FIVE-LINEAR SINGULAR INTEGRAL ESTIMATES OF BRASCAMP-LIEB TYPE

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ABSTRACT. We prove the full range of estimates for a five-linear singular integral of Brascamp-Lieb type. The study is methodology-oriented with the goal to develop a sufficiently general technique to estimate singular integral variants of Brascamp-Lieb inequalities that are not of Hölder type. The invented methodology constructs localized analysis on the entire space from local information on its subspaces of lower dimensions and combines such tensor-type arguments with the generic localized analysis. A direct consequence of the boundedness of the five-linear singular integral is a Leibniz rule which captures nonlinear interactions of waves from transversal directions.

1. Introduction

1.1. Background and Motivation. Brascamp-Lieb inequalities refer to inequalities of the form

\begin{equation}
\int_{\mathbb{R}^n} \left| \prod_{j=1}^{m} F_j(L_j(x)) \right| dx \leq BL(L,p) \prod_{j=1}^{m} \left( \int_{\mathbb{R}^{n_j}} |F_j|^{p_j} \right)^{\frac{1}{p_j}},
\end{equation}

where \( BL(L,p) \) represents the Brascamp-Lieb constant depending on \( L := (L_j)_{j=1}^{m} \) and \( p := (p_j)_{j=1}^{m} \). For each \( 1 \leq j \leq m \), \( L_j : \mathbb{R}^n \to \mathbb{R}^{n_j} \) is a linear surjection and \( p_j \geq 1 \). One equivalent formulation of (1.1) is

\begin{equation}
\left( \int_{\mathbb{R}^n} \left| \prod_{j=1}^{m} F_j(L_j(x)) \right|^{r} dx \right)^{\frac{1}{r}} \leq BL(L,r,p) \prod_{j=1}^{m} \left( \int_{\mathbb{R}^{n_j}} |F_j|^{r p_j} \right)^{\frac{1}{r p_j}},
\end{equation}

for any \( r > 0 \). Brascamp-Lieb inequalities have been well-developed in [6], [5], [4], [3], [7]. Examples of Brascamp-Lieb inequalities consist of Hölder’s inequality and the Loomis-Whitney inequality.

Singular integral estimates corresponding to Hölder’s inequality have been studied extensively, including boundedness of single-parameter paraproducts [9] and multi-parameter paraproducts [20], [21], single-parameter flag paraproducts [19], bilinear Hilbert transform [16], multilinear operators of arbitrary rank [23], etc. But it is of course natural to ask if there are similar singular integral estimates corresponding to Brascamp-Lieb inequalities that are not necessarily of Hölder type. This question was asked to us by Jonathan Bennett during a conference in Matsumoto, Japan, in February 2016. Since then, we adopted the informal definition of singular integral estimate of Brascamp-Lieb type as the singular integral estimate which is reduced to a classical Brascamp-Lieb inequality when the kernels are replaced by Dirac distributions. For the readers familiar with the recent expository work of Durcik and Thiele in [11], this is similar to the generic estimate (2.3) from [11]. So far, to the best of our knowledge, the only research article in the literature where the term “singular Brascamp-Lieb” has been used is the recent work by Durcik and Thiele [10]. However, we would like to emphasize that the basic inequalities\(^1\) corresponding to the “cubic singular expressions” considered in [10] are still of Hölder type, and the term “singular Brascamp-Lieb” was used to underline that the necessary and sufficient boundedness condition (1.6) of [10] is of the same flavor as the one for classical Brascamp-Lieb inequalities stated as (8) in [5].

Techniques to tackle multilinear singular integral operators corresponding to Hölder’s inequality [9], [20], [21], [19], [16], [23] usually involve localizations on phase space subsets of the full-dimension. In contrast, the understanding of singular integral estimates corresponding to Brascamp-Lieb inequalities with \( k_j < n \) for some \( k_j \) in (1.2) (and thus not of Hölder scaling) is far beyond satisfaction. The ultimate goal would be to develop a general methodology to treat a large class of singular Brascamp-Lieb estimates that are not of Hölder type. It is natural to believe that such an approach would need to extract and integrate local information on subspaces of lower dimensions. Also due to its multilinear structure, localizations on the entire space could be necessary as well and a hybrid of both localized analyses would be demanded.

\(^{1}\)Basic inequalities refer to the inequalities obtained for Dirac kernels.
The subject of our study in this present paper is one of the simplest multilinear operators, whose complete understanding cannot be reduced to earlier results and which requires such a new type of analysis. More precisely, it is the five-linear operator defined by

\[
T_{K_1, K_2}(f_1^x, f_2^x, g_1^y, g_2^y, h^{x, y}) = \text{p.v.} \int_{\mathbb{R}^n} K_1((t_1^x, t_2^x), (s_1^y, s_2^y))K_2((s_1^y, s_2^y), (s_1^y, s_2^y))K((s_1^y, s_2^y), (s_1^y, s_2^y)).
\]

(1.3) \[
\int_{\mathbb{R}^n} f_1(x-t_1^x) f_2(x-t_2^x) g_1(y-t_1^y) g_2(y-t_2^y) h(x-y) \, dx \, dy.
\]

where \(t_i^x = (t_1^x, t_2^x), \quad s_j^y = (s_1^y, s_2^y)\) for \(i = 1, 2\) and \(j = 1, 2, 3\). In (1.3), \(K_1\) and \(K_2\) are Calderón-Zygmund kernels that satisfy

\[
|\nabla K_1(t_1^x, t_2^x)| \lesssim \frac{1}{|t_1^x, t_2^x|^{\alpha}} |t_1^x, t_2^x|^{\beta},
\]

\[
|\nabla K_2(s_1^y, s_2^y)| \lesssim \frac{1}{|s_1^y, s_2^y|^{\beta}} |s_1^y, s_2^y|^{\alpha}.
\]

As one can see, the operator \(T_{K_1, K_2}\) takes two functions depending on the \(x\) variable \((f_1, f_2)\), two functions depending on the \(y\) variable \((g_1, g_2)\) and one depending on both \(x\) and \(y\) (namely \(h\)) into another function of \(x\) and \(y\). Our goal is to prove that \(T_{K_1, K_2}\) satisfies the mapping property

\[
L^{p_1}(\mathbb{R}^x) \times L^{q_1}(\mathbb{R}_y) \times L^{q_2}(\mathbb{R}_y) \times L^{s}(\mathbb{R}^2) \rightarrow L^r(\mathbb{R}^2)
\]

for \(1 < p_1, q_1, q_2, s \leq \infty, \quad r > 0, \quad (p_1, q_1), (p_2, q_2) \neq (\infty, \infty)\) with

\[
\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{s} = \frac{1}{p_2} + \frac{1}{q_2} + \frac{1}{s} = \frac{1}{r}.
\]

To verify that the boundedness of \(T_{K_1, K_2}\) qualifies to be a singular integral estimate of Brascamp-Lieb type, one can remove the singularities by setting

\[
K_1(t_1^x, t_2^x) = \delta_0(t_1^x, t_2^x),
\]

\[
K_2(s_1^y, s_2^y) = \delta_0(s_1^y, s_2^y),
\]

and express its boundedness explicitly as

\[
\left\| f_1(x) f_2(x) g_1(y) g_2(y) h(x, y) \right\|_r \lesssim \left\| f_1 \right\|_{L^{p_1}(\mathbb{R}^x)} \left\| f_2 \right\|_{L^{p_2}(\mathbb{R}^x)} \left\| g_1 \right\|_{L^{q_1}(\mathbb{R}_y)} \left\| g_2 \right\|_{L^{q_2}(\mathbb{R}_y)} \left\| h \right\|_{L^{s}(\mathbb{R}^2)}.
\]

The above inequality follows from Hölder’s inequality and the Loomis-Whitney inequality, which, in this simple two dimensional case, is the same as Fubini’s theorem. Clearly, it is an inequality of the same type as (1.2), with a different homogeneity than Hölder. Moreover, this reduction shows that (1.4) is indeed a necessary condition for the boundedness exponents of (1.5) and thus of (1.3).

1.2. Connection with Other Multilinear Objects. The connection with other well-established multilinear operators that we will describe next justifies that \(T_{K_1, K_2}\) defined in (1.3) is a reasonably simple and interesting operator to study, with the hope of inventing a general method that can handle a large class of singular integral estimates of Brascamp-Lieb type with non-Hölder scaling.

Let \(M(\mathbb{R}^d)\) denote the set of all bounded symbols \(m \in L^{\infty}(\mathbb{R}^d)\) smooth away from the origin and satisfying the Marcinkiewicz-Hörmander-Mihlin condition

\[
|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{m_1}}
\]

for any \(\xi \in \mathbb{R}^d \setminus \{0\}\) and sufficiently many multi-indices \(\alpha\). The simplest singular integral operator which corresponds to the two-dimensional Loomis-Whitney inequality would be

\[
T_{m_1, m_2}(f, g)(x, y) := \int_{\mathbb{R}^2} m_1(\xi) m_2(\eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i \xi x} e^{2\pi i \eta y} d\xi d\eta,
\]

where \(m_1, m_2 \in M(\mathbb{R})\). (1.6) is a tensor product of Hilbert transforms whose boundedness are well-known. The bilinear variant of (1.6) can be expressed as

\[\textit{\footnote{Many cases of arbitrary complexity follow from the mixed-norm estimates for vector-valued inequalities in the paper by Benea and the first author [2].}}\]
From this perspective,\textup{(}1.8\textup{)} whose theory has been developed by Muscalu, Pipher, Tao and Thiele \cite{Muscalu19}, to avoid trivial tensor products of single-parameter paraproducts, one then completes \textup{(}1.7\textup{)} by adding a generic function of two variables thus obtaining
\begin{equation}
T_b(f^x_1, f^x_2, f^y_1, f^y_2, h^{x,y})(x, y) \quad \overset{\text{def}}{=} \int_{\mathbb{R}^4} b((\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)) f_1(\xi_1) f_2(\xi_2) g_1(\eta_1) g_2(\eta_2) e^{2\pi i x (\xi_1 + \xi_2)} e^{2\pi i y (\eta_1 + \eta_2)} d\xi_1 d\eta_1 d\xi_2 d\eta_2,
\end{equation}
where $m_1, m_2 \in \mathcal{M}(\mathbb{R}^2)$. It can be separated as a tensor product of single-parameter paraproducts whose boundedness are proved by Coifman-Meyer’s theorem \cite{Coifman78}. To avoid trivial tensor products of single-parameter paraproducts, one then replaces the singularity in each subspace by a flag singularity. In one dimension, the corresponding trilinear operator takes the form
\begin{equation}
T_{m_1 m_2}(f_1, f_2, f_3)(x) := \int_{\mathbb{R}^3} m_1(\xi_1, \xi_2) m_2(\xi_2, \xi_3) f_1(\xi_1) f_2(\xi_2) f_3(\xi_3) e^{2\pi i x (\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3,
\end{equation}
where $m_1 \in \mathcal{M}(\mathbb{R}^2)$ and $m_2 \in \mathcal{M}(\mathbb{R}^3)$. The operator \textup{(}1.9\textup{)} was studied by Muscalu \cite{Muscalu19} using time-frequency analysis which applies not only to the operator itself, but also to all of its adjoints. Miyachi and Tomita \cite{Miyachi18} extended the $L^p$-boundedness for $p > 1$ established in \cite{Muscalu19} to all Hardy spaces $H^p$ with $p > 0$. The single-parameter flag paraproduct and its adjoints are closely related to various nonlinear partial differential equations, including nonlinear Schrödinger equations and water wave equations as discovered by Germain, Masmoudi and Shatah \cite{Germain13}. Its bi-parameter variant is indeed related to the subject of our study and is equivalent to \textup{(}1.3\textup{)}:
\begin{equation}
T_{ab}(f^x_1, f^x_2, g^y_1, g^y_2, h^{x,y}) := \int_{\mathbb{R}^6} a((\xi_1, \eta_1), (\xi_2, \eta_2)) b((\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3)) f_1(\xi_1) f_2(\xi_2) g_1(\eta_1) g_2(\eta_2) \hat{h}(\xi_3, \eta_3) e^{2\pi i x (\xi_1 + \xi_2 + \xi_3)} e^{2\pi i y (\eta_1 + \eta_2 + \eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3,
\end{equation}
where
\begin{equation}
|\partial_{\xi_1}^{\alpha_1}(\xi_1, \xi_2) \partial_{\eta_1}^{\beta_1}(\eta_1, \eta_2) a| \lesssim \frac{1}{||\xi_1, \xi_2|| ||\eta_1, \eta_2||^{1-}},
\end{equation}
\begin{equation}
|\partial_{\xi_1}^{\alpha_2}(\xi_1, \xi_2) \partial_{\eta_1}^{\beta_2}(\eta_1, \eta_2, \eta_3) b| \lesssim \frac{1}{||\xi_1, \xi_2, \xi_3|| ||\eta_1, \eta_2, \eta_3||^{1-}},
\end{equation}
for sufficiently many multi-indices $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$. The equivalence can be derived with
\begin{equation}
a = \hat{K}_1,
\end{equation}
\begin{equation}
b = \hat{K}_2.
\end{equation}

The general bi-parameter trilinear flag paraproduct is defined on larger function spaces where the tensor products are replaced by general functions in the plane.\textsuperscript{3} From this perspective, $T_{ab}$ or equivalently $T_{K_1 K_2}$ defined in \textup{(}1.10\textup{)} and \textup{(}1.3\textup{)} respectively can be viewed as a trilinear operator with the desired mapping property
\begin{equation}
T_{ab} : L^p_x(L^q_y) \times L^p_x(L^q_y) \times L^s(\mathbb{R}^2) \to L^r(\mathbb{R}^2)
\end{equation}
for $1 < p_1, p_2, q_1, q_2, s \leq \infty, r > 0, (p_1, q_1), (p_2, q_2) \neq (\infty, \infty)$ and \begin{equation}
\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{s} = \frac{1}{p_2} + \frac{1}{q_2} + \frac{1}{r} = \frac{1}{r},
\end{equation}
where the first two function spaces are restricted to be tensor-product spaces. The condition that $(p_1, q_1), (p_1, q_2) \neq (\infty, \infty)$
\textsuperscript{3}Its boundedness is at present an open question, raised by the first author of the article on several occasions.
is inherited from single-parameter flag paraproducts and can be verified by the unboundedness of the operator when $f_1, f_2 \in L^\infty(\mathbb{R})$ are constant functions. Lu, Pipher and Zhang [17] showed that the general bi-parameter flag paraproduct can be reduced to an operator given by a symbol with better singularity using an argument inspired by Miyachi and Tomita [18]. The boundedness of the reduced multiplier operator still remains open. The reduction allows an alternative proof of $L^p$-boundedness for (1.10) as long as $p \neq \infty$. However, we emphasize again that we will not take this point of view now, and instead, we treat our operator $T_{ab}$ as a five-linear operator.

1.3. Methodology. As one may notice from the last section, the five-linear operator $T_{ab}$ (or $T_{K_1, K_2}$) contains the features of the bi-parameter paraproduct defined in (1.8) and the single-parameter flag paraproduct defined in (1.9), which hints that the methodology would embrace localized analyses of both operators. Nonetheless, it is by no means a simple concatenation of two existing arguments. The methodology includes

(1) **tensor-type stopping-time decomposition** which refers to an algorithm that first implements a one-dimensional stopping-time decomposition for each variable and then combines information for different variables to obtain estimates for operators involving several variables;

(2) **general two-dimensional level sets stopping-time decomposition** which refers to an algorithm that partitions the collection of dyadic rectangles such that the dyadic rectangles in each sub-collection intersect with a certain level set non-trivially;

and the main novelty lies in

(i) the construction of two-dimensional stopping-time decompositions from stopping-time decompositions on one-dimensional subspaces;

(ii) the hybrid of tensor-type and general two-dimensional level sets stopping-time decompositions in a meaningful fashion.

The methodology outlined above is considered to be robust in the sense that it captures all local behaviors of the operator. The robustness may also be verified by the entire range of estimates obtained. After closer inspection of the technique, it would not be surprising that the technique gives estimates involving $L^\infty$-norms. In particular, the tensor-type stopping-time decompositions process information on each subspace independently. As a consequence, when some function defined on some subspace lies in $L^\infty$, one simply “forgets” about that function and glues the information from subspaces in an intelligent way specified later.

1.4. Structure. The paper is organized as follows: main theorems are stated in Chapter 2 followed by preliminary definitions and theorems introduced in Chapter 3. Chapter 4 describes the reduced discrete model operators and estimates one needs to obtain for the model operators while the reduction procedure is postponed to Appendix II. Chapter 5 gives the definition and estimates for the building blocks in the argument - sizes and energies. Chapter 6 - 9 focus on estimates for the model operators in the Haar case. All four chapters start with a specification of the stopping-time decompositions used. Chapter 10 extends all the estimates in the Haar setting to the general Fourier case.

It is also important to notice that Chapter 6 develops an argument for one of the simpler model operators with emphasis on the key geometric feature implied by a stopping-time decomposition, that is the sparsity condition. Chapter 7 focuses on a more complicated model which requires not only the sparsity condition, but also a Fubini-type argument which is discussed in details. Chapter 8 and 9 are devoted to estimates involving $L^\infty$-norms and the arguments for those cases are similar to the ones in Chapter 6, in the sense that the sparsity condition is sufficient to obtain the results.

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2. Main Results

We state the main results in Theorem 2.1 and 2.2. Theorem 2.1 proves the boundedness when $p_i, q_i$ are strictly between 1 and infinity whereas Theorem 2.2 deals with the case when $p_i = \infty$ or $q_j = \infty$ for some $i \neq j$.

**Theorem 2.1.** Suppose $a \in L^\infty(\mathbb{R}^d)$, $b \in L^\infty(\mathbb{R}^d)$, where $a$ and $b$ are smooth away from $\{(\xi_1, \xi_2) = 0\}$ and $\{(\xi_1, \xi_2, \xi_3) = 0\}$ respectively and satisfy the following Marcinkiewicz conditions:

\[
|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \partial_{\xi_3}^\gamma a(\xi_1, \eta_1, \xi_2, \eta_2)| \lesssim \frac{1}{|\xi_1|^{\alpha_1 + \beta_1} |\xi_2|^{\alpha_2 + \beta_2}}
\]

\[
|\partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \partial_{\xi_3}^\gamma b(\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3)| \lesssim \frac{1}{|\xi_1|^{\alpha_1 + \beta_1 + \gamma_1} |\eta_1, \eta_2, \eta_3|^{\alpha_2 + \beta_2 + \gamma_2}}
\]

for sufficiently many multi-indices $\alpha_1, \alpha_2, \beta_1, \beta_2, \alpha_3, \beta_3, \gamma_1, \gamma_2 \geq 0$. For $f_1, f_2, g_1, g_2 \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^2)$ where $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^2)$ denote the Schwartz spaces, define

\[
T_{ab}(f_1^\gamma f_2^\gamma g_1^\gamma g_2^\gamma h^{x,y}) := \int_{\mathbb{R}^d} a(\xi_1, \eta_1, \xi_2, \eta_2) b(\xi_1, \eta_1, \xi_2, \eta_2, \xi_3, \eta_3) e^{2\pi i \xi_1 \eta_1 + 2\pi i \xi_2 \eta_2 + 2\pi i \xi_3 \eta_3} \partial_{\xi_1} d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3.
\]

Then for $1 < p_1, p_2, q_1, q_2, < \infty, 1 < s \leq \infty, r > 0$, $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p_2} + \frac{1}{q_2} + \frac{1}{s} = \frac{1}{r}$, $T_{ab}$ satisfies the following mapping property

\[
T_{ab} : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \times L^{q_2}(\mathbb{R}^d) \times L^s(\mathbb{R}^2) \rightarrow L^r(\mathbb{R}^d).
\]

**Theorem 2.2.** Let $T_{ab}$ be defined as (2.1). Then for $1 < p < \infty$, $1 < s \leq \infty, r > 0$, $\frac{1}{p} + \frac{1}{s} = \frac{1}{r}$, $T_{ab}$ satisfies the following mapping property

\[
T_{ab} : L^p(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \times L^\infty(\mathbb{R}^d) \times L^s(\mathbb{R}^2) \rightarrow L^r(\mathbb{R}^d).
\]

**Remark 2.3.** The cases (i) $q_1 = q_2 < \infty$ and $p_1 = p_2 = \infty$ (ii) $p_1 = q_2 < \infty$ and $p_2 = q_1 = \infty$ (iii) $q_1 = p_2 = \infty$ and $p_1 = q_2 = \infty$ follows from the argument by the symmetry.

2.1. Restricted Weak-Type Estimates. For the Banach estimates when $r > 1$, Hölder’s inequality involving hybrid square and maximal functions is sufficient. The argument resembles the Banach estimates for the single-parameter flag paraproduct. The quasi-Banach estimates when $r < 1$ is trickier and requires a careful treatment. In this case, we use multilinear interpolations and reduce the desired estimates specified in Theorem 2.1 and 2.2 to the following restricted weak-type estimates for the associated multilinear form.

**Theorem 2.4.** Let $T_{ab}$ denote the operator defined in (2.1). Suppose that $1 < p_1, p_2, q_1, q_2, < \infty, 1 < s < 2, 0 < r < 1, \frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{p_2} + \frac{1}{q_2} + \frac{1}{s} = \frac{1}{r}$. Then for any measurable set $F_1 \subseteq \mathbb{R}^d$, $F_2 \subseteq \mathbb{R}^d$, $G_1 \subseteq \mathbb{R}^d$, $G_2 \subseteq \mathbb{R}^d$, $E \subseteq \mathbb{R}^2$ of positive and finite measure and any measurable function $|f_1(x)| \leq \chi_{F_1}(x)$, $|f_2(x)| \leq \chi_{F_2}(x)$, $|g_1(y)| \leq \chi_{G_1}(y)$, $|g_2(y)| \leq \chi_{G_2}(y), h \in L^s(\mathbb{R}^2)$, there exists $E' \subseteq E$ with $|E'| > |E|/2$ such that the multilinear form associated to $T_{ab}$ satisfies

\[
|\Lambda(f_1^\gamma f_2^\gamma g_1^\gamma g_2^\gamma h^{x,y}, \chi_{E'})| \lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} |h|^{\frac{1}{r}}.
\]

**Theorem 2.5.** Let $T_{ab}$ denote the operator defined in (2.1). Suppose that $1 < p < \infty, 1 < s < 2$, $0 < r < 1, \frac{1}{p} + \frac{1}{s} = \frac{1}{r}$. Then for any measurable set $F_1 \subseteq \mathbb{R}^d$, $G_1 \subseteq \mathbb{R}^d$, $E \subseteq \mathbb{R}^2$ of positive and finite measure and every measurable function $|f_1(x)| \leq \chi_{F_1}(x)$, $|g_1(y)| \leq \chi_{G_1}(y), f_2 \in L^\infty(\mathbb{R}^d), g_2 \in L^\infty(\mathbb{R}^d), h \in L^r(\mathbb{R}^2)$, there exists $E' \subseteq E$ with $|E'| > |E|/2$ such that the multilinear form associated to $T_{ab}$ satisfies

\[
|\Lambda(f_1^\gamma f_2^\gamma g_1^\gamma h^{x,y}, \chi_{E'})| \lesssim |F_1|^{\frac{1}{p}} |G_1|^{\frac{1}{q}} |f_2|^{\frac{1}{r}} |g_2|^{\frac{1}{r}} |h|^{\frac{1}{r}}.
\]

- Multilinear form, denoted by $\Lambda$, associated to an $n$-linear operator $T(f_1, \ldots, f_n)$ is defined as $\Lambda(f_1, \ldots, f_{n+1}) := \langle T(f_1, \ldots, f_n) \rangle_{n+1}$.
Remark 2.6. Theorems 2.4 and 2.5 hint the necessity of localization and the major subset $E'$ of $E$ is constructed based on the philosophy to localize the operator where it is well-behaved.

The reduction of Theorems 2.1 and 2.2 to Theorem 2.4 and 2.5 respectively will be postponed to Appendix I. In brief, it depends on the interpolation of multilinear forms described in Lemma 9.6 of [22] and a tensor-product version of Marcinkiewicz interpolation theorem.

2.2. Application - Leibniz Rule. A direct corollary of Theorem 2.1 is a Leibniz rule which captures the nonlinear interaction of waves coming from transversal directions. In general, Leibniz rules refer to inequalities involving norms of derivatives. The derivatives are defined in terms of Fourier transforms. More precisely, for $\alpha \geq 0$ and $f \in S(\mathbb{R}^d)$ a Schwartz function in $\mathbb{R}^d$, define the homogeneous derivative of $f$ as

$$D^\alpha f := \mathcal{F}^{-1} \left( |\xi|^\alpha \hat{f}(\xi) \right).$$

Leibniz rules are closely related to boundedness of multilinear operators discussed in Section 1.2. For example, the boundedness of one-parameter paraproducts give rise to a Leibniz rule by Kato and Ponce [14].

For $f, g \in S(\mathbb{R}^d)$ and $\alpha > 0$ sufficiently large,

$$\|D^\alpha (fg)\|_r \lesssim \|D^\alpha f\|_p \|g\|_q + \|f\|_p \|D^\alpha g\|_q,$$

with $1 < p_i, q_i < \infty, \frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{r}, i = 1, 2$. The inequality in (2.4) generalizes the trivial and well-known Leibniz rule when $\alpha = 1$ and states that the derivative for a product of two functions can be dominated by the terms which involve the highest order derivative hitting on one of the functions. The reduction of (2.4) to the boundedness of one-parameter paraproducts is routine (see Chapter 2 in [22] for details) and can be applied to other Leibniz rules with their corresponding multilinear operators, including the boundedness of our operator $T_{ab}$ and its Leibniz rule stated in Theorem 2.7 below. The Leibniz rule stated in Theorem 2.7 deals with partial derivatives, where the partial derivative of $f \in S(\mathbb{R}^d)$ is defined, for $(\alpha_1, \ldots, \alpha_d)$ with $\alpha_1, \ldots, \alpha_d \geq 0$, as

$$D_1^{\alpha_1} \cdots D_d^{\alpha_d} f := \mathcal{F}^{-1} \left( |\xi_1|^{\alpha_1} \cdots |\xi_d|^{\alpha_d} \hat{f}(\xi_1, \ldots, \xi_d) \right).$$

Theorem 2.7. Suppose $f_1, f_2 \in S(\mathbb{R}_x)$, $g_1, g_2 \in S(\mathbb{R}_y)$ and $h \in S(\mathbb{R}^2)$. Then for $\beta_1, \beta_2, \alpha_1, \alpha_2 > 0$ sufficiently large and $1 < p_1, p_2, q_1, q_2, s_1 \leq \infty$, $r > 0$, $(p_1, q_1), (p_2, q_2) \neq (\infty, \infty), \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r}$, for each $j = 1, \ldots, 16$,

$$\|D_1^{\beta_1} D_2^{\beta_2} (D_1^{\alpha_1} D_2^{\alpha_2} (f_1 f_2 g_1 g_2) h^{x-y})\|_{L^r(\mathbb{R}^2)} \lesssim \text{sum of 16 terms of the forms:}$$

$$\|D_1^{\alpha_1+\beta_1} f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{q_1}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2}(\mathbb{R})} \|g_2\|_{L^{q_2}(\mathbb{R})} \|h\|_{L^r(\mathbb{R}^2)} +$$

$$\|f_1\|_{L^{p_1}(\mathbb{R})} \|D_1^{\alpha_1+\beta_1} f_2\|_{L^{q_1}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2}(\mathbb{R})} \|g_2\|_{L^{q_2}(\mathbb{R})} \|h\|_{L^r(\mathbb{R}^2)} +$$

$$\|D_1^{\alpha_1+\beta_1} f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{q_1}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2}(\mathbb{R})} \|g_2\|_{L^{q_2}(\mathbb{R})} \|D_2^{\beta_2} h\|_{L^r(\mathbb{R}^2)} + \cdots$$

Remark 2.8. The reasoning for the number “16” is that

(i) for $\alpha_1$, there are 2 possible distributions of highest order derivatives thus yielding 2 terms;

(ii) for $\alpha_2$, there are 2 terms for the same reason in (i);

(iii) for $\beta_1$, it can hit $h$ or some function which comes from the dominant terms of $D^{\alpha_1} (f_1 f_2)$ and which have two choices as illustrated in (i), thus generating $2 \times 2 = 4$ terms;

(iv) for $\beta_2$, there would be 4 terms for the same reason in (iii).

By summarizing (i)-(iv), one has the count $4 \times 4 = 16$.

Remark 2.9. As commented in the beginning of this section, $f_1$ and $f_2$ in Theorem 2.7 can be viewed as waves coming from one direction while $g_1$ and $g_2$ are waves from the orthogonal direction. The presence of $h$, as a generic wave in the plane, makes the interaction nontrivial.

3. Preliminaries

3.1. Terminology. We will first introduce some notations which will be useful throughout the paper.
**Definition 3.1.** Suppose \( I \in \mathbb{R} \) is an interval. Then we say a smooth function \( \phi \) is adapted to \( I \) if

\[
|\phi^{(l)}(x)| \leq C_l C_M \frac{1}{|I|^l} \frac{1}{(1 + \frac{|x-x_I|}{|I|})^M}
\]

for sufficiently many derivatives \( l \), where \( x_I \) denotes the center of the interval \( I \).

**Definition 3.2.** Suppose \( \mathcal{I} \) is a collection of dyadic intervals. Then a family of \( L^2 \)-normalized bump functions \( (\phi_I)_{I \in \mathcal{I}} \) is lacunary if and only if for every \( I \in \mathcal{I} \),

\[
\text{supp } \hat{\phi}_I \subseteq [-4|I|^{-1}, 1/4|I|^{-1}] \cup \left[ |I|^{-1}, 4|I|^{-1} \right].
\]

A family of \( L^2 \)-normalized bump functions \( (\phi_I)_{I \in \mathcal{I}} \) is non-lacunary if and only if for every \( I \in \mathcal{I} \),

\[
\text{supp } \hat{\phi}_I \subseteq [-4|I|^{-1}, |I|^{-1}].
\]

We usually denote bump functions in lacunary family by \((\psi_I)_I\) and those in non-lacunary family by \((\varphi_I)_I\).

A simplified variant of bump functions given in Definition 3.2 is specified as follows - Haar wavelets and indicator functions defined in Definition 3.1. The arguments will correspond to lacunary family of bump functions and \( L^2 \)-normalized indicator functions are analogous to non-lacunary family of bump functions.

**Definition 3.3.** Define

\[
\psi^H(x) := \begin{cases} 
1 & \text{for } x \in [0, \frac{1}{2}) \\
-1 & \text{for } x \in [\frac{1}{2}, 1).
\end{cases}
\]

Let \( I := [n2^k, (n+1) \cdot 2^k) \) denote a dyadic interval. Then the Haar wavelet on \( I \) is defined as

\[
\psi^H_I(x) := 2^{-\frac{k}{2}} \psi^H(2^{-k}x - n).
\]

The \( L^2 \)-normalized indicator function on \( I \) is expressed as

\[
\varphi^H_I(x) := |I|^{-\frac{1}{2}} \chi_I(x).
\]

We shall remark that the boundedness of the multilinear form described in Theorem 2.4 and 2.5 can be reduced to the estimates of discrete model operators which are defined in terms of bump functions of the form specified in Definition 3.2. The precise statements are included in Theorem 4.2 and 4.3 and the proof is discussed in Appendix II. However, we will first study the simplified model operators with the general bump functions replaced by Haar wavelets and indicator functions defined in Definition 3.3. The arguments for the simplified models capture the main challenges while avoiding some technical aspects. We will leave the generalization and the treatment of the technical details to Chapter 10. The simplified models would be denoted as Haar models and we will highlight the occasions when the Haar models are considered.

### 3.2. Useful Operators - Definitions and Theorems

We also give explicit definitions for the Hardy-Littlewood maximal function, the discretized Littlewood-Paley square function and the hybrid square-and-maximal functions that will appear naturally in the argument.

**Definition 3.4.** The Hardy-Littlewood maximal operator \( M \) is defined as

\[
M f(\vec{x}) = \sup_{\vec{x} \in B} \int_B |f(\vec{u})| d\vec{u}
\]

where the supremum is taken over all open balls \( B \subseteq \mathbb{R}^d \) containing \( \vec{x} \).

**Definition 3.5.** Suppose \( \mathcal{I} \) is a finite family of dyadic intervals and \( (\psi_I)_I \) a lacunary family of \( L^2 \)-normalized bump functions. The discretized Littlewood-Paley square function operator \( S \) is defined as

\[
S f(x) = \left( \sum_{I \in \mathcal{I}} \frac{|(f, \psi_I)^2}{|I|} \chi_I(x) \right)^{\frac{1}{2}}
\]

**Definition 3.6.** Suppose \( \mathcal{R} \) is a finite collection of dyadic rectangles. Let \((\phi_R)_{R \in \mathcal{R}}\) denote the family of \( L^2 \)-normalized bump functions with \( \phi_R = \psi_I \otimes \psi_J \) where \( R = I \times J \).
(1) the double square function operator $SS$ is defined as

$$SSh(x, y) = \left( \sum_{I \times J} \frac{|\langle h, \psi_I \otimes \psi_J \rangle|^2}{|I| |J|} \chi_{I \times J}(x, y) \right)^{1/2};$$

(2) the hybrid maximal-square operator $MS$ is defined as

$$MSh(x, y) = \sup_I \frac{1}{|I|^{1/2}} \left( \sum_J \frac{|\langle h, \varphi_J \otimes \psi_J \rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \chi_I(x);$$

(3) the hybrid square-maximal operator $SM$ is defined as

$$SMh(x, y) = \left( \sum_I \left( \sup_J \frac{|\langle h, \psi_I \otimes \varphi_J \rangle|^2}{|I|} \chi_J(y) \right) \chi_I(x) \right)^{1/2};$$

(4) the double maximal function $MM$ is defined as

$$MMh(x, y) = \sup_{(x, y) \in R} \frac{1}{|R|} \int_{R} |h(s, t)| ds dt,$$

where the supremum is taken over all dyadic rectangles in $R$ containing $(x, y)$.

The following theorem about the operators defined above is used frequently in the argument. The proof of the theorem and other contexts where the hybrid operators appear can be found in [22], [8] and [12].

**Theorem 3.7.**

1. $M$ is bounded in $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ and $M : L^1 \to L^{1, \infty}$.
2. $S$ is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$.
3. The hybrid operators $SS, MS, SM, MM$ are bounded in $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.

4. **Discrete Model Operators**

In this chapter, we will introduce the discrete model operators whose boundedness implies the estimates specified in Theorem 2.4 and Theorem 2.5. The reduction procedure follows from a routine treatment which has been discussed in [22]. The details will be enclosed in Appendix II for the sake of completeness. The model operators are usually more desirable because they are more “localizable”. The discrete model operators are defined as follows.

**Definition 4.1.** Suppose $I, J, K, L$ are finite collections of dyadic intervals. Suppose $(\phi_I^i)_{i \in I}$, $(\phi_J^j)_{j \in J}$, $(\phi_K^K)_{K \in K}$, $(\phi_L^l)_{l \in L}$, $i, j, k, l = 1, 2, 3$ are families of $L^2$-normalized bump functions adapted to $I, J, K, L$ respectively. We further assume that at least two families of $(\phi_I^i)_{i \in I}$, $i = 1, 2, 3$, are lacunary. Same conditions are assumed for families $(\phi_J^j)_{j \in J}$, $(\phi_K^K)_{K \in K}$ and $(\phi_L^l)_{l \in L}$. In some models, we specify the lacunary and non-lacunary families by explicitly denoting the functions in the lacunary family as $\psi$ and those in the non-lacunary family as $\varphi$. Let $\#_1, \#_2$ denote some positive integers. Define

1. $$\Pi^{\#_1 \otimes \text{paraproduct}}_{\text{flag}}(f_1^x, f_2^x, g_1^y, g_2^y, h^{x \times y}) := \sum_{I \times J \in I \times J} \frac{1}{|I|^{1/2} |J|} \langle B_I(f_1, f_2), \varphi_I^1 \rangle \langle g_1, \phi_J^1 \rangle \langle g_2, \phi_J^2 \rangle \langle h, \psi_I^1 \otimes \phi_J^3 \rangle \psi_I^1 \otimes \phi_J^3$$

where

$$B_I(f_1, f_2)(x) := \sum_{K \in K \mid |K| \geq |I|} \frac{1}{|K|^{1/2}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \phi_K^3(x).$$

2. $$\Pi^{\#_2 \otimes \text{paraproduct}}_{\text{flag}}(f_1^x, f_2^x, g_1^y, g_2^y, h^{x \times y}) := \sum_{J \times I \in J \times I} \frac{1}{|J|^{1/2} |I|} \langle B_J^{\#_2}(f_1, f_2), \varphi_J^1 \rangle \langle g_1, \phi_I^1 \rangle \langle g_2, \phi_I^2 \rangle \langle h, \psi_J^1 \otimes \phi_I^3 \rangle \psi_J^1 \otimes \phi_I^3.$$

The details will be enclosed in Appendix II for the sake of completeness.
Let
\[
\Pi_{\text{flag}^0 \otimes \text{flag}^0}(f_1^x, f_2^x, g_1^y, g_2^y, h^x; y) := \sum_{I \times J \in \mathcal{I} \times \mathcal{J}} \frac{1}{|I|^2 |J|^2} \langle B_1(f_1, f_2), \varphi_1 \rangle \langle \tilde{B}_J(g_1, g_2), \varphi_J \rangle \langle h, \psi_J^2 \otimes \psi_J^3 \rangle \psi_I^2 \otimes \psi_I^3
\]
where
\[
B_1(f_1, f_2)(x) := \sum_{K \in \mathcal{K} : |K| \geq |I|} \frac{1}{|K|^2} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \phi_K^3(x),
\]
\[
\tilde{B}_J(g_1, g_2)(y) := \sum_{L \in \mathcal{L} : |L| \geq |J|} \frac{1}{|L|^2} \langle g_1, \phi_L^1 \rangle \langle g_2, \phi_L^2 \rangle \phi_L^3(y).
\]

**Theorem 4.2.** Let \(\Pi_{\text{flag}^0 \otimes \text{paraproduct}}, \Pi_{\text{flag}^0 \otimes \text{paraproduct}}, \Pi_{\text{flag}^0 \otimes \text{flag}^0}, \Pi_{\text{flag}^0 \otimes \text{flag}^0} \) and \(\Pi_{\text{flag}^0 \otimes \text{flag}^0} \) be multilinear operators specified in Definition 4.1. Then all of them satisfy the mapping property stated in Theorem 2.4, where the constants are independent of \#1, \#2 and the cardinalities of the collections \(\mathcal{I}, \mathcal{J}, \mathcal{K}\) and \(\mathcal{L}.\)

**Theorem 4.3.** Let \(\Pi_{\text{flag}^0 \otimes \text{paraproduct}}, \Pi_{\text{flag}^0 \otimes \text{paraproduct}}, \Pi_{\text{flag}^0 \otimes \text{flag}^0}, \Pi_{\text{flag}^0 \otimes \text{flag}^0} \) and \(\Pi_{\text{flag}^0 \otimes \text{flag}^0} \) be multilinear operators specified in Definition 4.1. Then all of them satisfy the mapping property stated in Theorem 2.5, where the constants are independent of \#1, \#2 and the cardinalities of the collections \(\mathcal{I}, \mathcal{J}, \mathcal{K}\) and \(\mathcal{L}.\)

The following chapters are devoted to the proofs of Theorem 4.2 and 4.3 which would imply Theorem 2.4 and 2.5. We will mainly focus on discrete model operators defined in (3) (Chapter 7) and (5) (Chapter 6), whose arguments consist of all the essential tools that are needed for other discrete models.
5. Sizes and Energies

The notion of sizes and energies appear first in [24] and [25]. Since they will play important roles in the main arguments, the explicit definitions of sizes and energies are introduced and some useful properties are highlighted in this chapter.

**Definition 5.1.** Let $\mathcal{I}$ be a finite collection of dyadic intervals. Let $(\psi_I)_{I \in \mathcal{I}}$ denote a lacunary family of $L^2$-normalized bump functions and $(\varphi_I)_{I \in \mathcal{I}}$ a non-lacunary family of $L^2$-normalized bump functions. Define

1. $\text{size}_\mathcal{I}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \sup_{I \in \mathcal{I}} \frac{|\langle f, \varphi_I \rangle|}{|I|^\frac{1}{2}}$;

2. $\text{size}_\mathcal{I}((\langle f, \psi_I \rangle)_{I \in \mathcal{I}}) := \sup_{I \in \mathcal{I}} \frac{1}{|I_0|} \left\| \sum_{I \subseteq I_0} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \chi_I \right\|_{1,\infty}$;

3. $\text{energy}_\mathcal{I}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \sup_{n \in \mathbb{Z}} 2^n \sup_{D_n} \sum_{I \in D_n} |I|$ where $D_n$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying $\frac{|\langle f, \varphi_I \rangle|}{|I|^\frac{1}{2}} > 2^n$;

4. $\text{energy}_\mathcal{I}((\langle f, \psi_I \rangle)_{I \in \mathcal{I}}) := \sup_{n \in \mathbb{Z}} 2^n \sup_{D_n} \sum_{I \in D_n} |I|$ where $D_n$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying

$$\frac{1}{|I|} \left\| \sum_{I \subseteq I} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \chi_I \right\|_{1,\infty} > 2^n$$;

5. For $t > 1$, define

$\text{energy}_t^\mathcal{I}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \left( \sum_{n \in \mathbb{Z}} 2^{tn} \sup_{D_n} \sum_{I \in D_n} |I| \right)^\frac{1}{t}$ where $D_n$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying $\frac{|\langle f, \varphi_I \rangle|}{|I|^\frac{1}{2}} > 2^n$.

**5.1. Useful Facts about Sizes and Energies.** The following propositions describe facts about sizes and energies which will be heavily employed later on. Proposition 5.2 and 5.3 are routine and the proofs can be found in Chapter 2 of [22]. Proposition 5.4 consists of two parts - the first part is discussed in [22] while the second part is less standard. We will include the proof of both parts in Section 5.3 for the sake of completeness.

Proposition 5.10, Proposition 5.5 and Proposition 5.14 highlight the useful size and energy estimates involving the operators $B$ and $\tilde{B}$ in the Haar model. The emphasis on the Haar model assumption keeps track of the arguments we need to modify for the general Fourier case. It is noteworthy that Proposition 5.5 describes a “global” energy estimate while Proposition 5.10 and 5.14 take into the consideration that the operators $B$ and $\tilde{B}$ are localized to intersect certain level sets which carry crucial information for the estimates of the sizes and energies for $B$ and $\tilde{B}$. While the proof of Proposition 5.5 follows from the boundedness of paraproducts ([9], [22]), the arguments for Proposition 5.10 and Proposition 5.14 request localizations and more careful treatments that will be discussed in subsequent sections.
**Proposition 5.2** (John-Nirenberg). Let $\mathcal{I}$ be a finite collection of dyadic intervals. For any sequence $(a_I)_{I \in \mathcal{I}}$ and $r > 0$, define the BMO-norm for the sequence as

$$
\| (a_I)_I \|_{BMO(r)} := \sup_{I_0 \in \mathcal{I}} \frac{1}{|I_0|^r} \left\| \left( \sum_{I \subseteq I_0} \frac{|a_I|^2}{|I|} \chi_I(x) \right)^{\frac{1}{2}} \right\|_r.
$$

Then for any $0 < p < q < \infty$,

$$
\| (a_I)_I \|_{BMO(p)} \lesssim \| (a_I)_I \|_{BMO(q)}.
$$

**Proposition 5.3.** Suppose $f \in L^1(\mathbb{R})$. Then

$$
size_{\mathcal{I}}((f, \varphi_1)_I), size_{\mathcal{I}}((f, \psi_1)_I) \lesssim \sup_{I \in \mathcal{I}} \int_R \hat{\varphi}_M dx
$$

for $M > 0$ and the implicit constant depends on $M$. $\hat{\varphi}_I$ is an $L^\infty$-normalized bump function adapted to $I$.

**Proposition 5.4.**

1. Suppose $f \in L^1(\mathbb{R})$. Then

$$
\text{energy}_{\mathcal{I}}((f, \varphi_1)_I), \text{energy}_{\mathcal{I}}((f, \psi_1)_I) \lesssim \| f \|_1.
$$

2. Suppose $f \in L^r(\mathbb{R})$ for $t > 1$. Then

$$
\text{energy}_{\mathcal{I}}((f, \varphi_1)_I) \lesssim \| f \|_t.
$$

**Proposition 5.5** (Global Energy). Suppose that $F_1, F_2 \subseteq \mathbb{R}_x$ and $G_1, G_2 \subseteq \mathbb{R}_y$ are sets of finite measure and $|f_1| \leq \chi_{F_1}$, $|g_1| \leq \chi_{G_1}$, $i, j = 1, 2$. Suppose that $\mathcal{K}$ and $\mathcal{L}$ are finite collections of dyadic intervals. Define

$$
B(f_1, f_2)(x) := \sum_{k \in \mathcal{K}} \frac{1}{|k|^2} (f_1, \phi_k^1)(f_2, \phi_k^2)(x),
$$

$$
\tilde{B}(g_1, g_2)(y) := \sum_{l \in \mathcal{L}} \frac{1}{|l|^2} (g_1, \phi_l^1)(g_2, \phi_l^2)(y).
$$

1. Then for any $0 < \rho, \rho' < 1$, one has

$$
\text{energy}_{\mathcal{I}}((B(f_1, f_2), \varphi_1)_I) \lesssim |F_1|^\rho |F_2|^{1-\rho},
$$

$$
\text{energy}_{\mathcal{J}}((\tilde{B}(g_1, g_2), \varphi_1)_J) \lesssim |G_1|^\rho |G_2|^{1-\rho},
$$

2. Suppose that $t, s > 1$. Then for any $0 \leq \theta_1, \theta_2, \varsigma_1, \varsigma_2 < 1$, with $\theta_1 + \theta_2 = \frac{1}{t}$ and $\varsigma_1 + \varsigma_2 = \frac{1}{s}$, one has

$$
\text{energy}_{\mathcal{I}}((B(f_1, f_2), \varphi_1)_I) \lesssim |F_1|^\theta_1 |F_2|^{\theta_2},
$$

$$
\text{energy}_{\mathcal{J}}((\tilde{B}(g_1, g_2), \varphi_1)_J) \lesssim |G_1|^\varsigma_1 |G_2|^{\varsigma_2}.
$$

It is not difficult to observe that Proposition 5.5 follows immediately from Proposition 5.4 and the following lemma.

**Lemma 5.6.** Suppose that $1 < p_1, p_2 \leq \infty$ and $1 < q_1, q_2 \leq \infty$ with $(p_i, q_i) \neq (\infty, \infty)$ for $i = 1, 2$. Further assume that $\frac{1}{p_1} + \frac{1}{q_1} < 1$ and $\frac{1}{p_2} + \frac{1}{q_2} < 1$. Then for any $f_1 \in L^{p_1}$, $f_2 \in L^{q_1}$, $g_1 \in L^{p_2}$ and $g_2 \in L^{q_2}$,

$$
\|B(f_1, f_2)\|_{L^r} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{q_1}},
$$

$$
\|\tilde{B}(g_1, g_2)\|_{L^r} \lesssim \|g_1\|_{L^{p_2}} \|g_2\|_{L^{q_2}}.
$$

By identifying that $B$ and $\tilde{B}$ are one-parameter paraproducts, Lemma 5.6 is a restatement of Coifman-Meyer’s theorem on the boundedness of paraproducts [9].

We will now turn our attention to local size estimates for $((B_{e_1}^{\theta_1, H}, \varphi_1)_I)$ and $((\tilde{B}_{e_2}^{\theta_2, H}, \varphi_1)_I)$ and local energy estimates for $((B_{e_1}^{H}, \varphi_1)_I)$ and $((\tilde{B}_{e_2}^{H}, \varphi_1)_I)$ in the Haar model. The precise definitions for the operators $B_{e_1}^{\theta_1, H}, \tilde{B}_{e_2}^{\theta_2, H}, B_{e_1}^{H}$ and $B_{e_2}^{H}$ are stated as follows.
Definition 5.7. Suppose that $I$ and $J$ are fixed dyadic intervals and $\mathcal{K}$ and $\mathcal{L}$ are finite collections of dyadic intervals. Suppose that $(\phi^K_i)_{K \in \mathcal{K}}, (\phi^L_i)_{i \in \mathcal{L}}$ for $i, j = 1, 2$ are families of $L^2$-normalized bump functions. Further assume that $(\phi^3_K)_{K \in \mathcal{K}}$ and $(\phi^3_L)_{i \in \mathcal{L}}$ are families of Haar wavelets or $L^2$-normalized indicator functions. A family of Haar wavelets is considered to be a lacunary family and a family of $L^2$-normalized indicator functions to be a non-lacunary family. Suppose that at least two families of $(\phi^K_K)_{K \in \mathcal{K}}$ and $(\phi^3_K)_K$ are lacunary and that at least two families of $(\phi^L_L), (\phi^2_L)_L$ and $(\phi^3_L)_L$ are lacunary. Let

$$B^h_{f_1, f_2}(x) := \sum_{K \in \mathcal{K}, |L| = 2^{|K|} |I|} \frac{1}{|K|^\frac{3}{2}} \langle f_1, \phi^K, \phi^3 \rangle \langle f_2, \phi^K, \phi^3 \rangle, \quad (5.1)$$

$$\tilde{B}^h_{g_1, g_2}(y) := \sum_{L \in \mathcal{L}, |L| = 2^{|L|} |J|} \frac{1}{|L|^\frac{3}{2}} \langle g_1, \phi^L, \phi^3 \rangle \langle g_2, \phi^L, \phi^3 \rangle, \quad (5.2)$$

In the Haar model, for any fixed dyadic intervals $I$ and $K$, the only non-degenerate case $(\phi^3, \varphi^3) \neq 0$ is that $K \supseteq I$. Such observation provides natural localizations for the sequence $(\langle B^h_{f_1, f_2}, \varphi^3 \rangle)_{i \in \mathcal{I}}$ and thus for the sequences $(\langle f_1, \phi^K \rangle)_K$ and $(\langle f_2, \phi^K \rangle)_K$ as explicitly stated in the following lemma.

Lemma 5.8 (Localization of Sizes in the Haar Model). Suppose that $S$ is a measurable subset of $\mathbb{R}_x$ and $S'$ a measurable subset of $\mathbb{R}_y$. If $\mathcal{I}, \mathcal{J}'$ are finite collections of dyadic intervals such that $I \subset S \neq \emptyset$ for any $I \in \mathcal{I}$ and $J \cap S' \neq \emptyset$ for any $J \in \mathcal{J}'$, then

$$\text{size}_{\mathcal{I}}((\langle B^h_{f_1, f_2}, \varphi^3 \rangle)_{i \in \mathcal{I}}) \lesssim \sup_{K \cap S \neq \emptyset} \frac{|\langle f_1, \phi^K \rangle|}{|K|^\frac{3}{2}} \sup_{K \cap S' \neq \emptyset} \frac{|\langle f_2, \phi^K \rangle|}{|K|^\frac{3}{2}},$$

$$\text{size}_{\mathcal{J}'}((\langle \tilde{B}^h_{g_1, g_2}, \varphi^3 \rangle)_{j \in \mathcal{J}'}) \lesssim \sup_{L \cap S' \neq \emptyset} \frac{|\langle g_1, \phi^L \rangle|}{|L|^\frac{3}{2}} \sup_{L \cap S' \neq \emptyset} \frac{|\langle g_2, \phi^L \rangle|}{|L|^\frac{3}{2}}.$$
\[
\sup_{L \cap \mathcal{U} \neq \emptyset} \frac{|\langle g_1, \phi_L^1 \rangle|}{|L|^\frac{1}{\nu}} \lesssim \min(C_22^{n^2}|G_1|, 1),
\]
\[
\sup_{L \cap \mathcal{U} \neq \emptyset} \frac{|\langle g_2, \phi_L^2 \rangle|}{|L|^\frac{1}{\nu}} \lesssim \min(C_22^{n^2}|G_2|, 1).
\]

We will also explore the local energy estimates which are “stronger” than the global energy estimates. Heuristically, in the case when \(f_1 \in L^p\) and \(f_2 \in L^q\) with \(|f_1| \leq \chi_{F_1}\) and \(|f_2| \leq \chi_{F_2}\) for \(p_1, q_1 > 1\) and close to 1, the global energy estimates would not yield the desired boundedness exponents for \(|F_1|\) and \(|F_2|\) whereas one could take advantages of the local energy estimates to obtain the result. In the Haar model, a perfect localization can be achieved for energy estimates involving bilinear operators \(B^H_f\) and \(B^H_F\) specified in Definition 5.7(ii). In particular, the corresponding energy estimates can be compared to the energy estimates for \((B^{n_1,n_1}_0, \varphi_1)_{I \in \mathcal{I}}\) and \((\tilde{B}^{n_2,n_2}_0, \varphi_J)_{J \in \mathcal{J}}\), where \(B^{n_1,n_1}_0\) and \(\tilde{B}^{n_2,n_2}_0\) are localized operators defined as follows.

**Definition 5.11.** Let \(\mathcal{U}_{n_1,m_1}, \mathcal{U}'_{n_2,m_2}\) be defined as levels sets described in Proposition 5.10. And suppose that \(\mathcal{I}', \mathcal{J}'\) are finite collections of dyadic intervals such that \(I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset\) for any \(I \in \mathcal{I}'\) and \(J \cap \mathcal{U}'_{n_2,m_2} \neq \emptyset\) for any \(J \in \mathcal{J}'\). Define

\[
B^{n_1,n_1}_0(f_1, f_2)(x) := \left\{ \begin{array}{ll}
\sum_{K : K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset} \frac{1}{|K|^\frac{1}{\nu}} |\langle f_1, \psi_K^1 \rangle| |\langle f_2, \psi_K^2 \rangle| |\varphi_K^H(x)| & \text{if } \varphi_K^H \text{ is } L^2\text{-normalized indicator func.} \\
\sum_{K : K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset} \frac{1}{|K|^\frac{1}{\nu}} |\langle f_1, \varphi_K \rangle| |\langle f_2, \varphi_K \rangle| |\varphi_K^H(x)| & \text{if } \varphi_K^H \text{ is Haar wavelet,}
\end{array} \right.
\]

\[
\tilde{B}^{n_2,n_2}_0(g_1, g_2)(y) := \left\{ \begin{array}{ll}
\sum_{L : L \cap \mathcal{U}'_{n_2,m_2} \neq \emptyset} \frac{1}{|L|^\frac{1}{\nu}} |\langle g_1, \psi_L^1 \rangle| |\langle g_2, \psi_L^2 \rangle| |\varphi_L^H(y)| & \text{if } \varphi_L^H \text{ is } L^2\text{-normalized indicator func.} \\
\sum_{L : L \cap \mathcal{U}'_{n_2,m_2} \neq \emptyset} \frac{1}{|L|^\frac{1}{\nu}} |\langle g_1, \varphi_L \rangle| |\langle g_2, \varphi_L \rangle| |\varphi_L^H(y)| & \text{if } \varphi_L^H \text{ is Haar wavelet.}
\end{array} \right.
\]

**Remark 5.12.** We would like to emphasize that \(B^{n_1,n_1}_0\) and \(\tilde{B}^{n_2,n_2}_0\) are localized to intersect level sets \(\mathcal{U}_{n_1,m_1}\) and \(\mathcal{U}'_{n_2,m_2}\) nontrivially. It is not difficult to imagine that the energy estimates for \((B^{H_1}_0, \varphi_1)_{I \in \mathcal{I}'}\) and \((\tilde{B}^{H_2}_0, \varphi_J)_{J \in \mathcal{J}'}\) would be better than the “global” energy estimates (i.e. energy\((B(f_1, f_2), \varphi_I)\)) and energy\((\tilde{B}(g_1, g_2), \varphi_J)\)) since one can now employ the information about intersections with level sets to control

\[
\frac{|\langle f_1, \phi_I \rangle|}{|I|^\frac{1}{\nu}} = \frac{1}{|I|^\frac{1}{\nu}} |\langle f_2, \phi_I \rangle|, \quad \frac{|\langle g_1, \phi_J \rangle|}{|J|^\frac{1}{\nu}} = \frac{|\langle g_2, \phi_J \rangle|}{|J|^\frac{1}{\nu}}.
\]

The energy estimates for \((B^{H}_0, \varphi_I)_{I \in \mathcal{I}'}\) and \((\tilde{B}^H_0, \varphi_J)_{J \in \mathcal{J}'}\) can indeed be reduced to the energy estimates for \((B^{n_1,n_1}_0, \varphi_I)_{I \in \mathcal{I}'}\) and \((\tilde{B}^{n_2,n_2}_0, \varphi_J)_{J \in \mathcal{J}'}\), as stated in Lemma 5.13.

**Lemma 5.13.** (Localization of Energies in the Haar Model). Suppose that \(\mathcal{I}', \mathcal{J}'\) are finite collections of dyadic intervals such that \(I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset\) for any \(I \in \mathcal{I}'\) and \(J \cap \mathcal{U}'_{n_2,m_2} \neq \emptyset\) for any \(J \in \mathcal{J}'\). Then

\[
\text{energy}_{\mathcal{I}'}((B^{H}_0, \varphi_I^H)_{I \in \mathcal{I}'}) \leq \text{energy}_{\mathcal{I}'}((B^{n_1,n_1}_0, \varphi_I^H)_{I \in \mathcal{I}'})
\]
\[
(5.3)
\]
\[
\text{energy}_{\mathcal{J}'}((\tilde{B}^H_0, \varphi_J^H)_{J \in \mathcal{J}'}) \leq \text{energy}_{\mathcal{J}'}((\tilde{B}^{n_2,n_2}_0, \varphi_J^H)_{J \in \mathcal{J}'}).
\]

The following local energy estimates will play a crucial role in the proof of our main theorem.

**Proposition 5.14.** (Local Energy Estimates in the Haar Model). Suppose that \(F_1, F_2 \subseteq \mathbb{R}_x\) and \(G_1, G_2 \subseteq \mathbb{R}_y\) are sets of finite measure and \(|f_i| \leq \chi_{F_i}, |g_j| \leq \chi_{G_j}, i, j = 1, 2\). Assume that \(\mathcal{I}', \mathcal{J}'\) are finite collections of dyadic intervals such that \(I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset\) for any \(I \in \mathcal{I}'\) and \(J \cap \mathcal{U}'_{n_2,m_2} \neq \emptyset\) for any \(J \in \mathcal{J}'\). Further assume that \(\frac{1}{\nu_1} + \frac{1}{\nu_2} = \frac{1}{\nu} + \frac{1}{\nu} > 1\).

(i) Then for any \(0 \leq \theta_1, \theta_2 < 1\) with \(\theta_1 + \theta_2 = 1\) and \(0 \leq \xi, \eta, \xi' \leq 1\) with \(\xi + \xi' = 1\), one has

\[
\text{energy}_{\mathcal{I}'}((B^{H}_0, \varphi_I^H)_{I \in \mathcal{I}'}) \lesssim C_1 \left( \frac{1}{|F_1|^\frac{1}{\nu_1}} + \frac{1}{|F_2|^\frac{1}{\nu_2}} \right),
\]

\[
(5.4)
\]

\[
\text{energy}_{\mathcal{J}'}((\tilde{B}^H_0, \varphi_J^H)_{J \in \mathcal{J}'}) \lesssim C_2 \left( \frac{1}{|G_1|^\frac{1}{\nu_1}} + \frac{1}{|G_2|^\frac{1}{\nu_2}} \right).
\]
(ii) Suppose that $t, s > 1$. Then for any $0 \leq \theta_1, \theta_2, \zeta_1, \zeta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{t}$ and $\zeta_1 + \zeta_2 = \frac{1}{s}$, one has

\[
\begin{align*}
\text{energy}_{T'}((B_H^1, \varphi_I^H))_{1 \leq I} & \lesssim C_1^{1 + \frac{1}{2n} - \theta_1 - \theta_2} 2^{n_1(\frac{1}{2n} - \theta_1)2^{m_1}} |F_1|^{\frac{1}{n_1}} |F_2|^{\frac{1}{n_1}}, \\
\text{energy}_{T'}((\tilde{B}_{H}^1, \varphi_I^H))_{1 \leq I} & \lesssim C_2^{\frac{1}{2} + \frac{1}{2} - \zeta_1 - \zeta_2} 2^{n_2(\frac{1}{2} - \zeta_1)2^{m_2}} |G_1|^{\frac{1}{n_2}} |G_2|^{\frac{1}{n_2}}.
\end{align*}
\]

Remark 5.15. The condition that

\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} > 1
\]

is required in the proof the proposition. Moreover, the energy estimates in Proposition 5.14 are useful for the range of exponents specified as (5.4). A simpler argument without the use of Proposition 5.14 can be applied for the other case

\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} \leq 1.
\]

Remark 5.16. Thanks to the localization specified in Lemma 5.13, it suffices to prove that

\[
\begin{align*}
\text{energy}_{T'}((B_{0}^{n_1,m_1}, \varphi_I^H))_I, \\
\text{energy}_{T'}((\tilde{B}_{0}^{n_2,m_2}, \varphi_I^H))_I
\end{align*}
\]

satisfy the same estimates on the right hand side of the inequalities in Proposition 5.14, equivalently

(i') for any $0 \leq \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{t}$ and $\zeta_1 + \zeta_2 = \frac{1}{s}$,

\[
\begin{align*}
\text{energy}_{T'}((B_{0}^{n_1,m_1}, \varphi_I^H))_I & \lesssim C_1^{\frac{1}{2n} - \theta_1 - \theta_2} 2^{n_1} |F_1|^{\frac{1}{n_1}} |F_2|^{\frac{1}{n_1}}, \\
\text{energy}_{T'}((\tilde{B}_{0}^{n_2,m_2}, \varphi_I^H))_I & \lesssim C_2^{\frac{1}{2} - \zeta_1 - \zeta_2} 2^{n_2} |G_1|^{\frac{1}{n_2}} |G_2|^{\frac{1}{n_2}};
\end{align*}
\]

(ii') for any $0 \leq \theta_1, \theta_2, \zeta_1, \zeta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{t}$ and $\zeta_1 + \zeta_2 = \frac{1}{s}$,

\[
\begin{align*}
\text{energy}_{T'}((B_{0}^{n_1,m_1}, \varphi_I^H))_I & \lesssim C_1^{\frac{1}{2n} - \theta_1 - \theta_2} 2^{n_1} |F_1|^{\frac{1}{n_1}} |F_2|^{\frac{1}{n_1}}, \\
\text{energy}_{T'}((\tilde{B}_{0}^{n_2,m_2}, \varphi_I^H))_I & \lesssim C_2^{\frac{1}{2} - \zeta_1 - \zeta_2} 2^{n_2} |G_1|^{\frac{1}{n_2}} |G_2|^{\frac{1}{n_2}}.
\end{align*}
\]

Due to Proposition 5.4, the proofs of (i') and (ii') and thus of (i) and (ii) can be reduced to verifying Lemma 5.17.

Lemma 5.17. Suppose that $F_1, F_2 \subseteq \mathbb{R}_x$ and $G_1, G_2 \subseteq \mathbb{R}_y$ are sets of finite measure and $|f_i| \leq \chi_{F_i}$, $|g_j| \leq \chi_{G_j}$, $i, j = 1, 2$. Fix $t, s \geq 1$. Then for any $0 \leq \theta_1, \theta_2, \zeta_1, \zeta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{t}$ and $\zeta_1 + \zeta_2 = \frac{1}{s}$, one has

\[
\begin{align*}
\|B_{0}^{n_1,m_1}(f_1, f_2)\| & \lesssim C_1^{\frac{1}{2n} - \theta_1 - \theta_2} 2^{n_1} |F_1|^{\frac{1}{n_1}} |F_2|^{\frac{1}{n_1}}, \\
\|\tilde{B}_{0}^{n_2,m_2}(g_1, g_2)\| & \lesssim C_2^{\frac{1}{2} - \zeta_1 - \zeta_2} 2^{n_2} |G_1|^{\frac{1}{n_2}} |G_2|^{\frac{1}{n_2}}.
\end{align*}
\]

5.2. Proof of Proposition 5.4.

(i) One notices that there exists an integer $n_0$ and a disjoint collection of intervals, denoted by $\mathbb{D}_n^0$, such that

\[
\text{energy}_{T'}((f, \varphi_I))_I = 2^{n_0} \sum_{I \in \mathbb{D}_n^0} |I|,
\]

where for any $I \in \mathbb{D}_n^0$,

\[
\frac{|(f, \varphi_I)|}{|I|^s} > 2^{n_0}.
\]

Meanwhile for any $x \in I$,

\[
Mf(x) \geq \frac{|(f, \varphi_I)|}{|I|^s}.
\]
which implies that 
\[ I \subseteq \{ Mf(x) > 2^{n_0} \} \]
for any \( I \in \mathcal{I}' \) satisfying (5.8). Then by the disjointness of \( \mathcal{D}_{n_0} \), one can estimate the energy as follows
\[
\text{energy}^n(\{(f, \varphi_l)\}_{l \in \mathcal{I}'}) = \left( \sum_{n} 2^{tn} \sum_{l \in \mathcal{D}_{n_0}'} |I| \right)^{\frac{n}{t}}
\]
where for any \( I \in \mathcal{D}_{n_0}' \),
\[
\frac{|\langle f, \varphi_l \rangle|}{|I|^2} > 2^n.
\]
By the same reasoning in (i),
\[ I \subseteq \{ Mf(x) > 2^n \} \]
Then by the disjointness of \( \mathcal{D}_{n_0}' \), one can estimate the energy as follows
\[
\text{energy}^n(\{(f, \varphi_l)\}_{l \in \mathcal{I}'}) \leq \left( \sum_{n} 2^{tn} |\{ Mf(x) > 2^n \}| \right)^{\frac{n}{t}} \lesssim \| Mf \|_t.
\]
One can then apply the fact that the mapping property of maximal operator \( M : L^t \to L^t \) for \( t > 1 \) and derive
\[
\| Mf \|_t \lesssim \| f \|_t.
\]

5.3. Proof of Propositions 5.8. Without loss of generality, we will prove the first size estimate and the second follows from the same argument. One recalls the definition of
\[
\text{size}^n(\{(B_{I_0}^{i_1, H}, \varphi_{I_0}^H)\}_{l \in \mathcal{I}'}) = \frac{|\langle B_{I_0}^{i_1, H} (f_1, f_2), \varphi_{I_0}^H \rangle|}{|I_0|^2}
\]
for some \( I_0 \in \mathcal{I}' \) with the property that \( I_0 \cap S \neq \emptyset \) by the assumption. Then
\[
\frac{|\langle B_{I_0}^{i_1, H} (f_1, f_2), \varphi_{I_0}^H \rangle|}{|I_0|^2} \leq \frac{1}{|I_0|} \sum_{K \cap [K]_{i_1, H} \cap I_0 = \emptyset} \frac{1}{|K|^2} |\langle f_1, \varphi_{K}^1 \rangle| |\langle f_2, \varphi_{K}^2 \rangle| |\langle I_0 |\varphi_{I_0}^H, \varphi_{K}^{3,H} \rangle| = \frac{1}{|I_0|} \sum_{K \cap [K]_{i_1, H} \cap I_0 = \emptyset} \frac{|\langle f_1, \varphi_{K}^1 \rangle| |\langle f_2, \varphi_{K}^2 \rangle|}{|K|^2} |\langle I_0 |\varphi_{I_0}^H, |K|^2 |\varphi_{K}^{3,H} \rangle|.
\]
Since \( \varphi_{I_0}^H \) and \( \varphi_{K}^{3,H} \) are compactly supported on \( I_0 \) and \( K \) respectively with \( |I_0| \lesssim |K| \), one has
\[
\langle |I_0|^{\frac{1}{2}} |\varphi_{I_0}^H, |K|^{\frac{1}{2}} |\varphi_{K}^{3,H} \rangle \neq 0
\]
if and only if
\[ I_0 \subseteq K. \]
By the hypothesis that \( I_0 \cap S \neq \emptyset \), one derives that \( K \cap S \neq \emptyset \) and
\[
\frac{|\langle B_{I_0}^{i_1, H} (f_1, f_2), \varphi_{I_0}^H \rangle|}{|I_0|^2} \leq \frac{1}{|I_0|} \sup_{K \cap S \neq \emptyset} |\langle f_1, \varphi_{K}^1 \rangle| \sup_{K \cap S \neq \emptyset} |\langle f_2, \varphi_{K}^2 \rangle| \sum_{K \cap [K]_{i_1, H} \cap I_0 = \emptyset} |\langle I_0 |\varphi_{I_0}^H, |K|^{\frac{1}{2}} |\varphi_{K}^{3,H} \rangle| \lesssim \frac{1}{|I_0|} \sup_{K \cap S \neq \emptyset} |\langle f_1, \varphi_{K}^1 \rangle| \sup_{K \cap S \neq \emptyset} |\langle f_2, \varphi_{K}^2 \rangle| \cdot |I_0|,
\]
where the last inequality holds trivially given that \( |I_0|^{\frac{1}{2}} \varphi_{I_0}^H \) is an \( L^\infty \) normalized characteristic function of \( I_0 \) and \( |K|^{\frac{1}{2}} \varphi_{K}^{3,H} \) an \( L^\infty \) normalized characteristic function of \( K \). This completes the proof of the proposition.
5.4. Proof of Lemma 5.13. Suppose that for any \( I \in \mathcal{I}' \), \( I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset \). By definition of energy, there exists \( n \in \mathbb{Z} \) and a disjoint collection of dyadic intervals \( \mathbb{D}_n^0 \) such that

\[
\text{energy}_{\mathcal{X}'}(\langle B_I^H, \varphi_I^H \rangle_{I \in \mathcal{I}'}) := 2^n \sum_{I \in \mathbb{D}_n^0} |I|
\]

where

\[
\frac{|\langle B_I^H, \varphi_I^H \rangle|}{|I|^\frac{1}{2}} > 2^n.
\]

Case I. \((\phi_K^3)_K\) is lacunary. One recalls that in the Haar model,

\[
\langle B_I^H, \varphi_I^H \rangle := \sum_{K \subseteq I} \frac{1}{|K|^\frac{1}{2}} \|\varphi_K^1\| \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \langle \varphi_I^H, \psi_K^3 \rangle
\]

where \(\varphi_I^H\) is an \(L^2\)-normalized indicator function of \( I \) and \(\psi_K^3\) is a Haar wavelet on \( K \). It is not difficult to observe that

\[
\langle \varphi_I^H, \psi_K^3 \rangle \neq 0 \iff K \supseteq I.
\]

Given \( I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset \), one can deduce that \( K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset \). As a consequence,

\[
\langle B_I^H, \varphi_I^H \rangle = \sum_{K \subseteq I, K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \langle \varphi_I^H, \psi_K^3 \rangle
\]

Let

\[
B_0^{n_1,m_1}(f_1, f_2)(x) := \sum_{K \subseteq I, K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \psi_K^3(x).
\]

Then

\[
\langle B_I^H, \varphi_I^H \rangle = \langle B_0^{n_1,m_1}, \varphi_I^H \rangle.
\]

Remark 5.18 (Biest trick). In the Haar model, equation (5.10) trivially holds due to (5.9). Such technique of replacing the operator defined in terms of \( I \) (namely \( B_I^H \)) by another operator independent of \( I \) (namely \( B_0^{n_1,m_1} \)) is called biest trick which allows neat energy estimates for

\[
\text{energy}((\langle B_0^{n_1,m_1}, \varphi_I \rangle)_{I \in \mathcal{I}}),
\]

\[
\text{energy}((\langle B_0^{n_2,m_2}, \varphi_J \rangle)_{J \in \mathcal{J}}),
\]

and yields a local energy estimates described in Proposition 5.14.

Case II: \((\phi_K^3)_K\) is non-lacunary. Since \(\phi_K^3\) and \(\varphi_I^H\) are \(L^2\)-normalized indicator functions of \( K \) and \( I \) respectively, \(|K| \leq |I|\) implies that \( K \supseteq I \). As a result, \( K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset \) given \( I \cap \mathcal{U}_{n_1,m_1} \neq \emptyset \). Then

\[
\frac{|\langle B_I^H, \varphi_I^H \rangle|}{|I|^\frac{1}{2}} \leq \sum_{K \subseteq I, K \cap \mathcal{U}_{n_1,m_1} \neq \emptyset} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle |\langle \varphi_I^H, \psi_K^3 \rangle|.
\]
One can drop the condition $K \supseteq I$ in the sum and bound the above expression by
\[
\frac{|\langle B^H_i, \varphi^H_j \rangle|}{|I|^{\frac{1}{2}}} \leq \frac{1}{|I|} \sum_{K \in \mathcal{K}} \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \phi^1_K \rangle| |\langle f_2, \phi^2_K \rangle| |\langle \varphi^H_j, \psi_K \rangle|.
\]

One can define the localized operator in this case
\[
B^{n_1, m_1}_0(x) := \sum_{K \in \mathcal{K}} \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \phi^1_K \rangle| |\langle f_2, \phi^2_K \rangle| |\varphi^H_j(x)|.
\]

The discussion above yields that
\[
\frac{|\langle B^H_i, \varphi^H_j \rangle|}{|I|^{\frac{1}{2}}} \leq \frac{|\langle B^{n_1, m_1}_0, \varphi^H_j \rangle|}{|I|^{\frac{1}{2}}}
\]
and therefore
\[
\text{energy}_{\mathcal{F}}(\langle B^H_i, \varphi^H_j \rangle_{i \in \mathcal{F}}) \leq \text{energy}_{\mathcal{F}}(\langle B^{n_1, m_1}_0, \varphi^H_j \rangle_{i \in \mathcal{F}}).
\]

This completes the proof of the lemma.

Remark 5.19. $B^{n_1, m_1}_0$ is perfectly localized in the sense that the dyadic intervals (that matter) intersect with $\mathcal{U}_{n_1, m_1}$ nontrivially given that $I \cap \mathcal{U}_{n_1, m_1} \neq \emptyset$. As will be seen from the proof of Lemma 5.17, such localization is essential in deriving desired estimates. In the general Fourier case, more efforts are needed to create similar localizations as will be discussed in Chapter 10.

5.5. **Proof of Lemma 5.17.** The estimates described in Lemma 5.17 can be obtained by a very similar argument for proving the boundedness of one-parameter paraproducts discussed in Chapter 2 of [22]. We would include the customized proof here since the argument depends on a one-dimensional stopping-time decomposition which is also an important ingredient for our tensor-type stopping-time decompositions that will be introduced in later chapters.

5.5.1. **One-dimensional stopping-time decomposition - maximal intervals.** Given finiteness of the collection of dyadic intervals $\mathcal{K}$, there exists some $K_1 \in \mathbb{Z}$ such that
\[
\frac{|\langle f_1, \varphi_K \rangle|}{|K|^{\frac{1}{2}}} \leq C_i 2^{K_1} \text{energy}_K((f_1, \varphi_K))_{K \in \mathcal{K}}.
\]

We can pick the largest interval $K_{\text{max}}$ such that
\[
\frac{|\langle f_1, \varphi_{K_{\text{max}}} \rangle|}{|K_{\text{max}}|^{\frac{1}{2}}} > C_i 2^{K_1 - 1} \text{energy}_K((f_1, \varphi_K))_{K \in \mathcal{K}}.
\]

Then we define a tree
\[
U := \{ K \in \mathcal{K} : K \subseteq K_{\text{max}} \},
\]
and let $K_U := K_{\text{max}}$, usually called as tree-top. Now we look at $\mathcal{K} \setminus U$ and repeat the above step to choose maximal intervals and collect their subintervals in their corresponding sets. Since $\mathcal{K}$ is finite, the process will eventually end. We then collect all $U$’s in a set $U_{K_{-1}}$. Next we repeat the above algorithm to $\mathcal{K} \setminus \bigcup_{U \in U_{K_{-1}}} U$. We thus obtain a decomposition $\mathcal{K} = \bigcup_k \bigcup_{U \in U_k} U$. If, otherwise, the sequence is formed in terms of bump functions in lacunary family, then the same procedure can be performed to
\[
\frac{1}{|K|} \left\| \left( \sum_{K' \subseteq K} \frac{|\langle f_2, \psi_{K'} \rangle|^2}{|K'|} \chi_{K'} \right)^{\frac{1}{2}} \right\|_{1, \infty}.
\]

The next proposition summarizes the information from the stopping-time decomposition and the details of the proof are included in Chapter 2 of [22].
Proposition 5.20. Suppose $\mathcal{K} = \bigcup_{k} \bigcup_{U \in \mathcal{U}_k} U$ is a decomposition obtained from the stopping-time algorithm specified above, then for any $k \in \mathbb{Z}$, one has

$$2^{k-1} \text{energy}_\mathcal{K}((f_1, \phi_K))_{K \in \mathcal{K}} \leq \text{size}_{\bigcup_{U \in \mathcal{U}_k}} U((f_1, \phi_K))_{K \in \mathcal{K}} \leq \min(2^k \text{energy}_\mathcal{K}((f_1, \phi_K))_{K \in \mathcal{K}}),$$

size_{\bigcup_{U \in \mathcal{U}_k}} U((f_1, \phi_K))_{K \in \mathcal{K}}

In addition,

$$\sum_{U \in \mathcal{U}_k} |K_U| \lesssim 2^{-k}.$$
(2) **Estimate of** $\|B_0^{n_1,m_1}\|_t$ **for** $t > 1$. We will first prove restricted weak-type estimates for $B_0^{n_1,m_1}$ specified in Claim 5.22 and then the strong-type estimates in Claim 5.23 follows from the standard interpolation technique.

**Claim 5.22.** $\|B_0^{n_1,m_1}(f_1,f_2)\|_{l,\infty} \lesssim C_1^{1+\frac{n_1}{p_1}-\theta_1}2^{n_1(\frac{1}{p_1}-\theta_1)}2^{m_1(\frac{1}{q_1}-\theta_2)}|F_1|^\frac{1}{p_1}|F_2|^\frac{1}{q_1}$, where $\theta_1 + \theta_2 = \frac{1}{t}$ and $t \in (t-\delta,t+\delta)$ for some $\delta > 0$ sufficiently small.

**Proof of Claim 5.22.** It suffices to apply the dualization and prove that for any $\chi_S \in L^p$, 

$$|\langle B_0^{n_1,m_1}, \chi_S \rangle| \lesssim 2^{n_1(\frac{1}{p_1}-\theta_1)}2^{m_1(\frac{1}{q_1}-\theta_2)}|F_1|^\frac{1}{p_1}|F_2|^\frac{1}{q_1}|S|^\frac{1}{p_1},$$

where $\theta_1 + \theta_2 = \frac{1}{t}$.

The multilinear form can be estimated using a similar argument described in the proof of (i). In particular, let 

$$K' := \{ K \in K : K \cap U_{n_1,m_1} \neq \emptyset \}.$$

Then

$$|\langle B_0^{n_1,m_1}, \chi_S \rangle| \lesssim \text{size}_{K'}((f_1, \phi_K^1)_K)^{1-\theta_1} \text{size}_{K'}((f_2, \phi_K^2)_K)^{1-\theta_2} \text{size}_{K'}((\chi_S, \phi_K^3)_K)^{1-\theta_3},$$

(5.14)

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. The size and energy estimates involving $f_1, f_2$ in part (1) are still valid. Here $\phi_K^i$ are defined differently in Case I and II. However, one applies the same straightforward estimates that

$$\text{size}_{K'}((\chi_S, \phi_K^3)_K) \lesssim 1,$$

(5.15)

$$\text{energy}_{K'}((\chi_S, \phi_K^3)_K) \lesssim |S|.$$

By plugging in the above estimate (5.15) and (5.11) into (5.14), one has

$$|\langle B_0^{n_1,m_1}, \chi_S \rangle| \lesssim (C_1|F_1|)^{\alpha_1(1-\theta_1)}(C_1|F_2|)^{\beta_1(1-\theta_2)}|F_1|^\theta_1|F_2|^\theta_2|S|^\theta_3,$$

$$= C_1^{\alpha_1(1-\theta_1)+\beta_1(1-\theta_2)}2^{n_1(\alpha_1(1-\theta_1))}2^{m_1(\beta_1(1-\theta_2))}|F_1|^\alpha_1(1-\theta_1)|F_2|^\beta_1(1-\theta_2)+|S|^\theta_3,$$

for any $0 \leq \alpha_1, \beta_1 \leq 1$. Let $\theta_3 = \frac{1}{t}$, then $\theta_1 + \theta_2 = \frac{1}{t}$. One can then conclude

$$\|B_0^{n_1,m_1}\|_{l,\infty} \lesssim C_1^{\alpha_1(1-\theta_1)+\beta_1(1-\theta_2)}2^{n_1(\alpha_1(1-\theta_1))}2^{m_1(\beta_1(1-\theta_2))}|F_1|^\alpha_1(1-\theta_1)|F_2|^\beta_1(1-\theta_2)+|S|^\theta_3.$$

Since $\frac{1}{p_1} + \frac{1}{q_1} > 1$, one can choose $0 \leq \alpha_1, \beta_1 \leq 1$ and $\theta_1, \theta_2$ with $\theta_1 + \theta_2 = \frac{1}{t}$ such that

$$\alpha_1(1-\theta_1) + \theta_1 = \frac{1}{p_1},$$

$$\beta_1(1-\theta_2) + \theta_2 = \frac{1}{q_1},$$

the claim then follows. \qed

6. **Proof of Theorem 4.2 for $\Pi_{\text{flag}#1@\text{flag}#2}$ - Haar Model**

In this chapter, we will first specify the localization for the discrete model $\Pi_{\text{flag}#1@\text{flag}#2}$, which can be viewed as a starting point for the stopping-time decompositions. Then we will introduce different stopping-time decompositions used in the estimates. Finally, we will discuss how to apply information from the multiple stopping-time decompositions to obtain estimates. The organizations of Chapters 7-9 will follow the same scheme.
6.1. Localization. The definition of the exceptional set, which settles the starting point for the stopping-time decompositions, is expected to be compatible with the stopping-time algorithms involved. There would be two types of stopping-time decompositions undertaken for the estimates of $\Pi_{\text{flag}^#1 \oplus \text{flag}^#2} - $ one is the tensor-type stopping-time decomposition and the other one the general two-dimensional level sets stopping-time decomposition. While the second algorithm is related to a generic exceptional set (denoted by $\Omega^2$), the first algorithm aims to integrate information from two one-dimensional decompositions, which corresponds to the creation of a two-dimensional exceptional set (denoted by $\Omega^1$) as a Cartesian product of two one-dimensional exceptional sets.

One defines the exceptional set, denoted by $\tilde{\Omega}$, as follows. Let

$$\tilde{\Omega} := \{ M \chi_\Omega > \frac{1}{100} \}$$

with

$$\Omega := \Omega^1 \cup \Omega^2,$$

where

$$\Omega^1 := \bigcup_{n \in \mathbb{Z}} \{ Mf_1 > C_1 2^n |F_1| \} \times \{ Mg_1 > C_2 2^{-n} |G_1| \} \cup \bigcup_{\tilde{n} \in \mathbb{Z}} \{ Mf_2 > C_1 2^{\tilde{n}} |F_2| \} \times \{ Mg_2 > C_2 2^{-\tilde{n}} |G_2| \} \cup \bigcup_{\tilde{n} \in \mathbb{Z}} \{ Mf_1 > C_1 2^{\tilde{n}} |F_1| \} \times \{ Mg_2 > C_2 2^{-\tilde{n}} |G_2| \} \cup \bigcup_{\tilde{n} \in \mathbb{Z}} \{ Mf_2 > C_1 2^{\tilde{n}} |F_2| \} \times \{ Mg_1 > C_2 2^{-\tilde{n}} |G_1| \},$$

$$\Omega^2 := \{ SSf > C_3 \| h \|_{L^p(\mathbb{R}^2)} \}.$$

Remark 6.1. Given by the boundedness of the Hardy-Littlewood maximal operator and the double square function operator, it is not difficult to check that if $C_1, C_2, C_3 \gg 1$, then $|\tilde{\Omega}| \ll 1$. For different model operators, we will define different exceptional sets based on different stopping-time decompositions to employ. Nevertheless, their measures can be controlled similarly using the boundedness of the maximal operator and hybrid maximal and square operators.

By scaling invariance, we will assume without loss of generality that $|E| = 1$ throughout the paper. Let (6.1)

$$E' := E \setminus \tilde{\Omega},$$

then $|E'| \sim |E|$ and thus $|E'| \sim 1$. Our goal is to show that (2.2) holds with the corresponding subset $E' \subseteq E$ (which will be different for each discrete model operator). In the current setting, this is equivalent to proving that the multilinear form

$$\Lambda_{\text{flag}^#1 \oplus \text{flag}^#2, \chi_{E'}} := \langle \Pi_{\text{flag}^#1 \oplus \text{flag}^#2} (f_1^x, f_2^x, g_1^y, g_2^y, h^{x,y}, \chi_{E'}) \rangle$$

satisfies the following restricted weak-type estimate

$$|\Lambda_{\text{flag}^#1 \oplus \text{flag}^#2}| \lesssim |F_1|^\frac{1}{p_1} |G_1|^\frac{1}{p_2} |F_2|^\frac{1}{p_3} |G_2|^\frac{1}{q} \| h \|_{L^p(\mathbb{R}^2)}.$$  (6.3)

Remark 6.2. It is noteworthy that the discrete model operators are perfectly localized to $E'$ in the Haar model. In particular,

$$\langle \Pi_{\text{flag}^#1 \oplus \text{flag}^#2} \chi_{E'} \rangle := \sum_{I \times J \in \mathbb{R}} \frac{1}{|I|^\frac{1}{p_1} |J|^\frac{1}{p_2}} \langle B_I (f_1, f_2), \phi^H_1 (B_J (g_1, g_2), \phi^H_2 (h, \phi^H_1 \otimes \phi^H_2) \chi_{E'}, \phi^H_1 \otimes \phi^H_2) \rangle$$

(6.4)

$$= \sum_{I \times J \in \mathbb{R}} \frac{1}{|I|^\frac{1}{p_1} |J|^\frac{1}{p_2}} \langle B_I (f_1, f_2), \phi^H_1 (B_J (g_1, g_2), \phi^H_2 (h, \phi^H_1 \otimes \phi^H_2) \chi_{E'}, \phi^H_1 \otimes \phi^H_2) \rangle,$$
because for $I \times J \cap \hat{\Omega}^c = \emptyset$, $I \times J \cap E' = \emptyset$ and thus $\langle \chi_E', \phi^3_{I} \otimes \phi^3_{J} \rangle = 0$, which means that dyadic rectangles satisfying $I \times J \cap \hat{\Omega}^c = \emptyset$ do not contribute to the multilinear form. In the Haar model, we would heavily rely on the localization (6.4) and consider only the dyadic rectangles $I \times J \in \mathcal{R}$ such that $I \times J \cap \hat{\Omega}^c \neq \emptyset$.

6.2. **Tensor-type stopping-time decomposition I - level sets.** The first tensor-type stopping time decomposition, denoted by the tensor-type stopping-time decomposition $I$, will be performed to obtain estimates for $\Pi_{\text{flag}^n \otimes \text{flag}^m}$. It aims to recover intersections with two-dimensional level sets from intersections with one-dimensional level sets for each variable. Another tensor-type stopping-time decomposition, denoted by the tensor-type stopping-time decomposition $II$, involves maximal intervals and plays an important role in the discussion for $\Pi_{\text{flag}^n \otimes \text{flag}^m}$. We will focus on the tensor-type stopping-time decomposition $I$ in this chapter.

6.2.1. **One-dimensional stopping-time decompositions - level sets.** One can perform a one-dimensional stopping-time decomposition on $\mathcal{I} := \{I : I \times J \in \mathcal{R}\}$. Let

$$\Omega^n_{I_1} := \{M_{f_1} > C_1 2^{N_1} \mid F_1\},$$

for some $N_1 \in \mathbb{Z}$ and

$$\mathcal{I}_{N_1} := \{I \in \mathcal{I} : |I \cap \Omega^n_{I_1}| > \frac{1}{10} |I|\}.$$

Define

$$\Omega^n_{I_1 - 1} := \{M_{f_1} > C_1 2^{N_1 - 1} \mid F_1\},$$

and

$$\mathcal{I}_{N_1 - 1} := \{I \in \mathcal{I} \setminus \mathcal{I}^{N_1} : |I \cap \Omega^n_{I_1 - 1}| > \frac{1}{10} |I|\}.$$

The procedure generates the sets $(\Omega^n_{I_1})_{n_1}$ and $(\mathcal{I}_{N_1})_{n_1}$. Independently define

$$\Omega^n_{M_1} := \{M_{f_2} > C_1 2^{M_1} |F_2|\},$$

for some $M_1 \in \mathbb{Z}$ and

$$\mathcal{I}_{M_1} := \{I \in \mathcal{I} : |I \cap \Omega^n_{M_1}| > \frac{1}{10} |I|\}.$$

Define

$$\Omega^n_{M_1 - 1} := \{M_{f_2} > C_1 2^{M_1 - 1} |F_2|\},$$

and

$$\mathcal{I}_{M_1 - 1} := \{I \in \mathcal{I} \setminus \mathcal{I}^{M_1} : |I \cap \Omega^n_{M_1 - 1}| > \frac{1}{10} |I|\}.$$

The procedure generates the sets $(\Omega^n_{M_1})_{m_1}$ and $(\mathcal{I}_{M_1})_{m_1}$. Now define $\mathcal{I}_{n_1, m_1} := \mathcal{I}_{n_1} \cap \mathcal{I}_{m_1}$ and the decomposition on $\mathcal{I} = \bigcup_{n_1, m_1} \mathcal{I}_{n_1, m_1}$.

Same algorithm can be applied to $\mathcal{J} := \{J : I \times J \in \mathcal{R}\}$ with respect to the level sets in terms of $M_{g_1}$ and $M_{g_2}$, which produces the sets

(i) $(\Omega^n_{m_2})_{n_2}$ and $(\mathcal{J}_{n_2})_{n_2}$, where

$$\Omega^n_{m_2} := \{M_{g_1} > C_2 2^{n_2} |G_1|\},$$

and

$$\mathcal{J}_{n_2} := \{J \in \mathcal{J} \setminus \mathcal{J}_{n_2 + 1} : |J \cap \Omega^n_{m_2}| > \frac{1}{10} |J|\}.$$

(ii) $(\Omega^n_{m_2})_{m_2}$ and $(\mathcal{J}_{m_2})_{m_2}$, where

$$\Omega^n_{m_2} := \{M_{g_2} > C_2 2^{m_2} |G_2|\},$$

and

$$\mathcal{J}_{m_2} := \{J \in \mathcal{J} \setminus \mathcal{J}_{m_2 + 1} : |J \cap \Omega^n_{m_2}| > \frac{1}{10} |J|\}.$$ 

One thus obtains the decomposition $\mathcal{J} = \bigcup_{n_2, m_2} \mathcal{J}_{n_2, m_2}$, where $\mathcal{J}_{n_2, m_2} := \mathcal{J}_{n_2} \cap \mathcal{J}_{m_2}$.
6.2.2. **Tensor product of two one-dimensional stopping-time decompositions - level sets.** If we assume that all dyadic rectangles satisfy \( I \times J \cap \tilde{\Omega}^c \neq \emptyset \) as in the Haar model, then we have the following observation.

**Observation 1.** If \( I \times J \in \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2} \), then \( n_1, m_1, n_2, m_2 \in \mathbb{Z} \) satisfies \( n_1 + n_2 < 0 \) and \( m_1 + m_2 < 0 \).

(Equivalently, \( \forall I \times J \cap \tilde{\Omega}^c \neq \emptyset, I \times J \in \mathcal{I}_{-n_2,-m_2} \times \mathcal{J}_{n_2,m_2}, \) for some \( n_2, m_2 \in \mathbb{Z} \) and \( n, m > 0 \).)

**Remark 6.3.** The observation shows how a rectangle \( I \times J \) intersects with a two-dimensional level sets is closely related to how the corresponding intervals intersect with one-dimensional level sets (namely \( I \times J \subset \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2} \) with \( n_1 + n_2 < 0 \) and \( m_1 + m_2 < 0 \)), as commented in the beginning of the section.

**Proof.** Given \( I \in \mathcal{I}_{n_1} \), one has \( |I \cap \{ M_{f_1} > C_1 2^{n_1} |F_1| \}| > \frac{1}{|I|} |I| \); similarly, \( J \in \mathcal{J}_{n_2} \) implies that \( |J \cap \{ M_{g_1} > C_2 2^{n_2} |G_1| \}| > \frac{1}{|J|} |J| \). If \( n_1 + n_2 \geq 0 \), then \( \{ M_{f_1} > C_1 2^{n_1} |F_1| \} \times \{ M_{g_1} > C_2 2^{n_2} |G_1| \} \subseteq \Omega_1 \subseteq \Omega \). Then \( |I \times J \cap \Omega| > \frac{1}{100} |I \times J| \), which implies that \( I \times J \subseteq \tilde{\Omega} \) and contradicts the assumption. Same reasoning applies to \( m_1 \) and \( m_2 \).

6.3. **General two-dimensional level sets stopping-time decomposition.** With the assumption that \( R \cap \tilde{\Omega}^c \neq \emptyset \), one has that

\[
|R \cap \Omega^2| \leq \frac{1}{100} |R|,
\]

where

\[
\Omega^2 = \{ SSh > C_3 \|h\|_s \}.
\]

Then define

\[
\Omega^2_{-1} := \{ SSh > C_3 2^{-1} \|h\|_{L^s} \}
\]

and

\[
\mathcal{R}_{-1} := \{ R \in \mathcal{R} : |R \cap \Omega^2_{-1}| > \frac{1}{100} |R| \}.
\]

Successively define

\[
\Omega^2_{-2} := \{ SSh > C_3 2^{-2} \|h\|_{L^s} \}
\]

and

\[
\mathcal{R}_{-2} := \{ R \in \mathcal{R} \setminus \mathcal{R}_{-1} : |R \cap \Omega^2_{-2}| > \frac{1}{100} |R| \}.
\]

This two-dimensional stopping-time decomposition generates the sets \( (\Omega^2_{k})_{k \leq 0} \) and \( (\mathcal{R}_{k})_{k \leq 0} \).

Independently one can apply the same algorithm involving \( SSh_{X e} \) which generates \( (\Omega^2_{k})_{k_2 \leq K} \) and \( (\mathcal{R}_{k})_{k_2 \leq K} \) where \( K \) can be arbitrarily large. The existence of \( K \) is guaranteed by the finite cardinality of the collection of dyadic rectangles.

6.4. **Sparsity condition.** One important property followed from the **tensor-type stopping-time decomposition I - level sets** is the sparsity of dyadic intervals at different levels. Such geometric property plays an important role in the arguments for the main theorems.

**Proposition 6.4.** Suppose that \( \mathcal{J} = \bigcup_{n_2 \in \mathbb{Z}} \mathcal{J}_{n_2} \) is a decomposition of dyadic intervals with respect to \( M_{g_1} \) as specified in Section 6.3. For any fixed \( n_2 \in \mathbb{Z} \), suppose that \( J_0 \in \mathcal{J}_{n_2 - 10} \). Then

\[
\sum_{J \in \mathcal{J}_{n_2}, J \cap J_0 \neq \emptyset} |J| \leq \frac{1}{2} |J_0|.
\]

To prove the proposition, one would need the following claim about point-wise estimates for \( M_{g_1} \) on \( J \in \mathcal{J}_{n_2} \):

**Claim 6.5.** Suppose that \( \bigcup_{n_2} \mathcal{J}_{n_2} \) is a partition of dyadic intervals generated from the stopping-time decomposition described above. If \( J \in \mathcal{J}_{n_2} \), then for any \( y \in J \),

\[
M_{g_1}(y) > 2^{-7} \cdot C_2 2^{n_2} |G_1|.
\]
Proof on Claim $\implies$ Proposition 6.4. We will first explain why the proposition follows from the claim and then prove the claim. One recalls that all the intervals are dyadic, which means if $J \cap J_0 \neq \emptyset$, then either

\[ J \subseteq J_0 \]

or

\[ J_0 \subseteq J. \]

If $J_0 \subseteq J$, then the claim implies that

\[ J_0 \subseteq J \subseteq \{ M g_1 > C 2^{n_2 - 7} |G_1| \}. \]

But $J_0 \in \mathcal{J}_{n_2 - 10}$ infers that

\[ |J_0 \cap \{ M g_1 > C 2^{n_2 - 7} \}| < \frac{1}{10} |J_0|, \]

which is a contradiction. If $J \subseteq J_0$ and suppose that

\[ \sum_{J \in \mathcal{J}_{n_2} \atop J \subseteq J_0} |J| > \frac{1}{2} |J_0|. \]

Then one can derive from $J \in \mathcal{J}_{n_2}$ that

\[ |J \cap \{ M g_1 > C 2^{n_2} |G_1| \}| > \frac{1}{10} |J|. \]

Therefore

\[ \sum_{J \in \mathcal{J}_{n_2} \atop J \subseteq J_0} |J \cap \{ M g_1 > C 2^{n_2} |G_1| \}| > \frac{1}{10} \sum_{J \in \mathcal{J}_{n_2} \atop J \subseteq J_0} |J| > \frac{1}{20} |J_0|. \]

But by the disjointness of $(J)_{J \in \mathcal{J}_{n_2}}$,

\[ \sum_{J \in \mathcal{J}_{n_2} \atop J \subseteq J_0} \left| J \cap \{ M g_1 > C 2^{n_2} |G_1| \} \right| \leq \left| J_0 \cap \{ M g_1 > C 2^{n_2} |G_1| \} \right|. \]

Thus

\[ |J_0 \cap \{ M g_1 > C 2^{n_2} |G_1| \}| > \frac{1}{20} |J_0|, \]

Now the claim, with slight modifications, implies that $J_0 \subseteq \{ M g_1 > C 2^{n_2 - 8} |G_1| \}$. But $J_0 \in \mathcal{J}_{n_2 - 10}$, which gives the necessary condition that

\[ |J_0 \cap \{ M g_1 > C 2^{n_2} |G_1| \}| \leq \frac{1}{10} |J_0| \]

and reaches a contradiction. \qed

We will now prove the claim.

Proof of Claim. Without loss of generality, we assume that $g$ is non-negative since if it is not, we can always replace it by $|g|$ where $M g = M(|g|)$. We prove the claim case by case:

Case (i): $\forall y \in \{ M g_1 > C 2^{n_2} |G_1| \}$, there exists $J_y \subseteq J$ such that $\text{ave}_{J_y}(g_1) > C \cdot 2^{n_2} |G_1|$;

Case (ii): There exists $y_0 \in \{ M g_1 > C 2^{n_2} |G_1| \}$ and $J_{y_0} \not\subseteq J$ such that $\text{ave}_{J_{y_0}}(g_1) > C \cdot 2^{n_2} |G_1|$ and

- Case (iia): $\frac{1}{20} |J| \leq |J_{y_0} \cap J|$ and $|J_{y_0}| \leq |J|$;
- Case (iib): $\frac{1}{20} |J| \leq |J_{y_0} \cap J|$ and $|J_{y_0}| > |J|$;
- Case (iic): $|J_{y_0} \cap J| < \frac{1}{20} |J|$.

Proof of (i): In Case (i), one observes that $\{ M g_1 > C 2^{n_2} |G_1| \} \cap J$ can be rewritten as $\{ M(g_1 \cdot \chi_J) > C 2^{n_2} |G_1| \} \cap J$. Thus

\[ C 2^{n_2} |G_1| \{ M g_1 > C 2^{n_2} |G_1| \} \cap J = C 2^{n_2} |G_1| \{ \| M(g_1 \chi_J) > C 2^{n_2} |G_1| \} \cap J \leq \| g_1 \chi_J \|_1. \]

One recalls that $\{ M g_1 > C 2^{n_2} |G_1| \} \cap J \geq \frac{1}{10} |J|$, which implies that

\[ C 2^{n_2} |G_1| \cdot \frac{1}{10} |J| \leq \| g_1 \chi_J \|_1, \]
or equivalently, 
\[
\|g_1 \chi_J\|_1 \geq \frac{1}{10} C \cdot 2 |J| |G_1|.
\]

Therefore \(Mg_1 > 2^{-4} C \cdot 2^n |G_1|\).

**Proof of (ii):** We will prove that if either (iia) or (iib) holds, then \(Mg_1 > 2^{-7} C \cdot 2^n |G_1|\). If neither (iia) nor (iib) happens, then (iic) has to hold and in this case, \(Mg_1 > 2^{-7} C \cdot 2^n |G_1|\).

If there exists \(y_0 \in \{Mg_1 > C \cdot 2^n |G_1|\}\) such that (iia) holds, then
\[
\|g_1 \chi_{J_{y_0}}\|_1 \leq \|g_1 \chi_{J_{y_0} \cup J}\|_1 \leq \|g_1 \chi_{J_{y_0} \cap J}\|_1 \leq \|g_1 \chi_{J_{y_0} \cup J}\|_1 \leq \frac{1}{20} |J|,
\]
where the last inequality follows from \(\frac{1}{20} |J| \leq |J_{y_0} \cap J|\). Moreover, \(|J_{y_0}| \leq |J|\) and \(y \in J_{y_0} \cap J \neq \emptyset\) infer that \(|J_{y_0} \cup J| \leq 2|J|\). Thus
\[
\frac{\|g_1 \chi_{J_{y_0} \cup J}\|_1}{|J_{y_0} \cup J|} \leq \frac{\|g_1 \chi_{J_{y_0} \cup J}\|_1}{|J_{y_0} \cup J|}.
\]
which implies
\[
\|g_1 \chi_{J_{y_0} \cup J}\|_1 > \frac{1}{80} C \cdot 2^n |G_1|,
\]
and as a result \(Mg_1 > 2^{-7} C \cdot 2^n |G_1|\) on \(J\).

If there exists \(y \in \{Mg_1 > C \cdot 2^n |G_1|\}\) such that (iib) holds, then
\[
\|g_1 \chi_{J_{y_0}}\|_1 \leq \|g_1 \chi_{J_{y_0} \cup J}\|_1 \leq \frac{2\|g_1 \chi_{J_{y_0} \cup J}\|_1}{2|J_{y_0}|} \leq \frac{2\|g_1 \chi_{J_{y_0} \cup J}\|_1}{|J_{y_0} \cup J|},
\]
where the last inequality follows from \(|J_{y_0}| > |J|\). As a consequence,
\[
\frac{2\|g_1 \chi_{J_{y_0} \cup J}\|_1}{|J_{y_0} \cup J|} > C \cdot 2^n |G_1|,
\]
and \(Mg_1 > 2^{-1} C \cdot 2^n |G_1|\) on \(J\).

If neither (i), (iia) nor (iib) happens, then for \(S_{(iic)} := \{y : Mg_1(y) \geq C \cdot 2^n |G_1|\} \text{ and (i) does not hold}\), one direct geometric observation is that \(|S_{(iic)} \cap J| \leq \frac{1}{40} |J|\). In particular, suppose \(y \in S_{(iic)}\), then any \(J_{y_0}\) with \(\text{ave}_{J_{y_0}}(g_1) > C \cdot 2^n |G_1|\) has to contain the left endpoint or right endpoint of \(J\), which we denote by \(J_{\text{left}}\) and \(J_{\text{right}}\). If \(J_{\text{left}} \in J_{y_0}\), then the assumption that neither (iia) nor (iib) holds implies that
\[
|J_{y_0} \cap J| < \frac{1}{40} |J|,
\]
and thus
\[
|J_{\text{left}}, y| < \frac{1}{40} |J|.
\]
Same implication holds true for \(y \in S_{(iic)}\) with \(J_{\text{right}} \in J_{y_0}\). Therefore, for any \(y \in S_{(iic)}\), \(|J_{\text{left}}, y| < \frac{1}{40} |J|\) or \(|y, J_{\text{right}}| < \frac{1}{40} |J|\), which can be concluded as
\[
|S_{(iic)} \cap J| < \frac{1}{20} |J|.
\]

Since \(|\{Mg_1 > C \cdot 2^n |G_1|\} \cap J| > \frac{1}{10} |J|\),
\[
\left|\left\{\{Mg_1 > C \cdot 2^n |G_1|\} \setminus S_{(iic)}\right\} \cap J\right| > \frac{1}{20} |J|,
\]
in which case one can apply the argument for (i) with \({\{Mg_1 > C \cdot 2^n |G_1|\}}\) replaced by \({\{Mg_1 > C \cdot 2^n |G_1|\}} \setminus S_{(iic)}\) to conclude that
\[
Mg_1 > 2^{-5} C \cdot 2^n |G_1|.
\]
This ends the proof for the claim.
**Proposition 6.6.** Given an arbitrary collection of dyadic rectangles $\mathcal{R}_0$. Define $\mathcal{J} := \{ J : R = I \times J \in \mathcal{R}_0 \}$.
Suppose that $\mathcal{J} = \bigcup_{n_2 \in \mathbb{Z}} J_{n_2}$ is a decomposition of dyadic rectangles with respect to $M g_1$ as specified in Section 6.3 so that $\mathcal{R}_0 = \bigcup_{n_2 \in \mathbb{Z}} \bigcup_{R = I \times J \in \mathcal{R}_0 \cap J \in J_{n_2}} R$ is a decomposition of dyadic rectangles in $\mathcal{R}_0$. Then

$$\sum_{n_2 \in \mathbb{Z}} \left| \bigcup_{R = I \times J \in \mathcal{R}_0 \cap J \in J_{n_2}} R \right| \lesssim \left| \bigcup_{R \in \mathcal{R}_0} R \right|.$$ 

**Proof of Proposition 6.6.** Proposition 6.4 gives a sparsity condition for intervals in the $y$-direction, which is sufficient to generate sparsity for dyadic rectangles in $\mathbb{R}^2$. In particular,

$$\sum_{n_2 \in \mathbb{Z}} \left| \bigcup_{R = I \times J \in \mathcal{R}_0 \cap J \in J_{n_2}} R \right| = \sum_{i=0}^{9} \sum_{n_2 \equiv i \mod 10} \left| \bigcup_{R = I \times J \in \mathcal{R}_0 \cap J \in J_{n_2}} R \right| \lesssim \sum_{i=0}^{9} \left| \bigcup_{n_2 \equiv i \mod 10} \bigcup_{R = I \times J \in \mathcal{R}_0 \cap J \in J_{n_2}} R \right| \leq 10 \left| \bigcup_{R \in \mathcal{R}_0} R \right|,$$

where the second inequality follows from the sparsity condition in Proposition 6.4.

**Remark 6.7.** The picture below illustrates from a geometric point of view why the two-dimensional sparsity condition (Proposition 6.6) follows naturally from the one-dimensional sparsity (Proposition 6.4). In the figure, $A_1, A_2 \in I \times J_{n_2+20}$, $B \in I \times J_{n_2+10}$ and $C \in I \times J_{n_2}$ for some $n_2 \in \mathbb{Z}$.

---

6.5. **Summary of stopping-time decompositions.**

I. Tensor-type stopping-time decomposition I on $I \times J$ \[ \rightarrow I \times J \in \mathcal{I}_{n_1, m_1} \times \mathcal{J}_{n_2, m_2} \] \[ (n_1 + n_2 < 0, m_1 + m_2 < 0) \]

II. General two-dimensional level sets stopping-time decomposition \[ \rightarrow I \times J \in \mathcal{R}_{k_1, k_2} \] \[ (k_1 < 0, k_2 \leq K) \]
6.6. Application of stopping-time decompositions. With the stopping-time decompositions specified above, one can rewrite the multilinear form as

\[
| \Lambda_{\text{flag}#1 \oplus \text{flag}#2} | \\
= \sum_{n_1+n_2<0} \sum_{m_1+m_2<0} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2}} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{R}_{k_1,k_2}} \frac{1}{|I|^7 |J|^7} | \langle B_{I}^{#1,H} (f_1,f_2), \varphi_{I}^{1,H} \rangle | \langle \hat{B}_{J}^{#2,H} (g_1,g_2), \varphi_{J}^{1,H} \rangle | \langle h, \psi_{1}^{2,H} \otimes \psi_{J}^{2,H} \rangle | \langle \chi_{E'}, \psi_{I}^{3,H} \otimes \psi_{J}^{3,H} \rangle | |I||J| |I||J|.
\]

One can therefore rewrite the multilinear form as

\[
| \Lambda_{\text{flag}#1 \oplus \text{flag}#2} | \\
= \sum_{n_1+n_2<0} \sum_{m_1+m_2<0} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2}} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{R}_{k_1,k_2}} \frac{1}{|I|^7 |J|^7} | \langle B_{I}^{#1,H} (f_1,f_2), \varphi_{I}^{1,H} \rangle | \langle \hat{B}_{J}^{#2,H} (g_1,g_2), \varphi_{J}^{1,H} \rangle | \langle h, \psi_{1}^{2,H} \otimes \psi_{J}^{2,H} \rangle | \langle \chi_{E'}, \psi_{I}^{3,H} \otimes \psi_{J}^{3,H} \rangle | |I||J| |I||J| |I||J| |I||J|.
\]

One recalls the general two-dimensional level sets stopping-time decomposition that \( I \times J \in \mathcal{R}_{k_1,k_2} \) only if

\[
| I \times J \cap (\Omega_{k_1}^{c}) | \geq \frac{99}{100} | I \times J |
\]

with \( \Omega_{k_1}^{c} := \{ SS > C_{3} 2^{k_1} \| h \|_{s} \} \), and \( \Omega_{k_2}^{c} := \{ SS \chi_{E'} > C_{3} 2^{k_2} \} \). As a result,

\[
| I \times J | \approx | I \times J \cap (\Omega_{k_1}^{c}) \cap (\Omega_{k_2}^{c}) |.
\]

One can therefore rewrite the multilinear form as

\[
| \Lambda_{\text{flag}#1 \oplus \text{flag}#2} | \\
= \sum_{n_1+n_2<0} \sum_{m_1+m_2<0} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2}} \sum_{k_1<k_2 \leq K} \sum_{I \in J \in \mathcal{R}_{k_1,k_2}} \frac{1}{|I|^7 |J|^7} \cdot | I \times J \cap (\Omega_{k_1}^{c}) \cap (\Omega_{k_2}^{c}) | \cdot | \langle B_{I}^{#1,H} (f_1,f_2), \varphi_{I}^{1,H} \rangle | \langle \hat{B}_{J}^{#2,H} (g_1,g_2), \varphi_{J}^{1,H} \rangle | \langle h, \psi_{1}^{2,H} \otimes \psi_{J}^{2,H} \rangle | \langle \chi_{E'}, \psi_{I}^{3,H} \otimes \psi_{J}^{3,H} \rangle | |I||J| |I||J| |I||J| |I||J| |I||J| |I||J| |I||J| |I||J|.
\]

(6.5)

We will now estimate each components in (6.5) separately for clarity.

6.6.1. Estimate for the integral. One can apply the Cauchy-Schwartz inequality to the integrand and obtain

\[
\int_{(\Omega_{k_1}^{c}) \cap (\Omega_{k_2}^{c})} | \langle h, \psi_{1}^{2,H} \otimes \psi_{J}^{2,H} \rangle | \langle \chi_{E'}, \psi_{I}^{3,H} \otimes \psi_{J}^{3,H} \rangle | \chi_{I}(x) \chi_{J}(y) dxdy.
\]
\[ \leq \int_{(\Omega_{k_1}^2)^c \cap (\Omega_{k_2}^2)^c} \left( \sum_{I \times J \in \mathcal{I}_{n_1, m_1} \times \mathcal{J}_{n_2, m_2}} \frac{|(h, \psi^2_H, \psi^2_J)|^2}{|I||J|} \chi_I(x) \chi_J(y) \right)^{\frac{1}{2}} \]

\[ \leq \int_{(\Omega_{k_1}^2)^c \cap (\Omega_{k_2}^2)^c} SSH(x, y)SS \chi_{E'}(x, y) \cdot \chi_{\bigcup I \times J \in \mathcal{I}_{n_1, m_1} \times \mathcal{J}_{n_2, m_2} I \times J(x, y)} dxdy. \] 

(6.6)

Based on the general two-dimensional level sets stopping-time decomposition, the hybrid functions have point-wise control on the domain for integration. In particular, for any \((x, y) \in (\Omega_{k_1}^2)^c \cap (\Omega_{k_2}^2)^c,\)

\[ SSh(x, y) \lesssim C_3 2^{k_1} \|h\|_s, \]

\[ SS \chi_{E'}(x, y) \lesssim C_3 2^{k_2}. \]

As a result, the integral can be estimated by

\[ C_3^2 2^{k_1} \|h\|_s 2^{k_2} \left| \bigcup_{I \times J \in \mathcal{I}_{n_1, m_1} \times \mathcal{J}_{n_2, m_2}} I \times J \right|. \]

6.6.2. Estimate for \(\sup_{I \in \mathcal{I}_{n_1, m_1}} \frac{|(B^{\#_1 H}(f_1, f_2, \varphi^1_H))|}{|I|^{\frac{1}{2}}} \) and \(\sup_{J \in \mathcal{J}_{n_2, m_2}} \frac{|(B^{\#_2 H}(g_1, g_2, \varphi^2_H))|}{|J|^{\frac{1}{2}}} \). One recalls the algorithm in the tensor type stopping-time decomposition \(I\) - level sets, which incorporates the following information.

\[ I \in \mathcal{I}_{n_1, m_1} \]

implies that

\[ |I \cap \{M f_1 < C_1 2^{m_1}|F_1|\}| \geq \frac{9}{10} |I|, \]

\[ |I \cap \{M f_2 < C_1 2^{m_1}|F_2|\}| \geq \frac{9}{10} |I|, \]

which translates into

\[ I \cap \{M f_1 < C_1 2^{m_1}|F_1|\} \cap \{M f_2 < C_1 2^{m_1}|F_2|\} \neq \emptyset. \]

Then one can recall Proposition 5.10 to estimate

\[ \sup_{I \in \mathcal{I}_{n_1, m_1}} \frac{|(B^{\#_1 H}(f_1, f_2, \varphi^1_H))|}{|I|^{\frac{1}{2}}} \lesssim C_1^2 (2^{m_1}|F_1|)^{\alpha_1} (2^{m_1}|F_2|)^{\alpha_2}, \]

for any \(0 \leq \alpha_1, \alpha_2 \leq 1\). Similarly, one can apply Proposition 5.10 with \(\mathcal{U}_{n_2, m_2} := \{M g_1 < C_2 2^{m_2}|G_1|\} \cap \{M g_2 < C_2 2^{m_2}|G_2|\}\) to conclude that

\[ \sup_{J \in \mathcal{J}_{n_2, m_2}} \frac{|(B^{\#_2 H}(g_1, g_2, \varphi^2_H))|}{|J|^{\frac{1}{2}}} \lesssim C_2^2 (2^{m_2}|G_1|)^{\beta_1} (2^{m_2}|G_2|)^{\beta_2}, \]

for any \(0 \leq \beta_1, \beta_2 \leq 1\). By choosing \(\alpha_1 = \frac{1}{p_1}, \alpha_2 = \frac{1}{q_1}, \beta_1 = \frac{1}{p_2}, \beta_2 = \frac{1}{q_2}\), the multilinear form can therefore be estimated by

\[ |\Lambda_{\text{flag}^{\#_1} \otimes \text{flag}^{\#_2}}| \lesssim C_1^2 C_2^2 C_3^2 \sum_{n_1 + n_2 < 0, m_1 + m_2 < 0, k_1 < 0, k_2 \leq K} 2^{n_1 + m_1 + n_2 + m_2 + k_1} |F_1|^{\frac{n_1}{p_1}} |F_2|^{\frac{n_2}{p_2}} |G_1|^{\frac{m_1}{q_1}} |G_2|^{\frac{m_2}{q_2}} . k_1 \cdot 2^{k_1} \cdot \|h\|_s \cdot 2^{k_2} \cdot \left| \bigcup_{R \in \mathcal{R}_{k_1, k_2}} R \right|. \]
(6.7)

One recalls that
\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}
\]
then
\[
2^{n_1 - \frac{n}{p_1}} 2^{m_1 - \frac{m}{p_1}} 2^{n_2 - \frac{n}{q_1}} 2^{m_2 - \frac{m}{q_1}} = 2^{n_1 - \frac{n}{p_1}} 2^{m_1 - \frac{m}{p_1}} 2^{n_2 - \frac{n}{q_1}} 2^{m_2 - \frac{m}{q_1}} 2^m 2^n
\]
(6.8)

By the definition of exceptional sets, $2^{n_1 + n_2} \lesssim 1, 2^{m_1 + m_2} \lesssim 1, 2^{m_1 + n_2} \lesssim 1, 2^{m_1 + m_2} \lesssim 1$. Then
\[
n := -(n_1 + n_2) \geq 0,
m := -(m_1 + m_2) \geq 0.
\]

Without loss of generality, one further assumes that $\frac{1}{q_2} \geq \frac{1}{q_1}$ (with $q_1$ and $q_2$ swapped in the opposite case), which implies that
\[
(2^{n_1 + m_2})^{\frac{1}{p_2}} - \frac{1}{q_1} \lesssim 1.
\]

Now (6.7) can be bounded by
\[
|\Lambda_{\text{flag}^1 \otimes \text{flag}^2}|
\]
(6.9)
\[
\lesssim C_1^2 C_2^2 C_3 \sum_{n > 0} \sum_{n_2 \in \mathbb{Z}} \sum_{m > 0} \sum_{m_2 \in \mathbb{Z}} 2^{-n \frac{n}{p_1}} 2^{-m \frac{m}{p_1}} |F_1|^{\frac{n}{p_1}} |F_2|^{\frac{m}{p_1}} |G_1|^{\frac{n}{q_1}} |G_2|^{\frac{m}{q_1}} \cdot 2^{k_1} \| h \|_{L^2} 2^{k_2} \cdot \bigg| \bigcup_{R \in I_{-n_2 - m_2} \times J_{n_2, m_2}} R \bigg|.
\]

With $k_1, k_2, n, m$ fixed, one can apply the sparsity condition (Proposition 6.6) repeatedly and obtain the following bound for the expression
\[
(6.10)
\]
\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \bigg| \bigcup_{R \in I_{-n_2 - m_2} \times J_{n_2, m_2}} R \bigg| \lesssim \sum_{m_2 \in \mathbb{Z}} \bigg| \bigcup_{R \in I_{-m_2} \times J_{m_2}} R \bigg| \lesssim \bigg| \bigcup_{R \in \mathcal{R}_{k_1}} R \bigg| \leq \min\left( \bigg| \bigcup_{R \in \mathcal{R}_{k_1}} R \bigg|, \bigg| \bigcup_{R \in \mathcal{R}_{k_2}} R \bigg| \right).
\]

Remark 6.8. The arbitrariness of the collection of rectangles in Proposition 6.6 provides the compatibility of different stopping-time decompositions. In the current setting, the notation $\mathcal{R}_0$ in Proposition 6.6 is chosen to be $\mathcal{R}_{k_1, k_2}$. The sparsity condition allows one to combine the tensor-type stopping-time decomposition I and general two-dimensional level sets stopping-time decomposition and to obtain information from both stopping-time decompositions.

Remark 6.9. The readers who are familiar with the proof of single-parameter paraproducts [22] or bi-parameter paraproducts [20], [22] might recall that (6.10) employs a different argument from the previous ones [20], [22]. In particular, by previous reasonings, one would fix $n_2, m_2 \in \mathbb{Z}$ and obtain
\[
\sum_{n_2 \in \mathbb{Z}} \bigg| \bigcup_{R \in \mathcal{R}_{k_1, k_2}} R \bigg| \lesssim \min\left( \bigg| \bigcup_{R \in \mathcal{R}_{k_1}} R \bigg|, \bigg| \bigcup_{R \in \mathcal{R}_{k_2}} R \bigg| \right).
\]

However, the expression on the right hand side of (6.11) is independent of $n_2$ or $m_2$, which gives a divergent series when the sum is taken over all $n_2, m_2 \in \mathbb{Z}$. This explains the novelty and necessity of the sparsity condition (Proposition 6.6) for our argument.

To estimate the right hand side of (6.10), one recalls from the general two-dimensional level sets stopping-time decomposition that $R \in \mathcal{R}_{k_1}$ implies
\[
|R \cap \Omega_{k_1 - 1}^2| > \frac{1}{100} |R|,
\]
or equivalently
\[ \bigcup_{R \in \mathcal{R}_k} R \subseteq \{ M(\chi \Omega_{k-1}^2) > \frac{1}{10} \}. \]
As a result,
\[ (6.12) \quad \bigcup_{R \in \mathcal{R}_k} R \leq \| M(\chi \Omega_{k-1}^2) > \frac{1}{10} \| \leq |\Omega_{k-1}^2| = \| SS h > C_3 2^{k_1} \|_{L^1} \| h \|_{L^1} \leq C_3^{-s} 2^{-k_1 s}, \]
where the last inequality follows from the boundedness of the double square function described in Proposition 3.7. By a similar reasoning and the fact that \( |E'| \sim 1 \),
\[ (6.13) \quad \bigcup_{R \in \mathcal{R}_k} R \leq |\{ M(\chi \Omega_{k-2}^2) > \frac{1}{10} \} \| \leq |\Omega_{k-2}^2| = \| SS(\chi E') > C_3 2^{k_2} \| \leq C_3^{-\gamma} 2^{-k_2 \gamma}, \]
for any \( \gamma > 1 \). Interpolation between (6.12) and (6.13) yields
\[ (6.14) \quad \bigcup_{R \in \mathcal{R}_k} R \leq 2^{-\frac{4k_1}{3}} 2^{-\frac{k_2 \gamma}{2}}, \]
and by plugging (6.14) into (6.10), one has
\[ (6.15) \quad \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \bigcup_{R \in \mathcal{R}_n, k_2} R \leq 2^{-\frac{4k_1}{3}} 2^{-\frac{k_2 \gamma}{2}}, \]
for any \( \gamma > 1 \). One combines the estimates (6.15) and (6.9) to obtain
\[ |A_{\text{flag}^1 \otimes \text{flag}^2}| \lesssim C_1^2 C_2^2 C_3^2 \sum_{n \geq 0, m \geq 0, k_1 \leq 0, k_2 \leq K} 2^{-n} 2^{-m} 2^{-\frac{4k_1}{3}} 2^{-\frac{k_2 \gamma}{2}} |F_1|^{\frac{1}{3}} |F_2|^{\frac{1}{3}} |G_1|^{\frac{1}{3}} |G_2|^{\frac{1}{3}} 2^{k_1(1-\frac{4}{3})} \|h\|_{L^1} 2^{k_2(1-\frac{\gamma}{2})}. \]

The geometric series \( \sum_{k_1 < 0} 2^{-k_1(1-\frac{4}{3})} \) is convergent given that \( s < 2 \). For \( \sum_{k_2 \leq K} 2^{k_2(1-\frac{\gamma}{2})} \), one can choose \( \gamma > 1 \) to be sufficiently large for the range \( 0 \leq k_2 \leq K \) and \( \gamma > 1 \) and close to 1 for \( k_2 < 0 \). One thus concludes that
\[ |A_{\text{flag}^1 \otimes \text{flag}^2}| \lesssim C_1^2 C_2^2 C_3^2 |F_1|^{\frac{1}{3}} |F_2|^{\frac{1}{3}} |G_1|^{\frac{1}{3}} |G_2|^{\frac{1}{3}} \|h\|_{L^1}. \]

Remark 6.10. One important observation is that thanks to Lemma 5.8, the sizes
\[ \sup_{I \in \mathcal{I}_{n_1, m_1}} \frac{|\langle B_{j_1}^1, H \rangle_{f_1, f_2}^{1, H} |_{I}^{\frac{1}{3}}}{|I|^{\frac{1}{2}}} \]
and
\[ \sup_{J \in \mathcal{J}_{n_2, m_2}} \frac{|\langle B_{j_2}^2, H \rangle_{g_1, g_2}^{1, H} |_{J}^{\frac{1}{3}}}{|J|^{\frac{1}{2}}} \]
can be estimated in the exactly same way as
\[ \text{size}_{\mathcal{I}_{n_1}} ((f_1, \phi_I) t) \cdot \text{size}_{\mathcal{J}_{n_1}} ((f_2, \phi_I) t) \]
and
\[ \text{size}_{\mathcal{J}_{n_2}} ((g_1, \phi_J) t) \cdot \text{size}_{\mathcal{J}_{n_2}} ((g_2, \phi_J) t) \]
respectively. Based on this observation, it is not difficult to verify that the discrete model \( \Pi_{\text{flag}^1 \otimes \text{paraproduct}} \cdot \Pi_{\text{paraproduct} \otimes \text{paraproduct}} \) can be estimated by a essentially same argument as \( \Pi_{\text{flag}^1 \otimes \text{flag}^2} \). In addition, \( \Pi_{\text{flag}^0 \otimes \text{flag}^2} \) can be studied similarly as \( \Pi_{\text{flag}^0 \otimes \text{paraproduct}} \).
7. Proof of Theorem 4.2 for $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ - Haar Model

The argument in Chapter 6 is not sufficient for $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ because the localized size

$$\sup_{I \cap S \neq \emptyset} \frac{|[B_I^H, \varphi_I^{1,H}]|}{|I|^{\frac{3}{2}}}.$$

$$\sup_{J \cap S' \neq \emptyset} \frac{|[\tilde{B}_J^H, \varphi_J^{1,H}]|}{|J|^{\frac{3}{2}}}$$

cannot be controlled without information about corresponding level sets. In particular, one needs to impose the additional assumption that

$$I \cap \{ MB^H \leq C_1 2^{|I|} \| B^H \|_1 \} \neq \emptyset,$$

$$J \cap \{ M \tilde{B}^H \leq C_2 2^{l_2} \| \tilde{B}^H \|_1 \} \neq \emptyset,$$

where

$$B^H(f_1, f_2)(x) := \sum_{K \in \mathcal{K}} \frac{1}{|K|^\frac{3}{2}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \phi_K^3 H(x),$$

$$\tilde{B}^H(g_1, g_2)(y) := \sum_{L \in \mathcal{L}} \frac{1}{|L|^\frac{3}{2}} \langle g_1, \phi_L^1 \rangle \langle g_2, \phi_L^2 \rangle \phi_L^3 H(y).$$

However, while the sizes of $B^H$ and $\tilde{B}^H$ can be controlled in this way, they lose the information from the localization (e.g. $K \cap \{ M f_1 \leq C_1 2^{m_1} |F_1| \} \neq \emptyset$ for some $n_1 \in \mathbb{Z}$) and are thus far away from satisfaction. It is indeed the energies which capture such local information and compensate for the loss from size estimates in this scenario.

7.1. Localization. As one would expect from the definition of the exceptional set, the tensor-type stopping-time decompositions and the general two-dimensional level sets stopping-time decomposition are involved in the argument. We define the set

$$\Omega := \Omega^1 \cup \Omega^2,$$

where

$$\Omega^1 := \bigcup_{n_1 \in \mathbb{Z}} \{ M f_1 > C_1 2^{m_1} |F_1| \} \times \{ M g_1 > C_2 2^{-n_1} |G_1| \} \cup$$

$$\bigcup_{m_1 \in \mathbb{Z}} \{ M f_2 > C_1 2^{m_1} |F_2| \} \times \{ M g_2 > C_2 2^{-m_1} |G_2| \} \cup$$

$$\bigcup_{l_1 \in \mathbb{Z}} \{ M B^H > C_1 2^{|I|} \| B^H \|_1 \} \times \{ M \tilde{B}^H > C_2 2^{-l_1} \| \tilde{B}^H \|_1 \},$$

$$\Omega^2 := \{ SS \sigma > C_3 \| \sigma \|_{L^1} \},$$

and

$$\hat{\Omega} := \{ M \chi_\Omega > \frac{1}{100} \}.$$

Let

$$E' := E \setminus \hat{\Omega}.$$

Then similar argument in Remark 6.1 yields that $|E'| \sim |E|$ where $|E|$ can be assumed to be 1 by scaling invariance. We aim to prove that the multilinear form

$$\Lambda_{\text{flag}^0 \otimes \text{flag}^0}(f_1^x, f_2^x, g_1^y, g_2^y, h^x, h^y, \chi_{E'}) := \langle \Pi_{\text{flag}^0 \otimes \text{flag}^0}(f_1^x, f_2^x, g_1^y, g_2^y, h^x, h^y, \chi_{E'}) \rangle$$

satisfies the following restricted weak-type estimate

$$\Lambda_{\text{flag}^0 \otimes \text{flag}^0} \lesssim |F_1|^{\frac{2}{2}} |G_1|^{\frac{2}{2}} |F_2|^{\frac{2}{2}} |G_2|^{\frac{2}{2}} \| h \|_{L^1(\mathbb{R}^2)}.$$

7.2. Tensor-type stopping-time decomposition II - maximal intervals.
7.2.1. One-dimensional stopping-time decomposition - maximal intervals. One applies the stopping-time decomposition described in Section 5.5.1 to the sequences

\[
\left(\frac{|\langle B^H_i(f_1,f_2), \psi^R_j \rangle|}{|I|^\frac{1}{2}}\right)_{I \in \mathcal{I}}
\]

and

\[
\left(\frac{|\langle \hat{B}^H_i(g_1,g_2), \psi^R_j \rangle|}{|J|^\frac{1}{2}}\right)_{J \in \mathcal{J}}
\]

We will briefly recall the algorithm and introduce some necessary notations for the sake of clarity. Since \( \mathcal{I} \) is finite, there exists some \( L_1 \in \mathbb{Z} \) such that

\[
\frac{|\langle B^H_{\text{max}}(f_1,f_2), \psi^R_{\text{max}} \rangle|}{|I_{\text{max}}|^\frac{1}{2}} \geq C_1 2^{L_1} \cdot \|B^H\|_1.
\]

Then we define a tree

\[
T := \{I \in \mathcal{I} : I \subseteq I_{\text{max}}\}
\]

and the corresponding tree-top

\[
I_T := I_{\text{max}}.
\]

Now we repeat the above step on \( \mathcal{I} \setminus T \) to choose maximal intervals and collect their subintervals in their corresponding sets, which will end thanks to the finiteness of \( \mathcal{I} \). Then collect all \( T \)'s in a set \( T_{L_1-1} \) and repeat the above algorithm to \( \mathcal{I} \setminus \bigcup\limits_{T \in T_{L_1-1}} T \). Eventually the algorithm generates a decomposition \( \mathcal{I} = \bigcup\limits_{l_1} \bigcup\limits_{l_2} T \).

One simple observation is that the above procedure can be applied to general sequences indexed by dyadic intervals. One can thus apply the same algorithm to \( \mathcal{J} := \{J : I \times J \subseteq \mathcal{R}\} \). We denote the decomposition as \( \mathcal{J} = \bigcup\limits_{l_2} \bigcup\limits_{S \in S_{l_2}} T \) with respect to the sequence \( \left(\frac{|\langle B^H_{l_1}(g_1,g_2), \psi^R_{l_2} \rangle|}{|J|^\frac{1}{2}}\right)_{J \in \mathcal{J}} \), where \( S \) is a collection of dyadic intervals analogous to \( T \) and is denoted by tree. And \( J_S \) represents the corresponding tree-top analogous to \( I_T \).

7.2.2. Tensor product of two one-dimensional stopping-time decompositions - maximal intervals.

**Observation 2.** If \( I \times J \cap \hat{\Omega}^c \neq \emptyset \) and \( I \times J \subseteq T \times S \) with \( T \in \mathcal{T}_{l_1} \) and \( S \subseteq S_{l_2} \), then \( l_1, l_2 \in \mathbb{Z} \) satisfies \( l_1 + l_2 < 0 \). Equivalently, \( I \times J \subseteq T \times S \) with \( T \in \mathcal{T}_{-l_1-l_2} \) and \( S \subseteq S_{l_2} \) for some \( l_2 \in \mathbb{Z}, l > 0 \).

**Proof.** \( I \in T \) with \( T \in \mathcal{T}_{l_1} \) means that \( I \subseteq I_T \) where

\[
\frac{|\langle B^H_{l_1}(f_1,f_2), \psi^R_{l_2} \rangle|}{|I_T|^\frac{1}{2}} > C_1 2^{l_1} \cdot \|B^H\|_1.
\]

By the best trick,

\[
\frac{|\langle B^H_{l_1}(f_1,f_2), \psi^R_{l_2} \rangle|}{|I_T|^\frac{1}{2}} \leq \frac{\|B^H(x)\|}{|I_T|^\frac{1}{2}} \leq MB^H(x) \text{ for any } x \in I_T. \quad \text{Thus } I_T \subseteq \{MB^H > C_1 2^{l_1} \cdot \|B^H\|_1\}. \]

By a similar reasoning, \( J \in S \) with \( S \subseteq S_{l_2} \) implies that \( J \subseteq J_S \subseteq \{MB^H > C_2 2^{l_2} \cdot \|B^H\|_1\} \). If \( l_1 + l_2 \geq 0 \), then \( \{MB^H > C_1 2^{l_1} \cdot \|B^H\|_1\} \times \{MB^H > C_2 2^{l_2} \cdot \|B^H\|_1\} \subseteq \Omega^1 \subseteq \Omega \). As a consequence, \( I \times J \subseteq \Omega \subseteq \hat{\Omega}^c \), which is a contradiction. \( \square \)

7.3. Summary of stopping-time decompositions. The notions of tensor-type stopping-time decomposition I and general two-dimensional level sets stopping-time decomposition introduced in Chapter 6 will be applied without further specifications.

I. Tensor-type stopping-time decomposition I on \( I \times J \) \( \rightarrow I \times J \in \mathcal{I}_{-n_2-m_2} \times \mathcal{J}_{n_2,m_2} \) \( (n_2,m_2 \in \mathbb{Z}, n > 0) \)

II. Tensor-type stopping-time decomposition II on \( I \times J \) \( \rightarrow I \times J \in T \times S, \text{with } T \in \mathcal{T}_{-l_1-l_2}, S \subseteq S_{l_2} \) \( (l_2 \in \mathbb{Z}, l > 0) \)

III. General two-dimensional level sets stopping-time decomposition on \( I \times J \) \( \rightarrow I \times J \in \mathcal{R}_{k_1,k_2} \) \( (k_1 < 0, k_2 \leq K) \)
7.4. Application of stopping-time decompositions. One first rewrites the multilinear form with the partition of the dyadic rectangles specified in the stopping-time algorithm:

\[
|A_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \frac{1}{|I|^\frac{1}{2} |J|^\frac{1}{2}} |\langle B^H_j(f_1, f_2), \varphi^1_{j}\rangle| |\langle \tilde{B}^H_j(g_1, g_2), \varphi^1_{j}\rangle| \cdot |\langle h, \psi^2_{j} \otimes \psi^2_{j}\rangle| |\langle \chi_E, \psi^3_{j} \otimes \psi^3_{j}\rangle|.
\]

One can now apply the exactly same argument in Section 6.6.1 to estimate the multilinear form by

\[
|A_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \sup_{l \in T} \frac{|\langle B^H_j(f_1, f_2), \varphi^1_{j}\rangle|}{|I|^\frac{1}{2}} \cdot \sup_{J \in S} \frac{|\langle \tilde{B}^H_j(g_1, g_2), \varphi^1_{j}\rangle|}{|J|^\frac{1}{2}} \cdot 2^{k_1} \cdot \|h\|_{L^2} \cdot 2^{k_2} \cdot \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \sup_{l \in T} \frac{|\langle B^H_j(f_1, f_2), \varphi^1_{j}\rangle|}{|I|^\frac{1}{2}} \cdot \sup_{J \in S} \frac{|\langle \tilde{B}^H_j(g_1, g_2), \varphi^1_{j}\rangle|}{|J|^\frac{1}{2}} \cdot 2^{k_1} \cdot \|h\|_{L^2} \cdot \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \left( 1 \times J \right).
\]

Fix \(-l - l_2\) and \(T \in \mathbb{T}_{-l - l_2}\), one recalls the tensor-type stopping-time decomposition II to conclude that

\[
\sup_{l \in T} \frac{|\langle B^H_j(f_1, f_2), \varphi^1_{j}\rangle|}{|I|^\frac{1}{2}} \lesssim C_1 2^{-l - l_2} \|B^H\|_1.
\]

By the similar reasoning,

\[
\sup_{J \in S} \frac{|\langle \tilde{B}^H_j(g_1, g_2), \varphi^1_{j}\rangle|}{|J|^\frac{1}{2}} \lesssim C_2 2^{l_2} \|\tilde{B}^H\|_1.
\]

By applying the estimates (7.4) and (7.5) to (7.3),

\[
|A_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim C_1 C_2 C_3^2 \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} 2^{-l} \|B^H\|_1 \cdot \|\tilde{B}^H\|_1 \cdot 2^{k_1} \cdot \|h\|_{L^2} \cdot 2^{k_2} \cdot \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \sup_{l \in T} \frac{|\langle B^H_j(f_1, f_2), \varphi^1_{j}\rangle|}{|I|^\frac{1}{2}} \cdot \sup_{J \in S} \frac{|\langle \tilde{B}^H_j(g_1, g_2), \varphi^1_{j}\rangle|}{|J|^\frac{1}{2}} \cdot 2^{k_1} \cdot \|h\|_{L^2} \cdot \sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \left( 1 \times J \right).
\]

7.5. Estimate for nested sum of dyadic rectangles. One can estimate the nested sum (7.7) in two approaches - one with the application of the sparsity condition and the other with a Fubini-type argument which will be introduced in Section 7.5.2.

\[
\sum_{n > 0} \sum_{m > 0} \sum_{l > 0} \sum_{k_1 < 0} \sum_{k_2 \leq K} \left( 1 \times J \right).
\]

Both arguments aim to combine different stopping-time decompositions and to extract useful information from them. Generically, the sparsity condition argument employs the geometric property, namely Proposition 6.6, of the tensor-type stopping-time decomposition I and applies the analytical implication from the general two-dimensional level sets stopping-time decomposition. Meanwhile, the Fubini-type argument focuses on the hybrid of the tensor-type stopping time decomposition I - level sets and the tensor-type stopping-time decomposition II - maximal intervals. As implied by the name, the Fubini-type argument attempts to estimate measures of two dimensional sets by the measures of its projected one-dimensional sets. The approaches to estimate projected one-dimensional sets are different depending on which tensor-type stopping decomposition is in consideration.
7.5.1. **Sparsity condition.** The first approach relies on the sparsity condition which mimics the argument in the Chapter 6. In particular, fix $n, m, l, k_1$ and $k_2$, one estimates (7.7) as follows.

\[
\sum_{l_2} \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| \leq \sup_{l_2} \left( \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| \right) \dagger
\]

\[
\cdot \sum_{l_2} \left( \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| \right)^\frac{1}{2}.
\]

**Estimate of SC - I.** Fix $l, n, m, k_1, k_2$ and $l_2$. Then by the one-dimensional stopping-time decomposition - maximal intervals, for any $I \in \mathbb{T}$ and $I' \in \mathbb{T}'$ such that $T, T' \in \mathbb{T}^{l_2 - l_2}$ and $T \neq T'$, one has $I \cap I' = \emptyset$. Hence for any fixed $n_2$ and $m_2$, one can rewrite

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| = \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right|,
\]

where the right hand side of (7.8) can be trivially bounded by

\[
\left| \sum_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right|.
\]

One can then recall the sparsity condition highlighted as Proposition 6.6 and reduce the nested sum of measures of unions of rectangles to the measure of the corresponding union of rectangles. More precisely,

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| \sim \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right),
\]

where the right hand side can be estimated by

\[
\left| \bigcup_{n_2 \in \mathbb{Z}} \left( \bigcup_{m_2 \in \mathbb{Z}} I \times J \right) \right| \leq \left| \bigcup_{I \times J \in \mathbb{R}_{k_1, k_2}} I \times J \right| \leq \min(2^{-k_1}, 2^{-k_2}).
\]

for any $\gamma > 1$. The last inequality follows directly from (6.15). Since the above estimates hold for any $l_2 \in \mathbb{Z}$, one can conclude that

\[
SC - I \lesssim \min(2^{-k_1}, 2^{-k_2}).
\]

**Estimate of SC - II.** One invokes (7.8) and Proposition 6.6 to obtain

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left| \sum_{S \in S_{l_2}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right) \right| \sim \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left( \bigcup_{I \times J \in \mathbb{T}_{l_2} \cap S} I \times J \right).
\]
One enlarges the collection of the rectangles by forgetting about the restriction that the rectangles lie in $\mathcal{R}_{k_1,k_2}$ and estimate the right hand side of (7.10) by

\[(7.11) \quad \left| \bigcup_{m_2 \in \mathbb{Z}} \bigcup_{T \in T_{-l-2}} \bigcup_{S \in S_{l_2}} I \times J \times S \right|,
\]

which is indeed the measure of the union of the rectangles collected in the tensor-type stopping-time decomposition II - maximal intervals at a certain level. In other words,

\[
\left| \bigcup_{m_2 \in \mathbb{Z}} \bigcup_{T \in T_{-l-2}} \bigcup_{S \in S_{l_2}} I \times J \times S \right| = \left| \bigcup_{T \times S \in T_{-l-2} \times S_{l_2}} I_T \times J_S \right|.
\]

Then

\[(7.12) \quad SC - II \leq \sum_{l_2 \in \mathbb{Z}} \left| \bigcup_{T \times S \in T_{-l-2} \times S_{l_2}} I_T \times J_S \right|^\frac{1}{2},
\]

whose estimate follows a Fubini-type argument that plays an important role in the proof. We will focus on the development of this Fubini-type argument in a separate section and discuss its applications in other useful estimates for the proof.

7.5.2. **Fubini argument.** Alternatively, one can apply a Fubini-type argument to estimate (7.7) in the sense that the measure of some two-dimensional set is estimated by the product of the measures of its projected one-dimensional sets. To introduce this argument, we will first look into (7.12) which requires a simpler version of the argument.

**Estimate of (7.12) - Introduction of Fubini argument.** As illustrated before, one first rewrites the measure of two dimensional-sets in terms of the measures of two one-dimensional sets as follows.

\[
\sum_{l_2 \in \mathbb{Z}} \left| \bigcup_{T \times S \in T_{-l-2} \times S_{l_2}} I_T \times J_S \right|^\frac{1}{2} \leq \left( \sum_{l_2 \in \mathbb{Z}} \left| \bigcup_{T \in T_{-l-2}} I_T \right| \right)^\frac{1}{2} \left( \sum_{l_2 \in \mathbb{Z}} \left| \bigcup_{S \in S_{l_2}} J_S \right| \right)^\frac{1}{2},
\]

(7.13)

where the last step follows from the Cauchy-Schwartz inequality. To estimate the measures of the one-dimensional sets appearing above, one can convert them to the form of “global” energies and apply the energy estimates specified in Proposition 5.5. In particular, (7.13) can be rewritten as

\[
\left( \sum_{l_2 \in \mathbb{Z}} \left( C_1 2^{-l-2l_2} \| B^H \|_1 \right)^{1+\delta} \bigcup_{T \in T_{-l-2}} I_T \right)^\frac{1}{2} \left( \sum_{l_2 \in \mathbb{Z}} \left( C_2 2^{l_2} \| \hat{B}^H \|_1 \right)^{1+\delta} \bigcup_{S \in S_{l_2}} J_S \right)^\frac{1}{2} 
\]

\[
\cdot 2^\left(1+\delta\right) \| B^H \|_1 \| \hat{B}^H \|_1, \quad l_2 \in \mathbb{Z}
\]

(7.14)

for any $\delta > 0$. One notices that for fixed $l$ and $l_2$, 

\[
\{I_T : T \in T_{-l-2}\}
\]

is a disjoint collection of dyadic intervals according to the one-dimensional stopping-time decomposition - maximal interval. Thus

\[
(7.15) \quad \sum_{l_2 \in \mathbb{Z}} \left( C_1 2^{-l-2l_2} \| B^H \|_1 \right)^{1+\delta} \bigcup_{T \in T_{-l-2}} I_T = \sum_{l_2 \in \mathbb{Z}} \left( C_1 2^{-l-2l_2} \| B^H \|_1 \right)^{1+\delta} \sum_{T \in T_{-l-2}} |I_T|
\]

is indeed a “global” $L^{1+\delta}$-energy for which one can apply the energy estimates to obtain the bound

\[
|F_1|^{\mu_1(1+\delta)} |F_2|^{\mu_2(1+\delta)},
\]
where \( \delta, \mu_1, \mu_2 > 0 \) with \( \mu_1 + \mu_2 = \frac{1}{1+\delta} \). Similarly, one can apply the same reasoning to the measure of the set in the \( y \)-direction to derive

\[
\sum_{l_2 \in \mathbb{Z}} \left( C_2 \| \hat{B}^H \|_1 \right)^{1+\delta} \bigg| \bigcup_{S \in \mathcal{S}_{l_2}} J_S \bigg| \lesssim |G_1|^{\nu_1(1+\delta)} |G_2|^{\nu_2(1+\delta)},
\]

for any \( \nu_1, \nu_2 > 0 \) with \( \nu_1 + \nu_2 = \frac{1}{1+\delta} \). By applying (7.15) and (7.16) into (7.14), one derives that

\[
\sum_{l_2 \in \mathbb{Z}} \bigg| \bigcup_{T \in \mathcal{T}_{-l-1}, S \in \mathcal{S}_{l_2}} I_T \times J_S \bigg|^\frac{1}{2} \lesssim 2 \left( \sum_{l_2 \in \mathbb{Z}} \| \hat{B}^H \|_1 \right)^{\frac{1+\delta}{2}} \left| G_1 \right|^{\nu_1(1+\delta)} \left| G_2 \right|^{\nu_2(1+\delta)} \| \hat{B}^H \|_1^{\frac{1+\delta}{2}} \| \hat{B}^H \|_1^{-\frac{1+\delta}{2}},
\]

for any \( \delta, \mu_1, \mu_2, \nu_1, \nu_2 > 0 \) with \( \mu_1 + \mu_2 = \nu_1 + \nu_2 = \frac{1}{1+\delta} \).

**Remark 7.1.** The reason for leaving the expressions \( \| B^H \|_1^{1+\delta} \) or \( \| \hat{B}^H \|_1^{1+\delta} \) will become clear later. In short, \( \| B^H \|_1 \) and \( \| \hat{B}^H \|_1 \) will appear in estimates for other parts. We will keep them as they are for the exponent-counting and then use the estimates for \( \| B^H \|_1 \) and \( \| \hat{B}^H \|_1 \) at last.

By combining the estimates \( SC - I \) and \( SC - II \), one can conclude that (7.7) is majorized by

\[
2^{-\frac{1+\delta}{2}} 2^{\nu_1(1+\delta)} \left| F_1 \right|^{\nu_1(1+\delta)} \left| F_2 \right|^{\nu_2(1+\delta)} \left| G_1 \right|^{\nu_1(1+\delta)} \left| G_2 \right|^{\nu_2(1+\delta)} \| B^H \|_1^{\frac{1+\delta}{2}} \| \hat{B}^H \|_1^{-\frac{1+\delta}{2}},
\]

for \( \gamma > 1, \delta, \mu_1, \mu_2, \nu_1, \nu_2 > 0 \) with \( \mu_1 + \mu_2 = \nu_1 + \nu_2 = \frac{1}{1+\delta} \).

**Remark 7.2.** The framework for estimating the measure of two-dimensional sets by its corresponding one-dimensional sets, as illustrated by (7.13), is the so-called “Fubini-type” argument which we will heavily employ from now on.

**Estimate of (7.7) - Application of Fubini argument.** It is not difficult to observe that (7.7) can also be estimated by

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{n \in \mathcal{I}_{-l-1}} \bigg| \bigcup_{S \in \mathcal{S}_{l_2}} I \times J \bigg| = \sum_{n_2 \in \mathbb{Z}} \sum_{n \in \mathcal{I}_{-l-1}} \bigg| \bigcup_{S \in \mathcal{S}_{l_2}} I \bigg| \left| \bigcup_{J \in \mathcal{J}_{n_2, m_2}} J \bigg|. \tag{7.18}
\]

One now rewrites the above expression and separates it into two parts. Both parts can be estimated by the Fubini-type argument whereas the methodologies to estimate projected one-dimensional sets are different. More precisely, (7.18) can be separated as

\[
\sup_{n_2 \in \mathbb{Z}} \sum_{n \in \mathcal{I}_{-l-1}} \left( \sum_{S \in \mathcal{S}_{l_2}} \bigg| \bigcup_{I \in \mathcal{I}_{-n_2, m_2}} I \bigg| \right)^\frac{1}{2} \left( \sum_{J \in \mathcal{J}_{n_2, m_2}} \bigg| \bigcup_{J \in \mathcal{J}_{n_2, m_2}} J \bigg| \right)^\frac{1}{2} \times \tag{A}
\]

\[
\sup_{n_2 \in \mathbb{Z}} \sum_{n \in \mathcal{I}_{-l-1}} \left( \sum_{S \in \mathcal{S}_{l_2}} \bigg| \bigcup_{I \in \mathcal{I}_{-n_2, m_2}} I \bigg| \right)^\frac{1}{2} \left( \sum_{J \in \mathcal{J}_{n_2, m_2}} \bigg| \bigcup_{J \in \mathcal{J}_{n_2, m_2}} J \bigg| \right)^\frac{1}{2} \tag{B}.
\]

To estimate (A), one first notices that for for any fixed \( n, m, n_2, m_2, l, l_2 \) and a fixed tree \( T \in T^{-l-l_2} \), a dyadic interval \( I \in \mathcal{T}_{-n_2, m_2} \) means that

(i) \( I \subseteq I_T \) where \( I_T \) is the tree-top interval as implied by the one-dimensional stopping-time decomposition - maximal interval;

(ii) \( I \cap \{ M f_1 \leq C_1 2^{-n-n_2+1} |F_1| \} \cap \{ M f_2 \leq C_1 2^{-m-m_2+1} |F_2| \} \neq \emptyset \).

By (i) and (ii), one can deduce that

\[
I_T \cap \{ M f_1 \leq C_1 2^{-n-n_2+1} |F_1| \} \cap \{ M f_2 \leq C_1 2^{-m-m_2+1} |F_2| \} \neq \emptyset.
\]
As a consequence,

\[(7.19) \quad \sum_{T \in \mathcal{T}_{n-1,2}} \left| \bigcup_{I \in \mathcal{I}} I \right| \leq \sum_{T \in \mathcal{T}_{n-1,2}} |I_T|.
\]

A similar reasoning applies to the term involving intervals in the $y$-direction and generates

\[(7.20) \quad \sum_{S \in \mathcal{S}_{l_2}} \left| \bigcup_{J \in \mathcal{J}} J \right| \leq \sum_{S \in \mathcal{S}_{l_2}} |J_S|.
\]

By applying the Cauchy-Schwartz inequality together with (7.19) and (7.20), one obtains

\[(7.21) \quad \mathcal{A} \leq \sup_{n_2 \in \mathbb{Z}} \left( \sum_{l_2 \in \mathbb{Z}} \sum_{T \in \mathcal{T}_{n,2}} |I_T| \right)^{\frac{1}{2}} \cdot \left( \sum_{l_2 \in \mathbb{Z}} \sum_{S \in \mathcal{S}_{l_2}} |J_S| \right)^{\frac{1}{2}}.
\]

One then “completes” the expression (7.21) to produce localized energy-like terms as follows.

\[
\mathcal{A}^1 \leq \sup_{n_2 \in \mathbb{Z}} \left( \sum_{l_2 \in \mathbb{Z}} \sum_{T \in \mathcal{T}_{n,2}} |I_T| \right)^{\frac{1}{2}} \cdot \left( \sum_{l_2 \in \mathbb{Z}} \sum_{S \in \mathcal{S}_{l_2}} |J_S| \right)^{\frac{1}{2}} \cdot 2^\frac{1}{2} ||B^H||_1^{-1} \cdot \tilde{B}^H_1^{-1}.
\]

It is not difficult to recognize that $\mathcal{A}^1$ and $\mathcal{A}^2$ are $L^2$-energies. Moreover, they follow stronger local energy estimates described in Proposition 5.14. $\mathcal{A}^1$ is indeed an $L^2$ energy localized to $\{Mf_1 \leq C_12^{-n_2}|F_1|\} \cap \{Mf_2 \leq C_12^{-m_2}|F_2|\}$. Then Proposition 5.14 gives the estimate

\[(7.22) \quad \mathcal{A}^1 \lesssim (C_12^{-n_2})^{\frac{1}{2p_1}} (C_12^{-m_2})^{\frac{1}{2p_2}} |F_1|^\frac{1}{p_1} |F_2|^\frac{1}{p_2},
\]

for any $0 \leq \theta_1, \theta_2 < 1$ satisfying $\theta_1 + \theta_2 = \frac{1}{2}$. One applies the same reasoning to $\mathcal{A}^2$ to deduce that

\[(7.23) \quad \mathcal{A}^2 \lesssim C_2^2 2^{n_2(\frac{1}{p_2} - \varsigma_1)} \cdot 2^{m_2(\frac{1}{p_2} - \varsigma_2)} |G_1|^\frac{1}{p_2} |G_2|^\frac{1}{p_2},
\]

where $0 \leq \varsigma_1, \varsigma_2 < 1$ and $\varsigma_1 + \varsigma_2 = \frac{1}{2}$. One can now combine the estimates for $\mathcal{A}^1$ (7.22) and $\mathcal{A}^2$ (7.23) to derive

\[
\mathcal{A} \lesssim C_1^2 C_2^2 \sup_{n_2 \in \mathbb{Z}} 2^{n_2(\frac{1}{p_1} - \varsigma_1)} \cdot 2^{m_2(\frac{1}{p_2} - \varsigma_2)} |F_1|^\frac{1}{p_1} |F_2|^\frac{1}{p_2} |G_1|^\frac{1}{p_2} |G_2|^\frac{1}{p_2} \cdot 2^\frac{1}{2} ||B^H||_1^{-1} \cdot \tilde{B}^H_1^{-1}.
\]

One observes that the following two conditions are equivalent:

\[(7.24) \quad \frac{1}{p_1} - \theta_1 = \frac{1}{q_1} - \varsigma_1 \iff \frac{1}{q_1} - \theta_2 = \frac{1}{q_2} - \varsigma_2.
\]

The equivalence is imposed by the fact that

\[(7.25) \quad \frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{q_1} = \frac{1}{q_2} = \frac{1}{p_2} = \frac{1}{q_2},
\]

\[(7.26) \quad \mathcal{A} \lesssim C_1^2 C_2^2 2^{-n(\frac{1}{p_2} - \varsigma_1)} \cdot 2^{-m(\frac{1}{p_2} - \varsigma_2)} |F_1|^\frac{1}{p_2} |F_2|^\frac{1}{p_2} |G_1|^\frac{1}{p_2} |G_2|^\frac{1}{p_2} \cdot 2^\frac{1}{2} ||B^H||_1^{-1} \cdot \tilde{B}^H_1^{-1}.
\]
Remark 7.3. (7.24) and (7.25) together imposes a condition that

\[(7.27)\quad \left| \frac{1}{p_1} - \frac{1}{p_2} \right| = \left| \frac{1}{q_1} - \frac{1}{q_2} \right| < \frac{1}{2}.\]

Without loss of generality, one can assume that \(\frac{1}{p_1} \geq \frac{1}{p_2}\) and \(\frac{1}{q_1} \leq \frac{1}{q_2}\). Then either (7.27) holds or

\[\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_2} - \frac{1}{q_1} > \frac{1}{2},\]

which implies

\[\left| \frac{1}{p_1} - \frac{1}{q_2} \right| = \left| \frac{1}{p_2} - \frac{1}{q_1} \right| < \frac{1}{2}.\]

Then one can switch the role of \(g_1\) and \(g_2\) to “pair” the functions as \(f_1\) with \(g_2\) and \(f_2\) with \(g_1\). A parallel argument can be applied to obtain the desired estimates.

One can apply another Fubini-type argument to estimate \(B\) with \(l, n\) and \(m\) fixed. Such argument again relies heavily on the localization. First of all, for any fixed \(l_2 \in \mathbb{Z}\),

\[\{ I : I \in T, T \in T_{-l_2} \}\]

is a disjoint collection of dyadic intervals. Thus

\[\sum_{T \in T_{-l_2}} \left| \bigcup_{I \in T} I \right| \leq \left| \bigcup_{I \in I_{-n_2,-m_2}} I \right| .\]

One then recalls the point-wise estimate stated in Claim 6.5 to deduce

\[\bigcup_{I \in I_{-n_2,-m_2}} I \subseteq \{ M f_1 \geq C_1 2^{-n_2-10} |F_1| \} \cap \{ M f_2 \geq C_1 2^{-m_2-10} |F_2| \},\]

and for arbitrary but fixed \(l_2 \in \mathbb{Z}\),

\[(7.28)\quad \sum_{T \in T_{-l_2}} \left| \bigcup_{I \in T} I \right| \leq \left| \{ M f_1 > C_1 2^{-n_2-10} |F_1| \} \cap \{ M f_2 > C_1 2^{-m_2-10} |F_2| \} \right| .\]

A similar reasoning applies to the intervals in the \(y\)-direction and yields that for any fixed \(l_2 \in \mathbb{Z}\),

\[(7.29)\quad \sum_{S \in S_{l_2}} \left| \bigcup_{J \in J_{S_2}} J \right| \leq \left| \{ M g_1 > C_2 2^{n_2-10} |G_1| \} \cap \{ M g_2 > C_2 2^{m_2-10} |G_2| \} \right| .\]

To apply the above estimates, one notices that there exists some \(\tilde{l}_2 \in \mathbb{Z}\) possibly depending \(n, m, l, n_2, m_2\) such that

\[B = \sum_{n_2 \in \mathbb{Z}} \left( \sum_{m_2 \in \mathbb{Z}} \left( \sum_{T \in T_{-l_2}} \left| \bigcup_{I \in T} I \right| \right)^{\frac{1}{2}} \left( \sum_{S \in S_{l_2}} \left| \bigcup_{J \in J_{S_2}} J \right| \right)^{\frac{1}{2}} \right).\]

One can further “complete” \(B\) in the following manner for appropriate use of the Cauchy-Schwartz inequality.

\[B = \sum_{n_2 \in \mathbb{Z}} \left( (C_1 2^{-n_2-10} |F_1|)^{\mu(1+c)} (C_1 2^{-m_2-10} |F_2|)^{(1-\mu)(1+c)} \sum_{T \in T_{-l_2}} \left| \bigcup_{I \in T} I \right| \right)^{\frac{1}{2}} \times \left( (C_2 2^{n_2-10} |G_1|)^{\mu(1+c)} (C_2 2^{m_2-10} |G_2|)^{(1-\mu)(1+c)} \sum_{S \in S_{l_2}} \left| \bigcup_{J \in J_{S_2}} J \right| \right)^{\frac{1}{2}} \times 2^n \frac{1}{2}(1+c) 2^m \frac{1}{2}(1-\mu)(1+c) |F_1|^{-\frac{1}{2}(1+c)} |F_2|^{-\frac{1}{2}(1-\mu)(1+c)} |G_1|^{-\frac{1}{2}(1+c)} |G_2|^{-\frac{1}{2}(1-\mu)(1+c)} \]
for any $\epsilon > 0$, $0 < \mu < 1$, where the second inequality follows from the Cauchy-Schwartz inequality.

To estimate $B^1$, one recalls (7.28) - which holds for any fixed $l_2 \in \mathbb{Z}$ - to obtain

$$B^1 \lesssim \left[ \sum_{n_2 \in \mathbb{Z}} (C_1 2^{-n-n_2} |F_1|)^{1/1+\epsilon} (C_1 2^{-m-m_2} |F_2|)^{(1-\mu)(1+\epsilon)} \sum_{T \in T_{-1-l_2}} \bigg| \bigcup_{I \in \mathcal{I}} f_I \bigg| \right]^{1/2} \cdot 2^{n \frac{\mu}{1+\epsilon} m} 2^{n(1-\mu)(1+\epsilon)} |G_1|^{-\frac{\mu}{1+\epsilon}} |G_2|^{-\frac{\mu}{1+\epsilon}} (1+\epsilon),$$

where the last step follows from Hölder’s inequality. One can now use the mapping property for the Hardy-Littlewood maximal operator $M : L^p \to L^p$ for any $p > 1$ and deduces that

$$\left( \int (M f_1(x))^{1+\epsilon} dx \right)^\mu \lesssim \|f_1\|_{L^{1+\epsilon}}^{1+\epsilon} = |F_1|^\mu,$$

(7.31)

$$\left( \int (M f_2(x))^{1+\epsilon} dx \right)^{1-\mu} \lesssim \|f_2\|_{L^{1+\epsilon}}^{1+\epsilon} = |F_2|^{1-\mu}.$$ 

By plugging the estimate (7.31) into (7.30),

(7.32) 

$$B^1 \lesssim |F_1|^{\frac{\mu}{1+\epsilon}} |F_2|^{\frac{1-\mu}{1+\epsilon}}.$$ 

By the same argument with $-n - n_2$ and $-m - m_2$ replaced by $n_2$ and $m_2$ correspondingly, one obtains

(7.33) 

$$B^2 \lesssim |G_1|^{\frac{\mu}{1+\epsilon}} |G_2|^{\frac{1-\mu}{1+\epsilon}}.$$ 

Combination of the estimates for $B^1$ (7.32) and $B^2$ (7.33) yields

(7.34) 

$$B \lesssim |F_1|^{\frac{\mu}{1+\epsilon}} |F_2|^{\frac{1-\mu}{1+\epsilon}} |G_1|^{\frac{\mu}{1+\epsilon}} |G_2|^{\frac{1-\mu}{1+\epsilon}} 2^n 2^m 2^{\frac{\mu}{1+\epsilon}} (1+\epsilon).$$ 

By applying the results for both $A$ (7.26) and $B$ (7.34), one concludes with the following estimate for (7.7).

(7.35) 

$$\sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} \sum_{T \in T_{-1-l_2}} \bigg| \bigcup_{I \times J \in \mathcal{J}} I \times J \bigg| \lesssim C_1 C_2 2^{-n(\frac{\mu}{1+\epsilon} - \frac{1}{2})} 2^{m(\frac{1}{2} - \frac{1}{1+\epsilon} - \frac{\mu}{1+\epsilon})} 2^n 2^m 2^{\frac{\mu}{1+\epsilon}} 2 \|B^H\|_1^{-1} \|B^H\|_1^{-1},$$

for any $0 \leq \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{2}$, $0 < \mu < 1$ and $\epsilon > 0$.

One can now interpolate between the estimates obtained with two different approaches, namely (7.17) and (7.33), to derive the following bound for (7.7).

$$C_1 C_2^{\frac{\mu}{1+\epsilon}} 2^{-n(\frac{\mu}{1+\epsilon} - \frac{1}{2} - \frac{1}{1+\epsilon} - \frac{\mu}{1+\epsilon})} 2^m 2^{\frac{\mu}{1+\epsilon}} 2 \|B^H\|_1^{-1} \|B^H\|_1^{-1},$$

for any $0 \leq \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 = \frac{1}{2}$, $0 < \mu < 1$ and $\epsilon > 0$.
Using the fact that \(0 < \lambda \leq 1\) and select \(\gamma > 1\) in each case to make the series about \(2^{k_2}\) converge. Therefore, one can estimate the multilinear form by

\[
\Lambda_{\text{flag}^0 \otimes \text{flag}^0} \lesssim C_1^2 C_3^2 \|h\|_{L^\infty} \| B_1^H \|_1^{\Lambda(1- \frac{1}{2})} \| B_2^H \|_1^{\Lambda(1- \frac{1}{2})} \cdot |F_1|^{\Lambda(\frac{1}{p_1} + \frac{1}{q_1} \lambda(1- \frac{1}{2} - \rho)} + (1- \lambda)(\frac{p_1}{q_1} - \frac{\rho}{q_1} \epsilon)} |F_2|^{\Lambda(\frac{1}{p_2} + \frac{1}{q_2} \lambda(1- \frac{1}{2} - \rho)} + (1- \lambda)(\frac{p_2}{q_2} - \frac{\rho}{q_2} \epsilon)} |G_1|^{\Lambda(\frac{1}{p_1} + \frac{1}{q_1} \lambda(1- \frac{1}{2} - \rho)} + (1- \lambda)(\frac{p_1}{q_1} - \frac{\rho}{q_1} \epsilon)} |G_2|^{\Lambda(\frac{1}{p_2} + \frac{1}{q_2} \lambda(1- \frac{1}{2} - \rho)} + (1- \lambda)(\frac{p_2}{q_2} - \frac{\rho}{q_2} \epsilon)},
\]

where one can apply Proposition 5.6 to derive

\[
\| B_1^H \|_1 \lesssim |F_1|^{\rho} |F_2|^{1- \rho},
\]

\[
\| B_2^H \|_1 \lesssim |G_1|^{\rho'} |G_2|^{1- \rho'},
\]

with the corresponding exponent to be positive as guaranteed by the fact that \(0 < \lambda, \delta < 1\). One thus obtains

\[
| \Lambda_{\text{flag}^0 \otimes \text{flag}^0} | \lesssim C_1^2 C_3^2 \|h\|_{s} |F_1|^{\Lambda(\frac{1}{p_1} + \frac{1}{q_1} \lambda(1- \frac{1}{2} - \rho)) + \rho \lambda(1- \frac{1}{2} - \rho)} |F_2|^{\Lambda(\frac{1}{p_2} + \frac{1}{q_2} \lambda(1- \frac{1}{2} - \rho)) + (1- \rho) \lambda(1- \frac{1}{2} - \rho)},
\]
\[ (7.39) \quad |G_1|^{2\gamma(1+\delta)} |G_2|^{2\gamma(1+\delta)} \]

With a little abuse of notation, we use \( \tilde{p}_i \) and \( \tilde{q}_i \), \( i = 1, 2 \) to represent \( p_i \) and \( q_i \) in the above argument. And from now on, \( p_i \) and \( q_i \) stand for the boundedness exponents specified in the main theorem. One has the freedom to choose \( 1 < \tilde{p}_i, \tilde{q}_i < \infty, 0 < \mu, \lambda < 1 \) and \( \epsilon > 0 \) such that

\[
\begin{align*}
\lambda^{\frac{\mu_1(1+\delta)}{2}} + (1 - \lambda)(\frac{1}{\tilde{p}_1} - \frac{\mu}{2}) + \rho(1 - \frac{1+\delta}{2}) &= \frac{1}{\tilde{p}_1} \\
\lambda^{\frac{\mu_2(1+\delta)}{2}} + (1 - \lambda)(\frac{1}{\tilde{q}_1} - \frac{\mu}{2}) + (1 - \rho)\lambda(1 - \frac{1+\delta}{2}) &= \frac{1}{\tilde{q}_1} \\
\lambda^{\frac{\mu_1(1+\delta)}{2}} + (1 - \lambda)(\frac{1}{\tilde{p}_2} - \frac{\mu}{2}) + \rho\lambda(1 - \frac{1+\delta}{2}) &= \frac{1}{\tilde{p}_2} \\
\lambda^{\frac{\mu_2(1+\delta)}{2}} + (1 - \lambda)(\frac{1}{\tilde{q}_2} - \frac{1}{2}) + (1 - \rho')\lambda(1 - \frac{1+\delta}{2}) &= \frac{1}{\tilde{q}_2}.
\end{align*}
\]

(7.40)

**Remark 7.5.** To see that above equations can hold, one can view the parts without \( \tilde{p}_i \) and \( \tilde{q}_i \) as perturbations which can be controlled small. More precisely, when \( 0 < \delta < 1 \) is close to \( 0 \),

\[ \lambda(1 - \frac{1+\delta}{2}) \ll 1. \]

When \( 0 < \lambda < 1 \) is close to \( 0 \), one has

\[ \lambda^{\frac{\mu_1(1+\delta)}{2}}, \lambda^{\frac{\mu_2(1+\delta)}{2}}, \lambda^{\frac{\mu_1(1+\delta)}{2}}, \lambda^{\frac{\mu_2(1+\delta)}{2}} \ll 1 \]

and

\[ \frac{1}{\tilde{p}_i} - (1 - \lambda)(\frac{1}{\tilde{p}_i} - \frac{\mu}{2}) \ll 1, \]

\[ \frac{1}{\tilde{q}_i} - (1 - \lambda)(\frac{1}{\tilde{q}_i} - \frac{1}{2}) \ll 1, \]

for \( i = 1, 2 \).

It is also necessary to check is that \( \tilde{p}_i \) and \( \tilde{q}_i \) satisfy the conditions which have been used to obtain (7.39), namely

(7.41)

\[ \frac{1}{\tilde{p}_1} + \frac{1}{\tilde{q}_1} = \frac{1}{\tilde{p}_2} + \frac{1}{\tilde{q}_2} > 1. \]

One can easily verify the first equation and the second inequality by manipulating (7.40). As a result, we have derived that

\[ |\Lambda_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim C_1^2 C_2^2 C_3^3 |F_1|^{\frac{1}{\tilde{p}_1}} |F_2|^{\frac{1}{\tilde{p}_2}} |G_1|^{\frac{1}{\tilde{q}_1}} |G_2|^{\frac{1}{\tilde{q}_2}}. \]

8. PROOF OF THEOREM 4.3 FOR \( \Pi_{\text{flag}^1 \otimes \text{flag}^2} \) - HAAR MODEL

One can mimic the proof in Chapter 6 with a change of perspectives on size estimates. More precisely, one applies some trivial size estimates for functions \( f_2 \) and \( g_2 \) lying in \( L^\infty \) spaces while one needs to pay respect to the fact that \( f_1 \) and \( g_1 \) could lie in \( L^p \) space for any \( p > 1 \). Such perspective is demonstrated in the stopping-time decomposition and in the definition of the exceptional set.

8.1. LOCALIZATION. One defines

\[ \Omega := \Omega^1 \cup \Omega^2, \]

where

\[ \Omega^1 := \bigcup_{n_1 \in \mathbb{Z}} \{ Mf_1 > C_1 2^{n_1} \|f_1\|_p \} \times \{ Mg_1 > C_2 2^{-n_1} \|g_1\|_p \}, \]

\[ \Omega^2 := \{ SSH > C_3 \|h\|_{L^s} \}, \]
and
\[ \hat{\Omega} := \{ M \chi_{\Omega} > \frac{1}{100} \}. \]
Let
\[ E' := E \setminus \hat{\Omega}. \]
It is not difficult to check that given \( C_1, C_2 \) and \( C_3 \) are sufficiently large, \( |E'| \sim |E| \) where \( |E| \) can be assumed to be 1. It suffices to prove that the multilinear form defined as
\[ (8.1) \quad \Lambda_{\text{flag} \#1 \otimes \text{flag} \#2}^{\#1} = (\Pi_{\text{flag} \#1 \otimes \text{flag} \#2}^{\#1} (f_1^x, f_2^x, g_1^y, g_2^y, h^{x,y}, \chi_{E'})) \]
satisfies the restricted weak-type estimate
\[ (8.2) \quad |\Lambda_{\text{flag} \#1 \otimes \text{flag} \#2}^{\#1}| \lesssim |F_1|^{\frac{1}{m_1}} |G_1|^{\frac{1}{m_2}} |F_2|^{\frac{1}{m_3}} |G_2|^{\frac{1}{m_4}} \|h\|_{L^\infty(\mathbb{R}^2)}. \]

8.2. Summary of stopping-time decompositions.

I. Tensor-type stopping-time decomposition I on \( I \times J \rightarrow I \times J_{n_2} \times J_{n_2}' \)

II. General two-dimensional level sets stopping-time decomposition on \( I \times J \)

where
\[ I_{n_2} := \{ I \in I \setminus I_{n_2} + 1 : |I \cap \Omega_{n_2}^x| > \frac{1}{10} |I| \}, \]
\[ J_{n_2}' := \{ J \in J \setminus J_{n_2} + 1 : |I \cap \Omega_{n_2}^y| > \frac{1}{10} |J| \}, \]
with
\[ \Omega_{n_2}^x := \{ M f_1 > C_1 2^{-n_2} \| f_1 \|_p \}, \]
\[ \Omega_{n_2}^y := \{ M g_1 > C_2 2^{-n_2} \| g_1 \|_p \}. \]

8.3. Application of stopping-time decompositions. One can now apply the stopping-time decompositions and follow the same argument in Section 6 to deduce that
\[ \sum_{n_2 \in \mathbb{Z}} \sum_{k_1 < 0} \sum_{k_2 \leq K} \frac{1}{|I|^2 |J|^2} \left| \langle B_{\#1}^{1,1}(f_1, f_2, \varphi_1^{1,1}) \rangle \right| \sup_{J \in J_{n_2}'} \left| \langle B_{\#2}^{1,1}(g_1, g_2, \varphi_1^{1,1}) \rangle \right| C_3 2^{k_1} \|h\|_{L^2} 2^{k_2}. \]

(8.3)

To estimate \( \sup_{I \times J \in I_{n_2} \times J_{n_2}'} \left| \langle B_{\#1}^{1,1}(f_1, f_2, \varphi_1^{1,1}) \rangle \right| \frac{1}{|I|^2} \), one can now apply Lemma 5.8 with \( S := \{ M f_1 \leq C_1 2^{-n_2} \| f_1 \|_p \} \)
and obtain
\[ \sup_{I \times J \in I_{n_2} \times J_{n_2}'} \left| \langle B_{\#1}^{1,1}(f_1, f_2, \varphi_1^{1,1}) \rangle \right| \frac{1}{|I|^2} \lesssim \sup_{K \cap S \neq \emptyset} \left| \langle f_1, \varphi_1^{1,1} \rangle \right| \frac{1}{|K|^2} \sup_{K \cap S \neq \emptyset} \left| \langle f_2, \varphi_2^{1,1} \rangle \right| \frac{1}{|K|^2}, \]
where by the definition of \( S \),
\[ \sup_{K \cap S \neq \emptyset} \left| \langle f_1, \varphi_1^{1,1} \rangle \right| \frac{1}{|K|^2} \lesssim C_1 2^{-n_2} \| f_1 \|_p, \]
and by the fact that \( f_2 \in L^\infty \),

\[
\sup_{K \cap S \neq \emptyset} \frac{|\langle f_2, \phi_K \rangle|}{|K|^\frac{3}{2}} \lesssim \|f_2\|_{L^\infty}.
\]

As a result,

\[
(8.4) \quad \sup_{I \in I_{n-n_2}} \frac{|\langle B_1^H (f_1, f_2), \varphi_i^H \rangle|}{|I|^\frac{3}{2}} \lesssim C_1 2^{-n-n_2} \|f_1\|_p \|f_2\|_\infty.
\]

By a similar reasoning,

\[
(8.5) \quad \sup_{J \in J'_{n_2}} \frac{|\langle \tilde{B}_2^H (g_1, g_2), \varphi_j^H \rangle|}{|J|^\frac{3}{2}} \lesssim C_2 2^{n_2} \|g_1\|_p \|g_2\|_\infty.
\]

When combining the estimates (8.4) and (8.5) into (8.3), one concludes that

\[
|A_{\text{flag}} \otimes \text{flag}^2| \lesssim C_1 C_2 C_3 \sum_{n>0} 2^{-n} \|f_1\|_p \|g_1\|_p C_3 2^{k_1} \|h\|_{L^p} 2^{k_2} \cdot \sum_{n_2 \in \mathbb{Z}} \left( \bigcup_{R \in \mathcal{R}_{k_1, k_2}} \bigcup_{I \times J \in I'_{n-n_2-m_1-n_2}} \bigcup_{J'_{n_2}} \right) \lesssim C_1 C_2 C_3 \sum_{n>0} 2^{-n} \|f_1\|_p \|g_1\|_p C_3 2^{k_1 (1-\frac{t}{2})} \|h\|_{L^p} 2^{k_2 (1-\frac{t}{2})},
\]

where the last inequality follows from the sparsity condition. With proper choice of \( \gamma > 1 \), one obtains the desired estimate.

9. Proof of Theorem 4.3 for \( \Pi_{\text{flag}} \otimes \text{flag}^0 \)-Haar Model

One interesting fact is that when

\[
(9.1) \quad \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} \leq 1,
\]

Theorem 4.2 can be proved by a simpler argument as remarked in Chapter 7. And Theorem 4.3 for the model \( \Pi_{\text{flag}} \otimes \text{flag}^0 \) can be viewed as a sub-case and proved by the same argument. The key idea is that in the case specified in (9.1), one no longer needs the localization of the operator \( B \) in the proof.

Let

\[
\frac{1}{t} := \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2},
\]

and the condition of the exponents (9.1) translates to

\[
t \geq 1.
\]

9.1. Localization. One first defines

\[
\Omega := \Omega^1 \cup \Omega^2,
\]

where

\[
\Omega^1 := \bigcup_{l_2 \in \mathbb{Z}} \{ MB > C_1 2^{-l_2} \|B\|_1 \} \times \{ MB \tilde{B} > C_2 2^{l_2} \|\tilde{B}\|_1 \},
\]

\[
\Omega^2 := \{ SS h > C_3 \|h\|_{L^p} \},
\]

and

\[
\hat{\Omega} := \{ M \chi \Omega > \frac{1}{100} \}.
\]

Let

\[
E' := E \setminus \hat{\Omega}.
\]

Remark 9.1. We shall notice that \( t \geq 1 \) allows one to use the mapping property of the Hardy-Littlewood maximal operator, which plays an essential role in the estimate of \( |\Omega| \).
A straightforward computation shows $|E'| \sim |E|$ given that $C_1, C_2$ and $C_3$ are sufficiently large. It suffices to assume that $|E'| \sim |E| = 1$ and to prove that the multilinear form

$$\Lambda_{\text{flag}^0 \otimes \text{flag}^0}(f_1, f_2, g_1, g_2, h, \chi_E) := \langle \Pi_{\text{flag}^0 \otimes \text{flag}^0}(f_1, f_2, g_1, g_2, h, \chi_E) \rangle$$

satisfies the following restricted weak-type estimate

$$|\Lambda_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim |F_1|^{\frac{1}{p}} |G_1|^{\frac{1}{q}} |F_2|^{\frac{1}{r}} |G_2|^{\frac{1}{s}} \|h\|_{L^t}.$$  

9.2. Summary of stopping-time decompositions.

General two-dimensional level sets stopping-time decomposition

$$I \times J \in \mathcal{R}_{k_1,k_2} \quad (k_1 < 0, k_2 \leq K)$$

One performs the general two-dimensional level sets stopping-time decomposition with respect to the hybrid maximal-square functions as specified in the definition of the exceptional set. It would be evident from the argument below that there is no stopping-time decomposition necessary for the maximal functions involving $B$ and $\tilde{B}$. One brief explanation is that only “averages” for $B$ and $\tilde{B}$ are required while the measurement of the set where the averages are attained is not. As a consequence, the macro-control of the averages would be sufficient and the stopping-time decompositions, which can be seen as a more delicate “slice-by-slice” or “level-by-level” partition, is not compulsory. More precisely,

$$|\Lambda_{\text{flag}^0 \otimes \text{flag}^0}| = \sum_{k_1 < 0} \sum_{I \times J \in \mathcal{R}_{k_1,k_2}} \frac{1}{|I|^{\frac{1}{2}} |J|^{\frac{1}{2}}} \langle B_I(f_1, f_2), \varphi^{1,H}_I \rangle \langle \tilde{B}_J(g_1, g_2), \varphi^{1,H}_J \rangle \langle h, \psi^{2,H} \rangle \langle \chi_E \rangle$$

$$\lesssim \sum_{k_1 < 0} \sup_{k_2 \leq K} \left( \frac{|B_I(f_1, f_2)| |\tilde{B}_J(g_1, g_2)|}{|I|^{\frac{1}{2}} |J|^{\frac{1}{2}}} \right) \cdot C_3^{k_1,2} \|h\|_s^2 \|k\|_s^2 \bigcup_{I \times J \in \mathcal{R}_{k_1,k_2}} I \times J.$$

By the same reasoning applied in previous chapters, one has

$$\left| \bigcup_{I \times J \in \mathcal{R}_{k_1,k_2}} I \times J \right| \lesssim \min(C_3^1 2^{1-k_1}, C_3^{-1} 2^{1-k_2})$$

for any $\gamma > 1$. Meanwhile, an argument similar to the proof of Observation 2 in Section 7.2.2 implies that

$$\frac{|B_I(f_1, f_2)| |\tilde{B}_J(g_1, g_2)|}{|I|^{\frac{1}{2}} |J|^{\frac{1}{2}}} \lesssim C_1 C_2 \|B\|_t \|\tilde{B}\|_t,$$

for any $I \times J \cap \tilde{J} \neq \emptyset$ as assumed in the Haar model. As a consequence,

$$|\Lambda_{\text{flag}^0 \otimes \text{flag}^0}| \lesssim C_1 C_2 C_3^2 \sum_{k_1 < 0} \sum_{k_2 \leq K} \|B\|_t \|\tilde{B}\|_t C_3^{k_1,1} \|h\|_{L^t} 2^{k_2(1-\gamma)} \lesssim \|B\|_t \|\tilde{B}\|_t,$$

with appropriate choice of $\gamma > 1$. One can now invoke Lemma 5.6 to complete the proof of Theorem 4.2. In particular,

$$\|B\|_t \lesssim \|f_1\|_{p_1} \|f_2\|_{q_1},$$

while the case described in Theorem 4.3 is when $q_1 = q_2 = \infty$ and (9.5) can be rewritten as

$$\|B\|_p \lesssim \|f_1\|_p \|f_2\|_s,$$

$$\|\tilde{B}\|_p \lesssim \|g_1\|_p \|g_2\|_s.$$

Remark 9.2.

1. One notices that Theorem 4.3 in the Haar model is proved directly with generic functions in $L^p$ and $L^s$ spaces for $1 < p < \infty, 1 < s < 2$.

2. The above argument proves Theorem 4.2 in the Haar model for $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ with the range of exponents described as (9.1), which completes the proof of Theorem 4.2 - Haar model.
10. Generalization to Fourier Case

We will first highlight where we have used the assumption about the Haar model in the proof and then modify those partial arguments to prove the general case.

We have used the following implications specific to the Haar model.

H(I). Let \( \chi_{E^c} := \chi_{E^c \setminus \tilde{\Omega}} \). Then

\[
\langle \chi_{E^c}, \phi_1^H \otimes \phi_2^H \rangle \neq 0 \iff I \times J \cap \tilde{\Omega}^c \neq \emptyset.
\]

As a result, what contributes to the multilinear forms in the Haar model are the dyadic rectangles \( I \times J \in \mathbb{R} \) satisfying \( I \times J \cap \tilde{\Omega}^c \neq \emptyset \), which is a condition we heavily used in the proofs of the theorems in the Haar model.

H(II). For any dyadic intervals \( K \) and \( I \) with \( |K| \geq |I| \),

\[
\langle \phi^K_3, \phi_I^H \rangle \neq 0
\]

if and only if \( K \supseteq I \).

Therefore, the non-degenerate case imposes the condition on the geometric containment we have employed for localizations of the operator \( B \).

H(III). In the case \( \langle \phi^K_3, \phi_I^H \rangle \) is a family of Haar wavelets, the observation highlighted as (5.9) generates the biest trick (5.10) which is essential in the energy estimates.

We will focus on how to generalize proofs of Theorem 4.2 for \( \Pi_{\text{flag}^\#1 \otimes \text{flag}^\#2} \) and \( \Pi_{\text{flag}^\#1 \otimes \text{flag}^\#2} \) and discuss how to tackle restrictions listed as \( H(I), H(II) \) and \( H(III) \). The generalizations of arguments for other model operators and for Theorem 4.3 follow from the same ideas.

10.1. Generalized Proof of Theorem 4.2 for \( \Pi_{\text{flag}^\#1 \otimes \text{flag}^\#2} \).

10.1.1. Localization and generalization of \( H(I) \). The argument for \( \Pi_{\text{flag}^\#1 \otimes \text{flag}^\#2} \) in Chapter 6 takes advantage of the localization of spatial variables, as stated in \( H(I) \). The following lemma allows one to decompose the original bump function into bump functions of compact supports so that a perfect localization in spatial variables can be achieved, which can be viewed as generalized \( H(I) \) and whose proof is included in Chapter 3 of [22].

**Lemma 10.1.** Let \( I \subseteq \mathbb{R} \) be an interval. Then any smooth bump function \( \phi_I \) adapted to \( I \) can be decomposed as

\[
\phi_I = \sum_{\tau \in \mathbb{N}} 2^{100\tau} \phi^\tau_I
\]

where for each \( \tau \in \mathbb{N} \), \( \phi^\tau_I \) is a smooth bump function adapted to \( I \) and \( \text{supp}(\phi^\tau_I) \subseteq 2^\tau I \). If \( \int \phi_I = 0 \), then the functions \( \phi^\tau_I \) can be chosen such that \( \int \phi^\tau_I = 0 \).

The multilinear form associated to \( \Pi_{\text{flag}^\#1 \otimes \text{flag}^\#2} \) in the general case can now be rewritten as

\[
\Lambda_{\text{flag}^\#1 \otimes \text{flag}^\#2}(f_1^x, f_2^x, g_1^y, g_2^y, h_{xy}, \chi_{E'}) := \sum_{\tau_1, \tau_2 \in \mathbb{N}} 2^{-100(\tau_1 + \tau_2)} \sum_{I \times J \subseteq \mathbb{R}} \frac{1}{|I|^{|\tau_1|} |J|^{|\tau_2|}} \langle B_1^\#(f_1, f_2), \phi_{1,\tau_1}^I \rangle \langle \tilde{B}^\#(g_1, g_2), \phi_{2,\tau_2}^J \rangle \langle h, \phi_{1,\tau_1}^I \otimes \phi_{2,\tau_2}^J \rangle \langle \chi_{E'}, \phi_3^3 \rangle.
\]

(10.2)

For \( \tau_1, \tau_2 \in \mathbb{N} \) fixed, define

\[
\Lambda_{\text{flag}^\#1 \otimes \text{flag}^\#2}^{\tau_1, \tau_2}(f_1^x, f_2^x, g_1^y, g_2^y, h_{xy}, \chi_{E'}) := \sum_{I \times J \subseteq \mathbb{R}} \frac{1}{|I|^{|\tau_1|} |J|^{|\tau_2|}} \langle B_1(f_1, f_2), \phi_{1,\tau_1}^I \rangle \langle \tilde{B}(g_1, g_2), \phi_{2,\tau_2}^J \rangle \langle h, \phi_{1,\tau_1}^I \otimes \phi_{2,\tau_2}^J \rangle \langle \chi_{E'}, \phi_3^3 \rangle.
\]

(10.3)

It suffices to prove that for any fixed \( \tau_1, \tau_2 \in \mathbb{N} \),

\[
|\Lambda_{\text{flag}^\#1 \otimes \text{flag}^\#2}^{\tau_1, \tau_2}| \lesssim (2^{\tau_1 + \tau_2})^\Theta |F_1|^\frac{1}{\tau_1} |F_2|^\frac{1}{\tau_2} |G_1|^\frac{1}{\tau_1} |G_2|^\frac{1}{\tau_2},
\]

for some \( 0 < \Theta < 100 \), thanks to the fast decay \( 2^{-100(\tau_1 + \tau_2)} \) in the decomposition of the original multilinear form (10.2).
One first re-defines the exceptional set with the replacement of $C_1$, $C_2$ and $C_3$ by $C_1 2^{10 \tau_1}$, $C_2 2^{10 \tau_2}$ and $C_3 2^{10 \tau_1 + 10 \tau_2}$ respectively. In particular, let

$$
C_1^{\tau_1} := C_1 2^{10 \tau_1}, \\
C_2^{\tau_2} := C_2 2^{10 \tau_2}, \\
C_3^{\tau_1, \tau_2} := C_3 2^{10 \tau_1 + 10 \tau_2}.
$$

Then define

$$
\Omega_1^{\tau_1, \tau_2} := \bigcup_{n \in \mathbb{Z}} \{Mf_1 > C_1^{\tau_1} 2^n |F_1|\} \times \{Mg_1 > C_2^{\tau_2} 2^{-\hat{n}} |G_1|\} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{Mf_2 > C_1^{\tau_1} 2^n |F_2|\} \times \{Mg_2 > C_2^{\tau_2} 2^{-\hat{n}} |G_2|\} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{Mf_1 > C_1^{\tau_1} 2^n |F_1|\} \times \{Mg_2 > C_2^{\tau_2} 2^{-\hat{n}} |G_2|\} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{Mf_2 > C_1^{\tau_1} 2^n |F_2|\} \times \{Mg_1 > C_2^{\tau_2} 2^{-\hat{n}} |G_1|\},
$$

$$
\Omega_2^{\tau_1, \tau_2} := \{SSh > C_3^{\tau_1, \tau_2} \|h\|_{L^2(\mathbb{R})}\}.
$$

One also defines

$$
\Omega_1^{\tau_1, \tau_2} := \Omega_1^{\tau_1, \tau_2} \cup \Omega_2^{\tau_1, \tau_2}, \\
\tilde{\Omega}_1^{\tau_1, \tau_2} := \{M(\chi_{\Omega_1^{\tau_1, \tau_2}}) > \frac{1}{100}\}, \\
\tilde{\tilde{\Omega}}_1^{\tau_1, \tau_2} := \{M(\chi_{\tilde{\Omega}_1^{\tau_1, \tau_2}}) > \frac{1}{2^{10 \tau_1 + 10 \tau_2}}\},
$$

and finally

$$
\tilde{\Omega} := \bigcup_{\tau_1, \tau_2 \in \mathbb{N}} \tilde{\tilde{\Omega}}_1^{\tau_1, \tau_2}.
$$

**Remark 10.2.** It is not difficult to verify that $|\tilde{\Omega}| \ll 1$ given that $C_1, C_2$ and $C_3$ are sufficiently large. One can then define $E' := E \setminus \tilde{\Omega}$, where $|E'| \sim |E|$ as desired. For such $E'$, one has the following simple but essential observation.

**Observation 3.** For any fixed $\tau_1, \tau_2 \in \mathbb{N}$ and any dyadic rectangle $I \times J$,

$$
\langle \chi_{E'}, \phi_I^{3, \tau_1} \otimes \phi_J^{3, \tau_2} \rangle \neq 0
$$

implies that

$$
I \times J \cap (\tilde{\Omega}^{\tau_1, \tau_2})^c \neq \emptyset.
$$

**Proof.** We will prove the equivalent contrapositive statement. Suppose that $I \times J \cap (\tilde{\Omega}^{\tau_1, \tau_2})^c = \emptyset$, or equivalently $I \times J \subseteq \tilde{\Omega}^{\tau_1, \tau_2}$, then

$$
|2^{\tau_1} I \times 2^{\tau_2} J \cap \tilde{\Omega}^{\tau_1, \tau_2}| > \frac{1}{2^{2 \tau_1 + 2 \tau_2}} |2^{\tau_1} I \times 2^{\tau_2} J|,
$$

which infers that

$$
2^{\tau_1} I \times 2^{\tau_2} J \subseteq \tilde{\Omega}^{\tau_1, \tau_2} \subseteq \tilde{\Omega}.
$$

Since $E' \cap \tilde{\Omega} = \emptyset$, one can conclude that

$$
\langle \chi_{E'}, \phi_I^{3, \tau_1} \otimes \phi_J^{3, \tau_2} \rangle = 0,
$$

which completes the proof of the observation. \(\square\)
Remark 10.3. Observation 3 settles a starting point for the stopping-time decompositions with fixed parameters $\tau_1$ and $\tau_2$. More precisely, suppose that $\mathcal{R}$ is an arbitrary finite collection of dyadic rectangles. Then with fixed $\tau_1, \tau_2 \in \mathbb{N}$, let $\mathcal{R} := \bigcup_{n_1, n_2 \in \mathbb{N}} \mathcal{T}_{n_1} \times \mathcal{J}_{n_2}$ denote the tensor-type stopping-time decomposition $I$ - level sets introduced in Chapter 6. Now $\mathcal{T}_{n_1}$ and $\mathcal{J}_{n_2}$ are defined in the same way as $I_{n_1}$ and $\mathcal{J}_{n_2}$ with $C_1$ and $C_2$ replaced by $C_{1, \tau_1}$ and $C_{2, \tau_2}$. By the argument for Observation 1 in Chapter 6, one can deduce the same conclusion that if for any $I \times J \in \mathcal{R}$, $I \times J \cap \Omega^c \neq \emptyset$, then $n_1 + n_2 < 0$.

Due to Remark 10.3, one can perform the stopping-time decompositions specified in Chapter 6 with $C_1, C_2$ and $C_3$ replaced by $C_{1, \tau_1}$, $C_{2, \tau_2}$ and $C_{3, \tau_3, \tau_4}$ respectively and adopt the argument without issues. The difference that lies in the resulting estimate is the appearance of $O(2^{50\tau_3})$, $O(2^{50\tau_4})$ and $O(2^{50\tau_3 + 50\tau_4})$, which is not of concerns as illustrated in (10.21). The only “black-box” used in Chapter 6 is the local size estimates (Proposition 5.10), which needs a more careful treatment and will be explored in the next subsection.

10.1.2. Local size estimates and generalization of $H(II)$. We will focus on the estimates of size$(\langle B_{l}^{\#_1}, \varphi_I \rangle)_I$, whose argument applies to size$(\langle B_{l}^{\#_2, \varphi} \rangle)_I$ as well. It suffices to prove Lemma 5.8 in the Fourier case and the local size estimates described in Proposition 5.10 follow immediately. One first attempts to apply Lemma 10.1 to create a setting of compactly-supported bump functions so that the same localization described in Chapter 5 can be achieved. Suppose that for any $I \in I'$, $I \cap S \neq \emptyset$, then

$$\text{size}_{I'}((\langle B_{l}^{\#_1, H}, \varphi_I \rangle)_I = \frac{|\langle B_{l}^{\#_1}(f, f), \varphi_{l}^{1}_{I_0} \rangle|}{|I_0|^{\frac{q}{2}}},$$

for some $I_0 \in I'$ such that $I_0 \cap S \neq \emptyset$. Consider

$$(10.5) \quad \frac{|\langle B_{l}^{\#_1}(f, f), \varphi_{l}^{1}_{I_0} \rangle|}{|I_0|^{\frac{q}{2}}} \leq \sum_{\tau_1, \tau_2 \in \mathbb{N}} 2^{100\tau_3} 2^{-100\tau_4} \sum_{K:|K|<2^{\tau_1}|I_0|} \frac{1}{|K|^{\frac{q}{2}}} \langle f, f, \varphi_{K}^{1, \tau_3} \rangle \langle f, f, \varphi_{K}^{2, \tau_4} \rangle \langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle,$$

where $\varphi_{l}^{1}$ denotes the $L^2$ smooth bump function adapted to $I$, $\varphi_{l}^{1, \tau_3}$ is an $L^2$-normalized bump function adapted to $I$ with $\text{supp}(\varphi_{l}^{1, \tau_3}) \subseteq 2^{\tau_3} I$, and $\varphi_{K}^{1, \tau_3}$ is an $L^2$-normalized bump function with $\text{supp}(\varphi_{K}^{1, \tau_3}) \subseteq 2^{\tau_3} K$. With the property of being compactly supported, one has that if

$$\langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle \neq 0,$$

then

$$2^{\tau_3} I \cap 2^{\tau_3} K \neq \emptyset.$$

One also recalls that $I \cap S \neq \emptyset$ and $|I| \leq |K|$, it follows that

$$(10.6) \quad \frac{\text{dist}(K, S)}{|K|} \lesssim 2^{\tau_3 + \tau_4}.$$

Therefore, one can apply (10.6) and rewrite (10.5) as

$$\frac{|\langle B_{l}^{\#_1}(f, f), \varphi_{l}^{1}_{I_0} \rangle|}{|I_0|^{\frac{q}{2}}} \lesssim \sum_{\tau_1, \tau_2 \in \mathbb{N}} 2^{-100\tau_3} 2^{-100\tau_4} \frac{1}{|I_0|} \sum_{K:|K|<2^{\tau_1}|I_0|} \frac{|\langle f, f, \varphi_{K}^{1} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{2} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}},$$

$$\leq \sum_{\tau_1, \tau_2 \in \mathbb{N}} 2^{-100\tau_3} 2^{-100\tau_4} \frac{1}{|I_0|} \sup_{K:|K|<2^{\tau_1}|I_0|} \frac{|\langle f, f, \varphi_{K}^{1} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{2} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}},$$

$$\leq \sum_{\tau_1, \tau_2 \in \mathbb{N}} 2^{-100\tau_3} 2^{-100\tau_4} \frac{1}{|I_0|} \sup_{K:\text{dist}(K, S) \leq 2^{\tau_3 + \tau_4}} \frac{|\langle f, f, \varphi_{K}^{1} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{2} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}}. $$

One notices that

$$\sup_{K:|K|<2^{\tau_3 + \tau_4}} \frac{|\langle f, f, \varphi_{K}^{1} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{2} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle \varphi_{l}^{1, \tau_3}, \varphi_{K}^{3, \tau_4} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}} \frac{|\langle f, f, \varphi_{K}^{3} \rangle|}{|K|^{\frac{q}{2}}}. $$
fails to generate the local energy estimates. In particular, for \( \Pi \) and \( \phi \)

\[
\text{Generalized Proof of Theorem}
\]

\[
The localization has been obtained in (10.11). Similarly,
\]

\[
(10.9)
\]

where \( K' := 2^{r_3+r_4}K \), the interval with the same center as \( K \) with the radius \( 2^{r_3+r_4}|K| \). Similarly,

\[
(10.10)
\]

Morever,

\[
(10.10)
\]

By combining (10.8), (10.9) and (10.10), one can estimate (10.7) as

\[
\langle B_{f_0}^{\Phi^I}(f_1, f_2), \varphi^I_{I_0} \rangle
\]

\[
\lesssim \frac{1}{|I_0|} \sum_{\tau_3, \tau_4 \in \mathbb{N}} 2^{100\tau_3 - 100\tau_4} \sum_{K : |K| \geq |I|} \sup_{K' \cap S \neq \emptyset} \frac{|\langle f_1, \phi_K^{\tau_3} \rangle|}{|K'|^{\frac{1}{2}}} \sup_{K' \cap S \neq \emptyset} \frac{|\langle f_2, \phi_K^{\tau_4} \rangle|}{|K'|^{\frac{1}{2}}},
\]

which is exactly the same estimate for the corresponding term in Lemma 5.8. This completes the proof of Theorem 4.2 and 4.3 for \( \Pi_{\text{flag}^{r_1} \otimes \text{flag}^{r_2}} \).

10.2. Generalized Proof of Theorem 4.2 for \( \Pi_{\text{flag}^{r_1} \otimes \text{flag}^{r_2}} \).

10.2.1. Local energy estimates and generalization of \( H(III) \). The delicacy of the argument for \( \Pi_{\text{flag}^{r_1} \otimes \text{flag}^{r_2}} \) with the lacunary family \( (\phi_K^{\tau_3})_K \) lies in the localization and the application of the biest trick for the energy estimates. It is worthy to note that Lemma 10.1 fails to generate the local energy estimates. In particular, one can decompose

\[
\langle B_1(f_1, f_2), \varphi^I_{I_0} \rangle = \sum_{\tau_3, \tau_4 \in \mathbb{N}} 2^{100\tau_3 - 100\tau_4} \sum_{K : |K| \geq |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \phi_K^{\tau_3} \rangle \langle f_2, \phi_K^{\tau_4} \rangle \langle \varphi^I_{I_0} \rangle.
\]

Then by the geometric observation (10.6) implied by the non-degenerate condition \( \langle \varphi^I_{I_0}, \psi_K^{\tau_3} \rangle \neq 0 \),

\[
(10.11)
\]

\[
(10.11)
\]

The localization has been obtained in (10.11). Nonetheless, for each fixed \( \tau_3 \) and \( \tau_4 \), one cannot equate the terms

\[
\sum_{K : |K| \geq |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \phi_K^{\tau_3} \rangle \langle f_2, \phi_K^{\tau_4} \rangle \langle \varphi^I_{I_0} \rangle \neq \sum_{K : |K| \geq |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \phi_K^{\tau_3} \rangle \langle f_2, \phi_K^{\tau_4} \rangle \langle \varphi^I_{I_0} \rangle.
\]
The reason is that $\varphi_1^1$, $\varphi_1^3$ and $\psi_3^1$, $\psi_3^3$ are general $L^2$-normalized bump functions instead of Haar wavelets and $L^2$-normalized indicator functions. The “if and only if” condition
\begin{equation}
\langle \varphi_1^1, \psi_3^1 \rangle \neq 0 \iff |K| \geq |I|
\end{equation}
is no longer valid and is insufficient to derive the biest trick. The biest trick is crucial in the local energy estimates which shall be evident from the previous analysis. In order to use the biest trick in the Fourier case, one needs to exploit the compact Fourier supports instead of the compact supports for spatial variables in the Haar model. As a consequence, one cannot simply apply Lemma 10.1 to localize the energy term involving $B$ as (10.11) since the bump functions $\varphi_1^1$, $\psi_3^1$ are compactly supported in space and cannot be compactly supported in frequency due to the uncertainty principle.

To achieve the biest trick, one needs to apply a generalized localization. One first recalls that the Littlewood-Paley decomposition imposes that $\text{supp}(\varphi_1^1) \subseteq \omega_I$ and $\text{supp}(\psi_3^1) \subseteq \omega_K$ where $\omega_I$ and $\omega_K$ behave as follows in the frequency space:

\[\begin{array}{cccc}
\omega_I & \omega_K & \omega_I & \omega_K \\
-4|I| & -4|K|^{-1} & -\frac{1}{4}|I|^{-1} & 0 \\
4|I|^{-1} & 4|K|^{-1} & 0 & \frac{1}{4}|K|^{-1} \\
4|I|^{-1} & 4|K|^{-1} & 0 & \frac{1}{4}|K|^{-1}
\end{array}\]

As one may notice,
\begin{equation}
\langle \varphi_1^1, \psi_3^1 \rangle \neq 0 \iff \omega_K \subseteq \omega_I \iff |K| \geq |I|,
\end{equation}
which yields the biest trick as desired. Meanwhile, we would like to attain some localization for the energy. In particular, fix any $n_1$, $m_1$, define the level set
\[\Omega_{x_{n_1,m_1}}^x := \{Mf_1 > C_12^{m_1}|F_1|\} \cap \{Mf_2 > C_12^{m_1}|F_2|\},\]
then one would like to reduce energy($\mathcal{B}_I^1, \varphi_1^1$) to energy($\mathcal{B}_0^{n_1,m_1}, \varphi_1^1$) where
\[B_0^{n_1,m_1} := \sum_{K \in \mathcal{K}_d} \frac{1}{|K|^2} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \psi_K^3.\]

One observes that since $\psi_K^3$ and $\varphi_1^3$ are not compactly supported in $K$ and $I$ respectively, one cannot deduce that $K \cap \Omega_{x_{n_1,m_1}}^x \neq \emptyset$ given that $|K| \geq |I|$. The localization in the Fourier case is attained in a more analytic fashion. One decomposes the sum
\begin{equation}
\left| \frac{\langle B_I, \varphi_1^1 \rangle}{|I|^2} \right| = \frac{1}{|I|^2} \sum_{K \in \mathcal{K} \cap |K| \geq |I|} \frac{1}{|K|^2} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_1^1, \psi_K^3 \rangle.
\end{equation}

where
\[\mathcal{K}_d^{n_1,m_1} := \{K : 1 + \frac{\text{dist}(K, \Omega_{x_{n_1,m_1}})}{|K|} \sim 2^d\},\]
and
\[\mathcal{K}_0^{n_1,m_1} := \{K : K \cap \Omega_{x_{n_1,m_1}} \neq \emptyset\}.\]
Ideally, one would like to "omit" the former term, which is reasonable once

$$\sum_{d > 0} \sum_{|K| \geq |I|} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle \ll \sum_{|K| \geq |I|} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle$$

so that one can apply the previous argument discussed in Chapter 6. In the other case when

$$\sum_{d > 0} \sum_{|K| \geq |I|} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle \gtrsim \sum_{|K| \geq |I|} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle,$$

local energy estimates are not necessary to achieve the result. The following lemma generates estimates for the former term and provides a guideline about the separation of cases. The notations in the lemma are consistent with the previous discussion.

**Lemma 10.4.** Suppose that $d > 0$. Then

$$\frac{1}{|I|^\frac{1}{2}} \sum_{|K| \geq |I|} \frac{1}{|K|^2} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle \bigg| \lesssim 2^{-Nd} (C_1 2^{m_1} |F_1|)^{\alpha_1} (C_1 2^{m_1} |F_2|)^{\beta_1},$$

for any $0 \leq \alpha_1, \beta_1 \leq 1$ and some $N \gg 1$.

**Remark 10.5.**

1. One simple but important fact is that for any fixed $d > 0$, $n_1$ and $m_1$, $K_d^{n_1, m_1}$ is a disjoint collection of dyadic intervals.

2. Aware of the first comment, one can apply the exactly same argument in Section 10.1 to prove the lemma.

Based on the estimates described in Lemma 10.4, one has that

$$\frac{1}{|I|^\frac{1}{2}} \sum_{d > 0} \sum_{|K| \geq |I|} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle \bigg| \lesssim \sum_{d > 0} 2^{-Nd} (C_1 2^{n_1} |F_1|)^{\alpha_1} (C_1 2^{m_1} |F_2|)^{\beta_1}$$

(10.16)

$$\lesssim (C_1 2^{n_1} |F_1|)^{\alpha_1} (C_1 2^{m_1} |F_2|)^{\beta_1},$$

for any $0 \leq \alpha_1, \beta_1 \leq 1$. One can then use the upper bound in (10.16) to proceed the discussion case by case.

**Case I: There exists $0 \leq \alpha_1, \beta_1 \leq 1$ such that** $\frac{|\langle B_1, \varphi_1^1 \rangle|}{|I|^\frac{1}{2}} \gg (C_1 2^{n_1} |F_1|)^{\alpha_1} (C_1 2^{m_1} |F_2|)^{\beta_1}$. In Case I, (10.15) holds and the dominant term in expression (10.14) has to be

$$\frac{1}{|I|^\frac{1}{2}} \sum_{K \in K_d^{n_1, m_1}} \frac{1}{|K|^\frac{1}{2}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \langle \varphi_K^1, \psi_K^2 \rangle,$$

which provides a localization for energy estimates involving $B$. In particular, in the current case

$$\text{energy}((\langle B_1, \varphi_1^1 \rangle)_t) \lesssim \text{energy}((B_0^{n_1, m_1}, \varphi_1)_t)_{|t| \cap \Delta_{n_1, m_1} \neq \emptyset},$$

$$\text{energy}^t((\langle B_1, \varphi_1^1 \rangle)_t) \lesssim \text{energy}^t((B_0^{n_1, m_1}, \varphi_1)_t)_{|t| \cap \Delta_{n_1, m_1} \neq \emptyset},$$

for any $t > 1$. Furthermore,

$$\text{energy}((B_0^{n_1, m_1}, \varphi_1)_t)_{|t| \cap \Delta_{n_1, m_1} \neq \emptyset} \lesssim \|B_0^{n_1, m_1}\|_1,$$

$$\text{energy}^t((B_0^{n_1, m_1}, \varphi_1)_t)_{|t| \cap \Delta_{n_1, m_1} \neq \emptyset} \lesssim \|B_0^{n_1, m_1}\|_t,$$

for any $t > 1$, where $\|B_0^{n_1, m_1}\|_1$ and $\|B_0^{n_1, m_1}\|_t$ follow from the same estimates for their Haar variants described in Chapter 5. We will explicitly state the local energy estimates in this case.

**Proposition 10.6 (Local Energy Estimates in Fourier Case in x-Direction).** Suppose that $n_1, m_1 \in \mathbb{Z}$ are fixed and suppose that $I'$ is a finite collection of dyadic intervals such that for any $I \in I'$, $I$ satisfies
(1) \( I \in I_{n_1, m_1} \);
(2) \( I \in T \) with \( T \in T_{l_1} \) for some \( l_1 \) satisfying the condition that there exists some \( 0 \leq \alpha_1, \beta_1 \leq 1 \) such that

\[
2^{l_1} \| B \|_1 \gg (C_1 2^{n_1}|F_1|)^{\alpha_1} (C_1 2^{m_1}|F_2|)^{\beta_1}.
\]

(i) Then for any \( 0 \leq \theta_1, \theta_2 < 1 \) with \( \theta_1 + \theta_2 = 1 \), one has

\[
\text{energy}_{T'}((B_1, \varphi_1))_{I \in T'} \lesssim C_1 \frac{1}{\theta_1 + \theta_2} - \theta_1 - \theta_2 2^{n_1 (\frac{1}{\theta_1} - \theta_1)} 2^{m_1 (\frac{1}{\theta_2} - \theta_2)} |F_1|^\frac{1}{\theta_1} |F_2|^\frac{1}{\theta_2}.
\]

(ii) Suppose that \( t > 1 \). Then for any \( 0 \leq \theta_1, \theta_2 < 1 \) with \( \theta_1 + \theta_2 = \frac{1}{t} \), one has

\[
\text{energy}_{T'}((B_1, \varphi_1))_{I \in T'} \lesssim C_1 \frac{1}{\theta_1 + \theta_2} - \theta_1 - \theta_2 2^{n_1 (\frac{1}{\theta_1} - \theta_1)} 2^{m_1 (\frac{1}{\theta_2} - \theta_2)} |F_1|^\frac{1}{\theta_1} |F_2|^\frac{1}{\theta_2}.
\]

A parallel statement holds for dyadic intervals in the \( y \)-direction, which will be stated for the convenience of reference later on.

**Proposition 10.7** (Local Energy Estimates in Fourier Case in \( y \)-Direction). Suppose that \( n_2, m_2 \in \mathbb{Z} \) are fixed and suppose that \( J' \) is a finite collection of dyadic intervals such that for any \( J \in J' \), \( J \) satisfies

(1) \( J \in J_{n_2, m_2} \);
(2) \( J \in S \) with \( S \in S_{l_2} \) for some \( l_2 \) satisfying the condition that there exists some \( 0 \leq \alpha_2, \beta_2 \leq 1 \) such that

\[
2^{l_2} \| \tilde{B} \|_1 \gg (C_1 2^{n_2}|G_1|)^{\alpha_2} (C_1 2^{m_2}|G_2|)^{\beta_2}.
\]

(i) Then for any \( 0 \leq \zeta_1, \zeta_2 < 1 \) with \( \zeta_1 + \zeta_2 = 1 \), one has

\[
\text{energy}_{J'}((\tilde{B}_1, \varphi_1))_{J \in J'} \lesssim C_2 \frac{1}{\zeta_1 + \zeta_2} - \zeta_1 - \zeta_2 2^{n_2 (\frac{1}{\zeta_1} - \zeta_1)} 2^{m_2 (\frac{1}{\zeta_2} - \zeta_2)} |G_1|^\frac{1}{\zeta_1} |G_2|^\frac{1}{\zeta_2}.
\]

(ii) Suppose that \( s > 1 \). Then for any \( 0 \leq \zeta_1, \zeta_2 < 1 \) with \( \zeta_1 + \zeta_2 = \frac{1}{s} \), one has

\[
\text{energy}_{J'}((\tilde{B}_1, \varphi_1))_{J \in J'} \lesssim C_2 \frac{1}{\zeta_1 + \zeta_2} - \zeta_1 - \zeta_2 2^{n_2 (\frac{1}{\zeta_1} - \zeta_1)} 2^{m_2 (\frac{1}{\zeta_2} - \zeta_2)} |G_1|^\frac{1}{\zeta_1} |G_2|^\frac{1}{\zeta_2}.
\]

**Remark 10.8.** We would like to highlight that the localization of energies is attained under the additional conditions (10.17) and (10.18), in which case one obtains the local energy estimates stated in Proposition 10.6 and 10.7 that can be viewed as analogies of Proposition 5.14.

**Case II:** For any \( 0 \leq \alpha_1, \beta_1 \leq 1 \), \( \frac{\| (B_1, \varphi_1) \|}{|I|^\frac{1}{2}} \lesssim (C_1 2^{n_1}|F_1|)^{\alpha_1} (C_1 2^{m_1}|F_2|)^{\beta_1} \).

In this alternative case, the size estimates are favorable and a simpler argument can be applied without invoking the local energy estimates.

**10.2.2. Proof Part 1 - Localization.** In this last section, we will explore how to implement the case-by-case analysis and generalize the argument in the proof of Theorem 4.2 for \( \Pi_{\Pi}^\theta \circ \Pi^\theta \Pi^\rho \Pi^\rho \) when \( (\phi_k^\theta)_K \) and \( (\phi_l^\rho)_L \) are lacunary families and \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} > 1 \), which is the most tricky part to generalize for the argument in Chapter 7. The generalized argument can be viewed as a combination of the discussions in Chapter 6 and 7. One first defines the exceptional set \( \Omega \) as follows: For any \( \tau_1, \tau_2 \in \mathbb{N} \), define

\[
\Omega^{\tau_1, \tau_2} := \Omega^{\tau_1, \tau_2}_1 \cup \Omega^{\tau_1, \tau_2}_2
\]

with

\[
\Omega^{\tau_1, \tau_2}_1 := \bigcup_{\hat{n} \in \mathbb{Z}} \{ Mf_1 > C_1^{\tau_1} 2^{\hat{n}} |F_1| \} \times \{ Mg_1 > C_2^{\tau_1} 2^{\hat{n}} |G_1| \} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{ Mf_2 > C_1^{\tau_2} 2^{\hat{n}} |F_2| \} \times \{ Mg_2 > C_2^{\tau_2} 2^{\hat{n}} |G_2| \} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{ Mf_1 > C_1^{\tau_1} 2^{\hat{n}} |F_1| \} \times \{ Mg_2 > C_2^{\tau_1} 2^{\hat{n}} |G_2| \} \cup \\
\bigcup_{\hat{n} \in \mathbb{Z}} \{ Mf_2 > C_1^{\tau_2} 2^{\hat{n}} |F_2| \} \times \{ Mg_1 > C_2^{\tau_2} 2^{\hat{n}} |G_1| \} \cup
\]

The argument then proceeds similarly to the one in Chapter 7.
and
\[
\Omega_{\tau_1, \tau_2} := \{ S h > C_{\lambda}^{\tau_1, \tau_2} \| h \|_{L^r(\mathbb{R}^2)} \},
\]
and finally
\[
\tilde{\Omega} := \bigcup_{\tau_1, \tau_2 \in \mathbb{N}} \tilde{\Omega}_{\tau_1, \tau_2}.
\]
Let
\[
E' := E \setminus \tilde{\Omega},
\]
where \(|E'| \sim |E| = 1\) given that \(C_1, C_2\) and \(C_3\) are sufficiently large constants. Our goal is to prove that
\[
\Lambda_{\text{flag}^0 \odot \text{flag}^0}(f_1^x, f_2^x, g_1^y, g_2^y, h^{xy}, \chi_{E'}) := \sum_{\tau_1, \tau_2 \in \mathbb{N}} \frac{1}{2^{-100(\tau_1 + \tau_2)}} \sum_{I \times J \in \mathcal{R}} \frac{1}{|I|^\frac{1}{2} |J|^\frac{1}{2}} (B_1^{\tau_1} (f_1, f_2), \phi_{\tau_1}^{\tau_1}) (B_2^{\tau_2} (g_1, g_2), \phi_{\tau_2}^{\tau_2}) \cdot (h, \phi_{\tau_1}^{\tau_1} \otimes \phi_{\tau_2}^{\tau_2}) (\chi_{E'}, \phi_{\tau_1}^{\tau_1} \otimes \phi_{\tau_2}^{\tau_2})
\]
satisfies the restricted weak-type estimates
\[
(10.19) \quad |\Lambda_{\text{flag}^0 \odot \text{flag}^0}| \lesssim |F_1|^\frac{\gamma_1}{2} |F_2|^\frac{\gamma_2}{2} |G_1|^\frac{\gamma_1}{2} |G_2|^\frac{\gamma_2}{2}.
\]
For \(\tau_1, \tau_2 \in \mathbb{N}\) fixed, let
\[
(10.20) \quad \Lambda_{\text{flag}^0 \odot \text{flag}^0}(f_1^x, f_2^x, g_1^y, g_2^y, h^{xy}, \chi_{E'}) := \sum_{I \times J \in \mathcal{R}} \frac{1}{|I|^\frac{1}{2} |J|^\frac{1}{2}} (B_1 (f_1, f_2), \phi_{\tau_1}^{\tau_1}) (B_2 (g_1, g_2), \phi_{\tau_2}^{\tau_2}) \cdot (h, \phi_{\tau_1}^{\tau_1} \otimes \phi_{\tau_2}^{\tau_2}) (\chi_{E'}, \phi_{\tau_1}^{\tau_1} \otimes \phi_{\tau_2}^{\tau_2})
\]
then (10.19) can be reduced to proving that for any fixed \(\tau_1, \tau_2 \in \mathbb{N}\),
\[
(10.21) \quad |\Lambda_{\text{flag}^0 \odot \text{flag}^0}| \lesssim (2^{\tau_1 + \tau_2})^\Theta |F_1|^\frac{\gamma_1}{2} |F_2|^\frac{\gamma_2}{2} |G_1|^\frac{\gamma_1}{2} |G_2|^\frac{\gamma_2}{2}
\]
for some \(0 < \Theta < 100\).

10.2.3. Proof Part 2 - Summary of stopping-time decompositions. For any fixed \(\tau_1, \tau_2 \in \mathbb{N}\), one can carry out the exactly same stopping-time algorithms in Chapter 7 with the replacement of \(C_1, C_2\) and \(C_3\) by \(C_{\tau_1}^{\tau_1, \tau_2}\), \(C_{\tau_2}^{\tau_1, \tau_2}\) and \(C_{\tau_1}^{\tau_1, \tau_2}\) respectively. The resulting level sets, trees and collections of dyadic rectangles will follow the similar notations as before with extra indications of \(\tau_1\) and \(\tau_2\).

I. Tensor-type stopping-time decomposition I on \(I \times J\) \(\rightarrow\) \(I \times J \in \mathcal{T}_{n-m-n_2-m-m_2}^{\tau_1} \times \mathcal{J}_{n_2-m_2}^{\tau_2}
\)
\((m_2, m_2 \in \mathbb{Z}, n > 0)\)

II. Tensor-type stopping-time decomposition II on \(I \times J\) \(\rightarrow\) \(I \times J \in T \times S\) with \(T \in \mathcal{T}_{n_2}^{\tau_1}, S \in \mathcal{J}_{n_2}^{\tau_2}
\)
\((l_2 \in \mathbb{Z}, l > 0)\)

III. General two-dimensional level sets stopping-time decomposition on \(I \times J\) \(\rightarrow\) \(I \times J \in \mathcal{R}_{k_1, k_2}^{\tau_1, \tau_2}
\)
\((k_1 < 0, k_2 \leq K)\)
10.2.4. Proof Part 3 - Application of stopping-time decompositions. As one may recall, the multilinear form is estimated based on the stopping-time decompositions, the sparsity condition and the Fubini-type argument.

\[
|A_{\tau_1,\tau_2}^{\tau_1,\tau_2}(a,b,c)| = \sum_{l>0} \sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} \sum_{I \in J_1, l_2} \frac{1}{|I|^2} |(b_1, b_2)\langle \tilde{a}(g_1, g_2), \phi^1_1 \rangle \langle h, \phi^0 \otimes \phi^0_2 \rangle (\chi_{E'}, \phi^3_1 \otimes \phi^3_2)|
\]

\[
\leq C_{1} \gamma_{C_{T_2}} C_{T_2} (C_{T_2})^2 \sum_{l>0} \sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} |h|_{l} \sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} 2^{-I - \delta_2} \sum_{I \in J_1, l_2} \sum_{I \times J} \bigcup_{I \times J} \Bigg| \int_{T \in \mathbb{R}^2} \chi_{E'}(x) \phi^3_1 (x) \phi^3_2 (x) dx \Bigg|
\]

(10.22)

The nested sum

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} 2^{-I - \delta_2} \sum_{I \in J_1, l_2} \sum_{I \times J} \bigcup_{I \times J} \Bigg| \int_{T \in \mathbb{R}^2} \chi_{E'}(x) \phi^3_1 (x) \phi^3_2 (x) dx \Bigg|
\]

(10.23)

can be estimated using the same sparsity condition for (7.17) and a modified Fubini argument as discussed in the following two subsection.

10.2.5. Proof Part 4 - Sparsity condition. One invokes the sparsity condition Theorem 6.4 and argument in Chapter 6 to obtain the following estimate for (10.23).

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} 2^{-I - \delta_2} \sum_{I \in J_1, l_2} \sum_{I \times J} \bigcup_{I \times J} \Bigg| \int_{T \in \mathbb{R}^2} \chi_{E'}(x) \phi^3_1 (x) \phi^3_2 (x) dx \Bigg|
\]

(10.24)

For any \(0 < \delta \ll 1\), Lemma 5.6 implies that

\[
\|B\|_1 \leq |F_1|^+ \leq |F_2|^+ \leq |G_1|^+ \leq |G_2|^+ \leq |B|^+ \leq |B|^+.
\]

Therefore, (10.24) can be majorized by

\[
2^{-I - \delta_2} \sum_{n_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} 2^{-I - \delta_2} \sum_{I \in J_1, l_2} \sum_{I \times J} \bigcup_{I \times J} \Bigg| \int_{T \in \mathbb{R}^2} \chi_{E'}(x) \phi^3_1 (x) \phi^3_2 (x) dx \Bigg|
\]

(10.25)

10.2.6. Proof Part 5 - Fubini argument. The separation of cases based on the levels of the stopping-time decompositions for \((|B_1|, |B_2|, |B_2|^+)) \in J\) and \((|B_1|^+, |B_2|^+)) \in J\), in particular the ranges of \(l_2\) in the tensor-type stopping-time decomposition \(T\), plays an important role in the modified Fubini-type argument. With \(l \in \mathbb{N}\) fixed, the ranges of exponents \(l_2\) are defined as follows:

\[
\mathcal{E}_{\mathcal{X}_{\mathcal{P}}^{l}}(-n-n_2, m-m_2, m_2, m_2) := \{l_2 \in \mathbb{Z} : \text{for any } 0 \leq \alpha_1, \beta_1, \alpha_2, \beta_2 \leq 1, 2^{-l_2} |B_1| \leq (C_{1}^2 m^{-n_2} |F_1|)^{\alpha_1} (C_{1}^2 m^{-m_2} |F_2|)^{\beta_1} \text{ and } 2^{l_2} |B_2| \leq (C_{2}^2 m^{-m_2} |G_1|)^{\alpha_2} (C_{2}^2 m^{-m_2} |G_2|)^{\beta_2}\}
\]

\[
\mathcal{E}_{\mathcal{X}_{\mathcal{P}}^{l}}(-n-n_2, m-m_2, m_2, m_2) := \{l_2 \in \mathbb{Z} : \text{there exists } 0 \leq \alpha_1, \beta_1 \leq 1 \text{ such that } 2^{-l_2} |B_1| \geq (C_{1}^2 m^{-n_2} |F_1|)^{\alpha_1} (C_{1}^2 m^{-m_2} |F_2|)^{\beta_1}\}
\]
energy estimates are indeed not necessary. In particular,

\[ \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{i, n, n_2, -n - m, m_2} := \{ l_2 \in \mathbb{Z} : \text{ for any } 0 \leq \alpha_1, \beta_2 \leq 1, \]

\[ 2^{-l_2} \| \mathcal{B} \|_{2} \lesssim (C_2^{\tau_2} 2^{n_2} |G_1|)^{\alpha_1} (C_2^{\tau_2} 2^{m_2} |G_2|)^{\beta_2}, \]

\[ \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{i, n, n_2, -n - m, m_2} = \{ l_2 \in \mathbb{Z} : \text{ for any } 0 \leq \alpha_1, \beta_2 \leq 1, \]

\[ 2^{-l_2} \| \mathcal{B} \|_{1} \lesssim (C_1^{\tau_1} 2^{-n_2} |F_1|)^{\alpha_1} (C_1^{\tau_1} 2^{-m_2} |F_2|)^{\beta_1} \text{ and there exists } 0 \leq \alpha_2, \beta_2 \leq 1 \text{ such that} \]

\[ 2^2 \| \mathcal{B} \|_{1} \lesssim (C_2^{\tau_2} 2^{n_2} |G_1|)^{\alpha_2} (C_2^{\tau_2} 2^{m_2} |G_2|)^{\beta_2}, \]

One decomposes the sum into four parts based on the ranges specified above:

\[
\begin{aligned}
&\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathbb{Z}} 2^{-l_2} \| B \|_{1, 2} \| \mathcal{B} \|_1 \\
&\bigcup_{S \in \mathcal{S}, T \in \mathcal{Y}} \bigcup_{I \times J \in \mathcal{R}_{k_1, k_2}} I \times J \\
&= \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{1, n_2, -n - m, m_2}} + \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{2, n_2, -n - m, m_2}} + \\
&\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{3, n_2, -n - m, m_2}} + \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{4, n_2, -n - m, m_2}} .
\end{aligned}
\]

One denotes the four parts by I, II, III and IV and will derive estimates for each part separately. The multilinear form can thus be decomposed correspondingly as follows:

\[ |A_{\text{flag}^{\eta_1, \eta_2}}^{\tau_1, \tau_2}| \lesssim C_1^{\tau_1} C_2^{\tau_2} (C_3^{\tau_1, \tau_2})^2 \sum_{l_2 > 0} 2^{k_1} \| h \|_{1, 2} 2^{k_2} \cdot I + C_1^{\tau_1} C_2^{\tau_2} (C_3^{\tau_1, \tau_2})^2 \sum_{l_2 > 0} 2^{k_1} \| h \|_{1, 2} 2^{k_2} \cdot II + \\
\hline
A_{\tau_1, \tau_2}^{\tau_1, \tau_2}
\hline
C_1^{\tau_1} C_2^{\tau_2} (C_3^{\tau_1, \tau_2})^2 \sum_{l_2 > 0} 2^{k_1} \| h \|_{1, 2} 2^{k_2} \cdot III + C_1^{\tau_1} C_2^{\tau_2} (C_3^{\tau_1, \tau_2})^2 \sum_{l_2 > 0} 2^{k_1} \| h \|_{1, 2} 2^{k_2} \cdot IV .
\]

It would be sufficient to prove that each part satisfies the bound

\[ (C_1^{\tau_1} C_2^{\tau_2} C_3^{\tau_1, \tau_2})^{\eta_1} |F_1|^{\frac{1}{\eta_1}} |F_2|^{\frac{1}{\eta_1}} |G_1|^{\frac{1}{\eta_1}} |G_2|^{\frac{1}{\eta_2}} \]

for some constant \( 0 < \Theta < 100 \).

With little abuse of notation, we will simplify \( \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{i, n_2, -n - m, m_2} \) by \( \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{i} \), for \( i = 1, 2, 3, 4 \).

**Estimate of** \( \Lambda_{\tau_1, \tau_2}^{\tau_1, \tau_2} \). Though for I, the localization of energies cannot be applied at all, one observes that energy estimates are indeed not necessary. In particular,

\[
I \lesssim \sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \sum_{l_2 \in \mathcal{E}^* \mathcal{X} \mathcal{P}^d_{i, n_2, -n - m, m_2}} 2^{-l_2} \| B \|_{1, 2} \| \mathcal{B} \|_1 \bigcup_{S \in \mathcal{S}, T \in \mathcal{Y}} \bigcup_{I \times J \in \mathcal{R}_{k_1, k_2}} I \times J
\]

for any \( 0 \leq \alpha_2, \beta_2 \leq 1 \),

\[ 2^2 \| \mathcal{B} \|_{1} \lesssim (C_2^{\tau_2} 2^{n_2} |G_1|)^{\alpha_2} (C_2^{\tau_2} 2^{m_2} |G_2|)^{\beta_2} . \]
\begin{align}
(10.26) & \leq \sum_{\substack{n_2 \in \mathbb{Z} \\
m_2 \in \mathbb{Z}}}
\left( \sup_{l_2 \in \mathcal{XP}_1} 2^{-(l_2-n_2)\|B\|_1} \right) \left( \sup_{l_2 \in \mathcal{XP}_1} \sum_{T \in \mathcal{T}^{-1}_{n_2-n_2},m_2-m_2} \bigcup_{S \in \mathcal{S}_{T}^{J_{n_2},m_2}} I \times J \right) \left( \sum_{l_2 \in \mathcal{XP}_1} 2^{l_2\|\hat{B}\|_1} \right).
\end{align}

We will estimate the expressions in the parentheses separately.

(i) It is trivial from the definition of $\mathcal{XP}_1$ that for any $0 \leq \alpha_1, \beta_1 \leq 1$,
\begin{align}
\sup_{l_2 \in \mathcal{XP}_1} 2^{-(l_2-n_2)\|B\|_1} \lesssim \left( C_1^2 2^{n_2} |F_1| \right)^{\alpha_1} \left( C_1^2 2^{m_2} |F_2| \right)^{\beta_1},
\end{align}
for any $0 \leq \alpha_2, \beta_2 \leq 1$.\]

(ii) The last expression is a geometric series with the largest term bounded by
\begin{align}
(10.27) & \left( C_2^2 2^{n_2} |G_1| \right)^{\alpha_2} \left( C_2^2 2^{m_2} |G_2| \right)^{\beta_2},
\end{align}
for any $0 \leq \alpha_2, \beta_2 \leq 1$ according to the definition of $\mathcal{XP}_1$. As a result,
\begin{align}
\sup_{l_2 \in \mathcal{XP}_1} 2^{l_2\|\hat{B}\|_1} \lesssim \left( C_2^2 2^{n_2} |G_1| \right)^{\alpha_2} \left( C_2^2 2^{m_2} |G_2| \right)^{\beta_2},
\end{align}
for any $0 \leq \alpha_2, \beta_2 \leq 1$.

(iii) For any fixed $-n-n_2, m-m_2, n_2, m_2, l_2, t_1, t_2$,
\begin{align}
\{ I_T : I_T \in \mathcal{T}^{-n-n_2,-m-m_2} \quad \text{and} \quad T \in \mathcal{T}^{-1_{l_2}} \},
\end{align}
are disjoint collections of dyadic intervals. Therefore
\begin{align}
(10.28) & \sup_{l_2 \in \mathcal{XP}_1} \left( \sum_{T \in \mathcal{T}^{-1_{l_2}}} \bigcup_{S \in \mathcal{S}_{T}^{J_{n_2},m_2}} I \times J \right) \lesssim \sup_{l_2 \in \mathcal{XP}_1} \left( \bigcup_{T \in \mathcal{T}^{-1_{l_2}}} \bigcup_{S \in \mathcal{S}_{T}^{J_{n_2},m_2}} I \times J \right) \lesssim \left( \bigcup_{I \times J} I \times J \right).\end{align}

One can now plug in the estimates (10.27), (10.28) and (10.29) into (10.26) and derive that for any $0 \leq \alpha_1, \beta_1, \alpha_2, \beta_2 \leq 1$,
\begin{align}
I \lesssim (C_1^1)^2 (C_2^2)^2 \sum_{n_2 \in \mathbb{Z}} \left( 2^{-n-n_2} |F_1| \right)^{\alpha_1} \left( 2^{-m-m_2} |F_2| \right)^{\beta_1} \left( 2^{n_2} |G_1| \right)^{\alpha_2} \left( 2^{m_2} |G_2| \right)^{\beta_2} \bigcup_{I \times J} I \times J.
\end{align}

(10.30)
\begin{align}
\leq (C_1^1)^2 (C_2^2)^2 \sum_{n_2 \in \mathbb{Z}} \left( 2^{-n-n_2} |F_1| \right)^{\alpha_1} \left( 2^{-m-m_2} |F_2| \right)^{\beta_1} \left( 2^{n_2} |G_1| \right)^{\alpha_2} \left( 2^{m_2} |G_2| \right)^{\beta_2} \bigcup_{I \times J} I \times J.
\end{align}

By letting $\alpha_1 = \frac{1}{p_1}$, $\alpha_2 = \frac{1}{q_1}$, $\beta_1 = \frac{1}{p_2}$ and $\beta_2 = \frac{1}{q_2}$ and the argument for choice of indices in Chapter 6, one has
\begin{align}
I \lesssim (C_1^1)^2 (C_2^2)^2 \sum_{n_2 \in \mathbb{Z}} \left( 2^{-n-n_2} |F_1| \right)^{\alpha_1} \left( 2^{-m-m_2} |F_2| \right)^{\beta_1} \left( 2^{n_2} |G_1| \right)^{\alpha_2} \left( 2^{m_2} |G_2| \right)^{\beta_2} \bigcup_{I \times J} I \times J.
\end{align}
for any $\gamma > 1$. The estimate is a direct application of the sparsity condition described in Proposition 6.6 that has been extensively used before. One can now apply (10.31) to conclude that

$$|\Lambda_{I_1T_2}| = C_1^2 C_2^2 (C_3^{T_1 T_2})^2 \sum_{l > 0} 2^k_1 \|h\|_2 2^{k_2} \cdot I$$

$$\lesssim (C_1^{T_1} C_2^{T_2} C_3^{T_1 T_2})^6 \sum_{l > 0} 2^k_1 (1+\frac{2}{\gamma}) \|h\|_2 2^{k_2 (1+\frac{2}{\gamma})} 2^{-n} \frac{1}{\gamma} 2^{-m} \frac{1}{\gamma} |F_1| \frac{1}{\gamma} |F_2| \frac{1}{\gamma} |G_1| \frac{1}{\gamma} |G_2| \frac{1}{\gamma},$$

and achieves the desired bound with appropriate choice of $\gamma > 1$.

**Estimate of $\Lambda_{I_1T_2}$**. One first observes that the estimates for $\Lambda_{I_1T_2}$ apply to $\Lambda_{III}$ due to the symmetry. One shall notice that

$$II \leq \sum_{n_2 \in \mathbb{Z}} \left( \sum_{l_2 \in \mathcal{J}_{I_1 T_2}} 2^{l_2} \|\tilde{B}\|_1 \right) \left( \sup_{l_2 \in \mathcal{J}_{I_1 T_2}} 2^{-l_2} \|B\|_1 \sum_{l_2 \in \mathcal{J}_{I_1 T_2}} |I_{l_2}| \right) \left( \sup_{S \in \mathcal{S}_{I_1 T_2}} \sum_{J \in \mathcal{J}_{n_2 m_2}} |J_S| \right).$$

(i) The first expression is a geometric series which can be bounded by

$$\left( C_1^{2n_2} |G_1| \right)^{\alpha_2} \left( C_2^{2m_2} |G_2| \right)^{\beta_2},$$

for any $0 \leq \alpha_2, \beta_2 \leq 1$ (up to some constant as discussed in the estimate of $I$).

(ii) The second term in (10.32) can be considered as a localized $L^{1,\infty}$-energy. In addition, given by the restriction that $l_2 \in \mathcal{J}_{I_1 T_2}$, one can apply the localization and the corresponding energy estimates described in Proposition 10.6. In particular, for any $0 \leq \theta_1, \theta_2 < 1$ with $\theta_1 + \theta_2 = 1$,

$$\sup_{l_2 \in \mathcal{J}_{I_1 T_2}} 2^{-l_2} \|B\|_1 \sum_{l_2 \in \mathcal{J}_{I_1 T_2}} |I_{l_2}|$$

$$\lesssim (C_1^{T_1} 2^{-n-n_2}) \frac{1}{\gamma} - \theta_1 (C_1^{T_1} 2^{-m-m_2}) \frac{1}{\gamma} - \theta_2 |F_1| \frac{1}{\gamma} |F_2| \frac{1}{\gamma}.$$

(iii) For any fixed $n_2, m_2, l_2$ and $T_2$, $\{J_S : J_S \in \mathcal{J}_{n_2 m_2} \}$ and $S \in \mathcal{S}_{I_1 T_2}$ is a disjoint collection of dyadic intervals, which implies that

$$\sup_{l_2} \sum_{S \in \mathcal{S}_{I_1 T_2}} |J_S| \leq \sum_{J \in \mathcal{J}_{n_2 m_2}} |J_S|$$

$$\lesssim \left[ \{Mg_1 > C_1^{2n_2} 2^{-10} |G_1| \} \cap \{Mg_2 > C_2^{2m_2} 2^{-10} |G_2| \} \right].$$

By combining (10.33), (10.34) and (10.35), one can majorize (10.32) as

$$II \lesssim \sum_{n_2 \in \mathbb{Z}} \left( C_1^{2n_2} |G_1| \right)^{\alpha_2} \left( C_2^{2m_2} |G_2| \right)^{\beta_2} (C_1^{T_1} 2^{-n-n_2}) \frac{1}{\gamma} - \theta_1 (C_1^{T_1} 2^{-m-m_2}) \frac{1}{\gamma} - \theta_2 |F_1| \frac{1}{\gamma} |F_2| \frac{1}{\gamma} \cdot \left[ \{Mg_1 > C_1^{2n_2} 2^{-10} |G_1| \} \cap \{Mg_2 > C_2^{2m_2} 2^{-10} |G_2| \} \right]$$

$$\leq \sup_{n_2 \in \mathbb{Z}} \left( C_1^{T_1} 2^{-n-n_2} \right)^{\alpha_2 - (1+\epsilon) (1-\mu)} (C_2^{T_2} 2^{m_2})^{\beta_2 - (1+\epsilon) \mu} \|G_1\|^{\alpha_2 - (1+\epsilon) (1-\mu)} |G_2|^{\beta_2 - (1+\epsilon) \mu}.$$
To sum up, one has the following estimate for \( II \):

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left( C_2^{\tau_2} 2^{n_2} |G_1| \right)^{(1+\epsilon)(1-\mu)} \left( C_2^{\tau_2} 2^{m_2} |G_2| \right)^{(1+\epsilon)\mu} \{ M g_1 > C_2^{\tau_2} 2^{n_2-10} |G_1| \} \cap \{ M g_2 > C_2^{\tau_2} 2^{m_2-10} |G_2| \}.
\]

By the Hölder-type argument introduced in Chapter 7, one can estimate the expression

\[
\sum_{n_2 \in \mathbb{Z}} \sum_{m_2 \in \mathbb{Z}} \left( C_2^{\tau_2} 2^{n_2} |G_1| \right)^{(1+\epsilon)(1-\mu)} \left( C_2^{\tau_2} 2^{m_2} |G_2| \right)^{(1+\epsilon)\mu} \{ M g_1 > C_2^{\tau_2} 2^{n_2-10} |G_1| \} \cap \{ M g_2 > C_2^{\tau_2} 2^{m_2-10} |G_2| \}
\]

\[
\lesssim |G_1|^{1-\mu} |G_2|^\mu.
\]

Therefore, by plugging in (10.37) and some simplifications, (10.36) can be majorized by \( II \)

\[
\lesssim (C_1^{\tau_1} C_2^{\tau_2})^2 \sup_{n_2 \in \mathbb{Z}} \sup_{m_2 \in \mathbb{Z}} (2^{-n-n_2} \frac{1}{n_1^{\theta_1}} (2^{-m-m_2}) \frac{1}{m_1^{\theta_2}} F_1 |\nabla^m h_1| F_2 |\nabla^{m_2} h_2| \{ G_1 \} |_{\nu_1-1} \{ G_2 \} |_{\nu_2-1})^2.
\]

One would like to choose \( 0 \leq \alpha_2, \beta_2 \leq 1, 0 < \mu < 1 \) and \( \epsilon > 0 \) such that

\[
\alpha_2 - \epsilon (1 - \mu) = \frac{1}{p_2},
\]

\[
\beta_2 - \epsilon \mu = \frac{1}{q_2}.
\]

Meanwhile, one can also achieve the equalities

\[
\frac{1}{p_1} - \frac{1}{q_1} = \alpha_2 - (1 + \epsilon) (1 - \mu),
\]

\[
\frac{1}{p_2} - (1 - \theta_1) = \frac{1}{q_2} - \mu.
\]

which combined with (10.38), yield

\[
\frac{1}{p_1} - \frac{1}{q_1} = \frac{1}{p_2} - (1 - \mu),
\]

\[
\frac{1}{q_1} - (1 - \theta_1) = \frac{1}{q_2} - \mu.
\]

Thanks to the condition that

\[
\frac{1}{p_1} - \frac{1}{p_2} = \theta_1 - (1 - \mu),
\]

one only needs to choose \( 0 < \theta_1, \mu < 1 \) such that

\[
\frac{1}{p_1} - \frac{1}{p_2} = \theta_1 - (1 - \mu).
\]

To sum up, one has the following estimate for \( II \):

\[
II \lesssim (C_1^{\tau_1} C_2^{\tau_2})^2 2^{-n(n-\theta_1)} 2^{-m(m-\theta_2)} |F_1| \nabla^m h_1 |F_2| \nabla^{m_2} h_2 |G_1| \nabla^{\nu_1-1} |G_2| \nabla^{\nu_2-1}.
\]

Last but not least, one can interpolate between the estimates (10.39) and (10.25) obtained from the sparsity condition to conclude that

\[
|A_II| = C_1^{\tau_1} C_2^{\tau_2} (C_3^{-\tau_1-\tau_2})^2 \sum_{l_1 > 0} \sum_{n > 0} \sum_{m > 0} \sum_{k_1 < 0} \sum_{k_2 < 0} 2^{k_1} \|h_1\|_2 2^{k_2} \cdot II
\]

\[
\lesssim (C_1^{\tau_1} C_2^{\tau_2} C_3^{-\tau_1-\tau_2})^6 \sum_{l_1 > 0} \sum_{n > 0} \sum_{m > 0} \sum_{k_1 < 0} \sum_{k_2 < 0} 2^{k_1} \|h_1\|_2 2^{k_2} (1-\frac{1}{\nu_1}) 2^{-\lambda(1-(\frac{1}{\nu_1}) - (1 - \theta_1))} 2^{-n(1-\lambda)(\frac{1}{\nu_1} - \theta_1)} 2^{-m(1-\lambda)(\frac{1}{\nu_2} - (1 - \theta_1))}
\]
\begin{align}
& |F_1| (1-\lambda) \frac{1}{n_1} + \lambda \frac{\nu_1 (1+\nu_1)}{2} + \lambda \rho (1-\frac{1-\nu_1}{2}) |F_2| (1-\lambda) \frac{1}{n_2} + \lambda \frac{\nu_2 (1+\nu_2)}{2} + \lambda (1-\rho)(1-\frac{1-\nu_2}{2}) \\
& |G_1| (1-\lambda) \frac{1}{n_1} + \lambda \frac{\nu_1 (1+\nu_1)}{2} + \lambda \rho (1-\frac{1-\nu_1}{2}) |G_2| (1-\lambda) \frac{1}{n_2} + \lambda \frac{\nu_2 (1+\nu_2)}{2} + \lambda (1-\rho)(1-\frac{1-\nu_2}{2}).
\end{align}

(10.40)

One has enough degree of freedom to choose the indices and obtain the desired estimate:

(i) for any $0 < \lambda, \delta < 1$, the series $\sum_{l>0} 2^{-l \Delta(1-\frac{1-\nu_1}{2})}$ is convergent;

(ii) one notices that for $0 < \theta_1 < 1$, \(\sum_{n>0} 2^{-n(1-\lambda)(\frac{1}{n_1} - \theta_1)}\) and $\sum_{m>0} 2^{-m(1-\lambda)(\frac{1}{n_1} - (1-\theta_1))}$ converge if $$\frac{1}{p_1} - \theta_1 > 0, \quad \frac{1}{q_1} - (1- \theta_1) > 0,$$

which implies

$$\frac{1}{p_1} + \frac{1}{q_1} > 1.$$

This would be the condition we impose on the exponents $p_1$ and $q_1$. The proof for range $\frac{1}{p_1} + \frac{1}{q_1} \leq 1$ follows a simpler argument.

(iii) One can identify (10.40) with (7.39) and choose the indices to match the desired exponents for $|F_1|, |F_2|, |G_1|$ and $|G_2|$ in the exactly same fashion.

**Estimate of $\Lambda_{IV}^{\psi_1 \psi_2}$.** When $l_2 \in \mathcal{E} \mathcal{X} P_4$, one has the localization that the main contribution of

$$\sum_{|K| \geq |l|} \frac{1}{|K|^2} \langle f_1, \varphi_K \rangle \langle f_2, \psi_K \rangle \langle \chi_{E'}, \psi_K \rangle$$

comes from

$$\sum_{|K| \geq |l|} \frac{1}{|K|^2} \langle f_1, \varphi_K \rangle \langle f_2, \psi_K \rangle \langle \chi_{E'}, \psi_K \rangle$$

as in the Haar model. As a consequence, it is not difficult to check that the argument in Section 7 applies to the estimate of $IV$, where one employs the local energy estimates stated in Proposition 10.6 and 10.7 instead of Proposition 5.14 and derive that

\begin{align}
IV \lesssim (C_1^2 C_2^2)^2 2^{-n(\frac{1}{p_1} - \theta_1 - \frac{1}{2}(1+\epsilon))} 2^{-m(\frac{1}{p_1} - \theta_2 - \frac{1}{2}(1+\epsilon))} |F_1| \frac{1}{p_1} - \frac{1}{2} \epsilon |F_2| \frac{1}{p_1} - \frac{1}{2} \epsilon |G_1| \frac{1}{p_2} - \frac{1}{2} \epsilon |G_2| \frac{1}{p_2} - \frac{1}{2} \epsilon.
\end{align}

(10.41)

By interpolating between (10.41) and (10.24) which agree with the estimates for the nested sum using the Fubini argument and the sparsity condition developed in Section 7, one achieves the desired bound.

**Remark 10.9.** When only one of the families $(\phi_K)_{K \in \mathcal{K}}$ and $(\phi_L)_{L \in \mathcal{L}}$ is lacunary, a simplified argument is sufficient. Without loss of generality, we assume that $(\psi_K)_{K \in \mathcal{K}}$ is a lacunary family while $(\varphi_L)_{L \in \mathcal{L}}$ is a non-lacunary family. One can then split the argument into two parts depending on the range of the exponents $l_2$:

(i) $l_2 \in \{l_2 \in \mathbb{Z} : 2^{l_2} \parallel B_1 \parallel \lesssim (C_1^2 C_2^2 |G_1|)^{\alpha_2} (C_1^2 C_2^2 |G_2|)^{\beta_2}\}$

(ii) $l_2 \in \{l_2 \in \mathbb{Z} : 2^{l_2} \parallel B_1 \parallel \gg (C_1^2 C_2^2 |G_1|)^{\alpha_2} (C_1^2 C_2^2 |G_2|)^{\beta_2}\}$

where Case (i) can be treated by the same argument for $II$ and Case (ii) by the reasoning for $IV$. This completes the proof of Theorem 4.2 for $\Pi_{\text{flag}} \otimes \Pi_{\text{flag}}$ in the general case.

As commented in the beginning of this section, the argument for Theorem 4.2 and 4.3 developed in the Haar model can be generalized to the Fourier setting, which ends the proof of the main theorems.

**11. Appendix I - Multilinear Interpolations**

This chapter is devoted to various multilinear interpolations that allow one to reduce Theorem 2.1 to 2.4 (and Theorem 2.2 to 2.5 correspondingly). We will start from the statement in Theorem 2.4 and implement interpolations step by step to reach Theorem 2.1. Throughout this chapter, we will consider $T_{ab}$ as a trilinear operator with first two function spaces restricted to tensor-product spaces.
11.1. Interpolation of Multilinear forms. One may recall that Theorem 2.4 covers all the restricted weak-type estimates except for the case $2 \leq s \leq \infty$. We will apply the interpolation of multilinear forms to fill in the gap. In particular, Let $T_{ab}^*$ denote the adjoint operator of $T_{ab}$ such that

$$\langle T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h), l \rangle = \langle T_{ab}^*(f_1 \otimes g_1, f_2 \otimes g_2, l), h \rangle$$

Due to the symmetry between $T_{ab}$ and $T_{ab}^*$, one concludes that the multilinear form associated to $T_{ab}^*$ satisfies

$$|\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, l)| \lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} |H|^{\frac{1}{r}} |L|^{\frac{1}{s}}$$

for every measurable set $F_1, F_2 \subseteq \mathbb{R}_x$, $G_1, G_2 \subseteq \mathbb{R}_y$, $H, L \subseteq \mathbb{R}^2$ of positive and finite measure and every measurable function $|f_i| \leq \chi_{F_i}$, $|g_j| \leq \chi_{G_j}$, $|h| \leq \chi_H$ and $|l| \leq \chi_L$ for $i, j = 1, 2$. The notation and the range of exponents agree with the ones in Theorem 2.4. One can now apply the interpolation of multilinear forms described in Lemma 9.6 of [22] to attain the restricted weak-type estimate with $1 < s \leq \infty$:

$$(11.1) \quad |\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, l)| \lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} |H|^{\frac{1}{r}} |L|^{\frac{1}{s}}$$

where $\frac{1}{s} = 0$ if $s = \infty$. For $1 \leq s < \infty$, one can fix $f_1, g_1, f_2, g_2$ and apply linear Marcinkiewicz interpolation theorem to prove the strong-type estimates for $h \in L^s(\mathbb{R}^2)$ with $1 < s < \infty$. The next step would be to validate the same result for $h \in L^\infty$. One first rewrites the multilinear form associated to $T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h)$ as

$$\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'}) := \langle T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h), \chi_{E'} \rangle$$

$$= \langle T_{ab}^*(f_1 \otimes g_1, f_2 \otimes g_2, h), \chi_{E'} \rangle.$$

(11.2)

Let $Q_N := [-N, N]^2$ denote the cube of length $2N$ centered at the origin in $\mathbb{R}^2$, then (11.2) can be expressed as

$$\lim_{N \to \infty} \int_{Q_N} T_{ab}^*(f_1 \otimes g_1, f_2 \otimes g_2, \chi_{E'}) (x) h(x) dx$$

$$= \lim_{N \to \infty} \int_{Q_N} T_{ab}^*(f_1 \otimes g_1, f_2 \otimes g_2, \chi_{E'}) (x) (h \cdot \chi_{Q_N}) (x) dx$$

$$= \lim_{N \to \infty} \int_{Q_N} T_{ab}^*(f_1 \otimes g_1, f_2 \otimes g_2, h \cdot \chi_{Q_N}) (x) \chi_{E'} (x) dx$$

$$= \lim_{N \to \infty} \Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h \cdot \chi_{Q_N}, \chi_{E'}).$$

Let $\tilde{h} := \frac{h \chi_{Q_N}}{\|h \chi_{Q_N}\|}$, where $|\tilde{h}| \leq \chi_{Q_N}$ with $|Q_N| \leq N^2$. One can thus invoke (11.1) to conclude that

$$|\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{Q_N}, \chi_{E'})| = \|h \|_{\infty} \cdot |\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, \tilde{h}, \chi_{E'})|$$

$$\lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} \|h\|_{\infty} \|E\|^{\frac{1}{s}}.$$

As the bound for the multilinear form is independent of $N$, passing to the limit when $N \to \infty$ yields that

$$|\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'})| \lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} \|h\|_{\infty} \|E\|^{\frac{1}{s}}.$$

Combined with the statement in Theorem 2.4, one has that for any $1 < p_1, p_2, q_1, q_2 < \infty$, $1 < s \leq \infty$, $0 < r < \infty$, $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{r} = \frac{1}{s}$,

$$\lim_{N \to \infty} \int_{Q_N} \left| \Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'}) \right| dx \lesssim |F_1|^{\frac{1}{p_1}} |G_1|^{\frac{1}{q_1}} |F_2|^{\frac{1}{p_2}} |G_2|^{\frac{1}{q_2}} \|h\|_{s}$$

(11.3)

for every measurable set $F_1, F_2 \subseteq \mathbb{R}_x$, $G_1, G_2 \subseteq \mathbb{R}_y$ of positive and finite measure and every measurable function $|f_i| \leq \chi_{F_i}$, $|g_j| \leq \chi_{G_j}$ for $i, j = 1, 2$.

11.2. Tensor-type Marcinkiewicz Interpolation. The next and final step would be to attain strong-type estimates for $T_{ab}$ from (11.3). We first fix $h \in L^s$ and define

$$T^h(f_1 \otimes g_1, f_2 \otimes g_2) := T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h)$$

One can then apply the following tensor-type Marcinkiewicz interpolation theorem to each $T^h$ so that Theorem 2.1 follows.
Theorem 11.1. Let \( 1 < p_1, p_2, q_1, q_2 < \infty \) and \( 0 < t < \infty \) such that \( \frac{1}{p_1} + \frac{t}{q_1} = \frac{1}{p_2} + \frac{t}{q_2} = \frac{1}{t} \). Suppose a multilinear tensor-type operator \( T(f_1 \otimes g_1, f_2 \otimes g_2) \) satisfies the restricted weak-type estimates for any \( \tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2 \) in a neighborhood of \( p_1, p_2, q_1, q_2 \) respectively with \( \frac{1}{\tilde{p}_1} + \frac{t}{\tilde{q}_1} = \frac{1}{\tilde{p}_2} + \frac{t}{\tilde{q}_2} = \frac{1}{t} \), equivalently

\[ \| T(f_1 \otimes g_1, f_2 \otimes g_2) \|_{t, \infty} \lesssim |F_1|^\frac{1}{p_1} |G_1|^\frac{1}{q_1} |F_2|^\frac{1}{p_2} |G_2|^\frac{1}{q_2} \]

for any measurable sets \( F_1 \subseteq \mathbb{R}_1, F_2 \subseteq \mathbb{R}_2, G_1 \subseteq \mathbb{R}_3, G_2 \subseteq \mathbb{R}_4 \) of positive and finite measure and any measurable function \( |f_1(x)| \leq \chi_{F_1}(x), |f_2(x)| \leq \chi_{F_2}(x), |g_1(y)| \leq \chi_{G_1}(y), |g_2(y)| \leq \chi_{G_2}(y) \). Then \( T \) satisfies the strong-type estimate

\[ \| T(f_1 \otimes g_1, f_2 \otimes g_2) \|_t \lesssim \| f_1 \|_{p_1} \| g_1 \|_{p_2} \| f_2 \|_{q_1} \| g_2 \|_{q_2} \]

for any \( f_1 \in L^{p_1}(\mathbb{R}_1), f_2 \in L^{q_1}(\mathbb{R}_2), g_1 \in L^{p_2}(\mathbb{R}_3), g_2 \in L^{q_2}(\mathbb{R}_4) \).

Remark 11.2. The proof of the theorem resembles the argument for the multilinear Marcinkiewicz interpolation (see [1]) with small modifications.

12. Appendix II - Reduction to Model Operators

12.1. Littlewood-Paley Decomposition.

12.1.1. Set Up. Let \( \varphi \in \mathcal{S}(\mathbb{R}) \) be a Schwartz function with \( \text{supp} \varphi \subseteq [-2, 2] \) and \( \varphi(\xi) = 1 \) on \([-1, 1]\). Let

\[ \hat{\psi}(\xi) = \hat{\varphi}(\xi) - \hat{\varphi}(2\xi) \]

so that \( \text{supp} \hat{\psi} \subseteq [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] \). Now for every \( k \in \mathbb{Z} \), define

\[ \hat{\psi}_k := \hat{\psi}(2^{-k}\xi) \]

One important observation is that

\[ \sum_{k \in \mathbb{Z}} \hat{\psi}_k(\xi) = 1 \]

We will adopt the notation lacunary for \( (\psi_k)_k \) and non-lacunary for \( (\varphi_k)_k \).

12.1.2. Special Symbols. We will first focus on a special case of the symbols and the general case will be studied as an extension afterwards. Suppose that

\[ a(\xi_1, \eta_1, \eta_2) = a_1(\xi_1, \xi_2) a_2(\eta_1, \eta_2) \]

\[ b(\xi_1, \eta_1, \eta_2, \xi_3, \eta_3) = b_1(\xi_1, \xi_2, \xi_3) b_2(\eta_1, \eta_2, \eta_3) \]

where

\[ a_1(\xi_1, \xi_2) = \sum_{k_1} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_2}(\xi_2) \]

\[ b_1(\xi_1, \xi_2, \xi_3) = \sum_{k_2} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_2}(\xi_2) \hat{\phi}_{k_3}(\xi_3) \]

At least one of the families \( (\phi_{k_1}(\xi_1))_{k_1} \) and \( (\phi_{k_2}(\xi_2))_{k_2} \) is lacunary and at least one of the families \( (\phi_{k_3}(\xi_3))_{k_3} \) is lacunary. Moreover,

\[ a_2(\eta_1, \eta_2) = \sum_{j_1} \hat{\phi}_{j_1}(\eta_1) \hat{\phi}_{j_1}(\eta_2) \]

\[ b_2(\eta_1, \eta_2, \eta_3) = \sum_{j_2} \hat{\phi}_{j_2}(\eta_1) \hat{\phi}_{j_2}(\eta_2) \hat{\phi}_{j_2}(\eta_3) \]

where at least one of the families \( (\phi_{j_1}(\eta_1))_{j_1} \) and \( (\phi_{j_2}(\eta_2))_{j_2} \) is lacunary and at least one of the families \( (\phi_{j_2}(\eta_3))_{j_3} \) is lacunary. Then

\[ a_1(\xi_1, \xi_2) b_1(\xi_1, \xi_2, \xi_3) = \sum_{k_1, k_2} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_2}(\xi_2) \hat{\phi}_{k_3}(\xi_1) \hat{\phi}_{k_3}(\xi_2) \hat{\phi}_{k_3}(\xi_3) \]

\[ = \sum_{k_1 \approx k_2} + \sum_{k_1 \ll k_2} + \sum_{k_1 \gg k_2} . \]
Case $I^1$ gives rise to the symbol of paraproduct. More precisely,

$$I^1 = \sum_k \hat{\phi}_k(\xi_1)\hat{\phi}_k(\xi_2)\hat{\phi}_k(\xi_3)$$

where $\hat{\phi}_k(\xi_1) := \hat{\phi}_{k_1}(\xi_1)\hat{\phi}_{k_2}(\xi_1)$ and $\hat{\phi}_k(\xi_2) := \hat{\phi}_{k_1}(\xi_2)\hat{\phi}_{k_2}(\xi_2)$ when $k := k_1 \approx k_2$. The above expression can be completed as

$$I^1 = \sum_k \hat{\phi}_k(\xi_1)\hat{\phi}_k(\xi_2)\hat{\phi}_k(\xi_3)\hat{\phi}_k(\xi_1 + \xi_2 + \xi_3)$$

and at least two of the families $(\hat{\phi}_k(\xi_1))_k$, $(\hat{\phi}_k(\xi_2))_k$, $(\hat{\phi}_k(\xi_3))_k$, $(\hat{\phi}_k(\xi_1 + \xi_2 + \xi_3))_k$ are lacunary.

Case $II^1$ and $III^1$ can be treated similarly. In Case $II^1$, the sum is non-degenerate when $(\hat{\phi}_{k_1}(\xi_1))_{k_1}$ and $(\hat{\phi}_{k_2}(\xi_2))_{k_2}$ are non-lacunary. In particular, one has

$$II^1 = \sum_{k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1)\hat{\phi}_{k_1}(\xi_2)\hat{\phi}_{k_2}(\xi_1)\hat{\phi}_{k_2}(\xi_2)\tilde{\psi}_{k_2}(\xi_3)$$

In the case when the symbols are assumed to take the special form, the above expression can be rewritten as

$$\sum_{k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1)\hat{\phi}_{k_1}(\xi_2)\tilde{\psi}_{k_2}(\xi_3),$$

which can be “completed” as

$$(12.1) \sum_{k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1)\hat{\phi}_{k_1}(\xi_2)\hat{\phi}_{k_2}(\xi_1 + \xi_2)\tilde{\psi}_{k_2}(\xi_1 + \xi_2)\tilde{\psi}_{k_2}(\xi_1 + \xi_2 + \xi_3)$$

The exact same argument can be applied to $a_2(\eta_1, \eta_2)b_2(\eta_1, \eta_2, \eta_3)$ so that the symbol can be decomposed as

$$\sum_{j_1 \ll j_2} + \sum_{j_1 \ll j_2} + \sum_{j_1 \gg j_2}$$

where

$$I^2 = \sum_j \tilde{\phi}_j(\eta_1)\tilde{\phi}_j(\eta_2)\tilde{\phi}_j(\eta_3)\tilde{\phi}_j(\eta_1 + \eta_2 + \eta_3)$$

with at least two of the families $(\tilde{\phi}_j(\eta_1))_j$, $(\tilde{\phi}_j(\eta_2))_j$, $(\tilde{\phi}_j(\eta_3))_j$ and $(\tilde{\phi}_j(\eta_1 + \eta_2 + \eta_3))_j$ are lacunary. Case $II^2$ and $III^2$ have similar expressions, where

$$II^2 = \sum_{j_1 \ll j_2} \tilde{\phi}_{j_1}(\eta_1)\tilde{\phi}_{j_1}(\eta_2)\tilde{\phi}_{j_1}(\eta_1 + \eta_2)\tilde{\phi}_{j_2}(\eta_1 + \eta_2)\tilde{\phi}_{j_2}(\eta_3)\tilde{\phi}_{j_2}(\eta_1 + \eta_2 + \eta_3).$$

One can now combine the decompositions and analysis for $a_1, a_2, b_1$ and $b_2$ to study the original operator:

$$T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h) = T_{ab}^{II^1} + T_{ab}^{II^2} + T_{ab}^{III^1} + T_{ab}^{III^2} + T_{ab}^{III^1 + III^2} + T_{ab}^{II^1 + III^2} + T_{ab}^{II^1 + III^1} + T_{ab}^{II^1 + II^2 + II^1 + III^1 + III^2}.$$
where at least two of the families \((\phi_{k_1})_{k_1}\) are lacunary and at least two of the families \((\phi_{j_1})_{j_1}\) are lacunary.

\[
T_{ab}^{II_1II_2} = \sum_{k_1 \ll k_2, j_1 \ll j_2} \int \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\phi}_{k_2}(\xi_1 + \xi_2) \hat{\phi}_{k_2}(\xi_3) \hat{\psi}_{k_2}(\xi_1 + \xi_2 + \xi_3) \\
\quad \times \hat{\phi}_{j_1}(\eta_1) \hat{\phi}_{j_1}(\eta_2) \hat{\phi}_{j_2}(\eta_1 + \eta_2) \hat{\phi}_{j_2}(\eta_3) \hat{\psi}_{j_2}(\eta_1 + \eta_2 + \eta_3) \\
\quad \times \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{g}_1(\eta_1) \hat{g}_2(\eta_2) \hat{h}(\xi_3, \eta_3) \cdot e^{2\pi i (\xi_1 + \xi_2 + \xi_3)} e^{2\pi i (\eta_1 + \eta_2 + \eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3 \\
= \sum_{k_1 \ll k_2, j_1 \ll j_2} \left( (f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_2} \right) \left( (g_1 * \phi_{j_1})(g_2 * \phi_{j_1}) * \phi_{j_2} \right) \\
\quad \cdot (h * \psi_{k_2} \otimes \psi_{j_2}) * \psi_{k_2} \otimes \psi_{j_2},
\]

where at least two of the families \((\phi_{k_1})_{k_1}\) are lacunary and at least two of the families \((\phi_{j_1})_{j_1}\) are lacunary.

12.1.3. **General Symbols.** The extension from special symbols to general symbols can be treated as specified in Chapter 2.13 of [22]. With abuse of notations, we will proceed the discussion as in the previous section with recognition of the fact that bump functions do not necessarily equal to 1 on their supports, which prevents simple manipulation as before.

One notices that \(I^t\) generates bi-parameter paraproduct as previously. In Case \(II_1\), since \(k_1 \ll k_2, \hat{\phi}_{k_2}(\xi_1)\) and \(\hat{\phi}_{k_2}(\xi_2)\) behave like \(\hat{\phi}_{k_2}(\xi_1 + \xi_2)\). One could obtain (12.1) as a result. To make the argument rigorous, one considers the Taylor expansions

\[
\hat{\phi}_{k_2}(\xi_1) = \hat{\phi}_{k_2}(\xi_1 + \xi_2) + \sum_{l_1 > 0} \frac{\hat{\phi}^{(l_1)}(\xi_1 + \xi_2)}{l_1!} (-\xi_2)^{l_1} \\
\hat{\phi}_{k_2}(\xi_2) = \hat{\phi}_{k_2}(\xi_1 + \xi_2) + \sum_{l_2 > 0} \frac{\hat{\phi}^{(l_2)}(\xi_1 + \xi_2)}{l_2!} (-\xi_1)^{l_2}
\]

There are some abuse of notations in the sense that \(\hat{\phi}_{k_2}(\xi_1 + \xi_2)\) in both equations do not represent for the same function - they correspond to \(\hat{\phi}_{k_2}(\xi_1)\) and \(\hat{\phi}_{k_2}(\xi_2)\) respectively, and share the common feature that \((\phi_{k_2}(\xi_1))_{k_2}\) and \((\phi_{k_2}(\xi_2))_{k_2}\) are non-lacunary families of bump functions. Let \(\hat{\phi}_{k_2}(\xi_1 + \xi_2)\) denote the product of the two and one can rewrite \(II_1^t\) as

\[
\sum_{k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\phi}_{k_2}(\xi_1 + \xi_2) \hat{\psi}_{k_2}(\xi_3) + \\
\sum_{0 < l_1 + l_2 \leq M, k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \frac{\hat{\phi}^{(l_1)}(\xi_1 + \xi_2)}{l_1!} \frac{\hat{\phi}^{(l_2)}(\xi_1 + \xi_2)}{l_2!} (-\xi_1)^{l_1} (-\xi_2)^{l_2} \hat{\psi}_{k_2}(\xi_3) + \\
\sum_{l_1 + l_2 > M, k_1 \ll k_2} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \frac{\hat{\phi}^{(l_1)}(\xi_1 + \xi_2)}{l_1!} \frac{\hat{\phi}^{(l_2)}(\xi_1 + \xi_2)}{l_2!} (-\xi_1)^{l_1} (-\xi_2)^{l_2} \hat{\psi}_{k_2}(\xi_3),
\]

where \(M \gg |\alpha_1|\).

One observes that \(II_0^t\) can be “completed” to obtain (12.1) as desired.
One can simplify $I^1_{1}$ as

$$I^1_{1} = \sum_{0<l_1,l_2 \leq M} \sum_{\mu=100}^{\infty} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2}^{(l_1)}(\xi_1+\xi_2) \hat{\varphi}_{k_2}^{(l_2)}(\xi_1+\xi_2) \frac{(\xi_1+\xi_2)^{l_1}(-\xi_1)^{l_2}(-\xi_2)^{l_1}}{l!} \psi_{k_2}(\xi_3)$$

$$= \sum_{0<l_1,l_2 \leq M} \sum_{\mu=100}^{\infty} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) 2^{-k_2 l_1} \hat{\varphi}_{k_2,l_1}(\xi_1+\xi_2) 2^{-k_2 l_2} \hat{\varphi}_{k_2,l_2}(\xi_1+\xi_2) (-\xi_1)^{l_2}(-\xi_2)^{l_1} \psi_{k_2}(\xi_3)$$

$$\sim \sum_{0<l_1,l_2 \leq M} \sum_{\mu=100}^{\infty} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) 2^{-k_2 l_1} \hat{\varphi}_{k_2,l_1}(\xi_1+\xi_2) 2^{-k_2 l_2} \hat{\varphi}_{k_2,l_2}(\xi_1+\xi_2) 2^{k_1 l_1} 2^{k_1 l_2} \psi_{k_2}(\xi_3)$$

$$= \sum_{0<l_1,l_2 \leq M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2,l_1}(\xi_1+\xi_2) \hat{\varphi}_{k_2,l_2}(\xi_1+\xi_2) \psi_{k_2}(\xi_3)$$

$$= \sum_{0<l_1,l_2 \leq M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2) \psi_{k_2}(\xi_3)$$

where $\hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2)$ denotes an $L^\infty$-normalized non-lacunary bump function with Fourier support at scale $2^{k_2}$. One notices that $I^1_{1,\mu}$ has a form similar to (12.1) and can be rewritten as

$$\sum_{k_2=k_1+\mu} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2) \hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2) \psi_{k_2}(\xi_3)$$

Meanwhile,

$$I^1_{\text{rest}} = \sum_{l_1,l_2 > M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \sum_{k_2=k_1+\mu}^{\infty} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2) \psi_{k_2}(\xi_3)$$

$$\leq \sum_{\mu=100}^{\infty} 2^{-\mu M} \sum_{k_2=k_1+\mu}^{\infty} \sum_{l_1,l_2 > M} \hat{\phi}_{k_1}(\xi_1) \hat{\phi}_{k_1}(\xi_2) \hat{\varphi}_{k_2,l_1,l_2}(\xi_1+\xi_2) \psi_{k_2}(\xi_3),$$

where $m^1_\mu := I^1_{\text{rest,\mu}}$ is a Coifman-Meyer symbol satisfying

$$|\partial^{\alpha_1} m^1_\mu| \lesssim 2^{\mu|\alpha_1|} \frac{1}{|\xi_1,\xi_2|^{\alpha_1}}$$

for sufficiently many multi-indices $\alpha_1$.

Same procedure can be applied to study $a_2(\eta_1,\eta_2) b_2(\eta_1,\eta_2,\eta_3)$. One can now combine all the arguments above to decompose and study

$$T_{ab} = T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab} + T^{I_1 I_2}_{ab}$$

where each operator takes the form

$$\int_{\mathbb{R}^6} \text{symbol} \cdot \hat{f}(\xi_1) \hat{f}(\xi_2) \hat{g}(\eta_1) \hat{g}(\eta_2) \hat{h}(\xi_3,\eta_3) e^{2\pi i (\xi_1+\xi_2+\xi_3)} e^{2\pi i (\eta_1+\eta_2+\eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3$$

with the symbol for each operator specified as follows.

1. $T^{I_1 I_2}_{ab}$ is a bi-parameter paraproduct as in the special case.

2. $T^{I_1 I_2}_{ab} : (I^1_{0} + I^1_{1} + I^1_{\text{rest}}) \otimes I^2$ where the operator associated with each symbol can be written as
\(T^{II_0^1 T^2} := \sum_{k_1 \ll k_2, j_1 \ll j_2} \left( (f_1 \ast \phi_{k_1})(f_2 \ast \phi_{k_1}) \ast \phi_{k_2} \right) \left( (g_1 \ast \tilde{\phi}_j)(g_2 \ast \tilde{\phi}_j) \ast \psi_{k_2} \otimes \phi_j \right) \ast \psi_{k_2} \otimes \phi_j \)

(ii) \n
\(T^{II_1^1 T^2} := \sum_{0 \ll l_1, l_2 \leq M} \sum_{\mu=100}^\infty 2^{-\mu(l_1+l_2)}T^{II_1^1 T^2} \)

with \n
\(T^{II_1^1 T^2}_{\text{rest}} := \sum_{k_2=k_1+\mu, j_1 \ll j_2} \left( (f_1 \ast \phi_{k_1})(f_2 \ast \phi_{k_1}) \ast \tilde{\phi}_{k_2, l_1, l_2} \right) \left( (g_1 \ast \tilde{\phi}_j)(g_2 \ast \tilde{\phi}_j) \ast \psi_{k_2} \otimes \phi_j \right) \ast \psi_{k_2} \otimes \phi_j \)

(iii) \n
\(T^{II_1^1 T^2}_{\text{rest}} := \sum_{\mu=100}^\infty 2^{\mu M}T^{II_1^1 T^2}_{\text{rest}} \)

One notices that \(II_0^1 - II_0^1\) and \(T^2\) are Coifman-Meyer symbols. \(T^{II_1^1 T^2}_{\text{rest}}\) is therefore a bi-parameter paraproduct and one can apply the Coifman-Meyer theorem on paraproducts to derive the bound of type \(O(2^{l_1} 2^{l_2})\), which would suffice due to the decay factor \(2^{-\mu M}\). 

(3) \(T^{II_1^1 T^2}_0 : (II_0 + II_1^1) \otimes (II_0^1 + II_2) \)

where the operator associated with each symbol can be written as 

(i) \n
\(T^{II_1^1 T^2}_0 := \sum_{k_1 \ll k_2, j_1 \ll j_2} \left( (f_1 \ast \phi_{k_1})(f_2 \ast \phi_{k_1}) \ast \phi_{k_2} \right) \left( (g_1 \ast \phi_{j_1})(g_2 \ast \phi_{j_1}) \ast \phi_{j_2} \right) \ast \psi_{k_2} \otimes \psi_{j_2} \)

(ii) \n
\(T^{II_1^1 T^2}_0 := \sum_{0 \ll l_1, l_2 \leq M} \sum_{\mu=100}^\infty 2^{-\mu(l_1+l_2)}T^{II_1^1 T^2}_0 \)

with \n
\(T^{II_1^1 T^2}_{\text{rest}} := \sum_{k_2=k_1+\mu, j_1 \ll j_2} \left( (f_1 \ast \phi_{k_1})(f_2 \ast \phi_{k_1}) \ast \tilde{\phi}_{k_2, l_1, l_2} \right) \left( (g_1 \ast \phi_{j_1})(g_2 \ast \phi_{j_1}) \ast \phi_{j_2} \right) \ast \psi_{k_2} \otimes \psi_{j_2} \)

(iii) \n
\(T^{II_1^1 T^2}_{\text{rest}} := \sum_{\mu=100}^\infty 2^{\mu M}T^{II_1^1 T^2}_{\text{rest}} \)

where \(T^{II_1^1 T^2}_{\text{rest}} \) is a multiplier operator with the symbol 

\(m^1_\mu \otimes II_0^2 \)

which generates a model similar as \(T^{II_1^1 T^2}_0\) or, by symmetry, \(T^{II_1^1 T^2}\). 

(iv) \n
\(T^{II_1^1 T^2}_0 := \sum_{0 \ll l_1, l_2 \leq M} \sum_{\mu=100}^\infty 2^{-\mu(l_1+l_2)}2^{\mu'(l_1'+l_2')}T^{II_1^1 T^2}_0 \)

with \n
\(T^{II_1^1 T^2}_{\text{rest}} := \sum_{k_2=k_1+\mu, j_2=j_1+\mu'} \left( (f_1 \ast \phi_{k_1})(f_2 \ast \phi_{k_1}) \ast \tilde{\phi}_{k_2, l_1, l_2} \right) \left( (g_1 \ast \phi_{j_1})(g_2 \ast \phi_{j_1}) \ast \phi_{j_2} \right) \ast \psi_{k_2} \otimes \psi_{j_2} \)
\[(v)\]
\[T^{II_1}_{\text{rest}, I_1}^2 := \sum_{\mu=100}^{\infty} 2^{\mu \gamma} T^{II_1}_{\text{rest, }\mu} I_1^2 \]
where \(T^{II_1}_{\text{rest, }\mu} I_1^2\) has the symbol
\[m_1^\gamma \otimes I_1^2\]
which generates a model similar as \(T^{I_1} I_1^2\) or \(T^{III_1} I_1^2\).

\[(vi)\]
\[T^{II_1}_{\text{rest, }\text{rest}}^2 := \sum_{\mu, \mu'}^{100} 2^{\mu \gamma} 2^{\mu' \gamma} T^{II_1}_{\text{rest, }\mu} I_1^2 \]
where \(T^{II_1}_{\text{rest, }\mu} I_1^2\) is associated with the symbol
\[m_1^\gamma \otimes m_1^\gamma\]
which generates a model similar as \(T^{II_1}_{\text{rest, }\mu} I_1^2\), \(T^{III_1} I_1^2\) or \(T^{III_1} I_1^2\).

(4) \(T^{II_1} I_1^2\), \(T^{III_1} I_1^2\) and \(T^{III_1} I_1^2\) can be studied by the exact same reasoning for \(T^{II_1} I_1^2\), \(T^{III_1} I_1^2\) and \(T^{III_1} I_1^2\) by the symmetry between symbols II and III.

12.2. Discretization. With discretization procedure specified in Chapter 2.2 of [22], one can reduce the above operators into the following discrete model operators listed in Theorem (2.4):

\[T^{II_1} I_1^2 \rightarrow \Pi_{\text{flag}} \otimes \text{paraproduct}\]
\[T^{II_1}_{\mu} I_1^2 \rightarrow \Pi_{\text{flag}} \otimes \text{paraproduct}\]
\[T^{II_1} I_1^2 \rightarrow \Pi_{\text{flag}} \otimes \text{flag}\]
\[T^{II_1}_{0} I_1^2 \rightarrow \Pi_{\text{flag}} \otimes \text{flag}\]
\[T^{II_1}_{1} I_1^2, \mu' \rightarrow \Pi_{\text{flag}} \otimes \text{flag}\]

References

[1] Benea, C. and Muscalu, C. Quasi-Banach valued Inequalities via the helicoidal method, J. Funct. Anal. 273, no. 4, 1295-1353, [2017].
[2] Benea, C. and Muscalu, C. Mixed norm estimates via the helicoidal method, Preprint [2020].
[3] Bennett, J., Bez, N., Buschenhenke, S. and Flock, T. C. The nonlinear Brascamp-Lieb inequality for simple data, Preprint, arXiv:1801.05214.
[4] Bennett, J., Bez, N., Cowling, M. G. and Flock, T. C. Behaviour of the Brascamp-Lieb constant, Bull. Lond. Math. Soc. 49, no. 3, 512-518, [2017].
[5] Bennett, J., Carbery, A., Christ, M. and Tao, T. The Brascamp-Lieb inequalities: finiteness, structure and extremals, Geom. Funct. Anal. 17, 1343-1415, [2007].
[6] Bennett, J., Carbery, A., Christ, M. and Tao, T. The Brascamp-Lieb inequalities: finiteness, structure and extremals, Geom. Funct. Anal. 17, 1343-1415, [2007].
[7] Brenner, H. J. and Lieb, E. H. Best constants in Young’s inequality, its converse, and its generalization to more than three functions, Adv. Math. 20, 151-173, [1976].
[8] Carbery, A., Hänninen, T. S. and Valdimarsson, S. Multilinear duality and factorisation for Brascamp-Lieb-type inequalities with applications, arXiv:1809.02449.
[9] Chang, S.-Y. A. and Fefferman, R. Some recent developments in Fourier analysis and \(H^p\) theory on product domains, Bull. Amer. Math. Soc., vol. 12, 1-43, [1985].
[10] Christ, M. and Nagel, A. On the local and global well-posedness theory for the KP-I equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 21, 827-838, [2004].
[11] Coifman, R. R. and Meyer, Y. Operateurs multilinéaires, Hermann, Paris, [1991].
[12] Durcik, P. and Thiele, C. Singular Brascamp-Lieb inequalities with cubical structure, arXiv:1809.08688.
[13] Fefferman, C. and Stein, E. Some maximal inequalities, Amer. J. Math., vol. 93, 107-115, [1971].
[14] Germain, P., Masmoudi, N. and Shatah, J. Global Solutions for the Gravity Water Waves Equation in Dimension 3, [2009].
[15] Kato, T. and Ponce, G. Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41, 891-907, [1988].
[16] Kenig, C. On the local and global well-posedness theory for the KP-I equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 21, 827-838, [2004].
[17] Lacey, M. and Thiele, C. On Calderón’s conjecture, Ann. of Math. (2), 149(2):475-496, [1999].
[18] Miyachi, A. and Tomita, N. Estimates for trilinear flag paraproducts on \(L^{\infty}\) and Hardy spaces, Math. Z. 282, 577-613, [2016].
[19] Miyachi, A. and Tomita, N. Estimates for trilinear flag paraproducts on \(L^{\infty}\) and Hardy spaces, Math. Z. 282, 577-613, [2016].
Muscalu, C. Paraproducts with flag singularities I. A case study, Rev. Mat. Iberoamericana, vol. 23 705-742, [2007]. 1, 3

Muscalu, C., Pipher, J., Tao, T. and Thiele, C. Bi-parameter paraproducts, Acta Math., vol. 193, 269-296, [2004]. 1, 3, 28

Muscalu, C., Pipher, J., Tao, T. and Thiele, C. Multi-parameter paraproducts, Rev. Mat. Iberoamericana, pp. 963-976, [2006]. 1

Muscalu, C. and Schlag, W. Classical and Multilinear Harmonic Analysis, [2013]. 6, 8, 10, 17, 18, 28, 44, 58, 61, 64

Muscalu, C., Tao, T. and Thiele, C. Multi-linear operators given by singular multipliers, J. Amer. Math. Soc. 15, 469-496, [2002]. 1

Muscalu, C., Tao, T. and Thiele, C. $L^p$ estimates for the biest I: The Walsh case, Math. Ann. 329, 401-426, [2004]. 10

Muscalu, C., Tao, T. and Thiele, C. $L^p$ estimates for the biest II: The Fourier case, Math. Ann., 329, 427-461, [2004]. 10

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