Taylor expansions on Lefschetz thimbles
(and not only that)

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Abstract

Thimble regularisation is a possible solution to the sign problem, which is evaded by formulating quantum field theories on manifolds where the imaginary part of the action stays constant (Lefschetz thimbles). A major obstacle is due to the fact that one in general needs to collect contributions coming from more than one thimble. Here we explore the idea of performing Taylor expansions on Lefschetz thimbles. We show that in some cases we can compute expansions in regions where only the dominant thimble contributes to the result in such a way that these (different, disjoint) regions can be bridged. This can most effectively be done via Padé approximants. In this way multi-thimble simulations can be circumvented. The approach can be trusted provided we can show that the analytic continuation we are performing is a legitimate one, which thing we can indeed show. We briefly discuss two prototypal computations, for which we obtained a very good control on the analytical structure (and singularities) of the results. All in all, the main strategy that we adopt is supposed to be valuable not only in the thimble approach, which thing we finally discuss.

1 Introduction: thimble regularisation and single thimble dominance

Lattice regularisation provides an effective framework for a non-perturbative definition of Quantum Field Theories. It also enables numerical computations: in the euclidean formulation, a lattice-regularised QFT resembles a statistical physics problem, the functional integral defines a decent probability measure and Monte Carlo simulations are viable. However, this is not always the case. When a complex action is in place, we have no probability measure to start with and there is no obvious way to set up a Monte Carlo scheme. This is one consequence of what is known as the sign problem. Among other theories, QCD at non-zero chemical potential is plagued by a sign problem and at the moment we have no
effective way to tackle the investigation of its (supposedly rich) phase diagram by lattice methods (for a clean explanation of the sign problem in the framework of QCD see [1]; for an up-to-date account of the status of the efforts to constrain the QCD phase diagram see [2]).

Thimble regularisation is an elegant (attempt at a) solution to the sign problem. Following seminal papers by Witten [3, 4], it has been introduced with the idea that the sign problem can be beaten at a fundamental level. Thimble regularisation is conceptually simple: one changes the domain of integration by complexifying the degrees of freedom and defines the theory on manifolds where the imaginary part of the action stays constant. These manifolds are the so-called Lefschetz thimbles [5, 6]. In practice, the method has many subtleties. A major one has to do with a fundamental feature, i.e. the so-called thimble decomposition. While there are cases in which a single contribution (attached to the so-called dominant thimble) accounts for the solution of the problem at hand, in general a given quantity is computed summing contributions attached to many thimbles. This feature in turn has to do with the occurrence of Stokes phenomena. The latters take place at given points in the space of the parameters which define the theory: for certain values of the parameters there is no meaningful thimble decomposition. While at those points there is no thimble decomposition, across those points there is a discontinuity in thimble decompositions, which nevertheless does not generally imply a discontinuity in physical results. This basic feature will play a major role in the following.

In the original formulation [5] a single thimble dominance conjecture was put forward. The underlying idea is very simple: one defines the theory as the functional integral restricted to the dominant thimble, i.e. the one associated to the absolute minimum of the real action. From very general (semiclassical) arguments, this contribution is expected to be more and more enhanced in the thermodynamic limit. At a more fundamental level, the regularisation of a field theory on the dominant thimble defines a local quantum field theory with exactly the same symmetries, the same number of degrees of freedom (belonging to the same representations of the symmetry groups) and the same local interactions as the original theory. Moreover, the perturbative expansion on the dominant thimble is exactly the same computed in standard perturbation theory in the original formulation. Quite interestingly, in the case of the relativistic Bose gas this approximation proved to work very well [7]. The Bose gas was of course only one success. Needless to say, if the dominant thimble dominance hypothesis held true for a wide range of theories, that would be a major success: numerical simulations for thimble regularisation would be in the end not that difficult. The Thirring model was shown to be a (first) counterexample: in this case the dominant thimble does not capture the (complete) correct result [8, 9]. This was a major motivation for the exploration of alternative formulations somehow inspired by thimbles. The idea of deforming the original domain of integration is indeed a very general one (from this point of view the complexification of the degrees of freedom is quite an obvious thing to do). Alternatives to thimble regularisation were put forward, e.g. the holomorphic flow [9] or various definitions of sign-optimised manifolds, possibly selected by deep-learning techniques [10, 11, 12]; for a recent, nice review of most of these ideas see [13].

In this work we want to explore the idea of performing Taylor expansions on Lefschetz thim-
bles. All in all, the main idea is to compute Taylor expansions in different regions of the parameter space of a given theory, namely around points where only the dominant thimble contributes to the result one is interested in. This could seem somehow a lucky scenario, but we argue that that this can quite often be the case. Through multiple expansions these (different, disjoint) regions can be bridged and in this way multi-thimbles simulations can be circumvented. This bridging can most effectively be obtained via Padé approximants. The question then should be: “Can we trust this bridging?” The answer is yes if we can prove that we are going through a legitimate analytic continuation. Needless to say, working with Taylor expansions we are aiming at a control on analytic contributions to the result and we will be blind to any non-perturbative effect in the expansion parameter. Most noticeably, in the simple examples we preliminarily discuss in this work, we show that we can have a very good control on the analytic structure (and singularities) of the results. We stress that this good control is coming from multiple Taylor expansions at different points (in the end, different values for the chemical potential) and Padé approximants. While all this is discussed in the framework of (and motivated by) thimble regularisation, it is important to recognise in what we do an overall strategy that does not apply only to this regularisation. This is in a sense a strategy which can go well beyond thimbles.

The paper is organised in a such a way that a minimal prior knowledge of the subject is assumed. In section 2 we collect the basics of thimble regularisation; in particular, we define what a thimble decomposition is and we mention a couple of approaches to multiple thimbles simulations we have been testing in the last few years; the main content of the section is the focus on Stokes phenomena. Section 3 contains the basic description of the computational method we propose together with a discussion of preliminary results for a couple of theories, which (while admittedly simple) are supposed to be valuable prototypes. Conclusions are meant to recognise to which extent the strategy we discuss can go beyond the application to thimble regularisation.

\footnote{We stress that the non-perturbative effects we are talking about are not the ones in the coupling constant, \textit{i.e.} the ones we are most often concerned with.}
2 Thimble decomposition and Stokes phenomena: a story of continuity and discontinuities

2.1 Basics of thimble decomposition and basic multiple thimbles computations

Let us summarise the general problem by writing

\[ \langle O \rangle = Z^{-1} \int dx \ e^{-S(x)} O(x) \tag{1a} \]

\[ = \sum_{\sigma} n_{\sigma} e^{-iS_{I}(p_{\sigma})} \int_{J_{\sigma}} dz \ e^{-S_{R}(z)} O(z) \ e^{i\omega(z)} \]

\[ = \sum_{\sigma} n_{\sigma} e^{-iS_{I}(p_{\sigma})} Z_{\sigma} \langle O e^{i\omega} \rangle_{\sigma} \tag{1b} \]

\[ = \sum_{\sigma} n_{\sigma} e^{-iS_{I}(p_{\sigma})} Z_{\sigma} \langle X \rangle_{\sigma} \tag{1c} \]

It is assumed that \( S(x) = S_{R}(x) + iS_{I}(x) \). \( x \) stands collectively for the real degrees of freedom of our original problem and \( z \) for the degrees of freedom that are the complexification of the latter. (1a) is the original formulation of the problem, while (1b) is the thimble decomposition as expected from Lefschetz/Picard theory. \( p_{\sigma} \) are critical points where \( \partial z S = 0 \). The thimbles \( J_{\sigma} \) attached to each critical point are the union of all the Steepest Ascent paths (SA) which are the solutions of

\[ \frac{d}{dt} \tilde{z}_{i} = \partial \tilde{S} \]

stemming from a given critical point (initial condition). Both numerator and denominator (i.e. the partition function) of (1a) are rewritten as a linear combination of integrals computed on the thimbles attached to critical points; the sum is formally extended to all of them, but the coefficients \( n_{\sigma} \) can be zero for possibly many critical points. Actually \( n_{\sigma} = 0 \) for a critical point when the associated unstable thimble (the union of the Steepest Descent paths stemming from the critical point) does not intersect the original integration manifold. While the (constant) phase \( e^{-iS_{I}(p_{\sigma})} \) is factored in front of each integral, yet another phase enters the integrands. This is the so-called residual phase \( (e^{i\omega}) \) which accounts for the orientation of the thimbles with respect to the embedding manifold\(^2\).

In (1c) \( \langle O \rangle \) is rewritten by defining

\[ \langle X \rangle_{\sigma} \equiv \frac{\int_{J_{\sigma}} dz \ e^{-S_{R}} X}{\int_{J_{\sigma}} dz \ e^{-S_{R}}} = \frac{\int_{J_{\sigma}} dz \ e^{-S_{R}} X}{Z_{\sigma}}. \]

Stated in this way, the thimble decomposition is a linear combination of expectation values computed on single thimbles, with coefficients proportional to the \( Z_{\sigma} \). As a result, multiple thimbles simulations amount to a given prescription for obtaining, in one way or another, (a) the contribution attached to each given thimble \( \langle O \rangle_{\sigma} \) contributing to the result and (b) the relative weights in (1c). Actually, (b) turns out to be a harder task than (a). Despite the difficulties, there are cases in which we could attempt and succeed in multiple thimbles simulations.

\(^2\)Thimbles are manifolds of the same (real) dimension of the original manifold the theory was formulated on, but they are embedded in a manifold of twice that dimension.
simulations. While these are admittedly preliminary steps in an interesting direction, they are worth mentioning, if only to appreciate the peculiar circumstances under which they worked out.

There are cases in which not only it turns out that non-null contributions come from a limited number of thimbles, but also a few of the latter are related to each other due to symmetries. One possible strategy in such cases is the one which was successful for QCD in $0 + 1$ dimensions \[15\]. In that case (due to a symmetry which is in place) the correct result was obtained by taking into account only two contributions according to

$$\langle O \rangle = \frac{\langle O e^{i\omega} \rangle_{\sigma_1} + \alpha \langle O e^{i\omega} \rangle_{\sigma_2}}{\langle e^{i\omega} \rangle_{\sigma_1} + \alpha \langle e^{i\omega} \rangle_{\sigma_2}}. \quad (2)$$

(2) is yet another rewriting of the thimble decomposition. All in all, all our ignorance of relative weights is in such a case coded in one single parameter, \textit{i.e.} $\alpha$. The value of the latter can be fixed assuming one known measurement as a normalisation point. We can then predict the value of other observables. It is obvious that in such a way we give up the hope of a first principles derivation of relative weights. This is very much in the spirit of general frameworks for (non-perturbative) renormalisation. Quite interestingly, this appears to be possible also in the framework of the Thirring model \[16\].

It should be stressed that relative weights are quite easy to obtain in a semiclassical approximation, which is also referred to as the \textit{gaussian approximation}. This suggests another strategy: one starts with the relative weights as computed in such an approximation and then compute corrections as the simulations proceed. Once again, this is not expected to work efficiently in every case. A case of success was a minimal version of the so-called Heavy Dense approximation for QCD \[17\]. Yet another proposal for “rewriting Lefschetz Thimbles” was put forward in \[18\].

2.2 Deeper into the problem: Stokes phenomena

Multiple thimbles simulations are a hard problem and solutions so far have been admittedly a partial success. Our goal here is to find an alternative to them which goes beyond the naive single thimble prescription. In order to proceed, let’s have a look at Stokes phenomena: these control the basic mechanism of the thimble decomposition. The main lesson to take home is that a thimble decomposition is never given once and forever. In the following we provide a simplified, informal discussion. The interested reader is strongly referred to \[19\] for a nice discussion of the subject in the context of the Thirring model.

Loosely speaking, we have a thimble decomposition when the union of a given number of thimbles is a convenient deformation of the original domain of integration, just like in standard applications of the Cauchy theorem. It is clear that the deformation provided by thimbles is not the only possible one (this is \textit{e.g.} the spirit of \[9,10,11,12\]). Strictly speaking, thimbles provide a basis of the relative homology group which the integration
cycle we are interested in belongs to. It is a very convenient basis because the imaginary part of the action stays constant on thimbles. It is also a very convenient basis because the coefficients in the linear combination reconstructing a given path (the $n_\sigma$) are integers. Moreover (as already said) we have a criterion to establish which thimbles do not enter a decomposition: $n_\sigma = 0$ whenever the unstable thimble associated to a given critical point does not intersect the original domain of integration\(^3\). This has an important consequence, which is quite clear pictorially. As they are different solutions of the same (first order) differential equation subject to different initial conditions, different thimbles can not cross each other. Put in a simple-minded (but maybe effective) way, they act as barriers to each other: when the union of a given number of thimbles provide a correct decomposition of the original integration contour, other thimbles are simply kept out.

\[\begin{align*}
\text{Figure 1: Thimbles structure for a 0-dim } \phi^4 \text{ toy model: continuous (blue) lines are stable thimbles; dashed (red) lines are unstable thimbles. The three panels refer to different points in the parameter space of the theory. In the middle, an example of a Stokes phenomenon. For further details, see [20].}
\end{align*}\]

In order to gain some insight, in Figure we plot what the problem looks like in a simple toy model, \textit{i.e.} the 0-dim $\phi^4$ theory (the problem amounts to computing a simple real integral; see [20]). In the left panel, we can see the correct thimble decomposition at a given point of the parameter space of the theory; stable thimbles are depicted as continuous (blue) lines: as it is manifest, one single thimble is enough to get a correct deformation of the original domain of integration (the latter being the real axis\(^4\)). Probing different points in the parameter space, one finds that critical points and associated thimbles do move around in the manifold embedding the original one \textit{(i.e.} the complexified manifold). As they smoothly move around, they are always subject to the constraint of not crossing each other. Thus the thimbles that contribute to the decomposition of the original domain of integration keep on keeping the others out.

There is only one way thimbles can cross each other: we need two thimbles to sit on top of each other. This means that two different critical points are connected by a SA/SD path, \textit{i.e.} the stable thimble of one sits on top of the unstable thimble of the other. When this occurs we are in presence of a Stokes phenomenon: see the central panel. As it is clear from the figure, \textit{at a Stokes phenomenon the thimble decomposition fails.}\n
After a Stokes phenomenon has occurred, the relative arrangement of thimbles can change

\(^3\text{It can be shown that the } n_\sigma \text{ have the meaning of intersection numbers.}\)

\(^4\text{Note how only one unstable thimble intersects the original domain of integration; unstable thimbles are depicted as dashed (red) lines.}\)
and a different thimble decomposition is in place (see right panel).

The toy model we referred to is a trivial example: one can recognise the occurrence of a Stokes phenomenon by direct inspection. One could then think that tracking Stokes phenomena could be an almost impossible task in a typical case. This is not the case, since we have a clear signal that a Stokes phenomenon can occur: when two critical points are connected as described above, the imaginary part of the action takes the same value. To look for Stokes phenomena, we change the values of the parameters describing the theory; let’s denote collectively these parameters by $\xi$. As the $\xi$ vary, a critical point $p_\sigma$ moves around and the value of the imaginary part of the action associated to it (and to the stable and unstable thimbles attached to it) describes a curve $S^I_\lambda(\sigma)(\xi)$. In order that a Stokes phenomenon occurs, two curves need to intersect, i.e. $S^I_\lambda(\sigma_0) = S^I_\lambda(\sigma_0')$. For a beautiful description of the procedure which we just sketched, we refer the interested reader to [19].

We end this sketchy discussion with a trivial, but in practice relevant observation. Reconstructing the correct thimble decomposition is not necessarily the end of story. It can well be that one (or more) thimble(s) entering the correct thimble decomposition is (are) so damped with respect to other contributions that its (their) contribution is de facto negligible. There are cases in which this is evident from the semi-classical approximation: this has to do with the $e^{-S_R(p_\sigma)}$ that can be factored in front of the integral associated to a critical point $p_\sigma$ and which was one of the main rationales for the single thimble dominance hypothesis.

2.3 Discontinuities vs continuity

Stokes phenomena mark discontinuities in the thimble decomposition and one is left with the problem of fixing the values of the $n_\sigma$. With this respect, points where Stokes phenomena occur act as borders. In our simple example (Figure 1): on one side of the border (left panel: one single thimble is relevant) we have no $n_\sigma$ around; on the other side (right panel: three thimbles) we need to fix the correct values of three $n_\sigma$. A key point is that discontinuity in the thimble decomposition does not imply a discontinuity in a physical observable we can be interested in. Therefore, one way of fixing the $n_\sigma$ values is to ensure continuity of an observable (or possibly more than one). The interested reader is referred to [20] for a discussion in the case of the 0-dim $\phi^4$ toy model.

The determination of the $n_\sigma$ is not the only point we want to make, nor the most important one. The main message we want to deliver is that

- across points where Stokes phenomena occur, we generally have discontinuity of thimble decompositions and continuity of physical observables and this continuity can build a bridge over different regions;

- the natural candidates to build the bridge are Taylor expansions.
3 Taylor expansions on thimbles

To circumvent multiple thimbles simulations by computing Taylor expansions we want to

1. find at least two points where one single thimble contributes to the result, either strictly
   speaking or *de facto*, and compute Taylor expansions at those points;

2. bridge the gap in between the regions where the previous computations are performed:
   in principle one could show that the Taylor expansions join smoothly; in practice, at
   a given order of the expansions, Padé approximants can do (much) better;

3. show that the Padé approximants provide a fairly good control on the singularity
   structure in the complex plane; first of all, this can confirm the correctness of the
   procedure (remember the big question: do the radii of convergence enable the analytic
   continuation we have been trying?); also, this can possibly provide *in se* extra insight
   into the theory at hand.

The approach is based on Taylor expansions (which we eventually trade for Padé approximants) and those are expansions in $\mu X$, where $X$ is a dimensionful parameter, *i.e.* a mass $m$ or the temperature $T$. It is therefore clear that we will be blind to any non-perturbative

... effect in $\mu X$. We stress that the latter are not the non-perturbative effects we are most often concerned with (*i.e.* those in the coupling constant).

To show that our program can indeed be accomplished we present two examples. Although simple, we will argue that they display general enough features to support the hope that the method has a potential for further applications.

**1-dim Thirring model** We can now fill the gap that was originally pointed out in [8, 9].

The action of the 1-dim Thirring model is

$$ S = \beta \sum_n (1 - \cos(x_n)) - \log(\det D) \quad \det D = \frac{1}{2^{L-1}} \left( \cosh(L\hat{\mu} + i \sum_n x_n) + \cosh(L \sinh^{-1}(\hat{m})) \right) $$

The chemical potential $\hat{\mu}$ and the mass $\hat{m}$ are given in lattice units. We work at fixed value of mass $m$ (and of $\beta$) and there is a single parameter controlling the sign problem, namely $\mu$. We can obtain a dimensionless quantity by taking the ratio $\frac{\hat{\mu}}{m} = \frac{\mu}{m}$. Since the analytic result is known, the single thimble approximation was shown not to account for the correct result on the entire $\frac{\hat{\mu}}{m}$ axis. In our new approach the problem is solved and in Figure 2 we display the essential features of our results: as an example, we show results for the chiral condensate $\langle \bar{\chi} \chi \rangle$ (parameters are $L = 8$, $\beta = 1$, $m = 2$). We can argue that all the requirements of the program that we sketched above can be met. There is a preliminary point we have to make. For real $\beta$ a Stokes phenomenon is potentially present up to a given value of $\frac{\hat{\mu}}{m}$: this involves the dominant thimble $p_{\sigma_0}$ and another critical point. We denote the latter $p_{\sigma_0}$, following the notation of [19]. The problem can be easily solved by adding a small imaginary part to $\beta$: in this way a Stokes phenomenon does not take place, a thimble decomposition is in place...
and while \( p_{\sigma_0} \) could in principle give a contribution to the result, this is \textit{de facto} negligible due to the huge difference \( S_R(p_{\sigma_0}) \gg S_R(p_{\sigma_0}). \) This solves the problem and any further reference to this point will be omitted in the following.

1. A first value of \( \frac{\mu}{m} \) for which only the dominant thimble \( p_{\sigma_0} \) accounts for the correct result can be found in a very fundamental, yet simple way. The range of values \( S_I \) can take on the real axis depends on the values of \( \hat{\mu} \) and \( \hat{m} \) and, below a given value of \( \frac{\mu}{m} \), this range is limited. By explicit computation of the \( S_I^{(\sigma)}(\frac{\mu}{m}) \) we can show that no unstable thimble associated to a critical point \( p_\sigma \) other that the dominant one can intersect the original domain of integration below a given value \( \frac{\mu_0}{m} \). Thus for \( \frac{\mu}{m} < \frac{\mu_0}{m} \) we can easily select a first point at which the dominant thimble provides the only contribution to the result. We picked \( \frac{\mu}{m} = 0.4 \) and computed the Taylor expansion up to the second derivative.

We now need to find a second value of \( \frac{\mu}{m} \) at which the dominant thimble accounts for the complete result and compute the Taylor expansion on it. In principle we could study the crossing mechanism between the different curves \( S_I^{(\sigma)}(\frac{\mu}{m}) \) (see subsection 2.2). In practice there is a much simpler way to proceed. First of all, we point out that the asymptotic value of \( \langle \bar{\chi}\chi \rangle \) is known: for large enough values of \( \mu \) the chiral condensate is zero. We notice that for \( \frac{\mu}{m} = 1.4 \) the value of \( \langle \bar{\chi}\chi \rangle \) computed on the dominant thimble is very close to zero. By inspecting the values of \( S_R(p_\sigma) \) for thimbles other than the fundamental one, we find that, for \( \frac{\mu}{m} = 1.4, S_R(p_\sigma) \gg S_R(p_{\sigma_0}) \) for all the critical points but three, that we denote \( \sigma_1, \sigma_1, \sigma_2 \). Two of them (\( \sigma_1 \) and \( \sigma_2 \)) have values of the real action which are lower than \( S_{\text{min}} \), which is the minimum value \( S_R \) takes on the original domain of integration: because of this, the unstable thimbles associated to them can’t intersect the original domain of integration. As for \( \sigma_1 \), in this simple model it does not take that much to show that the unstable thimble attached to it does not intersect the original domain of integration (see the left panel of Figure 2). We conclude that the dominant thimble \( \sigma_0 \) can account for the complete result at this value of \( \frac{\mu}{m} \). We have thus selected the second point we were looking for; at this point the series has been computed up to the fifth derivative. One might object that we made use of the explicit query for intersections between the original domain of integration and a given unstable thimble, which thing is quite hard to do in a less simple theory. In the second example we will proceed in a different way: in principle one could follow the same approach also in this case.

2. In order to bridge the gap in between the two values of \( \frac{\mu}{m} \) at which Taylor expansions on the dominant thimble have been computed, one could try to show that the two Taylor expansions do smoothly join. This would actually ask for computing quite a large number of derivatives at the lower value of the chemical potential (as one can see, the curve is quite flat nearby). As we have already pointed out, a Padé approximant can do better. In the middle panel of Figure 2 we plot the interpolation we got from a Padé approximant on top of the analytic result. In order to appreciate how this solves

\[5\] The value of \( \hat{m} \) is held fixed.

\[6\] We once again adhere to the notation of [19].

\[7\] We will see that we need to have a known value (which one can trust as correct) and reconstruct the latter by our Taylor expansion. In this case the asymptotic value at large enough values of \( \frac{\mu}{m} \) is the natural candidate (possibly to be reached in a two/three steps procedure).
the problem of the inconsistency of single thimble computations we refer the reader to the figures in [8].

Figure 2: (Left panel) The flow lines highlighting the thimbles structure of the 1-dim Thirring model at $\mu_m = 1.4$: stable thimbles are depicted in blue, unstable thimbles in magenta. The dominant thimble is associated to the critical point sitting at $\Re(z) = 0$. The critical point $\sigma_1$ is the closest to the latter to the right (there is a mirror image to the left as well): notice that the unstable thimble associated to it does not intersect the original domain of integration (which is on the real axis). (Center panel) The chiral condensate as obtained from the analytic solution (continuous black line) and from our Padé approximant (we plot points instead of a continuum line so that the size of errors are easier to spot.). The points providing input to the evaluation of Padé are marked as triangles. (Right panel) Singularity of the solution in the complex plane: red point computed from the analytic solution, green point is the only pole of our Padé approximant. We plot the radii of convergence which are relevant for the expansions at hand: our analytic continuation indeed stands on firm ground.

3. From the middle panel of Figure 2 one can see that actually four points were taken into account in our Padé procedure (see the four triangles in the plot). On top of the two points we discussed previously, two other points enter and act as extra constraints: the values of the condensate at $\mu_m = 0$ and for $\mu_m$ high enough are known and thus they can be taken into account[8]. The right panel in the figure provides us with a confirmation that the overall procedure is under good control. Since the analytic solution is known, we know that the latter displays a singularity in the complex plane. As one can see, our Padé approximant study captures it quite well: see the green and red points practically on top of each other. The detection of the (expected) singularity is indeed very stable with respect to variations in the number of orders that we take into account. One can thus inspect the convergence radii of our Taylor expansions and it is indeed confirmed that what we got is a legitimate analytic continuation. This is not the only relevant point. The knowledge of the analytic structure of the solution provides in se extra insight into the theory at hand. Needless to say, this is quite often one of the main piece of information we will be interested in.

The present computation is essentially only a proof of concept; the extraction of continuum limit on a line of constant physics has been obtained as well and will be reported elsewhere [21].

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8 Notice that at $\mu_m = 0$ there is no sign problem and thus one could even quite easily compute a high order Taylor expansion.
In subsection 2.1 we mentioned the multiple thimbles simulation of Heavy Dense QCD. We now go back to that theory via the application of the Taylor expansion method we suggest in this paper; the interested reader can compare results with what we discussed in [17]. We are dealing with the effective formulation that can be obtained from QCD by a combined strong-coupling and hopping parameter expansion. One ends up with a 3d effective theory, whose only degrees of freedom are Polyakov loops [22, 23, 24]. We tackle the simplest version of the theory, described by the action

\[ S = S_G + S_F = -\lambda \sum_{<x,y>} (\text{Tr} \, W_x \text{Tr} \, W_y^\dagger + \text{Tr} \, W_x^\dagger \text{Tr} \, W_y) \]

\[ -2 \sum_x \ln \left( 1 + h_1 \text{Tr} \, W_x + h_2^2 \text{Tr} \, W_x^\dagger + h_3^2 \right). \]

Here \( \lambda = u^{N_t} e^{N_t \cdot (14u^4 + 12u^5 - 14u^6 - 36u^7 + \ldots)} \), \( h_1 = (2ke^{\hat{\mu}})^{N_t} \), \( u \approx \beta_{18} \), \( k \) is the hopping parameter and \( W_x = \prod_{t=1}^{N_t} U_0(x, t) \) is the Polyakov loop. The theory can be refined by adding to the fermionic part a sum extended over nearest neighbours, which couples degrees of freedom sitting at different lattice points. The truncation at hand amounts to neglecting \( O(k^2) \) terms in the hopping parameter expansion. Moreover we are concerned with the cold regime, where \( N_t \gg 1, \lambda \approx 0 \); in these conditions the gauge term of the action is negligible (and is indeed neglected). With no interaction among different degrees of freedom \( W_x \), one can write the sum over the degrees of freedom in the action simply as \( \sum_{x=1}^{L} \ln(1 + \ldots) \); we studied the model with \( L = 8 \). Despite its simplicity, the theory displays a sign problem. Also, it displays features that are interesting in our approach: we will have the chance to do something different from what we did in the context of the Thirring model. We stress that also in this case there is a single parameter controlling the sign problem. We work at fixed values of \( T \) and \( m \), so that only \( \mu \) varies and it enters the game through the combination which defines \( h_1 = (2ke^{\hat{\mu}})^{N_t} e^{-\left(\frac{m-\mu}{T}\right)} \). Notice that (being \( m \) and \( T \) fixed) this parameter can be \( h_1 < 1 \) or \( h_1 > 1 \) depending on \( \mu \) being \( \mu < m \) or \( \mu > m \). Figure 3 displays our results for the quark number density (normalised to 1). We plot results versus the ratio \( \mu/m \), the value of \( m \) being fixed by \( k = 0.0000887 \). Also in this case, we can argue that all the requirements we formulated for the application of our new program can be met.

1. To find two points at which a Taylor expansion can be computed on the dominant thimble, we revert this time to a different strategy. We notice that at \( \frac{\mu}{m} = 1 \) (the point that sits in the middle of the relevant region) there is no sign problem at all; thus computing a Taylor expansion poses no problem. We pick two points, to the right (\( \mu > m \)) and to the left (\( \mu < m \)) at which we will compute Taylor expansions on the dominant thimble. We argue that computations on the dominant thimble at those points provide us with the complete result by checking that the results obtained by Taylor expansions smoothly (we would say perfectly actually) join the result we get at \( \frac{\mu}{m} = 1 \) (of whose correctness we are certain). Remember once again that we are concerned with analytic contributions and we will be blind to any non-perturbative

9Also in this case, we use the hat notation for lattice (dimensionless) quantities.

10Notice that in the limit in which we work, with interaction among degrees of freedom missing, dimensionality is a somehow odd concept: \( L = 8 \) can be interpreted as a tiny 2d 3d system (this is the canonical HDQCD, coming from actual QCD), but this is not the only possible interpretation. One could think of a \( L = 8 \) 1d system (in which case one would have started from 1+1 QCD).
effect in the expansion parameter. As an extra confirmation, in the left panel of Figure 3 we plot the relative weights of different thimbles as computed from the semiclassical (gaussian) approximation at those values of $\frac{\mu}{m}$ (the dominant thimble virtually saturates the normalisation; the weight of the least depressed thimble - other than the dominant one - is hardly visible in the figure). In the center panel of Figure 3 one can inspect the location of the two points we selected (once again, they are marked as triangles).

2. While we plot results as a function of $\frac{\mu}{m}$, our expansions are not computed in powers of the latter variable. The natural parameter for the expansion is $h_1 = e^{-\frac{(\mu-m)}{T}}$: we actually expand in $h_1$ (for $\mu < m$) and in $h_1^{-1}$ (for $\mu > m$). More precisely, we expand up to the second derivative with respect to $h_1$ at $\frac{\mu}{m} = 0.9995$ and we supplement as extra constraints the values of the observable and its first derivative at $\frac{\mu}{m} = 1$; we expand up to the second derivative with respect to $h_1^{-1}$ at $\frac{\mu}{m} = 1.0005$ and we supplement as extra constraints the values of the observable and its first derivative at $\frac{\mu}{m} = 1$ and at a value of $\frac{\mu}{m}$ large enough (which is fixed by saturation). Also in this case, we plot in Figure 3 (center panel) the result obtained out of a Padé approximant.

3. Knowing the analytic solution, also in this case we know of a singularity in the complex plane. In the right panel of Figure 3 we display how this is fairly well reconstructed. The conclusion is once again that the procedure we followed is a posteriori proved correct: the convergence radii of the expansions on which we build our construction indeed show that the latter is a legitimate analytic continuation.

4 Conclusions and outlook

We discussed a new strategy to circumvent multiple thimbles simulations in the Lefschetz thimble regularisation of a lattice field theory. The idea is to explore the space of the
parameters describing the theory and find (at least) two points at which the dominant thimble accounts for the full result: as we saw, there could be different strategies to attain this. We do expect that in between these two points Stokes phenomena can occur, so that a non trivial thimble decomposition can be in place. While Stokes phenomena introduce discontinuities in the thimble decomposition of the integrals we are interested in, they do not determine in general discontinuities in physical results. Taylor expansions can thus bridge the different (disjoint) regions where we can compute on the dominant thimble only. We are thus aiming at computing the analytic dependence of our result on the parameter expansion, being blind to any possible non-perturbative contribution\footnote{We stress once again that these are not the standard non-perturbative effects in the coupling}. Not surprisingly, Padé approximants turn out to be the most effective tool to implement our program. In particular, they give us the chance to inspect the analytic structure of our results in the complex plane and thus the convergence radius of our expansions, so that our construction can be \textit{a posteriori} proved to be a legitimate analytic continuation. In a quite natural way, computing multiple Taylor expansions in the complex plane in order to get informations out of Padé approximants can be a legitimate approach beyond thimbles, no matter what is the strategy used to obtain the different expansions. In the end, singularities in the complex plane are quite often one of the most valuable pieces of information we look for.

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