ON ASYMPTOTICS FOR $C_0$-SEMIGROUPS

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Abstract. We stretch the spectral bound equal growth bound condition along with a generalized Lyapunov stability theorem, known to hold for $C_0$-semigroups of normal operators on complex Hilbert spaces, to $C_0$-semigroups of scalar type spectral operators on complex Banach spaces. For such semigroups, we obtain exponential estimates with the best stability constants. We also extend to a Banach space setting a celebrated characterization of uniform exponential stability for $C_0$-semigroups on complex Hilbert spaces and thereby acquire a characterization of uniform exponential stability for scalar type spectral and eventually norm-continuous $C_0$-semigroups.

1. Introduction

Based on the recently established fact that $C_0$-semigroups of scalar type spectral operators on complex Banach spaces are subject to a precise weak spectral mapping theorem [18], we stretch to such semigroups the spectral bound equal growth bound condition along with a generalized Lyapunov stability theorem (see Preliminaries), known to hold for $C_0$-semigroups of normal operators on complex Hilbert spaces, and further obtain exponential estimates with the best stability constants for them. We also extend to a Banach space setting the celebrated Gearhart-Prüss-Greiner characterization of uniform exponential stability for $C_0$-semigroups on complex Hilbert spaces [9, Theorem V.3.8] (see [10,11,25]) and thereby acquire a characterization of uniform exponential stability for scalar type spectral and eventually norm-continuous $C_0$-semigroups.

2. Preliminaries

Here, we outline certain essential preliminaries.

2.1. Resolvent Set and Spectrum.

For a closed linear operator $A$ in a complex Banach space $X$, the set

$$\rho(A) := \left\{ \lambda \in \mathbb{C} \mid \exists R(\lambda, A) := (A - \lambda I)^{-1} \in L(X) \right\}$$

($I$ is the identity operator on $X$, $L(X)$ is the space of bounded linear operators on $X$) and its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ are called its resolvent set and spectrum, respectively.

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2.2. $C_0$-Semigroups.

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space $(X, \| \cdot \|)$ with generator $A$ subject to a weak spectral mapping theorem

\[(\text{WSMT}) \quad \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \setminus \{0\}, \; t \geq 0, \]

($\sigma$ is the closure of a set) satisfies the following spectral bound equal growth bound condition:

\[(\text{SBeGB}) \quad s(A) = \omega_0, \]

where

\[s(A) := \sup \{\Re \lambda | \lambda \in \sigma(A)\} \quad (s(A) := -\infty \text{ if } \sigma(A) = \emptyset)\]

is the spectral bound of the generator $A$ and

\[\omega_0 := \inf \{\omega \in \mathbb{R} | \exists M = M(\omega) \geq 1 : \|T(t)\| \leq Me^{\omega t}, \; t \geq 0\}\]

(see also \cite{22}).

Generally,

\[-\infty \leq s(A) \leq \omega_0 = \inf_{t > 0} \frac{1}{t} \ln \|T(t)\| = \lim_{t \to \infty} \frac{1}{t} \ln \|T(t)\| < \infty\]

(see, e.g., \cite[Proposition V.1.22]{9}).

An eventually norm-continuous $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space, i.e., such that, for some $t_0 > 0$, the operator function

\[[t_0, \infty) \ni t \mapsto T(t) \in L(X),\]

is continuous relative to the operator norm, is subject to the following stronger version of \(\text{WSMT}\):

\[(\text{SMT}) \quad \sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \; t \geq 0,\]

called a spectral mapping theorem (see \cite[Proposition V.2.3]{9} and \cite[Theorem V.2.8]{9}). The class of eventually norm-continuous $C_0$-semigroups encompasses $C_0$-semigroups with certain regularity properties, such as eventually compact and eventually differentiable, in particular analytic and uniformly continuous (see \cite[Section II.5]{9}).

The asymptotic behavior of a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space with generator $A$ satisfying spectral bound equal growth bound condition \(\text{SBeGB}\) is governed by the spectral bound of its generator and, in particular, is subject to the subsequent generalization of the classical Lyapunov Stability Theorem \cite[Theorem V.3.6]{9} (see also \cite{15}).

**Theorem 2.1** (Generalized Lyapunov Stability Theorem).

A $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space $X$ with generator $A$, for which spectral bound equal growth bound condition \(\text{SBeGB}\) holds, is uniformly exponentially stable, i.e., \(\omega_0 < 0\), or equivalently,

\[\exists \omega < 0, \; \exists M = M(\omega) \geq 1 : \|T(t)\| \leq Me^{\omega t}, \; t \geq 0,\]
iff 

\[ s(A) < 0. \]

Cf. [9, Definition V.3.1], [9, Proposition V.3.5], and [9, Theorem V.3.7].

The following statement [10,11,25] characterizes uniform exponential stability for \( C_0 \)-semigroups on complex Hilbert spaces.

**Theorem 2.2** (Gearhart-Prüss-Greiner Characterization [9, Theorem V.3.8]).

A \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on a complex Hilbert space space \((X, \langle \cdot, \cdot \rangle, \| \cdot \|)\) with generator \( A \) is uniformly exponentially stable iff

\[ \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \subseteq \rho(A) \quad \text{and} \quad \sup_{\text{Re} \lambda \geq 0} \| R(\lambda, A) \| < \infty. \]

A \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) (of normal operators) on a complex Hilbert space generated by a normal operator \( A \) is subject to the following precise version of weak spectral mapping theorem (WSMT):

\[(\text{PWSMT}) \quad \sigma(T(t)) = e^{t\sigma(A)}, \quad t \geq 0,\]

[9, Corollary V.2.12] without being a priori eventually norm continuous, and hence, to spectral bound equal growth bound condition (SBeGB) along with the Generalized Lyapunov Stability Theorem (Theorem 2.1).

2.3. **Scalar Type Spectral Operators.**

A **scalar type spectral operator** is a densely defined closed linear operator \( A \) in a complex Banach space with strongly \( \sigma \)-additive spectral measure (the resolution of the identity) \( E_A(\cdot) \), which assigns to the Borel sets of the complex plane projection operators on \( X \) and has the operator’s spectrum \( \sigma(A) \) as its support [4,5,8].

Associated with such an operator is the **Borel operational calculus**, assigning to each Borel measurable function \( F : \sigma(A) \to \mathbb{C} \) (\( \mathbb{C} := \mathbb{C} \cup \{ \infty \} \) is the extended complex plane) with \( E_A(\{ \lambda \in \mathbb{C} \mid F(\lambda) = \infty \}) = 0 \) a scalar type spectral operator

\[ F(A) := \int_{\sigma(A)} F(\lambda) \, dE_A(\lambda) \]

in \( X \), whose spectral measure is the image of \( E_A(\cdot) \) under \( F(\cdot) \), i.e.,

\[ E_{F(A)}(\delta) = E_A(F^{-1}(\delta)), \quad \delta \in \mathcal{B}(\mathbb{C}), \]

(\( \mathcal{B}(\mathbb{C}) \) is the Borel \( \sigma \)-algebra on \( \mathbb{C} \)), with

\[ A = \int_{\sigma(A)} \lambda \, dE_A(\lambda) \]

[1,4,5,8].

On a complex finite-dimensional Banach space, scalar type spectral operators are those linear operators, which furnish an eigenbasis for the space, i.e., allow a diagonal matrix representation (see, e.g., [4,5,8]).

In a complex Hilbert space, scalar type spectral operators are those that are similar to normal operators [26] (see also [14,16]), the latter being the scalar type spectral
operators for which the corresponding spectral measure projections are orthogonal (see, e.g., [7, 24]).

Various examples of scalar type spectral operators, including differential operators arising in the study of linear systems of partial differential equations, in particular perturbed Laplacians, can be found in [8].

Due to its strong $\sigma$-additivity, the spectral measure is uniformly bounded, i.e.,

\[(2.2) \exists M \geq 1 \forall \delta \in \mathcal{B} : \|E_A(\delta)\| \leq M\]

(see, e.g., [6]).

By [8, Theorem XVIII.2.11 (c)], for a Borel measurable function $F : \sigma(A) \to \overline{\mathbb{C}}$, the operator $F(A)$ is bounded iff $F(\cdot)$ is $E_A$-essentially bounded, i.e.,

\[E_A\text{-ess sup}_{\lambda \in \sigma(A)} |F(\lambda)| < \infty,\]

where $M \geq 1$ is from (2.2).

A scalar type spectral $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (i.e., a $C_0$-semigroup of scalar type spectral operators) on a complex Banach space $X$ is generated by a scalar type spectral operator [3, 23], which is the case iff

\[s(A) < \infty\]

with

\[T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \geq 0,\]

[20, Proposition 3.1], the orbit maps of the semigroup

\[T(t)f = e^{tA}f, \ t \geq 0, f \in X,\]

being the weak solutions (also called the mild solutions) of the associated abstract evolution equation

\[y'(t) = Ay(t), \ t \geq 0,\]

[21] (see also [2, 9]).

### 3. SBeGB Condition and Exponential Estimates

By the Precise Weak Spectral Mapping Theorem [18, Theorem 5.1], a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator $A$ is subject to precise weak spectral mapping theorem (PWSMT). Thus, by [9, Proposition V.2.3] (see Preliminaries), we arrive at the following

**Corollary 3.1 (SBeGB Condition).**

For a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator $A$, spectral bound equal growth bound condition (SBeGB) holds.
Remarks 3.1.

- Considering that, for a scalar type spectral operator $A$ in a complex Banach space, $\sigma(A) \neq \emptyset$, when such an operator generates a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$,
  $$-\infty < s(A) = \omega_0 < \infty$$
  (see Preliminaries).

- For a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$, the exponential estimates
  \[(EE) \exists \omega \in \mathbb{R}, \exists M = M(\omega) \geq 1: \|T(t)\| \leq Me^{\omega t}, \ t \geq 0,\]
  hold for all $\omega > \omega_0$ with some $M = M(\omega) \geq 1$ (see Preliminaries).

However, the best stability constants, i.e., the smallest numbers $\omega$ and $M$ for which $(EE)$ is valid, (cf. [13]) need not exist, i.e., the infimum in (2.1) may not be attained even when $\omega_0 \in \mathbb{R}$ (see [9, Section I.1] and Example 3.1).

Proposition 3.1 (Exponential Estimate for Scalar Type Spectral $C_0$-Semigroups).

Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup (of scalar type spectral operators) on a complex Banach space $(X, \| \cdot \|)$ generated by a scalar type spectral operator $A$ with spectral measure $E_A(\cdot)$. Then

\[(3.1) \quad \|T(t)\| \leq 4M_0e^{\omega_0 t}, \ t \geq 0,\]

where $\omega_0 = s(A)$ is the best stability constant in the exponent and

\[(3.2) \quad M_0 := \sup_{\delta \in \mathbb{R}(\mathbb{C})} \|E_A(\delta)\| \geq 1.\]

Proof. Since

$$T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \ t \geq 0,$$

(see Preliminaries), by (2.3), for any $t \geq 0$,

$$\|T(t)\| = \|e^{tA}\| = \left\| \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda) \right\| \leq 4M \|E_A\| - \text{ess sup} \|e^{t\lambda}\| \leq 4M \sup_{\lambda \in \sigma(A)} \|e^{t\lambda}\|$$

$$= 4M \sup_{\lambda \in \sigma(A)} e^{t\Re \lambda} \leq 4Me^{s(A)t} \quad \text{in view of } s(A) = \omega_0;$$

where $M \geq 1$ is from (2.2).

Therefore,

$$\|T(t)\| = \|e^{tA}\| \leq 4M_0e^{\omega_0 t}, \ t \geq 0,$$

where $\omega_0 = s(A)$ (see Corollary 3.1) is the best stability constant in the exponent and $M_0$, defined by (3.2), is the smallest $M \geq 1$ for which (2.2) holds. \qed
Remarks 3.2.

- Thus, for a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator $A$ with spectral measure $E_A(\cdot)$, $\omega_0 = s(A)$ is the best stability constant in the exponent and the other best stability constant satisfies the estimate

$$1 \leq \min \{M \geq 1 \mid \|T(t)\| \leq M e^{\omega_0 t}, \ t \geq 0\} \leq 4 \sup_{\delta \in \mathbb{C}} \|E_A(\delta)\|.$$

- As the next example demonstrates, for a non-scalar-type-spectral $C_0$-semigroup on a complex Banach space, even under spectral bound equal growth bound condition (SBeGB), the best stability constants need not exist.

Example 3.1. On the complex Banach space $l^{(2)}_p$ ($1 \leq p \leq \infty$), the bounded linear operator $A$ of multiplication by the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which is nonzero nilpotent, and hence, not scalar type spectral (see, e.g., [8]), generates the uniformly continuous (not scalar type spectral) semigroup of its exponentials

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \ t \geq 0,$$

where $e^{tA}, t \geq 0$, is the bounded linear operator on $l^{(2)}_p$ of multiplication by the matrix

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

(see, e.g., [9, 19]).

Although, in the considered case,

$$s(A) = \omega_0 = 0,$$

the exponential estimate

$$\|T(t)\| \leq M e^{\omega_0 t} = M, \ t \geq 0,$$

holds for no $M \geq 1$ (cf. [9]).

For $C_0$-semigroups of normal operators on complex Hilbert spaces exponential estimate (3.1) can be refined as follows.

**Proposition 3.2** (Exponential Estimate for Normal $C_0$-Semigroups).

Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup (of normal operators) on a complex Hilbert space $(X, (\cdot, \cdot), \|\cdot\|)$ generated by a normal operator $A$ with spectral measure $E_A(\cdot)$. Then

$$\|T(t)\| \leq e^{\omega_0 t}, \ t \geq 0,$$

(3.3)

with $\omega_0 = s(A)$ and $M_0 := 1$ being the best stability constants.
Proof. Since
\[ T(t) = e^{tA} := \int_{\sigma(A)} e^{t\lambda} dE_A(\lambda), \quad t \geq 0, \]
(see, e.g., \([9, 24]\)), for arbitrary \( t \geq 0 \) and \( f \in X \), by the properties of the Borel operational calculus,
\[
\|T(t)f\| = \|e^{tA}f\| = \left( \int_{\sigma(A)} |e^{t\lambda}|^2 d(E_A(\lambda)f, f) \right)^{1/2}
\leq \left( \int_{\sigma(A)} e^{2t \text{Re}\lambda} d(E_A(\lambda)f, f) \right)^{1/2}
\leq \left[ e^{2s(A)t} \|E_A(\sigma(A))f\|^2 \right]^{1/2}
\]
since \( E_A(\sigma(A)) = I \);
\[
= \left[ e^{2s(A)t} \|f\|^2 \right]^{1/2} = e^{s(A)t} \|f\|
\]
in view of \( s(A) = \omega_0 \);
\[
= e^{\omega_0 t} \|f\|.\]
Whence, we obtain exponential estimate (3.3), in which \( \omega_0 = s(A) \) (see Corollary 3.1) and \( M_0 := 1 \) are the best stability constants. \(\square\)

4. Generalized Lyapunov Stability Theorem

From the SBeGB Condition Corollary (Corollary 3.1) and the Exponential Estimate for Scalar Type Spectral \( C_0 \)-Semigroups (Proposition 3.1), we arrive at the following version of the Generalized Lyapunov Stability Theorem (Theorem 2.1) for scalar type spectral \( C_0 \)-semigroups.

**Theorem 4.1** (GLST for Scalar Type Spectral \( C_0 \)-Semigroups).
A \( C_0 \)-Semigroup \( \{T(t)\}_{t \geq 0} \) (of scalar type spectral operators) on a complex Banach space generated by a scalar type spectral operator \( A \) is uniformly exponentially stable iff
\[ s(A) < 0, \]
in which case
\[ \|T(t)\| \leq 4M_0 e^{\omega_0 t}, \quad t \geq 0, \]
where
\[ \omega_0 = s(A) \quad \text{and} \quad M_0 := \sup_{\delta \in \partial(\mathbb{C})} \|E_A(\delta)\| \geq 1. \]

For the \( C_0 \)-semigroups of normal operators on complex Hilbert spaces, in view of the Exponential Estimate for Normal \( C_0 \)-Semigroups (Proposition 3.2), the exponential estimate in the prior statement can be refined as follows.
Corollary 4.1 (GLST for Normal \( C_0 \)-Semigroups).

A \( C_0 \)-Semigroup \( \{T(t)\}_{t \geq 0} \) (of normal operators) on a complex Hilbert space generated by a normal operator \( A \) is uniformly exponentially stable iff

\[
s(A) < 0,
\]

in which case

\[
\|T(t)\| \leq e^{\omega_0 t}, \; t \geq 0,
\]

where \( \omega_0 = s(A) \).

5. Uniform Exponential Stability

The following statement extends the Gearhart-Prüss-Greiner Characterization (Theorem 2.2) \[10, 11, 25\] to a Banach space setting.

Theorem 5.1 (Characterization of Uniform Exponential Stability).

For a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on a complex Banach space \((X, \| \cdot \|)\) with generator \( A \) to be uniformly exponentially stable it is necessary and, provided spectral bound equal growth bound condition (SBeGB) holds, sufficient that

\[
\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \subseteq \rho(A) \text{ and } \sup_{\text{Re} \lambda \geq 0} \|R(\lambda, A)\| < \infty.
\]

Proof.

Necessity. Suppose that a \( C_0 \)-semigroup \( \{T(t)\}_{t \geq 0} \) on a complex Banach space \((X, \| \cdot \|)\) is uniformly exponentially stable. Then

\[
\exists \omega < 0, \exists M = M(\omega) \geq 1 : \|T(t)\| \leq Me^{\omega t}, \; t \geq 0.
\]

Therefore,

\[
\{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq 0 \} \subseteq \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda > \omega \} \subseteq \rho(A).
\]

Also, for arbitrary \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > \omega \) and \( f \in X \),

\[
R(\lambda, A)f = -\int_0^\infty e^{-\lambda t}T(t)f \, dt
\]

(see, e.g., \[9, 12\]) and

\[
\|R(\lambda, A)f\| = \left\| -\int_0^\infty e^{-\lambda t}T(t)f \, dt \right\| \leq \int_0^\infty \|e^{-\lambda t}T(t)f\| \, dt \\
\leq \int_0^\infty e^{-\text{Re} \lambda t} \|T(t)\| \|f\| \, dt \\
\leq M \int_0^\infty e^{-(\text{Re} \lambda - \omega) t} \|f\| \, dt = \frac{M}{\text{Re} \lambda - \omega} \|f\|,
\]

which implies that

\[
\sup_{\text{Re} \lambda \geq 0} \|R(\lambda, A)\| \leq \sup_{\text{Re} \lambda \geq 0} \frac{M}{\text{Re} \lambda - \omega} = -\frac{M}{\omega} < \infty,
\]

completing the proof of the necessity.

Sufficiency. Let us prove this part by contradiction.
Suppose that a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on a complex Banach space $(X, \| \cdot \|)$ with generator $A$ is subject to spectral bound equal growth bound condition (SBeGB) and conditions (5.1) but is not uniformly exponentially stable, which, by the Generalized Lyapunov Stability Theorem (Theorem 2.1), is equivalent to the fact that

\[(5.3) \quad s(A) := \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \} \geq 0.\]

Observe that this implies, in particular, that $\sigma(A) \neq \emptyset$ (see Preliminaries).

In view of the inclusion

\[\{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \} \subseteq \rho(A),\]

(5.3) implies that

\[(5.4) \quad s(A) = 0.\]

The operator $A$ being closed, for an arbitrary $\lambda \in \rho(A)$,

\[\|R(\lambda, A)\| \geq \frac{1}{\dist(\lambda, \sigma(A))},\]

where

\[\dist(\lambda, \sigma(A)) := \inf_{\mu \in \sigma(A)} |\mu - \lambda|,\]

(see, e.g., [6, 19]), which, in view of (5.4), implies that

\[\sup_{\Re \lambda \geq 0} \|R(\lambda, A)\| \geq \sup_{\Re \lambda \geq 0} \frac{1}{\dist(\lambda, \sigma(A))} = \frac{1}{\inf_{\Re \lambda \geq 0} \dist(\lambda, \sigma(A))} = \infty,\]

contradicting (5.1).

The obtained contradiction completes the proof of the sufficiency, and hence, of the entire statement. \hfill \Box

Remarks 5.1.

- Thus, the necessity of the Gearhart-Prüss-Greiner characterization of the uniform exponential stability [9, Theorem V.3.8] holds in a Banach space setting.

- As the next example shows, the requirement that the semigroup be subject to spectral bound equal growth bound condition (SBeGB) in the sufficiency of the prior characterization is essential and cannot be dropped.

Example 5.1. As discussed in [9, Counterexample V.1.26], the left translation $C_0$-semigroup

\[\{T(t)f\}(x) := f(x+t), \quad t, x \geq 0,\]

on the complex Banach space

\[X := \left\{ f \in C(\mathbb{R}_+) \mid \lim_{x \to -\infty} f(x) = 0 \text{ and } \int_{0}^{\infty} |f(x)|e^x \, dx < \infty \right\}\]

($\mathbb{R}_+ := [0, \infty)$) with the norm

\[X \ni f \mapsto \|f\| := \|f\|_\infty + \|f\|_1 = \sup_{x \geq 0} |f(x)| + \int_{0}^{\infty} |f(x)|e^x \, dx\]
is generated by the differentiation operator
\[ Af := f' \]
with the domain
\[ D(A) := \{ f \in X \mid f \in C^1(\mathbb{R}_+), f' \in X \} , \]
for which
\[ \sigma(A) = \{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \leq -1 \} , \]
and hence \( s(A) = -1. \)
The semigroup is not uniformly exponentially stable since
\[ \| T(t) \| = 1, \ t \geq 0, \]
and hence, \( \omega_0 = 0. \)
Thus,
\[ s(A) \neq \omega_0. \]
However, the resolvent
\[ R(\lambda, A)f = -\int_0^\infty e^{-\lambda t}T(t)f \, dt, \ f \in X, \]
of the generator \( A \) exists for all \( \lambda \in \mathbb{C} \) with \( \text{Re } \lambda > -1 \) and
\[ \sup_{\text{Re } \lambda \geq 0} \| R(\lambda, A) \| < \infty \]
(see [9, Comments V.3.9]).

By the SBeGB Condition Corollary (Corollary 3.1) and [9, Corollary V.2.9] (see also [9, Corollary V.2.10]), we arrive at

**Corollary 5.1** (Characterization of Uniform Exponential Stability).
*For a \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) on a complex Banach space \( (X, \| \cdot \|) \) with generator \( A \) to be uniformly exponentially stable it is necessary and, provided the semigroup is scalar type spectral or eventually norm-continuous, in particular eventually compact, eventually differentiable, analytic, or uniformly continuous, sufficient that*
\[ \{ \lambda \in \mathbb{C} \mid \text{Re } \lambda \geq 0 \} \subseteq \rho(A) \quad \text{and} \quad \sup_{\text{Re } \lambda \geq 0} \| R(\lambda, A) \| < \infty. \]

On the regularity of scalar type spectral \( C_0 \)-semigroups, see [17].

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