Critical stability of three-body relativistic bound states with zero-range interaction

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\textbf{Abstract.} For zero-range interaction providing a given mass $M_2$ of the two-body bound state, the mass $M_3$ of the relativistic three-body bound state is calculated. We have found that the three-body system exists only when $M_2$ is greater than a critical value $M_c$ (\approx 1.43\,m for bosons and \approx 1.35\,m for fermions, $m$ is the constituent mass). For $M_2 = M_c$ the mass $M_3$ turns into zero and for $M_2 < M_c$ there is no solution with real value of $M_3$.

\section{Introduction}

Zero-range two-body interaction provides an important limiting case which qualitatively reflects the characteristic properties of nuclear and atomic few-body systems. In the nonrelativistic three-body system it results in the Thomas collapse \cite{1}. The latter means that the three-body binding energy tends to $-\infty$, when the interaction radius tends to zero.

When the binding energy or the exchanged particle mass is not negligible in comparison to the constituent masses, the nonrelativistic treatment becomes invalid and must be replaced by a relativistic one. Two-body calculations show that in the scalar case, relativistic effects are repulsive (see e.g. \cite{2}). Relativistic three-body calculations with zero-range interaction have been performed in a minimal relativistic model \cite{3} and in the framework of the Light-Front Dynamics \cite{4}. It was concluded that, due to relativistic repulsion, the three-body binding energy remains finite and the Thomas collapse is consequently avoided. However, in these works a cutoff was implicitly introduced. Because of that, it was not clear, to what degree the finite binding energy results from the relativistic repulsion, and to what degree – from the cutoff. The latter can be imposed by many ways and it is evident in advance that one can always find an enough strong cutoff making the binding energy finite. Therefore we are interested in a net effect of relativistic zero range interaction, without any cutoff.

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We present here our solution \[5\] of the problem of three equal mass (\(m\)) bosons interacting via zero-range forces. In addition, we consider also the three-fermion system. We show that the existence of the three-body system depends on the strength two-body interaction. For strong enough interaction, instead of the Thomas collapse its relativistic counterpart takes place. Namely, when the two-body bound state mass \(M_2\) decreases, the mass \(M_3\) of the three-body system decreases as well and vanishes at some critical value of \(M_2 = M_c (\approx 1.43 m\) for three bosons and \(\approx 1.35 m\) for three fermions). For \(M_2 < M_c\) there are no solutions with real value of \(M_3\), what means – from physical point of view – that the three-body system no longer exists.

2 Three-boson system

We use the explicitly covariant formulation of the Light-Front Dynamics (see for a review \[6\]). The wave function is defined on the light-front plane given by the equation \(\omega \cdot x = 0\), where \(\omega\) is a four-vector with \(\omega^2 = 0\), determining the light-front orientation. In the particular case \(\omega = (1, 0, 0, -1)\) we recover the standard approach.

The three-body equation we consider is written for the vertex function \(\Gamma\), related to the wave function \(\psi\):

\[
\psi(1, 2, 3) = \frac{\Gamma(1, 2, 3)}{M_0^2 - M_3^2}, \quad M_0^2 = (k_1 + k_2 + k_3)^2,
\]

where \(M_3\) is the three-body bound state mass.

The Faddeev amplitudes \(\Gamma_{ij}\) are introduced in the standard way:

\[
\Gamma(1, 2, 3) = \Gamma_{12}(1, 2, 3) + \Gamma_{23}(1, 2, 3) + \Gamma_{31}(1, 2, 3)
\]

and one obtains a system of three coupled equations for them. With the symmetry relations \(\Gamma_{23}(1, 2, 3) = \Gamma_{12}(2, 3, 1)\) and \(\Gamma_{31}(1, 2, 3) = \Gamma_{12}(3, 1, 2)\), the system is reduced to a single equation for one of the amplitudes, say \(\Gamma_{12}\).

For zero-range forces, the interaction kernel in momentum space is replaced by a constant \(\lambda\). This is precise meaning of the relativistic zero-range interaction. For a given two-body bound state mass \(M_2\) the constant \(\lambda\) is expressed through \(M_2\) and disappears from the problem.

Equation for \(\Gamma_{12}\) can be rewritten in variables \(R_{i\perp}, x_i\), \((i = 1, 2, 3)\), where \(R_{i\perp}\) is the spatial component of the four-vector \(R_i = k_i - x_i p\) orthogonal to \(\omega\) and \(x_i = \frac{\omega \cdot k_i}{\omega \cdot p}\) \[8\]. In general, \(\Gamma_{12}\) depends on all variables \((R_{i\perp}, x_i)\), constrained by the relations \(R_{1\perp} + R_{2\perp} + R_{3\perp} = 0, x_1 + x_2 + x_3 = 1\), but for contact kernel it depends only on \((R_{3\perp}, x_3)\) \[4\]. The equation for the Faddeev amplitude reads:

\[
\Gamma_{12}(R_{\perp}, x) = F(M_{12}) \frac{1}{(2\pi)^3} \int_0^1 dx' \int_0^\infty d^2 R_{\perp}' \frac{\Gamma_{12}(R_{\perp}', x'(1-x))}{(R_{\perp} - x'R_{\perp})^2 + m^2 - x'(1-x'M_{12}^2).}
\]

The factor \(F(M_{12})\) is the two-body off-shell scattering amplitude. It corresponds to the fixed two-body bound state mass \(M_2\) and depends on the off-shell
two-body effective mass $M_{12}$. For $0 \leq M_{12}^2 < 4m^2$ the calculation gives:

$$F(M_{12}) = \frac{8\pi^2}{\arctan \frac{y_{M_{12}}}{y_{M_{2}}} - \arctan \frac{y_{M_{2}}}{y_{M_{12}}}} ,$$

where $y_{M_{12}} = \frac{M_{12}}{\sqrt{4m^2 - M_{12}^2}}$ and similarly for $y_{M_{2}}$. If $M_{12}^2 < 0$, the amplitude obtains the form:

$$F(M_{12}) = \frac{8\pi^2}{\frac{1}{2y'_{M_{12}}} \log \frac{1 + y'_{M_{12}}}{1 - y'_{M_{12}}}} - \arctan \frac{y_{M_{2}}}{y_{M_{12}}} ,$$

where $y'_{M_{12}} = \frac{\sqrt{-M_{12}^2}}{\sqrt{4m^2 - M_{12}^2}}$. It has the pole at $M_{12} = M_{2}$.

The two-body mass squared $M_{12}^2$ is expressed through the three-body variables as:

$$M_{12}^2 = (1 - x)M_3^2 - \frac{R_{\perp}^2 + (1 - x)m^2}{x}.$$  

The three-body mass $M_3$ enters the equation (1) through $M_{12}^2$. The arguments of $\Gamma_{12}$ run the values $0 \leq R_{\perp} < \infty, 0 \leq x \leq 1$. The variable $M_{12}^2$ becomes negative at $x \to 0$ (the square is understood in the sense of Minkowsky metric). By a replacement of variables the equation (1) can be transformed to the form of equation (11) from [4] except for the integration limits. In the papers [3, 4] the variable $M_{12}^2$ was constrained by positive values, that strongly restricts the domain of variables $R_{\perp}, x$ and, in this way, introduces a cutoff.

Being interested in studying the zero-range interaction, we do not cut the variation domain of variables $R_{\perp}, x$. As we will see, this point turns out to be crucial for the appearance of the relativistic collapse.

3 Three-fermion system

The zero-range two-fermion kernel can be constructed using many different spin couplings. Our main interest is the influence of the antisymmetrization of the wave function which should be taken into account for any kernel. Therefore we solve the problem for a simplified kernel:

$$K_{\sigma_1\sigma_2} (1, 2; 1', 2') = \lambda \tilde{K}_{\sigma_1\sigma_2} (1, 2) K^{\sigma_1'\sigma_2'} (1', 2'),$$

where we denote:

$$\tilde{K}_{\sigma_1\sigma_2} (1, 2) = \frac{m}{\omega (k_1 + k_2)} [\bar{u}_{\sigma_1} (k_1) i\tilde{\omega} \gamma_5 U_c \bar{u}_{\sigma_2} (k_2)]$$

and $K^{\sigma_1'\sigma_2'} (1', 2') = \tilde{K}^{\sigma_1'\sigma_2'} (1', 2')$. The matrix $\tilde{\omega} = \gamma^\mu \omega_\mu$ appears in the contact interaction of fermions [6]. This kernel is factorized relative to initial and final states. The divergence of two-body scattering amplitude (at fixed $\lambda$) is logarithmic. At fixed value of the two-body mass $M_{2}$ the amplitude becomes finite, like in the boson case. In nonrelativistic limit, kernel (2) corresponds to interaction in the $^1S_0$ state only.
The equation for the Faddeev component is generalized for the three-fermion case by adding the spin indices. Its solution has the form:

\[ \Gamma_{\sigma_1 \sigma_2 \sigma_3}^{\sigma}(1, 2, 3) = \bar{K}_{\sigma_1 \sigma_2}^{\sigma}(1, 2) G_{\sigma_3}^{\sigma}(3), \]

where \( \bar{K}_{\sigma_1 \sigma_2}(1, 2) \) is defined in (3). It is antisymmetric relative to permutation \( 1 \leftrightarrow 2 \), whereas the sum of three Faddeev components is antisymmetric relative to permutation of any pair.

The \( 2 \times 2 \)-matrix \( G_{\sigma_3}^{\sigma}(3) \) can be decomposed as:

\[ G_{\sigma_3}^{\sigma}(3) = g_1 \bar{u}_{\sigma_3}(k_3) S_1 u_\sigma(p) + g_2 \bar{u}_{\sigma_3}(k_3) S_2 u_\sigma(p) \]

with the basis matrices

\[ S_1 = \left[ 2x_3 - (m + x_3 M_3) \frac{\hat{\omega}}{\omega p} \right], \quad S_2 = m \frac{\hat{\omega}}{\omega p}. \]

We get system of two equations for the scalar functions \( g_1, g_2 \). One of the equations contains only \( g_1 \) and namely it determines the three-fermion bound state mass \( M_3 \).

4 Numerical results

The results of solving equation (1) for three bosons and corresponding equation for three fermions are presented in what follows. Calculations were carried out with constituent mass \( m = 1 \) and correspond to the ground state. We represent in Fig. 1 the three-body bound state mass \( M_3 \) as a function of the two-body one \( M_2 \) (solid line) together with the dissociation limit \( M_3 = M_2 + m \).

![Figure 1](image.png)

**Figure 1.** Three-boson bound state mass \( M_3 \) versus the two-body one \( M_2 \) (solid line). Results obtained with integration limits (4) are in dash line. Dots values are taken from [7].

Our results corresponding to integration limits (4) are included in Fig. 1 (dash line) for comparison. Values from [7] (corrected relative to [4]) are indicated by dots. In both cases the three-body binding energy is finite and the Thomas collaps
is absent, like it was already found in [3]. However, except for the zero binding limit, solid and dash curves strongly differ from each other. In the two-body zero binding limit, the three-boson binding energy (solid line) is $B_3 \approx 0.012 \, m$.

![Figure 2. Three-boson bound state mass squared $M_3^2$ versus $M_2$.](image)

When $M_2$ decreases, the three-body mass $M_3$ decreases very quickly and vanishes at the two-body mass value $M_2 = M_c \approx 1.43$. This result was reproduced in Ref. [5]. Whereas the meaning of collapse as used in the Thomas paper implies unbounded nonrelativistic binding energies and cannot be used here, the zero bound state mass $M_3 = 0$ constitutes its relativistic counterpart. Indeed, for two-body masses below the critical value $M_c$, the three-body system no longer exists.

We would like to remark that for $M_2 \leq M_c$, equation (1) possesses square integrable solutions with negative values of $M_3^2$. They have no physical meaning but $M_3^2$ remains finite in all the two-body mass range $M_2 \in [0, 2]$. The results of $M_3^2$ are given in Fig. [2] When $M_2 \to 0$, $M_3^2$ tends to $\approx -11.6$.

The results for three-fermion system are shown in Fig. [3] Qualitatively they are similar to the three-boson case with the curve shifted to smaller $M_2$ values. As a consequence, the critical value is $M_c \approx 1.35$ instead of 1.43 for bosons. This value may however depend on the particular type of spin coupling used in the two-body kernel. Contrary to the boson case, the three-fermion system is unbound in the two-body zero binding limit. The binding appears when the two-fermion system is already bound by $B_2 = 0.1$, that is, for an interaction strong enough to compensate the Pauli repulsion.

5 Conclusion

In summary, we have considered the relativistic problem of three equal-mass bosons and fermions, interacting via zero-range forces constrained to provide finite two-body mass $M_2$. The Light-Front Dynamics equations have been derived and solved numerically.

We have found that the three-body bound state exists for two-body mass val-
values in the range $M_c \approx 1.43 m \leq M_2 \leq 2 m$ for bosons and $M_c \approx 1.35 m \leq M_2 \leq 1.9 m$ for fermions. The Thomas collapse is avoided in the sense that three-body mass $M_3$ is finite, in agreement with [3, 4]. However, another kind of catastrophe happens. Removing infinite binding energies, the relativistic dynamics generates zero three-body mass $M_3$ at a critical value $M_2 = M_c$. For stronger interaction, i.e. when $0 \leq M_2 < M_c$, there are no physical solutions with real value of $M_3$. In this domain, $M_2^2$ becomes negative and the three-body system cannot be described by zero range forces, as it happens in nonrelativistic dynamics. This fact can be interpreted as a relativistic collapse.

Acknowledgement. This work is partially supported by the French-Russian PICS and RFBR grants Nos. 1172 and 01-02-22002. Numerical calculations were performed at CGCV (CEA Grenoble) and IDRIS (CNRS).

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