Existence of the Gauge for Fractional Laplacian Schrödinger Operators

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Abstract
Let \( \Omega \subseteq \mathbb{R}^n \) be an open set, where \( n \geq 2 \). Suppose \( \omega \) is a locally finite Borel measure on \( \Omega \). For \( \alpha \in (0, 2) \), define the fractional Laplacian \( (-\Delta)^{\alpha/2} \) via the Fourier transform on \( \mathbb{R}^n \), and let \( G \) be the corresponding Green’s operator of order \( \alpha \) on \( \Omega \). Define \( T(u) = G(u\omega) \). If \( \|T\|_{L^2(\omega)\rightarrow L^2(\omega)} < 1 \), we obtain a representation for the unique weak solution \( u \) in the homogeneous Sobolev space \( L^{\alpha/2,2}(\Omega) \) of
\[
(-\Delta)^{\alpha/2} u = u\omega + v \quad \text{on} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Omega^c,
\]
for \( v \) in the dual Sobolev space \( L^{-\alpha/2,2}(\Omega) \). If \( \Omega \) is a bounded \( C^{1,1} \) domain, this representation yields matching exponential upper and lower pointwise estimates for the solution when \( v = \chi_\Omega \). These estimates are used to study the existence of a solution \( u_1 \) (called the “gauge”) of the integral equation \( u_1 = 1 + G(u_1\omega) \) corresponding to the problem
\[
(-\Delta)^{\alpha/2} u = u\omega \quad \text{on} \quad \Omega, \quad u \geq 0 \quad \text{on} \quad \Omega, \quad u = 1 \quad \text{on} \quad \Omega^c.
\]
We show that if \( \|T\| < 1 \), then \( u_1 \) always exists if \( 0 < \alpha < 1 \). For \( 1 \leq \alpha < 2 \), a solution exists if the norm of \( T \) is sufficiently small. We also show that the condition \( \|T\| < 1 \) does not imply the existence of a solution if \( 1 < \alpha < 2 \). The condition \( \|T\| \leq 1 \) is necessary for the existence of \( u_1 \) for all \( 0 < \alpha \leq 2 \).

Dedication: For Guido Weiss, my postdoctoral advisor, with appreciation—Michael W. Frazier.
For Professor Guido Weiss, a wonderful mathematician and human being—Igor E. Verbitsky.
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1 Introduction

Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, is a non-empty open set (possibly the whole space), $\omega$ is a locally finite (positive) Borel measure on $\Omega$, and $\alpha \in (0, 2)$. We consider the problems:

$$\begin{cases} \frac{(-\Delta)^{\alpha/2}u}{2} = \omega u + v & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c. \end{cases} \quad (1.1)$$

where $\Omega^c = \mathbb{R}^n \setminus \Omega$, and

$$\begin{cases} \frac{(-\Delta)^{\alpha/2}u}{2} = \omega u & \text{in } \Omega, \\ u = 1 & \text{on } \Omega^c. \end{cases} \quad (1.2)$$

Here the fractional Laplacian $(-\Delta)^{\alpha/2}$ is the non-local operator defined in $\mathbb{R}^n$ via the Fourier transform by $((-\Delta)^{\alpha/2}u)(\xi) = |\xi|^{\alpha} \hat{u}(\xi)$, when $|\xi|^{\alpha} \hat{u}(\xi) \in S'(\mathbb{R}^n)$ (for example when $u$ belongs to the Sobolev space $H^\alpha(\mathbb{R}^n)$). For sufficiently nice $u$ (e.g., $u$ in the Schwartz class), there is a pointwise representation

$$(-\Delta)^{\alpha/2}h(x) = C(\alpha, n) \text{ p.v.} \int_{\mathbb{R}^n} \frac{h(x) - h(y)}{|x - y|^{n+\alpha}} dy, \quad (1.3)$$

see [8], Proposition 3.3. See [17] for equivalent definitions on $\mathbb{R}^n$. There are other interpretations of the fractional Laplacian restricted to a domain; see for example [22] and [20], where the interpretation above is called the restricted fractional Laplacian $(-\Delta|_\Omega)^{\alpha/2}$ ([22]) or the Dirichlet fractional Laplacian $(-\Delta|_\Omega)^{\alpha/2}_D$ ([20]). Problems (1.1) with $v = \chi_\Omega$ and (1.2) were considered for the classical Laplacian ($\alpha = 2$) in [11].

The (minimal) solution $u = u_1$ to (1.2) is called the gauge function (the Feynman–Kac gauge or simply the gauge) in the probability literature. In the case $d\omega = q(x) \, dx$, $q \in L^1_{loc}(\Omega)$, it can be expressed in the form

$$u_1(x) = E^x \left( e^{\int_0^{\tau_\Omega} q(X_s) ds} \right),$$

where $X_t$ is the (scaled) Brownian motion if $\alpha = 2$, or a symmetric $\alpha$-stable Lévy process if $0 < \alpha < 2$, starting at $x$, and $\tau_\Omega = \inf\{t > 0 : X_t \in \Omega^c\}$. If $q$ lies in the corresponding Kato class, then the so-called Gauge Theorem says (see [4], Theorem 4.19, if $\alpha = 2$, and [2], Theorem 2.9, if $0 < \alpha < 2$) that, for a bounded domain $\Omega$,
$u_1$ is uniformly bounded, provided $(\Omega, q)$ is “gaugeable”, which is equivalent to the condition $||T|| < 1$ discussed below for general $\omega \geq 0$.

We do not impose any conditions of Kato type on $\omega \geq 0$, and consequently in this general setup the gauge is no longer uniformly bounded. As we will show below, it is finite a.e. only under additional conditions in the case $1 < \alpha \leq 2$.

Let $G(x, y) = G^{(\alpha)}(x, y)$ be the Green’s function for $(-\Delta)^{\alpha/2}$ on the domain $\Omega_1$, defined as in [18], Ch. IV.5. Then $G$ is a non-negative, symmetric function on $\mathbb{R}^n \times \mathbb{R}^n$.

We also denote by $G = G^{(\alpha)}$ the corresponding Green’s operator of order $\alpha$, acting on a measure $\mu$ on $\Omega_1$ by

$$G\mu(x) = \int_{\Omega_1} G(x, y) d\mu(y), \quad x \in \mathbb{R}^n.$$  \hfill (1.4)

If $d\mu = f \, dx$, we write $Gf$ instead of $G\mu$.

For $f \in C^\infty_0(\Omega_1)$, the equation $(-\Delta)^{\alpha/2} u = f$ on $\Omega$, with $u = 0$ on $\Omega^c$, has solution $u = Gf$. By applying $G$ to both sides of the equation $(-\Delta)^{\alpha/2} u = \omega u + \nu$, we obtain the corresponding integral equation

$$u = G(\omega u) + G\nu.$$  \hfill (1.5)

Let $T$ be the operator

$$Tf(x) = G(f \omega)(x) = \int_{\Omega} G(x, y) f(y) \, d\omega(y).$$  \hfill (1.6)

Then (1.5) becomes $u = Tu + G\nu$, which has the formal solution

$$u = (I - T)^{-1} G\nu = \sum_{j=0}^{\infty} T^j G\nu.$$  \hfill (1.7)

Our main assumption is that $\|T\| \equiv \|T\|_{L^2(\omega) \to L^2(\omega)} < 1$. This assumption is equivalent (see Lemma 2.5 below) to the existence of some $\beta < 1$ such that

$$\|h\|_{L^2(\omega)} \leq \beta \|(-\Delta)^{\alpha/2} h\|_{L^2(\mathbb{R}^n)}, \quad \text{for all } h \in C^\infty_0(\Omega).$$  \hfill (1.8)

We inductively define kernels $G_j(x, y)$ on $\Omega \times \Omega$ for $j \geq 1$ by setting $G_1 = G$ and, for $j \geq 2$,

$$G_j(x, y) = \int_{\Omega} G_{j-1}(x, z) G(z, y) \, d\omega(z).$$  \hfill (1.9)

Then $T^j f(x) = \int_{\Omega} G_j(x, y) f(y) \, d\omega(y)$ for $j \geq 1$, by an application of Fubini’s theorem. We define the fractional Green’s function $G\mathcal{G}$ of order $\alpha$ associated to $\omega$ and $\Omega$:
\[ G(x, y) = \sum_{j=1}^{\infty} G_j(x, y), \quad (1.9) \]

and the corresponding operator \( G(v)(x) = \int_{\Omega} G(x, y) \, dv(y) \). Note that each \( G_j \), and hence \( G \), is symmetric and non-negative. Another use of Fubini’s theorem gives
\[ T^j G(v)(x) = \int_{\Omega} G_j(x, y) \, dv(y). \]
We give the name \( u_0 \) to our formal solution above, and note that
\[ u_0(x) = \sum_{j=0}^{\infty} T^j G(v)(x) = \sum_{j=0}^{\infty} \int_{\Omega} G_j(x, y) \, dv(y) = G(v)(x). \quad (1.10) \]

If \( v \) is a positive measure, then \( u_0 \) satisfies \( u = Tu + G(v) \) at every point, in the sense of functions with possibly infinite values. We say \( u_0 \) is a pointwise solution of \((1.5)\) if \( u_0 \prec \infty \) Lebesgue-a.e. on \( \Omega \).

The homogeneous Sobolev space \( L_0^{\alpha/2,2}(\Omega) \) is defined, for \( \alpha \in (0, 2) \), to be the closure of \( C_0^\infty(\Omega) \) with respect to the norm
\[ \|u\|_{L_0^{\alpha/2,2}(\Omega)} = \|(-\Delta)^{\alpha/4} u\|_{L^2(\mathbb{R}^n)}. \quad (1.11) \]

We also define \( L^{-\alpha/2,2}(\Omega) \) to be the dual of \( L_0^{\alpha/2,2}(\Omega) \).

If \( \|T\| < 1 \), then, by the Lax–Milgram Theorem, there is a unique weak solution \( u \in L_0^{\alpha/2,2}(\Omega) \) to \((1.1)\) for each \( v \in L^{-\alpha/2,2}(\Omega) \) (see §2). The following proposition shows that this weak solution is realized via \( G \), and that the condition \( \|T\| < 1 \) on \( \omega \) is close to being necessary for the existence of a solution.

**Proposition 1.1** Suppose \( n \geq 2 \) and \( 0 < \alpha < 2 \). Let \( \Omega \subseteq \mathbb{R}^n \) be open and let \( \omega \) be a (positive) Borel measure on \( \Omega \).

(A) Suppose \( \|T\| < 1 \). If \( v \in L^{-\alpha/2,2}(\Omega) \) is a (positive) measure, then \( G(v) \) is a non-negative pointwise solution to \((1.5)\), and \( G(v) \in L_0^{\alpha/2,2}(\Omega) \), with
\[ \|G(v)\|_{L_0^{\alpha/2,2}(\Omega)} \leq \frac{1}{1 - \|T\|} \|v\|_{L^{-\alpha/2,2}(\Omega)}. \quad (1.12) \]

Also, \( G \) extends to be a bounded operator from \( L^{-\alpha/2,2}(\Omega) \) to \( L_0^{\alpha/2,2}(\Omega) \) with norm at most \((1 - \|T\|)^{-1}\). For a general \( v \in L^{-\alpha/2,2}(\Omega) \), \( G(v) \) is the weak solution to \((1.1)\).

(B) If \((1.5)\) has a non-trivial non-negative pointwise solution \( u \) for some positive measure \( v \), then \( \|T\| \leq 1 \). If also \( u \in L_0^{\alpha/2,2}(\Omega) \), then \( \|v\| \in L^{-\alpha/2,2}(\Omega) \).

**Remark** For \( \alpha = 2 \), the results of Proposition 1.1 hold for \( n \geq 3 \) by the same methods. For \( n = 2 \), they hold for domains \( \Omega \) with a non-trivial non-negative Green’s function. See also [5], [11] and the references given there for \( d\omega = q \, dx \), where \( q \in L_1^{1/\alpha}(\Omega) \), in bounded smooth domains \( \Omega \).
The existence of the solution operator \( \mathcal{G} \) in Proposition 1.1 and its mapping properties follow easily from the Lax–Milgram Theorem. The specific representation of \( \mathcal{G} \) and (B) in Proposition 1.1 may be new. This representation will allow us to use the results of [12] to obtain pointwise estimates for the solution to (1.1) in the case where \( \nu = \chi_{\Omega} \), which we will then relate to equation (1.2).

Although Proposition 1.1 holds for a general open set \( \Omega \), for further conclusions we require some additional conditions. First, we must have \( \chi_{\Omega} \in L^{-\alpha/2, 2}(\Omega) \). This condition holds whenever \( |\Omega| < \infty \). This follows from (3.2) and Lemma 2.4, because (3.2) shows that \( \int_{\Omega} G \chi_{\Omega} dx < \infty \). Alternately, note that \( \chi_{\Omega} \in L^q(\mathbb{R}^n) \) for any \( q \), in particular for \( q^* = 2n/(n + \alpha) \), the conjugate index to \( p^* = 2n/(n - \alpha) \). By the Sobolev imbedding theorem, \( L^{\alpha/2, 2}(\Omega) \) imbeds in \( L^{p^*} \) continuously (see, e.g., [8], Theorem 6.5), so \( L^{p^*} \) imbeds continuously in the dual space \( L^{-\alpha/2, 2}(\Omega) \).

In addition, to apply [12], we require the Green’s function of \( \Omega \) to have certain properties (see §3). These properties hold if \( \Omega \) is a bounded \( C^{1, 1} \) domain.

**Theorem 1.2** Suppose \( n \geq 2 \) and \( 0 < \alpha < 2 \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^{1, 1} \) domain and let \( \omega \) be a (positive) Borel measure on \( \Omega \).

(i) Suppose \( \|T\| < 1 \). For \( x \in \Omega \), let \( m(x) = \delta(x)^{\alpha/2} \), where \( \delta(x) \) is the distance from \( x \) to the boundary of \( \Omega \). Let \( u_0 = \mathcal{G} \chi_{\Omega} \) be the solution of (1.1) when \( dv = \chi_{\Omega} dx \). Then there exist constants \( C = C(\Omega, \alpha, \|T\|) > 0 \), and \( C_1 = C_1(\Omega, \alpha) > 0 \), such that

\[
|u_0(x)| \leq C_1 m(x)e^{C \frac{Tm(x)}{m(x)}}, \text{ for all } x \in \Omega. \tag{1.13}
\]

(ii) Conversely, if \( u \) is any non-negative solution of (1.5) with \( dv = \chi_{\Omega} dx \), then there exist positive constants \( c = c(\Omega, \alpha) \) and \( c_1 = c_1(\Omega, \alpha) \) such that

\[
|u(x)| \geq c_1 m(x)e^{c \frac{Tm(x)}{m(x)}}, \text{ for a.e. } x \in \Omega. \tag{1.14}
\]

In fact, in statement (ii), an estimate with more precise constants is proved in [14]: any non-negative solution \( u \) satisfies the lower bound

\[
|u(x)| \geq s(x)e^{\frac{Ts(x)}{s(x)}}, \text{ for a.e. } x \in \Omega,
\]

where \( s(x) = G \chi_{\Omega}(x) \approx m(x) \).

For \( dv = \chi_{\Omega} dx \), our conclusions in Proposition 1.1 and Theorem 1.2 are very similar to the results for \( \alpha = 2 \) in [11]. On the other hand, our conclusions for the solution of (1.2) are very different from those for \( \alpha = 2 \) in [11]. To formulate the integral equation corresponding to (1.2), let \( v = u - 1 \). Then \( v \) satisfies \( (-\Delta)^{\alpha/2} v = v \omega + \omega \) on \( \Omega \), with \( v = 0 \) on \( \partial \Omega \). Applying \( \mathcal{G} \) to both sides gives \( v = \mathcal{G}(v \omega) + G\omega = T v + G\omega \). Therefore we consider \( v = (I - T)^{-1}G\omega = \sum_{j=1}^{\infty} T^j G\omega = \mathcal{G}\omega \). Thus the integral equation analogue of (1.2) is

\[
u = 1 + \mathcal{G}(\nu \omega) \tag{1.15}
\]
with solution (at every point, but with values that might be $+\infty$)

$$u_1(x) = 1 + G_\omega(x) = 1 + \int_{\Omega} G(x, y) \, d\omega(y), \quad (1.16)$$

for $x \in \Omega$, and $u_1 = 1$ on $\Omega^c$. We say that $u_1$ is a pointwise solution of $u = 1 + G(\omega)$ if $u_1 < \infty$ Lebesgue-a.e. on $\Omega$.

For $x \in \Omega$ and $y \in \Omega^c$, let $P(x, y) = P^\alpha(x, y)$ be the Poisson kernel of order $\alpha$ for $\Omega$ (see §3).

**Theorem 1.3** Suppose $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and let $\omega$ be a (positive) Borel measure on $\Omega$.

(i) Suppose $0 < \alpha < 1$ and $\|T\| < 1$. Then $u_1 \in L^1(\Omega, \, dx)$ and hence $u_1$ is a pointwise solution of (1.15). Moreover, $u_1 - 1 \in L^{\alpha/2,2}_0(\Omega)$. (ii) Suppose $1 \leq \alpha < 2$. Then there exists a constant $\gamma = \gamma(\alpha, \Omega) \in (0, 1)$ such that if $\|T\| < \gamma$, then $u_1 \in L^1(\Omega, \, dx)$ and hence $u_1$ is a solution of (1.15). When $0 < \alpha < 2$ and $\|T\| < 1$, there exist constants $C_3(\Omega, \alpha), C_4(\Omega, \alpha, \|T\|)$, such that

$$u_1(x) \leq C_3 \int_{\Omega^c} e^{C_4 \int_{\Omega} G(x, y) \frac{P(y, z)}{P(x, z)} \, d\omega(y)} \, P(x, z) \, dz. \quad (1.17)$$

Also, if $u$ is a non-negative solution to $u = 1 + G(\omega)$, then

$$u(x) \geq \int_{\Omega^c} e^{\int_{\Omega} G(x, y) \frac{P(y, z)}{P(x, z)} \, d\omega(y)} \, P(x, z) \, dz. \quad (1.18)$$

The next result states that in the case $1 < \alpha < 2$, the condition $\|T\| < 1$ is not sufficient to obtain that $u_1$ is finite a.e. (as is the case when $\alpha = 2$, by the results in [11]). From the proof, we will also see that $\gamma \to 0$ as $\alpha \to 2^-$ in Theorem 1.3.

**Theorem 1.4** Suppose $n \geq 2$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose $1 < \alpha < 2$. Then there exists a positive measure $\omega \in L^1_{loc}(\Omega)$ such that $\|T\| < 1$, but $u_1$ is identically $+\infty$ on $\Omega$.

In [11], it was shown for $\alpha = 2$ that $u_1$ is finite a.e. if $\|T\| < 1$ and $\omega$ satisfies an additional boundary condition (the exponential integrability of the balayage of $\delta d\omega$), and conversely if there is a solution, then $\|T\| \leq 1$ and the balayage of $\delta d\omega$ is exponentially integrable with a different constant. No such boundary conditions on $\omega$ appear in the case $0 < \alpha < 2$.

### 2 Proof of Proposition 1.1

We start by summarizing some background. For $0 < \alpha < n$, the function $k_\alpha(x) = c_{\alpha, n} |x|^{\alpha-n}$, where $c_{\alpha, n}$ is an appropriate normalization constant, has Fourier transform
\( \hat{k}_\alpha(\xi) = |\xi|^{-\alpha}. \) On \( \mathbb{R}^n \), the Riesz potential \( I_\alpha \) of order \( \alpha \) acts on a Borel measure \( \mu \) by

\[
I_\alpha \mu(x) = k_\alpha \ast \mu(x) = c_{\alpha,n} \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}. \tag{2.1}
\]

Thus \( I_\alpha \) serves as the Green's operator \( G(\alpha) \) on \( \mathbb{R}^n \), since \( (-\Delta)^{\alpha/2} I_\alpha \varphi = \varphi \) for sufficiently nice functions \( \varphi \). Note that \( I_\alpha \) is self-adjoint and satisfies the semi-group property: \( I_{\alpha+\beta} = I_\alpha I_\beta \) for \( \alpha, \beta > 0 \) such that \( \alpha + \beta < n \). See, e.g., [1], § 1.2.2, for these facts. For \( 0 < \alpha < n \), define the homogeneous Sobolev space

\[
L^{\alpha/2,2}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : (-\Delta)^{\alpha/4} u \in L^2(\mathbb{R}^n) \},
\]

with norm

\[
\| u \|_{L^{\alpha/2,2}(\mathbb{R}^n)} = \| (-\Delta)^{\alpha/4} u \|_{L^2(\mathbb{R}^n)}. \tag{2.2}
\]

Note that each \( u \in L^{\alpha/2,2}(\mathbb{R}^n) \) can be written as \( u = I_{\alpha/2} f \) for \( f = (-\Delta)^{\alpha/4} u \in L^2(\mathbb{R}^n) \), with \( \| f \|_{L^2(\mathbb{R}^n)} = \| u \|_{L^{\alpha/2,2}(\mathbb{R}^n)}. \)

Also, define

\[
L^{-\alpha/2,2}(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) : |\xi|^{-\alpha/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \},
\]

with norm \( \| u \|_{L^{-\alpha/2,2}(\mathbb{R}^n)} = \| |\xi|^{-\alpha/2} \hat{u}(\xi) \|_{L^2(\mathbb{R}^n)}. \) A Borel signed measure \( \mu \) on \( \mathbb{R}^n \) has finite \( \alpha \)-energy if and only if \( \mu \in L^{-\alpha/2,2}(\mathbb{R}^n) \), and

\[
\int_{\mathbb{R}^n} I_\alpha \mu \, d\mu = \| I_{\alpha/2} \mu \|^2_{L^2(\mathbb{R}^n)} = \| \mu \|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)}. \tag{2.3}
\]

For \( \mu \in L^{-\alpha/2,2}(\mathbb{R}^n) \),

\[
\| I_\alpha \mu \|_{L^{\alpha/2,2}(\mathbb{R}^n)} = \| (-\Delta)^{\alpha/4} I_\alpha u \|_{L^2(\mathbb{R}^n)} = \| \mu \|_{L^{-\alpha/2,2}(\mathbb{R}^n)}. \tag{2.4}
\]

In fact, \( I_\alpha \) maps \( L^{-\alpha/2,2}(\mathbb{R}^n) \) isometrically onto \( L^{\alpha/2,2}(\mathbb{R}^n) \).

Let \( L^{\alpha/2,2}_0(\mathbb{R}^n) \) (\( 0 < \alpha < n \)) be the homogeneous Sobolev space defined as the completion of \( C_0^\infty(\mathbb{R}^n) \) with respect to the \( L^{\alpha/2,2}(\mathbb{R}^n) \)-norm. Then \( L^{\alpha/2,2}_0(\Omega) \) is a closed subspace of \( L^{\alpha/2,2}(\mathbb{R}^n) \) with the inherited norm. Then \( L^{-\alpha/2,2}(\mathbb{R}^n) \) is the dual space of \( L^{\alpha/2,2}_0(\mathbb{R}^n) \).

For \( E \subset \mathbb{R}^n \), we define the \( \alpha \)-capacity of \( E \) relative to \( \mathbb{R}^n \) by

\[
\text{cap}_\alpha(E) = \inf \{ \| f \|^2_{L^2(\mathbb{R}^n)} : I_{\alpha/2} f \geq \chi_E, f \geq 0, f \in L^2(\mathbb{R}^n) \}.
\]
This capacity is sometimes denoted by \( \text{cap}_{\alpha/2, \alpha} (\cdot) \), and is a special case of the “nonlinear” capacity \( \text{cap}_{s, p} (\cdot) \) defined by

\[
\text{cap}_{s, p} (E) = \inf \{ \| f \|_{L^p (\mathbb{R}^n)}^p : I_s f \geq \chi_E, f \geq 0, f \in L^p (\mathbb{R}^n) \},
\]

for \( 1 < p < \infty \) and \( 0 < s < n/p \). (See [1], Sec. 2.3.)

A property holds quasi-everywhere (abbreviated q.e., or \( \alpha \)-q.e. if the value of \( \alpha \) is not clear from context) if it holds except on a set of \( \alpha \)-capacity 0.

A function \( f \) defined on an open set \( \Omega \subset \mathbb{R}^n \) (or quasi-everywhere in \( \mathbb{R}^n \)) is said to be \( \alpha \)-quasicontinuous on \( \Omega \) if, for every \( \epsilon > 0 \), there exists an open set \( G \) such that \( \text{cap}_\alpha (G) < \epsilon \), and \( f \) is continuous outside \( G \). If \( g \in L^2 (\mathbb{R}^n) \) and \( \alpha > 0 \), then \( I_{\alpha/2} g \) is \( \alpha \)-quasicontinuous (in particular \( I_{\alpha/2} g \) is finite \( \alpha \)-q.e.). Hence every function \( f \in L^{\alpha/2, 2} (\mathbb{R}^n) \) has an \( \alpha \)-quasicontinuous representative \( \tilde{f} = I_{\alpha/2} g \), for \( g = (-\Delta)^{\alpha/4} f \in L^2 (\mathbb{R}^n) \). Also, if \( \mu \in L^{-\alpha/2, 0} (\mathbb{R}^n) \) is a positive measure, then \( I_{\alpha/2} \mu \in L^2 (\mathbb{R}^n) \), and hence \( I_{\alpha} \mu = I_{\alpha/2} (I_{\alpha/2} \mu) \) is \( \alpha \)-quasicontinuous. Moreover, if two \( \alpha \)-quasicontinuous functions coincide a.e., then they coincide \( \alpha \)-q.e. (see [1], Sec. 6.1 for these facts about quasicontinuity in the case of the inhomogeneous Sobolev spaces; the proof for the homogeneous spaces is virtually the same). If \( \{ f_j \}_{j=1}^\infty \) is a sequence of functions in \( L^{\alpha/2, 2} (\mathbb{R}^n) \) converging in norm to \( f \), then there is a subsequence \( f_{jk} \) converging \( \alpha \)-q.e. to a \( \alpha \)-quasicontinuous representative \( \tilde{f} \) of \( f \) in \( L^{\alpha/2, 2} (\mathbb{R}^n) \). (This fact is a restatement of [1], Proposition 2.3.8, by considering \( g_j \in L^2 (\mathbb{R}^n) \) such that \( f_j = I_{\alpha/2} g_j \).) In particular, if \( f_j \) also converges to \( f \) \( \alpha \)-q.e., then \( f \) is \( \alpha \)-quasicontinuous.

Note that if \( \lambda \in L^{-\alpha/2, 0} (\mathbb{R}^n) \) is a signed Borel measure, and \( E \subset \mathbb{R}^n \) is a Borel set with \( \text{cap}_\alpha (E) = 0 \), then \( \lambda (E) = 0 \), as follows. We may assume \( \lambda \) is a positive measure. Let \( \mathcal{A} \) denote the class of functions in the infimum defining \( \text{cap}_\alpha \). Then for all \( f \in \mathcal{A} \),

\[
\lambda (E) \leq \int_{\mathbb{R}^n} I_{\alpha/2} f d\lambda = \int_{\mathbb{R}^n} f I_{\alpha/2} \lambda d\lambda \leq \| f \|_{L^2 (\mathbb{R}^n)} \| \lambda \|_{L^{-\alpha/2, 2} (\mathbb{R}^n)},
\]

so taking the infimum over \( f \in \mathcal{A} \) gives \( \lambda (E) = 0 \). Observe that for \( \lambda \in L^{-\alpha/2, 2} (\mathbb{R}^n) \) a signed measure and \( f \in L^{\alpha/2, 2}_0 (\mathbb{R}^n) \), the quantity \( \int_{\mathbb{R}^n} f d\lambda \) is not well defined, because changing \( f \) on a set of measure 0 but positive \( \alpha \)-capacity may change the integral. However, if \( |\lambda| \in L^{-\alpha/2, 2} (\mathbb{R}^n) \) and \( \tilde{f} \) an \( \alpha \)-quasicontinuous representative of the equivalence class of \( f \) in \( L^{\alpha/2, 2}_0 (\mathbb{R}^n) \), then \( \int_{\mathbb{R}^n} \tilde{f} d\lambda \) is well defined (i.e., independent of the choice of \( \alpha \)-quasicontinuous representative) and the duality pairing between \( \lambda \) and \( f \) is \( \langle \lambda, f \rangle = \int_{\mathbb{R}^n} \tilde{f} d\lambda \) (see [1], Sec. 7.1, equation (7.1.2)).

We also define the capacity of a compact set \( E \subset \Omega \) relative to \( \Omega \):

\[
\text{cap}_\alpha (E, \Omega) = \inf \{ \| u \|_{L^{\alpha/2, 2}_0 (\Omega)}^2 : u \geq 1 \text{ on } E, u \in C_0^\infty (\Omega) \}.
\]

In the case \( \Omega = \mathbb{R}^n \), we have \( \text{cap}_\alpha (E, \mathbb{R}^n) = \text{cap}_\alpha (E) \) ([1], Proposition 2.3.13). For an open set \( G \subset \mathbb{R}^n \), we set \( \text{cap}_\alpha (G, \Omega) = \sup \{ \text{cap}_\alpha (E, \Omega) : E \subset G, E \text{ compact} \} \).

The following is a dual form of Deny’s Theorem ([6]; also see Theorem 9.1.7 in [1]).
Lemma 2.1 Suppose $f \in L^{a/2,2}(\mathbb{R}^n)$ and $f$ has an $\alpha$-quasicontinuous representative $\tilde{f}$ such that $\tilde{f} = 0 \alpha$-q.e. on $\Omega^c$. Then $f \in L^{a/2,2}_0(\Omega)$.

Proof Let $f$ be as in the assumptions and let $\mu \in L^{-a/2,2}(\mathbb{R}^n)$ be a (positive) measure with supp $\mu \subseteq \Omega^c$. Then $\langle f, \mu \rangle = \int_{\mathbb{R}^n} f \, d\mu = 0$, where here $\langle, \rangle$ denotes the pairing between $L^{a/2,2}(\mathbb{R}^n)$ and its dual $L^{-a/2,2}(\mathbb{R}^n)$. By a theorem of Deny ([6], p. 143, or see [1], Corollary 9.1.7 and the remarks in Section 9.13) any distribution $T \in L^{-a/2,2}(\mathbb{R}^n)$ with support in $\Omega^c$ can be approximated in $L^{-a/2,2}(\mathbb{R}^n)$ by linear combinations of positive measures in $L^{-a/2,2}(\mathbb{R}^n)$, with support in $\Omega^c$. Then $\langle f, T \rangle = 0$ for all such $T$. Every such $T$ vanishes on $C_0^\infty(\Omega)$ by the support assumption, and hence vanishes on the closure $L^{a/2,2}_0(\Omega)$. By the Hahn–Banach theorem, if $f \notin L^{a/2,2}_0(\Omega)$, there would be a $T \in L^{-a/2,2}(\mathbb{R}^n)$ vanishing on $L^{a/2,2}_0(\Omega)$ but not on $f$, which we have seen is impossible. Hence $f \in L^{a/2,2}_0(\Omega)$. \hfill \Box

For $\mu$ a finite positive measure on $\Omega$, there exists a positive measure $\mu'$, called the balayage of $\mu$ onto $\Omega^c$, such that $\mu'$ is supported in $\Omega^c$,

$$I_\alpha \mu(x) \geq I_\alpha \mu'(x), \quad \text{for all } x \in \mathbb{R}^n,$$

$$I_\alpha \mu(x) = I_\alpha \mu'(x), \quad \alpha - \text{q.e. on } \Omega^c,$$

and

$$G\mu(x) = I_\alpha \mu(x) - I_\alpha \mu'(x).$$

(See [18], Sec. IV.6, no. 24-25, and Sec. V.1, no. 2). Also $\mu'$ is a finite measure with $\mu'\mathbb{R}^n) \leq \mu(\Omega)$; this fact follows from [18], equation (4.5.5), and p. 263, lines 11 and 13. By (2.6), $G\mu = 0 \alpha$-q.e. on $\Omega^c$. If $\mu \in L^{-a/2,2}(\mathbb{R}^n)$, then $\mu' \in L^{-a/2,2}(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} I_\alpha \mu' \, d\mu' \leq \int_{\mathbb{R}^n} I_\alpha \mu \, d\mu = \int_{\mathbb{R}^n} I_\alpha \mu' \, d\mu \leq \int_{\mathbb{R}^n} I_\alpha \mu \, d\mu < \infty,$$

by (2.3)-(2.5). Hence if $\mu \in L^{-a/2,2}(\mathbb{R}^n)$, then $I_\alpha \mu$ and $I_\alpha \mu'$ are $\alpha$-quasicontinuous, so $G\mu$ is $\alpha$-quasicontinuous.

Most of the conclusions of Lemmas 2.2-2.4 are more or less implicit in [18], but in somewhat different language from what we require. See also the recent paper by Fuglede and Zorii [13].

Lemma 2.2 Suppose $\mu \in L^{-a/2,2}(\mathbb{R}^n)$ is a finite positive measure on $\Omega$. Then $G\mu \in L^{a/2,2}_0(\Omega)$, $\int_\Omega G\mu \, d\mu < \infty$, and

$$\|G\mu\|^2_{L^{a/2,2}_0(\Omega)} = \int_\Omega G\mu \, d\mu.$$

\hfill \Box
Proof Since $\mu, \mu' \in L^{-\alpha/2,2}(\mathbb{R}^n)$ and $I_\alpha$ maps $L^{-\alpha/2,2}(\mathbb{R}^n)$ into $L^{\alpha/2,2}(\mathbb{R}^n)$, we have $G\mu \in L^{\alpha/2,2}(\mathbb{R}^n)$ by (2.7). Then $G\mu \in L_0^{\alpha/2,2}(\Omega)$ by (2.6), (2.7), and Lemma 2.1, because $G\mu$ is quasicontinuous as noted above. Applying (2.6)-(2.7), (2.3), and the fact that $I_\alpha$ maps $L^{-\alpha/2,2}(\mathbb{R}^n)$ isometrically to $L^{\alpha/2,2}(\mathbb{R}^n)$, we have

$$\int_\Omega G\mu \, d\mu = \int_{\mathbb{R}^n} I_\alpha (\mu - \mu') \, d\mu = \int_{\mathbb{R}^n} I_\alpha (\mu - \mu') \, d(\mu - \mu')$$

$$= \| \mu - \mu' \|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)}$$

$$= \| I_\alpha (\mu - \mu') \|^2_{L^{\alpha/2,2}(\mathbb{R}^n)} = \| G\mu \|^2_{L^{\alpha/2,2}(\Omega)}.$$  

\[\square\]

The following lemma is an analogue of (2.3) for $\Omega$. The proof is more complicated because $G^{(\alpha)}$ does not satisfy a semi-group property.

Lemma 2.3 Let $\mu$ be a finite positive Borel measure which is compactly supported in $\Omega$. Let $\alpha \in (0,2)$ and let $G = G^{(\alpha)}$ be the Green’s operator of order $\alpha$. Then the following are equivalent:

(i) $\mu \in L^{-\alpha/2,2}(\mathbb{R}^n)$;
(ii) $\mu \in L^{-\alpha/2,2}(\Omega)$;
(iii) $\int_\Omega G\mu \, d\mu < \infty$.

If these conditions hold, then

$$\int_\Omega G\mu \, d\mu = \| \mu \|^2_{L^{-\alpha/2,2}(\Omega)},$$  

(2.9)

Proof Since $L_0^{\alpha/2,2}(\Omega) \subseteq L_0^{\alpha/2,2}(\mathbb{R}^n)$, (i) implies (ii). To show that (ii) implies (i), suppose $\mu \in L^{-\alpha/2,2}(\Omega)$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$. Let $\gamma \in C_0^\infty(\Omega)$ satisfy $\gamma = 1$ on the support of $\mu$. Then by Plancherel’s theorem,

$$\| \varphi \gamma \|^2_{L^{\alpha/2,2}(\mathbb{R}^n)} = \| \xi \|^2_{L^{\alpha/2}(\mathbb{R}^n)} \| \gamma \|^2_{L^{\alpha/2}(\mathbb{R}^n)} \leq c_{\gamma,a,n} \| \xi \|^2_{L^{\alpha/2}(\mathbb{R}^n)} \| \gamma \|^2_{L^{\alpha/2}(\mathbb{R}^n)} = c_{\gamma,a,n} \| \varphi \|^2_{L^{\alpha/2,2}(\mathbb{R}^n)},$$

where $M$ is the Hardy–Littlewood maximal function (see, e.g., [21], p. 63). The second inequality holds because the maximal function is bounded on $L^2(\mathbb{R}^n)$, $|\xi|^\alpha \, d\xi$ because $|\xi|^\alpha$ is an $A_2$-weight, since $\alpha < n$ (see [19]). Since $\varphi \gamma = \varphi$ on the support of $\mu$, and $\varphi \gamma \in C_0^\infty(\Omega)$,

$$\left| \int_{\mathbb{R}^n} \varphi \, d\mu \right| = \left| \int_{\mathbb{R}^n} \varphi \gamma \, d\mu \right| \leq \| \mu \|_{L^{-\alpha/2,2}(\Omega)} \| \varphi \gamma \|_{L_0^{\alpha/2,2}(\mathbb{R}^n)}$$

$$\leq c_{\gamma,a,n} \| \mu \|_{L^{-\alpha/2,2}(\Omega)} \| \varphi \|^2_{L^{\alpha/2,2}(\mathbb{R}^n)},$$

Hence $\mu$ extends from the dense subspace $C_0^\infty(\mathbb{R}^n)$ to define a bounded linear functional on $L_0^{\alpha/2,2}(\mathbb{R}^n)$. Thus (i) holds.
If $\mu \in L^{-\alpha/2,2}(\mathbb{R}^n)$, then by (2.7) and (2.3),
\[ \int_{\Omega} G\mu \, d\mu \leq \int_{\mathbb{R}^n} I_\alpha \mu \, d\mu = \|\mu\|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)}. \]

Hence (i) implies (iii).

To show that (iii) implies (i), assume that $\int_{\Omega} G\mu \, d\mu < \infty$. We show that $\int_{\mathbb{R}^n} I_\alpha \mu \, d\mu < \infty$, hence, by (2.3), $\mu \in L^{-\alpha/2,2}(\mathbb{R}^n)$. By (2.5) and (2.7),
\[ \int_{\mathbb{R}^n} I_\alpha \mu \, d\mu = \int_{\Omega} G\mu \, d\mu + \int_{\Omega} I_\alpha \mu' \, d\mu, \]
so it suffices to show that $\int_{\mathbb{R}^n} I_\alpha \mu \, d\mu' = \int_{\mathbb{R}^n} I_\alpha \mu' \, d\mu < \infty$. Let $K$ denote the support of $\mu$ and let $O$ be an open set such that $K \subseteq O$ and $\overline{O} \subseteq \Omega$. Select a point $x_0 \in O$ such that $I_\alpha \mu(x_0) < \infty$; such a point exists because $I_\alpha \mu$ is finite a.e. ([18], p. 61).

Since the distance between $O$ and $\Omega^c$ is positive, then for $x \in O$ and $y \in \Omega^c$, we have $|x - y| \approx |x_0 - y|$ with constants uniform over $x$, $y$, and hence there is a constant $C$ such that for all $y \in \Omega^c$,
\[ I_\alpha \mu(y) = c_{n,\alpha} \int_{K} \frac{d\mu(x)}{|x - y|^{n-\alpha}} \leq C \int_{K} \frac{d\mu(x)}{|x_0 - y|^{n-\alpha}} = \frac{C \mu(K)}{|x_0 - y|^{n-\alpha}}. \]

Hence, using (2.5) and the fact that $\mu'$ is supported in $\Omega^c$,
\[ \int_{\mathbb{R}^n} I_\alpha \mu \, d\mu' \leq C \mu(K) \int_{\mathbb{R}^n} \frac{d\mu'(y)}{|x_0 - y|^{n-\alpha}} = C_1 \mu(K) I_\alpha \mu'(x_0) \leq C_1 \mu(K) I_\alpha \mu(x_0) < \infty. \]

To prove (2.9) for $\mu \in L^{-\alpha/2,2}(\mathbb{R}^n)$, as in the proof of Lemma 2.2, we have
\[ \int_{\Omega} G\mu \, d\mu = \|\mu - \mu'\|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)}. \]

Let $A$ denote the set of linear combinations of positive measures belonging to $L^{-\alpha/2,2}(\mathbb{R}^n)$ and supported in $\Omega^c$. We claim that
\[ \|\mu - \mu'\|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)} = \inf_{\lambda \in A} \|\mu - \lambda\|^2_{L^{-\alpha/2,2}(\mathbb{R}^n)}, \]
(i.e., that $\mu'$ is extremal for the right side of (2.10)). A similar statement where the supremum is over positive measures $\lambda$ is proved in [18], Lemma 4.4, but the extension involving linear combinations is not difficult, as follows. Let $\langle \cdot, \cdot \rangle_*$ denote the inner product in $L^{-\alpha/2,2}(\mathbb{R}^n)$:
\[ \langle v_1, v_2 \rangle_* = \int_{\mathbb{R}^n} I_{\alpha/2} v_1 I_{\alpha/2} v_2 \, dx = \int_{\mathbb{R}^n} I_\alpha v_1 \, d\mu. \]
Let $\lambda \in \mathcal{A}$. Then
\[
\|\mu - \lambda\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2 = \|\mu - \mu'\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2 + 2\langle \mu - \mu', \mu' - \lambda \rangle_* + \|\mu' - \lambda\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2.
\]
However,
\[
\langle \mu - \mu', \mu' - \lambda \rangle_* = \int_{\mathbb{R}^n} I_\alpha(\mu - \mu')d(\mu' - \lambda) = 0,
\]
by (2.6), because $\mu', \lambda \in L^{-\alpha/2,2}(\mathbb{R}^n)$ are supported in $\Omega^c$. Hence $\|\mu - \lambda\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2 \geq \|\mu - \mu'\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2$, which establishes (2.10).

By Deny’s theorem, as noted above, every distribution $\lambda \in L^{-\alpha/2,2}(\mathbb{R}^n)$ supported in $\Omega^c$ can be approximated in norm by linear combinations of positive measures supported in $\Omega^c$. Therefore we can replace the class $\mathcal{A}$ with the class $\mathcal{B} = \{\lambda \in L^{-\alpha/2,2}(\mathbb{R}^n) : \text{supp } \lambda \subseteq \Omega^c\}$, to obtain
\[
\int_{\Omega} G \mu d\mu = \inf_{\lambda \in \mathcal{B}} \|\mu - \lambda\|_{L^{-\alpha/2,2}(\mathbb{R}^n)}^2.
\]
By the Hahn–Banach theorem, the last infimum is the square of the norm of $\mu$ in $L^{-\alpha/2,2}(\Omega)$. Thus (2.9) holds.

We will need a few more facts about $G$ for the next proof. By [2], p. 15 and 20-21, for $x \in \Omega$, the function $x \rightarrow G(x, y)$ is an $\alpha$-harmonic (hence $C^2$) function of $y$ on $\Omega \setminus \{x\}$, and satisfies a Harnack inequality: for $K$ a compact subset of $\Omega \setminus \{x\}$, there exists a constant $C(K, \Omega, \alpha) > 0$ such that $G(x, y_1) \leq C(K, \Omega, \alpha)G(x, y_2)$ for all $y_1, y_2 \in K$. (That this inequality holds even if $x$ and $y$ are in different connected components of $K$ is a remarkable feature that does not hold in the classical case $\alpha = 2$.) The definition of the Green’s function shows that $\lim_{y \to x} G(x, y) = +\infty$. This fact and the Harnack inequality show that we have the strict inequality $G(x, y) > 0$ for each $x, y \in \Omega$. Then applying Harnack’s inequality again shows that for any $x \in \Omega$ and $K \subset \Omega$ compact, $G(x, y)$ is bounded away from 0 on $K$: more precisely,
\[
C(x, K) \equiv \inf_{y \in K} G(x, y) > 0.
\]
In particular, if $\mu$ is a positive measure on $\Omega$ and $\mu_K$ is the restriction of $\mu$ to $K$, then
\[
G\mu_K(x) = \int_K G(x, y) d\mu(y) \geq C(x, K)\mu(K).
\]
(2.11)

Also, $C(x, K)$ is a measurable function of $x \in \Omega$, since, for a countable dense subset $\{y_i\}$ of $K$ and $t \in \mathbb{R}$, we have $\{x \in \Omega : C(x, K) < t\} = \cup_i \{x \in \Omega : G(x, y_i) < t\}$, by the continuity of $G(x, y)$ for $y \neq x$.

We can remove the compact support and/or finiteness assumptions on $\mu$ for parts of the last two lemmas by limiting arguments.
Lemma 2.4 (A) Suppose μ is a positive measure on Ω. Then the following are equivalent:

(i) \( \mu \in L^{-\alpha/2} \mathcal{L}_{\Omega} \);
(ii) \( \int \mu G d\mu < \infty \);
(iii) \( G\mu \in L^{\alpha/2} \mathcal{L}_{\Omega} \).

If these conditions hold, then \( G\mu \) is \( \alpha \)-quasicontinuous and

\[
\| G\mu \|^{2}_{L^{\alpha/2}_{0} \mathcal{L}_{\Omega}} = \int \mu \, G d\mu = \| \mu \|^{2}_{L^{-\alpha/2} \mathcal{L}_{\Omega}}. 
\tag{2.12}
\]

Also, if \( \mu_1, \mu_2 \) are positive measures belonging to \( L^{-\alpha/2} \mathcal{L}_{\Omega} \), then (2.12) holds for \( \mu = \mu_1 - \mu_2 \) as well.

(B) We can extend \( G \) by continuity to an isometry from \( L^{-\alpha/2} \mathcal{L}_{\Omega} \) onto \( L^{\alpha/2} \mathcal{L}_{0} \mathcal{L}_{\Omega} \). Then for all \( \mu \in L^{-\alpha/2} \mathcal{L}_{\Omega} \),

\[
\| G\mu \|^{2}_{L^{\alpha/2}_{0} \mathcal{L}_{\Omega}} = \langle \mu, G\mu \rangle = \| \mu \|^{2}_{L^{-\alpha/2} \mathcal{L}_{\Omega}},
\tag{2.13}
\]

where \( \langle , \rangle \) denotes the duality pairing between \( L^{-\alpha/2} \mathcal{L}_{\Omega} \) and \( L^{\alpha/2} \mathcal{L}_{0} \mathcal{L}_{\Omega} \).

Proof (A) First suppose \( \mu \in L^{-\alpha/2} \mathcal{L}_{\Omega} \) is a positive measure. Let \( K \) be any compact subset of \( \Omega \) and let \( \mu_K \) be the restriction of \( \mu \) to \( K \). Then we can find \( \varphi \in C_{0}^{\infty} (\mathcal{L}_{\Omega}) \) such that \( \varphi \geq 0 \) and \( \varphi = 1 \) on \( K \). Then

\[
\mu (K) \leq \int \varphi \, d\mu \leq \| \mu \|_{L^{-\alpha/2} \mathcal{L}_{\Omega}} \| \varphi \|_{L^{\alpha/2} \mathcal{L}_{0} \mathcal{L}_{\Omega}} < \infty.
\]

Hence \( \mu \) is finite on compact subsets of \( \Omega \).

We claim that \( \mu_K \in L^{-\alpha/2} \mathcal{L}_{\Omega} \). Since \( \mu_K \) is finite and compactly supported in \( \Omega \), it suffices to show that \( \mu_K \in L^{-\alpha/2} \mathcal{L}_{\mathbb{R}^n} \), by Lemma 2.3. Using (2.3),

\[
\| \mu_K \|_{L^{-\alpha/2} \mathcal{L}_{\mathbb{R}^n}} = \| I_{\alpha/2} \mu \|_{L^{2} \mathcal{L}_{\mathbb{R}^n}} = \sup_{\varphi \in \mathcal{A}} \int \mathbb{R}^n (I_{\alpha/2} \mu) \varphi \, dx,
\]

where \( \mathcal{A} = \{ \varphi \in C_{0}^{\infty} (\mathbb{R}^n) : \varphi \geq 0 \text{ and } \| \varphi \|_{L^{2} \mathcal{L}_{\mathbb{R}^n}} \leq 1 \} \). Since \( I_{\alpha/2} \) has a symmetric kernel,

\[
\| \mu_K \|_{L^{-\alpha/2} \mathcal{L}_{\mathbb{R}^n}} = \sup_{\varphi \in \mathcal{A}} \int \mathbb{R}^n I_{\alpha/2} \varphi \, d\mu_K.
\]

Let \( \gamma \in C_{0}^{\infty} (\Omega) \) satisfy \( \gamma \geq 0 \) and \( \gamma = 1 \) on \( K \). Then

\[
\| \mu_K \|_{L^{-\alpha/2} \mathcal{L}_{\mathbb{R}^n}} \leq \sup_{\varphi \in \mathcal{A}} \int \mathbb{R}^n \gamma I_{\alpha/2} \varphi \, d\mu_K \leq \sup_{\varphi \in \mathcal{A}} \int \mathbb{R}^n \gamma I_{\alpha/2} \varphi \, d\mu,
\]

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since the integrand is non-negative on $\Omega$. By the same argument as in the proof of Lemma 2.3, we have

$$\|\gamma I_\alpha/2\varphi\|_{L^{a/2,2}(\mathbb{R}^n)} \leq c_{\gamma,\alpha,n} \|I_\alpha/2\varphi\|_{L^{a/2,2}(\mathbb{R}^n)} = c_{\gamma,\alpha,n} \|\varphi\|_{L^2(\mathbb{R}^n)} \leq c_{\gamma,\alpha,n}.$$ 

By Lemma 2.1, then, $\gamma I_\alpha/2 \varphi \in L^{a/2,2}_0(\Omega)$. Hence

$$\|\mu_K\|_{L^{-a/2,2}(\mathbb{R}^n)} \leq \sup_{\varphi \in \mathcal{A}} \|\mu\|_{L^{-a/2,2}(\Omega)} \|\gamma I_\alpha/2\varphi\|_{L^{a/2,2}(\mathbb{R}^n)} = \|\gamma\|_{L^{a/2,2}(\mathbb{R}^n)} \|\mu\|_{L^2(\mathbb{R}^n)} < \infty.$$ 

Since $\mu_K \in L^{-a/2,2}(\Omega)$, Lemma 2.2 gives that $G\mu_K \in L^{a/2,2}_0(\Omega)$, and $\|G\mu_K\|_{L^{a/2,2}(\mathbb{R}^n)}^2 = \int_\Omega G\mu_K \ d\mu_K < \infty$. Hence

$$\int_\Omega G\mu_K \ d\mu_K \leq \|\mu\|_{L^{a/2,2}(\Omega)} \left(\int_\Omega G\mu_K \ d\mu_K \right)^{1/2}.$$ 

Dividing and squaring gives

$$\int_\Omega G\mu_K \ d\mu_K \leq \|\mu\|_{L^{a/2,2}(\Omega)}^2.$$ 

Let $\{K_j\}_{j=1}^\infty$ be an increasing sequence of compact subsets of $\Omega$ with $\bigcup_{j=1}^\infty K_j = \Omega$. Let $\mu_j$ be the restriction of $\mu$ to $K_j$. Then by our last conclusion, we have $\int_\Omega G\mu_j \ d\mu_j \leq \|\mu\|_{L^{a/2,2}(\Omega)}^2$. The monotone convergence theorem shows that

$$\int_\Omega G\mu \ d\mu \leq \|\mu\|_{L^{a/2,2}_0(\Omega)}^2.$$ 

Hence (i) implies (ii).

Now suppose $\mu$ is a positive measure on $\Omega$ and $\int_\Omega G\mu \ d\mu < \infty$. We first note that $\mu$ is finite on any compact subset $K$ of $\Omega$. Assume $\mu(K) > 0$ and let $\mu_K$ be the restriction of $\mu$ to $K$. Let $C(x, K) = \inf_{y \in K} G(x, y)$. We noted earlier that $C(x, K) > 0$ for all $x \in \Omega$. Hence $C(K) = \int_K C(x, K) \ d\mu(x) > 0$. By (2.11),

$$C(K)\mu(K) = \int_K C(x, K)\mu(K) \ d\mu(x) \leq \int G\mu_K \ d\mu_K \leq \int G\mu \ d\mu < \infty.$$ 

Hence $\mu(K) < \infty$.

Let $K_j$ and $\mu_j$ be as above. Since $\int_\Omega G\mu_j \ d\mu_j < \infty$, Lemma 2.3 implies that $\mu_j \in L^{-a/2,2}(\Omega)$ and

$$\|\mu_j\|_{L^{a/2,2}(\Omega)}^2 = \int_\Omega G\mu_j \ d\mu_j.$$ (2.14)
We claim that \( \{\mu_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( L^{-\alpha/2,2}(\Omega) \). Suppose \( \ell > j \). Then by Lemma 2.3 applied to the finite positive measure \( \mu_{\ell} - \mu_j \) (which is \( \mu \) restricted to \( K_{\ell} \setminus K_j \)),

\[
\|\mu_{\ell} - \mu_j\|_{L^{-\alpha/2,2}(\Omega)}^2 = \int_{\Omega} G(\mu_{\ell} - \mu_j) d(\mu_{\ell} - \mu_j) = \int_{\Omega} G(\mu \chi_{K_\ell} \setminus K_j) \chi_{K_\ell} \setminus K_j d\mu \leq \int_{\Omega} G(\mu \chi_{\Omega} \setminus K_j) \chi_\Omega \setminus K_j d\mu \to 0
\]
as \( j \to \infty \), by the dominated convergence theorem, with dominating function \( G\mu \). Hence \( \mu_j \) converges to some \( \mu_0 \in L^{-\alpha/2,2}(\Omega) \). Let \( \varphi \in C_0^\infty(\Omega) \). For \( j \) large enough that \( \text{supp} \varphi \subseteq K_j \), we have \( \langle \mu_j, \varphi \rangle = \langle \mu, \varphi \rangle \), so \( \langle \mu, \varphi \rangle = \lim_{j \to \infty} \langle \mu_j, \varphi \rangle = \langle \mu_0, \varphi \rangle \). Hence \( \mu = \mu_0 \) in \( \mathcal{D}'(\Omega) \). Therefore \( \mu \in L^{-\alpha/2,2}(\Omega) \). Thus (ii) implies (i).

Hence (i) and (ii) are equivalent, and if either holds, letting \( j \to \infty \) in (2.14) shows that \( \int_{\Omega} G\mu \ d\mu = \|\mu\|^2_{L^{-\alpha/2,2}(\Omega)} \).

Now suppose (i) and (ii) hold. Note that \( \mu_j \in L^{-\alpha/2,2}(\mathbb{R}^n) \) since \( \int_{\Omega} G\mu_j \ d\mu_j \leq \int_{\Omega} G\mu \ d\mu < \infty \). By the same argument as above, only using Lemma 2.2 instead of Lemma 2.3, \( \{G\mu_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( L_{0}^{a/2,2}(\Omega) \) and hence \( G\mu_j \) converges in \( L_{0}^{a/2,2}(\Omega) \) to some \( h \). Since \( L_{0}^{a/2,2}(\mathbb{R}^n) \) imbeds continuously in \( L^{p^*}(\mathbb{R}^n) \), where \( p^* = 2n/(n-\alpha) \) (see, e.g., [8], Theorem 6.5), \( G\mu_j \) converges to \( h \) in \( L^{p^*}(\mathbb{R}^n) \), and hence a subsequence converges almost everywhere to \( h \). But \( G\mu_j(x) \) increases to \( G\mu(x) \) at every \( x \), hence \( h = G\mu \). Therefore \( G\mu \in L_{0}^{a/2,2}(\Omega) \). By Lemma 2.2,

\[
\|G\mu_j\|_{L_{0}^{a/2,2}(\Omega)}^2 = \int_{\Omega} G\mu_j \ d\mu_j,
\]

and taking the limit as \( j \to \infty \) and applying the monotone convergence theorem on the right side gives the identity on the left of (2.12). Hence (i) and (ii) imply (iii).

We now show that (i) and (ii) imply that \( G\mu \) is \( \alpha \)-quasicontinuous. Indeed, each \( G\mu_j \) is quasicontinuous, and \( G\mu_j \) converges in \( L_{a/2}^2(\Omega) \) norm and pointwise to \( G\mu \), so the quasicontinuity of \( G\mu \) follows from our earlier remarks.

Let us now assume that \( G\mu \in L_{0}^{a/2,2}(\Omega) \). Then, as we noted above, \( G\mu \in L^{p^*}(\Omega) \), where \( p^* = 2n/(n-\alpha) \). For any \( h \in C_0^\infty(\Omega) \), we have (see [2], p. 14)

\[
G \left( (-\Delta)^{\alpha/2} h \right) (x) = h(x), \quad x \in \mathbb{R}^n.
\]

By Fourier inversion and (1.3), we see easily that

\[
|(-\Delta)^{\alpha/2} h(x)| \leq c(1 + |x|)^{-n-\alpha}, \quad x \in \mathbb{R}^n.
\]
Therefore \((-\Delta)^{\alpha/2} h \in L^r(\mathbb{R}^n)\) for \(r > n/(n + \alpha)\). In particular, \((-\Delta)^{\alpha/2} h \in L^{q^*}(\mathbb{R}^n)\), where \(q^* = 2n/(n + \alpha)\) is the conjugate index to \(p^*\). It follows that

\[
\int_{\mathbb{R}^n \times \Omega} G(x, y) \left| (-\Delta)^{\alpha/2} h(x) \right| d\mu(y) dx = \int_{\mathbb{R}^n} \left| (-\Delta)^{\alpha/2} h(x) \right| G\mu(x) dx < \infty.
\]

Hence, by Fubini’s theorem

\[
\left| \int_{\Omega} h \, d\mu \right| = \left| \int_{\Omega} G \left( (-\Delta)^{\alpha/2} h \right) \, d\mu \right| = \left| \int_{\mathbb{R}^n} \left( -\Delta \right)^{\alpha/2} h \, G\mu \, dx \right|
\]

\[
= \langle G\mu, (-\Delta)^{\alpha/2} h \rangle \leq \|G\mu\|_{L_0^{a/2} (\mathbb{R}^n)} \|(-\Delta)^{\alpha/2} h\|_{L^{-a/2,2} (\mathbb{R}^n)}
\]

Thus, \(\mu \in L^{-a/2,2} (\Omega)\), and so (iii) implies (i).

Now suppose \(\mu_1, \mu_2\) are positive measures belonging to \(L^{-a/2,2} (\Omega)\). By (2.12),

\[
\|G(\mu_1 + \mu_2)\|_{L_0^{a/2,2} (\Omega)}^2 = \int_{\Omega} G(\mu_1 + \mu_2) \, d(\mu_1 + \mu_2).
\]

Expanding and using (2.12) to cancel the non-diagonal terms, we obtain

\[
\langle G\mu_1, G\mu_2 \rangle_{a/2} + \langle G\mu_2, G\mu_1 \rangle_{a/2} = \int_{\Omega} G\mu_1 \, d\mu_2 + \int_{\Omega} G\mu_2 \, d\mu_1, \quad (2.15)
\]

where \(\langle \cdot, \cdot \rangle_{a/2}\) denotes the inner product in \(L^{a/2,2} (\Omega)\). To obtain the first identity in (2.12) for \(\mu = \mu_1 - \mu_2\), expand both sides as above, only with \(\mu_1 - \mu_2\) in place of \(\mu_1 + \mu_2\), and use (2.15), and (2.12) for \(\mu_1\) and \(\mu_2\). The second identity in (2.12) is proved in the same way.

(B) We now have that \(G\) is an isometry from the linear combinations of positive measures in \(L^{-a/2,2} (\Omega)\) to \(L_0^{a/2,2} (\Omega)\). By Deny’s theorem ([6], Theorem II.2), the linear combinations of positive measures in \(L^{-a/2,2} (\Omega)\) are dense. Hence we can extend \(G\) to be an isometry from all of \(L^{-a/2,2} (\Omega)\) to \(L_0^{a/2,2} (\Omega)\). To prove (2.13), suppose \(\mu \in L^{-a/2,2} (\Omega)\). Then there exists a sequence \(\mu_j\) of linear combinations of positive measures converging in \(L^{-a/2,2} (\Omega)\) norm to \(\mu\). Then \(G\mu_j\) converges to \(G\mu\) in \(L_0^{a/2,2} (\Omega)\), hence

\[
\langle \mu, G\mu \rangle = \lim_{j \to \infty} \langle \mu_j, G\mu_j \rangle = \lim_{j \to \infty} \int_{\Omega} G\mu_j \, d\mu_j
\]

\[
= \lim_{j \to \infty} \|\mu_j\|_{L^{-a/2,2} (\Omega)}^2 = \|\mu\|_{L^{-a/2,2} (\Omega)}^2.
\]

The other identity in (2.13) follows now because \(G\) is an isometry.

To prove that \(G\) maps \(L^{-a/2,2} (\Omega)\) onto \(L_0^{a/2,2} (\Omega)\), suppose otherwise. Then by the Hahn–Banach theorem, there exists \(v \in L^{-a/2,2} (\Omega)\) which is not identically zero.
on $L^{\alpha/2,2}_{0}(\Omega)$, but vanishes on the image of $L^{-\alpha/2,2}(\Omega)$, hence on $G\nu$. Therefore

$$\|\nu\|_{L^{-\alpha/2,2}(\Omega)}^2 = \langle \nu, G\nu \rangle = 0,$$

so $\nu = 0$ in $L^{-\alpha/2,2}(\Omega)$, a contradiction.

\[\square\]

The last result defines $G\mu$, for $\mu \in L^{-\alpha/2,2}(\Omega)$, as an element of the Sobolev space $L^{\alpha/2,2}_{0}(\Omega)$, hence a.e. When considering $G\mu$ pointwise, we now define $G\mu \alpha$-q.e. by choosing an $\alpha$-quasicontinuous representative $\tilde{G}\mu$ of the equivalence class of $G\mu$ in $L^{\alpha/2,2}_{0}(\Omega)$, and defining $G\mu$ to be $\tilde{G}\mu$. Any other $\alpha$-quasicontinuous representative will agree $\alpha$-q.e., by [1], Ch. 6.1, so $G\mu$ is now defined as an equivalence class under the equivalence relation of equality q.e. This convention will allow us to avoid replacing $G\mu$ with $\tilde{G}\mu$ at several points later, and, more importantly, will allow us to interpret the identity $u = G(u\omega + \nu)$ as holding pointwise $\alpha$-q.e. rather than just a.e. If $\mu$ is a positive measure, $G\mu$ is defined for all $x$ as $\int_{\Omega} G(x, y) d\mu(y)$, which is finite $\alpha$-q.e. and is $\alpha$-quasicontinuous, by Lemma (2.4). If $\mu \in L^{-\alpha/2,2}(\Omega)$ is a linear combination of positive measures, $G\mu$ is defined where all of the measures in the linear combination are finite, hence $\alpha$-q.e., and $G\mu$ is $\alpha$-quasicontinuous. Hence our pointwise definition of $G\mu$ for general $\mu \in L^{-\alpha/2,2}(\Omega)$ is consistent with pointwise definitions considered previously.

**Lemma 2.5** Let $T$ be the operator in (1.6). Let $\beta > 0$. The following are equivalent:

(i) $T$ maps $L^2(\omega)$ to itself boundedly with $\|T\| \leq \beta^2$;

(ii) (1.7) holds, i.e.,

$$\|h\|_{L^2(\omega)} \leq \beta \|h\|_{L_{0}^{\alpha/2,2}(\Omega)}, \text{ for all } h \in C_0^\infty(\Omega);$$

(iii)

$$\|\tilde{u}\|_{L^2(\omega)} \leq \beta \|u\|_{L_{0}^{\alpha/2,2}(\Omega)}, \text{ for all } u \in L_{0}^{\alpha/2,2}(\Omega),$$

(2.16)

where $\tilde{u}$ denotes any quasicontinuous representative of $u$ in $L_{0}^{\alpha/2,2}(\Omega)$;

(iv)

$$\|h\omega\|_{L^{-\alpha/2,2}(\Omega)} \leq \beta \|h\|_{L^2(\omega)}, \text{ for all } h \in L^2(\omega).$$

(2.17)

**Proof** First we show the equivalence of (ii) and (iii). Suppose (ii) holds. Let $u \in L_{0}^{\alpha/2,2}(\Omega)$. There exists a sequence $\{h_n\}_{n=1}^\infty$ with $h_n \in C_0^\infty(\Omega)$ such that $\|h_n - u\|_{L_{0}^{\alpha/2,2}(\Omega)} \to 0$. Then, as noted earlier, there exists a subsequence $h_{n_k} \to \tilde{u}$ q.e. in $\mathbb{R}^n$, where $\tilde{u}$ is a quasicontinuous representative of $u$. By (ii), $h_{n_k}$ is a Cauchy sequence in $L^2(\omega)$, hence $h_{n_k} \to u_0$ in $L^2(\omega)$ for some $u_0 \in L^2(\omega)$. Hence, replacing $h_{n_k}$ with a further subsequence, we see that $h_{n_k} \to u_0$, $d\omega$ a.e., and at the same time $h_{n_k} \to \tilde{u}$ q.e.
Let $K \subset \Omega$ be compact. If $\varphi \in C^\infty_0(\Omega)$ and $\varphi \geq 1$ on $K$, then by (1.7),
\[ \omega(K) \leq \|\varphi\|^2_{L^2(\omega)} \leq \beta^2 \|\varphi\|^2_{L^{\alpha/2,2}_0(\Omega)}, \]
hence, taking the infimum over such $\varphi$,
\[ \omega(K) \leq \beta^2 \text{cap}_\alpha(K, \Omega). \quad (2.18) \]
For compact sets $K \subset \Omega$, we have
\[ \text{cap}_\alpha(K, \Omega) \preceq \text{cap}_\alpha(K, \mathbb{R}^n), \]
where the constants of equivalence depend on $\text{dist}(K, \Omega^c)$, by the same argument as at the beginning of the proof of Lemma 2.3. In particular, $\omega$ is absolutely continuous with respect to $\text{cap}_\alpha(\cdot, \mathbb{R}^n)$.

It follows that $h_{nk} \rightarrow \tilde{u} \text{ d}\omega \text{-a.e. on } K$, and consequently $u_0 = \tilde{u} \text{ d}\omega \text{-a.e. on } K$, for any compact set $K \subset \Omega$. Thus, using (1.7) with $h = h_{nk}$, and letting $n_k \rightarrow \infty$, we arrive at
\[ \|\tilde{u}\|_{L^2(\omega_K)} \leq \beta \|u\|_{L^{\alpha/2,2}_0(\Omega)}. \quad (2.19) \]
Since $K$ is an arbitrary compact subset of $\Omega$, we deduce that (2.16) holds. Hence (ii) implies (iii). The converse is trivial.

Next we show that (ii) and (iv) are equivalent. Suppose (ii) holds, $h \in L^2(\omega)$, and, to begin with, that $h \geq 0$. Let
\[ A = \{ \varphi \in C^\infty_0(\Omega) : \|\varphi\|^2_{L^{\alpha/2,2}_0(\Omega)} \leq 1 \} \]
and let $\langle \cdot, \cdot \rangle$ denote the pairing between $L^{\alpha/2,2}_0(\Omega)$ and its dual $L^{-\alpha/2,2}(\Omega)$. Then
\[ \|h\omega\|_{L^{-\alpha/2,2}(\Omega)} = \sup_{\varphi \in A} |\langle h\omega, \varphi \rangle| = \sup_{\varphi \in A} \left| \int_{\Omega} \varphi h \text{ d}\omega \right|. \]
For $\varphi \in A$, we have $\|\varphi\|_{L^2(\omega)} \leq \beta$, by (ii). Hence
\[ \|h\omega\|_{L^{-\alpha/2,2}(\Omega)} \leq \sup_{g : \|g\|_{L^2(\omega)} \leq \beta} \left| \int_{\Omega} gh \text{ d}\omega \right| = \beta \|h\|_{L^2(\omega)}. \]
The same argument holds if $h \leq 0$ (since we still have $|h\omega| = -h\omega \in L^{-\alpha/2,2}(\Omega)$, which is needed to justify the identity $\langle h\omega, \varphi \rangle = \int_{\Omega} \varphi h \text{ d}\omega$). For a general $h \in L^2(\omega)$, we have $h^+, h^- \in L^2(\omega)$, so by what we have just shown, $h^+\omega, h^-\omega \in L^{-\alpha/2,2}(\Omega)$, hence $|h\omega| \in L^{-\alpha/2,2}(\Omega)$. Then the same argument yields (2.17) in the general case.
Now suppose (iv) holds and \( h \in C_0^\infty(\Omega) \). If \( f \in L^2(\omega) \), then \( |f \omega| \in L^{-\alpha/2,2}(\Omega) \) by (iv), so

\[
\|h\|_{L^2(\omega)}^2 = \sup_{\|f\|_{L^2(\omega)} \leq 1} \left| \int_\Omega hf \, d\omega \right| = \sup_{\|f\|_{L^2(\omega)} \leq 1} |\langle f \omega, h \rangle| \\
\leq \sup_{\|f\|_{L^2(\omega)} \leq 1} \|f \omega\|_{L^{-\alpha/2,2}(\Omega)} \|h\|_{L^2(\omega)} = \beta \|h\|_{L^2(\omega)}^2,
\]

by (iv). Thus (ii) holds.

We now prove the equivalence of (i) and (iv). For either direction, we observe that since \( T \) is self-adjoint,

\[
\|T\|_{L^2(\omega) \to L^2(\omega)} = \sup_{g : \|g\|_{L^2(\omega)} \leq 1} |\langle Tg, g \rangle_{\omega}| = \sup_{g : \|g\|_{L^2(\omega)} \leq 1} \left| \int_\Omega \int_\Omega G(x, y)g(y)g(x) \, d\omega(x) \, d\omega(y) \right|.
\]

For \( g \in L^2(\omega) \), let \( g^+ = g \chi_{\{x \in \Omega : \omega(x) > 0\}} \) and \( g^- = -g \chi_{\{x \in \Omega : \omega(x) < 0\}} \).

Suppose (i) holds. Then

\[
\int_\Omega G(g^+ \omega) g^+ \, d\omega = \int_\Omega \int_\Omega G(x, y)g^+(y)g^+(x) \, d\omega(x) \, d\omega(y) \\
\leq \|T\| \|g^+\|_{L^2(\omega)}^2 < \infty,
\]

and similarly for \( g^- \). By Lemma 2.4, \( g^+ \omega, g^- \omega \in L^{-\alpha/2,2}(\mathbb{R}^n) \), so \( g\omega \in L^{-\alpha/2,2}(\mathbb{R}^n) \), and

\[
\|T\| = \sup_{\|g\|_{L^2(\omega)} \leq 1} \int_\Omega G(g\omega)g \, d\omega = \sup_{\|g\|_{L^2(\omega)} \leq 1} \|g\omega\|_{L^{-\alpha/2,2}(\Omega)}^2,
\]

(2.20)

Hence

\[
\|g\omega\|_{L^{-\alpha/2,2}(\Omega)}^2 \leq \|T\| \|g\|_{L^2(\omega)}^2 \leq \beta^2 \|g\|_{L^2(\omega)}^2,
\]

(2.21)

for all \( g \in L^2(\omega) \).

Conversely, if (iv) is assumed, and \( g \in L^2(\omega) \), then \( g^+ \omega, g^- \omega \in L^{-\alpha/2,2}(\Omega) \). Then (2.20) holds by Lemma 2.4. Then, using (2.17) again, \( \|T\| \leq \beta^2 \).

\( \square \)

**Remark** We note that the equivalence of (ii) and (iii) holds in the range \( 0 < \alpha \leq 2 \), \( \alpha < n \). If \( \alpha = 2 \) and \( n = 2 \), then we have to assume that the domain \( \Omega \) is a Green domain (with non-trivial Green’s function), since otherwise the inequality fails for constant \( h \in L^1_{0,2}(\Omega) \), unless \( \omega = 0 \). Then the same proof works if the Riesz capacity \( \text{cap}_\sigma(\cdot, \mathbb{R}^2) \) is replaced with the Bessel capacity \( \text{Cap}_\sigma(\cdot, \mathbb{R}^2) \). This case was considered for \( \sigma \in L^1_{\text{loc}}(\omega) \), without providing details, in [11].
The same observation applies to the case \( \alpha = n \) for \( n > 2 \), where Bessel capacities \( \text{Cap}_\alpha (\cdot, \mathbb{R}^n) \) can be used if the domain \( \Omega \) is \( n \)-Green. However, we consider here only the case \( \alpha < n \).

For \( v \in L^{-\alpha/2,2} (\Omega) \), we say that \( u \) is a weak solution of equation (1.1) if \( u \in L^{\alpha/2,2}_{0} (\Omega) \) and

\[
\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} \varphi \, dx = \int_{\mathbb{R}^n} \tilde{u} \tilde{\varphi} \, d\omega + \langle v, \varphi \rangle, \tag{2.22}
\]

for all \( \varphi \in L^{\alpha/2,2}_{0} (\Omega) \), where \( \tilde{u} \) and \( \tilde{\varphi} \) are quasicontinuous representatives of \( u \) and \( \varphi \), respectively, in \( L^{\alpha/2,2}_{0} (\Omega) \) and \( \langle \cdot, \cdot \rangle \) is the pairing between \( L^{-\alpha/2,2} (\Omega) \) and \( L^{\alpha/2,2}_{0} (\Omega) \). A standard application of the Lax–Milgram Theorem shows that if \( \| T \| < 1 \), then for each \( v \in L^{-\alpha/2,2} (\Omega) \), there exists a unique weak solution \( u \) of (1.1), as follows. Following the formulation of the Lax–Milgram Theorem in [9], §6.2.1, let \( H = L^{\alpha/2,2}_{0} (\Omega) \) and define the bilinear form

\[
B (u, v) = \int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} v \, dx - \int_{\mathbb{R}^n} \tilde{u} \tilde{v} \, d\omega \tag{2.23}
\]

on \( H \times H \). Let \( \| T \|^{1/2} = \beta < 1 \). Then (2.16) holds, which shows that \( \int_{\mathbb{R}^n} \tilde{u} \tilde{\varphi} \, d\omega \) is well defined, i.e., if \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are quasicontinuous representatives of the same equivalence class in \( L^{\alpha/2,2}_{0} (\Omega) \), then \( \tilde{u}_1 = \tilde{u}_2 \) \( \omega \)-a.e. By the Cauchy–Schwarz inequality,

\[
| B (u, v) | \leq \| u \|_{L^{\alpha/2,2}_{0} (\Omega)} \| v \|_{L^{\alpha/2,2}_{0} (\Omega)} + \| \tilde{u} \|_{L^2 (\omega)} \| \tilde{v} \|_{L^2 (\omega)} \leq (1 + \beta^2) \| u \|_{L^{\alpha/2,2}_{0} (\Omega)} \| v \|_{L^{\alpha/2,2}_{0} (\Omega)}.
\]

Equation (2.16) also implies the coercivity of \( B \):

\[
B (u, u) = \| u \|_{L^{\alpha/2,2}_{0} (\Omega)}^2 - \| \tilde{u} \|_{L^2 (\omega)}^2 \geq (1 - \beta^2) \| u \|_{L^{\alpha/2,2}_{0} (\Omega)}^2.
\]

Then the Lax–Milgram Theorem gives, for each \( v \in L^{-\alpha/2,2} (\Omega) \), the existence of a unique \( u \in L^{\alpha/2,2}_{0} (\Omega) \) such that \( B (u, \varphi) = \langle v, \varphi \rangle \) for all \( \varphi \in L^{\alpha/2,2}_{0} (\Omega) \), which is (2.22).

**Proof of Proposition 1.1 (A)** First suppose \( v \in L^{-\alpha/2,2}_{0} (\Omega) \) is a positive measure. Then \( G v \in L^{\alpha/2,2}_{0} (\Omega) \) and \( G v \) is \( \alpha \)-quasicontinuous, by Lemma 2.4. By (2.16) with \( \beta = \| T \| \) and Lemma 2.5, it follows that \( G v \in L^2 (\omega) \). Since \( \| T \| < 1 \), we have \( u_0 \equiv G v = (I - T)^{-1} G v \in L^2 (\omega) \). Then by (2.17) with \( \beta = \| T \| \), we have \( u_0 \omega \in L^{-\alpha/2,2} (\Omega) \). By Lemma 2.4 again, we have \( G (u_0 \omega) \in L^{\alpha/2,2}_{0} (\Omega) \) and \( G (u_0 \omega) \) is quasicontinuous. Recall that \( u_0 = G (u_0 \omega) + G v \) holds pointwise at all points, if we allow infinite values. Since \( G (u_0 \omega) \) and \( G v \) belong to \( L^{\alpha/2,2}_{0} (\Omega) \) and are quasicontinuous, we obtain that \( u_0 \in L^{\alpha/2,2}_{0} (\Omega) \) and \( u_0 \) is quasicontinuous.
Recall that by Sobolev imbedding (as noted in the proof of Lemma 2.4), it follows that $u_0 \in L^{p^*}(\Omega)$, where $p^* = 2n/(n-\alpha)$. Hence $u_0 < \infty$ a.e., so $u_0$ is a pointwise solution of $u_0 = G(u_0\omega) + Gv$. Now $\|Gv\|_{L_0^{a/2,2}(\Omega)} = \|v\|_{L^{-a/2,2}(\Omega)}$ by Lemma 2.4, and, following the estimates in the above results,

$$\|G(u_0\omega)\|_{L_0^{a/2,2}(\Omega)} = \|u_0\omega\|_{L^{-a/2,2}(\Omega)} \leq \|T\|^{1/2}\|u_0\|_{L^{2}(\omega)}$$

$$= \|T\|^{1/2}\|(I - T)^{-1}Gv\|_{L^{2}(\omega)} \leq \frac{\|T\|^{1/2}}{1 - \|T\|}\|Gv\|_{L^{2}(\omega)}$$

$$\leq \frac{\|T\|}{1 - \|T\|}\|Gv\|_{L_0^{a/2,2}(\Omega)} = \frac{\|T\|}{1 - \|T\|}\|v\|_{L^{-a/2,2}(\Omega)}.$$  

Hence

$$\|u_0\|_{L_0^{a/2,2}(\Omega)} = \|G(u_0\omega) + Gv\|_{L_0^{a/2,2}(\Omega)}$$

$$\leq \left(\frac{\|T\|}{1 - \|T\|} + 1\right)\|v\|_{L^{-a/2,2}(\Omega)}$$

$$= \left(\frac{1}{1 - \|T\|}\right)\|v\|_{L^{-a/2,2}(\Omega)}.$$ 

We define $Gv$ by linearity when $v$ is a linear combination of positive measures in $L^{-a/2,2}(\Omega)$; then $Gv$ is defined q.e. (in fact whenever each term in the sum defining $Gv$ is finite) and $u_0 = Gv \in L_0^{a/2,2}(\Omega)$ is quasicontinuous. By Lemma 2.4, we still obtain (1.12), by the same steps, and the equation $u_0 = G(u_0\omega) + Gv$ holds q.e. and as elements of $L_0^{a/2,2}(\Omega)$.

Since the linear combinations of positive measures are dense in $L^{-a/2,2}(\Omega)$, we can extend the map $G$ to a bounded map (with the same bound) from $L^{-a/2,2}(\Omega)$ into $L_0^{a/2,2}(\Omega)$. As for $G$, for $\mu \in L^{-a/2,2}(\Omega)$, we further define $G\mu$ pointwise q.e. to be a quasicontinuous representative of its equivalence class in $L_0^{a/2,2}(\Omega)$. To show that $u_0 \equiv Gv$ satisfies $u_0 = G(u_0\omega) + G(v)$ for a general $v \in L^{-a/2,2}(\Omega)$, let $v_j$ be a sequence of linear combinations of positive measures converging to $v$ in $L^{-a/2,2}(\Omega)$. Let $u_{0,j} = G(v_j)$. Then

$$u_{0,j} = G(u_{0,j}\omega) + Gv_j,$$  

for each $j$. By continuity of $G$ and $G$, we have that $Gv_j$ converges to $Gv$, and $u_{0,j} = Gv_j$ converges to $Gv = u_0$ in $L_0^{a/2,2}(\Omega)$. By (2.16), $u_{0,j}$ converges to $u_0$ in $L^2(\omega)$, since $u_{0,j} = Gv_j$ and $u_0 = Gv$ are quasicontinuous (without this convention we would need to replace them with quasicontinuous representatives at this point). Then by (2.17), $u_{0,j}\omega$ converges to $u_0\omega$ in $L^{-a/2,2}(\Omega)$. By the boundedness of $G$, then, $G(u_{0,j}\omega)$ converges to $G(u_0\omega)$ in $L_0^{a/2,2}(\Omega)$. Hence taking the limit as $j \to \infty$ in (2.24), we see that $u_0 = G(u_0\omega) + G(v)$ holds in the sense of equality in $L_0^{a/2,2}(\Omega)$, hence a.e., and therefore q.e. since both sides of the equation are quasicontinuous.
We now show that \( u_0 \) is the weak solution of (1.1). We claim that for any \( \mu \in L^{-\alpha/2,2}(\Omega) \) and any \( \varphi \in L^0_{\alpha/2,2}(\Omega) \),

\[
\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} G \mu(-\Delta)^{\alpha/4} \varphi \, dx = \langle \mu, \varphi \rangle. \tag{2.25}
\]

First suppose \( \mu \in L^{-\alpha/2,2}(\Omega) \) is a finite positive measure on \( \Omega \) and \( \varphi \in C^\infty_0(\Omega) \). Let \( \mu' \in L^{-\alpha/2,2}(\Omega) \) be as in (2.5)-(2.7). Then \( I_{\alpha/2} \mu \in L^2(\mathbb{R}^n) \), and hence \((-\Delta)^{\alpha/4} I_{\alpha} \mu = I_{\alpha/2} \mu \) by Fourier transform, and similarly for \( \mu' \). By (2.7),

\[
\int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} G \mu(-\Delta)^{\alpha/4} \varphi \, dx = \int_{\mathbb{R}^n} I_{\alpha/2}(-\Delta)^{\alpha/4} \varphi \, dx \\
= \int_{\mathbb{R}^n} I_{\alpha/2}(-\Delta)^{\alpha/4} \varphi \, d\mu \\
- \int_{\mathbb{R}^n} I_{\alpha/2}(-\Delta)^{\alpha/4} \varphi \, d\mu' \\
= \int_{\mathbb{R}^n} \varphi \, d\mu - \int_{\mathbb{R}^n} \varphi \, d\mu' \\
= \int_{\mathbb{R}^n} \varphi \, d\mu = \langle \mu, \varphi \rangle,
\]

where the intermediate steps are justified via the Fourier transform, and \( \int_{\mathbb{R}^n} \varphi \, d\mu' = 0 \) because \( \mu' \) is supported in \( \Omega' \). Since

\[
\left| \int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} G \mu(-\Delta)^{\alpha/4} \varphi \, dx \right| \leq \| G \mu \|_{L^2_{\alpha/2,2}(\Omega)} \| \varphi \|_{L^2_{\alpha/2,2}(\Omega)},
\]

by the Cauchy–Schwarz inequality, we can extend (2.25) to all \( \varphi \in L^{\alpha/2,2}(\Omega) \) by applying (2.25) to a sequence \( \{ \varphi_j \}_{j=1}^\infty \) of elements of \( C^\infty_0(\Omega) \) converging to \( \varphi \) in \( L^{\alpha/2,2}(\Omega) \) and letting \( j \to \infty \).

Next, suppose \( \mu \in L^{-\alpha/2,2}(\Omega) \) is a positive measure, not necessarily finite. Define \( \{ \mu_j \}_{j=1}^\infty \) as in the proof of Lemma 2.4. In that proof, we saw that each \( \mu_j \) is finite, and \( \mu_j \to \mu \) in the norm on \( L^{-\alpha/2,2}(\Omega) \). Hence \( G \mu_j \) converges to \( G \mu \) in \( L^0_{\alpha/2,2}(\Omega) \). Therefore, applying (2.25) to \( \mu_j \) and taking the limit, we obtain (2.25) for \( \mu \) and all \( \varphi \in L^{\alpha/2,2}(\Omega) \). This result then extends to linear combinations of positive measures in \( L^{-\alpha/2,2}(\Omega) \). Then, by Deny’s Theorem again, such linear combinations are dense in \( L^{-\alpha/2,2}(\Omega) \), so another passage to the limit implies (2.25) for all \( v \in L^{-\alpha/2,2}(\Omega) \) and all \( \varphi \in L^{\alpha/2,2}(\Omega) \).

Now for \( v \in L^{-\alpha/2,2}(\Omega) \), we have that \( u_0 = \mathcal{G} v \in L^0_{\alpha/2,2}(\Omega) \) is quasicontinuous. Then \( u_0 \in L^2(\omega) \), by Lemma 2.5 (iii). Then by Lemma 2.5 (iv), \( u_0\omega \in L^{-\alpha/2,2}(\Omega) \). We also have that \( u_0 = G(u_0\omega + v) \) as elements of \( L^0_{\alpha/2,2}(\Omega) \). By (2.25),
\[ \int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} u_0 (-\Delta)^{\alpha/4} \varphi \, dx = \int_{\mathbb{R}^n} (-\Delta)^{\alpha/4} G(u_0 \omega + v) (-\Delta)^{\alpha/4} \varphi \, dx = \langle u_0 \omega + v, \varphi \rangle. \]

Since \( u_0 \in L^2(\omega) \), we have \( u_0^+, u_0^- \in L^2(\omega) \). By Lemma 2.5 (iv), we obtain \( u_0^+ \omega, u_0^- \in L^{-\alpha/2,2}(\Omega) \), hence \( |u_0 \omega| \in L^{-\alpha/2,2}(\Omega) \). This allows us to conclude that

\[ \langle u_0 \omega, \varphi \rangle = \int_{\Omega} u_0 \tilde{\varphi} \, d\omega = \int_{\Omega} \tilde{u}_0 \tilde{\varphi} \, d\omega \]

for all \( \varphi \in L^{\alpha/2,2}(\Omega) \), because \( \tilde{u}_0 = u_0 \) since \( u_0 \) is quasicontinuous. Therefore \( u_0 \) is the weak solution of (1.1).

(B) Now suppose that (1.5) has a non-negative solution \( u \) for some non-trivial positive measure \( \nu \) (i.e., \( \nu(\Omega) > 0 \)). Since \( \nu \) is non-trivial, there exists a compact subset \( K \) of \( \Omega \) such that \( \nu(K) > 0 \), hence by (2.11), \( G \nu(x) \geq G \nu_K(x) > 0 \). Since \( Tu \geq 0 \), we get that \( u \geq G \nu > 0 \) on \( \Omega \). So \( 0 < u < \infty \) a.e. on \( \Omega \) and satisfies \( u = Tu + G \nu \), hence \( Tu \leq u \) on \( \Omega \). Schur’s Lemma for integral operators implies that \( \|T\| \leq 1 \).

Now suppose also that \( u = G(u \omega + \nu) \in L^{\alpha/2,2}(\Omega) \). Then \( \int_{\Omega} G(u \omega + \nu) (u \, d\omega + d\nu) < \infty \), by Lemma 2.4. Consequently \( \int_{\Omega} G \nu \, d\nu < \infty \). Hence \( \nu \in L^{-\alpha/2,2}(\Omega) \), by Lemma 2.4 again.

\[ \square \]

## 3 Proofs of Theorems 1.2, 1.3, and 1.4

For the remainder of this paper, we assume that \( \Omega \) is a bounded \( C^{1,1} \) domain. For such domains, the following estimates for \( G \) are obtained in [16] and [3]:

\[ G(x, y) \approx \frac{\delta(x)^{\alpha/2} \delta(y)^{\alpha/2}}{|x - y|^{n-\alpha}(|x - y| + \delta(x) + \delta(y))^{\alpha}}, \]  

(3.1)

where “\( \approx \)” means that the ratio of the left and right sides is bounded above and below by positive constants depending only on \( \alpha \) and \( \Omega \). A useful consequence of (3.1) is

\[ G1(x) \approx \delta(x)^{\alpha/2}. \]  

(3.2)

The lower bound follows because the quantities in the denominator of the right side of (3.1) are bounded above on a bounded domain, so \( G1(x) \geq c \delta(x)^{\alpha/2} \int_{\Omega} \delta(y)^{\alpha/2} \, dy \).

The upper bound is similar: by (3.1), we have \( G(x, y) \leq c \delta(x)^{\alpha/2} |x - y|^{-n+\alpha/2}; \) then \( \sup_{x \in \Omega} \int_{\Omega} |x - y|^{-n+\alpha/2} \, dy < \infty \) since \( \Omega \) is bounded.

As for the classical Laplacian considered in [11], our results are based on the estimates in [12] for quasi-metric kernels. We refer to [12] (or the summary in §3
of [11]) for the definitions and details. The equivalence (3.1) shows that for \( m(x) = \delta(x)^{a/2} \), the kernel \( K(x, y) = \frac{G(x, y)}{m(x)m(y)} \) is a quasi-metric kernel. Let \( v_0 = \sum_{j=0}^{\infty} T^j m \). Then Corollary 3.5 of [12] states that there exists \( c > 0 \) depending only on the quasi-metric constant of \( K \) such that

\[
v_0 \geq me^{Tm/m}.
\] (3.3)

If we assume in addition that \( \|T\| < 1 \), the same result states that there exists \( C > 0 \) depending only on \( \|T\| \) and the quasi-metric constant of \( K \) such that

\[
v_0 \leq me^{CTm/m}.
\] (3.4)

**Proof of Theorem 1.2**

(i) First suppose that \( \|T\| < 1 \). Since \( \nu \in L^{-a/2,2}(\Omega) \) (see the remarks in the Introduction before the statement of Theorem 1.2), by Proposition 1.1 we have \( u_0 = G\nu \in L^{a/2,2}(\Omega) \) and \( u_0 \) is a solution of \( u = G(u\omega) + G1 \). Because of (3.2) we obtain \( u_0 = \sum_{j=0}^{\infty} T^j G1 \approx \sum_{j=0}^{\infty} T^j m = v_0 \), so the estimate (1.13) follows from (3.4).

(ii) As in [11], p. 1405, \( u_0 \) is the minimal positive solution of (1.5). Hence \( u \geq u_0 \). Since \( u_0 \approx v_0 \), (1.14) follows from (3.3).

\[\square\]

Turning to equation (1.2), we first recall that for \( v = u - 1 \), equation (1.2) becomes \((-\Delta)^{a/2} v = \omega v + \omega \) on \( \Omega \) with \( v = 0 \) on \( \Omega^c \), which is equation (1.1) with \( v \) replaced by \( \omega \). Therefore if we assume \( \omega \in L^{-a/2,2}(\Omega) \) and \( \|T\| < 1 \), we obtain \( v_1 = G\omega \in L^{a/2,2}(\Omega) \), and \( v_1 \) is the unique weak solution guaranteed by the Lax–Milgram Theorem. However, the Lax–Milgram Theorem applies only when \( \omega \in L^{-a/2,2}(\Omega) \), whereas the integral formulation (1.15) allows us to consider more general \( \omega \). In Remark 3.1 we give an example where the integral equation holds a.e., but \( \omega \notin L^{-a/2,2}(\Omega) \).

As in the case \( \alpha = 2 \) in [11], the functions \( u_0 = G1 \) and \( u_1 = 1 + G\omega \) are related by the identity

\[
\int_{\Omega} u_1 \, dx = \int_{\Omega} 1 \, dx + \int_{\Omega} \int_{\Omega} G(x, y) \, d\omega(y) \, dx = |\Omega| + \int_{\Omega} u_0 \, d\omega,
\] (3.5)

using the symmetry of \( G \) and Fubini’s theorem.

We make some remarks about the Poisson kernel. For bounded domains with the outer cone property (in particular, bounded Lipschitz domains; see [2], pp. 16-17, or [3], p. 468) the Poisson kernel of order \( \alpha \) satisfies \( \int_{\Omega^c} P(x, y) \, dy = 1 \) for all \( x \in \Omega \) and can be written as

\[
P(x, y) = P^{(\alpha)}(x, y) = A_{\alpha,n} \int_{\Omega} \frac{G(x, z)}{|y - z|^{n+\alpha}} \, dz,
\]
for \( x \in \Omega \) and \( y \in \Omega^c \), where \( A_{\alpha,n} \) is a constant. For \( x \in \Omega \), define
\[
\phi(x) = A_{\alpha,n} \int_{\Omega^c} \frac{1}{|x - z|^{n+\alpha}} \, dz. \tag{3.6}
\]

Hence
\[
G\phi(x) = \int_{\Omega^c} P(x, y) \, dy = 1 \text{ for all } x \in \Omega. \tag{3.7}
\]

We note that there exist positive constants \( c(\alpha, \Omega) \) and \( C(\alpha, n) \) such that
\[
c(\alpha, \Omega) \leq \phi(x) \leq C(\alpha, n) \alpha^{-1} \frac{1}{\delta(x)^{\alpha}}, \text{ for all } x \in \Omega, \tag{3.8}
\]
where \( \delta(x) \) is the distance from \( x \) to \( \Omega^c \). The upper bound in (3.8) is elementary, whereas the lower bound follows because \( \Omega^c \) domains (in fact NTA domains) have the property that there are constants \( c > 0 \) and \( r_0 > 0 \) such that
\[
|B(y, r) \cap \Omega^c| \geq c|B(y, r)|, \text{ for all } y \in \partial\Omega \text{ and } 0 < r < r_0.
\]

We will also use the well-known equivalence ( [3], Theorem 1.5)
\[
P(x, z) \approx \frac{\delta(x)^{\alpha/2}}{\delta(z)^{\alpha/2}(1 + \delta(z))^{\alpha/2}|x - z|^n}, \tag{3.9}
\]
where here \( \delta(z) = \text{dist}(z, \partial\Omega) \), with equivalence constants independent of \( x \) and \( z \).

**Proof of Theorem 1.3** First suppose \( 0 < \alpha < 1 \). By (3.7), \( G\phi = \chi_\Omega \). By (3.8), we have
\[
\int_\Omega G\phi(x) \phi(x) \, dx = \int_\Omega \phi(x) \, dx \leq C(\alpha, n) \int_\Omega \delta(x)^{-\alpha} \, dx < +\infty
\]
for \( 0 < \alpha < 1 \), for a broad class of domains \( \Omega \) (e.g., Ahlfors regular domains; in particular, bounded Lipschitz domains). By Lemma 2.4, \( \chi_\Omega \in L^{2/\alpha, 2}(\Omega) \) for all \( 0 < \alpha < 1 \). Then by (2.16), \( \chi_\Omega \in L^2(\omega) \), or \( \omega(\Omega) < \infty \); i.e., \( \omega \) is a finite measure. By Theorem 1.1 and (2.16), \( u_0 \in L^2(\omega) \), so by the finiteness of \( \omega \), we have \( u_0 \in L^1(\omega) \). Thus by (3.5), \( u_1 \in L^1(\Omega, dx) \).

Notice that \( G\omega = T(\chi_\Omega) \), hence
\[
\int_\Omega G\omega \, d\omega \leq (\omega(\Omega))^{1/2} \|T(\chi_\Omega)\|_{L^2(\omega)} \leq \beta^2 \omega(\Omega) < \infty.
\]

Therefore \( \omega \in L^{-\alpha/2, 2}(\Omega) \), by Lemma 2.4. By Theorem 1.1, \( u_1 - 1 = G\omega \in L^{\alpha/2, 2}(\Omega) \).

Now suppose \( 1 \leq \alpha < 2 \). By (3.7), for any non-negative Borel measure \( \nu \) on \( \Omega \), we have
\[
\nu(\Omega) = \int_\Omega G\nu(x) \phi(x) \, dx,
\]
by Fubini’s theorem. Applying this fact with \( dv = u_0 d\omega \), where \( u_0 \) is defined by (1.10) and satisfies \( u_0 = G(u_0 \omega) + G1 \), yields

\[
\int_{\Omega} u_0 d\omega = \int_{\Omega} G(u_0 \omega)\phi \, dx = \int_{\Omega} u_0 \phi \, dx - \int_{\Omega} G1 \cdot \phi \, dx. \tag{3.10}
\]

By (3.2) and (3.8), \( G1 \cdot \phi \approx \delta^{-\alpha/2} \), and hence \( \int_{\Omega} G1 \cdot \phi \, dx < \infty \). Our goal is to show that \( \int_{\Omega} u_0 \phi \, dx < \infty \) for \( \|T\| \) sufficiently small.

Recall that if \( \|T\| < 1 \), then for \( m = \delta^{\alpha/2} \), we have

\[
u_0 \leq C_1 me^{CTm/m},
\]

by Theorem 1.2, where \( C = C(\Omega, \alpha, \|T\|) \). Choose and fix \( p > \frac{2+\alpha}{2-\alpha} \), which guarantees that \( \frac{\alpha(p+1)}{2(p-1)} < 1 \). Let \( C_2 = C(\Omega, \alpha, 1/p) \); that is, \( C_2 \) is the constant \( C \) when \( \|T\| = 1/p \). Let \( c \) be the constant from (1.14); note that \( c \leq C_2 \) (e.g., by (1.4) in [12]). Define

\[
\gamma = \frac{c}{C_2 p}.
\]

Note that \( \gamma \leq 1/p \). Now suppose \( \|T\| < \gamma \). Then \( u_0 \leq C_1 me^{C_2 Tm/m} \). Hence, by Hölder’s inequality (using (3.8))

\[
\int_{\Omega} u_0 \phi \, dx \leq C(\alpha, n) \int_{\Omega} \frac{u_0}{m^2} \, dx \leq C_1 C(\alpha, n) \int_{\Omega} \frac{1}{m} e^{C_2 Tm/m} \, dx
\]

\[
\leq C_1 C(\alpha, n) \left( \int_{\Omega} m^{-\frac{n+1}{p-1}} \, dx \right)^{1-\frac{1}{p}} \left( \int_{\Omega} me^{C_2 p Tm/m} \, dx \right)^{1/p}.
\]

Since \( m^{-\frac{n+1}{p-1}} = \delta^{-\frac{\alpha(p+1)}{2(p-1)}} \), the first integral on the previous line is finite. To show that the second integral is finite, let \( \omega_1 = \gamma^{-1} \omega \). We apply Theorem 1.2 with \( \omega_1 \) in place of \( \omega \), but with \( G \) and \( m \) unchanged. Define \( T_1 = \gamma^{-1} T \); note that \( T_1 f(x) = G(\omega_1 f) \), and

\[
\|T_1\|_{L^2(\omega_1) \to L^2(\omega_1)} = \|T_1\|_{L^2(\omega_2) \to L^2(\omega_2)} = \gamma^{-1} \|T\|_{L^2(\omega) \to L^2(\omega_2)} < 1.
\]

Define \( u_0^* = \sum_{j=0}^{\infty} T_1^j G1 \). By Theorem 1.2, \( c_1 me^{CT_1 m/m} \leq u_0^* \) and \( u_0^* \in L^{\alpha/2,2}(\Omega) \subseteq L^{p^*}(\Omega, \, dx) \) (since \( \Omega \) is bounded), for \( p^* = 2n/(n-\alpha) \). But \( C_2 p Tm = cT_1 m \), so

\[
\int_{\Omega} me^{C_2 p Tm/m} \, dx = \int_{\Omega} me^{C_1 T_1 m/m} \, dx \leq c_1^{-1} \int_{\Omega} u_0^* \, dx < \infty.
\]

We have shown \( \int_{\Omega} u_0 \phi \, dx < \infty \), hence \( u_0 \in L^1(\omega) \), by (3.10). By (3.5), we have \( u_1 \in L^1(\Omega, \, dx) \).
We now turn to the pointwise bounds (1.17) and (1.18). Their proofs are similar to the proofs of (1.12) and (1.14) in [11]. By the same argument as on p. 1413 of [11], using (3.7) we have

\[ u_1(x) = \int_{\Omega^c} \sum_{j=0}^{\infty} T^j(P(\cdot, z))(x) \, dz. \]  

(3.11)

Define the quasi-metric

\[ d(x, y) = |x - y|^{n - \alpha} \left[ |x - y|^2 + \delta(x)^2 + \delta(y)^2 \right]^{\alpha/2}, \quad x, y \in \mathbb{R}^n. \]

Note that for \( x \in \Omega^c \) and \( z \in \Omega^c \), we have \( d(x, z) \approx |x - z|^n \), because \( \delta(x), \delta(z) \leq |x - z| \). Momentarily fixing \( z \in \Omega^c \), let \( m(x) = P(x, z) \). By (3.1) and (3.9), it follows that

\[ K(x, y) \equiv G(x, y) m(x) \approx c(z) \frac{d(x, z)d(y, z)}{d(x, y)}, \]

where \( c(z) = (1 + \delta(z))^\alpha \). From a lemma due to Hansen and Netuka ( [15]), quoted as Lemma 3.4 in [11], it follows that \( K(x, y) \) is a quasi-metric kernel on \( \Omega^c \) with quasi-metric constant independent of \( z \). By Corollary 3.5 in [12] (essentially (3.4)),

\[ \sum_{j=0}^{\infty} T^j(P(\cdot, z))(x) \leq C_3P(x, z)e^{C_4 \int_{\Omega^c} G(x, y) P(y, z) \, d\omega(y)}, \]

with constants independent of \( z \). Using (3.11), then, we obtain (1.17). We also have the lower estimate (1.18) for \( u_1 \), by Theorem 1.2 in [14], which gives an estimate in the opposite direction in the preceding inequality, with sharper constants than Corollary 3.5 in [12]. Since \( u_1 \) is the minimal positive solution of \( u = 1 + G(u\omega) \), we obtain (1.18) for \( u \) as well. \( \square \)

**Proof of Theorem 1.4** Let \( d\omega = \phi \, dx \), where \( \phi \) is defined by (3.6). By (3.7),

\[ T^j1(x) = G\phi(x) = 1 \text{ for all } x \in \Omega. \]

Hence \( T^j1 = 1 \) for all \( j = 1, 2, \ldots \), so \( u_1 = 1 + \sum_{j=1}^{\infty} T^jG\phi \equiv +\infty \) on \( \Omega \).

It follows (see, e.g., [8], Proposition 3.4 and the proof of Lemma 5.1) from Plancherel’s theorem that for, say, \( u \in C_0^\infty(\Omega) \),

\[
\|u\|_{L^{0/2,2}_0(\Omega)}^2 = \|u\|_{L^{0/2,2}_0(\mathbb{R}^n)}^2 = \frac{A_{\alpha,n}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} \, dx \, dy
\]

\[
= \frac{A_{\alpha,n}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+\alpha}} \, dx \, dy
\]

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\[ + \int_{\Omega} u^2(\phi(x)) \, dx, \]

where \( A_{\alpha,n} \) is the constant from (3.8). The following version of Hardy’s inequality holds for bounded Lipschitz domains if \( 1 < \alpha < 2 \) ([7], Theorem 1.1):

\[ \int_{\Omega} u^2 \delta^{-\alpha} \, dx \leq C_1(\alpha, \Omega) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n + \alpha} \, dx \, dy, \quad \text{for all } u \in C_0^\infty(\Omega). \]

Recalling (3.8), we obtain

\[ \int_{\Omega} u^2 \phi \, dx \leq \frac{c_n A_{\alpha,n}}{\alpha} \int_{\Omega} u^2 \delta^{-\alpha} \, dx \leq \frac{c_n C_1(\alpha, \Omega) A_{\alpha,n}}{\alpha} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^n + \alpha} \, dx \, dy. \]

Therefore

\[ \left( 1 + \frac{\alpha}{2c_n C_1(\alpha, \Omega)} \right) \int_{\Omega} u^2 \phi \, dx \leq \|u\|^2_{L^{n/2,2}_0(\Omega)}, \quad \text{for all } u \in C_0^\infty(\Omega). \]

Hence, for \( d\omega = \phi \, dx \), (1.7) holds with \( \beta = \left( 1 + \frac{\alpha}{2c_n C_1(\alpha, \Omega)} \right)^{-1/2} < 1 \). By Lemma 2.5, we have \( \|T\| \leq \beta \).

By [10], p. 115, for convex domains \( \Omega \) the constant \( C_1(\alpha, \Omega) \) depends only on \( \alpha \) and the dimension \( n \), and the value

\[ C_1(\alpha, \Omega) = \frac{\alpha \Gamma\left( \frac{n+\alpha}{2} \right)}{2^{\alpha-2} \pi^{\frac{n-2}{2}} \Gamma\left( 1 - \frac{\alpha}{2} \right) \Gamma^2\left( \frac{\alpha+1}{2} \right)} \]

is sharp. In that case, as \( \alpha \to 2^- \), we have \( C_1(\alpha, \Omega) \to 0 \), so in the above proof, \( \beta \to 0 \). Consequently, the value of \( \gamma \) in Theorem 1.3 must converge to 0 as \( \alpha \to 2^- \).

**Remark 3.1** We observe that, for \( d\omega = \gamma \phi \, dx \), where \( \phi \) is defined by (3.6) as above and \( \gamma \in (0, 1) \), we have \( u_1(x) = 1/(1 - \gamma) \) in \( \Omega \), and \( u_1(x) = 1 \) in \( \Omega^c \). Then \( u_1 = G(u_1 \omega) + 1 \) in \( \mathbb{R}^n \), and \((-\Delta)^{\alpha/2} u_1 = \omega u_1 \) in \( D'(\Omega) \). However, in contrast to the case \( 0 < \alpha < 1 \), we have \( \omega \notin L^{-\alpha/2,2}(\Omega) \) and \( u_1 - 1 = G\omega \notin L^{\alpha/2,2}_0(\Omega) \) for all \( 1 \leq \alpha < 2 \); in fact, \( u_1 \) obviously does not have a quasicontinuous representative in \( \mathbb{R}^n \) since \( \text{cap}_\alpha(\partial \Omega) > 0 \) in this case.

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