'Superluminal' tunnelling, quantum measurements and the speed of information transfer

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Abstract

We exploit the analogy between the transfer of a pulse across a scattering medium and Aharonov’s weak measurements to resolve the long standing paradox between the impossibility to exceed the speed of light and the seemingly ‘superluminal’ behaviour of a tunnelling particle in the barrier or a photon in a ‘fast-light’ medium. We demonstrate that ‘superluminality’ occurs when the value of the duration $\tau$ spent in the barrier is uncertain, whereas when $\tau$ is known accurately, no ‘superluminal’ behavior is observed. In all cases only subluminal durations contribute to the transmission which precludes faster-than-light information transfer, as observed in a recent experiment.

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Recent experiments [1] on transmitting information-containing features of an optical pulse across the 'fast-light' medium, in which the group velocity exceeds the vacuum speed of light $c$, have renewed the interest in the so-called 'superluminal' propagation phenomenon. It is well known that a wavepacket transmitted across the potential barrier, undersized waveguide or fast-light medium may arrive in a detector ahead of the one that propagates freely, as if it has crossed the scatter infinitely fast. This phenomenon, often referred to as 'apparent superluminality of quantum tunnelling' was first noticed more than seventy years ago [2] and often discussed since (for Reviews see [3,4], a recent selection of different views on the subject can be found in [5]). It seems to raise the question about the possibility of faster-than-light travel, e.g., in classically forbidden regions, inaccessible to a classical particle. It is, however, broadly understood that the paradox results from an incorrect identification of the transmitted peak with the incident one, since the incident pulse undergoes severe reshaping in the barrier region. Although it has been suggested [6], that a barrier may, in some manner, carve the transmitted pulse from the forward tail of the incident one, the question of how exactly this is achieved remains open. The precise mechanism of this reshaping, its relation to Aharonov’s 'weak' measurements [7–9] of the time delay and the implications for the information transfer, studied in [1], are the subjects of this Letter. Earlier work on the weak nature of the tunnelling times can be found in [10]-[12], and a more recent approach relating 'superluminality' to superoscillations is given in [13].

We start by revisiting [7,8] the analysis of a quantum system prepared at $t = 0$ in the initial state $|I>$ and then post-selected (observed) at $t = T$ in the final state $|F>$,

$$
|I> = \sum_{\nu} a_{\nu}|\nu> = |F> = \sum_{\nu} b_{\nu}|\nu> , \quad \hat{A}|\nu> = A_{\nu}|\nu> .
$$

(1)

If at some $t$, $0 < t < T$, the system is subjected to a von Neumann-type measurement [14] of an operator $\hat{A}$, the state of the pointer with position $\tau$ after post-selection is given by [7,8]

$$
<\tau|M >= \sum_{\nu} G(\tau - A_{\nu})\eta_{\nu} \quad \eta_{\nu} \equiv b_{\nu}^{*}a_{\nu}
$$

(2)
where $G(\tau)$ is the initial (e.g., Gaussian) state of the meter at $t = 0$ and $\sum_{\nu}$ must be replaced by an integral $\int d\nu$ if $\hat{A}$ has a continuous spectrum. The meter is then read, i.e., the pointer position is accurately determined. For an accurate 'strong' measurement, the width $\Delta \tau$ is small (compared to the separation between the eigenvalues or, if the spectrum is continuous, to the scale on which $\eta_\nu$ varies considerably) and the meter’s readings occur close to the eigenvalues $A_\nu$. For an inaccurate, or ‘weak’, measurement, $\Delta \tau$ is large, and the interference between overlapping Gaussians in Eq.(2) may, for a special choice of $|I>$ and $|F>$, produce anomalous readings in the regions where $\hat{A}$ has no eigenvalues [7,8]. Next we show that when using the coordinate of the tunnelled particle to estimate the time delay it has experienced in the barrier, $\tau$, one, in fact, performs a weak measurement of $\tau$. The anomalously small value of $\tau$ that is obtained, results, just as an Aharonov’s weak value, from the quantum uncertainty inherent to the procedure.

Consider a one-dimensional wave packet transmitted across a short-ranged potential barrier $V(x)$, contained inside the region $0 < x < b$. At $t = 0$ the particle is prepared as an incident wave packet with a mean momentum $k_0$, centred at $x = x_I < 0$

$$\Psi_0(x) \equiv <x|I> = \int_{-\infty}^{\infty} C(k-k_0) \exp(ikx) dk.$$  \hspace{1cm} (3)

where the factor $C(k-k_0)$ insures that only positive momenta contribute to the integral. At some large $t = T$, it is post-selected in the transmitted state ($\hbar = 1$)

$$\Psi_T(x) \equiv <x|F> = \int_{-\infty}^{\infty} T(k)C(k-k_0) \exp[ikx - iE(k)T] dk,$$  \hspace{1cm} (4)

where $T(k)$ is the barrier transmission amplitude. Using the convolution property of the Fourier integral, it is convenient to rewrite Eq.(4) as

$$<x|F> = \int \Psi_T(x-x')\xi(x')dx'$$  \hspace{1cm} (5)

where

$$\Psi_T(x) = \int_{-\infty}^{\infty} C(k-k_0) \exp[ikx - iE(k)T] dk,$$ \hspace{1cm} (6)
is the state that would evolve from the initial one under free propagation, and \( \xi(x') \) is the Fourier transform of \( T(k) \). We can continue the discussion in terms of the ‘tunnelling times’ by identifying (\( v_0 \) is the velocity corresponding to the wavevector \( k_0 \))

\[
\tau(x') \equiv -x'/v_0
\]  

with the delay experienced by the particle in the barrier. Changing the variables in Eq.(5) and separating the inessential phase associated with the free motion, we obtain

\[
\Psi_F = \exp[ik_0x - iE(k_0)T] \int G(\tau(x) - \tau)\eta(\tau)d\tau
\]

where \( G \) is the envelope of \( \Psi_T \),

\[
G(\tau) \equiv \exp[iE(k_0)T + ik_0v_0\tau]\Psi_T(-v_0\tau)
\]

and

\[
\eta(\tau) \equiv -(2\pi)^{-1}v_0 \int T(k) \exp[i(k_0 - k)v_0\tau]dk.
\]

Comparing Eq.(8) with Eq.(2) shows that the relation between the time delay \( \tau \) and the particle’s position \( x \) is that between the measured quantity, whose amplitude distribution is \( \eta(\tau) \) and the position of the pointer, whose initial state is determined by the envelope of the initial pulse. As in the original Aharonov’s approach, the particle is post-selected in its transmitted state. However, unlike in Ref. [7]-[9], no external pointer variable is employed and its role is played by the particle’s own position. Equivalently, registering the transmitted particle at a location \( x \) amounts to measuring the time delay \( \tau \) of a particle with the momentum \( k_0 \). Finding the particle roughly the the width of the barrier \( b \) ahead of the free one, as it happens in tunnelling [2], corresponds to a negative time delay of \( \approx -b/v_0 \). Importantly, the accuracy to which the time delayed is evaluated is limited (if the spreading of the wavepacket is neglected [15]) by the uncertainty \( \Delta x \) of the particle’s position in its initial state [16], \( \Delta \tau \approx \Delta x/v_0 \).

In the case of tunnelling, the momentum spread of the initial wavepacket must be at least
small enough for all its components to tunnel, rather than to pass over the barrier. Such wavepackets, broad in the coordinate space, correspond to inaccurate weak measurements which may produce anomalous 'superluminal' readings, even when the amplitude distribution $\eta(\tau)$ contains only non-negative time delays. Indeed, in the complex $k$-plane, $T(k)$ may only have poles on the positive imaginary axis and in the lower half-plane [18]. The poles of the first kind (I) correspond to the bound states supported by $V(x)$, while those of the second kind (II) correspond to scattering resonances. Closing the contour of integration in Eq.(10) as appropriate, we obtain

$$\eta(\tau) = 2\pi i \sum_{I} \text{Res}_n T \exp(ik_n v_0 \tau), \quad \tau < 0$$

(11)

$$= -2\pi i \sum_{II} \text{Res}_n T \exp(-ik_n v_0 \tau), \quad \tau > 0.$$  

(12)

where $\text{Res}_n T$ denotes the residue of $T(k)$ at the $n$-th pole. Thus, the poles (I), if present, produce negative time delays and are responsible, in the classical limit, for the speed up of a particle passing above a potential well. It is interesting to note that a potential well too shallow to support a bound state would not speed up a passing wavepacket. If, on the other hand, no bound states are present, then

$$\eta(\tau) \equiv 0 \text{ for } \tau < 0$$

(13)

and the 'spectrum' of the time delays in Eq.(8) is confined in the $0 \geq \tau < \infty$ semi-axis. Note that Eq.(13) demonstrates the causal nature of the scattering process, since the condition $\text{Im}k_n < 0$, used in its derivation, also ensures that $\text{Im}E(k_n) < 0$ for $\text{Re}k_n > 0$ and the resonance states containing outgoing waves, $\text{Re}k_n > 0$, are emptied, rather than filled up, as the time increases [18].

It is clear now that with a careful choice of $T(k)$, and, therefore, $\eta(\tau)$ the 'superluminal' pulse can be produced, in an explicitly causal manner, from the front tails of $\Psi_T(x - x')$, all delayed relative to free propagation. One such system is a particle tunnelling across a potential barrier. However, our approach only relies on the analyticity of the transmission amplitude $T(k)$ and can also be applied to evanescent propagation in waveguides [5] and
optical propagation through 'fast-light' media (see [3], [1]), which have additional advantage of dealing with wavepackets not subject to spreading in vacuum.

Information transfer is often associated with propagation of non-analytic features, such as cut-offs [1] and next we will show that Eq.(13) ensures that it cannot be transferred faster than light. For a simple example, consider the propagation of an electromagnetic pulse across two narrow semi-transparent mirrors, broadly similar to the setup studied in [19]. If the mirrors are modelled by δ-functions of magnitude Ω located at \(x = 0\) and \(x = b\), respectively, \(T(k)\) is given by the multiple scattering expansion

\[
T(k) = \sum_{m=0}^{\infty} T^{(m)}(k) \equiv (1 + R(k)) \sum_{m=0}^{\infty} R(k)^{2m} \exp(2imkb)
\]

where

\[
R(k) = -i\Omega/(2k + i\Omega)
\]

is the reflection amplitude for a single δ-function placed at the origin \(x = 0\). Accordingly, the distribution \(\eta(\tau)\) is decomposed into subamplitudes \(\eta_m(\tau)\), \(\eta_m(\tau) \equiv 0\) for \(\tau < 2mb/c\), each peaked near \(\tau_m = 2mb/c\) (Fig.1a). For a large \(\Omega, \Omega b \gg 1\), the widths can be neglected and the incident pulse is split into a number of discrete path modes corresponding to \(2m, m = 0, 1, 2...\) additional reflections experienced by the ray between \(x = 0\) and \(x = b\) [19], and Eq.(4) reduces to Eq.(2) for a variable with a discrete spectrum \(\{\tau_m\}\),

\[
\Psi_F(x) \approx \sum_m G(\tau(x) - \tau_m)T^{(m)}(k_0).
\]

A numerical evaluation of \(\Psi_F(x)\), for \(\Omega b = 100\), corresponding to a weak Gaussian measurement,

\[
G(x) = \exp(-x^2/\Delta x^2), \quad \Delta x > b,
\]

shows (see Fig1.b) how a set of nearly Gaussian shapes, each delayed by \(\tau_m > 0\), interfere to produce a negative time delay \(\approx -b/c\). Note that this is the best speed up which can be achieved with the model Eq.(14) [20].

It is now straightforward to show that this speed up effect cannot be used to send information
faster than light. Aharonov and co-workers have already demonstrated [8] that a weak von Neumann pointer cannot be used for this purpose. Rather, they argued, the meter acts as a filter, extracting, in a non-trivial manner, the signal, otherwise hidden by a noise. The same argument applies to 'superluminal' propagation. If the incident wave packet is chosen to be the rear half of a Gaussian with a sharp front, $G_-(x) = [1 - \theta(x)] \exp(-x^2/\Delta x^2)$, Eqs.(8) and (13) show that the transmitted field will vanish outside the causal boundary $x_B = cT + x_I$, or for $\tau < 0$, as shown in Fig.1c. Note that in Fig.1c the narrow spikes near $\tau = \tau_m$ result from large oscillations of $\eta_m(\tau)$ clearly visible in Fig1.a. Equally unsuccessful would be an attempt at superluminal transfer of information encoded in a sharp cut-off at the rear of the incident pulse, $G_+(x) = \theta(x) \exp(-x^2/\Delta x^2)$. Figure 1d shows that by inspecting the field at $x > cT$, an observer cannot decide whether the whole Gaussian, or only its front half was incident on the barrier, and must await the arrival of the information-carrying part of the signal.

Recent experiments, in which a detector was to distinguish between a cut ($G_+$) and an uncut ($G$) signals, [1] have shown that the information detection time for pulses propagating through the fast-light medium is somewhat longer than that in vacuum, even though the group velocity in the medium is in the highly superluminal regime. As in Fig.1d, the absence of superluminal information transfer has a simple explanation. Since the medium (in this case, the potassium vapor) does not bind photons, the transmission amplitude cannot have poles in the upper half of the $k$-plane and $\xi(x')$ in Eq.(5) must vanish for $x' > 0$. The most advanced term in Eq.(5), $\Psi_T(x)$ contains a cutoff at $x = x_B \equiv cT + x_I$, and the 'superluminal' ($x > x_B$) part of the transmitted field in Fig.1d builds up from the front tails of $\Psi_T(x - x')$, $x' \leq 0$, unaffected by the cut-off, and for this reason is the same as the advanced uncut field (cf. Fig.2a of Ref. [1]). For $x < x_B$ the field has a complicated shape and the actual delay in detecting the back face of the pulse is likely to be caused by its deformation while propagating through the medium. To advance the cut-off beyond $x_B$, and achieve a truly superluminal information transfer, one requires a one-dimensional system capable of supporting both bound and scattering states of a photon, in the way a
finite-depth potential well supports bound states of an electron. At present, we are not aware of the existence of such systems.

In summary, the notion of 'superluminality' in wavepacket propagation is based on relating the final position $x$ of the transmitted particle to the time $\tau$ it is supposed to have spent in the scatterer. Quantally, $\tau$ and $x$ are related as the measured quantity and the pointer position in a measurement, whose accuracy is determined by the coordinate width of the pulse. In cases where apparent 'superluminality' is observed, e.g., in tunnelling or optical propagation through 'fast-light' media, the measurement is inevitably weak. Even if no negative time delays contribute to the transmission ($\eta(\tau) \equiv 0$ for $\tau < 0$), it may, therefore, produce an anomalous reading by constructing a 'superluminal' pulse from the front tails of the components of the transmitted pulse, all delayed relative to free propagation. Just as an improvement in the accuracy destroys anomalous weak values [9], a choice of a narrow incident pulse destroys superluminal propagation by making higher incident momenta pass over the barrier, or spill outside the anomalous dispersion region. As a result, a 'strong' measurement registers only the 'subluminal' time delays, as illustrated in Fig.1e for the simple model described above. The contradiction between the impossibility of faster-than-light travel and observing an apparently 'superluminal' pulse is, therefore, resolved in a typically quantum mechanical fashion: when 'superluminality' is present, one does not know the delay, and cannot claim that the duration spent in the scatterer is shorter than $b/c$. Conversely, when the delay is known, no 'superluminal' transmission is observed. The absence of negative virtual delays in optical propagation does, however, limit the speed of information transfer to $c$ and below, as a non-analytic feature (e.g., a cut-off) of the initial pulse may only travel as far as the most advanced component in Eq.(5), i.e., at most by $cT$. Beyond this point the field will either vanish if the front part of the pulse was discarded, or remain identical to the uncut field if its the rear tail has been removed as the recent experiments by Stenner et al [1] show.

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REFERENCES

[1] M.D. Stenner, D.J.Gauthier and M.A.Neifeld, Nature, 425, 695 (2003).

[2] L.A. MacColl, Phys.Rev. 40, 621 (1932)

[3] R.Y. Chiao and A.M. Steinberg, Prog. Opt. 37, 345 (1997)

[4] Time in Quantum Mechanics, edited by J.G.Muga, R.Sala-Mayato and I.L. Egusquiza
   (Springer, 2002)

[5] Special Issue, Ann.Phys.(Leipzig),7, 662 (1998)

[6] J.M. Deutch and F.E.Low, Ann.Phys.228, 184 (1993)

[7] Y. Aharonov, D.Z. Albert and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988)

[8] Y. Aharonov and L. Vaidman, Phys.Rev.A 41, 11 (1990)

[9] Y. Aharonov and L. Vaidman, in [4], pp. 369-412

[10] A.M. Steinberg, Phys.Rev.Lett., 74,2405 (1995)

[11] A.M. Steinberg, Phys.Rev.A, 52,32 (1995)

[12] G. Iannaccone, in Time and Its Implications, edited by D. Mugnai, A. Ranfagni and
    L.S. Schulman (World Scientific, 1997), pp. 292-309

[13] Y. Aharonov, N. Erez and B. Reznik, Phys.Rev.A 65, 052124 (2002); J.Mod.Opt.50,
    pp.1139-1150 (2003).

[14] J. von Neumann, Mathematical Foundations of Quantum Mechanics, (Princeton Univer-
    sity, 1955)

[15] For a massive particle, the wavepacket would spread and the accuracy of the measure-
    ment would deteriorate further, as the time progresses. We, however, assume that
    the transmitted particle is detected before significant spreading has occurred. No such
restriction is necessary for a photon, whose wavepacket retains its shape in free propagation.

[16] Consider, for example, free motion, in which case \( \eta(\tau) \approx v_0^{-1}\delta(\tau) \) and different values of \( \tau(x) \) arise solely from the spread in the particle’s initial position.

[17] A more stringent condition, \( \partial_k \text{Re} \{\ln T(k_0)\} / \Delta x \ll 1 \) is needed if one also wants to avoid distortion.

[18] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, Oxford, 1977).

[19] Wang Yun-ping and Zhang Dian-lin, Phys.Rev.A, 52, 2597 (1995)

[20] For \( \Omega b \gg 1 \) and \( k \ll \Omega \), the transmission probability is small, \( |T(k)| \ll 1 \), except near equally spaced resonances at \( k_j = \pi j/b, j = 1, 2, \ldots \). Between the resonances we have \( (\phi' \equiv \partial_k \phi) \phi'(k) \approx -b, \phi''(k) \ll 1 \). Thus for the transmitted pulse with \( k_0 \approx \pi(j + 1/2) \) we obtain \( \langle x|F \rangle \approx \exp[ik_0x - iE(k_0)T(k_0)\int [iqx + iq\phi'(k_0)] \approx T(k_0)G_T(x+b) \) so that \( \tau = -b/c \) is the best speed up achieved with this model. In addition, we must ensure that the momentum width of the incident pulse is small compared to the spacings between the resonances, \( \Delta k \ll \pi \). If this is not the case, the wings of the momentum distribution are caught in the neighbouring resonances, and the barrier loses its ‘filtering’ property, transmitting, instead of a single pulse, an extended wave, sustained by the decay of the resonances. The minimal coordinate width of a Gaussian pulse is, therefore, limited by the condition \( 4\pi \Delta x/b \gg 1 \).
FIGURES

FIG. 1.
Figure captions

Fig. 1:

a) Real (solid) and imaginary (dashed) part of the time delay amplitude distribution, $\eta(\tau)$, for $\Omega b = 100$;
b) Transmitted field $\psi_F(x)$ (thick solid) and its components $\psi_m$, corresponding to different terms in Eq.(14), for $\Omega b = 100$, and a Gaussian incident pulse $k_0b = 1.4\pi$ and $\Delta x/b = 2.85$;
c) Same as b) but for an incident Gaussian pulse truncated at front.
d) Same as b) but for an incident Gaussian pulse truncated at rear.
e) Same as b) but for $k_0b = 7\pi$ and $\Delta x = 0.57$. 