Emergent ballistic transport of Bose–Fermi mixtures in one dimension

Sheng Wang¹,², Xiangguo Yin³, Yang-Yang Chen¹,⁴, Yunbo Zhang⁵ and Xi-Wen Guan¹,⁶,⁷

¹ State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, Wuhan 430071, People’s Republic of China
² University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
³ Institute of Theoretical Physics, Shanxi University, Taiyuan 030006, People’s Republic of China
⁴ Shenzhen Institute for Quantum Science and Engineering, and Department of Physics, Southern University of Science and Technology, Shenzhen 518055, People’s Republic of China
⁵ Key Laboratory of Optical Field Manipulation of Zhejiang Province and Physics Department of Zhejiang Sci-Tech University, Hangzhou 310018, People’s Republic of China
⁶ NSFC-SPT Peng Huanwu Center for Fundamental Theory, Xian 710127, People’s Republic of China
⁷ Department of Theoretical Physics, Research School of Physics and Engineering, Australian National University, Canberra ACT 0200, Australia

E-mail: ybzhang@zstu.edu.cn and xwe105@wipm.ac.cn

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Abstract

The degenerate Bose–Fermi (BF) mixtures in one dimension present a novel realization of two decoupled Luttinger liquids with bosonic and fermionic degrees of freedom at low temperatures. However, the transport properties of such decoupled Luttinger liquids of charges is little known. Here we report on the transport properties of one-dimensional (1D) BF mixtures with delta-function interactions. The initial state is set up as the semi-infinite halves of two 1D BF mixtures with different temperatures, joined together at the time \( t = 0 \) and the junction point \( x = 0 \). Using the Bethe ansatz solution, we first rigorously prove the existence of conserved charges for both the bosonic and fermionic degrees of freedom, preserving the Euler-type continuity equations. Applying generalized hydrodynamics, we then analytically obtain the
distributions of the densities and currents of the local conserved quantities which solely depend on the ratio $\xi = x/t$. The left and right moving quasiparticle excitations of the two halves form multiple segmented light-cone hydrodynamics that display ballistic transport of the conserved charge densities and currents in different degrees of freedom. Such profiles reveal a novel dynamical separation of the two Luttinger liquids of fermionic and bosonic atoms in 1D. Our analytical results provide a deep understanding of the role of interaction and quantum statistical effects in quantum transport.

Keywords: Bose–Fermi mixture, generalized hydrodynamics, nonequilibrium dynamics, ballistic transport, thermodynamic Bethe ansatz

(Some figures may appear in colour only in the online journal)
local conserved quantities for the bosons and fermions as a function of the ratio $\xi = x/t$. Here $x$ is the distance from the junction and $t$ is the time. The quasiparticle excitations in the two halves elegantly determine the segmented multiple light-cone hydrodynamics that displays the ballistic transport of the densities and currents of the conserved charges. In contrast to the quantum transport of 1D XXZ spin chain [9], the nonlinear aspect of spectrum gives rise to different broadening features of the energy bands in distributions of densities and currents of bosons and fermions. Some distinctive behaviours of fermionic and bosonic quasi-particles are observed. In particular, the quasiparticles with bosonic degree of freedom leads to a back flow in the boson density current at the light cone. The subtle segmented charge–charge separation emerges in the steady state of the system within the light cones of the two degrees of freedom.

The model and conserved charges. A Bose–Fermi mixture system is described by the Hamiltonian [47, 48]

$$H = \int_0^L dx \left( \frac{\hbar^2}{2m_b} \partial_x \psi_b^\dagger \partial_x \psi_b + \frac{\hbar^2}{2m_f} \partial_x \psi_f^\dagger \partial_x \psi_f + \frac{g_{bf}}{2} \psi_b^\dagger \psi_b \psi_f^\dagger \psi_f + g_{bd} \psi_b^\dagger \psi_f^\dagger \psi_b \psi_f \right),$$  

(1)

where $\psi_b$ and $\psi_f$ are boson and fermion field operators and $m_b$ and $m_f$ are the masses of the bosons and fermions, respectively. We denote by $g_{bb}$ and $g_{bf}$ the boson–boson and boson–fermion interaction strengths, noting that their fermion-fermion counterpart is zero due to the Pauli exclusion statistics of spinless fermions. When $m_b = m_f = m$ and $g_{bb} = g_{bf} = g = -(2\hbar^2/ma_{1D})$, here $a_{1D}$ is an effective 1D scattering length, the system is integrable [44–48].

We consider a system with $N$ total particles of which $M$ are bosons and $N - M$ are spinless fermions. The model can be solved using a nested Bethe Ansatz in the framework of the quantum inverse scattering method, see appendix A. The spectrum $E = \sum_{i=1}^N k_i^2$ of the Hamiltonian (1) is determined by the quasi-momenta $\{k_i\}$ with $i = 1, \ldots, N$ and rapidities $\{\Lambda_n\}$ with $n = 1, \ldots, M$ subject to the Bethe Ansatz equations (BAE) $\prod_{i=1}^M \theta (k_i - \Lambda_n) = e^{\frac{ik_iL}{a_{1D}}}$, and $\prod_{j=1}^N \theta (k_j - \Lambda_n) = 1$, where $\theta(x) = (x + \frac{c}{2}) / (x - \frac{c}{2})$ with $c = -2/a_{1D}$. The solutions of the BAE give all the eigenstates.

We first prove the existence of two independent sets of conserved quantities of the BF mixture based on the polynomial form of its quantum transfer matrix, see appendix A. The eigenvalues of the quantum transfer matrix are given by the nested Bethe Ansatz

$$t(u) = \prod_{\alpha=1}^M \left( \frac{u - \Lambda_{\alpha} + ic}{u - \Lambda_{\alpha}} \right) \left( 1 - \frac{\prod_{j=1}^N (u - k_j)}{\prod_{j=1}^N (u - k_j - ic)} \right),$$  

(2)

exhibiting them as rational functions of the spectral parameter $u$. In general, the quantum transfer matrix $t(u)$ is the generating function of all conserved quantities for the integrable system. Therefore, all the coefficients of $t(u)$, when expanded as a polynomial in $u$ and $u^{-1}$, are conserved quantities of the system. These coefficients may be expressed as a product of two parts. One involves only the $\{k_i\}$, whereas the other involves only the $\{\Lambda_n\}$. The integrability suggests [47, 48] that there are $N$ conserved quantities of the first type. Thus the remaining $M$ coefficients, i.e. those involving only the $\{\Lambda_n\}$, are the independent conserved quantities in the bosonic degree of freedom.

In the thermodynamic limit, the BAE [54] can be written in terms of the distributions of particles $\rho(k)$, $\sigma_1(\Lambda)$ and holes $\rho_b(k)$, $\sigma_b(\Lambda)$ in the total charge and bosonic sectors
\[ \rho_\sigma(k) = \frac{1}{2\pi} + a(k) * \sigma(\Lambda), \]
\[ \sigma(\Lambda) = a(\Lambda) * \rho(k), \]
where \( a(x) = \frac{1}{2\pi} \delta(x - k) \) and \(*\) denotes the convolution \( a(x) * f(y) = \int_{-\infty}^{\infty} a(x - y)f(y)dy \). Following the Yang-Yang thermodynamic approach [55], we obtain the thermodynamic Bethe Ansatz equations (TBAE) for equilibrium states at finite temperatures \( T \) [54, 56],
\[ \varepsilon(k) = k^2 - \mu_\sigma - a(k) * f_\sigma(\Lambda), \]
\[ \varphi(\Lambda) = \mu_\sigma - \mu_\rho - a(\Lambda) * f_\rho(k), \]
where we denoted by \( \varepsilon(k) \) and \( \varphi(\Lambda) \) the energies of the excitations, \( f_{\rho,\sigma}(x) = T \ln \left( 1 + e^{-\frac{x}{T}} \right) \) and \( \mu_\rho \) and \( \mu_\sigma \) are the chemical potentials of the fermions and bosons. The BAE (3) and the TBAE (4) provide a convenient setting for determining the densities and currents of all conserved charges in the two degrees of freedom.

**Generalized hydrodynamics.** We apply the GHD [6, 7, 9] to calculate the distributions of densities and currents of all conserved charges of the model. To this end, we denote the two sets of single-particle conserved quantities by \( h_\rho^0(k) \) and \( h_\sigma^0(\Lambda) \). For example: \( h_\rho^0(k) = 1; \)
\[ h_\sigma^0(\Lambda) = \rho(k) = k; \]
\[ h_\rho^0(k) = \varepsilon(k) = k^2; \]
\[ h_\sigma^0(\Lambda) = 1, \]
Analogously, under the local equilibrium approximation, a non-equilibrium state (NES) of an integrable system can be identified by the distributions of all the conserved quantities \( q_1(x), q_2(x), \ldots, q_\rho(x), \ldots, x \in [0, L] \), more detailed description is given in appendix B. Then, we have expressions for the densities of all the conserved quantities \( \rho_\sigma(x,t) = \int dk b(k,x,t)h_\sigma^0(k) \), \( b = \rho, \sigma \).
Following the GHD approach [6, 7, 36], the expectation value of the currents is given by \( j_\sigma^0(x,t) = \int dk b(k,x,t)v_\sigma(k,x,t)h_\sigma^0(k) \) for a homogeneous stationary state. Here \( v_\sigma \) (with \( b = \rho, \sigma \)) are the sound velocities of the excitations in the total charge and the bosonic degrees of freedom with \( v_\rho(k) = \frac{\partial \varepsilon(k)}{\partial \rho_\rho^0(k)} \frac{\sigma(\Lambda)}{\rho_\rho^0(\Lambda)} \) and \( v_\sigma(\Lambda) = \frac{\partial \varphi(\Lambda)}{\partial \rho_\sigma^0(\Lambda)} \frac{\sigma(\Lambda)}{\rho_\sigma^0(\Lambda)} \). Here \( \rho_\sigma^0(k) \) and \( \rho_\sigma^0(\Lambda) \) denote the dressed momenta of the excitations in the two degrees of freedom. While the prime denotes the derivative of these dressed momenta and energy, explicitly
\[ p_\sigma^0(k) = 2\pi[\rho(k) + \rho_\rho(k)], \]
\[ p_\sigma^0(\Lambda) = 2\pi[\sigma(\Lambda) + \sigma_\sigma(\Lambda)]. \]

Building on the densities and currents of all conserved quantities in terms of Bethe ansatz densities via the above definitions of \( q_\sigma^0(x,t) \) and \( j_\sigma^0(x,t) \), and using the BAE (3) and TBAE (4), we can prove the continuity equations for the total charge and the bosonic degrees of freedom
\[ \partial_t n_\rho(k,x,t) + v_\rho(k,x,t) \partial_x n_\rho(k,x,t) = 0, \]
\[ \partial_t n_\sigma(\Lambda,x,t) + v_\sigma(\Lambda,x,t) \partial_x n_\sigma(\Lambda,x,t) = 0, \]
where we defined the occupation numbers \( n_\rho(k) = \frac{\rho_\rho(k)}{\rho_\rho(0) + \rho_\rho(k)} \) and \( n_\sigma(\Lambda) = \frac{\sigma(\Lambda)}{\sigma(0) + \sigma(\Lambda)} \), see appendix B.

We choose a special initial state, i.e. two semi-infinite 1D systems in equilibrium with \( T_L > T_R \) that are suddenly joined together, and then we let the whole system evolve under the Hamiltonian \( H \). Obviously, the initial state is not in equilibrium and the conserved charges transport from left to right. In view of the fact that the relaxation times in the fluid cells are much less than variational scales of the physical properties [6, 7, 9], the system has
a special scaling invariant property, i.e. under the scaling transformation \((x, t) \rightarrow (\alpha x, \alpha t)\), the GHD equation \((6)\) remain invariant. We then have the solution

\[
\begin{align*}
(v_p(k, \xi) - \xi)\partial_\xi n_p(k, \xi) &= 0 \\
(v_p(\Lambda, \xi) - \xi)\partial_\xi n_p(\Lambda, \xi) &= 0
\end{align*}
\]

with \(\xi = x/t\). From the initial state, we have the boundary conditions \(\lim_{\xi \rightarrow \pm \infty} n_p(k, \xi) = n_p^{RL}(k)\), and \(\lim_{\xi \rightarrow \pm \infty} n_p(\Lambda, \xi) = n_p^{RL}(\Lambda)\), which give the flowing solutions of equation \((7)\)

\[
\begin{align*}
n_p(k, \xi) &= n_p^k(k)H(v_p(k, \xi) - \xi) + n_p^R(k)H(\xi - v_p(k, \xi)), \\
n_p(\Lambda, \xi) &= n_p^k(\Lambda)H(v_p(\Lambda, \xi) - \xi) + n_p^R(\Lambda)H(\xi - v_p(\Lambda, \xi)),
\end{align*}
\]

where \(H(x)\) denotes the Heaviside step function, \(H(x) = 1\) for \(x \geq 0\) and zero otherwise. The equation \((8)\) indicates a ballistic transport of left- and right-moving quasiparticles in the transient region. This solution is not an explicit form because the whole solution relies on the GHD description and the exact Bethe ansatz solution, i.e. The equation \((8)\) depends on velocities which in turn depend on the occupation numbers \(n_{p\sigma}\).

In figure 1, we show occupation numbers \((8)\) in the charge and bosonic sectors for different values of \(\xi\). In \((b)–(d)\) and \((f)–(h)\), the filling areas show the occupation numbers of the total charges and bosons in both equilibrium states and NES, determining the density profiles in transport, also see figure 4(a). It is interesting to note that figure 1(a) shows that the effective velocity of excitations in the charge sector monotonously increases as the quasi-momentum increases, where the positive slope of the velocity in charge sector indicates positive effective mass in the nonlinear dispersion. However, in figure 1(e), the slope of the effective velocity of the bosons is observed to be negative for large \(|\Lambda|\), indicating negative effective mass in the nonlinear dispersion of quasiparticles. Such a subtle negative mass leads to the opposite tendencies of the velocity and the momentum in the nonlinear dispersion of quasiparticles. This leads to a back flow in the boson density current at the light cone \(\xi = v_p(\Lambda_0)\), see figure 4(b). Such a difference is mainly caused by different excitation spectra of fermionic and bosonic quasiparticles at higher momentum. In the long wave limit, i.e. for a small momentum, both velocities are always monotonously growing with momentum. More subtle different behaviour induced from the fermionic and bosonic degrees of freedom could be found in the dynamical evolution of correlation function. In order to understand quantum statistical effects in hydrodynamics of the 1D Bose–Fermi mixture, we will further investigate the GHD by the Haldane generalized exclusion statistics. This research will be published elsewhere.

**Emergent ballistic transport.** Calculating transport properties in terms of the GHD equation \((8)\) always imposes a big theoretical difficulty. At low temperatures, the distributions of the densities and currents of all conserved charges, i.e. both the charge and bosonic degrees of freedom, can be expressed as polynomials in the temperature, see appendix C. After a tedious analytical calculation, the excitation energy density distribution relative to the ground state is given by \(\delta E/L = \sum_{i=c,b} \delta v_i W_i\), where \(v_c = v_p(k_0)\) and \(v_b = v_p(\Lambda_0)\) are the Fermi velocities at \(T = 0\). The functions \(W_i\) are given by \(W_i = T_R^i H(\xi - v_i) + T_L^i H(-v_i - \xi) + \frac{1}{2}(T_L^i + T_R^i) H(v_i - |\xi|)\) with \(i = c, b\). We thus immediately obtain the excitation energy density and current for the steady state for \(\xi = 0\)

\[
\frac{\delta E_{steady}}{L} = \frac{1}{2} (T_L^2 + T_R^2) \sum_{i=c,b} \frac{\pi}{6v_i}.
\]
Figure 1. (a) and (e) show the velocities of the charge and the bosonic excitations, respectively. Here the asterisks correspond to the zero and the Fermi quasimomenta \( k = 0, \pm k_0 \) and \( \Lambda = 0, \pm \Lambda_0 \), respectively. (b)–(d) The occupation number \( n_\rho \) of the total charge is symmetric for the equilibrium state in the two halves: the blue solid line indicates the right half at \( T_R = 0.01 \) and the red circles indicate the left half at \( T_L = 2 \). The occupation number of the total charges is asymmetric for the NES. The green dashed lines show the occupation numbers determined by equation (8): (b) for \( \xi = v_\rho (k_0) \), (c) for \( \xi = 0 \) and (d) for \( \xi = v_\rho (-k_0) \), respectively. (f)–(h) The occupation number of the bosons shows a symmetric feature for the equilibrium states in the two halves: again a blue solid line for the right half at \( T_R = 0.01 \) and red circles for the left half at \( T_L = 0.2 \). The green dashed lines show the occupation numbers determined by the NES equation (8): (f) for \( \xi = v_\sigma (\Lambda_0) \), (g) for \( \xi = 0 \) and (h) for \( \xi = v_\sigma (-\Lambda_0) \), respectively. The filling areas for the NES (green dashed lines) in (b) and (h) are larger than the areas for the equilibrium states in the right and left halves, respectively, indicating peaks in the density profiles. On the other hand, the filling areas for the NES (green dashed lines) in (d) and (f) are smaller than the areas for the equilibrium states in the left and right halves, respectively, indicating valleys in the density profiles. (c) and (g) Show the occupation numbers for the steady state, indicating the average in the density profiles. \( \mu_b = 11, \mu_f = 12, c = 10 \) for all diagrams.

\[
\bar{J}^E = \frac{\pi^2}{6} (T_L^2 - T_R^2) \sum_{i=b, r} H (v_i - |\xi|). \tag{10}
\]

These show that the multiple light cone evolutions of the energy density and current are determined by the ballistic transport of the quasiparticles with local velocities \( v_\rho (k) \) and \( v_\sigma (\Lambda) \) in the two sectors. Solving the GHD equation (8), the BAE (3) and the TBAE (4) by iteration, we may obtain the distributions of energy, densities and currents of the conserved charges. Figure 2, elegantly presents the ballistic transport of the energy current within different light cones. In this figure, we clearly observe the crossover of broadening in the vicinities of the light cones. We observe that both the Bose–Ferm gas model (1) and spin-1/2 Fermi gas [13] share the similar light cone structure. However, the occupation numbers, particle densities and currents of the two models are quite different, see figures 1 and 4. In general, the statistics of quasiparticles in low energy would significantly affect the transport properties of conserved charges, depending on the effective masses of quasiparticles. Moreover, figure 3 further shows
Figure 2. The light cone diagram of the energy current of the 1D BF mixture after suddenly joining the two semi-infinite halves at different low temperatures. Here we set $T_L = 1$, $T_R = 0.5$, $\mu_b = 11$, $\mu_f = 12$ and $c = 10$. The steady state regions (constant value) and transition regions (peaks or valleys) show the distinguished features of the separated linear and non-linear Luttinger liquids, respectively, see the text.

Figure 3. Figures (a) and (b) respectively show the distributions of the density and current of the excitation energy at low temperatures. The dashed lines show the results from the analytical low-temperature expansion (9) and (10) which are in excellent agreement with the numerical results (solid lines), see appendix. In contrast the distributions of the densities and currents of the particles, see figure 4, there are no peak and valley in the distributions of the density and current of the excitation energy. These results were obtained with $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_f = 12$ and $c = 10$.

the transition and broadening in the vicinities of the light cones of the energy density and current, confirming the prediction from Luttinger liquid theory [9].

The change in the density of total charges is obtained in a unified form

$$\delta N/L = \sum_{i=c,b} F_i W_i$$

for the regions $|\xi - rv_i| > T_L |r|/m_i^* v_i$ with $i = c, b$ and $r = \pm 1$, see appendix C. In the above equations $F_i = \frac{\epsilon_i}{3k_i v_i} \left[ f_i'(\lambda_0) - \left( \frac{\partial}{\partial \lambda} + A(\lambda_0) \right) f_i(\lambda_0) \right]$. Here $K_c = \epsilon_0'(k_0)$, $K_b = \epsilon_0'(\Lambda_0)$, $v_i' = (v_i')'(k_0)$ and $v_i'' = (v_i'')'(\Lambda_0)$, while $f_i'(\lambda_0) = \left[ \partial f_i(\lambda)/\partial \lambda \right]_{\lambda=\lambda_0}$ and $f_i(\lambda)$ and $A_i(\lambda)$
Figure 4. Upper panel (a) and (b) (lower panel (c) and (d)) respectively show the distributions of the density and current of bosons (fermions) at low temperatures. The dashed lines show the results from the first order of the analytical low-temperature expansion (11) and (14) and are in excellent agreement with the numerical results (solid lines), see appendix. Good agreement is also seen for the round peaks (13) and (15) near the points $\xi = \pm v_{c,b}$. The back flows were observed at the points $\xi = \pm v_b$. These results were obtained with $T_L = 0.04$, $T_R = 0.01$, $\mu_b = 11$, $\mu_t = 12$ and $c = 10$.

satisfy the following equations

$$f_i(\lambda) = f_{i}^{\text{bare}} + \int_{-\lambda_0}^{\lambda_0} a(\lambda - \Lambda) f_i(\Lambda) d\Lambda,$$

$$\frac{K_i}{K_i} A_i(\lambda) = \int_{-\lambda_0}^{\lambda_0} a(\lambda - \Lambda) A_i(\Lambda) d\Lambda - \sum_{\ell = \pm 1} a(\lambda + \ell \lambda_0)$$

with $f_{i(c,b)}^{\text{bare}} = 1(0)$, and $\bar{c} = b$, $\bar{b} = c$, respectively.

In contrast, for the crossover region $|\xi - r v_i| < T_{L(R)}/m_i^* v_i$ with $i = c, b$ and $r = \pm 1$ (that denote right and left Fermi points, respectively), the nonlinear dispersion effect gives rise to the peak/valley densities of charges

$$\frac{\delta N_i}{L} = \sum_{r = \pm 1} \text{sgn} (r m_i^*) \frac{T_L f_i(\lambda_0)}{2\pi v_i} D_\eta \left( \frac{m_i^* v_i}{T_L} (\xi - r v_i) \right)$$

with $D_\eta (z) \equiv \ln (1 + e^z) - \eta \ln \left( 1 + e^{z/\eta} \right)$. Here the effective mass is defined by $m_i^* = K_i/(v_i^2 v_i)$. The equation (13) shows a universal broadening of the light cones in the low-temperature hydrodynamics of a 1D interacting BF mixture in both the charge and bosonic degrees of freedom.

We can moreover derive the particle current for the total charges in the transient region $|\xi - r v_i| > T_{L(R)}/m_i^* v_i$ with $i = c, b$ and $r = \pm 1$,

$$J = \left( T_L^2 - T_R^2 \right) \sum_{i = c,b} G_i H \left( v_i - |\xi| \right)$$
with \( G_i = \frac{\pi}{i2\Gamma} \left[ f'_i(\lambda_0) - A_i(\lambda_0) f_i(\lambda_0) \right] \), see appendix C. Similarly, in the vicinity of the light cones \( |\xi - rv_i| < T_{L(R)}/m_i^* v_i \) with \( i = c, b \) and \( r = \pm 1 \), the peaks (or valleys) of the particle current are given by

\[
J_i = \sum_{r=\pm} \text{sgn} \left( m_i^* \right) \frac{T_L f_i(\lambda_0)}{2\pi} D_{T_L} \left[ m_i^* v_i \left( \xi - rv_i \right) \right].
\]

(15)

From the results (11), (13), (14) and (15), we can directly obtain the distributions of the density and current of bosons and fermions which show excellent agreement with numerical calculations, see figure 4. Here we noted that the total particle number is conserved and the function \( f^{\text{bare}}_{c(b)} = 0(1) \) in equation (12) for the bosonic degree of freedom.

In summary, we have presented universal emergent transport properties of the 1D BF mixture, in which two decoupled Luttinger liquids are formed in the bosonic and fermionic degrees of freedom, respectively. We have analytically obtained the distributions of the densities and currents of the local conserved quantities that show the ballistic transport of the quasiparticles in both charge degrees. At low temperatures, our analytical expressions for the transport properties provide deep insights into understanding ballistic transport for integrable systems with both bosonic and fermionic fields. Subtle dynamical separation of charge-charge Luttinger liquids has been observed in terms of densities and currents of conserved quantities. This reveals the role of interaction and quantum statistical effects in quantum transport of multi-component Luttinger liquids. Although both the Bose–Fermi gas model (1) and spin-1/2 Fermi gas \[13\] share the similar light cone structure, the occupation numbers, particle densities and currents of the two models are quite different. The statistics of quasiparticles significantly affect the transport properties and dynamical evolution of correlation function of the model. Moreover, our approach can be applied to the study of quantum transport of 1D quantum gases with high spin symmetries. Such fluid-like behavior can be observed in current experiment with ultracold atoms, see the recent observation of the hydrodynamics in a single 1D cloud of \(^{87}\text{Rb}\) atoms trapped on an atom chip after a quench of the longitudinal trapping potential \[32\].

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Appendix A. Bethe ansatz and quantum inverse scattering method

Here, we give the exact solutions of Bose–Fermi mixture by nested Bethe Ansatz \[44–48, 57–59\]. The Bethe ansatz equations for a more general 1D Bose–Fermi mixture with \( SU(2|1) \) symmetry was given By Hu et al \[60\] in 2006. Here we present a detailed derivation of the Bethe ansatz equations for the 1D Bose–Fermi mixture with \( SU(1|1) \) symmetry by means of the quantum inverse scattering method. The Wavefunction of the system is supposed to be symmetric with respect to insices \( i = 1, \ldots, M \) (bosons) and antisymmetric with respect to \( i = M + 1, \ldots, N \) (fermions). We assume the coordinate Bethe wave function have the following
form: for $0 < x_{Q_1} < x_{Q_2} < \ldots < x_{Q_N} < L$

$$\Psi_\sigma(x) = \sum_P A_\sigma(P, Q)e^{i\sum k_P x_{Q_k}} ,$$

(16)

where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$, with $\sigma_j$ denoting the $SU(1|1)$ components of the $j$th particle; $P, Q$ are arbitrary permutations of the $S_N$.

By solving the Schrödinger equation, we have the boundary conditions

$$\left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) \Psi_{x_j=x_k^+} = \frac{2c}{\Psi_{x_j=x_k^-}}.$$  

(17)

From the continuity condition of the wave function, we have

$$\Psi_{x_j=x_k^-} = \Psi_{x_j=x_k^+}.$$  

(18)

Suppose $Q$ and $Q'$ are two permutations, such that

$$Q = (\ldots Q_aQ_b \ldots), \quad Q' = (\ldots Q_bQ_a \ldots), \quad b = a + 1.$$  

(19)

Similarly, $P$ and $P'$ are two permutations,

$$P = (\ldots P_aP_b \ldots), \quad P' = (\ldots P_bP_a \ldots).$$  

(20)

Then, for two different regions $Q$ and $Q'$, we have the Bethe wave functions as the following forms

$$Q: \Psi_\sigma(x) = \sum \left\{ A_\sigma(P, Q)e^{-ik_{Q_a}x_{Q_a}+ik_{Q_b}x_{Q_b}+\cdots} ight. 
+ A_\sigma(P', Q)e^{-ik_{Q_b}x_{Q_a}+ik_{Q_a}x_{Q_b}+\cdots} \right\},$$

(21)

$$Q': \Psi_\sigma(x) = \sum \left\{ A_\sigma(P, Q')e^{-ik_{Q_a}x_{Q_a}+ik_{Q_b}x_{Q_b}+\cdots} ight. 
+ A_\sigma(P', Q')e^{-ik_{Q_b}x_{Q_a}+ik_{Q_a}x_{Q_b}+\cdots} \right\}.$$  

(22)

Using the two conditions above, we have

$$A_\sigma(P, Q) = \frac{(kp_a - kp_b)P_{Q_aQ_b} + ic}{kp_a - kp_b - ic} A_\sigma(P', Q),$$  

(23)

where

$$A_\sigma(P', Q') = P_{Q_aQ_b}A_\sigma(P', Q).$$  

(24)

For simplifying the equation, we set

$$y_{P_aP_b}^{P_aP_b} = \frac{(kp_a - kp_b)P_{Q_aQ_b} + ic}{kp_a - kp_b - ic}.$$  

(25)
We can also define the permutation operator \( P_{\sigma_0, \sigma_0'} \) of two \( \sigma_0, \sigma_0' \), such that
\[
P_{\sigma_0, \sigma_0'} P_{\sigma_0, \sigma_0'} = 1, \quad \text{ (identical principle)}
\] (26)

\[
A_{\sigma_0, \sigma_0'}(P', Q) = P_{\sigma_0, \sigma_0'} P_{\sigma_0, \sigma_0'} A_{\sigma_0, \sigma_0'}(P', Q)
\]
\[
= P_{\sigma_0, \sigma_0'} A_{\sigma_0, \sigma_0'}(P', Q)
\] (27)

Then we have
\[
A_\sigma(P, Q) = \sum_{P_0} A_\sigma(P_0, Q_0) e^{ik_{P_0} x_{Q_0}}
\]
\[
= \sum_{P_0} A_\sigma(P_0, Q_0) e^{ik_{P_0} x_{Q_0}}
\] (28)

Finally, we have the two-body scattering operator,
\[
S_{Q_0, Q_0'}(k_{P_0} - k_{P_0}) = \frac{(k_{P_0} - k_{P_0}) - i e P_{\sigma_0, \sigma_0} A_\sigma(P', Q')}{k_{P_0} - k_{P_0} + i c}
\] (29)

To solve the problem of \( N \) particles in a box of length \( L \), we need the boundary conditions. Here we apply the periodic boundary conditions
\[
\Psi(x_1, \ldots, x_N) = \Psi(x_1, \ldots, x_i + L, \ldots, x_N)
\] (30)

on the wave function with a length \( L \) for \( i \)th particle with \( 1 \leq i \leq N \).

Without losing generality, we can take the position of the \( i \)th particle to be \( 0, x_i = 0 \). For the wave function to be confined in the region \( 0 \leq x_1 \leq x_2 \leq \ldots \leq x_N \leq L \), we have
\[
\Psi(x_i = 0, x_1, \ldots, x_N) = \Psi(x_1, \ldots, x_N, x_i = L)
\] (31)

Writing out both wave functions explicitly,
\[
\sum_P A_\sigma(P_1, \ldots, P_N; Q_1, \ldots, Q_N) e^{ik_{P_1} x_{Q_1} + \cdots + k_{P_N} x_{Q_N}}
\]
\[
= \sum_{P_0} A_\sigma(P_0, Q_1, \ldots, Q_N) e^{ik_{P_0} x_{Q_0}}
\] (32)

To compare terms on both sides of the equation with the same exponent, we have
\[
A_\sigma(P_1, \ldots, P_N; Q_1, \ldots, Q_N) = e^{ik_{P_1} A_\sigma(P_1, \ldots, P_N; P_0, Q_1, \ldots, Q_N)}
\] (33)

LHS
\[
= S_{i_1}(k_{i_1} - k_{i_1}) S_{i_2}(k_{i_2} - k_{i_2}) \ldots S_{i_{-1}}(k_{i_{-1}} - k_{i_{-1}}) A_\sigma(P, Q)
\]

RHS
\[
= S_{i_1}(k_{i_1} - k_{i_1}) S_{i_2}(k_{i_2} - k_{i_2}) \ldots S_{i_{-1}}(k_{i_{-1}} - k_{i_{-1}}) e^{ik_{i_1} A_\sigma(P, Q)}
\] (34)

Then we have
\[
S_{i_1}(k_{i_1} - k_{i_1}) \ldots S_{i_{-1}}(k_{i_{-1}} - k_{i_{-1}}) A_\sigma(P, Q)
\]
\[
e^{ik_{i_1} A_\sigma(P, Q)}
\] (35)

In order to obtain the exact solution of the system, we need to solve the eigenvalue equation above. By means of the algebraic Bethe Ansatz, this goal can be achieved. Considering the local
space as the Hilbert space $h_n = C^2$, then the entire state space of the system with $N$ particles is given by $H_N = \prod_{n=1}^{N} h_n$.

Let us first introduce the operator $R_j(u)$

\[
R_j(u) = \frac{u - ic \text{P}_{ij}}{u + ic}
\]  

which acts on the space $V_i \otimes V_j$. $V$ is auxiliary space, $V = C^2$, and $P_{ij}$ is the super permutation operator. Here the super tensor product is given by

\[
[A \otimes_s B]_{\alpha\beta,\gamma\delta} = (-1)^{p(\alpha) + p(\gamma)} A_{\alpha\beta} B_{\gamma\delta}
\]

where $p(i)$ with $i = 1, 2$ are the Grassmann parities $P(1) = 0$ and $P(2) = 1$.

Denoting the fermion and boson states as

\[
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then, all the operators can be expressed as the form of matrices.

For Bose–Fermi mixture system, we give

\[
P_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
R_j(u) = \frac{u - ic \text{P}_{ij}}{u + ic} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{u}{u + ic} & \frac{-ic}{u + ic} & 0 \\ 0 & \frac{u + ic}{u + ic} & \frac{u}{u + ic} & 0 \\ 0 & 0 & 0 & \frac{u + ic}{u + ic} \end{pmatrix}.
\]

Let us denote

\[
f(u) = \frac{u}{u + ic}; \quad g(u) = \frac{-ic}{u + ic}; \quad h(u) = \frac{u - ic}{u + ic}.
\]

The definition of the Lax operator on $j$ site involves the local quantum space $h_j$ and the auxiliary space $V$ which is $C^2$. It acts on $h_j \otimes V$ and its explicit expression is given by

\[
L_j(k_j - u) = f(k_j - u) + g(k_j - u)P_{jr} = \begin{pmatrix} a(k_j - u) & b(k_j - u) \\ c(k_j - u) & d(k_j - u) \end{pmatrix},
\]

where $a, b, c, d$ are functions of $k_j$.
where \(a(k_j - u), b(k_j - u), c(k_j - u), d(k_j - u)\) are operators which act in the space of \(h_j\).

\[
L_j(k_j - u)\uparrow > = \begin{pmatrix} 1 & g(k_j - u)\sigma_j^- \\ 0 & f(k_j - u) \end{pmatrix} \uparrow >.
\]  

(44)

The Monodromy operator of the system is defined as

\[
T_N(u) = L_N(k_N - u) \ldots L_1(k_1 - u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.
\]  

(45)

For Bose–Fermi mixture system, the Lax operator satisfies the graded Yang–Baxter relation and the Monodromy operator satisfies the graded RTT relation

\[
R_{12}(u - v)^1_{TN}(u)^2_{TN}(v) = \frac{2}{R_{12}(u - v)}TN(v)^1_{TN}(u)^2_{TN}.
\]  

(46)

The graded transfer matrix is defined as

\[
\tau(u) = s \text{ tr} T_N(u) = A(u) - D(u)
\]  

(47)

Then we can obtain the equivalent equation to (36) as following,

\[
\tau(k_j)A_{\sigma}(P, Q) = e^{ik_jL}A_{\sigma}(P, Q).
\]  

(48)

What we need to do next is to diagonalize \(\tau(u)\).

The definition of the reference state is given by

\[
|\Omega > = \uparrow >^N \ldots \uparrow >_1
\]  

(49)

\[
T_N(u)|\Omega > = \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^N f(k_j - u) \end{pmatrix} |\Omega >.
\]  

(50)

From the GRTT relation, we can obtain the commutation relations of the matrix elements of the monodromy matrix

\[
[B(u), B(v)] = 0,
\]

\[
A(u)B(v) = \frac{1}{f(u - v)}B(v)A(u) - \frac{g(u - v)}{f(u - v)}B(u)A(v),
\]

\[
D(u)B(v) = \frac{h(v - u)}{f(v - u)}B(v)D(u) + \frac{g(v - u)}{f(v - u)}B(u)D(v).
\]  

(51)

We assume that an arbitrary eigenstate of the system has the form \(B(\Lambda_1) \ldots B(\Lambda_M)|\Omega >\), a creation of \(M\) Bosons on the reference state \(\Omega >\) (the vacuum state). Then let the transfer matrix act on the eigenstate
\[
\tau(u) \prod_{j=1}^{M} B(\Lambda_j) | \Omega > = \left( \prod_{\alpha=1}^{M} \frac{1}{f(u - \Lambda_{\alpha})} - \prod_{\alpha=1,\alpha \neq \beta}^{M} \frac{h(\Lambda_{\alpha} - u)}{f(\Lambda_{\alpha} - u)} \prod_{j=1}^{N} f(k_j - u) \right) \prod_{j=1}^{M} B(\Lambda_j) | \Omega > \\
+ \prod_{\beta=1}^{M} g(\Lambda_{\beta} - u) \left( \prod_{\alpha=1,\alpha \neq \beta}^{M} \frac{1}{f(\Lambda_{\alpha} - \Lambda_{\beta})} - \prod_{\alpha=1,\alpha \neq \beta}^{M} \frac{h(\Lambda_{\alpha} - \Lambda_{\beta})}{f(\Lambda_{\alpha} - \Lambda_{\beta})} \prod_{j=1}^{N} f(k_j - \Lambda_{\beta}) \right) \\
\times B(u) \prod_{j=1,\beta \neq \beta}^{M} B(\Lambda_j) | \Omega > .
\] 

(52)

Based on the analyticity of the eigenvalue of the transfer matrix, thus we have

\[
t_2 = \prod_{\alpha=1,\alpha \neq \beta}^{M} \frac{1}{f(\Lambda_{\beta} - \Lambda_{\alpha})} - \prod_{\alpha=1,\alpha \neq \beta}^{M} \frac{h(\Lambda_{\alpha} - \Lambda_{\beta})}{f(\Lambda_{\alpha} - \Lambda_{\beta})} \prod_{j=1}^{N} f(k_j - \Lambda_{\beta}) = 0.
\]

(53)

The eigenvalue of the transfer matrix is given by

\[
t_1(u) = \prod_{\alpha=1}^{M} \frac{1}{f(u - \Lambda_{\alpha})} - \prod_{\alpha=1}^{M} \frac{h(\Lambda_{\alpha} - u)}{f(\Lambda_{\alpha} - u)} \prod_{j=1}^{N} f(k_j - u) \\
= \prod_{\alpha=1}^{M} \frac{u - \Lambda_{\alpha} + ic}{u - \Lambda_{\alpha}} - \prod_{\alpha=1}^{M} \frac{u - \Lambda_{\alpha} + ic}{u - \Lambda_{\alpha}} \prod_{j=1}^{N} \frac{u - k_j}{u - k_j + ic}.
\]

(54)

So that we have the Bethe Ansatz equations

\[
\begin{cases}
    t_1(k_j) = e^{ik_jL} \\
    t_2 = 0
\end{cases}
\]

(55)

which are

\[
\prod_{\alpha=1}^{M} \frac{k_j - \Lambda_{\alpha} + ic}{k_j - \Lambda_{\alpha}} = e^{ik_jL},
\]

(56)

\[
\prod_{j=1}^{N} \frac{k_j - \Lambda_{\beta} + ic}{k_j - \Lambda_{\beta}} = 1.
\]

Making a shift in \( \Lambda, \Lambda \to \Lambda + \frac{c}{2} \), then as same as the results of [47, 48], we obtain the Bethe Ansatz equations

\[
\prod_{\alpha=1}^{M} \frac{k_j - \Lambda_{\alpha} + \frac{c}{2}}{k_j - \Lambda_{\alpha} - \frac{c}{2}} = e^{ik_jL},
\]

(57)

\[
\prod_{j=1}^{N} \frac{k_j - \Lambda_{\beta} + \frac{c}{2}}{k_j - \Lambda_{\beta} - \frac{c}{2}} = 1.
\]
Appendix B. Generalized hydrodynamics

From the continuity equations
\[ \partial_t q_i^\rho(x, t) + \partial_x j_i^\rho(x, t) = 0, \]
\[ \partial_t q_i^\sigma(x, t) + \partial_x j_i^\sigma(x, t) = 0. \]  \hspace{1cm} (58)

we have
\[ \int dk \left( \partial_t \rho(k, x, t) + \partial_x (v_x \rho(k, x, t) \rho(k, x, t)) \right) h_i^\rho(k) = 0, \]
\[ \int d\Lambda \left( \partial_t \sigma(\Lambda, x, t) + \partial_x (v_x \sigma(\Lambda, x, t) \sigma(\Lambda, x, t)) \right) h_i^\sigma(\Lambda) = 0. \]  \hspace{1cm} (59)

For Bose–Fermi integrable systems, when we consider the excited states, the derivatives of momenta and energies have the following forms
\[ p_i^\rho(k) = 1 + \int_{-\infty}^{\infty} d\Lambda a(k, \Lambda) n_\sigma(\Lambda) p_i^\rho(\Lambda), \]
\[ p_i^\sigma(\Lambda) = \int_{-\infty}^{\infty} dk a(k, \Lambda) n_\rho(k) p_i^\sigma(k), \]  \hspace{1cm} (60)
\[ \epsilon'(k) = \epsilon'(k) + \int_{-\infty}^{\infty} d\Lambda a(k, \Lambda) n_\sigma(\Lambda) \varphi'(\Lambda), \]
\[ \varphi'(\Lambda) = \int_{-\infty}^{\infty} dk a(k, \Lambda) n_\rho(k) \epsilon'(k). \]  \hspace{1cm} (61)

From (60), we can easily find that
\[ p_i^\rho(k) = 2\pi[\rho(k) + \rho_h(k)], \]
\[ p_i^\sigma(\Lambda) = 2\pi[\sigma(\Lambda) + \sigma_h(\Lambda)]. \]  \hspace{1cm} (62)

These two equations give us not only analytical access to the dispersions but also the numerical results of sound velocities (64) for given \( n_\rho(k) \) and \( n_\sigma(\Lambda) \). Here we use them to simplify (59). We see that \( h_i^\rho(k) \) and \( h_i^\sigma(\Lambda) \) are the basis vectors of two function spaces. Therefore the two differential equations can be obtained from (59)
\[ \partial_t \rho(k, x, t) + \partial_x (v_x \rho(k, x, t) \rho(k, x, t)) = 0, \]
\[ \partial_t \sigma(\Lambda, x, t) + \partial_x (v_x \sigma(\Lambda, x, t) \sigma(\Lambda, x, t)) = 0. \]  \hspace{1cm} (63)

These two equations can be further simplified. The proof can be found below.

From (62) and the definition
\[ v_x^\rho(k) = \frac{\partial \epsilon(k)}{\partial p_i^\rho(k)} = \frac{\epsilon'(k)}{p_i^\rho(\Lambda)}, \]
\[ v_x(\Lambda) = \frac{\partial \varphi(\Lambda)}{\partial p_i^\sigma(\Lambda)} = \frac{\varphi'(\Lambda)}{p_i^\rho(\Lambda)}. \]  \hspace{1cm} (64)
we have
\[
\partial_\rho \rho_c(k, x, t) + \partial_\rho v_c(k, x, t) \rho_c(k, x, t) = \frac{1}{2\pi} \partial_\lambda \rho_c(k, x, t) p_{\rho c}^{\text{bar}}(k, x, t) + \frac{1}{2\pi} \partial_\lambda \rho_c(k, x, t) \varepsilon'(k, x, t)
\]
\[
= \frac{1}{2\pi} p_{\rho c}^{\text{bar}}(k, x, t) \left[ \partial_\lambda n_c(k, x, t) + v_c(k, x, t) \partial_\lambda n_c(k, x, t) \right]
\]
\[
+ \frac{1}{2\pi} n_c(k, x, t) \left[ \partial_\lambda p_{\rho c}^{\text{bar}}(k, x, t) + \partial_\lambda \varepsilon'(k, x, t) \right] = 0.
\]
\[
\partial_\sigma \sigma(\Lambda, x, t) + \partial_\sigma v_c(\Lambda, x, t) \sigma(\Lambda, x, t) = \frac{1}{2\pi} \partial_\lambda \sigma(\Lambda, x, t) p_{\sigma}^{\text{bar}}(\Lambda, x, t) + \frac{1}{2\pi} \partial_\lambda \sigma(\Lambda, x, t) \varepsilon'(\Lambda, x, t)
\]
\[
= \frac{1}{2\pi} p_{\sigma}^{\text{bar}}(\Lambda, x, t) \left[ \partial_\lambda n_c(\Lambda, x, t) + v_c(\Lambda, x, t) \partial_\lambda n_c(\Lambda, x, t) \right]
\]
\[
+ \frac{1}{2\pi} n_c(\Lambda, x, t) \left[ \partial_\lambda p_{\sigma}^{\text{bar}}(\Lambda, x, t) + \partial_\lambda \varepsilon'(\Lambda, x, t) \right] = 0.
\]  
(65)

From (60) and (61), we have the relations
\[
\partial_\lambda p_{\rho c}^{\text{bar}}(k, x, t) + \partial_\lambda \varepsilon'(k, x, t) = \int d\lambda \, a(k, \Lambda) \left[ \partial_\lambda n_c(\Lambda, x, t) p_{\rho c}^{\text{bar}}(\Lambda, x, t) \right]
\]
\[
+ \partial_\lambda n_c(\Lambda, x, t) \varepsilon'(\Lambda, x, t) = 0,
\]
\[
\partial_\lambda p_{\sigma}^{\text{bar}}(\Lambda, x, t) + \partial_\lambda \varepsilon'(\Lambda, x, t) = \int dk \, a(\Lambda, k) \left[ \partial_\lambda n_c(k, x, t) p_{\sigma}^{\text{bar}}(k, x, t) \right]
\]
\[
+ \partial_\lambda n_c(k, x, t) \varepsilon'(k, x, t) = 0.
\]  
(66)

Finally, as same as the results of given in [6, 7, 13, 14, 16, 22, 30], we obtain
\[
\partial_\lambda n_c(k, x, t) + v_c(k, x, t) \partial_\lambda n_c(k, x, t) = 0,
\]
\[
\partial_\lambda n_c(\Lambda, x, t) + v_c(\Lambda, x, t) \partial_\lambda n_c(\Lambda, x, t) = 0.
\]  
(67)

Appendix C. Analytical calculation of the quantum transport properties

In this section, we will use the explicit forms of the equations to calculate the distributions of densities and currents of conserved quantities. Now we proceed the calculation of the densities and currents of some conserved quantities at low temperatures. For our convenience in calculation, we first unify the TBA equation (4) in the main text for dressed energies of the total charge and bosons
\[
f_i^d(\lambda) = f_i^\text{bar}(\lambda) - T \int_{-\infty}^{\infty} a(\lambda - \lambda') \ln \left( 1 + e^{-\frac{\lambda' - \lambda}{T}} \right) d\lambda',
\]  
(68)

where we denote the dressed charges \(f_i^d(\lambda)\) with \(i = c, b\) for the charge and the bosonic degrees of freedom, respectively. In the above equations, the notations \(\hat{c} = b, b = c; \lambda = k, \Lambda\) and \(\hat{k} = \Lambda, \Lambda = k\) were implied and the bare quasi-particle excitations are given by \(f_i^\text{bar}(k) = k^2 - \mu_i\) and \(f_i^\text{bar}(\Lambda) = \mu_i - \mu_b\), respectively. At zero temperature, the unified TBA equation (68) become
\[
f_i^d(\lambda) = f_i^\text{bar}(\lambda) + \int_{-\lambda_0}^{\lambda_0} a(\lambda - \lambda)f_i^d(\lambda)d\lambda,
\]  
(69)
which will be essentially used in the calculation of densities and currents of all conserved quantities. The explicit forms of the equation (69) reads

\[ f^d_c(k) = f^\text{bare}_c(k) + \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) f^d_c(\Lambda) d\Lambda, \]
\[ f^d_b(\Lambda) = f^\text{bare}_b(\Lambda) + \int_{-\Lambda_0}^{\Lambda_0} a(\Lambda - k) f^d_c(k) dk \]

(70)

where we denote \( c \) and \( b \) for the charge and bosonic degrees of freedom, respectively.

Considering the expressions of the distributions of the densities and currents of the conserved quantities

\[ q(\xi) = \int_{-\infty}^{\infty} n_c(k, \xi) f^\text{bare}_c(k) \rho_c(k, \xi) dk + \int_{-\infty}^{\infty} n_s(\Lambda, \xi) f^\text{bare}_b(\Lambda) \sigma(\Lambda, \xi) d\Lambda, \]
\[ j(\xi) = \int_{-\infty}^{\infty} n_c(k, \xi) f^\text{bare}_c(k) v_c(k, \xi) \rho_c(k, \xi) dk + \int_{-\infty}^{\infty} n_s(\Lambda, \xi) f^\text{bare}_b(\Lambda, \xi) \sigma(\Lambda, \xi) d\Lambda, \]

we observe that analytical calculation of the GHD will involve the expressions of the densities and currents of conserved quantities, like the form \( \int_{-\infty}^{\infty} n_s(\Lambda, \xi) \mathcal{O}(\Lambda, \xi) d\Lambda \), here \( \mathcal{O}(\Lambda, \xi) \) stands for the functions involving in the densities and currents of conserved quantities. In addition, \( y \) denotes either the quasiparticle distribution functions of charge \( \rho \) or bosons \( \sigma \).

For the steady state reason, \( n_s(\Lambda, \xi) \sigma(\Lambda, \xi) = 0 \) \( \frac{\phi(\lambda)}{e^{\phi(\lambda)/2} + 1} \), where we denote \( \phi(\lambda) \) for dressed energy, then we can expand \( \int_{-\infty}^{\infty} n_s(\Lambda, \xi) \mathcal{O}(\Lambda, \xi) d\Lambda \) by the Sommerfeld expansion as following

\[ \int_{0}^{\infty} \frac{\mathcal{O}(\lambda)}{e^{\phi(\lambda)/2} + 1} d\lambda = \int_{0}^{\lambda'} \frac{d\lambda}{\phi(\lambda)} + \frac{\pi^2 T^2}{6} \frac{\mathcal{O}(\lambda')}{(\phi'(\lambda'))^2} + O(T^4), \quad (\phi'(\lambda')) = 0; \ T \to 0. \]

(73)

Here, \( \lambda' \) is the shifted Fermi point of the dressed energy at the finite temperatures. But what we want to achieve is that all the coefficients of the polynomials are only related to the ground state properties. Therefore we need to find the relations between \( \lambda' \) and \( \lambda_0 \), where \( \lambda_0 \) is the Fermi point of the dressed energy at zero temperature.

To this end, from the thermodynamic Bethe Ansatz equation (4), we have

\[ (\varepsilon(k) - \varepsilon(0)) = -T \int_{-\infty}^{\infty} a(k - \Lambda) \ln(1 + e^{-\varepsilon(\lambda)/T}) d\Lambda - \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) \varepsilon(\lambda) d\Lambda \]

\[ = \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) (\varepsilon(\lambda) - \varepsilon(0) d\Lambda + \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) \varepsilon(\lambda) d\Lambda \]

\[ + \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) \varepsilon(\lambda) d\Lambda - T \int_{-\infty}^{\infty} a(k - \Lambda) \ln(1 + e^{-|\varepsilon(\lambda)/T}) d\Lambda \]

\[ \approx \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) (\varepsilon(\lambda) - \varepsilon(0) d\Lambda - \frac{\pi^2 T^2}{6 \varepsilon'\lambda_0} (a(k - \lambda_0) + a(k + \lambda_0)) \]

(74)
Similarly, we have
\[
\varphi(\Lambda) - \varphi_0(\Lambda) = \int_{-k_0}^{k_0} a(\Lambda - k)(\varepsilon(k) - \varepsilon_0(k))dk
- \frac{\pi^2 T^2}{6\varepsilon'_0(k_0)}(a(\Lambda - k_0) + a(\Lambda + k_0)). \tag{75}
\]

We then define the following equations to simplify the above expressions,
\[
\varepsilon(k) - \varepsilon_0(k) = \frac{\pi^2 T^2}{6\varepsilon'_0(k_0)}A_c(k),
\]
\[
\varphi(\Lambda) - \varphi_0(\Lambda) = \frac{\pi^2 T^2}{6\varepsilon'_0(\Lambda_0)}A_b(\Lambda). \tag{76}
\]
From equations (74) and (75), we have the relations
\[
\frac{\varphi'_0(\Lambda_0)}{\varepsilon'_0(k_0)}A_c(k) = \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda)A_b(\Lambda)d\Lambda - (a(k - \Lambda_0) + a(k + \Lambda_0)),
\]
\[
\frac{\varepsilon'_0(k_0)}{\varepsilon'_0(\Lambda_0)}A_b(\Lambda) = \int_{-k_0}^{k_0} a(\Lambda - k)A_c(k)dk - (a(\Lambda - k_0) + a(\Lambda + k_0)). \tag{77}
\]
These are still regarded as the definitions of the functions of $A_c(k)$ and $A_b(\Lambda)$. Taking expansion with the left hand sides of equation (76) at the Fermi points, we obtain the differences between the shifted and zero temperature Fermi points, namely,
\[
k'_0 - k_0 = - \frac{\pi^2 T^2}{6\varepsilon'_0(k_0)}A_c(k_0), \tag{78}
\]
\[
\Lambda'_0 - \Lambda_0 = - \frac{\pi^2 T^2}{6\varepsilon'_0(\Lambda_0)}A_b(\Lambda_0),
\]
from which we see that the second and third terms in equation (74) can be safely neglected. Finally, we obtain the proper form which is presented as
\[
\int_{-\infty}^{\infty} n_i(\lambda)O(\lambda)d\lambda = \int_{-\Lambda_0}^{\Lambda_0} O(\lambda)d\lambda + \frac{\pi^2 T^2}{6\varepsilon'_0(\Lambda_0)} \left[ O'(\lambda_0) - O'(-\lambda_0)
- \left( \frac{\phi''_0(\lambda_0)}{\phi'_0(\Lambda_0)} + A_i(\lambda_0) \right) (O(\lambda_0) + O(-\lambda_0)) \right], \tag{79}
\]
where $i = c$ or $i = b$.

Now, let us process the low-temperature expansion for the NES equation (7) in the main text with the given initial state, which is set up as two semi-infinite halves with different temperatures joined together at $t = 0$. We first consider the steady state regions $|\xi - rv_i| > T_{L[R]/m_i^*v_i}$, where $v_i$ is fermi velocity at zero temperature, namely, $v_c = v_{p,0}(k_0)$ and $v_b = v_{\sigma,0}(\Lambda_0)$ with the subscript ‘0’ represents zero temperature. $m_i^* = K_i/(v_i^*v_i)$ is the effective mass with $i = c, b$ and $r = \pm 1$, $v'_c = (v_{p,0})'(k_0)$, $v'_b = (v_{\sigma,0})'(\Lambda_0)$, $K_c = \varepsilon'_0(k_0)$ and $K_b = \varepsilon'_0(\Lambda_0)$. equation (79) with the density $n_j$ given by the NES equation (7) can be used to calculate the low-temperature
expansion of the distributions of densities of the conserved quantities (71)

\[
q(\xi) = \int_{-\infty}^{\infty} \rho_{c}(k, \xi) f_c^\text{bare}(k) \rho_{c}(k, \xi) \, dk + \int_{-\infty}^{\infty} \rho_{\sigma}(\Lambda, \xi) f_b^\text{bare}(\Lambda) \rho_{\sigma}(\Lambda, \xi) \, d\Lambda
\]

\[
= q_0 + \int_{-k_0}^{k_0} f_c^\text{bare}(k) (\rho_{c}(k, \xi) - \rho_{\sigma,0}(k)) \, dk + \int_{-\Lambda_0}^{\Lambda_0} f_b^\text{bare}(\Lambda) (\rho_{\sigma}(\Lambda, \xi) - \sigma_{\sigma,0}(\Lambda)) \, d\Lambda
\]

\[
+ \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(v_c - \xi)
\]

\[
+ \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(-k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(-v_c - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
= q_0 + \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(v_c - \xi)
\]

\[
+ \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(-k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(-v_c - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
\text{The terms } \rho_{c}(k, \xi) - \rho_{\sigma,0}(k) \text{ and } \sigma_{\sigma}(\Lambda, \xi) - \sigma_{\sigma,0}(\Lambda) \text{ in (80) are needed to calculated from the Bethe Ansatz equations at finite and zero temperatures, namely}
\]

\[
\rho_{c}(k, \xi) - \rho_{\sigma,0}(k) = \int_{-\infty}^{\infty} a(k - \Lambda) n_{\sigma}(\Lambda, \xi) \sigma_{\sigma}(\Lambda, \xi) \, d\Lambda - \int_{-\infty}^{\infty} a(k - \Lambda) \sigma_{\sigma,0}(\Lambda) \, d\Lambda
\]

\[
= \int_{-\Lambda_0}^{\Lambda_0} a(k - \Lambda) (\sigma_{\sigma}(\Lambda, \xi) - \sigma_{\sigma,0}(\Lambda)) \, d\Lambda + \frac{\pi T^2_{1}}{12K_b v_b} \left( -a'(k - \Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) a(k - \Lambda_0) \right) H(v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( -a'(k - \Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) a(k - \Lambda_0) \right) H(-v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( a'(k - \Lambda_0) - \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) a(k - \Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( -a'(k - \Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) a(k + \Lambda_0) \right) H(\Lambda_0 + v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( a'(k + \Lambda_0) - \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) a(k + \Lambda_0) \right) H(\Lambda_0 + v_b - \xi)
\]

\[
= q_0 + \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(v_c - \xi)
\]

\[
+ \frac{\pi T^2_{1}}{12K_c v_c} \left( f_c^\text{bare}(-k_0) - \left( \frac{\Omega}{v_c} + A_c(k_0) \right) f_c^\text{bare}(-k_0) \right) H(-v_c - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
+ \frac{\pi T^2_{2}}{12K_b v_b} \left( f_b^\text{bare}(\Lambda_0) - \left( \frac{\Omega}{v_b} + A_b(\Lambda_0) \right) f_b^\text{bare}(\Lambda_0) \right) H(\Lambda_0 - v_b - \xi)
\]

\[
(81)
\]
Finally, from the equation (80) we have

\[
\sigma_l(\Lambda, \xi) = \sigma_{l,0}(\Lambda) = \int_{-\infty}^{\infty} a(\Lambda - k) \rho(\xi, k) \rho_0(k, \xi) \text{d}k - \int_{-k_0}^{k_0} a(\Lambda - k) \rho_0(k) \text{d}k
\]

\[
= \int_{-k_0}^{k_0} a(\Lambda - k)(\rho(k, \xi) - \rho_0(k)) \text{d}k + \frac{\pi T_R^2}{12 K_c v_c} \left( a'(\Lambda - k_0) - \left( \frac{\nu_f}{v_c} + A_c(k_0) \right) a(\Lambda - k_0) \right) H(\nu_c - \xi)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( -a'(\Lambda + k_0) - \left( \frac{\nu_f}{v_c} + A_c(k_0) \right) a(\Lambda + k_0) \right) H(-\nu_c - \xi)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( a'(\Lambda - k_0) - \left( \frac{\nu_f}{v_c} + A_c(k_0) \right) a(\Lambda - k_0) \right) H(\xi - \nu_c)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( -a'(\Lambda + k_0) - \left( \frac{\nu_f}{v_c} + A_c(k_0) \right) a(\Lambda + k_0) \right) H(\xi + \nu_c).
\]

(82)

After the multiplication and integration among the equations (70), (81) and (82), we have

\[
\int_{-k_0}^{k_0} f_{b}^{\text{bare}}(k)(\rho(k, \xi) - \rho_0(k)) \text{d}k + \int_{-\Lambda_0}^{\Lambda_0} f_{b}^{\text{bare}}(\Lambda)(\sigma_l(\Lambda, \xi) - \sigma_{l,0}(\Lambda)) \text{d}\Lambda
\]

\[
= \frac{\pi T_R^2}{12 K_c v_c} \left( f_{b}^{\text{bare}}(-\Lambda_0) - f_{b}^{\text{bare}}(\Lambda_0) \right) \left( \frac{\nu_f}{v_b} + A_b(\Lambda_0) \right) H(v_b - \xi)
\]

\[
+ \frac{\pi T_R^2}{12 K_b v_b} \left( f_{b}^{\text{bare}}(-\Lambda_0) - f_{b}^{\text{bare}}(\Lambda_0) \right) \left( \frac{\nu_f}{v_b} + A_b(\Lambda_0) \right) H(-v_b - \xi)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( f_{c}^{\text{bare}}(-\Lambda_0) - f_{c}^{\text{bare}}(\Lambda_0) \right) \left( \frac{\nu_f}{v_c} + A_c(\Lambda_0) \right) H(\xi + v_c)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( f_{c}^{\text{bare}}(-\Lambda_0) - f_{c}^{\text{bare}}(\Lambda_0) \right) \left( \frac{\nu_f}{v_c} + A_c(\Lambda_0) \right) H(\xi - v_c)
\]

\[
+ \frac{\pi T_R^2}{12 K_c v_c} \left( f_{c}^{\text{bare}}(-\Lambda_0) - f_{c}^{\text{bare}}(\Lambda_0) \right) \left( \frac{\nu_f}{v_c} + A_c(\Lambda_0) \right) H(\xi + v_c).
\]

(83)

Finally, from the equation (80) we have
\[ q(\xi) = q_0 + \frac{\pi}{12K_b v_b} \left( T^2 f^d_\xi(\Lambda) - T^2 f^d_\xi(-\Lambda) - (T^2 f^d_\xi(\Lambda) + T^2 f^d_\xi(-\Lambda)) \left( \frac{v'_c}{v_c} + A_\xi(\Lambda) \right) \right) H(v_b - |\xi|) + \frac{\pi}{12K_b v_b} \left( T^2 f^d_\xi(\Lambda) - T^2 f^d_\xi(-\Lambda) - (T^2 f^d_\xi(\Lambda) + T^2 f^d_\xi(-\Lambda)) \left( \frac{v'_c}{v_c} + A_\xi(\Lambda) \right) \right) H(-v_b - \xi) + \frac{\pi}{12K_b v_c} \left( T^2 f^d_\xi(k) - T^2 f^d_\xi(-k) - (T^2 f^d_\xi(k) + T^2 f^d_\xi(-k)) \left( \frac{v'_c}{v_c} + A_k(\Lambda) \right) \right) H(v_c - |\xi|) + \frac{\pi}{12K_b v_c} \left( T^2 f^d_\xi(k) - T^2 f^d_\xi(-k) - (T^2 f^d_\xi(k) + T^2 f^d_\xi(-k)) \left( \frac{v'_c}{v_c} + A_k(\Lambda) \right) \right) H(-v_c - \xi) \]

Now, we have the low-temperature expansion for the distributions of the densities of the conserved quantities, explicitly the dressed energies read

\[ f^d_\xi(k) = \varepsilon_0(k) = 0, \]
\[ f^d_\xi(\Lambda) = \varphi_0(\Lambda) = 0, \]
\[ f^d_\xi(k) = \varepsilon'_0(k) = K_c, \]
\[ f^d_\xi(\Lambda) = \varphi'_0(\Lambda) = K_b. \]

Using these dressed energies, we have the distributions of the density of energy as below,

\[ q_\varepsilon(\xi) = q_0 + \frac{\pi}{12K_b v_b} \left( T^2 f^d_\xi + T^2 f^d_\xi \right) H(v_b - |\xi|) + \frac{\pi T^2_b}{6v_b} H(-v_b - \xi) + \frac{\pi T^2_b}{6v_b} H(\xi - v_b) \]

\[ + \frac{\pi}{12K_b v_c} \left( T^2 f^d_\xi + T^2 f^d_\xi \right) H(v_c - |\xi|) + \frac{\pi T^2_b}{6v_c} H(-v_c - \xi) + \frac{\pi T^2_b}{6v_c} H(\xi - v_c). \]

Similarly, we calculate the distributions of the currents of the conserved quantities

\[ j(\xi) = \int_{-\infty}^{\infty} n_c(\xi, \Lambda, \varepsilon) j^\text{bare}_c(\Lambda, \xi) d\Lambda + \int_{-\infty}^{\infty} n_c(\Lambda, \varepsilon) j^\text{bare}_c(\Lambda, \xi) d\Lambda \]

\[ + \int_{-\infty}^{\infty} n_c(\Lambda, \varepsilon) j^\text{bare}_c(\Lambda, \xi) d\Lambda + \int_{-\infty}^{\infty} n_c(\Lambda, \varepsilon) j^\text{bare}_c(\Lambda, \xi) d\Lambda + \int_{-\infty}^{\infty} n_c(\Lambda, \varepsilon) j^\text{bare}_c(\Lambda, \xi) d\Lambda \]

\[ + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(v_c - \xi) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(-v_c - \xi) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(\xi - v_c) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(\xi + v_c). \]

\[ + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(v_b - \xi) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(-v_b - \xi) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(\xi - v_b) + \frac{\pi T^2_b}{12K_b} \left( f^\text{bare}_c(k) - A_k(\Lambda, \xi, \sigma) \right) H(\xi + v_b). \]

\[ \text{(87)} \]
Like the expansion of the distributions of the densities of conserved quantities, now we need to calculate $v_{\rho}(k, \xi) \rho_{\rho}(k, \xi)$ and $v_{\sigma}(\lambda, \xi) \sigma_{\lambda, \xi}$ and $v_{\sigma}(\lambda, \xi) \sigma_{\lambda, \xi} - v_{\rho}(k, \lambda) \rho_{\rho}(k, \lambda)$. From (64), (61) and (62), we have

$$
\begin{align*}
v_{\rho}(k, \xi) \rho_{\rho}(k, \xi) &= \frac{\varepsilon'_{\rho}(k, \xi)}{2\pi} = \frac{\varepsilon'_{\rho}(k)}{2\pi} + \int_{-\infty}^{\infty} a(k - \lambda) n_{\rho}(\lambda, \xi) v_{\rho}(\lambda, \xi) \sigma_{\lambda, \xi} d\lambda, \\
v_{\sigma}(\lambda, \xi) \sigma_{\lambda, \xi} &= \frac{\varepsilon'_{\sigma}(\lambda, \xi)}{2\pi} = \int_{-\infty}^{\infty} a(\lambda - k) n_{\rho}(k, \xi) v_{\rho}(k, \xi) \rho_{\rho}(k, \xi) d\lambda.
\end{align*}
$$

(88)

For the ground state, we have the following equation which is similar to the derivation of the equation (88)

$$
\begin{align*}
v_{\rho,0}(k) \rho_{\rho,0}(k) &= \frac{\varepsilon'_{\rho,0}(k)}{2\pi} = \frac{\varepsilon'_{\rho,0}(k)}{2\pi} + \int_{-\Lambda_0}^{\Lambda_0} a(k - \lambda) v_{\rho,0}(\lambda) \sigma_{\lambda,0}(\lambda) d\lambda, \\
v_{\sigma,0}(\lambda) \sigma_{\lambda,0}(\lambda) &= \frac{\varepsilon'_{\sigma,0}(\lambda)}{2\pi} = \int_{-\Lambda_0}^{\Lambda_0} a(\lambda - k) v_{\rho,0}(k) \rho_{\rho,0}(k) d\lambda.
\end{align*}
$$

(89)

Then it turns out

$$
\begin{align*}
v_{\rho}(k, \xi) \rho_{\rho}(k, \xi) - v_{\rho,0}(k) \rho_{\rho,0}(k) &= \int_{-\Lambda_0}^{\Lambda_0} a(k - \lambda) n_{\rho}(\lambda, \xi) v_{\rho}(\lambda, \xi) \sigma_{\lambda, \xi} d\lambda \\
&- \int_{-\Lambda_0}^{\Lambda_0} a(k - \lambda) v_{\rho,0}(\lambda) \sigma_{\lambda,0}(\lambda) d\lambda \\
&= \int_{-\Lambda_0}^{\Lambda_0} a(k - \lambda)(v_{\rho}(\lambda, \xi) \sigma_{\lambda, \xi} - v_{\rho,0}(\lambda) \sigma_{\lambda,0}(\lambda)) d\lambda \\
&+ \frac{\pi T^2}{24K_b} \left( a'(k - \Lambda_0) - A_b(\Lambda_0) a(k - \Lambda_0) \right) H(\nu_b - \xi), \\
&+ \frac{\pi T^2}{24K_b} \left( a'(k + \Lambda_0) + A_b(\Lambda_0) a(k + \Lambda_0) \right) H(-\nu_b - \xi) \\
&+ \frac{\pi T^2}{24K_b} \left( a'(k - \Lambda_0) - A_b(\Lambda_0) a(k - \Lambda_0) \right) H(\xi - \nu_b) \\
&+ \frac{\pi T^2}{24K_b} \left( a'(k + \Lambda_0) + A_b(\Lambda_0) a(k + \Lambda_0) \right) H(\xi + \nu_b).
\end{align*}
$$

(90)
\[ v_\sigma(\Lambda, \xi) \sigma_\tau(\Lambda, \xi) - v_{\sigma,0}(\Lambda) \sigma_{\tau,0}(\Lambda) = \int_{-\infty}^{\infty} a(\Lambda - k) n_\sigma(k, \xi) \rho_\tau(k, \xi) dk \]
\[ - \int_{-k_0}^{k_0} a(\Lambda - k) v_{\sigma,0}(k) \rho_\tau(0) dk \]
\[ = \int_{-k_0}^{k_0} a(\Lambda - k)(v_\sigma(k, \xi) \rho_\tau(k, \xi) - v_{\sigma,0}(k) \rho_\tau(0)) dk \]
\[ + \frac{2\pi T_1^2}{12K^2} \left( A_c(k_0) a_0(\Lambda - k_0) \right) H(v_c - \xi) \]
\[ + \frac{2\pi T_2^2}{12K^2} \left( A_c(k_0) a_0(\Lambda + k_0) \right) H(-v_c - \xi) \]
\[ + \frac{2\pi T_3^2}{12K^2} \left( A_c(k_0) a_0(\Lambda - k_0) \right) H(\xi - v_c) \]
\[ + \frac{2\pi T_4^2}{12K^2} \left( A_c(k_0) a_0(\Lambda + k_0) \right) H(\xi + v_c). \]

(91)

After multiplication and integration among the equations (70), (90) and (91), we have

\[ \int_{-k_0}^{k_0} f^{\text{bare}}(k)(v_\sigma(k, \xi) \rho_\tau(k, \xi) - v_{\sigma,0}(k) \rho_\tau(0)) dk + \int_{-k_0}^{k_0} f^{\text{bare}}(\Lambda)(v_\sigma(\Lambda, \xi) \sigma_\tau(\Lambda, \xi) - v_{\sigma,0}(\Lambda) \sigma_{\tau,0}(\Lambda)) d\Lambda \]
\[ = \frac{2\pi T_1^2}{12K^2} \left( f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0) \right) A_b(\Lambda_0)(f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0)) H(v_b - \xi) \]
\[ + \frac{2\pi T_2^2}{12K^2} \left( (f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0) + A_b(\Lambda_0)(f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0))) H(-v_b - \xi) \right) \]
\[ + \frac{2\pi T_3^2}{12K^2} \left( (f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0) + A_b(\Lambda_0)(f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0))) H(\xi - v_b) \right) \]
\[ + \frac{2\pi T_4^2}{12K^2} \left( (f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0) + A_b(\Lambda_0)(f^{\text{bare}}(\Lambda_0) - f^{\text{bare}}(\Lambda_0))) H(\xi + v_b) \right) \]
\[ + \frac{2\pi T_1^2}{12K^2} \left( f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0) - A_c(k_0)(f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0)) H(v_c - \xi) \right) \]
\[ + \frac{2\pi T_2^2}{12K^2} \left( f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0) + A_c(k_0)(f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0)) H(-v_c - \xi) \right) \]
\[ + \frac{2\pi T_3^2}{12K^2} \left( f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0) + A_c(k_0)(f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0)) H(\xi - v_c) \right) \]
\[ + \frac{2\pi T_4^2}{12K^2} \left( f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0) + A_c(k_0)(f^{\text{bare}}(k_0) - f^{\text{bare}}(k_0)) H(\xi + v_c) \right). \]

(92)

Finally, we have the low-temperature expansion for the currents of conserved quantities relative to the ground state.
\[ \Delta j(\xi) = \frac{\pi}{12K_b} \left( T_L^2 f^{dr}_b(\lambda_0) + T_R^2 f^{dr}_b(-\lambda_0) - (T_L^2 f^{dr}_b(\lambda_0) - T_R^2 f^{dr}_b(-\lambda_0))A_b(\lambda_0) \right) H(v_b - |\xi|) \\
+ \frac{\pi}{12K_b} \left( T_L^2 f^{dr}_b(\lambda_0) + T_R^2 f^{dr}_b(-\lambda_0) - (T_L^2 f^{dr}_b(\lambda_0) - T_R^2 f^{dr}_b(-\lambda_0))A_b(\lambda_0) \right) H(-v_b - \xi) \\
+ \frac{\pi}{12K_b} \left( T_L^2 f^{dr}_c(k_0) + T_R^2 f^{dr}_c(-k_0) - (T_L^2 f^{dr}_c(k_0) - T_R^2 f^{dr}_c(-k_0))A_c(k_0) \right) H(v_c - |\xi|) \\
+ \frac{\pi}{12K_c} \left( T_L^2 f^{dr}_c(k_0) + T_R^2 f^{dr}_c(-k_0) - (T_L^2 f^{dr}_c(k_0) - T_R^2 f^{dr}_c(-k_0))A_c(k_0) \right) H(-v_c - \xi) \\
+ \frac{\pi}{12K_c} \left( T_L^2 f^{dr}_c(k_0) + T_R^2 f^{dr}_c(-k_0) - (T_L^2 f^{dr}_c(k_0) - T_R^2 f^{dr}_c(-k_0))A_c(k_0) \right) H(\xi - v_c). \\
\] (93)

Moreover, substituting the dressed energy in the we obtain the distribution of the current of energy relative to the ground state

\[ \Delta j_e(\xi) = \frac{\pi}{12} \left( T_L^2 - T_R^2 \right) H(v_b - |\xi|) + \frac{\pi}{12} \left( T_L^2 - T_R^2 \right) H(v_c - |\xi|). \] (94)

We now consider the calculation of distributions of densities and currents of conserved quantities in the transition regions, i.e. \(|\xi - r v_i| < T_{Li}/m^*_i v_i\) with \(i = c, b\) and \(r = \pm 1\). In this case, the equation (79) fails. Instead, we have an equally important equation for the transition regions

\[ \int_{-\infty}^{\infty} d\lambda \, n_i(\lambda, \xi) \mathcal{O}(\lambda) = \int_{-\lambda_0}^{\lambda_0} \mathcal{O}(\lambda) d\lambda + \frac{\text{sgn}(v'_i) T_i}{K_i} \sum_{r=\pm 1} r \mathcal{O}(r \lambda_0) D_{r^2} \left( K_i \frac{\xi - r v_i}{T_i v'_i} \right), \] (95)

where \(D_\eta(z) \equiv \ln(1 + e^z - \eta) \ln(1 + e^{z/\eta})\) and \(i = b, c\). The process in calculating the low-temperature expansion in the transition regions is similar to what we did above for the steady state regions. Here, we just present the final results, for the regions \(|\xi - r v_i| < T_{Li}/m^*_i v_i\) with \(r = \pm 1\)

\[ \Delta q(\xi) = T_L \frac{\text{sgn}(v'_i)}{2 \pi v'_i} \left( f^{dr}_c(k_0) D_{r^2} \left( K_c \frac{\xi - v_c}{T_c v'_c} \right) - f^{dr}_c(-k_0) D_{r^2} \right) \times \left( K_c \frac{\xi + v_c}{T_c v'_c} \right) + O(T^2), \] (96)

\[ j(\xi) = T_L \frac{\text{sgn}(v'_i)}{2 \pi} \left( f^{dr}_c(k_0) D_{r^2} \left( K_c \frac{\xi - v_c}{T_c v'_c} \right) \\
+ f^{dr}_c(-k_0) D_{r^2} \left( K_c \frac{\xi + v_c}{T_c v'_c} \right) \right) + O(T^2). \] (97)

For the regions \(|\xi - r v_b| < T_{Li}/m^*_b v_b\) with \(r = \pm 1\)

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\[ q(\xi) = q_0 + T \frac{\text{sgn}(v'_b)}{2\pi v_b} \left( f'_b(\Lambda_0)D_T \frac{\xi - v_b}{T v'_b} - f'_b(-\Lambda_0)D_T \frac{\xi + v_b}{T v'_b} \right) \]
\[ \times \left( K_b \frac{\xi + v_b}{T v'_b} \right) + O(T^2), \] (98)

\[ j(\xi) = T \frac{\text{sgn}(v'_b)}{2\pi} \left( f'_b(\Lambda_0)D_T \frac{\xi - v_b}{T v'_b} \right) \]
\[ + f'_b(-\Lambda_0)D_T \frac{\xi + v_b}{T v'_b} \left( K_b \frac{\xi + v_b}{T v'_b} \right) + O(T^2). \] (99)

ORCID iDs

Xi-Wen Guan https://orcid.org/0000-0001-6293-8529

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