FULLY FAITHFUL FOURIER-MUKAI FUNCTORS AND GENERIC VANISHING

GIUSEPPE PARESCHI

Abstract. The aim of this mainly expository note is to point out that, given an Fourier-Mukai functor, the condition making it fully faithful is an instance of generic vanishing. We test this point of view on some fairly classical examples, including the strong simplicity criterion of Bondal and Orlov, the standard flip and the Mukai flop.

The aim of this mainly expository note is to point out that, given an Fourier-Mukai functor, the condition making it fully faithful is an instance of generic vanishing. We test this point of view on some fairly classical examples, including the strong simplicity criterion of Bondal and Orlov, the standard flip and the Mukai flop.

The notion of generic vanishing arose in work of Green and Lazarsfeld on irregular varieties ([GL1],[GL2]), where they showed that the sheaves of holomorphic differential forms, twisted with a generic topologically trivial line bundle, satisfy a cohomological vanishing of Kodaira-Nakano type. The natural environment for the notion of generic vanishing introduced by Green and Lazarsfeld is the Fourier-Mukai functor defined by the Poincaré line bundle. It makes sense to study the same kind of property for any FM functor ([PPo4],[Po]).

In this note we remark that a FM functor \( \Phi_{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y) \) is fully faithful if and only if \( O_Y \), the structure sheaf of \( Y \), satisfies Green-Lazarsfeld’s generic vanishing condition (in the current terminology: is a geometric GV-object) with respect to the FM functor

\[
\Phi_{E^c \rightarrow Y \times X} : D^b(Y) \rightarrow D^b(X \times X)
\]

(see below for the notation), plus an additional condition which is usually easier to check. In essence, to be fully faithful is very close to be a generic vanishing condition. While this is certainly not a new result, but just a restatement of well known basic facts, it is the author’s hope that this point of view can be an useful complement to the existing methods of investigating whether a given FM functor is fully faithful, in particular an equivalence. We test this by providing different proofs of some fairly classical full-faithfulness results.

Here is what the reader will find in this paper. The first section is background on fully faithful FM functors. The second section is background about generic vanishing conditions: they are usually stated in three equivalent ways, which we recall. In Section 3 we show that, in the context of full-faithfulness, the first equivalent condition essentially boils down to the strong simplicity criterion of Bondal and Orlov. Interestingly, the natural version of Bondal-Orlov’s criterion in this context works under weaker hypotheses (Prop. 3.1). In Section 4 we use the second equivalent condition to give, or outline, alternative proofs about full-faithfulness of the natural FM functors associated
coherent sheaves. For example \( \otimes \) means \( \otimes^L \), and the underived tensor product of coherent sheaves is denoted \( \text{tor}_0 \). Moreover \( H^i \) means hypercohomology.

(b) Unless otherwise stated, all varieties are assumed smooth and projective over an algebraically closed ground field.

This paper is dedicated to the memory of Alexandru Lascu, my teacher and adviser back when I was an undergraduate at the University of Ferrara.

1. Fully faithful Fourier-Mukai functors

Let \( X \) and \( Y \) be smooth projective varieties and

\[
\Phi^X_{\mathcal{E}}: D^b(X) \rightarrow D^b(Y)
\]

a Fourier-Mukai functor of kernel \( \mathcal{E} \in D^b(X \times Y) \). We denote

\[
\mathcal{E}^\vee = R\text{Hom}(\mathcal{E}, \mathcal{O}_{X \times Y})
\]

and \( p_X \) and \( p_Y \) the projections of \( X \times Y \). The functor

\[
\Phi^Y_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y}[\dim Y]: D^b(Y) \rightarrow D^b(X)
\]

is the left adjoint of \( \Phi^X_{\mathcal{E}} \). It follows that \( \Phi^X_{\mathcal{E}} \) is fully faithful if and only the natural morphism of functors

\[
\Phi^Y_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y}[\dim Y] \circ \Phi^X_{\mathcal{E}} \longrightarrow id_{D^b(X)}
\]

is an isomorphism. The functor \( \Phi^Y_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y}[\dim Y] \circ \Phi^X_{\mathcal{E}} \) is the FM functor of kernel

\[
\Phi^{Y \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)[\dim Y]}(\mathcal{O}_{\Delta_Y})
\]

(e.g. [In] Ex. 5.13(ii)). Therefore, since the unique kernel for \( id_{D^b(X)} \) is \( \mathcal{O}_{\Delta_X} \), the functor \( \Phi^X_{\mathcal{E}} \) is fully faithful if and only if

\[
\Phi^{Y \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)}(\mathcal{O}_{\Delta_Y}) = \mathcal{O}_{\Delta_X}[-\dim Y]
\]

Given \( \mathcal{E} \) and \( \mathcal{F} \) objects of \( D^b(X \times Y) \), let \( \mathcal{E} \boxtimes Y \mathcal{F} \) be the object of \( D^b(X \times X \times Y) \) defined as

\[
\mathcal{E} \boxtimes Y \mathcal{F} = p_{13}^* \mathcal{E} \otimes p_{23}^* \mathcal{F}
\]

where \( p_{13} \) and \( p_{23} \) are the two projections of \( (X \times Y) \times X \). Since

\[
\Phi^{Y \times X \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)}(\mathcal{O}_{\Delta_Y}) = \Phi^{Y \times X \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)}(\mathcal{O}_{\Delta_Y})
\]

the above condition \((1.3)\) can be also written as follows

\[
\Phi^{Y \times X \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)}(\mathcal{O}_{Y}) = \mathcal{O}_{\Delta_X}[-\dim Y]
\]

or also

\[
\Phi^{Y \times X \times X}_{\mathcal{E}^\vee \otimes (p_Y^* \omega_Y)}(\omega_Y) = \mathcal{O}_{\Delta_X}[-\dim Y]
\]
2. Generic vanishing

Let $Y$ and $Z$ be smooth projective varieties and $P \in D^b(Y \times Z)$. Let $z \in Z$ be a closed point and $k_z$ its residue field, seen a coherent sheaf on $Z$ supported at $z$. We have that $\Phi_P^Z Y(k_z) = i_z^* P$, where $i_z : Y \to Y \times Z$ is $i_z(y) = (y, z)$. We consider now the Fourier-Mukai functor in the opposite direction, $\Phi_P^Y Z : D^b(Y) \to D^b(Z)$. Given $F \in D^b(Y)$, we define its cohomological support loci with respect to $\Phi_P^Y Z$ as

$$V^i_P(Y, F) = \{ z \in Z \mid h^i(Y, F \otimes \Phi_P^Z Y(k_z)) > 0 \}$$

**Example 2.1.** (Green-Lazarsfeld sets) Let $Y$ be an irregular variety, $Z = \text{Pic}^0 Y$ and $P$ a Poincaré line bundle on $Y \times \text{Pic}^0 Y$. Then $z \in \text{Pic}^0 Y$ corresponds to a line bundle $P_z$ on $Y$, which is precisely $\Phi_P^{\text{Pic}^0 Y} Y(k_z)$. Given a coherent sheaf $F$ on $Y$, the cohomological support loci

$$V^i_P(Y, F) = \{ z \in \text{Pic}^0 Y \mid h^i(F \otimes P_z) > 0 \}$$

were introduced and studied by Green and Lazarsfeld for the sheaves $F = \Omega^i_Y$ (GL1, GL2) and subsequently studied for other relevant sheaves on abelian and/or irregular varieties, see e.g. [PPo1], [PPo2].

The following notion was introduced by Mihnea Popa in [Po1], Def. 3.7.

**Definition 2.2.** (Geometric GV-objects) An object $F \in D^b(Y)$ is called a geometric GV-object with respect to a functor $\Phi_P^Y Z$ if

(i) $V^i_P(Y, F) = \emptyset$ for $i < 0$ and

(ii) $\text{codim}_Z V^i_P(Y, F) \geq i$ for $i \geq 0$.

The following result is well known, see [PPo3], [PPo4] and especially [Po1], Th. 3.8, Remark 3.10 and Cor. 4.3 and references therein. See also Remark 2.6 below.

**Theorem 2.3.** In the above setting, the following are equivalent

(a) $F$ is a geometric GV-object with respect to the functor $\Phi_P^Y Z$;

(b) $\Phi_P^Y Z(F^\vee \otimes \omega_Y)$ is concentrated in cohomological degree $\dim Y$. That is:

$$\Phi_P^Y Z(F^\vee \otimes \omega_Y) = R^{\dim Y} \Phi_P^Y Z(F^\vee \otimes \omega_Y)[- \dim Y]$$

(c) If $A$ is a sufficiently high multiple of an ample line bundle on $Z$ then

$$H^i(Y, \Phi_P^Y Z(A) \otimes F^\vee \otimes \omega_Y) = 0 \quad \text{for all } i \neq \dim Y.$$
According to a terminology/notation due to Mukai, condition \((b)\) is sometimes referred to as the fact that \(F^\vee \otimes \omega_Y\) satisfies the weak index theorem with index \(i = \dim Y\). For short: WIT(\(\dim Y\)). If this is the case the sheaf \(R^{\dim Y} \Phi_{P^Y \rightarrow Z}(F^\vee \otimes \omega_Y)\) is denoted \(F^\vee \otimes \omega_Y\). In this notation condition \((b)\) is written as

\[
\Phi_{P^Y \rightarrow Z}(F^\vee \otimes \omega_Y) = F^\vee \otimes \omega_Y[-\dim Y]
\]

**Remark 2.4.** (On the assumptions for Theorem 2.3) Theorem 2.3 works under more general hypotheses: assuming that the kernel \(P\) is a perfect complex, for the equivalence between \((a)\) and \((b)\) \(Z\) need not to be projective, and both \(Y\) and \(Z\) need not to be smooth varieties, but only Cohen-Macaulay schemes of finite type over any field (but, if \(Z\) is not Gorenstein, in condition \((b)\) \(P^\vee\) has to be replaced with \(P^\vee \otimes p^*_Z \omega_Z\) see [Po] Remark 3.10). The equivalence with \((c)\) holds under the further assumption that \(Z\) is projective. We refer to [Po] and [PPo4].

**Remark 2.5.** (Conditions \((a)\) and \((b)\) for (hyper)cohomology) A simple-minded way to see the equivalence between \((a)\) and \((b)\) is as follows. Let \(Z\) be a point, denoted \(\{pt\}\), and \(P = \mathcal{O}_{Y \times \{pt\}}\). The two conditions of the previous theorem are reduced to:

\((a_0)\) \(H^i(Y, F) = 0\) for \(i \neq 0\);

\((b_0)\) \(H^i(Y, F^\vee \otimes \omega_Y) = 0\) for \(i \neq \dim Y\).

They are equivalent by Serre duality, and the meaning of the equivalence between \((a)\) and \((b)\) is that, for arbitrary Fourier-Mukai functors, they admit distinct equivalent generalizations: the generalization of \((a_0)\) is geometric-GV, namely the generic vanishing of a family of hypercohomology groups. The generalization of \((b_0)\) is WIT(\(\dim Y\)), that is the vanishing of the hyperdirect image sheaves \(R^k \Phi_{P^\vee}(F^\vee \otimes p^*_Y \omega_Y)\).

**Remark 2.6.** (Perverse sheaves) The geometric-GV and WIT(\(\dim Y\)) conditions are better stated in terms of \(t\)-structures and perverse sheaves. We refer to [Po] and [PoS] §6-7 for this. Briefly, it follows from a result of Kashiwara [K] that a geometric GV-object with respect to \(\Phi_{P^Y \rightarrow Z}\) is an object \(F\) of \(D^b(Y)\) such that \(\Phi_P F\) belongs to the heart of the dual \(t\)-structure on \(D^b(Z)\). This is the equivalence between \((a)\) and \((b)\) in Theorem 2.3

**Remark 2.7.** (Condition \((c)\) with ample sequences) By duality,

\[
(\Phi_{P \rightarrow Y}^Z(A))^\vee \cong \Phi_{\mathcal{O}^{\dim Z}}^Z(A^\vee \otimes \omega_Z)
\]

Therefore condition \((c)\) can be written as follows

\[
\text{Hom}(\Phi_{\mathcal{O}^{\dim Z}}^Z(A^{-1} \otimes \omega_Z), F^\vee \otimes \omega_Y[j]) = 0 \quad \text{for all } j \neq \dim Y
\]

Note that, if \(A\) is ample and \(k >>\) then \(L_k := A^{-k} \otimes \omega_Z\) is ample sequence in \(\text{coh}(Z)\). Condition \((c)\) of Theorem 2.3 can be stated more generally as follows: given an ample sequence \(\{L_k\}\) in \(\text{coh}(Z)\),

\[
\text{Hom}(\Phi_{\mathcal{O}^{\dim Z}}^Z(L_k), F^\vee \otimes \omega_Y[j]) = 0 \quad \text{for all } j \neq \dim Y\text{ and } k <<
\]

The equivalence with condition \((b)\) of Theorem 2.3 is proved in the same way.

3. **Full faithfulness via condition \((a)\)**

The relationship between full faithfulness and generic vanishing is in [1.5], which can be reformulated as follows:
\( \Phi_\mathcal{E} \) is fully faithful if and only if \( \omega_Y \) satisfies WIT(\( \dim Y \)) with respect to \( \Phi_{\mathcal{E} \boxtimes \mathcal{Y}, \mathcal{E}^\vee} \) — that is condition (b) of Theorem 2.3 — and, in addition, its transform is the sheaf \( \mathcal{O}_{\Delta_X} \) in cohomological degree \( \dim Y \).

Experience shows that, usually, the more difficult part to be checked is the WIT(\( \dim Y \)) condition, while the additional requirement is easier. With this in mind, Theorem 2.3 provides three distinct ways of checking full-faithfulness.

Condition (a) leads to the classical strong simplicity criterion of Bondal-Orlov (see [BO], [Br] and [Hu] §7.1). See also [HeLS] and [L] for generalizations). Actually one gets the result under weaker hypotheses. To do this, let us consider the loci

\[
W^i(Y) = \{ (x, x') \in X \times X \mid \text{Hom}(\Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_{x'}))[i] \neq 0 \}
\]

**Proposition 3.1.** Assume that \( \text{char} k = 0 \). Then \( \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y) \) is fully faithful if and only if the following conditions hold:

(a) \( W^i(Y) \) is empty for \( i < 0 \);

(b) \( \dim W^i(Y) \leq 2 \dim X - i \) for all \( i \geq 0 \);

(c) \( \text{Hom}(\Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_{x'})) = k \) if \( x = x' \) and 0 otherwise.

**Proof.** Since \( X \) is smooth

\[
(\Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x))^\vee \cong \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x)
\]

for all \( x \in X \). Therefore

\[
\text{Hom}(\Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_{x'}))[i]) = \text{Ext}^i(\Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_{x'})) \cong \\
\cong H^i(Y, \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x) \otimes \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_{x'})) = H^i(Y, \Phi_{\mathcal{E}^\vee}^{X \times X \rightarrow Y}(k_{x} \otimes k_{x'}))
\]

It follows that

\[
W^i(Y) = V_{\mathcal{E}^\vee}^{\mathcal{Y}, \mathcal{E}^\vee}(Y, \mathcal{O}_Y)
\]

Therefore (a) and (b) mean exactly that \( \mathcal{O}_Y \) is a geometric GV-object with respect to \( \Phi_{\mathcal{E}^\vee}^{X \times X \rightarrow Y} \).

By (a) \( \leftrightarrow \) (b) of Theorem 2.3 this is equivalent to the fact that \( \Phi_{\mathcal{E}^\vee}^{X \times X \rightarrow \mathcal{Y}}(\omega_Y) \) is a coherent sheaf on \( X \times X \) in cohomological degree \( \dim Y \) (note that \( (\mathcal{E} \boxtimes \mathcal{Y}, \mathcal{E}^\vee)^\vee \cong \mathcal{E}^\vee \boxtimes \mathcal{Y} \mathcal{E} \)). According to notation (2.1), we denote \( \hat{\omega}_Y \) this coherent sheaf. Hypotheses (a) and (c) of the present Theorem imply, by homology and base change, that this sheaf is in fact a line bundle on a possibly non-reduced variety supported on the diagonal \( \Delta_X \).

But in fact, as the ground field is assumed to be algebraically closed of characteristic zero, actually

\[
\hat{\omega}_Y = \delta_X^* L,
\]

where \( \delta_X : X \rightarrow X \times X \) is the diagonal embedding and \( L \) is a line bundle on \( X \); this is proved exactly as in Bridgeland’s account of Bondal-Orlov’s theorem, using the Kodaira-Spencer map ([Br] Lemmas 5.2-3 or [Hu] Steps 3 and 5 or the proof of the main result in [L]), so we won’t reproduce this argument here. This already proves that \( \Phi_\mathcal{E} \) is fully faithful (and, a posteriori, \( L = \mathcal{O}_X \)).

\footnote{In fact one known that, since \( \mathcal{O}_Y \) is geometric-GV with respect to \( \Phi_{\mathcal{E}^\vee}^{X \times X \rightarrow Y} \), the sheaf \( \hat{\omega}_Y \) has the “base-change property”, namely, in the present case, the natural map \( \text{tor}_0(\hat{\omega}_Y, k_{(x,x)}) \rightarrow H^{\dim Y}(Y, \omega_Y \otimes \Phi_{\mathcal{E}^\vee}^{X \times X \rightarrow Y}(k_{(x,x)})) \) is an isomorphism.}
Corollary 3.2. (Bondal-Orlov) $\Phi$ is fully faithful if and only if

$$\text{Hom}(\Phi^X \to Y(k_x), \Phi^X \to Y(k_{x'})[i]) = \begin{cases} k & \text{if } x = x' \text{ and } i = 0 \\ 0 & \text{if } x \neq x', \text{ or } i < 0, \text{ or } i > \dim X \end{cases}$$

Proof. The hypotheses can be restated as follows: $\text{Hom}(\Phi^X \to Y(k_x), \Phi^X \to Y(k_{x'})[i]) = k$ if $x = x'$ and $W^i_Y(Y) \subseteq \Delta_X$ for $0 \leq i \leq \dim X$

Therefore Proposition 3.1 implies the Corollary. □

Remark 3.3. (On the assumptions for Prop. 3.1 and Corollary 3.2) (i) As pointed out in [HeLS] Remark 1.25 and [L] the characteristic zero is necessary, unless one puts a supplementary hypothesis. (ii) Checking carefully more general assumptions for the validity of (1.2) and for the equivalence between (a) and (b) in Theorem 2.3 it follows that Prop. 3.1 and Corollary 3.2 work under more general hypotheses on $X$ and $Y$: $X$ needs to be smooth but not necessarily projective, while $Y$ needs to be projective but it is allowed to be singular (Cohen-Macaulay). This is a result in [HeLS], see also [L] and references therein.

4. Full faithfulness via condition (b)

In this section we will consider some examples where, from the point of view of generic vanishing, the easiest way of proving/disproving full faithfulness is given by condition (1.5) at once, which corresponds to condition (b) of Theorem 2.3. This amounts to

$$R^i \Phi_{E} \otimes (\omega_Y) = \begin{cases} 0 & \text{for } i \neq \dim Y \\ \mathcal{O}_{\Delta_X} & \text{for } i = \dim Y \end{cases}$$

This is certainly a bit old fashioned (for example, it is the way Mukai originally showed in [M] that the Poincaré kernel provides a derived equivalence between dual abelian varieties) but however the proofs below are easy, self-contained and conceptually clear, and might provide a complementary insight on some aspects of Kawamata’s conjecture $K$-equivalence $\Rightarrow$ $D$-equivalence.

Example 4.1. (Standard flip) We consider a standard flip

$$E = \mathbb{P}^d \times \mathbb{P}^k$$

where $N_{\mathbb{P}^d/X} = \mathcal{O}(-1)^{k+1}$ and $N_{\mathbb{P}^k/Y} = \mathcal{O}(-1)^{l+1}$, so that the dimension of the varieties $X$, $Y$ and $Z$ is $d = k + l + 1$. The morphism $\pi_X$ (resp. $\pi_Y$) is the blow up of $\mathbb{P}^d$ (resp. $\mathbb{P}^k$). Note that the
functor $\Phi_{O_Z}^X$ coincides with $q_\ast p^\ast$. The result, again due to Bondal and Orlov [BO], is that: $\Phi_{O_Z}^X : D^b(X) \to D^b(Y)$ is fully faithful if and only if $k \leq l$.

Let us prove this statement by verifying condition (4.1). We have that

$$(4.4)$$

We first compute

Next we compute the difference between $tor_i^{X \times Y \times Y} (O_{Z_1 \times Z_2})$ and $tor_i^{X \times Y \times Y} (O_{E_1 \times E_2})$. This is achieved by tensoring with $O_{X \times X \times \Delta_{34} Y}$ the two exact sequences

$$0 \to O_{Z_1} \boxtimes O_{Z_2} (-E_2) \to O_{Z_1 \times Z_2} \to O_{Z_1 \times E_2} \to 0$$

Claim 4.2. for $i > 0$ $tor_i^{X \times Y \times Y} (O_{Z_1 \times Z_2}, O_{X \times X \times \Delta_{34} Y}) = \Lambda^i (O_{\mathcal{P}^l \times \mathcal{P}^l \times \mathcal{P}^k}(0, 0, 1)^{\oplus l + 1})$

Proof. We first compute

$$tor_i^{Y \times Y \times Y} (O_{E_1 \times E_2}, O_{X \times X \times \Delta_{34} Y}) = tor_i^{Y \times Y \times Y} (O_{E_1 \times P^l \times \mathcal{P}^k}, O_{X \times X \times \Delta_{34} Y}) =$$

$$= p^*_{34} N^\vee_{p^l / Y} = p^*_{34} \Lambda^i (O(1)^{\oplus l + 1}) = \Lambda^i (O_{E_1 \times E_2}(0, 0, 1)^{\oplus l + 1})$$

The third equality follows from the general isomorphism $(\mathcal{F} \boxtimes \mathcal{G}) \otimes_{Y \times Y} \mathcal{O}_{\Delta_Y} = \mathcal{F} \otimes_Y \mathcal{G}$ (where for $\mathcal{F}, \mathcal{G} \in D(Y)$). Therefore $tor_i^{Y \times Y} (\mathcal{F} \boxtimes \mathcal{G}, \mathcal{O}_{\Delta_Y}) = tor_i^{Y} (\mathcal{F}, \mathcal{G})$. In our case

$$tor_Y^{Y \times Y} (O_{\mathcal{P}^l \boxtimes \mathcal{P}^k}, \mathcal{O}_{\Delta_Y}) = tor_Y^{Y} (O_{\mathcal{P}^l}, \mathcal{O}_{\mathcal{P}^k}) = \Lambda^i N_{p^l / Y}^\vee.$$
0 \to O_{Z_{13}}(-E_{13}) \boxtimes O_{E_{24}} \to O_{Z_{13} \times E_{24}} \to O_{E_{13} \times E_{24}} \to 0

The assertion follows after a little calculation with the first exact sequence.

Concerning the underived tensor product, since
\[
\text{tor}_0(O_{Z_{13}} \boxtimes O_{Z_{24}}, O_{X \times X \times \Delta_{34} Y}) = O_{(\Delta_{12,34} Z) \cup (P^l \times P^l \times \Delta_{34} P^k)}
\]
we have the "Mayer-Vietoris" exact sequence
\[
0 \to \text{tor}_0(O_{Z_{13}} \boxtimes O_{Z_{24}}, O_{X \times X \times \Delta_{34} Y}) \to O_{\Delta_{13,24} Z} \oplus (O_{P^l \times P^l \times \Delta_{34} P^k}) \to O_{\Delta_{12} P^l \times \Delta_{34} P^k} \to 0
\]
Since
\[
\omega_Z|_E = O(-l,-k)
\]
from the Claim we get that, for \( i > 0 \)
\[
(4.5) \quad \text{tor}_i^{X \times X \times Y \times Y}(O_{Z_{13}} \boxtimes \omega_{Z_{24}}, O_{X \times X \times \Delta_{34} Y}) = O_{P^l \times P^l \times P^l}(0,-l,-k+i)^{\oplus (l+1)}
\]
(in particular, it vanishes for \( i > l - 1 \)). For \( i = 0 \) we have the exact sequence
\[
(4.6) \quad 0 \to \text{tor}_0(O_{Z_{13}} \boxtimes \omega_{Z_{24}}, O_{X \times X \times \Delta_{34} Y}) \to \omega_{\Delta_{13,24} Z} \oplus O_{P^l \times P^l \times \Delta_{34} P^k}(0,-l,-k) \to O_{\Delta_{12} P^l \times \Delta_{34} P^k}(-l,-k) \to 0
\]
Applying \( p_{X \times X,*} \), i.e. \( p_{12,*} \), to \( (4.6) \) it follows easily that in any case
\[
(4.7) \quad R^i p_{12,*}(\text{tor}_0^{X \times X \times Y \times Y}(O_{Z_{13}} \boxtimes \omega_{Z_{24}}, O_{X \times X \times \Delta_{34} Y})) = R^i p_{12,*}(\omega_{\Delta_{13,24} Z}) = \begin{cases} \omega_{\Delta_{12} X} & \text{for } j = 0 \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}
\]
because \( p_{12} \) restricted to \( \Delta_{13,24} Z \) is simply the birational morphism \( p : Z \to X \). Hence the above \( \text{tor}_0 \) does not cause any obstruction to the validity of \( (4.3) \). On the other hand, applying \( p_{12,*} \) to \( (4.5) \) one sees that the vanishing
\[
(4.8) \quad R^i p_{12,*}(\text{tor}_i^{X \times X \times Y \times Y}(O_{Z_{13}} \boxtimes \omega_{Z_{24}}, O_{X \times X \times \Delta_{34} Y})) = 0 \quad \text{for all } i > 0 \text{ and all } j
\]
holds if and only if \( k \geq l \). Via an easy spectral sequence, \( (4.7) \) and \( (4.8) \) prove that \( (4.3) \), i.e. full-faithfulness of \( \Phi_{\Omega_Z}^{X \to Y} : D^b(X) \to D^b(Y) \), holds if \( k > l \). In a similar way it follows also that the full-faithfulness does not hold for \( k < l \).

**Example 4.3. (Mukai flop)** We follow the notation of \([\text{Hi}], \S 11.4\). We have the diagram
\[
(4.9) \quad \begin{array}{ccc} E & \subset & \mathbb{P} \times \mathbb{P}^\vee \\ \ \swarrow \pi_X & & \searrow \pi_Y \\ \mathbb{P} & \leftarrow & Z \\ & \leftarrow & \uparrow \downarrow \\ & X & \rightarrow & Y \\ & q & & p \\ \rightarrow & \mathbb{P}^\vee & \rightarrow & \mathbb{P}^\vee \\ \end{array}
\]
and \( N_{\mathbb{P}|X} = \Omega_{\mathbb{P}}, \ N_{\mathbb{P}^\vee|Y} = \Omega_{\mathbb{P}^\vee}. \) Here \( \dim X = 2n \) and \( \mathbb{P} = \mathbb{P}^n. \) The maps \( p \) (resp. \( q \)) is the blow-up of \( \mathbb{P} \) (resp. \( \mathbb{P}^\vee \)) and \( E = \mathbb{P}(\Omega_{\mathbb{P}}) \subset \mathbb{P} \times \mathbb{P}^\vee \) is the incidence correspondence point-hyperplane. It is well known, by a result of Kawamata and Namikawa \((\text{Ka1}, \text{N1})\) that: the functor \( q_* \circ p^* = \Phi_{\Omega_Z} : D^b(X) \to D^b(Y) \) is not fully faithful.
Let us check this within the method of the previous example. Exactly as above, the condition for full-faithfulness is \((4.3)\), and one has to compute
\begin{equation}
\text{tor}^Y_X \otimes Y \times Y (O_{Z_{13}} \otimes \omega_{Z_{24}}, O_X \times \Delta_{34, Y})
\end{equation}
Again the intersection in \(X \times X \times Y \times Y\) of \(Z_{13} \times Z_{24}\) and \(X \times X \times \Delta_{34}\) is the fibered product \(\Delta \times \Delta Z\), which has the two irreducible components:
\[(Z_{13} \times Z_{24}) \cap (X \times X \times \Delta_{34}) = \Delta \times \Delta Z = \Delta_{13, 24} \cup (E_{13} \times \Delta_{24}) E_{24}\]
One can compute all \(\text{tor}\) sheaves \((4.10)\), as in Claim \(4.2\) and the result is similar. It happens that higher \(\text{tor}\)’s (i.e. the sheaves \((4.10)\) for \(i > 0\) don’t affect condition \((4.3)\), namely
\begin{equation}
R^p\text{p}_{12*} \left( \text{tor}^Y_X \otimes Y \times Y (O_{Z_{13}} \otimes \omega_{Z_{24}}, O_X \times \Delta_{34, Y}) \right) = 0 \quad \text{for all } i > 0 \text{ and all } j
\end{equation}
We leave this to the reader.

The reason why \((4.3)\) is not satisfied is in the underived tensor product
\[
\text{tor}^Y_X \otimes Y \times Y (O_{Z_{13}} \otimes \omega_{Z_{24}}, O_X \times \Delta_{34, Y})
\]
As in the previous example this sits in the exact sequence
\begin{equation}
0 \rightarrow \text{tor}_0(O_{Z_{13}} \otimes \omega_{Z_{24}}, O_X \times \Delta_{34, Y}) \rightarrow \omega_{\Delta_{13, 24}} \oplus \left( (p^*_2 \omega_z)|_{E_{13} \times \Delta_{24}, E_{24}} \right) \rightarrow (\omega_{\Delta_{13, 24}}|_{\Delta_{13, 24}}E \rightarrow 0
\end{equation}
We apply \(p_{X \times X^*}\), i.e. \(p_{12*}\), to the above exact sequence. Since \((\omega_X)|_P\) is trivial, we have that
\begin{equation}
(\omega_Z)|_E = \omega_E(-E) = O_{\mathbb{P}X \mathbb{P}^v}(-n, -n)|_E \otimes O_{\mathbb{P}X \mathbb{P}^v}(1, 1)|_E = O_{\mathbb{P}X \mathbb{P}^v}(-(n - 1), -(n - 1)|_E
\end{equation}
It follows that, for all \(i\), \(R^i\text{p}_{12*}\) applied to the sheaf on the right of the exact sequence \((4.12)\) is zero for all \(i\). Therefore, to compute the higher direct images \(R^i\text{p}_{12*}\) of the \(\text{tor}_0\) on the left, it is enough to compute \(R^i\text{p}_{12*}\) of the sheaf in the middle. This has two summands. Concerning the first one, as in the previous example there is nothing contradicting \((4.3)\), since
\begin{equation}
R^i\text{p}_{12*}(\omega_{\Delta_{13, 24}}) = \begin{cases} 
\omega_{\Delta_{13}X} & \text{for } i = 0 \\
0 & \text{otherwise}
\end{cases}
\end{equation}
Concerning the second summand, note that the fiber of the projection
\begin{equation}
p_{12} : E_{13} \times \Delta_{34, \mathbb{P}^v} E_{24} \rightarrow \mathbb{P} \times \mathbb{P} \subset X \times X
\end{equation}
over a pair \((x, x') \in \mathbb{P} \times \mathbb{P}\), with \(x \neq x'\), is the intersection of the two hyperplanes of \(\mathbb{P}^v\) corresponding to \(x\) and \(x'\), that is a \(\mathbb{P}^{n-2} \subset \mathbb{P}^v\). Now \((4.13)\) tells that \(p^*_2 \omega_{\Delta_{13}X}\), restricted to a general fiber of \((4.15)\) is \(O_{\mathbb{P}^{n-2}(-(n - 1))}\). Therefore \(R^i\text{p}_{12*}\), applied to the second summand of the middle part of sequence \((4.12)\) is zero for \(i < n - 2\) and non-zero and supported on \(\mathbb{P} \times \mathbb{P}\) for \(i = n - 2\). By an easy spectral sequence this, together with \((4.11)\) and \((4.13)\), yields that \(R^{n-2}\Phi_{O_{Z \mathbb{P}}} O_{Z \mathbb{P}Y \omega_{\Delta_{13}}} (O_{\Delta_{13}})\) is non-zero. Therefore \((4.3)\) is not verified and \(\Phi_{O_{Z}} : D^b(X) \rightarrow D^b(Y)\) is not fully faithful.

With a similar, but more complicated, calculation one can prove directly the result of Kawai-mata and Namikawa ([Ka], [N], see also [Hu]) that \(\Phi_{O_{Z}} : D^b(X) \rightarrow D^b(Y)\) is fully faithful, where \(\overline{Z} = Z \cup (\mathbb{P} \times \mathbb{P}^v)\). As a disclaimer, we should point out that this method, applied to the stratified Atiyah flop and Mukai flop ([C], [Ka2], [Ma], [N2]) becomes much more complicated.
5. Full-faithfulness via condition (c)

In this section we use condition (c) of Theorem 2.3 to provide another way to check full-faithfulness. So far condition (c) has proven to be extremely useful for detecting generic vanishing when the kernel is a Poincaré line bundle. In fact, if \( X \) is an abelian variety, \( P \) a Poincaré line bundle on \( X \times \hat{X} \) (where \( \hat{X} \) is the dual abelian variety) and \( A \) is an ample line bundle on \( \hat{X} \), the object \( \Phi^{\hat{X} \to X} (A) \), has a peculiar description, which can be seen as an effect of the "abelianity" of the context: \( \Phi^{\hat{X} \to X} (A) \) is a locally free sheaf which is, up to pullback via the isogeny \( \varphi_A : \hat{X} \to X \) associated to \( A \), sum of copies of the line bundle \( A^\vee \) (see e.g. [M], Prop. 3.11(1)). Therefore, in the case of Poincaré kernel on dual abelian varieties, condition (c) is a very effective way of reducing the GV condition to vanishing theorems. This idea, due to Hacon ([H]), is extremely fruitful in the study of the geometry of irregular varieties. It is an interesting problem to find an adequate description of the objects \( \Phi^{E \to Y} (A) \) in other cases.

In the present context, condition (c) of Theorem 2.3 leads to the full-faithfulness criterion below. Due to the above reason, at present its range of applicability is confined to abelian or irregular varieties.

**Proposition 5.1.** Let \( A \) be a sufficiently high power of an ample line bundle on \( X \times X \). Then \( \Phi_E : D^b(X) \to D^b(Y) \) is fully faithful if and only if

\[
\tag{5.1}
H^i(Y, \Phi^{X \times X \to Y} (A) \otimes \omega_Y) = \begin{cases} 
0 & \text{for } i \neq \dim Y \\
h^0(A \otimes O_{\Delta_X}) & \text{for } i = \dim Y
\end{cases}
\]

**Proof.** By the equivalence between (b) and (c) of Theorem 2.3 the first line above means that \( \Phi^{\hat{X} \to X} (A) \) is a sheaf in cohomological degree \( \dim Y \), denoted \( \hat{\omega}_Y [-\dim Y] \) (according to notation (2.1)). Therefore the adjunction morphism (1.2) is, up to a shift

\[
\Phi^{X \to Y}_{\omega_Y} \to \Phi^{X \to X}_{\Delta_X}
\]

This induces a morphism of \( O_{X \times X} \)-modules

\[
\tag{5.2}
\hat{\omega}_Y \to O_{\Delta_X}
\]

which is surjective, since, for all \( x \in X \), the adjunction morphism \( \Phi^{X \to X}_{\omega_Y} (k_x) \to k_x \) is non-zero, hence surjective.

We stop for a moment, to recall from [PPo4], Lemma 2.1 the functorial isomorphism, for all \( i \) and for all objects \( G \) (resp. \( A \)) of \( D^b(Y) \) (resp. \( D^b(Z) \))

\[
\tag{5.3}
H^i(Y, G \otimes \Phi^{X \times X \to Y} (A)) \cong H^i(X \times X, \Phi^{Y \times X \to X}_{\omega_Y} (G) \otimes A)
\]

Note that, via duality, (5.3) is a restatement of the description of the adjoints of Fourier-Mukai functors, but it is more simply proved by the fact that

\[
R\Gamma(Y, G \otimes \Phi^{X \times X \to Y} (A)) \cong R\Gamma(Y \times X \times X, p_Y^* G \otimes (E \otimes \omega_Y ) \otimes p_X^* (A) \cong R\Gamma(X \times X, \Phi^{Y \times X \times X}_{E \cdot \omega_Y} (G) \otimes A)
\]

by Leray isomorphism and projection formula.

---

4 this is also the key ingredient in the proof of the equivalence between (b) and (c) of Theorem 2.3
Going back to our proposition, given a line bundle $A$ which is a sufficiently high power of an ample line bundle, from (5.3) and the second line of (5.1) we get

$$h^0(Y, \Phi^{X \times X \to Y}_{\hat{E} \otimes \hat{E}' \otimes \hat{E}'} (A) \otimes \omega_Y) = h^0(X \times X, \hat{\omega} \otimes A) = h^0(X \times X, \mathcal{O}_{\Delta_X} \otimes A)$$

Therefore, since the morphism (5.2) is surjective, Serre’s vanishing applied to its kernel yields that (5.1) is an isomorphism. Hence (1.5) is verified. This proves that (5.1) implies full-faithfulness of $\Phi^{X \to Y}_{\hat{E}}$. The other implication follows immediately from (5.3) and Serre’s vanishing.

**Example 5.2. (Mukai’s theorem on the Poincaré kernel)** To illustrate Prop. 5.1 let us take $X$ an abelian variety, $Y = \hat{X}$ and as $\mathcal{P}$ a Poincaré line bundle. We will show that $\Phi^{X \to \hat{X}}_P$ is fully faithful. From this it follows that it is in fact an equivalence. Moreover, since $\mathcal{P}^\vee = (-id, id)^* = (id, -id)^* \mathcal{P}$, this proves also that

$$\Phi^{X \to \hat{X}}_P \circ \Phi^{\hat{X} \to X}_P = (-id)^*[- \dim X]$$

i.e. the theorem of Mukai in [M]. As in [Hu] Prop. 9.19, in characteristic zero one has a much easier proof, using Bondal-Orlov’s strong simplicity criterion (see §3). The present proof works in any characteristic, as well as Mukai’s original proof.

Let $L = \mathcal{O}_X(n \Theta)$ be a sufficiently high power ample line bundle on $X$. We take $A = L \otimes L$. By Proposition 5.1 it is sufficient to prove that

$$h^i(\hat{X}, \Phi^{X \times X \to \hat{X}}_{\hat{P} \otimes \hat{P} \otimes \hat{P}^\vee} (L \otimes L)) = \begin{cases} 
0 & \text{if } i \neq \dim X \\
\h^0(X, L^2) & \text{if } i = \dim X
\end{cases}$$

Let

$$\varphi_L : X \to \hat{X}$$

the isogeny associated to $L$. We have that

$$(id, \varphi_L)^* \mathcal{P} = (p_1 + p_2)^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

where $p_1 + p_2 : X \times X \to X$ is the group law. Therefore, letting $\mathcal{F} = (p_1 + p_2)^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$, by flat base change we have that

$$\varphi_L^* (\Phi^{X \times X \to \hat{X}}_{\hat{P} \otimes \hat{P} \otimes \hat{P}^\vee} (L \otimes L)) = \Phi^{X \times X \to X}_{\mathcal{F} \otimes \mathcal{F}^\vee} (L \otimes L)$$

where $U \boxtimes X V$ means $p_{12}^* U \otimes p_{23}^* V$, where $p_{12}$ and $p_{23}$ are the two projections of $(X \times X) \times_X (X \times X) = X \times X \times X$ (the fibred product is with respect to the second projection of the first factor and the first projection of the second factor). Therefore we must compute the right-hand side of (5.5). We have that

$$\mathcal{F} \otimes_X \mathcal{F}^\vee = (p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L \otimes p_2^* L^{-1}$$

and by Serre vanishing (actually this is not needed in the case) the outcome is a sheaf:

$$p_{2*}((p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2) = R^0 p_{2*}((p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2)[0]$$
Via the automorphism \((p_1, p_1 + p_2, p_2 + p_3) : X \times X \times X \to X \times X \times X\), i.e. \((x, y, z) \mapsto (x, x + y, y + z)\), the line bundle \((p_1 + p_2)^*L \otimes (p_2 + p_3)^*L^{-1} \otimes p_1^*L^2\) is identified to \(L^2 \otimes L \otimes L^{-1}\). Hence

\[
h^i(X, \Phi_{\mathcal{F} \to \hat{\mathcal{F}}}(L \otimes L)) = h^i(X \times X \times X, L^2 \otimes L \otimes L^{-1}) = \begin{cases} h^0(X, L)^2 h^0(X, L^2) & \text{for } i = \dim X \\ 0 & \text{otherwise} \end{cases}
\]

Since the degree of the isogeny \(\varphi_L\) is \(h^0(X, L)^2\), we get (5.4).

**References**

[BO] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties, arXiv:alg-geom/9506012

[Br] T. Bridgeland, Equivalences of triangulated categories and Fourier-Mukai functors, Bull. London Math. Soc. 31 (1999), 25–34

[C] S. Cautis, Equivalences and stratified flops, Compos. Math., 148 (2012) 185–208

[GL1] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90 (1987), 389–407.

[GL2] M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, J. Amer. Math. Soc. 1 (1991), no.4, 87–103.

[H] Ch. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math. 575 (2004), 173–187.

[HeLS] D. Hernández Ruipérez, A. C. López Martín, F. Sancho de Salas, Fourier-Mukai transform for Gorenstein schemes Adv. Math. 211 (2007), 594–620

[Hu] D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Clarendon Press - Oxford (2006)

[K] M. Kashiwara, t-structures on the derived categories of holonomic \(D\)-modules and coherent \(\mathcal{O}\)-modules, Moscow Math. J. 4 (2004), no. 4, 847–868

[Ka1] Y. Kawamata, D-equivalence and K-equivalence, J. Diff. Geom. 61 (2002) 147–171

[Ka2] Y. Kawamata, Derived equivalence for stratified Mukai flop on \(G(2,4)\), in *Mirror symmetry. V*, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc. (2006), 285–294.

[L] A.C. López-Martín, Fully faithfulness criteria, appendix to: M. Melo, A. Rapagnetta, F. Viviani, Fourier-Mukai and autoduality for compactified jacobians, I, (version 2) arXiv:alg-geom/1207.7233v2

[Ma] E. Markman, Brill-Noether duality for moduli spaces of sheaves on K3 surfaces, J. Algebraic Geom. 10 (2001), 623–694.

[M] S. Mukai, Duality between \(D(X)\) and \(D(\hat{X})\) with its application to Picard sheaves, Nagoya Math. J. 81 (1981), 153–175.

[N1] Y. Namikawa, Mukai flops and derived categories, J. Reine Angew. Math. 560 (2003), 65–76

[N2] Y. Namikawa, Mukai flops and derived categories II, in *Algebraic structures and moduli spaces*, CRM Proc. Lecture Notes, 38, Amer. Math. Soc. (2004), 149–175

[PPo1] G. Pareschi and M. Popa, Regularity on abelian varieties I, J. Amer. Math. Soc. 16 (2003), 2857302.

[PPo2] G. Pareschi and M. Popa, Generic vanishing and minimal cohomology classes on abelian varieties, Math. Ann. 340 no. 1 (2008), 209–222

[PPo3] G. Pareschi and M. Popa, Strong generic vanishing and a higher-dimensional Castelnuovo-de Franchis inequality, Duke Math. J. 150 (2009), 269–285.

[PPo4] G. Pareschi and M. Popa, GV-sheaves, Fourier-Mukai transform, and Generic Vanishing, Amer. J. of Math. 133 (2011), 235–271

[Po] M. Popa, Generic vanishing filtrations and perverse objects in derived categories of coherent sheaves in *Derived categories in algebraic geometry*, EMS (2012), 251–277

[PoS] M. Popa and Ch. Schnell, Generic vanishing theory via mixed Hodge modules, Forum of Mathematics, Sigma 1 (2013), 1–60

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA, TOR VERGATA, V.LE DELLA RICERCA SCIENTIFICA, I-00133 ROMA, ITALY

E-mail address: pareschi@mat.uniroma2.it