Quantum Decoys

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Alice communicates with words drawn uniformly amongst \[\{\psi_1, \psi_2\}\] of the two non-orthogonal equiprobable states ensemble and Peres[11], who tackled the seemingly simple case tradeoffs themselves have remained stubbornly difficult have been given, Information Gain versus Disturbance Yet while numerous protocol-specific proofs of security can thus be viewed as the power engine behind quantum cryptographic protocols. Moreover the methods employed in this article should be of strong interest to anyone concerned with the old but fundamental problem of how much information may be gained about a system, versus how much this will disturb the system, in quantum mechanics.

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I. MOTIVATION AND CLAIM

The key principle of quantum cryptography could be summarized as follows. Honest parties play using quantum states. To the eavesdropper these states are random and non-orthogonal. In order to gather information she must measure them, but this may cause irreversible damage. Honest parties seek to detect her mischief by checking whether certain quantum states are left intact. The tradeoff between the eavesdropper’s information gain (about an ensemble of quantum states), and the disturbance she necessarily induces (upon this ensemble), can thus be viewed as the power engine behind quantum cryptographic protocols.

Yet while numerous protocol-specific proofs of security have been given, Information Gain versus Disturbance tradeoffs themselves have remained stubbornly difficult to quantify. The problem was first taken over by Fuchs and Peres[11], who tackled the seemingly simple case of the two non-orthogonal equiprobable states ensemble \[\{(1/\sqrt{2})|\psi_0\rangle, (1/\sqrt{2})|\psi_1\rangle\}. A geometrical derivation of their result can be found in [2]. For discrete distributions this is just about the only result available. Of lesser interest for cryptography, but very important in terms of its methods is the work by Banaszek[3], who quantified the tradeoff for the continuous uniform n-dimensional ensemble. Barnum[5] makes several accurate qualitative remarks upon the same ensemble, suggesting the tradeoff remains unchanged for a uniform distribution over mutually unbiased states.

In the present work we quantify the disturbance induced upon the uniform ensemble of n-dimensional states \[\{(1/n^2, \rho_{jk})\}, where j and k range from 1 to n, and \rho_{jk} stands for the density matrix of pairwise superpositions \[\langle j | \pm i | k \rangle \langle j | \pm i | k \rangle\ (note that when j = k this is simply the basis state \(j \), \langle j | j \rangle). When making use of non-orthogonal states this is no doubt a natural distribution to consider, and thus an important building block for n-dimensional cryptographic protocols. Its \(\pi/2\) phase renders this ‘pairing ensemble’ indistinguishable from the canonical ensemble \[\{(1/n, |j\rangle \langle j|)\}, for they both have density matrix \(Id/n\) (the maximally mixed state). This feature enables the honest parties to hide the pairwise superpositions within classical messages as means of securing those, i.e. to use the superpositions as ‘quantum decoys’. In such situations the eavesdropper seeks to gather information about the classical messages, not the decoys. Therefore we quantify her information gain with respect to the canonical ensemble \[\{(1/n, |j\rangle \langle j|)\}, as suits the following scenario best:

Scenario 1 (Quantum decoys) Consider a quantum channel for transmitting n-dimensional systems having canonical orthonormal basis \(\{|j\rangle\}. Suppose Alice’s message words are drawn from the canonical ensemble \[\{(1/n, |j\rangle \langle j|)\}_{j=1...n}, whilst her quantum decoys are drawn from the pairing ensemble \[\{(1/n^2, \rho_{jk})\}_{j,k=1...n}, with \(\rho_{jk} = \frac{\langle j | + i | k \rangle \langle j | - i | k \rangle}{\sqrt{2}}\). Alice sends Bob, over the quantum channel, either a message word or a decoy. Suppose that Bob, whenever a quantum decoy \(\rho_{jk}\) gets sent, measures

\[P_{\text{intact}} = \left(\frac{|j \rangle + i | k \rangle}{\sqrt{2}}\right)\left(\frac{|j \rangle - i | k \rangle}{\sqrt{2}}\right), \ P_{\text{tamper}} = \mathbb{I} - P_{\text{intact}}\]

so as to check for tampering. Suppose Eve is eavesdropping the quantum channel, and has an interest in deter-
mining Alice’s message words.

Results on cryptographic protocols involving discrete distributions in n-dimensional quantum systems (where n is left to vary), remain relatively scarce to this day \cite{1, 7, 8, 13} and tend to focus on mutually unbiased states. We hope our main result will prove a useful contribution to this difficult line of research:

**Claim 1 (Statement of security)** Referring to Scenario 1, suppose Eve performs an individual attack such that, whenever a message word gets sent, she is able to identify which with probability \( G \) (mean estimation fidelity).

Then, whenever a quantum decoy gets sent, the probability \( D \) (induced disturbance) of Bob detecting the tampering is bounded below under the following tight inequality:

\[
D \geq \frac{1}{2} - \frac{1}{2n} \left( \sqrt{G} + \sqrt{(n-1)(1-G)} \right)^2
\]  

(2)

For optimal attacks \( G \) varies from \( \frac{1}{n} \) to 1 as \( D \) varies from 0 to \( \frac{1}{2} - \frac{1}{2n} \).

The reminder of this paper is dedicated to proving the above statement. The method is just as important as the result, since it seems applicable to several similar problems in quantum cryptography.

In section II we provide the necessary mathematical results required to prove Claim 1. We recall, in particular, a key inequality regarding scalar products of vectors (first obtained in \cite{7}), as well as some powerful formulae arising from the state-operator correspondence (first obtained in \cite{8}). Section III exploits the latter formula to express the probability of Bob not detecting the tampering (induced fidelity) as a linear functional upon the positive matrix corresponding to Eve’s attack. This brings about crucial simplifications, finally placing us in a position to apply the inequality. We do so in Section IV and prove our claim.

II. MATHEMATICAL METHODS

*Notations.* We denote by \( M_d(\mathbb{C}) \) the set of \( d \times d \) matrices of complex numbers, and by \( \text{Herm}_+^d(\mathbb{C}) \) its subset of positive matrices, also referred to as the (non-normalized) states of a \( d \)-dimensional quantum system. In this section we let \( \{ |i\rangle \} \) and \( \{ |j\rangle \} \) be orthonormal bases of \( \mathbb{C}^m \) and \( \mathbb{C}^n \) respectively, which we will refer to as canonical.

The following result is a minor generalization of some steps by Banaszek \cite{8}.

**Proposition 1 (Inequality)** Consider a vector of complex numbers \( v = (a_{jr})_{jr} \) together with a function \( j : \mathbb{N} \rightarrow \mathbb{N} \). We then have:

\[
f \leq \left( \sqrt{g} + \sqrt{(m-1)(n-g)} \right)^2
\]

And subject to \( ||v||^2 = n \).

**Proof.** Further let

\[
v_j = (a_{jr})_r \quad ; \quad v_{jr} = (a_{j(r)r})_r \quad v'_j = (a_{jr})_r \quad \text{with } r \text{ such that } j(r) \neq j
\]

and notice that \( g = ||v_{jr}||^2, f = \sum_{ij} v_i v_j^* \). The Cauchy-Schwartz inequality yields:

\[
vi v_j^* \leq ||v_i|| ||v_j|| \quad f \leq \left( \sum_{j=0}^{m-1} ||v_j|| \right)^2
\]

(3)

The quadratic/arithmetic mean inequality yields:

\[
\frac{1}{m-1} \sum_{j=0}^{m-1} ||v'_j|| \leq \sqrt{\frac{1}{m-1} \sum_{j=0}^{m-1} ||v'_j||^2} \leq \sqrt{\frac{n-g}{m-1}}
\]

(4)

Combining Inequalities 3 and 4 yields the lemma. \( \square \)

Next we remind the reader of an isomorphism from quantum states to quantum operations, which in quantum information theory dates back to the work of Jamiołkowski \cite{12} and Choi \cite{10}. The correspondence was subsequently reviewed and taken further in \cite{8}, where Proposition 2 appears. First we relate vectors of \( \mathbb{C}^m \otimes \mathbb{C}^n \) to endomorphisms from \( \mathbb{C}^n \) to \( \mathbb{C}^m \).

**Isomorphism 1** The following linear map

\[
\cdot : \mathbb{C}^m \otimes \mathbb{C}^n \rightarrow \text{End}(\mathbb{C}^n \rightarrow \mathbb{C}^m)
\]

\[
A \mapsto \hat{A}
\]

\[
\sum_{ij} A_{ij} |i\rangle |j\rangle \rightarrow \sum_{ij} A_{ij} |i\rangle |j\rangle
\]

where \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), is an isomorphism taking \( mn \times mn \) vectors \( A \) into \( m \times n \) matrices \( \hat{A} \).

Second we relate elements of \( M_{mn}(\mathbb{C}) \) to linear maps from \( M_n(\mathbb{C}) \) to \( M_m(\mathbb{C}) \).
Isomorphism 2 The following linear map:

\[ \hat{\psi} : \mathbb{C}^{mn} \otimes (\mathbb{C}^{mn})^\dagger \rightarrow \text{End}(M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C})) \]

\[ \rho \mapsto \hat{\psi}(\rho) \]

such that \( AB^\dagger \rightarrow [\rho \mapsto A\rho B^\dagger] \)

\[ \sum_{ijkl} A_{ij} B_{kl}^* |i\rangle |j\rangle \langle k| \langle l| \rightarrow [\rho \mapsto \sum_{ijkl} A_{ij} B_{kl}^* |i\rangle \langle j| |k\rangle \langle l|] \]

where \( i, k = 1, \ldots, m \) and \( j, l = 1, \ldots, n \), is an isomorphism.

Definition 1 A linear map \( \Omega : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Completely Positive-preserving if and only if for all \( \rho \) and for all \( \rho \in \text{Herm}_m^+(\mathbb{C}) \), \( (\Omega \otimes \text{Id}_n)(\rho) \) belongs to \( \text{Herm}_m^+(\mathbb{C}) \).

Completely Positive-preserving linear maps from quantum states in \( \text{Herm}_m^+(\mathbb{C}) \) to quantum states in \( \text{Herm}_m^+(\mathbb{C}) \) are exactly those which are physically allowable. They correspond, via Isomorphism \( 2 \) to quantum states in \( \text{Herm}_m^+(\mathbb{C}) \).

Theorem 1 \[ \Omega \] The linear operation \( \hat{\psi} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Completely Positive-preserving if and only if \( \hat{\psi} \) belongs to \( \text{Herm}_m^+(\mathbb{C}) \).

Definition 2 A linear map \( \hat{\psi} : M_n(\mathbb{C}) \rightarrow M_m(\mathbb{C}) \) is Trace-preserving if and only if for all \( \rho \in M_n(\mathbb{C}) \), \( \text{Tr}(\hat{\psi}(\rho)) = \text{Tr}(\rho) \).

Completely Positive-preserving linear maps having unit probability of occurrence on every input quantum state are exactly those which are Trace-preserving. They correspond, via Isomorphism \( 2 \) to quantum states in \( \text{Herm}_m^+(\mathbb{C}) \) verifying

\[ \text{Tr}_1(\hat{\psi}) = \text{Id}_n. \] (5)

Proposition 2 (State-operator formulae) \[ \hat{\psi} \]

a linear map from \( M_n(\mathbb{C}) \) to \( M_m(\mathbb{C}) \), \( \sigma, \rho \) two elements of \( M_n(\mathbb{C}) \), \( \kappa, \tau \) two elements of \( M_m(\mathbb{C}) \). Then we have:

\[ \hat{\psi}(\rho \sigma) \tau = \text{Tr}_2((\kappa \otimes \rho') \hat{\psi}(\tau \otimes \sigma')) \]

where \( \text{Tr}_2 \) denotes the partial trace over the second system \( \mathbb{C}^n \) in \( \mathbb{C}^m \otimes \mathbb{C}^n \). In particular this implies that for all \( \rho \in M_n(\mathbb{C}) \) and \( \kappa \in M_m(\mathbb{C}) \),

\[ \text{Tr}(\hat{\psi}(\rho)) = \text{Tr}((\kappa \otimes \rho') \hat{\psi}(\rho)). \]

As with many quantum cryptographic problems our analysis will require a careful optimization of the fidelity induced by a quantum operation \( \hat{\psi} \). By means of the above formulae we shall be able to write the induced fidelity as a linear functional upon \( \hat{\psi} \). This step is crucial to the next section (Lemma 3).

III. PRELIMINARY CALCULATIONS

A. Information Gain

There exists several well-motivated methods with which to quantify Eve’s information gain. The one we shall adopt focuses on her ability to make a guess after the measurement. Compared with Shannon’s mutual entropy this measure is advantageously close in nature to the notion of disturbance.

Definition 3 (Mean estimation fidelity) The mean estimation fidelity of a generalized measurement \{\( \hat{A}_r \)\} with guesses \{\( |\psi_r\rangle \)\} w.r.t to an ensemble \{(\( p_i, |\phi_i\rangle \langle \phi_i| \))\} is defined by:

\[ G = \sum_{r,i} p_i (\langle \psi_r | \hat{A}_r | \phi_i \rangle - \langle \phi_i | \psi_r \rangle)^2 \]

\[ = \sum_{r,i} p_i (\langle \psi_r | \hat{A}_r | \phi_i \rangle - \langle \phi_i | \psi_r \rangle)^2 \]

The mean estimation fidelity is to be understood as the average fidelity between the measurer’s guess knowing outcome \( r \) occurred (the \( |\psi_r\rangle \)’s) and the \( i^{th} \) state which was indeed originally sent to him (the \( |\phi_i\rangle \)’s).

Notice it is justified to consider that Eve’s preferred attack is a generalized measurement. In general she could perform a quantum operation, which leaves her the possibility to regroup several measurement outcomes into one likelier outcome. But there is no information to be gained by ignoring the break-up of the likelier measurement outcome. In fact this would simply force some of the \( |\psi_r\rangle \)’s to be equal: the induced disturbance can only be made worse.

In our scenario Eve gathers information about the canonical ensemble \{(\( 1/n, |j\rangle \langle j| \))\}_{j=1 \ldots n}, for which one obtains

\[ G = \frac{1}{n} \sum_r \text{Tr}(\langle j| \hat{A}_r | j \rangle |j| |\psi_r\rangle \langle \psi_r|) \]

Clearly Eve’s optimal guess knowing outcome \( r \) occurred is \( |j(r)\rangle \) such that \( \langle j(r)| \hat{A}_r | j(r) \rangle = \max_j (\langle j| \hat{A}_r | j \rangle) \).

As a consequence

\[ G = \frac{1}{n} \sum_r \text{Tr}(\hat{A}_r | j(r) \rangle \langle j(r)| \hat{A}_r^\dagger) \]

\[ = \frac{1}{n} \sum_r \text{Tr}(\hat{A}_r | j(r) \rangle \langle j(r)| \hat{A}_r) \]

where we applied Proposition 2. This yields:

Lemma 1 (Estimation as a linear functional) Let \( \hat{\psi} \equiv \{\hat{A}_r\} \) be a generalized measurements with best guess \( |j(r)\rangle \), and \( \hat{\psi} \equiv \{\hat{A}_r\} \) its corresponding quantum state. Further let

\[ \mathcal{E} = \frac{1}{n} \sum_r \text{Tr}(\hat{A}_r | j(r) \rangle \langle j(r)| \hat{A}_r | j(r) \rangle \langle j(r)| \hat{A}_r) \]


With \( j_{(r)} \) such that \( \langle j_{(r)} | A_r^\dagger A_r | j_{(r)} \rangle = \max_j \langle j | A_r^\dagger A_r | j \rangle \). Then the mean estimation fidelity of \( \hat{\mathcal{S}} \) with respect to the canonical ensemble is given by

\[
G = \sum_r \text{Tr}(\mathcal{E}(A_r \otimes |r\rangle \langle r|)A_r \otimes |r\rangle \langle r|)^1). \tag{6}
\]

As we have seen the generalized measurement is equivalently described, using Isomorphism \( \mathcal{H} \) by \( \{ A_r \} \), a set of non-zero non-normalized \( n^2 \)-dimensional vectors. Further consider the larger vector \( v = (A_{j(jr)}|j_{(jr)} \rangle \), i.e. with \( r \) itself an index of the complex components. The trace-preserving condition upon the generalized measurement is easily seen to imply that \( \|v\|_2^2 \) should be equal to \( n \). From Lemma \( \mathcal{H} \) it is clear that when seeking an upper bound for \( G \) under this fixed norm constraint, we may assume \( v \) to take the form \( v = (A_{j(jr)}|j_{(jr)} \rangle \), because of the identity matrix on the first subsystem of \( \mathcal{E} \). As we shall explain in subsection \( \mathcal{H} \), this can be done at no cost for the mean induced fidelity. This way we reach the following Lemma:

**Lemma 2 (Information)** Consider a generalized measurement \( \{ A_r \} \). \( \sum_r A_r^\dagger A_r = 1d, \ A_r \) diagonal for all \( r \), acting upon an \( n \)-dimensional system. Then the mean estimation fidelity w.r.t the canonical ensemble verifies

\[
G \leq \frac{1}{n^2} g
\]

with \( g = \sum_r |A_{j_{(r)}}|_{(r)} |r\rangle^2 \) and \( j_{(r)} \) such that \( |A_{j_{(r)}}|_{(r)} |r\rangle^2 = \max_j |A_{j_{(jr)}}|^2 \).

**B. Disturbance**

The notion of disturbance refers to Bob’s chances of detecting Eve’s alteration of the state originally sent. For this purpose Bob can, at best, project the received state upon the span of the original state. Thus the disturbance verifies \( D = 1 - F \), where \( F \) is the induced fidelity.

**Definition 4 (Induced fidelity)** The fidelity induced by a quantum operation \( \hat{\mathcal{S}} \) upon an ensemble \( \{(\phi_i,|\phi_i\rangle \langle \phi_i|)\} \) is defined by:

\[
F = \sum_i p_i \text{Tr}(|\phi_i\rangle \langle \phi_i| \hat{\mathcal{S}}(|\phi_i\rangle \langle \phi_i|))
\]

The induced fidelity is to be understood as the average fidelity between the output of the quantum operation (the \( \hat{\mathcal{S}}(|\phi_i\rangle \langle \phi_i|) \)’s) and its input (the \( |\phi_i\rangle \langle \phi_i| \)’s). A straightforward application of Proposition \( 2 \) yields: (with * denoting componentwise complex conjugation as usual)

\[
F = \sum_i p_i \text{Tr}(\langle \phi_i | \langle \phi_i^* \otimes |\phi_i\rangle \langle \phi_i^* | \rangle \hat{\mathcal{S}}) \tag{7}
\]

In our scenario Eve is tested on the pairing ensemble \( \{(1/n^2, \rho_{jk})\}_{j=1...n} \) with \( \rho_{jk} = \frac{(|j\rangle + i|k\rangle)(|j\rangle - i|k\rangle)}{2} \), for which one obtains:

\[
4 \rho_{jk} \otimes \rho_{jk}^* = \langle jj \rangle |jj\rangle + \langle jj \rangle |kk\rangle + i\langle jj \rangle |jk\rangle - i\langle jj \rangle |kj\rangle
\]

\[
+ \langle kk \rangle |jj\rangle - i\langle kk \rangle |jk\rangle - i\langle kk \rangle |kj\rangle + i\langle kj \rangle |jj\rangle + i\langle kj \rangle |kk\rangle - |kj\rangle |jk\rangle + |kj\rangle |kj\rangle
\]

\[
\rho_{jk} \otimes \rho_{jk}^* + \rho_{kj} \otimes \rho_{kj}^* = \frac{1}{2}(\langle jj \rangle + \langle kk \rangle)(\langle jj \rangle + \langle kk \rangle)
\]

\[
+ \frac{1}{2}(\langle kj \rangle - \langle kj \rangle)(\langle kj \rangle - \langle kj \rangle)
\]

\[
\sum_{jk} \rho_{jk} \otimes \rho_{jk}^* = \frac{1}{4} \sum_{jk} \left(\langle jj \rangle + \langle kk \rangle\right)
\]

\[
= 2 \sum_{jk} \left(\langle jj \rangle + \langle jj \rangle\right)
\]

\[
= 2n \sum_{j} \langle jj \rangle + 2\left(\sum_{j} \langle jj \rangle\right)
\]

As regards the subspace of non-repeated indices the vectors \( |j\rangle - |k\rangle \) are already orthogonal to each other, so long as we maintain \( j < k \). Combining our newly found spectral decomposition with Equation \( 4 \) yields:

**Lemma 3 (Fidelity as a linear functional)** Let \( \hat{\mathcal{S}} \) be a quantum operation, and \( \mathcal{S} \) its corresponding quantum state. Further let

\[
\mathcal{L} = \frac{1}{2n} P_{\text{rep}} + \frac{1}{2n} P_{\beta} P_{\text{rep}}
\]

\[
+ \frac{1}{n^2} \sum_{j<k} \left(\frac{\langle j \rangle - \langle k \rangle}{\sqrt{2}}\right)
\]

\[
\left(\frac{\langle k \rangle - \langle j \rangle}{\sqrt{2}}\right) P_{\text{nonrep}}
\]

With \( P_{\text{rep}} = \sum_{j} |j\rangle \langle j| \langle j\rangle \) \( P_{\text{nonrep}} = 1d - P_{\text{rep}} \)

\[
|\beta\rangle = \frac{1}{\sqrt{n}} \sum_{j} |jj\rangle \quad \text{and} \quad P_{\beta} = |\beta\rangle \langle \beta|
\]

Then the fidelity induced by \( \hat{\mathcal{S}} \) upon the pairing ensemble is given by

\[
F = \text{Tr}(\mathcal{L} \hat{\mathcal{S}}). \tag{9}
\]

Using Theorem \( 2 \) \( \hat{\mathcal{S}} \) is positive and may be thus be written \( \hat{\mathcal{S}} = \sum A_r A^\dagger_r \), with \( \{ A_r \} \) a set of non-zero non-normalized \( n^2 \)-dimensional vectors. Further consider the larger vector \( v = (A_{j(jr)}|j_{(jr)} \rangle \). The trace-preserving condition upon \( \hat{\mathcal{S}} \) is easily seen to imply, by Equation \( 6 \), that \( \|v\|_2^2 \)
should be equal to $n$. From Lemma 3 it is clear that, when seeking an upper bound for $F$ under this fixed norm constraint and if $n \geq 2$, we may assume $v$ to lie in the subspace of projector $P_{\text{rep}} \otimes \text{Id}$. In other words $v$ takes the form $v = (A_{jjr})_{jjr}$. As we have explained in subsection III A this can be done at no cost for the mean estimation fidelity. We then have, using Lemma 3 still:

$$F = \frac{1}{2n} \sum_{ir} |A_{jjr}|^2 + \frac{1}{2n^2} \sum_{jkr} A_{jjr} A_{kkr}^*$$

This way we reach the following Lemma:

**Lemma 4 (Disturbance)** Consider a generalized measurement $\{\hat{A}_r\}$, $\sum_r \hat{A}_r^* \hat{A}_r = \text{Id}$, $\hat{A}_r$ diagonal for all $r$, acting upon an $n$-dimensional system. Then the disturbance induced upon the pairing ensemble verifies

$$D \geq \frac{1}{2} - \frac{1}{2n^2} f$$

with $f = \sum_r |\sum_{j=0}^{n-1} A_{jjr}|^2$.

**IV. OPTIMIZATION AND CONCLUSION**

We are now set to prove Claim 1. From Proposition 1 we immediately have

$$\frac{1}{2} - \frac{1}{2n^2} f \geq \frac{1}{2} - \frac{1}{2n^2} (\sqrt{g} + \sqrt{(n-1)(n-g)})^2.$$  

Applying Lemma 2 and 4 yields

$$D \geq \frac{1}{2} - \frac{1}{2n^2} (\sqrt{ng} + \sqrt{n(n-1)(1-G)})^2$$

which in turn is nothing but Inequality (2). A plot of the curve is shown in the Figure below. As was the case with the continuous uniform ensemble this the generalized measurement family

$$\{\hat{A}_r\}, \quad \hat{A}_r = \sqrt{G} |r\rangle \langle r| + \sqrt{\frac{1-G}{n-1}} (\text{Id} - |r\rangle \langle r|)$$

saturates the tradeoff for any fixed $G = 1/n .. 1$. This may come as no surprise since the corresponding $n^2$ vectors $A_r$ verify

$$A_r = \lambda |r\rangle + \mu |\beta\rangle, \quad \lambda = \sqrt{G} - \sqrt{\frac{1-G}{n-1}}, \quad \mu = \sqrt{\frac{1-G}{n-1}}.$$

In other words these unit vectors $\{A_r\}$ can be thought of as superpositions of Eve’s two extreme attacks: on the one hand $\lambda = 1$ yields the projective measurement $\{|r\rangle \langle r|\}$ maximizing the mean estimation fidelity, whilst on the other hand $\mu = 1$ yields the ‘do nothing’ measurement $\{\text{Id}\}$ minimizing the disturbance. Viewed from the perspective of Lemma 3 Eve, as she seeks to be more conservative, increases her component in the subspace of $P_{\beta}$.

The generalized measurement family ‘measure $\{|r\rangle \langle r|\}$ with probability $p$ else leave it alone’ does not saturate the tradeoff, but linear combinations of pure states corresponding to measurement elements do. Stated in this simple manner, our result suggests the state-operator correspondence method developed in this paper could establish itself as a very natural procedure for deriving quantum cryptographic security bounds in general.

Finally we wish to point out that cryptographic applications of quantum decoys have recently been investigated. An asymmetric variant of secure computation, whereby Alice gets Bob to compute some well-known function $f$ upon her input $x$, but wants to prevent Bob from learning anything about $x$, makes crucial use of the artefact for its security [4].

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