WEAK-STRONG UNIQUENESS OF INCOMPRESSIBLE MAGNETO-VISCOELASTIC FLOWS

WENJING ZHAO
Department of Mathematics
College of Sciences, Northeastern University
Shenyang 110819, China

(Communicated by Zhen Lei)

Abstract. Our aim in this paper is to prove the weak-strong uniqueness property of solutions to a hydrodynamic system that models the dynamics of incompressible magneto-viscoelastic flows. The proof is based on the relative energy approach for the compressible Navier-Stokes system.

1. Introduction. We consider the following magneto-viscoelastic system:

\[ \begin{align*}
v_t + (v \cdot \nabla)v - \mu \Delta v + \nabla p &= \nabla \cdot (FF^T) - \nabla \cdot (\nabla^T M \nabla M) + \nabla^T H_{ext} M, \\
\nabla \cdot v &= 0, \\
F_t + (v \cdot \nabla)F &= \nabla v F, \\
M_t + (v \cdot \nabla)M &= \Delta M - \frac{1}{\alpha^2}(|M|^2 - 1)M + H_{ext},
\end{align*} \]

for \((x, t) \in \Omega \times (0, T)\), where \(T > 0\) and \(\Omega = \mathbb{T}^d, \quad d = 2\text{ or } 3\). Here, \(v(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}^d\) is the velocity field, \(p(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}\) is the pressure, \(F : \Omega \times (0, T) \rightarrow \mathbb{R}^{d \times d}\) denotes the deformation gradient and \(M : \Omega \times (0, T) \rightarrow \mathbb{R}^3\) describes the magnetization vector. The magneto-viscoelastic fluid is sometimes exposed to an external effective magnetic field \(H_{ext}(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}^3\). For the sake of simplicity, the fluid viscosity denoted by \(\mu\) is assumed to be a positive constant. Besides, \(\alpha > 0\) stands for a parameter that controls the strength of penalization on the deviation of \(|M|\) from 1.

The magneto-viscoelastic system (1)–(4) can be viewed as a simplification of the general model derived in [8] based on an energetic variational approach (see, e.g., [13]), where in the general magneto-viscoelastic model, the evolution of the magnetization \(M\) is governed by a Landau-Lifshitz-Gilbert (LLG) system with advection (see [1, 2, 8, 14, 16] and the references therein). On the other hand, from the mathematical point of view, the system (1)–(4) consists of two subsystems that have been extensively studied in the literature: one is the incompressible viscoelastic system (i.e., \(M \equiv 0\), see [4, 12, 17, 18, 20, 21]), while the other one is the simplified Ericksen-Leslie (E-L) system for incompressible nematic liquid crystal flows with Ginzburg-Landau approximation (i.e., \(F \equiv 0\), see [10, 19, 25]). In [8], the author

2010 Mathematics Subject Classification. Primary: 76W05, 76B03; Secondary: 35Q35.

Key words and phrases. Weak-strong uniqueness, magneto-viscoelastic flows, relative energy.
proved the existence of global weak solutions to the system (1) − (4) with an additional regularizing term \( \kappa \Delta F \) \((\kappa > 0)\) in equation (3) for the deformation tensor \( F \). Later, uniqueness of solutions to the same regularized problem with slightly different boundary conditions was studied in [23]. The authors proved the uniqueness of global weak solutions when the spatial dimension is two and also a weak-strong type of uniqueness result in three dimensions. Here we are interested in mathematical analysis of the system (1) − (4) without the artificial regularizing term \( \kappa \Delta F \). Due to its highly nonlinear coupling structure and in particular, the absence of damping mechanism in the deformation equation (3), it is a rather challenging task to study well-posedness of the magneto-viscoelastic system (1) − (4). Recently, existence and uniqueness of local-in-time classical solutions to the system (1) − (4) was obtained in [27] in the periodic setting, while some blow-up criteria have been derived as well. The study on existence and uniqueness of global weak solutions turns out to be more subtle. We note that if the magnetic effect is neglected, i.e., \( M \equiv 0 \), the system (1) − (4) reduces to a well-known model for incompressible viscoelastic flows [17, 20]. Even for the viscoelastic system, existence of global weak solutions with finite energy is still an outstanding open problem. Among those attempts in this direction, we refer to [11] for the existence of global weak solutions with small energy in two dimensions, and to [15] for global existence of the so-called dissipative solutions in two and three dimensions.

Our aim in this paper is to establish a weak-strong type of uniqueness result (cf. [24]) for the system (1) − (4). We recall that several weak-strong uniqueness results have been obtained for problems that are closely related to our system, see, for instance, [12, 15] for the incompressible viscoelastic flow and [5, 19] for the Ericksen-Leslie system of incompressible liquid crystal flow. In order to deal with the complicated coupling structure of system (1) − (4), we shall employ the concept of a relative energy approach for compressible Navier-Stokes systems (see [6, 7, 9]). Besides, we overcome the possible difficulty due to nonconvexity of the free energy by using the idea derived in [5] for the general Ericksen-Leslie system [3, 26].

The rest of this paper is organized as follows. In Section 2, we reformulate the magneto-viscoelastic system (1) − (4), recall some important features of it and then state the main result of this paper (see Theorem 2.4). In Section 3, we prove Theorem 2.4 by using the relative energy approach.

2. Preliminaries and main result. We denote by \( L^p(\Omega) \), \( W^{m,p}(\Omega) \) the usual Lebesgue and Sobolev spaces on the domain \( \Omega \), with norms \( \| \cdot \|_p \), \( \| \cdot \|_{W^{m,p}} \) respectively. For \( p = 2 \), we simply denote \( H^m(\Omega) = W^{m,2}(\Omega) \) with norm \( \| \cdot \|_{H^m} \). The norm and inner product on \( L^2(\Omega) \) will be denoted by \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively. For simplicity, we do not distinguish functional spaces when scalar-valued, vector-valued or matrix-valued functions are involved. We denote by \( C \) a generic positive constant throughout this paper, which may vary at different places. Its spatial dependence will be indicated explicitly if necessary.

In this paper, we study the magneto-viscoelastic system (1) − (4) in the periodic setting \( \Omega = \mathbb{T}^d \) \((d = 2, 3)\). Following the notation in [17], we define the usual strain tensor in the form of

\[
E = F - I,
\]

where \( I \) is the \( d \times d \) identity matrix. Besides, in the remaining part of the paper, we always take \( H_{ext} = 0 \) in (1) − (4) for the sake of simplicity. Then the system
(1) – (4) can be rewritten into the following form:
\[
\begin{align*}
&v_t + (v \cdot \nabla) v - \mu \Delta v + \nabla p \\
&= \nabla \cdot (EE^T) + \nabla \cdot E - \nabla \cdot (\nabla^T M \nabla M), \\
\n&\nabla \cdot v = 0, \\
&\nabla \cdot v E = \nabla v E + \nabla v, \\
&\nabla \cdot M = \Delta M - \frac{1}{\alpha^2} (|M|^2 - 1) M,
\end{align*}
\]
for \((x, t) \in \mathbb{T}^d \times (0, T)\). Here, we use the following notations
\[
(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (\nabla v E)_{ij} = (\nabla v)_{ik} E_{kj}, \quad (\nabla \cdot E)_i = \partial_j E_{ij}.
\]

The system (1)–(4) is subject to the following periodic initial data:
\[
v(x, 0) = v_0(x), \quad F(x, 0) = I + E_0(x), \quad M(x, 0) = M_0(x), \quad \forall x \in \mathbb{T}^d,
\]
which satisfy the following constraints
\[
\begin{cases}
\nabla \cdot v_0 = 0, & \det(I + E_0) = 1, & \nabla \cdot E_0^T = 0, \\
\nabla_m E_{0ij} - \nabla_j E_{0im} = E_{0lj} \nabla_l E_{0im} - E_{0lm} \nabla_l E_{0ij}.
\end{cases}
\]

Concerning (6), we recall the following important properties of the strain tensor
\(E\) (see [17, Proposition 2.2]).

**Lemma 2.1.** Assume that \((v, E, M)\) is the solution of the periodic initial-boundary value problem (1) – (5), satisfying the constraint (6). Then the following identities hold

\[
\begin{align*}
&\det(I + E) = 1, \\
&\nabla \cdot E^T = \nabla \cdot E = 0, \\
&\nabla_m E_{ij} - \nabla_j E_{im} = E_{lj} \nabla_l E_{im} - E_{lm} \nabla_l E_{ij},
\end{align*}
\]
for all time \(t \geq 0\). Here \(E_{ij} = E_{ij}\) stands for the \(j\)-th column of the matrix \(E\).

Next, we recall the basic energy law of the system (1) – (5) (see [27, Proposition 1]).

**Lemma 2.2** (Basic energy law). Let \((v, E, M)\) be a classical solution of the periodic initial-boundary value problem (1) – (5) on \(\Omega \times [0, T]\). Then we have
\[
\frac{1}{2} \frac{d}{dt} \left[ \|v\|^2 + \|E\|^2 + \|
abla M\|^2 + \int_\Omega 2F(M)dx \right] + \mu \int_\Omega |\nabla v|^2 dx + \int_\Omega |\Delta M - F'(M)|^2 dx = 0,
\]
for all \(t \in (0, T)\). Here, we denote \(F(M) = \frac{1}{4\mu}(M^2 - 1)^2\), and \(F'(M) = \frac{1}{4\mu}(M^2 - 1) M\).

We note that the equations (1) and (3) can be written as (cf. [12])
\[
\begin{align*}
v_t &+ v \cdot \nabla v - \mu \Delta v + \nabla p \\
&= \nabla \cdot (E_t \otimes E_t) + \nabla \cdot E - \nabla \cdot (\nabla M_t \otimes \nabla M_t), \\
E_{jt} &+ \nabla \cdot (E_j \otimes v - v \otimes E_j) = \nabla_j v,
\end{align*}
\]
where \((a \otimes b)_{ij} = a_i b_j\), and repeated indicators represent summation. Then we introduce the notion of weak solutions to the system (1) – (5).

**Definition 2.3** (Suitable weak solution). We say that \((v, E, M)\) is a suitable weak solution of the periodic initial-boundary value problem (1) – (5) on \(\Omega \times [0, T]\) if the following hold:

\[
\begin{align*}
v &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \\
E &\in L^\infty(0, T; L^2(\Omega)), \\
M &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\end{align*}
\]

and satisfy the system (1) – (4) in the sense of distributions, i.e., \(\nabla \cdot v = 0\) in the distributional sense and for all test functions \(\varphi, \psi, \eta \in C^1([0, T]; D(\Omega))\) with \(\nabla \cdot \varphi = 0\), we have

\[
\int_\Omega v(t, \cdot) \cdot \varphi(t, \cdot) dx - \int_\Omega v_0 \cdot \varphi(0) dx = \int_0^t \int_\Omega v \cdot \varphi_x dxd\tau + \int_0^t \int_\Omega (v \otimes v - \mu \nabla v) : \nabla \varphi dx d\tau
\]

\[
- \int_0^t \int_\Omega (E_l \otimes E_l + E) : \nabla \varphi dx d\tau - \int_0^t \int_\Omega \nabla \cdot (\nabla M_l \otimes \nabla M_l) \cdot \varphi dx d\tau
\]

\[
\int_\Omega E(t, \cdot) : \psi(t, \cdot) dx - \int_\Omega E_0 : \psi(0) dx
= \int_0^t \int_\Omega E : \psi_x dx d\tau + \int_0^t \int_\Omega (E_l \otimes v - v \otimes E_l) : \nabla \psi dx d\tau + \int_0^t \int_\Omega \nabla v : \psi dx d\tau
\]

\[
\int_\Omega M(t, \cdot) \cdot \eta(t, \cdot) dx - \int_\Omega M_0 \cdot \eta(0) dx
= \int_0^t \int_\Omega M : \eta_x dx d\tau + \int_0^t \int_\Omega (-v \cdot \nabla M + \Delta M - F'(M)) \cdot \eta dx d\tau
\]

where \(A : B = \sum_{i,j} A_{ij} B_{ij}\). Moreover, the following energy inequality holds for a.e. \(t \in [0, T]\), that is

\[
\mathcal{E}(v, E, M; t) + \int_0^t F(M(t)) dx + \int_0^t (\mu \|\nabla v\|^2 + \|\Delta M - F'(M)\|^2) d\tau
\]

\[
\leq \mathcal{E}(v, E, M; 0) + \int_0^t F(M_0) dx,
\]

where \(\mathcal{E}(v, E, M; t) = \frac{1}{2}(\|v(t)\|^2 + \|E(t)\|^2 + \|\nabla M(t)\|^2)\).

Now we state the main result of this paper.

**Theorem 2.4** (Weak-Strong Uniqueness). Suppose that \(\Omega = T^d\), \(d = 2, 3\). Let \(v_0, E_0 \in H^s(\Omega)\) and \(M_0 \in H^{s+1}(\Omega)\) \((s \geq 3)\), satisfying the constraints in (6). Assume that \((v, E, M)\) is a strong solution of the periodic initial-boundary value problem (1) – (5) and \((\bar{v}, \bar{E}, \bar{M})\) is a suitable weak solution (in the sense of Definition 2.3) with the same initial data, then there holds \((v, E, M) \equiv (\bar{v}, \bar{E}, \bar{M})\) on the time interval of existence.
Remark 1. We note that existence of local strong solutions to problem (1) – (5) subject to regular initial data have been proved in [27, Theorem 1.1], while existence of the suitable weak solution of the magneto-viscoelastic system is still an open question. Nevertheless, in the two dimensional case, it is possible to employ the arguments in [11, 19] to prove the existence of global weak solutions, provided that the initial deformation tensor is close to the identity matrix and the initial velocity is small. This issue will be illustrated in a forthcoming paper.

Remark 2. Quite recently, global existence of a different notion of weak solution, i.e., the so-called dissipative solution, was proved for the incompressible viscoelastic system with arbitrary finite initial energy in [15], where the author also showed the coincidence of a dissipative solution and a strong solution emanating from the same initial data. We note that it is not difficult to extend that result to our magneto-viscoelastic system (1) – (5) and obtain the weak-strong uniqueness result by using a similar argument in this paper.

3. Proof of Theorem 2.4. Set
\[ v^* = \bar{v} - v, \quad E^* = \bar{E} - E \quad \text{and} \quad M^* = \bar{M} - M. \]
We consider the relative energy
\[ E(v^*, E^*, M^*; t) + \frac{1}{4\alpha^2}||(\bar{M}^2 - 1) - (M^2 - 1)||^2 \]
\[ = E(\bar{v}, \bar{E}, \bar{M}; t) + E(v, E, M; t) - (\bar{v}, v) - (\bar{E}, E) - (\nabla \bar{M}, \nabla M) \]
\[ + \int_{\Omega} F(\bar{M}(t))dx + \int_{\Omega} F(M(t))dx - \frac{1}{2\alpha^2}(\bar{M}^2 - 1, M^2 - 1), \quad (17) \]
which plays a role of “distance” between a strong solution \((v, E, M)\) and a generic triplet \((\bar{v}, \bar{E}, \bar{M})\). Using the basic energy law (2.2) for \((v, E, M)\) and the energy inequality (16) for \((\bar{v}, \bar{E}, \bar{M})\), we obtain
\[ E(v^*, E^*, M^*; t) + \frac{1}{4\alpha^2}||(\bar{M}^2 - 1) - (M^2 - 1)||^2 \]
\[ \leq ||v_0||^2 - (\bar{v}, v) + ||E_0||^2 - (\bar{E}, E) + ||\nabla M_0||^2 - (\nabla \bar{M}, \nabla M) \]
\[ + \frac{1}{2\alpha^2}||M_0^2 - 1||^2 - \frac{1}{2\alpha^2}(\bar{M}^2 - 1, M^2 - 1) - \mu \int_0^t (||\nabla \bar{v}||^2 + ||\nabla v||^2)dt \]
\[ - \int_0^t (||\Delta \bar{M} - F'(\bar{M})||^2 + ||\Delta M - F'(M)||^2)dt. \quad (18) \]
In the variational formulation (13) for the weak solution \(\bar{v}\), taking \(\varphi = v\) as the test function and use the equation (1) for \(v\), we get that
\[ ||v_0||^2 - (\bar{v}, v) \]
\[ = - \int_0^t \int_{\Omega} \bar{v} \cdot v_t d\tau - \int_0^t \int_{\Omega} (\bar{v} \otimes \bar{v} - \mu \nabla \bar{v}) : \nabla v dx d\tau \]
\[ + \int_0^t \int_{\Omega} (\bar{E}_t \otimes \bar{E}_t + \bar{E}) : \nabla v dx d\tau + \int_0^t \int_{\Omega} \nabla \cdot (\nabla \bar{M}_t \otimes \nabla \bar{M}_t) \cdot v dx d\tau \]
\[ = 2\mu \int_0^t \int_{\Omega} \nabla \bar{v} : \nabla v dx d\tau + \int_0^t \int_{\Omega} (\bar{v} \otimes v - \bar{v} \otimes \bar{v}) : \nabla v dx d\tau \]
\[ + \int_0^t \int_{\Omega} (E_t \otimes E_t : \nabla \bar{v} + \bar{E}_t \otimes \bar{E}_t : \nabla v) dx d\tau + \int_0^t \int_{\Omega} (\nabla \bar{v} : E + \bar{E} : \nabla v) dx d\tau \]
that we obtain

Next, we take \( \psi = E \) as the test function in (14) and use the equation (3) for \( E \), so that we obtain

\[
\|E_0\|^2 - (\bar{E}, E) = -\int_0^t \int_\Omega \bar{E} : E_t dxdt - \int_0^t \int_\Omega [(\bar{E}_t \otimes \bar{v} - \bar{E}_t \otimes \bar{E}_t) : \nabla E_t] dxdt - \int_0^t \int_\Omega \nabla \bar{v} : E dxdt
\]

\[
= \int_0^t \int_\Omega (\bar{E}_t \otimes v : \nabla E_t - \bar{E}_t \otimes E_t : \nabla v) dxdt - \int_0^t \int_\Omega \bar{E} : \nabla v dxdt
\]

\[
= \int_0^t \int_\Omega (-\bar{E} : \nabla v - \nabla \bar{v} : E) dxdt
\]

\[
+ \int_0^t \int_\Omega [(\bar{E}_t \otimes v - \bar{E}_t \otimes \bar{v}) : \nabla E_t - \bar{E}_t \otimes E_t : \nabla v - \bar{E}_t \otimes \bar{E}_t : \nabla \bar{v}] dxdt. \tag{20}
\]

Then in the variational formulation (15) for the weak solution \( \bar{M} \), taking \( \eta = \Delta M \) as the test function in (15) and using the equation (4) for \( M \) we obtain that

\[
\|\nabla M_0\|^2 - (\nabla \bar{M}, \nabla M) = \int_0^t \int_\Omega [\bar{M} \cdot \Delta M_t + (-\bar{v} \cdot \nabla \bar{M} + \Delta \bar{M} - F'(\bar{M}) \cdot \Delta M)] dxdt
\]

\[
= \int_0^t \int_\Omega [\Delta \bar{M} \cdot (-\bar{v} \cdot \nabla M + \Delta M - F'(M)) dxdt
\]

\[
+ \int_0^t \int_\Omega (-\bar{v} \cdot \nabla \bar{M} + \Delta \bar{M} - F'(\bar{M})) \cdot \Delta M dxdt
\]

\[
= \int_0^t \int_\Omega [\Delta \bar{M} \cdot (\Delta M - F'(M)) + \Delta M \cdot (\Delta \bar{M} - F'(\bar{M}))] dxdt
\]

\[
- \int_0^t \int_\Omega (\bar{v} \cdot \nabla M_0 \Delta \bar{M}_t + \bar{v} \cdot \nabla \bar{M}_t \Delta M_t) dxdt. \tag{21}
\]

Similarly, taking \( \eta = -\frac{\bar{M}(M^2 - 1)}{\alpha^2} \), we get

\[
\frac{1}{\alpha^2} \int_\Omega M_0^2(M^2 - 1) dx - \frac{1}{\alpha^2} \int_\Omega \bar{M}^2(M^2 - 1) dx
\]

\[
= -\int_0^t \int_\Omega \bar{M} \frac{d}{d\tau} \left( \frac{\bar{M}(M^2 - 1)}{\alpha^2} \right) dx dt
\]
which implies

\[ + \int_0^t \int_{\Omega} (\vec{v} \cdot \nabla \tilde{M} - \Delta \tilde{M} + F'(\tilde{M})) \tilde{M} \frac{(M^2 - 1)}{\alpha^2} dxd\tau \]

\[ = - \int_0^t \int_{\Omega} \left\{ \frac{1}{2\alpha^2} \frac{d}{d\tau} \left[ (\tilde{M}^2 - 1)(M^2 - 1) \right] - \frac{M^2 - 1}{2\alpha^2} \frac{d}{d\tau} (M^2 - 1) \right. \]

\[ + \tilde{M}^2 \frac{d}{d\tau} \left( \frac{M^2 - 1}{\alpha^2} \right) \left\} dxd\tau \]

\[ + \int_0^t \int_{\Omega} (\vec{v} \cdot \nabla \tilde{M} - \Delta \tilde{M} + F'(\tilde{M})) \tilde{M} \frac{(M^2 - 1)}{\alpha^2} dxd\tau \]

\[ = \frac{1}{2\alpha^2} \int_{\Omega} (M^2 - 1)^2 dx - \frac{1}{2\alpha^2} \int_{\Omega} (\tilde{M}^2 - 1)(M^2 - 1) dxd\tau + \frac{1}{\alpha^2} \int_{\Omega} (M^2 - 1) dx \]

\[ - \frac{1}{\alpha^2} \int_{\Omega} (M^2 - 1) dx - \int_0^t \int_{\Omega} \frac{M^2 - 1}{\alpha^2} \frac{d}{d\tau} (M^2 - 1) dx d\tau \]

\[ + \int_0^t \int_{\Omega} (\vec{v} \cdot \nabla \tilde{M} - \Delta \tilde{M} + F'(\tilde{M})) \tilde{M} \frac{(M^2 - 1)}{\alpha^2} dxd\tau, \quad (22) \]

which implies

\[ \frac{1}{2\alpha^2} \| M_0^2 - 1 \|^2 - \frac{1}{2\alpha^2} (\tilde{M}_0^2 - 1, M^2 - 1) \]

\[ = - \int_0^t \int_{\Omega} \tilde{M}^2 - 1 \frac{d}{d\tau} (M^2 - 1) dxd\tau \]

\[ + \int_0^t \int_{\Omega} (\vec{v} \cdot \nabla \tilde{M} - \Delta \tilde{M} + F'(\tilde{M})) \tilde{M} \frac{(M^2 - 1)}{\alpha^2} dxd\tau \]

\[ = - \int_0^t \int_{\Omega} \tilde{M}^2 - 1 \frac{d}{d\tau} M(-v \cdot \nabla M + \Delta M - F'(M)) dxd\tau \]

\[ + \int_0^t \int_{\Omega} (\vec{v} \cdot \nabla \tilde{M} - \Delta \tilde{M} + F'(\tilde{M})) \tilde{M} \frac{(M^2 - 1)}{\alpha^2} dxd\tau \]

\[ = - \int_0^t \int_{\Omega} [F'(\tilde{M})(\Delta M - F'(M)) + F'(M)(\Delta \tilde{M} - F'(\tilde{M}))] dxd\tau \]

\[ - \int_0^t \int_{\Omega} [-v \cdot \nabla MF'(\tilde{M}) - \vec{v} \cdot \nabla \tilde{M} F'(M)] dxd\tau \]

\[ + \int_0^t \int_{\Omega} \tilde{M}^2 - 1 \frac{d}{d\tau} M'(-v \cdot \nabla M + \Delta M - F'(M)) dxd\tau \]

\[ - \int_0^t \int_{\Omega} \tilde{M}^2 - 1 \frac{d}{d\tau} M'(-\vec{v} \cdot \nabla \tilde{M} + \Delta \tilde{M} - F'(\tilde{M})) dxd\tau. \quad (23) \]

Then it follows from \((19) - (23)\) that

\[ E(v^*, E^*, M^*; t) + \frac{1}{4\alpha^2} \|(M^2 - 1) - (M^2 - 1)\|^2 \]

\[ \leq -\mu \int_0^t ||\nabla v^*||^2 d\tau - \int_0^t ||\Delta M^* - F'(\tilde{M}) + F'(M)||^2 d\tau + R_1 + R_2 + R_3, \quad (24) \]

where the reminder terms \(R_1, R_2\) and \(R_3\) are given by

\[ R_1 = \int_0^t \int_{\Omega} (\vec{v} \otimes v - \vec{v} \otimes \vec{v}) : \nabla v dxd\tau, \]
\[ R_2 = \int_0^t \int_\Omega (E_t \otimes E_t : \nabla \bar{v} + \tilde{E}_t \otimes \tilde{E}_t : \nabla v) d\tau d\sigma \]
\[ + \int_0^t \int_\Omega [(\tilde{E}_t \otimes v - \tilde{E}_t \otimes \bar{v}) : \nabla E_t - \tilde{E}_t \otimes E_t : \nabla v - E_t \otimes \tilde{E}_t : \nabla \bar{v}] d\tau d\sigma, \]

\[ R_3 = \int_0^t \int_\Omega (\bar{v} \cdot \nabla M_t \Delta M_t + v \cdot \nabla M_t \Delta M_t) d\tau d\sigma \]
\[ - \int_0^t \int_\Omega (v \cdot \nabla M_t \Delta M_t + \bar{v} \cdot \nabla \tilde{M}_t \Delta \tilde{M}_t) d\tau d\sigma \]
\[ - \int_0^t \int_\Omega (-v \cdot \nabla M F'(\tilde{M}) - \bar{v} \cdot \nabla \tilde{M} F'(\tilde{M})) d\tau d\sigma \]
\[ + \int_0^t \int_\Omega \frac{M^2 - 1}{\alpha^2} M^* (-v \cdot \nabla M + \Delta M - F'(M)) d\tau d\sigma \]
\[ - \int_0^t \int_\Omega \frac{M^2 - 1}{\alpha^2} M^*(-\bar{v} \cdot \nabla \tilde{M} + \Delta \tilde{M} - F'\tilde{M})) d\tau d\sigma. \]

Using the incompressible condition, we compute that

\[ R_1 = -\int_0^t \int_\Omega \nabla \times \nabla v d\sigma d\tau \leq C \int_0^t \| \nabla v \|_{L^\infty} \| v^* \|^2 d\tau, \quad (25) \]

\[ R_2 = \int_0^t \int_\Omega (-E_t \otimes E_t^* : \nabla \bar{v} + \tilde{E}_t \otimes E_t^* : \nabla v) d\tau d\sigma \]
\[ = -\int_0^t \int_\Omega (-E_t \otimes E_t^* : \nabla v + E_t^* \otimes E_t^* : \nabla v) d\tau d\sigma \]
\[ \leq \frac{\mu}{2} \int_0^t \| \nabla v^* \|^2 d\tau + C \int_0^t (\| E^* \|^2_{L^\infty} + \| \nabla v \|_{L^\infty} + \| \nabla E \|_{L^\infty})(\| v^* \|^2 + \| E^* \|^2) d\tau, \quad (26) \]

and

\[ R_3 = \int_0^t \int_\Omega (-\bar{v} \cdot \nabla M_t^* \Delta M_t + v \cdot \nabla M_t^* \Delta M_t) d\tau d\sigma \]
\[ - \int_0^t \int_\Omega (v \cdot \nabla M^*(F'(\tilde{M}) - F'(M)) d\tau d\sigma \]
\[ + \int_0^t \int_\Omega (M^2 - 1 - (M^2 - 1)) M^* M_t d\tau d\sigma \]
\[ + \int_0^t \int_\Omega \frac{M^2 - 1}{\alpha^2} M^*(v \cdot \nabla M + \bar{v} \cdot \nabla \tilde{M}) d\tau d\sigma \]
\[ - \int_0^t \int_\Omega \frac{M^2 - 1}{\alpha^2} M^*(\Delta M^* - F'(\tilde{M}) + F'(M)) d\tau d\sigma \]
\[ = \int_0^t \int_\Omega -v^* \cdot \nabla M_t^* \Delta M_t d\tau d\sigma \]
\[ + \int_0^t \int_\Omega \nabla M_t^* (\Delta M_t^* - F'(\tilde{M}) + F'(M)) d\tau d\sigma \]
\[ + \int_0^t \int_\Omega \nabla M^* F'(M) d\tau d\sigma + \int_0^t \int_\Omega \frac{(M^2 - 1) - (M^2 - 1)}{\alpha^2} M^* M_t d\tau d\sigma \]
Then we infer from the above estimates and the inequality (24) that

\[
E \leq -\frac{1}{2} \int_0^t \|\Delta M^* - F'(\bar{M}) + F'(M)\|^2 d\tau
\]

Using the incompressibility condition again, we can derive the following estimates

\[
\frac{1}{2} \int_0^t \|\Delta M^* - F'(\bar{M}) + F'(M)\|^2 d\tau
\]

In order to control the \(L^6\)-norm of \(M^*\) in the last line of the above inequality, thanks to the Sobolev embedding theorem, we only need to estimate the \(L^2\)-norm of \(M^*\). In fact, \(M^*\) satisfies the following equation in distributional sense:

\[
M^*_t + \bar{v} \cdot \nabla \bar{M} - v \cdot \nabla M = \Delta M^* - (F'(\bar{M}) - F'(M)).
\]

Noting Definition 2.3, we can take \(L^2\) inner product with the above equation with \(M^*\). After integration by parts, we obtain

\[
\frac{1}{2} \int \frac{d}{dt} \|M^*\|^2 + \|\nabla M^*\|^2
\]

Using the incompressibility condition again, we can derive the following estimates by the Cauchy and Hölder inequalities

\[
\int (v \cdot \nabla MM^* - \bar{v} \cdot \nabla \bar{M}M^*) dx = \int (v \cdot \nabla M\bar{M} + \bar{v} \cdot \nabla \bar{M}M) dx
\]

and

\[
\int (F'(\bar{M}) - F'(M))M^* dx \leq \|M^*\|^2 + \|\bar{M} - M\|^2.
\]
Recalling that the weak solution \( \bar{v}, \bar{E}, \bar{M} \) satisfies the energy inequality (16), then we infer from the above inequality and Gronwall’s lemma that

\[
\leq C(1 + \|\bar{M}\|_{H^2}^2)\|M^*\|^2 + \frac{C}{\alpha^4} \|M\|_{H^2}^2(\|\bar{M}^2 - 1\| - (M^2 - 1))
\]

Then we deduce that

\[
\frac{d}{dt}\|M^*\|^2 + \|\nabla M^*\|^2 \leq C \left( \|v^*\|^2 + \frac{1}{\alpha^4} \|(\bar{M}^2 - 1) - (M^2 - 1)\| \right) + C \left( 1 + \|\bar{M}\|^2 \right) \|M^*\|^2.
\]

Recalling that the weak solution \((\bar{v}, \bar{E}, \bar{M})\) satisfies the energy inequality (28), then we infer from the above inequality and Gronwall’s lemma that

\[
\|M^*(t)\|^2 \leq C \int_0^t e^{C(t-\tau)} \left( \|v^*\|^2 + \frac{1}{\alpha^4} \|(\bar{M}^2 - 1) - (M^2 - 1)\| \right) d\tau
\]

Set

\[
g(t) = \mathcal{E}(v^*, E^*, M^*; t) + \frac{1}{4\alpha^4} \|(\bar{M}^2 - 1) - (M^2 - 1)\|
\]

and

\[
h(t) = \|\nabla v\|_L^\infty + \|\nabla E\|_L^\infty + \|E\|_L^\infty + \|\Delta M\|_L^\infty + \|M_t\|_L^3
\]

\[
+ \|\nabla M\|_L^5 \|M^2 - 1\|_L^6 + \|v\|_L^6 \|M^2 - 1\|_L^6 + \|M^2 - 1\|_L^3 + \|F'(M)\|_L^\infty.
\]

Using the Sobolev embedding theorem, for any \( t \leq T < +\infty \), we can write the inequality (28) as

\[
g(t) \leq g(0) + C \int_0^t h(\tau) \left[ g(\tau) + \int_0^\tau e^{C(t-s)} g(s) ds \right] d\tau
\]

\[
= C \int_0^t \left( h(\tau) + \int_\tau^t h(s)e^{s-\tau} ds \right) g(\tau) d\tau. \tag{31}
\]

Since \((v, E, M)\) is a strong solution of the system (1) – (4), it follows from the Sobolev embedding theorem that \(h(t) \in L^1(0,T)\). Thus by Gronwall’s lemma, we infer from the fact \(g(0) = 0\) that \(g(t) = 0\) on \([0,T]\). As a consequence, we obtain the weak-strong uniqueness for solutions of problem (1) – (4).

The proof of Theorem 2.4 is complete.

Acknowledgments. The author would like to express her sincere gratitude to Prof. Jiaxing Hong for his encouragement and support. Part of the work was done during the author’s visit to School of Mathematical Sciences at Fudan University, whose hospitality is gratefully acknowledged.

REFERENCES

[1] B. Benešová, J. Forster, C. García-Cervera, C. Liu and A. Schlömerkemper, Analysis of the flow of magnetoelastic materials, Proc. Appl. Meth. Mech., 16 (2016), 663–664.

[2] B. Benešová, J. Forster, C. Liu and A. Schlömerkemper, Existence of weak solutions to an evolutionary model for magnetoelasticity, SIAM J. Math. Anal., 50 (2018), 1200–1236.

[3] C. Cavaterra, E. Rocca and H. Wu, Global weak solution and blow-up criterion of the general Ericksen-Leslie system for nematic liquid crystal flows, J. Differ. Equ., 255 (2013), 24–57.

[4] Y. Chen and P. Zhang, The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions, Comm. Partial Differ. Equ., 31 (2006), 1793–1810.

[5] E. Emmrich and R. Lasarzik, Weak-strong uniqueness for the general Ericksen-Leslie system in three dimensions, Discrete Contin. Dyn. Syst., 38 (2018), 4617–4635.
WEAK-STRONG UNIQUENESS OF IMVF

[6] E. Feireisl, Y. Lu and A. Novotný, Weak-strong uniqueness for the compressible Navier-Stokes equations with a hard-sphere pressure law, Sci. China Math., 61 (2018), 2003–2016.

[7] E. Feireisl, B. J. Jin and A. Novotný, Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system, J. Math. Fluid Mech., 14 (2012), 717–730.

[8] J. Forster, Variational Approach to the Modeling and Analysis of Magnetoelastic Materials, Ph.D thesis, University of Würzburg, 2016.

[9] P. Germain, Weak-strong uniqueness for the isentropic compressible Navier-Stokes system, J. Math. Fluid Mech., 13 (2011), 137–146.

[10] M. Grasselli and H. Wu, Long-time behavior for a nematic liquid crystal model with asymptotic stabilizing boundary condition and external force, SIAM J. Math. Anal., 45 (2013), 965–1002.

[11] X. P. Hu and F. H. Lin, Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data, Commun. Pure Appl. Math., 69 (2016), 372–404.

[12] X. P. Hu and H. Wu, Long-time behavior and weak-strong uniqueness for incompressible viscoelastic flows, Discrete Contin. Dyn. Syst., 35 (2015), 3437–3461.

[13] Y. Hyon, D. Y. Kwak and C. Liu, Energetic variational approach in complex fluids: maximum dissipation principle, Discrete Contin. Dyn. Syst., 26 (2010), 1291–1304.

[14] N. Jing, H. Liu, and Y. L. Luo, Global classical solutions to an evolutionary model for magnetoelasticity, preprint, arXiv:1904.09531v1.

[15] M. Kalousek, On dissipative solutions to a system arising in viscoelasticity, preprint, arXiv:1903.03635.

[16] M. Kalousek, J. Kortum and A. Schlömerkemper, Mathematical analysis of weak and strong solutions to an evolutionary model for magnetoviscoelasticity, preprint, arXiv:1904.07179.

[17] Z. Lei, C. Liu and Y. Zhou, Global solutions for incompressible viscoelastic fluids, Arch. Ration. Mech. Anal., 188 (2008), 371–398.

[18] Z. Lei and Y. Zhou, Global existence of classical solutions for 2D Oldroyd model via the incompressible limit, SIAM J. Math. Anal., 37 (2005), 797–814.

[19] F. H. Lin and C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Commun. Pure Appl. Math., 48 (1995), 501–537.

[20] F. H. Lin, C. Liu and P. Zhang, On hydrodynamics of viscoelastic fluids, Commun. Pure Appl. Math., 58 (2005), 1437–1471.

[21] F. H. Lin and P. Zhang, On the initial boundary value problem of the incompressible viscoelastic fluid system, Commun. Pure Appl. Math., 61 (2008), 539–558.

[22] C. Liu and N. J. Walkington, An Eulerian description of fluids containing visco-hyperelastic particles, Arch. Ration. Mech. Anal., 159 (2001), 229–252.

[23] A. Schlömerkemper and J. Żabensky, Uniqueness of solutions for a mathematical model for magneto-viscoelastic flows, Nonlinearity, 31 (2018), 2989–3012.

[24] J. Serrin, On the interior regularity of weak solutions of Navier-Stokes equations, Arch. Ration. Mech. Anal., 9 (1962), 187–195.

[25] H. Wu, Long-time behavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst., 26 (2010), 379–396.

[26] H. Wu, X. Xu and C. Liu, On the general Ericksen-Leslie system: Parodi’s relation, well-posedness and stability, Arch. Ration. Mech. Anal., 208 (2013), 59–107.

[27] W. J. Zhao, Local well-posedness and blow-up criteria of magneto-viscoelastic flows, Discrete Contin. Dyn. Syst., 38 (2018), 4637–4655.

Received August 2019; revised September 2019.

E-mail address: zhaowenjing@mail.neu.edu.cn