THE $C_2^n$ BOREL DUAL STEENROD ALGEBRA

NICK GEORGAKOPOULOS

Abstract. In this very short note, we expand the Hu-Kriz computation of the $G$-equivariant Borel dual Steenrod algebra in characteristic 2, from the group $G = C_2$ to all power-2 cyclic groups $G = C_{2^n}$.

Contents

1. Introduction 1
2. Conventions and notations 1
3. Borel cohomology 2
4. The Borel dual Steenrod algebra 3
5. Comparison with Greenlees’s description 4
References 5

1. Introduction

In this companion piece to [Geo21a], we show that the $C_2$-equivariant Borel dual Steenrod algebra computation in [HK96] generalizes to all groups $G = C_{2^n}$. More precisely, we give an explicit description of the $RO(C_{2^n})$-graded ring of the homotopy fixed points $(H \mathbb{F}_2 \wedge H \mathbb{F}_2)^{hC_{2^n}}$ as a Hopf algebroid over $(H \mathbb{F}_2)^{hC_{2^n}}$, where $\mathbb{F}_2$ stands for the constant $C_{2^n}$-Green functor associated to the field of two elements. We also compare our description to the dual description of the Borel Steenrod algebra of [Gre88].

Acknowledgment. We would like to thank Peter May for his numerous editing suggestions, including the idea to split off this paper from [Geo21a].

2. Conventions and notations

We will use the letter $k$ to denote the field $\mathbb{F}_2$ with trivial $G = C_{2^n}$ action, the constant $G$-Mackey functor $k = \mathbb{F}_2$ and the corresponding equivariant Eilenberg-MacLane spectrum $Hk$. The meaning should always be clear from the context.

Henceforth all our co/homology will be in $k$ coefficients. We use $k^\bullet(X)$ to denote the $RO(G)$-graded Mackey functor of $G$-equivariant homology in $k$-coefficients. The value of $k^\bullet(X)$ on the $G/H$ orbit is denoted by $k^H(X)$.

The real representation ring $RO(C_{2^n})$ is spanned by the irreducible representations $1, \sigma, \lambda_{s,k}$ where $\sigma$ is the 1-dimensional sign representation and $\lambda_{s,m}$ is the 2-dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1 \leq m$ dividing $2^{n-2}$ and odd $1 \leq s < 2^n/m$. Note that 2-locally, $S^{\lambda_{s,m}} \simeq S^{\lambda_{1,m}}$ as
C_{2n}\text{-equivariant spaces, by the s-power map. Therefore, to compute } k^G_\star(X) \text{ for } \star \in RO(C_{2n}) \text{ it suffices to only consider } \star \text{ in the span of } 1, \sigma, \lambda_k := \lambda_{1,2k} \text{ for } 0 \leq k \leq n - 2 (\lambda_n = 2\sigma \text{ and } \lambda_n = 2). 

For } V = \sigma \text{ or } V = \lambda_n, \text{ denote by } a_V \in k_{C_{2n}^V} \text{ the Euler class induced by the inclusion of north and south poles } S^0 \hookrightarrow S^V; \text{ also denote by } u_V \in k_{\overline{V}|V|-V} \text{ the orientation class generating the Mackey functor } k_{|V|-V} = k ([HHR16]).

3. Borel cohomology 

Let } EG \text{ be a contractible free } G\text{-space and } \bar{E}G \text{ be the cofiber of the collapse map } EG_+ \to S^0. \text{ For a spectrum } X \text{ we use the notation } X_h = EG_+ \wedge X, \ X^h = \bar{E}G \wedge X^h; \text{ there is a cofiber sequence } 

\[ X_h \to X^h \to X^t \]

The } G\text{-fixed points of } X_h, X^h, X^t \text{ are the nonequivariant spectra of homotopy orbits } X_{hG}, \text{ homotopy fixed points } X^{hG} \text{ and Tate fixed points } X^{IG} \text{ respectively.}

The orientation classes } u_V : k \wedge S^{|V|} \to k \wedge S^V \text{ are nonequivariant equivalences, hence induce } G\text{-equivalences in } X_h, X^h, X^t \text{ for a } k\text{-module } X, \text{ so they act invertibly on } X_h\star, X^h\star \text{ and } X^t\star. \text{ This implies that } 

\[ X_h\star \cong X_h|\star, \ X^h\star = X^h|\star, \ X^t\star = X^t|\star \]

and the } RO(G) \text{ graded part is determined by the integer graded part.}

Proposition 3.1. \text{ For } G = C_{2n} \text{ and } n > 1: 

\[ k^hG = k[a_\sigma, a_{\lambda_0}, u_{\sigma^2}, u_{\lambda_0}, ..., u_{\lambda_{n-2}}] / a_\sigma^2 \]

\[ k^IG = k[a_\sigma, a_{\lambda_0}^2, u_{\lambda_0}^2, u_{\lambda_0}^2, ..., u_{\lambda_{n-2}}] / a_\sigma^2 \]

and } k_{hG}\star = \Sigma^{-1} k^G / k^G \star (\text{forgetting the ring structure}). \text{ The map } k_{hG}\star \to k^G \star \text{ is trivial.}

Proof. \text{ The homotopy fixed point spectral sequence becomes: } 

\[ H^*(G;k)|u_{\sigma^2}, u_{\lambda_0}, ..., u_{\lambda_{n-2}}| \implies k^G \star \]

We have } H^*(G;k) = k^*BG = k[a]/a^2 \otimes k[b] \text{ where } |a| = 1 \text{ and } |b| = 2. \text{ The spectral sequence collapses with no extensions and we can identify } a = a_\sigma u_{\sigma^{-1}} \text{ and } b = a_{\lambda_0}u_{\lambda_0}. \text{ Finally, } \bar{E}G = S^{\infty\lambda_0} = \lim(S^{\lambda_0} \xrightarrow{a_{\lambda_0}} S^{\lambda_0} \xrightarrow{a_{\lambda_0}} \cdots) \text{ so to get } k^G \star \text{ we are additionally inverting } a_{\lambda_0}. \]

For } G = C_2 \text{ we have } 

\[ k^hC_2 = k[a_\sigma, u_\sigma^{-1}] \]

\[ k^{C_2} = k[a_{\sigma}, u_{\sigma}^2] \]

and } k_{hC_2}\star = \Sigma^{-1} k^{C_2} / k^{C_2} \star (\text{forgetting the ring structure}). \text{ The map } k_{hC_2}\star \to k^{C_2} \star \text{ is trivial.} \]
4. The Borel dual Steenrod algebra

The G-Borel dual Steenrod algebra is

\[(k \wedge k)^G\]

This is a Hopf algebroid over \(k^G\).

We will implicitly be completing it at the ideal generated by \(a_\sigma\) for \(G = C_2\), and at the ideal generated by \(a_{\lambda_0}\) for \(G = C_{2^n}\), \(n > 1\) (see [HK96] pg. 373 for more details in the case of \(G = C_2\)). With this convention, Hu-Kriz computed the \(C_2\)-Borel dual Steenrod algebra to be

\[(k \wedge k)^{hC_2} = k^{hC_2}[\xi_i] \]

for \(|\xi_i| = 2^i - 1 (\xi_0 = 1)\). The generators \(\xi_i\) restrict to the Milnor generators in the nonequivariant dual Steenrod algebra and

\[\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k\]

\[e(\xi_i) = 0, i \geq 1\]

\[\eta_R(a_\sigma) = a_\sigma\]

\[\eta_R(u_\sigma)^{-1} = \sum_{i=0}^{\infty} a_\sigma^{2^i - 1} u_\sigma^{-2^i} \xi_i\]

Proposition 4.1. For \(G = C_{2^n}\), \(n > 1\),

\[(k \wedge k)^{hG} = k^{hG}[\xi_i]\]

for \(|\xi_i| = 2^i - 1\) restricting to the \(C_{2^{n-1}}\) generators \(\xi_i\), with

\[\Delta(\xi_i) = \sum_{j+k=i} \xi_j^{2^k} \otimes \xi_k\]

\[e(\xi_i) = 0, i \geq 1\]

\[\eta_R(a_\sigma) = a_\sigma, \eta_R(a_{\lambda_0}) = a_{\lambda_0}\]

\[\eta_R(u_\sigma) = u_\sigma + a_\sigma \xi_1\]

\[\eta_R(u_{\lambda_m}) = u_{\lambda_m}, m > 0\]

\[\eta_R(u_{\lambda_0})^{-1} = \sum_{i=0}^{\infty} a_{\lambda_0}^{2^i - 1} u_{\lambda_0}^{-2^i} \xi_i\]

Proof. The computation of \((k \wedge k)^{hG} = (k \wedge k)^* (BG)\) follows from the computation of \(k^{hG} = k^* (BG) = k[a]/a^2 \otimes k[b]\) and the fact that nonequivariantly, \(k \wedge k\) is a free \(k\)-module. To see that the homotopy fixed point spectral sequence for \(k \wedge k\) converges strongly, let \(F^iBG\) be the skeletal filtration on the Lens space \(BG = S^m/C_{2^n}\); we can then compute directly that \(\lim (k \wedge k)^* (F^iBG) = \lim (k[a]/a^2 \otimes k[b]/b^i) = 0\).

Thus we get \((k \wedge k)^{hG} = k^{hG}[\xi_i]\) and the diagonal \(\Delta\) and augmentation \(e\) are the same as in the nonequivariant case. The Euler classes \(a_\sigma, a_{\lambda_0}\) are maps of spheres so they are preserved under \(\eta_R\). The action of \(\eta_R\) on \(u_\sigma, u_{\lambda_0}\) can be
computed through the right coaction on $k^hG$: The (completed) coaction of the nonequivariant dual Steenrod algebra on $k^*(BG) = k[a]/a^2 \otimes k[b]$ is

$$a \mapsto a \otimes 1$$

$$b \mapsto \sum_i b^{2i} \otimes 2^i$$

To verify the formula for the coaction on $b$ we need to check that $Sq^1(b) = 0$ (the alternative is $Sq^1(b) = ab$). From the long exact sequence associated to $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, we can see that the vanishing of the Bockstein on $b$ follows from $H^2(C_{2^n}; \mathbb{Z}/4) = \mathbb{Z}/4$ ($n > 1$).

After identifying $a = a_\sigma u^{-1}_\sigma$ and $b = a_\lambda u^{-1}_\lambda$ we get the formula for $\eta_R(u_\lambda)$ and also that

$$\eta_R(u_\sigma) = u_\sigma + \epsilon a_\sigma \xi_1$$

where $\epsilon$ is either 0 or 1. This is equivalent to

$$\eta_R(u^{-1}_\sigma) = u^{-1}_\sigma + \epsilon a_\sigma u^{-2}_\sigma \xi_1$$

and to see that $\epsilon = 1$ we use the map $k^hC_2 = k^h(C_{2^n}/C_{2^{n-1}}) \to k^hC_2^m$ that sends $a_\sigma, u_\sigma$ to $a_\sigma, u_\sigma$ respectively. Finally, to compute $\eta_R(u_\lambda)$ for $m > 0$ note that

$$k^hC_{2^{n-m}} = k^hC_{2^n} / C_{2^m} \to k^hC_2^m$$

sends $a_\lambda, u_\lambda$ to $a_\lambda, 0, u_\lambda$, respectively.

\[\square\]

5. Comparison with Greenlees' description

We now compare our result with the description of the Borel Steenrod algebra given in [Gre88], which is dual to our calculation.

In our notation, the $G$-spectrum $b$ of [Gre88] is $b = k^h$ and $b^V(X)$ corresponds to $(k^h)^V_G(X)$; to get $(k^h)^V_G(X)$ we need to multiply with the invertible element $u_V \in k^h_{V-V}$. The Borel Steenrod algebra is $b^G \ast b = (k^h)^G_G(k^h)$ and the Borel dual Steenrod algebra is $b^G \ast b = (k^h)^{G}\ast (k^h) = (k \land k)^{G}$.

Greenlees proves that the Borel Steenrod algebra is given by the Massey-Peterson twisted tensor product ((MP65)) of the nonequivariant Steenrod algebra $k^*k$ and the Borel cohomology of a point $(k^h)^{G} \ast = k^hG$. The twisting has to do with the fact that the action of the Borel Steenrod algebra on $x \in (k^h)^{G}(X)$ is given by:

$$(\theta \otimes a)(x) = \theta(ax)$$

where $\theta \in k^*k$ and $a \in k^hG$. The product of elements $\theta \otimes a$ and $\theta' \otimes a'$ in the Borel Steenrod algebra is not $\theta \theta' \otimes a a'$, since $\theta$ does not commute with cup-products, but rather satisfies the Cartan formula:

$$\theta(ab) = \sum_i \theta'_i(a) \theta''_i(b), \Delta \theta = \sum_i \theta'_i \otimes \theta''_i$$

Therefore:

$$(\theta \otimes a)(\theta' \otimes a')(x) = \theta(a \theta'(a' x)) = \sum_i \theta'_i(a)(\theta''_i \theta')(a' x)$$
so

\[(\theta \otimes a)(\theta' \otimes a') = \sum_i \theta'_i(a)(\theta'^{\prime}_i \theta' \otimes a')\]  

(1)

(we have ignored signs as we are working in characteristic 2).

So the Borel Steenrod algebra is \(k^* \otimes k^G\) with twisted algebra structured defined by (1).

Moreover, Greenlees expresses the action of \(k^*k\) on \((k^h)^G\) in terms of the action of \(k^*k\) on the orientation classes \(u\) and the usual (nonequivariant) action of \(k^*k\) on \((k^h)^G(X) = k^*(X \wedge G EG)\). This is done through the Cartan formula: If \(x \in (k^h)^G(X)\) then \(u^{-1}x \in (k^h)^G(X)\) and

\[\theta(x) = \theta(uu^{-1}x) = \sum_i \theta'_i(\theta'^{\prime}_i (u^{-1}x))\]

What remains to compute is \(\theta'_i(\theta'^{\prime}_i)\), namely the action of \(k^*k\) on orientation classes.

In our case, for \(G = C_{2n}\), we can see that:

**Proposition 5.1.** The action of \(k^*k\) on orientation classes is determined by:

\[Sq^i(u_{\sigma}) = \begin{cases} u_{\sigma} & i = 0 \\ a_{\sigma} & i = 1 \\ 0 & \text{otherwise} \end{cases}\]

\[Sq^i(u_{\lambda_m}) = \begin{cases} u_{\lambda_m} & i = 0 \\ a_{\lambda_0} & i = 2, m = 0 \\ 0 & \text{otherwise} \end{cases}\]

**Proof.** Compare with the proof of Proposition 4.1.

The twisting in the case of the Borel dual Steenrod algebra corresponds to the fact that \((k \wedge k)^G\) is a Hopf algebroid and not a Hopf algebra; computationally this amounts to the formula for \(\eta_R\) of Proposition 4.1.

**References**

[Geo21a] N. Georgakopoulos, *The RO\((C_4)\)* cohomology of the infinite real projective space*, available here

[Gre88] J.P.C. Greenlees, *Stable maps into free G-spaces*, Transactions of the American Mathematical Society, Volume 310, Number 1, November 1988

[HHR16] M. A. Hill, M. J. Hopkins, D. C. Ravenel, *On the non-existence of elements of Kervaire invariant one*, Annals of Mathematics, Volume 184 (2016), Issue 1

[HK96] P. Hu and I. Kriz, *Real-oriented homotopy theory and an analogue of the Adams Novikov spectral sequence*, Topology 40 (2001), no. 2, 317–399

[MP65] W. S. Massey and F. P. Peterson, *The cohomology structure of certain fibre spaces I*, Topology 4 (1965), 47-65

[Wil19] D. Wilson, *C\(_2\)*-equivariant Homology Operations: Results and Formulas, arXiv:1905.00058

**Department of Mathematics, University of Chicago**

*E-mail: nickg@math.uchicago.edu*

*Website: math.uchicago.edu/~nickg*