Many skyrmion wave functions and skyrmion statistics in quantum Hall ferromagnets

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Abstract

We determine the charge and statistical angle of skyrmions in quantum Hall ferromagnets by performing Berry phase calculations based on the microscopic variational wave functions for many-skyrmion states. We find, in contradiction to a recent claim by Dziarmaga, that both the charge and the statistical angle of a skyrmion are independent of its spin (size), and are identical to those of Laughlin quasiparticles at the same filling factor. We discuss some subtleties in the use of these variational wave functions.

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Recently there has been considerable interest in quantum Hall systems that spontaneously develop spin ferromagnetism, as well as in double layer quantum Hall systems with spontaneous interlayer phase coherence, where the layer degrees of freedom play a role similar to that of spins of the electrons. Of special interest is the realization that the low energy charged excitations in these systems are highly collective topological solitons called skyrmions. It is now well understood that the charge carried by a skyrmion is equal to its topological charge times the Landau level filling factor of the system, while its energy and spin (number of electrons whose spins are flipped in the core region of the skyrmion) depend sensitively on the strength of Zeeman field. These theoretical predictions have been confirmed experimentally for the case of filling factor $\nu = 1$.

In reformulating the low energy physics near ferromagnetic fillings in terms of skyrmions, the issue of what statistics to assign them becomes important. While several previous authors have addressed this starting from the bosonic Chern-Simons approach to the quantum Hall effect, consensus has not been reached. Sondhi \textit{et al.} and Nayak and Wilczek have argued that there is a Hopf term in the effective $\sigma$ model that describes the low energy spin dynamics of the system, which enforces, for the skyrmions, statistics identical to those of the polarized, Laughlin quasiparticles at the same filling factor. The same conclusion was reached by Moon \textit{et al.} based on an analysis of the $CP^1$ model and duality transformations. Recently however, Dziarmaga has claimed that the statistical angle of the skyrmions depends on their spin (or core size). Were this to be the case, the consequences would be striking: both even and odd denominator states would arise from the standard hierarchy construction carried out with skyrmions, and their realization in a given sample would depend sensitively on the value of the Zeeman energy.

In this paper we show that this does not happen and that both the charge and the statistical angle of a skyrmion depend \textit{only} on its topological properties, and are identical to those of the Laughlin quasiparticles at the same filling factor. We show this by performing direct Berry phase calculations with generalizations of the \textit{microscopic} variational wave functions for skyrmions proposed by Moon \textit{et al.} in close analogy to the approach taken by
Arovas et al.\textsuperscript{3} in computing the charge and statistics of Laughlin quasiparticles in the spin polarized quantum Hall effect. Before turning to the details of the calculations we discuss the wave functions appropriate to a system with many skyrmions.

**Many Skyrmion Wave Functions:** In the symmetric gauge, the ground state of a quantum Hall system at filling factor $\nu = 1/m$ (where $m$ is an odd integer) is well approximated by the wave function:

$$
\psi_g(\chi_1, z_1, \ldots, \chi_N, z_N) = \prod_{i=1}^{N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \psi_m(z_1, \ldots, z_N),
$$

where

$$
\psi_m(z_1, \ldots, z_N) = \prod_{j<k} (z_j - z_k)^m \exp(-\sum_{i} |z_i|^2/4\ell^2)
$$

is the Laughlin wave function that involves only the spatial coordinates of the electrons. Here $\chi_j$ is the spin wave function of the $j$th electron, $z_j = x_j + iy_j$ is the complex coordinate of the $j$th electron, $\ell$ is the magnetic length, and $N$ is the total number of electrons in the system. It is the exact ground state wave function for $m = 1$ within the lowest Landau level (LLL) approximation. We have assumed that the Zeeman field has polarized the spins of all electrons into the up direction. In the absence of Zeeman coupling, the electron spins are fully polarized spontaneously but the polarization vector may point in any direction. The system has a macroscopic ground state degeneracy in this case.

Variational wavefunctions for states with arbitrary distributions of skyrmions can be written down as appropriate transcriptions of the known time independent classical solutions of the 2+1 dimensional scale invariant $O(3)$ non-linear $\sigma$ model.\textsuperscript{4} Recall that the latter can be written in the unnormalized, spinorial form $(f^g)$, where $f$ and $g$ are analytic functions of the complex two dimensional co-ordinate $z$. The corresponding LLL wave function for a textured state takes the form

$$
\psi_{f,g} = \prod_{i}^{N} \begin{pmatrix} f(z_i) \\ g(z_i) \end{pmatrix} \psi_m.
$$
It is not hard to see by counting powers of $z$ that the total physical charge in these states is equal to the filling factor times the maximum degree of $f$ and $g$ in $z$. A nicer argument, which makes the connection to the $\sigma$ model manifest, is available for slowly varying textures. For these one can use the plasma interpretation of the squared modulus of the wavefunction to show that the local spin density and charge density of the quantum state are precisely the spin density and topological density of the the $\sigma$ model solution and that its charge density is indeed just $\nu = 1/m$ times the topological density of the texture. Our assertion about the charge of our wavefunctions then follows from the known result for the net topological charge for the $\sigma$ model solutions.

In particular, the variational wave function for a skyrmion with topological charge $n$, located at $z_0$ with core parameter $\lambda$ takes the form

$$\psi^\text{z}_m = A N \prod_i \left( \frac{(z_i - z_0)^n}{\lambda^n} \right)_i \psi_m, \tag{4}$$

where, in $A$, we have included a normalization factor. $\lambda$ is a complex number; its magnitude sets the core size and its phase determines the internal spin orientation within the core region of the skyrmion. This state carries extra positive charge (compared with the ground state) near $z_0$ (i.e., the skyrmion is hole-like since electrons carry negative charge), within spatial range $|\lambda|$; spins are rotated away from the up direction in the same region. In the limit $\lambda = 0$ it reduces to the Laughlin quasihole wave function. The optimal core size is determined by the competition between Zeeman energy and Coulomb exchange energy.

Wave functions for electron-like, anti-skyrmions can be written down in the same spirit. In this paper we focus on hole like skyrmions for convenience. The conclusions we reach, however, apply to both skyrmions and anti-skyrmions.

The wavefunction for two skyrmions located at $z_a$ and $z_b$, with core parameters $\lambda_a$ and $\lambda_b$, and winding numbers $n_a$ and $n_b$ is

$$\psi^{z_a,z_b}_m = A' \prod_i \left( \frac{(z_i - z_a)^{n_a}(z_i - z_b)^{n_b}}{\lambda_a^{n_a}(z_i - z_b)^{n_b} + \lambda_b^{n_b}(z_i - z_a)^{n_a}} \right)_i \psi_m. \tag{5}$$
This is the appropriate two skyrmion wave function for as \( z_i \sim z_a \) the wave function reduces to

\[
\chi \sim \begin{pmatrix}
(z_i - z_a)^{n_a}(z_i - z_b)^{n_b} \\
\lambda^{n_a}_a (z_i - z_b)^{n_b}
\end{pmatrix} \sim \begin{pmatrix}
(z_i - z_a)^{n_a} \\
\lambda^{n_a}_a
\end{pmatrix},
\] (6)

which is the same as that of a single skyrmion at \( z_a \) with winding number \( n_a \) and core parameter \( \lambda_a \). Similarly near \( z_b \) the spinor wave function is identical to that of a single skyrmion of winding number \( n_b \) and core parameter \( \lambda_b \). An important difference between (5) and (6) is that in (6) the both up and down spin components of the wave function depend on the location of the skyrmions, while in (5) only up spin component depends on the location of the skyrmion.

Finally, we record the explicit form of the wavefunction for a system with \( M \) skyrmions, located at complex coordinates \( \xi_1, \xi_2, \ldots, \xi_M \), with core parameters and winding numbers \( \lambda_1, \ldots, \lambda_M \) and \( n_1, \ldots, n_M \) respectively:

\[
\Psi = \prod_i^n \left[ \prod_j^M (z_i - \xi_j)^{n_j} \left( \frac{1}{\sum_k^M (z_j - \xi_k)^{n_k}} \right) \right] \psi_m.
\] (7)

Clearly, as \( z_i \sim \xi_j \), the wave function reduces to

\[
\chi \sim \begin{pmatrix}
(z_i - \xi_j)^{n_j} \\
\lambda^{n_j}_j
\end{pmatrix},
\] (8)

which is that of a single skyrmion of the appropriate scale and winding number.

**Berry Phase Calculations:** To determine the charge and statistics of the skyrmions, we take the same approach used by Arovas *et al.* to determine the same quantities for Laughlin quasiparticles and quasiholes. We first consider the case where there is a single skyrmion in the system, and consider the Berry phase for moving it around a closed loop. This can be written as an integral over the skyrmion co-ordinate \( z_0 \),

\[
\gamma = i \oint dz_0 \langle \psi_m^{z_0} | \partial_{z_0} \psi_m^{z_0} \rangle.
\] (9)

From Eq. (8) we have
\[
\frac{\partial \psi^{z_0}}{\partial z_0} = \left\{ \sum_{j=1}^{N} \begin{pmatrix} \frac{n}{z_0-z_j} & 0 \\ 0 & 0 \end{pmatrix} \right\} \psi^m, 
\] (10)

where the $2 \times 2$ matrix with subscript $j$ acts on the spinor wave function of the $j$th electron. We thus find that

\[
\gamma = i \oint dz_0 \left\langle \psi^m \left| \sum_{j} \begin{pmatrix} \frac{n}{z_0-z_j} & 0 \\ 0 & 0 \end{pmatrix} \right| \psi^m \right\rangle. 
\] (11)

Recognizing that the average density of up spin electrons at point $z$ is

\[
\rho^{z_0}_\uparrow(z) = \left\langle \psi^m \left| \sum_{j} \begin{pmatrix} \delta(z_j-z) & 0 \\ 0 & 0 \end{pmatrix} \right| \psi^m \right\rangle, 
\] (12)

we obtain

\[
\gamma = i \oint dz_0 \int dxdy \rho^{z_0}_\uparrow(z) \frac{n}{z_0-z}. 
\] (13)

We write $\rho^{z_0}_\uparrow(z) = \rho^0_\uparrow + \delta \rho^{z_0}_\uparrow(z)$, where $\rho^0_\uparrow = 1/(2\pi m \ell^2)$ is the density of up spin electrons in the ground state, and $\delta \rho^{z_0}_\uparrow$ is the extra density of up spin electrons due to the existence of the skyrmion itself. If $z_0$ moves around a circle of radius $R$ clockwise, the contribution to $\gamma$ due to the $\rho^0_\uparrow$ term is

\[
\gamma_0 = -2\pi n \langle N \rangle_R = -(2\pi n/m)(\Phi/\Phi_0), 
\] (14)

where $\langle N \rangle_R$ is the average number of up spin electrons in inside the circle, and $\Phi$ is the amount of magnetic flux enclosed by the circle. The $\delta \rho$ term does not contribute to $\gamma$ due to the fact that $\delta \rho^{z_0}(z)$ is symmetric about the point $z_0$. Since the phase picked up by a charged particle with charge $q$ after moving along a closed loop enclosing flux $\Phi$ is $-2\pi q\Phi/\Phi_0$, we find from Eq. (14) that the charge of a skyrmion described by the wave function (4) is $q = ne/m$, identical to that of the Laughlin quasiholes and independent of its core size, as expected. This result has recently been obtained by Stone using the effective $\sigma$ model.
We now consider the situation where there are two skyrmions located at $z_a$ and $z_b$, with core parameters $\lambda_a$ and $\lambda_b$, winding numbers $n_a$ and $n_b$, separated by a distance $R = |z_a - z_b| \gg |\lambda_{a,b}|$ and $R \gg \ell$, described by wave function (5). We let $z_a$ stand still, and $z_b$ move along a closed loop enclosing $a$. We find that the Berry phase picked up by the system is

$$\gamma = i \oint dz_b \langle \psi_{m_a}^{z_a,z_b} | \partial_{z_b} \psi_{m_a}^{z_a,z_b} \rangle$$

$$= im_b \oint dz_b \int dxdy \left[ \rho_{\uparrow}^{m_a}(z) \frac{1}{z_b - z} - \rho_{\downarrow}^{m_a}(z) \right] \frac{\lambda_{a}^{n_a}(z - z_b)^{n_b - 1}}{\lambda_{a}^{n_a}(z - z_b)^{n_b} + \lambda_{b}^{n_b}(z - z_a)^{n_a}}. \quad (15)$$

When the distance between the two skyrmions is much larger than the skyrmion core sizes, we have $\rho_{\uparrow}^{m_a}(z) = \rho_{\uparrow}^{0} + \delta \rho_{\uparrow}^{a}(z) + \delta \rho_{\uparrow}^{b}(z)$, and $\rho_{\downarrow}^{m_a}(z) = \delta \rho_{\downarrow}^{a}(z) + \delta \rho_{\downarrow}^{b}(z)$, where $\delta \rho_{\uparrow,\downarrow}^{a,b}$ are the extra charge density for up (down) spin electrons due to skyrmions $a(b)$, which is non zero only near $z_a(z_b)$. The $\rho_{\uparrow}^{0}$ term is identical to that in Eq. (14), which is due to the charge of skyrmion $b$ moving in a magnetic field. For the same reason as before, the $\delta \rho_{\uparrow}^{b}$ term vanishes. Thus the additional phase due to the existence of skyrmion $a$ is

$$\Delta \gamma = in_b \oint dz_b \int dxdy \left[ \frac{\delta \rho_{\downarrow}^{a}(z)}{z_b - z} - (\delta \rho_{\downarrow}^{a}(z) + \delta \rho_{\downarrow}^{b}(z)) \right] \frac{\lambda_{a}^{n_a}(z - z_b)^{n_b - 1}}{\lambda_{a}^{n_a}(z - z_b)^{n_b} + \lambda_{b}^{n_b}(z - z_a)^{n_a}}. \quad (16)$$

In the large $R$ limit, $\delta \rho_{\uparrow,\downarrow}^{a,b}$ can be treated as $\delta$ functions localized at $z_a(z_b)$, with corrections to the above expression vanishing at least as $1/R$. For $n_b > 1$ or $n_b = 1, n_a > 1$ the $\delta \rho_{\downarrow}^{b}$ term vanishes in the large $R$ limit. Thus we obtain

$$\Delta \gamma = in_b \oint dz_b \int dxdy \frac{\delta \rho_{\downarrow}^{a}(z) + \delta \rho_{\downarrow}^{b}(z)}{z_b - z} = 2\pi n_b q_a/e = 2\pi n_a n_b / m. \quad (17)$$

This additional phase is due to the statistical interaction between the skyrmions. We thus find that the statistical angle of the skyrmions, which is the additional phase accumulated when two identical hole like skyrmions with winding number $n$ are exchanged clockwise, to be

$$\phi_n = \Delta \gamma / 2 = n^2 \pi / m, \quad (18)$$

again identical to the Laughlin quasiholes and independent of the core size (or spin).
Careful readers may worry at this point that the variational wave functions (4) and (5) we use here are not eigenstates of the total spin and its $z$ component which are both good quantum numbers. On general grounds, we believe this is not a limitation on the validity of our results. The difference between variational “classical” skyrmions and exact eigenstates is the inclusion of symmetry related configurations. The Berry phases we compute are geometric phases and will not be affected by the inclusion of these additional configurations. For example, it is straightforward to project our variational wave functions onto states of definite $z$ component of spin $\frac{1}{2}$ and show that this does not change our results.

**Skyrmions with Topological Charge 1:** There is a subtlety in our above considerations that we have glossed over thus far. This is that in the special case of $n_a = n_b = 1$, the large separation asymptotics used above in calculating the statistical phase $\delta \gamma$, no longer apply. In contrast to the result for all $n > 1$, we find that $\Delta \gamma$ depends on the relative size and spin orientation of the two skyrmions which is clearly unphysical for well separated skyrmions.

This problem can be traced to the power law tails of the variational skyrmion wavefunctions, which are a consequence of the scale invariance of the $O(3)$ model. For $n > 1$, the power is large enough that two skyrmions decouple at large distances but for $n = 1$ they never really do. Indeed, the single $n = 1$ skyrmion wavefunction is pathological on its own; the number of down spin particles in it (4) diverges logarithmically with system size leading to a divergent Zeeman energy in the presence of Zeeman field.

It is known that in the problem with Coulomb interactions, where their competition with the Zeeman energy sets a scale for the skyrmions, the form of the skyrmions is modified at large distances; this issue will be further addressed elsewhere. What is important for our purposes is recognizing, that this will lead to a decoupling of the $n = 1$ skyrmions at large distances and hence a result for their statistics consistent with those for the decoupled, $n > 1$ skyrmions (18) obtained above.

In summary, we have obtained the charge and statistical angle of quantum Hall ferromagnet skyrmions by calculating Berry phases with microscopic wave functions. We find they do not depend on the spin of the skyrmions, and are identical to those of the Laughlin
quasiholes at the same filling factor.

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When \( f \) and \( g \) have nodes at the same positions, the situation is somewhat different. In this case \( f \) and \( g \) must contain common factors and we have

\[
\left( \frac{f(z)}{g(z)} \right) = h(z) \left( \frac{\tilde{f}(z)}{\tilde{g}(z)} \right),
\]

where \( h, \tilde{f} \) and \( \tilde{g} \) are all analytic in \( z \), and the total topological charge is given by the maximum degree of \( \tilde{f} \) and \( \tilde{g} \) in \( z \). The physical charge however has contributions from both the spin textures and quasiholes introduced by \( h \) in this case, and is still equal to the the maximum degree of \( f \) and \( g \) in \( z \) times the filling factor.

Strictly speaking the wavefunction (5) is an appropriate wave function only when \(|z_a - z_b| \gg \lambda_a, \lambda_b\), i.e., it should only be used to describe two distant skyrmions. As \( z_a \) approaches \( z_b \), (5) becomes a state with a skyrmion and a (multi)quasihole at the same position. Although it still is a valid wave function, the energetically most favorable state in this case should be a single skyrmion with topological charge \( n = n_a + n_b \), described by (4).

F. D. M. Haldane, unpublished.

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This problem cannot be remedied by projection onto eigenstates of the \( z \) component of the total spin. After projection the state will have finite Zeeman energy cost, but in such a state the down spin electrons are not bound to the center of the skyrmion, hence it does not properly describe a skyrmion. For further discussion see Ref. [22].

E. H. Rezayi and S. L. Sondhi, in preparation.