Riesz Transform Characterization of Hardy Spaces Associated with Ball Quasi-Banach Function Spaces

Fan Wang¹ · Dachun Yang¹ · Wen Yuan¹

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Abstract
Let \( X \) be a ball quasi-Banach function space satisfying some mild assumptions and \( H_X(\mathbb{R}^n) \) the Hardy space associated with \( X \). In this article, the authors introduce both the Hardy space \( H_X(\mathbb{R}^n_+^1) \) of harmonic functions and the Hardy space \( \mathbb{H}_X(\mathbb{R}^n_+^1) \) of harmonic vectors, associated with \( X \), and then establish the isomorphisms among \( H_X(\mathbb{R}^n) \), \( H_X,2(\mathbb{R}^n_+^1) \), and \( \mathbb{H}_X,2(\mathbb{R}^n_+^1) \), where \( H_X,2(\mathbb{R}^n_+^1) \) and \( \mathbb{H}_X,2(\mathbb{R}^n_+^1) \) are, respectively, certain subspaces of \( H_X(\mathbb{R}^n_+^1) \) and \( \mathbb{H}_X(\mathbb{R}^n_+^1) \). Using these isomorphisms, the authors establish the first order Riesz transform characterization of \( H_X(\mathbb{R}^n) \). The higher order Riesz transform characterization of \( H_X(\mathbb{R}^n) \) is also obtained. The results obtained in this article have a wide range of generality and can be applied to classical Hardy spaces, weighted Hardy spaces, variable Hardy spaces, Herz–Hardy spaces, Lorentz–Hardy spaces, mixed-norm Hardy spaces, local generalized Herz–Hardy spaces, and mixed-norm Herz–Hardy spaces and all the obtained results on the aforementioned last five Hardy-type spaces are completely new.

Keywords  Riesz transform characterization · Ball quasi-Banach function space · Hardy space · Poisson integral

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Dachun Yang
dcyang@bnu.edu.cn
Fan Wang
fanwang@mail.bnu.edu.cn
Wen Yuan
wenyuan@bnu.edu.cn

¹ Laboratory of Mathematics and Complex Systems (Ministry of Education of China), School of Mathematical Sciences, Beijing Normal University, Beijing 100875, The People’s Republic of China
1 Introduction

Let $S(\mathbb{R}^n)$ be the set of all Schwartz functions on $\mathbb{R}^n$, equipped with the well-known topology determined by a countable family of norms. Recall that, for any $j \in \{1, \ldots, n\}$, the $j$-th Riesz transform $R_j$ is defined by setting, for any $f \in S(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$R_j(f)(x) := \lim_{\delta \to 0^+} c(n) \int_{\{y \in \mathbb{R}^n : |y| > \delta\}} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy,$$

here and thereafter, $\delta \to 0^+$ means that $\delta \in (0, \infty)$ and $\delta \to 0$ and

$$c(n) := \frac{\Gamma[(n+1)/2]}{\pi^{(n+1)/2}}$$

with $\Gamma$ denoting the Gamma function. It is well known that Riesz transforms are natural generalizations of the Hilbert transform to the Euclidean space of higher dimension and the most typical examples of Calderón–Zygmund operators and have many interesting and useful properties (see, for instance, [23, 51, 52] and their references). Indeed, they are the simplest, nontrivial, and “invariant” operators under the acting of the group of rotations on the Euclidean space $\mathbb{R}^n$, and they also constitute typical and important examples of Fourier multipliers. Moreover, they can be used to mediate between various combinations of partial derivatives of functions. All these properties make Riesz transforms ubiquitous in mathematics; see [51] for more details on their applications.

Let $\mathcal{M}(\mathbb{R}^n)$ be the set of all measurable functions on $\mathbb{R}^n$. Recall that the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (0, \infty)$ is defined by setting

$$L^p(\mathbb{R}^n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{L^p(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \, dx \right]^{1/p} < \infty \right\}.$$

A very classical result about Riesz transforms is that they are bounded on the Lebesgue space $L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$. However, when $p \in (0, 1)$, the Riesz transform is not bounded on $L^p(\mathbb{R}^n)$ anymore. As a natural generalization and a natural substitute of $L^p(\mathbb{R}^n)$ with $p \in (0, 1]$, the Hardy space $H^p(\mathbb{R}^n)$ was originally initiated by Stein and Weiss [53] and then systematically investigated by Fefferman and Stein [21]. The celebrated articles [21, 53] inspire many new ideas for the real-variable theory of function spaces. For instance, various real-variable characterizations of classical Hardy spaces reveal the important connections among various concepts in harmonic analysis, such as harmonic functions, various maximal functions and various square functions.

It is well known that Riesz transforms are not only bounded on Hardy spaces, but also characterize Hardy spaces. As a famous feature of Hardy spaces, the characterization via Riesz transforms was first studied by Fefferman and Stein [21] in 1972 and further extended by Wheeden [63] to the weighted Hardy spaces. From then on, there exists a series of studies on the Riesz transform characterization of various Hardy type...
spaces. As a recent progress in this direction, we recall that Cao et al. [6] established the Riesz transform characterization of Musielak–Orlicz Hardy spaces and Yang et al. [67] established the Riesz transform characterization of variable exponent Hardy spaces. We also refer the reader to [3, 13, 18, 19, 31, 38] for some recent developments on the study of Riesz transforms and to [4, 22, 29, 33, 59, 68] for some research of Riesz transforms associated with operators.

A key step in establishing the Riesz transform characterization of Hardy spaces $H^p(\mathbb{R}^n)$ is to extend the elements of $H^p(\mathbb{R}^n)$ to the upper half space $\mathbb{R}^n_+ := \mathbb{R}^n \times (0, \infty)$ via the Poisson integral. This extension in turn has a close relation with the analytical definition of $H^p(\mathbb{R}^n)$, which is the key starting point of studying the Hardy space, before people paid attention to the real-variable theory of $H^p(\mathbb{R}^n)$ (see, for instance, [50, 53]). Recall also that the real-variable theory of $H^p(\mathbb{R}^n)$ and their weighted versions play very important roles in analysis, such as harmonic analysis and partial differential equations; see, for instance [24, 52].

On the other hand, along with the development of various analysis areas, there appear many classes of function spaces which are more inclusive and exquisite than Lebesgue spaces, such as weighted Lebesgue spaces, Lorentz spaces, variable Lebesgue spaces, Orlicz spaces, Morrey spaces, and (quasi-)Banach function spaces. For their definitions and properties of (quasi-)Banach function spaces, see, for instance, [2, Chapter 1] and also Definition 2.1 below. These spaces and the Hardy-type spaces based on them have been investigated extensively in recent decades.

It is known that Lebesgue spaces, Lorentz spaces, variable Lebesgue spaces, and Orlicz spaces are (quasi-)Banach function spaces. However, weighted Lebesgue spaces, Herz spaces, Morrey spaces and Musielak–Orlicz spaces might not be quasi-Banach function spaces (see, for instance [49, 62, 69] for more details and examples). Therefore, in this sense, the concept of (quasi-)Banach function spaces is restrictive. To establish a general framework including also weighted Lebesgue spaces, Herz spaces, Morrey spaces, and Musielak–Orlicz spaces, Sawano et al. [46] introduced the so-called ball quasi-Banach function space (see Definition 2.2 below). Moreover, based on a given ball quasi-Banach function space $X$, Sawano et al. [46] also developed a real-variable theory of the related Hardy-type space $H_X(\mathbb{R}^n)$ (see Definition 2.6 below or [46, Definition 6.17]), which was originally defined via the maximal function of Petree type. The equivalent characterizations of $H_X(\mathbb{R}^n)$, respectively, in terms of Lusin-area functions, atoms, and radial or non-tangential maximal functions, were also established in [46]. More recently, Wang et al. [61] characterized the space $H_X(\mathbb{R}^n)$ via Littlewood–Paley $g$-functions or $g^*_\lambda$-functions (see also [7]) and obtained the boundedness of Calderón–Zygmund operators on $H_X(\mathbb{R}^n)$. Besides, Yan et al. [66] established the intrinsic square function characterization of $H_X(\mathbb{R}^n)$. We also refer the reader to [14, 15, 56, 57, 64, 65] for some latest progress on both ball quasi-Banach function spaces and their related Hardy-type spaces.

A natural and interesting question is whether or not such a general Hardy-type space $H_X(\mathbb{R}^n)$ can also be characterized by the Riesz transform and the main purpose of this article is to give an affirmative answer to this question. To be precise, let $X$ be a ball quasi-Banach function space satisfying Assumptions 2.8 and 2.10 below with both the same $s \in (0, 1]$ and $\theta \in (0, s)$. We establish the first order Riesz transform characterization of $H_X(\mathbb{R}^n)$ when $\theta \in \left[ \frac{n-1}{n}, s \right)$ and the higher order Riesz transform...
characterization of $H_X(\mathbb{R}^n)$ when $\theta \in (0, \frac{n-1}{n})$; see Theorems 3.15 and 4.4 below for more details.

It is worth pointing out that the results obtained in this article have a wide range of generality. More precisely, the Hardy type space $H_X(\mathbb{R}^n)$ considered in this article generalizes and unifies many known Hardy-type spaces, including classical Hardy spaces, weighted Hardy spaces, Herz–Hardy spaces, Lorentz–Hardy spaces, mixed-norm Hardy spaces, local generalized Herz–Hardy spaces, mixed-norm Herz–Hardy spaces, and Morrey–Hardy spaces. Moreover, to the best of our knowledge, even for Herz–Hardy spaces, Lorentz–Hardy spaces, mixed-norm Hardy spaces, local generalized Herz–Hardy spaces, mixed-norm Herz–Hardy spaces, and Morrey–Hardy spaces, the results obtained in this article are completely new (see Sect. 5 below).

We also point out that the range $\theta \in [\frac{n-1}{n}, s)$ in Theorem 3.15 is the best possible [see Remark 3.17(ii) below for more details]. These obviously reveal both the generality and the flexibility of the main results of this article and hence more applications to some newfound function spaces are predictable.

The organization of the remainder of this article is as follows.

In Sect. 2, we first recall some basic concepts of the ball quasi-Banach function space $X$ and the Hardy space $H_X(\mathbb{R}^n)$ associated with $X$. We also recall the concept of Poisson integrals and present some basic properties of Poisson integrals of distributions in $H_X(\mathbb{R}^n)$.

Section 3 is devoted to establishing the characterization of $H_X(\mathbb{R}^n)$ via the first order Riesz transform. To this end, we first introduce both the Hardy space $H_X(\mathbb{R}^{n+1})$ of harmonic functions associated with $X$ and the Hardy space $\mathcal{H}_X(\mathbb{R}^{n+1})$ of harmonic vectors associated with $X$. Then we establish the isomorphisms among $H_X(\mathbb{R}^n)$, $H_{X,2}(\mathbb{R}^{n+1})$, and $\mathcal{H}_{X,2}(\mathbb{R}^{n+1})$, where $H_{X,2}(\mathbb{R}^{n+1})$ and $\mathcal{H}_{X,2}(\mathbb{R}^{n+1})$ are, respectively, certain subspaces of $H_X(\mathbb{R}^{n+1})$ and $\mathcal{H}_X(\mathbb{R}^{n+1})$; see Theorem 3.13 below. Using these isomorphisms, we then obtain the first order Riesz transform characterization of $H_X(\mathbb{R}^n)$; see Theorem 3.15 below. Differently from both the case of Musielak–Orlicz–Hardy spaces in [6] and the case of variable Hardy spaces in [67], since the norm of the space $H_X(\mathbb{R}^n)$ has no explicit expression, the methods used in both [6, 67] are not feasible anymore. To overcome this essential difficulty, we assume that $X$ supports a vector-valued maximal inequality (see Assumption 2.8 below) and then we embed $X$ into a weighted Lebesgue space (see Lemma 2.23 below). Moreover, using this assumption, we also find surprisingly that the parameter $\theta$ appearing in Assumption 2.8, which felicitously characterizes the boundedness of the fractional Hardy–Littlewood maximal operator on $X$, plays the same role in some sense as that the parameter $p$ plays in $L^p(\mathbb{R}^n)$; see both (ii) and (iii) of Remark 3.17 below.

In Sect. 4, we turn to establishing the higher order Riesz transform characterization of $H_X(\mathbb{R}^n)$. We first recall some basic concepts associated with tensor products and then introduce the higher order Riesz–Hardy space. Using these spaces, we generalize Theorem 3.15 and obtain higher order Riesz transform characterization of $H_X(\mathbb{R}^n)$; see Theorem 4.4 below.

In Sect. 5, we apply our main results obtained in Sects. 3 and 4 to four concrete examples of ball quasi-Banach function spaces, namely Lorentz spaces (Subsect. 5.1),
mixed-norm Lebesgue spaces (Subsect. 5.2), local generalized Herz spaces (Subsect. 5.3), mixed-norm Herz spaces (Subsect. 5.4), and Morrey spaces (Subsect. 5.5).

Finally, we make some conventions on notation. Throughout this article, let \( N := \{1, 2, \ldots \} \), \( Z_{+} := N \cup \{0\} \), and \( 0 \) be the origin of \( \mathbb{R}^{n} \). We always use \( C \) to denote a positive constant, independent of the main parameters involved, but perhaps varying from line to line. Moreover, we also use \( C(\alpha, \beta, \ldots) \) to denote a positive constant depending on the parameters \( \alpha, \beta, \ldots \). The symbol \( f \lesssim g \) means that \( f \leq Cg \) and if \( f \lesssim g \) and \( g \lesssim f \), we then write \( f \sim g \). If \( f \leq Cg \) and \( g = h \) or \( g \leq h \), we then write \( f \lesssim g \sim h \) or \( f \lesssim g \lesssim h \). The symbol \( |s| \) for any \( s \in \mathbb{R} \) denotes the smallest integer not less than \( s \) and the symbol \( \lfloor s \rfloor \) for any \( s \in \mathbb{R} \) denotes the largest integer not greater than \( s \). For any subset \( E \) of \( \mathbb{R}^{n} \), we denote by \( E^{0} \) the set \( \mathbb{R}^{n} \setminus E \) and by \( 1_{E} \) its characteristic function. For any multi-index \( \alpha := (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{Z}_{+}^{n} := (\mathbb{Z}_{+})^{n} \), let \( |\alpha| := \alpha_{1} + \cdots + \alpha_{n} \). The operator \( M \) always denotes the Hardy–Littlewood maximal operator, which is defined by setting, for any \( f \in L_{\text{loc}}^{1}(\mathbb{R}^{n}) \) (the set of all locally integrable functions on \( \mathbb{R}^{n} \)) and \( x \in \mathbb{R}^{n} \),

\[
M(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy,
\]

where \( B(x, r) \) denotes the ball with the center \( x \) and the radius \( r \). The symbol \( \lim \) means \( \lim \inf \). We use \( S(\mathbb{R}^{n}) \) to denote the Schwartz space equipped with the well-known classical topology determined by a countable family of norms, while \( S'(\mathbb{R}^{n}) \) denotes its topological dual space equipped with the weak-* topology. The symbol \( \mathcal{M}(\mathbb{R}^{n}) \) denotes the set of all measurable functions on \( \mathbb{R}^{n} \). For any \( \varphi \in S(\mathbb{R}^{n}) \) and \( t \in (0, \infty) \), let \( \varphi_{t}(\cdot) := t^{-n} \varphi(\cdot/t) \). For any \( q \in [1, \infty) \), we denote by \( q' \) its conjugate exponent, that is, \( 1/q + 1/q' = 1 \). Also, when we prove a theorem (or the like), in its proof we always use the same symbols as in the statement itself of that theorem (or the like).

### 2 Ball Quasi-Banach Function Spaces and Poisson Integrals

In this section, we recall the concepts of ball quasi-Banach function spaces and Poisson integrals, as well as some of their basic properties. Let us begin with the concept of quasi-Banach function spaces (see, for instance, [2, Chapter 1] for more details). Recall that \( \mathcal{M}(\mathbb{R}^{n}) \) denotes the set of all measurable functions on \( \mathbb{R}^{n} \).

**Definition 2.1** Let \( X \subset \mathcal{M}(\mathbb{R}^{n}) \) be a quasi-normed linear space equipped with a quasi-norm \( \| \cdot \|_{X} \) which makes sense for the whole \( \mathcal{M}(\mathbb{R}^{n}) \). Then \( X \) is called a quasi-Banach function space if it satisfies

(i) if \( f \in \mathcal{M}(\mathbb{R}^{n}) \), then \( \|f\|_{X} = 0 \) implies that \( f = 0 \) almost everywhere;

(ii) if \( f, g \in \mathcal{M}(\mathbb{R}^{n}) \), then \( |g| \lesssim |f| \) in the sense of almost everywhere implies that \( \|g\|_{X} \leq \|f\|_{X} \);

(iii) if \( \{f_{m}\}_{m \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^{n}) \) and \( f \in \mathcal{M}(\mathbb{R}^{n}) \), then \( 0 \leq f_{m} \uparrow f \) as \( m \to \infty \) in the sense of almost everywhere implies that \( \|f_{m}\|_{X} \uparrow \|f\|_{X} \) as \( m \to \infty \);

(iv) \( 1_{E} \in X \) for any measurable set \( E \subset \mathbb{R}^{n} \) with finite measure.
Moreover, a quasi-Banach function space $X$ is called a *Banach function space* if it satisfies

(v) for any $f, g \in X$, $\|f + g\|_X \leq \|f\|_X + \|g\|_X$;

(vi) for any measurable set $E \subset \mathbb{R}^n$ with finite measure, there exists a positive constant $C_{(E)}$, depending on $E$, such that, for any $f \in X$,

$$\int_E |f(x)| \, dx \leq C_{(E)} \|f\|_X.$$ 

As is mentioned in the introduction, Lebesgue spaces, Lorentz spaces, variable Lebesgue spaces, and Orlicz spaces are (quasi-)Banach function spaces, but weighted Lebesgue spaces, Herz spaces, Morrey spaces, and Musielak–Orlicz spaces might not be quasi-Banach function spaces (see, for instance, [49, 62, 69] for more details). To give a more general framework containing all the aforementioned spaces, Sawano et al. [46] introduced the following ball quasi-Banach function spaces.

**Definition 2.2** Let $X \subset \mathcal{M}(\mathbb{R}^n)$ be a quasi-normed linear space. Then $X$ is called a *ball quasi-Banach function space* (for short, BQBF space) if it satisfies (i–iii) of Definition 2.1 and

(vii) $1_B \in X$ for any ball $B \subset \mathbb{R}^n$.

A ball quasi-Banach function space $X$ is called a *ball Banach function space* if the norm of $X$ satisfies

(viii) for any $f, g \in X$,

$$\|f + g\|_X \leq \|f\|_X + \|g\|_X;$$

(ix) for any ball $B \subset \mathbb{R}^n$, there exists a positive constant $C_{(B)}$ such that, for any $f \in X$,

$$\int_B |f(x)| \, dx \leq C_{(B)} \|f\|_X.$$ 

**Remark 2.3** (i) Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$. By [64, Remark 2.6(i)] or [65, Remark 2.5(i)], we conclude that, for any $f \in \mathcal{M}(\mathbb{R}^n)$, $\|f\|_X = 0$ if and only if $f = 0$ almost everywhere.

(ii) As was mentioned in [64, Remark 2.6(ii)] or [65, Remark 2.5(ii)], we obtain an equivalent formulation of Definition 2.2 via replacing any ball $B$ by any bounded measurable set $E$ therein.

(iii) We should point out that, in Definition 2.2, if we replace any ball $B$ by any measurable set $E$ with finite measure, we obtain the definition of (quasi-)Banach function spaces which were originally introduced in [2, Chapter 1, Definitions 1.1 and 1.3]. Thus, a (quasi-)Banach function space is also a ball (quasi-)Banach function space and the converse is not necessary to be true.

(iv) By [16, Theorem 2], we conclude that both (ii) and (iii) of Definition 2.2 imply that any ball quasi-Banach function space is complete and the converse is not necessary to be true.
(v) In Definition 2.2, if we replace (vii) by the following saturation property:

(a) for any measurable set $E \subset \mathbb{R}^n$ of positive measure, there exists a measurable set $F \subset E$ of positive measure satisfying that $1_F \in X$,

then we obtain the definition of quasi-Banach function spaces in Lorist and Nieraeth [37]. Moreover, by [71, Proposition 2.5] (see also [39, Proposition 4.22]), we find that, if the quasi-normed vector space $X$ under consideration satisfies the extra assumption that the Hardy–Littlewood maximal operator is bounded on its convexification, then the definition of quasi-Banach function spaces in [37] coincides with the definition of ball quasi-Banach function spaces. Thus, under this extra assumption, working with ball quasi-Banach function spaces in the sense of Definition 2.2 or quasi-Banach function spaces in the sense of [37] would yield exactly the same results.

From Definition 2.2, we deduce the following Fatou property of $X$. In what follows, we denote $\lim_{k \to \infty} \inf_{j \geq k}$ simply by $\lim_{k \to \infty}$.

**Lemma 2.4** Let $X$ be a BQBF space. If a sequence $\{f_k\}_{k \in \mathbb{N}} \subset X$ and an $f \in X$ satisfy

$$\lim_{k \to \infty} |f_k(x)| = |f(x)|$$

for almost every $x \in \mathbb{R}^n$, then

$$\|f\|_X \leq \lim_{k \to \infty} \|f_k\|_X.$$

**Proof** For any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let $g_k(x) := \inf_{j \geq k} |f_j(x)|$. Then we find that, for any $k \in \mathbb{N}$ and almost every $x \in \mathbb{R}^n$,

$$g_k(x) \leq |f_k(x)|, \quad g_k(x) \leq g_{k+1}(x), \quad \text{and} \quad \lim_{k \to \infty} g_k(x) = |f(x)|. \quad (2.1)$$

From Definition 2.1(ii), we deduce that $\|f\|_X = \|f\|_X$ and $\|f_k\|_X = \|f_k\|_X$. This, together with (2.1) and both (ii) and (iii) of Definition 2.1, further implies that

$$\|f\|_X = \|f\|_X = \lim_{k \to \infty} \|g_k\| \leq \lim_{k \to \infty} \|f_k\|_X = \lim_{k \to \infty} \|f_k\|_X,$$

which completes the proof of Lemma 2.4. \qed

We now recall the concepts of both the $p$-convexification and the convexity of $X$ (see, for instance, [42, Chapter 2] and [34, Definition 1.d.3] for more details).

**Definition 2.5** Let $X$ be a BQBF space and $p \in (0, \infty)$.

(i) The $p$-convexification $X^p$ of $X$ is defined by setting

$$X^p := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X \right\},$$

equipped with the quasi-norm $\|f\|_{X^p} := \|f|^p\|_X^{1/p}$. 
(ii) The space \( X \) is said to be \( p \)-convex if there exists a positive constant \( C \) such that, for any \( \{ f_j \}_{j \in \mathbb{N}} \subset X^{1/p} \),

\[
\left\| \sum_{j=1}^{\infty} |f_j| \right\|_{X^{1/p}} \leq C \sum_{j=1}^{\infty} \| f_j \|_{X^{1/p}}.
\]

In particular, when \( C = 1 \), \( X \) is said to be strictly \( p \)-convex.

Based on BQBF spaces, Sawano et al. [46] also introduced the following Hardy-type spaces.

**Definition 2.6** Let \( X \) be a BQBF space. Let \( \Phi \in \mathcal{S}(\mathbb{R}^n) \) satisfy \( \int_{\mathbb{R}^n} \Phi(x) \, dx \neq 0 \) and \( b \in (0, \infty) \) be sufficiently large. Then the Hardy space \( H_X(\mathbb{R}^n) \) associated with \( X \) is defined by setting

\[
H_X(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{H_X(\mathbb{R}^n)} := \| M_b^{**}(f, \Phi) \|_X < \infty \right\},
\]

where \( M_b^{**}(f, \Phi) \) is defined by setting, for any \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M_b^{**}(f, \Phi)(x) := \sup_{(y, t) \in \mathbb{R}_+^{n+1}} \frac{|\Phi_t \ast f(x - y)|}{(1 + t^{-1}|y|)^b}.
\]

**Remark 2.7** Let all the symbols be the same as in Definition 2.6. Assume that there exists an \( r \in (0, \infty) \) such that the Hardy–Littlewood maximal operator \( M \) is bounded on \( X^{1/r} \). If \( b \in (n/r, \infty) \), then, by [46, Theorem 3.1], we find that the Hardy space \( H_X(\mathbb{R}^n) \) is independent of the choice of \( b \).

Denote by \( L^1_{\text{loc}}(\mathbb{R}^n) \) the set of all locally integrable functions on \( \mathbb{R}^n \). For any \( \theta \in (0, \infty) \), the powered Hardy–Littlewood maximal operator \( M(\theta) \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M(\theta)(f)(x) := \left\{ M \left( |f|^\theta \right)(x) \right\}^{\frac{1}{\theta}}.
\]

Moreover, we also need some basic assumptions on \( X \) as follows (see also [46, (2.8) and (2.9)]).

**Assumption 2.8** Let \( X \) be a BQBF space. Assume that, for some \( \theta, s \in (0, 1] \) such that \( \theta < s \), there exists a positive constant \( C \) such that, for any \( \{ f_j \}_{j=1}^{\infty} \subset L^1_{\text{loc}}(\mathbb{R}^n) \),

\[
\left\| \left\{ \sum_{j=1}^{\infty} \left[ M(\theta)(f_j) \right]^s \right\}^{\frac{1}{s}} \right\|_{X} \leq C \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^s \right\}^{\frac{1}{s}} \right\|_{X}. \tag{2.2}
\]
Remark 2.9 The inequality (2.2) is called the Fefferman–Stein vector-valued maximal inequality on $X$. If $X := L^p(\mathbb{R}^n)$ with $p \in (1, \infty)$, $\theta = 1$, and $s \in (1, \infty]$, the inequality (2.2) was originally established by Fefferman and Stein [20, Theorem 1]. See, for instance, [61, Remark 2.4] for some examples of ball quasi-Banach function spaces satisfying (2.2).

To state the next assumption on $X$, we need the concept of the associate space. For any ball Banach function space $X$, the associate space (also called the Köthe dual) $X'$ of $X$ is defined by setting

$$X' := \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{X'} < \infty \},$$

where, for any $f \in \mathcal{M}(\mathbb{R}^n)$,

$$\| f \|_{X'} := \sup \{\| fg \|_{L^1(\mathbb{R}^n)} : g \in X, \| g \|_X = 1\}$$

(see, for instance, [2, Chapter 1, Section 2] for the details). Recall that, for any given ball Banach function space $X$, $X'$ is also a ball Banach function space (see [46, Proposition 2.3]).

Assumption 2.10 Let $X$ be a BQBF space. Assume that there exists an $s \in (0, 1]$ such that $X^{1/s}$ is also a ball Banach function space and there exists a $q \in (1, \infty]$ and a $C \in (0, \infty)$ such that, for any $f \in (X^{1/s})'$,

$$\| M^{(q/s)'}(f) \|_{(X^{1/s})'} \leq C \| f \|_{(X^{1/s})'}.$$

(2.3)

Remark 2.11 (i) We point out that, in [46, Theorems 2.10, 3.7 and 3.21], one needs the extra assumption that there exists an $s \in (0, 1]$ such that $X^{1/s}$ is a ball Banach function space. Indeed, this assumption ensures that $(X^{1/s})'$ is also a ball Banach function space, which further implies that, for any $f \in (X^{1/s})'$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and hence the Hardy-Littlewood maximal operator can be defined on $(X^{1/s})'$.

(ii) We refer the reader to Lorist and Nieraeth [36, Theorem 3.1] for two equivalent characterizations on the boundedness of the Hardy–Littlewood maximal operator $M$ on the associate space of the convexification of $X$.

Next, we recall some basic concepts associated with the Poisson integral.

Definition 2.12 (i) A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is called a bounded distribution if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\phi \ast f \in L^\infty(\mathbb{R}^n)$, where

$$L^\infty(\mathbb{R}^n) := \{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{L^\infty(\mathbb{R}^n)} := \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| < \infty \}.$$

(ii) For any $(x, t) \in \mathbb{R}^n_{++}$,

$$P_t(x) := c(n) \frac{t}{(t^2 + |x|^2)^{(n+1)/2}}$$
is called the Poisson kernel, where \(c(n)\) is the same as in Eq. (1.1).

(iii) Assume that \(f \in S'(\mathbb{R}^n)\) is a bounded distribution. The non-tangential maximal function \(M(f; P)\) of \(f\) is defined by setting, for any \(x \in \mathbb{R}^n\),

\[
M(f; P)(x) := \sup_{\{(y, t) \in \mathbb{R}^n_+^1 : |y - x| < t\}} |P_t * f(y)|. \tag{2.4}
\]

**Remark 2.13** If \(f \in L^p(\mathbb{R}^n)\) with \(p \in [1, \infty]\), then, by the Young inequality, it is easy to show that \(f\) is a bounded distribution. Moreover, if \(f\) is a bounded distribution, then \(P_t * f\) is a well-defined, bounded, and smooth harmonic function on \(\mathbb{R}^n_+^1\) (see, for instance, [52, p. 90]).

The following theorem establishes the Poisson integral characterization of a bounded distribution in \(H_X(\mathbb{R}^n)\).

**Theorem 2.14** Let \(X\) be a BQBF space such that \(M\) is bounded on \(X\) for some \(r \in (0, \infty)\). Assume that \(f \in S'(\mathbb{R}^n)\) is a bounded distribution. Then \(f \in H_X(\mathbb{R}^n)\) if and only if \(M(f; P) \in X\), where \(M(f; P)\) is the same as in Eq. (2.4). Moreover, there exist positive constants \(C_1\) and \(C_2\), independent of \(f\), such that

\[
C_1 \|f\|_{H_X(\mathbb{R}^n)} \leq \|M(f; P)\|_X \leq C_2 \|f\|_{H_X(\mathbb{R}^n)}.
\]

**Proof** We first prove the necessity. To this end, fix a bounded distribution \(f \in S'(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)\). For any \(N \in \mathbb{N}\) and \(\phi \in S(\mathbb{R}^n)\), let

\[
p_N(\phi) := \sum_{\alpha \in \mathbb{Z}^n_+, |\alpha| < N} \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \phi(x)|
\]

and

\[
\mathcal{F}_N := \{ \phi \in S(\mathbb{R}^n) : p_N(\phi) \leq 1 \}.
\]

For any \(f \in S'(\mathbb{R}^n)\), the grand maximal function \(M_N(f)\) is defined by setting, for any \(x \in \mathbb{R}^n\),

\[
M_N(f)(x) := \sup \left\{ |\phi_t * f(y)| : (x, t) \in \mathbb{R}^n_+^1, |x - y| < t, \phi \in \mathcal{F}_N \right\}.
\]

By [46, Theorem 3.1(i)], we find that there exists an \(N \in \mathbb{N}\) such that

\[
\|M_N(f)\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)} < \infty. \tag{2.5}
\]

On the other hand, from an argument similar to that used in the estimation of [24, (2.1.39)], we infer that

\[
M(f; P) \lesssim M_N(f).
\]
Combining this, Definition 2.1(ii), and Eq. (2.5), we obtain
\[ \| M(f; P) \|_X \lesssim \| M_N(f) \|_X \lesssim \| f \|_{H^1(\mathbb{R}^n)} < \infty. \]

This finishes the proof of the necessity.

Next, we show the sufficiency. To this end, fix a bounded distribution \( f \in S'(\mathbb{R}^n) \) such that \( \| M(f; P) \|_X < \infty \). By the same argument as that used in [52, p. 99], we find that there exists a \( \Phi \in S(\mathbb{R}^n) \), satisfying \( \int_{\mathbb{R}^n} \Phi(x) \, dx \neq 0 \), such that, for any \( x \in \mathbb{R}^n \),
\[ M(f; \Phi)(x) \lesssim \sup_{t \in (0, \infty)} |P_t * f(x)| \lesssim M(f; P)(x), \]
where
\[ M(f; \Phi)(x) := \sup_{t \in (0, \infty)} |\Phi_t * f(x)|. \tag{2.6} \]

This, together with both [46, Theorem 3.1(ii)] and Definition 2.1(ii), further implies that
\[ \| f \|_{H^1(\mathbb{R}^n)} \lesssim \| M(f; \Phi) \|_X \lesssim \| M(f; P) \|_X < \infty, \]
which completes the proof of the sufficiency and hence Theorem 2.14. \( \Box \)

**Remark 2.15** In [46, Theorem 3.3], Sawano et al. also established the Poisson integral characterization of \( H^1(\mathbb{R}^n) \). We point out that Theorem 2.14 is different from [46, Theorem 3.3] in the following sense.

(i) In [46, Theorem 3.3], Sawano et al. used the quantity \( \sup_{t \in (0, \infty)} |P_t * f| \) instead of \( M(f; P) \).

(ii) In [46, Theorem 3.3], Sawano et al. needed an extra assumption that there exists a positive constant \( C \) such that
\[ \inf_{x \in \mathbb{R}^n} \| 1_{B(x, 1)} \|_X \geq C. \tag{2.7} \]
This is a very strong condition because Eq. (2.7) does not hold true even when \( X \) is some weighted Lebesgue space.

(iii) In Theorem 2.14, we assume, a priori, that \( f \in S'(\mathbb{R}^n) \) is a bounded distribution. This is because, for an arbitrary \( f \in H^1(\mathbb{R}^n) \), we cannot show that \( f \) is a bounded distribution without any extra assumptions (see also [6, Remark 2.5]).

Now, we introduce the concept of the Poisson integral \( P_t * f \), where \( f \in S'(\mathbb{R}^n) \) is a limit in \( S'(\mathbb{R}^n) \) of a sequence \( \{ f_k \}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \).

**Definition 2.16** For any \( f \in S'(\mathbb{R}^n) \) and \( \{ f_k \}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) satisfying \( \lim_{k \to \infty} f_k = f \) in \( S'(\mathbb{R}^n) \) and for any \( (x, t) \in \mathbb{R}^{n+1}_+ \), define
\[ P_t * f(x) := \lim_{k \to \infty} P_t * f_k(x) \tag{2.8} \]
pointwisely.

The following lemma indicates that Eq. (2.8) is well defined for any \( f \in \mathcal{S}'(\mathbb{R}^n) \).

**Lemma 2.17** Let \( f \in \mathcal{S}'(\mathbb{R}^n) \). Then the following statements hold true.

(i) There exists a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in \( \mathcal{S}'(\mathbb{R}^n) \).

(ii) If \( \{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) satisfies \( \lim_{k \to \infty} f_k = f \) in \( \mathcal{S}'(\mathbb{R}^n) \), then, for any \((x, t) \in \mathbb{R}^n_+\), \( \lim_{k \to \infty} P_t * f_k(x) \) exists.

(iii) If \( \{f_k\}_{k \in \mathbb{N}}, \{g_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) satisfy that \( \lim_{k \to \infty} f_k = f = \lim_{k \to \infty} g_k \) in \( \mathcal{S}'(\mathbb{R}^n) \), then, for any \((x, t) \in \mathbb{R}^n_+\),

\[
\lim_{k \to \infty} P_t * f_k(x) = \lim_{k \to \infty} P_t * g_k(x). \tag{2.9}
\]

**Proof** (i) is just [23, Proposition 2.3.23]. To show (ii), fix an \( f \in \mathcal{S}'(\mathbb{R}^n) \) and a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in \( \mathcal{S}'(\mathbb{R}^n) \). Notice that, by [52, p. 90], we find that there exist \( \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \) and an \( h \in L^1(\mathbb{R}^n) \) such that, for any \((x, t) \in \mathbb{R}^n_+\),

\[
P_t(x) = h_t * \phi_t(x) + \psi_t(x).
\]

Thus, for any \( k, j \in \mathbb{N} \) and \((x, t) \in \mathbb{R}^n_+\), we have

\[
|P_t * f_k(x) - P_t * f_j(x)| = |h_t * \phi_t * (f_k - f_j)(x) + \psi_t * (f_k - f_j)(x)|.
\]

Since \( \{f_k\}_{k \in \mathbb{N}} \) converges to \( f \) in \( \mathcal{S}'(\mathbb{R}^n) \), it follows that, for any \( \epsilon \in (0, \infty) \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), there exists a \( K \in \mathbb{N} \), depending on \( \epsilon \) and \( \varphi \), such that, for any \( k > K \) and \( j > K \),

\[
|\langle f_k - f_j, \varphi \rangle| < \epsilon.
\]

Thus, for any \((x, t) \in \mathbb{R}^n_+\), we can find a \( K_{(x, t)} \in \mathbb{N} \), depending on \( \phi \) and \( \psi \) and hence the Poisson kernel, such that, for any \( k > K_{(x, t)} \) and \( j > K_{(x, t)} \),

\[
|\phi_t * (f_k - f_j)(x)| < \epsilon \quad \text{and} \quad |\psi_t * (f_k - f_j)(x)| < \epsilon
\]

and hence

\[
|P_t * f_k(x) - P_t * f_j(x)| < (|h_t| * \epsilon)(x) + \epsilon = \left[ \|h\|_{L^1(\mathbb{R}^n)} + 1 \right] \epsilon. \tag{2.10}
\]

This shows that \( \{P_t * f_k(x)\}_{k \in \mathbb{N}} \) is a Cauchy sequence for any \((x, t) \in \mathbb{R}^n_+\). Therefore,

\[
\lim_{k \to \infty} P_t * f_k(x)
\]

exists, which completes the proof of (ii).
Finally, we show (iii). To this end, by an argument similar to that used in the estimation of Eq. (2.10) via $f_j$ replaced by $g_k$, we find that, for any $\varepsilon \in (0, \infty)$ and $(x, t) \in \mathbb{R}^{n+1}_+$, there exists a $K_{(x, t)} \in (0, \infty)$ such that, for any $k > K_{(x, t)}$,

$$|P_t \ast f_k(x) - P_t \ast g_k(x)| < \left( \|h\|_{L^1(\mathbb{R}^n)} + 1 \right) \varepsilon,$$

which further implies that Eq. (2.9) holds true. This finishes the proof of (iii) and hence Lemma 2.17.

Next, we recall the concept of absolutely continuous quasi-norms as follows (see, for instance, [2, Chapter 1, Definition 3.1]).

**Definition 2.18** Let $X$ be a BQBF space. A function $f \in X$ is said to have an **absolutely continuous quasi-norm** in $X$ if $\|f 1_{E_j}\|_X \downarrow 0$ as $j \to \infty$ whenever $\{E_j\}_{j=1}^\infty$ is a sequence of measurable sets that satisfy $E_j \supset E_{j+1}$ for any $j \in \mathbb{N}$ and $\bigcap_{j=1}^\infty E_j = \emptyset$. Moreover, $X$ is said to have an **absolutely continuous quasi-norm** if, for any $f \in X$, $f$ has an absolutely continuous quasi-norm in $X$.

**Remark 2.19** We point out that many function spaces, such as Lebesgue spaces, Lorentz spaces, weighted Lebesgue spaces, Herz spaces, variable exponent Lebesgue spaces, and Orlicz-slice space, have absolutely continuous quasi-norms, but the Morrey space might have no absolutely continuous quasi-norm (see [46, p. 10] and [61, Remark 3.4] for more details).

The following conclusion is a combination of [46, Theorems 3.6 and 3.7 and Remark 3.12]; we omit the details here.

**Lemma 2.20** Assume that $X$ is a BQBF space and satisfies both Assumptions 2.8 and 2.10. If $f \in H_X(\mathbb{R}^n)$, then there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \cap H_X(\mathbb{R}^n)$ and a positive constant $C$ such that

$$\lim_{k \to \infty} f_k = f \quad (2.11)$$

in $\mathcal{S}'(\mathbb{R}^n)$ and, for any $k \in \mathbb{N}$, $\|f_k\|_{H_X(\mathbb{R}^n)} \leq C \|f\|_{H_X(\mathbb{R}^n)}$. Moreover, if $X$ has an absolutely continuous quasi-norm, then Eq. (2.11) holds true in $H_X(\mathbb{R}^n)$.

From both Definition 2.16 and Lemma 2.20, we deduce that, if $X$ is a BQBF space and satisfies both Assumptions 2.8 and 2.10, then, for any $f \in H_X(\mathbb{R}^n)$, $P_t \ast f$ is well defined. Moreover, if $X$ has an absolutely continuous quasi-norm, then we have the following conclusion.

**Proposition 2.21** Assume that $X$ is a BQBF space, satisfies both Assumptions 2.8 and 2.10, and has an absolutely continuous quasi-norm. Then, for any compact set $K \subset \mathbb{R}^{n+1}_+$, Eq. (2.8) converges uniformly on $K$.

To prove Proposition 2.21, we need the following concept of weighted Lebesgue spaces.
Definition 2.22 For any \( q \in [1, \infty] \), denote by \( A_q(\mathbb{R}^n) \) the class of all Muckenhoupt weights (see, for instance, [23, Definitions 7.1.1 and 7.1.3] for its definition). For any \( p \in (0, \infty) \) and \( w \in A_\infty(\mathbb{R}^n) \), the weighted Lebesgue space \( L^p_w(\mathbb{R}^n) \) is defined by setting

\[
L^p_w(\mathbb{R}^n) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{L^p_w(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right]^{1/p} < \infty \right\}.
\]

The following lemma is just [7, Lemma 4.7].

Lemma 2.23 Let \( X \) be a BQBF space. Assume that there exists an \( s \in (0, \infty) \) such that \( X^{1/s} \) is a ball Banach function space and \( M \) is bounded on \( (X^{1/s})' \). Then there exists an \( \varepsilon \in (0, 1) \) such that \( X \) continuously embeds into \( L^s_w(\mathbb{R}^n) \) with \( w := [M(1_B(0,1))]^\varepsilon \).

Remark 2.24 (i) Let \( X \) be a BQBF space satisfying Assumption 2.10. Then, by both [7, Lemma 2.10] and Eq. (2.3), we find that \( M \) is bounded on \( (X^{1/s})' \).

(ii) Let \( w \) be the same as in Lemma 2.23. By [23, Theorem 7.2.7], we find that \( w \in A_1(\mathbb{R}^n) \). For any measurable set \( E \subset \mathbb{R}^n \), let

\[
\mu(E) := \int_E w(x) \, dx.
\]

From [23, Proposition 7.1.5], it follows that \( \mu \) is a doubling measure. For any \( p \in (0, \infty) \), let

\[
L^p(\mu) := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \| f \|_{L^p(\mu)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x) \right]^{1/p} < \infty \right\}.
\]

It is easy to show that \( L^p(\mu) = L^p_w(\mathbb{R}^n) \) with equivalent quasi-norms.

Next, we show Proposition 2.21.

Proof of Proposition 2.21 To show this proposition, fix an \( (x_0, t_0) \in \mathbb{R}^{n+1}_+ \) and an \( f \in H_X(\mathbb{R}^n) \). By Lemma 2.20, we find a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \cap H_X(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in both \( H_X(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \). We first claim that Eq. (2.8) converges uniformly on \( B(x_0, t_0/4) \times (3t_0/4, 5t_0/4) \). Indeed, for any \( x, y \in B(x_0, t_0/4) \) and \( t \in (3t_0/4, 5t_0/4) \), we have

\[
|x - y| \leq |x - x_0| + |x_0 - y| < \frac{t_0}{2} < t.
\]

From this, we infer that, for any \( x, y \in B(x_0, t_0/4), t \in (3t_0/4, 5t_0/4), \) and \( k, j \in \mathbb{N} \),

\[
|P_t * f_k(x) - P_t * f_j(x)| = |P_t * (f_k - f_j)(x)| \leq \sup_{\{(z, \ell) \in \mathbb{R}^{n+1}_+: |z - y| < t\}} |P_t * (f_k - f_j)(z)|
\]

\[
= M(f_k - f_j; P)(y),
\]

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where $M(f_k - f_j; P)$ is the same as in Eq. (2.4) via replacing $f$ by $f_k - f_j$. For any $k, j \in \mathbb{N}$, let
\[ E_{k,j} := \left\{ y \in B \left( x_0, \frac{t_0}{4} \right) : M(f_k - f_j; P)(y) \leq 1 \right\}. \]

By this, Eq. (2.12), Lemma 2.23, Theorem 2.14, and the fact that $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $H_X(\mathbb{R}^n)$, we conclude that
\[
\mu \left( B \left( x_0, \frac{t_0}{4} \right) \setminus E_{k,j} \right) = \int_{B(x_0, t_0/4) \setminus E_{k,j}} w(y) \, dy \\
\leq \int_{\mathbb{R}^n} [M(f_k - f_j; P)(y)]^s \, w(y) \, dy \\
= \| M(f_k - f_j; P) \|_{L^s_w(\mathbb{R}^n)}^s \lesssim \| M(f_k - f_j; P) \|_X \\
\lesssim \| f_k - f_j \|_{H_X(\mathbb{R}^n)} \to 0
\]
as $k, j \to \infty$, where $w$ and $s$ are the same as in Lemma 2.23. This further implies that there exists a positive constant $\tilde{L}$, depending on both $x_0$ and $t_0$, such that, for any $k > \tilde{L}$ and $j > \tilde{L}$,
\[
\mu \left( B \left( x_0, \frac{t_0}{4} \right) \setminus E_{k,j} \right) < \frac{1}{2} \mu \left( B \left( x_0, \frac{t_0}{4} \right) \right)
\]
and hence
\[
\mu (E_{k,j}) \geq \frac{1}{2} \mu \left( B \left( x_0, \frac{t_0}{4} \right) \right). 
\tag{2.14}
\]

Since $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $H_X(\mathbb{R}^n)$, it follows that, for any $\epsilon \in (0, \infty)$, there exists an $\tilde{L}_0 \in (\tilde{L}, \infty)$ such that, for any $k > \tilde{L}_0$ and $j > \tilde{L}_0$,
\[
\| f_k - f_j \|_{H_X(\mathbb{R}^n)} < \epsilon.
\]

Using this, Eqs. (2.13) and (2.14), Lemma 2.23 and Theorem 2.14, we find that, for any $x \in B(x_0, t_0/4)$, $t \in (3t_0/4, 5t_0/4)$, $k > \tilde{L}_0$, and $j > \tilde{L}_0$,
\[
| P_t \ast f_k(x) - P_t \ast f_j(x) | \\
= \left\{ \frac{1}{\mu(E_{k,j})} \int_{E_{k,j}} | P_t \ast f_k(x) - P_t \ast f_j(x) |^s w(y) \, dy \right\}^{1/s} \\
\leq \left\{ \frac{1}{\mu(E_{k,j})} \int_{E_{k,j}} | M(f_k - f_j; P)(y) |^s w(y) \, dy \right\}^{1/s} \\
\lesssim \| M(f_k - f_j; P) \|_{L^s_w(\mathbb{R}^n)} \lesssim \| M(f_k - f_j; P) \|_X \\
\lesssim \| f_k - f_j \|_{H_X(\mathbb{R}^n)} \lesssim \epsilon,
\]
where the implicit positive constants may depend on both $x_0$ and $t_0$. This, together with [45, Theorem 7.8], further implies that Eq. (2.8) converges uniformly on $B(x_0, t_0/4) \times (3t_0/4, 5t_0/4)$ and hence finishes the proof of the above claim.

Now, for any compact set $K \subset \mathbb{R}^{n+1}$, from

$$K \subset \bigcup_{(x, t) \in K} \left[ B \left( x, \frac{t}{4} \right) \times \left( \frac{3t}{4}, \frac{5t}{4} \right) \right],$$

it follows that there exists an $m \in \mathbb{N}$ and a finite number of points, $\{(x_i, t_i)\}_{i=0}^m \subset K$, such that

$$K \subset \bigcup_{i=0}^m \left[ B \left( x_i, \frac{t_i}{4} \right) \times \left( \frac{3t_i}{4}, \frac{5t_i}{4} \right) \right].$$

By the above claim, for any $\epsilon \in (0, \infty)$ and $i \in \{1, \ldots, m\}$, we find an $L_i \in (0, \infty)$ such that, for any $(x, t) \in B(x_i, t_i/4) \times (3t_i/4, 5t_i/4)$, $k > L_i$, and $j > L_i$,

$$\left| P_t * f_k(x) - P_t * f_j(x) \right| < \epsilon. \quad (2.15)$$

Let $L := \max\{L_1, \ldots, L_m\}$. Then, for any $(x, t) \in K$, $k > L$, and $j > L$, Eq. (2.15) holds true. From this and [45, Theorem 7.8], we deduce that Eq. (2.8) converges uniformly on $K$. This finishes the proof of Proposition 2.21. \hfill \Box

Using both Lemma 2.17 and Proposition 2.21, we obtain the following conclusions.

**Corollary 2.25** Assume that $X$ is a BQBF space and satisfies both Assumptions 2.8 and 2.10.

(i) Then, for any $(x, t) \in \mathbb{R}^{n+1}$ and $f, g \in H_X(\mathbb{R}^n)$,

$$P_t * (f + g)(x) = P_t * f(x) + P_t * g(x).$$

(ii) There exists a positive constant $C$ such that, for any $t \in (0, \infty)$ and $f \in H_X(\mathbb{R}^n)$,

$$\| P_t * f \|_X \leq C \| f \|_{H_X(\mathbb{R}^n)}.$$

(iii) If $X$ has an absolutely continuous quasi-norm, then, for any $f \in H_X(\mathbb{R}^n)$, $u(x, t) := P_t * f(x)$ is harmonic on $\mathbb{R}^{n+1}$.

**Proof** We first prove (i). By Lemma 2.17(i), we find two sequences $\{f_k\}_{k \in \mathbb{N}}, \{g_k\}_{k \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ such that

$$\lim_{k \to \infty} f_k = f \text{ and } \lim_{k \to \infty} g_k = g \tag{2.16}$$

in $\mathcal{S}'(\mathbb{R}^n)$, which further imply that

$$\lim_{k \to \infty} (f_k + g_k) = f + g.$$
in \( S'(\mathbb{R}^n) \). From this, Definition 2.16, and Eq. (2.16), we infer that, for any \((x, t) \in \mathbb{R}^n_+\),

\[
P_t * (f + g)(x) = \lim_{k \to \infty} P_t * (f_k + g_k)(x) = \lim_{k \to \infty} P_t * f_k(x) + \lim_{k \to \infty} P_t * g_k(x) = P_t * f(x) + P_t * g(x).
\]

This finishes the proof of (i).

Next, we show (ii). To this end, fix an \( f \in H_X(\mathbb{R}^n) \). By Lemma 2.20, we choose a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in \( S'(\mathbb{R}^n) \) and

\[
\| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)}.
\]

From this, Definition 2.16, Definition 2.1(ii), Lemma 2.4, and Theorem 2.14, we deduce that, for any \( t \in (0, \infty) \),

\[
\| P_t * f \|_X = \left\| \lim_{k \to \infty} P_t * f_k \right\|_X = \left\| \lim_{k \to \infty} P_t * f_k \right\|_X \leq \lim_{k \to \infty} \| P_t * f_k \|_X \leq \lim_{k \to \infty} \| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)},
\]

which completes the proof of (ii).

Finally, we show (iii). To this end, fix an \( f \in H_X(\mathbb{R}^n) \). By Lemma 2.20, we choose a sequence \( \{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in both \( S'(\mathbb{R}^n) \) and \( H_X(\mathbb{R}^n) \). Using Definition 2.16, we find that, for any \((x, t) \in \mathbb{R}^n_+\),

\[
u(x, t) = \lim_{k \to \infty} P_t * f_k(x).
\]

For any \( k \in \mathbb{N} \), from both \( f_k \in L^2(\mathbb{R}^n) \) and Remark 2.13, we infer that \( u_k(x, t) := P_t * f_k(x) \) is harmonic on \( \mathbb{R}^n_+ \). Using this, Proposition 2.21, and [55, p. 42, Corollary 1.8], we conclude that \( u \) is harmonic on \( \mathbb{R}^n_+ \). This finishes the proof of (iii) and hence Corollary 2.25.

### 3 First Order Riesz Transform Characterization

In this section, we establish the first order Riesz transform characterization of \( H_X(\mathbb{R}^n) \). In order to achieve this goal, we need to introduce both the Hardy space \( H_X(\mathbb{R}^n_+) \) of harmonic functions and the Hardy space \( \mathcal{H}_X(\mathbb{R}^n_+) \) of harmonic vectors on the upper half space \( \mathbb{R}^n_+ \) and to clarify their relations with \( H_X(\mathbb{R}^n) \). Let us begin with
the concept of the Hardy type space $H^p(R^{n+1}_+)$ of harmonic functions. Recall that a function $u$ on $\mathbb{R}^{n+1}_+$ is said to be harmonic if, for any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} u(x, t) + \frac{\partial^2}{\partial t^2} u(x, t) = 0.$$ 

**Definition 3.1** Let $p \in (0, \infty)$. The Hardy space $H^p(R^{n+1}_+)$ of harmonic functions is defined to be the set of all the harmonic functions $u$ on $\mathbb{R}^{n+1}_+$ such that

$$\|u\|_{H^p(R^{n+1}_+)} := \sup_{t \in (0, \infty)} \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} < \infty.$$ 

Now, we introduce the concept of Hardy spaces $H_X(\mathbb{R}^{n+1}_+)$ of harmonic functions associated with a BQBF space $X$.

**Definition 3.2** Let $X$ be a BQBF space.

(i) Let $u$ be a function on $\mathbb{R}^{n+1}_+$. Its nontangential maximal function $u^*$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$u^*(x) := \sup_{\{(y,t)\in \mathbb{R}^{n+1}_+; |y-x|<t\} } |u(y,t)|.$$ 

(ii) The Hardy space $H_X(\mathbb{R}^{n+1}_+)$ of harmonic functions associated with $X$ is defined to be the set of all the harmonic functions $u$ on $\mathbb{R}^{n+1}_+$ such that $u^* \in X$. Moreover, for any $u \in H_X(\mathbb{R}^{n+1}_+)$, its (quasi-)norm $\|u\|_{H_X(\mathbb{R}^{n+1}_+)}$ is defined by setting

$$\|u\|_{H_X(\mathbb{R}^{n+1}_+)} := \|u^*\|_X.$$ 

(iii) The subspace $H_{X,2}(\mathbb{R}^{n+1}_+)$ is defined to be the set of all the functions $u \in H_X(\mathbb{R}^{n+1}_+)$ satisfying that there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^{n+1}_+) \cap H^2(\mathbb{R}^{n+1}_+)$ such that

$$u = \lim_{k \to \infty} u_k$$

in $H_X(\mathbb{R}^{n+1}_+)$. Moreover, for any $u \in H_{X,2}(\mathbb{R}^{n+1}_+)$, let $\|u\|_{H_{X,2}(\mathbb{R}^{n+1}_+)} := \|u\|_{H_X(\mathbb{R}^{n+1}_+)}$.

The following proposition establishes the relation between $H_X(\mathbb{R}^n)$ and $H_{X,2}(\mathbb{R}^{n+1}_+)$ via the Poisson integral.

**Proposition 3.3** Assume that $X$ is a BQBF space, satisfies both Assumptions 2.8 and 2.10, and has an absolutely continuous quasi-norm. Let $u$ be a harmonic function on
Then \( u \in H_X(\mathbb{R}_+^{n+1}) \) if and only if there exists an \( f \in H_X(\mathbb{R}_+^n) \) such that, for any \( (x, t) \in \mathbb{R}_+^{n+1} \),
\[
u(x, t) = P_t \ast f(x),
\]
where \( P_t \ast f \) is the same as in Eq. (2.8); moreover, there exist positive constants \( C_1 \) and \( C_2 \), independent of both \( f \) and \( u \), such that
\[
C_1 \| f \|_{H_X(\mathbb{R}_+^n)} \leq \| u \|_{H_X(\mathbb{R}_+^{n+1})} \leq C_2 \| f \|_{H_X(\mathbb{R}_+^n)}.
\]

To prove Proposition 3.3, we need the following lemma on the convergence.

**Lemma 3.4** Let \( X \) be a BQBF space. Assume that there exists an \( s \in (0, \infty) \) such that \( X^{1/s} \) is a ball Banach function space and \( M \) is bounded on \((X^{1/s})'\). If \( \{f_k\}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}_+^{n+1}) \) and \( f \in H_X(\mathbb{R}_+^{n+1}) \) satisfy \( \lim_{k \to \infty} f_k = f \) in \( H_X(\mathbb{R}_+^{n+1}) \) and if \( \{\epsilon_k\}_{k \in \mathbb{N}} \subset (0, \infty) \) satisfies \( \lim_{k \to \infty} \epsilon_k = 0 \), then, for any \( (x, t) \in \mathbb{R}_+^{n+1} \),
\[
\lim_{k \to \infty} f_k(x, t + \epsilon_k) = f(x, t). \tag{3.1}
\]

**Proof** Assume that \( \{f_k\}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}_+^{n+1}) \) and \( f \in H_X(\mathbb{R}_+^{n+1}) \) satisfy
\[
\lim_{k \to \infty} f_k = f \tag{3.2}
\]
in \( H_X(\mathbb{R}_+^{n+1}) \). Since \( f \in H_X(\mathbb{R}_+^{n+1}) \), it follows that \( f \) is harmonic. Thus, to prove Eq. (3.1), it suffices to show that, for any \( (x, t) \in \mathbb{R}_+^{n+1} \),
\[
\lim_{k \to \infty} f_k(x, t + \epsilon_k) = \lim_{k \to \infty} f(x, t + \epsilon_k). \tag{3.3}
\]
Observe that, for any \( k \in \mathbb{N}, (x, t) \in \mathbb{R}_+^{n+1}, \) and \( y \in B(x, t) \),
\[
|f_k(x, t + \epsilon_k) - f(x, t + \epsilon_k)| \leq \sup_{(z, s) \in \mathbb{R}_+^{n+1}: |z-y|<\epsilon} |f_k(z, s) - f(z, s)| = (f_k - f)^*(y).
\]
From this, Lemma 2.23, and Eq. (3.2), we deduce that, for any \( (x, t) \in \mathbb{R}_+^{n+1} \),
\[
|f_k(x, t + \epsilon_k) - f(x, t + \epsilon_k)| \leq [\mu(B(x, t))]^{-\frac{1}{2}} \left[ \int_{B(x, t)} |f_k(x, t + \epsilon_k) - f(x, t + \epsilon_k)|^s w(y) \, dy \right]^{\frac{1}{2}} \leq [\mu(B(x, t))]^{-\frac{1}{2}} \left\{ \int_{B(x, t)} [(f_k - f)^*(y)]^s w(y) \, dy \right\}^{\frac{1}{2}} \leq [\mu(B(x, t))]^{-\frac{1}{2}} \|f_k - f\|^s_{L^s_w(\mathbb{R}^n)} \lesssim [\mu(B(x, t))]^{-\frac{1}{2}} \|f_k - f\|^s_X.
\]
as \( k \to \infty \), where \( w \) and \( s \) are the same as in Lemma 2.23. This further implies that Eq. (3.3) holds true and hence finishes the proof of Lemma 3.4. \( \Box \)

Next, we show Proposition 3.3.

**Proof of Proposition 3.3** We first prove the necessity. To this end, fix a \( u \in H_{X,2}(\mathbb{R}^{n+1}_+) \). By Definition 3.2(iii), we find a sequence \( \{u_k\}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^{n+1}_+) \cap H^2(\mathbb{R}^{n+1}_+) \) such that

\[
\lim_{k \to \infty} u_k = u
\]

in \( H_X(\mathbb{R}^{n+1}_+) \). For any \( k \in \mathbb{N} \) and \( \epsilon \in (0, \infty) \), let \( f_{k,\epsilon}(\cdot) := u_k(\cdot, \epsilon) \). Since \( u_k \in H^2(\mathbb{R}^{n+1}_+) \), from [55, p. 51, Lemmas 2.6 and 2.7], it follows that, for any \( (x, t) \in \mathbb{R}^{n+1}_+ \),

\[
P_t \ast f_{k,\epsilon}(x) = u_k(x, t + \epsilon).
\]

Using this, Eq. (2.4), and Theorem 2.14, we conclude that

\[
\sup_{\epsilon \in (0, \infty)} \| f_{k,\epsilon} \|_{H_X(\mathbb{R}^n)} \sim \sup_{\epsilon \in (0, \infty)} \| M(f_{k,\epsilon}; P) \|_X \sim \sup_{\epsilon \in (0, \infty)} \sup_{(y, t) \in \mathbb{R}^{n+1}_+: |y-t|<t} |P_t \ast f_{k,\epsilon}(y)| \bigg|_{X} (3.6)
\]

\[
\sim \sup_{\epsilon \in (0, \infty)} \sup_{(y, t) \in \mathbb{R}^{n+1}_+: |y-t|<t} |u_k(y, t + \epsilon)| \bigg|_{X} \lesssim \sup_{\epsilon \in (0, \infty)} \| u_k \|^*_X \sim \| u \|_{H_X(\mathbb{R}^{n+1}_+)},
\]

where \( M(f_{k,\epsilon}; P) \) is the same as in Eq. (2.4) via \( f \) replaced by \( f_{k,\epsilon} \). Thus, for any \( k \in \mathbb{N} \), \( \{f_{k,\epsilon}\}_{\epsilon \in (0, \infty)} \) is a bounded set in \( H_X(\mathbb{R}^n) \). Moreover, by [32, Lemma 4.8.18], we find that \( H_X(\mathbb{R}^n) \) continuously embeds into \( \mathcal{S}'(\mathbb{R}^n) \), which further implies that \( \{f_{k,\epsilon}\}_{\epsilon \in (0, \infty)} \) is a bounded set in \( \mathcal{S}'(\mathbb{R}^n) \). By the weak compactness of \( \mathcal{S}'(\mathbb{R}^n) \) (see, for instance, [52, p. 119]), we find an \( f_k \in \mathcal{S}'(\mathbb{R}^n) \) and a sequence \( \{\epsilon_j\}_{j \in \mathbb{N}} \), satisfying \( \lim_{j \to \infty} \epsilon_j = 0 \), such that \( \lim_{j \to \infty} f_{k,\epsilon_j} = f_k \) in \( \mathcal{S}'(\mathbb{R}^n) \), which further implies that, for any \( \Phi \in \mathcal{S}(\mathbb{R}^n) \) satisfying \( \int_{\mathbb{R}^n} \Phi(y) \, dy \neq 0 \) and for any \( (x, t) \in \mathbb{R}^{n+1}_+ \),

\[
\lim_{j \to \infty} [f_{k,\epsilon_j} \ast \Phi_t(x)] = f_k \ast \Phi_t(x).
\]

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and hence, for any \( x \in \mathbb{R}^n \),

\[
M(f_k; \Phi)(x) = \sup_{t \in (0, \infty)} \lim_{j \to \infty} |f_{k, \epsilon_j} * \Phi_t(x)| \\
\leq \sup_{t \in (0, \infty)} \lim_{j \to \infty} \sup_{t \in (0, \infty)} |f_{k, \epsilon_j} * \Phi_t(x)| \\
\leq \lim_{j \to \infty} M(f_{k, \epsilon_j}; \Phi)(x),
\]

where \( M(f_{k, \epsilon_j}; \Phi) \) is the same as in Eq. (2.6) via replacing \( f \) by \( f_{k, \epsilon_j} \). From this, [36, Theorem 3.1], Lemma 2.4, and Eq. (3.6), we deduce that, for any \( x \), and hence, for any \( x \in \mathbb{R}^n \),

\[
\| f_k \|_{H_X(\mathbb{R}^n)} \sim \| M(f_k; \Phi) \|_X \lesssim \lim_{j \to \infty} \| M(f_{k, \epsilon_j}; \Phi) \|_X \\
\sim \lim_{j \to \infty} \| f_{k, \epsilon_j} \|_{H_X(\mathbb{R}^n)} \lesssim \| u \|_{H_X(\mathbb{R}^n)}^{n+1}),\]

which further implies that \( \{ f_k \}_{k \in \mathbb{N}} \) is also a bounded sequence in \( H_X(\mathbb{R}^n) \) and hence in \( S'(\mathbb{R}^n) \). By the weak compactness of \( S'(\mathbb{R}^n) \) again and the diagonal principle, we find an \( f \in S'(\mathbb{R}^n) \) and a subsequence \( \{ f_{k_j, \epsilon_j} \}_{j \in \mathbb{N}} \) of \( \{ f_{k, \epsilon_j} \}_{j \in \mathbb{N}} \) such that

\[
\lim_{j \to \infty} k_j = \infty, \quad \lim_{j \to \infty} \epsilon_j = 0, \quad \text{and} \quad \lim_{j \to \infty} f_{k_j, \epsilon_j} = f.
\]

in \( S'(\mathbb{R}^n) \). Applying an argument similar to that used in the estimation of Eq. (3.7), we conclude that \( \| f \|_{H_X(\mathbb{R}^n)} \lesssim \| u \|_{H_X(\mathbb{R}^n)}^{n+1}) \) and hence \( f \in H_X(\mathbb{R}^n) \). From Eq. (3.4), Lemma 3.4, Eqs. (3.5) and (3.8), Lemma 2.17, and Proposition 2.21, we infer that, for any \( (x, t) \in \mathbb{R}^{n+1}_+ \),

\[
u(x, t) = \lim_{j \to \infty} u_{k_j}(x, t + \epsilon_j) = \lim_{j \to \infty} P_t * f_{k_j, \epsilon_j}(x) = P_t * f(x).
\]

This finishes the proof of the necessity.

Now, we show the sufficiency. To this end, assume that \( f \in H_X(\mathbb{R}^n) \) such that, for any \( (x, t) \in \mathbb{R}^{n+1}_+ \), \( u(x, t) = P_t * f(x) \). Then, by Corollary 2.25(i), we conclude that \( u \) is harmonic on \( \mathbb{R}^{n+1}_+ \). Also, by both Lemma 2.20 and Eq. (2.8), we find a sequence \( \{ f_k \}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) such that, for any \( (x, t) \in \mathbb{R}^{n+1}_+ \),

\[
u(x, t) = \lim_{k \to \infty} P_t * f_k(x).
\]

For any \( k \in \mathbb{N} \) and \( (x, t) \in \mathbb{R}^{n+1}_+ \), let \( u_k(x, t) := P_t * f_k(x) \). From [35, Theorem 2.1(a)], we deduce that \( \{ u_k \}_{k \in \mathbb{N}} \subset H^2(\mathbb{R}^{n+1}_+) \). Moreover, by Theorem 2.14, we find
that, for any $k \in \mathbb{N}$,
\[
\|u_k\|_{H_X(\mathbb{R}_+^{n+1})} = \sup_{(y,t) \in \mathbb{R}_+^{n+1} : |y| < t} |P_t \ast f_k(y)| \|_{X} = \|M(f_k; P)\|_X \lesssim \|f_k\|_{H_X(\mathbb{R}^n)},
\]
which further implies that $\{u_k\}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}_+^{n+1}) \cap H^2(\mathbb{R}_+^{n+1})$. Finally, we show that $u \in H_X(\mathbb{R}_+^{n+1})$ and $\lim_{k \to \infty} u_k = u$ in $H_X(\mathbb{R}_+^{n+1})$. Indeed, observe that $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $H_X(\mathbb{R}^n)$. Thus, for any $\epsilon \in (0, \infty)$, we find a $K \in \mathbb{N}$ such that, for any $k > K$ and $j > K$,
\[
\|f_k - f_j\|_{H_X(\mathbb{R}^n)} < \epsilon.
\]
From this, Definition 3.2(ii), Eq. (3.9), Lemma 2.4 and Theorem 2.14, we infer that, for any $k > K$,
\[
\|u_k - u\|_{H_X(\mathbb{R}_+^{n+1})} = \sup_{(y,t) \in \mathbb{R}_+^{n+1} : |y| < t} |u_k(y,t) - u(y,t)| \|_{X} = \sup_{(y,t) \in \mathbb{R}_+^{n+1} : |y| < t} \lim_{j \to \infty} |P_t \ast (f_k - f_j)(y)| \|_{X} \leq \lim_{j \to \infty} \|M(f_k - f_j; P)\|_X \leq \lim_{j \to \infty} \|f_k - f_j\|_{H_X(\mathbb{R}^n)} \lesssim \epsilon,
\]
which further implies that $u \in H_X(\mathbb{R}_+^{n+1})$ and $\lim_{k \to \infty} u_k = u$ in $H_X(\mathbb{R}_+^{n+1})$. Thus, $u \in H_{X,2}(\mathbb{R}_+^{n+1})$. This then finishes the proof of the sufficiency and hence Proposition 3.3.

Next, we introduce the Hardy space $H_X(\mathbb{R}_+^{n+1})$ consisting of vectors of harmonic functions. To be precise, let $F := (u_0, u_1, \ldots, u_n)$ be a harmonic vector on $\mathbb{R}_+^{n+1}$, that is, for any $k \in \{0, 1, \ldots, n\}$, $u_k$ is harmonic on $\mathbb{R}_+^{n+1}$. The vector $F$ is said to satisfy the generalized Cauchy–Riemann equation if
\[
\sum_{j=0}^{n} \frac{\partial u_j}{\partial x_j} = 0, \quad \frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j}, \quad \forall j, k \in \{0, 1, \ldots, n\},
\]
where, for any $(x, t) \in \mathbb{R}_+^{n+1}$, $x := (x_1, \ldots, x_n)$, and $x_0 := t$. 
Definition 3.5 Let $X$ be a BQBF space.

(i) The Hardy space $\mathbb{H}_X(\mathbb{R}^{n+1})$ consisting of vectors of harmonic functions is defined to be the set of all the harmonic vectors $F := (u_0, u_1, \ldots, u_n)$ on $\mathbb{R}^{n+1}$ satisfying Eq. (3.10) and

$$\|F\|_{\mathbb{H}_X(\mathbb{R}^{n+1})} := \sup_{t \in (0, \infty)} \|F(\cdot, t)\|_X < \infty,$$

where, for any $(x, t) \in \mathbb{R}^{n+1}$,

$$|F(x, t)| := \left[ \sum_{j=0}^n |u_j(x, t)|^2 \right]^{1/2}.$$

When $X := L^p(\mathbb{R}^n)$ with $p \in (0, \infty)$, the Hardy space $\mathbb{H}_{L^p}(\mathbb{R}^n)(\mathbb{R}^{n+1})$ is simply denoted by $\mathbb{H}^p(\mathbb{R}^{n+1})$.

(ii) The subspace $\mathbb{H}_{X,2}(\mathbb{R}^{n+1})$ is defined to be the set of all the vectors $F \in \mathbb{H}_X(\mathbb{R}^{n+1})$ satisfying that there exists a sequence $\{F_k\}_{k \in \mathbb{N}} \subset \mathbb{H}_X(\mathbb{R}^{n+1}) \cap H^2(\mathbb{R}^{n+1})$ such that

$$F = \lim_{k \to \infty} F_k$$

in $\mathbb{H}_X(\mathbb{R}^{n+1})$. Moreover, for any $F \in \mathbb{H}_{X,2}(\mathbb{R}^{n+1})$, let $\|F\|_{\mathbb{H}_{X,2}(\mathbb{R}^{n+1})} := \|F\|_{\mathbb{H}_X(\mathbb{R}^{n+1})}$.

For any $F \in \mathbb{H}_X(\mathbb{R}^{n+1})$, we have the following conclusion.

Lemma 3.6 Let $X$ be a BQBF space. Assume that there exists an $s \in (\frac{n-1}{n}, \infty)$ such that $X^{1/s}$ is a ball Banach function space and $M$ is bounded on $(X^{1/s})'$. Then, for any $a \in [0, \infty)$, $q \in [\frac{n-1}{n}, s)$, $F \in \mathbb{H}_X(\mathbb{R}^{n+1})$, and $(x, t) \in \mathbb{R}^{n+1}$,

$$|F(x, t + a)|^q \leq P_t \ast (g_a)^q(x), \quad (3.11)$$

where $g_a(\cdot) := \lim_{t \to 0^+} |F(\cdot, t + a)|$. Moreover, for any $a \in [0, \infty)$, $g_a \in X$ and

$$\|g_a\|_X \leq \|F\|_{\mathbb{H}_X(\mathbb{R}^{n+1})}.$$
where $\omega_n$ is the unit spherical measure of $\mathbb{R}^{n+1}$ and $S^n$ is the unit sphere of $\mathbb{R}^{n+1}$.

To show Lemma 3.6, we also need the following so-called $B_p(\mathbb{R}^n)$-condition of Muckenhoupt weights (see, for instance, [6, (2.21)]).

**Lemma 3.8** Let $p \in [1, \infty)$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a positive constant $C$ such that, for any $(x, t) \in \mathbb{R}_+^{n+1},$

$$\int_{\mathbb{R}^n} \frac{w(y)}{(t + |x - y|)^{np}} dy \leq Ct^{-np} \int_{B(x,t)} w(y) dy.$$  

Now, we show Lemma 3.6.

**Proof of Lemma 3.6** For any $a \in [0, \infty)$ and $t \in (0, \infty)$, let

$$F_a(\cdot, t) := F(\cdot, t + a)$$

and

$$K(|F_a|^{q\eta}, t) := \int_{\mathbb{R}^n} \frac{|F(x, t + a)|^{q\eta}}{(|x| + 1 + t)^{n+1}} dx,$$

where $\eta \in (1, s/q)$ (the existence of such an $\eta$ is guaranteed by $q \in \left[\frac{n-1}{n}, s\right]$). We claim that $K(|F_a|^{q\eta}, \cdot)$ is bounded on $(0, \infty)$. To show this, for any $t \in (0, \infty)$, let

$$E_t := \{x \in \mathbb{R}^n : |F(x, t + a)| \leq 1\}.$$  

Then we write

$$K(|F_a|^{q\eta}, t) = \int_{E_t} \frac{|F(x, t + a)|^{q\eta}}{(|x| + 1 + t)^{n+1}} dx + \int_{E_t^c} \cdots =: I + II.$$  

We first estimate $I$. By the definition of $E_t$, we conclude that

$$I \leq \int_{\mathbb{R}^n} \frac{1}{(|x| + 1 + t)^{n+1}} dx \lesssim \int_0^\infty \frac{1}{(r + 1 + t)^{n+1}} r^{n-1} dr \sim \frac{1}{1 + t}.$$  

Next, we estimate $II$. Letting $q_0 := s/(q \eta)$ and $w$ be the same as in Lemma 2.23, then we find that $q_0 \in (1, \infty)$. By the Hölder inequality, we obtain

$$II \leq \left\{ \int_{\mathbb{R}^n} |F(x, t + a)|^{s} w(x) dx \right\}^{\frac{1}{q_0}} \left\{ \int_{\mathbb{R}^n} \frac{[w(x)]^{-\frac{q_0}{q_0}}} {(|x| + 1 + t)^{(n+1)q_0}} dx \right\}^{\frac{1}{q_0}}$$

$$=: III \times IV.$$

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From Lemma 2.23 and $F \in \mathbb{H}_X(\mathbb{R}^{n+1}_+)$, we deduce that

$$\text{III} \leq \| F(\cdot, t + a) \|^q_{L^q_w(\mathbb{R}^n)} \lesssim \| F(\cdot, t + a) \|^q_{X} \lesssim \| F \|^q_{\mathbb{H}_X(\mathbb{R}^{n+1}_+)}.$$  

By Remark 2.24, $q_0 \in (1, \infty)$, and [23, Proposition 7.1.5(6)], we conclude that $w \in A_{q_0}(\mathbb{R}^n)$. Moreover, from [23, Proposition 7.1.5(4)], we further infer $w^{-q_0/q_0} \in A_{q_0}(\mathbb{R}^n)$. Using this and Lemma 3.8 with $p := q_0'$, $w := w^{-q_0/q_0}$, and $(x, t) := (0, 1)$, we find that

$$\text{IV} \leq \frac{1}{1 + t} \left\{ \int_{\mathbb{R}^n} \left[ \frac{w(x)^{-q_0/q_0}}{(|x| + 1)^{nq_0}} \right]^{1/q_0'} \right\} \sim \frac{1}{1 + t}.$$

Combining this and the estimates of I, II, and III, we then finish the proof of the above claim.

On the other hand, by $F \in \mathbb{H}_X(\mathbb{R}^{n+1}_+)$ and [55, p. 234, Theorem 4.14], we find that $|F_a|^q$ is subharmonic for any $q \in \left(\frac{n-1}{n}, 1\right)$. From [41, Theorem 8], we deduce that $|F_a|^q \leq P_t * (g_a)^q$, where, for any $x \in \mathbb{R}^n$,

$$g_a(x) := \lim_{t \to 0^+} |F_a(x, t)| = \lim_{t \to 0^+} |F(x, t + a)|. \quad (3.12)$$

This shows that Eq. (3.11) holds true.

Finally, by both Lemma 2.4 and Eq. (3.12), we obtain, for any $a \in [0, \infty)$,

$$\| g_a \|_X = \lim_{t \to 0^+} \| F(\cdot, t + a) \|_X \leq \lim_{t \to 0^+} \| F(\cdot, t + a) \|_X \leq \| F \|_{\mathbb{H}_X(\mathbb{R}^{n+1}_+)}.$$

This further implies $g_a \in X$ and hence finishes the proof of Lemma 3.6. \qed

Using Lemma 3.6, we obtain the following conclusion.

**Proposition 3.9** Let $X$ be a BQBF space. Assume that $X$ satisfies both Assumptions 2.8 and 2.10 with $s \in (\frac{n-1}{n}, 1]$ and $\theta \in [\frac{n-1}{n}, s)$. If $F := (u_0, u_1, \ldots, u_n) \in \mathbb{H}_X(\mathbb{R}^{n+1}_+)$, then $u_0 \in H_X(\mathbb{R}^{n+1}_+)$ and there exists a positive constant $C$, independent of $F$, such that

$$\| u_0 \|_{H_X(\mathbb{R}^{n+1}_+)} \leq C \| F \|_{\mathbb{H}_X(\mathbb{R}^{n+1}_+)}.$$

**Proof** Assume that $F \in \mathbb{H}_X(\mathbb{R}^{n+1}_+)$. By Lemma 3.6, we find that there exists a non-negative function $h \in X$ such that, for any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$|F(x, t)|^q \leq P_t * h^q(x).$$
where $q \in [\frac{n-1}{n}, \theta]$. From this, we infer that, for any $x \in \mathbb{R}^n$,

$$[u_0^q(x)]^q = \sup_{(y,t) \in \mathbb{R}^{n+1}: |y-x| \leq t} |u_0(y,t)|^q \leq \sup_{(y,t) \in \mathbb{R}^{n+1}: |y-x| \leq t} |F(y,t)|^q \leq \sup_{(y,t) \in \mathbb{R}^{n+1}: |y-x| \leq t} P_t \ast h^q(y).$$ \hspace{1cm} (3.13)

Observe that, for any $x \in \mathbb{R}^n$ and $(y,t) \in \mathbb{R}^{n+1}$ with $|y-x| < t$,

$$P_t \ast h^q(y) \sim \int_{\mathbb{R}^n} \frac{t[h(z)]^q}{(t^2 + |y-z|^2)^{(n+1)/2}} \, dz$$

$$\sim \int_{B(x,2t)} \frac{t[h(z)]^q}{(t^2 + |y-z|^2)^{(n+1)/2}} \, dz + \sum_{j=1}^{\infty} \int_{B(x,2^{j+1}t) \setminus B(x,2jt)} [h(z)]^q \, dz$$

$$\lesssim \frac{1}{t^n} \left[ \int_{B(x,2t)} [h(z)]^q \, dz + \sum_{j=1}^{\infty} \frac{1}{2^{(j+1)(n+1)}} \int_{B(x,2^{j+1}t)} [h(z)]^q \, dz \right]$$

$$\lesssim M(h^q)(x),$$

which, together with Eq. (3.13), further implies that

$$(u_0^q)^q \lesssim M(h^q).$$ \hspace{1cm} (3.15)

By Assumption 2.8, $q \in [\frac{n-1}{n}, \theta]$, and [7, Lemma 2.9], we find that $M$ is bounded on $X^{1/q}$. Combining this, Eq. (3.15), and Lemma 3.6, we conclude that

$$\|u_0\|_{H_X(\mathbb{R}^{n+1})} = \| (u_0^q)^{1/q} \|_{X^{1/q}} \lesssim \| M(h^q) \|_{X^{1/q}}$$

$$\lesssim \| h^q \|_{X^{1/q}} \sim \| h \|_X \lesssim \| F \|_{H_X(\mathbb{R}^{n+1})}.$$ \hspace{1cm} (3.16)

This finishes the proof of Proposition 3.9. \hspace{1cm} $\square$

Using Proposition 3.9, we obtain the following corollary.

**Corollary 3.10** Let $X$ be a BQBF space. Assume that $X$ satisfies both Assumptions 2.8 and 2.10 with $s \in (\frac{n-1}{n}, 1]$ and $\theta \in [\frac{n-1}{n}, s)$. If $F := (u_0, u_1, \ldots, u_n) \in H_{X,2}(\mathbb{R}^{n+1})$, then $u_0 \in H_{X,2}(\mathbb{R}^{n+1})$ and there exists a positive constant $C$, independent of $F$, such that

$$\|u_0\|_{H_X(\mathbb{R}^{n+1})} \leq C \| F \|_{H_X(\mathbb{R}^{n+1})}.$$ 

**Proof** Fix an $F := (u_0, u_1, \ldots, u_n) \in H_{X,2}(\mathbb{R}^{n+1})$. By Definition 3.5(ii), for any $k \in \mathbb{N}$, we find an

$$F_k := (u_{0,k}, u_{1,k}, \ldots, u_{n,k}) \in H_X(\mathbb{R}^{n+1}) \cap H^2(\mathbb{R}^{n+1}) \hspace{1cm} (3.17)$$
such that
\[ F = \lim_{k \to \infty} F_k \]  
(3.18)
in \( \mathbb{H}_X(\mathbb{R}_{n+1}^+) \) and, for any \( k \in \mathbb{N} \),
\[ \| F_k \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \lesssim \| F \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)}. \]  
(3.19)

From both Proposition 3.9 and Eq. (3.19), we deduce that
\[ u_0 \in H_X(\mathbb{R}_{n+1}^+) \]  
and \( \| u_0 \|_{H_X(\mathbb{R}_{n+1}^+)} \lesssim \| F \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \)
and, for any \( k \in \mathbb{N} \),
\[ u_{0,k} \in H_X(\mathbb{R}_{n+1}^+) \]  
and \( \| u_{0,k} \|_{H_X(\mathbb{R}_{n+1}^+)} \lesssim \| F_k \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \lesssim \| F \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \).

On the other hand, by both Definitions 3.1 and 3.5 and by Eq. (3.17), we find that, for any \( k \in \mathbb{N} \),
\[ \| u_{0,k} \|_{H^2(\mathbb{R}_{n+1}^+)} \leq \sup_{t \in (0, \infty)} \| F_k(\cdot, t) \|_{L^2(\mathbb{R}^n)} = \| F_k \|_{\mathbb{H}^2(\mathbb{R}_{n+1}^+)} \]
and hence \( u_{0,k} \in H_X(\mathbb{R}_{n+1}^+) \cap H^2(\mathbb{R}_{n+1}^+) \). Moreover, using the definition of \( \mathbb{H}_X(\mathbb{R}_{n+1}^+) \), Proposition 3.9, and Eq. (3.18), we conclude that, as \( k \to \infty \),
\[ \| u_0 - u_{0,k} \|_{H_X(\mathbb{R}_{n+1}^+)} \lesssim \| F - F_k \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \to 0. \]

This further implies that \( u_0 \in H_{X,2}(\mathbb{R}_{n+1}^+) \) and hence finishes the proof of Corollary 3.10. \( \square \)

Now, we establish the relation between \( H_X(\mathbb{R}^n) \) and \( H_{X,2}(\mathbb{R}_{n+1}^+) \).

**Proposition 3.11** Assume that \( X \) is a BQBF space, satisfies both Assumptions 2.8 and 2.10, and has an absolutely continuous quasi-norm. If \( f \in H_X(\mathbb{R}^n) \), then there exists an
\[ F := (u_0, u_1, \ldots, u_n) \in H_{X,2}(\mathbb{R}_{n+1}^+) \]
such that \( F \) satisfies Eq. (3.10) and, for any \( (x, t) \in \mathbb{R}_{n+1}^+ \), \( u_0(x, t) = P_t \ast f(x) \). Moreover, there exists a positive constant \( C \), independent of \( f \), such that \( \| F \|_{\mathbb{H}_X(\mathbb{R}_{n+1}^+)} \leq C \| f \|_{H_X(\mathbb{R}^n)}. \)

To show Proposition 3.11, we need the following boundedness of Riesz transforms on \( H_X(\mathbb{R}^n) \) which is a combination of both [61, Theorem 3.14] and Lemma 2.23; we omit the details here.
Lemma 3.12 Assume that $X$ is a BQBF space and satisfies both Assumptions 2.8 and 2.10. Then, for any $j \in \{1, \ldots, n\}$, the Riesz transform $R_j$ is bounded on $H_X(\mathbb{R}^n)$.

Next, we show Proposition 3.11.

Proof of Proposition 3.11 Assume $f \in H_X(\mathbb{R}^n)$. By Lemma 2.20, we find a sequence $(f_k)_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that
\[
\lim_{k \to \infty} f_k = f
\]
in $H_X(\mathbb{R}^n)$ [and hence in $S'(\mathbb{R}^n)$ because $H_X(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ (see [32, Lemma 4.8.19])] and, for any $k \in \mathbb{N}$,
\[
\|f_k\|_{H_X(\mathbb{R}^n)} \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
\]
For any $k \in \mathbb{N}$ and $(x, t) \in \mathbb{R}^{n+1}_+$, let
\[
u_{0,k}(x, t) := P_t \ast f_k(x)
\]
and, for any $j \in \{1, \ldots, n\}$,
\[
u_{j,k}(x, t) := Q_t^{(j)} \ast f_k(x),
\]
where, for any $j \in \{1, \ldots, n\}$, $Q_t^{(j)}$ is the $j$-th conjugate Poisson kernel defined by setting, for any $(x, t) \in \mathbb{R}^{n+1}_+$,
\[
Q_t^{(j)}(x) := c_{(n)} \frac{x_j}{(t^2 + |x|^2)^{(n+1)/2}}
\]
with $c_{(n)}$ in Eq. (1.1). Moreover, for any $k \in \mathbb{N}$, let
\[
F_k := (\nu_{0,k}, \nu_{1,k}, \ldots, \nu_{n,k}).
\]
From $f_k \in L^2(\mathbb{R}^n)$ and [55, p. 236, Theorem 4.17(i)], we infer that, for any $k \in \mathbb{N}$, $F_k \in H^2(\mathbb{R}^{n+1}_+)$ and satisfies Eq. (3.10). Using Theorem 2.14 and Eq. (3.21), we conclude that, for any $k \in \mathbb{N}$ and $t \in (0, \infty)$,
\[
\|\nu_{0,k}(\cdot, t)\|_X \lesssim \|f_k\|_{H_X(\mathbb{R}^n)} \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
\]
Moreover, from [51, p. 65, Theorem 3 and p. 78, Item 4.3], we deduce that, for any $k \in \mathbb{N}$, $j \in \{1, \ldots, n\}$, and $(x, t) \in \mathbb{R}^{n+1}_+$,
\[
Q_t^{(j)} \ast f_k(x) = P_t \ast R_j(f_k)(x).
\]
This, together with Theorem 2.14, Lemma 3.12, and Eq. (3.21), further implies that, for any \( k \in \mathbb{N} \), \( j \in \{1, \ldots, n\} \), and \( t \in (0, \infty) \),

\[
\| u_{j,k}(\cdot, t) \|_X = \| P_t \ast R_j(f_k) \|_X \lesssim \| R_j(f_k) \|_{H^X(\mathbb{R}^n)} \\
\lesssim \| f_k \|_{H^X(\mathbb{R}^n)} \lesssim \| f \|_{H^X(\mathbb{R}^n)},
\]

which, combined with Eq. (3.22), shows that

\[
\| F_k \|_{H^X(\mathbb{R}^n+1)} = \sup_{t \in (0, \infty)} \| F_k(\cdot, t) \|_X \lesssim \sup_{t \in (0, \infty)} \sum_{j=0}^n \| u_{j,k}(\cdot, t) \|_X \lesssim \| f \|_{H^X(\mathbb{R}^n)}.
\]

(3.23)

Thus, for any \( k \in \mathbb{N} \),

\[
F_k \in \left[ H^X(\mathbb{R}^n+1) \cap H^2(\mathbb{R}^n+1) \right].
\]

(3.24)

On the other hand, by Eq. (3.20), the well-known boundedness of Riesz transforms on \( L^2(\mathbb{R}^n) \), and Lemma 3.12, we find that, for any \( j \in \{1, \ldots, n\} \),

\[
\{ R_j(f_k) \}_{k \in \mathbb{N}} \subset H^X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \text{ and } \lim_{k \to \infty} R_j(f_k) = R_j(f)
\]

in \( H^X(\mathbb{R}^n) \). For any \((x, t) \in \mathbb{R}^n+1 \), let

\[
F(x, t) := (P_t \ast f(x), P_t \ast R_1(f)(x), \ldots, P_t \ast R_n(f)(x)).
\]

Using Proposition 2.21, we find that, for any \( j \in \{1, \ldots, n\} \) and \((x, t) \in \mathbb{R}^n+1 \),

\[
\lim_{k \to \infty} P_t \ast R_j(f_k)(x) = P_t \ast R_j(f)(x)
\]

uniformly on any compact set of \( \mathbb{R}^n+1 \). From this and the fact that \( F_k \) satisfies Eq. (3.10), we infer that \( F \) also satisfies Eq. (3.10) and, for any \((x, t) \in \mathbb{R}^n+1 \),

\[
| F(x, t) | = \lim_{k \to \infty} | F_k(x, t) |.
\]

By this, Lemma 2.4, and Eq. (3.23), we conclude that

\[
\| F \|_{H^X(\mathbb{R}^n+1)} = \sup_{t \in (0, \infty)} \| \lim_{k \to \infty} | F_k(x, t) | \|_X \lesssim \sup_{t \in (0, \infty)} \lim_{k \to \infty} \| F_k(x, t) \|_X \lesssim \| f \|_{H^X(\mathbb{R}^n)}.
\]
Thus, \( F \in \mathbb{H}_X(\mathbb{R}^{n+1}_+) \). Finally, we show that \( F \in \mathbb{H}_{X,2}(\mathbb{R}^{n+1}_+) \). Indeed, using the definitions of both \( F \) and \( F_k \), both (ii) and (iii) of Corollary 2.25, and Lemma 3.12, we find that
\[
\| F - F_k \|_{\mathbb{H}_X(\mathbb{R}^{n+1}_+)} \lesssim \sup_{t \in (0, \infty)} \| P_t * f - P_t * f_k \|_X + \sum_{j=1}^n \| P_t * R_j(f_k) - P_t * R_j(f) \|_X
\]
\[
\lesssim \sup_{t \in (0, \infty)} \| P_t * (f - f_k) \|_X + \sup_{t \in (0, \infty)} \sum_{j=1}^n \| P_t * R_j(f - f_k) \|_X
\]
\[
\lesssim \| f - f_k \|_{\mathbb{H}_X(\mathbb{R}^{n}_+)} \to 0
\]
as \( k \to \infty \). This, together with Eq. (3.24), proves that \( F \in \mathbb{H}_{X,2}(\mathbb{R}^{n+1}_+) \) and hence finishes the proof of Proposition 3.11.

Combining Proposition 3.3, Corollary 3.10, and Proposition 3.11, we obtain the following isomorphism theorem on \( H_X(\mathbb{R}^{n}_+), H_{X,2}(\mathbb{R}^{n+1}_+), \) and \( H_{X,2}(\mathbb{R}^{n+1}_+) \); we omit the details here.

**Theorem 3.13** Assume that \( X \) is a BQBF space, satisfies both Assumptions 2.8 and 2.10 with \( s \in \left( \frac{n-1}{n}, 1 \right] \) and \( \theta \in \left[ \frac{n-1}{s}, s \right) \), and has an absolutely continuous quasi-norm. Then the following statements are equivalent:

(i) \( u \in H_{X,2}(\mathbb{R}^{n+1}_+) \);

(ii) there exists an \( f \in H_X(\mathbb{R}^{n}_+) \) such that \( u(x, t) = P_t * f(x) \) for any \( (x, t) \in \mathbb{R}^{n+1}_+ \);

(iii) there exist harmonic functions \( \{ u_1, \ldots, u_n \} \) on \( \mathbb{R}^{n+1}_+ \) such that
\[
F := (u, u_1, \ldots, u_n) \in H_{X,2}(\mathbb{R}^{n+1}_+).
\]

Moreover,
\[
\| u \|_{H_{X,2}(\mathbb{R}^{n+1}_+)} \sim \| f \|_{H_X(\mathbb{R}^{n}_+)} \sim \| F \|_{H_{X,2}(\mathbb{R}^{n+1}_+)},
\]
where the positive equivalence constants are independent of \( u, f, \) and \( F \).

Now, we establish the first order Riesz transform characterization of \( H_X(\mathbb{R}^{n}_+) \). Let us first introduce the following Riesz–Hardy space associated with \( X \).

**Definition 3.14** Let \( X \) be a BQBF space. The Riesz–Hardy space \( H_{X,Riesz}(\mathbb{R}^{n}_+) \) associated with \( X \) is defined to be the set of all the \( f \in \mathcal{S}'(\mathbb{R}^{n}_+) \) satisfying that there exists a positive constant \( A \) and a sequence \( \{ f_k \}_{k \in \mathbb{N}} \subset L_2(\mathbb{R}^{n}_+) \) such that \( \lim_{k \to \infty} f_k = f \) in \( \mathcal{S}'(\mathbb{R}^{n}_+) \) and, for any \( k \in \mathbb{N} \),
\[
\| f_k \|_X + \sum_{j=1}^n \| R_j(f_k) \|_X \leq A.
\]
Moreover, for any $f \in H_{X, \text{Riesz}}(\mathbb{R}^n)$,
\[ \|f\|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} := \inf \{ A : A \text{ satisfies Eq. (3.25)} \} . \]

**Theorem 3.15** Assume that $X$ is a BQBF space and satisfies both Assumptions 2.8 and 2.10 with $s \in (\frac{n-1}{n}, 1]$ and $\theta \in [\frac{n-1}{n}, s)$. Then the following statements hold true.

(i) There exists a positive constant $C \in [0, \infty)$ such that, for any $f \in L^2(\mathbb{R}^n)$,
\[ \|f\|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} \leq C \|f\|_{H_X(\mathbb{R}^n)}. \]

(ii) There exists a positive constant $C \in [0, \infty)$ such that, for any $f \in S'(\mathbb{R}^n)$,
\[ \|f\|_{H_X(\mathbb{R}^n)} \leq C \|f\|_{H_{X, \text{Riesz}}(\mathbb{R}^n)}. \]

(iii) Moreover, if $X$ has an absolutely continuous quasi-norm, then
\[ H_X(\mathbb{R}^n) = H_{X, \text{Riesz}}(\mathbb{R}^n) \]
with equivalent (quasi-)norms.

To prove Theorem 3.15, we need the following concept of radial decreasing functions.

**Definition 3.16** A function $f$ on $\mathbb{R}^n$ is said to be radial decreasing if $f$ satisfies:

(i) for any $x, y \in \mathbb{R}^n$ with $|x| = |y|$, $f(x) = f(y)$;

(ii) for any $t \in (0, \infty)$, let $\tilde{f}(t) := f(x)$, where $x \in \mathbb{R}^n$ is such that $|x| = t$. Then $\tilde{f}$ is decreasing on $(0, \infty)$.

Now, we show Theorem 3.15.

**Proof of Theorem 3.15** We first show (i). To this end, fix an $f \in L^2(\mathbb{R}^n)$. If $\|f\|_{H_X(\mathbb{R}^n)} = \infty$, then (i) obviously holds true. In what follows, we assume that $\|f\|_{H_X(\mathbb{R}^n)} < \infty$. Choose a $\Phi \in S(\mathbb{R}^n)$ to be positive and radial decreasing such that $\int_{\mathbb{R}^n} \Phi(x) \, dx = 1$. Since $f \in L^2(\mathbb{R}^n)$, from [17, Corollary 2.9], it follows that
\[ |f| \leq M(f; \Phi) \quad \text{and} \quad |R_j(f)| \leq M(R_j(f); \Phi), \tag{3.26} \]
where $M(f; \Phi)$ is the same as in Eq. (2.6) and $M(R_j(f); \Phi)$ is the same as in Eq. (2.6) via $f$ replaced by $R_j(f)$. Using this, Definition 2.1(ii), [46, Theorem 3.1(i)], and Lemma 3.12, we conclude that
\[ \|f\|_X + \sum_{j=1}^{n} \|R_j(f)\|_X \]
\[ \leq \|M(f; \Phi)\|_X + \sum_{j=1}^{n} \|M(R_j(f); \Phi)\|_X \lesssim \|f\|_{H_X(\mathbb{R}^n)}, \tag{3.27} \]
which further implies that \( f \in H_{X, \text{Riesz}}(\mathbb{R}^n) \) and \( \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)} \). This finishes the proof of (i).

We next prove (ii). To achieve this, fix an \( f \in S'(\mathbb{R}^n) \). If \( \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} = \infty \), then (ii) obviously holds true. In what follows, we assume that \( \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} < \infty \). By Definition 3.14, we find a sequence \( \{ f_k \}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n) \) such that

\[
\lim_{k \to \infty} f_k = f
\]

(3.28)
in \( S'(\mathbb{R}^n) \) and, for any \( k \in \mathbb{N} \),

\[
\| f_k \|_X + \sum_{j=1}^{n} \| R_j(f_k) \|_X \lesssim \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)}.
\]

(3.29)

For any \( k \in \mathbb{N} \) and \( (x, t) \in \mathbb{R}^{n+1}_+ \), let

\[
F_k(x, t) := (P_t * f_k(x), P_t * R_1(f_k)(x), \ldots, P_t * R_n(f_k)(x)).
\]

Using \( f_k \in L^2(\mathbb{R}^n) \) and [55, p. 236, Theorem 4.17(i)], we conclude that, for any \( k \in \mathbb{N} \), \( F_k \in \mathbb{H}^2(\mathbb{R}^{n+1}_+). \) From this and Lemma 3.6, we deduce that, for any \( q \in [\frac{n-1}{n}, \theta] \) and \( (x, t) \in \mathbb{R}^{n+1}_+ \),

\[
|F_k(x, t)|^q \leq P_t * [h_k]^q(x),
\]

(3.30)

where, for any \( x \in \mathbb{R}^n \), \( h_k(x) := \lim_{t \to 0^+} |F_k(x, t)|. \) By both \( f_k \in L^2(\mathbb{R}^n) \) and [17, Corollary 2.9], we obtain, for almost every \( x \in \mathbb{R}^n \),

\[
h_k(x) = \lim_{t \to 0^+} \left[ |P_t * f_k(x)|^2 + \sum_{j=1}^{n} |P_t * R_j(f_k)(x)|^2 \right]^{1/2}
\]

(3.31)

\[= \left[ \| f_k(x) \|^2 + \sum_{j=1}^{n} \| R_j(f_k)(x) \|^2 \right]^{1/2}.\]

Using [7, Lemma 2.9], Assumption 2.8, and \( q \in [\frac{n-1}{n}, \theta] \), we find that \( M \) is bounded on \( X^{1/q} \). This, combined with Definition 2.1(ii) and Eqs. (3.30), (3.14), (3.31) and (3.29), further implies that, for any \( k \in \mathbb{N} \),

\[
\| F_k \|_{\mathbb{H}^2(\mathbb{R}^{n+1}_+)} = \sup_{t \in (0, \infty)} \| F_k(\cdot, t) \|^q_{X^{1/q}} \leq \| P_t * [h_k]^q \|^q_{X^{1/q}}
\]

\[\lesssim \| M \|_{X^{1/q}} \| h_k \|_X
\]

\[\lesssim \| f_k \|_X + \sum_{j=1}^{n} \| R_j(f_k) \|_X \lesssim \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)}.
\]
Thus, \( \{ F_k \}_{k \in \mathbb{N}} \subset \mathbb{H}_X(\mathbb{R}^{n+1}_+) \, \cap \, \mathbb{H}^2(\mathbb{R}^{n+1}_+) \). From this and Proposition 3.9, we infer that, for any \( k \in \mathbb{N} \), \( P_r * f_k \in H_X(\mathbb{R}^{n+1}_+) \) and

\[
\| P_r * f_k \|_{H_X(\mathbb{R}^{n+1}_+)} \lesssim \| F_k \|_{\mathbb{H}_X(\mathbb{R}^{n+1}_+)} \lesssim \| f \|_{H_X, \text{Riesz}(\mathbb{R}^n)},
\]

which, together with Remark 2.13, Theorem 2.14, Eq. (2.4), and both (i) and (ii) of Definition 3.2, further implies that, for any \( k \in \mathbb{N} \), \( f_k \in H_X(\mathbb{R}^n) \) and

\[
\| f_k \|_{H_X(\mathbb{R}^n)} \sim \| M(f_k; P) \|_X \sim \| P_r * f_k \|_{H_X(\mathbb{R}^{n+1}_+)} \lesssim \| f \|_{H_X, \text{Riesz}(\mathbb{R}^n)}.
\]

Using this, [46, Theorem 3.1], Eq. (3.28), and Lemma 2.4, we obtain

\[
\| f \|_{H_X(\mathbb{R}^n)} \lesssim \| M(f; \Phi) \|_X \sim \sup_{t \in (0, \infty)} \| f * \Phi_t \|_X \sim \sup_{t \in (0, \infty)} \left[ \lim_{k \to \infty} \| f_k * \Phi_t \|_X \right] \lesssim \lim_{k \to \infty} \| M(f_k; \Phi) \|_X \lesssim \lim_{k \to \infty} \| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X, \text{Riesz}(\mathbb{R}^n)}.
\]

This finishes the proof of (ii).

Finally, we show (iii). To achieve this, by (ii), it suffices to prove that

\[
H_X(\mathbb{R}^n) \subset H_{X, \text{Riesz}}(\mathbb{R}^n).
\]

To this end, fix an \( f \in H_X(\mathbb{R}^n) \). By Lemma 2.20, we find a sequence \( \{ f_k \}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^n) \, \cap \, L^2(\mathbb{R}^n) \) such that \( \lim_{k \to \infty} f_k = f \) in both \( H_X(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) and \( \| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)} \) for any \( k \in \mathbb{N} \). Using this and Eq. (3.27), we conclude that, for any \( k \in \mathbb{N} \),

\[
\| f_k \|_X + \sum_{j=1}^n \| R_j(f_k) \|_X \lesssim \| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)},
\]

which further implies that \( f \in H_{X, \text{Riesz}}(\mathbb{R}^n) \) and \( \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)} \). This finishes the proof of Eq. (3.32) and hence Theorem 3.15.

**Remark 3.17** (i) We point out that, when \( X := L^p(\mathbb{R}^n) \) with \( p \in (\frac{n-1}{n}, 1] \), Theorem 3.15 in this case was obtained in [21] (see also [52, p. 123, Proposition 3]). When \( X := L^1_{\text{loc}}(\mathbb{R}^n) \), Theorem 3.15 in this case was obtained in [63]. When \( X \) is a variable Lebesgue space, Theorem 3.15 in this case was obtained in [67, Theorem 1.5]. When \( X \) is a Musielak–Orlicz space, Theorem 3.15 in this case was obtained in [6, Theorem 1.5]. To the best of our knowledge, when \( X \) is a Lorentz space, a mixed-norm Lebesgue space, a local generalized Herz space, or a mixed-norm Herz space, the results obtained in Theorem 3.15 are new. When \( X \) is a Morrey space, the results obtained in (i) and (ii) of Theorem 3.15 are new. Observe that the
proof of Theorem 3.15(iii) strongly depends on the density of $H_X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $H_X(\mathbb{R}^n)$. Therefore, it is unclear whether or not Theorem 3.15(iii) still holds true when $X$ is a Morrey space.

(ii) We also point out that the range $\theta \in [\frac{n-1}{n}, s)$ in Theorem 3.15 is the best possible in the sense that, for any $\theta \in (0, \frac{n-1}{n})$, Theorem 3.15 does not hold true anymore. Indeed, let $X := L^p(\mathbb{R}^n)$ satisfy both Assumptions 2.8 and 2.10 with both $\theta \in (0, \frac{n-1}{n})$ and $s \in (\theta, 1]$ and, by [61, Remark 2.4(a)], we find that $p \in (0, \theta) \subset (0, \frac{n-1}{n})$. However, it is known that, for any $p \in (0, \frac{n-1}{n})$, $H^p(\mathbb{R}^n)$ can no longer be characterized by the first order Riesz transforms but can be characterized by the higher order Riesz transforms (see, for instance, [21, p. 168] for more details). This further implies that Theorem 3.15 does not hold true in this case. Thus, the range $\theta \in [\frac{n-1}{n}, s)$ in Theorem 3.15 is the best possible.

(iii) It is worth pointing that, when $X := L^p(\mathbb{R}^n)$, the range of index $p$ plays a vital role in considering the first order Riesz transform characterization of the Hardy space $H^p(\mathbb{R}^n)$. That is because, for a harmonic function $u$, only when $p \in (\frac{n-1}{n}, \infty)$, $|u|^p$ is subharmonic. This fact also results in that we have to find an appropriate range of $q \in [\frac{n-1}{n}, \infty)$ in the proof of Lemma 3.6. Notice that, when we establish the first order Riesz transform characterization of $H_X(\mathbb{R}^n)$ associated with a BQBF space $X$, an essential difficulty is that the quasi-norm of the space $X$ under consideration has no explicit expression. Moreover, a key tool used in the proof of Proposition 3.9, which strongly depends on Lemma 3.6, is the boundedness of the Hardy–Littlewood maximal function on a convexification of $X$ [see Eq. (3.16) above], which follows from Assumption 2.8. Notice that, if $f_j \equiv 0$ for any $j \in \mathbb{N} \cap [2, \infty)$ in Eq. (2.2), then Eq. (2.2) becomes
\[
\|M(f)\|_{X^{1/\theta}} \lesssim \|f\|_{X^{1/\theta}},
\]
which further implies that $s$ plays no role in this case. Observe that, if $X := L^p(\mathbb{R}^n)$, then, in this case, $M$ is bounded on $X^{1/\theta}$ if and only if $p \in (\theta, \infty]$ and hence $\theta$ is the critical index of $p$. Thus, in some sense, $\theta$ can play the part of $p$ when $p$ is not available, that is, when the quasi-norm of $X$ under consideration has no explicit expression. The conclusion of Theorem 3.15 confirms this observation.

4 Higher Order Riesz Transform Characterization

In this section, we establish the higher order Riesz transform characterization of $H_X(\mathbb{R}^n)$. Let us begin with recalling the concept of the tensor product of $m$ copies of $\mathbb{R}^{n+1}$.

**Definition 4.1** Let $m \in \mathbb{N}$ and $\{e_0, e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. The tensor product of $m$ copies of $\mathbb{R}^{n+1}$ is defined to be the set
\[
\underbrace{\mathbb{R}^{n+1} \times \cdots \times \mathbb{R}^{n+1}}_{m} := \left\{ \xi := \sum_{j_1, \ldots, j_m = 0}^{n} \xi_{j_1, \ldots, j_m} e_{j_1} \otimes \cdots \otimes e_{j_m} : \{\xi_{j_1, \ldots, j_m}\}_{j_1, \ldots, j_m = 0}^{n} \subseteq \mathbb{C} \right\},
\]
\[\text{Birkhäuser} \]
where $e_{j_1} \otimes \cdots \otimes e_{j_m}$ denotes the tensor product of $e_{j_1}, \ldots, e_{j_m}$ and

$$
\sum_{j_1, \ldots, j_m=0}^{n} := \sum_{j_1=0}^{n} \cdots \sum_{j_m=0}^{n} .
$$

Each $\xi \in \bigotimes_{k=1}^{m} \mathbb{R}^{n+1}$ is called a tensor of rank $m$.

Let $F : \mathbb{R}^{n+1}_+ \to \bigotimes_{k=1}^{m} \mathbb{R}^{n+1}_+$ be a tensor-valued function of rank $m$ on $\mathbb{R}^{n+1}_+$, that is, it has the form that, for any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$
F(x, t) := \sum_{j_1, \ldots, j_m=0}^{n} F_{j_1, \ldots, j_m}(x, t) e_{j_1} \otimes \cdots \otimes e_{j_m} , \quad (4.1)
$$

where, for any $j_1, \ldots, j_m \in \{0, \ldots, n\}$, $F_{j_1, \ldots, j_m}$ is a function from $\mathbb{R}^{n+1}_+$ to $\mathbb{C}$. A tensor-valued function $F$ of rank $m$ is said to be symmetric if, for any permutation $\sigma$ on $\{1, \ldots, m\}$, any $j_1, \ldots, j_m \in \{0, \ldots, n\}$, and any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$
F_{j_1, \ldots, j_m}(x, t) = F_{\sigma(j_1), \ldots, \sigma(j_m)}(x, t).
$$

A symmetric tensor-valued function $F$ of rank $m$ is said to be of trace zero if, for any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$
\begin{align*}
\sum_{j=0}^{n} F_{j, j, j_3, \ldots, j_m}(x, t) &= 0, \quad \forall j_3, \ldots, j_m \in \{0, \ldots, n\} \text{ if } m \geq 3, \\
\sum_{j=0}^{n} F_{j, j}(x, t) &= 0 \quad \text{ if } m = 2.
\end{align*}
$$

We make the convention that a tensor-valued function $F$ of rank 1 is always of trace zero. Let $F$ be the same as in Eq. (4.1) and, for any $j_1, \ldots, j_m \in \{0, \ldots, n\}$, $F_{j_1, \ldots, j_m}$ be differentiable. Then the gradient of $F$,

$$
\nabla F : \mathbb{R}^{n+1}_+ \to \bigotimes_{k=1}^{m+1} \mathbb{R}^{n+1}_+ ,
$$

is a tensor-valued function of rank $m + 1$ of the form that, for any $(x, t) \in \mathbb{R}^{n+1}_+$,

$$
\nabla F(x, t) = \sum_{j=0}^{n} \frac{\partial F}{\partial x_j}(x, t) \otimes e_j = \sum_{j=0}^{n} \sum_{j_1, \ldots, j_m=0}^{n} \frac{\partial F_{j_1, \ldots, j_m}}{\partial x_j}(x, t) e_{j_1} \otimes \cdots \otimes e_{j_m} \otimes e_j.
$$
Here and thereafter, we always let $x_0 := t$. A tensor-valued function $F$ is said to satisfy the generalized Cauchy–Riemann equation if both $F$ and $\nabla F$ are symmetric and of trace zero. Obviously, if $m = 1$, this definition of the generalized Cauchy–Riemann equation is equivalent to that in Eq. (3.10). For more details on the generalized Cauchy–Riemann equation on tensor-valued functions, we refer the reader to [43, 54].

**Remark 4.2** Let $u$ be a harmonic function on $\mathbb{R}^{n+1}_+$ and, for any $m \in \mathbb{N}$,

$$
\nabla^m u := \{ \partial^\alpha u \}_{\alpha \in \mathbb{Z}_+^{n+1}, |\alpha|=m},
$$

where, for any $\alpha := (\alpha_0, \ldots, \alpha_n) \in \mathbb{Z}_+^{n+1}$, $|\alpha| := \sum_{j=0}^n \alpha_j$ and

$$
\partial^\alpha := \left( \frac{\partial}{\partial x_0} \right)^{\alpha_0} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.
$$

By [67, Remark 2.13], we find that $\nabla^m u$ satisfies the generalized Cauchy–Riemann equation.

Now, we establish the higher order Riesz transform characterization of $H_X(\mathbb{R}^n)$. We first introduce the concept of higher order Riesz–Hardy spaces.

**Definition 4.3** Let $m \in \mathbb{N}$ and $X$ be a BQBF space. The $m$-th Riesz–Hardy space $H_X^{m, \text{Riesz}}(\mathbb{R}^n)$ associated with $X$ is defined to be the set of all the $f \in S'(\mathbb{R}^n)$ satisfying that there exists a positive constant $A$ and a sequence $\{ f_k \}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n)$ such that

$$
\lim_{k \to \infty} f_k = f
$$
in $S'(\mathbb{R}^n)$ and, for any $k \in \mathbb{N}$, $l \in \{1, \ldots, m\}$, and $j_1, \ldots, j_l \in \{1, \ldots, n\}$,

$$
f_k \in X, \ R_{j_l} \cdots R_{j_1}(f_k) \in X
$$
and

$$
\| f_k \|_X + \sum_{l=1}^m \sum_{j_1, \ldots, j_l=1}^n \| R_{j_l} \cdots R_{j_1}(f_k) \|_X \leq A. \tag{4.3}
$$

Moreover, for any $f \in H_X^{m, \text{Riesz}}(\mathbb{R}^n)$,

$$
\| f \|_{H_X^{m, \text{Riesz}}(\mathbb{R}^n)} := \inf \{ A : A \text{ satisfies Eq. (4.3)} \}.
$$

Obviously, when $m = 1$, then $H_X^{1, \text{Riesz}}(\mathbb{R}^n)$ coincides with $H_X,\text{Riesz}(\mathbb{R}^n)$ in Definition 3.14. The following theorem establishes the higher Riesz transform characterization of $H_X(\mathbb{R}^n)$.
Theorem 4.4 Let \( m \in \mathbb{N} \cap [2, \infty) \). Assume that \( X \) is a BQBF space and satisfies both Assumptions 2.8 and 2.10 with \( s \in (\frac{n-1}{n+m-1}, 1] \) and \( \theta \in [\frac{n-1}{n+m-1}, s) \). Then the following statements hold true.

(i) There exists a positive constant \( C \in [0, \infty) \) such that, for any \( f \in L^2(\mathbb{R}^n) \),
\[
\| f \|_{H^m_X, \text{Riesz}}(\mathbb{R}^n) \leq C \| f \|_{H^m_X}(\mathbb{R}^n).
\]

(ii) There exists a positive constant \( C \in [0, \infty) \) such that, for any \( f \in S'(\mathbb{R}^n) \),
\[
\| f \|_{H^m_X}(\mathbb{R}^n) \leq C \| f \|_{H^m_X, \text{Riesz}}(\mathbb{R}^n).
\]

(iii) Moreover, if \( X \) has an absolutely continuous quasi-norm, then
\[
H^m_X(\mathbb{R}^n) = H^m_{X, \text{Riesz}}(\mathbb{R}^n)
\]
with equivalent (quasi-)norms.

To prove Theorem 4.4, we need two lemmas. The following lemma is just [5, Theorem 1].

Lemma 4.5 Let \( m \in \mathbb{N} \) and \( u \) be a harmonic function on \( \mathbb{R}^{n+1}_+ \). Then, for any \( q \in [\frac{n-1}{n+m-1}, \infty) \), \( |\nabla^m u|^q \) is subharmonic, where \( \nabla^m u \) is the same as in Eq. (4.2) and
\[
|\nabla^m u| := \left[ \sum_{\alpha \in \mathbb{Z}^{n+1}_+, |\alpha|=m} |\partial^\alpha u|^{2} \right]^{1/2}.
\]

The following lemma is just [60, Theorem 14.3] (see also [55]).

Lemma 4.6 Let \( m \in \mathbb{N} \cap [2, \infty) \) and \( F \) be a tensor-valued function of rank \( m \) satisfying that both \( F \) and \( \nabla F \) are symmetric and that \( F \) is of trace zero. Then there exists a harmonic function \( u \) on \( \mathbb{R}^{n+1}_+ \) such that \( \nabla^m u = F \), that is, for any \( j_1, \ldots, j_m \in \{0, 1, \ldots, n\} \) and \( (x, t) \in \mathbb{R}^{n+1}_+ \),
\[
\frac{\partial}{\partial x_{j_1}} \cdots \frac{\partial}{\partial x_{j_m}} u(x, t) = F_{j_1,\ldots,j_m}(x, t).
\]

Next, we introduce the concept of the Hardy space \( H^m_X(\mathbb{R}^{n+1}_+) \) of tensor-valued functions of rank \( m \) associated with \( X \).

Definition 4.7 Let \( X \) be a BQBF space. The Hardy space \( H^m_X(\mathbb{R}^{n+1}_+) \) of tensor-valued functions of rank \( m \) associated with \( X \) is defined to be the set of all the tensor-valued functions \( F \) of rank \( m \) on \( \mathbb{R}^{n+1}_+ \) satisfying the generalized Cauchy–Riemann equation and
\[
\| F \|_{H^m_X(\mathbb{R}^{n+1}_+)} := \sup_{t \in (0, \infty)} \| F(\cdot, t) \|_X < \infty,
\]
where, for any \((x, t) \in \mathbb{R}_+^{n+1}\),

\[
|F(x, t)| := \left\{ \sum_{j_1, \ldots, j_m=0}^{n} |F_{j_1, \ldots, j_m}(x, t)|^2 \right\}^{1/2}.
\]

**Remark 4.8** Observe that, in both Lemma 3.6 and Proposition 3.9, we need the restriction \(\theta \in [\frac{n-1}{n}, s]\) just because, for any \(q \in [\frac{n-1}{n}, \infty)\), the \(q\)-power of the absolute value of the first order gradient, \(|\nabla u|^q\), of a harmonic function \(u\) on \(\mathbb{R}_+^{n+1}\) is subharmonic. By Lemma 4.5, we find that, for any \(m \in \mathbb{N}\) and \(q \in [\frac{n-1}{n+m-1}, \infty)\), \(|\nabla^m H|^q\) is subharmonic on \(\mathbb{R}_+^{n+1}\). Therefore, the restriction \(\theta \in [\frac{n-1}{n}, s]\) in both Lemma 3.6 and Proposition 3.9 can be relaxed to \(\theta \in [\frac{n-1}{n+m-1}, s]\) when we deal with the Hardy space \(\mathbb{H}_X^n(\mathbb{R}_+^{n+1})\) of tensor-valued functions of rank \(m\) instead of the Hardy space \(\mathbb{H}_X(\mathbb{R}_+^{n+1})\) of harmonic vectors. Then we can show that both Lemma 3.6 and Proposition 3.9 still hold true for any \(q \in [\frac{n-1}{n+m-1}, \theta]\); we omit the details. Moreover, for any \(\text{BQBF space } X\) satisfying both Assumptions 2.8 and 2.10 with \(s \in (0, 1]\) and \(\theta \in (0, s)\), we can always find a sufficiently large \(m\) such that \(\theta \in [\frac{n-1}{n+m-1}, s]\).

Now, we show Theorem 4.4.

**Proof of Theorem 4.4** We first show (i). To this end, fix an \(f \in L^2(\mathbb{R}^n)\). If \(\|f\|_{H_X(\mathbb{R}^n)} = \infty\), then (i) obviously holds true. In what follows, we assume that \(\|f\|_{H_X(\mathbb{R}^n)} < \infty\). Choose a \(\Phi \in S(\mathbb{R}^n)\) to be positive and radial decreasing such that \(f \ast \Phi(x)\, dx = 1\). By this, Eq. (3.26), Definition 2.1(ii), [46, Theorem 3.1(i)], and Lemma 3.12, we conclude that

\[
\|f\|_{X} + \sum_{k=1}^{m} \sum_{j_1, \ldots, j_k=1}^{n} \|R_{j_1 \ldots R_{j_k}}(f)\|_{X} \\
\leq \|M(f; \Phi)\|_{X} + \sum_{k=1}^{m} \sum_{j_1, \ldots, j_k=1}^{n} \|M(R_{j_1 \ldots R_{j_k}}(f); \Phi)\|_{X} \\
\leq \|f\|_{H_X(\mathbb{R}^n)} + \sum_{k=1}^{m} \sum_{j_1, \ldots, j_k=1}^{n} \|R_{j_1 \ldots R_{j_k}}(f)\|_{H_X(\mathbb{R}^n)} \lesssim \|f\|_{H_X(\mathbb{R}^n)}.
\]

(4.4)

This finishes the proof of (i).

Next, we prove (ii). To achieve this, fix an \(f \in S'(\mathbb{R}^n)\). If \(\|f\|_{H_X^{\text{Riesz}}(\mathbb{R}^n)} = \infty\), then (ii) obviously holds true. In what follows, we assume that \(\|f\|_{H_X^{\text{Riesz}}(\mathbb{R}^n)} < \infty\). By Definition 4.3, we find a sequence \(\{f_k\}_{k \in \mathbb{N}} \subset L^2(\mathbb{R}^n)\) such that

\[
\lim_{k \to \infty} f_k = f
\]
in $S'(\mathbb{R}^n)$ and, for any $k \in \mathbb{N}$,

$$\| f_k \|_X + \sum_{k=1}^m \sum_{j_1, \ldots, j_k=1}^n \| R_{j_1} \cdots R_{j_k} (f_k) \|_X \lesssim \| f \|_{H^m_{X, \text{Riesz}}(\mathbb{R}^n)}. \tag{4.5}$$

For any $k \in \mathbb{N}, j_1, \ldots, j_m \in \{0, \ldots, n\}$, and $(x, t) \in \mathbb{R}^{n+1}_+$, write

$$F_{j_1, \ldots, j_m, k}(x, t) := P_t \ast (R_{j_1} \cdots R_{j_m} (f_k))(x),$$

where $R_0 := I$ is the identity operator, and let

$$F_k(x, t) := \sum_{j_1, \ldots, j_m=0}^n F_{j_1, \ldots, j_m, k}(x, t) e_{j_1} \otimes \cdots \otimes e_{j_m}.$$

From this and the proof of [60, Lemma 17.1], we deduce that, for any $k \in \mathbb{N}$, $F_k$ satisfies the generalized Cauchy–Riemann equation. Using this and [60, Lemma 17.2], we have, for any $k \in \mathbb{N}$, $q \in [\frac{n-1}{n+m-1}, \theta]$, and $(x, t) \in \mathbb{R}^{n+1}_+$,

$$|F_k(x, t)|^q \leq P_t \ast |h_k|^q(x), \tag{4.6}$$

where, for any $x \in \mathbb{R}^n$,

$$h_k(x) := \{ R_{j_1} \cdots R_{j_m} (f_k)(x) \}_{j_1, \ldots, j_m \in \{0, \ldots, n\}}$$

and

$$|h_k(x)| := \left\{ \sum_{j_1, \ldots, j_m=0}^n |R_{j_1} \cdots R_{j_m} (f_k)(x)|^2 \right\}^{1/2}.$$

By this, Definition 2.1(ii), and Eqs. (4.6) and (4.5), we conclude that, for any $k \in \mathbb{N}$,

$$\| F_k \|_{\mathbb{E}^m_X(\mathbb{R}^{n+1})} = \sup_{t \in (0, \infty)} \| F_k(\cdot, t) \|_q^{1/q} \| X^{1/q} \| \leq \sup_{t \in (0, \infty)} \| P_t \ast |h_k|^q \|_X^{1/q} \| X^{1/q} \| \lesssim \| M(|h_k|)^q \|_X^{1/q} \lesssim \| h_k \|_X,$$

$$\lesssim \| f_k \|_X + \sum_{j_1, \ldots, j_m=0}^n \| R_{j_1} \cdots R_{j_m} (f_k) \|_X \lesssim \| f \|_{H^m_{X, \text{Riesz}}(\mathbb{R}^n)},$$

which further implies that $F_k \in \mathbb{E}^m_X(\mathbb{R}^{n+1})$. Combining this and Remark 4.8 (on the higher order counterparts of both Lemma 3.6 and Proposition 3.9), we obtain, for any $k \in \mathbb{N}$ and $t \in (0, \infty)$,

$$\| P_t \ast f_k \|_{H^m_{X, \mathbb{R}^{n+1}}(\mathbb{R}^n)} \lesssim \| f_k \|_{\mathbb{E}^m_X(\mathbb{R}^{n+1})} \lesssim \| f \|_{H^m_{X, \text{Riesz}}(\mathbb{R}^n)}.$$
Applying this and an argument similar to that used in the proof of Theorem 3.15(ii), we find that $f \in H_X(\mathbb{R}^n)$ and $\| f \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)}$. This finishes the proof of (ii).

Finally, we show (iii). To this end, by (ii), it suffices to prove that $H_X(\mathbb{R}^n) \subset H_{X, \text{Riesz}}^m(\mathbb{R}^n)$. (4.7)

To this end, fix an $f \in H_X(\mathbb{R}^n)$. By Lemma 2.20, we find a sequence $\{ f_k \}_{k \in \mathbb{N}} \subset H_X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\lim_{k \to \infty} f_k = f$ in both $H_X(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ and $\| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)}$. From this and Eq. (4.4), we infer that, for any $k \in \mathbb{N}$,

$$\| f_k \|_X + \sum_{k=1}^m \sum_{j_1, \ldots, j_k=1}^n \| R_{j_1} \ldots R_{j_k} f_k \|_X \lesssim \| f_k \|_{H_X(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)},$$

which further implies that $f \in H_{X, \text{Riesz}}(\mathbb{R}^n)$ and $\| f \|_{H_{X, \text{Riesz}}(\mathbb{R}^n)} \lesssim \| f \|_{H_X(\mathbb{R}^n)}$. This finishes the proof of Eq. (4.7) and hence Theorem 4.4.

**Remark 4.9** We point out that, when $X := L^p(\mathbb{R}^n)$ with $p \in (\frac{2n-1}{n}, 1]$, Theorem 4.4 in this case was obtained in [21] (see also [52, p. 133, Item 5.16]). When $X$ is a variable Lebesgue space, Theorem 4.4 in this case was obtained in [67, Theorem 1.6]. When $X$ is a Musielak–Orlicz space, Theorem 4.4 in this case was obtained in [6, Theorem 1.7]. To the best of our knowledge, when $X$ is a Lorentz space, a mixed-norm Lebesgue space, a local generalized Herz space, or a mixed-norm Herz space, the results obtained in Theorem 4.4 are new. Observe that the proof of Theorem 4.4 strongly depends on the density of $H_X(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $H_X(\mathbb{R}^n)$. Thus, it is unclear whether or not Theorem 4.4 still holds true when $X$ is a Morrey space.

### 5 Applications

In this section, we apply our main results, Theorems 3.15 and 4.4, respectively, to five concrete examples of ball quasi-Banach function spaces, namely Lorentz spaces (Subsect. 5.1), mixed-norm Lebesgue spaces (Subsect. 5.2), local generalized Herz spaces (Subsect. 5.3), mixed-norm Herz spaces (Subsect. 5.4), and Morrey spaces (Subsect. 5.5), and give the Riesz transform characterization of Hardy type spaces based on these ball quasi-Banach function spaces; all these obtained results are completely new. These examples indicate both the practicality and the operability of the main results of this article and more applications to newfound function spaces are obviously possible.

#### 5.1 Lorentz Spaces

In this section, we apply our main results to Lorentz spaces. Let us begin with the following concept of Lorentz spaces.
Definition 5.1 Let $p \in (0, \infty)$ and $r \in (0, \infty]$. The Lorentz space $L^{p,r}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ satisfying that, when $p, r \in (0, \infty)$,
\[ \|f\|_{L^{p,r}(\mathbb{R}^n)} := \left\{ \int_0^\infty \left( t^{1/p} f^*(t) \right)^r \frac{dt}{t} \right\}^\frac{1}{r} < \infty \]
and, when $p \in (0, \infty)$ and $r = \infty$,
\[ \|f\|_{L^{p,\infty}(\mathbb{R}^n)} := \sup_{t \in (0, \infty)} \left\{ t^{1/p} f^*(t) \right\} < \infty, \]
where $f^*$ denotes the decreasing rearrangement function of $f$, which is defined by setting, for any $t \in [0, \infty)$,
\[ f^*(t) := \inf\{s \in (0, \infty) : \mu_f(s) \leq t\} \]
with $\mu_f(s) := \left| \{ x \in \mathbb{R}^n : |f(x)| > s \} \right|$. Applying both Theorems 3.15 and 4.4, we have the following Riesz transform characterization of Lorentz–Hardy spaces.

Theorem 5.2 Let $m \in \mathbb{N}$, $p \in (\frac{n-1}{n+m-1}, \infty)$, $r \in (0, \infty)$, and $X := L^{p,r}(\mathbb{R}^n)$. Then $H_X(\mathbb{R}^n) = H^m_{X, \text{Riesz}}(\mathbb{R}^n)$ with equivalent (quasi-)norms.

Proof By [12, Theorem 2.3(iii)], we conclude that $L^{p,r}(\mathbb{R}^n)$ satisfies both Assumptions 2.8 and 2.10 with $s \in (\frac{n-1}{n+m-1}, \min\{1, p\})$, $\theta \in [\frac{n-1}{n+m-1}, s)$, and $q \in (\max\{1, p, r\}, \infty)$. Moreover, from [61, Remark 3.4(iii)], we deduce that $L^{p,r}(\mathbb{R}^n)$ has an absolutely continuous quasi-norm. Then, using both Theorems 3.15(iii) and 4.4(iii) with $X := L^{p,r}(\mathbb{R}^n)$, we obtain the desired conclusion, which completes the proof of Theorem 5.2.

5.2 Mixed-Norm Lebesgue Spaces

The mixed-norm Lebesgue space $L^p(\mathbb{R}^n)$ was studied by Benedek and Panzone [1] in 1961, which can be traced back to Hörmander [25]. Later on, in 1970, Lizorkin [35] further developed both the theory of multipliers of Fourier integrals and estimates of convolutions in the mixed-norm Lebesgue spaces. Particularly, in order to meet the requirements arising in the study of the boundedness of operators, partial differential equations, and some other analysis subjects, the real-variable theory of mixed-norm function spaces has rapidly been developed in recent years (see, for instance [9–11, 26–28]).

Definition 5.3 Let $p := (p_1, \ldots, p_n) \in (0, \infty]^n$. The mixed-norm Lebesgue space $L^p(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ such that
\[ \|f\|_{L^p(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}} \cdots \left[ \int_{\mathbb{R}} |f(x_1, \ldots, x_n)|^{p_1} dx_1 \right]^{p_2/p_1} \cdots dx_n \right\}^{\frac{1}{p}}. \]
with the usual modifications made when \( p_i = \infty \) for some \( i \in \{1, \ldots, n\} \), is finite. Moreover, let

\[
p_- := \min\{p_1, \ldots, p_n\} \quad \text{and} \quad p_+ := \max\{p_1, \ldots, p_n\}.
\]

(5.1)

**Remark 5.4** Let \( p := (p_1, \ldots, p_n) \in (0, \infty)^n \) and both \( p_− \) and \( p_+ \) be the same as in Eq. (5.1). By Definition 5.3, we easily conclude that \( L^p(\mathbb{R}^n) \) is a ball quasi-Banach space, but it is worth pointing out that \( L^p(\mathbb{R}^n) \) may not be a quasi-Banach function space (see, for instance \([69, \text{Remark 7.20}]\)).

On the mixed-norm Hardy space, we have the following Riesz transform characterization.

**Theorem 5.5** Let \( m \in \mathbb{N} \), \( p := (p_1, \ldots, p_n) \in (0, \infty)^n \), \( p_− \) be the same as in Eq. (5.1), and \( X := L^p(\mathbb{R}^n) \). If \( p_− \in \left(\frac{n-1}{n+m-1}, \infty\right) \), then \( H^m_X(\mathbb{R}^n) = H^m_{X, \text{Riesz}}(\mathbb{R}^n) \) with equivalent (quasi-)norms.

**Proof** Choose an \( s \in \left(\frac{n-1}{n+m-1}, \min\{p_-, 1\}\right) \), a \( \theta \in \left[\frac{n-1}{n+m-1}, s\right) \), and a \( q \in (\max\{p_+, 1\}, \infty) \), where \( p_+ \) is the same as in (5.1). Then, by [26, Lemma 3.7] (see also [40, Theorem 1.7]), the dual theorem of \( L^p(\mathbb{R}^n) \) (see \([1, \text{p. 304, Theorem 1.a}]\)), and [26, Lemma 3.5], we conclude that \( L^p(\mathbb{R}^n) \) satisfies both Assumptions 2.8 and 2.10 with \( s, \theta, \) and \( q \) chosen as above. Moreover, from the dominated convergence theorem, we infer that \( L^p(\mathbb{R}^n) \) has an absolutely continuous quasi-norm. Then, using Theorems 3.15(iii) and 4.4(iii) with \( X := L^p(\mathbb{R}^n) \), we obtain the desired conclusion, which completes the proof of Theorem 5.5. \( \square \)

### 5.3 Local Generalized Herz Spaces

In this section, we apply our main results to local generalized Herz spaces (see, for instance \([32, 44]\)). Let us begin with the concepts of the almost decreasing function (see, for instance, \([30, \text{p. 30}]\)) and the function class \( M(\mathbb{R}^+) \) (see, for instance, \([44, \text{Definition 2.1}]\)).

**Definition 5.6** Let \( \mathbb{R}^+ := (0, \infty) \). A nonnegative function \( \omega \) on \( \mathbb{R}^+ \) is said to be **almost decreasing** on \( \mathbb{R}^+ \) if there exists a constant \( C \in [1, \infty) \) such that, for any \( t, \tau \in (0, \infty) \) satisfying \( t \geq \tau \),

\[
\omega(t) \leq C\omega(\tau).
\]

**Definition 5.7** Let \( \mathbb{R}^+ := (0, \infty) \). The **function class** \( M(\mathbb{R}^+) \) is defined to be the set of all the positive functions \( \omega \) on \( \mathbb{R}^+ \) such that, for any \( 0 < \delta < N < \infty \),

\[
0 < \inf_{t \in (\delta, N)} \omega(t) \leq \sup_{t \in (\delta, N)} \omega(t) < \infty
\]

and there exist four constants \( \alpha_0, \beta_0, \alpha_\infty, \beta_\infty \in \mathbb{R} \) such that

(i) for any \( t \in (0, 1) \), \( \omega(t)t^{-\alpha_0} \) is almost increasing and \( \omega(t)t^{-\beta_0} \) is almost decreasing;
(ii) for any $t \in [1, \infty)$, $\omega(t)t^{-\alpha_{\infty}}$ is almost increasing and $\omega(t)t^{-\beta_{\infty}}$ is almost decreasing.

Now, we recall the concept of local generalized Herz spaces introduced in [44, Definition 2.1].

**Definition 5.8** Let $p, r \in (0, \infty)$ and $\omega \in M(\mathbb{R}_+)$. The local generalized Herz space $\dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n)} := \left\{ \sum_{k \in \mathbb{Z}} \|f \mathbf{1}_{B(0,2^k)}\|_{L^p(\mathbb{R}^n)}^r \right\}^{\frac{1}{r}} < \infty.$$  

We also recall the following concept of Matuszewska–Orlicz indices; see, for instance, [44] and [32, Definition 1.1.4].

**Definition 5.9** Let $\omega$ be a positive function on $\mathbb{R}_+$. Then the Matuszewska–Orlicz indices $m_0(\omega), M_0(\omega), m_{\infty}(\omega)$, and $M_{\infty}(\omega)$ of $\omega$ are defined, respectively, by setting, for any $h \in (0, \infty)$,

$$m_0(\omega) := \sup_{t \in (0,1)} \frac{\ln(\lim_{h \to 0^+} \frac{\omega(ht)}{\omega(h)})}{\ln t}, \quad M_0(\omega) := \inf_{t \in (0,1)} \frac{\ln(\lim_{h \to 0^+} \frac{\omega(ht)}{\omega(h)})}{\ln t},$$

$$m_{\infty}(\omega) := \sup_{t \in (1,\infty)} \frac{\ln(\lim_{h \to \infty} \frac{\omega(ht)}{\omega(h)})}{\ln t},$$

and

$$M_{\infty}(\omega) := \inf_{t \in (1,\infty)} \frac{\ln(\lim_{h \to \infty} \frac{\omega(ht)}{\omega(h)})}{\ln t}.$$  

**Remark 5.10** From [32, Theorem 1.2.20], we deduce that $\dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n)$ is a ball quasi-Banach function space with $p, r \in (0, \infty)$ and $\omega \in M(\mathbb{R}_+)$ satisfying $m_0(\omega) \in (-\frac{n}{p}, \infty)$ and $M_{\infty}(\omega) \in (-\frac{n}{p}, \infty)$. However, it is worth pointing out that $\dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n)$ with $p, r \in (0, \infty)$ and $\omega \in M(\mathbb{R}_+)$ may not be a quasi-Banach function space (see, for instance, [8, Remark 4.13]).

On the local generalized Herz–Hardy space, we have the following Riesz transform characterization.

**Theorem 5.11** Let $m \in \mathbb{N}$, $p \in (\frac{n-1}{n+m-1}, \infty)$, $r \in (\frac{n-1}{n+m-1}, \infty]$, $\omega \in M(\mathbb{R}_+)$ satisfy $m_0(\omega) \in (-\frac{n}{p}, \infty)$ and $m_{\infty}(\omega) \in (-\frac{n}{p}, \infty)$, and $X := \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n)$. Assume that

$$\frac{n}{\max\{M_0(\omega), M_{\infty}(\omega)\} + n/p} \in \left( \frac{n-1}{n+m-1}, \infty \right).$$

Then $H_X(\mathbb{R}^n) = H^m_{X,\text{Riesz}}(\mathbb{R}^n)$ with equivalent (quasi-)norms.
Proof By [32, Theorems 1.2.20 and 1.4.1], we conclude that \( \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n) \) is a ball quasi-Banach space with an absolutely continuous quasi-norm. To prove the desired conclusion, it suffices to show that \( \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n) \) satisfies all the assumptions of both Theorems 3.15 and 4.4. Indeed, \( \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n) \) satisfies Assumption 2.8 with

\[
s \in \left( \frac{n-1}{n+m-1}, \min \left\{ \frac{n}{\max \{M_0(\omega), M_{\infty}(\omega)\} + n/p} \right\} \right]
\]

and

\[
\theta \in \left[ \frac{n-1}{n+m-1}, s \right];
\]

see [32, Lemma 4.3.9]. Moreover, from [32, Lemma 1.8.5], we infer that Assumption 2.10 with \( X := \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n) \) holds true for any given

\[
q \in \left( \max \left\{ 1, p, \frac{n}{\min \{m_0(\omega), m_{\infty}(\omega)\} + n/p} \right\}, \infty \right).
\]

Then, using Theorems 3.15(iii) and 4.4(iii) with \( X := \dot{K}^{p,r}_{\omega,0}(\mathbb{R}^n) \), we obtain the desired conclusion, which completes the proof of Theorem 5.11.

\[\square\]

5.4 Mixed-Norm Herz Spaces

In this section, we apply our main results to mixed-norm Herz spaces (see [70]).

Definition 5.12 Let \( p := (p_1, \ldots, p_n), q := (q_1, \ldots, q_n) \in (0, \infty]^n \), and \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \). The mixed-norm Herz space \( E_q^{\alpha,p}(\mathbb{R}^n) \) is defined to be the set of all the functions \( f \in \mathcal{M}(\mathbb{R}^n) \) such that

\[
\|f\|_{E_q^{\alpha,p}(\mathbb{R}^n)} := \left\{ \sum_{k_n \in \mathbb{Z}} 2^{k_n} p_n^{\alpha_n} \left[ \int_{R_{k_n}} \cdots \left[ \sum_{k_1 \in \mathbb{Z}} 2^{k_1} p_1^{\alpha_1} \right] \left( \int_{R_{k_1}} |f(x_1, \ldots, x_n)|^{q_1} \frac{dx_1}{p_1^{q_1}} \right)^{\frac{p_1}{q_1}} \cdots \frac{dx_n}{p_n^{q_n}} \right]^{\frac{1}{\sum_{i} \frac{1}{p_i}}} \right\}^{\frac{1}{\sum_{i} \frac{1}{p_i}}},
\]

with the usual modifications made when \( p_i = \infty \) or \( q_j = \infty \) for some \( i, j \in \{1, \ldots, n\} \), is finite.

Remark 5.13 Let \( p, q \in (0, \infty]^n \) and \( \alpha \in \mathbb{R}^n \). By [70, Propositions 2.8 and 2.22], we conclude that \( E_q^{\alpha,p}(\mathbb{R}^n) \) is a ball quasi-Banach space. However, from [70, Remark 2.4], we deduce that, when \( p = q \) and \( \alpha = 0 \), the mixed-norm Herz space \( E_q^{\alpha,p}(\mathbb{R}^n) \) coincides with the mixed-norm Lebesgue space \( L^p(\mathbb{R}^n) \) defined in Definition 5.3.
Using Remark 5.4, we conclude that $L^p(\mathbb{R}^n)$ may not be a quasi-Banach function space and hence $\dot{E}^{\alpha,p}_q(\mathbb{R}^n)$ may not be a quasi-Banach function space.

On the mixed-norm Herz–Hardy space, we have the following Riesz transform characterization.

**Theorem 5.14** Let $m \in \mathbb{N}$, $p := (p_1, \ldots, p_n)$, $q := (q_1, \ldots, q_n) \in (0, \infty)^n$, $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with $\alpha_i \in (-\frac{1}{q_i}, \infty)$ for any $i \in \{1, \ldots, n\}$, and $X := \dot{E}^{\alpha,p}_q(\mathbb{R}^n)$. Assume that $p_-, q_- \in \left(\frac{n-1}{n+m-1}, \infty\right)$, where $p_-$ and $q_-$ are the same as in Eq. (5.1), and, for any $i \in \{1, \ldots, n\}$,

$$\left(\alpha_i + \frac{1}{q_i}\right)^{-1} \in \left(\frac{n-1}{n+m-1}, \infty\right).$$

Then $H_X(\mathbb{R}^n) = H^{m}_{X,Riesz}(\mathbb{R}^n)$ with equivalent (quasi-)norms.

**Proof** By [70, Propositions 2.8 and 2.22], we conclude that $\dot{E}^{\alpha,p}_q(\mathbb{R}^n)$ is a ball quasi-Banach space with an absolutely continuous quasi-norm. By [70, Lemma 5.3(i)], we conclude that $\dot{E}^{\alpha,p}_q(\mathbb{R}^n)$ satisfies Assumption 2.8 with

$$s \in \left(\frac{n-1}{n+m-1}, \min \left\{1, p_-, q_-, \left(\alpha_1 + \frac{1}{q_1}\right)^{-1}, \ldots, \left(\alpha_n + \frac{1}{q_n}\right)^{-1}\right\}\right)$$

and

$$\theta \in \left[\frac{n-1}{n+m-1}, s\right).$$

Let $p_+$ and $q_+$ be the same as in Eq. (5.1) and

$$q \in \left(\max \left\{1, p_+, q_+, \left(\alpha_1 + \frac{1}{q_1}\right)^{-1}, \ldots, \left(\alpha_n + \frac{1}{q_n}\right)^{-1}\right\}, \infty\right).$$

Then, by [70, Lemma 5.3(ii)] and its proof, we conclude that Assumption 2.10 holds true with $X := \dot{E}^{\alpha,p}_q(\mathbb{R}^n)$. Then, using Theorems 3.15(iii) and 4.4(iii) with $X := \dot{E}^{\alpha,p}_q(\mathbb{R}^n)$, we obtain the desired conclusion, which completes the proof of Theorem 5.14.

\[\Box\]

### 5.5 Morrey Spaces

In this section, we apply our main results to Morrey spaces. Let us recall the concept of Morrey spaces.
Definition 5.15 Let $0 < p \leq r \leq \infty$. The Morrey space $\mathcal{M}_p^r(\mathbb{R}^n)$ is defined to be the set of all the $f \in L_p^\text{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_p^r(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} |B|^\frac{1}{r-1} \left[ \int_B |f(y)|^p \, dy \right]^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

On the Morrey–Hardy space, we have the following Riesz transform characterization.

Theorem 5.16 Let $m \in \mathbb{N}, 0 < p \leq r \leq \infty$, and $X := \mathcal{M}_p^r(\mathbb{R}^n)$. If $p \in \left(\frac{n-1}{n+m-1}, \infty\right)$, then the following two statements hold true.

(i) There exists a positive constant $C \in [0, \infty)$ such that, for any $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{H^m_{X, \text{Riesz}}(\mathbb{R}^n)} \leq C \|f\|_{H_X(\mathbb{R}^n)}.$$

(ii) There exists a positive constant $C \in [0, \infty)$ such that, for any $f \in S'(\mathbb{R}^n)$,

$$\|f\|_{H_X(\mathbb{R}^n)} \leq C \|f\|_{H^m_{X, \text{Riesz}}(\mathbb{R}^n)}.$$

Proof By [47, Theorem 2.4] and [58, Lemma 2.5] (see also [61, Remarks 2.4(e) and 2.7(e)]), we conclude that $X$ is a ball quasi-Banach space and satisfies Assumption 2.8 with

$$s \in \left(\frac{n-1}{n+m-1}, \min\{1, p\}\right) \text{ and } \theta \in \left[\frac{n-1}{n+m-1}, s\right].$$

Let

$$q \in (\max\{1, p\}, \infty).$$

Then, from [48, Theorem 4.1] (see also [61, Remark 2.7(e)]), we infer that Assumption 2.10 holds true with $X := \mathcal{M}_p^r(\mathbb{R}^n)$. Using (i) and (ii) of Theorem 3.15 and (i) and (ii) of Theorem 4.4 with $X := \mathcal{M}_p^r(\mathbb{R}^n)$, we obtain the desired conclusion, which then completes the proof of Theorem 5.16. \(\square\)

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