Interdimensional degeneracies
for a quantum $N$-body system in $D$ dimensions

Xiao-Yan Gu * and Zhong-Qi Ma †

CCAST (World Laboratory), P.O.Box 8730, Beijing 100080, China

and Institute of High Energy Physics, Beijing 100039, China

Jian-Qiang Sun ‡

Institute of High Energy Physics, Beijing 100039, China

Complete spectrum of exact interdimensional degeneracies for a quantum $N$-body system in $D$-dimensions is presented by the method of generalized spherical harmonic polynomials. In an $N$-body system all the states with angular momentum $[\mu + n]$ in $(D - 2n)$ dimensions are degenerate where $[\mu]$ and $D$ are given and $n$ is an arbitrary integer if the representation $[\mu + n]$ exists for the SO($D - 2n$) group and $D - 2n \geq N$. There is an exceptional interdimensional degeneracy for an $N$-body system between the state with zero angular momentum in $D = N - 1$ dimensions and the state with zero angular momentum in $D = N + 1$ dimensions.

For a quantum few-body system in $D$ dimensions, one of the characteristic features is the presence of exact interdimensional degeneracies. Perhaps first noticed by Van Vleck [1], an isomorphism exists between angular momentum $l$ and dimension $D$ such that each unit increment in $l$ is equivalent to two-unit increment in $D$ for any central force problem in $D$ dimensions. For a two-body system (e.g. one-electron atom) states related by the dimensional link $D, l \leftrightarrow (D - 2), (l + 1)$ are exactly degenerate [2,3]. For three-body system (e.g. two-electron atom) Herrick and Stillinger found exact

*Electronic address: guxy@mail.ihep.ac.cn
†Electronic address: mazq@sun.ihep.ac.cn
‡Electronic address: sunjq@mail.ihep.ac.cn
interdimensional degeneracies between the states $^{1,3}P^e$ and $^{1,3}D^o$ in $D = 3$ and the states $^{3,1}S^e$ and $^{3,1}P^o$ in $D = 5$, respectively [3]. For a four-body system (e.g. three-electron atom) Herrick [2] found an exceptional interdimensional degeneracy that the triply excited $2p^3 \, ^4S$ fermion state of the lithium atom is exactly degenerate with the spinless boson $1s^3$ ground state for $D = 5$ ($D = 5$ was misprinted as $D = 3$ in Ref. [2]). In 1961 Schwartz [4] proved by the recursion relation that for a three-body system in three-dimensional space, any angular momentum state can be expanded in a complete set of the independent bases whose number is finite. Recently, by the method of the generalized Schwartz expansion [5,6], Dunn and Watson showed some exact interdimensional degeneracies of two-electron system in an arbitrary $D$-dimensional space [7,8]. To our knowledge, no theoretical method has yet dealt with interdimensional degeneracies when $N > 3$.

Recently, we proved the Schwartz expansion again by the method of generalized spherical harmonic polynomials, and presented a new development for separating completely the global rotational degrees of freedom from the internal ones for the $N$-body Schrödinger equation in three-dimensional space [9] as well as in $D$ dimensions [10,11]. We found a complete set of base functions for angular momentum in the system. Any wave function with a given angular momentum can be expanded with respect to them where the coefficients, called the generalized radial functions, depend only upon the internal variables. The generalized radial equations satisfied by the generalized radial functions are derived from the Schrödinger equation without any approximation [9]. The exact interdimensional degeneracies in a three-body system [12] were obtained directly from the generalized radial equations. In this Letter we study interdimensional degeneracies for an $N$-body system in $D$-dimensional space.

For a quantum $N$-body system in an arbitrary $D$-dimensional space, we denote the position vectors and the masses of $N$ particles by $r_k$ and by $m_k$, $k = 1, 2, \ldots, N$, respectively. $M = \sum_k m_k$ is the total mass. The Schrödinger equation for the $N$-body system with a spherically symmetric potential $V$ is

$$-\frac{1}{2} \sum_{k=1}^{N} m_k^{-1} \nabla_{r_k}^2 \Psi + V \Psi = E \Psi, \quad (1)$$

where $\nabla_{r_k}^2$ is the Laplace operator with respect to the position vector $r_k$. For simplicity,
the natural units $\hbar = c = 1$ are employed throughout this Letter. Replace the position vectors $r_k$ with the Jacobi coordinate vectors $R_j$:

$$R_0 = M^{-1/2} \sum_{k=1}^{N} m_k r_k, \quad R_j = \left( \frac{m_{j+1}M_j}{M_{j+1}} \right)^{1/2} \left( r_{j+1} - \sum_{k=1}^{j} \frac{m_k r_k}{M_j} \right),$$

$$1 \leq j \leq (N-1), \quad M_j = \sum_{k=1}^{j} m_k, \quad M_N = M,$$

where $R_0$ describes the position of the center of mass, $R_1$ describes the mass-weighted separation from the second particle to the first particle, $R_2$ describes the mass-weighted separation from the third particle to the center of mass of the first two particles, and so on. In the center-of-mass frame, $R_0 = 0$, the $N$-body Schrödinger equation reduces to a differential equation with respect to $(N-1)$ Jacobi coordinate vectors $R_j$:

$$\nabla^2 \Psi_{M}^{[\mu]}(R_1, \ldots, R_{N-1}) = -2 \left\{ E - V(\xi) \right\} \Psi_{M}^{[\mu]}(R_1, \ldots, R_{N-1}),$$

$$\nabla^2 = \sum_{j=1}^{N-1} \nabla^2_{R_j},$$

where $[\mu]$ stands for the angular momentum as discussed later.

In a $D$-dimensional space it needs $(D-1)$ vectors to determine the body-fixed frame. When $D \geq N$, all Jacobi coordinate vectors are used to determine the body-fixed frame, and all internal variables can be chosen as

$$\xi_{jk} = R_j \cdot R_k, \quad 1 \leq j \leq k \leq N - 1.$$  (4)

We call the set of internal variables (4) the first set. The numbers of the rotational variables and the internal variables are $(N-1)(2D-N)/2$ and $N(N-1)/2$, respectively. When $D < N$, only $(D-1)$ Jacobi coordinate vectors are involved to determine the body-fixed frame, and the first set of internal variables is not complete because it could not distinguish two configurations with different directions of, say $R_D$ reflecting to the superplane spanned by the first $(D-1)$ Jacobi coordinate vectors. In this case we need to use the second set of internal variables:

$$\xi_{jk} = R_j \cdot R_k, \quad \zeta_\alpha = \sum_{a_1 \ldots a_D} \epsilon_{a_1 \ldots a_D} R_{a_1 a_2} \ldots R_{(D-1)a_{D-1}} R_{a a_D},$$

$$1 \leq j \leq D - 1, \quad j \leq k \leq N - 1, \quad D \leq \alpha \leq N - 1.$$  (5)
The numbers of the rotational variables and the internal variables are $D(D - 1)/2$ and $D(2N - D - 1)/2$, respectively.

For an $N$-body system in $D$-dimensions, the angular momentum is described by an irreducible representation of $SO(D)$. When $D \geq N$ the irreducible representation is denoted by an $(N - 1)$-row Young pattern $[\mu] \equiv [\mu_1, \mu_2, \ldots, \mu_{N-1}]$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{N-1}$. Due to the traceless condition, the representation $[\mu]$ of $SO(D)$ exists only if the sum of boxes of the first two columns on the left of the Young pattern $[\mu]$ is not larger than $D$. Some selfdual representations, antiselfdual ones, and the equivalent ones may occur when $N \leq D \leq 2(N - 1)$. They only change the explicit forms of the base functions. The reader is suggested to refer our previous paper for detail [11].

Due to the rotational symmetry, one only needs to discuss the eigenfunctions of angular momentum with the highest weight. The independent base function for the angular momentum $[\mu]$ with the highest weight is $Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})$ where $(q)$ contains $(N-1)(N-2)/2$ parameters $q_{jk}, 1 \leq k \leq j \leq N-2$, and determines a standard Young tableau. A Young tableau is obtained by filling the digits 1, 2, ..., $N - 1$ arbitrarily into a given $(N - 1)$-row Young pattern $[\mu]$. A Young tableau is called standard if the digit in every column of the tableau increases downwards and the digit in every row does not decrease from left to right. The parameter $q_{jk}$ denotes the number of the digit "$j$" in the $k$th row of the standard Young tableau. $q_{jk}$ should satisfy the following constraints:

$$
\sum_{j=k}^{r} q_{jk} \leq \sum_{j=k-1}^{r-1} q_{j(k-1)}, \quad \mu_{k+1} \leq \sum_{j=k}^{N-2} q_{jk} \leq \mu_k, \quad 1 \leq k \leq N-2, \quad k \leq r \leq N-2. \quad (6)
$$

The number of the independent base functions $Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})$ is equal to the dimension $d_{[\mu]}[SU(N-1)]$ of the irreducible representation $[\mu]$ of the $SU(N-1)$ group.

The explicit form of $Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})$ for the given standard Young tableau $(q)$ is very easy to write. In the Young tableau, in correspondence to each column with the length $t$, filled by digits $j_1 < j_2 < \ldots < j_t$, $Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})$ contains a determinant as a factor. The $r$th row and $s$th column in the determinant is $R_{j_r(2s-1)} + iR_{j_r(2s)}$, where $R_{ja}$ is the $a$th component of $R_j$, if $D > 2(N - 1)$. $Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})$ also contains a numerical coefficient for convenience. When $N \leq D \leq 2(N - 1)$, the explicit form of
\(Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})\) is a little bit changed [11], but it will not affect the generalized radial equations as well as the interdimensional degeneracies. When \(D < N\), only the first \((D - 1)\) Jacobi coordinate vectors are involved in the base functions \(Q_{(q)}^{[\mu]}(R_1, \ldots, R_{D-1})\), which are the same as those for smaller \(N = D\). \(Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})\) is a homogeneous polynomial of degrees \(\sum_k q_{jk}\) and \(\sum_j \mu_j - \sum_{jk} q_{jk}\) with respect to the components of respectively the Jacobi coordinate vectors \(R_j\) and \(R_{N-1}\), and satisfies the generalized Laplace equations

\[
\nabla_{R_j} \cdot \nabla_{R_k} Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1}) = 0, \quad 1 \leq j \leq k \leq N - 1.
\]

There is a one-to-one correspondence between base functions \(Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})\) and \(Q_{(q')}^{[\mu+n]}(R_1, \ldots, R_{N-1})\), where \([\mu] \equiv [\mu_1, \ldots, \mu_{N-1}], [\mu+n] \equiv [\mu_1+n, \ldots, \mu_{N-1}+n]\), and \(q'_{jk} = q_{jk} + n\delta_{jk}\). As a matter of fact, each standard Young tableau for \([\mu+n]\) can be obtained from a corresponding standard Young tableau for \([\mu]\) by adhering from its left \(n\) columns with \(N - 1\) rows where the boxes in the \(j\)th row are filled with \(j\), \(1 \leq j \leq N - 1\). From viewpoint of group theory, two representation \([\mu]\) and \([\mu+n]\) of \(SU(N-1)\) are equivalent to each other and their dimensions \(d_{[\mu+n]}[SU(N-1)] = d_{[\mu]}[SU(N-1)]\).

When \(D \geq N\), any wave function \(\Psi_{M}^{[\mu]}(R_1, \ldots, R_{N-1})\) with the given angular momentum \([\mu]\) can be expanded with respect to the complete and independent base functions \(Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1})\)

\[
\Psi_{M}^{[\mu]}(R_1, \ldots, R_{N-1}) = \sum_{(q)} \psi_{(q)}^{[\mu]}(\xi) Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1}). \tag{8}
\]

The coefficients \(\psi_{(q)}^{[\mu]}(\xi)\), called the generalized radial functions, only depends upon the internal variables. When \(D < N\), \(\psi_{(q)}^{[\mu]}(\xi)\) and \(Q_{(q')}^{[\mu]}(R_1, \ldots, R_{N-1})\) in Eq. (8) have to be replaced with \(\psi_{(q)}^{[\mu]}(\xi, \zeta)\) and \(Q_{(q')}^{[\mu]}(R_1, \ldots, R_{D-1})\), respectively. Substituting Eq. (8) into the \(N\)-body Schrödinger equation (3), one is able to obtain the generalized radial equations. The main calculation is to apply the Laplace operator to the wave function \(\Psi_{M}^{[\mu]}(R_1, \ldots, R_{N-1})\). The calculation consists of three parts. The first part is to apply the Laplace operator to the generalized radial functions \(\psi_{(q)}^{[\mu]}(\xi)\) which can be calculated by replacement of variables. When \(D \geq N\), we have

\[
\nabla^2 \psi_{(q)}^{[\mu]}(\xi) = \sum_{j=1}^{N-1} \left[ 4\xi_{jj} \partial_{\xi_{jj}}^2 + 2D \partial_{\xi_{jj}} \right]
\]
\[ \sum_{j=1}^{N-1} \sum_{k=j+1}^{N-1} \left[ (\xi_{jj} + \xi_{kk}) \partial_{\xi_{jk}}^2 + 4\xi_{jk} \left( \partial_{\xi_{jj}} + \partial_{\xi_{kk}} \right) \partial_{\xi_{jk}} \right] \]
\[ + 2 \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \sum_{j=1}^{N-1} \sum_{k=1}^{N-1} \xi_{kj} \partial_{\xi_{jk}} \partial_{\xi_{jt}} \right) \psi_{(q)}^{[\mu]}(\xi), \]

where \( \xi_{jk} = \xi_{kj} \) and \( \partial \xi \) denotes \( \partial / \partial \xi \) and so on. The second part is to apply the Laplace operator to the generalized spherical harmonic polynomials \( Q_{(q)}^{[\mu]}(R_1, \ldots, R_{N-1}) \), which is vanishing due to Eq. (7). The third part is the mixed application:

\[ 2 \sum_{j=1}^{N-1} \left\{ \left( \partial_{\xi_{jj}} \psi_{(q)}^{[\mu]} \right) 2R_j + \sum_{j\neq k=1}^{N-1} \left( \partial_{\xi_{jk}} \psi_{(q)}^{[\mu]} \right) R_k \right\} \cdot \nabla R_j Q_{(q)}^{[\mu]}(\xi). \]

The second term is invariant under transformation \( \mu_j \rightarrow \mu_j + n \) and \( q_{jk} \rightarrow q_{jk} + n\delta_{jk} \).

The first term is equal to

\[ Q_{(q)}^{[\mu]} \left\{ 4 \sum_{j=1}^{N-2} \left( \sum_{k=1}^{j} q_{jk} \right) \partial_{\xi_{jj}} \psi_{(q)}^{[\mu]}(\xi) + 4 \left( \sum_{j=1}^{N-1} \mu_j - \sum_{j=1}^{N-2} \sum_{k=1}^{j} q_{jk} \right) \partial_{\xi,(N-1)(N-1)} \psi_{(q)}^{[\mu]}(\xi) \right\}. \]

Under the above transformation it produces the additional terms to the generalized radial equations

\[ \sum_{j=1}^{N-1} 4n\partial_{\xi_{jj}} \psi_{(q)}^{[\mu]}(\xi), \]

which exactly cancel with the additional term from Eq. (9) if \( D \) is replaced with \( D-2n \) at the same time.

From the above proof we come to the conclusion for the complete spectrum of the exact interdimensional degeneracies for an arbitrary \( N \)-body system with a spherically symmetric potential that all the states in the system with the angular momentum \([\mu + n]\) in \( (D - 2n) \) dimensions are degenerate where \([\mu]\) and \( D \) are given and \( n \) is an arbitrary integer if the representation \([\mu + n]\) exists for the \( \text{SO}(D-2n) \) group and \( D - 2n \geq N \), because those states are described by the wave functions with the same number of the generalized radial functions depending upon the same set of internal variables and satisfying the same generalized radial equations.

Now, we turn to discuss the case of \( D < N \). The base functions and the internal variables for \( D < N \) depend upon \( D \) and are very different to those for \( D \geq N \) \([9,11]\) so that, generally speaking, there is no interdimensional degeneracy when \( D < N \) but only one exception when \( D = N - 1 \).
When $D = N - 1$, $\zeta_D$ in the second set of internal variables happens to be proportional to $Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1})$ with $\mu_j = 1$, $q_{kt} = \delta_{kt}$, $1 \leq j \leq D$, and $1 \leq t \leq k \leq D - 1$. We denote this $Q^{[\mu]}_{(q)}(R_1, \ldots, R_{N-1})$ as $Q_0$. Note that $Q_0$ corresponds to a standard Young pattern of one column with $D$ rows describing the identity representation of $SO(D)$. Due to the traceless condition the Young pattern for $Q_0$ is the only Young pattern with $D$ rows for $SO(D)$. $\zeta_D^2$ can be expressed by the first set of internal variables. If a wave function with zero angular momentum for an $N$-body system in $D = N - 1$ dimensions can be expressed as a product of $\zeta_D$ and a function $f(\xi_{jk}, \zeta_D^2)$, we can rewrite it as a product of a base function $Q_0$ and a generalized radial function depending upon the first set of internal variables. Thus, we compare this state with the state of zero angular momentum in $D = N + 1$ dimensions, the number of the generalized radial function (one), the internal variables (the first set), and the generalized radial equation (see the proof for the cases of $D \geq N$) are all the same, respectively. Therefore, we obtain an exceptional interdimensional degeneracy between these two states.

In this Letter we have provided a systematic procedure for analysis of observed degeneracies among different states in different dimensions and yielded considerable insight into the energy spectra of an $N$-body system. Since the generalized radial equations for a quantum $N$-body system in an arbitrary $D$-dimensional space with a spherically symmetric potential $V$ are derived without any approximation [11], the interdimensional degeneracies given here are exact and general.

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