Crepant Terminalisations and Orbifold Euler Numbers for SL(4) Singularities

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Abstract. Let X and Y be two analytic canonical Gorenstein orbifolds. A resolution of singularities $Y \to X$ is called an Euler resolution if $Y$ and $X$ have the same orbifold Euler number. If $Y$ is only terminal rather than smooth, it is called an Euler terminalisation.

It is proved that Euler terminalisations exist for toric varieties in any dimension, for 4-dimensional toroidal varieties, and for singularities $\mathbb{C}^4/G$ where $G$ belongs to certain classes of SL(4) subgroups. The method of proof is expected to be applicable to a sizeable number of finite SL(4) subgroups and to lead to a generalisation of the Dixon-Harvey-Vafa-Witten orbifold Euler number conjecture to dimension 4.

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0. Introduction

An analytic $n$-fold $X$ will be called a canonical Gorenstein orbifold if it has at most canonical Gorenstein singularities and is such that, for each $x \in X$, there exists a finite group $\pi_x < \text{SL}(n)$ such that

$$(X, x) \cong (\mathbb{C}^n / \pi_x, 0)$$

as germs of analytic spaces.

0.1. The Orbifold Euler Number. The orbifold Euler number of $X$ is defined as the (finite) sum

$$\chi_{\text{orb}}(X) := \sum_{k \geq 1} k \chi(m^{-1}(k)), \quad (1)$$

where $\chi$ is the ordinary Euler number and

$$m: X \longrightarrow \mathbb{Z} \quad \quad x \longmapsto |\text{Cl}(\pi_x)|$$

is the upper semi-continuous map assigning to each point $x$ the number of conjugacy classes of $\pi_x$. It is easy to show [Roa90] that if $M$ is an $n$-fold admitting a $G$-action whose non-trivial elements’ fixed-point loci have codimension at least two, and such that $M/G$ has only Gorenstein singularities, then

$$\chi_{\text{orb}}(M/G) = \chi_{\text{DHVW}}(M; G), \quad (2)$$

$$:= \sum_{[g] \in \text{Cl}(G)} \chi(M^g / N^G_g) \quad (3)$$

where the right-hand side denotes the Dixon-Harvey-Vafa-Witten Euler number proposed in [DHVW85, DHVW86].

If $X$ and $Y$ are canonical Gorenstein orbifolds and $Y \rightarrow X$ is a bi-meromorphic map such that $\chi_{\text{orb}}(Y) = \chi_{\text{orb}}(X)$, then $Y$ will be called
an *Euler blow-up* of $X$. If $Y$ is in addition smooth, then $Y$ will be called an *Euler resolution*. Restated in the above terminology, the Dixon-Harvey-Vafa-Witten Euler conjecture [DHVW85] is

Every 3-dimensional canonical Gorenstein orbifold has an Euler resolution.

It took ten years to give a positive answer to the conjecture [MOP87, Roa91, Mar93, Roa93, Itô94, Roa94].

0.2. **Euler Terminalisations.** Right from the start it was recognized that the analogous conjecture in dimension 4 (Do all 4-dimensional canonical Gorenstein orbifolds have Euler resolutions?) is trivially false: the simplest non-smooth example $\mathbb{C}^4/\langle -1 \rangle$ is already terminal.

However, rephrased slightly, the conjecture can be made to look much more promising. For this, note that the existence of Euler resolutions in dimension 3 is equivalent to saying that the minimal models for these singularities are smooth. In other words “smooth” is equivalent to “terminal” for 3-dimensional Gorenstein finite quotient singularities.

**Definition 0.1.** A Euler blow-up $Y \to X$ such that $Y$ has only terminal singularities is called an *Euler terminalisation* of $X$. The property $\text{Term}(X)$ is defined to be true if and only if such a $Y$ exists.

**Definition 0.2.** The property $\text{Term}(n)$ is defined to be true if and only if $\text{Term}(X)$ is true for all $n$-dimensional canonical Gorenstein orbifolds $X$.

Thus, $\text{Term}(2)$ is true in virtue of classical work and $\text{Term}(3)$ is true by the recent work mentioned above.

**Question 1.** Is $\text{Term}(n)$ true?

The next open case is of course $\text{Term}(4)$. As in the case of dimension 3, the question reduces to the problem of constructing Euler terminalisations for the local singularities $\mathbb{C}^4/G$ for all the small subgroups $G$ of $\text{SL}(4)$.

We shall use the following terminology. A subgroup $G < \text{SL}(V)$ will be called *reducible* if $V$ is reducible as a $G$-module. The *type* of a group $G < \text{SL}(n)$ denotes the dimensions of the irreducible representations of $G$ appearing in the chosen special linear representation $\mathbb{C}^n$. For instance, irreducible groups have type $(n)$ and abelian groups have type $(1,1,\ldots,1)$.

For any $n$, denote by $Z_n$ the cyclic central subgroup of $\text{SL}(n)$. For any element $g \in G$, $N_g^G$ denotes the centralizer of $g$ in $G$, namely $\{ h \in G | h^{-1}gh = g \}$
0.3. Main Results.

0.3.1. Toric and Toroidal cases. The first result is that Term($X$) is true in all dimensions for toric varieties. Furthermore, it holds also for toroidal varieties (analytic varieties which are locally isomorphic to toric varieties) in dimension 4 and in dimension $n$ if termination of flips can be proved.

**Theorem 0.3.** All simplicial toric canonical Gorenstein orbifolds have Euler terminalisations. Furthermore, if flips terminate in dimension $n$, then all $n$-dimensional toroidal canonical Gorenstein orbifolds have Euler terminalisations. In particular, this is the case in dimension 4.

The proof of the toric case is straightforward; crepant blow-ups of $X$ correspond to subdividing the first quadrant in $\mathbb{R}^n$ by rays whose generators all lie in the same hyper-plane; the orbifold Euler number of any cone is equal to its volume (meaning the volume of the simplex spanned by its generators), and since the sum of the volumes of the cones in the subdivision is equal to the total volume of the original quadrant, the orbifold Euler number is seen to remain invariant under crepant blow-ups. The fact that among all the crepant blow-ups there exists a terminal one is a consequence of the Toric Minimal Model Program [Rei83].

There is no minimal model program as yet for non-toric varieties in dimensions 4 and over. However, in the toroidal case, a well-known technique makes it is possible to construct flips by patching together local toric flips and using uniqueness. Thus if termination of flips is also known, (as it is in dimension 4 [KMM87]), the existence of the terminal model follows and this again must have the same orbifold Euler number as the original variety.

0.3.2. SL(4) subgroups of type (3,1). The second set of results concerns 4-dimensional non-abelian singularities created by finite SL(4) subgroups of type (3,1).

Let $G < \text{SL}(4)$ be a finite subgroup which stabilises a line $V^2 \subset V = \mathbb{C}^4$ and let $V^1$ be a $G$-submodule such that $V = V^1 \oplus V^2$. Denote by $\eta$ the generic point of the line $\{0\} \times V^2$, and by $G_\eta$ the stabiliser of $\eta$. Note that $G_\eta$ is a subgroup of $\text{SL}(V^1) \times \{1\} \cong \text{SL}(3)$.

**Theorem 0.4.** If $G < \text{SL}(4)$ fixes a line and is such that the group $G_\eta$ does not contain $\mathbb{Z}_3$ as a subgroup, then $\mathbb{C}^4/G$ has an Euler terminalisation with only toric singularities.

The method of proof essentially consists in using the results of Roan [Roa90] to construct a 3-dimensional Euler resolution of $\mathbb{C}^3/G_\eta$ which is equivariant under the larger group $G$. 
As mentioned in the next section, the assumption that $G_\eta$ does not contain the group $Z_3$ is not essential to the method. However, so far the author has been unable to construct the equivariant resolutions without it. An attempt is made in Section 4.5.

0.3.3. SL(4) subgroups containing $Z_4$. The third set of results reduces the question $\text{Term}(\mathbb{C}^4/G)$ for the irreducible $G$ which contain $Z_4$ to a conjecture regarding the existence of the equivariant resolutions of SL(3) mentioned above. The conjecture is proved for irreducible subgroups of SL(3) which do not contain $Z_3$, but remains open in the general case.

Remark 0.5. The material presented here can no doubt be pushed further, but the author has so far been unable to do so. Nevertheless, it is hoped that the attentive reader will be able to perceive a direction in which to proceed. Is is also conceivable that the general strategy that emerges from this approach may be applicable to an understanding of quotient singularities in higher dimensions.

The idea here is as follows. Suppose that $G \subset \text{SL}(n)$ acts on $\mathbb{C}^n$ and contains the centre $Z_n$, and write $\tilde{G} := G/Z_n$ and $\tilde{V} := \text{Bl}_0 V/Z_n$. Then Lemma 3.3 implies that $\tilde{V}/\tilde{G}$ is another canonical Gorenstein orbifold and $\tilde{V}/G \rightarrow \tilde{V}/\tilde{G}$ is an Euler blow-up. With the aid of another Lemma (The Patching Lemma 3.4), the problem is thus reduced to the construction and patching of Euler blow-ups of local neighbourhoods of $\tilde{V}/\tilde{G}$. The advantage of these is that the stabilisers of $\tilde{G}$ in the tangent space to the blowup $\tilde{V}$ are simpler than those of $G$ (because the stabiliser of $\tilde{G}$ must fix a line in the tangent space to $\tilde{V}$, so its type must be $(t,1)$, where $t$ is a partition of $n-1$).

This approach is spelt out in Section 4 for irreducible subgroups of SL(4) which contain $Z_4$; it reduces the problem to constructing and patching together equivariant SL(3) resolutions.

0.3.4. Discussion. What if, on the other hand, the group $G$ is irreducible, but doesn’t contain the centre of SL(V)?

A complete answer to Question 3 for all finite SL(4) singularities seems out of reach of the methods suggested here, if only because the cases when $G$ does not contain $Z_4$ include simple groups, such as the alternating group $A_5$; if dimension 3 is any indication [Mar93, Roa93], it seems that ad-hoc methods will be necessary to construct a terminalisation.

However, the possibility is open that, as a general rule, the non-simple finite subgroups SL(n) not containing $Z_n$ are few in number and relatively amenable in form. For instance, if dim $V = 2$ no cases
occur. For dim $V = 3$, one has — apart from the simple groups (H) and (I) which require ad-hoc methods — “half” the groups of type (C) and “half” of those of type (D). It turns out that under the assumption $Z_3 \not< G$ these are semi-direct products of abelian groups with the alternating group $A_3$ and the symmetric group $S_3$ respectively. This allows one to construct their Euler resolutions from toric resolutions — see [Roa90].

Extension of this method to the SL(4) case would seem to be feasible. Table 1 outlines the state of the Term($\mathbb{C}^4/G$) problem.

| Type of Group | Method | Status |
|---------------|--------|--------|
| Irred. (4)    | $G \not< Z_4$ Simple | Ad hoc method? |
|               | Non-Simple | Small # of cases? Semi-direct products of abelian groups with $A_4$ and $S_4$? |
|               | $G > Z_4$ 4-d Euler blow-up of origin. Patch together equivariant lower-d blowups. | $\mathbb{R}_4$ ? |
| Red. (3,1)    | $G_\eta \not< Z_3$ Construct equivariant 3-d Euler resolution. Reduces to the toric case. | $\mathbb{R}_{2.3}$ OK |
|               | $G_\eta > Z_3$ Euler blow-up of fixed line. Reduce to 2-d equivariant blow-up. | $\mathbb{R}_2$ ? |
| (2,2)         | Use 2-d results | $\mathbb{R}_{5.1}$ ? |
| (2,1,1)       | Use 2&3-d results | $\mathbb{R}_{5.2}$ ? |
| (1,1,1,1) (Abelian) | Toric MMP | $\mathbb{R}_1$ OK |

Table 1. Constructing Euler Terminalisations of $\mathbb{C}^4/G$ for $G < \text{SL}(4)$. (Bold numbers in parentheses indicate the type of the group. In the cases (3,1), the group $G_\eta$ denotes the stabiliser in $G$ of a generic point of the line fixed by $G$.)
The \textit{Status} column indicates the section of this paper which makes some contribution to the problem. Questions which are solved in this paper are indicated by the mention “OK”. A question mark indicates that the question is still open and that methods of this paper are relevant. Finally, where no results are known, nothing is indicated in the Status column, but some guesses are given as to the likely situation, based on the present state of knowledge.

0.4. \textbf{Open Problems.} Interesting open problems arise in relation to recent work of Ito and Reid [IR94] which establishes a one-one correspondence between crepant divisors of $V/G$ and conjugacy classes “of weight 1” in (the dual group to) $G$.

One interesting question is how this correspondence behaves under the Euler blow-ups which are constructed for the SL(4) singularities mentioned above. A deeper understanding of this (apart from being of interest in itself) would also no doubt allow one to say more about the singularities of the terminalisation.

For instance, for toric and toroidal varieties, the Euler blow-up can be made projective rather than just analytic. But in general, the gluing process in the Patching Lemma 3.4 does not in itself guarantee that the blow-up $X$ will be projective, because a divisor which is reducible when considered locally in the neighbourhood of one of the points $[\xi]$ might have two of its components identified by the gluing process. Such troublesome cases might conceivably be ruled out by a deeper understanding of the correspondence between divisors and conjugacy classes.

0.5. \textbf{Outline.} Section 1 deals with the construction of Euler terminalisations for the toric and toroidal cases. Section 2 deals with the finite SL(4) groups of type (3,1). Section 3 presents a blowing-up construction and some other technical lemmas which are then used in Section 4 in dealing with irreducible finite SL(4) subgroups containing $Z_4$. Finally, Section 5 makes some comments regarding the finite SL(4) groups of type (2,2) and (2,1,1).

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1. Toric and Toroidal Cases

Theorem 1.1 (Toric Minimal Model Program). Let $Y$ be a toric variety and $X$ be a simplicial toric variety which admits a projective birational toric morphism $f: X \to Y$. Then there exists a sequence $X \xrightarrow{h} Z \xrightarrow{g} Y$ such that

1. $h$ is a composite of toric divisorial contractions or toric flips.
2. $g$ is a projective morphism and $Z$ is a simplicial toric variety with terminal singularities such that $K_Z$ is relatively nef for $g$.

Note that if $Y$ has canonical singularities, then $K_Z = g^*K_Y$ in the sense of $\mathbb{Q}$-Cartier divisors, i.e. $g$ is crepant.

Proof. See [Rei83, Theorem 0.2], where the result is proved under the assumption that $Y$ is projective. As remarked in [Rei83], this assumption is not essential and the result is valid for non-complete toric varieties also (the easiest way to see this is to reduce the non-projective case to the projective one by completing the fan in an appropriate way).

Corollary 1.2. All toric canonical Gorenstein orbifolds admit (toric) Euler terminalisations.

Proof. Note that a toric variety has at most orbifold singularities if and only if it is simplicial. It is therefore sufficient to prove that any crepant blow-up of a simplicial Gorenstein toric variety must have the same orbifold Euler number as the original. But this is true because the orbifold Euler number of a simplicial toric variety is just the volume\(^1\) of the cone, meaning the volume of the simplex defined by the cone’s generators; a crepant blow-up corresponds to a fan subdivision by one-dimensional rays whose primitive generators all belong to the same plane, and therefore the sum of the volumes of the cones in such a subdivision is equal to the volume of the original cone.

\[^1\]Also called the multiplicity of the cone.

The minimal model program for general varieties is at present only proved in dimension 3. In dimension 4, although termination has been shown [KMM87], existence of flips remains a problem. Nevertheless, a technique well-known to minimal model program specialists allows one to use the Theorem above and the termination result to say something about toroidal varieties, i.e. varieties which are only locally isomorphic to toric varieties. The argument can phrased for general $n$, even though at present, termination has only been proved for $n \leq 4$. 
Theorem 1.3. Assume that flips terminate in dimension $n$. Then all $n$-dimensional toroidal canonical Gorenstein orbifold admit Euler terminalisations (which are themselves toroidal). In particular, this is true in dimension 4.

Proof. Let $Y$ be a canonical Gorenstein orbifold locally isomorphic to a toric variety and let $p: X \to Y$ be any resolution obtained by toric blowups.

Suppose that $K_X$ is not $p$-nef, and let $c: X \to W$ be an extremal contraction. If it is a divisorial contraction, then replace $X$ by $W$. If $c$ is a small contraction, consider its restriction $c^+: U_X^+ \to U_W$ over each neighbourhood $U_W$. A flip being unique if it exists, the local flips patch together on the overlaps to form a global flip $c^+: X^+ \to W$. Thus, existence of flips is guaranteed in this case. Applying the same procedure repeatedly (and using the termination hypothesis) results in a projective morphism $p: Z \to Y$ such that $Z$ has $Q$-factorial terminal singularities with $K_Z$ being $p$-nef, which means that $p$ is crepant, since $Y$ is canonical. Furthermore, any toric terminalisation which is crepant must have the same orbifold Euler number by the volume argument in Corollary 1.2. □

Corollary 1.2 and Theorem 1.3 together give Theorem 1.3.

2. Groups $G$ of type $(3,1)$

2.1. Notation. Let $G < SL(n+1)$ be a finite subgroup such that $V = \mathbb{C}^{n+1}$ decomposes into two irreducible $G$-modules: $V = V_1 \oplus V_2$, with $V_1$ of dimension $n$, and $V_2$ of dimension 1.

Denote by $\eta$ the generic point of the line $\{0\} \times V_2$, and by $G_\eta$ the stabiliser of $\eta$. Note that $G_\eta$ is a subgroup of $SL(n) \times \{1\} \cong SL(n)$. The quotient $C := G/G_\eta$ is cyclic, being a naturally a subgroup of $GL(V_2) \cong \mathbb{C}^*$. The canonical quotient map is denoted $\pi: G \to C$ and the induced map on conjugacy classes is denoted $\pi_*: Cl(G) \to Cl(C)$.

Note that $G$ can be considered as a subgroup of $GL(V_1) = GL(n)$ by forgetting about the last row and column of any matrix element. Let $h \in G$ be an element such that $G = \langle G_\eta, h \rangle$ — i.e. a representative for a generator of $C$ — and denote by $h_1$ the $n \times n$ sub-matrix consisting of the first $n$ rows and columns of $h$. Let $\lambda_h$ be a complex $n$-th root of $det(h_1)^{-1}$. Then $h' := \lambda_h h_1 \in SL(n)$ and normalises $G_\eta$. Hence $G_\eta$ is a normal subgroup of $G' := \langle G_\eta, h' \rangle < SL(n)$ with quotient a cyclic subgroup $C'$. Note that $G'$ and $C'$ are defined up to a choice of the $n$-th root $\lambda_h$. 
The canonical quotient map is denoted $\pi': G' \to C'$ and the induced map on conjugacy classes is denoted $\pi'_*: \text{Cl}(G') \to \text{Cl}(C')$.

2.2. Equivariant Resolution. From now on, let $n = 3$, i.e. consider the case where $G < \text{SL}(4)$ fixes a line in $\mathbb{C}^4$ (and therefore $G_\eta < \text{SL}(3)$).

Conjecture 1. Let $G < \text{SL}(4)$ be a finite subgroup which stabilises a line $V^2 \subset V = \mathbb{C}^4$ and let $G_\eta$ denote the stabiliser in $G$ of the generic point of $V^2$ and let $G'$ and $C'$ be defined as in Section 2.1 above.

Then there exists a $G'$-invariant Euler resolution $W^1 \to V^1/G_\eta$, satisfying

$$\chi((W^1)^{\epsilon'}/C') = |\pi'^{-1}([\epsilon'])|,$$

for all $\epsilon' \in C'$.

If true, we shall see that this conjecture implies that $\text{Term}(\mathbb{C}^4/G)$ is true for groups stabilizing a line (i.e. types $(3, 1)$ and $(2, 1, 1)$) and (cf. Section 4) for all irreducible $G$ which contain $Z_4$.

Proposition 2.1. Suppose that Conjecture 1 is true for $G$. Then $W^1 \times V^2/C \to V/G$ is an Euler blowup, and $V/G$ has an Euler terminalisation with only toric singularities.

Proof. First, note that since $V^1/G_\eta$ has dimension at most three, it has a minimal model. The fact that there are only a finite number of distinct minimal models implies that $\mathbb{C}^*$ acts on any of them: for if the action of some element $\lambda \in \mathbb{C}^*$ produced a different minimal model, then, by continuity, one could produce countably many distinct minimal models by acting with a countable family of distinct neighbours of $\lambda$.

Second, note that the equality (4) would follow from the same equality with $C'$ replaced by the group $C$. For if $\phi$ is any element of $G$, one can find $\lambda \in \mathbb{C}^*$ such that $\phi' = \lambda \phi \in G'$. The invariant sets $(W^1)^{\phi}$ and $(W^1)^{\phi'}$ have the same homotopy type, by an easy application of Bialynicki-Birula’s well-known decomposition theorem [BB73, Thm. 4.1] to the smooth variety $W^1$. Hence $\chi((W^1)^{\phi}) = \chi((W^1)^{\phi'})$ and so, averaging,

$$\chi((W^1)^{\phi}/C') = \chi((W^1)^{\phi'}/C').$$

On the other hand, if $c$ and $c'$ denote the images in $C$ and $C'$ of $\phi$ and $\phi'$ respectively,

$$|\pi^{-1}([c])| = |\pi'^{-1}([\epsilon'])|,$$

where $\pi$ denotes the projection $\pi: G \to C$ and $\pi_*: \text{Cl}(G) \to \text{Cl}(C)$ the induced map on conjugacy classes, and similarly for the primed symbols.
Hence one can assume formula (4) to be valid for the group $C$. The
variety $(W_1 \times V^2)/G$ is a blow-up of $V/G$ having only cyclic
quotient singularities resulting from the residual action of $C = G/G_\eta$ on $W_1 \times V^2$.

Its orbifold Euler number can be expressed as a sum:

$$\chi_{\text{orb}}(W_1 \times V^2/C) = \sum_{[c] \in \text{Cl}(C)} \chi((W_1 \times V^2)^c/C), \quad \text{since } N_c^C = C$$

$$= \sum_{[c] \in \text{Cl}(C)} \chi((W_1)^c/C), \quad \text{since } (V^2)^c \text{ is contractible.} \quad (7)$$

On the other hand, one has:

$$|\text{Cl}(G)| = \sum_{[c] \in \text{Cl}(C)} |\pi^{-1}([c])|, \quad (9)$$

which agrees with the previous sum term-by-term. Hence, $Y := (W_1 \times V^2)/G \rightarrow (V_1 \times V^2)/G$ is an Euler blow-up with only toric (cyclic) singularities.

Applying the minimal model program (Theorem 1.3) to $Y$, one obtains a crepant terminalisation $t : Z \rightarrow Y$ which satisfies $\chi_{\text{orb}}(Z) = \chi_{\text{orb}}(Y) = \chi_{\text{orb}}(V/G)$, and has only toric singularities. \quad $\square$

Thus in the case where $G$ fixes a line in $V = \mathbb{C}^4$ it suffices to prove
the existence of a $G'$-equivariant Euler resolution $W_1 \rightarrow V_1/G_\eta$ which satisfies equation (4).

Remark 2.2. The same method as above can be used to deal with the
easier case when $n = 2$ and $G < \text{SL}(3)$ and fixes a line in $\mathbb{C}^3$. In 2
dimensions, the minimal model is unique, so there is no need to check
$G'$-stability of the Euler resolution of $V_1/G_\eta$.

2.3. Case where $G_\eta$ doesn’t contain $Z_3$. In this section, Conjecture \[\text{(1.1)}\]
is proved in the case where $G_\eta$ is irreducible and does not contain $Z_3$.

In order to construct a $G$-equivariant Euler resolution $W_1 \rightarrow V_1/G_\eta$, the cases to be considered are first restricted using the following lemma
(which makes use of the classification of small finite sub-groups of $\text{SL}(3)$ — see [YY93], although the notation adopted here is that of [Roa90],
which is slightly different).

Lemma 2.3. Suppose that

$$1 \rightarrow G_\eta \rightarrow G' \xrightarrow{\pi'} C' \rightarrow 1, \quad (10)$$
is an exact sequence of finite groups of SL(3) such that $C'$ is cyclic and non-trivial, and $G_\eta$ is non-abelian and doesn’t contain $Z_3 = \langle \omega_3 \rangle$. Then exactly one of the following is true.

1. $G_\eta$ is of type (B) and $G'$ is of type (B).
2. $G_\eta$ is of type (C), (D), (H) or (I) and $G' = \langle G_\eta, \omega_3 \rangle$.
3. $G_\eta$ is of type (C) and $G'$ is of type (D), with $G' = \langle G_\eta, R \rangle$ or $G' = \langle G_\eta, \omega_3 R \rangle$.

Proof. The groups of type (E), (F), (G), (H$^*$), (I$^*$) all contain $Z_3$, so do not occur as the group $G_\eta$ by assumption. The only finite subgroup of SL(3) containing the simple group (H) (resp. the simple group (I)) as a normal subgroup is (H$^*$) (resp. (I$^*$)) \cite[p.36]{YY93}. Thus for these, the result follows immediately.

If $G_\eta$ has type (B), a simple argument \cite[§1.4, p.18]{YY93} shows that $G'$ must also have type (B).

It remains to deal with the case where $G_\eta$ has type (C) or (D). Throughout the rest of this proof, write

$$T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

for the element which, together with a diagonal group, generates a group of type (C). To get a group of type (D), recall that one must add to a group of type (C) an element of the form

$$\phi = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & c \\ 0 & b & 0 \end{pmatrix} \quad \text{with} \quad abc = -1. \quad (11)$$

Claim 2.4. If $G_\eta$ has type (C) or (D), then $G'$ must also be of type (C) or (D) (though not necessarily the same type as $G_\eta$).

Proof. Denote by $x_1, x_2, x_3$ the standard coordinates on $\mathbb{C}^3$. If $G_\eta$ is of type (C) or (D) and does not contain the centre $Z_3$, then the monomial $x_1x_2x_3$ is invariant under $G_\eta$ up to scale \cite[§1.3]{YY93}. It follows from an easy argument that $G'$ must also leave $\mathbb{C}x_1x_2x_3$ invariant. Thus $G'$ cannot be primitive. This means that $G'$ must be of type (C) or (D). \qed

The next step is to study the normal diagonal subgroups of $G_\eta$ and $G'$, which are denoted by $H_\eta$ and $H'$ respectively.

Notation 2.5. The standard toric notation for diagonal matrices will be used:

$$\frac{1}{d}(r_1, r_2, \ldots, r_n) := \left[ \exp\left( \frac{2\pi i r_1}{d} \right), \exp\left( \frac{2\pi i r_2}{d} \right), \ldots, \exp\left( \frac{2\pi i r_n}{d} \right) \right]$$


Claim 2.6. If $H'$ contains an element of order 3 then that element must be $\omega_3$ or $\omega_2^3$. As a consequence, all the elements of $H_\eta$ have orders prime to 3.

Proof. If $x \in H'$ has order 3 and its does not belong to the centre $Z_3$ then it can be chosen to be of the form $x = \frac{1}{3}(i, i + 1, i + 2)$ for some $i \in \{0, 1, 2\}$. But then

$$(TxT^{-1})x^{-1} = \frac{1}{3}(i + 1, i + 2, i) - \frac{1}{3}(i, i + 1, i + 2)$$

$$= \frac{1}{3}(1, 1, -2)$$

$$= \frac{1}{3}(1, 1, 1) = \omega_3$$

On the other hand, since $x$ normalises $G_\eta$, one has $xT^{-1}x^{-1} \in G_\eta$ and so $\omega_3 = TxT^{-1}x^{-1} \in G_\eta$, which contradicts the hypothesis of the lemma. Thus the only elements of order 3 in $H'$ are $\omega_3$ and $\omega_2^3$. As a consequence, $H_\eta$ has no elements of order 3, since $Z_3 \not< H_\eta$. The claim for $H_\eta$ follows immediately from this. \qed

Claim 2.7. $H' < \langle H_\eta, \omega_3 \rangle$, i.e. $H'$ is either equal to $H_\eta$ or equal to $\langle H_\eta, \omega_3 \rangle$.

Proof. Let $\varphi \in H'$. I begin by showing that $\varphi^3 \in H_\eta$.

Since $H'$ is normal in $G'$, the element $T\varphi T^{-1}$ is diagonal, and so, therefore, is $f := \varphi^{-1}T\varphi T^{-1}$. One has $fT = \varphi^{-1}T \varphi \in G_\eta$, since $G_\eta$ is normal in $G'$, so $f$ belongs to $G_\eta$. Since $f$ is also diagonal, it follows that $f \in H_\eta$.

Writing $\varphi = \frac{1}{d}(a, b, -a - b)$, one has

$$f = \varphi^{-1}(T\varphi T^{-1}) = \frac{1}{d}(-a, -b, a + b) + \frac{1}{d}(b, -a - b, a)$$

$$= \frac{1}{d}(b - a, -a - 2b, 2a + b)$$

and

$$T^{-1}fT = \frac{1}{d}(2a + b, b - a, -a - 2b).$$

Dividing the second element by the first gives

$$T^{-1}fTf^{-1} = \frac{1}{d}(3a, 3b, -3(a + b)) = \varphi^3,$$

so $\varphi^3 \in H_\eta$.

Let $x := \varphi^{-3} \in H_\eta$. By Claim 2.6, the order of $x$ is prime to 3, so there exists an integer $l$ such that $x^l = x$. Writing $\alpha := x^l = (\varphi^{-3l}) \in$
$H_\eta$, one has \((\alpha \varphi)^3 = 1\), and so Claim \ref{claim:2.6} again implies that \(\alpha \varphi = \omega_3\) or \(\omega_3^2\). Thus \(H' = \langle H_\eta, \omega_3 \rangle\). \hfill \Box

Now one can deal with the groups \(G_\eta\) and \(G'\) themselves. To begin with, since the quotient \(C' = G'/G_\eta\) is cyclic, \(G' = \langle G_\eta, \phi \rangle\) for some \(\phi \in G' \setminus G_\eta\). Now any element \(\phi\) in a group of type (C) or (D) has associated to it a permutation \(\sigma(\phi) \in S_3\), defined according to how it permutes the coordinates \(x_1, x_2, x_3\). If \(\sigma(\phi)\) is the identity, then \(\phi\) is diagonal, whereas if \(\sigma(\phi)\) is a permutation of order 3 then \(\phi T\) or \(\phi T^{-1}\) is diagonal. In these cases, since \(T \in G_\eta\), it follows from the claim above that \(G' = \langle G_\eta, \omega_3 \rangle\).

The only remaining possibility is that \(\sigma(\phi)\) equals a transposition or order 2, which can be assumed to be the transposition \((12)\), by multiplying \(\phi\) by a suitable power of \(T\). Thus \(\phi\) is of the form \((11)\).

Claim 2.8. For any \(\phi\) of the form \((11)\), define

\[
\tilde{\phi} := T^{-1} \phi^2 T\phi = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & C \\ 0 & B & 0 \end{pmatrix},
\]

with \(A = -1\), \(B = b^2 c\) and \(C = -B^{-1}\). Suppose that \(Z_3 \not\subset \langle \tilde{\phi}, T \rangle\). Then there exists an element \(t \in \langle T, \tilde{\phi} \rangle\) such that \(t \tilde{\phi} = R\), where

\[
R := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

Proof. Define \(f := \tilde{\phi} T \tilde{\phi}^{-1} T = [-B, -B, B^{-2}]\) and \(f' := f T^{-1} f T = [1, -B^3, -B^{-3}]\). If the order of \(B\) is a multiple of 3, say \(m = 3k\), then \(f^{2k} = \omega_3\) or \(\omega_3^2\), so \(Z_3 \subset \langle \tilde{\phi}, T \rangle\). Thus if \(Z_3 \not\subset \langle \tilde{\phi}, T \rangle\) then the order of \(B\) is prime to three, and a suitable power of \(f'\) gives the required element \(t = [-1, -B^{-1}, -B]\), which satisfies \(t \tilde{\phi} = R\). \hfill \Box

Now the element \(\phi^2\) is diagonal, so belongs to \(\langle H_\eta, \omega_3 \rangle\), by claim \ref{claim:2.6}. Note also that \(\phi = \tilde{\phi}(T \phi^{-2} T^{-1}) \in \tilde{\phi} H'\).

Case 1: \(\phi^2 \in H_\eta\). In this case, \(C'\) has order 2, so \(\omega_3 \notin H'\) — i.e. \(H' = H_\eta\). By Claim \ref{claim:2.8}, \(t \tilde{\phi} = R\) for some element \(t \in \langle T, \tilde{\phi} \rangle < G_\eta\). Thus \(G' = \langle G_\eta, \phi \rangle = \langle G_\eta, \tilde{\phi}, H' \rangle = \langle G_\eta, R \rangle\), since \(H' = H_\eta\).

Case 2: \(\phi^2 \notin H_\eta\). Then \(\phi^2 = \varphi \omega_3^k\) for \(k = 1\) or 2 and \(\varphi \in H_\eta\). Let \(\phi' := \omega_3^k \phi\). Then \((\phi')^2 = \varphi^2 \in H_\eta\), so by the preceding case, one may assume \(t \phi' = R\) for some \(t \in \langle T, \tilde{\phi} \rangle < G_\eta\). Now \(\tilde{\phi} = \tilde{\phi}'\), so \(G' = \langle G_\eta, \phi \rangle = \langle G_\eta, \tilde{\phi}, H' \rangle = \langle G_\eta, R t^{-1}, H' \rangle = \langle G_\eta, R, \omega_3 \rangle\).

This completes the proof of the lemma. \hfill \Box
Proposition 2.9. Conjecture \([\text{4}]\) is true when \(G_\eta\) is irreducible and doesn’t contain \(Z_3\).

Proof. Recall from the proof of Prop. \([2.1]\), that \(\mathbb{C}^*\) acts on any Euler resolution. Thus, in all cases where \(G' = \langle G_\eta, \omega_3 \rangle\), any smooth crepant resolution of \(V^1/G_\eta\) admits a \(G'\)-action — indeed a \(G\)-action, as remarked in the proof of Prop. \([2.1]\), since \(G < \langle G', \mathbb{C}^* \rangle\).

Hence, Lemma \([2.3]\) implies that it suffices to deal with the cases where \(G = \langle G_\eta, \omega_3 \rangle\) or \(G_\eta = \langle H_\eta, T \rangle\) is of type (C) with \(G'\) of type (D), either equal to \(\langle G_\eta, R \rangle\) or equal to \(\langle G_\eta, R, \omega_3 \rangle\).

In these cases (see \([\text{Roa90}]\)) an Euler resolution of \(\mathbb{C}^3/G_\eta\) is obtained by taking a toric Euler resolution of \(\widetilde{\mathbb{C}^3} \rightarrow \mathbb{C}^3/H_\eta\) which is \(T\)-stable and then resolving the singularities of \(\mathbb{C}^3/H_\eta/(T)\). The existence of a \(T\)-stable toric Euler resolution follows from the fact that \(H_\eta\) is normal in \(G_\eta\). However, \(H_\eta\) is also normal in \(G'\), so \(\mathbb{C}^3/H_\eta\) can also be chosen to be \(R\)-stable. The singularities of \(\mathbb{C}^3/H_\eta/(T)\) are fixed points of \(R\), so resolving them gives the desired \(G'\)-invariant resolution of \(\mathbb{C}^3/G_\eta\).

The proof that \(\chi((W^1)^{G'}/C') = |\pi'^{-1}([c'])|\) is done by treating case by case the three possibilities for \(G'\) given by Lemma \([2.3]\).

Case 1: \(G' = \langle G_\eta, \omega_3 \rangle\). In this case, since \(\omega_3 \in \mathbb{C}^*\), the same argument as the second paragraph of the proof of Proposition \([2.1]\) gives \(\chi((W^1)^{\omega_3}) = \chi(W^1) = |\text{Cl}(G_\eta)|\). On the other hand, \(G'\) is just a direct product of \(G_\eta\) and \(Z_3\), so \(\pi'_s\) is everywhere \(\chi(G_\eta) : 1\).

Case 2: \(G' = \langle G_\eta, R \rangle\). If we denote by \(H_\eta\) and \(H'\) the normal diagonal subgroups of \(G_\eta\) and \(G'\) respectively, then they are equal. Since they do not contain \(\omega_3\), their order is \(d^2\) (for some \(d\) prime to 3) and they are a semi-direct factor in \(G'\) \([\text{Roa94}, \text{Lemma 10}]\). The inverse image of the trivial class in \(C'\) is the number of \(G'\)-conjugate elements in \(G_\eta\). Since \(G_\eta\) is normal, this is the same as the number of \(G_\eta\)-conjugate elements in \(G_\eta\), i.e. equal to \(\text{Cl}(G_\eta) = \chi(W^1)\). For the non-trivial class \([R], \text{[Roa94], Formula (32)]\) implies that \(\chi((W^1)^{R}/(R)) = d\) and the proof of \([\text{Roa94}, \text{Lemma 10}]\) again gives \(\pi'^{-1}([R]) = |Z_R \cap H'| = d\).

Case 3: \(G' = \langle G_\eta, \omega_3 R \rangle\). As we remarked in Case 1, the Euler number does not depend on scalar factors, so \(\chi((W^1)^{\omega_3 R}) = \chi((W^1)^{R})\). On the other hand, the discussion in the second paragraph of this proof implies that \(|\pi'^{-1}([\omega_3 R])| = |\pi^{-1}([R])|\). The result thus follows from Case 2. \(\Box\)

Hence, if \(G_\eta\) is irreducible and does not contain \(Z_3\), \(V/G\) has an Euler terminalisation with only toric singularities.
Question 2. The proof of Propositions 2.9 depends on the classification of SL(3) groups. Is it possible to find a proof which doesn’t depend on the classification?

2.4. Case where $G_\eta$ contains $Z_3$. Unfortunately, the author was not able to prove the corresponding result to Conjecture 1 in the case where $G_\eta$ is irreducible but contains $Z_3$. A method is suggested in Section 4.5, but requires further work.

If the conjecture can be proved in all cases, the work above implies that $\text{Term}(\mathbb{C}^4/G)$ is true for all groups of type $(3, 1)$ and $(2, 1, 1)$. The results of the next section imply that in that case, $\text{Term}(\mathbb{C}^4/G)$ is true for all irreducible $G$ which contain $Z_4$.

3. Blowing up in the presence of the centre $Z_n$

3.1. Invariant sets in projective space. First, a lemma about the Euler number of a invariant sets in projective space.

Lemma 3.1. Let $H$ be a finite abelian group acting linearly on $\mathbb{P}^n$. Then

$$\chi((\mathbb{P}^n)^H) = \chi(\mathbb{P}^n) = n + 1.$$  

Proof. Suppose $H$ has order $r$. Diagonalise the action of $H$, and order the weights of the action so that they form a non-decreasing sequence of elements of $\{0, \ldots, r - 1\}$. The sequence will consist of $d_1$ occurrences of the smallest weight $w_1$, followed by $d_2$ occurrences of the second smallest weight $w_2$, and so on, ending with $d_s$ occurrences of the greatest weight $w_s$. Since there are $n + 1$ (not necessarily distinct) weights in the sequence, the multiplicities $d_i$ sum to $n + 1$.

Computing the invariant part of $\mathbb{P}^n$ with respect to the $H$-action, one sees that it consists of a disjoint union over all $i \in \{1, \ldots, s\}$ of projective spaces $\mathbb{P}^{d_i - 1}$. Taking the sum of the Euler numbers of the invariant components, and using the fact that the Euler number of $\mathbb{P}^d$ is $d + 1$, one obtains the value $\sum d_i$, which by the previous paragraph indeed coincides with the Euler number $n + 1$ of $\mathbb{P}^n$. \hfill \Box

3.2. Blowing up the origin. Now let $V = \mathbb{C}^n$ and let $\text{Bl}_0 V$ be the blow-up of $V$ at the origin. This has a natural $G$-action and one has the following commutative diagram

$$\begin{array}{ccc}
\text{Bl}_0 V & \xrightarrow{\alpha_0} & V \\
\downarrow & & \downarrow \\
\text{Bl}_0 V/G & \xrightarrow{\alpha'_0} & V/G.
\end{array}$$

A standard discrepancy calculation yields the following result.
**Lemma 3.2.** The morphism $\text{Bl}_0 V/G \to V/G$ is crepant if and only if $G$ contains $Z_n = \langle \omega_n \rangle$.

The following lemma contains the basic idea to constructing Euler blow-ups.

**Lemma 3.3.** Assume that $G$ contains $Z_n$ and write $\bar{G} := G/Z_n$. Then $\bar{V} := \text{Bl}_0 V/Z_n$ is smooth, and $\bar{V}/\bar{G} \to V/G$ is a projective Euler blow-up.

**Proof.** The only place where singularities of $\bar{V}$ could arise is on the image $\bar{E} = E/Z_n$ of the exceptional divisor $E$ of $\sigma_0$. Identifying $E$ with $\mathbb{P}(V)$, a local chart for $\text{Bl}_0 V$ at a point $\xi \in E$ is given in suitable local coordinates by

$$(x_1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}),$$

so therefore $\omega_n$ acts there as $(\omega_n, 1, 1, 1)$, i.e. as a pseudo-reflection.

The Euler number computation goes as follows:

$$\chi_{\text{orb}}(\bar{V}/\bar{G}) = \sum_{[\bar{g}] \in \text{Cl}(\bar{G})} \chi(\mathbb{P}(V)^{\bar{g}}/N_{\bar{G}}),$$

since $\bar{V} \sim E \cong \mathbb{P}(V)$ (homotopy)

$$= \sum_{[\bar{g}] \in \text{Cl}(\bar{G})} \chi(\mathbb{P}(V)),$$

averaging and using Lemma 3.1,

$$= |\text{Cl}(\bar{G})| n = |\text{Cl}(G)|$$

$$= \chi_{\text{orb}}(V/G).$$

Thus if $G$ contains $Z_n$ then $\text{Term}(V/G) \equiv \text{Term}(\bar{V}/\bar{G})$.

### 3.3. Patching

In many cases, an Euler blow-up is constructed by patching together local Euler blow-ups. The following lemma summarises the necessary conditions to carry this out this procedure.

**Lemma 3.4 (Patching Lemma).** Let $G$ be a finite subgroup of $\text{SL}(n)$ which contains $Z_n$, so that $Y := V/G \to V/G$ is an Euler blow-up. Let $\bar{E} := E/Z_n$, where $E$ is the exceptional divisor of $\text{Bl}_0 V \to V$ and denote by $p: \bar{V} \to \bar{E}$ the projection.

There exists a finite collection of points $y \in \bar{E}/\bar{G}$ and corresponding analytic neighbourhoods $E_{\xi} \subset \bar{E}/\bar{G}$ such that $Y$ is covered by $\{Y_{y}\}$, where $Y_y := p^{-1}(E_y)$. 

Suppose that for each \(y\), there exists an Euler blow-up \(\varphi_y: X_y \to Y_y\), such that if \(y \neq y'\), one nevertheless has
\[
X_{y|Y_y \cap Y_{y'}} = X_{y'|Y_y \cap Y_{y'}}. \tag{12}
\]
Then the analytic canonical Gorenstein orbifold \(X\) obtained by gluing together all the \(X_y\) is an Euler blow-up of \(Y\).

**Proof.** The existence of the finite covering \(\{\overline{E}_y\}\) follows because \(\overline{E}\) is compact. Since, all the \(X_y\) are birational to each other above the overlaps, equation (12) implies that \(X\) is well-defined. Furthermore, since no crepant divisors are introduced during the local blow-ups and since the orbifold Euler numbers of \(X_y\) and \(Y_y\) are the same, \(X\) is crepant over \(Y\) and has the same orbifold Euler number. \(\square\)

### 4. Irreducible \(G\) which contain \(Z_4\)

#### 4.1. Notation.
When \(G\) contains \(Z_4\), \(\overline{V}/\overline{G} \to V/G\) is an Euler blow-up. Let \(\overline{E}\) be the exceptional divisor of \(\overline{V} \to V/Z_4\), and let \(p: \overline{V} \to \overline{E}\) be the projection. Let \(\xi \in \overline{E}\) be a point in the base, and consider the tangent space of \(\overline{V}\) at \(\xi\). This decomposes into \(\overline{G}\)-modules
\[
V^1_{\xi} \oplus V^2_{\xi},
\]
where \(V^1_{\xi}\) is the tangent space to \(\overline{E}\) and \(V^2_{\xi}\) is the line tangent to the fibre of \(p\), and stabilised by \(\overline{G}\). Let \(\xi' \in V^2_{\xi}\) be the generic point, so that its stabiliser \(\overline{G}_{\xi'}\) is a subgroup of \(\text{SL}(3)\).

#### 4.2. Local Blow-ups.
Let \(\xi \in \overline{V}\) and let \(\xi\) denote its image in \(Y = \overline{V}/\overline{G}\). A local analytic neighbourhood of \(\xi\) is isomorphic to
\[
Y_\xi := (V^1_{\xi} \oplus V^2_{\xi})/\overline{G}_\xi = (V^1_{\xi}/\overline{G}_{\xi'}) \oplus V^2_{\xi}/C_\xi,
\]
where \(C_\xi\) denotes the quotient \(C_\xi := \overline{G}_\xi/\overline{G}_{\xi'}\), which is cyclic, being a naturally a subgroup of \(\text{GL}(V^2_{\xi}) \cong \mathbb{C}^*\).

Restricting attention to the “base” \(\overline{E}\), one has, corresponding to each \(\xi\) in \(\overline{E}\), a quotient singularity \(\overline{E}_\xi = V^1_{\xi}/\overline{G}_{\xi'}\) which is an \(\text{SL}(3)\)-singularity. Since \(\overline{E}/\overline{G}\) is compact, the choice of a finite number of points \(\xi\) is sufficient for \(\bigcup_\xi Y_\xi\) to cover the whole of \(\overline{V}/\overline{G}\).

#### 4.3. Gluing.

**Lemma 4.1.** Let \(W^1_{\xi} \to V^1_{\xi}/\overline{G}_{\xi'}\), be the \(\overline{G}_\xi\)-equivariant Euler resolution such as that in Conjecture [4]. Define
\[
X_\xi := (W^1_{\xi} \times V^2_{\xi})/\overline{G}_\xi.
\]
Then \(\varphi_\xi: X_\xi \to Y_\xi\) are Euler blowups which have only cyclic quotient singularities, and for each \(\xi, \xi'\), \(X_\xi\) and \(X_{\xi'}\) agree above the inverse
image of $Y_\xi \cap Y_\xi'$. Thus by Lemma 3.4, they glue to form a complex analytic Euler blow-up $\tilde{Y} \to \tilde{V}/\tilde{G}$ which has only cyclic quotient singularities.

Proof. One must check that $\chi_{\text{orb}}(X_\xi) = \chi_{\text{orb}}(Y_\xi)$ and that, on $\varphi_\xi^{-1}(Y_\xi \cap Y_\xi')$ the blow-ups corresponding to $\xi$ and $\xi'$ agree. The orbifold Euler number equality is checked in Proposition 2.1. For the agreement of the blow-ups on the overlaps, knowledge of the $SL(3)$ singularities implies that these overlaps only occur over curves of 2-dimensional singularities (the components of the non-isolated singularities are all curves for $SL(3)$). Over these, the resolutions which are being glued-in are trivial families of minimal resolutions: they are therefore unique, and so resolutions coming from neighbourhoods corresponding to different $\xi$'s will agree.

This gives a crepant analytic blow-up $\tilde{Y}$ which is locally analytically isomorphic to a cyclic quotient (and hence locally analytically $\mathbb{Q}$-factorial).

4.4. Terminalisation and the Orbifold Euler Number. Since the orbifold Euler number can be calculated by summing the contributions of the various analytic neighbourhoods, the equality

$$\chi_{\text{orb}}(X) = \chi_{\text{orb}}(\tilde{V}/\tilde{G})$$

will follow by showing that the resolutions glued in above preserve the orbifold Euler number. This is proved in Proposition 2.9. Thus $X$ is an Euler blow-up with only cyclic singularities.

Applying the minimal model program to $X$ (Theorem 1.3), one obtains a crepant terminalisation $t: Z \to X$ which satisfies $\chi_{\text{orb}}(Z) = \chi_{\text{orb}}(X) = \chi_{\text{orb}}(V/G)$, and has only toric singularities.

Remark 4.2. By studying which toric flips can occur, one might be able to prove that the singularities of $T$ are in fact at most cyclic. They would then have to necessarily be isolated. For if a 4–dimensional Gorenstein cyclic singularity consisted of a curve of singularities, these would also have to be (3-dimensional) terminal Gorenstein cyclic quotients. But the classification of 3-dimensional terminal cyclic quotients [MS84] shows that they are all of the form $\frac{1}{r}(1, -1, a)$, and so can only be Gorenstein if they are smooth.

4.5. Case where $G$ is of type $(3, 1)$ revisited. A method similar to the one above can be applied to the case treated in Section 2.4, namely the case where $G$ is a group of type $(3, 1)$ and the stabiliser group $G_\eta < SL(3)$ is irreducible and contains $Z_3$. This goes some way towards a solution of Conjecture 1.
Since \( G_{\eta} > Z_3 \), there exists an Euler blow-up \( \bar{V}^1/\bar{G}_{\eta} \rightarrow V^1/G_{\eta} \) and this is equivariant under the \( G \) (and hence \( G' \)) action, since it is obtained from blowing up the origin of \( V^1 \), which is of course fixed by \( G \). Its singularities are of the type \((2,1)\) and \((1,1,1)\) and the singular locus is invariant under \( G \).

An analytic resolution of \( \bar{V}^1/\bar{G}_{\eta} \) can be constructed by the same gluing procedure as in Lemma 4.1: in any local analytic neighbourhood of \( \bar{V}^1/\bar{G}_{\eta} \), construct an Euler resolution, doing this equivariantly under the \( G' \)-action. These glue together, since they can only intersect over smooth points. This gives a resolution \( W^1 \rightarrow \bar{V}^1/\bar{G}_{\eta} \).

It remains to show that it admits a \( \bar{G} \)-action and that it satisfies the Euler number property of equation (4).

5. Groups \( G \) of type \((2,2)\) and \((2,1,1)\)

5.1. Groups of type \((2,2)\). Let \( \mathbb{C}^4 = V^1 \oplus V^2 \) with \( V^i \) 2-dimensional irreducible \( G \)-modules and denote by \( \eta_i \) the generic point of \( V^i \). If \( G < SL(2) \times SL(2) \) then the following lemma constrains the stabilisers of \( \eta_i \).

**Lemma 5.1.** Suppose that \( V = V^1 \oplus V^2 \) and that \( G < SL(V^1) \times SL(V^2) < SL(V) \). Denote by \( G_i \) the stabiliser of the generic point \( \eta_i \in V^i \) for \( i = 1, 2 \). If both stabilisers \( G_1 \) and \( G_2 \) are trivial, then \( V/G \) must be terminal.

**Proof.** This can be proved by an easy discrepancy calculation, or equivalently, by using the concept of “weights” for the group action \([IR9]\) as follows. The number of crepant divisors of \( V/G \) is equal to the “number of elements of \( G(1) \) of weight one”. Here, \( G(1) := \text{Hom}(\mu_r, G) \), where \( r \) is the least common multiple of the orders of the elements of \( G \). The **weight** \( \text{wt}(\hat{g}) \) of \( \hat{g} \in G(1) \) is defined by evaluating \( \hat{g} \) on a primitive generator \( \epsilon \) of \( \mu_r \), diagonalising the resulting matrix \( \hat{g}(\epsilon) \) and expressing the diagonal elements in terms of powers of \( \epsilon \) ranging between 0 and \( r - 1 \). Because \( G < SL(V) \), the sum of these powers divided by \( r \) is a non-negative integer, called the **weight** of \( \hat{g} \). Note that if we simply want to calculate the **number** of elements of \( G(1) \) of a given weight, we can identify \( G \) with \( G(1) \) by fixing a primitive generator of \( \mu_r \), and pretend to be calculating the weights of the elements of \( G \).

Suppose \( V/G \) is not terminal, so that there exists and element \( g \in G \) of weight one. For each \( i = 1, 2 \), denote by \( g_i \) the part of the matrix of \( g \) which represents its action on the module \( V^i \). Since \( \text{wt}(g) = \text{wt}(g_1) + \text{wt}(g_2) \), one of the \( g_j \)’s must be equal to the identity (whereas the other one must be a non-trivial matrix). But \( G_1 = 1 \) and \( g_j = 1 \)
for some \( j \) would imply that \( g = 1 \), so one of the two stabilisers must be non-trivial.

Thus, \( G < \text{SL}(2) \times \text{SL}(2) \) then \( V/G \) not terminal implies (Lemma 5.1) that one of \( G_{\eta_{1}} \) must be non-trivial, say \( G_{\eta_{1}} = G_{\eta_{2}} \neq 1 \). Denoting by \( W^{1} \to V^{1}/G_{\eta_{1}} \) the minimal resolution, one obtains a crepant blow-up \( (W^{1} \times V^{2})/(G/G_{\eta_{1}}) \to V/G \) with singularities of type \( (2, 1, 1) \) or \( (1, 1, 1, 1) \).

In order to prove that the orbifold Euler number remains unchanged under the blowup \( (W^{1} \times V^{2})/G \to V/G \), one must prove a formula similar to that of equation (4), namely:

\[
\chi((W^{1})^{d}/Z_{d}) = |\pi_{3}^{-1}([d])|,
\]

for all \( d \in D := G/G_{\eta} < \text{GL}(V^{1}) \).

This could presumably be achieved by examining the finite \( \text{SL}(2) \) and \( \text{GL}(2) \) subgroups and determining how many fit into an exact sequence of the form given in equation (10), with the cyclic group \( C \) replaced by the \( \text{GL}(2) \) subgroup \( D \).

**Question 3.** What happens if \( G \) is not a subgroup of \( \text{SL}(2) \times \text{SL}(2) \)?

### 5.2. Groups of type \((2, 1, 1)\)

Suppose the irreducible decomposition of \( \mathbb{C}^{4} \) is \( V^{1} \oplus V^{2} \oplus V^{3} \) with \( \dim V^{1} = 2 \) and \( \dim V^{i} = 2 \) for \( i = 2, 3 \). Denote by \( \eta_{i} \) the generic point of \( V^{i} \), by \( G_{3} \) the stabiliser of \( \eta_{3} \) in \( G \) and by \( G_{32} \) the stabiliser of \( \eta_{2} \) in \( G_{3} \). Then we have exact sequences

\[
1 \to G_{3} \to G \xrightarrow{\pi_{3}} C \to 1,
\]

and

\[
1 \to G_{32} \to G_{3} \xrightarrow{\pi_{32}} D \to 1,
\]

with \( C \) and \( D \) cyclic subgroups, \( G_{32} < \text{SL}(2) \) and \( G_{3} < \text{SL}(3) \).

The analytic germ of \( V/G \) is isomorphic to

\[
(V^{1} \times V^{2} \times V^{3})/G \cong ((V^{1}/G_{32} \times V^{2})/D \times V^{3})/C.
\]

The term \( V^{1}/G_{32} \) has a minimal resolution \( Z^{1} \) which is unique and therefore admits an action of \( G_{3} \). The germ \( (Z^{1} \times V^{2})/D \) is a cyclic \( \text{SL}(3) \) singularity and has an Euler resolution \( W^{1} \). It remains to be shown that \( W^{1} \) admits a \( G \)-action and that \( \chi_{\text{orb}}((W^{1} \times V^{3})/C) = \chi_{\text{orb}}(V/G) \). As in Section 4, the later statement would follow from the equalities

\[
\chi((W^{1})^{c'}/C') = |\pi_{3}^{c'-1}([c'])|,
\]

for \( c' \in C' \), where the primed objects are defined similarly to those in Section 4.
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