Trivariate gamma regression

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Abstract. Regression analysis is a method for determining a causal relationship between the response and predictor variables. The regression model has been developed in various ways, one of them is based on the distribution of the response variables. In this study, the response variables follow trivariate gamma distribution, such that the regression model developed is Trivariate Gamma Regression (TGR). The purposes of this study are to obtain the parameter estimators, test statistics, and hypothesis testing on parameters are significance (overall and partial) of the TGR model. The parameter estimators are obtained using the Maximum Likelihood Estimation (MLE). The overall test for the model’s significance is using Maximum Likelihood Ratio Test (MLRT), and the partial test is using the Z test. Based on the results of this study, it can be inferred that the parameter estimators obtained from the MLE are not closed form. Hence a numerical method is needed. In this study, the algorithm of numerical optimization used is BFGS quasi-Newton.

1. Introduction

The normal distribution has a broad application in various fields, but in reality, there are situations where the results of events indicate a distribution that is not symmetrical or does not show a symmetrical tendency. In such cases, the normal distribution model cannot provide the right results if used. For events that the results show a distribution that has a significant variation in the size of the inclination, the gamma distribution is one alternative that is widely used.

Gamma distribution is one continuous distribution and generalization form of exponential distribution [1]. Gamma distribution comes from the gamma function which has been widely known and studied in various fields. In [2] conducted a study of the estimation of the bivariate gamma distribution parameters using the MLE and moments. In parameters estimation of the bivariate gamma distribution, the MLE is more frequently used than the moment. The results of this study informed that MLE is more efficient than moments. The difference between the two methods is very significant for a high correlation coefficient ($r$).

One statistical analysis that can be used for data follows gamma distribution is regression analysis. Regression analysis used is generally a linear regression analysis that requires the fulfillment of several standard assumptions, one of which is the error follow the normal distribution. The normal error assumption is right for most types of data because most phenomena that occur spread normally. However, for certain cases, the pattern of error distribution is not able to fulfill this assumption because it forms a pattern. The regression that does not meet normal assumptions, because there is a nonlinear relationship between the response variables to one or more parameters that follow a particular functional form is called nonlinear regression [3].

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One form of nonlinear regression is gamma regression that explains the relationship between one or more predictor variables with response variables that follow a gamma distribution. The most commonly used method to obtain the parameter estimators of the gamma regression model is MLE. However, if the solution is not closed-form, then a numerical method is needed to obtain the parameter estimators.

Some researchers who examined gamma regression, including [4], conducted a study of the maximum likelihood estimator in the gamma regression model. This research considered a nonlinear regression model, in which the dependent variable has the gamma distribution. The model considered that the shape parameter of the random variable is the sum of continuous and algebraically independent functions. This research proves that there is exactly one maximum likelihood estimator for the gamma regression model.

In [5] conducted a maximum penalized likelihood estimation in a gamma-frailety model. That research shows how maximum penalized likelihood estimation can be applied to the nonparametric estimation of a continuous hazard function in a shared gamma-frailety model using right-censored and left-truncated data.

Based on several theories above, the researcher developed a statistical approach about continuous response variables’ data follows trivariate gamma distribution through TGR. The problem to be discussed is how to construct the TGR model and obtain the parameter estimators, then construct test statistics, parameters hypothesis testing of the TGR model simultaneously using MLRT and partially using the Z test. For further research plans, this method will be applied to the case studies of the Human Development Index (HDI) in Java.

2. Material and Methodology

The gamma distribution function was firstly introduced by Swiss Mathematician, Leonhard Euler (1707-1783). The gamma distribution is an exponential distribution family. The following are explanations of the gamma distribution, gamma regression, parameter estimation, and parameter hypothesis testing of the gamma regression model.

2.1. Gamma Distribution

The random variable \( Y \) is univariate gamma distribution with one parameter (\( \alpha \) or shape parameter), denoted by \( Y \sim \text{Gamma}(\alpha) \), if \( Y \) has a probability density function (pdf) as follows [6]:

\[
f(y|\alpha) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y} & ; \alpha > 0, 0 < y < \infty \\
0 & ; \text{otherwise}
\end{cases}
\]  

(1)

The random variable \( Y \) is univariate gamma distribution with two parameters (\( \alpha, \gamma \) or scale parameter), denoted by \( Y \sim \text{Gamma}(\alpha,\gamma) \), if \( Y \) has the pdf as follows [7]:

\[
f(y|\alpha,\gamma) = \begin{cases} 
\frac{1}{\gamma^n \Gamma(\alpha)} y^{\alpha-1} e^{-y/\gamma} & ; \alpha,\gamma > 0, 0 < y < \infty \\
0 & ; \text{otherwise}
\end{cases}
\]  

(2)

The random variable \( Y \) is univariate gamma distribution with three parameters (\( \alpha, \gamma, \lambda \) or location parameter), denoted by \( Y \sim \text{Gamma}(\alpha,\gamma,\lambda) \), if \( Y \) has the pdf as follows [8]:

\[
f(y|\alpha,\gamma,\lambda) = \begin{cases} 
\frac{1}{\gamma^n \Gamma(\alpha)} (y\cdot\lambda)^{\alpha-1} e^{-y/\gamma} & ; \alpha,\gamma,\lambda > 0, y > \lambda \\
0 & ; \text{otherwise}
\end{cases}
\]  

(3)
Based on the pdf of the random variable $Y$ is univariate gamma distribution with three parameters, as shown in (3), the pdf can be formed for random variables $Y_1, Y_2,$ and $Y_3$ with trivariate gamma distribution with three parameters as follows [9]:

$$f(y_1, y_2, y_3) = \frac{\gamma^{-\alpha}}{\prod_{i=1}^{3} \Gamma(\alpha_i)} \left(\frac{y_i}{\gamma}\right)^{\alpha_i-1}e^{-y_i/\gamma}$$  \hspace{1cm} (4)

with $\alpha_i > 0, \gamma > 0, \lambda_i \in R, y_1 > \lambda_1, y_2 > \lambda_2, y_3 > \lambda_3$, $\alpha_3 = \alpha_1 + \alpha_2 + \alpha_3$, otherwise $f(y_1, y_2, y_3) = 0$.

2.2. Gamma Regression

The univariate gamma regression model means that the single response variable follows a gamma distribution. The matrix form of the response variables, predictor variables, and univariate gamma regression parameters are denoted as follows:

$$y = [y_1, y_2, \ldots, y_i]^T, x = [1, X_1, X_2, \ldots, X_f]_T^T$$
$$\beta = [\beta_0, \beta_1, \beta_2, \ldots, \beta_h]_h^T$$

with $l$ is the number of observations, $l = 1, 2, \ldots, n$, and $f$ is the number of predictor variables, $f = 1, 2, \ldots, h$.

The univariate gamma regression model with two parameters ($\alpha$ and $\gamma$) is:

$$E(Y) = \alpha\gamma = e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_h x_h}$$  \hspace{1cm} (5)

Based on (5), the parameter $\gamma$ is formulated as:

$$\gamma = \frac{e^{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_h x_h}}{\alpha}$$  \hspace{1cm} (6)

By substituting the formula $\gamma$ in (6) into (2), a pdf for $y_i$ is formulated as:

$$f(y_i) = \frac{1}{\frac{\gamma}{\alpha}} \left(\frac{y_i^{\alpha-1}e^{-y_i/\gamma}}{\Gamma(\alpha)}\right); \alpha > 0, 0 < y < \infty.$$  \hspace{1cm} (7)

2.3. Parameter Estimation of the Gamma Regression Model

Parameter estimators of the gamma regression model are obtained using the MLE as one of the parameter estimation methods that maximize the likelihood function. Flexibility in determining the likelihood function makes it easy to estimate parameters using the MLE.

The steps for determining the parameter estimators of the univariate gamma regression model are:

1. Specifying the univariate gamma regression model as shown in (5) and the pdf in (7)
2. Having $n$ random samples, with $l$ denote the number of observations, $l = 1, 2, \ldots, n$
3. Constructing the likelihood function based on the pdf of univariate gamma in (7)

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{1}{\frac{\gamma}{\alpha} \Gamma(\alpha)} y_i^{\alpha-1}e^{-y_i/\gamma} = \left[\frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{n} y_i^{\alpha-1}e^{-y_i/\gamma}\right) \left(\prod_{i=1}^{n} y_i^{\alpha-1}e^{-y_i/\gamma}\right)\right]$$  \hspace{1cm} (8)

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(y_i) = \prod_{i=1}^{n} \frac{1}{\frac{\gamma}{\alpha} \Gamma(\alpha)} y_i^{\alpha-1}e^{-y_i/\gamma} = \left[\frac{1}{\Gamma(\alpha)} \left(\sum_{i=1}^{n} y_i^{\alpha-1}e^{-y_i/\gamma}\right) \left(\prod_{i=1}^{n} y_i^{\alpha-1}e^{-y_i/\gamma}\right)\right]$$  \hspace{1cm} (8)
4. Operating natural log function to the likelihood in (8) to construct the ln-likelihood function

\[ \ln L(\alpha, \beta) = \ln \left( \frac{1}{\Gamma(\alpha)} \left( \frac{\sum_{i=1}^{n} x_i^\beta}{\alpha} \right)^{n} \left( \prod_{i=1}^{n} x_i^{\alpha-1} \right) \right) \]

\[ = -n \ln \Gamma(\alpha) - a \sum_{i=1}^{n} x_i^\beta + a(n \ln \alpha + (\alpha - 1) \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} x_i^{\alpha} \frac{x_i^\beta}{\alpha} \]  

(9)

5. Constructing the first derivative of the ln-likelihood function in (9) for each parameter. The parameters are \( \alpha \) and \( \beta \).

The maximum value of \( L(\alpha, \beta) \) will be achieved if: \( \frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = 0 \) and \( \frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = 0 \).

The first derivative of the ln-likelihood function with respect to \( \alpha \) and \( \beta \) parameters, respectively, are:

\[ \frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = -n \Gamma'(\alpha) - \sum_{i=1}^{n} x_i^\beta + n \ln(\alpha) + n + \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} x_i^{\alpha} \frac{x_i^\beta}{\alpha} = 0 \] 

(10)

\[ \frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = -a \sum_{i=1}^{n} x_i^\alpha + \alpha^{-1} \sum_{i=1}^{n} x_i^\beta y_i e^{-\frac{x_i^\beta}{\alpha}} = 0 \] 

(11)

The solutions in (10) and (11) are not closed form, so a numerical method is needed to obtain the solution, i.e. parameter estimate \( \hat{\alpha} \) and \( \hat{\beta} \). One of the numerical methods that can be employed is BFGS quasi-Newton formulated as:

\[ \hat{\theta}_{p+1} = \hat{\theta}_p + \lambda_p k_p, \quad p = 0, 1, 2, \ldots \]

(12)

with \( \lambda_p = \frac{g^T(\hat{\theta}_p)k_p}{k_p^T k_p} \) and \( k_p = -H^{-1}(\hat{\theta}_p)g(\hat{\theta}_p) \). \( \hat{\theta}^{(0)} = \left[ \hat{\alpha}^{(0)} \beta^{(0)} \right]^T \), and \( \beta^{(0)} \) are obtained using Ordinary Least Square (OLS).

\[ H(\hat{\theta}_p) = \begin{bmatrix} \frac{\partial^2 \ln L(\alpha, \beta)}{\partial \alpha^2} & \frac{\partial^2 \ln L(\alpha, \beta)}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L(\alpha, \beta)}{\partial \beta^2} & \frac{\partial^2 \ln L(\alpha, \beta)}{\partial \beta^2} \end{bmatrix} \]

and

\[ g(\hat{\theta}_p) = \begin{bmatrix} \frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} \\ \frac{\partial \ln L(\alpha, \beta)}{\partial \beta} \end{bmatrix} \]

6. Starting the BFGS quasi-Newton iteration using (12). The iteration stops at the \( p \)-th iteration if satisfy \( \|g(\hat{\theta}_p)\| < \varepsilon \). If the iteration stops, the parameter estimates \( \hat{\alpha} \) and \( \hat{\beta} \) are obtained.

2.4. Parameter Hypothesis Testing of Gamma Regression Model

The parameter hypothesis testing aims to determine whether or not the influence of predictor variables on response variables both simultaneously and partially. The simultaneous test of parameter hypothesis can be done using MLRT. The hypotheses used are:

\( H_0: \beta_1 = \beta_2 = \cdots = \beta_f = \cdots = \beta_h = 0 \)

\( H_1: \) At least one \( \beta_f \neq 0 \); \( f = 1, 2, \ldots, h \)
Let $\Omega$ be a set of parameters under the population with $\Omega = \{\alpha, \beta_0, \beta_1, \ldots, \beta_h\}$, and $\omega$ be a set of parameters under the null-hypothesis with $\omega = \{\alpha, \beta_0\}$. Next is to maximize the likelihood function under the population and under the null-hypothesis. The function $L(\Omega)$ is the maximum likelihood value for the complete model, and the function $L(\omega)$ is the maximum likelihood value for the model without including the predictor variables. Next, calculating the odds ratio ($\Lambda$) and test statistics ($G^2$) using (13) and (14).

$$\Lambda = \frac{L(\omega)}{L(\Omega)}$$

(13)

The test statistics is:

$$G^2 = -2 \ln \Lambda = -2\ln(L(\omega) - L(\Omega))$$

(14)

Reject $H_0$ if $G^2 > \chi^2_{\alpha, df}$ with $df = n(\Omega) - n(\omega)$, $n(\Omega)$ is total of parameters under the population, and $n(\omega)$ is total of parameters under the null-hypothesis.

The partial hypothesis testing can be done using the $Z$ test. The hypotheses used are:

$H_0: \beta_f = 0$

$H_1: \beta_f \neq 0, f = 0, 1, 2, \ldots, h$

The test statistics is:

$$Z = \frac{\hat{\beta}_f}{SE(\hat{\beta}_f)} \text{ with } SE(\hat{\beta}_f) = \sqrt{\text{var}(\hat{\beta}_f)}$$

(15)

Reject $H_0$ if $|Z| > Z_{\alpha/2}$.

2.5. Procedure

The steps to determine the parameter estimators of the TGR model using the MLE and the hypothesis testing of the TGR model parameters simultaneously and partially are:

a. Determining the parameter estimators of the TGR model using the MLE
   a.1 Specifying the TGR model
   a.2 Having $n$ random samples, with $l$ denotes the number of observations, $l = 1, 2, \ldots, n$
   a.3 Constructing the likelihood function of the TGR model
   a.4 Operating natural log function to the likelihood in a.3 to construct the ln-likelihood function
   a.5 Constructing the first derivative of the ln-likelihood function in a.4 for each parameter, then equal to zero
   a.6 - If the results of step a.5 are closed form, the parameter estimators are obtained
   - If the results of step a.5 are not closed form, a numerical method is needed to obtain the parameter estimators

b. Determining the simultaneous test statistics of the TGR model using MLRT
   b.1 Determining the null-hypothesis and the alternative hypothesis
   b.2 Specifying the set of parameters under the population ($\Omega$)
   b.3 Specifying the set of parameters under the null-hypothesis ($\omega$)
   b.4 Constructing the likelihood function for the set of parameters under the population
   b.5 Constructing the likelihood function for the set of parameters under the null-hypothesis
   b.6 Operating $L(\Omega) = \max_{\Omega} L(\Omega)$ and $L(\omega) = \max_{\omega} L(\omega)$
   b.7 Calculating the odds ratio value using the formula in (13)
      Reject $H_0$ if $\Lambda < \Lambda_0$, $0 < \Lambda_0 < 1$
b.8 Determining the test statistics ($G^2$) using the formula in (14)
b.9 Determining the distribution of $G^2$
b.10 Determining the area of rejection
b.11 Determining the conclusion
c. Determining the partial test statistics of the TGR model using $Z$ test
c.1 Determining the null-hypothesis and the alternative hypothesis
c.2 Calculating the test statistics ($Z$) using the formula in (15)
c.3 Determining the area of rejection
c.4 Determining the conclusion

### 3. Results and Discussions

Known $Y_{11}, Y_{21}, \ldots, Y_{i1}$, $Y_{12}, Y_{22}, \ldots, Y_{i2}$, $Y_{13}, Y_{23}, \ldots, Y_{i3}$, $Y_{i3}$, $Y_{i3}$, $Y_{i3}$, $Y_{i3}$ are the response variables data follows trivariate gamma distribution, $X_{11}, X_{21}, \ldots, X_{i1}, \ldots, X_{n1}, X_{12}, X_{22}, \ldots, X_{n2}, X_{i1n}, X_{i2n}, \ldots, X_{in}, \ldots, X_{inh}$, $X_{inh}$ are the predictor variables; $n$ is the number of observations with $i=1,2,\ldots,n$; $k$ is the number of response variables with $i=1,2,\ldots,k$; and $h$ is the number of predictor variables with $i=1,2,\ldots,h$. For response variables follows trivariate gamma distribution can be denoted by $(Y_i, Y_{i2}, Y_{i3}) \sim \text{Trivariate Gamma} \left( \alpha_i, \gamma, \lambda_i \right)$.

#### 3.1. Construct the TGR Model

The TGR model can be stated in the following equation.

$$E(Y_i) = \gamma \alpha_i + \lambda_i = e^{x_i \beta_i}$$

(16)

with $\alpha_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i$, $\lambda_i = \lambda_1 + \lambda_2 + \ldots + \lambda_i$, $i=1,2,\ldots,k$

Based on (16), formulas for $E(Y_1), E(Y_2)$, and $E(Y_3)$ can be formed as follows.

$$E(Y_1) = \gamma \alpha_1 + \lambda_1 = e^{x_1 \beta_1}$$

(17)

$$E(Y_2) = \gamma(\alpha_1 + \alpha_2) + \lambda_1 + \lambda_2 = e^{x_2 \beta_2}$$

(18)

$$E(Y_3) = \gamma(\alpha_1 + \alpha_2 + \alpha_3) + \lambda_1 + \lambda_2 + \lambda_3 = e^{x_3 \beta_3}$$

(19)

Then the values of $\alpha_1$, $\alpha_2$, and $\alpha_3$ shown in (17), (18), and (19) are substituted into the pdf of trivariate gamma distribution in (4), so the pdf for $y_1, y_2, y_3$ is:

$$f(y_1, y_2, y_3) = \frac{\gamma^{\alpha_1}{\beta_1}^{y_1-1}{\lambda_1}^{\gamma-1} \cdot \gamma^{\alpha_2}{\beta_2}^{y_2-1}{\lambda_2}^{\gamma-1} \cdot \gamma^{\alpha_3}{\beta_3}^{y_3-1}{\lambda_3}^{\gamma-1} \cdot e^{-\gamma^{\sum_{i=1}^{3} \lambda_i}}} {y_1^\alpha \cdot \prod_{i=1}^{3} \Gamma(\alpha_i)}$$

(20)

with $\alpha > 0, \gamma > 0, \lambda_1 < y_1 < \infty, \lambda_2 < y_2 < \infty, \lambda_3 < y_3 < \infty, \alpha^* = \alpha_1 + \alpha_2 + \alpha_3$,

otherwise $f(y_1, y_2, y_3) = 0.$
3.2. Parameter Estimation of the TGR Model

The likelihood function of the TGR model in (20) is:

\[
L(\gamma, \lambda_1, \lambda_2, \beta_1, \beta_2, \beta_3) = \prod_{i=1}^{n} f(y_{i1}, y_{i2}, y_{i3}) = \prod_{i=1}^{n} \left( \frac{e^{\gamma \beta_3 - \lambda_3}}{\gamma^\gamma} \left( \frac{y_{i1} - \lambda_3}{\gamma} \right)^{\frac{y_{i1} - \lambda_3}{\gamma}} \frac{e^{\gamma \beta_2 - \lambda_2}}{\gamma^\gamma} \left( \frac{y_{i2} - \lambda_2}{\gamma} \right)^{\frac{y_{i2} - \lambda_2}{\gamma}} \frac{e^{\gamma \beta_1 - \lambda_1}}{\gamma^\gamma} \left( \frac{y_{i3} - \lambda_1}{\gamma} \right)^{\frac{y_{i3} - \lambda_1}{\gamma}} \right)^{\frac{y_{i1} + y_{i2} + y_{i3}}{\gamma}}
\]

(21)

The ln-likelihood function based on (21) is:

\[
\ln L(\gamma, \lambda_1, \lambda_2, \beta_1, \beta_2, \beta_3) = \ln \left( \prod_{i=1}^{n} \left( \frac{e^{\gamma \beta_3 - \lambda_3}}{\gamma^\gamma} \left( \frac{y_{i1} - \lambda_3}{\gamma} \right)^{\frac{y_{i1} - \lambda_3}{\gamma}} \frac{e^{\gamma \beta_2 - \lambda_2}}{\gamma^\gamma} \left( \frac{y_{i2} - \lambda_2}{\gamma} \right)^{\frac{y_{i2} - \lambda_2}{\gamma}} \frac{e^{\gamma \beta_1 - \lambda_1}}{\gamma^\gamma} \left( \frac{y_{i3} - \lambda_1}{\gamma} \right)^{\frac{y_{i3} - \lambda_1}{\gamma}} \right)^{\frac{y_{i1} + y_{i2} + y_{i3}}{\gamma}} \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{e^{\gamma \beta_3} - e^{\gamma \beta_2} - e^{\gamma \beta_1} - e^{-\gamma \lambda_3} \ln(y_{i1} - \lambda_3) + \frac{e^{\gamma \beta_3} - e^{\gamma \beta_2} - e^{\gamma \beta_1} - e^{-\gamma \lambda_2} \ln(y_{i2} - \lambda_2) + \frac{e^{\gamma \beta_3} - e^{\gamma \beta_2} - e^{\gamma \beta_1} - e^{-\gamma \lambda_1} \ln(y_{i3} - \lambda_1)}{\gamma} - \sum_{i=1}^{n} \frac{y_{i1} - \lambda_1 - \lambda_2}{\gamma} \right)
\]

\[
\sum_{i=1}^{n} \frac{e^{\gamma \beta_3} - \lambda_3 - \lambda_2 - \lambda_1}{\gamma} \ln \gamma \sum_{i=1}^{n} \ln \left( \frac{x_{i1}^{e^{\gamma \beta_3}}}{\gamma} \right) - \sum_{i=1}^{n} \ln \left( \frac{x_{i2}^{e^{\gamma \beta_2}}}{\gamma} \right) - \sum_{i=1}^{n} \ln \left( \frac{x_{i3}^{e^{\gamma \beta_1}}}{\gamma} \right)
\]

(22)

After obtaining the likelihood function from the TGR model in (22), the next step is to construct the first derivative of the likelihood function for each parameter, then equal to zero. By defining \( \theta = (\gamma, \lambda_1, \lambda_2, \beta_1, \beta_2, \beta_3) \), the first derivative of the ln-likelihood function for each parameter are:

\[
\frac{\partial \ln L(\theta)}{\partial \beta_1} = \sum_{i=1}^{n} x_{i1} x_{i2}^{\gamma \beta_1} \left( \ln(y_{i1} - \lambda_3) - \ln(y_{i2} - \lambda_2) \right)
\]

(23)

\[
\frac{\partial \ln L(\theta)}{\partial \beta_2} = \sum_{i=1}^{n} x_{i1} x_{i2}^{\gamma \beta_2} \left( \ln(y_{i2} - \lambda_3) - \ln(y_{i3} - \lambda_1) \right)
\]

(24)

\[
\frac{\partial \ln L(\theta)}{\partial \beta_3} = \sum_{i=1}^{n} x_{i1} x_{i2}^{\gamma \beta_3} \left( \ln(y_{i3} - \lambda_3) - \ln(y_{i1} - \lambda_2) \right)
\]

(25)

\[
\frac{\partial \ln L(\theta)}{\partial \lambda_1} = \sum_{i=1}^{n} \left( -\frac{1}{\gamma} \ln(y_{i1} - \lambda_1) \right) \ln \gamma - \frac{1}{\gamma} \ln \left( \gamma \right) \left( \frac{\lambda_1}{\gamma} \right)
\]

(26)

\[
\frac{\partial \ln L(\theta)}{\partial \lambda_2} = \sum_{i=1}^{n} \left( -\frac{1}{\gamma} \ln(y_{i2} - \lambda_2) \right) \ln \gamma - \frac{1}{\gamma} \ln \left( \gamma \right) \left( \frac{\lambda_2}{\gamma} \right)
\]

(27)
\[
\frac{\partial \ln L(\theta)}{\partial \beta_3} = \sum_{i=1}^{n} \left( -\frac{1}{\gamma} \ln \left( y_{i1} - \lambda_3 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_3}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_3}{\gamma} \right)
\]

\[
\frac{\partial \ln L(\theta)}{\partial \beta_2} = \sum_{i=1}^{n} \left( -\frac{1}{\gamma^2} e^{\beta_2} \ln \left( y_{i2} - \lambda_2 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_2}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_2}{\gamma} \right) + \sum_{i=1}^{n} \left( \frac{e^{\beta_2} - e^{\beta_2} + \lambda_2}{\gamma^2} \ln \left( y_{i2} - y_{i1} - \lambda_2 \right) \right)
\]

\[
\frac{\partial \ln L(\theta)}{\partial \beta_1} = \sum_{i=1}^{n} \left( -\frac{1}{\gamma^2} e^{\beta_1} \ln \left( y_{i1} - \lambda_1 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_1}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_1}{\gamma} \right) + \sum_{i=1}^{n} \left( \frac{e^{\beta_1} - e^{\beta_1} + \lambda_1}{\gamma^2} \ln \left( y_{i1} - y_{i2} - \lambda_1 \right) \right)
\]

\[
\sum_{i=1}^{n} \left( -\frac{1}{\gamma^2} e^{\beta_1} \ln \left( y_{i1} - \lambda_1 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_1}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_1}{\gamma} \right) + \sum_{i=1}^{n} \left( \frac{e^{\beta_1} - e^{\beta_1} + \lambda_1}{\gamma^2} \ln \left( y_{i1} - y_{i2} - \lambda_1 \right) \right)
\]

\[
\sum_{i=1}^{n} \left( -\frac{1}{\gamma^2} e^{\beta_2} \ln \left( y_{i2} - \lambda_2 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_2}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_2}{\gamma} \right) + \sum_{i=1}^{n} \left( \frac{e^{\beta_2} - e^{\beta_2} + \lambda_2}{\gamma^2} \ln \left( y_{i2} - y_{i1} - \lambda_2 \right) \right)
\]

\[
\sum_{i=1}^{n} \left( -\frac{1}{\gamma^2} e^{\beta_3} \ln \left( y_{i3} - \lambda_3 \right) - 2 \right) \frac{1}{\gamma} \ln \left( \frac{-\beta_3}{\gamma} \right) - \frac{1}{\gamma} \ln \left( \frac{\beta_3}{\gamma} \right) + \sum_{i=1}^{n} \left( \frac{e^{\beta_3} - e^{\beta_3} + \lambda_3}{\gamma^2} \ln \left( y_{i3} - y_{i2} - \lambda_3 \right) \right)
\]

with \( \Psi(\cdot) = \Gamma'(\cdot)/\Gamma(\cdot) \)

Based on the first derivative of the ln-likelihood for each parameter as shown in (23) until (29), it can be seen that the first derivative form is not closed form. Therefore a numerical method is needed to solve these equations. One numerical method that can be used to solve not closed form equations is BFGS quasi-Newton, as shown in (12). The iteration process stops at the \( p \)-th iteration with the criteria \( \| g(\hat{\theta}_p) \| < \varepsilon \). If the iteration stops, the parameter estimates \( \hat{\gamma}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\beta}_1, \hat{\beta}_2, \) and \( \hat{\beta}_3 \) are obtained.

3.3. Parameter Hypothesis Testing of TGR Model

The hypothesis used in the simultaneous test is:

\( H_0: \beta_{11} = \beta_{21} = \ldots = \beta_{1f} = \ldots = \beta_{h1} = \beta_{h2} = \ldots = \beta_{f2} = \ldots = \beta_{f3} = \ldots = \beta_{h3} = 0 \)

\( H_1: \text{At least one } \beta_{fi} \neq 0; f = 1, 2, \ldots, h, i = 1, 2, 3 \)

The set of parameters under the population is: \( \Omega = \{ \gamma, \lambda_1, \lambda_2, \lambda_3, \beta_1, \beta_2, \beta_3 \} \)

The set of parameters under the null-hypothesis is: \( \omega = \{ \gamma, \lambda_1, \lambda_2, \lambda_3, \beta_{01}, \beta_{02}, \beta_{03} \} \)

After determining the set of parameters under the population and under the null-hypothesis, the next step is determining the likelihood function under the population:
\[
\ln L(\Omega) = \ln \left[ \prod \left( y \frac{e^{\theta_i y - \theta_i}}{\gamma} \right) \right] = \sum \left( y_i - \lambda_i \right) - \sum \frac{y_i - \lambda_i}{\gamma} - \sum \ln \gamma - \sum \ln \Gamma \left( \frac{y_i - \lambda_i}{\gamma} \right) - \sum \ln \Gamma \left( \frac{y_i - \lambda_i}{\gamma} \right) - \sum \ln \Gamma \left( \frac{y_i - \lambda_i}{\gamma} \right) - \sum \ln \Gamma \left( \frac{y_i - \lambda_i}{\gamma} \right)
\]

The next step is to determine \( L(\hat{\Theta}) = \max_{\Theta} L(\Omega) \) and \( L(\hat{\omega}) = \max_{\omega} L(\omega) \) using the MLE so that the parameter estimator values for the set of parameters is obtained under the population are \( \hat{\gamma}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 \), and the parameter estimator values for the parameter set under the null-hypothesis are \( \hat{\gamma}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 \). Then determine the value of test statistics \( G^2 \) using the formula in (14). The decision to reject the null-hypothesis if \( G^2 > \chi^2_{\alpha; df} \) with \( df = n(\Omega) - n(\omega) \). \( n(\Omega) \) is the total of parameters under the population, and \( n(\omega) \) is the total of parameters under the null-hypothesis. The hypothesis used in the partial test is:

\( H_0: \beta_f = 0 \)

\( H_1: \beta_f \neq 0; f = 0,1,2,...,h; i = 1,2,3 \)

The test statistics is: \( Z = \frac{\hat{\beta}_f}{SE(\hat{\beta}_f)} \text{ with } SE(\hat{\beta}_f) = \sqrt{\text{var}(\hat{\beta}_f)} \) (32)

The decision to reject the null-hypothesis if \(|Z| > Z_{a/2} \).
4. Conclusions

Based on the results of the analysis and discussions, some conclusions are obtained as follows. The first, the estimation of the parameters of the TGR model using the MLE. The first derivative of the likelihood function is not closed form, so a numerical method is needed to solve these equations. One of the numerical methods that can be used is BFGS quasi-Newton. If the iteration has stopped (convergent), each parameter estimator is obtained. The second, the testing hypothesis for model significance (the overall test) and parameter significance (the partial test) for the TGR model are respectively using the MLRT and the Z test.

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