On the structure of Schubert modules and filtration by Schubert modules

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Abstract

In this paper, we investigate some properties of Schubert modules introduced by Kraśkiewicz and Pragacz. We find a generating set for the annihilator ideal of the lowest vector in a Schubert module. We use this computation to derive a projectivity result on Schubert modules. Using this, we obtain some characterizations of modules having a filtration by Schubert modules.

Keywords: Schubert polynomials, Schubert functors, Schubert modules

1 Introduction

Though Schubert polynomials originate from the cohomology ring of flag varieties, they also have combinatorial interests apart from the geometry of flag varieties. Since Schubert polynomials are a kind of generalizations of Schur functions, it is natural to consider analogues of several Schur positivity problems for Schubert polynomials. For example, it is a classical result that $S_u S_v$ is a positive sum of Schubert polynomials, which is proved using the cohomology ring of flag varieties. The problem we are currently concerned about is the Schubert-positivity question for the “plethysm” of a Schur function with a Schubert polynomial. For a symmetric function $s$ and a polynomial $f = x^{\alpha} + x^{\beta} + \cdots$, the plethysm of $s$ and $f$ is defined as $s[f] = s(x^{\alpha}, x^{\beta}, \ldots)$ ([6, §I.8]). The question is: is $s_{\sigma}[S_w]$ a positive sum of Schubert polynomials, for all partitions $\sigma$ and permutations $w$? In this paper we provide some new results which we hope to be useful in attacking this problem.

One of the possible methods for studying Schubert polynomials is through Schubert modules introduced by Kraśkiewicz and Pragacz ([4]). For a permutation $w$, Kraśkiewicz and Pragacz defined a certain representation $\mathcal{S}_w$ of the Lie algebra $\mathfrak{b}$ of all upper triangular matrices, which admits a weight space decomposition with respect to the subalgebra $\mathfrak{h}$ of all diagonal matrices and has the property that the character of $\mathcal{S}_w$ is equal to the Schubert polynomial $\mathcal{S}_w$. So, the problem concerning Schubert positivity is deeply related to the class of modules having a filtration by Schubert modules. For instance, the Schubert positivity of $\mathcal{S}_u \mathcal{S}_v$ and $s_{\sigma}[\mathcal{S}_w]$ are related to the question asking whether
The module $S_w$ is generated by its lowest weight vector $u_w$. We show in Section 3 that the annihilator ideal $\text{Ann}_U(n^+)(u_w)$, where $n^+$ is the Lie subalgebra of all strictly upper triangular matrices, is generated by the elements $e_{ij}^{m_{ij}(w)+1}$ ($1 \leq i < j \leq n$) for some integers $m_{ij}(w)$ which can be read off from $w$, where $e_{ij}$ denotes the $(i,j)$-th matrix unit. This result can be seen as a generalization of a classical result which states that the finite dimensional irreducible representation of $\mathfrak{gl}_n$ with lowest weight $-\lambda$ can be presented as $U(n^+)/\langle e^{\lambda_{i,j}+1}_{i,j} \rangle_{1 \leq i \leq n-1}$ as a $U(n^+)$-module. This result can also be seen as an analog of the result on Demazure modules, given by Joseph (3), which states, in the $\mathfrak{gl}_n$-case, that the annihilator of the generator of the Demazure module with lowest weight $\lambda \in \mathbb{Z}^n$ is generated by the elements $e_{ij}^{\max(0,\lambda_j-\lambda_i)}$ ($1 \leq i < j \leq n$). Since the presentation obtained for $S_w$ gives a projective presentation (as a weight $b$-module) for $S_w$, we can get some information about extensions of Schubert modules. In particular, we see that a certain ordering, introduced in Section 5, on the weight lattice is very useful in studying Schubert modules, and prove an analogue of Polo’s theorem (originally for Demazure modules: see [5], [11] §3) for Schubert modules using this ordering. Finally, using the results obtained so far, we obtain some criteria for a module to have a filtration by Schubert modules, in a way similar to the argument given by van der Kallen ([10], [11] §3) for Demazure modules.

The paper is organized as follows. In Section 2 we recall and define some basic notations and results about Schubert polynomials and Schubert modules. In Sections 3 and 4 we give a generating set for the annihilator ideal of the lowest weight vector in a Schubert module. In Section 5 we introduce a new ordering on the weight lattice and show some results relating Schubert modules with this ordering. In Sections 6 and 7 we obtain some characterizations of modules having a filtration by Schubert modules, using the results of the previous sections. Section 8 serves as a concluding remark by stating some future problems.

Acknowledgement. I would like to thank Katsuyuki Naoi for giving the author information on related materials.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and let $\mathbb{Z}_{\geq 0}$ be the set of all nonnegative integers. A permutation $w$ is a bijection from $\mathbb{N}$ to itself which fixes all but finite points. Let $S_{\infty}$ denote the group of all permutations. For a positive integer $N$, let $S_N = \{ w \in S_{\infty} : w(i) = i \ (i > N) \}$ and $S_{\infty}^{(N)} = \{ w \in S_{\infty} : w(N+1) < w(N+2) < \cdots \}$. We sometimes write a permutation in its one-line form; i.e., if $w \in S_N$, we may write $[w(1) \ w(2) \cdots w(N)]$ to mean $w$. For $i < j$, let $t_{ij}$ denote the permutation which exchanges $i$ and $j$ and fixes all other points. Let $s_i = t_{i,i+1}$. For a permutation $w$, let $\ell(w) = \# \{ i < j : w(i) > w(j) \}$ and $\text{sgn}(w) = (-1)^{\ell(w)}$. For $w \in S_{\infty}^{(N)}$, we define $\text{inv}(w) = (\text{inv}(w)_1, \ldots, \text{inv}(w)_N) \in \mathbb{Z}_{\geq 0}^N$ by $\text{inv}(w)_i = \# \{ j : i < j, w(i) > w(j) \}$, and if $\lambda = \text{inv}(w)$ we write $w = \text{perm}(\lambda)$. The Rothe diagram of $w \in S_{\infty}$ is defined as $D(w) = \{ (i, w(j)) : i < j, w(i) > w(j) \}$. We picture $D(w)$ as in the figure below, and use the words $S_i \otimes S_j$ and $s_\alpha(S_w)$ (here $s_\alpha$ denote the Schur functor), respectively, have such filtrations or not.
row, column, box, etc. to describe certain concepts about Rothe diagrams.

For a polynomial \( f = f(x_1, x_2, \ldots) \) and \( i \in \mathbb{N} \), we define \( \partial_i f = \frac{f(x_1, \ldots, x_i + 1, \ldots)}{x_i} \).

For \( w \in S_\infty \) we can assign its Schubert polynomial \( \mathcal{S}_w \in \mathbb{Z}[x_1, x_2, \ldots] \), which is recursively defined by

- \( \mathcal{S}_{w_0} = x_1^{N-1}x_2^{N-2} \cdots x_{N-1} \) if \( w_0 = [N \ 1 \ \cdots \ 1] \in S_N \), and
- \( \mathcal{S}_{w_{si}} = \partial_i \mathcal{S}_w \) if \( \ell(ws_i) < \ell(w) \).

We note the fact (see eg. [5]) that if \( w \in S_\infty \) (resp. \( S_\infty^{(N)} \)) then \( \mathcal{S}_w \) is a linear combination of \( x_1^{i_1} \cdots x_N^{i_N} \) with \( i_j \in \{ 0, \ldots, N - j \} \) (resp. a polynomial in \( x_1, \ldots, x_N \)).

Schubert polynomials satisfy the following identity:

**Proposition 2.1 ([5] (4.16)).** Let \( w \in S_\infty \setminus \{ \text{id} \} \). Let \( j \in \mathbb{N} \) be the maximal index such that \( w(j) > w(j + 1) \) and take \( k > j \) maximal with \( w(j) > w(k) \).

Let \( v = wt_{jk} \). Let \( i_1 < \cdots < i_A \) be the all integers less than \( j \) such that \( \ell(vt_{i_a j}) = \ell(v) + 1 \), and let \( w^{(a)} = vt_{i_a j} \). Then

\[
\mathcal{S}_w = x_j \mathcal{S}_v + \sum_{a=1}^{A} \mathcal{S}_{w^{(a)}}.
\]

Note that if \( w \in S_\infty^{(N)} \), \( v \) and \( w^{(1)}, \ldots, w^{(A)} \) are also in \( S_\infty^{(N)} \).

Hereafter in this paper, we fix a positive integer \( n \). Let \( K \) be a field of characteristic zero. Let \( \mathfrak{b} \) be the Lie algebra of all \( n \times n \) upper triangular \( K \)-matrices and let \( \mathfrak{h} \subset \mathfrak{b} \) be the subalgebra of all diagonal matrices. Let \( \mathcal{U}(\mathfrak{b}) \) be the universal enveloping algebra of \( \mathfrak{b} \). For a \( \mathcal{U}(\mathfrak{b}) \)-module \( M \) and \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \), let \( M_\lambda = \{ m \in M : bm = \langle \lambda, h \rangle m \ \forall h \in \mathfrak{h} \} \) where \( \langle \lambda, h \rangle = \sum \lambda_i h_i \). \( M_\lambda \) is called the weight space of weight \( \lambda \) or \( \lambda \)-weight space, and elements of \( M_\lambda \) are said to have weight \( \lambda \). If \( M_\lambda \neq 0 \) then \( \lambda \) is said to be a weight of \( M \). If \( M \) is the direct sum of its weight spaces and each weight space has finite dimension, then \( M \) is said to be a weight module and we define \( \text{ch}(M) = \sum_{\lambda} \dim M_\lambda x^\lambda \) where \( x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \). For \( 1 \leq i < j \leq n \), let \( e_{ij} \in \mathfrak{b} \) be the matrix with 1 at the \((i, j)\)-position and all other coordinates 0. It is easy to see that if \( M \) is a \( \mathcal{U}(\mathfrak{b}) \)-module and \( x \in M_\lambda \), then \( e_{ij} x \in M_{\lambda + e_i - e_j} \), where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at the \( i \)-th position.
For \( \lambda \in \mathbb{Z}^n \), let \( K_\lambda \) denote the one-dimensional \( \mathcal{U}(b) \)-module where \( h \in \mathfrak{h} \) acts by \( \langle \lambda, h \rangle \) and \( e_{ij} \) acts by 0. Note that every finite-dimensional weight module admits a filtration by these one dimensional modules.

In [4], Kraśkiewicz and Pragacz defined certain \( \mathcal{U}(b) \)-modules which we call here Schubert modules. Here we use the following definition. Let \( w \in S^{(n)}_w \). Let \( K^n = \bigoplus_{1 \leq i \leq n} Ku_i \) be the vector representation of \( \mathfrak{h} \). For each \( p \in \mathbb{N} \), let \( \{ i : (i, p) \in D(w) \} = \{ i_1, \ldots, i_p \} \) \( (i_1 < \cdots < i_p) \), and let \( u_w^{(p)} = u_{i_1} \wedge \cdots \wedge u_{i_p} \) as an element of \( (K^n)^{\otimes p} \). Let \( u_w = u_w^{(1)} \otimes u_w^{(2)} \otimes \cdots \in (K^n)^{\otimes |D(w)|} \). Then this Schubert module \( S_w \) associated to \( w \) is defined as \( S_w = \mathcal{U}(b)u_w \). Then the Schubert module \( S_w \) associated to \( w \) is defined as \( S_w = \mathcal{U}(b)u_w \). Then this Schubert module \( S_w \) associated to \( w \) is defined as \( S_w = \mathcal{U}(b)u_w \).

### Theorem 2.2 ([4] Remark 1.6, Theorem 4.1)

\( S_w \) is a weight module and \( \text{ch}(S_w) = \mathcal{S}_w \).

In this paper we have to slightly extend the notion of Schubert polynomials and Schubert modules. For \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \), we define the Schubert polynomial and the Schubert module associated to \( \lambda \) as follows. For \( \lambda \in \mathbb{Z}^n \), let \( \mathcal{S}_\lambda = \mathcal{S}_w \) and \( \lambda_\lambda = \lambda_w \) where \( w = \text{perm}(\lambda) \). For a general \( \lambda \in \mathbb{Z}^n \), take \( k \in \mathbb{Z} \) so that \( \lambda + k1 \in \mathbb{Z}^n_0 \), where \( 1 = (1, \ldots, 1) \), and we define \( \mathcal{S}_\lambda = x^{-k1} \mathcal{S}_{\lambda+k1} \) and \( \mathcal{S}_k = K_{-k1} \otimes \mathcal{S}_{\lambda+k1} \). Note that this definition does not depend on the choice of \( k \), since if \( \text{perm}(\lambda) = w \), then \( \text{perm}(\lambda + 1) = [1w(1) + 1 \cdots w(n) + 1 1 w(n+1) + 1 \cdots] =: \tilde{w}, \) and \( \mathcal{S}_{\tilde{w}} = x^1 \mathcal{S}_{\tilde{w}} \) and \( \mathcal{S}_{\tilde{w}} = K_{-k1} \otimes \mathcal{S}_{\lambda+k1} \) hold for them. It then follows from the theorem above that \( \mathcal{S}_\lambda \) is a weight module and \( \text{ch}(\mathcal{S}_\lambda) = \mathcal{S}_\lambda \) for all \( \lambda \in \mathbb{Z}^n \). Note that from this fact if \( x^\mu \) appears in \( \mathcal{S}_\lambda \) with nonzero coefficient then \( \mu \geq \lambda \), where \( \geq \) denote the dominance order: \( \mu \geq \lambda \) iff \( \mu - \lambda = \sum_{i=1}^n a_i (\epsilon_i - \epsilon_{i-1}) \) for some \( a_1, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0} \). We also note here that for any \( \mu, \nu \in \mathbb{Z}^n \), the number of \( \lambda \in \mathbb{Z}^n \) with \( \mu \geq \lambda \geq \nu \) is finite.

### 3 Annihilator of the lowest weight vector

For \( w \in S^{(n)}_w \) and \( 1 \leq i < j \leq n \), let \( m_{ij}(w) = \# \{ k > j : w(i) < w(k) < w(j) \} \) (in particular, \( m_{ij}(w) = 0 \) if \( w(i) > w(j) \)). Since there exists exactly \( m_{ij}(w) \) integers \( p \) such that \( (i, p) \notin D(w) \) and \( (j, p) \in D(w) \), \( e_{ij}^{m_{ij}(w)+1} \) annihilates \( u_w \) by the definition of \( u_w \). From this and the fact that \( \mathcal{S}_w \) is generated by an element \( u_w \) of weight \( \text{inv}(w) \), if we let \( I_w \) denote the left ideal of \( \mathcal{U}(b) \) generated by \( h - \text{inv}(w), h \) \( (h \in \mathfrak{h}) \) and \( e_{ij}^{m_{ij}(w)+1} (i < j) \), there is a unique surjective morphism of \( \mathcal{U}(b) \)-modules from \( \mathcal{U}(b)/I_w \) to \( \mathcal{S}_w \) sending 1 mod \( I_w \) to \( u_w \). We show the following:

### Theorem 3.1

The surjection above is an isomorphism.

### Remark 3.2

It is also possible to define \( u_D \) and \( \mathcal{S}_D \) for a general finite subset \( D \subset \{1, \ldots, n\} \times \mathbb{N} \) as in the same way we defined Schubert modules (\( \mathcal{S}_D \) is often called the flagged Schubert module) associated to \( D \), see e.g. [7] §7; the equivalence of the definition there and our definition can be checked by the same argument as in [4] Remark 1.6). Again in this setting, if we let \( m_{ij}(D) = \# \{ p : (i, p) \notin D, (j, p) \in D \} \) and \( \lambda_i = \# \{ p : (i, p) \in D \}, e_{ij}^{m_{ij}(D)+1} (i < j) \) and \( h - \langle \lambda, h \rangle (h \in \mathfrak{h}) \) annihilates \( u_D \), and therefore we have a surjective morphism \( \mathcal{U}(b)/I_D \to \mathcal{S}_D \) where \( I_D \) is the left ideal generated by these elements. But
this is not an isomorphism for general $D$: for example, if $D = \{(2,1),(3,2)\}$, then $\text{ch}(U(b)/ID) = x_2x_3 + x_1x_3 + x_2^2 + 2x_1x_2 + x_1^2 + x_1x_2x_3^2$ while $\text{ch}(SD) = x_2x_3 + x_1x_3 + x_2^2 + 2x_1x_2 + x_1^2$.

To prove Theorem 3.3 it suffices to prove the following:

**Lemma 3.3.** Let $w \in S_{\infty}^{(n)} \setminus \{\text{id}\}$ and take $j,v,w^{(1)},\ldots,w^{(A)}$ as in Proposition 2.1 then there is a filtration $0 = F_0 \subset F_1 \subset \cdots \subset FA \subset U(b)/I_w$ of $U(b)$-module such that there exist surjective morphisms $U(b)/I_w(a) \twoheadrightarrow F_a/F_{a-1}$ ($a = 1,\ldots,A$) and $(U(b)/I_w) \otimes K_{e_j} \twoheadrightarrow (U(b)/I_w)/FA$.

In fact, using this lemma and Proposition 2.1 we have $\dim U(b)/I_w \geq \dim S_w / \dim \Theta_w(1,1,\ldots,1)$. So dim $U(b)/I_w = \dim S_w$ must hold and this shows Theorem 3.1.

To prove Lemma 3.3 let us first make some observations. A filtration $0 = F_0 \subset F_1 \subset \cdots \subset FA \subset U(b)/I_w$ corresponds to a sequence of left ideals $I_w = I^{(1)}(1) \subset \cdots \subset I^{(a)} \subset U(b)$ such that $I_w = I^{(a)}(a)/I_w$. Since $\text{inv}(w) = \text{inv}(v) + e_j$, a surjection $(U(b)/I_w) \otimes K_{e_j} \twoheadrightarrow (U(b)/I_w)/FA$ exists if $I_w^{(a)} \subset I^{(a)}(a) \subset FA$ where $I_w^{(a)}$ is a left ideal generated by $h = \langle \text{inv}(w), h \rangle$ ($h \in h$) and $e_j^{m_{ij}^{(v)}}$ ($i < j$) so $U(b)/I_w^{(a)} \cong U(b)/I_w \otimes K_{e_j}$.

A surjection $U(b)/I_w(a) \twoheadrightarrow F_a/F_{a-1}$ exists if $I_w(a) = I^{(a-1)} + U(b)x_a$ for some $x_a \in U(b)$ such that $I_w(a)x_a \subset I^{(a-1)}$. Thus, Lemma 3.3 follows from the following lemma, which we prove in the next section:

**Lemma 3.4.** Let $w \in S_{\infty}^{(n)} \setminus \{\text{id}\}$ and take $j,v,w^{(1)},\ldots,w^{(A)}$ as in Proposition 2.1 then $I^{(0)} = I_w$ and $I^{(a)} = I^{(a-1)} + U(b)x_a$ for $a = 1,\ldots,A$ where $x_a = e_j^{m_{ij}^{(v)}}$. Then $I_w = (I^{(a-1)}) \subset I_w(a)x_a \subset (I^{(a-1)})$ for $a = 1,\ldots,A$.

4 Proof of Lemma 3.4

Throughout this section, let $w \in S_{\infty}^{(n)} \setminus \{\text{id}\}$ and take $j,v,w^{(1)},\ldots,w^{(A)}$ as in Proposition 2.1. Take $x_1,\ldots,x_a$ and $I^{(0)},\ldots,I^{(a)}$ as in Lemma 3.4. Let $m_{pq}(v)$ for $1 \leq p < q \leq n$. For $x,y,z \in U(b)$, let $(x,y,z)$ denote the left ideal generated by $x,y,z$.

To make the calculations simple, we use the following basic fact from the representation theory of the general linear Lie algebras:

**Proposition 4.1.** Let $n_3^+ = Ke_{12} \oplus Ke_{13} \oplus Ke_{23}$ be the Lie algebra of all $3 \times 3$ strictly upper triangular matrices which acts on $K^3 = Ku_1 \oplus Ku_2 \oplus Ku_3$ and $\Lambda^2 K^3 = K(u_1 \wedge u_2) \oplus K(u_1 \wedge u_3) \oplus K(u_2 \wedge u_3)$ in the usual way. Then for $a,b \geq 0$, the $U(n_3^+)$-module generated by $(u_2 \wedge u_3)^a \otimes u_3^b$ is isomorphic to $U(n_3^+)/I_{a,b}$ where $I_{a,b}$ is the left ideal generated by $e_j^{a+1}$ and $e_j^{b+1}$.

From this proposition, we have the following:

**Lemma 4.2.** Let $f(x,y,z)$ be a non-commutative polynomial and let $a,b \geq 0$. If $f(e_{12},e_{13},e_{23})(u_2 \wedge u_3)^a \otimes u_3^b = 0$, then for $1 \leq p < q < r \leq n$, $f(e_{pq},e_{pr},e_{qr}) \in \langle e_{pq}^{a+1}, e_{qr}^{b+1} \rangle$. 

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Proof. From Proposition 4.2, we have \( f(e_{12}, e_{13}, e_{23}) \in U(n_{3}^{2})e_{12}^{a_{2}+1} + U(n_{3})e_{23}^{b_{3}+1} \). Since \( e_{pq}, e_{pr}, e_{qr} \) also satisfy the relations satisfied by \( e_{12}, e_{13}, e_{23} \), the lemma follows.

With this lemma in hand, it is easy to prove the following:

**Lemma 4.3.** For \( 1 \leq p < q < r \leq n \),

1. \( e_{pq}^{N}e_{qr}^{M} \equiv 0 \pmod{(e_{pq}^{N}, e_{qr}^{M})} \) for \( N + M > N' + M' - 2 \).
2. \( e_{pq}^{N}e_{pr}^{M} \equiv 0 \pmod{(e_{pq}^{N}, e_{pr}^{M})} \) for \( N + M > N' + M' - 2 \).
3. \( e_{pr}^{N} \equiv \frac{1}{N}e_{pq}^{N}e_{qr}^{N} \pmod{(e_{pq}^{N}, e_{qr}^{N})} \).
4. \( e_{pq}^{N} \equiv \frac{1}{N}e_{pq}^{N}e_{qr}^{N} \pmod{(e_{pq}^{N}, e_{qr}^{N})} \).
5. \( e_{pq}^{N} = 0 \pmod{(e_{pq}^{N}, e_{qr}^{M+1})} \).
6. \( e_{qr}^{N} = 0 \pmod{(e_{pq}^{N}, e_{pr}^{M+1})} \).

Proof. (1)-(5) follows from straightforward calculations checking the condition of Lemma 4.2. (6) also follows from the previous lemma, since \( e_{pq}^{3}e_{pr}^{1} = (\text{const.}) \cdot u_{1}^{u_{2}^{2}}u_{3}^{u_{2}} = (\text{const.}) \cdot e_{23}^{3}e_{13}^{u_{3}}e_{12}^{u_{3}} = (\text{const.}) \cdot e_{12}^{M}e_{13}^{N}e_{pr}^{N} \in (e_{pq}^{N}, e_{qr}^{N}) \).

Let us move on to the proof of Lemma 4.4. First we prove \( I_{v}^{'} \subset I^{(A)} \). Since \( h = (\text{inv}(w), h) \in I_{w} \subset I^{(A)} \), it suffices to show \( e_{pq}^{m_{pq}+1} \in I^{(A)} \) for all \( 1 \leq p < q \leq n \). If \( q \neq j \), \( q \neq j \) and \( v(p) > v(j) \), we have \( m_{pq} = m_{pq}(w) \) so \( e_{pq}^{m_{pq}+1} \in I_{w} \subset I^{(A)} \). If \( q = j \) and \( p = q \), we have \( e_{pq}^{m_{pq}+1} = x_{a} \in I^{(A)} \). Otherwise, the conclusion follows from the following lemma:

**Lemma 4.4.** Let \( p \neq i_{1}, \ldots, i_{A} \). If \( a \in \{1, \ldots, A\} \) is the maximal index such that \( v(i_{a}) > v(p) \), then \( e_{ij}^{m_{pq}+1} \in I^{(A)} \).

Proof. Let \( i = i_{a} \). From the assumption we see that there exists no \( r \) such that \( i < r < j \) and \( v(p) < v(r) < v(i) \). Thus \( m_{pq} = \# \{ r > i : v(p) < v(r) < v(i) \} = \# \{ r > j : v(p) < v(r) < v(i) \} = m_{pq} - m_{ij} \). So from Lemma 4.1, \( e_{pq}^{m_{pq}+1} \in (e_{pq}^{m_{pq}+1}, e_{ij}^{m_{ij}+1}) \). Since \( e_{ij}^{m_{ij}+1} \in I_{w} \subset I^{(a)} \) and \( e_{ij}^{m_{ij}+1} = x_{a} \in I^{(a)} \) we are done.

Let us now prove \( I_{w}^{(a)}x_{a} \subset I^{(a-1)} \) (\( a = 1, \ldots, A \)). Fix \( a \in \{1, \ldots, A\} \) and let \( i = i_{a} \). We want to prove \( (h - (\text{inv}(w^{(a)}), h))x_{a} \in I^{(a-1)} \) for all \( h \in \mathfrak{h} \) and \( m_{pq}(w^{(a)}+1)x_{a} \in I^{(a-1)} \) for all \( p < q \). The first one is easy: \( \text{inv}(w^{(a)}) = \text{inv}(v) + (m_{ij} + 1), e_{ij} = \text{inv}(w) + (m_{ij} + 1)(e_{i} - e_{j}) \) is just the weight of \( x_{a} \) mod \( I^{(a-1)} \in U(\mathfrak{h})/I^{(a-1)} \), since 1 mod \( I^{(a-1)} \) has weight \( \text{inv}(w) \).

We check \( m_{pq}(w^{(a)}+1)x_{a} = m_{pq}(w^{(a)}+1)e_{ij} \in I^{(a-1)} \) for all \( 1 \leq p < q \leq n \). If \( q = j \) and \( v(p) < v(i) \), and in such case \( e_{pq}^{m_{pq}+2} \in I^{(a-1)} \).
• $q > j$ : In this case we have $m_{pq}(u^{(a)}) = 0 = m_{pq}(w)$, since both $w$ and $u^{(a)}$ are increasing from $(j + 1)$-th position and thus there are no $r > q$ with $w(r) < w(q)$ or $w^{(a)}(r) < w^{(a)}(q)$. If $p \neq j$, $e_{pq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq} \in I^{(a)−1}$ since $e_{pq} \in I^{(a)−1}$. If $p = j$, $e_{jq}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{jq} = (m_{ij} + 1)e_{ij}^{m_{ij}}e_{iq} \in I^{(a)−1}$ since $e_{jq}, e_{iq} \in I^{(a)−1}$.

• $p = i$ and $q = j$ : Trivial from $m_{ij}(u^{(a)}) = 0$ and $e_{ij}^{m_{ij}(u^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+2} \in I^{(a)−1}$.

Hereafter we assume $p < q \leq j$ and $(p, q) \neq (i, j)$.

• $\{p, q\} \cap \{i, j\} = \emptyset$ : If $m_{pq}(u^{(a)}) = m_{pq}$ the proof is trivial since in this case $e_{pq}^{m_{pq}(u^{(a))+1}} \in I^{(a)−1}$ and $e_{pq}^{m_{pq}(u^{(a))+1}}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{pq}^{m_{pq}(u^{(a))+1}}$. Otherwise, wee see that $p < i < q < j$ and $v(i) < v(p) < v(j) < v(q)$ must hold. (More precisely, the argument here goes as follows. If $m_{pq}(u^{(a)}) \neq m_{pq}$, $v(p) < v(q)$ must hold, $q$ must be larger than $i$, and exactly one of $v(i)$ and $v(j)$ must lie between $v(p)$ and $v(q)$. Since $i < q < j$, the case $v(p) < v(i) < v(q) < v(j)$ cannot occur. So $v(i) < v(p) < v(j) < v(q)$. Then we have $p < i$ by the same reason.) Here $m_{pq}(u^{(a)}) = m_{pq} − 1$. Using the fact that there exists no $i < r < j$ with $v(i) < v(r) < v(j)$, we obtain $m_{iq} = m_{ij} = m_{pq} = m_{pq}$.

We have
\[
e_{ij}^{m_{ij}+1}e_{ij} = \frac{1}{m_{pq}+1}e_{pq}^{m_{pq}+1}e_{ij}^{m_{ij}+1}e_{iq}^{m_{ij}+1} \pmod{I^{(a)−1}}\]
by Lemma 4.3(3) since $e_{ij}^{m_{ij}+1}e_{ij}^{m_{ij}+1} \in I^{(a)−1}$. Since the RHS is a linear combination of $e_{ij}^{m_{ij}+1}e_{ij}^{m_{ij}+1}e_{iq}^{m_{ij}+1} (\nu \geq 0)$, it suffices to show that these elements are in $I^{(a)−1}$ for each $\nu$. If $\nu > m_{ij}$ it is clear from $[e_{pq}, e_{iq}] = 0$ and $e_{pq}^{m_{pq}+1} \in I^{(a)−1}$. Otherwise, it suffices to show $e_{pq}^{m_{pq}−\nu}e_{iq}^{m_{ij}+1} \in I^{(a)−1}$ since $[e_{pq}, e_{pq}] = 0$. But this follows from $e_{pq}^{m_{pq}−m_{pq}}e_{ij}^{m_{ij}+1} = e_{pq}^{m_{pq}−m_{ij}−1}e_{ij}^{m_{ij}+1} \in I^{(a)−1}$, which can be deduced from $e_{pq}^{m_{pq}+1}e_{iq}^{m_{ij}+1} \in I^{(a)−1}$ using Lemma 4.3(1).

• $p = i$ : Since $i < q < j$ here, the case $v(i) < v(q) < v(j)$ cannot occur. If $v(q) < v(i)$, we have $m_{iq}(u^{(a)}) = 0$, and thus $e_{ij}^{m_{ij}(u^{(a)})+1}e_{ij}^{m_{ij}+1} = e_{ij}^{m_{ij}+1}e_{iq}^{m_{ij}+1} \in I^{(a)−1}$ since $e_{iq} \in I^{(a)−1}$. If $v(q) > v(j)$, $m_{iq}(u^{(a)}) = m_{iq} − m_{ij} − 1$ since $\{r > q : w^{(a)}(i) < w^{(a)}(r) < w^{(a)}(q)\} = \{r > q : v(i) < v(r) < v(q)\} \cup \{r > j : v(i) < v(r) < v(j)\}$, so we want to show $e_{ij}^{m_{ij}−m_{ij}−1}e_{ij}^{m_{ij}+1} \in I^{(a)−1}$. This follows from Lemma 4.3(2) since $e_{ij}^{m_{ij}+1}, e_{ij} \in I^{(a)−1}$.

• $q = i$ : Here we have three cases to consider. If $v(p) < v(i)$, we have $m_{pi}(u^{(a)}) = m_{pi} + m_{ij} + 1$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > i : v(p) < v(r) < v(i)\} \cup \{r > j : v(i) < v(r) < v(j)\}$, and so we want to show $e_{pi}^{m_{pi}+m_{ij}+2}e_{ij}^{m_{ij}+1} \in I^{(a)−1}$. This follows from Lemma 4.3(5) since $e_{pi}^{m_{pi}+1}, e_{ij}^{m_{ij}+2} \in I^{(a)−1}$. If $v(i) < v(p) < v(j)$, we have $m_{pq}(u^{(a)}) = m_{pq}$ since $\{r > i : w^{(a)}(p) < w^{(a)}(r) < w^{(a)}(i)\} = \{r > j : v(p) < v(r) < v(j)\}$ and so we want to show $e_{pq}^{m_{pq}+1}e_{ij}^{m_{ij}+1} \in I^{(a)−1}$. This follows from
Lemma 5.1 (cf. [11, Lemma 3.1.1]) for Schubert modules. This can be seen as an analog of Polo’s theorem ([8, Corollary 2.5], [11, Theorem 3.1.10]) for Schubert modules.

5 Projectivity of Schubert modules

In this section we prove certain projectivity results about Schubert modules. This can be seen as an analog of Polo’s theorem ([8, Corollary 2.5], [11, Theorem 3.1.10]) for Schubert modules.

Let \( C \) be the category of all weight modules. For \( \Lambda \subset \mathbb{Z}^n \), let \( C_\Lambda \) be the full subcategory of \( C \) consists of all weight modules whose weights are in \( \Lambda \). Note that if \( |\Lambda| < \infty \) and \( \Lambda' = \{ \rho - \lambda : \lambda \in \Lambda \} (\rho = (n-1, n-2, \ldots, 0)) \), then \( C_{\Lambda'} \cong C_{\Lambda}^{op} \) by \( M \mapsto M^* \otimes K_\rho \).

Lemma 5.1 (cf. [11, Lemma 3.1.1]). For any \( \Lambda \subset \mathbb{Z}^n \), \( C_\Lambda \) has enough projectives.
Proof. Let \( P_\lambda = U(\mathfrak{n}^+) \otimes K_\lambda \) (here \( U(\mathfrak{n}^+) \) is seen as an \( U(\mathfrak{b}) \)-module by the adjoint action) and let \( P_\lambda^A \) be the largest quotient of \( P_\lambda \) which is in \( C_\lambda \). Then \( P_\lambda^A \) is projective in \( C_\lambda \) since for \( N \in C_\lambda \), \( \text{Hom}(P_\lambda^A, N) = \text{Hom}(P_\lambda, N) = N_\lambda \). This shows that \( C_\lambda \) is enough projective.

Note that, since the head of \( P_\lambda^A \) is \( K_\lambda \), \( P_\lambda^A \) is the projective cover of \( K_\lambda \) in \( C_\lambda \).

We introduce two orderings (other than dominance order) on \( \mathbb{Z}^n \) as follows. For two permutations \( w, v \in S_\infty \), we write \( w \leq v \) if \( w = v \) or there exists an \( i \in \mathbb{N} \) such that \( w(j) = v(j) \) for all \( j < i \) and \( w(i) < v(i) \). Likewise, we write \( w \preceq v \) if \( w = v \) or there exists an \( i \in \mathbb{N} \) such that \( w(j) = v(j) \) for all \( j > i \) and \( w(i) < v(i) \). For the choice of \( \lambda \), since the other implication follows by exchanging \( \lambda \) for \( \mu \) and \( \lambda \preceq \mu \) for \( \lambda \leq \mu \), for general \( \lambda \) and \( \mu \) in \( \mathbb{Z}^n \), take \( k \) so that \( \lambda + k \mathbf{1} \) and \( \mu + k \mathbf{1} \) are in \( \mathbb{Z}^n \), and define \( \lambda \geq \mu \iff \lambda + k \mathbf{1} \geq \mu + k \mathbf{1} \). Note that this definition does not depend on the choice of \( k \) since \( \text{perm}(\lambda)^{-1} \leq \text{perm}(\mu)^{-1} \iff \text{perm}(\lambda + 1)^{-1} \leq \text{perm}(\mu + 1)^{-1} \) for \( \lambda, \mu \in \mathbb{Z}^n \). We define the other ordering \( \geq' \) in the same way, except that we use \( \preceq' \) instead of \( \preceq \). We prepare the following two lemmas about these orderings:

**Lemma 5.2.** For \( \lambda, \mu \in \mathbb{Z}^n \), \( \lambda \geq \mu \) if and only if \( \rho - \lambda \geq' \rho - \mu \).

Proof. We may assume \( |\lambda| = |\mu| \). We only need to prove the “only if” direction since the other implication follows by exchanging \( \lambda \) and \( \mu \). Take integers \( L \) and \( M \) so that \( \lambda + L \mathbf{1}, \mu + L \mathbf{1}, \rho - \lambda + M \mathbf{1}, \rho - \mu + M \mathbf{1} \in \mathbb{Z}^n \). Let \( w = \text{perm}(\lambda + L \mathbf{1}), v = \text{perm}(\mu + L \mathbf{1}), w' = \text{perm}(\rho - \lambda + M \mathbf{1}) \) and \( v' = \text{perm}(\rho - \mu + M \mathbf{1}) \). Then \( w, v, w', v' \in S_\infty \cap S_N \), and these permutations are related by \( w'(i) = N + 1 - w(i), v'(i) = N + 1 - v(i) \) for \( i = 1, \ldots, n \), where \( N = n + L + M \). Thus, for \( p \in \{1, \ldots, N\} \), \( w'^{-1}(p) \leq n \) if and only if \( w^{-1}(N + 1 - p) \leq n \) (and the same for \( v \)) and \( w'^{-1}(p) = \begin{cases} w^{-1}(N + 1 - p) & (w^{-1}(p) \leq n) \\ n + N + 1 - w^{-1}(N + 1 - p) & (w^{-1}(p) > n) \end{cases} \) and \( v'^{-1}(p) = \begin{cases} v^{-1}(N + 1 - p) & (v^{-1}(p) \leq n) \\ n + N + 1 - v^{-1}(N + 1 - p) & (v^{-1}(p) > n) \end{cases} \).

Now assume \( w \leq v \). If \( w = v \) we have nothing to prove so we assume that there is an \( i \) such that \( w^{-1}(1) = v^{-1}(1), \ldots, w^{-1}(i - 1) = v^{-1}(i - 1), w^{-1}(i) < v^{-1}(i) \). By the above description of \( w' \) and \( v' \) it is clear that \( w'^{-1}(j) = v'^{-1}(j) \) for \( j > N + 1 - i \). We show \( w'^{-1}(N + 1 - i) < v'^{-1}(N + 1 - i) \). If \( w^{-1}(i) < v^{-1}(i) \leq n \) we have \( w^{-1}(N + 1 - i) = w^{-1}(i) < v^{-1}(i) = v^{-1}(N + 1 - i) \). If \( w^{-1}(1) \leq n < v^{-1}(i) \) we have \( w^{-1}(N + 1 - i) \leq n < v^{-1}(N + 1 - i) \). The case \( n < w^{-1}(i) \) cannot occur, since in such case \( w^{-1}(i) = n + 1 + \# \{ j < i : w^{-1}(j) > n \} \), \( v^{-1}(i) = n + 1 + \# \{ j < i : v^{-1}(j) > n \} \) and \( \{ j < i : w^{-1}(j) > n \} = \{ j < i : v^{-1}(j) > n \} \).

**Lemma 5.3.** For any \( \lambda \in \mathbb{Z}^n \), the set \( \{ v : v \leq \lambda \} \) is finite and linearly ordered by \( \leq \).

Proof. Linear-orderedness is clear from the definition of \( \leq \). We claim that if \( \lambda, \mu \in \mathbb{Z}^n \) and \( \min_\lambda \lambda_i > \min_\mu \mu_i \), then \( \lambda \leq \mu \) (this shows the lemma since
there exists only finitely many \( v \in \mathbb{Z}^n \) with \(|v| = |\lambda|\) and \( \min_{i} v_i \geq \min_{i} \lambda_i \).
We may assume that \( \lambda, \mu \in \mathbb{Z}_{\geq 0}^n \). Let \( m = \min_{i} \mu_i \). Then \( w = \text{perm}(\lambda) \) and \( v = \text{perm}(\mu) \) satisfy \( w^{-1}(1) = v^{-1}(1) = n + 1, \ldots, w^{-1}(m) = v^{-1}(m) = n + m \) and \( w^{-1}(m + 1) > n \geq v^{-1}(m + 1) \). Thus \( w^{-1} \geq v^{-1} \).

We define \( C_{\leq \lambda} = C_{\{v, v \leq \lambda\}} \), \( C_{< \lambda} = C_{\{v, v < \lambda\}} \) and \( C_{\leq \lambda} = C_{\{v, v \leq \lambda\}} \). The main result of this section is the following proposition:

**Proposition 5.4.** For \( \lambda \in \mathbb{Z}^n \), the modules \( S_{\lambda} \) and \( S_{\rho} \otimes K_\rho \) are in \( C_{\leq \lambda} \), Moreover, \( S_{\lambda} \) is projective and \( S_{\rho} \otimes K_\rho \) is injective in \( C_{\leq \lambda} \).

Note that, by the remark before Lemma 5.1, the last claim is equivalent to the claim that \( S_{\rho} \) is projective in \( C_{\{v, v < \lambda\}} \). Moreover, since the head of \( S_{\lambda} \) is \( K_\lambda \), this proposition claims that \( S_{\lambda} \) is the projective cover of \( K_\lambda \) in both \( C_{\leq \lambda} \) and \( C_{< \lambda} \), i.e., \( S_{\lambda} \cong P_{\lambda} \cong P_{\lambda} \) (we write \( P_{\lambda} \) and \( P_{\lambda} \) for \( P_{\lambda} \) and \( P_{\lambda} \) respectively). We also note that the proposition implies the same statement for \( \leq \) instead of \( \leq \).

To prove Proposition 5.4 we have to prove the following four facts: for every \( \lambda, \mu \in \mathbb{Z}^n \), (1) \( (S_{\lambda})_\mu \neq 0 \) implies \( \lambda \geq \mu \), (2) \( (S_{\rho} \otimes K_\rho)_\mu \neq 0 \) (which is equivalent to \( (S_{\rho} \otimes K_\rho)_\mu \neq 0 \)) implies \( \lambda \geq \mu \), (3) \( \text{Ext}^1(S_{\lambda}, K_\rho) \neq 0 \) implies \( \lambda < \mu \) (here \( \text{Ext}^1 \) is taken in either \( \mathcal{O} \) or \( C_{\leq \lambda} \), which does not matter since \( C_{\leq \lambda} \) is closed under extension), and (4) \( \text{Ext}^1(S_{\rho} \otimes K_\rho) \neq 0 \) implies \( \lambda < \mu \). Note that each of these claim is invariant under \( \lambda, \mu \mapsto \lambda + k1, \mu + k1 \) for any \( k \in \mathbb{Z} \).

We first make an observation about the weights of a Schubert module. In general, for \( 1 \leq i_1 < \cdots < i_r \leq n \), the \( \mathcal{U}(b) \)-module generated by \( u_{i_1} \wedge \cdots \wedge u_{i_r} \) is contained in (in fact, equal to) the span of the elements \( u_{i_1} \wedge \cdots \wedge u_{i_s} \) (\( j_s \leq i_s \)). Therefore if we let \( \{i : (i, p) \in D(w)\} = \{i_{1p}, \ldots, i_{rp}\} \) \( (i_{1p} \leq \cdots < i_{rp}) \) for each \( p \in \mathbb{N} \), then \( S_w \) is contained in the space spanned by the elements \( u_{i_{1p}} \wedge \cdots \wedge u_{i_{rp}} \) \( (j_k \leq i_{kp}) \). In particular, every weight of \( S_w \) appears as a weight of the latter space.

We can picture this situation as follows. A *labelling of* \( D(w) \) is a map \( T : D(w) \to \mathbb{N} \). A labelling \( T \) is called *column-strict* if \( T(i, p) \neq T(i', p) \) for any \( (i, p), (i', p) \in D(w) \) with \( i \neq i' \). A labelling \( T \) is called *flagged* if \( T(i, p) \leq i \) for any \( (i, p) \in D(w) \). The *weight* of a labelling \( T \) is defined as \( (\lambda_1, \lambda_2, \ldots) \) where \( \lambda_k \in \mathbb{Z}_{\geq 0} \) is the number of \( (i, p) \in D(w) \) with \( T(i, p) = k \). Then, from the observation above, if \( (S_w)_\mu \neq 0 \) then \( \mu \) is the weight of some column-strict flagged labelling of \( D(w) \).

**Remark 5.5.** \( \text{ch}(S_w) = S_w \) in fact enumerates all column-strict flagged labellings with certain condition: see [9].

Let us now move on to the proof of Proposition 5.4. First we prove (1) and (2).

(1): We may assume that \( \lambda \) and \( \mu \) are in \( \mathbb{Z}^n_{\geq 0} \). We define \( \mu_{n+1} = \mu_{n+2} = \cdots = 0 \) (same for the proofs below). We prove the stronger statement: if \( \mu \) is the weight of some column-strict flagged labelling of \( D(\text{perm}(\lambda)) \) then \( \lambda \geq \mu \).
Let \( w = \text{perm}(\lambda) \) and \( v = \text{perm}(\mu) \). Take \( N \) so that \( w \) and \( v \) are both in \( S_N \).

We first show \( w^{-1}(1) \leq v^{-1}(1) \). Let \( i = w^{-1}(1) \). Then we have \( w(1), \ldots, w(i-1) \geq 2 \) and so the first column of \( D(w) \) consists of boxes \( (1,1), \ldots, (i-1,1) \). Since \( T \) is column-strict and flagged, the only possible labelling for this first
column is $T(j, 1) = j$ for all $1 \leq j \leq i - 1$. Therefore $\mu_1, \ldots, \mu_{i-1} \geq 1$ and so
$i \leq \min\{j : \mu_j = 0\} = v^{-1}(1)$.

Next we consider what happens when $w^{-1}(1) = v^{-1}(1)$. In this case we have $\mu_i = 0$, i.e., the labelling $T$ does not use $i$. We define a new labelling $T'$ of $D(w')$ ($w' = [w(1) - 1 \cdots w(i - 1) - 1 w(i + 1) - 1 \cdots w(N) - 1] \in S_{N-1}$) as follows. For $(i', p') \in D(w')$, we have $(\sigma_i(i'), p' + 1) \in D(w)$ where $\sigma_i(i') = \begin{cases} i' & (i' < i) \\ i' + 1 & (i' > i) \end{cases}$. Since $T$ does not use the label $i$, $T(\sigma_i(i'), p' + 1)$ is either strictly smaller or strictly larger than $i$. Define $T'(i', p') = T(\sigma_i(i'), p' + 1)$ in the former case and $T(i', p') = T(\sigma_i(i'), p' + 1) - 1$ in the latter case. It is easy to check that $T'$ is column-strict and flagged. Moreover, $T'$ has weight $\mu' = (\mu_1, \ldots, \mu_{i-1} - 1, \mu_{i+1}, \ldots, \mu_N)$. Therefore, by an inductive argument, we have $\text{perm}(\mu')^{-1} = [v(1) - 1 \cdots v(i - 1) - 1 v(i + 1) - 1 \cdots v(N) - 1]^{-1} \leq w'^{-1}$.

This shows $w^{-1}_\lambda \leq v^{-1}$.

(2): We prove the stronger statement: if $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$, and $\mu$ is the weight of some column-strict flagged labelling of $D(\text{perm}(\lambda))$ then $\rho - \lambda \geq \rho - \mu$, or equivalently, $\lambda \geq \mu$. Let $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$. Take $N$ so that $w, v \in S_N$. Take a column-strict flagged labelling $T$ of $D(w)$ with weight $\mu$.

We first show $w^{-1}(N) \leq v^{-1}(N)$. Let $i = w^{-1}(N)$ and let $j < i$. In the $w(1), \ldots, w(j)$-th and the $N$-th columns, there exist no boxes weakly below the $j$-th row. Thus the label $j$ cannot appear in these columns since $T$ is flagged. Therefore $T$ contains the label $j$ at most $N - j - 1$ times since $T$ is column-strict. This shows $i \leq \min\{j : \mu_j = N - j\} = v^{-1}(N)$.

Now consider the case $w^{-1}(N) = v^{-1}(N)$. In this case, the label $i$ is used $N - i$ times in $T$. The columns in which the label $i$ can be used are the columns $w(i + 1), \ldots, w(N)$, since the other columns have boxes below the $i$-th row. Therefore, for each $i + 1 \leq j \leq N$, the label $i$ must appear somewhere in the $w(j)$-th column. If $i$ appears strictly below the $i$-th row in the $w(j)$-th column (say at the box $(i', w(j))$, then swapping the labels of $(i, w(j))$ and $(i', w(j))$ gives another column-strict flagged labelling of $D(w)$. We therefore may assume that all the labels $i$ used in $T$ are in the $i$-th row of $D(w)$. Then we can define a new labelling $T'$ of $D(w')$ ($w' = [w(1) - 1 \cdots w(i - 1) - 1 w(i + 1) - 1 \cdots w(N)] \in S_{N-1}$) by $T'(i', p') = \begin{cases} T(\sigma_i(i'), p' + 1) & (T(i, p') < i) \\ T(\sigma_i(i'), p' + 1) - 1 & (T(i, p') > i) \end{cases}$. It is easy to check that $T'$ is column-strict and flagged and has weight $\mu' = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N)$. So $\text{perm}(\mu')^{-1} = [v(1) - 1 \cdots v(i - 1) - 1 v(i + 1) - 1 \cdots v(N)]^{-1} \leq w'^{-1}$ by an inductive argument. This shows $w^{-1}_\lambda \leq v^{-1}$.

Before we prove (3) and (4), we note the following things. By Theorem 3.3, for a permutation $w$ there is a projective resolution of $S_w$ in $C$ starting as $\cdots \to P_1 \to P_0 \to S_w \to 0$, with $P_0 = P_{\text{inv}(w)}$, $P_1 = \bigoplus_{p < q} P^\text{inv}(w)_{\text{pr}(w) + 1(m_{pq}(w) + 1)\epsilon_p - \epsilon_q}$. Here by Remark 3.3 we can take $P_1$ as the direct sum over all $p < q$ such that there does not exist $p < r < q$ with $m_{pq}(w) = m_{pr}(w) + m_{rq}(w)$. Thus $\text{Ext}^1(S_w, K_p) = 0$ unless $v = \text{inv}(w) + (m_{pq}(w) + 1)\epsilon_p - \epsilon_q$ for some $p < q$ with the property above.

(3): We may assume that $\lambda, \mu \in \mathbb{Z}^n_{\geq 0}$. Let $w = \text{perm}(\lambda)$ and $v = \text{perm}(\mu)$. Take $N$ so that $w, v \in S_N$. We have $\mu = \lambda + (m_{pq}(w) + 1)\epsilon_p - \epsilon_q$ for some
For any $\lambda \in C_{\leq \lambda}$, and $i \geq 0$, $\text{Ext}^i_{\leq \lambda}(M, N) \cong \text{Ext}^i_{\leq \lambda}(M, N)$. Here Ext$^i_{\leq \lambda}$ is short for Ext$^i_{\leq \lambda}$.

**Proof.** It is enough (by Lemma 5.3) to prove $\text{Ext}^i_{\leq \lambda}(M, N) = \text{Ext}^i_{\leq \lambda}(M, N)$ for $M, N \in C_{\leq \lambda}$. Take a projective resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ such that each $P_i$ is a direct sum of some copies of $P^{\leq \lambda}_{\mu}$ with $\mu \leq \lambda$ (in fact, the only indecomposable projectives in $C_{\leq \lambda}$ are $P^{\leq \lambda}_{\mu}$, so this condition is superfluous).

For $L \in C_{\leq \lambda}$, let $\overline{L}$ be the largest quotient of $L$ which is in $C_{\leq \lambda}$, i.e., $\overline{L}$ is the quotient of $L$ by the submodule generated by the weight space $L_\lambda$ of weight $\lambda$. 

Note that if $P_i = P^{\leq \lambda}_{\mu} \oplus \cdots$, then $\overline{P_i} = P^{\leq \lambda}_{\mu} \oplus \cdots$, where $P^{\leq \lambda}_{\mu}$ is the largest quotient of $P_{\mu}$ which is in $C_{\leq \lambda}$. We are done if we show that $\cdots \rightarrow \overline{P_1} \rightarrow \overline{P_0} \rightarrow M \rightarrow 0$ is a projective resolution of $M$, since $\text{Hom}(\overline{P_i}, N) = \text{Hom}(P_i, N)$. 


Let $\ker_1$ be the kernel of $P_1 \to \overline{P}_1$. Since $\cdots \to P_3 \to P_2 \to P_1 \to M \to 0$ is exact, the exactness of $\cdots \to \overline{P}_3 \to \overline{P}_2 \to \overline{P}_1 \to \overline{M} \to 0$ is equivalent to that of $\cdots \to \ker_1 \to \ker_0 \to 0$.

For any $\mu, \nu \leq \lambda$, we have a morphism $(P_{\mu}^{\leq \lambda})_\lambda \otimes (P_{\nu}^{\leq \lambda})_\nu \to (P_{\nu}^{\leq \lambda})_\nu$ of vector spaces defined by $xu_\mu \otimes yu_\lambda \mapsto yxu_\mu$ for $x, y \in U(n^+)$ where $u_\mu$ is the image of $1 \otimes 1 \in U(n^+) \otimes K, \mu = P_\mu$ by $P_\mu \to P_{\mu}^{\leq \lambda}$ (this definition does not depend on the choice of $y$ since the submodule of $P_{\mu}^{\leq \lambda}$ generated by $u_\mu$ is a quotient of $P_{\mu}^{\leq \lambda}$ by definition). This morphism defines a surjection from $(P_{\mu}^{\leq \lambda})_\lambda \otimes (P_{\nu}^{\leq \lambda})_\nu$ to the kernel of $(P_{\mu}^{\leq \lambda})_\nu \to (P_{\nu}^{\leq \lambda})_\nu$. We are done if we show that this surjection is in fact an isomorphism, since if we show this we have $(P_1)_\lambda \otimes (P_\lambda^{\leq \lambda})_\nu \cong (\ker_1)_\nu$, and thus the exactness of $\cdots \to \ker_1 \to \ker_0 \to 0$ follows from that of $\cdots \to (P_1)_\lambda \to (P_0)_\lambda \to 0$.

We have $(P_{\mu}^{\leq \lambda})_\nu \cong (S_{\lambda})_\nu$ by the result of the previous section. The result from the previous section also shows that $(P_{\mu}^{\leq \lambda})_\lambda \cong (S_{\nu-\rho})_{\rho-\mu}$, since as vector spaces $(P_{\mu}^{\leq \lambda})_\lambda \cong U(n^+)_{\lambda-\mu}/\Span_K\{xy : x \in U(n^+)_{\lambda-\mu}, y \in U(n^+)_{\kappa-\mu} (\exists \kappa s.t. \kappa \leq \lambda)\}$ and $(S_{\nu-\rho})_{\rho-\mu} \cong (P_{\rho-\lambda})_{\rho-\mu} \cong U(n^+)_{\lambda-\mu}/\Span_K\{xy : x \in U(n^+)_{\lambda-\mu}, y \in U(n^+)_{\kappa-\mu} (\exists \kappa s.t. \rho - \kappa \leq \rho - \mu)\}$, and thus the algebra antiautomorphism on $U(n^+)$ defined by $X \mapsto X (X \in n^+)$ induces an isomorphism between these two spaces by Lemma 5.2.

By the observations above, if $\lambda \in \mathbb{Z}^n, \mu, \nu \leq \lambda$ and $\kappa^{(1)} < \cdots < \kappa^{(r)}$ are the weights less than or equal to $\lambda$, then there exists a quotient filtration $(P_{\mu}^{\leq \lambda})_\nu = (P_{\mu}^{\leq \kappa^{(r)}})_\nu \to (P_{\mu}^{\leq \kappa^{(r-1)}})_\nu \to \cdots \to (P_{\mu}^{\leq \kappa^{(1)}})_\nu \to 0$ with each subquotient being a quotient of $(S_{\mu})_{\rho-\mu} \otimes (S_{\kappa^{(i)}})_\nu$. Here we used a notation that if $\mu$ or $\nu$ is greater than $\kappa$ then $(P_{\mu}^{\leq \kappa})_\nu = 0$. So, if we show that $\dim(P_{\mu}^{\leq \kappa})_\nu$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{\kappa \leq \lambda} \langle S_{\mu-\kappa}(x)S_{\nu}(y) \rangle$, then the desired isomorphism $(S_{\nu-\rho})_{\rho-\mu} \otimes (S_{\lambda})_\nu \cong \ker((P_{\mu}^{\leq \kappa})_\nu \to (P_{\mu}^{\leq \kappa})_\nu)$ follows for all $\kappa \leq \lambda$. Since $\dim(P_{\mu}^{\leq \kappa})_\nu = \dim(U(n^+))_{\nu-\rho}$ for a sufficiently large $\lambda$, the proof of this lemma is reduced to the following lemma:

**Lemma 6.2.** For $\mu, \nu \in \mathbb{Z}^n, \dim(U(n^+))_{\nu-\rho}$ is equal to the coefficient of $x^{\rho-\mu}y^\nu$ in $\sum_{\kappa \in \mathbb{Z}^n} \langle S_{\mu-\kappa}(x)S_{\nu}(y) \rangle$.

Let us prove this lemma. We use the following result from [9]:

**Lemma 6.3** ([9] Lemma 6.2 and Corollary 9.2, reformulated). For a positive integer $N$, define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by $\langle x^\alpha, x^\beta \rangle = \delta_{\alpha, \beta}$. Then for $w, v \in S_N$, $\langle \langle w, \langle w_0, v \rangle(x_1^{-1}, \ldots, x_N^{-1}) \rangle \rangle_{1 \leq i,j \leq N}(x_i-x_j) = \delta_{wv}$, where $w_0 = \{N N-1 \cdots 1\} \in S_N$.

We slightly modify this lemma into one which is more suitable for our use:

**Lemma 6.4.** If we define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$ by $\langle x^\alpha, x^\beta \rangle = \delta_{\alpha, \beta}$, then for $\lambda, \mu \in \mathbb{Z}^n$, $\langle \langle \langle \langle \langle \rangle \rangle \rangle \rangle \rangle \rangle$.

Proof. We may assume that $\lambda, \mu \in \mathbb{Z}^n$. Let $w = \perm(\lambda), v = \perm(\mu)$. Take $N$ so that $w, v \in S_N$. Then by the previous lemma, we have

$$\langle \langle \langle \langle \langle \rangle \rangle \rangle \rangle \rangle = \delta_{wv} = \delta_{\lambda, \mu} \cdots (\ast)$$

where $w_0 = \{N N-1 \cdots 1\} \in S_N$. 13
Since \( S_{w,v}(x_1^{-1}, \ldots, x_N^{-1}) \) is a linear combination of \( \prod x_i^{-a_i} \) with \( 0 \leq a_i \leq N - i \), the total degree in variables \( x_{n+1}, \ldots, x_N \) of all of its terms is greater than or equal to \(-((N - n - 1) + (N - n - 2) + \cdots + 1)\). On the other hand, \( \prod_{1 \leq i < j \leq N} (x_i - x_j) = \sum_{g \in S_N} \text{sgn}(g)x^{g\rho(N)} \) where \( \rho(N) = (N - 1, N - 2, \ldots, 0) \in \mathbb{Z}^N \), and the total degree of the summand is always greater than or equal to \((N - n - 1) + (N - n - 2) + \cdots + 1\). Thus, since \( S_w \) is a polynomial in \( x_1, \ldots, x_n \), if we denote by \( f \) the sum of all terms in \( S_{w,v}(x_1^{-1}, \ldots, x_N^{-1}) \) whose total degree in \( x_{n+1}, \ldots, x_N \) is exactly \(-((N - n - 1) + (N - n - 2) + \cdots + 1)\), the LHS of (1) is equal to

\[
(\langle S_w, f \rangle, \sum_{g \in S_N} \text{sgn}(g)x^{g\rho(N)})
\]

\[
= (\langle S_w, f \rangle, (x_1 \cdots x_n)^{N-n}x_{n+1}^{-N-n-1}x_{n+2}^{-N-n-2} \cdots x_{N-1}^{-1} \prod_{1 \leq i < j \leq n} (x_i - x_j))
\]

So we are done if we show

\[
f = (x_1 \cdots x_n)^{N-n}x_{n+1}^{-N-n-1}x_{n+2}^{-N-n-2} \cdots x_{N-1}^{-1} S_{\rho-\mu}(x_1^{-1}, \ldots, x_n^{-1}).
\]

We show the equivalent claim, i.e., the sum \( \tilde{f} \) of all terms in \( S_{w,v} \) whose total degree in \( x_{n+1}, \ldots, x_N \) is exactly \((N - n - 1) + \cdots + 1\) is equal to

\[
(x_1 \cdots x_n)^{N-n}x_{n+1}^{-N-n-1}x_{n+2}^{-N-n-2} \cdots x_{N-1}^{-1} S_{\rho-\mu}.
\]

In fact, if we consider \( \partial_{N-1} \partial_{N-2} \cdots (\partial_{n+1} \cdots \partial_{N-1})S_{w,v} = S_{\rho,v} \) where \( g = s_{N-1} \cdots s_{n+1} \cdots s_{N-2} s_{N-1} \cdots s_1 \), then we can show \( S_{\rho,v} = x_{n+1}^{-N-n-1}x_{n+2}^{-N-n-2} \cdots x_{N-1}^{-1} \tilde{f} \). This gives the desired result since \( \text{inv}(g(v)) = (N - n) + 1 + \rho - \mu \).

Let us get back to the proof of Lemma 6.2. Essentially this is a “Cauchy formula” for the dual bases \( \{x_k\} \) and \( \{\Theta_{\rho-\mu}(x_1^{-1}, \ldots, x_n^{-1})\} \), but since we are dealing with an infinite-dimensional space a careful justification is needed. Let \( c_{\alpha,\beta} \) be the coefficient of \( x^\alpha y^\beta \) in \( \sum_{\kappa \in \mathbb{Z}^n} \Theta_{\rho-\kappa}(x)\Theta_{\kappa}(y) \). We observe that if \( c_{\rho-\mu,\nu} \neq 0 \), then \( \rho - \mu \geq \kappa \geq \nu \) and \( \nu \geq \kappa \) must hold and so \( \nu \geq \rho \geq \kappa \). So \( c_{\rho-\mu,\nu} = 0 \) for \( \nu \geq \rho \). Using this as the base case, if we show \( \sum_{g \in S_N} \text{sgn}(g)c_{\rho-\mu,\nu-\rho+g\rho} = \delta_{\mu,\nu} \), or equivalently, \( \sum_{g \in S_N} \text{sgn}(g)c_{\kappa,\beta+g\rho} = \delta_{\alpha,\beta-\rho} \), then we can show \( c_{\rho-\mu,\nu} = \text{dim}(U(n)^{\nu-\mu}) \) by induction on \( \nu \).

Since \( c_{\alpha,\beta+g\rho} = c_{\alpha+g1,\beta+g}\rho-k1 \), we may assume that \(-\beta \in \mathbb{Z}_{\geq 0}^n\). We may further assume that if \( \kappa \in \mathbb{Z}^n \) satisfies \( \alpha \geq \kappa \geq \beta + \rho - g \rho \) for some \( g \in S_N \), then \( \kappa \in \mathbb{Z}_{\geq 0}^n \), i.e., the remark at the end of Section 2. It is sufficient to consider the case \( |\alpha| = -|\beta| \). Let \( d = |\alpha| \). Let \( V \) be the space of all ordinary polynomials in \( x_1, \ldots, x_n \) which are homogeneous of degree \( d \). Equip \( V \) with a bilinear form \( \langle x^\alpha, x^\beta \rangle = \delta_{\alpha,\beta} \). Then by Lemma 6.4 the bases \( \{x_k\} : \kappa \in \mathbb{Z}_{\geq 0}^n \} \) and \( \{\Theta_{\rho-\kappa}(x_1^{-1}, \ldots, x_n^{-1})\} \) coincide.
are dual of each other. Here for \( f \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), \([f]\) is the sum of all terms in \( f \) which do not contain any negative powers of \( x_1, \ldots, x_n \). Thus we have

\[
\sum_{\kappa \in \mathbb{Z}^n_{\geq 0}, |\kappa| = d} \sum_{\gamma \in \mathbb{Z}^n_{\geq 0}, |\gamma| = d} \xi_{\gamma}(x_1, \ldots, x_n) \xi_{\gamma - \kappa}(y_1^{-1}, \ldots, y_n^{-1}) \prod_{1 \leq i < j \leq n} (y_i - y_j)
\]

modulo terms containing some negative powers of some \( y_i \) (recall that for any finite-dimensional vector space \( V \), the sum \( \sum \phi_i \otimes \phi_i^* \in V \otimes V^* \) does not depend on the choice of dual bases \( \{\phi_i\} \subset V, \{\phi_i^*\} \subset V^* \). Since \( -\beta \in \mathbb{Z}^n_{\geq 0} \), the coefficient of \( x^\alpha y^{-\beta} \) is equal for both side. The coefficient for the LHS is \( \delta_{\alpha, -\beta} \). Moreover, if \( \kappa \in \mathbb{Z}^n \) and \( \xi_{\kappa}(x_1, \ldots, x_n) \xi_{\kappa - \kappa}(y_1^{-1}, \ldots, y_n^{-1}) \) contains some \( x^\alpha y^{-\beta - g\rho} \) (\( g \in S_n \)) with nonzero coefficients, then such \( \kappa \) does appear in the sum above, since such \( \kappa \) must satisfy \( \alpha \geq \kappa \) and \( \beta + g\rho \geq \mu - \kappa \). So the coefficient of \( x^\alpha y^{-\beta} \) in the RHS is \( \sum_{g \in S_n} \text{sgn}(g) c_{\alpha, \beta + g\rho} \) and we are done. 

**Remark 6.5.** This proof, together with some results from the previous section, in fact shows that \( C_{\leq \lambda} \) can be equipped with a structure of highest-weight category (\([11]\) Theorem 3.2.2) for Schubert modules.

From Lemma 5.1, we obtain the following corollary. This can be seen as an analog of “Strong form of Polo’s theorem” (\([11]\) Theorem 3.2.2) for Schubert modules.

**Corollary 6.6.** For \( \lambda \in \mathbb{Z}^n, \mu, \nu \leq \lambda \) and \( i \geq 1 \), \( \text{Ext}_{\leq \lambda}^i(S_\mu, S_{\mu - \nu} \otimes K_\nu) = 0 \).

**Proof.** By Lemma 6.1, it suffices to prove \( \text{Ext}_{\leq \max(\mu, \nu)}^i(S_\mu, S_{\mu - \nu} \otimes K_\nu) = 0 \). If \( \mu \geq \nu \), this follows from the projectivity of \( S_\mu \in C_{\leq \mu} \) since \( S_{\mu - \nu} \otimes K_\nu \in C_{\leq \mu} \). Otherwise it follows from the injectivity of \( S_{\mu - \nu} \otimes K_\nu \in C_{\leq \nu} \) since \( S_\mu \in C_{\leq \nu} \).

### 7 Filtration by Schubert modules

Using the results obtained so far, we can obtain a criterion for a module to have a filtration by Schubert modules, using the similar argument from \([11] \) §3. Hereafter, \( \text{Ext}^i \) means \( \text{Ext}_{\leq \lambda}^i \) for certain \( \lambda \) (by Lemma 5.1, this does not depend on the choice of \( \lambda \)).

**Theorem 7.1.** Let \( \lambda \in \mathbb{Z}^n, M \in C_{\leq \lambda} \) and assume that \( \text{Ext}^i(M, S_{\mu - \nu} \otimes K_\nu) = 0 \) for all \( \mu \leq \lambda \). Then \( M \) has a filtration such that each of its subquotients is isomorphic to some \( S_\mu \) (\( \nu \leq \lambda \)).

Note that the converse obviously holds since \( \text{Ext}^1(S_\nu, S_{\mu - \nu} \otimes K_\nu) = 0 \).

**Proof.** Let \( \{\nu : \nu \leq \lambda\} = \{\nu^{(1)} < \nu^{(2)} < \cdots < \nu^{(r)}\} \). Let \( M_i \) be the largest quotient of \( M \) whose weights are in \( \{\nu^{(1)}, \ldots, \nu^{(i)}\} \) (so \( M_0 = 0 \) and \( M_r = M \)). By definition, we have a natural surjection \( M_i \twoheadrightarrow M_j \) for \( i > j \). We show that \( \text{Ker}(M_i \twoheadrightarrow M_{i-1}) \) is a direct sum of some copies of \( S_{\nu} \) by the induction on \( i \). This will show that \( M = M_r \twoheadrightarrow M_{r-1} \twoheadrightarrow \cdots \twoheadrightarrow M_0 = 0 \) gives a quotient filtration with desired property.
Fix $i$ and let $\nu = \nu^{(i)}$. It is sufficient to show that $\text{Ker}(M_i \to M_{i-1})$ is the projective cover of its $\nu$-weight space $\text{Ker}(M_i \to M_{i-1})_\nu$ in $\mathcal{C}_\leq \nu$, since $\mathcal{S}_\nu$ is the projective cover of $K_\nu$ in $\mathcal{C}_\leq \nu$. Since $\text{Ker}(M_i \to M_{i-1})$ is generated by $\text{Ker}(M_i \to M_{i-1})_\nu$, it suffices to show that $\text{Ker}(M_i \to M_{i-1})$ is projective in $\mathcal{C}_\leq \nu$, that is, $\text{Ext}^1(\text{Ker}(M_i \to M_{i-1}), K_\mu) = 0$ for all $\mu \leq \nu$.

Let $\mu \leq \nu$. We have an exact sequence $\text{Ext}^1(M, S^*_\rho \otimes K_\rho) \to \text{Ext}^1(\text{Ker}(M \to M_{i-1}), S^*_\rho \otimes K_\rho) \to \text{Ext}^2(M_{i-1}, S^*_\rho \otimes K_\rho)$. Here $\text{Ext}^1(M, S^*_\rho \otimes K_\rho) = 0$ by the hypothesis. Moreover, $\text{Ext}^1(M_{i-1}, S^*_\rho \otimes K_\rho) = 0$ by Corollary 6.6, since $M_{i-1}$ has a filtration by modules $S_\kappa$ ($\kappa < \nu$) by the induction hypothesis. Therefore $\text{Ext}^1(\text{Ker}(M \to M_{i-1}), S^*_\rho \otimes K_\rho) = 0$.

We have an exact sequence $\text{Hom}(\text{Ker}(M \to M_i), S^*_\rho \otimes K_\rho) \to \text{Ext}^1(\text{Ker}(M \to M_{i-1}), S^*_\rho \otimes K_\rho) = 0$. But here $\text{Hom}(\text{Ker}(M \to M_i), S^*_\rho \otimes K_\rho) = 0$, since the weights of $S^*_\rho \otimes K_\rho$ are all less than or equal to $\mu$ and therefore $\leq \nu$, while $\text{Ker}(M \to M_i)$ is generated by the elements whose weights are $> \nu$. Therefore $\text{Ext}^1(\text{Ker}(M \to M_{i-1}), S^*_\rho \otimes K_\rho) = 0$.

We have an exact sequence $\text{Hom}(\text{Ker}(M_i \to M_{i-1}), (S^*_\rho \otimes K_\rho)/K_\mu) \to \text{Ext}^1(\text{Ker}(M_i \to M_{i-1}), K_\mu) \to \text{Ext}^1(\text{Ker}(M_i \to M_{i-1}), S^*_\rho \otimes K_\rho) = 0$. But since the weights of $(S^*_\rho \otimes K_\rho)/K_\mu$ are strictly less than $\mu$ and thus $< \nu$ while $\text{Ker}(M_i \to M_{i-1})$ is generated by its $\nu$-weight space, $\text{Hom}(\text{Ker}(M_i \to M_{i-1}), (S^*_\rho \otimes K_\rho)/K_\mu) = 0$. So $\text{Ext}^1(\text{Ker}(M_i \to M_{i-1}), K_\mu) = 0$ and we are done.

Another criterion for filtration can be also derived:

**Theorem 7.2.** Let $\lambda \in \mathbb{Z}^n$ and $M \in \mathcal{C}_{\leq \lambda}$. Then $\text{ch}(M) \leq \sum_{\nu \leq \lambda} \dim (\text{Hom}(M, S^*_\rho \otimes K_\rho))_{\mathcal{S}_\nu}$, and the equality holds if and only if $M$ has a filtration such that each of its subquotients is isomorphic to some $\mathcal{S}_\nu$ ($\nu \leq \lambda$). Here $\sum a_\nu x^\nu \leq \sum b_\nu x^\nu$ is defined as $a_\nu \leq b_\nu$ ($\forall \nu$).

**Proof.** Let $\nu : \nu \leq \lambda = \{\nu^{(1)} < \nu^{(2)} < \cdots < \nu^{(r)}\}$. By the proof of Theorem 7.1 $M$ has a desired filtration if and only if $\text{Ker}(M_i \to M_{i-1})$ is a direct sum of some copies of $S^{(i)}_{\nu^{(i)}}$, where $M_i$ is the largest quotient of $M$ whose weight are in $\{\nu^{(1)}, \ldots, \nu^{(i)}\}$.

We have $\text{ch}(M) = \sum_{i} \text{ch}(\text{Ker}(M_i \to M_{i-1}))$. Since $\text{Ker}(M_i \to M_{i-1})$ is generated by its $\nu^{(i)}$-weight space $(M_i)_{\nu^{(i)}}$, if we let $d_i$ denote the dimension of this weight space, we have a surjection from $(P_{\nu^{(i)}}^{\leq \nu^{(i)}})^{\oplus d_i}$ to $\text{Ker}(M_i \to M_{i-1})$, where $P_{\nu^{(i)}}^{\leq \nu^{(i)}}$ is the largest quotient of $P_{\nu^{(i)}}$ whose weights are all $\leq \nu^{(i)}$, which is $S_{\nu^{(i)}}$ by Proposition 5.4. Thus $\text{ch}(M) \leq \sum_{i} \dim ((M_i)_{\nu^{(i)}})S_{\nu^{(i)}}$ and the equality holds when and only when each kernel is a direct sum of some copies of $S_{\nu^{(i)}}$, i.e. $M$ has a desired filtration.

For each $i$, we have $\text{Hom}(M_i, S^*_\rho \otimes K_\rho) \cong \text{Hom}(S^*_\rho, M^*_i \otimes K_\rho) \cong (M^*_i \otimes K_\rho)^{\otimes (M^*_i \otimes K_\rho)}$ where $\nu = \nu^{(i)}$, since $M^*_i \otimes K_\rho \in \mathcal{C}_{\rho \leq \nu} = \mathcal{C}_{\rho \leq \nu}$, and $S^*_\rho$ is the projective cover of $K^*_\rho$ in this category. Thus the theorem follows.
8 Questions

Question 8.1. For $\lambda, \mu \in \mathbb{Z}^n$, does $S^\lambda \otimes S^\mu$ have a filtration by Schubert modules, i.e., does it have a filtration whose every subquotient is isomorphic to some $S^\nu$ ($\nu \in \mathbb{Z}^n$)?

By the criteria obtained above, this question is equivalent to ask:

- whether $\text{Ext}^1(S^\lambda \otimes S^\mu, S^\nu \otimes K^\rho) = 0$ or not, or
- whether the dimension of $\text{Hom}(S^\lambda \otimes S^\mu, S^\nu \otimes K^\rho)$ is equal to the coefficient of $\mathcal{G}_\nu$ in the expansion of $\mathcal{G}_\lambda \mathcal{G}_\mu$ into a linear combination of Schubert polynomials.

Question 8.2. Let $s_\sigma$ denote the Schur functor corresponding to a partition $\sigma$ and let $\lambda \in \mathbb{Z}^n$. Then, does $s_\sigma(S^\lambda)$ have a filtration by Schubert modules?

As explained in the introduction, positive answer for this question implies that the “plethysm” $s_\sigma[\mathcal{G}_\lambda]$ is a positive sum of Schubert polynomials.

We note the following connection between these two problems.

Proposition 8.3. Suppose that the answer to Question 8.1 is yes. Then the answer to Question 8.2 is yes.

Proof. By iteratively using 8.1, we see that $S^{\lambda(1)} \otimes \cdots \otimes S^{\lambda(r)}$ has a filtration by Schubert modules for any $\lambda^{(1)}, \ldots, \lambda^{(r)} \in \mathbb{Z}^n$. Especially, $S^\lambda \otimes K^\rho$ has a filtration by Schubert modules for any $\lambda$ and $k$. Therefore $\text{Ext}^1((S^\lambda)^\otimes k, S^\nu \otimes K^\rho) = 0$ for any $\nu$. Since $s_\sigma(S^\lambda)$ is a direct sum factor of $(S^\lambda)^\otimes [\sigma]$, $\text{Ext}^1(s_\sigma(S^\lambda), S^\nu \otimes K^\rho) = 0$. Thus $s_\sigma(S^\lambda)$ has a desired filtration by Theorem 7.1. \qed

So the main problem we pursued, which asks the Schubert-positivity of the plethysm, is somewhat reduced to the study of generalized LR coefficients.

References

[1] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math., 391:85–99, 1988.

[2] Sergey Fomin, Curtis Greene, Victor Reiner, and Mark Shimozono. Balanced labellings and Schubert polynomials. Eur. J. Comb., 18(4):373–389, 1997.

[3] A. Joseph. On the Demazure character formula. Ann. Sci. École Norm. Sup. (4), 18(3):389–419, 1985.

[4] Witold Krasikievicz and Piotr Pragacz. Schubert functors and Schubert polynomials. Eur. J. Comb., 25(8):1327–1344, 2004.

[5] I. G. Macdonald. Notes on Schubert Polynomials. LACIM, Universite du Quebec a Montreal, 1991.

[6] I. G. Macdonald. Symmetric Functions and Hall Polynomials, second edition. Oxford University Press, 1999.
[7] Peter Magyar. Four new formulas for Schubert polynomials. http://math.msu.edu/~magyar/papers/FourFormulas.pdf.

[8] Patrick Polo. Variétés de Schubert et excellentes filtrations. Astérisque, (173-174):10–11, 281–311, 1989. Orbites unipotentes et représentations, III.

[9] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. J. Algebraic Combin., 29:133–74, 2009.

[10] Wilberd van der Kallen. Longest weight vectors and excellent filtrations. Math. Z., 201(1):19–31, 1989.

[11] Wilberd van der Kallen. Lectures on Frobenius Splittings and B-modules. Springer, 1993.