Solving Kepler’s equation via nonlinear sequence transformations

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Since more than three centuries Kepler’s equation continues to represent an important benchmark for testing new computational techniques. In the present paper, the classical Kapteyn series solution of Kepler’s equation originally conceived by Lagrange and Bessel will be revisited from a different perspective, offered by the relatively new and still largely unexplored framework of the so-called nonlinear sequence transformations.

The main scope of the paper is to provide numerical evidences supporting the fact that the Kapteyn series solution of Kepler’s equation could be a Stieltjes series. To support such a conjecture, two types of Levin-type sequence transformations, namely Levin $\delta$- and Weniger $\delta$-transformations, will be employed to sum up several wildly divergent series derived by the Debye representation of Bessel functions. As an interesting byproduct of this analysis, an effective recursive algorithm to generate arbitrarily higher-order Debye’s polynomials will be developed.

Such a “Stieltjesness” conjecture will also be numerically validated by directly employing the Levin-type transformations to accelerate the complex Kapteyn series solution of the Kepler equation. Both $\delta$- and $\delta$-transformations display exponential convergence, whose rate will be numerically estimated.

A few conclusive words, together with some hints for future extensions in the direction of more general class of Kapteyn series, eventually close the paper.

I. INTRODUCTION

Likely, there are a very few equations in the world (if any) which can boast such a large number of solving strategies as much as the celebrated Kepler equation (KE henceforth). Since 1650 a huge number of different methods have been conceived to numerically solve KE. In the classical textbook by Colwell [1], most of such methods have been listed and briefly described. However, since 1993 such a list dramatically increased and it is really hard to give account for all strategies developed so far. Just to give an idea, it is worth quoting some of the papers dealing with KE which have been published within the last ten years [2–40].

Despite its apparent mathematical simplicity, solving KE continues to play a pivotal role in the science of computation. As it was pointed out again in [1],

Any new technique for the treatment of transcendental equations should be applied to this illustrious case; any new insight, however slight, lets its conceiver join an eminent list of contributors.

In the present paper we intend to give a further contribution to the subject, by focusing our attention on a celebrated semi-analytic approach to solve KE, based on a Bessel function series expansion [1]. A similar approach has recently been applied to search solutions of the so-called post-Newtonian KE’s [21]. Although the Bessel solution of KE has been abandoned for practical, computational purposes, it presents interesting unexplored features that, in our opinion, deserve to be investigated.

The Lagrange/Bessel solution of KE was the first example of a class of expansions called Kapteyn series (KS henceforth) [11], which in recent times have gained a role of considerable importance [12–47]. One of the scopes of the present paper is to provide numerical evidences suggesting that a particular type of Kapteyn series, which is strictly connected to the Bessel solution of KE, is a Stieltjes series. This, in turns, would imply the whole computational machinery developed for resumming Stieltjes asymptotic series to be employable in order to retrieve the KE solution. This will be done in the second part of the paper, where two examples of the so-called Levin-type nonlinear sequence transformation will be used to accomplish such a computational task.

The paper is structured as follows: in Sec. II a brief resume of the Bessel solution of KE’s equation is outlined, together with a few words about Stieltjes functions and Stieltjes series. Section III is devoted to numerically explore the Stieltjes nature of the Bessel KS, while in Sec. IV the results of several numerical experiments carried out on directly applying Levin-type transformations on the Bessel KS are shown. Some conclusive words, together with some hints to extend our approach to deal with more general classes of Kapteyn series, are given in Sec. V.

II. PRELIMINARIES

A. The Bessel solution of Kepler’s equation

The (elliptic) KE is formally defined as follows:

$$M = \psi - \varepsilon \sin \psi,$$

where the meaning of the quantities $M$, $\psi$, and $\varepsilon$ can be grasped on referring to Fig. I.

Consider the motion of a planet around the star $S$ along an elliptic orbit with eccentricity $\varepsilon$ [28]. Suppose $P$ to be the position of the planet at the time $t$, which is measured on assuming the apocenter $A$ as the initial (i.e., at $t = 0$) position (motion is counterclockwise). The
mean angular speed \( \omega = 2\pi/T \), with \( T \) being the orbital period, is introduced in such a way that \( M = \omega t \), which is called the mean anomaly. To retrieve the position of the planet along the orbit at time \( t > 0 \), the angle \( \psi \) is first evaluated by solving KE \([1]\), which gives the position of the point \( Q \) along the circumscribed circle. \( P \) is then immediately obtained by the geometric construction depicted in Fig. 1.

In the present paper our attention will be focused on the following explicit Fourier series representation of the KE solution: \([1] \) Ch. 3:

\[
\psi = M + S(\epsilon; M),
\]

where the function \( S(\epsilon; M) \) is defined as

\[
S(\epsilon; M) = \sum_{n=1}^{\infty} \frac{2J_n(n \epsilon)}{n} \sin nM.
\]

Although the above series converges for any \( \epsilon \in [0, 1) \), it turns out that such a convergence is extremely slow, especially when \( \epsilon \to 1 \). To give a single numerical example of such a slow convergence, in Tab. I the behaviour of the partial sums of the series into Eq. (3) are shown as a function of their order for \( \epsilon = 9/10 \) and \( M = \pi/4 \).

It is not difficult to understand why such bad convergence features, together with the fact that a considerable number of Bessel function evaluations has to be implemented, unavoidably brought the Fourier series to be definitely abandoned as far as practical applications of KE are concerned. Nevertheless, the Bessel series expansion \([3]\) presents features that make it a subject of considerable interest in math as well as in theoretical physics. The scope of the present paper is to explore some of these features. To this end, function \( S \) is first recast as the imaginary part of the complex function \( S(\epsilon; M) \) defined through the following KS:

\[
S(\epsilon; M) = \sum_{n=1}^{\infty} \frac{2J_n(n \epsilon)}{n} \exp(nM),
\]

which is nothing but a complex power series expansion evaluated across the unit circle. Its convergence could be proved, for example, on invoking the following Bessel function asymptotics, valid for \( \epsilon < 1 \) \([48, \text{Sec. 8.4}]\):

\[
J_n(n \epsilon) \sim \frac{1}{\sqrt{2\pi \sqrt{1 - \epsilon^2}}} \frac{\rho^n}{\sqrt{n}}, \quad n \to \infty,
\]

where

\[
\rho = \exp \left( \sqrt{1 - \epsilon^2} \right) \frac{1 - \sqrt{1 - \epsilon^2}}{\epsilon},
\]

is a positive parameter less than unity when \( \epsilon \in (0, 1) \). On substituting from Eq. (5) into Eq. (4) and on introducing the complex parameter \( z = \rho \exp(\text{i}M) \), it turns out that the single terms of the series \([1]\) asymptotically approach those of the paradigmatic model series

\[
\sum_{n=1}^{\infty} \frac{z^n}{n^{3/2}},
\]

which, for \( |z| < 1 \), is nothing but the Dirichlet series representation of so-called polylogarithm function \( \text{Li}_{3/2}(z) \), where

\[
\text{Li}_\nu(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\nu}, \quad |z| < 1.
\]

For \( |z| \geq 1 \), the series into Eq. (8) can be analytically extended to the whole complex plane but the half-axis

| Partial sum order | Estimate of \( \psi^a \) |
|-------------------|----------------------|
| 0                 | 1.35949...           |
| 1                 | 1.66564...           |
| 2                 | 1.78539...           |
| 3                 | 1.78539...           |
| 4                 | 1.73032...           |
| 5                 | 1.67194...           |
| 10                | 1.70076...           |
| 15                | 1.66772...           |
| 20                | 1.68367...           |
| 25                | 1.68138...           |
| 30                | 1.67725...           |
| 35                | 1.68210...           |
| 40                | 1.67933...           |
| 45                | 1.67968...           |
| 50                | 1.68076...           |
| 55                | 1.67945...           |
| 60                | 1.68023...           |
| 65                | 1.68014...           |
| 70                | 1.67978...           |

\(^a\)The exact value given by numerically solving Eq. (1) is 1.6800337357880455291...
Re\{z\} > 1. More importantly, polylogarithm \( \text{Li}_\nu(z) \) is an example of Stieltjes function. For reader’s convenience, the main definitions of Stieltjes functions and of Stieltjes series will now be briefly summarized. Interested readers can find some useful references for instance in Ref. [19].

A function \( f(z) \), with \( z \in \mathbb{C} \), is called Stieltjes function if it admits the following integral representation:

\[
f(z) = \int_0^\infty \frac{d\mu(t)}{t + z}, \quad |\arg(z)| < \pi ,
\]

where \( \mu(t) \) is a bounded, nondecreasing function taking infinitely many different values on the interval \( 0 \leq t < \infty \). The integral into Eq. (9) is called Stieltjes integral. On introducing the moment sequence \( \{\mu_m\}_{m=0}^\infty \), whose elements are defined as

\[
\mu_m = \int_0^\infty t^m \, d\mu , \quad m \geq 0 ,
\]

it turns out that they are nonnegative for all values of \( m \). Any Stieltjes integral [19] can be associated to a so-called Stieltjes series, which is asymptotic to \( f(z) \) as \( z \to \infty \), in the Poincaré sense, as follows:

\[
f(z) \sim \sum_{m=0}^{\infty} \frac{(-1)^m \mu_m}{z^{m+1}} , \quad z \to \infty .
\]

Whether such a series converges or diverges depends ultimately on the asymptotic behavior of the moment sequence \( \{\mu_m\}_{m=0}^\infty \) as \( m \to \infty \).

Now, proving the Stieltjes nature of polylogarithm functions is almost trivial. It is sufficient to start from the following integral representation of \( \text{Li}_\nu \) [50 25.12.11]:

\[
\text{Li}_\nu(z) = \frac{z}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} \exp(-x)}{z} \, dx = -\frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} \exp(-x)}{\exp(-x) - 1/z} \, dx ,
\]

and set \( t = \exp(-x) \) to have

\[
\text{Li}_\nu(z) = -\frac{1}{\Gamma(\nu)} \int_0^1 \frac{(-\log t)^{\nu-1}}{t - 1/z} \, dt .
\]

Apart from the overall minus sign, the function \( \text{Li}_\nu \) can then be written in the form of an Stieltjes integral [19], with \(-1/z\) in place of \( z \) and with the measure, say \( \mu_\nu(t) \), defined as

\[
\mu_\nu(t) = \begin{cases} 
\Gamma(\nu, -\log t) / \Gamma(\nu) , & 0 \leq t \leq 1 , \\
1 , & t > 1 ,
\end{cases}
\]

where symbols \( \Gamma(\cdot) \) and \( \Gamma(\cdot, \cdot) \) denote the gamma and the incomplete gamma functions, respectively [50].

The above described “asymptotic connection” between series [1] and [7] via Eq. (5) would at first sight suggest the function \( S(\epsilon; M) \) to be Stieltjes too. Although we are not able to provide a rigorous proof about such a conjecture, in the next section some strong numerical evidences supporting it will be given.

III. IS \( S(\epsilon; M) \) A STIELTJES SERIES?

A. Debye’s expansion

The Bessel function asymptotics given into Eq. (5) is the leading term of a complete series expansion known as Debye’s expansion. More explicitly, we have [50 formula (10.19.3)]

\[
J_\nu(n \epsilon) = \frac{\rho^n}{\sqrt{2\pi} \sqrt{1 - \epsilon^2}} \sum_{k=0}^{\infty} \frac{1}{n^{k+1/2}} U_k \left( \frac{1}{\sqrt{1 - \epsilon^2}} \right) ,
\]

where the symbol \( U_k(\cdot) \) denotes the so-called \( k \)th-order Debye polynomial.

Debye polynomials are still largely unexplored mathematical objects. They are formally defined through the following integro-differential recurrence rule [50 formula (10.41.9)]:

\[
U_{k+1}(t) = \frac{t^2}{2} (1 - t^2) \frac{dU_k}{dt} + \frac{1}{8} \int_0^t (1 - 5\xi^2) U_k(\xi) \, d\xi ,
\]

with the initial condition \( U_0(t) = 1 \). Now, from a mere computational point of view, definition (16) makes the generation of higher-order Debye polynomials considerably time demanding, even if powerful symbolic platforms are employed. To give an idea, the commercial software Mathematica 12.0 requires, on a DELL workstation, about 10 seconds to generate \( U_{11}(t) \) but more than 12 minutes to generate \( U_{13}(t) \).

Since a deep numerical analysis of the Debye expansion [15] is mandatory for the scopes of the present paper, in Appendix A an effective recursive algorithm to compute only the coefficients of \( U_k(t) \), starting from the definition into Eq. (16), has been outlined. In this way, to generate Debye polynomials having degrees of the order of hundreds or even of thousands takes only a few seconds. For instance, to produce \( U_{1000}(t) \) on the above described computing machine and with the same symbolic platform only 25 seconds are required.

Equipped with such a powerful computational tool, we are now ready to start our numerical exploration of the Debye expansion [15]. To begin, in Fig. 2 the behavior, as a function of \( k \), of the modulus of the single terms of the series into Eq. (15) is shown for \( n = 10 \) and for \( \epsilon = 5/10 \) (a) and \( \epsilon = 9/10 \) (b). Both sequences appear to be clearly divergent.

Such a divergence should entirely be ascribed to the asymptotic behaviour of the Debye polynomials for large values of their orders (for \( \epsilon < 1 \) their argument is always greater that one). Unfortunately, the mathematical properties of Debye polynomials are still largely unknown. On the other hand, their exploration is certainly beyond the scopes of the present paper. For such reason we shall limit ourselves to grasp the asymptotics of \( U_k(t) \) for \( k \to \infty \) and \( t \gg 1 \), in such a way that \( U_k(t) \sim a_{3k} t^{3k} \).
FIG. 2: Behavior, as a function of $k$, of the modulus of the single terms of the series in Eq. (15), shown for $n = 10$ and $\epsilon = 5/10$ (a), $\epsilon = 9/10$ (b).

In particular, in Appendix B it is shown that

$$a_{3k}^k \sim C \left( -\frac{3}{2} \right)^k k!, \quad k \to \infty, \quad (17)$$

where $C$ denotes a positive constant that can be estimated empirically. To this aim, in Fig. 3 the behaviour of $|a_{3k}^k|$ versus $k$ is shown (dots). The solid curve represents the asymptotic law (17), evaluated with $C = 10^4$.

The reasonably good agreement, at least at a visual level, suggests that also the single terms of the series (15) could asymptotically follow, at least in the limiting case $\epsilon \to 1$, the factorial law

$$\rho^n C \frac{C}{\sqrt{2\pi n}} \frac{3}{\sqrt{1 - \epsilon^2}} \frac{(3/2)^n}{n^3/2} k!, \quad k \to \infty. \quad (18)$$

In Fig. 4 the modulus of the single terms of the series (15) is shown as a function of the index $k$ (dots) together with Eq. (18) (solid curve), when $n = 10$ and $\epsilon = 99/100$.

Once the factorially divergent character of the expansion (15) has been established (at least on an empirical basis), it remains to see whether and how the quantity $J_n(\epsilon n C)$ could be retrieved starting from the corresponding partial sum sequence. To this end, it must be recalled that several classes of factorially divergent series can be efficiently decoded by using suitable transformation schemes, the most known of them being Borel summation [51] and Padé approximants [52]. However, new resummation algorithms more effective than Borel and Padé have been proposed in the last three decades for decoding factorially divergent alternating series. In the second part of our work these relatively new computational algorithms, called Levin-type nonlinear sequence transformations [53], will directly be applied to the partial sum sequences of the series in Eq. (15).

For reader’s convenience, in the next section a brief survey of Levin-type transformations will now be given.

B. Levin-type nonlinear sequence transformations

Several nonlinear transformations have been conceived to convert the sequence of the partial sums of a diverging series into new sequences converging to the so-called generalized limit of the series itself, namely the (finite) value “hidden” within the divergent partial sum sequence. The probably most known transformation scheme is the so-called Wynn or $\epsilon$-algorithm [54], which implements Padé approximants through a surprisingly easy recursive scheme.
Consider the partial sum sequence, say \( \{ s_n \}_{n=0}^{\infty} \), of a given series \( \sum_{k=0}^{\infty} a_k \), i.e.,

\[
s_n = \sum_{k=0}^{n} a_k, \quad n = 0, 1, 2, \ldots \tag{19}
\]

A typical feature of Wynn’s algorithm (and of several other sequence transformations) is that only the input of the numerical values of a finite substring of the sequence \( \{ s_n \}_{n=0}^{\infty} \) is required to achieve resummation. In other words, no further information is needed to compute approximations of the limit (or the antilimit) of (19), i.e., of the series sum.

However, in some cases additional information on the index dependence of the truncation errors is available. For instance, the magnitudes of the truncation errors of Stieltjes series are bounded by the first term neglected in the partial sum and they also display the same sign patterns. In 1973, Levin \cite{53} introduced a sequence transformation which utilizes such a valuable \textit{a priori} structural information to enhance the efficiency and the convergence speed of the transformed sequence. For a general discussion about the construction principles of Levin-type sequence transformations, readers are encouraged to go through the paper by Weniger \cite{55} as well as through Ref. \cite{70}, which contains a slightly upgraded and formally improved version of the former. In the same paper it was also rigorously proved that Levin-type transformations are able to sum what is considered to be the paradigmatic factorially divergent series in physics, the Euler series, with convergence rates considerably higher than Padé approximants.

In the following, two special Levin-type transformations will be employed to numerically explore the Stieltjes nature of the \( S(\epsilon; M) \). The first is the so-called Levin \textit{d}-transformation, which maps the partial sum sequence \( \{ s_n \} \) into the new sequence \( \{ d_k \} \) defined as follows \cite{55}:

\[
d_k = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} (1 + j)^{k-1} \frac{s_j}{a_{j+1}}, \quad k = 1, 2, \ldots \tag{20}
\]

The other transformation is the so-called Weniger transformation, also called \textit{δ}-transformation, which maps the sequence \( \{ s_n \} \) into the sequence \( \{ \delta_k \} \) defined as follows \cite{55}:

\[
\delta_k = \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k}{j} (1 + j)^{k-1} \frac{s_j}{a_{j+1}}, \quad k = 1, 2, \ldots \tag{21}
\]

where \((\cdot)_k\) denotes Pochhammer symbol. It is worth nothing that in order to pass from the \( \{ d_k \} \) sequence to the \( \{ \delta_k \} \) sequence it is sufficient to formally replace the exponential term \((1 + j)^{k-1}\) by the Pochhammer term \((1 + j)\) for \( k = 1, 2, \ldots \), and \textit{viceversa}.

Despite their unusual and first sight apparently complicated aspects, sequence transformations into Eqs. (20) and (21) are of straightforward implementation. For instance, in the last twenty years they have profitably been employed to solve several optical problems, showing an extraordinary effectiveness in summing divergent as well as slowly convergent series \cite{57,68}.

Our first numerical experiment is aimed at appreciating the powerfulness of \textit{d}- and \textit{δ}-transformations in resumming the divergent Debye expansion \cite{15} to retrieve, from the partial sum sequences represented in Figs. 2a and 2b, the “coded” values of \( J_{10}(5) \) and \( J_{10}(9) \), respectively. Table II refers to \( J_{10}(5) \): the values of the partial sum sequence \( \{ s_n \} \) of the series \( \{ J_{10} \} \) are listed in the second column, whereas the third and the fourth columns contain the values of the sequences \( \{ d_k \} \) and \( \{ \delta_k \} \) obtained by applying, on the sequence \( \{ s_n \} \), maps (20) and (21), respectively (the corresponding orders are reported in the first column). It can be appreciated that both sequences converge (approximately with the same speed) toward the correct limit \( J_{10}(5) = 0.001467802647 \ldots \) which is only partially touched by the (diverging) partial sum sequence.

Things are much more evident as far as the evaluation of \( J_{10}(9) \) is concerned. This is shown in Table III where it is easier to grasp the alternating divergent character of the partial sum sequence, dramatically evident since order 5. At the same time, it is now possible to appreciate the different convergence speed of \textit{d}- and \textit{δ}-transformations in retrieving the correct value \( J_{10}(9) = 0.1246940928 \ldots \)

C. Is Kepler Kapteyn series Stieltjes?

Equipped with the above mathematical tools (Debye expansion and Levin-type transformations), we are now ready to explore the Stieltjes nature of the KS \cite{5}. To this end, Eq. (15) is first substituted into Eq. (4) to give

\[
S(\epsilon; M) = \sqrt{\frac{2}{\pi \sqrt{1-\epsilon^2}}} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{z^n}{n^{k+3/2}} U_k \left( \frac{1}{\sqrt{1-\epsilon^2}} \right),
\]

which, on interchanging the order of the \( n \)- and of the \( k \)-series, leads to

\[
S(\epsilon; M) = \sqrt{\frac{2}{\pi \sqrt{1-\epsilon^2}}} \sum_{k=0}^{\infty} U_k \left( \frac{1}{\sqrt{1-\epsilon^2}} \right) \sum_{n=1}^{\infty} \frac{z^n}{n^{k+3/2}},
\]

where
and finally, on taking Eq. (8) into account, to

\[ S(\epsilon; M) = \sqrt{\frac{2}{\pi \sqrt{1 - \epsilon^2}}} \sum_{k=0}^{\infty} U_k \left( \frac{1}{\sqrt{1 - \epsilon^2}} \right) \text{Li}_{k+3/2}(z). \]  

(24)

The subsequent step consists in substituting the integral representation of polylogarithm function given by Eq. (13) into Eq. (24) and then to interchange the series with the integral. After simple algebra the following integrals representation of \( S(\epsilon; M) \) can then be established:

\[ S(\epsilon; M) = -\frac{2}{\pi \sqrt{1 - \epsilon^2}} \int_0^1 U \left( -\log t, \frac{1}{\sqrt{1 - \epsilon^2}} \right) \frac{1}{t - 1/z} \, dt, \]  

with the function \( U(\cdot, \cdot) \) being defined as follows:

\[ U(x, y) = \sum_{k=0}^{\infty} \frac{x^{k+1/2}}{\Gamma \left( k + \frac{3}{2} \right)} U_k(y), \quad x \geq 0, \ y > 1. \]  

(25)

In order to prove the “Stieltjesness” of \( S \), it would be mandatory to show that, for any given \( y > 1 \), both \( U \) and \( \partial U/\partial x \) to be nonnegative for \( x \in [0, \infty) \). Presently we do not possess a rigorous proof of it, so that we shall limit ourselves to provide strong numerical evidences of such a conjecture. In doing so, Levin-type transformations will be again employed to decode the series into Eq. (26) which, thanks to the above described asymptotics of Debye’s polynomials, is expected to display a factorial divergence. The

| order | Partial sums | \( d_k \) | \( \delta_k \) |
|-------|-------------|---------|---------|
| 1     | 0.001492003408 | 0.1246940928 |
| 2     | 0.001465682591 | 0.001467770986 |
| 3     | 0.001468263281 | 0.001467802631 |
| 4     | 0.001467586862 | 0.001467802641 |
| 5     | 0.001467863379 | 0.001467802647 |
| 6     | 0.001467770986 | 0.001467802647 |
| 7     | 0.001467822501 | 0.001467802647 |
| 8     | 0.001467788088 | 0.001467802647 |
| 9     | 0.001467814880 | 0.001467802647 |
| 10    | 0.001467791058 | 0.001467802647 |
| 11    | 0.001467814875 | 0.001467802647 |
| 12    | 0.001467788427 | 0.001467802647 |
| 13    | 0.001467820725 | 0.001467802647 |
| 14    | 0.001467777704 | 0.001467802647 |
| 15    | 0.001467873974 | 0.001467802647 |
| 16    | 0.001467743345 | 0.001467802647 |
| 17    | 0.001467903836 | 0.001467802647 |
| 18    | 0.001467618939 | 0.001467802647 |
| 19    | 0.001468156252 | 0.001467802647 |
| 20    | 0.001467083337 | 0.001467802647 |
| 21    | 0.001469344663 | 0.001467802647 |
| 22    | 0.00146327946 | 0.001467802647 |
| 23    | 0.001476013517 | 0.001467802647 |
| 24    | 0.00147498726 | 0.001467802647 |
| 25    | 0.001520240513 | 0.001467802647 |
| 26    | 0.001326611312 | 0.001467802647 |
| 27    | 0.001863491524 | 0.001467802647 |
| 28    | 0.0003153551669 | 0.001467802647 |
| 29    | 0.0004951150350 | 0.001467802647 |
| 30    | -0.009444360750 | 0.001467802647 |

and finally, on taking Eq. (8) into account, to

\[ S(\epsilon; M) = \sqrt{\frac{2}{\pi \sqrt{1 - \epsilon^2}}} \sum_{k=0}^{\infty} U_k \left( \frac{1}{\sqrt{1 - \epsilon^2}} \right) \text{Li}_{k+3/2}(z). \]  

(24)

The subsequent step consists in substituting the integral representation of polylogarithm function given by Eq. (13) into Eq. (24) and then to interchange the series with the integral. After simple algebra the following inte-
result of a preliminary numerical experiment is shown in Tab. IV where \( d \) and \( \delta \) were employed to retrieve the value of \( U\left(\log 2, \frac{100}{\sqrt{199}}\right) \), corresponding to the pair \( (t, \epsilon) = (1/2, 99/100) \). Also in that case the convergence of both \( d \)-transformed and \( \delta \)-transformed sequences is more than evident. However, the performances of the \( d \)-transformation appears to be slightly superior with respect those of \( \delta \). Such a superiority has been confirmed during other several tests carried out on different pairs \( (t, \epsilon) \) (not shown here for evident room reasons). For such reason, in the rest of the paper only the Levin \( d \)-transformation will be used to provide evidences of the “Stieltjes conjecture”.

To this end, in Fig. 5 the behaviour of \( U\left(-\log t, \frac{1}{\sqrt{1-\epsilon^2}}\right) \) has been plotted for \( t \in (0, 1] \) and for \( \epsilon = 99/100 \). The four curves correspond to different values of the \( d \)-transformation order, precisely 6 (solid thin curve), 10 (dashed curve), 20 (dotted curve), and 40 (thick solid curve). The last two curve are practically coincident, which confirms the rapid convergence of \( d \)-transformation for all \( t \in (0, 1] \). More importantly, the shape of the graphic of the generating function \( U \) seems to possess those features which are mandatory for integral \( [25] \) to be a *Stieltjes integral*.

![FIG. 5: Behaviour of \( U(-\log t, 1/\sqrt{1-\epsilon^2}) \) against \( t \) for different values of the \( d \)-transformation order.](image)

![FIG. 6: Behaviour of \( U(-\log t, 1/\sqrt{1-\epsilon^2}) \) against \( t \) for for \( \epsilon = 1/10 \) (solid thin curve), \( \epsilon = 5/10 \) (dashed curve), \( \epsilon = 7/10 \) (dotted curve), and \( \epsilon = 9/10 \) (solid thick curve). All curves have been generated by setting the \( d \)-transformation order at 40.](image)

Smaller values of \( \epsilon \) are expected to give much less numerical troubles that 99/100. In Fig. 5 the behaviour of \( U\left(-\log t, \frac{1}{\sqrt{1-\epsilon^2}}\right) \) is still represented vs. \( t \) for \( \epsilon = 1/10 \) (solid thin curve), \( \epsilon = 5/10 \) (dashed curve), \( \epsilon = 7/10 \) (dotted curve), and \( \epsilon = 9/10 \) (solid thick curve). All curves have been generated by setting the \( d \)-transformation order at 40, a value which guarantees a rapid convergence for any \( t \), as confirmed by the results of several numerical tests not reported here. Moreover, further numerical experiments have also been carried out for different values of \( \epsilon \) and of the \( d \)-transformation order.

Figures 5 and 6 are the main results of the present

![TABLE IV: Using Levin-type transformations to retrieve the value of \( U\left(\log 2, \frac{100}{\sqrt{199}}\right) \) via the divergent expansion given in Eq. (26).](table)

| order | Partial sums | \( d_k \) | \( \delta_k \) |
|-------|-------------|-------|-------|
| 1     | 0.9394372787 | –     | –     |
| 2     | -30.89260169 | 0.08325018113 | 0.5787695135 |
| 3     | 4951.945127  | 0.18043920913 | 0.5112075442 |
| 4     | 2.620274608 \times 10^8 | 0.3855265022 | 0.4605014333 |
| 5     | -5.44468076 \times 10^20 | 0.4123795954 | 0.4154058936 |
| 6     | 1.87304379 \times 10^{-33} | 0.4131211762 | 0.4119027710 |
| 7     | -5.70995144 \times 10^{-45} | 0.4128774275 | 0.4125818326 |
| 8     | 3.65003960 \times 10^{-58} | 0.4128567548 | 0.4129445672 |
| 9     | -8.13835802 \times 10^{-71} | 0.4128573130 | 0.4130283827 |
| 10    | 9.40035017 \times 10^{-83} | 0.4128574819 | 0.4130014720 |
| 11    | -5.03078710 \times 10^{-96} | 0.4128574565 | 0.4129603691 |
| 12    | 2.75995841 \times 10^{-109} | 0.4128574649 | 0.4129237187 |
| 13    | -1.54290197 \times 10^{-122} | 0.4128574648 | 0.4128982813 |
| 14    | 8.75710192 \times 10^{-134} | 0.4128574648 | 0.4128819512 |
| 15    | -5.03576430 \times 10^{-147} | 0.4128574648 | 0.4128718730 |
| 16    | 2.92692032 \times 10^{-160} | 0.4128574648 | 0.4128657968 |
| 17    | -1.76729904 \times 10^{-173} | 0.4128574648 | 0.4128621941 |
| 18    | 1.014189268 \times 10^{-186} | 0.4128574648 | 0.4128600897 |
| 19    | -6.039885436 \times 10^{-199} | 0.4128574648 | 0.4128588796 |
| 20    | 3.616269264 \times 10^{-211} | 0.4128574648 | 0.4128581966 |
| 21    | -2.176637686 \times 10^{-224} | 0.4128574648 | 0.4128582020 |
| 22    | 1.316300235 \times 10^{-237} | 0.4128574648 | 0.4128576192 |
| 23    | -7.993851066 \times 10^{-249} | 0.4128574648 | 0.4128575168 |
| 24    | 4.873145754 \times 10^{-262} | 0.4128574648 | 0.4128574683 |

...
IV. SOLVING KEPLER’S EQUATION VIA LEVIN-TYPE TRANSFORMATIONS

As it was said before, nonlinear sequence transformations in general and Levin-type transformation in particular have proved to have extraordinary capabilities as far as Stieltjes asymptotic series decoding is concerned. However, differently from Padé approximants, at present a convergence theory of nonlinear sequence transformations on Stieltjes series is far from being complete. Accordingly, all numerical experiments within such a perspective should be welcomed. In the previous sections the Stieltjes nature of the function $S$ has been conjectured and numerically “validated” on the basis of several experiments. It is now time to analyze the performances of $d$- and $\delta$-transformations when they are directly applied on the KS series in Eq. (1) to solve Kepler’s Equation (1).

Consider as a first example the solution of KE for $M = \pi/2$ and for different values of $\epsilon$. In Fig. 7 the relative error, evaluated through Eqs. (20) and (21), with respect to the value obtained by directly solving Eq. (1) via standard numerical methods, against the $\delta$- (a) and the $d$- (b) transformation order $k$. The open circles correspond to $\epsilon = 2/10$, the black circles to $\epsilon = 6/10$, the open squares to $\epsilon = 9/10$, and the black squares to $\epsilon = 99/100$.

In order to extract a quantitative information about the convergence rates of the $\delta$-transformation, in Fig. 8 the behaviour of the relative transformation error is plotted, as a function of the transformation order $k$, for $M = \pi/2$ and $\epsilon = 99/100$ (open circles). The solid curve represents the analytical fit $\exp(-\alpha k\nu)$, where the parameter $\nu$ provides a meaningful measure of the convergence rate.

A similar analysis was carried out in [70] to estimate the convergence rate, in the limit of $k \gg 1$, of both $\delta$- and $d$-transformations, together with Padé approximants, when they were applied to the Euler series. It was there found that $\nu = 2/3$ for the $\delta$-transformation and $\nu \approx 3/4$ for the $d$-transformation, both of them greater than the convergence rate of Padé approximants, $\nu = 1/2$ [70]. For the Kepler equation a similar analysis can be carried out, although necessarily from a numerical perspective, as it is shown for example in Fig. 8 corresponding to the case $M = \pi/2$ and $\epsilon = 99/100$. The fit is evaluated on the tail (i.e., for $k > 10$), and gives $\nu \approx 1$. Similarly as it was done in [70], Fig. 9 shows the same as in Fig. 8 but for the Levin $d$-transformation in place of the $\delta$-transformation.

In the present case the convergence rate of $\delta$-transformation is slightly greater than that of $d$-transformation. However, a more detailed numerical investigation reveals that the latter is on average more efficient than the former, as reported in Fig. 10 where the behaviour of $\nu$ is plotted against $\epsilon \in [1/10, 99/100]$, for $M = \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4$. Black curves correspond to Weniger $\delta$-transformation, gray curves to Levin $d$-transformation.
need of developing new methods to achieve summability of Kapteyn series of both first and second kind have already been invoked in the past [39]. In this respect, we believe the results presented here could be of some help. In fact, it would be rather natural ask whether Levin-type transformations would still be so effective in resumming more general type of Kapteyn series, similarly as done in [43] on using Wynn’s ε-algorithm. Just to give an idea, consider what follows.

Several numerical tests (not shown here) carried out on the Kapteyn series \( \sum_{m \geq 1} \frac{z^m}{m} J_m(me) \), showed Levin-type transformations to be able in resumming the series not only for \(|z| = 1\), which corresponds to the KE case, but also for \(|z| > 1\), provided that \(|\arg(z)| < \pi\). This could be a direct consequence of the Stieltjes conjecture. In other words, the Kapteyn series could be “numerically continued” to a function \( F(z; \epsilon) \) living, for \( \epsilon \in [0, 1] \), on the whole complex plane but the infinite interval \((1, \infty)\). However, things are more complex than they could appear. In Tab. V the performances of Levin \( d\)- and of Weniger \( \delta\)-transformations in resumming the Kapteyn series for \( \epsilon = 9/10 \) and \( z = 10 \exp(i\pi/3) \) are shown. The indisputable superiority of \( \delta \) over \( d \) is now more than evident, the former being quickly convergent toward a precise limit. But, is such a limit meaningful? And if so, what should be its meaning?

![Relative Error Graph](image)

**FIG. 9:** The same as in Fig. 8 but for the Levin’s \( d \)-transformation [20]. Our results indicate \( \nu \simeq 9/10 \).

![Bar Graph](image)

**FIG. 10:** Behaviour of the convergence parameter \( \nu \) vs. 
for \( M = \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4 \). Black solid curves: \( d \)-transformation. Gray solid curves: \( d \)-transformation.

### V. DISCUSSION AND CONCLUSIONS

The classical Bessel solution of Kepler’s equation has here been revisited from a new perspective, offered by the still largely unexplored framework of the nonlinear sequence transformations.

After introducing a complex version of the original Kapteyn series [4] representing KE’s solution, numerical evidences supporting the Stieltjes conjecture for such Kapteyn series were provided. Our principal tool was the (divergent) Debye expansion of Bessel functions [15], whose numerical exploration was carried out by using two different Levin-type nonlinear sequence transformations, the Levin \( d \) and the Weniger \( \delta \). As a potentially interesting byproduct of our analysis, an effective recursive algorithm to generate arbitrarily high-order Debye’s polynomials has also been developed.

Exploring the Stieltjes conjecture for the Kepler equation Bessel solution could also reveal new relevant aspects concerning the more general framework of Kapteyn series, which continues to play a fundamental role in several areas of theoretical physics and mathematics. The

![Graph](image)

**TABLE V:** Resummation of the (divergent) Kapteyn series

\[
\sum_{m \geq 1} \frac{z^m}{m} J_m(me) \text{ for } z = 10 \exp(i\pi/3).
\]

A possible answer can be given on using again Kepler’s equation (1) and, in particular, by suitably extending its “complexification” procedure. To this end, the complex parameter \( z = \exp(iM) \) is again introduced, in such a way that \( M = -i\log z \), but now \( z \) is let to span the whole complex plane. Accordingly, Eq. (3) takes on the form

\[
S(\epsilon; M) = \frac{1}{i} \sum_{m=1}^{\infty} \frac{z^m - z^{-m}}{m} J_m(m\epsilon),
\]

and then Eq. (2) becomes

\[
\sum_{m=1}^{\infty} \frac{z^m}{m} J_m(m\epsilon) - \sum_{m=1}^{\infty} \frac{z^{-m}}{m} J_m(m\epsilon) = \Psi - \log z,
\]
where now $\Psi$ represents the (complex) solution of a modified version of the Kepler equation, precisely

$$\log z = \Psi - \epsilon \sinh \Psi.$$  \hspace{1cm} (29)

Coming back to the case in Tab. VII in Fig. 11 the result of a numerical experiment carried out on Eqs. (28) and (29) is plotted for $z = 10 \exp(1\pi/3)$. Both series in the r.h.s. of Eq. (28) have been computed via Levin $d$- and via Weniger $\delta$-transformations and the relative error with respect to the l.h.s. of the same equation, evaluated through standard method from Eq. (29), is plotted against the transformation order for both Levin (black circles) and Weniger (open circles).

![Graph showing relative error vs. k](image)

**FIG. 11:** Both series in the r.h.s. of Eq. (28) have been computed via Levin $d$- and via Weniger $\delta$-transformations of the same order $k$. The relative error with respect to the l.h.s. of the same equation, evaluated through standard method from Eq. (29), is plotted against the transformation order for both Levin (black circles) and Weniger (open circles).

From the figure it appears that the performances of the two transformations are nearly identical for small values of the transformation order $k$ (up to about ten). After a relative error of the order of $10^{-5}$ has been reached, Levin $d$- seems to be no longer able to further reducing the error itself, differently from Weniger transformation which continues to work, even though with a smaller convergence rate.

Padé approximants, together with Wynn $\epsilon$-algorithm, continue to be the most favorite mathematical tool to resum divergent as well as slowly convergent series in theoretical physics. This is in part due to the fact that a solid convergence theory for Padé approximants has been developed in the past, especially in the case of Stieltjes functions and Stieltjes asymptotic series [52, 71, 73]. Differently from Wynn’s $\epsilon$-algorithm, Levin-type transformations have specifically been designed for resumming alternating factorially divergent asymptotic series. As a consequence, their numerical performances turn out to be considerably superior with respect to the performances of the former. Unfortunately, a convergence theory for Levin-type transformations is still largely missing within the present mathematical literature, despite their indisputable superiority over Padé in several cases, the probably most paradigmatic being the Euler series [63, 70]. As a consequence, to build up such a convergence theory any new “experimental” result would be helpful.

In this respect, the deep numerical exploration of the Stieltjes nature of the complex Kapteyn series 1 operated via $d$- and $\delta$-transformations should certainly be welcomed, not only to convince skeptic readers about the extraordinary retrieving capabilities of Levin-type nonlinear sequence transformations. Another clue supporting our “Stieltjness” conjecture is the exponential convergence displayed by both $d$- and $\delta$-transformations when they were employed to solve KE via “brutal force”.

From the perspective of somebody who only wants to employ Levin-type transformations as computational tools, the situation is not really bad. However, our theoretical understanding of the convergence properties of Levin-type transformations is far from satisfactory, especially with respect to the summation of factorially divergent series. In the case of Padé approximants, if the input data are the partial sums of a so-called Stieltjes series, it can be proved rigorously that certain subsequences of the Padé table converge to the value of the corresponding Stieltjes function [52, Chapter 5]. Proving that a divergent power series is a Stieltjes series implies that such a series would be Padé summable. For instance, in [74] numerical evidences of the fact that the factorially divergent perturbation expansion for an energy eigenvalue of the PT-symmetric Hamiltonian $H(\lambda) = p^2 + 1/4x^2 + i\lambda x^3$ is a Stieltjes series and thus Padé summable, were given. Such a conjecture was later proven rigorously in [75].

It would be interesting to explore a similar possibility also for Levin-type nonlinear transformations. In other words, under which conditions (if any) a Stieltjes series can be summed up to the correct limit by using, for instance, Weniger $\delta$ transformation? Presently, it seems not only a definite answer but also weaker conjectures concerning such a question are out of our possibilities. We hope what is contained in the present paper could stimulate the interest of physicists and mathematicians about the importance, both theoretical and practical, of Levin-type nonlinear sequence transformations which, in our opinion, have not yet gained the recognition they would definitely deserve.

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Appendix A: A recursive algorithm for generating Debye’s polynomials

Let us write the explicit expression of the \(k\)th-order Debye polynomial \(U_k(t)\) as

\[
U_k(t) = \sum_{m=0}^{3k} a_m^k t^m, \quad (A1)
\]

where the superscript \(k\) in the symbol \(a_m^k\) refers to the polynomial order. Similarly we write

\[
U_{k+1}(t) = \sum_{m=0}^{3k+3} a_m^{k+1} t^m, \quad (A2)
\]

so that the task is to find the relationship between the sequences \(\{a_m^k\}\) and \(\{a_m^{k+1}\}\). To this aim, all we have to do is to implement the recursive definition (16) together with Eqs. (A1) and (A2). First of all we have

\[
U'_k(t) = \sum_{m=1}^{3k} ma_m^k t^{m-1} = \sum_{m=0}^{3k-1} (m+1)a_{m+1}^k t^m, \quad (A3)
\]

which, after some algebra, leads to

\[
\frac{t^2(1-t^2)}{2} U'_k(t) = \sum_{m=2}^{3k+3} \frac{m-1}{2} a_{m-1}^k t^m - \sum_{m=4}^{3k+3} \frac{m-3}{2} a_{m-3}^k t^m. \quad (A4)
\]

Similarly we have

\[
\frac{1}{8} \int_0^t (1-5\xi^2) U_k(\xi) \, d\xi = \sum_{m=0}^{3k} \frac{a_m^k}{8(m+1)} t^{m+1} - \sum_{m=0}^{3k} \frac{5a_m^k}{8(m+3)} t^{m+3} = \sum_{m=1}^{3k+1} \frac{a_m^{k-1}}{8m} t^m - \sum_{m=3}^{3k+3} \frac{5a_m^{k-3}}{8m} t^m, \quad (A5)
\]

so that, on taking Eq. (A4) into account,

\[
\frac{t^2(1-t^2)}{2} U'_k(t) + \frac{1}{8} \int_0^t (1-5\xi^2) U_k(\xi) \, d\xi = \frac{a_0^k}{8} t^2 + \frac{9a_1^k}{16} t^2 + \frac{5}{24} (5a_2^k - a_0^k) t^2
\]

\[
- \frac{3(2k+1)(6k-1)}{8(3k+2)} a_{3k-1}^k t^{3k+2}
\]

\[
- \frac{36k(k+1)+5}{24(k+1)} a_{3k}^k t^{3k+3}
\]

\[
+ \sum_{m=4}^{3k+3} \left[ \frac{(2m-1)^2}{8m} a_m^{k-1} - \frac{4m(m-3)+5}{8m} a_m^{k-3} \right] t^m. \quad (A6)
\]

Finally, on substituting from Eqs. (A2) and (A6) into Eq. (16), after straightforward algebra the following recurrence relation is obtained at once:

\[
a_0^{k+1} = 0
\]

\[
a_1^{k+1} = \frac{a_0^k}{8}
\]

\[
a_2^{k+1} = \frac{9a_1^k}{16}
\]

\[
a_3^{k+1} = \frac{5}{24} (5a_2^k - a_0^k)
\]

\[
a_m^{k+1} = \frac{(2m-1)^2}{8m} a_m^{k-1} - \frac{4m(m-3)+5}{8m} a_m^{k-3}
\]

\[4 \leq m \leq 3k+1
\]

\[
a_{3k+2}^{k+1} = - \frac{3(2k+1)(6k-1)}{8(3k+2)} a_{3k-1}^k
\]

\[
a_{3k+3}^{k+1} = - \frac{36k(k+1)+5}{24(k+1)} a_{3k}^k
\]

\[k \to \infty, \quad (A7)
\]

Appendix B: Proof of Eq. (17)

It is sufficient to start from the last row of Eq. (A7), i.e.,

\[
a_{3k+2}^{k+1} = - \frac{36k(k+1)+5}{24(k+1)} a_{3k}, \quad (B1)
\]

which, in the limit of \(k \gg 1\), becomes

\[
a_{3k+2}^{k+1} \sim - \frac{3}{2} k a_{3k}^k, \quad k \to \infty, \quad (B2)
\]
and whose explicit solution is

\[ a_{3k}^k \sim C \left( -\frac{3}{2} \right)^{k-1} (k-1)! , \quad k \to \infty , \quad (B3) \]

i.e., Eq. (17).

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