On an tangent equation by primes

S. I. Dimitrov

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Abstract

In this paper we introduce a new diophantine equation with prime numbers. Let \([\cdot]\) be the floor function. We prove that when \(1 < c < \frac{23}{21}\) and \(\theta > 1\) is a fixed, then every sufficiently large positive integer \(N\) can be represented in the form

\[
N = [p_1^c \tan^\theta(\log p_1)] + [p_2^c \tan^\theta(\log p_2)] + [p_3^c \tan^\theta(\log p_3)],
\]

where \(p_1, p_2, p_3\) are prime numbers. We also establish an asymptotic formula for the number of such representations.

Keywords: Diophantine equation · Tangent equation · Primes

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1 Introduction and main result

Analytical number theorists remember 1937 well when Vinogradov [14] proved the ternary Goldbach problem. He showed that every sufficiently large odd integer \(N\) can be represented in the form

\[
N = p_1 + p_2 + p_3,
\]

where \(p_1, p_2, p_3\) are prime numbers.

Source of detailed proof of Vinogradov’s theorem, beginning with an historical perspective along with an overview of essential lemmas and theorems, can be found in monograph of Rassias [11].

In 1995 Laporta and Tolev [10] investigated an analogue of the Goldbach-Vinogradov theorem. They considered the diophantine equation

\[
N = [p_1^c] + [p_2^c] + [p_3^c],
\]
where \( p_1, p_2, p_3 \) are primes. For \( 1 < c < \frac{17}{16} \) they showed that for the sum

\[
R(N) = \sum_{N=\lfloor p_1^c \rfloor + \lfloor p_2^c \rfloor + \lfloor p_3^c \rfloor} \log p_1 \log p_2 \log p_3
\]

the asymptotic formula

\[
R(N) = \frac{\Gamma^3(1 + 1/c)}{\Gamma(3/c)} N^{3/c-1} + O\left( N^{3/c-1} \exp \left(- (\log N)^{1/3-\epsilon} \right) \right)
\]  

holds.

Afterwards the result of Laporta and Tolev was improved by Kumchev and Nedeva \[9\] to \( 1 < c < \frac{12}{11} \), by Zhai and Cao \[15\] to \( 1 < c < \frac{258}{235} \), by Cai \[3\] to \( 1 < c < \frac{137}{119} \), by Zhang and Li \[16\] to \( 1 < c < \frac{3113}{2703} \), by Baker \[1\] to \( 1 < c < \frac{3581}{3106} \) and this is the best result up to now.

On the other hand recently the author \[4\] showed that when \( 1 < c < \frac{10}{9} \) and \( N \) is a sufficiently large positive number, then for any fixed \( \theta > 1 \), the tangent inequality

\[
\left| p_1^c \tan^\theta(\log p_1) + p_2^c \tan^\theta(\log p_2) + p_3^c \tan^\theta(\log p_3) - N \right| < \left( \frac{2^\theta N}{3^\theta+3} \right)^{\frac{1}{2 \theta}}\left( c^{\frac{1}{2 \theta}} \right)
\]

has a solution in prime numbers \( p_1, p_2, p_3 \).

Motivated by these results in this paper we introduce new diophantine equation with prime numbers. Let \( N \) is a sufficiently large positive integer and \( X \) is an arbitrary solution of the equation

\[
\pi \left[ \frac{\log X}{\pi} \right] + \arctan 2 = \frac{1}{c} \log \frac{N}{2^\theta} .
\]  

(2)

Define

\[
\Delta_1 = e^{\pi \left[ \frac{\log X}{\pi} \right]} + \arctan 1 ;
\]  

(3)

\[
\Delta_2 = e^{\pi \left[ \frac{\log X}{\pi} \right]} + \arctan 2 ;
\]  

(4)

\[
\Gamma = \sum_{\Delta_1 < p_1, p_2, p_3 \leq \Delta_2} \log p_1 \log p_2 \log p_3 .
\]  

(5)

**Theorem 1.** Let \( N \) is a sufficiently large positive integer, \( \theta > 1 \) is a fixed and \( X \) is an arbitrary solution of the equation \( \left( 2 \right) \). Then for any fixed \( 1 < c < \frac{2\theta}{2\theta - 1} \), the asymptotic formula

\[
\Gamma = \frac{\Delta_2^{1-c}}{2^\theta c + 5\theta 2^\theta - 1} X^2 + O\left( X^{3-c} \exp \left(- (\log X)^{1/3-\epsilon} \right) \right)
\]  

holds. Here \( \Delta_2 \) is defined by \( \left( 4 \right) \).
In addition we have the following conjecture.

**Conjecture 1.** Let $N$ is a sufficiently large positive integer and $\theta > 1$ is a fixed. There exists $c_0 > 1$ such that for any fixed $1 < c < c_0$, the tangent equation

$$N = \left[ p_1^c \tan^\theta (\log p_1) \right] + \left[ p_2^c \tan^\theta (\log p_2) \right]$$

has a solution in prime numbers $p_1$, $p_2$.

### 2 Notations

Assume that $N$ is a sufficiently large positive integer. By $\varepsilon$ we denote an arbitrary small positive constant, not the same in all appearances. The letter $p$ with or without subscript will always denote prime number. We denote by $\Lambda(n)$ von Mangoldt’s function. Moreover $e(y) = e^{2\pi i y}$. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of $t$. We recall that $t = [t] + \{t\}$ and $||t|| = \min(\{t\}, 1 - \{t\})$. We denote by $\tau_k(n)$ the number of solutions of the equation $m_1 m_2 \ldots m_k = n$ in natural numbers $m_1$, $\ldots$, $m_k$. Throughout this paper we suppose that $1 < c < \frac{23}{21}$. Assume that $\theta > 1$ is a fixed. Consider the function $t(y)$ defined by

$$t = y^c \tan^\theta (\log y)$$

for

$$y \in [\Delta_1, \Delta_2].$$

The first derivative of $y$ as implicit function of $t$ is

$$y' = \frac{y^{1-c}}{(c \tan(\log y) + \theta \sec^2(\log y)) \tan^{\theta-1}(\log y)}.$$  

Denote

$$\tau = X^{1-c-\varepsilon};$$

$$N_1 = \Delta_1^c \tan^\theta (\log \Delta_1);$$

$$S(\alpha) = \sum_{\Delta_1 < p \leq \Delta_2} e(\alpha [p^c \tan^\theta (\log p)]) \log p;$$

$$\Theta(\alpha) = \sum_{N_1 < m \leq N} \frac{y^{1-c}(m)}{(c \tan (\log y(m)) + \theta \sec^2 (\log y(m))) \tan^{\theta-1} (\log y(m))} e(m \alpha);$$
\begin{align*}
\Gamma_1 &= \int_{-\tau}^{\tau} S^3(\alpha) e(-N\alpha) \, d\alpha; \\
\Gamma_2 &= \int_{\tau}^{1-\tau} S^3(\alpha) e(-N\alpha) \, d\alpha; \\
\Psi_k &= \int_{-1/2}^{1/2} \Theta^k(\alpha) e(-N\alpha) \, d\alpha, \quad k = 1, 2, 3, \ldots; \\
\tilde{\Psi} &= \int_{-\tau}^{\tau} \Theta^3(\alpha) e(-N\alpha) \, d\alpha.
\end{align*}

3 Lemmas

**Lemma 1.** Let \( f(x) \) be a real differentiable function in the interval \([a, b]\). If \( f'(x) \) is a monotonous and satisfies \( |f'(x)| \leq \theta < 1 \). Then we have

\[
\sum_{a < n \leq b} e(f(n)) = \int_a^b e(f(x)) \, dx + O(1).
\]

*Proof.* See ([12], Lemma 4.8). \qed

**Lemma 2.** For any complex numbers \( a(n) \) we have

\[
\left| \sum_{a < n \leq b} a(n) \right|^2 \leq \left( 1 + \frac{b-a}{Q} \right) \sum_{|q| < Q} \left( 1 - \frac{|q|}{Q} \right) \sum_{a < n, n+q \leq b} a(n+q)\overline{a(n)},
\]

where \( Q \) is any positive integer.

*Proof.* See ([8], Lemma 8.17). \qed

**Lemma 3.** Let \( k \geq 0 \) be an integer. Suppose that \( f(t) \) has \( k+2 \) continuous derivatives on \( I \), and that \( I \subseteq (N, 2N) \). Assume also that there is some constant \( F \) such that

\[
|f^{(r)}(t)| \asymp FN^{-r}
\]

for \( r = 1, \ldots, k+2 \). Let \( Q = 2^k \). Then

\[
\left| \sum_{n \in I} e(f(n)) \right| \ll F^{1+\frac{1}{4Q-2}} N^{1-\frac{k+2}{4Q-2}} + F^{-1} N.
\]

The implied constant depends only upon the implied constants in ([18]).
Proof. See ([3], Theorem 2.9).

**Lemma 4.** Let $x, y \in \mathbb{R}$ and $H \geq 3$. Then the formula

$$e(-x\{y\}) = \sum_{|h| \leq H} c_h(x)e(hy) + O\left(\min\left(1, \frac{1}{H\|y\|}\right)\right)$$

holds. Here

$$c_h(x) = \frac{1 - e(-x)}{2\pi i(h + x)}.$$

Proof. See ([2], Lemma 12).

**Lemma 5.** Let $G(n)$ be a complex valued function. Assume further that

$$P > 2, \quad P_1 \leq 2P, \quad 2 \leq U < V \leq Z \leq P,$$

$$U^2 \leq Z, \quad 128UZ^2 \leq P_1, \quad 2^{18}P_1 \leq V^3.$$

Then the sum

$$\sum_{P < n \leq P_1} \Lambda(n)G(n)$$

can be decomposed into $O\left(\log^6 P\right)$ sums, each of which is either of Type I

$$\sum_{M < m \leq M_1} a(m) \sum_{L < l \leq L_1} G(ml)$$

and

$$\sum_{M < m \leq M_1} a(m) \sum_{L < l \leq L_1} G(ml) \log l,$$

where

$$L \geq Z, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P$$

or of Type II

$$\sum_{M < m \leq M_1} a(m) \sum_{L < l \leq L_1} b(l)G(ml)$$

where

$$U \leq L \leq V, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log P, \quad b(l) \ll \tau_5(l) \log P.$$ 

Proof. See ([6]).
Lemma 6. For any real number $t$ and $H \geq 1$, there holds
\[ \min\left(1, \frac{1}{H\|t\|}\right) = \sum_{h=-\infty}^{+\infty} a_h e(ht), \]
where
\[ a_h \ll \min\left(\frac{\log 2H}{H}, \frac{1}{|h|}, \frac{H}{|h|^2}\right). \]

Proof. See ([7], p. 245).

Lemma 7. For the sum denoted by (12) we have
\( i \) \[ \int_{-\tau}^{\tau} |S(\alpha)|^2 \, dt \ll X^{2-c} \log^2 X, \]
\( ii \) \[ \int_{0}^{1} |S(\alpha)|^2 \, dt \ll X \log X. \]

Proof. It follows from the arguments used in ([4], Lemma 8).

4 Proof of the Theorem

From (5), (12), (14) and (15) we have
\[ \Gamma = \int_{0}^{1} S^3(\alpha)e(-N\alpha) \, d\alpha = \Gamma_1 + \Gamma_2. \]
(19)

4.1 Estimation of $\Gamma_1$

We write
\[ \Gamma_1 = (\Gamma_1 - \bar{\Psi}) + (\bar{\Psi} - \Psi_3) + \Psi_3. \]
(20)

Bearing in mind (13) and (16) we obtain
\[ \Psi_1 = \int_{-1/2}^{1/2} \Theta(\alpha)e(-N\alpha) \, d\alpha = y^{1-c}(N) \left( \frac{1}{(c \tan \left( \log y(N) \right) + \theta \sec^2 \left( \log y(N) \right))} \right) \tan^{\theta-1} \left( \log y(N) \right). \]
Suppose that
\[ \Psi_k = \frac{y^{1-c}(N)}{\left( c \tan \left( \log y(N) \right) + \theta \sec^2 \left( \log y(N) \right) \right) \tan^\theta \left( \log y(N) \right)} X^{k-1} + O \left( X^{k-2} \right) \]  
(21)
for
\[ k \geq 2. \]

From (2) and (7) it follows that
\[ y(N) = \Delta_2. \]  
(22)
Now (3), (4), (7), (8), (11), (21) and (22) yields
\[ \Psi_{k+1} = \sum_{N_1 < m \leq N} \frac{y^{1-c}(m)}{\left( c \tan \left( \log y(m) \right) + \theta \sec^2 \left( \log y(m) \right) \right) \tan^\theta \left( \log y(m) \right)} \times \left( \sum_{N_1 < m_1 \leq N - m} \sum_{N_1 < m_2 \leq N - m} \frac{y^{1-c}(m_1)}{\left( c \tan \left( \log y(m_1) \right) + \theta \sec^2 \left( \log y(m_1) \right) \right) \tan^\theta \left( \log y(m_1) \right)} \right) \right. 
\[ \vdots \]
\[ \left( \frac{y^{1-c}(m_2)}{\left( c \tan \left( \log y(m_2) \right) + \theta \sec^2 \left( \log y(m_2) \right) \right) \tan^\theta \left( \log y(m_2) \right)} \right) \right. 
\[ = \sum_{N_1 < m \leq N} \frac{y^{1-c}(m)}{\left( c \tan \left( \log y(m) \right) + \theta \sec^2 \left( \log y(m) \right) \right) \tan^\theta \left( \log y(m) \right)} \times \left( \frac{y^{1-c}(N - m)}{\left( c \tan \left( \log y(N - m) \right) + \theta \sec^2 \left( \log y(N - m) \right) \right) \tan^\theta \left( \log y(N - m) \right)} X^{k-1} + O \left( X^{k-2} \right) \right) \right. 
\[ = \sum_{N_1 < m < N - N_1} \frac{y^{1-c}(m)}{\left( c \tan \left( \log y(m) \right) + \theta \sec^2 \left( \log y(m) \right) \right) \tan^\theta \left( \log y(m) \right)} \times \frac{y^{1-c}(N - m)}{\left( c \tan \left( \log y(N - m) \right) + \theta \sec^2 \left( \log y(N - m) \right) \right) \tan^\theta \left( \log y(N - m) \right)} X^{k-1} + O \left( X^{k-1} \right) \right. 
\[ = \frac{y^{1-c}(N)}{\left( c \tan \left( \log y(N) \right) + \theta \sec^2 \left( \log y(N) \right) \right) \tan^\theta \left( \log y(N) \right)} X^{k} + O \left( X^{k-1} \right). \]

Consequently the supposition (21) is true. Bearing in mind (11), (21) and (22) we deduce
\[ \Psi_k = \frac{\Delta_2^{1-c}}{2^\theta c + 5\theta 2^{\theta - 1}} X^{k-1} + O \left( X^{k-2} \right) \]  
(23)
for \( k \geq 2. \)
Now the asymptotic formula (23) gives us

$$\Psi_3 = \Delta_2^{1-c} \frac{X^2}{2^\theta c + 5^\theta \theta^{-1}} + O(X).$$

(24)

From (14) and (17) we get

$$|\Gamma_1 - \tilde{\Psi}| \ll \int_{-\tau}^{\tau} \left| S^3(\alpha) - \Theta^3(\alpha) \right| d\alpha$$

$$\ll \max_{|\alpha| \leq \tau} \left| S(\alpha) - \Theta(\alpha) \right| \left( \int_{-\tau}^{\tau} |S(\alpha)|^2 d\alpha + \int_{-1/2}^{1/2} |\Theta(\alpha)|^2 d\alpha \right).$$

(25)

By (3), (4), (7), (8), (11), (13) and (22) we obtain

$$\int_{-1/2}^{1/2} |\Theta(\alpha)|^2 d\alpha \ll X^{2-c}.$$  

(26)

Now we shall estimate from above $|S(\alpha) - \Theta(\alpha)|$ for $|\alpha| \leq \tau$. Using (3), (4), (10) and (12) we write

$$S(\alpha) = \sum_{\Delta_1 < p \leq \Delta_2} e(\alpha p^c \tan^\theta (\log p)) \log p + O(\tau X)$$

$$= \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n) e(\alpha n^c \tan^\theta (\log n)) + O(X^{1/2}) + O(\tau X)$$

$$= \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n) e(\alpha n^c \tan^\theta (\log n)) + O(X^{1-\varepsilon}).$$

(27)

From $|\alpha| \leq \tau$, $y \geq N_1$ and Lemma 4 we have that

$$\sum_{N_1 < m \leq y} e(m\alpha) = \int_{N_1}^{y} e(\alpha t) dt + O(1).$$

(28)
Using (3), (4), (7), (8), (9), (10), (11), (13), (22), (28) and partial summation we find

\[ \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n)e(\alpha n^c \tan^\theta (\log n)) = \int_{\Delta_1}^{\Delta_2} e(\alpha y^c \tan^\theta (\log y)) \, dy + O\left( X \exp\left( - (\log X)^{1/3} \right) \right) \]

\[ = \int_{N_1}^{N} e(\alpha t) \frac{y^{1-c}(t)}{(c \tan (\log y(t)) + \theta \sec^2 (\log y(t))) \tan^{\theta-1} (\log y(t))} \, dt \]

\[ + O\left( X \exp\left( - (\log X)^{1/3} \right) \right) \]

\[ = \int_{N_1}^{N} \frac{y^{1-c}(t)}{(c \tan (\log y(t)) + \theta \sec^2 (\log y(t))) \tan^{\theta-1} (\log y(t))} \, dt \left( \sum_{N_1 < m \leq t} e(m \alpha) + O(1) \right) \]

\[ + O\left( X \exp\left( - (\log X)^{1/3} \right) \right) \]

\[ = \sum_{N_1 < m \leq N} \frac{y^{1-c}(m)}{(c \tan (\log y(m)) + \theta \sec^2 (\log y(m))) \tan^{\theta-1} (\log y(m))} e(m \alpha) \]

\[ + O\left( X \exp\left( - (\log X)^{1/3} \right) \right) \]

\[ = \Theta(\alpha) + O\left( X \exp\left( - (\log X)^{1/3} \right) \right). \] 

By (27) and (29) it follows that

\[ \max_{|\alpha| \leq \tau} |S(\alpha) - \Theta(\alpha)| \ll X \exp\left( - (\log X)^{1/3} \right). \] 

(30)

Taking into account (25), (26), (30) and Lemma 7 we get

\[ \Gamma_1 - \tilde{\Psi} \ll X^{3-c} \exp\left( - (\log X)^{1/3-\epsilon} \right). \] 

(31)
Using (3), (4), (7), (8), (9), (10), (11), (13), (16), (17), (22) and working as in ([13], Lemma 2.8) we deduce

\[ |\Psi_3 - \tilde{\Psi}| \ll \int_{\tau \leq |\alpha| \leq 1/2} |\Theta(\alpha)|^3 d\alpha \ll \int_{\tau} \alpha^{-\frac{3}{\varepsilon}} d\alpha \ll X^{3-c-\varepsilon}. \]  

(32)

Summarizing (20), (24), (31) and (32) we obtain

\[ \Gamma_1 = \Delta_{2}^{1-c} \frac{\Delta_{1}}{2^c + 5\theta 2^{\theta-1}X^2} + O\left( X^{3-c} \exp \left( - (\log X)^{1/3-\varepsilon} \right) \right). \]  

(33)

4.2 Estimation of \( \Gamma_2 \)

**Lemma 8.** Assume that \( \tau \leq \alpha \leq 1 - \tau \).

\[ \text{Set} \quad S_I = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq M_1} a(m) \sum_{\Delta_1 < ml \leq \Delta_2} e ((h + \alpha) m^{c} l^{c} \tan^{\theta} (\log(ml))) \]  

(35)

and

\[ S'_I = \sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq M_1} a(m) \sum_{\Delta_1 < ml \leq \Delta_2} e ((h + \alpha) m^{c} l^{c} \tan^{\theta} (\log(ml))) \log l, \]  

(36)

where

\[ L \geq X^{\frac{22}{45}}, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log \Delta_1, \quad H = X^\frac{4-3c}{15} \]  

(37)

and \( c_h(\alpha) \) denote complex numbers such that \( |c_h(\alpha)| \ll (1 + |h|)^{-1} \).

Then

\[ S_I, S'_I \ll X^{\frac{11+3c}{15}+\varepsilon}. \]

**Proof.** First we notice that (3), (4), (35) and (37) imply

\[ LM \asymp X. \]  

(38)

Denote

\[ f_h(l, m) = m^{c} l^{c} \tan^{\theta} (\log(ml)). \]  

(39)
By (3), (35), (37) and (39) we write

\[ S_I \ll X^\varepsilon \max_{|\eta| \in [\tau, H+1]} \sum_{M < m \leq M_1} \left| \sum_{L' < l \leq L'_1} e(\eta f(l, m)) \right|, \tag{40} \]

where

\[ L' = \max \left\{ L, \frac{\Delta_1}{m} \right\}, \quad L'_1 = \min \left\{ L_1, \frac{\Delta_2}{m} \right\}. \tag{41} \]

From (37) and (41) for the sum in (40) it follows

\[ \left| \sum_{M < m \leq M_1} \sum_{L' < l \leq L'_1} e\left( \frac{\eta f(l, m)}{m} \right) \right|, \tag{42} \]

On the other hand for the function defined by (39) we find

\[ \frac{\partial f(l, m)}{\partial l} = m^c l^{c-1} \tan^{\theta-1}(\log(ml)) \left( c \tan(\log(ml)) + \theta \sec^2(\log(ml)) \right) \tag{43} \]

and

\[ \frac{\partial^2 f(l, m)}{\partial l^2} = m^c l^{c-2} \tan^{\theta-2}(\log(ml)) \left( 2\theta \sec^2(\log(ml)) + c^2 - c \right) \tan(\log(ml)) \]

\[ + (2c - 1)\theta \sec^2(\log(ml)) \tan(\log(ml)) + (\theta^2 - \theta) \sec^4(\log(ml)) \]. \tag{44} \]

Now (3), (4), (37), (42), (43) and (44) yields

\[ \frac{\partial f(d, l)}{\partial l} \simeq M^c L^{c-1} \tag{45} \]

and

\[ \frac{\partial^2 f(d, l)}{\partial l^2} \simeq M^c L^{c-2}. \tag{46} \]

Proceeding in the same way we get

\[ \frac{\partial^3 f(d, l)}{\partial l^3} \simeq M^c L^{c-3}. \tag{47} \]

Using (10), (34), (37), (38), (40), (42), (45), (46), (47) and Lemma with \( k = 1 \) we obtain

\[ S_I \ll X^\varepsilon \max_{|\eta| \in [\tau, H+1]} \sum_{M < m \leq M_1} \left( |\eta|^{\frac{1}{2}} L^\frac{1}{2} + |\eta|^{-1} M^{-\varepsilon} L^{1-c} \right) \]

\[ \ll X^\varepsilon \left( M^{\frac{1}{2}} H^\frac{1}{2} X^{\frac{1}{2}} + \tau^{-1} X^{1-c} \right) \]

\[ \ll X^{\frac{31}{15} + \varepsilon}. \]
To estimate the sum defined by (36) we apply Abel’s summation formula and proceed in the same way to deduce

\[ S_I' \ll X^{\frac{11+3c}{10}+\varepsilon}. \]

This proves the lemma.

**Lemma 9.** Assume that

\[ \tau \leq \alpha \leq 1 - \tau. \]  
(48)

Set

\[ S_{II} = \sum_{|h| \leq H} c_h(\alpha) \sum_{M \leq m \leq M_1} \sum_{\Delta_1 \leq ml \leq \Delta_2} a(m) b(l) e((h + \alpha)ml^c \tan^\theta (\log(ml))) \]  
(49)

where

\[ 2^{11} X^{\frac{3}{8}} \leq L \leq 2^7 X^{\frac{3}{4}}, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \]

\[ a(m) \ll \tau_5(m) \log \Delta_1, \quad b(l) \ll \tau_5(l) \log \Delta_1, \quad H = X^{\frac{4+3c}{10}} \]  
(50)

and \( c_h(\alpha) \) denote complex numbers such that \(|c_h(\alpha)| \ll (1 + |h|)^{-1}\).

Then

\[ S_{II} \ll X^{\frac{11+3c}{10}+\varepsilon}. \]

**Proof.** First we notice that (3), (4), (49) and (50) give us

\[ LM \asymp X. \]  
(51)

From (3), (39), (49), (50), (51), Cauchy’s inequality and Lemma 2 with \( Q = X^{\frac{8-6c}{10}} \) it follows

\[ S_{II} \ll \sum_{|h| \leq H} |c_h(\alpha)| \sum_{M \leq m \leq M_1} \sum_{\Delta_1 \leq ml \leq \Delta_2} a(m) b(l) e((h + \alpha)ml^c \tan^\theta (\log(ml))) \]

\[ \ll \sum_{|h| \leq H} |c_h(\alpha)| \left( \sum_{M \leq m \leq M_1} |a(m)|^2 \right)^{\frac{1}{2}} \left( \sum_{L \leq l \leq L_1} \sum_{\Delta_1 \leq ml \leq \Delta_2} b(l) e((h + \alpha)f_h(l, m)) \right)^{\frac{1}{2}} \]

\[ \ll M^{\frac{1}{2}+\varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left( \sum_{M \leq m \leq M_1} \frac{L}{Q} \sum_{|q| \leq Q} \left( 1 - \frac{q}{Q} \right) \sum_{\Delta_1 \leq ml \leq \Delta_2} b(l + q)b(l) e(f_h(l, m, q)) \right)^{\frac{1}{2}} \]  

\[ \ll M^{\frac{1}{2}+\varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left( \sum_{M \leq m \leq M_1} \frac{L}{Q} \sum_{|q| \leq Q} \left( 1 - \frac{q}{Q} \right) \sum_{\Delta_1 \leq ml \leq \Delta_2} b(l + q)b(l) e(f_h(l, m, q)) \right)^{\frac{1}{2}} \]
\[ \ll M^{\frac{1}{2} + \varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{L}{Q} \sum_{M < m \leq M_1} \left( L^{1 + \eta} \right) \right) \]
\[ \quad + \sum_{1 \leq |q| < Q} \left( 1 - \frac{q}{Q} \right) \sum_{L < l, l + q \leq L_1} \sum_{\Delta_1 < m(l + q) \leq \Delta_2} b(l + q) \overline{b(l)} e(f_h(l, m, q)) \] \[ \ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + \frac{X}{Q} \sum_{1 \leq |q| < Q} \sum_{L < l, l + q \leq L_1} \sum_{M' < m \leq M'_1} e(f_h(l, m, q)) \right) \] \[ \text{where} \]
\[ M' = \max \left\{ M, \frac{\Delta_1}{l}, \frac{\Delta_1}{l + q} \right\}, \quad M' = \min \left\{ M_1, \frac{\Delta_2}{l}, \frac{\Delta_2}{l + q} \right\} \]
\[ f_h(l, m, q) = (h + \alpha) m^c \left( (l + q)^c \tan^\theta \log(m(l + q)) - l^c \tan^\theta \log(ml) \right) \] .

From (50) and (53) for the sum in (52) it follows
\[ L < l, l + q \leq L_1 \\
M' < m \leq M'_1 \\
\Delta_1 < ml \leq \Delta_2 \\
\Delta_1 < m(l + q) \leq \Delta_2 \\
(M', M'_1) \subseteq (M, 2M) \]

We have
\[ \frac{\partial f_h(l, m, q)}{\partial m} = (h + \alpha)(q + l)^c m^{c-1} \tan^{\theta-1} \log((q + l)m) \]
\[ \times \left( c \tan \log((q + l)m) + \theta \sec^2 \log((q + l)m) \right) \]
\[ - (h + \alpha) l^c m^{c-1} \tan^{\theta-1} \log(ml) \left( c \tan \log(ml) \right) + \theta \sec^2 \log(ml) \] (55)

and
\[ \frac{\partial^2 f_h(l, m, q)}{\partial m^2} = (h + \alpha)(q + l)^c m^{c-2} \tan^{\theta-2} \log((q + l)m) \]
\[ \times \left( 2\theta \sec^2 \log((q + l)m) + c^2 - c \right) \tan^2 \log((q + l)m) \]
\[ + (2c - 1) \theta \sec^2 \log((q + l)m) \tan \log((q + l)m) \]
\[ + (\theta^2 - \theta) \sec^4 \log((q + l)m) \right) \]
\[ - (h + \alpha) l^c m^{c-2} \tan^{\theta-2} \log(ml) \left( 2\theta \sec^2 \log(ml) + c^2 - c \right) \tan^2 \log(ml) \]
\[ + (2c - 1) \theta \sec^2 \log(ml) \tan \log(ml) + (\theta^2 - \theta) \sec^4 \log(ml) \right) . \] (56)
From (3), (4), (50), (54), (55) and (56) we obtain
\[
\frac{\partial f_h(l, m, q)}{\partial m} \asymp |h + \alpha|L^c M^{c-1}
\]
(57)
and
\[
\frac{\partial^2 f_h(l, m, q)}{\partial m^2} \asymp |h + \alpha|L^c M^{c-2}.
\]
(58)
Now (10), (48), (50), (51), (52), (54), (57), (58) and Lemma 3 with \(k = 0\) imply
\[
S_{II} \ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + X \sum_{1 \leq q \leq Q} \sum_{L < l \leq L_1} \left( |h + \alpha| \frac{1}{2}L^\frac{c}{2} M^{\frac{c}{2}} + |h + \alpha|^{-1}L^{-c}M^{1-c} \right) \right) \frac{1}{2}
\]
\[
\ll X^\varepsilon \sum_{|h| \leq H} |c_h(\alpha)| \left( \frac{X^2}{Q} + X \left( H^{-\frac{c}{2}} L X^{\frac{c}{2}} + \tau^{-1} L^{1-c} M^{1-c} \right) \right) \frac{1}{2}
\]
\[
\ll X^{\frac{11+3c}{15} + \varepsilon} \sum_{|h| \leq H} |c_h(\alpha)| \ll X^{\frac{11+3c}{15} + \varepsilon} \sum_{|h| \leq H} \frac{1}{1 + |h|}
\]
\[
\ll X^{\frac{11+3c}{15} + \varepsilon}.
\]
This proves the lemma.

\[\square\]

**Lemma 10.** Let
\[
\tau \leq \alpha \leq 1 - \tau.
\]
Then for the exponential sum denoted by (12) we have
\[
S(\alpha) \ll X^{\frac{11+3c}{15} + \varepsilon}.
\]

**Proof.** In order to prove the lemma we will use the formula
\[
S(\alpha) = S^*(\alpha) + O\left( X^{\frac{c}{2}} \right),
\]
(59)
where
\[
S^*(\alpha) = \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n)e\left( \alpha n^c \tan^\theta(\log n) \right).
\]
(60)
By (60) and Lemma 4 with \(x = \alpha, y = n^c, H = X^{\frac{11+3c}{15}}\) we get
\[
S^*(\alpha) = \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n)e\left( \alpha n^c \tan^\theta(\log n) \right)
\]
\[
= \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n)e\left( \alpha n^c \tan^\theta(\log n) \right) e\left( - \alpha \left( n^c \tan^\theta(\log n) \right) \right)
\]
\[
= \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n)e\left( \alpha n^c \tan^\theta(\log n) \right)
\]

14
\[
\times \left( \sum_{|h| \leq H} c_h(\alpha) e\left( \alpha n^c \tan^\theta (\log n) \right) + O\left( \min\left( 1, \frac{1}{H_n^c \tan^\theta (\log n)} \right) \right) \right)
\]

\[
= \sum_{|h| \leq H} c_h(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n) e\left( (h + \alpha) n^c \tan^\theta (\log n) \right)
\]

\[
+ O\left( \log X \sum_{\Delta_1 < n \leq \Delta_2} \min\left( 1, \frac{1}{H_n^c \tan^\theta (\log n)} \right) \right)
\]

\[
= S^*_0(\alpha) + O\left( \log X \sum_{\Delta_1 < n \leq \Delta_2} \min\left( 1, \frac{1}{H_n^c \tan^\theta (\log n)} \right) \right),
\]

where

\[
S^*_0(\alpha) = \sum_{|h| \leq H} c_h(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} \Lambda(n) e\left( (h + \alpha) n^c \tan^\theta (\log n) \right).
\]

Taking into account (3) and (4) we have that

\[
\Delta_2 < 2\Delta_1, \quad 2^{-4} X < \Delta_1 < 2^2 X, \quad 2^{-3} X < \Delta_2 < 2^2 X.
\]

Let

\[
U = 2^{-11} X^{\frac{1}{15}}, \quad V = 2^7 X^{\frac{1}{3}}, \quad Z = X^{\frac{22}{35}}.
\]

According to Lemma 5 the sum \( S^*_0(\alpha) \) can be decomposed into \( O\left( \log^6 \Delta_1 \right) \) sums, each of which is either of Type I

\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq M_1} a(m) \sum_{\Delta_1 < ml \leq \Delta_2} e\left( (h + \alpha) ml^c \tan^\theta (\log(ml)) \right)
\]

and

\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq M_1} a(m) \sum_{\Delta_1 < ml \leq \Delta_2} e\left( (h + \alpha) ml^c \tan^\theta (\log(ml)) \right) \log l,
\]

where

\[
L \geq Z, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log \Delta_1
\]

or of Type II

\[
\sum_{|h| \leq H} c_h(\alpha) \sum_{M < m \leq M_1} a(m) \sum_{\Delta_1 < ml \leq \Delta_2} b(l) e\left( (h + \alpha) ml^c \tan^\theta (\log(ml)) \right),
\]

where

\[
U \leq L \leq V, \quad M_1 \leq 2M, \quad L_1 \leq 2L, \quad a(m) \ll \tau_5(m) \log \Delta_1, \quad b(l) \ll \tau_5(l) \log \Delta_1.
\]
Using (62), Lemma 8 and Lemma 9 we obtain

\[ S_0^*(\alpha) \ll X^{1 + \frac{3\epsilon}{15} + \epsilon}. \]  

(64)

We have

\[ \frac{\partial k y^c \tan^\theta (\log y)}{\partial y} = k y^{-1} \tan^{-1} (\log y) \left( c \tan (\log y) + \theta \sec^2 (\log y) \right) \]  

(65)

and

\[ \frac{\partial^2 k y^c \tan^\theta (\log y)}{\partial y^2} = k y^{-2} \tan^{-2} (\log y) \left( (2\theta \sec^2 (\log y) + c^2 - c) \tan (\log y) + (2c - 1) \theta \sec^2 (\log y) \tan (\log y) + (\theta^2 - \theta) \sec^4 (\log y) \right). \]  

(66)

From (3), (4), (63), (65) and (66) it follows

\[ \frac{\partial k y^c \tan^\theta (\log y)}{\partial y} \asymp |k| \Delta_1^{c-1} \quad \text{for} \quad y \in [\Delta_1, \Delta_2] \]  

(67)

and

\[ \frac{\partial^2 k y^c \tan^\theta (\log y)}{\partial y^2} \asymp |k| \Delta_1^{c-2} \quad \text{for} \quad y \in [\Delta_1, \Delta_2]. \]  

(68)

Using (3), (4), (63), (67), (68), Lemma 3 with \( k = 0 \) and Lemma 6 we write

\[ \sum_{\Delta_1 < n \leq \Delta_2} \min \left( 1, \frac{1}{H \| n^c \tan^\theta (\log n) \|} \right) \]

\[ = \sum_{\Delta_1 < n \leq \Delta_2} \sum_{k = -\infty}^{+\infty} a_k e \left( k n^c \tan^\theta (\log n) \right) \ll \sum_{k = -\infty}^{+\infty} |a_k| \sum_{\Delta_1 < n \leq \Delta_2} e \left( k n^c \tan^\theta (\log n) \right) \]

\[ \ll \frac{X \log 2H}{H} + \sum_{1 \leq k \leq H} \frac{1}{k} \left| \sum_{\Delta_1 < n \leq \Delta_2} e \left( k n^c \tan^\theta (\log n) \right) \right| + \sum_{k > H} \frac{H}{k^2} \left| \sum_{\Delta_1 < n \leq \Delta_2} e \left( k n^c \tan^\theta (\log n) \right) \right| \]

\[ \ll \frac{X \log 2H}{H} + \sum_{1 \leq k \leq H} \frac{1}{k} \left( k^{\frac{3}{2}} X^{\frac{3}{2}} + k^{-1} X^{1-c} \right) + \sum_{k > H} \frac{H}{k^2} \left( k^{\frac{3}{2}} X^{\frac{3}{2}} + k^{-1} X^{1-c} \right) \]

\[ \ll X^\epsilon \left( H^{-1} X + H^{\frac{3}{2}} X^{\frac{3}{2}} + X^{1-c} \right) \ll X^{1 + \frac{3\epsilon}{15} + \epsilon}. \]  

(69)

Summarizing (59), (61), (64) and (69) we establish the statement in the lemma.

\[ \square \]

**Lemma 11.** Let \( 0 < \alpha < 1 \). Set

\[ A(\alpha) = \sum_{\Delta_1 < n \leq \Delta_2} e \left( \alpha [n^c \tan^\theta (\log n)] \right). \]  

(70)

Then

\[ A(\alpha) \ll X^{1 + \frac{3\epsilon}{15} + \epsilon} + \frac{X^{1-c}}{\alpha}. \]
Proof. By (70) and Lemma 4 with \(x = \alpha, y = n^c\), \(H_0 = X^{\frac{2+c}{4}}\) we find

\[
A(\alpha) = \sum_{\Delta_1 < n \leq \Delta_2} e\left(\alpha n^c \tan^\theta (\log n) - \alpha \{n^c \tan^\theta (\log n)\}\right)
\]

\[
= \sum_{\Delta_1 < n \leq \Delta_2} e\left(\alpha n^c \tan^\theta (\log n)\right) e\left(- \alpha \{n^c \tan^\theta (\log n)\}\right)
\]

\[
= \sum_{\Delta_1 < n \leq \Delta_2} e\left(\alpha n^c \tan^\theta (\log n)\right)
\]

\[
\times \left(\sum_{|h| \leq H_0} c_h(\alpha) e\left(\alpha n^c \tan^\theta (\log n) + O\left(\min (1, \frac{1}{H_0 \|n^c \tan^\theta (\log n)\|})\right)\right)\right)
\]

\[
= \sum_{|h| \leq H_0} c_h(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} e\left((h + \alpha) n^c \tan^\theta (\log n)\right)
\]

\[
+ O\left(\sum_{\Delta_1 < n \leq \Delta_2} \min \left(1, \frac{1}{H_0 \|n^c \tan^\theta (\log n)\|}\right)\right)
\]

\[
= A_0(\alpha) + O\left(\sum_{\Delta_1 < n \leq \Delta_2} \min \left(1, \frac{1}{H_0 \|n^c \tan^\theta (\log n)\|}\right)\right), \quad (71)
\]

where

\[
A_0(\alpha) = \sum_{|h| \leq H_0} c_h(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} e\left((h + \alpha) n^c \tan^\theta (\log n)\right). \quad (72)
\]

From (3), (4), (63), (67), (68), (72) and Lemma 3 with \(k = 0\) we deduce

\[
A_0(\alpha) = c_0(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} e\left(\alpha n^c \tan^\theta (\log n)\right) + \sum_{1 \leq |h| \leq H_0} c_h(\alpha) \sum_{\Delta_1 < n \leq \Delta_2} e\left((h + \alpha) n^c \tan^\theta (\log n)\right)
\]

\[
\ll X^{\frac{c}{2}} + \alpha^{-1} X^{1-c} + \sum_{1 \leq |h| \leq H_0} \frac{1}{h} (h + \alpha)^{\frac{3}{4}} X^{\frac{c}{4}} + (h + \alpha)^{-1} X^{1-c}
\]

\[
\ll X^{\frac{c}{2}} + \alpha^{-1} X^{1-c} + H_0^{\frac{1}{4}} X^{\frac{c}{4}} + X^{1-c} \ll X^{1+c} + \alpha^{-1} X^{1-c}. \quad (73)
\]

On the other hand (69) gives us

\[
\sum_{\Delta_1 < n \leq \Delta_2} \min \left(1, \frac{1}{H_0 \|n^c \tan^\theta (\log n)\|}\right) \ll X^\varepsilon \left(H_0^{-1} X + H_0^{\frac{1}{4}} X^{\frac{c}{4}}\right) \ll X^{\frac{1+c}{4} + \varepsilon}. \quad (74)
\]

Now the lemma follows from (71), (73) and (74).

We are now in a good position to estimate \(\Gamma_2\) defined by (15). We use Cai's
argument. From (3), (4) and (15) we write

$$|Γ_2| = \left| \sum_{Δ_1<p≤Δ_2} (\log p) \int_{τ}^{1/τ} S^2(α)e(α[p^c \tan^θ(\log p)] - Nα) \, dα \right| \leq \sum_{Δ_1<p≤Δ_2} (\log p) \left| \int_{τ}^{1/τ} S^2(α)e(α[p^c \tan^θ(\log p)] - Nα) \, dα \right| \leq (\log X) \sum_{Δ_1<n≤Δ_2} \left| \int_{τ}^{1/τ} S^2(α)e(α[n^c \tan^θ(\log n)] - Nα) \, dα \right|. \quad (75)$$

By (3), (4), (75) and Cauchy’s inequality we deduce

$$|Γ_2|^2 \leq X(\log X)^2 \sum_{Δ_1<n≤Δ_2} \left| \int_{τ}^{1/τ} S^2(α)e(α[n^c \tan^θ(\log n)] - Nα) \, dα \right|^2 \leq X(\log X)^2 \sum_{Δ_1<n≤Δ_2} \int_{τ}^{1/τ} S^2(α)e(α[n^c \tan^θ(\log n)] - Nα) \, dα \leq X(\log X)^2 \int_{τ}^{1/τ} S^2(β)e(β[n^c \tan^θ(\log n)] - Nβ) \, dβ \leq X(\log X)^2 \int_{τ}^{1/τ} S^2(β)e(β-n^c) \, dβ \int_{τ}^{1/τ} S^2(α)A(α - β)e(-Nα) \, dα \leq X(\log X)^2 \int_{τ}^{1/τ} |S(β)|^2 \, dβ \int_{τ}^{1/τ} |S(α)|^2 |A(α - β)| \, dα. \quad (76)$$

Using Lemma 7, Lemma 10 and Lemma 11 we obtain

$$\int_{τ}^{1/τ} |S(α)|^2 |A(α - β)| \, dα \ll \int_{τ}^{1/τ} \frac{|S(α)|^2 |A(α - β)| \, dα}{|α-β|≤X^{-c}} + \int_{τ}^{1/τ} |S(α)|^2 \left(\frac{X^{1-c}}{|α-β|} + \frac{X^{1-c}}{|α-β|} \right) \, dα$$

18
\[
\ll X \max_{\tau \leq \alpha \leq 1 - \tau} |S(\alpha)|^2 \int_{|\alpha - \beta| \leq X^{-c}} d\alpha + X^{\frac{1+c}{4}+\varepsilon}\int_{\tau}^{1-\tau} |S(\alpha)|^2 d\alpha \\
+ X^{1-c} \max_{\tau \leq \alpha \leq 1 - \tau} |S(\alpha)|^2 \int_{X^{-c} < |\alpha - \beta| \leq 2-2\tau} \frac{1}{|\alpha - \beta|} d\alpha \\
\ll X^{1-c+\varepsilon} \max_{\tau \leq \alpha \leq 1 - \tau} |S(\alpha)|^2 + X^{\frac{3}{15}+\varepsilon} \\
\ll X^{3-c-\varepsilon}.
\]

Bearing in mind \((76)\) and \((77)\) and Lemma \(7\) we get

\[
\Gamma_2 \ll X^{3-c-\varepsilon}.
\]

4.3 The end of the proof

Summarizing \((19)\), \((33)\) and \((78)\) we establish the asymptotic formula \((6)\).

The Theorem is proved.

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S. I. Dimitrov
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
8, St.Kliment Ohridski Blvd.
1756 Sofia, BULGARIA
e-mail: sdimitrov@tu-sofia.bg