A NOTE ON SIMPLE MODULES OVER QUASI-LOCAL RINGS.

PAULA A.A.B. CARVALHO, CHRISTIAN LOMP, AND PATRICK F. SMITH

Dedicated to John Clark

Abstract. Matlis showed that the injective hull of a simple module over a commutative Noetherian ring is Artinian. Many non-commutative Noetherian rings whose injective hulls of simple modules are locally Artinian have been extensively studied in [1–4, 7, 8, 12, 15, 17]. This property had been denoted by property (⋄). In this paper we investigate, which non-Noetherian semiprimary commutative quasi-local rings \((R, m)\) satisfy property (⋄). For quasi-local rings \((R, m)\) with \(m^3 = 0\), we prove a characterisation of this property in terms of the dual space of \(\text{Soc}(R)\). Furthermore, we show that \((R, m)\) satisfies (⋄) if and only if its associated graded ring \(\text{gr}(R)\) does.

Given a field \(F\) and vector spaces \(V\) and \(W\) and a symmetric bilinear map \(\beta : V \times V \rightarrow W\) we consider commutative quasi-local rings of the form \(F \times V \times W\), whose product is given by \((\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1\lambda_2, \lambda_1v_2 + \lambda_2v_1, \lambda_1w_2 + \lambda_2w_1 + \beta(v_1, v_2))\) in order to build new examples and to illustrate our theory. In particular we prove that any quasi-local commutative ring with radical cube-zero does not satisfy (⋄) if and only if it has a factor, whose associated graded ring is of the form \(F \times V \times F\) with \(V\) infinite dimensional and \(\beta\) non-degenerated.

1. Introduction

The structure and in particular finiteness conditions of injective hulls of simple modules have been widely studied. Rosenberg and Zelinsky’s work [16] is one of the earliest studies of finiteness conditions on the injective hull of a simple module. Matlis showed in his seminal paper [13] that any injective hull of a simple module over a commutative Noetherian module is Artinian. Jans in [11] has termed a ring \(R\) to be left co-noetherian if every simple left \(R\)-module has an Artinian injective hull. Vamos showed in [21] that a commutative ring \(R\) is co-noetherian if and only if \(R_m\) is Noetherian for any maximal ideal \(m \in \text{Max}(R)\) - generalizing in this way Matlis’ result. In connection with the Jacobson Conjecture for non-commutative Noetherian rings, Jategaonkar showed in [12] (see also [5, 18]) that the injective hulls of simple modules are locally Artinian, i.e. any finitely generated submodule is Artinian, provided the ring \(R\) is fully bounded Noetherian. We say that a ring \(R\) satisfies condition (⋄) if

\[
(\diamond)
\]

Injective hulls of simple left \(R\)-modules are locally Artinian.

In this paper we study (⋄) for, not necessarily Noetherian, quasi-local commutative rings \(R\) with maximal ideal \(m\) such that \(m^3 = 0\). A description of such rings is given in terms of the dual space of \(\text{Soc}(R)\) seen as a vector space over \(R/m\) (Theorem 12). Furthermore, we
relate property $(\diamond)$ of $(R, \mathfrak{m})$ with its associated graded ring $\text{gr}(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2$ in Corollary 14. Given a field $F$ and vector spaces $V$ and $W$ and a symmetric bilinear map $\beta : V \times V \to W$ we consider commutative quasi-local rings of the form $F \times V \times W$, whose product is given by

$$(\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1 \lambda_2, \lambda_1 v_2 + \lambda_2 v_1, \lambda_1 w_2 + \lambda_2 w_1 + \beta(v_1, v_2))$$

to build new examples and to illustrate our theory. In particular we prove in Proposition 17 that a quasi-local commutative ring with radical cube-zero does not satisfy $(\diamond)$ if and only if it has a factor whose associated graded ring is of the form $F \times V \times F$ with $V$ of infinite dimension and $\beta$ non-degenerate.

2. Preliminaries

The following Lemma shows that condition $(\diamond)$ is intrinsically linked to Krull’s intersection Theorem:

**Lemma 1.** Let $R$ be a (not necessarily commutative) ring with Jacobson radical $J$, such that finitely generated Artinian modules have finite length. If $R$ has property $(\diamond)$, then for any left ideal $I$ of $R$ one has

$$\bigcap_{n=0}^{\infty} (I + J^n) = I$$

**Proof.** Let $I$ be any left ideal of $R$. Then $R/I$ embeds into a product of cyclic modules $R/K_i$ with essential simple socle by Birkhoff’s theorem, where $I \subseteq K_i$ and the intersection $\bigcap K_i = I$. By hypothesis each of these modules $R/K_i$ is Artinian and hence has finite length. Thus there exists a number $n_i \geq 1$ such that $J^{n_i}R/K_i = 0 \iff J^{n_i} \subseteq K_i$. Hence $I + J^{n_i} \subseteq K_i$ for all $i$ and as the intersection of the $K_i$'s is $I$, we have

$$I = \bigcap_i K_i \supseteq \bigcap_i (I + J^{n_i}) \supseteq \bigcap_n (I + J^n) \supseteq I.$$

$\square$

**Remark 2.**

1. Assuming the hypotheses of Lemma 1, one can easily adapt the above proof to show that $\bigcap_{n=0}^{\infty} J^n M = 0$ for any finitely generated left $R$-module $M$. Furthermore, if $M$ is a finitely generated essential extension of a simple left $R$-module, then there exists $n > 0$ such that $J^n M = 0$.

2. Finitely generated Artinian left $R$-modules have finite length if for example $R$ is left Noetherian or if $R$ is commutative. For the later case let $M$ be an Artinian module over a commutative ring $R$ generated by $x_1, \ldots, x_k$. Then

$$R/\text{Ann}(M) \to R(x_1, \ldots, x_k) \subseteq M^k$$

is an embedding. Since $M^k$ is Artinian, $R/\text{Ann}(M)$ is Artinian and by the Hopkins-Levitzki’s Theorem $R/\text{Ann}(M)$ is Noetherian. As $M$ is finitely generated over $R/\text{Ann}(M)$, $M$ is also Noetherian, i.e. has finite length.
(3) We follow the terminology of commutative ring theory and call a commutative ring \( R \) quasi-local if it has a unique maximal ideal \( \mathfrak{m} \). A local ring is a commutative Noetherian quasi-local ring. From Lemma 7 we see that any commutative quasi-local ring \((R, \mathfrak{m})\), that satisfies \((\diamond)\), is separated in the \( \mathfrak{m} \)-adic topology. Moreover if \( \mathfrak{m}^n \) is idempotent for some \( n \geq 1 \), then \( \mathfrak{m}^n = 0 \).

Recall that a ring \( R \) with Jacobson radical \( J \) is called semilocal if \( R/J \) is semisimple. A semilocal ring with nilpotent Jacobson radical is called semiprimary. The second socle of a module \( M \) is the submodule \( \text{Soc}_2(M) \) of \( M \) with \( \text{Soc}(M/\text{Soc}(M)) = \text{Soc}_2(M)/\text{Soc}(M) \). For an ideal \( K \) of \( R \), denote by \( \text{Ann}_M(K) \) the set of elements \( m \in M \) such that \( Km = 0 \). For a ring \( R \) with \( R/J \) semisimple, it is well-known that \( \text{Ann}_R(J) = \text{Soc}(M) \) and that \( \text{Ann}_R(J^2) = \text{Soc}_2(M) \). For a left ideal \( I \) set \( I : K = \{ r \in R \mid Kr \subseteq I \} \).

**Proposition 3.** The following statements are equivalent, for a semiprimary ring \( R \) with Jacobson radical \( J \).

(a) \( R \) has property \((\diamond)\).

(b) \( \text{Soc}_2(M) \) has finite length, for any left \( R \)-module \( M \) with \( \text{Soc}(M) \) finitely generated.

(c) \( (I : J^2)/I \) is finitely generated, for any left ideal \( I \) of \( R \) with \((I : J)/I \) finitely generated.

**Proof.** Note first that since \( R \) is semiprimary, there exists \( n \geq 0 \) such that \( J^n = 0 \). Moreover since \( R \) is semilocal, \( \text{Soc}(M) = \text{Ann}_M(J) = \{ m \in M \mid Jm = 0 \} \) for any left \( R \)-module \( M \).

In particular any left \( R \)-module has an essential socle, since any left \( R \)-module \( M \) has a finite socle series:

\[
0 = J^n M \subseteq J^{n-1} M \subseteq \cdots \subseteq JM \subseteq M.
\]

(a) \( \Rightarrow \) (b) If \( R \) satisfies \((\diamond)\) then any finitely generated module with finitely generated (essential) socle is Artinian. Hence if \( \text{Soc}(M) \) is finitely generated, \( M \) must be Artinian and hence \( M/\text{Soc}(M) \) is Artinian.

(b) \( \Rightarrow \) (c) for \( M = R/I \) one has \( \text{Soc}(R/I) = (I : J)/I \) as mentioned above. Moreover, \( (I : J^2) = ((I : J) : J) \) and therefore \( (I : J^2)/(I : J) = \text{Soc}(R/(I : J)) = \text{Soc}(M/\text{Soc}(M)) \).

Thus the statement follows from (b).

(c) \( \Rightarrow \) (a) is clear since if \( I \) is a left ideal such that \( M = R/I \) is a cyclic essential extension of a simple left \( R \)-module, then \((I : J)/I \) is cyclic and by assumption \((I : J^2)/I \) is finitely generated. Hence \((I : J^2)/I \) has finite length. Applying our hypothesis to \( I' = (I : J) \), we can conclude that \((I : J^3)/I \) has finite length. Continuing we have also that \( R/I = (I : J^n)/I \) has finite length. \( \square \)

A sufficient condition for a ring to satisfy \((\diamond)\) is given by the following Lemma.

**Lemma 4.** Any ring \( R \), with \( R/\text{Soc}(R) \) Artinian, satisfies \((\diamond)\).

**Proof.** Suppose \( I \subseteq K \subseteq R \) are ideals such that \( K/I \) is a simple \( R \)-module and essential in \( R/I \). If \( \text{Soc}(R) \subseteq I \), then \( R/I \) is a factor of \( R/\text{Soc}(R) \) and hence Artinian. If \( \text{Soc}(R) \not\subseteq I \), then \((\text{Soc}(R) + I)/I \) is a semisimple submodule of \( R/I \) and hence must equal \( K/I \), i.e. \( \text{Soc}(R) + I = K \). As a quotient of \( R/\text{Soc}(R) \), the module \( R/K = R/(\text{Soc}(R) + I) \) is an Artinian and so is \( R/I \). \( \square \)
Clearly it is not necessary for a ring $R$ with $(\diamond)$ to satisfy $R/Soc(R)$ being Artinian. Moreover, Example 16 shows that there are commutative rings $R$ such that $R/Soc(R)$ satisfies $(\diamond)$, but $R$ does not.

In recent papers [2-4, 8], several non-commutative Noetherian rings have been shown to satisfy $(\diamond)$. In this note we intend to study condition $(\diamond)$ for non-Noetherian commutative rings.

3. Local-Global Argument

Jans in [11] defined a ring $R$ to be left co-noetherian if for every simple left $R$-module its injective hull is Artinian. Vamos has shown in [21] that a commutative ring $R$ is co-noetherian if and only if $R_m$ is Noetherian for all $m \in \text{Max}(R)$.

The following Lemma shows the relation between co-Noetherianess and condition $(\diamond)$ for commutative quasi-local rings. The proof follows the ideas of [20, Theorem 1.8].

**Lemma 5.** The following statements are equivalent for a commutative quasi-local ring $R$ with maximal ideal $m$.

(a) $R$ is Noetherian.
(b) $R$ is co-noetherian.
(c) $R$ satisfies $(\diamond)$ and $m/m^2$ is finitely generated.
(d) $R$ satisfies $\bigcap_{n=0}^{\infty}(I + m^n) = I$ for all ideals $I$ of $R$ and $m/m^2$ is finitely generated.

**Proof.** (a) $\iff$ (b) follows from Vamos’ result [21, Theorem 2].

(a) $\Rightarrow$ (c) is clear and (c) $\Rightarrow$ (d) follows from Lemma [1].

(d) $\Rightarrow$ (a) There exists a finitely generated ideal $B$ of $R$ such that $m = B + m^2$. Then

$$m = B + m^2 = B + (B + m^2)^2 \subseteq B + m^3 \subseteq m,$$

and in general $m = B + m^n$ for every positive integer $n$. It follows that $m = B$ and hence $m$ is finitely generated. Suppose that $R$ is not Noetherian. Let $Q$ be maximal among the ideals $C$ of $R$ such that $C$ is not finitely generated. Then $Q$ is a prime ideal of $R$ by a standard argument (see [6, Theorem 2]). Clearly $Q \neq m$. Let $p \in m$ with $p \notin Q$. By the choice of $Q$ the ideal $Q + Rp$ is finitely generated, say

$$(1) \quad Q + Rp = R(q_1 + r_1p) + \cdots + R(q_k + r_kp),$$

for some positive integer $k$, $q_i \in Q(1 \leq i \leq k)$, $r_i \in R(1 \leq i \leq k)$. Let $D = Rq_1 + \cdots + Rq_k \subseteq Q$. Let $q \in Q \setminus D$. Then by equation (1) there exist $s_1, \ldots, s_k \in R$ and $d \in D$ such that $q - d = (s_1r_1 + \cdots + s_kr_k)p \in Q$. Since $Q$ is prime and $p \notin Q$, $(s_1r_1 + \cdots + s_kr_k) \in Q$, i.e. $Q = D + Qp$. Now $Q = D + Qp^t$ for every positive integer $t$ and hence

$$Q = \bigcap_{s=1}^{\infty}(D + Qp^s) \subseteq \bigcap_{s=1}^{\infty}(D + m^s) = D.$$

It follows that $Q = D$ and hence $Q$ is finitely generated, a contradiction. Thus $R$ is Noetherian.

□
The diamond condition is equivalent to the condition that any injective hull of a simple \(R\)-module is locally Artinian, i.e., finitely generated submodules are Artinian. As seen in Remark 2(2) a commutative ring satisfies \((\diamond)\) if and only if any injective hull \(E\) of a simple \(R\)-module is locally of finite length, i.e., any finitely generated submodule of \(E\) has finite length.

The following Proposition is well-known and can be found for example in [19, Proposition 5.6]

**Proposition 6.** Let \(R\) be a commutative ring, \(\mathfrak{m}\) a maximal ideal of \(R\) and denote by \(R_{\mathfrak{m}}\) the localisation of \(R\) by \(\mathfrak{m}\). Then the injective hull \(E = E(R/\mathfrak{m})\) of \(R/\mathfrak{m}\) as \(R\)-module is also the injective hull of \(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}\) as \(R_{\mathfrak{m}}\)-module.

It would be good to have a kind of local-global argument for condition \((\diamond)\) in comparison to Vamos’ result on co-Noetherian rings. The following Proposition intends to find this kind of argument.

**Proposition 7.** Let \(R\) be a commutative ring, \(\mathfrak{m} \in \text{Max}(R)\) and \(E = E(R/\mathfrak{m})\) the injective hull of \(R/\mathfrak{m}\). The following statements are equivalent:

(a) \(E\) is locally of finite length as \(R_{\mathfrak{m}}\)-module;

(b) \(E\) is locally of finite length as \(R\)-module and the \(R\)-submodule generated by an element \(x \in E\) and the \(R_{\mathfrak{m}}\)-submodule generated by \(x\) coincide.

**Proof.** (a) \(\Rightarrow\) (b) Suppose \(E\) is locally of finite length as \(R_{\mathfrak{m}}\)-module. Let \(0 \neq x \in E\). By hypothesis \(R_{\mathfrak{m}}x\) has finite length as \(R_{\mathfrak{m}}\)-module. Hence there exist \(k > 0\) such that \(\mathfrak{m}^k x \subseteq (\mathfrak{m}R_{\mathfrak{m}})^k x = 0\). For any \(a \in R \setminus \mathfrak{m}\) we have \(R = Ra + \mathfrak{m}\), which implies also \(R = Ra + \mathfrak{m}^k\). Thus there exists \(b \in R\) such that \(1 - ab \in \mathfrak{m}^k\), i.e., \(x = abx\). Hence \(a^{-1}x = bx \in Rx\) shows that \(R_{\mathfrak{m}}x = Rx\). Hence the \(R\)-submodule generated by any set of elements of \(E\) coincides with the \(R_{\mathfrak{m}}\)-submodule generated by that set. In particular any finitely generated \(R\)-submodule of \(E\) is also a finitely generated \(R_{\mathfrak{m}}\)-submodule of \(E\), which has finite length.

(b) \(\Rightarrow\) (a) The condition that \(R\)-submodules and \(R_{\mathfrak{m}}\)-submodules generated by a set coincide means that the lattice of submodules of \(RE\) and \(R_{\mathfrak{m}}E\) are identical. Hence \(E\) is locally of finite length as \(R\)-module implies also that it is locally of finite length as \(R_{\mathfrak{m}}\)-module.

**Corollary 8.** The following statements are equivalent for a commutative ring \(R\).

(a) \(R_{\mathfrak{m}}\) satisfies \((\diamond)\) for all \(\mathfrak{m} \in \text{Max}(R)\).

(b) \(R\) satisfies \((\diamond)\) and the \(R\)-submodule generated by an element \(x \in E(R/\mathfrak{m})\) and the \(R_{\mathfrak{m}}\)-submodule generated by \(x\) coincide.

**Question:** Does there exist a commutative ring \(R\) that satisfies \((\diamond)\), but \(R_{\mathfrak{m}}\) does not satisfy \((\diamond)\), for some \(\mathfrak{m} \in \text{Max}(R)\)?

4. **Commutative semiprimary quasi-local rings**

A quasi-local commutative ring is co-noetherian if and only if it is Noetherian. Recall, that an ideal \(I\) of a ring \(R\) is called **subdirectly irreducible** if \(R/I\) has an essential simple
4.1. Commutative quasi-local rings with square-zero maximal ideal. Given any vector space $V$ over a field $F$, the trivial extension (or idealization) is defined on the vector space $R = F \times V$ with multiplication given by $(a, v)(b, w) = (ab, aw + vb)$, for all $a, b \in F$ and $v, w \in V$. Any such trivial extension $R$ is a commutative quasi-local ring that satisfies $\Diamond$. However $R$ is Noetherian if and only if $V$ is finite dimensional.

Lemma 9. Any commutative quasi-local ring with square-zero maximal ideal satisfies $\Diamond$.

Proof. Let $(R, m)$ be a commutative quasi-local ring with $m^2 = 0$. Since $m$ is a vector space over $R/m$ it is semisimple. Let $K$ be any subdirectly irreducible ideal of $R$. If $K = m$, then $R/m$ is simple. So assume $K \subset m$. Then there exists a complement $L$ such that $m = L \oplus K$ and $\text{Soc}(R/K) = m/K \simeq L$ is simple.

$$0 \longrightarrow m/K \longrightarrow R/K \longrightarrow R/m \longrightarrow 0$$

is a short exact sequence. Hence $R/K$ has length 2. \qed

4.2. Commutative quasi-local rings with cube-zero maximal ideal. In this section we will characterise commutative quasi-local rings $(R, m)$ with $m^3 = 0$ satisfying $\Diamond$. Recall that $\text{Soc}(R) = \text{Ann}(m) = \{r \in R : rm = 0\}$. Hence $m^2 \subseteq \text{Soc}(R)$. We start with a simple observation.

Lemma 10. Let $(R, m)$ be a commutative quasi-local ring with $m^3 = 0$.

1. If $m/\text{Soc}(R)$ is finitely generated, then $R$ satisfies $\Diamond$.
2. If $\text{Soc}(R)$ is finitely generated, then $R$ satisfies $\Diamond$ if and only if $m/\text{Soc}(R)$ is finitely generated.

Proof. (1) If $m/\text{Soc}(R)$ is finitely generated, then $R/\text{Soc}(R)$ is Artinian and by Lemma 9, $R$ satisfies $\Diamond$.

(2) If $\text{Soc}(R)$ is finitely generated, then by Proposition 3 $R$ satisfies $\Diamond$ if and only if $\text{Soc}_2(R) = m$ is Artinian if and only if $m/\text{Soc}(R)$ is finitely generated. \qed

The last Lemma raises the question, whether the converse of (1) holds? That is, whether $m/\text{Soc}(R)$ needs to be finitely generated for a commutative quasi-local ring $R$ with $m^3 = 0$ and satisfying property $\Diamond$? As we will see in Example 18, this need not be the case.

Lemma 11. Let $(R, m)$ be a commutative quasi-local ring with residue field $F = R/m$. Suppose $m^3 = 0$. Then there exists a correspondence between subdirectly irreducible ideals of $R$ that do not contain $\text{Soc}(R)$ and non-zero linear maps $f : \text{Soc}(R) \to F$. Each corresponding pair $(I, f)$ satisfies

$$\text{Soc}(R) + I = V_f := \{a \in m \mid f(ma) = 0\}.$$

Proof. Let $I$ be a subdirectly irreducible ideal that does not contain $\text{Soc}(R)$, then $\text{Soc}(R/I) = (\text{Soc}(R) + I)/I$ is simple. Thus $\text{Soc}(R) = Fx \oplus (\text{Soc}(R) \cap I)$, for a non-zero element
contradicting essentiality. Hence

$$x \in \text{Soc}(R).$$

Let \( f : \text{Soc}(R) \to F \) be the linear map such that \( f|_{\text{Soc}(R) \cap I} = 0 \) and \( f(x) = 1. \) Clearly \( \text{Soc}(R) + I \subseteq V_f, \) because \( m(\text{Soc}(R) + I) = mI \subseteq m^2 \cap I \subseteq \text{Soc}(R) \cap I. \)

To show that \( V_f = \text{Soc}(R) + I \) we use the essentiality of \( \text{Soc}(R/I) = (\text{Soc}(R) + I)/I \) in \( R/I. \) For any \( a \in V_f \setminus I, \) there exists \( r \in R \) such that \( ra + I \) is a non-zero element of \( \text{Soc}(R/I) = (\text{Soc}(R) + I)/I. \) Note that \( r \notin m \) since otherwise \( f(ra) = 0 \) and hence \( ra \in \text{Ker}(f) \subseteq I. \) Therefore \( r \) is invertible and \( a + I = r^{-1}ra + I \in \text{Soc}(R/I), \) i.e. \( V_f = \text{Soc}(R) + I. \)

On the contrary, let \( f \) be any non-zero element \( f \in \text{Hom}_F(\text{Soc}(R), F). \) Then there exists a non-zero element \( x \in \text{Soc}(R) \) with \( f(x) = 1 \) and hence \( \text{Soc}(R) = Fx \oplus \text{Ker}(f). \) Let \( I \) be an ideal of \( R \) that contains \( \text{Ker}(f) \) and that is maximal with respect to \( x \notin I. \) Thus \( I \) is subdirectly irreducible and \( \text{Soc}(R/I) = \text{Soc}(R) + I = (Fx \oplus I)/I \) is simple and essential in \( R/I. \) By construction \( \text{Ker}(f) = I \cap \text{Soc}(R). \)

Note that \( m(\text{Soc}(R) + I) = mI \subseteq m^2 \cap I \subseteq \text{Soc}(R) \cap I = \text{Ker}(f), \) i.e. \( \text{Soc}(R) + I \subseteq V_f. \) To show the converse, let \( a \in V_f \setminus I, \) then by essentiality there exists \( r \in R \) with \( ra = x + y \in \text{Soc}(R) + I \) and \( y \in I \) and \( ra \notin I. \) If \( r \in m, \) then \( ra \in \text{Ker}(f) \subseteq I, \) contradicting essentiality. Hence \( r \notin m \) and \( a \in \text{Soc}(R) + I, \) i.e. \( V_f = \text{Soc}(R) + I. \)

**Theorem 12.** Let \( (R, m) \) be a commutative quasi-local ring with residue field \( F \) and \( m^3 = 0. \) Then \( R \) satisfies \((\diamond)\) if and only if \( m/V_f \) is finite dimensional for any \( f \in \text{Hom}_F(\text{Soc}(R), F). \)

**Proof.** Suppose that \( R \) satisfies \((\diamond)\) and let \( f \in \text{Hom}(\text{Soc}(R), F). \) If \( f = 0, \) then \( V_f = m \) and \( m/V_f \) has dimension zero. If \( f \neq 0, \) then by Lemma [11] there exists a subdirectly irreducible ideal \( I \) with \( V_f = \text{Soc}(R) + I \) and \( I \) not containing \( \text{Soc}(R). \) As \( R \) satisfies \((\diamond)\), \( R/I \) is Artinian and as a subquotient \( m/V_f = (m/I)/(V_f/I) \) is also Artinian. As \( R \)-module \( m/V_f \) must be finite dimensional.

Suppose \( m/V_f \) is finite dimensional for any \( f \in \text{Hom}(\text{Soc}(R), F). \) Let \( I \) be a subdirectly irreducible ideal of \( R. \) If \( \text{Soc}(R) \subseteq I, \) then \( R/I \) is an \( R/m^2 \)-module. Since \( R/m^2 \) is a quasi-local ring with square-zero radical, we have by Lemma [9] that \( R/m^2 \) satisfies \((\diamond)\). Hence \( R/I \) must be Artinian. If \( \text{Soc}(R) \not\subseteq I, \) then by Lemma [11] there exists a non-zero map \( f : \text{Soc}(R) \to F \) such that \( \text{Soc}(R) + I = V_f. \) By hypothesis \( m/V_f \) is finite dimensional and is therefore Artinian as \( R \)-module. As \( R/m \) and \( V_f/I \) are simple modules, also \( R/I \) is Artinian, proving that \( R/I \) is Artinian for any subdirectly irreducible ideal \( I \) of \( R, \) i.e. \( R \) satisfies \((\diamond)\). \( \square \)

Let \( (R, m) \) be any commutative quasi-local ring. The **associated graded ring** of \( R \) with respect to the \( m \)-filtration is the commutative ring \( \text{gr}(R) = \bigoplus_{n \geq 0} m^n/m^{n+1} \) with multiplication given by

\[(a + m^{i+1})(b + m^{j+1}) = ab + m^{i+j-1}, \quad \forall a \in m^i, b \in m^j, \text{ and } i, j \geq 0.\]

For any ideal \( I \) of \( R, \) the associated graded ideal is \( \text{gr}(I) = \bigoplus_{n \geq 0} (I \cap m^n + m^{n+1})/m^{n+1}. \) In particular \( \text{gr}(m) = \bigoplus_{n \geq 1} m^n/m^{n+1} \) is the unique maximal ideal of \( \text{gr}(R). \) Hence \((\text{gr}(R), \text{gr}(m))\) is a commutative quasi-local ring with residue field \( F = R/m. \) Furthermore, \( \text{gr}(\text{Soc}(R)) \) is contained in \( \text{Soc}(\text{gr}(R)). \)
Lemma 13. Let \( R, m \) be a commutative quasi-local ring with residue field \( F \) and \( m^3 = 0 \). For any \( f \in \text{Hom}_F(\text{gr}(\text{Soc}(R)), F) \) there exists \( g \in \text{Hom}_F(\text{Soc}(R), F) \) such that

\[
V_f = \{ a \in \text{gr}(m) : f(\text{gr}(m)a) = 0 \} = \text{gr}(V_g),
\]

where \( V_g \) is the ideal of \( R \) defined by Lemma 11.

Proof. Let \( f : \text{gr}(\text{Soc}(R)) \rightarrow F \) and denote by \( \pi : \text{Soc}(R) \rightarrow m^2 \) the projection onto \( m^2 \), since \( m^2 \) is a direct summand of \( \text{Soc}(R) \). Define \( g : \text{Soc}(R) \rightarrow F \) by \( g(a) = f(0, a, \pi(a)) \) for all \( a \in \text{Soc}(R) \). Then \( (0, x, y) \in V_f \) if and only if \( f(0, 0, tx) = 0 \), for all \( t \in m \). Since \( tx = \pi(tx) \in m^2 \), the later is equivalent to \( g(tx) = 0 \) for all \( t \in m \), i.e. \( x \in V_g \) (in the ring \( R \)). Hence \( V_f = \{ (0, x, y) \in \text{gr}(m) : x \in V_g, y \in m^2 \} = \text{gr}(V_g) \). \( \square \)

Corollary 14. Let \( R, m \) be a quasi-local ring with \( m^3 = 0 \). Then \( R \) satisfies \( (\diamond) \) if and only if its associated graded ring \( \text{gr}(R) \) does.

Proof. If \( R \) satisfies \( (\diamond) \) and \( f : \text{Soc}\,(\text{gr}(R)) \rightarrow F \) is a non-zero map, then by Lemma 13 there exists \( g : \text{Soc}(R) \rightarrow F \) such that \( V_f = \text{gr}(V_g) \). Since \( V_g \) contains \( \text{Soc}(R) \) and hence \( m^2 \), we have \( V_f = \text{gr}(V_g) = 0 \times V_g / m^2 \times m^2 \). Thus, \( \text{gr}(m)/V_f \simeq m/V_g \). By Theorem 12, \( m/V_g \) is finite dimensional as \( R \) satisfies \( (\diamond) \). Hence \( \text{gr}(m)/V_f \) is finite dimensional for all \( f \in \text{Soc}(\text{gr}(R))^* \). Again by Theorem 12, \( \text{gr}(R) \) satisfies \( (\diamond) \).

Let \( f : \text{Soc}(R) \rightarrow F \) be any non-zero linear map and let \( V_f = \{ a \in m : f(ma) = 0 \} \). If \( m^2 \subseteq \ker(f) \), then \( V_f = m \). If \( m^2 \nsubseteq \ker(f) \), then there exists \( x \in m^2 \) with \( f(x) = 1 \). Note that \( V_f \subseteq \ker(f) \), hence \( I = 0 \times V_f / m^2 \times \ker(f) \) is a subdirectly irreducible ideal of \( \text{gr}(R) \). To see this note that \( E = 0 \times 0 \times Fx \) is a simple submodule of \( \text{gr}(R)/I \simeq F \times m/V_f \times Fx \). We will show that \( E \) is essential in \( \text{gr}(R)/I \). Let \( (0, \overline{a}, \overline{b}) \in \text{gr}(R)/I \). If \( a \notin V_f \), there exists \( c \in m \) such that \( f(ac) \neq 0 \) and \( ac - f(ac)x \in \ker(f) \). Hence \( (0, \overline{a}, \overline{b})(0, \overline{c}, 0) = (0, 0, f(ac)x) \in E \) is non-zero. If \( a \in V_f \), i.e. \( \overline{a} = 0 \), and \( \overline{b} \neq 0 \), then \( (0, 0, \overline{b}) \) is a non-zero element of \( E \). Hence \( E \) is an essential simple submodule of \( \text{gr}(R)/I \) and if \( \text{gr}(R) \) satisfies \( (\diamond) \), the quotient \( \text{gr}(R)/I \) and therefore also \( m/V_f \) must be Artinian, thus finite dimensional. By Theorem 12, \( R \) satisfies \( (\diamond) \). \( \square \)

5. Examples

Let \( R, m \) be a commutative quasi-local ring with \( m^3 = 0 \). The associated graded ring \( \text{gr}(R) \) is of the form \( \text{gr}(R) = F \oplus V \oplus W \) where \( F = R/m \) and \( V = m/m^2 \) and \( W = m^2 \) are spaces over \( F \). Moreover, the multiplication of \( R \) induces a symmetric bilinear map \( \beta : V \times V \rightarrow W \). Hence \( \text{gr}(R) \) is uniquely determined by \( (F, V, W, \beta) \) and its multiplication.
can be identified with the multiplication of a generalized matrix ring. Writing the elements of \( S = F \times V \times W \) as 3-tuples \((\lambda, v, w)\) we have that the multiplication is given by

\[
(\lambda_1, v_1, w_1)(\lambda_2, v_2, w_2) = (\lambda_1\lambda_2, \lambda_1v_2 + \lambda_2v_1, \lambda_1w_2 + \lambda_2w_1 + \beta(v_1, v_2)).
\]

The units are precisely the elements \((\lambda, v, w)\) with \(\lambda \neq 0\) and the unique maximal ideal of \( S \) is given by \( \text{Jac}(S) = 0 \times V \times W \). Let

\[
V_{\beta}^+ = \{a \in V \mid \beta(V, a) = 0\},
\]

then \( \text{Soc}(S) = 0 \times V_{\beta}^+ \times W \), while \( \text{Jac}(S)^2 = 0 \times 0 \times \text{Im}(\beta) \).

Recall, that \( \beta \) is called non-degenerate or non-singular if \( V_{\beta}^+ = 0 \). In general \( \beta \) need not be non-degenerate:

**Example 15.** Let \( F \) be any field and \( V \) be any vector space over \( F \) with countably infinite basis \( \{v_0, v_1, v_2, \ldots\} \). Define a symmetric bilinear form as \( \beta : V \times V \to F \) with \( \beta(v_0, v_0) = 1 \) and \( \beta(v_i, v_j) = 0 \) for any \((i, j) \neq (0, 0)\). Then \( S = F \times V \times F \) is a commutative quasi-local ring that satisfies \((\phi)\), because \( V_{\beta}^+ = \text{span}\langle v_i \mid i > 0 \rangle \) and hence

\[
\text{Jac}(S)/\text{Soc}(S) = (0 \times V \times W)/(0 \times V_{\beta}^+ \times F) \cong F
\]

is one-dimensional. By Lemma 10, \( S \) satisfies \((\phi)\). Note that \( S \) is not Artinian. Moreover, \( S = \text{gr}(R) \), where \( R = F[x_0, x_1, x_2, \ldots]/(x_0^{3}, x_i x_j \mid (i, j) \neq (0, 0)) \).

The bilinear form of the last example was not non-degenerate. Since \( 0 \times V_{\beta}^+ \times 0 \) is always an ideal, we can pass to \( F \times V/V_{\beta}^+ \times W \) where the bilinear form \( \beta \) is now non-degenerate. The following is a natural example of such a ring with non-degenerate bilinear form:

**Example 16.** Let \( F = \mathbb{R} \) and let \( V = C([0, 1]) \) be the space of continuous real valued functions on \([0, 1]\). Set

\[
\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \quad \forall f, g \in C([0, 1]).
\]

Then \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) is a non-degenerate symmetric bilinear form on \( V \). Hence, by Lemma 10, \( R = \mathbb{R} \times V \times \mathbb{R} \) is a commutative quasi-local ring with cube-zero radical, that does not satisfy \((\phi)\), because its socle \( \text{Soc}(R) = 0 \times 0 \times \mathbb{R} \) is one-dimensional, hence finitely generated, but \( \text{Jac}(R)/\text{Soc}(R) \) is infinite dimensional, hence not finitely generated as \( R \)-module.

These kind of rings must occur as the associated graded ring of a quotient of a commutative quasi-local ring \((R, m)\) with \( m^3 = 0 \) that does not satisfy \((\phi)\).

**Proposition 17.** A commutative quasi-local ring \((R, m)\) with \( m^3 = 0 \) and residue field \( F \) does not satisfy \((\phi)\) if and only if it has a factor \( R/I \) whose associated graded ring \( \text{gr}(R/I) \) is of the form \( F \times V \times F \) with \( \dim(V) = \infty \) and non-degenerate bilinear form \( \beta : V \times V \to F \).
Proof. By Theorem 12 $R$ does not satisfy $(\circ)$ if and only if there exists $f \in \text{Soc}(R)^*$ such that $m/V_f$ has infinite dimension. Note that $f \neq 0$ since $V_f \neq m$. By Lemma 11 there exists a subdirectly irreducible ideal $I$ of $R$ such that $V_f = \text{Soc}(R) + I$. In particular $(m/I)^2 = V_f/I = \text{Soc}(R/I) \simeq F$, because if $m^2 \subseteq I$, then $m/I \subseteq \text{Soc}(R/I) = V_f/I$ and hence $m = V_f$, contradicting $V_f \neq m$. Hence $m^2 \not\subseteq I$ and $(m/I)^2 = \text{Soc}(R/I)$ as $R/I$ has a simple socle. Moreover, $\text{gr}(R/I) = F \times m/V_f \times F$, with bilinear form $\beta : m/V_f \times m/V_f \to F$, which is non-degenerate by the definition of $V_f$.

On the other hand if $R$ has a factor $R/I$ whose associated graded ring $\text{gr}(R/I)$ is of the form $F \times V \times F$ with infinite dimensional vector space $V$ and non-degenerate bilinear form $\beta : V \times V \to F$, then $\text{Soc}(\text{gr}(R/I)) = 0 \times 0 \times F$ is a simple submodule of $\text{gr}(R/I)$, which is essential since $\beta$ is non-degenerate. As $V$ is infinite dimensional, the semisimple $\text{gr}(R/I)$-module $(0 \times V \times F)/(0 \times 0 \times F)$ is not artinian and hence $\text{gr}(R/I)$ does not satisfy $(\circ)$. By Corollary 14 $R/I$ does not satisfy $(\circ)$ and therefore also $R$ does not. \qed

Let $V = A$ be any unital commutative $F$-algebra. Consider the multiplication of $A$ as a symmetric non-degenerate bilinear map $\mu : A \times A \to A$ and form the ring $S = F \times A \times A$ as before. In order to apply Theorem 12 recall that $m = 0 \times A \times A$ and $\text{Soc}(S) = 0 \times 0 \times A$, as the multiplication of $A$ is non-degenerate. Hence elements of $\text{Soc}(S)^*$ can be identified with elements of $A^*$. For any $f \in A^*$ we defined

$$V_f = \{(0, a, b) \in m \mid f(Aa) = 0\} = 0 \times I(f) \times A,$$

where $I(f)$ is the largest ideal of $A$ that is contained in $\text{Ker}(f)$. Theorem 12 says that $S$ satisfies $(\circ)$ if and only if $m/V_f \simeq A/I(f)$ is finite dimensional for any $f \in A^*$. From the theory of coalgebras, we borrow the notion of the finite dual $A^\circ$ of an algebra, which is the subspace of $A^*$ consisting of the elements $f \in A^*$ that contain an ideal of finite codimension in their kernel. Hence $S$ satisfies $(\circ)$ if and only if $A^\circ = A^*$.

Example 18. The trivial extension $A = F \times V$ of a vector space $V$ (see [2, 7]) is an example of an algebra $A$ satisfying $A^\circ = A^*$. To see this, note that for any linear subspace $U$ of $A$, $U \cap V$ is an ideal of $A$. Thus, if $f \in A^*$, then $\text{Ker}(f) \cap V$ is an ideal of codimension less or equal to 2 and $f \in A^\circ$. In particular, for such $A$, $S = F \times A \times A$ satisfies $(\circ)$. However if $V$ is infinite dimensional, then $m/\text{Soc}(S) \simeq V$ is not finitely generated as $S$-module, which shows that the converse of Lemma 10(1) does not hold.

However, it might happen that the kernel of an element $f \in A^*$ does not contain an ideal of finite codimension as the following example shows.

Example 19. Let $A$ be a commutative unital $F$-algebra with multiplication $\mu$ and $f \in A^*$. Note that the composition $\beta = f \circ \mu$ is a non-degenerate bilinear form if and only if $f(Aa) \neq 0$, for all non-zero $a \in A$. Or, in other words, $\beta$ is non-degenerate if and only if $\text{Ker}(f)$ does not contain any non-zero ideal of $A$. Such a map $f$ can be constructed in case $F$ has characteristic zero and $A$ has a countably infinite multiplicative basis $\{b_n\}_{n \in \mathbb{N}}$, i.e. $b_nb_m = b_{n+m}$ for all $n, m \in \mathbb{N}$. This implies in particular that $b_0 = 1$. Suppose $f : A \to F$ is a linear map and let $0 \neq a \in A$ such that $f(Aa) = 0$. For $a = \sum_{i=0}^n \lambda_i b_i \in A$ and any
m \geq 0$, we have
\[ f(b_m a) = \sum_{i=0}^{n} \lambda_i f(b_{m+i}). \]

Thus \( v = (\lambda_0, \lambda_1, \ldots, \lambda_n) \) is in the kernel of the linear map given by the matrix:
\[
B_n = \begin{pmatrix}
  f(b_0) & f(b_1) & f(b_2) & \cdots & f(b_n) \\
  f(b_1) & f(b_2) & f(b_3) & \cdots & f(b_{n+1}) \\
  f(b_2) & f(b_3) & f(b_4) & \cdots & f(b_{n+2}) \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  f(b_n) & f(b_{n+1}) & f(b_{n+2}) & \cdots & f(b_{2n+1})
\end{pmatrix}
\]

In particular \( \det(B_n) = 0 \). Hence if the sequence \( (f(b_n))_{n \in \mathbb{N}} \) produces a sequence of matrices \( (B_n) \) that have all non-zero determinant, then for each \( 0 \neq a = \sum_{i=0}^{n} \lambda_i b_i \in A \), there exists \( 0 \leq m \leq n \) such that \( f(b_m a) \neq 0 \), i.e. \( \beta = f \circ \beta \) is non-degenerate.

Matrices of the form of \( B_n \) are called Toeplitz or Hankel matrices. A particular example of such a matrix is the Hilbert matrix, which is the matrix
\[
B_{n-1} = \begin{pmatrix}
  1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\
  \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\
  \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1}
\end{pmatrix}
\]
in case \( F \) has characteristic 0. In 1894, Hilbert computed that \( \det(B_{n-1}) = \frac{c_n^4}{c_{2n}} \), where \( c_n = \prod_{i=1}^{n-1} i^{n-i} \) (see [9]). Hence if we define \( f(b_n) = \frac{1}{n+1} \) for any \( n \geq 0 \), then the kernel of \( f \) does not contain any non-zero ideal and the bilinear form \( \beta = f \circ \mu \) is non-degenerate. An algebra with a multiplicative basis as above is for example the polynomial algebra \( A = F[x] \).

Thus \( S = F \times F[x] \times F[x] \) does not satisfy \( (\diamond) \).

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