The Einstein-Hilbert type action on almost $k$-product manifolds

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Abstract

A Riemannian manifold endowed with $k > 2$ orthogonal complementary distributions (called a Riemannian almost $k$-product structure) appears in such topics as multiply warped products, the webs composed of several foliations, and proper Dupin hypersurfaces of real space-forms. In the paper we consider the mixed scalar curvature of such structure for $k > 2$, derive Euler-Lagrange equations for the Einstein-Hilbert type action with respect to adapted variations of metric, and present them in a nice form of Einstein equation.

Keywords: Almost $k$-product manifold, mixed scalar curvature, Einstein-Hilbert action

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Introduction

Many examples of Riemannian metrics come (as critical points) from variational problems, a particularly famous of which is the Einstein-Hilbert action, given on a smooth closed manifold $M$ by

$$J : g \rightarrow \int_M \left\{ \frac{1}{2a} (S - 2\Lambda) + \mathcal{L} \right\} \text{dvol}_g,$$

(1)

e.g., [5]. Here, $g$ is a pseudo-Riemannian metric on $M$, $S$ – the scalar curvature of $(M, g)$, $\Lambda$ – a constant (the “cosmological constant”), $\mathcal{L}$ – Lagrangian describing the matter contents, and $a = 8\pi G/c^4$ – the coupling constant involving the gravitational constant $G$ and the speed of light $c$. To deal also with non-compact manifolds (“spacetimes”), it is assumed that the integral above is taken over $M$ if it converges; otherwise, one integrates over arbitrarily large, relatively compact domain $\Omega \subset M$, which also contains supports of variations of $g$. The Euler-Lagrange equation for (1) (called the Einstein equation) is

$$\text{Ric} - \frac{1}{2} S \cdot g + \Lambda g = a \Theta$$

(2)

with the Ricci curvature $\text{Ric}$ and the energy-momentum tensor $\Theta$ (generalizing the stress tensor of Newtonian physics), given in a coordinates by $\Theta_{\mu\nu} = -2 \partial \mathcal{L} / \partial g^{\mu\nu} + g_{\mu\nu} \mathcal{L}$. The solution of (2) is a metric $g$, satisfying this equation, where the tensor $\Theta$ (describing a specified type of matter) is given. The classification of solutions of (2) is a deep and largely unsolved problem [5].

Distributions on a manifold (i.e., subbundles of the tangent bundle) appear in various situations, e.g., [4, 8] and are used to build up notions of integrability, and specifically of a foliated manifold. On a manifold equipped with an additional structure (e.g., almost product [8] or contact [6]), one can consider an analogue of (1) adjusted to that structure. This approach was taken in [2, 3, 9, 12, 13], for $M$ endowed with a distribution $\mathcal{D}$ or a foliation (that can be viewed as an integrable distribution).

In this paper, a similar change in (1) is considered on a connected smooth $n$-dimensional manifold endowed with $k \geq 2$ pairwise orthogonal $n_i$-dimensional distributions with $\sum n_i = n$. The notion of a multiple warped product, e.g., [7], is a special case of this structure, which

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can be also viewed in the theory of webs composed of foliations of different dimensions, see [1]. The mixed Einstein-Hilbert action on \((M, \mathcal{D}_1, \ldots, \mathcal{D}_k)\),

\[
J_D : g \mapsto \int_M \left\{ \frac{1}{2a} \left( S(\mathcal{D}_1, \ldots, \mathcal{D}_k) - 2\Lambda \right) + \mathcal{L} \right\} d\text{vol}_g, \tag{3}
\]

is an analog of [1], where \(S\) is replaced by the mixed scalar curvature \(S(\mathcal{D}_1, \ldots, \mathcal{D}_k)\), see [6]. The geometrical part of (3) is \(J_D^g : g \mapsto \int_M S(\mathcal{D}_1, \ldots, \mathcal{D}_k) d\text{vol}_g\) - the total mixed scalar curvature. The mixed scalar curvature is the simplest curvature invariant of a pseudo-Riemannian almost \(k\)-product structure, which can be defined as an averaged sum of sectional curvatures of planes that non-trivially intersect with both of the distributions. Investigation of \(S(\mathcal{D}_1, \ldots, \mathcal{D}_k)\) can lead to multiple results regarding the existence of foliations and submersions with interesting geometry, e.g., integral formulas and splitting results, curvature prescribing and variational problems, see [10] for \(k = 2\). Varying (3) as a functional of adapted metric \(g\), we obtain the Euler-Lagrange equations in the form of Einstein equation (2), i.e.,

\[
\mathcal{R}ic_D - (1/2) S_D \cdot g + \Lambda g = a \Theta \tag{4}
\]

(for \(k = 2\), see [2, 12, 13]), where the Ricci tensor and the scalar curvature are replaced by the Ricci type tensor \(\mathcal{R}ic_D\), see (22), and its trace \(S_D\). By the equality

\[
S = S(\mathcal{D}_1, \ldots, \mathcal{D}_k) + \sum_i S(D_i),
\]

where \(S(D_i)\) is the scalar curvature along the distribution \(\mathcal{D}_i\), one can combine the action [1] with (3) to obtain the perturbed Einstein-Hilbert action on \((M, \mathcal{D}_1, \ldots, \mathcal{D}_k)\):

\[
J_{\varepsilon} : g \mapsto \int_M \left\{ \frac{1}{2a} \left( S + \varepsilon S(\mathcal{D}_1, \ldots, \mathcal{D}_k) - 2\Lambda \right) + \mathcal{L} \right\} d\text{vol}_g
\]

with \(\varepsilon \in \mathbb{R}\), whose critical points describe the “space-times” in extended theory of gravity.

In the paper we consider the geometrical part of the Einstein-Hilbert type action [3] (called the total mixed scalar curvature), derive Euler-Lagrange equations with respect to adapted variations of metric (which generalize equations for \(k = 2\) in [12]), and present them in a nice form of Einstein equation. We delegate the following for further study: a) generalize our results for more general variations of metrics; b) extend our results for variations of connections (as in Einstein-Cartan theory); c) find more applications of our results in geometry and physics.

## 1 Preliminaries

Let \(M\) be a smooth connected \(n\)-dimensional manifold with the Levi-Civita connection \(\nabla\) and the curvature tensor \(R\). A pseudo-Riemannian metric \(g = \langle \cdot, \cdot \rangle\) of index \(q\) on \(M\) is an element \(g \in \text{Sym}^2(M)\) (of the space of symmetric \((0, 2)\)-tensors) such that each \(g_x (x \in M)\) is a non-degenerate bilinear form of index \(q\) on the tangent space \(T_x M\). For \(q = 0\) (i.e., \(g_x\) is positive definite) \(g\) is a Riemannian metric and for \(q = 1\) it is called a Lorentz metric. Let \(\text{Riem}(M) \subset \text{Sym}^2(M)\) be the subspace of pseudo-Riemannian metrics of a given signature.

A distribution \(\mathcal{D}\) on \(M\) is non-degenerate, if \(g_x\) is non-degenerate on \(\mathcal{D}_x \subset T_x M\) for all \(x \in M\); in this case, the orthogonal complement of \(\mathcal{D}\) is also non-degenerate.

Let \(M\) be endowed with \(k \geq 2\) pairwise transversal \(n_i\)-dimensional distributions \(\mathcal{D}_i (1 \leq i \leq k)\) with \(\sum n_i = n\). Denote by \(\text{Riem}(M, \mathcal{D}_1, \ldots, \mathcal{D}_k) \subset \text{Riem}(M)\) the subspace of pseudo-Riemannian metrics making \(\{\mathcal{D}_i\}\) pairwise orthogonal and non-degenerate. A \((M; \mathcal{D}_1, \ldots, \mathcal{D}_k)\) with a compatible metric \(g\) will be called a Riemannian almost \(k\)-product manifold, see [11]. Let \(P_i : TM \to \mathcal{D}_i\) be the orthoprojector, and \(P_i^\perp = \text{id}_{TM} - P_i\) be the orthoprojector onto
For a function $f$, the self-adjoint (1,1)-tensor $\Psi$ to $D$ and the integrability tensors of distributions $D_1$, $D_2$, $D_3$, $D_4$, $D_5$, $D_6$, $D_7$, $D_8$, $D_9$, $D_{10}$ are defined by

$$h_i(X,Y) = \frac{1}{2} P_i^\perp(\nabla_X Y + \nabla_Y X), \quad T_i(X,Y) = \frac{1}{2} P_i^\perp(\nabla_X Y - \nabla_Y X) = \frac{1}{2} P_i^\perp [X,Y].$$

Similarly, $h_i^\perp$, $H_i^\perp = \text{Tr}_g h_i^\perp$, $T_i^\perp$ are the second fundamental forms, mean curvature vector fields and the integrability tensors of distributions $D_i^\perp$ in $M$. Note that $H_i = \sum_{j\neq i} P_j H_i$, etc. Recall that a distribution $D_i$ is called integrable if $T_i = 0$, and $D_i$ is called totally umbilical, harmonic, or totally geodesic, if $h_i = (H_i/n_i) g$, $H_i = 0$, or $h_i = 0$, respectively.

The “musical” isomorphisms $\sharp$ and $\flat$ will be used for rank one and symmetric rank 2 tensors. For example, if $\omega \in \Lambda^1(M)$ is a 1-form and $X, Y \in \mathfrak{X}_M$ then $\langle \omega, Y \rangle = \langle \omega^\sharp, Y \rangle$ and $X^\flat(Y) = \langle X, Y \rangle$. For arbitrary (0,2)-tensors $B$ and $C$ we also have $\langle B, C \rangle = \text{Tr}_g(B^\sharp C^\sharp) = \langle B^\perp, C^\perp \rangle$.

The shape operator $(A_i)_Z$ of $D_i$ with $Z \in D_i^\perp$, and the operator $(T_i)_Z^2$ are defined by

$$\langle (A_i)_Z(X), Y \rangle = h_i(X,Y), \quad \langle (T_i)_Z^2(X), Y \rangle = \langle T_i(X,Y), Z \rangle, \quad X, Y \in D_i.$$

Given $g \in \text{Riem}(M, D_1, \ldots, D_k)$, there exists a local $g$-orthonormal frame $\{E_1, \ldots, E_n\}$ on $M$, where $\{E_1, \ldots, E_{n_1}\} \subset D_1$ and $\{E_{n_1+1}, \ldots, E_{n_i}\} \subset D_i$ for $2 \leq i \leq k$, and $\varepsilon_a = \langle E_a, E_a \rangle \in \{-1,1\}$. All quantities defined below using an adapted frame do not depend on the choice of this frame. The squares of norms of tensors are obtained using

$$\langle h_i, h_i \rangle = \sum_{n_{i-1} < a,b \leq n_i} \varepsilon_a \varepsilon_b \langle h_i(E_a, E_b), h_i(E_a, E_b) \rangle, \quad \langle T_i, T_i \rangle = \sum_{n_{i-1} < a,b \leq n_i} \varepsilon_a \varepsilon_b \langle T_i(E_a, E_b), T_i(E_a, E_b) \rangle.$$

The divergence of a vector field $X \in \mathfrak{X}_M$ is given by $(\text{div} X) d\text{vol}_g = \mathcal{L}_X (d\text{vol}_g)$, where $d\text{vol}_g$ is the volume form of $g$. One may show that $\text{div} X = \text{Tr}(\nabla X) = \text{div}_i X + \text{div}_{i}^\perp X$, where

$$\text{div}_i X = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a \langle \nabla_a X, E_a \rangle, \quad \text{div}_{i}^\perp X = \sum_{b \neq n_{i-1}, n_i} \varepsilon_b \langle \nabla_b X, E_b \rangle.$$

Observe that for $X \in D_i$ we have

$$\text{div}_{i}^\perp X = \text{div} X + \langle X, H_i^\perp \rangle.$$

(5)

For a $(1,2)$-tensor $Q$ define a $(0,2)$-tensor $\text{div}_{i}^\perp Q$ by

$$(\text{div}_{i}^\perp Q)(X,Y) = \sum_{b \neq n_{i-1}, n_i} \varepsilon_b \langle (\nabla_b Q)(X,Y), E_b \rangle, \quad X, Y \in \mathfrak{X}_M.$$

For a $D_i$-valued $(1,2)$-tensor $Q$, similarly to $[3]$, we have $\text{div}_{i}^\perp Q = \text{div} Q + \langle Q, H_i \rangle$, where

$$(\text{div} Q)(X,Y) = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a \langle (\nabla_a Q)(X,Y), E_a \rangle = -\langle Q(X,Y), H_i \rangle,$$

and $\langle Q, H_i \rangle(X,Y) = \langle Q(X,Y), H_i \rangle$ is a $(0,2)$-tensor. For example, $\text{div}_{i}^\perp h_i = \text{div} h_i + \langle h_i, H_i \rangle$.

For a function $f$ on $M$, we use the notation $P_i^\perp(\nabla f)$ for the projection of $\nabla f$ onto $D_i^\perp$.

The $D_i$-deformation tensor $\text{Def}_{D_i} Z$ of $Z \in \mathfrak{X}_M$ is the symmetric part of $\nabla Z$ restricted to $D_i$,

$$2 \text{Def}_{D_i} Z(X,Y) = \langle \nabla_X Z, Y \rangle + \langle \nabla_Y Z, X \rangle, \quad X, Y \in D_i.$$

The self-adjoint $(1,1)$-tensors: $A_i$ (the Casorati type operator) and $T_i$ and the symmetric $(0,2)$-tensor $\Psi_i$, see $[3] [12]$, are defined using $A_i$ and $T_i$ by

$$A_i = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a (\langle A_i \rangle_{E_a})^2, \quad T_i = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a (\langle T_i \rangle_{E_a}^2)^2,$$

$$\Psi_i(X,Y) = \text{Tr}(\langle A_i \rangle_{Y} (A_i)_X + \langle T_i \rangle_{Y} (T_i)_X), \quad X, Y \in D_i^\perp.$$
We define a self-adjoint $(1,1)$-tensor $K_i$ by the formula with Lie bracket,

$$K_i = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a [(T^a_i) E_a, (A_i) E_a] = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a ((T_i) E_a (A_i) E_a - (A_i) E_a (T_i) E_a).$$

For any $(1,2)$-tensors $Q_1, Q_2$ and a $(0,2)$-tensor $S$ on $TM$, define the following $(0,2)$-tensor $\Upsilon_{Q_1, Q_2}$:

$$\langle \Upsilon_{Q_1, Q_2}, S \rangle = \sum_{\lambda, \mu} \varepsilon_{\lambda} \varepsilon_{\mu} \left[ S(Q_1(e_{\lambda}, e_{\mu}), Q_2(e_{\lambda}, e_{\mu})) + S(Q_2(e_{\lambda}, e_{\mu}), Q_1(e_{\lambda}, e_{\mu})) \right],$$

where on the left-hand side we have the inner product of $(0,2)$-tensors induced by $g$, $\{e_{\lambda}\}$ is a local orthonormal basis of $TM$ and $\varepsilon_{\lambda} = \langle e_{\lambda}, e_{\lambda} \rangle \in \{-1,1\}$.

**Remark 1.** If $g$ is definite then $\Upsilon_{h_i, h_i} = 0$ if and only if $h_i = 0$. Indeed, for any $X \in D_i$ we have

$$\langle \Upsilon_{h_i, h_i}, X^\flat \otimes X^\flat \rangle = 2 \sum_{a,b} \langle X, h_i(E_a, E_b) \rangle^2.$$  

The above sum is equal to zero if and only if every summand vanishes. This yields $h_i = 0$. Thus, $\Upsilon_{h_i, h_i}$ is a “measure of non-total geodesy” of $D_i$. Similarly, if $\Upsilon_{T_i, T_i} = 0$ then

$$\langle \Upsilon_{T_i, T_i}, X^\flat \otimes X^\flat \rangle = 2 \sum_{a,b} \langle X, T_i(E_a, E_b) \rangle^2.$$  

Hence, if $g$ is definite then the condition $\Upsilon_{T_i, T_i} = 0$ is equivalent to $T_i = 0$. Therefore, $\Upsilon_{T_i, T_i}$ can be viewed as a “measure of non-integrability” of $D_i$.

### 2 The mixed scalar curvature

A plane in $TM$ spanned by two vectors belonging to different distributions $D_i$ and $D_j$ will be called *mixed*, and the its sectional curvature is called mixed. Similarly to the case of $k = 2$, the mixed scalar curvature of $(M, g; D_1, \ldots, D_k)$ is defined as an averaged mixed sectional curvature.

**Definition 1** (see [II]). Given $g \in \text{Riem}(M; D_1, \ldots, D_k)$ with $k \geq 2$ non-degenerate distributions, the following function on $M$ will be called the *mixed scalar curvature*:

$$S(D_1, \ldots, D_k) = \sum_{i<j} S(D_i, D_j), \quad (6)$$

where

$$S(D_i, D_j) = \sum_{n_{i-1} < a \leq n_i, \ n_{j-1} < b \leq n_j} \varepsilon_a \varepsilon_b \langle R(E_a, E_b) E_a, E_b \rangle, \quad i \neq j.$$  

The following symmetric $(0,2)$-tensor $r$ will be called the *partial Ricci tensor*:

$$r(X, Y) = \frac{1}{2} \sum_{i=1}^k r_i(X, Y),$$

where the partial Ricci tensor related to $D_i$ is

$$r_i(X, Y) = \sum_{n_{i-1} < a \leq n_i} \varepsilon_a \langle R_{E_a, P_i} X E_a, P_i Y \rangle, \quad X, Y \in \mathfrak{X}_M. \quad (7)$$

**Proposition 1.** We have

$$S(D_1, \ldots, D_k) = \frac{1}{2} \sum_i S(D_i, D_i^\perp) = \text{Tr}_g r.$$

**Proof.** This directly follows from definitions (6) and (7) and equality $\text{Tr}_g r_i = S(D_i, D_i^\perp)$.  


The following formula for a Riemannian manifold \((M, g)\) endowed with two complementary orthogonal distributions \(D\) and \(D^\perp\), see [13]:

\[
\text{div}(H + H^\perp) = S(\mathcal{D}, \mathcal{D}^\perp) + \langle h, h \rangle + \langle h^\perp, h^\perp \rangle - \langle H, H \rangle - \langle H^\perp, H^\perp \rangle - \langle T, T \rangle - \langle T^\perp, T^\perp \rangle,
\]

has many interesting global corollaries (e.g., decomposition criteria using the sign of \(S\), [14]). In [11], we generalized (8) to (11) and isometric immersions of manifolds, in particular, multiply warped products.

The following formula for a Riemannian manifold \((M, g)\) is valid, see [3, 12]:

\[
S(\mathcal{D}_i, \mathcal{D}_i^\perp) = \sum_{a_i - 1 < a_i \leq n_i, \ n_i \neq \{n_i - 1, n_i\}} \varepsilon_a \varepsilon_b (R_{E_a, E_b} E_a, E_b).
\]

If \(\mathcal{D}_i\) is spanned by a unit vector field \(N\), i.e., \(\langle N, N \rangle = \varepsilon_N \in \{-1, 1\}\), then \(S(\mathcal{D}_i, \mathcal{D}_i^\perp) = \varepsilon_N \text{Ric}_{N, N}\), where \(\text{Ric}_{N, N}\) is the Ricci curvature in the \(N\)-direction. We have \(S(\mathcal{D}_i, \mathcal{D}_i^\perp) = \text{Tr}_g r_i = \text{Tr}_g r_i^\perp\). If \(\dim\mathcal{D}_i = 1\) then \(r_i = \varepsilon_N R_N\), where \(R_N = R(N, \cdot)N\) is the Jacobi operator, and \(r_i^\perp = \text{Ric}_{N, N} g_i^\perp\), where the symmetric \((0, 2)\)-tensor \(g_i^\perp\) is defined by \(g_i^\perp(X, Y) = \langle P_i^\perp X, P_i^\perp Y \rangle\) for \(X, Y \in \mathfrak{X}_M\). The following presentation of the partial Ricci tensor in [7] is valid, see [3, 12]:

\[
r_i = \text{div} h_i + \langle h_i, H_i \rangle - A_i^\perp - T_i^0 - \Psi_i^\perp + \text{Def}_{D_i} H_i^\perp.
\]

Tracing (10) over \(\mathcal{D}\) and applying the equalities

\[
\text{Tr}_g (\text{div} h_i) = \text{div} H_i, \quad \text{Tr}_g \langle h_i, H_i \rangle = \langle H_i, H_i \rangle, \quad \text{Tr}_g \Psi_i^\perp = \langle h_i^\perp, h_i^\perp \rangle - \langle T_i^\perp, T_i^\perp \rangle,
\]

we get (8) with \(\mathcal{D} = \mathcal{D}_i\).

**Theorem 1 ([11]).** For a Riemannian almost \(k\)-product manifold \((M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)\) we have

\[
\text{div} \sum_{i} \left( H_i + H_i^\perp \right) = 2S(\mathcal{D}_1, \ldots, \mathcal{D}_k) + \sum_{i} \left( \langle h_i, h_i \rangle - \langle H_i, H_i \rangle - \langle T_i, T_i \rangle + \langle h_i^\perp, h_i^\perp \rangle - \langle H_i^\perp, H_i^\perp \rangle - \langle T_i^\perp, T_i^\perp \rangle \right).
\]

**Example 1.** To illustrate the proof of (11) for \(k > 2\), consider the case of \(k = 3\). Using (8) for two distributions, \(\mathcal{D}_1\) and \(\mathcal{D}_1^\perp = \mathcal{D}_2 \oplus \mathcal{D}_3\), according to (8) we get

\[
\text{div}(H_1 + H_1^\perp) = 2S(\mathcal{D}_1, \mathcal{D}_1^\perp) + \left( \langle h_1, h_1 \rangle - \langle H_1, H_1 \rangle - \langle T_1, T_1 \rangle \right) + \left( \langle h_1^\perp, h_1^\perp \rangle - \langle H_1^\perp, H_1^\perp \rangle - \langle T_1^\perp, T_1^\perp \rangle \right),
\]

and similarly for \((\mathcal{D}_2, \mathcal{D}_2^\perp)\) and \((\mathcal{D}_3, \mathcal{D}_3^\perp)\). Summing 3 copies of (12), we obtain (11) for \(k = 3\):

\[
\text{div} \sum_{i} \left( H_i + H_i^\perp \right) = 2S(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) + \sum_{i} \left( \langle h_i, h_i \rangle - \langle H_i, H_i \rangle - \langle T_i, T_i \rangle + \langle h_i^\perp, h_i^\perp \rangle - \langle H_i^\perp, H_i^\perp \rangle - \langle T_i^\perp, T_i^\perp \rangle \right).
\]

**Remark 2.** Using Stokes’ Theorem for (11) on a closed manifold \((M, g; \mathcal{D}_1, \ldots, \mathcal{D}_k)\) yields the integral formula for any \(k \in \{2, \ldots, n\}\), which for \(k = 2\) directly follows from (8),

\[
\int_M \left( 2S(\mathcal{D}_1, \ldots, \mathcal{D}_k) + \sum_{i} \left( \langle h_i, h_i \rangle - \langle H_i, H_i \rangle - \langle T_i, T_i \rangle + \langle h_i^\perp, h_i^\perp \rangle - \langle H_i^\perp, H_i^\perp \rangle - \langle T_i^\perp, T_i^\perp \rangle \right) \right) \text{dvol}_g = 0.
\]
3 Variations of metric

We consider smooth 1-parameter variations \( \{g_t \in \text{Riem}(M) : |t| < \varepsilon \} \) of the metric \( g_0 = g \).

Let the infinitesimal variations, represented by a symmetric \((0,2)\)-tensor

\[
B(t) \equiv \partial g_t / \partial t,
\]

be supported in a relatively compact domain \( \Omega \) in \( M \), i.e., \( g_t = g \) and \( B_t = 0 \) outside \( \Omega \) for \( |t| < \varepsilon \). A variation \( g_t \) is volume-preserving if \( \text{Vol}(\Omega, g_t) = \text{Vol}(\Omega, g) \) for all \( t \). We adopt the notations \( \partial_t \equiv \partial / \partial t, \ B \equiv \partial_t g_t |_{t=0} = \dot{g} \), but we shall also write \( B \) instead of \( B_t \) to make formulas easier to read, wherever it does not lead to confusion. Since \( B \) is symmetric, then \( \langle C, B \rangle = (\text{Sym}(C), B) \) for any \((0,2)\)-tensor \( C \). Denote by \( \otimes \) the product of tensors and use the symmetrization operator to define the symmetric product of tensors: \( B \otimes C = \text{Sym}(B \otimes C) = \frac{1}{2} (B \otimes C + C \otimes B) \).

Definition 2. A family of metrics

\[
\{g(t) \in \text{Riem}(M, D_1, \ldots, D_k) : |t| < \varepsilon \}
\]

such that \( g_0 = g \) will be called an adapted variation. In other words, \( D_i \) and \( D_j \) are \( g_t \)-orthogonal for all \( i \neq j \) and \( t \). An adapted variation \( g_t \) will be called a \( D_j \)-variation (for some \( j \in [1, k] \)) if

\[
g_t(X, Y) = g_0(X, Y), \quad X, Y \in D_j^\perp, \quad |t| < \varepsilon.
\]

For an adapted variation we have \( g_t = \bigoplus_{j=1}^k g_j(t) \), where \( g_j(t) = g_t|_{D_j} \). Thus, the tensor \( B_t = \partial_t g_t \) of an adapted variation of metric \( g \) on \((M; D_1, \ldots, D_k)\) can be decomposed into the sum of derivatives of \( D_j \)-variations; namely, \( B_t = \sum_{j=1}^k B_j(t) \), where \( B_j(t) = \partial_t g_j(t) = B_j |_{D_j} \).

Lemma 1. Let a local adapted frame \( \{E_a\} \) evolve by \( g_t \in \text{Riem}(M, D_1, \ldots, D_k) \) according to

\[
\partial_t E_a = -(1/2) B_t^a(E_a).
\]

Then, \( \{E_a(t)\} \) is a \( g_t \)-orthonormal adapted frame for all \( t \).

Proof. For \( \{E_a(t)\} \) we have

\[
\partial_t \langle g_t(E_a, E_b) \rangle = g_t(\partial_t E_a(t), E_b(t)) + g_t(E_a(t), \partial_t E_b(t)) + (\partial_t g_t)(E_a(t), E_b(t))
\]

\[
= B_t(E_a(t), E_b(t)) - \frac{1}{2} g_t(B_t^a(E_a(t)), E_b(t)) - \frac{1}{2} g_t(E_a(t), B_t^a(E_b(t))) = 0.
\]

From this the claim follows. \( \square \)

The following proposition was proved in [13] when \( k = 2 \).

Proposition 2. If \( g_t \) is a \( D_j \)-variation of \( g \) on \((M; D_1, \ldots, D_k)\), then

\[
\partial_t \langle h_j^+, h_j^+ \rangle = -\langle(1/2) \Upsilon_{h_j^+, h_j^+}, B_j \rangle,
\]

\[
\partial_t \langle h_j, h_j \rangle = \langle \text{div} h_j + \mathcal{K}_j, B_j \rangle - \text{div} \langle h_j, B_j \rangle,
\]

\[
\partial_t \langle H_j^+, H_j^+ \rangle = -\langle H_j^+ \otimes H_j^+, B_j \rangle,
\]

\[
\partial_t \langle H_j, H_j \rangle = \langle (\text{div} H_j) g_j, B_j \rangle - \text{div}((\text{Tr}_{D_j} B_j^2) H_j),
\]

\[
\partial_t \langle T^+_j, T^+_j \rangle = \langle (1/2) \Upsilon_{T^+_j, T^+_j}, B_j \rangle,
\]

\[
\partial_t \langle T_j, T_j \rangle = \langle 2 T_j^2, B_j \rangle,
\]
Theorem 2

for the action (3) with respect to volume-preserving adapted variations of metric if and only if

\[ \partial_t \langle h_i^+ + (K_i^+)^\lambda, B_j \rangle = \langle \text{div} \, h_i^+ + (K_i^+)^\lambda, B_j \rangle, \]

\[ \partial_t \langle h_i, h_i \rangle = \langle -(1/2) \Upsilon_{h_i, h_i}, B_j \rangle, \]

\[ \partial_t g(H_i^+, H_i^+) = \langle (\text{div} \, H_i^+) g_i^+, B_j \rangle - \text{div}((\text{Tr} \, D_i^+(B_j)) H_i^+), \]

\[ \partial_t g(H_i, H_i) = -(H_i \otimes H_i, B_j), \]

\[ \partial_t (T_i^+, T_i^+) = \langle 2 (T_i^+) + , B_j \rangle, \]

\[ \partial_t (T_i, T_i) = \langle (1/2) \Upsilon_{T_i, T_i}, B_j \rangle. \]

For any variation \( g_t \) of metric \( g \) on \( M \) with \( B = \partial_t g \) we have

\[ \partial_t \left( \text{d} \, \text{vol}_g \right) = \frac{1}{2} \langle \text{Tr}_g B, \text{d} \, \text{vol}_g \rangle. \]  \hfill (13)

Differentiating the well-known formula \( \text{div} \, X \cdot \text{d} \, \text{vol}_g = \mathcal{L}_X (\text{d} \, \text{vol}_g) \) and using (13), we obtain

\[ \partial_t \left( \text{div} \, X \right) = \text{div}(\partial_t X) + \frac{1}{2} X (\text{Tr}_g B) \]  \hfill (14)

for any variation \( g_t \) of metric and a \( t \)-dependent vector field \( X \) on \( M \). By (14) and (13), using the Divergence Theorem, we have

\[ \frac{d}{dt} \int_M (\text{div} \, X) \, \text{d} \, \text{vol}_g = \int_M \text{div} (\partial_t X + \frac{1}{2} (\text{Tr}_g B) X) \, \text{d} \, \text{vol}_g = 0 \]  \hfill (15)

for any variation \( g_t \) with \( \langle \partial_t g \rangle \subset \Omega \), and \( t \)-dependent \( X \in \mathfrak{X}_M \) with \( \langle \partial_t X \rangle \subset \Omega \).

The following theorem allows us to restore the partial Ricci curvature (22). It is based on calculating the variations with respect to \( g \) of components in (8) and using (15) for divergence terms. By this theorem and Definition 3 we conclude that an adapted metric \( g \) is critical for the action \( \Lambda \) with respect to volume-preserving adapted variations of metric if and only if (14) holds.

**Theorem 2** (see [13]). A metric \( g \in \text{Riem}(M, \mathcal{D}_1, \ldots, \mathcal{D}_k) \) is critical for the geometrical part of (3) (i.e., \( \Lambda = 0 = \Theta \)) with respect to volume-preserving adapted variations if and only if

\[ \sum_{i \neq j} \left( \text{div} \, h_i^+ + (K_i^+)^\lambda \right) H_i \otimes H_i - \frac{1}{2} \Upsilon_{h_i, h_i} - \frac{1}{2} \Upsilon_{T_i, T_i} - 2 (T_i^+)^\lambda \]

\[ + \text{div} \, h_j + K_j^\lambda - \frac{1}{2} \Upsilon_{h_j, h_j} + H_j \otimes H_j - \frac{1}{2} \Upsilon_{T_j, T_j} - 2 T_j^\lambda \]

\[ = \left( S(\mathcal{D}_1, \ldots, \mathcal{D}_k) - \text{div}(H_j + \sum_{i \neq j} H_i^+) + \lambda_j \right) g \]  \hfill (16)

for some \( \lambda_j \in \mathbb{R} \) and \( 1 \leq j \leq k \).

**Proof.** Let \( g_t \) be a \( \mathcal{D}_j \)-variation for some \( j \), and let

\[ Q(g) := S(\mathcal{D}_1, \ldots, \mathcal{D}_k) - \text{div} \sum_i (H_i + H_i^+). \]

Then

\[ \frac{d}{dt} J_{\mathcal{D}, \Omega}(g_t) \big|_{t=0} = \frac{d}{dt} \int_{\Omega} Q(g_t) \, \text{d} \, \text{vol}_{g_t} \big|_{t=0} + \frac{d}{dt} \int_{\Omega} \text{div} \sum_i (H_i + H_i^+) \, \text{d} \, \text{vol}_{g_t} \big|_{t=0}. \]

For adapted variations of \( g \) supported in \( \Omega \), both fields \( \partial_t \sum_i (H_i + H_i^+) \) and \( \langle \text{Tr} \, B^2 \rangle \sum_i (H_i + H_i^+) \) vanish on \( \partial \Omega \), and by (14) we get \( \frac{d}{dt} \int_{\Omega} \text{div} \sum_i (H_i + H_i^+) \, \text{d} \, \text{vol}_{g_t} = 0 \). Thus, we have

\[ \frac{d}{dt} J_{\mathcal{D}, \Omega}(g_t) \big|_{t=0} = \frac{d}{dt} \int_{\Omega} Q(g_t) \, \text{d} \, \text{vol}_{g_t} \big|_{t=0}. \]
and $Q(g)$ can be presented using (8) and (11) as

$$Q(g) = \frac{1}{2} \sum_i (\langle T_i, T_i \rangle - \langle h_i, h_i \rangle + \langle H_i, H_i \rangle + \langle T_i^+, T_i^+ \rangle - \langle h_i^+, h_i^+ \rangle + \langle H_i^+, H_i^+ \rangle).$$

By Proposition 2, we get

$$\partial_t (\langle h_i^+, h_i^+ \rangle - \langle h_i, h_j \rangle + g(H_i^+, H_j^+)) + g(H_i, H_j) + \langle T_i^+, T_j^+ \rangle + \langle T_i, T_j \rangle)
\begin{align*}
= & \langle - \text{div} h_j - K_j - H_j^+ \otimes H_j^+ + (1/2) \nabla h_j, h_j^+ \rangle + (1/2) \nabla T_j^+, T_j^+ + 2 T_j^+ + (\text{div} H_j) g_j, B_j \\
& + \text{div}(\langle h_i^+, B_j \rangle - (\text{Tr} D_i^+ B_i^\otimes) H_j),
\end{align*}
\quad \text{and for } i \neq j \text{ we have}

$$\partial_t (\langle h_i^+, h_i^+ \rangle - \langle h_i, h_i \rangle + g(H_i^+, H_i^+)) + g(H_i, H_i) + \langle T_i^+, T_i^+ \rangle + \langle T_i, T_i \rangle)
\begin{align*}
= & \langle - \text{div} h_i^+ - (K_i^+) - H_i \otimes H_i + (1/2) \nabla h_i, h_i \rangle + (1/2) \nabla T_i^+, T_i^+ + 2 (T_i^+) + (\text{div} H_i) g_i, B_j \\
& + \text{div}(\langle h_i^+, B_j \rangle - (\text{Tr} D_i^+ (B_i^\otimes) H_i^+).\]
\end{align*}
\quad \text{We use the above to derive } \partial_t Q(g). \text{ Removing integrals of divergences of vector fields compactly supported in } \Omega, \text{ we get}

$$\int_\Omega \partial_t Q(g) |_{t=0} \, d \text{vol}_g = \frac{1}{2} \int_\Omega \left( \sum_{i \neq j} \langle - \text{div} h_i^+ - (K_i^+) - H_i \otimes H_i + \frac{1}{2} \nabla h_i, h_i \rangle + \frac{1}{2} \nabla T_i^+, T_i^+ + 2 T_i^+ + (\text{div} H_i) g_i, B_j \rangle \right) \, d \text{vol}_g,
\quad \text{where } B_j = \partial_t g_t |_{t=0}.
\begin{align*}
\text{by (17) and (13), we have}
\frac{d}{dt} J_{D, \Omega}(g_t) |_{t=0} = \int_\Omega \partial_t Q(g_t) |_{t=0} \, d \text{vol}_g + \int_\Omega Q(g) \left( \partial_t \text{d vol}_{g_t} |_{t=0} \right),
\end{align*}
$$\frac{d}{dt} J_{D, \Omega}(g_t) |_{t=0} = \int_\Omega \left[ \langle - \text{div} h_j - K_j - H_j^+ \otimes H_j^+ + \frac{1}{2} \nabla h_j, h_j^+ \rangle + \frac{1}{2} \nabla T_j^+, T_j^+ + 2 T_j^+ \right]
\begin{align*}
& + \sum_{i \neq j} \langle - \text{div} h_i^+ - (K_i^+) - H_i \otimes H_i + \frac{1}{2} \nabla h_i, h_i \rangle + \frac{1}{2} \nabla T_i^+, T_i^+ + 2 (T_i^+) \right]
\begin{align*}
& + \left( S(D_1, \ldots, D_k) - \text{div}(H_j + \sum_{i \neq j} H_i^+) g_j, B_j \right) \right] \, d \text{vol}_g.
\end{align*}
\quad \text{If } g \text{ is critical for } J_{D, \Omega} \text{ with respect to } D_j\text{-variations of } g, \text{ then the integral in (18)} \text{ is zero for any symmetric (0, 2)-tensor } B_j. \text{ This yields the } D_j\text{-component of Euler-Lagrange equation}

$$\begin{align*}
\text{div} h_j + K_j + H_j^+ \otimes H_j^+ - \frac{1}{2} \nabla h_j, h_j^+ - \frac{1}{2} \nabla T_j^+, T_j^+ - 2 T_j^+ \right]
\begin{align*}
& + \sum_{i \neq j} \langle \text{div} h_i^+ + (K_i^+) - H_i \otimes H_i - \frac{1}{2} \nabla h_i, h_i \rangle - \frac{1}{2} \nabla T_i^+, T_i^+ - 2 (T_i^+) \right]
\begin{align*}
& = \left( S(D_1, \ldots, D_k) - \text{div}(H_j + \sum_{i \neq j} H_i^+) \right) g_j.
\end{align*}
\end{align*}
\quad \text{For volume-preserving } D_j\text{-variations, the Euler-Lagrange equation (of the geometrical part of (3) with respect to volume-preserving adapted variations) will be (10) instead of (19).} \quad \Box
Remark 3. Using the partial Ricci tensor (7) and replacing div $h_j$ and div $h_i^\perp$ for $i \neq j$ in (10) according to (10), we can rewrite (16) as
\[
\sum_{i \neq j} (r_i^+ - \langle h_i^+, H_i^+ \rangle + (A_i^+)\mathbf{b} + (T_i^+)\mathbf{b} + \Psi_i - \text{Def}_{D_i} h_i + (K_i^+)\mathbf{b})
+ H_i \otimes H_i - \frac{1}{2} T_{h_i, h_i} - \frac{1}{2} T_{T_i, T_i} - 2(T_i^+)\mathbf{b} + r_j - \langle h_j, H_j \rangle + A_j^\perp + T_j^+ \\
+ \Psi_j^+ - \text{Def}_{D_j} h_j^\perp + K_j^+ + H_j^\perp \otimes H_j^\perp - \frac{1}{2} Y_{h_i^+, h_i^+} - \frac{1}{2} T_{h_i, h_i} - 2T_j^+ \\
= (S(D_1, \ldots, D_k) - \text{div}(H_j + \sum_{i \neq j} H_{i^+})) g_j + \lambda_j g_j, \quad j = 1, \ldots, k.
\] (20)

Example 2. A pair $(D_i, D_j)$ with $i \neq j$ of distributions on a Riemannian almost $k$-product manifold $(M, g; D_1, \ldots, D_k)$ is called mixed integrable, if $T_{i,j}(X, Y) = 0$ for all $X \in D_i$ and $Y \in D_j$, see (11). Let $(M, g; D_1, \ldots, D_k)$ with $k > 2$ has integrable distributions $D_1, \ldots, D_k$ and each pair $(D_i, D_j)$ is mixed integrable. Then $T_j^+(X, Y) = 0$ for all $l \leq k$ and $X \in D_l, Y \in D_j$ with $i \neq j$, see (11) Lemma 2. In this case, (20) reads as
\[
\sum_{i \neq j} (r_i^+ - \langle h_i^+, H_i^+ \rangle + (A_i^+)\mathbf{b} + \Psi_i - \text{Def}_{D_i} h_i + H_i \otimes H_i - \frac{1}{2} T_{h_i, h_i})
+ r_j - \langle h_j, H_j \rangle + A_j^\perp + \Psi_j^+ - \frac{1}{2} T_{h_j^+, h_j^+} + H_j^\perp \otimes H_j^\perp - \text{Def}_{D_j} h_j^\perp \\
= (S(D_1, \ldots, D_k) - \text{div}(H_j + \sum_{i \neq j} H_{i^+})) g_j + \lambda_j g_j, \quad j = 1, \ldots, k.
\]

Definition 3. The Ricci type symmetric $(0, 2)$-tensor $\text{Ric}_D$ in (4) is defined by its restrictions $\text{Ric}_{D|D_j \times D_j}$ on $k$ subbundles $D_j \times D_j$ of $TM \times TM$,
\[
\text{Ric}_{D|D_j \times D_j} = r_j - \langle h_j, H_j \rangle + A_j^\perp + T_j^+ + \frac{1}{2} T_{h_j, h_j} + K_j^+ + H_j^\perp \otimes H_j^\perp \\
- \frac{1}{2} T_{h_i^+, h_i^+} - \frac{1}{2} T_{T_i, T_i} + \sum_{i \neq j} (r_i^+ - \langle h_i^+, H_i^+ \rangle + (A_i^+)\mathbf{b} + (T_i^+)\mathbf{b})
+ \Psi_i - \text{Def}_{D_i} h_i + (K_i^+)\mathbf{b} + H_i \otimes H_i - \frac{1}{2} Y_{h_i, h_i} - \frac{1}{2} T_{T_i, T_i} - 2(T_i^+)\mathbf{b} + \mu_j g_j.
\]

In other words,
\[
\text{Ric}_{D|D_j \times D_j} = U_j + \mu_j g_j,
\]
where $U_j$ is the LHS of (10) and $(\mu_j)$ are uniquely determined so (see Theorem 3 below) that critical metrics satisfy Einstein type equation (4).

Theorem 3. A metric $g \in \text{Riem}(M, D_1, \ldots, D_k)$ is critical for the geometrical part of (3) (i.e., $\Lambda = 0 = \Theta$) with respect to adapted variations if and only if $g$ satisfies Einstein type equation (4), where the tensor $\text{Ric}_D$ is given in Definition 3 with some (uniquely defined) $\mu_i \in \mathbb{R}$.

Proof. The Euler-Lagrange equations (16) consist of $D_j \times D_j$-components. Thus, for the geometrical part of (3) we obtain (21). If $n = 2$ and $k = 2$, then we take $\mu_1 = \mu_2 = 0$, see (9). Assume that $n > 2$. Substituting (21) with arbitrary $(\mu_1, \ldots, \mu_k)$ into (4) along $D_j$, we conclude that if the Euler Lagrange equations $U_j = b_j g_j$ $(1 \leq j \leq k)$ hold, then
\[
\text{Ric}_D - (1/2) S_D : g = 0,
\]
see (4) with $\Lambda = 0 = \Theta$, if and only if $(\mu_j)$ satisfy the following linear system:
\[
\sum_{i} n_i \mu_i - 2 \mu_j = a_j, \quad j = 1, \ldots, k,
\] (21)
with coefficients $a_j = 2 b_j - \text{Tr} \sum_{i} U_i$. The matrix of (21) is
\[
A = \begin{pmatrix}
  n_1 - 2 & n_2 & \cdots & n_{k-1} & n_k \\
  n_1 & n_2 - 2 & \cdots & n_{k-1} & n_k \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  n_1 & n_2 & \cdots & n_{k-1} & n_k - 2
\end{pmatrix}.
\]

Its determinant: \( \det A = 2^{k-1}(2-n) \) is negative when \( n > 2 \). Hence, the system (21) has a unique solution \( (\mu_1, \ldots, \mu_k) \). It is given by \( \mu_i = -\frac{1}{2n-1} \left( \sum_j (a_i - a_j) n_j - 2 a_i \right) \).

**Example 3** (see [9]). The symmetric Ricci type tensor \( \mathcal{R}_{icD} \) in (4) with \( k = 2 \), is defined by its restrictions on two subbundles of \( TM \times TM \),

\[
\mathcal{R}_{icD}\big|_{D \times D^\perp} = r - \langle h^\perp, H^\perp \rangle + (A^\perp)^b - (T^\perp)^b + \Psi - \text{Def}_D H + (\mathcal{K}^\perp)^b \\
+ H^b \otimes H^b - \frac{1}{2} \mathcal{T}_{h,h} - \frac{1}{2} \mathcal{T}_{T,T} + \mu_1 g^\perp, \\
\mathcal{R}_{icD}\big|_{D\times D} = r - \langle h, H \rangle + A^b - T^b + \Psi^\perp - \text{Def}_{D^\perp} H^\perp + \mathcal{K}^b \\
+ (H^\perp)^b \otimes (H^\perp)^b - \frac{1}{2} \mathcal{T}_{h^\perp,h^\perp} - \frac{1}{2} \mathcal{T}_{T^\perp,T^\perp} + \mu_2 g^\perp,
\]

(22)

where \( \mu_1 = -\frac{n-1}{n-2} \text{div}(H^\perp - H) \) and \( \mu_2 = \frac{n-1}{n-2} \text{div}(H^\perp - H) \). Here (22)_2 is dual to (22)_1 with respect to interchanging distributions \( D \) and \( D^\perp \), and their last terms vanish if \( n_1 = n_2 = 1 \). Also, \( S_D := \text{Tr}_g \mathcal{R}_{icD} = S(D, D^\perp) + \frac{n-1}{n-2} \text{div}(H^\perp - H) \).

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