On Hyperholomorphic Bergman Type Spaces in Domains of $\mathbb{C}^2$

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Abstract
Quaternionic analysis is a branch of classical analysis referring to different generalizations of the Cauchy-Riemann equations to the quaternion skew field $\mathbb{H}$ context. In this work we deal with $\mathbb{H}$—valued $(\theta, u)$—hyperholomorphic functions related to elements of the kernel of the Helmholtz operator with a parameter $u \in \mathbb{H}$, just in the same way as the usual quaternionic analysis is related to the set of the harmonic functions. Given a domain $\Omega \subset \mathbb{H} \cong \mathbb{C}^2$, our main goal is to discuss the Bergman spaces theory for this class of functions as elements of the kernel of $\theta_u \mathcal{D}[f] = \theta \mathcal{D}[f] + uf$ with $u \in \mathbb{H}$ defined in $C^1(\Omega, \mathbb{H})$, where

$$\theta \mathcal{D} := \frac{\partial}{\partial \bar{z}_1} + ie^{i\theta} \frac{\partial}{\partial \bar{z}_2} j = \frac{\partial}{\partial \bar{z}_1} + ie^{i\theta} j \frac{\partial}{\partial \bar{z}_2}, \quad \theta \in [0, 2\pi).$$

Using as a guiding fact that $(\theta, u)$—hyperholomorphic functions includes, as a proper subset, all complex valued holomorphic functions of two complex variables we obtain some assertions for the theory of Bergman spaces and Bergman operators in domains of $\mathbb{C}^2$, in particular, existence of a reproducing kernel, its projection and their covariant and invariant properties of certain objects.

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1 Introduction

The history of Bergman spaces goes back to the book [2] in the early fifties by S. Bergman, where the first systematic treatment of the subject for holomorphic functions of one complex variable was given, and since then there have been a lot of papers on the subject in different function classes. Some standard works here are [3, 8, 13] and the references therein, which contain a broad summary and historical notes of the subject, that frees us from referring to missing details.

Nowadays, quaternionic analysis is regarded as a broadly accepted branch of classical analysis offering a successful generalization of complex holomorphic function theory, the most renowned examples are Sudbery’s paper [21] and several already published books see e.g., [11]. It relies heavily on results on functions defined on domains in $\mathbb{R}^4$ with values in the skew field of real quaternions $\mathbb{H}$ associated to a generalized Cauchy-Riemann operator by using a general orthonormal basis in $\mathbb{R}^4$ (a so-called structural set $\psi$ of $\mathbb{H}$, see [16]). This theory is centered around the concept of $\psi-$hyperholomorphic functions, see [15, 17, 18].

There has been a great deal of work done over the recent years on weighted Bergman spaces for $\psi-$hyperholomorphic functions in $\mathbb{R}^4$, see [10, 19]. For a natural development of Bergman kernel function in Clifford analysis we refer the reader to [4, 20].

Despite the fact that theory of $\psi-$hyperholomorphic functions has been generalized and studied extensively to that of $(\theta; u)-$hyperholomorphic functions with a quaternionic parameter $u$ (see [9, 14] and the references given there), to the best of the authors knowledge, a Bergman theory for such functions classes is still open and this is precisely the main goal of the present work.

The outline of the paper reads as follows. After this short introduction, we briefly review in Sect. 2 some basic notation and definitions related to quaternionic analysis based on properties of $\psi-$hyperholomorphic functions, where $\psi$ is the structural set. In Sect. 3 we review the basic definitions and results of $(\theta; u)-$hyperholomorphic theory and it is also introduced the corresponding Bergman spaces. Section 4 discusses the case of $(\theta; u)-$hyperholomorphic Bergman type spaces in domains of $\mathbb{C}^2$. Section 5 gives some concluding remarks.

2 Preliminaries

Consider the skew field of real quaternions $\mathbb{H}$ with its basic elements $1, i, j, k$. Thus any element $x$ from $\mathbb{H}$ is of the form $x = x_0 + x_1i + x_2j + x_3k, x_s \in \mathbb{R}, s = 0, 1, 2, 3$. The basic elements define arithmetic rules in $\mathbb{H}$: by definition $i^2 = j^2 = k^2 = -1,$
two complex variables, we will identify $H$ of stating $H$ with $C^2$ (or $R^4$) by mean of the mapping

$$x_0 + x_1i + x_2j + x_3k \rightarrow (x_0 + i x_1) + (x_2 + i x_3)j \rightarrow (x_0, x_1, x_2, x_3). \quad (1)$$

From now on, we will use this fact in essential way.

The set of elements of the form $q = z_1 + z_2j, z_1, z_2 \in C$, endowed both with a component-wise addition and with the associative multiplication is then another way of stating $H$. In particular, the two quaternions $q = z_1 + z_2j$ and $ξ = ξ_1 + ξ_2j$ are multiplied according the rule: $q \ ξ = (z_1 ξ_1 - z_2 ξ_2) + (z_1 ξ_2 + z_2 ξ_1)j$. The quaternion conjugation gives: $z_1 + z_2j := \bar{z}_1 - z_2j$ and $q \ \bar{q} = \bar{q} = |z_1|^2 + |z_2|^2$.

Quaternionic analysis is regarded as a broadly accepted branch of classical analysis offering a successful generalization of complex holomorphic function theory. It relies heavily on results on functions defined on domains in $R^4$ with values in $H$ associated to a generalized Cauchy-Riemann operator by using a general orthonormal basis in $R^4$ (a so-called structural set $ψ = (1, ψ_1, ψ_2, ψ_3)$ of $H^4$, see [16]).

This theory is centered around the concept of $ψ$–hyperholomorphic functions, i.e., the collection of all null solutions of the generalized Cauchy-Riemann operator

$$ψD = \frac{∂}{∂x_0} + ψ_1 \frac{∂}{∂x_1} + ψ_2 \frac{∂}{∂x_2} + ψ_3 \frac{∂}{∂x_3},$$

see [15, 17, 18] for more details.

The $ψ$–hyperholomorphic function theory by itself is not much of a novelty since it can be reduced by an orthogonal transformation to the standard case. But despite this, the picture changed completely in the study of some important operators which involve a pair of different orthonormal basis, as was considered for instance in [1]. On the other hand, the possibility to study simultaneously several conventional known theories, which can be embedded usefully each of them into a corresponding version of $ψ$–hyperholomorphic function theory, again can not be reduced to the standard context and reveal indeed the importance of the $ψ$–hyperholomorphic function theory. The best reference here is [17].

Functions $f$ defined in a bounded domain $Ω \subset R^4 \cong C^2$ with value in $H$ are considered. They may be written as: $f = f_0 + f_1i + f_2j + f_3k$, where $f_s, s = 0, 1, 2, 3$, are $R$-valued functions in $Ω$. Properties as continuity, differentiability, integrability and so on, which as ascribed to $f$ have to be posed by all components $f_s$. We will follow standard notation, for example $C^1(Ω, H)$ denotes the set of continuously differentiable $H$-valued functions defined in $Ω$. 


Given $0 \leq \theta < 2\pi$, introduce the left Cauchy–Riemann operator

$$\theta D = 2 \left\{ \frac{\partial}{\partial \bar{z}_1} + ie^{i\theta} \frac{\partial}{\partial z_2} j \right\} = 2 \left\{ \frac{\partial}{\partial \bar{z}_1} + ie^{i\theta} j \frac{\partial}{\partial \bar{z}_2} \right\}$$

defined on $C^1(\Omega, \mathbb{H})$, where $z = x_0 + x_1i + ie^{i\theta} j x_2 + e^{i\theta} j x_3 = z_1 + ie^{i\theta} j z_2$ with $z_1 = x_0 + x_1 i$ and $z_2 = x_2 + x_3 i$. It represents the complex form of the quaternionic operator

$$\theta D := \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + ie^{i\theta} j \frac{\partial}{\partial x_2} + e^{i\theta} j \frac{\partial}{\partial x_3},$$

which is $\psi D$ associated to the structural set $\psi = (1, i, ie^{i\theta} j, e^{i\theta} j)$.

If $\theta D$ acts on the right it is then denoted by $\theta D_r$. These operators decompose the four-dimensional Laplace operator:

$$\theta D \circ \theta D = \theta D \circ \theta D = \theta D_r \circ \theta D_r = \theta D_r \circ \theta D_r = \Delta_{\mathbb{R}^4}.$$

The elements of the sets $\theta M(\Omega, \mathbb{H}) = ker \theta D$ and $\theta M_r(\Omega, \mathbb{H}) = ker \theta D_r$ are called left (respectively right) $\theta$-hyperholomorphic functions on $\Omega$.

These spaces $\theta M$ and $\theta M_r$ are right-quaternionic Banach modules, although they are also real linear spaces (from both sides). This has the disadvantage that $\theta D_r$ acts on them only as a real linear operator, not a quaternionic linear operator, but in our context it will be good enough since the principal operator under consideration is $\theta D$.

In addition, $f = f_1 + f_2 j \in \theta M(\Omega, \mathbb{H})$, where $f_1, f_2 : \Omega \to \mathbb{C}$, if and only if

$$\begin{cases}
\frac{\partial f_1}{\partial \bar{z}_1} = ie^{i\theta} \frac{\partial f_2}{\partial \bar{z}_2}, \\
\frac{\partial f_1}{\partial \bar{z}_2} = -ie^{i\theta} \frac{\partial f_2}{\partial \bar{z}_1}.
\end{cases}$$

Let us consider the following spaces of holomorphic maps:

$$Hol(\Omega, \mathbb{C}) = \{ f \in C^1(\Omega, \mathbb{C}) : \frac{\partial f}{\partial \bar{z}_1} = 0, \frac{\partial f}{\partial \bar{z}_2} = 0 \}$$

and more generally

$$Hol(\Omega, \mathbb{C}^2) = \{ f = (f_1, f_2) : f_1, f_2 \in Hol(\Omega, \mathbb{C}) \}.$$

The relation

$$Hol(\Omega, \mathbb{C}^2) = \bigcap_{0 \leq \theta < 2\pi} \theta M(\Omega)$$

can be found in [15].
Given \( x, y \in \mathbb{H} \), denote \( (x, y)_\theta = \sum_{k=0}^{3} x_k y_k \), where \( x_\theta = x_0 + x_1 i + x_2 e^{i\theta} j + x_3 e^{i\theta} j \) and \( y_\theta = y_0 + y_1 i + y_2 e^{i\theta} j + y_3 e^{i\theta} j \) with \( x_k, y_k \in \mathbb{R} \) for all \( k \).

Set \( z_\theta = x_0 + x_1 i + i e^{i\theta} j x_2 + e^{i\theta} j x_3 = z_1 + i e^{i\theta} j z_2 \), for \( z \in \mathbb{H} \), where \( z_1 = x_0 + x_1 i, z_2 = x_2 + x_3 i \in \mathbb{C} \). The mapping \( z_\theta \rightarrow (z_1, z_2) \) establishes the following operations in \( \mathbb{C}^2 \), according to the quaternionic algebraic structure.

- \( z \bar{z} = j \bar{z} \) for all \( z \in \mathbb{C} \),
- \( (z_1, z_2) \pm (\zeta_1, \zeta_2) = (z_1 \pm \zeta_1, z_2 \pm \zeta_2) \),
- \( (z_1, z_2)(\zeta_1, \zeta_2) = (z_1 \zeta_1 \bar{\zeta}_2, \bar{\zeta}_1 \zeta_2 + z_2 \zeta_1) \),
- \( (z_1, z_2) = (\bar{z}_1, -\bar{z}_2) \).
- \( |z_1 + z_2 j|^2 = |z_1|^2 + |z_2|^2 = (z_1, z_2)(\bar{z}_1, -\bar{z}_2) \).
- Topology in \( \mathbb{C}^2 \) is determined by the metric \( d(\zeta, z) := |\zeta - z| \) for \( z, \zeta \in \mathbb{C}^2 \).

For abbreviation we write \( z \) instead of \( z_\theta \) for the elements of the domains of our functions and in case that we were using the representation \( z = z_1 + z_2 j \) to indicate that the mapping \( z_1 + z_2 j = x_1 + y_1 i + (x_2 + y_2 i) j = x_1 + y_1 i + x_2 j + y_2 k \in \mathbb{H} \) we shall write it explicitly.

Let \( \Omega \subset \mathbb{C}^2 \) be a bounded domain with its boundary \( \partial \Omega \) a compact 3—dimensional sufficiently smooth hypersurface (co-dimension 1 manifold). The following formulas can be found in many sources (see, for example \([10, 15]\)).

Given \( f, g \in C^1(\Omega, \mathbb{H}) \) we have the quaternionic Stokes formulas

\[
\begin{align*}
\int_{\partial \Omega} (g \sigma^\theta_\zeta f) &= g \theta D[f] + D^\theta [g] f, \\
\int_{\partial \Omega} g \sigma^\theta_\zeta f &= \int_{\Omega} \left( g \theta D[f] + D^\theta [g] f \right) d\mu,
\end{align*}
\]

and the quaternionic Borel-Pompieiu formula

\[
\begin{align*}
\int_{\partial \Omega} (K_\theta(\zeta - z) \sigma^\theta_\zeta f(\zeta) + g(\zeta) \sigma^\theta_\zeta K_\theta(\zeta - z)) \\
- \int_{\Omega} (K_\theta(\zeta - z) \theta D[f](\zeta) + D^\theta [g](\zeta) K_\theta(\zeta - z)) d\mu_\zeta \\
= \left\{ \begin{array}{ll}
f(z) + g(z), & z \in \Omega, \\
0, & z \in \mathbb{H} \setminus \overline{\Omega},
\end{array} \right.
\end{align*}
\]

where

\[
K_\theta(\zeta - z) := \frac{1}{2\pi^2 |\zeta - z|^4} [(\bar{\zeta}_1 - \bar{\zeta}_1) - ie^{-i\theta}(\bar{\zeta}_2 - \bar{\zeta}_2) j]
\]

is called \( \theta \)—hyperholomorphic Cauchy kernel and

\[
\sigma^\theta_\zeta := \frac{1}{2} [d\bar{\zeta}_2] \wedge d\zeta - i e^{i\theta} d\zeta_{[1]} \wedge d\zeta j,
\]

where \( d\zeta_{[1]} \) denotes as usual \( d\zeta_0 \wedge d\zeta_1 \wedge d\zeta_2 \wedge d\zeta_3 \) omitting the factor \( d\zeta_i \), represents the \( \mathbb{H} \)—valued area form to \( \partial \Omega \) and \( d\mu_\zeta \) stands for the 4—dimensional volume element in
\[ \Omega \). Here \( C^1(\overline{\Omega}, \mathbb{H}) \) denotes the subclass of functions which can be extended smoothly to an open set containing the closure \( \overline{\Omega} \).

From the above we can see that the \( \theta \)-hyperholomorphic Cauchy kernel generates the following important operator.

\[ \theta T[f](z) := \int_{\Omega} K_{\theta}(\xi - z) f(\xi) d\mu_{\xi} \quad (5) \]

on \( L^2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}) \) and meets \( \theta D \circ \theta T = I \). This can be found in [10, 15].

Given a one-to-one correspondence \( g : \Xi_1 \to \Omega \) and \( h : \Omega \to \mathbb{H} \), denote \( W_g[f] = f \circ g \) and \( h M[f] = hf \) for \( f \) in a function space associated to \( \Omega \).

The Möbius transformations in \( \mathbb{R}^4 \cong \mathbb{C}^2 \) can be represented by quaternionic fractional-linear transformations:

\[ T(z) = (az + b)(cz + d)^{-1}, \quad (6) \]

where \( a, b, c, d \in \mathbb{H} \) satisfy \( ad \neq 0 \) if \( c = 0 \) and \( b - ac^{-1}d \neq 0 \) if \( c \neq 0 \), see [7, 12].

Let \( \Omega, \Xi \subset \mathbb{R}^4 \) two conformal equivalent domains, i.e., there exists \( T \) given by (6) such that \( T(\Xi) = \Omega \). Denote \( \zeta = T(z) \in \Omega \) for \( z \in \Xi \) and define the following functions:

\[ A_T(z) := \begin{cases} \bar{d}, & \text{if } c = 0; \\ \frac{cz(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1}}{(b - ac^{-1}d)} |cz(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1}|^4, & \text{if } c \neq 0, \end{cases} \quad (7) \]

\[ B_T(\zeta) := \begin{cases} \bar{a}, & \text{if } c = 0; \\ -c(\zeta - ac^{-1})|\zeta - ac^{-1}|^4, & \text{if } c \neq 0, \end{cases} \quad (8) \]

\[ C_T(z) := \begin{cases} |a|^2 d^{-1}, & \text{if } c = 0, \\ \frac{|c|^2}{|b - ac^{-1}d|} (b - ac^{-1}d)^{-1} \frac{cz(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1}}{|cz(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1}|^4}, & \text{if } c \neq 0, \end{cases} \quad (9) \]

\[ \rho_T(z) := \begin{cases} 1, & \text{if } c = 0, \\ \frac{1}{|cz(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1}|^2}, & \text{if } c \neq 0. \end{cases} \quad (10) \]

The \( \theta \)-hyperholomorph \( \text{Bergman space associated to } \Omega \subset \mathbb{H}, \) is defined by

\[ \theta A(\Omega) := \theta M(\Omega, \mathbb{H}) \cap L^2(\Omega, \mathbb{H}) \]
and is a quaternionic right-Hilbert space equipped with the inner product and norm inherited from $L_2(\Omega, \mathbb{H})$. Moreover, there were proved the following assertions:

1. Analogue of the chain rule

$$\theta D_z[A_T f \circ T] = (B_T \circ T) \theta D_z[f] \circ T,$$

(11)

for all $f \in C^1(\Omega, \mathbb{H})$.

2. Define

$$P_T[f](z) = C_T(z) f(T(z)), \quad \forall z \in \Xi,$$

for all $f \in C^1(\Omega, \mathbb{H})$ and it is well-known that $P_T : L_2(\Omega, \mathbb{H}) \rightarrow L_2(\rho_T(\Xi, \mathbb{H}))$ is an isometric isomorphism of quaternionic right-Hilbert spaces. What is more,

$$\int_{\Xi} P_T[f] P_T[g] \rho_T d\mu = \int_{\Omega} \bar{f} \bar{g} d\mu,$$

(12)

for all $f, g \in L_2(\Omega, \mathbb{H})$.

The identity (11) is an usual phenomena in hypercomplex analysis whose antecedent within complex analysis is the formula used when deriving a composition of analytic functions, called chain rule, and the functions $A_T$ and $B_T$ are known as conformal weights. For example, to preserve the hyperholomorphy of a function in composition with the quaternionic inversion according to

$$D_x\left[\frac{\bar{x}}{|x|^4} f \circ T\right] = -\frac{x}{|x|^6} D_y[f](T(x)),$$

it is necessary the conformal weight $\frac{\bar{x}}{|x|^4}$.

On the other hand, the weight function $\rho_T$ used in (12) arises from computations such as the relationship between the quaternionic inversion and the space $L_2$. Set $x^{-1} = \frac{\bar{x}}{|x|^2} = T(x) = y$ and from the representation of the differential of volume in terms of differential forms we obtain that

$$d\mu_y = dy_0 \wedge dy_1 \wedge dy_2 \wedge dy_3 = \frac{1}{|x|^8} dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = \frac{1}{|x|^8} d\mu_x.$$

Therefore,

$$\langle f, g \rangle_{L_2(T(\Xi), \mathbb{H})} = \int_{T(\Xi)} f(y) g(y) d\mu_y = \int_{\Xi} \left(\frac{\bar{x}}{|x|^4} f \circ T(x)\right) \frac{\bar{x}}{|x|^4} g \circ T(x) \rho(x) d\mu_x$$

$$= \langle P_T[f], P_T[g] \rangle_{L_2(\rho_T(\Xi, \mathbb{H}))},$$
where $C_T(x) = \frac{\bar{x}}{|x|^4}$ and $\rho(x) = \frac{1}{|x|^2}$ for all $x \in \mathbb{R}$. Note that $\frac{1}{|x|}^8$ was conveniently decomposed in the conformal weights to preserve the hyperholomorphy and in the weight $\rho$ of our image function space.

The same reasoning idea is applied to rotations, dilations and translations. Finally, the general case with $c \neq 0$ is a consequence of the composition these operator according to the identity

$$(ax + b)(cx + d)^{-1} = (cx(b - ac^{-1}d)^{-1} + d(b - ac^{-1}d)^{-1})^{-1} - ac^{-1}.$$ 

If $c = 0$ we use $(ax + b)(cx + d)^{-1} = axd^{-1} + bd^{-1}.$

The standard work here is [10].

### 3 $(\theta, \, u)$-Hyperholomorphic Bergman Type Spaces

Before introduce the Bergman type space to be studied, we present briefly the basic definitions and results of $(\theta, \, u)$-hyperholomorphic theory necessary for our purpose. For more information, we refer the reader to [14].

Let $\Omega \subset \mathbb{H}$ be a domain and $u \in \mathbb{H}$. The Helmholtz type operators with a quater-nionic wave number $u^2$ that we are going to consider, which act on $C^2(\Omega, \mathbb{H})$, must be left and right:

$$u^2 \Delta = \Delta_{\mathbb{R}^4} + u^2 f$$

and

$$u^2, r \Delta = \Delta_{\mathbb{R}^4} + f u^2,$$

respectively.

Consider the left- and the right- $(\theta, \, u)$-Cauchy-Riemann type operators defined by

$$\theta^u D[f] := \theta D[f] + uf$$

and

$$\theta D_{r,u}[f] := \theta D_r[f] + fu,$$

for all $f \in C^1(\Omega, \mathbb{H})$ respectively.

Thus

$$\theta D_{r,u} \circ \theta D[f] = \theta D_r[\theta D[f]] + (\theta D[f])u$$

$$= \theta D_r[\theta D[f] + uf] + (\theta D[f] + uf)u$$

$$= \theta D_r \circ \theta D[f] + \theta D_r[uf] + \theta D[f]u + uf u.$$
\[= \overline{\theta D} \circ_{\theta} D[f] + \theta D[uf] + \theta D[f]u + uf \bar{u}.\]

\[
\overline{\theta D} \circ_{\theta} D[f] = \overline{\theta D}[u_{\theta}D[f]] + u_{\theta}^{\theta}D[f])
= \overline{\theta D} \circ_{\theta} D[f] + \theta D[uf] + \theta D[f] |u|^2 f
= \Delta_{\mathbb{R}^4}[f] + |u|^2 f + \theta D[uf] + \theta D[f].
\]

In the particular case when \( f(x) \) be independent from the scalar part of \( x \) and \( u \in \mathbb{R} \) we have

\[
\overline{\theta D} \circ_{\theta} D[f] = \Delta_{\mathbb{R}^4}[f] + |u|^2 f.
\]

In analogy with the notion of \( \theta \)-hyperholomorphic function, consider the following definition of \( (\theta, u) \)-hyperholomorphic functions:

We will denote by \( \theta \mathcal{M}(\Omega, \mathbb{H}) = C^1(\Omega, \mathbb{H}) \cap Ker_{\theta} D \) the quaternionic right-linear space of the left \( (\theta, u) \)-hyperholomorphic functions, \( (\theta, u) \)-hyperholomorphic functions for brevity, associated to \( \Omega \subset \mathbb{H} \).

The quaternionic left-linear space of the \( r - (\theta, u) \)-hyperholomorphic functions on \( \Omega \) will be denoted by \( \theta \mathcal{M}_{r, u}(\Omega, \mathbb{H}) = C^1(\Omega, \mathbb{H}) \cap Ker_{\theta} D_{r, u} \).

**Remark 1** Set \( u = u_1 + ie^{\theta_j}j u_2 \) for \( u_1, u_2 \in \mathbb{C} \). Then, \( f = f_1 + f_2 j \in \theta \mathcal{M}(\Omega, \mathbb{H}) \) if and only if

\[
\begin{cases}
\frac{\partial f_1}{\partial \bar{z}_1} + u_1 f_1 = ie^{\theta_j} \left( \frac{\partial f_2}{\partial \bar{z}_2} + u_2 f_2 \right), \\
\frac{\partial f_1}{\partial \bar{z}_2} + u_2 f_1 = -ie^{\theta_j} \left( \frac{\partial f_2}{\partial \bar{z}_1} + u_1 f_2 \right).
\end{cases}
\]

We can now state and prove the analogues of Stokes and Borel-Pompeiu formulas for the \( (\theta, u) \)-hyperholomorphic functions:

**Proposition 3.1** Let \( \Omega \subset \mathbb{H} \) be a bounded domain with \( \partial \Omega \) a 3-dimensional compact sufficiently smooth surface. Then

\[
d(f(z)u_{\theta}(z)g(z)) = (\theta D_{r, u}[f](z)g(z) + f(z)^{\theta} D[g](z))e^{2(a,z)\theta},
\]

\[
\int_{\partial \Omega} f(z)u_{\theta}(z)g(z) = \int_{\Omega} (\theta D_{r, u}[f](\xi)g(\xi) + f(\xi)^{\theta} D[g](\xi)) d\lambda_{\theta}(\xi)
\]

and

\[
\int_{\partial \Omega} (K_{\theta}(\xi - z)g(\xi) + g(\xi)K_{\theta}(\xi - z))
- \int_{\Omega} (K_{\theta}(\xi - z)u_{\theta} D[f](\xi) + u_{\theta} D_{r, u}[g](\xi)K_{\theta}(\xi - z)) d\mu_{\xi}
\]

\[
= \begin{cases}
f(z) + g(z), & z \in \Omega, \\
0, & z \in \mathbb{H} \setminus \Omega,
\end{cases}
\]
for all \( f, g \in C^1(\Omega, \mathbb{H}) \cap C(\overline{\Omega}, \mathbb{H}) \) is obtained, where \( d\lambda^\theta_u(\zeta) = e^{2(\theta, \zeta)} \, d\mu_\zeta \), \( v^\theta_u(\zeta) = e^{2(\theta, \zeta)} \sigma^\theta_\zeta \) and \( K^\theta_u(\zeta - z) = e^{[\theta, \zeta - z]}K_\theta(\zeta - z) \) is a \((\theta, u)\)-hyperholomorphic reproducing function.

**Proof** Note that

\[
\theta D[e^{(u,z)\theta} f] = (y_0 e^{(u,z)\theta} f + e^{(y,z)\theta} \frac{\partial f}{\partial x_0}) + i(y_1 e^{(u,z)\theta} f + e^{(y,z)\theta} \frac{\partial f}{\partial x_1}) \\
+ i e^{i\theta}(y_2 e^{(u,z)\theta} f + e^{(y,z)\theta} \frac{\partial f}{\partial x_2}) + e^{i\theta}(y_3 e^{(u,z)\theta} f + e^{(y,z)\theta} \frac{\partial f}{\partial x_3})
\]

for all \( f \in C^1(\Omega, \mathbb{H}) \), where \( u = y_0 + y_1 i + y_2 e^{i\theta} j + y_3 e^{i\theta} j \) and \( z = x_0 + x_1 i + x_2 e^{i\theta} j + x_3 e^{i\theta} j \) with \( y_k, x_k \in \mathbb{R} \) for all \( k \).

The following identity is established by analogous reasoning

\[
\theta D_r[e^{(u,z)\theta} f] = e^{(u,z)\theta} \theta D_r u[f],
\]

for all \( f \in C^1(\Omega, \mathbb{H}) \), see also [9].

The proof is completed by applying formulas (2), (3) and (4) to \( e^{(u,z)\theta} f, e^{(u,z)\theta} g \) and making use of (16) and (17).

**Remark 2** The analogous fact, for the \( \psi \)-hyperholomorphy, to the previous one appears in [9, Proposition 3.3]:

\[
\psi D_x[e^{(u,x)\psi} f] = e^{(u,x)\psi} \left( u f(x) + \psi D_x[f] \right).
\]

Particularly, for \( u = 0 \) we see \( \psi D_x[e^{(0,x)\psi} f] = \psi D_x[f] \).

**Corollary 3.2** Cauchy-type formula for \((\theta, u)\)-hyperholomorphic functions.

\[
\int_{\partial \Omega} K^\theta_u(\zeta - z) \sigma^\theta_\zeta f(\zeta) = \begin{cases} f(z), & z \in \Omega, \\
0, & z \in \mathbb{H} \setminus \overline{\Omega}, \end{cases}
\]

for all \( f \in \mathcal{M}(\Omega) \).

**Proof** It is a direct consequence from (15).

Let \( f \in L^2_2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}) \). Consider the operator

\[
\theta_T[f](z) := \int_{\Omega} K^\theta_u(\zeta - z) f(\zeta) d\mu_\zeta.
\]
Proposition 3.3 \( \theta_u D \circ \theta_u T = I. \)

**Proof** As \( \theta_D \circ \theta T = I \) on \( L_2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}) \) then using (16) we obtain

\[
e^{(u,z)}_\theta f(z) = \theta_D [e^{(u,z)}_\theta \int_\Omega e^{-(u,z)}_\theta K^\theta (\xi - z) e^{(u,z)}_\theta f(\xi) d\mu_\xi]
\]

for all \( f \in L_2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}). \)

\[\square\]

**Remark 3** Let \( f = f_1 + f_2 j, \) where \( f_1, f_2 : \Omega \rightarrow \mathbb{C} \) and define

\[
\theta_u T_1(f_1, f_2)(z) = \int_\Omega \frac{e^{(u,z)}_\theta}{2\pi^2 |\xi - z|^4} \left[ (\bar{\xi}_1 - \bar{z}_1) f_1(\xi) + i e^{-i\theta}(\bar{\xi}_2 - \bar{z}_2) f_2(\xi) \right] d\mu_\xi
\]

\[
\theta_u T_2(f_1, f_2)(z) = \int_\Omega \frac{e^{(u,z)}_\theta}{2\pi^2 |\xi - z|^4} \left[ (\bar{\xi}_1 - \bar{z}_1) f_2(\xi) - i e^{-i\theta}(\bar{\xi}_2 - \bar{z}_2) f_1(\xi) \right] d\mu_\xi,
\]

where \( \xi = \xi_1 + i e^{i\theta} j \xi_2, \) \( z = z_1 + i e^{i\theta} j z_2 \in \Omega, \) with \( \xi_k, z_k \in \mathbb{C} \) for \( k = 1, 2. \)

Therefore, \( \theta_u T(f) = \theta_u T_1(f_1, f_2) + \theta_u T_1(f_1, f_2) j \) for all \( f = f_1 + f_2 j \in L_2(\Omega, \mathbb{H}) \cup C(\Omega, \mathbb{H}) \) and the previous proposition becomes

\[
\frac{\partial}{\partial \bar{z}_1} \theta_u T_1(f_1, f_2) - i e^{i\theta} \frac{\partial}{\partial \bar{z}_2} \theta_u T_2(f_1, f_2) = f_1
\]

\[
\frac{\partial}{\partial \bar{z}_1} \theta_u T_2(f_1, f_2) + i e^{i\theta} \frac{\partial}{\partial \bar{z}_2} \theta_u T_1(f_1, f_2) = f_2,
\]

on \( \Omega. \)

**Proposition 3.4** (Chain rule or the conformal co-variant property of \( \theta_u D \)) Given \( u, v \in \mathbb{H} \) and \( \Omega, \Xi \subset \mathbb{H} \) two conformal equivalent domains; i.e., \( T(\Xi) = \Omega, \) where \( T \) is given by (6) and let \( \zeta = T(z) \in \Omega \) for \( z \in \Xi. \) Then

\[
\theta_u D_\zeta [e^{(v-u,z)}_\theta A_T f \circ T] = e^{(v-u,z)}_\theta (B_T \circ T)(\theta_{\delta_T} D_\zeta [f]) \circ T, \quad \forall f \in C^1(\Omega, \mathbb{H}),
\]

where the index \( \zeta \) of \( \theta_u D_\zeta \) denotes the quaternionic variable of differentiation, the function \( \delta_T = (B_T \circ T)^{-1} v (A_T \circ T^{-1}) \) on \( \Omega, \) with \( (B_T \circ T)^{-1} \) being the multiplicative inverse of \( B_T \circ T \) and \( T^{-1} \) the inverse mapping of \( T \) and \( \theta_{\delta_T} D_\zeta [f] := \theta D_\zeta [f] + \delta_T f. \)

**Proof** From (11) we have

\[
\theta_u D_\zeta [e^{(v-u,z)}_\theta A_T f \circ T] = u e^{(v-u,z)}_\theta A_T f \circ T + \theta D_\zeta [e^{(v-u,z)}_\theta A_T f \circ T]
\]

\[
= u e^{(v-u,z)}_\theta A_T f \circ T + (v - u) e^{(v-u,z)}_\theta A_T f \circ T
\]

\[
+ e^{(v-u,z)}_\theta \theta D_\zeta [A_T f \circ T]
\]
\[ e^{(v-u,z)\theta} v \circ T + e^{(v-u,z)\theta} (B_T \circ T) \theta D_\xi \{ f \} \circ T \]
\[ = e^{(v-u,z)\theta} (B_T \circ T) \times \left\{ (B_T \circ T)^{-1} v (A_T \circ T^{-1}) f + \theta D_\xi \{ f \} \right\} \circ T. \]

See [9] for more details. \[ \square \]

**Corollary 3.5** Denote \( \phi_{\delta_T} \mathcal{M}(\Omega) \) the quaternionic right linear space of \( f \in C^1(\Omega, \mathbb{H}) \) such that \( \theta D[f] + \delta_T f = 0 \) on \( \Omega \). Then
\[ f \in \phi_{\delta_T} \mathcal{M}(\Omega) \iff e^{(v-u,z)\phi} A_T f \circ T \in \theta u \mathcal{M}(\Xi), \tag{19} \]

or equivalently \( f = f_1 + f_2 j \) we see that
\[
\begin{cases}
\frac{\partial f_1}{\partial \xi_1} + \delta_1 f_1 = ie^{i\theta} \left( \frac{\partial f_2}{\partial \xi_2} + \delta_2 f_2 \right), \\
\frac{\partial f_1}{\partial \xi_2} + \delta_2 f_1 = -ie^{i\theta} \left( \frac{\partial f_2}{\partial \xi_1} + \delta_1 f_2 \right),
\end{cases}
\]

hold on \( \Omega \) if and only if
\[
\begin{cases}
\frac{\partial g_1}{\partial \bar{z}_1} + u_1 g_1 = ie^{i\theta} \left( \frac{\partial g_2}{\partial \bar{z}_2} + u_2 g_2 \right), \\
\frac{\partial g_1}{\partial \bar{z}_2} + u_2 g_1 = -ie^{i\theta} \left( \frac{\partial g_2}{\partial \bar{z}_1} + u_1 g_2 \right),
\end{cases}
\]
on \( \Xi \), where \( u = u_1 + ie^{i\theta} j u_2 \) with \( u_1, u_2 \in \mathbb{C}, \delta_T(\xi) = \delta_1(\xi) + ie^{i\theta} j \delta_2(\xi) \) for all \( \xi \in \Omega \) with \( \delta_1, \delta_2 : \Omega \to \mathbb{C} \), and \( e^{(v-u,z)\theta} A_T(z) f \circ T(z) = g_1(z) + g_2(z) j \) for all \( z \in \Xi \) with \( g_1, g_2 : \Xi \to \mathbb{C} \).

**Proposition 3.6** Let \( \Omega \subset \mathbb{H} \) be a bounded domain.

1. If \( f \in \theta u \mathcal{M}(\Omega) \) and \( g \in \theta \mathcal{M}_{r,u}(\Omega) \) then
\[
\int_{\partial \Omega} (K^\theta_u(\zeta - z) g(\zeta) K^\theta_u(\zeta - z)) = \begin{cases} f(z) + g(z), & z \in \Omega, \\ 0, & z \in \mathbb{H} \setminus \Omega. \end{cases}
\]

2. If \( z \in \Omega \) and \( \epsilon > 0 \) are such that \( \overline{B(z, \epsilon)} \subset \Omega \) then there exists a constant \( k_\epsilon > 0 \) such that
\[ |f(z)| \leq k_\epsilon \left( \int_{B(z, \epsilon)} |f(\zeta)|^2 d\mu_{\zeta} \right)^{\frac{1}{2}}, \]
for all \( f \in \theta u \mathcal{M}(\Omega) \).
**Proof**  

1. Use (15).

2. From (18) and Stokes integral formula, see (14), we see that

\[
|f(z)| \leq \frac{1}{2\pi^2\epsilon^4} \int_{B(z,\epsilon)} |e^{2(u,\zeta+\zeta^\theta)} D_{r,u}[e^{-(u,\zeta+\zeta^\theta)} ((\tilde{\zeta}_1 - \bar{\zeta}_1) - i e^{-i\theta}(\tilde{\zeta}_2 - \bar{\zeta}_2)) |f(\xi)| d\mu(\xi) \leq \frac{1}{2\pi^2\epsilon^4} \int_{B(z,\epsilon)} |e^{2(u,\zeta+\zeta^\theta)} D_{r,u}[e^{-(u,\zeta+\zeta^\theta)} ((\tilde{\zeta}_1 - \bar{\zeta}_1) - i e^{-i\theta}(\tilde{\zeta}_2 - \bar{\zeta}_2)) |^2 d\mu(\xi) \right)^{1/2}.
\]

The mapping

\[(\zeta_1, \zeta_2) \mapsto e^{2(u,\zeta+\zeta^\theta)} D_{r,u}[e^{-(u,\zeta+\zeta^\theta)} ((\tilde{\zeta}_1 - \bar{\zeta}_1) - i e^{-i\theta}(\tilde{\zeta}_2 - \bar{\zeta}_2))]
\]

is a continuous functions on $\mathbb{C}^2$ thus it is a bounded function on $\mathbb{B}(z, \epsilon)$ and there exists $M_\epsilon > 0$ such that

\[|e^{2(u,\zeta+\zeta^\theta)} D_{r,u}[e^{-(u,\zeta+\zeta^\theta)} ((\tilde{\zeta}_1 - \bar{\zeta}_1) - i e^{-i\theta}(\tilde{\zeta}_2 - \bar{\zeta}_2))]| < M_\epsilon,
\]

for all $(\zeta_1, \zeta_2) \in \mathbb{B}(z, \epsilon)$. Therefore,

\[|f(z)| \leq \frac{1}{2\pi^2\epsilon^4} M_\epsilon \left( \frac{\pi^2\epsilon^4}{2} \right)^{1/2} \left( \int_{\mathbb{B}(z,\epsilon)} |f(\xi)|^2 d\mu(\xi) \right)^{1/2}.
\]

Denote $k_\epsilon = \frac{M_\epsilon}{2\sqrt{2}\pi \epsilon^2}$. 

\[\square\]

**Definition 3.7** The quaternionic right linear space $\mathcal{A}(\Omega) = \mathcal{A}(\Omega) \cap L_2(\Omega, \mathbb{H})$ is called $(\theta, u)$—hyperholomorphic Bergman space associated to $\Omega$ and is denoted by

\[\|f\|_{\mathcal{A}(\Omega)}^\theta = \left( \int_{\Omega} |f|^2 d\mu \right)^{1/2},\]

for all $f \in \mathcal{A}(\Omega)$.

**Remark 4** The previous sentence allows to see that the quaternionic right-linear space $\mathcal{A}(\Omega)$ equipped with the scalar product

\[\langle f, g \rangle_{\mathcal{A}(\Omega)}^\theta = \int_{\Omega} f \overline{g} d\mu, \quad \forall f, g \in \mathcal{A}(\Omega)
\]

is a quaternionic right-Hilbert space and from deeply similar computations to those presented in [10] and [19] we can see that the valuation functional $f \mapsto f(z)$, for $z \in \Omega$, is bounded on $\mathcal{A}(\Omega)$. 


Riesz representation theorem for quaternionic right Hilbert space, see [4], shows that there exists \( \theta_i B_z \in \theta_i A(\Omega) \) such that \( f(z) = \langle \theta_i B_z, f \rangle \) for all \( f \in \theta_i A(\Omega) \). Therefore the Bergman kernel of \( \theta_i A(\Omega) \) is \( \theta_i B \Omega(z, \zeta) = \theta_i B_i(z) \) and satisfies:

\[
f(z) = \int_{\Omega} \theta_i B \Omega(z, \zeta) f(\zeta) d\mu_{\zeta}, \quad \forall f \in \theta_i A(\Omega).
\]

The Bergman projection associated to \( \theta_i A(\Omega) \) is

\[
\theta_i B \Omega [f](z) := \int_{\Omega} \theta_i B \Omega(z, \zeta) f(\zeta) d\mu_{\zeta}, \quad \forall f \in L^2(\Omega, \mathbb{H}).
\]

What is more, repeating the reasoning given in [10, 19], we obtain:

1. \( \theta_i B \Omega \) is hermitian.
2. The mapping \( z \mapsto \theta_i B \Omega(z, \zeta) \) belongs to \( \theta_i M(\Omega) \) for each \( \zeta \in \Omega \) and \( \zeta \mapsto \theta_i B \Omega(z, \zeta) \) belongs to \( M_i(\Omega) \) for each \( z \in \Omega \).
3. \( \theta_i B \Omega(\cdot, \cdot) \) is the unique function with the previous properties.
4. \( \theta_i B \Omega \) is a continuous symmetric operator.
5. \( \theta_i B \Omega[L^2(\Omega, \mathbb{H})] = \theta_i A(\Omega) \).
6. \( \theta^2_i B \Omega ) = \theta_i B \Omega \).

**Remark 5** Particularly, for \( u = 0 \), or equivalently \( u_1 = u_2 = 0 \), we obtain the theory of hyperholomorphic functions associated to domains in \( \mathbb{C}^2 \) since the equations (13) become the Cauchy equations in \( \mathbb{C}^2 \):

\[
\begin{align*}
\frac{\partial f_1}{\partial \bar{z}_1} &= i e^{i\theta} \frac{\partial f_2}{\partial \bar{z}_2}, \\
\frac{\partial f_1}{\partial \bar{z}_2} &= -i e^{i\theta} \frac{\partial f_2}{\partial \bar{z}_1}.
\end{align*}
\]

The Bergman space \( \theta_0 A(\Omega) = \theta_0 M(\Omega) \cap L^2(\Omega, \mathbb{H}) \) was studied in [10].

**Proposition 3.8** Given \( u, v \in \mathbb{H} \) and let \( \Omega, \Xi \subset \mathbb{H} \) be two conformally equivalent domains with \( T(\Xi) = \Omega \), where \( T \) is given by (6). Define \( \delta_T A(\Omega) := \delta_T M(\Omega) \cap L^2(\Omega, \mathbb{H}) \) equipped with the inner product inherited from the weighted space \( L^2_T(\Omega, \mathbb{H}) \), where \( \delta_T \) is defined in Proposition 3.4.

Denote \( \gamma_T(z) = e^{-2(v-u, z)} \rho_T(z) \) for all \( z \in \Xi \) and define

\[
\theta_u A_{\gamma_T}(\Xi) = \theta_u M(\Xi) \cap L^2_{\gamma_T}(\Xi, \mathbb{H})
\]

equipped with the inner product inherited from the weighted space \( L^2_{\gamma_T}(\Xi, \mathbb{H}) \), i.e.,

\[
\langle f, g \rangle_{\theta_u A_{\gamma_T}(\Xi)} = \int_{\Xi} \bar{f} g \gamma_T d\mu,
\]
for all $f, g \in \theta^u A_{\gamma_T}(\mathbb{E})$. Then $e^{(v-u,z)\theta} C_T M \circ W_T \mid_{\delta_T A(\Omega)} : \delta_T A(\Omega) \to \theta^u A_{\gamma_T}(\mathbb{E})$ is an isometric isomorphism of quaternionic right-Hilbert spaces, where $C_T$ and $\rho_T$ are given by (12). What is more,

$$\begin{align*}
\theta^u \mathcal{B}_{\mathbb{E}, \gamma_T}(z, w) &= e^{(v-u,z+w)\theta} C_T(z) \delta_T \mathcal{B}_{\Omega}(T(z), T(w)) C_T(w), \\
\theta^u \mathcal{B}_{\mathbb{E}, \gamma_T} &= e^{(v-u,z)\theta} C_T M \circ W_T \circ \delta_T \mathcal{B}_{\Omega} \circ \left( e^{(v-u,z)\theta} C_T M \circ W_T \right)^{-1},
\end{align*}$$

where $\theta^u \mathcal{B}_{\mathbb{E}, \gamma_T}$ and $\theta^u \mathcal{B}_{\Omega}$ are the Bergman kernels of $\theta^u A_{\gamma_T}(\mathbb{E})$ and $\theta^u A(\Omega)$ respectively, and $\theta^u \mathcal{B}_{\mathbb{E}, \gamma_T}$ and $\theta^u \mathcal{B}_{\Omega}$ are the Bergman projections of $\theta^u A_{\gamma_T}(\mathbb{E})$ and $\theta^u A(\Omega)$.

**Proof** From Cauchy and Stokes formula recently showed for $(u, \theta)$—hyperholomorphic functions on $\Omega$ we have that

$$f(z) = \int_{\mathcal{B}(z, \epsilon)} K_u^\theta(w - z) f(w) d\mu_w = \int_{\mathcal{B}(z, \epsilon)} e^{-2(u,w)\theta} K_u^\theta(w - z) f(w)$$

$$= \frac{1}{2\pi^2 \epsilon ^4} \int_{\mathcal{B}(z, \epsilon)} \theta^u D_r [e^{(u,-w-z)\theta} ([\bar{w}_1 - \bar{z}_1] - ie^{-i\theta}(\bar{w}_2 - \bar{z}_2) j)] f(w) d\lambda_\mu^\theta(w)$$

$$= \frac{1}{2\pi^2 \epsilon ^4} \int_{\mathcal{B}(z, \epsilon)} e^{2(u,w)\theta} \theta^u D_r [e^{(u,-w-z)\theta} ([\bar{w}_1 - \bar{z}_1] - ie^{-i\theta}(\bar{w}_2 - \bar{z}_2) j)] f(w) d\lambda_\mu^\theta(w)$$

$$= \frac{1}{2\pi^2 \epsilon ^4} \int_{\mathcal{B}(z, \epsilon)} e^{2(u,w)\theta} \theta^u D_r [e^{(u,-w-z)\theta} ([\bar{w}_1 - \bar{z}_1] - ie^{-i\theta}(\bar{w}_2 - \bar{z}_2) j)] f(w) \frac{1}{\gamma_T^2(w)} d\mu_w,$$

for all $f \in \theta^u A_{\rho_T}(\mathbb{E})$, where $\mathcal{B}(z, \epsilon) \subset \Omega$, and the Cauchy-Schwarz inequality gives us that

$$|f(z)| \leq \frac{1}{2\pi^2 \epsilon ^4} \int_{\mathcal{B}(z, \epsilon)} \left| e^{2(u,w)\theta} \theta^u D_r [e^{(u,-w-z)\theta} ([\bar{w}_1 - \bar{z}_1] - ie^{-i\theta}(\bar{w}_2 - \bar{z}_2) j)] f(w) \frac{1}{\gamma_T^2(w)} d\mu_w \right|^2$$

$$\leq \frac{1}{2\pi^2 \epsilon ^4} \left( \int_{\mathcal{B}(z, \epsilon)} \left| f(w) \right|^2 \gamma_T^2(w) d\mu_w \right)^{\frac{1}{2}},$$

for all $f \in \theta^u A_{\gamma_T}(\mathbb{E})$. Therefore, there exists $K_\epsilon > 0$, that depends of $\epsilon > 0$ such that $|f(z)| \leq K_\epsilon \|f\|_{\theta^u A_{\gamma_T}(\mathbb{E})}$. Using the previous inequality we obtain that $\theta^u A_{\gamma_T}(\mathbb{E})$ is a quaternionic right-Hilbert space whose valuation functional is bounded and as a consequence $\theta^u A_{\gamma_T}(\mathbb{E})$ has a reproducing kernel and a projection.
Due to $C_T = \lambda_T A_T$ where $\lambda_T$ is a positive real constant, Corollary 3.5 gives us $e^{(v-u, z)\theta} C_T f \circ T \in \mu_M(\Xi)$ iff $f \in \delta_T \mathcal{M}(\Omega)$ and

$$\int_{\Xi} e^{(v-u, z)\theta} C_T(z) f \circ T(z) e^{(v-u, z)\theta} C_T(z) g \circ T(z) e^{2(v-u, z)\theta} \rho_T(z) d\mu_z = \int_{\Omega} f(\xi)g(\xi) d\mu_\xi,$$

obtained from (12), we conclude that

$$\langle e^{(v-u, z)\theta} C_T M \circ W_T[f], e^{(v-u, z)\theta} C_T M \circ W_T[g]\rangle_{\mu_A(\Xi)} = \langle f, g\rangle_{\mu_A(\Omega)},$$

for all $f, g \in \delta_T A(\Omega)$. Therefore, $e^{(v-u, z)\theta} C_T M \circ W_T$ can be used for to show that $\delta_T A(\Omega)$ has the same properties that $\delta_T A(\Xi)$, i.e., $\delta_T A(\Omega)$ is a quaternionic right-Hilbert space with a reproducing kernel and a projection.

Finally, if $h \in \delta_T A(\Xi)$ then

$$\left(\left(e^{(v-u, z)\theta} C_T M \circ W_T\right)^{-1}[h]\right)(\xi) = \int_{\Omega} \delta_T B_{\Xi, \gamma_T}(\xi, z) \left(e^{(v-u, z)\theta} C_T M \circ W_T\right)^{-1}[h](z) d\mu_z$$

$$= \delta_T B_{\Xi, \gamma_T}(\xi) \left(e^{(v-u, z)\theta} C_T M \circ W_T\right)^{-1}[h]_{\mu_A(\Omega)}$$

$$= \langle e^{(v-u, z)\theta} C_T M \circ W_T[\delta_T B_{\Xi, \gamma_T}(\xi, \cdot)], h\rangle_{\mu_A(\Xi)}$$

$$= \int_{\Xi} e^{(v-u, z)\theta} C_T M \circ W_T[\delta_T B_{\Xi, \gamma_T}(\xi, \cdot)] h(y_T) d\mu_y.$$

Apply $e^{(v-u, z)\theta} C_T M \circ W_T$ for to see that

$$h(z) = \int_{\Xi} e^{(v-u, z+w)\theta} C_T(z) (\delta_T B_{\Xi}(T(z), T(w)) C_T(w) h(w) y_T(w)) d\mu_w.$$

The relationships between the kernels and the projections are consequences of the identity given above. Similar computations were presented in [9].

Remark 6 If in the pervious proposition we consider $\mu = v = 0$ we shall see that $\delta_T = 0$ and $\delta_T A(\Omega) = \delta \mathcal{M}(\Omega) \cap L_2(\Omega, \mathbb{H})$ and the inner product is inherited from $L_2(y_T(\Omega, \mathbb{H})$, where $y_T(z) = \rho_T(z)$ for all $z \in \Xi$. Also

$$\delta_0 A_{\rho_T}(\Xi) = \delta \mathcal{M}(\Xi) \cap L_2(y_T(\Xi, \mathbb{H})$$

whose inner product is

$$\langle f, g\rangle_{\delta_0 A_{\rho_T}(\Xi)} = \int_{\Xi} \tilde{f} g \rho_T d\mu,$$

for all $f, g \in \delta_0 A_{\rho_T}(\Xi)$. 
Then the previous proposition concludes that \( C_T M \circ W_T \mid_{\mathcal{O}} : \mathcal{O} \to \mathcal{O} \) is an isometric isomorphism of quaternionic right-Hilbert spaces. What is more,

\[
\mathcal{O} B\mathcal{O}_\gamma (z, w) = C_T(z) \mathcal{O} B\mathcal{O}_T(T(z), T(w)),
\]

\[
\mathcal{O} B\mathcal{O}_\rho (z, w) = C_T M \circ W_T \circ \mathcal{O} B\mathcal{O} \circ (C_T M \circ W_T)^{-1},
\]

which are the same results presented in [10, Theorem 4.3]. From similar reasoning we can see that several previous results in case \( u = 0 \) are the same presented in [10].

There exists another kind of weighted quaternionic Bergman spaces induced by \( \mathcal{K}_0(D) \).

**Definition 3.9** Given a domain \( \Omega \subset \mathbb{H} \) define

\[
\lambda^\theta \mathcal{O} = \{ f \in \lambda^\theta \mathcal{O}(\Omega) \mid \int_\Omega |f(z)|^2 d\lambda^\theta_u(z) < \infty \}.
\]

\[
\| f \|_{\lambda^\theta \mathcal{O}}^2 := \int_\Omega |f(z)|^2 d\lambda^\theta_u, \quad \langle f, g \rangle_{\lambda^\theta \mathcal{O}} := \int_\Omega \bar{f} \bar{g} d\lambda^\theta_u,
\]

\[
\lambda^\theta_\mu S[f](z) := e^{2(u, \cdot)_\theta d\mu_z}, \quad \forall z \in \Omega,
\]

for all \( f, g \in \lambda^\theta \mathcal{O} \). Let us recall that \( d\lambda^\theta_u(\cdot) = e^{2(u, \cdot)_\theta d\mu_\cdot} \).

We abbreviate \( \lambda^\theta \mathcal{O} \) to \( \mathcal{O} \) the quaternionic right-Hilbert space, written \( \mathcal{O} \) in [10].

**Proposition 3.10** Given \( u, v \in \mathbb{H} \) then \( \lambda^\theta_\mu S : \lambda^\theta \mathcal{O}(\Omega) \to \lambda^\theta \mathcal{O}(\Omega) \) and \( \lambda^\theta_{u-v} S : \lambda^\theta \mathcal{O}(\Omega) \to \lambda^\theta \mathcal{O}(\Omega) \) are isometric isomorphisms of quaternionic right-Hilbert spaces and

\[
\lambda^\theta_{u-v} S \mid_{\lambda^\theta \mathcal{O}} = e^{2(u-v, \cdot)_\theta d\mu_{u-v}},
\]

where \( \lambda^\theta B\mathcal{O} \) and \( \lambda^\theta B\mathcal{O} \) are the kernel and the projection of \( \lambda^\theta \mathcal{O}(\Omega) \) respectively and \( \lambda^\theta B\mathcal{O} \) and \( \lambda^\theta B\mathcal{O} \) are the kernel and the projection of \( \lambda^\theta \mathcal{O}(\Omega) \).

**Proof** If \( f, g \in \lambda^\theta \mathcal{O}(\Omega) \) then \( \lambda^\theta_\mu S[f], \lambda^\theta_\mu S[g] \in \lambda^\theta \mathcal{O}(\Omega) \), see (17), and

\[
\langle \lambda^\theta_\mu S[f], \lambda^\theta_\mu S[g] \rangle_{\theta \mathcal{O}(\Omega)} = \int_\Omega \lambda^\theta_\mu S[f] \lambda^\theta_\mu S[g] d\mu = \int_\Omega \bar{f}(x) g(x) e^{2(u, x)_\theta d\mu_x}
\]

\[
= \langle f, g \rangle_{\theta \mathcal{O}(\Omega)}.
\]

As \( \lambda^\theta_\mu S : \lambda^\theta \mathcal{O}(\Omega) \to \lambda^\theta \mathcal{O}(\Omega) \) is a bijective quaternionic right-linear operator that preserves the inner product we conclude that \( \lambda^\theta \mathcal{O}(\Omega) \) is a copy of \( \mathcal{O} \), as quaternionic
right-Hilbert spaces, and (21) is used to prove the relationships between the reproducing kernels and between the projections. For the rest of conclusions note that

\[(\vartheta, S)^{-1} = \vartheta_{-v} S\]

and

\[\vartheta_{u-v} S = \vartheta_{v} S \circ \vartheta_{u} S.\]

\[\square\]

**Remark 7**

1. If there exists \(\lambda \in \mathbb{R}\) such that \(\Omega \subset \{z \in \mathbb{H} \mid \langle u, z \rangle_{\vartheta} < \lambda\}\). Then \(\vartheta_{u} A(\Omega) \subset \vartheta_{\lambda} A(\Omega)\) as function sets.
2. If there exists \(\lambda \in \mathbb{R}\) such that \(\Omega \subset \{z \in \mathbb{H} \mid \langle u, z \rangle_{\vartheta} > \lambda\}\). Then \(\vartheta_{\lambda} A(\Omega) \subset \vartheta_{u} A(\Omega)\) as function sets.
3. If \(\Omega\) is a bounded domain hence \(\vartheta_{\lambda} A(\Omega) = \vartheta_{u} A(\Omega)\) as function sets.

4 The \((\theta, u)\)–Hyperholomorphic Bergman Type Spaces in Domains of \(\mathbb{C}^2\)

To illustrate the main ideas and motivation of this paper, we develop a theory of Bergman spaces for certain family of holomorphic function on the setting of bounded smooth domains in \(\mathbb{C}^2\), equipped with the topology induced by its usual norm.

**Definition 4.1** Let \(\Omega \subset \mathbb{C}^2 \cong \mathbb{H}\) be a domain and set \(\alpha, \beta \in \mathbb{C}\). The real linear space of \((\alpha, \beta)\)–holomorphic functions on \(\Omega\), denoted by \(\alpha, \beta Hol(\Omega, \mathbb{C})\), consists of all \(f \in C^{1}(\Omega, \mathbb{C})\) such that

\[\frac{\partial f}{\partial \bar{z}_1} = -\alpha f \quad \frac{\partial f}{\partial \bar{z}_2} = -\beta f.\]

(22)

This definition derives from the identity

\[\theta D_{r,(\alpha+i\epsilon^j \beta)}[f] = 0\]

with \(f \in C^{1}(\Omega, \mathbb{C})\).

Note that solutions of (22) can be thought as \((\theta, u)\)–hyperholomorphic functions, when \(u = \alpha + i\epsilon^j \beta\) using (13).

It is the fact that \(\alpha, \beta Hol(\Omega, \mathbb{C}) = Hol(\Omega, \mathbb{C})\) that makes notation \(\alpha, \beta Hol(\Omega, \mathbb{C})\) allowable for a natural generalization of the space of holomorphic functions.

Application of \(\theta D_{r,(\alpha+i\epsilon^j \beta)}\) to \(g \in C^{1}(\Omega, \mathbb{C})\) gives

\[\theta D_{r,(\alpha+i\epsilon^j \beta)}[g] = \frac{\partial g}{\partial \bar{z}_1} + \alpha g + i\epsilon^j \beta \left(\frac{\partial \bar{g}}{\partial \bar{z}_2} + \beta \bar{g}\right)\]
and hence \( g \in \text{Ker } \theta D_{r,(\alpha+ie^\theta j \beta)} \cap C^1(\Omega, \mathbb{C}) \) if and only if
\[
\frac{\partial g}{\partial \bar{z}_1} = -\alpha g \\
\frac{\partial g}{\partial z_2} = -\beta g.
\]
(23)

Unfortunately, here appears implicitly a condition of non-homogeneous anti-holomorphy.

Now we shall consider Stokes and Borel-Pompieu types formulas induced by equations (22) and (23). Also Cauchy type formula for \( \alpha, \beta \in \text{Hol}(\Omega, \mathbb{C}) \) and a kind inverse of operator \( \left( \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_2} \right) \) are presented.

**Corollary 4.2** Let \( \Omega \subset \mathbb{C}^2 \) be a bounded domain with \( \partial \Omega \) a 3-dimensional compact sufficiently smooth surface.

1. **Integral Stokes formula**
\[
\int_{\partial \Omega} f v_1 g = \int_{\Omega} \left[ \frac{\partial f}{\partial \bar{z}_1}(z) + \alpha f(z) g(z) + f(z) \left( \frac{\partial g}{\partial \bar{z}_1}(z) + \alpha g(z) \right) \right] e^{Re(\alpha \bar{z}_1 + \beta \overline{z}_2)} d\mu_z
\]
and
\[
\int_{\partial \Omega} \bar{f} v_2 g = \int_{\Omega} \left[ \frac{\partial f}{\partial z_2}(z) + \beta f(z) g(z) + f(z) \left( \frac{\partial g}{\partial z_2}(z) + \beta g(z) \right) \right] e^{Re(\alpha \bar{z}_1 + \beta \overline{z}_2)} d\mu_z,
\]
for all \( f, g \in C^1(\Omega, \mathbb{C}) \cap C(\overline{\Omega}, \mathbb{C}) \), where \( z = z_1 + ie^\theta j z_2 \in \Omega \). The complex differential forms \( v_1 \) and \( v_2 \) are such that \( v^\theta_{\alpha+ie^\theta j \beta} = v_1 + ie^{i\theta} j v_2 \).

2. **Borel-Pompieu Formula**
\[
\int_{\partial \Omega} \left( K_1(\xi - z) \sigma_1(\xi) + K_2(\xi - z) \sigma_2(\xi) \right) f(\xi)
- \int_{\Omega} \left( K_1(\xi - z) \left( \frac{\partial f}{\partial \bar{z}_1}(\xi) + \alpha f(\xi) \right) + K_2(\xi - z) \left( \frac{\partial f}{\partial z_2}(\xi) + \beta f(\xi) \right) \right) d\mu_\xi
\]
= \{ f(z), z \in \Omega, 0, z \in \mathbb{H} \setminus \overline{\Omega}, \}
for all \( f \in C^1(\Omega, \mathbb{C}) \cap C(\overline{\Omega}, \mathbb{C}) \), where
\[
K_1(\xi - z) = \frac{e^{Re(\bar{\alpha}(\xi_1 - z_1) + \bar{\beta}(\xi_2 - z_2))}}{2\pi^2 |\xi - z|^4}(\bar{\xi}_1 - z_1),
\]
\[
K_2(\xi - z) = \frac{e^{Re(\bar{\alpha}(\xi_1 - z_1) + \bar{\beta}(\xi_2 - z_2)) - 2i\theta}}{2\pi^2 |\xi - z|^4}(\bar{\xi}_2 - z_2).
\]
and $\sigma_1, \sigma_2$ are complex differential forms such that $\sigma_\zeta^\theta = \sigma_1(\zeta) + i e^{i\theta} j \sigma_2(\zeta)$ and $\zeta = \zeta_1 + i e^{i\theta} j \zeta_2$

3. **Cauchy type formula in $\alpha,\beta Hol(\Omega, \mathbb{C})$.**

$$
\int_{\partial \Omega} \left( K_1(\zeta - z)\sigma_1(\zeta) + K_2(\zeta - z)\sigma_2(\zeta) \right) f(\zeta) = \begin{cases}
  f(z), & z \in \Omega, \\
  0, & z \in \mathbb{H} \setminus \Omega,
\end{cases}
$$

for $f \in \alpha,\beta Hol(\Omega)$. Hence, the pair $(K_1, K_2)$ are our reproducing functions.

4. **(A kind of inverse operators of $\left( \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_2} \right)$).** For any $f \in L^2(\Omega, \mathbb{C}) \cup C(\Omega, \mathbb{C})$ there holds that

$$
\frac{\partial}{\partial \bar{z}_1} \alpha,\beta T_1(f) + \frac{\partial}{\partial z_2} \alpha,\beta \bar{T}_2(f) = f
$$

and

$$
-\frac{\partial}{\partial \bar{z}_1} \alpha,\beta T_2(f) + \frac{\partial}{\partial z_2} \alpha,\beta \bar{T}_1(f) = 0,
$$

on $\Omega$, where

$$
\alpha,\beta T_1(f)(z) = \int_{\Omega} K_1(\zeta - z) f(\zeta) d\mu_\zeta
$$

$$
\alpha,\beta T_2(f)(z) = \int_{\Omega} K_2(\zeta - z) \bar{f}(\zeta) d\mu_\zeta,
$$

where $\zeta = \zeta_1 + i e^{i\theta} j \zeta_2$, $z = z_1 + i e^{i\theta} j z_2 \in \Omega$, with $\zeta_k, z_k \in \mathbb{C}$ for $k = 1, 2$.

**Proof** Items 1., 2. and 3. are consequences of Proposition 3.1 and making use of

$$
\langle \alpha + i e^{i\theta} j \beta, z_1 + i e^{i\theta} j z_2 \rangle = \left( \frac{\alpha + \bar{\alpha}}{2} \right) \left( \frac{z_1 + \bar{z}_1}{2} \right) + \left( \frac{\alpha - \bar{\alpha}}{2i} \right) \left( \frac{z_1 - \bar{z}_1}{2i} \right)
$$

$$
+ \left( \frac{\beta + \bar{\beta}}{2} \right) \left( \frac{z_2 + \bar{z}_2}{2} \right) + \left( \frac{\beta - \bar{\beta}}{2i} \right) \left( \frac{z_2 - \bar{z}_2}{2i} \right)
$$

$$
= \frac{1}{2} \left( \alpha \bar{z}_1 + \bar{\alpha} z_1 + \beta \bar{z}_2 + \bar{\beta} z_2 \right)
$$

$$
= \text{Re} \left( \bar{\alpha} z_1 + \bar{\beta} z_2 \right)
$$

and

$$
d\lambda_{\alpha + i e^{i\theta} j \beta}^\theta(z) = e^{\text{Re}(\bar{\alpha} z_1 + \bar{\beta} z_2)} d\mu_z.
$$

Finally, item 4. follows by Proposition 3.3 and Remark 3. \qed
Remark 8 Equation (24) can be rewritten as follows:

\[
\begin{pmatrix}
\frac{\partial}{\partial \bar{z}_1} & \frac{\partial}{\partial \bar{z}_2} \\
\frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1}
\end{pmatrix}
\begin{pmatrix}
\alpha, \beta T_1(f) \\
\alpha, \beta T_2(f)
\end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}
\]
on \Omega, for any \( f \in L_2(\Omega, \mathbb{C}) \cup C(\Omega, \mathbb{C}) \).

Corollary 4.3 Consider \( \alpha, \beta, \chi, \xi \in \mathbb{C} \) and \( \Omega, \Xi \subset \mathbb{C}^2 \approx \mathbb{H} \) such that \( T(\Xi) = \Omega \), where \( T \) is given by (6) for \( c = 0 \) and \( d \in \mathbb{C} \).

Denote \( \zeta_1 + i e^{i \theta} j \zeta_2 = T(z_1 + i e^{i \theta} j z_2) \in \Omega \) for \( z_1 + i e^{i \theta} j z_2 \in \Xi \). Then \( f \in C^1(\Omega, \mathbb{C}) \) satisfies

\[
\begin{align*}
\frac{\partial f}{\partial \bar{\zeta}_1} &= -\delta_1 f \\
\frac{\partial f}{\partial \bar{\zeta}_2} &= -\delta_2 f.
\end{align*}
on \Omega \text{ if and only if}
\]

\[
\begin{align*}
\frac{\partial (e^{Re((\chi - \alpha)z_1 + (\xi - \beta)z_2)} A_T f \circ T)}{\partial \bar{z}_1} &= -\alpha e^{Re((\chi - \alpha)z_1 + (\xi - \beta)z_2)} A_T f \circ T \\
\frac{\partial (e^{Re((\chi - \alpha)z_1 + (\xi - \beta)z_2)} A_T f \circ T)}{\partial \bar{z}_2} &= -\beta e^{Re((\chi - \alpha)z_1 + (\xi - \beta)z_2)} A_T f \circ T.
\end{align*}
on \Xi, \text{ where } \delta_T = (B_T \circ T)^{-1}(\chi + i e^{i \theta} j \xi)(A_T \circ T^{-1}) = \delta_1 + i e^{i \theta} \delta_2 \text{ and } \delta_1, \delta_2 : \Omega \to \mathbb{C}.
\]

Proof We have only to use the fact that conditions on \( T \) imply \( A_T \) is a complex valued function and application of Proposition 3.4. \( \square \)

We will present some properties of the Bergman space induced by (22).

Definition 4.4 The complex linear space \( \alpha, \beta A(\Omega) = \alpha, \beta \text{Hol}(\Omega, \mathbb{C}) \cap L_2(\Omega, \mathbb{C}) \) is called \( \alpha, \beta \)-holomorphic Bergman space associated to \( \Omega \) and denote

\[
\|f\|_{\alpha, \beta A(\Omega)} = \left( \int_{\Omega} |f|^2 d\mu \right)^{\frac{1}{2}}, \quad \langle f, g \rangle_{\alpha, \beta A(\Omega)} = \int_{\Omega} f \bar{g} d\mu,
\]
for all \( f, g \in \alpha, \beta A(\Omega) \).

Remark 9 Due to \( \alpha, \beta A(\Omega) \) is a closed subset of \( \alpha + i e^{i \theta} \beta A(\Omega) \) and that

\[
\langle f, g \rangle_{\alpha, \beta A(\Omega)} = \langle f, g \rangle_{\alpha + i e^{i \theta} \beta A(\Omega)}
\]
for all \( f, g \in \alpha, \beta A(\Omega) \) we obtain that \( (\alpha, \beta A(\Omega), \langle \cdot, \cdot \rangle_{\alpha, \beta A(\Omega)}) \) is a complex Hilbert space.

As the valuation functional is bounded on \( \alpha+ie^\theta \beta A(\Omega) \) so it is bounded on \( \alpha, \beta A(\Omega) \). Therefore, Riesz representation theorem for complex linear spaces establishes the existence of a reproducing kernel \( \alpha, \beta B_\Omega : \Omega \times \Omega \to \mathbb{C} \) such that

\[
\int_\Omega \alpha, \beta B_\Omega(z, w) f(w) d\mu_w, \quad \forall f \in \alpha, \beta A(\Omega)
\]

and the Bergman projection associated to \( \alpha, \beta A(\Omega) \) is given by

\[
\alpha, \beta B_\Omega[f](z) = \int_\Omega \alpha, \beta B_\Omega(z, \zeta) f(\zeta) d\mu_\zeta, \quad \forall f \in L^2(\Omega, \mathbb{C}).
\]

The space \( 0, 0 A(\Omega) \) was studied in [10]. Thus this paper is an extension, preserving the structure, of [10].

The following properties of \( \alpha, \beta B_\Omega \) and \( \alpha, \beta B_\Omega \) can be directly verified. This follows by the same method as in [10].

1. \( \alpha, \beta B_\Omega \) is hermitian in complex sense.
2. The mapping \( z \mapsto \alpha, \beta B_\Omega(z, \zeta) \) belongs to \( \alpha, \beta Hol(\Omega, \mathbb{C}) \) for each \( \zeta \in \Omega \) fix and \( h(\zeta) = \alpha B_\Omega(z, \zeta) \) satisfies

\[
\frac{\partial h}{\partial \zeta_1} = -\bar{\alpha} h,
\quad \frac{\partial h}{\partial \zeta_2} = -\bar{\beta} h,
\]

on \( \Omega \) for each \( z \in \Omega \).
3. \( \alpha, \beta B_\Omega(\cdot, \cdot) \) is the unique reproducing kernel holding the previous properties.
4. \( \alpha, \beta B_\Omega \) is a continuous symmetric operator such that

\[
\alpha, \beta B_\Omega[L^2(\Omega, \mathbb{C})] = \alpha, \beta A(\Omega),
\]

\[
(\alpha, \beta B_\Omega)^2 = \alpha, \beta B_\Omega.
\]

**Corollary 4.5** Let \( \Omega, \Xi \subset \mathbb{C}^2 \approx \mathbb{H} \) such that \( T(\Xi) = \Omega \), where \( T \) is given by (6) such that and \( c = 0 \) and \( d \in \mathbb{C} \). Given \( \alpha, \beta, \chi, \xi \in \mathbb{C} \). Consider the following Bergman type space

\[
\delta_T A(\Omega) = \delta_1, \delta_2 Hol(\Omega, \mathbb{C}) \cap L^2(\Omega, \mathbb{C})
\]

and

\[
\alpha, \beta A_{\gamma_T}(\Xi) = \alpha, \beta Hol(\Xi, \mathbb{C}) \cap L^2, \gamma_T(\Xi, \mathbb{C}).
\]
Recall that
\[ \delta_T = (B_T \circ T)^{-1}(\chi + ie^{i\theta} j \xi)(A_T \circ T^{-1}) = \delta_1 + ie^{i\theta} \delta_2, \]
\[ y_T(z) = e^{-Re((\chi-\alpha)z_1 + (\xi-\beta)z_2)} \rho_T(z). \]

Then
\[ J_T : e^{Re((\chi-\alpha)(z_1 + \zeta_1) + (\xi-\beta)(z_2 + \zeta_2))} CT M \circ WT |_{\delta_1, \delta_2 A(\Omega)} : \delta_1, \delta_2 A(\Omega) \to \alpha, \beta A y_T (\Xi) \]
is an isometric isomorphism of complex Hilbert spaces, where \( C_T, \rho_T \) are given by (12).

In addition, denoting \( \zeta_1 + ie^{i\theta} j \xi_2 = T(z_1 + ie^{i\theta} j \xi_2) \in \Omega \) for \( z_1 + ie^{i\theta} j \xi_2 \in \Xi \) we have that
\[ \alpha, \beta B_\Xi, y_T (z, \xi) = e^{Re((\chi-\alpha)(z_1 + \zeta_1) + (\xi-\beta)(z_2 + \zeta_2))} C_T(z) \delta_1, \delta_2 B_\Omega(T(z), T(\xi)) \overline{C_T(\xi)}, \]
\[ \alpha, \beta B_\Xi, y_T = J_T \circ \delta_1, \delta_2 B_\Omega \circ (J_T)^{-1}, \]
where \( \alpha, \beta B_\Xi, y_T \) and \( \delta_1, \delta_2 B_\Omega \) are the Bergman kernels of \( \alpha, \beta A y_T (\Xi) \) and \( \delta_1, \delta_2 A(\Omega) \), respectively, meanwhile \( \alpha, \beta B_\Xi, y_T \) and \( \delta_1, \delta_2 B_\Omega \) are the Bergman projections of \( \alpha, \beta A y_T (\Xi) \) and \( \delta_1, \delta_2 A(\Omega) \) respectively.

**Proof** It suffices to use Proposition 3.4 together with the observation that \( C_T \) is a \( \mathbb{C} \)-valued function.

Finally, we shall introduce another weighted Bergman type spaces in two complex variables induce by the system (22).

**Definition 4.6** Given a domain \( \Omega \subset \mathbb{C}^2 \approx \mathbb{H} \) and \( \alpha, \beta \in \mathbb{C} \). Set \( A_{\alpha,\beta}(\Omega) \) consists of \( f \in \alpha, \beta Hol(\Omega, \mathbb{C}) \) such that
\[ \int_\Omega |f(z)|^2 t_{\alpha, \beta}(z) d\mu_z < \infty \]
where \( t_{\alpha, \beta}(z) = e^{Re(\overline{\alpha} z_1 + \overline{\beta} z_2)} \) and \( z = z_1 + ie^{i\theta} j z_2 \) and define
\[ \|f\|_{A_{\alpha, \beta}(\Omega)}^2 := \int_\Omega |f(z)|^2 t_{\alpha, \beta}(z) d\mu_z, \quad (f, g)_{A_{\alpha, \beta}(\Omega)} := \int_\Omega \overline{f(z)} g(z) t_{\alpha, \beta}(z) d\mu_z, \]
\[ \alpha, \beta P[f](z) := e^{Re(\overline{\alpha} z_1 + \overline{\beta} z_2)} f(z), \quad \forall z = z_1 + ie^{i\theta} j z_2 \in \Omega, \]
for all \( f, g \in A_{\alpha, \beta}(\Omega) \).

The particular case \( A_{0,0}(\Omega) := A(\Omega) \) was considered in [10].

**Corollary 4.7** Given \( \alpha, \beta, \chi, \xi \in \mathbb{C} \). Then the operators \( \alpha, \beta P : A_{\alpha, \beta}(\Omega) \to A(\Omega) \) and \( \alpha-\chi, \beta-\xi P : A_{\alpha, \beta}(\Omega) \to A_{\alpha, \beta}(\Omega) \) are isometric isomorphisms of complex Hilbert spaces and
\[ i_{\chi, \xi} B_\Omega(z, \xi) = e^{Re(\overline{\alpha-\chi}(z_1 + \zeta_1) + (\overline{\beta-\xi}(z_2 + \zeta_2) - (\alpha-\chi) z_1 + (\beta-\xi) z_2)} i_{\alpha, \beta} B_\Omega(z, \xi), \]
\[ i_{\chi, \xi} B_\Omega = (\alpha-\chi, \beta-\xi P) \circ i_{\alpha, \beta} B_\Omega \circ (\alpha-\chi, \beta-\xi P), \]
where $\alpha, \beta B_{\Omega}$ and $\alpha, \beta B_{\Omega}$ are the kernel and the projection of $A_{\alpha, \beta} (\Omega)$ respectively, and $\chi, \xi B_{\Omega}$ and $\chi, \xi B_{\Omega}$ are the kernel and the projection of $A_{\chi, \xi} (\Omega)$.

**Proof** It follows from Proposition 3.10. 

**Remark 10**

1. If

$$\Omega \subset \{ z = z_1 + ie^{i\theta} j z_2 \in \mathbb{H} \mid z_1, z_2 \in \mathbb{C}, \Re(\bar{\alpha}z_1 + \bar{\beta}z_2) < 0 \}$$

then $\alpha, \beta A(\Omega) \subset A_{\alpha, \beta} (\Omega)$ as function sets.

2. If

$$\Omega \subset \{ z = z_1 + ie^{i\theta} j z_2 \in \mathbb{H} \mid z_1, z_2 \in \mathbb{C}, \Re(\bar{\alpha}z_1 + \bar{\beta}z_2) > 0 \}$$

then $A_{\alpha, \beta} (\Omega) \subset \alpha, \beta A(\Omega)$ as function sets.

3. If $\Omega$ is a bounded domain then $A_{\alpha, \beta} (\Omega) = \alpha, \beta A(\Omega)$ as function sets.

5 Concluding Remarks

This paper is an extension of previous studies developed in [10, 19, 20]. Our contribution here was the design of a methodology for investigation of Bergman spaces for certain perturbed conventional function theories, which are associated with the solutions of first order linear partial differential equation systems and can be embedded into one of the $(\theta, u)$—hyperholomorphic functions classes. It seems reasonable to expect that this viewpoint will prove to be rather promising and long-range. A discussion on the specific example of $(\alpha, \beta)$—holomorphic functions in two complex variables that such embedding proves to be quite useful for the study of Bergman type spaces in domains of $\mathbb{C}^2$ has been illustrative.

There exists another conventional function theory to be considered. This is the case of the Cimmino system, which was introduced in 1941 by G. Cimmino [5], see also [6]. The next step to our work would be to develop of Bergman spaces theory induced by the Cimmino system, whose solutions form a proper subset of the $\pi/2$—hyperholomorphic functions class.

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