The multi-degree of coverings on Lie groups

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Abstract

We associate to each covering map of simple Lie groups a sequence of integers, called the multi-degree of the covering; develop a unified method to evaluate the sequence; and apply the results to solve two extension problems of mathematical physics.

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1 Main results

Covering maps on Lie groups are essential to extend the constructions and calculations of various physical models associated with simply connected Lie groups, to that associated with non-simply connected Lie groups. Naturally, topological invariants of the coverings are useful to formulate, or to solve, the relevant extension problems. In this paper we introduce for each covering map of Lie groups an invariant, called the multi-degree of the covering; extend Schubert calculus to evaluate the invariant; and apply the results to two outstanding topological problems arising from the studies of the Wess–Zumino–Witten models and the topological Gauge theories [6, 16, 25]. The main tool in our approach is the Chow rings of Lie groups, introduced by Grothendieck [19] in 1958.

To clarify the concept of multi-degree of a covering on Lie groups we recall the classical results of Hopf and Chevalley on the real cohomology of Lie groups. The Lie groups \( G \) under consideration will be compact and connected. For a maximal torus \( T \) on \( G \) the dimension \( n := \dim T \) is an invariant of \( G \), called the rank of \( G \). The multiplication \( \mu : G \times G \to G \) on \( G \) defines the co–product on the real cohomology algebra \( H^*(G; \mathbb{R}) \)

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For a Lie group \(G\) the following result may be seen as an integral refinement of Theorem 1.1.

In particular, the multiplication map on the torsion free parts of the cohomologies induces ring map \(f: \mathcal{P}G \to \mathcal{R}^n\), passing to the quotients yields the (graded) ring map \(f\) isomorphism is given by the Kunneth formula. An element \(y \in H^*(G; \mathbb{R})\) is called primitive if the relation \(\mu^*(y) = y \otimes 1 + 1 \otimes y\) is satisfied. Since the linear combinations of primitive elements are also primitive such elements form a subspace \(P(G; \mathbb{R})\) of the algebra \(H^*(G; \mathbb{R})\). Hopf [18] has shown that

**Theorem 1.1.** For a Lie group \(G\) with rank \(n\) we have \(\dim P(G; \mathbb{R}) = n\).

Moreover, if \(\{y_1, \cdots, y_n\}\) is a basis of the space \(P(G; \mathbb{R})\), then the real cohomology algebra \(H^*(G; \mathbb{R})\) is the exterior algebra generated by \(y_1, \cdots, y_n\):

\[
H^*(G; \mathbb{R}) = \Lambda_{\mathbb{R}}(y_1, \cdots, y_n), \quad \deg y_i \equiv 1 \text{ mod } 2.\]

By (1.2) the sequence \(I_G = \{r_1, \cdots, r_n\}\) of integers defined by

\[
r_i := \frac{1}{2} (\deg y_i + 1), \quad 1 \leq i \leq n,
\]

is also an invariant of the group \(G\), which has been shown by Chevalley [3, 27] (3.2) Theorem to be the degree sequence of the basic Weyl invariants of \(G\). We may assume that the sequence \(I_G\) is ordered by \(r_1 \leq \cdots \leq r_n\). In particular, if the group \(G\) is simple one has \(2 = r_1 < \cdots < r_n\) (see in Table 1).

In general, for a CW–complex \(X\) the torsion subgroup \(\tau(X)\) of the integral cohomology \(H^*(X)\) is an ideal and therefore, defines the (graded) quotient ring \(\mathcal{F}(X) := H^*(X)/\tau(X)\). In view of the obvious additive decomposition

\[
H^*(X) = \mathcal{F}(X) \oplus \tau(X)
\]

we may call \(\mathcal{F}(X)\) the torsion free part of the integral cohomology \(H^*(X)\). Furthermore, if \(f: X \to Y\) is a continuous map between two CW–complexes, the induce ring map \(f^*\) on the cohomologies preserves the torsion ideals. Therefore, passing to the quotients yields the (graded) ring map \(f^\# : \mathcal{F}(Y) \to \mathcal{F}(X)\). In particular, the multiplication map \(\mu\) on \(G\) furnishes the ring \(\mathcal{F}(G)\) with a co–product

\[
\mu^\#: \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G).
\]

The following result may be seen as an integral refinement of Theorem 1.1.

**Theorem A.** For a Lie group \(G\) with \(I_G = \{r_1, \cdots, r_n\}\), there exist \(n\) elements \(x_1, \cdots, x_n \in \mathcal{F}(G)\), \(\deg x_i = 2r_i - 1\), so that

i) \(\mathcal{F}(G) = \Lambda(x_1, \cdots, x_n)\) (i.e. the exterior ring over \(\mathbb{Z}\))

ii) \(\mu^\#(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad 1 \leq i \leq n\).

In addition, if the group \(G\) is simple, then the generators \(x_1, \cdots, x_n\) satisfying i) and ii) are unique up to the sign \(\pm\) convention.

Let \(G\) be a simple Lie group with non–trivial center \(Z(G)\). The quotient map \(c : G \to PG := G/Z(G)\) is a covering map of Lie groups, and induces a ring map on the torsion free parts of the cohomologies

\[
c^\#: \mathcal{F}(PG) = \Lambda(y_1, \cdots, y_n) \to \mathcal{F}(G) = \Lambda(x_1, \cdots, x_n),
\]

\[2\]
where \( \deg y_i = \deg x_i \) by the well known fact \( I_G = I_{PG} \) of invariant theory. Granted with Theorem A we shall show that

**Theorem C.** We shall show that

i) the map \( c^\# \) is given by \( c^\#(y_i) = a_i \cdot x_i \), \( 1 \leq i \leq n \);

ii) the product \( a_1 \cdots a_n \) is the order \( |Z(G)| \) of the center.

The unique sequence \( \{a_1, \ldots, a_n\} \) obtained by Theorem B, written \( D(G, PG) \) and called the multi–degree of the covering \( c \), will be the main concern of this work. To be precise we tabulate all the simply connected simple Lie groups \( G \) with non–trivial centers \( Z(G) \), together with the degree sequences \( I_G \) of their basic Weyl invariants, using the table below.

| Table 1. The simply connected simple Lie groups with non–trivial centers |
|---|
| \( G \) | \( \text{rank} G \) | \( I_G \) | \( Z(G) \) |
| \( SU(n) \) | \( n - 1 \) | \( \{2, 3, \ldots, n\} \) | \( \mathbb{Z}_n \) |
| \( Sp(n) \) | \( n \) | \( \{2, 4, \ldots, 2n\} \) | \( \mathbb{Z}_2 \) |
| \( Spin(2n) \) | \( n \) | \( \{2, 2, \ldots, 2(n-1), 0\} \Pi \{n\} \) | \( \mathbb{Z}_2 \) |
| \( Spin(2n), n \equiv 1 \mod 2 \) | \( n \) | \( \{2, 4, \ldots, 2(n-1), 0\} \Pi \{n\} \) | \( \mathbb{Z}_4 \) |
| \( Spin(2n), n \equiv 0 \mod 2 \) | \( n \) | \( \{2, 2, \ldots, 2(n-1), 0\} \Pi \{n\} \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) |
| \( E_6 \) | 6 | \( \{2, 5, 6, 8, 9, 12\} \) | \( \mathbb{Z}_3 \) |
| \( E_7 \) | 7 | \( \{2, 6, 8, 10, 12, 14, 18\} \) | \( \mathbb{Z}_2 \) |

Let \( J(PG) \) be the subring of \( H^*(PG) \) generated multiplicatively by the second cohomology group \( H^2(PG) \), together with the multiplicative unit 1 \( G \in H^0(PG) \). By computing with the Chow rings \([19]\) of the adjoint Lie groups \( PG \) we shall show that

**Theorem C.** For each simply connected Lie group \( G \) in Table 1 we have

\[
(1.4) \quad H^2(PG) = Z(G).
\]

Moreover, there exist generators

\[
\omega \in H^2(PG) \quad \text{for} \quad G \neq Spin(2n) \quad \text{with} \quad n \equiv 0 \mod 2; \quad \text{or}
\]

\[
\omega_1, \omega_2 \in H^2(PG) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for} \quad G = Spin(2n) \quad \text{with} \quad n \equiv 0 \mod 2
\]

so that the ring \( J(PG) \) has the following presentation:

i) \( J(PSU(n)) = \mathbb{Z}^{|\omega|}_{b_{n,r} \cdot \omega (1+\frac{1}{n+1})} \), where \( b_{n,r} = \text{g.c.d.} \{\binom{n}{r}\} \); \n
ii) \( J(PSp(n)) = \mathbb{Z}^{|\omega|}_{2^m \cdot \omega^{2m+1}} \), where \( n = 2^m (2k + 1) \); \n
iii) \( J(PSpin(2n+1)) = \mathbb{Z}^{|\omega|}_{2^m \cdot \omega^{2m+1}} \), where \( 2^m \leq n < 2^{m+1} \); \n
iv) \( J(PSpin(2n)) = \mathbb{Z}^{|\omega|}_{(2^m \cdot 2^{2m+1})} \), if \( n \equiv 1 \mod 2 \); \n
\( \mathbb{Z}^{|\omega|}_{(2^m \cdot 2^{2m+1})} \), if \( n \equiv 0 \mod 2 \),

where \( n = 2^m (2k + 1) \), \( 2^r < n < 2^{r+1} \); \n
v) \( J(PE_6) = \mathbb{Z}^{|\omega|}_{(2^m \cdot \omega^{2m+1})} \); \n
vi) \( J(PE_7) = \mathbb{Z}^{|\omega|}_{(2^m \cdot \omega^{2m+1})} \).
Our main result implies that, the multi-degree sequence $D(G, PG)$ is determined entirely by the structure of the ring $J(PG)$ given by Theorem C (see Theorem 4.4).

**Theorem D.** For a Lie group $G$ in Table 1 the multi-degree $D(G, PG)$ is

| $G$                  | $D(G, PG)$                                      |
|----------------------|-------------------------------------------------|
| $SU(n)$              | $\{a_1, \ldots, a_{n-1}\}, \ a_k = \frac{1}{2^{n-1}}$ |
| $Sp(n)$, $n = 2^t(2b + 1)$ | $\{1, \ldots, 1, 2(2^{2t}), 1 \cdots, 1\}$       |
| $Spin(2n+1)$, $2^s \leq n < 2^{s+1}$  | $\{1, \ldots, 1, 2(2^{2s}), 1 \cdots, 1\}$       |
| $Spin(2n)$, $2^s < n = 2b + 1 \leq 2^{s+1}$  | $\{2, 1, \ldots, 1, 2(2^{2s}), 1 \cdots, 1\}$ |
| $Spin(2n), 2^s < n \leq 2^t(2b + 1) \leq 2^{s+t}, t \geq 1$ | $\{1, \ldots, 1, 2(2^{2s}), 1 \cdots, 1\}$       |
| $E_6$ or $E_7$       | $\{1, 1, 1, 3, 1\}$ or $\{2, 1, 1, 1, 1, 1\}$ |

where the notation $2_{(k)}$ stands for $a_k = 2$, and where if $G = SU(n)$ and the integer $n$ has the prime factorization $n = p_1^{s_1} \cdots p_t^{s_t}$, then (see [10, Lemma 3.1])

\begin{equation}
(1.5) \quad a_k = \begin{cases} 
p_i & \text{if } k = p_i^t \text{ with } 1 \leq i \leq t \text{ and } 1 \leq s \leq r_i; \\
1 & \text{otherwise.}
\end{cases}
\end{equation}

Theorem D is a natural extension of [10, Theorem 1.1], where the multi-degree of the universal covering $SU(n) \to PSU(n)$ on the projective unitary group $PSU(n)$ has been computed. On the other hand, E.B. Dynkin [14, 15] raised the problem to determine induce actions of Lie group homomorphisms $G \to G'$ on the cohomologies, which is essentially solved by Theorem D for the cases of universal coverings of the adjoint Lie groups. Nevertheless, the present paper is motivated by two topics from mathematical physics. To start with we note by Theorem D that

**Corollary 1.2.** For a Lie group $G$ in Table 1 the leading term $a_1$ of the sequence $D(G, PG)$ is either 1 or 2, where $a_1 = 2$ occurs if and only if $G$ is isomorphic to one of the following groups

i) $SU(n)$ with $n \equiv 0 \text{ mod } 2$;

ii) $Sp(n)$ with $n \equiv 1 \text{ mod } 2$;

iii) $Spin(2^t(2b + 1))$ with $t = 1, 2$;

iv) $E_7$. $\square$

In the study on the Wess–Zumino–Witten models with simple Lie groups Felder, Gawedzki and Kupiainen obtained the short exact sequence

\begin{equation}(1.6) \quad 0 \to H_3(G)(\equiv \mathbb{Z}) \xrightarrow{\beta} H_3(PG) \to Z(G) \to 0,
\end{equation}

and investigated its extension problem [16, Appendix 1], where $G$ is simply connected and simple. This problem was emphasized by Dijkgraaf and Witten in the work [6] on the topological Gauge theories, and by Mathai and Rosenberg [25] in the study on the relationship between Langlands duality and T–duality for compact Lie groups. Since the map $c_*$ in (1.6) is Kronecker dual to the map $c^*$ on $H^3(PG)$ we get from Corollary 1.2 the following result.

**Corollary 1.3.** For the Lie groups $G$ given in Table 1 we have
$H_3(PG) = \mathbb{Z} \oplus \mathbb{Z}(G)$

with the following exceptions

i) $H_3(PSU(2m)) = \mathbb{Z} \oplus \mathbb{Z}_m$, where $m \in \mathbb{Z}$;

ii) $H_3(PSp(2b + 1)) = H_3(PE_7) = \mathbb{Z}$, where $b \in \mathbb{Z}$;

iii) $H_3(PSpin(2t(2b + 1))) = \mathbb{Z} \oplus \mathbb{Z}_2$, where $t = 1, 2, b \in \mathbb{Z}$. □

**Remark 1.4.** In the inspiring work [25] Mathai and Rosenberg have computed the leading terms $a_1$ of the sequences $D(G, PG)$. In the cases of $G = SU(n)$ with $n$ divisible by 4 and $G = Spin(2(2b + 1))$, our results in Corollary 1.2 are different with the ones stated in [25] Theorem 1, (1), (3)]. We note that these are precisely the cases where the integral cohomologies $H^*(G)$ have torsion elements of the order 4. Therefore, working with the $\mathbb{Z}_2$ algebra $H^*(G; \mathbb{Z}_2)$ alone may not suffice to decide $a_1$. □

Let $B_G$ be the classifying space of a simple Lie group $G$. By the naturality of the cohomology suspension $\tau_G : H^*(B_G) \to H^{-1}(G)$ [17] p.22 in the universal $G$-bundle

$G \hookrightarrow E_G \to B_G$

the covering $c : G \to PG$ induces the commutative diagram

(1.7) $\begin{array}{ccc}
H^4(BPG) & \xrightarrow{\tau_{PG}} & H^3(PG) \\
\downarrow Be^* & & \downarrow e^* \\
H^4(B_G) & \xrightarrow{\sim \tau_G} & H^3(G)
\end{array}$ (compare with [6] (4.14))

where, as being pointed out in [6] that, the four vertices groups in (1.7) are all isomorphic to $Z$, and where the suspension $\tau_G$ at the bottom is an isomorphism when $G$ is simply connected. Let $\omega$ and $\tilde{\omega}$ denote the generators of respectively $H^4(BPG)$ and $H^4(B_G)$, $\xi$ and $\tilde{\xi}$ denote the generators of respectively $H^3(PG)$ and $H^3(G)$. In [6] Section 4.3] Dijkgraaf and Witten raised the interesting problem to decide the pair $(\alpha, \beta)$ of integers characterized by the relations

$Be^*(\omega) = \alpha \cdot \tilde{\omega}$, $\tau_{PG}(\omega) = \beta \cdot \xi$,

where the pair $(\alpha, \beta)$ has shown to be $(\frac{2n}{(2, n - 1)}, n)$ for the case $G = SU(n)$.

On the other hand, by the commutativity of the diagram (1.7) one has $\alpha = a_1 \cdot \beta$, where $a_1$ is the leading term of the sequence $D(G, PG)$. Therefore, combining results of Corollary 1.2 with Dijkgraaf–Witten’s formula [6] (4.18) evaluating the integer $\alpha$ we obtain the following results.

**Corollary 1.5.** For a Lie group $G$ in Table 1 the pair $(\alpha, \beta)$ of integers is given by the table:

| $G$           | $(\alpha, \beta)$ |
|---------------|--------------------|
| $SU(n)$       | $(\frac{2n}{(2, n - 1)}, n)$ |
| $Spin(2n + 1)$| $(2, 2)$           |
| $Spin(n), n = 2t(2b + 1)$ | $(8, 4), (4, 4), (2, 2)$ or $(1, 1)$ for $t = 0, 1, 2 \text{ or } 3$ |
| $Spin(2n), n = 2t(2b + 1)$ | $(8, 4), (4, 4)$ or $(2, 2)$ for $t = 0, 1 \text{ or } 2$ |
| $E_6$         | $(3, 3)$           |
| $E_7$         | $(4, 2)$           |
The paper is arranged as follows. In Section 2 we develop fundamental properties of the Serre spectral sequence of the torus fibration \( \pi : G \to G/T \)
where \( T \) is a maximal torus on \( G \). The results are applied in Section 3 to show Theorems A, B and D. Section 4 is devoted to an exact sequence associated to the cyclic coverings of Lie groups, by which the proof of Theorem D is reduced to computing with the Chow rings of Lie groups.

In this paper the cohomologies and spectral sequences are over the ring \( \mathbb{Z} \) of integers, unless otherwise stated. Given a subset \( S \) of a ring \( A \) the notion \( \langle S \rangle \) stands for the ideal generated by \( S \), while \( A/\langle S \rangle \) denotes the quotient ring. In addition, the elements in a graded ring or algebra are assumed to be homogeneous.

**Remark 1.6.** The problem of computing the cohomology of Lie groups was raised by E. Cartan in 1929, which was solved only for the cohomologies with field coefficients, see Reeder [27] and Kač [21] for accounts about the history. As for the task of the present work general results on the integral cohomology of Lie groups, such as Theorems A and B, are requested.

In his problem 15 Hilbert asked for a rigorous foundation of Schubert calculus. Van der Waerden [28] and A. Weil [29, p.331] attributed the problem to the determination of the cohomology rings of the flag manifolds \( G/T \). The proofs of Theorems C and D illustrate how Schubert calculus could be extended to computing with the integral cohomologies of Lie groups [13, Remark 6.3]. □

## 2 The integral cohomology of Lie groups

For a Lie group \( G \) with a maximal torus \( T \) consider the torus fibration

\[
(2.1) \quad T \to G \overset{\pi}{\to} G/T.
\]

on the group \( G \). The Borel transgression in \( \pi \) is the composition

\[
\tau = (\pi^*)^{-1} \circ \delta : H^1(T) \overset{\delta}{\to} H^2(G, T) \overset{(\pi^*)^{-1}}{\cong} H^2(G/T) \quad (\text{[9]}),
\]

where \( \delta \) is the connecting homomorphism in the cohomological exact sequence of the pair \((G, T)\), and where the map \( \pi^* \) from \( H^2(G/T) \) to \( H^2(G, T) \) is always an isomorphism. By the Leray-Serre theorem [24, p.135] we have that

**Lemma 2.1.** The second page of the Serre spectral sequence \( \{E_r^{*,*}(G), d_r\} \) of the fibration \( \pi \) is the Koszul complex

\[
(2.2) \quad E_2^{*,*}(G) = H^*(G/T) \otimes H^*(T) \quad (\text{see [24] p.259})
\]

on which the differential \( d_2 \) is determined by the transgression \( \tau \) as

i) \( d_2(x \otimes 1) = 0, \) \( d_2(1 \otimes t) = \pi(t) \otimes 1, \) \( t \in H^1(T), \)

ii) \( d_2(z \cdot z') = d_2(z) \cdot z' + (-1)^{\deg z} z \cdot d_2(z'), \) \( z, z' \in E_2^{*,*}(G). \)

In particular, if \( \dim G/T = m \) and \( \dim T = n \), then

\[
(2.3) \quad E_2^{m,n}(G) = H^m(G) \otimes H^n(T) = E_1^{m,n}(G) = H^{n+m}(G) = \mathbb{Z}, \quad r \geq 2;
\]
Proof. The base manifold $G/T$ of $\pi$ is the complete flag manifold associated to the Lie group $G$, hence is simply connected. Formula (2.2), together with properties i) and ii), are standard [24, p.259].

By formula (2.2) we have that

$$E_{m,n}^2(G) = H^m(G/T) \otimes H^n(T) = \mathbb{Z}$$

and that $E_{s,t}^2(G) = 0$ if either $s > m$ or $t > n$. It implies that any differential $d_r$ that acts or lands on the group $E_{r}^{m,n}(G)$ must be trivial, confirming the isomorphisms in (2.3).

Finally, by formula i) the differential

$$d_2 : E_2^{*,1}(G) = H^*(G/T) \otimes H^1(T) \to E_2^{*,0}(G) = H^*(G/T)$$

is $d_2(x \otimes t) = x \cup \tau(t)$, $t \in H^1(T)$. We get formula (2.4) from

$$E_{1}^{*,0}(G) = E_2^{*,0}(G)/\text{Im } d_2 = H^*(G/T)/\langle \text{Im } \tau \rangle.$$

□

The Koszul complex $\{E_{2}^{*,0}(G), d_2\}$ has now been well understood by the following works:

a) the base manifold $G/T$ is a flag variety which has a canonical decomposition into the Schubert cells on $G/T$ [2, 5];

b) presentation of the ring $H^*(G/T)$ by special Schubert classes on $G/T$ has been completed by Duan and Zhao in [11];

c) with respect to the Schubert basis on $H^2(G/T)$ formula of the transgression $\tau$ is available in [9, Theorem 2.5].

Combining these results explicit construction of the ring $E_{3}^{*,*}(G)$ has been carried out in [8, 12], among which we shall only need the following two results.

In [19] Grothendieck defined the Chow ring of a Lie group $G$ to be the subring $A^*(G) := \pi^*(H^*(G/T))$ of $H^*(G)$, where he has also shown

$$A^*(G) = H^*(G/T)/\langle \text{Im } \tau \rangle$$

([19, p.21, Rem.2]).

Comparing this with formula (2.4) we obtain that

**Lemma 2.2.** The map $\pi$ in (2.1) induces an isomorphism

$$A^*(G) = E_{3}^{*,0}(G).$$

In particular, for the Lie groups $G$ given in Table 1, the rings $A^*(G)$ and $A^*(PG)$ admit the following presentations (in terms of generators–relations):

$$A^*(SU(n)) = \mathbb{Z}, \quad A^*(PSU(n)) = \mathbb{Z}[\omega_{b,n}]_{1 \leq r \leq m}, \quad b_{n,r} = \text{g.c.d.} \{ \binom{n}{1}, \ldots, \binom{n}{r} \};$$

$$A^*(Sp(n)) = \mathbb{Z}, \quad A^*(PSp(n)) = \mathbb{Z}[\omega_{2n,2d+1}], \text{ where } n = 2^r(2k + 1);$$

$$A^*(Sp(n)) = \mathbb{Z}, \quad A^*(PSp(n)) = \mathbb{Z}[\omega_{2n,2d+1}], \text{ where } n = 2^r(2k + 1);$$

$$A^*(Sp(n)) = \mathbb{Z}, \quad A^*(PSp(n)) = \mathbb{Z}[\omega_{2n,2d+1}], \text{ where } n = 2^r(2k + 1);$$

$$A^*(Sp(n)) = \mathbb{Z}, \quad A^*(PSp(n)) = \mathbb{Z}[\omega_{2n,2d+1}], \text{ where } n = 2^r(2k + 1);$$
where the generators $\omega, x_i$

\[
\text{Proof.}
\]

On the other hand for the group $P\text{SU}(G)$

\[\langle \, 2x_{2i-1}, x_{2i} \rangle \]

\[\langle 2x_{2i-1}, x_{2i} \rangle \]

\[
\text{(2.9) } A^*(Spin(2n)) = \left\{ \begin{array}{ll}
\langle x_1x_3\cdots x_i[\frac{1}{2}] - 1 \rangle & \text{if } n \equiv 1 \mod 2, \\
\langle x_1x_3\cdots x_i[\frac{1}{2}] - 1 \rangle & \text{if } n = 2^h(2b + 1), h \geq 1; \\
\end{array} \right.
\]

\[
\text{(2.10) } A^*(E_6) = \left\{ \begin{array}{ll}
\langle 2x_3, 3x_4, x_5, x_6 \rangle & \\
\langle 3x_3, 3x_4, (x_3 + \omega)^2, \omega^0, x_4 \rangle & \\
\end{array} \right.
\]

\[
\text{(2.11) } A^*(E_7) = \left\{ \begin{array}{ll}
\langle x_3, x_4, x_5, x_6 \rangle & \\
\langle 3x_3, 3x_4, x_5, x_6, x_7, x_8, x_9 \rangle & \\
\end{array} \right.
\]

where the generators $\omega, x_i$ (deg $\omega = 2$, deg $x_i = 2k$) are the $\pi^*$–images of certain Schubert classes on $G/T$ specified in [11], and where the power $k_i$'s appearing in the denominator of the quotient (2.7) or (2.8) are respectively

\[
k_i = 2^{\lceil \frac{n - 1}{2} \rceil} \text{ or } k_i = 2^{\lceil \frac{n - 1}{2} \rceil} + 1.
\]

\[
\text{Proof.}
\]

Combining the results mentioned in b) and c) above it is straightforward to evaluate the Chow rings $A(G)$ and $A^*(PG)$ using formula (2.4). As examples we show the results for $G = PSU(n)$ and $P\text{Spin}(2n)$, and refer the remaining cases to [11] Corollary 6.2; formula (6.13) and [23] p.11.

For $G = SU(n)$ the ring $H^*(G/T)$ has the presentation

\[
H^*(G/T) = \frac{\mathbb{Z}[t_1, t_2, \ldots, t_n]}{(e_1, \ldots, e_n)} \quad \text{(by Borel),}
\]

where deg $t_i = 2$, $e_r$ is the $r$th elementary symmetric polynomial in the $t_1, \cdots, t_n$. On the other hand for the group $PSU(n)$ we have

\[
\text{Im } \tau = \{t_1 - t_2, t_2 - t_3, \cdots, t_{n-1} - t_n\} \quad \text{by [9] Corollary 3.2.}
\]

From (2.4) we get

\[
A^*(PSU(n)) = \frac{\mathbb{Z}[t_1, t_2, \ldots, t_n]}{(e_1, \cdots, e_n)} \mid t_i = t_j = \frac{Z[\omega]}{(\langle \frac{e_r}{\omega^r} \rangle, 1 \leq r \leq n)},
\]

where $\omega := t_1 = \cdots = t_n$. It implies that the order of the power $\omega^r$ in the ring $A^*(PSU(n))$ is precisely $b_{n,r} = g.c.d. \{\binom{n}{r}, \cdots, \binom{n}{r}\}$, showing (2.6).

For $G = \text{Spin}(2n)$ we have by Marlin [23] p.20, Theorem 3] that

\[
H^*(\text{Spin}(2n)/T) = \frac{\mathbb{Z}[t_1, t_2, \ldots, t_n, x_1, x_2, \ldots, x_{n-1}]}{(e_1, \cdots, e_n, n_1, \cdots, q_{n-1})}, \text{ deg } t_i = 2, \text{ deg } x_r = 2r,
\]

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where, with \( x_k = 0 \) for \( k \geq n \) being understood,

\[
\delta_r = 2x_r - e_r(t_1, \cdots, t_n), \quad 1 \leq i \leq n - 1; \quad \delta_n = e_n(t_1, \cdots, t_n),
\]

\[
q_r = x_{2r} + 2 \sum_{1 \leq i \leq r-1} (-1)x_i x_{2r-i} + (-1)^r x_r^2, \quad 1 \leq r \leq n - 1,
\]

and where \( e_r(t_1, \cdots, t_n) \) is the \( r^{th} \) elementary symmetric polynomials in \( t_1, \cdots, t_n \).

On the other hand, for the group \( P\text{Spin}(2n) \) we have

\[
\text{Im } \tau = \{ t_1 - t_2, \cdots, t_{n-1} - t_n, 2t_n \} \text{ by [9] Corollary 3.2].}
\]

The second formula in (2.9) is verified by

\[
\begin{align*}
\mathcal{A}^*(P\text{Spin}(2n)) &= H^*(\text{Spin}(2n)/T) / (\text{Im } \tau) \\
&= H^*(\text{Spin}(2n)/T) |_{t_1 = \cdots = t_n, 2t_n = 0},
\end{align*}
\]

where \( x_1 = \frac{1}{2}e_1 \) (by the relation \( \delta_1 \) ), and where \( \omega := t_1 = \cdots = t_n. \)

With the product inherited from that on \( E_3^{*,*}(G) \) the third page \( E_3^{*,*}(G) \)

is a bi-graded ring [30, P.668]. In particular, \( E_3^{*,1}(G) \) can be considered as a module

over the subring \( \mathcal{A}^*(G) = E_3^{*,0}(G) \subset E_3^{*,*}(G) \).

**Lemma 2.3.** For each compact Lie group \( G \) with \( I_G = \{ r_1, \cdots, r_n \} \) there exist \( n \) elements \( \rho_k \in E_3^{2r_k-2,1}(G) \), \( 1 \leq k \leq n \), so that

i) as a \( \mathcal{A}^*(G) \)-module \( E_3^{*,1}(G) \) is spanned additively by \( \{ \rho_1, \cdots, \rho_n \} \);

ii) the product \( \rho_1 \cdots \rho_n \) generates the group \( E_3^{m,*}(G) = \mathbb{Z} \) (see (2.3)).

**Example 2.4.** A classical result of Leray [9 Example 1.4] states that for the cohomology with real coefficients one has isomorphism of algebras

\[
H^*(G; \mathbb{R}) = E_3^{*,*}(G; \mathbb{R}) = \Lambda \langle y_1, \cdots, y_n \rangle
\]

where the generators \( y_1, \cdots, y_n \) is a basis of the subspace \( E_3^{*,1}(G; \mathbb{R}) \) constructed from the basic Weyl invariants of the group \( G \). The elements \( \rho_1, \cdots, \rho_n \in E_3^{*,1}(G) \) asserted by Lemma 2.3 may be seen as the integral lifts of the classes \( y_1, \cdots, y_n \), and will be called a basis of the \( \mathcal{A}^*(G) \)-module \( E_3^{*,1}(G) \) (for the latter convenience). Note that \( E_3^{*,1}(G) \) may fail to be a free module over the ring \( \mathcal{A}^*(G) \), e.g. see [9] Example 1.4 for the case \( G = \text{PSU}(8) \).

Without the loss of generalities we can assume in Lemma 2.3 that the Lie group \( G \) is simple. In this case the basis \( \{ \rho_1, \cdots, \rho_n \} \) of the module \( E_3^{*,1}(G) \)

has been constructed uniformly for all simply connected \( G \) in [12], and for the non-simply connected \( G \) in [8][9]. In this paper knowing the degrees of these elements is sufficient for our purpose.

It is worth to mention that, in the course to compute the integral cohomology of the spinor group \( \text{Spin}(n) \) Pittie [26] has constructed a basis \( \{ \rho_1, \rho_2, \cdots \} \) of the module \( E_3^{*,1}(G) \) for the classical groups \( G = U(n), SO(n) \) and \( \text{Spin}(n) \).

To see the implication of Lemma 2.3 let \( F_p \) be the filtration on the integral cohomology \( H^*(G) \) defined by the map \( \pi \). That is ([24] P.146)

\[
0 = F^{r+1}(H^*(G)) \subset F^r(H^*(G)) \subset \cdots \subset F^0(H^*(G)) = H^*(G)
\]
with

\[ E^p_0(G) = F^p(H^{p+q}(G))/F^{p+1}(H^{p+q}(G)). \]

The fact \( H^{2k+1}(G/T) = 0 \) due to Bott and Samelson \[3\] implies that

\( a) \ E^0_{2k+1} = 0 \) for \( k \geq 0; \quad b) \ E^4_{2k+2} = \cdots = E^4_{\infty} \).

From \( F^{2s+1}(H^{2s+1}(G)) = F^{2s+2}(H^{2s+1}(G)) = 0 \) by a) one finds that

\[ E^2_{\infty}(G) = F^{2s}(H^{2s+1}(G)) \subset H^{2s+1}(G). \]

Combining this with the routine relation \( d_r(E^r_{s-1}) = 0 \) for \( r \geq 3 \) yields the composition

\[ (2.12) \ \kappa : E^1_{s}(G) \to E^2_{s}(G) \to \cdots \to E^\infty_{s}(G) \subset H^s(G) \]

that interprets elements of \( E^1_{s} \) directly as cohomology classes of \( G \). Carrying on the results of Lemma 2.3 we obtain the following characterization of the integral cohomology \( H^s(G) \) in terms of its free part and torsion ideal.

**Lemma 2.5.** The integral cohomology of \( G \) has the additive presentation

\[ (2.13) \ H^s(G) = \Delta(\xi_1, \cdots, \xi_n) \oplus \tau(G) \text{ with } \xi_i := \kappa(\rho_i) \in H^s(G), \]

where \( \Delta(\xi_1, \cdots, \xi_n) \) denotes the free \( \mathbb{Z} \)-module with the basis

\[ \Phi := \{1, \xi_1 = \bigcup_{i \in I} \xi_i \in H^s(G) \mid I \subseteq \{1, \cdots, n\}\}. \]

**Proof.** Let \( \{E^s_*=^s(G; \mathbb{R}, d_s)\} \) be the Serre spectral sequence of \( \pi \) with real coefficients. According to Leray \[22\] the algebra \( E^s_*=^s(G; \mathbb{R}) \) is generated multiplicatively by its subspace \( E^1_*=^1(G; \mathbb{R}) \), while the map \( \kappa \) in (2.12) induces

i) an isomorphism \( E^1_*=^1(G; \mathbb{R}) \cong P(G; \mathbb{R}) \) of vector spaces, and

ii) an isomorphism \( E^s_*=^s(G; \mathbb{R}) \cong H^s(G; \mathbb{R}) \) of algebras,

respectively. Assume that \( \{\rho_1, \cdots, \rho_n\} \) is a basis of the \( A^s(G) \)-module \( E^1_*=^1(G) \), and that \( \dim G/T = m \), \( \dim T = n \). In addition to \( \xi_i := \kappa(\rho_i) \) we put

\[ \xi^R_1 := \xi_i \otimes 1 \in H^*(G; \mathbb{R}) = H^*(G) \otimes \mathbb{R}, \ 1 \leq i \leq n. \]

Since \( A^*(G) \otimes \mathbb{R} = \mathbb{R} \) the space \( E^s_*=^s(G; \mathbb{R}) = E^s_*=^s(G) \otimes \mathbb{R} \) has the basis \( \{\rho_1 \otimes 1, \cdots, \rho_n \otimes 1\} \) \text{ by i) of Lemma 2.3. By the isomorphisms i) and ii) above}

\[ H^*(G; \mathbb{R}) = \Lambda_\mathbb{R}(\xi^R_1, \cdots, \xi^R_n) \text{ with } \xi^R_i \in P(G; \mathbb{R}), \]

implying that the set \( \Phi \) of \( 2^n \) monomials is linearly independent in \( H^*(G) \). It remains to show that the set \( \Phi \) spans a direct summand of \( H^*(G) \).

Assume, on the contrary, that there exist a monomial \( \xi_1 \in \Phi \), an integral class \( \varsigma \in H^*(G) \), as well as some integer \( a > 1 \), so that a relation of the form \( \xi_1 = a \cdot \varsigma \) holds in \( H^*(G) \). Multiplying both sides by \( \xi_j \) with \( J \) the complement of \( I \subseteq \{1, \cdots, n\} \) yields that

\[ \xi_1 \cup \cdots \cup \xi_n = (-1)^r a \cdot (\varsigma \cup \xi_j) \text{ (for some } r \in \{0, 1\}). \]

However, in view of the identification (2.3) the map \( \kappa \) in (2.12) transforms the generator \( \rho_1 \cdots \rho_n \) of \( E^1_*=^n(G) = \mathbb{Z} \) to the generator \( \xi_1 \cup \cdots \cup \xi_n \) of \( H^{n+m}(G) = \mathbb{Z} \) (by \( \xi_i := \kappa(\rho_i) \)). The proof is completed by this contradiction. \( \square \)
3 Proofs of Theorems A, B and C

Let \( \{\rho_1, \ldots, \rho_n\} \) be a basis of the \( \mathcal{A}^*(G) \)-module \( E_{\mathcal{A}}^*(G) \). With \( \xi_i = \kappa(\rho_i) \) and \( \xi^R_i = \xi_i \otimes 1 \) we have by the proof of Lemma 2.5 that

\[
(3.1) \quad H^*(G; \mathbb{R}) = \Lambda R(\xi^R_1, \ldots, \xi^R_n),
\]

where \( \xi^R_i \in P(G; \mathbb{R}) \).

**Proof of Theorem A.** Let \( q : H^*(G) \to \mathcal{F}(G) = H^*(G)/\tau(G) \) be the quotient map and put \( x_i := q(\xi_i) \). Then \( \mathcal{F}(G) = \Delta(x_1, \ldots, x_n) \) by (2.13), where

\[
\deg x_i = \deg \xi_i = 2r_i - 1, 1 \leq i \leq n, \quad I_G = \{r_1, \ldots, r_n\}.
\]

In addition, with \( \deg \xi_i \equiv 1 \mod 2 \) we get \( x_i^2 = 0 \) from \( \xi_i^2 \in \tau(G) \), showing

\[
(3.2) \quad \mathcal{F}(G) = \Lambda(x_1, \ldots, x_n) \quad \text{(i.e. the formula i) of Theorem A)}.
\]

For the co–product \( \mu^* \) on \( H^*(G) \) we can assume, in general, that

\[
\mu^*(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + z_i + d_i, \quad 1 \leq i \leq n,
\]

where \( z_i = \Sigma a_j \otimes b_j \in \mathcal{F}^+(G) \times \mathcal{F}^+(G) \) is a mixed term, \( d_i \in \tau(G \times G) \). With \( \xi^R_i \in P(G; \mathbb{R}) \) by (3.1) there must be \( z_i = 0 \). Thus, applying \( q \times q \) to this equality verifying the assertion ii) of Theorem A

\[
\mu^#(x_i) = x_i \otimes 1 + 1 \otimes x_i.
\]

Finally, suppose that the group \( G \) is simple, and that in addition to (3.2) one has \( \mathcal{F}(G) = \Lambda(x_1', \ldots, x_n') \) with

\[
\deg x_i = \deg x_i' \quad \text{and} \quad \mu^#(x_i') = x_i' \otimes 1 + 1 \otimes x_i', \quad 1 \leq i \leq n.
\]

Let \( D \subseteq \mathcal{F}(G) \) be the subring consists of the decomposable elements in the positive degrees. Since both \( \{x_1', \ldots, x_n'\} \) and \( \{x_1, \ldots, x_n\} \) are basis of the free \( \mathbb{Z} \)-module \( \mathcal{F}(G)/D \), and since \( 2 = r_1 < \cdots < r_n \), the transition functions on \( \mathcal{F}(G) \) between these two sets of generators take the forms

\[
x_i' = \pm x_i + d_i \quad \text{with} \quad d_i \in D, \quad 1 \leq i \leq n.
\]

Applying the co–product \( \mu^# \) to both sides, and using the relations \( \mu^#(y) = y \otimes 1 + 1 \otimes y \) for \( y = x_i \) or \( x_i' \) to simplify the resulting equation, yields that

\[
d_i \otimes 1 + 1 \otimes d_i = \mu^#(d_i).
\]

With \( d_i \in D \) this implies \( d_i = 0 \), completing the proof of Theorem A. ∎

**Proof of Theorem B.** Let \( G \) be a simple Lie group with maximal torus \( T \) and non–trivial center \( \mathcal{Z}(G) \). Since the covering \( c : G \to PG = G/\mathcal{Z}(G) \) carries \( T \) to the maximal torus \( T' := c(T) \) on \( PG \), it can be viewed as a bundle map over the identity of \( G/T \)

\[
\begin{array}{c}
T \xleftarrow{c} T' \\
\cap \quad \cap \\
G \xrightarrow{\pi} PG \\
\pi' \downarrow \quad \downarrow \\
G/T = PG/T'
\end{array}
\]

(11)
hence induces a map \( c^* : E_2^{\ast}(PG) \rightarrow E_2^{\ast}(G) \) of Koszul complexes. Assume by Lemma 2.3 that \( \{\rho_1', \cdots, \rho_n'\} \) is a basis of the \( \mathcal{A}(PG) \)-module \( E_2^{1}(PG) \), and that \( \{\rho_1, \cdots, \rho_n\} \) is a basis of the \( \mathcal{A}(G) \)-module \( E_2^{1}(G) \). Then, by the proof of Theorem A

\[
\mathcal{F}(PG) = \Lambda(y_1, \cdots, y_n) \quad \text{with} \quad y_i := q \circ \kappa(\rho_i') \in \mathcal{F}(PG),
\]

\[
\mathcal{F}(G) = \Lambda(x_1, \cdots, x_n) \quad \text{with} \quad x_i := q \circ \kappa(\rho_i) \in \mathcal{F}(G).
\]

On the other hand, for the degree reason \( \deg \rho_i = \deg \rho_i' \) we can assume by i) and Lemma 2.4 that

\[
(3.3) \quad c^*(\rho_i') = a_i \rho_i + b_{i-1} \rho_{i-1} + \cdots + b_1 \rho_1, \quad a_i, b_j \in \mathcal{A}^+(G),
\]

where \( \mathcal{A}^+(G) \) denotes the subring of \( \mathcal{A}(G) \) consisting of elements in the positive degrees. Since the ring \( \mathcal{A}^+(G) \) is always finite (3.3) implies that \( c^* \) on the quotient \( \mathcal{F}(G) \), where we can assume \( a_i \geq 0 \) by modifying the sign of the generator \( x_i \) whenever is necessary, to get the assertion i) of Theorem B.

Finally, applying \( c^* \) to the generator \( \rho_1' \cdots \rho_n' \) of \( E_3^{m-n}(PG) = H^{m-n}(PG) = \mathbb{Z} \) (see Lemma 2.3 and (2.3)), and noting that the partial sum \( b_{i-1} \rho_{i-1} + \cdots + b_1 \rho_1 \) in (3.3) is of finite order, we get

\[
(3.4) \quad c^*(\rho_1' \cdots \rho_n') = (a_1 \cdots a_n) \rho_1 \cdots \rho_n \quad \text{on} \quad E_3^{m-n}(G) = H^{m-n}(G) = \mathbb{Z}.
\]

Since the mapping degree of \( c \) is the order \( |\mathcal{Z}(G)| \) of the center, and since the product \( \rho_1 \cdots \rho_n \) generates the group \( E_3^{m-n}(G) = H^{m-n}(G) = \mathbb{Z} \) by Lemma 2.3, we get from (3.4) that \( a_1 \cdots a_n = |\mathcal{Z}(G)| \), completing the proof of Theorem B.\( \square \)

**Proof of Theorem C.** Since the second homotopy group of \( PG \) is trivial, the homotopy exact sequence of \( \pi' \) contains the free resolution of the group \( \pi_1(PG) \)

\[
0 \rightarrow \pi_2(G/T) \rightarrow \pi_1(T') \xrightarrow{j} \pi_1(PG) = \mathbb{Z}(G) \rightarrow 0,
\]

where \( j \) is the inclusion of the maximal torus. Applying the co-functor \( \text{Hom}(,\mathbb{Z}) \) to this sequence, and using the Hurewicz isomorphisms

\[
\pi_2(G/T) = H_2(G/T), \quad \pi_1(T') = H_1(T'), \quad \pi_1(PG) = H_1(PG)
\]

to substitute the relevant groups, one obtains the exact sequence

\[
(3.6) \quad 0 \rightarrow H^1(PG) \xrightarrow{j^*} H^1(T') \xrightarrow{j^*} H^1(G/T) \xrightarrow{\alpha} \text{Tor} H^2(PG) = \mathcal{Z}(G) \rightarrow 0
\]

in cohomologies, where \( \pi' \) is the transgression in \( \pi' \). It follows that

\[
H^2(G/T)/\text{Im} \pi' \cong H^2(PG) = \text{Tor} H^2(PG) = \mathcal{Z}(G),
\]

where the first equality follows from \( H^2(PG) \odot \mathbb{R} = 0 \) by Theorem 1.1. With \( \mathcal{A}^2(PG) = H^2(G/T)/\text{Im} \pi' \) this implies that the inclusion \( \mathcal{A}(PG) \subseteq H^*(PG) \) restricts to an isomorphism in degree 2. Therefore, the subring \( J(PG) \) of \( H^*(PG) \) generated by \( H^2(PG) \) agrees with the subring of \( \mathcal{A}(PG) \) generated by \( \mathcal{A}^2(PG) \). In particular, one reads out \( J(PG) \) from the formulae of the ring \( \mathcal{A}(PG) \) presented in Lemma 2.2, showing Theorem C.\( \square \)
4 An exact sequence for cyclic coverings

As a covering \( c : G \to G' \) on Lie groups is a group homomorphism we have \( \ker c \subseteq \mathbb{Z}(G) \). The covering \( c \) is called cyclic if \( \ker c \) is a cyclic subgroup. We note that

a) if \( G \) is one of the simply connected Lie groups in Table 1 with \( G \neq \text{Spin}(2n) \), then the covering \( G \to PG \) is always cyclic;

b) if \( G = \text{Spin}(2n) \) the covering \( G \to PG \) can be decomposed into the composition \( \text{Spin}(2n) \xrightarrow{\varphi} \text{SO}(2n) \xrightarrow{\psi} \text{PSpin}(2n) \) of two cyclic ones, both with order 2.

In addition, for a cyclic covering \( G \to G' \) between simple Lie groups, the multi–degree \( D(G,G') \) is also defined by the proof of Theorem B. Summarizing, to show Theorem D it suffices to compute the invariant \( D(G, G') \) for the cyclic coverings. In this section we establish an exact sequence, that reduces this task to the calculation with the Chow ring \( \mathbb{A}^*(G') \) of the group \( G' \).

Assume that \( c : G \to G' \) is a cyclic covering on simple Lie groups. The central extension of \( c \) is the principle \( U(1) \)-fibration over the group \( G' \)

\[ (4.1) \quad 0 \to U(1) \to \tilde{G} := G \rtimes_{\ker c} U(1) \xrightarrow{\iota} G' \to 0, \]

where \( \ker c \) acts on the cycle \( U(1) \) as the anti–clockwise rotation through the angle \( 2\pi/|\ker c| \), and where \( \tilde{G} \) is furnished with the obvious group structure. Moreover, fix once for all a maximal torus \( T \) on \( G \), \( \dim T = n \). Then both

\[ \tilde{T} := T \times_{\ker c} U(1) \subset \tilde{G} \]

are respectively maximal torus of the corresponding Lie groups, while the \( U(1) \)-fibration \( C \) can be regarded as a bundle map between two torus fibrations:

\[ (4.2) \quad \begin{array}{ccc}
U(1) & \xrightarrow{\iota} & \tilde{T} \\subset & \tilde{G} \\
\parallel & & \subset & \subset \\
U(1) & \xrightarrow{\iota} & \tilde{G} \quad \subseteq \quad G' \quad \text{(see (2.1)).}
\end{array} \]

Since both \( \tilde{T} \) and \( T' \) are torus groups, the restriction \( C' \) of \( C \) on \( \tilde{T} \) is splittable, implying that the group \( H^1(\tilde{T}) \) admits a basis \( \{ t_0, t_1, \ldots, t_n \} \) so that the induced map \( C' \) carries the ring \( H^*(T') \) isomorphically to the subring \( \Lambda(t_1, \ldots, t_n) \) of \( H^*(\tilde{T}) = \Lambda(t_0, t_1, \ldots, t_n) \). It follows that

**Lemma 4.1.** For a cyclic covering \( c : G \to G' \) of simple Lie groups we have

\[ E_2^{*=*}(\tilde{G}) = H^*(G/T) \otimes \Lambda(t_0, t_1, \ldots, t_n); \]

\[ E_2^{*=*}(G') = H^*(G/T) \otimes \Lambda(t_1, \ldots, t_n), \]

on which \( C \) induces a map \( C^{*} : E_2^{*=*}(G') \to E_2^{*=*}(\tilde{G}) \) of Koszul complexes with

i) \( C^{*}(z \otimes t) = z \otimes t \), where \( z \in H^*(G/T), \ t \in H^*(T') \);

ii) the transgression \( \tau' \) of \( \pi' \) is the restriction of the transgression \( \tilde{\tau} \) of \( \tilde{\pi} \) to the subgroup \( H^1(T') \subset H^1(\tilde{T}) \);

iii) the Euler class of the \( U(1) \)-fibration \( C \) over \( G' \) is \( \omega = [\tilde{\tau}(t_0) \otimes 1] \in E_3^{2,0}(G') = H^2(G') \).
where \( [x] \) denotes the cohomology class of a \( d_2^{\ast} \)-cocycle \( x \in E^{2,k}_2(G') \), and where the identification \( E^{2,0}_3(G') = H^2(G') \) follows from the proof of Theorem C \( \square \)

By Lemma 4.1 the map \( C^* \) fits into the short exact sequence

\[
0 \to E^{*,k}_2(G') \overset{C^*}{\to} E^{*,k}_2(\tilde{G}) \overset{\theta}{\to} E^{*,k-1}_2(G') \to 0,
\]

of Koszul complexes, where the map \( \theta \) is evaluated by the simple rule:

\[
\theta(x \otimes y) = x \otimes y_2 \text{ if } y = y_1 + t_0 \cdot y_2 \text{ with } y_1, y_2 \in H^*(T').
\]

Moreover, with \( \text{rank} H^2(G/T) = n \) and \( \dim \tilde{T} = n + 1 \), the transgression \( \tilde{\tau} \) satisfies that \( \ker \tilde{\tau} = \mathbb{Z} \subseteq H^1(\tilde{T}) \). Taking a generator \( s \in \ker \tilde{\tau} \) and noting that

\[
1 \otimes s \in E^{0,1}_2(\tilde{G}) \text{ is } \tilde{d}_2 \text{-closed by i) of Lemma 2.1, we get the cohomology class}
\]

\[
\rho_0 = [1 \otimes s] \in E^{0,1}_2(\tilde{G}).
\]

**Theorem 4.2.** For a cyclic covering \( c : G \to G' \) one has the exact sequence

\[
0 \to E^{*,*}_3(G')/\langle \omega \rangle \overset{C^*}{\to} E^{*,*}_3(\tilde{G}) \overset{\theta}{\to} E^{*,*}_3(G') \overset{\omega \cdot \cdot}{\to} \rangle E^{*,*}_3(G') \to 0,
\]

in which

\[
E^{*,*}_3(\tilde{G}) = E^{*,*}_3(G) \otimes \Lambda(\rho_0),
\]

where \( \omega \) is the Euler class of the central extension \( C \) of \( c \), and where for any \( z \in E^{*,*}_3(G') \), \( y \in E^{*,*}_3(G) \),

i) \( \theta(C^*(z) \cdot y) = z \cdot \theta(y) \);

ii) \( \theta(\rho_0) = \lvert \ker c \rvert \in E^{0,0}_3(G') = \mathbb{Z} \);

iii) \( \omega(z) = \omega \cdot z \);

iv) \( C^*(z) \equiv c^*(z) \otimes 1 \mod \rho_0 \) (with respect to (4.6)).

**Proof.** The short exact sequence (4.3) of Koszul complexes yields the long exact sequence in cohomologies

\[
\ldots \to E^{*,r}_3(G') \overset{C^*}{\to} E^{*,r}_3(\tilde{G}) \overset{\theta}{\to} E^{*,r-1}_3(G') \overset{\omega \cdot \cdot}{\to} \rangle E^{*,r-1}_3(G') \overset{C^*}{\to} \ldots
\]

for which properties i), ii) and iii) follows easily from Lemma 4.1. Since \( \ker C^* = \omega \cdot E^{*,*}_3(G') \subset E^{*,*}_3(G') \) (by the exactness) we obtain (4.5) from

\[
\text{Im } \omega = \omega \cdot E^{*,*}_3(G') \text{ and } \ker \omega = E^{*,*}_3(G')/\text{Im } \omega = E^{*,*}_3(G')/\langle \omega \rangle.
\]

It remains to establish the decomposition (4.6), together with the relation iv). In addition to (4.2) the group \( \tilde{G} \) also fits into the fibrations

\[
\begin{align*}
0 & \to T \quad \cap \quad \tilde{T} \quad \cap \quad U(1) \quad \to 0 \\
0 & \to G \quad \cap \quad \tilde{G} \quad \cap \quad U(1) \quad \to 0 \\
G/T & = \quad \pi \quad \cap \quad \tilde{\pi} \quad \cap \quad G/T
\end{align*}
\]

\[ \tag{4.7} \]
where $g$ is the quotient of the projection $G \times U(1) \to U(1)$ by ker $c$; $g'$ is the restriction of $g$ on $\widetilde{T}$; and where the upper two rows are short exact sequences of Lie groups. It follows from Lemma 2.1 that $E_2^{s,t}(\widetilde{G}) = E_2^{s,t}(G) \otimes A(s)$. With $\rho_0 = [1 \otimes s]$ by our convention we obtain (4.6) from the Künneth formula. Finally, the relation iv) is transparent, since the inclusion $i$ in (4.7) identifies $G$ with the normal subgroup $q^{-1}(1)$ of $\widetilde{G}$ that satisfies the relation $C \circ i = c$. □

Recall that a $U(1)$–fibration $E \to X$ over a CW–complex $X$ is classified by its Euler class $\omega \in H^2(X)$. The following result, fairly transparent in the context of [1], provides a geometric interpretation of the generators of the rings $J(PG)$ given in Theorem C.

**Lemma 4.3.** Let $G$ be a simply connected simple Lie group given in Table 1.

i) if $G \neq Spin(2\cdot 2b+1))$ with $t \geq 2$, the Euler class of the central extension of the cyclic covering $c : G \to PG$ is the generator $\omega \in J(PG)$;

ii) if $G = Spin(2n)$ with $n \equiv 0 \mod 2$, the Euler class of the central extension of the 2 sheets covering $c : SO(2n) \to PSpin(2n)$ is the generator $\omega_1 \in J(PSpin(2n))$. □

It is crucial to note in (4.5) that, with respect to the bi–gradation on $E_3^{s,t}$ imposed by the "base degrees $s$" and "fiber degrees $t$", the map $C^*$ preserves both the base and fiber degrees; the map $\theta$ preserves the base degrees, but reduces the fiber degrees by 1; and that the map $\omega$ increases the base degrees by 2, and preserves the fiber degrees. In particular, for each $k \geq 0$ one has by (4.5) the exact sequences with four terms

\[(4.8) \ 0 \to E_3^{2k,1}(PG)/\omega \xrightarrow{C^*} E_3^{2k,1}(\widetilde{G}) \xrightarrow{\theta} A^{2k}(PG) \xrightarrow{\omega} \omega \cdot A^{2k}(PG) \to 0\]

where the groups $E_3^{2k,0}(PG) = A^{2k}(PG)$ and $E_3^{2k,0}(\widetilde{G}) = A^{2k}(G)$ have been decided by Lemma 2.2. Furthermore, if $\{\rho_1, \cdots, \rho_n\}$ is a basis of the $A^*(PG)$–module $E_3^{1,1}(PG)$, and $\{\rho_1, \cdots, \rho_n\}$ is a basis of the $A^*(G)$–module $E_3^{1,1}(G)$, then, with $A(PG)/\omega = A(G)$ by Lemma 2.2,

\[(4.9) \ E_3^{1,1}(\widetilde{G}) \text{ is a } A^*(G)\text{–module with basis } \{\rho_0, \rho_1, \cdots, \rho_n\} \text{ by (4.6):}\]

\[E_3^{1,1}(PG)/\omega \text{ is a } A^*(G)\text{–module with basis } \{\rho_1, \cdots, \rho_n\}.\]

**Proof of Theorem D.** Let $G$ be a simply connected Lie group given in Table 1, and assume that $D(G, PG) = \{a_1, \cdots, a_n\}$. By ii) of Theorem B we have

\[(4.10) \ a_1 \cdots a_n = |Z(G)|.\]

In addition, by the naturality of the map $\kappa$ in (2.12) with respect to bundle maps, we have the commutative diagram

\[(4.11) \ E_3^{2k,1}(PG) \xrightarrow{\kappa} H^*(PG) \xrightarrow{\eta} F(PG) \xrightarrow{\eta} C^# \]

allowing us to calculate the eigenvalues $a_1, \cdots, a_n$ of $c^#$ (see i) of Theorem B) by computing with the action of $C^*$ on $E_3^{2k,1}(PG)$. Granted with the relations (4.8), (4.9) and (4.10), together with the diagram (4.11), the proof of Theorem D will be given in the following order
G = SU(n), Sp(n), E7, E6, Spin(2n + 1) and Spin(2n),
where \( \mathbb{Z}_m\{x\} \) (resp. \( \mathbb{Z}\{x\} \)) denotes the cyclic group of order \( m \) (resp. of order \( \infty \)) with generator \( x \).

**Case 1.** \( G = SU(n) \). For each \( 1 \leq k \leq n - 1 \) we have by (4.8) the exact sequence

\[
0 \to \mathbb{Z}\{\theta^k\} \to \mathbb{Z}\{\theta_{r}\} \to \mathbb{Z}_{b_{n,k}}\{\omega^{k}\} \to 0,
\]

where \( A^{2k}(PG) = \mathbb{Z}_{b_{n,k}}\{\omega^{k}\} \) by (2.6), and where

\[
E_{3,4}^{2k,1}(PG)/\langle \omega \rangle = \mathbb{Z}\{\theta_{r}\},
\]

As the order of the power \( \omega^{r} \) is precisely \( b_{n,r} \), we get by the exactness that \( \ker \omega = \mathbb{Z}_{b_{n,k}/b_{n,k+1}}\{\omega\} \), showing \( a_{k} = \frac{b_{n} - b_{n,k+1}}{b_{n,k+1}} \), as that stated in Theorem D.

**Case 2.** \( G = Sp(n) \) with \( n = 2^{r}(2b + 1) \). Taking in (4.8) that \( k = 2^{r+1} - 1 \) we get the exact sequence

\[
0 \to \mathbb{Z}\{\theta_{2^{r+1}}\} \to \mathbb{Z}\{\theta_{2}\} \to \mathbb{Z}_{2}\{\omega^{2^{r+1} - 1}\} \to 0,
\]

where \( A^{2k}(PG) = \mathbb{Z}_{2}\{\omega^{k}\} \) and \( \omega \cdot \mathbb{Z}_{2}\{\omega^{k}\} = 0 \) by (2.7), and where

\[
E_{3,4}^{2k,1}(\tilde{G}) = \mathbb{Z}\{\rho_{2}\}, E_{3,4}^{2k,1}(PG)/\langle \omega \rangle = \mathbb{Z}\{\rho_{2}\} \text{ by (4.9).}
\]

This shows that \( a_{2^{r}} = 2 \). Moreover, with \( |Z(G)| = 2 \) the relation (4.10) forces \( a_{i} = 1 \) for \( i \neq 2^{r} \), verifying Theorem D for \( G = Sp(n) \).

**Case 3.** \( G = E_{7} \). Taking \( k = 1 \) in (4.8) we obtain the short exact sequence

\[
0 \to \mathbb{Z}\{\theta_{1}\} \to \mathbb{Z}\{\theta_{1}\} \to \mathbb{Z}_{2}\{\omega\} \to 0
\]

where \( A^{2}(PG) = \ker \omega = \mathbb{Z}_{2}\{\omega\} \) by (2.11), and where

\[
E_{3,4}^{1,1}(\tilde{G}) = \mathbb{Z}\{\rho_{1}\}, E_{3,4}^{1,1}(PG)/\langle \omega \rangle = \mathbb{Z}\{\rho_{1}\} \text{ by (4.9).}
\]

It shows that \( a_{1} = 2 \). Consequently, \( a_{2} = \cdots = a_{7} = 1 \) by (4.10).

**Case 4.** \( G = E_{6} \). Taking \( k = 8 \) in (4.8) we get the short exact sequence

\[
0 \to E_{3,4}^{16,1}(PG)/\langle \omega \rangle \to E_{3,4}^{16,1}(\tilde{G}) \to \mathbb{Z}_{3}\{\omega^{8}\} \to 0
\]

where \( A^{16}(PG) = \mathbb{Z}_{3}\{\omega^{8}\} \oplus \mathbb{Z}_{3}\{x_{3}^{2}\} \) with \( \ker \omega = \mathbb{Z}_{3}\{\omega^{8}\} \) by (2.10), and where

\[
E_{3,4}^{16,1}(\tilde{G}) = \mathbb{Z}\{\rho_{3}\} \oplus x_{3}\cdot \mathbb{Z}\{\rho_{3}\} \oplus x_{4}\cdot \mathbb{Z}\{\rho_{2}\},
\]

\[
E_{3,4}^{16,1}(PG)/\langle \omega \rangle = \mathbb{Z}\{\rho_{3}\} \oplus x_{3}\cdot \mathbb{Z}\{\rho_{3}\} \oplus x_{4}\cdot \mathbb{Z}\{\rho_{2}\} \text{ by (4.9).}
\]

Moreover, by the relation i) of Theorem 4.2, together \( \text{Im} \theta = \mathbb{Z}_{3}\{\omega^{8}\} \),

\[
\theta(x_{3}\cdot \rho_{3}) = x_{3}\cdot \theta(\rho_{3}) = 0; \theta(x_{4}\cdot \rho_{2}) = x_{4}\cdot \theta(\rho_{2}) = 0.
\]
These imply, by the exactness, that $\theta(\rho_5) = \omega^3$. Since $\theta(\rho_5)$ is of order 3 there must be $C^*(\rho_5') = 3\rho_5$, showing $a_5 = 3$. As result $a_i = 1$ for $i \neq 5$ by (4.10).

**Case 5.** $G = Spin(2n + 1)$, $2^s \leq n < 2^{s+1}$. By the formula (2.8) of $A^*(PG)$

$$\ker\{ A^{2k}(PG) \Rightarrow \omega \cdot A^{2k}(PG) \} = \begin{cases} 0 \text{ if } 0 \leq k < 2^{s+1} - 1; \\ \mathbb{Z}_2[\omega] \text{ if } k = 2^{s+1} - 1, \end{cases}$$

where $\omega = x_1$. By (4.8) we have the exact sequences

(4.12) $0 \to E_3^{2k,1}(PG)/\langle \omega \rangle \xrightarrow{\theta} E_3^{2k,1}(\tilde{G}) \to 0$ for $1 \leq k < 2^{s+1} - 1$;

(4.13) $0 \to E_3^{2(2^{s+1} - 1),1}(PG)/\langle \omega \rangle \xrightarrow{\theta} E_3^{2(2^{s+1} - 1),1}(\tilde{G}) \to \mathbb{Z}_2[\omega^2] \to 0$.

In particular, by (4.12)

(4.14) $\theta(\rho_i) = 0$, $1 < i < 2^s$.

Moreover, by (4.9) the groups $E_3^{2(2^{s+1} - 1),1}(\tilde{G})$ and $E_3^{2(2^{s+1} - 1),1}(PG)/\langle \omega \rangle$ in (4.13) are spanned respectively by the following elements

$\rho_{2^s} = a_i \cdot \rho_i$, $1 \leq i < 2^s$, $a_i \in A^{4(2^s - i)}(G)$ and

$\rho_{2^s}' = a_i \cdot \rho_i'$, $1 \leq i < 2^s$, $a_i \in A^{4(2^s - i)}(G),

with respect to them, by i) of Lemma 4.2 and by (4.14),

$$\theta(a_i \cdot \rho_i) = a_i \cdot \theta(\rho_i) = 0, 1 \leq i < 2^s,$$

By the exactness of the sequence (4.13) we get

$$\theta(\rho_{2^s}) = \omega^{2^{s+1} - 1},$$

hence $C^*(\rho_{2^s}) = 2\rho_{2^s}$,

showing $a_{2^s} = 2$. With $Z(G) = \mathbb{Z}_2$ we get $a_i = 1$ for $i \neq 2^s$ by (4.10).

**Case 6.** $G = Spin(2n)$, $2^s < n = 2b + 1 \leq 2^{s+1}$. According to (2.9) we have

$$A^*(PG) = \mathbb{Z}[\omega, x_1, x_2, \ldots, x_{2^s}]/\langle 4^s \cdot 2^2 \cdot \omega^{2^{s+1}-1} \cdot 2^{x_{2^s-1}} \cdot 2^{2^s-1} : 2 \leq i \leq \frac{2^s}{2} \rangle,$$

where $\omega = x_1$. It implies that

$$\ker\{ A^{2k}(PG) \Rightarrow \omega \cdot A^{2k}(PG) \} = \begin{cases} \mathbb{Z}_2[2\omega] \text{ if } k = 1; \\ \mathbb{Z}_2[\omega^{2^{s+1}} - 1] \text{ if } k = 2^{s+1} - 1. \end{cases}$$

By (4.8) we have the exact sequences

$$\omega = x_1.$$
implying respectively that $a_1 = 2$ and $a_{2^s} = 2$ (by an argument similar to the previous case). With $Z(G) = Z_4$ we get $a_i = 1$ for $i \neq 1$ or $2^s$ by (4.10).

**Case 7.** $G = \text{Spin}(2n)$, $2^s < n = 2^t(2b + 1) \leq 2^{s+1}$, $t \geq 1$. Decompose the covering $c : G \to PG$ into the composition of two cyclic coverings of order 2

$$c = c_2 \circ c_1 : \text{Spin}(2n) \to SO(2n) \to P\text{Spin}(2n),$$

where, in addition to formulae (2.9) of $A^*(G)$ for $G = \text{Spin}(2n)$, $P\text{Spin}(2n)$,

$$A^*(SO(2n)) = \mathbb{Z}[x_1, x_3, x_5, \ldots, x_{2^s}]_{2^s}, \deg x_k = 2k, h_i = 2^{\lfloor \frac{n-1}{2^i} \rfloor + 1}.$$

by Marlin [23]. The same calculation as that in Case 5 shows that

$$\mathcal{D}(\text{Spin}(2n), SO(2n)) = \{1, \ldots, 1, 2_{(2^t)}, 1, \ldots, 1\},$$

$$\mathcal{D}(SO(2n), P\text{Spin}(2n)) = \{1, \ldots, 1, 2_{(2^t-1)}, 1, \ldots, 1\}.$$

For $G = \text{Spin}(2n)$ Theorem D is verified by the multiplicative property of multi-degree with respect to the composition of coverings. \hfill $\Box$

For a cyclic covering $c : G \to G'$ on simple Lie groups let $J(\omega) \subseteq H^*(G')$ be the subring generated by the Euler class $\omega$ of the central extension $\tilde{G} \to G'$, and let $b_r$ be the order of the power $\omega^r$ in $J(\omega)$. Define the characteristic of the ring $J(\omega)$ to be the sequence $Ch(J(\omega)) := \{b_1, b_2, \cdots \}$ of integers. It is clear that

i) $b_{r+1}$ divides $b_r$ for $r \geq 1$; and

ii) $J(\omega) = J(PG)$ for $G \neq \text{Spin}(2n)$ with $n \equiv 1 \mod 2$

by Theorem C. For examples, we get from ii) that

$$Ch(J(\omega)) = \{3, \cdots, 3, 1\}$$

for the three sheets covering $E_6 \to PE_6$

$$Ch(J(\omega)) = \{2, 1, \cdots, 1\}$$

for the two sheets covering $E_7 \to PE_7$.

Since the proof of Theorem D goes through (essentially) all the possible cyclic coverings between simple Lie groups, it implies the following result that is independent of the types of simple Lie groups.

**Theorem 4.4.** For any cyclic covering $c : G \to G'$ between simple Lie groups the two invariants $\mathcal{D}(G, G')$ and $Ch(J(\omega))$ are related by the equalities

$$a_k = b_{r_k}/b_{r_{k+1}}, 1 \leq k \leq n, \text{ where } I_G = \{r_1, \cdots, r_n\}. \hfill \Box$$

**Remarks 4.5.** The passage from a cyclic covering $c : G \to G'$ to its central extension $\tilde{G} \to G'$ is an useful construction in geometry. As examples, for the cyclic covering $SU(n) \to PSU(n)$ of order $n$ we have $\tilde{G} = U(n)$, the unitary group of rank $n$; for the cyclic covering $Spin(n) \to SO(n)$ of order 2 we get the spin$^c$ group $\tilde{G} = \text{Spin}^c(n)$.

In the context of Schubert calculus the integral cohomologies of all simply connected simple Lie groups $G$ have been constructed in [12]. In [11, 8] the exact sequence (4.5) has been applied to extend the construction to the integral cohomology of the adjoint Lie groups $PG$. \hfill $\Box$
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