GENERALIZED EXTERNAL CONE CONDITION FOR
DOMAINS IN RIEMANNIAN MANIFOLDS

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ABSTRACT. The aim of this note is to present an alternative proof for an
already known result relative to the solvability of the Dirichlet problem
in Riemannian manifolds (see remark 0.1). In particular, we discuss the
\( p \)-regularity (regularity relative to the \( p \)-laplacian) of domains of the form
\( I = \Omega \setminus K \), where \( \Omega \) is a regular domain and \( K \) is a regular submanifold
of variable codimension (see theorem 4.4). In theorem 5.1 we prove a
sort of generalized external cone condition for the regularity of domains
in Riemannian manifolds giving a geometric and intuitive proof of this
fact.

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my master thesis, and to prof. Stefano Pigola and prof. Jana Bjorn for their
kindness and the useful hints they gave me.

INTRODUCTION

The solvability and regularity of the Dirichlet problem is a well studied
subject in mathematics. The \( p \)-Dirichlet problem consists in finding a \( p \)-
harmonic function in a domain \( \Omega \) which is continuous in \( \overline{\Omega} \) and assumes a
prescribed value on the boundary, but there are at least two different ways of
specifying this values. In the more classical approach, a continuous function
\( f : \partial \Omega \to \mathbb{R} \) is fixed and one asks whether it exists a function
\( u \in H(\Omega) \cap C(\overline{\Omega}) \quad u|_{\partial \Omega} = f \)

Another approach is the Sobolev-Dirichlet one, in which it is requested that
\( f \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \quad u - f \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega}) \)

In general, the two problems lead to different results, in fact not every
function \( f \in C(\partial \Omega) \) can be seen as the restriction to the boundary of
\( \tilde{f} \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \), and for this reason not every solution of the classical
Dirichlet problem belongs to \( W^{1,p}(\Omega) \), but only to \( W^{1,p}_{loc}(\Omega) \). Anyway,
regular points for the classical Dirichlet and the Sobolev-Dirichlet problem coincide, as proved in section 5.3 of [10], so without loss of generality we will treat only the latter problem, since its theory is easier to manage.

The aim of this paper is to give some practical geometrical conditions for the regularity of a point \( x_0 \in \partial \Omega \) that extend the classical exterior cone condition to cases when \( V(x_0) \cap \Omega^C \) has positive codimension. The main reference we will use is [7], in particular chapters 2, 3 and 6 offer a robust background for our considerations. Even if this paper is intended to be as self-contained as possible, many theorems and properties will be cited without proof.

Remark 0.1. I used a geometrical argument to prove this result, but prof. Jana Bjorn kindly suggested that an argument based on the Wiener criterion and estimates on capacity and Hausdorff measure is the standard method to treat this problem. These estimates can be found in theorem 5.1.13 in [1]. Anyway, although it is surely not the best way to get the result, since the proof might be interesting in itself, we present it in the following.

We stress that we treat only boundary points for which there exists a neighborhood \( V \) such that \( \tilde{V} \equiv V \cap \Omega^C \) has some regularity. In particular we assume that \( \tilde{V} \) contains a regular submanifold \( K \) of codimension \( c < p \) for which the generalized external cone condition holds at \( x_0 \). Some regularity conditions with less restrictive assumptions are known, for example in [3] it is proved that for the standard laplace operator in \( \mathbb{R}^2 \), \( x_0 \in \partial \Omega \) is regular if the component of \( \Omega^C \) containing \( x_0 \) has more than 1 point (see discussion in pag. 26-27, just before section 2.9). However our results are stated for a generic dimension \( n \), and deal with the \( p \)-laplace equation, not only the standard \( (2-) \)-laplacian. For this reason we hope this paper can be of some interest when it comes to solving some Dirichlet problems, as happened to us while studying the existence of Evans potential on parabolic manifolds.

In the following \( R \) denotes a generic Riemannian manifold of dimension \( m \) with metric tensor \( g^{ij} \), and \( \sqrt{g} \) is the square root of the determinant of the matrix \( g \).

1. Preliminaries

In this article, we use the standard notation for \( \mathcal{A} \)-harmonic and \( \mathcal{A} \)-sub and superharmonic functions on \( \mathbb{R}^n \), and give for granted their basic properties like the lower semicontinuity of \( \mathcal{A} \)-superharmonic functions, the essliminf-property, the local nature of \( \mathcal{A} \)-harmonicity and the comparison principle. A good reference for these properties is [7]. The definition of these functions can be easily extended also to Riemannian manifolds. Let \( \Omega \) be an open domain in a Riemannian manifold \( R \), an operator \( \mathcal{A} : T(\Omega) \to T(\Omega) \) (\( T(\Omega) \)
is considered locally as the product of $\Omega$ and $\mathbb{R}^n$) is said to be an $p$-operator if:

1. the mapping $(x,v) \rightarrow A(x,v)$ is measurable with respect to the $x$ variable and continuous with respect to $v$
2. $A(x,\lambda v) = \lambda |\lambda|^{p-2} A(x,v)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$
3. $\langle A(x,v) - A(x,w) | v - w \rangle > 0$ for all $v \neq w$ and any $x \in \Omega$
4. locally in $\Omega$, there exists two positive constants $0 < \alpha \leq \beta < \infty$ such that:
   $$\langle A(x,v) | v \rangle \geq \alpha \|v\|^p \quad \|A(x,v)\| \leq \beta \|v\|^{p-1}$$

A function $u \in W^{1,p}_{loc}(\Omega)$ is said to be $A$-harmonic if for every $\phi \in C^\infty(\Omega)$:

$$\int_{\Omega} \langle A(x, \nabla u) | \nabla \phi(x) \rangle \, dV(x) = 0$$

With standard cutoff and partition of unity techniques, it is easily seen that $A$-harmonicity is a local property also on Riemannian manifolds, and so $u$ is $A$-harmonic in $\Omega$ if and only if every $x \in \Omega$ has a neighborhood where $u$ is $A$-harmonic. For this reason, it is interesting to find out what equation does the local representative $\tilde{u}$ of an $A$-harmonic function satisfy. An easy computation leads to the following: if $(U, \phi)$ is a local chart for $R$ with $\Omega \subset U$, given $u \in W^{1,p}(\Omega)$ and its local representative $\tilde{u} : \phi(\Omega) \rightarrow \mathbb{R}$, $\tilde{u}(x) = u(\phi^{-1}(x))$, then $u$ is $A$-harmonic in $\Omega$ if and only if:

$$\int_{\Omega} \sqrt{g^{ij} A_i(\nabla u) \partial_j \phi} \, dx = 0 \quad \forall \phi \in C^\infty(\Omega)$$

So $u$ is $A$-harmonic if and only if $\tilde{u}$ is $A'$-harmonic, where $A' : \phi(\Omega) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the operator:

$$A'_j(x, \vec{v}) = \sqrt{g(x)} g^{ij}(x) A_i(x, \vec{v})$$

For example, if we take $A(x,v) = \|v\|^{p-2} v$, $A$-harmonic functions relative to this special operator are called $p$-harmonic functions and locally in $\mathbb{R}^n$ they are $A'$-harmonic with:

$$A'_j(x, \vec{v}) = \sqrt{g(x)} \left( g^{st}(x) v_s v_t \right)^{\frac{p-2}{2}} g^{ij}(x) v_i$$

Thanks to the properties of the metric tensor $g^{ij}$, in particular to its continuity and positive definiteness, we see that all local representatives $A'$ of $A$ operators satisfy conditions (1)-(4) in $\mathbb{R}^m$, so all the theory developed in $\mathbb{R}^m$ can be applied also to functions on Riemannian manifolds, and all local results still remain valid. As we will see, regularity of a boundary point $p \in \partial \Omega$ is a local property of $\partial \Omega$, for this reason as far as we are concerned there is no substantial difference between $\mathbb{R}^m$ and a generic $R$. 
We now briefly recall the definition of the Sobolev-Dirichlet problem for an $A$-operator.

**Definition 1.1.** Given a domain $\Omega \subset \mathbb{R}$ and a function $f \in W^{1,p}(\Omega)$, we say that $u \in W^{1,p}(\Omega)$ solves the Sobolev-Dirichlet problem relative to $f$ if:

1. $u$ is $A$-harmonic in $\Omega$
2. $u - f \in W^{1,p}_0(\Omega)$

We recall that for any bounded domain $\Omega$, there always exists a unique solution to this problem.

2. **The barrier condition**

In the Sobolev-Dirichlet problem, one might ask additional conditions that the solution $u$ must satisfy, for example, if $f$ is continuous in $x_0 \in \partial \Omega$, is the function $u$ continuous in $x_0$ with $u(x_0) = f(x_0)$?

**Definition 2.1.** A point $x_0 \in \partial \Omega$ is said to be regular for the Sobolev-Dirichlet problem if for every $f \in W^{1,p}(\Omega)$ continuous in $x_0$, the solution $u$ to the problem 1.1 is continuous in $x_0$ and $u(x_0) = f(x_0)$. A domain $\Omega$ is regular if every $x \in \partial \Omega$ is regular.

**Definition 2.2.** Given a point $x_0 \in \partial \Omega$, a lower semicontinuous function $\beta : \Omega \to \mathbb{R}$ is called a **barrier** if:

1. $\beta$ is an $A$-supersolution in $\Omega$
2. $\lim_{x \to x_0, \, x \in \Omega} \beta(x) = 0$
3. $\beta|_{\Omega \setminus \{x_0\}} > 0$

a function with the same properties but defined only in $\overline{\Omega} \cap V(x_0)$, where $V(x_0)$ is a neighborhood at $x_0$, is called a **local barrier**.

**Remark 2.3.** If a local barrier $\tilde{\beta}$ for a point $x_0$ exists, then a global barrier $\beta$ for $x_0$ also exists.

**Proof.** The proof of this statement is quite simple. Let $\tilde{\beta}$ be a local barrier defined on $\overline{\Omega} \cap V(x_0)$ (which can be assumed to be compact). Let $U(x_0) \subset \overline{U(x_0)} \subset V(x_0)^\circ \subset V(x_0)$ be an open neighborhood of $x_0$ and $m$ be the minimum of $\tilde{\beta}$ in $V(x_0) \setminus U(x_0)$, which by compactness is strictly positive. Then

$$\beta(x) = \begin{cases} \min\{\tilde{\beta}(x), m\} & x \in \overline{\Omega} \cap V(x_0) \\ m & x \in \overline{\Omega} \cap V(x_0)^\circ \end{cases}$$

is a global barrier for $x_0$. \hfill $\square$

The reason for the definition of barriers lies in the following theorem.
Theorem 2.4. Given a bounded domain $\Omega \subset \mathbb{R}$ and $x_0 \in \partial \Omega$, $x_0$ is regular for the Sobolev-Dirichlet problem if and only if there exists a barrier $\beta$ relative to this point.

Proof. For the standard Laplace operator in $\mathbb{R}^n$, this theorem is very well known, and a simple proof can be found in [2], theorem 11.7 and 11.10. Since the only property of harmonic function needed for this proof is the comparison principle (valid for every $p > 1$), the proof remains valid even in the nonlinear case on Riemannian manifolds. Another proof can be found in theorem 9.8 in [7]. \qed

Remark 2.5. Looking carefully at the proof of the previous theorem, one sees that instead of asking for one single barrier at $x_0 \in \partial \Omega$, we can ask that there exists a family $\{\beta_\epsilon\}_{\epsilon > 0}$ of non-negative lower semicontinuous functions such that:

1. $\beta_\epsilon$ is a supersolution in $\Omega$
2. $\lim_{x \to x_0, x \in \Omega} \beta_\epsilon(x) = 0$
3. $\beta_\epsilon|_{\{x \in \Omega \text{ t.c. } d(x,x_0) \geq \epsilon\}} > 0$

and obtain the same conclusion.

Remark 2.6. The regularity of $x_0 \in \partial \Omega$ is a local property of $\Omega$.

Since the existence of a global barrier is equivalent to the existence of a local one, the regularity of a domain $\Omega$ is a local problem, that is the regularity of $x_0 \in \partial \Omega$ depends only on the behaviour of $\partial \Omega$ in a small neighborhood of $x_0$. For this reason, the solvability of the Sobolev-Dirichlet problem in a Riemannian manifold shares many properties with the same problem in $\mathbb{R}^n$. In fact, with the barrier condition it is easily verified that given a domain $\Omega \subset \mathbb{R}$, a point $x_0 \in \partial \Omega$ and a local chart $(U, \phi)$ centered at $x_0$, $x_0$ is regular for $\Omega$ with respect to the $p$-laplace operator if and only if $\phi(x_0)$ is regular for $\phi(\Omega \cap U)$ with respect to the operator $A$ defined in equation 2.

3. Capacity, Hausdorff dimension and the Wiener criterion

In this section we briefly describe the connection between capacity of a set and its Hausdorff dimension and introduce also the Wiener criterion, another necessary and sufficient condition for the regularity of a point $p \in \partial \Omega$. In the following, $\text{cap}_p(K, \Omega)$ is the capacity of the couple $K \subset \Omega$ and $d_h(E)$ is the Hausdorff dimension of the set $E$. Recall that, by definition, for $E$ bounded we say that $\text{cap}_p(E) = 0$ if and only if $\text{cap}_p(E, \Omega) = 0$ for any open set $\Omega \supset E$ or equivalently for a single open bounded $\Omega \supset E$.

Theorem 3.1. Let $E \subset \mathbb{R}$ and $1 < p < n$. Then

$$d_h(E) \leq n - p \iff \text{cap}_p(E) = 0$$
Moreover if \( p = n \), then \( \text{cap}_p(E) = 0 \Rightarrow d_h(E) = 0 \).

**Proof.** This result is proved in the setting of \( \mathbb{R}^n \) in theorems 2.26 and 2.27 in [7]. The generalization to a generic Riemannian manifold is quite straightforward, so we only sketch its proof.

If \( E \subset U \) where \( (U, \phi) \) is a local chart, then it is very easy to see that there is no substantial difference with the standard Euclidean setting. If \( E \) is not contained in a local chart, it suffices to remember that \( \text{cap}_p(E) = 0 \) if and only if \( \text{cap}_p(E_n) = 0 \) for every \( n \), where \( E_n = E \cap \Omega_n \) with \( \Omega_n \subset U_n \) and \( (U_n, \phi_n) \) is a locally finite atlas for \( \mathbb{R}^n \).

We recall that if \( E \) is a regular submanifold of \( \mathbb{R}^n \) (with or without boundary), then its Hausdorff dimension coincides with its dimension as a manifold. For more informations on the Hausdorff dimension, we refer the reader to [5] and [9].

Now we are ready to state the Wiener criterion.

**Theorem 3.2.** Given a domain \( \Omega \subset \mathbb{R}^n \), a point \( x_0 \in \partial \Omega \) and an operator \( \mathcal{A} \), \( x_0 \) is regular for \( \Omega \) if and only if:

\[
W_p(R \setminus \Omega, x_0) \equiv \int_0^1 \left( \frac{\text{cap}_p(\Omega^C \cap B(x_0, t), B(x_0, 2t))}{\text{cap}_p(B(x_0, t), B(x_0, 2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \infty
\]

where the integral is taken in any right neighborhood of 0.

**Proof.** A well detailed proof of this theorem and a brief description of the history of this proof can be found in [10] (theorem 1.1). \( \square \)

For simplicity, we define \( \eta(\Omega, x, t) \equiv \left( \frac{\text{cap}_p(\Omega^C \cap B(x_0, t), B(x_0, 2t))}{\text{cap}_p(B(x_0, t), B(x_0, 2t))} \right)^{\frac{1}{p-1}} \). Even if for our purposes this form of the Wiener criterion does the job, we mention that there are formulation of this criterion in more general settings than Riemannian manifolds. For the interested reader, we mention [3] and [4].

To apply Wiener’s criterion, one needs some sort of estimates on the function \( \eta(\Omega, x, t) \), estimates that are not easy to find in concrete problems. Anyway the Wiener criterion has a very interesting corollary:

**Theorem 3.3.** Given a domain \( \Omega \subset \mathbb{R}^n \) and \( x_0 \in \partial \Omega \), \( x_0 \) is regular for an \( \mathcal{A} \)-operator if and only if it is regular for any other \( \mathcal{A} \)-operator with the same index \( p \).

**Proof.** The proof is straightforward, it suffices to notice that in condition \( \mathbb{K} \) \( p \) is the only characteristic of \( \mathcal{A} \) which plays a role. \( \square \)

This is corollary 1.2 in [10], but that this equivalence was established before this article for some particular cases. For example, if \( \mathcal{A} \) is a uniformly elliptic
operator \((p = 2)\) this equivalence is stated in theorem 36.3 and its corollary in \([9]\), and is proved with a completely different approach. The same result is present in \([11]\).

This equivalence is very powerful, because if you need to prove the regularity of \(x_0 \in \partial \Omega\) for an operator \(A\) which may have a very complicated form, you can always change \(A\) with a simpler operator, for example one with more symmetries.

Since regularity depends only on \(p\), we may call it \(p\)-regularity to underline this property.

4. Regularity of domains

This section is the heart of the article. The goal is to prove the regularity of \(I\)-type domains with respect to \(A\)-harmonic functions of index \(p\) on Riemannian manifolds (for example the standard \(p\)-harmonic functions).

We first assume that \(p < n\). Let \(R\) be a Riemannian \(n\)-dimensional manifold, \(\Omega\) an open bounded domain in \(R\) with smooth boundary and let \(K \subset \Omega\) be a regular submanifold (possibly with smooth boundary) with dimension strictly greater than \(n - p\) (=codimension strictly lower than \(p\)). Then any set of the form \(\Omega \setminus K\) is \(p\)-regular. Moreover if the codimension of \(K\) (for convenience \(c(K)\)) is greater than equal to \(p\), then not only \(\Omega \setminus K\) is not \(p\)-regular, but the set \(K\) is in some sense negligible. We also briefly consider the case \(p \geq n\).

We start with the case \(p < n\) and \(c(K) > n - p\).

We begin with a lemma that will be the starting point for all our future considerations.

**Lemma 4.1.** Fix \(0 < c < n \in \mathbb{N}\), let \(B = B(\bar{x}, R) \subset \mathbb{R}^n\) and let \(D = D(\bar{x}, r, c)\) be the \(c\)-codimensional disk of radius \(r < R\) centered in \(\bar{x}\), explicitly:

\[
D(\bar{x}, r, c) \equiv \left\{ \bar{x} \in \mathbb{R}^n \text{ t.c. } \bar{x} - \bar{x} = (x_1, \ldots, x_{n-c}, 0, \ldots, 0), \sum_{i=1}^{n-c} x_i^2 \leq r^2 \right\}
\]

then if \(c < p\), there exists a function \(f\) such that:

1. \(f\) is \(p\)-harmonic in \(B \setminus D\)
2. \(f\) is continuous in \(\overline{B}\)
3. \(f|_D = 1\) and \(f|_{\partial B} = 0\)

This function is the \(p\)-potential of \((D, B)\).

**Proof.** By homogeneity of \(\mathbb{R}^n\) and of the \(p\)-laplace operator, we can assume without loss of generality that \(x_0 = 0\) and \(R = 1\).\footnote{this class is the class of \(I\)-type domains}
The $p$-capacity potential of the couple $(D, B)$ is the candidate for $f$. So let $f$ be the solution to the Sobolev-Dirichlet problem:

$$\Delta_p f \equiv \text{div}(|\nabla f|^{p-2} \nabla f) = 0 \quad \text{in} \quad B \setminus D \quad f - \psi \in W_0^{1,p}(B \setminus D)$$

where $\psi \in C_0^\infty(B)$ is identically 1 in a neighborhood of $D$. Since regularity is a local property, every point in $\partial B$ is $p$-regular for $B \setminus D$, so that $f$ is continuous in $B \setminus D$ and is zero on $\partial B$.

Now we turn our attention to the continuity of $f$ in a neighborhood of $D$. Since $B$ is bounded, we know from theorem 3.1 that

$$0 < \text{cap}_p(D, B) = \int_B |\nabla f|^p \, dx$$

so that $f$ cannot be identically 0, and by the minimum principle, $f(x) > 0$ for all $x \in B \setminus D$. Since it is evident that $f \leq 1$ everywhere, and since $f$ enjoys the “ess lim inf” property (see theorem 3.63 in [7]), to prove that $f$ is continuous in a neighborhood of $D$ and $f|_D = 1$, it suffices to show that for every $y \in D$

$$\liminf_{x \to y} f(x) = \text{ess lim inf}_{x \to y} f(x) = \liminf_{x \to y, x \in B \setminus D} f(x) = L(y) = 1$$

To this end, fix $\bar{y} \in D$. Given any real number $\lambda$, define $\lambda \ast \bar{y}$ as the homotopy of parameter $\lambda$ centered in $\bar{y}$, i.e. for any set $S \subset \mathbb{R}^n$:

$$\lambda \ast \bar{y}(S) \equiv (1 - \lambda)\bar{y} + \lambda S = \{(1 + \lambda)y + \lambda x \quad s.t. \quad x \in S\}$$

In the following when there’s no risk of confusion we will write for simplicity $\lambda \ast \bar{y} = \lambda \ast$.

Let $\lambda$ be such that $\lambda \ast (\partial B) \subset B \setminus D$ (see figure 1). By continuity on a compact set, $f$ attains its minimum (say $0 \leq m \leq 1$) on $\lambda \ast (\partial B)$, and by the maximum principle, $0 < m < 1$. Define a new function:

$$\tilde{f}(x) \equiv (1 - m)f \left(\frac{x}{\lambda} - \frac{1 - \lambda}{\lambda} \bar{y}\right) + m$$

by the homogeneity of $\mathbb{R}^n$ and of the $p$–Laplace operator, $\tilde{f}$ is a $p$-harmonic function in $\lambda \ast (B \setminus D)$, moreover it is evident that:

$$\liminf_{x \to \bar{y}, x \in \lambda(B \setminus D)} \tilde{f}(x) = (1 - m) \liminf_{x \to \bar{y}, x \in B \setminus D} f(x) + m \equiv (1 - m)L(\bar{y}) + m$$

Now let’s compare the two functions $f$ and $\tilde{f}$ on the set $\lambda \ast (B \setminus D)$. First of all, $\tilde{f}$ is $p$-harmonic while $f$ is a $p$-supersolution. Moreover on $\partial(\lambda \ast B)$ both functions are continuous and $\tilde{f} = m \leq f$, and both $f$ and $\tilde{f}$ have Sobolev-boundary value 1 on $\lambda \ast D$. Thanks to the comparison principle we can conclude that $f \geq \tilde{f}$ on $\lambda \ast (B \setminus D)$, and so:

$$L(\bar{y}) \geq (1 - m)L(\bar{y}) + m \quad \implies \quad L(\bar{y}) \geq 1$$
Remark 4.2. We notice that the only properties of $D$ needed to prove the last theorem are its codimension (that ensures a positive capacity for $(D, B)$ thanks to theorem 3.1) and its convexity, which is necessary and sufficient to guarantee that $\lambda \ast D \subset D$. In fact, one can substitute $D$ with any other convex submanifold of the same codimension.

With the help of the last lemma, it’s easy to prove that:

**Theorem 4.3.** Fixed $0 < c < n \in \mathbb{N}$, let $B = B(\bar{x}, R) \subset \mathbb{R}^n$ and let $D = D(\bar{x}, r, c)$ be the $c$-codimensional disk of radius $r < R$ centered in $\bar{x}$, namely:

$$D(\bar{x}, r, c) \equiv \{ \bar{x} \in \mathbb{R}^n \text{ t.c. } \bar{x} - \bar{x} = (x_1, \ldots, x_{n-c}, 0, \ldots, 0), \sum_{i=1}^{n-c} x_i^2 \leq r^2 \}$$

then if $c < p$ the set $B \setminus D$ is $p$-regular.

**Proof.** As before, we may assume without loss of generality that $r < R = 1$. Every point in $\partial B$ is easily seen to be $p$-regular, for the points in $\partial D = D$ we use remark 2.5.

Let $y \in D$, and for every $\epsilon$ small enough consider a point $z$ such that $y \in D(z, \epsilon, c) \subset D(\bar{x}, r, c)$. Then if $u_\epsilon$ is the $p$-potential of the pair

$$(D(z, \epsilon, c), B(z, 1))$$

the family $\{ \beta_\epsilon \equiv 1 - u_\epsilon \}$ satisfies all the properties required in remark 2.5. □
We are now ready to prove our main theorem:

**Theorem 4.4.** If $1 < p < n$, let $\Omega$ be a $p$-regular domain in the $n$-dimensional Riemannian manifold $R$, and let $K$ be a closed $c$-codimensional submanifold possibly with smooth boundary contained in $\Omega$. Then the set $\Omega \setminus K$ is $p$-regular if and only if $c < p$.

**Proof.** The “if” part follows rapidly from remark 2.6 and theorem 4.3. It is evident that only the case $c \geq 1$ is of interest. We will prove the $p$-regularity of $\Omega \setminus K$ by proving that for any $x_0 \in K$, there exists a local $p$-superharmonic barrier centered at $x_0$. Thanks to theorem 3.3, this is equivalent to the regularity relative to any $A$-type operator with index $p$. In the following, $\Omega \setminus K \equiv \Omega'$

Let $x_0 \in \partial K = K$ and $(U, \phi)$ a local Fermi chart for $K$ centered in $x_0$, i.e.:

$$\phi(x_0) = 0 \quad \phi(K \cap U) \subset \{(x_1, \cdots, x_{n-c}, 0, \cdots, 0)\}$$

We divide the proof in two cases: $x_0$ is an interior point in the submanifold sense of $K$ and $x_0$ is a boundary point in the submanifold sense of $K$. For both cases let $A$ be the operator defined in equation 2, i.e. a sort of local representation for the Riemannian $p$-laplacian in $R$. Thanks to remark 2.6, we only need to prove that $\phi(x_0)$ is $A$-regular for $\phi(\Omega' \cap U)$.

In the first case, if $x$ is an interior point of $K$ in the submanifold sense, there exists $\epsilon > 0$ such that $D = D(x_0, \epsilon, c) \subset \phi(\Omega' \cap U)$ and $B = B(x_0, 2\epsilon) \subset \phi(U)$. Consider the smooth function $f_{x_0}(y) \equiv |y - x_0|^2$. Then since the set $(D, B)$ is $A$-regular (as stated in theorem 4.3), there exists a unique function $u$ such that:

1. $u$ is $A$-harmonic in $B \setminus D$
2. $u$ is continuous in $\overline{B}$
3. $u - f_{x_0} \in W^{1,p}_0(B)$, i.e. $u|_D = f|_D$, $u|_{\partial B} = f|_{\partial B}$.

The minimum principle assures that $u > 0$ in $B \setminus D$, so it is straightforward to see that this function is a local barrier for the point $x_0$, so by theorem 2.4 $x_0$ is regular.

The second case is proved in quite the same way, one only needs to be more careful in the choice of $D$. We need that:

$$x_0 \in D \subset \phi(K \cap U)$$

and since $x_0$ is a boundary point (in the submanifold sense), for every $\epsilon > 0$, $D(x_0, \epsilon, c) \not\subset \phi(K \cap U)$. But we have assumed that the boundary of $K$ is smooth, so finding a suitable $D$ is always possible (if $\partial K$ is $C^2$, finding for every $x \in \partial K$ a ball $B$ such that $x_0 = B \cap \partial K$ and $B \subset K$ is a standard problem, for a complete proof see for example the proof of corollary 11.13 in

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1. for example let $\partial \Omega$ be smooth
Once we find a suitable $D$, let $B$ be any ball concentric to $D$ and containing $D$. With the same argument as before, we can build an $A$-harmonic local barrier for $x_0$, proving its regularity.

Let’s show that if $c \geq p$, then $\Omega \setminus K$ is not regular. If $c \geq p$, then theorem 3.1 shows that $\text{cap}_p(K, \Omega) = 0$, and according to theorem 7.36 in [7] any bounded $A$-harmonic function defined in $\Omega \setminus K$ has a unique extension to $\Omega$. This proves that $\Omega \setminus K$ is not regular. In fact, let $f, g$ be two functions in $W^{1,p}(\Omega \setminus K) \cap C(\overline{\Omega})$ such that $f|_{\partial\Omega} = g|_{\partial\Omega}$, and let $u, v$ be the solutions to the Sobolev-Dirichlet problems:

$$-\text{div}(A(\nabla u)) = -\text{div}(A(\nabla v)) = 0$$

$$u - f \in W^{1,p}_0(\Omega \setminus K) \quad v - g \in W^{1,p}_0(\Omega \setminus K)$$

Since both $u$ and $v$ have a unique $A$-harmonic extension to $\Omega$ (which for simplicity we will denote with the same name), and since

$$(u - v) - (f - g) = (u - f) - (v - g) \in W^{1,p}_0(\Omega \setminus K) = W^{1,p}_0(\Omega)$$

$u = v$ by the comparison principle ($W^{1,p}_0(\Omega \setminus K) = W^{1,p}_0(\Omega)$ since $K$ is a closed set of $p$-capacity zero, see theorem 2.43 in [7]). This proves that the solution $u$ is independent on the values that $f$ assumes on $K$, so any $x_0$ on $K$ cannot be regular.

For completeness, we need to consider the two cases $p = n$ and $p > n$.

Let $p = n$. The difference between this case and the case $p < n$ is in theorem 3.1. In fact, since it is not possible to argue that if $c \geq p$ then $\text{cap}_p(K) = 0$, the reverse implication in theorem 4.4 cannot be proved with the same technique used if $p < n$, anyway the proof of the other implication is still valid. However this problem is easily solved if we restrict our attention only to $c$ codimensional submanifolds and set aside more general sets. If $p = n$, the only submanifolds of codimension $c \geq n$ are points, and since points are set of $p$-capacity zero, it is straightforward to see that $\Omega \setminus \{x_0\}$ is not a regular domain.

The case $p > n$ is even easier, since in this case any boundary point of any set is $p$-regular. In fact, let $x_0 \in \partial\Omega$, then by standard capacity estimates
(see for example section 2.11 in [7]) we have:

\[
W_p(\mathbb{R}^n \setminus \Omega, x_0) \equiv \int_0^1 \left( \frac{\text{cap}_p(\Omega^C \cap B(x_0, t), B(x_0, 2t))}{\text{cap}_p(B(x_0, t), B(x_0, 2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} \geq \int_0^1 \left( \frac{\text{cap}_p(\{x_0\}, B(x_0, 2t))}{\text{cap}_p(B(x_0, t), B(x_0, 2t))} \right)^{\frac{1}{p-1}} \frac{dt}{t} = \int_0^1 \left( \frac{2^{n-p}t^{n-p}}{(2^{\frac{p}{p-1}} - 1)^{1-p}t^{n-p}} \right) \frac{dt}{t} = \infty
\]

and the Wiener criterion proves our statement.

Summing up, we have just proved that in theorem 4.4 the hypothesis \(1 < p < n\) can be replaced by \(1 < p < \infty\), even if the really interesting cases are \(1 < p \leq n\).

5. Generalized external cone condition

As noticed in remark 2.5, the only properties of \(D\) needed to make the proof of 4.1 work are its convexity and its codimension, so if \(D\) is a truncated cone of the right codimension all the theorems above are still valid. The next theorem summarizes the results proved in this note in a more general form than the one presented before for the sake of simplicity. Its proof is just a reformulation of the proofs presented before.

**Theorem 5.1.** Let \(R\) be an \(n\)-dimensional Riemannian manifold and \(\Omega\) an open domain in \(R\). Consider \(x_0 \in \partial \Omega\). Then if there exists a local chart \((U, \phi)\) centered in \(x_0\) and a truncated closed cone \(C\) of codimension \(c < p\) such that:

\[
x_0 \in C \subset \phi(\Omega^C \cap U)
\]

then \(x_0\) is a regular boundary point for \(\Omega\) with respect to the \(p\)-laplace operator.

This theorem is a sort of generalization for the external cone condition (see for example proposition 11.16 in [2]). In fact it is not necessary for the regularity of a point \(x_0 \in \partial \Omega\) to be the vertex of an \(n\)-dimensional truncated cone contained in \(\Omega^C\), but the cone can have codimension \(c < p\).

**References**

1. David R. Adams and Lars Inge Hedberg, *Function spaces and potential theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 314, Springer-Verlag, Berlin, 1996. MR 1411441 (97j:46024)
2. Sheldon Axler, Paul Bourdon, and Wade Ramey, *Harmonic function theory*, second ed., Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001. MR 1805196 (2001j:31001)
3. Jana Björn, *Wiener criterion for Cheeger $p$-harmonic functions on metric spaces*, Potential theory in Matsue, Adv. Stud. Pure Math., vol. 44, Math. Soc. Japan, Tokyo, 2006, pp. 103–115. MR 2277826 (2007i:31013)

4. , *Necessity of a Wiener type condition for boundary regularity of quasiminimizers and nonlinear elliptic equations*, Calc. Var. Partial Differential Equations 35 (2009), no. 4, 481–496. MR 2496653 (2010c:35060)

5. M. Maurice Dodson and Simon Kristensen, *Hausdorff dimension and Diophantine approximation*, Fractal geometry and applications: a jubilee of Benoît Mandelbrot. Part 1, Proc. Sympos. Pure Math., vol. 72, Amer. Math. Soc., Providence, RI, 2004, pp. 305–347. MR 2112110 (2005m:11139)

6. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364 (2001k:35004)

7. Juha Heinonen, Tero Kilpeläinen, and Olli Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications. MR 1207810 (94e:31003)

8. R.-M. Hervé, *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier (Grenoble) 12 (1962), 415–571. MR 0139756 (25 #3186)

9. Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941. MR 0006493 (3,312b)

10. Tero Kilpeläinen and Jan Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. 172 (1994), no. 1, 137–161. MR 1264000 (95a:35050)

11. W. Littman, G. Stampacchia, and H. F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. Pisa (3) 17 (1963), 43–77. MR 0161019 (28 #4228)

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