Advances on the fixed point results via simulation function involving rational terms

Erdal Karapınar\textsuperscript{1,2,3*}, Chi-Ming Chen\textsuperscript{4}, Maryam A. Alghamdi\textsuperscript{5} and Andreea Fulga\textsuperscript{6}

Abstract
In this paper, we propose two new contractions via simulation function that involves rational expression in the setting of partial b-metric space. The obtained results not only extend, but also generalize and unify the existing results in two senses: in the sense of contraction terms and in the sense of the abstract setting. We present an example to indicate the validity of the main theorem.

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1 Introduction and preliminaries
The origin of the fixed point theory goes back a century, to the pioneer work of Banach. Since the first study of Banach, researchers have been extended, improved, and generalized this very simple stated but at the same time very powerful theorem. For this purpose, the terms of the contraction inequality and the abstract structure of Banach’s theorem have been investigated. In this paper, we shall combine these two trends and introduce two new type contraction via simulation functions involving rational terms in the more general setting, partial-b-metric space.

For the sake of the completeness of the manuscript, we shall recall some basic results and concepts here.

Theorem 1 ([1]) Let \((A,\delta)\) be a complete metric space and \(O : A \to A\) be a mapping. If there exist \(k_1, k_2 \in [0,1)\), with \(k_1 + k_2 < 1\) such that

\[
\delta(Ov, Ow) \leq k_1 \cdot \delta(v, Ow) + k_2 \cdot \delta(v, w),
\]

for all \(v, w \in A\), then \(O\) has a unique fixed point \(u \in A\) and the sequence \(\{O^n x\}\) converges to the fixed point \(u\) for all \(x \in A\).
**Theorem 2** ([2]) Let \((\mathcal{A}, \delta)\) be a complete metric space and \(\mathcal{O} : \mathcal{A} \to \mathcal{A}\) be a continuous mapping. If there exist \(\kappa_1, \kappa_2 \in [0, 1)\), with \(\kappa_1 + \kappa_2 < 1\) such that

\[
\delta(Ov, Ow) \leq \kappa_1 \frac{\delta(v, Ow) \delta(Ov, Ow)}{\delta(v, \omega)} + \kappa_2 \cdot \delta(v, \omega),
\]

for all distinct \(v, \omega \in \mathcal{A}\), then \(\mathcal{O}\) possesses a unique fixed point in \(\mathcal{A}\).

We mention that over the last few years many interesting and different generalizations for rational contractions have been provided; see, for example [3–8].

Let \(\Gamma\) be the set of all non-decreasing and continuous functions \(\psi : [0, +\infty) \to [0, +\infty)\) such that \(\psi(0) = 0\).

**Definition 1** ([9]) A function \(\eta : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}\) is a \(\psi\)-simulation function if there exists \(\psi \in \Gamma\) such that the following conditions hold:

1. \((\eta_1)\) \(\eta(r, 0) < \psi(t) - \psi(r)\) for all \(r, t \in \mathbb{R}^+\);
2. \((\eta_2)\) if \(\{r_n\}, \{t_n\}\) are two sequences in \([0, +\infty)\) such that \(\lim_{n \to +\infty} r_n = \lim_{n \to +\infty} t_n > 0\), then

\[
\limsup_{n \to +\infty} \eta(r_n, t_n) < 0. \tag{1.3}
\]

We will denote by \(\mathcal{Z}_\psi\) the family of all \(\psi\)-simulation functions; see e.g. [10–22]. It is clear, due to the axiom \((\eta_1)\), that

\[
\sigma(r, t) < 0 \quad \text{for all } r > 0. \tag{1.4}
\]

**Definition 2** ([23]) On a non-empty set \(\mathcal{A}\), a function \(\rho : \mathcal{A} \times \mathcal{A} \to \mathbb{R}_0^+\) is a partial metric if the following conditions hold:

1. \((\rho_1)\) \(v = \omega\) if \(\rho(v, v) = \rho(\omega, \omega) = \rho(\omega, v)\);
2. \((\rho_2)\) \(\rho(v, v) \leq \rho(v, \omega)\);
3. \((\rho_3)\) \(\rho(v, \omega) = \rho(\omega, v)\);
4. \((\rho_4)\) \(\rho(v, \omega) \leq \rho(v, z) + \rho(z, \omega) - \rho(z, z)\);

hold for all \(v, \omega, z \in \mathcal{A}\).

The pair \((\mathcal{A}, \rho)\) is called a partial-metric space.

Every partial metric \(\rho\) on \(\mathcal{A}\) generates a \(T_0\) topology on \(\mathcal{A}\), that has a base of the set of all open balls \(B_\rho(v)\), where an open ball for a partial metric \(\rho\) on \(\mathcal{A}\) is defined [23] as

\[
B_\rho^e(v) = \{\omega \in \mathcal{A} : \rho(v, \omega) < \rho(v, v) + e\},
\]

for each \(v \in \mathcal{A}\) and \(e > 0\).

If \((\mathcal{A}, \rho)\) is a partial-metric space and \(\{v_n\}\) a sequence in \(\mathcal{A}\), then:

- \(\{v_n\}\) is convergent to a limit \(u \in \mathcal{A}\), if \(\lim_{m \to +\infty} \rho(v_m, u) = \rho(u, u)\);
- \(\{v_n\}\) is a Cauchy sequence if \(\lim_{m,n,q \to +\infty} \rho(v_m, v_q)\) exists and is finite.

Moreover, we say that the partial-metric space \((\mathcal{A}, \rho)\) is complete if every Cauchy sequence \(\{v_n\}\) in \(\mathcal{A}\) converges to a point \(u \in \mathcal{A}\), that is,

\[
\lim_{m \to +\infty} \rho(v_m, u) = \rho(u, u) = \lim_{m,q \to +\infty} \rho(v_m, v_q).
\]
Remark 1. The limit in a partial metric space may not be unique. For a sequence \( \{v_m\} \) on \((A, \rho)\), we denote by \( L(\{v_m\}) \) the set of limit points (if there exist any),

\[
L(\{v_m\}) = \left\{ u \in A : \lim_{m \to +\infty} \rho(v_m, u) = \rho(u, u) \right\}.
\]

We recall some results in the context of partial-metric spaces, necessary in our following considerations.

Lemma 1. Let \((A, \rho)\) be a partial-metric space and \(\{v_m\}\) be a sequence in \(A\) such that

\[
\lim_{m \to +\infty} \rho(v_m, v_{m+1}) = 0. \text{If} \lim_{m,n \to +\infty} \rho(v_m, v_{m+n}) \neq 0, \text{then there exist} \ e > 0 \text{and subsequences} \ \{v_{m_l}\}, \ {v_{q_l}} \text{ of} \ \{v_m\} \text{such that}
\]

\[
\lim_{l \to +\infty} \rho(v_{m_l}, v_{q_l}) = \lim_{l \to +\infty} \rho(v_{m_l}, v_{q_l}+1) = \lim_{l \to +\infty} \rho(v_{m_l+1}, v_{q_l})
\]

\[
= \lim_{l \to +\infty} \rho(v_{m_l+1}, v_{q_l+1}) = e.
\]

Lemma 2 ([24]). Let \(\{v_m\}\) be a Cauchy sequence on a complete partial-metric space \((A, \rho)\). If there exists \(\chi \in L(\{v_m\})\) with \(\rho(\chi, \chi) = 0\), then \(\chi \in L(\{v_m\})\), for every subsequence \(\{v_{m_l}\}\) of \(\{v_m\}\).

Lemma 3 ([25]). If \(\{v_m\}, \ \{w_m\}\) are two sequences in a partial-metric space \((A, \rho)\) such that

\[
\lim_{m \to +\infty} \rho(v_m, \chi) = \lim_{m \to +\infty} \rho(v_m, v_m) = \rho(\chi, \chi),
\]

\[
\lim_{m \to +\infty} \rho(w_m, y) = \lim_{m \to +\infty} \rho(w_m, w_m) = \rho(y, y),
\]

then \(\lim_{m \to +\infty} \rho(v_m, w_m) = \rho(\chi, y)\). Moreover, \(\lim_{m \to +\infty} \rho(v_m, u) = \rho(\chi, u)\), for each \(u \in A\).

On a partial-metric space \((A, \rho)\), a mapping \(O : A \to A\) is continuous at \(v_0\) if and only if for every \(e > 0\), there exists \(\delta > 0\) such that

\[
O(B^\rho_{\delta}(v_0)) \subseteq B^\rho_e(O(v_0)).
\]

(\(O\) is continuous if it is continuous at every point \(v \in A\).)

Lemma 4 ([24]). On a complete partial-metric space \((A, \rho)\), let \(O\) be a continuous mapping and \(\{v_m\}\) be a Cauchy sequence in \(A\). If there exists \(\chi \in L(\{v_m\})\) with \(\rho(\chi, \chi) = 0\), then \(O\chi \in L(\{v_m\})\).

Definition 3 ([26]). Let \(A\) be a non-empty set and \(s \geq 1\). A function \(\rho_b : A \times A \to \mathbb{R}^+_0\) is a partial \(b\)-metric with a coefficient \(s\) if the following conditions hold for all \(v, \omega, z \in A\)

\begin{align*}
(\rho_b1) \quad & v = \omega \text{ iff } \rho_b(v, v) = \rho_b(\omega, \omega); \\
(\rho_b2) \quad & \rho_b(v, \omega) \leq s \rho_b(v, z) + \rho_b(z, \omega); \\
(\rho_b3) \quad & \rho_b(v, \omega) = \rho_b(\omega, v); \\
(\rho_b4) \quad & \rho_b(v, \omega) \leq s[\rho_b(v, z) + \rho_b(z, \omega)] - \rho_b(z, z).
\end{align*}

In this case, we say that \((A, \rho_b, s)\) is a partial \(b\)-metric space.

Example 1 ([26]). Let \(A\) be a non-empty set and \(v, \omega \in A\).
• if \( \rho \) is a partial metric on \( \mathcal{A} \), then the function \( \rho_b \) defined as

\[
\rho_b(v, \omega) = \left[ \rho(v, \omega) \right]^\lambda
\]

is a partial \( b \)-metric on \( \mathcal{A} \), with \( s = 2^{\lambda - 1} \), for \( \lambda > 1 \).

• if \( b \) is a \( b \)-metric and \( \rho \) is a partial metric on \( \mathcal{A} \), then the function

\[
\rho_b(v, \omega) = \rho(v, \omega) + b(v, \omega)
\]

is a partial \( b \)-metric on \( \mathcal{A} \).

A sequence \( \{v_m\} \) in a partial \( b \)-metric space \( (\mathcal{A}, \rho_b, s) \) is said to be \( \rho_b \)-convergent to a point \( u \in \mathcal{A} \) if

\[
\lim_{m \to +\infty} \rho_b(v_m, u) = \rho_b(u, u). \tag{1.8}
\]

If the limit \( \lim_{m,q \to +\infty} \rho_b(v_m, v_q) \) exists and it is finite, the sequence \( \{v_m\} \) is said to be \( \rho_b \)-Cauchy. Moreover, if every \( \rho_b \)-Cauchy sequence in \( \mathcal{A} \) is \( \rho_b \)-convergent to \( u \in \mathcal{A} \), that is

\[
\lim_{m,q \to +\infty} \rho_b(v_m, v_q) = \lim_{m \to +\infty} \rho_b(v_m, u) = \rho_b(u, u), \tag{1.9}
\]

we say that the partial \( b \)-metric space \( (\mathcal{A}, \rho_b, s) \) is \( \rho_b \)-complete.

**Remark 2** In [27] it is proved that a partial \( b \)-metric induces a \( b \)-metric, say \( \delta_b \), with

\[
\delta_b(v, \omega) = 2\rho_b(v, \omega) - \rho_b(v, v) - \rho_b(\omega, \omega), \tag{1.10}
\]

for all \( v, \omega \in \mathcal{A} \).

On the other hand, in [28], the notion of \( 0-\rho_b \)-completeness was introduced and the relation between \( 0-\rho_b \)-completeness and \( \rho_b \)-completeness of a partial \( b \)-metric was established.

**Definition 4** ([28]) A sequence \( \{v_m\} \) on a partial \( b \)-metric space \( (\mathcal{A}, \rho_b, s) \) is \( 0-\rho_b \)-Cauchy if \( \lim_{m,q \to +\infty} \rho_b(v_m, v_q) = 0 \). Moreover, the space \( (\mathcal{A}, \rho_b, s) \) is said to be \( 0-\rho_b \)-complete if for each \( 0-\rho_b \)-Cauchy sequence in \( \mathcal{A} \), there is \( u \in \mathcal{A} \), such that

\[
\lim_{m,q \to +\infty} \rho_b(v_m, v_q) = \lim_{m \to +\infty} \rho_b(v_m, u) = \rho_b(u, u) = 0. \tag{1.11}
\]

**Lemma 5** ([28]) If the partial \( b \)-metric space \( (\mathcal{A}, \rho_b, s) \) is \( \rho_b \)-complete, then it is \( 0-\rho_b \)-complete.

**Lemma 6** ([29]) Let \( (\mathcal{A}, \rho_b, s) \) be a partial \( b \)-metric space. If \( \rho_b(v, \omega) = 0 \) then \( v = \omega \) and \( \rho_b(v, \omega) > 0 \) for all \( v \neq \omega \).

The next result is important in our future considerations.
Lemma 7 ([30]) Let \((A, \rho_b, s \geq 1)\) be a partial \(b\)-metric space, \(O : A \rightarrow A\) a mapping and a number \(\kappa \in [0, 1)\). If \(\{v_n\}\) is a sequence in \(A\), where \(v_m = Ov_{m-1}\) and
\[
\rho_b(v_m, v_{m+1}) \leq \kappa \rho_b(v_{m-1}, v_m), \tag{1.12}
\]
for each \(m \in \mathbb{N}\), then the sequence \(\{v_n\}\) is \(0-\rho_b\)-Cauchy.

2 Main results
We start with the definition of simulation function for partial \(b\)-metric spaces.

Definition 5 Let \((A, \rho_b, s \geq 1)\) be a partial \(b\)-metric space. A \(b-\psi\)-simulation function is a function \(\eta_b : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}\) satisfying:
- \((\eta_{b1})\) \(\eta_b(r, t) < \psi(t) - \psi(r)\) for all \(r, t \in \mathbb{R}^+\);
- \((\eta_{b2})\) if \(\{r_n\}, \{t_n\}\) are two sequences in \([0, +\infty)\), such that for \(p > 0\)
\[
\limsup_{n \to +\infty} t_n = s^p \lim_{n \to +\infty} r_n > 0, \tag{2.1}
\]
then
\[
\limsup_{n \to +\infty} \eta_b(s^p r_n, t_n) < 0. \tag{2.2}
\]

We shall denote by \(\mathcal{Z}_{\psi_b}\) the family of all \(b-\psi\)-simulation functions.

Example 2 Let \(\psi \in \Gamma\) and \(\gamma : [0, +\infty) \rightarrow [0, +\infty)\) be a function such that \(\limsup_{t \to t_0} \gamma(t) < 1\) for every \(t_0 > 0\) and \(\phi(t) = 0\) if and only if \(t = 0\). Then \(\eta_b(r, t) = \gamma(t)\psi(t) - \psi(r)\), for \(r, t \geq 0\) is a \(b-\psi\)-simulation function.

Example 3 Let \(\psi \in \Gamma\) and \(\phi : [0, +\infty) \rightarrow [0, +\infty)\) be a function such that \(\lim_{t \to t_0} \phi(t) > 0\) for every \(t_0 > 0\) and \(\phi(t) = 0\) if and only if \(t = 0\). Then \(\eta_b(r, t) = \psi(t) - \phi(t) - \psi(r)\), for \(r, t \geq 0\) is a \(b-\psi\)-simulation function.

Obviously, \((\eta_{b1})\) holds. Now, considering two sequences \(\{r_n\}\) and \(\{t_n\}\) in \((0, +\infty)\) such that \((2.1)\) holds, we have
\[
\lim_{n \to +\infty} \eta_b(s^p r_n, t_n) = \lim_{n \to +\infty} \psi(t_n) - \phi(t_n) - \psi(s^p r_n) \leq -\phi(t_n) < 0.
\]
Thus, also \((\eta_{b2})\) holds, that is \(\eta_b \in \mathcal{Z}_{\psi_b}\).

Definition 6 Let \((A, \rho_b, s \geq 1)\) be a partial \(b\)-metric space. A mapping \(O : A \rightarrow A\) is called \((\eta_b)\)-rational contraction of type \(A\) if there exists a function \(\eta_b \in \mathcal{Z}_{\psi_b}\) such that
\[
\frac{1}{25} \min\{\rho_b(v, Ov), \rho_b(Ov, O\omega)\} \leq \rho_b(v, \omega), \tag{2.3}
\]
which implies
\[
\eta(s^p \rho_b(Ov, O\omega), D_A(v, \omega)) \geq 0, \tag{2.4}
\]
for every \(v, \omega \in A\), where \(D_A\) is defined as
\[
D_A(v, \omega) = \max\left\{\delta(v, \omega), \delta(v, Ov), \delta(\omega, O\omega), \frac{\delta(\omega, O\omega)[1 + \delta(v, O\omega)]}{1 + \delta(v, \omega)}\right\}.
\]
With the purpose to simplify the demonstrations, we prefer in the sequel, to discuss separately, the cases.

**Theorem 3** Let \((A, \rho_b, s > 1)\) be a \(\rho_b\)-complete partial \(b\)-metric space and \(O : A \to A\) be a \((\eta_b)\)-rational contraction of type \(A\). Then \(O\) admits exactly one fixed point.

**Proof** Let \(v_0 \in A\) be an arbitrary but fixed point and \(\{v_m\}\) be the sequence in \(A\) defined as follows:

\[
v_m = Ov_{m-1}, \quad \forall m \in \mathbb{N}. \tag{2.5}
\]

Thus, we can assume that \(v_{m-1} \neq v_m\) for every \(m \in \mathbb{N}\). Indeed, if we suppose that there exists \(m_0 \in \mathbb{N}\) such that \(v_{m_0-1} = v_{m_0}\), then we get \(v_{m_0-1} = Ov_{m_0-1}\), that is, \(v_{m_0-1}\) is a fixed point of \(O\). Therefore, substituting \(v = v_{m-1}\) and \(\omega = v_m\) in (2.4), we have

\[
\mathcal{D}_A(v_{m-1}, v_m) = \max \left\{ \rho_b(v_{m-1}, v_m), \rho_b(v_{m-1}, Ov_{m-1}), \rho_b(v_{m-1}, Ov_m) \right\} = \max \left\{ \rho_b(v_{m-1}, v_m), \rho_b(v_{m-1}, v_m) \right\} = \rho_b(v_{m-1}, v_m).
\]

Moreover, by (2.3) we get

\[
\frac{1}{2^5} \min \left\{ \rho_b(v_{m-1}, Ov_{m-1}), \rho_b(v_m, Ov_m) \right\} = \frac{1}{2^5} \min \left\{ \rho_b(v_{m-1}, v_m), \rho_b(v_m, v_{m+1}) \right\} \leq \rho_b(v_{m-1}, v_m), \quad \text{for all } m \in \mathbb{N},
\]

which implies

\[
\eta_b(\rho_b, Ov_{m-1}, Ov_m, \mathcal{D}_A(v_{m-1}, v_m)) \geq \eta_b(\rho_b, v_{m-1}, v_{m+1}).
\]

Now, taking into account \((\eta_{b1})\), the above inequality yields

\[
0 < \psi(\mathcal{D}_A(v_{m-1}, v_m)) - \psi(\rho_b, Ov_{m-1}, Ov_m)) = \psi(\rho_b, v_{m-1}, v_{m+1}) \leq \rho_b(v_{m-1}, v_m),
\]

or, equivalently,

\[
\psi(\rho_b, v_{m-1}, v_{m+1}) < \psi(\mathcal{D}_A(v_{m-1}, v_m)) = \psi(\max \left\{ \rho_b(v_{m-1}, v_m), \rho_b(v_{m-1}, v_{m+1}) \right\}).
\]

Consequently, due to the monotony of the function \(\psi\), we obtain

\[
\rho_b(v_{m-1}, v_{m+1}) < \max \left\{ \rho_b(v_{m-1}, v_m), \rho_b(v_{m-1}, v_{m+1}) \right\}. \tag{2.6}
\]
If there exists \( m_1 \in \mathbb{N} \) such that \( \max\{\rho_b(v_{m_1}, v_{m_1}), \rho_b(v_{m_1}, v_{m_1+1})\} = \rho_b(v_{m_1}, v_{m_1+1}) \), (2.6) becomes \( s^p \rho_b(v_{m_1}, v_{m_1+1}) < \rho_b(v_{m_1}, v_{m_1+1}) \), which is a contradiction (because \( s > 1 \)). Therefore, for any \( m \in \mathbb{N} \) we have

\[
s^p \rho_b(v_{m^*}, v_{m+1}) < \rho_b(v_{m^*}, v_{m}),
\]

or

\[
\rho_b(v_{m^*}, v_{m+1}) < \frac{1}{s^p} \rho_b(v_{m^*}, v_{m}). \quad (2.7)
\]

Denoting \( \frac{1}{s^p} \) by \( \kappa \), we have \( \rho_b(v_{m^*}, v_{m+1}) < \kappa \rho_b(v_{m^*}, v_{m}) \), with \( 0 \leq \kappa < 1 \). Thus, by Lemma 7 we see that the sequence \( \{v_p\} \) is a \( 0^p \)-Cauchy sequence on the \( \rho_b \)-complete partial b-metric space. Since by Lemma 5, the space is also \( 0^p \)-complete, it follows that there exists \( u \in \mathcal{A} \) such that

\[
\lim_{m \to +\infty} \rho_b(v_m, u) = \lim_{m \to +\infty} \rho_b(v_m, u) = \rho_b(u, u) = 0. \quad (2.8)
\]

Now, we claim that

\[
\frac{1}{2^s} \rho_b(v_m, v_{m+1}) \leq \rho_b(v_m, u) \quad \text{or} \quad \frac{1}{2^s} \rho_b(v_{m+1}, v_{m+2}) \leq \rho_b(v_{m+1}, u).
\]

Assuming the contrary, we can find \( m_0 \in \mathbb{N} \) such that

\[
\rho_b(v_{m_0}, v_{m_0+1}) \leq \frac{1}{2^s} \left[ \rho_b(v_{m_0}, u) + \rho_b(u, v_{m_0+1}) \right] - \rho_b(u, u) < \frac{1}{2^s} \left[ \rho_b(v_{m_0}, v_{m_0+1}) + \frac{1}{2^s} \rho_b(v_{m_0+1}, v_{m_0+2}) \right]
\]

\[
= \frac{1}{2} \left[ \rho_b(v_{m_0}, v_{m_0+1}) + \rho_b(v_{m_0+1}, v_{m_0+2}) \right] \quad \text{(taking (2.7) into account)}
\]

\[
< \rho_b(v_{m_0}, v_{m_0+1}),
\]

which is a contradiction. Thus, there exists a subsequence \( \{v_{m_l}\} \) of \( \{v_m\} \) such that

\[
\frac{1}{2^s} \min\{\rho_b(v_{m_l}, \mathcal{O}v_{m_l}), \rho_b(u, \mathcal{O}u)\} = \frac{1}{2^s} \rho_b(v_{m_l}, v_{m_l+1}) \leq \rho_b(v_{m_l}, u),
\]

which implies

\[
\eta_b(s^p \rho_b(\mathcal{O}v_{m_l}, \mathcal{O}u), \mathcal{D}_A(v_{m_l}, u)) \geq 0,
\]

where

\[
\rho_b(u, \mathcal{O}u) \leq \mathcal{D}_A(v_{m_l}, u) = \max \left\{ \rho_b(v_{m_l}, u), \rho_b(v_{m_l}, \mathcal{O}v_{m_l}, \rho_b(u, \mathcal{O}u), \rho_b(\mathcal{O}v_{m_l}, \mathcal{O}u), \rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u), \rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u), \rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u), \right\}
\]

\[
= \max \left\{ \rho_b(v_{m_l}, u), \rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u), \right\}
\]

\[
\rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u), \rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u).
\]

\[
\rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u).
\]

\[
\rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u).
\]

\[
\rho_b(v_{m_l}, v_{m_l+1}), \rho_b(u, \mathcal{O}u).
\]
Therefore, letting \( l \to +\infty \) and keeping (2.8) in mind we get

\[
\lim_{l \to +\infty} D_A(v_{m(l)}, u) = \rho_b(u, \mathcal{O} u).
\]

(2.9)

On one hand, without loss of generality, we assume that \( v_m \neq u \), for infinitely many \( m \in \mathbb{N} \). Thus,

\[
\eta_b(s^p \rho_b(\mathcal{O} v_m, \mathcal{O} u), D_A(v_m, u)) \geq 0,
\]

which by \((\eta_b)\) leads us to

\[
\psi(s^p \rho_b(\mathcal{O} v_m, \mathcal{O} u)) < \psi(D_A(v_m, u)).
\]

Taking into account the non-decreasing property of \( \psi \)

\[
s^p \rho_b(\mathcal{O} v_m, \mathcal{O} u) < D_A(v_m, u).
\]

On the other hand,

\[
\rho_b(u, \mathcal{O} u) \leq s \left[ \rho_b(u, \mathcal{O} v_m) + \rho_b(\mathcal{O} v_m, \mathcal{O} u) \right] - \rho_b(\mathcal{O} v_m, \mathcal{O} v_m)
\]

\[
\leq s \rho_b(u, \mathcal{O} v_m) + s^p \rho_b(\mathcal{O} v_m, \mathcal{O} u) - \rho_b(v_{m+1}, v_{m+1})
\]

\[
< s \rho_b(u, \mathcal{O} v_m) + D_A(v_m, u).
\]

Letting \( m \to +\infty \) in the above inequality and keeping in mind (2.8) and (2.9) we get

\[
\rho_b(u, \mathcal{O} u) \leq s^p \lim_{m \to +\infty} \rho_b(\mathcal{O} v_m, \mathcal{O} u) < \lim_{m \to +\infty} D_A(v_m, u) = \rho_b(u, \mathcal{O} u).
\]

Therefore, \( s^p \lim_{m \to +\infty} \rho_b(\mathcal{O} v_m, \mathcal{O} u) = \rho_b(u, \mathcal{O} u) \). Thus, letting \( t_m = \rho_b(\mathcal{O} v_m, \mathcal{O} u) \) and \( t_m = D_A(v_m, u) \), by \((\eta_b)\), it follows \( \lim \sup_{m \to +\infty} \eta_b(s^p t_m, t_m) < 0 \), which is a contradiction. Then \( \rho_b(u, \mathcal{O} u) = 0 = \rho_b(u, u) \), that is, \( u \) is a fixed point of \( \mathcal{O} \).

As a last step, we establish uniqueness of the fixed point. Indeed, if we can find another point, \( z \in \mathcal{A} \), \( z \neq u \) such that \( z = \mathcal{O} z \),

\[
0 = \frac{1}{2s} \min \left\{ \rho_b(z, \mathcal{O} z), \rho_b(u, \mathcal{O} u) \right\} \leq \rho_b(z, u),
\]

which implies

\[
0 \leq \eta_b(s^p \rho_b(\mathcal{O} z, \mathcal{O} u), D_A(z, u)) < \psi(D_A(z, u)) - \psi(s^p \rho_b(\mathcal{O} z, \mathcal{O} u))
\]

\[
= \psi(\rho_b(z, u)) - \psi(s^p \rho_b(z, u)),
\]

which is a contradiction. Thus, \( u = z \). \( \square \)

**Example 4** Let the set \( \mathcal{A} = \{10, 11, 12, 13\} \) and \( \rho_b \) be the partial \( b \)-metric on \( \mathcal{A} \) \( (s = 2) \), where \( \rho_b(v, w) = \begin{cases} 0 & \text{for } v = w, \\ 0.000002 & \text{for } v = w = 13, \\ 10 & \text{for } v \in \{10, 11, 12\}, \\ 11 & \text{for } v = 13. \end{cases} \) We define the mapping \( \mathcal{O} : \mathcal{A} \to \mathcal{A} \), \( \mathcal{O} v = \frac{v + \phi(t)}{2} \). We choose \( \phi \in \Gamma \), \( \phi(t) = \frac{t}{2} \) and \( \eta_b(r, t) = \frac{12^t r}{t^2} \). It is easy to see that \( \eta_b \in \mathcal{Z}_{\phi_b} \) (by taking \( \gamma(t) = \frac{12^t}{t^2} \) in Example 2). We have
and shall consider the following cases:

1. For \( v, \omega \in \{10, 11, 12\} \), we have \( \rho_b(v, v) = 0 \), and then

\[
\frac{1}{25} \min \{ \rho_b(v, v), \rho_b(\omega, \omega) \} \leq 1 \leq \rho_b(v, \omega),
\]

which implies

\[
2 \rho_b(v, \omega) = 0 \leq \frac{15}{16} D_b(v, \omega).
\]

2. For \( v = 10, \omega = 13 \) we have \( \rho_b(v, \omega) = 9 \), \( \rho_b(10, 10) = 0 \), \( \rho_b(13, 13) = \rho_b(13, 11) = 4 \), \( \rho_b(10, 11) = 1 \) and then

\[
\frac{1}{4} \min \{ \rho_b(10, 10), \rho_b(13, 13) \} = 0 < 9 = \rho_b(v, \omega),
\]

which implies

\[
2 \rho_b(10, 13) = 2 \leq \frac{135}{16} = \frac{15}{16} \cdot \rho_b(10, 13).
\]

3. For \( v = 11, \omega = 13 \) we have \( \rho_b(v, \omega) = 4 \), \( \rho_b(11, 11) = 1 \), \( \rho_b(13, 13) = \rho_b(13, 11) = 4 \), \( \rho_b(10, 11) = 1 \) and then

\[
\frac{1}{4} \min \{ \rho_b(11, 11), \rho_b(13, 13) \} = \frac{1}{4} < 4 = \rho_b(v, \omega),
\]

which implies

\[
2 \rho_b(11, 13) = 2 \leq \frac{15}{4} = \frac{15}{16} \cdot \rho_b(11, 13).
\]

4. For \( v = 12, \omega = 13 \) we have \( \rho_b(v, \omega) = 1 \), \( \rho_b(12, 12) = 4 \), \( \rho_b(13, 13) = \rho_b(13, 11) = 4 \), \( \rho_b(10, 11) = 1 \) and then

\[
\frac{1}{4} \min \{ \rho_b(12, 12), \rho_b(13, 13) \} = 1 \rho_b(v, \omega),
\]

which implies

\[
2 \rho_b(12, 13) = 2 \leq \frac{75}{16} = \frac{15}{16} \cdot \frac{\rho_b(12, 12)(1 + \rho_b(13, 13))}{1 + \rho_b(12, 13)} \leq \frac{15}{16} D_A(12, 13).
\]

Thus, the hypothesis of Theorem 3 are satisfied and \( v = 10 \) is the fixed point of the mapping \( \Omega \).
Definition 7 Let \((A, \rho_b, s > 1)\) be a partial \(b\)-metric space. The mapping \(O : A \to A\) is said to be a \((\eta_b)\)-rational contraction of type \(B\) if there exists \(\eta_b \in \mathcal{P}_b\) such that

\[
\frac{1}{2s} \min\{\rho_b(v, Ov), \rho_b(\omega, O\omega)\} \leq \rho_b(v, \omega),
\]

which implies

\[
\eta_b(\varphi^s \rho_b(Ov, O\omega), D_B(v, \omega)) \geq 0,
\]

for all \(v, \omega \in A\), \(\rho_b(v, \omega) > 0\), where

\[
D_B(v, \omega) = \max \left\{ \rho_b(v, \omega), \rho_b(v, Ov), \frac{\rho_b(O\omega, O\omega) + \rho_b(v, \omega)}{\rho_b(v, Ov) + \rho_b(O\omega, O\omega)} \right\}.
\]

Theorem 4 On a \(\rho_b\)-complete partial \(b\)-metric space \((A, \rho_b, s > 1)\) any continuous \((\eta_b)\)-rational contraction of type \(B\), \(O : A \to A\) admits exactly one fixed point.

Proof Let the sequence \(\{v_m\}\) be defined by (2.5). Since \(v_{m-1} \neq v_m\), for each \(m \in \mathbb{N}\) (by similar reasoning as in the proof of Theorem 3), we have

\[
\frac{1}{2s} \min\{\rho_b(v_m, Ov_m), \rho_b(v_{m+1}, Ov_{m+1})\} = \frac{1}{2s} \min\{\rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2})\}
\]

\[
\leq \rho_b(v_m, v_{m+1}),
\]

which implies

\[
0 \leq \eta_b(\varphi^s \rho_b(Ov_m, Ov_{m+1}), D_B(v_m, v_{m+1}))
\]

\[
< \psi(D_B(v_m, v_{m+1})) - \psi(\varphi^s \rho_b(Ov_m, Ov_{m+1})),
\]

where

\[
D_B(v_m, v_{m+1}) = \max \left\{ \rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2}), \frac{\rho_b(v_{m+1}, v_{m+2}) + \rho_b(v_m, v_{m+1})}{\rho_b(v_{m+1}, v_{m+2}) + \rho_b(v_m, v_{m+1})} \right\}
\]

\[
\leq \max \left\{ \rho_b(v_m, v_{m+1}), \frac{\rho_b(v_{m+1}, v_{m+2}) + \rho_b(v_m, v_{m+1})}{\rho_b(v_{m+1}, v_{m+2}) + \rho_b(v_m, v_{m+1})} \right\}
\]

\[
\leq \max \left\{ \rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2}) \right\}.
\]

Therefore

\[
\psi(\varphi^s \rho_b(v_{m+1}, v_{m+2})) < \psi(D_B(v_m, v_{m+1})) \leq \psi(\max\{\rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2})\})
\]

and since the function \(\psi\) is non-decreasing, we get, for any \(m \in \mathbb{N}\),

\[
\varphi^s \rho_b(v_{m+1}, v_{m+2}) < \max\{\rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2})\}.
\]

Moreover, if \(\max\{\rho_b(v_m, v_{m+1}), \rho_b(v_{m+1}, v_{m+2})\} = \rho_b(v_{m+1}, v_{m+2})\) we get a contradiction, and then it follows that

\[
\rho_b(v_{m+1}, v_{m+2}) < \frac{1}{\varphi^s} \rho_b(v_m, v_{m+1})
\]
and by Lemma (7), we conclude that \( \{ v_n \} \) is a \( 0, \rho_b \)-Cauchy on a \( \rho_b \)-complete \( b \)-partial-metric space, and there exists \( u \in A \) such that \( \lim_{n \to +\infty} v_n = u \).

Taking into account the continuity of the mapping \( O \), we have

\[
u = \lim_{m \to +\infty} v_{m+1} = \lim_{m \to +\infty} O \left( \lim_{m \to +\infty} v_m \right) = Ou,
\]

that is, \( u \) is a fixed point of the mapping \( O \).

We claim that the fixed point of \( O \) is unique. Let \( u, z \in A \) be two different fixed point of \( O \). Then

\[
0 = \frac{1}{2s} \min \{ \rho_b(u, Ou), \rho_b(z, Oz) \} < \rho_b(u, z),
\]

which implies

\[
0 \leq \eta \left( s\rho (ou, oz), D(b, u, z) \right) < \psi \left( D(b, u, z) \right) - \psi \left( s\rho (ou, oz) \right) = \psi \rho (u, z) - \psi \left( s\rho (u, z) \right),
\]

which is a contradiction. Therefore, \( \rho_b(u, z) = 0 \), that is (by Lemma 6), \( u = z \).

\( \square \)

**Example 5** Let the set \( A = [0, 1] \), and \( \rho_b : A \times A \to [0, +\infty) \), \( \rho_b(v, \omega) = (\max \{ v, \omega \})^2 \) be a partial \( b \)-metric on \( A \). Let the continuous mapping \( O : A \to A \) be defined by \( Ov = \left\{ \begin{array}{ll} v^2 & \text{for} \ v \in [0, \frac{2}{3}] \setminus r \omega, \\
\frac{2}{3} & \text{for} \ v \in \left( \frac{2}{3}, 1 \right] \setminus r \omega \end{array} \right. \) and the functions \( \psi \in \Gamma, \eta \in \mathcal{Z}_{\phi_b} \), where \( \psi(t) = \frac{1}{2} \) and \( \eta \left( t, \frac{2}{3} \right) = \frac{8}{9} \left( \frac{1}{2} \right) - \frac{1}{2} \).

We verify that \( O \) is a \( (\eta_b) \)-\( \psi \)-rational contraction of type B.
1. For \( v, \omega \in [0, 2/3] \), if \( v > \omega \), (the case \( v \leq \omega \) is similar), we have

\[
\rho_b(v, \omega) = (\max \{ v, \omega \})^2 = v^2, \quad \rho_b(v, \omega) = (\max \{ v, v^2 \})^2 = v^2,
\]

\[
\rho_b(\omega, \omega) = \omega^2, \quad \rho_b(\omega, v) = (\max \{ v^2, \omega^2 \})^2 = v^2.
\]

Therefore,

\[
\frac{1}{4} \min \{ \rho_b(v, \omega), \rho_b(\omega, \omega) \} = \frac{1}{4} v^2 \leq v^2 = \rho_b(v, \omega),
\]

which implies

\[
2 \rho_b(\omega, v) = 2 v^4 \leq \frac{8}{9} v^2 \leq \frac{8}{9} \rho_b(v, \omega).
\]

2. For \( v, \omega \in (2/3, 1] \), if \( v > \omega \), (the case \( v \leq \omega \) is similar), we have

\[
\rho_b(v, \omega) = (\max \{ v, \omega \})^2 = v^2, \quad \rho_b(v, \omega) = (\max \{ v, \frac{4}{9} \})^2 = v^2,
\]

\[
\rho_b(\omega, \omega) = \omega^2, \quad \rho_b(\omega, v) = \frac{16}{81}.
\]

Therefore,

\[
\frac{1}{4} \min \{ \rho_b(v, \omega), \rho_b(\omega, \omega) \} = \frac{1}{4} v^2 \leq v^2 = \rho_b(v, \omega),
\]
which implies
\[
2 \rho_b(Ov, Ow) = \frac{32}{81} \leq \frac{8}{9} v^2 \leq \frac{8}{9} D_B(v, \omega).
\]

3. For \( v \in [0, 2/3], \omega \in (2/3, 1] \), we have
\[
\rho_b(v, \omega) = \left( \max\{v, \omega\} \right)^2 = \omega^2, \quad \rho_b(v, \omega) = v^2,
\]
\[
\rho_b(\omega, Ow) = \omega^2, \quad \rho_b(Ov, Ow) = \frac{16}{81}.
\]

Therefore,
\[
\frac{1}{4} \min\{\rho_b(v, \omega), \rho_b(\omega, Ow)\} = \frac{1}{4} \omega^2 \leq \omega^2 = \rho_b(v, \omega),
\]
which implies
\[
2 \rho_b(Ov, Ow) = \frac{32}{81} \leq \frac{8}{9} \omega^2 = \frac{8}{9} \rho_b(\omega, Ow) \leq \frac{8}{9} D_B(v, \omega).
\]

Therefore, all the hypotheses of Theorem 2.10 are satisfied and \( v = 0 \) is the unique fixed point of \( O \).

Removing the condition \( \frac{1}{4} \min\{\rho_b(v, \omega), \rho_b(\omega, Ow)\} \leq \rho_b(v, \omega) \) in Theorem 3, respectively, Theorem 4, we immediately obtain the next results.

**Corollary 1** Let \((A, \rho_b, s > 1)\) be a \( \rho_b \)-complete partial \( b \)-metric space and \( O : A \to A \) be a mapping such that there exists \( \eta_b \in \mathcal{Z}_{\psi_b} \) such that
\[
\eta_b \left( s \rho_b(Ov, Ow), D_A(v, \omega) \right) \geq 0
\]
for all \( v, \omega \in A \), where \( D_A \) is defined by (2.4). Then \( O \) has a unique fixed point.

**Corollary 2** Let \((A, \rho_b, s > 1)\) be a \( \rho_b \)-complete partial \( b \)-metric space and \( O : A \to A \) be a continuous mapping such that there exists \( \eta_b \in \mathcal{Z}_{\psi_b} \) such that
\[
\eta_b \left( s \rho_b(Ov, Ow), D_B(v, \omega) \right) \geq 0
\]
for all distinct \( v, \omega \in A \), where \( D_B \) is defined by (2.11). Then \( O \) has a unique fixed point.

**Corollary 3** Let \( O : A \to A \) be a mapping on a \( \rho_b \)-complete partial \( b \)-metric space \((A, \rho_b, s > 1)\). Suppose that \( \psi \in \Gamma \) and \( \phi : [0, +\infty) \to [0, +\infty) \) is a function such that
\[
\lim \inf_{t \to t_0} \phi(t) > 0, \text{ for } t_0 > 0 \text{ and } \phi(t) = 0 \iff t = 0. \]
If for every \( r, t \in A \)
\[
\frac{1}{25} \min\{\rho_b(v, \omega), \rho_b(\omega, Ow)\} \leq \rho_b(v, \omega),
\]
which implies
\[
\psi \left( s \rho_b(Ov, Ow) \right) \leq \psi \left( D_A(v, \omega) \right) - \phi \left( D_A(v, \omega) \right)
\]
then \( O \) admits a unique fixed point.
Proof Let \( \eta_b(r, t) = \psi(t) - \phi(t) - \psi(r) \) in Theorem 3 and take into account Example 2.

Corollary 4 Let \( O : A \to A \) be a continuous mapping on a \( \rho_b \)-complete partial \( b \)-metric space \((A, \rho_b, s > 1)\). Suppose that \( \psi \in \Gamma \) and \( \phi : [0, +\infty) \to [0, +\infty) \) is a function such that \( \liminf_{t \to t_0} \phi(t) > 0 \) for \( t_0 > 0 \) and \( \phi(t) = 0 \Leftrightarrow t = 0 \). If for every distinct \( r, t \in A \)

\[
\frac{1}{2s} \min \{\rho_b(v, ov), \rho_b(o, ow)\} \leq \rho_b(v, w),
\]

which implies

\[
\psi(\rho_b(ov, ow)) \leq \psi(D_B(v, w)) - \phi(D_B(v, w))
\]

then \( O \) admits a unique fixed point.

Proof Let \( \eta_b(r, t) = \psi(t) - \phi(t) - \psi(r) \) in Theorem 4 and take into account Example 3.

Corollary 5 Let \( O : A \to A \) be a mapping on a \( \rho_b \)-complete partial \( b \)-metric space \((A, \rho_b, s > 1)\). Suppose that \( \psi \in \Gamma \) and \( \gamma : [0, +\infty) \to [0, 1) \) is a function such that \( \limsup_{t \to t_0} \gamma(t) < 1 \) for \( t_0 > 0 \) and \( \gamma(t) = 0 \Leftrightarrow t = 0 \). If for every \( r, t \in A \)

\[
\frac{1}{2s} \min \{\rho_b(v, ov), \rho_b(o, ow)\} \leq \rho_b(v, w),
\]

which implies

\[
\psi(\rho_b(ov, ow)) \leq \gamma(A(v, w)) \psi(D_B(v, w))
\]

then \( O \) admits a unique fixed point.

Proof Let \( \eta_b(r, t) = \gamma(t)\psi(t) - \psi(r) \) in Theorem 3 and take into account Example 2.

Corollary 6 Let \( O : A \to A \) be a continuous mapping on a \( \rho_b \)-complete partial \( b \)-metric space \((A, \rho_b, s > 1)\). Suppose that \( \psi \in \Gamma \) and \( \gamma : [0, +\infty) \to [0, 1) \) is a function such that \( \limsup_{t \to t_0} \gamma(t) < 1 \) for \( t_0 > 0 \) and \( \gamma(t) = 0 \Leftrightarrow t = 0 \). If for every \( r, t \in A \), with \( \rho_b(v, w) > 0 \),

\[
\frac{1}{2s} \min \{\rho_b(v, ov), \rho_b(o, ow)\} \leq \rho_b(v, w),
\]

which implies

\[
\psi(\rho_b(ov, ow)) \leq \psi(D_B(v, w)) - \phi(D_B(v, w))
\]

then \( O \) admits a unique fixed point.

Proof Let \( \eta_b(r, t) = \gamma(t)\psi(t) - \psi(r) \) in Theorem 4 and take into account Example 2.

We will prove below results similar to those stated in Theorems 3, 4 that can be formulated for the case \( s = 1 \).
**Theorem 5** Let \((A, \rho)\) be a \(\rho_0\)-complete partial-metric space and \(O : A \to A\) be a mapping. If there exists a function \(\eta \in \mathcal{Z}_\psi\) such that

\[
\frac{1}{2} \min \left\{ \rho(v, Ov), \rho(\omega, \omega) \right\} \leq \rho(v, \omega), \quad \text{which implies}
\]

\[
\eta(\rho(Ov, \omega), \mathcal{D}_A^1(v, \omega)) \geq 0,
\]

(2.13)

for every distinct \(v, \omega \in A\), where \(\mathcal{D}_A^1\) is defined as

\[
\mathcal{D}_A^1(v, \omega) =\max \left\{ \frac{\rho(v, \omega), \rho(v, Ov), \rho(\omega, \omega)[1 + \rho(v, Ov)]}{1 + \rho(v, \omega)} \right\},
\]

(2.14)

then \(O\) admits exactly one fixed point.

**Proof** For \(v_0 \in A\), let \(\{v_n\}\) be the sequence defined by (2.5), \(\rho(v_m, v_{m+1}) > 0\), for any \(m \in \mathbb{N}\). First of all, we claim that \(\lim_{n \to +\infty} \rho(v_m, v_{m+1}) = 0\). From (2.13), we have

\[
\frac{1}{2} \min \left\{ \rho(v_{m-1}, Ov_{m-1}), \rho(v_m, Ov_m) \right\} \leq \frac{1}{2} \min \left\{ \rho(v_{m-1}, v_m), \rho(v_m, v_{m+1}) \right\} \leq \rho(v_m, v_{m+1}),
\]

which implies

\[
0 \leq \eta(\rho(Ov_{m-1}, Ov_m), \mathcal{D}_A^1(v_{m-1}, v_m)) < \psi(\mathcal{D}_A^1(v_{m-1}, v_m)) - \psi(\rho(Ov_{m-1}, Ov_m)).
\]

Consequently, we get

\[
\psi(\rho(Ov_{m-1}, Ov_m)) < \psi(\mathcal{D}_A^1(v_{m-1}, v_m)),
\]

which, since \(\psi\) is non-decreasing, implies

\[
\rho(v_m, v_{m+1}) = \rho(Ov_{m-1}, Ov_m) < \mathcal{D}_A^1(v_{m-1}, v_m) = \max \left\{ \rho(v_{m-1}, v_m), \rho(v_m, v_{m+1}) \right\}.
\]

Therefore, the sequence \(\{\rho(v_m, v_{m+1})\}\) is decreasing, so, we can find \(\theta \geq 0\) such that \(\lim_{m \to +\infty} \rho(v_m, v_{m+1}) = \theta\). On the other hand, it is easy to see that \(\lim_{m \to +\infty} \mathcal{D}_A^1(v_{m-1}, v_m) = \theta\), as well. Assuming that \(\theta > 0\), from (\(\eta_2\)) and (2.13) it follows that

\[
0 \leq \lim_{m \to +\infty} \eta(\rho(v_m, v_{m+1}), \mathcal{D}_A^1(v_{m-1}, v_m)) < 0,
\]

which is a contradiction. So, we found that

\[
\theta = \lim_{m \to +\infty} \rho(v_m, v_{m+1}) = 0.
\]

(2.15)

We claim that \(\{v_m\}\) is a Cauchy sequence. If we suppose that \(\lim_{m,q \to +\infty} \rho(v_m, v_q) \neq 0\), there exist two subsequences \(\{v_{m_l}\}, \{v_{q_l}\}\) of the sequence \(\{v_m\}\) and a number \(e > 0\) such that \(\rho(v_{m_l}, v_{q_l}) > e\).

Moreover, by Lemma 1, we have

\[
\lim_{l \to +\infty} \rho(v_{m_l}, v_{q_l-1}) = e = \lim_{l \to +\infty} \rho(v_{m_{l+1}}, v_{q_l}).
\]

(2.16)
Looking on the definition of the function $D_{1}^{A}$, we have

$$
\rho(v_{m}, v_{q-1}) \leq D_{1}^{A}(v_{m}, v_{q-1}) = \max \left\{ \rho(v_{m}, v_{q-1}), \frac{\rho(v_{m}, v_{m+1})}{\rho(v_{q-1}, v_{q})}, \frac{\rho(v_{m}, v_{ml}) \rho(v_{ml}, v_{ml}+1)}{1+\rho(v_{m}, v_{q-1})} \right\} \quad (2.17)
$$

and keeping in mind (2.15) and (2.16) we get

$$
\lim_{l \to +\infty} D_{1}^{A}(v_{m}, v_{q-1}) = e. \quad (2.18)
$$

Now, letting $r_{l} = \rho(v_{m+1}, v_{q})$ and $t_{l} = D_{1}^{A}(v_{m}, v_{q-1})$, by (η2) we get

$$
\limsup_{l \to +\infty} \eta(\rho(\Omega v_{m}, \Omega v_{q-1}), D_{1}^{A}(v_{m}, v_{q-1})) < 0. \quad (2.19)
$$

On the other hand, by (2.15), we have

$$
\rho(v_{m}, v_{m+1}) < \frac{e}{2} \quad \text{and} \quad \rho(v_{q-1}, v_{q}) < \frac{e}{2}. \quad (2.20)
$$

Thus, by the triangle inequality and taking into account (2.20), we get

$$
e < \rho(v_{m}, v_{q}) \leq \rho(v_{m}, v_{q-1}) + \rho(v_{q-1}, v_{q}) > \rho(v_{m}, v_{q-1}) + \frac{e}{2},
$$

and then $\frac{e}{2} < \rho(v_{m}, v_{q-1})$. Therefore,

$$
\frac{1}{2} \min \{ \rho(\Omega v_{m}, \Omega v_{m}), \rho(v_{q-1}, v_{q}) \} = \frac{1}{2} \min \{ \rho(v_{m}, v_{m+1}), \rho(v_{q-1}, v_{q}) \} \leq \frac{e}{4} < \frac{e}{2} < \rho(v_{m}, v_{q-1}),
$$

which implies

$$
0 \leq \eta(\rho(\Omega v_{m}, \Omega v_{q-1}), D_{1}^{A}(v_{m}, v_{q-1})),
$$

which contradicts (2.19). Thus,

$$
\lim_{m, q \to +\infty} \rho(v_{m}, v_{q}) = 0
$$

and \( \{ v_{n} \} \) is a Cauchy sequence in the complete partial-metric space \( (A, \rho) \). This implies that there exists \( u \in A \) such that

$$
\lim_{m, q \to +\infty} \rho(v_{m}, v_{q}) = 0 = \lim_{m \to +\infty} \rho(v_{m}, u) = \rho(u, u). \quad (2.21)
$$

We shall prove that \( u = \Omega u \). By \( (\rho_{2}) \), we get

$$
\frac{1}{2} \min \{ \rho(\Omega v_{m}, \Omega v_{m}), \rho(u, \Omega u) \} \leq \rho(v_{m}, u),
$$
Let \( u \) which is a contradiction. Thus, we conclude that
\[
\rho(\alpha v_{m}, \alpha u) = 1
\]
and using (2.21) we get \( \alpha = 0 \). Thus, \( u = \alpha u \) and \( u \) is a fixed point of \( \alpha \).

In order to show the uniqueness of the fixed point, let \( u, z \in A \) such that \( u = \alpha u \) and \( z = \alpha z \). We have
\[
0 = \frac{1}{2} \min \rho(u, \alpha u), \quad \rho(z, \alpha z) \leq \rho(u, z),
\]
which implies
\[
0 \leq \eta \left( \rho(\alpha u, \alpha z), D_{\alpha}(u, z) \right)
\]
\[
< \psi \left( \max \left\{ \rho(u, z) , \rho(u, \alpha u) , \rho(z, \alpha z) , \frac{\rho(z, \alpha z) [1 + \rho(u, \alpha u)]}{1 + \rho(u, z)} \right\} \right)
\]
\[
- \psi (\rho(\alpha u, \alpha z))
\]
\[
= \rho(u, z) - \rho(u, z),
\]
which is a contradiction. Thus, we conclude that \( u \) is the unique fixed point of \( \alpha \). \( \square \)

**Theorem 6** Let \((A, \rho)\) be a \( \rho_{0} \)-complete partial-metric space and \( \alpha : A \to A \) be a continuous mapping. If there exists a function \( \eta \in \Theta_{\psi} \) such that
\[
\frac{1}{2} \min \{ \rho(v, \alpha v), \rho(\alpha v, \alpha \alpha v) \} \leq \rho(v, \alpha v), \quad \text{which implies}
\]
\[
\eta \left( \rho(\alpha v, \alpha \alpha v), D_{\alpha}^1(\alpha v, \alpha \alpha v) \right) \geq 0,
\]
holds for every \( v, \alpha v \in A, \rho(v, \alpha v) > 0 \) where \( D_{\alpha}^1 \) is defined as
\[
D_{\alpha}^1(v, \alpha v) = \max \left\{ \rho(v, \alpha v), \rho(v, \alpha v), \rho(\alpha v, \alpha \alpha v), \frac{\rho(v, \alpha v) + \rho(\alpha v, \alpha \alpha v)}{\rho(\alpha v, \alpha \alpha v)} \right\},
\]
then \( \alpha \) admits exactly one fixed point.
Proof Let \( \eta_0 \in \mathcal{A} \) and consider the sequence \( \{v_n\} \), with \( v_n = O(v_{n-1}) \). We assume that 
\( \rho(v_m, v_{m-1}) > 0 \) for each \( m \in \mathbb{N} \) because we remark that, on the contrary, if there exits \( l_0 \) such that \( v_0 = v_{l_0+1} = Ov_{l_0} \), that is \( v_0 \) is a fixed point for the mapping \( O \), then by (2.23), for any terms \( v = v_m \) and \( \omega = v_{m+1} \) we have

\[
\begin{align*}
D_B^1(v_m, v_{m+1}) &= \max \left\{ \frac{\rho(v_m, v_{m+1}), \rho(v_m, Ov_m), \rho(v_{m+1}, Ov_{m+1})}{\rho(Ov_m, Ov_{m+1})}, \frac{\rho(Ov_m, Ov_{m+1})}{\rho(v_m, Ov_{m+1})} \right\} \\
&= \max \left\{ \frac{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})}{\rho(v_m, v_{m+1})}, \frac{\rho(v_m, v_{m+1})}{\rho(v_{m+1}, v_{m+2})} \right\} \\
&\leq \max \left\{ \frac{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})}{\rho(v_m, v_{m+1})}, \frac{\rho(v_m, v_{m+1})}{\rho(v_{m+1}, v_{m+2})} \right\}
\end{align*}
\]

On the other hand, by (2.22),

\[
\frac{1}{2} \min \{\rho(v_m, Ov_m), \rho(v_m, Ov_{m+1})\} = \frac{1}{2} \min \{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})\} \leq \rho(v_m, v_{m+1}),
\]

which implies

\[
0 \leq \eta(\rho(Ov_m, Ov_{m+1}), D_B^1(v_m, v_{m+1})) < \psi(D_B^1(v_m, v_{m+1})) - \psi(\rho(v_m, v_{m+1})).
\]

But \( \psi \in \Gamma \) and then

\[
\rho(v_{m+1}, v_{m+2}) < D_B^1(v_m, v_{m+1}) \leq \max \{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})\}. \tag{2.24}
\]

If for some \( m \), \( \max \{\rho(v_{m+1}, v_{m+2}), \rho(v_m, v_{m+1})\} = \rho(v_m, v_{m+1}) \) then (2.24) becomes \( \rho(v_{m+1}, v_{m+2}) < \rho(v_m, v_{m+1}) \), which is a contradiction. Then, for each \( m \geq 0 \), \( \max \{\rho(v_{m+1}, v_{m+2}), \rho(v_m, v_{m+1})\} = \rho(v_m, v_{m+1}) \), the inequality (2.24) yields

\[
\rho(v_{m+1}, v_{m+2}) < \rho(v_m, v_{m+1}).
\]

Thus, the sequence \( \{\rho(v_m, v_{m+1})\} \) is decreasing, so it is convergent (being bounded from below). In this case, we can find a real number \( u \geq 0 \) such that \( \lim_{m \to +\infty} \rho(v_m, v_{m+1}) = u \). Assume that \( u > 0 \), let \( r_m = \rho(v_{m+1}, v_{m+2}) \) and \( t_m = D_B^1(v_m, v_{m+1}) \). Since

\[
\lim_{m \to +\infty} r_m = \lim_{m \to +\infty} t_m = u,
\]

from (\( \eta_2 \)) we have

\[
0 \leq \lim_{m \to +\infty} \sup_{m \to +\infty} \eta(r_m, t_m) < 0.
\]

This is a contradiction, so that

\[
\lim_{m \to +\infty} \rho(v_m, v_{m+1}) = 0. \tag{2.25}
\]
As a next step, we claim that \( \{v_m\} \) is a Cauchy sequence in \((A, \rho)\). Reasoning by contradiction, we suppose that \( \lim_{m,q \to +\infty} \rho(v_m, v_q) \neq 0 \). Then, by Lemma 1, there exist the subsequences \( \{v_{m_1}\}, \{v_{q_1}\} \) of the sequence \( \{v_m\} \), with \( q_1 > m_1 > l \), and a number \( e > 0 \) such that \( \rho(v_{m_1}, v_{q_1}) \geq e \) and

\[
\lim_{l \to +\infty} \rho(v_{m_1}, v_{q_1}) = e = \lim_{l \to +\infty} \rho(v_{m_1}, v_{q_1}).
\]

Now, according to (2.25), there exists \( n_1 \in \mathbb{N} \), such that

\[
\rho(v_{m_1-1}, v_{m_1}) < \frac{e}{2}, \quad \text{for any } l > n_1
\]

and \( n_2 \in \mathbb{N} \), such that

\[
\rho(v_{q_1}, v_{q_1+1}) < \frac{e}{2}, \quad \text{for any } l > n_2.
\]

Therefore, for \( l > \max\{n_1, n_2\} \) we have

\[
e \leq \rho(v_{m_1}, v_{q_1}) \leq \rho(v_{m_1}, v_{m_1-1}) + \rho(v_{m_1-1}, v_{q_1}) - \rho(v_{m_1-1}, v_{m_1-1}) \leq \rho(v_{m_1-1}, v_{q_1}) + \frac{e}{2} - \rho(v_{m_1-1}, v_{m_1-1})
\]

and we can conclude \( \frac{e}{2} \leq \rho(v_{m_1-1}, v_{q_1}) \). Thus,

\[
\frac{1}{2} \min\{\rho(v_{m_1-1}, v_{m_1}), \rho(v_{q_1}, v_{q_1+1})\} \leq \frac{e}{4} \leq \frac{e}{2} \leq \rho(v_{m_1-1}, v_{q_1}),
\]

which implies

\[
0 \leq \lim_{l \to +\infty} \eta(\rho(Ov_{m_1-1}, Ov_{q_1}), D_B(v_{m_1-1}, v_{q_1})).
\]

On the other hand,

\[
\lim_{l \to +\infty} D_B^1(v_{m_1-1}, v_{q_1}) = \lim_{l \to +\infty} \max_{l \to +\infty} \left\{ \frac{\rho(v_{m_1-1}, v_{q_1}), \rho(v_{m_1-1}, v_{m_1}), \rho(v_{q_1}, v_{q_1+1})}{\rho(v_{m_1-1}, v_{m_1}), \rho(v_{m_1-1}, v_{q_1}), \rho(v_{q_1}, v_{q_1+1})} \right\} = e
\]

and \((\eta_2)\) implies

\[
\lim_{l \to +\infty} \eta(\rho(Ov_{m_1-1}, Ov_{q_1}), D_B(v_{m_1-1}, v_{q_1})) < 0,
\]

which contradicts (2.26). Therefore, \( \{v_m\} \) is a Cauchy sequence in a \( \rho \)-complete partial-metric space \((A, \rho)\) and there exists \( u \in A \) such that

\[
\rho(u, u) = \lim_{m \to +\infty} \rho(v_m, u) = \lim_{m,q \to +\infty} \rho(v_m, v_q) = 0.
\]

On the other hand, due to the continuity of the mapping \( O \), we get

\[
\lim_{m \to +\infty} \rho(Ov_{m+1}, Ou) = \lim_{m \to +\infty} \rho(Ov_m, Ou) = 0.
\]
Consequently, from (2.27), (2.28), on account of Lemma 3, we see that \( u \) is a fixed point of \( O \). The uniqueness of the fixed point follows immediately as in the previous theorem. □

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**Author details**
1 Division of Applied Mathematics, Thu Dau Mot University, 820000, Binh Duong Province, Vietnam. 2 Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey. 3 Department of Medical Research, China Medical University, Taichung, Taiwan. 4 Institute for Computational and Modeling Science, National Tsing Hua University, 521 Nan-Dah Road, Hsinchu City, Taiwan. 5 Department of Mathematics, University of Jeddah, College of Science, Jeddah, Saudi Arabia. 6 Department of Mathematics and Computer Science, Transilvania University of Brașov, Brașov, Romania.

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