ON THE FUNDAMENTAL GROUP OF COMPLETE MANIFOLDS WITH ALMOST EUCLIDEAN VOLUME GROWTH

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Abstract. It is proved that the fundamental group of a complete Riemannian manifold with nonnegative Ricci curvature and certain volume growth conditions is trivial or finite.

1. Introduction

Throughout the paper $M$ denotes a complete noncompact Riemannian $n$-manifold with nonnegative Ricci curvature. Let $V_p(r)$ be the volume of the metric ball $B_p(r)$ origin at $p$ with radius $r$ in $M$. By Bishop-Gromov volume comparison, \( \frac{V_p(r)}{\omega_n r^n} \) is a decreasing function, where $\omega_n$ is the volume of unit ball in $\mathbb{R}^n$. So the limit is existent as $r$ goes to infinite.

Denote the volume growth of $M$ by

$$\alpha_M := \lim_{r \to \infty} \frac{V_p(r)}{\omega_n r^n}.$$ 

The $\alpha_M$ is independent of $p$ and so a global geometric invariant. Moreover, the volume comparison also implies that $0 \leq \alpha_M \leq 1$ and $\alpha_M = 1$ if and only if $M$ is isometric to $\mathbb{R}^n$. We say that $M$ has Euclidean volume growth (or large volume growth) if $\alpha_M > 0$.

The main result of this note is

**Theorem 1.1.** Given $n$, there is a constant $C(n) < 2^n$ such that if an open $n$-manifold $M$ satisfies

1) (1.1) $$\frac{V_p(2r)}{V_p(r)} > C(n)$$\n
for some $p \in M$ and all $r > 0$, then $M$ is simple connected.

2) (1.2) $$\liminf_{r \to \infty} \frac{V_p(2r)}{V_p(r)} > C(n),$$\n
for some $p \in M$, then the fundamental group $\pi_1(M)$ is finite.

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We should note that even though $M$ has Euclidean volume growth, one cannot deduce that $M$ is simple connected. So formula (1.1) holding for all $r > 0$ is important.

Set $\epsilon(n) = n - \log_2 \frac{C(n+x)}{2}$. We see immediately that 2) of Theorem 1.1 implies the following

**Corollary 1.2.** Given $n$, there is a constant $\epsilon(n)$ such that if an open $n$-manifold $M$ satisfies

\[
\lim_{r \to \infty} \frac{V_p(r)}{p^{n-\epsilon}} > 0,
\]

for some $p \in M$, $0 \leq \epsilon < \epsilon(n)$, then $\pi_1(M)$ is finite.

This shows a gap phenomenon for a well-known result of Peter Li [5] and Anderson [1] states that $\pi_1(M)$ is finite provided $M$ has Euclidean volume growth.

On the other hand, Anderson has proved that (see Theorem 1.1 in [1]) under condition (1.3), every finitely generated subgroup of $\pi_1(M)$ is actually of polynomial growth of order $\leq \epsilon < 1$. In [6] Bingye Wu proved that under condition (1.3) $\pi_1(M)$ is finitely generated. But every infinite group of finitely generated has polynomial growth of order at least 1 (I thank the referee for pointing out this fact. See Section 3). So Corollary 1.2 is also a consequence of Anderson and Wu’s results.

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2. A related volume ratio

In this section we prove an estimate on the volume ratio $V_p(2r)/V_p(r)$ related to certain generated elements of $\pi_1(M)$ (Lemma 2.2 below). The main ingredients are Sormani’s uniform cut lemma [5] and some ideas due to Shen [4].

2.1. A uniform estimate. Let $g \in \pi_1(M, p)$ and $\pi : \tilde{M} \to M$ be the universal cover. Following [5], we say that $\gamma$ is a minimal representative geodesic loop (based at $p$) of $g$ if $\gamma = \pi \circ \gamma_\tilde{p}$, where $\gamma_\tilde{p}$ is a minimal geodesic connecting $\tilde{p}$ and $g\tilde{p}$. So $L(\gamma) = d_{\tilde{M}}(\tilde{p}, g\tilde{p})$.

Given a group $G$, we say that $\{g_1, g_2, g_3, \cdots\}$ is an ordered set of independent generators of $G$ if every $g_i$ can not be expressed as the previous generators and their inverses, $g_1, g_1^{-1}, \cdots, g_{i-1}, g_{i-1}^{-1}$.

In [5] Sormani proved the following two lemmas.

**Lemma 2.1.** (halfway lemma) There exists an ordered set of independent generators $\{g_1, g_2, \cdots\}$ of $\pi_1(M, p)$ with minimal representative geodesic loops $\gamma_k$ of length $d_k$ such that

\[
d_M(\gamma_k(0), \gamma_k(d_k/2)) = d_k/2.
\]

In particular, we have a sequence of such generators if $\pi_1(M, p)$ is infinitely generated.

**Lemma 2.2.** (uniform cut lemma) Let $M^n$ $(n \geq 3)$ be a complete manifold with $\text{Ric} \geq 0$. Let $\gamma$ be a noncontractible geodesic loop based at $p_0$ of length $L(\gamma) = 2D$ such that

1. If $\sigma$ based at $p_0$ is a loop homotopic to $\gamma$, then $L(\sigma) \geq 2D$;
2. The $\gamma$ is minimal on $[0, D]$ and $[D, 2D]$.

Then there is a constant $S_n$ depending on $n$ such that if $x \in \partial B_{p_0}(rD)$ where $r \geq 1 + 2S_n$, then

\[
d(x, \gamma(D)) \geq (r - 1)D + 4S_n D.
\]
The main idea of proof of uniform cut lemma is to lift geodesic loop to the universal covering space and research carefully the related excess function. It contains a nice application of Abresch-Gromoll’s estimate on excess function [2]. The above two lemmas allow her to show that Milnor conjecture holds for the manifold with so called small linear diameter growth.

Let \( \gamma \) be a minimal representative geodesic loops based at \( p \) of \( L(\gamma) = d \) satisfying Lemma 2.1. The below estimate is important for our purpose.

**Lemma 2.2.** Let \( \sigma \) be a geodesic issuing from \( p \) such that \( \sigma(t) \) is minimal on \([0, d]\). Then there is a constant \( S(n) \) such that

\[
h \triangleq d_M(\gamma(d/2), \sigma|_{[0,d)}) \geq S(n)d.
\]

**Proof.** We set \( h_1 \triangleq d_M(\gamma(d/2), \sigma|_{[0,d/2)}) \) and \( h_2 \triangleq d_M(\gamma(d/2), \sigma|_{[d/2,d)}) \). By Lemma 2.2, we have

\[
H = d_M(\gamma(d/2), \sigma(d/2)) \geq 2S(n)d,
\]

where \( S(n) \) is a universal constant,

\[
S(n) = \frac{1}{n} \cdot \frac{1}{1 + (\frac{d}{n} - 1)^{n-1}}.
\]

Suppose that \( h_1 = d_M(\gamma(d/2), \sigma(r_0)) \). By the triangle inequality one has

\[
h_1 \geq \frac{d}{2} - r_0
\]

and

\[
h_1 \geq H - \frac{d}{2} - r_0.
\]

Then

\[
h_1 \geq H/2 \geq S(n)d.
\]

We also note that \( d_M(\gamma(d/2), \sigma(d)) \geq d/2 \). So similarly one has

\[
h_2 \geq H/2 \geq S(n)d.
\]

It follows that

\[
h = \min(h_1, h_2) \geq S(n)d.
\]

\[\square\]

2.2. A volume’s ratio. Continuing with notations \( p, d \) in Lemma 2.3 we shall prove

**Lemma 2.4.** We have the following ratio of volume

\[
\frac{V_p(2d)}{V_p(d)} \leq (1 - \frac{2S(n)}{3})^n(2^n - 1) + 1.
\]

Before giving the proof of Lemma 2.4 (following [4]) we introduce some necessary notations. Let \( \Sigma_p \) be a close subset of unit tangent sphere \( S_pM \subset T_pM \). Let \( B_{\Sigma_p}(r) \) be the set of points \( x \in B_p(r) \) such that there exists a minimal geodesic \( \gamma \) from \( p \) to \( x \) with \( \gamma(0) \in \Sigma_p \). We write \( V_{\Sigma_p}(r) \) for the volume of \( B_{\Sigma_p}(r) \).

We denote by \( \Sigma_p(r) \) the set of unit vectors \( v \in S_pM \) such that \( \gamma(t) = \exp_p(tv) \) is minimal on \([0, r]\).

**Proof. of Lemma 2.4** We write \( \gamma = \gamma(d/2) \). Since \( h \leq d/2 \), we have \( B_p(d) \supset B_p(h) \cup B_{\Sigma_p(d)}(d) \). By the definition of \( h \), this gives \( V_p(d) \geq V_p(h) + V_{\Sigma_p(d)}(d) \), i.e.

\[
1 \geq \frac{V_p(h)}{V_p(d)} + \frac{V_{\Sigma_p(d)}(d)}{V_p(d)}.
\]

(2.1)
Claim 1:

(2.2) \[ \frac{V_p(h)}{V_p(d)} \geq \left( \frac{2S(n)}{3} \right)^p. \]

By the Bishop–Gromov comparison theorem, \( \mu_\nu V_p(r) \geq V_p(\mu r) \) for \( \mu \geq 1 \). By Lemma 2.3, we have \( h \geq S(n)d \). So

\[ V_p(h) \geq V_p(S(n)d) \geq \left( \frac{2S(n)}{3} \right)^p V_p(3d/2). \]

Since \( B_r(3d/2) \supset B_p(d) \), we have \( V_p(3d/2) \geq V_p(d) \). Thus we obtain (2.2).

Claim 2:

(2.3) \[ \frac{V_{\Sigma_p(2r/d)}(d)}{V_p(d)} \geq \frac{1}{2^n-1} (\frac{V_p(2d)}{V_p(d)} - 1). \]

Following the observation of Shen (c.f. [4] Lemma 2.4), we see that

\[ B_p(2r) \setminus B_p(r) \subset B_{\Sigma_p(2r)}(d) \setminus B_{\Sigma_p(r)}(d). \]

Then we have

\[ V_p(2r) - V_p(r) \leq V_{\Sigma_p(r)}(2r) - V_{\Sigma_p(r)}(r) \leq (2^n-1)V_{\Sigma_p(r)}(r). \]

The second inequality follows from the generalized volume comparison (Lemma 2.2 of [4]). Thus

\[ \frac{V_{\Sigma_p(r)}(r)}{V_p(r)} \geq \frac{1}{2^n-1} \left( \frac{V_p(2r)}{V_p(r)} - 1 \right). \]

Joining formulas (2.1), (2.2) and (2.3), we establish the lemma. \( \square \)

3. A PROOF OF THEOREM 1.1

We set

\[ C(n) = (1 - \frac{2S(n)}{3})^p(2^n - 1) + 1. \]

If \( \frac{V_p(2r)}{V_p(r)} > C(n) \) for all \( r > 0 \), then there is no nontrivial generator satisfying Lemma 2.4. So \( M \) is simple connected. Thus the first part of Theorem 1.1 is proved.

The proof of second part of Theorem 1.1 is divided into two steps.

Firstly, \( \pi_1(M, p) \) is finitely generated. We argue by contradiction. Assume \( \pi_1(M, p) \) is infinitely generated, then by Lemma 2.3, there is a sequence \( \{d_k\} \), \( d_k \to \infty \) as \( k \to \infty \) satisfying Lemma 2.4, i.e.

\[ \frac{V_p(2d_k)}{V_p(d_k)} \leq C(n), \]

for all \( k \geq 1 \). This contradicts to condition (1.2).

Secondly, condition (1.2) implies that \( V_p(r) \geq C \cdot r^{n-\varepsilon} \) for some \( \varepsilon < 1 \) and sufficiently large \( r \). So by Anderson’s result [1], \( \pi_1(M) \) has polynomial growth of order \( \leq \varepsilon < 1 \).

The form of \( C(n) \) allows us to write \( C(n) \approx 2^{n-\varepsilon}, \varepsilon < 1 \). By condition (1.2), there exists \( r_0 > 0 \), for all \( r \geq r_0 \), one has

\[ V_p(2r) \geq 2^{r-\varepsilon} V_p(r). \]

So

\[ V_p(r_0) \leq (2^{n-\varepsilon})^{-1} V_p(2r_0) \leq \cdots \leq (2^{n-\varepsilon})^{-k} V_p(2^k r_0), \]
For any $r \geq r_0$, we can assume $r \in [2^kr_0, 2^{k+1}r_0]$ for some $k \in \mathbb{N}$. Then
\[ V_p(r) \geq V_p(2^kr_0) \geq \frac{V_p(r_0)}{r_0^{n-\epsilon}}(2^kr_0)^{n-\epsilon} \geq \frac{V_p(r_0)}{r_0^{n-\epsilon}}(2^kr_0 \cdot \frac{r}{2^{k+1}r_0})^{n-\epsilon} = \frac{V_p(r_0)}{(2r_0)^{n-\epsilon}}r^{n-\epsilon}.

It follows that $V_p(r) \geq C \cdot r^{n-\epsilon}$ for all $r > r_0$.

An algebraic fact: If $\Gamma$ is an infinite group with generators $S = \{g_1, \cdots, g_k\}$, then $\#U(r) \geq r$ for all $r \in \mathbb{N}$, where $U(r)$ is the set of elements with word length $\leq r$ with respect to $S$. In particular, $\Gamma$ has polynomial growth of order at least 1. (This proof is provided by referee)

To see this we argue by contradiction. Let $r$ be the smallest integer so that $\#U(r) < r$, then $r - 1 \leq \#U(r - 1) \leq \#U(r) < r$. This shows $U(r) = U(r - 1)$. In other words, any word of length $r$ can be expressed as a word of length $\leq r - 1$. It follows that $\Gamma = U(r - 1)$, which is finite, a contradiction.

The second part of Theorem 1.1 follows from above immediately.

Remark 3.1. Our proof of finite generation of $\pi_1(M)$ is much different to Wu’s arguments under condition (1.3). Wu’s proof was based on the estimate of ordered set of independent generators with minimal representative geodesic loops.

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