On the Approximate Eigenstructure of Time–Varying Channels

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Abstract

In this article we consider the approximate description of doubly–dispersive channels by its symbol. We focus on channel operators with compactly supported spreading, which are widely used to represent fast fading multipath communication channels. The concept of approximate eigenstructure is introduced, which measures the accuracy $E_p$ of the approximation of the channel operation as a pure multiplication in a given $L_p$–norm. Two variants of such an approximate Weyl symbol calculus are studied, which have important applications in several models for time–varying mobile channels. Typically, such channels have random spreading functions (inverse Weyl transform) defined on a common support $U$ of finite non–zero size such that approximate eigenstructure has to be measured with respect to certain norms of the spreading process. We derive several explicit relations to the size $|U|$ of the support. We show that the characterization of the ratio of $E_p$ to some $L_q$–norm of the spreading function is related to weighted norms of ambiguity and Wigner functions. We present the connection to localization operators and give new bounds on the ability of localization of ambiguity functions and Wigner functions in $U$. Our analysis generalizes and improves recent results for the case $p = 2$ and $q = 1$.

Index Terms

Doubly–dispersive channels, time–varying channels, Weyl calculus, Wigner function, ambiguity function

I. INTRODUCTION

Optimal signaling through linear time–varying (LTV) channels is a challenging task for future communication systems. For a particular realization of the time–varying channel
operator the transmitter and receiver design, which avoids interference is related to "eigen-signaling". Eigen–signaling simplifies much of the information theoretic treatment of communication in dispersive channels. However, it is well known that for an ensemble of channels, which are dispersive in time and frequency such a joint diagonalization can not be achieved because the eigen–decompositions can differ from one to another channel realization. Several approaches like for example the "basis expansion model" (BEM) [1] and the canonical channel representation [2] are proposed to describe eigen–signaling in some approximate sense. Then a necessary prerequisite is the characterization of remaining approximation errors.

A typical scenario commonly encountered in wireless communication, is signaling through a random time-varying and frequency selective (doubly–dispersive) channel, which in general is represented by a pseudo-differential operator \( H \). The abstract random channel operating on an input signal \( s : \mathbb{R} \rightarrow \mathbb{C} \) can be expressed (at least in the weak sense) in the form of a random kernel, symbol or spreading function. The signal \( r : \mathbb{R} \rightarrow \mathbb{C} \) at the time instant \( t \) at the output of the time–varying channel is then:

\[
r(t) = (Hs)(t)
\]

It is a common assumption that knowledge of \( H \) at the receiver can be obtained up to certain accuracy by channel estimation, which will allow for coherent detection. However, channel knowledge at the transmitter simplifies equalization and detection complexity at the receiver and can increase the link performance. It can be used to perform a diagonalizing operation (i.e. eigen–signaling) and allocation of resources in this domain (e.g. power allocation). We shall call the first part of this description from now on as the eigenstructure of \( H \). Signaling through classes of channels having common eigenstructure could be, in principle, interference–free and would allow for simple information recovering algorithms based on the received signal \( r(t) \). However, for \( H \) being random, random eigenstructure has to be expected in general such that the design of the transmitter and the receiver has to be performed jointly for ensembles of channels having different eigenstructures. Nevertheless, interference then can not be avoided in the communication chain. For such interference scenarios it is important to have bounds on the distortion of a particular selected signaling scheme. Refer for example to [3] for a recent application in information theory.

Initial results in this field can be found in the literature on pseudo-differential operators [4], [5] where the overall operator was split up into a main part to be studied and a ”small” operator to be controlled. More recent results with direct application to time–varying channels...
were obtained by Kozek [6], [7] and Matz [8] which resemble the notion of underspread channels. They investigated the approximate symbol calculus of pseudo-differential operators in this context and derived bounds for the $L_2$–norm of the distortion which follow from the approximate product rule in terms of Weyl symbols. We will present more details on this approach in Section III-C. Controlling this approximation intimately scales with the "size" of the spreading of the contributing channel operators. For operators with compactly supported spreading such a scale is $|U|$ – the size of the spreading support $U$. Interestingly this approximation behavior breaks down in their framework at a certain critical size. Channels below this critical size are called according to their terminology underspread and otherwise overspread. However, we found that previous bounds can be improved and generalized in several directions by considering the problem of approximate eigenstructure from another perspective, namely investigating directly the $L_p$–norm $E_p$ of the error $\mathcal{H}s – \lambda r$ for well known choices of $\lambda$. We shall focus on the case where $\lambda$ is the symbol of the operator $\mathcal{H}$ and on the important case where $\lambda$ is the orthogonal distortion which can be understood as the $L_2$–minimizer. We believe that extensions to $p \neq 2$ are important when further statistical properties of the spreading process of the random channel operator are at hand\(^1\). Our approach will also show the connection to well known fidelity and localization criteria related to pulse design [6], [9], [10]. In particular, the latter is also related to the notion of localization operators [11]. The underspread property of doubly–dispersive channels occurs also in the context of channel measurement and identification [12]. In addition refer to the following recent articles [13], [14] for rigorous treatments of channel identification based on Gabor (Weyl–Heisenberg) frame theory. The authors connect the critical time–frequency sampling density immanent in this theory to the stability of the channel measurement. A relation between these different notions of underspreadness has to be expected but is beyond the scope of this paper.

The paper is organized as follows: In Section II we shall give an introduction into the basics from time–frequency analysis including the Weyl correspondence and the spreading representation of doubly–dispersive channels. In Section III of the paper we shall consider the problem of approximate eigenstructure for operators with spreading functions, which are supported on a common set $U$ in the time–frequency plane having non–zero and finite Lebesgue measure $|U|$. We present the approach for $E_2$ followed by a summary of the main

\(^1\)We provide further motivation and arguments in Remark 2 at the end of the paper.
results of our analysis on $E_p$. The detailed analysis for $E_p$ will be presented in Section IV. Finally, Section V contains a numerical verification of our results.

A. Notation and Some Definitions

We present certain notation and definitions that shall be used through the paper. For $1 \leq p < \infty$ and functions $f : \mathbb{R}^n \to \mathbb{C}$ the functionals $\|f\|_p := (\int |f(t)|^p dt)^{1/p}$ are then usual notion of $p$–norms ($dt$ is the Lebesgue measure on $\mathbb{R}^n$). Furthermore for $p = \infty$ is $\|f\|_\infty := \text{ess sup} |f(t)|$. If $\|f\|_p$ is finite $f$ is said to be in $L^p(\mathbb{R}^n)$. The inner product $\langle \cdot, \cdot \rangle$ on the Hilbert space $L^2(\mathbb{R}^n)$ is given as $\langle x, y \rangle := \int_{\mathbb{R}^n} \overline{x}(t)y(t)dt$ where $\overline{x}(t)$ denotes complex conjugate of $x(t)$. A particular dense subset of $L^p(\mathbb{R}^n)$ is the class of Schwartz functions $S(\mathbb{R}^n)$ (infinite differentiable rapidly decreasing functions). The notation $p'$ denotes always the dual index of $p$, i.e. $1/p + 1/p' = 1$ with $p' = \infty$ if $p = 1$ (and the reverse).

II. Time–Frequency Analysis

A. Phase Space Displacements and Ambiguity Functions

Several physical properties of time–varying channels (like delay and Doppler spread) are in general related to a time–frequency view on operators $\mathcal{H}$. Time-frequency representations itself are important tools in signal analysis, physics and many other scientific areas. Among them are the Woodward cross ambiguity function [15] and the Wigner distribution. Ambiguity functions can be understood as inner product representations of time–frequency shift operators. More generally, a displacement (or shift) operator for functions $f : \mathbb{R}^n \to \mathbb{C}$ can be defined as:

$$ (S_\mu f)(x) := e^{i2\pi \mu_2 \cdot x}f(x - \mu_1) $$

(1)

where $\mu = (\mu_1, \mu_2) \in \mathbb{R}^{2n}$ and $\mu_1, \mu_2 \in \mathbb{R}^n$. In general $\mathbb{R}^{2n}$ is called phase space. Later on we shall focus on $n = 1$, where we have that the functions $f$ are signals in time and $\mu$ is a displacement in time and frequency. Then the phase space is also called time–frequency plane and the operators $S_\mu$ are time–frequency shift operators. There is an ambiguity as to which displacement should be performed first where (1) corresponds to the separation $S_\mu = S_{(0, \mu_2)}S_{(\mu_1, 0)}$. However, it is well known that a generalized view can be achieved by considering so–called $\alpha$–generalized displacements:

$$ S^{(\alpha)}_\mu := S_{(0, \mu_2(1/2 + \alpha))}S_{(\mu_1, 0)}S_{(0, \mu_2(1/2 - \alpha))} = e^{-i2\pi(1/2 - \alpha)\zeta(\mu, \mu)}S_\mu $$

(2)
where $\zeta(\mu, \nu) = \mu_1 \cdot \nu_2$ (inner product on $\mathbb{R}^n$) and then set $S_\mu = S_\mu^{(1/2)}$. Usually $\alpha$ is called *polarization*. The operators in (2) act isometrically on all $L_p(\mathbb{R}^n)$, hence are unitary on $L_2(\mathbb{R}^n)$. Furthermore, they establish\(^2\) unitary representations (Schrödinger representation) of the Weyl–Heisenberg group on $L_2(\mathbb{R}^n)$ (see for example [5]). In physics it is common to choose the most symmetric case $\alpha = 0$ and the operators are usually called Weyl operators or Glauber displacement operators. If we define the symplectic form as $\eta(\mu, \nu) := \zeta(\mu, \nu) - \zeta(\nu, \mu)$, we have the following well known *Weyl commutation relation*:

$$S_\mu^{(\alpha)} S_\nu^{(\beta)} = e^{-i2\pi\eta(\mu, \nu)} S_\nu^{(\beta)} S_\mu^{(\alpha)}$$  \hspace{1cm} (3)

for arbitrary polarizations $\alpha$ and $\beta$. In this way a generalized (cross) ambiguity function can be defined as:

$$A_{g\gamma}^{(\alpha)}(\mu) \overset{\text{def}}{=} \langle g, S_\mu^{(\alpha)} \gamma \rangle = \int_{\mathbb{R}^n} \bar{g}(x + (\frac{1}{2} - \alpha) \mu_1) \gamma(x - (\frac{1}{2} + \alpha) \mu_1) e^{i2\pi\mu_2 \cdot x} dx$$  \hspace{1cm} (4)

The function $A_{g\gamma}^{(1/2)}$ is also known as the *Short–time Fourier transform* (sometimes also windowed Fourier transform or Fourier–Wigner transform) of $g$ with respect to a window $\gamma$. This function is continuous for $g \in S(\mathbb{R}^n)$ and $\gamma \in S'(\mathbb{R}^n)$ (the dual of $S(\mathbb{R}^n)$, i.e. the tempered distributions). Well known relations of these functions, which follow directly from definition (4) are:

$$|A_{g\gamma}^{(\alpha)}(\mu)| = |\langle g, S_\mu^{(\alpha)} \gamma \rangle| \leq \|g\|_2 \|\gamma\|_2 = \|A_{g\gamma}^{(\alpha)}\|_2$$  \hspace{1cm} (5)

where the right hand side (rhs) is sometimes also called the radar uncertainty principle. For particular weight functions $m : \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$ the weighted $p$–norms $\|A_{g\gamma}^{(\alpha)}\|_{M^p m}$ are also called the modulation norms $\|\gamma\|_{M^p m}$ of $\gamma$ with respect to Schwartz function $g \in S(\mathbb{R}^n)$ ($M^p m$ is then corresponding modulation space [16]). Let the symplectic Fourier transform $\mathcal{F}_s F$ of a function $F : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ be defined as:

$$(\mathcal{F}_s F)(\mu) = \int_{\mathbb{R}^{2n}} e^{-i2\pi\eta(\mu, \nu)} F(\nu) d\nu$$  \hspace{1cm} (6)

The symplectic Fourier transform of the (cross) ambiguity function $\mathcal{F}_s A_{g\gamma}^{(\alpha)}$ is called the (cross) Wigner distribution of $g$ and $\gamma$ in polarization $\alpha$.

\(^2\)up to unitary equivalence
B. Weyl Correspondence and Spreading Representation

The operational meaning of pseudo-differential operators can be stated with a (distributional) kernel, coordinate-based in the form of infinite matrices or in some algebraic manner (see for example [17, Chapter 14]). The kernel based description is usually written in a form like:

\[
(\mathcal{H}\gamma)(t) = \int_{\mathbb{R}^n} h(t, t')\gamma(t')dt'
\]  

with a kernel \( h : \mathbb{R}^{2n} \to \mathbb{C} \) (for two Schwartz functions \( \gamma, g \in S(\mathbb{R}^n) \) the kernel \( h \) exists even as a tempered distribution, i.e. Schwartz kernel theorem states \( h \in S'(\mathbb{R}^{2n}) \) with \( \langle g, \mathcal{H}\gamma \rangle = \langle h, \bar{g} \otimes \gamma \rangle \), see for example [17, Thm. 14.3.4]). However, the abstract description of \( \mathcal{H} \) as superpositions of time–frequency shifts is important and quite close to the physical modeling of time–varying channels. We will adopt this time-frequency framework to describe the channel operators. Let us denote with \( \mathcal{T}_\infty \) the set of compact operators, i.e. for \( X \in \mathcal{T}_\infty \) holds \( X = \sum_k s_k \langle x_k, \cdot \rangle y_k \) with singular values \( \{s_k\} \) and two orthonormal bases (singular functions) \( \{x_k\} \) and \( \{y_k\} \). For \( p \leq \infty \) the \( p \)th Schatten class is the set of operators \( \mathcal{T}_p := \{ X \mid \|X\|_p := \text{Tr}((X^*X)^{p/2}) = \sum_k |s_k|^p < \infty \} \) where \( \text{Tr}(\cdot) \) is the usual meaning of the trace (e.g. evaluated in a particular basis). Then \( \mathcal{T}_p \) for \( 1 \leq p < \infty \) are Banach spaces and \( \mathcal{T}_1 \subset \mathcal{T}_p \subset \mathcal{T}_\infty \) (see for example [18]). The sets \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are called trace class and Hilbert–Schmidt operators. Hilbert–Schmidt operators form itself a Hilbert space with inner product \( \langle Y, X \rangle_{\mathcal{T}_2} := \text{Tr}(Y^*X) \). For \( X \in \mathcal{T}_1 \) it holds by properties of the trace that \( \|\langle Y, X \rangle_{\mathcal{T}_2}\| \leq \|X\|_1 \|Y\| \), where \( \|\cdot\| \) denotes the operator norm. Hence for \( Y = S_\mu^{(\alpha)} \) given by (2) one can define analogously to the ordinary Fourier transform [19], [20] a mapping \( \mathcal{T}_1 \to L_2(\mathbb{R}^{2n}) \) via:

\[
\Sigma_X^{(\alpha)}(\mu) \overset{\text{def}}{=} \langle S_\mu^{(\alpha)}, X \rangle_{\mathcal{T}_2} \tag{8}
\]

In essence, the kernel \( h \) of the channel operator \( \mathcal{H} \) is given as the (inverse) Fourier transform in the \( \mu_2 \) variable (see for example [17, Chapter 14]). Note that \( \Sigma_X^{(\alpha)}(0) = \text{Tr}(X) \) and \( |\Sigma_X^{(\alpha)}(\mu)| \leq \|X\|_1 \) (because \( \|S_\mu^{(\alpha)}\| = 1 \)). The function \( \Sigma_X^{(\alpha)} \) is sometimes called the "non–commutative" Fourier transform [21], characteristic function, inverse Weyl transform [22] or \( \alpha \)–generalized spreading function of \( X \) [6], [23]. From (2) it follows that \( \Sigma_X^{(\alpha)} = e^{i2\pi(1/2-\alpha)\zeta(\mu,\mu)}\Sigma_X^{(1/2)} \).

**Lemma 1 (Spreading Representation)** Let \( X \in \mathcal{T}_2 \). Then there it holds:

\[
X = \int_{\mathbb{R}^{2n}} \langle S_\mu^{(\alpha)}, X \rangle_{\mathcal{T}_2} S_\mu^{(\alpha)} d\mu \tag{9}
\]
where the integral is meant in the weak sense\(^3\).

The extension to the Hilbert–Schmidt operators \(\mathcal{T}_2\) is due to continuity of the mapping in (8) and density of \(\mathcal{T}_1\) in \(\mathcal{T}_2\). A complete proof of this lemma can be found in many books on Weyl calculus (for example matched to our notation in [21, Chapter V] and [24]). Furthermore, the following important shift–property:

\[
\Sigma^{(\alpha)}_{\mathcal{I},(\beta)} X \Sigma^{(\nu)}_{\mathcal{I}}(\nu) = e^{-i2\pi\eta(\mu,\nu)} \Sigma^{(\alpha)}_{X}(\nu) \tag{10}
\]

can be verified easily using (2) and (3). The composition of the symplectic Fourier transform \(\mathcal{F}_s\) as defined in (6) with the mapping in (8) establishes the so called Weyl correspondence [22] in a particular polarization \(\alpha\) (for this generalized approach in signal processing see also [23]). The function \(L^{(\alpha)}_X = \mathcal{F}_s \Sigma^{(\alpha)}_X\) is called (generalized) Weyl symbol of \(X\). The original Weyl symbol is \(L^{(0)}_X\). The cases \(\alpha = \frac{1}{2}\) and \(\alpha = -\frac{1}{2}\) are also known as Kohn–Nirenberg symbol (or Zadeh’s time–varying transfer function) and Bello’s frequency–dependent modulation function [25]. The Parseval identities are:

\[
\langle X, Y \rangle_{\mathcal{T}_2} = \langle \Sigma^{(\alpha)}_X, \Sigma^{(\alpha)}_Y \rangle = \langle L^{(\alpha)}_X, L^{(\alpha)}_Y \rangle \tag{11}
\]

for \(X, Y \in \mathcal{T}_2\). For a rank–one operator \(X = \langle \gamma, \cdot \rangle g\) it follows that \(\Sigma^{(\alpha)}_X = \tilde{A}^{(\alpha)}_{\gamma}\) such that (11) reads in this case as: \(\langle g, Y \gamma \rangle = \langle \tilde{A}^{(\alpha)}_{\gamma}, \Sigma^{(\alpha)}_Y \rangle\).

### III. Problem Statement and Main Results

In this section we will establish a concept, which we have called the ”approximate eigenstructure”. The latter are sets of signals and coefficients which fulfill a particular property of singular values and functions up to certain approximation error \(E_p\) measured in \(p\)–norm. Part III-A motivates this concept for a single channel operator. In part III-B of this section we will then extent this framework to a time–frequency formulation for ”random” time–varying channels with a common support of the spreading functions. We consider on how approximate eigenstructure behavior scales with the respect to the particular spreading functions, which is the main problem of this paper. Recent results in this direction are for \(p = 2\) and based on estimates on the approximate product rule of Weyl symbols. We will give a general formulation of this approach and an overview over the known results for \(E_2\) in part III-C of this section. After that we present in III-D a new (direct) approach for upperbounding

\[^3\text{For } \|\Sigma_X^{(\alpha)}\|_1 < \infty \text{ (9) is a Bochner integral. Weak interpretation of (9) as } \langle g, X \gamma \rangle \text{ extents the meaning of this integral to tempered distributions [5, Chapter 2] or [17, Chapter 14.3].}\]
yielding for our setup also improved and more general estimates for \( p = 2 \). This part contains a summary of the main results, where the more detailed analysis is in Section IV.

A. The Approximate Eigenstructure

It is a common approach to describe a given channel operator \( \mathcal{H} \) on a superposition of signals where \( \mathcal{H} \) act rather simple. As already mentioned, compact operators on a Hilbert space can be formally represented as 

\[
\mathcal{H} = \sum_{k=1}^{\infty} s_k \langle x_k, \cdot \rangle y_k
\]

with the singular values \( \{s_k\} \) and singular functions \( \{x_k\} \) and \( \{y_k\} \). Transmitting an information bearing complex data symbol \( c \) for example in the form of the signal 

\[
s = c \cdot x_k
\]

through \( \mathcal{H} \) we known that with proper channel measurement (obtaining \( s_k \)) the information can be coherently ”recovered” from the estimate 

\[
\langle y_k, \mathcal{H}s \rangle = s_k \cdot c.
\]

The crucial point here is that the transmitting device has to know and implement \( \{x_k\} \) before. However, in practical implementation \( \{x_k\} \) is required to be fixed and structured to some sense (for example in the form of filterbanks). But in general, also the singular functions depend explicitly on the operator \( \mathcal{H} \), i.e. they vary from one realization to another. They can be very unstructured and it is difficult to relate properties of \( \mathcal{H} \) in such representations to physical measurable quantities.

Hence, instead of requiring \( \mathcal{H}x_k = s_k y_k \) we would like to have that \( \mathcal{H}x_k - s_k y_k \) is ”small” in some sense. Usually, approximation in the \( L_2 \)-norm seems to be of most interest in the signal design. However, there are certain problems as peak power and stability issues where stronger results are required. Furthermore intuitively we are aware that the approximation of the singular behavior of \( \{s_k, y_k, x_k\} \) has to be ”uniform” in more than one particular norm.

In this paper we consider the \( L_p \) norms for the approximation, thus we have the following formulation for the Hilbert space \( L_2(\mathbb{R}^n) \):

**Definition 2 (Approximate Eigenstructure)** Let \( \epsilon \) be a given positive number. Consider \( \lambda \in \mathbb{C} \) and two functions \( g, \gamma \in L_2(\mathbb{R}^n) \) with \( \|g\|_2 = \|\gamma\|_2 = 1 \). If

\[
E_p := \|\mathcal{H}\gamma - \lambda g\|_p \leq \epsilon
\]

we call \( \{\lambda, g, \gamma\} \) a \( L_p \)-approximate eigenstructure of \( \mathcal{H} \) with bound \( \epsilon \).

The set of \( \lambda \)'s for which exists \( g_\lambda \) such that \( \{\lambda, g_\lambda, \gamma\} \) is a \( L_2 \)-approximate eigenstructure for a common fixed \( \epsilon \) is also called the \( \epsilon \)-pseudospectrum\(^4\) of \( \mathcal{H} \). More generally, we will allow

\(^4\)Thanks to T. Strohmer for informing me about this relation.
also $g \neq \gamma$ such that the term "approximate singular" functions is suited for our approach as well. Obviously for the "true eigenstructure" $\{s_k, y_k, x_k\}$ as defined above we have that $\epsilon = 0$ for each $p$ and $k$. On the other hand, for given $g$ and $\gamma$ the minimum of the left hand side (lhs) of (12) is achieved for $p = 2$ at $\lambda = \langle g, \mathcal{H} \gamma \rangle$ such that $E_p$ for $\{\langle g, \mathcal{H} \gamma \rangle, g, \gamma\}$ describes the amount of orthogonal distortion caused by $\mathcal{H}$ measured in the $p$–norm.

B. The Problem Statement for Channels with Compactly Supported Spreading

It is of general importance to what degree the Weyl symbol or a smoothed version of it approaches the eigen–value (or more generally singular value) characteristics of a given channel operator $\mathcal{H}$. Inspired from the ideas in [7] we will consider now the following question: What is the error $E_p(\mu)$ if we approximate the action of $\mathcal{H}$ on $S_{\mu} g$ by $\lambda(\mu)$? Hence, instead of the "true" eigenstructure consisting of the singular values and functions of $\mathcal{H}$ we shall consider a more structured family $\{\lambda(\mu), S_{\mu} g, S_{\mu} \gamma\}$. The latter will intuitively probe the operator $\mathcal{H}$ locally in a phase space (time–frequency) meaning if $g$ and $\gamma$ are in some sense time–frequency localized around the origin. The validity of this approximate picture, in which the function $\lambda : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ now serves as a multiplicative channel is essentially described by $E_p(\mu)$.

For example, in wireless communication $S_{\mu} g$ and $S_{\mu} \gamma$ could be well–localized prototype filters at time–frequency slot $\mu$ of the receive and transmit filterbanks of a particular communication device and $\lambda(\mu)$ is an effective channel coefficient to be equalized. However, with this application in mind, we are typically confronted with random channel operators $\mathcal{H}$ characterized by random spreading functions $\Sigma_{\mathcal{H}}^{(n)}$ having a common (Lebesgue measurable) support $U$ of non–zero and finite measure $|U|$, i.e. $0 < |U| < \infty$. The assumption of a known support seems to be the minimal apriori channel knowledge that enters practically the system design (e.g. of a communication device). For example, a typical doubly–dispersive channel model ($n = 1$) for this application is that spreading occurs in $U = [0, \tau_d] \times [-B_D, B_D]$ where $\tau_d$ and $B_D$ are the maximum delay spread and Doppler frequency. It is then desirable to have common prototype filters for all these channel realizations. It is clear that in this direction Definition 2 is not yet adequate enough. We have to measure the approximation error with respect to a certain scale of the particular random spreading functions. In this paper we measure the approximate eigenstructure with respect to its $L_q$–norm. We believe that this approach is important to have reasonable estimates for the various statistical fading and scattering environments. An example of such an application is given in Remark 2 in
Section V.

We consider only bounded spreading functions such that the operator $\mathcal{H}$ is of Hilbert Schmidt type, i.e. $\mathcal{H} \in T_2$. To this end, let us call this set of channel operators as $\text{OP}(U)$, i.e.

$$\text{OP}(U) := \{ \mathcal{H} | \text{supp}(\Sigma^{(\alpha)}_{\mathcal{H}}) \subseteq U \text{ and } \sup_{\mu \in U} |\Sigma^{(\alpha)}_{\mathcal{H}}(\mu)| < \infty \}$$  \hspace{1cm} (13)

As already discussed for example in [14] the operator class $\text{OP}(U)$ does not include limiting cases of doubly–dispersive channels like the time–invariant channel or the identity. Generalizations, for example in the sense of tempered distributions, are beyond the scope of this paper. We aim at an extension of Definition 2 for the approximate eigenstructure which is meaningful and suited for this class of channels. We will formulate this as our main problem of this paper:

**Problem:** Consider two functions $g, \gamma : \mathbb{R}^n \to \mathbb{C}$ with $\|g\|_2 = \|\gamma\|_2 = 1$. Let be $1 \leq q \leq \infty$, $1 \leq p < \infty$ and $0 < \delta < \infty$ such that for all operators $\mathcal{H} \in \text{OP}(U)$ it holds:

$$E_p(\mu) := \|\mathcal{H}S_{\mu} \gamma - \lambda(\mu)S_{\mu}g\|_p \leq \delta \cdot \|\Sigma^{(\alpha)}_{\mathcal{H}}\|_q$$  \hspace{1cm} (14)

where the $p$–norm is with respect to the argument of the function $\mathcal{H}S_{\mu} \gamma - \lambda(\mu)S_{\mu}g$. Then $\{\lambda(\mu), S_{\mu}g, S_{\mu} \gamma\}$ is an $L_p$–approximate eigenstructure for all $\mathcal{H} \in \text{OP}(U)$, each of them with individual bound $\epsilon = \delta \cdot \|\Sigma^{(\alpha)}_{\mathcal{H}}\|_a$. How small can we choose the scale $\delta$ given $g$, $\gamma$, $U$, $p$ and $q$? What can be said about $\inf_{g,\gamma}(\delta)$?

Note that, independently of the polarization $\alpha$, the operator $S_{\mu}$ can be replaced in (14) with any $\beta$–polarized shift $S^{(\beta)}_{\mu}$ without change of $E_p(\mu)$. Furthermore, as already stated in the definition of $E_p$ in (12) $\|g\|_2 = \|\gamma\|_2 = 1$, throughout the rest of the paper. Summarizing: How much could $\{S_{\mu}g, S_{\mu} \gamma\}$ serve as common approximations (measured in $p$–norm) to the singular functions of the operator class $\text{OP}(U)$ for fixed $U$?

**C. Results Based on the Approximate Product Rule**

In previous work [6], [26], [8], [27] results were provided for $g = \gamma$ and (apart of [28]) $\lambda = L^{(\alpha)}_{\mathcal{H}}$ for the case $p = 2$. These are obtained if one considers the problem from view of symbolic calculus and can be summarized in the following lemma:

**Lemma 3** Let $\gamma = g$ and $\lambda = L^{(\alpha)}_{\mathcal{H}}$. It holds:

$$E_2(\mu) \leq \left( |L^{(\alpha)}_{\mathcal{H}}(\mu) - L^{(\alpha)}_{\mathcal{H}}(\mu)|^2 + \|\Sigma^{(\alpha)}_{\mathcal{H}}\|_1 + 2|L^{(\alpha)}_{\mathcal{H}}(\mu)| \cdot \|\Sigma^{(\alpha)}_{\mathcal{H}}\|_1 \right)^{\frac{1}{2}}$$  \hspace{1cm} (15)
where \( \Omega = |A_{\gamma \gamma}^{(\alpha)} - 1| \).

Note that the lemma not yet necessarily requires \( \mathcal{H} \in \text{OP}(U) \). The proof is provisioned in Appendix A. This bound is motivated by the work of W. Kozek [6]. However it has been formulated in a more general context. The first term of the bound in (15) contains the Weyl symbol \( L_{XY}^{(\alpha)} \) of the composition \( XY \) of two operators \( X \) and \( Y \) (\( \mathcal{H}^* \) and \( \mathcal{H} \) in this case), which is the twisted multiplication [29] of the symbols of the operators \( X \) and \( Y \). On the level of spreading functions\(^5\), \( \Sigma_{XY}^{(\alpha)} \) is given by the so called \textit{twisted convolution} \( \natural \phi \) of \( \Sigma_{X}^{(\alpha)} \) and \( \Sigma_{Y}^{(\alpha)} \) [24], [30]:

\[
\Sigma_{XY}^{(\alpha)}(\rho) = \int_{\mathbb{R}^{2n}} \Sigma_{X}^{(\alpha)}(\mu) \Sigma_{Y}^{(\alpha)}(\rho - \mu) e^{-i2\pi\phi(\mu,\rho)} d\mu = (\Sigma_{X}^{(\alpha)} \natural \phi \Sigma_{Y}^{(\alpha)}) (\rho) \quad (16)
\]

with \( \phi(\mu, \rho) = (\alpha + \frac{1}{2})\zeta(\mu, \rho) + (\alpha - \frac{1}{2})\zeta(\rho, \mu) - 2\alpha\zeta(\mu, \mu) \). For the polarization \( \alpha = 0 \) it follows \( \phi(\mu, \rho) = \eta(\mu, \rho)/2 \) and conventional convolution is simply \( \natural \). Expanding \( \exp(-i2\pi\phi(\mu, \rho)) \) in \( \mu \) as a Taylor series reveals that twisted convolutions are weighted sums of \( \natural \)-convolutions [5] related to moments of \( \Sigma_{X}^{(\alpha)} \) and \( \Sigma_{Y}^{(\alpha)} \). Hausdorff–Young inequality with sharp constants \( c_p^2 = p^\frac{1}{p}/p^\frac{1}{p'} \) (and \( c_1 = c_\infty = 1 \)) gives for \( 1 \leq p \leq 2 \) estimates on the following ”approximate product rule” of Weyl symbols: \( \|L_{XY}^{(\alpha)} - L_{X}^{(\alpha)} L_{Y}^{(\alpha)}\|_{p'} \leq 2c_p^2 \|F\|_p \) where \( F(\rho) = \int_{\mathbb{R}^{2n}} |\Sigma_{X}^{(\alpha)}(\mu)\Sigma_{Y}^{(\alpha)}(\rho - \mu)\sin(2\pi\phi(\mu, \rho))| d\mu \). In particular, for \( p = 1 \) we get:

\[
|L_{\mathcal{H}^* \mathcal{H}}^{(\alpha)} - L_{\mathcal{H}}^{(\alpha)}| \leq 2\|F\|_1 \quad \text{a.e.} \quad (17)
\]

Let us assume now that \( \mathcal{H} \in \text{OP}(U) \). With \( \chi_U \) we shall denote the characteristic function of \( U \) (its indicator function). Kozek [6, Thm. 5.6] has considered the case \( \alpha = 0 \) obtaining the following result:

**Theorem 4 (W. Kozek [6])** Let \( U = [-\tau_0, \tau_0] \times [-\nu_0, \nu_0] \) and \( \alpha = 0 \). If \( |U| = 4\tau_0\nu_0 \leq 1 \) then

\[
E_2(\mu) \leq \left( 2\sin\left(\frac{\pi |U|}{4}\right) \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(0)}\|_1^2 + \epsilon_\gamma \left( \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(0)}\|_1 + 2\|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(0)}\|_1^2 \right) \right)^\frac{1}{2} \quad (18)
\]

where \( \epsilon_\gamma = \|(A_{\gamma \gamma}^{(0)} - 1) \chi_U\|_\infty \).

The proof can be found in [6] or independently from Lemma 3 with \( \epsilon_\gamma = \|\mathcal{H} \chi_U\|_\infty \) and \( \|L_{\mathcal{H}^* \mathcal{H}}^{(\alpha)}\|_\infty \leq \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(\alpha)}\|_1 \). Further utilizing the fact that \( \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(\alpha)}\|_1 \leq \|\Sigma_{\mathcal{H}^* \mathcal{H}}^{(\alpha)}\|_1^2 \), Equation (18) can

\(^5\)Symbols (spreading functions) in \( \mathcal{L}_0(\mathbb{R}^{2n}) \) with twisted multiplication (convolution) are *-isomorphic to the algebra of Hilbert–Schmidt operators.
be written as:

\[
\frac{E_2(\mu)}{\|\Sigma^{(0)}_{\mu}\|_1} \leq \left(2\sin\left(\frac{\pi |U|}{4}\right) + 3\|A^{(0)}_{\gamma\gamma} - 1\|_{\infty}\right)^{\frac{1}{2}}
\]  

(19)

which gives an initial answer to the problem formulated in section III-B. Theorem 4 was further extended in [8, Thm. 2.22] by G. Matz (see also [27]) to a formulation in terms of weighted 1–moments of (not necessarily compactly supported) spreading functions. His approach includes also different polarizations \(\alpha\). For a spreading function as in Theorem 4 and \(\alpha = 0\) the results agree with (19). Equation (19) could be interpreted in such a way that only the second term can be controlled by \(\gamma\) (e.g. pulse shaping) where the first term of the rhs of (19) is only related to the overall spread \(|U|\). However we shall show in the next section that the first term can be eliminated from the bound and the second (shape–independent) term can further tightened.

D. Results Based on a Direct Approach

We have considered the function \(\lambda\) as the Weyl symbol exclusively for the exponents \(p = 2\) and \(a = 1\) in III-C. This approach is in line with prior work of Kozek, Matz and Hlawatsch and provides results on the approximate eigenstructure problem established in Section III-B. To obtain further results for different values of \(p, a\) and \(\lambda\), we shall now restart the analysis from a different perspective. In the following we present the main results, through most of the analysis will be presented in Section IV. We use a ”smoothed” version of the Weyl symbol:

\[
\lambda = \mathcal{F}_s(\Sigma^{(\alpha)}_{\mu} \cdot B)
\]

(20)

where \(B : \mathbb{R}^{2n} \to \mathbb{C}\) is a bounded function. We consider two important cases:

**Case C1:** Let \(B = A^{(\alpha)}_{g\gamma}\) such that (20) reads as \(\lambda = L^{(\alpha)}_{\gamma\mu} \ast \mathcal{F}_s A^{(\alpha)}_{g\gamma}\) where \(\ast\) denotes convolution. This corresponds to the well known smoothing with the cross Wigner function \(\mathcal{F}_s A^{(\alpha)}_{g\gamma}\) and was already considered in [28] (for averages over WSSUS\(^6\) channels ensembles).

In particular this is exactly the orthogonal distortion:

\[
\lambda(\mu) = \langle S_{\mu\gamma}, \mathcal{H}S_{\mu\gamma}\rangle = \left(\mathcal{F}_s(\Sigma^{(\alpha)}_{\mu\mu} \cdot A^{(\alpha)}_{g\gamma})\right)(\mu)
\]

(21)

as already mentioned in III-A and corresponds to the choice of the \(E_2–\)minimizer. Since \(\lambda\) depends in this case on \(g\) and \(\gamma\) we consider here how accurately the action of operators \(\mathcal{H}\) on the family \(\{S_{\mu\gamma}\}\) can be described as multiplication operators on the family \(\{S_{\mu\gamma}\}\). From

\(^6\)Wide–sense stationary uncorrelated scattering (WSSUS) channel model [25]
the rule in (2) the definition of the cross ambiguity function in (4) and the non–commutative Fourier transform in (8) it is clear that this choice is also independent of the polarization α. Recall that the Weyl symbol of a rank-one operator is the Wigner distribution, such that with this approach we again effectively compare twisted with ordinary convolution.

**Case C2:** Here we consider $B = 1$ such that $\lambda = L^{(\alpha)}_{\gamma\gamma}$ is the Weyl symbol. The function $\lambda$ is now independent of $g$ and $\gamma$. Thus in contrast to C1 this case is related to the ”pure” symbol calculus. Obviously, we have to expect now a dependency on the polarization $\alpha$.

Furthermore, this was the approach considered for $p = 2$ in the previous part of this section.

The first theorem parallels Theorem 4 and its consequence (19). We shall not yet restrict ourselves to the cases C1 and C2. Instead we only require that $B$ has to be essentially bounded.

**Theorem 5** Let $H \in OP(U)$, $g, \gamma \in L_\infty(\mathbb{R}^n)$ and $B \in L_\infty(\mathbb{R}^{2n})$. For $2 \leq p < \infty$ and $1 \leq q \leq \infty$ (with the usual meaning for $q = \infty$):

$$\frac{E_p(\mu)}{\|\Sigma^{(\alpha)}_{\mathcal{H}}\|_q} \leq C^{\frac{2}{p}} \cdot \|1 + |B|^2 - 2Re\{A^{(\alpha)}_{\gamma\gamma} \bar{B}\}\chi_U\|_{q'/p}$$

(22)

where $C$ is a constant depending on $g$, $\gamma$ and $B$. The minimum of this bound over $B$ is achieved in the case of $p = 2$ for C1.

**Proof:** The proof follows from the middle term of (39) in Lemma 9 if we set $C$ as:

$$C = \text{ess sup}_{x \in \mathbb{R}^n, \nu \in U} |(S^{(\alpha)}_{\nu\gamma})(x) - B(\nu)g(x)| \leq \|\gamma\|_\infty + \|B\|_\infty \|g\|_\infty$$

(23)

In Lemma 9 the range of $p$ is $1 \leq p < \infty$. However, from the discussion in Section IV-B it is clear that (22) gives only for $p \geq 2$ a reasonable bound. □

**Comparison to the bound of Kozek:** With $|1 - \text{Re}\{A^{(\alpha)}_{\gamma\gamma}\}| \leq |1 - A^{(\alpha)}_{\gamma\gamma}|$ we can transform the result of the last theorem for C2 with settings $p = 2$, $q = 1$ and $g = \gamma$ into a form comparable to (18) and (19) which is:

$$\frac{E_2(\mu)}{\|\Sigma^{(\alpha)}_{\mathcal{H}}\|_1} \leq (2\|1 - A^{(\alpha)}_{\gamma\gamma}\chi_U\|_\infty)^{\frac{1}{2}}$$

(24)

Hence this technique improves the previous bounds. It includes different polarizations $\alpha$ and does not require any shape or size constraints on $U$. Interestingly the offset in (19), which

\[\text{However, also there the same methodology as in (20) could be applied as well.}\]
does not depend on \((g, \gamma)\) and in an initial glance seems to be related to the notion of underspreadness, has been disappeared now.

**Discussion of the critical size:** The behavior of the bound in (22) on \(|U|\) depends in general on the choice of the function \(B\). For example for the case C1, \(p = q = 2\) and with (5) it follows that the rhs of (22) is the square root of \(|U| - \langle |A_{g\gamma}^{(\alpha)}|, \chi_U \rangle\) and again with (5) we have that:

\[
\sqrt{|U| - \min(|U|, 1)} \leq \text{rhs of (22)} \leq \sqrt{|U|}.
\]  

(25)

This implies that this term is of the same order as \(\sqrt{|U|}\) for \(|U| \gg 1\) (see also Lemma 11 later on). The lhs of the inequality suggests that for \(|U| \leq 1\) the scaling behavior might alter, i.e. \(|U| = 1\) is in this sense a critical point between over- and underspread channels as introduced in [6]. On the other hand the lhs of the last equation is not zero for \(0 < |U| \leq 1\). Indeed from Theorem 14 we have an improved version as follows:

\[
\sqrt{|U| - \min(|U|e^{-\frac{|U|}{e}}, 1)} \leq \text{rhs of (22)} \leq \sqrt{|U|}.
\]

(26)

which suggest that at \(|U| = e\) the behavior changes.

**Restriction to the cases C1 and C2:** If we further restrict ourselves to \(q > 1\) (i.e. \(q' < \infty\)) we can establish the relation to weighted norms of ambiguity functions [10]. For simplicity let us consider now the two cases C1 \((k = 1)\) and C2 \((k = 2)\). We define therefore the functions \(A_k : \mathbb{R}^{2n} \rightarrow \mathbb{R}\) for \(k = 1, 2\) as:

\[
A_1 := |A_{g\gamma}^{(\alpha)}|^2 \quad \text{and} \quad A_2 := \text{Re}\{A_{g\gamma}^{(\alpha)}\}
\]

(27)

We then have the following result:

**Theorem 6** Let \(\mathcal{H} \in \mathcal{OP}(U)\) and \(g, \gamma \in \mathcal{L}_\infty(\mathbb{R}^n)\). For \(2 \leq p < \infty\), \(1 < q \leq \infty\) and \(|U| \leq 1\) it holds:

\[
\frac{E_p(\mu)}{\|\Sigma_{\mathcal{H}}^{(\alpha)}\|_q} \leq C_{k,p}^{\frac{p-2}{p}} k (|U| - \langle A_k, \chi_U \rangle)^{1/\max(p', p)}
\]

(28)

where \(k = 1\) for C1 and \(k = 2\) for C2.

**Proof:** We now use the bound (40) in Lemma 9 with the uniform estimates \(C_{bp} \leq k\) from Lemma 10. Again, as follows from the discussion in Section IV-B, we consider only \(p \geq 2\).

The assumption \(|U| \leq 1\) is only used to simplify the bound. Improved estimates follow from Lemma 9 directly. From the positivity of \(A_1\) we observe that the orthogonal distortion (the case C1) is always related to weighted 2–norms of the cross ambiguity function (the weight
is in this case only $\chi_U$). For the case C2 this can be turned into weighted 1–norm if $A_2$ is positive on $U$ or fulfill certain cancellation properties. Furthermore, the case C2 depends obviously on the polarization $\alpha$. For particular symmetries of $U$ explicit values can be found as shown with the following theorem (for simplicity we consider $n = 1$):

**Theorem 7** If $\mathcal{H} \in \text{OP}(U)$ with $\chi_U(\mu) = \chi_U(-\mu)$ and $g, \gamma \in \mathcal{L}_\infty(\mathbb{R})$ and $B \in \mathcal{L}_\infty(\mathbb{R}^2)$.

For $2 \leq p < \infty$, $1 < q \leq \infty$ and $\|U\| \leq 1$ it holds:

$$\frac{E_p(\mu)}{\|\Sigma^{(\alpha)}_{\mathcal{H}}\|_q} \leq 32 \frac{p^2}{\pi} k (k\|U\| (1 - L))^{1/\max(q', p)} \tag{29}$$

In general, $L \geq \lambda_{\max}(Q^*Q)^{1/k}$ and $Q$ is an operator with spreading function $\chi_U/|U|$ in polarization 0. Furthermore, $L \geq l(|U|/2/k)$ with $l(x) = 2(1 - e^{-x/2})/x$ for $U$ being a disc and $l(x) = 2 \cdot \text{erf}(\sqrt{\pi x/8})^2/x$ for $U$ being a square.

**Proof:** We combine Lemma 9 and Lemma 13 with the uniform estimates $C_{bp} \leq k$ from Lemma 10 such that

$$\frac{E_p(\mu)}{\|\Sigma^{(\alpha)}_{\mathcal{H}}\|_q} \leq 32 \frac{p^2}{\pi} k (k\|U\| (1 - \lambda_{\max}(Q^*Q)^{1/k}))^{1/\max(q', p)} \tag{30}$$

where $Q$ is a compact operator with spreading function $\chi_U/|U|$ in polarization $\alpha$. From Lemma 15 we know that our assumptions imply that $Q$ is Hermitian for $\alpha = 0$. In general, therefore it holds that $L = \lambda_{\max}(Q^*Q)^{1/k} = \lambda_{\max}(Q)^{2/k}$ where $\lambda_{\max}(Q)$ is at least as the value of the integral (56) over the first Laguerre function. We abbreviate $L = l(|U|^{2/k})$ such that for a disc of radius $\sqrt{|U|/\pi}$ this integral is $l(x) = 2(1 - e^{-x/2})/x$ and for a square of length $\sqrt{U}$ we get $l(x) = 2 \cdot \text{erf}(\sqrt{\pi x/8})^2/x$. However, Lemma 15 asserts this as an upper bound achieved in this case with Gaussians $g(x) = 2^{1/4} e^{-\pi(x,x)}$ which is tight for C2 but not for C1. This means, for C1 this can be further improved by direct evaluation on Gaussians. Indeed, from the proof of Corollary 16 we know that $l(|U| \cdot 2/k) \geq l(|U|^{2/k})$ is achievable. For $\|V\|_\infty$ we get $\|V\|_\infty = 2\|g\|_\infty = 32^{1/4}$.

This result can be extended in part to regions $U$ which are canonical equivalent to discs and squares centered at the origin (see the discussion at the beginning of Section IV-D). However, this holds in principle only for $p = 2$ because such canonical transformations will change the constants in (29).

**IV. General Analysis and Proofs**

With the following lemma we separate the support of the spreading function $\Sigma^{(\alpha)}_{\mathcal{H}}$ from the quantity $E_p(\mu)$. We shall make use of the non–negative function $V : \mathbb{R}^n \times \mathbb{R}^{2n} \rightarrow \mathbb{R}_+$,
defined as:

\[ V(x, \nu) := |(S^{(\alpha)}(\nu))(x) - B(\nu)g(x)| \cdot \chi_U(\nu) \geq 0 \]  

and of the functionals \( V_p(\nu) := \|V(\cdot, \nu)\|_p \), i.e. the usual \( p \)-norms in the first argument. For simplifying our analysis we shall restrict ourselves to indicator weights \( \chi_U \) (the characteristic function of \( U \)). However, the same can be repeated with slight abuse of notation using more general weights.

**Lemma 8** Let \( \mathcal{H} \in OP(U) \), \( 1 \leq p < \infty, 1 \leq q \leq \infty \). If \( V(\cdot, \nu) \in L_p(\mathbb{R}^n) \) for all \( \nu \in U \) then it holds:

\[ E_p(\mu) \| \Sigma_{\mathcal{H}}^{(\alpha)} \|_q \leq \| V_p \|_{q'} \]  

whenever \( \Sigma_{\mathcal{H}}^{(\alpha)} \in L_q(\mathbb{R}^{2n}) \) and \( V_p \in L_{q'}(\mathbb{R}^{2n}) \).

**Proof:** Firstly, using Weyl’s commutation rule (3) and the definition of \( \lambda \) in (20) gives:

\[ E_p(\mu) = \left\| \int_{\mathbb{R}^{2n}} d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) S^\mu_{\nu} \gamma - \lambda(\mu) S^\mu_{\nu} g \right\|_p \]

\[ = \left\| S^\mu_{\nu} \left( \int_{\mathbb{R}^{2n}} d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) e^{-i2\pi\eta(\nu,\mu)} S^\nu_{\nu} \gamma - \lambda(\mu) g \right) \right\|_p \]  

\[ = \left\| \int_{\mathbb{R}^{2n}} d\nu \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) e^{-i2\pi\eta(\nu,\mu)} (S^\nu_{\nu} \gamma - B(\nu)g) \right\|_p . \]  

Note that the \( p \)-norm is with respect to the argument of the functions \( g \) and \( S^\nu_{\nu} \gamma \). The last step follows because \( S^\mu_{\nu} \) acts isometrically on all \( L_p(\mathbb{R}^n) \). Let \( f : \mathbb{R}^n \times \mathbb{R}^{2n} \rightarrow \mathbb{C} \) be the function defined as:

\[ f(x, \nu) := e^{-i2\pi\eta(\nu,\mu)} \Sigma_{\mathcal{H}}^{(\alpha)}(\nu) [(S^\nu_{\nu} \gamma)(x) - B(\nu)g(x)] \]  

From \( \mathcal{H} \in OP(U) \) (bounded spreading functions) and \( V(\cdot, \nu) \in L_p(\mathbb{R}^n) \) for all \( \nu \in U \) it follows that \( f(\cdot, \nu) \in L_p(\mathbb{R}^n) \). Then (33) reads for \( 1 \leq p < \infty \) by Minkowski (triangle) inequality

\[ E_p(\mu) = \left\| \int_{\mathbb{R}^{2n}} d\nu f(\cdot, \nu) \right\|_p \leq \int_{\mathbb{R}^{2n}} d\nu \left\| f(\cdot, \nu) \right\|_p = \| \Sigma_{\mathcal{H}}^{(\alpha)} \cdot V_p \|_1 \leq \| \Sigma_{\mathcal{H}}^{(\alpha)} \|_q \| V_p \|_{q'} \]  

In the last step we used Hölder’s inequality, such that the claim of this lemma follows. 

A. The Relation to Ambiguity Functions

In the next lemma we shall show that \( \| V_p \|_{q'} \) can be related to ambiguity functions, which occur for \( p = 2 \). We introduce \( R : \mathbb{R}^{2n} \to \mathbb{R}_+ \) as the non–negative function:

\[
R := V_2^2 = \left( 1 + |A_{g\gamma}^{(\alpha)} - B|^2 - |A_{g\gamma}^{(\alpha)}|^2 \right) \cdot \chi_U \geq 0
\]

(36)

and abbreviate \( R_s := \| R \|_s \). Using (27) we can write (36) for the cases C1 and C2 as

\[
R = k(1 - A_{k}) \cdot \chi_U \text{ for } k = 1, 2.
\]

From the non–negativity of \( R \) follows that:

\[
R_1 = |U| + \|(A_{g\gamma}^{(\alpha)} - B)\chi_U\|^2_2 - \|A_{g\gamma}^{(\alpha)}\chi_U\|^2_2
\]

(37)

Hence, \( R_1 \) reflects an interplay between two localization criteria in the phase space. In particular, we get for C1 and for C2:

\[
R_1 = k(|U| - \langle A_{k}, \chi_U \rangle)
\]

(38)

With the following lemma we shall explicitly provide the relation between the bound \( \| V_p \|_{q'} \) in Lemma 8 to the quantity \( R_1 \).

**Lemma 9** For \( 1 \leq p < \infty \) and \( 1 \leq q' \leq \infty \) it holds (with the usual meaning for \( q' = \infty \)):

\[
\| V_p \|_{q'} \leq \| V \|_\infty^{p-2} \cdot R^{1/p}_{q'/p}
\]

(39)

Equality is achieved for \( p = 2 \) and then the minimum over \( B \) of the rhs is achieved for C1. For \( q' < \infty \) let \( C_{pq} = R_{\infty}^{\frac{q}{q'} - \frac{p}{q'}} \) for \( p \leq q' \) and \( C_{pq} = |U|^{\frac{p}{q' - p}} \) else. Then it holds further that:

\[
\| V_p \|_{q'} \leq \| V \|_\infty^{p-2} \cdot C_{pq} \cdot R_1^{1/\max(p,q')}
\]

(40)

with equality for \( q' = p = 2 \).

The proof can be found in the Appendix B. The main reason for this lemma, in particular for the second part, is that it opens up for case C1 the relation to weighted norms of ambiguity function (i.e. localization of \( A_k \) on \( U \)). However, for C2 we are also concerned with the question of positivity (and cancellation properties) in \( U \). We shall study these relations in more detail in Section IV-C.
B. Uniform Estimates

As already mentioned before, for "true" eigenstructure we have $E_p = 0$ for all $p$, such that the notion of approximate eigenstructure should be in some sense uniform in $q$ and $p$. In the first step it is therefore necessary to validate uniform bounds for $C_{pq}$. We observe that $\|V\|_{\infty}^{\frac{p-2}{p}}$ will then restrict the application of Lemma 9 only to $p \geq 2$ because $\|V\|_{\infty}$ will be in general small. For example for C2 and $g = \gamma$ let $|U| \to 0$ in (31). This behavior has to be expected because the ambiguity function is a $L_2$–related construction and from $L_2$ boundedness one can only with further decay conditions infer $L_p$–boundedness for $p < 2$. Consequentially we shall restrict the following analysis to $2 \leq p < \infty$ such that $\sup \|V\|_p^{-2} \leq \max(\|V\|_\infty, 1)$.

For $\|V\|_\infty$ we can use for example a worst case estimate of the form $\|V\|_\infty \leq \|\gamma\|_\infty + \|B\|_\infty \cdot \|g\|_\infty \leq \|\gamma\|_\infty + \|g\|_\infty$ which is valid for C1 ($\|B\|_\infty = \|A_{g\gamma}\|_\infty \leq 1$ by (5)) and C2.

**Lemma 10 (Uniform Bounds for $C_{pq}$)** For $2 \leq p < \infty$ and $1 < q \leq \infty$ it holds the uniform estimate $C_{pq} \leq k$ if $q' \geq p$ where $k = 1$ for C1 and $k = 2$ for C2. If $q' < p$ then it holds $C_{pq} \leq \max(|U|, 1)$.

**Proof:** It is easily verified that $\sup |U|^{\frac{p-q'}{q'}} = \max(|U|, 1)$ where the supremum is over all $1 \leq q' < p$ and $2 \leq p < \infty$. The same can be found also for $1 \leq p < \infty$. Similarly we get for the quantity $R_{q'p}^{q'p}$ the uniform estimate $\sup R_{q'p}^{q'p} \leq \max(\sqrt{R_{q'p}}_\infty, 1)$ where $p \leq q' \leq \infty$ $2 \leq p < \infty$. For $1 \leq p < \infty$ we would get instead $\max(R_{q'p}, 1)$. From the non–negativity of $R$ it follows that:

$$R_{\infty} = k(1 - \text{ess inf } A_k(\nu))$$

From (5) it follows that the inequality $R_{\infty} \leq 1$ is always fulfilled for C1. For the case C2 this gives instead that $R_{\infty} \leq 4$, in general.

The following lemma provides a simple upper bound on $E_p/\|\Sigma^{(\alpha)}_{\gamma}\|_q$ which is for $p = 2$ uniformly in $g$ and $\gamma$. Thus, it will serve as a benchmark.

**Lemma 11 (Uniform Bound for $E_p/\|\Sigma^{(\alpha)}_{\gamma}\|_q$)** For $1 \leq p < \infty$ and $1 < q \leq \infty$ it holds:

$$\frac{E_p(\mu)}{\|\Sigma^{(\alpha)}_{\gamma}\|_q} \leq \|V\|_{\infty}^{\frac{p-2}{p}} \cdot k^{2/p} \cdot |U|^{1/q'}$$

with $k = 1$ for C1 and $k = 2$ for C2.

**Proof:** We use $R_{\infty} \leq \max |U|$ in (39) of Lemma 9 and the uniform estimates $R_{\infty} \leq k$. from Lemma 10.
This bound can not be related to ambiguity functions, i.e. will give no insight on possible improvements due to localization.

C. Weighted Norms of Ambiguity Functions and Localization

In the previous section we have shown that $R_1$ is a relevant term, which controls the approximate eigenstructure. In the following analysis we shall further investigate $R_1$. We are interested in

$$\inf_{g,\gamma}(R_1)$$

which is:

$$\inf_{g,\gamma}(R_1) = k |U| \left( 1 - \sup_{g,\gamma}\langle A_k, C' \rangle \right)$$

(43)

where $C' := \chi_U / |U|$. Thus, (43) is a particular case of a more general problem, where $C$ is some arbitrary weight (non–negative) function $C$. Thus, let us consider $\sup_{g,\gamma}\langle A_k, C' \rangle$ and let us focus first only on $A_1 = |A_{g\gamma}^{(a)}|^2$ which is also positive. Since $A_1$ is quadratic in $\gamma$ we can rewrite $\langle A_1, C' \rangle = \langle \gamma, L_{C,g} \rangle$ where this quadratic form defines (weakly) an operator $L_{C,g}$. Such operators are also called localization operators [11] and it follows that $\sup_{\gamma}\langle A_1, C' \rangle = \lambda_{\text{max}}(L_{C,g})$. The eigen–values and eigen–functions of Gaussian ($g$ is set to be a Gaussian) localization operators on the disc ($U$ is a disc) are known to be Hermite functions (more generally this holds if $C$ has elliptical symmetry). Kozek [6], [7] found that for elliptical symmetry also the joint optimization results in Hermite functions$^8$. For $C$ being Gaussian the joint optimum ($g$ and $\gamma$) is known explicitly [10]. The last result is based on a theorem, formulated in [10], which we will need also in this paper. Let us consider for simplicity once again the one–dimensional case (the generalizations for $n > 1$ are similar), i.e. for $n = 1$ we have:

**Theorem 12** Let $\|g\|_2 = \|\gamma\|_2 = 1$ and $s, r \in \mathbb{R}$. Furthermore let $C \in L_{s'}(\mathbb{R}^2)$. Then the inequality:

$$\langle |A_{g\gamma}^{(a)}|^r, C' \rangle \leq \left( \frac{2}{rs} \right)^{\frac{1}{r}} \|C\|_{s'}$$

(44)

holds for each $s \geq \max\{1, \frac{2}{r}\}$.

From (2) follows that (44) does not depend on the polarization $\alpha$. The proof can be found in [10] and is based on a result of E. Lieb [31]. Note that apart from the normalization

$^8$Kozek considered $g = \gamma$. However one can show that for elliptical symmetry around the origin the optimum has also this property.
constraint the bound in Theorem 12 does not depend anymore on $g$ and $\gamma$. Hence for any given $C$ the optimal bound $N_r(C)$ can be found by

$$N_r(C) := \min_{R \ni s \geq \max \{1, \frac{1}{2}\}} \left( \frac{2}{rs} \right)^{\frac{1}{2}} \|C\|_{s'}$$

(45)

The equality case in Theorem 12 is given for $g, \gamma$ and $C$ being Gaussians (see [10] for more details). The following lemma states lower and upper bounds on the optimal achievable values of the quantities $\langle A_k, C \rangle$.

**Lemma 13** Let be $C : \mathbb{R}^{2n} \to \mathbb{R}_+$ a non–negative weight function with $\|C\|_1 = 1$. Then it holds:

$$\lambda_{\max}(Q^*Q) \leq \sup_{g, \gamma} \langle A_1, C \rangle \leq N_2(C)$$

(46)

for case C1 and equivalently for case C2:

$$\lambda_{\max}(Q^*Q)^{1/2} = \max_{g, \gamma} \langle A_2, C \rangle \leq N_1(C)$$

(47)

where $Q$ is the operator with spreading function $C$ in polarization $\alpha$.

**Proof:** Considering first the case C1 (that is $k = 1$), which is independent of the polarization $\alpha$. The corresponding term $\langle A_1, C \rangle$ is relevant in the theory of WSSUS pulse shaping [9] where $C$ is called the scattering function. In [32] we have already pointed out that a lower bound can be obtained from convexity. We have:

$$|\langle g, Q\gamma \rangle|^2 \leq \langle A_1, C \rangle \leq N_2(C)$$

(48)

where $Q$ is a compact (follows from normalization) operator with spreading function $C$. The uniform upper bound is according to (45). The optimum of the lower bound is achieved for $g$ and $\gamma$ being the eigen–functions of $Q^*Q$ and $QQ^*$ corresponding to the maximal eigen–value $\lambda_{\max}(Q^*Q)$, such that for the supremum over $g$ and $\gamma$ it follows that:

$$\lambda_{\max}(Q^*Q) \leq \sup_{g, \gamma} \langle A_1, C \rangle \leq N_2(C)$$

(49)

For the case C2 ($k = 2$) we proceed as follows. For a given $\gamma$ we have:

$$\langle A_2, C \rangle = \frac{1}{2} (\langle Q\gamma, g \rangle + \langle g, Q\gamma \rangle) \leq \|Q\gamma\|_2$$

(50)

with equality in the last step for $g = Q\gamma/\|Q\gamma\|_2$. Choosing $\gamma$ from the eigen–space of $Q^*Q$ related to the maximal eigen–value, we get:

$$\lambda_{\max}(Q^*Q)^{1/2} = \max_{g, \gamma} \langle A_2, C \rangle \leq N_1(C)$$

(51)
because $\langle A_2, C \rangle \leq \langle |A_2|, C \rangle \leq N_1(C)$ where again $N_1$ is from (45).

For the particular weight function of interest in this paper, i.e. for $C = \chi_U/|U|$ the upper bounds can be calculated explicitly. For $n = 1$ we get the following result:

**Corollary 14 (Norm Bounds for Flat Scattering)** Let be $C := \chi_U/|U|$. Then it holds that:

$$\langle |A^{(\alpha)}_g|, C \rangle < N_r(C) = \begin{cases} e^{-\frac{|U|}{2e}} & |U| \leq 2e/r^* \\ \left(e^{-\frac{|U|}{r/r^*}} \right)^{r/r^*} & \text{else} \end{cases}$$

where $r^* = \max\{r, 2\}$. It is not possible to achieve equality.

The proof is obviously independent of $\alpha$ and available in [10].

**Remark 1** When using the WSSUS model [25] for doubly–dispersive mobile communication channels one typically assumes time–frequency scattering within a shape $U = [0, \tau_d] \times [-B_d, B_d]$ such that $|U| = 2B_d\tau_d \ll 1 < e$, where $B_d$ denotes maximum Doppler bandwidth $B_d$ and $\tau_d$ is maximum delay spread. Then (52) predicts for a $L_1$–normalized scattering function $C := |U|^{-1}\chi_U$, that the best (mean) correlation response ($r = 2$) in using filter $g$ at the receiver and $\gamma$ at the transmitter is bounded above by $e^{-2B_d\tau_d/e}$.

From the definition of $R_1$ in (37) and from (52) of Corollary 14 we know that for $|U| \leq ke$ we have the estimate:

$$k|U|(1 - e^{-\frac{|U|}{2e}}) < \inf_{g, \gamma}(R_1) \leq k|U|(1 - \lambda_{\text{max}}(Q^*Q)^{1/k})$$

which are implicit inequalities for $|U|$. The restriction $|U| \leq e$ for the lower bound can be removed if the second alternative in (52) of Corollary 14 is further studied. However, for simplicity we have considered only the first region which is suited to our application (small $|U|$). In particular, with $R_1 \leq R_\infty|U|$ we have also $R_\infty \geq k(1 - e^{-\frac{|U|}{2e}})$. This proves also the assertion in [33], i.e. a necessary condition for $R_\infty \leq 1$ is that $|U| \leq 2e \ln 2$. Furthermore for $R_\infty \to k$ the size constraint on $U$ vanishes.

**D. Even Spreading Functions and Laguerre Integrals**

Simple estimates for $\langle |A^{(\alpha)}_g|^r, C \rangle$ (and therefore also for $\lambda_{\text{max}}(Q^*Q)$) can be found if $C$ exhibits certain symmetries upon canonical transformations. Let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the transformation $T(\nu) = L \cdot \nu + c$ with a $2n \times 2n$ symplectic matrix $L$ and a phase space

---

9 This means that $\eta(L\mu, L\mu) = \eta(\mu, \mu)$ for all $\mu$. In particular this means that $|\det(L)| = 1$ such that the measure $|U|$ is invariant under $L$. 

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translation $c \in \mathbb{R}^{2n}$. It is well known that $|A_{g\gamma}| = |A_{\tilde{g}\tilde{\gamma}} \circ T|$, where $\tilde{g}$ and $\tilde{\gamma}$ are related to $g$ and $\gamma$ by unitary transforms which depend on $T$. See for example [5, Chapter 4] for a review on metaplectic representation. We have then: $\langle |A_{g\gamma}|^r, C \rangle = \langle |A_{\tilde{g}\tilde{\gamma}}|^r, C \circ T^{-1} \rangle$. In particular this means, that we can always rotate, translate and (jointly) scale $C$ to simple prototype shapes. For example, elliptical (rectangular) shapes can always be transformed to discs (squares) centered at the origin. Further symmetries can be exploited as shown exemplary in the following lemma (for simplicity we consider only $n = 1$):

**Lemma 15** Let be $Q$ the operator with spreading function $\chi_U$. If the shape of $U$ has the symmetry $\chi_U(\mu) = \chi_U(-\mu)$ then for each $m \geq 0$ it holds that:

$$\lambda_{\text{max}}(Q^*Q) \geq \left( \frac{1}{|U|} \int_U l_m(\pi(|\mu|^2))d\mu \right)^2$$

(54)

where $|\mu|^2 = \mu_1^2 + \mu_2^2$ and $l_m$ is the $m$th Laguerre function.

**Proof:** The calculation of $\lambda_{\text{max}}(Q^*Q)$ simplifies much for normal operators which involves the investigation of $Q$ only, i.e. $\lambda_m(Q^*Q) = |\lambda_m(Q)|^2$. For an arbitrary operator $Y$ it follows that $\Sigma_Y^{(\alpha)}(\mu) = \overline{\Sigma_Y^{(\alpha)}(-\mu)} e^{-i4\pi\alpha\zeta(\mu,\mu)}$ is the spreading function of $Y^*$ in polarization $\alpha$. Hence, on the level of spreading functions the normality of $Y$ is equivalent to:

$$\Sigma_Y^{(\alpha)}(\mu) \overline{\Sigma_Y^{(\alpha)}(\nu)} = \Sigma_Y^{(\alpha)}(-\mu) \Sigma_Y^{(\alpha)}(-\nu) \cdot e^{i4\pi\alpha(\zeta(\mu,\mu)+\zeta(\nu,\nu))}$$

(55)

which can be verified using the rules for $S_\mu^{(\alpha)}$ like (2) and (3). The operator $Q$ has by definition the real spreading function $\chi_U$. Hence the desired symmetry is fulfilled for $\alpha = 0$. Let be $h_m$ the $m$th Hermite function. It is known that the ambiguity functions of Hermite functions are given by the Laguerre functions [34] (see for example also [5]). Obviously, the maximal eigen–value fulfills:

$$\lambda_{\text{max}}(Q) \geq \langle h_m, Q h_m \rangle = \frac{1}{|U|} \int_U \langle h_m, S_\mu^{(0)} h_m \rangle d\mu = \frac{1}{|U|} \int_U l_m(\pi|\mu|^2) d\mu$$

(56)

where $l_m(t) = e^{-t/2} L_m^{(0)}(t)$ are the Laguerre functions and $L_m^{(0)}$ are the $0$th Laguerre polynomials.

### 5. Gaussian Signaling and the Corresponding Bounds

The previous part of this section indicates that approximate eigen–functions have to be "Gaussian–like". Hence it makes sense to consider Gaussian signaling explicitly. For simplicity we do this for the time–frequency symmetric case $g = \gamma = 2^\frac{1}{2} e^{-\pi t^2}$ and $n = 1$. We
have the relation:

\[
(S^{(\alpha)}_\nu) g(x) = e^{-\pi(2i\eta(x) + i(1-2\alpha)\zeta(x) + \nu^2)} g(x)
\]  

(57)

if we let \( e := (1, i) \). According to (33) the error \( E_p(\mu) \) can be calculated as:

\[
E_p(\mu) = \| \int_{\mathbb{R}^2} d\nu \Sigma^{(\alpha)}_\nu(v) e^{-i2\pi\eta(\nu, \mu)} \cdot g \cdot f(\nu, \cdot) \|_p
\]  

(58)

The function \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C} \) is defined for a particular polarization \( \alpha \) as:

\[
f(\nu, x) = \begin{cases} e^{-\pi(2i\eta(x)+i(1-2\alpha)\zeta(x)+\nu^2)} - e^{-\pi(\frac{|\mu\nu|}{2}-2i\alpha\zeta(x, \nu))]} & \text{for } C1 \\
 e^{-\pi(2i\eta(x)+i(1-2\alpha)\zeta(x)+\nu^2)} - 1 & \text{for } C2
\end{cases}
\]  

(59)

where we have used that the ambiguity function in polarization \( \alpha \) is \( A^{(\alpha)}_g(\nu) = e^{-\frac{\pi}{2} s_\alpha(\nu)} \) and \( s_\alpha(\nu) := \nu \cdot \nu + 4i\alpha\zeta(\nu, \nu) \). The following Corollary contains the bounds specialized to the Gaussian case:

**Corollary 16 (Gaussian Bounds)** For the case C1 \((k = 1)\) and for the case C2 \((k = 2)\) in polarization \( \alpha = 0 \) it holds for any \( 1 \leq p < \infty \) and \( 1 \leq q \leq \infty \) that:

\[
\frac{E_p(\mu)}{\| \Sigma^{(\alpha)}_\nu \|_q} \leq 32^{\frac{p-2}{4p}} \cdot \| k(1 - e^{-\frac{\pi}{2} s_0}) \|_{q'/p}^{1/p}
\]  

(60)

where \( s_0(\nu) := \langle \nu, \nu \rangle \). For \( q > 1 \) it follows from (60) also:

\[
\frac{E_p(\mu)}{\| \Sigma^{(\alpha)}_\nu \|_q} \leq 32^{\frac{p-2}{4p}} \cdot k \cdot (k|U| (1 - \langle C, e^{-\frac{\pi}{2} s_0} \rangle)^{1/\max(q',p)}
\]  

(61)

where \( C = \chi U/|U| \).

**Proof:** We use the abbreviation \( A_1 = e^{-\pi s_0} \) and \( A_2 = \text{Re} \{ e^{-\frac{\pi}{2} s_\alpha} \} \) as introduced in (27). Only for \( \alpha = 0 \) the case C2 provides an Euclidean distance measure in phase space. Equation (60) of the claim follows from Lemma 8 and from (39) of Lemma 9 together with \( \| V \|_\infty \leq 2 \| g \|_\infty = 32^{1/4} \). If \( q > 1 \) we can relate this further by (40) of Lemma 9 to weighted norms of ambiguity functions. Using the uniform bound \( C_{pq} \leq k \) from Lemma 10 and the relation for \( R_1 \) in (38) we get Gaussian integrals of the form (61) which can now be solved analytically for some cases. For example, if \( U \) is a centered disc of radius \( \sqrt{|U|/\pi} \) we get \( \langle C, e^{\frac{\pi}{2} s_0} \rangle = l(2|U|/k) \) where \( l(x) = 2(1-e^{-x/2})/x \). For a centered square of length \( \sqrt{|U|} \) we have instead \( l(x) = 2erf(\sqrt{x/8})^2/x \).

\[ \square \]
V. Numerical Verification

In this part we shall establish a spreading model with a finite number of random parameters. We shall need this model to verify numerically the bounds derived in this paper. Since several (iterated) integrals are involved which partially can only be computed numerically we have evaluate the achieved accuracy. We aim at computing $E_p(\mu)/\|\Sigma^{(\alpha)}\|_q$ up to a desired accuracy $\Delta$. In our derivation we will assume that single definite integrals can be computed within a given predefined error (for example in using Simpson quadrature).

A. Spreading Model with Finite Number of Parameters

Let us consider a doubly–dispersive channel model with a finite number of fading parameters $c_k$, where $k \in \mathbb{Z}_K^2$ and $\mathbb{Z}_K = \{0 \ldots K - 1\}$. Each fading contribution has its own doubly–dispersive operation on the input signal, hence the model is different from the usual (distributional) models having a finite number of separated paths with fixed Doppler frequencies. The spreading function $\Sigma^{(\alpha)}$ should be of the form:

$$\Sigma^{(\alpha)}(\nu) = \sum_{k \in \mathbb{Z}_K^2} c_k \chi_u(\nu - u(k + o)) = \sum_{k \in \mathbb{Z}_K^2} c_k \chi_1(\nu/u - k + o)$$  \hspace{1cm} (62)

where $\chi_u(y) = \chi_{[0,u]}(y_1)\chi_{[0,u]}(y_2)$ is the characteristic function of the square $[0,u] \times [0,u] =: [0,u]^2$ and $o = (\frac{1}{2}, \frac{1}{2})$. Thus the latter is a disjoint partition of the square $[0,Ku]^2$ with area $(Ku)^2$. In other words, if we fix the support of the spreading function to be $|U|$, then it follows for a $K^2$–sampling of this area that $u = \sqrt{|U|}/K$. For such a model the $q$–norm of the spreading function as needed for the calculation of the ratio $E_p/\|\Sigma^{(\alpha)}\|_q$ is:

$$\|\Sigma^{(\alpha)}\|_q = u^{2/q}\|c\|_q$$  \hspace{1cm} (63)

where $\|c\|_q := (\sum_k |c_k|^q)^{1/q}$ is simply the $q$th vector norm of the vector $c = (\ldots, c_k, \ldots) \in \mathbb{C}^{K^2}$. Let us abbreviate $l = l(k) = k + o$. With (59) we get for the integrand in (58):

$$\int_{\mathbb{R}^2} \Sigma^{(\alpha)}(\nu)e^{-i2\pi\eta(\nu,\mu)}g(x)f(\nu, x)d\nu = \sum_{k \in \mathbb{Z}_K^2} c_k \cdot g(x) \int_{\mathbb{R}^2} \chi_1(\nu/u - l)e^{-i2\pi\eta(\nu, \mu)}f(\nu, x)d\nu$$  \hspace{1cm} (63)

The approximate eigenstructure error reads now as $E_p(\mu) = \|\sum_{k \in \mathbb{Z}_K^2} c_k \cdot g \cdot F_k\|_p$. For $\alpha = 1/2$ and case C2 the integral in $F_k(x)$ can be calculated explicitly. In general, however, $F_k(x)$ has to be computed numerically up to a certain accuracy $\delta$ (it is a well–defined and definite integral). Thus, let the computed value $\bar{F}_k(x)$ be such that pointwise $|\bar{F}_k(x) - F_k(x)| \leq \delta$ for all $x$ and $k$. We would like to use $\bar{F}_k(x)$ instead of $F_k(x)$ to compute the approximation $\bar{E}_p(\mu)$ on $E_p(\mu)$. However we have to restrict the remaining indefinite integral over $x$ to a
finite interval $I := [-L, L]$. With $J$ we denote its complement in $\mathbb{R}$, i.e. $J := \mathbb{R} \setminus I$. Observe from (59) that $|f| \leq 2$, hence $|F_k| \leq 2u^2$ in (63) and that for a Gaussian $\|g\|^p_p = 1/\sqrt{p}$. If we choose $\pi L \geq \max(\sqrt{\log(2u^2/\delta)}, 1)$ we have:

$$
\|F_k g \cdot \chi_J\|_p \leq 2u^2 \|g \cdot \chi_J\|_p = 2u^2 \text{erfc}(\sqrt{\pi p L})^{1/p} \leq \frac{2u^2}{(\pi \sqrt{p L})^{1/p}} e^{-\pi L^2}\tag{64}
$$

For such a chosen $L$ the integration with respect to $x$ over the interval $I = [-L, L]$ can be performed again within an accuracy of $\delta$. This yields for the overall calculation error:

$$
|E_p(\mu) - \tilde{E}_p(\mu)| \leq \delta + \sum_k |c_k| \left(\|\chi_J\|_p + \|F_k g \cdot \chi_J\|_p\right)
$$

If we choose $\delta = \Delta \cdot \|\Sigma^{(\alpha)}_q\|_{\infty} \cdot (1 + 2\|c\|_1 \cdot p^{-\frac{1}{2p}})^{-1}$ (and $L$ respectively) we can guarantee that the error on $E_p(\mu)/\|\Sigma^{(\alpha)}_q\|_{\infty}$ is below $\Delta$.

**Remark 2 (Interference Estimates for Statistical Models)** Consider the following example: The transmitter sends the signal $S_{\mu, \gamma}$ through the unknown channel $\mathcal{H}$. Let us again for simplicity use the finite–parameter spreading model (62) for a support $U$ of square shape. The receiver already knows the vector of fading parameters $c$ for the spreading function $\Sigma_q^{(\alpha)}$ of the channel, the pulse $g$ and $\gamma$ and the time–frequency slot $\mu$. The normalized $q$–norms $c_q = \|c\|_q \cdot K^{-2/q}$ of the $K^2$ fading coefficients characterize the statistical model for the spreading such that $\|\Sigma_q^{(\alpha)}\|_q = |U|^{1/q} \cdot c_q$. If the contribution of this particular slot $\mu$ is removed from the signal it remains $e := \mathcal{H} S_{\mu, \gamma} - \lambda(\mu) S_{\mu, g}$. Let us assume that the receiver expects another information in the span of the function $f$ (for example $f = S_{\nu, g}$ could be another slot $\nu$). The interference will be $\langle f, e \rangle$. Let be $A_f(p) = \|f\|_{p'} \cdot 32^{\frac{p-2}{2p}}$. We have

$$
|\langle f, e \rangle| \leq E_p(\mu) \cdot \|f\|_{p'} < A_f(p) \cdot (|U| (1 - L^2))^{1/\max(q', p')} \cdot |U|^{1/q} \cdot c_q\tag{66}
$$

With the assumption that $|U| \leq 1$ we use $|U|^{1/\max(q', p') + 1/q} \leq |U|$ such that:

$$
|\langle f, e \rangle| < A_f(p) \cdot (1 - L^2)^{1/\max(q', p')} \cdot |U| \cdot c_q\tag{67}
$$

This means, for different statistical models (characterized by $c_q$) and functions $f$ (characterized by $\|f\|_{p'}$ in the quantity $A_f(p)$) we can characterize the amount of interference.
B. Numerical Experiments

We will consider now the case where the coefficients $c_k$ of the vector $c \in \mathbb{C}^{K^2}$ are identical, independent and normal distributed which refers to the doubly–dispersive Rayleigh fading channel. The square shaped support $U$ has a random size $|U|$ taken from a distribution uniformly on the interval $[10^{-3}, 10^{-2}]$ corresponding to values of the time–frequency spread relevant in mobile communication. Each realization of the fading factor $c$ and $u = \sqrt{|U|}$ parameterize via (62) a random spreading function $\Sigma^{(\alpha)}_{H}$ in a given polarization $\alpha$ which give itself rise to a random channel operator $\mathcal{H}$ by Lemma 1. On this random channel we have evaluated $E_p(\mu)$ for Gaussian signaling as described previously in Section IV-E. For each realization we have taken $\mu$ uniformly from $[-5, 5]^2$. We have calculated $N = 1000$ Monte Carlo (MC) runs for different values of $p$ and $q$. For each run $E_p(\mu)/\|\Sigma^{(\alpha)}_{H}\|_q$ has been computed (corresponding to one point in Fig.1 and Fig.2) up to an accuracy of $\Delta = 10^{-8}$. The computed values $E_p(\mu)$ are compared to the uniform bound in (42) of Lemma 11 which depends only on the support and is valid for any normalized $g$ and $\gamma$. Improved bounds are valid only for particular $g$ and $\gamma$ like the Laguerre/Gauss (GL) bound from Theorem (7). Fig.1 shows the case $C1$ for $p = q = 2$, where we expect the most tight results. The GL bound improves the uniform estimates approximately by a factor of 10. However the computed MC values are still below this estimate by a factor of approximately two. The latter estimate degrades to a factor of approximately 10 for $p = 3$ and $q = 3/2$ as displayed in Fig.2.

VI. Conclusions

In this paper we have considered doubly–dispersive channels with compactly supported spreading. We have shown to what level of approximation error a description as simple multiplication operators is valid. We have focused on two well known choices of such a description, i.e. the multiplication with the (generalized) Weyl symbol of the operator and the case of Wigner smoothing. We found that in both cases the approximation errors can be bounded by the size of the support of the spreading function. Our estimates improve and generalize recent results in this direction. Furthermore we have drawn the relation to localization operators and fidelity measures known from the theory of pulse shaping. Finally, we have verified our estimates using Monte Carlo methods with a precise control of the numerical uncertainties.
Fig. 1. *Approximate Eigenstructure for the case C1, p = 2, q = 2*: Verification of 1000 Monte Carlo runs with the uniform bound in Lemma 11 and the optimized Laguerre/Gauss bound of Theorem 7.

Fig. 2. *Approximate Eigenstructure for the case C1, p = 3, q = 1.5*: Verification of 1000 Monte Carlo runs with the uniform bound in Lemma 11 and the optimized Laguerre/Gauss bound of Theorem 7.
The author would like to thank Gerhard Wunder, Thomas Strohmer and Holger Boche for helpful discussions on this topic. The author extents a special thanks to the anonymous reviewers for their constructive comments, which have improved the paper.

APPENDIX

A. Proof of Lemma 3

The following proof is motivated by [6].

Proof: For each complex Hilbert space with \( \|x\|_2^2 = \langle x, x \rangle \) the following inequality

\[
\|x - y\|_2^2 \leq \|x\|_2^2 - \|y\|_2^2 + 2 |\langle y, x - y \rangle| \tag{68}
\]

holds. Now let \( x = \mathcal{H} S_{\mu}^{(a)} \gamma \) and \( y = L_{\mathcal{H}}^{(a)} (\mu) S_{\mu}^{(a)} \gamma \). Using (10) the following upper bounds

(a) \( = \langle \gamma, \left( S_{\mu}^{(a)} \mathcal{H}^* \mathcal{H} S_{\mu}^{(a)} - L_{\mathcal{H}}^{(a)} \right) \gamma \rangle \)

\[
\leq |L_{\mathcal{H}}^{(a)}|^2 - |L_{\mathcal{H}}^{(a)}| \gamma \| \left( \frac{S_{\mu}^{(a)} \mathcal{H}^* \mathcal{H} S_{\mu}^{(a)} - L_{\mathcal{H}}^{(a)} \gamma \| \right) |
\]

\[
= |L_{\mathcal{H}}^{(a)}| - |L_{\mathcal{H}}^{(a)}|^2 + \int_{\mathbb{R}^2} \Sigma_{\mathcal{H}}^{(a)}(\nu) e^{-i2\pi \gamma(\nu, \mu)}(A^{(a)}(\nu) - 1) d\nu |
\]

\[
\leq |L_{\mathcal{H}}^{(a)}| - |L_{\mathcal{H}}^{(a)}|^2 + \| \Sigma_{\mathcal{H}}^{(a)} \|_1 \tag{69}
\]

(b) \( = L_{\mathcal{H}}^{(a)}(\mu) \| \left( \frac{S_{\mu}^{(a)} \mathcal{H}^* \mathcal{H} S_{\mu}^{(a)} - L_{\mathcal{H}}^{(a)} \gamma \| \right) \gamma \rangle \)

\[
= |L_{\mathcal{H}}^{(a)}(\mu)| \int_{\mathbb{R}^2} \Sigma_{\mathcal{H}}^{(a)}(\nu) e^{-i2\pi \gamma(\nu, \mu)}(A^{(a)}(\nu) - 1) d\nu | \leq |L_{\mathcal{H}}^{(a)}(\mu)| \| \Sigma_{\mathcal{H}}^{(a)} \|_1 \]

will give the proposition. 

B. Proof of Lemma 9

Proof: Firstly – note that Hölder’s inequality for the index pair \((1, \infty)\) gives \( V_p^p \leq V_{p-2} V_2^2 \) with equality for \( p = 2 \); and in turn \( \|V_p\|_{q'} \leq \|V_{\infty}^{p-2} \cdot V_2^{2q'/p} \|_{q'} \). For \( q > 1 \) we can rewrite this and use again Hölders inequality. We get:

\[
\|V_p\|_{q'} \leq \|V_{\infty}^{p-2} \cdot V_2^{2q'/p} \|_{1/q'} \leq \|V_{\infty}^{p-2} \cdot V_2^{2q'/p} \|_{1/q'} = \|V_{\infty}^{p-2} \cdot R_{q'/p}^{1/p} \|_{q'} \tag{70}
\]

For \( q = 1 \) (i.e. \( q' = \infty \)) we obtain rhs of the last equation directly:

\[
\|V_p\|_{\infty} = \|V_{\infty}^{p-2} \cdot V_2^{2} \|_{\infty} \leq \|V_{\infty}^{p-2} \cdot R_{\infty}^{1/p} \|_{\infty} \tag{71}
\]
which proves (39) of this lemma. From the definition of \( R \) in (36) it is obvious that the minimum of the bounds is taken at \( B(U) = A_{q^{1/p}}^{(3)}(U) \) which is provided by C1. Because there it holds always equality for \( p = 2 \) this is also the optimizer for \( \|V\|_{q'} \) for any \( q \). From (70) we get further for \( p \leq q' < \infty \):

\[
R_{q'/p}^{1/p} = R_\infty^{1/p} \|R/\chi_U\|_{q'/p}^{1/p} \leq R_\infty^{q'/p} R_1^{1/q'}
\]  

(72)

because in this case \((R(\nu)/R_\infty)^{q'/p} \leq R(\nu)/R_\infty\) for all \( \nu \in U \). For \( q' = p \) equality occurs in the last inequality. This proves (39) of this lemma for \( q' \geq p \). For \( q' < p \) we use the concavity of \( R_{q'/p} \), i.e. we proceed instead as follows:

\[
R_{q'/p}^{1/p} = \left( \|U| \cdot \|R_{q'/p}^{1/p} \chi_U/\|U\|_1 \right)^{1/q'}
\]

\[
\leq |U|^{1/q'} \cdot \|R \chi_U/\|U\|_1^{1/p} \leq |U|^{\frac{q'}{q'} \cdot R_{1/p}^{1/p}}
\]

The bounds (72) and (73) agree for \( q' = p \) and are tight for \( q = p = 2 \).

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