A Liouville comparison principle for solutions of semilinear parabolic second-order partial differential inequalities.

Vasilii V. Kurta.

May 2, 2014

Abstract

We obtain a new Liouville comparison principle for entire weak solutions \((u, v)\) of semilinear parabolic second-order partial differential inequalities of the form

\[
 u_t - \mathcal{L}u - |u|^{q-1}u \geq v_t - \mathcal{L}v - |v|^{q-1}v
\]  

in the half-space \(\mathbb{S} = \mathbb{R}^1_+ \times \mathbb{R}^n\). Here \(n \geq 1, q > 0\) and

\[
 \mathcal{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[ a_{ij}(t, x) \frac{\partial}{\partial x_j} \right],
\]

where \(a_{ij}(t, x), i, j = 1, \ldots, n,\) are functions defined, measurable and locally bounded in \(\mathbb{S}\), and such that \(a_{ij}(t, x) = a_{ji}(t, x)\) and

\[
 \sum_{i,j=1}^{n} a_{ij}(t, x) \xi_i \xi_j \geq 0
\]

for almost all \((t, x) \in \mathbb{S}\) and all \(\xi \in \mathbb{R}^n\). The critical exponents in the Liouville comparison principle obtained, which responsible for the non-existence of non-trivial (i.e., such that \(u \not\equiv v\)) entire weak solutions to (*) in \(\mathbb{S}\), depend on the behaviour of the coefficients of the operator \(\mathcal{L}\) at infinity. As direct corollaries we obtain a new Fujita comparison principle for entire weak solutions \((u, v)\) of the Cauchy problem for the inequality (*), as well as new Liouville-type and Fujita-type theorems for non-negative entire weak solutions \(u\) of the inequality (*) in the case when \(v \equiv 0\). All the results obtained are new and sharp.
Introduction and preliminaries

This work is devoted to a new Liouville comparison principle of elliptic type for entire weak solutions to parabolic inequalities of the form

\[ u_t - Lu - |u|^{q-1}u \geq v_t - Lv - |v|^{q-1}v \]  \hspace{1cm} (1)

in the half-space \( \mathbb{S} = (0, +\infty) \times \mathbb{R}^n \), where \( n \geq 1 \) is a natural number, \( q > 0 \) is a real number and \( L \) is a linear second-order partial differential operator in divergence form defined by the relation

\[ L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left[ a_{ij}(t,x) \frac{\partial}{\partial x_j} \right]. \]  \hspace{1cm} (2)

Here and in what follows, we assume that the coefficients \( a_{ij}(t,x), i,j = 1,\ldots,n \), of the operator \( L \) are functions defined, measurable and locally bounded in \( \mathbb{S} \), and such that \( a_{ij}(t,x) = a_{ji}(t,x), i,j = 1,\ldots,n \), for almost all \((t,x) \in \mathbb{S} \). Also, we assume that the corresponding quadratic form satisfies the conditions

\[ 0 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leq A(t,x)|\xi|^2 \]  \hspace{1cm} (3)

for all \( \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n \) and almost all \((t,x) \in \mathbb{S} \), with \( A(t,x) \) some function defined, measurable, non-negative and locally bounded in \( \mathbb{S} \).

It is important to note that if \( u = u(t,x) \) and \( v = v(t,x) \) satisfy inequalities

\[ u_t \geq Lu + |u|^{q-1}u \]  \hspace{1cm} (4)

and

\[ v_t \leq Lv + |v|^{q-1}v, \]  \hspace{1cm} (5)

then the pair \((u,v)\) satisfies the inequality (1). Thus, all the results obtained in this paper for solutions of (1) are valid for the corresponding solutions of the system (4)–(5).

Under entire solutions of inequalities (1), (4) and (5) we understand solutions defined in the whole half-space \( \mathbb{S} \), and under Liouville results of elliptic type for solutions of evolution inequalities (1), (4) and (5) in the half-space \( \mathbb{S} \) we understand Liouville-type results which, in their formulations, have no
restrictions on the behaviour of solutions to these inequalities on the hyper-plane $t = 0$. Also, we would like to underline that we impose neither growth conditions on the behaviour of solutions to inequalities (1), (4) and (5) or on that of any of their partial derivatives at infinity.

In the case when the coefficients of the operator $\mathcal{L}$ are globally bounded in $\mathbb{S}$, a Liouville comparison principle of elliptic type for entire weak solutions $(u, v)$ of the inequality (1), as well as Liouville-type and Fujita-type theorems for non-negative entire weak solutions $u$ of the inequality (4), were obtained in [7]. In those results, a critical exponent which is responsible for the non-existence of non-trivial (i.e., such that $u \neq v$) entire weak solutions $(u, v)$ to the inequality (1), as well as non-trivial (i.e., such that $u \neq 0$) non-negative entire weak solutions $u$ to the inequality (4), coincides with the well-known Fujita critical blow-up exponent for non-trivial non-negative entire classical solutions to the Cauchy problem for the equation

$$u_t - \Delta u = |u|^{q-1}u,$$

which was established in [2], [4] and [8]. However, it is intuitively clear that the character of the behaviour of the coefficients $a_{ij}(t, x)$ of the operator $\mathcal{L}$ as $|x| \to +\infty$ must manifest itself in Liouville-type and Fujita-type results. In particular, a potential critical exponent in a Liouville comparison principle for entire weak solutions of (1), which is responsible for the non-existence of non-trivial entire weak solutions to the inequality (1) must depend on the behaviour of the coefficients of the operator $\mathcal{L}$ as $|x| \to +\infty$.

In order to trace this dependence we consider the value

$$A(R) = \text{ess sup}_{(t,x) \in (0, +\infty) \times \{R/2 < |x| < R\}} A(t, x)$$

for any $R > 0$ and assume that the coefficients of the operator $\mathcal{L}$ satisfy the condition

$$A(R) \leq cR^{2-\alpha},$$

with some real constant $\alpha$ and some real positive constant $c$, for all $R > 1$. It is clear that if $\alpha < 2$, then the coefficients of the operator $\mathcal{L}$ may be unbounded in $\mathbb{S}$, if $\alpha = 2$, the coefficients of the operator $\mathcal{L}$ are globally bounded in $\mathbb{S}$, and if $\alpha > 2$, they must vanish as $|x| \to +\infty$. Our main concern in this paper is the cases when $\alpha \neq 2$.

We also introduce a special function space, which is directly associated to the linear partial differential operator $\mathcal{P} = \frac{\partial}{\partial t} - \mathcal{L}$, and assume that entire
weak solutions of inequalities (1), (4) and (5) belong to this function space only locally in $S$.

2 Definitions

**Definition 1** Let $n \geq 1$, $q > 0$ and $\hat{q} = \max\{1, q\}$, let $\mathcal{L}$ be a differential operator defined by (2), and let $\Pi$ be an arbitrary bounded domain in $S$. By $W^{\mathcal{L}, \hat{q}}(\Pi)$ we denote the completion of the function space $C^\infty(\Pi)$ with respect to the norm

$$
\|w\|_{W^{\mathcal{L}, \hat{q}}(\Pi)} = \int_\Pi |w_t| dtdx + \left( \int_\Pi \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} dtdx \right)^{1/2} + \left( \int_\Pi |w|^{\hat{q}} dtdx \right)^{1/\hat{q}},
$$

where $C^\infty(\Pi)$ is the space of all functions defined and infinitely differentiable in $\Pi$.

**Definition 2** Let $n \geq 1$ and $q > 0$, and let $\mathcal{L}$ be a differential operator defined by (2). A function $w = w(t, x)$ belongs to the function space $W^{\mathcal{L}, q}_{\text{loc}}(S)$ if $w$ belongs to $W^{\mathcal{L}, \hat{q}}(\Pi)$ for any bounded domain $\Pi$ in $S$.

**Definition 3** Let $n \geq 1$ and $q > 0$, and let $\mathcal{L}$ be a differential operator defined by (2). A pair $(u, v)$ of functions $u = u(t, x)$ and $v = v(t, x)$ is called an entire weak solution to the inequality (1) in $S$, if these functions are defined and measurable in $S$, belong to the function space $W^{\mathcal{L}, q}_{\text{loc}}(S)$ and satisfy the integral inequality

$$
\int_S \left[ u_t \varphi + \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} - |u|^{q-1} u \varphi \right] dtdx \geq

\int_S \left[ v_t \varphi + \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \varphi}{\partial x_i} \frac{\partial v}{\partial x_j} - |v|^{q-1} v \varphi \right] dtdx
$$

(9)

for every function $\varphi \in C^\infty(S)$ with compact support in $S$, where $C^\infty(S)$ is the space of all functions defined and infinitely differentiable in $S$.

**Remark 1** We understand the inequality (9) in the sense discussed, e.g., in [10] or [15].

Analogous definitions of solutions to inequalities (4) and (5), as special cases of the inequality (1) for $v \equiv 0$ or $u \equiv 0$, follow immediately from Definition 3.
3 Results

**Theorem 1** Let $n \geq 1$, $\alpha > 0$ and $1 < q \leq 1 + \frac{\alpha}{n}$, let $L$ be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given $\alpha$ and some $c > 0$, and let $(u, v)$ be an entire weak solution of the inequality (1) in $\mathbb{S}$ such that $u \geq v$. Then $u = v$ in $\mathbb{S}$.

As we have observed above, since any solutions $u = u(t, x), v = v(t, x)$ of inequalities (4), (5) is a solution $(u, v)$ of the inequality (1), then the following statement is a direct corollary of Theorem 1.

**Theorem 2** Let $n \geq 1$, $\alpha > 0$ and $1 < q \leq 1 + \frac{\alpha}{n}$, let $L$ be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given $\alpha$ and some $c > 0$, and let $u = u(t, x)$ be an entire weak solution of the inequality (4) and $v = v(t, x)$ be an entire weak solution of the inequality (5) in $\mathbb{S}$ such that $u \geq v$. Then $u = v$ in $\mathbb{S}$.

The results in Theorems 1 and 2, which evidently have a comparison principle character, we term a Liouville-type comparison principle, since in particular cases when either $u \equiv 0$ or $v \equiv 0$, it becomes a Liouville-type theorem for solutions of inequality (5) or (4), respectively. We formulate here only the case when $v \equiv 0$.

**Theorem 3** Let $n \geq 1$, $\alpha > 0$ and $1 < q \leq 1 + \frac{\alpha}{n}$, let $L$ be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given $\alpha$ and some $c > 0$, and let $u = u(t, x)$ be a non-negative entire weak solution of the inequality (4) in $\mathbb{S}$. Then $u = 0$ in $\mathbb{S}$.

Since in Theorems 1 and 2 we impose no conditions on the behaviour of entire weak solutions of inequalities (1), (4) and (5) on the hyper-plane $t = 0$, we can formulate, as a direct corollary of the Liouville comparison principle in Theorems 1 and 2, a comparison principle, which in turn one can term a Fujita comparison principle, for entire weak solutions of the Cauchy problem with arbitrary initial data for $u$ and $v$ for inequalities (1), (4) and (5) in $\mathbb{S}$.

**Theorem 4** Let $n \geq 1$, $\alpha > 0$ and $1 < q \leq 1 + \frac{\alpha}{n}$, let $L$ be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given $\alpha$ and some $c > 0$, and let $(u, v)$ be an entire weak solution of the Cauchy problem, with arbitrary initial data for $u = u(t, x)$ and $v = v(t, x)$, for the inequality (1) in $\mathbb{S}$ such that $u \geq v$. Then $u = v$ in $\mathbb{S}$. 
Note that the initial data for \( u = u(t, x) \) and \( v = v(t, x) \) in Theorem 4 may be different.

**Theorem 5** Let \( n \geq 1, \alpha > 0 \) and \( 1 < q \leq 1 + \frac{\alpha}{n} \), let \( L \) be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given \( \alpha \) and some \( c > 0 \), and let \( u = u(t, x) \) be an entire weak solution of the Cauchy problem, with arbitrary initial data, for the inequality (4) and \( v = v(t, x) \) be an entire weak solution of the Cauchy problem, with arbitrary initial data, for the inequality (5) in \( S \) such that \( u \geq v \). Then \( u = v \) in \( S \).

It is clear that in a particular case when \( u \equiv 0 \) or \( v \equiv 0 \), the Fujita comparison principle in Theorems 4 and 5 becomes a Fujita-type theorem for entire weak solutions of the Cauchy problem for inequality (5) or (4), respectively. As before, we formulate here only the case when \( v \equiv 0 \).

**Theorem 6** Let \( n \geq 1, \alpha > 0 \) and \( 1 < q \leq 1 + \frac{\alpha}{n} \), let \( L \) be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given \( \alpha \) and some \( c > 0 \), and let \( u = u(t, x) \) be a non-negative entire weak solution of the Cauchy problem, with arbitrary initial data, for the inequality (4) in \( S \). Then \( u = 0 \) in \( S \).

As we have mentioned above, if the coefficients of the operator \( L \) are globally bounded in \( S \), then the condition (8) for these coefficients is fulfilled with \( \alpha = 2 \) and some constant \( c > 0 \), and, therefore, the results obtained (here we restrict ourselves only with Theorem 1) may be formulated in the following form:

**Corollary 1** Let \( n \geq 1 \) and \( 1 < q \leq 1 + \frac{2}{n} \), let \( L \) be a differential operator defined by (2), the coefficients of which are globally bounded in \( S \), and let \((u, v)\) be an entire weak solution of the inequality (1) in \( S \) such that \( u \geq v \). Then \( u = v \) in \( S \).

So, in a particular case when \( \alpha = 2 \), the critical blow-up exponent in Theorems 1–6 coincides with the well-known Fujita critical blow-up exponent, and the well-known Fujita theorem on blow-up of non-trivial non-negative entire classical solutions to the Cauchy problem with arbitrary initial data for the equation (6) proved in [2], [4] and [8] is a direct corollary of Theorem 6 when \( \alpha = 2 \). Also, as we have mentioned above, similar results to those in Theorems 1–6 when \( \alpha = 2 \) were obtained in [7]. The difference between the
results in Theorems 1–6 when \( \alpha = 2 \) and those obtained in [7] consists of the fact that in the present paper we study solutions to inequalities (1), (4) and (5) in the function space \( \mathcal{W}_{L,q}^{L,\text{loc}}(\mathbb{S}) \) which is, generally speaking, wider than that considered in [7]. Thus, all the results in Theorems 1–6 are new, with new critical blow-up exponents in the cases when \( \alpha \neq 2 \). We demonstrate their sharpness by the following examples.

**Example 1** Let \( n \geq 1, +\infty > \alpha > -\infty \) and \( q \leq 1 \), and let \( \mathcal{L} \) be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given \( \alpha \) and some \( c > 0 \). It is evident that the function \( u(t,x) = e^t \) is a positive entire classical solution of the inequality (4) in \( \mathbb{S} \). Also, it is clear that the function \( v = -u(t,x) \) is a negative entire classical solution of the inequality (5) in \( \mathbb{S} \), and, thus, the pair of the functions \( u = u(t,x) \) and \( v = v(t,x) \) is a non-trivial entire classical solution of the system (4)–(5) and, therefore, of the inequality (1) in \( \mathbb{S} \) such that \( u(t,x) > v(t,x) \).

**Example 2** Let \( n \geq 1, \alpha > 0 \) and \( q > 1 + \frac{\alpha}{n} \). Consider the operator \( \mathcal{L} \) defined by (2) with the coefficients given by the formula

\[
a_{ij}(t,x) = (1 + |x|^2)^{\frac{2-\alpha}{2}} \delta_{ij},
\]

where \( \delta_{ij} \) are Kronecker’s symbols and \( i,j = 1,\ldots,n \). It is easy to see that the condition (8) is fulfilled for these coefficients with the given \( \alpha \) and some \( c > 0 \). Also, for the given \( \alpha \) and \( q \), let \( \beta = \frac{1}{q-1}, \frac{1}{\alpha n(q-1)} < \gamma \leq \left(\frac{1}{\alpha}\right)^2 \), \( 0 < \kappa \leq \left(\alpha n \left(\gamma - \frac{1}{\alpha n(q-1)}\right) \right)^{1/(q-1)} \) and

\[
u(t,x) = \kappa(t+1)^{-\beta} \exp \left( -\gamma \frac{(1 + |x|^2)^{\frac{\alpha}{2}}}{t+1} \right).
\]

Making necessary calculations, it is not difficult to verify that the function \( u = u(t,x) \) defined by the formula (11) is a positive entire classical solution of the inequality (4) in \( \mathbb{S} \), with \( a_{ij}(t,x) \), the coefficients of the operator \( \mathcal{L} \), defined by (10). Also, it is clear that the function \( v = -u(t,x) \) is a negative entire classical solution of the inequality (5) in \( \mathbb{S} \), with \( a_{ij}(t,x) \) in (2) defined by (10), and, thus, the pair of the functions \( u = u(t,x) \) and \( v = v(t,x) \) is a non-trivial entire classical solution of the system (4)–(5) and, therefore, of the inequality (1) in \( \mathbb{S} \) such that \( u(t,x) > v(t,x) \), with \( a_{ij}(t,x) \) in (2) defined by (10).
Note that a positive entire classical sub-solution of the equation (6) in a form similar to that given by the formula (11) with \( \alpha = 2 \) was constructed in [17, p. 283].

**Example 3** Let \( n \geq 1, \alpha \leq 0, q > 1 + \frac{\alpha}{n} \) and \( q > 1 \). Consider the operator \( L \) defined by (2) with the coefficients given by the formula

\[
a_{ij}(t, x) = (1 + |x|^2)^{\frac{2\alpha}{2}} \delta_{ij},
\]

where \( \delta_{ij} \) are Kronecker's symbols and \( i, j = 1, \ldots, n \). As in Example 2, it is easy to see that \( A(R) \leq CR^{2-\alpha} \) for all \( R > 1 \), with \( C \) some positive constant which possibly depends on \( \alpha \) and \( n \), and, therefore, the condition (8) is fulfilled for these coefficients with the given \( \alpha \) and some \( c > 0 \). Also, for the given \( \alpha \) and \( q \), let \( \beta = \frac{1}{q-1}, \alpha n(q-1) < \gamma \leq \left( \frac{1}{\alpha} \right)^2, 0 < \kappa \leq \left( \frac{\alpha n}{\alpha n(q-1)} \right)^{1/(q-1)} \) and

\[
u(t, x) = \kappa(t + 1)^{-\beta} \exp \left( -\gamma \frac{(1 + |x|^2)^{\frac{\alpha}{2}}}{t + 1} \right). \tag{13}\]

Again as in Example 2, it is not difficult to verify that the function \( u = u(t, x) \) defined by the formula (13) is a positive entire classical solution of the inequality (4) in \( S \), with \( a_{ij}(t, x) \) in (2) defined by (12). Also, it is clear that the function \( v = -u(t, x) \) is a negative entire classical solution of the inequality (5) in \( S \), with \( a_{ij}(t, x) \), the coefficients of the operator \( L \), defined by (12), and, thus, the pair of the functions \( u = u(t, x) \) and \( v = v(t, x) \) is a non-trivial entire classical solution of the system (4)–(5) and, therefore, of the inequality (1) in \( S \) such that \( u(t, x) > v(t, x) \), with \( a_{ij}(t, x) \) in (2) defined by (12).

**Remark 2** For the case when \( n \geq 1, \alpha \leq 0 \) and \( 1 \geq q > 1 + \frac{\alpha}{n} \), see Example 1.

Finally, we would like to note that elliptic analogues of the results in Theorems 1–6 were obtained in [12] and [13]. To prove the results obtained in the present work we further develop an approach proposed in [11]. That approach was subsequently used and developed in the same framework by E. Mitidieri, S. Pokhozhaev and many others, almost none of which cite the original research.
For a survey of the literature on the asymptotic behaviour and blow-up of solutions to the Cauchy problem for nonlinear parabolic equations or inequalities we refer to [1], [3], [5], [14], [16] and [17].

4 Proofs

Proof of Theorem 1. Let \( n \geq 1, \alpha > 0 \) and \( 1 < q \leq 1 + \frac{\alpha}{n} \), let \( \mathcal{L} \) be a differential operator defined by (2), the coefficients of which satisfy the condition (8) with the given \( \alpha \) and some \( c > 0 \), and let \((u, v)\) be an entire weak solution of the inequality (1) in \( \mathbb{S} \) such that \( u \geq v \). By the well-known inequality

\[
(|u|^{q-1}u - |v|^{q-1}v)(u - v) \geq 2^{1-q}|u - v|^{q+1}
\]

which holds for any \( q \geq 1 \) and any \( u, v \in \mathbb{R}^1 \), see, e.g., [6], we obtain from (9) the relation

\[
\int_{\mathbb{S}} \left[ (u - v) \phi + \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \phi}{\partial x_i} \frac{\partial (u - v)}{\partial x_j} \right] dtdx \geq 2^{1-q} \int_{\mathbb{S}} (u - v)^q \phi dtdx
\]

which holds for every function \( \phi \in C^\infty(\mathbb{S}) \) with compact support in \( \mathbb{S} \). Let \( \tau > 0, R > 1 \) and \( T > 0 \) be real numbers. Let \( \eta : [0, +\infty) \to [0, 1] \) be a \( C^\infty \)-function which has the non-negative derivative \( \eta' \) and equals 0 on the interval \([0, \tau]\) and 1 on the interval \([2\tau, +\infty)\), and let \( \zeta : [0, +\infty) \times \mathbb{R}^n \to [0, 1] \) be a \( C^\infty \)-function which equals 1 on \([0, T/2] \times \overline{B(R/2)} \) and 0 on \( \{(0, +\infty) \times \mathbb{R}^n\} \setminus \{[0, T] \times \overline{B(R)}\} \), where \( B(R) \) is the ball in \( \mathbb{R}^n \) centered at the origin of \( \mathbb{R}^n \) with radius \( R \). Let

\[
\varphi(t, x) = (w(t, x) + \varepsilon)^{-\nu} \zeta^s(t, x) \eta^2(t),
\]

where \( w(t, x) = u(t, x) - v(t, x), \varepsilon > 0 \) and the positive constants \( s > 1 \) and \( 1 > \nu > 0 \) will be chosen below. Substituting the function \( \varphi \) in (14) and then integrating by parts there we obtain

\[
-\frac{s}{1 - \nu} \int_0^T \int_{B(R)} (w + \varepsilon)^{1-\nu} \zeta \xi^{s-1} \eta^2 dtdx - \frac{2}{1 - \nu} \int_0^T \int_{B(R)} (w + \varepsilon)^{1-\nu} \zeta^s \eta' \eta dtdx
\]

\[
-\nu \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{1-\nu} \zeta^s \eta^2 dtdx
\]
\[ + s \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \xi}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu} \zeta s^{-1} \eta^2 dtdx \]

\[ \equiv I_1 + I_2 + I_3 + I_4 \geq \int_0^T \int_{B(R)} w^q (w + \varepsilon)^{-\nu} \zeta s \eta^2 dtdx. \quad (15) \]

In (15), first observe that \( I_3 \) is non-positive and then estimate \( I_4 \) in terms of \( I_3 \). Namely, since

\[ |I_4| = \left| s \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \xi}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu} \zeta s^{-1} \eta^2 dtdx \right| \]

\[ \leq \int_0^T \int_{B(R)} s \left( \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right)^{\frac{1}{2}} \left( \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_j} \right)^{\frac{1}{2}} (w + \varepsilon)^{-\nu} \zeta s^{-1} \eta^2 dtdx, \quad (16) \]

we estimate, first, the right-hand side of (16) by using Young’s inequality

\[ AB \leq \rho A^2 + \rho^{-1} B^2, \]

with \( \rho = \frac{\nu}{2} \),

\[ A = \left( \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right)^{\frac{1}{2}} (w + \varepsilon)^{-\frac{1+\nu}{2}} \zeta s \eta \]

and

\[ B = s \left( \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_j} \right)^{\frac{1}{2}} (w + \varepsilon)^{\frac{1-\nu}{2}} \zeta s^{-1} \eta. \]

As a result, we arrive at

\[ |I_4| \leq \frac{\nu}{2} \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu - 1} \zeta s \eta^2 dtdx \]

\[ + \int_0^T \int_{B(R)} \frac{2s^2}{\nu} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \xi}{\partial x_i} \frac{\partial \xi}{\partial x_j} (w + \varepsilon)^{1-\nu} \zeta s^{-2} \eta^2 dtdx. \quad (17) \]
Further, since $I_2$ in (15) is also non-positive, the inequality

$$\int_0^T \int_{B(R)} \frac{s}{1-\nu} (w + \varepsilon)^{1-\nu} \zeta_t |\zeta| \zeta^{s-1} \eta^2 dt dx$$

$$+ \int_0^T \int_{B(R)} \frac{2s^2}{\nu} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} (w + \varepsilon)^{1-\nu} \zeta^{s-2} \eta^2 dt dx$$

$$\geq \int_0^T \int_{B(R)} w^q (w + \varepsilon)^{-\nu} \zeta^{s} \eta^2 dt dx$$

$$+ \frac{\nu}{2} \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-1} \zeta^{s} \eta^2 dt dx$$

(18)

easily follows from (15) and (17). Estimating both integrands on the left-hand side of (18) by Young’s inequality

$$AB \leq \rho A^{\frac{\beta}{\beta-1}} + \rho^{1-\beta} B^\beta,$$

respectively, with $\rho = \frac{1}{4}, \beta = \frac{q-\nu}{q-1}$.

$$A = (w + \varepsilon)^{1-\nu} \zeta \frac{s(q-1)}{q-\nu} \eta \frac{2(1-\nu)}{q-1- \nu}$$

$$B = \frac{s}{1-\nu} |\zeta_t| \zeta \frac{s(q-1)}{q-\nu} \eta \frac{2(q-1)}{q-1- \nu}$$

and with $\rho = \frac{1}{4}, \beta = \frac{q-\nu}{q-1}$,

$$A = (w + \varepsilon)^{1-\nu} \zeta \frac{s(q-1)}{q-\nu} \eta \frac{2(1-\nu)}{q-1- \nu}$$

$$B = \frac{2s^2}{\nu} \zeta \frac{s(q-1)}{q-\nu} \eta \frac{2(q-1)}{q-1- \nu} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j}$$

we obtain

$$\frac{1}{4} \int_0^T \int_{B(R)} (w + \varepsilon)^{q-\nu} \zeta^{s} \eta^2 dt dx + c_1 \int_0^T \int_{B(R)} |\zeta_t| \frac{q-\nu}{q-1} \zeta^{s-\frac{q-\nu}{q-1}} \eta^2 dt dx$$

11
Here and what follows, we use the symbols $c_i, i = 1, 2, \ldots,$ to denote constants depending possibly on $c, n, q, s, \alpha$ and $\nu$, but not on $\varepsilon, \tau$ and $R$.

At this point, using the inequality (19), we obtain an upper bound on the integral

\[ + \frac{1}{4} \int_0^T \int_{B(R)} (w + \varepsilon)^{q-\nu} \zeta^s \eta^2 dt dx \]

\[ + c_2 \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{q-\nu}{q-1}} \zeta^s \eta^{-\frac{2(\nu-1)}{q-1}} \eta^2 dt dx \]

\[ \geq \int_0^T \int_{B(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dt dx \]

\[ + \frac{\nu}{2} \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx. \] (19)

As before, it is easy to see that the second term on the left-hand side of (20)
is non-positive and thus (20) yields

\[ s \int_0^T \int_{B(R)} w|\zeta|^{\zeta^s-1} \eta^2 dt dx + s \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 dt dx \]
\[ \geq \int_0^T \int_{B(R)} w^q \zeta^s \eta^2 dt dx. \quad (21) \]

Estimating now the first integral on the left-hand side of (21) by Hölder’s inequality, we arrive at

\[ s \left( \int_0^{T/2} \int_{B(R)} w^q \zeta^s \eta^2 dt dx \right)^\frac{1}{q} \left( \int_0^T \int_{B(R)} |\zeta|^{\frac{s}{1-q}} \zeta^{s-\frac{q}{1-q}} \eta^2 dt dx \right)^\frac{q-1}{q} \]
\[ + s \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 dt dx \geq \int_0^T \int_{B(R)} w^q \zeta^s \eta^2 dt dx. \quad (22) \]

Further, since

\[ \left| \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \right| \leq \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \right)^\frac{1}{2} \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^\frac{1}{2}, \]

we have

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 dt dx \]
\[ \leq \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^\frac{1}{2} \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right)^\frac{1}{2} \zeta^{s-1} \eta^2 dt dx. \quad (23) \]
Estimating the right-hand side of (23) by Hölder’s inequality, we obtain the relation

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 \, dt \, dx \]

\[ \leq \left( \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} (w + \varepsilon)^{1+\nu} \zeta^{s-2} \eta^2 \, dt \, dx \right)^{1/2} \]

\[ \times \left( \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 \, dt \, dx \right)^{1/2} \]  

(24)

which holds for any \( \varepsilon > 0 \) and any \( \nu \in (0, 1) \). Further, it is easy to see that the inequality

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} (w + \varepsilon)^{1+\nu} \zeta^{s-2} \eta^2 \, dt \, dx \]

\[ \leq \left( \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{d}{d-1}} \zeta^{s-\frac{2d}{d-1}} \eta^2 \, dt \, dx \right)^{\frac{d-1}{d}} \]

\[ \times \left( \int_0^T \int_{B(R) \setminus B(R/2)} (w + \varepsilon)^{d(1+\nu)} \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{d}} \]  

(25)

holds for any \( d > 1 \). In (25), choosing for any sufficiently small \( \nu \in (0, 1) \) the parameter \( d \) such that \( d(1 + \nu) = q \), we obtain from (24) and (25) the relation

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 \, dt \, dx \]

\[ \leq \left( \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{d}{d-1}} \zeta^{s-\frac{2d}{d-1}} \eta^2 \, dt \, dx \right)^{\frac{d-1}{2d}} \]
\[ \times \left( \int_0^T \int_{B(R) \setminus B(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dtdx \right)^{1/2} \]

which holds for any \( \varepsilon > 0 \) and any sufficiently small \( \nu \in (0, 1) \). In (26), estimating the last term on the right-hand side by virtue of (19), we have

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} (w + \varepsilon)^{-\nu - 1} \zeta^s \eta^2 dtdx \]

\[ \leq \left( \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{d}{\nu - 1}} \zeta^{s - \frac{2d}{\nu - 1}} \eta^2 dtdx \right)^{\frac{d - 1}{\nu}} \]

\[ \times \left( \int_0^T \int_{B(R) \setminus B(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dtdx \right)^{1/2} \left( \frac{1}{\nu} \int_0^T \int_{B(R)} (w + \varepsilon)^{-\nu - 1} \zeta^s \eta^2 dtdx \right) \]

\[ - \frac{2}{\nu} \int_0^T \int_{B(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dtdx + c_3 \int_0^T \int_{B(R)} |\zeta|^{\frac{2\nu - 1}{q - 1}} \zeta^{s - \frac{2(\nu - 1)}{q - 1}} \eta^2 dtdx \]

\[ + c_4 \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{q - \nu}{q - 1}} \zeta^{s - \frac{2(q - \nu)}{q - 1}} \eta^2 dtdx \right)^{\frac{1}{2}} \]

In (27), passing to the limit as \( \varepsilon \to 0 \) as justified by Lebesgue’s theorem (see, e.g., [9, p. 303]), we obtain for any sufficiently large \( s \) the inequality

\[ \int_0^T \int_{B(R)} \sum_{i,j=1}^{n} a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial w}{\partial x_j} \zeta^{s-1} \eta^2 dtdx \]
In turn, (22) and (28) yield the inequality

\[
\left( \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{4}{q+1}} \eta^2 \right)^{\frac{q+1}{4}} \leq \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \right)^{\frac{q}{4}} \times \left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q}{q+1}} \eta^2 \right)^{\frac{1}{q}}
\]

which holds for any sufficiently large \( s \) and any sufficiently small \( \nu \in (0, 1) \).

Further, by the condition (3) on the coefficients of the operator \( \mathcal{L} \), we obtain

\[
\left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q}{q+1}} \eta^2 \right)^{\frac{1}{q}} \leq \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \right)^{\frac{q}{4}} \times \left( \int_0^T \int_{B(R)} \left( \sum_{i,j=1}^n a_{ij}(t,x) \frac{\partial \zeta}{\partial x_i} \frac{\partial \zeta}{\partial x_j} \right)^{\frac{4}{q+1}} \eta^2 \right)^{\frac{q+1}{4}} \leq \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \right)^{\frac{q}{4}}
\]

which holds for any sufficiently large \( s \) and any sufficiently small \( \nu \in (0, 1) \).
from (29) the inequality

\[ \int_0^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \leq c_5 \left( \int_{T/2}^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}} \left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q}{q-1}} \eta^2 \, dt \, dx \right)^{\frac{q-1}{q}} \]

\[ + c_5 \left( A(T, R) \right)^{\frac{1}{q}} \left( \int_0^T \int_{B(R)} |\nabla \zeta|^{\frac{2d}{q-1}} \eta^2 \, dt \, dx \right)^{\frac{d-1}{2d}} \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q-\nu}{q}} \eta^2 \, dt \, dx + (A(T, R))^{\frac{q-\nu}{q-1}} \int_0^T \int_{B(R) \setminus B(R/2)} (|\nabla \zeta|^{\frac{2(q-\nu)}{q-1}} \eta^2) \, dt \, dx \right)^{\frac{1}{2}}, \]

where

\[ A(T, R) = \text{ess sup}_{(t, x) \in (0, T) \times (B(R) \setminus B(R/2))} A(t, x), \]

which in turn by the condition (8) yields

\[ \int_0^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \]

\[ \leq c_6 \left( \int_{T/2}^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}} \left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q}{q-1}} \eta^2 \, dt \, dx \right)^{\frac{q-1}{q}} \]

\[ + c_6 R^{2\alpha} \left( \int_0^T \int_{B(R)} |\nabla \zeta|^{\frac{2d}{q-1}} \eta^2 \, dt \, dx \right)^{\frac{d-1}{2d}} \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{2}} \]

\[ \times \left( \int_0^T \int_{B(R)} |\zeta_t|^{\frac{q-\nu}{q-1}} \eta^2 \, dt \, dx + R^{\frac{(2-\alpha)(q-\nu)}{q-1}} \int_0^T \int_{B(R) \setminus B(R/2)} |\nabla \zeta|^{\frac{2(q-\nu)}{q-1}} \eta^2 \, dt \, dx \right)^{\frac{1}{2}}. \]
Now, for arbitrary \( (t, x) \in S \), \( R > 1 \) and \( T > 0 \), we choose in (30) the function \( \zeta(t, x) \) in the form
\[
\zeta(t, x) = \psi\left(\frac{t}{T}\right) \psi\left(\frac{2|x|^2}{R^2}\right),
\]
where \( \psi : [0, +\infty) \to [0, 1] \) is a \( C^\infty \)-function which equals 1 on \([0, 1/2]\) and 0 on \([1, +\infty)\) and such that the inequalities
\[
|\zeta_t| \leq c_T T^{-1} \quad \text{and} \quad |\nabla_x \zeta| \leq c_T R^{-1}
\]
hold. Note that it is always possible to find such a function \( \zeta \). Indeed, this can be easily verified by direct calculation of the corresponding derivatives of the function \( \zeta \) defined by (31). Also, in what follows we let
\[
T = R^\alpha.
\]
Since \(|\eta| \leq 1\), by (32) and (33), we have from (30) the inequality
\[
\begin{align*}
\int_0^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx & \leq c_8 \left( R^{n+\alpha-\frac{d(q-\nu)}{q-1}} \right)^{\frac{q-1}{q}} \left( \int_{T/2 B(R)}^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}} \\
+ c_8 R^{\frac{d-q}{q}} \left( R^{n+\alpha-\frac{2d}{d-1}} + R^{(\alpha-\frac{d-q}{d-1})} R^{n+\alpha-\frac{2(q-\nu)}{q-1}} \right)^{\frac{1}{d}} \\
\times \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{d}}.
\end{align*}
\]
Making simple calculations in (34) we obtain
\[
\begin{align*}
\int_0^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx & \leq c_8 \left( R^{n+\alpha-\frac{d(q-\nu)}{q-1}} \right)^{\frac{q-1}{q}} \left( \int_{T/2 B(R)}^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}} \\
+ c_8 \left( R^{n+\alpha-\frac{d(q-\nu)}{q-1}} \right)^{\frac{d-q}{d}} \left( R^{n+\alpha-\frac{2(q-\nu)}{q-1}} \right)^{\frac{1}{d}} \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{d}}.
\end{align*}
\]
In turn, since $d(1 + \nu) = q$, i.e., $\frac{2d}{d-1} = \frac{2q}{q-1-\nu}$, the relation (35) implies the inequality

$$
\int_0^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx
\leq c_8 R^{\frac{n}{q-1}[q-1-\frac{\alpha}{n}] + \frac{1}{q}} \left( \int_{T/2}^T \int_{B(R)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}}
$$

$$
+ c_8 R^{\frac{n(2q - 1 - \nu)}{2q(q - 1)}} \left[ q - 1 - \frac{\alpha}{n} \right] \left( \int_0^T \int_{B(R) \setminus B(R/2)} w^q \zeta^s \eta^2 \, dt \, dx \right)^{\frac{1}{q}}
$$

(36)

which holds for any sufficiently small $\nu \in (0, 1)$. Now, since $q > 1$, $d > 1$, and for any $\nu \in (0, 1)$ both quantities

$$
\frac{n}{q - 1} \quad \text{and} \quad \frac{n(2q - 1 - \nu)}{2q(q - 1)}
$$

are positive, it follows from (36), where we remind that $T = R^\alpha$, by passing $R \to +\infty$ that the relation

$$
\int_S w^q \eta^2 \, dt \, dx = 0
$$

(37)

holds for

$$
1 < q < 1 + \frac{\alpha}{n}.
$$

We show now that (37) also holds for

$$
q = 1 + \frac{\alpha}{n}.
$$

(38)

Indeed, since $q > 1$ and $d > 1$, by (38) we have from (36) the estimate

$$
\int_S w^q \eta^2 \, dt \, dx < +\infty.
$$

(39)
In turn, by Fubini’s theorem (see, e.g., [9, p. 317]), we obtain from (39) the relations

$$\int_{T_k/2}^{T_k} \int w^q \eta^2 dt dx \to 0$$

and

$$\int_0^{+\infty} \int_{B(R_k)\setminus B(R_k/2)} w^q \eta^2 dt dx \to 0$$

which hold for any sequences $R_k$ and $T_k$ such that $R_k \to +\infty$ and $T_k \to +\infty$. On the other hand, from (36) we have the inequality

$$\int_0^{T/2} \int_{B(R/2)} w^q \eta^2 dt dx \leq c_8 R^{\frac{n}{q-1}} \left[ \int_0^{+\infty} \int_{B(R)\setminus B(R/2)} w^q \eta^2 dt dx \right]^{\frac{1}{q}}$$

$$+ c_8 R^{\frac{(2q-1-\nu)}{(q-1)}} \left[ q-1-\frac{\alpha}{n} \right]^{\frac{1}{q-\nu}} \left( \int_0^{+\infty} \int_{B(R)\setminus B(R/2)} w^q \eta^2 dt dx \right)^{\frac{1}{q}}$$

which, together with (40) and (41), where we choose $T = R^\alpha = T_k = (R_k)^\alpha$, implies the relation

$$\int_0^{(R_k/2)^\alpha} \int_{B(R_k/2)} w^q \eta^2 dt dx \to 0$$

which holds for any sequence $R_k \to +\infty$. In turn, (43) yields the relation (37) for $q$ given by (38). Thus, we prove that the relation (37) holds for any

$$1 < q \leq 1 + \frac{\alpha}{n},$$

where $\eta : [0, +\infty) \to [0, 1]$ is a $C^\infty$-function which equals 1 on the interval $[2\tau, +\infty)$. In (37), passing to the limit as $\tau \to 0$, we obtain that $u(t, x) = v(t, x)$ in $S$. 

20
References

[1] K. Deng and H.A. Levine, *The role of critical exponents in blow-up theorems: the sequel*, J. Math. Anal. Appl. (2000), vol. 243, no. 1, 85–126.

[2] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* \( u_t = \Delta u + u^{1+\alpha} \), J. Fac. Sci. Univ. Tokyo, Sect. I, (1966), v. 13, 109–124.

[3] M.-H. Giga, Y. Giga, J. Saal, *Nonlinear partial differential equations. Asymptotic behavior of solutions and self-similar solutions*, Progress in Nonlinear Differential Equations and their Applications, 79. Birkhäuser Boston, Inc., Boston, MA (2010), 294 pp.

[4] K. Hayakawa, *On nonexistence of global solutions of some semilinear parabolic differential equations*, Proc. Japan Acad. (1973), vol. 49, 503–505.

[5] A.S. Kalashnikov, *Some problems of the qualitative theory of second-order nonlinear degenerate parabolic equations*, Uspekhi Mat. Nauk 42 (1987), 135–176.

[6] A.G. Kartsatos and R.D. Mabry, *Controlling the space with preassigned responses*, J. Optim. Theory Appl., 54 (1987), no. 3, 517–540.

[7] A.G. Kartsatos and V.V. Kurta, *On a comparison principle and the critical Fujita exponents for solutions of semilinear parabolic inequalities*, J. London Math. Soc. (2) 66 (2002), no. 2, 351–360.

[8] K. Kobayashi, T. Sirao, and H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan (1977), vol. 29, no. 3, 407–424.

[9] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 403 pp.

[10] V.A. Kondrat′ev and E.M. Landis, *Semilinear second-order equations with nonnegative characteristic form*, (Russian) Mat. Zametki 44 (1988), 457–468.
[11] V.V. Kurta, *Some problems of qualitative theory for nonlinear second-order equations*, Doctoral Dissert., Steklov Math. Inst., Moscow, 1994.

[12] V.V. Kurta, *On the absence of positive solutions to semilinear elliptic equations*, (Russian) Tr. Mat. Inst. Steklova 227 (1999), Issled. po Teor. Differ. Funkts. Mnogikh Perem. i ee Prilozh. 18, 162–169; translation in Proc. Steklov Inst. Math. 1999, no. 4 (227), 155–162.

[13] V. Kurta, *Liouville comparison principles for solutions of semilinear elliptic second-order partial differential inequalities*. Complex Variables and Elliptic Equations 10 (2012), no. 6, 1747–1762.

[14] H.A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. (1990), vol. 32, no. 2, 262–288.

[15] O.A. Oleinik and E.V. Radkevich, *Second order equations with nonnegative characteristic form*, Translated from the Russian by Paul C. Fife. Plenum Press, New York-London, 1973, 259 pp.

[16] P. Quittner and P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel (2007), 584 pp.

[17] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in Quasi-linear Parabolic Equations*, Walter de Gruyter, Berlin (1995), 535 pp.
Authors’ addresses:

Vasilii V. Kurta
Mathematical Reviews
416 Fourth Street, P.O. Box 8604
Ann Arbor, Michigan 48107-8604, USA
e-mail: vkurta@umich.edu, vvk@ams.org