ON THE POINTWISE CONVERGENCE OF THE CUBIC AVERAGE WITH MULTIPLICATIVE OR VON MANGOLDT WEIGHTS

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Abstract. It is shown that the cubic nonconventional ergodic averages of any order with a bounded aperiodic multiplicative function or von Mangoldt weights converge almost surely.

1. Introduction.

The purpose of this note is motivated by the recent great interest on the Möbius function from the dynamical point view, and by the problem of the multiple recurrence which goes back to the seminal work of Furstenberg [16]. This later problem has nowadays a long history.

The dynamical study of Möbius function was initiated recently by Sarnak in [33]. Therein, Sarnak made a conjecture that the Möbius function is orthogonal to any deterministic dynamical sequence. Notice that this precise the definition of a reasonable sequence in the Möbius randomness law mentioned by Iwaniec-Kowalski in [26, p.338]. Sarnak further mentioned that Bourgain’s approach can be used to prove that for almost all point \( x \) in any measurable dynamical system \((X, \mathcal{A}, T, \mathbb{P})\), the Möbius function is orthogonal to any dynamical sequence \( f(T^n x) \), where \( f \) is a square integrable function. For simple proofs and other related results, see [1] and [10].

Here, we are interested in the pointwise convergence of cubic nonconventional ergodic averages with bounded aperiodic multiplicative functions weight and von Mangoldt weight.

The convergence of cubic nonconventional ergodic averages was initiated by Bergelson in [5], where convergence in \( L^2 \) was shown for order 2 and under the extra assumption that all the transformations are equal. Under the same assumption, Bergelson’s result was extended by Host and Kra for cubic averages of order 3 in [22], and for arbitrary order in [23]. Assani proved that pointwise convergence of cubic nonconventional ergodic averages of order 3 holds for not necessarily commuting maps in [3],

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and further established the pointwise convergence for cubic averages of arbitrary order when all the maps are equal. In [8], Chu and Frantzikinakis completed the study and established the pointwise convergence for the cubic averages of arbitrary order. Very recently, Huang-Shao and Ye [25] gave a topological-like proof of the pointwise convergence of the cubic nonconventional ergodic average when all the maps are equal. They further applied their method to obtain the pointwise convergence of nonconventional ergodic averages for a distal system.

Here, we establish that the cubic averages of any order with the aperiodic bounded multiplicative function weight converge to zero almost surely. The proof depends heavily on the Gowers inverse theorem. As a consequence, we obtain that the cubic averages of any order with Möbius or Liouville weights converge to zero almost surely. Moreover, we establish that the cubic nonconventional ergodic averages weighted with von-Mangoldt function converge. This result is obtained as a consequence of the very recent result of Ford-Green-Konyagin-Tao [14] combined with the Gowers inverse theorem and the sieve methods trick called $W$-trick.

The paper is organized as follows. In Section 2, we state our main results and we recall the main ingredients needed for the proofs. In Section 3, we prove our first main result. In Section 4, we give a proof of our second main result.

2. Basic definitions and tools.

In this section we will recall some Basic definitions and tools we will use in this paper, and state our main results.

2.1. The multiplicative or von Mangoldt functions. Recall that the Liouville function $\lambda : \mathbb{N}^* \rightarrow \{-1, 1\}$ is defined by

$$\lambda(n) = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the number of prime factors of $n$ counted with multiplicities with $\Omega(1) = 1$. Obviously, $\lambda$ is completely mutiplicative, that is, $\lambda(nm) = \lambda(n)\lambda(m)$, for any $n, m \in \mathbb{N}^*$. The integer $n$ is said to be not square-free if there is a prime number $p$ such that $n$ is in the class of $0 \mod p^2$. The Möbius function $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$ is define as follows

$$\mu(n) = \begin{cases} 
\lambda(n), & \text{if } n \text{ is square-free}; \\
1, & \text{if } n = 1; \\
0, & \text{otherwise}.
\end{cases}$$

This definition of Möbius function establishes that the the restriction of Liouville function and Möbius function to set of square free numbers coincident. Nevertheless, the Möbius function is only mutiplicative, that is, $\mu(nm) = \mu(m)\mu(n)$ whenever $n$ and $m$ are coprime.
We further remind that the von Mangoldt function and its cousin $\Lambda'$ are given by

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^\alpha, \text{ for some prime } p \text{ and } \alpha \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\Lambda'(n) = \begin{cases} \log(n) & \text{if } n \text{ is a prime,} \\ 0 & \text{if not.} \end{cases}$$

We will denote as customary by $\pi(N)$ the number of prime less than $N$ and by $\mathcal{P}$ the set of primes. We remind that for $x > 0$ the Chebyshev functions are defined by

$$\Psi(x) = \sum_{n \leq x} \Lambda(n) \quad \text{and} \quad \upsilon(x) = \sum_{n \leq x} \Lambda'(n).$$

By the Prime Number Theorem with Reminder (PNTR), it is well-known that $\pi(N)$ is related to the function $Li$ given by

$$Li(N) = \int_2^N \frac{1}{\log(t)} dt.$$ 

We further have, by Selberg’s estimation,

$$\pi(N) = \frac{N}{\log(N)} + O\left(\frac{N}{(\log(N))^2}\right),$$

This estimation is equivalent to the following relation

$$\upsilon(N) = N + O\left(\frac{N}{(\log(N))}\right).$$

We say that the multiplicative function $\nu$ is aperiodic if

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \nu(an + b) = 0,$$

for any $(a, b) \in \mathbb{N}^* \times \mathbb{N}$. By Davenport’s theorem [12] and Batman-Chowla’s theorem [4], the Möbius and Liouville functions are aperiodic.

We need the following notion of statistical orthogonality.

**Definition 1.** Let $(a_n), (b_n)$ be two sequences of complex numbers. The sequences $(a_n)$ and $(b_n)$ are said to be statistically orthogonal (or just orthogonal) if

$$\left| \frac{1}{N} \sum_{n=1}^{N} a_n \overline{b_n} \right| = o\left( \left( \frac{1}{N} \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{n=1}^{N} |b_n|^2 \right)^{\frac{1}{2}} \right).$$
2.2. Cubic averages and related topics. Let \((X, \mathcal{B}, \mathbb{P})\) be a Lebesgue probability space and given three measure preserving transformations \(T_1, T_2, T_3\) on \(X\). Let \(f_1, f_2, f_3 \in L^\infty(X)\). The cubic nonconventional ergodic averages of order 2 with weight \(A\) are defined by

\[
\frac{1}{N^2} \sum_{n, m=1}^N A(n)A(m)f_1(T_1^n x)f_2(T_2^m x)f_3(T_3^{n+m} x).
\]

This nonconventional ergodic average can be seen as a classical one as follows

\[
\frac{1}{N^2} \sum_{n, m=1}^N \tilde{f}_1(\tilde{T}_1^n (A, x))\tilde{f}_2(\tilde{T}_2^m (A, x))\tilde{f}_3(\tilde{T}_3^{n+m} (A, x)),
\]

where \(\tilde{f}_i = \pi_0 \otimes f_i, \tilde{T}_i = (S \otimes T_i), i = 1, 2, 3\) and \(\pi_0\) is define by \(x = (x_n) \mapsto x_0\) on the space \(Y = \mathbb{C}^N\) equipped with some probability measure.

More generally, let \(k \geq 1\) and put

\[C^* = \{0, 1\}^k \setminus \{(0, \cdots, 0)\}.\]

Consider the family \((T_e)_{e \in C^*}\) of transformations measure preserving on \(X\) and for each \(e \in C^*\) let \(f_e\) be in \(L^\infty(X)\). The cubic nonconventional ergodic averages of order \(k\) with weight \(A\) are given by

\[(1) \quad \frac{1}{N^k} \sum_{n \in [1,N]^k} \prod_{e \in C^*} A(n.e)f_e(T_e^{n.e} x) \xrightarrow{N \to +\infty} 0,
\]

where \(n = (n_1, \cdots, n_k), e = (e_1, \cdots, e_k), n.e\) is the usual inner product.

The study of the cubic averages is closely and strongly related to the notion of seminorms introduced in [17] and [23]. They are nowadays called Gowers-Host-Kra’s seminorms.

Assume that \(T\) is an ergodic measure preserving transformation on \(X\). Then, for any \(k \geq 1\), the Gowers-Host-Kra’s seminorms on \(L^\infty(X)\) are defined inductively as follows

\[
\|f\|_1 = \left| \int f \, d\mu \right|;
\]

\[
\|f\|_{k+1} = \lim_{N \to +\infty} \frac{1}{N^k} \sum_{n \in [1,N]^k} \|T^n f \|_{k}.
\]

For each \(k \geq 1\), the seminorm \(\|\cdot\|_k\) is well defined. For details, we refer the reader to [23] and [21]. Notice that the definitions of Gowers-Host-Kra’s seminorms can be easily extended to non-ergodic maps as it was mentioned by Chu and Frantzikinakis in [8].
The importance of the Gowers-Host-Kra’s seminorms in the study of the nonconventional multiple ergodic averages is due to the existence of a $T$-invariant sub-$\sigma$-algebra $Z_{k-1}$ of $X$ that satisfies

$$\mathbb{E}(f|Z_{k-1}) = 0 \iff \|f\|_k = 0.$$  

This was proved by Host and Kra in [23]. The existence of the factors $Z_k$ was also showed by Ziegler in [36]. We further notice that Host and Kra established a connection between the $Z_k$ factors and the nilsystems in [23].

### 2.3. The main results

At this point we are able to state our main results

**Theorem 1.** The cubic nonconventional ergodic averages of any order with a bounded aperiodic multiplicative function weight converge almost surely to zero, that is, for any $k \geq 1$, for any $(f_e)_{e \in C^*} \subset L^\infty(X)$, for almost all $x$, we have

$$\frac{1}{N^k} \sum_{n \in [1,N]^k} \prod_{e \in C^*} \nu(n.e) f_e(T_e^{n.e}x) \xrightarrow{N \to +\infty} 0,$$

where $n = (n_1, \cdots, n_k)$, $e = (e_1, \cdots, e_k)$, $C^* = \{0,1\}^k \setminus \{(0, \cdots, 0)\}$, $n.e$ is the usual inner product, and $\nu$ is the bounded aperiodic multiplicative function.

Our second main result can be stated as follows

**Theorem 2.** The cubic nonconventional ergodic averages of any order with von Mangoldt function weight converge almost surely.

Notice that Theorem 2 is related in some sense to the weak correlation of von Mangoldt function. Moreover, the study of the correlations of von Mangoldt function is of great importance in number theory, since it is related to the famous old conjecture of the twin numbers and more generaly to Hard-Littlewood k-tuple conjecture. It is also related to Riemann hypothesis and the Goldbach conjectures.

### 2.4. Some tools from the theory of Nilsystems and nilsequences

The nilsystems are defined in the setting of homogeneous space 1. Let $G$ be a Lie group, and $\Gamma$ a discrete cocompact subgroup (Lattice, uniform subgroup) of $G$. The homogeneous space is given by $X = G/\Gamma$ equipped with the Haar measure $h_X$ and the canonical complete $\sigma$-algebra $\mathcal{B}_c$. The action of $G$ on $X$ is by the left translation, that is, for any $g \in G$, we have $T_g(x\Gamma) = gx\Gamma = (gx)\Gamma$. If further $G$ is a nilpotent Lie group of order $k$, $X$ is said to be a $k$-step nilmanifold. For any fixed $g \in G$, the dynamical system $(X, \mathcal{B}_c, h_X, T_g)$ is called a $k$-step nilsystem. The basic $k$-step nilsequences on $X$ are defined by $f(g^n x \Gamma) = (f \circ T_g^n)(x \Gamma)$, where $f$ is a continuous function of $X$. Thus, $(f(g^n x \Gamma))_{n \in \mathbb{Z}}$ is any element of $\ell^\infty(\mathbb{Z})$, the space of bounded sequences, equipped with uniform norm $\|(a_n)\|_\infty = \sup_{n \in \mathbb{Z}} |a_n|$. A $k$-step nilsequence is a uniform limit of basic $k$-step nilsequences. For more details on the nilsequences we refer the

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1 For a nice account of the theory of the homogeneous space we refer the reader to [11], [28, pp.815-919].
Recall that the sequence of subgroups \((G_n)\) of \(G\) is a filtration if \(G_1 = G\), \(G_{n+1} \subset G_n\), and \([G_n, G_p] \subset G_{n+p}\), where \([G_n, G_p]\) denotes the subgroup of \(G\) generated by the commutators \([x, y] = x y x^{-1} y^{-1}\) with \(x \in G_n\) and \(y \in G_p\). The lower central filtration is given by \(G_1 = G\) and \(G_{n+1} = [G, G_n]\). It is well know that the lower central filtration allows to construct a Lie algebra \(\text{gr}(G)\) over the ring \(\mathbb{Z}\) of integers. \(\text{gr}(G)\) is called a graded Lie algebra associated to \(G\) \([7, \text{p.38}]\). The filtration is said to be of degree or length \(l\) if \(G_{l+1} = \{e\}\), where \(e\) is the identity of \(G\). We denote by \(G^e\) the identity component of \(G\). Since \(X = G/\Gamma\) is compact, we can assume that \(G/G^e\) is finitely generated \([29]\).

If \(G\) is connected and simply-connected with Lie algebra \(\mathfrak{g}\) \(^3\), then \(\exp : G \to \mathfrak{g}\) is a diffeomorphism, where \(\exp\) denotes the Lie group exponential map. We further have, by Mal’cev’s criterion, that \(\mathfrak{g}\) admits a basis \(\mathcal{X} = \{X_1, \cdots, X_m\}\) with rational structure constants \([30]\), that is,

\[ [X_i, X_j] = \sum_{n=1}^{m} c_{ijn} X_n, \text{ for all } 1 \leq i, j \leq k, \]

where the constants \(c_{ijn}\) are all rational.

Let \(\mathcal{X} = \{X_1, \cdots, X_m\}\) be a Mal’cev basis of \(\mathfrak{g}\), then any element \(g \in G\) can be uniquely written in the form \(g = \exp(t_1 X_1 + t_2 X_2 + \cdots + t_m X_m), t_i \in \mathbb{R}\), since the map \(\exp\) is a diffeomorphism. The numbers \((t_1, t_2, \cdots, t_k)\) are called the Mal’cev coordinates of the first kind of \(g\). In the same manner, \(g\) can be uniquely written in the form \(g = \exp(s_1 X_1) \cdot \exp(s_2 X_2) \cdots \cdot \exp(s_m X_m), s_i \in \mathbb{R}\). The numbers \((s_1, s_2, \cdots, s_k)\) are called the Mal’cev coordinates of the second kind of \(g\). Applying Baker-Campbell-Hausdorff formula, it can be shown that the multiplication law in \(G\) can be expressed by a polynomial mapping \(\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m\) \([32, \text{p.55}], [18]\). This gives that any polynomial sequence \(g\) in \(G\) can be written as follows

\[ g(n) = \gamma_1^{p_1(n)} \cdots \gamma_m^{p_m(n)}, \]

where \(\gamma_1, \cdots, \gamma_m \in G, p_i : \mathbb{N} \to \mathbb{N}\) are polynomials \([18]\). Given \(n, h \in \mathbb{Z}\), we put

\[ \partial_h g(n) = g(n + h) g(n)^{-1}. \]

This can be interpreted as a nilsequencediscrete derivative on \(G\). Given a filtration \((G_n)\) on \(G\), a sequence of polynomial \(g(n)\) is said to be adapted to \((G_n)\) if \(\partial_{h_1} \cdots \partial_{h_1} g\) takes values in \(G_i\) for all positive integers \(i\) and for all choices of \(h_1, \cdots, h_i \in \mathbb{Z}\). The set of all polynomial sequences adapted to \((G_n)\) is denoted by \(\text{poly}(\mathbb{Z}, (G_n))\).

\(^2\)The term ‘nilsequence’ was coined by Bergelson-Host and Kra in 2005 \([6]\).

\(^3\)By Lie’s fundamental theorems and up to isomorphism, \(\mathfrak{g} = T_e G\), where \(T_e G\) is the tangent space at the identity \(e\) \([27, \text{p.34}]\).
Furthermore, given a Mal’cev’s basis \( \mathcal{X} \) one can induce a right-invariant metric \( d_\mathcal{X} \) on \( X \) [18]. We remind that for a real-valued function \( \phi \) on \( X \), the Lipschitz norm is defined by

\[
\|\phi\|_L = \|\phi\|_\infty + \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d_\mathcal{X}(x, y)}.
\]

The set \( \mathcal{L}(X, d_\mathcal{X}) \) of all Lipschitz functions is a normed vector space, and for any \( \phi \) and \( \psi \) in \( \mathcal{L}(X, d_\mathcal{X}) \), \( \phi \psi \in \mathcal{L}(X, d_\mathcal{X}) \) and \( \|\phi \psi\|_L \leq \|\phi\|_L \|\psi\|_L \). We thus get, by Stone-Weierstrass theorem, that the subsalgebra \( \mathcal{L}(X, d_\mathcal{X}) \) is dense in the space of continuous functions \( C(X) \) equipped with uniform norm \( \|\cdot\|_\infty \). It turns out that for a Lipschitz function, the extension from an arbitrary subset is possible without increasing the Lipschitz norm, thanks to Kirszbraun-Mcshane extension theorem [13, p.146].

In this setting, we remind the following fundamental Green-Tao’s theorem on the strong orthogonality of the Möbius function to any \( m \)-step nilsequence, \( m \geq 1 \).

**Proposition 2.1.** [19, Theorem 1.1]. Let \( G/\Gamma \) be a \( m \)-step nilmanifold for some \( m \geq 1 \). Let \( (G_p) \) be a filtration of \( G \) of degree \( l \geq 1 \). Suppose that \( G/\Gamma \) has a \( Q \)-rational Mal’cev basis \( \mathcal{X} \) for some \( Q \geq 2 \), defining a metric \( d_\mathcal{X} \) on \( G/\Gamma \). Suppose that \( F : G/\Gamma \to [-1, 1] \) is a Lipschitz function. Then, for any \( A > 0 \), we have the bound,

\[
\sup_{g \in \partial \phi(g, (G_p))} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) F(g(n)\Gamma) \right| \leq C \frac{(1 + \|F\|_L)}{\log^A N},
\]

where the constant \( C \) depends on \( m, l, A, Q, N \geq 2 \).

We further need the following decomposition theorem due to Chu-Frantzikinakis and Host from [9, Proposition 3.1].

**Proposition 2.2** (NSZE-decomposition theorem [9]). Let \( (X, \mathcal{A}, \mu, T) \) be a dynamical system, \( f \in L^\infty(X) \), and \( k \in \mathbb{N} \). Then for every \( \varepsilon > 0 \), there exist measurable functions \( f_{ns}, f_z, f_e \), such that

(a) \( \|f_\kappa\|_\infty \leq 2\|f\|_\infty \) with \( \kappa \in \{ns, z, e\} \).

(b) \( f = f_{ns} + f_z + f_e \) with \( \|f_z\|_{k+1} = 0 \); \( \|f_e\|_1 < \varepsilon \); and

(c) for \( \mu \) almost every \( x \in X \), the sequence \( (f_{ns}(T^n x))_{n \in \mathbb{N}} \) is a \( k \)-step nilsequence.

3. **Proof of the first main result (Theorem 1).**

The proof of our first main result (Theorem 1) for \( k \geq 2 \) is essentially based on the inverse Gowers norms theorem due to Green, Tao and Ziegler [20] combined with the recent result of Host-Frantzikinakis [15]. We need also the following result due to T. Tao.

**Proposition 3.1** (Uniform Discrete inverse theorem for Gowers norms, [34]). Let \( N \geq 1 \) and \( s \geq 1 \) be integers, and let \( \delta > 0 \). Suppose \( f : \mathbb{Z} \to [-1, 1] \) is a function supported on \( \{1, \cdots, N\} \) such that

\[
\frac{1}{N^{s+2}} \sum_{(n,n) \in [1,N]^{s+1}} \prod_{e \in \{0,1\}^{s+1}} f(n + n.e) \geq \delta.
\]
Then there exists a filtered nilmanifold $G/\Gamma$ of degree $\leq s$ and complexity $O_{\delta}(s)$, a polynomial sequence $g : \mathbb{Z} \rightarrow G$, and a Lipschitz function $F : G/\Gamma \rightarrow \mathbb{R}$ of Lipschitz constant $O_{\delta}(s)$ such that

$$\frac{1}{N} \sum_{n=1}^{N} f(n)F(g(n)\Gamma) \gg_{s,\delta} 1.$$  

For the definition of complexity, we refer to [20] and for the proof of Proposition 3.1 we refer to [34]. Using this version of the discrete inverse theorem for Gowers norms, T. Tao established the continuous version of the inverse theorem for Gowers norms. According to this version, we notice that the nilsequence is independent of $N$. However, we warn the reader that the version of the inverse theorem for Gowers norms in [20] allows the nilsequence to depend on $N$.

At this point, let us give a proof of our first main result.

**Proof of Theorem 1.** By our assumption $\nu$ is aperiodic, Therefore, by Theorem 2.2 from [15], for any nilsequence $(u_n)$, we have

$$\frac{1}{N} \sum_{n=1}^{N} \nu(n)u_n \rightarrow 0.$$  

Now, for the case $k = 1$, we refer to [1, Section 3]. Let us assume from now that $k \geq 2$.

We proceed by contradiction. Assume that (2) does not hold. Then, there exist $\delta > 0$, a functions $f_e, e \in V_k$ and $\mu(A) > 0$ such that for each $x \in A$

$$\limsup_{N \rightarrow +\infty} \frac{1}{N^k} \sum_{n \in [1,N]^k} \prod_{e \in C^*} \nu(n.e) f_e(T_e^{n.e}x) \geq \delta,$$

where $\nu$ is an aperiodic bounded multiplicative function. Whence, by [8, Proposition 3.2], we have

$$|||\nu(n)f_e \circ T_e^n(x)|||_{U_k+1} \geq \delta,$$

for some $e \in C^*$ and for any $x \in A$.

This combined with Proposition 3.1 yields that there exist a nilmanifold $G/\Gamma$, a filtration $(G_p)$, a polynomial sequence $g$ and a Lipschitz function $F$ such that

$$\frac{1}{N} \sum_{n=1}^{N} \nu(n)f_e(T_e^{n}x)F(g(n)\Gamma) \gg_{k,\delta} 1.$$  

Applying the decomposition theorem (Proposition 2.2), we may write $f_e = f_{e,ns} + f_{e,z} + f_{e,e}$ and may assume that we have $||f_{e,z}(T_e^n(x))||_{U_k+1} = 0$. Notice that we further have
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\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e,n}(T_n^m x) F(g(n)\Gamma) \xrightarrow{N \to +\infty} 0,
\]

since the pointwise product of two nilsequences is a nilsequence and \( \nu \) is aperiodic combined with the fact that the Gowers norms \( \|\nu\|_{U^s(N)} \xrightarrow{N \to +\infty} 0, s \geq 2 \), by Theorem 2.5 from [15].

We thus get, up to some small error, that

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e}(T_n^m x) F(g(n)\Gamma) \sim \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e,z}(T_n^m x) F(g(n)\Gamma).
\]

Whence, by the spectral theorem \(^4\), we can write

\[
\left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e,z}(T_n^m x) F(g(n)\Gamma) \right\|_2 \leq \left\| \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e,z}(T_n^m x) F(g(n)\Gamma) \right\|_{L^2(\sigma_{f_{e,z}})} \gg k, \delta 1.
\]

Letting \( N \to +\infty \), we get a contradiction since \( \|\nu\|_{U^s(N)} \xrightarrow{N \to +\infty} 0, s \geq 2 \). The proof of the theorem is complete.

4. PROOF OF OUR SECOND MAIN RESULT (THEOREM 2)

In the same spirit as in the proof of our first main result we start by proving the following

**Proposition 4.1.** The cubic nonconventional ergodic averages of any order with von Mangoldt function weight converge almost surely provided that the systems are nilsystems.

The proof of Proposition 4.1 is largely inspired from Ford-Green-Konyagin-Tao’s proof of the main theorem in [14]. It used also some elementary fact on Gowers uniformity semi-norms.

As it is mentioned in [14], the fundamental ingredients in the proof of the main theorem are the Möbius disjointness of the nilsequences combined with the inverse Gowers theorem (Proposition 3.1) and the so-called “W-trick”. For the W-trick, we define

\[
W = \prod_{\substack{p \in \mathcal{P} \leq \omega(N) \backslash \{p\}}} p,
\]

\(^4\)The spectral measure \( \sigma_{f_{e}} \) is a finite measure on the circle given by \( \hat{\sigma}_{f_{e}}(n) = \int f_{e}(T_n^m x) f_{e}(x) d\mu(x) \).
where \(\omega(N) \leq \frac{1}{2} \log\left(\log(N)\right)\) for large \(N \in \mathbb{N}\). Therefore, by the PNT, we have \(W = O(\sqrt{\log(N)})\). For \(r < W\) and coprime to \(W\), we put

\[
\Lambda_{r,\omega}'(n) = \frac{\phi(W)}{W} \Lambda'(Wn + r), \quad n \in \mathbb{N}.
\]

Let \((a_n)\) be a given sequence, then, by the sieve methods (see for instance [26]), we have

\[
\frac{1}{WN} \sum_{n=1}^{WN} \Lambda'(n) a_n = \frac{1}{W} \sum_{\gcd(r,W)=1} \frac{1}{N} \sum_{n=1}^{N} \Lambda'(Wn + r) a_{Wn+r} + o_W(1)
\]

\[
= \frac{1}{\phi(W)} \sum_{\gcd(r,W)=1} \frac{1}{N} \sum_{n=1}^{N} \Lambda_{r,\omega}' a_{Wn+r} + o_W(1)
\]

Whence

\[
\frac{1}{WN} \sum_{n=1}^{WN} \Lambda'(n) a_n = \frac{1}{\phi(W)} \sum_{\gcd(r,W)=1} \frac{1}{N} \sum_{n=1}^{N} (\Lambda_{r,\omega}' - 1) a_{Wn+r}
\]

\[
+ \frac{1}{\phi(W)} \sum_{\gcd(r,W)=1} \frac{1}{N} \sum_{n=1}^{N} a_{Wn+r} + o_W(1).
\]

(3)

Moreover, if the sequence \((a_n)\) satisfy \(a_n = o(n)\) then the convergence of the sequence \(\left(\frac{1}{WN} \sum_{n=1}^{WN} a_n\right)\) implies the convergence of the sequence \(\left(\frac{1}{W} \sum_{n=1}^{N} a_n\right)\).

We need also the following technical lemma, and we present it in a form which is more precise than we need here, since we hope it may find other applications.

**Lemma 1.** Let \((a_n)\) be a bounded sequence of complex numbers. Then, we have

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime, } p \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) a_n \right| \leq \frac{8C}{\log N} + \frac{6C^2}{(\log N)^2} + \frac{(\log N)^2}{2\sqrt{N} \log 2},
\]

where \(C\) is some absolute positive constant.

For the proof of Lemma 1 we need the following classical result due to Chebyshev [2, p. 76]

**Proposition 4.2.** For any \(x > 0\),

\[
0 \leq \frac{\Psi(x) - v(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x} \log x}.
\]
Proof of Lemma 1. We assume Without lost of generalities that \((a_n)\) is bounded by 1. By the triangle inequality, we have

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime} \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) a_n \right| \leq \left| \frac{1}{\pi(N)} \sum_{p \text{ prime} \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) a_n \right| + \left| \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) a_n - \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) a_n \right|.
\]

Therefore

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime} \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) a_n \right| \leq \frac{(\log N)^2}{2\sqrt{N} \log N},
\]

by Proposition 4.2. Now, we need to estimate the first term in the RHS. For that, we observe that

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime} \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda(n) a_n \right| \leq \frac{1}{N} \sum_{p \text{ prime} \leq N} \left| \frac{N}{\pi(N)} - \log(p) \right|
\]

But, by PNTR, we have

\[
\left| \frac{N}{\pi(N)} - \log(N) \right| \leq \frac{C.N}{\pi(N). \log(N)},
\]

where \(C\) is an absolute positive constant. Moreover, for any \(N \geq 2\), we have \(\pi(N) \leq 6\frac{N}{\log(N)}\). We can thus rewrite (4) as follows

\[
\left| \frac{N}{\pi(N)} - \log(N) \right| \leq 6C.
\]
This combined with the triangle inequality gives

\[
\frac{1}{N} \sum_{\ell \leq N, \ell \text{ prime}} \left| \frac{N}{\pi(N)} - \log(\ell) \right| \leq \frac{1}{N} \sum_{\ell \leq N, \ell \text{ prime}} \left| \frac{N}{\pi(N)} - \log(N) \right| + \frac{1}{N} \sum_{\ell \leq N, \ell \text{ prime}} \left( \log(N) - \log(\ell) \right),
\]

Whence

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime}, p \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n)a_n \right| \leq \frac{\pi(N)}{N} \left| \frac{N}{\pi(N)} - \log(N) \right| + \frac{\log(N) \cdot \pi(N)}{N} - \frac{\nu(N)}{N} \leq \frac{8C}{\log(N)} + \frac{6C^2}{(\log(N))^2}.
\]

\[\square\]

From the proof of Lemma 1 the estimation of the sum over the prime is reduced to the estimation of the following quantity

\[
\frac{1}{N} \sum_{n=1}^{N} \Lambda(n)a_n \text{ or } \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n)a_n.
\]

Indeed, we have proved the following

\[
\left| \frac{1}{\pi(N)} \sum_{n \leq N} \Lambda(n)a_n - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n)a_n \right| \leq \frac{(\log(N))^2}{2\sqrt{N} \log(2)},
\]

and

\[
\left| \frac{1}{\pi(N)} \sum_{p \text{ prime}, p \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n)a_n \right| \leq \frac{8C}{(\log(N))} + \frac{6C^2}{(\log(N))^2},
\]

for some absolute positive constant \(C\).

**Proof of Proposition 4.1.** For the case \(k = 1\). The result holds for any dynamical system. Indeed, the result follows from Lemma 1 combined with the main result from [35]. Let us assume from now that \(k \geq 2\). Then, as in Ford-Green-Konyagin-Tao’s proof, it suffices to see that the following holds

\[
\frac{1}{N^k} \sum_{\vec{n} \in [1,N]^k} \prod_{e \in C^*} \left( \Lambda'_{b_e,W}(\vec{n}_e) - 1 \right) \prod_{e \in C^*} f_e(T_{\vec{n}_e,e}), \quad (NCSM)
\]
where \( b_i \in [1, W] \), \( i = 1, \cdots, \phi(W) \), coprime to \( W \). But \( (\Lambda_{b_i,W} - 1) \) is orthogonal to the nilsequences by Green-Tao result. Therefore the limit of the quantity \( (NCSM) \) is zero.

We are now able to prove our second main result.

**Proof of Theorem 2.** We proceed as before by claiming that

\[
\lim_{N \to +\infty} \frac{1}{N^k} \sum_{\vec{n} \in [1, N]^k} \prod_{e \in C^*} \left( \Lambda'_{b_i,W}(\vec{n}.e) - 1 \right) \prod_{e \in C^*} f_e(T_{\vec{n}.e}^n x) \to 0.
\]

If not then there exist \( \delta > 0 \) a functions \( f_e, e \in V \) and a Borel set \( A \) with positive measure such that for each \( x \in A \)

\[
\limsup_{N \to +\infty} \frac{1}{N^k} \sum_{\vec{n} \in [1, N]^k} \prod_{e \in C^*} \left( \Lambda'_{b_i,W}(\vec{n}.e) - 1 \right) f_e(T_{\vec{n}.e}^n x) \geq \delta.
\]

Whence, by the same arguments as in the proof of Theorem 1, we get that there exist a nilmanifold \( G/\Gamma \), a filtration \( (G_p) \), a polynomial sequence \( g \) and a Lipschitz function \( F \) such that

\[
\frac{1}{N} \sum_{n=1}^N \left( \Lambda'_{b_i,W}(n) - 1 \right) f_e(T_{e}^n x) F(g(n)\Gamma) \gg_{k, \delta} 1.
\]

We way further assume without lost of generalities that

\[
\left\| f_e(T_e^n x) \right\|_{L^2(\sigma_{f_e})} = 0,
\]

and write, up to some small error, that

\[
\frac{1}{N} \sum_{n=1}^N \left( \Lambda'_{b_i,W}(n) - 1 \right) z^n F(g(n)\Gamma) \gg_{k, \delta} 1.
\]

Integrating with respect to \( x \) and applying the spectral theorem, we see that

\[
\left\| \frac{1}{\phi(W)} \sum_{i=1}^{\phi(W)} \frac{1}{N} \sum_{n=1}^N \left( \Lambda'_{b_i,W}(n) - 1 \right) z^n F(g(n)\Gamma) \right\|_{L^2(\sigma_{f_e})} \gg_{k, \delta} 1.
\]

Whence, by letting \( W, N \to +\infty \), we get

\[
\limsup \left\| \frac{1}{\phi(W)} \sum_{i=1}^{\phi(W)} \frac{1}{N} \sum_{n=1}^N \left( \Lambda'_{b_i,W}(n) - 1 \right) z^n F(g(n)\Gamma) \right\|_{L^2(\sigma_{f_e})} \gg_{k, \delta} 1,
\]

which contradicts Green-Tao Theorem since the sequence

\[
\left( \frac{1}{\phi(W)} \sum_{i=1}^{\phi(W)} \frac{1}{N} \sum_{n=1}^N \left( \Lambda'_{b_i,W}(n) - 1 \right) z^n F(g(n)\Gamma) \right)
\]

is bounded by the PNT, Lemma 1 and the triangle inequality, and it converges to zero. The proof of the theorem is complete.
Remark 1. Let us stress that according to Ford-Green-Konyagin-Tao's Theorem \[14\], if all the functions \( f_e \) are constant (say equal 1), then the limit is given by the following explicit expression

\[
\prod_p \beta_p, \quad \text{where } \beta_p = \frac{1}{p^d} \sum_{\vec{n} \in (\mathbb{Z}/p\mathbb{Z})^d} \prod_{e \in C^*} \Lambda_{\mathbb{Z}/p\mathbb{Z}}(\vec{n}.e).
\]

The function \( \Lambda_{\mathbb{Z}/p\mathbb{Z}} \) is the local von Mangoldt function, that is, the \( p \)-periodic function defined by setting \( \Lambda_{\mathbb{Z}/p\mathbb{Z}}(b) = \frac{p}{p-1} \) when \( b \) is coprime to \( p \) and \( \Lambda_{\mathbb{Z}/p\mathbb{Z}}(b) = 0 \) otherwise.

Our proof suggests the following questions.

**Question 1.** Let \((X, \mathcal{A}, T, \mu)\) a dynamical system, let \( f \in L^1(X) \), do we have that there exists a Borel set \( X' \) with full measure such that for any nilsequence \((a_n)\), for any \( x \in X' \), the following averages

\[
\frac{1}{\pi(N)} \sum_{p \in \mathcal{P}, p < N} a_p f(T^p x),
\]

converge? In other word, can one establish Wiener-Wintner version of the prime ergodic theorem?

Let us notice further that the Gowers uniformity norm of \( \mu \) is small, that is, \( \|\mu\|_{U^k(N)} \xrightarrow{N \to +\infty} 0 \). This suggests the following generalization of Sarnak’s conjecture.

**Question 2.** Do we have that for any multiplicative function \((\nu(n))\) with small Gowers norms for any dynamical flow on a compact set \((X, T)\) with topological entropy zero, for any continuous function \( f \), for all \( x \in X \),

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) \xrightarrow{N \to +\infty} 0?
\]

By our assumption, it is obvious that for any nilsystem \((X, T)\), for any continuous function \( f \), for any \( x \in X \), we have

\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f(T^n x) \xrightarrow{N \to +\infty} 0.
\]

Let’s stress that by the recent result of L. Matthiesen [31] there is a class of not necessarily bounded multiplicative functions with a small Gowers norms.

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ON THE POINTWISE CONVERGENCE OF THE CUBIC . . .

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