Analyzing critical propagation in a reaction-diffusion-advection model using unstable slow waves

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Abstract. The effect of advection on the propagation and in particular on the critical minimal speed of traveling waves in a reaction-diffusion model is studied. Previous theoretical studies estimated this effect on the velocity of stable fast waves and predicted the existence of a critical advection strength below which propagating waves are not supported anymore. In this paper, an analytical expression for the advection-velocity relation of the unstable slow wave is derived. In addition, the critical advection strength is calculated taking into account the unstable slow wave solution. We also analyze a two-variable reaction-diffusion-advection model numerically in a wide parameter range. Due to the new control parameter (advection) we can find stable wave propagation in the otherwise non-excitable parameter regime, if the advection strength exceeds a critical value. Comparing theoretical predictions to numerical results, we find that they are in good agreement. Theory provides an explanation for the observed behaviour.

1 Introduction

Traveling waves are basic patterns emerging in excitable media and are observed in many physical, chemical, and biological systems. In chemical systems, propagating excitation waves can be found in the Belousov-Zhabotinsky (BZ) reaction \cite{1,2}. Many important examples of excitation waves are found in biological systems, in particular, neuronal systems, such as the action potential, a wave of electrical depolarization that propagates along the membrane of a nerve cell axon with constant shape and velocity \cite{3}, or spreading depression (SD), a wave of sustained cell and tissue depolarization caused by a massive release of Gibbs free energy that propagates through gray matter tissue \cite{4,5}. Furthermore, intracellular calcium waves have been observed \cite{6,7}. In physical systems, a large variety of spatiotemporal patterns have been shown to occur during the oxidation of CO on a Pt(110) surface \cite{8–10}.

The above cases are described by reaction-diffusion processes, which possess reflection symmetry in space. This symmetry is broken by advection processes. Advection processes appear in a wide variety of systems. E.g., the advection term models the mean flow of ions driven by an externally applied constant electrical field \cite{11,12}. This case has been studied in the chemical BZ reaction \cite{13,14} and in some preliminary studies in cortical SD \cite{15}. Electrical and magnetic stimulation of brain tissue is also discussed in migraine therapy \cite{16,17}. Besides, advection processes take part in electrodifusion and osmotic water flow through cells and tissues \cite{18} and in cable models for neuronal dendrites \cite{19}. Reaction-diffusion-advection systems are also used to describe heterogeneous catalysis with in- and outflow of chemical species \cite{20–23}. Generic features of reaction-diffusion-advection models have been subject to detailed mathematical analysis, e.g., conditions for the existence, uniqueness and asymptotic stability of time periodic traveling wave solutions have been found \cite{24} and the influence of non-local coupling on dynamics of reaction-diffusion-advection systems has been analyzed \cite{25–28}. Besides, critical properties of traveling waves affected by advection have been discussed \cite{12,29,30}. In addition, in two-dimensional reaction-diffusion media, a small curvature of a wave front can also formally lead to an advection term under some approximations resulting in a reduced reaction-diffusion-advection description in one dimension \cite{12,29,31}. Front curvature effects have been observed in the BZ reaction \cite{32,33}.

As a model for traveling waves, we consider excitable media of activator-inhibitor type. This macroscopic description is used to study the generic behavior of traveling waves in reaction-diffusion-advection systems.

It has been shown that advection can have destructive and constructive effects on traveling waves, namely, slowing them down and annihilating them at a critical speed,
and accelerating them [29]. Here we show that in the non-excitatory parameter regime, which does not support traveling waves, advection can even induce stable propagation. We provide an analytical approximation and compare our results with numerical simulations.

2 Model

2.1 FitzHugh-Nagumo model

Let us firstly consider excitable media of activator-inhibitor type in one spatial dimension with diffusion,

\[
\begin{align*}
\frac{\partial u}{\partial t} &= f(u, v) + D \frac{\partial^2 u}{\partial x^2}, \\
\frac{\partial v}{\partial t} &= \varepsilon g(u, v),
\end{align*}
\]

with FitzHugh-Nagumo dynamics \([34–36]\) \(f(u, v) = 3u - u^3 - v\) and \(g(u, v) = (u + \beta + \gamma v)\). FitzHugh-Nagumo dynamics is chosen, as it provides a mathematically tractable excitatory medium of activator-inhibitor type.

The system has two variables \(u(x, t)\) and \(v(x, t)\) called activator and inhibitor, respectively, that depend on time \(t\) and space \(x\). The parameter \(D\) is the diffusion coefficient of activator \(u\). Inhibitor diffusion is assumed to be slow and hence negligible. The parameter \(\varepsilon\) is the time scale ratio between \(u\) and \(v\). \(\varepsilon\) has to be chosen much smaller than unity, because only slow inhibitor kinetics render dynamics excitatory. In the following, \(\gamma\) is set equal to zero.

The spatially homogeneous FHN system \((D = 0)\) has a single fixpoint, which is stable for \(\beta > 1\). Stability of the fixpoint is necessary for excitability. At \(\beta = 1\), a limit cycle occurs in a supercritical Hopf bifurcation, the fixpoint becomes unstable. Thus, for \(-1 < \beta < 1\), the system exhibits self-sustained periodic oscillations. In the following, we only consider the excitatory regime \((\beta < \beta < \sqrt{3})\).

The excitability behaviour of the system is, besides the strong separation of time-scales between the two variables \((\varepsilon \ll 1)\), crucially determined by the cubic nonlinearity of the activator equation. When the homogeneous system is perturbed by a sufficiently large (super-threshold) stimulus, it undergoes a large excursion in phase space, corresponding to a characteristic spike in the time evolution of the \(u\)-variable. Thereby, it starts and ends in the stable fixpoint (rest state). Small (sub-threshold) stimulation will result in fast relaxation. In principle, the transition from small to large amplitude excitation is continuous; in fact, however, phase space excursions of intermediate amplitude are very rare and extremely sensitive to the stimulus, i.e., slight changes in the stimulus induce large changes in model response. This causes the threshold-like behaviour of the FHN model.

In the following, we consider the spatially 1D extended system (eqs. (1)-(2) with \(D \neq 0\)). In the excitatory regime \((\beta > 1\) and \(\varepsilon\) and \(\beta\) below a critical value), the system has, besides the homogeneous rest state, two spatially inhomogeneous solutions, an unstable slow traveling wave and a stable fast traveling wave, i.e., waves that propagate with constant velocity \(c\) and constant wave profile \(u(x, t), v(x, t)\), see fig. 1.

![Fig. 1. Snapshots of the profile of activator \(u\) as a function of space \(x\) of a stable fast wave (red, solid line) and an unstable slow wave (blue, dotted line) propagating in negative \(x\)-direction numerically computed from eqs. (1)-(2). \(\varepsilon = 0.022, \beta = 1.6\).](image)

Traveling waves are stationary solutions in the co-moving coordinate \(\xi = x + ct\). Without loss of generality, we only consider waves propagating in negative \(x\)-direction. In co-moving coordinates, eqs. (1)-(2) can be transformed to

\[
\begin{align*}
\frac{\partial u}{\partial \xi} &= 3u - u^3 - v + D \frac{\partial^2 u}{\partial \xi^2}, \\
\frac{\partial v}{\partial \xi} &= \varepsilon (u + \beta).
\end{align*}
\]

2.2 Differential advection

An advection term, added to eq. (1) or eq. (3), may arise through different mechanisms.

First, an advection term in an 1D medium is an approximation of curved reaction-diffusion waves in spatially 2D media [29]. Propagating curved wave segments with \(R \ll L\), where \(L\) is the width of the wave segment and \(R\) is the curvature radius of the front, can be locally approximated by

\[
\begin{align*}
\frac{c(A)}{\partial \xi} &= 3u - u^3 - v + D \frac{\partial^2 u}{\partial \xi^2} + A \frac{\partial u}{\partial \xi}, \\
\frac{c(A)}{\partial \xi} &= \varepsilon (u + \beta),
\end{align*}
\]

with \(A = \frac{\partial}{\partial x}\), see appendix A. This effective 1D description is useful to calculate the properties of curved 2D wave segments, e.g., the dependency of the propagation velocity on the curvature or the maximal possible curvature of a wave front. Examples of stable curved reaction-diffusion waves are target pattern, spiral waves [37] or ring-shaped autowaves, that propagate on a torus and thus exhibit positive and negative Gaussian curvature [38, 39]. It has been shown, that the ring-shaped autowaves break up, when the negative Gaussian curvature exceeds a critical value.
Second, the same set of equations, i.e., eqs. (5)-(6), can be obtained, if one considers advection due to a constant external driving force. Both, activator $u$ and inhibitor $v$ can be associated with particles of different mobilities $\mu_u$ and $\mu_v$.

Particle motion can then be affected by a homogeneous external field, which is applied parallel to the propagation direction, e.g., ions or charged macromolecules are influenced by a homogeneous external electric field of strength $E$. This has been experimentally studied in the chemical BZ reaction with spiral waves [40], and Turing patterns influenced by external electrical fields have been studied in the chlorine dioxide-iodine-malonic acid reaction [41]. Then, eqs. (3)-(4) read

$$c(\xi) \frac{\partial u}{\partial \xi} = 3u - u^3 - v + D \frac{\partial^2 u}{\partial \xi^2} + \mu_u F \frac{\partial u}{\partial \xi},$$  
(7)

$$c(\xi) \frac{\partial v}{\partial \xi} = \varepsilon(u + \beta) + \mu_v F \frac{\partial v}{\partial \xi},$$  
(8)

where $F$ is the strength of the field and $\varepsilon E = -F$ with the valence $z$ of the ion.

In ref. [12] it is proposed to change the velocity of the co-moving frame to $\bar{c} = c - \mu_v F$. For $c = c(A)$, this yields

$$c(A) \frac{\partial u}{\partial \xi} = 3u - u^3 - v + D \frac{\partial^2 u}{\partial \xi^2} + A \frac{\partial u}{\partial \xi},$$  
(9)

$$c(A) \frac{\partial v}{\partial \xi} = \varepsilon(u + \beta),$$  
(10)

where $\xi = x - (c - \mu_v F)t$ and $A = F(\mu_u - \mu_v)$. Now, eqs. (9)-(10) and eqs. (5)-(6) are the same. In stationary coordinates, this reads

$$\frac{\partial u}{\partial \bar{t}} = 3u - u^3 - v + D \frac{\partial^2 u}{\partial \bar{x}^2} + A \frac{\partial u}{\partial \bar{x}},$$  
(11)

$$\frac{\partial v}{\partial \bar{t}} = \varepsilon(u + \beta).$$  
(12)

Let us briefly remark that in the excitable parameter regime, eqs. (11)-(12) have four wave solutions (two stable fast and two unstable slow ones) propagating in opposite direction. In the following, we only consider the two waves propagating in negative $x$-direction. This can be done without losing information, as the two waves propagating in positive $x$-direction influenced by advection of strength $A$ show the same behaviour as the two waves propagating in negative $x$-direction influenced by advection of strength $-A$.

In the remainder, $D$ is set to unity, what accords to scaling $x$ and $A$ with $\frac{1}{\sqrt{D}}$.

### 3 Theory

In this section, we derive an approximation for the critical velocity and the corresponding critical advection strength, sect. 3.3. To this end, we first define the propagation boundary, sect. 3.1, and then derive the advection-velocity relation for unstable waves in sect. 3.2.

#### 3.1 Propagation boundary

FitzHugh-Nagumo system without advection (eqs. (1)-(2)) $(1 < \beta < \sqrt{3}$ and $\varepsilon$ sufficiently small) have a stable fast wave solution and an unstable slow wave solution (fig. 1) which correspond to homoclinic orbits of the related ODE problem (eqs. (3)-(4)), see ref. [42]. There exists a critical line $\partial P$ in the $(\varepsilon, \beta)$ space, at which the fast wave branch coalesces with the slow wave branch, see fig. 2. For values of $\beta$ and $\varepsilon$ above this critical line, propagation of traveling waves cannot be obtained. These properties carry over to the case of finite advection strength $A$.

Thus it is reasonable to take into account the slow wave solution when calculating the critical properties, i.e., the critical surface in the $(\varepsilon, \beta, A)$ space, which separates the excitable and the non-excitable parameter regime and a critical velocity $c_{cr}$ depending on advection strength $A$.

#### 3.2 Advection-velocity relation for the fast and slow wave solution

The advection-velocity relation of the slow wave can be derived in the same way as the known advection-velocity relation of the fast wave (nonlinear Eikonal equation [29]). Introducing $c^*$ and $\varepsilon^*$

$$c^* = c(A) - A,$$  
(13)

$$\varepsilon^* = \varepsilon \frac{c^*}{c(A)},$$  
(14)

in eqs. (9)-(10) yields

$$c^* \frac{\partial u}{\partial \xi} = 3u - u^3 - v + D \frac{\partial^2 u}{\partial \xi^2},$$  
(15)

$$c^* \frac{\partial v}{\partial \xi} = \varepsilon^*(u + \beta),$$  
(16)
which has the same form as the FitzHugh-Nagumo model without advectio (eqs. (3)-(4)). Thus \( c^* \) has the same dependency on \( \varepsilon^* \) and \( \beta \) as the propagation velocity \( c|_{A=0} \) (see eqs. (3)-(4)) on \( \varepsilon \) and \( \beta \). The velocity \( c|_{A=0} \) for the fast and the slow wave can then approximately be calculated using a singular perturbation theory \[44\]. The propagation velocity of the fast, \( c^f|_{A=0} \), and the slow, \( c^s|_{A=0} \), wave is then obtained of

\[
\begin{align*}
c^f|_{A=0} &= c_0 + \varepsilon c^f_1, \\
c^s|_{A=0} &= \sqrt{\varepsilon} c^s_1,
\end{align*}
\]

which see fig. 2. The derivations of the expressions for \( c_0, c^f_1 \) and \( c^s_1 \) are provided in appendix C.1 and C.2, respectively.

For \( c^* \) (see eqs. (15)-(16)) we, therefore, obtain the expressions

\[
\begin{align*}
c^{f*} &= c_0 + \varepsilon c^f_1, \\
c^{s*} &= \sqrt{\varepsilon} c^s_1.
\end{align*}
\]

Inserting \( c^* \) and \( \varepsilon^* \) (eqs. (13)-(14)) and solving eq. (19) for \( c^f(A) \), we obtain the so-called nonlinear Eikonal equation

\[
c^f(A) = \frac{1}{2} \left( (A + c_0 + \varepsilon c_1) \pm \sqrt{(A + c_0 + \varepsilon c_1)^2 - 4\varepsilon A c_1} \right),
\]

where \( c^f(A) \) is the valid advection-velocity relation, see ref. \[29\].

To get the advection-velocity relation of the slow wave, we solve eq. (20) for \( c^s(A) \). The three solutions are

\[
\begin{align*}
c^s_+ (A) &= \frac{1}{2} \left( A + \sqrt{A^2 + 4\varepsilon c^2_1} \right), \\
c^s_0 (A) &= \frac{1}{2} \left( A - \sqrt{A^2 + 4\varepsilon c^2_1} \right), \\
c^s_0 (A) &= A.
\end{align*}
\]

The valid advection-velocity relation for the slow wave (with \( c^s(A) > 0 \)) is \( c^s_+ (A) \), because \( c^s|_{A=0} \equiv \sqrt{\varepsilon} c^s_1 \).

### 3.3 Critical velocity and critical advection strength

In the \((\varepsilon, \beta, A)\) parameter space, there exists a critical surface \((\varepsilon_{cr}, \beta_{cr}, A_{cr})\) of co-dimension one, which separates the excitable and the non-excitable parameter regime. At \((\varepsilon_{cr}, \beta_{cr}, A_{cr})\), the single holomorphic solution of eqs. (9)-(10) corresponds to the connection between the fast wave branch and the slow wave branch, and the propagation velocity of the stable fast wave is minimal \((c_{cr}(A_{cr}))\). This critical velocity \((c_{cr}(A_{cr}))\) is calculated here. Also an expression for \( A_{cr} \) is captured.

We want to mention that in ref. \[29\] an analytical expression for the critical velocity \( c_{cr} \) and the critical advection strength derived from nonlinear Eikonal equation (B.3) is proposed (see appendix B). This mathematical framework provides a good approximation for \( A < 0 \), but fails for \( A > 0 \), see fig. 3.

![Fig. 3. Critical advection strength \( A_{cr} \) as a function of threshold size \( \beta \). The grey dashed line shows the results from eq. (B.3), which was derived from the nonlinear Eikonal equation. The blue solid line shows the results computed from eqs. (11)-(12); the propagation boundary \( \partial P_{A=0} \) is computed from eqs. (1)-(2). \( \varepsilon = 0.022 \) in all cases.](image)

To calculate \( c_{cr}(A_{cr}) \) and \( A_{cr} \), we start from FitzHugh-Nagumo model without advection (eqs. (1)-(2)). The critical surface of co-dimension one is a line in the \((\varepsilon, \beta)\) parameter space. At this critical line \((\varepsilon_{cr}, \beta_{cr})\), the propagation velocity of the fast wave is minimal \((c_{cr}|_{A=0})\).

The critical time scale ratio \( \varepsilon_{cr} \) as a function of \( \beta \) can be approximated by solving

\[
\begin{align*}
c^f|_{A=0} &= c^f|_{A=0}, \\
\end{align*}
\]

for \( \varepsilon_{cr} \), where \( c^f|_{A=0} \) and \( c^s|_{A=0} \) are calculated using singular perturbation theory (eqs. (17)-(18)). This yields

\[
\varepsilon_{cr}(\beta) = -\frac{2c_0 c^f_1 + c^2_1 \pm \sqrt{-4c_0 c^f_1 c^2_1 + c^4_1}}{2c^2_1},
\]

where \( \varepsilon_{cr} < c^+ \) and thus \( \varepsilon_{cr} = \varepsilon_{cr}^- \).

For the critical velocity \( c_{cr}|_{A=0} \) as a function of \( \beta \) we obtain from eqs. (17)-(18)

\[
\begin{align*}
c_{cr}|_{A=0} &= c_0 + \varepsilon c^s_1 = \sqrt{\varepsilon_{cr}} c^s_1.
\end{align*}
\]

Advection changes the critical velocity. To obtain an analytical expression for \( c_{cr}(A_{cr}) \), we again start from eqs. (15)-(16), which have the same form as FitzHugh-Nagumo model without advection eqs. (3)-(4). Substituting \( c^* \) for \( c_{cr}|_{A=0} \) and \( \varepsilon^* \) for \( \varepsilon_{cr} \), the holomorphic solution of eqs. (15)-(16) ceases to exist at the connection between the fast wave branch and the slow wave branch. Thus, the critical velocity \( c_{cr}(A_{cr}) \) in systems affected by advection can be derived from eqs. (13)-(14) by setting \( c^* = c_{cr}|_{A=0} \) and \( \varepsilon^* = \varepsilon_{cr} \). With \( c^* = c(A) - A \) and \( \varepsilon^* = \varepsilon \frac{c^*}{c(A)} \) it follows, that

\[
\begin{align*}
c_{cr}|_{A=0} &= c_{cr}(A_{cr}) - A_{cr}, \\
\varepsilon_{cr} &= \frac{\varepsilon_{cr}|_{A=0}}{c_{cr}(A_{cr})},
\end{align*}
\]
where \( c_{cr}|_{A=0} \) is the minimal propagation velocity of the fast wave for \( A = 0 \) (eq. (27)) and \( c_{cr}(A_c) \) is the minimal propagation velocity of the fast wave, that can be achieved by influencing the system with critical advection strength \( A_{cr} \).

Solving eq. (29) for \( c_{cr}(A_{cr}) \) and eq. (28) for \( A_{cr} \), we finally obtain

\[
\begin{align*}
    c_{cr}(A_{cr}) &= \frac{\varepsilon}{c_{cr}}|_{A=0}, \\
    A_{cr} &= c_{cr}(A_{cr}) - c_{cr}|_{A=0} = c_{cr}|_{A=0} \left( \frac{\varepsilon}{c_{cr}} - 1 \right). 
\end{align*}
\]

Be aware that \( c_{cr}|_{A=0} \) eq. (27) as well as \( c_{cr} \) eq. (26) are fully determined by \( \beta \). Thus eq. (31) is an approximation for the critical surface in the \((\varepsilon, \beta, A)\) space, above which propagating waves are not supported. As a function of \( A \) and \( \beta \) it reads

\[
\varepsilon = \frac{(A + c_{cr}|_{A=0}) \varepsilon_{cr}}{c_{cr}|_{A=0}}. \tag{32}
\]

For values of \( \varepsilon \) above this critical surface, wave propagation is impossible, see fig. 4.

### 4 Numerical validation

Here, the analytical advection-velocity relation for the slow wave eq. (23) as well as the nonlinear Eikonal equation (21), which provides the advection-velocity relation in the fast wave, are compared with the propagation velocity of the fast, \( c'(A) \), and slow wave \( c^s(A) \) numerically obtained from eqs. (11)-(12) as a function of \( A \), (fig. 5a, b). Referring to systems without advection, the propagation velocity \( c(A) \) is decelerated for negative advection \( A < 0 \) and accelerated for positive advection \( A > 0 \). We find, that eq. (23) is in good accordance with numerical results. Close to the point where the fast wave branch and the slow wave branch meet, the advection-velocity relation for the slow wave deviates from numerical results, because perturbation theory does not capture the bifurcation behaviour. The deviation of the fast wave velocity calculated from eq. (21) from the numerically obtained results is relatively large. This deviation is a consequence of the inaccuracy of the fast wave velocity \( c'(A) \) calculated from eq. (17), which is a singular perturbation approximation up to first order of \( \varepsilon \), see fig. 2.

Furthermore, in sect. 3.3, we found an analytical expression for the critical propagation velocity \( c_{cr}(A_{cr}) \) (eq. (30)), which predicts an acceleration of the critical propagation velocity \( c_{cr}(A_{cr}) \) at larger threshold value \( \beta \), see fig. 6. For comparison, the propagation velocity of the fast, \( c'(A) \), and slow wave \( c^s(A) \) affected by advection of different strength \( A \) are numerically calculated as a function of \( \beta \) from eqs. (11)-(12), see fig. 6. The fast and the slow wave branch meet at a critical velocity \( c_{cr}(A_{cr}) \). We find that eq. (30) provides the same characteristic trend as numerical results.

Besides, numerical results in fig. 6 show a shift of the propagation boundary \( \partial P \) (connection between fast and slow wave branch) to smaller threshold \( \beta \) for negative advection \( A < 0 \) and to larger threshold \( \beta \) for positive advection \( A > 0 \). This behaviour is predicted by eq. (31),
see fig. 7. The critical line in the \((\beta, A)\) parameter space separates the excitable \((A > A_{cr})\) and the non-excitable \((A < A_{cr})\) parameter regime. We find, that eq. (31) provides the same characteristic trend as numerical results, but deviates strongly from numerical line for large negative advection strength \(A < 0\). This is due to the fact, that in this case \(\varepsilon^* = \varepsilon(1 - \frac{A}{c_{cr}(A)})\) (eq. (14)) is very large, and thus the singular perturbation theory is inaccurate.

A theoretical explanation of the stabilizing effect of positive advection has been found: every parameter point in the \((\varepsilon, \beta)\) space can be allocated a critical velocity (eq. (30)). Media without advection are excitable, if the propagation velocity of the fast wave is larger than this critical velocity (parameter regime above the critical line \(\varepsilon_{cr}(\beta)\) eq. (26)) and non-excitable, if the propagation velocity of the fast wave is smaller than this critical velocity (parameter regime below the critical line \(\varepsilon_{cr}(\beta)\)). Negative advection \(A < 0\) causes a deceleration of traveling waves, which in turn can induce a destabilization of an originally stable wave, if the fast wave is decelerated below the critical velocity \(c_{cr}(A_{cr})\) [29]. On the contrary, positive advection \(A > 0\) causes an acceleration of traveling waves, which in fact can induce stable propagation in the former non-excitable parameter regime, if the fast wave is accelerated above the critical velocity \(c_{cr}(A_{cr})\).

**5 Conclusion**

In this work, we described the dependency of the propagation velocity of an unstable slow traveling wave \(c^s(A)\) on advection of strength \(A\) analytically (eq. (23)) and numerically. Furthermore, we have shown, that positive advection \(A > 0\), corresponding to a constant field externally applied parallel to the propagation direction respectively corresponding to a small positive curvature (V-shaped pattern), can induce stable propagation of traveling waves in the non-excitable parameter regime. This behaviour is explained analytically: Every point in the \((\varepsilon, \beta)\) space, where \(\varepsilon\) is the time scale ratio and \(\beta\) is a measure for the threshold of the system, is related to a critical velocity \(c_{cr}(A_{cr})\) (eq. (30)). \(c_{cr}(A_{cr})\) is the propagation velocity at a saddle-node bifurcation of an unstable slow and a stable fast traveling wave solution, thus the minimal possible velocity of the fast wave solution. Stable wave propagation in the non-excitable parameter regime now is induced by accelerating the fast wave velocity above the critical velocity by affecting it with advection larger than a critical advection strength \(A_{cr}\) (eq. (31)). We derived an analytical approximation of a critical surface in the \((\varepsilon, \beta, A)\) space (eq. (32)), above which wave propagation is impossible. Finally, we confirmed numerically, that the calculated dependencies of the critical velocity \(c_{cr}(A_{cr})\) and the critical advection strength \(A_{cr}\) on \(\beta\) and \(\varepsilon\) are valid.

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**Appendix A. 1D description of curved 2D waves**

As described in [45], FHN system in two spatial dimensions

\[
\frac{\partial u}{\partial t} = f(u, v) + D \frac{\partial^2 u}{\partial x^2} + D \frac{\partial^2 u}{\partial y^2}, \quad (A.1)
\]

\[
\frac{\partial v}{\partial t} = \varepsilon g(u, v), \quad (A.2)
\]

can be written in polar coordinates with the variables \(r\) and \(\varphi\),

\[
\frac{\partial u}{\partial t} = f(u, v) + D \frac{\partial^2 u}{\partial r^2} + D \frac{\partial u}{r \partial r} + D \frac{\partial^2 u}{r^2 \partial \varphi^2}, \quad (A.3)
\]

\[
\frac{\partial v}{\partial t} = \varepsilon g(u, v). \quad (A.4)
\]

The curvature of pulse segments can be approximated by a section of a circle. Then, the front of a pulse segment lies at \(r = R\), with \(R\) being the curvature radius. As the pulse segment is locally symmetrical in \(\varphi\), \(\frac{\partial^2 u}{\partial \varphi^2} = 0\). The gradient of the activator \(\frac{\partial u}{R \partial r}\) only has considerable values at the location of the pulse segment, elsewhere \(\frac{\partial u}{R \partial r}\) is negligible. If the width of the pulse \(L\) is much smaller than the curvature radius \(R\), the approximation \(\frac{\partial u}{R \partial r} \approx \frac{\partial u}{\partial r}\) is valid, and thus curved 2D wave segments can be described by the 1D approximation

\[
\frac{\partial u}{\partial t} = f(u, v) + D \frac{\partial^2 u}{\partial r^2} + \frac{D}{R} \frac{\partial u}{\partial r}, \quad (A.5)
\]

\[
\frac{\partial v}{\partial t} = \varepsilon g(u, v). \quad (A.6)
\]
Appendix B. Critical advection strength derived from nonlinear Eikonal equation

The nonlinear Eikonal equation is given by (see eq. (21))

\[ c_±^f(A) = \frac{1}{2} \left( (A + c_0 + \varepsilon c_1) \pm \sqrt{(A + c_0 + \varepsilon c_1)^2 - 4\varepsilon A c_1} \right). \]  

(B.1)

The propagation velocity \( c_+^f(A) \) remains real only if the discriminant is larger than zero. Hence the limiting allowable advection strength \( A_{cr} \) is determined by

\[ (A_{cr} + c_0 + \varepsilon c_1)^2 - 4\varepsilon A_{cr}c_1 = 0. \]  

(B.2)

Solving eq. (B.2) for \( A_{cr} \) yields

\[ A_{cr}^± = - \left( c_0 - \varepsilon c_1 \pm 2\sqrt{-c_0\varepsilon c_1} \right). \]  

(B.3)

The critical advection strength \( A_{cr} \) is \( A_{cr}^+ \), because \( |A_{cr}^±| > |A_{cr}^+| \).

Appendix C. Calculating \( c^f|_{A=0} \) and \( c^i|_{A=0} \) using a singular perturbation theory

As proposed in [44], singular perturbation theory is used to find an approximation for the propagation velocity of the fast wave \( c^f|_{A=0} \) and the slow wave \( c^i|_{A=0} \). Since \( \varepsilon \) is a small parameter, the velocities \( c^f|_{A=0} \), \( c^i|_{A=0} \) and the profiles of the activator \( u(\xi) \) and the inhibitor \( v(\xi) \) can be represented as power series in \( \varepsilon \) and \( \sqrt{\varepsilon} \), respectively.

Appendix C.1. Fast wave velocity \( c^f|_{A=0} \)

In power series of \( \varepsilon \), \( c^f|_{A=0}, u(\xi) \) and \( v(\xi) \) read

\[ c^f|_{A=0} \approx c_0 + \varepsilon c_1^f + \varepsilon^2 c_2^f + O(\varepsilon^3), \]  

\[ u(\xi) \approx u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + O(\varepsilon^3), \]  

\[ v(\xi) \approx v_0(\xi) + \varepsilon v_1(\xi) + \varepsilon^2 v_2(\xi) + O(\varepsilon^3). \]  

Substituting this expressions into eqs. (3)-(4) and equating terms with the same power series of \( \varepsilon \) gives in zeroth and first order

\[ D \frac{\partial^2 u_0}{\partial \xi^2} - c_0 \frac{\partial u_0}{\partial \xi} + f(u_0) = v_0, \]  

(C.4)

\[ c_0 \frac{\partial v_0}{\partial \xi} = 0, \]  

(C.5)

\[ D \frac{\partial^2 u_1}{\partial \xi^2} - c_0 \frac{\partial u_1}{\partial \xi} + \frac{\partial f}{\partial u} u_1 = v_1 + c_1^f \frac{\partial u_0}{\partial \xi}, \]  

(C.6)

\[ c_0 \frac{\partial v_1}{\partial \xi} = u_0, \]  

(C.7)

with \( f(u) = 3u - u^3 \).

For \( c_0 \neq 0 \), \( v(\xi) = \text{const.} \) Thus, eq. (C.4) equals the so-called Schlögl-equation, which has an exact analytical solution.

For \( u_2^* < u_1^* + u_2^* \) with \( u_1^* \), \( u_2^* \) and \( u_3^* \) being the intersection points of the \( u \)-nullcline with the inhibitor fixpoint \( v_0 = -3\beta + \beta^3, u_1^* = -\beta, u_2^* = \frac{\beta}{2} - \sqrt{3-3/4\beta^2} \), and \( u_3^* = \frac{\beta}{2} + \sqrt{3-3/4\beta^2} \), and with the boundary conditions \( u_0(-\infty) = u_1^*, u_0(\infty) = u_3 \), eq. (C.4) describes a wave front that propagates to negative \( x \)-direction, with the profile

\[ u_0(\xi) = \frac{u_1^* + u_3^*}{2} + \frac{u_1^* - u_3^*}{2} \tanh \left( \sqrt{\frac{1}{2D}} \left( \frac{u_1^* - u_3^*}{2} \right) \right). \]  

(C.8)

From multiplying eq. (C.4) with \( \frac{\partial u_0}{\partial \xi} \) and integrating over \( \xi \) there follows

\[ c_0 = \frac{\int_{-\infty}^{u_3^*} f(u_0)du_0}{\int_{-\infty}^{\infty} \left( \frac{\partial u_0}{\partial \xi} \right)^2 d\xi}. \]  

(C.9)

This yields

\[ c_0 = \sqrt{\frac{D}{2} (u_1^* + u_3^* - 2u_2^*)}. \]  

(C.10)
To calculate $c_1^2$, eq. (C.6) is analyzed. The parameter $c_1$ is an eigenvalue. Since $\frac{\partial u_0}{\partial \xi}$ is an eigensolution of the corresponding homogeneous equation, the right-hand side is subject to the orthogonality condition

$$c_1 \int_{-\infty}^{\infty} \left( \frac{\partial u_0}{\partial \xi} \right)^2 e^{-c_0 \xi} \, d\xi + \int_{-\infty}^{\infty} \frac{\partial u_0}{\partial \xi} e^{-c_0 \xi} \, d\xi = 0.$$  

(C.11)

The correction to first order of $\varepsilon$ of the propagation velocity of the inner stable fast wave solution considering solitary waves thus is

$$c_1' = -\frac{\int_{-\infty}^{\infty} \frac{\partial u_0}{\partial \xi} e^{-c_0 \xi} \, d\xi}{\int_{-\infty}^{\infty} \left( \frac{\partial u_0}{\partial \xi} \right)^2 e^{-c_0 \xi} \, d\xi}. \quad (C.12)$$

$v_1$, the correction to first order of $\varepsilon$ of the inhibitor concentration (inner solution) of the fast wave, can be derived from eq. (C.7),

$$v_1(\xi) = \frac{1}{c_0} \int_{-\infty}^{\xi} (u_0(\mu) - u_0(-\infty)) \, d\mu \quad (C.13)$$

$$v_1(\xi) = \frac{1}{c_0} (u_3 - u_1) \left( \xi + \left( \varepsilon^{\frac{1}{2}} - \frac{\varepsilon^{\frac{1}{2}}}{u_3 - u_1} \ln \left( 1 + e^{-\frac{u_3 - u_1}{\varepsilon^{\frac{1}{2}}} \xi} \right) \right) \right). \quad (C.14)$$

**Appendix C.2. Slow wave velocity $c_{\text{sl}}'|_{\varepsilon=0}$**

By checking various possibilities, one finds that the following assumptions yield reasonable results:

$$c_{\text{sl}}'|_{\varepsilon=0} \approx c_0 + \sqrt{\varepsilon} c_1' + O(\varepsilon),$$  

(C.15)

$$u(\xi) \approx u_0(\xi) + \sqrt{\varepsilon} u_1(\xi) + O(\varepsilon),$$  

(C.16)

$$v(\xi) \approx v_0(\xi) + \sqrt{\varepsilon} v_1(\xi) + O(\varepsilon).$$  

(C.17)

We know, that $c_{\text{sl}}'|_{\varepsilon=0} = 0$ (critical nucleus solution of Schlögl model has velocity zero).

Substituting this expressions into eqs. (3)-(4) and equating terms with the same power series of $\sqrt{\varepsilon}$ gives

$$D \frac{\partial^2 u_0}{\partial \xi^2} + f(u_0) = v_0,$$  

(C.18)

$$D \frac{\partial^2 u_1}{\partial \xi^2} + \left. \frac{\partial f}{\partial u} \right|_{u=u_0} u_1 = v_1 + c_1' \frac{\partial u_0}{\partial \xi},$$  

(C.19)

$$c_1' \frac{\partial u_0}{\partial \xi} = u_0 + \beta.$$  

(C.20)

To solve this for $c_1'$, an orthogonality condition on the right-hand side of eq. (C.19) has to be imposed (this is possible, as the homogeneous equation corresponding to eq. (C.19) has $\frac{\partial u_0}{\partial \xi}$ as a solution), and $c_1$ has to be replaced by integration of eq. (C.20). This yields

$$(c_1')^2 = \frac{\int_{-\infty}^{\infty} u_0^2 \, d\xi}{\int_{-\infty}^{\infty} \left( \frac{\partial u_0}{\partial \xi} \right)^2 \, d\xi}. \quad (C.21)$$

The positive value of the square root should be used, thus

$$c_1^2 = \frac{2\sqrt{2m} - 2\ln \alpha}{\frac{2m}{\sqrt{2m}} - \frac{\alpha}{2}\sqrt{2m} + \frac{(l(2m-2))}{2} \ln \alpha}, \quad (C.22)$$

where $\alpha = \frac{\sqrt{\ln \frac{l\sqrt{2m}}{1-\sqrt{2m}}} - \frac{\alpha}{2}\sqrt{2m}}{l\sqrt{2m}}$ and $l = \frac{2}{3}(-2u_1 + u_2 + u_3)$ and $m = (u_2 - u_1)(u_3 - u_1)$. For details, see [44].

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