The chord-length distribution of a polyhedron

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Abstract

The chord-length distribution function \([\gamma''(r)]\) of any bounded polyhedron has an elementary algebraic form, the expression of which changes in the different subdomains of the \(r\)-range. In each of these, the \(\gamma''(r)\) expression only involves, as transcendental contributions, inverse trigonometric functions of argument equal to \(R[r, \Delta_1]\), \(\Delta_1\) being the square root of a 2nd-degree \(r\)-polynomial and \(R[x, y]\) a rational function. Besides, as \(r\) approaches one boundary point \((\delta)\) of each \(r\)-subdomain, the derivative of \(\gamma''(r)\) can only show singularities of the forms \((r - \delta)^{-n}\) and \((r - \delta)^{-m+1/2}\) with \(n\) and \(m\) appropriate positive integers. Finally, the explicit algebraic expressions of the primitives are also reported.

Keywords: small-angle scattering, stochastic geometry, integral geometry, chord-length distribution, polyhedron, asymptotic behavior

1
1. Introduction

Since long the correlation function (CF) has become a useful theoretical tool in almost all scientific disciplines as, to mention just a few, the small-angle scattering\(^1,^2\), the signal theory\(^3\), the pattern recognition theory\(^4\) and the stochastic geometry\(^5\). It also happens that its name changes with the discipline, since it is often referred to as covariogram in the last two disciplines. Its use being so general, no wonder that it is studied both from the viewpoint of practical applications and from that of establishing the rigorous conditions that ensure its existence. In fact, given a one dimensional physical quantity \(\eta(x)\), its CF is defined as

\[
\gamma(r) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \eta(x + r) \eta(x) dx.
\]

(1)

The definition clearly involves a limit procedure with the consequent problem of establishing the conditions on \(\eta(x)\) for the limit to exist. Wiener\(^6\) was one of the first mathematicians to afford this problem. In particular he got a result\(^6\) that we like to mention for its generality and elegance: "if \(\gamma(r)\) exists for any \(r\) and if it continuous at \(r = 0\) then it is everywhere continuous" (see, also, reference \(7\)).

In the case of small-angle scattering theory it happens that \(\eta(r)\), the so-called scattering density fluctuation, can fairly be looked at as a two value function and the CF is defined as

\[
\gamma(r) = \lim_{V \to \infty} \frac{1}{4\pi V \langle \eta^2 \rangle_V} \int \int d\hat{\omega} \int_V \eta(r_1 + r \hat{\omega}) \eta(r_1) dv_1.
\]

(2)

Here, \(dv_1\) is the volume infinitesimal element set at position \(r_1\), \(\langle \eta^2 \rangle_V\) denotes the mean value of function \(\eta^2(r)\) evaluated over the volume \(V\), and \(\hat{\omega}\) is a unit vector which can takes all possible orientations over which the first integral, accounting for \((4\pi)^{-1}\) factor, evaluates the angular mean. Definition (2) ensures that \(\gamma(0) = 1\). To make clear the reason why definition (2) is related the covariogram of a geometrical body we first recall that the covariogram measures the angular average of the overlapping volume of the body with its image, resulting by a translation of the body by \(r \hat{\omega}\). Assume that \(\eta(r)\) refers to a collection of particles, having the same shape and size and randomly distributed in the space, and that the collection is dilute. Besides, let \(\eta(r)\) be equal to one inside the particles and to zero elsewhere. The dilution and the randomness conditions allow us to approximate integral (2) by

\[
\gamma(r) \approx \lim_{V \to \infty} \frac{N}{4\pi V \langle \eta^2 \rangle_V} \int \int_{V_\infty} d\hat{\omega} \int_{V_\infty} \eta(r_1 + r \hat{\omega}) \eta(r_1) dv_1,
\]

(3)
where $N$ denotes the number of the particles inside the volume $V$ and $V_p$ the spatial set (with volume $V_p$) occupied by a single particle. Since $\langle \eta^2 \rangle_V$ is equal to $N V_p/V$, equation (4), in the limit $V \to \infty$, becomes

$$
\gamma_p(r) = \frac{1}{4 \pi V_p} \int d\hat{\omega} \int_{V_p} \eta(r_1 + r \hat{\omega}) \eta(r_1) dv_1,
$$

(4)

that is the particle covariogram definition. The second derivative of this function, adapting to the present case the general expression obtained by Ciccariello et al. [8], has the following integral expression

$$
\gamma''(r) = -\frac{1}{4\pi V} \int d\hat{\omega} \int_S dS_1 \int_S (\hat{\nu}_1 \cdot \hat{\omega})(\hat{\nu}_2 \cdot \hat{\omega}) \delta(r_1 + r \hat{\omega} - r_2) dS_2.
$$

(5)

Here, for simplicity, we omitted suffix $p$. Besides, $S$ denotes the boundary surface of the particle, $\delta(\cdot)$ the three-dimensional Dirac function, $\hat{\nu}_1$ ($\hat{\nu}_2$) the unit normal to the infinitesimal surface element $dS_1$ ($dS_2$) located at the point $r_1$ ($r_2$). It is also assumed that $S$ is an orientable surface and that the considered unit normals point externally to the particle. $\gamma''(r)$ also represents the probability density that, randomly tossing a stick of length $r$, the ends of the stick lie on $S$. This property explains why $\gamma''(r)$ is investigated in the realm of stochastic geometry where $\gamma''(r)$ is often referred as chord length distribution (CLD). At the same time, the study of $\gamma''(r)$ is also relevant to the integral geometry [9] that investigates the relations existing between the geometry of a body and some integrals over the latter.

These considerations explain why efforts to get the CF or the CLD of particles with a well definite shape are valuable. So far we explicitly know the CFs of the sphere [1], the cube [10], the right parallelepiped [11], the tetrahedron [12], the octahedron [13] and the circular cylinder [14,15]. For all these shapes, except the cylindrical one, the CFs turn out to be simple algebraic functions, while the cylinder CF involves elliptical integral functions. If we recall that the two-dimensional CF of any bounded plane polygon also has an algebraic form [16,17] we are led to conjecture that the CF of any bounded polyhedron, whatever its shape, has an algebraic expression.

In this paper we show the truth of a weaker form of this conjecture in so far it certainly applies to the CLD of any polyhedron. [As yet, we do not know if it also applies to the CF.] In fact, we prove the following property:

**P** <sub>0</sub> - the CLD of any bounded polyhedron can be expressed in terms of elementary algebraic functions and inverse trigonometric functions depending on rational functions of the two variables: $r$ and $\sqrt{P_2(r)}$, with $P_2(r)$ equal to a 2nd degree $r$-polynomial.

The plan of the paper is as follows. In the next section we show that a decomposition of the facets of the polyhedron into an union of triangles allows
us to consider the integration superficial domains, present in (5), as triangular ones. First we consider the case where the two triangles are not parallel. Section 3 shows that the six-dimensional integral (5) can be converted in a two-dimensional one. Section 4 explicitly performs a further quadrature so as to convert (5) into a one-dimensional integral. The values of this integral depends on the bounds of the integration domain that must be determined by reducing the inequalities present in the integrand definition. Section 5 performs the main steps of this task and section 6 shows that the resulting integrand is a rational function \( R(x, y) \) with \( y \) equal to the square root of a second degree polynomial of the integration variable \( x \). In this way property \( P_0 \) is proved. Section 7 analyzes the case where the two integration triangles are parallel confirming that the aforesaid result holds also true in this case. Section 8 draws the final conclusions. The appendix reports the explicit algebraic expressions of the primitives which must be evaluated at the appropriate end-points of the last one-dimensional integral to get the explicit expression of the polyhedron CLD.

2. Basic mathematical definitions

Our task consists in evaluating integral (5) knowing that \( S \) is the surface bounding a polyhedron of arbitrary shape having however a finite maximal chord. For any polyhedron, the bounding surface \( S \) is made up of plane polygons \( S_1, S_2, \ldots, S_N \), so as to have \( S = \bigcup_{i=1}^{N} S_i \). Consequently, integral (5) becomes

\[
\gamma''(r) = \sum_{i,j=1}^{N} g_{i,j}(r)
\]  

(6)

with

\[
g_{i,j}(r) \equiv -\frac{1}{4\pi V} \int d\hat{\omega} \int_{S_i} \int_{S_j} (\hat{\nu}_1 \cdot \hat{\omega})(\hat{\nu}_2 \cdot \hat{\omega}) \delta(r_1 + r\hat{\omega} - r_2) dS_2
\]  

(7)

The prime on the summation symbol indicates that the cases \( i = j \) can there be omitted because \( g_{i,i}(r) \equiv 0 \). In fact, if \( i = j \), the unit vectors \( \hat{\nu}_1 \) and \( \hat{\nu}_2 \) are equal and the end points of \( r_1 \) and \( r_2 \) lie onto the considered facet’s plane. Since the Dirac function requires that \( r\hat{\omega} = r_2 - r_1 \), it follows that \( \hat{\omega} \) also lies onto the aforesaid plane and is, therefore, orthogonal both to \( \nu_1 \) and to \( \nu_2 \). Then the integrand of (7) vanishes and the property is proved. Thus, owing to equation (6), the problem of evaluating \( \gamma''(r) \) becomes that of evaluating the \( g_{i,j}(r) \)s with \( i \neq j \). Hereafter, for notational simplicity, we shall put \( i = 1 \) and \( j = 2 \). Let \( \pi_1 \) denote the plane on which \( S_1 \) lies and \( \pi_2 \) that relevant to \( S_2 \). Consider first the case where \( \pi_1 \) and \( \pi_2 \) intersect each other along a line that we choose as
the $x$ axis of a Cartesian orthogonal frame (see Fig. 1). Let $V_1, V_2, \cdots, V_M$ denote the vertices of the polygon $S_1$. We draw along each of the $V_i$s a straight line parallel to $x$. In this way, $S_1$ is divided into a set of trapezia, even though some of these can be simple triangles (see Fig. 2). Considering the only trapezia, each of these, by considering one of its diagonals, splits into two triangles. By so doing, we have split $S_1$ into the union of $N_1$ triangles $T_i$ (each of these having one side parallel to axis $x$), i.e. $S_1 = \bigcup_{i=1}^{N_1} T_i$. A similar decomposition applies to $S_2$, i.e. $S_2 = \bigcup_{i=1}^{N_2} T_i'$. Once we use the above two decompositions of $S_1$ and $S_2$ into equation (7), $g_{1,2}(r)$ becomes a sum of terms that have the same structure of (7) with the only change that $S_1$ and $S_2$ are now simple triangles. Hence, our task is that of evaluating the following integral expression

$$g(r) \equiv -\frac{1}{4\pi V} \int d\hat{\omega} \int_{T} dS_1 \int_{T'} (\hat{\nu}_1 \cdot \hat{\omega})(\hat{\nu}_2 \cdot \hat{\omega}) \delta(r \hat{\omega} - r_1 + r_2) dS_2,$$  

(8)
where, for greater notational simplicity, we omitted indices on the considered triangles and on the $g(r)$ symbol.

3. Reduction of integral (8) to a two-dimensional integral

To evaluate integral (8), we first define the most convenient Cartesian frame. To this aim we refer to Fig. 1 that shows the two triangles $T = ABC$ and $T' = A'B'C'$. As already anticipated, we choose the $x$-axis along the intersection line of the planes containing the two triangles. At first, we arbitrarily choose one of the two possible orientations for $x$. Then, the oriented $y$-axis is chosen perpendicularly to $x$ and in such a way that $T$ fully lies in the region $y \geq 0$. The oriented $z$-axis is chosen perpendicularly to $x$ and $y$ and in such a way that the resulting system $Oxyz$ be right-handed. Now, we anti-clock-wisely rotate around $x$ the half-plane containing $T$ by an angle $\beta$ till it superposes to the half-plane containing $T'$. We always can choose $Oxyz$ in such a way that $0 < \beta < \pi$. In fact, if proceeding as just said, we find that $\beta$ exceeds $\pi$, then we choose for $x$ and $z$ the opposite directions and in this way the resulting $Oxyz$ is still right-handed and the resulting $\beta$ obeys to $0 < \beta < \pi$. We also observe that it is not restrictive to assume that both $\hat{\nu}_1$ and $\hat{\nu}_2$ point towards the interior of the dihedral angle $\beta$ so as to have

$$\hat{\nu}_1 = (0, 0, 1) \quad \text{and} \quad \hat{\nu}_2 = (0, \sin \beta, -\cos \beta).$$

(9)

In fact, whenever one or both of these conditions were not realized, we change the direction(s) of the normal(s) that points (point) outside and, at the end of the integral evaluation, we change the sign of the result in the only case where one normal has undergone a change of direction.

With respect to the chosen $Oxyz$ frame, the components of $r_1$ are $(x_1, y_1, 0)$ that we find it more convenient to rename as $(x, y, 0)$. It is also convenient to introduce a further cartesian frame $OXYZ$ having axis $x$ coinciding with $x$, axis $Y$ orthogonal to $X$ and oriented in such a way that triangle $T'$ lies in the region $Y \geq 0$, and, finally, $Z$ orthogonal to both $X$ and $Y$ (see Fig. 1) and with orientation such as to ensure the right-handedness of $OXYZ$. With respect to this frame the components of $r_2$ are $(X, Y, 0)$. These, converted to the $Oxyz$ frame, become $(X, Y \cos \beta, Y \sin \beta)$. Finally, choosing $z$ as polar axis and axis $x$ as origin of the longitudinal angle, the $Oxyz$ components of $\hat{\omega}$ are $\hat{\omega} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. Besides, we have $d\hat{\omega} = \sin \theta d\theta d\varphi$, $dS_1 = dxdy$ and $dS_2 = dXdY$. By these definitions follows that the Dirac
function requires the fulfillment of the following three equalities:

\[
X = x + r \sin \theta \cos \varphi, \tag{10}
\]

\[
Y \cos \beta = y + r \sin \theta \sin \varphi, \tag{11}
\]

\[
Y \sin \beta = r \cos \theta. \tag{12}
\]

We solve these equations with respect to \(Y, X\) and \(y\) and denote the solutions as

\[
Y \equiv Y(r, \theta) \equiv r \cos \theta / \sin \beta, \tag{13}
\]

\[
y \equiv y(r, \theta, \varphi) \equiv r \cot \beta \cos \theta - r \sin \theta \sin \varphi, \tag{14}
\]

\[
X \equiv X(x, r, \theta, \varphi) \equiv x + r \sin \theta \cos \varphi. \tag{15}
\]

The Dirac function allows us to explicitly perform the integrations with respect to \(X, Y\) and \(y\) in equation (8). We obtain

\[
g(r) = -\frac{1}{4\pi V \sin \beta} \int d\hat{\omega} \int dx (\hat{\nu}_1 \cdot \hat{\omega}) (\hat{\nu}_2 \cdot \hat{\omega}) \Theta T(\cdots). \tag{16}
\]

Here \(\Theta T(\cdots)\) denotes a product of Heaviside functions that, as it will be expounded below, ensures that solutions (13)-(14) fall inside triangles \(T\) and \(T'\). In fact, the condition that \(r_2 \in T'\) implies that:

\[
Y_m \leq Y \leq Y_M, \quad \text{and} \quad \ell'(Y) \leq X \leq \mathcal{L}'(Y), \tag{17}
\]

where \(Y_m\) and \(Y_M\) respectively denote the \(Y\) ordinate of the side \(A'B'\) points and that of vertex \(C'\) (see Fig.1), while

\[
\ell'(Y) = X_{A'} + \frac{X_{C'} - X_{A'}}{Y_M - Y_m} (Y - Y_m) \equiv a' + b'Y \tag{18}
\]

is the \(OXY\) equation of the side \(A'C'\) and

\[
\mathcal{L}'(Y) = X_{B'} + \frac{X_{C'} - X_{B'}}{Y_M - Y_m} (Y - Y_m) \equiv A' + B'Y \tag{19}
\]

that of the side \(B'C'\). The rightmost sides of the above two equations represent a more compact notation of the equations and the values of constants \(a', b', A'\) and \(B'\) are easily obtained by the polynomial identity principle. We note that, whenever \(T'\) have a form such that \(C'\) were closer to axis \(x\) than the side \(A'B'\), then \(Y_m\) and \(Y_M\) are the \(Y\)-ordinates of \(C'\) and \(A'B'\), respectively, and \(Y_m\) and \(Y_M\) must be interchanged in the middle sides of (18) and (19). Thus, in general, \(Y_m\) (\(Y_M\)) denotes the smallest (largest) \(Y\)-ordinate of
the points of $T'$. Quite similarly, the $(x, y)$ ordinates of the points of $T$ obey the inequalities

$$y_m \leq y \leq y_M, \quad \text{and} \quad \ell(y) \leq x \leq L(y), \quad (20)$$

where, now, $\ell(y) \equiv a + by$ and $L(y) \equiv A + B y$ respectively are the Oxy equations of sides $AC$ and $BC$ of triangle $T$ and are defined quite similarly to equations (18) and (19). It follows that the solutions of equations (10)-(12) must obey inequalities (17) and the first of (20), i.e.:

$$Y_m \leq \overline{Y}(r, \theta) \leq Y_M, \quad (21)$$

$$y_m \leq \overline{y}(r, \theta, \phi) \leq y_M, \quad (22)$$

$$\ell'(\overline{Y}) \leq \overline{X}(x, r, \theta, \phi) \leq L'(\overline{Y}). \quad (23)$$

Owing to equation (15), inequality (23) can be converted into an inequality for $x$ since it can be written as

$$\overline{v}(r, \theta, \phi) \leq x \leq \overline{L}'(r, \theta, \phi), \quad (24)$$

where we have put

$$\overline{v} = \ell'(r, \theta, \phi) \equiv \ell'(\overline{Y}) - r \sin \theta \cos \phi = a' + r(b' \frac{\cos \theta}{\sin \beta} - \sin \theta \cos \phi), \quad (25)$$

and

$$\overline{L} = L'(r, \theta, \phi) \equiv L'(\overline{Y}) - r \sin \theta \cos \phi = A' + r(B' \frac{\cos \theta}{\sin \beta} - \sin \theta \cos \phi). \quad (26)$$

Combining inequalities (24) and the second of (20), evaluated at $y = \overline{y}$, we obtain the final inequalities for variable $x$, namely

$$\max[\ell(\overline{y}), \overline{v}(r, \theta, \phi)] \leq x \leq \min[\overline{L}(\overline{y}), \overline{L}'(r, \theta, \phi)], \quad (27)$$

where we have put

$$\overline{v} = \ell(\overline{y}) = a + b r (\cot \beta \cos \theta - \sin \theta \sin \phi) \quad (28)$$

and

$$\overline{L} = L(\overline{y}) = A + B r (\cot \beta \cos \theta - \sin \theta \sin \phi). \quad (29)$$

We explicit note a property that will often be used later under the name of substitution property: the substitution $\{a \rightarrow A, b \rightarrow B\}$ and its inverse $\{A \rightarrow a, B \rightarrow b\}$ respectively transform $\ell$ into $\overline{L}$ and $\ell'$ into $\overline{L}'$. Similarly, $\{a' \rightarrow A', b' \rightarrow B'\}$ transforms $\overline{v}$ into $\overline{L}'$ and $\{A' \rightarrow a', B' \rightarrow b'\}$ transforms
Using relation (27), integral (16) can be further integrated with respect to \( x \) to yield

\[
g(r) = -\frac{1}{4\pi V \sin \beta} \int d\hat{\omega} (\hat{\nu}_1 \cdot \hat{\omega})(\hat{\nu}_2 \cdot \hat{\omega})(\min[\overline{L}, \overline{L}'] - \max[\overline{\ell}, \overline{\ell}']) \Theta_T(\cdots),
\]

where \( \Theta_T(\cdots) \) denotes that the integration variables \( \theta \) and \( \varphi \) must be restricted to the subdomain \( \mathcal{D}_o(r) \) of \( \mathcal{D}_o \) where inequalities (21), (22) and (27) are obeyed. [Domain \( \mathcal{D}_o \) denotes the outset integration domain defined as \( \mathcal{D}_o \equiv \{ \theta, \varphi | 0 < \theta < \pi; \varphi \in \Phi_o \} \), with \( \Phi_o \equiv \{ \varphi | -\pi/2 < \varphi < 3\pi/2 \} \). The last \( \varphi \) choice will turn out to be convenient later.]

The fact that the CLD of a polyhedron is a sum of contributions having the form of equation (30) allows us to state the property:

\textbf{P}_1 - the CLD of any polyhedron can always be written as a sum of two-dimensional integrals.

4. Reduction of the CLD to a single quadrature

The shape of \( \mathcal{D}(r) \) can exactly be determined by reducing inequalities (21), (22) and (27), a task in principle simple but in practice long and boring to be carried out. We shall later do some steps in this reduction analysis. For the moment, we only note that \( \mathcal{D}(r) \) changes its shape with \( r \) and that it may consist of disjoined sets (see, e.g., Ref. [11]). In any case, \( \mathcal{D}(r) \) can be partitioned into the union of smaller sets \( \mathcal{D}_j(r) \) with \( j = 1, \ldots, M_D \) such that each \( \mathcal{D}_j(r) \) can be written as \( \{ \theta, \varphi | \theta_j,m < \theta < \theta_j,M, \phi_{j,m}(\theta) < \varphi < \phi_{j,M}(\theta) \} \), where the dependence of the bounds \( \theta_j,m, \theta_j,M, \phi_{j,m}(\theta) \) and \( \phi_{j,M}(\theta) \) on \( r \) is omitted for simplicity. The determination of these bounds rests upon the reduction of the aforesaid three inequalities. Hence, in a given \( \mathcal{D}_j(r) \), we know if \( \min[\overline{L}, \overline{L}'] \) is equal to \( \overline{L} \) or to \( \overline{L}' \) and if \( \max[\overline{\ell}, \overline{\ell}'] \) is equal to \( \overline{\ell} \) or to \( \overline{\ell}' \) and that inequality \( \min[\overline{L}, \overline{L}'] > \max[\overline{\ell}, \overline{\ell}'] \) is obeyed.

From these considerations we draw the following general property:

\textbf{P}_2 - the determination of the CLD of a polyhedron always reduces to a quadrature problem.

To prove this statement, we recall that the determination of the CLD reduces to a sum of contributions of the form (11) where the integration domains are simple triangles and that all these contributions have the form of the two-dimensional integral (30). It is now convenient to introduce the following quantities

\[
\Psi(\theta, \varphi) \equiv (\hat{\nu}_1 \cdot \hat{\omega})(\hat{\nu}_2 \cdot \hat{\omega}) = \cos \theta [\sin \beta \sin \theta \sin \varphi - \cos \beta \cos \theta],
\]

(31)
\[ F_1(r, \theta, \varphi) \equiv \Psi(\theta, \varphi) \ell(r, \theta, \varphi) = -\cos \beta \cos^2 \theta (a + b r \cot \beta \cos \theta) + \sin \varphi \sin \theta \cos \theta \times (a \sin \beta + 2b r \cos \beta \cos \theta) - b r \sin \beta \sin^2 \varphi \cos \theta \sin^2 \theta, \]

\[ F_2(r, \theta, \varphi) \equiv \Psi(\theta, \varphi) \widetilde{L}(r, \theta, \varphi) = F_1(r, \theta, \varphi) \bigg|_{(a \to A, b \to B)}, \] (33)

\[ F_3(r, \theta, \varphi) \equiv \Psi(\theta, \varphi) \ell'(r, \theta, \varphi) = - (a' \cos \beta + b' r \cot \beta \cos \theta) \cos^2 \theta + r \cos \beta \cos \varphi \sin \theta \cos^2 \theta + \sin \varphi \cos \theta \sin \theta (a' \sin \beta + b' r \cos \theta) - r \sin \beta \cos \varphi \sin \varphi \cos \theta \sin^2 \theta, \]

\[ F_4(r, \theta, \varphi) \equiv \Psi(\theta, \varphi) \widetilde{L}'(r, \theta, \varphi) = F_3(r, \theta, \varphi) \bigg|_{(a' \to A', b' \to B')}. \] (35)

Here, equation (33) follows from equation (31) and the $\hat{\omega}$ expression reported above equation (10) while the right hand sides of (32)-(35) follow from equation (31) and definitions (28), (29), (25) and (26). Besides, on the rightmost sides of (33) and (35) we used the substitution property reported below equation (29). Recalling how the $\mathcal{D}_j(r)$s have been defined and using definitions (32)-(35), after putting

\[ \mathcal{J}_k(r, \theta, \varphi) \equiv \int F_k(r, \theta, \varphi) d\varphi, \quad k = 1, \ldots, 4, \] (36)
equation (30) becomes

\[ g(r) = -\sum_{j=1}^{M_P} \frac{1}{4\pi V \sin \beta} \int_{\theta_{j,m}}^{\theta_{j,m}} \sin \theta d\theta \times \left[ \left( J_{a_j}(r, \theta, \phi_{j,M}(\theta)) - J_{b_j}(r, \theta, \phi_{j,M}(\theta)) \right) - \left( J_{a_j}(r, \theta, \phi_{j,m}(\theta)) - J_{b_j}(r, \theta, \phi_{j,m}(\theta)) \right) \right], \] (37)

where the values of indices $a_j$ and $b_j$ are determined by the set $\mathcal{D}_j(r)$. Note that $a_j$ is equal to 2 or 4 and index $b_j$ to 1 or 3. Definitions (32)-(35) make it evident that the four $\mathcal{J}_k(r, \theta, \varphi)$ functions are explicitly known because the integrands are polynomial functions in the variables $\sin \varphi$ and $\cos \varphi$. Then equation (37) proves that property $P_2$ is true. Moreover, relation (37) shows that if

\[ \int \mathcal{J}_k(r, \theta, \phi_B(\theta)) \sin \theta d\theta \] (38)
is an algebraic function for any $k = 1, \ldots, 4$ and for $\phi_B(\theta)$ equal to one of the possible $\phi_{j,M}(\theta)$s or $\phi_{j,m}(\theta)$s, then $g(r)$ also is an algebraic function and property $P_0$ is proved.
5. Reduction of inequalities (21), (22) and (27)

Hence, the proof of $P_0$ amounts to prove that integral (38) is an algebraic function. To this aim we need first to know how $\phi_{j,m}$ and $\phi_{j,M}$ depend on $\theta$ and we must therefore elaborate inequalities (21), (22) and (27).

Inequality (21) only concerns variable $\theta$ and yields

\[
Y_m \sin \beta/r < \cos \theta < Y_M \sin \beta/r. \quad (39)
\]

This inequality only exists if $r > Y_m \sin \beta$. Thus, we see that the inequality reduction also restricts the range of the acceptable $r$ values. Assuming the aforesaid constraint on $r$ obeyed, inequality (39) restricts variable $\theta$ to lie within the interval $\Theta_I$ defined as

\[
\Theta_I \equiv [\arccos(Y_M \sin \beta/r), \arccos(Y_m \sin \beta/r)], \quad (40)
\]

that must be interpreted *cum grano salis* in the sense that the left bound must be set equal to zero whenever $Y_M \sin \beta/r > 1$. We observe that $\Theta_I$ is a subset of the interval $[0, \pi/2]$ so that quantities $\cos \theta$ and $\sin \theta$, encountered in the following analysis, always are non-negative.

Inequality (22), solved with respect to $\varphi$, yields

\[
\frac{A_{II,L} + B_{II} \cos \theta}{\sin \theta} < \sin \varphi < \frac{A_{R,II} + B_{II} \cos \theta}{\sin \theta} \quad (41)
\]

with

\[
A_{II,L} \equiv -y_M/r, \quad A_{R,II} \equiv -y_m/r \quad \text{and} \quad B_{II} \equiv \cot \beta. \quad (42)
\]

(Hereafter, subscripts L and R will always refer to left and right bounds.)

For inequality (41) to exist it is necessary that

\[
(A_{II,L} + B_{II} \cos \theta)/\sin \theta \leq 1 \quad \text{and} \quad (A_{R,II} + B_{II} \cos \theta)/\sin \theta \geq -1. \quad (43)
\]

The last two inequalities only constraint variables $\theta$ and $r$ and we shall not further analyze their implications because we are only interested in the bounds on variable $\varphi$. Put

\[
\phi_{II,L,1} \equiv \arcsin\left(\frac{A_{II,L} + B_{II} \cos \theta}{\sin \theta}\right) \quad (44)
\]

and

\[
\phi_{II,R,1} \equiv \arcsin\left(\frac{A_{R,II} + B_{II} \cos \theta}{\sin \theta}\right), \quad (45)
\]
then inequality (41) is equivalent to the validity of the following two inequalities

\[ \phi_{II,L,1} \leq \varphi \leq \phi_{II,R,1}, \]  
\[ (46) \]

\[ \phi_{II,L,2} \equiv \pi - \phi_{II,R,1} \leq \varphi \leq \phi_{II,R,2} \equiv \pi - \phi_{II,L,1}. \]  
\[ (47) \]

[Similarly to the remark reported below equation (40), bounds \( \phi_{II,L,1} \) and \( \phi_{II,R,1} \) must respectively be set equal to \(-\pi/2\) if \((\mathcal{A}_{II,L} + \mathcal{B}_0 \cos \theta)/\sin \theta < -1\) and to \(\pi/2\) if \((\mathcal{A}_{II,R} + \mathcal{B}_1 \cos \theta)/\sin \theta > 1\). Similar substitutions will be understood in all the following relations involving the arcsin function whenever the latter argument is smaller than \(-1\) or greater than \(1\).] Hence, inequality (22) requires that variable \( \varphi \) be always confined to the set \( \Phi_{II} \) defined as

\[ \Phi_{II} \equiv [\phi_{II,L,1}, \phi_{II,R,1}] \cup [\phi_{II,L,2}, \phi_{II,R,2}]. \]  
\[ (48) \]

We turn now to the analysis of the third inequality, \textit{i.e.} condition (27). This involves the following cases:

\[ \ell < \ell' < L < L', \]  
\[ (49) \]

\[ \ell < \ell' < L < L', \]  
\[ (50) \]

\[ \ell' < \ell < L < L', \]  
\[ (51) \]

\[ \ell < \ell' < L < L'. \]  
\[ (52) \]

These inequalities are reduced with respect to \( \varphi \) also requiring that \( \varphi \in \Phi_{II} \) for inequality (22) to be obeyed.

We begin by considering the leftmost inequality of (49) and (50). We first determine the \( \varphi \) range, momentarily denoted by \( \Phi \), where it results \( \ell(\gamma) < \ell(r, \theta, \varphi) \). Owing to definitions (32) and (33), which associate \( \ell \) to \( F_1 \) and \( \ell' \) to \( F_3 \), this case will be referred to as the “1 3” case. Using definitions (28) and (25), inequality \( \ell(\gamma) < \ell(r, \theta, \varphi) \) can be written as

\[ a - a' + r \cos \theta (b \cot \beta - \frac{b' \sin \beta}{\sin \beta}) < r \sin \theta (b \sin \varphi - \cos \varphi). \]  
\[ (53) \]

Putting

\[ \cos f_{13} \equiv b/\sqrt{1 + b^2}, \quad \sin f_{13} \equiv 1/\sqrt{1 + b^2}, \]  
\[ (54) \]

and

\[ \mathcal{A}_{13} \equiv \frac{a - a'}{r \sqrt{1 + b^2}}, \quad \mathcal{B}_{13} \equiv \frac{b \cos \beta - b'}{\sin \beta \sqrt{1 + b^2}}, \]  
\[ (55) \]

inequality (53) takes a form similar to (11), \textit{i.e.}

\[ \sin(\varphi - f_{13}) > \frac{\mathcal{A}_{13} + \mathcal{B}_{13} \cos \theta}{\sin \theta}. \]  
\[ (56) \]
We conclude that \( \ell < \ell' \) is obeyed within the interval
\[
\Phi_{13} = [\phi_{13,L}, \phi_{13,R}],
\]
with
\[
\phi_{13,L} \equiv f_{13} + \arcsin\left( \frac{A_{13} + B_{13} \cos \theta}{\sin \theta} \right), \tag{58}
\]
\[
\phi_{13,R} \equiv \pi + f_{13} - \arcsin\left( \frac{A_{13} + B_{13} \cos \theta}{\sin \theta} \right). \tag{59}
\]

From this result follows that the leftmost inequality of (51) and (52), i.e., \( \ell > \ell' \), is obeyed within the set \( \Phi_{13}^C \) that is the complement of \( \Phi_{13} \) to the set \( \Phi \). In this way, the analysis of case "13" is accomplished.

We analyze now the inequality related to the rightmost sides of (49) and (51), i.e., the condition \( \ell < \ell' \). This will be referred to as the "24" case owing to definitions (33) and (35). The rightmost sides of these definitions show that inequality \( \ell < \ell' \) follows from \( \ell < \ell' \) once we apply to this the transformation \( (a \rightarrow A, b \rightarrow B, a' \rightarrow A', b' \rightarrow B') \). Then, we conclude that inequality \( \ell < \ell' \) is obeyed within the \( \phi \) interval \( \Phi_{24} \) that is obtained applying the previous transformation to \( \Phi_{13} \), i.e.
\[
\Phi_{24} = \Phi_{13}'(a \rightarrow A, b \rightarrow B, a' \rightarrow A', b' \rightarrow B'). \tag{60}
\]

The inequality \( \ell > \ell' \), related to the rightmost sides of (50) and (52), is fulfilled within the set \( \Phi_{24}^C \) that is the complementary set of \( \Phi_{24} \) to \( \Phi \), as well as the set obtained by applying the substitution \( (a \rightarrow A, b \rightarrow B, a' \rightarrow A', b' \rightarrow B') \) to \( \Phi_{13}^C \). In this way, the analysis of case "24" also is fully accomplished.

To complete the reduction of inequalities (49)-(52) we must analyze the inequalities present in the middle of each of them.

We start by considering (49) where the middle inequality is \( \ell < \ell' \) that, owing to definitions (28) and (26), will be referred to as case "23". It is more convenient to analyze first the opposite inequality, i.e., \( \ell > \ell' \). This reads
\[
r \sin \theta (B \sin \varphi - \cos \varphi) > A - a' + r \cos \theta (B \cot \beta - b' \csc \beta). \tag{61}
\]

Putting
\[
\cos f_{23} \equiv B/\sqrt{1 + B^2}, \quad \sin f_{23} \equiv 1/\sqrt{1 + B^2}, \tag{62}
\]
\[
A_{23} \equiv \frac{A - a'}{r \sqrt{1 + B^2}} \quad \text{and} \quad B_{23} \equiv \frac{B \cos \beta - b'}{\sin \beta \sqrt{1 + B^2}}, \tag{63}
\]

13
inequality (61) can be written in a form similar to (56), i.e.

\[ \sin(\varphi - f_{23}) > \frac{A_{23} + B_{23} \cos \theta}{\sin \theta}. \]  

(64)

Thus, inequality $\ell' > L$ is obeyed within the interval

\[ \Phi_{23} = [\phi_{23,L}, \phi_{23,R}] \]  

(65)

with

\[
\begin{align*}
\phi_{23,L} &\equiv f_{23} + \arcsin \left( \frac{A_{23} + B_{23} \cos \theta}{\sin \theta} \right), \\
\phi_{23,R} &\equiv \pi + f_{23} - \arcsin \left( \frac{A_{23} + B_{23} \cos \theta}{\sin \theta} \right),
\end{align*}
\]

(66) \hspace{1cm} (67)

while the inequality $\ell' < L$ is obeyed within $\Phi_{C_{23}}$, the complementary set of $\Phi_{23}$ to $\Phi$. We conclude that relation (49) is obeyed within the $\varphi$-set:

\[ \Phi_A \equiv \Phi_{13} \cap \Phi_{C_{24}} \cap \Phi_{II}, \]  

(68)

We pass now to analyze inequality $\ell' < L$ present in the middle of relation (50). This inequality will be referred to as the "3 4" case. Since it reads

\[ r (B' - b') \cos \theta > (a' - A') \sin \beta, \]  

(69)

it puts no constraints on $\varphi$. Thus relation (50) is obeyed within the $\varphi$-set

\[ \Phi_B \equiv \Phi_{13} \cap \Phi_{C_{24}} \cap \Phi_{II}, \]  

(70)

We reduce now inequality $\ell < L$, present in the middle of relation (51). This case is referred to as the "1 2" case. The inequality reads

\[ r (b - A) \sin \theta \sin \varphi > (a - A) + r (b - A) \cot \beta \cos \theta. \]  

(71)

Assuming that $b > A$, and putting

\[
\begin{align*}
A_{12} &\equiv \frac{a - A}{r (b - A)}, & B_{12} &\equiv \cot \beta,
\end{align*}
\]

(72)

the set of the allowed $\varphi$ values is

\[ \Phi_{12} = [\phi_{12,L}, \phi_{12,R}] \]  

(73)
with
\[
\phi_{12, L} \equiv \arcsin \left( \frac{A_{12} + B_{12} \cos \theta}{\sin \theta} \right),
\]
\[
\phi_{12, R} \equiv \pi - \arcsin \left( \frac{A_{12} + B_{12} \cos \theta}{\sin \theta} \right),
\]
(74)

Whenever \( b < B \), inequality \( (71) \) becomes
\[
\sin \varphi < \frac{(a - A)/[r (b - B)] + \cot \beta \cos \theta}{\sin \theta}.
\]
(76)

and the allowed \( \varphi \)-set is \( \Phi^C_{12} \). Thus, we conclude that relation \( (51) \) is obeyed within the set
\[
\Phi_C \equiv \begin{cases} 
\Phi^C_{13} \cap \Phi_{12} \cap \Phi_{24} \cap \Phi_{II} & \text{if } b > B, \\
\Phi^C_{13} \cap \Phi^C_{12} \cap \Phi_{24} \cap \Phi_{II} & \text{if } b < B.
\end{cases}
\]
(77)

Finally, we analyze the inequality present in the middle of \( (52) \), i.e. \( \ell < \overline{L} \), referred to as the "1 4" case. Proceeding as in the three previous cases, after putting
\[
\cos f_{14} \equiv b/\sqrt{1 + b^2}, \quad \sin f_{14} \equiv 1/\sqrt{1 + b^2},
\]
\[
A_{14} \equiv \frac{a - A'}{r \sqrt{1 + b^2}} \quad \text{and} \quad B_{14} \equiv \frac{b \cos \beta - B'}{\sin \beta \sqrt{1 + b^2}},
\]
(78)

we find that inequality \( \overline{\ell} < \overline{L} \) becomes
\[
\sin(\varphi + f_{14}) > \frac{A_{14} + B_{14} \cos \theta}{\sin \theta},
\]
(80)

and the \( \varphi \)-range where it is obeyed is
\[
\Phi_{14} = [\phi_{14, L}, \phi_{14, R}]
\]
(81)

with
\[
\phi_{14, L} \equiv f_{14} + \arcsin \left( \frac{A_{14} + B_{14} \cos \theta}{\sin \theta} \right),
\]
\[
\phi_{14, R} \equiv \pi + f_{14} - \arcsin \left( \frac{A_{14} + B_{14} \cos \theta}{\sin \theta} \right).
\]
(82)

Thus, the \( \varphi \)-range where relation \( (45) \) holds true is
\[
\Phi_D \equiv \Phi^C_{13} \cap \Phi_{14} \cap \Phi^C_{24} \cap \Phi_{II}.
\]
(84)
In this way we have determined the \( \varphi \)-domains, \( i.e. \) \( (68) \), \( (70) \), \( (77) \) and \( (84) \), where inequalities \( (21) \), \( (22) \) and \( (27) \) can be fulfilled. In fact, one should still analyze if the relevant intersections do not yield void sets. This analysis, to be performed, requires the enumerations of all possible order relations among geometrical parameters: \( r \), \( y_m \), \( a \) and so on. However the proof of the validity of property \( P_0 \) does not require this further analysis. To this aim, it is sufficient to note that the end-points of domains \( \Phi_A \), . . . , \( \Phi_D \) necessarily are either of the form \( f + \arcsin[(\mathfrak{A} + \mathfrak{B} \cos \theta)/\sin \theta] \) or are equal to constants as \( \pm \pi/2 \) or \( 3\pi/2 \).

6. Integral \( (38) \) is an algebraic function

We begin the proof of this point by reporting first the expression of the integrand of \( (36) \), for the cases \( k = 1, 3 \), in terms of the new variables

\[
t = \cos \theta \quad \text{and} \quad u = \sin \phi.
\]

(85)

Functions \( J_1[r, \theta, \phi(\theta)] \) and \( J_3[r, \theta, \phi(\theta)] \) respectively become

\[
\mathcal{F}_1[r, t, u(t)] = \phi(t) \mathcal{F}_{1,A}(r, t) + \mathcal{F}_{1,B}[r, t, u(t)],
\]

(86)

with

\[
\mathcal{F}_{1,A}(r, t) \equiv -\left( a + b r t \cot \beta \right) t^2 \cos \beta - \left( b r t/2 \right) (1 - t^2) \sin \beta,
\]

(87)

\[
\mathcal{F}_{1,B}[r, t, u(t)] \equiv \left( b r/2 \right) t (1 - t^2) u \sqrt{1 - u^2} \sin \beta - t \sqrt{1 - t^2} \sqrt{1 - u^2} (a \sin \beta + 2 b r t \cos \beta),
\]

(88)

and

\[
\mathcal{F}_3[r, t, u(t)] = \phi(t) \mathcal{F}_{3,A}(r, t) + \mathcal{F}_{3,B}[r, t, u(t)],
\]

(89)

with

\[
\mathcal{F}_{3,A}(r, t) \equiv -t^2 \left( a' \cos \beta + b' r t \cot \beta \right),
\]

(90)

\[
\mathcal{F}_{3,B}[r, t, u(t)] \equiv r t^2 u \sqrt{1 - t^2} \cos \beta - (r t/2)(1 - t^2) u^2 \sin \beta - t \sqrt{1 - t^2} \sqrt{1 - u^2} (a' \sin \beta + b' r t).
\]

(91)

Functions \( \mathcal{F}_2(r, t, u) \) and \( \mathcal{F}_4(r, t, u) \), generated by \( J_2(r, \theta, \phi) \) and \( J_4(r, \theta, \phi) \), as well as their components \( \mathcal{F}_{2,A}(r, t), \mathcal{F}_{2,B}(r, t, u) \) and \( \mathcal{F}_{4,A}(r, t), \mathcal{F}_{4,B}(r, t, u) \), are respectively obtained from \( (80)-(88) \) and \( (89)-(91) \) by the parameter transformations reported in \( (38) \) and \( (35) \). [All these \( \mathcal{F}_\ldots(\ldots) \) expressions have been obtained setting \( \cos \phi = \sqrt{1 - u^2} \), implicitly assuming that \( -\pi/2 < \phi < \pi/2 \).]
\[ \pi / 2. \] Whenever one had \( \pi / 2 < \phi < 3\pi / 2 \) one should set \( \cos \phi = -\sqrt{1 - u^2} \) but the following conclusions remain unchanged.

The above \( \mathcal{F} \) (. . .) functions must be evaluated at each end-point \( \phi(\theta) \) of the allowed \( \varphi \)-range and then integrated with respect to \( t \) [note that \( \sin \theta d\theta \rightarrow dt \)]. Consider first the case where \( \phi = \text{const} \). Then, \( u(= \sin \phi) \) also is equal to a constant and all the \( \mathcal{F} \) s are polynomial functions of the only \( t \) and \( \sqrt{1 - t^2} \). Consequently, their \( t \)-primitives are algebraic functions of \( t \).

We consider now to the more compliate case where the end points of the allowed \( \varphi \)-range are of the form \( \phi = f + \arcsin \left[ A + B \cos \theta \sin \theta \right] \).

We see that the above two expressions are rational functions of two radicals, namely \( \Delta \equiv \sqrt{1 - t^2} \) and \( \Delta_1 \equiv \sqrt{1 - t^2} - (A + B t)^2 \). Consequently the substitution of (94) and (95) into the previous \( \mathcal{F}_{j,B} \) \( (j = 1, \ldots, 4) \) definitions, that we shall first analyze, yields rational functions of \( t \) and two radicals. From this finding one would hastily conclude that the \( t \)-primitives of the \( \mathcal{F}_{j,B} \)s are elliptical integral functions and not simple algebraic functions. However, a more careful consideration of the \( \mathcal{F}_{j,B} \)s shows that these involve variable \( u \) in the forms: \( u^2, u\sqrt{1 - u^2} \) and \( \sqrt{1 - t^2}\sqrt{1 - u^2} \). Thus, substituting here equations (94) and (95) one obtains expressions which only involve the radical \( \Delta_1 \). In fact, the explicit substitution of (94) and (95) into (88) and (91) yields

\[ \mathcal{F}_{1,B}[r, t, u(t)] = C_{1,B,1}(t) + C_{1,B,2}(t) \Delta_1 \]
with
\[
\mathcal{C}_{1,B,1}(t) \equiv (t/2) \sin f \left[ 4 br(t(A + B)t) \cos \beta + \left( 2a(A + B)t - b r(2A^2 - 1 + 4A B t + (1 + 2B^2)t^2) \cos f \right) \sin \beta \right].
\]

\[
\mathcal{C}_{1,B,2}(t) \equiv (t/2) \left[ b r(A + B)t \cos f \sin \beta - br(A + B)t \sin^2 f \sin \beta \right. - (97)
\]
\[
2 \cos f(2 br t \cos \beta + a \sin \beta)
\]

and
\[
\mathfrak{F}_{3,B}[r, t, u(t)] = \mathcal{C}_{3,B,1}(t) + \mathcal{C}_{3,B,2}(t) \Delta_1
\]

with
\[
\mathcal{C}_{3,B,1}(t) \equiv (t/2) \left[ 2 r t(A + B)t \cos f \cos \beta - r (A + B)t^2 \cos f \sin^2 \beta + \sin f \left( 2b' t r(A + B)t + \left( 2a'(A + B)t + r (A^2 - 1 + 2A B t + (1 + B^2)t^2) \sin f \right) \sin \beta \right) \right] - (98)
\]
\[
\mathcal{C}_{3,B,2}(t) \equiv -t \left[ -r t \cos \beta \sin f + \cos f \left( b' r t + \sin \beta (a' + r (A + B)t) \sin f \right) \right].
\]

The expressions of \( \mathfrak{F}_{2,B}(r, t, u(t)) \) and \( \mathfrak{F}_{4,B}(r, t, u(t)) \) are respectively obtained from the above two ones by the aforesaid transformations (83) and (85) of coefficients \( a, b, \ldots \). One concludes that the \( t \)-primitives of \( \mathcal{C}_{j,B,1}(t) \), for \( j = 1, \ldots, 4 \), simply are \( t \)-polynomials of the 4th-degree. The primitives of \( \mathcal{C}_{j,B,2}(t) \Delta_1 \) (for \( j = 1, \ldots, 4 \)) are obtained by the procedure expounded in Ref. [18]. For definiteness we take \( j = 1 \) and we observe that \(- (1 + B^2)\), the coefficient of \( t^2 \) inside radical \( \Delta_1 \), is negative. Since the integral must be real, variable \( t \) must vary between the lowest \( (\mu_1) \) and the largest root \( (\mu_2) \) of the equation \( 1 - t^2 - (A + B)t^2 = 0 \). We change the integration variable \( t \) into the new variable \( \xi \) according to
\[
t = t(\xi) = \frac{\mu_2 + \mu_1 \xi^2}{1 + \xi^2}.
\]

the inverse of which is \( \xi = \sqrt{(\mu_2 - t)/(t - \mu_1)} = \frac{\Delta_1}{(t - \mu_1) \sqrt{\mu_2 - t}} \). We find that
\[
\Delta_1 = \frac{(\mu_2 - \mu_1) \xi}{1 + \xi^2} \quad \text{and} \quad \left| \frac{dt}{d\xi} \right| = \frac{2(\mu_2 - \mu_1) \xi}{(1 + \xi^2)^2}. \tag{103}
\]
The new $\xi$-integrand, also accounting for the Jacobian, is

$$2 \sqrt{1 + B^2 C_1 B_2(t)} \left( \mu_2 - \mu_1 \right)^2 \xi^2 \left(1 + \xi^2 \right)^4$$

that is a rational function of $\xi^2$. The relevant $\xi$-primitive is a rational function of $\xi$ plus a contribution proportional to $\arctan \xi$. Hence, the primitive of $C_j B_2(t) \Delta_1$ is a rational function of $t$ and $\Delta_1$ plus an inverse trigonometric contribution function of the same two variables.

To complete the proof of $P_0$ for the case of non-parallel facets, we must show that the other contributions of the form $\arcsin[u(t)] \tilde{F}_1 A(r, t)$ also are integrable in algebraic form. For definiteness we again consider the case $j = 1$. An integration by parts of $\arcsin[u(t)] F_1 A(r, t)$ yields

$$P_{1,A}(r, t) \arcsin[u(t)] - \int P_{1,A}(r, t) \left( \frac{d \arcsin[u(t)]}{dt} \right) dt$$

(104)

where

$$P_{1,A}(r, t) \equiv \int \tilde{F}_{1,A}(r, t) dt,$$

(105)

certainly is an algebraic $t$-function since it is a 4th-degree $t$-polynomial. The $\arcsin[u(t)]$ derivative reads

$$\frac{d \arcsin[u(t)]}{dt} = \frac{B + A t}{(1 - t^2) \Delta_1}.$$  

(106)

It involves the only radical $\Delta_1$. Then, the integrand present in (104) is a rational function of $t$ and of $\Delta_1$ and its primitive can explicitly be evaluated as reported in the first part of the appendix. In this way we have completed the proof that each of the integrals present in equation (38) has an algebraic form and the proof of $P_0$ for the case of polyhedrons with no pair of parallel facets is achieved.

7. The case of the parallel facets

To complete the proof of $P_0$ we must show that, whenever the considered polyhedron has parallel facets, the contributions of these to the CLD also are algebraic functions.

With reference to Fig. 2, let ABCD and $A'C'B'D'E'F'$ be the parallel facets of the given polyhedron. Denote them by $S_1$ and $S_2$ and the planes where these lie by $\pi$ and $\pi'$. We choose one of their sides as the direction for carrying out the "triangularization" of the facets. In Fig. 2 we have chosen side AB. Through each vertex we draw a line parallel to AB till intersecting
Figure 2: ABCD and A’C’B’D’E’F’ represent the polyhedron’s facets that lie on parallel planes \( \pi \) and \( \pi' \). The figure shows how to decompose the two facets into a set of triangles.

the opposite side and then the resulting trapezia are bisected by one of their diagonals. (The cutting lines are broken in Fig.2.) By so doing, ABCD is decomposed into three triangles and A’C’B’D’E’F’ into six. Contribution \( \mathcal{S}_1 \), due to the present \( S_1 \) and \( S_2 \), is equal to the sum of contributions of the form \( \mathcal{S} \) where \( \mathcal{T} \) and \( \mathcal{T}' \) respectively span all the triangles of \( S_1 \) and of \( S_2 \). Consider, for definiteness, the contribution where \( \mathcal{T} = \text{ABC} \) and \( \mathcal{T}' = \text{A'B'C'} \).

We choose the right-handed cartesian frame \( Oxyz \) by taking axis \( x \) along AB, axis \( y \) lying onto \( \pi \) and orthogonal to \( x \) and axis \( z \) orthogonal to \( \pi \) and \( \pi' \). Besides, eventually renaming some vertices of the triangles, we can always have the case depicted in the figure where the origin coincides with A and the \( y \)-ordinate of C as well as the \( z \)-ordinate of \( \pi' \) are both positive.

Furthermore, we can always assume that \( \hat{\nu}_1 = \hat{\nu}_2 = (0,0,1) \) by eventually changing the sign of integral \( \mathcal{S} \). We denote by \( h \) the distance between the two parallel planes and by \( \mathbf{r}_1 = (x,y,0) \) and \( \mathbf{r}_2 = (X,Y,h) \) the coordinates of a generic point of \( \mathcal{T} \) and of \( \mathcal{T}' \), respectively. We also agree that sides AC, BC have equations

\[
\ell(y) = a + b \, y \quad \text{and} \quad \mathcal{L}(y) = \mathbf{A} + \mathbf{B} \, y \quad \quad (107)
\]

and sides A’C’ and B’C’

\[
\ell'(Y) = a' + b' \, Y \quad \text{and} \quad \mathcal{L}'(Y) = \mathbf{A}' + \mathbf{B}' \, Y. \quad (108)
\]
Then, for triangles $\mathcal{T}$ and $\mathcal{T}'$, we respectively have that:

$$\ell(y) < L(y) \quad \text{if} \quad 0 < y < y_M; \quad (109)$$

$$\ell'(Y) < L'(Y) \quad \text{if} \quad Y_m < Y < Y_M. \quad (110)$$

Equation (8) takes now the form

$$g_p(r) = -\frac{1}{4\pi V} \int d\varphi \int \cos^2 \theta \sin \theta d\theta \int_0^{y_M} dy \int_{\ell(y)}^{L(y)} dx \times (111)$$

$$\int_{Y_m}^{Y_M} dY \int_{\ell'(Y)}^{L'(Y)} dX \cos^2 \theta \delta(r_1 + r_\hat{\omega} - r_2),$$

while the Dirac function yields the equations

$$x + r \sin \theta \cos \varphi - X = 0, \quad (112)$$

$$y + r \sin \theta \sin \varphi - Y = 0, \quad (113)$$

$$r \cos \theta - h = 0. \quad (114)$$

[Suffix $p$ on the left side of (111) recall us that the integral refers to parallel facets. Polar coordinates $(\theta, \varphi)$ of $\hat{\omega}$ have been defined as in section 3.] We solve with respect to $X, Y$ and $\theta$. The solutions are:

$$\bar{\theta} = \arccos(h/r), \quad (115)$$

$$\bar{X} = \bar{X}(x) \equiv x + \sqrt{r^2 - h^2} \cos \varphi, \quad (116)$$

$$\bar{Y} = \bar{Y}(y) \equiv y + \sqrt{r^2 - h^2} \sin \varphi. \quad (117)$$

Performing the integrations with respect to $\theta, X$ and $Y$, integral (111) becomes

$$g_p(r) = -\frac{h^2}{4\pi r^3 V} \int d\varphi \int_0^{y_M} dy \int_{\ell(y)}^{L(y)} \Theta(\cdot) dx, \quad (118)$$

where $\Theta(\cdot) = 1$ if: $r > h$, $Y_m < \bar{Y}(y) < Y_M$ and $\ell'(\bar{Y}) < \bar{X} < L'(\bar{Y})$, and equal to zero elsewhere. Once we substitute, in the last inequality, $\bar{X}$ with expression (116) we obtain a further inequality for $x$ beside the one specified in the integral over $x$. The combination of these two inequalities yields

$$\max[\ell(y), \ell'(y)] < x < \min[L(y), \bar{L}'(y)] \quad (119)$$

where we have put

$$\bar{\ell}(y) \equiv \ell'(Y(y)) - \sqrt{r^2 - h^2} \cos \varphi, \quad (120)$$

$$\bar{L}'(y) \equiv L'(Y(y)) - \sqrt{r^2 - h^2} \cos \varphi. \quad (121)$$
Owing to (119) the integral over $x$ can be performed and yields

$$g_p(r) = -\frac{\hbar^2}{4\pi r^3 V} \int d\varphi \int_0^{y_M} \left[ \min[\mathcal{L}(y), \mathcal{L}'(y)] - \max[\ell(y), \ell'(y)] \right] dy. \quad (122)$$

This integral is similar to (30). Thus, we apply a procedure similar to that described in sections 4-6 in order to show that $g_p(r)$ also has an algebraic form. In this case the analysis is somewhat simpler. First, we observe that the integrand of (122) involves the same four cases reported in Eqs. (49)-(52) and its value is respectively equal to $\mathcal{L}(y) - \ell(y)$, $\mathcal{L}'(y) - \ell'(y)$, $\mathcal{L}(y) - \ell(y)$ and $\mathcal{L}'(y) - \ell(y)$. Thus, the integrand always is a linear function of $y$, $\cos \varphi$ and $\sin \varphi$. We must also require that $Y_m < \bar{Y}(y) < Y_M$ in agreement with (111). This inequality combined with $0 < y < y_M$ yields

$$\max[0, Y_m - \sqrt{r^2 - h^2 \sin \varphi}] < y < \min[y_M, Y_m - \sqrt{r^2 - h^2 \sin \varphi}] \quad (123)$$

The reduction of the inequalities, related to the left and right sides of (123), bounds the $\varphi$-range, the end-points of which are of the form

$$\varphi < f_1 + \arcsin[R_1(\sqrt{r^2 - h^2}, \ldots)] \quad \text{or} \quad \varphi > f_2 + \arcsin[R_2(\sqrt{r^2 - h^2}, \ldots)], \quad (124)$$

where $f_1$ and $f_2$ denote suitable constants, $R_1$ and $R_2$ rational functions and the dots other geometrical parameters as $y_m, \ldots, Y_M$. The remaining $y$-inequalities, associated to (123) and to (49)-(52), determine the corresponding intervals of the acceptable $y$-values. Since they are linear in $y$ it follows that the left and right bound of each interval are of the form

$$y_B = \mathcal{C}(\ldots) + \mathcal{D}(\ldots) \sin \varphi, \quad (125)$$

where $\mathcal{C}$ and $\mathcal{D}$ are functions of some of the geometrical parameters, generically denoted by the dots in (125). We remark that $y_B$ is a linear function of $\sin \varphi$. Thus, in taking the intersection of the previous $y$-intervals, we need to compare the $y$-bounds among themselves. This comparison restricts the $\varphi$-range but we are certain that the resulting $\varphi$-bounds are of the form $f + \arcsin[R(\sqrt{r^2 - h^2}, \ldots)]$, where $R$ again denotes a rational functions and the dots the involved geometrical parameters. We can now proceed to establish the functional form of integral (122). We already noted that the integrand always is a linear $y$-function. Thus, its primitive is a 2nd-degree $y$-polynomial. Once this is evaluated at the ends of the $y$-integration domain, these points are of the form (125). Consequently, the primitive value is a 2nd-degree polynomial of $\sin \varphi$ and the resulting $\varphi$-primitive will be a 2nd-degree polynomial of $\sin \varphi$ and $\cos \varphi$ plus a term linear in $\varphi$. [The explicit expression is reported at the end of the appendix.] This proves that $g_p(r)$ also has an algebraic form and statement $\mathbf{P}_0$ is now fully proved.
8. Conclusions

We have shown that the CLD of any polyhedron is a sum of integrals of the form (7) that, in turns, converts into a sum of contributions of the forms (30) and (118) for each pair of non-parallel and parallel facets, respectively. After having reduced the corresponding inequalities with respect to the variable of the innermost integrals, we have shown that the final primitives, with respect to $t$ [for definiteness we consider the case of the non-parallel facets, but the results also apply to the parallel ones], always are a sum of a rational functions and an inverse trigonometrical function, the argument of which is a rational function of $t$ and the square root of a 2nd-degree polynomial of $t$. The primitives have to be evaluated at the end points of the allowed range of $t$. These bounds depend on $r$ and the parameters that describe the geometry of the facets. Even though we have not fully determined their explicit expressions, we have shown that they are of the form $\arcsin[R(r, \ldots)]$, $R[\cdot]$ being a rational function and the dots denoting the appropriate geometrical parameters. The presence of the $\arcsin$ functions both in the bounds and in the primitive implies that the (higher order) $r$-derivatives of the final CLD can only show singularities of the form $|r - \delta_j|^{-n}$ and $|r - \delta_j|^{-m+1/2}$ with $n$ and $m$ positive integers and $\delta_j$ depending on the geometrical parameters. The first kind of singularity follows from the rational function contributions and the second from the inverse trigonometric functions as well as from the fact that $R$ depends on the square root of a polynomial. Finally, on physical grounds, it can be stated that each of the above $\delta_j$s must be one of end-points of the different $r$-subdomains where the CLD has a specific functional form. Hence, the following general property:

**P$_3$** - the derivatives of the CLD of any polyhedron, as $r$ approaches one of the end points (say $\delta_j$) of the $r$-subdomains, can only show algebraic singularities of form $(r - \delta_j)^{-n}$ or $(r - \delta_j)^{-m+1/2}$ with $n$ and $m$ positive integers.

Illustrations of this property can easily be found by looking at the derivatives of the algebraic known CLDs of the first three Platonic solids that we mentioned earlier. Besides **P$_3$** is useful in deriving the sub-leading terms of the asymptotic expansion of the Fourier transform of the CLD (see, e.g., Ref. [19]).
Appendix: Primitives’ explicit expressions

For completeness we report below the explicit expressions of the $t$-primitives.

The non parallel case

We consider first the case of the non-parallel facets. We recall that, after having reduced all the inequalities, in each of the resulting cases we must perform an integration over variable $\theta$ of an integrand which, according to equation (37), is the algebraic sum of four contributions of the form (38) where, as shown in section 5, the most general form of $\phi_B(\theta)$ is: $f + \arcsin\left[\frac{A + B \cos \theta}{\sqrt{1 - u^2}}\right]$. Passing to the new integration variable $t = \cos \theta$, the integrand functions become the functions $F_1[r, t, u(t)]$, $F_2[r, t, u(t)]$, $F_3[r, t, u(t)]$ and $F_4[r, t, u(t)]$. The first and the third are respectively defined by equations (86) and (89), with $u(t) = \sin \phi(t)$ and $\phi(t) = f + \arcsin\left[\frac{A + B \sqrt{1 - u^2}}{\sqrt{1 - t^2}}\right]$ [see (85) and (93)], while the remaining two are obtained from the former ones by applying the appropriate substitutional relation. According to equations (86) and (89), both $F_1[r, t, u(t)]$ and $F_3[r, t, u(t)]$ are sums of two functions characterized by the further index $A$ or $B$. The explicit calculations, carried out with the help of the MATHEMATICA software along the lines expounded in section 6, show that the primitive of $\left[\phi(t) F_{1,A}(r, t)\right]$ is equal to

$$P_{1,A,a}(t) + P_{1,A,b,0}(t) + P_{1,A,b,1}(t) + P_{1,A,b,2}(t) + P_{1,A,b,3}(t),$$

and that of $F_{1,B}[r, t, u(t)]$ to

$$P_{1,B,a}(t) + P_{1,B,A}(t) + P_{1,B,B}(t).$$

Similarly, the primitive of $\left[\arcsin[u(t)] F_{3,A}(r, t)\right]$ is given by

$$P_{3,A,a}(t) + P_{3,A,b,0}(t) + P_{3,A,b,1}(t) + P_{3,A,b,2}(t) + P_{3,A,b,3}(t),$$

and that of $F_{3,B}[r, t, u(t)]$ by

$$P_{3,B,a}(t) + P_{3,B,A}(t) + P_{3,B,B}(t).$$

All the above $P_{\ldots}(t)$ functions are reported below. The primitives of $F_2[r, t, u(t)]$ and $F_4[r, t, u(t)]$ are respectively obtained from that of $F_1[r, t, u(t)]$ and that of $F_2[r, t, u(t)]$ by the appropriate substitutional rule. It is also noted that the sign ambiguity reported below equation (131) does not invalidate the derivation of the above $t$-primitives. In fact, whenever we had to put $\cos \varphi = -\sqrt{1 - u^2}$ into equation (88) [or (89)], it is sufficient to change the signs of $a$ and $b$ [$a'$ and $b'$] to recover the
integrand form to which [127] [129] applies. The primitive expressions, with the definitions of the related quantities, read as follows:

\[
\Delta \equiv \sqrt{1 - t^2}, \quad \Delta_1 \equiv \sqrt{1 - t^2 - (\mathcal{A} + \mathcal{B} t)^2},
\]

\[
\Lambda_1 \equiv \sqrt{1 + \mathcal{B}^2}, \quad \Lambda_2 \equiv \sqrt{1 - \mathcal{A}^2 + \mathcal{B}^2},
\]

\[
c_{1,A,a,1} = -(1/4) b r \sin \beta, \quad c_{1,A,a,2} = -(1/3) a \cos \beta,
\]

\[
c_{1,A,a,3} = -(1/16) b r [1 + 3 \cos(2\beta)] \csc \beta,
\]

\[
\mathcal{P}_{1,A,a}(t) = \sum_{j=1}^{3} c_{1,A,a,j} t^{3-j}
\]

\[
c_{1,A,b,I,1} = -(1/4) b r \sin(\beta), \quad c_{1,A,b,I,2} = -(1/3) a \cos(\beta),
\]

\[
c_{1,A,b,I,3} = -(1/16) b r [1 + 3 \cos(2\beta)] \csc \beta,
\]

\[
\mathcal{P}_{1,A,b,I}(t) = \arcsin \left[ \frac{\mathcal{A} + \mathcal{B} t}{\Delta} \right] \sum_{j=1}^{3} c_{1,A,b,I,j} t^{j+1},
\]

\[
c_{1,A,b,0,1,1} = 8 \mathcal{A} \mathcal{B} \Lambda_1^2 [2 + 2 \mathcal{B}^2 - 3 \Lambda_1^2 + 3 \Lambda_2^2] \cos \beta,
\]

\[
c_{1,A,b,0,1,2} = \mathcal{A} \mathcal{B} \{2 [8 \Lambda_1^4 + 15 \Lambda_2^4 - \Lambda_1^2 (2 + 11 \Lambda_2^2)] \csc \beta + 3 \Lambda_1^2 (2 + 11 \Lambda_2^2) - 4 \Lambda_1^4 - 15 \Lambda_2^4 \sin \beta\},
\]

\[
c_{1,A,b,0,2,1} = 8 \mathcal{A} \mathcal{B} \Lambda_1^2 \cos \beta,
\]

\[
c_{1,A,b,0,2,2} = \mathcal{B} \mathcal{B} \Lambda_1^2 (5 \Lambda_2^2 - 2 \Lambda_1^2) (2 \csc \beta - 3 \sin \beta),
\]

\[
c_{1,A,b,0,3,1} = 0, \quad c_{1,A,b,0,3,2} = \mathcal{A} \mathcal{B} \Lambda_1^4 [1 + 3 \cos(2\beta)] \csc \beta,
\]

\[
\mathcal{P}_{1,A,b,0}(t) = \frac{\Delta_1}{48 \Lambda_1^2} \sum_{i=1}^{2} \sum_{j=1}^{2} c_{1,A,b,0,i,j} t^{i-1} r^{j-1},
\]

\[
\Omega_{1,A,b,1} = \frac{\Lambda_1 \Delta_1}{\mathcal{A} \mathcal{B} + \Lambda_2 + \Lambda_1^2 t^2},
\]

\[
c_{1,A,b,1,1} = 16 \mathcal{A} \mathcal{B} [3 - \mathcal{A}^2 + (3 + 2 \mathcal{A}^2) \mathcal{B}^2] \Lambda_1^3 \cos \beta,
\]

\[
c_{1,A,b,1,2} = 3 \mathcal{B} \mathcal{B} (\Lambda_1^2 - \mathcal{A}^2) [7 - 3 \mathcal{A}^2 + (13 + 2 \mathcal{A}^2) \mathcal{B}^2 + 6 \mathcal{B}^2 + 5 - 9 \mathcal{A}^2 + (7 + 6 \mathcal{A}^2) \mathcal{B}^2 + 2 \mathcal{B}^4] \cos(2\beta) \csc \beta,
\]

\[
\mathcal{P}_{1,A,b,1}(t) = \frac{\arctan(\Omega_{1,A,b,1})}{48 \Lambda_1^4} \sum_{j=1}^{2} c_{1,A,b,1,j} r^{j-1},
\]
\[ \Omega_{1,A,b,2} = \frac{(A + B)\Delta_1}{1 - A B - A^2 - \Lambda_2 - (1 + A B + B^2 - \Lambda_2) t}, \quad (148) \]

\[ \Psi_{1,A,b,2}(t) = \frac{\arctan(\Omega_{1,A,b,2}) \csc \beta}{48} \times \{8 a \sin(2 \beta) + 3 b r [3 + \cos(2 \beta)]\}; \quad (149) \]

\[ \Omega_{1,A,b,3} = \frac{(A - B)\Delta_1}{1 - A^2 + A B + \Lambda_2 + (1 - A B + B^2 + \Lambda_2) t}, \quad (150) \]

\[ \Psi_{1,A,b,3}(t) = -\frac{\arctan(\Omega_{1,A,b,3}) \csc \beta}{48} \times \{8 a \sin(2 \beta) - 3 b r [3 + \cos(2 \beta)]\}; \quad (151) \]

\[ c_{1,B,a,1} = \frac{\sin f \sin \beta}{4} [2 a A + (1 - 2 A^2) b r \cos f], \quad (152) \]

\[ c_{1,B,a,2} = \frac{\sin f}{3} [2 A b r \cos \beta + B (a - 2 A b r \cos f) \sin \beta], \quad (153) \]

\[ c_{1,B,a,3} = -\frac{b r \sin f}{8} [(1 + 2 B^2) \cos \beta \sin \beta - 4 B \cos \beta], \quad (154) \]

\[ \Psi_{1,B,a}(t) = \sum_{j=1}^{3} c_{1,B,a,j} t^{j+1}; \quad (155) \]

\[ c_{1,B,A,1,1} = 8 a [2 - 2 A^2 + (2 + A^2) B^2] \Lambda_1^2 \cos f \sin \beta, \quad (156) \]

\[ c_{1,B,A,1,2} = -A b \{4 B [2 A^2 (2 B^2 - 13) + 13 A^2] \cos \beta + [8 + 3 B^2 - 5 B^4 + A^2 (9 B^2 + 2 B^4 - 8)] \sin \beta \cos(2f)\}, \quad (157) \]

\[ c_{1,B,A,2,1} = -8 a A B \Lambda_1^4 \cos f \sin \beta, \quad (158) \]

\[ c_{1,B,A,2,2} = b \Lambda_1^2 \{4 A^2 (2 B^2 - 3) + 3 A^2 \} \cos \beta + B [2 A^2 (7 + 2 B^2) - 3 \Lambda_1^2] \cos(2f) \sin \beta, \quad (159) \]

\[ c_{1,B,A,3,1} = -16 a A \Lambda_1^6 \cos f \sin \beta, \quad (160) \]

\[ c_{1,B,A,3,2} = 2 A B \Lambda_1^4 [(4 + 5 B^2) \cos(2f) \sin \beta - 4 B \cos f \cos \beta], \quad (161) \]

\[ c_{1,B,A,4,1} = 0, \quad (162) \]

\[ c_{1,B,A,4,2} = 6 b \Lambda_1^6 [B \cos(2f) \sin \beta - 4 \cos f \cos \beta], \quad (163) \]

\[ \Psi_{1,B,A}(t) = \frac{\Delta_1}{48 \Lambda_1^6} \sum_{i=1}^{4} \sum_{j=1}^{2} c_{1,B,A,i,j} t^{i-1} r^{j-1}; \quad (164) \]
\[ c_{1,B,B,1} = -8 \mathfrak{a} \mathfrak{B} \Lambda_1^2 \cos f \sin \beta, \]

\[ c_{1,B,B,2} = 4 \mathfrak{b} [4 \Lambda_1^4 + 5 \Lambda_2^2 - 4 \Lambda_1^2 (1 + \Lambda_2^2)] \cos f \cos \beta + \]

\[ \mathfrak{b} \mathfrak{B} (4 \Lambda_1^2 - 5 \Lambda_2^2) \cos(2f) \sin \beta, \]  

\[ \mathfrak{P}_{1,B,B}(t) = \frac{\Delta_2}{8 \Lambda_1^2} \arctan \left[ \frac{\Delta_1 \Lambda_1}{\mathfrak{a} \mathfrak{B} + \Lambda_2 + t \Lambda_1^2} \right] \sum_{j=1}^{2} c_{1,B,B,j} r^{-1}; \]  

\[ \mathfrak{P}_{3,A,a}(t) = -\frac{f^3 \cos \beta}{12} \left( 4 a' + 3 b' r t \frac{\sin \beta}{\sin \beta} \right); \]

\[ \mathfrak{P}_{3,A,b,l}(t) = -\frac{f^3 \cos \beta}{12} \arcsin \left[ \frac{\mathfrak{a} + \mathfrak{B} t}{\Delta} \right] \left( 4 a' + 3 b' r t \frac{\sin \beta}{\sin \beta} \right); \]

\[ c_{3,A,b,1} = \mathfrak{a} (8 \Lambda_1^4 - 2 \Lambda_1^2 + 15 \Lambda_2^2 - 11 \Lambda_1^2 \Lambda_2^2), \]

\[ c_{3,A,b,2} = -\mathfrak{B} \Lambda_1^2 (2 \Lambda_1^2 - 5 \Lambda_2^2), \quad c_{3,A,b,3} = 2 \mathfrak{a} \Lambda_1^4, \]

\[ \mathfrak{P}_{3,A,b,0}(t) = \frac{\Delta_1 \cos \beta}{24 \Lambda_1^2} \left( \frac{r b'}{\sin \beta} \sum_{j=1}^{3} c_{3,A,b,j} t^{j-1} + 4 a' \Lambda_1^2 \left[ \mathfrak{B} (2 \Lambda_1^2 - 3 \mathfrak{a}^2) + \mathfrak{a} \Lambda_1^2 t \right] \right); \]

\[ \Omega_{3,A,1} = \frac{\Delta_1 \Lambda_1}{\mathfrak{a} \mathfrak{B} + \Lambda_2 + t \Lambda_1^2}, \]

\[ c_{3,A,1} = 4 \mathfrak{a} a' \Lambda_1^2 (2 \Lambda_1^4 + 3 \Lambda_2^2 - 2 \Lambda_1^2 \Lambda_2^2), \]

\[ c_{3,A,2} = 3 \mathfrak{B} b' r \Lambda_2^2 [4 \Lambda_1^4 + 5 \Lambda_2^2 - 2 \Lambda_1^2 (2 + \Lambda_2^2)], \]

\[ \mathfrak{P}_{3,A,1}(t) = -\frac{\arctan[\Omega_{3,A,1}] \cos \beta}{12 \Lambda_1^2} \sum_{j=1}^{2} c_{3,A,j} \sin^{1-j} \beta; \]

\[ \Omega_{3,A,2} = \frac{(\mathfrak{a} + \mathfrak{B}) \Delta_1}{\mathfrak{a} \mathfrak{B} + \Lambda_1^2 + \Lambda_2 - \Lambda_2^2 - 1 + t (\mathfrak{a} \mathfrak{B} + \Lambda_1^2 - \Lambda_2)}, \]

\[ \mathfrak{P}_{3,A,2}(t) = \frac{-\arctan[\Omega_{3,A,2}] \cos \beta}{12} \left[ 4 a' + 3 b' r \frac{\sin \beta}{\sin \beta} \right]; \]

\[ \Omega_{3,A,3} = \frac{(\mathfrak{a} - \mathfrak{B}) \Delta_1}{(1 + \mathfrak{a} \mathfrak{B} - \Lambda_1^2 + \Lambda_2 + \Lambda_2^2 + t (\Lambda_1^2 + \Lambda_2 - \mathfrak{a} \mathfrak{B})}, \]

\[ \mathfrak{P}_{3,A,3}(t) = \frac{-\arctan[\Omega_{3,A,3}] \cos \beta}{12} \left[ 4 a' - 3 b' r \frac{\sin \beta}{\sin \beta} \right]; \]
\[
c_{3,B,a,1} = \frac{\sin \beta}{8} \{4 \mathfrak{A} \mathfrak{a} \sin f - r [1 - \cos(2f) + 2 \mathfrak{A}^2 \cos(2f)] \}, \quad (181)
\]
\[
c_{3,B,a,2} = \left\{ r \mathfrak{A} [\cos f \cos \beta + \mathfrak{b} \sin f - \mathfrak{B} \cos(2f)] \sin \beta \right\} + \mathfrak{a}' \mathfrak{B} \sin f \sin \beta) / 3, \quad (182)
\]
\[
c_{3,B,a,3} = \frac{r}{16} \left\{ 4 \mathfrak{B} \cos f \cos \beta + 4 \mathfrak{B} \mathfrak{b}' \sin f + \sin \beta [1 - (1 + 2 \mathfrak{B}^2) \cos(2f)] \right\}, \quad (183)
\]
\[
\mathcal{P}_{3,B,a}(t) = \sum_{j=1}^{3} c_{3,B,a,j} t^{j+1}; \quad (184)
\]
\[
c_{3,B,A,1,1} = 4 \mathfrak{a}^2 \mathfrak{L}_1^2 [\mathfrak{A}^2 (\mathfrak{B}^2 - 2) + 2 \mathfrak{L}_2^2] \cos \mathfrak{f} \sin \beta, \quad (185)
\]
\[
c_{3,B,A,1,2} = \mathfrak{A} \mathfrak{B} \{3 \mathfrak{A}^2 (2 \mathfrak{B}^2 - 13) + 13 \mathfrak{L}_2^2 \} (\cos \beta \sin \mathfrak{f} - \mathfrak{b}' \cos \mathfrak{f}) + \mathfrak{A} [8 + 3 \mathfrak{B}^2 - 5 \mathfrak{B}^4 + 2 \mathfrak{A}^2 (9 \mathfrak{B}^2 + 2 \mathfrak{B}^4 - 8)] \cos \mathfrak{f} \sin \mathfrak{f} \sin \beta, \quad (186)
\]
\[
c_{3,B,A,2,1} = -4 \mathfrak{A} \mathfrak{a}^3 \mathfrak{B} \frac{3 \mathfrak{L}_2^2 - \mathfrak{A}^2 (7 + 2 \mathfrak{B}^2)}{\cos \mathfrak{f} \sin \mathfrak{f} \sin \beta), \quad (187)
\]
\[
c_{3,B,A,2,2} = \mathfrak{B} [3 \mathfrak{L}_2^2 - \mathfrak{B}^2 (2 \mathfrak{B}^2 - 3)] (\mathfrak{b}' \cos \mathfrak{f} - \cos \beta \sin \mathfrak{f}) + \mathfrak{B} \cos \beta \sin \mathfrak{f}, \quad (188)
\]
\[
c_{3,B,A,3,1} = -8 \mathfrak{a} \mathfrak{L}_1^2 \cos \mathfrak{f} \sin \beta, \quad (189)
\]
\[
c_{3,B,A,3,2} = -2 \mathfrak{A} \mathfrak{L}_1^2 \{ \cos \mathfrak{f} [\mathfrak{B} \mathfrak{b}' + (4 + 5 \mathfrak{B}^2) \sin \mathfrak{f} \sin \beta] - \mathfrak{B} \cos \beta \sin \mathfrak{f} \}, \quad (190)
\]
\[
c_{3,B,A,4,1} = 0, \quad (191)
\]
\[
c_{3,B,A,4,2} = -6 \mathfrak{L}_1^2 ((\mathfrak{b}' + \mathfrak{B} \sin \mathfrak{f} \sin \beta) \cos \mathfrak{f} - \cos \beta \sin \mathfrak{f}), \quad (192)
\]
\[
\mathcal{P}_{3,B,A}(t) = \frac{\Delta_1}{24 \Delta_1^6} \sum_{i=1}^{3} \sum_{j=1}^{2} c_{3,B,A,i,j} t^{i-1} r^{j-1}; \quad (193)
\]
\[
\Omega_{3,B,3} = \frac{\mathfrak{L}_1 \Delta_1}{\mathfrak{A} \mathfrak{B} + \Delta_2 + \Delta_1^2 t}, \quad (194)
\]
\[
c_{3,B,B,3,1} = -4 \mathfrak{a} \mathfrak{a}' \mathfrak{B} \mathfrak{L}_1^2 \cos \mathfrak{f} \sin \beta, \quad (195)
\]
\[
c_{3,B,B,3,2} = \mathfrak{B} (5 \mathfrak{L}_2^2 - 4 \mathfrak{L}_1^2) \cos \mathfrak{f} \sin \mathfrak{f} \sin \beta + [4 \mathfrak{L}_1^2 + 5 \mathfrak{L}_2^2 - 4 \mathfrak{L}_1^2 (1 + \mathfrak{L}_2^2)] (\mathfrak{b}' \cos \mathfrak{f} - \cos \beta \sin \mathfrak{f}), \quad (196)
\]
\[
\mathcal{P}_{3,B,B}(t) = \frac{\mathfrak{L}_1^2 \arctan(\Omega_{3,B,3})}{4 \mathfrak{L}_1^2} \sum_{j=1}^{2} c_{3,B,B,3,j} r^{j-1}. \quad (197)
\]

The parallel case

In this case, the CLD takes the form of equation [122] and this integral always is a linear combination of two integrals the integrands of which, as it
results from equations \(107\), \(120\), \(121\) and \(117\), have the form

\[
\mathcal{f}_p(y, \varphi) \equiv A + B \cos \varphi + C \sin \varphi + D y
\]  

(198)

where constants \(A, B, C, D\) depend on the contribution dictated by the inequality fulfillment. For instance, if we had to consider \(L'(y)\), we would find:

\[
A = A', \quad B = -\sqrt{r^2 - h^2}, \quad C = B' \sqrt{r^2 - h^2} \quad \text{and} \quad D = B'.
\]

The \(y\)-primitive of (198) is (setting the integration constant equal to zero)

\[
\int \mathcal{f}_p(y, \varphi) \equiv (A + B \cos \varphi + C \sin \varphi) y + D y^2/2.
\]  

(199)

Each of the end points of the \(y\)-integration domain has the general form: \(C + D \sin \varphi\) with \(C\) and \(D\) suitable constants depending on the geometrical parameters. Hence, the explicit determination of integral \(122\) is certainly possible if we explicitly know the \(\varphi\)-primitive of \(\mathcal{f}_p(C + D \sin \varphi, \varphi)\). The explicit expression of this primitive reads

\[
\Psi_p(\varphi) = \frac{4 A C + 2 D C^2 + 2 C D + D D^2}{4} \varphi +
\]

\[
B C \sin \varphi - \frac{D(2 C + D D)}{8} \sin(2\varphi) -
\]

\[
[C C + (A + D C) D] \cos \varphi - \frac{B D}{4} \cos(2\varphi).
\]  

(200)
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