Benchmark results for flows in helical pipes

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Abstract

Benchmark results for steady flow in helical pipes and their stability are reported. The calculations are performed by the finite difference method. The problem is formulated in Germano coordinates in two equivalent but different forms of the momentum equation, so that results obtained using both formulations cross verify each other. Grid independence of the results is established and Richardson extrapolation is applied. The calculated steady flows agree well with experimental measurements and previous numerical results. The computed critical parameters corresponding to onset of oscillatory instability are also converged, but mostly disagree with known experimental and numerical data, showing necessity for more independent research.
1. Introduction

Helical pipe is widely used as an effective mixing tool [1]. The Dean vortices necessarily appear in the helical pipe flow driven by an applied pressure drop regardless the flow intensity. These vortices effectively mix either heat or mass without any need of additional mixing means. Contrarily to flows in straight ducts, circular or rectangular, no analytic solutions, similar to the Poiseuille profile, can be found for the helical pipe flow. Therefore, the numerical modeling is called for even at low and moderate values of the Reynolds number. The computation of steady flows is followed by the problem of their stability and transition to turbulence.

For a long time, the helical pipe flow was considered as three-dimensional until in 1982 Germano [2] showed that a two-dimensional formulation is possible in a specially tailored system of curvilinear orthogonal coordinates. Since then computations of this flow became affordable (see, e.g., [6-10] and references therein). However, to the best of the author’s knowledge, all the numerical studies address different pipe geometries, and no comparison between independent numerical results was ever reported.

In this paper we report results obtained using two different formulations of governing equations in Germano coordinates. For calculation of steady states and the linear stability studies we apply the methodology of [11]. The results obtained for each formulations are convergent and agree between themselves. For the benchmarking purposes we propose, at first, computation of steady flow at moderate Reynolds numbers, basing on parameters considered in [6]. We show also that at moderate Reynolds numbers the present result agree well with the experimental measurements of [12].

For the second benchmarking task we propose computation of the eigenvalues of the linearized equation, and then calculation of the critical Reynolds number corresponding to the linear instability onset. The critical Reynolds number should be accompanied by the critical wavenumber along the pipe centerline, and the frequency of oscillations for oscillatory instability. Along with the obtained results we report comparison with the previous experimental and numerical studies, most of which exhibit a quite striking disagreement with the present results, as well as between themselves. This is in complete contradiction with the previous experience of the author regarding, e.g., similar computations of swirling flows in cylindrical coordinates, where an excellent agreement with both experimental and independent numerical studies was established [11,13]. In the author’s opinion this illustrates how this part
of the proposed benchmark is challenging, in spite of the stability problem considered is quasi-two-dimensional, and nowadays can be effectively solved.

2. Coordinate system

To describe the system of orthogonal curvilinear coordinates introduced by Germano [2], we start from definitions related to the pipe centerline, which is a helical curve defined parametrically as

\[ \mathbf{R}_0(t) = \{x(t), y(t), z(t)\} = \{c \cdot \cos(t), c \cdot \sin(t), bt\}, \]

where \( c \) is radius of the helix, and \( 2\pi b \) is distance between coils (see Fig. 1). The curvature and the torsion of the helical curve are defined as

\[ \kappa = \frac{c}{b^2 + c^2}, \quad \tau = \frac{b}{b^2 + c^2}, \]

respectively. In the following we use also their ratio \( \lambda = \frac{\tau}{\kappa} = \frac{b}{c} \).

The most natural way to define position of the point inside a helical pipe with circular cross-section was proposed in [3]. The point position is defined by its location \( s \) with respect to the pipe centerline and by polar coordinates \((r, \theta)\) inside an orthogonal to the centerline pipe cross-section (Fig. 1). However, as shown in [2,3] these coordinates are not orthogonal. This inconvenience was removed by Germano [2], who proposed to rotate the position of \( \theta = 0 \) along with the pipe centerline as (assuming \( \tau \) is a constant)

\[ \xi = \theta - \int_{s_0}^{s} \tau ds = \theta - \tau(s - s_0) \]

The resulting coordinate system \((r, \xi, s)\) is orthogonal. The Lame coefficients of these coordinates are \( H_r = 1, \ H_\xi = r, \ H_s = 1 + krs\sin(\xi) \). Note that for the constant torsion

\[ \frac{d}{ds} = \frac{\partial}{\partial s} + \frac{\partial}{\partial \xi} \frac{\partial}{\partial s} = \frac{\partial}{\partial s} - \tau \frac{\partial}{\partial \xi} = \frac{\partial}{\partial s} - k\lambda \frac{\partial}{\partial \xi} \]

In the coordinates \((r, \xi, s)\) we can assume that the three fluid velocity components and the pressure are independent of the cross-section at the pipe centerline, i.e., independent on \( s \). Thus, we arrive to a two-dimensional formulation for the velocity and the pressure dependent only on \( r \) and \( \xi \).

3. Governing equations

We consider flow of incompressible fluid in a helical pipe of the inner radius \( a \), radius of the coil \( c \), and a constant distance between the coils equal to \( 2\pi b \). The pipe is sketched in Fig. 1. The flow is created by a pressure gradient, which is constant along the pipe centreline.
and is governed by the continuity and momentum equations. The flow is characterized by the three dimensionless parameters, which are dimensionless curvature $\varepsilon = a\kappa$, torsion to curvature ratio $\lambda$, and the Reynolds number $Re_d = 2a\bar{U}/\nu$, where $d = 2a$ is the pipe diameter, $\nu$ is the kinematic viscosity, and $\bar{U}$ is the flow mean velocity. The Reynolds number sometimes is replaced by the Dean number $De_d = Re_d\sqrt{\varepsilon}$.

The above definition of the Reynolds (Dean) number requires mean velocity value, which is convenient for experimental studies. For a numerical study, the mean velocity is a result of calculations. Since it is not known a priori, its use in the problem formulation causes certain inconvenience. Thus, to use this traditional formulation, a non-linear problem making dimensionless $\bar{U}$ equal to unity was solved in [4]. To make an alternative and more convenient non-dimensionalization, we use the pressure gradient based scales introduced in [5]. Assuming that the pressure gradient $G$ is known a priori, we define the scales of length, time, velocity, and pressure as $a, (\rho a/G)^{1/2}, (Ga/\rho)^{1/2}$, and $Ga$. The resulting system of the dimensionless continuity and momentum equations reads

$$\nabla \cdot \mathbf{v} = 0, \quad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \frac{1}{R_G} \Delta \mathbf{v}$$

(6,7)

Where the dimensionless parameter $R_G = (Ga^2/\rho \nu^2)^{1/2}$ replaces the Reynolds number. The equations (6,7) are solved together with the no-slip condition

$$\mathbf{v}(r = a, \xi, s) = 0.$$  

(8)

After the flow is computed its dimensionless mean velocity $\bar{V}_G$ can be easily obtained. Then the dimensional mean velocity is $\bar{U} = \bar{V}_G(Ga/\rho)^{1/2}$, and the resulting Reynolds number is calculated as $Re_d = 2a\bar{U}/\nu = 2a\bar{V}_G(Ga/\rho)^{1/2}/\nu = 2\bar{V}_GR_G$.

The friction factor is defined as ($L$ is the pipe length)

$$f = \frac{\Delta p}{\rho \bar{U}^2/\nu} \left(\frac{2a}{L}\right) = 4\frac{\Delta p}{L} \frac{a}{\rho \bar{V}_G^2(Ga/\rho)} = \frac{4}{\bar{V}_G^2}$$

(9)

For a “two-dimensional” flow independent on the coordinate $s$, the continuity equation in the above introduced helical coordinates reads

$$\nabla \cdot \mathbf{v} = \frac{1}{\tau H_s} \left\{ \frac{\partial}{\partial \tau} [\tau H_s \mathbf{v}_r] + \frac{\partial}{\partial \xi} [H_s \mathbf{v}_\xi] - \varepsilon \lambda r \frac{\partial \mathbf{v}_s}{\partial \xi} \right\} = 0.$$  

(10)

It can be satisfied by introducing a function $\hat{\psi}$ as
\[ r H_s v_r = \frac{\partial \hat{\psi}}{\partial \xi}, \quad H_s v_\xi = \epsilon \lambda r, \quad v_s = -\frac{\partial \hat{\psi}}{\partial r} \]  

(11)

This is not a "real" stream function since \( v \neq \text{rot}[\hat{\psi} e_s] \). As is noted by several authors (e.g., \[2,6\]), the quasi-two-dimensional flows are defined by two scalar functions \( v_s(r, \xi) \) and \( \hat{\psi}(r, \xi) \).

To make the stream function more meaningful physically, we modify it as 

\[ \psi = \hat{\psi}/(1 + \epsilon r \sin(\xi)), \] 

which leads to 

\[ v_r = \frac{1}{r H_s} \frac{\partial [H_s \psi]}{\partial \xi}, \quad v_\xi - \frac{\epsilon \lambda r}{(1 + \epsilon r \sin(\xi))} v_s = -\frac{\partial [1 + \epsilon r \sin(\xi)] \psi}{\partial r}, \]  

(12)

so that now 

\[ v_r e_r + \left[ v_\xi - \frac{\epsilon \lambda r}{H_s} v_s \right] e_\xi = \text{rot}[\psi e_s] \]  

(13)

Noticing that \( v_\xi = r \dot{\xi} \) and \( v_s = H_s \dot{s} \), we can consider transformation from the coordinate \( \xi \) to a new polar angle \( \phi \):

\[ \phi = \xi - \epsilon \lambda s, \quad \dot{\phi} = \dot{\xi} - \epsilon \lambda \dot{s}, \quad v_\phi = r \dot{\phi} = r \dot{\xi} - \epsilon \lambda r \dot{s} = v_\xi - \frac{\epsilon \lambda r}{H_s} v_s \]  

(14)

Thus, isolines \( \psi = \text{const} \) are tangent to vectors \( \{v_r, v_\phi, 0\} \), which can be interpreted as a two-dimensional divergence-free velocity in new polar coordinates \( (r, \phi) \). In the following we call \( \psi \) pseudo – streamfunction.

The momentum equations are written in general orthogonal coordinates as in [14]

\[
\begin{align*}
\frac{\partial v_1}{\partial t} + \frac{v_1 \partial v_1}{H_1 \partial x_1} + \frac{v_2 \partial v_2}{H_2 \partial x_2} + \frac{v_3 \partial v_3}{H_3 \partial x_3} + \frac{v_1 v_2 \partial H_1}{H_1 H_2 \partial x_2} + \frac{v_1 v_3 \partial H_1}{H_1 H_3 \partial x_3} + \frac{v_2 v_3 \partial H_2}{H_2 H_3 \partial x_3} + \frac{v_1 v_3 \partial H_1}{H_1 H_3 \partial x_3} &= \frac{1}{H_1 \partial x_1} \times \\
+ \frac{1}{R_0} &\left( -\frac{\partial}{\partial x_2} \left[ H_3 \frac{\partial (H_2 v_2)}{\partial x_1} \right] + \frac{\partial}{\partial x_2} \left[ H_3 \frac{\partial (H_1 v_1)}{\partial x_2} \right] + \frac{\partial}{\partial x_3} \left[ H_2 \frac{\partial (H_1 v_1)}{\partial x_3} \right] - \frac{\partial}{\partial x_3} \left[ H_2 \frac{\partial (H_3 v_3)}{\partial x_1} \right] \right). \end{align*}
\]  

(15)

Here the indices 1, 2 and 3 stay for \( r, \xi, \) and \( s \), respectively. Two other equations are obtained by the cyclic permutation of the indices. These equations contain mixed second derivatives, which may cause certain inconvenience and loss of accuracy at the discretization stage. To avoid this, the mixed derivatives are eliminated using the continuity equation (10). The resulting set of momentum equations is detailed in Appendix. In the following, applying eqs. (14) is called Formulation 1, while applying eqs. (A1)-(A6) from the Appendix is called Formulation 2. Mathematical equivalence of both formulations is verified by symbolic computations on a computer.

4. Numerical technique
The problem was solved on staggered grids using central finite differences with linear interpolation between the nodes where necessary. The Newton iteration was applied for calculation of steady flows. Application of the Newton method is identical to [5], and is based on the LU decomposition of the sparse Jacobian matrix with further analytic solution for the Newton corrections.

Consideration of the linear stability of calculated steady flows requires consideration of infinitesimally small disturbances that can be periodic along the pipe centerline direction $s$. The perturbations were represented in the form $\{\tilde{v}(r,\xi), \tilde{p}(r,\xi)\} \exp[\sigma t + iks]$, where $\sigma$ is the complex time increment, $k$ is the wavenumber along the centerline and infinitesimally small perturbation amplitude is denoted by tilde. The linearization procedure is standard, except of derivatives in the $s$–direction, for which eq. (4) must be replaced, for the dimensionless variables, by

$$\frac{d}{ds} = -\varepsilon\frac{\partial}{\partial \xi} + ik \quad .$$ \hspace{1cm} (15)

The linear stability problem reduces to the generalized eigenvalue problem

$$\sigma \mathbf{B}(\tilde{v}, \tilde{p})^T = \mathbf{J}(\tilde{v}, \tilde{p})^T$$ \hspace{1cm} (16)

Here $\mathbf{J}$ is the Jacobian matrix that defines r.h.s. of the linearized problem, and $\mathbf{B}$ is the diagonal matrix such that its diagonal elements corresponding to the time derivatives of $\tilde{v}$ are equal to one, while the elements corresponding to $\tilde{p}$ and the boundary conditions are zeros, so that $\det \mathbf{B} = 0$. Thus, the generalized eigenproblem (16) cannot be transformed into a standard eigenproblem. To study stability of an axisymmetric steady flow state for a given set of the governing parameters it is necessary to compute the eigenvalue $\hat{\sigma}$ having the largest real part for all real wavenumbers $k$. This $\hat{\sigma}$ is called leading eigenvalue. Apparently, $\hat{\sigma} = \max_k \{\text{Real}[\sigma(k)]\} > 0$ means instability of the axisymmetric steady flow state. The value of the wavenumber yielding the maximum of $\text{Real}[\sigma(k)]$ we call critical and denote as $k_{cr}$. The imaginary part of the leading eigenvalue we call critical frequency and denote as $\omega_{cr} = \text{Im}[k_{cr}]$. The corresponding eigenvector of (16) defines the most unstable perturbation of the base state.

The eigenproblem (16) is solved by the Arnoldi iteration in the shift-and-invert mode

$$(\mathbf{J} - \sigma_0 \mathbf{B})^{-1} \mathbf{B}(\tilde{v}, \tilde{p})^T = \theta(\tilde{v}, \tilde{p})^T, \quad \theta = \frac{1}{\sigma - \sigma_0}$$ \hspace{1cm} (17)
where \( \sigma_0 \) is a complex shift. The Arnoldi method realized in the ARPACK package [15] is used. Following [11] and similarly to the Newton method, we calculate LU decomposition of the complex matrix \((J - \sigma_0 B)^{-1}\), so that calculation of the next Krylov vector for the Arnoldi method is reduced to one backward and on forward substitutions. It should be noted that the Jacobian matrices for the Newton iteration and the stability analysis are different, since it contains the terms depending on the wavenumber \( k \) that can also be complex. The Jacobian matrices were calculated directly from the numerical schemes. The corresponding parts of the code were verified by numerical differentiation of the equations right hand sides.

To calculate the leading eigenvalue \( \hat{\sigma} \) it is necessary to choose the shift \( \sigma_0 \) close to \( \hat{\sigma} \) and to calculate 10-20 eigenvalues with the largest absolute value. In the following calculations we fix \( \text{Real}(\sigma_0)=0 \) and vary \( \text{Im}(\sigma_0) \) until the leading eigenvalue \( \hat{\sigma} \) is computed. Then we calculate the instability point with \( \hat{\sigma} = (0, \text{Im}(\hat{\sigma})) \) and vary \( \text{Im}(\sigma_0) \) further to ensure that there is no another eigenvalue with larger real part. After that we vary the wavenumber \( k \) to find at which \( k = k_{cr} \) the instability takes places at the lowest \( R_G = R_{G,cr} \). For a given pair of the geometrical governing parameters \( \varepsilon \) and \( \lambda \), the result of stability study is defined by the critical values \( R_{G,cr} \), \( k_{cr} \), \( \omega_{cr} \) and the leading eigenvector. The critical Reynolds number then can be calculated as \( Re_{d,cr} = 2\overline{V}_c R_{G,cr} \), and the dimensionless critical frequency scaled by \((2a)/\overline{U}\) is \( \omega_{d,cr} = 4\omega_{cr} R_{G,cr} / Re_{d,cr} \).

5. Results

5.1. Steady flows: comparison with experiment

To validate the formulation, the numerical approach and the code we compare with the experimental results of [11] [12]. The computed and measured friction factors as functions of the Reynolds numbers are sown in Fig. 10 for four different helical pipes used in the experiments. It is seen that at Reynolds numbers lower than \( \approx 3000 \), the calculations and the measurements coincide. At larger Reynolds number the results differ, which happen due to the laminar – turbulent transition, similarly to the straight pipes. At the same time, at \( Re_d \leq 3000 \) the non-linear terms already become significantly large, which allows us to assume that our code reproduces laminar flows correctly.

5.2. Steady flows: convergence and comparison with an independent solution
For the convergence study and comparison with independent results we chose work of Yamamoto et al [6], where ratios of mass fluxes in helical and straight pipes were reported for various curvatures and torsions. It can be easily seen that the mean velocity of the Poiseuille flow in a straight pipe is \( \overline{w}_{\text{Poiseuille}} = \frac{R_G}{8} \), so that the flux ratio is equal to \( \frac{\overline{U}}{\overline{w}_{\text{Poiseuille}}} \). The velocity in [6] was rendered dimensionless using the viscous scale \( \nu/a \), which resulted in the Dean number \( Dn = R_G^2 \sqrt{2\varepsilon} \). The two geometric parameters were \( \delta = \varepsilon \) and \( \beta = \lambda/\sqrt{2\varepsilon} \). The solution was computed on \( N_r \times N_\xi = 35 \times 60 \) collocation points.

Comparison with the results of [6] is shown in Table 1. We report results obtained using both above formulations and uniform grids having 50 × 100 and 100 × 200 nodes. The Richardson extrapolations basing on these grids and made for both formulations coincide to within the third decimal place. Taking into account that the grid used in [6] was rather coarse, the agreement with this work is considered as good. To allow for a more detail comparison, the maximal and minimal values of the pseudo-streamfunction and the friction factor also are reported in Table 2.

The calculated steady flows are shown in Figs. 3 and 4. In the limit and imaginary case of zero torsion, the helical pipe turns into a torus, and the Dean vortices are symmetric. The third, \( s \)-component, of velocity, is advected by these vortices so that its maximum is shifted toward the outer side of the pipe. The Dean vortices symmetry is broken when the torsion is non-zero. With further growth of the torsion the two Dean vortices merge in a single one. Note that the meridional motion does not vanish due to the non-zero curvature. An interesting observation is that the friction factor is maximal at certain torsion (see Table 2) and then decreases with further increase of the torsion.

5.3. Stability of steady flows: benchmark results

As mentioned above, study of stability of steady flows is usually more computationally demanding than computation of steady flow states. This is also the case for flows in helical pipes, which is illustrated in Table 3. There we report leading eigenvalues computed for the parameters of [6] for the wavenumbers \( k = 0 \) and 0.5. The calculations were carried out for the uniform grids of 50×100, 100×200, and 200×400 nodes in the \( r \) and \( \xi \) directions respectively. Only Richardson extrapolations computed for a pair of grids are reported in Table 3. For the computed eigenvalues we observe convergence for only two decimal places. The convergence slightly slows down with the increasing torsion. This can be expected since more terms of the governing equations (see Appendix) at the non-zero torsion affect the final result. The
convergence becomes most problematic when the leading eigenvalue real part is small, e.g., at \( \delta = 0.4 \) and \( = 0.5 \). Since the instability threshold corresponds to the zero real part, one can expect a slow convergence of the critical parameters.

The calculated critical parameters that we propose for comparison are presented in Table 4. For this example we fixed the curvature \( \varepsilon = \delta = 0.4 \) and varied torsion to curvature ratio \( \lambda \). We observe that the critical parameters computed using both above formulations, and after the Richardson extrapolation, coincide in 2 – 3 first decimal places. To provide more information on the instability, the absolute values of the dominant eigenvectors computed at the critical points are shown in Figs. 5 and 6. Note, that the eigenvectors are computed to within multiplication by a complex constant, so that their real and imaginary parts depend on the phase. Their absolute values are defined within multiplication by a real constant. The maximal field values shown in Figs. 5 and 6 have only relative meaning. These patterns can help to explain physics of the instability threshold, however this is beyond the scope of present study. We report these patterns to enable also qualitative comparison of calculated instabilities.

5.4. Stability of steady flows: comparison with experiment and independent computations

Comparison with experiments and independent numerical results for stability studies is not as good as for the steady flows. It is well known that in a straight pipe the transition to turbulence takes place without any linear instability, and the classical Poiseuille flow in a circular pipe is believed to be linearly stable. When the pipe is coiled, the flow appears to be linearly unstable at finite Reynolds numbers, which are rather large at small curvatures, but decrease when the curvature increases. The experimental evidence for this is discussed in [16]. It is expected that the experimentally observed transition in slightly coiled pipes will be below the theoretically found linear limit. At the same time, it can be expected that in strongly coiled pipes the instability appears in agreement with predictions of the linear stability theory.

One of the earliest experiments on instability was conducted by G. I. Taylor [17] who considered two helical pipe configurations with \( \varepsilon \approx 0.0314, \lambda \approx 0.01 \) and \( \varepsilon \approx 0.06, \lambda \approx 0.019 \). The measured critical Reynolds numbers were 5010 and 5830, respectively. Present linear stability analysis yields, respectively, \( Re_{cr} = 8133 \) and 5970, so that at smaller curvature the theoretical \( Re_{cr} \) is noticeably above the experimental one, while the two results become close when the curvature is almost doubled. This single comparison allows one to assume that a good agreement between experimental and numerical results can be expected for \( \varepsilon > 0.06 \). Unfortunately, this is not exactly the case, which is illustrated in Fig. 7, where we compare
present results with the critical Reynolds numbers directly measured in several experimental studies [16-19] and computed in [4] and [20].

Figure 7 is based on the results of [18] where the authors distinguished three different “instabilities” and described them as follows. The critical Reynolds number of “upper critical” corresponded to appearance of the intermittency near the inner pipe wall; “conservative” critical Reynolds number corresponded to appearance of the first burst near the outer wall; and “liberal” critical Reynolds number corresponded to appearance of turbulence everywhere in the chosen cross-section. Following representation of [18] (see Fig. 6 there), we plot the critical Reynolds numbers versus the ratio of the pipe and the coil radii $a/c$, which is close, but not equal to the dimensionless curvature. The torsion was different in different measurements, but was very small and only slightly affected the results. This can be seen also from the present numerical results shown by circles in Fig. 7. The calculations were performed for the exact experimental parameters, for which the torsion was different but small. All the results are located near the black curve that corresponds to the parameters of [18].

We observe that the “upper critical” and “liberal” Reynolds numbers of [18] grow with the increasing radii ratio while it remains small, $a/c < 0.06$. This instability most probably corresponds to the by-pass transition observed in a straight pipe. When the pipe is coiled stronger, starting form $a/c \approx 0.06$, the “liberal” critical number decreases. The “conservative” $Re_{cr}$ appears at $a/c \approx 0.03$ and also decreases with increasing radii ratio. At larger radii ratio, $\varepsilon > 0.08$, both “liberal” and “conservative” critical numbers are close. At these larger radii ratios, meaning larger curvatures, we would expect the instability to appear in accordance with the predictions of the linear theory. However, the experimental critical numbers of [18] remain considerably lower than those calculated in the present study. The linear instability can be subcritical, but this issue is beyond the scope of the present study.

An interesting observation is that the results of [17] are very close to the “liberal” critical curve of [18], while the result of [19] falls very close to the “conservative” critical curve (Fig. 7). The described above good agreement between experiment [17] and present computation at $\varepsilon \approx 0.06$ can be due to intersection of numerical and “liberal” critical curves near this value, so that the agreement may be coincidental. The frequency of supercritical flow oscillations, which also can be compared with numerical data, was reported in [19] as 0.85 Hz. We were unable to computationally find an instability with such frequency for $Re < 8000$.

The critical Reynolds numbers measured in the most recent study [16] are noticeably smaller than all measured before for the interval $0.04 \leq a/c \leq 0.08$. For $a/c > 0.08$ they become close
to the “conservative” neutral curve of [18]. Needless to say that that no agreement between results of [16] and present results was found. At the same time, recently published results of [4] obtained for the toroidal pipe, which is the zero torsion limit of the present case, agree very well with results of [16] for the interval $0.03 \leq a/c \leq 0.08$. When $0.1 \leq a/c \leq 0.12$ the numerical results of [4] are also very close to the “conservative” curve of [18]. At larger radii ratios the critical Reynolds numbers of [4] grow and become larger than those calculated in the present study (not shown on the graph).

Seeking for a possible source of so striking disagreement with computations of [4], we reproduced steady states shown in Fig. 2 of this paper. The agreement was perfect. Therefore, the problem exists at the stage of stability study, and can be connected with a correct numerical resolution of the eigenmodes. To illustrate the complicacy of this issue additionally, we recall results of [20] who found for $a/c = 0.1$ and $0.3$, respectively, $Re_{cr} = 5175$ and 4575. The results of [4] are 3330 and 3380, while the present results are 4615 and 2572, respectively, so that all the three studies are in an apparent disagreement.

Another interesting result, yet to be validated, is a steep decrease of the critical Reynolds with the increasing torsion, as is seen from Table 4.

4. Concluding remarks

In this study we report benchmark results for computational modeling of flows in helical pipes. The results consist of two separate parts. The first part relates to the steady flow states, for which our results agree well with those calculated independently in [6]. As mentioned, we also obtained a very good comparison with steady flows depicted in Fig. 2 of [4] for the zero torsion. These make us confident in the correctness of this part of the results.

The second part of results relates to the linear instability of steady flows. These results cannot be confirmed by comparison with the existing experimental data, as well as with the recently published numerical study [4]. To verify the present numerical approach and to eliminate possible programming errors, we performed computations in two different formulations that arrived to very close results. We also examined correctness of the analytic calculation of the Jacobian matrix by finite difference differentiation. Also, we verified that computed eigenvalues and eigenvalues satisfy the initial eigenvalue problem to within machine precision. Nevertheless, due to the apparent disagreement with the previous studies, the question about correctness of these results remains open.
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Figure captions

**Figure 1.** Sketch of a helical pipe (left) and illustration of helical coordinates of introduced by Germano [2].

**Figure 2.** Comparison of measured and calculated friction factors. Lines – calculations, symbols – results of [11].

**Figure 3.** Isolines of the $s$–velocity and pseudo–streamfunction $\psi$ for parameters of Yamamoto et al [6] $Dn = 1000$, $\delta = 0.4$, and $0 \leq \delta \leq 0.6$.

**Figure 4.** Isolines of the $s$–velocity and pseudo–streamfunction $\psi$ for parameters of Yamamoto et al [6] $Dn = 1000$, $\delta = 0.4$, and $1 \leq \delta \leq 2.5$.

**Figure 5.** Amplitudes of leading perturbations of (from left to right) $v_r$, $v_\xi$ and $v_z$ for $\varepsilon = 0.4$ and (a) $\delta = 0$, (b) $\delta = 0.5$, and (c) $\delta = 1$. Maximal values of the amplitudes are shown below the plots.

**Figure 6.** Amplitudes of leading perturbations of (from left to right) $v_r$, $v_\xi$ and $v_z$ for $\varepsilon = 0.4$ and (a) $\delta = 1.5$, (b) $\delta = 2$, and (c) $\delta = 2.5$. Maximal values of the amplitudes are shown below the plots.

**Figure 7.** Comparison of present linear stability results with the experimental and independent numerical data.
Appendix

For the “two-dimensional” flow depending only on the coordinates \( r \) and \( \xi \), the momentum equation in an alternative form with the eliminated mixed second derivatives reads

\[
\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\xi \frac{\partial v_r}{\partial \xi} - \frac{\epsilon \lambda}{H_s} \frac{\partial v_r}{\partial \xi} - \frac{v_\xi^2}{r} - \frac{\epsilon \sin(\xi)}{H_s} v_s^2 =
\]

\[
= - \frac{\partial p}{\partial r} + \frac{1}{R_c r H_s} \left\{ \frac{\partial^2}{\partial r^2} [r H_s v_r] + \frac{1}{r} \frac{\partial}{\partial r} \left[ H_s \frac{\partial v_r}{\partial r} \right] + \epsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi^2} \left[ \frac{1}{H_s} \frac{\partial v_r}{\partial \xi} \right] - \frac{10 \partial v_r}{r \frac{\partial \xi}{\partial \xi}} - \frac{\epsilon \lambda}{H_s} \frac{\partial v_r}{\partial \xi} \right\}
\]

(A1)

\[
\frac{\partial v_\xi}{\partial t} + v_r \frac{\partial v_\xi}{\partial r} + v_\xi \frac{\partial v_\xi}{\partial \xi} + \frac{v_\xi v_s}{r} - \frac{\epsilon \sin(\xi)}{H_s} v_s^2 = - \frac{\partial p}{\partial \xi} + \frac{1}{R_c r H_s} \left\{ \frac{\partial^2}{\partial r^2} [r H_s v_\xi] + \epsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi^2} \left[ \frac{1}{H_s} \frac{\partial v_\xi}{\partial \xi} \right] + \epsilon \cos(\xi) \frac{\partial v_\xi}{\partial \xi} + \epsilon^2 \lambda \frac{\partial}{\partial \xi} \left[ \frac{\cos(\xi)}{H_s} v_s \right] \right\}
\]

(A2)

\[
\frac{\partial v_s}{\partial t} + v_r \frac{\partial v_s}{\partial r} + v_\xi \frac{\partial v_s}{\partial \xi} - \frac{\epsilon \lambda}{H_s} \frac{\partial v_s}{\partial \xi} + \frac{\epsilon \sin(\xi)}{H_s} v_s^2 = - \frac{\partial p}{\partial s} + \frac{1}{R_c} \left\{ \frac{1}{r \frac{\partial}{\partial r}} \left[ H_s \frac{\partial v_s}{\partial r} \right] + \frac{\partial}{\partial \xi} \left[ \frac{1}{H_s} \frac{\partial v_s}{\partial \xi} \right] + \epsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi^2} \left[ \frac{1}{H_s} \frac{\partial v_s}{\partial \xi} \right] + \epsilon \cos(\xi) \frac{\partial v_s}{\partial \xi} + \epsilon^2 \lambda \frac{\partial}{\partial \xi} \left[ \frac{\cos(\xi)}{H_s} v_s \right] \right\}
\]

(A3)

The equations linearized near the steady state flow \( \{V_r(r,\xi),V_\xi(r,\xi),V_s(r,\xi),P(r,\xi)\} \) that govern infinitely small disturbances \( \{u_r(r,\xi),u_\xi(r,\xi),u_s(r,\xi),p(r,\xi)\} \exp[i\omega t + ik\xi] \) are

\[
\lambda u_r + V_r \frac{\partial u_r}{\partial r} + u_r \frac{\partial V_r}{\partial r} + \frac{\epsilon \lambda}{H_s} \frac{\partial u_r}{\partial \xi} + \frac{\epsilon \sin(\xi)}{H_s} u_s^2 = - \frac{\partial p}{\partial r} + \frac{1}{R_c r H_s} \left\{ \frac{\partial^2}{\partial r^2} [r H_s u_r] + \frac{\partial}{\partial \xi} \left[ H_s \frac{\partial u_r}{\partial \xi} \right] + \epsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi^2} \left[ \frac{1}{H_s} \frac{\partial u_r}{\partial \xi} \right] - \frac{10 \partial u_r}{r \frac{\partial \xi}{\partial \xi}} - \frac{\epsilon \lambda}{H_s} \frac{\partial u_r}{\partial \xi} - k^2 u_r - \frac{2 \epsilon \sin(\xi)}{r} V_s u_s \right\}
\]

(A4)

\[
\lambda u_\xi + V_r \frac{\partial u_\xi}{\partial r} + u_\xi \frac{\partial V_r}{\partial r} + \frac{\epsilon \lambda}{H_s} \frac{\partial u_\xi}{\partial r} + \frac{\epsilon \sin(\xi)}{H_s} u_s^2 = - \frac{\partial p}{\partial \xi} + \frac{1}{R_c r H_s} \left\{ \frac{\partial^2}{\partial r^2} [r H_s u_\xi] + \frac{\partial}{\partial \xi} \left[ H_s \frac{\partial u_\xi}{\partial \xi} \right] + \epsilon^2 \lambda^2 \frac{\partial^2}{\partial \xi^2} \left[ \frac{1}{H_s} \frac{\partial u_\xi}{\partial \xi} \right] - \frac{10 \partial u_\xi}{r \frac{\partial \xi}{\partial \xi}} - \frac{\epsilon \lambda}{H_s} \frac{\partial u_\xi}{\partial \xi} - k^2 u_\xi + \frac{2 \epsilon \sin(\xi)}{r} V_s u_s + \frac{ik}{H_s} V_s u_\xi \right\}
\]

(A5)
\[ \lambda v_s + V_r \frac{\partial u_s}{\partial r} + u_r \frac{\partial v_s}{\partial r} + \frac{V_s \partial u_s}{r} + \frac{u_r \partial v_s}{r} - \epsilon \lambda \frac{u_r \partial v_s}{r} + \epsilon \lambda \frac{V_s \partial u_s}{r} + \frac{\epsilon \sin(\xi)}{H_s} (V_r u_s + u_r V_s) - \frac{\epsilon \cos(\xi)}{H_s} \]

\[ \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial (H_s u_s)}{\partial r} \right] + \frac{1}{r} \frac{\partial (H_s u_s)}{\partial r} \right\} + \epsilon^2 \lambda^2 \frac{\partial}{\partial \xi} \left[ \frac{1}{r^2} \frac{\partial u_r}{\partial \xi} \right] + \frac{1}{r} \frac{\partial u_r}{\partial \xi} - \frac{\epsilon \lambda \cos(\xi)}{H_s} \frac{\partial}{\partial r} \left[ \frac{r^2}{H_s^2} u_r \right] - 2 \epsilon \lambda \frac{\partial}{\partial \xi} \left[ \frac{\sin(\xi)}{H_s^2} u_r \right] - \epsilon^2 \lambda \frac{\partial}{\partial \xi} \left[ \frac{\cos(\xi)}{H_s^2} u_r \right] \]

(A6)
Figure 1. Sketch of a helical pipe (left) and illustration of helical coordinates of introduced by Germano [2].
Figure 2. Comparison of measured and calculated friction factors. Lines – calculations, symbols – results of [10].
Figure 3. Isolines of the $s$ – velocity and pseudo – streamfunction $\psi$ for parameters of Yamamoto et al [6] $Dn = 1000$, $\delta = 0.4$, and $0 \leq \delta \leq 0.6$. 
Figure 4. Isolines of the s-velocity and pseudo-streamfunction $\psi$ for parameters of Yamamoto et al. [6] $Dn = 1000$, $\delta = 0.4$, and $1 \leq \delta \leq 2.5$.
Figure 5. Amplitudes of leading perturbations of (from left to right) $v_r$, $v_\xi$, and $v_\zeta$ for $\varepsilon = 0.4$ and (a) $\delta = 0$, (b) $\delta = 0.5$, and (c) $\delta = 1$. Maximal values of the amplitudes are shown below the plots.
Figure 6. Amplitudes of leading perturbations of (from left to right) $v_r, v_\xi$ and $v_z$ for $\varepsilon = 0.4$ and (a) $\delta = 1.5$, (b) $\delta = 2$, and (c) $\delta = 2.5$. Maximal values of the amplitudes are shown below the plots.
Figure 7. Comparison of present linear stability results with the experimental and independent numerical data.
Table 1. Values of $Q/Q_{Poiseuille}$ for the parameters $Dn=1000$, $\beta=0.4$ of [6]. Convergence using the two formulations and comparison with the results of [6]. Uniform grid.

| $\delta$ | [6]   | Formulation 1 | Richardson extrapolation | Formulation 2 | Richardson extrapolation |
|----------|-------|---------------|-------------------------|---------------|-------------------------|
| 0        | 0.5995| 0.5981        | 0.5984                  | 0.5987        | 0.5983                  | 0.5985 |
| 0.2      | 0.5917| 0.5902        | 0.5905                  | 0.5908        | 0.5905                  | 0.5907 |
| 0.4      | 0.5705| 0.5688        | 0.5692                  | 0.5696        | 0.5692                  | 0.5694 |
| 0.6      | 0.5424| 0.5403        | 0.5408                  | 0.5413        | 0.5409                  | 0.5411 |
| 0.8      | 0.5208| 0.5182        | 0.5192                  | 0.5202        | 0.5194                  | 0.5199 |
| 1.0      | 0.5600| 0.5625        | 0.5647                  | 0.5669        | 0.565                   | 0.5662 |
| 1.2      | 0.6555| 0.6576        | 0.6605                  | 0.6634        | 0.6593                  | 0.6616 |
| 1.4      | 0.7312| 0.7397        | 0.7426                  | 0.7455        | 0.7410                  | 0.7435 |
| 1.6      | 0.8030| 0.806        | 0.8090                  | 0.8090        | 0.8042                  | 0.8068 |
| 1.8      | 0.8506| 0.8535        | 0.8564                  | 0.8564        | 0.8517                  | 0.8543 |
| 2.0      | 0.8861| 0.8889        | 0.8917                  | 0.8917        | 0.8871                  | 0.8896 |
| 2.5      | 0.9403| 0.9428        | 0.9453                  | 0.9453        | 0.9413                  | 0.9434 |
Table 2. Minimal and maximal values of the stream function, and the friction factor, obtained as Richardson extrapolations using the uniform grids 50×100 and 100×200. The parameters $Dn=1000$, $\beta=0.4$ of [6].

| $\delta$ | Formulation 1 | | | Formulation 2 | | |
|-----------|----------------|---|---|----------------|---|---|
|           | $\psi_{\text{min}}$ | $\psi_{\text{max}}$ | $f$ | $\psi_{\text{min}}$ | $\psi_{\text{max}}$ | $f$ |
| 0         | -0.2425 | 0.2425 | 0.6388 | -0.2421 | 0.2426 | 0.63875 |
| 0.2       | -0.2589 | 0.2313 | 0.656  | -0.2586 | 0.2315 | 0.65575 |
| 0.4       | -0.2836 | 0.2243 | 0.7059 | -0.2833 | 0.2244 | 0.7057  |
| 0.6       | -0.3243 | 0.2194 | 0.7816 | -0.3241 | 0.2195 | 0.7813  |
| 0.8       | -0.4066 | 0.2119 | 0.8463 | -0.4061 | 0.212  | 0.84575 |
| 1.0       | -0.6684 | 0.1309 | 0.7122 | -0.6626 | 0.1306 | 0.7113  |
| 1.2       | -1.1989 | 0.030875 | 0.5203 | -1.1856 | 0.030873 | 0.5196 |
| 1.4       | -1.8094 | 0.0061745 | 0.4119 | -1.7955 | 0.0062225 | 0.4114 |
| 1.6       | -2.3961 | 0.00038183 | 0.3499 | -2.3836 | 0.00039378 | 0.3495 |
| 1.8       | -2.9442 | 0 | 0.3121 | -2.9338 | 0 | 0.2759 |
| 2.0       | -3.4577 | 0 | 0.288 | -3.4487 | 0 | 0.2877 |
| 2.5       | -4.6258 | 0 | 0.2563 | -4.6211 | 0 | 0.2561 |
Table 3. Leading eigenvalues obtained as Richardson extrapolations using the uniform grids $50\times100$, $100\times200$ and $200\times400$ of $Dn=1000$, $\beta=0.4$ of [6].

| $\delta$ | Formulation 1 | Formulation 2 |
|----------|----------------|----------------|
|          | $\sigma(k=0)$ 50x100 & 100x200 | $\sigma(k=0)$ 100x200 & 200x400 | $\sigma(k=0.5)$ 50x100 & 100x200 | $\sigma(k=0.5)$ 100x200 & 200x400 | $\sigma(k=0)$ 50x100 & 100x200 | $\sigma(k=0)$ 100x200 & 200x400 | $\sigma(k=0.5)$ 50x100 & 100x200 | $\sigma(k=0.5)$ 100x200 & 200x400 |
| 0        | (-0.4858, 1.4597) | (-0.4860, 1.4600) | (-0.5487, -3.5748) | (-0.5472, -3.5601) | (-0.4858, 1.4956) | (-0.4862, 1.4603) | (-0.5553, -3.56) |
| 0.5      | (-0.5391, 0) | (-0.5399, 0) | (-0.5218, -0.5672) | (-0.5217, -0.5678) | (-0.5392, 0) | (-0.5400, 0) | (-0.5237, -0.56) |
| 1.0      | (0.0121, 2.6227) | (-0.001902, 2.6541) | (-0.205, -0.9735) | (-0.2014, -0.9644) | (-0.06831, 2.648) | (0.031628, 2.648) | (-0.2027, -0.97) |
| 1.5      | (-0.2315, 0) | (-0.2349, 0) | (-0.2876, 1.5866) | (-0.2875, 1.6052) | (-0.2339, 0) | (-0.2332, 0) | (-0.2912, 1.57) |
| 2.0      | (-0.1993, 0) | (-0.1995, 0) | (0.02955, 4.5359) | (0.02379, 4.5434) | (-0.1991, 0) | (-0.1986, 0) | (0.03707, 4.52) |
| 2.5      | (-0.1723, 0) | (-0.193, 0) | (-0.8882, 7.3918) | (-0.8874, 7.4093) | (-0.1862, 0) | (-0.1858, 0) | (-0.8876, 7.39) |
Table 4. Critical parameters obtained as Richardson extrapolations using the uniform grids 100×200 and 200×400

| \( \lambda \) | \( R_{G,cr} \) | \( \omega_{cr} \) | \( k_{cr} \) | \( Re_{d,cr} \) | \( R_{G,cr} \) | \( \omega_{cr} \) | \( k_{cr} \) |
|---|---|---|---|---|---|---|---|
| 0 | 220.3 | -2.854 | 0.5762 | 2457 | 217.0 | -2.854 | 0.5761 |
| 0.5 | 205.4 | -13.35 | 2.367 | 2238 | 205.0 | -13.45 | 2.386 |
| 1.0 | 101.5 | -16.21 | 3.727 | 794.2 | 101.9 | -15.83 | 3.641 |
| 1.5 | 76.05 | -17.152 | 4.859 | 497.4 | 76.50 | -17.08 | 4.836 |
| 2.0 | 63.58 | -11.38 | 4.057 | 401.7 | 63.62 | -11.22 | 4.003 |
| 2.5 | 27.26 | 2.180 | 0.1321 | 126.0 | 27.30 | 2.188 | 0.1313 |