The Krause-Hegselmann Consensus Model
with Discrete Opinions

Santo Fortunato

Fakultät für Physik, Universität Bielefeld, D-33501 Bielefeld, Germany

e-mail: fortunat@physik.uni-bielefeld.de

Abstract

The consensus model of Krause and Hegselmann can be naturally
extended to the case in which opinions are integer instead of real
numbers. Our algorithm is much faster than the original version and
thus more suitable for applications. For the case of a society in which
everybody can talk to everybody else, we find that the chance to reach
consensus is much higher as compared to other models; if the number
of possible opinions $Q \leq 7$, in fact, consensus is always reached, which
might explain the stability of political coalitions with more than three
or four parties. For $Q > 7$ the number $S$ of surviving opinions is
approximately the same independently of the size $N$ of the population,
as long as $Q < N$. We considered as well the more realistic case of a
society structured like a Barabási-Albert network; here the consensus
threshold depends on the outdegree of the nodes and we find a simple
scaling law for $S$, as observed for the discretized Deffuant model.

Keywords: Sociophysics, Monte Carlo simulations, scale free networks.

1 Introduction

Can statistical mechanics help to describe opinion dynamics? The last few
years have witnessed several attempts in this direction [1, 2, 3, 4, 5, 6] and
Monte Carlo simulations have become an important part of sociophysics [7],
enlarging the new field of interdisciplinary applications of statistical physics
[8]. The starting point is represented by a random distribution of opinions,
which can be integer or real numbers, among a group of persons, or agents.
Next, some simple dynamical mechanism is introduced so that, due to in-
teractions between the agents, the opinion of each agent changes over the
time until, at some stage, a configuration is attained where the opinions are no longer modified by the dynamics and then remain the same. It is then interesting to study the possible stable opinion configurations at the end of the process to see, for instance, if it is possible that all agents stick to one and the same opinion ("consensus") or whether a polarization around fewer dominant opinions takes place. The most studied consensus models are those of Sznajd [4], Galam [5], Deffuant et al. [2] and the one of Krause and Hegselmann (KH) [3]. The crucial differences between the models are the dynamical mechanisms to update the opinions, but they differ as well in other important aspects. For instance, in the original versions of the models of Deffuant and KH, the opinion variable is a real number between 0 and 1, whereas in the models of Sznajd and Galam it is an integer (this possibility was also studied for Deffuant in [10]). Besides, there are as well differences in the way the topology of the society is conceived. In the Sznajd model the agents sit on a lattice and can have opinion-affecting interactions only with their lattice neighbours; in the models of Deffuant and KH, instead, one assumes a society in which each agent has the same probability to interact with everybody, although recently scale free network topologies have also been considered for Deffuant [9, 10]. Among the above-mentioned consensus models, those of Sznajd and Deffuant are meanwhile quite well-known, as the convergence to the final configuration is relatively quick, which allows to simulate populations with millions of agents [6].

In this paper we focus instead on the KH model, which has not been investigated by many people so far. In its original version [3] one introduces a real parameter $\epsilon$, called confidence bound. At every Monte Carlo step, the randomly selected agent $i$ with opinion $s_i$ takes the average of the opinions of those agents $j$ such that $|s_i - s_j| < \epsilon$. This averaging process makes the algorithm very time-consuming compared, for instance, to Deffuant, and is essentially the reason why most people of the computational sociophysics community do not find it attractive. However, the real interest behind consensus models is whether they are able to describe real situations, and it is not said that the faster the algorithms the better they are; the Sznajd model could effectively simulate the distribution of votes among candidates in Brazilian and Indian elections [11, 12].

We study here a modified version of the KH algorithm, where opinions take integer values so that each individual has a finite number of possible choices. This is most often the case in real life; thinking for instance about elections, the voters have a limited number of possible parties and/or candi-
dates among which to choose. We have used several values for the number of opinions $Q$ and checked how many different opinions survive in the final configuration. Initially we have assumed a society in which every agent interacts with all the others. This is however quite an unrealistic situation; therefore we have as well checked what happens if the personal relationships within the society form a scale free network, with few people having lots of friends and many having just a few. To build the network we adopted the popular ”rich get richer” strategy proposed by Barabási and Albert \cite{13}.

2 The model

Our opinions can take the values $1, 2, 3, \ldots Q$. As far as the confidence bound $\epsilon$ is concerned, we will assume in this paper that the agents are influenced only by the individuals whose opinions differ by at most one unit from theirs. This corresponds to the case $\epsilon = 1/Q$ in the original KH model. Actually, in order to go smoothly to the continuous limit one should introduce another parameter $L$, which is the discrete confidence bound, i.e. the maximal distance between compatible opinions (for us $L = 1$), and take the limit $L, Q \to \infty$ by keeping $\epsilon = L/Q$ fixed.

The algorithm starts by randomly distributing the opinions among the agents. The use of integer-valued opinions spoils the original KH concept of ”average of compatible opinions”, because such average in most cases would not be an integer.

A possible way out is to re-interpret the spirit of the original KH model in a probabilistic fashion. Suppose we want to update the status of agent $i$, which has opinion $k$. The number of agents with compatible opinions are $n_{k-1}$, $n_k$ and $n_{k+1}$ (respectively for opinions $k-1$, $k$ and $k+1$). If the total number of compatible individuals is $n = n_{k-1} + n_k + n_{k+1}$, we say that agent $i$ takes opinion $k-1$, $k$ or $k+1$ with probability $p_{k-1} = n_{k-1}/n$, $p_k = n_k/n$ and $p_{k+1} = n_{k+1}/n$, respectively. This is to our mind a natural extension of the KH model to discrete opinions, and is the version we have used here. The status of the agents is updated sequentially, in an ordered sweep over the whole population; the program stops if no agent changed opinion during an iteration.

The fact that the opinions are discretized allowed us to speed up the algorithm compared to the continuous case. In the latter the time to complete
an iteration goes as $N^2$ ($N$ is the size of the population), because for each agent to update one needs to make a sweep over the whole population to look for compatible individuals and calculate the average of their opinions. Here only the probabilities $p_k$ matter, so we keep an array where the opinion histogram $n_k$ is stored ($k = 1, 2, ..., Q$). When we update the agent $i$ with opinion $k$, from the histogram we derive directly the probabilities for the agent to take its next opinion. Suppose that agent $i$ takes opinion $j$, what we need to do is to increase $n_j$ and to decrease $n_k$ by one unit, to get the new opinion distribution, after that we can proceed to update a new agent. In this way we avoid to count each time the number of compatible individuals, which is very time consuming, and the time needed to complete the iteration goes as $(2L + 1)N$ (for each agent one needs to make a sweep over the $2L + 1$ compatible opinion channels to get the probabilities, in our case $2L + 1 = 3$).
Figure 2: Fraction of final configurations in which no consensus is reached, as a function of the number $N$ of agents, for $Q = 7, 8$.

With our algorithm systems with millions of agents, unreachable by standard KH, can be simulated (it took us less than six hours to simulate one million agents on a PC).

We found that the convergence to a stable configuration is much slower than in the original KH model. In Fig. 1 we plot the average number of sweeps necessary for convergence in the two cases, as a function of $Q$ (we use the relation $\epsilon = 1/Q$ to make a correspondence between the two models). We see that, except for very high values of $Q$, the number of evolution steps for our algorithm is an order of magnitude higher than for the continuous model. This is most likely due to the stochastic character of our procedure; the opinion distributions vary more slowly if we allow jumps from one opinion channel to the neighbouring ones with some probability instead of systematically shifting every agent to the average channel. Moreover, the standard deviation of the average evolution time is much larger in the discrete than in the continuous model, which hints to the presence of wild fluctuations.
Figure 3: Number of surviving opinions $S$ as a function of $Q$ for a society where everybody interacts with everybody. We averaged over 1000 realizations.

3 Results

The main result of our simulations is the existence of a threshold $Q_c$, such that, for $Q \leq Q_c$, consensus is always reached. This is true for both social topologies we have considered. For a society where each agent has relationships with all others, we find that $Q_c = 7$ as we can see in Fig. 2. Here we plot the probability of having polarization as a function of the number $N$ of agents, for $Q = 7, 8$. By polarization we mean that more than just a single opinion survive in the final configuration. The probability is given by the fraction of configurations with polarization. For $Q = 7$ this probability decreases strongly with $N$ and for $N = 10000$ all samples presented a single final opinion. For $Q = 8$, instead, the probability for polarization is basically one for $N = 10000$. We remark that our threshold is higher than in the Sznajd model, where $Q_c$ lies between 3 and 4 [6], and in the discretized version.
of Deffuant\(^1\), where \(Q_c = 2\). This shows that the dynamics of the KH model is the most suitable to explain how competing factions can find an agreement and to justify the stability of political coalitions with several parties like in Italy.

In Fig. 3 we show how the number of surviving opinions \(S\) varies with \(Q\), for several \(N\)’s. We see that, as long as \(Q \ll N\), so that no finite size effects take over, \(S\) is approximately the same independently of the number of agents; the result holds for the Deffuant model as well \[14\].

Let us now check what happens if we put the agents on a scale free network a la Barabási-Albert. To build the network we must specify the outdegree \(m\) of the nodes, i.e. the number of edges which originate from a node. The procedure is dynamic; one starts from \(m\) nodes which are all connected to each other and adds further \(N - m\) nodes one at a time. When

\(^1\)In the original continuous versions the transition value of the confidence bound for consensus is \(\epsilon_c \sim 0.4\) for Deffuant and \(\epsilon_c \sim 0.21\) (our estimate) for KH.
a new node is added, it selects $m$ of the preexisting nodes as neighbours, so that the probability to get linked to a node is proportional to the number of its neighbours. In all networks created in this way the number of agents with degree $k$, i.e. having $k$ neighbours, is proportional to $1/k^3$ for $k$ large, independently of $m$.

In our simulations we took the network as undirected, so that communication between two neighbouring agents can take place in both directions. We find again that there is a threshold $Q_c$ for the system to evolve to complete consensus. Interestingly, $Q_c$ depends on the outdegree $m$, as shown in Fig. 4. In fact, the final stable configurations are those in which each agent is surrounded only by agents which share its opinion or are incompatible, a solution of a special graph colouring problem; only in this case each agent will maintain its opinion in the future with probability one. If $m$ is small the average degree is small and it is easier to reach such configurations even when just a few opinions are available. Looking at Fig. 4 we see that $Q_c = 3, 4$
Figure 6: Scaled plot of the same data as in Fig. 5. Apart from deviations for very small $Q$, where $S$ is the same independently of $N$, a reasonable scaling is observed. The two straight lines represent the two situations where few opinions survive (horizontal) and every agent keeps its own opinion (skew).

for $m = 1, 2$, respectively\(^2\). No polarization is possible for $Q = 2$ because, the network being connected, there would be at least two clusters sharing a border. As each agent has at least $m$ neighbours by construction, for large $m$ we expected to reach the threshold $Q_c = 7$ that we have found in the case in which everybody is connected to everybody; this is indeed true for $m$ larger than about 40.

The analysis of the number of surviving opinions $S$ is instead relatively independent of the outdegree $m$. We chose $m = 3$, in order to make comparisons with corresponding results for the discretized Deffuant model \([10]\). Fig. 5 shows $S$ as a function of $Q$ for different population sizes. The pattern looks very similar to the one observed in \([10]\) for Deffuant: for $Q$ not too small, $S$

\(^2\)As a matter of fact, in the special case $m = 1$ we find that consensus is not complete for $Q = 3$, but is reached in about 80% of the cases.
equals $Q$ and only when $Q$ gets close to $N$, finite size effects take over and $S$ converges towards $N$. In the latter case, the simulation stops very early because most agents have different opinions and therefore the chance for an agent to change its mind is small. In [10] a simple scaling behaviour of $S$ with $Q$ and $N$ was observed. The ansatz was

$$S = (Q - 1)f(Q/N); \quad f(x \to 0) = 1, f(x \to \infty) = 1/x \quad (1)$$

In Fig. 6 we rescaled the data of Fig. 5 according to the ansatz of Eq. 1. The ”teeth” below the horizontal line refer to small values of $Q$, and we know that here $S$ equals one or is close to one, independently of $N$; otherwise the scaling is good.

4 Conclusions

We have studied an extension of the Krause-Hegselmann consensus model to integer-valued opinions, both when all agents talk to each other and when they sit on the nodes of an undirected Barabási-Albert network. We assumed that only agents with opinions differing by at most one unit can influence each other. A non-trivial implementation is necessarily probabilistic and many more iterations are required for convergence compared to the standard model. On the other hand, our algorithm is much faster than the continuous version and therefore larger population sizes can be explored. In a society where each agent can interact with all others, when no more than seven different opinions/positions are possible, the system always evolves towards consensus if the size $N$ of the population is large enough (of the order of $10^4$ agents or more). On the other hand, on a network-structured society, the threshold $Q_c$ depends on the minimal number of friends an agent can have; if this minimum is less than 30, consensus is more difficult and $Q_c$ can lower up to 3. This versatility of the model makes it more suitable than others in order to explain how consensus can be reached in a variety of situations. In Italy, for instance, the 80’s were the years of the so-called ”pentapartito”, a government’s coalition of five parties. The stability of this coalition could be justified neither by the Sznajd model nor by Deffuant, but it is natural in our model (although the concept of political stability in a country where fifty-seven governments alternated in fifty-eight years is questionable!).
In a fully connected society, the number of final opinions \( S \) is an intensive quantity, i.e. independent of \( N \) for large \( N \).

On a network \( S \) grows with \( N \) if \( Q \) and \( N \) increase so that the ratio \( Q/N \) is constant, but is intensive if \( Q \) is kept fixed when \( N \to \infty \). When \( Q \) is not too small \( S \) is well described by the same simple scaling function that reproduces the data for the discretized Deffuant model [10]. Moreover, we find that \( S = Q \) for \( Q \) of the order of ten or larger, so in a realistic society with more than ten different opinions/positions, the dynamics of the KH model is unable to suppress any of them.

Acknowledgements

I am indebted to D. Stauffer for introducing me into this fascinating field and for many suggestions and comments. I gratefully acknowledge the financial support of the DFG Forschergruppe under grant FOR 339/2-1.

References

[1] R. Axelrod, J. Conflict Resolut. 41, 203 (1997).

[2] G. Deffuant, D. Neau, F. Amblard and G. Weisbuch, Adv. Complex Syst. 3, 87 (2000); G. Weisbuch, G. Deffuant, F. Amblard, and J.-P. Nadal, Complexity 7, 2002; G. Deffuant, F. Amblard, G. Weisbuch and T. Faure, Journal of Artificial Societies and Social Simulations 5, issue 4, paper 1 (jasss.soc.surrey.ac.uk) (2002).

[3] R. Hegselmann and U. Krause, Journal of Artificial Societies and Social Simulation 5, issue 3, paper 2 (jasss.soc.surrey.ac.uk) (2002) and Physics A, in press (2004); U. Krause, Soziale Dynamiken mit vielen interakteuren. Eine Problemskizze. In U. Krause and M. Stöckler (Eds.), Modellierung und Simulation von Dynamiken mit vielen interagierenden Akteuren, 37-51, Bremen University, Jan. 1997.

[4] K. Sznajd-Weron and J. Sznajd, Int. J. Mod. Phys. C 11, 1157 (2000).

[5] S. Galam, J. Stat. Phys. 61, 943 (1990) and Physica A 238, 66 (1997).

[6] D. Stauffer, The Monte Carlo Method on the Physical Sciences, edited by J. E. Gubernatis, AIP Conf. Proc. 690, 147 (2003), cond-mat/0307133
[7] W. Weidlich, *Sociodynamics; A Systematic Approach to Mathematical Modelling in the Social Sciences*. Harwood Academic Publishers, 2000.

[8] S. Moss de Oliveira, P. M. C. de Oliveira and D. Stauffer, *Evolution, Money, War and Computers*, Teubner, Stuttgart and Leipzig (1999).

[9] D. Stauffer and H. Meyer-Ortmanns, cond-mat/0308231, Int. J. Mod. Phys. C 15, issue 2; G. Weisbuch, cond-mat/0311279, Eur. Phys. J. B in press.

[10] D. Stauffer, A. O. Sousa and C. Schulze, cond-mat/0310243, accepted for J. Artificial Societies and Social Simulation, June 2004.

[11] A. T. Bernardes, D. Stauffer and J. Kertész, Eur. Phys. J. B 25, 123 (2002), cond-mat/0111147

[12] M. C. Gonzalez, A. O. Sousa and H. J. Herrmann, Int. J. Mod. Phys. C 15, No. 1 (2004), cond-mat/0307537

[13] R. Albert and A. L. Barabási, Rev. Mod. Phys. 74, 47 (2002).

[14] E. Ben-Naim, P. Krapivsky and S. Redner, Physica D 183, 190-204 (2003).