Affine Structures on Jet and Weil Bundles

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Abstract

Weil algebra morphism induce natural transformations between Weil bundles. In some well known cases, a natural transformation is endowed with a canonical structure of affine bundle. We show that this structure arises only when the Weil algebra morphism is surjective and its kernel has null square. Moreover, in some cases, this structure of affine bundle is passed down to Jet spaces. We give a characterization of this fact in algebraic terms. This algebraic condition also determines an affine structure between the groups of automorphisms of related Weil algebras.

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Introduction

The theory of Weil bundles and Jet spaces is developed in order to understand the geometry of PDE systems. C. Ehresmann formalized contact elements of S. Lie, introducing the spaces of jets of sections; simultaneously A. Weil showed in [8] that the theory of S. Lie could be formalized easily by replacing the spaces of contact elements by the more formal spaces of “points proches”, known as Weil bundles. The general theory of jet spaces [6] recovers the classical spaces of contact elements $J^l_m M$ of S. Lie from the ideas and methodology of A. Weil.

In the theory of Weil bundles, morphisms $A \to B$ of Weil algebras induce natural transformations [4] between Weil bundles. There are well known cases in which these natural transformations are affine bundles that often appear in differential geometry [4]. In [5] I. Kolár showed that this is the behaviour of $M^{A_l} \to M^{A_l-1}$. In this paper we characterize natural transformations that are affine bundles. It is done easily by adopting a different point of view of the tangent space of $M^A$ as done in [6]. Our result is as follows: there is a canonical affine structure for natural transformations $M^A \to M^B$ induced by a surjective morphisms $A \to B$ whose kernel has null square. This is true for $M^{A_l} \to M^{A_k}$ with $2k + 1 \geq l > k \geq 0$. 

1
In some cases the natural transformations induce maps between Jet spaces. This situation holds in the cases studied in [5]. We characterize this situation, and moreover, we will determine when an affine structure on the Weil bundle morphism is passed down to the Jet space morphism. In addition to that, we will prove that in this case there also exist an affine structure in the morphism between the groups of automorphisms of the Weil algebras. This is true for spaces $J^l_m M \to J^k_m$ with $l > k > 0$ and $3k + 1 \geq 2l$.

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**Notation and Conventions**

All manifolds and maps are assumed to be infinitely differentiable. All results involving a manifold $M$ assume that it is not empty and all results involving jet spaces $J^A M$ assume that the jet space $J^A M$ is also not empty (in such case, maybe the algebraic conditions for the existence of affine structure are satisfied, but no structure exists).

1 **Weil Bundles**

By a *Weil algebra* we mean a finite dimensional, local, commutative $\mathbb{R}$-algebra with unit. If $A$ is a Weil algebra, let us denote by $m_A$ its maximal ideal. If $A$ and $B$ are Weil algebras, by a morphism $A \to B$ we mean a $\mathbb{R}$-algebra morphism.

**Example 1** Let $\mathbb{R}[[\xi_1, \ldots, \xi_m]]$ be the ring of formal series with real coefficients and free variables $\xi_1, \ldots, \xi_m$. Let $m$ be the maximal ideal spanned by $\xi_1, \ldots, \xi_m$. Then, for any non-negative integer $l$, the ring

$$\mathbb{R}_m^l = \mathbb{R}[[\xi_1, \ldots, \xi_m]]/m^{l+1}$$

is a Weil algebra.

For every Weil algebra $A$, there is a non-negative integer $l$ such that $m_A \neq 0$ but $m_A = 0$; we say that $l$ is the *height* of $A$. The *width* of $A$ is the dimension of the vector space $m_A/m_A^2$. Thus, $\mathbb{R}_m^l$ is a Weil algebra of height $l$ and width $m$. If $A$ is of height $l$ and width $m$ there exists a surjective morphism $\mathbb{R}_m^l \to A$ (see [5], [6]).

**Definition 1** Let $M$ be a smooth manifold and $A$ a Weil algebra. The set $M^A$ of the $\mathbb{R}$-algebra morphisms

$$p^A : C^\infty(M) \to A,$$
is the so-called space of near-points of type $A$ of $M$, also called $A$-points of $M$.

Let us consider a basis $\{a_k\}$ of $A$. For each $f \in C^\infty(M)$ we define real valued functions $\{f_k\}$ on $M^A$ by setting:

$$p^A(f) = \sum_k f_k(p^A) a_k.$$ 

We say that the $\{f_k\}$ are the real components of $f$ relative to the basis $\{a_k\}$.

**Theorem 1 (6)** The space $M^A$ is endowed with a unique structure of a smooth manifold such that the real components of smooth functions on $M$ are smooth functions on $M^A$.

**Example 2** It is well known that each morphism $C^\infty(M) \to \mathbb{R}$ is a point of $M$. Since the real components on $M^R$ of smooth functions coincide with the functions themselves we know that $M^R = M$.

**Example 3** For each Weil algebra $\mathbb{R}^l : m$ let us denote by $M^l_m$ the space of near points of type $\mathbb{R}^l_m$. Then $M^l_1$ is the tangent bundle $TM$. In general $M^l_m$ is the space of germs at the origin of smooth maps $\mathbb{R}^m \to M$ up to order $l$ (see [8]).

A Weil algebra morphism $\phi : A \to B$ induces by composition a smooth map $\hat{\phi} : M^A \to M^B$ [4] [6] which is called a natural transformation, $\hat{\phi}(p^A) = \phi \circ p^A$.

**Example 4** Let us notice that a Weil algebra is provided with a unique morphism $\mathbb{R} \to A$. It induces a canonical map $M^A \to M$ which is a fibre bundle. This bundle is the so-called Weil bundle of type $A$ over $M$. Let $p^A$ be in $M^A$ and $p$ its projection in $M$. Then we will say that $p^A$ is an $A$-point near $p$. For each smooth function $f$ the value $p^A(f)$ depends only on the germ at $p$ of $f$.

A smooth map $f : M \to N$ of smooth manifolds induces by composition a $\mathbb{R}$-algebra morphism

$$f^* : C^\infty(N) \to C^\infty(M), \quad f^*(g) = g \circ f.$$ 

We can compose this morphism $f^*$ with $A$-points of $M$ obtaining $A$-points of $N$. Then, Weil algebra morphisms and smooth maps transform near-points by composition, and this implies a functorial behaviour of Weil bundles with respect to those transformations.

We can formalize this situation in the following way. Let us consider $\mathcal{M}$ the category of smooth manifolds, and $\mathcal{W}$ the category of Weil algebras. It the direct product category $\mathcal{M} \times \mathcal{W}$, objects are pairs $(M, A)$ and morphisms are pairs $(f, \phi)$. We define $w(M, A) = M^A$. Thus, the natural way of defining the natural image of the morphism $(f, \phi)$,

$$f : M \to N, \quad \phi : A \to B,$$
is 
\[ w(f, \phi) : M^A \to N^B, \quad p^A \mapsto \phi \circ p^A \circ f^*. \]

The following result follows easily:

**Proposition 1** The assignment
\[ w : M \times W \to M, \quad (M, A) \mapsto M^A, \]
is a covariant functor.

**Remark 1** There are two remarkable cases of induced maps:
- If \( X \subset M \) is an embedding then for each Weil algebra \( A \) the induced map \( X^A \to M^A \) is also an embedding.
- If \( A \to B \) is a surjective morphism then for all \( M \) the induced natural transformation \( M^A \to M^B \) is a fibre bundle.

**Example 5** Let \( A \) be a of height \( l \). For each \( k \leq l \) let us define \( A_k = A/m_A^{k+1} \). Then \( A_k \) is a Weil algebra of height \( k \) and \( A_l = A \). For \( k \geq r \) we have a natural projection \( M^{A_k} \to M^A \) which is a bundle. In particular we have canonical bundles \( M_k^\prime \to M_k^r \).

### 1.1 Tangent Structure

Given \( p^A \in M^A \), let us denote by \( \text{Der}_{p^A}(C^\infty(M), A) \) the space of derivations of the ring \( C^\infty(M) \) into the module \( A \), where the structure of \( C^\infty(M) \)-module of \( A \) is induced by the point \( p^A \) itself. By derivations we mean \( \mathbb{R} \)-linear maps \( \delta : C^\infty(M) \to A \) satisfying Leibniz’s formula:

\[ \delta(f \cdot g) = p^A(f) \cdot \delta(g) + p^A(g) \cdot \delta(f). \quad (1) \]

If \( D \) is a tangent vector to \( M^A \) at \( p^A \) then it defines a derivation give by:

\[ \delta(f) = \sum_k (Df_k) a_k, \]

and it is easy to prove that the spaces \( \text{Der}_{p^A}(C^\infty(M), A) \) and \( T_{p^A}(M^A) \) are identified in this way [6]. From now on we will assume this identification. It applies not just to the vector spaces, but it is also compatible with Proposition 1. The following theorem resume some results in [6].

**Theorem 2 (Muñoz, Rodríguez, Muriel [6])** Let us consider a smooth map \( f : M \to N \), a Weil algebra morphism \( \phi : A \to B \), and the induced smooth map:

\[ w(f, \phi) : M^A \to N^B, \quad p^A \to q^B = \phi \circ p^A \circ f^*. \]

Then, the linearized map \( w(f, \phi)' : T_{p^A}(M^A) \to T_{q^B}(N^B) \) coincides (under the identification assumed above) with the map:

\[ \text{Der}_{p^A}(C^\infty(M), A) \to \text{Der}_{q^B}(C^\infty(N), B), \quad \delta \mapsto \phi \circ \delta \circ f^*. \]
1.2 Affine Structure

In this section we will analyze the structure of the fibre bundle induced by a surjective morphism \( A \to B \) which has been introduced in Remark 1. In some specific cases it has been proved that those bundles are endowed with a canonical structure of affine bundles. We will see that this structure has its foundation in the algebraic construction of the spaces of near-points. Indeed, it is an easy task to give an algebraic characterization of this fact. A morphism will induce an affine structure if and only if its kernel ideal has null square.

The key point is to consider both, near-points and tangent vectors to \( M^A \) as \( \mathbb{R} \)-linear maps from \( C^\infty(M) \) to a Weil algebra. Thus, they are provided with the addition law of \( \mathbb{R} \)-linear maps. Under some adequate assumptions we will obtain a new near-point when we add a derivation to a near-point.

**Lemma 1** Let us consider \( p^A \in M^A \) and \( D \in T_{p^A}(M^A) \). The sum \( p^A + D \) is an \( A \)-point of \( M \) if an only if \( (\text{Im}(D))^2 = 0 \).

**Proof.** Let us define \( \tau = p^A + D \). Then, for all \( f, g \in C^\infty(M) \),

\[
\tau(f \cdot g) = \tau(f) \cdot \tau(g) - D(f) \cdot D(g);
\]

since it is a \( \mathbb{R} \)-linear map, it is an algebraic morphism if and only if for any pair \( f \) and \( g \) of smooth functions in \( M \) we have \( D(f) \cdot D(g) = 0 \). \( \text{q.e.d.} \)

**Lemma 2** Let us consider \( p^A \) and \( q^A \) in \( M^A \). The difference \( \delta = q^A - p^A \) is a derivation and belongs to \( T_{p^A}(M^A) \) if and only if \( (\text{Im}(\delta))^2 = 0 \).

**Proof.** For all \( f \) and \( g \) in \( C^\infty(M) \) we have

\[
\delta(f \cdot g) = p^A(f) \cdot \delta(g) + p^A(g) \cdot \delta(f) + \delta(f) \cdot \delta(g).
\]

Thus, \( \delta \) satisfies Leibniz’s formula if and only if for all \( f \) and \( g \) in \( C^\infty(M) \) the product \( \delta(f) \cdot \delta(g) \) vanishes. \( \text{q.e.d.} \)

**Lemma 3** Let \( I \) be an ideal of a \( k \)-algebra \( A \) with \( k \) a field of characteristic different from 2. The square \( I^2 \) of the ideal vanishes if an only if for all \( x \in I \) its square \( x^2 \) also vanishes.

**Proof.** If \( I^2 \) vanishes it is clear that \( x^2 = 0 \) for all \( x \in I \). Conversely, let us assume that the square of all elements of \( I \) vanish. Let \( x \) and \( y \) be in \( I \), then:

\[
0 = (x + y)^2 = x^2 + y^2 + 2xy = 2xy.
\]

Hence, \( xy = 0 \) and \( I^2 \) vanish. \( \text{q.e.d.} \)

Let \( \phi: A \to B \) be a surjective morphism of Weil algebras, and let \( I \) be its kernel ideal. Let us consider a smooth manifold \( M \), and the induced fibre bundle \( \hat{\phi}: M^A \to M^B \). The linearization \( \hat{\phi}' \) gives rise to an exact sequence

\[
0 \to TV_{p^A}(M^A) \to T_{p^A}(M^A) \to T_{p^A}(M^B) \to 0
\]
which defines the vertical tangent sub-bundle \( TV^\hat{\varphi}(M^A) \subset T(M^A) \). Taking into account that tangent vectors are derivations from \( C^\infty(M) \) to \( A \), we will notice that \( D \in T_{p^A}(M^A) \) belongs to \( TV^\hat{\varphi}(M^A) \) if and only if \( \text{Im}(D) \subseteq I \). Then \( TV^\hat{\varphi}(M^A) \) is the space of derivations from \( C^\infty(M) \) to \( I \), where the structure of \( C^\infty(M) \)-module in \( I \) is given by the morphism \( p^A: C^\infty(M) \to A \).

Let us assume that \( I^2 \) vanishes. Let us consider \( p^A \) and \( q^A \) in the same fibre of the bundle, \( \text{id est } \hat{\varphi}(p^A) = \hat{\varphi}(q^A) = p^B \). Thus, \( p^A \) and \( q^A \) induce the same structure of \( C^\infty(M) \)-module in \( I \). Hence, the space of derivations \( TV^\hat{\varphi}(M^A) \) is canonically isomorphic to \( TV^\hat{\varphi}(M^A) \). We will denote this space by \( TV^\hat{\varphi}_{p^A}(M^B) \).

Using Lemma 1 and 2 we conclude that for any pair of \( A \)-points \( p^A \) and \( q^A \) in the fibre of \( p^B \), as above the difference \( p^A - q^A \) is a derivation which belongs to the space \( TV^\hat{\varphi}_{p^A}(M^B) \). And also, for any derivation \( D \in TV^\hat{\varphi}_{p^A}(M^B) \), the sum \( p^A + D \) is a near-point of type \( A \) in fiber of \( p^B \). Thus, the natural law of addition of linear maps,

\[
\hat{\phi}^{-1}(p^B) \times TV^\hat{\varphi}_{p^A}(M^B) \to \hat{\phi}^{-1}(p^B), \quad (p^A, D) \mapsto p^A + D
\]

induces an affine structure on the fibre \( \hat{\phi}^{-1}(p^B) \) associated with the vector space \( TV^\hat{\varphi}_{p^A}(M^B) \) of derivations from \( C^\infty(M) \) to \( I \).

We define the vector bundle \( TV^\hat{\varphi}(M^B) \) on \( M^B \),

\[TV^\hat{\varphi}(M^B) \to M^B,\]

whose fibre over a \( B \)-point \( p^B \) is the space \( TV^\hat{\varphi}_{p^A}(M^B) \). Hence, \( TV^\hat{\varphi}(M^B) \) is the vector bundle associated with the affine bundle \( \hat{\phi}: M^A \to M^B \),

\[M^A \times_M M^B TV^\hat{\varphi}(M^B) \to M^A, \quad (p^A, D) \mapsto p^A + D.\]

On the other hand, if \( I^2 \) does not vanish, by applying Lemma 3 we find a derivation \( D: C^\infty \to I \) such that \( (\text{Im}(D))^2 \) does not vanish. Hence, \( p^A + D \) does not belong to \( M^A \). We have proved the following:

**Theorem 3** Let \( \phi: A \to B \) be a surjective Weil algebra morphism, and let \( I \) be its kernel ideal. For any manifold \( M \), the natural addition law of linear maps induces an structure of affine bundle in the fibre bundle \( \hat{\phi}: M^A \to M^B \) if and only if \( I^2 = 0 \).

Be means of some elementary computations on the algebras \( A_k \) and \( \mathbb{R}^m \) we deduce the following corollaries to the Theorem 3.

**Corollary 1** Let \( A \) be a Weil algebra of height \( l \). The natural projection \( M^A \to M^{Ak} \) is endowed with a canonical structure of affine bundle if and only if \( 2k + 1 \geq l \).

**Corollary 2** The natural projection of spaces of frames, \( M^I_m \to M^I_m \) is endowed with a canonical structure of affine bundle if and only if \( 2k + 1 \geq l \).
2 Jet Spaces

Definition 2 A jet of \( M \) is an ideal of differentiable functions \( \mathfrak{p} \subset C^\infty(M) \) such that the quotient algebra \( A_\mathfrak{p} = C^\infty(M)/\mathfrak{p} \) is a Weil algebra. A jet \( \mathfrak{p} \) is said to be of type \( A \), or an \( A \)-jet, if \( A_\mathfrak{p} \) is isomorphic to \( A \). The set \( J^A M \) of \( A \)-jets of \( M \) is the so called \( A \)-jet space of \( M \).

An \( A \)-point \( p^A \) of \( M \) is said to be regular if it is a surjective morphism. The set of regular \( A \)-points of \( M \) is denoted by \( \tilde{M}^A \). It is a dense open subset of \( M^A \). It is obvious that an \( A \)-point is regular if and only if its kernel is an \( A \)-jet. Thus, we have a surjective map:

\[ \ker: \tilde{M}^A \to J^A M. \] (2)

Let us consider \( \text{Aut}(A) \), the group of automorphisms of \( A \). It is a linear algebraic group, as can be seen easily by representing it as a subgroup of \( GL(\mathfrak{m}_A) \) (see [3]). This group \( \text{Aut}(A) \) acts on \( \tilde{M}^A \) by composition. Two \( A \)-points related by an automorphism have the same kernel ideal. Moreover, two \( A \)-points with the same kernel ideal are related by an automorphism. In this way \( J^A M \) is identified with the space of orbits \( \tilde{M}^A/\text{Aut}(A) \), and its manifold structure is determined in this way (see [1]).

Example 6 The group \( G^l_m \) of automorphisms of \( \mathbb{R}^l_m \) is called the \( l \)-th prolongation of the linear group of rank \( m \) (see [7]), also called jet group. In particular \( G^1_m \) is the linear group of rank \( m \). The group \( G^l_m \) is the group of transformations of \( \mathbb{R}^l \) around a fixed point up to order \( l \).

Theorem 4 (Alonso-Blanco [1]) There is a unique structure of smooth manifold on \( J^A M \) such that the map \( \ker \) (appearing in equation (2)) is a principal bundle with structural group \( \text{Aut}(A) \).

Example 7 Let us denote by \( J^l_m M \) the space of jets of type \( \mathbb{R}^l_m \) of \( M \). Thus, \( J^l_m M \) is the space of germs of \( m \)-submanifolds of \( M \) up to order \( l \).

The space \( J^A M \) is a bundle over \( M \). We will say that that \( \mathfrak{p} \) is a jet over the point \( p \) if \( \mathfrak{p} \subset \mathfrak{m}_p \), where \( \mathfrak{m}_p \) is the ideal of smooth functions vanishing at \( p \). If \( p^A \) is an \( A \)-point near \( p \) then \( \ker(p^A) \) is a jet over \( p \).

2.1 Functorial behaviour

In contrast with Weil bundles, jet spaces do not show a functorial behaviour. A smooth map \( f: M \to N \) induces a smooth map on jet spaces, but in the general case it is defined only on an open dense subset of \( J^A M \), which depends on \( f \). There is no natural object associated to a Weil algebra morphism \( A \to B \). There is a natural, highly interesting, object associated to a pair \( (A,B) \) of Weil algebras: the Lie correspondence. This is a submanifold \( \Lambda_{A,B} M \) of the fibred product \( J^A M \times_M J^B M \),

\[ \Lambda_{A,B} M = \{(p, \bar{p}) \in J^A M \times_M J^B M : p \subset \bar{p}\}. \]
The Lie correspondence is empty if and only if there does not exist any surjective morphism from $A$ to $B$. There is a special case to be analyzed in which it is the graph of a bundle $J^A M \rightarrow J^B M$.

Let $I$ be an ideal of $A$. Then, for each automorphism $\sigma$ of $A$, the space $\sigma(I)$ is another ideal of $A$; the group $\text{Aut}(A)$ acts in the set of ideals of $A$. We say that $I$ is an invariant ideal of $A$ if for all $\sigma \in \text{Aut}(A)$ we have $I = \sigma(I)$. For each positive integer $k$ the $k$-th power of the maximal ideal $m_A^k$ is an invariant ideal, and any other ideals obtained from these by general processes of division and derivation are also invariant; some examples are shown in [2]. Let $I \subset A$ be an invariant ideal and $\phi: A \rightarrow A/I = B$ the canonical projection into the quotient algebra $B$. Let $p^A$ be an $A$-point and $p$ be its kernel. It is obvious that that the kernel ideal $\overline{p}$ of the composition $\phi \circ p^A$ is the unique $B$-jet containing $p$. Let us denote by $\hat{\phi}$ the restriction of $\phi$ to the space of regular points $\hat{M}^A$.

We have a commutative diagram:

\[
\begin{array}{ccc}
\hat{M}^A & \xrightarrow{\hat{\phi}} & \hat{M}^B \\
\downarrow & & \downarrow \\
J^A M & \xrightarrow{\phi^J} & J^B M \\
\downarrow & & \downarrow \\
p & \xrightarrow{\phi(p^A)} & p
\end{array}
\] (3)

The Lie correspondence is precisely the set

$$\Lambda_{A,B} = \{(p, \phi^J(p)) : p \in J^A M\}.$$ 

Summarizing, the following result holds:

**Theorem 5** If $I \subset A$ is an invariant ideal and $B$ is the quotient algebra $A/I$ then there is a canonical bundle structure $J^A M \rightarrow J^B M$.

### 2.2 Tangent structure

In order to study the linearization of $\phi^J$ in diagram (3) we need some characterization of the tangent space to $J^A M$ at a jet $p$.

**Theorem 6** (1, 6) The space $T_p(J^A M)$ realizes itself canonically as a quotient of the space of derivations $C^\infty(M) \rightarrow A_p$. A derivation $\delta$ defines the null vector if and only if $\delta(p) = 0$. Thus,

$$T_p(J^A M) \cong \text{Der}(C^\infty(M), A_p)/\text{Der}(A_p, A_p),$$

In order to a better understanding let us give some sketch of the proof. Let us remind that the Lie algebra of $\text{Aut}(A)$ is the space of derivations $\text{Der}(A, A)$ as can be shown in a matrix representation of the group (see [3, 4]). Taking $p^A \in M^A$ such that $\text{ker}(p^A) = p$, the representation of $\text{Der}(A, A)$ as fundamental vector fields of the action of $\text{Aut}(A)$ on $M^A$ gives rise to an exact sequence:

$$0 \rightarrow \text{Der}(A, A) \xrightarrow{\text{fun. vec. fields}} T_{p^A}(M^A) \xrightarrow{\ker'} T_p J^A(M) \rightarrow 0.$$
Let us take into account that $T_{p^A}(M^A)$ is the space of derivations $\text{Der}_{p^A}(C^\infty(M), A)$ and that $p^A$ induces an isomorphism of $C^\infty$-algebras between $A$ and $A_p$. Thus, it follows the isomorphism of the theorem. This isomorphism does not depend on the A-point $p^A$ representing the A-jet $p$. It can be seen by means of the principal structure stated in Theorem 4.

3 Affine Structure on Jet Spaces

3.1 Space of Regular Points

Let $I$ be an ideal of the Weil algebra $A$, and $\phi: A \to B$ the canonical projection onto the quotient algebra $B = A/I$.

Lemma 4 (6) A finite set $\{a_1, \ldots, a_m\} \subset m_A$ is a system of generators of $A$ if and only if the set of their classes $\{\bar{a}_1, \ldots, \bar{a}_m\}$ in $m_A/m_A^2$ is a basis of $m_A/m_A^2$.

Lemma 5 If $I \not\subset m_A^2$ then there exists a non trivial subalgebra $S \subset A$ such that $S/(S \cap I) \cong B$.

Proof. If $I \not\subset m_A^2$ then the canonical projection $m_A/m_A^2 \to m_B/m_B^2$ has nontrivial kernel. There exists a finite set $\{a_1, \ldots, a_r\}$ such that $\{\phi(a_1), \ldots, \phi(a_r)\}$ is a basis of $m_B/m_B^2$ but $\{\bar{a}_1, \ldots, \bar{a}_r\}$ is not a basis of $m_A/m_A^2$. Then $S = \mathbb{R}[a_1, \ldots, a_r]$ is a proper subalgebra and verifies $S/(S \cap I) = B$. q.e.d.

Note that each subalgebra of a $A$ is a Weil algebra. For each subset $X \subset m_A$, $\mathbb{R}[X]$ is a Weil algebra and its maximal ideal is spanned by $X$.

Lemma 6 The following conditions are equivalent:

1. $I \subset m_A^2$.
2. $\hat{\phi}^{-1}(\hat{M}^B) \subset \hat{M}^A$.

Proof. Let us assume $I \subset m_A^2$, and consider $p^A \in M^A$ such that $\hat{\phi}(p^A)$ is a regular $B$-point. There are differentiable functions $f_1, \ldots, f_m$ in $M$ such that $\{\phi(p^A(f_1)), \ldots, \phi(p^A(f_m))\}$ is a system of generators of $B$. Then their classes modulo $m_B^2$ form a basis $\{\phi(p^A(f_1)), \ldots, \phi(p^A(f_m))\}$ of $m_B/m_B^2$. Since $I$ is contained in $m_A^2$ and $m_B = m_A/I$ we have that $m_B/m_B^2 \cong m_A/m_A^2$. Then $\{\phi(p^A(f_1)), \ldots, \phi(p^A(f_m))\}$ is a basis of $m_A/m_A^2$ and $\{p^A(f_1), \ldots, p^A(f_m)\}$ is a system of generators of $A$. Thus, $A$ is regular.

Conversely, let us assume $I \not\subset m_A^2$. Consider a subalgebra $S \subset A$ as in Lemma 5. Then $M^S \to M^B$ is a bundle. Let $p^B \in M^B$ be a regular $B$-point, and $p^S$ any preimage of $p^B$. Hence, $p^S$ is a $S$-point, and thus a non regular $A$-point, but $\hat{\phi}(p^S) = p^B$. q.e.d.
From now on we will consider the annihilator ideal of $I$,

$$\text{Ann}(I) = \{a \in A : \forall b \in I \; ab = 0\}.$$  

Let us notice that $I \subseteq \text{Ann}(I)$ if and only if $I^2 = 0$.

**Theorem 7** The bundle $\hat{\phi} : \hat{M}^A \to \hat{M}^B$ is endowed with a canonical structure of affine bundle (given by the addition law of morphisms and derivations) if and only if $I \subseteq m_A^2 \cap \text{Ann}(I)$.

**Proof.** Suppose that $I \subseteq m_A^2 \cap \text{Ann}(I)$. Then the addition law of $A$-points and derivations induces an affine structure on $\hat{\phi}$. Let $p^B \in \hat{M}^B$ be a regular $B$-point. In view of Lemma 6 the fibre $\hat{\phi}^{-1}(p^B)$ consist of regular points. Thus, $\hat{\phi}^{-1}(p^B) = \hat{\phi}^{-1}(p^B)$ so that the bundle $\hat{M}^A \to \hat{M}^B$ is the restriction of $M^A \to M^B$ to the open submanifold $\hat{M}^B$; which is an affine bundle.

On the other hand, let us assume that $I \not\subseteq (m_A^2 \cap \text{Ann}(I))$. If $I \not\subseteq \text{Ann}(I)$, then the addition of an $A$-points and a derivation is not in general an $A$-opint and there is no affine structure. Finally, let us assume that $I \subseteq \text{Ann}(I)$ but $I \not\subseteq m_A^2$. Then there is an affine structure on $\hat{\phi}$. However, by Lemma 6 there is a non-regular $A$-point $p^A M$ such that its projection $p^B$ is regular. Let us consider $q^A \in \hat{\phi}^{-1}(p^B)$, and $D = p^A - q^A \in TV_{q^A} \hat{M}^A$. Thus $q^A + D \in \hat{M}^A$, and there is not affine structure. q.e.d.

**Corollary 3** Let $A$ be of height $l$. Then for each $l > k > 0$ the natural projection $\hat{M}^A \to \hat{M}^k$ is an affine bundle if and only if $2k + 1 \geq l$.

**Corollary 4** For any $l > k > 0$, the natural projection $\hat{M}^l_m \to \hat{M}^k_m$ is an affine bundle if and only if $2k + 1 \geq l$.

### 3.2 Affine Structure on the Group of Automorphisms

Let $\subset A$ be an invariant ideal of the Weil algebra $A$, and $\phi : A \to B$ the canonical projection into the quotient algebra. Each automorphism $\sigma \in \text{Aut}(A)$ verifies $\sigma(I) = I$, thus it induces an automorphism $\phi_\sigma(\sigma) \in \text{Aut}(B)$.

**Definition 3** We will call the affine sequence associated to $I$ the following sequence of algebraic groups:

$$K(I) \to \text{Aut}(A) \xrightarrow{\phi_*} \text{Aut}(B),$$

where

$$K(I) = \{\sigma \in \text{Aut}(A) : \sigma(a) - a \in I, \sigma(b) = b, \forall a \in A \forall b \in I\},$$

is the subgroup of automorphisms of $A$ inducing the identity both in $B$ and $I$.  

10
We will say that the affine sequence is *exact on the left side* in \( K(I) = \ker \phi_* \). Analogously, we will say that it is *exact on the right side* if \( \phi_* \) is surjective. Note that if it is exact both on the right and left sides then it is an exact sequence.

Let us notice that if \( I \subseteq \text{Ann}(I) \) then the \( A \)-module \( I \) is also a \( B \)-module. By composition we have a canonical immersion \( \text{Der}(B, I) \subseteq \text{Der}(A, I) \) identifying derivations from \( B \) to \( I \) with derivations from \( A \) to \( I \) which vanish on \( I \subset A \).

**Proposition 2** Let us assume \( I \subseteq \text{Ann}(I) \). The affine sequence associated to \( I \) is exact on the left side if and only if \( \text{Der}(B, I) = \text{Der}(A, I) \).

**Proof.** Assuming that the affine sequence is exact on the left side, let us consider the sequence of Lie algebras induced by the sequence of algebraic groups associated to \( I \). The Lie algebra of \( K(I) \) is, by the definition of \( K(I) \), the space of derivations from \( A \) to \( I \) which vanish on \( I \). Thus, it is identified with the space \( \text{Der}(B, I) \). On the other hand the kernel of the Lie algebra morphism induced by \( \phi_* \) is the space \( \text{Der}(A, I) \). If the affine sequence is exact on the left side, then the Lie algebra of \( K(I) \) coincides with this last one, and \( \text{Der}(B, I) = \text{Der}(B, A) \).

Conversely, let us assume that \( \text{Der}(A, I) = \text{Der}(B, I) \), i.e. all derivations from \( A \) to \( I \) vanish on \( I \). Let \( \sigma \) be an automorphism of \( A \). The difference \( \text{Id}_A - \sigma \) is a derivation from \( A \) to \( I \). It vanishes on \( I \), and thus for any \( a \in I \) we have \( \sigma(a) = a \), and then \( \sigma \) induces the identity in \( I, \sigma \in K(I) \). q.e.d.

**Theorem 8** If \( I \subseteq \text{Ann}(A) \cap m_A^2 \) and the affine sequence is exact, then \( \phi_* \) is endowed with a natural structure of affine bundle associated with the space \( \text{Der}(A, I) \) with the following addition law:

\[
\sigma \oplus D = \sigma + \sigma \circ D.
\]

**Proof.** Let \( D \) be a derivation from \( A \) to \( I \). Then, \( \text{Id}_A + D \) is an automorphism of \( A \). Conversely, let \( \sigma \) be an automorphism of \( A \) such that \( \phi_*(\sigma) = \text{Id}_B \). In such case \( \sigma - \text{Id}_A \) is a derivation and it takes values in \( I \). We have:

\[
\text{Der}(A, I) = \ker(\phi_*).
\]

By definition of the addition law we have that \( \sigma \oplus D = \sigma(\text{Id} + D) \), so that \( \sigma \oplus \text{Der}(A, I) = \sigma \circ \ker(\phi_*) \). Finally let us see that the addition law of the bundle is compatible with the vector space structure of \( \text{Der}(A, I) \):

\[
(\sigma \oplus D) \oplus D' = \sigma \oplus (D + D').
\]

From Proposition 2 we have that,

\[
(\sigma \oplus D \oplus D' = \sigma + \sigma \circ D(\sigma + \sigma \circ D) \circ D' = \sigma \oplus (D + D') + \sigma \circ D \circ D'.
\]

And because of each derivation vanish on \( I \) we have that \( D \circ D \) vanish. q.e.d.

**Lemma 7** If \( I \subseteq \text{Ann}(I)^2 \) then the affine sequence associated to \( I \) is exact on the left side.
Proof. Let us consider a derivation $D: A \to I$, and $a$ in $I$; thus $a$ is also in in Ann($I$)$^2$ and the we can write $a = \sum b_k c_k$ for suitable $b_k$ and $c_k$ in Ann($I$). We have,

$$D(a) = \sum b_k D(c_k) + c_k D(b_k) = 0.$$ 

Thus, $D$ anihilates $I$. We conclude that Der($A, I$) = Der($B, I$). Our assertion follows directly from Proposition 2 q.e.d.

Corollary 5 If the natural numbers $l > r > 0$ verify $3r + 1 \geq 2l$ then the natural projection $G^l_m \to G^r_m$ is an affine bundle.

Proof. In general $G^l_m \to G^k_m$ is a surjective morphism. We apply the Lemma 7 to the case $A = \mathbb{P}^l_m, I = m^{-1}_A$. Then, Ann($I$) = $m^{-1}_A$ and $m^{-1}_A \subseteq$ Ann($m^{-1}_A$). If and only if $k + 1 \geq 2(l - k)$. q.e.d.

3.3 Affine structure on Jet bundles

Let $I \subseteq A$ be an invariant ideal with $I \subseteq$ Ann($I$) $\cap m^2_A$ and let us denote by $B$ the quotient algebra $A/I$ as above. For each $p \in J^l A$ let us denote by $\pi_p: C^\infty(M) \to A_p$ the canonical projection and $\bar{p} = \phi^l(p) \in J^B M$. Then $A_p \simeq p$ and $\bar{p}/p \simeq I$. For each $D \in$ Der($C^\infty(M), \bar{p}/p$) let us define,

$$p + D = \ker(\pi_p + D).$$

In such case, because $I \subseteq$ Ann($I$) we have that $\pi_p + D$ is an $A_p$-point. It is regular because $I \subseteq m^2_A$. Hence, $p + D$ is an $A$-jet. We also have that $\phi^l(p + D) = \bar{p}$, because $D$ takes values in $\bar{p}/p$.

Lemma 8 Each derivation $D: C^\infty(M) \to \bar{p}/p$ which vanishes on $p$ also vanishes on $\bar{p}$ if and only if the affine sequence associated to $I$ is exact on the left side.

Proof. A derivation $C^\infty(M) \to \bar{p}/p$ which anihilates $p$ factorizes through a derivation $A_p \to \bar{p}/p$. Then, the claim es equivalent to Lemma 2 q.e.d.

Theorem 9 The addition law (1) defines an affine structure on the bundle $\phi^l: J^A M \to J^B M$ for any smooth manifold $M$ if and only if the affine sequence associated to $I$ is exact.

Proof. A derivation $D: C^\infty(M), \bar{p}/p$ defines a tangent vector $[D] \in T_p(J^A M)$ as it is shown in Theorem 8. Moreover, we have that $[D] \in TV_{\bar{p}}^{\phi^l} J^A M$, because $D$ takes values in $\bar{p}/p$. Let us prove that the following conditions, which are equivalent to the assertion of the theorem, hold if and only if the affine sequence associated to $I$ is exact.

1. If two derivations $D$ and $D'$ from $C^\infty(M)$ to $A_p$ define the same tangent vector at $p$ then $p + D = p + D'$.
2. The natural projection \( \text{Der}(C^\infty(M), \mathfrak{p}/\mathfrak{p}) \to TV^\mathfrak{p}_{\phi'}(J^A M) \) is surjective.

3. For each \( q \subset \overline{p} \) there is a unique \([D] \in TV^\mathfrak{p}_{\phi'}(J^A M)\) such that \( p + [D] = q \).

4. For each \( A \)-jet \( q \) contained in the \( B \)-jet \( \overline{p} \) there is a canonical isomorphism \( TV^\mathfrak{p}_{\phi'}(J^A M) \cong TV_q^{\phi'}(J^A M) \).

Condition (1.) holds if and only if the affine sequence is exact on the left side.

Let \( D \) and \( D' \) define the same tangent vector \([D] \in TV^\mathfrak{p}_{\phi'}(J^A M)\). In such case the difference \( \delta = D - D' \) vanishes on \( p \). By Lemma [5] each derivation vanishing on \( p \) also vanish on \( \overline{p} \) if and only if the affine sequence associated to \( I \) is exact on the left side. In the case of exact affine sequence we have:

\[
\ker(\pi_p + D) = \ker(\pi_p + D').
\]

If the affine sequence is exact on the right side then condition (2.) holds.

It is an application of the classical Snake Lemma. We have a natural diagram of exact columns and arrows:

![Diagram](image)

According to the Snake Lemma, if \( \text{coker}(\psi) \) vanishes then we have an exact sequence

\[
\ldots \to 0 \to \text{coker}(\overline{\psi}) \to 0 \to \ldots
\]

and vice-versa. Hence, \( \text{coker}(\psi) \) vanish if and only if \( \text{coker}(\overline{\psi}) \) vanishes. Note that the natural mapping \( \psi \) is the linearization of the algebraic group morphism \( \text{Aut}(A_p) \to \text{Aut}(A_{\overline{p}}) \). Since \( A_p \cong A \) and \( A_{\overline{p}} \cong B \) we conclude that if the affine sequence associated to \( I \) is exact on the right side then (2.) holds.

Condition (3.) holds if and only if the affine sequence associated to \( I \) is exact on the right side.
Let us consider any other $A$-jet $q \subset \bar{p}$ and an isomorphism $\tau: A_q \to A_p$. Thus, we have diagram (not commutative):

$$
\begin{array}{ccc}
C^\infty(M) & \xrightarrow{\pi_q} & A_p \\
\downarrow{\pi_q} & \tau & \downarrow{\bar{\pi}_p} \\
A_q & \xrightarrow{\pi_q} & A_{\bar{p}} \\
\end{array}
$$

Let us prove the following assertion: for each $q$ as above we can find an isomorphism $\tau$ such that $\bar{\pi}_p \circ \tau = \bar{\pi}_q$ if and only if the affine sequence associated to $I$ is exact on the right side. First, let us assume that the affine sequence is exact on the right side. Let us consider $p^A$ and $q^A$ two $A$-jet representing $p$ and $q$ respectively. The $B$-points, $\phi(p^A)$ and $\phi(q^A)$ represent the same $B$-jet $\bar{p}$. If the affine sequence associated to $I$ is exact on the right side, then $\text{Aut}(B)$ is a quotient of $\text{Aut}(A)$. Hence, $\tau_1$ lifts to an automorphism $\tau_2$ of $A$. This automorphism induces the isomorphism $\tau$ when we replace $A$ for $A_p$ and $A_q$. Conversely, if the affine sequence is not exact on the right side, we can choose $q$ and $A$-points $p$ and $q$ such that $\bar{\phi}(p^A)$ and $\bar{\phi}(p^B)$ are related by an automorphism which can not be lifted to $A$. In such case, we can not find such isomorphism $\tau$.

Now, let us consider that the affine sequence is exact on the right side and let $\tau$ be as above. Then $\pi_p$ and $\tau \circ \pi_p$ are regular $A_p$-points that are projected onto the same $A_{\bar{p}}$-point. Then, $D = \pi_p - \tau \circ \pi_q$ is a derivation of $C^\infty(M)$ and it take values in $\bar{p}/p$. It defines a vertical vector $[D] \in TV^\phi(J^A M)$ and it follows that:

$$
P + [D] = q.
$$

If the affine sequence is exact then condition (4.) holds

If the affine sequence is exact, we can find $\tau$ and $\bar{\tau}: A_q \to A_p$ as above. Then, $\sigma = \tau \circ \bar{\tau}$ is an automorphism of $A_p$ which induces the identity on $A_{\bar{p}}$. Since the affine sequence is exact on the right side we have that $\sigma$ induces the identity map on $\bar{p}/p$. it follows that the restriction of $\tau$ to the space $\bar{p}/q$ is canonical and does not depend on $\tau$. This canonical identification $\tau: \bar{p}/q \to \bar{p}/p$ induces canonical isomorphisms:

$$
\begin{array}{ccc}
\text{Der}(A_{\bar{p}}, \bar{p}/q) & \xrightarrow{\sigma} & \text{Der}(C^\infty(M), \bar{p}/q) \\
\downarrow & & \downarrow \tau^* \\
\text{Der}(A_{\bar{p}}, \bar{p}/p) & \xrightarrow{\sigma} & \text{Der}(C^\infty(M), \bar{p}/p) \\
\end{array}
\xrightarrow{\phi} \begin{array}{ccc}
TV^\phi(J^A M) & \xrightarrow{\tau^*} & TV^\phi(J^A M) \\
\tau^* & & \tau^*
\end{array}
$$

Thus, condition (4.) is satisfied.

If the affine sequence is exact then the vector space $TV^\phi(J^A M)$ depends only on the base $B$-jet $\bar{p}$. Those spaces define a vector bundle $TV^\phi(J^B M) \to
$J^B M$ and the composition law:

$$J^A M \times_{J^B M} TV^{\phi^j}(J^B M) \to J^A M, \quad (p + [D]) \mapsto p + [D].$$

is an affine structure on the bundle $\phi^j$. \textbf{q.e.d.}

**Corollary 6** Let $A_l$ be of height $l$, and $l > k > 0$. The natural projection $J^{A_l} M \to J^{A_k} M$ is endowed with a canonical structure of affine bundle if and only if $3k + 1 \geq 2l$ and $\text{Aut}(A_l) \to \text{Aut}(A_k)$ is surjective.

**Corollary 7** The natural projection $J^l M \to J^r M$ for $l > r > 0$ is endowed with a canonical structure of affine bundle if and only if $3r + 1 \geq 2l$.

**Remark 2** Those results extend the well known affine structure of the spaces of jets of sections. First, they show that this structure arises not only for the projection by lower order one-by-one, but if follows an arithmetic formula which is also different from the expected one of duplicating orders. Second, this affine structure is inherent to the spaces $J^l M$ as spaces of ideals, it does not depend on their realization as spaces of sections of fibre bundles.

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