On a stratification defined by real roots of polynomials

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Abstract

We consider the family of polynomials $P(x, a) = x^n + a_1 x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbb{R}$, and the stratification of $\mathbb{R}^n \cong \{(a_1, \ldots, a_n) | a_i \in \mathbb{R}\}$ defined by the multiplicity vector of the real roots of $P$. We prove smoothness of the strata and a transversality property of their tangent spaces.

Key words: multiplicity vector; multiplicity surplus

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1 Formulation of the result

For $n \in \mathbb{N}^*$ fixed consider the family of polynomials $P(x, a) = x^n + a_1 x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbb{R}$. A multiplicity vector (MV) is a vector whose components are the multiplicities of the real roots of $P$ (for a fixed) listed in increasing order. E.g. for $n = 9$ the MV $[3,1,2,1]$ means that for the real roots $x_i$ one has $x_1 = x_2 = x_3 < x_4 < x_5 = x_6 < x_7$ and there is a complex conjugate couple about whose real and imaginary part the MV gives no information.

Define the length (resp. the multiplicity surplus) of a MV $[r_1, r_2, \ldots, r_q]$ as the integer $l = r_1 + \ldots + r_q$ (resp. $r = \sum_{j=1}^{q} (r_j - 1) = l - q$). The number of complex conjugate couples of roots of $P$ (counted with the multiplicities) equals $(n - l)/2$.

Stratify the space $\mathbb{R}^n \cong \{(a_1, \ldots, a_n) | a_i \in \mathbb{R}\}$ – each MV with multiplicity surplus $r$ defines a stratum of codimension $r$.

Remarks 1

1) It is clear that if $S_1$, $i = 1, 2$, are two strata defined by the MVs $V_1$ and if $S_1$ belongs to the closure of $S_2$, then for the lengths $l_i$ of the MVs $V_i$ one has $l_1 \geq l_2$. Indeed, to pass from $S_1$ to $S_2$ one or several real roots must bifurcate. If all new roots are real, then $l_1 = l_2$. If there is at least one complex conjugate couple, then $l_1 > l_2$.

2) If the MVs $V_i$ are of the same length, then $S_1$ belongs to the closure of $S_2$ if and only if $V_1$ is obtained from $V_2$ by replacing groups of consecutive multiplicities by their sums (call this an operation of type A; it corresponds to confluence of real roots).

3) An operation of type B consists either in adding a component equal to 2 at an arbitrary place of the MV (a complex conjugate couple becomes a double real root different from the other real roots of $P$) or in increasing of one of the components of the MV by 2 (a complex conjugate couple becomes a double real root which coincides with one of the other real roots). If for the lengths $l_i$ of $S_1$ one has $l_1 = l_2 + 2k$, $k \in \mathbb{N}$, then $S_1$ belongs to the closure of $S_2$ if and only if $V_1$ is obtained from $V_2$ either by an operation of type A followed by $k$ operations of type B or just by $k$ operations of type B.

4) Given a stratum $T$ of dimension $d < n$, with MV $\bar{v}$, one can obtain all MVs of strata of dimension $d+1$ adjacent to $T$ either by replacing some component $m > 1$ of $\bar{v}$ by two consecutive components $m' \geq 1, m'' \geq 1, m' + m'' = m$ or by deleting from $\bar{v}$ a component equal to 2 (which
means that a double real root becomes a complex conjugate couple). Indeed, to pass to a stratum of next dimension adjacent to the given one a root must bifurcate into two roots. If these two roots are complex conjugate, then they cannot be multiple because the stratification does not take into account the multiplicities of the complex roots.

The first aim of the present paper is to prove the following

**Theorem 2** A stratum of codimension \( r \) is a smooth real contractible algebraic variety of dimension \( n - r \). It is the graph of a smooth \( r \)-dimensional vector-function defined on the projection of the stratum in \( Oa_1 \ldots a_{n-r} \). The field of tangent spaces to the stratum is continuously extended to the strata belonging to its closure. The extension is everywhere transversal to the space \( Oa_{n-r+1} \ldots a_n \).

The theorem is proved in Section 4. It is illustrated by an example in Section 2. We discuss in Section 3 (in view of the theorem and of previous results from [Ko1]) the mutual disposition of adjacent strata.

**Remarks 3**

1) The theorem generalizes Theorem 1.8 from [Ko2]. The latter treats the case when \( P \) is hyperbolic, i.e. all roots are real.

2) The above stratification (or a similar one) has been considered (at least in some aspects) by other authors as well, see for instance [AVG-Z], [AVaGL] and their bibliographies, [Ka], [ShKh] and [ShW]; the list is anything but exhaustive.

2 An example

On Fig. 1. we show for \( n = 4 \), \( a_1 = 0 \), the well-known picture of the swallowtail, i.e. the surface \( \Sigma = \{(a_2, a_3, a_4) \in \mathbb{R}^3 | \text{Res}(P, P') = 0 \} \). The three strata of dimension 3 and their respective MVs are the open subset of \( \mathbb{R}^3 \) “above” \( \Sigma \), with empty MV (no real roots), its open subset “below” \( \Sigma \), with MV \([1, 1]\), and the interior of the curvilinear pyramid \( \Pi = OABC \), with MV \([1, 1, 1, 1]\).

The two-dimensional stratum \( S \) defined by the MV \([2]\) is the swallowtail without the boundary of \( \Pi \). The limits of the tangent spaces to \( S \) are different on the different sides of the self-intersection curve \( AO \) (excepting \( O \) where the limit is unique), see Remark 4. The strata \( ABO \) and \( ACO \) (without the boundaries) are defined respectively by the MVs \([2, 1, 1]\) and \([1, 1, 2]\); the stratum \( BCO \) (without the boundary) is defined by the MV \([1, 2, 1]\).

The one-dimensional strata \( BO \) and \( CO \) (without the point \( O \)) are defined respectively by the MVs \([3, 1]\) and \([1, 3]\) while \( AO \) (without the point \( O \)) is defined by the MV \([2, 2]\). Finally, the point \( O \) is defined by the MV \([4]\).

The self-intersection curve \( AO \) has an analytic continuation \( OD \) (represented by a dashed line on Fig. 1) which consists of polynomials of the form \((x^2 + b^2)^2\), \( b \in \mathbb{R} \), i.e. having a conjugate complex couple of roots of multiplicity 2. This curve is not a stratum of the stratification (although it belongs to \( \Sigma \)) because the multiplicities of the complex roots are not taken into account, see part 4) of Remarks 4.

**Remark 4** The following proposition is proved in [Md], p. 52-53.

**Proposition 5** If \( P, P_1, \ldots, P_r \) are monic polynomials where \( P = P_1P_2 \ldots P_r \) and \( P_i \) have two by two no root in common, then there exist neighbourhoods \( U, U_1, \ldots, U_r \) of \( P, P_1, \ldots, P_r \) such that the product map \( U_1 \times \ldots \times U_r \to U \), \((Q_1, \ldots, Q_r) \to Q_i\), is a diffeomorphism.
This result implies that at any point of the open arc $AO$ the swallowtail is locally diffeomorphic to two intersecting planes in $\mathbb{R}^3$. Indeed, on $AO$ the polynomial $P$ has two double real roots, hence, is of the form $(x-a)^2(x+a)^2$, $a \in \mathbb{R}$. One can set $P_1 = (x-a)^2$, $P_2 = (x+a)^2$ and $Q_1 = (x-a)^2 + \alpha(x-a) + \beta$, $Q_2 = (x+a)^2 - \alpha(x+a) + \gamma$. The two planes are defined respectively by the variables $(\alpha, \beta)$ and $(\alpha, \gamma)$.

Proposition 3 can be applied in the general case as well (i.e. when $P$ has any MV) to understand what the set $\{(a_1, \ldots, a_n) \in \mathbb{R}^n | \text{Res}(P, P') = 0 \}$ is locally like up to a diffeomorphism.

3 On the mutual disposition of strata

Consider a point $A$ of a stratum $U$ of dimension $s \leq n - 2$. Intersect $U$ by the affine space $F$ of dimension 2 containing $A$ and parallel to $Oa_{s+1}a_{s+2}$. By Theorem 2 the intersection is the point $A$. The intersections with $F$ of the strata of dimension $s + 1$ are curves containing $A$ in their closures and having non-vertical limits at $A$ of their tangent lines (Theorem 2). Each such curve (considered locally, at $A$) projects only “to the left” or “to the right” of $A$ on $Oa_{s+1}$. The intersections with $F$ of the strata of dimension $s + 2$ are sectors delimited by these curves.

It is explained in [Ko], Subsection 1.2, how the above curves are situated near $A$ in the case when $U$ is a stratum of hyperbolic polynomials. We generalize here these results for the case of arbitrary stratum $U$.

Denote by $\vec{v} = [r_1, \ldots, r_q]$ the MV of the stratum $U$. Denote by $U_{ij}$ the stratum with MV obtained from $\vec{v}$ by replacing the component $r_i$ by two components $-j$, $r_i - j$ – where $j = 1, \ldots, r_i - 1$. If $r_i = 2$, denote by $V_i$ the stratum whose MV is obtained from $\vec{v}$ by deleting the component $r_i$. By part 4) of Remarks 1 these are all strata of dimension $s + 1$ adjacent to $U$. We use the notation $U_{ij}$, $V_i$ also for the intersections of these strata with $F$. In what follows we assume that the equations of the limits at $A$ of the tangent lines to the curves $U_{ij}$ are given in the form $a_{s+2} = k_i a_{s+1} + \theta_i$.

Lemma 6 The slopes of the limits at $A$ of the tangent lines to the curves $U_{i_1,j_1}$, $U_{i_2,j_2}$, are the same for $i_1 = i_2$ and different for $i_1 \neq i_2$.

The lemma is proved by full analogy with Lemma 16 from [Ko] (using Proposition 3). In what follows we assume that the equations of the limits at $A$ of the tangent lines to the curves $U_{ij}$ are given in the form $a_{s+2} = k_i a_{s+1} + \theta_i$. Here $k_i$ is the slope of the limit of the tangent line.

Lemma 7 If $r_i = 2$, then the tangent lines to $U_{i,1}$ and $V_i$ are the same. The curves $U_{i,1}$, $V_i$ and the point $A$ are parts of one and the same curve smooth at $A$.

Proof:

In the particular case when $P = x^2 + \lambda$ the strata $U_{i,1}$ and $V_i$ are the half-lines $\{\lambda < 0\}$ and $\{\lambda > 0\}$. In the general case the lemma is proved by analogy with Lemma 16 from [Ko] (using Proposition 3). □

Lemma 8 For the slopes $k_i$ of the limits at $A$ of the tangent lines to the curves $U_{ij}$ one has $k_1 > \ldots > k_q$.

Proof:

1. In the case when the stratum $U$ consists of hyperbolic polynomials the lemma is proved in [Ko], see Lemma 22 there. Suppose that $U$ does not consist of hyperbolic polynomials.
Denote by $U'$ the stratum whose MV is obtained from $\vec{v}$ by adding to the right $n - l$ components equal to 1. Hence, $U'$ consists of hyperbolic polynomials.

20. Choose a point $B \in U'$. Connect it with $A$ by a continuous curve (parametrized by $\sigma \in [0, 1]$) passing only through strata with MVs whose first $q$ components are the same as the ones of $\vec{v}$ and such that for $(n - l)/2$ distinct values $\sigma_\nu$ of $\sigma$ the greatest two of the real roots become equal after what they form a complex conjugate couple. The slopes $k_i$ can be defined for any $\sigma \in [0, 1]$. For different $i$ they remain different throughout the deformation (even for $\sigma = \sigma_\nu$ which can be proved like Lemma 6). For all $\sigma$ they are finite. Hence, their order is the same at $A$ and at $B$. $\blacksquare$

**Lemma 9** If $i$ is even, then the projection on $Oa_{s+1}$ of $U_{i,j}$ is “to the right” of the one of $A$; if $i$ is odd, then “it is on its left”.

**Proof:**
In the case when $U$ consists of hyperbolic polynomials this is Lemma 18 from [Ko1]. In the general case the lemma is proved like the previous one – being “to the left or to the right” does not change throughout the deformation, because it depends continuously on $\sigma$, i.e. in fact it does not depend on $\sigma$. $\blacksquare$

**Remark 10** It follows from Lemmas 7 and 9 that if $r_i = 2$ and if the projection of $U_{i,1}$ on $Oa_{s+1}$ is “to the right” (resp. “to the left”) of the one of $A$, then the projection of $V_i$ on $Oa_{s+1}$ is “to the left” (resp. “to the right”) of the one of $A$.

**Lemma 11** For $i$ fixed the curve $U_{i,j_1}$ is “above” the curve $U_{i,j_2}$ if and only if either $i$ is odd and $j_1 > j_2$ or $i$ is even and $j_1 < j_2$.

**Proof:**
If the stratum $U$ consists of hyperbolic polynomials, then this is Lemma 20 from [Ko1]. If not, then use the same deformation as in the proof of Lemma 8. For any value of $\sigma$ and for any $i$ fixed the curves $U_{i,j}$ have the same mutual disposition. Indeed, one can apply Proposition 3 – the curves $U_{i,j}$ with one and the same $i$ correspond to one and the same neighbourhood $U_i$ and to the respective curves constructed after the polynomial $(x - a)^{r_i}$. The latter’s mutual disposition does not depend on $\sigma$ or on $a$. $\blacksquare$

**Remark 12** On Fig. 2 we show the curves $U_{i,j}$ and $V_i$ in the case when $l = 10$, $n = 10 + 2h$, $h \in \mathbb{N}$. The reader can check the above lemmas on this example.

**Remark 13** If for some $i$ one has $r_i = r_{i+1} = 2$, then the MVs of the strata $V_i$ and $V_{i+1}$ are the same. Hence, this is one and the same stratum but it gives rise to two (or more) different curves $V_i$ at $A$, with different slopes of their tangent lines at $A$ Lemma 6). The simplest example of such a situation is the one of the previous section (the stratum defined by the MV $[2]$ admitting two different limits of the tangent space along the curve $AO$).
4 Proof of Theorem 2

1. Contractibility follows from the contractibility of the parameter space (the roots play the role of parameters and define the coefficients $a_i$ via the Vieta formulas; these formulas define a homeomorphism).

Further we prove some statements of the theorem not for the stratification of $\mathbb{R}^n$, the space of the coefficients $a_i = (-1)^i \sigma_i$ (where $\sigma_i$ is the $i$-th symmetric function of the roots of $P$ counted with the multiplicities), but for the space of the Newton functions $b_i$ which are the sums of the $i$-th powers of the roots. The statements formulated for the two spaces (of the quantities $a_i$ or $b_i$) are equivalent because there exist polynomials $q_j, q_j^*$ such that

$$ja_j = -nb_j + q_j(b_1, \ldots, b_{j-1}) \quad nb_j = -ja_j + q_j^*(a_1, \ldots, a_{j-1}) .$$

2. Denote by $(x_1, \ldots, x_{n-r})$ the roots of $P$ where the real roots are distinct while the complex ones might not be all distinct. The Jacobian matrix corresponding to the mapping $(x_1, \ldots, x_{n-r}) \mapsto (b_1, \ldots, b_{n-r})$ is obtained from the Vandermonde matrix $W(x_1, \ldots, x_{n-r})$ by multiplying the columns of the real roots by their respective multiplicities. Hence, if all complex roots are distinct, then the determinant $A$ of the matrix $W$ is nonzero and at such a point the mapping is a local diffeomorphism which means that the stratum is locally of dimension $n - r$. At such a point one can express the roots $x_i$ as smooth functions of $(b_1, \ldots, b_{n-r})$, and then express $(b_{n-r+1}, \ldots, b_n)$ as smooth functions of $x_i$; hence, as smooth functions of $(b_1, \ldots, b_{n-r})$.

3. Prove the smoothness of the stratum regardless of whether the complex roots are all distinct or not. (We use the same ideas here as in [Me], p. 52-53.) To this end set $P = QR,$ $Q = x^{2k} + c_1 x^{2k-1} + \ldots + c_{2k},$ $R = x^{2k+1} + d_1 x^{2k-1} + \ldots + d_{n-2k}$ where all roots of $Q$ are complex and all roots of $R$ are real. The mapping

$$(c_1, \ldots, c_{2k}, d_1, \ldots, d_{n-2k}) \mapsto (a_1, \ldots, a_n)$$

(where $c_i, d_j$ are regarded as free parameters) is a local diffeomorphism. Indeed, its Jacobian matrix is the Sylvester matrix of $Q$ and $R$; the latter’s determinant equals $\text{Res}(Q, R)$ which is nonzero because $Q$ and $R$ have no root in common.

4. The field of tangent spaces to the given stratum is continuously extended to the strata of lower dimension belonging to its closure and to the points where some complex roots coincide. The extension is everywhere transversal to the space $Oa_{n-r+1} \ldots a_n$.

The rest of the theorem follows from the statement. The latter implies in particular that even in a neighbourhood of a point of the stratum where some complex roots coincide, the stratum is locally the graph of a smooth $r$-dimensional function defined on the projection of the stratum in $Oa_1 \ldots a_{n-r}$, see 2. - 3.

5. To prove the statement from 4 compute the partial derivatives $\partial b_k/\partial b_u, k \geq n-r + 1, u \leq n-r$ bearing in mind that $b_j$ is the $j$-th Newton function of the roots $x_i$. (We follow here the same ideas as the ones used in the proof of Theorem 1.8 from [Ko2].) Denote by $m_i$ the quantity equal to the multiplicity of $x_i$ if $x_i$ is a real root and to 1 if it is a complex one. One has

$$\partial b_k/\partial b_u = \sum_{i=1}^{n-r} (\partial b_k/\partial x_i)(\partial x_i/\partial b_u) = k \sum_{i=1}^{n-r} (m_i x_i^{k-1})(\partial x_i/\partial b_u) = k \sum_{i=1}^{n-r} (m_i x_i^{k-1} A_{u,i})/w$$  \hspace{1cm} (1)

where $w = \text{det} \parallel \partial b_j/\partial x_v \parallel = g \prod_{q<v} (x_q - x_v), \ g \neq 0$, and $A_{u,i}$ is the cofactor of the element $\partial b_u/\partial x_i$ in the matrix $\parallel \partial b_j/\partial x_v \parallel$.  

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Suppose first that the roots $x_\mu, x_\nu$ are complex. Then one has $m_\mu = m_\nu = 1$ and $m_\mu A_{u,\mu} + m_\nu A_{u,\nu} = A_{u,\mu} + A_{u,\nu} = 0$ (to be checked directly). This means that the numerator of the right hand-side of (1) is 0 when $x_\mu = x_\nu$, i.e. it is representable in the form $wh(x_1, \ldots, x_{n-r})$ for some polynomial $h$. Hence,

$$\partial b_k / \partial b_u = h$$  \hspace{1cm} (2)

If the roots $x_\mu, x_\nu$ are real, then one again checks directly that $m_\mu A_{u,\mu} + m_\nu A_{u,\nu} = 0$ and again one has (3).

7). The closure of the stratum can be defined by a continuous parametrization of the roots $x_i$ by some parameters $z$ (one can choose as such parameters part of the variables $b_j$; in general, these variables are more than the parameters needed). By (2), the partial derivatives $\partial b_k / \partial b_u$ are bounded continuous functions of the parameters $z$. Hence, the limits of these partial derivatives exist on the closure of the stratum. This proves the statement.

The theorem is proved. \hfill \square

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