(an + b)-COLOR COMPOSITIONS

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Abstract. For \(a, b \in \mathbb{N}_0\), we consider \((an + b)\)-color compositions of a positive integer \(\nu\) for which each part of size \(n\) admits \(an + b\) colors. We study these compositions from the enumerative point of view and give a formula for the number of \((an + b)\)-color compositions of \(\nu\) with \(k\) parts. Our formula is obtained in two different ways: 1) by means of algebraic properties of partial Bell polynomials, and 2) through a bijection to a certain family of weak compositions that we call domino compositions. We also discuss two cases when \(b\) is negative and give corresponding combinatorial interpretations.

1. Introduction

A composition of a positive integer \(\nu\) with \(k\) parts is an ordered \(k\)-tuple \((j_1, \ldots, j_k)\) of positive integers called parts such that \(j_1 + \cdots + j_k = \nu\).

Given a sequence of nonnegative integers \(w = (w_n)_{n \in \mathbb{N}}\), we define a \(w\)-color composition of \(\nu\) to be a composition of \(\nu\) such that part \(n\) can take on \(w_n\) colors. If \(w_n = 0\), it means that we do not use the integer \(n\) in the composition. Such colored compositions have been considered by many authors and continue to be of current interest. For a comprehensive account on the subject, we refer to the book by S. Heubach and T. Mansour [2].

If we let \(W_n\) be the number of \(w\)-color compositions of \(n\), Moser and Whitney [4] observed that the generating functions \(w(t) = \sum_{n=1}^{\infty} w_n t^n\) and \(W(t) = \sum_{n=1}^{\infty} W_n t^n\) satisfy the relation \(W(t) = \frac{w(t)}{1-w(t)}\), which means that the sequence \((W_n)_{n \in \mathbb{N}}\) is the invert transform of \((w_n)_{n \in \mathbb{N}}\).

In this paper, we consider the sequence of colors \(w_n = an + b\) for \(n \geq 1\), with \(a, b \in \mathbb{N}_0\). Thus \(w(t) = \sum_{n=1}^{\infty} (an + b)t^n\), and we have

\[
w(t) = \sum_{n=1}^{\infty} (an + b)t^n = \frac{at}{(1-t)^2} + \frac{bt}{1-t} = \frac{(a + b)t - bt^2}{(1-t)^2}.
\]

Therefore, \(W(t) = \frac{w(t)}{1-w(t)} = \frac{(a+b)t - bt^2}{1-(a+b+2)t + (b+1)t^2}\), and so the number \(W_\nu\) of \((an + b)\)-color compositions of \(\nu\) satisfies the recurrence relation

\[
W_\nu = (a + b + 2)W_{\nu-1} - (b + 1)W_{\nu-2} \quad \text{for} \quad \nu > 2, \tag{1.1}
\]

with initial conditions \(W_1 = a + b\) and \(W_2 = (a + b)^2 + (2a + b)\).
2. Colored compositions with \( k \) parts

Let \( c_{n,k}(w) \) be the number of \( w \)-color compositions of \( n \) with exactly \( k \) parts. In \([3]\), Hoggatt and Lind derived the formula

\[
c_{n,k}(w) = \sum_{\pi_k(n)} \frac{k!}{k_1! \cdots k_n!} w_1^{k_1} \cdots w_n^{k_n}, \tag{2.1}
\]

where the sum runs over all \( k \)-part partitions of \( n \), i.e. over all solutions of

\[
k_1 + 2k_2 + \cdots + nk_n = n \text{ such that } k_1 + \cdots + k_n = k
\]

with \( k_j \in \mathbb{N}_0 \) for all \( j \). Observe that the right-hand side of (2.1) is precisely the \((n,k)\)-th partial Bell polynomial \( B_{n,k}(1!w_1,2!w_2,\ldots) \) multiplied by the factor \( k!/n! \). Thus (2.1) may be written as

\[
c_{n,k}(w) = \frac{k!}{n!} B_{n,k}(1!w_1,2!w_2,\ldots), \tag{2.2}
\]

and the total number of such compositions of \( n \) is \( W_n = \sum_{k=1}^n c_{n,k}(w) \).

**Proposition 1.** Let \( x = (x_n)_{n \in \mathbb{N}} \) and \( y = (y_n)_{n \in \mathbb{N}} \) be sequences of non-negative integers, and let \( a, b \in \mathbb{Z} \). Letting \( c_{0,0}(w) = 1 \) and \( c_{m,j}(w) = 0 \) for \( m < j \), we have

\[
c_{n,k}(ax + by) = \sum_{m=0}^n \sum_{j=0}^k \binom{k}{j} a^j b^{k-j} c_{m,j}(x)c_{n-m,k-j}(y).
\]

**Proof.** Since \( c_{n,k}(w) = \frac{k!}{n!} B_{n,k}(1!w_1,2!w_2,\ldots) \), we can use basic properties of the partial Bell polynomials (see e.g. \([1]\) Sec. 3.3) together with the notation \( !w = (n!w_n) \) to get

\[
c_{n,k}(ax + by) = \frac{k!}{n!} B_{n,k}(1!(ax + by))
\]

\[
= \frac{k!}{n!} \sum_{m=0}^n \sum_{j=0}^k \binom{n}{m} B_{m,j}(1!(ax)) B_{n-m,k-j}(1!(by))
\]

\[
= \frac{k!}{n!} \sum_{m=0}^n \sum_{j=0}^k \binom{n}{m} a^j B_{m,j}(1!) b^{k-j} B_{n-m,k-j}(1!y)
\]

\[
= \frac{k!}{n!} \sum_{m=0}^n \sum_{j=0}^k \binom{n}{m} a^j b^{k-j} \frac{m!}{j!} c_{m,j}(x) \frac{(n-m)!}{(k-j)!} c_{n-m,k-j}(y)
\]

\[
= \sum_{m=0}^n \sum_{j=0}^k \binom{k}{j} a^j b^{k-j} c_{m,j}(x)c_{n-m,k-j}(y).
\]

\( \Box \)
Theorem 2. The number of \((an + b)\)-color compositions of \(\nu\) with \(k\) parts is given by

\[
c_{\nu,k}(an + b) = \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \left( \frac{\nu + j - 1}{\nu - k} \right).
\]

Thus the total number of \((an + b)\)-color compositions of \(\nu\) is

\[
W_{\nu} = \sum_{k=1}^{\nu} \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \left( \frac{\nu + j - 1}{\nu - k} \right).
\] (2.3)

Proof. We use the above proposition with the sequences \(x_n = n\) and \(y_n = 1\). Then

\[
c_{\nu,k}(an + b) = \sum_{m=0}^{\nu} \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} c_{\nu-m,k-j}(1)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \sum_{m=0}^{\nu} \left( \frac{m+j-1}{m-j} \right) \left( \frac{\nu-m-1}{k-j-1} \right)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \sum_{\ell=0}^{k-j} \left( \frac{\ell+2j-1}{\ell} \right) \left( \frac{\nu-\ell-j-1}{k-j-1} \right)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \sum_{\ell=0}^{k} \left( \frac{\ell+2j-1}{\ell} \right) \left( \frac{\nu-\ell-j-1}{\nu-k-\ell} \right)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} (-1)^{\nu-k} \sum_{\ell=0}^{\nu-k} \left( -2j \right) \left( \frac{j-k}{\nu-k-\ell} \right)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} (-1)^{\nu-k} \left( \frac{j-k}{\nu-k} \right)
\]

\[
= \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \left( \frac{\nu+j-1}{\nu-k} \right).
\]

Example 3. For some values of \(a\) and \(b\), (2.3) gives nice formulas for the following sequences, listed in the OEIS [5]:

| Compositions | Sequence | Compositions | Sequence |
|--------------|----------|--------------|----------|
| \(n\)-color  | A001906  | \((2n - 1)\)-color | A003946  |
| \((n + 1)\)-color | A003480 | \(2n\)-color | A052530 |
| \((n + 2)\)-color | A010903 | \((2n + 1)\)-color | A060801 |
| \((n + 3)\)-color | A010908 | \((3n - 1)\)-color | A055841 |
3. Combinatorial interpretation: Domino compositions

In this section, an \( n \)-domino is a tile of the form
\[
\alpha | \beta
\]
with \( 0 < \alpha \leq n \) and \( 0 \leq \beta \leq n \). (3.1)

A domino with \( \beta = 0 \) will be called a zero \( n \)-domino. An \( n \)-domino composition of \( \nu \) is a weak composition of \( \nu \) using \( n \)-dominos of the form (3.1). For example,
\[
1 1 4 0 1 3
\]
is a domino composition of 10 with 4-dominos corresponding to the weak composition \((1,1,4,0,1,3)\).

**Definition 4.** For \( a, b \in \mathbb{N}_0 \), \( n, k \in \mathbb{N} \), and \( j \leq k \), let \( T^{a,b}_{n,j}(n,k) \) be the set of \( n \)-domino compositions of \( n+j \) with \( j \) nonzero \( n \)-dominos, available in \( a \) different colors, and \( k-j \) zero \( n \)-dominos available in \( b \) different colors. Let

\[
T^{a,b}_{n,j}(n,k) = \bigcup_{n_j=0}^{n} T^{a,b}_{n_j}(n,k).
\]

**Lemma 5.**
\[
|T^{a,b}_{n,j}(n,k)| = a^j b^{k-j} \binom{k}{j} \binom{n+j-1}{n-k}.
\]

**Proof.** Having \( k \) dominos, there are \( \binom{k}{j} \) ways to choose the \( j \) nonzero dominos. Once the dominos are chosen, there are \( 2^j + (k-j) = k+j \) spaces to place positive numbers whose sum is \( n+j \). These are compositions of \( n+j \) with \( k+j \) parts and there are \( \binom{n+j-1}{k+j-1} = \binom{n+j-1}{n-k} \) of them. Since the nonzero dominos come in \( a \) colors and the zero dominos in \( b \) colors, we need to multiply by \( a^j b^{k-j} \) to account for all of the possibilities. \( \square \)

**Theorem 6.** For any given \( a, b \in \mathbb{N}_0 \), there is a bijection \( \varphi \) between \( T^{a,b}(\nu,k) \) and the set of \((an+b)\)-color compositions of \( \nu \) with \( k \) parts.

**Proof.** We start by discussing the case when \( a = 1 \). Let \( (D_1, \ldots, D_k) \) be an element of \( T^{1,b}(\nu,k) \) with \( j \) nonzero \( \nu \)-dominos. For a nonzero domino \( D \), we define \( \varphi(D) \) by
\[
\begin{array}{c|c}
\alpha & \beta \\
\hline
\end{array} \rightarrow (\alpha + \beta - 1)_{\beta},
\]
where the notation \((i)_{\ell}\) means part \( i \) with color \( \ell \). For a zero domino \( D \) with color \( \delta \leq b \), we define \( \varphi(D) \) by
\[
\begin{array}{c|c}
\alpha & 0 \\
\hline
\end{array} \delta \rightarrow (\alpha)_{\ell}, \text{ where } \ell = \alpha + \delta.
\]
If we denote the nonzero dominos by \((\alpha_1, \beta_1), \ldots, (\alpha_j, \beta_j)\), and the zero dominos by \((\alpha_{j+1}, 0), \ldots, (\alpha_k, 0)\), then by definition \((\alpha_1 + \beta_1) + \cdots + (\alpha_j + \beta_j) + \alpha_{j+1} + \cdots + \alpha_k = \nu + j\), and therefore, \((\alpha_1 + \beta_1 - 1) + \cdots + (\alpha_j + \beta_j - 1) + \alpha_{j+1} + \cdots + \alpha_k = \nu\). In other words, \((\varphi(D_1), \ldots, \varphi(D_k))\) is an \((n+b)\)-color composition of \( \nu \) with \( k \) parts.
Conversely, let \((i_1)_{\ell_1}, \ldots, (i_k)_{\ell_k}\) be an \((n + b)\)-color composition of \(\nu\) such that \(j\) of its parts are of the form \((i)_{\ell}\) with \(\ell \leq i\). If \((i)_{\ell}\) is such a part, then we define \(\psi((i)_{\ell})\) by

\[
(i)_{\ell} \rightarrow \begin{array}{c}
\alpha_i \\
\ell
\end{array}, \quad \text{where } \alpha_i = i - \ell + 1,
\]

and if part \((i)_{\ell}\) is such that \(\ell = i + \delta > i\), then we define \(\psi((i)_{\ell})\) by

\[
(i)_{\ell} \rightarrow \begin{array}{c}
\alpha_i \\
0 \delta
\end{array}.
\]

In particular, the components of a domino \(\psi((i)_{\ell})\) add to \(i + 1\) if \(\ell \leq i\) or they add to \(i\) if \(\ell > i\). Since \(i_1 + \cdots + i_k = \nu\), we get that \((\psi((i_1)_{\ell_1}), \ldots, \psi((i_k)_{\ell_k}))\) is a \(\nu\)-domino composition in \(T^{1,b}_1(\nu, k)\). Clearly, \(\psi\) is the inverse of \(\phi\).

For \(a > 1\) the argument is similar. In this case, for a nonzero domino \(D_\gamma \in T^{a,b}(\nu, k)\) with color \(1 \leq \gamma \leq a\), we define \(\varphi(D_\gamma)\) by

\[
\begin{array}{c}
\alpha \\
\beta
\end{array}_\gamma \rightarrow (\alpha + \beta - 1)_{\ell}, \quad \text{where } \ell = (\alpha + \beta - 1)(\gamma - 1) + \beta,
\]

and for a zero domino \(D_\delta\) with color \(1 \leq \delta \leq b\), we define \(\varphi(D_\delta)\) by

\[
\begin{array}{c}
\alpha \\
0 \delta
\end{array} \rightarrow (\alpha)_{\ell}, \quad \text{where } \ell = a\alpha + \delta.
\]

The inverse map is obtained as follows. For a part \(i\) with color \(\ell\), \(1 \leq \ell \leq ai + b\), write \(\ell = qi + r\) with \(0 < r \leq i\) and define a \(\nu\)-domino as follows:

- if \(q < a\) : \((i)_{\ell} \rightarrow \begin{array}{c}
\alpha_i \\
q + 1
\end{array}\)
- if \(q = a\) : \((i)_{\ell} \rightarrow \begin{array}{c}
i \\
0
\end{array}.

where the subscript outside the domino indicates its color.

\[\square\]

**Example 7.** In the context of \((n + 2)\)-color compositions, we have

\[
\begin{array}{c|c|c|c}
1_1 & 1 & 2 & 3 \\
2_1 & 2 & 3 & 1 \\
3_1 & 3 & 1 & 2 \\
1_2 & 1 & 0 & 1 \\
2_2 & 2 & 0 & 1 \\
3_2 & 3 & 0 & 1 \\
1_3 & 1 & 0 & 2 \\
2_3 & 2 & 0 & 2 \\
3_3 & 3 & 0 & 2 \\
\end{array}
\]

For example, the composition \((3_5, 1_2, 3_2)\) of 7 corresponds to

\[
\begin{array}{c|c|c|c|c|c}
3 & 0 & 2 & 1 & 0 & 1 \\
2 & 2 & & & &
\end{array}
\]

As a direct consequence of Theorem 6 and Lemma 5 we obtain:
Corollary 8. The number of \((an + b)\)-color compositions of \(\nu\) with \(k\) parts is given by
\[
c_{\nu,k}(an + b) = \sum_{j=0}^{k} a^j b^{k-j} \binom{k}{j} \binom{\nu + j - 1}{\nu - k}.
\]

4. Other examples

We finish with two examples related to \((n - 1)\)-color and \((n - 2)\)-color compositions.

Example 9 \((a = 1, b = -1)\). In this case, we have that
\[
c_{\nu,k}(n - 1) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu + j - 1}{\nu - k}
\]
is the number of compositions of \(\nu\) with \(k\) parts with no part 1 and such that each part \(i > 1\) may be colored in \(i - 1\) different ways. This is also the number of \(n\)-color compositions of \(\nu\) with \(k\) parts and no color 1.

Example 10 \((a = 1, b = -2)\). Let \(\mathcal{C}_{n-2}(\nu, k)\) be the set of compositions of \(\nu\) with \(k\) parts such that:
- there is no part 2
- each part \(i > 2\) maybe colored in \(i - 2\) different ways.

If \(\mathcal{C}_{n-2}^{1,\text{even}}(\nu, k)\) denotes the set of compositions in \(\mathcal{C}_{n-2}(\nu, k)\) with an even number of 1’s, and \(\mathcal{C}_{n-2}^{1,\text{odd}}(\nu, k)\) is the set of compositions with an odd number of 1’s, then we have
\[
c_{\nu,k}(n - 2) = |\mathcal{C}_{n-2}^{1,\text{even}}(\nu, k)| - |\mathcal{C}_{n-2}^{1,\text{odd}}(\nu, k)|,
\]
which implies
\[
c_{\nu,\nu}(n - 2) = (-1)\nu,
\]
\[
c_{\nu,k}(n - 2) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu - k - 1}{2j - 1} \text{ for } k < \nu.
\]

Moreover, by (1.1), the sequence defined by \(W_{\nu} = \sum_{k=1}^{\nu} c_{\nu,k}(n - 2)\) satisfies the recurrence relation
\[
W_1 = -1, \quad W_2 = 1,
\]
\[
W_{\nu} = W_{\nu-1} + W_{\nu-2} \text{ for } \nu > 2.
\]

In other words, \(W_{\nu}\) is the Fibonacci number \(F_{\nu-3}\) and we get the identity
\[
F_{\nu-3} = \sum_{k=1}^{\nu} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \binom{\nu - k - 1}{2j - 1}.
\]
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