Inequalities and \( p \)th moment exponential stability of impulsive delayed Hopfield neural networks

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Abstract

In this paper, the \( p \)th moment exponential stability for a class of impulsive delayed Hopfield neural networks is investigated. Some concise algebraic criteria are provided by a new method concerned with impulsive integral inequalities. Our discussion neither requires a complicated Lyapunov function nor the differentiability of the delay function. In addition, we also summarize a new result on the exponential stability of a class of impulsive integral inequalities. Finally, one example is given to illustrate the effectiveness of the obtained results.

Keywords: \( p \)th moment exponential stability; Integral inequality; Impulse; Hopfield neural networks; Delay

1 Introduction

In the past few years, the artificial neural networks introduced by Hopfield \([1, 2]\) have become a significant research topic due to their wide applications in various areas such as signal and image processing, associative memory, combinatorial optimization, pattern classification, etc. \([3–5]\). All the applications of Hopfield neural networks (HNNs) depend on qualitative behavior such as stability, existence and uniqueness, convergence, oscillation, and so on \([6–10]\). Particularly, the stability property is a major concern in the design and applications of neural networks. Therefore, many researchers have been paying much attention to the stability study of HNNs.

In addition, since time delays are frequently encountered for the finite switching speed of neurons and amplify in implementation of neural networks, it is meaningful to discuss the effect of time delays on the stability of HNNs. Consequently, the scientists put forward the model of delayed Hopfield neural networks (DHNNs) and made great efforts for the stability research (see e.g. \([11, 12]\)).

Furthermore, it is worth noting that impulsive effects are also a common phenomenon in many engineering systems, that is, instantaneous jump or reset of system states of automobile industry, network control, video coding, etc. Hence, the model of impulsive delayed Hopfield neural networks (IDHNNs) is more representative, and it is necessary to probe
the stability of IDHNNs theoretically and practically. So far, there have been a number of research achievements (see e.g. [13–17])

Among the existing stability results of impulsive delayed systems, one powerful technique is Lyapunov method (see e.g. [18–24]). Wei et al. [18] studied the global exponential stability in the mean-square sense of a class of stochastic impulsive reaction-diffusion systems with S-type distributed delays based on a Lyapunov–Krasovskii functional and an impulsive inequality. Ren et al. [19] considered the mean-square exponential input-to-state stability for a class of delayed stochastic neural networks with impulsive effects driven by G-Brownian motion by constructing an appropriate G-Lyapunov–Krasovskii functional, mathematical induction approach, and some inequality techniques.

It should be pointed out that the key to the Lyapunov method is to construct a suitable Lyapunov function or functional. However, finding a suitable Lyapunov function or functional often involves some mathematical difficulties.

On the other hand, an alternative technique for stability analysis of impulsive delayed systems has been developed based on the fixed point theorem (see e.g. [25–29]). Zhang et al. [25] studied the application of the fixed point theory to the stability analysis of a class of impulsive delayed neural networks. By employing the contraction mapping principle, some novel and concise sufficient conditions have been presented to ensure the existence and uniqueness of solution and the global exponential stability of the considered system.

However, the fixed point method has its disadvantage due to using Holder inequalities at an inappropriate time.

Motivated by the above discussion, we attempt to study the stability of IDHNNs by a new method different from the Lyapunov method and the fixed point method. As we all know, there are many works focused on discussion to mean-square stability of complex dynamical systems. However, mean-square stability is actually a special case of \( p \)th moment stability by choosing \( p = 2 \), so the study of \( p \)th moment stability will be more representative. In our paper, we investigate the \( p \)th moment exponential stability of IDHNNs with the help of impulsive integral inequalities. Compared with the Lyapunov method and the fixed point theory, our method has two advantages. One is no demand of Lyapunov functions and the differentiability of the delay function. The other is no demand of seeking the appropriate time to use Holder inequalities. Furthermore, a new criterion for the exponential stability of impulsive integral inequalities is provided based on our discussion.

The contents of this paper are organized as follows. In Sect. 2, some notations, the model description, and a useful lemma are introduced. In Sect. 3, we consider the \( p \)th moment exponential stability of IDHNNs and obtain some new sufficient conditions. Inspired by Sect. 3, we discuss the exponential stability of a class of impulsive integral inequalities in Sect. 4 and give an algebraic criterion. In Sect. 5, one example is given to illustrate the effectiveness of our results.

2 Preliminaries

Notations: Let \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space. \( | \cdot | \) represents the Euclidean norm for vectors or absolute value for real numbers. \( \mathcal{N} \triangleq \{1, 2, \ldots, n\} \). \( \mathbb{R}_+ = [0, \infty) \). \( C[X, Y] \) stands for the space of continuous mappings from the topological space \( X \) to the topological space \( Y \). For some \( \tau > 0 \), let \( C[[-\tau, 0], \mathbb{R}] \) be the family of all continuous real-valued functions \( \phi \) defined in \([-\tau, 0]\) equipped with the norm \( \|\phi\| = \sup_{s \in [-\tau, 0]} |\phi(s)| \).
Consider a class of impulsive delayed Hopfield neural network described by

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau_j(t))), \quad t \geq 0, t \neq t_k,$$

$$\Delta x_i(t_k) = x_i(t_k^+ - 0) - x_i(t_k^-) = I_{ik}(x_i(t_k^+)).$$

where $i \in \mathcal{N}$ and $n$ is the number of neurons in the neural network. $x_i(t)$ stands for the state of the $i$th neuron at time $t$. $f_j(\bullet), g_j(\bullet) \in C[R, R], f_j(x_j(t))$ is the activation function of the $j$th neuron at time $t$ and $g_j(x_j(t - \tau_j(t)))$ is the activation function of the $j$th neuron at time $t - \tau_j(t)$, where $\tau_j(t) \in C[R^+, R^+]$ denotes the transmission delay along the axon of the $j$th neuron and satisfies $0 \leq \tau_j(t) \leq \theta_j$ ($\theta_j$ is a constant). The constant $a_i > 0$ stands for the rate with which the $i$th neuron will reset its potential to the resting state when disconnected from the network and external inputs. The constant $b_{ij}$ represents the connection weight of the $j$th neuron on the $i$th neuron at time $t$. The constant $c_{ij}$ denotes the connection strength of the $j$th neuron on the $i$th neuron at time $t - \tau_j(t)$. The fixed impulsive moments $t_k (k = 1, 2, \ldots)$ satisfy $0 = t_0 < t_1 < t_2 < \ldots$ and $\lim_{k \to \infty} t_k = \infty$. $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and left-hand limit of $x_i(t)$ at time $t_k$, respectively. $I_{ik}(x_i(t_k))$ shows the abrupt change of $x_i(t)$ at the impulsive moment $t_k$ and $I_{ik}(\bullet) \in C[R, R]$, $\varphi_i(s) \in C[[-\tau, 0], R]$ and $\tau = \max_{j \in \mathcal{N}}(\theta_j)$.

Denote by $x(t; \varphi) = (x_1(t; \varphi_1), \ldots, x_n(t; \varphi_n))^T \in R^n$ the solution of system (2.1), where $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))^T \in R^n$. The solution $x(t; \varphi)$ of system (2.1) is, for time variable $t$, a piecewise continuous vector-valued function with the first kind discontinuity at the points $t_k (k = 1, 2, \ldots)$, where it is left-continuous i.e. the following relations are valid:

$$x_i(t_k^-) = x_i(t_k^0), \quad x_i(t_k^0 + 0) = x_i(t_k^- + 0) = x_i(t_k^-) + I_{ik}(x_i(t_k^-)), \quad i \in \mathcal{N}, k = 1, 2, \ldots.$$

Throughout this paper, we always assume that $f_j(0) = g_j(0) = I_{ik}(0) = 0$ for $j \in \mathcal{N}$ and $k = 1, 2, \ldots$. Then system (2.1) admits a trivial solution with initial value $\varphi = 0$.

**Definition 2.1** The trivial solution of system (2.1) is said to be $p$th ($p \geq 1$) moment exponentially stable if there exists a pair of positive constants $\lambda$ and $C$ such that

$$\left\|x_i(t; \varphi)\right\|^p \leq C \max_{i \in \mathcal{N}}\left\{\left\|\varphi_i\right\|^p\right\} e^{\lambda t}, \quad t \geq 0,$$

holds for any $\varphi_i(s) \in C[[-\tau, 0], R]$ and $i \in \mathcal{N}$.

**Lemma 2.1** Suppose $0 < \theta < 1$ and $\lambda \theta(t - s) < 1 - \theta$. Then $\int_s^t e^{\lambda x} dx > \theta(t - s)e^{\lambda t}$ holds for $t > s$ and $\lambda > 0$.

**Proof** Construct function $F(t) = \int_s^t e^{\lambda x} dx - \theta(t - s)e^{\lambda t}$. For fixed $s$, it is easy to find that $F(s) = 0$ and

$$F(t) = e^{\lambda t} - \theta e^{\lambda t} - \lambda \theta(t - s)e^{\lambda t} = e^{\lambda t}\left[1 - \theta - \lambda \theta(t - s)\right] > 0.$$

So, $F(t) > F(s) = 0$ as $t > s$, which means $\int_s^t e^{\lambda x} dx > \theta(t - s)e^{\lambda t}(t > s)$. \(\square\)
3 pth moment exponential stability of IDHNNs

In this section, we develop a new method to discuss the pth moment exponential stability of system (2.1). Before proceeding, we introduce some hypotheses listed as follows:

(H1) There exist nonnegative constants $\alpha_j$ such that, for any $x_j^{(1)}, x_j^{(2)} \in \mathbb{R}$,

$$\left| f_j(x_j^{(1)}) - f_j(x_j^{(2)}) \right| \leq \alpha_j |x_j^{(1)} - x_j^{(2)}|, \quad j \in \mathcal{N}.$$  

(H2) There exist nonnegative constants $\beta_j$ such that, for any $x_j^{(1)}, x_j^{(2)} \in \mathbb{R}$,

$$\left| g_j(x_j^{(1)}) - g_j(x_j^{(2)}) \right| \leq \beta_j |x_j^{(1)} - x_j^{(2)}|, \quad j \in \mathcal{N}.$$  

(H3) There exist nonnegative constants $P_{jk}$ such that, for any $x_j^{(1)}, x_j^{(2)} \in \mathbb{R}$,

$$\left| I_{jk}(x_j^{(1)}) - I_{jk}(x_j^{(2)}) \right| \leq P_{jk} |x_j^{(1)} - x_j^{(2)}|, \quad j \in \mathcal{N}, k = 1, 2, \ldots.$$  

**Theorem 3.1** Suppose that

(i) there exist constants $\mu > 0$ and $\theta \in (0, 1)$ such that $\inf_{k=1,2,\ldots} \{\theta(t_k - t_{k-1})\} \geq \mu$ and $\max_{k=1,2,\ldots} \{t_k - t_{k-1}\} < \frac{\ln \theta}{\ln \mu}$,

(ii) there exist constants $P_i$ such that $P_{ik} \leq P_i \mu$ for $i \in \mathcal{N}$ and $k = 1, 2, \ldots$,

(iii) $\exists^{p-1} \left\{ a_{i-1} - \mu \left( \sum_{j=1}^{n} |b_{ij}\alpha_j| \right)^{p} + a_{i-1} - \mu \left( \sum_{j=1}^{n} |c_{ij}\beta_j| \right)^{p} + a_{i-1} - \mu P_i^p \right\} < a_i$.

Then system (2.1) is globally exponentially stable in the pth ($p \geq 1$) moment.

**Proof** Multiplying both sides of system (2.1) with $e^{\theta \iota t}$ and integrating from $t_{k-1} + \varepsilon$ ($\varepsilon > 0$) to $t \in (t_{k-1}, t_k)$ yields

$$x_i(t)e^{\theta \iota t} = x_i(t_{k-1} + \varepsilon)e^{\theta \iota (t_{k-1} + \varepsilon)} + \int_{t_{k-1} + \varepsilon}^{t} e^{\theta \iota s} \left\{ \sum_{j=1}^{n} b_{ij}\alpha_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}\beta_j(x_j(s - \tau_j(s))) \right\} ds. \quad (3.1)$$

Letting $\varepsilon \to 0^+$ in (3.1), we have, for $t \in (t_{k-1}, t_k)$ ($k = 1, 2, \ldots$),

$$x_i(t)e^{\theta \iota t} = x_i(t_{k-1} + 0)e^{\theta \iota t_{k-1}} + \int_{t_{k-1}}^{t} e^{\theta \iota s} \left\{ \sum_{j=1}^{n} b_{ij}\alpha_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}\beta_j(x_j(s - \tau_j(s))) \right\} ds. \quad (3.2)$$

Setting $t = t_k - \varepsilon'$ ($\varepsilon' > 0$) in (3.2), we get

$$x_i(t_k - \varepsilon')e^{\theta \iota (t_k - \varepsilon')} = x_i(t_{k-1} + 0)e^{\theta \iota t_{k-1}} + \int_{t_{k-1}}^{t_k - \varepsilon'} e^{\theta \iota s} \left\{ \sum_{j=1}^{n} b_{ij}\alpha_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}\beta_j(x_j(s - \tau_j(s))) \right\} ds,$$
which generates, by letting \( \varepsilon' \to 0^+ \),
\[
x_i(t_k - 0)e^{\alpha_{t_k}} = x_i(t_k - 1 + 0)e^{\alpha_{t_k-1}}
\]
\[+ \int_{t_{k-1}}^{t_k} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds. \tag{3.3}
\]

As \( x_i(t_k - 0) = x_i(t_k) \), (3.3) can be rearranged as
\[
x_i(t_k)e^{\alpha_{t_k}} = x_i(t_k - 1 + 0)e^{\alpha_{t_k-1}}
\]
\[+ \int_{t_{k-1}}^{t_k} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds. \tag{3.4}
\]

Combining (3.2) and (3.4), we derive, for \( t \in (t_{k-1}, t_k) \) \( (k = 1, 2, \ldots) \),
\[
x_i(t)e^{\alpha_t} = x_i(t_{k-1})e^{\alpha_{t_{k-1}}} + \int_{t_{k-1}}^{t} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds
\]
\[+ I_{i(k-1)}(x_i(t_{k-1}))e^{\alpha_{t_{k-1}}}.
\]

Hence,
\[
x_i(t_{k-1})e^{\alpha_{t_{k-1}}} = x_i(t_{k-2})e^{\alpha_{t_{k-2}}} + \int_{t_{k-2}}^{t_{k-1}} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds
\]
\[+ I_{i(k-2)}(x_i(t_{k-2}))e^{\alpha_{t_{k-2}}},
\]
\[\vdots\]
\[
x_i(t_2)e^{\alpha_{t_2}} = x_i(t_1)e^{\alpha_{t_1}} + \int_{t_1}^{t_2} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds
\]
\[+ I_{i1}(x_i(t_1))e^{\alpha_{t_1}},
\]
\[
x_i(t_1)e^{\alpha_{t_1}} = \phi_i(0) + \int_{0}^{t_1} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds.
\]

By induction, we obtain that, for \( t > 0 \),
\[
x_i(t) = \phi_i(0)e^{-\alpha_t} + e^{-\alpha_t} \int_{0}^{t} e^{\alpha_s} \left\{ \sum_{j=1}^{n} b_{ij}(i(s)) + \sum_{j=1}^{n} c_{ij}(x_j(s - \tau_j(s))) \right\} \, ds
\]
\[+ e^{-\alpha_t} \sum_{0 < t_2 < t} \{ I_{i_k}(x_i(t_2))e^{\alpha_{t_2}} \}.
\]
From (H1)–(H3), we know, for \( t > 0 \),
\[
|\psi(t)| \leq |\psi(0)|e^{-\alpha t} + e^{-\alpha t} \sum_{j=1}^n |b_j\alpha_j| \int_0^t e^{\alpha t} |x_j(s)| \, ds \\
+ e^{-\alpha t} \sum_{j=1}^n |c_j\beta_j| \int_0^t e^{\alpha t} \sup_{s-t(u) \leq s} |x_j(s)| \, ds + e^{-\alpha t} \sum_{0 < c_k < t} \{ P_{ik} |x_i(t)| e^{\alpha t_k} \}.
\]

Denote
\[
I_{i1} = |\psi(0)|e^{-\alpha t}, \quad I_{i2} = e^{-\alpha t} \sum_{j=1}^n |b_j\alpha_j| \int_0^t e^{\alpha t} |x_j(s)| \, ds,
\]
\[
I_{i3} = e^{-\alpha t} \sum_{j=1}^n |c_j\beta_j| \int_0^t e^{\alpha t} \sup_{s-t(u) \leq s} |x_j(s)| \, ds, \quad I_{i4} = e^{-\alpha t} \sum_{0 < c_k < t} \{ P_{ik} |x_i(t)| e^{\alpha t_k} \}.
\]

Condition (iii) implies that there exists \( \chi \in (0, 1) \) such that
\[
\frac{3^{p-1}}{(1-\chi)^{p-1}} \left\{ a_i^{1-p} \left( \sum_{j=1}^n |b_j\alpha_j| \right)^p + a_i^{1-p} \left( \sum_{j=1}^n |c_j\beta_j| \right)^p + a_i^{1-p} P_i^p \right\} < a_i.
\] (3.5)

By employing Holder’s inequality, we get
\[
|\psi(t)|^p \leq \chi^{1-p} I_{i1}^p + (1-\chi)^{1-p} (I_{i2} + I_{i3})^p \\
\leq \chi^{1-p} I_{i1}^p + 3^{p-1} (1-\chi)^{1-p} P_i^p + 3^{p-1} (1-\chi)^{1-p} P_i^p + 3^{p-1} (1-\chi)^{1-p} P_i^p.
\]

Moreover, it follows from Holder’s inequality that
\[
P_{i2}^p = \left\{ \sum_{j=1}^n |b_j\alpha_j| \frac{p}{p-1} \left( \int_0^t e^{-\alpha(t-s)} |x_j(s)| \, ds \right)^{p-1} \right\}^p \\
\leq \left( \sum_{j=1}^n |b_j\alpha_j| \right)^{p-1} \sum_{j=1}^n |b_j\alpha_j| \left( \int_0^t e^{-\alpha(t-s)} |x_j(s)| \, ds \right)^p \\
= \left( \sum_{j=1}^n |b_j\alpha_j| \right)^{p-1} \sum_{j=1}^n |b_j\alpha_j| \left[ \int_0^t e^{-\alpha(t-s)} \left( \int_0^t e^{-\alpha(t-s)} |x_j(s)| \, ds \right) \, ds \right]^p \\
\leq \left( \sum_{j=1}^n |b_j\alpha_j| \right)^{p-1} \left\{ \int_0^t e^{-\alpha(t-s)} \, ds \right\} \left( \sum_{j=1}^n |b_j\alpha_j| \int_0^t e^{-\alpha(t-s)} |x_j(s)| \, ds \right)^{p-1} \\
\leq a_i^{1-p} \left( \sum_{j=1}^n |b_j\alpha_j| \right)^{p-1} \left( \sum_{j=1}^n |b_j\alpha_j| \int_0^t e^{-\alpha(t-s)} |x_j(s)| \, ds \right)^p.
\]

Similarly, we get
\[
P_{i3}^p \leq a_i^{1-p} \left( \sum_{j=1}^n |c_j\beta_j| \right)^{p-1} \left( \sum_{j=1}^n |c_j\beta_j| \int_0^t e^{-\alpha(t-s)} \sup_{s-t(u) \leq s} |x_j(s)| \, ds \right)^p.
\]
In addition, Lemma 2.1, conditions (i)–(ii), and Hlder’s inequality yield

\[
I_{\alpha}^p \leq \left\{ \sum_{0 \leq k < t} \left( \int_0^t |x(t_k) - x(t_{k-1})| e^{-\alpha(t_{k-1})} ds \right)^p \right\}^{\frac{1}{p}} \leq \left( \int_0^t e^{-\alpha(t_{k-1})} |x(s)| ds \right)^p \\
= P_t^p \left( \int_0^t e^{-\alpha(t_{k-1})} |x(s)| ds \right)^p \\
\leq P_t^p \left( \int_0^t e^{-\alpha(t_{k-1})} |x(s)| ds \right)^{p-1} \left( \int_0^t e^{-\alpha(t_{k-1})} |x(s)|^p ds \right) \\
\leq a_i^{1-p} P_t^p \left( \int_0^t e^{-\alpha(t_{k-1})} |x(s)|^p ds \right). 
\]

Therefore,

\[
|x_i(t)|^p \leq \chi^{1-p}|\psi_i(0)|^p e^{-\alpha_i t} + 3^{p-1}(1-\chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^n |b_j \alpha_j| \right)^{p-1} \\
\times \left( \sum_{j=1}^n |b_j \alpha_j| \int_0^t e^{-\alpha_j(t_{k-1})} |x_j(s)|^p ds \right) \\
+ 3^{p-1}(1-\chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^n |c_j \beta_j| \right)^{p-1} \\
\times \left( \sum_{j=1}^n |c_j \beta_j| \int_0^t e^{-\alpha_j(t_{k-1})} \sup_{s \leq t_{k-1} \leq t} |x_j(s)|^p ds \right) \\
+ 3^{p-1}(1-\chi)^{1-p} a_i^{1-p} P_t^p \left( \int_0^t e^{-\alpha_i(t_{k-1})} |x_i(s)|^p ds \right). \tag{3.6}
\]

For each \( i \in N \), define the following function:

\[
G_i(\lambda) = (\lambda - a_i) + 3^{p-1}(a_i(1-\chi))^{1-p} \left( \sum_{j=1}^n |b_j \alpha_j| \right)^p \\
+ 3^{p-1}(a_i(1-\chi))^{1-p} \left( \sum_{j=1}^n |c_j \beta_j| \right)^{p-1} \left( \sum_{j=1}^n |c_j \beta_j| e^{\lambda_j} \right) + 3^{p-1}(a_i(1-\chi))^{1-p} P_t^p. 
\]

From (3.5), we know \( G_i(0) < 0 \). Further, \( G_i(\lambda) \) is continuous on \( R _ + \), \( G_i(\pm \infty) = + \infty \), and \( G_i'(\lambda) > 0 \) for \( \lambda \in R _ + \), so for each \( i \in N \), the equation \( G_i(\lambda) = 0 \) has a unique solution \( \lambda_i \in R _ + \). Choosing \( \theta = \min_{i \in N} \{ \lambda_i \} \), we get, for \( i \in N \),

\[
\frac{3^{p-1}}{(1-\chi)^{p-1}} \left\{ a_i^{1-p} \left( \sum_{j=1}^n |b_j \alpha_j| \right)^p + a_i^{1-p} \left( \sum_{j=1}^n |c_j \beta_j| \right)^{p-1} \left( \sum_{j=1}^n |c_j \beta_j| e^{\lambda_j} \right) + a_i^{1-p} P_t^p \right\} \\
\leq a_i - \theta. \tag{3.7}
\]
Let $u(t) = \chi^{1-p} \max_{c \in \mathcal{N}} \{ \| \psi_j \|^p \} e^{-\theta t}$, $t \in [-\tau, +\infty)$. Obviously, $u(s) = u(t)e^{\theta(t-s)}$ holds for $-\tau \leq s \leq t$, and $\sup_{\tau \leq s \leq t} u(s) \leq u(t)e^{\theta \tau}$ is true for each $j \in \mathcal{N}$ and $t \in \mathbb{R}$. Denote

$$
L_1 = u(0)e^{-\alpha_1 t},
$$

$$
L_2 = 3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^{n} |b_j\alpha_j| \right)^{p-1} \left( \sum_{j=1}^{n} |b_j\alpha_j| \right) \int_{0}^{t} e^{-\alpha_1(t-s)} u(s) \, ds,
$$

$$
L_3 = 3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^{n} |c_j\beta_j| \right)^{p-1} \left( \sum_{j=1}^{n} |c_j\beta_j| \right) e^{-\alpha_1 t} \int_{0}^{t} e^{-\alpha_1(t-s)} u(s) \, ds,
$$

$$
L_4 = 3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} P_i \int_{0}^{t} e^{-\alpha_1(t-s)} u(s) \, ds.
$$

As

$$
L_1 = u(0)e^{-\alpha_1 t} = u(t)e^{\theta(-\alpha_1)t},
$$

$$
L_2 = u(t)3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^{n} |b_j\alpha_j| \right)^{p-1} \int_{0}^{t} e^{(\theta-\alpha_1)t} u(s) \, ds,
$$

$$
L_3 \leq u(t)3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^{n} |c_j\beta_j| \right)^{p-1} \left( \sum_{j=1}^{n} |c_j\beta_j| e^{\theta t} \right) \int_{0}^{t} e^{(\theta-\alpha_1)t} u(s) \, ds,
$$

$$
L_4 = u(t)3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} P_i \int_{0}^{t} e^{(\theta-\alpha_1)t} u(s) \, ds,
$$

we obtain from (3.7) that

$$
L_1 + L_2 + L_3 + L_4
\leq u(t)e^{\theta(-\alpha_1)t} + u(t)(1 - e^{\theta(-\alpha_1)t}) = u(t). \tag{3.8}
$$

Finally, we prove that $|x_i(t)|^p \leq u(t)$ for all $t \geq -\tau$ and $i \in \mathcal{N}$ by a contradiction. Obviously, $|x_i(t)|^p \leq u(t)$ holds for $t \in [-\tau, 0]$ and $i \in \mathcal{N}$. For each $i$, assume that there exist $t_i > 0$ and $\varepsilon > 0$ such that $|x_i(t_i)|^p < u(t_i) + \varepsilon$ as $t \in [0, t_i]$ and $|x_i(t_i)|^p = u(t_i) + \varepsilon$. Choose $t^- \equiv t^* = \min_{t \in \mathcal{N}} \{ t_i \}.$ Obviously, $\chi^{1-p}|\psi_i(0)|^p < u(0) + \varepsilon$, from (3.6) and (3.8), we get

$$
|x_i(t)|^p - u(t^*)
\leq 3^{p-1}(1 - \chi)^{1-p} a_i^{1-p} \left( \sum_{j=1}^{n} |b_j\alpha_j| \right)^{p-1}
\times \left( \int_{0}^{t^*} e^{-\alpha_1(s-t^*)} \left[ |x_i(s)|^p - u(s) \right] ds \right)
$$
Theorem 3.2

\[ \sum_{n=1}^{\infty} |c_{ij}| \leq 1, \]

research shows that system (2.1) is globally exponentially stable on the condition that

\[ 3.1 \text{ In [25], the fixed point theory was employed to study system (2.1), and the} \]

Remark means that (iii) in Theorem 3.2 is weaker.

\[ \text{Let } \theta = 1 \] and \( k = 1,2, \ldots , \) which is a contradiction. This shows that \( |x_i(t)|^p \leq u(t) \) for all \( t \geq -\tau \) and \( i \in \mathcal{N}, \) which means that \( |x_i(t; \varphi)|^p \leq \chi^{1-p} \text{ max}_{i \in \mathcal{N}} \{ \| \varphi_i \|^p \} e^{-\theta t} \text{ for } t \in [-\tau, +\infty). \]

As a special case, we give the following theorem.

**Theorem 3.2** Suppose that

(i) there exists constant \( \mu > 0 \) such that \( \inf_{k=1,2, \ldots} \{ \frac{k-\mu-1}{2} \} \geq \mu \) and \( \max_{k=1,2, \ldots} \{ t_k - t_{k-1} \} < \frac{1}{a_1}, \)

(ii) there exist constants \( P_i, \) such that \( P_{ik} \leq P_i \mu \) for \( i \in \mathcal{N} \) and \( k = 1,2, \ldots , \)

(iii) \( -a_i + \sum_{j=1}^{n} |b_{ij} \alpha_j | + \sum_{j=1}^{n} |c_{ij} \beta_j | + P_i < 0. \)

Then system (2.1) is globally exponentially stable.

**Proof** Let \( p = 1 \) and \( \theta = \frac{1}{2} \) in Theorem 3.1.

**Remark 3.1** In [25], the fixed point theory was employed to study system (2.1), and the research shows that system (2.1) is globally exponentially stable on the condition that

\[ \text{Remark (iii) in Theorem 3.2 is weaker.} \]
4 Exponential stability of impulsive integral inequalities

Consider the following impulsive integral inequalities:

\[ y_i(t) \leq C\phi_i(0)e^{-\alpha_1 t} + \sum_{j=1}^{n} \alpha_j \int_0^t e^{-\alpha_j (t-s)} y_j(s) ds + \sum_{j=1}^{n} \beta_{i,j} \int_0^t e^{-\beta_{i,j} (t-s)} \sup_{s+\tau \leq u \leq t} y_j(u) ds \]

\[ + \sum_{0 \leq i_1 < \cdots < i_k \leq t} \{ P_{i_1i_2 \cdots i_k} (t) \} e^{-\lambda_{i_1i_2 \cdots i_k} (t-t_k)} \sum_{j=1}^{n} \mu_{i_1i_2 \cdots i_k} \int_0^t e^{-\mu_{i_1i_2 \cdots i_k} (t-s)} \sup_{s+\tau \leq u \leq t} y_j(u) ds \]

where \( C \geq 1 \), and for each \( i, j \in N \), \( y_i(t) \geq 0 \) for \( t \geq -\tau \), \( 0 \leq \tau_j(s) \leq \tau \) for \( s \geq 0 \), and \( a_i > 0, \alpha_j \geq 0, \beta_{i,j} \geq 0, \) \( P_{i_1i_2 \cdots i_k} \geq 0, k = 1, 2, \ldots \).

**Theorem 4.1** Suppose that

(i) there exist constants \( \mu > 0 \) and \( \theta \in (0,1) \) such that \( \inf_{t \geq 1, 2, \ldots} \{ \theta (t_k - t_{k-1}) \} \geq \mu \) and \( \max_{k=1, 2, \ldots} \{ t_k - t_{k-1} \} < \frac{1-\mu}{\theta \mu} \),

(ii) there exist nonnegative constants \( P \) such that \( P \geq P_i \) for \( i \in N \) and \( k = 1, 2, \ldots \),

(iii) \( a_i + \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_{i,j} + P_i < 0 \).

Then there exist positive constant \( C \) and \( \lambda^* \) such that

\[ \max_{i \in N} y_i(t) \leq C \max_{i \in N} \| \phi_i \| e^{-\lambda^* t}, \quad t \in [-\tau, +\infty), \]

where \( \lambda^* \) is the minimum solution of the following equations:

\[ \lambda - a_i + \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_{i,j} e^{\lambda^* t} + P_i = 0. \]

**Proof** For each \( i \in N \), define the following function:

\[ F_i(\lambda) = \lambda - a_i + \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_{i,j} e^{\lambda t} + P_i. \]

Note that \( F_i(\lambda) \) is continuous on \( R_+ \), \( F_i(0) = -a_i + \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{n} \beta_{i,j} + P_i < 0 \), \( F_i(+\infty) = +\infty \), and \( F_i'(\lambda) > 0 \) for \( \lambda \in R_+ \), so for each \( i \in N \), the equation \( F_i(\lambda) = 0 \) has a unique solution \( \lambda_i \in R_+ \). Choosing \( \lambda^* = \min_{i \in N} \{ \lambda_i \} \), we get

\[ \lambda^* - a_i + \sum_{j=1}^{n} |\beta_{i,j}| + \sum_{j=1}^{n} |c_{i,j}| \lambda^* + P_i \leq 0, \quad i \in N. \]

Let \( u(t) = C \max_{i \in N} \{ \| \phi_i \| e^{-\lambda_i t} \}, t \in [-\tau, +\infty) \). Similar to Theorem 3.1, we get

\[ e^{-a_i t} u(0) + e^{-a_i t} \sum_{j=1}^{n} \alpha_j \int_0^t e^{a_i s} u(s) ds + e^{-a_i t} \sum_{j=1}^{n} \beta_{i,j} \int_0^t e^{\beta_{i,j} s} \sup_{s+\tau \leq u \leq t} u(s) ds \]

\[ + e^{-a_i t} \sum_{0 \leq i_1 < \cdots < i_k \leq t} \{ P_{i_1i_2 \cdots i_k} (t) \} e^{\lambda_{i_1i_2 \cdots i_k} t} \leq u(t), \quad t \geq 0. \]
Finally, we prove that $y_i(t) \leq u(t)$ for all $t \geq -\tau$ and $i \in \mathcal{N}$ by a contradiction. This is similar to the proof of Theorem 3.1, so we omit it here now. This shows that $y_i(t) \leq u(t)$ for all $t \geq -\tau$ and $i \in \mathcal{N}$, which means that $\max_{i \in \mathcal{N}} y_i(t) \leq \max_{i \in \mathcal{N}} \{ \| \phi_i \| \} e^{-\lambda^* t}$ for $t \in [-\tau, +\infty)$.

**Remark 4.1** Inequalities (4.1) can be considered as multidimensional Halanay inequalities with impulses. In [30–32], the authors used the one-dimensional Halanay inequality to consider the stability of delayed neural networks. However, they did not consider the impulse effect, and they needed to construct a complicated Lyapunov function [30, 31] or define a complicated matrix norm [32]; in addition, their results are not easy to verify in practice. The advantages of our multidimensional Halanay inequalities with impulses are that we take into account the impulse effects, and we neither require to construct a complicated Lyapunov function nor to define the adaptive matrix form; furthermore, our results are easy to verify.

**5 Example**

Consider the following two-dimensional impulsive delayed Hopfield neural network:

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} c_{ij} g_j(x_j(t-\tau_j(t))), \quad t \geq 0, t \neq t_k,$$

$$\Delta x_i(t_k) = x_i(t_k + 0) - x_i(t_k) = I_{ik}(x_i(t_k)), \quad t_k = 0.25k, k = 1, 2, \ldots,$$

with the initial conditions $x_1(s) = \cos(s), x_2(s) = \sin(s)$ on $-\tau \leq s \leq 0$, where $a_1 = a_2 = 4, b_{11} = 0, b_{12} = 0.1, b_{21} = -0.2, b_{22} = 0, c_{11} = 0.2, c_{12} = 0, c_{21} = 0, c_{22} = -0.1, f_j(s) = g_j(s) = (|s + 1| - |s - 1|)/2$ ($j = 1, 2$), $I_{ik}(x_i(t_k)) = \arctan(0.45x_i(t_k))$ for $i = 1, 2$ and $k = 1, 2, \ldots (k = 1, 2, \ldots)$. It is easy to find that $\mu = 0.125, \alpha_j = \beta_j = 1$, and $P_{ik} = 0.45$. Select $P_1 = 3.6$ and then compute $-a_i + \sum_{j=1}^{n} |b_{ij}|\alpha_j + \sum_{j=1}^{n} |c_{ij}|\beta_j + P_1 = -0.1 < 0$. From Theorem 3.2, we know this system is globally exponentially stable (Fig. 1).
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