RAYNAUD-MUKAI CONSTRUCTION AND CALABI-YAU
THREEFOLDS IN POSITIVE CHARACTERISTIC

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Abstract. In this article, we study the possibility of producing a Calabi-Yau
threefold in positive characteristic which is a counter-example to Kodaira vanish-
ing. The only known method to construct the counter-example is so called in-
ductive method such as the Raynaud-Mukai construction or Russel construction.
We consider Mukai’s method and its modification. Finally, as an application of
Shepherd-Barron vanishing theorem of Fano threefolds, we compute $H^1(X, H^{-1})$
for any ample line bundle $H$ on a Calabi-Yau threefold $X$ on which Kodaira van-
ishing fails.

1. Introduction

Although every K3 surface in positive characteristic can be lifted to characteristic
0 [2], there are some non-liftable Calabi-Yau threefolds, namely a smooth threefold
$X$ with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. If a Calabi-Yau
polarized threefold $(X, L)$ over the field $k$ of char($k$) = $p$ ≥ 3 is a counter-example
to Kodaira vanishing, i.e., $H^i(X, L^{-1}) \neq 0$ for $i = 1$ or $i = 2$, $X$ is non-liftable to the
second Witt vector ring $W_2(K)$ (and the Witt vector ring $W(k)$) by the cerebrated
Raynaud-Deligne-Illusies version of Kodaira vanishing theorem [3]. But this does
not necessarily imply that $X$ cannot be liftable to characteristic 0. Moreover, a
non-liftable variety is not necessarily a counter-example to Kodaira vanishing and
as far as the author is aware, it is not known whether Kodaira vanishing holds for
the non-liftable Calabi-Yau threefolds [6, 7, 8, 16, 4, 1] that have been found so far.
We do not even know whether Kodaira vanishing holds for all Calabi-Yau threefolds.
Thus Kodaira type vanishing for Calabi-Yau threefolds is an interesting problem,
which is independent from but seems to be closely related to the lifting problem.

A counter-example to Kodaira vanishing has been given by M. Raynaud, which
is a surface over a curve [14]. This example was extended to arbitrary dimension
by S. Mukai [11, 12], which we will call the Raynaud-Mukai construction or, simply,
Mukai construction.

The idea is, so to say, an inductive construction. Namely, we start from a polarized
smooth curve $(C, D)$. The ample divisor $D$ satisfies a special condition, which is a
sufficient condition for the non-vanishing $H^1(X, \mathcal{O}_X(-D)) \neq 0$, and called a (pre-
)Tango structure. Then we give an algorithm to construct from a variety $X$ with
a (pre-)Tango structure $D$ a new variety $\tilde{X}$ with a higher dimensional (pre-)Tango
structure $\tilde{D}$ such that $\dim \tilde{X} = \dim X + 1$, using cyclic cover technique. There is
another way of constructing counter-examples using quotient of $p$-closed differential
forms [15, 19]. But this is also an inductive construction and the obtained varieties
are the same as the Raynaud-Mukai construction [19]. As far as the author is aware, non-inductive construction of higher dimensional counter-examples is not yet found.

In this paper, we consider the problem of whether we can construct a Calabi-Yau threefold with Kodaira non-vanishing by Mukai construction or by its modification. Section 2 presents the Raynaud-Mukai construction. For \( p \geq 5 \), Raynaud-Mukai varieties are of general type so that the only possibility resides in the cases of \( p = 2, 3 \). Then in section 3, we will see that Mukai construction does not produce any K3 surfaces or Calabi-Yau threefolds (Corollary 9 and Corollary 10). Then we consider possible modifications of the Raynaud-Mukai construction: we keep the inductive construction but give up obtaining a (pre-)Tango structure. We show that if there exists a surface \( X \) of general type together with a (pre-)Tango structure \( \tilde{D} \) satisfying some property (this is not obtained by Mukai construction), we can construct a Calabi-Yau threefold \( \tilde{X} \) with a (pre-)Tango structure \( \tilde{D} \) (Corollary 11) and describe the cohomology \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \) in certain situations (Proposition 13). Unfortunately, we could not prove or disprove existence of such a polarized surface \((X,D)\).

Finally, in section 3 we show that if Kodaira non-vanishing \( H^1(X, L^{-1}) \neq 0 \) holds for a polarized Calabi-Yau threefold \((X,L)\) over the field \( k \) of char \( k = p \geq 5 \) satisfying the condition that \( L^\ell \) is a Tango-structure for some \( \ell \geq 1 \), we compute the cohomology \( H^1(X, H^{-1}) \) for any ample line bundle \( H \) of \( X \) (Theorem 18, Corollary 19).

2. The Raynaud-Mukai construction

In this section, we present the Raynaud-Mukai construction. Although [12] is available now, we prefer to use the version described in [11], which is slightly different from the 2005 version. As 1979 version is only available in Japanese, we present some details for the readers convenience.

The idea is to construct from a counter-example to Kodaira vanishing, i.e., a polarized variety \((X,L)\) with \( H^1(X, L^{-1}) \neq 0 \) a new counter-example \((\tilde{X}, \tilde{L})\) with \( \dim \tilde{X} = \dim X + 1 \). This inductive construction starts from a polarized curve \((X,L)\) called a Tango-Raynaud curve.

2.1. pre-Tango structure and Kodaira non-vanishing.

**Definition 1** (pre-Tango structure). Let \( X \) be a smooth projective variety. Then an ample divisor \( D \), or an ample line bundle \( L = \mathcal{O}_X(D) \), is called a pre-Tango structure if there exists an element \( \eta \in k(X) \backslash k(X)^p \), where \( k(X) \) denotes the function field of \( X \), such that the Kähler differential is \( d\eta \in \Omega_X(-pD) \), which will be simply denoted as \( (d\eta) \geq pD \). In this paper, the element \( \eta \) will be called a justification of the pre-Tango structure.

Existence of a pre-Tango structure implies Kodaira non-vanishing. In fact, consider the absolute Frobenius morphism

\[
F : \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X(-pD)
\]

such that \( F(a) = a^p \) for \( a \in \mathcal{O}_X \) and set \( B_X(-D) := \text{Coker } F \). Then we have

\[
0 \longrightarrow H^0(X, B_X(-D)) \longrightarrow H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F} H^1(X, \mathcal{O}_X(-pD))\]
and then we can show

Proposition 2. \( H^0(X, B_X(-D)) = \{ df \in k(X) \mid (df) \geq pD \} \).

Thus, if there exists a pre-Tango structure \( D \) and \( \dim X \geq 2 \), then we have Kodaira non-vanishing: \( H^1(X, \mathcal{O}_X(-D)) \neq 0 \).

Notice that the inclusion \( H^0(X, B_X(-D)) \subset H^1(X, \mathcal{O}_X(-D)) \) may be strict, so that there is a possibility that a non pre-Tango structure \( L \) causes a Kodaira non-vanishing. However, since the iterated Frobenius map

\[
F^e : H^1(X, L^{-1}) \rightarrow H^1(X, L^{-p^e})
\]

is trivial for \( e \gg 0 \), \( L^n \) is a pre-Tango structure for sufficiently large \( n \in \mathbb{N} \).

Pre-Tango structure for curves are characterized by the Tango-invariant \([21, 20]\).

Let \( C \) be a smooth projective curve of genus \( g \geq 2 \). Then the Tango-invariant is defined as

\[
n(C) = \max \left\{ \deg \left[ \frac{df}{p} \right] : f \in k(X)/k(X)^p \right\}
\]

where \([\cdot \cdot \cdot]\) denotes the round up. We easily know that

\[
0 \leq n(C) \leq \frac{2(g-1)}{p}.
\]

Then, \( C \) has a pre-Tango structure \( D \) if \( n(C) > 0 \). We just set \( D = \left[ \frac{(df)}{p} \right] \) and then \( D \) is ample on \( C \) such that \( (df) \geq pD \).

In the following, we will call the pair \((X, L)\) in Definition \(1\) a pre-Tango polarization. The Raynaud-Mukai construction is an algorithm to make a new pre-Tango polarization from a pre-Tango polarization whose dimension is lower by one.

2.2. purely inseparable cover. From a pre-Tango polarized variety \((X, L)\) we can construct a reduced and irreducible purely inseparable cover \( \tau : G \rightarrow X \) of degree \( p \). Conversely, existence of such a cover implies existence of a pre-Tango polarization.

2.2.1. Construction and characterization. Given a pre-Tango polarized variety \((X, L = \mathcal{O}_X(D))\), choose an element \((0 \neq) \eta \in H^0(X, B_X(-D))(= \ker F)\). Then we have a corresponding non-split short exact sequence

\[
(1) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow L \rightarrow 0
\]

where \( E \) is a rank 2 vector bundle on \( X \). Taking the Frobenius pull-back, we obtain an exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow E^{(p)} \rightarrow L^{(p)} \rightarrow 0.
\]

where, for example, \( E^{(p)} = E \otimes_{\mathcal{O}_X} \mathcal{O}_X^{p} \) with \( F : \mathcal{O}_X \rightarrow \mathcal{O}_X \) the Frobenius morphism. Notice that the new sequence corresponds to \( F(\eta) = 0 \) so that it splits and by using the split maps, we obtain the sequence with the reverse arrows

\[
0 \leftarrow \mathcal{O}_X \leftarrow E^{(p)} \leftarrow L^{(p)} \leftarrow 0.
\]

Tensoring by \( L^{(p)-1} \) over \( \mathcal{O}_X \), we finally obtain the sequence

\[
(2) \quad 0 \rightarrow \mathcal{O}_X \rightarrow E^{(p)} \otimes L^{(p)-1} \rightarrow L^{(p)-1} \rightarrow 0.
\]
Now we consider the $\mathbb{P}^1$-fibration
\[ \pi : P = \mathbb{P}(E) \rightarrow X \]

together with the canonical section $F \subset P$, which is defined by the image of $1 \in \mathcal{O}_X$ in $E$, and

\[ \pi^{(p)} : P^{(p)} = \mathbb{P}(E^{(p)} \otimes L^{(p)^{-1}}) \cong \mathbb{P}(E^{(p)}) \rightarrow X \]

together with the canonical section $F^{(p)} \subset P^{(p)}$ which is the image of $1 \in \mathcal{O}_X$ in $E^{(p)} \otimes L^{(p)^{-1}}$, corresponding to (1) and (2). Moreover, we consider the relative Frobenius morphism $\psi : P \rightarrow P^{(p)}$ over $X$. On an open set $U \subset X$ such that $E|U \cong \mathcal{O}^r_U$ with $r = \text{rank } E$, $\psi$ is induced by the local morphism $E^{(p)}|_U \cong \mathcal{O}^r_U \otimes \mathcal{O}_U$

$\mathcal{O}_U \rightarrow E|_U$ sending $\sum_i a_i \otimes f = \sum_i 1 \otimes a_i^p f \in \mathcal{O}^r_U$ to $\sum_i a_i^p f \in \mathcal{O}^r_U$. Thus, on a fiber $\pi^{-1}(x) \cong \mathbb{P}^1$, $\psi|_{\pi^{-1}(x)} : \pi^{-1}(x) \rightarrow \pi^{(p)}(x)$ is the Frobenius pull-back, i.e., $\psi(a,b) = (a^p,b^p)$ for every projective coordinate $(a,b) \in \pi^{-1}(x)$. Now consider the scheme theoretic inverse image of $F^{(p)}$ inside $P$:

\[ G := \psi^{-1}(F^{(p)}) \subset P \]

Then we can show

**Proposition 3.**

1. $G \cap F = \emptyset$,
2. $\mathcal{O}_P(G) \cong \mathcal{O}_P(p) \otimes \pi^* L^{-p} \cong \mathcal{O}_P(pF - p\pi^* D)$, and
3. $\rho = \pi|_{G} : G \rightarrow X$ is a purely inseparable cover of degree $p$.

We can show that existence of such a $G$ characterizes pre-Tango structure. To summarize, we have

**Theorem 4** (See Proposition 1.1 in [12]). Let $X$ be a smooth projective variety of characteristic $p > 0$ and $L$ be an ample line bundle. Then the following are equivalent:

1. $L$ is a pre-Tango structure.
2. There exists a $\mathbb{P}^1$-bundle $\pi : P \rightarrow X$ and a reduced irreducible effective divisor $G \subset P$ such that
   (a) $\rho : G \rightarrow X$ is a purely inseparable cover of degree $p$
   (b) $P = \mathbb{P}(E)$ where $E$ is a rank 2 vector bundle on $X$ such that

\[ 0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow L \rightarrow 0 \]

2.2.2. smoothness. For smoothness of the purely inseparable cover $G$, we have

**Theorem 5** (S. Mukai [12]). Let $(X,D)$ be a pre-Tango polarized variety over the field of characteristic $p > 0$ and $G$ is the purely inseparable cover constructed from a justification $(0 \neq) \eta \in k(X) \setminus k(X)^p$. Then $G$ is smooth if and only if $(d\eta) = pD$. This means that for the multiplication by $d\eta$

\[ \mathcal{O}_X(pD) \xrightarrow{d\eta} \Omega_X \rightarrow \text{Coker}(d\eta), \]

$\text{Coker} d\eta$ is locally free at every $x \in X$.

**Proof.** For a proof in the case of $\text{dim } X = 2$, see Theorem 3 [18]. □
Definition 6 (Tango structure). Let $X$ be a smooth projective variety with a pre-Tango structure $L = \mathcal{O}_X(D)$. Then, $D$, or $L$, is called a Tango structure if and only if a justification $\eta \in k(X) \setminus k(X)^p$ satisfies $(d\eta) = pD$. In this case, the pre-Tango polarization $(X, L)$ or $(X, D)$ will be called a Tango polarization.

A smooth projective curve $X$ of genus $g \geq 2$ with a Tango structure $D$ is called a Tango-Raynaud curve. For examples of Tango-Raynaud curves, see for example [14, 11, 12].

2.3. cyclic cover. Let $(X, D)$ be a pre-Tango polarization and $D$ is divided by $k \in \mathbb{N}$ with $(p, k) = 1$ and we have $D = kD'$. If $X$ is a curve, we can divide $D$ by any natural number $k$ dividing $\text{deg} D$ using the theory of Jacobian variety (cf. page 62 of [13]). But the condition $(p, k) = 1$ is necessary for the covering to be cyclic.

Now we construct a $k$th cyclic cover of the $\mathbb{P}^1$-fibration $\pi : P \rightarrow X$ ramified over $F + G$, which means that $\pi$ is ramified at the reduced preimage of $F + G$. There are at least two well-known constructions.

The first one is rather explicit and is suitable for computing cohomologies (cf. [18]). We first choose $m \in \mathbb{N}$ such that $k | (p + m)$ and set $\mathcal{M} = \mathcal{O}_P(-\frac{m}{p}F) \otimes \pi^*\mathcal{O}_X(pD')$. Then we have $\mathcal{M}^\otimes k = \mathcal{O}_P(-mF) \otimes \mathcal{O}_P(-pF) \otimes \pi^*\mathcal{O}_X(pD) = \mathcal{O}_P(-mF - G)$ by Proposition 3. Then we can introduce $\bigoplus_{i=0}^{k-1} \mathcal{M}^\otimes i$ the structure of a graded $\mathcal{O}_P$-algebra by defining multiplication $\mathcal{M}^\otimes i \times \mathcal{M}^\otimes j \rightarrow \mathcal{M}^\otimes i+j$ s. t. $(a,b) \mapsto a \otimes b$ if $i + j < k$ and $\mathcal{M}^\otimes i \times \mathcal{M}^\otimes j \rightarrow \mathcal{M}^\otimes i+j \rightarrow \mathcal{M}^\otimes i+j-k$ s. t. $(a,b) \mapsto a \otimes b \rightarrow a \otimes b \otimes \xi$ if $i + j \geq k$ where we choose a non-trivial element $\xi \in \mathcal{O}_P(mF + G)$ such that $mF + G$ is the zero locus of $\xi$. Now we consider the affine morphism $X' := \text{Spec} \bigoplus_{i=0}^{k-1} \mathcal{M}^\otimes i \rightarrow P$ and this is the cyclic cover ramified over $mF + G$. Since $X$ is smooth, $F \cong X$ is also smooth. Moreover if $D$ is a Tango structure and $G$ is smooth by Theorem 5, then $X'$ is smooth if and only if $m = 1$; if $m > 1$ then $X'$ is singular along $F$, which may cause non-normality of $X'$. Normalization of $X'$, if necessary, is carried out by Esnault-Viehweg’s method (see § 3 of [5]). $\bar{X} = \text{Spec} \bigoplus_{i=0}^{k-1} \mathcal{M}^\otimes i \otimes \mathcal{O}_P \left( \frac{i(mF+G)}{k} \right)$ and this is smooth if $D$ is Tango. We note that this normalization procedure highly depends on the condition $(p, k) = 1$ since we use the $k$th root of unity. Then we set the natural morphism $\varphi : \bar{X} \rightarrow X' \rightarrow P$.

The second construction uses normalization. Since we have linear equivalence $G \sim pF - p\pi^*(D)$ there exist a function $R \in \mathcal{O}(P)$ such that $(R) = G - (pF - \pi^*(D)) = G - (pF - pk\pi^*(D'))$. Then let $\tilde{X}$ be the normalization of $P$ in the finite extension $k(P)(R^{1/k})$ of $k(P)$ and $\varphi : \tilde{X} \rightarrow P$ be the normalization morphism. Then we set $f = \pi \circ \varphi$. Now if we work locally we know that there exist divisors $G$ and $F$ on $\tilde{X}$ such that $\varphi^*F = k\tilde{F}$ and $\varphi^*G = k\tilde{G}$. Moreover, we have $\tilde{G} \sim p\tilde{F} - pf^*(D')$ on $\tilde{X}$. We note that the condition $(p, k) = 1$ is necessary to assure the existence of $\tilde{F}$, division of $F$ by $k$. Otherwise, if $k = p^\ell r$ with $\ell \geq 1$ and $(p, r) = 1$ we have $F \subset \tilde{X}$ such that $\varphi^*F = k'\tilde{F}$ with $k' = p^{\ell-1}r = k/p$. $\tilde{X}$ is smooth if $D$ is Tango.
Now we set \( f := \pi \circ \varphi : \tilde{X} \longrightarrow X \), which is actually a fibration of rational curves with moving singularities, i.e., rational curves with cusp singularity of type \( x^p = y^t \) at \( \tilde{G} \).

2.4. **polarization.** The cyclic cover \( \tilde{X} \) of the \( \mathbb{P}^1 \)-fibration is a counter-example to Kodaira vanishing because the polarization \( \tilde{D} = (k - 1)\tilde{F} + f^* (D') \) causes non-vanishing \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{D})) \neq 0 \). In fact, \( \tilde{D} \) is ample (see Sublemma 1.6 [12]) and we have

**Proposition 7.** Suppose \( \tilde{X} \) is as above, then \( \tilde{D} \) is a Tango structure of \( \tilde{X} \) and in particular we have Kodaira non-vanishing \( H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{D})) \neq 0 \).

This result is stated in [11] without proof and in the case of \( k \equiv 1 \mod p \) a proof using Maruyama’s elementary transformation [10] is given in [12]. We give here a proof of general case.

**Proof.** Let \( \tilde{\eta} = R^{1/k} \in k(\tilde{X}) \). Since \( (\tilde{\eta}) = \tilde{G} - p\tilde{F} + pf^*(D') \), \( \tilde{\eta} \) is locally described as \( \tilde{\eta} = g(\delta \phi^{-1})^p \) where \( g, \phi \) and \( \delta \) are local equations defining \( G, \tilde{F} \) and \( f^*(D') \). Then its Kähler differential is

\[
(3) \quad d\tilde{\eta} = (\delta \phi^{-1})^pdg = (\phi^{k-1}\delta)^p\phi^{-pk}dg.
\]

Now we consider \( dg \). As a Cartier divisor we describe \( D = \{(U_i, g_i)\}_i \) for an open cover \( X = \bigcup U_i \) and \( g_i \in k(X) \). Since \( D \) is a pre-Tango structure, there exists a justification \( \eta \in k(X) \) such that \( (d\eta) \geq pD \), which locally means that we have \( \eta|_{U_i} = g_i^p c_i \) for some \( c_i \in \mathcal{O}_{U_i} \) so that we have \( (d\eta)|_{U_i} = g_i^p dc_i \). Then, as in Proposition 1 [13], \( G \subset P \) is locally described as

\[
\text{Proj} \mathcal{O}_{U_i}[x, y]/(c_i x^p + y^p)
\]

where \( x \) is the (local) coordinate corresponding to the canonical section \( F \) of \( \pi : P \longrightarrow X \). Hence the local defining equation of \( G \subset P \) is \( c_i x^p + y^p \), and since \( \phi^* F = k\tilde{F} \) and \( \phi^* G = kG \), the defining equation of \( G \) is \( g = c_i Z^{kp} + W^{kp} \), where \( Z \) is the local coordinate of \( \tilde{X} \) corresponding to \( \tilde{F} \), namely \( Z = \varepsilon \phi \) with some local unit \( \varepsilon \). Thus we have

\[
(4) \quad dg = \varepsilon^{pk} \phi^{pk} dc_i.
\]

Thus by (3) and (4) we obtain

\[
d\tilde{\eta} = (\delta \phi^{-1})^pdg = \varepsilon^{pk}(\phi^{k-1}\delta)^p dc_i \text{ so that}
\]

\[
(d\tilde{\eta}) \geq p((k - 1)\tilde{F} + f^* D') = p\tilde{D}
\]

where the equality holds if \( (d\eta) = pD, \) i.e., if \( D \) is a Tango structure. \( \square \)

3. **Calabi-Yau threefolds and the Raynaud-Mukai construction**

3.1. **Raynaud-Mukai varieties cannot be Calabi-Yau.** The aim of this section is to show that Mukai construction does not produce K3 surfaces or Calabi-Yau threefolds. Notice that Raynaud-Mukai variety is always of general type for \( p \geq 5 \) (cf. Prop. 7 [11] or Prop. 2.6 [12]) so that the only possibility is the case \( p = 2, 3 \).
Now let \((X, D), D = kD' (k \in \mathbb{N})\), \(\pi : P \rightarrow X, \ F, G \subset P, (\tilde{X}, \tilde{D}), \varphi : \tilde{X} \rightarrow P\) and \(f : \tilde{X} \rightarrow X\) be as in the previous section. The canonical divisor of \(X\) will be simply denoted by \(K\). Now we have

**Proposition 8** (cf. Prop. 7 [11]). Let \(\tilde{K}\) be the canonical divisor of \(\tilde{X}\). Then we have

\[
\tilde{K} \sim (pk - p - k - 1)\tilde{F} + f^*(K - (pk - p - k)D')
\]

**Proof.** Since the finite morphism \(\varphi : \tilde{X} \rightarrow P\) is ramified at \(\tilde{F} = (\varphi^*(F))_{\text{red}}\) and \(\tilde{G} = (\varphi^*(G))_{\text{red}}\) with the same ramification index \(k\) and \(F + G \sim (p+1)F - pk\pi^*D'\), we compute

\[
\tilde{K} \sim \varphi^*K_p + (k-1)(\tilde{F} + \tilde{G}) \quad \text{by ramification formula}
\]

\[
\sim \varphi^*K_p + (k-1)\frac{1}{k}\varphi^*(F + G)
\]

\[
\sim \varphi^*K_{p} + (k-1)((p+1)\tilde{F} - pf^*D').
\]

Moreover, since \(E\) is the rank 2 vector bundle satisfying

\[
0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X(kD') \rightarrow 0,
\]

we have \(K_p \sim -2F + \pi^*(K + kD')\). Then we obtain the required formula. \(\square\)

We notice that since \(\text{Pic } P \cong \mathbb{Z} \cdot [F] \oplus \pi^*\text{Pic } X\) and \(\varphi\) is finite, we have \(\text{Pic } \tilde{X} \cong \mathbb{Z} \cdot [\tilde{F}] \oplus f^*\text{Pic } X\). This fact will be used implicitly in the following discussion.

**Corollary 9.** A Raynaud-Mukai surface can never be a K3 surface.

**Proof.** Assuming \(\text{dim } \tilde{X} = 2\), we have only to show that we never have \(\tilde{K} \sim 0\). Assume that we have \(\tilde{K} \sim (pk - p - k - 1)\tilde{F} + \varphi^*(K - (pk - p - k)D') \sim 0\), from which have two relations \(pk - p - k = 0\) and \(K - (pk - p - k)D' = 0\). By the first relation, we have \(k = \frac{p+1}{p-1} \in \mathbb{N}\), so that we must have \(p = 2\) and \(k = 3\) or \(p = 3\) and \(k = 2\). This implies \(K = D'\) by the second relation. However, since \((X, D)\) is a (pre-)Tango polarized curve, we have \((d\eta) \geq pD\) for some justification \(\eta \in k(X)\), namely \(D' = K \geq pD = pkD'\), which is impossible unless \(pk = 1\). \(\square\)

By a similar discussion to the proof of Corollary 9, we can also show

**Corollary 10.** A Raynaud-Mukai threefold can never be Calabi-Yau.

**Proof.** Let \(\tilde{X}\) be a Mukai threefold obtained from a Mukai surface \(X\) with a (pre-)Tango structure \(D = kD'\) as a \(k\)th cyclic cover of the \(\mathbb{P}^1\)-fibration \(P\) and assume that \(\tilde{K} \sim 0\). Then as in the proof of Corollary 9 we have \((p, k) = (2, 3)\) or \((3, 2)\) and

\[
K \sim D'.
\]

Now we will consider the situation whose dimensions are all lower by one. Namely, let the surface \(X\) be constructed from a (pre-)Tango polarized curve \((X_1, D_1)\) with \(D_1 = k_1D'_1\). We have the \(k_1\)th cyclic cover \(\varphi_1 : X \rightarrow P_1\) of the \(\mathbb{P}^1\)-fibration \(\pi_1 : P_1 \rightarrow X_1\) ramified over \(F_1 + G_1\) and \(\tilde{F}_1 = (\varphi_1^*(F_1))_{\text{red}}\) and \(\tilde{G}_1 = (\varphi_1^*(G_1))_{\text{red}}\)
have the same ramified index \( k_1 \). We set \( f_1 = \pi_1 \circ \varphi_1 \). Then by Proposition 8, we have

\[ K \sim (pk_1 - p - k_1 - 1)\tilde{F}_1 + f_1^*(K_1 - (pk_1 - p - k_1)D_1') \]

Since we have \((kD' = )D = (k_1 - 1)\tilde{F}_1 + f_1^*(D_1')\) by definition, the condition (5) entails

\[ \left(pk_1 - p - k_1 - 1 - \frac{k_1 - 1}{k}\right)\tilde{F}_1 + f_1^* \left(K_1 - (pk_1 - p - k_1 + \frac{1}{k})D_1'\right) \sim 0. \]

Then the coefficient of \( \tilde{F}_1 \) must be 0 so that we have

\[ k_1 = \frac{k(p + 1) - 1}{k(p - 1) - 1} = \begin{cases} 4 & \text{if } p = 2 \\
\frac{7}{3} & \text{if } p = 3 \end{cases} \]

But since we must have \( k_1 \in \mathbb{N} \) and \((k_1, p) = 1\), these values of \( k_1 \) are not allowed. \( \square \)

3.2. a modification of the Raynaud-Mukai construction. The Raynaud-Mukai construction is an algorithm to construct from a given (pre-)Tango polarization \((X, D)\) with \( D = kD' \) a new (pre-)Tango polarization \((\tilde{X}, \tilde{D})\) with \( \dim X = \dim \tilde{X} - 1 \) by taking a \( k \)th cyclic cover. We apply this procedure inductively starting from a (pre-)Tango polarized curve. We have seen in the previous subsection that the essential reason that the Raynaud-Mukai construction does not produce Calabi-Yau threefolds is that we cannot find the degree \( k \) cyclic covers with \((p, k) = 1\) in all inductive steps.

Now we will consider some modification of the Raynaud-Mukai construction. There are following two possibilities.

(I) Let \((X, D)\) be a (pre-)Tango polarized surface obtained by a method other than Mukai construction. Then apply the Raynaud-Mukai construction to obtain a (pre-)Tango polarized threefold \((\tilde{X}, \tilde{D})\).

(II) Let \((X, D)\) be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then we construct a Calabi-Yau threefold in a similar way to Mukai construction. Namely, we do not assume the condition \((p, k) = 1\) for the degree \( k \) of “cyclic cover “.

The Calabi-Yau threefolds obtained by (I) are counter-examples to Kodaira vanishing. The surface \( X \) required in (I) is precisely as follows:

**Corollary 11.** Let \((X, D)\) a (pre-)Tango polarized surface with \( D = kD' \) for some \( k \in \mathbb{N} \). Then the Raynaud-Mukai construction gives a polarized Calabi-Yau threefold \((\tilde{X}, \tilde{D})\) by a \( k \)th cyclic cover if and only if

(i) \((p, k) = (2, 3)\) or \((3, 2)\), and

(ii) \( D = kD' \) for some ample \( D' \) and \( K_X \sim D' \).

In particular, \( X \) is a surface of general type.

**Proof.** By the same discussion as in the proof of Corollary 9 and 10. \( \square \)

Unfortunately we do not know how to construct a polarized surface \((X, D)\) as in Corollary 11. But Theorem 12(i) below seems to indicate a possibility.
Remark 14. Using another spectral sequence and 5-term exact sequence we can show the inclusion $H^0(C, R^1 g_* (f_* O_{\tilde{X}})) \subset H^0(C, R^1 h_* O_{\tilde{X}})$ but the equality does not hold in general.

Theorem 12 (S. Mukai [11]). Let $X$ be a (smooth) surface over the field $k$ of char $k = p > 0$. Assume that Kodaira vanishing fails on $X$. Then we have

(i) $X$ is of general type or quasi-elliptic surface with Kodaira dimension 1 (if $p = 2, 3$).

(ii) There exists a surface $X'$ birationally equivalent with $X$ such that there is a morphism $g : X' \to C$ to a curve $C$ whose fibers are all connected and singular.

It is proved that, in the case of surfaces, Kodaira (non-)vanishing is preserved in birational equivalence (see Corollary 8 [20]). Thus by Theorem 12 (ii) it seems to be reasonable to consider a fibration $p : X \to C$ to a curve.

For a Calabi-Yau threefold, we often assume simple connectedness which implies $H^1(\tilde{X}, O_{\tilde{X}}) = 0$ for our example. For this property, we have the following.

Proposition 13. Assume that the surface $X$ in Corollary 11 has a fibration over a curve $C$: $g : X \to C$ and set $h : \tilde{X} \xrightarrow{f} X \xrightarrow{g} C$. Then we have $H^1(\tilde{X}, O_{\tilde{X}}) \cong H^1(C, g_* O_X) \oplus H^0(C, R^1 h_* O_{\tilde{X}})$.

Proof. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(C, R^q h_* O_{\tilde{X}}) \Rightarrow H^{p+q}(\tilde{X}, O_{\tilde{X}}).$$

Then by the 5-term exact sequence we have

$$0 \to H^1(C, h_* O_{\tilde{X}}) \to H^1(\tilde{X}, O_{\tilde{X}}) \to H^0(C, R^1 h_* O_{\tilde{X}}) \to H^2(C, h_* O_{\tilde{X}})$$

where the last term $H^2(C, h_* O_{\tilde{X}})$ vanishes since dim $C < 2$. Thus we have

$$H^1(\tilde{X}, O_{\tilde{X}}) \cong H^1(C, h_* O_{\tilde{X}}) \oplus H^0(C, R^1 h_* O_{\tilde{X}}).$$

On the other hand, we have $(p, k) = (2, 3)$ or $(3, 2)$ by Corollary 11 and the explicit construction of the cyclic cover gives

$$\tilde{X} = \left\{ \begin{array}{ll}
\text{Spec } \bigoplus_{i=0}^2 \mathcal{O}_P(-i) \otimes \pi^*(2iD') & \text{if } (p, k) = (2, 3) \\
\text{Spec } \bigoplus_{i=0}^2 \mathcal{O}_P(-2i) \otimes \pi^*(3iD') & \text{if } (p, k) = (3, 2)
\end{array} \right.$$%

where $\pi : P \to X$ is the $\mathbb{P}^1$-fibering. Thus we compute

$$h_* O_{\tilde{X}} = (g \circ \pi \circ \varphi)_* O_{\tilde{X}} = (g \circ \pi)_* (\varphi_* O_{\tilde{X}})$$

$$= \left\{ \begin{array}{ll}
(g \circ \pi)_* \left( \bigoplus_{i=0}^2 \mathcal{O}_P(-i) \otimes \pi^*(2iD') \right) & \text{if } (p, k) = (2, 3) \\
(g \circ \pi)_* \left( \bigoplus_{i=0}^2 \mathcal{O}_P(-2i) \otimes \pi^*(3iD') \right) & \text{if } (p, k) = (3, 2)
\end{array} \right.$$%

$$= \left\{ \begin{array}{ll}
g_*(\pi_* \mathcal{O}_P) \oplus g_*(\pi_* \mathcal{O}_P(-1) \otimes \mathcal{O}_X(2D')) & \text{if } (p, k) = (2, 3) \\
g_*(\pi_* \mathcal{O}_P(-2) \otimes \mathcal{O}_X(4D')) & \text{if } (p, k) = (3, 2)
\end{array} \right.$$%

Now since $\pi_* \mathcal{O}_P = \mathcal{O}_X$ and $\pi_* \mathcal{O}_P(-i) = 0$ for $i > 0$ we obtain $h_* O_{\tilde{X}} = g_* O_X$. \quad \Box

Remark 14. Using another spectral sequence and 5-term exact sequence we can show the inclusion $H^0(C, R^1 g_* (f_* O_{\tilde{X}})) \subset H^0(C, R^1 h_* O_{\tilde{X}})$ but the equality does not hold in general.
Next we consider the construction (II), whose algorithm is as follows: Given a (pre-)Tango curve, we make a (pre-)Tango polarized surface \((X, D)\) and a \(\mathbb{P}^1\)-bundle \(\pi : P \to X\) with the canonical section \(F \subset P\) together with a purely inseparable cover \(\pi|_G : G \to X\) of degree \(p\) corresponding to \(D\). Then choose \(k = p^fr\) with \((p, r) = 1\) and \(\ell \geq 1\) and let \(\varphi : \tilde{X} \to P\) be the normalization of \(P\) in \(k(P)(R^{1/k})\) where \(R \in K(P)\) is such that \((R) = G - (pF - p\pi^*(D))\).

**Lemma 15.** Let \((X, D)\), \(D = kD'\) with \((2 \leq k) \in \mathbb{N}\), be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then the construction (II) gives a Calabi-Yau threefold if and only if \((p, k, K) = (2, 4, 2D')\) or \((3, 3, D)\).

**Proof.** Let \((X, D)\) be a (pre-)Tango polarized surface by the Raynaud-Mukai construction. Then we obtain a \(\mathbb{P}^1\)-bundle \(\pi : P \to X\) together with the canonical section \(F\) and the purely inseparable cover \(G \to X\) of degree \(p\) (see Theorem 4).

In Mukai construction, we take a \(k\)th cyclic cover of \(P\) where \((k, p) = 1\). This does not work as we have seen in Corollary 10. Thus we assume \((k, p) \neq 1\) and set \(k = p^fr\) with \((p, r) = 1\), \(\ell \geq 1\). Since we have \(D = kD'\) and \(G \sim pF - p\pi^*(D)\), there exists \(R \in k(P)\) such that \((R) = G - pF + p\pi^*(kD')\). Now let \(\varphi : \tilde{X} \to P\) be the normalization of \(P\) in \(k(P)(R^{1/k})\). Then, if we set \(\tilde{F} = (\varphi^*(F))_{\text{red}}\) and \(\tilde{G} = (\varphi^*(G))_{\text{red}}\), we have \(\varphi^*(G) = k\tilde{G}\) and \(\varphi^*(F) = (k/p)\tilde{F}\) and \(\tilde{G} \sim \tilde{F} - pf^*(D')\) where \(f = \pi \circ \varphi\). Notice that we do not have the coefficient \(p\) for \(\tilde{F}\) as in the case of \((p, k) = 1\). Now as in proof of Proposition 8 we compute

\[
\tilde{K} \sim \varphi^*K_P + (k - 1)\tilde{G} + \left(\frac{k}{p} - 1\right)\tilde{F}
\sim \varphi^*K_P + \left(k + \frac{k}{p} - 2\right)\tilde{F} - p(k - 1)f^*D'
\sim (p^fr - p^{f-1}r - 2)\tilde{F} + f^*(K + (p^fr - p(p^fr - 1))D').
\]

Then if \(\tilde{X}\) is a Calabi-Yau threefold, i.e., \(\tilde{K} \sim 0\), we must have \(p^fr - p^{f-1}r - 2 = 0\) and \(K + (p^fr - p(p^fr - 1))D' \sim 0\), from which we have \((\ell, r, p, k) = (1, 1, 3, 3)\) or \((2, 1, 2, 4)\) and

\[
K \sim \left\{ \begin{array}{ll} 2D' & \text{if } (p, k) = (2, 4) \\ 3D'(= D) & \text{if } (p, k) = (3, 3) \end{array} \right\}
\]

Now we can show

**Proposition 16.** Calabi-Yau threefolds cannot be obtained by the construction (II).

**Proof.** We assume that the (pre-)Tango polarized surface \((X, D)\) is a fibration \(f_1 : X \to X_1\) over a Tango polarized curve \((X_1, D_1)\) with \(D_1 = k_1D'_1\), which is a \(k_{1th}\) cyclic cover \(\varphi_1 : X \to P_1\) of a \(\mathbb{P}^1\)-fibration \(\pi_1 : P_1 \to X_1\) ramified over \(F_1 + G_1 \subset P_1\) and we set \(\tilde{F}_1 = (\varphi_1^*(F_1))_{\text{red}}\). In this situation, we have

\[
K \sim (pk_1 - p - k_1 - 1)\tilde{F}_1 + f_1^*(K_{X_1} - (pk_1 - p - k_1)D'_1)
\]
by Proposition 5. We have $D = (k_1 - 1)\tilde{F}_1 + f_1^*D'_1$ by definition. Now we first consider the case $(p, k) = (2, 4)$. By Lemma 15 we have

$$2D' = \frac{1}{2}D = \frac{1}{2}(k_1 - 1)\tilde{F}_1 + \frac{1}{2}f_1^*D'_1 \sim K = (k_1 - 3)\tilde{F}_1 + f_1^*(K_{X_1} - (k_1 - 2)D'_1)$$

or otherwise

$$\frac{5 - k_1}{2}\tilde{F}_1 + f_1^*(\frac{2k_1 - 3}{2}D'_1 - K_{X_1}) \sim 0,$$

which entails $k_1 = 5$ and $K_{X_1} = \frac{7}{2}D'_1$. But since $D_1$ is a (pre-)Tango structure we must have $\frac{7}{2}D'_1 = K_{X_1} \geq pD_1 = 2 \cdot 4D'_1 = 8D'_1$, a contradiction.

The case of $(p, k) = (3, 3)$ is similar. Since we must have $D \sim K$, we have $k_1 = 3$ and $K_{X_1} = 4D'$. But, since $(X_1, D_1)$ is a Tango-Raynaud curve, we must have $4D'_1 = K_{X_1} \geq pD_1 = 3k_1D'_1 = 9D'_1$, a contradiction. □

4. Cohomology of Calabi-Yau threefold with Tango-structure

In this section, we compute the cohomology $H^1(X, H^{-1})$ for arbitrary ample $H$ under the assumption that $X$ is a Calabi-Yau threefold on which Kodaira vanishing fails.

**Theorem 17** (N. Shepherd-Barron [17]). Let $X$ be a normal locally complete intersection Fano threefold over the field $k$ of char $k = p \geq 5$ and $L$ be an ample line bundle on $X$. Then we have $H^1(X, L^{-1}) = 0$.

Recall that, for a polarized smooth variety $(X, L)$, Kodaira non-vanishing $H^1(X, L^{-1}) \neq 0$ does not necessarily imply $L$ is a (pre-)Tango structure. But by Enriques-Severi-Zariski’s theorem, there exists $\ell > 0$ such that we have $H^1(X, L^{-\ell p+1}) = 0$ but $H^1(X, L^{-\ell p}) \neq 0$. Then such $L^\ell$ is at least a pre-Tango structure. Now based on these observations, we obtain

**Theorem 18.** Let $(X, L)$ be a smooth Calabi-Yau threefold over a field $k$ of char $k = p \geq 5$ with Kodaira non-vanishing $H^1(X, L^{-1}) \neq 0$. If $L^\ell$ is a Tango structure for some $\ell \geq 1$, then we have

$$H^1(X, H^{-1}) = H^0(X, H^{-1} \otimes (\rho_*\mathcal{O}_Y/\mathcal{O}_X))$$

for every ample line bundle $H$ on $X$, where $\rho : Y \to X$ is a purely inseparable cover of degree $p$ corresponding to the Tango structure as in Theorem 7.

**Proof.** By taking a sufficiently large power $L^\ell$, $\ell \gg 0$, we can assume from the beginning that $H^1(X, L^{-p}) = 0$. Also, by the assumption we can assume that $L$ is a Tango structure. Then by Theorem 4 we have a purely inseparable cover $\rho : Y \to X$ of degree $p$ and $\omega_Y \cong \rho^*(\omega_X \otimes L^{-p+1}) \cong (\rho^*L)^{-p+1}$, see II 6.1.6 [9]. Since $\rho$ is a finite morphism and $L$ is ample, $\rho^*L$ is also ample. Thus we know that $Y$ is an integral Fano threefold. Also since $L$ is a Tango structure, $Y$ is smooth by Theorem 5. Now let $H$ be an arbitrary ample line bundle on $X$. Then, since $\rho$ is surjective, we have the following exact sequence

$$0 \to H^{-1} \to H^{-1} \otimes \rho_*\mathcal{O}_Y \to H^{-1} \otimes \rho_*\mathcal{O}_Y/\mathcal{O}_X \to 0,$$
from which we obtain the long exact sequence

\[ H^0(X, \rho_* \rho^* H^{-1}) \to H^0(X, H^{-1} \otimes \rho_* \mathcal{O}_Y / \mathcal{O}_X) \to H^1(X, H^{-1}) \to H^1(X, \rho_* \rho^* H^{-1}). \]

Now, we have \( H^0(X, \rho_* \rho^* H^{-1}) = H^0(Y, \rho^* H^{-1}) = 0 \) since \( \rho \) is finite and \( H \) is ample. Also \( H^1(X, \rho_* \rho^* H^{-1}) = H^1(Y, \rho^* H^{-1}) \) and this is 0 by Theorem [17].

Recall that for a purely inseparable cover \( p : Y \to X \) of degree \( p \) there exists a \( p \)-closed rational vector field \( D \) on \( X \) such that \( (\rho_* \mathcal{O}_Y)^D := \{ f \in \rho_* \mathcal{O}_Y : D(f) = 0 \} = \mathcal{O}_X \) (cf. [15]). Thus we have

**Corollary 19.** Under the same assumption as Theorem [18], we have

\[ H^1(X, H^{-1}) = H^0(X, H^{-1} \otimes D(\rho_* \mathcal{O}_Y)) \]

where \( D \) is a \( p \)-closed rational vector field on \( X \) corresponding to the purely inseparable cover \( \rho \).

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