ARITHMETIC PROPERTIES FOR \((r, s)\)-REGULAR PARTITION FUNCTIONS WITH DISTINCT PARTS

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Abstract. For any relatively prime integers \(r\) and \(s\), let \(a_{r,s}(n)\) denote the number of \((r, s)\)-regular partitions of a positive integer of \(n\) into distinct parts. Prasad and Prasad (2018) proved many infinite families of congruences modulo 2 for \(a_{3,5}(n)\). In this paper, we establish families of congruences modulo 2 and 4 for \(a_{r,s}(n)\) with \((r, s) \in \{(2,5), (2,7), (4,5), (4,9)\}\). For example, we show that for all \(\beta \geq 0\) and \(n \geq 0\), we have

\[
a_{2,5}\left(4 \cdot 5^{2\beta+1}n + \frac{37 \cdot 5^{2\beta} - 1}{6}\right) \equiv 0 \pmod{4}.
\]

1. Introduction

The partition of a positive integer \(n\) is a non-increasing sequence of positive integers whose sum is equal to \(n\). If \(p(n)\) denote the number of partitions of a positive integer \(n\) and we adopt the convention \(p(0) = 1\), then the generating function for \(p(n)\) satisfies the identity

\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty},
\]

where

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).
\]

Throughout this paper, we write

\[
f_k := (q^k; q^k)_\infty, \quad \text{for any integer } k \geq 1.
\]

Ramanujan [11] established the following beautiful congruences for all \(n \geq 0\):

\[
p(5n+4) \equiv 0 \pmod{5}, \quad p(7n+5) \equiv 0 \pmod{7} \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}.
\]

Ramanujan’s congruences on \(p(n)\) have motivated many mathematicians to seek similar results for restricted partition functions. One example is the \(\ell\)-regular
partition function $b_\ell(n)$, which counts the number of partitions of $n$ in which no part is divisible by $\ell$ and whose generating function satisfies the identity

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3}$$

Many results on the arithmetic of $b_\ell(n)$ have been established (see, for example, [2, 9, 12]).

For relatively prime integers $r$ and $s$, an $(r,s)$-regular partition is one in which none of the parts is divisible by $r$ or $s$. Denote by $a_{r,s}(n)$, the number of $(r,s)$-regular partitions of $n$ into distinct parts. For example, $a_{2,5}(13) = 2$ since the $(2,5)$-regular partitions of 13 into distinct parts are 13 and $9+3+1$. The generating function for $a_{r,s}(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} a_{r,s}(n)q^n = (-q; q)_{\infty}(-q^{r,s}; q^{r,s})_{\infty}(-q^{r}; q^{r})_{\infty}(-q^{s}; q^{s})_{\infty}. \tag{1.4}$$

Prasad and Prasad [10] proved many infinite families of congruences modulo 2 for $a_{3,5}(n)$.

In this paper, we establish families of congruences modulo 2 for $a_{2,5}(n)$, $a_{2,7}(n)$, $a_{4,5}(n)$ and $a_{4,9}(n)$. We also prove congruences modulo 4 for $a_{2,5}(n)$. The congruences are listed in the following theorems:

**Theorem 1.1.** For every $n \geq 0$, we have

$$a_{2,5}(4n + 2) \equiv 0 \pmod{2} \tag{1.5}$$

and

$$a_{2,5}(4n) \equiv \begin{cases} 1 \pmod{2}, & \text{if } n \text{ is a pentagonal number} \\ 0 \pmod{2}, & \text{otherwise}. \end{cases} \tag{1.6}$$

**Theorem 1.2.** Let $p > 5$ be a prime with $(-\frac{10}{p}) = -1$ and $1 \leq j \leq p - 1$. Then for all $\gamma \geq 0$, we have

$$\sum_{n=0}^{\infty} a_{2,5}\left(4 \cdot p^{2\gamma} n + \frac{7 \cdot p^{2\gamma} - 1}{6}\right)q^n \equiv f_2 f_5 \pmod{2}, \tag{1.7}$$

$$a_{2,5}\left(4 \cdot p^{2\gamma+1}(pn + j) + \frac{7 \cdot p^{2\gamma+2} - 1}{6}\right) \equiv 0 \pmod{2}, \tag{1.8}$$

$$\sum_{n=0}^{\infty} a_{2,5}\left(20 \cdot p^{2\gamma} n + \frac{55 \cdot p^{2\gamma} - 1}{6}\right)q^n \equiv f_1 f_{10} \pmod{2} \tag{1.9}$$

and

$$a_{2,5}\left(20 \cdot p^{2\gamma+1}(pn + j) + \frac{55 \cdot p^{2\gamma+2} - 1}{6}\right) \equiv 0 \pmod{2}. \tag{1.10}$$
Theorem 1.3. If \( w_1 \in \{13, 37\} \) and \( w_3 \in \{41, 89\} \). Then for all \( \beta \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_{2,5}(4 \cdot 5^{2\beta} n + \frac{13 \cdot 5^{2\beta} - 1}{6}) q^n \equiv 2q^2 f_1 f_3^2 \pmod{4},
\]

\[
\sum_{n=0}^{\infty} a_{2,5}(4 \cdot 5^{2\beta+1} n + \frac{17 \cdot 5^{2\beta+1} - 1}{6}) q^n \equiv 2f_3 f_5 \pmod{4},
\]

\[
a_{2,5}\left(4 \cdot 5^{2\beta+1} n + \frac{w_1 \cdot 5^{2\beta} - 1}{6}\right) \equiv 0 \pmod{4}
\]

and

\[
a_{2,5}\left(4 \cdot 5^{2(\beta+1)} n + \frac{w_3 \cdot 5^{2\beta+1} - 1}{6}\right) \equiv 0 \pmod{4}.
\]

Theorem 1.4. Let \( p > 7 \) be a prime with \( \left(\frac{-14}{p}\right) = -1 \) and \( 1 \leq j \leq p - 1 \). Then for all \( \alpha \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_{2,7}(2 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha} - 1}{4}) q^n \equiv f_1 f_{14} \pmod{2},
\]

\[
a_{2,7}\left(2 \cdot p^{2\alpha+1}(pn + j) + \frac{5 \cdot p^{2\alpha+2} - 1}{4}\right) \equiv 0 \pmod{2},
\]

\[
\sum_{n=0}^{\infty} a_{2,7}(14 \cdot p^{2\alpha} n + \frac{21 \cdot p^{2\alpha} - 1}{4}) q^n \equiv f_2 f_7 \pmod{2}
\]

and

\[
a_{2,7}\left(14 \cdot p^{2\alpha+1}(pn + j) + \frac{21 \cdot p^{2\alpha+2} - 1}{4}\right) \equiv 0 \pmod{2}.
\]

Theorem 1.5. If \( w \in \{13, 17\} \), then for all \( \alpha \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_{4,5}(2 \cdot 5^{\alpha} n + \frac{5^{\alpha} - 1}{2}) q^n \equiv f_1 f_5 \pmod{2}
\]

and

\[
a_{4,5}\left(2 \cdot 5^{\alpha+1} n + \frac{w \cdot 5^{\alpha} - 1}{2}\right) \equiv 0 \pmod{2}.
\]

Theorem 1.6. Let \( p > 5 \) be a prime with \( \left(\frac{-5}{p}\right) = -1 \) and \( 1 \leq j \leq p - 1 \). Then for all \( \alpha \geq 0 \), we have

\[
\sum_{n=0}^{\infty} a_{4,5}(2 \cdot p^{2\alpha} n + \frac{p^{2\alpha} - 1}{2}) q^n \equiv f_1 f_5 \pmod{2}
\]

and

\[
a_{4,5}\left(2 \cdot p^{2\alpha+1}(pn + j) + \frac{p^{2\alpha+2} - 1}{2}\right) \equiv 0 \pmod{2}.
\]
Theorem 1.7. Let \( w_1 \in \{3, 5\} \), \( w_2 \in \{13, 25, 37\} \) and \( \alpha \geq 0 \). Then

\[
a_{4,9}(6n + w_1) \equiv 0 \pmod{2},
\]

\[
a_{4,9}(24n + 19) \equiv 0 \pmod{2},
\]

\[
a_{4,9}(6 \cdot 4^{\alpha+2}n + 20 \cdot 4^{\alpha+1} - 1) \equiv 0 \pmod{2},
\]

\[
a_{4,9}(48n + w_2) \equiv 0 \pmod{2}
\]

and

\[
a_{4,9}(48n + 1) \equiv \begin{cases} 1 \pmod{2} & \text{if } n \text{ is a pentagonal number}, \\ 0 \pmod{2} & \text{otherwise}. \end{cases}
\]

2. Preliminaries

In this section, we collect the \( q \)-series identities that are used in our proofs. Recall that Ramanujan’s general theta-function \( f(a, b) \) is defined by

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.
\]

Important special cases of \( f(a, b) \) [1, p. 36, Entry 22 (i), (ii), (iii)] are the theta-functions \( \phi(q) \), \( \psi(q) \) and \( f(-q) \), which satisfy the identities

\[
\phi(q) := f(q, q) = \sum_{n=0}^{\infty} q^{n^2} = (-q; q^2)^2 \frac{q^2}{(q^2; q^2)_{\infty}} = \frac{f_2^5}{f_1^2 f_4^2},
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{f_2^2}{f_1},
\]

and

\[
f(-q) := f(-q, -q^2) = \sum_{n=0}^{\infty} (-1)^{n} q^{n(3n-1)/2} = (q; q)_{\infty} = f_1.
\]

In terms of \( f(a, b) \), Jacobi’s triple product identity [1, Entry 19, p.35] is given by

\[
f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.
\]

Lemma 2.1. [3, Theorem 2.2] For any prime \( p \geq 5 \), we have

\[
f_1 = \sum_{k=(p-1)/2}^{k=(p^*-1)/2} \sum_{k\neq (p^*-1)/6}^{k=-p-1/2} (-1)^k q^{(3k^2+k)/2} f \left( -q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)
\]

\[+ (-1)^{(p^*-1)/6} q^{(p^2-1)/24} f_{p^2},
\]
where
\[ p^* = \begin{cases} \, p, & \text{if } p \equiv 1 \pmod{6} \\ -p, & \text{if } p \equiv 5 \pmod{6} \end{cases} \]

Furthermore, if
\[ \frac{-(p - 1)}{2} \leq k \leq \frac{(p - 1)}{2} \] and \( k \neq \frac{(p^* - 1)}{6} \),
then
\[ \frac{3k^2 + k}{2} \not\equiv \frac{p^2 - 1}{24} \pmod{p}. \]

**Lemma 2.2.** [1, p. 303, Entry 17(v)] We have that
\[ f_1 = f_{49} \left( \frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \]
where \( A(q) = f(-q^3, -q^4) \), \( B(q) = f(-q^2, -q^5) \) and \( C(q) = f(-q, -q^6) \).

**Lemma 2.3.** [5] We have that
\[ f_1 = f_{25}(R(q^5) - q - q^2 R(q^5)^{-1}), \]
where
\[ R(q) = \frac{(q^2; q^5)\infty (q^3; q^5)\infty}{(q; q^5)\infty (q^4; q^6)\infty}. \]

**Lemma 2.4.** We have
\[ \frac{1}{f_1^2} = \frac{f_5^2}{f_2 f_1^2} + 2q \frac{f_4^3 f_6^3}{f_2^3 f_8}, \]
\[ \frac{f_9}{f_1} = \frac{f_3^2 f_6}{f_2 f_6 f_36} + q \frac{f_2^2 f_6 f_36}{f_2^3 f_1}, \]
\[ f_1 f_5^3 = f_2^3 f_10 - q \frac{f_2^2 f_10 f_20}{f_1} + 2q^2 f_4 f_3^3 - 2q^3 \frac{f_4^3 f_10 f_40}{f_2 f_8^2}, \]
\[ \frac{f_5}{f_1} = \frac{f_8 f_20}{f_2 f_40} + q \frac{f_4^3 f_10 f_40}{f_2 f_8 f_20}, \]
\[ \frac{f_3^3}{f_1} = \frac{f_3^2 f_6}{f_2 f_6 f_12} + q \frac{f_{12}^3}{f_4}, \]
\[ f_1 f_7 = \frac{f_2 f_{14} f_16 f_5^5}{f_4 f_8 f_{28} f_{112}^2} - q f_4 f_{28} + q^6 \frac{f_2 f_8^5 f_{14} f_{112}^2}{f_4^2 f_16 f_{28} f_{56}}. \]
For the proof of (2.9), see Hirschhorn [7, p.40]. Equation (2.10) was proved by Xia and Yao [13]. For the proof of (2.11), see Naika et al [8]. Equation (2.12) was proved by Hirschhorn and Sellers [6]. Equation (2.13) was proved by Hirschhorn et al [4]. Equation (2.14) was proved by Xia [14, Lemma 3.14].

To end this section, we record the following congruence which can be easily proved using the binomial theorem: For all positive integers \( t \) and \( m \) we have

\[
(2.15) \quad f_t^{2m} \equiv f_{2t}^m \pmod{2}.
\]

### 3. Proof of Theorems 1.1-1.3

**Proof of Theorem 1.1:**

**Proof.** Setting \((r, s) = (2, 5)\) in \((1.4)\) and using elementary \(q\)-operations, we obtain

\[
(3.1) \quad \sum_{n=0}^{\infty} a_{2,5}(n)q^n = \frac{f_2^2 f_5 f_{20}}{f_1 f_{10} f_{20}}.
\]

Combining (2.12) and (3.1), we find that

\[
(3.2) \quad \sum_{n=0}^{\infty} a_{2,5}(n)q^n = \frac{f_8 f_{20}^3}{f_4 f_{10} f_{20}} + q \frac{f_4^2 f_{40}}{f_2 f_8 f_{10}}.
\]

Extracting the terms involving even powers of \(q\) of (3.2), we obtain

\[
(3.3) \quad \sum_{n=0}^{\infty} a_{2,5}(2n)q^n = \frac{f_4 f_{10}^3}{f_2 f_8^2 f_{20}}.
\]

In view of (2.15), (3.3) can be written as

\[
(3.4) \quad \sum_{n=0}^{\infty} a_{2,5}(2n)q^n \equiv f_2 \pmod{2}.
\]

Extracting the terms involving odd powers of \(q\) from (3.4) yields (1.5). Finally, extracting the terms involving even powers of \(q\) from both sides of (3.4) and using (2.4) yields (1.6). \(\square\)

**Proof of Theorem 1.2:**

**Proof.** Extracting the terms involving odd powers of \(q\) from both sides of (3.2), we obtain

\[
(3.5) \quad \sum_{n=0}^{\infty} a_{2,5}(2n+1)q^n = \frac{f_2^2 f_{20}}{f_1 f_4 f_5}.
\]

In view of (2.15), we can rewrite (3.5) as

\[
(3.6) \quad \sum_{n=0}^{\infty} a_{2,5}(2n+1)q^n \equiv \frac{f_1 f_{10}^3}{f_2} \pmod{2}.
\]
Combining (2.11) and (3.6), we find that

\[(3.7) \quad \sum_{n=0}^{\infty} a_{2,5}(2n + 1)q^n \equiv f_2^2 f_{10} - q \frac{f_2 f_{10}^2 f_{20}}{f_4} \quad (\text{mod } 2).\]

Extracting the terms involving even powers of \(q\) from both sides of (3.7), we obtain

\[(3.8) \quad \sum_{n=0}^{\infty} a_{2,5}(4n + 1)q^n \equiv f_2 \quad (\text{mod } 2).\]

Equation (3.8) is the \(\gamma = 0\) case of (1.7). Now suppose that (1.7) holds for some \(\gamma \geq 0\). Using (2.6) in (1.7), we deduce that

\[(3.9) \quad \sum_{n \geq 0} a_{2,5}\left(4 \cdot p^{2\gamma} n + \frac{7 \cdot p^{2\gamma} - 1}{6}\right)q^n \equiv \left[\sum_{k=-(p-1)/2}^{k=(p-1)/2} q^{3k^2 + k} f\left(-q^{3p^2 + (6k+1)p}, -q^{3p^2 - (6k+1)p}\right)\right.\
\left. + q^{(p^2-1)/12} f_{2p^2}\right] \times \left[\sum_{m=-(p-1)/2}^{m=(p-1)/2} q^{5(3m^2 + m)/2} f\left(-q^{5(3p^2 + (6m+1)p)/2}, -q^{5(3p^2 - (6m+1)p)/2}\right)\right.\
\left. + q^{5(p^2-1)/24} f_{5p^2}\right] \quad (\text{mod } 2).\]

Consider the congruence

\[3k^2 + k + 5\frac{(3m^2 + m)}{2} \equiv \frac{7(p^2 - 1)}{24} \quad (\text{mod } p),\]

which is equivalent to

\[(12k + 2)^2 + 10(6m + 1)^2 \equiv 0 \quad (\text{mod } p).\]

Since \(\left(\frac{-10}{p}\right) = -1\), the only solution of this congruence is \(k = m = \frac{(p^* - 1)}{6}\).

Therefore, extracting the terms involving \(q^{m+7(p^2-1)/24}\) from both sides of (3.9), dividing by \(q^{7(p^2-1)/24}\) and then replacing \(q^p\) by \(q\), we find that

\[(3.10) \quad \sum_{n=0}^{\infty} a_{2,5}\left(4 \cdot p^{2\gamma+1} n + \frac{7 \cdot p^{2\gamma+2} - 1}{6}\right)q^n \equiv f_{2p} f_{5p} \quad (\text{mod } 2),\]

which yields

\[(3.11) \quad \sum_{n=0}^{\infty} a_{2,5}\left(4 \cdot p^{2(\gamma+1)} n + \frac{7 \cdot p^{2(\gamma+1)} - 1}{6}\right)q^n \equiv f_2 f_5 \quad (\text{mod } 2),\]
which is the $\gamma + 1$ case of (1.7). On the other hand, extracting the terms involving $q^{pn+j}$ ($1 \leq j \leq p-1$) from (3.10), we arrive at (1.8). Employing (2.8) in (1.7) and then extracting the terms involving $q^{5n+2}$ yields (1.9). Next, using (2.6) in (1.9) and proceeding as in the proof of (1.7), we arrive at

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(20 \cdot p^{2\gamma+1} n + \frac{55 \cdot p^{2\gamma+2} - 1}{6}) q^n \equiv f_p f_{10} \pmod{2}.
\end{equation}

Finally, (1.10) follows from extracting the terms involving $q^{pn+j}$ ($1 \leq j \leq p-1$) from (3.12).

**Proof of Theorem 1.3:**

*Proof.* Combining (2.9) and (3.3), we find that

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(2n)q^n = \frac{f_4 f_5^5}{f_2 f_6 f_20 f_8^3} + 2q^2 \frac{f_4 f_{20} f_8^3}{f_2 f_6 f_20}.
\end{equation}

Extracting the terms involving odd powers of $q$ from both sides of (3.13), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(4n + 2)q^n = 2q^2 \frac{f_2 f_{10} f_8^3}{f_1 f_5^3 f_20}.
\end{equation}

In view of (2.15), (3.14) can be written as

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(4n + 2)q^n \equiv 2q^2 f_1 f_3^3 f_20 \pmod{4},
\end{equation}

which is the $\beta = 0$ case of (1.11). Now assume that (1.11) holds for some $\beta \geq 0$. Employing (2.8) in (1.11), we arrive at

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(4 \cdot 5^{2\beta} n + \frac{13 \cdot 5^{2\beta} - 1}{6}) q^n \equiv 2q^2 f_2 f_{20} f_25 \left( R(q^5) - q - q^2 R(q^5)^{-1} \right) \pmod{4}.
\end{equation}

Extracting the terms involving $q^{5n+3}$ from (3.16) yields (1.12). Next, using (2.8) in (1.12) and then extracting the terms involving $q^{5n+2}$, we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{2,5}(4 \cdot 5^{2(\beta+1)} n + \frac{13 \cdot 5^{2(\beta+1)} - 1}{6}) q^n \equiv 2q^2 f_1 f_3^3 f_{20} \pmod{4},
\end{equation}

which is the $\beta + 1$ case of (1.11). Employing (2.8) in (1.11) and then extracting the terms involving $q^{5n+j}$ for $j \in \{0, 1\}$ yields (1.13). Finally, using (2.8) in (1.12) and then extracting terms involving $q^{5n+j}$ for $j \in \{1, 3\}$ yields (1.14). \qed

4. **Proof of Theorem 1.4**

*Proof.* Setting $(r, s) = (2, 7)$ in (1.4) and using elementary $q$-operations, we have

\begin{equation}
\sum_{n=0}^{\infty} a_{2,7}(n) q^n = \frac{f_2^2 f_7 f_{28}}{f_1 f_4 f_{14}^3}.
\end{equation}
In view of (2.15), we can rewrite (4.1) as

(4.2) \[ \sum_{n=0}^{\infty} a_{2,7}(n)q^n \equiv \frac{f_1f_7}{f_2} \pmod{2}. \]

Combining (2.14) and (4.2), we find that

(4.3) \[ \sum_{n=0}^{\infty} a_{2,7}(n)q^n \equiv \frac{f_{14}f_2^2f_5^2}{f_4f_8f_3f_{12}^2} - q\frac{f_4f_{28}}{f_2} + q^6\frac{f_6^2f_{14}f_7^2}{f_4f_5^2f_{28}f_{56}} \pmod{2}. \]

Extracting the terms involving odd powers of \( q \) from both sides of (4.3), we obtain

(4.4) \[ \sum_{n=0}^{\infty} a_{2,7}(2n+1)q^n \equiv f_1f_{14} \pmod{2}, \]

which is the \( \alpha = 0 \) case of (1.15). Using (2.6) in (1.15) and proceeding as in the proof of (1.7), we arrive at

(4.5) \[ \sum_{n=0}^{\infty} a_{2,7}(2 \cdot p^{2n+1} + 5 \cdot p^{2n+2} - 1) \frac{4}{4} q^n \equiv f_{p}f_{14p} \pmod{2}, \]

which yields

(4.6) \[ \sum_{n=0}^{\infty} a_{2,7}(2 \cdot p^{2n+1} + 5 \cdot p^{2n+2} - 1) \frac{4}{4} q^n \equiv f_{p}f_{14p} \pmod{2}, \]

which is the \( \alpha + 1 \) case of (1.15). On the other hand, extracting the terms involving \( q^{pn+j} \) \( (1 \leq j \leq p - 1) \) from (4.5), we arrive at (1.16). Next, using (2.7) in (1.15) and then extracting the terms involving \( q^{2n+2} \) yields (1.17). Now employing (2.6) in (1.17) and proceeding as in the proof of (1.15), we arrive at

(4.7) \[ \sum_{n=0}^{\infty} a_{2,7}(14 \cdot p^{2n+1} + 21 \cdot p^{2n+2} - 1) \frac{4}{4} q^n \equiv f_{2p}f_{7p} \pmod{2}. \]

Finally, (1.18) follows from extracting the terms involving \( q^{pn+j} \) \( (1 \leq j \leq p - 1) \) from (4.7).

5. Proof of Theorems 1.5-1.6

Proof of Theorem 1.5:

Proof. Setting \( (r,s) = (4,5) \) in (1.4) and using elementary \( q \)-operations, we obtain

(5.1) \[ \sum_{n=0}^{\infty} a_{4,5}(n)q^n = \frac{f_2f_4f_5f_{40}}{f_1f_8f_{10}f_{20}}. \]

Combining (2.12) and (5.1), we find that

(5.2) \[ \sum_{n=0}^{\infty} a_{4,5}(n)q^n = \frac{f_{4}f_{20}}{f_{2}f_{10}} + q\frac{f_4^2f_{40}^2}{f_2^2f_8^2f_{20}^2}. \]
Extracting the terms involving even powers of $q$ from both sides of (5.2), we obtain

\[(5.3) \quad \sum_{n=0}^{\infty} a_{4,5}(2n)q^n = \frac{f_2f_{10}}{f_1f_5}.\]

In view of (2.15), we can rewrite (5.3) as

\[(5.4) \quad \sum_{n=0}^{\infty} a_{4,5}(2n)q^n \equiv f_1f_5 \pmod{2},\]

which is the $\alpha = 0$ case of (1.19). Now assume that (1.19) holds for some $\alpha \geq 0$. Using (2.8) in (1.19), we find that

\[(5.5) \quad \sum_{n=0}^{\infty} a_{4,5}(2\cdot 5^\alpha n + \frac{5^\alpha - 1}{2})q^n \equiv f_5f_{25}(R(q^5) - q - q^2R(q^5)^{-1}) \quad \pmod{2}.\]

Extracting the terms involving $q^{5n+1}$ from both sides of (5.5), we arrive at

\[(5.6) \quad \sum_{n=0}^{\infty} a_{4,5}(2\cdot 5^{\alpha+1}n + \frac{5^{\alpha+1} - 1}{2})q^n \equiv f_1f_5 \pmod{2},\]

which is the $\alpha + 1$ case of (1.19). Finally, using (2.8) in (1.19) and then extracting the terms involving $q^{5n+j}$ for $j \in \{3, 4\}$ yields (1.20).

**Proof of Theorem 1.6:**

Proof. Congruence (5.4) is the $\alpha = 0$ case of (1.21). Now suppose that (1.21) holds for some $\alpha \geq 0$. Using (2.6) in (1.21) and proceeding as in the proof of (1.7), we arrive at

\[(5.7) \quad \sum_{n=0}^{\infty} a_{4,5}(2\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2} - 1}{2})q^n \equiv f_pf_{5p} \pmod{2},\]

which yields

\[(5.8) \quad \sum_{n=0}^{\infty} a_{4,5}(2\cdot p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)} - 1}{2})q^n \equiv f_1f_5 \pmod{2},\]

which is the case $\alpha + 1$ of (1.21). On the other hand, extracting the terms involving $q^{pn+j}$ ($1 \leq j \leq p - 1$) from (5.7), we arrive at (1.22).

**□**

6. **Proof of Theorem 1.7**

Proof. Setting $(r,s) = (4,9)$ in (1.4) and using elementary $q$-operations, we obtain

\[(6.1) \quad \sum_{n=0}^{\infty} a_{4,9}(n)q^n = \frac{f_2f_4f_9f_{72}}{f_1f_8f_{18}f_{36}}.\]

Combining (2.10) and (6.1), we find that

\[(6.2) \quad \sum_{n=0}^{\infty} a_{4,9}(n)q^n = \frac{f_4f_3^2f_{72}}{f_2f_6f_8f_{36}^2} + q\frac{f_3^2f_6f_{72}}{f_2^2f_8f_{12}f_{18}}.\]
Extracting the terms involving odd powers of $q$ from (6.2) and then employing (2.15), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(2n+1)q^n \equiv \frac{f_3^3}{f_3} \pmod{2}.
\end{equation}

Comparing the terms involving $q^{3n+j}$, for $j \in \{1, 2\}$ from both sides of (6.3) yields (1.23). Next, extracting the terms involving $q^{3n}$ from both sides of (6.3), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(6n+1)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}.
\end{equation}

Combining (2.13) and (6.4), we find that

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(6n+1)q^n \equiv \frac{f_3^3 f_6^2}{f_2^5 f_{12}} + qf_3^3 f_1 \pmod{2}.
\end{equation}

Extracting the terms involving odd powers of $q$ from (6.5), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(12n+7)q^n \equiv \frac{f_6^3}{f_2} \pmod{2}.
\end{equation}

Extracting the terms involving odd powers of $q$ from (6.6) yields (1.24). Next, extracting the terms involving even powers of $q$ from both sides of (6.6), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(24n+7)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}.
\end{equation}

Combining (6.4) and (6.7), we find that

\begin{equation}
a_{4,9}(24n+7) \equiv a_{4,9}(6n+1) \pmod{2}.
\end{equation}

From (6.8) and by mathematical induction, we have

\begin{equation}
a_{4,9}\left(6 \cdot 4^{\alpha+1}n + 2 \cdot 4^{\alpha+1} - 1\right) \equiv a_{4,9}(6n+1) \pmod{2}.
\end{equation}

Using (6.9) and congruence (1.24), we arrive at (1.25). On the other hand, extracting the terms involving even powers of $q$ from both sides of (6.5), we obtain

\begin{equation}
\sum_{n=0}^{\infty} a_{4,9}(12n+7)q^n \equiv f_4 \pmod{2}.
\end{equation}

Extracting the terms involving $q^{4n+j}$ for $j \in \{1, 2, 3\}$ from (6.10) yields (1.26). On the other hand, extracting the terms involving $q^{4n}$ from both sides of (6.10) and using (2.4) yields (1.27).

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