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Quasiconformal extension for harmonic mappings on finitely connected domains

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Abstract. We prove that a harmonic quasiconformal mapping defined on a finitely connected domain in the plane, all of whose boundary components are either points or quasicircles, admits a quasiconformal extension to the whole plane if its Schwarzian derivative is small. We also make the observation that a univalence criterion for harmonic mappings holds on uniform domains.

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1. Introduction

Let \( f \) be a harmonic mapping in a planar domain \( D \) and let \( \omega = \frac{f_z}{f_{\bar{z}}} \) be its dilatation. According to Lewy’s theorem the mapping \( f \) is locally univalent if and only if its Jacobian \( J_f = |f_z|^2 - |f_{\bar{z}}|^2 \) does not vanish. Duren’s book [4] contains valuable information about the theory of planar harmonic mappings.

The Schwarzian derivative of \( f \) was defined by Hernández and Martín [9] as
\[
S_f = \rho_{zz} - \frac{1}{2} (\rho_z)^2, \quad \text{where} \quad \rho = \log J_f.
\]

When \( f \) is holomorphic this reduces to the classical Schwarzian derivative. Another definition, introduced by Chuaqui, Duren and Osgood [3], applies to harmonic mappings which admit a lift to a minimal surface via the Weierstrass–Enneper formulas. However, focusing on the planar theory in this note we adopt the definition (1).

We assume that \( \mathbb{C} \setminus D \) contains at least three points, so that \( D \) is equipped with the hyperbolic metric, defined by
\[
\lambda_D (\pi(z)) |\pi'(z)| |dz| = \lambda_{\mathbb{D}} (z) |dz| = \frac{|dz|}{1 - |z|^2}, \quad z \in \mathbb{D},
\]
where $\mathbb{D}$ is the unit disk and $\pi : \mathbb{D} \to D$ is a universal covering map. The size of the Schwarzian derivative of a mapping $f$ in $D$ is measured by the norm

\[ \|S_f\|_D = \sup_{z \in D} \lambda_D(z)^{-2} |S_f(z)|. \]

A domain $D$ in $\overline{\mathbb{C}}$ is a $K$-quasidisk if it is the image of the unit disk under a $K$-quasiconformal self-map of $\overline{\mathbb{C}}$, for some $K \geq 1$. The boundary of a quasidisk is called a quasicircle.

According to a theorem of Ahlfors [1], if $D$ is a $K$-quasidisk then there exists a constant $c > 0$, depending only on $K$, such that if $f$ is analytic in $D$ with $\|S_f\|_D \leq c$ then $f$ is univalent in $D$ and has a quasiconformal extension to $\overline{\mathbb{C}}$. This has been generalized by Osgood [12] to the case when $D$ is a finitely connected domain whose boundary components are either points or quasicircles. Further, the univalence criterion was generalized to uniform domains (see Section 4 for a definition) by Gehring and Osgood [7] and, subsequently, the quasiconformal extension criterion was generalized to uniform domains by Astala and Heinonen [2].

For harmonic mappings and the definition (1) of the Schwarzian derivative, a univalence and quasiconformal extension criterion in the unit disk $\mathbb{D}$ was proved by Hernández and Martín [8]. This was recently generalized to quasidisks by the present author in [5]. Moreover, in [5] it was shown that if all boundary components of a finitely connected domain $D$ are either points or quasicircles then any harmonic mapping in $D$ with sufficiently small Schwarzian derivative is injective. The main purpose of this note is to prove the following theorem.

**Theorem 1.** Let $D$ be a finitely connected domain whose boundary components are either points or quasicircles and let also $d \in [0,1)$. Then there exists a constant $c > 0$, depending only on the domain $D$ and the constant $d$, such that if $f$ is analytic in $D$ with $\|S_f\|_D \leq c$ and with dilatation $\omega$ satisfying $|\omega(z)| \leq d$ for all $z \in D$ then $f$ admits a quasiconformal extension to $\overline{\mathbb{C}}$.

As mentioned above, for the case when $D$ is a (simply connected) quasidisk this was shown in [5] while, on the other hand, for the case $d = 0$ (when $f$ is analytic) this was proved by Osgood in [12]. Osgood’s proof amounts to proving a univalence criterion in $f(D)$. Such an approach does not seem to work here since for a holomorphic $\phi$ on $f(D)$ the composition $\phi \circ f$ is not, in general, harmonic.

Since isolated boundary points are removable for quasiconformal mappings (see [10, Ch. I, § 8.1]), we may assume for the proof of Theorem 1 that $\partial D$ consists of $n$ non-degenerate quasicircles. Our proof will be based on the following theorem of Springer [13] (see also [10, Ch.II, § 8.3]).

**Theorem 2 ([13]).** Let $D$ and $D'$ be two $n$-tuply connected domains whose boundary curves are quasicircles. Then every quasiconformal mapping of $D$ onto $D'$ can be extended to a quasiconformal mapping of the whole plane.

Hence, to prove Theorem 1 it suffices that we show that the boundary components of $f(D)$ are quasicircles. We prove this in Section 3. It relies on Osgood’s [12] quasiconformal decomposition, which we briefly present in Section 2. In Section 4 we give a univalence criterion on uniform domains.

### 2. Quasiconformal Decomposition

Let $D$ be a domain in $\overline{\mathbb{C}}$. A collection $\mathcal{D}$ of domains $\Delta \subset D$ is called a $K$-quasiconformal decomposition of $D$ if each $\Delta$ is a $K$-quasidisk and any two points $z_1, z_2 \in D$ lie in the closure of some $\Delta \in \mathcal{D}$. This definition was introduced by Osgood in [12], along with the following lemma.

**Lemma 3 ([12]).** If $D$ is a finitely connected domain and each component of $\partial D$ is either a point or a quasicircle then $D$ is quasiconformally decomposable.

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We now present, almost verbatim, the construction proving Lemma 3. We focus on the parts of the construction we will be needing, maintaining the notation of [12] and skipping all the relevant proofs. The interested reader should consult [12] for further details.

As we mentioned earlier, we may assume that \( \partial D \) consists of non-degenerate quasicircles \( C_0, C_1, \ldots, C_{n-1} \), for \( n \geq 2 \). Let \( F \) be a conformal mapping of \( D \) onto a circle domain \( D' \). Then, with an application of Theorem 2 to \( F^{-1} \), it will be sufficient to find a quasiconformal decomposition of \( D' \). Hence we may assume that \( D' \) itself is a circle domain with boundary circles \( C_j, j = 0, \ldots, n-1 \).

If \( n = 2 \) then we may assume that \( D \) is the annulus \( 1 < |z| < R \). Then the domains

\[
\Delta_1 = \left\{ z \in D : 0 < \arg(z) < \frac{4\pi}{3} \right\}, \quad \Delta_2 = e^{2\pi i/3} \Delta_1, \quad \Delta_3 = e^{4\pi i/3} \Delta_1
\]

make a quasiconformal decomposition of \( D \).

Let \( n \geq 3 \). Then there exists a conformal mapping \( \Psi \) of the circle domain \( D \) onto a domain \( D' \) consisting of the entire plane minus \( n \) finite rectilinear slits lying on rays emanating from the origin. The mapping can be chosen so that no two distinct slits lie on the same ray. The boundary behavior of \( \Psi \) is the following: it can be analytically extended to \( \overline{D} \), and the two endpoints of the slit \( C'_j = \Psi(C_j) \) correspond to two points on the circle \( C_j \) which partition \( C_j \) into two arcs, each of which is mapped onto \( C'_j \) in a one-to-one fashion.

Let \( \xi_j \) be the endpoint of \( C'_j \) furthest from the origin and let \( Q'_j \) be the part of the ray that joins \( \xi_j \) to infinity. Let also \( S'_j \) be the sector between \( C'_j \) and \( C'_{j+1} \). Let \( \omega'_j \) be the midpoint of \( C'_j \) and let \( P'_j \) be a polygonal arc joining \( \omega'_j \) to \( \omega'_{j+1} \) that, except for its endpoints, lies completely in \( S'_j \). Then

\[
P' = \bigcup_{j=0}^{n-1} P'_j
\]

is a closed polygon separating 0 from \( \infty \) that does not intersect any of the \( Q'_j \). Let \( G'_0 \) and \( G'_1 \) be the components of \( D' \setminus P' \) that contain 0 and \( \infty \), respectively. Now define

\[
\Delta'_{0j} = G'_0 \cup S'_j, \quad \Delta'_j = D' \setminus \Delta'_{0j}
\]

and

\[
\mathfrak{D}' = \left\{ \Delta'_{0j} : j = 0, 1, \ldots, n-1 \right\} \cup \left\{ \Delta'_j : j = 0, 1, \ldots, n-1 \right\}.
\]

This collection has the covering property for \( D' \).

We denote the various parts of \( D \) corresponding under \( \Psi^{-1} \) to those of \( D' \) by the same symbol without the prime. Then

\[
\mathfrak{D} = \left\{ \Delta_{0j} : j = 0, 1, \ldots, n-1 \right\} \cup \left\{ \Delta_j : j = 0, 1, \ldots, n-1 \right\}
\]

is a quasiconformal decomposition of \( D \).

3. Proof of Theorem 1

Let \( f \) be a mapping in \( D \) as in Theorem 1. By [5, Theorem 2], \( f \) is injective if \( c \) is sufficiently small. Also, \( f \) extends continuously to \( \partial D \) since every boundary point of \( D \) belongs to \( \partial \Delta \) for some \( \Delta \) in the collection \( \mathfrak{D} \) and, by [5, Theorem 1], the restriction of \( f \) on \( \Delta \) admits a homeomorphic extension to \( \overline{C} \).

Let \( \Psi \) be a conformal mapping of \( D \) onto the slit domain \( D' \) of the previous section. Let \( C_j \) be a boundary quasicircle of \( D \). We first prove that \( f(C_j) \) is a Jordan curve. The slit \( C'_j \) is divided by its midpoint \( \omega'_j \) into two line segments, which we denote by \( \Sigma'_j(m), m = 1, 2 \), so that

\[
\Sigma'_j(1) = \left\{ z \in C'_j : |z| \leq \left| \omega'_j \right| \right\} \quad \text{and} \quad \Sigma'_j(2) = \left\{ z \in C'_j : |z| \geq \left| \omega'_j \right| \right\}.
\]
Let $\Sigma_j'(m)^\pm$ denote the two sides of $\Sigma_j'(m)$, so that a point $z_0$ on $\Sigma_j'(m)^-$ is reached only by points $z \in S_j'-1$, meaning that $\arg z \to (\arg z_0)^-$ when $z \to z_0$. Similarly, a point $z_0$ on $\Sigma_j'(m)^+$ is reached only by points $z \in S_j'$, so that $\arg z \to (\arg z_0)^+$ when $z \to z_0$. Corresponding under $\Psi^{-1}$ are four disjoint -except for their endpoints- arcs on the quasicircle $C_j$, denoted without the prime by $\Sigma_j(m)^\pm$, $m = 1, 2$. Now consider the domains $\Delta_{0,j-1}, \Delta_0$ and $\Delta_k$ in the collection $D$, for some $k \neq j - 1, j$; see Figure 1 for their images under $\Psi$. By [5, Theorem 1] $f$ is injective up to the boundary of each $\Delta \in D$. Note that the arcs $\Sigma_j(1)^-, \Sigma_j(1)^+$ and $\Sigma_j(2)^-$ are subsets of $\partial \Delta_{0,j-1}$, so that their images under $f$, except for their endpoints, are disjoint. It remains to show that the images of these three arcs under $f$ are not intersected by the remaining image $f(\Sigma_j(2)^+)$. Note that the arcs $\Sigma_j(1)^-, \Sigma_j(1)^+$ and $\Sigma_j(2)^-$ are subsets of $\partial \Delta_0$, so that $f(\Sigma_j(2)^+)$ does not intersect $f(\Sigma_j(1)^-)$ nor $f(\Sigma_j(1)^+)$. What remains to be seen is that $f(\Sigma_j(2)^-)$ and $f(\Sigma_j(2)^+)$ are disjoint and this follows from the fact that the arcs $\Sigma_j(2)^-$ and $\Sigma_j(2)^+$ are subsets of $\partial \Delta_k$.

To see that the Jordan curve $f(C_j)$ is actually a quasicircle note that each point of $f(C_j)$ belongs to some open subarc of $f(C_j)$ which is entirely included in the boundary of either $f(\Delta_{0,j-1}), f(\Delta_0)$ or $f(\Delta_k)$. These three domains are quasidisks by [5, Theorem 3]. Now the assertion that $f(C_j)$ is a quasicircle follows by an application of Theorem 8.7 in [10, Ch. II, § 8.9].

4. Remarks on uniform domains

A domain $D$ in $\mathbb{C}$ is called uniform if there exist positive constants $a$ and $b$ such that each pair of points $z_1, z_2 \in D$ can be joined by an arc $\gamma \subset D$ so that for each $z \in \gamma$ it holds

$$\ell(\gamma) \leq a |z_1 - z_2|$$

and

$$\min_{j=1,2} \ell(\gamma_j) \leq b \operatorname{dist}(z, \partial D),$$

where $\gamma_1, \gamma_2$ are the components of $\gamma \setminus \{z\}$, $\operatorname{dist}(z, \partial D)$ denotes the euclidean distance from $z$ to the boundary of $D$ and $\ell(\cdot)$ denotes euclidean length. Uniform domains were introduced by Martio and Sarvas [11]; see also, e.g., [7] for this equivalent definition. In [11] it was shown that all boundary components of a uniform domain are either points or quasicircles. The converse of this is also true for finitely connected domains, but not, in general, for domains of infinite connectivity; see [6, § 3.5]. The following univalence criterion was proved in [11].

**Theorem 4 ([7, 11]).** If $D$ is a uniform domain then there exists a constant $c > 0$ such that every analytic function $f$ in $D$ with $\|S_f\|_D \leq c$ is injective.

Gehring and Osgood [7] gave a different proof of Theorem 4 by providing a characterization of uniform domains. They showed that a domain $D$ is uniform if and only if it is quasiconformally decomposable in the following weaker (than the one we saw in Section 2) sense: there exists a constant $K$ with the property that for each $z_1, z_2 \in D$ there exists a $K$-quasidisk $\Delta \subset D$ for which
$z_1, z_2 \in \Delta$. Note that, in contrast to Osgood’s \cite{Osgood:1980} decomposition, here $\Delta$ depends on the points $z_1, z_2$. However, this can readily be used to generalize the implication (i) $\Rightarrow$ (iii) of \cite[Theorem 2]{Efraimidis:2020}, according to which a univalence criterion for harmonic mappings holds on finitely connected uniform domains. The following theorem extends it to all uniform domains.

**Theorem 5.** Let $D$ be a uniform domain in $\mathbb{C}$. Then there exists a constant $c > 0$ such that if $f$ is harmonic in $D$ with $\|S_f\|_D \leq c$ then $f$ is injective.

**Proof.** Assume that there exist distinct points $z_1, z_2 \in D$ for which $f(z_1) = f(z_2)$. By \cite{Gehring:1979}, there exists a $K$-quasidisk $\Delta \subset D$ for which $z_1, z_2 \in \Delta$. The domain monotonicity for the hyperbolic metric shows that

$$\|S_f\|_\Delta \leq \|S_f\|_D \leq c.$$  

But the homeomorphic extension of \cite[Theorem 1]{Efraimidis:2020} shows that if $c$ is sufficiently small then $f$ is injective up to the boundary of $\Delta$, a contradiction. \qed

Regarding quasiconformal extension, Astala and Heinonen \cite{Astala:1988} proved the following theorem.

**Theorem 6 (\cite{Astala:1988}).** If $D$ is a uniform domain then there exists a constant $c > 0$ such that every analytic function $f$ in $D$ with $\|S_f\|_D \leq c$ admits a quasiconformal extension to $\mathbb{C}$.

This evidently implies Theorem 4 and was also proved in substantially greater generality, but we omit it here. It is not clear how to generalize Theorem 6 to the setting of harmonic mappings. Therefore, we propose the following problem.

**Problem.** Let $D$ be a uniform domain. Does there exist a constant $c > 0$ such that if $f$ is harmonic in $D$ with $\|S_f\|_D \leq c$ and with dilatation $\omega$ satisfying $\sup_{z \in D} |\omega(z)| < 1$ then $f$ admits a quasiconformal extension to $\mathbb{C}$?

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