Light Cone Quantization of the $c = 2$ Matrix Model

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ABSTRACT

We study the large $N$ limit of an interacting two-dimensional matrix field theory, whose perturbative expansion generates the sum over planar random graphs embedded in two dimensions. In the light cone quantization the theory possesses closed string excitations which become free as $N \to \infty$. If the longitudinal momenta are discretized, then the calculation of the free string spectrum reduces to finite matrix diagonalization, the size of the matrix growing as the cut-off is removed. Our numerical results suggest that, for a critical coupling, the light cone string spectrum becomes continuous. This would indicate the massless dynamics of the Liouville mode of two-dimensional gravity, which would constitute a third dimension of the string theory.
1. Introduction

Recently there has been considerable renewed interest in large-\(N\) matrix models. The one-dimensional hermitian matrix chain models have been solved in the double-scaling limit and identified with the \(c < 1\) minimal models coupled to two-dimensional quantum gravity\cite{1, 2}. In the same fashion the hermitian matrix quantum mechanics, which describes the \(c = 1\) theory, has been solved both for non-compact \cite{3} and for circular target space \cite{4}. The \(c = 1\) model can be interpreted in terms of \(D = 2\) string theory \cite{5}, where the role of the extra dimension is played by the conformal factor of the world sheet quantum gravity. One should keep in mind that the models with \(c > 1\) are of much greater interest because they are expected to correspond to string theories in \(D > 2\). These theories should have a much richer structure than in \(D = 2\) because strings can exhibit transverse oscillations. Unfortunately, although the \(c \leq 1\) models have been solved exactly to all orders in the genus expansion, little is known about the \(c > 1\) theories.

In this paper we begin to fill this gap by studying a \(c = 2\) matrix model, which is defined in the euclidean space as a two-dimensional field theory with the action

\[
S = \int d^2x \, \text{Tr} \left( \frac{1}{2} (\partial_\alpha M)^2 + \frac{1}{2} \mu M^2 - \frac{1}{3\sqrt{N}} \lambda M^3 \right),
\]

where \(M(x^0, x^1)\) is an \(N \times N\) hermitian matrix field. The connection of this matrix model with triangulated two-dimensional quantum gravity follows, as usual, after identifying the Feynman graphs with the graphs dual to triangulations. The lattice link factor is the two-dimensional scalar propagator,

\[
G(\vec{x}_i, \vec{x}_j) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{i\vec{p} \cdot (\vec{x}_i - \vec{x}_j)}}{p^2 + \mu} = \frac{1}{2\pi} K_0(\sqrt{\mu} |\vec{x}_i - \vec{x}_j|),
\]

where \(K_0\) is a modified Bessel function. Thus, at the leading order in \(N\), we obtain a sum over the planar triangulated random surfaces embedded in two dimensions.\footnote{Note that, as for \(c = 1\), the coordinates are specified at the centers of the triangles.} The logarithmic divergence of \(G\) at small separations is very mild. If the tadpole graphs are discarded, as they should be because they do not correspond to good triangulations, then the entire perturbative
expansion of the theory (1) is finite. This is similar to what we find in the matrix models for $c \leq 1$. Therefore, it is sensible to look for a singularity in the sum over the planar graphs as a function of the dimensionless parameter $\lambda/\mu$. The primary purpose of this paper is to look for such critical behaviour, and to discuss the non-critical string theory that results.

The arguments above suggest that, although the $c = 2$ matrix model of eq. (1) is certainly more complex than the matrix quantum mechanics, we should still be able to take advantage of the simplifications that distinguish the two-dimensional field theories. The method to pursue this that we believe is particularly convenient, is to continue $x^0 \to ix^0$ and to carry out light cone quantization of the resulting $(1 + 1)$-dimensional field theory. This is often the most efficient way to study a theory. Its additional advantage here is that, in the large-$N$ limit, one can formulate a linear light cone Schroedinger equation satisfied by free string states. After a cut-off is introduced in the form of discretized longitudinal momentum [6, 7], this equation can be solved for the free string spectrum. Subsequently, this spectrum can be studied as the cut-off is being removed.

The procedure outlined above was used in a paper by Susskind and one of the authors [8] to study a $c > 2$ light cone lattice gauge theory of a large-$N$ complex matrix. It was shown that this theory can be driven into a phase where the longitudinal momentum of each string bit assumes the smallest possible value. In this phase, each of the $c - 2$ transverse dimensions becomes a free massless scalar field in the parameter space of the string. Then one recovers the free critical string spectrum [8, 9] without any necessity of taking the transverse lattice spacing to zero.

We could look for this “critical string phase” in the model of eq. (1), but the result would turn out uninteresting, as the theory has $c = 2$ and would end up with no degrees of freedom. In this paper we instead look for a totally different phase of light cone large-$N$ matrix models. In this phase, the longitudinal momentum of each string bit is not frozen at its minimum value, but is instead allowed to fluctuate. Therefore, the light cone dynamics of the theory (1) is quite non-trivial, being described by fluctuations in the number of string bits and their longitudinal momentum. We will formulate the details of these dynamics in section 2. Our intuitive expectation is that, as $\lambda/\mu$ is driven to its critical value, these fluctuations implement the effects of the conformal factor of the continuum two-dimensional quantum gravity, producing a continuous spectrum of the light cone hamiltonian. Then the model represents a $D = 3$ string theory. This would be an explicit demonstration that the
rule \( D = c + 1 \) \cite{10} applies to non-critical string theory with \( c > 1 \).

It is appropriate to call the phase we are after the “non-critical string phase” of the theory \(1\). Our study of this phase will be based on exact numerical diagonalizations of the light cone Hamiltonian with discretized longitudinal momenta. This is a convenient cut-off because it renders the total number of single-string states finite. However, the number of states grows rapidly as the cut-off is being removed, and the calculations we have managed to carry out are rather far from the continuum limit. We cannot yet draw any conclusions, but the results that we present in section 3 do encourage us to believe that the theory has a critical point where the spectrum is continuous due to the Liouville mode.

2. Light Cone Quantisation

In light cone quantisation we treat \( x^+ = (x^0 + x^1)/\sqrt{2} \) as the time variable, and from the Minkowskian version of the action \( S \) in eq. (1) derive;

\[
P^+(x^+) = \int dx^- \text{Tr}(\partial_- M)^2, \\
P^-(x^+) = \int dx^- \text{Tr}(\frac{1}{2} \mu M^2 - \frac{\lambda}{3\sqrt{N}} M^3).
\]

The canonical commutation relations are imposed at equal \( x^+ \) lines;

\[
[M_{ij}(x^-), \partial_- M_{kl}(\tilde{x}^-)] = \frac{i}{2} \delta(x^- - \tilde{x}^-) \delta_{il} \delta_{jk}.
\]

We may expand in Fourier modes \(\star\);

\[
M_{ij} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} (a_{ij}(k^+)e^{-i k^+ x^-} + a_{ji}^\dagger(k^+)e^{i k^+ x^-})
\]

to obtain

\[
[a_{ij}(k^+), a_{ik}^\dagger(\tilde{k}^+)] = \delta(k^+ - \tilde{k}^+) \delta_{il} \delta_{jk}.
\]

\(\star\) The symbol \(\dagger\) is always understood to have purely quantum meaning and never acts on indices.
After substituting eq. (5) into eq. (3), we obtain

$$P^+ := \int_0^\infty dk^+ k^+ a_{ij}(k^+) a_{ij}(k^+),$$

$$P^- := \frac{1}{2\mu} \int_0^\infty \frac{dk^+}{k^+} a_{ij}(k^+) a_{ij}(k^+) - \frac{\lambda}{4\sqrt{N\pi}}$$

$$\times \int_0^\infty \frac{dk^+_1 dk^+_2}{\sqrt{k^+_1 k^+_2 (k^+_1 + k^+_2)}} \left\{ a_{ij}^\dagger (k^+_1 + k^+_2) a_{ik}(k^+_2) a_{kj}(k^+_1) + a_{ik}^\dagger (k^+_1) a_{kj}(k^+_2) a_{ij}(k^+_1 + k^+_2) \right\}$$

(repeated indices are summed over). The normal ordering is equivalent to removing the tadpole graphs in the Lagrangian approach. The states of the Fock space are of the form;

$$a_{i_1j_1}^\dagger (k^+_1) \cdots a_{i_Bj_B}^\dagger (k^+_B)|0>$$

and the vacuum $|0>$ is an eigenstate of the fully interacting light cone Hamiltonian $P^-$. The key simplifying feature of light cone quantisation is that positive energy quanta require $k^+ > 0$. In this way if the allowed $k^+$ are discrete and non-zero, then one can enumerate all the states of some given total longitudinal momentum $P^+ [6, 7]$. In order to implement the discretisation we shall compactify the $x^-$ direction to a circle of length $L$, so that the matrix field is periodic, $M_{ij}(x^-) = M_{ij}(x^- + L)$. The allowed values of $k^+_b$ are then $2\pi n_b/L$. Note also from (7) that quanta of $k^+_b = 0$ are at infinite energy if $\mu > 0$ and so the integers $n_b$ are restricted to be positive. For $P^+ = 2\pi K/L$, we find $\sum_{b=1}^B n_b = K$. The positive integer $K$ plays the role of a cut-off, and sending it to infinity corresponds to removing the cut-off in the discretized light cone quantization [7].

Before proceeding we need to elaborate upon the light cone string picture of our analysis. In matrix models the states in (8) which have an unambiguous description as single closed strings are of the form;

$$N^{-B/2} \text{Tr}[a_{i_1j_1}^\dagger (k^+_1) \cdots a_{i_Bj_B}^\dagger (k^+_B)]|0>$$

in other words the singlet states under the $M \rightarrow \Omega M \Omega^\dagger$ global $SU(N)$ symmetry of the action $S$. This represents a boundary of length $B$ in the dynamical triangulation. Each
$a^\dagger(k^\pm)$ creates a string bit carrying longitudinal momentum $k^\pm$. It is important to realise that there is no particular reason to believe that the non-singlet states, which have no simple string identification, are unimportant as the cutoff is removed; they may well have energies of the same order as those of the states (9). This may be a crucial flaw in such a matrix model as we are considering here, to which we will return in section 4.

In the limit $N \to \infty$ the light cone Hamiltonian $P^-$ takes single closed string states to single closed string states. One can easily check that the terms that convert one closed string into two closed strings (two oscillator traces acting on the vacuum) are of order $1/N$. Thus, as expected, the string coupling constant is $\sim 1/N$, and sending it to zero allows us to study the spectrum of free closed string states. Because $P^-$ is $SU(N)$ invariant it does not couple these closed strings to the non-singlet states and we shall ignore the latter sector of the theory. A more fundamental reason for taking $1/N \to 0$ is that one expects simple bosonic string theories at $c > 1$ to be tachyonic, so while there is no objection to studying the free theory, the vacuum would collapse if we did not turn off the interactions.

Now in the light cone formalism of critical string theory the longitudinal momentum supported between two points on the string is proportional to the amount of $\sigma$-space between these points. We can adopt a similar co-ordinate system for the non-critical strings (9). Indeed, fixing a particular bit as origin, we can define a positive scalar field on this $\sigma$-space by $X = \Delta b/\Delta \sigma$, where $b$ is the distance, measured in number of bits, from the origin. As we remove the cutoff on the longitudinal momentum allowed for bits, and hence on discreteness of $\sigma$-space, the scalar field will generically take constant values almost everywhere in $\sigma$-space. We would like to be able to tune the theory to a critical point where the scalar field is in a long wavelength regime. Then it would be natural to identify it as an effective extra dimension in which the string fluctuates, an interpretation commonly given to the Liouville mode. In the next section we will attempt to identify where such a critical point might exist. Of course it may not exist, but there is some hope that an extra massless field arises since there is a known example of $c = 2$ free string theory where the spectrum is basically that of a single free massless field. This picture emerges both from the naive extension of Liouville theory to $c > 1$ [11] and a naive extension of dual string theory to $c < 25$ [12]. The relationship between such a free field and the scalar defined above would presumably be quite complicated however.
3. Exact Diagonalizations and Numerical Results

Having discretised the longitudinal momenta $k^+ = nP^+/K$, we define new discrete mode expansions

$$M_{ij} = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( A_{ij}(n)e^{-iP^+n^-/K} + A_{ji}^\dagger(n)e^{iP^+n^-/K} \right). \quad (10)$$

The oscillator algebra is

$$[A_{ij}(n), A_{lk}^\dagger(n')] = \delta_{nn'}\delta_{il}\delta_{jk}. \quad (11)$$

A normalized closed string state of $B$ bits is of the form

$$\frac{1}{NB\sqrt{s}} \text{Tr}[A_1^\dagger(n_1) \cdots A_B^\dagger(n_B)]|0\rangle. \quad (12)$$

The states are defined by ordered partitions of $K$ into $B$ positive integers, modulo cyclic permutations. Therefore, the closed strings are oriented. If $(n_1, n_2, \ldots, n_B)$ is taken into itself by $s$ out of $B$ possible cyclic permutations, then the corresponding state receives a normalisation factor $1/\sqrt{s}$. In the absence of special symmetries, $s = 1$. The light cone Hamiltonian for strings of momentum $P^+$ becomes

$$:P^- := \frac{K}{2P^+} \left( \mu V - \frac{\lambda}{2\sqrt{\pi}} T \right). \quad (13)$$

The potential term;

$$V = \sum_{n=1}^{\infty} \frac{1}{n} A_{ij}^\dagger(n)A_{ij}(n), \quad (14)$$

measures the tensional energy of a string without changing its state. The kinetic term;

$$T = \frac{1}{\sqrt{N}} \sum_{n_1, n_2=1}^{\infty} \frac{A_{ij}^\dagger(n_1 + n_2)A_{ik}(n_2)A_{kj}(n_1) + A_{ik}^\dagger(n_1)A_{kj}^\dagger(n_2)A_{ij}(n_1 + n_2)}{\sqrt{n_1n_2(n_1 + n_2)}}, \quad (15)$$

changes string states by splitting bits into two and joining adjacent bits. Let us rewrite eq.
(13) as
\[
\frac{2P^+P^-}{\mu} = K (V - xT) \tag{16}
\]
where \(x = \lambda/(2\mu\sqrt{\pi})\) is the dimensionless parameter of the problem. Compare eq. (16) with the usual form for the light cone string dispersion relation,
\[
2P^+ P^- \alpha' = P_+^2 \alpha' + 4r - \frac{D-2}{6}, \tag{17}
\]
where \(r\) is a non-negative integer. \(\mu\) is the quantity of dimension mass\(^2\) which in our approach plays the role of string tension \(1/\alpha'\). The most basic evidence for the presence of transverse dimensions is the continuous spectrum of the operator \(2P^+ P^-/\mu\). For a finite cut-off \(K\), it is a finite dimensional symmetric real matrix whose spectrum is necessarily discrete. We can, however, look for the critical value of \(x\) where the spectrum becomes dense in the \(K \to \infty\) limit. In practice, we work with finite \(K\) and look for increasing density of eigenvalues.

To illustrate the calculation with a simple example, consider \(K = 3\) where there are only three closed string states. Working to leading order in \(N\), the normalized states are
\[
|1> = \frac{1}{N^{3/2} \sqrt{3}} \Tr[A^\dagger(1)A^\dagger(1)A^\dagger(1)] |0> , \quad |2> = \frac{1}{N} \Tr[A^\dagger(2)A^\dagger(1)] |0> , \quad |3> = \frac{1}{N^{1/2}} \Tr[A^\dagger(3)] |0> . \tag{18}
\]
A quick calculation gives
\[
K(V - xT) = \begin{pmatrix}
9 & -3\sqrt{\frac{3}{2}} x & 0 \\
-3\sqrt{\frac{3}{2}} x & \frac{9}{2} & -\sqrt{6} x \\
0 & -\sqrt{6} x & 1
\end{pmatrix} . \tag{19}
\]

As \(K\) increases, the number of states grows rapidly. We used Mathematica to generate the states, evaluate eq. (16) and diagonalise it over the range \(0 < x < 1\). Figure 1 shows the eigenvalues at \(K = 7\) (19 states, the lowest 16 of which are shown) as a function of \(x\). This illustrates the general qualitative features. The number of states is always odd. \(V\) is a diagonal matrix with positive entries; \(T\) is a (relatively sparse) symmetric matrix with zeros on the diagonal, having eigenvalues symmetrically distributed about zero. As \(x\) is increased.
from zero, many eigenvalues begin to decrease in such a way that the density of low-lying eigenvalues increases. On the other hand, for large $x$ the spectrum is approximately that of the kinetic term multiplied by $x$, and the gaps become large. Therefore, there seems to be a possibility of fine tuning $x$ so that the potential and the kinetic term are carefully balanced to produce a denser and denser spectrum as $K$ increases. To search for such behavior, we plotted the mean square separation between adjacent energy levels for the lower 50% of the states. This is shown in Figure 2 (solid curves). The dashed curve shows the lowest eigenvalue in each case. These plots clearly show a maximum density of eigenvalues around $x_c \sim 0.64 - 0.7$, with a slight downward drift of $x_c$ with increasing $K$. Since the solid curves of Figure 2 are quite bumpy due to many levels crossing we also plotted the difference between the lowest two eigenvalues, shown in Figure 3 for $K = 8, 10, 12$. This demonstrates more dramatically the decrease in level separation around $x_c$. Although the approach to any sort of critical regime seems to be slow, we must keep in mind the low cut-off employed. Moreover one is reminded that at $c = 1$ the approach to criticality was logarithmic. If the minima in Fig. 2 are indicating an approach to a continuous density of eigenvalues, then it is significant that the ground state seems consistently tachyonic at that point, as one expects for any $D > 2$ string theory.

Clearly, at the moment the data is at best suggestive and calculations with higher $K$ are needed. A clear advantage of the discretized light cone quantization is that it renders the problem perfectly regularized and numerically tractable. A disadvantage is in the rapid growth of the size of the matrices to be diagonalized. However, if we are only interested in the low-lying spectrum, exact diagonalizations may be replaced by approximate techniques which could allow significantly larger $K$ to be probed.

4. Discussion

In the preceding sections we have presented some, albeit not entirely conclusive, evidence that the light cone singlet spectrum of the infinite $N$ matrix model (1) is continuous. Thus, the model may give us useful information about a free $D = 3$ string theory. If this conjecture is confirmed by more detailed numerical studies, is there any hope that, for a finite $N$ eq.

* The few eigenvalues that monotonically increase with $x$ are presumably artifacts of the discretized problem that decouple in the continuum limit.
(1) defines a consistent interacting $D = 3$ string theory? Unfortunately, we believe that the answer is negative.

One problem with this theory is that, on general grounds, it is expected to be tachyonic. Our numerical results appear to agree with that, because at the coupling strength where the spectrum has the highest density there is always a negative eigenvalue of $P^-$. Another problem, which is perhaps even more severe, comes from the states that transform non-trivially under $SU(N)$, such as the adjoint representation states of the form

$$N^{(1-B)/2} A^+_i(n_1) A^+_j(n_2) \ldots A^+_r(n_{B-1}) A^+_s(n_B)|0>.$$  \hspace{1cm} (20)

As discussed previously, the $SU(N)$ non-singlet states cannot be identified with closed strings. One indication of that is their diverging degeneracy factors in the $N \rightarrow \infty$ limit. Perhaps the non-singlets can be thought of as closed strings that have disintegrated into separate bits. Of course, one way of preventing this is through a confinement phenomenon which would push their energies to infinity in the continuum limit. In fact, confinement is at work in the $c = 1$ string theory, where the non-singlet energies diverge logarithmically in the cut-off \cite{13, 14}. The phase transition at a critical value of the target space radius, which was observed in the matrix model \cite{4}, corresponds physically to the Kosterlitz-Thouless deconfinement of the world sheet vortices \cite{15}.

Could the $c = 2$ model of eq. (1) also expel the non-singlet states to infinite energy? To address this question, we have studied the action of the light cone hamiltonian on the states in the adjoint representation of $SU(N)$. Once again, we find a Schrödinger equation in the $N \rightarrow \infty$ limit. We calculated its energy levels exactly for low values of the cut-off $K$. We find that the lowest eigenvalue of $2P^+ P^- / \mu$ starts at 1 for $x = 0$ and decreases monotonically with increasing $x$. Although it is always higher than the lowest singlet eigenvalue, it is impossible for this gap to diverge as $K \rightarrow \infty$. In fact, a variational upper bound of 1 easily follows for the lowest eigenvalue, if we consider $A^+_i(K)|0>$ as the variational state. Therefore, there is no confinement, and the non-singlet states probably cause additional problems for the interacting string theory. Some heuristic arguments with similar conclusions were made in the context of $c > 1$ models with discrete target spaces \cite{13}. The proliferation of the non-singlets was associated with the Kosterlitz-Thouless vortices which wind around the target space plaquettes of lattice size. An advantage of the light cone approach is that the
non-singlet spectrum can be found with the same technique as the singlet spectrum. It would be nice to carry out a more detailed numerical study of the spectrum in the adjoint representation.

The lack of confinement of the non-singlet states can be traced to the fact that the action (1) has only the global $SU(N)$ symmetry, which is not gauged. Gauging the $SU(N)$ will certainly lead to the confinement, and the non-singlets will be pushed to infinite energy. The effect of gauging the $SU(N)$ symmetry on the two-dimensional theory of eq. (1) is to give the action

$$S_{gauged} = \int d^2x \Tr \left( \frac{1}{4g^2} F_{\alpha\beta}^2 + \frac{1}{2} (\partial_\alpha M + i [A_\alpha, M])^2 + \frac{1}{2} \mu M^2 - \frac{1}{3} \frac{\lambda}{\sqrt{N}} M^3 \right)$$

This theory may have a better chance than (1) of describing a sensible $c = 2$ non-critical string. Actually, eq. (21) is somewhat related to the well-known Weingarten model [16], formulated for non-critical strings [17]. This issue, as well as the possibility of a light cone solution of (21) will be left for the future work.

In conclusion, we believe that the light cone quantization has provided us with some new insights into the structure of large-$N$ matrix models and of non-critical string theories with $c > 1$. Further progress may be achieved through numerical solutions of the discretized light cone Schroedinger equation with higher values of the cut-off $K$. We hope that the goal of constructing a sensible interacting non-critical string model with $c > 1$ is not beyond reach.

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Figure Captions.

Figure 1. – The variation with $x$ of the lowest 16 eigenvalues (out of a total of 19) for cutoff $K = 7$.

Figure 2. – The solid curves show the variation with $x$ of the mean square separation between eigenvalues for the lower half of the states. The total number of states for $K = 9, 10, 11, 12$ is 59, 107, 187, 351 respectively. The dashed curve shows the variation with $x$ of the lowest eigenvalue in each case.

Figure 3. – The variation with $x$ of the difference between the lowest two eigenvalues for $K = 8, 10, 12$. The well becomes deeper with increasing $K$.  

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