Quantum entanglement, after playing a significant role in the development of the foundations of quantum mechanics, has been recently rediscovered as a new physical resource with potential commercial applications such as, for example, quantum cryptography, better frequency standards or quantum-enhanced positioning and clock synchronization. On the mathematical side the studies of entanglement have revealed very interesting connections with the theory of positive maps. The capacity to generate entangled states is one of the basic requirements for building quantum computers. Hence, efficient experimental methods for detection, verification and estimation of quantum entanglement are of great practical importance. Here, we propose an experimentally viable, direct detection of quantum entanglement which is efficient and does not require any a priori knowledge about the quantum state. In a particular case of two entangled qubits it provides an estimation of the amount of entanglement. We view this method as a new form of quantum computation, namely, as a decision problem with quantum data structure.

Suppose we are given \( n \) pairs of particles, all in the same quantum state described by some density operator \( \rho \), which is unknown. We need to decide whether the particles in each pair are entangled or not. From a mathematical point of view we need to assert whether the Hilbert spaces associated with each particle are of finite dimensions \( d \) (taken to be the same for the two particles), so that one can always find \( k \leq d^2 \). If \( \rho \) were known then we could try either to find the decomposition (1) directly or to use one of the mathematical separability criteria (2). For sufficiently large \( n \) we may indeed start with the quantum state estimation, however, this involves estimating \( d^4 - 1 \) real parameters of \( \rho \), most of which are irrelevant in the context of the entanglement detection. In the following we describe a direct method of detecting quantum entanglement without invoking the state estimation.

We construct a measurement which can be performed on all copies of \( \rho \) and which is as powerful in detecting quantum entanglement as the best mathematical test based on positive maps (2). The measurement can be viewed as two consecutive physical operations: firstly, we construct a transformation which maps \( \rho \) into an appropriate state \( \rho' \) and, secondly, we measure the lowest eigenvalue of \( \rho' \). This eigenvalue alone serves as a separability indicator.

A convenient starting point for our construction is the most powerful, albeit purely mathematical and not directly implementable, separability criterion proposed to date. It is based on mathematical properties of linear positive maps acting on matrices. Let \( M_d \) be a space of matrices of dimension \( d \); recall that \( \Lambda : M_d \rightarrow M_d \) is called positive if \( X \geq 0 \) implies \( \Lambda(X) \geq 0 \) (expression \( X \geq 0 \) means that the matrix \( X \) has a nonnegative spectrum). If the induced map \( I \otimes \Lambda \) is also positive then \( \Lambda \) is called completely positive, and, as such, it represents a physically allowed transformation of density operators (here \( I \) denotes the identity map on an auxiliary system of any dimension). Using this terminology the separability criterion reads (2): \( \rho \) is separable iff

\[
[I \otimes \Lambda](\rho) \geq 0,
\]

for all positive but not completely positive maps \( \Lambda : M_d \rightarrow M_d \) acting on the second particle. In fact it is sufficient to consider only positive maps \( \Lambda \) such that the maximum of \( \text{Tr} \Lambda(\rho) \) over all \( \rho \) is equal to unity. Other positive maps differ only by a positive multiplicative factor which does not affect the condition (2).

Furthermore, in some cases, instead of scanning all positive maps, we can choose just one. For example, all entangled states of two qubits can be detected by choosing \( \Lambda \) to be transposition (1). The snag is that positive maps \( \Lambda \), such as an anti-unitary transposition, and the induced maps \( I \otimes \Lambda \) cannot be implemented in a laboratory. Thus, the criterion (2) tacitly assumes prior knowledge of \( \rho \). However, there is a way to modify it so that it becomes experimentally viable without involving any state estimation.

If we mix in an appropriate proportion \( [I \otimes \Lambda] \) with a depolarizing map that turns any density matrix into a maximally mixed state then the resulting map can be completely positive. This is because the lowest negative
The eigenvalues generated by the induced map \([\mathbb{I} \otimes I] \otimes (\mathbb{I} \otimes \Lambda)\) can be offset by the positive eigenvalues of the maximally mixed state generated by the depolarizing map. The most negative eigenvalue \(-\Lambda < 0\) is obtained when \((\mathbb{I} \otimes \mathbb{I}) \otimes (\mathbb{I} \otimes \Lambda)\) acts on the maximally entangled state of the form \(\frac{1}{\sqrt{d^2}} \sum_{i=1}^{d^2} |i\rangle \langle i|\), where each state \(|i\rangle\) pertains to a \(d^2\) dimensional subsystem which itself is composed of two \(d\) dimensional parts. Thus the map

\[
\hat{\Lambda} = \frac{1}{d^2} \sum_{i=1}^{d^2} \rho_{ii} - \frac{1}{d^2}
\]

is completely positive and therefore physically implementable when the induced map \(([\mathbb{I} \otimes \mathbb{I}) \otimes (\mathbb{I} \otimes \Lambda)]\) is positive, which happens for \(p \geq (d^2\lambda)/(d^2\lambda + 1)\) [1]. By inserting the threshold value \(p = (d^1\lambda)/(d^1\lambda + 1)\) into (2) we can modify the criterion (2) as follows: \(\rho\) is separable iff for all positive maps \(\Lambda\),

\[
[\mathbb{I} \otimes \Lambda]\rho \geq \frac{d^2\lambda}{d^2\lambda + 1},
\]

i.e., when the minimal eigenvalue of the transformed state \(\rho' = [\mathbb{I} \otimes \Lambda]\rho\) is greater than \((d^2\lambda)/(d^2\lambda + 1)\). In general, for some maps \(\Lambda\), the related completely positive maps \(\mathbb{I} \otimes \Lambda\) are not trace-preserving and require postselection in their physical implementations. Maps such as \(\mathbb{I} \otimes \Lambda\) have been referred to as "structural" physical approximations of unphysical maps \(\mathbb{I} \otimes \Lambda\) [11].

For example, if we take \(\Lambda\) to be transposition \(T\), (the first positive map used for detecting entanglement), we obtain

\[
[\mathbb{I} \otimes T]\rho = \frac{d}{d^3 + 1} I \otimes I + \frac{1}{d^3 + 1} [\mathbb{I} \otimes T]\rho.
\]

In the two qubit case, where the partial transposition is a sharp test for entanglement, we obtain,

\[
[\mathbb{I} \otimes T]\rho = \frac{2}{9} I \otimes I + \frac{1}{9} [\mathbb{I} \otimes T]\rho,
\]

which can be represented and implemented as

\[
\frac{1}{3} \Lambda_1 \otimes \Lambda_2 + \frac{2}{3} \otimes \sigma_x \sigma_y \Lambda_1 \sigma_x \sigma_y,
\]

with the two channels defined as:

\[
\Lambda_1(\rho) = \frac{1}{3} \sum_{i=x,y,z} \sigma_i \rho \sigma_i, \quad \Lambda_2(\rho) = \frac{1}{4} \sum_{i=x,y,z} \sigma_i \rho \sigma_i.
\]

The map can be implemented by applying selected products of unitary (Pauli) transformations with the prescribed probabilities. The map \(\Lambda\) is trace-preserving hence any postselection in experimental realizations is avoided.

Thus, in order to detect entanglement of an arbitrary two-qubit state \(\rho\) it is enough to estimate a single parameter, i.e. the minimal eigenvalue of \([\mathbb{I} \otimes T]\rho\). The state \(\rho\) is separable iff this eigenvalue satisfies \(\lambda_{\text{min}} \geq \frac{2}{d^2}\).

Let us also point out an extra bonus: \(\lambda_{\text{min}}\) gives us \(-\lambda'\), the most negative eigenvalue of \([\mathbb{I} \otimes T]\rho\), which enters the expression for the upper and lower bounds for the entanglement of formation,

\[
H \left( \frac{1 + \sqrt{1 - 4\lambda^2}}{2} \right) \leq E(\rho) \\
\leq H \left( \frac{1 + \sqrt{1 - 4(\sqrt{2\lambda^2 + \lambda' - \lambda})}}{2} \right),
\]

where \(H(x)\) is the Shannon entropy. The above formula can be derived from the estimations of the concurrence provided by Verstrate et al [12].

Suppose for a moment that \(\mathbb{I} \otimes \Lambda\) is trace-preserving, e.g. the transposition case. The first part of our entanglement detection measurement is accomplished by applying \(\mathbb{I} \otimes \Lambda\) to each of the \(n\) pairs to obtain \(n\) copies of \(\rho' = [\mathbb{I} \otimes \Lambda]\rho\). Then, following the criterion (4), we need to measure the lowest eigenvalue of \(\rho'\).

This can be viewed as a special case of the spectrum estimation and possible approaches depend a lot on particular physical realizations of \(\rho'\). Here, we provide two general solutions. The first one, based on quantum interferometry, is conceptually simple and relies on estimating \(d^2 - 1\) parameters from which the spectrum of \(\rho'\) can be calculated (this is a significant gain over the state estimation which involves \(d^3 - 1\) parameters). The second solution is a joint measurement on all copies of \(\rho'\) which gives directly the estimate of the lowest eigenvalue.

We start with the quantum interferometry, presented here as a quantum network shown in Fig.(1). A typical interferometric set-up for a single qubit — the Hadamard gate, phase shift \(\phi\), the Hadamard gate, followed by a measurement — is modified by inserting in between the Hadamard gates a controlled-\(U\) operation, with its control on the qubit and with \(U\) acting on a quantum system described by some unknown density operator \(\rho\). (N.B. we do not assume anything about the form of \(\rho\), it can, for example, describe several entangled or separable subsystems.) The action of the controlled-\(U\) on \(\rho\) modifies the interference pattern by the factor,

\[
\text{Tr} \rho U = \text{ve}^{i\alpha},
\]

where \(v\) is the new visibility and \(\alpha\) is the shift of the interference fringes, also known as the Pancharatnam phase [13]. Formula (10) has been derived, in the context of geometric phases, by Sjöqvist et al. [14].
Thus measuring the average values of \( V^{(k)} \) for \( k = 2, 3, \ldots, m \) gives us effectively the spectrum of \( \rho \). In particular, in our case, we obtain the spectrum (and the lowest eigenvalue) of \( \rho' = [I \otimes \Lambda](\rho) \) by estimating \( d^2 - 1 \) parameters \( \text{Tr} \rho'^k \), where \( k = 2, \ldots, d^2 \). Again, the phase in the interferometry can be fixed at \( \phi = 0 \).

The interferometric scheme described above is conceptually simple and experimentally viable, however, if the simplicity of the implementation is not an issue then we can measure the estimate of the lowest eigenvalue directly. This requires a joint measurement on all of the \( n \) pairs. We use the Keyl and Werner spectrum estimation method \([13]\), which, in the entanglement detection context, works as follows. The \( n \) copies of the \( m \times m \) state \( \rho' \) (in our case \( m = d^2 \)) form an operator on the \( n \)-fold tensor product space which can be decomposed according to irreps of \( SU(m) \), so that each summand, including multiplicities, is labelled by a Young tableau, i.e. \( n \) boxes arranged in rows of decreasing length (c.f. \([6]\) for the \( SU(2) \) case). The tableaus give a family of projectors for the spectrum estimation measurement. The normalized row lengths of each tableau are taken as estimates of the ordered sequence of eigenvalues of \( \rho' \). The probability that the error is greater than some fixed \( \epsilon \) decreases exponentially with \( n \) \([3]\). In our particular case, we are interested only in the lowest eigenvalue. We modify the Keyl-Werner scheme by adding together all projectors corresponding to Young tableaux with the fixed length of the last row. The measurement determined by these projectors gives directly the estimate of only one parameter — the lowest eigenvalue of \( \rho' \). Such a measurement can be represented as a quantum network implementing projections on the symmetric and on partially symmetric subspaces (see \([15]\) for the network projecting on the symmetric subspace).

Our considerations remain valid, with some minor modifications, when \( I \otimes \Lambda \) is not trace-preserving. In this case experimental implementations require postselections, which result in \( n' = n \text{Tr}(I \otimes \Lambda(\rho)) \) copies of normalized states \( I \otimes \Lambda(\rho)/\text{Tr}(I \otimes \Lambda(\rho)) \). The spectrum estimation procedure is not affected, however, before checking the condition \([3]\) the lowest eigenvalue has to be rescaled by the factor \( \text{Tr}(I \otimes \Lambda(\rho)) \).

Let us summarize our findings. Given \( n \) copies of a bipartite \( d \otimes d \) system described by some unknown density operator \( \rho \) we can test for entanglement either by estimating \( \rho \) and applying criterion \([2]\), or, more directly, by performing the measurements we have just described. The state estimation involves estimating \( d^4 - 1 \) parameters of \( \rho \), most of which are of no relevance for the entanglement detection. The optimal state estimations rely on joint measurements on all copies of \( \rho \). However, one can construct less efficient but simpler state estimation methods which involve measurements only on individual copies. Our more direct, interferometry based, method requires estimations of only \( d^2 - 1 \) parameters and joint operations on \( d \) copies of \( \rho' \). The most demanding, from the experimental point of view, is our second method. It
is a measurement with an outcome which is an estimate of just one parameter, but, like the optimal state estimation, the measurement involves joint operations on all copies of ̺. Both direct and indirect entanglement detections have their own merits. Depending on the context, applications, and technologies involved one can choose one or the other.

Direct entanglement detections, can be employed as sub-routines in quantum computation. For example, one may consider performing or not performing a quantum operation on a given quantum system conditioned on some part of quantum data being entangled or not. In fact direct entanglement detections can be viewed as quantum computations solving an inherently quantum decision problem: given as an input n copies of ̺ decide whether ̺ is entangled. Here the input data is quantum and such a decision problem cannot even be even formulated for classical computers. Nonetheless the problem is perfectly well defined for quantum computers. Finally, let us add that the method presented here can be easily generalized to cover all linear maps tests for arbitrary multiparticle entanglement [8] and the so called k-positive map tests detecting Schmidt numbers of density matrices [13]. Modification of the method to the case of two distant labs (i.e. under restrictions to local operations and classical communication [8]) will be considered elsewhere.

For the sake of completeness we should also mention here that there are two-particle observables, called entanglement witnesses which can detect quantum entanglement in some special cases (see [20]). They have positive mean values on all separable states and negative on some entangled states. Therefore any individual entanglement witness leaves many entangled states undetected. When ̺ is unknown we need to check infinitely many witnesses, which effectively reduces this approach to the quantum state estimation. However, let us point out any witness defines a positive map which can be used in our test.

To conclude, we have demonstrated that direct and physically implementable methods of entanglement detection are possible. They are equivalent to the most powerful mathematical separability criteria known to date [8]. We have described their possible realizations in the generic terms of quantum gates and networks. These generic components can be implemented using several experimental techniques ranging from trapped ions to quantum dots [21].

This research was supported in part by the Polish Committee for Scientific Research, European Commission, Elsag Spa, EPSRC, and the Royal Society, London.

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