation. We further determine the Poisson brackets between constraints. We show that the algebra of these constraints takes exactly the same form as in [10–12]. In other words, we explicitly prove the consistency of Lagrange modified theories of gravity from the Hamiltonian point of view.

Let us summarize our results. We study the Lagrange multiplier modified theories with emphasis on their Hamiltonian formalism. We find that the resulting Hamiltonian is again given as a linear combination of the first-class constraints. We show that the Poisson brackets of these constraints have the same form as in general relativity.

This paper is organized as follows. In section 2, we perform the Hamiltonian formulation of the general relativity action together with the Lagrange multiplier modified scalar field action. We find the corresponding Hamiltonian and diffeomorphism constraints and calculate their algebra. In section 3, we study the Lagrange multiplier modified action introduced in [3]. We again determine the corresponding Hamiltonian. Then we calculate the Poisson brackets of the secondary constraints and we find that they take exactly the same form as in general relativity.

2. Lagrange multiplier modified scalar field action

In this section, we develop the Hamiltonian formalism for Lagrange multiplier modified scalar field action. We study the form of the action that was introduced in [3]:

$$S = \int d^{(D+1)}x \sqrt{-\hat{g}} \left[ (D+1) R(\hat{g}) - \frac{\omega(\phi)}{2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \lambda \left[ \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + U(\phi) \right] \right].$$ (1)

Let us explain our notation. We consider a $(D+1)$-dimensional manifold $\mathcal{M}$ with the coordinates $x^\mu$, $\mu = 0, \ldots, D$ and where $x^\mu = (t, \mathbf{x})$, $\mathbf{x} = (x^1, \ldots, x^D)$. We presume that this spacetime is endowed with the metric $\hat{g}_{\mu\nu}(x^\rho)$ with signature $(-, +, \ldots, +)$. Suppose that $\mathcal{M}$ can be foliated by a family of space-like surfaces $\Sigma_t$ defined by $t = x^0$. Let $g_{ij}$, $i, j = 1, \ldots, D$, denote the metric on $\Sigma_t$ with inverse $g^{ij}$ so that $g_{ij} \hat{g}^{hk} = \delta^i_j$. We introduce the future-pointing unit normal vector $n^\mu$ to the surface $\Sigma_t$. In ADM variables, we have $n^0 = \sqrt{-\hat{g}^{00}}$, $n^i = -\hat{g}^{0i}/\sqrt{-\hat{g}^{00}}$. We also define the lapse function $N = 1/\sqrt{-\hat{g}^{00}}$ and the shift function $N^i = -\hat{g}^{ik}/\sqrt{-\hat{g}^{00}}$. In terms of these variables, we write the components of the

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\(^1\) For reviews, see [17, 19, 20].

\(^2\) For related works, see [24, 25].
metric $\hat{g}_{\mu\nu}$ as
\[
\begin{align*}
\hat{g}_{00} &= -N^2 + N_i g^{ij} N_j, \\
\hat{g}_{i0} &= N_i, \\
\hat{g}_{ij} &= g_{ij} - N_i N_j \tag{2}
\end{align*}
\]
Then it is easy to see that
\[
\sqrt{-\det \hat{g}} = N \sqrt{\det g},
\]
and for the momentum conjugate to $\phi$ and $\lambda$:
\[
p_\phi = \sqrt{g}(\omega + \lambda) \nabla^i \phi, \quad p_\lambda \approx 0 \tag{5}
\]
Finally we write the Hamiltonian for general relativity part of the action
\[
H^{GR} = \int d^Dx \mathcal{H}^{GR}, \quad \mathcal{H}^{GR} = N \mathcal{H}^{GR}_T + N^i \mathcal{H}^{GR}_i \tag{7}
\]
where
\[
\begin{align*}
\mathcal{H}^{GR}_T &= \frac{1}{\sqrt{g}} \pi^{ij} g_{ik} g_{jl} \pi_{kl} - \frac{1}{\sqrt{g}} R^{(D)} - \frac{1}{\sqrt{g}} R \Omega^2, \\
\mathcal{H}^{GR}_i &= -2 g_{ik} \nabla_j \pi_{kj},
\end{align*}
\]
and where $\pi^{ij}$ is the momentum conjugate to $g_{ij}$ with non-trivial Poisson brackets
\[
\{ g_{ij}(x), \pi^{kl}(y) \} = \frac{1}{2} \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \delta(x - y) \tag{9}
\]
and where $\pi = \pi^{ij} g_{ij}$.

In summary, the total Hamiltonian is $H = H^\phi + H^{GR}$. The preservation of the primary constraints $p_N \approx 0$ and $p_\lambda \approx 0$ implies the secondary ones
\[
\mathcal{H}_T = \mathcal{H}^{GR}_T + \mathcal{H}^\phi_T \approx 0, \quad \mathcal{H}_i = \mathcal{H}^{GR}_i + \mathcal{H}^{GR}_{\phi} \approx 0 \tag{10}
\]
It is useful to introduce the smeared form of these constraints
\[
\begin{align*}
T_T(N) &= T_T^{GR}(N) + T_T^\phi(N), \\
T_S(N^i) &= T_S^{GR}(N^i) + T_S^\phi(N^i),
\end{align*}
\]
where
\[
\begin{align*}
T_T^{GR}(N) &= \int d^Dx N T_T^{GR}, \\
T_T^\phi(N) &= \int d^Dx N T_T^\phi, \\
T_S^{GR}(N^i) &= \int d^Dx N^i T_S^{GR}, \\
T_S^\phi(N^i) &= \int d^Dx (N^i T_T^\phi + N^i p_\lambda \partial_\lambda) \tag{12}
\end{align*}
\]
where we included the primary constraint \( p_{\lambda} \approx 0 \) into the definition of \( T^{\phi}_{S}(N^{i}) \) in order to ensure the correct form of the Poisson bracket between the diffeomorphism generator \( T^{\phi}_{S}(N^{i}) \) and the scalar field \( \lambda \).

It is well known that the Poisson brackets between the smeared form of the general relativity constraints take the form [10–12]

\[
\{ T^{GR}_{S}(N), T^{GR}_{S}(M) \} = T^{GR}_{S}(g^{ij}(N_{i} M_{j} - M_{i} N_{j})),
\]

\[
\{ T^{GR}_{S}(N^{i}), T^{GR}_{S}(M^{j}) \} = T^{GR}_{S}(N^{i} M^{j} - M^{i} N^{j}).
\]

(13)

On the other hand, we have to determine the Poisson brackets between constraints corresponding to the scalar field. First of all it is easy to see that

\[
\{ T^{\phi}_{S}(N^{i}), T^{\phi}_{S}(M^{j}) \} = T^{\phi}_{S}(N^{j} \partial_{j} M_{i} - M^{j} \partial_{j} N_{i}).
\]

(14)

On the other hand, the Poisson bracket between \( T^{S}_{S}(N^{i}) \) and \( T^{\phi}_{T}(M) \) is equal to

\[
\{ T^{S}_{S}(N^{i}), T^{\phi}_{T}(M) \} = \int d^{D}x (-N^{k} \partial_{k} H^{\phi}_{T} - \partial_{k} N^{k} H^{\phi}_{T})
\]

\[
= \int d^{D}x N^{k} \partial_{k} H^{\phi}_{T} = T^{\phi}_{S}(N^{k} \partial_{k} M)
\]

(15)

using

\[
\{ T^{S}_{S}(N^{i}), g_{ij} \} = -N^{k} \partial_{k} g_{ij} - \partial_{i} \xi^{k} g_{kj} - g_{ik} \partial_{j} \xi^{k},
\]

\[
\{ T^{S}_{S}(N^{i}), \sqrt{g} \} = -N^{k} \partial_{k} \sqrt{g} - \sqrt{g} \partial_{k} N^{k}.
\]

(16)

Note that the presence of the term \( N^{i} p_{\lambda} \partial_{i} \lambda \) in the definition of \( T^{\phi}_{S}(N^{i}) \) was crucial for deriving the correct form of the Poisson bracket (15). Finally we calculate the Poisson bracket between \( T^{\phi}_{T}(N) \) and \( T^{\phi}_{T}(M) \) and after some algebra we obtain the desired result

\[
\{ T^{\phi}_{T}(N), T^{\phi}_{T}(M) \} = T^{\phi}_{S}(g^{ij}(N_{i} M_{j} - M_{i} N_{j})).
\]

(17)

It is also easy to show that

\[
\{ T^{GR}_{S}(N), T^{\phi}_{T}(M) \} + \{ T^{\phi}_{S}(N), T^{GR}_{S}(M) \} = 0
\]

(18)

due to the fact that \( H^{\phi}_{T} \) depends on \( g \) and not on its derivatives. If we combine these results, we find that the Poisson brackets of the constraints \( T^{S}_{T}(N), T^{S}_{S}(N^{i}) \) have the desired form (13).

As the next step, we analyze the stability of the primary constraint \( p_{\lambda} \approx 0 \). The requirement of its stability implies the secondary constraint

\[
\partial_{t} p_{\lambda}(x) = \{ p_{\lambda}(x), H \} = \frac{1}{2 \sqrt{g}(\omega + \lambda)^{2}} p_{\phi}^{2} - \frac{1}{2 \sqrt{g}} g^{ij} \partial_{i} \phi \partial_{j} \phi - \sqrt{g} U \equiv G_{\lambda}(x) \approx 0.
\]

(19)

We observe that

\[
\{ p_{\lambda}(x), G_{\lambda}(y) \} = \frac{1}{\sqrt{g}(\omega + \lambda)^{2}} p_{\phi}^{2}(x) \delta(x - y).
\]

(20)

In other words \( p_{\lambda} \) and \( G_{\lambda} \) are the second-class constraints. However, there are additional non-zero Poisson brackets. The first one is

\[
\{ G_{\lambda}(x), G_{\lambda}(y) \} = -2 \frac{1}{\sqrt{g}(\omega + \lambda)^{2}} g^{ij} H^{\phi}_{T}(x) \partial_{i} \delta(x - y)
\]

\[
- \partial_{i} \left[ \frac{1}{\sqrt{g}(\omega + \lambda)^{2}} g^{ij} H^{\phi}_{T}(x) \right] \delta(x - y).
\]

(21)
It is also clear from the structure of the constraint $G_i$ that there is a non-zero Poisson bracket between $G_i$ and $H$ defined by (10)
\[
\{G_i(x), H\} \neq 0,
\]
where $H = N\mathcal{H}_T + N\mathcal{H}_I$. Note that the explicit form of this Poisson bracket is not important for us.

Using these results we can proceed to the study of the stability of the secondary constraints. Following the standard analysis of the constraint systems, we introduce the total Hamiltonian as
\[
H_T = H + \int d^Dx (\alpha p_\alpha + \beta G_\beta),
\]
where $\alpha, \beta$ are Lagrange multipliers and analyze the stability of the constraints $\mathcal{H}, G_\alpha, p_\alpha$.

First we have
\[
\partial_t \mathcal{H} = \{\mathcal{H}(x), H\} + \int d^Dy (\beta \{\mathcal{H}(x), G_\beta(y)\} + \alpha \{\mathcal{H}(x), p_\alpha(y)\})
\]
\[
\approx \int d^Dy \beta \{\mathcal{H}(x), G_\beta(y)\} \neq 0,
\]
where we used the fact that the Poisson brackets between $\mathcal{H}$ and $H$ weakly vanish. Then the requirement of stability of the constraint $H \approx 0$ determines the value of the Lagrange multiplier $\beta$ to be equal to 0. On the other hand, the time evolution of the constraint $G_\lambda$ is given by the equation
\[
\partial_t G_\lambda(x) = \{G_\lambda(x), H\} + \int d^Dy \alpha \{G_\lambda(x), p_\beta(y)\} \approx 0.
\]

Due to the fact that $\{G_\lambda, H\} \neq 0$ and $\{G_\lambda, p_\beta\} \neq 0$, the equation above can be solved for $\alpha$ at least in principle. Then using these results it is easy to see that the constraint $p_\beta \approx 0$ is preserved during the time evolution of the system. Further, $p_\beta$ and $G_\lambda$ are the second-class constraints that can be solved for $\lambda$ and $p_\beta$ so that the reduced phase space is spanned by $(g_{ij}, \pi^{ij})$ and $(\phi, p_\phi)$ and the symplectic structure is given by the Dirac brackets between these variables. In order to find their form, we introduce the following notation for the Poisson brackets of the second-class constraints $p_\beta, G_\lambda$:
\[
\Delta_{11}(x, y) = \{p_\beta(x), p_\beta(y)\} = 0, \quad \Delta_{12}(x, y) = \{p_\beta(x), G_\lambda(y)\} \neq 0,
\]
\[
\Delta_{21}(x, y) = \{G_\lambda(x), p_\beta(y)\} \neq 0, \quad \Delta_{22}(x, y) = \{G_\lambda(x), G_\lambda(y)\} \neq 0
\]
and denote the inverse matrix as $(\Delta^{-1})^{AB}(x, y)$. This matrix by definition obeys the equation
\[
\int dx \Delta_{AC}(x, z)(\Delta^{-1})^{CB}(z, y) = \delta^B_A (x - y).
\]

It can be shown that the matrix $(\Delta^{-1})$ has the following structure:
\[
(\Delta^{-1}) = \begin{pmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
where $*$ denotes non-zero elements. It is important for the calculation of the Dirac brackets that $(\Delta^{-1})^{22} = 0$. Explicitly, the Dirac bracket between $\phi$ and $p_\phi$ takes the form
\[
\{\phi(x), p_\phi(y)\}_D = \{\phi(x), p_\phi(y)\}
\]
\[
- \int d^Dz d^Dz' [\phi(x), \Phi_A(z)](\Delta^{-1})^{AB}(z, z')[\Phi_B(z'), p_\phi(y)]
\]
\[
= \{\phi(x), p_\phi(y)\} - \int d^Dz d^Dz' [\phi(x), G_\lambda(z)](\Delta^{-1})^{22}(z, z')[G_\lambda(z'), p_\phi(y)]
\]
\[
= \{\phi(x), p_\phi(y)\},
\]
where $\Phi_A = (p_\beta, G_\lambda)$ is the common notation for the second-class constraints.
We are now ready to completely eliminate the second-class constraints $\Phi_\lambda$. The constraint $G_\lambda = 0$ can be solved for $\omega + \lambda$

$$(\omega + \lambda) = \frac{P_\phi}{\sqrt{g} \sqrt{g^{ij} \partial_i \phi \partial_j \phi} + 2U}.$$  

(30)

Inserting this result into the Hamiltonian constraint (6), we find that it takes the form

$$\mathcal{H}_I^\phi = p_\phi \sqrt{g^{ij} \partial_i \phi \partial_j \phi} + 2U + \sqrt{g^\omega U} - \sqrt{g^\omega U}.$$  

(31)

We observe that this Hamiltonian density is linear in momenta. Then the equation of motion for $\phi$ takes the form

$$\partial_t \phi = \{\phi, H\} = N \sqrt{g^{ij} \partial_i \phi \partial_j \phi} + 2U + N^j \partial_j \phi.$$  

(32)

that shows that the time evolution of $\phi$ does not depend on $p_\phi$. Such systems were extensively studied in the past in the context of ’t Hooft’s deterministic approach to quantum mechanics [13–16] and it is really interesting that the Hamiltonian with the similar structure arises in the Lagrange modified multiplier theory. More precisely, let us review basic facts considering such a system, following [18]. Let us consider the Hamiltonian system

$$H = p_i f^i(q) + U(q), \quad i = 1, \ldots, N.$$  

(33)

From (33) we determine the equations of motion for $q^i$:

$$\partial_t q^i = \{q^i, H\} = f^i(q).$$  

(34)

This equation for $q^i$ is autonomous, i.e. it is decoupled from the conjugate momenta $p_i$. Further it is impossible to perform the Legendre transformation to the Lagrangian since $H_{ij} = \frac{\partial^2 U}{\partial q^i \partial q^j} = 0$. However, it is possible to find the Lagrangian that gives the equation of motion (34) when we introduce the auxiliary fields $\lambda_i$ and write the Lagrangian as

$$L = \lambda_i (\dot{q}^i - f^i(q)) - U(q).$$  

(35)

No we show that from (35) we can derive Hamiltonian (33). The momenta conjugate to $\lambda_i$ and $q^i$ take the form

$$p^i_\lambda = \frac{\delta L}{\delta \dot{\lambda}_i} \approx 0, \quad p^i_{q} = \frac{\delta L}{\delta \dot{q}^i} = \lambda_i$$  

(36)

so that we have two sets of primary constraints:

$$\Phi^i_\lambda = p^i_\lambda \approx 0, \quad \Phi^i_q = p^i_q - \lambda_i \approx 0.$$  

(37)

The extended Hamiltonian that follows from (35) takes the form

$$H_E = H + \omega^i_q \Phi^i_\lambda + \omega^i_q \Phi^i_q, \quad H = \lambda_i f^i + U(q).$$  

(38)

Then we study the stability of the constraints $\Phi^i_\lambda$, $\Phi^i_q$:

$$\partial_t \Phi^i_\lambda = \{\Phi^i_\lambda, H_E\} = -f^i + \omega^i_q = 0$$

$$\partial_t \Phi^i_q = \{\Phi^i_q, H_E\} = -\lambda_j \frac{\partial f^j}{\partial q^i} - \omega^i_q = 0.$$  

(39)

From these equations, we can in principle determine the Lagrange multipliers. In other words, the constraints $\Phi^i_\lambda$ and $\Phi^i_q$ are the second class that should strongly vanish. Solving these constraints, we obtain the Hamiltonian

$$H = p^i_q f^i + U(q)$$  

(40)

that coincides with Hamiltonian (33). Further, it can be easily shown that the Dirac brackets between $q^i$ and $p_i$ coincide with their Poisson brackets. However, the problem with Hamiltonian
(33) is that it is not bounded from below which is due to the absence of a leading kinetic term quadratic in the momenta \((p_j)^2\).

After this short review of the Hamiltonian analysis of the autonomous system, we complete our analysis by the calculation of the Poisson bracket between \(H^f_\phi\) given in (31) and the spatial diffeomorphism constraint \(T_3(N^i)\). Using

\[
\{T_3(N^i), H^f_\phi\} = -N^i \partial_\phi (g^{ij} \partial_\phi \partial_j \phi),
\]

we easily obtain

\[
\{T_3(N^i), H^f_\phi\} = -N^i \partial_\phi H^f_\phi - \partial_i N^i H^f_\phi,
\]

where \(H^f_\phi\) was given in (31). The analysis of the remaining Poisson brackets is the same as above with conclusion that the smeared form of the constraints obeys the algebra of constraints given in (13). In other words, we show that the Lagrangian multiplier modified scalar action together with general relativity action obeys the basic rules of geometrodynamics.

3. Hamilton analysis of \(F(R)\) theories with Lagrange multipliers

It turns out that the Lagrange multiplier modified \(F(R)\) gravity possesses many interesting properties. For example, the reconstruction program can be more easily performed in Lagrange multiplier modified gravity [3]. In usual \(F(R)\) gravity, we need to solve the complicated differential equation to realize the reconstruction program; for a recent review, see [23]. It was demonstrated in [3] that the presence of the constraint significantly simplifies the reconstruction scenario. It was also shown there that the presence of the Lagrange multiplier implies that it is necessary to include the second \(F(R)\) function into action.

The action introduced in [3] takes the form

\[
S = \int \sqrt{-g} \left[ F_1 (A) - \lambda \left( \frac{1}{2} \partial_\mu g_{\mu\nu} \partial^\nu A + F_2 (A) \right) - B(A) \right].
\]

Introducing two auxiliary fields \(A, B\), we can rewrite action (43) into the form

\[
S = \int \sqrt{-g} \left[ F_1 (A) - \lambda \left( \frac{1}{2} \partial_\mu A g_{\mu\nu} \partial^\nu A + F_2 (A) \right) + B^{(D+1)} - A \right].
\]

It is easy to see that the integration of \(A, B\) from (44) leads to (43). Our goal is to find the Hamiltonian from (44) implementing the \(D+1\) formalism. In fact, using (4) it is easy to see that action (44) takes the form

\[
S = \int d^D x \sqrt{\hat{g}} N (F_1 (A) - \lambda (-\nabla_n A \nabla_n A + g^{ij} \partial_i A \partial_j A + F_2 (A)) - B(A))
\]

\[
+ \int d^D x \sqrt{\hat{g}} N B (K_{ij} \hat{g}^{ij} K_{kl} + R^{(D)} - A)
\]

\[
- 2 \int d^D x \sqrt{\hat{g}} \partial_i (\sqrt{\hat{g}} \partial_i B - N^i \partial_i B) A + 2 \sqrt{\hat{g}} \partial_i B g^{ij} \partial_j N).
\]

where we performed integration by parts and ignored boundary terms. From (45), we easily find momenta conjugate to canonical variables \(g_{ij}, N, N_i, A\) and \(B\)

\[
\pi_{ij} = \sqrt{\hat{g}} B \hat{g}^{ij} K_{kl} - \sqrt{\hat{g}} \nabla_n B g^{ij}, \quad p_N \approx 0, \quad p^i \approx 0,
\]

\[
p_B = -2 \sqrt{\hat{g}} K, \quad p_A = 2 \sqrt{\hat{g}} \nabla_n A, \quad p_k \approx 0.
\]

Note that the Lagrange multiplier implies that \(A\) is a dynamical field which is different from the standard \(F(R)\) theory of gravity where \(A\) remains the auxiliary field. Then after some effort, we derive the Hamiltonian density in the form

\[
\mathcal{H} = N \mathcal{H}_T + N^i \mathcal{H}_i, \quad \mathcal{H}_T = \hat{g}^{ij} \partial_\phi \partial_i \phi, \quad \mathcal{H}_i = \frac{1}{2} \partial_\phi \partial_i \phi.
\]
where
\[ H_T = \frac{1}{\sqrt{g}} B g^{ij} g^{kl} \pi_{ij} \pi_{kl} - \frac{1}{\sqrt{g}} B D \pi^2 - \frac{\pi p_B}{\sqrt{g}} \]
\[ + \frac{B}{4 \sqrt{g} D} (D - 1) p_B^2 - \sqrt{g} B R^{(D)} + 2 \partial_i [\sqrt{g} g^{ij} \partial_j B] \]
\[ + \frac{1}{4 \sqrt{g} \lambda} p_A^2 + \sqrt{g} B A - \sqrt{g} [F_1(A) - \lambda (g^{ij} \partial_i A \partial_j A + F_2(A))]. \] (48)
and where
\[ H_i = p_A \partial_i A + p_B \partial_i B + p_\lambda \partial_\lambda - 2 g_{ik} \nabla_j \pi^{jk}. \] (49)

For further purposes, we split \( H_T \) into two parts as \( H_T = H_T^{\text{GR}} + H_T^A \) where
\[ H_T^{\text{GR}} = \frac{1}{\sqrt{g}} B g^{ij} g^{kl} \pi_{ij} \pi_{kl} - \frac{1}{\sqrt{g}} B D \pi^2 - \frac{\pi p_B}{\sqrt{g}} \]
\[ + \frac{B}{4 \sqrt{g} D} (D - 1) p_B^2 - \sqrt{g} B R^{(D)} + 2 \partial_i [\sqrt{g} g^{ij} \partial_j B] \]
\[ H_T^A = \frac{1}{4 \sqrt{g} \lambda} p_A^2 + \sqrt{g} B A - \sqrt{g} [F_1(A) - \lambda (g^{ij} \partial_i A \partial_j A + F_2(A))]. \] (50)

The theory possesses \( 2 + D \) primary constraints
\[ \pi_N \approx 0, \quad \pi_i \approx 0, \quad \pi_\lambda \approx 0. \] (51)
The preservation of the primary constraints \( \pi_N \) and \( \pi_i \) implies the secondary constraints \( \mathcal{H}_T \approx 0 \) and \( H_i \approx 0 \), while the preservation of \( \pi_\lambda \) leads to the secondary constraint
\[ G_\lambda = \frac{1}{4 \sqrt{g} \lambda} p_A^2 - \sqrt{g} (g^{ij} \partial_i A \partial_j A + F_2(A)) \approx 0. \] (52)
We see that it takes the same form as the secondary constraint (19). Clearly, \( p_\lambda \) together with \( G_\lambda \) is the second-class constraint. The properties of these constraints were analyzed in the previous section and results derived there can be used in this section as well.

On the other hand, the form of the Hamiltonian constraint \( H_T^{\text{GR}} \) is new and we have to check that this constraint is preserved during the time evolution of the system. In other words, we have to calculate the Poisson brackets of the smeared form of these constraints\(^3\)
\[ T_T^{\text{GR}}(N) = \int d^D x N(x) \mathcal{H}_T^{\text{GR}}(x), \quad T_S^{\text{GR}}(N') = \int d^D x N'(x) \mathcal{H}_T^{\text{GR}}(x). \] (53)
Let us now outline the strategy of the calculations of these Poisson brackets. In the process of their calculations, several delta functions occur. However, it turns out that the non-zero contributions give terms that contain derivatives of these delta functions. Such expressions arise for example from the following Poisson bracket:
\[ \{ \pi^{ij}(x), (\sqrt{g} R^{(D)})(y) \} = - \frac{\delta (\sqrt{g} R^{(D)})(y)}{\delta g_{ij}(x)}. \] (54)
The right-hand side of this equation can be calculated using the formulas
\[ \delta R^{(D)} = -(R^{(D)})^{ij} \delta g_{ij} + \nabla^i \nabla^j \delta g_{ij} - g^{ij} \nabla^k \delta g_{ji}, \quad \delta g = g^{ij} \delta g_{ij}. \] (55)
\(^3\) In [22], a similar analysis has been performed in the context of a non-projectable version of Hořava–Lifshitz \( F(R) \) gravity.
Now we are ready to perform these calculations. It turns out that the following non-zero Poisson brackets contribute to the final result:

\[
\left\{ \int d^Dx N \frac{1}{\sqrt{g}} \pi^{ij} g_{ik} g_{jl} \pi^{kl}, \int d^Dy M \sqrt{\mathcal{R}} R^{(D)} \right\}
- \left\{ \int d^Dy N \sqrt{\mathcal{R}} R^{(D)}, \int d^Dx M \frac{1}{\sqrt{g}} \pi^{ij} g_{ik} g_{jl} \pi^{kl} \right\}
= 2 \int d^Dx (N \nabla_i \nabla_j M - M \nabla_i \nabla_j N) \pi^{ij} - 2 \int d^Dx \pi (N \nabla_i \nabla^i M - M \nabla_i \nabla^i N)
+ 4 \int d^Dx \pi^{ij} (N \nabla_i M - M \nabla_i N) \frac{1}{B} \nabla_j B
- 4 \int d^Dx \pi (N \nabla_i M - M \nabla_i N) \frac{1}{B} \nabla^i B,
\]

\( (56) \)

\[
\left\{ \int d^Dy \frac{N}{\sqrt{g}} B \pi^2, \int d^Dx M \sqrt{\mathcal{R}} R^{(D)} \right\} + \left\{ \int d^Dx N \sqrt{\mathcal{R}} R^{(D)} B M, \int d^Dy \frac{M}{\sqrt{g}} \pi^2 \right\}
= - \frac{2}{D} \int d^Dx \pi (N \nabla_i \nabla^i M - M \nabla_i \nabla^i N) + 2 \int d^Dx \pi (N \nabla_i \nabla^i M - M \nabla_i \nabla^i N)
\]

\( (57) \)

and

\[
\left\{ \int d^Dx N \frac{\pi^p B}{\sqrt{g}}, \int d^Dy \frac{\sqrt{\mathcal{R}}}{\sqrt{g}} R^{(D)} B M, \int d^Dy \frac{M}{\sqrt{g}} \pi^2 \right\}
= \left\{ \int d^Dx N \frac{\pi^p B}{\sqrt{g}}, \int d^Dy \frac{\sqrt{\mathcal{R}}}{\sqrt{g}} R^{(D)} B N, \int d^Dx \frac{\pi M}{\sqrt{g}} \right\}
= - \frac{(1 - D)}{D} \int d^Dx \pi^p B (N \nabla_i \nabla^i M - M \nabla_i \nabla^i N)
\]

\( (58) \)

\[
\left\{ \int d^Dx \frac{N}{\sqrt{g}} B \pi^{ij} g_{ik} g_{jl} \pi^{kl}, \int dy 2 M \partial_{i} \left[ \sqrt{g} \delta^{ij} \partial_{j} B \right] \right\}
+ \left\{ \int dy 2 N \partial_{i} \left[ \sqrt{g} \delta^{ij} \partial_{j} B \right], \int d^Dx \frac{M}{\sqrt{g}} \pi^{ij} g_{ik} g_{jl} \pi^{kl} \right\}
= 2 \int d^Dx \pi \frac{1}{B} (N \nabla_i M - M \nabla_i N) g^{ij} \nabla_j B
\]

\( (59) \)

and

\[
\left\{ \int d^Dx \frac{N}{\sqrt{g} BD} \pi^2, \int d^Dy M 2 \partial_m \left[ \sqrt{g} \delta^{mn} \partial_n B \right] \right\}
- \left\{ \int d^Dx M 2 \partial_m \left[ \sqrt{g} \delta^{mn} \partial_n B \right], \int d^Dx \frac{M}{\sqrt{g} BD} \pi^2 \right\}
= \frac{2(2 - D)}{D} \int d^Dx \frac{\pi}{B} (N \nabla_i M - M \nabla_i N) \nabla^i B
\]

\( (60) \)
and
\[- \left\{ \int d^Dx \frac{N}{\sqrt{g}} \pi p_B, \int d^Dy 2M \partial_m [\sqrt{g} g^{mn} \partial_n B] \right\} \]
\[- \left\{ \int d^Dy 2N \partial_m [\sqrt{g} g^{mn} \partial_n B], \int d^Dx \frac{M}{\sqrt{g}} \partial_m \pi p_B \right\} \]
\[= \frac{(2 - D)}{D} \int d^Dx (N \nabla_m M - M \nabla_m N) \pi p_B \nabla^m B \]
\[+ \frac{2}{D} \int d^Dx \pi (N \nabla_m \nabla^m M - M \nabla_m \nabla^m N), \quad (61)\]
\[
\left\{ \int d^Dx \frac{NB}{4\sqrt{g}} (D - 1) p_B^2, 2 \int d^Dy M \partial_m [\sqrt{g} g^{mn} \partial_n B] \right\}
\[+ \left\{ 2 \int d^Dy N \partial_m [\sqrt{g} g^{mn} \partial_n B], \int d^Dy \frac{MB}{4\sqrt{g}} (D - 1) p_B^2 \right\} \]
\[= - \frac{D - 1}{D} \int d^Dx p_B B (N \nabla_m \nabla^m M - M \nabla_m \nabla^m N). \quad (62)\]

Collecting all these terms together, we find that almost all contributions cancel and the final result takes the form
\[\{ T_{GR}^T (M), T_{GR}^T (N) \} = T_{GR}^S ((N \partial^j M - M \partial^j N) g^{ji}). \quad (63)\]

In other words, the Poisson bracket of the smeared form of the Hamiltonian constraints (50) has the same form as in general relativity and hence it is in agreement with the basic principles of geometrodynamics. Alternatively, it has the form that is expected for fully diffeomorphism invariant theory. Note also that the Poisson bracket between the smeared form of the diffeomorphism and Hamiltonian constraint takes the standard form that follows from the fact that the Hamiltonian is manifestly invariant under spatial diffeomorphism. Then it is clear that the diffeomorphism and Hamiltonian constraints are preserved during the time evolution of the system.

Now it is straightforward to finish the analysis of the Poisson brackets of the constraints of the Lagrange multiplier modified gravity. Since the Poisson brackets of the constraints corresponding to the gravity part of the action are the same as in general relativity and since the scalar part of the constraints has exactly the same form as in the previous section, we immediately find that the Poisson brackets of the Lagrange multiplier modified $F(R)$ gravity take the form
\[\{ T_f^T (N), T_f^T (M) \} = T_f^S (g^{ij} (N \partial_j M - M \partial_j N)), \]
\[\{ T_f^S (N^i), T_f^T (M) \} = T_f^S (N^i \partial_j M), \quad (64)\]
\[\{ T_f^S (N^i), T_f^S (M^j) \} = T_f^S (N^i \partial_j M^j - M^j \partial_j N^i).\]

where
\[\mathcal{H}_f = \mathcal{H}_f^{GR} + \mathcal{H}_f^A, \quad \mathcal{H}_f = - g_{il} \nabla_k \pi^{lk} + p_A \partial_i A, \quad (65)\]

where $\mathcal{H}_f^{GR}$ is given in (48). Note that $\mathcal{H}_f^A$ is equal to
\[\mathcal{H}_f^A = p_A \sqrt{\frac{g^{ij}}{\sqrt{g}} \partial_j A \partial_j A + F_2 (A) + \sqrt{g} B A - \sqrt{g} F_1 (A)} \quad (66)\]

after solving the second-class constraint $G_1$ given in (52) with respect to $\lambda$.
\[\lambda = \frac{p_A}{2 \sqrt{\frac{\sqrt{g}}{F_2 (A) + g^{ij} \partial_j \phi \partial_j \phi}}}. \quad (67)\]
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