A Linear Response relation for perturbations in compact stars and the classical Einstein-Langevin formalism

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We give a linear response relation for perturbations in relativistic stars and classify them in terms of implicit and induced perturbations. The focus in this article is on the induced perturbations which may arise due to internal sources in dense matter compact objects. Based on the linear response relation, an Einstein Langevin equation is given and its solutions for a simple case are obtained in closed analytical form as a first exercise. Our results show how perturbations in relativistic stars can arise due to a cumulative effect of mesoscopic scale fluctuations in the dense matter fluids, which otherwise seem to be screened off at hydrodynamic scales due to averaging. We discuss the relevance of such a framework and its potential towards building up a theme of research in asteroseismology using a first principles approach.

I. INTRODUCTION

The study of perturbations in relativistic stars is of importance for stability issues and equilibrium properties [1, 2], gravitational collapse scenario [3, 4] and asteroseismology [5–7] which give insights to the interiors and dynamical properties of these compact objects. The observational interests of the astronomers, find connections with the available and well established theoretical framework that is understood in terms of various signals which can be received/filtered out from the electromagnetic spectrum and the gravitational radiation, up to possible detectable precision. The theoretical base for asteroseismology was formed by developing the framework of linearized first order perturbations of the Einstein’s Equations [1, 8]. For the compact objects the matter fields are modelled by fluid stress-energy tensors in the strong gravity regions. The perturbed Einstein’s equations are written as

$$\delta G_{ab}(x)[\delta h] = 8\pi \delta T_{ab}(x)[h, \xi]$$

(1)

where the background metric is $g_{ab}$ and its perturbations are given by $h_{ab}$, for the fluid $\xi_a$ denotes the displacement vector of the fluid trajectories. The fundamental variables describing the perturbed relativistic star are the Lagrangian displacement vector $\xi_a$ and the metric perturbations $h_{ab}$. The perturbing agency whose details are insignificant and ignored often lies outside the configuration of interest, plays a role to set up these oscillations in the system. Of interest is the mode analysis with details of oscillations of the perturbed compact object, in order to characterize its internal structure. This has gained more importance due to recent achievements in Gravitational wave detection [9, 10]. We would term these as implicit oscillations in a system. The other possibilities are internal dynamics like differential rotation in fluid and chemical processes in hot dense matter which are active and may be responsible for setting up a different set of perturbations due to large scale kinematics.

However the sources of perturbations that may arise inside the compact objects, have often been ignored in the theoretical framework and hence also for the observational consequences. Such sources are necessarily indirect for the models that have been built up, or may be minute, compared to the macroscopic scales of observational interest till date. The modelling for such purposes is done using suitable gauges and approximations for the perturbed Einstein’s equations. For non-radial perturbations often Cowling approximation is used, which mathematically simplifies the equations for obtaining solutions.

If one is to consider any sources which are interior to the relativistic star, and can cause perturbations similar to any external agency, the model described by perturbed Einstein’s equations should include them. It is with this motivation that we have recently proposed a classical Einstein Langevin equation for applications to relativistic stars in [11, 12]. In the present article we make an attempt to elaborate further on this and give an improved version of the classical Einstein-Langevin equation. The details of the Langevin noise are open to phenomenological modelling, based on first principles which should capture the essential statistical features of the dense matter in a mesoscopic description of the system.

Our interest lies in exploring the dense matter inside these objects and its nature, in more detail than the idealized hydrodynamic macroscopic models which are currently in use. Though equation of state of dense matter in compact objects is of active interest in research [13], it does not yet have a precise theoretical way of deciphering its inherent nature completely. Hence it leaves scope to find out new constructs which may enhance the existing models, also at a basic theoretical level. Whether it is only the equation of state of matter that suffices to understand the dense matter, or there can be more features that one can explore, can be an open question. Such a question depends on the way one can add more refined details to the existing models or proposes new ones meaningfully.

The theme that we propose to develop, is based on ideas from the well established literature of statistical
mechanics/physics \[14\ 15\], in order to probe through the mesoscopic scales lying a little below the smooth hydro-scales of dense matter content. This would enable us to understand better, the physical, thermodynamical, dynamical as well as statistical properties in the exotic matter. For such a purpose one needs modification over the regular theoretical formulation in statistical physics, since the underlying spacetime structure for the astrophysical purposes has to be taken into account. The first idea that we can extend from the theory of statistical mechanics is a linear response relation. In this article, we present and work over a new linear response relation which enables a distinct form of the perturbed Einstein’s equations driven by a source term. We work out the solution in complete analytical form for a simple model.

We also restrict to the case of induced radial perturbations in the spherically symmetric relativistic star in this article. The reason for restricting to the radial perturbations only as a first exercise is to correctly work out the very basic results in closed analytical form as a mathematically simpler exercise, which is essential to be able to do further developments correctly with more realistic cases. However for the radial perturbations, the results that we get are important for a general view of the new insight and possibilities that such a formulation can give. As the first example of matter content, we would show how induced perturbations, for the cold dense matter can be built up due to non-thermal fluctuations. The two types of sources, external and internal can be expected to give rise to the two types of perturbations, implicit and induced respectively, which have a linear combination. The theory for what we term as implicit perturbations, is very well developed and studied in the standard asterosiesmology literature. We focus on developing the theory for the induced perturbations. In this first article, we obtain the form of these via the simplest possible theoretical model of a spherically symmetric relativistic star as the first example.

In section \[III\] we give a linear response relation for perturbations, and a complete form of Einstein Langevin equation suitable for perturbations in compact stars with a fluid model of matter. This forms the base of work in this article and for further developments. In section \[IV\] a source term which is responsible for inducing perturbations in the system is described in detail. A particular model of background noise in the gravitating fluid is given which we use to solve for the configuration here. We also mention in brief about averaging in general relativity in the section. The subsections \[IIIA\] and \[IIIB\] describe the spherically symmetric relativistic star model which we work on for this article using our approach. The explicit forms of induced radial perturbations for this system by solving the Einstein Langevin equations are obtained in section \[IV\]. The results are presented in detail, while conclusions and further direction are given in section \[V\].

\[\text{II. A LINEAR RESPONSE RELATION FOR PERTURBATIONS IN RELATIVISTIC STARS}\]

We begin with the environment-system split in the language of statistical mechanics, which for the compact astrophysical objects fits in naturally for matter-spacetime. Thus viewing matter fields as environment and spacetime geometry (metric) as the system. We study then, the linear response of the metric to disturbances in the environment. In this section we give a linear response relation between these two in terms of perturbations.

Here we lay foundations for such a study via a linear response theory suitable for the configuration which in general can be of the form.

\[
F[h; x] = \int K(x - x')f[\xi, x']dx'
\]

where \(F[h; x]\) and \(f[\xi, x]\) denote perturbed quantities in term of the metric perturbations denoted by \(h\) and the fluid displacement vector denoted by \(\xi\). The kernel \(K(x - x')\) connects the two and decides the response.

Writing the above linear response relation in terms of the perturbed Einstein’s equations gives it the explicit form

\[
\delta G_{ab}[h; x] - 8\pi \delta T_{ab}[h; x] = 8\pi \int K(x - x')\delta T_{ab}[\xi, x']dx'
\]

We see that the perturbations on the l.h.s are a linear response to the fluid being displaced, the cause/source of which may lie outside or inside the gravitating body. We model this linear response relation with the condition, that a \(\delta\)-correlation of the response kernel \(K(x - x') = \delta(x - x')\) leads to the usual perturbed Einstein’s equations. In this case, the source of perturbations lies outside the relativistic star. The other possibility is that some component of the noise that we show below, is not a part of the internal source that is perturbing the system. Then for those components of the Einstein’s equations in perturbed form we need to take the delta correlated response function. The latter aspect is important for our considerations for the set of Einstein’s equations, as the background noise tensor may have zero and non-zero components. This would become clearer as we proceed on
the specific model and do the first exercise in the following sections. For the sources lying outside the astronomical object, the case reduces to the regular developments in asteroseismology [2 3]. Then our linear response relation above reduces very easily to the regular perturbed Einstein’s equations, and hence shows consistency.

Our interest lies in an analysis, where the source of perturbations necessarily lies within the dense matter of the body, it could be a thermal or non-thermal effect in general also due to micro-dynamical or microscopic phenomena which act as seeds for a cumulative effect at an intermediate/mesoscopic scales. For example recent results show the effect of quantum noise on macroscopic objects [16]. The details of such sources in dense matter are open to investigation and further considerations would depend on specific cases of interest to study. We work with some essential general features of such sources in terms of fluid variables fluctuating/oscillating in the background spacetime.

To complete the formalism, we add a source term (having origin inside the bulk of dense matter) in the above equation, which is defined on the background spacetime \( g_{ab} \). Thus we can model our system with the following Einstein Langevin equation in its full form given by

\[
\delta G_{ab}[h; x] - 8\pi \delta T_{ab}[h; x] = 8\pi \int K(x-x')\delta T_{ab}[\xi; x']dx' = \tau_{ab}[g, x) \quad (5)
\]

The above equation is linear in perturbations and hence covariantly conserved with respect to the background metric \( g_{ab} \) as is usually assumed for linear first order perturbations in general relativity. Thus \( \nabla_{a}\tau_{ab}[g, x] = 0 \) is essentially to be satisfied for the source term. The above equation is fundamentally different from the perturbed Einstein’s equations which form the theoretical base of asteroseismology. Our model of the perturbations is not meant to address the implicit perturbations of the system, but induced ones only. In fact one can address with our development, issues other than or in addition to the implicit perturbations. The implicit ones do not in fact have origin in this kind of a linear response relation, dividing fluid and metric perturbations with a lag or a retarded effect between them due any reasons. This is the essential difference between the two, to be noticed.

The source term in the E-L equation denoted by \( \tau_{ab} \) has been defined in [11, 12] as

\[
\tau_{ab}(x) = T_{ab}(x) - < T_{ab}(x) > \quad (6)
\]

where \( < T_{ab}(x) > \) is the average over the fluid stress tensor. We discuss on the averaging of the fluid stress tensor in the gravitating system in the next section. Keeping the above definition of \( \tau_{ab} \) as a general guideline, it can be somewhat modified or framed in slightly different ways according to the physical considerations, but should satisfy the general covariance property necessarily for all valid models.

### III. MODELLING SOURCE TERMS IN THE EINSTEIN LANGEVIN EQUATION

The source term \( \tau_{ab} \) in the Einstein–Langevin equation has been motivated by the theoretical developments in semiclassical stochastic gravity [17]. Recently a crude form of the classical Einstein Langevin equation as a first attempt in [11, 12] with an explicit form of stochastic noise in a perfect fluids has been proposed. In the present article we have improved on the classical Einstein Langevin equation, and presented it in a complete and better form, keeping the same definition of the source term or Langevin noise as in the earlier reference. We can further specify source for two possible cases in relativistic stars composed of dense fluid matter.

1. The term defined as \( \tau_{ab}(x) = T_{ab}(x) - < T_{ab}(x) > \) (which may be also be represented as \( \delta T_{ab}(x) \)) in general, the 's over delta to label it as "source" in the unperturbed background \( g_{ab} \) ), being deterministic. In that case \( \tau_{ab}(x) \) technically is not a Langevin source. Then the Einstein Langevin equation becomes a deterministic equation, more appropriately called a perturbed Einstein’s equation with internal driving sources. Modelling such sources in terms of the classical matter field variables at mesoscopic scales correctly is important, we carry one example of this in the present article and discuss its implications.

2. The term \( \tau_{ab}(x) \) is probabilistic, i.e stochastic fluctuations of the fluid variables are responsible for giving it this nature, which then satisfies the Langevin noise conditions, defined by \( < \tau_{ab}(x) > = 0 \) and \( < \tau_{ab}(x)\tau_{cd}(x') > = N_{abcd}(x, x') \), as a two point noise kernel which is a bitensor. This is then of interest to study statistical correlations of the perturbations and analysis based on the probabilistic framework. The averages denoted by \( <> \) in such a case are statistical averages over the variables of matter. This case will be taken up in all details in an upcoming article, where we would state more elaborately a need to generalize stochasticity for a spacetime structure, an effort towards this has recently been introduced in [18].

Before we go further, we discuss briefly, averaging of the matter field stress tensors in general relativity. The averaging for the stress tensor w.r.t to the spacetime vari-
ables, for a $3 + 1$ split can be associated with the temporal variable or the spatial variables, separately. Thus Einstein’s equations can have the form

$$G_{ab}(x) = <T_{ab}(x)> \ (\text{w.r.t spatial sector})$$

$$G_{ab}(x) = T_{ab}(x) \ (\text{w.r.t temporal variable})$$

This has been discussed recently in a different context, that of filtering and averaging in general relativity for a fibration picture [20] for modelling eddies in relativistic fluids. Thus averaging in general relativity is open to detailed modelling in various contexts. For the purpose that we aim here, we consider averages such that the sources as defined above can be accommodated with the theoretical model consistently. In this article, we would consider deterministic sources only. However, we would see that the source still satisfies the condition that the time average, $\overline{\tau_{ab}(x)} = 0$, hence this can still be termed as a deterministic Langevin source or noise in the background satisfying the conditions of the E-L equation.

The astrophysical configuration that we intend to work upon, in order to explain and show our framework is detailed in the subsections below.

A. Model of spherically symmetric relativistic star

A static spherically symmetric spacetime in Schwarzschild coordinates is of the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2d\Omega^2$$

where the four-velocity $u^a = e^{-\nu}(1, 0, 0, 0)$. For the above case of the metric, the perfect fluid stress tensor describing the interior has the explicit form

$$T_{ab} = (\epsilon + p)u_a u_b + g_{ab}p$$

The base model of the relativistic star is that of a static fluid, with

$$u^a = (e^{-\nu}, 0, 0, 0)$$

The motion of the fluid is described in terms of $v = e^{\lambda - \nu}\dot{x}$, such that the four-velocity has components

$$u^a = \frac{1}{\sqrt{1 - v^2}}(e^{-\nu}, e^{-\lambda}v, 0, 0)$$

Accordingly, the components of field equation are given as

$$G_i^i = 8\pi T_i^i : \quad e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2}{r} \lambda'\right) - \frac{1}{r^2} = -8\pi\epsilon \frac{1}{1 - v^2}$$

$$G_r^r = 8\pi T_r^r : \quad e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2}{r} \nu'\right) - \frac{1}{r^2} = 8\pi \epsilon \frac{v^2}{1 - v^2} + p$$

$$G_\theta^\theta = 8\pi e^{2\lambda}T_\theta^\theta : \quad e^{2\lambda}\left(\nu'' + \nu'^2 - \nu' \lambda' + \frac{1}{r}(\nu' - \lambda')\right) = 8\pi e^{2\lambda}p$$

From the above, one can obtain the relation

$$\nu' + \lambda' = 4\pi(\epsilon + p)e^{2\lambda}p$$

The velocity $v$ is introduced in the above, in order to account for the perturbations in radial velocity, such that even in the static background fluid where $v = 0$, the perturbations $\delta v$ can be considered. The picture is that of a static fluid getting displaced which needs motion of the fluid particles to trigger up from a zero value. In the above Einstein’s equations, one can set back $v = 0$ after getting the perturbed expressions from them, in order to account for the static background structure correctly.

Thus one can relate $\delta v$ to the fluid displacement vector by the relation

$$\delta v = e^{\lambda - \nu}\dot{\xi}$$

Note that we have described only the unperturbed and the perturbed quantities here, not the sources yet. Thus the perturbed velocity $\delta v$ and $\xi$ should not be confused with the sources that induce these perturbations, which we address now, in the subsection below.

B. Model of source term in the background fluid

The model of source term depends on the nature of fluid, we consider the simplest case of an ideal fluid, as described in the section above. These sources that enhance the model of ideal fluid, are expected to be present at scales below the macroscopic hydrodynamic limit of the cold dense matter but above the microscopic scales of quantum physics description, that may be needed for degenerate matter in neutron stars. Our scale of interest lies in between these two scales and addresses mesoscopic effects, which show up as small variations/deviations in
terms of the hydrodynamic fluid variables of the system. The details of such mesoscopic effects may be attributed to more refined structure/effects, which we can say are coarse grained remanants of the microscopic details and are open to investigations. However without being able to trace the details to microscopic level, our effort here is to build up the model based on simpler first principles in a different way. We begin with the robust and well established foundations in other areas in physics, which can be framed for dense fluid models.

In such a dense matter, which is characterized by its pressure and energy density, we introduce additional oscillatory contributions in the background. As the ideal fluid is cold matter, we do not associate any thermal effects here or heat transfer, hence the nature of matter does not change. The usual hydrodynamic scale that is considered for such fluid is an idealization, we look into deviations from such an idealization of smoothness in a very general way without assuming any details of the microscopic structure in advance. These can be considered as averaged out or filtered effects of the microscopic details which are nonvanishing. Since we consider a static background to begin with, the stress tensor itself $T_{ab}(x)$ does not depend on $t$. The adiabatic perturbations that we intend to set in the fluid are such that the spherical symmetry of the star is still maintained. For this special case, the source that triggers perturbations in the system can be modelled by

$$\tau_{ab}(x) = \tau_{ab}(\vec{x}) e^{i\omega \cdot x} = \delta_s(T_{ab}(\vec{x}) e^{i\omega \cdot x}$$

Hence the background with static configuration is enhanced with the seeds of oscillations of pressure and energy density of the medium present at mesoscopic scales. The overall hydrodynamic scales may still screen them off and one can define averaged values of pressure and energy density in the usual way as static in the background at macroscopic scales. But we would see how such an addition to the otherwise idealized static model of a perfect fluid in compact astrophysical objects may show some interesting effects.

Considering the averaging over the "time" coordinate, it is clear that in the above $\tau_{ab}(x) = 0$, which satisfies the Langevin equation condition for the source term. For a spherically symmetric configuration,

$$\tau_{t}^t(r) e^{i\omega \cdot t} = \delta_s(r) e^{i\omega \cdot t}$$

$$\tau_{\phi}^\phi(r) e^{i\omega \cdot t} = \tau_{\phi}^\phi(r) e^{i\omega \cdot t} = \delta_s(r) e^{i\omega \cdot t} \delta^p(r) e^{i\omega \cdot t}$$

are the non-vanishing components of $\tau_{ab}$ in the relativistic star fluid. With $\delta^p(r)$ and $\delta_x(r)$ denoting amplitudes of the oscillatory parts of pressure and energy density forming the physical sources of fluctuations in the background fluid, the frequency being given by $\omega_r$ with a radial dependence, to account for each radial layer separately.

We would like to emphasise, that for the model of source in this article, we do not assume the 3-velocity to have similar oscillatory contributions in the background, as we do not begin with oscillatory motion of the fluid elements. But we rather show, how they can be induced by other component sources, specific to the matter content in the star. Also the fact, that the pressure and energy density in dense matter, do not arise due to 3- velocity of any observer, but rather characterize the nature of matter, is an essential point to claim this. For example the pressure in neutron stars is due to the neutron degeneracy and has a quantum mechanical origin. Also the gravitational pull over the shells of the relativistic stars which balances the pressure, has origin other than thermodynamics and hence not related to kinetic energy or any heat flow as in thermal systems. Since we address the cold matter ideal fluid in this article, one can rule out any thermal contributions to the pressure. Therefore we do not consider the 3-velocity fluctuations here along with these oscillating pressure and energy density element as source of perturbations in the bulk of the unperturbed configuration.

IV. THE INDUCED PERTURBATIONS IN THE SYSTEM

The background source $\tau_{ab}(g; x)$ then induces perturbations in the system. Beginning with the Einstein-Langevin equation (5).

$$\delta G_{ab}[h; x] - 8\pi \delta T_{ab}[h; x] - 8\pi \int K(x - x') \delta T_{ab}[\xi, x'] dx' = \tau_{ab}[g, x]$$ (22)

For a spherically symmetric system, and radial perturbations, the above can be easily solved using equations 14, 15 and 16. Only the $t-t$, $t-r$, $r-r$, $\theta-\theta$, $\phi-$ $\phi$ components of equation (22) are non-zero for the case.

As we have discussed earlier, and it is clear from the above equation, that the perturbations of the fluid stress tensor $T_{ab}(x)$ has two parts (known to be of standard form), one corresponding to the fluid displacement vector $\xi_a$ and other to the metric perturbations $h_{ab}$. For the perfect fluid stress tensor $T_{ab} = u_a u_b (\xi + p) + g_{ab} p$ these are explicitly given in terms of $\delta p$, $\delta \epsilon$ and $\delta u_a$, and are
of the standard form
\[ \delta u^a = q_0^a \mathcal{L}_u \xi^b + \frac{1}{2} u^a u^c u^d h_{cd} \text{ (where } q_0^a = u^a u_b + \delta_b^a) \]
\[ \delta \epsilon = -\frac{1}{2} (\epsilon + p) q^{ab} (h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a) - \xi \cdot \nabla \epsilon \]
\[ \delta p = -\frac{1}{2} \Gamma_{p q}^{a b} (h_{a b} + \nabla_a \xi_b + \nabla_b \xi_a) - \xi \cdot \nabla p \]
(23)

then for the radial perturbations here, \( \xi_a \) has only one non-zero component \( \xi_r(x) \), defined by (19). The above perturbations reduce to (separated into \( \delta \lambda, \delta \nu \))
\[ \delta p[\xi] = -\Gamma_{1 p} \frac{e^{-\lambda}}{r^2} [e^{\lambda} r^2 \xi'] - \xi p' \] \( \delta \epsilon[\xi] = -(p + \epsilon) \frac{e^{-\lambda}}{r^2} [e^{\lambda} r^2 \xi'] - \xi \xi' \] \[ \delta u^r[\xi] = -\mathcal{L}_u \xi^r = \mathcal{L}_u \xi^r = e^{-\nu} \xi \] \( \delta \nu \text{ is a complex number in general and the factor } e^{\gamma_r t} \text{ decides the} \]

For the static background where \( v = 0 \) implying \( u_r = 0 \), \( \delta u_r \) has contributions only due to the displacement vector of the fluid. We work out the induced perturbations in the system by solving the Einstein-Langevin equation with all the above information. Choosing for simplicity \( K(x - x') = \delta(r - r') K(t - t') \) which has only the radial and time dependence due to the symmetry of the system, the useful non-zero components of E-L equation are as following.

Proceeding with the \( t - t \) component of equation (22), which reads
\[ \delta G^t_t[\xi(r, t), t] - 8\pi \delta T^t_t[\xi(r, t)] - 8\pi \int \delta T^t_t[\xi(r, t')] K(t - t') dt' = \tau^t_t[\xi](r, t) \] \( -2e^{-2\lambda(r)} \frac{\delta \lambda(r, t)}{r} (1 - \nu'(r) - \lambda(r)) - \frac{2}{r} \delta \lambda'(r, t) e^{-2\lambda(r)} - 8\pi \int K(t - t') (\tau(r) + p(r)) \xi(r, t') dt' \]
\[ + \epsilon'(r) \xi(r, t') dt' = 8\pi \delta e(r) e^{\gamma_r t} \]
(30)

The source which is described in the previous section, is then responsible to feed these perturbations. The induced fluid displacement \( \xi \) can be assumed to be of a reasonable general form \( \xi(r) e^{\gamma_r t} \), such that \( \gamma_r \) is a complex process, such that
\[ \int \int K(t - t') e^{-\gamma_r t} dt' = -e^{-\gamma_r t} \int K(\tau) e^{-\gamma_r \tau} d\tau \]
\[ = -e^{\gamma_r t} \hat{K}(\gamma_r) \] \( \hat{K}(\gamma_r) \) gives the susceptibility of the spacetime metric to get perturbed due to the fluid displacement building up, and is specified by the \( \gamma_r \) radial spectrum. Similarly the \( r - r \) component reads,
\[
\begin{align*}
\delta G'_r[r,t] - 8\pi\delta T'_{r}[h](r,t) - 8\pi \int \delta T'_r[\xi](r',t')K(t-t')dt' &= \tau'_r[g](r,t) \\
2e^{-2\lambda(r)}\left(\frac{\delta\nu'(r,t) + \frac{2\nu'}{r}}{r}\right) + \frac{\delta\lambda(r,t)}{r} + 8\pi \Gamma_p\delta\lambda(r,t) - 8\pi \Gamma_1 p\delta\lambda(r,t) &= 0 \quad \text{(34)}
\end{align*}
\]

For the \( t \rightarrow r \) component of E-L equation
\[
\begin{align*}
\delta G'_t[r,t] - 8\pi\delta T'_{r}[h](r,t) - 8\pi \int \delta T'_r[\xi](r',t')K(t-t')dt' &= \tau'_t[g](r,t) \\
\end{align*}
\]  

From the model of the noise which we use here, \( \tau'_r(r,t) = 0 \), hence there is no source term corresponding to \( t \rightarrow r \) component, the above equation reduces to the simple perturbed case such that \( K(t-t') = \delta(t-t') \), also \( \delta T'_t[h](r,t) = 0 \) as \( \delta u_r[h] = 0 \), which is discussed earlier, then we get
\[
\begin{align*}
-\frac{2}{r}e^{-2\nu(r)}(\delta\lambda(r,t)) &= 8\pi e^{2\lambda(r)-2\nu(r)}(\epsilon(r) + p(r))\xi(r,t) \\
\end{align*}
\]  

which gives
\[
\begin{align*}
\xi(r,t) &= -\frac{\delta\lambda(r,t)}{(\nu'(r) + \lambda'(r))} \\
\end{align*}
\]  

This also ensures that the \( e^{\gamma_0 t} \) dependence is same for \( \xi(r,t) \) and \( \lambda(r,t) \). Since the \( t \rightarrow r \) component of the E-L equation seems to be source free, it is not the case that the perturbations vanish here or occur due to any source external to the system. For this the other components of the E-L equation show how the induced perturbations are related to the source term correctly. The \( t \rightarrow r \) component above is one of the set of equations, and shows a relation between \( \delta\lambda \) and \( \xi \), at a given \( (r,t) \). This is useful to obtain the explicit relation between the source terms and the perturbations in simple and a direct way. Using this relation between \( \xi(r,t) \) and \( \delta\lambda(r,t) \) to solve equation (32),
\[
\begin{align*}
\xi(r,t) &= 4\pi(\nu'(r) + \lambda'(r))e^{\nu_0 t} \int_{\tau_1}^{t} e^{3\lambda(r')-\nu(r')} \delta_\epsilon(r') e^{i\omega r' t} dr' \\
\end{align*}
\]  

Similarly from equation (33) one gets,
\[
\begin{align*}
\delta\nu(r,t) &= \int Y_1(r',\gamma_{\tau_1})(\int e^{3\lambda(r')-\nu(r')} \delta_\epsilon(r'') e^{i\omega r'' t} dr'') dr' + \int Y_2(r',\gamma_{\tau_1}) \delta_\epsilon(r') e^{i\omega r' t} dr' \\
\end{align*}
\]  

The limits of integration over \( r' \) and \( r'' \) would be such that the causality conditions are not violated. And the
The appearance of the speed of sound in the above solutions, points to the possibility of associating the issue of the disturbances in pressure and energy traveling from shells separated by larger distances inside the star and the causal effect to be related. This speed of sound may decide up to which depth the contributions from the sources can be taken. And thus give a more stringent criteria on the valid limits of integration \([r_1, r]\) for the r.h.s of equations (40) (41) and (42).

The equations (40) (41) (42) are the main results and solutions of the Einstein-Langevin equation (22) for a spherically symmetric relativistic star with a static background, enhanced with the sources \(21\) in the dense fluid model which induce perturbations in the system. One can see from equation (40) that the metric potential \(\delta \lambda (r, t)\) is obtained by the integrated effect of the source term \(\delta \sigma (r')\) oscillating with \(e^{i\omega t}\) over shells of radius \(r'\) for an interval \([r_1, r]\). Thus \(\delta \lambda\) is the perturbation induced at the shell \(r\), due to the integrated effect of the small amplitude energy density oscillations over a certain radial depth in the star. We see the appearance of susceptibility of the spacetime \(K(\gamma_r)\) on the rhs, which determines how the spacetime metric or potentials respond to these fluid energy density disturbances. The susceptibility \(K(\gamma_r)\) is the characteristic of the spacetime geometry and not that of the fluid. However \(\gamma_r\) shows radial dependence for this \(\), which says that the response felt by the metric perturbations has radial dependence in the spherically symmetric star. We see similar results follow for \(\xi (r, t)\) which is the fluid displacement vector and the other metric potential \(\delta \nu\).

Thus the perturbations are sourced by the oscillating pressure and energy density seeds in the background system. As discussed earlier in detail, such background oscillations in the matter may be present due to various reasons, even in the cold dense matter stars. Various reasons that can be responsible for their existence can be investigated in detail further. What we intend to emphasize here, is to rule out any thermal/thermodynamical effects as sources in the background cold matter. Thus we want to address the fluctuations which are non-thermal, either quantum mechanical in origin giving characteristics of the dense matter itself, or due to other mechanical/microdynamical effects in the gravitating systems. The details of these may be yet unexplored in astrophysics, thus, this opens up a way for new findings. However, even with the unknown details, formulating any disturbances in terms of the physical parameters of the matter like pressure and energy density of the fluid, which are coarse grained effects, is the first simple approach. Hence one is not flawed by such assumptions at this stage. The indications towards this are few recent interesting results, which touch upon foundations of quantum theory with observational consequences \([10]\), where the magnitude of quantum fields is responsible for bulk properties like the radiation pressure in lasers, with noise such that it shows observational effects on macroscopic objects. Also current research in dark matter puts questions over the quantum nature of matter and its partial quantum nature to be retained in macroscopic bulk fluid description \([19]\).

\[ Y_1 (r, \gamma_r) = -4\pi [\bar{K}(\gamma_r)C_s^2 (2e^{r') + (1 + e^{r')}(r' - \lambda')) + (\lambda' + \nu')C_s^2 + 2\nu' r + e^{r)}] \]
\[ Y_2 (r, \gamma_r) = -4\pi r e^{2\lambda (r)}K(\gamma_r)C_s^2 \]  

\[ (43) \]

\[ (44) \]

The equations (40) (41) (42) are the main results and solutions of the Einstein-Langevin equation (22) for a spherically symmetric relativistic star with a static background, enhanced with the sources (21) in the dense fluid model which induce perturbations in the system. One can see from equation (40) that the metric potential \(\delta \lambda (r, t)\) is obtained by the integrated effect of the source term \(\delta \sigma (r')\) oscillating with \(e^{i\omega t}\) over shells of radius \(r'\) for an interval \([r_1, r]\). Thus \(\delta \lambda\) is the perturbation induced at the shell \(r\), due to the integrated effect of the small amplitude energy density oscillations over a certain radial depth in the star. We see the appearance of susceptibility of the spacetime \(K(\gamma_r)\) on the rhs, which determines how the spacetime metric or potentials respond to these fluid energy density disturbances. The susceptibility \(K(\gamma_r)\) is the characteristic of the spacetime geometry and not that of the fluid. However \(\gamma_r\) shows radial dependence for this \(\), which says that the response felt by the metric perturbations has radial dependence in the spherically symmetric star. We see similar results follow for \(\xi (r, t)\) which is the fluid displacement vector and the other metric potential \(\delta \nu\).

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V. CONCLUSION AND FURTHER DIRECTIONS

In this article we have given a formulation of the classical Einstein- Langevin equation, based on a linear response relation for matter in strong gravity regions. This is the first simple model solved in all details to confirm the mathematical and astrophysical aspects in this regard. The radial perturbations are of importance to stability issues, but we can progress on the same lines with these theoretical developments towards non-radial perturbations induced due to matter fluctuations or disturbances in the fluid of the star. We aim to explore several configurations of relativistic stars for further work, including hot matter fluids with heat flux and anisotropies, stationary and non-stationary fluids, and more involved configuration. The main difference in our aim from the conventional approach is our interest in exploring the perturbations in the system which arise due to internal sources in relativistic stars. Such a study finally leads to a different way and a closer look through the internal layers of the compact stars, thus giving us access to the nature and behaviour of the matter in a new way. This is namely equilibrium and non-equilibrium statistical properties with information pertaining to mesoscopic range inside the matter of relativistic stars. The known approaches towards studying the dense matter concentrate either on the macroscopic hydrodynamic scales about the nature of dense matter, or the astroparticle picture and the microscopics. The mesoscopic scale interest in exploring the dense matter, arises not just to fill in the
scale gaps, but also to address the transition from the hydrodynamic scale down to the microscopic scales and see the connections and phases that the exotic matter can undergo. Thus it enhances the physical picture of interiors of the relativistic stars, the area which is gaining more importance for observational purposes as well, since the detection of gravitational waves took place.

Our efforts at present are to establish the theory in full detail, step by step in this direction. One can expect that the observational consequences may be varied for different configurations. Even for the electromagnetic spectra received for massive stars, this cannot be ruled out, as small parameter changes may be reflected in the data for such observations. However such interests can be addressed correctly and more precisely, once the basic rigorous theoretical framework is set up as initial stages in this research program. Some models may eventually require numerical methods to obtain solutions of the Einstein-Langevin equation. As of now we concentrate on the simpler analytical closed form results that can be obtain for few of the cases to combine together the basic concepts from general relativity, astrophysics and statistical physics with new minor or major constructs and modifications as may be needed at each step.

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