Spin-statistics relation for abelian quantum Hall states

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We prove a generic spin-statistics relation for the fractional quasiparticles that appear in abelian quantum Hall states on the disk. The proof is based on an efficient way for computing the Berry phase acquired by a generic quasiparticle translated in the plane along a circular path, and on the crucial fact that once the gauge-invariant generator of rotations is projected onto a Landau level, it fractionalizes among the quasiparticles and the edge. Using these results we define a measurable quasiparticle fractional spin that satisfies the spin-statistics relation. As an application, we predict the value of the spin of the composite-fermion quasielectron proposed by Jain; our numerical simulations agree with that value. We also show that Laughlin’s quasielectrons satisfy the spin-statistics relation, but carry the wrong spin to be the anyons of Laughlin’s quasiholes. We conclude highlighting the fact that the statistical angle between two quasiparticles can be obtained by measuring the angular momentum whilst merging the two quasiparticles.

Introduction — The spin-statistics theorem is one of the pillars of our description of the world and classifies all quantum particles into bosons and fermions according to their spin, integer or half-integer [1]. It was early noted that in two spatial dimensions this relation is modified and intermediate statistics exist, called ayonic [2,3]. These objects too satisfy a generalised spin-statistics relation (SSR), and it is common nowadays to speak of fractional spin and statistics [4,5]. This type of SSR, which we also consider, arises in a non-relativistic, non-field-theoretic context [6,7].

The quantum Hall effect (QHE) [8,9] is the prototypical setup where anyons have been studied, and several of their remarkable properties have also been experimentally observed [10,11]. Whereas the notion of fractional statistics has been early applied to the localised quasiparticles of the QHE [12,13], the notion of spin has been more controversial. The existence of a fractional spin satisfying a SSR has been established for setups defined on curved spaces thanks to the coupling to the curvature of the surface [15,20]. The extension of this notion to planar surfaces has required more care and it is not completely settled yet [21,23].

In this letter we prove an SSR for the abelian quasiparticles of the QHE on a planar surface under the generic assumption that a QHE state satisfies the screening property. It does not require the notion of curvature and identifies an observable spin that is an emergent collective property and is not related to the physical SU(2) spin of electrons or atoms. Several applications are presented. First, we apply it to the quasielectron (QE) wavefunctions proposed by Jain [24] and by Laughlin [25] for the filling factor \( \nu = 1/M \). Second, we show how the fractional statistics affects the total angular momentum of the setup leading to observable consequences. Finally, we remark on an intrinsic ambiguity in the definition of the spin.

The QHE model — We consider a two-dimensional (2D) system of \( N \) quantum particles with mass \( m \) and charge \( q > 0 \) traversed by a uniform and perpendicular magnetic field \( \vec{B} = B\hat{e}_z \), \( B > 0 \). The cyclotron frequency and the magnetic length read \( \omega = qB/m \) and \( \ell_B = \sqrt{\hbar c/(qB)} \). We adopt the standard parametrisation of the plane \( z_j = x_j + iy_j = |z_j|e^{i\phi_j} \).

The Hamiltonian is:

\[
H_0 = \sum_{\alpha=1}^{N} \left( \frac{\pi_{\alpha,x}^2 + \pi_{\alpha,y}^2}{2m} + v(|z_\alpha|) \right) + \sum_{i<j} V_{\text{int}}(|z_i - z_j|),
\]

where \( \pi_{\alpha,a} = p_{\alpha,a} - (q/e)A_\alpha(z_\alpha) \) and \( v(|z|) \) is a central confining potential. We assume that the interaction potential \( V_{\text{int}}(|z|) \) is rotationally invariant, as the Coulomb interaction relevant for electrons [26] and the contact interaction relevant for cold gases [27]. We also assume that the ground state of [11] is not degenerate and realizes an incompressible QHE state characterised by the screening property, so that the specific form of \( v(|z|) \) is not important as long as we are only interested in the bulk.

We assume the presence of \( N_{\text{qp}} \) pinning potentials located at positions \( s_\alpha \); using the complex-plane parametrisation \( \eta_\alpha = s_{\alpha,x} + is_{\alpha,y} = |\eta_\alpha|e^{i\theta_\alpha} \) we write:

\[
H_1(\eta) = \sum_{\alpha=1}^{N_{\text{qp}}} \sum_{i=1}^{N} V_\alpha(|z_i - \eta_\alpha|),
\]

where \( \eta \) is a shorthand for \( \eta_1, \ldots, \eta_{N_{\text{qp}}} \). Since the pinning potentials might be different, we keep the subscript \( V_\alpha \); they are all assumed to be rotationally invariant.

The ground state of the model \( H = H_0 + H_1(\eta) \) is \( |\Psi_\eta\rangle \); we assume that it is unique, that it satisfies the screening property, and that it localises \( N_{\text{qp}} \) quasiparticles at \( \eta_\alpha \). Since the pinning potentials can be different, the quasiparticles need not be of the same kind. The set of \( \eta_\alpha \) is completely arbitrary and rotational invariance
With a quasiparticle at $\eta$, the ground state satisfies the Schrödinger equation $H_\eta |\Psi_\eta\rangle = E_\eta |\Psi_\eta\rangle$. However, $E_\eta$ can only depend on $|\eta|$, and not on $\theta$; thus, $\partial_\theta E_\eta = 0$. We conclude that $H_\eta U_\beta |\Psi_\eta\rangle = U_\beta E_\eta |\Psi_\eta\rangle = E_\eta U_\beta |\Psi_\eta\rangle$, namely that $U_\beta |\Psi_\eta\rangle$ is an eigenvector of $H_\eta$ with energy $E_\eta$. If the ground state is unique, it must be an eigenvector of $U_\beta$ and of its generator $\hat{L}$; we dub the eigenvalue of the latter $\ell_\eta$. For example, in the case of the normalised Laughlin state with a quasi-hole (QH), $N(|\eta|)^{-1/2} \sum_i (z_i - \eta) \sum_{j<k} (z_j - z_k)^4 e^{-\sum_i |z_i|^2/4}\bar{\eta}_i$, the eigenvalue $\ell_\eta = \frac{M}{N} N(N-1) + N$ is the degree of the polynomial in $z_i$ and $\eta$.

We now perform a gauge transformation that unwinds the generalised angular momentum $\ell_\eta$ moving along a trajectory at fixed $|\eta|$: $|\tilde{\Psi}_\eta\rangle = e^{i\tilde{g}(|\eta|)}|\Psi_\eta\rangle$ with $\tilde{g}(\eta) = -\int_0^\theta \ell_0 d\theta'$. In the aforementioned case, the Laughlin state gets multiplied by the phase $(\eta/|\eta|)^{-\frac{3M}{2} N(N-1)-\frac{M}{2}}$. Let us show that:

$$\hat{L}|\tilde{\Psi}_\eta\rangle = 0.$$  \hspace{1cm} (4)

By definition, $\hat{L}|\tilde{\Psi}_\eta\rangle = e^{i\tilde{g}(|\eta|)} \hat{L}|\Psi_\eta\rangle + (L_e e^{i\tilde{g}(\eta)}) |\Psi_\eta\rangle$. The first term of the sum is $\ell_n|\tilde{\Psi}_\eta\rangle$, the second term is obtained by differentiating the exponential, and equals $-\ell_n|\Psi_\eta\rangle$. This concludes the proof of the statement that one can find a gauge such that $\hat{L}$ annihilates the ground state. Note that for a state satisfying Eq. (4), it is also true that $U_\beta|\tilde{\Psi}_\eta\rangle = |\tilde{\Psi}_\eta\rangle$ for any angle $\beta$. Choosing $\beta = 2\pi$ we obtain that this state is single-valued in the $\eta$ coordinate because $U_{2\pi}|\tilde{\Psi}_\eta\rangle$ is also equal to $|\tilde{\Psi}_\eta\rangle$.

This reasoning can be easily extended to the case of several quasiparticles. We can define a reference angle $\theta_0$ and express $\theta_\alpha = \theta_0 + \Delta\theta_\alpha$, treating the variables $\Delta\theta_\alpha = \theta_\alpha - \theta_0$ as independent from $\theta_0$. The operator $-i\partial_{\theta_0}$ generates the group $e^{i\alpha x(-i\partial_{\theta_0})}$ that modifies the quasiparticle polar angles as follows: $\theta_\alpha \rightarrow \theta_\alpha + \beta$, leaving the radial distance unchanged; thus: $\eta_\alpha \rightarrow \eta_\alpha e^{i\beta}$. This is exactly the action of $L_{\alpha}'$, and thus we conclude that $L_{\alpha}' = -i\partial_{\theta_0}$. With arguments paralleling those for one quasiparticle, one can (i) show that $\hat{L}|\tilde{\Psi}_\eta\rangle = \ell_n|\tilde{\Psi}_\eta\rangle$, (ii) make the dependence on $\theta_0$, the $\Delta\theta_\alpha$ and the $|\eta_\alpha|$ explicit by writing $\ell_0, \Delta\theta_\alpha, |\eta_\alpha|$, and (iii) define $|\tilde{\Psi}_\eta\rangle = e^{i\tilde{g}(\eta)|\Psi_\eta\rangle}$ with $g(\eta) = -\int_0^\theta \ell_0 d\theta'$. In the kernel of $\hat{L}$.

**Berry phase for the translation of the quasiparticles along a circle** — We now compute the Berry phase corresponding to the translation along a closed circular path of the $N_{qp}$ quasiparticle coordinates via $\theta_0 \rightarrow \theta_0 + 2\pi$ generated by $L_{\alpha}'$ leaving all the $\Delta\theta_\alpha$ and $|\eta|_\alpha$ invariant. Using the fact that $|\tilde{\Psi}_\eta\rangle$ is single-valued in $\eta$, this Berry phase is $\gamma_\eta = \int_0^{2\pi} \langle\tilde{\Psi}_\eta| i\partial_{\theta_0} |\tilde{\Psi}_\eta\rangle d\theta_0$, where only the $\theta_0$ coordinate is changed in the state inside the integral. Employing the definitions of $L$ and $L_{\alpha}'$, and using

![Image of action of \( \hat{L} \). Panel (a): contour plot in \( z \) space of the function \( f(z, \eta) = \exp[-\frac{1}{2}\Re(z - \eta)^2 - 2\Im(z - \eta)^2] \) for \( \eta = 2 \). Panel (b): contour plot of \( f(ze^{i\beta}, \eta) \) for \( \beta = 2\pi/3 \); with respect to (a), the plot is translated and rotated. Panel (c): contour plot of \( f(z, \eta e^{i\beta}) \); this time, the plot is only translated. Panel (d): contour plot of \( f(ze^{i\beta}, \eta e^{i\beta}) \): the composition of the two is a just a rotation and thus \( \hat{L} \) generates the self-rotations of the quasiparticles in the \( z \) plane.](image-url)
we get
\[ \gamma_\eta = \int_0^{2\pi} \langle \Psi_\eta | L_z | \Psi_\eta \rangle d\theta_0 = \int_0^{2\pi} \langle \Psi_\eta | L_z | \Psi_\eta \rangle d\theta_0. \]

The matrix element in the integral is manifestly gauge-independent, as the \( L_z \) operator does not act on the \( \Psi_\eta \); one can thus also use the original states. Finally, let us note that the integrand cannot be a function of \( \theta_0 \), and thus we have an even simpler expression:
\[ \gamma_\eta = 2\pi \langle \Psi_\eta | L_z | \Psi_\eta \rangle. \]

Like any operator projected onto the lowest Landau level (LLL), the angular momentum \( L_z \) is a function of the guiding-center operators \( R_{j,x} = x_j + (\ell_B^2/\hbar)\pi_{j,y} \) and \( R_{j,y} = y_j - (\ell_B^2/\hbar)\pi_{j,x} \), with \( [R_{j,x}, R_{j',y}] = -i\ell_B^2\delta_{jj'} \), and it reads \( L_z = \sum_j (R_{j,y}^2 - R_{j,x}^2 - 1)/2 \). Written in this projected form, \( L_z \) is the gauge-invariant generator of rotations, and it is just a function of the density of the gas \( \rho_\eta(z) \), which through the screening property can be split into a bulk contribution \( \rho_\eta(z) \) (the state without quasiparticles) and an edge contribution \( \rho_\eta(z) \) (the difference at the edge with respect to the state without quasiparticles) and a quasiparticle contribution localised around the \( \eta_\alpha, \eta_\beta, \eta(\eta) \). We split the integrand into three parts:
\[ \langle \Psi_\eta | L_z | \Psi_\eta \rangle = L_{\eta} + L_c(N_{\eta}) + L_{\eta}(\eta); \]

as long as the quasiparticles are far from the edge, the screening property ensures that \( L_c \) can only depend on their number (more precisely: on how many quasiparticles of each kind), but not on their positions; in fact, it also does not change when two of them are put close by or stacked on top of each other.

Notice that \( L_\eta \) is an integer because in a circularly-symmetric problem the angular momentum is quantized; therefore, it contributes the Berry phase \( \gamma_\eta \) with an integer multiple of \( 2\pi \) that we disregard. The only relevant information is contained in the remaining pieces, which indeed depend, directly or indirectly, on the quasiparticles, and this constitutes the first main result of the letter:
\[ \gamma_\eta = 2\pi \times (L_c(N_{\eta}) + L_{\eta}(\eta)). \]

Compared to the direct evaluation of the integral, Eq. \( \gamma_\eta \) is a way to compute the Berry phase that in most cases is simpler to evaluate.

Let us consider now the case of a single quasiparticle at \( \eta \): on the basis of very general arguments, \( \gamma_\eta \) should be the Aharonov-Bohm (AB) phase \( gQ|\eta|^2 B/(\hbar c) \), where \( Q \) is the charge of the quasiparticle in units of \( e \). Let us compare Eq. \( \gamma_\eta \) with this widely-accepted result. In very general terms, the angular momentum of a rotationally-invariant quasiparticle \( L_{\eta}(\eta) \) can be split into an orbital part \( Q|\eta|^2/\ell_B^2 \) and an intrinsic part \( J_{\eta}(\eta) \) (computed with respect to the center of mass). It follows that:
\[ \gamma_\eta = \pi Q|\eta|^2/\ell_B^2 + 2\pi(L_{\eta}(1) + J_{\eta}(\eta)). \]

We recognise the AB phase, to which an apparently spurious contribution has been added; yet, we can show that it is an integer multiple of \( 2\pi \), and thus inessential. Indeed, the angular momentum of the rotationally-invariant system with a quasiparticle put exactly at the centre is \( L_\eta + J_{\eta}(1) \) which is an integer. Since \( L_\eta \) is an integer, \( J_{\eta}(1) \) is an integer, and our thesis follows. A similar relation can be obtained also for \( n \) identical and stacked quasiparticles, \( J_{\eta}(n) + L_c(n) \in \mathbb{Z} \), and even for many quasiparticles of different kinds. Very generally, the gauge-invariant generator of rotations fractionalizes between the bulk quasiparticles and the edge, implying that the spin is robust to local circularly-symmetric perturbations.

**Spin-statistics relation** — We consider two identical quasiparticles placed at opposite positions \( \eta \) and \( -\eta \) and far from each other and from the edge. In order to compute the statistical parameter \( \kappa \), we consider a double exchange, that gives a gauge-invariant expression and avoids any discussion on the identity of the pinning potentials \( \tilde{B} \). Accordingly, we study the difference between the Berry phase for exchanging two opposite particles and the single-particle AB phases \( \gamma_\eta \):
\[ \kappa_{\eta\eta} = \frac{1}{2\pi} (\gamma_{\eta,-\eta} - 2\gamma_\eta). \]

Using Eq. \( \gamma_\eta \), we write \( \kappa_{\eta\eta} = L_c(2) + L_{\eta}(\eta,-\eta) - 2L_{\eta}(1) - 2L_{\eta}(\eta) \). We observe that \( L_{\eta}(\eta,-\eta) = 2L_{\eta}(\eta) \), and by using \( L_{\eta}(n) + J_{\eta}(n) \in \mathbb{Z} \) we obtain the SSR:
\[ \kappa_{\eta\eta} = -J_{\eta\eta} + 2J_{\eta\eta} \pmod{1}. \]

This result allows us to identify the intrinsic angular momentum with the fractional spin associated to the fractional statistics, and constitutes the second main result of the letter. Interestingly, we have linked the statistics to a local property of the quasiparticles: if we assume screening, the fine details of the boundary do not matter, and one could probably prove \( \kappa_{\eta\eta} \) without requiring that \( v(|z|) \) is a central potential.

With similar arguments, the SSR can be extended to the situation where the two quasiparticles are different: calling \( J_{ab} \) and \( J_{\eta\eta} \) their spins, and \( J_{ab} \) the spin of the composite quasiparticle obtained by stacking them at the same place, we obtain the mutual statistics parameter:
\[ \kappa_{a-b} = -J_{a-b} + J_{a} + J_{b} \pmod{1}. \]

In the theory of modular tensor categories (see see for instance \[29\]), a relation of this type is called a ribbon identity. Moreover, the fractionalisation property allows us to read the phase \( \kappa \) directly at the edge; indeed, one easily obtains \( \kappa_{\eta\eta} = L_{\eta}(2) - 2L_{\eta}(1) \) and \( \kappa_{a-b} = L_{\eta}(a,b) - L_{\eta}(a) - L_{\eta}(b) \).

**The spin of the QE** — As a first application of our SSR \( \kappa_{\eta\eta} \), we consider the QE of the Laughlin state at filling \( \nu = 1/M \). Several numerical studies have highlighted that the composite-fermion wavefunction for the QE proposed by Jain \[23, 30\] has the correct statistical properties when the QE is braided with another QE \( (\kappa_{qe} = 1/M) \) or with a QH \( (\kappa_{qe-qh} = -1/M) \) \[31, 36\]. Previous articles have already shown that LLL quasipar-
TABLE I. The spin $J_p$ of Jain’s QE for a Laughlin state at several filling factors $\nu$.

| $p$ | $\nu = 1/2$ | $\nu = 1/3$ | $\nu = 1/4$ |
|-----|-------------|-------------|-------------|
| -1  | $-1$        | $-1$        | $-1$        |
| 0   | $-1$        | $-1$        | $-1$        |
| 1   | $-1$        | $-1$        | $-1$        |

FIG. 2. Calculation of the QE spin via the integral $J(r) = \int_0^1 \left( \frac{\nu}{2} \frac{r^2}{R^2} - 1 \right) \rho_{qp}(r) 2\pi r dr$; the spin coincides with the plateau appearing when $r$ is far from the center and the boundary; $R_{cl} = \sqrt{2N/\nu}$ is the classical radius of the droplet. Panel (a): the spin of a single Jain’s QE for $\nu = 1/2$, 1/3 and 1/4. Panel (b): the case of two stacked Jain’s QEs. Panels (c) and (d): the same for one and two Laughlin’s QEs, respectively. Theoretical predictions following from the SSR relation in Table I are marked with dashed lines and are only compatible with the spin of Jain’s QE. Dashed-dotted lines in panels (c) and (d), together with their values, highlight the position of the spin plateau for Laughlin’s QE.

Fig. 3. Angular momentum $L(R_1, R_2)$ of a Laughlin state ($N = 25$, $\nu = 1/2$) with two QHs at distances $R_1$ and $R_2$ from the center. Panel (a): displacement of the first QH; the angular momentum variation $L(R, R_0) - L(R_0, R_0)$ is plotted in black circles, and it is a quadratic function of $R$ that agrees with the theory prediction $-\epsilon (R^2 - R_0^2)$ (red line). Panel (b): displacement of the second QH; the variation $L(0, R) - L(0, R_0)$ is plotted in brown triangles and it is a quadratic function of $R$ only at large $R$; when the QHs fuse a deviation sets in that equals $-\kappa$, the statistical parameter.

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when the two QH fuse, their total spin changes. In fact, the final value is \( L_0 + 2\epsilon R^2_0 - \kappa \). We verify this result with numerical simulations reported in Fig. 3. This provides an experimental procedure for measuring the mutual statistics of two generic quasiparticles in a controllable quantum simulator of the QHE.

**Alternative spins** — Our definition of spin follows directly from the physical angular momentum \( L = L_R + L_\pi \), where \( L_R = (R_x^2 + R_y^2)/2 \), and \( L_\pi = -(\pi^x_\ell + \pi^y_\ell)/2\hbar^2 \); it is manifestly gauge-invariant and is the generator of 2D rotations, because it satisfies \( [L_j, R_k] = i\epsilon_{jk} R_k \) and \( [L, \pi_j] = i\epsilon_{jk} \pi_k \), with \( \epsilon_{jk} \) the Levi-Civita tensor. The definition is ambiguous: any operator \( L_c = L + c \), with \( c \) a \( c \)-number, has indeed the correct commutation properties: this is a peculiarity of \( U(1) \) rotational quantum simulator of the QHE.

**Conclusions** — We have presented a SSR for the abelian quasiparticles of the QHE on planar surfaces. We have shown that the quasiparticles fractionalise the gauge-invariant generator of rotations and that this quantity can be used to define a measurable spin. The fractional statistical properties of the quasiparticles follow from that. It would be extremely interesting to relax the assumption of non-degenerate ground state, and to move to the non-abelian case [41]. Additionally, it would be interesting to extend our results to a situation where \( H_0 \) is not rotational invariant. The discussion of pinning potentials that are not rotationally symmetric seems also of physical relevance in the context of disorder physics [20] and is left for future work. We observed that Laughlin’s and Jain’s QE have a different spin on the disk. It would be interesting to determine their spin on the sphere from an explicit Berry phase calculation, to check which one matches the expected value \( J_p \), obtained in [17] for \( p = \pm 1 \).

**Acknowledgements** — We acknowledge discussions with I. Carusotto, T. Comparin, B. Estienne, H. Hansson, M. Hermanns, A. Polychronakos and N. Regnault. A.N. thanks Université Paris-Saclay and LPTMS for warm hospitality. L.M. has been supported by LabEx PALM (ANR-10-LABX-0039-PALM).

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Appendix A: The spin of the quasielectron

We tested the spin of two paradigmatic quasielectron (QE) wavefunctions: Laughlin’s [25] and Jain’s [24], both for single and double QE. The spin of lowest Landau level projected wavefunctions is computed according to [23]

$$J_{qe}(R) = \int_{0}^{R} r dr \left( \frac{r^2}{2} - 1 \right) (\rho_{qe}(r) - \rho(r)), \quad (S.1)$$

$\rho_{qe}(r)$ being the density of a QE state placed at the origin and $\rho(r)$ the background density of the fractional quantum Hall state hosting the QE excitation. When $1 \ll r \ll R_{cl}$, $J_{qe}(R)$ has a plateau at the spin value. The knowledge of the analytical form of the wavefunctions allows us to compute the spin by Monte Carlo sampling [42] the spin integral (S.1).

In the following paragraphs, some information on the numerics is given. The results for the spin (S.1) are shown in the main text; here in Fig. S1 we complement by showing the density of the states at filling fraction $\nu = \frac{1}{2}$ and the excess charge with respect to the bulk Laughlin liquid.

a. Single Jain’s quasielectron — Jain’s composite fermion approach to the fractional quantum Hall states suggests [24]

$$\Psi_{JQE} = \hat{P}_{LLL} \prod_{i < j} \left( z_i - z_j \right)^{m-1} (S.2)$$

as a candidate wavefunction for the QE on top of a Laughlin state at filling $\nu = \frac{1}{m}$. Here and in the following, Gaussian factors will be left implicit. Carrying out standard projection onto the lowest Landau level [43] gives (apart for constant proportionality factors)

$$\Psi_{JQE} = \sum_{i} \left( \frac{1}{\prod_{l \neq i} z_i - z_l} \sum_{j \neq i} \frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j)^m \quad (S.3)$$

which has already been shown to carry the correct fractional charge [31] and having the correct exchange statistics [32, 36].

b. Double Jain’s quasielectron — Jain’s composite fermion approach suggests the following wavefunction for a doubly charged QE at the centre of a circularly symmetric droplet

$$\Psi_{J2QE} = \hat{P}_{LLL} \prod_{i < j} (z_i - z_j)^{m-1}. \quad (S.4)$$

Notice that this wavefunction is not the most energetically-favourable double quasielectron state [30], which is realized by promoting two composite fermions to their first Landau level. We study the composite fermion antiparticle of the double-quasihole state wavefunction because it has the same angular momentum as the Laughlin’s double quasielectron
FIG. S1. Comparison of Jain’s and Laughlin’s QE densities $\rho_{qp}(r)$ and charges $Q(r) = \int_0^r (\rho_{qp}(r') - \rho_L(r')) r' dr'$, where $\rho_{qp}(r)$ is the QE density and $\rho_L(r)$ the background Laughlin density. The charge $Q$ of the QE coincides with the plateau appearing when $r$ is far from the center and the boundary; $R_{cl} = \sqrt{2N/\nu}$ is the classical radius of the droplet. Panel (a) (/c): comparison between the densities $\rho_{qp}(r)$ of a single (/double) Jain’s QE (full lines) compared to that of Laughlin’s QEs (dashed lines), for $\nu = 1/2, 1/3$ and 1/4. Horizontal dashed-dotted lines represent the bulk Laughlin state density $\rho_b = \nu/2\pi l_B^2$. Panel (b) (/d): comparison between the QE charge $Q(r)$ of a single (/double) Jain’s QE (full lines) compared to that of Laughlin’s QEs (dashed lines). Horizontal dashed-dotted lines represent the charge of Laughlin’s quasiparticles, $Q = \nu$.

We use standard lowest Landau level projection, although different inequivalent projection methods have been proposed. After some tedious algebra (and again dropping constant proportionality factors) we find

$$
\Psi_{J2QE} = \sum_{i \neq j} \left( \frac{(z_i - z_j) \Gamma_{ij}}{\prod_{k \neq i} (z_k - z_i) \prod_{k \neq j} (z_k - z_j)} \right) \prod_{i < j} (z_i - z_j)^m
\tag{S.5}
$$

where

$$
\Gamma_{ij} = (m - 1)^2 A_i^2 A_j - (m - 1) B_i A_j + \frac{2(m - 1) A_i}{(z_i - z_j)^2} - \frac{2}{(z_i - z_j)^3}
\tag{S.6}
$$

and

$$
\begin{align*}
A_i &= \sum_{j \neq i} \frac{1}{z_i - z_j}, \\
B_i &= \sum_{j \neq i} \frac{1}{(z_i - z_j)^2}.
\end{align*}
\tag{S.7}
$$

c. Single Laughlin’s quasielectron — Laughlin proposed a QE wavefunction by generalizing his successful quasihole (QH) wavefunction [25]

$$
\psi_{LQE} = \left( \prod_i \frac{\partial}{\partial z_i} \right) \prod_{i < j} (z_i - z_j)^m,
\tag{S.8}
$$

which however - unlike the QH counterpart - is not easy to deal with from the computational point of view, due to the $N$-th order derivative term. We are interested in computing expectation values of local, single-particle observables...
\[ \hat{O} = \sum_i \hat{o}_i \]
\[ \langle \hat{O} \rangle = \int \mathcal{D}z \psi^*_\text{LQE} O(z, z^*) \psi_{\text{LQE}} \]  
(S.9)

where \( \mathcal{D}z = \prod_i d^2 z_i \) and \( z \) is a shorthand for all the particles' coordinates. To simplify the expressions we assume \( O(z, z^*) = O(|z|^2) \). By performing integration by parts we then get

\[ \langle \hat{O} \rangle = \frac{1}{Z} \int \mathcal{D}z \left| \prod_{i<j}(z_i - z_j)^m \right|^2 \prod_i \left( |z_i|^2 - 2 \right) \sum_i A_i(|z_i|^2) \]  
(S.10)

where \( A_i \) is related to the observable \( o_i \) through

\[ A(r^2) = o(r^2) - 4 \frac{r^2 - 1}{r^2 - 2} \frac{\partial o(r^2)}{\partial r^2} + 4 \frac{r^2}{r^2 - 2} \frac{\partial^2 o(r^2)}{\partial (r^2)^2} \]  
(S.11)

but crucially involves its derivatives. The normalization factor \( Z \) can be found by looking at \( \hat{O} = I \).

As an example, we expand here the expressions for the observable being the charge up to radius \( R \)

\[ o(r^2) = \theta(R^2 - r^2). \]  
(S.12)

where \( \theta \) is the step function. The spin case (S.1) is perfectly analogous but the expressions are more lengthy because of the \( r^2/2 - 1 \) factor multiplying the step function. The integrals involve derivatives of the \( \delta \) function. It is convenient to take these out of the integrals and rearrange (S.10) in the following form

\[ Q(R) = I_0(R^2) + 4 I_1(R^2) + 4 \frac{\partial I_2(R^2)}{\partial R^2} \]  
(S.13)

where

\[
\begin{aligned}
I_0(R^2) &= \frac{1}{Z} \int \mathcal{D}z \left| \prod_{i<j}(z_i - z_j)^m \right|^2 \prod_i \left( |z_i|^2 - 2 \right) \sum_i \theta(R^2 - r_i^2) \\
I_1(R^2) &= \frac{1}{Z} \int \mathcal{D}z \left| \prod_{i<j}(z_i - z_j)^m \right|^2 \prod_i \left( |z_i|^2 - 2 \right) \sum_i \delta(R^2 - r_i^2) \frac{r_i^2 - 1}{r_i^2 - 2} \\
I_2(R^2) &= \frac{1}{Z} \int \mathcal{D}z \left| \prod_{i<j}(z_i - z_j)^m \right|^2 \prod_i \left( |z_i|^2 - 2 \right) \sum_i \delta(R^2 - r_i^2) \frac{r_i^2}{r_i^2 - 2}.
\end{aligned}
\]  
(S.14)

Taking derivatives of noisy observables is tricky; to circumvent the problem we Fourier transform the relevant quantities

\[ \tilde{I}(k) = \int_0^\infty I(R) J_0(kR) R dR \]  
(S.15)

where \( J_0 \) is the order 0 Bessel function of the first kind. We then filter out the “high-wavevector” noise superimposed to the “low-wavevector” signal \( \tilde{I}_l(k) = c(k) \tilde{I}(k) \), with some suitably chosen cut-off function \( c(k) \), and invert the transform

\[ \frac{\partial^n I_e(R)}{\partial (R^2)^n} = \int_0^\infty \tilde{I}_e(k) \frac{\partial^n J_0(kR)}{\partial (R^2)^n} k dk. \]  
(S.16)

Derivatives of \( J_0 \) can be expressed in closed compact form in terms of the \( _0F_1 \) hypergeometric function, thus avoiding the computation of finite differences.

\textit{d. Double Laughlin’s quasielectron} — A doubly charged Laughlin’s QE can be placed at the origin as \( \psi_{\text{LQED}} = \prod_i \frac{\partial}{\partial z_i} \prod_{i<j} (z_i - z_j)^m \).

Again, carrying out the derivatives explicitly seems not to be feasible, however it is simpler to look at local observables as \( \hat{O} \). Repeated integration by parts yields

\[ \langle \hat{O} \rangle = \frac{1}{Z} \int \mathcal{D}z \left| \prod_{i<j}(z_i - z_j)^m \right|^2 \prod_i \left( 8 - 8|z_i|^2 + |z_i|^4 \right) \sum_i A_i(|z_i|^2) \]  
(S.18)
with
\[
A(r^2) = o(r^2) - \frac{4}{8} - 6r^2 + r^4 \frac{\partial o(r^2)}{\partial r^2} + \frac{8}{8 - 8r^2 + r^4} \frac{\partial^4 o(r^2)}{\partial(r^2)^2} - 32 \frac{-2r^2 + r^4}{8 - 8r^2 + r^4} \frac{\partial^3 o(r^2)}{\partial(r^2)^2} + 16 \frac{r^4}{8 - 8r^2 + r^4} \frac{\partial^4 o(r^2)}{\partial(r^2)^4}. 
\]
(S.19)

Once again, the expressions for the observables involve derivatives of the observable itself; whenever dealing with derivatives of the delta function, we adopted the procedure outlined in the previous paragraph.

### Appendix B: The MPS formulation on the cylinder

In the main text, the spin of the single and double QEs was obtained for the disk geometry. Here, we report results on the cylinder, using the spin of the single and double QEs was obtained for the $\nu = \frac{1}{M}$ fermionic Laughlin state. We refer to [15, 16] for more information on the MPS formulation of quantum Hall states in general. The MPS formulation of a single Jain QE was given in detail in [39]. Here, we provide the MPS matrices for a QH of size $p$, i.e., a QH with charge $\frac{p}{M}$, as well as for QEs with various sizes, i.e., the cases with $p$ negative.

As was discussed in detail in [39], if one wants to be able to consider a state with several QHs and QEs, it is necessary to introduce two chiral boson fields $\varphi$ and $\tilde{\varphi}$, in order that the various operators have the correct statistics with one another. However, in the case that one is interested in simulating only one QE (which can be a single QE of arbitrary size), without any QHs, a single chiral boson field suffices.

The CFT operators for an electron, a size $p > 0$ QH and a the modified electron operator for the size $p < 0$ QH (on the disk) are given by (in terms of the single chiral boson field $\varphi(z)$)
\[
V_{\text{el}}(z) = e^{i\varphi(z)\sqrt{M}} : , \\
V_{\text{qh},p}(\eta) = e^{i\varphi(\eta)p/\sqrt{M}} : , \\
\tilde{V}_{\text{el},\mu}(z) = \frac{\partial}{\partial z} : e^{i\varphi(z)(M-|p|)/\sqrt{M}} : ,
\]
where we note that $p < 0$ in the QE case. In addition, one has the following constraint on the size of the QH, $0 < |p| < M$.

Without going into the details, we here state the MPS matrices for an empty site, a site occupied by an electron, the matrix for a size $p$ QH, as well as the modified electron operators corresponding to a size $p$ QH.

- **The matrices of the empty sites and ordinary electron operator** — The circumference of the cylinder is denoted by $L$. Using the notation of [39], the MPS matrix for an empty site is given by
\[
B^{[0]} = \delta_{Q',Q-1} \delta_{P',P} \delta_{\mu',\mu} e^{-\left(\frac{2\pi}{2 M}\right)^2 \left(\frac{Q'^2}{M^2} + \frac{Q'}{2M} + P'\right)}. 
\]
(S.2)

The matrix for an site occupied by an electron is given by
\[
B^{[1]} = \delta_{Q',Q+M-1} \delta_{P',P-Q} A_{\mu',\mu}^{\sqrt{M}} e^{-\left(\frac{2\pi}{2 M}\right)^2 \left(\frac{Q'^2}{M^2} + \frac{Q'}{2M} + P'\right)},
\]
(S.3)

where $A_{\mu',\mu}^{\sqrt{M}}$ is given by
\[
A_{\mu',\mu}^{\sqrt{M}} = \prod_{j=1}^{\infty} \sum_{s=0}^{m_j} \sum_{r=0}^{m_j} \frac{\delta_{m_j-r,m_j-s}}{\sqrt{r+1}} \left(\frac{\beta}{\sqrt{2}}\right)^{r+s} \sqrt{\binom{m_j}{s} \binom{m_j}{r}}. 
\]
(S.4)

- **The matrices for the size $p$ QH operator** — We consider the matrix for the size $p$ QH operator, where it is assumed that we only consider states with one QH (which can be of arbitrary size $p > 0$). The operator is inserted between orbitals $l-1$ and $l$. We denote this operator as $H_{l,p}(\eta)$, where $\eta$ is the position of the QH on the cylinder. We find
\[
H_{l,p}(\eta) = (-1)^{l+p} \delta_{Q',Q+l} \delta_{P',P} A_{\mu',\mu}^{\sqrt{M}} e^{-\left(\frac{2\pi}{2 M}\right)^2 \left(\frac{Q'^2}{M^2} + \frac{Q'}{2M} + P'\right)} e^{-\left(\frac{2\pi}{2 M}\right)^2 \left(\tau_l(\eta)\left(\frac{Q'^2}{M^2} + \frac{Q'}{2M} + P'\right)\right)}. 
\]
(S.5)

- **The matrices for the modified electron for the size $p$ QE operator** — As explained in detail in [39], a QE is created by modifying an elektron operator, to create a delocalized (angular momentum eigenstate, with angular momentum $k$) QE. The latter are used to create a localized QE at $\xi$, by means of applying a localizing kernel, weighing
FIG. S2. The particle density profile $\rho(r)$ (in units of $1/\ell_B^2$) as a function of the radius (in units of $\ell_B$), for a single/double quasihole (upper/lower left panel) and for a single/double Jain quasielectron (upper/lower right panel). The dotted lines represent the background density $\rho_0 = \nu/(2\pi)$ with $\nu = 1/3$.

d. Results from the MPS formulation — We used the MPS formulation for the QH and QE states, to obtain the density profiles, the access charge $Q(r)$ and the spin $J(r)$ of the excitations. In all cases, we used the following parameters for the MPS. The number of electrons $N_e = 100$; the circumference of the cylinder $L = 18\ell_B$. The maximum value of the momentum for the non-zero modes $p_{\text{max}} = 12$. In all cases, the excitations were placed in the middle of the cylinder.
FIG. S3. The excess charges $Q(r)$ (in units of $q$) as defined in the caption of Fig. S1 for a single/double quasihole (upper/lower left panel) and for a single/double quasielectrons (upper/lower right panel). The dotted lines represent the expected values $\pm \frac{1}{3}$ and $\pm \frac{2}{3}$.

We first display the obtained density profiles in Fig. S2. In Fig. S3 we display the excess charges $Q(r)$ (as defined above), as a function of the radial distance (in units of the magnetic length $\ell_B$). Finally, in Fig. S4 we display the spin $J(r)$ as defined in the main text.

In calculating the integrals to obtain $Q(r)$ and $J(r)$, we assumed that the quasiparticles have cylindrical symmetry, and integrated only along the cylinder. The exception is the double QE. In that case, we performed the full 2-dimensional integral up to radius $r = \ell_B/2$. For $r > \ell_B/2$, we again assumed cylindrical symmetry for the double QE. The reason for doing this, is that in the case of the double QE, one needs a higher cutoff value $p_{\text{max}}$ to obtain a cylindrically symmetric double QE.

The results we obtain using the MPS formulation of the quasihole and quasielectron states are fully consistent with the results obtained using Monte Carlo for the disk geometry. We should note, however, that the double quasielectron we simulate using the MPS formulation, has different short distance properties in comparison to the double Jain quasielectron studied in the main text. It has been noted before in the literature, see for instance [30, 39], that there are different ways to create double quasielectron excitations. The double quasielectron studied in the MPS formulation, has a charge profile that is more concentrated, leading to a higher value of the spin, in comparison to the double Jain quasielectron studied in the main text. Because the difference between the two spins is an (even) integer, the results are nevertheless fully compatible with one another, the spin-statistics relation is satisfied in both cases.

For the references cited in the Supplementary Material, we refer the reader to the bibliography at the end of the main text.
FIG. S4. The spin $J(r)$ (in units of $\hbar$) for a single/double quasihole (upper/lower left panel) and for a single/double quasielectron (upper/lower right panel). The dotted lines represent the expected values, namely $\frac{1}{3}$ for both the single and double quasihole, $-\frac{2}{3}$ for the single quasielectron and $\frac{1}{3}$ for the double quasielectron.