Stability of a vortex in a trapped Bose-Einstein condensate

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Based on the method of matched asymptotic expansion and on a time-dependent variational analysis, we study the dynamics of a vortex in the large-condensate (Thomas-Fermi) limit. Both methods as well as an analytical solution of the Bogoliubov equations show that a vortex in a trapped Bose-Einstein condensate has formally unstable normal mode(s) with positive normalization and negative frequency, corresponding to a precession of the vortex line around the center of the trap. In a rotating trap, the solution becomes stable above an angular velocity Ω, characterizing the onset of metastability with respect to small transverse displacements of the vortex from the central axis.

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Recent experimental observations of Bose-Einstein condensation in trapped alkali-atom gases at ultra-low temperatures have raised the question of vortex states similar to those in superfluid helium. Unlike superfluid helium, however, the harmonic trap makes the alkali Bose condensates nonuniform.

Numerical studies of a vortex in small to medium axisymmetric condensates have determined the condensate density, along with the low-lying excitation spectrum. These analyses found a nontrivial eigenmode of a vortex-free condensate by a small amount of vortex-line motion. Separate studies have considered vortex formation by a moving object or a rotating force.

The existence of a positive-norm eigenstate with negative-excitation energy formally implies an instability of the vortex state because the condensate itself corresponds to zero energy. The present work obtains explicit analytical negative-energy solutions for a vortex in a large condensate, holding when the interatomic and trap interactions predominate and the kinetic energy of the nonuniform density is small. This TF limit is valid for \( R/d \approx 15(Na/d)^{1/5} > 1 \), where \( R \) is the mean condensate radius, \( d = (\bar{x} d_0 \bar{z})^{1/3} \) is the mean harmonic oscillator length, \( \omega_j = \hbar/M \omega_j \), with \( M \) the atomic mass and \( \omega_j \) the corresponding trap frequency \( (j = x, y, z) \), \( N \) is the number of atoms in the condensate, and \( a > 0 \) is the s-wave scattering length. Typically \( d \approx \) a few \( \mu m \), \( a \approx \) a few nm, and the experimental value \( N \gtrsim 10^6 \) amply satisfies the TF criterion that \( N \gg 10^3 \). In the TF limit \( (R \gg d) \), the vortex core radius \( \xi \sim d^2/R \) is small compared to both \( d \) and \( R \).

In this paper we assume that most of the particles are in the condensate, neglecting scattering with thermal background atoms, as confirmed by recent experimental studies of the lowest-lying normal modes. The condensate containing a vortex quantized vortex in an axisymmetric trap is known to have only stable normal density modes \( |m| \geq 2|q| \) (so that unstable modes occur only for \( 0 < |m| < 2|q| \)). Here, we exhibit unstable solutions with \( |m| = 1 \) that have positive norm and negative frequency in which the vortex line precesses around the center of the trap.

Dynamics of a vortex in the trapped condensate —Let us consider a condensate in a nonaxisymmetric trap that rotates with an angular velocity \( \Omega \) around the \( z \) axis. At zero temperature in a frame rotating with the angular velocity \( \Omega \), the evolution of the condensate wave function \( \Psi \) is described by the time-dependent Gross-Pitaevskii equation

\[
\left( -\frac{\hbar^2}{2M} \nabla^2 + V_{tr} + g|\Psi|^2 - \mu(\Omega) + i\hbar \Omega \partial_t \right) \Psi = i\hbar \frac{\partial \Psi}{\partial t},
\]

(1)

where \( V_{tr} = \frac{1}{2} M (\omega_0^2 x^2 + \omega_0^2 y^2 + \omega_z^2 z^2) \) is the external trap potential, \( g = 4\pi\hbar^2 a/M > 0 \) is the effective interparticle interaction strength, \( \mu(\Omega) \) is the chemical potential in the rotating frame, and \( \phi \) is the azimuthal angle in cylindrical polar coordinates. We assume that the condensate contains a \( q \)-fold quantized vortex parallel to the \( z \) axis located near the center of the trap at the transverse position \( \rho_0(t) \). In this section we use the method of matched asymptotic expansion to determine the vortex velocity as a function of the local gradient of the trap potential and angular velocity \( \Omega \), generalizing two-dimensional results obtained by Rubinstein and Pismen to the case of a three-dimensional rotating harmonic potential. The method applies when the external potential does not change significantly on distances comparable with the core size \( |q| \xi \ll R \) (TF limit); it matches the outer asymptotics of the solution of Eq. (1) in the vortex-core region \( (|\rho - \rho_0| \leq |q|\xi) \) with the short-distance behavior of the solution in the region far from the vortex core \( (|\rho - \rho_0| \gg |q|\xi) \). Recently the method was used to analyze the drift of an optical vortex soliton in a slowly...
varying background field [21].

To find the solution in the vortex core region, one may consider Eq. (1) in a coordinate frame centered on the vortex line that moves with the vortex velocity \( \mathbf{V} \) (\( \mathbf{V} \perp \hat{z} \)). The solution is assumed to be stationary in the comoving frame and satisfies the equation:

\[
\left( -\frac{\hbar^2}{2M} \nabla^2 + V_{tr} + g|\Psi|^2 - \mu(\Omega) + i\hbar\Omega\partial_\phi \right) \Psi = -i\hbar \mathbf{V} \cdot \nabla \Psi. \tag{2}
\]

In the vortex core region we may seek a solution in the form of an expansion in the small parameter \( \xi/R_\perp \):

\[
\Psi = [\Psi_0(\rho, z)] - \chi(\rho, z) \cos \phi \ e^{i q \rho - i n(\rho, z) \sin \phi}, \tag{3}
\]

where \( \Psi_0 \) is the condensate wave function with \( V_{tr} \) replaced by \( V_0 = \frac{1}{2} M \omega_z^2 z^2 \) and \( \chi \) and \( \eta \) characterize the perturbation in the absolute value and phase. Physically \( \Psi_0 \) is the analogous wave function for an unbounded condensate in the \( xy \) plane with the same chemical potential \( \mu(\Omega) \). The polar angle \( \phi \) is measured from the direction of the local potential gradient \( \nabla \Psi_{tr}(\rho_0) \), and \( \rho \) is the local radial cylindrical coordinate. For \( \rho \gg |q| \xi \) the perturbations have the asymptotic form:

\[
\eta \approx \frac{q |\nabla V_{tr}(\rho_0)|}{2g(|\Psi_0|^2) \rho} \ln (A\rho), \quad \chi \approx \frac{|\nabla \Psi_{tr}(\rho_0)|}{2g(|\Psi_0|^2)} \rho. \tag{4}
\]

Here we omit terms containing \( \Omega \) explicitly because they are proportional to the small parameter \( \hbar \Omega/\mu \). The parameter \( A \) must be determined by matching the solutions [4] with those far from the vortex core; in fact, \( A \) depends on \( \Omega \).

To lowest order in the small parameter \( \xi/R_\perp \), Eq. (3) far from the vortex core reduces to an equation for the condensate phase only (\( \Psi = |\Psi| e^{i \phi} \)):

\[
|\Psi_T|^2 \nabla^2 S + \nabla |\Psi_T|^2 \cdot \nabla S - \frac{M \Omega}{\hbar} \partial_\phi |\Psi_T|^2 = 0, \tag{5}
\]

where \( g|\Psi_T|^2 \approx \mu - \frac{1}{2} M \omega_z^2 z^2 \) and \( \mu \equiv \mu(\Omega) + \hbar \Omega Q \).

Comparing the solution of Eq. (3) for \( \rho \ll R_\perp \) with the outer asymptotic form (1) of the core solution gives the value of the parameter \( A \). With logarithmic accuracy we find

\[
\ln (Ae) = \ln \left( \frac{1}{R_\perp} \right) + \frac{4 \Omega g |\Psi_0|^2}{q \hbar (\omega_z^2 + \omega_y^2)}, \tag{6}
\]

where \( R_\perp \) is the mean transverse dimension of the condensate. To find the velocity \( \mathbf{V} \) of the vortex, one can use a solvability condition (Fredholm’s alternative) of the first-order equation in the small parameter (see [20] for details). In the frame rotating with the trap, the vortex velocity is found to be orthogonal to the direction of the local gradient of the trap potential \( \nabla \Psi_{tr} \):

\[
\mathbf{V} = \frac{3q \hbar}{4 M \mu} \left[ \ln \left( \frac{R_\perp}{|q| \xi} \right) - \frac{8 \mu \Omega}{3 q \hbar (\omega_z^2 + \omega_y^2)} \right] (\hat{z} \times \nabla \Psi_{tr}), \tag{7}
\]

where \( \mu \) can be taken as the chemical potential for a nonrotating trap. The vortex follows an elliptic trajectory along the line \( V_{tr} = \text{const} \). For \( q > 0 \) and \( \Omega < \Omega_m = 3[q \hbar (\omega_z^2 + \omega_y^2) \ln (R_\perp/|q| \xi)/8 \mu] \), the vortex moves counter-clockwise in the positive sense. With increasing rotation frequency \( \Omega \) of the trap, the vortex velocity (as seen in the rotation frame) decreases towards zero and vanishes at \( \Omega = \Omega_m \). At this rotation frequency, the effective potential for the vortex becomes flat near the trap center [see Eq. (23) below]. For \( \Omega > \Omega_m \) the apparent motion of the vortex becomes clockwise.

From Eq. (1) it is straightforward to obtain the vortex position as a function of time:

\[
x_0 = \varepsilon_0 R_x \sin(\omega t + \phi_0), \tag{8}
\]

\[
y_0 = \pm \varepsilon_0 R_y \cos(\omega t + \phi_0), \tag{9}
\]

where

\[
\omega = \pm \left[ \frac{2 \omega_x \omega_y \Omega}{(\omega_x^2 + \omega_y^2)} - \frac{3q \hbar \omega_x \omega_y}{4 \mu} \ln \left( \frac{R_\perp}{|q| \xi} \right) \right]. \tag{10}
\]

This vortex motion near the trap center can be considered a collective excitation of the condensate with an eigenfrequency \( \omega \). The two different frequencies given by (10) correspond to solutions with different sign of normalization: in terms of the Bogoliubov amplitudes \( u(\mathbf{r}), v(\mathbf{r}) \), these solutions have the normalization \( \int d^3r \left( |u|^2 - |v|^2 \right) = \pm \text{sgn}(q) \). The solution with positive norm has the energy

\[
E = \text{sgn}(q) \left[ \frac{2 \omega_x \omega_y \Omega}{(\omega_x^2 + \omega_y^2)} - \frac{3q \hbar^2 \omega_x \omega_y}{4 \mu} \ln \left( \frac{R_\perp}{|q| \xi} \right) \right]. \tag{11}
\]

If \( \Omega = 0 \), the excitation energy is negative and hence formally unstable. This negative energy implies the existence of a state with lower energy that can become macroscopically occupied. Nevertheless, the condensate will be unstable only if there is a mechanism to transfer the system to the lower energy state [22], which requires the presence of a reservoir to conserve energy and angular momentum. Thus a vortex in a nonrotating harmonic trap can remain stable at low enough temperatures when the noncondensate atoms are negligible. If the trap rotates, Eq. (11) shows that the solution becomes stable at \( |\Omega| \geq \Omega_m \), which coincides with the angular velocity at which the vortex at the center becomes metastable (see below). Note that the precession frequency (10) in a nonrotating trap involves the product \( \omega_x \omega_y \), whereas the metastable rotation frequency \( \Omega_m \) instead involves the combination \( \frac{1}{2}(\omega_z^2 + \omega_y^2) \).
Analysis of the Bogoliubov equations for an axisymmetric trap — The collective excitation energies $E$ of the (in general, nonuniform) condensate are the eigenvalues of the Bogoliubov equations for the coupled amplitudes $u(r)$, $v(r)$

\[
\begin{pmatrix}
\frac{-\hbar^2}{2M} \nabla^2 + V(r) + 2g|\Psi|^2 - \mu(\Omega)
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
 i\hbar\Omega\partial_{\phi} - g\Psi^2 \\
-g\Psi^2 - i\hbar\Omega\partial_{\phi}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix} = E \begin{pmatrix}
u \\
-v\n\end{pmatrix}.
\]

We use Eqs. (17)-(18) as a zero-order approximation for a perturbation expansion of Eq. (13). The first-order shift of $E$ requires the matrix elements $(U_\pm|\tilde{V}|U_\mp)$. With logarithmic accuracy, their main contribution arises from the integration region $|q|<\rho < R_\perp$, where $|q|\xi$ is the vortex core radius. For this region in the TF limit, we have

\[
g|\Psi|^2 \approx \mu - \frac{\hbar^2q^2}{2M\rho^2} - V(r) - \frac{\hbar^2\omega_z^2}{2g|\Psi_{TF}|^2} + \frac{\hbar^2}{2M} \frac{\partial_{zz}^2}{|\Psi_{TF}|^2},
\]

where $g|\Psi_{TF}|^2 \approx \mu - \frac{1}{2} M \omega_z^2 \rho^2$ and $\mu \equiv \mu(\Omega) + \hbar\Omega q$. The last two terms in [19] reflect the gradient terms in the Gross-Pitaevskii equation and are small in the TF limit. Nevertheless, these small terms give the main contribution to the matrix elements. The same expression [19] holds for the density of the unbounded condensate $|\Psi_0|^2$ but with $\omega_\perp = 0$. Thus the perturbation operator $\tilde{V}$ becomes

\[
\tilde{V} = \left(\frac{1}{2} M \omega_z^2 \left(x^2 + y^2\right) + \frac{\hbar^2 \omega_z^2}{2g|\Psi_{TF}|^2} \left(e^{-2iq\phi} - 1\right)
\right)
\]

and the off-diagonal matrix elements $(U_\pm|\tilde{V}|U_\mp)$ vanish. With logarithmic accuracy, the last term in Eq. (20) gives the main contribution to

\[
\langle U_\pm|\tilde{V}|U_\mp \rangle \approx - \frac{\hbar^2 \omega_z^2}{g^2} \int dq \frac{q^2}{p^2}.
\]

Integrating over $dp$ gives a logarithmic factor $\ln (R_\perp/|q|\xi)$. As a result, we obtain two solutions for the excitation energy. They are normalized to the value $\text{sgn}(\mp q)$, and the solution with positive norm has an energy

\[
E = \text{sgn}(q)\hbar\Omega - \frac{3|q|\hbar^2 \omega_z^2}{4\mu} \ln (R_\perp/|q|\xi),
\]

where $\mu$ can be taken as the chemical potential for a non-rotating trap. Formula (22) agrees with (11) obtained on the basis of the method of matched asymptotic expansion (for $\omega_z = \omega_y = \omega_\perp$).

Energy of a vortex in a Bose-Einstein condensate in a rotating trap — In a frame rotating with angular velocity $\Omega \hat{z}$, the energy functional of the system is

\[
E(\Psi) = \int dV \left( \frac{\hbar^2}{2M} |\nabla \Psi|^2 + V_c |\Psi|^2 + \frac{g}{2} |\Psi|^4 + \Psi^* i\hbar \Omega \partial_{\phi} \Psi \right).
\]

(23)

Consider a $q$-fold quantized vortex displaced from the center of the trap, with transverse coordinates $x = x_0$, $y = y_0$, and $z = z_0$. For a $q$-fold quantized vortex at the center of the trap, the unperturbed condensate wave function has the form $\Psi_0 = e^{iq\phi} |\Psi_0|$, and Eq. (14) has the following exact pair of solutions $|\Psi_\pm|^2$

\[
\begin{pmatrix}
\Psi_+ \\
\Psi_-
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{2}} e^{iq\phi} (- \partial_\rho \mp q/\rho) \\
\frac{1}{\sqrt{2}} e^{-iq\phi} (\partial_\rho \pm q/\rho)
\end{pmatrix} |\Psi_0|.
\]

where $\rho$ is the radial cylindrical coordinate, and $I^2 = 4\pi|q| \int dz |\Psi_0|^2 \approx 16\sqrt{2}\pi|q|\rho^3/3\hbar\omega_\perp \sqrt{M}$ in the TF limit. These solutions are normalized so that $\int d^{3}r (|\Psi_\pm|^2 - |\Psi_0|^2)^2 = \text{sgn}(\mp q)$. They generalize those obtained by Pitaevskii [18] to the case of an axial trap with potential $V_c$ and describe the transverse motion of the vortex line as a whole relative to the condensate. It is obvious that the solutions $U_+$ and $U_-$ are orthogonal.
y = y_0. In the TF limit, the vortex-induced change in the condensate density is negligible. Hence, the main contribution to the change $\Delta E(x_0, y_0)$ in the energy of the system due to the vortex arises from the condensate’s superfluid motion, and we can take $|\Psi|^2$ in Eq. (23) to be the density of the condensate without a vortex. A transverse shift of the Cartesian coordinate system places the origin on the vortex axis and yields

$$\Psi = |\Psi| e^{iq\phi + iS_0},$$

(24)

$$g|\Psi|^2 = \mu - \frac{1}{2} M \left[ \omega_x^2 (x + x_0)^2 + \omega_y^2 (y + y_0)^2 + \omega_z^2 z^2 \right],$$

(25)

where $\phi$ is a polar angle around the vortex line and $S_0$ is a periodic function of $\phi$.

Varying the functional (23) gives an Euler-Lagrange equation for $S_0$. For a vortex-free condensate, we get

$$S_0 = -\frac{M\Omega}{\hbar} \left( \frac{\omega_x^2 - \omega_y^2}{\omega_x^2 + \omega_y^2} \right) (x + x_0)(y + y_0).$$

(26)

If the condensate has a $q$-fold quantized vortex, then the contribution to the free energy from the phase factor $iS_0$ in Eq. (24) can be neglected with respect to the contribution from $iq\phi$ for small angular velocity $\Omega$. Only for $|\Omega| > |\hbar(\omega_x^2 + \omega_y^2)/4\mu| do the contributions from these two phase factors become comparable. For such large angular velocities, however, the function $S_0$ is insensitive to the presence of the vortex. Thus Eq. (24) with $S_0$ given by Eq. (26) provides a good approximation for the condensate wave function for general angular velocity $\Omega$.

Substituting Eqs. (24), (26) into the energy functional (23) and integrating with logarithmic accuracy over the volume of condensate yield the following expression

$$\Delta E(x_0, y_0) = \frac{8\pi}{3} \mu R_z \xi^2 n_0(0) \left( 1 - \frac{x_0^2}{R_x^2} - \frac{y_0^2}{R_y^2} \right)^{3/2} \left[ q^2 \ln \left( \frac{R_\perp}{|q|\xi} \right) - \frac{8q\mu\Omega}{5\hbar} \left( 1 - \frac{x_0^2}{R_x^2} - \frac{y_0^2}{R_y^2} \right) \right],$$

(27)

where $R_j^2 = 2\mu/M\omega_j^2$ fixes the condensate’s dimensions, $R_\perp^2 = M(\omega_x^2 + \omega_y^2)/4\mu = \frac{1}{2}(R_x^2 + R_y^2)$ characterizes the mean transverse dimension, $\xi^2 = \hbar^2/2M\mu$, and $n_0(0) = \mu/g$ is the density at the center of the vortex-free condensate. For small displacements of the vortex ($x_0 \ll R_x$, $y_0 \ll R_y$), we have:

$$\Delta E(x_0, y_0) = \frac{8\pi}{3} \mu R_z \xi^2 n_0(0) \left[ q^2 \ln \left( \frac{R_\perp}{|q|\xi} \right) - \frac{8q\mu\Omega}{5\hbar} \left( \frac{x_0^2}{R_x^2} + \frac{y_0^2}{R_y^2} \right) \right],$$

(28)

where $R_j^2 = 2\mu/M\omega_j^2$ fixes the condensate’s dimensions, $R_\perp^2 = M(\omega_x^2 + \omega_y^2)/4\mu = \frac{1}{2}(R_x^2 + R_y^2)$ characterizes the mean transverse dimension, $\xi^2 = \hbar^2/2M\mu$, and $n_0(0) = \mu/g$ is the density at the center of the vortex-free condensate. For small displacements of the vortex ($x_0 \ll R_x$, $y_0 \ll R_y$), we have:

For $|\Omega| > \Omega_c = \frac{5q\hbar}{8\mu} \left( R_\perp/|q|\xi \right) / 8\mu = \frac{5q\hbar}{8\mu} \left( R_\perp/MR_z^2 \right) \ln \left( R_\perp/|q|\xi \right)$, the central vortex is stable because $\Delta E(0, 0)$ is negative (23). Furthermore, $\Delta E(x_0, y_0)$ has a local minimum at the origin for $|\Omega| > \Omega_c$, showing that the central vortex is metastable in the interval $\Omega_m < |\Omega| < \Omega_c$. Since $\hbar\omega_{x,y,z} \ll \mu$ in the TF limit, the thermodynamic critical angular velocity $\Omega_c$ for vortex formation and $\Omega_m$ are both much smaller than the trap frequencies.

**Time-dependent variational analysis** — Instead of solving the time-dependent Gross-Pitaevskii equation (1) directly, one may consider a variational problem for the action obtained from the effective Lagrangian (we assume $\Omega = 0$ in this section):

$$L(t) = \int d^3 r \left[ \frac{i}{2} \hbar \left( \Psi^\ast \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^\ast}{\partial t} \right) - \frac{\hbar^2}{2M} |\nabla \Psi|^2 - V_0(r) |\Psi|^2 - \frac{1}{2g} |\Psi|^4 \right].$$

(29)

With a suitable trial function containing time-dependent parameters, the time evolution follows from the associated Lagrange equations for these parameters. Although not exact, this method gives a clear physical picture of the system’s behavior. It determined the low-energy excitations of a trapped zero-temperature condensate [24], predicting the low-energy excitation spectrum of the condensate and, in the TF limit, reproducing the formulas derived by Stringari [25]. In this present paper we use this method as an alternative way to study the normal modes of a condensate with a vortex.

We are interested in the relative motion of a vortex inside the condensate, so it is natural to take a time-dependent trial function in the form:

$$\Psi(r, t) = B(t) f [r - \rho_0(t)] F[r - \eta_0(t)] \times \prod_{j=x,y,z} \exp \left[ i x_j \alpha_j(t) + i x_j^2 \beta_j(t) \right].$$

(30)

Here the function $f(r)$ characterizes the vortex inside the trap and has the approximate form $f(r) = e^{iq\phi}$ for $\rho \gg |q|\xi$; the function $F(r)$ describes the condensate density distribution and, in the TF limit, has the form

$$F^2(r) = 1 - \frac{x^2}{R_x^2(t)} - \frac{y^2}{R_y^2(t)} - \frac{z^2}{R_z^2(t)}.$$

(31)

The time-dependent vector $\eta_0(t) = (\eta_{0x}, \eta_{0y}, \eta_{0z})$ describes the motion of the center of the condensate, while $\rho_0(t) = (x_0, y_0, 0)$ describes the motion of the vortex line in the $xy$ plane. The other variational parameters are the amplitude $B$, the width of the condensate $R_j$, and the set $\alpha_j$ and $\beta_j$ that describe the condensate’s motion. These parameters are real functions of time, characterizing the macroscopic wave function of the condensate. Substitution of the trial wave function (31) into Eq. (29) yields
an effective Lagrangian as a function of the variational parameters (and their first time derivatives).

We focus on small oscillations, when the displacement $\rho_0(t)$ of the vortex from the trap center and the displacement $\eta_0(t)$ of the condensate are small relative to the width of the condensate; expanding for small $\rho_0(t)$ and $\eta_0(t)$, we keep only the first nonzero corrections. The resulting equations for the motion of the vortex line (that is, for $x_0, y_0$) are coupled only with $\eta_{0x}, \eta_{0y}$ and $\alpha_x, \alpha_y$. This system of six first-order ordinary differential equations can be solved explicitly. Apart from the well-known dipole-mode solutions that describe the rigid oscillation of the condensate and the vortex as a whole, these equations have another solution that corresponds to the motion of the vortex relative to the condensate. For this solution the vortex motion is described by formulas (8), (9), while the displacement of the condensate is given by

$$\eta_{0x} = -\frac{15\varepsilon_0 \xi^2}{2R_y} \ln \left( \frac{R_\perp}{|q| \xi} \right) \frac{R_x}{R_x + R_y} \sin(\omega t + \phi_0), \quad (32)$$

$$\eta_{0y} = \frac{15\varepsilon_0 \xi^2}{2R_x} \ln \left( \frac{R_\perp}{|q| \xi} \right) \frac{R_y}{R_x + R_y} \cos(\omega t + \phi_0), \quad (33)$$

where

$$\omega = \mp \frac{3q \hbar \omega \xi}{4\mu} \ln \left( \frac{R_\perp}{|q| \xi} \right). \quad (34)$$

The quantity $x_0^2/R_x^2 + y_0^2/R_y^2 = \varepsilon_0^2$ remains constant as the vortex line follows an elliptic trajectory around the center of a trap along the line $V_{tr} = const$, and the energy of the system is conserved [as follows from Eq. (23)]. These results agree with those obtained from the method of matched asymptotic expansion. The condensate also precesses with relative phase shift $\pi$ at the same frequency, but the amplitude of the condensate motion is smaller than that of the vortex line by a factor $\sim \xi^2 \ln (R_\perp/|q| \xi)/R_x R_y$. As in the discussion of (10), these solutions have the normalization $-\text{sgn}(\omega)$, and the positive normalization corresponds to the solution with negative frequency. In the present limit of logarithmic accuracy, the structure of the vortex core is inessential for the excitation spectrum. The precession frequency obtained in the variational approach coincides with formula (10) (for $\Omega = 0$).

In conclusion, we have used the method of matched asymptotic expansion (for a general three-dimensional rotating nonaxisymmetric trap), analysis of the Bogoliubov equations and a time-dependent variational procedure to study particular normal modes of a trapped condensate containing a vortex. These modes have negative energy and positive norm, with the vortex line moving around the $z$ axis. The excitation frequency in a rotating trap becomes positive when the angular velocity $\Omega$ reaches the value that makes the central vortex metastable.

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