Thermal ensemble of string gas in a magnetic field

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ABSTRACT

We study the thermal ensemble of a gas of free strings in presence of a magnetic field. We find that the thermodynamical partition function diverges when the magnetic field exceeds some critical value $B_{cr}$, which depends on the temperature. We argue that there is a first-order phase transition with a large latent heat. At the critical value an infinite number of states -all states in a Regge trajectory- seem to become massless, which may be an indication of recuperation of spontaneously broken symmetries.
Among the mysteries clouding a fundamental formulation of string theory, the Hagedorn transition is, probably, the most arcane. It is found that the thermodynamical partition function of a free string gas diverges at some finite temperature \[1\]. The analogy with large \( N \) QCD \[2\] suggests that the Hagedorn temperature may not be a limiting temperature but rather an indication of another phase of the theory, perhaps where the description of physics in terms of strings is inadequate. This interpretation is supported by the results of refs. \[3\], where it is shown that at the Hagedorn transition a certain mode becomes massless. By studying the effective field theory near the Hagedorn temperature, Atick and Witten \[4\] argued that the transition should be first order with a large latent heat, due to a genus-zero contribution to the free energy above the Hagedorn temperature.

In quantum chromodynamics a clear evidence of partons appears in high-energy scattering processes. High-energy scattering also provides a way to recognize spontaneously broken symmetries. In the case of string theory, diverse studies in this direction were made in refs. \[5-8\]. In particular, in ref. \[7\] it was argued that in the high-energy limit string scattering amplitudes obey an infinite number of linear relations that are valid order by order in perturbation theory. This suggests the existence of an enormous symmetry which is restored at high energy.

In superconductivity, the restoration of the U(1) symmetry is achieved either by increasing the temperature or by increasing the magnetic field. This effect gives rise to the well-known Meissner curve separating the superconducting phase from the normal phase in type I superconductors. The analog of this phenomenon in the context of particle physics was explored in ref. \[9\], and more recently in ref. \[10\], where it was argued that spontaneously broken symmetries by the Higgs mechanism can be restored in the presence of a strong magnetic field.

In this paper we will further explore the Hagedorn transition by considering a string gas in a magnetic field. We will argue that a phenomenon analog to the case of superconductivity takes place in string theories. We will also argue that there are genus zero contributions to the free energy above the critical magnetic
field, and obtain the critical $B-T$ curve. In addition we will consider the heterotic string theory and find that an infinite number of physical particles become massless for approximately the same critical value of the magnetic field.\footnote{In the context of (zero-temperature) open string theory this was first pointed out in ref. \[11\], where it was also argued that this fact indicates a phase transition with possible restoration of symmetries.} We will find that this value of the magnetic field is precisely the critical value beyond which the thermodynamical partition function diverges. This result supports the view that there should exist an infinite-dimensional symmetry group governing string interactions, and it may provide a clue on the organization of multiplets.

Let us first derive an asymptotic formula for the level density of states with mass $M$ and angular momentum $J$ in the case of the bosonic open string theory. This calculation was done in \[12\] and we refer to this paper for details. Here we will present a simple, alternative derivation. We add to the world-sheet Hamiltonian a term containing the angular momentum in the $z$ direction with a Lagrange multiplier,

$$H = \sum_{n=1}^{\infty} \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i + \lambda J , \quad J = -i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1) .$$

The Hamiltonian can be diagonalized by $\alpha_n^1 = \sqrt{n/2} (a_n + b_n)$, $\alpha_n^2 = -i \sqrt{n/2} (a_n - b_n)$. One obtains for the partition function, $Z = \text{tr} \left[ e^{-\beta H} \right]$, the following expression:

$$Z = \prod_{n=1}^{\infty} \left[ (1 - w^n)^{-D+4} (1 - cw^n)^{-1} (1 - \frac{w^n}{c})^{-1} \right] ,$$

where $w \equiv e^{-\beta}$ and $c \equiv e^{\beta \lambda}$. Let us define $G = \log Z$ and consider

$$\frac{\partial G}{\partial \log c} = (c - c^{-1}) \sum_{k=1}^{\infty} \frac{w^k}{(1 - cw^k)(1 - c^{-1}w^k)} .$$
Inserting $c = e^{\beta \lambda}$ and taking the limit $\beta \to 0$ we obtain

$$\frac{\partial G}{\partial \lambda} = 2\lambda \sum_{k=1}^{\infty} \frac{1}{k^2 - \lambda^2}.$$  \hspace{1cm} (4)

The summation in eq. (4) can be performed explicitly. Indeed

$$\frac{\partial G}{\partial \lambda} = -\sum_{k=1}^{\infty} \left( \frac{1}{k + \lambda} - \frac{1}{k - \lambda} \right) = \psi(1 + \lambda) - \psi(1 - \lambda)$$

$$= \int_0^1 dx \frac{x^\lambda - x^{-\lambda}}{x - 1} = \frac{1}{\lambda} - \pi \cotg(\pi \lambda).$$ \hspace{1cm} (5)

Thus

$$G(\lambda) = \log \frac{\lambda}{\sin(\pi \lambda)} + G(0).$$ \hspace{1cm} (6)

The term $G(\lambda = 0)$ can be obtained by writing $Z$ in the following way:

$$Z(w, c) = \exp \left[ \sum_{m=1}^{\infty} \frac{1}{m} (c^m + c^{-m} + D - 4) \frac{w^m}{(1 - w^m)} \right].$$ \hspace{1cm} (7)

and setting $c = 1$. The leading term as $\beta \to 0$ is $Z(w, 0) = e^{a^2 \beta}$, $a \equiv \pi \sqrt{(D - 2)/6}$.

In this process subleading, power-like factors are neglected. Thus we obtain

$$Z(\beta, \lambda) \cong \text{const.} e^{\frac{a^2}{\beta}} \frac{\lambda}{\sin(\pi \lambda)}.$$ \hspace{1cm} (8)

This estimate is in agreement with the more accurate calculation of ref. [12].

By expanding $Z$, $Z(w, k) = \sum_{n,J} d_{n,J} w^n e^{i k J}$, $k = -i \beta \lambda$, then $d_{n,J}$ can be found by

$$d_{n,J} = \frac{1}{2\pi i} \oint \frac{dw}{w^{n+1}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikJ} Z(w, k).$$ \hspace{1cm} (9)

where the contour goes over a small circle around $w = 0$. These integrals were
carried out in ref. [12], with the result

\[ d_{n,J} \cong \text{const.} e^{(n+1)\beta+\alpha^2/\beta} \frac{1}{\cosh^2(\beta J/2)}, \quad \beta \equiv \frac{\alpha}{\sqrt{n+1-|J|}}. \]  

(10)

The constant can be chosen so that \( d_{n,J} = 1 \) on the Regge trajectories \( J = \pm n \).

Let us consider a canonical ensemble of free string gas in the presence of a magnetic field in the \( z \) direction. We have

\[ Z(T, B) = \int_0^\infty dn \int_{-n}^n dJ d_{n,J} e^{-\beta E}, \] 

(11)

where

\[ E^2 = m^2 + 2qB(l + 1/2) - 2JqB + O(B^2), \quad \alpha'm^2 = n - 1 \]

(12)

\( l \) represent the Landau level, and \( q \) is the total Chan-Paton charge of the open string. A sum over \( l \) is also understood in eq. (11). In eq. (12) we have used the fact that all physical states in open string theory have gyromagnetic factor equal 2 [13, 12]. The terms of \( O(B^2) \) have various origins. In particular, a magnetic field generates non-trivial corrections to sigma-model backgrounds starting from \( O(B^2) \). Let us disregard for the moment \( O(B^2) \) terms. From eq. (12) there seems to be a critical magnetic field at which some states become tachyonic, as noted in ref. [11]. The first states to become tachyonic are those with maximum spin, \( J = n \), at a magnetic field

\[ qB_{\text{cr}} \cong \frac{1}{2\alpha' - \frac{1}{m^2}}. \]

(13)

The state on the Regge trajectory with \( \alpha'm^2 = 1 \) will become tachyonic at \( qB \cong 1/\alpha' \). For large \( m^2 \) all the states on the first Regge trajectory will become tachyonic at \( qB_{\text{cr}} \cong 1/2\alpha' \). Massless states with spin 1 will be tachyonic for an infinitesimal value of the magnetic field. This last effect may be separated, as will be done in the case of the heterotic string theory.
The emergence of tachyons is a clear sign of instability. It is energetically more favorable to produce pairs with maximum spin aligned with the magnetic field. As a result, the partition function (11) will diverge for $B$ exceeding the critical value. This can be combined with another well-known effect. At zero magnetic field, the partition function diverges if the temperature is above the Hagedorn temperature, which for the bosonic open string theory is $T_H = 1/4\pi\sqrt{\alpha'}$. A critical curve $B - T$, analogous to the Meissner curve in superconductivity, can be obtained by inserting eq. (10) into eq. (11) and analysing the convergence properties. For this purpose power-like dependence can be neglected relative to the exponential factors. Let us write $J = \alpha n$ and analyse the integrand of (11) in the region $-1 < \alpha < 1$, $n \to \infty$. By using eqs. (12) and (10) we obtain the conditions

$$\alpha = 0 : \quad d_n J e^{-\beta E} \to 0 \iff T < T_H, \quad T_H = 1/4\pi\sqrt{\alpha'}$$

$$\alpha = 1 : \quad d_n J e^{-\beta E} \to 0 \iff B < B_{cr}, \quad B_{cr} = 1/2\alpha'$$

One can verify that these are necessary and sufficient conditions for convergence in all the region $-1 < \alpha < 1$, $n \to \infty$.

The phase diagram is displayed in Fig. 1. Our analysis only provides a rough estimate. It is plausible that a more careful and systematic inclusion of $O(B^2)$ and other effects will smooth out the critical curve. In particular, if the transition is first order one expects that the actual critical temperature at zero field is below the Hagedorn temperature [4]. The derivative of $B$ in the coexistence curve $\frac{dB_{cr}}{dT}$ at $T = 0$ should remain zero in a full treatment. There is a thermodynamical reason for this. Indeed, on the coexistence curve the chemical potentials of the phases must be equal, and thus one has $S_n - S_s \propto \frac{dB_{cr}}{dT}$. But the third law of thermodynamics implies that $S_n - S_s \to 0$ as $T \to 0$.

In any case, it is possible that the critical line is actually not well defined, since the notion of temperature may cease to be valid at the Planck scale.
In superconductivity the critical magnetic field $B_{cr}$ is related thermodynamically to the difference of the Helmholtz free energy density of the two phases. One has

$$\frac{B_{cr}^2}{8\pi} = \frac{F_n(T) - F_s(T)}{V}.$$  

The curve $B_{cr}(T)$ is quite well approximated by a parabolic law, $B_{cr}(T) \approx B_{cr}(0)\left[1 - \left(\frac{T}{T_{cr}}\right)^2\right]$.

![Diagram](image)

**FIGURE 1.** Phase diagram $B - T$.

By using eq. (10), in the case of the heterotic string theory one gets ($n_L \equiv n_R \equiv n$)

$$d_{n,J_L,J_R} \approx \text{const.} \frac{e^{(n+1)(\beta_L + \beta_R) + a_L^2/\beta_L + a_R^2/\beta_R}}{\cosh^2(\beta_L J_L/2) \cosh^2(\beta_R J_R/2)} \frac{a_L}{\sqrt{n + 1 - J}} \frac{a_R}{\sqrt{n + 1 - |J|}},$$

(14)

where $a_L = 2\pi$ and $a_R = \sqrt{2}\pi$ and $J_{L,R}$ represent the left and right contribution to the angular momentum. Thus

$$d_{n,J} = \int_{-n}^{n} dJ_L d_{n,J_L,J - J_L} = \int_{-n}^{n} dJ_R d_{n,J_R,J - J_R}.$$  

(15)
The energy formula can be obtained by using the gyromagnetic coupling derived in ref. [12]. One obtains

\[ E^2 = m^2 + 2qB(l + \frac{1}{2}) - 2J_R qB + O(B^2), \quad \alpha'm^2/2 \cong 2n, \quad n \gg 1. \quad (16) \]

Again there appears to be a limiting or critical value for the magnetic field. It may be convenient to go away from the self-dual point, so that all the particles with gyromagnetic coupling have mass. When

\[ qB_{cr} \cong \frac{2}{\alpha' - \frac{2}{m^2}}, \quad (17) \]

all the states on the Regge trajectory with \( J_R = n \) become massless. Let us consider the partition function

\[ Z(T, B) = \int_{-\infty}^{\infty} dn \int_{-n}^{n} dJ_L \int_{-n}^{n} dJ_Re^{-\beta E} \quad (18) \]

The peculiar coupling of the magnetic field with the right contribution to the angular momentum is characteristic of heterotic string theories where the gauge quantum numbers solely arise from the left sector [12]. A similar analysis as in the open string case leads to the following convergence region:

\[ qB < 2/\alpha', \quad T < T_H^{\text{het}}, \quad (19) \]

with \( T_H^{\text{het}} = 1/(\sqrt{\alpha'}(a_L + a_R)) = \frac{1}{\pi \sqrt{\alpha'}}(1 - \frac{1}{\sqrt{2}}). \)

A natural question is whether the critical magnetic field is not an artifact of having ignored \( O(B^2) \) terms. An complete treatment including all orders in the \( \alpha' \) expansion (i.e. including all powers of \( B \)) would be necessary in order to answer this question. Certainly it would be important to verify the existence of a critical magnetic field in some specific, exactly solvable example. However, it
should be stressed that there is no reason why higher orders in $B^2$ should stabilize the vacuum. It does not occur in the case of superconductivity and it does not seem to occur in the case of the electroweak model [9, 10]. $O(B^2)$ corrections to the geometry, etc. can only lead to further instabilities of different nature, as e.g. gravitational collapse.

Another way to derive a $B - T$ curve is by considering heterotic string with a compactified (euclidean) time dimension. Let us consider the right-moving modes in the NS sector (the analysis at $B = 0$ was carried out in ref. [4]). In this case the energy is given by

$$\alpha' E^2 = -3 + \frac{4\alpha' \pi^2 n_m^2}{\beta^2} + \frac{n_w^2 \beta^2}{4\alpha' \pi^2} - 2J_R\alpha' qB + 2N + 2\tilde{N} + O(B^2),$$

(20)

where $n_m$ and $n_w$ respectively denote quantized momentum and winding number. At zero field the first state to become tachyonic is that with $N = \tilde{N} = 0$ and $n_w = \pm 1, n_m = \pm 1/2$. This has $J_R = 0$ and hence it does not have gyromagnetic coupling. The curve is simply a vertical line at the Hagedorn temperature. The first charged state with $J_R \neq 0$ is given by $\tilde{N} = 1, N = 1/2, n_m = 0$ and $n_w = 2$ ($n_w = 1$ is excluded by GSO projection -for odd $n_w$ it is reversed relative to the standard projection). One obtains the curve $qB = 1/(\alpha' \pi T)^2 + O(B^2)$. This curve lies entirely above the coexistence curve (19) and so it is not very relevant; the phase transition already occurs for smaller fields. Above this curve the thermodynamical partition function, obtained by calculating genus $\geq 1$ contributions in the finite temperature theory ($X^0 = X^0 + \beta$), will develop another divergence in virtue of the appearance of the new tachyon.

In superconductivity the transition at zero magnetic field at $T_{cr}$ is second order, but in the presence of a magnetic field there is a discontinuous change in the thermodynamical state of the system with an associated latent heat, and the transition is of first order. By a similar analysis as in ref. [4], using the effective field theory of a state of level $n$ on the Regge trajectory that becomes tachyonic, one
obtains that the free energy for $B > B_{cr}$ is given by

$$F \sim -\text{const.} \frac{n^2}{g^2} (4/\alpha' - 2qB)^2,$$

where $g$ is the string coupling. This represents a genus zero contribution, just as it happens at zero field above the Hagedorn temperature. A genus-zero contribution cannot arise on simply connected Riemann surfaces. In large $N$ QCD, above the deconfining transition, continuum Riemann surfaces have to be replaced by Feynman diagrams.

In refs. [4, 7] it was argued that string theory might describe the spontaneously broken phase of a highly symmetric theory. This view is supported by what we have found here: the partition function diverges and an infinite number of particles become massless at approximately the same value of the magnetic field. This suggests that an enormous gauge symmetry is being restored. This symmetry would relate higher spin particles, somehow circumventing the Coleman-Mandula theorem, which asserts that the maximum spin of a conserved charge cannot exceed 1, but assumes that the number of particles with masses below any given scale is finite. This hypothesis does not apply in the $B \to B_{cr}$ limit, where symmetries would be recuperated, since infinitely many particles are becoming massless simultaneously. An interesting problem, which could unravel symmetries, is deriving an effective field theory for all particles in the Regge trajectory in a sigma-model background $B$ near $B_{cr}$.

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