A Husimi Rhombus Lattice with Random Angles to Present the Space Stochasticity in Exact Thermodynamic Calculation

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Square unit in Husimi lattice (the Bethe square lattice) is generalized to be rhombus with randomly variable angles. The independence feature of unit cells in recursive lattice makes the random angle conformation possible in the model construction, which is unfeasible in the conventional lattice. Since the randomness of the conformations in real system is naturally introduced into the model, this new lattice methodology can describe the off-crystal metastable states without artificial randomness. With reasonable simplification, a coefficient $A(\theta)$ is formulated to present the effect of angle in the rhombus unit. A “visit and count” recursive technique is developed to numerically calculate the thermodynamics. While the computation randomizes a quenched configuration in each iteration, the calculation counts and averages a large number of random units to deal with a system in equilibrium with annealed randomness at particular temperature. The critical temperature $T_c$ of spontaneous magnetization transition is lowered with the presence of angle randomness, which implies a less stable system. Besides consistent results to the regular lattice, the random-angled lattice features a distribution of solutions and the thermal fluctuation with exact calculation. The effects of the variation of energy and ground state parameters on thermodynamics are investigated.

I. INTRODUCTION

Except some well-arranged periodical conformations of particle in nature, e.g. the crystal structure, lattice models are usually employed in physics to quantize the continuous space for convenient and simplified modeling. With appropriate approximation, the lattice methodology works well to describe real systems or to investigate the physics insight, classical examples include the Flory-Huggins model to describe the thermodynamics of polymer solution, Ising model, Potts model and so on. Nevertheless the balance between the lattice approximation and the space continuity of real system is always an important concern in physics modeling. Among a number of sacrifices of lattice modeling, one notable compromise is the variable angle between interacting bonds.

Imagine a free-arranged 2D interacting monatomic system, in which a particle interacts with four other neighbors, then a quadrilateral obtained by sampling any four neighbor particles has a random-angled conformation with variable energy, while in a regular square lattice any four neighbor particles must form a square with the angle $90^\circ$. Although the lattice methodology can count in the angle effect under some circumstances, for example to adjust the related bond interactions to count the effect of off-$90^\circ$ angle, or non-square quadrilateral with particular angles can serve as the basic unit to construct a lattice in a periodical fashion, the problem is the uniformity of lattice cells: however has the lattice been setup to fit the angle effects, the fitting must be a uniform property of all the unit cells, while ironically the angles actually exhibit the feature of randomness in real system.

According to the above concern, we developed a lattice model to handle the random angles via the unique properties of recursive lattice. Recursive lattices such as the classical Bethe and Husimi lattice have been studied for decades and proven to be a reliable methodology to describe the real system or to approximate the regular lattice. In a recursive lattice, the unit cells only interact to each other on the joint site and the conformation of one cell is independent, which makes individual conformation with variable angles possible. The ensemble of individually-shaped units raises a non-uniform lattice to better demonstrate the space randomness in real system. In this random lattice, the conformation of each unit is a rhombus randomized and fixed by the program, therefore locally it has a quenched randomness, nevertheless the calculation counts and averages the statistical variables over

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a large number of random units to deal with the system in equilibrium at particular temperature (details will be discussed below), subsequently the system can be treated as with annealed randomness in general.

Ferromagnetic Ising model on this new lattice is chosen to investigate the thermodynamics. The first reason is that, it is a classical system that has been thoroughly studied for many years\cite{12,21}, we can easily compare and validate the calculation to previous studies. Secondly, this is an exactly solvable thermodynamic model\cite{12,22,32}, and the thermal fluctuation indicated by the exact calculation imprints the point of stochasticity in an exactly-solvable deterministic system that we would like to address in this work, i.e. the thermodynamic calculation still follows a deterministic way however it gives out the randomness in the real system. And another significance of applying Ising spins on this random lattice, is that we can combine the idea of spin system and structural glass model to investigate the off-crystal metastable state. Since the amorphousness of glass can be presented by the random conformation of lattice, the spins can form a glassy state with their well-defined orientation, that is, a glass of spins without the concept of “spin glass”. This model is then able to describe some particular systems, for example the metallic glass ferromagnet \cite{33}, in which the materials is in glassy state but the well-defined orientation of atoms gives superparamagnetism.

Other than the application in metallic glass ferromagnetism, we believe this methodology could be a decent contribution to this field, and has the potential to be applied in various physics problems, for example the glass thermodynamics, vibration of crystal lattice, and quasicrystal. In this paper we will explore the random-angled conformation in Husimi square lattice. More specifically, the unit cell is generalized to be rhombus with only one independent angle variable for convenience. It should be easy to extend our work to be a more general quadrilateral lattice, or other polygon and 3D cases. The Hamiltonian formulation and thermodynamic calculations on this lattice will be discussed in details, and we will investigate the thermal behavior, phase transitions and the effects of energy and ground state parameters setup.

II. LATTICE CONSTRUCTION

The conformation of a unit cell is independent in a recursive lattice, therefore a Husimi square lattice can easily transform to be an recursive ensemble of random-angled rhombus units. Fig.2 shows the idea of the recursive rhombus lattice, the structure is still in a Husimi recursive fashion but with the basic unit cells as random rhombuses. In a more general view, the basic unit cell of this recursive lattice can be called as a “conformation distribution cloud” instead of a fixed geometry unit (Fig.1b). The “cloud” collapses to be a particular rhombus conformation when it is visited during the calculation.

Note that in the lattice all the edges are in the same length to represent the neighbor interactions, the different size in Fig.1a is for possible drawing on the plane page. With the fixed edge length only one independent angle $\theta$ is necessary to determine the structure of a rhombus. By defining $\theta_L$ as the lower limit, we have $\theta \in [\theta_L, 180^\circ - \theta_L]$. In this paper, the angles are randomized with even probability distribution in the region $[\theta_L, 180^\circ - \theta_L]$, and $\theta_L$ is set to be $20^\circ$.

For the angle $\theta$ between two bonds in the rhombus, we must have one or more the lowest energy angles denoted as $\theta_o$ at which the $E(\theta_o) = E_{\min}$, i.e. the most stable state. For example with $\theta_o = 90^\circ$ the square unit will be the most preferred configuration. In this work we will only discuss the case that $\theta_o$ has a single value in the range $(0, 180^\circ)$. However a multi-stable states system implied by many values of $\theta_o$ is possible.

A. Modeling Angle Energy

With the ground state angle $\theta_o$, a higher energy $E(\theta)$ of the angle $\theta \in [\theta_L, \pi - \theta_L]$ rises with the deviation from $\theta_o$, and then we should have $E(\theta) = f(\theta, \theta_o)$.

Previous studies on Husimi lattice usually consider only the interaction $J$ between nearest-neighbor sites, while in several works the model also counted in other interactions, e.g. the diagonal interaction \cite{31}. Following this consideration, by safely assuming that the angle does not affect the nearest-neighbor interactions, then if the angles would do anything on the system energy it should be either the angle bending energy itself, or effects on the diagonal interactions in the rhombus, which relates to the opposite angles with the law of cosines. Since an artificial bending energy on the angle is similar to an external field applying on the lattice sites and not quite interesting, we will take the second choice to formulate the angle energy.

In homogeneous lattice the interaction energy of a pair of diagonal spins $(S_j, S_j')$ is $E_p = -J_p \cdot S_j S_j'$, where $J_p$ is the diagonal energy parameter. Assume the diagonal interaction is linearly proportional to the diagonal length, according to the law of cosines $c^2 = a^2 + b^2 - 2ab\cos\theta$, with $c$ as the diagonal opposite to angle $\theta$ and the side length...
FIG. 1: (a) A sample demonstration of Husimi rhombus lattice with random-angled units. (b) The basic rhombus unit with random angle in the range of $[\theta_L, 180^\circ - \theta_L]$ with the lowest energy angle $\theta_o$, in this work the $\theta_L$ is set to be 20° unless specially notified.

Define $A(\theta)$ as the angle energy coefficient:

$$A(\theta) = \sqrt{\frac{1 - \cos \theta}{1 - \cos \theta_o}},$$

(2)

as a natural choice for the quadrilateral, in this work we take $\theta_o = \pi/2$ for the general discussion. In this way, we
then have
\[ E_p = -J_p \cdot A(\theta) \cdot S_j S_{j'}. \]

where \( A(\theta) = \sqrt{1 - \cos \theta} \) for a random angle \( \theta \). The effect of various \( \theta \) will be discussed later.

It should be addressed here that the above formulation of angle energy is counter-intuitive because the assumption “diagonal interaction linearly proportional to the length” implies that the farther the diagonal pairs particles are, the larger the interaction is, which is not a common case in nature. The reason of such formulation is that when we combine the energies of two diagonal pairs in one rhombus, a higher total energy (smaller weights) of any off-angle configuration can be guaranteed with a larger magnitude of energy change on the other term: \( J_p \cdot A(\pi - \theta) \). In another word, the formulation based on \( A \propto \sqrt{(1 - \cos \theta_0)/(1 - \cos \theta)} \) will lead any off-\( \theta_0 \) configuration corresponding to a larger energy (lower weights), and the structure with \( \theta = \theta_0 \) is the most stable configuration with the largest weights.

We formulate the angle energy in this way merely for convenience to preliminarily explore the nature of random-angle lattice. It should be easy to extend the methodology with other angle energy formulations for particular circumstances.

B. The Hamiltonian

Adding the angle term introduced in Eq.3, the energy of a lattice unit containing four magnetic energies, four neighbor and two diagonal interactions is:
\[ e = -\sum_{<i,j>}^4 J \cdot S_i S_j - H \sum_{i=1}^4 S_i - J_p (A(\theta_i) \cdot S_j S_{j'} + A(\theta_j) \cdot S_i S_{i+2}), \]

and its Boltzmann weights is,
\[ w(\gamma) = e^{-\beta e}, \]

where \( \gamma \) denotes the conformation of unit.

In this paper the interaction parameter \( J \) will be set as +1 for the ferromagnetic case. Being consistent to \( J \), \( J_p \) is always set as positive. The effect of \( J_p \) variation from 0.1 to 0.6 will be discussed and \( J_p = 0.2 \) is taken to be the reference case. The magnetic field is not concerned with \( H = 0 \). In the third term \( S_j \) and \( S_{j'} \) denotes to the spins neighbor to \( S_i \) and they assemble a diagonal pair with opposition to the angle \( \theta_i \) on the site \( S_i \). The Hamiltonian of the entire lattice is the summation over all the units energies
\[ E = \sum_{\alpha} e_{\alpha}, \]

where \( \alpha \) denotes the index of unit cells.

III. CALCULATION

The calculation principally follows the same methodology of our previous works[29, 31, 32, 34] on Ising model on Husimi lattice. Imagine the entire lattice has an original site \( S_0 \) on the 0th level, then the sites nearest to \( S_0 \) are marked as \( S_1 \) (or \( S'_1 \) to distinguish the two sites in the same unit), the site diagonal to \( S_0 \) is marked as \( S_2 \), and so on. The situation of \( n \)th level is shown in Fig.2. On one rhombus unit there are three sub-trees \( T_{n+1} \) and \( T_{n+2} \) contributing to the sites \( S_n, S'_{n+1} \) and \( S_{n+2} \), and a larger sub-tree \( T_n \) is synthesized by adding the weights of the local unit. With the particular spin state of \( S_n = \pm 1 \), define the partial partition function (PPF) \( Z_n(S_n) \) on level \( n \) to count the contribution of sub-tree to the site \( S_n \) (not included), then the \( n \)th level PPF is the function of PPFs on the previous level \( n + 1, n + 2 \) and the weights of local unit \( w(\gamma) \):
\[ Z_n(+) = \sum_{\gamma=1}^8 Z_{n+1}(S_{n+1}) Z_{n+1}(S'_{n+1}) Z_{n+2}(S_{n+2}) w(\gamma), \]
FIG. 2: Four sites $S_i$ in a rhombus unit are labeled with levels. The $\mathcal{T}_i$ denotes the sub-tree contributions from level $i$ to the local unit. Solutions $x_i$ on corresponding site $S_i$ are calculated accordingly.

\[ Z_n(-) = \sum_{\gamma=9}^{16} Z_{n+1}(S_{n+1}) Z_{n+1}(S'_{n+1}) Z_{n+2}(S_{n+2}) w(\gamma). \] (8)

And the partition function (PF) of the whole sub-tree on $n$th level is given by

\[ Z_n = Z_n(+) e^{\beta H S_n} + Z_n(-) e^{-\beta H S_n}, \] (9)

which summers all the weights at level $n$ including the spin $S_n$.

By introducing the ratios

\[ x_n = \frac{Z_n(+)}{Z_n(+)+Z_n(-)}, \] (10)

\[ y_n = \frac{Z_n(-)}{Z_n(+)+Z_n(-)}, \] (11)

and a compact note

\[ z_n(S_n) = \begin{cases} x_n \text{ if } S_n = +1 \\ y_n \text{ if } S_n = -1 \end{cases} \] (12)

As the weights ratio of PPFs with the spin state of $S_n$, $x$ or $y$ denotes the sub-tree contribution to the magnetization of the site $S_n$, i.e. the cavity contribution. Therefore, we can use these ratios, which we can the “solution” of the model in the following, to indicate the magnetization of a site. In terms of

\[ B_n = Z_n(+) + Z_n(-), \]

we have $Z_n(+) = B_n x_n$ and $Z_n(-) = B_n y_n$. With Eq(7) [8] and (12) it gives

\[ B_n z_n(\pm) = \sum B_{n+1} z_{n+1}(S_{n+1}) B_{n+1} z_{n+1}(S'_{n+1}) B_{n+2} z_{n+2}(S_{n+2}) w(\gamma), \]

\[ z_n(\pm) = \sum z_{n+1} z_{n+1}(S_{n+1}) z_{n+2}(S_{n+2}) w(\gamma)/Q(x_{n+1}, x_{n+2}), \]
where the sum is over $\gamma = 1, 2, 3, \ldots, 8$ for $S_n = +1$, and over $\gamma = 9, 10, \ldots, 16$ for $S_n = -1$, and where

$$Q(x_{n+1}, x_{n+2}) = \frac{B_n}{B_{n+1}^2} B_{n+2};$$

it relates to the polynomials

$$Q_+(x_{n+1}, x_{n+2}) = \sum_{\gamma=1}^{8} z_{n+1}(S_{n+1}) z_{n+1}(S'_{n+1}) z_{n+2}(S_{n+2}) w(\gamma),$$

$$Q_-(x_{m+1}, x_{m+2}) = \sum_{\gamma=9}^{16} z_{n+1}(S_{n+1}) z_{n+1}(S'_{n+1}) z_{n+2}(S_{n+2}) w(\gamma),$$

according to

$$Q(x_{n+1}, x_{n+2}) = Q_+(x_{n+1}, x_{n+2}) + Q_-(y_{n+1}, y_{n+2}).$$

In terms of the above polynomials, we can have the recursive relation for the ratio $x_n$ in terms of $x_{n+1}$ and $x_{n+2}$:

$$x_n = \frac{Q_+(x_{n+1}, x_{n+2})}{Q(x_{n+1}, x_{n+2})}. \quad (13)$$

Similarly for $y_n = 1 - x_n$:

$$y_n = \frac{Q_-(y_{n+1}, y_{n+2})}{Q(y_{n+1}, y_{n+2})}. \quad (14)$$

Therefore the solution $x$ is a function of PPFs and subsequently a function of $x$s on the previous levels:

$$x_n = \frac{Z_n(\gamma^+)}{Z_n(\gamma^+) + Z_n(\gamma^-)} = f(x_{n+1}, x_{n+2}) \quad (15)$$

then starting with two initial guesses of $x$s we can recursively calculate the $x$ on lower level.

In the homogeneous lattice the recursive calculation is done for many iterations until reaching a fixed cycling solution $x_k = x_{k+1}$ (the fixed point solution). However in the random lattice discussed here, the feature of configuration randomness and variable local weights $w(\gamma)$ in each iteration makes the exact fix-point solution unachievable. After a number of iterations the calculation will provide an oscillating solution $s$ instead of the fixed $w$.

The “visit and count” method implies that the calculation process deals with a quenched randomness while a
FIG. 3: (a) The reference average solution $\bar{x}$ with $J_p = 0.2$ and $\theta_o = 90^\circ$. (b) The distribution of twenty sample solutions $x$ at $T = 2.0$.

A. Calculation of Thermodynamics

The Helmholtz free energy as the function of PF is given by $F = -T \log Z$. Although the lattice is infinitely large and it has no sense to calculate $Z$ or $F$, the free energy of the local unit on $n$th level can be achieved by the difference of free energies on successive levels $F_{\text{local},n} = F_n(Z_n) - 2F_{n+1}(Z_{n+1}) - F_{n+2}(Z_{n+2})$. Then we have

$$F_{\text{local},n} = -T \log \frac{Z_n}{Z_{n+1}^2 Z_{n+2}}$$  

(16)
FIG. 4: The thermodynamics of reference solution $\bar{x}$ with $J_p = 0.2$, $\theta_o = 90^\circ$.

With the derivation on PF, PPF and $x$s introduced above, the local free energy $F_{\text{local},n}$ can be simply calculated as a function of solutions $x_n$, $x_{n+1}$, $x_{n+2}$ and temperature $T$. The same “visit and count” method is applied on the free energy. In an iteration once the $F_{\text{local},n}$ was obtained, it is averaged back with previous results:

$$\bar{F}_n = \frac{1}{2} \cdot \frac{F_{n+1} \cdot (n - 1) + F_{\text{local},n}}{n}.$$  \hspace{1cm} (17)

The $1/2$ is to divide the number of sites in one unit (four half-shared sites). Again in the calculation practice, we average the last 2000 results out of 6000 iterations in the program.

The energy per site, i.e. energy density is calculated with the conformation probability. A rhombus unit with 4 spins on each site has sixteen possible conformations $\gamma$. The probability of one conformation is

$$P(\gamma) = \frac{[Z_{n+1}(S_{n+1})Z_{n+1}(S'_{n+1})Z_{n+2}(S_{n+2})] \cdot w(\gamma)}{Z_n}.$$  \hspace{1cm} (18)

Then the energy density is defined as the summation over the product of the energy and the probability of a conformation state:

$$E = \frac{1}{2} \sum e(\gamma)P(\gamma)$$  \hspace{1cm} (19)

where $e$ is from the Eq.4.

The average entropy per site is derived from $\bar{F}$ and $\bar{E}$:

$$\bar{S} = \frac{\bar{E} - \bar{F}}{T}.$$  \hspace{1cm} (20)

A reference thermodynamics of $J_p = 0.2$ and $\theta_o = 90^\circ$ in a wide temperature range is shown in Fig.4. The transition can be observed on the entropy or energy curve, on which a slope change at $T = 3.096$ indicates a phase transition from disordered to ordered state with decreasing $T$. When the $T$ is close to zero, the entropy approaches to zero as expected and the energy or free energy at zero is given as $\sim 2.180$. This ground energy quantitatively measures the effect of angle/diagonal energies presence. Referring from our previous work\cite{ref}, the recursive lattice of coordination number $q$ has the ground energy $E(T = 0) = \frac{1}{2}J \cdot q$ if there is no other interaction terms. Therefore the contribution of angle/diagonal energies to the ground energy is $\sim 0.180$. 

IV. RESULTS AND DISCUSSION

A. Spontaneous Magnetization and Critique Temperature

Fig. 3a shows clear spontaneous magnetization transition with the presence of angle randomness. In the high temperature range the solution is 0.5 indicating that all sites have 50% probability to be occupied by either +1 or −1 spin (disordered state). With cooling process till the critique temperature \( T_c \), the solution decreases and approaches to zero, i.e. at low temperature the system prefers more -1 spins and in the near-zero range all the sites will be occupied by −1 spins (ordered state). It should be clarified that although in Fig. 3a the transition is very sharp between \( T = 3.0 \) and \( T = 3.1 \), it is not a discontinuous transition and the detail is shown in Fig. 5. Depending on the initial seeds of \( x \) the solution below \( T_c \) may either symmetrically bend up and approaching to 1. That is the magnetization to the other direction and has the same thermodynamics therefore it is not shown in the figure.

In the quantitative aspect, presence of angle randomness decreases the \( T_c \). Fig. 5 shows the solutions of deterministic case and one sample of stochastic case zoomed in around the transition range. The deterministic solution is calculated with the same formulation except setting \( \theta \) always to be \( \theta_o \) in the program. The \( T_{c,deter} \) is 3.131 and \( T_{c,stocho} \) is 3.096. This is because \( \theta_o \) corresponds to the lowest energy rhombus conformation, subsequently the homogeneous lattice with universal \( \theta = \theta_o \) must be the most stable system. Meanwhile the presence of off-angle configurations, even the larger the off-angle degree the smaller weights they have, will generate higher energy to unstablize the system to be easier to experience the phase transition, i.e. the lower \( T_c \).

Another interesting phenomenon is that the stochasticity does not randomize \( T_c \). Five samples of stochastic solutions in the around \( T_c \) range are shown in Fig. 6. While below \( T_c \) all sets of solutions behaviors as a random distribution, the transition temperatures where solutions converge to the 0.5 are the same. This phenomenon can be understood in Eq. 4 and 5, the driving force of temperature applies on all the energy terms in Eq. 4 to obtain the weights, no matter what the randomized angles one rhombus unit have, the weights preferences on particular configurations under certain temperature are the same. In another word, regardless of the shape of the rhombus, when temperature decreases till the weights of the three and four −1 spins on the four sites are higher, the spontaneous magnetization occurs and the solution \( x \) heads to 0, or vice versa.

B. Thermal Fluctuation

From Fig. 5 we can more clearly observe that while the deterministic solution is a fix-valued line along the temperature axis the stochastic case provides a distribution. The solutions fluctuates off the equilibrium state (the equilibrium state is an imaginary line assembled by the solutions \( x \) averaged over infinite \( x \)s, it can be expected to be a single line likewise the deterministic solution, however cannot be calculated and displayed in the figure). Furthermore the
thermodynamics obtained from solutions also exhibit thermal fluctuation. More specifically it is possible to have a higher entropy at lower $T$, which is not possible in the exact calculation on homogeneous lattices.

Unless displaying several samples together on the same graph, observing the fluctuation with large temperature increment is difficult, e.g. in the Fig.3 and Fig.4. In the following discussion we call the $T$ increment as “resolution”, and usually we set $\Delta T \leq 0.01$ to observe the fluctuation/distribution. The fluctuation degree can be characterized by a “middle-value-based” deviation which will be detailed in next section.

1. Entropy Behavior

Fig.7 is the entropy with $J_p = 0.2, \theta_o = 90^\circ$ and resolution $\Delta T = 0.001$, the entropy of homogeneous lattice is also displayed for comparison. Besides the similar behavior, entropies of two systems distinguish from each other as a distribution and fix-valued neat line respectively like the solutions in Fig.5. In the stochastic case we can locate many neighbor pairs of data points that the entropy value at lower $T$ is larger than the other at higher $T$, this thermal fluctuation at equilibrium is not treated as disobeying the second law[36] and can be observed in both experiments and simulations. Nevertheless exact calculation usually cannot present the fluctuation and that raises an advantage of our model.
FIG. 8: Six samples of entropies exhibiting fluctuation and negative in the near zero range.

2. Near-Zero Temperature Range

The fluctuation exists rigorously in the range from a $T$ slightly higher than zero to the $T_c$. The solution is consistent 0.5 above $T_c$ and 0 in the close-zero area, in these two ranges the driving force of temperature is too strong to allow large fluctuations. Within the fluctuation range, there is a more significant fluctuation found in the near-zero temperature range. Fig 8 displays six sample solutions with the resolution $\Delta T = 0.01$, the calculation obtains negative entropy in a few higher $T$ range. Even it is easy to explain this behavior by the randomness in the free energy and energy density calculation, this phenomenon is still interesting if we consider that the drive force of temperature rapidly pulls up the entropy away from zero above a bit higher $T\left( T \gtrsim 0.55 \right)$, and firmly fix the entropy down to zero below a bit lower $T\left( T \lesssim 0.20 \right)$, but leave a range in which the randomness leads to a fluctuation with negative entropy. Several trials have been done and it seems that this negative $S$ range is inevitable. The reason of this phenomenon is not clear yet.

C. Effect of $J_p$

The effect of $J_p$ in homogeneous lattice has been discussed in reference [31]. Generally the $J_p$ of the same sign of $J$ enhances the weights of same spins configuration and increase the stability of system. In the random lattice this principle also works. The $T_c$ variation with $J_p = 0.1$ to 0.6 are summarized in table II.

| $J_p$ | $T_{c,\text{deter}}$ | $T_{c,\text{stoch}}$ |
|-------|----------------------|----------------------|
| 0.1   | 3.105                | 2.949                |
| 0.2   | 3.132                | 3.096                |
| 0.3   | 3.307                | 3.261                |
| 0.4   | 3.475                | 3.414                |
| 0.5   | 3.638                | 3.564                |
| 0.6   | 3.799                | 3.710                |

TABLE I: The effects of $J_p$ on $T_c$ in homogeneous and random lattice.

The effect of $J_p$ on the fluctuation degree is a more intriguing finding. A larger $J_p$ will stress the weights of random terms in Eq 4 thus we can expect a more rigorous fluctuation. For a single set solutions we developed a “middle-value-based deviation” $\delta_m$ to quantitatively characterize the fluctuation degree. For the solutions series $\{x_i(T_i)\}$, the middle value $x_{\text{m}}$ is calculated for each pair of two successive solutions $x_i$ and $x_{i+1}$: $x_{\text{m}} = \frac{x_i + x_{i+1}}{2}$, then the deviation $\delta_m$
FIG. 9: The solution distributions in around $T_c$ range of $J_p = 0.1$ and $J_p = 0.5$ case. The $J_p = 0.5$ curves is shifted by $T = -0.6$ along the temperature axis to overlap with $J_p = 0.1$ for comparison. The middle-value-based deviations $\delta_m$ are calculated for the gray area. The temperature resolution is $\Delta T = 0.0001$. The width of gray window is 0.02.

$\delta_m = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - x_{i,m})^2}$ (21)

The reliability of this deviation highly depends on the temperature resolution. Except the sharp transition range the results are acceptable with $\delta T \leq 0.01$ according to our experience. Fig.9 demonstrates the distributions of solutions with the $J_p = 0.1$ and 0.5 in around $T_c$ range with resolution of $\Delta T = 0.0001$. The $J_p = 0.5$ curves is shifted by $T = -0.6$ for overlapping. The “middle-value-based” deviations of both cases are calculated in a selected range below $T_c$. The results $\delta_m(J_p = 0.5) = 0.001289$ and $\delta_m(J_p = 0.1) = 0.000419$ agrees to the intuitive observation on the graph that higher $J_p$ corresponds to a larger fluctuation.

D. Effect of $\theta_o$

So far the ground state angle $\theta_o$ is set to be 90° as a natural choice for the quadrilateral lattice. But it is possible to have situations with non-90° $\theta_o$, i.e. the quadrilateral cannot be the lowest energy state of particles. For example the six-carbon ring is known to be the most stable ring structure for organic carbon compound (e.g. benzene or cyclohexane), however the four-carbon ring is also thermodynamically possible as a metastable compound (e.g. cyclobutane or cyclobutadiene). We will investigate the effect of $\theta_o \neq 90^\circ$ on system’s thermodynamics in this section.

The $J_p$ is fixed to be 0.2 for the $\theta_o$ variation. The $\theta_o$ value of 30°, 60°, 90°, 120° and 150° have been investigated. The transition temperature and energy density at zero are summarized in table II. The entropy behaviors are shown in Fig 10.

| $\theta_o$ (deg) | $T_c$ | $E(T = 0)$ |
|------------------|-------|------------|
| 30               | 3.622 | -2.492     |
| 60               | 3.226 | -2.255     |
| 90               | 3.096 | -2.180     |
| 120              | 3.038 | -2.147     |
| 150              | 3.011 | -2.133     |

TABLE II: The effects of $\theta_o$ on $T_c$ and ground energy.

FIG. 10: The entropies with various $\theta_o = 30^\circ, 60^\circ, 90^\circ, 120^\circ$ and $150^\circ$ with $J_p = 0.2$.

FIG. 11: The solution distributions in around $T_c$ range of $\theta_o = 30^\circ, 60^\circ, 90^\circ, 120^\circ$ and $150^\circ$. The shifts of the first four curves are $T = -0.6, -0.21, -0.06$ and $-0.025$ to overlap with the $150^\circ$ for comparison. The middle-value-based deviations $\delta_m$ are calculated for the gray window of 0.005 width. The temperature resolution is $\Delta T = 0.0001$.

The results of $\theta_o = 90^\circ, 120^\circ$ and $150^\circ$ basically agrees to the angle energy setup in Eq(2) and (3) that off-angle configurations will make the system less stable and lower the transition temperature. For $120^\circ$ and $150^\circ$ cases any randomized configuration must be an off-angle one therefore the lower $T_c$ and higher ground energy are expectable, although this effect is very small that in Fig(10) the three curves are very close.

However $\theta_o = 30^\circ$ and $60^\circ$ exhibit a higher $T_c$ and lower ground energy. That counter-intuitively implies if we force four particles, which is stable in equilateral triangle configuration, to form a quadrilateral, that will be more difficult to decompose. So far to the author’s knowledge there is no linear covalent bonds prefers an angle as small as $60^\circ$. Phosphorus white crystal has $60^\circ$ bond angles but that is in 3D with regular tetrahedron structure. Due to lack of analogs in nature, it is not known if this phenomenon is realistic or it is simply a defect of our angle energy formulation.

Fluctuation degrees of various $\theta_o$ are presented in the Fig(11) calculated by Eq(21) we have:

\[
\begin{align*}
\delta_m(\theta_o = 30^\circ) &= 0.002400, \\
\delta_m(\theta_o = 60^\circ) &= 0.001771, \\
\delta_m(\theta_o = 90^\circ) &= 0.001060, \\
\delta_m(\theta_o = 120^\circ) &= 0.000959, \\
\delta_m(\theta_o = 150^\circ) &= 0.000970.
\end{align*}
\]

The results follows the same principle in Fig(10) that the larger off-angle energy is, the larger the fluctuation is.
We introduced the random angle variation in Husimi square lattice and generalized it to be a rhombus lattice, this is made possible by the independence property of recursive units. Considering a real system with coordination number of 4 in 2D, if we randomly sample four neighboring particles they are certainly in a quadrilateral configuration with random bond length and angles, even the system follows a deterministic dynamics. By this meaning we constructed a stochastic recursive lattice with Ising model on it, while no artificial randomness (the general “Chaos” or “Noise” in many models) on the dynamics is necessary to present the stochasticity in the system.

For simplified treatment and easier calculation to explore the random lattice, the model is confined into one angle randomness and diagonal-proportional interaction. The Spontaneous Magnetization of ferromagnetic Ising model is presented in our model, but with a lower transition temperature $T_c$ since the off-angle configurations contribute a higher energy making the system less stable. The thermal fluctuation is induced by stochastic configurations and we introduced the quantity “middle-value-based” deviation $\delta_m$ to characterized the degree of fluctuation.

The effects of the variation of diagonal interaction parameter $J_p$ and ground angle $\theta_o$ are investigated. Both $T_c$ and $\delta_m$ are behaved as expected based on the features of each variable, except the phenomena observed with $\theta_o < 90^\circ$ are intriguing and may imply further interesting investigations.

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