A motivated proof of the Göllnitz-Gordon-Andrews identities

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Abstract

We present what we call a “motivated proof” of the Göllnitz-Gordon-Andrews identities. A similar motivated proof of the Rogers-Ramanujan identities was previously given by G. E. Andrews and R. J. Baxter, and was subsequently generalized to Gordon’s identities by J. Lepowsky and M. Zhu. We anticipate that the present proof of the Göllnitz-Gordon-Andrews identities will illuminate certain twisted vertex-algebraic constructions.

1 Introduction

The Göllnitz-Gordon-Andrews identities form a family of partition identities somewhat analogous to the Gordon-Andrews generalizations of the Rogers-Ramanujan identities. As presented in Chapter 7 of [A3] (which also gives the Rogers-Ramanujan and Gordon-Andrews identities), these identities state that for any \( k \geq 2 \) (for \( k = 1 \), one gets the trivial identity \( 1 = 1 \)) and \( i = 1, \ldots, k \),

\[
\prod_{\substack{m \geq 1, \, m \not\equiv 2 \pmod{4}, \, m \not\equiv 0, \, 2k \pm (2i-1) \pmod{4k}}} \frac{1}{1-q^m} = \sum_{n \geq 0} d_{k,i}(n) q^n,
\]

where \( d_{k,i}(n) \) is the number of partitions \((b_1, \ldots, b_s)\) of \( n \) (with \( b_p \geq b_{p+1} \)), satisfying the following conditions:

1. No odd parts are repeated,
2. \( b_p - b_{p+k-1} \geq 2 \) if \( b_p \) is odd,
3. \( b_p - b_{p+k-1} > 2 \) if \( b_p \) is even, and
4. at most \( k - i \) parts are equal to 1 or 2.

Here we have replaced \( i \) by \( k - i \) in the statement of these identities in Theorem 7.11 of [A3]. Also, here and below, \( q \) is a formal variable.

For \( k = 2 \), these identities were independently discovered by H. Göllnitz [Göl] and B. Gordon [G2], and were subsequently generalized to all \( k \) by G. E. Andrews [A2]. As noted in [SW], L. J. Slater had published in [S] analytic counterparts of these identities even earlier than [Göl] and [G2], and it has recently been pointed out by A. Sills that two analytic identities equivalent to the analytic Göllnitz-Gordon identities were actually recorded by Ramanujan in his “Lost Notebook” long before Slater rediscovered them (cf. [ABe], page 37).

In this paper, we shall present what we call a “motivated proof” of these identities. (This designation has become a technical term in our work, and we shall omit the quotation marks.)
To set the stage, we begin by explaining the motivated proof of the classical Rogers-Ramanujan identities carried out by Andrews and R. J. Baxter in [AB]. Recall the Rogers-Ramanujan identities:

\[
\prod_{m \geq 1, m \not\equiv 0, \pm 2 \pmod{5}} \frac{1}{1 - q^m} = \sum_{n \geq 0} p_1(n)q^n
\]

and

\[
\prod_{m \geq 1, m \not\equiv 0, \pm 1 \pmod{5}} \frac{1}{1 - q^m} = \sum_{n \geq 0} p_2(n)q^n,
\]

where

\[
p_1(n) = \text{the number of partitions of } n \text{ for which adjacent parts have difference at least 2}
\]

and

\[
p_2(n) = \text{the number of partitions of } n \text{ for which adjacent parts have difference at least 2 and in which 1 does not appear.}
\]

Note that the first product is the generating function for the number of partitions with parts congruent to \(\pm 1 \pmod{5}\), and the second product gives the number of partitions with parts congruent to \(\pm 2 \pmod{5}\).

Upon examining the two sum sides and assuming the truth of the Rogers-Ramanujan identities, one sees that if one subtracts the second Rogers-Ramanujan product, called \(G_2(q)\) in [AB], from the first, called \(G_1(q)\), then one gets a power series with nonnegative coefficients. An explanation of this phenomenon using only the product sides was asked for by Leon Ehrenpreis. While answering this question, Andrews and Baxter were naturally led to a proof of the Rogers-Ramanujan identities themselves. Their idea was to form a sequence of power series as follows: Let

\[
G_3(q) = (G_1(q) - G_2(q))/q, \\
G_4(q) = (G_2(q) - G_3(q))/q^2, \\
\text{and more generally, } G_i(q) = (G_{i-2}(q) - G_{i-1}(q))/q^{i-2}.
\]

The key point was that one then observes *empirically* that for all \(i\), \(G_i(q)\) is a formal power series with constant term 1 and that \(G_i(q) - 1\) is divisible by \(q^i\). This “Empirical Hypothesis” was then proved using only the product sides of the Rogers-Ramanujan identities, and the Empirical Hypothesis immediately led to a proof of the identities themselves. This proof was in fact closely related to Baxter’s proof in [B] and also to Rogers’s and Ramanujan’s proof in [RR]. The truth of the Empirical Hypothesis also gave a (motivated) answer to Ehrenpreis’s question as well, by means of a variant of the same argument.

In [LZ], the Andrews-Baxter motivated proof of the Rogers-Ramanujan identities was generalized to the setting of Gordon’s family of identities [G1] (see also Theorem 7.5 of [A3]), and in the present paper, we provide a motivated proof of the Göllnitz-Gordon-Andrews identities analogous to the motivated proof in [LZ]. A certain “shelf” picture, implicit in [AB], became transparent in [LZ]. Moreover, in [LZ] the appropriate Empirical Hypothesis was not actually observed empirically, but was instead proved directly from the product sides and this in turn was used to prove Gordon’s identities. This will be the case for the present paper as well. Therefore, as in [LZ], we shall use the term “Empirical Hypothesis” as a technical term. As we discuss below, this Empirical Hypothesis approach to discovering and proving new identities is expected to be valuable when the “sum sides” are not known.

The “shelf” picture, as clarified in [LZ], amounts to the following: Imagine that the given product sides form a “0th shelf” of formal power series in \(q\). Now one constructs a “1st shelf”
by taking certain judiciously chosen linear combinations, over the field of fractions of the ring of polynomials in the formal variable $q$, of the power series on the $0^{th}$ shelf. Using analogous linear combinations, one repeats this process recursively, to successively build higher shelves of formal power series. At every step, one ensures that the power series appearing on the $j^{th}$ shelf are of the form $1 + q^{j+1}f(q)$, where $f(q) \in \mathbb{Z}[[q]]$. This is essentially the Empirical Hypothesis of Andrews-Baxter and of Lepowsky-Zhu.

For the present context of the Göllnitz-Gordon-Andrews identities, and actually for Gordon’s identities as well, one way of discovering the linear combinations is to use linear combinations predicted by the $(a, x, q)$-recursions in Lemma 7.2 of [A3], with $a$ suitably specialized as in [A3], but now with $x$ specialized to successively higher powers of $q$, to construct the higher shelves (although this was not the method used in [LZ] for discovering the appropriate linear combinations). We elaborate on this process in Appendices A and C below. This issue was already pointed out in the context of the Rogers-Ramanujan identities in [AB], and it is one more reason why we use “Empirical Hypothesis” as a purely technical term. For the Göllnitz-Gordon-Andrews identities and Gordon’s identities, these linear combinations are “easy,” in the sense that each power series on the $j^{th}$ shelf is obtained by multiplying certain power series on the $(j-1)^{st}$ shelf by certain Laurent polynomials and then adding. In fact, for the Göllnitz-Gordon-Andrews identities, these linear combinations can be simplified further if one allows recursion on the $j^{th}$ shelf as well. In retrospect, therefore, it might not have been too difficult to guess the choice of these linear combinations using pure experimentation with the aim of producing a sound Empirical Hypothesis.

As is well known, partition identities of Rogers-Ramanujan type are intimately related to the representation theory of vertex operator algebras, and it turns out that the philosophy of “motivated proofs” fits extremely well with vertex-operator-theoretic investigations of such identities. As is explained in [LZ], one starts with the product sides of such identities as “given,” as was the case in [LM], and then the problem is to exhibit and explain the corresponding combinatorial sum sides in such a way that the relevant numbers of partitions arise as the dimensions of vector spaces constructed from natural generalized vertex operators. In early vertex operator theory, which was indeed motivated by this very problem, the combinatorial sum sides were built using monomials in principally twisted $Z$-operators, as developed in [LW2]-[LW4] using [LW1] as a starting point. In fact, the $Z$-algebraic structure developed in [LW2]-[LW4] was used to give a vertex-operator-theoretic interpretation of a certain family of Gordon-Andrews-Bressoud identities, as well as a vertex-operator-theoretic proof of the Rogers-Ramanujan identities, in the context of the affine Lie algebra $A_1^{(1)}$. A related vertex-operator-theoretic proof of the Gordon-Andrews-Bressoud identities was given in [MP]. Untwisted analogues of the $Z$-operators were developed and exploited in [LP]. See [L] for a number of related developments. All of these algebraic structures came to be understood in a general vertex-algebraic framework based on generalized vertex operator algebras, modules, twisted modules and intertwining operators; see for instance [DL], [H], [CLM1], [CLM2], [Ca1], [Ca2], [CaLM1]-[CaLM4] and [Sa]. However, in the setting of the principally twisted $Z$-algebras, a natural mechanism for interpreting the formal variable $x$ (referred to above), which counts the number of parts, is yet to be found. Therefore, proofs involving only the formal variable $q$, which is concerned with the integer being partitioned, are desirable. After such proofs are found, one can conceive of interpreting the steps in the proofs by means of twisted intertwining operators among modules for generalized vertex operator algebras; this is related to an early idea of J. Lepowsky and A. Milas. The desire to better understand the connection between partition identities and vertex operator algebra theory is the main incentive for seeking “motivated proofs” as in the present work. See the Introduction in [LZ] for further discussion of these issues.
An important feature of such motivated proofs is that they do not require knowledge of Andrews’s generalization of Rogers’s (somewhat mysterious) expression, formula (7.2.2) in [A3], which after appropriate specializations of $a$ and $x$ and the use of the Jacobi triple product identity gives the corresponding product sides of the relevant identities, and which on the other hand leads to recursions (Lemma 7.2 in [A3]) whose solutions yield the sum sides, as we recalled above. This important feature will be useful for the investigation of more complicated identities arising from the theory of vertex operator algebras. In fact, a number of additional works on motivated proofs are underway.

It was in his work on statistical mechanics that Baxter independently discovered the Rogers-Ramanujan identities [B]. There is a vast literature on connections between Rogers-Ramanujan-type identities and statistical mechanics, including [ABP] as well as our motivating paper [AB] and work of A. Berkovich, B. McCoy, A. Schilling, S. O. Warnaar and many others; we refer the reader to the references in [GOW] for a large number of relevant works, including connections between Rogers-Ramanujan-type identities and a variety of other fields. In [GOW], M. Griffin, K. Ono and Warnaar have given a framework extending the Rogers-Ramanujan identities, incorporating the Hall-Littlewood polynomials, in order to deduce new arithmetic properties of certain $q$-series.

This paper is organized as follows: In Section 2 we recall the use of the Jacobi triple product identity to re-express the product sides of the Göllnitz-Gordon-Andrews identities. The resulting formal power series form our $0^{th}$ shelf. In Section 3 we determine closed-form expressions for our higher shelves. We use these closed-form expressions to formulate and prove our Empirical Hypothesis in Section 4. We recast the relevant recursions in an elegant matrix formulation and prove one form of our main theorem in Section 5. We complete the motivated proof in Section 6. In Appendix A we compare our method with the proof in [A3]. Finally, the developments in the present work suggested to us some natural enhancements of [LZ], which we proceed to give in Appendices B and C.

Just as in [LZ], throughout this paper we treat power series as purely formal series rather than as convergent series (in suitable domains) in complex variables.

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## 2 The formal series $G_l(q)$

Throughout this paper, we fix $k \geq 2$. The main tool we will use for the motivated proof is an infinite sequence $G_l(q)$, $l \geq 1$, of formal power series that we shall generate recursively from the product sides of the Göllnitz-Gordon-Andrews identities. We will want to arrange these formal series in “shelves” as follows: for each $j \geq 0$, the $j^{th}$ shelf will consist of the $k$ formal series

$$G_{(k-1)j+i}(q)$$

for $1 \leq i \leq k$. Note that the shelves will overlap, since

$$(k - 1)j + k = (k - 1)(j + 1) + 1,$$  \quad (2.1)

that is, the first series in the $j + 1^{st}$ shelf is defined to be the last series of the $j^{th}$ shelf.

We start with the case $j = 0$ by defining the series $G_l(q)$ for $1 \leq i \leq k$ as the product sides of the Göllnitz-Gordon-Andrews identities as presented in Theorem 7.11 of [A3]:

$$G_l(q) = \prod_{m \geq 1, m \not\equiv 2 (\text{mod } 4), m \not\equiv 0, 2k \pm (2l-1) (\text{mod } 4k)} (1 - q^m)^{-1},$$  \quad (2.2)
From the sequence $G_i(q)$ of formal series, we shall recover the Göllnitz-Gordon-Andrews identities. For $k = 2$, $i = 1, 2$, these are the Göllnitz-Gordon identities, and the most general form of these identities, that is, for all $k \geq 2$, is due to Andrews.

**Remark 2.1.** Observe that for $1 \leq i \leq k$, $G_i(q)$ is the generating function for partitions into parts not congruent to 2 (mod 4) and not congruent to $0, 2k \pm (2i - 1)$ (mod 4k).

Now for $j \geq 1$, we define the series $G_{(k-1)j+i}(q)$, $1 \leq i \leq k$, recursively. As mentioned in (2.1),

$$G_{(k-1)j+1}(q) = G_{(k-1)(j-1) + k}(q)$$

tautologically, and for $2 \leq i \leq k$, we define

$$G_{(k-1)j+i}(q) = \frac{G_{(k-1)(j-1)+k-i+1}(q) - G_{(k-1)(j-1)+k-i+2}(q)}{q^{2j(i-1)}} - q^{-1}G_{(k-1)j+i-1}(q). \quad (2.3)$$

**Remark 2.2.** The recursions (2.3) can be predicted by Lemma 7.2 in [A3], as mentioned in the Introduction. Alternatively, they could be derived from knowledge of the sum sides of the Göllnitz-Gordon-A ndrews identities; see Sections 5 and 6 below.

The product definition (2.2) of the 0th-shelf series $G_i(q)$ for $1 \leq i \leq k$ is not useful for studying the series in higher shelves, so we will need to rewrite the $G_i(q)$ using the Jacobi triple product identity, and work with the resulting alternating sums. The Jacobi triple identity states (see for example Theorem 2.8 in [A3]):

$$\sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n^2} = \prod_{m \geq 0} (1 - q^{2m+2})(1 - zq^{2m+1})(1 - z^{-1}q^{2m+1}).$$

We rewrite the sum side:

$$\sum_{n \in \mathbb{Z}} (-1)^n z^n q^{n^2} = \sum_{n \geq 0} (-1)^n(z^n q^{n^2} - z^{-n-1}q^{(n+1)^2}) = \sum_{n \geq 0} (-1)^n z^n q^{n^2} (1 - z^{-2n-1}q^{2n+1}).$$

Specializing $q \mapsto q^{2k}$ and then $z \mapsto q^{2i-1}$, we have:

$$\prod_{m \geq 0} (1 - q^{4k(m+1)})(1 - q^{2k(2m+1)+2i-1})(1 - q^{2k(2m+1)-2i+1})$$

$$= \prod_{m \equiv 0, 2k \pm (2i-1) \pmod{4k}} (1 - q^m)$$

$$= \sum_{n \geq 0} (-1)^n q^{2kn^2+(2i-1)n} (1 - q^{(2n+1)(2k-2i+1)})$$

$$= \sum_{n \geq 0} (-1)^n q^{4k\left(\binom{n}{2}\right)+(2k+2i-1)n} (1 - q^{(2k-2i+1)(2n+1)}). \quad (2.4)$$

We shall use the notation

$$F(q) = \prod_{m \equiv 2 \pmod{4}} (1 - q^m) \quad (2.5)$$

here and below. Dividing the product and sum sides of (2.4) by $F(q)$, we obtain

$$G_i(q) = \frac{1}{F(q)} \sum_{n \geq 0} (-1)^n q^{4k\left(\binom{n}{2}\right)+(2k+2i-1)n} (1 - q^{(2k-2i+1)(2n+1)}). \quad (2.6)$$

It is this form of the series $G_i(q)$ that we will always work with, instead of their product expressions.
3 Closed-form determination of the $G_l(q)$

This section contains the technical heart of the motivated proof, which is determining a closed-form expression for all the $G_l(q)$, $l \geq 1$, in terms of certain alternating sums. The following theorem is analogous to Theorem 2.1 in [LZ], but the closed-form expressions are necessarily more complicated and the proof is more involved. As in [LZ], the proof is by induction on the shelf. We have:

**Theorem 3.1.** For any $j \geq 0$ and $1 \leq i \leq k$,

$$G_{(k-1)j+i}(q) = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2n} (1 - q^{2(n+j+1)}) \quad (3.1)$$

**Proof.** The expression (3.1) gives two different formulas for the “edge” cases — $i = 1$ for $j \geq 1$ and $i = k$ for $j = 1$ — as shown in (2.1), and we prove first that they are compatible. That is, we show that for any $j \geq 1$, the two formulas given for

$$G_{(k-1)(j-1)+k}(q) = G_{(k-1)j+1}(q)$$

agree. Using $\tilde{G}_{j,k}(q)$ to denote the right-hand side of (3.1), we show that

$$\tilde{G}_{j,k}(q) = \tilde{G}_{j+1,1}(q) \quad (3.2)$$

for any $j \geq 0$. We will call this equality “edge-matching.” We have:

$$\tilde{G}_{j,k}(q) = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2n} (1 - q^{2(n+j+1)})$$

$$= \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2n} (1 - q^{2(n+j+1)})$$

$$= \frac{1}{F(q)} \sum_{n \geq 1} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2n} (1 - q^{2(n+j+1)})$$

$$+ \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2n} (1 - q^{2(n+j+1)})$$

$$= -\frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2(n+1)+1})(1 + q^{2(n+2)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2(n+1)}$$

$$+ \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2(n+1)}$$

$$= \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2(n+1)}$$

$$\cdot (q^{4kn+2(k+2)+2i-2(n+1)+1} + q^{2(n+2)+1} + (1 + q^{2(n+1)+1})$$

$$+ \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k^{(i)}_j} + (2(k+2) + 2i - 1)n(1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} q^{-2(n+1)}$$

$$\cdot (q^{4kn+2(k+2)+2i-2(n+1)+1} + q^{2(n+2)+1} + (1 + q^{2(n+1)+1})$$
so we obtain

\[ 1 \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)}) (1 - q^{2(n+j+1)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\cdot (1 - q^{2k(2n+j+2)} + q^{2(n+j+1)+1}(1 - q^{2k(2n+j+2) - 2(2n+1) - 2(j+1)})
\]

\[ = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)}) (1 - q^{2(n+j+1)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\cdot (1 - q^{2k(2n+j+2)} + q^{2(n+j+1)+1}(1 - q^{2(k-1)(2n+j+2)})
\]

which is \( G_{j+1,1}(q) \), as desired.

We now prove (3.1) by induction on \( j \); we shall need the “edge-matching” formula (3.2) in this
argument. For the case \( j = 0 \) we multiply the numerator and denominator of (2.6) by \( 1 + q^{2n+1} \) to obtain

\[ G_i(q) = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k+2i-1)n} (1 - q^{2(n+1)}) (1 - q^{2(k-2i+1)(2n+1)})}{1 + q^{2n+1}}
\]

\[ = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k+2i-1)n} (1 - q^{2(k-i+1)(2n+1)}) + q^{2n+1} (1 - q^{2(k-i)(2n+1)})}{1 + q^{2n+1}}
\]

for \( 1 \leq i \leq k \), as desired.

Take \( j \geq 0 \). Assume that (3.1) holds for all \( i = 1, \ldots, k \). We shall show that it holds for
\( j + 1 \) and all \( i = 1, \ldots, k \) using induction on \( i \). The case \( i = 1 \) holds by “edge-matching” since
\( G_{(k-1)(j+1)+1} = G_{(k-1)j+k} \).

Now assume that (3.1) holds for \( j + 1 \) and \( i - 1 \) where \( i = 2, \ldots, k - 1 \). By definition and the
inductive hypothesis on \( j \),

\[ G_{(k-1)(j+1)+i}(q) + q^{-1}G_{(k-1)(j+1)+i-1}(q) = \frac{G_{(k-1)j+k-i+1}(q) - G_{(k-1)j+k-i+2}(q)}{q^{2(j+1)(i-1)}}
\]

\[ = \frac{1}{q^{2(j+1)(i-1)} F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\cdot \left( q^{-2i}(1 - q^{2(i-1)n}) + q^{2(n+j)+1}(1 - q^{2(i-1)(2n+j+1)})
\right.

\[ - q^{-2(i-1)n} (1 - q^{2(i-1)(2n+j+1)}) + q^{2(n+j)+1}(1 - q^{2(i-2)(2n+j+1)})
\right).
\]

(3.3)

The term in parentheses in the last two lines of (3.3) may be rewritten as

\[ q^{-2i}(1 - q^{2(i-1)n}) - q^{-2(i-1)n} (1 - q^{2(i-1)(2n+j+1)})
\]

\[ + q^{2(i-1)n} (1 - q^{2(i-1)(2n+j+1)}) - q^{-2(i-2)n} (1 - q^{2(i-2)(2n+j+1)})
\]

\[ = q^{-2i}(1 - q^{2n}) + q^{2(i-1)(n+j+1)} (1 - q^{2(n+j+1)})
\]

\[ + q^{2(i-1)n} (1 - q^{2n}) + q^{2(i-2)(n+j+1)} (1 - q^{2(n+j+1)})
\]

so we obtain

\[ G_{(k-1)(j+1)+i}(q) + q^{-1}G_{(k-1)(j+1)+i-1}(q)
\]

\[ = \frac{1}{q^{2(j+1)(i-1)} F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n) + (2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\]
\[ \cdot \left( q^{-2i} - q^{2i} \right) + q^{2(i-1)(n+j+1)} \left( 1 - q^{2(n+j+1)} \right) \]
\[ + \frac{q^{2j+1}}{q^{2(j+1)(i-1)} F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+2(2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\cdot \left( q^{-2i} - q^{2i} \right) + q^{2(2i)(n+j+1)} \left( 1 - q^{2(n+j+1)} \right) \right). \tag{3.4} \]

We now analyze the first sum on the right-hand side of (3.3):
\[
\sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+2(2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})} q^{-2in} \]
\[ + \sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+2(2k(j+2)+1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)}) (1 - q^{2(n+j+1)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})} q^{2(i-1)(n+j+1)} \]
\[ = \sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+4kn+2(2k(j+2)+2i-1)n} (1 - q^{2(n+1)}) (1 - q^{2(n+2)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})} \]
\[ \cdot \left( q^{-2in} + q^{2(2i)(n+j+1)} \left( 1 - q^{2(n+j+1)} \right) \right) \tag{3.5} \]

We can simplify the term in parentheses in the last line of (3.5) to obtain
\[-q^{2n(2k-2i+1)+2k(j+1)+2k-2i+1} (1 + q^{2n+1}) + q^{2(j+1)(i-1)} (1 + q^{2(n+j+1)+1}) \]
\[ = q^{2(j+1)(i-1)} \left( q^{-(2n+1)(2k-2i+1)+2(j+1)(k-1)} + 1 + q^{2(n+j+1)+1} \right) \]
\[ = q^{2(j+1)(i-1)} \left( 1 - q^{2(k-1)(2n-1)+j+1} + q^{2(n+j+1)+1} ; q^{2(k-1)(2n-j)+2(j+1)(k-1)} \right) \]
\[ = q^{2(j+1)(i-1)} (1 - q^{2(k-i)(2n+j+2)} + q^{2(n+j+1)+1} \left( 1 - q^{2(k-1)(2n+j+2)} \right)) \]

The second sum in (3.4) is exactly the same as the first except that \( i \) is replaced by \( i - 1 \), so we now see that (3.4) becomes
\[ G_{(k-1)(j+1)+i}(q) + q^{-1} G_{(k-1)(j+1)+i-1}(q) \]
\[ = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+2(2k(j+2)+2i-1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})}
\cdot \left( q^{2(j+1)(i-1)} (1 - q^{2(2n+1)(k-1)} + q^{2(n+j+1)+1} \left( 1 - q^{2(k-1)(2n+j+2)} \right) \right) \]
\[ + \frac{q^{2(j+1)(i-2)}}{q^{2(j+1)(i-1)} F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(n)+2(2k(j+2)+2i-1)n} (1 - q^{2(n+1)}) \ldots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \ldots (1 + q^{2(n+j)+1})} \]
\[
(1 - q^2) \cdots (1 - q^{2j})(1 - q^{2(k-i+1)(j+1)} + q^{2j+1}(1 - q^{2(k-i)(j+1)})/ (1 + q)(1 + q^3) \cdots (1 + q^{2j+1}) \prod_{m \not\equiv 2 \pmod{4}} (1 - q^m) \tag{4.1}
\]

Note first that
\[
1 - q^{2(k-i+1)(j+1)} + q^{2j+1}(1 - q^{2(k-i)(j+1)}) = 1 + q^{2j+1}h(q)
\]
where \(h(q)\) is a polynomial in \(q\). Now, the remaining factors in the numerator of (4.1) can be expressed as:
\[
(1 - q^2) \cdots (1 - q^{2j}) = \left( \prod_{1 \leq m \leq 2j} (1 - q^m) \right) \left( \prod_{1 \leq m \leq j} (1 - q^m)(1 + q^m) \right), \tag{4.2}
\]

The theorem now follows because the second term on the right-hand side of (3.6) is \(q^{-1}G_{(k-1)(j+1)+i-1}(q)\) by the inductive hypothesis on \(i\).

Remark 3.2. It is important to note that the common factor \(F(q)\) plays no role in the proof here, except for the identification with the original \(G_i(q), i = 1, \ldots, k\). The factor \(F(q)\) could have been replaced with any nonzero formal power series in \(q\), and every step of the proof would have been identical (beyond the identification with the original \(G_i(q), i = 1, \ldots, k\)), and equivalent to the existing step. However, \(F(q)\) is crucial for the “Empirical Hypothesis,” which, in fact, as we shall see, uniquely determines this factor.

Remark 3.3. The nested induction (on \(i\) as well as on \(j\)) is a feature of the proof of Theorem 3.1 that is not present in the proof of Theorem 2.1 of [LZ].

4 The Empirical Hypothesis

As a consequence of Theorem 3.1, we are now in a position to formulate and prove the Empirical Hypothesis, which will be the main ingredient in the motivated proof of the Göllnitz-Gordon-Andrews identities.

Theorem 4.1 (Empirical Hypothesis). For any \(j \geq 0\) and \(i = 1, \ldots, k\),
\[
G_{(k-1)j+i}(q) = 1 + q^{2j+1}\gamma(q)
\]
for some
\[
\gamma(q) \in \mathbb{C}[[q]].
\]

Remark 4.2. Note that since \(G_{(k-1)j+k}(q) = G_{(k-1)(j+1)+1}(q)\), for \(i = k\) Theorem 4.1 implies that we can write \(G_{(k-1)j+k}(q) = 1 + q^{2j+3}\gamma(q)\) where \(\gamma(q)\) is some formal power series.

Proof. Since
\[
4k\binom{n}{2} + (2k(j+1)+2i-1)n \geq 2k(j+1)+1 \geq 2j+3
\]
for \(n \geq 1\), it suffices to examine the \(n = 0\) term in (4.1), which is

\[
(1 - q^2) \cdots (1 - q^{2j+1})(1 - q^{2(k-i+1)(j+1)} + q^{2j+1}(1 - q^{2(k-i)(j+1)}))/ (1 + q)(1 + q^3) \cdots (1 + q^{2j+1}) \prod_{m \not\equiv 2 \pmod{4}} (1 - q^m) \tag{4.1}
\]
while the denominator of (4.1) can be rewritten
\[
(1 + q)(1 + q^3) \cdots (1 + q^{2j+1}) \prod_{m \not\equiv 2 \pmod{4}} (1 - q^m) =
\]
\[
\left( \prod_{1 \leq m \leq 2j+1 \atop m \text{ odd}} (1 + q^m)(1 - q^m) \right) \left( \prod_{1 \leq m \leq 2j \atop m \equiv 0 \pmod{4}} (1 - q^m) \right) \left( \prod_{m > 2j+1 \atop m \not\equiv 2 \pmod{4}} (1 - q^m) \right) =
\]
\[
\left( \prod_{1 \leq m \leq j \atop m \equiv 0 \pmod{4}} (1 + q^m)(1 - q^m) \right) \left( \prod_{j < m \leq 2j+1 \atop m \text{ odd}} (1 - q^{2m}) \right) \left( \prod_{1 \leq m \leq 2j \atop m \equiv 0 \pmod{4}} (1 - q^m) \right) \left( \prod_{m > 2j+1 \atop m \not\equiv 2 \pmod{4}} (1 - q^m) \right).
\]

Thus by cancellation we see that (4.1) becomes
\[
1 + q^{2j+1} h(q) = 1 + q^{2j+1} g(q)
\]
\[
\left( \prod_{j < m \leq 2j+1 \atop m \equiv 0 \pmod{4}} (1 - q^{2m}) \right) \left( \prod_{m > 2j+1 \atop m \not\equiv 2 \pmod{4}} (1 - q^m) \right)
\]

where \( g(q) \) is a formal power series.

**Remark 4.3.** It is clear from the proof of Theorem 4.1 that \( F(q) \) (recall (2.5)) is the unique formal power series in the denominator of \( G_{(k-1)j+i}(q) \) that yields the Empirical Hypothesis.

**Remark 4.4.** Analyzing the polynomial \( h(q) \) in the proof above, we can strengthen the Empirical Hypothesis as follows: Observe that
\[
h(q) = -q^{2(k-1)(j+1)+1} + 1 - q^{2(k-1)(j+1)},
\]
so that for \( i \neq k \) and \( j \geq 0 \),
\[
G_{(k-1)j+i}(q) = 1 + q^{2j+1} + \cdots, \tag{4.3}
\]
and for \( i = k \),
\[
G_{(k-1)j+k}(q) = G_{(k-1)(j+1)+1}(q) = 1 + q^{2j+3} + \cdots. \tag{4.4}
\]

We call (4.3) and (4.4) collectively the **Strong Empirical Hypothesis**.

**Remark 4.5.** As we shall see in the proof of Theorem 6.4 below, the only form of the Empirical Hypothesis that is logically required for the proof of the Göllnitz-Gordon-Andrews identities is a weaker one, which states that for any positive integer \( l \), there exists a positive integer \( f(l) \) such that
\[
G_l(q) \in 1 + q^{f(l)} \mathbb{C}[[q]]
\]
with
\[
\lim_{l \to \infty} f(l) = \infty.
\]
We call this the **Weak Empirical Hypothesis**.

**Remark 4.6.** All of the successively sharper forms of Empirical Hypothesis follow easily from the combinatorial sum sides of the Göllnitz-Gordon-Andrews identities, but the point of Theorems 3.1 and 4.1 is to obtain them from the product sides.
5 Matrix interpretation and consequences

As in [LZ], we now study the recursions (2.3) using a matrix approach, which is suggested by rewriting (2.3) in the form

\[ q^{-1} G_{(k-1)j+i-1}(q) + G_{(k-1)j+i}(q) = q^{2j(i-1)} G_{(k-1)(j-1)+k-i+1}(q) - q^{2j(i-1)} G_{(k-1)(j-1)+k-i+2}(q) \]  

(5.1)

for \( j \geq 1 \) and \( i = 2, \ldots, k \). Recall also the edge-matching tautology

\[ G_{(k-1)j+1}(q) = G_{(k-1)(j-1)+k}(q) \]  

(5.2)

for \( j \geq 1 \).

Define the vector

\[ \mathbf{G}(0) = \begin{bmatrix} G_1(q) \\ \\ \vdots \\ G_k(q) \end{bmatrix} \]

and more generally for each \( j \geq 0 \) define the vector

\[ \mathbf{G}(j) = \begin{bmatrix} G_{(k-1)j+1}(q) \\ \\ \vdots \\ G_{(k-1)j+k}(q) \end{bmatrix}. \]

For each \( j \geq 1 \) set

\[ \mathbf{B}(j) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & q^{-2j} & -q^{-2j} \\ 0 & 0 & \cdots & q^{-4j} & -q^{-4j} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & q^{-2(k-2)j} & \cdots & 0 & 0 & 0 \\ q^{-2(k-1)j} & -q^{-2(k-1)j} & \cdots & 0 & 0 & 0 \end{bmatrix} \]

and set

\[ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ q^{-1} & 1 & 0 & \cdots & 0 & 0 \\ 0 & q^{-1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & q^{-1} & 1 \end{bmatrix}. \]

Then (5.1) and (5.2) can be combined in the single matrix equation

\[ \mathbf{CG}(j) = \mathbf{B}(j) \mathbf{G}(j-1) \]  

(5.3)

for each \( j \geq 1 \).

It is easy to check that the inverse of \( \mathbf{B}(j) \) is the matrix

\[ \mathbf{A}(j) = \mathbf{B}(j)^{-1} = \begin{bmatrix} 1 & q^{2j} & q^{4j} & \cdots & q^{2(k-1)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & q^{2j} & q^{4j} & \cdots & 0 \\ 1 & q^{2j} & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}. \]  

(5.4)
Setting
\[ A'_j = A_j C \]
for each \( j \geq 1 \), we have:
\[
A'_j = A_j C = \begin{bmatrix}
1 + q^{2j-1} & q^{2j} + q^{4j-1} & \cdots & q^{2(k-2)j} + q^{2(k-1)j-1} & q^{2(k-1)j} \\
1 + q^{2j-1} & q^{2j} + q^{4j-1} & \cdots & q^{2(k-2)j} & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 + q^{2j-1} & q^{2j} & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}. \tag{5.5}
\]

For the remainder of this paper, we fix an integer
\[ J \geq 0, \tag{5.6} \]
which will indicate a “starting shelf” that need not be the 0th shelf. We will use \([5.3]\) to express \( G_j \) in terms of the \( G_j \), \( j \geq J \), and we will then use the Empirical Hypothesis to determine an expression for \( G_j \) which, in the case \( J = 0 \), will be different from the original definition \([2.2]\).

For any \( j \geq J \), repeated application of \([5.3]\) gives us
\[ G_j = A'_{(j+1)} \cdots A'_j G_j. \]
That is,
\[
\begin{bmatrix}
G_{(k-1)J+1}(q) \\
\vdots \\
G_{(k-1)J+k}(q)
\end{bmatrix} = A'_{(j+1)} \cdots A'_j G_j = J_h^{(j)} G_j, \tag{5.7}
\]
where
\[ J_h^{(j)} = A'_{(j+1)} \cdots A'_j \]
for \( j > J \). (We take \( J_h^{(j)} \) to be the empty product of matrices, i.e., the identity matrix.)

For each \( j \geq J \) and \( i = 1, \ldots, k \), define the row vector \( J_i h^{(j)}(q) \) to be the \( i \)th row of \( J_h^{(j)} \), and for \( l = 1, \ldots, k \), define \( J_i h^{(j)}_{l}(q) \) to be the \( l \)th component of \( J_i h^{(j)}(q) \). Then it is clear from \([5.7]\) that for any \( j \geq J \) and \( i = 1, \ldots, k \),
\[
G_{(k-1)J+i}(q) = J_i h^{(j)}_1(q) G_{(k-1)J+1}(q) + \cdots + J_i h^{(j)}_k(q) G_{(k-1)J+k}(q). \tag{5.8}
\]
Moreover, the \( J_i h^{(j)}_l(q) \) are polynomials in \( q \) with nonnegative integral coefficients, since the entries of \( A'_j \) are polynomials in \( q \) with nonnegative integral coefficients for each \( j \) by \([5.5]\). In addition, we have the following proposition, which will be needed for the combinatorial interpretation of the polynomials \( J_i h^{(j)}_l(q) \) in the next section:

**Proposition 5.1.** The polynomials \( J_i h^{(j)}_l(q) \) are determined by the initial conditions
\[ J_i h^{(j)}_{l}(q) = \delta_{i,l} \tag{5.9} \]
and the recursions
\[
J_i h^{(j)}_{l}(q) = q^{2j-l+1} \sum_{m=1}^{k-l+1} J_i h^{(j-1)}_{m}(q) + q^{2j-l} \sum_{m=1}^{k-l} J_i h^{(j-1)}_m(q) \tag{5.10}
\]
for \( j > J \).
Proof. The initial conditions (5.9) amount to the definition of $J_h^{(J)}$, and the recursions (5.10) are simply the equation

$$J_h^{(j)}(q) = J_h^{(j-1)}(q)A_j'$$

in component form.

Let us note a few consequences of (5.8), (5.10) and the Empirical Hypothesis. We first explain what we mean by the limit of a sequence $\{H_j(q)\}_{j=1}^{\infty}$ of formal power series: We say that

$$\lim_{j \to \infty} H_j(q)$$

exists if for each $i \geq 0$, there is some $J_i > 0$ such that the coefficients of $q^i$ in each series $H_j(q)$ for $j \geq J_i$ are equal. In other words, the limit exists if the coefficient of each power of $q$ in $H_j(q)$ stabilizes as $j \to \infty$. In this case, the limit of the $H_j(q)$ is the formal power series for which the coefficient of each power of $q$ stabilizes as $j \to \infty$. In this case, the limit of the $H_j(q)$ is the formal power series for which the coefficient of each power of $q$ stabilizes as $j \to \infty$. Hence,

$$\lim_{j \to \infty} J_h^{(j)}(q)$$

exists, which we denote by

$$J_h^{(\infty)}(q).$$

We have in fact proved more, which we record as a theorem:

**Theorem 5.2.** For any $J \geq 0$ and $i = 1, \ldots, k$,

$$G_{(k-1)J+i}(q) = J_h^{(\infty)}(q).$$

**Proof.** It follows from (5.11) and the Empirical Hypothesis that

$$G_{(k-1)J+i}(q) = \left(\lim_{j \to \infty} J_h^{(j)}(q) \right) \left( \lim_{j \to \infty} G_{(k-1)J+j}(q) \right) + \cdots + \left( \lim_{j \to \infty} J_h^{(k)}(q) \right) \left( \lim_{j \to \infty} G_{(k-1)J+k}(q) \right)$$

$$= J_h^{(\infty)} \cdot 1 + 0 + \cdots + 0$$

$$= J_h^{(\infty)}.$$

**Remark 5.3.** Note that “sum sides” of the Göllnitz-Gordon-Andrews identities have not emerged in Theorem 5.2 yet. In other words, we have yet to attach combinatorial meaning to the right-hand side of (5.12) (we will do this in the following section.). Nonetheless, Theorem 5.2 is the essence of the “motivated proof” since the left-hand side of (5.12) is concerned with certain congruence conditions (recall Remark 2.1) and these congruence conditions are completely invisible on the right-hand side of (5.12). Of course, this remark also holds in the context of the Rogers-Ramanujan identities (cf. [AB]) and in the more general setting of the Gordon identities as well (cf. [LZ] and Remark B.6).
6 Combinatorial interpretation of the sequence of expressions for $G_1(q)$

We are now ready to complete our motivated proof. We will deduce combinatorial interpretations of the formal power series on all of the shelves, and in particular, we will prove the Göllnitz-Gordon-Andrews identities, which correspond to the $0^{th}$ shelf, i.e., $J = 0$. We proceed as follows: We first present a combinatorial interpretation of the polynomials $\tilde{J} h_i^{(j)}(q)$, then we take the limit $j \to \infty$, and finally we use Theorem $5.11$.

**Proposition 6.1.** For $j \geq J + 1$, the polynomial $\tilde{J} h_i^{(j)}(q)$ is the generating function for partitions with parts $(b_1, \ldots, b_s)$ (with $b_p \geq b_{p+1}$), satisfying the following conditions:

1. No odd parts are repeated,
2. $b_p - b_{p+k-1} \geq 2$ if $b_p$ is odd,
3. $b_p - b_{p+k-1} > 2$ if $b_p$ is even,
4. the smallest part $b_s > 2J$,
5. there are no more than $k - i$ parts equal to $2J + 1$ or $2J + 2$, and
6. $b_1 \leq 2j$ and there are exactly $l - 1$ parts equal to $2j$.

**Proof.** Use $\tilde{J} h_i^{(j)}(q)$ to denote the generating function for partitions satisfying the conditions of the proposition. It is enough to verify that $\tilde{J} h_i^{(j+1)}(q) = \tilde{J} h_i^{(j+1)}(q)$ and that for $j \geq J + 2$, $\tilde{J} h_i^{(j)}(q)$ satisfies the recursion $\tilde{J} h_i^{(j+1)}(q)$.

In the case $j = J + 1$, there are two partitions satisfying the conditions of the proposition if $l \leq k - i$: one with $l - 1$ parts equal to $2J + 2$ and no parts equal to $2J + 1$, and one with $l - 1$ parts equal to $2J + 2$ and one part equal to $2J + 1$. If $l = k - i + 1$, only the partition with $l - 1$ parts equal to $2J + 2$ and no parts equal to $2J + 1$ satisfies the conditions of the proposition, and if $l > k - i + 1$, no partitions satisfy the conditions. Thus

$$\tilde{J} h_i^{(j+1)}(q) = \begin{cases} q^{2(l-1)(J+1)} + q^{2(l+1)-1} & \text{if } l \leq k - i + 1 \\ q^{2(J+1)(l-1)} & \text{if } l = k - i + 1 \\ 0 & \text{if } l > k - i + 1, \end{cases}$$

which agrees with $\tilde{J} h_i^{(j+1)}(q)$ by inspection of $\tilde{J} h_i^{(j)}(q)$ (setting $j = J + 1$).

Now consider $j \geq J + 2$. For convenience, we say that a partition is of type $(k - 1, 2J, k - i)$ if it satisfies the first five conditions of the proposition. Then the partitions of type $(k - 1, 2J, k - i)$ having largest part at most $2j$ and having exactly $l - 1$ parts equal to $2j + 2$ can be divided into two sets: the ones having no parts equal to $2j - 1$ and the ones having one part equal to $2j - 1$. In the first case, condition 3 of the proposition implies that at most $k - l$ parts can be equal to $2j - 2$, which means that the number of partitions in the first case are enumerated by the generating function $q^{2(l-1)} \sum_{m=1}^{k-l+1} \tilde{J} h_i^{(j-1)}.$

In the second case, the partitions in question have the form $((2j)^l, 2j - 1, b_{l+1}, \ldots, b_s)$, where $(b_{l+1}, \ldots, b_s)$ is a partition of type $(k - 1, 2J, k - i)$ having largest part at most $2j - 2$. By condition 3 of the proposition, at most $k - l - 1$ parts of $(b_{l+1}, \ldots, b_s)$ can equal $2j - 2$. Conversely, we note that $(b_{l+1}, \ldots, b_s)$ can be any partition of type $(k - i, 2J, k - 1)$ having largest part at most $2j - 2$ and at most $k - l - 1$ parts equal to $2j - 2$: the fact that at most $k - l - 1$ parts are equal to $2j - 2$.
for \((b_{l+1}, \ldots, b_s)\) implies that \(b_k < 2j - 2\), which in turn implies conditions 2 and 3 for the larger partition \(((2j)^{l-1}, 2j - 1, b_{l+1}, \ldots, b_s)\), while conditions 1, 4, 5 and 6 for the larger partition follow immediately from the same conditions for \((b_{l+1}, \ldots, b_s)\). Thus the partitions in the second case are enumerated by the generating function \(q^{2j(l-1)+2j-1} \sum_{m=1}^{k-1} J_{m}^{(j-1)}(q)\).

In conclusion, we see that for any \(i, l = 1, \ldots, k\),

\[
J_{i}^{(j)}(q) = q^{2j(l-1)} \sum_{i=1}^{k-l+1} J_{i}^{(j-1)}(q) + q^{2j(l-1)+2j-1} \sum_{m=1}^{k-l} J_{i}^{(j-1)}(q),
\]

which is the recursion (5.10). This proves the proposition. \(\square\)

In the case \(l = 1\), condition 6 in Proposition 6.1 just says that the largest part of the partition is strictly less than \(2j\). Thus \(J_{1}^{(j)}(q)\) is the generating function for partitions satisfying conditions 1 – 5 of Proposition 6.1 with largest part less than or equal to \(2j - 1\). Since each coefficient of \(J_{1}^{(j)}(q)\) agrees with the corresponding coefficient of \(J_{i}^{(j)}(q)\) for sufficiently large \(q\), we have:

**Proposition 6.2.** The formal power series \(J_{1}^{(\infty)}(q)\) is the generating function for partitions satisfying conditions 1 – 5 of Proposition 6.1 that is, \(J_{1}^{(\infty)}(q)\) is the generating function for partitions of type \((k - 1, 2J, k - i)\).

**Remark 6.3.** Note that it is obvious from conditions 1 and 4 of Proposition 6.1 that the Strong Empirical Hypothesis of Remark 4.4 holds for the series \(J_{1}^{(\infty)}(q)\) (recall Remark 4.6).

Now applying Theorem 5.2, we see that \(G_{(k-1),J+i}(q)\) for any \(J \geq 0\) and \(i = 1, \ldots, k\) is the generating function for partitions of type \((k - 1, 2J, k - i)\). In the special case \(J = 0\), we see that Remark 2.1, Theorem 5.2 and Proposition 6.2 immediately imply the Göllnitz-Gordon-Andrews identities:

**Theorem 6.4.** For any \(k \geq 1\) and \(i = 1, \ldots, k\), the number of partitions of a nonnegative integer \(n\) with parts not congruent to 2 mod 4 and not congruent to \(2k \pm (2i - 1)\ mod 4k\) is equal to the number of partitions \((b_1, \ldots, b_s)\) of \(n\) such that no odd parts are repeated, \(b_p - b_{p+k-1} \geq 2\) if \(b_p\) is odd, \(b_p - b_{p+k-1} \geq 2\) if \(b_p\) is even, and there are no more than \(k - i\) parts equal to 1 or 2.

**Remark 6.5.** The main ingredients in the proof of Theorem 6.4 are the recursive definition of the \(G_{i}(q)\)’s and the Weak Empirical Hypothesis (cf. Remark 4.5), but the determination of closed-form expressions for the \(G_{i}(q)\)’s could in principle be replaced with any other mechanism that yields the Weak Empirical Hypothesis.

**Remark 6.6.** (Cf. Remark 2.1 in [1Z]) As discussed in [AB], [R] and [A4], we can give an alternate, shorter proof of Theorem 6.4 using only the Empirical Hypothesis and without using Proposition 6.1. Let \(J_1(q), J_2(q), \ldots\) be a sequence of formal power series in \(q\) with constant term 1 satisfying the recursions (2.3) for \(j \geq 1\) (with \(J_p(q)\)’s in place of the \(G_p(q)\)’s), and suppose that the Empirical Hypothesis holds for \(J_1(q), J_2(q), \ldots\). In fact, \(J_1(q), J_2(q), \ldots, J_k(q)\), and therefore all \(J_1(q), J_2(q), \ldots\), are uniquely determined by the recursions (2.3) and the Empirical Hypothesis. If one recursively builds such a sequence of formal power series, say \(S_1(q), S_2(q), \ldots\), using (2.3) and starting from the combinatorial generating functions for the sum sides of the Göllnitz-Gordon-Andrews identities, then the Empirical Hypothesis is easily seen to hold for \(S_1(q), S_2(q), \ldots\). Since by Theorem 4.4 the Empirical Hypothesis holds for the sequence \(G_1(q), G_2(q), \ldots\), we must have that \(S_l(q) = G_l(q)\) for all \(l \geq 1\), by the uniqueness. For \(l = 1, \ldots, k\) this precisely gives the Göllnitz-Gordon-Andrews identities.
Appendices

A An \((x, q)\)-dictionary for the Göllnitz-Gordon-Andrews identities

The aim of this appendix is to compare the motivated proof of the Göllnitz-Gordon-Andrews identities given in this paper and the proof of these identities given in Chapter 7 of [A3]. Our main observation is the following: After setting up the correct dictionary between suitable specializations of the \(J_{k,i}(a, x, q)\) functions defined in (7.2.2) of [A3] and the \(G_i(q)\)'s defined in Section 2, the steps in the proof of Theorem 3.1 can be matched with those in the proofs of Lemmas 7.1 and 7.2 in [A3].

Let

\[
(y)_n = (1 - y) \cdots (1 - yq^{n-1}), \quad (y)_\infty = \prod_{m \geq 0} (1 - yq^m).
\]

We will use this notation in this appendix and in Appendix C.

Recall the definition (7.2.2) from [A3]:

\[
J_{k,i}(a, x, q) = \sum_{n \geq 0} x^{kn} q^{kn^2 + kn - in} a^n (1 - x^i q^{(2n+1)i})(axq^{n+2})_\infty (a^{-1})_\infty \frac{(q)_n (xq^{n+1})_\infty}{(q)_n (xq^{n+1})_\infty} - xqa \sum_{n \geq 0} x^{kn} q^{kn^2 + kn - (i-1)n} a^n (1 - x^{i-1} q^{(2n+1)(i-1)}) (axq^{n+2})_\infty (a^{-1})_\infty,
\]

for any \(k \geq 2\) and \(i = 0, \ldots, k + 1\). Note that for the “edge cases” \(i = 0\) and \(i = k + 1\), \(J_{k,i}(a, x, q)\) is still a formal power series in the variables \(x\) and \(q\), i.e., no negative powers of \(x\) or \(q\) appear.

Now, specializing \(q \mapsto q^2\), \(a \mapsto -q^{-1}\) and \(x \mapsto q^{2j}\), we get:

\[
J_{k,i}(-q^{-1}, q^{2j}, q^2) = \sum_{n \geq 0} q^{2(jkn + kn^2 + kn - in)} (-1)^n q^{-n} (1 - q^{2(j+2n+1)i}) (-q^{2(j+n+2)-1})_\infty (-q)_\infty \frac{(q^2)_n (q^{2(j+n+1)})_\infty}{(q^2)_n (q^{2(j+n+1)})_\infty} + q^{2(j+1)-1} \sum_{n \geq 0} q^{2(jkn + kn^2 + kn - (i-1)n)} (-1)^n q^{-n} (1 - q^{2(j+2n+1)(i-1)}) (-q^{2(j+n+2)-1})_\infty (-q)_\infty.
\]

Combining the sums and multiplying and dividing appropriately so that the denominator is

\[
(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1}) F(q)
\]

(recall the definition (2.5) of \(F(q)\)), we arrive at:

\[
J_{k,i}(-q^{-1}, q^{2j}, q^2) = \frac{1}{F(q)} \sum_{n \geq 0} \frac{(-1)^n q^{4k(j+2) + 2k(j+2) - 2i + 1} n (1 - q^{2(n+1)}) \cdots (1 - q^{2(n+j)})}{(1 + q^{2n+1})(1 + q^{2(n+1)+1}) \cdots (1 + q^{2(n+j)+1})} \cdot \left(1 - q^{2i(2n+1+j)} + q^{2(n+j)+1}(1 - q^{2(i-1)(2n+1+j)})\right).
\]
Comparing with (3.1), we obtain the following dictionary:

\[ G_{(k-1)j+i}(q) = J_{k,k-i+1}(-q^{i-1}, q^{2j}, q^2). \]  

(A.5)

The recursive definition of \( G_t(q) \) in (2.3) is motivated by Lemma 7.2 of [A3], which states that

\[ J_{k,i}(a,x,q) - J_{k,i-1}(a,x,q) = (xq)^{i-1}(J_{k,k-i+1}(a,x,q) - aJ_{k,k-i+2}(a,x,q)). \]

Making the replacements \( q \mapsto q^2, a \mapsto -q^{-1} \) and \( x \mapsto q^{2j} \) for \( j \geq 0 \), we obtain

\[ J_{k,i}(-q^{-1}, q^{2j}, q^2) - J_{k,i-1}(-q^{-1}, q^{2j}, q^2) = q^{2(j+1)(i-1)}(J_{k,k-i+1}(-q^{-1}, q^{2(j+1)}, q^2) + q^{-1}J_{k,k-i+2}(-q^{-1}, q^{2(j+1)}, q^2)). \]

Rearranging the terms, we see that

\[ J_{k,k-i+1}(-q^{-1}, q^{2(j+1)}, q^2) = \frac{J_{k,i}(-q^{-1}, q^{2j}, q^2) - J_{k,i-1}(-q^{-1}, q^{2j}, q^2)}{q^{2(j+1)(i-1)}} - q^{-1}J_{k,k-i+2}(-q^{-1}, q^{2(j+1)}, q^2). \]  

(A.6)

Under (A.5), edge-matching for the \( G_t(q)'s \) amounts to showing that for \( j \geq 0 \),

\[ J_{k,1}(-q^{-1}, q^{2j}, q^2) = J_{k,k}(-q^{-1}, q^{2(j+1)}, q^2). \]

Remark A.1. It is not hard to reverse-engineer the procedure of this section. That is, one can replace judiciously chosen instances of pure powers of \( q \) with appropriate powers of \( x \) in the proof of Theorem 3.1 and in this way our motivated proof would yield an “\((x,q)\)-proof” similar in spirit to the ones in Chapter 7 of [A3]. A similar observation was already made in Section 5 of [AB].

B Some remarks about [LZ]

The developments in the present paper suggest a number of enhancements to and comments on various aspects of [LZ], in which a motivated proof of Gordon’s generalization of the Rogers-Ramanujan identities was given. We proceed to describe these, referring in each case to particular sections, formulas, and/or results in [LZ].

B.1 The start of the induction in the proof of Theorem 2.1 of [LZ]

The first half of Theorem 2.1 of [LZ] entails proving equation (2.7) of [LZ] (recalled below in (B.1)), which gives closed forms for recursively defined series \( G_{(k-1)j+i}(q) \), where \( j \geq 0 \) and \( i = 1, \ldots, k \). (Note that the series \( G_{(k-1)j+i}(q) \) here are not the same as the \( G_{j+i}(q) \)'s defined in Section 2.) In this subsection, we comment on the step corresponding to \( j = 0 \) of the inductive proof of Theorem 2.1. For the inductive step corresponding to \( j = 0 \), the proof rests on equation (2.5) of [LZ], and for all higher \( j \)'s, the proof gives a variant of the argument in (2.5). However, the steps in the proof in fact hold for \( j = 0 \) also, thereby giving a slightly different proof of (2.5). In other words, an alternate way of presenting Theorem 2.1 would have been to omit the calculations in and around (2.5) and take \( j \geq 0 \) in the inductive step of the proof of Theorem 2.1. We now elaborate on this.

We first recall a few formulas from Section 2 of [LZ]. For \( i = 1, \ldots, k \), formula (2.2) in [LZ] is:

\[ G_i(q) = \frac{1 + \sum_{\lambda \geq 1} (-1)^\lambda q^{(2i+1)(\lambda)} \sum_{\lambda \geq 1} (1 + q^{2i-1}\lambda)}{\prod_{n \geq 1} (1 - q^n)}, \]

where
and formula (2.3) is:
\[
G_i(q) = \sum_{\lambda \geq 0} (-1)^{\lambda} q^{(2k+1)(\frac{\lambda}{2})+(k+i)\lambda} (1 - q^{(k-i+1)(2\lambda+1)}) \prod_{n \geq 1} (1 - q^n).
\]

Equation (2.2) of [LZ] was obtained by specializing formal variables in the Jacobi triple product identity and was used to obtain the closed-form expression for the series \(G_{k-1+i}(q)\) for \(i = 2, \ldots, k\) as in (2.5) of [LZ]:
\[
G_{k-1+i}(q) = \sum_{\lambda \geq 0} (-1)^{\lambda} q^{(2k+1)(\frac{\lambda}{2})+(k+i)\lambda} (1 - q^{\lambda+1})(1 - q^{(k-i+1)(2\lambda+2)}) \prod_{n \geq 1} (1 - q^n),
\]
where the series \(G_{k-1+i}(q)\) are defined recursively by:
\[
G_{k-1+i}(q) = \frac{G_{k-1+i-1} - G_{k-1+i+2}}{q^{i-1}}
\]
for \(i = 2, \ldots, k\).

We remark that although it was natural to use (2.2) to prove formula (2.5), it could have been omitted and all the computations could have been done starting with (2.3) instead of (2.2). In fact, formula (2.5) is a special case corresponding to \(j = 1\) of formula (2.7) in Theorem 2.1 in [LZ]:
\[
G_{(k-1)j+i}(q) = \sum_{\lambda \geq 0} (-1)^{\lambda} q^{(2k+1)(\frac{\lambda}{2})+[k(j+1)+i]\lambda} (1 - q^{\lambda+1}) \cdots (1 - q^{\lambda+j})(1 - q^{(k-i+1)(2\lambda+j+1)}) \prod_{n \geq 1} (1 - q^n).
\]
The proof of (2.7) used the analogue of (2.3) instead of the analogue of (2.2). As we mentioned, the proof of Theorem 2.1 refers to (2.5) for the truth of the case corresponding to \(j = 0\), but the steps in the proof hold for \(j = 0\) as well. As promised, the case \(j = 0\) gives an alternate proof of (2.5), this time using (2.3) instead of (2.2).

### B.2 Concerning the Empirical Hypothesis

Let us recall the formulation and proof (using Theorem 2.1 of [LZ]) of the Empirical Hypothesis from [LZ]. For \(j \geq 0\) and \(i = 1, \ldots, k\),
\[
G_{(k-1)j+i}(q) = \frac{1 - q^{(k-i+1)(j+1)}}{(1 - q^{j+1})(1 - q^{j+2}) \cdots} \sum_{\lambda \geq 1} (-1)^{\lambda} q^{(2k+1)(\frac{\lambda}{2})+[k(j+1)+i]\lambda} (1 - q^{\lambda+1}) \cdots (1 - q^{\lambda+j})(1 - q^{(k-i+1)(2\lambda+j+1)}) \prod_{n \geq 1} (1 - q^n).
\]

By (B.1),
\[
G_{(k-1)j+i}(q) = \begin{cases} 
1 + q^{j+1}\gamma_i^{(j+1)}(q) & \text{if } 1 \leq i \leq k - 1 \\
1 + q^{j+2}\gamma_k^{(j+2)}(q) & \text{if } i = k,
\end{cases}
\]
where
\[
\gamma_i^{(j)}(q) \in \mathbb{C}[[q]].
\]
In fact, (B.1) implies more: For \(i = 1, \ldots, k - 1\),
\[
\gamma_i^{(j+1)}(q) \in 1 + q\mathbb{C}[[q]],
\]
so that for \( i = 1, \ldots, k - 1, \)
\[
G_{(k-1)j+i}(q) = 1 + q^{j+1} + \cdots
\]
and for \( i = k, \)
\[
G_{(k-1)j+k}(q) = 1 + q^{j+2} + \cdots.
\]

Of course, analogues of Remarks 4.5, 4.6, and 6.5 in the present work are true in the setting of the Gordon identities:

**Remark B.1.** (Cf. Remark 4.5) Of the successively sharper forms of the Empirical Hypothesis we have arrived at, the only form that is really needed for the motivated proof is the weakest one, which states that for all positive integers \( l \) there exists a positive integer \( f(l) \) such that
\[
G_l(q) \in 1 + q^{f(l)}[[q]]
\]
with
\[
\lim_{l \to \infty} f(l) = \infty.
\]
We refer to this weakest form as the Weak Empirical Hypothesis.

**Remark B.2.** (Cf. Remark 6.5) The main facts used in the proof of the Gordon identities are the recursive definition of the \( G_l(q) \) and the Empirical Hypothesis. Therefore, in principle, the calculation of the closed-form expressions for the series \( G_l(q) \) could have been replaced with any other mechanism that entails the Empirical Hypothesis.

**Remark B.3.** (Cf. Remark 4.6) The (strongest form of the) Empirical Hypothesis is obvious from the sum sides of the Gordon identities. The point is to obtain it from only the corresponding product sides.

### B.3 J-generalization of statements in Proposition 2.1 of [LZ]

In the rest of this appendix, we give a series of \( J \)-generalizations (as in the body of the present paper; recall (5.6)) of the propositions and theorems in [LZ]. It is worth recalling that in [AB] — the special case of the Rogers-Ramanujan identities, i.e., \( k = 2 \) — the \( J = 3 \) case gave an answer to Ehrenpreis’s question (without resorting to the Rogers-Ramanujan identities themselves, demonstrating that the subtraction of the products, \( G_1(q) - G_2(q) \), has nonnegative coefficients), and the \( J = 1 \) and \( J = 2 \) cases led to a proof of the (two) Rogers-Ramanujan identities themselves.

Proposition 2.1 in [LZ] gives a recursion relation among the coefficients \( h^{(j)}_1(q) \) that arise when we express each \( G_i(q), 1 \leq i \leq k, \) in terms of \( G_{(k-1)j+i}(q), \ldots, G_{(k-1)j+k}(q) \) for arbitrary \( j \geq 0, \) as
\[
G_i(q) = h^{(j)}_1(q)G_{(k-1)j+1}(q) + \cdots + h^{(j)}_k(q)G_{(k-1)j+k}(q).
\]  
(B.2)

For each \( J \geq 0, \) \( G_{(k-1)j+i}(q) \) can be expressed analogously in such a way that the statements in [LZ] correspond to the case \( J = 0 \) and such that an analogue of Proposition 2.1 also holds.

Fix an integer \( J \geq 0. \) For each \( i = 1, \ldots, k \) and \( j \geq J, \) the use of (2.17) and (2.18) in [LZ] gives
\[
G_{(k-1)j+i}(q) = \sum_{j=1}^{k} \hat{h}^{(j)}_1(q)G_{(k-1)j+1}(q) + \cdots + \hat{h}^{(j)}_k(q)G_{(k-1)j+k}(q),
\]  
(B.3)
just as in (2.12). For each \( j \geq J \), the coefficients \( J_i h_j(q) \) form a \( k \times k \) matrix \( h_j(q) \) of polynomials in \( q \) with nonnegative integral coefficients. More explicitly, define row vectors
\[
J_i h_j(q) = [J_i h_j(q), \ldots, J_i h_k(q)].
\]
For \( j = J \) we have
\[
J_i h_J(q) = [0, \ldots, 1, \ldots, 0],
\]
with 1 in the \( i \)th position, so that \( J_i h_J(q) \) is the identity matrix. The \( J_i h_j(q) \)satisfy the same set of recursions with respect to \( j \), independently of \( i \). Explicitly:

**Proposition B.4.** Let \( j \geq J + 1 \). With the left subscript \( i \) suppressed, we have
\[
\begin{align*}
J_1 h_j(q) &= J_1 h_{j-1}(q) + \cdots + J_k h_{j-1}(q) + J_k h_{j-1}(q) \\
J_2 h_j(q) &= (J_1 h_{j-1}(q) + \cdots + J_k h_{j-1}(q))q^j \\
&\vdots \\
J_{k-1} h_j(q) &= (J_1 h_{j-1}(q) + J_2 h_{j-1}(q))q^{k-2j} \\
J_k h_j(q) &= J_1 h_{j-1}(q)q^{k-1j}
\end{align*}
\]
or in general,
\[
J_l h_j(q) = (J_1 h_{j-1}(q) + \cdots + J_{k-l+1} h_{j-1}(q))q^{(l-1)j}, \quad 1 \leq l \leq k. \tag{B.3}
\]
In matrix form, this is:
\[
J_h(q) = J h_{j-1} A_{(j)}, \tag{B.4}
\]
with \( J_h(q) \) the \( k \times k \) matrix defined above and with
\[
A_{(j)} = \begin{bmatrix}
1 & q^j & q^{2j} & \cdots & q^{(k-1)j} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & q^j & q^{2j} & \cdots & 0 \\
1 & q^j & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{B.5}
\]
In particular,
\[
J h_{(j)} = I \tag{B.6}
\]
(the identity matrix) and
\[
J h_{(j)} = A_{(j+1)} A_{(j+2)} \cdots A_{(j)} \tag{B.7}
\]
for all \( j > J \).

The proof is completely analogous to the proof of Proposition 2.1 in \( [LZ] \).

**Remark B.5.** The coefficients of the polynomials \( J_i h_j(q) \) for all \( j \geq J, i, l = 1, \ldots, k \) are nonnegative integers, as can be seen from equation (2.17) in \( [LZ] \) or, equivalently, from equations (B.6) and (B.7) in the matrix formulation above.
Remark B.6. From (B.3), (B.4) and the Empirical Hypothesis it is clear that

\[ G_{(k-1)J+i}(q) = \lim_{j \to \infty} J h_1^{(j)}(q) \left( \lim_{j \to \infty} G_{(k-1)j+1}(q) \right) + \cdots + \left( \lim_{j \to \infty} J h_k^{(j)}(q) \right) \left( \lim_{j \to \infty} G_{(k-1)j+k}(q) \right) \]

for all \( i = 1, \ldots, k \) (cf. Theorem 5.2 and Remark 5.3). This, combined with Remark B.5 immediately implies that the coefficients of all the \( G_i(q) \)'s are nonnegative integers. As mentioned earlier, for \( k = 2 \) and \( J = 3 \) this is precisely the answer to Ehrenpreis's question about the product sides of the (two) Rogers-Ramanujan identities.

B.4 \( J \)-generalization of statements in Proposition 2.2 of [LZ]

A similar generalization immediately works for Proposition 2.2 in [LZ]. Proposition 2.2 is the special case \( J = 0 \) of the following proposition:

**Proposition B.7.** For each \( j \geq J+1, i = 1, \ldots, k \) and \( l = 1, \ldots, k \), the polynomial \( J h_1^{(j)}(q) \) is the generating function for partitions with difference at least 2 at distance \( k-1 \), such that the smallest part is greater than \( J \), the part \( J+1 \) appears at most \( k-i \) times, the largest part is at most \( j \), and the part \( j \) appears exactly \( l-1 \) times.

**Proof.** Only trivial modification of the proof in [LZ] is needed. As in [LZ], we show that the combinatorial generating functions described here have the same initial values and recursions as the polynomials \( J h_1^{(j)}(q) \). We say that a partition is of type \((k-1, J, k-i)\) if it has difference at least 2 at distance \( k-1 \), smallest part larger than \( J \), and \( J+1 \) appearing at most \( k-i \) times. (Although we use the same notation, these are different from the partitions discussed in the proof of Proposition 6.1.) Similarly, (B.3) corresponds to the following combinatorial fact: For \( j \geq J+2 \),

the number of partitions of \( m \) of type \((k-1, J, k-i)\) such that the largest part is at most \( j \) and the part \( j \) appears exactly \( l-1 \) times

\[ = \sum_{p=1}^{k-l+1} \text{the number of partitions of } m - (l-1)j \text{ of type } (k-1, J, k-i) \text{ such that the largest part is at most } j-1 \text{ and the part } j-1 \text{ appears exactly } p-1 \text{ times}. \]

For \( J \geq 0 \), the initial values

\[ J h^{(J+1)} = [1, q^{J+1}, q^{2(J+1)}, \ldots, q^{(k-i)(J+1)}, 0, \ldots, 0] \]

also match those of the generating functions. \( \square \)

B.5 \( J \)-generalization of Theorem 2.2 of [LZ]

Correspondingly, Theorem 2.2 of [LZ] generalizes as follows:

**Theorem B.8.** For \( i = 1, \ldots, k \), \( G_{(k-1)J+i}(q) \) is the generating function for partitions with difference at least 2 at distance \( k-1 \) such that smallest part is greater than \( J \) and \( J+1 \) appears at most \( k-i \) times.

**Proof.** This follows immediately from (B.3), Proposition B.7 and the Empirical Hypothesis. \( \square \)
Remark B.9. It is interesting to note that Theorem B.8 is exactly Theorem 4.1 of [LZ], which was proved differently in [LZ].

Because of this remark, it is natural to give a $J$-generalization of Theorem 4.2 of [LZ] here:

**Theorem B.10.** For $l = 1, \ldots, k$, $J \geq 0$ and $j \geq J + 1$, the right-hand side of (B.3) expresses the generating function $G_{(k-1)J+l}(q)$ as the sum of its contributions corresponding to the number of times, namely, 0, 1, \ldots, $k - 1$, that the part $j$ appears in a partition.

**B.6 $J$-generalization of the matrix interpretation**

One can similarly express all the above $J$-generalizations in an illuminating matrix form:

Set

$$
G_{(0)} = \begin{bmatrix} G_1(q) \\ \vdots \\ G_k(q) \end{bmatrix}
$$

and in general,

$$
G_{(j)} = \begin{bmatrix} G_{(k-1)J+1}(q) \\ \vdots \\ G_{(k-1)J+k}(q) \end{bmatrix}
$$

for $j \geq 0$. Also set

$$
B_{(j)} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 1 \\
0 & 0 & \cdots & 0 & q^{-j} & -q^{-j} \\
0 & 0 & \cdots & q^{-2j} & -q^{-2j} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & 0 \\
q^{-(k-1)j} & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}
$$

for $j \geq 1$. Since

$$
G_{(j)} = B_{(j)} G_{(j-1)}
$$

for $j \geq 1$, one has

$$
G_{(j)} = B_{(j)} B_{(j-1)} \cdots B_{(J+1)} G_{(J)}
$$

for all $J \geq 0$ and $j \geq J$. Since

$$
B_{(j)} = (A_{(j)})^{-1}
$$

(recall (5.4)) we thus have

$$
G_{(j)} = A_{(J+1)} A_{(J+2)} \cdots A_{(j)} G_{(j)}
$$

for $j \geq J$. Defining $J h^{(j)}$ recursively by

$$
J h^{(j)} = I, \\
J h^{(j)} = J h^{(j-1)} A_{(j)}
$$

for $j \geq J$. Defining $J h^{(j)}$ recursively by
for \( j \geq J + 1 \), we have that
\[
J^j h(j) B(j) = J^{j-1} h^{j-1},
\]
\[
J^j h(j) = A_{(j+1)} \cdots A_{(j)}
\]
and
\[
G_{(j)} = J^j h^{(j)} G_{(j)}
\]
for each \( j \geq J \). Thus we have an “automatic” reformulation and proof of the \( J \)-generalized form of Proposition [B.21] including a \( J \)-generalized form of [B.3].

C An \((x, q)\)-dictionary for the Gordon identities

By analogy with what we did in Appendix A, we compare Section 2 of \([LZ]\) with Sections 7.1 and 7.2 of \([A3]\). Our main observation is the following (cf. Appendix A): After setting up the correct dictionary between suitable specializations of the formal series \( J_{k,i}(a, x, q) \) defined in (7.2.2) of \([A3]\) — \( x \) specialized to successively higher powers of \( q \) and \( a \) specialized to 0 — and the expressions \( G_l(q) \) defined in \([LZ]\), we can match the steps in the proof of Theorem 2.1 of \([LZ]\) with those in the proof of Lemma 7.1 in \([A3]\). We proceed to give this dictionary.

Fix \( k \geq 2 \) and let \( i = 0, \ldots, k+1 \). Let us first recall relevant notation from \([A3]\). Setting \( a \mapsto 0 \) in (7.2.1) and (7.2.2) of \([A3]\), we obtain:
\[
H_{k,i}(0, x, q) = \sum_{n \geq 0} (-1)^n x^{kn} q^{kn^2+n-i-n+\binom{n}{2}} (1 - x^i q^{2ni}) \frac{1}{(q)_n(xq^n)}\infty (C.1)
\]
\[
J_{k,i}(0, x, q) = H_{k,i}(0, xq, q) = \sum_{n \geq 0} (-1)^n x^{kn} q^{kn^2+n-i-n+\binom{n}{2}} (1 - x^i q^{i+2ni}) \frac{1}{(q)_n(xq^n)}\infty (C.2)
\]

Note that the expressions are well defined for the “edge cases” \( i = 0 \) and \( i = k + 1 \).

For \( j \geq 0 \), specializing \( x \mapsto q^j \) in (C.2), we obtain
\[
J_{k,i}(0, q^j, q) = H_{k,i}(0, q^{j+1}, q) = \sum_{n \geq 0} (-1)^n q^{jn} q^{kn^2+n-i-n+\binom{n}{2}} (1 - q^{j+i+2ni}) \frac{1}{(q)_n(q^{j+n+1})}\infty .
\]

We rewrite the sum so that the denominator in each summand is \((q)\infty\):
\[
J_{k,i}(0, q^j, q) = \sum_{n \geq 0} (-1)^n q^{n(2k+1)(\binom{n}{2})+(k(j+1)+k-i)n}(1-q^{n+1}) \cdots (1-q^{n+j})(1-q^{2n+j+1}) \frac{1}{(q)\infty} (C.3)
\]

and obtain
\[
J_{k,0}(0, q^j, q) = 0, \quad (C.4)
\]
\[
J_{k,k-i+1}(0, q^j, q) = \sum_{n \geq 0} (-1)^n q^{(2k+1)(\binom{n}{2})+(k(j+1)+i)n}(1-q^{n+1}) \cdots (1-q^{n+j})(1-q^{k-i+1}(2n+j+1)) \frac{1}{(q)\infty}. (C.5)
\]

It is now clear that for \( j \geq 0 \) and \( i = 0, \ldots, k + 1 \),
\[
G_{(1-k)j+i}(q) = J_{k,k-i+1}(0, q^j, q) \left( = H_{k,k-i+1}(0, q^{j+1}, q) \right), \quad (C.6)
\]

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and this is our desired dictionary.

With this setup it is now easy to match the steps in the proof of Lemma 7.1 of \[A3\] and the first half of the proof of Theorem 2.1 of \[LZ\], which is concerned with finding closed-form expressions for the series \(G_l(q)\).

We now give details about the “edge-matching,” which is the second assertion in Theorem 2.1 of \[LZ\]. Recall Lemma 7.2 of \[A3\], with \(a \mapsto 0:\)

\[
J_{k,i}(0, x, q) - J_{k,i-1}(0, x, q) = (xq)^{i-1}J_{k,k-i+1}(0, xq, q). \tag{C.7}
\]

The left-hand side of \( (C.6) \) exhibits the (tautological) “edge-matching” phenomenon

\[
G_{(k-1)j+k}(q) = G_{(k-1)(j+1)+1}(q).
\]

The same phenomenon for the right-hand side is demonstrated by specializing \( (C.7) \) with \( x \mapsto q^j \), \( i \mapsto 1 \) and noting \( (C.4) \):

\[
J_{k,1}(0, q^j, q) - J_{k,0}(0, q^j, q) = (q^j q)^{1-1}J_{k,k-1+1}(0, q^j+1, q)
\]

and hence

\[
J_{k,1}(0, q^j, q) = J_{k,k}(0, q^{j+1}, q).
\]

**Remark C.1.** An analogue of Remark \[A.1\] also holds in the context of the Gordon identities, with the reverse-engineering procedure applied to Theorem 2.1 of \[LZ\].

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