Thermalization of entangling power with arbitrarily weak interactions

Bhargavi Jonnadula\textsuperscript{1}, Prabha Mandayam\textsuperscript{2}, Karol Życzkowski\textsuperscript{3,4}, Arul Lakshminarayan\textsuperscript{2,5}

\textsuperscript{1}School of Mathematics, University of Bristol, Bristol, BS8 1TW, United Kingdom
\textsuperscript{2}Department of Physics, Indian Institute of Technology Madras, Chennai, India 600036
\textsuperscript{3}Smoluchowski Institute of Physics, Jagiellonian University, Cracow, Poland
\textsuperscript{4}Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, Poland and
\textsuperscript{5}Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany

(Dated: September 14, 2019)

The change of the entangling power of \( n \) fixed bipartite unitary gates, describing interactions, when interlaced with local unitary operators describing monopartite evolutions, is studied as a model of the entangling power of generic Hamiltonian dynamics. A generalization of the local unitary averaged entangling power for arbitrary subsystem dimensions is derived. This quantity shows an exponential saturation to the random matrix theory (RMT) average of the bipartite space, indicating thermalization of quantum gates that could otherwise be very non-generic and have arbitrarily small, but nonzero, entanglement. The rate of approach is determined by the entangling power of the fixed bipartite unitary, which is invariant with respect to local unitaries. The thermalization is also studied numerically via the spectrum of the reshuffled and partially transposed unitary matrices, which is shown to tend to the Girkó circle law expected for random Ginibre matrices. As a prelude, the entangling power \( e^t \) is analyzed along with the gate typicality \( g^t \) for bipartite unitary gates acting on two qubits and some higher dimensional systems. We study the structure of the set representing all unitaries projected into the plane \((e^t, g^t)\) and characterize its boundaries which contains distinguished gates including Fourier gate, CNOT and its generalizations, swap and its fractional powers. In this way a family of gates with extreme properties is identified and analyzed. We remark on the use of these operators as building blocks for many-body quantum systems.

I. INTRODUCTION

A clutch of quantities such as state entanglement, operator entanglement, operator scrambling, out-of-time-ordered correlators, and various measures of mutual information are being currently actively pursued as a means to understand information transport in complex quantum systems and to characterize quantum chaos [1–8]. Entangling power of the time evolution operator has been studied since its introduction as a state independent measure [9–11], and is the average entanglement an operator produces when acting on product, unentangled, states. Operator entanglement and entangling power has recently been applied to many-body systems, in particular in the context of spin-chains and conformal field theories [4, 12, 13], where it has been found useful to distinguish between integrable and non-integrable systems as well as in the analysis of the many-body-localization transition.

A fundamental concern is the growth of subsystem entropy and complexity of unitary and closed systems ranging from two large bipartite systems to quantum spins on lattices [14–27] and hence it is natural to study the entanglement and entangling power of the time evolution operator \( \exp(-iHt/\hbar) \), or its time-ordered version if the Hamiltonian \( H \) is a function of time. For the action of this operator on unentangled states generally creates entanglement, while conjugating operators with it results in operator scrambling. From the point of view of quantum computing [28], gate operations ordered in time are the source of information transfer and computing. Discrete products of unitary operators are then natural objects to study. Random quantum circuits with random unitary operators providing interaction among qubits have been studied in this context [29, 30] and provide efficient unitary \( t \)-designs that simulate Haar distributed unitaries, but have been studied in many other contexts including entanglement spreading, scrambling and many-body localization [7, 8, 31].

However, usually random quantum circuits are constructed by arbitrarily choosing pairs of quNits between which the interactions are taken to be random unitary matrices typically Haar distributed. We are primarily interested in the role of local random unitaries (local in the sense of acting on an irreducible subsystem) with non-random or fixed entangling gates or interactions and in the simplest context of bipartite systems, that is we do not have an extended lattice. This allows for detailed studies of the entangling power and related local invariants and could form building blocks for more general random quantum circuits with non-random entanglers and random locals. Indeed while local unitaries do not entangle, layering or interspersing it in time with entangling ones provides a crucial role for them [32]. Local unitary gates are easier to apply in an experiment and are thus naturally “cheaper” than nonlocal entangling gates and their effect in creating Haar random unitaries or thermalization is to our knowledge not sufficiently explored.

Another setting, in which products of unitaries appear, are in Floquet or time-periodic systems, wherein the object of interest could be powers of the Floquet operator \( U \) [4, 13, 15, 26, 27] which for nonintegrable systems then
We consider in this paper bipartite systems, which may each be a collection of many particles. Concentrating on the bipartition, we study operator entanglement of the propagator as a function of time $t$. We find that the natural quantities to study are indeed the entangling power and one that we have called “gate typicality” in [32]. In particular we are interested in the entangling power of $\Pi_j^{t} = U_j$, given those of $U_j$. If $U_j = (u_{A_j} \otimes u_{B_j}) U_{AB}$, the effect of the locals $u_{A_j, B_j}$ on the entangling power of the product is of interest, especially in relation to the entangling power of $U_{AB}$ and its powers.

We generalize earlier results [32] for the important case of two subsystems of different dimensions. In the central result we demonstrate exponential saturation of the entangling power $\langle e_p(\Pi_j^{t}) \rangle$ with time to that of a typical unitary operator. Here $\langle \cdot \rangle$ denotes an average over local unitary operators and $e_p$ is the entangling power, measured by the average linear entropy of product states transformed by the analyzed entangling gate. This is true for any fixed $U_{AB}$, provided it is not itself a product of local operators. Thus this illustrates the thermalization of the entangling power of $U_{AB}$ under time evolution with non-autonomous local evolutions. Under circumstances that need to be fully understood, these also seem to provide excellent approximations to autonomous Floquet systems [32]. Thus we expect applications not only for coupled chaotic systems such as the kicked top and the kicked rotor, but also to many-body systems such as the kicked and tilted field Ising models [4, 13].

Apart from the entangling power, the much less studied gate typicality [32] also has a simple exponential approach to the global RMT average. Both the entangling power and gate typicality are local invariants associated with the interaction $U_{AB}$. These invariants determine the complexity of products such as $\Pi_j^{t} = U_j$. Thus our first part is dedicated to mapping these invariants for typical bipartite gates in dimensions $N^2$. To our knowledge even this is not yet completely accomplished. For the case of qubits, $N = 2$, the picture is complete and we show in detail the various gates that makeup the “phase-space”. Of special interest are gates that maximize entangling power, in the sense that they saturate bounds set by the dimensionality $N$. It is known that qubits do not saturate this bound and that $CNOT$ and related gates (detailed below) have the maximum entangling power [9]. This is related to the fact that unitary matrices whose reshuffling and partial transposes (both permutations, see below for details) are both unitary do not exist in 4 dimensions. The fact that they do in any other dimension $N$ (except possibly $N = 6$ for which the status is unknown) is noteworthy.

The plan of this paper is as follows: Section (II) will introduce in detail all the relevant quantities, including operator entanglements and entangling power and the modifications when the dimensionality of the subsystems are unequal, Section (III) studies the allowed region of the invariants for the case of two qubits. We study this via the entangling-power, gate-typicality $(e_p - g_t)$ phase space and establish the boundaries of the allowed gates. Section (IV) discusses some special gates such as the Fourier and the fractional powers of swap in arbitrary dimensions, and give partial results for qutrits as well as conjecture that the fractional powers of swap form a boundary for all quNits. Finally in Section (V) we study time-evolution and prove the thermalization of entangling power (and gate-typicality) under certain conditions. Here we generalize and give an elegant proof of our earlier result in [32] as well as provide examples wherein the thermalization can be seen via approach of the partial transposed and reshuffled operators to the Girko-circle law and their (squared) singular values to the Marcenko-Pastur law. Section (VI) provides a summary and outlook.

II. LOCAL INVARIANTS OF OPERATORS: ENTANGLING POWER AND GATE TYPICALITY

A. Two sets of local unitary invariants and operator entanglement

Consider an unitary operator $U$ acting on the bipartite space $H^A_N \otimes H^B_N$ of two parts labeled $A$ and $B$. For simplicity we restrict attention to spaces whose dimensions are equal (and to $N$). These may be “gates” in the language of quantum circuits, or just quantum propagators describing evolution over some finite time. The fact that $U$ need not be of a product form $u_A \otimes u_B$, with $u_{A,B}$ acting on $H^A_N \otimes H^B_N$ in general implies that it is usually capable of creating entanglement when it acts on unentangled states. Let the operator Schmidt decomposition of $U$ be

$$U = \sum_{j=1}^{N^2} \sqrt{\lambda_j} M_{A_j} \otimes M_{B_j},$$

where the operators on the individual spaces $M_{A_j}$ and $M_{B_j}$, are in general not unitary themselves, but form an orthonormal basis for operators on their respective spaces, $\text{tr}(M_{A_j}^d M_{A_k}) = \text{tr}(M_{B_j}^d M_{B_k}) = \delta_{jk}$, where $\delta_{jk}$ is the Kronecker delta. Note that the vector $\lambda = \{\lambda_j\}_{j=1}^{N^2}$ is invariant under local unitary operations. Unitarity of $U$ implies that

$$\frac{1}{N^2} \sum_{j=1}^{N^2} \lambda_j = 1,$$

so the rescaled vector of Schmidt coefficients, $\{\lambda_j/N^2\}$, maybe treated as a discrete probability measure that characterizes the nonlocality of the operator $U$. To elaborate let

$$U \rightarrow U' = (u_{A_1} \otimes u_{B_1}) U (u_{A_2} \otimes u_{B_2}),$$
where \( u_{A_j B_k} \) are “local” unitary operators. In the language of dynamics, they constitute single-particle evolutions. The content of nonlocality of \( U \) and \( U' \) is identical and hence the measures characterizing their nonlocality must be the same. In the case of states, this constitutes the condition that all entanglement measures be local unitary invariants. It is clear from the definition of the operator Schmidt decomposition that the set \( \{ \lambda_j \} \) are \( N^2 \) such invariants, as \( M_{A_j} \rightarrow u_{A_j} M_{A_j} u_{A_j}^\dagger \) also constitute an operator basis consisting of orthonormal operators, and similarly for \( M_{B_j} \).

Another set of \( N^2 \) invariants are constructed from the operator Schmidt decompostion of the operator product \( U S \) where \( S \) is the swap (or flip) operator defined as

\[
S|\phi_A\rangle|\phi_B\rangle = |\phi_B\rangle|\phi_A\rangle, \quad \text{or} \quad S(u_A \otimes u_B)S = u_B \otimes u_A, \quad (4)
\]

for arbitrary states \( |\phi_{A,B}\rangle \) and operators \( u_{A,B} \). Let

\[
US = \sum_{j=1}^{N^2} \sqrt{\mu_j} \, M_{A_j} \otimes M_{B_j}, \quad (5)
\]

be its Schmidt decomposition. As \( S \) is unitary we also have that

\[
\frac{1}{N^2} \sum_{j=1}^{N} \mu_j = 1. \quad (6)
\]

That the set \( \{ \mu_j \} \) constitute \( N^2 \) invariants follows from the observation that

\[
U'S = (u_{A_1} \otimes u_{B_1}) U (u_{A_2} \otimes u_{B_2}) S
= (u_{A_1} \otimes u_{B_1}) U S (u_{B_2} \otimes u_{A_2}), \quad (7)
\]

and hence the Schmidt eigenvalues of \( US \), the \( \mu_j \), are the same as the Schmidt eigenvalues of \( U'S \). The product \( SU \) does not produce any newer invariants. This paper is focused on these two sets of invariants and quantities derived from them. In particular, their moments and entropies provide measures of how nonlocal the operator \( U \) is and we will be concerned with the entropies related to the second moments:

\[
E(U) = 1 - \frac{1}{N^4} \sum_{j=1}^{N^2} \lambda_j^2, \quad \text{and} \quad E(US) = 1 - \frac{1}{N^4} \sum_{j=1}^{N^2} \mu_j^2, \quad (8)
\]

which are the linear operator entanglements. They take values in \([0, 1 - 1/N^2]\), and \( E(U) = 0 \) iff \( U \) is a local product operator.

### B. Entangling power and a complementary quantity

Notice that \( E(U) \) and \( E(US) \) are in some sense complementary quantities, as for a product operator,

\[
E(u_A \otimes u_B) = 0, \quad \text{while} \quad E((u_A \otimes u_B)S) = E(S) = 1 - \frac{1}{N^2}. \quad (9)
\]

The last relation follows from the Schmidt decomposition of \( S \) which is

\[
S = \sum_{i,k=1}^{N} e_{ik} \otimes e_{ki}, \quad \text{where} \quad e_{ik} = |i\rangle\langle k|. \quad (10)
\]

Here \( \{ |i\rangle \}, 1 \leq i \leq N \} \) denotes any orthogonal basis and hence represents a continuous family of possible Schmidt decompositions, each with \( \lambda_j = 1 \) for \( 1 \leq j \leq N^2 \). The swap operator has the maximum operator entanglement entropy (according to any measure of entropy, including the linear one, \( E(S) \) as above), its complementary quantity vanishes, \( E(US) = E(S^2) = 0 \). In fact these two complementary quantities are combined in two measures that are extensively discussed in this paper and one of which is the already well-studied “entangling power”.

The entangling power \( e_p(U)[9, 10] \) of an operator \( U \in \mathcal{H}_N^A \otimes \mathcal{H}_B^N \) is defined as the average entanglement created when \( U \) acts on product state \( |\psi_A\rangle|\psi_B\rangle \) sampled according to the Haar measure on the individual spaces:

\[
e_p(U) = \left( \frac{N+1}{N-1} \right) E(U|\psi_A\rangle|\psi_B\rangle) \right)^{1/2}. \quad (11)
\]

Here the entanglement measure is the linear entropy \( E(|\psi\rangle) = 1 - \text{tr}^2(\rho^2_A) \) and \( \rho_A \) is the reduced density matrix \( \text{tr}_B(|\psi\rangle\langle\psi|) \). It has been shown in [10] that this is related to the more abstractly defined operator entanglements via

\[
e_p(U) = \frac{1}{E(S)} \left[ E(U) + E(US) - E(S) \right]. \quad (12)
\]

The range of \( e_p(U) \)

\[
0 \leq e_p(U) \leq 1 \quad (13)
\]

follows from the fact that the maximum value of \( E(U) \) is \( E(S) \). We have rescaled the definition of \( e_p(U) \) from that originally defined in [9] so that the maximum value is simply 1 independent of \( N \). If \( e_p(U) = 0 \) then \( U \) is either a product of local operators or locally equivalent to the swap. The fact that swap does not create any entanglement when acting on product states leads to \( e_p(S) = 0 \), but that it is highly nonlocal is reflected in its operator entanglement being maximum. This is one motivation for introducing the complementary quantity

\[
g_t(U) := \frac{1}{2E(S)} \left[ E(U) - E(US) + E(S) \right] \quad (14)
\]

where \( g_t \) is referred to as gate typicality in [32]. The range of \( g_t(U) \) is

\[
0 \leq g_t(U) \leq 1, \quad (15)
\]

and \( g_t(U) = 1 \) iff \( U \) is the swap or is locally equivalent to the swap. Again we have rescaled \( g_t \) from the original definition in [32] by a factor of 2 for complete parity with \( e_p \).
Thus while $e_p$ does not distinguish the local operators from the swap, $g_t$ does. It turns out that rather than discussing the pair \( \{ E(U), E(U^s) \} \) in several settings it seems more natural to work in the plane \( \{ e_p(U), g_t(U) \} \). The average of these measures when \( U \) is sampled uniformly from the space of unitary matrices constitutes the average over the circular unitary ensemble (CUE) and reads

\[
\bar{E} = \bar{e}_p = \frac{N^2 - 1}{N^2 + 1} = \frac{E(S)}{2 - E(S)}, \quad \bar{g}_t = \frac{1}{2}.
\]  

(16)

The fact that \( \bar{E} \) and \( \bar{e}_p \) are close to the maximal possible value equal to 1 implies that a typical Haar unitary gate has strong entangling properties [33], in analogy to the known fact that a generic bipartite pure states is strongly entangled [14, 34].

C. Bipartite unitary gates and four-party entangled pure states

The bijection between states on \( \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \) and operators on \( \mathcal{H}^{(1)} \cong \mathcal{H}^{(2)} \) is well-known. Any normalized bipartite pure state \( |\psi\rangle = \sum_{ij=1}^{N} x_{ij} |ij\rangle \) can be written as \( (X \otimes I) |\phi^+\rangle \), where \( |\phi^+\rangle = \sum_{j=1}^{N} |jj\rangle / \sqrt{N} \) is the maximally entangled state and \( |iX|j\rangle = \sqrt{N} x_{ij} |j\rangle \). Note that a state \( |\psi\rangle \) is maximally entangled, if and only if the matrix \( X \) is unitary, as then its partial trace is maximally mixed, \( \rho_A = Tr_B |\psi\rangle \langle \psi| = XX^\dagger/N = I/N \). It is often convenient to make use of this relation between the set \( U(N)/U(1) \) of unitary quantum gates of order \( N \) and the set of maximally entangled states in \( N \times N \) system [35].

The same relation can also be used in a more general set-up, if the system \( \mathcal{H}^{(1)} \) is composed and describes two subsystems of sizes \( N \) and \( M \geq N \), denoted \( A \) and \( B \) respectively. Note that in comparison to the previous section we generalize to the case when the subsystems could be of different dimensions. The system \( \mathcal{H}^{(2)} \) of the same size \( NM \) is also composed and contains two subsystems \( C \) and \( D \) of dimensions \( N \) and \( M \) respectively. The matrix \( X/\sqrt{NM} \) with elements \( x_{i\alpha,k\beta} \) describes now a 4-party pure state \( |\psi_{ABCD}\rangle = \sum_{ij=1}^{N} \sum_{\alpha\beta=1}^{M} x_{i\alpha,k\beta} |i\alpha k\beta\rangle \), and can be considered as a four-index tensor or a \( NM \times NM \) matrix with composite indices. We shall need the following operations on elements of such a matrix [36]: the partial transpose, \( X_{ij,\beta\alpha} = X_{i\alpha,j\beta} \), is also an \( NM \times NM \) matrix, and the reshuffling, \( X^R_{ij,\alpha\beta} = X_{i\alpha,j\beta} \), where \( X^R \) is of size \( N^2 \times M^2 \) dimensional array.

Any bi-partite matrix \( X \) acting on subsystems \( AB \), thus defines a four-party pure state, \( |\psi_{ABCD}\rangle = (X_{AB} \otimes I_{CD}) \phi_A^+ \otimes \phi_B^+ \).

(17)

Note that the above formula does not factorize, as the symbol \( \otimes \) denotes tensor products acting with respect to different partitions. If the bipartite matrix \( U = X \) acting on the subsystems \( AB \) is unitary then the corresponding four-party state \( |\psi_{ABCD}\rangle \) is maximally entangled with respect to the partition \( AB|CD \), so all the components of the corresponding Schmidt vector of length \( NM \), eigenvalues of \( \rho_{AB} = Tr_{CD} |\psi_{ABCD}\rangle \langle \psi_{ABCD}| = UU^\dagger/NM \), are equal to \( 1/NM \).

On the other hand, one can investigate whether this state is entangled with respect to two other possible partitions, \( AC|BD \) and \( AD|BC \). To this end one studies the partially reduced states \( \rho_{AC} = Tr_{BD} |\psi_{ABCD}\rangle \langle \psi_{ABCD}| \) with spectrum \( \lambda_i \), with \( 1 \leq i \leq N^2 \) and \( \rho_{AD} = Tr_{BC} |\psi_{ABCD}\rangle \langle \psi_{ABCD}| \) with spectrum \( \mu_j \), with \( 1 \leq j \leq NM \).

It is easy to check [36] that the vector \( \lambda \), equal to the spectrum of the positive matrix

\[
\rho_R(U) = \rho_{AC} = U_R^R(U_R)^\dagger/(NM).
\]

(18)

coincides with the vector defining the Schmidt decomposition of the matrix \( U \). Correspondingly, the vector \( \mu \), forming the spectrum of

\[
\rho_{TA}(U) = \rho_{AD} = U_{TA}^T(U_{TA})^\dagger/(NM),
\]

(19)

appears in the Schmidt decomposition of the operator \( U \) composed with the swap \( S \) for the symmetric case.

The most mixed state of \( AC \) is formed when \( \rho_{AC} = 1_{N^2}/N^2 \) and corresponds to the maximum entanglement in the \( AC|BD \) split. This in turn happens when the rearrangement \( U_R \) satisfies \( U_R^R(U_R)^\dagger = I_{N^2}(M/N) \), where \( I_{N^2} \) is the identity matrix of dimension \( N^2 \). In other words, for the symmetric case \( (N = M) \), if \( U_R \) is also unitary, then \( \rho_{R(U)} \) is maximally mixed and the subsystem \( AC \) is maximally mixed with \( BD \). Such matrices have also been referred to “self-dual” recently in an attempt to construct maximally chaotic soluble many-body systems [37]. The corresponding situation for maximal entanglement in the subsystem \( AD \) requires that the partial transposition \( U_{TA}^T \) is unitary.

Hence the linear entanglement entropy

\[
1 - \text{tr} \rho_{AC}^2 \]

based on the reshuffling of \( U \) can serve as a measure of the entanglement in the four-party state in Eq. (17) with respect to the partition \( AC|BD \), while the minimal quantity \( 1 - \text{tr} \rho_{AC}^2 \) based on the partial transpose of \( U \) characterizes entanglement in the splitting \( AD|BC \). Analyzing a bipartite unitary gate \( U \), described by a four-index matrix \( U_{a,b \alpha \beta} = u_{ia,j\beta} \), it is convenient to introduce the notion of multiunitarity [38]. For the symmetric case \( (N = M) \), a unitary matrix \( U \) of size \( N^2 \) is called 2-unitary, if two other matrices with interchanged entries, namely the partial transpose \( U_{TA}^T \) and the reshuffled matrix \( U_R^R \), are also unitary. The corresponding four-index tensor \( u_{ia,j\beta} \) of size \( N^2 \), is called perfect, if any matrix of size \( N^2 \) obtained from it by restructuring its entries into a matrix is unitary [39].

Any four-party pure state which is maximally entangled with respect to three possible partitions is called two-uniform [40] or absolutely maximally entangled (AME) [41]. Interestingly, such states do not exist in a four-qubit system [42], as the total size of the Hilbert
space is too small to find a state satisfying all necessary constraints. This is equivalent to the known fact [9, 43] that there is no unitary matrix of size $N^2 = 4$, for which the maximal value $e_p = 1$ of the entangling power is achieved. In the other notation, there are no 2-unitary matrices of order four [38].

On the other hand AME states exists for larger systems consisting of four qutrits, which is equivalent to the statement that there exists a 2-unitary matrix of size $N^2 = 9$, which maximizes the entangling power $e_p$ [43]. For any $N = 3, 4, 5$ and $N \geq 7$ there exist permutations matrices of size $N^2$ which are 2-unitary, and hence maximize the entangling power [43] and also correspond to AME states of four systems with $N$ levels each. For $N = 6$, the non-existence of any 2-unitary permutation matrix of order 36 is directly related to the famous problem of 36 officers by Euler and follows from the non-existence of two mutually orthogonal Latin Squares of size six. The more general question as to whether there exists a 2-unitary matrix of size $N^2 = 36$ (not necessarily a permutation) remains open. In the context of “self-dual” operators constructing solvable many-body models of quantum chaos [37], such 2-unitary matrices are special and correspond to models with maximal chaos.

The observations made in the above paragraphs are summarized for the symmetric case in the following proposition.

**Proposition 1.** For any unitary operator $U$ acting on a bipartite space $H_N^A \otimes H_N^B$, the following are equivalent.

(a) The unitary $U$ attains the global maximum of entangling power, that is, $e_p(U) = 1$, as both linear entanglement entropies $E(U)$ and $E(US)$ are maximal.

(b) The bipartite unitary matrix $U$ is 2-unitary. In other words both the transformed matrices $UR$ and $U^{TA}$ remain unitary.

(c) If $U_{AB} = U$, the pure state $|\psi_{ABCD}\rangle = (U_{AB} \otimes I_{CD})(\phi_{AC}^+) \otimes |\phi_{BD}^+\rangle$

defined in Eq. (17) is maximally entangled with respect to all possible bipartitions and thus forms an absolutely maximally entangled state of four quNits.

(d) The corresponding four-index tensor $u_{i\alpha, k\beta}$ whose elements describe the four-partite state $|\psi_{ABCD}\rangle = \sum_{i,j=1}^{N} \sum_{\alpha, \beta=1}^{N} u_{i\alpha, j\beta} |i\alpha j\beta\rangle$

is perfect.

### III. BOUNDARIES OF TWO-QUBIT GATES

Let us now focus our attention on two simple examples of two-qubit unitary gates, $N = M = 2$ and two-qutrit gates, $N = M = 3$. In particular, we study the structure of the set of unitary matrices, $U(N^2)$, projected into in the plane $\{e_p(U), g_4(U)\}$, which due to the normalization used can be restricted to the square $[0, 1]^2$. We will be interested in describing the boundary of the allowed area within the square and identifying particular gates corresponding to the distinguished points of the boundary.

In Fig. 1, gate typicality $g_t$ is plotted as a function of entangling power for two-qubit unitaries $U$ drawn at random from CUE(4). It is clear that $0 \leq e_p \leq 2/3$, reflecting the well-known fact that the maximum possible entangling power for a two-qubit gate is not 1 (with our choice of factors), but is only 2/3 [9]. This is related to the fact that absolutely maximally entangled states for a 4-qubit system do not exist [42].

Gate typicality is symmetric about its mean value $g_l(U) = 1/2$ and this is reflected by the following equality,

$$g_t(U) + g_t(US) = 1$$

(20)

Its maximal value $g_t = 1$ is attained only by the swap gate and its local equivalents, while the minimal value $g_t = 0$ corresponds to local operators. Therefore, it might be appropriate to call the operators with $1/2 \leq g_t \leq 1$ swap like.

The boundaries in Fig. 1 can be found using the limits of operator entanglement $E(U)$ and $E(US)$. Writing these quantities in terms of the entangling power $e_p$ and gate typicality $g_t$ of a two-qubit operator ($N = 2$ in Eq. (12) and Eq. (14)) leads to

$$E(U) = \frac{3}{8} [e_p(U) + 2g_t(U)]$$

$$E(US) = \frac{3}{8} [e_p(U) - 2g_t(U) + 2].$$

The upper bound of $E(U)$ and $E(US)$ (which is equal to 3/4) gives the region

$$e_p + 2g_t \leq 2 \quad \text{and} \quad g_t \geq e_p/2$$

(22)

which are the top and bottom lines in Fig. 1. The maximum value of $e_p = 2/3$ is reached by the CNOT gate and is an “optimal” gate in the terminology of [9]. The region is further restricted however and we will show below that the left boundary is given by the parabola $e_p = 2g_t(1 - g_t)$. We further show that this boundary in fact consists of gates of the form $S^\alpha$ with $0 \leq \alpha \leq 1$, that are rational powers of the swap operator $S$.

#### A. The Weyl chamber and various gates

While the lines in Eq. (22) are bounds, we identify the gates that make these actual boundaries of the allowed regions in the $e_p$ vs $g_t$ plot. It will be useful to work with the well known canonical form for two-qubit unitary operators. Any two-qubit operator $U \in SU(4)$, up to
left and right multiplication by local unitaries, can be expressed in terms of Euler angles \{c_1, c_2, c_3\} as \([44-47]\),

\[
U = \exp \left[ -i \left( \frac{c_1}{2} \sigma_1 \otimes \sigma_1 + \frac{c_2}{2} \sigma_2 \otimes \sigma_2 + \frac{c_3}{2} \sigma_3 \otimes \sigma_3 \right) \right],
\]

where \{\sigma_1, \sigma_2, \sigma_3\} are the Pauli matrices. In the standard computational basis (the eigenbasis of \(\sigma_3\)), any bipartite unitary operator can thus be written as,

\[
U = \begin{pmatrix}
e^{-\frac{i c_3}{2}} c^- & 0 & 0 & -i e^{-\frac{i c_3}{2}} s^- \\
0 & e^{\frac{i c_2}{2}} c^+ & -i e^{\frac{i c_2}{2}} s^+ & 0 \\
0 & -i e^{\frac{i c_2}{2}} s^+ & e^{\frac{i c_2}{2}} c^+ & 0 \\
e^{-\frac{i c_3}{2}} s^- & 0 & 0 & e^{-\frac{i c_3}{2}} c^-
\end{pmatrix}
\]

where,

\[
c^\pm = \cos[(c_1 \pm c_2)/2]; \quad s^\pm = \sin[(c_1 \pm c_2)/2].
\]

On imposing the constraint of local unitary equivalence, that is, if any two unitaries \(U\) and \(U' = (u_{A1} \otimes u_{B1})U(u_{A2} \otimes u_{B2})\) related by local unitaries are represented by the same set of Euler angles, the range of values gets restricted to \(c_3 < c_2 < c_1 < \pi/2\). This region in the \(c_1, c_2, c_3\) space containing the nonlocal two-qubit gates forms a tetrahedron known as the Weyl chamber [46].

In terms of the \(c_1, c_2, c_3\) parameterization, it is known [46, 48] one can define two quantities which are invariant under local unitary operations, namely,

\[
G_1 = \cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - \sin^2 c_1 \sin^2 c_2 \sin^2 c_3 + \frac{i}{4} \sin 2c_1 \sin 2c_2 \sin 2c_3,
\]

\[
G_2 = \cos 2c_1 + \cos 2c_2 + \cos 2c_3.
\]

The operator entanglements \(E(U)\) and \(E(US)\) can be written in terms of local invariants \(G_1\) and \(G_2\), as follows [49]:

\[
E(U) = 1 - \frac{1}{8} [3 + 2|G_1(U)| + G_2(U)],
\]

\[
E(US) = 1 - \frac{1}{8} [3 + 2|G_1(U)| - G_2(U)].
\]

Consequently, the entangling power and gate-typicality of any two-qubit gate \(U\) can be explicitly evaluated in terms of the angles \(\{c_1, c_2, c_3\}\) and takes on an elegant and simple form as,

\[
e_p(U) = \frac{2}{3} \left[ \sin^2 c_1 \cos^2 c_2 + \sin^2 c_2 \cos^2 c_3 + \sin^2 c_3 \cos^2 c_1 \right],
\]

\[
g_t(U) = \frac{1}{3} \left[ \sin^2 c_1 + \sin^2 c_2 + \sin^2 c_3 \right].
\]

This leads to the following restriction on the allowed region in the \(e_p - g_t\) plane for two-qubit gates.

**Theorem III.1 (Boundary of two-qubit gates).** The entangling power \(e_p(U)\) and gate-typicality \(g_t(U)\) for any two-qubit unitary \(U\) satisfy

\[
e_p(U) \geq 2g_t(U) \left[ 1 - g_t(U) \right].
\]

**Proof:** Using Eq. (26), we see that \(2g_t(U)(1 - g_t(U))\) is of the form,

\[
2g_t(U)(1 - g_t(U)) = \frac{2}{9} (x + y + z)(3 - (x + y + z)),
\]

where \(x \equiv \sin^2 c_1, y \equiv \sin^2 c_2, z \equiv \sin^2 c_3\) satisfy \(0 \leq x, y, z \leq 1\). Then, it is easy to see that,

\[
(x + y + z)(3 - (x + y + z)) = 3[x(1 - y) + y(1 - z) + z(1 - x)]
\]

\[
+ (xy + yz + zx) - (x^2 + y^2 + z^2)
\]

\[
\leq 3[x(1 - y) + y(1 - z) + z(1 - x)],
\]

since \(xy + yz + zx \leq x^2 + y^2 + z^2\), by Schwarz inequality. Using this in Eq. (28) above, we get,

\[
2g_t(U)(1 - g_t(U)) \leq \frac{2}{3} [x(1 - y) + y(1 - z) + z(1 - x)]
\]

\[
= \frac{2}{3} \left[ \sin^2 c_1 \cos^2 c_2 + \sin^2 c_2 \cos^2 c_3 + \sin^2 c_3 \cos^2 c_1 \right]
\]

\[
= e_p(U),
\]

as desired. \(\square\)

The inequality in Eq. (27) is tight as the \(S^\alpha\) family of gates with \(0 \leq \alpha \leq 1\) form the parabola. The CNOT gate \(C\) has the maximum entangling power of 2/3, furthermore, all members of the \(CS^\alpha\) family, with \(0 \leq \alpha \leq 1\) have maximum entangling power of 2/3 and form the rightmost vertical boundary in Figures 1 and 2. To see this, define

\[
U_t = \exp(itS) = 1 \cos t + i \sin t S,
\]
as $S^2 = 1$. Note that this is a route to defining fractional powers of $S$ as $\exp(i\pi S/2) = iS$ and therefore $(iS)^{(2\pi/\pi)}$ is the same as $\exp(itS)$ and the overall phase of $i^{2\pi/\pi}$ makes no difference to any of the subsequent calculations. Therefore $U_1$ is essentially $S^{2\pi/\pi}$. The rearranged matrix of $CS^\alpha$ is up to a constant phase given by $(CS^\alpha)^R = \cos(\pi\alpha/2)C^R + i\sin(\pi\alpha/2)(CS)^R$. The rearrangement of the CNOT gate is non-unitary being $|00\rangle\langle 00| + |01\rangle\langle 11| + |10\rangle\langle 01| + |11\rangle\langle 10|$, while $(CS)^R$ is again a permutation given by $|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 01| + |10\rangle\langle 10|$. A calculation then yields that

$$E(CS^\alpha) = \frac{1}{8} (5 - \cos(\pi\alpha))$$

(31)

Hence $e_p(CS^\alpha) = (E(CS^\alpha + E(CS^{\alpha+1} - E(S)))/E(S) = 2/3$ and $g_t(CS^\alpha) = 1/2 - \cos(\pi\alpha)/6$ interpolating between 1/3 and 2/3.

Several other standard two qubit gates are identified and their operator entanglement and entangling powers are in the Table I. We also identify gates in the Weyl chamber with different regions in $e_p$ vs $g_t$ plot. In the Figure 2, six edges of the tetrahedron forming a half of the chamber are shown. If $U$ is a two-qubit gate the “path” of the powers $U^n$ inside the Weyl chamber can be mapped to a tetrahedral billiard and could be ergodic [50].

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
Gate & $U$ & $E(U)$ & $E(US)$ & $e_p(U)$ & $g_t(U)$ \\
\hline
Local-gate & 0 & 3/4 & 0 & 0 & 0 \\
\hline
$\sqrt{\text{CNOT}}$ & $\pi/2$ & $\pi/4$ & $\pi/7$ & $\pi/6$ & \\
\hline
CNOT, B-GATE & $\pi/2$ & $\pi/4$ & $\pi/7$ & $\pi/6$ & \\
\hline
DCNOT & $\pi/2$ & $\pi/4$ & $\pi/7$ & $\pi/6$ & \\
\hline
Fourier & $\pi/2$ & $\pi/4$ & $\pi/7$ & $\pi/6$ & \\
\hline
$\sqrt{\text{SWAP}}$ & $\pi/6$ & $\pi/6$ & $\pi/7$ & $\pi/6$ & \\
\hline
SWAP & $\pi/4$ & 0 & 0 & 1 & \\
\hline
\end{tabular}
\caption{Nonlocal properties of some two-qubit gates ($N = 2$).}
\end{table}

**IV. BEYOND QUBITS AND THE ENTangling POWER OF SOME QUNIT GATES**

In this section, we discuss the entangling power of the Fourier gate and the fractional powers of the swap that form an important family of gates. As in the case of qubits, the fractional power of swap lie on a parabola that we provide numerical and analytical evidence remains the left boundary of the region. The rightmost point is at $e_p = 1$ and $g_t = 1/2$ and it is known that in all dimensions except $N = 6$ (and $N = 2$, which we have already dealt with) permutations exist which have these values. In the case $N = 3$ explicit examples of permutations which have $e_p = 1$ have been constructed [38, 43].

We examine two operators generalization to arbitrary symmetric bipartite spaces, the Fourier and the swap gate, the latter playing an important role in bounding the allowed region of the $e_p - g_t$ space beyond qubits and possibly forming the left boundary in general. The Fourier transform on the space $\mathcal{H}_N \otimes \mathcal{H}_N$ is given the unitary gate $F_{mn} = \exp(2\pi i m n/N^2)/N$. Explicitly in bipartite notation

$$\langle k\alpha | F | j\beta \rangle = \frac{1}{N} e^{2\pi i [(k + \alpha N)(j + \beta N)]},$$

(32)

where $0 \leq k, j, \alpha, \beta \leq N - 1$. It is then straightforward to verify that the reshuffled matrix $R^R$ is also unitary and hence the operator entanglement is maximum possible: $E(F) = 1 - 1/N^2$. In this sense the Fourier gate in arbitrary dimensions is self-dual, however, the partial transpose is not unitary and hence the Fourier does not have maximal entangling power. Equivalently $E(FS)$ is not the maximal possible, instead a calculation yields

$$E(FS) = 1 - \frac{1}{N^4} \left[ N^3 + 2 \sum_{k=1}^{N-1} k \frac{\sin^2 (k \pi/N)}{\sin^2 (\pi/N - k \pi/N^2)} \right] \approx 1 - \frac{2}{\pi^2} \int_0^1 \frac{x \sin^2 (\pi x)}{(1 - x)^2} dx \approx 0.344,$$

(33)
where the approximations are valid for large $N$. Thus the operator entanglement of $FS$ and the entangling power of the Fourier gate tends to $\approx 0.344$, about one-third of the maximum possible.

Let $S$ be the swap operator. As indicated above, fractional powers of $S$ up to phase factors are given by $U_t = \exp(i t S)$ and the fact that $S^R = S$ we get

$$U^R_t = 1_R \cos t + i \sin t S = N|\Phi^+\rangle\langle\Phi^+| \cos t + i \sin t S,$$

where

$$1_R = N|\Phi^+\rangle\langle\Phi^+|, \quad |\Phi^+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle,$$

is a maximally entangled state. Further, as $U_t S = S \cos t + i \sin t$, the following simple formulae follow for the fractional powers of the swap gate:

$$E(e^{itS}) = E(S)(1 - \cos^2 t), \quad E(e^{itS}) = E(S)(1 - \sin^4 t),$$

$$e_p(e^{itS}) = \frac{1}{2} \sin^2(2t), \quad g_1(e^{itS}) = \sin^2 t.$$

Note that it follows that $e_p(U) = 2g_1(U)(1 - g_1(U))$, when $U$ is a fractional power of $S$. If we define for general unitaries $u$, a function $f(u) = e_p(u) - 2g_1(u)(1 - g_1(u))$, then we can show that this function is stationary when $u$ is a fractional power of the swap.

**Lemma IV.1.** Define $f(u) \equiv e_p(u) - 2g_1(u)(1 - g_1(u))$, where $e_p(u)$ and $g_1(u)$ are respectively the entangling power and gate typicality of a bipartite unitary $u$. The function $f(u)$ is extremised whenever $u = e^{i t S}$ is a fractional power of the swap operator.

**Proof.** Operators close to $U$, with $0 < \epsilon \ll 1$ are

$$U_\epsilon = \exp(\epsilon i t S + i \epsilon H) \approx \exp(\epsilon i t S) (1 + i \epsilon H),$$

where $H$ is a Hermitian operator. We may require without loss of generality that $H$ is traceless, that is $\text{tr} H = 0$, as the overall phase will make no difference to calculations. We may also assume that $H$ is orthogonal to $S$, that is $\text{tr}(HS) = \text{tr}(SH) = 0$, as any overlap with $S$ will be equivalent to only shifting $t$ to a new value. The difference $\delta U = U_\epsilon - U$ is given by

$$\delta U = i \epsilon \left( \cos t H + i \sin t S H \right).$$

We will show that $\delta E(U) = 0$ and $\delta E(U S) = 0$, thus under such perturbations $\delta e_p(U) = 0$ and $\delta g_1(U) = 0$ and finally $\delta f(u) = 0$.

From

$$E(U) = 1 - \frac{1}{N^2} \text{tr} \left( U^R U^{R\dagger} \right)^2,$$

it follows that

$$\delta E(U) = -\frac{4}{N^2} \text{Re tr} \left( \delta U^R U^{R\dagger} U^R U^{R\dagger} \right).$$

From $U_e = U + \delta U$ we get $(U_e)^R = U^R + (\delta U)^R$. Thus $\delta U^R = (U_e)^R - U^R = (\delta U)_R$. From above expression for $\delta U$ we get

$$\delta U^R = (\delta U)^R = i \epsilon \left( \cos t H + i \sin t (S H) \right).$$

We need to find $\text{tr} (\delta U^R U^{R\dagger} U^R U^{R\dagger})$, which involves $\text{tr}(H^2 R^2), \text{tr}(H R S), \text{tr}(S H R^2)$ and $\text{tr}((S H) R S)$. It is easily seen that when $H$ is orthogonal to $S$ and is traceless, all these are zero. Thus it follows that $\text{tr} (\delta U^R U^{R\dagger} U^R U^{R\dagger}) = 0$. In a similar way it is easy to show also that $\text{tr} (\delta U^T a U^{T\dagger} U^T b U^{T\dagger}) = 0$. Thus $\delta f(u) = 0$ when $u = U$ is a power of $S$, except when $\delta u$ is also along $S$, but in that case $f(u) = 0$ strictly and there is no variation of $f$, establishing that $f(u)$ is an extremum when $u = e^{i t S}$, is a fractional power of the swap.

This Lemma establishes the possibility that the inequality in Eq. (27), that is $e_p(U) \geq 2 g_1(U)(1 - g_1(U))$ is valid not just for qubits, but for all dimensions, with the equality holding only for powers of the swap $S^\alpha$ and locally equivalent gates.

Finally, we perform numerical calculations that start with an operator of the form $S^\alpha$ and perturb it, while retaining the unitarity. There are many possible ways of doing it and they seem equivalent. For example one may deform $S^\alpha \rightarrow S^\alpha \exp(i \epsilon H)$ where $H$ is a random Hermitian matrix with unit variance and zero mean elements. Another approach is to take the ensemble $S^\alpha U_{CUE} U_{\alpha} U_{CUE}^\dagger$, where $U_{CUE}$ is a random unitary matrix, while $U_\alpha$ is a diagonal matrix of phases of the form $\exp(i \epsilon \xi)$ and $\xi$ is uniformly random in $[-\pi, \pi]$. Such perturbations always result in values of $\{ e_p, g_1 \}$ lying to the right of the parabola.

In Fig. 3, the neighborhood gates of several unitary quantum gates are generated for $N = 3$ and the corresponding phase space plot is shown. One of the perturbations with $\epsilon_p = 1$ defined in [38, 43] and denoted as $P_3$ in Fig. 3 forms the rightmost point. The Fourier as discussed above has maximal value of $E(U)$ and lies on the upper boundary formed by the line $x = 2(1 - y)$. The controlled unitary defined as

$$\text{CNOT1}_i \otimes |j\rangle = |i\rangle \otimes |i \oplus j\rangle, \quad i, j \in \mathbb{Z}_3,$$

where $\oplus$ denotes addition modulo 3, has maximal $E(\text{CNOT1}_i S)$ and lies on the boundary $y = 2x$. It is seen that the perturbations have the tendency to quickly approach the CUE “cloud” in the manner of a jet. The leftmost parabolic boundary is the $S^\alpha$ family and does seem to bound the allowed region in the $e_p - g_1$ plane.

V. TIME EVOLUTION AND MULTIPLE USES OF THE NONLOCAL OPERATORS

If $U$ is a bipartite quantum propagator, it is natural to consider a combination $U = (u_A \otimes u_B) U$ where $u_{A,B}$ is interpreted as “local dynamics” or single particle dynamics.
While we have presented related results earlier [32], this instance in terms of entangling power this case the local dynamics can play a crucial role. For evolution, is a different matter as the Schmidt coefficients nonlocal content of multiple applications \(U\) to play in the entanglement change, if any. However the entangled initial state, the local dynamics has no role either if systems that are periodically kicked in time: In particular Floquet propagator that evolves states across one period of the forcing, and its powers are time evolution.

It follows from the above discussions that the nonlocal content of \(U\) is the same as that of \(U\), for example \(e_p(U) = e_p(U)\). Thus if \(U\) contains only the interaction part, it follows that on one application of \(U\) to an unentangled initial state, the local dynamics has no role to play in the entanglement change, if any. However the nonlocal content of multiple applications \(U^\alpha\), that is time evolution, is a different matter as the Schmidt coefficients of an operator in general changes on taking powers. In this case the local dynamics can play a crucial role. For instance in terms of entangling power

\[
e_p(U^2) = e_p[U(u_A \otimes u_B)U] \neq e_p(U^2). \tag{43}
\]

One of the aims of this paper is to study this difference. While we have presented related results earlier [52], this paper contains an important generalization and a more elegant derivation that uses group theory. Note that we are interested in generic statements about average entanglement growth in time, a subject that has already a considerable literature and is still a topic of research.

### A. Thermalization of entangling power

The generalization allows for the subsystems to have different dimensions, say \(N \leq M\). The operator entanglement still follows from the Schmidt decomposition of \(U\) as in Eq. (1) and is determined by the singular values of the, in general rectangular \(N^2 \times M^2\), matrix \(U^R\). This gives the invariants \(\lambda_i\), while the other set of invariants which in the symmetric case came from the Schmidt decomposition of \(US\), in Eq. (5) can now be regarded as that of \((U^TA)^R\) and is determined by the singular values of the square matrix \(U^TA\).

The generalization of expression for entangling power based on the reshuffled and partially transposed matrix \(U\) was given in Wang and Zanardi [11]:

\[
e_p(U) = \frac{1}{M^2(N^2 - 1)} \left[ NM(NM + 1) - \text{tr}(U^R(U^R)^\dagger)^2 - \text{tr}(U^TA(U^TA)^\dagger)^2 \right]. \tag{44}
\]

Note that we use the normalization implying that the maximal entangling power is equal to unity, which is attained when \(\text{tr}(U^R(U^R)^\dagger)^2 = M^2\) and \(\text{tr}(U^TA(U^TA)^\dagger)^2 = NM\), and hence this differs from the expression in [11] by a factor

\[
e_p^{\text{max}} = \frac{M(N - 1)}{N(M + 1)}
\]

which is the unscaled maximum entangling power for a \(N \times M\) bipartite system. Thus we can consider a situation wherein the second subsystem is considerably larger, such as a thermal bath would be and the nonzero interaction can be arbitrarily small. In particular, we show further below that

\[
\left< e_p\left[U(u_A \otimes u_B)V\right]\right>_u{\text{A},u_B} = e_p(U) + e_p(V) - e_p(U)e_p(V)/\overline{e_p},
\]

\[
\left< g_t\left[U(u_A \otimes u_B)V\right]\right>_u{\text{A},u_B} = g_t(U) + g_t(V) - g_t(U)g_t(V)/\overline{g_t}. \tag{45}
\]

Here \(U\) and \(V\) are any two unitary operators and the angular brackets indicate averaging over the local unitary operations with \(u_{\text{A},B}\) sampled uniformly (Haar measure). The quantities \(\overline{e_p}\) and \(\overline{g_t}\) are the Haar averages over the \(MN\) dimensional space that generalize the ones in Eq. (16):

\[
\overline{e_p} = \frac{N(M^2 - 1)}{M(NM + 1)}, \quad \overline{g_t} = 1/2. \tag{46}
\]
A special case of this result when \( U = V^\dagger \) may be interpreted as the average entangling power on conjugation of product operators with the unitary entangling operator \( V \). This is an Heisenberg operator version of the usual entangling power defined via action of \( V \) on product states \([9, 10]\). As \( e_p(V^\dagger) = e_p(V) \) it follows that this is
\[
\langle e_p [V^\dagger (u_A \otimes u_B) V] \rangle_{u_A,u_B} = e_p(V) \left[ 2 - \frac{e_p(V)}{\overline{e_p}} \right].
\]  

This provides a way to quantify scrambling power of bipartite unitary operators. A number of works address operator scrambling, which measures the spread of an initially localized operator in a many-particle operators basis \([2, 7, 8]\) and generalizations of such scrambling power to multipartite systems is of interest.

That such a simple relation exists on averaging may be surprising, and similar relationships with \( E(U) \) and \( E(U,S) \) mix among themselves in a less transparent way. Although these are statements on performing local unitary averages they provide some immediate insights, for example it follows that if \( U \) and \( V \) are such that
\[
e_p(U) + e_p(V) - e_p(U)e_p(V)/\overline{e_p} > e_p(UV),
\]  
then there exist local unitaries such that they enhance the entangling power beyond a serial application of \( U \) and \( V \). The relations in Eq. (45) can be used to iterate, by inserting independent local operators between further nonlocal operators, for instance:
\[
\langle e_p [U(u_A \otimes u_B)V(u'_A \otimes u'_B)W] \rangle_{u_A,u_B,u'_A,u'_B} = e_p(U) + e_p(V) + e_p(W) - [e_p(U)e_p(V) + e_p(V)e_p(W)e_p(U)/(\overline{e_p})^2. \]

These indicate a certain “decoupling” that is induced by local dynamics. It is necessary that the local operators at each product be independent, else the correlations prevent such an expression. However we may expect, and is borne out from previous work \([32]\), that they provide very good approximations when the pairs \( u_A, u'_A \) and \( u_B, u'_B \) are even identical but are typical members of the CUE or quantum chaotic for dynamical systems.

An alternate proof is provided in the generalized setting of unequal dimensional subsystems for the local unitary averaged entangling power and gate typicality. The final formulæ remain the same as those displayed in \([32]\), indicating a certain universality in them.

**Theorem V.1.** Let \( U \) and \( V \) be unitary operators on \( \mathcal{H}_N^A \otimes \mathcal{H}_M^B \) and \( u_A, u_B \) be sampled from the groups \( U(N) \) and \( U(M) \) of unitary matrices according to their Haar measures, then
\[
\langle e_p [U(u_A \otimes u_B)V] \rangle_{u_A,u_B} = e_p(U) + e_p(V) - e_p(U)e_p(V)/\overline{e_p},
\]  
where \( \overline{e_p} = \langle e_p(W) \rangle_W \) and \( W \) is sampled according to the Haar measure on the unitary group \( U(NM) \).

**Proof.** Consider an extended Hilbert space \( \mathcal{H}_N^A \otimes \mathcal{H}_M^B \otimes \mathcal{H}_N^C \otimes \mathcal{H}_M^D \) where \( \mathcal{H}_N^C \) are \( \mathcal{H}_N^D \) are copies of \( \mathcal{H}_N^A \) and \( \mathcal{H}_M^B \). Using the identity \( \text{tr}(\rho_A \otimes \rho_C S_{AC}) = \text{tr}(\rho_A^2 \otimes S_{AC}) \) where \( \rho_C \) is a copy of \( \rho_A \) and \( S_{AC} \) is the swap operator, the entangling power of \( U \) acting on \( \mathcal{H}_N^A \otimes \mathcal{H}_M^B \) was written in \([9]\) as
\[
e_p(U) = \frac{2}{e_p} \text{tr}(U \otimes U \Omega^+U^\dagger \otimes U^{\dagger}) \Pi_{AC}
\]  
where \( \Omega_{AC}^+ = 2^{-1}(1 - S_{AC}) \) is the projector over the anti-symmetric subspace of \( \mathcal{H}_N^A \otimes \mathcal{H}_N^C \), and \( \Omega_{AC}^+ = \omega^+_{AC} \otimes \omega^+_{BD} \) and \( \omega^+_{AC} = \int d\mu(\psi_A)(|\psi_A\rangle \langle \psi_A| \otimes |\psi_A\rangle \langle \psi_A|), \) while \( \omega^+_{BD} \) is an identical operator. When \( d\mu(\psi_A) \) is the Haar measure on states in \( \mathcal{H}_N^A \), recognizing that \( \omega^+_{AC} \) has support only in the symmetric subspace and group theoretic arguments involving the Schur lemma was used in \([9]\) to show that \( \Omega_{AC}^+ = 4C_{AC}C_B \Pi_{AC} \Pi_{BD} \). Here \( C^{-1} = N(N+1), \) \( C_B = M(M+1), \) \( \Pi_{AC} = 2^{-1}(1 + S_{AC}) \) is the projector over the symmetric subspace of \( \mathcal{H}_N^A \otimes \mathcal{H}_N^C \), while \( \Pi_{BD} \) is a similar projector on \( \mathcal{H}_M^B \otimes \mathcal{H}_M^D \).

This forms a convenient starting point for us, as the local unitary averaged entangling power is
\[
\langle e_p(U(u_A \otimes u_B)V) \rangle_{u_A,u_B} = \frac{2}{e_p^{\max}} \text{tr}(U \otimes U \Omega^+U^\dagger \Pi_{AC})
\]  
where \( Q = V \otimes V \Omega^+U^\dagger \), and
\[
\langle Q \rangle = \int d\mu(u_A) d\mu(u_B) (u_A \otimes u_B)^{\otimes 2} V \otimes \Omega^+U^\dagger (u_A \otimes u_B)^{\otimes 2}.
\]

Since the local unitaries are sampled independently, the average over \( u_A, u_B \) can be done separately. Note that \( V \otimes V \) acts on \( AB \) and its copy \( CD \), while \( \Omega^+ \) acts on \( AC \) and \( BD \) independently. Note also that \( \langle Q \rangle \) is self-adjoint and hence diagonalizable. For any \( x_A \in U(N) \), \( \langle x_A \rangle^{\otimes 2}, \langle Q \rangle = 0 \) due to unitary invariance of Haar measure. With similar reasoning \( \langle x_A \rangle^{\otimes 2}, \langle x_B \rangle^{\otimes 2} \) acts irreducibly on the totally symmetric and anti-symmetric subspaces, it follows from the above commutation relations and Schur’s lemma \([31]\) that \( \langle Q \rangle \) can be written as a linear combinations of projectors on the symmetric and anti-symmetric subspaces,
\[
\langle Q \rangle = a_1 \Pi_{AC}^+ \Pi_{BD}^+ + a_2 \Pi_{AC}^1 \Pi_{BD}^- + a_3 \Pi_{AC}^1 \Pi_{BD}^+ + a_4 \Pi_{AC}^\dagger \Pi_{BD}^\dagger,
\]  
where \( a_l = |\text{tr}(\Pi_{AC}^l \Pi_{BD}^l)|^{-1} \text{tr}(Q \Pi_{AC}^l \Pi_{BD}^l); \ l = 1, \ldots, 4 \). That the operator \( Q \) can be used for finding \( a_l \) instead of \( \langle Q \rangle \) follows from the fact that \( (u_A \otimes u_A^\dagger) \Pi_{AC}^l (u_B \otimes u_C) = \Pi_{AC}^l \). Using Eq. (52) and the following relations (see the Appendix for details),
\[
\text{tr}(Q) = \text{tr}(Q(S_{AC} \otimes S_{BD})) = 1, \quad \text{tr}(Q(S_{AC}^\dagger \Pi_{BD}^\dagger) = 1 - \frac{e_p^{\max}}{e_p} e_p(V),
\]  
(55)
local unitary averaged entangling power is given by
\begin{equation}
\langle e_p[U(u_A \otimes u_B)V]\rangle_{u_A,u_B} = e_p(U) + \left[1 - \frac{e_p(U)}{e_p} \right] e_p(V),
\end{equation}
where \(\overline{e_p}\) is the CUE averaged entangling power in Eq. (46).

**Corollary V.1.1.** Let \(U^{(n)} = U(u_{A_{n-1}} \otimes u_{B_{n-1}}) \cdots (u_{A_1} \otimes u_{B_1}) U\), where \(u_{A_j} \in U(N)\) and \(u_{B_j} \in U(M)\) are unitary matrices. Let \(V = U^{(n-1)}\), so that \(U^{(n)} = U(u_{A_{n-1}} \otimes u_{B_{n-1}})V\), then from the theorem above
\begin{equation}
\langle e_p(U^{(n)})\rangle_W = e_p(U) + \left[1 - \frac{e_p(U)}{e_p} \right] \langle e_p(U^{(n-1)})\rangle_W^{\otimes n-2},
\end{equation}
where \(W\) denotes the set of local operators \(u_{A_{n-1}}, u_{B_{n-1}} \cdots u_{A_1}, u_{B_1}\) and the averaging is done over these assuming that they are independent and Haar distributed.

**Corollary V.1.2.** The local unitary averaged gate typicality is [32]
\begin{equation}
\langle g_t(U^{(n)})\rangle_W = 1 - [1 - g_t(U)]^n
\end{equation}

_Proof._ When \(M = N\), gate typicality in Eq. (14) is given by
\begin{equation}
g_t(U) = \text{tr}(U^{\otimes 2} \Omega^+ U^{\otimes 2} \Omega^-),
\end{equation}
where \(\Omega^+ = C_A \tilde{C}_B \Pi_{AC} \Pi_{BD}^+\), \(\tilde{C}_B = N(N - 1)\). Starting with the above relation for \(M \neq N\) and proceeding the same way as the proof of entangling power proves the corollary. \(\square\)

These constitute our main results and indicate that however small the entangling power of \(U\) may be, repeated applications with local unitaries sandwiched between them leads to typical gates in the sense that their entangling power and gate typicality are that of random CUE sampled operators on the product space. The result involves averaging over different local operators at each time step and may be considered a foil for quantities such as \(e_p[\langle (u_{A} \otimes u_{B})U\rangle^n]\) if \(u_{A,B}\) are sufficiently random and have no special relationship with \(U\). Thus while our result may be applicable for non-autonomous Floquet systems, they are also of relevance to autonomous ones. In the case of a many-body spin chain and for the symmetric case of \(N = M\) it has recently been found to hold in certain cases [13]. Our current generalization allows exploring different number of spins in each subsystem.

**B. Examples of Diagonal and CNOT nonlocal operators**

In [32] the entangling power of \(U^n\) and \(U^{(n)}\) was shown for a few gates \(U\), for the symmetric case \(M = N\). Here we augment these results significantly. Firstly, we numerically show that for \(N \neq M\), the thermalization of the entangling power to \(\overline{e_p}\) indeed holds for very small interactions in a model where we can control the strength of such interactions. In particular we test the results for the smallest interesting case of a qubit-qutrit system.

We then return to moderately larger spaces and equal dimensions to show numerically that the density of suitably rescaled eigenvalues of \(\rho_{RT}(U^{(n)})\) approach asymptotically the Marčenko-Pastur law. In fact the spectra of the non-unitary operators \(U^{(n)}R\) and \(U^{(n)}T_A\) (reshuffled and partially transposed matrices corresponding to \(U^{(n)}\), whose singular values are proportional to the spectra of the corresponding reduced density matrices), are also interesting and reflect a tendency to get increasingly uniform distribution of complex eigenvalues inside the unit disk (the so-called universal Girko circle law) along with the approach to typicality and the realization of the Marčenko-Pastur distribution. Let the local averaged purities with a \(n\) fold application of a fixed \(U\) along with \(n - 1\) applications of random locals be denoted as
\begin{equation}
X_n = \langle \text{tr}(\rho^-_{RT}(U^{(n)})) \rangle, \quad Y_n = \langle \text{tr}(\rho^+_{RT}(U^{(n)})) \rangle,
\end{equation}

If the eigenvalues of \(\rho\) are \(\lambda_i\) and \(\tilde{\lambda}_i N^2\) are distributed according to the Marčenko-Pastur law, \((2\pi)^{-1} \sqrt{(4 - x)/x}\), as may be expected in the large \(N\) limit, then both \(X_n\) and \(Y_n\) are expected to be of the order of \(2/N^2\) and this may be considered as a symptom of thermalization. A recursion relations for these operators starting from \((X_1, Y_1)\) was derived in [32] for the symmetric case, \(M = N\).

1. **Diagonal unitary operators**

Consider a diagonal unitary with matrix elements
\begin{equation}
(U_\epsilon)_{m,n;\alpha,\beta} = e^{2\pi \epsilon i \xi \delta_{mn} \delta_{\alpha \beta}} \text{ on } \mathcal{H}_N^\alpha \otimes \mathcal{H}_M^\beta
\end{equation}
where \(\epsilon \in [0, 1]\) and \(\xi\) is chosen randomly and uniformly from \([-1/2, 1/2]\). Such diagonal unitaries arise as interactions in many Floquet models and are important to study. While \(\epsilon = 0\) is evidently the case of zero interaction, \(\epsilon = 1\) is in a sense maximally interacting. As \(\xi\) is a random variable, it defines an ensemble of entangling gates and their entangling power was studied in [32] for the case \(\epsilon = 1\), while for general \(\epsilon\), it has been used in studies of spectral transitions and entanglement [20, 53, 54]. Let an overbar \(\overline{A}\) denote averages over this diagonal ensemble, note that we can also then perform ensemble averages not only over the local unitaries but additionally over the global diagonal gates.

For a fixed realization of the global diagonal \(U_\epsilon\), even if \(\epsilon\) is very small, \(\langle e_p(U_\epsilon^{(n)})\rangle_W\) reaches the Haar average \(\overline{e_p}\) due to interlacing of random locals \(W\), as illustrated.
The partial transpose of a diagonal unitary remains unchanged, hence
\[ \text{tr}(U^T_{\alpha}(U^T_{\alpha})^1)^2 = NM. \] (63)

Inserting Eq. (62) and Eq. (63) in Eq. (44) gives \( e_p(U_{\epsilon}) \approx N\epsilon^2/(N+1) \) and hence
\[ n^* \approx \frac{(N+1)(M^2-1)}{M(NM+1)} \epsilon^2 \] (64)

which is also numerically verified in Fig. 4 for \( N = 2, M = 3 \). The deviations, shown in the insets, are of the order of \( 1/\sqrt{n_W} \), where \( n_W \) denotes the number of realization of local gates over which the averaging is done and hence originate in finite sample size. The saturation value reached is indeed \( \tau_p \) which from Eq. (46) is \( 16/21 \approx 0.762 \). While the numerical test above is for the qubit-qutrit case, for large \( N \) and \( M \), it is possible to approximate the ensemble averaged \( \text{tr}(AA^1)^2 \) as \( N^2M^2\sin^2(\pi\epsilon) \approx (1-2\pi^2\epsilon^2/3)N^2M^2 \) where \( \sin(x) = \sin x/x \), hence \( e_p(U_{\epsilon}) \approx 2\pi^2\epsilon^2/3 \) and \( n^* \sim 3\tau_p/(2\pi^2\epsilon^2) \).

We turn to more detailed results about the spectra and singular values of the reshuffled and partial transposed operators at different times that determine the entangling power. To concentrate on the effect of time evolution itself, we set the interaction strength to the maximum, namely \( \epsilon = 1 \) and \( N = M \). When the non-local operator is from the diagonal unitary ensemble with \( \epsilon = 1 \), we denote these now as simply \( U_d \). The purities of the density matrices of the reshuffled and partial transposed operators (Eq. (60)), for \( n = 1 \) is
\[ X_1 = \frac{2N-1}{N^2} \quad \text{and} \quad Y_1 = \frac{1}{N^2}. \] (65)

The first is easy to derive from the reshuffled operator, see [52] and since the partial transpose of a diagonal unitary is again a diagonal matrix, \( \rho_{T_{\alpha}}(U_d) = I_{N^2}/N^2 \) hence, \( Y_1 = 1/N^2 \). Thus typical diagonal unitaries, even for \( \epsilon = 1 \), are far from being thermalized although their entangling power \( e_p(U_d) = (N-1)/(N+1) \) is large. This follows from Eq. (11), see also [52] which however adopts a different normalization of the entangling power.

However, with one pair of local unitary operators in between one obtains for \( n = 2 \),
\[ X_2 = \frac{6}{N^2+1} \quad \text{and} \quad Y_2 = \frac{2(N^4+N^2+1)}{N^4(N+1)^2}. \] (66)

Since \( Y_2 \sim 2/N^2 \), this quantity related to the partial transpose of \( U_d \) has almost reached its asymptotic value already after two applications of typical nonlocal diagonal operators. On the other hand, the dual quantity \( X_2 \) behaves as \( 6/N^2 \) which indicates significant deviations from typicality. This is seen in Fig. (5) in multiple ways. The eigenvalues of the \( (U^{(2)}_d)^R \) are not uniform in the unit circle, while the spectrum of \( (U^{(2)}_d)^{T_{\alpha}} \) behaves in this way. For the former operator there are several small eigenvalues which reflects the fact at \( n = 1 \), the
matrix $U_d^R$ is of rank $N$, rather than $N^2$. Even in the case of the partial transpose, there are visual deviations from uniformity in the radial distribution which are not seen for $n \geq 3$. Although $X_3 \sim 2/N^2$, there are still visual deviations in the radial distribution, which thus seems a sensitive indicator of thermalization. At $n = 4$ the properties of the partial transpose and the reshuffled matrix are close to those of matriced from the Ginibre ensemble of dimension $N^2$, so that the Marčenko-Pastur law is followed to a good approximation.

2. Controlled unitary operator

Consider a controlled unitary on a symmetric product space of the form

$$ U = P_{A_1} \otimes 1_B + P_{A_2} \otimes u_B $$

where $P_{A_1} + P_{A_2} = 1_N$, $P_{A_1} P_{A_2} = \delta_{ij} P_{A_i}$ are projectors and $u_B \in U(N)$. It is known any unitary that has rank-2 operator Schmidt decomposition is a controlled-unitary of this kind and that it can be implemented with LOCC (Local operations and classical communication) and a maximally entangled pair of qubits [55]. Thus this example may be considered the simplest entangling unitary. The reshuffling is given by

$$ U^R = |P_{A_1}^R \rangle \langle 1^R | + |P_{A_2}^R \rangle \langle u_B^R |,$$

where $|U^R \rangle$ reshapes the operator with elements $|iU|j\rangle$ into a column vector of length $N^2$ with entries $U_{ij}$. Noting that $\langle U^R_1 | U^R_2 \rangle = \text{tr}(U_1^T U_2)$, we get

$$ \rho_R = \frac{1}{N^2} U^R U^R = \frac{1}{N^2} \left[ N(|P_{A_1}^R \rangle \langle P_{A_1}^R | + |P_{A_2}^R \rangle \langle P_{A_2}^R |) + \text{tr}(u_B)^* |P_{A_1}^R \rangle \langle P_{A_1}^R | + \text{tr}(u_B) |P_{A_2}^R \rangle \langle P_{A_2}^R | \right], $$

which is only a rank-2 operator. In contrast, as $(u_A \otimes u_B)^T u_A \otimes u_B$ and as transposes of projectors remain projectors, $U^T u_B$ is also unitary and hence $\rho_T = 1/N^2$, a maximally mixed state. These then immediately imply that

$$ X_1 = \frac{1}{2} + \frac{1}{2N^2} \text{tr}(u_B)^2, \quad Y_1 = \frac{1}{N^2} $$

If we draw $u_B$ from the CUE ensemble of $U(N)$ matrices, then this defines an ensemble of controlled-unitary operators and we can find the averages of the purities over this ensemble. The CUE form factor is the average $\text{tr}(u_B)^2 = n$ if $n \leq N$ and $N$ for $n \geq N$. Denoting this additional averaging as in the case diagonal or unimodular ensemble by an overbar, $\overline{X}_1 \sim 1/2$ and $\overline{Y}_1 = 1/N^2$ while at $n = 2$, using the recursion relation in [32] and the CUE form factors results in

$$ X_2 = \frac{N^6 + 2N^4 - 6N^2 + 4}{4N^2(N^2 - 1)^2}, $$

$$ Y_2 = \frac{5N^4 - 10N^2 + 6}{4N^2(N^2 - 1)^2}. $$

FIG. 5: Distributions corresponding to a) of reshuffled matrix $(U_d^{(n)})_R$ and b) partially transposed matrix $(U_d^{(n)})_T$, are shown for times $n = 2, 3$ and 4. Here $U_d$ is a random diagonal unitary matrix of dimension $N^2$ with $N = 50$. Top row shows spectra in the complex plane, the middle row shows their radial density, while the bottom row shows the distribution of eigenvalues of the corresponding Wishart matrices compared with the Marcenko-Pastur distribution (solid curve).

The other details of $u_B$, are relevant to higher orders. It is also clear that $Y_n \approx \binom{1}{n}$, indicating that it takes a time $n^* \sim 4 \log_2(N)$ for the operators to thermalize, such that the operator entanglements are
comparable to that of the CUE on $U(N^2)$. Numerical data obtained for a typical controlled unitary gate presented in Figure 6 show that in this case the thermalization time is longer in comparison to random diagonal gates. Even at $n = 12$ one can see substantial deviations from the Girko circle law for the eigenvalues of the reshuffled matrix, while at $n = 14$, the fit to the Marcenko-Pastur law is good. Spectral properties of partially transposed matrix also reach typical behaviour around $n \sim 10$. These time scales are consistent with the $\log_2(N)$ scale and the thermalization of the controlled-unitary gates occurs more slowly, but surely.

VI. SUMMARY AND OUTLOOK

In this work we have investigated nonlocal properties of bipartite quantum gates acting on an $N \times M$ system. Representing them in the plane spanned by entangling power $e_p$ and gate typicality $g_t$, we have analyzed the boundary of the allowed set which enabled us to identify gates that correspond to critical points of the boundary and are distinguished by some particular properties. Making use of the Cartan decomposition and the canonical form of a two-qubit gate [44, 45] we have described the boundaries analytically, as they correspond to the edges and diagonals of the Weyl chamber. In the case of $N = 3$ such an approach does not work so only some parts of the boundary are known exactly. For instance, the structure of the set is still unknown in the vicinity of the right most point representing optimal gates, for which entangling power admits its maximal value, $e_p = 1$, and corresponds to maximally entangled states of a four–qutrit system. It is well known [42] that such a gate does not exist for $N = 2$, but the case of $N = 6$ still remains open.

The key issue addressed in this paper concerns nonlocal properties of a bipartite unitary gate applied sequentially. Although local unitary operations performed after a single usage of a nonlocal gate cannot change its entangling power, they do play a crucial role if the gate analyzed is performed several times. Our main result shows that an arbitrary small, but positive, entangling power of a nonlocal gate $U_{AB}$ is sufficient to assure that the gate $U_{AB}' = U_{AB}V_{loc}$ applied $n$ times will reach the entangling power typical to random unitary matrices exponentially fast. Here $V_{loc} = u_A \otimes u_B$ denotes a random local unitary, which is drawn independently at each time step. This statement illustrates the thermalization of non-local properties of bipartite gates with the interaction time and sheds more light into the properties of quantized chaotic dynamics, in which nonlocal kicks coupling both subsystems are interlaced by chaotic local evolution [56].

While the entangling power of a bipartite gate $U_{AB}$ determines the entangling power of time-evolutions augmented with local operators, it is interesting to note that it can possibly determine the complexity of the corresponding many-body systems built out of them in various architectures. As a concrete example, a product of $U_{AB} \otimes^L U_{AB}$ on a one-dimensional lattice of $2L$ sites and its translation by one-site was studied recently in terms of its correlation functions [37]. It is not hard to infer from their results that the case when $U_{AB}$ has maximal entangling power allowed by the dimensions, corresponds to the case of a maximally chaotic many-body system. It is interesting to observe that qubits do not satisfy this condition, while this is the case for qutrits [43]. Thus we are tempted to believe that our study is also relevant to a large body of recent work around understanding of...
quantum chaos for many-body systems.

ACKNOWLEDGMENTS

We would like to thank Dardo Goyeneche for several fruitful discussions. K. Z. acknowledges financial support by Narodowe Centrum Nauki under the grant number DEC-2015/18/A/ST2/00274.

Appendix A

The traces in Eq. (55) can be computed as follows (summation over repeated indices is assumed):

\[
\begin{align*}
\text{tr}(Q) &= \text{tr}(\Omega_p^{++}) = 1, \\
\text{tr}(Q S_{AC} \otimes S_{BD}) &= \frac{1}{N(N+1) M(M+1)} \left[ \text{tr}(V \otimes V (1 + S_{AC} + S_{BD}) + S_{AC} \otimes S_{BD}) V^\dagger \otimes V^\dagger S_{AC} \otimes S_{BD} \right] \\
&= N^2 M \\
&= N^2 M \\
\end{align*}
\]

Now,

\[
\begin{align*}
\text{tr} (V \otimes VS_{AC} V^\dagger \otimes V^\dagger S_{AC} \otimes S_{BD}) &= \langle i_1 \alpha_1 j_1 \beta_1 | V \otimes V | i_2 \alpha_2 j_2 \beta_2 | V^\dagger \otimes V^\dagger \rangle j_2 \beta_2 i_2 \alpha_2 \\
&= N^2 M \\
\end{align*}
\]

Similarly,

\[
\begin{align*}
\text{tr} (V \otimes VS_{AC} V^\dagger \otimes V^\dagger S_{AC} \otimes S_{BD}) &= N M^2 \\
\text{tr} (V \otimes VS_{AC} \otimes S_{BD} V^\dagger \otimes V^\dagger S_{AC} \otimes S_{BD}) &= N^2 M^2 \\
\end{align*}
\]

Combining these trace relations in Eq. (A1) gives,

\[
\text{tr}(Q S_{AC} \otimes S_{BD}) = 1 \\
\]

To compute \( \text{tr}(Q S_{AC}) \) and \( \text{tr}(Q S_{BD}) \), note that (see Sec. II C)

\[
\begin{align*}
\text{tr}(\rho_{AC}^2) &= \frac{1}{N^2 M^2} \text{tr} \left[ (V R (V R)^\dagger)^2 \right] \\
&= \frac{1}{N^2 M^2} \text{tr} \left( V \otimes V S_{AC} V^\dagger \otimes V^\dagger S_{AC} \right),
\end{align*}
\]

where the equality in the second line can be seen doing a similar calculation as in Eq. (A2) (see also [57]). Similarly,

\[
\begin{align*}
\text{tr}(\rho_{AD}^2) &= \frac{1}{N^2 M^2} \text{tr} \left[ (V T_A (V T_A)^\dagger)^2 \right] \\
&= \frac{1}{N^2 M^2} \text{tr} \left( V \otimes V S_{BD} V^\dagger \otimes V^\dagger S_{AC} \right).
\end{align*}
\]

Using Eq. (44), Eq. (A5), and Eq. (A6),

\[
\begin{align*}
\text{tr}(Q S_{AC}) &= \frac{1}{N(N+1) M(M+1)} \left[ \text{tr}(V \otimes V (1 + S_{AC} + S_{BD}) + S_{AC} \otimes S_{BD}) V^\dagger \otimes V^\dagger S_{AC} \otimes S_{BD} \right] \\
&= \frac{1}{N(N+1) M(M+1)} \left[ N M^2 + N^2 M \\
&\quad+ \text{tr} (V R (V R)^\dagger)^2 + \text{tr} (V T_A (V T_A)^\dagger)^2 \right] \\
&= 1 - e_p^{\max} e_p(V).
\end{align*}
\]

Similarly,

\[
\begin{align*}
\text{tr}(Q S_{AC}) &= 1 - e_p^{\max} e_p(V).
\end{align*}
\]

[1] B. Swingle, G. Bentsen, M. Schleier-Smith, and P. Hayden, arXiv:1602.06271 (2016).
[2] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X 7, 031016 (2017), URL https://link.aps.org/doi/10.1103/PhysRevX.7.031016.
[3] R.-Q. He and Z.-Y. Lu, Phys. Rev. B 95, 054201 (2017), URL https://link.aps.org/doi/10.1103/PhysRevB.95.054201.
[4] T. Zhou and D. J. Luitz, Phys. Rev. B 95, 094206 (2017), URL https://link.aps.org/doi/10.1103/PhysRevB.95.094206.
[5] A. Seshadri, V. Madhok, and A. Lakshminarayan, Phys. Rev. E 98, 052205 (2018), URL https://link.aps.org/doi/10.1103/PhysRevE.98.052205.
[6] P. Hosur, X.-L. Qi, D. A. Roberts, and B. Yoshida, Journal of High Energy Physics 2016, 4 (2016), ISSN 1029-8479, URL https://doi.org/10.1007/JHEP02(2016)004.
[7] A. Chan, A. De Luca, and J. T. Chalker, Phys. Rev. X 8, 041019 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.041019.
[8] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Phys. Rev. X 8, 021013 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.021013.
[9] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A 62, 030301 (2000).
[10] P. Zanardi, Phys. Rev. A 63, 040304 (2001), URL http://link.aps.org/doi/10.1103/PhysRevA.63.040304.
[11] X. Wang and P. Zanardi, Phys. Rev. A 66, 044303 (2002), URL http://link.aps.org/doi/10.1103/PhysRevA.66.044303.
[12] J. Dubail, Journal of Physics A: Mathematical and Theoretical 50, 234001 (2017), URL http://stacks.iop.org/1751-8121/50/i=23/a=234001.
[13] R. Pal and A. Lakshminarayan, Phys. Rev. B 98, 174304 (2018), URL https://link.aps.org/doi/10.
