Generalized $F(R, T)$ cosmological models with fermionic fields

Koblandy Yerzhanov, Gulnur Bauyrzhan, Ratbay Myrzakulov
Eurasian National University, Nur-Sultan 010008, Kazakhstan; Ratbay Myrzakulov Eurasian International Centre for Theoretical Physics, Nur-Sultan 010009, Kazakhstan

E-mail: yerzhanovkk@gmail.com

Abstract. We investigated the gravity model $F(R, T)$, which interacts with a fermion field in a uniform and isotropic flat spacetime FLRW. The main idea and purpose of the work done was to create a mathematical model and find a particular solution for the scale factor $a$, since it describes the dynamics of the evolution of the Universe. The solutions for this universe are obtained using the Noether symmetry method. With its help, a specific form of the Lagrangian is obtained. And the possible types of the scale factor were found. The evolution of the resulting cosmological model has been investigated.

1. Introduction

The universe is believed to have originated about 14 billion years ago after the Big Bang, when it began to expand rapidly. This period of exponential expansion of the universe is known as the period of initial inflation. Obviously, the inflationary cosmological model is closely related to the theory of gravity. Traditionally, cosmology uses the Einstein-Hilbert gravity model. But he does not describe the initial inflationary stage of the expansion of the Universe. Here it is necessary to use a generalized gravity model containing, for example, the quadratic and higher terms of the Ricci scalar $R$. As a more general model, a model is often considered that contains not only the curvature scalar, but also such a parameter as the torsion scalar $T$, as an element of torsional gravity, which also made it possible to give basic theoretical explanations regarding the acceleration of the Universe at a later time. This article discusses the most general type of gravity modification model - the $F(R, T)$ gravity model.

The fermion field is one of the possible reasons for the expansion of the Universe. the description of the evolution of the Universe in scalar field models is well developed. such models describe the accelerated expansion of early and late times. But there are not so many similar works on models with fermionic fields. At the same time, there are no works describing the evolution of the late time with gravity, written by a fermion field in a generalized form from the teleparallel of gravity. In this paper, we will consider cosmology with a gravitational model in its most general form with a fermion field.

2. $F(R, T)$ gravity with fermionic fields

The corresponding action for $F(R, T)$ model interacting with fermions field in FLRW metric is

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} h(u) F(R, T) + Y - V(u) \right\}$$

(1)
here $\psi = \frac{i}{2} \bar{\psi} \Gamma^\mu \left( \partial_\mu - \Omega_\mu \right) \psi - \bar{\psi} \left( \partial_\mu + \Omega_\mu \right) \Gamma^\mu \psi$ – canonical kinetic term of fermion field, $\Gamma^\mu_{\nu\lambda}$ – Levi-Civita connection, $\psi$ and its adjugate $\bar{\psi} = \psi^\dagger \psi^0$ denote the fermion field, dagger for its complex conjugation, function of time, $g$ is determinant of metric tensor, $R$ is Ricci scalar, $T$ is torsion scalar and $\Omega_\mu = -\frac{1}{4} g^{\sigma\nu} \left[ \Gamma^\nu_{\mu\lambda} - e^\nu_b \partial_\mu e^\lambda_b \right] \Gamma^\sigma \Gamma^\lambda$ is spin connection.

In FRLW metric $R_s = 6 \left( \ddot{a} a + \dot{a}^2 \right)$ and $T_s = 6 \dot{a}^2 a^2$, $\dot{a} = \frac{da}{dt}$. Then

$$S = \int d^4 x a^3 \left( \frac{1}{2} \hbar (u) F(R, T) - \lambda_1 \left( R - 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) - u \right) - \lambda_2 \left( T - 6 \frac{\dot{a}^2}{a^2} - v \right) + Y - V \right).$$

To find $\lambda_1$ we take from variation with respect to $R$ in the form

$$\lambda_1 = \frac{\hbar}{2} F_R.$$

To find $\lambda_2$ we take variation with respect to $T$:

$$\lambda_2 = \frac{\hbar}{2} F_T.$$

Then action take the form

$$S = \int d^4 x L,$$

were

$$L = \frac{\hbar a^3}{2} \left( F - TF_T - RF_R + uF_R + vF_R \right) + 3 \hbar a^2 (F_T - F_R) - 3 \hbar a^2 \left( \hat{R} F_{RR} + \hat{T} F_{RT} \right) -$$

$$-3 \hbar' \left( \bar{\psi} \psi + \bar{\psi} \psi^0 \right) \ddot{a} a F_R + a^3 Y - a^3 V.$$

3. Euler-Lagrange equations

The presence of the Lagrangian makes it possible to use the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\epsilon}} = \frac{\partial L}{\partial \epsilon},$$

where $\epsilon$ is familiar generalized coordinates.

Euler-Lagrange equation for the scale factor $a$ have next form

$$-\frac{F}{2} + \frac{1}{2} (TF_T + RF_R) - 3H^2 (F_T - F_R) + \frac{V}{\hbar} - \frac{Y}{\hbar} + 2 \frac{\dot{\hbar}}{\hbar} H (F_T - F_R) +$$

$$+2H (F_T \dot{\hbar} + F_T F_R) + \frac{1}{2} F_{TR} \dot{R} + F_{TR} \dot{\hbar} + F_{RR} \dot{R} - 2 \frac{\dot{\hbar}}{\hbar} \left( F_{RRR} \dot{R} + F_{RTT} \dot{\hbar} \right) -$$

$$-\dot{R} F_{RR} - F_{RRR} \dot{R}^2 - 2 F_{RRT} \dot{R} \dot{T} - \dot{T}^2 F_{RTT} - \dot{T} F_{RT} - \frac{\dot{\hbar}}{\hbar} F_R = 0.$$

For Ricci scalar $R$:

$$\frac{1}{2} (TF_T + RF_R) - 3 \dot{H} F_{RR} - 6H^2 F_{RR} - 3H^2 F_{TR} = 0.$$  

For torsion scalar $T$:
\[ \frac{1}{2} (TF_{TT} + RF_{RT}) - 3H F_{RT} - 6H^2 F_{RT} - 3H^2 F_{TT} = 0. \] (10)

Energy condition is

\[ E_L = 3H^2 (F_T - F_R) - \frac{1}{2} (F - TF_T - RF_R) - \frac{3}{h} H_F R - 3H \left( F_{RR} R + TF_{RT} \right) + \frac{V}{h} = 0. \] (11)

4. Noether symmetries

In theoretical physics, the Noether symmetry method is a good way to study models. In particular, it is an excellent approach for correcting cosmological models. The main idea of this method is that if the variational integral remains invariant with respect to a continuous group, then the group generator gives the corresponding conservation law. To do this, consider an infinitesimal generator for this model

\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \gamma \frac{\partial}{\partial T} + \delta \frac{\partial}{\partial \psi} + \epsilon \frac{\partial}{\partial \bar{\psi}} + \hat{\alpha} \frac{\partial}{\partial \dot{a}} + \hat{\beta} \frac{\partial}{\partial \dot{R}} + \hat{\gamma} \frac{\partial}{\partial \dot{T}} + \hat{\delta} \frac{\partial}{\partial \dot{\psi}} + \hat{\epsilon} \frac{\partial}{\partial \dot{\bar{\psi}}}. \] (12)

Here \( \alpha, \beta, \gamma, \delta, \epsilon \) are functions of \( a, R, T, \psi, \bar{\psi} \). Time derivatives are:

\[ \dot{\alpha} = \alpha_a \dot{a} + \alpha_R \dot{R} + \alpha_T \dot{T} + \alpha_{\psi} \dot{\psi} + \alpha_{\bar{\psi}} \dot{\bar{\psi}}, \] (13)

\[ \dot{\beta} = \beta_a \dot{a} + \beta_R \dot{R} + \beta_T \dot{T} + \beta_{\psi} \dot{\psi} + \beta_{\bar{\psi}} \dot{\bar{\psi}}, \] (14)

\[ \dot{\gamma} = \gamma_a \dot{a} + \gamma_R \dot{R} + \gamma_T \dot{T} + \gamma_{\psi} \dot{\psi} + \gamma_{\bar{\psi}} \dot{\bar{\psi}}, \] (15)

\[ \dot{\delta} = \delta_a \dot{a} + \delta_R \dot{R} + \delta_T \dot{T} + \delta_{\psi} \dot{\psi} + \delta_{\bar{\psi}} \dot{\bar{\psi}}, \] (16)

\[ \dot{\epsilon} = \epsilon_a \dot{a} + \epsilon_R \dot{R} + \epsilon_T \dot{T} + \epsilon_{\psi} \dot{\psi} + \epsilon_{\bar{\psi}} \dot{\bar{\psi}}. \] (17)

Thus, it is possible to construct the following system of equations

\[ \dot{a}^2 : \quad a_3 h(F_T - F_R) + b_3 h a(F_T T - F_{RR}) + \gamma_3 h a(F_{TT} - F_{RT}) + \alpha_6 h a (F_T - F_R) - \beta_3 h^2 a F_{RR} - \gamma_3 h^2 a F_{RT} - \delta_3 h^2 a^2 F_R - \epsilon_3 h^2 a^2 F_R = 0, \] (18)

\[ \dot{R}^2 : \quad 3a_R h^2 a F_{RR} = 0, \] (19)

\[ \dot{T}^2 : \quad 3a_T h^2 a F_{RT} = 0, \] (20)

\[ \dot{\psi}^2 : \quad \alpha_3 h a^2 F_R \bar{\psi} = 0, \] (21)

\[ \dot{\bar{\psi}}^2 : \quad \bar{\alpha}_3 h a^2 F_R \psi = 0, \] (22)

\[ \ddot{a} \dot{R} : \quad -a_6 h F_{RR} - b_3 h a F_{RR} - \gamma_3 h a^2 F_{RT} - \alpha_6 h a (F_T - F_R) - \beta_3 h^2 a F_{RR} - \gamma_3 h^2 a F_{RT} - \delta_3 h^2 a^2 F_R - \epsilon_3 h^2 a^2 F_R = 0, \] (23)

\[ \ddot{a} \dot{T} : \quad -a_6 h F_{RT} - b_3 h a F_{RT} - \gamma_3 h a^2 F_{RT} - \alpha_6 h a (F_T - F_R) - \beta_3 h^2 a F_{RR} - \gamma_3 h^2 a F_{RT} - \delta_3 h^2 a^2 F_R - \epsilon_3 h^2 a^2 F_R = 0, \] (24)
\begin{align}
\dot{a} \psi & : -\alpha 6 h' \psi a F_R - 3 h' a^2 F_R - 3 h' a^2 F_R T \psi - 3 h' a^2 F_R R \psi + \alpha 6 h a (F_T - F_R) - \\
\dot{a} \bar{\psi} & : -\alpha 6 h' \bar{\psi} a F_R - 3 h' a^2 F_R - 3 h' a^2 F_R T \bar{\psi} - 3 h' a^2 F_R R \bar{\psi} + \alpha 6 h a (F_T - F_R) - \\
\dot{R} T & : \alpha R 3 h a^2 F_R R - \alpha R 3 h a^2 F_R T = 0, \\
\dot{R} \psi & : \alpha R 3 h' \psi a^2 F_R - \alpha R 3 h a^2 F_R R = 0, \\
\dot{R} \bar{\psi} & : \alpha R 3 h' \bar{\psi} a^2 F_R - \alpha R 3 h a^2 F_R R = 0, \\
\dot{T} \psi & : -\alpha T 3 h' \psi a^2 F_R - \alpha R 3 h a^2 F_R T = 0, \\
\dot{T} \bar{\psi} & : -\alpha T 3 h' \bar{\psi} a^2 F_R - \alpha R 3 h a^2 F_R R = 0. \\
\end{align}

5. Solution

From $\dot{R}^2$ and $\dot{T}^2$ two possible solutions. First

\begin{equation}
F_{RR} = F_{RT} = 0.
\end{equation}

In this case we have presumable answer $F = s_1 R + s_2 T$, where $s_1$ and $s_2$ are functions of $(\psi, \bar{\psi})$.

Second possible solution from

\begin{equation}
\alpha R = \alpha T = 0.
\end{equation}

This possible if $\alpha$ have next form $\alpha = \alpha(a)$. From equations for $\dot{a} \dot{R}$ and $\dot{a} \dot{T}$ we can find new function

\begin{equation}
Z \left( a, \psi, \bar{\psi} \right) = -(2 \alpha - \alpha a a) F_R - \beta h a F_R - \gamma h a F_{RT}. 
\end{equation}

So it is possible to exclude $\beta$ and $\gamma$, which allows to get $\alpha$

\begin{equation}
\alpha = \left( \alpha_0 a \right)^n.
\end{equation}

All this give next solution for $F$

\begin{equation}
F = F_0 \ast (R^m + T^n) + F_{01},
\end{equation}

were $F_0$ and $F_{01}$ some function of $\psi, \bar{\psi}$, and $m, n$ is constants.

Now some solutions can be found for the scale factor $a$

\begin{equation}
a = a_{01} \left( t_0 1 t + t_0 2 \right)^q,
\end{equation}

\begin{equation}
a = a_{02} e^{t_0 a t + t_0 4},
\end{equation}

were $a_{01}, a_{02}, t_0 1, t_0 2, t_0 3, t_0 4, g$ - some constants.
6. Conclusion
In this work, the general model $F(R, T)$ of gravity is investigated, which interacts with a fermion field in a homogeneous and isotropic flat spacetime FLRW. The main idea and purpose of the work done was to create a mathematical model and find a particular solution for the scale factor $a$, which describes the dynamics of the evolution of the Universe.

For the calculation, the Euler-Lagrange equations and the Noether symmetry method were used. It is important to clarify that the solution gives both power and exponential solutions. What describes the accelerated expansion of the Universe of the initial and present stages of the evolution of the Universe. Here a solution is obtained, which is a consequence of the gravity model taken in the most general form. The result is a solution that satisfies both Einstein’s classical theory and Starobinsky’s model. It can be concluded that the fermionic field can be considered as a possible impetus to the expansion of the Universe and the observed effects.

7. Acknowledgments
This work was supported by the Ministry of Education and Science of the Republic of Kazakhstan. Grant AP09261147. "Study of the evolution of the Universe on the basis of generalized theories of gravity."

8. References
[1] Saridakis E.N., Shynaray Myrzakul S., Myrzakulov K., Yerzhanov K. Phys. Rev. D 102, 023525 (2020)
[2] R. Myrzakulov, K. Yerzhanov, G. Bauyrzhan. First Hermann-Minkowski meeting on the foundations of spacetime physics, 15-18 May 2017, Albena, Bulgaria.
[3] Bamba K., Myrzakulov R., Razina O., Yerzhanov K. Int. J. Mod. Phys. D, V. 22 (6), 1350023, 2013
[4] Razina O, Tsyba P, Meirbekov B, Myrzakulov R 2019 Int. J. Mod. Phys. D 28 1950126
[5] Razina O, Tsyba P and Sagidullayeva Z 2019 Bull. Univ. Karaganda Phys. 425 94
[6] Razina O V, Tsyba P Yu, Myrzakulov R, Meirbekov B, Shanina Z 2019 J. Phys. Conf. Ser 1391 012164
[7] Tsyba P, Razina O, Barkova Z, Bekov S and Myrzakulov R 2019 J. Phys. Conf. Ser 1391 012162