Well-Rounded Lattices for Reliability and Security in Rayleigh Fading SISO Channels

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Abstract—In the wiretap setting it is assumed that the same message is transmitted over two different channels, a channel to an intended/legitimate receiver Bob and a different channel to an eavesdropper Eve. Three contradicting objectives are simultaneously tried to be achieved: A high information rate between the sender and Bob, high reliability at the legitimate receiver, and minimal mutual information between the message and the output at the eavesdropper.

For many channel models asymptotically optimal coding schemes are known, but less effort has been put into actual realizations of wiretap codes for practical parameters. Bounds on the mutual information and error probability when using coset coding on a Rayleigh fading channel were recently established by Oggier and Belfiore, and the results in this paper build on their work. However, instead of using their ultimate inverse norm sum approximation, a more precise expression for the eavesdropper’s probability of correct decision is used in order to determine a general class of good coset codes. The code constructions are based on well-rounded lattices arising from simple geometric criteria. In addition to new coset codes and simulation results, novel number-theoretic results on well-rounded ideal lattices are presented.

I. INTRODUCTION

The current design criteria for wiretap channels, such as the inverse norm sum, are based on loose approximations and are not very reliable in terms of comparing different codes. Hence, it is desirable to derive new design criteria from tighter approximations. Motivated by this, we study the eavesdropper’s probability of correct decision and conclude that the coset codes used for the transmissions should arise from well-rounded (WR) lattices. By using WR sublattices of rotated $\mathbb{Z}^n$ lattices, one can simultaneously try to optimize for both low and high signal-to-noise ratio (SNR). Furthermore, this allows for the use of a skewed lattice for Eve while maintaining an orthogonal lattice structure for Bob, as suggested in [1].

A. Wiretap Channel

We consider a wiretap channel as described by Wyner [2]. A sender, Alice, transmits data over a possibly noisy channel to a receiver Bob and a second noisy channel to an eavesdropper Eve exists. The common assumption made is that the channel to the eavesdropper has lower SNR, or is in some other way more degraded, than the channel to the legitimate receiver. Wyner investigated the possibility of transmitting data to the receiver while having negligible mutual information with the eavesdropper. He could show that a non-zero secrecy capacity exists when a binary symmetric channel is considered. The mutual information between variables $X$ and $Y$ is defined in the usual way as

$$I(X, Y) := \sum_X \sum_Y P[X, Y] \log \left( \frac{P[X, Y]}{P[X]P[Y]} \right),$$

see [3] for more details. We will consider a single-input single-output (SISO) fast Rayleigh fading channel model [4], where the information is mapped to vectors in a codebook $x \in C \subset \mathbb{R}^n$ that are then component-wise sent through the channel. The vectors received by Bob and Eve are respectively given by

$$y = H_B x + e_B \text{ and } z = H_E x + e_E$$

where $H_x = \text{diag}(h_i^x)$ is a $n \times n$ diagonal matrix with $h_i^x$ being Rayleigh distributed fading coefficients with second raw moment $E[(h_i^x)^2] = 1$ and $e \in \mathbb{R}^n$ a Gaussian distributed error vector where each entry has variance $\sigma^2$.

B. Decoding Probability and Entropy

It is commonly assumed that Bob, by virtue of his superior channel, is able to decode correctly with high probability. All elements relevant to the security of a scheme are related to the channel to Eve and we will therefore from now on only consider it and suppress the index $E$. When Eve receives a vector $z$ she can translate this information into a probability distribution over all codewords

$$P[c] = \int_{V(c)} G(z, \sigma^2_E)$$

where $G$ is the $n$-dimensional Gaussian distribution centered at $z$ and $V(c)$ is the Voronoi cell around $c$ in the constellation $HC$. Even if the most likely codeword in this distribution is not equal to the sent codeword $x$, the eavesdropper might still gather information from it.

Example 1. Given the one dimensional example where $C = \{-2, -1, 0, 1, 2\}$, consider a channel where $H = 1$ and $x = 1$ was sent but Eve receives a vector closer to 2 due to the error term $e$. She decodes to 2, which is of course incorrect, but the mutual information is far from being negligible, since she still learned that the sent vector is highly unlikely to have been $-2, -1$ or 0.

The imbalance in this example is partly due to so called border effects.
In a perfect setup Eve would not be able to gather any information from the vector $z$. Every codeword would be equally likely to have been sent. Therefore a different strategy, coset coding, is employed where one message is represented by several different codewords. The probability of any message is given as the sum of the probabilities of the different codewords that represent that message. This strategy of course trades data rate for security. Information theoretically we try to increase the entropy for the random variable $z$ for fixed $x$.

II. NEW DESIGN CRITERIA FOR NESTED LATTICE COSET CODES

To achieve secret transmission of information, coset coding in nested lattices has been suggested by Oggier and Belfiore [5], based on ideas in [2] and [6]. We begin by introducing lattices and some necessary notation.

**Definition 2.** A lattice $\Lambda$ is the $\mathbb{Z}$-linear span of a set $\{b_1, \ldots, b_m\} \subset \mathbb{R}^n$ of linearly independent basis vectors. We call $\Lambda$ a full-rank lattice if $m = n$.

The squared length of a shortest (non-zero) vector in $\Lambda$, the *minimum distance*, is denoted by $\lambda_1(\Lambda)$. Each $x \in \Lambda$ such that $||x||^2 = \lambda_1(\Lambda)$ is called a minimal vector, and the set of minimal vectors of $\Lambda$ is denoted by $S(\Lambda)$. The volume of a lattice $\text{vol}(\Lambda) := |\det((b_i)_i)|$ where $(b_i)_i$ is any basis for $\Lambda$ is an important invariant. The Hermite constant in dimension $n$ is defined as

\[ \gamma_n := \max_{\Lambda \subset \mathbb{R}^n} \frac{\lambda_1(\Lambda)}{\text{vol}(\Lambda)^{\frac{1}{n}}}. \]

A. Nested Lattice Coset Coding

In nested lattice coset coding the codebook consists of vectors from a lattice $\Lambda_B$. This lattice is chosen such that Bob is able to decode correctly with high probability. A second lattice $\Lambda_E \subset \Lambda_B$ is then chosen and every possible message is mapped to an element in the quotient $\Lambda_B/\Lambda_E$. Consequently, the information rate is determined by the index $[\Lambda_B : \Lambda_E] := \frac{\text{vol}(\Lambda_B)}{\text{vol}(\Lambda_E)}$, not the actual codebook size. The vector $x$ is chosen as a random representative of the coset belonging to the intended message. Hence, a codeword $x$ is the sum of the message $m \in \Lambda_B/\Lambda_E$ (for a fixed shortest set of representatives) and a random vector $r \in \Lambda_E$. In practice we restrict $x$ to a finite region.

For a finite codebook $C \subseteq \mathbb{R}^n$ the data rate $R := \frac{1}{n} \log_2(\vert C \vert)$ in bits per channel use (bpcu) is split between the information rate $R_i$ from Alice to Bob, i.e., the actual amount of information transmitted, and random bits $R_c$ added to confuse the eavesdropper

\[ R = R_i + R_c = \frac{1}{n} \log_2([\Lambda_B : \Lambda_E]) + \frac{1}{n} \log_2 \left( \frac{|C|}{[\Lambda_B : \Lambda_E]} \right). \]

Increasing the number of coset representatives and thus increasing $R_c$ reduces border effects but increases the average energy by increasing the codebook size.

B. Correct Decoding Probability

In this setting Eve calculates the probability that a certain message $m$ was sent as

\[ P[m] = \sum_{r \in \Lambda_E} \int_{V(m+r)} G(z, \sigma_E^2) \]

when she receives the vector $z$. This ignores the implementation of a finite sending region and border effects, but yields a good approximation. It can be shown that the most likely message is in fact the message that was sent. It is clear that the probability to decode the correct message is lower bounded by the reciprocal of the number of possible messages $\frac{1}{[\Lambda_B : \Lambda_E]}$, since this is equivalent to randomly guessing the correct message. In this case Eve’s guess is completely independent from the message that was sent and $I(X,Z) = 0$. Thus, we can use Eve’s correct decoding probability $ECDP$ as a measure of how much information she can glean from $z$.

We point out that even if the probability would be quite high, the mutual information can still be zero, meaning Eve gains no information even though occasionally decoding correctly. Notice also that due to coset coding, there will be a coset representative close-by regardless of how big the noise is, as demonstrated by the lower bound $\frac{1}{[\Lambda_B : \Lambda_E]}$. This is where coset codes crucially differ from traditional lattice codes.

An analytic approximation for the correct decoding probability is developed in [7] as

\[ ECDP = (2\sigma_E)^{-n} \text{vol}(\Lambda_B) \prod_{i=1}^{n} \left( 1 + \left( \frac{r_i}{\sigma_E} \right)^2 \right)^{-\frac{3}{2}}. \] (1)

Here $n$ is the dimension of the lattices involved. The authors perform further approximations to come up with the so-called inverse norm sum, also investigated in, e.g., [8], [9]. Here, we do not take these further steps, but will analyze a more precise version of the probability expression as explained below.

We expand the product and bound from below by ignoring everything but the constant, the linear and the leading term

\[ \prod_{i=1}^{n} \left( 1 + \left( \frac{r_i}{\sigma_E} \right)^2 \right)^{-\frac{3}{2}} \leq \left( 1 + \sum_{i=1}^{n} \left( \frac{r_i}{\sigma_E} \right)^2 \right)^{-\frac{n}{2}} \left( 1 + \sum_{i=1}^{n} \left( \frac{r_i}{\sigma_E} \right)^2 \right) \]

(2)

The linear term corresponds to the squared length of the vector $\sigma_E^{-1}r$, while the leading term is given by the squared product distance $d_{p, \text{min}}(v) := \prod_{i=1}^{n} \vert v_i \vert$ of $v = \sigma_E^{-1}r$. Rotations of $\mathbb{Z}^n$ that maximize the minimum product distance have been investigated in several publications such as [4] or [10]. These increase reliability in channels with high SNR, or low $\sigma_E$, where the last term is dominant. In channels with lower SNR the linear term becomes more prominent and we should choose our lattice $\Lambda_E$ such that it is maximized.

Since the shortest vectors of $\Lambda_E$ contribute the most by the estimate (2) it is beneficial to maximize their length. This is equivalent to increasing the sphere packing radius of the lattice, i.e., maximizing the minimal length of its vectors. This reasoning of course ignores the number of shortest vectors in these lattices, e.g., $\mathbb{Z}^8$ has 16 vectors of minimal length 1 while $E_8$ has 240 vectors of minimal length 2 (both have volume 1).
Nevertheless, a short calculation shows that \( E_8 \) still presents us with a smaller lower bound for the ECDP when summing over the 240 shortest vectors in both lattices.

Although optimal sphere packing lattices would provide the longest minimal vectors achieving the Hermite constant, these are not always available or even the best choice. For instance, Harshan and Viterbo [11] showed that finite precision in practical decoders as well as quantization can heavily influence the performance. They hence proposed to use codes that only use integer coordinates, and we will also turn our attention to integer lattices. More precisely, we will be searching for (possibly rotated) sublattices of \( \mathbb{Z}^n \). The rotations are algebraic, so the lattices remain integral. We use the standard pulse amplitude modulation (PAM) for the lattice coordinates.

Even when an optimal integer lattice exists, \( \text{e.g.} \), \( D_4 \) or \( E_8 \) with suitable scaling, they only provide us with limited choices for indices. We therefore suggest a more general class of lattices, namely well-rounded lattices, which are available for many different parameters.

C. Well-Rounded Lattices

\textbf{Definition 3.} A lattice \( \Lambda \) is called well-rounded (abbreviated WR) if the set \( S(\Lambda) \) of minimal vectors contains \( n \) linearly independent vectors.

The set of minimal vectors \( S(\Lambda) \) does not necessarily form a basis for \( \Lambda \) [12, Chapter 2]. They are known to form a basis for all \( n \leq 4 \) as mentioned in [13]. This motivates the following stronger definition.

\textbf{Definition 4.} A lattice \( \Lambda \) is called (strongly) well-rounded if the set of minimal vectors \( S(\Lambda) \) generates \( \Lambda \).

If a lattice is WR, then the set of minimal vectors \( S(\Lambda) \) generates a sublattice of \( \Lambda \) that is also WR. Hence, from now on, we only work with WR lattices as in Definition 4.

More generally, WR lattices are of interest in investigations of sphere packing, sphere covering, and kissing number problems as well as in coding theory as shown in [13]–[15]. A particularly interesting class of WR lattices are the integral well-rounded (IWR) lattices. The properties of WR lattices in the plane which come from ideals in quadratic fields have been studied by Fukshansky \textit{et al.} In [16] and [17], the authors presented a characterization of WR ideal lattices in the plane and proved that even asymptotically a positive proportion of real and imaginary quadratic number fields contain ideals giving rise to WR lattices. We provide two new related results in the next section.

There is an easy criterion that relates the volume of a WR sublattice to the length of its minimal vectors using Hermite constants.

\textbf{Lemma 5.} For a full rank WR lattice of volume \( V \) it holds that \( V^{\frac{2}{n}} \leq \lambda_1 \leq \gamma_n V^{\frac{2}{n}} \), where \( \gamma_n \) is the Hermite constant for dimension \( n \).

III. WELL-ROUNDED IDEAL LATTICES

WR lattices that come from ideals of number fields are also called well-rounded ideal lattices. Well known examples of these lattices are the ring of integers and its ideals of cyclotomic fields [17]. In [16] and [17], the authors proved the existence of infinitely many real and imaginary quadratic fields that contain WR ideal lattices and studied their properties. A sufficient (resp. equivalent) condition for a positive square-free integer \( D \) such that the quadratic field \( \mathbb{Q}(\sqrt{D}) \) (resp. \( \mathbb{Q}(\sqrt{-D}) \)) contains WR ideal lattices was also given. However, for an arbitrary number field of degree at least three, the existence and structure of WR ideal lattices are unknown.

In this section, we further concentrate on real quadratic fields \( F \). Regarding computational aspects, the norms of ideals are frequently considered in comparison with the discriminant of \( F \). The relevance of the norm for our purposes arises from the fact that it corresponds to the nesting index and hence relates to the information rate, as we will see in this section. Here, we first present a result saying that WR ideal lattices of \( F \) have large norms compared to the discriminant. This is a nice result also from a practical point of view, since optimizing the minimum product distance is equivalent to minimizing the discriminant [4], whereas large norms are preferable due to their relation to the information rate.

In addition, we discuss a class of WR lattices that are generated from principal ideals. This also provides an answer to the question proposed in [16, Question 2] about the existence of WR principal ideal lattices of real quadratic fields.

\textbf{Definition 6.} Let \( F \) be a totally real number field of degree \( n \). There are exactly \( n \) distinct field homomorphisms \( \sigma_i : F \rightarrow \mathbb{R} \) for \( i = 1, \ldots, n \). The map \( \sigma : F \rightarrow \mathbb{R}^n \) defined by \( \sigma(\alpha) = (\sigma_1(\alpha), \ldots, \sigma_n(\alpha)) \) is called the canonical embedding of \( F \).

Let \( F \) be a totally real number field of degree \( n \) with the ring of integers \( O_F \). The images of ideals in \( O_F \) under \( \sigma \) are lattice in \( \mathbb{R}^n \). We denote by \( \Lambda_I = \sigma(I) \) the lattice in \( \mathbb{R}^n \) corresponding to the ideal \( I \) in \( O_F \) and \( \Lambda_F = \sigma(O_F) \). Then \( \Lambda_I \) is a sublattice of \( \Lambda_F \). Moreover, the norm \( N(I) \) of \( I \) is also the index of \( \Lambda_I \) in \( \Lambda_F \) and hence it gives the information rate as described below.

\textbf{Proposition 7.} Let \( \Lambda_E = \Lambda_I \) and \( \Lambda_B = \Lambda_F \). Then \( N(I) = |\Lambda_B : \Lambda_E| = 2^{\frac{n}{2}} \).

\textbf{Remark 8.} Under the canonical embedding, the algebraic integers are mapped to the lattice \( \Lambda_F \), which will be the underlying structure of our codes. Optimal rotations of \( \mathbb{Z}^n \) based on ideal lattices arising from canonical embeddings can be found in [18]. We shall use these algebraic rotations to rotate our coset codes for the simulations, to guarantee a large minimum product distance and hence good performance for Bob.

Let \( D \) be a positive, square-free integer and and let \( F = \mathbb{Q}(\sqrt{D}) \). The discriminant \( \Delta \) of \( F \) is equal to \( 4D \) or \( D \) depending on whether \( D \equiv 2, 3 \pmod{4} \) or \( D \equiv 1 \pmod{4} \) respectively. The canonical embedding \( \sigma(D) \) of \( D \) is determined by \( \sigma_i : F \rightarrow \mathbb{R} \) for \( i = 1, 2 \) where \( \sigma_1(\sqrt{D}) = \sqrt{D} \) and \( \sigma_2(\sqrt{D}) = -\sqrt{D} \) as in Definition 6. Each element \( x + y\sqrt{D} \in F \) with \( x, y \in \mathbb{Q} \) is mapped to the vector \( (\sigma_1(x + y\sqrt{D}), \sigma_2(x + y\sqrt{D})) = (x + y\sqrt{D}, x - y\sqrt{D}) \in \mathbb{R}^2 \).
via $\sigma$. Assume that $I$ has a $\mathbb{Z}$-basis $\{\alpha, \beta\}$. The vectors $\{b_1 = (\sigma_1(\alpha), \sigma_2(\alpha)), b_2 = (\sigma_1(\beta), \sigma_2(\beta))\}$ form a $\mathbb{Z}$-basis of $\Lambda_I$. In other words, the lattice $\Lambda_I$ can be represented as $\Lambda_I = \begin{bmatrix} \sigma_1(\alpha) & \sigma_1(\beta) \\ \sigma_2(\alpha) & \sigma_2(\beta) \end{bmatrix} \mathbb{Z}^2$.

The following new results are obtained. The proofs are omitted due to lack of space and will appear in an extended version of this paper [19].

**Proposition 9.** If $\Lambda_I$ is a WR ideal lattice in $O_F$, then $N(I) \geq \sqrt{3\Delta}$. In [16], the authors proved that for imaginary quadratic field $F = \mathbb{Q}(\sqrt{-D})$, the ring of integers contains principal WR ideals if and only if $D = 1, 3$. For real quadratic fields, we obtain the following result.

**Proposition 10.** There are infinitely many real quadratic fields $F = \mathbb{Q}(\sqrt{D})$ with a positive square-free integer $D \equiv 3 \pmod{4}$, respectively $D \equiv 1 \pmod{4}$, such that $O_F$ contains principal WR ideals.

**Remark 11.** Determining whether there exists an integer $D \equiv 2 \pmod{4}$ such that the quadratic field $\mathbb{Q}(\sqrt{D})$ has WR principal ideals is still an open problem. A computational search based on Lemma 5 showed that $\mathbb{Q}(\sqrt{D})$ does not contain any WR principal ideals of index at most $2D$ for any even integer $D \leq 10^3$.

The table below illustrates some examples of WR principal ideals $\Lambda_I$ of real quadratic fields $F = \mathbb{Q}(\sqrt{D})$ for some positive, square-free integer $D$ given in the first column. The second column contains a generator of principal ideals $I$. The index of the WR lattice $\Lambda_I$ is shown in the last column.

| $D$ | A generator of $I$ | Index |
|-----|-------------------|-------|
| 12  | $3 + \sqrt{3}$   | 6     |
| 15  | $5 + \sqrt{15}$  | 10    |
| 19  | $7 + \sqrt{19}$  | 15    |
| 143 | $13 + \sqrt{143}$| 26    |
| 195 | $15 + \sqrt{195}$| 30    |

**TABLE I**

Some WR principal ideals in $\mathbb{Q}(\sqrt{D})$

**IV. SIMULATION RESULTS**

**A. Probabilistic Lattice Search**

It is possible to find all WR sublattices of a given lattice and a given index by searching through all possible combinations of vectors of suitable lengths, as described by Lemma 5. Fortunately, WR lattices are common enough that a probabilistic algorithm often suffices.

Most lattices in the simulations were found after only minutes of randomly testing combinations of integer vectors of same length for linear independence and then using the LLL-algorithm to determine $\lambda_1$ for the lattice they generate.

**B. Well-Rounded Ideal Lattices**

We simulated the ECDP for two WR principal ideals and a sublattice of $\mathbb{Z}^2$, all having index 216. More details are given in the table below.

| Lattice | Field    | Ideal | $\lambda_1$ | $R_1$ | $R_2$ |
|---------|----------|-------|-------------|-------|-------|
| $\Lambda_1$ | $\mathbb{Q}([\sqrt{3}])$ | $18 + 6\sqrt{3}$ | 249.42 | 3.87744 | 1.12326 |
| $\Lambda_2$ | $\mathbb{Q}(\sqrt{5})$ | $18 + 6\sqrt{5}$ | 223.08 | 3.87744 | 1.13166 |
| $\Lambda_3 \subset \mathbb{Z}^2$ | - | - | 234 | 3.87744 | 1.12185 |

**TABLE II**

Parameters of lattices after normalization

A generator matrix for $\Lambda_3$ is given by $(\frac{3}{15}, \frac{15}{3})$. In Fig. 1, the simulation results for the ECDP are compared for these three lattices and it can be seen that they perform very similarly. We will therefore look at some higher dimensional examples that allow for significant differences in $\lambda_1$.

**C. $\mathbb{Z}^4$ sublattices**

In this section, we show simulation results for three different sublattices of $\mathbb{Z}^4$.

$\Lambda_1 = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$\Lambda_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\Lambda_3 = \begin{bmatrix} -4 & -1 & 0 & -1 \end{bmatrix}$

with parameters as shown in Table III.

These lattices are chosen because they provide a good sample of different types of lattices with the same information rate and showcase the importance of the $\lambda_1$ parameter. In Fig. 2, the simulation results for the correct decoding probability ECDP are shown. It can clearly be seen, that the lattice with larger $\lambda_1$ outperforms the other two and it is the only one that comes close to reaching the theoretical lower bound of $\frac{1}{\lambda_1}$ and hence negligible mutual information. Similar phenomenon...
can be observed for higher dimensional examples, but we omit the figures here due to lack of space.

| Lattice | $\lambda_1$ | WR index | $R_i$ | $R_c$ |
|---------|-------------|-----------|-------|-------|
| $\Lambda_1$ | 4 | no | 256 | 2 | 2 |
| $\Lambda_2$ | 16 | yes | 256 | 2 | 2 |
| $\Lambda_3$ | 20 | yes | 256 | 2 | 2 |

TABLE III
PARAMETERS FOR THREE DIFFERENT SUBLATTICES OF $\mathbb{Z}^4$

![Fig. 2. Simulation of ECIDP for three different $\mathbb{Z}^4$ sublattices](image)

**D. Cross-Packing**

In a recent paper, Sakzad, Trautmann, and Viterbo [20] suggest a new construction that aims at minimizing the error probability for Rician channels. This channel model is a generalization of the Gaussian and the Rayleigh channel models and features an additional parameter $K$. For $K = 0$ it simplifies to the Rayleigh channel observed here. For small $K$ they observe that the contour lines of the error probability are cross shaped and abstract a cross distance for integer vectors for which they design lattices of minimum cross distance $2t + 1$ for all dimensions $n \geq 2$. We compare their cross packing lattice $B_4$ with $t = 6$ with a WR sublattice $\Lambda$ of $\mathbb{Z}^4$ of the same index 302 (we use a LLL-reduced basis of $B_4$ here):

$$B_4 = \begin{pmatrix} -1 & 1 & 2 & 2 \\ -1 & 0 & 2 & -5 \\ 1 & -2 & 5 & -1 \\ -1 & -5 & 1 & 2 \end{pmatrix} \mathbb{Z}^4, \quad \Lambda = \begin{pmatrix} 1 & 1 & 3 & -2 \\ -4 & 1 & 0 & -4 \\ -1 & -2 & 3 & 1 \\ 2 & -4 & 2 & -1 \end{pmatrix} \mathbb{Z}^4$$

Fig. 3 shows that the WR lattice outperforms the cross packing. We also point out that the cross-packing construction is only available for indices $t^3 + 2t^2 + 2t + 2$ with $t \in \mathbb{N}$.

**ACKNOWLEDGMENT**

This work was partially supported by the Academy of Finland grants #276031, #282938, and #283262, and by a grant from Magnus Ehrnrooth Foundation, Finland. The support from the European Science Foundation under the ESF COST Action IC1104 is also gratefully acknowledged.

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