SINGULAR PERTURBATION BY BENDING FOR AN ADHESIVE OBSTACLE PROBLEM

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Abstract. A free boundary problem arising from materials science is studied in one-dimensional case. The problem studied here is an obstacle problem for the non-convex energy consisting of a bending energy, tension and an adhesion energy. If the bending energy, which is a higher order term, is deleted then “edge” singularities of the solutions (surfaces) may occur at the free boundary as Alt-Caffarelli type variational problems. The main result of this paper is to give a singular limit of the energy utilizing the notion of Γ-convergence, when the bending energy can be regarded as a perturbation. This singular limit energy only depends on the state of surfaces at the free boundary as seen in singular perturbations for phase transition models.

1. Introduction

1.1. Model and main results. Let us consider a non-convex higher order variational problem in one-dimensional case, which is the obstacle problem for the energy as proposed in [24]:

\[ \text{Minimize } E_\varepsilon[u] = \varepsilon^2 \int \kappa^2 ds + \int ds - \int_{\{u = \psi\}} (1 - \alpha) \, ds. \]  

Here a smooth function \( \psi : [a, b] \to \mathbb{R} \) called an obstacle (function), a constant coefficient \( \varepsilon > 0 \), and a continuous function \( \alpha : [a, b] \to (0, 1) \) are given. The function \( u \) is an admissible function constrained above the obstacle, and \( \kappa, s \) denote the curvature and the arclength of the graph of \( u \) respectively. The first term of the energy is called bending energy, the second one tension, the third one adhesion energy, and \( \alpha \) is called adhesion coefficient. The multiple constant of the tension is normalized to one. The adhesion coefficient \( \alpha \) can be inhomogeneous so that it may depend on a space variable. According to [24], this problem is motivated to determine the shape of membranes, interfaces or filaments on rippled surfaces (as Figure 1) in certain mesoscopic or nearly mesoscopic settings, and the coefficients \( \varepsilon, \alpha \) and the obstacle function \( \psi \) depend on the setting of materials, scaling and so on. In this case the graph of \( u \) is a membrane. In this paper we consider one dimensional model so when one considers a membrane it depends on only...
One direction and invariant in other direction in our setting. In addition, we regard the bending energy which is a higher order term as a perturbation, that is, only consider for sufficiently small $\varepsilon > 0$.

A characteristic point of this energy is to contain the adhesion energy. By this energy, surfaces shall reasonably adhere to the obstacle in order to decrease the energy. Accordingly, there may occur patterns as drawn in Figure 1 in this model. However, its law is complicated. One of physical and mathematical concerns is to perceive such pattern formation.

We shall first take the simplest approximation $\varepsilon = 0$ in order to consider the case $\varepsilon$ is sufficiently small. This approximation simplifies the problem (1.1), thus we can obtain many fine properties for minimizers (this is one of the important results of this paper, see Theorem 3.4). It is rigorously stated in Theorem 3.4, but roughly speaking the shape of any minimizer of $E_0$ is as drawn in Figure 2. There occur “edge” singularities at the free boundary and their angles are determined by the adhesion coefficient at the place of the free boundary, symbolically, $\cos \theta = \alpha$ (Young’s equation). This condition has been formally given in 24.

However, even if $\varepsilon = 0$, the energy is not convex and may admit that there exist multiple minimizers lacking consistency in their shape. That is to say, for instance, either of two different states as drawn in Figure 2 might be a minimizer for the same energy. Since this minimizing problem is considered to be a physical model, this non-uniqueness may be due to some small effect, perhaps, of higher order terms. Therefore, restoring the effect of the bending energy, it is expected to ameliorate the approximation.

Figure 1. membranes on rippled surface

Figure 2. minimizers in the case $\varepsilon = 0$
The main goal of this paper is a formulation of a singular perturbation by the bending energy, that is, to characterize the limit of minimizers of $E_\varepsilon$ as $\varepsilon \downarrow 0$ rigorously. To this end, we utilize the notion of $\Gamma$-convergence established by De Giorgi [13] in the 70's (more precisely, see [6], [7], [11], [12]) which is a convergence of energy functionals for minimizing problems. The idea of this convergence is to identify the first nontrivial term in an asymptotic expansion for the energy of perturbed problems. Our main singular limit result is rigorously stated in Theorem 4.2, but roughly speaking as follows:

**Theorem 1.1 (Singular limit).** If we define the singular limit energy $F$ by

$$F[u] := \int_{\partial\{u > \psi\}} 4(\sqrt{2} - \sqrt{1 + \alpha}) \, d\mathcal{H}^0,$$

then the $\Gamma$-convergence holds with respect to $W^{1,1}$-norm:

$$\frac{1}{\varepsilon}(E_\varepsilon - \inf_X E_0) \rightharpoonup^\Gamma F \quad \text{as} \quad \varepsilon \downarrow 0,$$

where $X$ is certain space of admissible functions $X \subset W^{1,1}(a,b)$.

From this theorem, the following asymptotic expansion is also valid for our problem in some sense (see [2]):

$$\inf E_\varepsilon = \inf E_0 + \varepsilon \inf F + o(\varepsilon).$$

Thus it means that $F$ is the main effect of the bending energy and its quantity only depends on the $\mathcal{H}^0$-measure (the number) of the free boundary and the adhesion coefficient at there. Since the adhesion coefficient at the free boundary determines the contact angle there ($\cos \theta = \alpha$) as mentioned above, it also means that the effect is determined by the number of “edge” singularities and their angles. Thanks to our singular limit result and geometrical intuition for the minimizing problem of $F$, we can easily find a key effect of the bending energy. For example, if $\alpha \equiv \text{const.}$ and there are multiple minimizers of $E_0$, then the one which has less number of edges shall be a minimizer of $E_\varepsilon$ when $\varepsilon > 0$ is sufficiently small. More precisely, by Theorem 1.1 and the fundamental theorem of $\Gamma$-convergence, we obtain the following:

**Corollary 1.2.** If $u_\varepsilon \to u$ in $W^{1,1}$ and any $u_\varepsilon$ is a minimizer of $E_\varepsilon$ in (1.1), then $u$ minimizes $F$ among minimizers of $E_0$ in (1.1).

It turns out that we can characterize the limit of minimizers of (1.1) as $\varepsilon \downarrow 0$ by our theorem.

An important point of our proof of Theorem 1.1 is to prove the liminf condition of $\Gamma$-convergence (see Definition 4.1). The proof is mainly separated into two parts. The first part is to prove that it suffices to consider more regular sequences which are “close” to a minimizer of $E_0$ in some sense. To state it rigorously, we introduce a notion called $\delta$-associate which explains closeness of functions. Especially, the part regarding how to coincide with
ψ is a key point because adhering or detaching leads to a discontinuous transition in our energy. We replace a general sequence by δ-associate of $W^{2,1}$-regularity converging to a minimizer of $E_0$ so that all quantities in the energy are well-defined with no increase of the energy. The second part is to obtain a lower estimate for functions δ-associated with a minimizer of $E_0$. By this procedure, we are able to handle the energy geometrically and establish a Modica-Mortola type inequality to prove the lim inf condition.

1.2. Related problems. Now let us briefly survey some mathematical problems related to our problem from two viewpoints.

1.2.1. Viewpoint of energies and settings. In the non-adhesive case $\alpha \equiv 1$, the energy in (1.1) is consisting of the total squared curvature functional and the length functional:

\[ \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma ds, \tag{1.2} \]

so-called Euler’s elastic energy. The critical points of the energy are called elastica (usually under boundary conditions and the length constraint for a curve $\gamma$). This problem is first considered by Euler in 1744 [14]. Numerous authors have considered this variational problem or related ones under various constraints in order to analyze the configuration of elastic bodies (see [20], [29], or [27] including a very well-written summary of the history of elastic problems by one section). However, there are still many unclear points in this problem because of difficulties of a higher order problem. Our problem is the variational problem for the Euler’s elastic energy with the adhesion term under an obstacle-type constraint, thus it may have similar difficulties. To circumvent this difficulty, we regard the bending energy as a perturbation in this paper.

The obstacle problem, which is the variational problem under an obstacle-type constraint, is one of the motivating problems invoking free boundaries and has been studied for a long time ([9], [25], [26]). A typical model is for the area functional (or the linearized one: Dirichlet energy):

\[ \text{Minimize } \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx \quad \text{or} \quad \frac{1}{2} \int_\Omega |
abla u|^2 \, dx \] 

under a boundary condition, where $u$ is a function on a bounded smooth domain $\Omega \subset \mathbb{R}^n$. This is so-called the unilateral Plateaux problem. This problem invokes the free boundary $\partial \{ u > \psi \}$ and in the non-coincidence set $\{ u > \psi \}$ the graph of a solution $u$ is a minimal surface (or harmonic). Generally, if an energy is convex, bounded and coercive in the sense as (1.3), then the classical variational inequality approach can work, thus we can obtain many fine properties of solutions (at least in the linearized case), for instance the uniquely existence, further the regularity of solutions and of its free boundary [9]. The problem (1.3) corresponds to the case non-bending $\varepsilon = 0$ and non-adhesion $\alpha \equiv 1$ in our problem (1.1). By the way, even if we
perturb it by bending, we only obtain a natural singular limit [26, Theorem 9.5]. Therefore, in non-adhesive obstacle problems, it seems that higher order terms can be neglected if $\varepsilon \ll 1$.

Our problem is closely related to the Alt-Caffarelli type variational problems [1] which is a model of cavitation:

$$\text{(1.4)} \quad \text{Minimize: } \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\{u > 0\}} Q^2 \, dx$$

under a boundary condition, where $Q$ is certain function. This is not an obstacle problem explicitly but the solutions are automatically constrained above 0, thus it is equivalent to the problem with the constraint $u \geq \psi \equiv 0$ (flat obstacle problem). This problem is one of the important interface models and generalized variously (for example, see [23] generalizing the first term, [3] the second term, or their references). Especially, Yamaura [31] considered a non-linearized case, that is, the first term is replaced to the area functional. Our problem is a generalization of (one-dimensional) non-linearized Alt-Caffarelli problem regarding the obstacle. Indeed, if $\varepsilon = 0$ and $\psi \equiv 0$ in (1.1) then it is equivalent to

$$\text{Minimize: } \int_{a}^{b} \sqrt{1 + u_x^2} \, dx + \int_{\{u > 0\}} (1 - \alpha) \, dx$$

since $u_x \equiv 0$ in $\{u = 0\}$. Thus the result in this paper is particularly valid for (1.4) with area functional instead of the Dirichlet energy and with continuous $Q : [a, b] \to (0, 1)$.

1.2.2. Viewpoint of singular perturbations. In view of singular perturbation by $\Gamma$-convergence for variational problems, there are some related results especially in phase transition models.

One of the most celebrated results is, owing to Modica-Mortola [22] (and [21], [30]), for the energy arising from the van der Waals-Cahn-Hilliard theory of fluid-fluid phase transitions:

$$\varepsilon \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(u) \, dx,$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ satisfying a volume constraint and $W$ is a double-well potential function, usually it is taken as $W(t) := (1 - t^2)^2 \, (t \in \mathbb{R})$. They proved that the singular limit ($\Gamma$-limit) of the energy is proportional to the area of a transition layer. These works are generalized to the vector-valued case, anisotropic cases and multi-well potential cases (see [5] and references cited there).

Furthermore, there are several higher order version results for the following energy:

$$\varepsilon \int_{\Omega} |\nabla^2 u|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} W(\nabla u) \, dx.$$
One of the earliest studies is for the functional arising from the theory of smectic liquid crystals introduced by Aviles-Giga [4]. It is considered the case that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ satisfies certain boundary conditions and $W$ is a “single-circle-well” potential $W(\xi) := (1 - |\xi|^2)^2$ ($\xi \in \mathbb{R}^2$).

The energy (1.5) or (1.6) or similar one also arises from, for example, the theory of solid-solid phase transitions, thin-films and magnetism, and there are several singular limit results for them ([10], [17], [18], [19], [28]). However, the number of results of higher order singular perturbations are limited compared with the first order cases because of its difficulty.

Our result is one of the higher order singular perturbations and means that the problem (1.1) can be regarded as a phase transition model as above. The cost of transition between the phase $\{u = \psi\}$ and $\{u > \psi\}$ is determined by the place of a transition layer.

As mentioned above, a key point in our proof of the liminf condition of $\Gamma$-convergence is to reduce a general sequence to a sequence which is easy to handle. We mention that this concept resembles, for example, the “slicing” technique used in [16].

1.3. Organization. This paper is organized as follows: In §2 we prepare some notations and definitions. In §3 we first consider the case $\varepsilon = 0$ and derive some properties of the minimizers of $E_0$. They are useful to prove our singular limit result. In §4 we state our main singular limit theorem (Theorem 4.2), and we prove it in §5 and §6.

2. Energy and function spaces

In this section, we give an energy functional by introducing several notations. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $\psi : \overline{\Omega} \to \mathbb{R}$ be a smooth function and $g : \overline{\Omega} \to \mathbb{R}$ be a smooth function satisfying $g \geq \psi$ on a given $\mathcal{H}^{n-1}$-measurable subset $\Sigma$ of the boundary $\partial \Omega$, where $\mathcal{H}^{n-1}$ denotes the $(n - 1)$-dimensional Hausdorff measure. For any positive $m \in \mathbb{Z}$ and $1 \leq p \leq \infty$, we define the space of admissible functions $X_{m,p}^\psi,g,\Sigma(\Omega)$ by

$$X_{m,p}^\psi,g,\Sigma(\Omega) := \left\{ u \in W_{m,p}(\Omega) \left| \begin{array}{c} u \geq \psi \text{ in } \Omega, \\ u = g \text{ on } \Sigma \end{array} \right. \right\}. \quad (2.1)$$

The boundary condition is in the sense of trace. The space $X_{m,p}^\psi,g,\Sigma(\Omega)$ is a non-empty, convex and closed set in the Sobolev space $W_{m,p}(\Omega)$. Usually it is assumed that the partial boundary $\Sigma$ is not empty, however we do not assume in this paper since our problem is not trivial even if $\Sigma = \emptyset$.

Let $\alpha : [a, b] \to \mathbb{R}$ be a continuous function satisfying $0 < \underline{\alpha} \leq \overline{\alpha} < 1$, where $\underline{\alpha} := \min \alpha$ and $\overline{\alpha} := \max \alpha$. For $\varepsilon \geq 0$, we define energy functionals $E_\varepsilon$ by

$$E_\varepsilon[u] := \varepsilon^2 \int_\Omega H^2_u \sqrt{1 + |\nabla u|^2} \, dx + \int_\Omega \overline{\alpha}[u] \sqrt{1 + |\nabla u|^2} \, dx, \quad (2.2)$$
where \( H_u \) is the mean curvature of \( u \):
\[
H_u := \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right),
\]
the coefficient \( \tilde{\alpha} \) is the redefined adhesion coefficient:
\[
\tilde{\alpha}[u](\cdot) := \vartheta(\cdot, u(\cdot) - \psi(\cdot))
\]
and \( \vartheta_\alpha : [a, b] \times \mathbb{R} \to [\underline{\alpha}, 1] \) such that
\[
\vartheta_\alpha(x, y) := \begin{cases} 
1 & (y > 0), \\
\alpha(x) & (y \leq 0). 
\end{cases}
\]

\( E_\varepsilon \) is well-defined on \( W^{2,1}(\Omega) \) for all \( \varepsilon \geq 0 \) and especially \( E_0 \) is well-defined on \( W^{1,1}(\Omega) \). Throughout this paper, we fix \( \Omega, \psi, g, \Sigma \) and \( \alpha \).

**Definition 2.1.** Let \( u \in X^{m,p}(\Omega) \cup C(\overline{\Omega}) \). We set
\[
\Omega_u^0 := \{ u = \psi \} = \{ x \in \Omega \mid u(x) = \psi(x) \} \subset \Omega,
\]
\[
\Omega_u^+ := \{ u > \psi \} = \{ x \in \Omega \mid u(x) > \psi(x) \} \subset \Omega,
\]
\[
\partial \Omega_u^+ := \partial \{ u > \psi \} \cap \Omega \subset \Omega.
\]
We call \( \Omega_u^0 \) coincidence set, \( \Omega_u^+ \) non-coincidence set and \( \partial \Omega_u^+ \) free boundary.

Note that \( \Omega_u^+ \) is open in \( \Omega \), \( \Omega_u^0 \) and \( \partial \Omega_u^+ \) are closed in \( \Omega \) and \( \Omega = \Omega_u^0 \cup \Omega_u^+ \).

3. Minimizers of \( E_0 \)

In this section, we derive some properties of the minimizers of \( E_0 \) in one-dimensional case. In §3.1 we verify the lower semicontinuity of \( E_0 \) and in §3.2 we derive properties to determine the shape of minimizers of \( E_0 \). These are useful to prove Theorem 4.2 which is our main singular limit result.

3.1. Lower semicontinuity of \( E_0 \).

**Lemma 3.1.** Let \( \psi : \overline{\Omega} \to \mathbb{R} \) be a smooth function, \( h : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty) \) be a Borel function and \( E[u] := \int_{\Omega} h(x, u - \psi, \nabla u) \, dx \). If \( h(x, \cdot, \cdot) \) is lower semicontinuous for a.e. \( x \in \Omega \), then \( E : W^{1,1}(\Omega) \to [0, \infty] \) is lower semicontinuous.

**Proof.** Fix any convergent sequence \( u^\varepsilon \to u \) in \( W^{1,1} \). For any subsequence, there exist a subsequence such that
\[
u^\varepsilon_j \to u \quad \text{and} \quad \nabla u^\varepsilon_j \to \nabla u \quad \text{a.e. in} \ \Omega.
\]
Then
\[
\liminf_{j \to \infty} E[u^\varepsilon_j] = \liminf_{j \to \infty} \int_{\Omega} h(x, u^\varepsilon_j - \psi, \nabla u^\varepsilon_j) \, dx \\
\geq \int_{\Omega} \liminf_{j \to \infty} h(x, u^\varepsilon_j - \psi, \nabla u^\varepsilon_j) \, dx \\
\geq \int_{\Omega} h(x, u - \psi, \nabla u) \, dx = E[u].
\]
The first inequality follows by Fatou’s lemma, and the second one follows by the lower semicontinuity of \( h \). Thus we get the consequence. \( \square \)

**Proposition 3.2.** \( E_0 : X^{1,1}(\Omega) \to [0, \infty) \) is lower semicontinuous.

**Proof.** By taking \( h(x, y, \xi) := \partial_\alpha(x, y) \sqrt{1 + |\xi|^2} \) in Lemma 3.1 and the lower semicontinuity of \( \partial_\alpha(x, \cdot) \).

\( \square \)

### 3.2. Property in one-dimensional case.

Now we assume \( n = 1 \) and denote \( \Omega = (a, b) \). Note that \( X^{m,p}(a, b) \subset C^{m-1}(a, b) \) in this case. To state the main theorem in this subsection, we introduce some terminologies.

**Definition 3.3** (Partitional regularity and contact angle).

1. Let \( N \in \mathbb{Z}_{\geq 0} \). We say that \( \{x_i\}_{i=0}^{N+1} \subset \mathbb{R} \) is a partition of an interval \((a, b)\) in \( \mathbb{R} \) if \( a = x_0 < x_1 < \cdots < x_{N+1} = b \).
2. Let \( u \in X^{1,1}(a, b) \). We say that \( u \) is partitionally regular if there exist a number \( N \in \mathbb{N} \) and a partition \( \{x_i\}_{i=0}^{N+1} \) of \((a, b)\) such that:
   - (a) \((x_0, x_1) \subset (a, b)_0^u \) or \((x_0, x_1) \subset (a, b)_+^u \),
   - (b) if \((x_{i-1}, x_i) \subset (a, b)_0^u \) then \((x_i, x_{i+1}) \subset (a, b)_0^u \),
   - (c) if \((x_{i-1}, x_i) \subset (a, b)_+^u \) then \((x_i, x_{i+1}) \subset (a, b)_+^u \).

Note that \( \partial(a, b)_+^u = \{x_1, \ldots, x_N\} \) and \( u \equiv \psi \) or \( u > \psi \) in \((x_i, x_{i+1})\) alternately. We denote such regularity by \([x_0; \ldots; x_{N+1}]\)-regular when we want to show the partition explicitly.
3. Let \( u \) be a function. If \( u_x \) exists at \( x \), we define a tangent angle \( \theta_u(x) = \arctan u_x(x) \).
4. Let \( u \in X^{1,1}(a, b) \) which is \([x_0; \ldots; x_{N+1}]\)-regular and \( N > 0 \). For \( 1 \leq i \leq N \) with \((x_i, x_{i+1}) \subset (a, b)_+^u \) (resp. \((a, b)_0^u \)), we define the contact angle \( \theta_i \in [0, \pi) \) of \( u \) with \( \psi \) at \( x_i \) by
   \[
   \theta_i := \theta_u(x_i+) - \theta_\psi(x_i) \quad (\text{resp. } \theta_\psi(x_i) - \theta_u(x_i-))
   \]
   if \( \theta_u(x_i\pm) := \lim_{x \to x_i\pm} \theta_u(x) \) can be defined.

Denote the set of the minimizers of a functional \( E : X^{m,p}(\Omega) \to [0, \infty] \) by

\[
\text{argmin}_{X^{m,p}} E := \left\{ u \in X^{m,p}(\Omega) \mid E[u] = \inf_{X^{m,p}(\Omega)} E \right\}.
\]

The following is the main theorem in this subsection.

**Theorem 3.4.** Let \( \Omega = (a, b) \). If \( \bar{u} \in \text{argmin}_{X^{1,1}} E_0 \) then there exists a partition \( \{\bar{x}_i\}_{i=0}^{N+1} \) of \((a, b)\) such that \( \bar{u} \) is \([\bar{x}_0; \ldots; \bar{x}_{N+1}]\)-regular, the graph of \( \bar{u} \) is a segment on any interval \((\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)^\bar{u}_+ \) and the contact angle \( \theta_i \) of \( \bar{u} \) at \( \bar{x}_i \) satisfies \( \cos \theta_i = \alpha(\bar{x}_i) \) for any \( 1 \leq i \leq N \).

**Remark 3.5.** This theorem especially means that the regularity of any minimizer of \( E_0 \) in \( X^{1,1}(a, b) \) is up to \( X^{1,\infty}(a, b) \) (further, piecewise smooth) and this is optimal. Moreover, it is remarkable that the number of the connected components of the non-coincidence set of a minimizer is finite in any case. This is not valid without the adhesion energy.
To prove Theorem 3.4, we prepare the following lemmas: Lemma 3.6–3.8.

**Lemma 3.6.** There exists $\delta_{\psi,\alpha} > 0$ determined by $\psi$ and $\alpha$ such that for any interval $I \in (a,b)$ whose width is less than $\delta_{\psi,\alpha}$ the inequality holds:

$$\inf_I \sqrt{1 + \psi_x^2} - \alpha \sup_I \sqrt{1 + \psi_x^2} \geq \frac{1 - \alpha}{2}.$$  

**Proof.** For any interval $I$, we have

$$\inf_I \sqrt{1 + \psi_x^2} - \alpha \sup_I \sqrt{1 + \psi_x^2} \geq \frac{1 - \alpha}{\alpha} \left( \frac{1}{\alpha} - \left( \frac{\inf_I \sqrt{1 + \psi_x^2} - \alpha \sup_I \sqrt{1 + \psi_x^2}}{\alpha} \right) \right).$$

Since $\psi$ is uniformly continuous, if the width of $I$ is sufficiently small then

$$\sup_I \sqrt{1 + \psi_x^2} - \inf_I \sqrt{1 + \psi_x^2} \leq \frac{1 - \alpha}{2\alpha},$$

thus we get the consequence. □

**Lemma 3.7.** Let $u \in X^{1,1}(a,b)$, $0 < \delta \leq \delta_{\psi,\alpha}$ and $I \subset (a,b)$ be an interval whose width is $\delta$. Suppose that $u = \psi$ on $\partial I$ and $u > \psi$ at some interior point of $I$. Then

$$v := \begin{cases} \psi & \text{in } I, \\ u & \text{in } (a,b) \setminus I, \end{cases}$$

is in $X^{1,1}(a,b)$ and satisfies $E_0[u] > E_0[v]$.

**Proof.** We first prove the case that $u > \psi$ at any interior point of $I$. $v \in X^{1,1}(a,b)$ is obvious since $u = \psi$ on $\partial I$. Denote the endpoints of $I$ by $y_0$, $y_1$. Then we have

$$E_0[u] - E_0[v] = \int_{y_0}^{y_1} \sqrt{1 + u_x^2} \, dx - \int_{y_0}^{y_1} \alpha \sqrt{1 + \psi_x^2} \, dx \geq \sqrt{|y_1 - y_0|^2 + |u(y_1) - u(y_0)|^2} - |y_1 - y_0| \alpha \max_{[y_0,y_1]} \sqrt{1 + \psi_x^2} \geq |y_1 - y_0| \left( \min_{[y_0,y_1]} \sqrt{1 + \psi_x^2} - \alpha \max_{[y_0,y_1]} \sqrt{1 + \psi_x^2} \right) \geq |y_1 - y_0| \left( \frac{\psi(y_1) - \psi(y_0)}{y_1 - y_0} - \frac{\psi(y_1) - \psi(y_0)}{y_1 - y_0} \alpha \max_{[y_0,y_1]} \sqrt{1 + \psi_x^2} \right) \geq |y_1 - y_0| \left( \frac{\psi(y_1) - \psi(y_0)}{y_1 - y_0} - \alpha \max_{[y_0,y_1]} \sqrt{1 + \psi_x^2} \right) \geq \frac{\delta \cdot (1 - \alpha)}{2} > 0.$$

The last inequality follows by Lemma 3.6.

Next we prove the general case. Since $u$ is continuous, we can decompose $I \cap (a,b)_+(\neq \emptyset)$ into at most countable disjoint open intervals $I_i$ ($1 \leq i \leq N$,}
$N$ is a positive integer or $\infty$). Note that $u = \psi$ at any endpoint of $I_i$. Denote $v_0 := u$, and for all $1 \leq i \leq N$ define $v_i \in X^{1,1}(a, b)$ by

$$v_i := \begin{cases} \psi & \text{in } I_i, \\ v_{i-1} & \text{in } (a, b) \setminus I_i. \end{cases}$$

By the above argument, we get $E_0[v_{i-1}] > E_0[v_i]$ for any $1 \leq i \leq N$. Therefore, if $N < \infty$ then $v = v_N$ and $E_0[u] > E_0[v]$. If $N = \infty$, then $v_i \to v$ in $X^{1,1}(a, b)$ and $E_0[u] > \liminf_{i \to \infty} E[v_i] \geq E_0[v]$ by Proposition 3.2. The proof is completed. \hfill \square

**Lemma 3.8.** If $\bar{u} \in \arg\min_{X^{1,1}} E_0$ then $\bar{u}_{xx} \equiv 0$ in $(a, b)_+^0$, i.e. the graph of $\bar{u}$ is a segment on any connected component of $(a, b)_+^0$.

**Proof.** Since $(a, b)_+^0$ is open, $\bar{u}$ is a minimal surface in $(a, b)_+^0$. \hfill \square

We now prove Theorem 3.4.

**Proof of Theorem 3.4.** We first prove by contradiction that the numbers of the connected components of $(a, b)_0^0$ and $(a, b)_0^0$ are finite. If either of them is infinite, then so is the other, thus there exist $\delta$ and $I$ as in Lemma 3.7, therefore $\bar{u}$ is not a minimizer. This is a contradiction.

Next we prove any connected component of $(a, b)_0^0$ is not a point (but an interval) by contradiction. If there is a connected component $\{x^*\} \subset (a, b)_0^0$, then there exists $r^* > 0$ such that $I_{r^*} := \{y| 0 < \pm(y - x^*) \leq r^*\} \subset (a, b)_0^0$. Note that $\bar{u}$ is straight on $I_{r^*} \cup \bar{u}_{x+}(x^*) \geq \psi_x(x^*) \geq \bar{u}_{x-}(x^*)$. If $\bar{u}_{x+}(x^*) > \bar{u}_{x-}(x^*)$ then the function $v^r$ which is equal to $\bar{u}$ out of $[x^* - r^*, x^* + r^*]$ and a segment in $[x^* - r^*, x^* + r^*]$ is contained in $X^{1,1}(a, b)$ and $E_0[\bar{u}] > E_0[v^r]$ holds. This contradicts the minimality of $\bar{u}$. Thus we can assume $\bar{u}_{x+}(x^*) = \psi_x(x^*) = \bar{u}_{x-}(x^*)$. For $0 < r < r^*$ if we take

$$v^r := \begin{cases} \bar{u} & \text{in } (a, x^*] \cup [x^* + r^*, b), \\ \psi & \text{in } (x^*, x^* + r], \\ \text{segment} & \text{in } (x^* + r, x^* + r^*), \end{cases}$$

then there exists a sequence $r' \downarrow 0$ such that $v^r' \in X^{1,1}(a, b)$ and we can see

$$E_0[v^r'] - E_0[\bar{u}] = \int_{x^*}^{x^* + r'} \alpha \sqrt{1 + \psi_x^2} \, dx$$

$$+ \sqrt{|r^* - r'|^2 + |\bar{u}(x^* + r^*) - \psi(x^* + r')|^2} - \sqrt{|r^*|^2 + |\bar{u}(x^*) - \bar{u}(x^*)|^2}. $$

Thus, noting $\bar{u}(x^*) = \psi(x^*)$, $\bar{u}_{x+}(x^*) = \psi_x(x^*)$ and $r^* \bar{u}_{x+}(x^*) = \bar{u}(x^* + r^*) - \bar{u}(x^*)$, we can compute as

$$\lim_{r' \downarrow 0} \frac{E_0[v^r'] - E_0[\bar{u}]}{r'} = \alpha(x^*) \sqrt{1 + \bar{u}_{x+}^2(x^*)} - 1 + \bar{u}_{x+}^2(x^*) < 0.$$ 

This also contradicts the minimality of $\bar{u}$. Therefore, we find any connected component of $(a, b)_0^0$ is an interval.
Finally, we compute the contact angles of \( \bar{u} \). Fix any \( 1 \leq i \leq N \). We only consider the case \( (\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)^2 \) (the case \( (\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)^2_0 \) is similar). From the above argument, there exists \( r_i^* > 0 \) such that \( [\bar{x}_i - r_i^*, \bar{x}_i] \subset (a, b)_0^\varepsilon \) and \( (\bar{x}_i, \bar{x}_i + r_i^*) \subset (a, b)^2_\varepsilon \). Moreover, \( \bar{u} \) is straight in \( (\bar{x}_i, \bar{x}_i + r_i^*) \) and \( \bar{u}_{x_+}(\bar{x}_i) > \psi_\alpha(\bar{x}_i) = \bar{u}_{x_-}(\bar{x}_i) \). Thus, by perturbing \( \bar{u} \) near \( \bar{x}_i \) for sufficiently small \( 0 < r < r_i^* \) as above, we can similarly compute as

\[
\lim_{r \to \pm 0} \frac{E_0[u^r] - E_0[\bar{u}]}{r} = \alpha(\bar{x}_i) \sqrt{1 + \psi_\alpha^2(\bar{x}_i)} - \frac{1 + \psi_\alpha(\bar{x}_i) \bar{u}_{x_+}(\bar{x}_i)}{\sqrt{1 + \bar{u}_{x_+}^2(\bar{x}_i)}}.
\]

Since \( \bar{u} \) is a minimizer, the right term has to be zero, thus we obtain

\[
\alpha(\bar{x}_i) = \frac{1 + \psi_\alpha(\bar{x}_i) \bar{u}_{x_+}(\bar{x}_i)}{\sqrt{1 + \psi_\alpha^2(\bar{x}_i) \sqrt{1 + \bar{u}_{x_+}^2(\bar{x}_i)}}}
\]

\[= \cos \theta_u(\bar{x}_i+) \cos \theta_\psi(\bar{x}_i) + \sin \theta_u(\bar{x}_i+) \sin \theta_\psi(\bar{x}_i)
\]

\[= \cos (\theta_u(\bar{x}_i+) - \theta_\psi(\bar{x}_i)) = \cos \theta_i.
\]

This completes the proof. \( \square \)

**Remark 3.9.** If \( \bar{u}(a) = \psi(a) \) and \( (x_0, x_1) \subset (a, b)^2 \) \((x_0 = a)\), then we can obtain \( \theta_u(a+) - \theta_\psi(a+) \geq \alpha(a) \) by the similar computation. Similarly we can obtain \( \theta_u(b-) - \theta_\psi(b-) \geq \alpha(b) \).

**Remark 3.10.** Since we only used local perturbations in the above arguments, the results in this section are also valid for local minimizers \( \bar{u} \) of \( E_0 \), that is, there exists \( \varepsilon > 0 \) such that \( E_0[u] \geq E_0[\bar{u}] \) for any \( \|u - \bar{u}\|_{H^{1,1}} < \varepsilon \).

## 4. \( \Gamma \)-CONVERGENCE

Now we rigorously state our main \( \Gamma \)-convergence result in one-dimensional setting. We set \( \Omega = (a, b) \) and let \( F_\varepsilon, F : X^{1,1}(a, b) \to [0, \infty] \) be defined by

\[
F_\varepsilon[u] := \begin{cases}
\varepsilon \int_a^b \kappa_u^2 \sqrt{1 + u_x^2} \, dx & (u \in X^{2,1}(a, b)), \\
\infty & (\text{otherwise}),
\end{cases}
\]

where \( \kappa_u = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} \) (curvature) and

\[
F[u] := \begin{cases}
\int_{\partial(a, b)^2} 4(\sqrt{2} - \sqrt{1 + \alpha}) \, dH^0 & (u \in \text{argmin}_{X^{1,1}} E_0), \\
\infty & (\text{otherwise}).
\end{cases}
\]

We begin by recalling the definition of \( \Gamma \)-convergence.

**Definition 4.1** (\( \Gamma \)-convergence). Let \( X \) be a metric space and \( F_\varepsilon, F : X \to [0, \infty] \). We say that \( F_\varepsilon \) \( \Gamma(X) \)-converges to \( F \) as \( \varepsilon \downarrow 0 \) if the following conditions hold:
(1) For any convergent sequence \( u_\varepsilon \to u \) \((\varepsilon \downarrow 0)\) in \(X\),
\[
\liminf_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon] \geq F[u].
\]
(2) For any \( u \in X \), there exists a convergent sequence \( u_\varepsilon \to u \) \((\varepsilon \downarrow 0)\) in \(X\) such that
\[
\limsup_{\varepsilon \to 0} F_\varepsilon[u_\varepsilon] \leq F[u].
\]
We denote such convergence by \( F_\varepsilon \Gamma \to F \) on \(X\).

We are now in position to state our main result on \(\Gamma\)-convergence.

Theorem 4.2. Let \( F_\varepsilon, F : X_{1,1}(a,b) \to [0, \infty] \) defined by (4.1) and (4.2).
Then \( F_\varepsilon \Gamma \to F \) on \(X_{1,1}(a,b)\) as \(\varepsilon \downarrow 0\).

This theorem is proved in \(\S\) 5 and \(\S\) 6.

It is one of the important properties of \(\Gamma\)-convergence that if \( F_\varepsilon \Gamma \)-converges to \(F\) and \(u_\varepsilon\) is a minimizer of \(F_\varepsilon\), then any cluster point of \(\{u_\varepsilon\}\) is a minimizer of \(F\). In our setting the minimizing problem of \(F_\varepsilon\) and of \(E_\varepsilon\) are equivalent, thus any cluster point of \(\{\bar{u}_\varepsilon\}\) which is a sequence of minimizers of \(E_\varepsilon\) is a minimizer of \(F\) (as mentioned in Corollary 1.2).

5. LIM-INF CONDITION OF \(\Gamma\)-CONVERGENCE

In this section, we prove the liminf condition of Theorem 4.2 (the first condition of Definition 4.1). The proof is separated into three parts: \(\S\) 5.1–5.3. In \(\S\) 5.1 we show that we only have to check special sequences. In \(\S\) 5.2 we impose an essential and stronger restriction for sequences in order to obtain a lower estimate. By this restriction, we are able to consider the energy \(F_\varepsilon\) geometrically. In \(\S\) 5.3 we obtain a lower bound for such restricted sequences. A summary of the overall proof is given in \(\S\) 5.4.

5.1. Assumption on sequences. We first give some simple assumptions for sequences.

Assumption 5.1. Assume that a sequence \(u^\varepsilon \to u\) in \(X_{1,1}(a,b)\) satisfies
1. \(\liminf_{\varepsilon \to 0} F_\varepsilon[u^\varepsilon] < \infty\),
2. \(F_\varepsilon[u^\varepsilon] < \infty\) for all \(\varepsilon > 0\) (especially \(\{u^\varepsilon\} \subset X_{2,1}(a,b)\)),
3. \(u = \bar{u} \in \text{argmin}_{X_{1,1}} E_0\) (especially \(\text{argmin}_{X_{1,1}} E_0 \neq \emptyset\)),
4. \(\bar{u}\) is \([\bar{x}_0; \ldots; \bar{x}_{N+1}]\)-regular for some \(N > 0\).

Proposition 5.2. If the liminf condition of Theorem 4.2 is valid for sequences \(u^\varepsilon \to u\) satisfying Assumption 5.1, then the liminf condition is fulfilled for any sequence.

Proof. (1) If \(\liminf_{\varepsilon \to 0} F_\varepsilon[u^\varepsilon] = \infty\) then the inequality is trivial.
(2) If there exists \(\varepsilon_0 > 0\) such that \(F_\varepsilon[u^\varepsilon] = \infty\) for any \(0 < \varepsilon < \varepsilon_0\), then
\[ \liminf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon] = \infty, \text{ thus it is trivial. If not, then there exists a subsequence } \{u^{\varepsilon'}\} \text{ such that } F_{\varepsilon'}[u^{\varepsilon'}] < \infty \text{ for all } \varepsilon' \text{ and} \]
\[ \liminf_{\varepsilon' \to 0} F_{\varepsilon'}[u^{\varepsilon'}] \leq \liminf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon], \]

thus we can assume (2) without loss of generally.

(3) If \( u^\varepsilon \to u \) and \( E_0[u] > \inf_{X^{1,1}} E_0 \), then \( \liminf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon] = \infty \) by Proposition 3.2.

(4) If \( \bar{u} \in X^{1,1}(a,b) \) is \([x_0; \ldots; x_{N+1}]\)-regular then

\[ \int_{\partial(a,b)^\varepsilon} 4(\sqrt{2} - \sqrt{1 + \alpha}) \, d\mathcal{H}^0 = \sum_{i=1}^{N} 4 \left( \sqrt{2} - \sqrt{1 + \alpha(x_i)} \right) , \]

thus it is trivial in the case \( N = 0 \) (the sum is defined to be zero when \( N = 0 \)).

5.2. \( \delta \)-associate. In this subsection, we reduce sequences in the liminf condition of Theorem 4.2 while not increasing the limit inferior. This reduction is important because it is relatively easy to obtain a lower estimate for such reduced sequences. To state the reduction result, we introduce a notation.

**Definition 5.3** (\( \delta \)-associate). Let \( \delta > 0 \) and \( u \in X^{1,1}(a,b) \). We say that \( v \in X^{1,1}(a,b) \) is \( \delta \)-associated with \( u \) if \( \|u - v\|_\infty \leq \delta \) and there exists a continuous strictly increasing operator \( S : [a,b] \to [a,b] \) such that \( |Sx - x| \leq \delta \) for all \( x \in [a,b] \) and \( S([a,b]^\varepsilon) = (a,b)^{\varepsilon+\delta} \).

\[ \text{Figure 3. } \delta \text{-associate} \]
Remark 5.4. For any partitional regular function $u$ and function $v$ which is $\delta$-associated with $u$, the numbers of the connected components of $(a,b)^u_\alpha$ and $(a,b)^v_\alpha$ are equal and $v$ is partitional regular.

The following proposition is the main reduction result of this subsection.

Proposition 5.5. Let $u^\varepsilon \to \bar{u}$ in $W^{1,1}(a,b)$ as in Assumption 5.1. Then there exists $\delta := \bar{\delta}(\psi,\alpha,\bar{u}) > 0$ satisfying the following: for any $0 < \delta \leq \bar{\delta}$ there exists a sequence $\{w^j_\delta\}_j \subset X^{2,1}(a,b)$ such that,

1. $\lim \inf_{j \to \infty} F_{\varepsilon_j}[w^j_\delta] \leq \lim \inf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon],$
2. $w^j_\delta$ is $\delta$-associated with $\bar{u}$ for any $j$.

Proof. This is a direct consequence of following three lemmas: Lemma 5.6 and Lemma 5.7.

Lemma 5.6. Let $u^\varepsilon \to \bar{u}$ in $W^{1,1}(a,b)$ as in Assumption 5.1 and $\delta_{\psi,\alpha}$ in Lemma 3.6. Then for any $0 < \delta \leq 3\delta_{\psi,\alpha}$ there exists a subsequence $\{u^{\varepsilon_j}\}_j \subset \{u^\varepsilon\}$ such that,

1. $\lim \inf_{j \to \infty} F_{\varepsilon_j}[u^{\varepsilon_j}] = \lim \inf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon],$
2. for any $\varepsilon_j > 0$ and interval $I \subset (a,b)_0^\bar{u}$ whose width is $\delta$ there exists $y \in I$ such that $u^{\varepsilon_j}(y) = \psi(y)$.

Proof. Fix any $0 < \delta \leq 3\delta_{\psi,\alpha}$. By definition of $\lim \inf$, we can take a subsequence $\{u^{\varepsilon_j}\}_j \subset \{u^\varepsilon\}$ such that

$$\lim_{\varepsilon \to 0} F_{\varepsilon}[u^{\varepsilon_j}] = \lim \inf_{\varepsilon \to 0} F_{\varepsilon}[u^\varepsilon] < \infty.$$ 

We prove that this sequence has a subsequence satisfying the second condition of Lemma 5.6 by contradiction. Note that it suffices to prove that, for any $I \subset (a,b)_0^\bar{u}$ whose width is $\delta/3$, there exists a subsequence (depending on $I$) touching $\psi$ somewhere in $I$. This is because, since $(a,b)_0^\bar{u}$ is covered by a finite number of intervals with width $\delta/3$, if we take such subsequences repeatedly then the obtained subsequence satisfies the condition.

Thus we suppose that for a given $I$ there is no such subsequence. In other words, suppose that there would exist $\varepsilon' > 0$ and some interval $I \subset (a,b)_0^\bar{u}$ whose width is $\delta/3$ such that

$$u^{\varepsilon'} > \psi \in I \quad \text{for} \quad 0 < \forall \varepsilon'' < \varepsilon'.$$

Then we would obtain

$$\varepsilon'' F_{\varepsilon''}[u^{\varepsilon''}] \geq E_0[u^{\varepsilon''}] - E_0[\bar{u}]$$

$$\geq \int_I \sqrt{1 + (u^{\varepsilon''}_x)^2} \, dx - \int_I \alpha \sqrt{1 + \psi^2_x} \, dx$$

$$+ \int_{(a,b) \setminus I} \tilde{\alpha}[u^{\varepsilon''}] \sqrt{1 + (u^{\varepsilon''}_x)^2} \, dx - \int_{(a,b) \setminus I} \tilde{\alpha}[\bar{u}] \sqrt{1 + \bar{u}^2_x} \, dx.$$ 

By Lemma 5.1 we get

$$\lim \inf_{\varepsilon'' \to 0} \int_a^b \chi_I \sqrt{1 + (u^{\varepsilon''}_x)^2} \, dx \geq \int_a^b \chi_I \sqrt{1 + \bar{u}^2_x} \, dx = \int_a^b \chi_I \sqrt{1 + \psi^2_x} \, dx.$$
and
\[
\liminf_{\varepsilon'' \to 0} \int_a^b \chi_{(a,b) \setminus I} \alpha(u^{\varepsilon''}) \sqrt{1 + (u^{\varepsilon''}_x)^2} \, dx \geq \int_a^b \chi_{(a,b) \setminus I} \alpha(\bar{u}) \sqrt{1 + \bar{u}_x^2} \, dx.
\]

Therefore, by \(0 < \delta/3 \leq \delta_{\psi,\alpha}\) and Lemma 3.6, we have
\[
\liminf_{\varepsilon'' \to 0} \int_a^b \chi_{(a,b) \setminus I} \alpha(u^{\varepsilon''}) \sqrt{1 + (u^{\varepsilon''}_x)^2} \, dx \geq \frac{\delta}{3} \left( \inf_I \sqrt{1 + \psi_x^2} - \alpha \sup_I \sqrt{1 + \psi_x^2} \right) \geq \frac{\delta}{3} \frac{1 - \alpha}{2} > 0,
\]
i.e. \(\liminf_{\varepsilon'' \to 0} F_{\varepsilon''}[u^{\varepsilon''}] = \infty\). This is a contradiction. \(\square\)

**Lemma 5.7.** Let \(u^\varepsilon \to \bar{u}\) in \(X^{1,1}(a,b)\) as in Assumption 5.1. Define
\[
\delta_{\psi,\alpha,\bar{u}} := \min \left\{ \delta_{\psi,\alpha}, \frac{\bar{x}_1 - x_0}{3}, \ldots, \frac{\bar{x}_N - x_N}{3} \right\}.
\]
Then for any \(0 < \delta \leq \delta_{\psi,\alpha,\bar{u}}\) there exists a sequence \(\{v^\varepsilon_j\}_j \subset X^{2,1}(a,b)\) such that,
1. \(v^\varepsilon_j \to \bar{u}\) in \(W^{1,1}(a,b)\),
2. \(\liminf_{j \to \infty} F_{\varepsilon_j}[v^\varepsilon_j] \leq \liminf_{j \to \infty} F_{\varepsilon}[u^\varepsilon_j]\),
3. for any \(\varepsilon_j > 0\) and \(0 \leq i \leq N\) with \((\bar{x}_i, \bar{x}_{i+1}) \subset (a,b)_{\bar{u}}\), \(v^\varepsilon_j \equiv \psi\) in \([\bar{x}_i + \delta, \bar{x}_{i+1} - \delta]\).

**Proof.** Fix \(0 < \delta \leq \delta_{\psi,\alpha,\bar{u}}\) and denote the subsequence obtained in Lemma 5.6 by \(\{u^\varepsilon_j\}_j\). Recall that, by Lemma 5.6, for any \(\varepsilon_j\) and \(0 \leq i \leq N\) with \((\bar{x}_i, \bar{x}_{i+1}) \subset (a,b)_{\bar{u}}\) there exist \(y^\varepsilon_j \in (\bar{x}_i, \bar{x}_i + \delta]\) and \(y^\varepsilon_{i+1} \in (\bar{x}_{i+1} - \delta, \bar{x}_{i+1})\) such that \(u^\varepsilon_j = \psi\) at \(y^\varepsilon_j\) and \(y^\varepsilon_{i+1}\). Moreover, since \(u^\varepsilon_j\) is continuously differentiable and \(u^\varepsilon_j \geq \psi\), we also have \((u^\varepsilon_j)_x = \psi_x\) at \(y^\varepsilon_j\) and \(y^\varepsilon_{i+1}\).

We define \(v^\varepsilon_j\) by replacing \(u^\varepsilon_j\) to be \(\psi\) in \((y^\varepsilon_j, y^\varepsilon_{i+1})\) for any \(i\) with \((\bar{x}_i, \bar{x}_{i+1}) \subset (a,b)_{\bar{u}}\). Then \(v^\varepsilon_j\) is in \(X^{2,1}(a,b)\) since \(u^\varepsilon_j = \psi\) and \((u^\varepsilon_j)_x = \psi_x\) at any \(y^\varepsilon_i\). Furthermore, \(\{v^\varepsilon_j\}\) satisfies all conditions in this lemma. Indeed, the first condition follows by
\[
\|v^\varepsilon_j - \bar{u}\|_{W^{1,1}} \leq \|u^\varepsilon_j - \bar{u}\|_{W^{1,1}},
\]
the second one by
\[
E_0[u^\varepsilon_j] - E_0[v^\varepsilon_j] = E_0[u^\varepsilon_j] - E_0[\bar{u}] \geq 0
\]
and
\[
F_{\varepsilon_j}[u^\varepsilon_j] - F_{\varepsilon_j}[v^\varepsilon_j] \geq -\varepsilon_j \int_a^b r^2 \psi^2 \sqrt{1 + \psi_x^2} \, dx \xrightarrow{j \to \infty} 0,
\]
and the third one by the definition of \(v^\varepsilon_j\). \(\square\)
Lemma 5.8. Let $v_{\delta}^{\varepsilon_j} \to \bar{u}$ and $\delta_{\psi,\alpha,\bar{u}}$ as in Lemma 5.7. Take a positive number $\delta_{\psi} > 0$ so that
\[
\sup \left\{ |\psi(x) - \psi(y)| \left| x, y \in [a, b], \ |x - y| \leq \delta_{\psi} \right. \right\} \leq \delta_{\psi,\alpha,\bar{u}}.
\]
Then for any $0 < \delta \leq \min \{ \delta_{\psi,\alpha,\bar{u}}, \delta_{\psi} \}$ there exists a sequence $\{w_{\delta}^{\varepsilon_j}\}_{j \in \mathbb{N}} \subset X^{2,1}(a, b)$ such that,

1. $\liminf_{j \to \infty} F_{\varepsilon_j}[w_{\delta}^{\varepsilon_j}] \leq \liminf_{j \to \infty} F_{\varepsilon_j}[v_{\delta}^{\varepsilon_j}]$,
2. there exists $M > 0$ such that $w_{\delta}^{\varepsilon_j}$ is $\delta$-associated with $\bar{u}$ for any $j \geq M$.

Proof. We first assume that $\bar{u} > \psi$ on $\partial(a, b)$. Fix $0 < \delta \leq \min \{ \delta_{\psi,\alpha,\bar{u}}, \delta_{\psi} \}$. Define
\[
I_{\delta}^i := \{ y \in (a, b) \mid |y - \bar{x}_i| \leq \delta \} \subseteq (a, b)
\]
for $1 \leq i \leq N$ and $I_{\delta} := \bigcup_{i=1}^{N} I_{\delta}^i$. Since
\[
\inf \left\{ \bar{u}(x) - \psi(x) \mid x \in (a, b)^a \setminus I_{\delta} \right\} > 0
\]
and $v_{\delta}^{\varepsilon_j} \to \bar{u}$ as $j \to \infty$ uniformly (by Sobolev embedding), there exists $M > 0$ such that $v_{\delta}^{\varepsilon_j} > \psi$ in $(a, b)^a \setminus I_{\delta}$ for any $j \geq M$. On the other hand, for any $\varepsilon_j > 0$ we find $v_{\delta}^{\varepsilon_j} \equiv \psi$ in $(a, b)^a \setminus I_{\delta}$ by the third condition of Lemma 5.7.

Therefore, to prove this lemma, we only have to reduce $v_{\delta}^{\varepsilon_j}$ in $I_{\delta}$. Define
\[
y_{\delta}^{\varepsilon_j} := \min \{ y \in I_{\delta}^i \mid v_{\delta}^{\varepsilon_j}(y) = \psi(y) \}
\]
for $i$ with $(\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)^a$. Then we find $v_{\delta}^{\varepsilon_j} = \psi$ on $\partial[y_{\delta}^{\varepsilon_j}, \bar{x}_i + \delta]$. Thus, by Lemma 3.7, the reduction replacing $v_{\delta}^{\varepsilon_j}$ to be $\psi$ only in $[y_{\delta}^{\varepsilon_j}, \bar{x}_i + \delta]$ does not increase $E_0$ and retains the function in $X^{2,1}(a, b)$. Similarly, for $i$ with $(\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)^a$, we can get the result reversed left and right. Then, defining $w_{\delta}^{\varepsilon_j}$ which is reduced for all $1 \leq i \leq N$ as above, we get the conclusion. Indeed, $w_{\delta}^{\varepsilon_j}$ is $\delta$-associated with $\bar{u}$ for any $j \geq M$ by the above argument and the definition of $w_{\delta}^{\varepsilon_j}$ (here $S\bar{x}_i = y_{\delta}^{\varepsilon_j}$). The first condition follows by $E_0[v_{\delta}^{\varepsilon_j}] - E_0[w_{\delta}^{\varepsilon_j}] \geq 0$ for any $\varepsilon_j$ and
\[
F_{\varepsilon_j}[v_{\delta}^{\varepsilon_j}] - F_{\varepsilon_j}[w_{\delta}^{\varepsilon_j}] \geq -\varepsilon_j \int_{a}^{b} \frac{\partial^2}{\partial x^2} \sqrt{1 + \psi^2} \, dx \to \infty \geq 0,
\]
Finally, we mention the case $\bar{u} = \psi$ at $\bar{x}_0(= a)$ or $\bar{x}_{N+1}(= b)$ or both. In addition to the above proof, we have only to reduce $v_{\delta}^{\varepsilon_j}$ near $\bar{x}_0$, $\bar{x}_{N+1}$. Let us only consider at $\bar{x}_0$ (it is similar at $\bar{x}_{N+1}$). If $\bar{u}(\bar{x}_0) = \psi(\bar{x}_0)$ and $(\bar{x}_0, \bar{x}_1) \subset (a, b)^a$, then the consequence follows by replacing $v_{\delta}^{\varepsilon_j}$ by $\psi$ in $(\bar{x}_0, \bar{x}_0 + \delta)$. If $\bar{u}(\bar{x}_0) = \psi(\bar{x}_0)$ and $(\bar{x}_0, \bar{x}_1) \subset (a, b)^a$, then $v_{\delta}^{\varepsilon_j}$ may touch $\psi$ near $\bar{x}_0$ even if $j$ is sufficiently large. Nevertheless, the consequence follows by replacing $v_{\delta}^{\varepsilon_j}$ by $\psi$ in $(\bar{x}_0, \bar{x}_0 + \delta)$, where
\[
y_{\delta}^{\varepsilon_j} := \max \{ y \in (\bar{x}_0, \bar{x}_0 + \delta) \mid v_{\delta}^{\varepsilon_j}(y) = \psi(y) \} \ (= S\bar{x}_0).
\]
\[\square\]
5.3. **Lower estimate for geometric energies.** In this subsection, we give a lower estimate for the functions as obtained in §5.2. In Proposition 5.11, for functions δ-associated with a minimizer of $E_0$, we rewrite the energy in order to consider geometrically, and obtain a key estimate in Proposition 5.12.

**Definition 5.9** (*W*^{2,1}-curve and geometrical energy).  
(1) We say that $\gamma : [0, 1] \to \mathbb{R}^2$ is a *(regular) W*^{2,1}-curve* if the two components $\gamma_1, \gamma_2$ are functions in $W^{2,1}(0, 1) \subset C^1([0, 1])$ and

$$|\dot{\gamma}| = \sqrt{(\dot{\gamma}_1, \dot{\gamma}_2)} = \sqrt{\gamma_1'^2 + \gamma_2'^2} > 0 \text{ in } [0, 1].$$

(2) For $\gamma$: *W*^{2,1}-curve, $\bar{\gamma}$ denotes the geodesic (segment) from $\gamma(0)$ to $\gamma(1)$. 
(3) For $\gamma$: *W*^{2,1}-curve, we denote the tension (length) by

$$\mathcal{L}[\gamma] := \int_0^1 |\dot{\gamma}(t)|dt.$$

(4) For $\gamma$: *W*^{2,1}-curve, we denote the bending energy by

$$\mathcal{B}[\gamma] := \int_0^1 |\kappa_\gamma(t)|^2|\dot{\gamma}(t)|dt,$$

where $\kappa_\gamma$ is the curvature of $\gamma$ defined by $\kappa_\gamma := |\dot{\gamma}|^{-3} |\gamma_1'' \gamma_2' - \gamma_2'' \gamma_1'|$. 
(5) For $\gamma$: *W*^{2,1}-curve, we define **boundary warp energy** by

$$\mathcal{W}[\gamma] := 4 \left( 2\sqrt{2} - \sqrt{1 + \cos \theta_0^1} - \sqrt{1 + \cos \theta_1^2} \right),$$

where $\theta_0^1$ and $\theta_1^2$ are **boundary warp angles** of $\gamma$ defined by $\angle(\gamma, \bar{\gamma}) := \arccos ((\dot{\gamma}, \bar{\gamma})/|\dot{\gamma}|)|\bar{\gamma}|) \in [0, \pi]$ at $\gamma(0)$ and $\gamma(1)$ respectively.

**Remark 5.10.** $\mathcal{L}$, $\mathcal{B}$, $\mathcal{W}$, $\theta_0^1$ and $\theta_1^2$ are well-defined for $W^{2,1}$-curves, i.e. they are invariant by $W^{2,1}$-reparameterization. They are also invariant with respect to translation, reflection and rotation. (To be more precise, see Appendix.) Moreover, for any $u \in W^{2,1}$ on a bounded interval, the graph of $u$ is a $W^{2,1}$-curve.

**Proposition 5.11.** Let $\bar{u} \in \text{argmin}_{X^{1,1}} E_0$ be $[\bar{x}_0; \ldots; \bar{x}_{N+1}]$-regular and $m \in \mathbb{N}$ be the number of the connected components of $(a, b)^u_\downarrow$. Then there exists $\delta(\psi, \bar{u}) > 0$ such that for any $\varepsilon > 0$, $0 < \delta \leq \delta(\psi, \bar{u})$ and $u \in X^{2,1}(a, b)$ which is $\delta$-associated with $\bar{u}$ the inequality holds:

$$F_\varepsilon[u] \geq \sum_{k=1}^m \varepsilon \mathcal{B}[\gamma^k] + \frac{1}{\varepsilon} (\mathcal{L}[\gamma^k] - \mathcal{L}[\bar{\gamma}^k]),$$

where $\gamma^k (1 \leq k \leq m)$ is a $W^{2,1}$-curve which is the graph of $u$ on the $k$-th connected component of $(a, b)^u_\downarrow$.

Moreover, the difference between the boundary warp angle of $\gamma^k$ at an end-point and the contact angle of $\bar{u}$ with $\psi$ at the corresponding endpoint of the $k$-th connected component of $(a, b)^u_\downarrow$ tends to be zero as $\delta \downarrow 0$ independently of $u$. 


Proof. Fix $\delta > 0$ and $u \in W^{2,1}(a, b)$ which is $\delta$-associated with $\tilde{u}$. We denote $x_i = Sx_i$ for $0 \leq i \leq N + 1$ (the shift operator $S$ is defined in Definition 5.3) and define $u_{\delta} \in W^{1,1}(a, b)$ by

$$u_{\delta}(x) := \begin{cases} \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} (x - x_i) + u(x_i), & (x \in (x_i, x_{i+1}) \subset (a, b), u), \\ \psi(x), & (otherwise). \end{cases}$$

Then, since any contact angle of $\tilde{u}$ with $\psi$ is positive by Theorem 3.4, there exists $\delta(\psi, \tilde{u}) > 0$ such that $u_{\delta}$ is in $X^{1,1}(a, b)$ and $[x_0; \ldots; x_{N+1}]$-regular for any $0 < \delta \leq \delta(\psi, \tilde{u})$. Therefore, since $E_0[u_{\delta}] \geq E_0[\tilde{u}]$ by the minimality of $\tilde{u}$, we have

$$F_{\varepsilon}[u] \geq \varepsilon \int_a^b \kappa[u]^2 \sqrt{1 + u_x^2} \, dx + \frac{1}{\varepsilon}(E_0[u] - E_0[u_{\delta}]).$$

Moreover, we can restrict the domain of integration of the right-hand term to the disjoint union of $(x_i, x_{i+1}) \subset (a, b)$ thus we have

$$F_{\varepsilon}[u] \geq \sum_{0 \leq i \leq N} \left( \varepsilon \int_{x_i}^{x_{i+1}} \kappa[u]^2 \sqrt{1 + u_x^2} \, dx \\ + \frac{1}{\varepsilon} \int_{x_i}^{x_{i+1}} \sqrt{1 + u_x^2} - \sqrt{1 + \tan^2 \tilde{\theta}_i} \, dx \right),$$

where $\tilde{\theta}_i := \arctan \left( \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \right)$.

By the above argument, if we take $\gamma^k$ as the statement then (5.2) holds. Now we take any $1 \leq i \leq N$ with $(x_i, x_{i+1}) \subset (a, b)$. The boundary warp angle of $\gamma^k$ at $(x_i, u(x_i))$ is $\theta_i - \theta_u(x_i) = \tilde{\theta}_i - \theta_\psi(x_i)$, that is,

$$\arctan \left( \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \right) - \theta_\psi(x_i).$$

and the contact angle of $\tilde{u}$ at $\bar{x}_i$ is $\theta_{\tilde{u}}(\bar{x}_i) - \theta_\psi(\bar{x}_i)$, that is,

$$\arctan \left( \frac{\tilde{u}(\bar{x}_{i+1}) - \tilde{u}(\bar{x}_i)}{\bar{x}_{i+1} - \bar{x}_i} \right) - \theta_\psi(\bar{x}_i).$$

Therefore, the difference of them tends to be zero as $\delta \downarrow 0$ not depending on $u$ (but only $\psi$) since $\|u - \tilde{u}\|_\infty \leq \delta$, $|x_i - \bar{x}_i| \leq \delta$, and $\arctan(\cdot), \theta_\psi$ are uniformly continuous. We can similarly consider for $1 \leq i \leq N$ with $(x_i, x_{i+1}) \subset (a, b)$, thus the proof is completed.

We now obtain a key estimate for the right-hand term of (5.2).

Proposition 5.12. For any $\varepsilon > 0$ and $W^{2,1}$-curve $\gamma$ with $\theta_0^\gamma, \theta_1^\gamma \in [0, \pi/2)$, the inequality holds:

$$\varepsilon B[\gamma] + \frac{1}{\varepsilon}(L[\gamma] - L[\tilde{\gamma}]) \geq W[\gamma].$$

This follows by Lemma 5.13 and Lemma 5.14.
Lemma 5.13. Let \( \varepsilon > 0 \). For any \( W^{2,1} \)-curve \( \gamma \) with \( \theta_0^\gamma, \theta_1^\gamma \in [0, \pi/2) \), there exist \( L_0, L_1 > 0 \) and a function \( \hat{u} \in W^{2,1}(-L_0, L_1) \) such that \( \hat{u}_x(0) = 0 \), \( \hat{u}(-L_0) = \tan \theta_0^\gamma \), \( \hat{u}(L_1) = \tan \theta_1^\gamma \) and

\[
\varepsilon B[\gamma] + \frac{1}{\varepsilon} (L[\gamma] - L[\hat{\gamma}]) \geq \int_{-L_0}^{L_1} \varepsilon \kappa_0^2 \sqrt{1 + \hat{u}_x^2} + \frac{1}{\varepsilon} \left( \sqrt{1 + \hat{u}_x^2} - 1 \right) \, dx.
\]

Proof. We can assume \( B[\gamma] < \infty \), \( \gamma(0) = (0, 0) \), \( \gamma(1) = (\gamma_1(1), 0) \) with \( \gamma_1(1) > 0 \) without loss of generally. Note that

\[
L[\gamma] = \gamma_1(1), \quad \tan \theta_0^\gamma = \gamma_1'(0)/|\gamma_2'(0)|, \quad \tan \theta_1^\gamma = \gamma_1'(1)/|\gamma_2'(1)|,
\]

and there exists \( 0 < t_\gamma < 1 \) such that \( \gamma_2'(t_\gamma) = 0 \). Now we fix \( \gamma : [0, 1] \to \mathbb{R}^2 \).

Take an angle \( \theta_1^\gamma \in (\max\{\theta_0^\gamma, \theta_1^\gamma, \pi/4\}, \pi/2) \) and for \( t \in [0, 1] \) define

\[
\hat{\gamma}(t) := \int_0^t \left( \max\{\gamma_1'(r) / (\cos 2\theta_1^\gamma r), \gamma_2'(r) + (\sin 2\theta_1^\gamma r)\} \right. \\
\min\{\gamma_2'(r), (\sin 2\theta_1^\gamma r)\gamma_1'(r) - (\cos 2\theta_1^\gamma r)\gamma_2'(r)\} \right) \, dr.
\]

Then \( \hat{\gamma} \) is a \( W^{2,1} \)-curve such that \( \hat{\gamma} \) is in the first quadrant of \( \mathbb{R}^2 \),

\[
\hat{\gamma}/|\hat{\gamma}| \in \{(\cos \theta, \sin \theta) \in \mathbb{S}^1 \mid 0 \leq \theta \leq \theta_1^\gamma \} \quad \text{in} \ [0, 1],
\]

and the following conditions hold:

\[
B[\gamma] = B[\hat{\gamma}], \quad L[\gamma] = L[\hat{\gamma}], \quad \hat{\gamma}(0) = (0, 0), \quad \hat{\gamma}_1(1) \geq \gamma_1(1), \quad \hat{\gamma}_2(t_\gamma) = 0,
\]

\[
\hat{\gamma}_1'(0)/\hat{\gamma}_2'(0) = \gamma_1'(0)/|\gamma_2'(0)|, \quad \hat{\gamma}_1'(1)/\hat{\gamma}_2'(1) = \gamma_1'(1)/|\gamma_2'(1)|.
\]

These conditions hold since it is only used translation or reflection partially in this transformation, and while

\[
\hat{\gamma}/|\hat{\gamma}| \in \{(\cos \theta, \sin \theta) \in \mathbb{S}^1 \mid |\theta| \leq \theta_1^\gamma \}
\]

only translation or one reflection \( (\gamma_1, \gamma_2) \to (\gamma_1, -\gamma_2) \). In addition, since \( \hat{\gamma}_1' > 0 \) in \([0, 1]\), we can define the inverse function of \( \hat{\gamma}_1 \) and it is in \( W^{2,1}(0, \gamma_1(1)) \). Thus, by taking

\[
\hat{u}(x) := \hat{\gamma}_2 \circ \hat{\gamma}_1^{-1}(x + \hat{\gamma}_1(t_\gamma)), \quad L_0 := \hat{\gamma}_1(t_\gamma), \quad L_1 := \hat{\gamma}_1(1) - \hat{\gamma}_1(t_\gamma),
\]

we obtain a desired function. Indeed, we have

\[
\hat{u}_x(0) = (\hat{\gamma}_2 \circ \hat{\gamma}_1^{-1})'(\hat{\gamma}_1(t_\gamma)) = \hat{\gamma}_2'(t_\gamma)/\hat{\gamma}_1'(t_\gamma) = 0,
\]

\[
\hat{u}_x(-L_0) = (\hat{\gamma}_2 \circ \hat{\gamma}_1^{-1})'(0) = \hat{\gamma}_2'(0)/\hat{\gamma}_1'(0) = \gamma_1'(0)/|\gamma_2'(0)| = \tan \theta_0^\gamma
\]

and \( \hat{u}_x(L_1) = \tan \theta_1^\gamma \) similarly, moreover the desired inequality follows by

\[
B[\gamma] = B[\hat{\gamma}] = \int \kappa_0^2 \sqrt{1 + \hat{u}_x^2}, \quad L[\gamma] = L[\hat{\gamma}] = \int \sqrt{1 + \hat{u}_x^2},
\]

\[
\int_{-L_0}^{L_1} \, dx = L_0 + L_1 = \hat{\gamma}_1(1) \geq \gamma_1(1) = L[\gamma].
\]

The proof is completed. \( \square \)
Lemma 5.14. Let \( v \in W^{2,1}(0, L) \) and \( \theta \in [0, \pi/2) \) satisfying \( v_x(0) = 0 \) and \( |v_x(L)| = \tan \theta \). Then the inequality holds:

\[
\int_0^L \varepsilon \kappa_t^2 \sqrt{1 + v_x^2} + \frac{1}{\varepsilon} \left( \sqrt{1 + v_x^2} - 1 \right) \, dx \geq 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta} \right). 
\]

**Proof.** For such \( v \in W^{2,1}(0, L) \), we have

\[
\int_0^L 2 \varepsilon \left( \frac{v_{xx}}{(1 + v_x^2)^{3/2}} \right)^2 \sqrt{1 + v_x^2} + \frac{1}{\varepsilon} \left( \sqrt{1 + v_x^2} - 1 \right) \, dx 
\geq \int_0^L \sqrt{1 + v_x^2} - 1 \, dx 
\geq \int_0^L |f'(v_x)v_{xx}| \, dx = |f(v_x(L)) - f(v_x(0))| = |f(v_x(L))|,
\]

where \( f \) is the even function given by

\[
f(y) := \int_0^y \frac{2 \sqrt{1 + z^2} - 1}{(1 + z^2)^{5/4}} \, dz.
\]

The first inequality follows by the easy trick \( \varepsilon X^2 + \varepsilon^{-1} Y^2 \geq 2XY \). Moreover, we have \( |f(v_x(L))| = f(|v_x(L)|) = f(\tan \theta) \) and

\[
f(\tan \theta) = 2 \int_0^{\tan \theta} \frac{\sqrt{1 + z^2} - 1}{(1 + z^2)^{5/4}} \, dz 
= 2 \int_0^\theta \sqrt{1 - \cos \varphi} \, d\varphi \quad (z = \tan \varphi) 
= 2 \int_{\cos \theta}^1 \frac{dw}{\sqrt{1 + w}} \quad (w = \cos \varphi) 
= 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta} \right),
\]

thus the proof is completed. \( \square \)

**Proof of Proposition 5.12**. Let \( \gamma \) be any \( W^{2,1} \)-curve and \( \hat{u} \) be the function obtained by Lemma 5.13. Using Lemma 5.14 for \( W^{2,1} \)-functions \( \hat{u}(-x) \) on \([0, L_0]\) and \( \hat{u}(x) \) on \([0, L_1]\), then we have

\[
\varepsilon B[\gamma] + \frac{1}{\varepsilon}(L[\gamma] - L[\bar{\gamma}]) \geq 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta_1^\gamma} \right) + 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta_0^\gamma} \right)
\]

and the right-hand term is nothing but \( W[\gamma] \). \( \square \)

5.4. **Completion of the proof of the liminf condition.** We shall prove the liminf condition of Theorem 4.2.

**Proof.** Take any sequence \( u^\varepsilon \to u \) in \( X^{1,1}(a, b) \) and fix it. We can assume Assumption 5.14 by Proposition 5.2 (thus \( u = \bar{u} \) is \( \bar{x}_0; \ldots; \bar{x}_{N+1} \)-regular).
Then for any sufficiently small $\delta > 0$, we have
\[
\liminf_{\epsilon \to 0} F_\epsilon[u^\epsilon] \geq \sum_{i=1}^{N} 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta_{i,\delta}} \right)
\]
by Proposition 5.5, 5.11 and 5.12, where $\theta_{i,\delta} \in (0,\pi/2)$ satisfying $\theta_{i,\delta} \to \theta_i$ as $\delta \downarrow 0$ and $\theta_i$ is the contact angle of $\bar{u}$ with $\psi$ at $\bar{x}_i$. Thus we conclude by taking $\delta \downarrow 0$ since $\cos \theta_i = \alpha(\bar{x}_i)$ and (5.1) hold.

\[\square\]

6. Lim-sup condition of $\Gamma$-convergence

Finally, we prove the limsup condition of Theorem 4.2.

**Proof.** We construct sequences concretely by modifying singularities of minimizers of $E_0$. To this end, for arbitrary $\epsilon > 0$ and $0 < \theta < \pi/2$ we consider the following ODE:

\[
(6.1) \quad \begin{cases} 
U'' = \frac{1}{\epsilon} (1 + (U')^2)^{5/4} \sqrt{1 + (U')^2} - 1 & \text{in } (-\infty, 0], \\
U(0) = 0, \ U'(0) = \tan \theta > 0.
\end{cases}
\]

This Cauchy problem has the unique solution $U_{\theta,\epsilon} : (-\infty, 0] \to \mathbb{R}$ satisfying $U_{\theta,\epsilon} < 0, U_{\theta,\epsilon}' > 0, U_{\theta,\epsilon}'' > 0$ in $(-\infty, 0)$. Moreover, $U_{\theta,\epsilon}(x) = \epsilon U_{\theta,1}(x/\epsilon)$, $\lim_{x \to -\infty} U_{\theta,1}(x) = 0$ and by definition we see

\[
\begin{align*}
\epsilon \int_{-\epsilon^{2/3}}^{0} \left( \frac{U_{\theta,\epsilon}''}{(1 + (U_{\theta,\epsilon}')^2)^{3/2}} \right)^2 \sqrt{1 + (U_{\theta,\epsilon}')^2} \, dx \\
+ \frac{1}{\epsilon} \int_{-\epsilon^{2/3}}^{0} \left( \sqrt{1 + (U_{\theta,\epsilon}')^2} - 1 \right) \, dx \\
= \int_{-\epsilon^{-1/3}}^{0} \left( \frac{U_{\theta,1}''}{(1 + (U_{\theta,1}')^2)^{3/2}} \right)^2 \sqrt{1 + (U_{\theta,1}')^2} \, dx \\
+ \int_{-\epsilon^{-1/3}}^{0} \left( \sqrt{1 + (U_{\theta,1}')^2} - 1 \right) \, dx \\
= \int_{-\epsilon^{-1/3}}^{0} 2 \left( \frac{U_{\theta,1}''}{(1 + (U_{\theta,1}')^2)^{5/4}} \sqrt{1 + (U_{\theta,1}')^2} - 1 \right) \, dx \\
= f(U_{\theta,1}(0)) - f(U_{\theta,1}(-\epsilon^{-1/3})) \xrightarrow{\epsilon \to 0} f(\tan \theta) = 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta} \right),
\end{align*}
\]

where $f$ is defined in (5.3).

Thus, for any two points in $\mathbb{R}^2$ and angles $\theta, \theta' \in (0,\pi/2)$, we can take a sequence of $W^{2,1}$-curves $\{\gamma_{\theta',\epsilon}\}_{\epsilon > 0}$ connecting the two points such that the boundary warp angles are $\theta, \theta'$ and the (rescaled) total energy of the curves...
converges to the boundary warp energy of them, that is,
\[
\lim_{\varepsilon \to 0} \left( \varepsilon B[\gamma^{\varepsilon}_{\theta,\theta'}] + \frac{1}{\varepsilon} \left( \mathcal{L}[\gamma^{\varepsilon}_{\theta,\theta'}] - \mathcal{L}[\gamma^{\varepsilon}_0]\right) \right) \\
= 4 \left( 2\sqrt{2} - \sqrt{1 + \cos \theta} - \sqrt{1 + \cos \theta'} \right),
\]
for example by rotating the graph of a $W^{2,1}$-function $V_{\varepsilon,\theta,\theta'}$ on some interval $[A, B]$ where
\[
V_{\varepsilon,\theta,\theta'}(x) := \begin{cases} 
U_{\theta,\varepsilon}(A - x) & x \in [A, A + \varepsilon^{2/3}], \\
\text{suitable function in } (A + \varepsilon^{2/3}, B - \varepsilon^{2/3}), \\
U_{\theta',\varepsilon}(x - B) & x \in [B - \varepsilon^{2/3}, B].
\end{cases}
\]
Indeed, in $[A, A + \varepsilon^{2/3}] \cup [B - \varepsilon^{2/3}, B]$ the total energy converges to
\[
4 \left( 2\sqrt{2} - \sqrt{1 + \cos \theta} - \sqrt{1 + \cos \theta'} \right)
\]
as $\varepsilon \downarrow 0$, and in $(A + \varepsilon^{2/3}, B - \varepsilon^{2/3})$, if we only use two arcs of the circle with radius $\varepsilon$ and central angles $\theta_\varepsilon, \theta'_\varepsilon$ tending to zero as $\varepsilon \downarrow 0$ and a segment suitably (as Figure 4), then the total energy of the circular parts (graphs on $I^{\varepsilon}_c \subset [A, B]$) tends to be zero by
\[
\varepsilon \int_{I^{\varepsilon}_c} \kappa^2 ds + \frac{1}{\varepsilon} \left( \int_{I^{\varepsilon}_c} ds - |I^{\varepsilon}_c| \right) = \left( \varepsilon \cdot \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \right) \int_{I^{\varepsilon}_c} ds - \frac{|I^{\varepsilon}_c|}{\varepsilon} \\
\approx O(\theta_\varepsilon) + O(\theta'_\varepsilon) \xrightarrow{\varepsilon \to 0} 0,
\]
and of the segment part (graph on $I^{\varepsilon}_s \subset [A, B]$) also tends to be zero by
\[
\varepsilon \int_{I^{\varepsilon}_s} \kappa^2 ds + \frac{1}{\varepsilon} \left( \int_{I^{\varepsilon}_s} ds - |I^{\varepsilon}_s| \right) = \frac{1}{\varepsilon} \left( \int ds - |I^{\varepsilon}_s| \right) \\
\leq \frac{\sqrt{|I^{\varepsilon}_s|^2 + (O(\varepsilon^{2/3}))^2} - |I^{\varepsilon}_s|}{\varepsilon} \\
\approx O(\varepsilon^{1/3}) \xrightarrow{\varepsilon \to 0} 0,
\]
since $U_{\varepsilon,\rho}(-\varepsilon^{2/3}) \approx U_{\varepsilon,\rho'}(-\varepsilon^{2/3}) \approx O(\varepsilon^{2/3})$ as $\varepsilon \downarrow 0$.

Therefore, for any $\bar{u} \in \arg\min_{X^{1,1}} E_0$ which is $[\bar{x}_0, \ldots, \bar{x}_{N+1}]$-regular, by modifying it as above in $(\bar{x}_i, \bar{x}_{i+1}) \subset (a, b)_+$, we can take $\{u^{\varepsilon}\} \subset X^{2,1}$ such that $u^{\varepsilon} \rightarrow \bar{u}$ in $X^{1,1}$ and we have
\[
\lim_{\varepsilon \to 0} F_\varepsilon[u^{\varepsilon}] = \sum_{i=1}^N 4 \left( \sqrt{2} - \sqrt{1 + \cos \theta_i} \right) = \sum_{i=1}^N 4 \left( \sqrt{2} - \sqrt{1 + \alpha(\bar{x}_i)} \right),
\]
where $\theta_i$ is the contact angle of $\bar{u}$ with $\psi$ at $\bar{x}_i$. Note that, the curves obtained by the modification can be represented as the graph of a $W^{2,1}$-function for any sufficiently small $\varepsilon > 0$, and the second derivative of the modified parts of $u^{\varepsilon}$ goes to infinity as $\varepsilon \downarrow 0$ near the free boundary of $\bar{u}$, thus $u^{\varepsilon} \geq \psi$ for any sufficiently small $\varepsilon > 0$. □
Appendix A.

Lemma A.1 (Change of variables). Let $F$ be a $L^1$-integrable or non-negative measurable function on $[a,b]$ and $\Phi$ be a $C^1$-diffeomorphism from $[A,B]$ to $[a,b]$. Then
\[
\int_a^b F(x) \, dx = \int_A^B F(\Phi(y))|\Phi'(y)| \, dy.
\]

Proof. The case $\|F\|_{L^1} < \infty$ is in [15, §3.3.3, Theorem 2]. If $\|F\|_{L^1} = \infty$ then the right hand term is also $\infty$ since $\Phi$ is $C^1$-diffeomorphism. □

Lemma A.2. Let $u \in W^{1,1}(a,b)$ and $\Phi$ be a $C^1$-diffeomorphism from $[A,B]$ to $[a,b]$. Then $u \circ \Phi \in W^{1,1}(A,B)$ and
\[(u \circ \Phi)' = (u' \circ \Phi) \Phi'.\]

Proof. By Lemma A.1. □

The following is nothing but [8, Corollary 8.11].

Lemma A.3. Let $u \in W^{1,1}(a,b)$ and $G \in C^1(\mathbb{R})$. Then $G \circ u \in W^{1,1}(a,b)$ and
\[(G \circ u)' = (G' \circ u)u'.\]

The above lemmas lead to Lemma A.4, further Lemma A.5, A.6

Lemma A.4. Let $u \in W^{2,1}(a,b)$, $\Phi \in W^{2,1}(A,B)$, $|\Phi'| > 0$ in $[A,B]$, $\Phi([A,B]) = [a,b]$ and $\Psi$ be the inverse function of $\Phi$. Then $u \circ \Phi \in W^{2,1}(A,B)$ and $\Psi \in W^{2,1}(a,b)$. Moreover,
\[(u \circ \Phi)' = (u' \circ \Phi) \Phi', \quad (u \circ \Phi)'' = (u'' \circ \Phi)(\Phi')^2 + (u' \circ \Phi) \Phi'',\]
\[\Psi' = \frac{1}{\Phi' \circ \Psi}, \quad \Psi'' = \frac{(\Phi'' \circ \Psi) \Psi' - (\Phi' \circ \Psi)^2}{(\Phi' \circ \Psi)^2} \cdot\]

Lemma A.5. Let $\gamma$ be a $W^{2,1}$-curve and $\Phi$ be a $W^{2,1}$-reparameterization, i.e. $\Phi \in W^{2,1}(0,1)$, $|\Phi'| > 0$ in $[0,1]$ and $\Phi([0,1]) = [0,1]$. Then $\gamma \circ \Phi$ is a $W^{2,1}$-curve. Moreover, $B$, $L$, $W$ are invariant by $W^{2,1}$-reparameterization.
Lemma A.6. $B$, $L$, $W$ are invariant with respect to translation, reflection and rotation.

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