ON THE RANGES OF BILINEAR PSEUDO-DIFFERENTIAL OPERATORS OF $S_{0,0}$-TYPE ON $L^2 \times L^2$

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ABSTRACT. In this paper, the ranges of bilinear pseudo-differential operators of $S_{0,0}$-type on $L^2 \times L^2$ are determined in the framework of Besov spaces. Our result improves the $L^2 \times L^2 \to L^1$ boundedness of those operators with symbols in the bilinear Hörmander class $BS^{m}_{0,0}$.

1. Introduction

In this paper, we use the following two symbol classes of $S_{0,0}$-type. One is the bilinear Hörmander class $BS^{m}_{0,0}$, $m \in \mathbb{R}$, consisting of all $\sigma(x, \xi_1, \xi_2) \in C^\infty((\mathbb{R}^n)^3)$ such that

$$|\partial_x^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi_1| + |\xi_2|)^m$$

for all multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n = \{0, 1, 2, \ldots \}^n$. The other is $BS^{(m_1, m_2)}_{0,0}$, $m_1, m_2 \in \mathbb{R}$, consisting of all $\sigma(x, \xi_1, \xi_2) \in C^\infty((\mathbb{R}^n)^3)$ such that

$$|\partial_x^\alpha \partial_{\xi_1}^\beta \partial_{\xi_2}^\gamma \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi_1|)^{m_1}(1 + |\xi_2|)^{m_2}.$$

For a symbol $\sigma$, the bilinear pseudo-differential operators $T_\sigma$ is defined by

$$T_\sigma(f_1, f_2)(x) = \frac{1}{(2\pi)^2n} \int_{(\mathbb{R}^n)^2} e^{ix\cdot(\xi_1 + \xi_2)} \sigma(x, \xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) d\xi_1 d\xi_2$$

for $f_1, f_2 \in S(\mathbb{R}^n)$.

In the linear case, the celebrated Calderón-Vaillancourt theorem \cite{C-V} states that if a symbol $\sigma(x, \xi) \in C^\infty((\mathbb{R}^n)^2)$ satisfies

\[(1.1) \quad |\partial_x^\alpha \partial_{\xi}^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta},\]

then the corresponding pseudo-differential operator is bounded on $L^2$. As a bilinear counterpart of this theorem, we naturally expect that the condition $\sigma \in BS^{0}_{0,0}$ ensures the $L^2 \times L^2 \to L^1$ boundedness of $T_\sigma$. However, Bényi-Torres \cite{B-T} pointed out that this boundedness does not hold in general. Michalowski-Rule-Staubach \cite{M-R-S} treated the subcritical case $m < -n/2$, and then Miyachi-Tomita \cite{M-T} showed that all bilinear pseudo-differential operators with symbols in $BS^{m}_{0,0}$ are bounded from $L^2 \times L^2$ to $L^1$ if and only if $m \leq -n/2$. It should be mentioned that the stronger $L^2 \times L^2 \to h^1$ boundedness was proved in \cite{M-T}, where $h^1$ is the local Hardy space. See also Miyachi-Tomita \cite{M-T2} for the classes of $S_{\rho,\rho}$-type, $0 \leq \rho < 1$.
Kato-Miyachi-Tomita recently proved that if \( m_1, m_2 < 0 \) and \( m_1 + m_2 = -n/2 \) and \( \sigma \in BS^{(m_1, m_2)}_{0, 0} \), then \( T_\sigma \) is bounded from \( L^2 \times L^2 \) to the amalgam space \( (L^2, \ell^1) \) ([10, Theorem 1.3, Example 1.4]). Here, we remark that

\[
BS^{-n/2}_{0, 0} \subset BS^{(m_1, m_2)}_{0, 0}, \quad m_1, m_2 \leq 0, \ m_1 + m_2 = -n/2.
\]

Hence, for \( 1 \leq p \leq 2 \), the \( L^2 \times L^2 \rightarrow L^p \) boundedness of \( T_\sigma \) with \( \sigma \in BS^{-n/2}_{0, 0} \) follows from the embedding \( (L^2, \ell^1) \rightarrow L^p \) (see Section 2). In the linear case, since the constant function satisfies (1.1), the \( L^2 \rightarrow L^p \) boundedness of pseudo-differential operators with symbols satisfying (1.1) holds if and only if \( p = 2 \). On the other hand, in the bilinear case, we have a choice to choose target spaces. Thus, the purpose of this paper is to determine the ranges of bilinear pseudo-differential operators with symbols in the critical class \( BS^{-n/2}_{0, 0} \) on \( L^2 \times L^2 \) in the framework of Besov spaces \( B^s_{p, q} \).

The main result of this paper is the following.

**Theorem 1.1.** Let \( m_1, m_2 < 0 \) and \( m_1 + m_2 = -n/2 \). If \( 1 \leq p \leq 2 \), then all bilinear pseudo-differential operators with symbols in \( BS^{(m_1, m_2)}_{0, 0} \) are bounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( B^0_{p, 1}(\mathbb{R}^n) \).

Since \( B^0_{0, 1} \hookrightarrow L^1 \) (in fact, \( B^0_{0, 1} \hookrightarrow h^1 \hookrightarrow L^1 \)), Theorem 1.1 is an improvement of the \( L^2 \times L^2 \rightarrow L^1 \) boundedness.

By (1.2), the following means the optimality of Theorem 1.1.

**Theorem 1.2.** Let \( 0 < p, q \leq \infty \). Then all bilinear pseudo-differential operators with symbols in \( BS^{−n/2}_{0, 0} \) are bounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( B^0_{p, q}(\mathbb{R}^n) \) if and only if \( 1 \leq p \leq 2 \) and \( 1 \leq q \leq \infty \).

By the “only if” part of Theorem 1.2 and the embedding \( B^{s}_{p, q_1} \hookrightarrow B^{s}_{p, q_2} \) for \( 0 < p, q_1, q_2 \leq \infty \) and \( s > 0 \), if the \( L^2 \times L^2 \rightarrow B^s_{p, q} \) boundedness of all \( T_\sigma \) with \( \sigma \in BS^{-n/2}_{0, 0} \) holds, then \( s \) must satisfy \( s \leq 0 \). This is the reason why we consider the case \( s = 0 \) in Theorems 1.1 and 1.2.

The following says that Theorem 1.1 cannot be compared with the result of [10].

**Proposition 1.3.** There is no embedding relation between the amalgam space \( (L^2, \ell^1)(\mathbb{R}^n) \) and the Besov spaces \( B^0_{p, 1}(\mathbb{R}^n) \), \( 1 \leq p \leq 2 \).

Related results to bilinear Fourier multiplier operators of \( S_{0, 0} \)-type on \( L^2 \times L^2 \) can be found in Grafakos-He-Slav’ıková [7] and Slav’ıková [17].

The contents of this paper are as follows. In Section 2, we give preliminary facts. In Section 3, we give basic estimates used in the proof of Theorem 1.1. In Sections 4, 5, and 6, we prove Theorems 1.1, 1.2, and Proposition 1.3, respectively.

## 2. Preliminaries

For two nonnegative quantities \( A \) and \( B \), the notation \( A \lesssim B \) means that \( A \leq CB \) for some unspecified constant \( C > 0 \), and \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \). For \( 1 \leq p \leq \infty \), \( p' \) is the conjugate exponent of \( p \), that is, \( 1/p + 1/p' = 1 \). The usual inner product of \( f, g \in L^2(\mathbb{R}^n) \) is denoted by \( \langle f, g \rangle \).

Let \( S(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) be the Schwartz space of rapidly decreasing smooth functions on \( \mathbb{R}^n \) and its dual, the space of tempered distributions, respectively. We define the Fourier transform \( \mathcal{F} f \) and the inverse Fourier transform \( \mathcal{F}^{-1} f \) of \( f \in S(\mathbb{R}^n) \).
by

\[
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \, d\xi.
\]

For \( m \in L^\infty(\mathbb{R}^n) \), the Fourier multiplier operator \( m(D) \) is defined by \( m(D)f = \mathcal{F}^{-1}[m\hat{f}] \) for \( f \in \mathcal{S}(\mathbb{R}^n) \).

Let \( \{\psi_\ell\}_{\ell \geq 0} \) be a sequence of Schwartz functions on \( \mathbb{R}^n \) satisfying

\[
\sup \psi_0 \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \},
\]

\[
\sup \psi_\ell \subset \{ \xi \in \mathbb{R}^n : 2^{\ell-1} \leq |\xi| \leq 2^{\ell+1} \}, \quad \ell \geq 1,
\]

\[
(2.1) \quad |\partial^\alpha \psi_\ell(\xi)| \leq C_{\alpha} 2^{-\ell|\alpha|}, \quad \ell \geq 0, \quad \alpha \in \mathbb{N}_0^n, \quad \xi \in \mathbb{R}^n,
\]

\[
\sum_{\ell=0}^\infty \psi_\ell(\xi) = 1, \quad \xi \in \mathbb{R}^n.
\]

For \( 0 < p, q \leq \infty \) and \( s \in \mathbb{R} \), the Besov space \( B^s_{p,q}(\mathbb{R}^n) \) consists of all \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\|f\|_{B^s_{p,q}} = \left( \sum_{\ell=0}^\infty 2^{\ell qs} \|\psi_\ell(D)f\|_{L^p}^q \right)^{1/q} < \infty
\]

with usual modification when \( q = \infty \). It is well known that the definition of Besov spaces \( B^s_{p,q} \) is independent of the choice of \( \{\psi_\ell\}_{\ell \geq 0} \) satisfying (2.1). For \( 0 < p, q_1, q_2 \leq \infty \) and \( s_1, s_2 \in \mathbb{R} \), the embedding \( B^s_{p,q_1} \hookrightarrow B^{s_2}_{p,q_2} \) holds if \( s_1 = s_2 \) and \( q_1 \leq q_2 \), or if \( s_1 > s_2 \). The dual space of \( B^s_{p,q} \) coincides with \( B^{-s}_{p',q'} \), where \( 1 \leq p, q < \infty \) and \( s \in \mathbb{R} \). See [19] for more details on Besov spaces.

For \( 1 \leq p, q \leq \infty \), the amalgam space \( (L^p, \ell^q)(\mathbb{R}^n) \) consists of all measurable functions \( f \) on \( \mathbb{R}^n \) such that

\[
\|f\|_{(L^p, \ell^q)} = \left\{ \sum_{\nu \in \mathbb{Z}^n} \left( \int_{\nu + [-1/2, 1/2]^n} |f(x)|^p \, dx \right)^{q/p} \right\}^{1/q} < \infty
\]

with usual modification when \( p \) or \( q \) is infinity. Obviously, \((L^p, \ell^p) = L^p\), and \((L^{p_1}, \ell^{p_1}) \hookrightarrow (L^{p_2}, \ell^{p_2})\) for \( p_1 \geq p_2 \) and \( q_1 \leq q_2 \). In particular, \((L^2, \ell^1) \hookrightarrow L^r\) for \( 1 \leq r < 2 \), and the stronger embedding \((L^2, \ell^1) \hookrightarrow h^1\) holds in the case \( r = 1 \), where \( h^1 \) is the local Hardy space (see [11] Section 2.3).

We end this section by quoting the following, which is called Schur’s lemma (see, e.g., [6] Appendix A).

**Lemma 2.1.** Let \( \{A_{k_1, k_2}\}_{k_1, k_2 \geq 0} \) be a sequence of nonnegative numbers satisfying

\[
\sup_{k_1 \geq 0} \sum_{k_2 \geq 0} A_{k_1, k_2} < \infty \quad \text{and} \quad \sup_{k_2 \geq 0} \sum_{k_1 \geq 0} A_{k_1, k_2} < \infty.
\]

Then

\[
\sum_{k_1, k_2 \geq 0} A_{k_1, k_2} b_{k_1} c_{k_2} \lesssim \left( \sum_{k_1 \geq 0} b_{k_1}^2 \right)^{1/2} \left( \sum_{k_2 \geq 0} c_{k_2}^2 \right)^{1/2}
\]

for all nonnegative sequences \( \{b_{k_1}\}_{k_1 \geq 0} \) and \( \{c_{k_2}\}_{k_2 \geq 0} \).
3. Basic estimates

For $R > 0$ and $L > n$, we set

$$S_R(f)(x) = R^n \int_{R^n} \frac{|f(y)|}{(1 + R|x - y|)^L} dy,$$

and simply write $S_1(f)(x) = S(f)(x)$.

The following lemma can be found in the proof of [9, Lemma 4.2], but we will give the proof for the reader’s convenience.

**Lemma 3.1.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, we have

$$\left( \sum_{\nu \in \mathbb{Z}^n} |\varphi(R^{-1}(D - \nu))f(x)|^2 \right)^{1/2} \lesssim R^{n/2} S_R(|f|^2)(x)^{1/2}$$

for all $R \geq 1$, where $\varphi(R^{-1}(D - \nu))f = \mathcal{F}^{-1} [\varphi(R^{-1}(-\nu))\hat{f}]$.

**Proof.** Since $\mathbb{R}^n = \cup_{\mu \in \mathbb{Z}^n} (2\pi \mu + [-\pi, \pi]^n)$, we have

$$\varphi(R^{-1}(D - \nu))f(x) = R^n \int_{\mathbb{R}^n} e^{i\nu(x-y)} \Phi(R(x-y)) f(y) dy$$

$$= R^n e^{iwx} \sum_{\mu \in \mathbb{Z}^n} \int_{2\pi \mu + [-\pi, \pi]^n} e^{-i\nu \cdot y} \Phi(R(x-y)) f(y) dy$$

$$= R^n e^{iwx} \int_{[-\pi, \pi]^n} e^{-i\nu \cdot y} \left( \sum_{\mu \in \mathbb{Z}^n} \Phi(R(x-y-2\pi\mu)) f(y+2\pi\mu) \right) dy,$$

where $\Phi = \mathcal{F}^{-1} \varphi$. Here, $\sum_{\mu \in \mathbb{Z}^n} \Phi(R(x-y-2\pi\mu)) f(y+2\pi\mu)$ is a $(2\pi \mathbb{Z})^n$-periodic function of the $y$-variable. Hence, it follows from Parseval’s identity that

$$\left( \sum_{\nu \in \mathbb{Z}^n} |\varphi(R^{-1}(D - \nu)) f(x)|^2 \right)^{1/2}$$

$$= (2\pi)^n/R^n \left( \int_{[-\pi, \pi]^n} \left| \sum_{\mu \in \mathbb{Z}^n} \Phi(R(x-y-2\pi\mu)) f(y+2\pi\mu) \right|^2 dy \right)^{1/2}.$$
In the rest of this section, we assume that $m_1, m_2 \in \mathbb{R}$ and $\sigma \in BS_{0,0}^{(m_1,m_2)}$, and shall consider the decomposition of $\sigma$. Let $\{\psi_j\}_{j \geq 0}$ be as in (2.4), and let $\varphi \in S(\mathbb{R}^n)$ be such that
\[
supp \varphi \subset [-1,1]^n \quad \text{and} \quad \sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu) = 1, \quad \xi \in \mathbb{R}^n.
\]
Using these functions, we decompose $\sigma$ as
\[
\sigma(x, \xi_1, \xi_2) = \sum_{j \in \mathbb{N}_0} \sum_{k \in (\mathbb{N}_0)^2} \sigma_{j,k}(x, \xi_1, \xi_2) = \sum_{j \in \mathbb{N}_0} \sum_{k \in (\mathbb{N}_0)^2} \sum_{\nu \in (\mathbb{Z}^n)^2} \sigma_{j,k,\nu}(x, \xi_1, \xi_2)
\]
with
\[
\sigma_{j,k}(x, \xi_1, \xi_2) = [\psi_j(D_x)\sigma](x, \xi_1, \xi_2)\psi_{k_1}(\xi_1)\psi_{k_2}(\xi_2)
\]
and
\[
\sigma_{j,k,\nu}(x, \xi_1, \xi_2) = \sigma_{j,k}(x, \xi_1, \xi_2)\varphi(\xi_1 - \nu_1)\varphi(\xi_2 - \nu_2).
\]

**Lemma 3.2.** (1) For each $\beta_1, \beta_2 \in (\mathbb{N}_0)^n$ and $N \in \mathbb{N}_0$, we have
\[
|\partial^{\beta_1}_{\xi_1} \partial^{\beta_2}_{\xi_2} \sigma_{j,k,\nu}(x, \xi_1, \xi_2)| \lesssim 2^{k_1 m_1 + k_2 m_2 - j N}
\]
for all $j \in \mathbb{N}_0$, $k = (k_1, k_2) \in (\mathbb{N}_0)^2$ and $\nu \in (\mathbb{Z}^n)^2$.

(2) For each $N \in \mathbb{N}_0$, we have
\[
|T_{\sigma_{j,k,\nu}}(f_1, f_2)(x)| \lesssim 2^{k_1 m_1 + k_2 m_2 - j N} S(f_1)(x) S(f_2)(x)
\]
for all $j \in \mathbb{N}_0$, $k = (k_1, k_2) \in (\mathbb{N}_0)^2$ and $\nu \in (\mathbb{Z}^n)^2$.

**Proof.** We first prove the assertion (1). Let $j \geq 1$. By the moment condition of $F^{-1} \psi_j$ and Taylor’s formula, we have
\[
\sigma_{j,k,\nu}(x, \xi_1, \xi_2) = \int_{\mathbb{R}^n} F^{-1} \psi_j(y) \left(\sigma_{k,\nu}(x - y, \xi_1, \xi_2) - \sum_{|\alpha| < N} \frac{(-y)^\alpha}{\alpha!} [\partial^\alpha \sigma_{k,\nu}](x, \xi_1, \xi_2)\right) dy
\]
and
\[
\sigma_{j,k,\nu}(x, \xi_1, \xi_2) = \int_{\mathbb{R}^n} F^{-1} \psi_j(y) \left(N \sum_{|\alpha| = N} \frac{(-y)^\alpha}{\alpha!} \int_0^1 (1 - t)^{N-1} [\partial^\alpha \sigma_{k,\nu}](x - ty, \xi_1, \xi_2) dt\right) dy,
\]
where
\[
\sigma_{k,\nu}(x, \xi_1, \xi_2) = \sigma(x, \xi_1, \xi_2)\psi_{k_1}(\xi_1)\psi_{k_2}(\xi_2)\varphi(\xi_1 - \nu_1)\varphi(\xi_2 - \nu_2).
\]
Since $1 + |\xi_i| \approx 2^{k_i}$ for $\xi_i \in \text{supp} \psi_{k_i}$, we see that
\[
|\partial^{\beta_1}_{\xi_1} \partial^{\beta_2}_{\xi_2} \sigma_{k,\nu}(x, \xi_1, \xi_2)| \lesssim 2^{k_1 m_1 + k_2 m_2}.
\]
Hence,
\[
|\partial^{\beta_1}_{\xi_1} \partial^{\beta_2}_{\xi_2} \sigma_{j,k,\nu}(x, \xi_1, \xi_2)| \lesssim \int_{\mathbb{R}^n} |F^{-1} \psi_j(y)| \left(\sum_{|\alpha| < N} |y|^\alpha \int_0^1 |[\partial^\alpha \sigma_{\xi_1} \partial^\alpha \sigma_{\xi_2} \sigma_{k,\nu}](x - ty, \xi_1, \xi_2)| dt\right) dy
\]
\[
\lesssim 2^{k_1m_1+k_2m_2} \int_{\mathbb{R}^n} |\mathcal{F}^{-1} \psi_j(y)| \|y\|^N dy \lesssim 2^{k_1m_1+k_2m_2-jN},
\]
where we used the inequality \(|\mathcal{F}^{-1} \psi_j(y)| \lesssim 2^{jn} (1 + 2^j |y|)^{-(N+n+1)}\). If we do not use the moment condition in the above argument, we have the same estimate with \(j = 0\). The proof of the assertion (1) is complete.

Next, we will show the assertion (2). We write

\[
T_{\sigma_j,k,\nu}(f_1,f_2)(x) = \int_{(\mathbb{R}^n)^2} K_{j,k,\nu}(x, x-y_1, x-y_2) f_1(y_1)f_2(y_2) \, dy_1dy_2,
\]
where \(K_{j,k,\nu}\) is defined by

\[
K_{j,k,\nu}(x, y_1, y_2) = \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} e^{i(y_1 \cdot \xi_1 + y_2 \cdot \xi_2)} \sigma_j,k,\nu(x, \xi_1, \xi_2) \, d\xi_1d\xi_2.
\]

It follows from integration by parts, the assertion (1) and the support condition \(\text{supp } \sigma_j,k,\nu(x, \cdot, \cdot) \subset \{ (\nu_1, \nu_2) + [-1,1]^{2n} \} \) for each \(x \in \mathbb{R}^n\) that

\[
|K_{j,k,\nu}(x, y_1, y_2)| \lesssim 2^{k_1m_1+k_2m_2-jN} (1 + |y_1|)^{-L} (1 + |y_2|)^{-L},
\]
which gives the desired result. \(\square\)

The following lemma plays a crucial role in the proof of Theorem 1.1 and its essence goes back to [14] Lemma 3.6).

**Lemma 3.3.** Let \(2 \leq r \leq \infty\), and let \(\Lambda\) be a finite subset of \(\mathbb{Z}^n\). For each \(N \geq 0\), we have

\[
\left( \sum_{\nu_1 \in \Lambda} \sum_{\nu_2 \in \mathbb{Z}^n} + \sum_{\nu_i \in \mathbb{Z}^n} \sum_{\nu_2 \in \Lambda} + \sum_{\mu \in \Lambda} \sum_{\nu_2 = \mu} \right) \| T_{\sigma_j,k,\nu}(f_1,f_2) \| g
\lesssim 2^{k_1m_1+k_2m_2-jN} |\Lambda|^{1/2} \| f_1 \|_{L^2} \| f_2 \|_{L^2} \| g \|_{L^r}
\]
for all \(j \in \mathbb{N}_0\) and \(k = (k_1,k_2) \in (\mathbb{N}_0)^2\), where \(|\Lambda|\) is the number of elements of \(\Lambda\).

**Proof.** Let \(\varphi\) be a Schwartz function such that \(\text{supp } \varphi \subset [-2,2]^n\) and \(\varphi(\xi) = 1\) on \([-1,1]^n\). For \(\nu_i \in \mathbb{Z}^n\), \(i = 1,2\), \(\mu \in \mathbb{Z}^n\) and \(j \geq 0\), we set

\[
f_{j,\nu_i} = \varphi(D-\nu_i)f_1, \quad g_{j,\mu} = \varphi(2^{-j+2}(D-\mu))g.
\]

Then, it follows from the identity \(\varphi \tilde{\varphi} = \varphi\) that

\[(3.2) \quad T_{\sigma_j,k,\nu}(f_1,f_2) = T_{\sigma_j,k,\nu}(f_1,\nu_1, f_2,\nu_2).\]

The Fourier transform of \(T_{\sigma_j,k,\nu}(f_1,f_2)\) can be written as

\[
\mathcal{F}[T_{\sigma_j,k,\nu}(f_1,f_2)](\zeta)
= \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} \psi_j(\zeta - (\xi_1 + \xi_2)) [\mathcal{F}\sigma](\zeta - (\xi_1 + \xi_2), \xi_1, \xi_2)
\times \psi_k(\xi_1)\psi_k(\xi_2)\varphi(\xi_1 - \nu_1)\varphi(\xi_2 - \nu_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \, d\xi_1d\xi_2,
\]
where \(\mathcal{F}\sigma\) denotes the partial Fourier transform of \(\sigma(x, \xi_1, \xi_2)\) with respect to the \(x\)-variable. Now, if \(\xi_i \in \text{supp } \varphi(-\nu_i)\) for \(i = 1,2\) and \(\zeta - (\xi_1 + \xi_2) \in \text{supp } \psi_j\), then \(\zeta \in \nu_1 + \nu_2 + [-2^{j+2}, 2^{j+2}]^n\). This implies that

\[(3.3) \quad \text{supp } \mathcal{F}[T_{\sigma_j,k,\nu}(f_1,f_2)] \subset \nu_1 + \nu_2 + [-2^{j+2}, 2^{j+2}]^n.\]
Hence, it follows from (3.2) that

\begin{equation}
\langle T_{\sigma, \mu, \nu}(f_1, f_2), g \rangle = \langle T_{\sigma, \mu, \nu}(f_1, f_2, g_j, \nu_1 + \nu_2) \\
= \langle T_{\sigma, \mu, \nu}(f_1, f_2, g_j, \nu_1 + \nu_2) \rangle.
\end{equation}

First, we consider the sum $\sum_{\nu_1 \in \Lambda} \sum_{\nu_2 \in \mathbb{Z}^n}$. By (3.4), Lemma 3.2 (2) with $N$ replaced by $N + n/2 + n/r$ and Schwarz’s inequality, we have

$$
\sum_{\nu_1 \in \Lambda} \sum_{\nu_2 \in \mathbb{Z}^n} \left| \langle T_{\sigma, \mu, \nu}(f_1, f_2), g \rangle \right| = \sum_{\nu_1 \in \Lambda} \sum_{\nu_2 \in \mathbb{Z}^n} \left| T_{\sigma, \mu, \nu}(f_1, \nu_1, f_2, \nu_2), g_j, \nu_1 + \nu_2 \right|
\leq 2^{k_1} 2^{k_2} 2^{k_2 - j(N + n/2 + n/r)} \int_{\mathbb{R}^n} S(f_1, \nu_1)(x) \left( \sum_{\nu_2 \in \mathbb{Z}^n} \left| S(f_2, \nu_2)(x) \right|^2 \right)^{1/2} \left( \sum_{\nu_2 \in \mathbb{Z}^n} \left| g_j, \nu_1 + \nu_2 \right|^2 \right)^{1/2} dx
\leq 2^{k_1} 2^{k_2} 2^{k_2 - j(N + n/2 + n/r)} \int_{\mathbb{R}^n} \left( \sum_{\nu_1 \in \mathbb{Z}^n} S(f_1, \nu_1)(x) \right)^{1/2} \left( \sum_{\nu_2 \in \mathbb{Z}^n} S(f_2, \nu_2)(x) \right)^{1/2} \left( \sum_{\nu_2 \in \mathbb{Z}^n} \left| g_j, \mu \right|^2 \right)^{1/2} dx.
$$

It follows from Lemma 3.1 that

$$
\left( \sum_{\nu_1 \in \mathbb{Z}^n} S(f_1, \nu_1)(x) \right)^{1/2} \lesssim \left( \sum_{\nu_1 \in \mathbb{Z}^n} S(|f_1, \nu_1|^2)(x) \right)^{1/2} = \left( \sum_{\nu_1 \in \mathbb{Z}^n} \left| f_1, \nu_1 \right|^2(x) \right)^{1/2}
\lesssim S(S(|f_1|^2))(x)^{1/2} = S(|f_1|^2)(x)^{1/2}
$$

and

$$
\left( \sum_{\mu \in \mathbb{Z}^n} \left| g_j, \mu \right|^2 \right)^{1/2} \lesssim 2^{(j+2)n/2} S_{2j+2}(|g|^2)(x)^{1/2}
\lesssim 2^n \left( \int_{\mathbb{R}^n} \frac{1}{(1 + 2^n|x-y|)^{Lq}} dy \right)^{1/q} \left( \int_{\mathbb{R}^n} |g(y)|^r \ dy \right)^{1/r}
\approx 2^n (1/(1+1/r)\|g\|_{L^r},
$$

where we used Hölder’s inequality with $1/q + 1/r = 1/2$ in the second inequality. Therefore, by Schwarz’s inequality and Young’s inequality, we obtain

$$
\sum_{\nu_1 \in \Lambda} \sum_{\nu_2 \in \mathbb{Z}^n} \left| \langle T_{\sigma, \mu, \nu}(f_1, f_2), g \rangle \right|
\lesssim 2^{k_1} 2^{k_2} 2^{k_2 - jN} \left( \int_{\mathbb{R}^n} |S(|f_1|^2)(x)|^{1/2} S(|f_2|^2)(x)^{1/2} dx \right) \|g\|_{L^r}
\lesssim 2^{k_1} 2^{k_2} 2^{k_2 - jN} \left( \int_{\mathbb{R}^n} \|S(|f_1|^2)\|_{L^r} \|S(|f_2|^2)\|_{L^r} \|g\|_{L^r} \right)
\approx 2^{k_1} 2^{k_2} 2^{k_2 - jN} \left( \int_{\mathbb{R}^n} \left| f_1 \right| dx \left| f_2 \right| dx \left| g \right| dx \right).$$

In the same way, we can estimate the sum $\sum_{\nu_1 \in \mathbb{Z}^n} \sum_{\nu_2 \in \Lambda}$. Next, we consider the sum $\sum_{\mu \in \Lambda} \sum_{\nu_1 + \nu_2 = \mu}$. By (3.4), Lemma 3.2 (2) and Schwarz’s inequality,

$$
\sum_{\mu \in \Lambda} \sum_{\nu_1 + \nu_2 = \mu} \left| \langle T_{\sigma, \mu, \nu}(f_1, f_2), g \rangle \right| = \sum_{\mu \in \Lambda} \sum_{\nu_1 + \nu_2 = \mu} \left| \langle T_{\sigma, \mu, \nu}(f_1, f_2, g_j, \nu_1 + \nu_2) \rangle \right|
\lesssim 2^{k_1} 2^{k_2} 2^{k_2 - jN} \left( \int_{\mathbb{R}^n} \left| f_1 \right| dx \left| f_2 \right| dx \left| g \right| dx \right).$$
Furthermore, we divide the sum as follows.

\[ \sum_{\ell \in A} \int_{\mathbb{R}^n} \left( \sum_{j_1, j_2 \in \mathbb{Z}^n} S(f_{1, \psi_1})(x) S(f_{2, \mu})(x) |g_{j, \mu}(x)| \right) dx \]

\[ \leq 2^{k_1 m_1 + k_2 m_2 - j(N + n/2 + n/r)} \sum_{\mu \in A} \int_{\mathbb{R}^n} S(f_{1, \psi_1})(x) S(f_{2, \mu})(x) |g_{j, \mu}(x)| \right) dx \]

\[ \leq 2^{k_1 m_1 + k_2 m_2 - j(N + n/2 + n/r)} \times \sum_{\mu \in A} \int_{\mathbb{R}^n} \left( \sum_{j_1, j_2 \in \mathbb{Z}^n} S(f_{1, \psi_1})(x) S(f_{2, \mu})(x) |g_{j, \mu}(x)| \right) dx \]

\[ \leq 2^{k_1 m_1 + k_2 m_2 - j(N + n/2 + n/r)} \parallel A \parallel^{1/2} \times \]

\[ \times \int_{\mathbb{R}^n} \left( \sum_{j_1, j_2 \in \mathbb{Z}^n} S(f_{1, \psi_1})(x) S(f_{2, \mu})(x) |g_{j, \mu}(x)| \right) dx \]

The rest of the proof is the same as before. The proof is complete. \( \square \)

4. Proof of Theorem 1.1

Let \( m_1, m_2, \sigma \) and \( p \) be the same as in Theorem 1.1. To obtain Theorem 1.1 by duality, it is sufficient to prove that

\[ \parallel \langle T_0(f_1, f_2), g \rangle \parallel_{L^2(L^2)} \parallel g \parallel_{B^p_{1, \infty}}. \]

From (3.1), we can write

\[ \langle T_0(f_1, f_2), g \rangle = \sum_{j \geq 0} \sum_{k \in \{k_0\}^2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, f_2), g \rangle \]

\[ = \sum_{j \geq 0} \sum_{k_1 \geq k_2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, f_2), g \rangle + \sum_{j \geq 0} \sum_{k_1 < k_2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, f_2), g \rangle. \]

By symmetry, we only consider the former sum in the last line, because the argument below works for the latter one.

Let \( \psi_k \in \mathcal{S}(\mathbb{R}^n), k \geq 0, \) be such that

\[ \text{supp} \psi_0 \subset \{ |\xi| \leq 4 \}, \quad \text{supp} \psi_k \subset \{ 2^{k-2} \leq |\xi| \leq 2^{k+2} \}, \quad k \geq 1, \]

\[ \text{supp} \psi_k = 1 \quad \text{on} \quad \text{supp} \psi_k, \quad k \geq 0. \]

Since \( \psi_k \psi_k = \psi_k \), it holds that

\[ \langle T_{\sigma_1, k, \nu}(f_1, f_2), g \rangle = \langle T_{\sigma_1, k, \nu}(f_1, k_1, f_2, k_2), g \rangle \]

with \( f_{i, k_i} = \psi_k(D) f_i, i = 1, 2. \) We also use the decomposition

\[ g = \sum_{\ell \geq 0} \psi_\ell(D) g = \sum_{\ell \geq 0} g_\ell, \]

where \( \{ \psi_\ell \}_{\ell \geq 0} \) is the same as in (2.1). Then, we can write

\[ \sum_{j \geq 0} \sum_{k_1 \geq k_2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, f_2), g \rangle = \sum_{j \geq 0} \sum_{k_1 \geq k_2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, k_1, f_2, k_2), g \rangle. \]

Furthermore, we divide the sum as follows.

\[ \sum_{j \geq 0} \sum_{k_1 \geq k_2} \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, k_1, f_2, k_2), g \rangle \]

\[ = \left( \sum_{j \geq 0} \sum_{k_1 \geq k_2} \right) \sum_{\nu \in \mathbb{Z}^n} \langle T_{\sigma_1, k, \nu}(f_1, k_1, f_2, k_2), g \rangle = A_1 + A_2. \]
Estimate for $A_1$. Since $\text{supp} \hat{g}_\ell \subset \{2^{\ell-1} \leq |\xi| \leq 2^{\ell+1}, \ell \geq 1$, and $\text{supp} F[T_{j,k,\nu}(f_1,k_1,f_2,k_2)] \subset \{|\xi| \leq 2^{j+6}\}, \ j \geq k_1 - 3, \ k_1 \geq k_2$ (see the argument around (3.3)), it follows that $\langle T_{j,k,\nu}(f_1,k_1,f_2,k_2), g_\ell \rangle = 0$ if $\ell \geq j + 7$. In addition, we see that if $\text{supp} \varphi(\cdot - \nu_2) \cap \text{supp} \psi_{k_2} = \emptyset$, then $\sigma_{j,k,\nu} = 0$, and consequently $\langle T_{j,k,\nu}(f_1,k_1,f_2,k_2), g_\ell \rangle = 0$. From these observations, $A_1$ can be written as

$$A_1 = \sum_{j \geq k_1 - 3} \sum_{\ell \geq j + 6} \sum_{k_1 \geq k_2} \langle T_{j,k,\nu}(f_1,k_1,f_2,k_2), g_\ell \rangle,$$

where

$$(4.1) \quad \Lambda_{k_2} = \{\nu_2 \in \mathbb{Z}^n : \text{supp} \varphi(\cdot - \nu_2) \cap \text{supp} \psi_{k_2} \neq \emptyset\}.$$

Note that the number of elements of $\Lambda_{k_2}$ satisfies $|\Lambda_{k_2}| \lesssim 2^{k_2n}$. Using Lemma 3.3 with $r = p'$ and $N \geq 1$, we obtain

$$|A_1| \leq \sum_{j \geq k_1 - 3} \sum_{\ell \geq j + 6} \sum_{k_1 \geq k_2} \langle T_{j,k,\nu}(f_1,k_1,f_2,k_2), g_\ell \rangle \|f_1,k_1\|_{L^2} \|f_2,k_2\|_{L^2} \|g_\ell\|_{L^{p'}}$$

$$\lesssim \sum_{j \geq k_1 - 3} \sum_{\ell \geq j + 6} \sum_{k_1 \geq k_2} 2^{k_1m_1 + k_2m_2 - jN} |\Lambda_{k_2}|^{1/2} \|f_1,k_1\|_{L^2} \|f_2,k_2\|_{L^2} \|g_\ell\|_{L^{p'}} \|
u_2\|^2 \lesssim \sum_{j \geq k_1 - 3} \sum_{\ell \geq j + 6} \sum_{k_1 \geq k_2} 2^{k_1m_1 + k_2m_2 + k_2n/2 - jN} \|f_1,k_1\|_{L^2} \|f_2,k_2\|_{L^2} \|g_\ell\|_{L^{p'}}.$$

By our assumptions $m_1, m_2 < 0$ and $m_1 + m_2 = -n/2$, it follows that $k_1m_1 + k_2m_2 + k_2n/2 \leq k_2m_1 + k_2m_2 + k_2n/2 = 0$ for $k_1 \geq k_2$. Hence, since $\sum_{k_1 \geq 0} \|\psi_{k_1}(\xi)\|^2 \lesssim 1$, the last quantity in (4.2) is estimated by

$$\lesssim \sum_{j \geq k_1 - 3} \sum_{\ell \geq j + 6} \sum_{k_1 \geq k_2} 2^{-jN} \|f_1,k_1\|_{L^2} \|f_2,k_2\|_{L^2} \|g_\ell\|_{L^{p'}} \|
u_2\|^2 \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \|g\|_{B^0_{p',\infty}},$$

which gives the desired result.

Estimate for $A_2$. We divide $A_2$ as follows.

$$A_2 = \left( \sum_{k_1 - 3 \leq k_2 \leq k_1} + \sum_{k_2 < k_1 - 3} \right) \sum_{j < k_1 - 3} \sum_{\ell \geq 0} \sum_{\nu \in \mathbb{Z}^n} \langle T_{j,k,\nu}(f_1,k_1,f_2,k_2), g_\ell \rangle = A_{2,1} + A_{2,2}.$$

We first consider the estimate for $A_{2,1}$. Since $\text{supp} F[T_{j,k,\nu}(f_1,k_1,f_2,k_2)] \subset \{|\xi| \leq 2^{k_1 + 3}\}, \ j < k_1 - 3, \ k_2 \leq k_1,$
it follows that $\langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle = 0$ if $\ell \geq k_1 + 4$. Furthermore, by (3.3) and the fact $\text{supp} \, \tilde{\varphi} \subset \text{supp} \, \psi_\ell$, we see that if $(\nu_1 + \nu_2 + [-2^{j+2}, 2^{j+2}]^n) \cap \text{supp} \, \psi_\ell = \emptyset$, then $\langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle = 0$. Combining these observations, we see that $A_{2,1}$ can be written as

$$A_{2,1} = \sum_{j < k_1 - 3} \sum_{\ell < k_1 + 3} \sum_{\mu : \nu_1 + \nu_2 = \mu} \langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle,$$

where

$$A_{j,\ell} = \{ \mu \in \mathbb{Z}^n : (\mu + [-2^{j+2}, 2^{j+2}]^n) \cap \text{supp} \, \psi_\ell \neq \emptyset \}.$$

The number of elements of $A_{j,\ell}$ can be estimated by $|A_{j,\ell}| \lesssim 2^{(j+\ell) n}$. Hence, it follows from Lemma 3.3 with $r = p'$ and $N > n/2$ that

$$|A_{2,1}| \lesssim \sum_{j < k_1 - 3} \sum_{\ell < k_1 + 3} \sum_{\mu : \nu_1 + \nu_2 = \mu} \langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle \lesssim \sum_{j < k_1 - 3} \sum_{\ell < k_1 + 3} 2^{k_1 m_1 + k_2 m_2 - j n} \left\| f_{1,k_1} \right\| L^2 \left\| f_{2,k_2} \right\| L^2 \left\| g_\ell \right\| L^{p'}$$

$$\lesssim \sum_{k_1 - 3 \leq k_2 \leq k_1} 2^{k_1 m_1 + k_2 m_2 - j n} 2^{(j+\ell) n} \left\| f_{1,k_1} \right\| L^2 \left\| f_{2,k_2} \right\| L^2 \left\| g \right\| B^{0}_{p',\infty}.$$

Since we can write $k_2 = k_1 + \tilde{k}$ with $\tilde{k} = -3, -2, -1, 0$ if $k_1 - 3 \leq k_2 \leq k_1$, the last sum can be written as

$$\sum_{k_1 \geq 0} 2^{-k_1 m_2 + (k_1 + \tilde{k}) m_2} \left\| f_{1,k_1} \right\| L^2 \left\| f_{2,k_1 + \tilde{k}} \right\| L^2$$

$$= \sum_{k_1 \geq 0} 2^{-k_1 m_2} \left\| f_{1,k_1} \right\| L^2 \left\| f_{2,k_1} \right\| L^2.$$

Hence, by Schwarz’s inequality, the right hand side of the above is estimated by

$$\sum_{-3 \leq k \leq 0} 2^{k_1 m_2} \left( \sum_{k_1 \geq 0} \left\| f_{1,k_1} \right\| L^2 \right)^{1/2} \left( \sum_{k_1 \geq 0} \left\| f_{2,k_1} \right\| L^2 \right)^{1/2} \lesssim \left\| f_1 \right\| L^2 \left\| f_2 \right\| L^2,$$

which implies the desired estimate.

Next, we consider the estimate for $A_{2,2}$. Since

$$\text{supp} \, \varphi_\infty \left( f_{1,k_1}, f_{2,k_2} \right) \subset \left\{ 2^{k_1 - 2} \leq |\zeta| \leq 2^{k_1 + 2} \right\}, \quad j < k_1 - 3, \quad k_2 < k_1 - 3,$$

it follows that $\langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle = 0$ if $\ell \leq k_1 - 3$ or $k_1 + 3 \leq \ell$. As before, if $\text{supp} \, \varphi_\infty \cap \text{supp} \, \psi_{k_2} = \emptyset$, then $\langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle = 0$. Therefore, we obtain

$$A_{2,2} = \sum_{j < k_1 - 3} \sum_{k_2 < k_1 - 3} \sum_{|\ell - k_1| \leq 2} \langle T_{\sigma_{j,k},\nu}(f_{1,k_1}, f_{2,k_2}), g_\ell \rangle.$$
where $\Lambda_{k_2}$ is the same as in (4.1). By applying Lemma 3.3 it follows that

$$|A_{2,2}| \leq \sum_{k_2 < k_1 - 3, |\ell - k_1| \leq 2} \sum_{\nu_1 \in \mathbb{Z}^n} \sum_{\nu_2 \in \Lambda_{k_2}} |\langle T_{\sigma_{\nu,\nu}}(f_{1,k_1}, f_{2,k_2}), g \rangle|$$

$$\lesssim \sum_{k_2 < k_1 - 3, |\ell - k_1| \leq 2} 2^{k_1} m_1 + k_2 m_2 - j N 2^{k_2} n/2 \|f_{1,k_1}\|_{L^2} \|f_{2,k_2}\|_{L^2} \|g\|_{B^0_{p',\infty}}$$

(4.3)

$$\lesssim \sum_{k_1, k_2 \geq 0} 2^{m_1(k_1 - k_2)} \|f_{1,k_1}\|_{L^2} \|f_{2,k_2}\|_{L^2} \|g\|_{B^0_{p',\infty}}$$

This completes the proof of Theorem 1.1.

5. PROOF OF THEOREM 1.2

In this section, we shall prove Theorem 1.2. The “if” part follows from Theorem 1.1 (1.2) and the embedding $B^0_{p,1} \hookrightarrow B^0_{p,q}$, $1 \leq q \leq \infty$. Thus, let us consider the “only if” part.

We assume that the $L^2 \times L^2 \rightarrow B^0_{p,q}$ boundedness of all $T_{\sigma}$ with $\sigma \in BS^{-n/2}_{0,0}$ holds throughout the rest of this section. We also take a function $\psi_0 \in S(\mathbb{R}^n)$ satisfying $\psi_0 = 1$ on $\{ |\xi| \leq 2^{1/4} \}$ and supp $\psi_0 \subset \{ |\xi| \leq 2^{3/4} \}$, and set $\psi_0(2^{-\ell}\xi) = \psi_0(2^{-\ell+1}\xi)$. Let $\ell \geq 1$. Then $\{ \psi_0 \}_{\ell \geq 0}$ satisfies (2.1) with the support conditions replaced by

$$\text{supp} \psi_0 \subset \{ |\xi| \leq 2^{3/4} \}, \quad \text{supp} \psi_0 \subset \{ 2^{\ell-3/4} \leq |\xi| \leq 2^{\ell+3/4} \}, \quad \ell \geq 1.$$

Combining this with the condition $\sum_{\ell \geq 0} \psi_0 = 1$, we see that

$$\psi_0 = 1 \quad \text{on} \quad \{ |\xi| \leq 2^{1/4} \}, \quad \psi_0 = 1 \quad \text{on} \quad \{ 2^{\ell-1/4} \leq |\xi| \leq 2^{\ell+1/4} \}, \quad \ell \geq 1,$$

and use this $\{ \psi_0 \}_{\ell \geq 0}$ as the partition of unity in the definition of Besov spaces.

The necessity of the condition $1 \leq p \leq 2$. First, we shall prove that $p \geq 1$. Let $\sigma \in S((\mathbb{R}^n)^2)$ satisfy $\sigma(\xi_1, \xi_2) = 1$ on $|\langle \xi_1, \xi_2 \rangle| \leq 1$, and obviously $\sigma \in BS^{-n/2}_{0,0}$. For $\varphi \in S(\mathbb{R}^n)$ satisfying supp $\varphi \subset \{ |\xi| \leq 1 \}$, we set

$$\widehat{f_{i,j}}(\xi) = 2^{jn/2} \varphi(2^j \xi), \quad i = 1, 2, \quad j \geq 1,$$

and note that $\|f_{i,j}\|_{L^2} \approx 1$. Since $|\langle \xi_1, \xi_2 \rangle| \leq 2^{-j+1/2} \leq 1$ for $|\xi_1|, |\xi_2| \leq 2^{-j}$ and $j \geq 1$, it follows that $\sigma(\xi_1, \xi_2) \varphi(2^j \xi_1) \varphi(2^j \xi_2) = \varphi(2^j \xi_1) \varphi(2^j \xi_2)$, and consequently

$$T_{\sigma}(f_{1,j}, f_{2,j})(x) = 2^{-jn}(\mathcal{F}^{-1} \varphi(2^{-j}x))^2.$$
By a change of variables, \(|T_\sigma(f_{1,j}, f_{2,j})|_{L^p} \approx 2^n(1/p - 1)|. Furthermore, we have 
\[ \text{supp} \mathcal{F}[T_\sigma(f_{1,j}, f_{2,j})] = \text{supp}[\varphi(2^j \cdot) \ast \varphi(2^j \cdot)] \subset \{|\xi| \leq 1\}. \]
Hence, it follows from (5.1) and (5.2) that \(\psi(D)T_\sigma(f_{1,j}, f_{2,j})\) equals \(T_\sigma(f_{1,j}, f_{2,j})\) if \(\ell = 0\), and 0 otherwise. From this, we obtain 
\[ \|T_\sigma(f_{1,j}, f_{2,j})|_{B^0_{p,q}} = \|T_\sigma(f_{1,j}, f_{2,j})|_{L^p} \approx 2^{jn(1/p - 1)}. \]
Therefore, by our assumption, the boundedness of \(T_\sigma\) gives 
\[ 2^{jn(1/p - 1)} \approx \|T_\sigma(f_{1,j}, f_{2,j})|_{B^0_{p,q}} \lesssim \|f_{1,j}\|_{L^2}\|f_{2,j}\|_{L^2} \approx 1. \]
The arbitrariness of \(j \geq 1\) implies that \(1/p - 1 \leq 0\), namely \(p \geq 1\).
Next we shall prove that \(p \leq 2\). Our argument is based on the proof of [10] Proposition 5.1. Let \(\psi, \varphi, \tilde{\psi}, \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^n)\) be such that 
\[ \text{supp} \tilde{\psi} \subset \{2^{-1/8} \leq |\xi_1| \leq 2^{1/8}\}, \quad \text{supp} \varphi \subset \{|\xi_2| \leq 2^{-j_0}\}, \]
\[ \text{supp} \tilde{\psi} \subset \{2^{-1/4} \leq |\xi_1| \leq 2^{1/4}\}, \quad \tilde{\psi} = 1 \text{ on supp } \psi, \]
\[ \text{supp} \tilde{\varphi} \subset \{|\xi_2| \leq 2^{-j_0+1}\}, \quad \tilde{\varphi} = 1 \text{ on supp } \varphi, \]
where \(j_0\) is a positive integer satisfying 
\[ 2^{-1/4} \leq 2^{-1/8} - 2^{-j_0} \quad \text{and} \quad 2^{1/8} + 2^{-j_0} \leq 2^{1/4}. \]
We set 
\[ \sigma(\xi_1, \xi_2) = \sum_{k \geq 1} 2^{-kn/2} \tilde{\psi}(2^{-k} \xi_1) \tilde{\varphi}(2^{-k} \xi_2), \]
\[ \tilde{f}_{1,j}(\xi_1) = 2^{-jn/2} \psi(2^{-j} \xi_1), \quad \tilde{f}_{2,j}(\xi_2) = 2^{-jn/2} \varphi(2^{-j} \xi_2), \quad j \geq 1. \]
It is not difficult to show that \(\sigma \in BS_{0,0}^{n/2}\) (since \(1 + |\xi_1| + |\xi_2| \approx 2^k\) for \(\xi_1 \in \text{supp} \tilde{\psi}(2^{-k} \cdot)\) and \(\xi_2 \in \text{supp} \tilde{\varphi}(2^{-k} \cdot)\) and \(\|f_{1,j}\|_{L^2}, \|f_{2,j}\|_{L^2} \approx 1\). Now, since \(\psi(2^{-k} \xi_1) \varphi(2^{-j} \xi_1) \varphi(2^{-j} \xi_2)\) equals \(\psi(2^{-j} \xi_1) \varphi(2^{-j} \xi_2)\) if \(k = j\), and 0 otherwise, it follows that 
\[ T_\sigma(f_{1,j}, f_{2,j})(x) = 2^{jn/2}[\mathcal{F}^{-1}\psi](2^j x)[\mathcal{F}^{-1}\varphi](2^j x). \]
From this, we have \(|T_\sigma(f_{1,j}, f_{2,j})|_{L^p} \approx 2^{jn(1/2 - 1/p)}\). Moreover, by the support conditions of \(\psi, \varphi\) and (5.3), we see that \(2^{-1/4} \leq |\xi_1 + \xi_2| \leq 2^{1/4}\) for \(\xi_1 \in \text{supp} \psi(2^{-j} \cdot)\) and \(\xi_2 \in \text{supp} \varphi(2^{-j} \cdot)\), and consequently 
\[ \text{supp} \mathcal{F}[T_\sigma(f_{1,j}, f_{2,j})] = \text{supp}[\psi(2^{-j} \cdot) \ast \varphi(2^{-j} \cdot)] \subset \{2^{-j/4} \leq |\xi| \leq 2^{j+1/4}\}. \]
Hence, it follows from (5.1) and (5.2) that \(\psi(D)T_\sigma(f_{1,j}, f_{2,j})\) equals \(T_\sigma(f_{1,j}, f_{2,j})\) if \(\ell = j\), and 0 otherwise. This yields that 
\[ \|T_\sigma(f_{1,j}, f_{2,j})|_{B^0_{p,q}} = \|T_\sigma(f_{1,j}, f_{2,j})|_{L^p} \approx 2^{jn(1/2 - 1/p)}\].
Therefore, by our assumption, the boundedness of \(T_\sigma\) gives 
\[ 2^{jn(1/2 - 1/p)} \approx \|T_\sigma(f_{1,j}, f_{2,j})|_{B^0_{p,q}} \lesssim \|f_{1,j}\|_{L^2}\|f_{2,j}\|_{L^2} \approx 1. \]
The arbitrariness of \(j \geq 1\) implies that \(1/2 - 1/p \leq 0\), namely \(p \leq 2\).

The necessity of the condition \(1 \leq q \leq \infty\). Finally, we shall prove that \(1 \leq q \leq \infty\). The basic idea of this part goes back to the proofs of [13] Lemma 6.3] and [13] Theorem 1.4]. Since there is nothing to prove if \(q = \infty\), we may assume that
0 < q < ∞. By the closed graph theorem, our assumption implies that there exists a positive integer \( N \) such that
\[
\| T_\sigma \|_{L^2 \times L^2 \to B^0_{p,q}} \lesssim \max_{|\alpha|,|\beta_1|,|\beta_2| \leq N} \| (1 + |\xi_1| + |\xi_2|)^{n/2} \partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma(x,\xi_1,\xi_2) \|_{L^{\infty}_{x,\xi_1,\xi_2}}
\]
for all \( \sigma \in B^{-n/2}_{0,0} \), where \( \| T_\sigma \|_{L^2 \times L^2 \to B^0_{p,q}} \) denotes the operator norm of \( T_\sigma \) (see [H Lemma 2.6]).

We use functions \( \varphi, \tilde{\varphi} \in \mathcal{S}(\mathbb{R}^n) \) such that
\[
supp \varphi \subset [-1/4,1/4]^n, \quad |\mathcal{F}^{-1} \varphi| \geq 1 \text{ on } [-1,1]^n,
\]
\[
supp \tilde{\varphi} \subset [-1/2,1/2]^n, \quad \tilde{\varphi} = 1 \text{ on } supp \varphi.
\]
Let \( \{ r_\nu(\omega) \}_{\nu \in \mathbb{Z}^n}, \omega \in [0,1]^n \), be a sequence of Rademacher functions (see [5 Appendix C]) enumerated in such a way that their index set is \( \mathbb{Z}^n \). For \( \omega \in [0,1]^n \) and \( \epsilon > 0 \), we set
\[
\sigma_\omega(\xi_1,\xi_2) = \sum_{k \geq k_0} \sum_{\nu \in \Lambda_k} \sum_{\nu_1+\nu_2=\nu} r_\nu(\omega)(1 + |\nu_1| + |\nu_2|)^{-n/2} \tilde{\varphi}(\xi_1 - \nu_1) \tilde{\varphi}(\xi_2 - \nu_2),
\]
\[
\tilde{f}_i(\xi_i) = \sum_{\mu_i \in \mathbb{Z}^n} \langle \mu_i \rangle^{-n/2}(\log(|\mu_i|))^{-(1+\epsilon)/2} \varphi(\xi_i - \mu_i), \quad i = 1,2,
\]
where \( \langle \mu_i \rangle = e + |\mu_i|, k_0 \) is a positive integer satisfying
\[
2^{k-1/4} \leq 2^{k-1/8} - \sqrt{n}/2, \quad 2^{k+1/8} + \sqrt{n}/2 \leq 2^{k+1/4}, \quad k \geq k_0,
\]
and
\[
\Lambda_k = \{ \nu \in \mathbb{Z}^n : 2^{k-1/8} \leq |\nu| \leq 2^{k+1/8} \}.
\]
The number of elements of \( \Lambda_k \) can be estimated by \( |\Lambda_k| \approx 2^{kn} \). It is easy to check that
\[
\max_{|\alpha|,|\beta_1|,|\beta_2| \leq N} \| (1 + |\xi_1| + |\xi_2|)^{n/2} \partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma_\omega(x,\xi_1,\xi_2) \|_{L^{\infty}_{x,\xi_1,\xi_2}} \lesssim 1,
\]
but it should be emphasized that the implicit constant in this inequality is independent of \( \omega \in [0,1]^n \). Using the facts that \( \{ \langle \mu_i \rangle^{-n/2}(\log(|\mu_i|))^{-(1+\epsilon)/2} \} \) belongs to \( \ell^2(\mathbb{Z}^n) \) and the supports of \( \varphi(\cdot - \mu_i), \mu_i \in \mathbb{Z}^n \), are mutually disjoint, we also see that \( \| f_i \|_{L^2} \lesssim 1, i = 1,2 \). Thus, by (5.4) and (5.5),
\[
\| T_{\sigma_\omega}(f_1,f_2) \|_{B^0_{p,q}} \lesssim \| f_1 \|_{L^2} \| f_2 \|_{L^2} \lesssim 1, \quad \omega \in [0,1]^n.
\]
Now, since \( \varphi(\xi_i - \nu_i) \tilde{\varphi}(\xi_i - \mu_i) \) is equal to \( \varphi(\xi_i - \nu_i) \) if \( \nu_i = \mu_i \), and 0 otherwise, \( i = 1,2 \), it follows that
\[
T_{\sigma_\omega}(f_1,f_2)(x) = \sum_{k \geq k_0} \sum_{\nu \in \Lambda_k} \sum_{\nu_1+\nu_2=\nu} r_\nu(\omega)(1 + |\nu_1| + |\nu_2|)^{-n/2} e^{i(\nu_1+\nu_2) \cdot x} (\mathcal{F}^{-1} \varphi(x))^2
\]
\[
\quad \times \langle \nu \rangle^{-n/2}(\log(|\nu|))^{-(1+\epsilon)/2}(\nu_1)^{-n/2}(\log(\nu_1))^{-(1+\epsilon)/2}
\]
\[
\quad \times \langle \nu \rangle^{-n/2}(\log(|\nu|))^{-(1+\epsilon)/2}(\nu_2)^{-n/2}(\log(\nu_2))^{-(1+\epsilon)/2}
\]
with \( \Phi = \mathcal{F}^{-1} \varphi \) and
\[
d_\nu = \sum_{\mu \in \mathbb{Z}^n} (1 + |\nu - \mu| + |\mu|)^{-n/2}
\]
\[
\quad \times \langle \nu - \mu \rangle^{-n/2}(\log(|\nu - \mu|))^{-(1+\epsilon)/2}(\mu)^{-n/2}(\log(\mu))^{-(1+\epsilon)/2}.
\]
We note that \( \text{supp} \varphi \ast \varphi \subset [-1/2, 1/2]^n \) and 
\[
\mathcal{F}[T_{\sigma_\nu}(f_1, f_2)](\zeta) = (2\pi)^{-n} \sum_{k \geq k_0} \sum_{\nu \in \Lambda_k} r_\nu(\omega)d\nu[\varphi \ast \varphi](\zeta - \nu).
\]
By (5.3), if \( \zeta - \nu \in [-1/2, 1/2]^n \), \( \nu \in \Lambda_k \) and \( k \geq k_0 \), then \( 2^{k - 1/4} \leq |\zeta| \leq 2^{k + 1/4} \). Hence, for \( k \geq k_0 \) and \( \nu \in \Lambda_k \), it follows from (5.1) and (5.2) that \( \psi_\nu(\zeta)[\varphi \ast \varphi](\zeta - \nu) \) is equal to \( |\varphi \ast \varphi| (\zeta - \nu) \) if \( k = \ell \), and 0 otherwise. This yields that 
\[
\psi_\ell(D)T_{\sigma_\nu}(f_1, f_2)(x) = \sum_{\nu \in \Lambda_\ell} r_\nu(\omega)d\nu e^{iu_\nu x} \Phi(x)^2, \quad \ell \geq k_0.
\]
Raising (5.7) to the \( q \)-th power and integrating over \( \omega \in [0, 1]^n \), we have 
\[
1 \geq \int_{[0,1]^n} \|T_{\sigma_\nu}(f_1, f_2)\|_{B_{p,q}}^q d\omega = \sum_{\ell \geq 0} \int_{[0,1]^n} \|\psi_\ell(D)T_{\sigma_\nu}(f_1, f_2)\|_{p,\nu}^q d\omega 
\]
\[
\geq \sum_{\ell \geq k_0} \left( \int_{[0,1]^n} \left( \sum_{\nu \in \Lambda_\ell} r_\nu(\omega)d\nu e^{iu_\nu x} \Phi(x)^2 \right)^{\min\{p,q\}} \right)^{q/p} d\omega 
\]
\[
\geq \sum_{\ell \geq k_0} \left( \int_{[-1,1]^n} \left( \sum_{\nu \in \Lambda_\ell} r_\nu(\omega)d\nu e^{iu_\nu x} \right)^{\min\{p,q\}} \right)^{q/p} d\omega,
\]
where we used Hölder’s inequality when \( p \leq q \), or Minkowski's inequality when \( p \geq q \) in the second inequality, and the condition \( |\Phi| = |\mathcal{F}^{-1}\varphi| \geq 1 \) on \([-1,1]^n\) in the third inequality. Furthermore, by Khintchine’s inequality (see, e.g., [5] Appendix C), the last quantity is equivalent to 
\[
\sum_{\ell \geq k_0} \left( \int_{[-1,1]^n} \left( \sum_{\nu \in \Lambda_\ell} |d_\nu e^{iu_\nu x}|^2 \right)^{q/p} d\omega \right)^{q/p} = 2^{nq/p} \sum_{\ell \geq k_0} \left( \sum_{\nu \in \Lambda_\ell} |d_\nu|^2 \right)^{q/2}.
\]
Since \( |\nu - \mu| \approx |\nu| \) for \( |\mu| \leq |\nu|/2 \), we have 
\[
|d_\nu| \gtrsim \sum_{|\mu| \leq |\nu|/2} (\nu)^{-3n/2}(\log |\nu|)^{-(1+\epsilon)} \approx (\nu)^{-n/2}(\log |\nu|)^{-(1+\epsilon)} \approx 2^{-\ell n/2} \ell^{-(1+\epsilon)}
\]
for \( \nu \in \Lambda_\ell \), and consequently 
\[
\left( \sum_{\nu \in \Lambda_\ell} |d_\nu|^2 \right)^{q/2} \gtrsim \left( \sum_{\nu \in \Lambda_\ell} 2^{-\ell n} \ell^{-2(1+\epsilon)} \right)^{q/2} \approx \ell^{-q(1+\epsilon)}.
\]
Combining the above estimates, we obtain
\[
\sum_{\ell \geq k_0} \ell^{-q(1+\epsilon)} \lesssim 1,
\]
which is possible only when \( q(1+\epsilon) > 1 \). Therefore, the arbitrariness of \( \epsilon > 0 \) implies that \( q \geq 1 \). The proof of the “only if” part of Theorem 1.2 is complete.

6. The amalgam space \((L^2, \ell^1)\) and Wiener amalgam space \(W_{1,2}\)

In this section, we shall prove Proposition 1.3. Let \( 1 \leq p \leq 2 \). It is easy to see that the embedding \( B_{0,0}^p \hookrightarrow (L^2, \ell^1) \) does not hold. In fact, it follows from the embedding \((L^2, \ell^1) \hookrightarrow L^q, 1 \leq q \leq 2 \), (see Section 2) that if \( B_{p,0}^q \hookrightarrow (L^2, \ell^1) \), then \( B_{p,0}^q \hookrightarrow L^q \) for all \( 1 \leq q \leq 2 \). However, since \( B_{p,0}^q \hookrightarrow L^q \) if and only if \( p = q \) (see, e.g., [10] Corollary 2.2.4/2)), this is a contradiction. Though we can directly
prove that the embedding \((L^2, \ell^1) \hookrightarrow B^0_{p,1}\) does not hold, we shall show it by using the characterization of \((L^2, \ell^1)\), Proposition [6.2] below. To do this, we recall the definitions of modulation spaces and Wiener amalgam spaces based on Feichtinger \[4\] and Gröchenig \[8\].

Fix a function \(\phi \in S(\mathbb{R}^n) \setminus \{0\}\) (called the window function). Then the short-time Fourier transform \(V_\phi f\) of \(f \in S'(\mathbb{R}^n)\) with respect to \(\phi\) is defined by

\[
V_\phi f(x, \xi) = \int_{\mathbb{R}^n} f(t) \overline{\phi(t-x)} e^{-i t \cdot \xi} dt, \quad x, \xi \in \mathbb{R}^n,
\]

where the integral is understood in the distributional sense. For \(1 \leq p, q \leq \infty\), the modulation space \(M_{p,q}(\mathbb{R}^n)\) consists of all \(f \in S'(\mathbb{R}^n)\) such that

\[
\|f\|_{M_{p,q}} = \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_\phi f(x, \xi)|^q \, dx \right)^{p/q} \, d\xi \right\}^{1/q} < \infty
\]

with usual modification when \(p = \infty\) or \(q = \infty\). On the other hand, the Wiener amalgam space \(W_{p,q}(\mathbb{R}^n)\) consists of all \(f \in S'(\mathbb{R}^n)\) such that

\[
\|f\|_{W_{p,q}} = \left\{ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_\phi f(x, \xi)|^q \, dx \right)^{p/q} \, d\xi \right\}^{1/p} < \infty.
\]

These definitions are independent of the choice of the window function \(\phi \in S(\mathbb{R}^n) \setminus \{0\}\), that is, different window functions yield equivalent norms. Since \(|V_\phi f(x, \xi)| = (2\pi)^n |\mathcal{F}^{-1} f(-\xi, x)|\), we see that

\[
(6.1) \quad \|f\|_{W_{p,q}} \approx \|\mathcal{F}^{-1} f\|_{M_{q,p}}.
\]

It is also known that if \(\varphi \in S(\mathbb{R}^n)\) has a compact support and if there exists a constant \(C > 0\) such that \(\sum_{\nu \in \mathbb{Z}^n} |\varphi(\xi - \nu)| \geq C\) for all \(\xi \in \mathbb{R}^n\), then

\[
(6.2) \quad \|f\|_{M_{p,q}} \approx \left( \sum_{\nu \in \mathbb{Z}^n} \|\varphi(D - \nu) f\|^2_{L^p} \right)^{1/2}.
\]

(see, e.g., [18 Theorem 3]).

Although the following may be well known to many people, we shall give a proof for the reader’s convenience.

**Proposition 6.1.** The amalgam space \((L^2, \ell^1)(\mathbb{R}^n)\) coincides with the Wiener amalgam space \(W_{1,2}(\mathbb{R}^n)\).

**Proof.** Let \(\varphi\) be a nonnegative function in \(S(\mathbb{R}^n)\) such that \(\text{supp} \varphi\) is compact and \(\varphi \geq 1\) on \([-1/2, 1/2]^n\). Then, we have the norm equivalence

\[
\|f\|_{(L^2, \ell^1)} \approx \sum_{\nu \in \mathbb{Z}^n} \|\varphi(\cdot - \nu) f\|_{L^2}
\]

(see, e.g., [11 Lemma 2.1]). Since \(\sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu) \geq 1\) for all \(\xi \in \mathbb{R}^n\), by Plancherel’s theorem, \[6.2\] and \[6.1\],

\[
\sum_{\nu \in \mathbb{Z}^n} \|\varphi(\cdot - \nu) f\|_{L^2} \approx \sum_{\nu \in \mathbb{Z}^n} \|\mathcal{F}^{-1}[\varphi(\cdot - \nu) f]\|_{L^2}
\]
\[
= \sum_{\nu \in \mathbb{Z}^n} \| \varphi(D - \nu) [\mathcal{F}^{-1} f] \|_{L^2} \approx \| \mathcal{F}^{-1} f \|_{M_{2,1}} \approx \| f \|_{W_{1,2}}.
\]

The proof is complete.

In the rest of this section, we shall prove that the embedding \((L^2, \ell^1) \hookrightarrow B_{p,1}^0\) does not hold, where \(1 \leq p \leq 2\). Let \(\phi, \varphi\) be nonnegative functions in \(\mathcal{S}(\mathbb{R}^n)\) such that \(\text{supp } \phi \subset [-1/4, 1/4]^n\), \(\text{supp } \varphi \subset [-3/4, 3/4]^n\) and \(\varphi = 1\) on \([-1/2, 1/2]^n\). We take a positive integer \(N_0\) satisfying

\[
2^{k-1/4} \leq 2^k - \sqrt{n}/4, \quad 2^k + \sqrt{n}/4 \leq 2^{k+1/4}, \quad k \geq N_0,
\]

and set

\[
f_N(x) = \sum_{k=N_0}^N e^{i2^k e_1 \cdot x} \mathcal{F}^{-1} \phi(x), \quad N \geq N_0,
\]

where \(e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n\).

We first consider the Besov space norm of \(f_N\). The support condition of \(\phi\) and \((6.3)\) give

\[
\text{supp } \phi(-2^k e_1) \subset \{ |\xi| - 2^k e_1 | \leq \sqrt{n}/4 \} \subset \{ 2^{k-1/4} \leq |\xi| \leq 2^{k+1/4} \}, \quad k \geq N_0.
\]

Then, since \(\mathcal{F}_N(\xi) = \sum_{k=N_0}^N \phi(\xi - 2^k e_1)\), we see that \(\psi_\ell(D)f_N(x)\) is equal to \(e^{i2^k e_1 \cdot x} \mathcal{F}^{-1} \phi(x)\) if \(N_0 \leq \ell \leq N\), and 0 otherwise, where \(\{ \psi_\ell \}\) is the same as in the beginning of Section 5.1 and we used \((5.1)\) and \((5.2)\). Hence,

\[
\| f_N \|_{B_{p,1}^0} = \sum_{\ell=N_0}^N \| e^{i2^\ell e_1 \cdot x} \mathcal{F}^{-1} \phi \|_{L^p} \approx N
\]

for sufficiently large \(N\).

We next consider the amalgam space norm of \(f_N\). By Proposition \((6.1)\) it is sufficient to estimate \(\| f_N \|_{W_{1,2}}\). Moreover, since \(\varphi\) is compact and \(\sum_{\nu \in \mathbb{Z}^n} \varphi(\xi - \nu) \geq 1\), we have by \((6.2)\)

\[
\| f_N \|_{(L^2, \ell^1)} \approx \left\| \left( \sum_{\nu \in \mathbb{Z}^n} |\varphi(D - \nu) f_N|^2 \right)^{1/2} \right\|_{L^1}.
\]

Noting that \(\text{supp } \phi(-2^k e_1) \subset 2^k e_1 + [-1/4, 1/4]^n\), \(\text{supp } \varphi(-\nu) \subset [-3/4, 3/4]^n\) and \(\varphi(-\nu) = 1\) on \(\nu + [-1/4, 1/4]^n\), we see that \(\varphi(D - \nu)f_N(x)\) is equal to \(e^{i2^k e_1 \cdot x} \mathcal{F}^{-1} \phi(x)\) if \(\nu = 2^k e_1\) and \(N_0 \leq k \leq N\), and 0 otherwise. Therefore,

\[
\| f_N \|_{(L^2, \ell^1)} \approx \left\| \left( \sum_{k=N_0}^N |e^{i2^k e_1 \cdot x} \mathcal{F}^{-1} \phi|^2 \right)^{1/2} \right\|_{L^1} \approx N^{1/2}
\]

for sufficiently large \(N\).

It follows from \((6.4)\) and \((6.5)\) that if \((L^2, \ell^1) \hookrightarrow B_{p,1}^0\), then

\[
N \approx \| f_N \|_{B_{p,1}^0} \leq \| f_N \|_{(L^2, \ell^1)} \approx N^{1/2}
\]

for all sufficiently large \(N\). However, this is a contradiction. The proof of Proposition \((1.3)\) is complete.
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