Abstract

This paper presents a formulation of the reactive dynamic user equilibrium problem in continuum form using a network-level Macroscopic Fundamental Diagram (MFD). Compared to existing continuum models for cities – all based in Hughes’ pedestrian model in 2002 – the proposed formulation (i) is consistent with reservoir-type models of the MFD literature, shedding some light into the connection between these two modeling approaches, (ii) can have destinations continuously distributed on the region, and (iii) can incorporate multi-commodity flows without additional numerical error. The proposed multi-reservoir numerical solution method treats the multi-commodity component of the model in Lagrangian coordinates, which is the natural representation to propagate origin-destination information (and any vehicle-specific characteristic) through the traffic stream. Fluxes between reservoir boundaries are computed in the Eulerian representation, and are used to calculate the speed of vehicles crossing the boundary. Simple examples are included that show the convergence of the model and its agreements with the available analytical solutions. We find that (i) when origins and destinations are uniformly distributed in a region, the distribution of the travel times can be approximated analytically, (ii) the magnitude of the detours from the optimal free-flow route due to congestion increase linearly with the inflow and decreases with the square of the speed, and (iii) the total delay of vehicles in the network converges to the analytical approximation when the size of reservoirs tends to zero.

1. Introduction

Dynamic traffic assignment (DTA) models using an aggregated representation of the network originate from the continuum pedestrian theory proposed by Hughes (2002) and have become increasingly popular after the empirical verification of a network-level Macroscopic Fundamental Diagram (MFD) on congested urban areas (Daganzo, 2007, Geroliminis and Daganzo, 2008). According to Aghamohammadi and Laval (2018) existing macroscopic DTA models can be classified into discrete-space and continuum-space models, depending on how the link-level network is represented.

Discrete-space models partition the network into disjoint subregions where the average link flow is a known function of the accumulation \( n \), the flow-MFD \( Q(n) \) (see e.g., Yildirimoglu and Geroliminis, 2014, Yildirimoglu et al., 2015). Dynamics are approximated using a reservoir model (Daganzo, 2007), see Fig. 1(a). The key to the reservoir model is that the outflow is assumed to be the network production divided by the average trip length.

Continuum-space models started earlier with the seminal work by Hughes (2002), and are typically formulated as a system of two partial differential equations (PDE), a two-dimensional (2-D) conservation law for the density, \( \rho \), coupled with a PDE for the route choice, typically of the Eikonal or Hamilton-Jacobi type, either for vehicular traffic (see e.g., Jiang et al., 2011, Du et al., 2013, 2015, Lin et al., 2017) or for pedestrian flow (see e.g., Huang et al., 2009a,b, Jiang et al., 2009, 2010, 2012, Hoogendoorn and Bovy, 2004, Hoogendoorn et al., 2015).

Unfortunately, the connection between these two modeling approaches is not well understood. To the best of our knowledge, there are only three references in the continuum literature that briefly mention the MFD to justify the
use of a speed-density relationship, \( V(\rho) \), in the conservation law \( Q(\rho) = \rho V(\rho) \), and no references in the MFD literature citing continuum models. Aghamohammadi and Laval (2018) delves into this gap in the literature and argues that the speed-density relationships in the continuum models need to consider the network effects and hence, it is valid to interpret them as MFD. However, it is still not clear how the flux function for the conservation law \( Q(\rho) = \rho V(\rho) \) and \( \bar{Q}(n) \) are related, or how the average trip length affects the continuum model. It is the purpose of this paper to fill this void by reformulating the theory such that both modeling approaches are consistent, in the sense that taking the limit of the reservoir size going to zero leads to the continuum model, and vice versa, the spatial integration of the continuum model leads to the reservoir model.

As pointed out in Aghamohammadi and Laval (2018), the numerical solution methods that have been proposed to solve both the conservation law and route choice component of continuum models correspond to the standard numerical solution methods for PDEs. These methods converge to exact solutions only when the mesh size of the numerical grid tends to zero, which could be problematic if fast computations are required; otherwise, significant numerical viscosity can be introduced even in the case of a single commodity (i.e. one type of users, all going to the same destination). For multi-commodity problems, the formulation becomes cumbersome and the numerical solution methods introduce additional errors that can lead to first-in-first-out (FIFO) violations (Jin and Jayakrishnan, 2005).

We argue that these challenges in current continuum models arise because they were formulated and solved numerically in Eulerian coordinates, which are attached to the infrastructure. A more natural representation of this problem is achieved in Lagrangian coordinates which are attached to the vehicle particles. In this representation, origin-destination information (and any vehicle-specific characteristic) is propagated naturally in the traffic stream (Leclercq et al., 2007, Laval and Leclercq, 2013). However, in a fully Lagrangian representation, it is cumbersome to keep track of variables that are fixed in space such as demand inflow or changes in the infrastructure, or the flux transfers between neighboring reservoirs, in particular, which are fixed in space.

Here, we propose using a semi-Lagrangian scheme that uses Eulerian and Lagrangian representations where it is most convenient. These schemes use an Eulerian computation grid where the motion of particles is tracked. Using these trajectories, at every time step we estimate the continuum variables at the grid points. This general idea has been applied copiously in the field of atmospheric sciences (Staniforth and Cote, 1991). In traffic flow, there have been efforts to combine these two coordinate systems in one spatial dimension for the kinematic wave model (Leclercq, 2007) using a microscopic car-following model for the road segments, and a macroscopic network junction model for merges. Here, we adapt Leclercq’s (2007) framework for the two-dimensional case and modify the microscopic components in the spirit of the MFD; i.e. instead of a car-following model where each vehicle can have a different speed, we impose that all vehicles traveling from one reservoir to an adjacent one have the same speed. The calculation of this speed is not trivial. We found that the conventional approach to assign the MFD speed \( V(\rho) \), as in Yildirimoglu and Geroliminis (2014), Yildirimoglu et al. (2015), Leclercq et al. (2017), might lead to stalling in the multi-reservoir setting. To avoid stalling, and in the spirit of the kinematic wave model, the speed is taken directly from the network junction model.

While the relevant literature contains no efforts to achieve a consistent formulation both in discrete and continuum space, the Eulerian components of our solution method are closely related to the framework proposed in Sossoe and Lebacque (2017), Hanseler et al. (2014, 2017), Sossoe and Lebacque (2017) proposes a multi-reservoir reactive DTA model where the flows between reservoirs are obtained with a network junction model (see e.g., Jin, 2012, Jin and
Zhang (2003) and references therein). This model considers only two possible paths in each cell for the route choice based on a logit model, where the cost of each path is assumed to be the instantaneous travel time. There is no mention of the MFD or of the corresponding continuum model, however. Hanseler et al. (2014) develops a discrete-time discrete-space pedestrian flow model, named PedCTM, which extends Daganzo’s Cell Transmission Model (CTM; Daganzo, 1994a, 1995) to 2 dimensions. The model in Hanseler et al. (2017) is analogous to the PedCTM model, where the cell-based fundamental diagram, potential fields and path choice are replaced by stream-based ones and anisotropy is also taken into account.

This paper is organized as follows: section 2 provides background on MFD theory and existing continuum models for cities, section 3 formulates the consistent continuum MFD formulation and discusses the properties of the solutions in both isotropic and anisotropic cases. Section 4 proposes the semi-Lagrangian solution method, two examples are provided in section 5 showing the convergence properties of the solution method, while section 6 includes a discussion.

2. Background

2.1. Discrete-space models based on MFD theory

For a given region on a traffic network, the MFD gives the average traffic state in the region as a function of the number of vehicles inside this region (accumulation). The main assumptions are: (i) congestion is homogeneously distributed across the network, i.e. there are no “hot spots” in the network, and (ii) each lane of the network obeys the LWR kinematic wave model (Lighthill and Whitham, 1955; Richards, 1956) with common fundamental diagram. However, the second assumption is only necessary for analytical derivation of MFD (see e.g., Daganzo and Geroliminis, 2008) and MFD can still exist in some cases even if this assumption is not met.

The shape of the MFD depends on network topology and control parameters such as block length, the existence of turn-only lanes, and traffic light settings. Laval and Castrillón (2015) show that (the probability distribution of) the MFD can be well approximated by a function of mainly two parameters: the density of traffic lights and the mean red to green ratio across the network. Since these are observable parameters, we now have a simple method for estimating the MFD on arbitrary road networks.

2.1.1. Reservoir models

If the trip length, $\ell$, is assumed to be identical for all commuters on a single region, one can construct a reservoir (or bathtub) model to approximate the evolution of the accumulation, $n$, of vehicles inside the reservoir, and estimate any average traffic variable of interest using the MFD (Daganzo, 2007), see Fig. 1(a). The reservoir model simply states the conservation of vehicles:

$$n'(t) = \Lambda(t) - M(n),$$

(1)

where $\Lambda(t)$ is the demand inflow into the network at time $t$, and $M(n)$ is the outflow MFD, i.e. the number of trip completions per unit time, and is given by:

$$M(n) = \bar{Q}(n) L/\ell,$$

(2)

(outflow MFD)

where $\bar{Q}(n)$ is the average network flow MFD and $L$ corresponds to the total lane-miles of the network. Fig. 1(b) illustrates a typical outflow-MFD.

A crucial point when the reservoir is congested is whether or not demand should be restricted to the average network outflow; i.e.,

$$\Lambda(t) \leq M(n(t)).$$

(demand constraint in congestion)

(3)

This appears to be a standard assumption in the literature, but as argued in Laval et al. (2017) this constraint negates the fact that distance traveled within the reservoir may increase with congestion. In addition, it seems reasonable to allow for short-lived demand surges; but if the surges last for a long time unrealistic gridlock might occur. Therefore, we argue that this constraint is optional as in the framework about to be presented.

Multi-reservoir models divide the city in a collection of finite regions, each one described as above, and add a DTA component for the route choice. The main challenge becomes accounting for the trip length in each of the sub-regions visited by the route from origin to destination for each trip (Yildirimoglu and Geroliminis, 2014; Yildirimoglu et al., 2015).
2.2. Continuum models for cities

Let a two-dimensional infrastructure be represented as a continuum with domain \( \Omega \subset \mathbb{R}^2 \), as seen in Fig. 2, where \( \Gamma_o \) denotes the outer spatial boundary, \( \Gamma_h \) is the hard boundary of any obstruction at which no traveler is allowed to enter or exit the walking facility, and \( \Gamma_d \) represents the boundary of each destination area, \( i \). Existing continuum-space models for cities have been derived from the seminal continuum pedestrian theory started by Hughes (2002). Although there exist many variations of this model in the literature (see e.g. Aghamohammadi and Laval, 2018), in this paper we focus on the simplest vehicular DTA version with a single commodity, i.e. one type of user, all going to the same destination.\(^\text{[1]}\) The basic equation is a 2-dimensional conservation law:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot f = \lambda, \tag{2D Conservation Law} \ (4)
\]

where boldface symbols indicate vector quantities, \( \rho(x, y, t) \) represents the time-varying density of travelers at point \((x, y)\) and time \(t\) (in units of \( \# \) per unit area), \( u = (u_1(x, y, t), u_2(x, y, t)) \) denotes the velocity vector, \( f(x, y, t) = (f_1(x, y, t), f_2(x, y, t)) \equiv \rho u \) is the flux vector (in flow per unit distance), \( \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \) is the spatial divergence of \( f \) and \( \lambda(x, y, t) \) represents the time-varying demand. Note that the term \( \lambda \) appears only in the vehicular DTA extensions of the model, where vehicles can enter network at any point inside the continuum; otherwise, it is set to zero in the original model (Hughes, 2002) and other extensions to the pedestrian flow, where the demand is assumed to enter the network only from its outer boundary, \( \Gamma_o \). If speed is an isotropic function of the density, \( ||u|| = V(\rho) \), then we have:

\[
||f|| = Q(\rho) = V(\rho) \rho, \tag{isotropic case} \ (5)
\]

where \( V(\rho) \) is the speed-density relationship and \( ||u|| \equiv \sqrt{u_1^2 + u_2^2} \) is the norm of the velocity vector. Notice that \( Q(\rho) \) in the continuum models is in units of flux and \( V(\rho) \) plays the role of MFD in the literature as mentioned in Du et al. (2013, 2015), Long et al. (2017).

The key to Hughes’ model is the incorporation of a route choice formulation that leads to the reactive dynamic traffic equilibrium (RDUE) conditions, as shown in Huang et al. (2009a). In analogy with streamlines in the motion of fluids, let the potential function, \( \phi(x, y, t) \), represent the cost of reaching the destination starting from \((x, y, t)\) along the minimum cost path. The pedestrian motion is therefore in the direction with maximum potential reduction, i.e. in the direction perpendicular to the isopotential curves. The gradient vector of the potential function, \( \nabla \phi \equiv \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \), represents the cost per unit distance of motion and is perpendicular to these isopotential curves in the direction of motion.

\(^\text{[1]}\)Without loss of generality, we assume the discomfort function \( g(\rho) = 1 \) in Hughes’ model.

Figure 2: Spatial representation of current continuum models with destinations and obstructions as closed areas.
maximum increase. It follows that the flux vector, and as a result the velocity vector, are parallel to the gradient of the cost potential but in the opposite direction. Thus, the velocity vector is given by:

\[ \mathbf{u} = -\frac{\nabla \phi}{\|\nabla \phi\|} V(\rho), \]  

(speed vector parallel to streamlines)

where \( \nabla \phi / \| \nabla \phi \| \) is the unit gradient vector of the potential function. If the cost potential is the travel time, the cost per unit distance is the pace, and we have:

\[ \| \nabla \phi \| = 1 / \| \mathbf{u} \| = 1 / V(\rho) \]  

which is an Eikonal PDE. With all, Hughes’ model can be expressed as

\[
\begin{cases}
\frac{\partial \rho}{\partial t} - \nabla \cdot (\rho V^2 \nabla \phi) = \lambda, \\
V = 1 / \| \nabla \phi \|,
\end{cases}
\]

(Hughes’ model) (8a) (8b)

with appropriate boundary conditions, typically:

\[
\begin{cases}
\rho(x, y, 0) = \rho_0(x, y), & \forall (x, y) \in \Omega, \\
f \cdot \mathbf{n} = 0, & \forall (x, y) \in \Gamma_o \cup \Gamma_h, \\
\phi = 0, & \forall (x, y) \in \Gamma_d,
\end{cases}
\]

(9a) (9b) (9c)

where \( \mathbf{n}(x, y) \) is the unit normal vector toward the boundary on the boundaries \( \Gamma_o \) and \( \Gamma_h \) and \( \rho_0(x, y) \) is the initial density inside \( \Omega \). Note that this is not a hyperbolic system, as customary in the 1-D traffic flow literature.

Hughes also identifies two limit cases and proposes simpler equations that the solution satisfies when \( \lambda = 0 \), and in steady state, where \( \partial \rho / \partial t = 0 \). At low free-flow densities, he argues that \( \| \nabla \rho \| >> \| \nabla \phi \| \) and the conservation law (8a) becomes

\[ \nabla \rho \cdot \nabla \phi \approx 0, \]  

(10)

which means that lines of constant density are almost perpendicular to lines of constant potential. At high congested densities Hughes notes that \( \| \nabla \rho \| << \| \nabla \phi \| \) and the conservation law becomes

\[ \nabla^2 \phi \approx 0, \]  

(11)

which corresponds to Laplace’s equation. This equation is well understood in the context of steady incompressible, irrotational fluid flow on the plane.

Finally, to incorporate multiple destinations and/or multiple types of users, multi-commodity extensions of Hughes’ model have been proposed exclusively in Eulerian coordinates for the multi-CBD city networks (Lin et al., 2017) and for the bi-directional pedestrian flow (see e.g., Huang et al., 2009b; Jiang et al., 2009, 2012). Separate potential functions are defined for each destination and the optimum movement direction of the travelers moving toward each of the destinations is found by the corresponding potential function. Similarly, as in the one-dimensional traffic flow literature, multi-commodity flows are cumbersome to implement in Eulerian coordinates and are prone to increase numerical viscosity and FIFO violations. In contrast, in this paper we will adopt a Lagrangian representation to incorporate multiple commodities.

3. Continuum MFD formulation

In this section, we will formulate a continuum-space model consistent with the assumptions behind the MFD theory. We will show that this consistent formulation is capable of handling randomly distributed destinations over the network. Later, the properties of solutions to this model in both isotopic and anisotropic cases will be discussed. Consistency in this section means that the spatial integration of the continuum model should lead to the reservoir
model. Notice that this formulation pertains only to the conservation law component of the model; the route choice component is arbitrary.

To see the relationship between the variables in the discrete- and continuum-space model formulations, let \( A \) be the region for which the reservoir model has been defined. Then,

\[
\begin{align*}
    n(t) &= \int_A \rho(x, y, t) \, dx \, dy, \\
    \Lambda(A, t) &= \int_A \lambda(x, y, t) \, dx \, dy, \quad \text{and} \\
    M(A, t) &= \int_A \mu(x, y, t) \, dx \, dy,
\end{align*}
\]

where \( \lambda(x, y, t) \) represents the time-varying demand and \( \mu(x, y, t) \) is the trip completion rate at \((x, y)\) at time \( t \), both in units of flow per unit area. It follows that the conservation law becomes:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot f = \lambda - \mu.
\]

This formulation is consistent with MFD theory because the reservoir model (11) can be obtained from (13) by recalling that \( \nabla \cdot f = 0 \) since the reservoir has no spatial dimensions, integrating over the region \( A \) and using (12).

Recall assumption (2) of MFD theory stating that the trip completion rate on a physical link is proportional to the flow on that link. In the continuum, this amounts to assuming that the trip completion rate is proportional to the norm of the flux, \( \mu = \|f\| / \ell \), which in the isotropic case is

\[
\mu = \frac{Q(\rho)}{\ell},
\]

which can be integrated over the region \( A \) to obtain

\[
\begin{align*}
    M(A, t) &= \int_A \frac{Q(\rho)}{\ell} \, dx \, dy, \\
    &\approx \frac{Q(\rho) |A|}{\ell}, \\
    &= \bar{Q}(n) \frac{L}{\ell}.
\end{align*}
\]

where approximation (15b) follows when both the flux MFD and the density are approximately constant in \( A \). Equality (15c) follows because the production in the continuum framework, \( Q(\rho) |A| \), and in the discrete framework, \( \bar{Q}(n) L \), are equal. To see this, one can use the fundamental traffic flow relationship in both frameworks to obtain \( \rho V(\rho) |A| \) and \( V(n) n \), respectively. Using the identity \( n = \rho |A| \) gives \( V(\rho) = V(n) \). Therefore, we have shown that the spatial integration of the continuum model leads to the reservoir model, as sought.

The introduction of the term \( \mu \) in conservation law (13) suggests that the vehicles can complete their trip and exit the network at any point. Therefore, the proposed formulation is intrinsically capable of handling multi-commodity

![Figure 3: Randomly distributed origins and destination over the continuum space in the proposed consistent model.](image-url)
flows and multiple destinations, which is more compatible with the nature of the real-life networks compared to the traditional continuum-space models, where there is only one or a few destination areas. When more than one destination is considered, the standard approach is to breakdown the total density into commodities, one per destination. As shown next, although some general insight might be derived, the general solution of such model can be cumbersome. Then, in section 3.2 we introduce a disaggregation by direction of travel, which is amenable for the semi-Lagrangian solution proposed in this paper.

3.1. Vehicles disaggregated by destination (isotropic case)

In the proposed formulation, any point on the continuum space can be a destination; see Fig. 3. To understand how each destination contributes to the conservation law of global quantities in (13), we let group $i = 1, 2, \ldots$ denote the vehicles bound towards destination point $x_i = (x_i, y_i)$, and subscript $i$ to denote group-specific quantities, e.g. $\rho_i$, $f_i$, etc. Clearly,

$$\rho = \sum_i \rho_i, \quad f = \sum_i f_i = \sum_i \rho_i u_i, \quad \text{and} \quad \lambda = \sum_i \lambda_i. \quad (16)$$

The speed vector of each group is now given by

$$u_i = -\frac{\nabla \phi_i}{\|\nabla \phi_i\|} V(\rho), \quad (17)$$

in the isotropic case, and notice that:

$$\|\nabla \phi_i\| = 1/\|u_i\| = 1/V(\rho), \quad (18)$$

as in (7). It follows that

$$f_i = -V(\rho)^2 \rho_i \nabla \phi_i. \quad (19)$$

The disaggregated outflow $\mu_i$ is less obvious because it should be identically zero everywhere except at the group’s destination $x_i$. Introducing the indicator function $\mathbb{1}_{x_i}(x)$ defined as:

$$\mathbb{1}_{x_i}(x) = \begin{cases} 1, & \text{if } x = x_i, \\ 0, & \text{otherwise}, \end{cases} \quad (20a)$$

we have:

$$\sum_i \mu_i \mathbb{1}_{x_i}(x) = \mu = V(\rho) \rho / \ell, \quad (21)$$

as required, and where the last equality follows from (21) and the fundamental traffic flow relationship $Q(\rho) = V(\rho) \rho$. With all, the conservation law of each group can be expressed as $\partial \rho_i / \partial t - \nabla \cdot f_i = \lambda_i - \mu_i \mathbb{1}_{x_i}(x)$, where summation over all the groups, we recover the global conservation law (13), as expected. Using (19) the conservation law for each group can also be expressed as:

$$\frac{\partial \rho_i}{\partial t} - \nabla \cdot \left( V(\rho)^2 \rho_i \nabla \phi_i \right) = \lambda_i - \mu_i \mathbb{1}_{x_i}(x). \quad (22)$$

Following Hughes (2002), we examine two limit cases of (22), light and heavy traffic, under steady-state conditions. Expanding the divergence term in (22), rearranging and considering locations $x \neq x_i$, we have:

$$- \left( V(\rho) \nabla \rho_i \cdot \nabla \phi_i + \rho_i \left( 2V'(\rho) \nabla \rho_i \cdot \nabla \phi_i + V(\rho) \nabla^2 \phi_i \right) \right) V(\rho) = \lambda_i. \quad (23)$$

At low free-flow densities, we can assume $\nabla^2 \phi_i \approx 0$, $\rho_i \approx \rho \approx 0$ and (23) becomes

$$\nabla \rho_i \cdot \nabla \phi_i \approx -\lambda_i / V^2(0), \quad (24)$$
which means that lines of constant density (or flow) are no longer perpendicular to lines of constant potential, as in Hughes model, unless the inflow $\lambda_i$ is zero. Instead, the angle between the two increases with $\frac{\lambda_i}{V^2(0)}$ (since the dot product is proportional to the cosine of the angle between two vectors). We can say that a local increase in the inflow of group $i$ increases the cost of the optimal route by imposing a detour.

At high congested densities, we have $||\nabla \rho|| \ll ||\nabla \phi||$, so that both $\nabla \rho \cdot \nabla \phi$ and $\nabla \rho_i \cdot \nabla \phi_i \approx 0$ and $(23)$ becomes

$$\nabla^2 \phi_i \approx -\frac{\lambda_i}{(V^2(\rho) \rho_i)},$$

(25) known as the Poisson equation, where the right-hand side is negative and probably large in magnitude (because speeds are low), which means that the cost potential is increasing very rapidly and nonlinearly away from the destination, possibly inducing gridlock near the destination. However, if the inflow is to be restricted by the outflow as discussed in section 2.1, it is likely that the the right-hand side would be $\approx 0$ and the steady-state solution should be close to Laplace’s equation $\nabla^2 \phi = 0$ as in Hughes model with zero inflow.

3.2. Vehicles disaggregated by direction (anisotropic case)

The multi-commodity formulation in the previous section is not easy to solve in Eulerian coordinates, especially when the number of destinations is large as in our case. Recognizing that in vehicular networks there are only a few possible movement directions at any given point (typically 4, one per cardinal direction), here we propose:

$$\frac{\partial \rho_a}{\partial t} + \nabla \cdot f_a = \lambda_a - \mu_a,$$

(26)

where $a = 1, 2, \ldots, m$ is an index for the direction of travel, and subscript $a$ is used to denote group-specific quantities. Naturally, summation of (26) over all directions yields the global conservation law (13), as expected. Notably, since the flux $f_a$ is in direction $a$ only, it suffices to consider a regular scalar function $f_a$, and therefore its divergence reduces to the scalar partial derivative:

$$\nabla \cdot f_a = \frac{\partial f_a}{\partial x_a},$$

(27)

where $x_a$ is the scalar distance in direction $a$. Replacing (27) into (26) gives the 1-D kinematic wave model with sink and source terms:

$$\frac{\partial \rho_a}{\partial t} + \frac{\partial f_a}{\partial x_a} = \lambda_a - \mu_a.$$

(28)

The main distinction is that in the current model at a given point the speed and flux in each direction, $v_a$ and $f_a$, are not only functions of $\rho_a$, but also of the densities in other directions as well. The next section will describe a semi-Lagrangian solution method to this model.

4. Multi-reservoir solution method

Here, we propose using a semi-Lagrangian solution scheme for the anisotropic case discussed earlier exploiting Eulerian or Lagrangian representations where it is most convenient. These schemes use an Eulerian computational grid (cells) over which the motion of particles is tracked. Using these trajectories, we estimate the continuum variables at the grid points. This general idea has been applied copiously in the field of atmospheric sciences [Staniforth and Côté, 1991]. In our case, we propose to use the Eulerian representation to compute the flux transfers between cells, and the Lagrangian representation to move discrete particles and compute the corresponding cell densities of each commodity; see Fig. 4. Each of the following subsections formulates each of the components in this figure.
4.1. The computational grid and network

In the proposed method, an Eulerian uniform computational grid is defined by spatial cells (reservoirs) of rectangular shape of sides $\Delta x, \Delta y > 0$ together with the time-step $\Delta t$, which is assumed to be small enough so as to satisfy the Courant-Friedrich-Lewy (CFL) condition:

$$\max \rho \left( \max \frac{|f_1'(\rho)|}{\Delta x}, \max \frac{|f_2'(\rho)|}{\Delta y} \right) \Delta t \leq 1,$$

which implies that a vehicle will not be able to traverse a cell in less than one time step. Assuming $\Delta y = \Delta x$ for simplicity, the numerical grid is then

$$x_i = i\Delta x, \quad y_j = j\Delta x, \quad t_k = k\Delta t, \quad i, j, k = 1, 2, \ldots$$

It will be convenient to denote a cell by the object $c$, and allow the dot notation $c.i$ and $c.j$ to represent the discrete spatial coordinates of the cell’s center. In this way, the numerical scheme will produce densities $\rho_k^c$ that approximate $\rho(x_c, y_c, t_k)$. For clarity in notation, from now on we drop the time index $k$ as much as possible.

On top of the Eulerian grid, we superimpose a Lagrangian network consisting of the directed links connecting every pair of adjacent cells; see Fig. 5. Each link, $a$, represents one of the four directions of travel, and contains the vehicles in the source cell, $a.s$, for which the minimum path to their destination passes by the adjacent target cell, $a.t$. Links do not have spatial dimensions, and the length of each link, $\ell_a$, is simply an attribute to represent the average trip length from source to target cells. At each time-step $k$, a link has speed $v_a^k$, flux $f_a^k$ and density $\rho_a^k$ in units of vehicles per unit area. The speed on link $a$ is computed using the fundamental traffic flow relationship:

$$v_a = f_a/\rho_a,$$

and all vehicles on link $a$ move at this speed. The key to our solution method is that fluxes $f_a$ also represent the flux at the boundary between source cell $a.s$ and target cell $a.t$, and are computed using Godunov’s method, as explained momentarily.

4.2. The route choice component

The role of the route choice component is the calculation of $\rho_a$, the density in cell $c$ willing to use outgoing link $a$. A popular method in the literature to calculate the $\rho_a$ is to use partial flows to represent each origin-destination commodity and specify turning proportions at each cell, typically with a discrete choice model of the logit type.
Figure 5: (a) Eulerian cells and Lagrangian links. Note that a link represents the distance traveled in the source cell only, as shown by the dashed arrow in (b). But we shift the beginning of the link to the centroid to avoid multiple overlaps.

(Hanseler et al., 2014; Yildirimoglu and Geroliminis, 2014; Hanseler et al., 2017). But as pointed out earlier, the solution to this problem in Eulerian coordinates is cumbersome, and here we adopt a Lagrangian approach instead.

We saw previously that to achieve RDUE conditions, vehicular motion should be in the direction with maximum potential reduction, i.e. in the direction of $-\nabla \phi / \| \nabla \phi \|$, which boils down to solving the Eikonal PDE (7). In the literature, this PDE is solved numerically in Eulerian coordinates with methods that are intimately connected with shortest path algorithms. This connection is not a coincidence since in a Lagrangian representation, we only need the direction of motion at each cell, which can be obtained with a standard shortest path routine on our Lagrangian network.

Once all shortest path trees have been computed, vehicles in cell $c$ are assigned to the outgoing link $a$, which is the first link of their shortest path. This allows the computation of $\rho_a$, simply by counting how many vehicles are assigned to each link (and dividing by $\Delta x^2$). The total density in cell $c$ is therefore:

$$\rho_c = \sum_{a \in O_c} \rho_a,$$

(32)

where $O_c$ is the set of directed links from $c$ to all its adjacent cells. The speed of each of the outgoing links is calculated with (31), where the flux on the link is determined by the Cell Transmission (CT) rule explained next.

4.3. The Cell Transmission rule

As customary in the 1-D traffic flow literature, Godunov scheme is a preferred method due to its simplicity and accuracy compared to the traditional methods. Godunov’s method was first used in traffic flow in the well-known Cell Transmission (CT) model (Daganzo, 1994b) to solve the 1-D traffic flow problem, i.e. a scalar conservation law in one spatial dimension with no source term. Here, we need a method in 2-D and with source term. Laval et al. (2016) proposed a Godunov-type scheme based on variational theory (Daganzo, 2005) to incorporate source terms in 1-D problems, and also showed that the CT scheme is able to cope with source terms, albeit not as accurately. For simplicity, our starting point here is the CT scheme, which can be expanded to two spatial dimensions as shown below.

The key to Godunov’s method is the computation of the fluxes at the boundary between adjacent cells, which are obtained by finding the entropy solution to the corresponding Riemann problems. To illustrate, let us assume for a moment that all vehicles in cell want to go to one of the adjacent cells, i.e. there is a single user class or commodity. The Riemann problem, in this case, is to obtain the maximum flux at the boundary between these two cells, given their initial densities. If these two cells are connected by link $a$, then the CT rule is precisely the solution to the Riemann problem:

$$f_a = \min \{D(\rho_a,s), S(\rho_a,t)\},$$

(CT rule, single commodity) (33)
where $D(\cdot)$ and $S(\cdot)$ are the well-known demand and supply flux functions in traffic flow, respectively:

$$D(\rho) = \begin{cases} \frac{Q(\rho)}{Q^*}, & \rho \leq \rho^* \\ \frac{Q^*}{Q^*}, & \rho > \rho^* \end{cases}$$

and

$$S(\rho) = \begin{cases} \frac{Q(\rho)}{Q^*}, & \rho \geq \rho^* \\ \frac{Q^*}{Q^*}, & \rho < \rho^* \end{cases}$$

(34)

where $\rho^*$ and $Q^*$ are the critical density and capacity of the flux MFD, $Q(\rho)$, respectively; see Fig. 6.

Nevertheless, flows are multi-commodity in two dimensions, one commodity per each origin-destination pair in the network. At the cell level, however, there are only a few commodities that matter, one per outgoing link $a$. The mapping from the origin-destination commodities to the outgoing link commodities is a result of the route choice model, which gives us the density willing to use each outgoing link $\rho_a$. The problem of deciding the corresponding flux through that link, $f_a$, given all the competing densities willing to travel to cell $a$, is identical to the network junction problem in the literature (see e.g., [Jin, 2012] and references therein). The solution to this problem can be expressed as the following multi-commodity CT rule:

$$f_a = \min[D_a, S_a],$$

(CT rule, multiple commodities)

(35)

and the speed on link $v_a$ can then be computed with (31). There has been multiple formulations of these commodity-specific demand and supply functions in the literature, in the context of intersection modeling (see e.g., [Jin, 2017] and references therein). Here, we use:

$$D_a = D(\rho_{a,s}) \cdot \frac{\rho_a}{\rho_{a,s}},$$

and

$$S_a = S(\rho_{a,t}) \cdot \text{IT}(b \in I_a),$$

(36)

where $I_a$ is the set of directed links from $c$’s adjacent cells into $c$. Equation (36) describes that demand and supply of link $a$ are a fraction of the total demand in source cell $a,s$ and supply in the target cell $a,t$, respectively. The demand fraction is proportional to the number of vehicles willing to use link $a$. For the supply fraction, we use the Incremental Transfer (IT) principle [Daganzo et al., 1997], where the total supply $S(\rho_{a,t})$ is viewed as a reservoir being filled gradually from links coming from each of the four adjacent cells, i.e., links $b \in I_a$. In this way, the IT principle states that each reservoir will send fluid downstream until upstream reservoirs are empty or the receiving reservoir is full.

The output of this component is the link speeds $v_a$, which are updated using (31), (35) and (36). Next, we update vehicle positions accordingly.

4.4. Vehicle position update

Let $v_n^k$ and $x_n^k$ be vehicle $n$’s speed and distance traveled along the current link $a$ at time-step $k$, respectively. Assuming that the speed remains constant during a time step, the position of this vehicle in the next time step will be:

$$x_n^{k} = x_n^{k-1} + \Delta t v_n^{k-1}.$$  

(37)
When at a time step \( t \), the distance traveled by the vehicle \( n \) on its current link becomes greater than the link length \( \ell_a \), the vehicle has crossed the cell boundary (i.e. entered cell \( a \) and one of its outgoing links based on the shortest path of the vehicle) at some time \( t_0 \), and all relevant variables are updated accordingly. In order to reduce the numerical error, we assume that the vehicle travels the remainder of time step after \( t_0 \) in the source cell, such as its position becomes:

\[
x_n^k = (t^k - t_0) v_n^{k-1},
\]

where \( x_n^k \) is now the distance traveled in cell \( a \) and \( v_n^{k-1} \) is the speed of the new link in cell \( a \) at the previous time step.

A crucial point is how to compute the speeds \( v_n^k \). In the spirit of the MFD one might postulate \( v_n^k = V(\rho_n^a) \), which is a common assumption in the literature (Yildirimoglu and Geroliminis [2014], Yildirimoglu et al. [2015], Leclercq et al. [2017]), but can lead to stalling (complete gridlock) on multi-reservoir configurations. To see this, notice that when a cell reaches jam density, both the speed and the outflow will be zero even when neighboring cells might be freely flowing. Instead, here we use the CT rule (35) to compute fluxes \( f_a \) and then use the fundamental traffic flow relationship to compute the speed on link, \( v_a \), per (31) and we assign this speed to all vehicles in the link, i.e.: \( v_n = v_a \).

This formulation avoids unrealistic stalling because if a cell is at jam density its flow will be determined by the supply on the downstream cell, which is zero only if that cell is also full.

4.5. Conservation of vehicles

After all vehicles’ positions and current cell have been updated, total cell densities \( \rho_c \) are updated simply by (i) counting the number of vehicles in each cell (ii) add the number of vehicles entering the network at cell \( c \) \( \Lambda_c \), (iii) subtract the number of vehicles whose destination is cell \( c \), and (iv) divide by the cell area \( \Delta x^2 \). Notice that it is customary to cap the inflow at the supply of the origin cell; i.e.,

\[
\Lambda_c = \min\{S(\rho_c), \lambda(x_{c,i}, y_{c,j}, t_k) \Delta x\} \Delta t \Delta x,
\]

since otherwise unrealistic gridlock can occur. At this point, we recalculate the minimum paths with the updated link speeds from (31).

5. Examples

In this section, we have used the aforementioned numerical solution method to simulate a square-shaped city of side \( b \) with parabolic homogeneous flux-MFD given by \( Q(\rho) = \rho u \cdot (1 - \rho/\rho_j) \), where \( u \) is the free-flow speed and \( \rho_j \) is the jam density (in units of vehicles per unit area). We eliminate these two parameters by measuring distance in units of \( \rho^{-1/2} \) and time in units of \( \rho^{1/2} u^{-1} \), which gives:

\[
Q(\rho) = \rho \cdot (1 - \rho).
\]

Total arrivals are given by a Poisson process of constant rate \( \lambda \) veh/hr-km\(^2\), while origins are uniformly distributed across the region. The average trip length inside each cell to all adjacent cells is assumed as the side length of the cell, \( \ell_a = \Delta x \). Two sets of experiments are performed differing only in the treatment of destinations: (i) a single region-destination, e.g. CBD, and (ii) multiple point-destinations.

5.1. A single-destination region

In this experiment, vehicles travel toward a single destination located at the center of the square region starting their trips from uniformly distributed origins. Different simulation runs have been done using different cell sizes, while the size of the destination area is kept constant across simulation runs in order to have the same maximum outflow rate, which makes us able to study the convergence of the model. This convergence is measured in terms of the delay with respect to the analytical solution, which is simple for this problem since the destination will serve at
constant capacity when congested. Therefore, we can find the theoretical cumulative departure curve and compare it with results of the experiment. Other assumptions for this experiment are summarized in Table 1.

The same origin-destination points are used for all simulations. Fig. 7 shows the cumulative arrivals, entrances and departures for a 10 by 10 and a 40 by 40 partitioning of the network area. The analytical cumulative departure curve is plotted by shifting the constant departure rate curve by the average free flow travel time of vehicles to the right. Although in the experiment with coarser partitioning we observe slight variations in the experimental departure curve whereas its slope is faintly lower than the expected maximum departure rate, the departure curve for the experiment with finer partitioning fits the analytical departure curve impeccably.

The distinction in the arrival and entrance curves arises due to the supply cap Eq. (40), i.e. the demand at the origin cell is equal or greater than its supply, therefore the vehicle has to wait for adequate supply to begin its trip. This phenomenon mostly arises at the cells near to the destination area, since these cells are utilizing the maximum supply they get, most of the time. As seen in Fig. 7, the entrance delay, i.e. the time that vehicles have to wait for sufficient supply to enter the network, is considerably lower in the experiment with finer partitioning.

Furthermore, Fig. 8 shows the average delay per vehicle for different number of cells versus the analytical approximation of average delay. It can be seen that the average delay converges to the analytical approximation with increasing the number of cells, i.e. decreasing the size of cells.

![Figure 7: Cumulative arrival, entrance, and departure curves for two different experiments](image-url)
The results suggest that the overall performance of the model improves with partitioning the region into smaller cells. However, by decreasing the cell size, the jam accumulation of each cell also decreases, which could result in stalling problem under congested conditions if the speed was directly taken from the MFD as in the discrete-space literature, which is not the case in the proposed model since the flux and speed are calculated using the supply and demand functions. To get a better insight into the performance of the model under congested conditions, in another experiment, the network is divided into 125-meter-long square cells, i.e., 40 cells in each direction, and it is loaded for 20 minutes with an average arrival rate of 5 vehicles per second, while the maximum outflow rate of the destination area is one vehicle per second. Fig. 9 shows the cell occupancies at time \( t = 1200 \) s, the end of loading duration, while the network is the most congested.

The interesting point in Fig. 9 is that the peak of density is not immediately on the borders of the destination area, rather it is observed at a distance from the destination area. An approximate steady-state analytical solution to this problem without inflow supply cap can be found in polar coordinates \((r, \theta)\), assuming that the destination is a circle of radius \( r_0 \). As shown in Newell (1980, p. 130), fluxes should be in the radial direction and rotationally symmetric, and therefore the conservation law in steady-state can be expressed as:

\[
\frac{\partial}{\partial r} (r Q(\rho)) = r \lambda. \tag{42}
\]

By imposing that far away from the destination the inflow vanishes, Newell shows that the flux decays with distance as \(1/r\), independently of the shape of the flux function. Here, we use the parabolic flux-MFD \(41\) and solve (42) to find:

\[
\rho(r) = \frac{1}{2} \pm \sqrt{2f_0 r_0 + r/2 - \lambda (r^2 - r_0^2)}, \tag{43}
\]

where \(\rho_0 = \rho(r_0)\) is the boundary condition at the destination’s perimeter, \(f_0 = Q(\rho_0)\) is the initial flux and \(\pm\) is the sign of \((\rho_0 - 1/2)\). Fig. 10 shows a cross-section of the density obtained with the simulation superimposed to the analytical approximation (43). It can be seen that despite some small variations, simulation results accord well with the analytical solution.

The flux as a function of distance can be readily obtained using (43):

\[
Q(\rho(r)) = \frac{\lambda (r^2 - r_0^2)}{2r} + \frac{f_0 r_0}{r}, \tag{44}
\]

1See Subsections 4.3 and 4.4 for more details.
Cell Occupancy, Cell Length= 125 m, t= 1200 s

where it can be seen that when the inflow vanishes ($\lambda = 0$) the flux indeed decays as $1/r$ as predicted by Newell, but if it is nonzero then eventually the flux will start increasing linearly with $r$; the minimum flux happens at location $r^*$ given by:

$$r^* = \sqrt{r_0 \left(2f_0/\lambda - r_0\right)}.$$

(45)

This result indicates that congestion is most severe at a distance $r^*$ from the destination, and that this distance increases as the square root of $f_0/\lambda$. 

Figure 9: Distribution of vehicles over the region partitioned into a 40 by 40 mesh

Figure 10: Density Cross-section: Analytical, Eq. (43) vs. Simulation

15
5.2. A Multi-destination region

In this experiment, the arrival rate and other assumptions are similar to the previous experiment but now both origins and destinations are uniformly distributed across the region. In steady-state, this problem is characterized by the fundamental traffic flow relationship:

\[ \rho = \frac{f}{v} = \frac{\lambda E[D]}{v}, \]  

(46)

where \( v \) is the (deterministic) steady-state speed in the network, and \( E[D] \) gives the expected value of the distance traveled by a driver, \( D \). Notice that the flux \( f = \lambda E[D] \) is a constant of this problem, and can be used to obtain the speed \( v \) using the flux-speed MFD provided we know the traffic regime. In our case we have:

\[ v(f) = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4f} \right), \]  

(47)

where \( \pm \) represents addition when in free-flow and subtraction in congestion.

In our case the Manhattan distance traveled by a driver is \( D = |x_1 - x_2| + |y_1 - y_2| \), where subscripts 1 and 2 refer to the coordinates of the origin and destination, respectively. For origins and destinations uniformly distributed on a square region of side \( b \), one can show that the probability density function (pdf) of \( D \), \( p_D(d) \), is:

\[ p_D(d) = \begin{cases} \frac{2}{b^2} \left( \frac{d}{b} - 2 \right)^2, & 1 \leq \frac{d}{b} \leq 2, \\ \frac{2}{b^2} \left( \frac{d}{b} \right)^2 - 6 \frac{d}{b} + 6, & 0 < \frac{d}{b} < 1, \\ 0, & \text{otherwise,} \end{cases} \]  

(pdf of \( D \))  

(48)

where one can verify that \( E[D] = 2b/3 \); therefore, we can express the speed as a function of the demand

\[ v(\lambda) = \frac{1}{2} \left( 1 - \sqrt{1 - 8\lambda b/3} \right). \]  

(49)

The travel time random variable is simply \( T = D/v \), and its pdf given by:

\[ p_T(t) = v p_D(v t). \]  

(travel time pdf)  

(50)

Fig. 11 shows travel time histograms from the simulation for different values of the demand parameter \( \lambda \). The thick lines correspond to the analytical pdf \( p_T(t) \), where \( v \) is obtained using (49). It can be seen that the agreements between analytical and simulated histograms is apparent. Although not surprising for the free-flow histogram, the good fit with the congested histogram indicates that (50) captures the correct shape of these density functions without the need of an extra parameter.

![Figure 11: Simulated and analytical (red) travel time histograms](image-url)
6. Discussion

Existing continuum-space models for cities do not consider the completion rate term $\mu$ in (13), perhaps because they are based on pedestrian models where it is customary to assume that travelers exit the network only at discrete destinations represented as small regions, as in Fig. 2. We argue that region-destinations are problematic, and should be avoided in general. The size of these regions is arbitrary and are devoid of flow. This means that the spatial integration will not lead to a consistent model unless some additional considerations are included. But even then one could argue that the size of the destinations might affect the shape of the MFD, which complicates matters even further. Additionally, we saw that in the isotropic case the missing term is $\mu = Q(\rho)/\ell$, and therefore in current formulations, the average trip length is also ignored. Incorporating destinations continuously distributed across the network would be a daunting task in the existing framework, but a very simple one if the MFD concepts are incorporated as in the formulation proposed here.

Section 3 presents the formulation of a consistent continuum MFD model, in the sense that as the size of the subregions tends to zero, the limit of the discrete model corresponds to the continuum model. The emergence of trip completion rate term, $\mu$, in the consistent conservation law (13) suggests that the proposed model is capable of incorporating randomly distributed destinations all over the continuum space, which is in more conformance with the nature of real-life networks. Our motivation is that existing multi-reservoir MFD models might not be consistent, which would imply that its predictions might change drastically upon changes in the partition of the network. The simplest way to accomplish consistency is to start with a numerical scheme to solve our problem and interpret each spatial cell as a subregion of the multi-reservoir model.

The numerical solution method presented here is based on the assumption that there are only a limited number of possible movement directions at each point in the vehicular networks, unlike the pedestrian flow models, where the pedestrians can move at any desired direction. By choosing rectangular cells, there are only four possible movement directions from each cell to each of the neighboring cells. Although the MFD is the same in all four cardinal directions, this does not mean that the speed $v_a$ in all directions is the same, necessarily. In fact, our commodity-specific demand and supply functions in (36) imply that the speed on all outgoing links are different in congestion, and determined by the downstream reservoir of each link. However, in congestion all incoming links will have the same speed since the IT principle splits the supply of the receiving cell evenly among incoming links. Of course, different priorities in the IT principle could be assigned to each cardinal direction to reflect uneven infrastructure or signal timing in each direction. This solution appears superior to imposing a different MFD for each direction altogether because extra conditions should be added to govern any correlation or interactions between the traffic states in all directions.

The proposed framework can be extended in a number of directions in a rather straightforward way, including reservoirs of arbitrary shapes, different cost functions, and inhomogeneous cities. Extensions such as users with different car-following behavior e.g. trucks, buses or automated vehicles, are more challenging because the different fleet compositions should change the MFD in ways that are not well understood yet. These and other extensions are being investigated by the authors.

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