Twisted and Non-Twisted Deformed Virasoro Algebra via Vertex Operators of $U_q(\hat{sl}_2)$

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Abstract

The work is devoted to a probably new connection between deformed Virasoro algebra and quantum $\hat{sl}_2$. We give an explicit realization of Virasoro current via vertex operators of level 1 integrable representation of $\hat{sl}_2$. The same is done for a twisted version of deformed Virasoro algebra.

1 Introduction

The work is devoted to a probably new connection between deformed Virasoro algebra and quantum $\hat{sl}_2$. More specifically, we use integrable representations $V(\Lambda_0)$ and $V(\Lambda_1)$ of quantum $\hat{sl}_2$ on level 1, they possess realization in terms of Heisenberg algebra. Also there are vertex operators $\Phi(z): V(\Lambda_i) \to V(\Lambda_1-i) \otimes V_z$, $\Psi(z): V(\Lambda_i) \to V_z \otimes V(\Lambda_1-i)$. (1.1)

for evaluation representation $V_z$ of $\hat{sl}_2$. The main result is realization of deformed Virasoro algebra in terms of these vertex operators see Theorem 3 and Theorem 4. Let us remark that deformed Virasoro algebra depend on parameters $q_1, q_2, q_3$ such that $q_1q_2q_3 = 1$. It turns out, that deformed Virasoro algebra is connected with $U_q(\hat{sl}_2)$ for $q = q_3^{1/2}$.

To be more precise, in Theorem 4 we have a realization of twisted deformed Virasoro algebra. This algebra was defined in [Shi04, (37)–(38)] but its bosonization was unknown.

In Theorem 3 we have constructed realization of ordinary (non-twisted) deformed Virasoro algebra defined in [SKAO96]. A bosonization of this algebra is known since [SKAO96], but our bosonization is a different one. The bosonization from [SKAO96] can be also realized by the same formula as in Theorem 3 but using another vertex operators [DI97] defined by (1.1) with respect to Drinfeld coproduct.

The existence of two realizations (our Theorems 3 and 4) is similar to existence of two choices of twist in XXZ model, see e.g. [MNN18, eq. 4.3].

Further development. Deformed Virasoro algebra is a particular case of $W_{q_1,q_2}(\hat{sl}_n)$ for $n = 2$. Twisted deformed Virasoro is a particular case of twisted $W$-algebras $W_{q_1,q_2}(\hat{sl}_n, n_{tw})$ for $n = 2$ and $n_{tw} = 1$. Generally $n_{tw}$ is a parameter of twist, i.e. for $n_{tw} = 0$ we obtain non-twisted $W$-algebra.

We expect that one can construct bosonization of $W_{q_1,q_2}(\hat{sl}_n, n_{tw})$ algebra from vertex operators of quantum $\hat{sl}_n$ on the level 1 (see [Koy94]).

Also we expect that tensor product of $W_{q_1,q_2}(\hat{sl}_n, n_{tw})$ with Heisenberg algebra $\text{Heis}$ are certain quotients of toroidal algebra $U_{q_1,q_2,q_3}(\hat{gl}_1)$; for non-twisted case such relation is known [FHS+10], [FFJ+11], [Neg18]. Hence a representation of $W_{q_1,q_2}(\hat{sl}_n, n_{tw}) \otimes \text{Heis}$ becomes a representation of $U_{q_1,q_2,q_3}(\hat{gl}_1)$ automatically. We expect that for $\gcd(n, n_{tw}) = 1$, the discussed above bosonized representation of $W_{q_1,q_2}(\hat{sl}_n, n_{tw})$ will lead to Fock modules of $U_{q_1,q_2,q_3}(\hat{gl}_1)$ with slope $n_{tw}/n$. For the case $q_3 = 1$ all this
was done in [BG19]. Let us remark $W_{q_1,q_2}(\mathfrak{sl}_n, m_{tw}) \otimes \text{Heis}$ acts on integrable level 1 representations of $U_q(\tilde{\mathfrak{gl}}_n) = U_q(\mathfrak{gl}_n) \otimes \text{Heis}$ if this holds without Heis factors.

One of our motivations for this project comes from [GN17]. It was conjectured in loc. cit. that there is an action (with certain properties) of $U_q\mathfrak{gl}_n$ on the Fock module of toroidal algebra $U_{q_1,q_2,q_3}(\tilde{\mathfrak{gl}}_1)$ with slope $n'/n$. As it was explained above, we also expect $U_q\mathfrak{gl}_n$ acts on the Fock module of toroidal algebra $U_{q_1,q_2,q_3}(\tilde{\mathfrak{gl}}_1)$. So we hope that both actions exist and coincide.

**Our methods.** The main technical tool of our paper is R-matrix relations (Theorems 1 and 2). One can find these relations without delta-function term in [JM95]. In loc. cit. the parameters of vertex operator are numbers, but in our paper the parameters are formal variables. Therefore our formulas are close to formulas in loc. cit., but have a different meaning and probably are new.

Technically, we write down formulas (4.5) and (4.24) for current of deformed Virasoro algebra $T(z)$ via vertex operators $\Phi(z)$ and $\Psi^+(z)$, and then we check relations of deformed Virasoro algebra using interchanging relations for vertex operators. Delta-function term on RHS of deformed Virasoro relation appears from the delta-function term in the R-matrix relation.

**Plan of the paper.** The paper is organized as follows

In Section 2 we recall bosonization of $U_q(\mathfrak{sl}_2)$ and its vertex operators following [JM95].

In Section 3 we study relations for vertex operators: interchanging relations (in particular R-matrix relations), and ‘special point relations’.

In Section 4 we construct realizations of twisted and non-twisted Virasoro algebra via vertex operators of $U_q(\mathfrak{sl}_2)$. A connection of obtained representations with Verma modules is studied.

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## 2 Bosonization of $U_q(\tilde{\mathfrak{sl}}_2)$ and its vertex operators

In this section we will recall the bosonization of the level 1 representations of $U_q(\tilde{\mathfrak{sl}}_2)$ and its vertex operators. All this can be found in [JM95, Chapters 5,6]. Our notation almost coincide with [JM95], however there are differences in normalization of vertex operators.

**Fock modules.** Algebra $U_q(\tilde{\mathfrak{sl}}_2)$ is generated by $a_k^\pm, a_l$ for $k \in \mathbb{Z}, l \in \mathbb{Z}_{\neq 0}$, $K^\pm$ and central elements $\gamma^{\pm1/2}$. These elements are called Drinfeld generators. The relations are [JM95, (5.3)-(5.7)], although let us recall

$$[a_k, a_l] = \delta_{k+l,0} \frac{2k}{k} \gamma^k - \gamma^{-k} q^{-1},$$

(2.1)

here $[n] = (q^n - q^{-n})/(q - q^{-1})$.

Denote by $\Lambda_0, \Lambda_1$ fundamental weights of $\tilde{\mathfrak{sl}}_2$ and by $\alpha$ root of $\mathfrak{sl}_2 \subset \tilde{\mathfrak{sl}}_2$ The algebra $U_q(\tilde{\mathfrak{sl}}_2)$ admits two basic representations $V(\Lambda_0)$ and $V(\Lambda_1)$. As vector spaces

$$V(\Lambda_i) = \mathbb{C}[a_{-1}, a_{-2}, \ldots] \otimes (\oplus_{\alpha} \mathbb{C} e^{\Lambda_i + i\alpha}).$$

(2.2)

As representations of Heisenberg subalgebra $a_k$, these modules are infinite sums of Fock modules

$$V_j = \mathbb{C}[a_{-1}, a_{-2}, \ldots] \otimes \mathbb{C} e^{\Lambda_i + i\frac{j}{2} \alpha} \quad \text{for} \ i \equiv j \mod 2.$$  

(2.3)

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Let us define operators $e^{\pm \alpha}$ and $\partial$ as follows

$$e^{\pm \alpha}(f \otimes e^\beta) = f \otimes e^{\beta \pm \alpha}, \quad \partial(f \otimes e^\beta) = (\alpha, \beta)f \otimes e^\beta.$$  

(2.4)

The action of other $U_q(\mathfrak{sl}_2)$ generators is given by

$$K = q^0, \quad \gamma = q, \quad X^\pm(z) = \exp \left( \pm \sum_{n=1}^\infty \frac{a-n}{[n]} q^{n/2} z^n \right) \exp \left( \pm \sum_{n=1}^\infty \frac{a-n}{[n]} q^{-n/2} z^{-n} \right) e^{\pm \alpha z \pm \partial},$$  

(2.6)

here $X^\pm(z) = \sum x_k^\pm z^{-k-1}$. Obtained representations are irreducible highest weight representations with highest vectors $|\Lambda_i\rangle = 1 \otimes e^{\lambda_i} \in V(\Lambda_i)$.

**Vertex operators.** Vertex operator of $U_q(\hat{\mathfrak{sl}}_2)$ are certain formal power series of operators

$$\Phi_{\pm}^{(1-i,i)}(z) : V(\Lambda_i) \rightarrow V(\Lambda_{1-i}), \quad \Psi_{\pm}^{(1-i,i)}(z) : V(\Lambda_i) \rightarrow V(\Lambda_{1-i}).$$  

(2.7)

Below we will abbreviate $\Phi_{\pm}(z) = \Phi_{\pm}^{(1-i,i)}(z)$, if a statement holds for both $i = 0, 1$.

A conceptual definition of these operators via certain intertwiner relations is given in [JM95, Chapter 6]. For us it is more convenient to give an ad hoc definition

$$\Phi_+(z) = \Phi_-(z), x_0^-, q^2 z \Phi_+(z) = \Phi_-(z), x_0^-, q^{-1},$$  

(2.10)

$$\Psi_-(z) = \Psi_+(z), x_0^+, (q^2 z)^{-1} \Psi_-(z) = \Psi_+(z), x_0^+, q^{-1}.$$  

(2.11)

here we use following notation $[A, B]_p = AB - pBA$.

We will also need the dual operator $\Psi^\pm_+(z) = \Psi^-_+(q^2 z)$. Then

$$\Psi^\pm_+(z) = \exp \left( - \sum_{n=1}^\infty \frac{a-n}{[n]} q^{n/2} z^n \right) \exp \left( \sum_{n=1}^\infty \frac{a-n}{[n]} q^{-n/2} z^{-n} \right) e^{-\alpha/2}(-q^2 z)^{-\partial/2}.$$  

(2.12)

Denote

$$\alpha_\phi(x) = \frac{(q^4 x; q^4)_\infty}{(q^2 x; q^4)_\infty}, \quad \alpha_\psi(x) = \frac{(q^2 x; q^4)_\infty}{(q x; q^4)_\infty}, \quad \beta(x) = \frac{(q x; q^4)_\infty}{(q^2 x; q^4)_\infty}.$$  

(2.13)

It is straightforward to check that

$$(-q^3 z)^{-1/2} \alpha_\phi(w/z) \Phi_-(z) \Phi_-(w) = : \Phi_-(z) \Phi_-(w) :,$$  

(2.14)

$$(-q z)^{-1/2} \alpha_\psi(w/z) \Psi_+(z) \Psi_+(w) = : \Psi_+(z) \Psi_+(w) :,$$  

(2.15)

$$(-q^3 z)^{1/2} \beta(w/z) \Phi_-(z) \Phi^*_-(w) = : \Phi_-(z) \Phi^*_-(w) :,$$  

(2.16)

$$(-q^3 w)^{1/2} \beta(z/w) \Phi^*_+(w) \Phi_-(z) = : \Phi^*_+(w) \Phi_-(z) :,$$  

(2.17)

here : : : stands for normal ordering in terms of Heisenberg algebra. Then

$$z^{-1/2} \alpha_\phi(w/z) \Phi^-_-(z) \Phi^-_-(w) = w^{-1/2} \alpha_\phi(z/w) \Phi^-_-(w) \Phi^-_-(z),$$  

(2.18)

$$z^{-1/2} \alpha_\psi(w/z) \Phi^*_-(z) \Phi^*_-(w) = w^{-1/2} \alpha_\psi(z/w) \Phi^*_-(w) \Phi^*_-(z),$$  

(2.19)

$$z^{1/2} \beta(w/z) \Phi^-_-(z) \Phi^*_-(w) = w^{1/2} \beta(z/w) \Phi^*_-(w) \Phi^-_-(z).$$  

(2.20)
Proof. Recall, that \( U_q(\hat{sl}_2) \) is also generated by \( e_i, f_i, t_i \) for \( i = 0, 1 \). These generators are called Chevalley generators. The connection with Drinfeld generators as follows

\[
t_1 = K, \quad x_0^+ = e_1, \quad x_0^- = f_1, \quad (2.21) \\
t_0 = \gamma K^{-1}, \quad x_1^- = e_0 t_1, \quad x_1^+ = t_1^{-1} f_0. \quad (2.22)
\]

Let us consider an exterior automorphism \( \pi \) of \( U_q(\hat{sl}_2) \) given by \( \pi(e_i) = e_{1-i}, \pi(f_i) = f_{1-i} \). Then \( \pi \) acts on the Drinfeld generators as follows

\[
\pi(K) = \gamma K^{-1}, \quad \pi(x_0^+) = x_1^- K^{-1}, \quad \pi(x_0^-) = K x_1^+, \quad (2.23) \\
\pi(x_1^-) = \gamma x_0^+ K^{-1}, \quad \pi(x_1^+) = \gamma^{-1} K x_0^- \quad (2.24)
\]

**Proposition 2.1.** There exist an involution \( \tilde{\pi} \) interchanging \( V(\Lambda_0) \) and \( V(\Lambda_1) \), such that \( \tilde{\pi} X \tilde{\pi} = \pi(X) \) for any \( X \in U_q(\hat{sl}_2) \).

**Proof.** \( V(\Lambda_0) \) and \( V(\Lambda_1) \) are irreducible highest weights representations and \( \pi \) preserves triangular decomposition. To finish the proof we notice that action of \( \pi \) interchange the highest weights of the representations. \( \square \)

To determine \( \tilde{\pi} \) uniquely we require \( \tilde{\pi} (|\Lambda_i\rangle) = |\Lambda_{1-i}\rangle \).

**Proposition 2.2.** Conjugation by involution \( \tilde{\pi} \) is expressed as follows

\[
\tilde{\pi} \left( \Phi_+^{(1-i,i)}(z) \right) = (-q^3)^{\frac{1}{2} - i} z^{-\frac{i}{2}} \Phi_-^{(1,i,i)}(z), \quad \tilde{\pi} \left( \Phi_-^{(1,i,i)}(z) \right) = (-q^3)^{\frac{1}{2} - i} z^{\frac{i}{2}} \Phi_+^{(1,i,i)}(z), \quad (2.25)
\]

\[
\tilde{\pi} \left( \Psi_+^{(1-i,i)}(z) \right) = -q \frac{1}{2} \Phi_-^{(1,i,i)}(z), \quad \tilde{\pi} \left( \Psi_-^{(1,i,i)}(z) \right) = -q \frac{1}{2} \Phi_+^{(1,i,i)}(z). \quad (2.26)
\]

**Sketch of a proof.** One can prove the formulas up to a constant via the intertwining properties [JM95, Chapter 6]

\[
\tilde{\pi} \left( \Phi_+^{(1,i,i)}(z) \right) \tilde{\pi} = c_1^{(i)} \Phi_-^{(1,i,i)}(z), \quad \tilde{\pi} \left( \Phi_-^{(1,i,i)}(z) \right) \tilde{\pi} = c_2^{(i)} \Phi_+^{(1,i,i)}(z), \quad (2.27)
\]

\[
\tilde{\pi} \left( \Psi_+^{(1-i,i)}(z) \right) \tilde{\pi} = c_1^{(i)} \Psi_-^{(1,i,i)}(z), \quad \tilde{\pi} \left( \Psi_-^{(1-i,i)}(z) \right) \tilde{\pi} = c_2^{(i)} \Psi_+^{(1,i,i)}(z), \quad (2.28)
\]

here \( c_1^{(i)} \) and \( c_2^{(i)} \) are some \( z \)-dependent scalars. Then one can find the constants by comparison with normalization [JM95, eq. (6.4), (6.5)]. \( \square \)

**Corollary 2.3.** The following relations hold

\[
z^{-1/2} \alpha_\phi(w/z) \Phi_+(z) \Phi_+(w) = w^{-1/2} \alpha_\phi(z/w) \Phi_+(w) \Phi_+(z), \quad (2.29)
\]

\[
z^{-1/2} \alpha_\psi(w/z) \Psi_+(z) \Psi_+(w) = w^{-1/2} \alpha_\psi(z/w) \Psi_+(w) \Psi_+(z), \quad (2.30)
\]

\[
z^{1/2} \beta(w/z) \Phi_+(z) \Phi_+(w) = w^{1/2} \beta(z/w) \Phi_+(w) \Phi_+(z). \quad (2.31)
\]

**Proof.** These relations are obtained from (2.18)–(2.20) after conjugation by \( \tilde{\pi} \). \( \square \)

### 3 Vertex operators relations revisited

The main results of this section are R-matrix relations (Theorems 1 and 2). One can find these relations without delta-function term in [JM95]. In loc. cit. parameters of vertex operator are numbers, but in our paper the parameters are formal variables. Although formulas below are close to formulas in loc. cit., they have a different meaning and can be considered as a new result.
3.1 R-matrix relations

R-matrix is an operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let $v_+, v_-$ be a basis of each $\mathbb{C}^2$. The matrix of this operator with respect to basis $v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-$ looks as follows

$$R(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{1-q^2} & \frac{1-q^2}{1-q^2} & 0 \\
0 & \frac{1-q^2}{1-q^2} & \frac{1}{1-q^2} & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}. \quad (3.1)
$$

This R-matrix is an important object in the representation theory of $U_q(\widehat{\mathfrak{sl}_2})$, though in this paper we will not use any information on R-matrix apart from (3.1). Below we will see that R-matrix encodes certain interchanging relations for vertex operators.

3.1.1 Interchanging relation on $\Phi$-vertex operators

Denote by $\delta(x,y) = \sum_{k+l=-1} x^k y^l$.

**Proposition 3.1.** Following relations hold

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w) - w^{-\frac{1}{2}}\alpha_\phi(z/w) \left( q \frac{1-z/w}{q^2-z/w} \Phi_+(w)\Phi_-(z) + \frac{(q^2-1)z/w}{q^2-z/w} \Phi_-(w)\Phi_+(z) \right) = (-1)^0(-q^3)^\frac{1}{2} q^{-2} \delta(z, q^2w), \quad (3.2)
$$

$$z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w) - w^{-\frac{1}{2}}\alpha_\phi(z/w) \left( q \frac{1-z/w}{q^2-z/w} \Phi_-(w)\Phi_+(z) + \frac{q^2-1}{q^2-z/w} \Phi_+(w)\Phi_-(z) \right) = -(-1)^0(-q^3)^\frac{1}{2} q^{-3} \delta(z, q^2w). \quad (3.3)
$$

**Proof.** Using (2.10), we obtain

$$[\Phi_-(z)\Phi_-(w), x_0]_q^2 = \Phi_-(z)[\Phi_-(w), x_0]_q + q[\Phi_-(z), x_0]_q \Phi_-(w) = \Phi_-(z)\Phi_+(w) + q\Phi_+(z)\Phi_-(w), \quad (3.4)
$$

$$[\Phi_-(z)\Phi_-(w), x_1]_q^{q-2} = \Phi_-(z)[\Phi_-(w), x_1]_q^{q-1} + q^{-1}[\Phi_-(z), x_1]_q^{-1}\Phi_-(w) = q^2 w \Phi_-(z)\Phi_+(w) + qz \Phi_+(z)\Phi_-(w). \quad (3.5)
$$

Solving the system of two linear equations, one can find

$$\Phi_-(z)\Phi_+(w) = \frac{z[\Phi_-(z)\Phi_-(w), x_0]_q^2 - [\Phi_-(z)\Phi_-(w), x_1]_q^{q-2}}{z(1-q^2w/z)}, \quad (3.6)
$$

$$\Phi_+(z)\Phi_-(w) = \frac{-qw[\Phi_-(z)\Phi_-(w), x_0]_q^2 + q^{-1}[\Phi_-(z)\Phi_-(w), x_1]_q^{q-2}}{z(1-q^2w/z)}. \quad (3.7)
$$

Using (2.14), we see that

$$(-q^3)^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w) = \frac{z[\Phi_-(z)\Phi_-(w); x_0]_q^2 - [\Phi_-(z)\Phi_-(w); x_1]_q^{q-2}}{z(1-q^2w/z)}, \quad (3.8)
$$

$$(-q^3)^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w) = \frac{-qw[\Phi_-(z)\Phi_-(w); x_0]_q^2 + q^{-1}[\Phi_-(z)\Phi_-(w); x_1]_q^{q-2}}{z(1-q^2w/z)}. \quad (3.9)
$$
Then
\[ z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_-(z)\Phi_+(w) - w^{-1/2}\alpha_\phi(z/w) \left( \frac{q(1-z/w)}{q^2-z/w} \Phi_+(w)\Phi_-(z) + \frac{(q^2-1)z/w}{q^2-z/w} \Phi_-(w)\Phi_+(z) \right) = (-q^3)^{\frac{1}{2}} \left( q^2w[:\Phi_-(q^2w)\Phi_-(w):;x_0] - \frac{1}{2}[:\Phi_-(q^2w)\Phi_-(w);x_1] \right) \delta(z,q^2w), \] (3.10)

\[ z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi_+(z)\Phi_-(w) - w^{-\frac{1}{2}}\alpha_\phi(z/w) \left( \frac{q(1-z/w)}{q^2-z/w} \Phi_+(w)\Phi_-(z) + \frac{q^2-1}{q^2-z/w} \Phi_+(w)\Phi_-(z) \right) = (-q^3)^{\frac{1}{2}} \left( -qw[:\Phi_+(q^2w)\Phi_-(w):;x_0] + q^{-1}[:\Phi_-(q^2w)\Phi_-(w);x_1] \right) \delta(z,q^2w). \] (3.11)

**Lemma 3.2.** Following relation holds
\[ q^2w[:\Phi_-(q^2w)\Phi_-(w);;x_0] - [:\Phi_-(q^2w)\Phi_-(w);;x_1] = (1)^{\frac{1}{2}}q^2. \] (3.12)

**Proof.** We will proof the lemma assuming w to be a number, not a formal variable; formal variable version follows. Let us consider two contours of integration \( C_+ = \{ y \mid |y| = R_+ \gg |w| \}, C_- = \{ y \mid |y| = R_- \ll |w| \} \).

Denote \( \Omega(w) = [:\Phi_-(q^2w)\Phi_-(w);. \) Note that
\[ [\Omega(w),x_0] = \int_{C_-} \Omega(w)X^- (y)dy - q^2 \int_{C_+} X^- (y)\Omega(w)dy, \] (3.13)
\[ [\Omega(w),x_1] = \int_{C_-} y\Omega(w)X^- (y)dy - q^2 \int_{C_+} yX^- (y)\Omega(w)dy. \] (3.14)

Hence
\[ q^2w[\Omega(w),x_0] - [\Omega(w),x_1] = \int_{C_-} (q^2w - y)\Omega(w)X^- (y)dy - \int_{C_+} (q^4w - q^-2y)X^- (y)\Omega(w)dy - \int_{C_-} \frac{q^2w - q^-2y}{y(q^4w - y)} X^- (y)\Omega(w)dy \]
\[ = \int_{C_-} \frac{q^2}{y - q^4w} :X^- (y)\Omega(w)dy - \int_{C_-} \frac{q^2}{y - q^4w} :X^- (y)\Omega(w)dy \]
\[ = \text{res}_{y=q^4w} q^2 wr^{-2} :X^- (y)\Omega(w)dy = (-1)^{\frac{1}{2}}q^{-2}. \]

Here we used \( :X^- (q^2w)\Omega(w) := (-1)^{\frac{1}{2}}. \)

To finish the proof of Proposition 3.1 we apply Lemma 3.2 to (3.10) and (3.11).

**Matrix notation** Denote \( \Phi(z) = \Phi_+(z) \otimes v_+ + \Phi_-(z) \otimes v_- \in \text{Hom}(V(A_i),V(A_{1-i})) \otimes \mathbb{C}^2 \). Denote products
\[ \Phi^{(1)}(z)\Phi^{(2)}(w) = \sum_{\epsilon_1,\epsilon_2=\pm} \Phi^{(1)}(z)\Phi^{(2)}(w) \otimes \Phi^{(1)}(w) \otimes \Phi^{(2)}(z), \] \[ \Phi^{(2)}(w)\Phi^{(1)}(z) = \sum_{\epsilon_1,\epsilon_2=\pm} \Phi^{(2)}(w)\Phi^{(1)}(z) \otimes \Phi^{(1)}(w) \otimes \Phi^{(2)}(z). \]

Finally denote by \( R^{-1}(z/w)\Phi^{(2)}(w)\Phi^{(1)}(z) \) the result of the action of \( R^{-1}(z/w) \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) tensor multiple of \( \text{Hom}(V(A_i),V(A_{1-i})) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \).

**Theorem 1.** The following relation holds
\[ z^{-\frac{1}{2}}\alpha_\phi(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w) = \]
\[ w^{-\frac{1}{2}}\alpha_\phi(z/w)R^{-1}(z/w)\Phi^{(2)}(w)\Phi^{(1)}(z) + (-1)^\frac{1}{2}(-q)^\frac{1}{2}(q^{-1}v_- \otimes v_+ - q^{-2}v_+ \otimes v_-) \delta(z,q^2w). \] (3.15)

**Proof.** The theorem is just a reformulation of (2.18), (2.29) and Proposition 3.1.
3.1.2 Interchanging relation for $\Psi$-vertex operators

**Proposition 3.3.** The following relations hold

\[
z^{-1/2} \alpha_{\psi}(w/z) \Psi_+(z) \Psi_-(w) - w^{-1/2} \alpha_{\psi}(z/w) \left( \frac{q(1 - z/w)}{1 - q^2 z/w} \Psi_-(w) \Psi_+(z) + \frac{1 - q^2}{1 - q^2 z/w} \Psi_+(w) \Psi_-(z) \right) = (-1)^{\delta} (-q)^{1/2} q^2 \delta(q^2 z, w), \quad (3.16)
\]

\[
z^{-1/2} \alpha_{\psi}(w/z) \Psi_-(z) \Psi_+(w) - w^{-1/2} \alpha_{\psi}(z/w) \left( \frac{q(1 - z/w)}{1 - q^2 z/w} \Psi_+(w) \Psi_-(z) + \frac{1 - q^2}{1 - q^2 z/w} \Psi_-(w) \Psi_+(z) \right) = (-1)^{\delta} (-q)^{1/2} q^2 \delta(q^2 z, w). \quad (3.17)
\]

**Proof.** Using (2.11) we obtain

\[
[\Psi_+(z) \Psi_+(w), x_0^+] q^2 = \Psi_+(z) [\Psi_+(w), x_0^+] q + q [\Psi_+(z), x_0^+] q \Psi_+(w) = \Psi_+(z) \Psi_-(w) + q \Psi_-(z) \Psi_+(w),
\]

\[
[\Psi_+(z) \Psi_+(w), x_{-1}^+]_{q^{-2}} = \Psi_+(z) [\Psi_+(w), x_{-1}^+]_{q^{-1}} + q^{-1} \Psi_+(z), x_{-1}^+]_{q^{-1}} \Psi_+(w)
\]

\[
= (q^2 w)^{-1} \Psi_+(z) \Psi_- (w) + (q^3 z)^{-1} \Psi_-(z) \Psi_+(w).
\]

Solving the system of linear equations, we obtain

\[
\Psi_+(z) \Psi_-(w) = \frac{w/z}{q^2 - w/z} 
\]

\[
\Psi_-(z) \Psi_+(w) = \frac{q [\Psi_+(z) \Psi_+(w), x_0^+] q^2 - q^3 w [\Psi_+(z) \Psi_+(w), x_{-1}^+]_{q^{-2}}}{q^2 - w/z}.
\]

Using (2.15), we see that

\[
(-qz)^{-1/2} \alpha_{\psi}(w/z) \Psi_+(z) \Psi_-(w) = \frac{w (-[\Psi_+(z) \Psi_+(w), x_{-1}^+]_{q^{-2}} + q^4 z [\Psi_+(z) \Psi_+(w), x_{-1}^+]_{q^{-2}})}{q^2 z (1 - \frac{w}{qz})}, \quad (3.20)
\]

\[
(-qz)^{-1/2} \alpha_{\psi}(w/z) \Psi_- (z) \Psi_+(w) = \frac{q [\Psi_+(z) \Psi_+(w), x_0^+] q^2 - q^3 w [\Psi_+(z) \Psi_+(w), x_{-1}^+]_{q^{-2}}}{q^2 (1 - \frac{w}{qz})}. \quad (3.21)
\]

Then

\[
z^{-1/2} \alpha_{\psi}(w/z) \Psi_+(z) \Psi_-(w) - w^{-1/2} \alpha_{\psi}(z/w) \left( \frac{1 - q^2}{1 - q^2 z/w} \Psi_+(w) \Psi_-(z) + \frac{q(1 - z/w)}{1 - q^2 z/w} \Psi_-(w) \Psi_+(z) \right)
\]

\[
= (-q)^{1/2} q^2 z \left[ \Psi_+(z) \Psi_+(q^2 z), x_0^+] q^2 + q^4 z [\Psi_+(z) \Psi_+(q^2 z), x_{-1}^+]_{q^{-2}} \right] \delta(q^2 z, w). \quad (3.22)
\]

\[
z^{-1/2} \alpha_{\psi}(w/z) \Psi_-(z) \Psi_+(w) - w^{-1/2} \alpha_{\psi}(z/w) \left( \frac{q(1 - z/w)}{1 - q^2 z/w} \Psi_+(w) \Psi_-(z) + \frac{1 - q^2}{1 - q^2 z/w} \Psi_-(w) \Psi_+(z) \right)
\]

\[
= (-q)^{1/2} q z \left[ \Psi_+(z) \Psi_+(q^2 z), x_0^+] q^2 - q^4 z [\Psi_+(z) \Psi_+(q^2 z), x_{-1}^+]_{q^{-2}} \right] \delta(q^2 z, w). \quad (3.23)
\]

**Lemma 3.4.** Following relation holds

\[
[\Psi_+(z) \Psi_+(q^2 z), x_0^+] q^2 - q^4 z [\Psi_+(z) \Psi_+(q^2 z), x_{-1}^+]_{q^{-2}} = (-1)^{\delta} z^{-1}.
\]
\textit{Proof.} Denote by 
\[ \Upsilon(z) = : \Psi_+(z) \Psi_+ (q^2 z) :. \] (3.25)

Note that
\[ [\Upsilon(z), x_0^+]_{q^2} = \int_{C_-} \Upsilon(z) X^+(y) dy - q^2 \int_{C_+} X^+(y) \Upsilon(z) dy, \] (3.26)
\[ [\Upsilon(z), x_1^+]_{q^{-2}} = \int_{C_-} y^{-1} \Upsilon(z) X^+(y) dy - q^{-2} \int_{C_+} y^{-1} X^+(y) \Upsilon(z) dy. \] (3.27)

Hence
\[ [\Upsilon(z), x_0^+]_{q^2} - q^4 z [\Upsilon(z), x_1^+]_{q^{-2}} = \int_{C_-} (1 - q^4 z/y) \Upsilon(z) X^+(y) dy - q^2 \int_{C_+} (1 - z/y) X^+(y) \Upsilon(z) dy \]
\[ = \int_{C_-} \frac{q^2 (1 - q^4 z/y)}{(q^2 z - y)(q^4 z - y)} : \Upsilon(z) X^+(y) : dy - q^2 \int_{C_+} \frac{(1 - z/y)}{(y - z)(y - q^2 z)} : X^+(y) \Upsilon(z) : dy \]
\[ = - \int_{C_-} \frac{q^2}{y(q^2 z - y)} : \Upsilon(z) X^+(y) : dy + \int_{C_+} \frac{q^2}{y(q^2 z - y)} : X^+(y) \Upsilon(z) : dy \]
\[ = \text{res}_{y = q^2 z} \frac{q^2}{y(q^2 z - y)} : X^+(y) \Upsilon(z) : dy = (-1)^3 z^{-1}. \]

Here we used \( X^+(q^2 z) \Upsilon(z) : = (-1)^3. \)

In terms of the operators \( \Psi^* \), Proposition 3.3 can be rewritten as follows.

\textbf{Corollary 3.5.} The following relation holds
\[ z^{-1/2} \alpha_{\Psi}(w/z) \Psi^*_-(z) \Psi^*_+(w) - w^{-1/2} \alpha_{\Psi}(z/w) \left( \frac{q(1 - z/w)}{1 - q^2 z/w} \Psi^*_+(w) \Psi^*_-(z) + \frac{1 - q^2}{1 - q^2 z/w} \Psi^*_-(w) \Psi^*_+(z) \right) \]
\[ = (-1)^3 (q^2 z/w), \] (3.28)
\[ z^{-1/2} \alpha_{\Psi}(w/z) \Psi^*_+(z) \Psi^*_-(w) - w^{-1/2} \alpha_{\Psi}(z/w) \left( \frac{q(1 - \frac{z}{w})}{1 - q^2 \frac{z}{w}} \Psi^*_-(w) \Psi^*_+(z) + \frac{1 - q^2 \frac{z}{w}}{1 - q^2 \frac{z}{w}} \Psi^*_+(w) \Psi^*_-(z) \right) \]
\[ = (-1)^3 (q^2 z/w). \] (3.29)

\textbf{Matrix notation} Denote \( \Psi^*(z) = \Psi^*_+(z) \otimes v^*_+ + \Psi^*_-(z) \otimes v^*_- \in \text{Hom}(V(\Lambda_i), V(\Lambda_{1-i})) \otimes (\mathbb{C}^2)^*. \) Let us emphasise that \(((\mathbb{C}^2)^*)^* \) is the dual space to \( \mathbb{C}^2 \), considered in the definition of \( \Phi(z) \).

Denote products
\[ \Psi^*(1)(z) \Psi^*(2)(w) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \Psi^*_{\epsilon_1}(z) \Psi^*_{\epsilon_2}(w) \otimes v^*_{\epsilon_1} \otimes v^*_{\epsilon_2}, \] (3.30)
\[ \Psi^*(2)(w) \Psi^*(1)(z) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \Psi^*_{\epsilon_2}(w) \Psi^*_{\epsilon_1}(z) \otimes v^*_{\epsilon_1} \otimes v^*_{\epsilon_2}. \] (3.31)

Finally denote by \( \Psi^*(2)(w) \Psi^*(1)(z) R(z/w) \) the result of the dual action of \( R(z/w) \) on \(((\mathbb{C}^2 \otimes \mathbb{C}^2)^*)^* \) tensor multiple of \( \text{Hom}(V(\Lambda_i), V(\Lambda_j)) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)^* \). In other words, we multiply the operator-valued \textit{row vector} on the matrix.
Theorem 2. The following relation holds
\[ z^{-\frac{1}{2}} \alpha_\psi(w/z) \Psi^{(1)}(z) \Psi^{(2)}(w) = w^{-\frac{1}{2}} \alpha_\psi(z/w) \Psi^{(2)}(w) \Psi^{(1)}(z) R(z/w) + (-1)^{\theta}(-q)^{\frac{1}{2}}(qv_+ \otimes v^*_+ - v^*_+ \otimes v_+) \delta(q^2 z, w). \] \tag{3.32}

Proof. The theorem is just a reformulation of (2.19), (2.30) and Corollary 3.5.

3.2 Special point relation

3.2.1 Special point for $\Phi$

Proposition 3.6. We have the following identity
\[ (-qz)^{-1/2} \alpha_\phi(w/z) (q\Phi_-(z)\Phi_+(w) - \Phi_+(z)\Phi_-(w)) \left|_{w=q^2 z} = \frac{(-1)^{\theta}}{zq^2(1-q^2)}. \tag{3.33} \right. \]

Remark 1. Note, that a priori LHS of (3.33) is not well defined since a coefficient of any power of $z$ is an infinite sum of operators. So we have to prove that the result of substitution exists as well as to find the result. Also note, that we substitute $w \mapsto q^2 z$ to the whole expression, not to the individual multiples; the result of substitution to the individual multiples does not have to exist.

Proof. Substituting $w \mapsto q^2 z$ to (3.8) and (3.9) we obtain
\[ (-q^3 z)^{-1/2} \alpha_\phi(w/z) (q\Phi_-(z)\Phi_+(w) - \Phi_+(z)\Phi_-(w)) \left|_{w=q^2 z} = \frac{q^2 z[\Phi_-(z)\Phi_-(q^2 z) \otimes x_0 \ominus \Phi_+(z)\Phi_-(q^2 z) \otimes x_1]}{q^2(1-q^2)}. \tag{3.34} \right. \]

To finish the proof we apply Lemma 3.2.

3.2.2 Special point for $\Psi$

Proposition 3.7. We have the following identity
\[ (-qz)^{-1/2} \alpha_\psi(w/z) (\Psi_-(z)\Psi_+(w) - q\Psi_+(z)\Psi_-(w)) \left|_{z=q^2 w} = \frac{qw^{-1}}{1-q^2}(-1)^{\theta}. \tag{3.35} \right. \]

Proof. Let us substitute $z \mapsto q^2 w$ to (3.20) and (3.21)
\[ (-qz)^{-1/2} \alpha_\psi(w/z) (\Psi_-(z)\Psi_+(w) - q\Psi_+(z)\Psi_-(w)) \left|_{z=q^2 w} = \frac{q[\Psi_+(q^2 w)\Psi_+(w) \ominus x_0 \ominus q^2 w; \Psi_+(q^2 w)\Psi_+(w) \ominus x_1]}{q^2 - 1}. \tag{3.36} \right. \]

To finish the proof we applying Lemma 3.4.

Corollary 3.8. For any $q_1 \in \mathbb{C}\setminus\{0\}$ we have the following identity
\[ (-q_1/z)^{1/2} \alpha_\psi(w/z) (\Psi_+(q_1 z)\Psi_+(q_1 w) - q\Psi_+(q_1 z)\Psi_+(q_1 w)) \left|_{z=q^2 w} = -\frac{w^{-1}}{1-q^2}(-1)^{\theta}. \tag{3.37} \right. \]
3.3 Interchanging relation on $\Phi$ and $\Psi$

Proposition 3.9. The following relation holds

$$z^{\frac{1}{2}} \beta(w/z) \Phi_{\epsilon_1}(z) \Psi_{\epsilon_2}^+(w) = w^{\frac{1}{2}} \beta(z/w) \Psi_{\epsilon_2}(w) \Phi_{\epsilon_1}(z).$$

(3.38)

Proof. We have already seen the cases $\epsilon_1 = \epsilon_2 = \pm$, see (2.20) and (2.31). To prove remaining cases, let us use a relation from [JM95, Section 6.3] and relation [JM95, (6.12)]

$$\Psi_{\epsilon_2}^+(x_0^- - x_0^- \Psi_{\epsilon_2}(z) = 0, \quad \Phi_{\epsilon_2}(x_0^+ - x_0^+ \Phi_{\epsilon_2}(z) = 0.$$

(3.39)

To be combined with (2.10) and (2.11), the relations yield

$$[\Phi_{\epsilon_2}(z) [\Phi_{\epsilon_2}^+(w), x_0^+]_q = [\Phi_{\epsilon_2}(z), x_0^-]_q \Psi_{\epsilon_2}^+(w) = \Phi_{\epsilon_2}(z) \Psi_{\epsilon_2}^+(w), \quad (3.40)$$

$$[\Phi_{\epsilon_2}(z) \Psi_{\epsilon_2}(w), x_0^+]_q = \Phi_{\epsilon_2}(z) [\Psi_{\epsilon_2}^+(w), x_0^+]_q = \Phi_{\epsilon_2}(z) \Psi_{\epsilon_2}^+(w). \quad (3.41)$$

Considering $q$-commutator of (2.20) with $x_0^-$ and $x_0^+$, we obtain cases $\epsilon_1 = +, \epsilon_2 = -$ and $\epsilon_1 = -, \epsilon_2 = +$ correspondingly. \qed

4 Realization of (Twisted) Deformed Virasoro algebra

In this section, we will consider two algebras: deformed Virasoro algebra and twisted deformed Virasoro algebra. Deformed Virasoro algebra is extensively studied. Twisted Virasoro was defined in [Shi04], though this algebra is considerably less famous.

The algebras depend on two parameters $q_1, q_2$. It is also convenient to consider $q_3$ such that $q_1 q_2 q_3 = 1$. In this section we study connection between the algebras and $U_q(\mathfrak{sl}_2)$ for $q^2 = q_3$.

To define (twisted) deformed Virasoro algebra, we need the following notation

$$\sum_{l=0}^{\infty} f_l x^l = f(x) = \exp \left( \sum_{n=1}^{\infty} \frac{1 - q_1^n (1 - q_2^n)}{1 + q_3^n} x^n \right).$$

(4.1)

Note that

$$f(x) = \frac{1}{1 - x} \beta (q_1 q x) \beta (q_1^{-1} q^{-1} x).$$

(4.2)

4.1 Deformed Virasoro algebra

Definition 4.1. Deformed Virasoro algebra $\text{Vir}_{q_1,q_2}$ is generated by $T_n$ for $n \in \mathbb{Z}$. The defining relation is

$$\sum_{l=0}^{\infty} f_l T_{n-l} T_{m+l} - \sum_{l=0}^{\infty} f_l T_{n-l} T_{m+l} = - \frac{(1 - q_1^n (1 - q_2^n)}{1 - q_3^n} (q_3^n - q_3^n) \delta_{n+m,0}.$$

(4.3)

Denote $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$, $\delta(x) = \sum_{k \in \mathbb{Z}} x^k$. Relation (4.3) is equivalent to

$$f(w/z) T(z) T(w) - f(z/w) T(w) T(z) = - \frac{(1 - q_1^n (1 - q_2^n)}{1 - q_3^n} \left( \delta \left( \frac{w}{q_3 z} \right) - \delta \left( \frac{q_3 w}{z} \right) \right).$$

(4.4)

Representation. Recall that $V_j$ were defined by (2.3).

Theorem 3. The formula below determines an action of $\text{Vir}_{q_1,q_2}$ on $V_j$ for all $j \in \mathbb{Z}$.

$$T(z) = z^{1/2} q_2^{3/2} (q_1^{1/2} - q_1^{-1/2}) \left( u \Psi_+ (q q_1 z) \Phi_+ (z) + u^{-1} \Psi_- (q q_1 z) \Phi_- (z) \right).$$

(4.5)
Denote the obtained representation by \( F_n^{[j]} \).

**Remark 2.** A bosonization of deformed Virasoro algebra is known since [SKA06], but our bosonization is a different one. In both cases current \( T(z) \) is presented as a sum two summands. Surprisingly, the first summands in both cases are ‘the same normally ordered exponent of Heisenberg \( a_k \); however, the second ones are different. In [SKA06] the second summand is also an exponent of the same Heisenberg, but this is not true for our bosonization. Note that \( \Psi^\ast (qq_1 z) \Phi_\ast (z) \) is an exponent of Heisenberg \( \pi (a_k) \), but not of \( a_k \).

**Proof.** The proof is basically verification of (4.4). Let us rewrite (4.5) in the matrix form

\[
T(z) = \frac{q^{3/2} (q_1^{1/2} - q_1^{-1/2})}{\beta(q/q_1)} \Psi^\ast (qq_1 z) \varepsilon_z \Phi(z), \text{ for } \varepsilon_z = z^{1/2} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}. \tag{4.6}
\]

Using (4.2), Proposition 3.9 and (2.13) we obtain

\[
f(w/z) (\Psi^\ast (qq_1 z) \varepsilon_z \Phi(z)) (\Psi^\ast (qq_1 w) \varepsilon_w \Phi(w)) = \frac{1}{1 - w/z} \beta \left( \frac{q q_1 w}{z} \right) \beta \left( \frac{q q_1 z}{w} \right) (\Psi^\ast (qq_1 z) \varepsilon_z \Phi(z)) (\Psi^\ast (qq_1 w) \varepsilon_w \Phi(w)) = \beta \left( \frac{z q q_1 w}{q_1 q z} \right) (qq_1 z w)^{1/2} \\
\times \left( z^{-\frac{1}{2}} \alpha_\Phi \left( \frac{w}{z} \right) \Psi^\ast (qq_1 z) \Psi^\ast (qq_1 w) \right) \varepsilon_z \otimes \varepsilon_w \left( z^{-\frac{1}{2}} \alpha_\Phi \left( \frac{w}{z} \right) \Phi(1) (z) \Phi(2) (w) \right). \tag{4.7}
\]

To continue the calculation, we apply Theorems 1 and 2. The RHS of (4.7) can be presented as sum of three summands. The first summand is

\[
= \beta \left( \frac{z q q_1 w}{q_1 q z} \right) \beta \left( \frac{w}{q_1 q z} \right) (qq_1 z w)^{1/2} \\
\times \left( w^{-\frac{1}{2}} \alpha_\Phi (z / w) \Psi^\ast (qq_1 w) \Psi^\ast (qq_1 z) \right) R(z / w) \varepsilon_z \otimes \varepsilon_w R^{-1} (z / w) \left( w^{-\frac{1}{2}} \alpha_\Phi (z / w) \Phi(2) (w) \Phi(1) (z) \right) \\
= \frac{1}{1 - z / w} \beta \left( \frac{z q q_1 w}{q_1 q z} \right) \beta \left( \frac{w}{q_1 q z} \right) \Psi^\ast (qq_1 w) \varepsilon_w \Phi(w) (\Psi^\ast (qq_1 z) \varepsilon_z \Phi(z)) \\
= f(z / w) (\Psi^\ast (qq_1 w) \varepsilon_w \Phi(w)) (\Psi^\ast (qq_1 z) \varepsilon_z \Phi(z)). \tag{4.8}
\]

Here we have used Proposition 3.9 and an important property

\[
R(z / w) \varepsilon_z \otimes \varepsilon_w R^{-1} (z / w) = \varepsilon_z \otimes \varepsilon_w. \tag{4.9}
\]

The second summand without factor \( \beta \left( \frac{z q q_1 w}{q_1 q z} \right) \beta \left( \frac{w}{q_1 q z} \right) \) is

\[
(q q_1 z w)^{1/2} \left( z^{-\frac{1}{2}} \alpha_\Phi \left( \frac{w}{z} \right) \Psi^\ast (qq_1 z) \Psi^\ast (qq_1 w) \right) \varepsilon_z \otimes \varepsilon_w \left( (-q)^{1/2} (q^{-1} v_- \otimes v_+ - q^{-2} v_+ \otimes v_-) \delta (z, q^2 w) \right) (-1)^{\vartheta} \\
= (q q_1)^{1/2} z w \left( z^{-\frac{1}{2}} \alpha_\Psi (w / z) \Psi^\ast (qq_1 z) \Psi^\ast (qq_1 w) \right) \left( (-q)^{1/2} (q^{-1} v_- \otimes v_+ - q^{-2} v_+ \otimes v_-) \delta (z, q^2 w) \right) (-1)^{\vartheta} \\
= q^{-1} z w (-q / z)^{1/2} \alpha_\Psi (w / z) \left( \Psi^\ast (qq_1 z) \Psi^\ast (qq_1 w) - \Psi^\ast (qq_1 q q_1 w) \Psi^\ast (qq_1 w) \right) (-1)^{\vartheta} \delta (z, q^2 w) \\
= q^{-1} z w w^{-1} \frac{1}{1 - q^2} \delta (z, q^2 w) = \frac{1}{q (1 - q^2)} \delta (z, q^2 w).
\]
Here we used Corollary 3.8.
The third summand without factor $\beta\left(\frac{x}{\eta q^{n}}\right)\beta\left(\frac{w}{\eta q^{2}}\right)$ is
\[
qq_{1}(zw)^{1/2} \left((q^{3/2}(v_{+}^{*}\otimes v_{-}^{*})-v_{+}^{*}\otimes v_{-}^{*})\delta(q^{3}q_{1}z,qq_{1}w)\right)\varepsilon_{z} \otimes \varepsilon_{w}\left(z^{-1/2}\alpha_{\phi}(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^{q}
= qq_{1}zw \left((q^{3/2}(v_{+}^{*}\otimes v_{-}^{*})-v_{+}^{*}\otimes v_{-}^{*})\delta(q^{3}q_{1}z,qq_{1}w)\right)\varepsilon_{z} \otimes \varepsilon_{w}\left(z^{-1/2}\alpha_{\phi}(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^{q}
= -q^{2}q_{1}zw \left((q^{3/2}(v_{+}^{*}\otimes v_{-}^{*})-v_{+}^{*}\otimes v_{-}^{*})\delta(q^{3}q_{1}z,qq_{1}w)\right)\varepsilon_{z} \otimes \varepsilon_{w}\left(z^{-1/2}\alpha_{\phi}(w/z)\Phi^{(1)}(z)\Phi^{(2)}(w)\right)(-1)^{q}
\]
Here we used Proposition 3.6.
When we calculated the second and the third summands, we have omitted the multiple
\[
\beta\left(\frac{q^{2}}{q^{2}}\right)\beta\left(\frac{q^{-2}}{q^{2}}\right) = \frac{1-q^{-1}}{1-q^{-2}} (\beta(q/q_{1}))^{2} = \frac{1-q_{2}}{1-q_{1}} (\beta(q/q_{1}))^{2}.
\] (4.10)
So the delta-function coefficient is
\[
\frac{1-q^{2}}{1-q^{-1}} (\beta(q/q_{1}))^{2} \times \frac{1}{q(1-q^{2})} = \frac{(1-q_{1})(1-q_{2})}{1-q_{3}^{-1}} \left(\frac{\beta(q/q_{1})}{q^{3/2}(q_{1}^{-1/2}-q^{-1/2})}\right)^{2}.
\] (4.11)
So we have proven
\[
f(w/z) (\Psi^{*}(qq_{1}z)\varepsilon_{z}\Phi(z)) (\Psi^{*}(qq_{1}w)\varepsilon_{w}\Phi(w)) - f(z/w) (\Psi^{*}(qq_{1}w)\varepsilon_{w}\Phi(w)) (\Psi^{*}(qq_{1}z)\varepsilon_{z}\Phi(z))
= (1-q_{1})(1-q_{2}) \left(\frac{\beta(q/q_{1})}{q^{3/2}(q_{1}^{-1/2}-q^{-1/2})}\right)^{2} \left(\frac{\beta(q/q_{1})}{q^{3/2}(q_{1}^{-1/2}-q^{-1/2})}\right)^{2} \left(\frac{z}{q^{2}}\right) - \delta\left(\frac{q^{2}z}{w}\right).
\] (4.12)
Evidently, this is equivalent to the theorem.

Connection with Verma module. Highest weight vector $|\lambda\rangle$ for $\text{Vir}_{q_{1}q_{2}}$ with highest weight $\lambda \in \mathbb{C}$ in a $\text{Vir}_{q_{1}q_{2}}$-module is defined by the following properties
\[
T_{0}|\lambda\rangle = \lambda|\lambda\rangle, \quad T_{n}|\lambda\rangle = 0 \text{ for } n > 0.
\] (4.13)
Denote by $|j\rangle = 1 \otimes \mathbb{C}e^{q_{1}z+1/2} \in V_{j}$ the highest weight vector with respect to Heisenberg algebra.

Proposition 4.1. Vector $|j\rangle \in F^{[j]}_{u}$ is a highest weight vector for $\text{Vir}_{q_{1}q_{2}}$ with the highest weight
\[
\lambda_{u,j} = (-q)^{1/2}(qq_{1})^{j/2}u + \left((-q)^{1/2}(qq_{1})^{j/2}u\right)^{-1}.
\] (4.14)
Proof. Using (2.17), we obtain
\[
\frac{q^{3/2}(q_{1}^{-1/2}-q^{-1/2})}{\beta(q/q_{1})} z^{1/2} \Psi^{*}(qq_{1}z)\Phi_{-}(z) = q^{3/2}(q_{1}^{-1/2}-q^{-1/2}) (-q^{3} \times q_{1}z) \frac{-1/2}{\beta(q/q_{1})} z^{1/2} \Psi^{*}(qq_{1}z)\Phi_{-}(z):
= q^{3/2}(q_{1}^{-1/2}-q^{-1/2}) (-q^{3} \times q_{1}z) \frac{-1/2}{\beta(q/q_{1})} \frac{1}{1-q_{1}} \Psi^{*}(qq_{1}z)\Phi_{-}(z): = (-q)^{-1/2} \Psi^{*}(qq_{1}z)\Phi_{-}(z):
\] (4.15)
Using (4.15) and formulas for explicit bosonization (2.8) and (2.12), we obtain
\[
\frac{z^{1/2} q^{3/2}(q_{1}^{-1/2}-q^{-1/2})}{\beta(q/q_{1})} \Psi^{*}(qq_{1}z)\Phi_{-}(z)|j\rangle = (-q)^{-1/2}(qq_{1})^{-j/2}|j\rangle + O(z).
\] (4.16)
Here $O(z)$ is a formal power series, which vanishes at $z=0$, i.e. $\sum_{n>0} \alpha_{n}z^{n}$. 

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Lemma 4.2. Vector $\tilde{\pi}|j\rangle$ coincides up to a scale with vector $|1 - j\rangle$.

Sketch of a proof. Let us consider two grading on $V(\Lambda_0) \oplus V(\Lambda_1)$

$$\deg_{pr} |j\rangle = j(j-1)/4 \quad \deg_K v = j \quad \text{iff} \quad Kv = q^j v$$

(4.17)

$$\deg_{pr} a_{-k} = k$$

(4.18)

One can check that

$$\deg_K (\pi(v)) = 1 - \deg_K v \quad \deg_{pr} (\pi(v)) = \deg_{pr} v$$

(4.19)

Up to a scale, vector $|j\rangle$ is the only vector with $\deg_K = j$ and $\deg_{pr} = j(j-1)/4$.

Let us apply $\pi$-involution to (4.16); Proposition 2.2 and Lemma 4.2 imply

$$z^{1/2} q^{3/2}(q_1^{1/2} - q_1^{-1/2})^{-1/2} \beta(q/q_1) (-q)^3 (1 - z) (q_1 q^3 z) \Psi_+^\dagger(\lambda_{-i})(z) |1 - j\rangle = (-q)^{1/2} (q_1)^{1/2} |j\rangle + O(z).$$

(4.20)

Replacing of $j \mapsto 1 - j$, we obtain

$$z^{1/2} q^{3/2}(q_1^{1/2} - q_1^{-1/2})^{-1/2} \beta(q/q_1) \Psi_+(q_1 z) \Phi_+(z) |j\rangle = (-q)^{1/2} (q_1)^{1/2} |j\rangle + O(z).$$

(4.21)

Comparison of (4.16) and (4.21) with (4.5) finishes the proof.

Verma module $M(\lambda)$ is a module with cyclic highest weight vector $|\lambda\rangle$ and without any other relations apart from (4.13). Verma module $M(\lambda)$ enjoys a universal property: it maps to any module with a highest weight vector of weight $\lambda$. According to Proposition 4.1 there is a natural map $\phi_{u,j}: M(\lambda_{u,j}) \to F_u^{[j]}$. We will say that $\lambda$ is generic if $\lambda \neq \pm (q_1^{1/2} q_2^{s/2} + q_1^{-1/2} q_2^{-s/2})$ for $r, s \in \mathbb{Z}_{\geq 1}$.

Proposition 4.3. For generic $\lambda$ the Verma module $M(\lambda)$ is irreducible. Dimension of $n$th graded component is $p(n)$, i.e. the number of partitions of $n$ elements.

This proposition follows from the fact that determinant of the Shapovalov form for such $\lambda$ is nonzero, this fact was proven in [BP98 Th. 3.3], using [SKAO96]. One can also deduce this from the irreducibility of tensor product of Fock modules of toroidal algebra $U_{q_1, q_2, q_3}(\mathfrak{g}_1)$ [FFJ+11] Lem 3.1 and relations to $W$-algebras [Neg18] [FHS+10].

We will say that pair $u, j$ is generic if the corresponding highest weight $\lambda_{u,j}$ is generic.

Corollary 4.4. For generic values of $u, j$ the module $F_u^{[j]}$ is irreducible. The natural map $\phi_{u,j}: M(\lambda_{u,j}) \to F_u^{[j]}$ is an isomorphism.

Proof. Note that dimensions of graded components of both $M(\lambda)$ and $F_u^{[j]}$ equals to $p(n)$, in particular they coincide. If $M(\lambda_{u,j})$ is irreducible, then the map $\phi_{u,j}: M(\lambda_{u,j}) \to F_u^{[j]}$ is an isomorphism.

Remark 3. As it was mentioned in Remark 2 another bosonization of $\text{Vir}_{q_1, q_2}$ was constructed in [SKAO96]. Moreover, their formula for the highest weight essentially coincides with our formula (4.14). Namely, in notation of [FF96 Sec. 3] the highest weight of the representation on the Fock space $\pi_\mu$ equals to $\lambda_{u,j}$ if $q^\mu$ in notation of loc. cit. equals to $(-q^3)^{1/2} (qq_1)^j u$ in notation of this paper (note that parameters $q, p$ in loc. cit. correspond to $q_1, q_3^{-1}$ in this paper). For generic $u, j$ these modules are isomorphic since they both are isomorphic to irreducible Verma module.
4.2 Twisted Deformed Virasoro algebra

**Definition 4.2.** Twisted deformed Virasoro algebra is generated by $T_r$ for $r \in 1/2 + Z$. The defining relation is

$$\sum_{l=0}^{\infty} f_l T_{r-l} T_{s+l} = \sum_{l=0}^{\infty} f_{l-r} T_{s-l} T_{r+l} = \frac{(1 - q_1)(1 - q_2)}{1 - q_3} (q^{-r} - q_3^r) \delta_{r+s,0}. \quad (4.22)$$

Denote $T(z) = \sum_{r \in 1/2 + Z} T_r z^{-r}$, $\delta_{\text{odd}}(x) = \sum_{r \in 1/2 + Z} x^r$. Relation (4.22) is equivalent to

$$f(w/z)T(z)T(w) - f(z/w)T(w)T(z) = -\frac{(1 - q_1)(1 - q_2)}{1 - q_3} \left( \delta_{\text{odd}} \left( \frac{w}{q_3z} \right) - \delta_{\text{odd}} \left( \frac{q_3w}{z} \right) \right). \quad (4.23)$$

**Theorem 4.** Formulas below determines an action of Twisted Deformed Virasoro algebra on $V(\Lambda_i)$ for $i = 0, 1$

$$T(z) = (-1)^{1/2} q^{3/2} (q^{1/2}_1 - q^{-1/2}_1) \text{Res}_{(q/q_1)} (z \Psi^*_{*}(qq_1z) \Phi(z) + \Psi^*_{+}(qq_1z) \Phi_{-}(z)). \quad (4.24)$$

Denote the obtained representation by $\mathcal{F}[i]$.

**Proof.** Let us rewrite (4.24) in the matrix form

$$T(z) = (-1)^{1/2} q^{3/2} (q^{1/2}_1 - q^{-1/2}_1) \text{Res}_{(q/q_1)} \Psi^*(qq_1z) \varepsilon_z \Phi(z), \text{ for } \varepsilon_z = \left( \begin{array}{c} 0 \\ z \\ 0 \end{array} \right). \quad (4.25)$$

The proof is very similar to the proof of Theorem 3. A crucial point is that (4.9) holds for the new $\varepsilon_z$. Hence RHS of (4.7) still can be presented as sum of three summands. The first summand is still given by (4.8). The second summand without factor $\beta \left( \frac{z}{q_1u} \right) \beta \left( \frac{w}{q_1v} \right)$

$$(qq_1wz)^{1/2} \left( z^{-\frac{1}{2}} \alpha^*_\psi(w/z) \Psi^*_{*}(1)(qq_1z) \Psi^*_{*}(2)(qq_1w) \right) \varepsilon_z \otimes \varepsilon_w \left( (q^{-1}v_+ \otimes v_+ - q^{-2}v_+ \otimes v_-) \delta(z, q^2w) \right)(-1)^0

= (-q)^{1/2} \left( z^{-\frac{1}{2}} \alpha^*_\psi(w/z) \Psi^*_{*}(1)(qq_1z) \Psi^*_{*}(2)(qq_1w) \right) \varepsilon_z \otimes \varepsilon_w \left( q^{-1}v_+ \otimes v_+ - q^{-2}v_+ \otimes v_- \right) \delta_{\text{odd}}(z, q^2w)(-1)^0

= (-q_1z)^{1/2} \left( q_1^2 \psi(w/z) \Psi^*_{*}(1)(qq_1z) \Psi^*_{*}(2)(qq_1w) \right) \left( q^{-1}wv_+ \otimes v_+ - q^{-2}wv_+ \otimes v_- \right) \delta_{\text{odd}}(z, q^2w)(-1)^0

= q_1^{-1}w \times (-q_1z)^{1/2} \left( \Psi^*_{*}(qq_1z) \Psi^*_{*}(qq_1w) \right) \left( q^{-1}wv_+ \otimes v_+ - q^{-2}wv_+ \otimes v_- \right) \delta_{\text{odd}}(z, q^2w)(-1)^0

= -q_1^{-1}w \frac{w^{-1}}{1 - q^2} \delta_{\text{odd}}(z, q^2w) = -1 \frac{1}{q(1 - q^2)} \delta_{\text{odd}}(z, q^2w).

The third summand without factor $\beta \left( \frac{z}{q_1w} \right) \beta \left( \frac{w}{q_1v} \right)$

$$(qq_1wz)^{1/2} \left( (-q)^{1/2} \left( q \psi_+ \otimes v_+ + v_+ \otimes v_+ \right) \delta(q^3q_1z, qq_1w) \right) \varepsilon_z \otimes \varepsilon_w \left( z^{-\frac{1}{2}} \alpha^*_\phi(w/z) \Phi^*_{*}(1)(z) \Phi^*_{*}(2)(w) \right)(-1)^0

= -(-q)^{1/2} \left( q \psi_+ \otimes v_+ - v_+ \otimes v_+ \right) \delta_{\text{odd}}(q^2z, w) \varepsilon_z \otimes \varepsilon_w \left( z^{-\frac{1}{2}} \alpha^*_\phi(w/z) \Phi^*_{*}(1)(z) \Phi^*_{*}(2)(w) \right)(-1)^0

= -(-q)^{1/2} \left( qzw_+ \otimes v_+ - w_+ \otimes v_+ \right) \left( z^{-\frac{1}{2}} \alpha^*_\phi(w/z) \Phi^*_{*}(1)(z) \Phi^*_{*}(2)(w) \right) \delta_{\text{odd}}(q^2z, w)(-1)^0

= qz(-q)^{1/2} \left( q \Phi^*_{*}(z) \Phi^*_{*}(w) - \Phi^*_{*}(z) \Phi^*_{*}(w) \right) \delta_{\text{odd}}(q^2z, w)(-1)^0

= qz \frac{1}{q^2(1 - q^2)} \delta_{\text{odd}}(q^2z, w).$$

The end of the proof is almost the same as in non-twisted case. □
Connection with Verma module. Highest weight vector $|∅⟩$ in a representation of twisted deformed Virasoro algebra is defined by the following properties

$$T^r|∅⟩ = 0 \quad \text{for } r > 0. \quad (4.26)$$

**Proposition 4.5.** The vectors $|Λ_i⟩ \in F^{[i]}$ are highest weight vectors.

**Proof.** Recall the grading $\deg_{pr}$ on $V(Λ_i)$ defined by (4.17) and (4.18). One can check that $F^{[i]}$ is a graded $Vir_{q_1,q_2}^{tw}$-module with respect to grading $\deg_{pr} T^r = r$. To finish the proof one has to note that $\deg_{pr} |0⟩ = \deg_{pr} |1⟩ = 0$ and $\deg_{pr} |j⟩ > 0$ for $j \neq 0,1$. □

Verma module $M^{tw}$ of twisted Virasoro algebra is a cyclic module with cyclic vector $|∅⟩$ and without any other relations apart from (4.26). Verma module enjoys a universal property: it maps to any module with a highest weight vector. Hence there exist a natural map $φ_i: M^{tw} → F^{[i]}$ such that $|∅⟩ → |Λ_i⟩$.

**Lemma 4.6.** Verma module $M^{tw}$ is spanned by

$$T_{-r_m} \cdots T_{-r_1}|∅⟩ \quad \text{for } 0 < r_1 ≤ r_2 ≤ \cdots ≤ r_m. \quad (4.27)$$

**Sketch of a proof.** One can prove that any element $T_{s_1} \cdots T_{s_k}|∅⟩$ can be presented as a linear combination of vectors (4.27) using (4.22) by induction. □

**Proposition 4.7.** For generic $q_3$ the Verma module $M^{tw}$ is irreducible. Natural maps $φ_i: M^{tw} → F^{[i]}$ are isomorphisms.

**Proof.** The representation $F^{[i]}$ for $q_3 = 1$ was considered in [BG19, Example 7.1]; it follows from [BG19, Section 7.3] that the representations are irreducible. Hence $F^{[i]}$ is irreducible for generic $q_3$. Then maps $φ_i: M^{tw} → F^{[i]}$ are surjective. Now recall the Gauss identity

$$\prod_{r=\frac{1}{2}+Z_{≥0}} \frac{1}{1-q^r} = \sum_{j\in i+2Z} \frac{q^{(j-1)/2}}{\prod_{n=1}^{∞}(1-q^n)^{j}} \quad \text{for } i = 0,1. \quad (4.28)$$

According to Lemma 4.6, dimensions of graded components of $M^{tw}$ do not exceed corresponding coefficient of LHS of (4.28). On the other hand, coefficients of RHS of (4.28) are equal to dimensions of graded components of $F^{[i]}$. Hence, it follows from surjectivity of $φ_i$ that $φ_i$ is an isomorphism. □

**Corollary 4.8.** For generic $q_3$ vectors (4.27) form a basis of $M^{tw}$.

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