Semiclassical 3-point function in WZW AdS$_3$ model

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1. Introduction

The AdS/CFT conjecture implies that the correlation functions in the boundary quantum field theory can be computed alternatively in string theory, i.e., by the methods of a two-dimensional theory. One may start with a semiclassical computation - for large ’t Hooft coupling $\lambda \gg 1$ (i.e., large worldsheet coupling $\sqrt{\lambda}$) the string path integral for the correlation functions is evaluated at the saddle-point of the action in the presence of sources, created by proper ”heavy” vertex operators. In the dual conformal theory they correspond to operators with dimensions of the order of $\sqrt{\lambda}$. Some recent results for correlators of three heavy operators were obtained by Janik and Wereszczynski [1]. They consider strings in $AdS_2 \times S^5$ and compute the $AdS_2$ contribution to the 3-point correlation function, exploiting the integrability of the model; see also [2].

Classically the relevant 2d actions are conformally invariant and presumably this symmetry is preserved on quantum level. This motivates the elaboration of semiclassical techniques in conformal field models. Here we review the derivation in [3] of the semiclassical 3-point correlator in the (Euclidean) $AdS_3$ model described by a WZW action. The alternatively derived formula reproduces the ”heavy charge” classical limit of the known quantum 3-point constant, exploited in the applications to the superstring on $AdS_3 \times S^3 \times M^4$ in the NS-NS background [4], [5].

2. Preliminaries on Euclidean $AdS_3$

The target space of the theory is the Euclidean $AdS_3$:

$$-X_1^2 + \sum_{i=1}^{3} X_i^2 = -1,$$

a coset $\simeq SL(2,\mathbb{C})/SU(2)$, parametrised in $SL(2,\mathbb{C})$ as

$$g(X) = X_1 1_2 - X_i \sigma_i = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^\phi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\gamma} & 1 \end{pmatrix}.$$  (2)

The WZW classical action given in terms of the coordinates $\phi(z,\bar{z}), \gamma(z,\bar{z}), \bar{\gamma}(z,\bar{z})$ reads

$$S_{AdS} = \frac{k}{\pi} \int d^2z (\partial_z \phi \partial_{\bar{z}} \phi + \partial_z \bar{\gamma} \partial_{\bar{z}} \gamma e^{2\phi}).$$  (3)

$^1$ Talk given by V.B. Petkova.
The equations of motion imply the conservation of the affine currents

\[ \partial_z J(z) = 0 = \partial_{\bar{z}} \tilde{J}(\bar{z}), \]
equivalent to two chiral first order equations for \( g(z, \bar{z}) \)

\[ k \partial_z g + J(z) g = 0, \quad k \partial_{\bar{z}} g + g \tilde{J}(\bar{z}) t^a = 0. \] (4)

The group \( SL(2, \mathbb{C}) \) acts as: \( g \to \omega g \omega^+ \); e.g. the translations shift \( \gamma \to \gamma - x \). The fundamental vertex operator is defined as projection on a shifted group element

\[ V = V_{j, m}(z, \bar{z}; x, \bar{x}) = e^{-\phi} + |\gamma - x|^2 e^\phi = (1, -x) g(z, \bar{z}) \left( \frac{1}{-\bar{x}} \right) \] (5)

and the generators \( t^a \) of the algebra \( sl(2) \) are realised by differential operators with respect to the isospin variable \( x \). The general vertex operators are given by

\[ V_j(z, \bar{z}, x, \bar{x}) = (e^{-\phi(z, \bar{z})} + |\gamma(z, \bar{z}) - x|^2 e^{\phi(z, \bar{z})})^{2j}. \] (6)

In the AdS/CFT correspondence the isospin variables \( x, \bar{x} \) play the role of the space-time coordinates of the dual theory and the expression (6), looked as a function of \( (x, \bar{x}) \) and of the \( AdS_3 \) variables \( (\phi, \gamma, \bar{\gamma}) \), represents the kernel of the integral boundary-to-bulk intertwining operator with boundary conformal dimension \( \Delta = -2j \)

\[ \Phi(\phi, \gamma, \bar{\gamma}) = \int d^2 x \ (e^{-\phi} + |\gamma - x|^2 e^\phi)^{-\Delta} \phi_0(x). \] (7)

In the quantum theory \( k \to k - 2 = 1/b^2 \) and furthermore a curvature term is added creating a "charge at infinity". The world sheet (Sugawara) scaling dimension of the vertex operator is

\[ \delta^{Su}(j) = -b^2 j(j + 1) = -\frac{a}{b} + a(Q - \alpha), \quad \alpha = -j b, \quad Q = \frac{1}{b} + b. \] (8)

Here \( \Delta_L(\alpha) = a(Q - \alpha) \) is the scaling dimension in the related, via Drinfeld- Sokolov reduction, \( c > 25 \) Virasoro (Liouville) theory. Accordingly the WZW 3-point constant is closely related to DOZZ formula in Liouville theory and is given up to an overall constant by [6]

\[ C(j_1, j_2, j_3) = \frac{\gamma_{j_1}(b)}{\gamma_{j_1}(a_{123} - b)} \prod_{i=1}^3 \frac{\gamma_{j_i}(2a_i)}{\gamma_{j_i}(a_{123} - 2a_i)}, \quad a_{123} = \alpha_1 + \alpha_2 + \alpha_3 \] (9)

where \( \gamma_b(x) \), expressed in terms of Barnes double Gamma function \( \Gamma_b(x) \), admits the integral representation

\[ \log \gamma_b(x) = \int_0^\infty \frac{dt}{t} \left( \frac{Q}{2} - x \right)^2 e^{-t} - \sinh^2 \left( \frac{Q}{2} - x \right) \frac{e^{-t}}{\sinh \left( \frac{t}{2} \right)} \] (10)

\bullet The semiclassical limit corresponds to \( b^2 = 1/(k - 2) \to 0 \) and "heavy" charges

\[ \Delta = -2j = \frac{2\eta}{b^2}, \quad \eta \text{ finite}, \] (11)

so that \( b^2 \delta^{Su}(j) \to -\eta^2 \). The limit of the 3-point function is reduced to that computed for Liouville theory in [7], using that the function \( \gamma_b \) goes to

\[ b^2 \log \gamma_b(\frac{\eta}{b}) \to F(\eta) : = \int_{1/2}^\infty dx \ \log \gamma(x) \quad ; \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \] (12)

We shall reproduce this limit by an alternative derivation - it follows and generalises the approach of [7] in the Liouville case which we will first sketch.
3. Liouville case main steps

The solutions of the classical equations of motion for the Liouville field

$$\partial \hat{\partial} \varphi(z, \bar{z}) = \pi \mu b^2 e^{2\varphi(z, \bar{z})}$$

(13)

can be recovered from the solutions of the second order chiral equations with the conserved energy - momentum tensor $\partial \ddot{T} = 0 = \ddot{T}$,

$$\partial_2^2 e^{-\varphi(z, \bar{z})} = (\partial_2 \varphi)^2 - \partial_2^2 e^{-\varphi(z, \bar{z})} = -\ddot{T}(z) e^{-\varphi(z, \bar{z})}.$$ \hfill (14)

It is given by the $b \to 0$ limit of the quantum tensor

$$b^2 \ddot{T}(z) = b^2(- (\partial_2 \varphi)^2 + Q \partial_2^2 \varphi) \to \ddot{T}(z), \quad (\varphi = b\phi, Q = b + \frac{1}{b}).$$ \hfill (15)

The equation (13) ensures the conservation of the classical energy-momentum tensor in (14) and vice versa. One looks for a solution of (13) in the presence of (three) sources created by vertex operators $V_{\alpha_i} = \alpha^2 e^{2\alpha_i \phi} = e^{2\eta_i} b^2$ with "heavy" charges $\alpha_i = \eta_i/b$, i.e., finite $\eta_i$, for $b \to 0$.

The semiclassical correlator is determined by the stationary point of the full action with the contribution of the sources accounted for

$$< V_{\alpha_1}(z_1, \bar{z}_1)V_{\alpha_2}(z_2, \bar{z}_2)V_{\alpha_3}(z_3, \bar{z}_3) > \sim e^{-\frac{1}{b^2} S_{cl}[\varphi; \eta_i]}.$$ \hfill (16)

The solutions of (14) in the presence of sources are reviewed, e.g., in [8], in relation to the uniformization of the three punctured sphere problem. The classical tensor $\ddot{T}_L$ is given through the limit $b^2 \to 0$ of the normalised correlator of $T$ with the three vertex operators,

$$\ddot{T}_L(z; z_a) = \lim_{b \to 0} b^2 T(z; z_a) =$$

$$\sum_i \left(\frac{h_i}{(z-z_i)^2} + \frac{\partial_i}{z-z_i} \log \frac{1}{(z_{12})^{h_{12}}(z_{23})^{h_{23}}(z_{13})^{h_{13}}}\right) =: (\partial_2 w)^2 \ddot{T}_L(w)$$ \hfill (17)

where

$$w = \frac{(z-z_1)z_{23}}{(z-z_2)z_{13}}.$$ \hfill (18)

and $h_i = h_i + h_j - h_k$, for the Liouville classical dimensions $h_i = \lim_{b \to 0} b^2 \alpha_i(Q - \alpha_i) = \eta_i(1 - \eta_i)$. The solution for $e^{-\varphi(z, \bar{z})}$ - now as a function also of the coordinates $z_i, \bar{z}_i$ of the sources is given, up to a prefactor, by a monodromy invariant diagonal combination

$$e^{-\varphi(z, \bar{z}; z_i, \bar{z}_i)} = \sqrt{\frac{e^{2\mu \beta b^2}}{(1-2\eta_1)} |(z-z_2)|^2 |z_{12}|^2 \mathcal{N}_L^2 \mathcal{G}_1(w)^2 - \mathcal{N}_L^2 \mathcal{G}_2(w)^2| = |y_1|^2 - |y_2|^2}$$ \hfill (19)

of holomorphic and anti-holomorphic solutions (hypergeometric functions). The permutation symmetry determines the relative constant $N_L$ while the overall constant is fixed by the Liouville equation (13) for $\varphi(z, \bar{z}; z_i, \bar{z}_i)$; the check of (13) given (19) requires the knowledge of the Wronskian of the two solutions, $A = y_1 \partial_z y_2 - y_2 \partial_z y_1$, a constant due to equation (14).

The chiral equation solved by (19) is the classical version of the Belavin-Polyakov-Zamolodchikov (BPZ) equation, resulting from the decoupling of a level 2 singular vector in the Vir module.
of (fundamental) h.w. \( \triangle_L(\alpha) = \triangle_L(-b/2) \), whence the semi-classical limit \( b \to 0 \) is identified with the semi-classical limit of \( (\text{fundamental}) \ h.w. \). The KZ equation (24) is equivalent to a pair of differential equations for the components \( G \) and one obtains the equation (written for the chiral constituents of \( \hat{sl}_2 \))

\[
e^{-\varphi(z; \bar{z}; x; \eta_i)} \sim \lim_{b \to 0} \langle V_{-\frac{b}{2}}(z, \bar{z})V_{a_1}(z_1)V_{a_2}(z_2)V_{a_3}(z_3) \rangle > \langle V_{a_1}(z_1)V_{a_2}(z_2)V_{a_3}(z_3) \rangle >
\]

\[
\varphi \sim -2\eta_1 \log |z - z_i|^2 + X_i + ....
\]

where \( X_i = X_i(|z_{ij}|; \eta_i) \) is the finite part and

\[
X_i = -\partial_{\eta_i} S_{cl}, \ i = 1, 2, 3.
\]

The integration of this system of equations recovers the semiclassical 3-point function in (16).

4. The WZW \( AdS_3 \) semiclassical case

In this case the BPZ equation is replaced by the (classical) Knizhnik-Zamolodchikov (KZ) equation. In the presence of three sources at \((z_i, x_i)\) the chiral equation for the field \( V_{j=1/2}(z, \bar{z}; x, \bar{x}) \) as a function of the coordinates \( \{z_i, \bar{z}_i, x_i, \bar{x}_i\} \) of the three sources becomes

\[
(\partial_z - j^a(z; z_i; x_i)t^a)V(z, x; z_i, x_i) = 0,
\]

where the current is defined through the classical limit of its 4-point function normalised with the 3-point function \( \langle V_{j_1}V_{j_2}V_{j_3} \rangle \)

\[
j^a(z; z_i, x_i) = \lim_{b \to 0} \frac{b^2}{\langle V_{j_1} \rangle \langle V_{j_2} \rangle \langle V_{j_3} \rangle} \sum_{z - z_i} \langle V_{j_1}V_{j_2}V_{j_3} \rangle.
\]

The \( sl(2) \) generators \( t^a \) are represented by the standard differential operators in \( x_i \). The projective invariance allows to write

\[
V(z, x; z_i, x_i) = \frac{(x - x_2)|x_{13}|}{|x_{12}| |x_{23}|} \hat{V}(w, y), \quad y = \frac{(x - x_1)x_{23}}{(x - x_2)x_{13}}
\]

and one obtains the equation (written for the chiral constituents of \( \hat{V} \))

\[
\partial_w G(w, y) = -\eta_{13}(\frac{y - w}{w - 1})^2 - 1 + 2\eta_1(\frac{y - w}{w - 1} + 1) + 2\eta_3(\frac{y - w}{w - 1} + 1))\partial_y G(w, y)
\]

\[
+ \left( \frac{(y - w)(y - w)}{w(w - 1)} + \frac{m}{w} + \frac{m}{w - 1} \right) G(w, y) .
\]

The KZ equation (24) is equivalent to a pair of differential equations for the components \( G_0, G_2 \) of

\[
G(w, y) = G_0(w) + (y - w)G_2(w),
\]
or, in matrix form for the vector \( G = (G_0, G_2)^T \)

\[
\frac{d}{dw} G(w) - \left( \begin{array}{c}
\frac{m}{w} + \frac{n_1}{w-1} & 1 - \eta_{123} \\
\frac{n_2}{w(w-1)} & -(\frac{m}{w} + \frac{n_3}{w-1})
\end{array} \right) G(w) = 0.
\]

(26)

The component \( G_0(w) = G(w, y = w) \) satisfies the second order equation (14) with the Liouville energy-momentum tensor \( \hat{T}_L(w) \) in (17) - a manifestation of the Drinfeld-Sokolov reduction of the WZW theory.

Combining the left and right solutions of (24) one obtains for \( V \)

\[
V(z, x; z_1, x_i) = \frac{|x-x_1|}{|x_23||x_12|}|N_{WZW}|^2 (|G_0^+(w, y)|^2 + N_{WZW}^2 |G_0^-(w, y)|^2),
\]

(27)

where the components of

\[
G^{(\pm)}(w, y) = G_0^{(\pm)}(w) + (y - w)G_2^{(\pm)}(w)
\]

(28)

are expressed by hypergeometric functions. The relative constant is related to that in the Liouville case

\[
N_{WZW}^2 = \frac{\gamma(\eta_{13})^2(2\eta_1)^2}{\gamma(\eta_{13})^2(\eta_{13})^2} = -\frac{(1 - 2\eta_1)^2}{(\eta_{123} - 1)^2} N_L^2
\]

(29)

and is determined from the requirement of permutation invariance. The leading singularities of the solutions in (27) for \( z \to z_1 \) \( (w \to 0) \) are given by

\[
G^{(+)}(w, y) \sim w^m (1 - \frac{\eta_2}{2\eta_1} y), \quad G^{(-)}(w, y) \sim w^{-m} (y - \frac{\eta_{23}}{1 - 2\eta_1} w).
\]

(30)

\[5\]
Rewritten in a matrix form (30) reads

\[
g(X) = hh^+, \quad h = \begin{pmatrix} u_1^{(+)}, & u_1^{(-)} \\ u_2^{(+)}, & u_2^{(-)} \end{pmatrix}.
\]  

(34)

The solution gives for \(\eta_1 > 0\) the leading contribution to \(\phi\) near the source at \(z = z_1\)

\[
2\phi \sim -2\eta_1 \log |z - z_1|^2 + X_1,
\]

(35)

where \(X_1\) is the finite part

\[
X_1 = 2\eta_1 \log \left| \frac{\eta_1 z_1 z_2 z_3}{z_2 z_3} \right|^2 - \log \left| \frac{x_1 x_2 x_3}{x_2 x_3} \right|^2 - \log \frac{\gamma(\eta_1 z_2)\gamma(\eta_2 z_3)\gamma(\eta_3 z_1)}{\gamma(\eta_1^2)\gamma(2\eta_1)^2}.
\]

(36)

Due to the symmetry of the classical solution analogous formulae hold in the vicinity of all three singular points with finite parts \(X_i\).

- Adapting the derivation in [7] in the Liouville case we add to the classical action \(S^{(cl)} = b^2 S'_{AdS}\) terms which account for the three vertex insertions

\[
\hat{S}^{(cl)} = S^{(cl)} + S^{(src)},
\]

\[
S^{(src)} = - \int d^2z \sum_i \delta^2(z - z_i, \bar{z} - \bar{z}_i)(2\eta_i \phi(z, \bar{z}) + 2\eta_i^2 \log |z - z_i|^2)
\]

\[
= - \sum_i \eta_i X_i.
\]

(37)

Here it is assumed that the integration in the first term \(S^{(cl)}\) is on the Riemann sphere with the points of insertion of the three sources excluded. The action is regularised, the logarithmic terms added compensate the singularities of the solution (35).

At the saddle point of \(\hat{S}^{(cl)}[\phi, \gamma, \bar{\gamma}]\), i.e., on a solution of the classical equations, only the source term in the action contributes to the derivative of the full action with respect to any of the charges \(\eta_i \Rightarrow \frac{\partial}{\partial \eta_i} \hat{S}^{(cl)} = -X_i, i = 1, 2, 3.\)

(38)

This set of equations for \(\hat{S}^{(cl)}\) integrates and one reproduces the 3-point function

\[
e^{-\hat{S}^{(cl)}} = C^{(cl)}(\eta_1, \eta_2, \eta_3) \left| x_{13}^{2j_1} x_{12}^{2j_2} x_{23}^{2j_3} \right| \left| z_{13}^{2\delta_{13}} z_{12}^{2\delta_{12}} z_{23}^{2\delta_{23}} \right|,
\]

(39)

where \(\delta_{ik} = \delta_i + \delta_k - \delta_l, \quad \delta_i = -\frac{\eta_i^2}{b^2}, \quad j_{ik} = j_i + j_k - j_l, \quad j_i = -\frac{\eta_i}{b^2} \).

(40)

The expression obtained for \(C^{(cl)}(\eta_1, \eta_2, \eta_3)\) reproduces up to an overall constant the classical limit of the quantum 3-point function (9)

\[-b^2 \log C(-\frac{\eta_1}{b^2}, -\frac{\eta_2}{b^2}, -\frac{\eta_3}{b^2}) \rightarrow -b^2 \log C^{(cl)}(\eta_1, \eta_2, \eta_3) = F(\eta_{123}) - F(0) + \sum_i (F(\eta_{123} - 2\eta_i) - F(2\eta_i))\]

(41)
with the function $F(\eta)$ given in (12).

- The semiclassical formula in [1] has the same structure but in terms of a different function, instead of $F(x)$. We note that the monodromy coefficients computed in [1], taken at the particular point of trivial spectral parameter, give an expression, compatible with a particular, differently normalised, basis of conformal blocks. Although the model considered in [1] is different, this coincidence remains to be explained.

Summary and discussion

We have presented an alternative derivation of the semiclassical 3-point function in the $AdS_3$ WZW model. It recovers its coordinate and isospin dependence and reproduces the $b \to 0$ limit of the quantum 3-point constant when all three charges (i.e., the conformal dimensions of the dual theory) are large $\Delta_i \sim 2\eta_i/b^2$.

Problem: extend the semiclassical computation to higher rank theories as the WZW related $AdS_5$, respectively $S^5$ model. For the 3-point constant of three scalar operators the fundamental $j = 1/2$ field is replaced by the $sl(4)$ fundamental weight $\lambda = (0, 1, 0)$.

Our consideration is not directly applicable to the string on $AdS_5 \times S^5$ since the bosonic part of the supersymmetric action [9] has no $B$ field (WZW term), whence no conserved chiral currents; rather the bosonic part is given by the standard sigma model - the realm of recent investigations based on integrability. The conformal invariance is ensured by the WZ term involving fermions. To make 2d CFT methods effective in the string part of the AdS/CFT correspondence one needs to develop relevant supersymmetry technique. One may start with the simpler $AdS_3 \times S^3$ model with R-R background, see, e.g., [10] for a recent discussion.

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