A remark on “Robust machine learning by median-of-means”

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1 Introduction

In this note we address regression function estimation, one of the most basic problems of statistical learning. Suppose $\mathcal{X}$ is some measurable space. Given an independent, identically distributed sample $D = (X_i, Y_i)_{i=1}^N$ of pairs of random variables with $X_i \in \mathcal{X}$ and $Y_i \in \mathbb{R}$, one wishes to select a function $\hat{f} : \mathcal{X} \to \mathbb{R}$ from a given class $F$ with small risk

$$E \left( (\hat{f}(X) - Y)^2 | D \right).$$

Tournament procedures were introduced in [4] and attain the optimal accuracy/confidence tradeoff in general prediction problems (see [4, 3, 5]). Roughly put, the idea behind tournaments is to select $n \leq N$ wisely, split the given sample $(X_i, Y_i)_{i=1}^N$ to $n$ blocks, each of cardinality $m = N/n$, and compare the statistical performance of every pair of functions on each block. The function that exhibits a superior performance on the majority of the blocks is the winner of the match, and in a perfect world, the procedure returns a function that wins all of its matches. However, the world is far from perfect: the outcome of a match between functions that are too close is not reliable. To address this issue, tournaments require an additional component: a data-dependent way of verifying when two functions are too close, allowing one to decide if the result of a match should be trusted.

Although tournaments attain the optimal accuracy/confidence tradeoff, they are far from being computationally feasible: comparing every pair of functions in a large, possibly infinite class is impossible. A natural question is therefore to find a more reasonable procedure that exhibits the same optimal statistical behaviour as tournaments and at the same time has at least a fighting chance of being computationally friendly. The authors of [1] claim that the procedure they suggest has these features and those claims are the subject of this note.

Before we explore the claims from [1], let us describe some technical facts that are at the heart of the results in [4, 5] and that will prove to be significant in what follows.

Given a class of functions $F$, let $f^* = \arg\min_{f \in F} E(f(X) - Y)^2$. Fix an integer $n$ and let $(I_j)_{j=1}^n$ be the natural decomposition of $\{1, \ldots, N\}$ of $n$ blocks of cardinality $m = N/n$. Set

$$Q_{f,h}(j) = \frac{1}{m} \sum_{i \in I_j} (f - h)^2(X_i), \quad M_{f,h}(j) = \frac{2}{m} \sum_{i \in I_j} (f - h)(X_i)(h(X_i) - Y_i)$$

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and note that

\[ B_{f,h}(j) = \frac{1}{m} \sum_{i \in I_j} (f(X_i) - Y_i)^2 - \frac{1}{m} \sum_{i \in I_j} (h(X_i) - Y_i)^2 = Q_{f,h}(j) + M_{f,h}(j). \]

The method developed in [4] was used there to show the following fact: given a convex class \( F \) that satisfies some minimal conditions, for the right choice of \( n \) and \( r \) (the choice of \( r \) depends on the geometry of the class \( F \) and on the parameters \( \gamma_1 \) and \( \gamma_2 \) appearing below), and for an absolute constant \( c_1 \), we have that, with probability \( 1 - 2 \exp(-c_1 n) \),

\begin{enumerate}
  \item for every \( f \in F \) such that \( \|f - f^*\|_{L_2} \geq r \), one has
    \[ B_{f,f^*}(j) \geq \gamma_1 \|f - f^*\|_{L_2}^2 \] (1.1)
    for 0.99\( n \) of the blocks;
  \item for every \( f \in F \) such that \( \|f - f^*\|_{L_2} < r \), one has
    \[ |M_{f,f^*}(j) - EM_{f,f^*}(j)| \leq \gamma_2 r^2 \] (1.2)
    for 0.99\( n \) of the blocks.
\end{enumerate}

Note that any fixed proportion of \( n \) is possible, for the price of slightly modified constants.

(1.1) and (1.2) are instrumental in our analysis of the performance of the procedure from [4].

The fact that (1.1) and (1.2) hold with probability at least \( 1 - 2 \exp(-c_1 n) \) is a reformulation of statements from [4]: (1) is Lemma 5.1 in [4] and (2) is Lemma 5.5 in [4].

**Remark 1.1.** Note that both facts are not enough to run a tournament procedure: although the identity of the winner of each match involving \( f^* \) is clear, it does not tell us whether the outcome of the match should be trusted. However, using the right notion of a distance oracle, these estimates suffice to ensure that with probability at least \( 1 - 2 \exp(-cn) \) the tournament procedure produces \( \widehat{f} \) for which

\[ \mathbb{E} \left( (\widehat{f}(X) - Y)^2 \mid (X_i, Y_i)_{i=1}^N \right) \leq \mathbb{E}(f^*(X) - Y)^2 + c'r'. \]

As mentioned previously, the motivation for this note is [4], where the authors suggest the following alternative to the tournament procedure: given a convex class \( F \) and a sample \( (X_i, Y_i)_{i=1}^N \), select \( \tilde{f} \) to be the minimizer in \( F \) of the functional

\[ \phi(f) = \max_{g \in F} \text{Med}(B_{f,g}), \]

where \( \text{Med}(B_{f,g}) \) is a median of the vector \( (B_{f,g}(j))_{j=1}^n \).

The claim in [4] is that for a well chosen number of blocks \( n \) (the same as in [4, 3]) \( \tilde{f} \) performs as well as the tournament procedure—both in the standard prediction framework.
of [4] and in the regularization framework of [3]; that it is robust to outliers and that it is computationally feasible.

To prove their claims, the authors make the same assumptions as in [4, 3], and essentially re-prove the technical machinery developed in [4]. This machinery is then used to analyze the performance of \( \tilde{f} \).

We show below that the theoretical contribution of [1] is somewhat overstated: the two main claims on the statistical performance of \( \tilde{f} \) are in fact almost obvious outcomes of (1.1) and (1.2), and thus the analysis of \( \tilde{f} \) is a direct and immediate corollary of Lemma 5.1 and Lemma 5.5 from [4]. Moreover, we show that the robustness to outliers exhibited by \( \tilde{f} \) is equally simple (this is one of the features of quantile-based procedures and therefore should not come as a major surprise).

Finally, [1] explores the issue of computational feasibility of \( \tilde{f} \). The claim is that one may use coordinate descent to find an approximate solution to the minimization problem that defines \( \tilde{f} \). We believe that at this point it is yet to be determined whether coordinate descent truly finds an approximate solution in polynomial time with the same theoretical properties as \( \tilde{f} \). Specifically, we question whether the optimal tradeoff between accuracy and confidence—the main novelty in tournaments—can be ensured by such an approximate solution.

2 The analysis of \( \tilde{f} \)

Assume that one is given a sample \( D = (X_i, Y_i)_{i=1}^N \) for which (1.1) and (1.2) hold. The analysis of the procedure \( \tilde{f} \) is straightforward: observe that as a minimizer, \( \phi(\tilde{f}) \leq \phi(f^*) \), and moreover, \( \phi(f^*) = \max_{g \in F} \text{Med}(-\mathbb{B}_{g,f^*}) \). Thanks to (1.1) and (1.2) one may provide an upper bound on \( \phi(f^*) \) and then use it to pin-point the location of \( \tilde{f} \) in \( F \).

**Theorem 2.1.** Assume that \( \gamma_1 > \gamma_2 \). Then for the given sample \( D \) one has

\[
\mathbb{E} \left( \left( \tilde{f}(X) - Y \right)^2 \mid D \right) \leq \mathbb{E}(f^*(X) - Y)^2 + (1 + 2\gamma_2)r^2.
\]

**Proof.** Let \( g \in F \) satisfy that \( \|g - f^*\|_{L_2} \geq r \). By (1.1), \( -\mathbb{B}_{g,f^*}(j) \leq -\gamma_1 r^2 \) on 0.99n of the blocks. On the other hand, if \( \|g - f^*\|_{L_2} \leq r \), we have that

\[
\mathbb{B}_{g,f^*}(j) = \mathbb{Q}_{g,f^*}(j) + \mathbb{M}_{g,f^*}(j) \geq \mathbb{M}_{g,f^*}(j).
\]

Since \( F \) is convex one has that \( \mathbb{E}\mathbb{M}_{g,f^*}(j) \geq 0 \). Indeed, this follows from the characterization of the nearest point map onto a convex set in a Hilbert space. Hence, by (1.2),

\[
\mathbb{M}_{g,f^*}(j) \geq \mathbb{E}\mathbb{M}_{g,f^*}(j) - \gamma_2 r^2 \geq -\gamma_2 r^2
\]

on 0.99n of the blocks.

Combining these two observations,

\[
\phi(f^*) = \max_{g \in F} \text{Med}(-\mathbb{B}_{g,f^*}) \leq \gamma_2 r^2.
\]

Now, as the minimizer, \( \phi(\tilde{f}) = \max_{g \in F} \text{Med}(\mathbb{B}_{\tilde{f},g}) \leq \gamma_2 r^2 \). In particular, \( \text{Med}(\mathbb{B}_{\tilde{f},f^*}) \leq \gamma_2 r^2 \), which forces \( \tilde{f} \) to be in a rather specific part of \( F \). To identify the location of \( \tilde{f} \), first observe
that \( \| \tilde{f} - f^* \|_{L^2} \leq r \). Otherwise, by (1.1), \( \mathbb{M}_{\tilde{f},f^*} \geq \gamma_1 r^2 \) on 0.99n of the blocks, which is impossible if \( \gamma_1 > \gamma_2 \). Finally, given that \( \| \tilde{f} - f^* \|_{L^2} \leq r \) then also \( \mathbb{E} \mathbb{M}_{\tilde{f},f^*} \leq 2\gamma_2 r^2 \), since otherwise, by (2.1), we would have

\[
\mathbb{B}_{\tilde{f},f^*}(j) \geq \mathbb{M}_{\tilde{f},f^*}(j) \geq \mathbb{E} \mathbb{M}_{\tilde{f},f^*}(j) - \gamma_2 r^2 > \gamma_2 r^2
\]
on 0.99n of the blocks, which is also impossible.

Therefore we have that

\[
\mathbb{E} \left( (\tilde{f}(X) - Y)^2 | D \right) - \mathbb{E}(f^*(X) - Y)^2 = \| \tilde{f} - f^* \|_{L^2}^2 + \mathbb{E} \mathbb{M}_{\tilde{f},f^*} \leq (1 + 2\gamma_2) r^2,
\]
as claimed.

**Remark 2.2.** The second claim from [1], that just like tournaments, \( \tilde{f} \) is robust to malicious corruption of the given data, is now completely clear: if (1.1) and (1.2) are to be believed, it means that 98\% of the values of both \( \mathbb{B}_{\tilde{f},f^*}(j) \) and \( \mathbb{M}_{\tilde{f},f^*}(j) \) are in the range we want. Even if another 40\% of those values are changed maliciously, the median of \( \{\mathbb{B}_{\tilde{f},f^*}(j)\}_{n=1} \) would still be in the right range, and the proof of Theorem 2.1 would still hold. As noted previously, this robustness to malicious changes is one of main features of quantiles.

It should be noted that the assumption made in [1] is that the number of “corrupted samples” is smaller than a small proportion of the number of blocks, as one would expect from the proof of Theorem 2.1.

### 2.1 Regularization

Next, let us turn to the question of regularization, where the analysis of \( \tilde{f} \) is equally simple and again, requires only the right versions of Lemma 5.1 and Lemma 5.5 from [4].

In the regularized tournament introduced in [3], the match between \( f \) and \( h \) is determined by the \( n \) values

\[
\mathbb{B}_{f,h}(j) + \lambda(\Psi(f) - \Psi(h)),
\]
where \( \Psi \) denotes the regularization function and \( \lambda \) is the regularization parameter. The alternative version of the regularized tournament suggested in [1] was to select \( \tilde{f}_\lambda \), set to be the minimizer of the functional

\[
\phi_\lambda(f) = \max_{g \in F} \text{Med}(\mathbb{B}_{f,g}) + \lambda(\Psi(f) - \Psi(g)).
\]

Before we explain how the performance of \( \tilde{f}_\lambda \) can be determined using (1.1) and (1.2) and in order to put our presentation in some context, let us briefly describe some ideas that are needed in the study of regularized procedures. For a more detailed description we refer the reader to [2].

The whole point in regularization is dealing with a problem that involves a convex class \( F \) that is simply ‘too big’. To address the size of \( F \), one assigns a ‘price-tag’ to each function in the class according to some prior belief, giving less favourable functions a higher price. That price is captured by the regularization function \( \Psi \), which, for the sake of simplicity is assumed to be a norm on a linear space \( E \) that contains \( F \).

As always, one would like to use the given random data \( (X_i, Y_i)_{i=1}^N \) to ‘distinguish’ between functions \( f \in F \) and \( f^* \), allowing one to rule-out functions whose statistical performance is
inferior to that of $f^*$. A good procedure is designed to select only functions that cannot be ‘distinguished’ from $f^*$, and the regularized procedures we consider are no exception.

The method of analysis we use here is based on ideas from [2]: given the class $F$, we first identify two radii, $\rho$ and $r$; consider $B_{f^*}(\rho)$—the $\Psi$-ball centred at $f^*$ and of radius $\rho$, and set $F_\rho = F \cap B_{f^*}(\rho)$. Now one proceeds with the following three steps:

1. Since the set $F_\rho$ is much smaller than $F$, standard methods of ‘distinguishing’ between $f^*$ and functions in $F_\rho$ is possible. Moreover, the difference between $\Psi(f)$ and $\Psi(f^*)$ is not that big—at most $\rho$. Hence, one would like to show that for functions in $F_\rho$ that satisfy $\|f - f^*\|_{L_2} \geq r$, the empirical component of the excess functional is ‘positive enough’ to overcome the contribution of the regularization terms, which is no worse than $-\lambda \rho$.

2. When considering functions in $F$ that satisfy $\Psi(f - f^*) = \rho$ and for which $\|f - f^*\|_{L_2} \leq r$ it is futile to expect that the empirical component of the excess functional is of any use. This is precisely when functions are ‘too close’ and cannot be distinguished using empirical data. Therefore, all the ‘hard work’ required to distinguish such functions from $f^*$ has to be based on properties of the regularization function $\Psi$. We describe how to obtain the wanted control in what follows.

3. Combining (1) and (2) we can distinguish between $f^*$ and any function that satisfies $\Psi(f - f^*) = \rho$, or between $f^*$ and functions in $F_\rho$ that satisfy $\|f - f^*\|_{L_2} \geq r$. Moreover, by a homogeneity argument, the estimate in the $\Psi$-sphere transfers for free to functions $f \in F$ that satisfy $\Psi(f - f^*) > \rho$, allowing us to distinguish between those and $f^*$.

Using the combination of the three components one may show that a procedure selects $h$ that satisfies both $\Psi(h - f^*) \leq \rho$ and $\|h - f^*\|_{L_2} \leq r$; showing that in addition, the excess risk of $h$ is smaller than $cr^2$ requires an additional argument, and we return to it later.

The key component in regularization happens to be (2): identifying features of the regularization function $\Psi$ that yield sufficient control when the empirical component of the functional fails. By now it is well understood that this property has to do with the smoothness of $\Psi$, as we briefly explain next.

Let $B_{\Psi^*}$ and $S_{\Psi^*}$ denote the unit ball and unit sphere in the dual space to $(E, \Psi)$, respectively. Therefore, $B_{\Psi^*}$ consists of all the linear functionals $z \in E^*$ for which $\sup_{\{x \in E : \Psi(x) = 1\}} |z(x)| \leq 1$. A linear functional $z^* \in S_{\Psi^*}$ is a norming functional for $f \in E$ if $z^*(f) = \Psi(f)$.

**Definition 2.3.** Let $\Gamma_f(\rho) \subset S_{\Psi^*}$ be the collection of functionals that are norming for some $v \in B_f(\rho/20)$. Set

$$\Delta_f(\rho, r) = \inf \inf_{f \in F} \sup_{h \in \Gamma_f(\rho)} z(h - f),$$

where the inner infimum is taken in the set

$$\{h \in F : \Psi(h - f) = \rho \text{ and } \|h - f\|_{L_2} \leq r\}.$$  \hspace{1cm} (2.4)

Several examples of regularization functions $\Psi$ and the resulting estimates on $\Delta_f(\rho, r)$ can be found in [2, 3]. Among the examples are standard sparsity-driven procedures like lasso and slope.

For our purposes, the crucial observation incorporates a lower bound on $\Delta_f(\rho, r)$ and a wise choice of the regularization parameter $\lambda$. This observation is not new as well: it is a
version of Lemma 4.5 from [3] and although its formulation is not identical to that lemma, its proof is — line for line. We present the proof in an the appendix for the sake of completeness.

**Lemma 2.4.** Fix $\gamma_1$ and $\gamma_2$ and some block $I_j$. Let $\rho$ and $r$ such that $\Delta_F(\rho, r) \geq 4\rho/5$ and set $\lambda$ to satisfy

$$3\gamma_2 \cdot \frac{r^2}{\rho} \leq \lambda \leq \frac{\gamma_1}{2} \cdot \frac{r^2}{\rho}. \quad (2.5)$$

Assume that for $h \in F_\rho$ such that $\|h - f^*\|_{L^2} \geq r$, one has

$$\mathfrak{B}_{h, f^*}(j) \geq \gamma_1 \|h - f^*\|_{L^2}^2; \quad (2.6)$$

and if $\|h - f^*\|_{L^2} \leq r$ then

$$|\mathfrak{M}_{h, f^*}(j) - \mathbb{E}\mathfrak{M}_{h, f^*}(j)| \leq \gamma_2 r^2. \quad (2.7)$$

Under these conditions the following holds:

1. If $h \in F_\rho$ and $\|h - f^*\|_{L^2} \geq r$ then

$$\mathfrak{B}_{h, f^*}(j) + \lambda(\Psi(h) - \Psi(f^*)) \geq \frac{\gamma_1}{2} \|h - f^*\|_{L^2}^2; \quad (2.8)$$

2. If $\Psi(h - f^*) = \rho$ and $\|h - f^*\|_{L^2} < r$ then

$$\mathfrak{B}_{h, f^*}(j) + \lambda(\Psi(h) - \Psi(f^*)) \geq \frac{\gamma_2}{2} r^2; \quad (2.9)$$

3. Let $f, h \in F$ satisfy that $\Psi(h - f^*) = \rho$ and $f = f^* + \alpha(h - f^*)$ for some $\alpha > 1$. If $\|h - f^*\|_{L^2} \geq r$ then

$$\mathfrak{B}_{f, f^*}(j) + \lambda(\Psi(f) - \Psi(f^*)) \geq \alpha \cdot \frac{\gamma_1}{2} \|h - f^*\|_{L^2}^2,$$

and if $\|h - f^*\|_{L^2} < r$ then

$$\mathfrak{B}_{f, f^*}(j) + \lambda(\Psi(f) - \Psi(f^*)) \geq \alpha \cdot \gamma_2 r^2.$$

Conditions (2.6) and (2.7) should not come as a surprise: they are simply (1.1) and (1.2) for the class $F_\rho$ and for one block. Hence, by Lemma 5.1 and Lemma 5.5 from [4], under minimal assumptions, with probability at least $1 - 2\exp(-c_1 n)$ for every $h \in F_\rho$, each condition holds for $0.99n$ of the blocks.

**Theorem 2.5.** Let $r, \rho$ and $\lambda$ be as in Lemma 2.4 and assume that for a sample $D = (X_i, Y_i)_{i=1}^N$, (1.1) and (1.2) hold in the class $F_\rho$. Then,

$$\Psi(\tilde{f}_\lambda - f^*) \leq c_1 \rho, \quad \|\tilde{f}_\lambda - f^*\|_{L^2} \leq c_2 r$$

and

$$\mathbb{E}\left( (\tilde{f}_\lambda(X) - Y)^2 | D \right) - \mathbb{E}(f^*(X) - Y)^2 \leq c_3 r^2$$

where $c_1, c_2$ and $c_3$ depend only on $\gamma_1$ and $\gamma_2$.  

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Proof. Recall the properties (1), (2) and (3) from Lemma 2.4 which we first use to obtain an upper estimate on

$$\phi_\lambda(f^*) = \max_{g \in F} \left[ - (\text{Med}(\mathbb{B}_{g,f^*}) + \lambda(\Psi(g) - \Psi(f^*))) \right]$$

by exploring all the possible options of $g \in F$.

If $g \in F\rho$ and $\|g - f^*\|_{L_2} \geq r$ then by (1), for 0.99$n$ of the blocks,

$$\mathbb{B}_{g,f^*}(j) + \lambda(\Psi(g) - \Psi(f^*)) \geq \frac{\gamma_1}{2} \|g - f^*\|^2_{L_2};$$

and if $\Psi(g - f^*) = \rho$ and $\|g - f^*\|_{L_2} < r$ then by (2), on 0.99$n$ of the blocks,

$$\mathbb{B}_{g,f^*}(j) + \lambda(\Psi(g) - \Psi(f^*)) \geq \frac{\gamma_2}{2} r^2.$$ 

Thus, together with the “super-linear” growth of $\mathbb{B}_{h,f^*}(j) + \lambda(\Psi(h) - \Psi(f^*))$ from (3), we have that if $\Psi(g - f^*) \geq \rho$ then

$$- (\text{Med}(\mathbb{B}_{g,f^*}) + \lambda(\Psi(g) - \Psi(f^*))) < 0.$$ 

All that remains is to study the case $\Psi(g - f^*) < \rho$ and $\|g - f^*\|_{L_2} \leq r$. Note that in that range, for 0.99$n$ of the blocks, $|M_{g,f^*}(j) - EM_{g,f^*}(j)| \leq \gamma_2 r^2$ and by the convexity of $F$, $EM_{g,f^*}(j) > 0$. Therefore,

$$\mathbb{B}_{g,f^*}(j) \geq M_{g,f^*}(j) \geq -\gamma_2 r^2 + EM_{g,f^*}(j) \geq -\gamma_2 r^2,$$

and with our choice of $\lambda$,

$$\mathbb{B}_{g,f^*}(j) + \lambda(\Psi(g) - \Psi(f^*)) \geq \gamma_2 r^2 - \lambda \rho \geq - \left( \gamma_2 + \frac{\gamma_1}{2} \right) r^2.$$ 

Hence, we have that $\phi_\lambda(f^*) \leq \left( \gamma_2 + \frac{\gamma_1}{2} \right) r^2$, and just as we did previously, we use this information to pin-point the location of $\tilde{f}_\lambda$.

By considering the choice $g = f^*$, we have

$$\phi_\lambda(f^*) \geq \text{Med}(\mathbb{B}_{f^*,f^*}) + \lambda(\Psi(\tilde{f}_\lambda) - \Psi(f^*)), $$

and let us rule out possible locations of $\tilde{f}_\lambda$. To that end, let $\alpha > 1$ to be specified later and set $f \in F$ such that $\Psi(f - e^*) = \alpha \rho$, i.e., $f = e^* + \alpha (h - e^*)$, where $h \in F$ and $\Psi(h - e^*) = \rho$. By the super-linear growth in (3), combined with (1) and (2), it follows that for 0.98$n$ of the blocks,

$$\mathbb{B}_{f,e^*}(j) + \lambda(\Psi(f) - \Psi(e^*)) \geq \frac{\alpha}{2} \min \{\gamma_1, \gamma_2\} r^2 > \left( \gamma_2 + \frac{\gamma_1}{2} \right) r^2$$

for the right choice of a large enough $\alpha$ that depends only on $\gamma_1$ and $\gamma_2$. Repeating this argument for larger values $\alpha' > \alpha$ rules out the possibility that $\Psi(\tilde{f}_\lambda - f^*) \geq \alpha \rho$, implying that $\Psi(\tilde{f}_\lambda - f^*) < \alpha \rho$.

Given that $\Psi(\tilde{f}_\lambda - f^*) \leq \alpha \rho$, let us estimate $\|f - f^*\|_{L_2}$. If $\|\tilde{f}_\lambda - f^*\|_{L_2} \geq \alpha r$, set

$$h = f^* + \frac{1}{\alpha}(\tilde{f}_\lambda - f^*) \quad (2.10)$$

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and observe that $\Psi(h - f^*) \leq \rho$ and $\|h - f^*\|_{L_2} \geq r$. Hence, by (1), on 0.99n of the blocks, 

$$B_{h, f^*}(j) + \lambda(\Psi(h) - \Psi(f^*)) \geq \frac{\gamma_1}{2} \|h - f^*\|_{L_2}^2,$$

and by (3) we have for the right choice of $\alpha$ that on the same blocks 

$$B_{f^*, f^*}(j) + \lambda(\Psi(\tilde{f}_\lambda) - \Psi(f^*)) \geq \alpha \frac{\gamma_1}{2} \|h - f^*\|_{L_2}^2 \geq \alpha \frac{\gamma_1}{2} r^2 > \left(\gamma_2 + \frac{\gamma_1}{2}\right) r^2,$$

which is impossible. Hence, $\Psi(\tilde{f}_\lambda - f^*) \leq \alpha \rho$ and $\|\tilde{f}_\lambda - f^*\|_{L_2} \leq \alpha r$. Moreover, $h$ defined in (2.10) satisfies that $\|h - f^*\|_{L_2} \leq r$.

Finally, let us show that 

$$\mathbb{E} M_{f, f^*}(j) \leq 2(\alpha + 1) \left(\gamma_2 + \frac{\gamma_1}{2}\right) r^2. \tag{2.11}$$

Indeed, assume that the reverse inequality holds. Since $\|h - f^*\|_{L_2} \leq r$ then by (2.7), on 0.99n of the blocks, $|M_{h, f^*}(j) - \mathbb{E} M_{h, f^*}(j)| \leq \gamma_2 r^2$; and, since $\alpha M_{h, f^*}(j) = M_{f^*, f^*}(j)$ it follows that on the same blocks, 

$$M_{f^*, f^*}(j) \geq \mathbb{E} M_{f^*, f^*}(j) - \alpha \gamma_2 r^2.$$

Thus, by our choice of $\lambda$, 

$$B_{f^*, f^*}(j) + \lambda(\Psi(\tilde{f}_\lambda) - \Psi(f^*)) \geq \mathbb{E} M_{f^*, f^*}(j) - \alpha \gamma_2 r^2 - \lambda \cdot \alpha \rho$$

$$\geq \mathbb{E} M_{f^*, f^*}(j) - \alpha \left(\gamma_2 + \frac{\gamma_1}{2}\right) r^2 > \left(\gamma_2 + \frac{\gamma_1}{2}\right) r^2,$$

which is impossible — confirming (2.11).

To conclude, we have shown that for the sample $D = (X_i, Y_i)_{i=1}^N$, 

$$\mathbb{E} \left( (\tilde{f}_\lambda(X) - Y)^2 | D \right) - \mathbb{E} f^*(X) - Y)^2 = \|\tilde{f}_\lambda - f^*\|_{L_2}^2 + \mathbb{E} M_{h, f^*} \leq c(\gamma_1, \gamma_2) r^2,$$

completing the proof.  

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A Proof of Lemma 2.4

Let us begin by examining

\[ (*) = \mathbb{B}_{h,f^*}(j) + \lambda (\Psi(h) - \Psi(f^*)) \]

in the set \( \{ f \in \mathcal{F} : \Psi(f - f^*) = \rho \} \), where one should consider two cases. First, if \( \|f - f^*\|_{L_2} \geq r \) and since \( \mathbb{B}_{h,f^*}(j) \geq \gamma_1 \|h - f^*\|^2_{L_2} \), then by the triangle inequality for \( \Psi \),

\[
(*) \geq \gamma_1 \|h - f^*\|^2_{L_2} - \lambda \Psi(f - f^*) \geq \gamma_1 \|h - f^*\|^2_{L_2} - \lambda \rho \geq \frac{\gamma_1}{2} \|h - f^*\|^2_{L_2} \quad (A.1)
\]

provided that

\[
\lambda \leq \frac{\gamma_1}{2} \cdot \frac{r^2}{\rho}. \quad (A.2)
\]

If, on the other hand, \( \|f - f^*\|_{L_2} \leq r \), then recalling that \( \mathbb{E} \mathbb{M}_{h,f^*}(j) \geq 0 \) we have \( \mathbb{B}_{h,f^*}(j) \geq -\gamma_2 r^2 \); therefore, \( (*) \geq -\gamma_2 r^2 + \lambda (\Psi(f) - \Psi(f^*)) \).

Fix \( v \in \mathcal{B}_{f^*}(\rho/20) \) and write \( f^* = u + v \); thus \( \Psi(u) \leq \rho/20 \). Set \( z \) to be a linear functional that is norming for \( v \) and observe that for any \( h \in E \),

\[
\Psi(h) - \Psi(f^*) \geq \Psi(h) - \Psi(v) - \Psi(u) \geq z(h - v) - \Psi(u) \geq z(h - f^*) - 2 \Psi(u)
\]

\[
\geq z(h - f^*) - \frac{\rho}{10} \quad (A.3)
\]

Hence, if \( f^* \in \mathcal{F} \), \( \Psi(h - f^*) = \rho \) and \( \|h - f^*\|_{L_2} \leq r \), then optimizing the choices of \( v \) and of \( z \), \( z(h - f^*) \geq \Delta_F(\rho,r) \); thus

\[
\Psi(h) - \Psi(f^*) \geq \Delta_F(\rho,r) - \frac{\rho}{10} \geq \frac{7}{10} \rho. \quad (A.4)
\]

And, if

\[
\lambda \geq 3 \gamma_2 \cdot \frac{r^2}{\rho}, \quad (A.5)
\]

we have that

\[
(*) \geq \frac{\gamma_2}{2} r^2.
\]

Next, if \( h \in F_\rho \) and \( \|h - f^*\|_{L_2} \geq r \), then

\[
\mathbb{B}_{h,f^*}(j) + \lambda (\Psi(f) - \Psi(f^*)) \geq \frac{\gamma_1}{2} \|h - f^*\|^2_{L_2};
\]

indeed, this follows from \( (A.1) \).

Finally, let us prove the super-linearity property when \( \Psi(f - f^*) > \rho \). Set \( \theta \in (0,1) \) and let \( h \in \mathcal{F} \) satisfy that

\[
\Psi(h - f^*) = \rho \quad \text{and} \quad \theta(f - f^*) = h - f^*.
\]

If \( \|h - f^*\|_{L_2} \geq r \), then by the triangle inequality for \( \Psi \) followed by \( (A.1) \),

\[
(*) \geq \frac{1}{\theta^2} \mathbb{Q}_{h,f^*}(j) + \frac{1}{\theta} (\mathbb{M}_{h,f^*}(j) - \lambda \Psi(h - f^*))
\]

\[
\geq \frac{1}{\theta} (\mathbb{B}_{h,f^*}(j) - \lambda \Psi(h - f^*)) \geq \frac{1}{\theta} \frac{\gamma_1}{2} \|h - f^*\|^2_{L_2}.
\]
If, on the other hand, \( \| h - f^* \|_{L_2} \leq r \), then using the argument from (A.3), (A.4) and the choice of \( \lambda \) from (A.5), we have

\[
(*) \geq \frac{1}{\theta} (M_{h,f^*}(j) + \lambda (z(h - f^*) - 2\theta \Psi(u))) \\
\geq \frac{1}{\theta} (M_{h,f^*}(j) + \lambda (z(h - f^*) - 2\Psi(u))) \geq \frac{1}{\theta} \cdot \frac{\gamma^2}{2} r^2,
\]

as claimed.