A fine tuning free resolution of the cosmological constant problem

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In a recent paper we discovered that a fermionic condensate is formed from gravitational interactions due to the covariant coupling of fermions in the presence of a torsion-fermion contact interaction. The condensate gap gives a negative contribution to the bare cosmological constant. In this letter, we show that the cosmological constant problem can be solved without fine tuning of the bare cosmological constant. We demonstrate how a universe with a large initial cosmological constant undergoes inflation, during which time the energy gap grows as the volume of the universe. Eventually the gap becomes large enough to cancel out the bare cosmological term, inflation ends and we end up in a universe with an almost vanishing cosmological term. We provide a detailed numerical analysis of the system of equations governing the self regulating relaxation of the cosmological constant.

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INTRODUCTION

There are many faces to the cosmological constant/dark energy problem. First, the naive perturbative theoretical evaluation of the vacuum energy of all particles in the standard model gives a result that disagrees with observations by 120 orders of magnitude. Second, a confluence of cosmological and astrophysical observations, such as the WMAP satellite and Type Ia supernovae, indicate that the cosmological constant or something very similar to it, currently dominates the universe.

Perhaps the most striking aspect of the cosmological constant problem is seen in the details of the inflationary paradigm. Inflation is driven by a constant part of the Energy-Momentum tensor of a scalar field, which is indistinguishable from a pure cosmological constant. Therefore, any mechanism which relaxes the cosmological constant would also prevent inflation from happening.

One way out of this possible conundrum is to do away with fundamental scalar fields, allow inflation to occur with a large cosmological constant and investigate any self consistent mechanism which negates the cosmological constant to almost zero. Such a mechanism would solve all three cosmological constant problems:

- The cosmological constant would be dynamically relaxed due to the non-trivial dynamics of inflation itself; it would be self regulating.
- Dark energy and the coincidence problem would be explained if a residual amount of cosmological constant would be left over by the end of inflation.
- Since inflation is not driven by a fundamental scalar fields, fine tuning of the cosmological constant is no longer needed.

Attempts at tackling the this problem via cosmological condensates include. More recently, Prokopec proposed a mechanism involving a Yukawa coupling between a scalar field and fermions.

A simple way to obtain inflation in the absence of matter is due to the presence of a non-zero, positive cosmological term on the right hand side of Einstein’s equation:

\[
G_{ab} = 8\pi G \Lambda_0 g_{ab}
\]

\[
\Rightarrow a(t) = a_0(t) e^{\sqrt{\frac{\Lambda_0}{3}} t}
\]

where we have used the FRW metric ansatz to obtain our solution. \(a(t)\) is the scale factor. From the solution it is clear that the Hubble rate \(H = \sqrt{\frac{\Lambda_0}{3}}\).

While this simple model gives us an inflating universe it is clearly not in line with reality because it does not predict an end to inflation. A way to get around this obstacle is to introduce matter, traditionally scalar fields, into the picture. Then the first Friedmann equation becomes:

\[
3 \left( \frac{\dot{a}}{a} \right)^2 = \Lambda_0 + \frac{1}{2} \dot{\phi}^2 + V(\phi)
\]

where \(\phi\) is the scalar field.

We also have the E.O.M for the scalar field:

\[
\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \frac{dV}{d\phi} = 0
\]
where \( V(\phi) \) is the scalar field potential. Such models typically require special initial conditions for the scalar field called the "slow-roll" conditions. The scalar field must start off at a large initial value and then start rolling slowly down an almost flat potential. This results in an inflationary universe. After a sufficient number of e-foldings, the scalar fields reaches the steeper part of the potential where it decays via parametric resonance leading to reheating and particle production after inflation has ended.

Unfortunately, such models have several shortcomings:

- The shape of the potential is arbitrary and we have no physical way of choosing the one that would correspond to our universe from an almost infinitely large family.
- We require the scalar field to start off at a large initial value. What mechanism would cause the scalar field to be "pumped up" to this value initially?
- The mass of the scalar field is an arbitrary parameter. It can be fixed once we fix the potential, but it remains a source of vagueness.
- Most importantly, from whence did this scalar field come? Perhaps if one tries to tackle this fundamental question head on the others might also be amenable to a solution.

In this letter we propose a dynamical solution to the CC problem assuming only the Standard Model and General Relativity. There are no fundamental scalar fields to tune. Therefore the universe will be dominated by a large cosmological constant, which naturally generates inflation. The non-trivial observation here is that the dynamics of inflation itself holds the key to relaxing the cosmological constant without fine tuning. How is this possible? The exponential time dependent behavior of de Sitter space counterintuitively enhances correlations between fermion pairs. These correlations become so strong that these fermions form a Cooper pair.

In a recent paper \[11\], we showed how the presence of torsion and fermionic matter in gravity naturally leads to the formation of a fermionic condensate with a gap which depends on the 3-volume. In this letter we will analyze explicitly the dynamics of the universe with a cosmological constant in the presence of this gap. Numerical calculations then show that with a large initial cosmological term and generic initial conditions for the scalar field and its momenta, we obtain a universe which undergoes an inflationary phase during which the gap grows as a function of \( \alpha^3 \), causing the effective cosmological term to diminish to a small positive value.

In Section 2 we discuss the E.O.M for our system. In Section 3 we present the numerical results and finally we conclude with some discussion or our results and what they imply for our understanding of inflation and the cosmological term.

**FRIEDMANN AND SCALAR EQUATIONS**

We briefly summarize the steps that were taken in \[11\]. We started with the Holst action for gravity with fermions:

\[
S_{H+D} = \frac{1}{2\kappa} \int d^4x e (e^\mu_I e^\nu_J R_{\mu\nu}^{IJ} - \frac{2}{3} \Lambda_0) - \frac{1}{2\kappa \gamma} \int d^4x e e^\mu_I e^\nu_J \star R_{\mu\nu}^{IJ} + \frac{i}{2} \int d^4x e (\bar{\Psi} e^\gamma_I e^\nu_J D_\mu \Psi - D_\mu \bar{\Psi} e^\gamma_I e^\nu_J \Psi) \quad (4)
\]

\( e^\mu_I \) is the tetrad field. \( R_{\mu\nu}^{IJ} \) is the curvature tensor. The second term in the above equation is analogous to the \( \Theta \) term in Yang-Mills theory and is required if we want to work with arbitrary values of the Immirzi parameter (\( \gamma \)). After varying the action w.r.t the connection \( A^\mu_{IJ} \) and solving the Gauss constraint we which that \( A^\mu_{IJ} = \omega^\mu_{IJ} + C_{IJ} \), where \( \omega^\mu_{IJ} \) is the tetrad compatible connection and \( C_{IJ} \) can be expressed in terms of the axial vector current:

\[
C_{IJ} = \frac{\kappa}{4 \gamma^2 + 1} j_a^M \left\{ \epsilon_{MJK} e^K_{\mu} - \frac{1}{2\gamma} \delta_{[IJ]} e^\mu_{\mu} \right\} \quad (5)
\]

where \( j_a^M = \bar{\Psi} \gamma_5 \gamma^M \Psi \). Inserting the torsion into the first order action we find the resulting second order action which now contains a four-fermi interaction and the tetrad is the only independent variable, the connection having already been solved for in the previous step.

\[
S[e, \Psi] = S_{H+D}[\omega(e)] - \frac{3}{2} \pi G \gamma^2 \gamma^2 \int d^4x e (j_a^I)^2 \quad (6)
\]

\(^1\) Which implies that the torsion is non-zero
Then we did the 3+1 decomposition of the action to find the Hamiltonian, which after making the ansatz of a FRW metric becomes:

\[
\mathcal{H} = -\frac{3}{\kappa} a^3 H^2 + a^3 \Lambda_0 + \frac{i}{a} (\psi_L^\dagger \gamma^a \partial_a \psi_L - \psi_R^\dagger \gamma^a \partial_a \psi_R) + \frac{3}{32 a^3 \gamma^2 + 1} \left[ \psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R \right]^2 = 0 \tag{7}
\]

We see that the right hand side is the sum of the gravitational, Dirac and interaction terms. \( \psi_L(\psi_R) \) is the spinor for left (right) handed fermions. \( \gamma \) is the Immirzi parameter. \( H = \frac{\dot{a}}{a} \) is the Hubble rate.

The key ingredient that dynamically cancels the cosmological constant arises from the four-fermion interaction in the r.h.s of (7). This effect arises from an interplay between general covariance and non-perturbative quantum mechanics. General covariance guarantees the four-fermion interaction. What about the non-perturbative quantum mechanics? We see that the effective coupling of the four-fermion interaction becomes large for small values of the scale factor (i.e. at early times). The form of this Hamiltonian maps directly into the BCS Hamiltonian of superconductivity, except it is the gravitational field that is playing the role of the phonons. As a result, just like in the BCS theory (see eg. [12]), an energy gap \( \Delta \) opens up which reflects the instability of the ground state associated with the bare cosmological constant. An effective cosmological constant with a lower energy is generated from the formation of the gap. To obtain the gap, we diagonalize the fermionic part of this Hamiltonian by expanding the fermions in normal modes and using a Boguliubov transformation. The resulting Hamiltonian is:

\[
\mathcal{H} = -\frac{3}{\kappa} a^3 H^2 + \frac{1}{\kappa} (\Lambda_0 - \Lambda_{corr}) + \int \frac{d^3k}{(2\pi)^3} \sqrt{E_k^2 + \Delta^2 (m_k + \bar{m} + n + \bar{n})} \tag{8}
\]

where the non-perturbative correction to the bare cosmological constant is\(^2\):

\[
\Lambda_{corr} = 2\Delta^2 = \frac{2 \hbar \omega_D \exp \frac{\Delta}{E_k}}{\exp \nu - 1}, \quad \left( \nu = \frac{2}{\kappa a^3 k^2} \frac{\gamma^2 + 1}{\gamma^2} \right) \tag{9}
\]

\(|k_f| \) is the fermi energy, \( E_k \) is the energy of the \( k \)th mode of the condensate and \( \gamma \) is the Immirzi parameter. \( m_k(n_k) \) and \( \bar{m}(\bar{n}) \) are the creation (annihilation) operators for the condensate of the left and right-handed fermions respectively. The \( a^3 \) factor in \( \Delta \) comes from the fact that the density of states in an expanding universe scales as the 3-volume. We see that the last term in the Hamiltonian constraint corresponds to the quantized expression for a scalar field condensate, \( \phi_c \), with mass \( \Delta^3 \). Replacing this with the classical expression for a scalar field we get:

\[
\mathcal{H} = -\frac{3}{\kappa} H^2 + \frac{1}{\kappa} (\Lambda_0 - 2\Delta(a)^2) + \frac{1}{2} \dot{\phi}_c^2 + \frac{1}{2} \Delta(a)^2 \phi_c^2 = 0 \tag{10}
\]

It is important to keep in mind that \( \phi_c \) is not a fundamental scalar field. Its annihilation and creation operators \( (m_k \) and \( n_k) \) correspond to excitations of the condensate. This leads to the first Friedmann equation with a time-dependent correction to the cosmological constant and a scalar field as our matter:

\[
3 \left( \frac{\dot{a}}{a} \right)^2 = \Lambda_0 - 2\Delta(a)^2 + \frac{1}{2} \dot{\phi}_c^2 + \frac{1}{2} \Delta(a)^2 \phi_c^2 \tag{11}
\]

after setting \( \kappa = 1 \).

The equation of motion for a scalar in a FRW background is:

\[
\ddot{\phi}_c + 3 \frac{\dot{a}}{a} \dot{\phi}_c + \Delta(a) \phi_c^2 = 0 \tag{12}
\]

We see that the energy gap [11] increases monotonically with \( a \). From this we can guess the qualitative behavior of the scale factor. As long as the initial value of the scale factor is such that \( 2\Delta^2 < \Lambda_0 \), then from (11) we see that the right hand side will be positive definite resulting in an inflating universe. The Hubble rate plays the role of friction for the scalar field. As time develops the friction will drive \( \phi_c \) to reach zero. From then until inflation ends, \( \phi_c \) will be a constant. Eventually the scale factor becomes large enough and the right side of (11) will start to decrease. \( H \) will then decrease and reach its minimum when:

\[^2 \Delta \text{ is obtained by solving the gap equation obtained in [11] in a self-consistent manner.}

\[^3 \text{To be precise we note that there are two scalar fields, corresponding to the two pairs of annihilation and creation operators. However in the following we use only one scalar field for simplicity. Noting that the left handed massless fermions are the antiparticles of the right handed ones, we can conjecture that this expression is the quantized form of a complex scalar field, which would imply that we are dealing with an axion.}\]
\[ \Lambda_0 = 2\Delta^2 - \frac{1}{2}\Delta^2 \phi_e^2 - \frac{1}{2}\hat{\phi}_e^2 \]  
(13)

\( \phi_e \) will then start rolling down the potential hill again, which is becoming steeper because \( a(t) \) and hence \( \Delta \) is still increasing. This presence of the scalar condensate coupling in the r.h.s of (13) means that when the system dynamically relaxes to \( \Lambda_{eff} = 0 \), it is in a state in which the energy density of the effective cosmological constant \( \Lambda_{eff} = \Lambda_0 - 2\Delta^2 \) traces the energy density of matter. This condition is similar to the relaxation mechanism due to backreaction of IR gravitational waves in which the backreaction effects ceases to negate the cosmological constant. A numerical calculation setting \( \Lambda_0 = M_{pl}^4 \sim 1 \). We find that initially the universe undergoes inflationary expansion (indicated by the constant value of the hubble rate). When the gap becomes large enough to cancel out the bare cosmological constant, inflation ceases. The behavior of the scalar field and momentum is in accord with the expectations outlined in the previous section. The scalar field increases or decreases depending on the sign of the initial value of the scalar field and radiation traces the effective cosmological constant. A numerical calculation setting \( \phi_e = \hat{\phi}_e = 0 \) confirms this. Fig. 2 shows the scalar field evolution for one set of initial values.

We must emphasize that the qualitative behavior is completely independent of the values of these parameters. In particular, if we set \( k_f = e^{-90} \) we would get 60 e-foldings.

\[ \Delta = 2\frac{\exp\left(\frac{1}{2}\alpha(t)^4\right)}{\exp\left(\frac{1}{2}\alpha(t)^4\right) - 1} \]  
(16)

We solved equations (12) and (11) numerically. An analytic solution is not possible because of non-analytic form of the gap (16). Fig. 1 shows the behavior of the scale factor and the hubble rate as a function of time.

**NUMERICAL SOLUTION AND RESULTS**

For our numerical calculation we work in Planck units \( (\kappa \sim M_{pl}^2 = 1) \). We set \( E_D \) and \( k_f \) to be \( M_{pl} \sim 1 \). We find that initially the universe undergoes inflationary expansion (indicated by the constant value of the hubble rate). When the gap becomes large enough to cancel out the bare cosmological constant, inflation ceases. The behavior of the scalar field and momentum is in accord with the expectations outlined in the previous section. The scalar field increases or decreases initially depending on the sign of the initial value of the scalar momentum. It quickly levels off to a constant value for the rest of the inflationary period, as the momentum is driven towards zero by a positive \( H \) and stays there until inflation ends. This behavior is independent of the initial values (which ranged from 0.5 to −0.5 in various runs) and shows that during inflation the Hubble rate during inflation is always \( \sqrt{\Lambda_0}/3 \). In fact, the scalar field plays no role in the relaxation of the bare cosmological constant. A numerical calculation setting \( \phi_e = \hat{\phi}_e = 0 \) confirms this. Fig. 2 shows the scalar field evolution for one set of initial values.
DISCUSSION

In a universe filled with fermions and with a positive cosmological constants is unstable. Their exists an interaction between fermions propagated by torsion at the level of the effective field theory. This interaction leads to the formation of Cooper pairs and a condensate forms whose free energy is lower than that of the deSitter background. Consequently the bare cosmological constant, which we identify to be the free energy of the deSitter background, is lowered by an amount proportional to the square of the condensate gap. We have cosmic expansion because initially the gap does not cancel out the bare cosmological constant completely. The size of the gap depends on the 3-volume. Hence as the expansion occurs the effective cosmological "constant" becomes smaller, until eventually after a period of inflation we emerge from the deSitter vacuum into flat Minkowski space, where $H \sim 0$.

The number of e-foldings during inflation is given by \( \text{(15)} \) and can be tuned by adjusting $E_D$ and $k_f$. The behavior of the scale factor is also independent of the scalar field evolution.

There are three free parameters in our model. The bare cosmological constant $\Lambda_0$, the fermi energy $k_f$ and the Debye energy $E_D$. In a condensate $E_D$ is the cutoff frequency and is determined by the lattice size. In a cosmological context therefore we can speculate that it should be $\sim M_{pl}$. $k_f$ can constrained according to \( \text{(15)} \) to be $\sim e^{-90}$. $\Lambda_0$ determines the Hubble rate during inflation. From the WMAP data \( \text{(2)} \), the upper limit on $H/M_{pl}^2$ is constrained to be $10^{-4}$. From this we can deduce that the bare cosmological constant, needs to be fixed by hand to be $\Lambda \sim H^2 \sim 10^{-8} M_{pl}^2$ in order to conform to observations.

We have presented here a non-perturbative mechanism which relaxes the bare cosmological constant to zero. As a bonus we find that the relaxation is accompanied by an inflationary period. The duration of inflation is determined by two parameters ($E_D$ and $k_f$) whose precise determination requires physics beyond the standard model. The lack of fine-tuning is demonstrated by the fact that the solution has an attractor with $H = 0$ independent of the values of the free parameters.

The scalar field discussed here is an emergent degree of freedom. After inflation, oscillations of this field can lead to reheating. However, to what extent this would be a viable description of the post-inflationary period remains to be seen in future work.

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