Large N QCD – Continuum reduction

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Numerical evidence combined with Eguchi-Kawai reduction indicate that there are no finite volumes effects in the large N limit of QCD as long as the linear extent of the four-torus is bigger than a critical size. This is referred to as continuum reduction. Since fermions in the fundamental representation are naturally quenched in the large N limit, as long as we only have a finite number of flavors, continuum reduction provides us with the exciting possibility to numerically solve large N QCD using chiral fermions and present day computers.

1. Introduction

Overlap fermions [1] enables one to compute quantities in the chiral limit of lattice QCD. But one has to work with \( \epsilon(\gamma_5 D_w(-m)) \) where \( D_w(-m) \) is the Wilson-Dirac operator in the supercritical region and \( \epsilon \) is the sign function of the operator. This makes the algorithms on the lattice computationally intensive and the situation is further complicated by the presence of very small eigenvalues of \( \gamma_5 D_w(-m) \) [2]. In spite of these difficulties several results have been obtained in the quenched approximation [3] and dynamical simulations using overlap fermions have been explored [4].

Large N QCD [5] provides an interesting alternative to dynamical simulations of SU(3) QCD since fermions in the fundamental representation are naturally quenched in the \( N \to \infty \) limit of large N QCD as long as the number of flavors are kept finite. It is useful to use the “double line” notation for the gauge propagator in large N QCD to denote that there are two color indices associated with the adjoint representation of the gauge field. In this notation, fermions in the fundamental representation have only one line in their propagator. Only planar diagrams survive in the limit of \( N \to \infty \) and the vacuum diagrams arising out of the fermionic part are suppressed by a factor of \( \frac{1}{N} \) and fermions are naturally quenched [5]. Large N QCD has all the qualitative features of \( N = 3 \) QCD and the spectrum of mesons has been calculated in two dimensions [6]. One also expects a non-trivial spectrum of mesons in the large N limit of four dimensional QCD but this model has not yet been exactly solved.

2. Continuum reduction

Eguchi and Kawai [7] proved that lattice QCD in the limit of \( N \to \infty \) can be reduced to QCD on a single site provided the global \( U_d(1) \) symmetries that multiplies the link matrices by \( U(1) \) phases are not broken. Unfortunately, this symmetry is broken for \( d > 2 \) [8] and one cannot reproduce QCD in the large N limit by working on a 1\( d \) lattice. But, the proof of Eguchi and Kawai also holds for an \( L^d \) lattice where \( L > 1 \). That is to say, the infinite space-time lattice can be reduced to a finite \( L^d \) lattice where \( L > 1 \). That is to say, the infinite space-time lattice can be reduced to a finite \( L^d \) lattice in the limit of \( N \to \infty \) provided the global \( U_d(1) \) symmetries that multiplies Polyakov loops by \( U(1) \) phases are not broken on the \( L^d \) lattice.

The only parameter in the lattice theory is the 't Hooft gauge coupling \( b = \frac{1}{g^2 N} \). This coupling has the dimensions of length in three dimensions and it is dimensionless in four dimensions. One can explicitly compute the location of the transition point \( b_c(L) \) such that the \( U_d(1) \) symmetries remain unbroken for \( b < b_c(L) \). If \( b_c(L) \) scales properly with \( L \) then one can define a critical size \( l_c \) in the continuum such that the \( U_d(1) \) symmetries remain unbroken for \( l > l_c \). The argument of Eguchi and Kawai will hold for all \( L \) as long as we keep \( b < b_c(L) \). There will be no dependence on the box size as long as the box size is bigger than...
the critical size. This is referred to as continuum reduction [9]. Physical results can be extracted by working on an $L^3$ lattice and keeping $b$ just below $b_c(L)$. Computations need to be done on two or three different values of $N$ to study the infinite N limit. Results for two or three different $L$ values can be used to study the effect of finite lattice spacing.

3. The bulk transition

One lattice artifact has to be taken into account when setting the parameters \{L, b < b_c(L)\} for physical computation. There is a bulk transition on the lattice in the large N limit that is associated with the spectrum of the single plaquette (1 × 1 Wilson loop). There is no phase transition for finite N but there is a cross-over that gets stronger with increasing N. The eigenvalues $e^{i\theta}$ of $U_p$, the parallel transporter along the plaquette, can be anywhere in the range $-\pi \leq \theta < \pi$. But the field strength is small close to the continuum limit and $\theta$ is expected to be in a small range close to zero. In fact, there is a value of coupling $b_B$ such that the distribution of $\theta$ develops a gap around $\pm \pi$ for $b > b_B$. This value of coupling does not depend on $L$ and therefore the transition is purely a lattice artifact. But one has to set $b > b_B$ to obtain the proper continuum limit. Only for some $L > L_B$ does $b_c(L)$ become bigger than $b_B$. For $L \leq L_B$ all one has is the bulk transition and the $U^d(1)$ symmetries also break at the same coupling. This is the reason why one cannot obtain continuum QCD in the large N limit using the single site Eguchi-Kawai model and there is a smallest lattice size that one has to use in \{L > L_B, b_B < b < b_c(L)\}.

4. Numerical results

In order to study QCD in the large N limit, one has to first obtain the appropriate set of parameters {L > L_B, b_B < b < b_c(L)}. The $U^2(1)$ symmetries are not broken in $d = 2$ and $b_c(L) = \infty$ for all $L \geq 1$. The bulk transition occurs at $b_B = 0.5$ [10] and Eguchi-Kawai reduction holds on a $1^2$ lattice.

Numerical study involves the generation of thermalized gauge field configurations and this is obtained using a combination of heat-bath and over-relaxation updates. The heat-bath updates are performed on all SU(2) subgroups and the over-relaxation updates are performed on the full SU(N) group [13]. A preliminary investigation in three dimensions show that $b_B = 0.4$ and $L_B = 2$ [9]. Therefore, one can obtain results in three dimensional large N QCD by working on as small a lattice as $3^3$. The breaking of the $U^3(1)$ symmetries occurs as follows. There is a $b_c(L)$ where one of the three $U^3(1)$ are broken and the other two remain unbroken. The eigenvalue distribution of the Polyakov lines in the unbroken direction are flat (random) but there is a peak in the distribution of the broken line. As one goes above $b_c(L)$ more lines gets ordered and there are two more transition points at any finite $L$. More investigation has to performed to study the behavior of the higher transitions as a function of $L$, but the lowest transition corresponding to one ordered Polyakov line scales properly with $L$. It is quite likely that all these transitions fuse as one goes to the continuum limit. An analysis of this transition shows that $0.6 < b_c(3) < 0.7$, $0.8 < b_c(4) < 0.9$, $1.0 < b_c(5) < 1.2$ and $1.2 < b_c(6) < 1.35$. When combined, these results imply

$$4.2 < \frac{L}{b_c(L)} < 5 \quad \text{for} \quad d = 3. \quad (1)$$

A study of the twisted Eguchi-Kawai model in four dimensions shows that $b_B = 0.36$ but that this transition is strongly first order [12]. Numerical investigation of the $U^4(1)$ breaking transition shows that only for $L > 7$ does $b_c(L)$ go above 0.36 [13]. Like in three dimensions, there are many transitions associated with $U^4(1)$ breaking on the lattice. Let 0c, 1c, 2c, 3c and 4c denote phases with 0,1,2,3 and 4 ordered Polyakov loops with $b$ above $b_B$. Let 0h denote the phase below $b_B$. One can start with a 1c phase and bring it down to a 0c phase and still be below $b = 0.36$. This is a metastable state. This metastable transition from 1c $\rightarrow$ 0c is shown in Figs. 1-8. The system is in a 1c state at $b = 0.359$ on a $7^4$ lattice with $N = 37$ and it goes down to a 0c state at $b = 0.355$. We know that both $b = 0.359$
and \( b = 0.355 \) are on the weak coupling side of the bulk transition since the eigenvalue distributions of the \( 1 \times 1 \) Wilson loop shown in Fig. 1 and Fig. 2 exhibit strong gaps at both values of \( b \). The breaking of the \( U^4(1) \) symmetry can be probed by looking at Polyakov loop observables. The Polyakov loops (as well as various Wilson loops) were built out of “smeared” \( \bar{U}_\mu(x) \) matrices, rather than the original link matrices \( U_\mu(x) \) since it suppresses ultraviolet fluctuations. Let \( \bar{P}_\mu \) be the smeared Polyakov loop in the \( \mu \) direction and let its eigenvalues be \( e^{i\theta_j}; i = 1, \ldots, n \). The quantity [8]

\[
p(\bar{P}_\mu) = \frac{1}{N^2} \left( \sum_{i,j=1}^N \sin^2 \frac{1}{2}(\theta_i - \theta_j) \right)
\]

averaged over the 3-plane perpendicular to \( \mu \) will be close to 0 if the \( U(1) \) symmetry in that direction is not broken. Breaking of \( U(1) \) symmetry in the \( \mu \) direction will open a gap in the spectrum of \( \bar{P}_\mu \) as well. This distribution should not have a gap and all six orientations of the Wilson loop should look the same at \( b = 0.359 \). Let \( \bar{P}_\mu \) be close to 0 if the \( U(1) \) symmetry is broken. Breaking of \( U(1) \) symmetry is not broken. Breaking of \( U \) symmetry in one direction will open a gap in the spectrum of \( \bar{P}_\mu \) (Fig. 3) shows that the \( U(1) \) symmetry is broken in one of the four directions but the time history at \( b = 0.355 \) (Fig. 4) shows that all four \( U(1) \) symmetries are preserved. Therefore \( b = 0.359 \) is in the 1c phase and \( b = 0.355 \) is in the 0c phase. Eigenvalue distributions of \( \bar{P}_\mu \) at \( b = 0.359 \) also exhibit the breaking of the \( U(1) \) symmetry in one of the directions as seen in Fig. 3. But all four eigenvalue distributions of \( \bar{P}_\mu \) at \( b = 0.355 \) look uniform as seen in Fig. 3. Further evidence for \( b = 0.359 \) in the 1c phase and \( b = 0.355 \) in the 0c phase can be seen by looking at the eigenvalue distribution of the \( L \times L \) Wilson loop. This distribution should not have a gap and all six orientations of the Wilson loop should look the same at \( b = 0.355 \) but three of the six Wilson loops should look different from the three others at \( b = 0.359 \). This is indeed the case as demonstrated in Fig. 7 and Fig. 8. The oscillations seen in the distributions are a finite \( N \) effect. Finally, one can see that the system stabilizes in the correct phase even if it started in the wrong phase. For example, a system started in the 1c phase on a \( 6^4 \) lattice at \( b = 0.351 \) ends up in the correct 0c phase after a few sweeps of the lattice as seen in Fig. 9. Similarly, a system started in the 0c phase on a \( 7^4 \) lattice at \( b = 0.3568 \) ends up in the correct 1c phase after a few sweeps of the lattice as seen in Fig. 10.

Such a metastable state can be used to reduce \( L \) where one realizes large \( N \) QCD. We find that one can go as low as \( L = 5 \) and remain in the 0c phase. The \( 1c \rightarrow 0c \) transitions are located at \( b_c(5) = 0.34775 \pm 0.00075, b_c(6) = 0.35125 \pm 0.00075, \) and \( b_c(7) = 0.3564 \pm 0.004 \). For \( L > 7 \), the transition from unbroken \( U^4(1) \) to one broken \( U(1) \) occurs in the stable region and \( b_c(8) = 0.3605 \pm 0.0015, b_c(9) = 0.364 \pm 0.001, \) and \( b_c(10) = 0.367 \pm 0.001 \).

The above values for the critical coupling in the range \( L = 5 \) to \( L = 10 \) can be used to study the scaling of the critical coupling in four dimensions [13]. The inverse function \( L_c(b) \) should scale according to

\[
L_c(b) \sim L_0 \left( \frac{11}{48\pi^2 b} \right)^\frac{11}{24} e^{\frac{2\pi^2}{24} b^4} \quad (3)
\]

if it is a truly continuum phenomenon. Since the above result is a two-loop result, one has to use the the “tadpole” [14] replacement

\[
b \rightarrow b_f \equiv b e(b) \quad e(b) = \frac{1}{N} \langle TrU_{\mu,\nu}(x) \rangle
\]

\[
L_c(b) \sim L_0 \left( \frac{11}{48\pi^2 b_f} \right)^\frac{11}{24} e^{\frac{2\pi^2}{24} b_f^4} \quad (5)
\]

to study scaling. Fig. 11 shows that the data does scale properly with \( L_0 \) between 0.245 and 0.275.

5. Conclusions and future work

The breaking of the \( U^d(1) \) symmetries for \( d > 2 \) implies that Eguchi-Kawai reduction does not work on a \( 1^d \) lattice. Quenched Eguchi-Kawai models [5] and twisted Eguchi-Kawai models [15] provide a way out of this problem. The problem of including fermions in the twisted version of the model has not been completely solved. It is possible to extract physical quantities in the large \( N \) limit of QCD using the quenched version of the model as shown in two dimensions [16]. But it is necessary to sample all the \( N^4 \) minima in the quenched model and this could prove to be numerically difficult in four dimensions. An alterna-
tive to the quenched Eguchi-Kawai model is provided by the idea of continuum reduction. Continuum reduction implies that there are no finite volume effects in the large N limit of QCD as long as one works with $L > L_B$ and $b_B < b < b_c(L)$.

The eigenvalue distribution of the single plaquette exhibits a gap and the space of gauge field configurations naturally splits into several disconnected pieces. It is natural to assume that the disconnected pieces correspond to different gauge field topologies and this can be explicitly verified by a numerical simulation. Since there are no finite volume effects, one should see evidence for a chiral condensate if the box size is bigger than the critical size and the chiral condensate should occur due to the lowest eigenvalues scaling like $1/N$ as $N$ gets large. This also can be verified using numerical simulations on fairly small lattices ($L = 8 - 10, N = 20 - 30$).

Since there are no finite volume effects it should be possible to compute meson propagators for arbitrarily small momenta as long as the box size is bigger than the critical size. Normally, it is not possible to work with momenta that are smaller than the inverse of the box size. This can be avoided in the large N limit of QCD since a constant $U(1)$ phase multiplying all the link matrices in a given direction can be interpreted as momentum carried by a quark line [8,13,17]. This interpretation works as long as the $U^d(1)$ symmetries are not broken since the pure gauge sector is invariant under a multiplication by a constant phase but the fermion propagator picks up a momentum equal to this constant phase. This is indeed the case if the box size is bigger than the critical size. A direct computation of the meson propagators in momentum space will lend credibility to the above argument.

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Figure 1. Eigenvalue distribution of the $1 \times 1$ Wilson loop at $b = 0.359$ shows a gap.

Figure 2. Eigenvalue distribution of the $1 \times 1$ Wilson loop at $b = 0.355$ shows a gap.

Figure 3. Time history of $p(\tilde{P}_a)$ shows that the $U(1)$ symmetry is broken in one of the four directions.

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Figure 4. Time history of $p(\tilde{P}_\mu)$ shows that the $U(1)$ symmetry is preserved in all four directions.

Figure 5. Eigenvalue distribution of $\tilde{P}_\mu$ at $b = 0.359$ shows that the $U(1)$ symmetry is broken in one of four directions.

Figure 6. Eigenvalue distribution of $\tilde{P}_\mu$ at $b = 0.355$ shows that the $U(1)$ symmetry is preserved in all four directions.

Figure 7. Eigenvalue distribution of the six different orientations of the $L \times L$ wilson loop at $b = 0.359$ showing that the $U(1)$ symmetry is broken in one of four directions.
Figure 8. Eigenvalue distribution of the six different orientations of the $L \times L$ wilson loop at $b = 0.355$ indicating that the $U(1)$ symmetry is preserved in all four directions.

Figure 9. Time history of $p(\tilde{P}_\mu)$ for each direction showing the evolution from $1c$ to $0c$.

Figure 10. Time history of $p(\tilde{P}_\mu)$ for each direction showing the evolution from $0c$ to $1c$.

Figure 11. Demonstration of scaling in four dimensional large N QCD.