VACUUM ENERGY: “IF NOT NOW, THEN WHEN?”

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Abstract

For a flat universe presently dominated by static or dynamic vacuum energy, cosmological constant (LCDM) or quintessence (QCDM), we calculate the asymptotic collapsed mass fraction as function of the present ratio of smooth energy to matter energy $R_0 = (1 - \Omega_m^0)/\Omega_m^0$. Identifying these collapsed fractions as anthropic probabilities, we find the observed present ratio $R_0 \sim 2$ to be likely in LCDM, but most likely in QCDM.

1 A Cosmological Constant or Quintessence?

Absent a known symmetry principle protecting its value, no theoretical reason for making the cosmological constant zero or small has been found. Inflation makes the universe flat, so that, at present, the vacuum or smooth energy density $\Omega_Q^0 = 1 - \Omega_m^0 < 1$, is $10^{120}$ times smaller than would be expected on current particle theories. To explain this small but non-vanishing present value, a dynamic vacuum energy, quintessence, has been invoked, which obeys the equation of state $w_Q \equiv P/\rho < 0$. (The limiting case, $w_Q = -1$, a static vacuum energy or Cosmological Constant, is homogeneous on all scales.)

The evidence for a flat low-density universe come from [1,2]: (1) The location of the first Doppler peak in the CBR anisotropy at $l \sim 200$: $\Omega_m + \Omega_Q = 1 \pm 0.2$; (2) The slow evolution of rich clusters, the mass power spectrum, the CBR anisotropy, the cosmic flow: $\Omega_m^0 = 0.3 \pm 0.05$; (3) Curvature in the SNIa Hubble diagram, dynamic age, height of first Doppler peak, cluster evolution: $\Omega_Q^0 = 1 - \Omega_m^0 \sim 2/3$. Of these, the SNIa evidence is most subject to systematic errors due to precursor intrinsic evolution and the possibility of grey dust extinction. The combined data nevertheless implies a flat, low-density universe with $\Omega_m^0 \sim 1/3$ and a smooth energy component with present energy density $\Omega_Q^0 \sim 2/3$ and negative pressure $-1 \leq w_Q \leq -1/2$.

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Accepting this small but non-vanishing value for static or dynamic vacuum energy, a flat Friedmann cosmology (CDM) is characterized by $\Omega_{m0}$, $\Omega_{Q0} = 1 - \Omega_{m0}$ or the present ratio

$$R_0 \equiv u_0^3 \equiv \Omega_{Q0}/\Omega_{m0} = (1 - \Omega_{m0})/\Omega_{m0},$$

and by the equation of state for the smooth energy. The Cosmic Coincidence problem now becomes pressing: Why do we live when the clustered matter density $\Omega(a)$, which is diluting as $a^{-3}$ with cosmic scale $a$, is just now comparable to the static vacuum energy or present value of the smooth energy i.e. when the ratio $R_0 \sim 2$?

In this paper, we distinguish the two limiting cases allowed \[1, 2\] for the smooth energy component: LCDM: Cosmological constant: $w_Q = -1$ and QCDM: Quintessence: $w_Q = -1/2$. In the next section, we compare the expansion of these two limiting low-density flat universes. In Section 3, we extend to QCDM the calculation of asymptotic mass fraction as function of a hypothetical continuous variable $\Omega_{m0}$ presented by Martel et al \[4, 5\] for QCDM. Finally, we statistically infer that, absent any prior information about $\Omega_{m0}$, the observed present ratio $R_0$ is reasonable for a LCDM universe, and most likely for a QCDM universe: “If not now, then when?” \[3\]

### 2 Expansion of a Low Density Flat Universe

The Friedmann equation in a flat universe with clustered matter and smooth energy density is

$$H^2(x) \equiv (\dot{a}/a)^2 = (8\pi G/3)(\rho_m + \rho_Q),$$

or, in units of $\rho_{cr}(x) = 3H^2(x)/8\pi G$, $1 = \Omega_m(x) + \Omega_Q(x)$, where the reciprocal scale factor $x \equiv a_0/a \equiv 1 + z \to \infty$ in the far past, $\to 0$ in the far future.

With the effective equation of state $w \equiv P/\rho = \text{constant}$, different kinds of energy density dilute at different rates $\rho \sim a^{-n}$, $n \equiv 3(1 + w)$, and contribute to the deceleration at different rates $(1 + 3w)/2$ shown in the table:

| substance     | $w$ | $n$ | $(1+3w)/2$ |
|---------------|-----|-----|------------|
| radiation     | 1/3 | 4   | 1          |
| NR matter     | 0   | 3   | 1/2        |
| quintessence  | -1/2| 3/2 | -1/4       |
| cosmolconst   | -1  | 0   | -1         |

Table 1: Energy Dilution for Various Equations of State

The expansion rate in present Hubble units is

$$H(x)/H_0 = (\Omega_{m0} x^3 + (1 - \Omega_{m0}) x_{Q0}^3)^{1/2}.$$

The Friedmann equation has an unstable fixed point in the far past and a stable attractor in the far future. (Note the tacit application of the anthropic principle: Why does our universe expand, rather than contract?)
The second Friedmann equation is \(-\ddot{a}/\dot{a}^2 = (1 + 3wQ\Omega_Q)/2\). The ratio of smooth energy to matter energy, \(R(a) = R_0(a_0/a)^{3wQ}\), increases as the cosmic expansion dilutes the matter density. A flat universe, characterized by \(R_0\), \(w_Q\), evolves out of an SCDM universe in the remote past towards a flat de Sitter universe in the future. As shown by the inflection points (O) on the middle curves of Figure 1, for fixed \(R_0\), QCDM expands faster than LCDM, but begins accelerating only at the present epoch. The top and bottom curves refer respectively to a de Sitter universe (\(\Omega_m = 0\)), which is always accelerating, and an SCDM universe (\(\Omega_m = 1\)), which is always decelerating.

The matter-smooth energy transition (“freeze-out”) \(\Omega_Q/\Omega_m = 1\) took place only recently at \(x_*-w_Q = R_0^{1/3} \equiv u_0\) or at \(x_* = 1 + z_* = u_0^2 = 1.59\) for QCDM and, even later, at \(x_* = 1 + z_* = u_0 = 1.26\) for LCDM. Because, for the same value of \(u_0\), a matter-QCDM freeze-out would take place earlier and more slowly than a matter-LCDM freeze-out, it imposes a stronger constraint on structure evolution. As summarized in the table below, quintessence dominance begins 3.6 Gyr earlier and more gradually than cosmological constant dominance. (In this table, the deceleration \(q(x) = -\ddot{a}/aH_0^2\) is measured in present Hubble units.)

\[
H_0t_L(z) = z - (1 + q_0)z^2 + ... \quad z < 1,
\]

where \(q_0 = 0\) for QCDM and \(-1/2\) for LCDM.

| event                                      | LCDM | QCDM |
|--------------------------------------------|------|------|
| Onset of Vacuum Dominance                  |      |      |
| reciprocal scale \(x_* = a_0/a = 1 + z\)  | \(u_0 = 1.260\) | \(u_0^2 = 1.587\) |
| age \(t(x*)/H_0^{-1}\)                    | 0.720 | 0.478 |
| in units \(h_{65}^{-1}\) Gyr              | 10.8 | 7.2  |
| horizon size in units \(cH_0^{-1}\)       | 2.39 | 1.58 |
| in units \(h_{65}^{-1}\) Gpc              | 11.0 | 7.24 |
| deceleration \(q(x^*)\) at freeze-out      | \(-0.333\) | \(0.333\) |
| Present Epoch                              |      |      |
| age \(t_0/H_0^{-1}\)                      | 0.936 | 0.845 |
| \(h_{65}^{-1}\) Gyr                       | 14.0 | 12.7 |
| horizon in units \(cH_0^{-1}\)            | 3.26 | 2.96 |
| in units \(h_{65}^{-1}\) Gpc              | 15.0 | 13.6 |
| present deceleration \(q_0\)               | \(-0.500\) | 0 |

Table 2: Comparative Evolution of LCDM and QCDM Universes

3 Evolution of Large Scale Structure

In this section, we extend to QCDM earlier LCDM calculations of the asymptotic mass fraction \(f_{c,\infty}\) that ultimately collapses into evolved galaxies. This is presumably a measure of the number density of galaxies like our own,
that are potentially habitable by intelligent life. We then compare the QCDM and LCDM asymptotic mass fraction distribution functions, as function of an assumed $\Omega_{m0}$.

The background density for large-scale structure formation is overwhelmingly Cold Dark Matter (CDM), consisting of clustered matter $\Omega_m$ and smooth energy or quintessence $\Omega_Q$. Baryons, contributing only a fraction to $\Omega_m$, collapse after the CDM and, particularly in small systems, produce the large overdensities that we see.

Structure formation begins and ends with matter dominance, and is characterized by two scales: The horizon scale at the first cross-over, from radiation to matter dominance, determines the power spectrum $P(k, a)$, which is presently characterized by a shape factor $\Gamma_0 = \Omega_{m0} h = 0.25 \pm 0.05$. The horizon scale at the second cross-over, from matter to smooth energy, determines a second scale factor, which for quintessence, is at $\sim 130$ Mpc, the scale of voids and superclusters. A cosmological constant is smooth at all scales.

Quasars formed as far back as $z \sim 5$, galaxies at $z \geq 6.7$, ionizing sources at $z = (10^{-30})$. The formation of any such structures, already sets an upper bound $x_* < 30$ or $(\Omega_\Lambda/\Omega_{m0}) < 1000, \Omega_{Q0} < 30$, for any structure to have formed. A much stronger upper bound, $u_0 < 5$, is set by when typical galaxies form i.e. by estimating the probability of our observing $R_0 = 2$ at the present epoch.

### 3.1 Asymptotic Collapsed Mass Parameter $\beta$

Martel et al. and Garriga et al. have already calculated the asymptotic mass fraction from the Press-Schechter formalism

$$f_{c,\infty} = \text{erfc}(\sqrt{\beta}) = \left(\frac{2}{\sqrt{\pi}}\right)\int_{\sqrt{\beta}}^{\infty} \exp(-t^2) \, dt,$$

depending only on

$$\beta \equiv \frac{\delta_{i,c}^2}{(2\sigma_i^2)},$$

where $\sigma_i^2$ is the variance of the density field, smoothed on some scale $R_G$, and $\delta_{i,c}$ is the minimum density contrast at recombination which will ultimately make a bound structure. This minimum density contrast grows with scale factor $a$, and is, except for a numerical factor of order unity, $\delta_{i,c} \sim x_*/(1 + z_i)$. Both numerator and denominator in $\beta$ refer to the epoch of recombination, but this factor $(1 + z_i)$ cancels out in the quotient. (MSW and MS have improved on the Press-Schechter formalism by assuming spherical collapse of Gaussian fluctuations or linear fluctuations that are surrounded by equal volumes of compensating under-density. Except in the limit $\beta \to 0$, the PS formula overestimates the collapsed mass by factor $\approx (1.70) * \beta^{0.085}$, or about 40% near $\Omega_{m0} = 1/3$. For simplicitly, this paper adheres to the PS formula with $R_G = 1$ Mpc. In a forthcoming paper, we will use the improved MSW formula for both $R_G = 1, 2$ Mpc.)

The variance of the mass power spectrum depends on the cosmological model ($\Omega_{m0}$) and on the relevant co-moving galactic size scale $R_G$, but is insensitive to $w_Q$, for $w_Q < -1/3$. For the QCDM model we consider, $\sigma_i^2(\Omega_{m0}, R_G)$ is therefore the same as that already calculated for LCDM, for a scale-invariant.
mass spectrum smoothed with a top-hat window function. For the observed ratio $R_0 = 2, \Omega_{m0} = 1/3$, at recombination $1000\sigma = 3.5, 2.4$, for comoving galactic size scale $R_G = 1, 2$ Mpc.

The numerical factor in $\delta_{i,c}$ is $9/5(4)^{1/3} = 1.1339$ for both $w_Q = -1$ and $w_Q = -1/2$, so that $\delta_{i,c} = 1.1339 x^*/(1 + z_i)$. Thus, the collapsed mass parameter $\sqrt{\beta} = 0.80 x^*/\sigma_i(R_G, u_0)$, depends explicitly on $u_0$ through $x^* = u_0, u_0^2$ for LCDM, QCDM respectively. It also depends implicitly on $u_0$ through $\sigma_i$. Nevertheless, in going from LCDM the argument of $f_{c,\infty}$ scales simply as $\sqrt{\beta}_{QCDM} = \sqrt{\beta}_{LCDM} \cdot u_0$.

Both asymptotic mass fractions are practically unity for large $\Omega_{m0}$, but fall off with increasing ratio $R_0 > 1$. For any $R_0 > 1$, QCDM always leads to a smaller asymptotic mass fraction than LCDM. For ratio $R_0 < 1$, $f_{c,\infty}$ changes slowly and the differences between QCDM and LCDM are not large. At the observed ratio $R_0 = 2$, the Press-Schechter asymptotic mass fractions are 0.696, 0.623 for LCDM, QCDM respectively.

3.2 Asymptotic Collapsed Mass Fraction Distribution Function

As function of the ratio $\Omega_{m0}$, the asymptotic mass fraction defines a distribution function

$$f_{c,\infty} = dP/dR_0.$$  

In Figure 2, instead of $f_{c,\infty}$ we plot the logarithmic distribution function in the ratio $R_0$

$$F(\Omega_{m0}) = R_0 \cdot f_{c,\infty} = dP/d\log R_0,$$

for LCDM and for QCDM and galactic size scale 1 Mpc. (Even for LCDM, this differs by a factor $\sigma_i^2(\Omega_{m0})$ from the logarithmic distribution in $\beta$, $dP/d\log(\beta^{3/2})$ that is plotted by MSW and GLV.) $F(\Omega_{m0})$ may be thought of as the ratio $R_0$ weighted by the number density of galaxies $f_{c,\infty}$.

The figure shows broad peaks in the logarithmic distributions in $\Omega_{m0}$ at $((\Omega_{m0}, F) = (0.23, 1.27) for LCDM and at (0.32, 1.78) for QCDM. At the observed $\Omega_{m0} = 1/3$, shown by circles (O), the asymptotic mass fraction logarithmic distribution in $R_0$ falls at 97% of the QCDM peak and at 78% of the LCDM peak.

4 $\Omega_Q \sim \Omega_m$ is Quite Likely for Our Universe

It is not surprising that our universe, containing at least one habitable galaxy, has $R_0 = O(1)$. What is impressive is that our observed low-density universe, is almost exactly that which will maximize the number of habitable galaxies. Our existence does not explain $\Omega_{m0}$, but the observed value makes our existence (and that of other evolved galaxies) most likely.

What epistemological inference should we draw from this remarkable coincidence between our observed universe and the possible asymptotic mass fractions
in either LCDM or QCDM universes? What should we infer statistically about any fundamental theory determining the parameters of our universe?

An anthropic interpretation has already been given [4, 5, 6, 7] to “explain” a non-vanishing cosmological constant, in an assumed universe of subuniverses with all possible values for the vacuum energy $\Omega_\Lambda = 1 - \Omega_{m0}$. In each of these subuniverses, the probability for habitable galaxies to have emerged before the present epoch, is a function of $\Omega_\Lambda$:

$$\mathcal{P}rob(\Omega_{m0}) \propto (\text{prior distribution in } \Omega_{m0}) \times F(\Omega_{m0}).$$

As always, the overall probability depends on the assumed prior. MSW, assuming nothing about initial conditions, take a prior flat in $\Omega_{m0}$. GLV argue that the prior should be determined by a theory of initial conditions and is not flat for most theories.

We prefer not to assume a distribution of real subuniverses, but to inversely apply Bayes’ Theorem to our own universe. In the absence of any physical explanation of the smooth energy, or until one is found, the partial information that intelligent astronomers exist tells us the observed smooth energy is just about what would be expected from equal a priori probabilities for $\Omega_{m0}$ in the very early universe. That our universe is realized at or near the maximum in the asymptotic mass distribution function confirms that the prior is flat or peaked at $\Omega_{m0} \sim 1/3$. Any phenomenologically viable fundamental theory must ultimately produce this value or be indifferent to the cosmological parameters.

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Figure 1: Scale evolution of LCDM and QCDM low-density flat universes in the recent past and near future. The lower curve shows the SCDM universe from which both LCDM and QCDM evolved in the far past. The upper curve shows the flat de Sitter universe towards which both LCDM and QCDM will evolve in the far past. The inflection points marked (O) show where first LCDM and later QCDM change over from decelerating to accelerating universes.
Figure 2: Logarithmic distribution function for the Press-Schechter asymptotic collapsed mass fraction as function of hypothetical present matter density $\Omega_{m0}$ (bottom scale) or smooth energy/matter ratio $R_0$ (top scale). Our observed universe (O) with $\Omega_{m0} \sim 1/3$, $R_0 \sim 2$ falls within the broad peak of the LCDM distribution and remarkably close to the peak of the QCDM distribution.