FINITE PERMUTATION GROUPS CONTAINING A REGULAR DIHEDRAL SUBGROUP

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Abstract. A characterization is given of finite permutation groups which contain a regular dihedral subgroup.

1. Introduction

A permutation group $G \leq \text{Sym}(\Omega)$ is called a $c$-group or a $d$-group if it contains a regular subgroup which is cyclic or dihedral, respectively. The study of these two classes of permutation groups has a long history, dating back to Burnside [2] who proved that a primitive c-group of composite prime-power degree is necessarily 2-transitive. Schur (1933, see [23]) completed Burnside’s work by proving that a primitive c-group of composite degree is 2-transitive. Based on the Classification of Finite Simple Groups, primitive simple c-groups were listed in [8, 9, 11]. More recently, Jones [10] and Li [14] completely determined primitive permutation c-groups, independently. Then in 2010, Li and Praeger [17] studied general c-groups.

Wielandt proved that primitive d-groups are 2-transitive, refer to [23], extended Schur’s result regarding c-groups to d-groups. Then in [16, 21, 22], quasiprimitive d-groups and bi-quasiprimitive d-groups are classified. The main theorem of this paper gives a recursive characterization of $d$-groups.

The recent research of c-groups and d-groups is often associated with the study of symmetrical graphs (directed or undirected). A circulant is a Cayley graph of a cyclic group, and a dihedrant is a Cayley graph of a dihedral group. An arc of a graph is an ordered pair of adjacent vertices, and a 2-arc is a triple of vertices one of which is adjacent to the other two. A graph is called arc-transitive or 2-arc-transitive if all of its arcs or 2-arcs, respectively, are equivalent under automorphisms. For arc-transitive circulants, recursive characterizations are obtained in [12, 15], and an explicit classification is given in [18]. For dihedrants, special classes of arc-transitive dihedrants are characterized in [5, 13, 19, 20]. However, the problem of classifying arc-transitive dihedrants is still an unsettled open problem. Theorem 1.1, we believe, would play an important role in the study of arc-transitive dihedrants.

To state the theorem, we need introduce a few concepts regarding permutation groups. Let $G$ be a transitive permutation group on a set $\Omega$. For a block system $\mathcal{B}$ and a block $B \in \mathcal{B}$, the group $G$ induces a transitive permutation group $G^B_\mathcal{B}$ on $\mathcal{B}$, and the stabilizer $G_B$ induces a transitive group $G_B^B$ on $B$. Then $G^B_\mathcal{B} \cong G/G_{(B)}$, and $G_B^B \cong G_B/G_{(B)}$, where $G_{(B)}$ is the kernel of $G$ on $B$, and $G_{(B)}$ fixes $B$ pointwise. A block system $\mathcal{B}$ is called maximal if the induced action $G^\mathcal{B}$ is primitive, and called minimal if, for a block $B \in \mathcal{B}$, the induced action $G_B^B$ is primitive.
An orbital graph of $G$ is a (di)graph with vertex set $\Omega$ and edge set $(\omega_1, \omega_2)^G$, where $\omega_1, \omega_2 \in \Omega$. An orbital graph is obviously arc-transitive. For a block system $\mathcal{B}$, an orbital graph $\Gamma$ has a quotient graph $\Gamma_{\mathcal{B}}$, which is an orbital graph of the induced permutation group $G^\mathcal{B}$. As usual, $K_n$ denotes a complete graph of order $n$, and $\overline{K}_n$ denotes the complement. For a graph $\Sigma$, denote by $\Sigma[\overline{K}_n]$ the lexicographic product of $\overline{K}_n$ by $\Sigma$.

Since primitive d-groups are 2-transitive and completely known by Theorem 2.5, the main result of this paper focuses on the imprimitive case.

**Theorem 1.1.** Let $G$ be an imprimitive d-group of degree $n$ and $\mathcal{B}$ a minimal block system, and let $K = G_{\{B\}}$ and $B \in \mathcal{B}$. Then each connected orbital graph is undirected, $G^\mathcal{B}$ is a c-group or a d-group, and further, one of the following holds:

1. $G \cong G^\mathcal{B}$, and $G$ is also a c-group of degree $n/2$, where $|B| = \frac{1}{2}|\Omega|$;
2. there exists an orbital graph for $G$ of the form $\Gamma_{\mathcal{B}}[\overline{K}_b]$, $K^\mathcal{B}$ is primitive and $K \not\cong K^\mathcal{B}$, where $b = |B|$ and $\Gamma_{\mathcal{B}}$ is an orbital graph of $G^\mathcal{B}$;
3. $G = (K \times H).\mathcal{O}$, where $K \cong K^\mathcal{B}$ is primitive, either of $K.\mathcal{O}$ and $H.\mathcal{O}$ is a c-group or a d-group, and $\mathcal{O} \leq \text{Out}(K)$;
4. $Z_2 = K = Z(G)$, and $G^\mathcal{B} \cong G/Z(G)$ is a d-group of degree $n/2$, and $K.G^\mathcal{B}$ is a non-split extension;
5. $K = \mathbb{Z}_p$ with $p$ being an odd prime, and $G = C_G(K).\mathbb{Z}_{2\ell}$, where $2\ell \mid (p - 1)$ and $\ell$ is odd, and $K.(C_G(K))^\mathcal{B}$ is a non-split extension;
6. $|B| = 4$, $R \cap K = \mathbb{Z}_2$, and $G = 2^d.G^\mathcal{B}$, where $G^\mathcal{B}$ is a c-group of degree $n/4$.

In subsequent work, it will be shown that an arc-transitive dihedrant either can be explicitly reconstructed from a smaller circulant or dihedrant, or is an orbital graph of a group in part (6).

**Remarks on Theorem 1.1:**

(a) For the case in Theorem 1.1 (1), the d-group $G$ of degree $n$ has a faithful permutation representation to be a c-group of degree $n/2$, which is better understood than the general case.
(b) In the viewpoint of arc-transitive dihedrants, the case in Theorem 1.1 (2) is completely reduced to smaller dihedrants.
(c) In case (3), as $\mathcal{O}$ is relatively very small, $G$ is essentially determined by a direct product of smaller c-groups or d-groups.
(d) Due to the classification of primitive c-groups and d-groups, the case in Theorem 1.1 (4) is completely reduced to a smaller case.
(e) The case in Theorem 1.1 (3) is reduced to c-groups.
(f) The case in Theorem 1.1 (5) is not well-understood as other cases, which seems challenge to obtain a nice description for this case.

The notation used in this paper is standard, see [1, 4] for example.

### 2. Preliminaries

We begin with a technical lemma for general permutation groups.

**Lemma 2.1.** Let $G$ be a transitive permutation group on $\Omega$ which has a connected orbital graph $\Gamma$. Let $\mathcal{B}$ be a minimal block system for $G$ on $\Omega$, $B \in \mathcal{B}$, and let $K = G_{\{B\}}$. Then one of the following holds:

1. $G \cong G^\mathcal{B}$,
(ii) $K \cong K^B$ is a transitive permutation group,
(iii) $\Gamma = \Gamma_B[\mathbf{K}_b]$, where $b = |B|$.

Proof. If $K = 1$, then $G^B \cong G/K \cong G$, as in part (i).

Assume that $K \neq 1$. Then $K^B \neq 1$ since the actions of $K$ on all of the blocks in $B$ are equivalent. Since $K^B \triangleleft G^B$ and $G^B$ is primitive, $K^B$ is transitive. If $K \cong K^B$, then $K$ has a faithful transitive representation on $B$, as in part (ii).

Finally, assume that $K \not\cong K^B$. Then the kernel $K(B) \neq 1$, and $K(B)$ is non-trivial on some block $B' \in B$. Since $\Gamma$ is connected, there is a path $\alpha_0, \alpha_1, \ldots, \alpha_{\ell}$ of $\Gamma$, where $\alpha_0 \in B$, $\alpha_i \in B_i$, and $\alpha_{\ell} \in B'$, and there exists $1 \leq k \leq \ell$ such that $K(B)$ is trivial on $B_i$ for $i \leq k - 1$ and non-trivial on $B_k$. Then $K(B_{k-1}) = K(B)$, and $1 \neq K(B) = K(B_{k-1}) < K^B_k$.

Since $K^B_k$ is primitive, $K^B_k(B)$ is transitive, and hence the induced subgraph on $[B_{k-1}, B_k]$ is isomorphic to $K^B_k(b, b)$ where $b = |B_{k-1}| = |B_k| = |B|$. Since $\Gamma$ is $G$-arc-transitive, it follows that $\Gamma = \Gamma_B[\mathbf{K}_b]$. \qed

In general, an orbital graph is not necessarily undirected. However, the next lemma shows that connected orbital graphs of a d-group are always undirected.

**Lemma 2.2.** Let $G$ be a d-group on $\Omega$, and $\Gamma$ an orbital graph of $G$. Then either

(i) $\Gamma$ is an undirected arc-transitive graph, or
(ii) $\Gamma$ is disconnected, and each component is a circulant.

Furthermore, is $\Gamma$ is connected, then $\Gamma$ is undirected.

Proof. Let $G$ be a d-group on $\Omega$ and $\Gamma$ a connected orbital graph of $G$. Then $G$ has a dihedral regular subgroup $D$, and $\Gamma$ is a Cayley graph of $D$, namely, $\Gamma = \text{Cay}(D, S)$ for some subset $S \subset D$. Let $D = \langle a \rangle : \langle z \rangle$, where $|z| = 2$ and $a^z = a^{-1}$.

Assume first that $\langle S \rangle \leq \langle a \rangle$. Then $\Gamma$ is disconnected, and each component is a Cayley graph of a subgroup of $\langle a \rangle$, and so it is a circulant, as in part (ii).

Now assume that $\langle S \rangle$ is not a subgroup of $\langle a \rangle$. Then $S$ contains an involution $g$ in $D \setminus \langle a \rangle$. The permutation of $\Omega$ induced by the right multiplication of $g$ is an element of $G$, and interchanges the arcs $(1_D, g)$ and $(g, 1_D)$, where $1_D$ is the identity of the group $D$. It follows that $\Gamma = \text{Cay}(D, S)$ is undirected. \qed

Next, we study d-groups. By the well-known Schur’s theorem ([23, Theorem 25.3]) and Wielandt’s theorem ([23, Theorem 25.6]), we have the following result.

**Theorem 2.3.** (Schur, Wielandt) If $G$ be a finite primitive c-group or d-group on a set $\Omega$, then either

(1) $|\Omega| = p$ with $p$ prime and $G \leq \text{AGL}(1, p)$, or
(2) $G$ is 2-transitive.

A transitive permutation group is called quasiprimitive if each of its non-trivial normal subgroup is transitive. Clearly, a primitive permutation group is quasiprimitive, but a quasiprimitive group is not necessarily primitive. However, for c-groups and d-groups, the quasiprimitivity and the primitivity are equivalent, and moreover, quasiprimitive c-groups and d-groups are classified, stated in the next two theorems. We remark that, although the proof of Theorem 2.3, consisting of both Schur’s work and Wielandt’s work, is ‘elementary’ and elegant, the following two theorems indeed depend on the Classification of Finite Simple Groups.
Theorem 2.4. (see [10, 14]) Let $X \leq \text{Sym}(\Omega)$ be a quasiprimitive $c$-group of degree $n$. Then $X$ is primitive on $\Omega$, and either $n = p$ is prime and $X \leq \text{AGL}(1, p)$, or $X$ is 2-transitive, listed in the following table.

| $X$     | $X_\omega$ | $n$ | condition                        |
|---------|------------|-----|----------------------------------|
| $A_n$   | $A_{n-1}$  | $n$ | is odd, $n \geq 4$               |
| $S_n$   | $S_{n-1}$  | $n$ | $\geq 4$                         |
| $\text{PGL}(d, q).o$ | $P_1(\text{parabolic})$ | $\frac{q^d-1}{q-1}$ | $o \leq \text{PGL}(d, q)/\text{PGL}(d, q)$ |
| $\text{PSL}(2, 11)$ | $A_5$ | $11$ |                                  |
| $M_{11}$ | $M_{10}$ | $11$ |                                  |
| $M_{23}$ | $M_{22}$ | $23$ |                                  |

Theorem 2.5. If $X \leq \text{Sym}(\Omega)$ is a quasiprimitive $d$-group with a regular dihedral subgroup $G$, then $X$ is 2-transitive on $\Omega$ and $(X, G, X_\omega)$ is one of the triples in the following table.

| $X$     | $G$         | $X_\omega$ | conditions                |
|---------|-------------|------------|---------------------------|
| $A_4$   | $D_4$       | $\mathbb{Z}_3$ |                                 |
| $S_4$   | $D_4$       | $S_3$      |                                 |
| $\text{AGL}(3, 2)$ | $D_8$ | GL$(3, 2)$ |                                 |
| $\text{AGL}(4, 2)$ | $D_{16}$ | GL$(4, 2)$ |                                 |
| $2^4 \times A_7$ | $D_{16}$ | $A_7$      |                                 |
| $2^4 \times S_6$ | $D_{16}$ | $S_6$      |                                 |
| $2^4 \times A_6$ | $D_{16}$ | $A_6$      |                                 |
| $2^4 \times S_5$ | $D_{16}$ | $S_5$      |                                 |
| $2^4 \times \Gamma L(2, 4)$ | $D_{16}$ | GL$(2, 4)$ |                                 |
| $M_{12}$ | $D_{12}$   | $M_{11}$   |                                 |
| $M_{22}.2$ | $D_{22}$ | PSL$(3, 4).2$ |                                 |
| $M_{24}$ | $D_{24}$   | $M_{23}$   |                                 |
| $S_{2m}$ | $D_{2m}$   | $S_{2m-1}$ |                                 |
| $A_{4m}$ | $D_{4m}$   | $A_{4m-1}$ |                                 |
| $\text{PSL}(2, r^f).o$ | $D_{r^f+1}$ | $\mathbb{Z}_f \times \mathbb{Z}_{r^f-1}.o$ | $r^f \equiv 3 \pmod{4}, o \leq \mathbb{Z}_2 \times \mathbb{Z}_f$ |
| $\text{PGL}(2, r^f).\mathbb{Z}_e$ | $D_{r^f+1}$ | $\mathbb{Z}_f \times \mathbb{Z}_{r^f-1} \times \mathbb{Z}_e$ | $r^f \equiv 1 \pmod{4}, e \mid f$ |

We now present a method for constructing d-groups by wreath product.

Construction 2.6. Let $H$ be a c-group on $\Delta$ of degree $k$, and label $\Omega = \{1, 2, \ldots, n\}$ ($n$ is even) , and $G$ contains a regular dihedral group on $\Omega$, say $D = D_{2n}$. Let

$$X = H \wr G = H^n \times G = (H_1 \times H_2 \times \cdots \times H_n) \rtimes G.$$ 

Let $\Sigma = \{ (\delta_1, \delta_2, \ldots, \delta_n) \mid \delta_i \in \Delta \}$, and let

$$L = ((H_1)_{\delta_1} \times H_2 \times \cdots \times H_n) \rtimes D.$$ 

Then $L$ is a stabilizer of the group $X$ acting on the set $\Sigma$.

Lemma 2.7. A group $X$ constructed in Construction 2.6 is a d-group on $\Sigma$ of degree $kn$. 
Proof. Let $n = 2m$. Without loss of generality, write $D = \langle a \rangle \times \langle b \rangle$, where $a = (12 \ldots m)(m + 1, m + 2, \ldots, 2m)$ and $b = (1, 2m)(2, 2m - 1) \ldots (m, m + 1)$. Let $x = (h_1, \ldots, h_n)$ such that $h_1 = 1$, $h_{m+1} = -1$, and other $h_i$ equal 0, and let
\[ y = xa = (1, 0, \ldots, 0, -1, 0, \ldots, 0)(12 \ldots m)(m + 1, m + 2, \ldots, 2m). \]
Then $y^m = (1, \ldots, 1, -1, \ldots, -1)$, and $|y| = km$. Furthermore,
\[ y^b = (0, \ldots, 0, -1, 0, \ldots, 0, 1)(2m, \ldots, m + 1)(m \ldots 1), \]
and
\[ y^{-1} = (xa)^{-1} = a^{-1}x^{-1} = (a^{-1}x^{-1}a)a^{-1} = (0, \ldots, 0, -1, 0, \ldots, 0, 1)(2m, \ldots, m + 1)(m \ldots 1) = y^b. \]
Thus $y^b = y^{-1}$, and $\langle y, b \rangle = D_{2kn}$. Since $y^m = (1, \ldots, 1, -1, \ldots, -1)$ generates a semiregular subgroup on $\Sigma$, the dihedral group $\langle y, b \rangle$ is regular on $\Sigma$, and so $X$ is a d-group.\qed

3. Proof of the main theorem

By Theorem 2.5, primitive d-groups are completely classified, and thus, to complete the proof of the main theorem, we only need to deal with the imprimitive case. We first make a hypothesis.

Hypothesis 3.1. Let $G$ be a d-group of degree $2n$, with a regular dihedral subgroup $D$. Let $B$ be a minimal block system and $B \in \mathcal{B}$. Let $K = G_{(B)}$, the kernel of $G$ acting on $\mathcal{B}$.

Since $D$ is a regular group on $\Omega$, the stabilizer $D_B$ induces a permutation group $D_B^B$, which is a regular group. Regarding the action of $G$ on the block system $\mathcal{B}$, the following lemma was given in [16, Lemma 6.5].

Lemma 3.2. Let $B \in \mathcal{B}$, and let $K = G_{(B)}$ be the kernel of $G$ acting on $\mathcal{B}$. Then one of the following holds:

(i) $D \cap K$ is regular on $B$, and $DK/K$ is regular on $B$, or
(ii) $D \cap K$ is semi-regular on $B$ with 2 orbits and $DK/K$ is transitive on $B$ with point stabilizer isomorphic to $\mathbb{Z}_2$. In the latter case, $D/(D \cap K)$ has a cyclic regular subgroup.

In particular, $|D \cap K| = |B|$ or $\frac{1}{2}|B|$.

Next, we further assume that $\mathcal{B}$ is a minimal block system. Then, for a block $B \in \mathcal{B}$, the induced permutation group $G_B^B$ is primitive.

Lemma 3.3. Using the notation defined above, the following hold:

(i) $D_B^B \leq G_B^B$ is regular, and $G_B^B$ is a c-group or a d-group,
(ii) either $G_B^B$ is 2-transitive, or $G_B^B = \mathbb{Z}_p \times \mathbb{Z}_\ell < \text{AGL}(1, p)$ with $|B| = p$ prime.

Proof. (i). Since $D$ is a regular group on $\Omega$ and $\mathcal{B}$ is a block system for $D$, it follows that $D_B^B \cong D_B$ is regular on $B$. Since $D$ is dihedral, a subgroup $D_B$ is cyclic or dihedral. Thus $G_B^B$ is a c-group or a d-group, respectively.

(ii). Since $\mathcal{B}$ is a minimal block system, $G_B^B$ is a primitive permutation group by definition. As $G_B^B$ is a c-group or a d-group by part (i), by Theorems 2.4 and 2.5,
Lemma 3.5. Assume that $G_B^B$ is 2-transitive, or $|B| = p$ is a prime and $\mathbb{Z}_p \leq G_B^B < AGL(1, p)$.

\[ \square \]

**Lemma 3.4.** If $K^B$ is an elementary abelian $p$-group with $p$, then so is $K$.

**Proof.** Since $K$ is normal in $G$, we have $K^B \triangleleft G_B^B$. Label $B = \{B_1, B_2, \ldots, B_m\}$. Then $K \leq K^{B_1} \times K^{B_2} \times \cdots \times K^{B_m}$, and $K^{B_1} \cong K^{B_2} \cong \cdots \cong K^{B_m}$. If $K^{B_1}$ is an elementary abelian $p$-group, then so is $K$. \[ \square \]

### 3.1. Small block case.

We handle the special case with $|B| = 2$. Let $Z$ be the cyclic subgroup of $D$ of index 2, and let $\sigma \in D \setminus Z$. Then $|\sigma| = 2$, and $D = Z \times \langle \sigma \rangle$.

**Lemma 3.5.** Assume that $|B| = 2$. Then one of the following holds:

(i) $K = 1$, $|B| = \frac{1}{2} |\Omega|$, and $G^B \cong G$ is a $c$-group; 

(ii) $K = \mathbb{Z}_2$, $G = \mathbb{Z}_2.G^B$, and either $D \cap K = 1$ and $G^B$ is a $c$-group, and $D \cap K = \mathbb{Z}_2$ and $G^B$ is a $d$-group; 

(iii) $K$ is an elementary abelian 2-group of order at least 4, and there exists an orbital graph $\Gamma_E[\mathcal{K}_2]$, where $\Gamma_E$ is an arc-transitive circulant or dihedral.

Moreover, for each of the three cases, there indeed are examples.

**Proof.** Suppose $K \cap D = 1$. Then the action of $D$ on $B$ is faithful. Thus the stabilizer of $B$ in $D$ is core-free in $D$, and so $D_B = \mathbb{Z}_2$. Since $D_B$ is regular on $B$, we have $|B| = 2$. As $D^B$ is transitive, the cyclic subgroup $\langle a \rangle$ is transitive and so regular on $B$. Thus $G^B$ is a $c$-group. In particular, if $K = 1$, then $G^B$ is a $c$-group, as in part (i).

Assume $K = \mathbb{Z}_2$. Then $G = K.G^B = \mathbb{Z}_2.G^B$. If $K \cap D = 1$, then $G^B$ is a $c$-group as shown above; if $K \cap D = \mathbb{Z}_2$, then $G^B$ is a $d$-group, as in part (ii).

Now assume $|K| > 2$. Since $\text{Sym}(B_i) = S_2$ and $K \leq \text{Sym}(B_1) \times \cdots \times \text{Sym}(B_m) = S_m^m$, we conclude that $K$ is an elementary abelian 2-group. Now the kernel of $K$ acting on a block $B$ is unfaithful, namely, $K_{\{B\}} \neq 1$. Thus $K_{\{B\}}$ is transitive on some $B' \in B$. Pick $\alpha \in B$ and $\beta \in B'$. Let $\Gamma$ be the orbital graph with arc set $(\alpha, \beta)^G$. Then $K_\alpha$ is transitive on $\Gamma(\alpha)$, and it follows that $\Gamma(\alpha) = B'$. Thus we conclude that $\Gamma = \Gamma_E[\mathcal{K}_2]$.

To complete the proof, we next construct examples of $d$-groups satisfying one of the conditions in the three cases.

**Examples for part (i):** Let $G = \text{PGL}(2, q)$, where $q = p^f$ with $p$ prime and $q \equiv 3 \pmod{4}$. Let $H = \mathbb{Z}_{p^f} \times \mathbb{Z}_{q-1}$, and $\Omega = [G : H]$. Then $H$ is of odd order, and $|\Omega| = 2(q + 1)$. Let $D = D_{2(q+1)}$ be a maximal subgroup of $G$. Then $H \cap D = 1$ and $|G| = |H||D|$. Hence $G = DH$, and $G$ is a $d$-group on $\Omega$ with $D$ being a dihedral regular subgroup.

Let $P$ be a maximal parabolic subgroup of $G$ which contains $H$, and let $\mathcal{B} = [G : P]$. Then $P = \mathbb{Z}_{p^f} \times \mathbb{Z}_{q-1}$, and $\mathcal{B}$ is block system for $G$ acting on $\Omega$. As $|P|/|H| = 2$, we conclude that $|\mathcal{B}| = \frac{1}{2}|\Omega|$ and a block $B \in \mathcal{B}$ has size 2. Moreover, since the almost simple group $G$ is faithful on $\mathcal{B}$, the $d$-group $G$ on $\Omega$ satisfies part (i) with kernel $K = 1$.

**Examples for part (ii):** Let $G = \text{PGL}(2, q) \times \langle c \rangle$, where $q = p^f \equiv 3 \pmod{4}$, and $|c| = \mathbb{Z}_2$. Let $P = \mathbb{Z}_{p^f} \times (\mathbb{Z}_{q-1} \times \langle z \rangle)$, a maximal parabolic subgroup of $G$, where $|z| = 2$. Let $H = \mathbb{Z}_{p^f} \times (\mathbb{Z}_{q-1} \times \langle zc \rangle) \cong P$, and let $\Omega = [G : H]$. Let $D = D_{2(q+1)}$
be a maximal subgroup of \( \operatorname{PGL}(2, q) \). Then \( H \cap D = 1 \) and \( |G| = |H||D| \). Hence \( G = DH \), and \( G \) is a d-group on \( \Omega \) with \( D \) being a dihedral regular subgroup.

Let \( M = H \times \langle c \rangle \), and let \( B = [G : M] \). Then, as \( H \) is a subgroup of \( M \) of index 2, \( B \) is block system for \( G \) acting on \( \Omega \) and \( |B| = \frac{1}{2} |\Omega| \). Since the normal subgroup \( \langle c \rangle \triangleleft G \) is contained in the stabilizer \( M \), we have \( G_{(B)} = \langle c \rangle \). Thus the d-group \( G \) on \( \Omega \) satisfies part (ii) with \( K = \mathbb{Z}_2 \) and \( D \cap K = 1 \).

**Examples for part (iii)** are given in Construction 2.6. This completes the proof of the lemma.

Finally, we present one more example for part (ii) of Lemma 3.5.

**Example 3.6.** Let \( p \equiv 3 \pmod{4} \) be a prime, and let \( \Delta = \{1, 2, \ldots, p\} \). Let \( G = \operatorname{Sym}(\Delta) \times \langle c \rangle \), where \( |c| = 2 \). Let \( H = \operatorname{Alt}(\Delta \setminus \{1\}) \), and let \( \Omega = [G : H] \). Then \( G \) is a permutation group on \( \Omega \) of degree 4\( p \). Let

\[
D = \langle (12 \ldots p), (2, p)(3, p - 1) \ldots \left( \frac{p + 1}{2}, \frac{p + 3}{2} \right) \rangle \times \langle c \rangle \cong \operatorname{D}_{2p} \times \mathbb{Z}_2 = \operatorname{D}_{4p}.
\]

Then \( D \) is regular on \( \Omega \), and \( G \) is a d-group on \( \Omega \).

Let \( M = H \times \langle c \rangle \), and let \( B = [G : M] \). Then, as \( H \) is a subgroup of \( M \) of index 2, \( B \) is block system for \( G \) acting on \( \Omega \) and \( |B| = \frac{1}{2} |\Omega| \). Since the normal subgroup \( \langle c \rangle \triangleleft G \) is contained in the stabilizer \( M \), we have \( G_{(B)} = \langle c \rangle \). Thus the d-group \( G \) on \( \Omega \) satisfies part (ii) with \( K = D \cap K = \mathbb{Z}_2 \).

### 3.2. Bi-primitive d-groups

We here determine another extremal case – bi-primitive d-groups. A transitive permutation group \( G \leq \operatorname{Sym}(\Omega) \) is said to be bi-\textit{primitive} if there is a minimal block system with exactly two blocks, namely, \( B = \{B_1, B_2\} \) such that \( G_{B_i} \) is primitive on \( B_i \). Similar to the definition of quasiprimitive groups, a permutation group \( G \leq \operatorname{Sym}(\Omega) \) is said to be bi-quasiprimitive if each minimal normal subgroup of \( G \) has exactly two orbits. Although the primitivity implies the quasiprimitivity, the bi-primitivity and the bi-quasiprimitivity are independent. Bi-quasiprimitive d-groups have been classified in [22]. The following proposition determines bi-primitive d-groups.

Let \( G \) be a bi-primitive permutation group on \( \Omega \), and let \( B \) be a block system. Then \( B \) contains exactly two blocks, namely, \( B = \{B, B'\} \). As usual, let \( G^+ = G_B = G_{B'} \). Then \( G^+ \) is primitive on both \( B \) and \( B' \), and \( G = G^+ \mathbb{Z}_2 \).

**Proposition 3.7.** Assume that \( G \) is a bi-primitive d-groups with notation defined above. Assume \( R \) is a regular d-group containing in \( G \). Then one of the following holds.

1. there exists an orbital graph \( K_{m,m} \);
2. \( G = G^+ \times \mathbb{Z}_2 \), and either
   a. \( G^+ = S_n \) or \( A_{4m+1} \) with \( n = 4m + 1 \), and \( R^+ = \mathbb{Z}_n \), or
   b. \( G^+ = \operatorname{PGL}(2, q).e \) and \( R^+ = D_n \), where \( n = \frac{q^d - 1}{q - 1} \) with \( e \mid f \) and \( q = p^f \);
3. \( m = p \), \( R = D_{2p} \), \( G = \mathbb{Z}_p \times \mathbb{Z}_2k \leq \operatorname{AGL}(1, p) \), and \( G_w = \mathbb{Z}_k \), where \( k \) is odd;
4. \( G = S_4 \) and \( G^+ = A_4 \).
(5) \( G \) is almost simple, and \((G,R,G_\omega)\) is one of the following triples.

\[
\begin{array}{|c|c|c|c|}
\hline
G & R & G_\omega & \text{condition} \\
\hline
S_n & D_{2n} & A_{n-1} & \\
\hline
\text{PGL}(2,q).e & D_{2(q+1)} & \mathbb{Z}_q^e \times \mathbb{Z}_{q^e-1}.Z_e & e \mid f \\
\hline
\text{PGL}(2,q).2e & D_{2(q+1)} & \mathbb{Z}_q^e \times \mathbb{Z}_{q^e-1}.2e & 2e \mid f, e \text{ odd} \\
\hline
\text{PGL}(d,q).e.2 & D_{2^d q^{d-1}} & P_1(\text{parabolic}) & d \geq 3, e \mid f \\
\hline
\text{PGL}(2,11) & D_{22} & A_5 & \\
\hline
\text{M}_{12}.2 & D_{24} & M_{11} & \\
\hline
\end{array}
\]

**Proof.** Assume first that \( G_B \) is unfaithful on \( B \). Then \( 1 \neq G(B) \triangleleft G_B \), and hence \( 1 \neq G_B^R \triangleleft G_B^R \). Since \( B \) is a minimal block system, \( G_B^R \) is primitive, and so \( G_B^R \) is transitive. It follows that there is an orbital graph \( K_{m,m} \), as in part (1).

Suppose that \( G^+ = G_B \trianglelefteq G_B^R \) is faithful on \( B \). Then \( G = G^+.Z_2 \), and since \( G_B^R \) is primitive by Theorem 2.3, either \( |B| = p \) and \( G^+ \leq AGL(1,p) \), or \( G^+ \) is a 2-transitive c-group or d-group.

**Case 1.** First, assume \( C_G(G^+) = 1 \). Then \( G \) is isomorphic to a subgroup of \( \text{Aut}(G^+) \), and thus \( G \) has a unique minimal normal subgroup, which is \( \text{soc}(G^+) \) and has exactly two orbits. Thus \( G \) is a bi-quasiprimitive d-group, and by [22, Theorem 1.1], one of parts (3)-(5) holds.

**Case 2.** Next, assume \( C_G(G^+) \neq 1 \). Since the center \( Z(G^+) \) is trivial, \( G^+ \cap C_G(G^+) = 1 \). As \( G = G^+.Z_2 \), we have \( C_G(G^+) \) is of order 2, so \( C_G(G^+) = \langle c \rangle = Z_2 \). Then \( G = G^+ \times \langle c \rangle \), where \( |c| = 2 \), and \( R^+ := R \cap G^+ \) is a cyclic group or a dihedral group of index 2 in \( R \).

Suppose that \( R^+ = \langle a \rangle = Z_n \) is cyclic. Then \( R = \langle a \rangle \times \langle gc \rangle = Z_n \times Z_2 = D_{2n} \) for some element \( g \in G^+ \). Thus \( a^{-1} = a^g c = a^g \), and so \( \langle a, g \rangle \cong D_{2n} \) and \( |g| = 2 \), that is, the c-group \( G^+ \) contains a transitive dihedral subgroup of order 2\( n \). Analyzing the groups listed in Theorem 2.4, we conclude that \( G^+ = S_n, A_{4m+1} \) with \( n = 4m + 1 \), or \( \text{PGL}(2,q).e \) with \( n = \frac{q^e - 1}{q - 1} \), where \( e \mid f \) with \( q = p^f \). This is displayed in part (2)(i).

Suppose that \( R^+ = D_n \) is dihedral. Then \( R = \langle R^+, gc \rangle = D_{2n} \), where \( g \in G^+ \). It follows that \( gc \) has order \( n \) or 2, and \( \langle R^+, gc \rangle \cong D_{2n} \). Thus \( |g| = 1, 2 \) or \( n \). If \( |g| = 1 \), then \( R = R^+ \times \langle c \rangle \cong D_{2n} \), and so \( R^+ = D_n \) with \( n/2 \) odd. Assume \( |g| \neq 1 \). Then there exists \( x \in R^+ \) such that \( xg \) is an involution, and \( \langle R^+, xg \rangle = D_{2n} \), namely, the d-group \( G^+ \) of degree \( n \) has a transitive dihedral subgroup \( \langle R^+, xg \rangle \) of order 2\( n \). It follows that the cyclic subgroup of \( \langle R^+, xg \rangle \) of order \( n \) is regular, and so \( G^+ \) is also a c-group. Arguing as in the previous paragraph, we have \( G^+ = S_n, A_{4m+1} \) with \( n = 4m + 1 \), or \( \text{PGL}(2,q).e \) with \( n = \frac{q^e - 1}{q - 1} \), where \( e \mid f \) with \( q = p^f \). In this case, \( R = \langle R^+, xgc \rangle = R^+ \times \langle xgc \rangle \). This is given in part (2)(ii).

Examples of d-groups for part (1) are classified in [6, 7]. The d-groups satisfying part (2) are given in the next example.

**Example 3.8.** (1) Let \( \Delta \) be the set of 1-spaces of \( \mathbb{F}_q^2 \), and let \( G_0 = \text{PGL}(2,q) \). Then \( G_0 \) is 2-transitive on \( \Delta \) of degree \( q + 1 \), and the point stabilizer is a parabolic subgroup \( P = \text{AGL}(1,q) \). Let \( D_0 = D_{2(q+1)} \) be a maximal subgroup of \( H \). Write \( D_0 = \mathbb{Z}_{q+1} \times \langle g \rangle \). Let

\[ G = G_0 \times \langle c \rangle, \; \Omega = [G : P], \; D = D_0 \times \langle gc \rangle. \]
Then $G$ is bi-primitive on $\Omega$, and $D$ is a regular dihedral subgroup of $G$ on $\Omega$, namely, $G = \mathrm{PGL}(2,q) \times \mathbb{Z}_2$ is a bi-primitive $d$-group of degree $2(q+1)$.

(b) Let $G_0 = A_4 = A_{4m+1}$ acting on $\Delta = \{1,2,\ldots,4m+1\}$ naturally. Let $a = (12\ldots4m+1)$, and $g = (1,4m)(2,4m-1)\ldots(2m,2m+1)$. Then $\langle a \rangle$ is regular on $\Delta$, and $\langle a, g \rangle = D_{2n}$ is transitive on $\Delta$. Let

$$G = G_0 \times \langle c \rangle, \quad \Omega = [G : A_{4m}], \quad D = D_0 \times \langle gc \rangle.$$  

Then $G$ is bi-primitive on $\Omega$, and $D$ is a regular dihedral subgroup of $G$ on $\Omega$, namely, $G = A_{4m+1} \times \mathbb{Z}_2$ is a bi-primitive $d$-group of degree $2(4m+1)$.

(c) Similarly, the symmetric group $S_n$ acting on $n$ points gives rise to a bi-primitive $d$-group of $2n$.

Examples of $d$-groups satisfying part (3) are all as follows.

**Example 3.9.** Let $G = \langle a \rangle \times \langle b \rangle = \mathbb{Z}_p \times \mathbb{Z}_\ell \leq AGL(1,p)$, where $p$ is an odd prime and $\ell$ is an even divisor of $p-1$. Let $H = \langle b^2 \rangle = \mathbb{Z}_\ell/2$, and let $\Omega = [G : H]$. Then $R = \langle a \rangle \times \langle b^{\ell/2} \rangle = D_{2p}$ is regular on $\Omega$, and $G$ is a $d$-group on $\Omega$. Let $M = \langle b \rangle$.

Then $H < M$, and $\mathcal{B} = [G : M]$ is a $G$-invariant partition of $\Omega$. Now $Z = \langle a \rangle = \mathbb{Z}_p$ is regular on $\mathcal{B}$, and $G^B \cong G$ is a $c$-group on $\mathcal{B}$.

### 3.3. General case.

In this subsection, we handle the general case. Let $\mathcal{B}$ be a minimal block system for $G$ acting on $\Omega$, satisfying Hypothesis 3.1 with $|B| > 2$ and $|\mathcal{B}| > 2$.

**Lemma 3.10.** The permutation group $K^B$ is a transitive subgroup of $G^B$, and $D \cap K$ is cyclic.

**Proof.** By Lemma 3.3, as $|B| > 2$, we have $|R \cap K| > 1$, and so $K \neq 1$. Thus $1 \neq K^B < G^B$. Since $G^B$ is primitive, the normal subgroup $K^B$ is transitive.

Since $R \cap K < R$ and $R/(R \cap K)$ had order divisible by $|B| > 2$, we conclude that $R \cap K$ is cyclic. \hfill $\square$

The following lemma characterizes the kernel $K = G_{(\mathcal{B})}$.

**Lemma 3.11.** One of the following holds:

(i) $K^B$ is 2-transitive,

(ii) $|B| = p$ with $p$ prime, and $\mathbb{Z}_p < K^B < G^B = \mathbb{Z}_p \times \mathbb{Z}_\ell < AGL(1,p)$,

(iii) $|B| = 4$ and $K$ is an elementary abelian 2-group.

**Proof.** By Lemma 3.3, the induced permutation group $G^B$ is a $c$-group or a $d$-group, and by Lemma 3.10, $K^B$ is a transitive normal subgroup of $G^B$. If $K^B$ is primitive, then by Theorems 2.4 and 2.5, either $K^B$ is 2-transitive, or $\mathbb{Z}_p < K^B < G^B = \mathbb{Z}_p \times \mathbb{Z}_\ell < AGL(1,p)$, as in part (i) or (ii).

Now assume that $K^B$ is imprimitive. Then $G^B$ is affine and the socle $\text{soc}(G^B)$ is not simple. By Theorem 2.5, we have $\text{soc}(G^B) = \mathbb{Z}_2$, or $\mathbb{Z}_2^2$, or $\mathbb{Z}_2$. Suppose that $\text{soc}(G^B) = \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. Then the block size $|B| = 8$ or 16, respectively. On the other hand, as $|\mathcal{B}| > 2$, the normal subgroup $R \cap K$ of the dihedral group $R$ is of index bigger than 2, and so $R \cap K$ is cyclic, say $K \cap R = \mathbb{Z}_\ell$, with $\ell \geq |B|/2 \geq 4$. Since $R$ is regular on $\Omega$, it follows that $K \cap R$ is faithful on $B$, and $K^B < G^B$ contains cyclic subgroup $(K \cap R)^B$ of order 4. By Theorem 2.5, the primitive permutation group $G^B = N.L$, where $N = \text{soc}(G^B)$ and $L$ is almost simple. Thus, $K^B = N.L_0$ is primitive where $L_0 \geq \text{soc}(L)$, which is a contradiction.
We therefore have $|B| = 4$, and $G_B^2 = A_4$ or $S_4$. It follows that $K$ is a $\{2, 3\}$-group. If $|K|$ is divisible by 3, then $K^B$ is of order divisible by 3, and as $K^B \triangleleft G_B^2$, we conclude that $K^B = A_4$ or $S_4$, and thus $K^B$ is primitive. This contradiction shows that $K$ is a 2-group. Suppose that $K$ has an element $g$ of order 4. Then $g^2$ acts non-trivially on some block, say $B$, and thus $g$ is an element of $K^B$ of order 4. Noticing that $K^B$ is a normal subgroup of the primitive permutation group $G_B^2$, we conclude that $K^B = G_B^2 = S_4$, which is a contradiction. Hence, every element of $K$ is of order 2, and $K$ is an elementary abelian 2-group.

We analyse the three cases separately.

**Lemma 3.12.** Assume that $K^B$ is primitive. Then one of the following holds:

1. $K$ is unfaithful on $B$, and there exists an orbital graph for $G$ of the form $\Gamma_B[K_b]$, where $b = |B|$;
2. $\mathbb{Z}_p \rtimes \mathbb{Z}_k = K \cong K^B \leq AGL(1, p)$, where $k$ is an odd divisor of $p - 1$, and either
   (i) $G = (\mathbb{Z}_p \rtimes \mathbb{Z}_k) \times H$. $\mathbb{Z}_p^H$, or
   (ii) $K = \mathbb{Z}_p$, and $G = H \rtimes \mathbb{Z}_2\ell$ where $H = C_G(K)$ is a non-split central extension of $\mathbb{Z}_p$ by $\mathbb{Z}_2\ell$,
   where $\ell$ is an odd divisor of $(p - 1)/k$ and $\mathbb{Z}_k \rtimes \mathbb{Z}_2\ell = \mathbb{Z}_k \rtimes \mathbb{Z}_2\ell \leq \mathbb{Z}_p$.
3. $K \cong K^B$ is non-solvable, and $G = (K \times H)\mathcal{O}$, where $\mathbb{Z}_2 \leq \mathcal{O} \leq \text{Out}(K)$.

**Proof.** If $K$ is unfaithful on $B$, then, by Lemma 2.1, there exists an orbital graph for $G$ of the form $\Gamma_B[K_b]$, where $b = |B|$, as in part (1). Assume that $K$ is faithful on $B$ in the following. Since $|B|, |B| > 2$, it follows from Lemma 3.2 that the intersection $K \cap D \neq 1$.

**Case 1.** First, assume that $\mathbb{Z}_p \leq K \cong K^B \leq AGL(1, p)$. Then $K = \mathbb{Z}_p \rtimes \mathbb{Z}_k$ with some integer $k \mid (p - 1)$. Let $H = C_G(K)$, the centralizer of $K$ in $G$. Since $K \triangleleft G$, we have $H, KH \triangleleft G$, and hence $G/(KH) \leq \text{Out}(K)$.

Suppose that $k \neq 1$. Then $K = \mathbb{Z}_p \rtimes \mathbb{Z}_k$ is a Frobenius group, and so $K \cap H = 1$. Thus $\text{Aut}(K) = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1} = AGL(1, p)$, and $\text{Out}(K) = \mathbb{Z}_{p-1}/\mathbb{Z}_k = \mathbb{Z}_{(p-1)/k}$. So $H \times K = HK \triangleleft G$, and

$$G = (H \times K)\mathbb{Z}_\ell,$$

where $\ell \mid (p - 1)/k$. Since $|B| > 2$, there are at least 3 orbits of $K \cap D$ on $\Omega$. Thus $K \cap D$ is cyclic and normal in $D$. It follows that $K \cap D = \mathbb{Z}_p$, and $H = C_G(K \cap D)$. Thus $R \cap (K \times H)$ is the cyclic subgroup of $D$ of index 2, and

$$G = (K \times H)\mathbb{Z}_2\ell = (\mathbb{Z}_p \rtimes \mathbb{Z}_k) \times K\mathbb{Z}_2\ell.$$

for some integer $\ell$, as in part (2)(i).

Now suppose that $k = 1$ and $K = \mathbb{Z}_p$. If $K.H^B$ is a split extension, then $KH = K \times H^B$, as in part (2)(i) with $k = 1$. If the extension is non-split, then the group $G$ satisfies part (2)(ii).

**Case 2.** Next, assume that $K \cong K^B$ is non-solvable. Let $H = C_G(K)$, the centralizer of $K$ in $G$. Since $K \cong K^B$ is 2-transitive, we have $K \cap H = 1$, and hence $KH = K \times H \triangleleft G$. The factor group $G/(KH) \leq \text{Out}(K)$. Suppose that $D \leq K \times H$. Then $D \leq D_1 \times D_2$, where $D_1$ is the projection of $D$ in $K$ and $D_2$ is the projection of $D$ in $H$. Since $D_1$ and $D_2$ are dihedral groups of order at least 4, this is not possible. Thus $D$ is not contained in $K \times H$, and $\mathcal{O} \geq \mathbb{Z}_2$. In particular,
Out($K$) $\geq Z_2$. By Theorems 2.4 and 2.5, either $K = A_4$ or $2^4 \times A_6$, or $K$ is one of the following almost simple groups:

$A_{2m+1}$, $M_{22}$, $A_{4m}$, $PSL(2,q).e$ with $q = r^f \equiv 3 \pmod{4}$ and $e \mid f$.

Moreover, since $K \cap D$ is cyclic, we conclude that $K = A_{2m+1}$, $M_{22}$ or $A_{4m}$. □

Lemma 3.13. If $|B| = 4$ and $K$ is an elementary abelian group, then $R \cap K = Z_2$, $G = 2^d.H$ where $H$ is a c-group.

Proof. Since $|B| \geq 4$, the intersection $R \cap K$ is cyclic, and so $R \cap K \cong Z_2$. By Lemma 3.2, the induced permutation group $G^B \cong G/K$ is a c-group of degree $n/4$. □

We end this subsection with some examples for the case where $|B| = 4$ and $K$ is an elementary abelian 2-group, $K \cap R = Z_2$, and $G^B$ is a c-group and contains a transitive dihedral subgroup.

Example 3.14. Let $G = Z_9^{3} \rtimes PTL(2,9)$, and let $H = Z_3^{3} \times 3^{2} \times [2^4]$. Let $\Omega = [G:H]$, degree 40. Let $M = Z_3^{3} \times 3^{2} \times [2^4] > H$. Then $B := [G:M]$ forms a $G$-invariant partition of $\Omega$, and $G^B = PTL(2,9)$, of degree 10, and $G^{B_9} = 3^{2} \times [2^4]$. Now $G^{B_9}$ has a cyclic regular subgroup $Z^{B_9} \cong Z_{10}$, and a transitive dihedral subgroup $R^{B_9} = D_{20}$.

Example 3.15. Let $G = 2^6 \rtimes (7 \times 6)$ is an irreducible subgroup of AGL(6,2). Let $G_5 = 2^4 \times 6$, and let $K = \text{soc}(G) = 2^B$. Then $\Omega := [G : G_5]$ is of degree 28, and $G$ has a dihedral subgroup $R \cong D_{28}$. Let $B$ be the set of $K$-orbits on $\Omega$. Then $|B| = 7$, and $B \in B$ has size 4. Furthermore, $G^B = 2^6 \times 6$, and $G^{B_9} = A_4$, and $G^B = 7 \times 6$.

3.4. Proof of Theorem 1.1.

We here summarize what we have obtained to complete the proof of Theorem 1.1.

Let $G$ be an imprimitive $d$-group on $\Omega$ of degree $2n$. Let $B$ be a minimal block system for $G$ on $\Omega$, and let $K = G_{(B)}$. We will list the candidates for $G$ according to the property of $K$.

If $K = 1$, then $|B| = 2$ by Lemma 3.10, and $G \cong G^B$ is also a c-group by Lemma 3.5, as in part (1).

Assume $K \neq 1$. If $K^B$ is imprimitive, then by Lemmas 3.11 and 3.13, we have that $|B| = 4$, and $K$ is an elementary abelian 2-group, and $G^B$ is a c-group of degree $n/4$, as in part (5). Suppose that $K^B$ is primitive. If $K_{(B)} \neq 1$, then there exists an orbital graph $\Gamma = \Gamma_B[K_b]$ where $b = |B|$, as in part (2).

Finally, suppose that $K \cong K^B$ is faithful and primitive. If $K = Z_p$ with $p$ prime, then by Lemmas 3.5 and 3.12, part (3) of Theorem 1.1 occurs. On the other hand, the centre $Z(K) = 1$ for $K \neq Z_p$ by Lemma 3.11. In this case, by Lemma 3.12, part (4) of Theorem 1.1 occurs. □

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge University Press, 2nd edition, New York, 1992.
[2] W. Burnside, On some properties of groups of odd order, Proc. London Math. Soc. 33 (1900) 162-185.
[3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of finite groups (Oxford University Press, 1985).
[4] J.D. Dixon and B. Mortimer, Permutation Groups (1996), New York: Springer-Verlag.
[5] S.F. Du, A. Malič, D. Marušić, Classification of 2-arc-transitive dihedrants, J. Combin. Theory Ser. B 98 (2008) 1349-1372.
[6] W.W. Fan, L. Dimitri, C.H. Li and J.M. Pan, Locally 2-arc-transitive complete bipartite graphs, *J. Combin. Theory Ser. A* **120** (2013), 683-699.

[7] W.W. Fan, C.H. Li and J.M. Pan, Finite locally-primitive complete bipartite graphs, *J. Group Theory* **17** (2014), 111-129.

[8] W. Feit, ‘Some consequences of the classification of finite simple groups’, The Santa Cruz Conference on Finite Groups, Santa Cruz, 1979 (ed. B. Cooperstein and G. Mason), Proceedings of Symposia in Pure Mathematics 37 (American Mathematical Society, Providence, RI, 1981) 175-181.

[9] D. Gorenstein, Finite simple groups (Plenum Press, New York, 1982).

[10] G. Jones, Cyclic regular subgroups of primitive permutation groups, *J. Group Theory*, **5** (4) (2002) 403-407.

[11] W. M. Kantor, ‘Some consequences of the classification of finite simple groups’, Finite groups – coming of age (ed. J. McKay, American Mathematical Society, Providence, RI, 1982).

[12] I. Kovács, Classifying arc-transitive circulants, *J. Algebraic Combin.* **20** (2004), 353-358.

[13] I. Kovács, D. Marušič and M. Muzychuck, On dihedrants admitting 2-regular group actions, *J. Algebraic Combin.* **35**(2011), 409-426.

[14] C. H. Li, The finite primitive permutation groups containing an abelian regular subgroup, *Proc. London Math. Soc.* **87** (2003), 725-748.

[15] C. H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.* **21** (2005), 131-136.

[16] C. H. Li, Finite edge-transitive Cayley graphs and rotary Cayley maps, *Trans. Amer. Math. Soc.* **358** (2006) 4605-4635.

[17] C. H. Li and C. E. Praeger, On finite permutation groups with a transitive cyclic subgroup, *J. Algebra* **349** (2012), 117-127.

[18] C. H. Li, B. Z. Xia and S. M. Zhou, Finite arc-transitive circulants, submitted (2019).

[19] D. Marušič, On 2-arc-transitivity of Cayley graphs, *J. Combin. Theory Ser. B* **87** (2003) 162-196; *J. Combin. Theory Ser. B* **96** (2006) 761-764 (Corrigendum).

[20] Jiangmin Pan, Locally primitive Cayley graphs of dihedral groups. *European J. Combin.* **36** (2014), 39-52.

[21] Jiangmin Pan, Xue Yu, Hua Zhang, and Zhaohong, Huang, Finite edge-transitive dihedrant graphs. *Discrete Math.* **312** (2012), no. 5, 1006-1012.

[22] S.J. Song, C.H. Li and H. Zhang, Finite permutation groups with a regular dihedral subgroup, and edge-transitive dihedrants, *J. Algebra*, **399**(2014), 948-959.

[23] H. Wielandt, *Finite Permutation Groups* (1964), Academic Press, New York.

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