Abstract

We present a class of magnetically charged brane solutions for the theory of Einstein-Maxwell gravity with a negative cosmological constant in $d \geq 4$ spacetime dimensions.

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1 Introduction

There has been lots of interest in the study of gravitational configurations in anti-de Sitter spaces. This, to a large extent, has been motivated by the conjectured equivalence between string theory on anti-de Sitter (AdS) spaces (times some compact manifold) and certain superconformal gauge theories living on the boundary of AdS [1]. Classical AdS gravity solutions can furnish important information on the dual gauge theory in the large $N$ limit, $N$ being the rank of the gauge group. Moreover, the AdS/CFT equivalence makes possible the microscopic analysis of the Bekenstein-Hawking entropy of asymptotically anti-de Sitter gravitational solutions. For example, the central charge of the AdS$_3$ asymptotic symmetry algebra [2] has been used to count the microstates giving rise to the BTZ black hole entropy [3].

Given the AdS/CFT correspondence, it becomes important to construct bulk gravitational solutions to the theory of Einstein gravity with a negative cosmological constant. Although many brane solutions in ungauged supergravity theories or gravity theories without a cosmological constant are known, the same can not be said concerning the corresponding objects in the AdS cases. Black hole solutions to Einstein’s equations with a negative cosmological constant were constructed a long time ago in [4]. Also, black holes with various topologies of the horizon (planar, toroidal or an arbitrary genus Riemann surface) were discussed in [5, 6, 7, 8, 9, 10, 11]. Generalizations to $d$ dimensional anti-de Sitter black holes with ‘horizons’ given by $(d - 2)$-dimensional compact Einstein spaces were also considered in [13].

In the context of gauged supergravity models, black holes and brane solutions were discussed in many places (see for example [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]). It must be noted that Einstein-Maxwell theories with a negative cosmological constant with spacetime dimensions less than six can be obtained as a consistent truncation of certain gauged supergravity models. The main purpose of this work is the study of magnetic brane solutions with various topologies in anti-de Sitter $d$-dimensional gravity coupled to a vector potential. We find brane solutions with a magnetic charge given in terms of the inverse of the cosmological constant. We also obtain magnetic solutions given in terms
of a product of a \((d-2)\)-dimensional solution for pure Einstein gravity with a negative cosmological constant and a two-dimensional Einstein space.

2 Branes in \(d\)-dimensional AdS E-M Gravity

We start our discussion by considering the anti-de Sitter Einstein-Maxwell theories in \(d\) \((\geq 4)\) dimensions. The action for our theory is given by

\[
S = -\frac{1}{16\pi G_d} \int_M d^dx \sqrt{-g} \left( R + (d-1)(d-2)l^2 - \frac{(d-2)}{2(d-3)} F^2 \right). \tag{2.1}
\]

Here \(G_d\) is Newton’s constant in \(d\) dimensions, \(R\) is the scalar curvature and \(F^2 = F_{\mu\nu}F^{\mu\nu}\), with \(F_{\mu\nu}\) being the field strength of an abelian gauge field \(A_\mu\). Here we have expressed the cosmological constant by \(\Lambda = -(d-1)(d-2)l^2\).

The equations of motion for the metric and the gauge field resulting from the action (2.1) are given by

\[
R_{\mu\nu} = \frac{(d-2)}{(d-3)} F_{\mu\rho} F^{\rho\nu} - \frac{1}{2(d-3)} g_{\mu\nu} F^2 - (d-1)l^2 g_{\mu\nu}, \tag{2.2}
\]

\[
\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0. \tag{2.3}
\]

We consider general \((d-4)\)-magnetic brane solutions whose metric can be written in the following form

\[
ds^2 = -e^{2V} (-dt^2 + \sum dz_{\alpha_i}^2) + e^{2U} dr^2 + N(r)^2 d\Omega^2, \tag{2.4}
\]

where the functions \(U\) and \(V\) are assumed to depend only on the transverse coordinate \(r\) and we consider either \(N(r) = r\) or \(N(r) = \mathcal{N} = \text{constant}\). The index \(\alpha_i\) labels the worldvolume coordinates with \(i\) ranging from 1 to \((d-4)\). Moreover, \(d\Omega^2\) denotes the metric of a two-manifold of a constant Gaussian curvature \(k\). In the following, and without loss of generality, we shall consider the cases \(k = 0, \pm 1\). Clearly our two-dimensional space is a quotient space of the universal coverings \(S^2 \ (k = 1)\), \(H^2 \ (k = -1)\) or \(E^2 \ (k = 0)\). Explicitly, we choose

\[
d\Omega^2 = d\theta^2 + f^2 d\phi^2, \tag{2.5}
\]
where

\[
     f(\theta) = \begin{cases} 
         \sin \theta, & k = 1, \\
         1, & k = 0, \\
         \sinh \theta, & k = -1. 
     \end{cases}
\]  

(2.6)

2.1 Solutions with Spherical, Flat and Hyperbolic Transverse Space

Using the general form of the Ricci curvature given in the appendix with \( N(r) = r \), we obtain

\[
    R_{tt} = -R_{zz} = e^{2V - 2U} \left[ (d - 3)V'' + (d - 3)V'V' + \frac{2}{r} V' \right],
\]

\[
    R_{rr} = -(d - 3)(V'' + V'^2 - U'V') + \frac{2}{r} U',
\]

\[
    R_{\theta\theta} = \frac{1}{F^2} R_{\phi\phi} = e^{-2U} \left[ -(d - 3)V' + rU' + ke^{2U} - 1 \right].
\]

(2.7)

Our brane solutions carry magnetic charge and for the gauge field strength we take

\[
    F_{\theta\phi} = \pm k q f,
\]

(2.8)

where \( q \) is a constant related to magnetic charge. Then using (2.7) and (2.8) in (2.2), we obtain the following differential equations

\[
    (d - 3)V'^2 + V'' - U'V' + \frac{2V'}{r} = e^{2U} \left[ \frac{k^2 q^2}{(d - 3)r^4 + (d - 1)l^2} \right],
\]

(2.9)

\[
    -(d - 3)(V'' + V'^2 - U'V') + \frac{2U'}{r} = -e^{2U} \left[ \frac{k^2 q^2}{(d - 3)r^4 + (d - 1)l^2} \right],
\]

(2.10)

\[
    -(d - 3)V' + rU' + ke^{2U} - 1 = e^{2U} \left[ \frac{k^2 q^2}{r^2} - r^2(d - 1)l^2 \right].
\]

(2.11)

To proceed in solving the resulting equations of motion we first write

\[
    V' = e^U W.
\]

(2.12)

Then by adding up the equations (2.9) and (2.10), we obtain the following differential equation

\[
    \frac{2}{r} W + (4 - d)W' + \frac{2}{r} U' e^{-U} = 0.
\]

(2.13)
Equation (2.11) provides an expression for $W$ in terms of $U$ which reads
\[
W = \frac{e^U}{(d-3)} \left[ -\frac{k^2 q^2}{r^3} + r(d-1)l^2 + \frac{1}{r}(k - e^{-2U}) + U'e^{-2U} \right]. \tag{2.14}
\]
Upon substituting (2.14) in (2.13), we finally obtain a differential equation for $U$,
\[
\frac{k^2 q^2 (10 - 3d)}{r^4} + (6 - d)(d-1)l^2 + (4-d)U' \left[ -\frac{k^2 q^2}{r^3} + r(d-1)l^2 + \frac{k}{r} \right] + e^{-2U} \left[ (4-d)(U'' - U'^2) + \frac{dU'}{r} + \frac{(2-d)}{r^2} \right] + \frac{k(d-2)}{r^2} = 0. \tag{2.15}
\]
The differential equation (2.13) admits a simple solution when the magnetic charge is expressed in terms of the cosmological constant by $q = \frac{1}{(d-2)l}$. In this case our solution is given by
\[
e^{-U} = lr + \frac{kq}{r} = lr + \frac{k}{(d-2)lr}. \tag{2.16}
\]
The function $W$ can be easily determined from equation (2.14) and we get
\[
W = \left[l - \frac{k}{(d-3)(d-2)lr^2} \right]. \tag{2.17}
\]
Using (2.12), (2.16) and (2.17) we can finally solve for $V$ and get
\[
e^V = lr \left[ 1 + \frac{k}{(d-2)l^2r^2} \right]^{\frac{1}{2}}(\frac{d-4}{d-3}). \tag{2.18}
\]
Therefore, $(d-4)$-magnetic branes with spherical transverse space are described by the metric and gauge field
\[
ds^2 = (lr)^2 \left[ 1 + \frac{1}{(d-2)l^2r^2} \right]^{\frac{d-2}{d-3}} \left( -dt^2 + \sum dz_{\alpha i}^2 \right) + \left[lr + \frac{1}{(d-2)lr} \right]^{-2} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]
\[F_{\theta\phi} = \pm \frac{\sin \theta}{(d-2)l}. \tag{2.19}\]
Note that for the $d = 5$ spherical case we obtain the magnetic string which has been considered in \[23\]. Also for $d = 4$ (in which case there are no $z_{\alpha i}$ coordinates), we obtain the “cosmic monopole” solution which was first discovered by Romans \[14\]. The
cases of four and five dimensions can be embedded in $N = 2$ gauged supergravity and the corresponding magnetic solutions were shown to preserve some supersymmetry \[14, 16, 23\]. For magnetic branes with hyperbolic transverse space (i.e., $f = \sinh \theta$) we have

$$ds^2 = (lr)^2 \left[ 1 - \frac{1}{(d-2)l^2r^2} \right]^{\frac{d-2}{d-3}} \left( -dt^2 + \sum dz_{\alpha_i}^2 \right) + \left[ lr - \frac{1}{(d-2)l} \right]^{-2} dr^2 + r^2 (d\theta^2 + \sinh^2 \theta d\phi^2).$$

(2.20)

These solutions are the generalizations to $d$ dimensions of the hyperbolic five-dimensional string solution constructed in \[24\]. We note that, unlike the spherical magnetic brane which contains a naked singularity, the hyperbolic black magnetic brane has an event horizon at $r = 1/(l\sqrt{d-2})$. This is analogous to the AdS$_4$ and AdS$_5$ cases discussed in \[15, 16, 23, 24\].

The solution with flat transverse space ($f = 1$), which is locally AdS$_d$, is a limiting case of a family of black branes, whose metric is given by

$$ds^2 = -e^{2V} dt^2 + e^{-2V} dr^2 + l^2 r^2 \sum dz_{\alpha_i}^2 + r^2 (d\theta^2 + d\phi^2),$$

(2.21)

where

$$e^{2V} = -\frac{m}{r^{d-3}} + l^2 r^2$$

and $m$ is a constant related to the mass of the magnetic brane. If $lz_{\alpha_i}$ is considered as a coordinate of the transverse space, then the solutions (2.21) can also be interpreted as black hole solutions which were actually constructed in \[13\].

2.2 Constant Warp Factor

Here we consider the case where the warp factor $N(r) = \mathcal{N} = \text{constant}$ and we take $F_{\theta\phi} = \pm qf$. Inspecting the equations of motion (2.2) in this case, we find that our solution factors into a product of two spaces $M_{d-2}$ and $M_2$ with metrics which we denote by $g_{mn}$ and $g_{\alpha\beta}$ respectively. The curvature tensors of $M_{d-2}$ and $M_2$ are respectively given

\[1\]It must be noted that the hyperbolic magnetic brane metric can be obtained from (2.19) by making the following substitutions:

$t \rightarrow it, \quad r \rightarrow ir, \quad z \rightarrow iz, \quad \theta \rightarrow i\theta, \quad \phi \rightarrow \phi$
by

\[ R_{mn} = -g_{mn} \left[ \frac{q^2}{(d-3)N^4} + (d-1)l^2 \right] = -c_1 g_{mn}, \quad (2.22) \]

\[ R_{\alpha\beta} = g_{\alpha\beta} \left[ \frac{q^2}{N^4} - (d-1)l^2 \right]. \quad (2.23) \]

This implies that \( M_{d-2} \) is given by a \((d-2)\)-dimensional Einstein space with a negative cosmological constant. For various topologies of \( M_2 \), \((k = 0, \pm 1)\), the magnetic charge is given by

\[ q^2 = (d-1)l^2N^4 + kN^2, \quad (2.24) \]

and the Ricci curvature of \( M_{d-2} \) is thus given by

\[ R_{mn} = -g_{mn} \left[ \frac{(d-1)(d-2)}{(d-3)}l^2 + \frac{k}{(d-3)N^2} \right]. \quad (2.25) \]

If we restrict ourselves to the metric ansatz (2.4) with constant warp factor, then the curvature tensor of \( M_{d-2} \) is given by

\[ R_{tt} = -R_{z\alpha}z_{\alpha} = e^{2V-2U} \left[ (d-3)V'' - U'V' + V'' \right], \]

\[ R_{rr} = - (d-3) \left( V'^2 + U'V' - V'' \right). \quad (2.26) \]

The equations of motion in this case are

\[ (d-3)V'^2 - U'V' + V'' = c_1 e^{2U}, \]

\[ (d-3) \left( V'^2 + U'V' - V'' \right) = c_1 e^{2U}. \quad (2.27) \]

As an example, we take \( q = \frac{1}{(d-2)l} \) and \( f = \sinh \theta \). This implies that

\[ N^2 = \frac{1}{(d-2)l^2}. \quad (2.28) \]

For this value of \( N \), one gets for the curvature of \( M_{d-2} \)

\[ R_{mn} = -g_{mn} \frac{(d-2)^2l^2}{(d-3)}, \quad (2.29) \]

and the equations of motion (2.27) can be seen to be satisfied for the choice

\[ V' = l \left( \frac{d-2}{d-3} \right) e^U. \quad (2.30) \]
The solution thus obtained describes the near horizon geometry of our hyperbolic magnetic brane solution which is given by $\text{AdS}_{(d-2)} \times H^2$.

As examples of $M_{d-2}$, which are solutions of Einstein’s equations with a negative cosmological constant in $(d-2)$ dimensions, we can take the solutions presented in [13]. For a $d'$-dimensional space, those spacetimes are given by

$$
\begin{align*}
\frac{ds^2}{(d' - 3)} &= \left(k' - \frac{m}{r^{d'-3}} + \lambda^2 r^2 \right) dt^2 + \left(k' - \frac{m}{r^{d'-3}} + \lambda^2 r^2 \right) dr^2 + r^2 h_{ij}(x) dx^i dx^j .
\end{align*}
$$

The metric function $h_{ij}$ is a function of the coordinates $x^i$ $(i = 1, ..., (d' - 2))$ and is referred to as the horizon metric. In [13], it was demonstrated that the above spacetimes are Einstein spaces with a negative cosmological constant, namely

$$
R_{\mu\nu} = -(d' - 1) \lambda^2 g_{\mu\nu} ,
$$

provided that the horizon metric is an Einstein space

$$
R_{ij}(h) = (d' - 3) k' h_{ij} , \quad k' = 0, \pm 1 .
$$

3 Summary

To sum up, we presented new magnetic $(d-4)$-brane solutions for the theory of $d$-dimensional Einstein-Maxwell theory with a negative cosmological constant. We discussed the cases of magnetically charged brane solutions for spherical, hyperbolic and flat transverse space. We found that the magnetic charge satisfies a “quantization relation” in terms of the inverse of the square of the cosmological constant. The solutions found are solitonic objects in the sense that the flat-space limit $l \to 0$ does not exist. Our solutions can be summarized by

$$
\begin{align*}
\frac{ds^2}{(d - 2)l^2 r^2} &= \left(1 + \frac{k^2}{(d - 2)l^2 r^2} \right)^{\frac{d - 2}{2}} \left(- dt^2 + \sum dz_{\alpha i}^2 \right) + \frac{1}{(l r)^2} \left[1 + \frac{k^2}{(d - 2)l^2 r^2} \right] ^2 dr^2 \\
&+ r^2 \left(d\theta^2 + f^2 d\phi^2 \right) .
\end{align*}
$$

with the gauge field strength given by $F_\theta \phi = \pm \frac{k f}{(d - 2)l}$. We note that it would be interesting to study more general solutions for which (3.1) are special limiting cases. We also
described the cases where we have a constant warp factor and there we found that our solution factors into a product of two spaces, one of which is given by a \((d - 2)\)-dimensional solution of the theory of Einstein gravity with a negative cosmological constant. The other factor of the product space is a two-dimensional Einstein manifold.
4 Appendix

Our conventions are as follows. General curved indices are labelled by \( \mu, \nu, \ldots \), time and transverse coordinates are labelled by \((t, r, \theta, \phi)\) with the corresponding flat indices labelled by \((0, 1, 2, 3)\). Worldvolume coordinates are labelled by \(z_\alpha, (with i = 1, \cdots, d-4)\) with corresponding flat indices given by \( \alpha_i \).

Consider the following metric

\[
ds^2 = e^{2V(r)}(-dt^2 + \sum dz_\alpha^2) + e^{2U(r)}dr^2 + N(r)^2(d\theta^2 + f^2d\phi^2). \tag{4.1}
\]

The dual basis of one-forms is given by

\[
\vartheta^0 = e^V dt, \quad \vartheta^{\alpha_i} = e^V dz^{\alpha_i}, \quad \vartheta^1 = e^U dr, \quad \vartheta^2 = N d\theta, \quad \vartheta^3 = N f d\phi. \tag{4.2}
\]

Exterior differentiation of the above forms gives

\[
d\vartheta^0 = V'e^{-U} \vartheta^1 \wedge \vartheta^0, \\
d\vartheta^{\alpha_i} = V'e^{-U} \vartheta^1 \wedge \vartheta^{\alpha_i}, \\
d\vartheta^1 = 0, \\
d\vartheta^2 = \frac{N'}{N} e^{-U} \vartheta^1 \wedge \vartheta^2, \\
d\vartheta^3 = \frac{N'}{N} e^{-U} \vartheta^1 \wedge \vartheta^3 + \frac{\partial_\theta f}{N f} \vartheta^2 \wedge \vartheta^3. \tag{4.3}
\]

Here the prime sign denotes differentiation with respect to \( r \). Using

\[
d\vartheta^a + w^a_b \wedge \vartheta^b = 0, \tag{4.4}
\]

we obtain for the connection forms the following

\[
w_1^0 = V'e^{-U} \vartheta^0, \quad w^{\alpha_i}_1 = V'e^{-U} \vartheta^{\alpha_i}, \quad w^2_1 = \frac{N'}{N} e^{-U} \vartheta^2, \\
w^3_1 = \frac{N'}{N} e^{-U} \vartheta^3, \quad w^2_2 = \frac{1}{N} \frac{\partial_\theta f}{f} \vartheta^3. \tag{4.5}
\]

Upon using

\[
dw^a_b + w^a_e \wedge w^e_b = \frac{1}{2} R^a_{bce} \vartheta^e \wedge \vartheta^f, 
\]
we obtain the components of the Ricci tensor $R_{ab} = g^{ef} R_{eafb}$

\[
R_{00} = -R_{\alpha\alpha} = e^{-2U} \left[ (d-3)V'^2 - U'V' + V'' + \frac{2V'N'}{N} \right],
\]
\[
R_{11} = -e^{-2U} \left[ (d-3) \left( V'^2 - U'V' + V'' \right) - \frac{2}{N} (U'N' - N'') \right],
\]
\[
R_{22} = R_{33} = \frac{1}{N} e^{-2U} \left[ -(d-3)V'N' + (U'N' - N'') + \frac{1}{N} (ke^{2U} - N'^2) \right].
\] (4.6)

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