Secrecy in prepare-and-measure CHSH tests with a qubit bound

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The security of device-independent (DI) quantum key distribution (QKD) protocols relies on the violation of Bell inequalities. As such, their security can be established based on minimal assumptions about the devices, but their implementation necessarily requires the distribution of entangled states. In a setting with fully trusted devices, any entanglement-based protocol is essentially equivalent to a corresponding prepare-and-measure protocol. This correspondence, however, is not generally valid in the DI setting unless one makes extra assumptions about the devices. Here we prove that a known tight lower bound on the min entropy in terms of the CHSH Bell correlator, which has featured in a number of entanglement-based DI QKD security proofs, also holds in a prepare-and-measure setting, subject only to the assumption that the source is limited to a two-dimensional Hilbert space.

The security of quantum key distribution (QKD) rests on tradeoffs inherent to quantum physics, such as the impossibility of state cloning, the measurement-disturbance tradeoff, or the monogamy of entanglement. Similarly, the security of device-independent (DI) QKD [1–3], which can be established with minimal assumptions about the internal functioning of the devices, is based on a fundamental tradeoff between the violation of Bell inequalities and the unpredictability of quantum measurements. The simplest setting in which this tradeoff can be stated involves two separate parties, Alice and Bob, sharing two subsystems in an entangled state on which they perform, respectively, one of two measurements \(x, y\) \(\in\{0,1\}\) yielding one of two outcomes \(a, b\) \(\in\{0,1\}\). In this setting, the expectation value

\[ S = \sum_{a,b \in \{0,1\}} P(ab | xy) \]

of the CHSH Bell correlator [4], where \(P(ab | xy)\) denotes the joint probabilities for outcomes \(a, b\) given measurements \(x, y\), implies the fundamental lower bound

\[ H_{\text{min}}(A | E) \geq 1 - \log_2(1 + \sqrt{2 - S^2/4}) \]

on the min entropy \(H_{\text{min}}(A | E)\) of Alice’s outcome conditioned on one of Alice’s inputs (say, \(x = 0\)) and the quantum side information \(E\) of any potential adversary. This relation is tight and is attained with equality with the optimal attack described in [3].

Contrarily to other tradeoffs used in standard QKD, which assume some level of trust and characterisation of the quantum systems, the bound (2) is device independent in the sense that it holds for any quantum state \(\rho_{A\mathcal{E}}\) and measurement operators \(M_{a|x}\) and \(M_{b|y}\) characterising Alice’s and Bob’s devices. The relation (2) was first derived in the context of DI-randomness certification [5] and has since featured as an ingredient in a number of DI QKD security proofs [6–8].

Since they are based on the violation of Bell inequalities, DI QKD protocols are entanglement-based (EB) protocols. Indeed, in the DI setting, entanglement is necessary to guarantee security with a minimal set of assumptions on the devices [9]. Implementations of traditional (non-DI) QKD protocols, such as BB84 [10], are, however, usually of the prepare-and-measure (PM) type. In a PM protocol, Alice uses a source to prepare certain states which are then transmitted through a quantum channel to Bob who performs measurements on them. PM schemes have the practical advantage that they do not require the manipulation of entanglement. For this same reason, however, they cannot be fully DI. Recent works have nevertheless considered the possibility of PM QKD schemes that are at least partially DI [11, 12].

In traditional QKD, a famous argument establishes an equivalence between the security of PM and EB protocols [13]. In the BB84 protocol, for instance, Alice could prepare the four BB84 states by preparing a \(\Phi^+\) Bell state \((|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)/\sqrt{2}\) (in some Hilbert space \(H_A \otimes H_{\mathcal{A}}\)) in her lab and measuring either in the computational \((|0\rangle_A, |1\rangle_A)\) basis or in the Hadamard \((|\pm\rangle_A, |\mp\rangle_A)\) basis in \(H_A\) and transmitting the projected state in \(H_{\mathcal{A}}\) to Bob. Since the security can only be reduced if the \(\Phi^+\) state is replaced by a state \(|\psi\rangle_{A\mathcal{E}}\) chosen by an adversary (Eve) and shared between Alice, Bob, and Eve (the situation considered in EB security proofs), a security proof of the EB version of the BB84 protocol automatically implies the security of the PM version.

In the DI setting one can similarly associate a corresponding PM scheme to any EB scheme. In particular, one can consider a PM version of the above CHSH scenario, as illustrated in Fig. 1. In this PM version, Alice possesses a source which can emit one of four different quantum states, noted \(\rho, \rho', \sigma, \sigma'\), depending on a respective choice of input \((x, a) = (0, 0), (0, 1), (1, 0), \) and \((1, 1)\). Alice randomly chooses \(x \in \{0,1\}\) (not necessarily equiprobably) and chooses \(a \in \{0,1\}\) randomly and

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My adversary may perform an arbitrary unitary operation \( \rho_{x,a} \) setting. During the transmission from Alice to Bob, the estimation of the CHSH correlator (3) need not be satisfied, and is actually explicitly relaxed in the PM setting.

We show here that the fundamental bound on Alice’s min entropy (2) nevertheless still holds in a semi-DI setting, with the PM version of the CHSH correlator (3) used in place of (1). Such a result can then be used to bring the semi-DI setting (for which security proofs are lacking) closer in line to known security results for DI QKD. In particular, the conditional min entropy can, for instance, be used to lower bound the Devetak-Winter key rate [14] in order to establish the security against collective attacks of a semi-DI QKD protocol based on the estimation of the CHSH correlator (3).

**Dimension assumption.**— Let us start by making precise the assumption that we need to derive (2) in the PM setting. During the transmission from Alice to Bob, an adversary may perform an arbitrary unitary operation on the states sent by Alice, with the intent of gaining some information about them. (More general quantum operations can be made unitary by enlarging the adversary’s Hilbert space, according to Stinespring’s dilation theorem.) Following this unitary attack, the emitted state \( \rho_{x,a} \) is now shared between Bob and Eve [15], i.e., acts on a Hilbert space \( \mathcal{H}_B \otimes \mathcal{H}_E \). We make the assumption that the two differences \( \rho - \rho' \) and \( \sigma - \sigma' \) between the source states (after the unitary attack) share their support on a common two-dimensional subspace \( \mathcal{H}_A \) of \( \mathcal{H}_B \otimes \mathcal{H}_E \). We refer to this condition as the *qubit source assumption*. We will later discuss the physical implications of this assumption; for now we simply take it as a mathematical condition satisfied by the states prepared by Alice’s box.

A simple example illustrates the necessity of the qubit source assumption. Specifically, if Alice’s source prepares pure states \( \rho_{x,a} = |\psi_{x,a}\rangle\langle\psi_{x,a}| \) which are duplicate copies of the BB84 states,

\[
|\psi_{00}\rangle = |0\rangle_B|0\rangle_E, \quad |\psi_{01}\rangle = |1\rangle_B|1\rangle_E, \quad |\psi_{10}\rangle = |+\rangle_B|+\rangle_E, \quad |\psi_{11}\rangle = |-\rangle_B|-\rangle_E,
\]

in which \( |0\rangle \) and \( |1\rangle \) are orthonormal and \( |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \), the maximal value \( S = 2\sqrt{2} \) can be attained while Eve always acquires exactly the same state as Bob. These states are not linearly independent (one can readily verify that \( |0\rangle |0\rangle + |1\rangle |1\rangle = |+\rangle |+\rangle + |−\rangle |−\rangle \) ) and span a three-dimensional Hilbert space, from which we see that the security of the semi-DI scenario is fully compromised if the qubit source assumption is not satisfied.

**Min entropy and Eve’s distinguishability.**— To prove the bound (2), let us first note that if the input \( a \) is chosen equiprobably, its min entropy, conditioned on the case \( x = 0 \) and on Eve’s quantum side information, is a function of the classical-quantum state

\[
\tau_{AE} = \frac{1}{2} \log \left( 1 + D(\rho_E, \rho'_E) \right),
\]

in which \( \rho_E \) and \( \rho'_E \) are Eve’s marginals of the states \( \rho \) and \( \rho' \) after some given unitary attack (in the rest of this article, subscripts indicate partial tracing in the obvious way, e.g., \( \rho_B = \text{Tr}_E[\rho] \)). Evaluated on (6), the min entropy can be expressed [16, 17] as

\[
H_{\text{min}}(A \mid E) = 1 - \log_2(1 + D(\rho_E, \rho'_E)),
\]

with the trace distance between \( \rho_E \) and \( \rho'_E \) defined by

\[
D(\rho_E, \rho'_E) = \frac{1}{2} \| \rho_E - \rho'_E \|_1 \quad \text{where} \quad \| A \|_1 = \text{Tr}[\sqrt{A^\dagger A}]
\]

denotes the trace norm of an operator \( A \). We will obtain the main result (2) by showing that the trace distance appearing in (7) is upper bounded by

\[
D(\rho_E, \rho'_E) \leq \sqrt{2 - S^2/4}
\]

in terms of \( S \).

**Outline of proof.**— We now outline the derivation of (8). The lengthier and more pedestrian parts of the proof are given in the appendix.
Let us first introduce operators $Z$ and $X$ defined such that
\begin{align}
\rho - \rho' &= \alpha Z , \quad (9) \\
\sigma - \sigma' &= \beta X , \quad (10)
\end{align}
with $\alpha = \frac{1}{2}\|\rho - \rho'\|_1$ and $\beta = \frac{1}{2}\|\sigma - \sigma'\|_1$ such that the (traceless qubit) operators $Z$ and $X$ satisfy $\frac{1}{2}\|Z\|_1 + \frac{1}{2}\|X\|_1 = 1$. Then, in terms of these operators, the CHSH expectation value (3) can be expressed as
\[ S = \frac{1}{2} \text{Tr}[U_B(\alpha Z_B + \beta X_B) + V_B(\alpha Z_B - \beta X_B)], \]
where $U_B = \sum_b (-1)^b M_B^{(b)}$ and $V_B = \sum_b (-1)^b M_B^{(b)}$ are Hermitian unitary operators acting on $\mathcal{H}_B$ describing the observables corresponding to the (without loss of generality, projective) measurements $y = 0$ and $y = 1$.

A general result for any pair of Hermitian unitaries is that they admit a common block diagonalisation in blocks of dimension no more than 2. We can thus set
\[ U_B = \bigoplus_k U^k_B , \quad V_B = \bigoplus_k V^k_B , \]
where $U^k_B$ and $V^k_B$ are still Hermitian and unitary and of dimension at most 2, $\forall k$ (the Jordan lemma [18], see Lemma 2 of [9] for a short proof). This reduces the problem to considering qubit subspaces on Bob’s side. For each subspace $k$, we can define the corresponding contribution to $S$ by
\[ S_k = \frac{1}{2} \alpha \text{Tr}[(U^k_B + V^k_B)Z_B] + \frac{1}{2} \beta \text{Tr}[(U^k_B - V^k_B)X_B], \]
with $\sum_k S_k = S$. Similarly, we define a probabilistic weight for each subspace $k$ by
\[ p_k = \frac{1}{2} \text{Tr}[I^k_B Z_B], \]
with $\sum_k p_k = 1$, defined in terms of the projection operator $I^k_B$ on the $k$th subspace (satisfying $I^k_B = (U^k_B)^* = (V^k_B)^*$) and the partial trace $Z_B = \text{Tr}_E[Z]$ of the identity on the space of source states (satisfying $Z = Z^2 = X^2$).

We now introduce an orthonormal basis $\{|y\rangle, |y'\rangle\}$ of the space of source states chosen such that $Y = |y\rangle \langle y| - |y'\rangle \langle y'|$ is orthogonal to the operators $Z$ and $X$, defined by Eqs. (9) and (10), on the Bloch sphere. In this basis, in an appropriate phase convention, $Z$ and $X$ can be expressed as
\begin{align}
Z &= e^{i \varphi} |y\rangle \langle y'| + e^{-i \varphi} |y'\rangle \langle y| , \\
X &= e^{-i \varphi} |y\rangle \langle y'| + e^{i \varphi} |y'\rangle \langle y|
\end{align}
with $\varphi$ a (priori unknown) angle $\varphi$, while the source space identity operator $Z$ takes the expression
\[ Z = |y\rangle \langle y| + |y'\rangle \langle y'| . \]
One can readily verify that $\{Z, Y\} = \{X, Y\} = 0$, that $[Z, X] = 2i \sin(\varphi) Y$, and that $\{Z, X\} = \cos(\varphi) Z$. An important step in the derivation of the trace-distance bound (8) consists in turning the value of $S$ into a constraint on the part $Y_B$ of the operator $Y$ accessible to Bob [19]. Specifically, in each subspace $k$ defined by the block diagonalisation (12), we prove in Appendix A that there exists a Hermitian unitary operator $W^k_B$ with the property that
\[ \alpha \frac{1}{2} \text{Tr}[W^k_B Y_B] \geq \sqrt{S_k^2 / 4 - p_k^2}, \]
where $S_k$ and $p_k$ are as defined in (13) and (14). Eq. (18) holds regardless of the value of $\beta$ appearing in (13) and of $\varphi$ in (15) and (16). Note that $\alpha$ can also be eliminated using that $\frac{1}{4} \text{Tr}[W^k_B Y_B] \geq \alpha \frac{1}{2} \text{Tr}[W^k_B Y_B]$.

In order to obtain the upper bound (8) on $D(\rho_E, \rho'_E)$, we also derive a tradeoff between the quantity $\frac{1}{2} \text{Tr}[W^k_B Y_B]$, appearing in (18), and the distinguishability of Eve’s states. Specifically, we prove in Appendix B that, for any Hermitian unitary $U_E$ acting on $\mathcal{H}_E$, the inequality
\[ \frac{1}{4} \text{Tr}[W^k_B Y_B]^2 + \frac{1}{2} \text{Tr}[(I^k_B \otimes U_E)Z]^2 \leq p_k^2 \]
holds in each subspace $k$.

We obtain (8) by taking for $U_E$ in (19) a Hermitian unitary such that $\frac{1}{2} \text{Tr}[U_E Y_E] = \frac{1}{2} \|Y_E\|_1$ [20]. Because $D(\rho_E, \rho'_E) = \frac{1}{2} \|Z_E\|_1 \leq \frac{1}{2} \|Z_E\|_1$, the trace distance is upper bounded by
\[ D(\rho_E, \rho'_E) \leq \sum_k \frac{1}{2} \text{Tr}[(I^k_B \otimes U_E)Z] . \]

Using (18) and (19) and omitting $\alpha$, we have
\[ \frac{1}{2} \text{Tr}[(I^k_B \otimes U_E)Z] \leq p_k \sqrt{2 - (S_k/p_k)^2 / 4}, \]
and substituting (21) into (20) and using that the function $S \mapsto \sqrt{2 - S^2 / 4}$ is concave, we finally obtain
\begin{align}
D(\rho_E, \rho'_E) &\leq \sum_k \frac{1}{2} \text{Tr}[(I^k_B \otimes U_E)Z] \\
&\leq \sum_k p_k \sqrt{2 - (S_k/p_k)^2 / 4} \\
&\leq \sqrt{2 - (\sum_k S_k)^2 / 4} \\
&= \sqrt{2 - S^2 / 4}. \quad (22)
\end{align}
Combining with the expression (7) for the min entropy, we obtain (2).

As with its EB counterpart, (8) and the resulting min-entropy bound are tight and are attained with a PM version of the optimal attack originally given in [3]; for completeness we have included a description of this attack in Appendix C.

**Discussion of the qubit assumption.** — Having proven our main result, let us now discuss the qubit source assumption in more detail. Note first that Alice’s “preparation” device may not in general actually prepare a new state from scratch, but instead implement a transformation on a preexisting qubit stored in her box, which
could be entangled with Eve’s system prior to the protocol. The presence of such prior entanglement between Alice’s device and Eve may completely break the security of a PM scheme, as noted in [11]. However, since our qubit assumption is formulated in the total space $\mathcal{H}_B \otimes \mathcal{H}_E$ including Eve’s Hilbert space, it naturally limits the amount of potential prior entanglement between Alice and Eve (or Alice and Bob) and thus a nice mathematical feature of our formulation is that we do not need to state this limitation on prior entanglement as a separate, additional assumption.

On the other hand, since our qubit assumption is formulated in the space $\mathcal{H}_B \otimes \mathcal{H}_E$ after Eve’s attack, it may not be possible to practically verify this assumption in a cryptographic setting (since Alice and Bob do not have access to Eve’s system). Note, however, that a sufficient condition for our assumption to be satisfied is that (i) there exists no prior entanglement between Alice and Eve or Bob (e.g., Alice’s preparation box has no quantum memory), and (ii) the states sent by Alice’s box, before going through the channel and suffering Eve’s (without loss of generality, unitary) attack, are such that $\rho - \rho'$ and $\sigma - \sigma'$ have support in the same two-dimensional subspace. Under these conditions, the states $\rho - \rho'$ and $\sigma - \sigma'$ after Eve’s unitary attack will still share the same two-dimensional support and thus our qubit source assumption will be satisfied. However, the condition that we use to derive the min-entropy bound (2) is formally weaker than the combination of (i) and (ii) as these only represent sufficient conditions for our assumption to be satisfied.

Another nice feature of our formulation is that the qubit assumption refers only to the differences $\rho - \rho'$ and $\sigma - \sigma'$ and not directly to the states $\rho_{x,a}$ themselves, which may live in a higher dimensional Hilbert space. For instance, in an optical implementation, each “qubit” may be a qubit encoded in the polarisation degree of freedom of a single photon, but may also possess a vacuum component and thus formally be a three-level system of the form $\rho_{x,a} = p|0\rangle\langle 0| + (1-p)\tilde{\rho}_{a,x}$, where $\tilde{\rho}_{a,x}$ is the one-photon polarised qubit part. Still, the differences $\rho - \rho'$ and $\sigma - \sigma'$ only involve the genuine qubit parts and thus satisfy our qubit source assumption.

Finally, let us remark that our assumption can immediately be weakened in two ways. First, using convexity arguments, it is easy to see that the min-entropy bound (2) still holds if Alice’s, Bob’s, and Eve’s systems share prior classical randomness, provided that for any value $\lambda$ of the shared randomness, the differences $\rho_{x} - \rho'_{x}$ and $\sigma_{x} - \sigma'_{x}$ satisfy the qubit assumption. Again, it may not be possible to practically verify this assumption in the most general DI setting (as Alice and Bob will not have access to the individual values of the shared randomness if their devices are uncharacterised). However, a sufficient condition for this assumption to be satisfied is if each of the averaged states $\rho_{x,a} = \sum_{\lambda} \rho_{x,a,\lambda}$ are contained in the same qubit space, a condition which does not require any knowledge of the shared randomness.

Second, we point out that the bound on the min entropy is also robust with respect to the qubit assumption; i.e., this assumption need only be approximately verified. Specifically, suppose that, instead of assuming (9) and (10), we assume that there exist traceless two-dimensional unit operators $\alpha \tilde{Z}$ and $\beta \tilde{X}$ such that $\frac{1}{4} ||(\rho - \rho') - \alpha \tilde{Z}||_1 \leq \epsilon$ and $\frac{1}{4} ||(\sigma - \sigma') - \beta \tilde{X}||_1 \leq \epsilon$. Then it is easy to see that $D(\rho_{E}, \rho_{E}^0) \leq \frac{1}{2} ||\alpha \tilde{Z}E||_1 + \epsilon$ and that the CHSH expectation value computed with $\alpha \tilde{Z}$ and $\beta \tilde{X}$ cannot differ from $S$ by more than $4\epsilon$. Small deviations from the qubit source assumption can thus be tolerated, with a bound on the min entropy no worse than

$$H_{\min}(A | E) \geq 1 - \log_2(1 + \sqrt{2 - (S - 4\epsilon)^2 / 4}) + \epsilon. \quad (23)$$

**Conclusion.** — We gave a proof that the fundamental lower bound (2) on the randomness of Alice’s outcomes as a function of the CHSH expression, originally derived in the context of device-independent QKD and randomness certification, still holds in a PM setting with a qubit assumption. Though the equivalence between EB and PM schemes in standard QKD may a priori suggest that this should naturally be the case, this is not at all immediate as this equivalence breaks in a DI setting. Indeed, the techniques that we have used here to establish the lower bound (2) in the PM setting are quite different from the ones used to establish the EB version of this bound.

This fundamental lower bound (2) can now, in principle, be used as a building block to prove the security of semi-DI QKD protocols, in the same way that it was used in the fully DI setting in Refs. [6–8].

We remark that in the EB scenario, an analogous tight bound for the Holevo quantity (or, equivalently, the conditional von Neumann entropy) instead of the min entropy had earlier been presented in [3] as part of a security proof against collective attacks. The conditional von Neumann entropy can likewise, in principle, be bounded in the PM scenario. A partial result for the von Neumann entropy, restricted to the case where Bob’s measurements are additionally assumed to be two-dimensional, is given in Ref. [21].

Finally, having shown that our min-entropy bound holds for Alice’s system conditioned on Eve, it would be interesting to investigate whether a similar result holds for the min entropy $H_{\min}(B | E)$ associated with Bob’s measurement outcome. In particular, a version of this result conditioned on just one of Alice’s state preparations would apply immediately to the problem of randomness certification [5, 22–24], which has similarly been investigated in PM scenarios [25, 26].

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Appendix A: Proof of (18)

As explained in the main text, after applying the Jordan lemma the optimal expectation value (11) of $S$ can be expressed as $S = \sum_k S_k$, with

\[
S_k = \frac{1}{2} \alpha \text{Tr}[(U_k + V_k)Z] + \frac{1}{2} \beta \text{Tr}[(U_k - V_k)X],
\]

where we have set $U_k = U_k^B \otimes I_E$ and $V_k = V_k^B \otimes I_E$, and $U_k^B$ and $V_k^B$ are the Hermitian unitary operators of dimension at most 2 appearing in (12) and (13).

Inserting the expressions (15) and (16) for the operators $Z$ and $X$, $S_k$ can be expressed as

\[
S_k = \alpha \text{Re}\left[(\gamma|e^{-i\gamma^\frac{\pi}{2}}(U_k + V_k)|\gamma')\right] + \beta \text{Re}\left[(\gamma|e^{i\gamma^\frac{\pi}{2}}(U_k - V_k)|\gamma')\right]
\]

in terms of the $y$ states. The only interesting case is where both $U_k^B$ and $V_k^B$ are two-dimensional and of eigenvalues $+1$ and $-1$. In this case, if $U_k^B$ and $V_k^B$ are separated by an angle $\gamma_k$ on the Bloch sphere, one can choose an orthonormal basis \{$(w_k^B, w_k^B')$\} in which

\[
U_k^B + V_k^B = 2 \cos\left(\frac{\gamma_k}{2}\right) (w_k^B)\langle w_k^B' | + (w_k^B')\langle w_k^B |)
\]

\[
U_k^B - V_k^B = 2 \sin\left(\frac{\gamma_k}{2}\right) (i|w_k^B\langle w_k^B' | - |w_k^B'\rangle w_k^B)\]

Inserting this into (A2), the expression for $S_k$ can be simplified to

\[
S_k/2 = \text{Re}\left[(\gamma| (\lambda_k|w_k^B\langle w_k^B' | + \mu_k|w_k^B'\rangle w_k^B) \otimes I_E)|\gamma')\right]
\]

where we have collected the various angles into

\[
\lambda_k = \cos\left(\frac{\gamma_k}{2}\right) e^{i\frac{\gamma_k}{2}} + i\beta \sin\left(\frac{\gamma_k}{2}\right) e^{i\frac{\gamma_k}{2}},
\]

\[
\mu_k = \cos\left(\frac{\gamma_k}{2}\right) e^{i\frac{\gamma_k}{2}} - i\beta \sin\left(\frac{\gamma_k}{2}\right) e^{i\frac{\gamma_k}{2}}
\]

for convenience. Introducing new vectors

\[
|A_k\rangle = (w_k^B \otimes I_E)|\gamma',
\]

\[
|A'_k\rangle = (w_k^B' \otimes I_E)|\gamma',
\]

\[
|B_k\rangle = (w_k^B \otimes I_E)|\gamma',
\]

\[
|B'_k\rangle = (w_k^B' \otimes I_E)|\gamma'
\]

in $\mathcal{H}_E$ in order to further simplify the notation, we obtain

\[
S_k/2 = \text{Re}\left[\lambda_k \langle A_k | A'_k \rangle + \mu_k \langle B_k | B'_k \rangle\right].
\]

Finally, in order to reexpress $S_k$ in a form better suited for determining an upper bound, we introduce new coefficients

\[
\xi_k = \cos\left(\frac{\gamma_k}{2}\right) + i \sin\left(\frac{\gamma_k}{2}\right),
\]

\[
\nu_k = \cos\left(\frac{\gamma_k}{2}\right) - i \sin\left(\frac{\gamma_k}{2}\right),
\]

and

\[
\gamma_k = \frac{1}{2} (\alpha \pm \beta),
\]

such that

\[
\lambda_k = \gamma_k + \xi_k + \gamma_k - \nu_k,
\]

\[
\mu_k = \gamma_k + \nu_k + \gamma_k - \xi_k.
\]

Inserting these into the expression for $S_k$, we arrive at

\[
S_k/2 = \text{Re}\left[\xi_k \langle A_k | A'_k \rangle + \nu_k \langle B_k | B'_k \rangle\right] - \lambda_k \langle A_k | A'_k \rangle - \mu_k \langle B_k | B'_k \rangle.
\]

In order to obtain a useful upper bound on $S_k$, we begin by taking the absolute value of the various terms in (A17), obtaining

\[
S_k/2 \leq |\xi_k| |\langle A_k | A'_k \rangle| + |\nu_k| |\langle B_k | B'_k \rangle| + |\lambda_k| |\langle A_k | A'_k \rangle| + |\mu_k| |\langle B_k | B'_k \rangle|.
\]

Applying the Cauchy-Schwarz inequality, using that $|\xi_k|^2 + |\nu_k|^2 = 2$, and developing,

\[
S_k/2 \leq 2 (|\langle A_k | A'_k \rangle|^2 + |\langle B_k | B'_k \rangle|^2)
\]

\[
+ 2 (|\langle A_k | A'_k \rangle| + |\langle B_k | B'_k \rangle|)^2
\]

\[
= 2 (\lambda_k^2 + \gamma_k^2) (|\langle A_k | A'_k \rangle|^2 + |\langle B_k | B'_k \rangle|^2)
\]

\[
+ 8 |\lambda_k + \gamma_k - |\langle A_k | A'_k \rangle| - |\langle B_k | B'_k \rangle|
\]

\[
\leq 2 (\lambda_k^2 + \gamma_k^2) (|\langle A_k | A'_k \rangle|^2 + |\langle B_k | B'_k \rangle|^2)
\]

\[
+ 8 |\lambda_k + \gamma_k - |\langle A_k | A'_k \rangle| - |\langle B_k | B'_k \rangle|
\]

where we used the Cauchy-Schwarz inequality again to substitute $|\langle A_k | A'_k \rangle| \leq \|A_k\|\|A'_k\|$ and $|\langle B_k | B'_k \rangle| \leq \|B_k\|\|B'_k\|$. Applying now that

\[
2 \|A_k\|^2 \leq \Sigma_A = \|A_k\|^2 + \|A'_k\|^2,
\]

\[
2 \|B_k\|^2 \leq \Sigma_B = \|B_k\|^2 + \|B'_k\|^2,
\]

we find that

\[
S_k/2 \leq \frac{1}{2} (\lambda_k^2 + \gamma_k^2) (\Sigma_A^2 + \Sigma_B^2) + 2 |\lambda_k + \gamma_k - \Sigma_A| |\Sigma_B|
\]

\[
= \min(\alpha^2, \beta^2) \frac{1}{2} (\Sigma_A^2 - \Sigma_B^2)
\]

\[
+ \max(\alpha^2, \beta^2) \frac{1}{2} (\Sigma_A + \Sigma_B)^2.
\]

Reinserting the definitions of the vectors $|A_k\rangle$, $|A'_k\rangle$, $|B_k\rangle$, and $|B'_k\rangle$, note that

\[
\Sigma_A - \Sigma_B = \text{Tr}[W_B^B Y_B],
\]

\[
\Sigma_A + \Sigma_B = \text{Tr}[I_B^B Z_B],
\]

\[
\Sigma_A - \Sigma_B = \text{Tr}[W_B^B Y_B],
\]

\[
\Sigma_A + \Sigma_B = \text{Tr}[I_B^B Z_B],
\]
where we recall that we defined
\begin{align}
I &= |y\rangle\langle y| + |y'\rangle\langle y'|, \\
Y &= |y\rangle\langle y| - |y'\rangle\langle y'|,
\end{align}
(A25)

\[ I_B = \text{Tr}[Z], \quad Y_B = \text{Tr}[Y], \]
and we have introduced
\begin{align}
I_B^w &= |w_k\rangle\langle w_k|_B + |w'_k\rangle\langle w'_k|_B, \\
Y_B^w &= |w_k\rangle\langle w_k|_B - |w'_k\rangle\langle w'_k|_B.
\end{align}
(A27)

In this way, we find that \( S_k \) is upper bounded by
\begin{align}
S_k^2 / 4 &\leq \min(\alpha^2, \beta^2) \frac{1}{2} \text{Tr}[W_B^w Y_B^w]^2 \\
&\quad + \max(\alpha^2, \beta^2) \frac{1}{4} \text{Tr}[I_B^w Z_B]^2.
\end{align}
(A29)

Finally, we substitute \( \min(\alpha^2, \beta^2) \leq \alpha^2, \max(\alpha^2, \beta^2) \leq 1 \), and \( p_k = \frac{1}{2} \text{Tr}[I_B^w Z_B] \) in order to obtain
\begin{align}
S_k^2 / 4 &\leq \alpha^2 \frac{1}{2} \text{Tr}[W_B^w Y_B^w]^2 + p_k^2,
\end{align}
(A30)

which rearranges to (18) in the main text. (If \( \frac{1}{2} \text{Tr}[W_B^w Y_B^w] \) is negative, we simply replace \( W_B^w \rightleftharpoons -W_B^w \).)

**Appendix B: Proof of (19)**

We start with the term \( \frac{1}{2} \text{Tr}[W_B^w Y_B^w] \). In an appropriate phase convention, the operator \( Y \) can be expressed as
\[ Y = e^{-i\frac{E}{2}} |z\rangle\langle z'| + e^{i\frac{E}{2}} |z'|\langle z|, \]
(B1)

where \( |z\rangle \) and \( |z'| \) are the eigenstates of \( Z \) such that \( |z\rangle = |z\rangle - |z'\rangle \). In terms of these \( z \) states,
\begin{align}
\frac{1}{2} \text{Tr}[W_B^w Y_B^w] &= \frac{1}{2} \text{Tr}[(W_B^w \otimes I_E)Y] \\
&= \text{Re}[e^{-i\frac{E}{2}} \langle z|W_B^w \otimes I_E|z'|] \\
&\leq \left| \langle z|W_B^w \otimes I_E|z'| \right|.
\end{align}
(B2)

We let \( U_E \) be any Hermitian unitary operator acting on \( H_E \). Such an operator can always be expressed as the difference between two orthogonal projectors, which we call \( P_E \) and \( Q_E \), such that \( U_E = P_E - Q_E \). Inserting \( I_E = P_E + Q_E \) into the last line of (B2) and developing, we obtain
\begin{align}
\frac{1}{2} \text{Tr}[W_B^w Y_B^w] &\leq \left| \langle z|W_B^w \otimes P_E|z'\rangle \right| + \left| \langle z|W_B^w \otimes Q_E|z'\rangle \right| \\
&\leq \sqrt{\langle z|I_B^w \otimes P_E|z\rangle \langle z'|I_B^w \otimes P_E|z'\rangle} \\
&\quad + \sqrt{\langle z|I_B^w \otimes Q_E|z\rangle \langle z'|I_B^w \otimes Q_E|z'\rangle} \\
&\leq \sqrt{\langle z|I_B^w \otimes P_E|z\rangle + \langle z'|I_B^w \otimes Q_E|z'\rangle} \times \sqrt{\langle z'|I_B^w \otimes P_E|z'\rangle + \langle z|I_B^w \otimes Q_E|z\rangle},
\end{align}
(B3)

in which we used the Cauchy-Schwarz inequality and that \( (W_B^w)^2 = I_B^w \) to obtain the second and third lines. Substituting \( P_E = \frac{1}{2}(I_E + U_E) \) and \( Q_E = \frac{1}{2}(I_E - U_E) \),
\begin{align}
\frac{1}{2} \text{Tr}[W_B^w Y_B^w] &\leq \sqrt{p_k + \frac{1}{2} \text{Tr}[(I_B^w \otimes U_E)Z]} \\
&\quad \times \sqrt{p_k - \frac{1}{2} \text{Tr}[(I_B^w \otimes U_E)Z]} \\
&= \sqrt{p_k^2 - \frac{1}{4} \text{Tr}[(I_B^w \otimes U_E)Z]^2},
\end{align}
(B4)

where we recovered \( I = |z\rangle\langle z| + |z'\rangle\langle z'| \), \( Z = |z\rangle\langle z| - |z'\rangle\langle z'| \), and \( p_k = \frac{1}{2} \text{Tr}[I_B^w Z_B] \). The end result rearranges to (19) in the main text.

**Appendix C: Tightness of the min-entropy bound**

The lower bound (2) on the conditional min-entropy is tight and is attained with a PM version of the optimal collective attack given in [3], which we describe here. We set the states \( \rho_{\sigma,a} \) (following Eve’s attack) to \( \rho = |\alpha\rangle\langle \alpha| \) and \( \rho' = |\alpha'\rangle\langle \alpha'| \), where
\[ |\alpha\rangle = |0\rangle_B|\psi\rangle_E, \quad |\alpha'\rangle = |1\rangle_B|\psi'\rangle_E, \]
(C1)

in which \( |0\rangle_B \) and \( |1\rangle_B \) are orthonormal and \( |\psi\rangle_E \) and \( |\psi'\rangle_E \) are normalised states whose inner product defines the specific attack. We set \( |\psi\rangle|\psi'\rangle = F_z \) for some real constant \( 0 \leq F_z \leq 1 \). We also set \( \sigma = |\beta\rangle\langle \beta| \) and \( \sigma' = |\beta'\rangle\langle \beta'| \) with
\begin{align}
|\beta\rangle &= \frac{1}{\sqrt{2}}(|\alpha\rangle + |\alpha'\rangle), \\
|\beta'\rangle &= \frac{1}{\sqrt{2}}(|\alpha\rangle - |\alpha'\rangle).
\end{align}
(C2)

(C3)

With these definitions, the source states span a qubit subspace. Note that \( \langle \alpha|\alpha'\rangle = \langle \beta|\beta'\rangle = 0 \), such that \( \alpha = \beta = 1 \).

From the above definitions, we have that \( \rho_E = |\psi\rangle\langle \psi|_E \) and \( \rho'_E = |\psi'\rangle\langle \psi'|_E \), and thus
\[ D(\rho_E, \rho'_E) = \sqrt{1 - F_z^2}. \]
(C4)

For the operators \( Z = |\alpha\rangle\langle -\alpha| + |\alpha'\rangle\langle \alpha'| \) and \( X = |\beta\rangle\langle \beta| - |\beta'\rangle\langle \beta'\rangle = |\alpha\rangle\langle \alpha| + |\alpha'\rangle\langle \alpha'| \), we find the partial traces
\[ Z_B = |0\rangle\langle 0|_B - |1\rangle\langle 1|_B = \sigma_z \]
(C5)

and
\[ X_B = F_z(|0\rangle\langle 1|_B + |1\rangle\langle 0|_B) = F_x \sigma_x. \]
(C6)

For optimal measurements on Bob’s side, we can write (11) as
\begin{align}
S &= \frac{1}{2} \|Z_B + X_B\|_1 + \frac{1}{2} \|Z_B - X_B\|_1 \\
&= \frac{1}{2} \|\sigma_z + F_x \sigma_x\|_1 + \frac{1}{2} \|\sigma_z - F_x \sigma_x\|_1 \\
&= 2\sqrt{1 + F_x^2},
\end{align}
(C7)

which rearranges to
\[ F_x = \sqrt{S^2 / 4 - 1}. \]
(C8)

Combining with (C4) confirms that we have described a family of attacks for which \( D(\rho_E, \rho'_E) = \sqrt{2 - S^2 / 4} \).
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