Towards the solution of some fundamental questions concerning group actions on the circle and codimension-one foliations

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September 9, 2014

Abstract

We consider finitely generated groups of real-analytic circle diffeomorphisms. We show that if such a group admits an exceptional minimal set (i.e., a minimal invariant Cantor set), then its Lebesgue measure is zero; moreover, there are only finitely many orbits of connected components of its complement. For the case of minimal actions, we show that if the underlying group is (algebraically) free, then the action is ergodic with respect to the Lebesgue measure. This provides first answers to questions due to É. Ghys, G. Hector and D. Sullivan.

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1 Introduction

1.1 Overview and statements of results

This work is mainly motivated by the next longstanding questions in codimension-one foliations theory. As far as we know, Question 1.1 and the first half of Question 1.2 below go back to D. Sullivan and É. Ghys, whereas the second half of Question 1.2 goes back to G. Hector.

**Question 1.1.** Let $F$ be a codimension-one foliation on a compact manifold $M$ that is transversally of class $C^2$. Assume that $F$ is minimal, that is, the closure of every leaf is the whole manifold. Is it true that $F$ is ergodic with respect to the transversal Lebesgue measure? In other words, if $A$ is a measurable union of leaves, is it true that either $\text{Leb}(A) = 0$ or $\text{Leb}(M \setminus A) = 0$?

**Question 1.2.** Let $F$ be a codimension-one foliation on a compact manifold that is transversally of class $C^2$. Assume that $F$ admits an exceptional minimal set. Does this set have zero Lebesgue measure, and does its complement have only finitely many connected components?

Many interesting examples of foliations are obtained by the suspension of a group action (see [2]). Actually, this provides a natural framework for testing potential conjectures for general foliations. In the codimension-one context, the underlying space (fiber) should be obviously the circle, and the questions above translate into the next ones:

**Question 1.3.** Let $G$ be a finitely generated group of circle diffeomorphisms of class $C^2$. Assume that $G$ acts minimally, that is, every orbit is dense in the circle. Is the action necessarily ergodic with respect to the Lebesgue measure? In other words, is it necessarily true that for every measurable $G$-invariant subset $A \subset S^1$, one has either $\text{Leb}(A) = 0$ or $\text{Leb}(S^1 \setminus A) = 0$?

**Question 1.4.** Let $G$ be a finitely generated group of circle diffeomorphisms of class $C^2$. Assume that the action of $G$ admits a minimal invariant Cantor set $\Lambda$ (also called an exceptional minimal set). Does $\Lambda$ necessarily have zero Lebesgue measure? Is the set of orbits of connected components of $S^1 \setminus \Lambda$ finite?

For simplicity, in what follows we will assume that all diffeomorphisms preserve the orientation. Notice that the general case reduces to this one, as the subgroup of orientation-preserving elements has index two in the original group, hence it carries all its relevant dynamical properties.

The aim of this work is to provide proofs of affirmative answers to Questions 1.3 and 1.4 for actions by real-analytic diffeomorphisms, with the extra hypothesis that the group $G$ is algebraically free for the first one. The reason for just treating free-group actions comes from that the very simple algebraic structure allows giving more elementary combinatorial arguments. Moreover, according to a theorem of Ghys to be discussed below, it still allows dealing with actions of non-necessarily free groups admitting an exceptional minimal set. Although our approach and techniques also
yield a road towards Question 1.2 and eventually to Question 1.1 more general cases are to be treated in separate works.

Finally, let us point out that translating our results and approach to the language of codimension-one foliations also seems to provide a road towards the well-known question of the topological invariance of the Godbillon–Vey class \[15\]. Namely, given a codimension-one foliation, one can ask whether its holonomy pseudogroup is (locally) discrete. In the case it is, an analogue of our Main Theorem, combined with Theorem 1.12 below would give us a rigid description of the dynamics that would probably suffice to describe the GV class. On the other hand, if the holonomy pseudogroup is not (locally) discrete, a topological conjugacy is often (transversally) analytic, which implies the invariance of the GV class almost immediately.

Let us begin by reviewing some of the literature on the subject. This will allow us to state our main (and somewhat technical) theorem from which the results announced above will directly follow.

First, in what concerns minimal actions, if the group is generated by a single diffeomorphism, then its ergodicity is guaranteed by a theorem independently proven by A. Katok in the $C^{1+bv}$ case \[21\], and by M. Herman \[17\] in the $C^{1+Lip}$ case:

**Theorem 1.5 (Katok–Herman).** The action of every minimal $C^2$ circle diffeomorphism (equivalently, of every $C^2$ diffeomorphism with irrational rotation number) is ergodic with respect to the Lebesgue measure.

This theorem easily extends to (minimal) group actions with an invariant probability measure (such a group necessarily contains an element of irrational rotation number -and is conjugated to a group of rotations- provided it is finitely generated). For richer actions, a partial answer to Question 1.4 comes from Sullivan’s exponential expansion strategy (see for instance \[26, 33\]):

**Theorem 1.6 (Sullivan).** Let $G$ be a group of $C^{1+\alpha}$ circle diffeomorphisms, with $\alpha > 0$. Assume that for all $x \in S^1$ there exists $g \in G$ such that $g'(x) > 1$. If the action of $G$ is minimal, then it is ergodic with respect to the Lebesgue measure.

A similar panorama arises when dealing with actions by $C^2$ diffeomorphisms with an exceptional minimal set (that is, a minimal invariant Cantor set) $\Lambda$. Indeed, in such a case, no invariant probability measure can exist provided the group is finitely generated (this is essentially a consequence of the Denjoy theorem). Moreover, if for every $x \in \Lambda$ there exists $g \in G$ such that $g'(x) > 1$, then the Lebesgue measure of $\Lambda$ can be easily shown to be zero.

The approach above was pursued by S. Hurder \[18\], who defined a Lyapunov expansion exponent for the action:

$$L(x) := \limsup_{n \to \infty} \frac{1}{n} \max_{g \in B(n)} \log(g'(x)),$$

where $B(n)$ denotes the ball of radius $n$ (centered at $id$) with respect to some fixed finite system of generators of $G$. He proved that if $G$ is made of $C^{1+\alpha}$ diffeomorphisms, $\alpha > 0$, then this exponent is constant Lebesgue almost everywhere on $S^1$ for minimal actions, and constant Lebesgue almost everywhere on $\Lambda$ in case of an exceptional
minimal set Λ. Moreover, the ergodicity in the former case and the zero Lebesgue measure of Λ in the latter one remain true whenever the exponent is positive. The main problem with this approach is that in all the examples we know where the Lyapunov expansion exponent is positive (including those of $\text{PSL}(2, \mathbb{Z})$ and Thompson’s group $T$ to be recalled later), expanding elements appear everywhere (resp., everywhere on Λ).

Actually, Corollary 1.16 below shows that, under certain hypothesis, this must be always the case.

It becomes clear from the previous discussion that a general obstacle for the application of the expansion strategy is the presence of non-expandable points:

**Definition 1.7.** A point $x \in S^1$ is non-expandable for the action of a subgroup $G \subset \text{Diff}^1_+(S^1)$ if for all $g \in G$ one has $g'(x) \leq 1$. The set of non-expandable points is denoted by $\text{NE} = \text{NE}(G)$ (we omit $G$ in this notation as the group is usually fixed).

The presence of non-expandable points is compatible with both minimality (with no invariant probability measure) and exceptional minimal sets for actions of finitely-generated groups. Such examples are known to exist in the real-analytic context, a relevant family being given by $\text{PSL}(2, \mathbb{Z})$ and other Fuchsian groups corresponding to surfaces with cusps (see [20]). There are also the Ghys-Sergiescu smooth realizations of Thompson’s group $T$ (see [11]), yet these are only $C^\infty$. We refer the reader to [7] for a general discussion on these two families of actions. Nevertheless, it is worth mentioning that a posteriori, such examples seem to be rare, forming a “thin” (though very interesting !) “boundary” between two “more stable” types of actions, namely, those having an exceptional minimal set, and those that are minimal and have a “rich” dynamics (e.g. non locally discrete, that is, having local flows in their local closures; c.f. Proposition 2.8).

In an attempt to handle (and understand) the case where NE is nonempty and intersects the minimal set (more rare in the minimal setting and easier to construct in the case of an exceptional minimal set), the following notions were introduced in [7].

**Definition 1.8.** A subgroup $G \subset \text{Diff}^2_+(S^1)$ satisfies property $(\star)$ if it is finitely generated, acts minimally, and for every $x \in \text{NE}$ there exist $g_+, g_-$ in $G$ such that $g_+(x) = g_-(x) = x$ and $x$ is an isolated-from-the-right (resp., isolated-from-the-left) point of the set of fixed points $\text{Fix}(g_+)$ (resp., $\text{Fix}(g_-)$).

**Definition 1.9.** A subgroup $G \subset \text{Diff}^2_+(S^1)$ satisfies property $(\Lambda \star)$ if Λ is a Cantor set in $S^1$ which is a minimal invariant set for the action of $G$ and for every $x \in \text{NE} \cap \Lambda$ there exist $g_+, g_-$ in $G$ such that $g_+(x) = g_-(x) = x$ and $x$ is an isolated-from-the-right (resp., isolated-from-the-left) point of $\text{Fix}(g_+)$ (resp., $\text{Fix}(g_-)$).

**Remark 1.10.** Notice that for subgroups of the group $\text{Diff}^\omega_+(S^1)$ of real-analytic diffeomorphisms of the circle, the conditions about fixed points above are equivalent to that for all $x \in \text{NE}$ (resp., $x \in \text{NE} \cap \Lambda$), there exists $g \in G \setminus \{\text{id}\}$ such that $g(x) = x$.

Whenever property $(\star)$ (resp., $(\Lambda \star)$) holds, one can still show that the action is ergodic with respect to the Lebesgue measure (resp., that $\text{Leb}(\Lambda) = 0$), as well as many other interesting properties. All of this is summarized in the next three theorems below. Essentially, this follows from a combination of Sullivan’s expansion strategy (performed away from non-expandable points) and classical parabolic expansion (performed close to these points).
Theorem 1.11 ([7]). Let $G \subset \text{Diff}^2_+(\mathbb{S}^1)$ be a group that satisfies property $\ast$. Then:

1) The action of $G$ is ergodic with respect to the Lebesgue measure.

2) The set $\text{NE}(G)$ is finite.

3) The set of points of bounded expansibility, that is,
\[ \{ x \in \mathbb{S}^1 \mid \exists C : \forall g \in G, \ g'(x) \leq C \}, \]

coincides with the union $G(\text{NE})$ of orbits of points of $\text{NE}$.

The following theorem also has a version for $C^2$ diffeomorphisms, but for simplicity we only refer here to the $C^\omega$ version.

Theorem 1.12 ([3, 10]). Let $G \subset \text{Diff}^\omega_+(\mathbb{S}^1)$ be a group satisfying property $\ast$ and such that $\text{NE}(G) \neq \emptyset$. Then:

1) There exists a Markov partition for the action, that is, a finite partition of the circle $\mathbb{S}^1$ into intervals $I_j$ each provided with an element $g_j \in G$ and a set $J_j$ of indexes so that
\[ g_j(I_j) = \bigcup_{k \in J_j} I_k \]
and for each $j$, at most one endpoint of $I_j$ is non-expandable. Moreover, for any $j$, one has $g_j'|I_j > 1$, and the strict inequality still holds at the endpoints of $I_j$ except for those which are non-expandable. If $x \in \partial I_j \cap \text{NE}$, then $g_j(x) = x$ and $g_j'(x) = 1$.

2) A generic $G$-orbit is a union of finitely many full orbits of the action of the locally non-strictly-expanding map $R$ defined by $R|I_j = g_j|I_j$.

3) The Lyapunov expansion exponent of $G$ vanishes Lebesgue almost-everywhere.

Remark 1.13. The notion of a Markov partition used here is weaker than the one introduced by J. Cantwell and L. Conlon in [3, 4]: whereas ours is adapted to the expansion procedure, the one in [3, 4] requires additionally that the full orbits of $R$ coincide with those of $G$. However, the obtained partition “almost” satisfies the definition in the Cantwell-Conlon sense. Indeed, conclusion 2) above says that orbits of $G$ are decomposed into at most a finite number of $R$-orbits.

Theorem 1.14 ([4]). Let $G \subset \text{Diff}^2_+(\mathbb{S}^1)$ be a group that satisfies property $(\Lambda \ast)$. Then:

1) The Lebesgue measure of $\Lambda$ is zero.

2) The set $\text{NE}(G) \cap \Lambda$ is finite.

3) The set of points in $\Lambda$ of bounded expansibility, that is,
\[ \{ x \in \Lambda \mid \exists C : \forall g \in G, \ g'(x) \leq C \}, \]

coincides with the union $G(\text{NE}) \cap \Lambda$ of orbits of points of $\text{NE} \cap \Lambda$. 
As far as we know, in the literature, for all minimal (resp., with an exceptional minimal set Λ) and sufficiently smooth actions of finitely-generated groups on the circle, property (⋆) (resp., (Λ⋆)) does hold. The main result of this work states that, at least under certain assumptions, this is necessarily the case.

**Main Theorem.** Let $G$ be a finitely-generated subgroup of $\text{Diff}^ω_+(S^1)$.

1) If $G$ is free of rank $\geq 2$ and acts minimally on the circle, then it satisfies property (⋆).

2) If $G$ acts on the circle with an exceptional minimal set $Λ$, then it satisfies property (Λ⋆).

Recall that a celebrated theorem of Mañé (see [23, Thm. A]) says that for a $C^2$ circle endomorphism that is not conjugated to an irrational rotation, every closed invariant set having no critical point and which is not hyperbolic must contain a parabolic fixed point. In a certain sense, our Main Theorem is an analogous of this result, though in our case we show that every point of the minimal set that cannot be expanded is a parabolic fixed point of one of the elements of the group.

**Corollary 1.15.** If $G \subset \text{Diff}^ω_+(S^1)$ is a finitely-generated free group acting minimally, then its action is ergodic with respect to the Lebesgue measure.

Indeed, for rank 1, this is Katok-Herman’s theorem, whereas for higher rank, this follows from the Main Theorem.1) together with Theorem 1.11.1).

**Corollary 1.16.** Let $G \subset \text{Diff}^ω_+(S^1)$ be a finitely-generated group that is free of rank at least 2. Assume that $G$ acts minimally and that NE($G$) is nonempty. Then the Lyapunov expansion exponent vanishes Lebesgue almost everywhere.

Again, this follows from the Main Theorem.1) together with Theorem 1.12.3).

**Corollary 1.17.** If $G \subset \text{Diff}^ω_+(S^1)$ is a finitely-generated group preserving an exceptional minimal set $Λ$, then $\text{Leb}(Λ) = 0$.

This follows from the Main Theorem.2) together with Theorem 1.14.1).

**Corollary 1.18.** If $G \subset \text{Diff}^ω_+(S^1)$ is a finitely-generated group preserving an exceptional minimal set $Λ$, then $S^1 \setminus Λ$ is the union of finitely many orbits of intervals.

This corollary corresponds to an analog of the famous Ahlfors’ finiteness theorem [1], and as we already mentioned, it answers in the affirmative a question of G. Hector. Its proof requires more discussion, hence it is postponed to §4, where we also further develop on the proof of the preceding corollary.

### 1.2 Sketch of the proof and plan of the article

Throughout the proof, we will constantly use classical tools of control of distortion which we briefly recall in §2.1. In most cases, we will use them without details (these are left to the reader), except whenever some precise and explicit estimate is needed.
In their original formulation, control of distortion techniques go back to the works of A. Denjoy [5], R. Schwartz [32] and R. Sacksteder [31]. These consist of tools that allow to compare a composition of “simple” $C^2$-diffeomorphisms, restricted to some “small” interval, to the corresponding affine map, provided that the sum of the lengths of the intermediate images of this interval is not too big. A nice view of this was later proposed by Sullivan [34], who cleverly noticed that the “right hypothesis” is the existence of an upper bound for the sum of intermediate derivatives at a single point $y$. In concrete terms, knowing that this sum does not exceed some constant $S$, then in a neighborhood of radius $\sim 1/S$ of $y$, the composition looks like an affine map (despite the number of the involved compositions could be very large!). We will also use certain results on composition of one-dimensional holomorphic maps, more specifically the commutators arguments and the vector fields technique. Both are recalled in §2.2.

We then proceed to the proof of the Main Theorem. We will mainly focus on item 1), as 2) will follow with some minor adjustments to be commented in §4. The proof is done by contradiction. We first assume that there is a non-expandable point $x_0 \in S^1$, and for each $n \in \mathbb{N}$ we consider all its possible images under the elements of $B(n) \subset G$, where $B(n) := \{g_k \circ \cdots \circ g_1 \mid \forall j : g_j \in \mathcal{G}, \ k \leq n\}$ is the ball of radius $n$ (centered at the identity) in the free group $G$, whose canonical system of generators is denoted by $\mathcal{G}$. Among these images, let us consider the one that is closest to $x_0$ on the right, and let us denote it by $x_n := f_n(x) \neq x_0$, where $f_n \in B(n)$. A first part of the proof consists in showing that the distance between $x_0$ and $x_n$ tends to zero exponentially. To do this, we first notice that all the images $g([x_0, x_n])$, where $g$ ranges over $B([n/2])$, are almost pairwise disjoint, in the sense that they may intersect but the number of overlappings grows at most linearly in $n$. This allows to apply the control of distortion techniques, thus estimating the length $|[x_0, x_n]|$ from above essentially by the inverse of the sum of the derivatives $g'(x_0)$, where $g$ ranges over $B([n/2])$. The desired estimate will then be proved in §3.1 by showing (assuming a quite technical proposition) that this sum tends to infinity exponentially. This proof involves several combinatorial arguments that are particular to the free group. In what concerns the aforementioned Proposition 3.2 (whose proof is postponed to §3.3), we will show that the failure of its conclusion implies property $(\ast)$, as well as some other complementary properties, as for instance the existence of a Markov partition for the action.

Next, in §3.2 we will first see how control of distortion arguments ensure that each map $f_n$ is very close to the identity on some relatively big neighborhood of $x_0$. Taking two such maps and applying to them the Ghys commutators technique and the Shcherbakov-Nakai-Loray-Rebelo dilatation argument, we will find local vector fields in the closure of the group. As a consequence, one can map any given point (as for instance $x_0$) arbitrarily close to any other one with a derivative bounded away from 0 and infinity. Nevertheless, this is in contradiction with (an extension of) Sacksteder’s theorem: there exist elements with hyperbolic fixed points (although the classical version of this is in the context of exceptional minimal sets, it extends to minimal actions without invariant probability measure; see [6]).

More technical issues and outcomes of the proof (see e.g. Remark 3.13) will be considered along the text.
2 Preliminaries

2.1 Control of distortion estimates

We begin by recalling several lemmas concerning control of distortion which are classical in the context of smooth one-dimensional dynamics. A more detailed discussion with (references to the) proofs may be found in [7].

Definition 2.1. Given two intervals $I, J$ and a $C^1$ map $g : I \to J$ which is a diffeomorphism onto its image, we define the distortion coefficient of $g$ on $I$ by

$$\kappa(g ; I) := \sup_{x, y \in I} \left| \log \left( \frac{g'(x)}{g'(y)} \right) \right|.$$

The distortion coefficient is subadditive under composition. Moreover, it satisfies

$$\kappa(g, I) = \kappa(g^{-1}, g(I)), \quad (1)$$

as well as

$$\kappa(g, I) \leq C \{g\} |I|,$$

where the constant $C \{g\}$ depends only on the $Diff^2$-norm of $g$ (indeed, one can take $C \{g\}$ as being the maximum of the absolute value of the derivative of the function $\log(g')$). This immediately implies the following

Proposition 2.2. Let $G$ be a subset of $Diff^2(S^1)$ that is bounded with respect to the $Diff^2$-norm. If $I$ is an interval of the circle and $g_1, \ldots, g_n$ are finitely many elements chosen from $G$, then

$$\kappa(g_n \circ \cdots \circ g_1 ; I) \leq C G^{n-1} \sum_{i=0}^{n-1} |g_i \circ \cdots \circ g_1(I)|,$$

where the constant $C_G$ depends only on $G$. (Here, $g_i \circ \cdots \circ g_1$ is the identity for $i = 0$.)

An almost direct consequence of this proposition is the next

Corollary 2.3. Under the assumptions of Proposition 2.2, let us fix a point $x_0 \in I$, and let us denote $f_i := g_i \circ \cdots \circ g_1$, $I_i := f_i(I)$, and $x_i := f_i(x_0)$. Then the following inequalities hold:

$$\exp \left( - C_G \sum_{j=0}^{i-1} |I_j| \right) \cdot \left| I_i \right| \leq f'_i(x_0) \leq \exp \left( C_G \sum_{j=0}^{i-1} |I_j| \right) \cdot \left| I_i \right|, \quad (2)$$

$$\sum_{i=0}^{n} |I_i| \leq |I| \exp \left( C_G \sum_{i=0}^{n-1} |I_i| \right) \sum_{i=0}^{n} f'_i(x_0). \quad (3)$$

Using this corollary, an inductive argument allows showing the following important

Proposition 2.4. Under the assumptions of Proposition 2.2, given a point $x_0 \in S^1$, let us denote $S := \sum_{i=0}^{n} f'_i(x_0)$. Then for every $\delta \leq \log(2)/2C_G S$, one has

$$\kappa(f_n, U_{\delta/2}(x_0)) \leq 2C_G S \delta,$$

where $U_{\delta/2}(x_0)$ denotes the $\delta/2$-neighborhood of $x_0$. 
Proof. By Proposition 2.2, it suffices to check that

\[ \sum_{i=0}^{n-1} |f_i(I)| \leq 2\delta S, \]

where \( I := U_{\delta/2}(x_0) \). To do this, we proceed by induction on \( n \). As \( f_0 \) is the identity and \( |I| = \delta \), for \( n = 1 \) we have \( S = 1 \) and

\[ |I| = \delta \leq 2\delta S, \]

hence the claim holds. Assume it holds for some \( n \), and let us check it for \( n + 1 \). Then by (3) we have

\[ \sum_{i=0}^{n} |f_i(I)| \leq \delta \cdot \exp \left( C \cdot \sum_{i=0}^{n-1} |f_i(I)| \right) \cdot \sum_{i=0}^{n} f_i'(x_0) \leq \exp(2C\delta) \cdot \delta S \leq \exp(\log 2) \cdot \delta S = 2\delta S, \]

where the last inequality is a consequence of the choice of \( \delta \). This closes the proof. \( \square \)

As an application of the previous discussion, we show

**Proposition 2.5.** Let \( G \) be a finitely-generated free group of \( C^2 \) circle diffeomorphisms acting minimally, and let \( \mathcal{G} \) be the set of its (standard) generators together with their inverses. Then for every point \( x \) of the circle, one can find elements (finite geodesics) \( g = \gamma_n \cdots \gamma_1 \), with \( \gamma_i \in \mathcal{G} \) and \( \gamma_i \neq \gamma_{i+1}^{-1} \) for each \( i < n \), with arbitrarily large sum of intermediate derivatives at \( x \).

**Proof.** Otherwise, due to Proposition 2.4, for a sufficiently small \( \delta > 0 \), the distortion on the neighborhood \( U_{\delta/2}(x) \) of all elements in the group would be uniformly bounded, say by a constant \( C > 0 \). Now since the action of \( G \) cannot preserve a probability measure (otherwise the group would be conjugate to a group of rotations, hence Abelian; see [27, Lemma 4.1.8]), the extension of Sacksteder’s theorem of [6] yields an element \( f \in G \) with an hyperbolically repelling fixed point \( y_0 \in U_{\delta/2} \). Up to changing \( f \) by some iterate if necessary, we may assume that \( f'(y_0) > \frac{1}{\delta e^C} \). The upper bound for the distortion then yields \( f'(y) > \frac{1}{\delta} \) for all \( y \in U_{\delta/2} \). However, this is impossible, as it would imply that the image of \( U_{\delta/2} \) under \( f \) is larger than the whole circle. \( \square \)

We will also need a “complex version” of Proposition 2.4 (with the obvious extensions of definitions), the proof of which is analogous to that of the classical one and is left to the reader.

**Proposition 2.6.** Suppose that \( \mathcal{G} \) is a finite subset of \( \text{Diff}_+^\omega(S^1) \). Then there exists a constant \( \rho > 0 \) depending only on \( \mathcal{G} \) such that the statement of Proposition 2.4 holds provided we add the condition \( \delta \leq \rho \). More precisely, for a certain constant \( C_\mathcal{G} > 0 \) and any point \( x_0 \in S^1 \), if we denote

\[ S := \sum_{i=0}^{n-1} f_i'(x_0), \]
where \( f_k := g_k \circ \cdots \circ g_1, \ g_i \in G, \) then for every \( \delta \leq \min\{\log(2)/2C_GS, \rho\}, \) one has
\[
\kappa(f_n, U^{C}_{\delta/2}(x_0)) \leq 2C_GS\delta,
\]
where \( U^{C}_{\delta/2}(x_0) \) denotes the complex \( \delta/2 \)-neighborhood of \( x_0 \).

### 2.2 Commutators and the vector fields technique

The next two results will be crucial to deal with maps that behave like translations on some intervals.

**Proposition 2.7** (Ghys [14, Prop. 2.7]). There exists \( \varepsilon_0 > 0 \) with the following property. Assume that the analytic local diffeomorphisms \( f_1, f_2 : U^{C}_1(0) \rightarrow \mathbb{C} \) are \( \varepsilon_0 \)-close (in the \( C^0 \) topology) to the identity, and let the sequence \( f_k \) be defined by the recurrence relation
\[
f_{k+2} = [f_k, f_{k+1}], \quad k = 1, 2, 3, \ldots
\]
Then all the maps \( f_k \) are defined on the disc \( U^{C}_{1/2}(0) \) of radius \( 1/2 \), and \( f_k \) converges to the identity in the \( C^1 \) topology on \( U^{C}_{1/2}(0) \).

The following proposition is in the spirit of results by A. Shcherbakov [8], I. Nakai [25], J. Rebelo [28, 30], and F. Loray and J. Rebelo [22]. However, though it seems to be well-known to specialists, the statement doesn’t appear in the form below in the literature. For the reader’s convenience, we provide a proof plus a short discussion.

**Proposition 2.8.** Let \( I \) be an interval on which certain real-analytic nontrivial diffeomorphisms \( f_k \in G \) are defined. Suppose that \( f_k \) converge to the identity in the \( C^\omega \) topology on \( I \), and let \( f \) be another \( C^\omega \) diffeomorphism having an hyperbolic fixed point on \( I \). Then there exists a (local) \( C^1 \) change of coordinates \( \varphi : I_0 \rightarrow [-1, 2] \) on some subinterval \( I_0 \subset I \) after which the pseudo-group \( G \) generated by the \( f_k \)'s and \( f \) contains in its \( C^1([0,1],[-1,2]) \)-closure a (local) translation subgroup:
\[
\{ \varphi \circ g \circ \varphi^{-1}[0,1] \mid g \in G \} \supset \{ x \mapsto x + s \mid s \in [-1,1] \}.
\]

**Proof.** By the Poincaré linearization theorem, we may assume that \( f \) is affine on a neighborhood of the hyperbolic fixed point \( p \). If \( f_k(p) \neq p \) for infinitely many \( k \), then the claim follows from [25, Proposition 3.1]. Assume \( f_k(p) = p \) for all but finitely many \( k \). If \( f \) does not commute with infinitely many \( f_k \), then the claim follows from [25, Section 3]. Otherwise, for all but finitely many \( k \) we have that \( f_k \) is affine around \( p \), with multiplier converging to 1. The corresponding affine flow is hence contained in the closure of \( G \) when restricted to the neighborhood; taking logarithmic coordinates, this becomes the desired translation flow.

**Remark 2.9.** In the proposition above, one may relax the convergence of \( f_k \) to the identity to hold only in class \( C^1 \). We sketch the proof for this case since it will have some relevance further on (see Remark 3.9).

**Sketch of the proof.** Again, we will assume that \( f \) is affine on a \( \varepsilon \)-neighborhood of the hyperbolic fixed point \( p = 0 \), say \( f(x) = \lambda x \) for \( x \in [-\varepsilon, \varepsilon] \), with \( \lambda > 1 \).
If \( f_k(0) = 0 \) holds for infinitely many \( k \), then we can still apply the above arguments (namely, Nakai’s result \([25]\) in case of noncommuting \( f, f_k \), and the affine flow argument in case of commuting elements). Otherwise, we may follow an argument of \([28]\). Namely, fix a positive \( \delta < \varepsilon/\lambda \), and let \( g_k := f^n_k f_k f^{-n_k} \), where \( n_k \) is to be defined.

Since \( f \) is linear on \([-\varepsilon, \varepsilon]\), for large-enough \( k \) we have

\[
\sup \{|g_k'(x) - 1| : x \in [-\varepsilon, \varepsilon]\} = \sup \{|f_k'(x) - 1| : x \in [-\varepsilon/\lambda n_k, \varepsilon/\lambda n_k]\}.
\]

Now, for the choice \( n_k := \left\lceil \log_\lambda \left( \frac{\delta}{|f_k(0)|} \right) \right\rceil \), for all \( k \) we have \( \frac{\delta}{\lambda} < |g_k(0)| \leq \delta \). Hence, for a certain subsequence of \( g_k \), we get the \( C^1 \) convergence to a translation (by a certain \( \pm t \), where \( t \in [\delta/\lambda, \delta] \)). As \( \delta \) can be chosen arbitrarily small, the \( C^1 \) local closure of the group contains arbitrarily small translations. \( \square \)

## 3 Proof of the theorem

### 3.1 Exponential growth estimates

In all what follows, unless otherwise explicitly stated, we assume that \( G \subset \text{Diff}^{\omega}_+(S^1) \) is a free group in finitely many (though at least two) generators. Let us denote by \( \mathcal{G} \) its standard generating system (in which we include the inverses of all the generators).

For simplicity, whenever we write an element \( g \in G \) in the form \( g = \gamma_n \cdots \gamma_1 \), we will implicitly assume that this is its reduced expression, that is, each \( \gamma_i \) belongs to \( G \) and \( \gamma_i+1 \neq \gamma_i^{-1} \). We will also think of these expressions as reduced words or geodesics. Since generators are composed (multiplied) from right to left, we will call a suffix of \( g \) an expression of the form \( \gamma_n \cdots \gamma_k, 1 \leq k \leq n \), while an expression like \( \gamma_k \cdots \gamma_1 \) will be a prefix of \( g \).

Given \( g = \gamma_n \cdots \gamma_1 \) in \( G \), we will call the cone based at \( g \) the set \( \mathcal{C}_g \) of elements of the form \( gg' \), where \( g = \tau_n \cdots \tau_1 \) satisfies \( \tau_1 \neq \gamma_n^{-1} \). (Notice that \( g \) does not belong to \( \mathcal{C}_g \).)

This paragraph is devoted to the proof of the exponential growth for the sum of the derivatives:

**Theorem 3.1.** Let \( G \subset \text{Diff}^{\omega}_+(S^1) \) be a finitely-generated free group acting minimally. Then either \( G \) satisfies property \((\ast)\), or there exist positive constants \( c, \lambda \) such that for all \( x \in S^1 \) and all \( n \geq 1 \),

\[
\sum_{g \in B(n)} g'(x) \geq ce^{\lambda n}.
\]

We will deduce this result from the following proposition, the proof of which we postpone until §3.3.

**Proposition 3.2.** Let \( G \) be as in the preceding theorem. Then either \( G \) satisfies property \((\ast)\), or for all \( x \in S^1 \) and all \( \gamma \in \mathcal{G} \), there exists \( g = \gamma_k \cdots \gamma_1 \in \mathcal{C}_\gamma \) satisfying

\[
\sum_{j=1}^k (\gamma_j \cdots \gamma_1)'(x) > 2.
\]
Notice that the conclusion of this proposition is not far away from that of Proposition 2.5. Indeed, from that proposition we know that for each point \( x \), there are geodesics with arbitrarily big sums of derivatives along the compositions. Nevertheless, here we stress this property (in absence of property \((\star)\)) by asking for such geodesics in each of the cones \( C_\gamma, \gamma \in G \).

Let us now deduce Theorem 3.1 from Proposition 3.2:

**Proof of Theorem 3.1.** Assume that \( G \) satisfies the assumptions of Theorem 3.1 but does not satisfy the property \((\star)\). Then, due to Proposition 3.2, for every point \( x \in S^1 \) and for every cone \( C_\gamma, \gamma \in G \), we can find a geodesic \( g = \gamma_k \cdots \gamma_1 \in C_\gamma \) such that the sum of the intermediate derivatives along \( \gamma \) exceeds 2:

\[
\sum_{j=1}^k (\gamma_j \cdots \gamma_1)'(x) > 2.
\] (5)

Obviously, this inequality still holds in a small neighborhood of the initial point \( x \). Hence, by the compactness of the circle, we can choose a finite set \( F \) of possible elements \( g \). In particular, we can assume that the length of each of these \( g \) does not exceed some constant \( L \).

We claim that the following exponential lower bound holds for each \( x \in S^1 \) and all \( n \):

\[
\sum_{g \in B(n)} g'(x) \geq 2^{[n/L]}.
\] (6)

To prove this, we will actually prove a stronger statement. Namely, let \( x \in S^1 \) be fixed. We will show that for every \( m \geq 0 \), there exist a subset \( \Gamma_m \subset B(mL) \) and a map \( \gamma : \Gamma_m \rightarrow G \) (both depending on \( x \)), so that the associated cones \( C_{\gamma(g)}g \), with \( g \in \Gamma_m \), start at \( g \) and satisfy:

1) For each \( g \in \Gamma_m \), the cone \( C_{\gamma(g)}g \) contains no point of \( \Gamma_m \) (hence these cones are all mutually disjoint).

2) One has \( \sum_{g \in \Gamma_m} g'(x) \geq 2^m \).

The proof proceeds by induction on \( m \). For \( m = 0 \), one can take \( \Gamma_0 := \{ \text{id} \} \) and any function \( \gamma : \Gamma_0 \rightarrow G \). Assume that for some \( m \), the set \( \Gamma_m \) and the map \( \gamma \) have been constructed, and let us construct them for \( \tilde{m} := m + 1 \). To do this, for each \( g \in \Gamma_m \), consider the point \( y := g(x) \), and take \( \overline{g} = \overline{\gamma_k \cdots \gamma_1} \) in \( C_{\gamma(g)} \cap \mathcal{F} \) (with \( k \leq L \)) so that inequality (5) holds for \( \overline{g} \) at the point \( y \). Next, take \( \Gamma_{m+1} \) to be the set of elements of the form \( \overline{\gamma_j \cdots \gamma_1}g \), where \( g \) runs over all possible elements in \( \Gamma_m \) and \( 1 \leq j \leq k \), with \( \overline{g} \) associated to \( g \) as above. We then define the new map \( \overline{\gamma} \) on \( \Gamma_{m+1} \) by letting \( \overline{\gamma(\overline{\gamma_j \cdots \gamma_1}g)} \) to be equal to any element of \( G \) different from \( (\overline{\gamma_j})^{-1} \) and (in case \( j < k \)) from \( \overline{\gamma_j \cdots \gamma_1} \). (See Fig. 1.)

By construction, \( \Gamma_{m+1} \subset B_{L(m+1)}(e) \). Disjointness of cones also follows from the
construction. Moreover, we have

\[
\sum_{g \in \Gamma_{m+1}} \bar{g}'(x) = \sum_{g \in \Gamma_m} \sum_{j=1}^{k} (\bar{g}_j \cdots \bar{g}_1)'(x) = \sum_{g \in \Gamma_m} \sum_{j=1}^{k} (\bar{g}_j \cdots \bar{g}_1)'(g(x)) \cdot g'(x) = \\
= \sum_{g \in \Gamma_m} g'(x) \sum_{j=1}^{k} (\bar{g}_j \cdots \bar{g}_1)'(g(x)) \geq 2 \sum_{g \in \Gamma_m} g'(x) \geq 2 \cdot 2^m = 2^\bar{m}.
\]

This concludes the inductive proof.

Finally,

\[
\sum_{g \in B(n)} g'(x) \geq \sum_{g \in \Gamma_{[n/L]}} g'(x) \geq 2^{[n/L]},
\]

which shows (6) and hence concludes the proof of the theorem. \(\square\)

Roughly speaking, the sets \(\Gamma\) constructed above are the sets of leaves of “growing trees”.

### 3.2 Proof of the Main Theorem for minimal actions

In this section, we prove the statement of the Main Theorem for minimal actions. The proof relies on the dichotomy given by Theorem 3.1. More precisely, we will assume that inequality (4) holds, and we will show that there is no non-expandable point. Actually, we will show a much stronger fact, namely, no non-expandable point can arise provided the sum of derivatives along balls grows faster than quadratically. As we will see in Example 3.14, quadratic growth is critical for this phenomenon.
Suppose that there is some point $x_0 \in \text{NE}$. To fix notation, let $S_n$ denote the function defined on the circle as

$$S_n(x) := \sum_{g \in B(n)} g'(x),$$

and let $S_n := S_n(x_0)$. We will find a contradiction just assuming that

$$\frac{S_n}{n^2} \to \infty \quad \text{as} \quad n \to \infty. \quad (7)$$

For each $n \geq 1$, let us consider the set $X_n := \{g(x_0) \mid g \in B(n) \setminus \{\text{id}\}\}$. Let $x_n$ be the point of $X_n$ that is closest to $x_0$ on the right. Then $x_n = f_n(x_0)$ for a unique $f_n \in B(n)$. Denote $I_n := [x_0, x_n)$.

**Case 1. The stabiliser of $x_0$ is trivial.**

We start with an elementary

**Lemma 3.3.** The images $g(I_n)$, where $g \in B([n/2])$, are pairwise disjoint.

*Proof.* If two such intervals $g_1([x_0, x_n])$ and $g_2([x_0, x_n])$ intersect, then the left endpoint of one of them must belong to the other one, say $g_2(x_0) \in g_1([x_0, x_n])$. Therefore, $g_1^{-1} \circ g_2(x_0) \in [x_0, x_n]$. If $g_1 \neq g_2$, then $g_1^{-1} \circ g_2(x_0)$ cannot be equal to $x_0$, because of the hypothesis on the stabilizer of $x_0$. Nevertheless, this is impossible, because $g_1^{-1} \circ g_2$ belongs to $B(n) \setminus \{\text{id}\}$, and we defined $x_n$ to be the image of $x_0$ under a map $g \in B(n) \setminus \{\text{id}\}$ that is the closest on the right among all such images. \[\square\]

**Corollary 3.4.** The distortion coefficients of all maps $g \in B([n/2])$ are uniformly bounded on $I_n$.

*Proof.* Write $g = \gamma_k \cdots \gamma_1$, so that the intermediate images $\gamma_j \cdots \gamma_1(I_n)$, $j \leq k$, are pairwise disjoint. Then the sum of their lengths does not exceed 1. A direct application of Proposition 2.2 thus concludes the proof. \[\square\]

**Lemma 3.5.** The length $|I_n|$ decreases to zero as $n$ tends to infinity. More precisely, there exists a positive constant $C_1$ such that for all $n \geq 1$,

$$|I_n| \leq \frac{C_1}{S_{[n/2]}}.$$

*Proof.* By Corollary 3.4 there exists a constant $c_1 > 0$ such that for all $g \in B([n/2])$, we have $|g(I_n)| \geq c_1 |I_n| |g(x_0)| |I_n|$. Hence,

$$\sum_{g \in B([n/2])} |g(I_n)| \geq c_1 |I_n| \sum_{g \in B([n/2])} |g'(x_0)| = c_1 |I_n| S_{[n/2]}.$$

On the other hand, since the intervals in the left-side expression are disjoint, the sum of their lengths does not exceed 1. Thus,

$$|I_n| \leq \frac{1}{c_1 S_{[n/2]}},$$

which concludes the proof. \[\square\]
Now, let us fix a sequence of positive numbers \( r_n \) such that \( r_n = o(1/n) \) and \( 1/S_{[n/2]} = o(r_n) \). For instance, \( r_n := 1/\sqrt{nS_{[n/2]}} \) will do, though we do not need to deal with any explicit formula. The following lemma says that for all sufficiently large \( n \), every map \( g \in B(n) \) is “almost affine” in a complex neighborhood of \( x_0 \) of radius \( r_n \).

**Lemma 3.6.** There exist \( C_2 > 0 \) and an integer \( n_1 \) such that for all \( n \geq n_1 \) and all \( g \in B(n) \), one has

\[
\kappa(g, U_{r_n}(x_0)) \leq C_2nr_n \xrightarrow{n \to \infty} 0.
\]

**Proof.** As the point \( x_0 \) is non-expandable, writing each \( g \in B(n) \) in its reduced form

\[
g = \gamma_k \cdots \gamma_1,
\]

we obtain the following upper bound for the sum of the intermediate derivatives:

\[
\sum_{j=0}^{k-1} (\gamma_j \cdots \gamma_1)'(x_0) \leq \sum_{j=0}^{k-1} 1 = k \leq n.
\]

A direct application of Proposition 2.6 then shows the lemma. \( \square \)

The next two lemmas are devoted to show that the maps \( f_n \) are in fact close to the identity on the \( r_n \)-neighborhood of \( x_0 \).

**Lemma 3.7.** The maps \( \tilde{f}_n(y) := \frac{1}{r_n} \left( f_n(x_0 + r_n y) - x_0 \right) \) converge to the identity in the \( C^1 \) topology on \( U_1^C(0) \).

**Proof.** By Lemma 3.5, we have that

\[
\tilde{f}_n(0) = \frac{1}{r_n} (x_n - x_0) = O\left( \frac{1}{r_n S_{[n/2]}} \right) \xrightarrow{n \to \infty} 0.
\]

Hence, it suffices to check that the derivatives \( \tilde{f}_n' \) tend to 1. Moreover, due to the control of distortion guaranteed by Lemma 3.6, it suffices to check such a convergence at a single point.

To do this, let us consider the map \( f_n^{-1} \in B(n) \). On the one hand, \( (f_n^{-1})'(x_0) \leq 1 \), as \( x_0 \) is a non-expandable point. On the other hand,

\[
(f_n^{-1})'(x_n) = (f_n^{-1})'(f_n(x_0)) = \frac{1}{f_n'(x_0)} \geq 1.
\]

Since \( ||x_0, x_n|| = O(1/S_{[n/2]}) = o(r_n) \), for all \( n \) sufficiently large, we have \( x_n \in U_{r_n}(x_0) \). Control of distortion then implies that these two derivatives are close to each other:

\[
\left| \log \left( \frac{(f_n^{-1})'(x_n)}{(f_n^{-1})'(x_0)} \right) \right| \leq C_2nr_n.
\]

Thus, both derivatives are \( o(1) \)-close to 1, and hence the same holds for \( f_n'(x_0) \). \( \square \)

Next, in order to apply Ghys’ commutators technique, we need to find two elements in \( G \) that generate a non-solvable group and are close enough to the identity on some interval. This is done by the following
Lemma 3.8. The sequence $f_n$ contains an infinite subsequence $f_{n_i}$ such that for each $i$, the elements $f_{n_i}$ and $f_{n_i+1}$ generate a free group.

Proof. It is well known that any two elements of a free group either generate a free subgroup, or belong to the same cyclic subgroup. Thus, if the conclusion of this lemma didn’t hold, all the maps $f_n$ with sufficiently large $n$ would be powers of some fixed $h \in G$, say $f_n = h^{k(n)}$ for all $n \geq m$. Since $f_n$ belongs to $B(n)$, we must necessarily have $|k(n)| \leq n$.

Now for each $n \geq m$, let us consider the segment of $h$-orbit $Y_n := \{h^j(x_0) \mid j = -n, \ldots, n\}$. Let $\bar{x}_n$ be the point of $Y_n \setminus \{x_0\}$ that is closest to $x_0$ on the right. The interval $\bar{I}_n := [x_0, \bar{x}_n)$ is contained in $I_n$. As a consequence, $x_0$ is (nonperiodic and) recurrent under the action of $h$, hence the rotation number $\tau(h)$ must be irrational.

Next, notice that there are arbitrarily large values of $n$ with the following property: the intervals $h^j(\bar{I}_n)$, $j = 0, 1, \ldots, 2n$, cover the whole circle with multiplicity at most two. Indeed, such a property is invariant under topological conjugacy, thus it suffices to check it for the Euclidean rotation of angle $\tau(h)$. Now, for this particular case, it can be easily verified for each integer of the form $n = q_i - 1$, where $\frac{h_i}{q_i}$ is the sequence of good rational approximations of $\tau(h)$.

Finally, control of distortion shows that for each of the integers $n$ above and $0 \leq j \leq 2n$,

$$|h^j(\bar{I}_n)| \leq C(h^j)'(x_0)|\bar{I}_n| \leq C|\bar{I}_n| \leq \frac{C C_2}{S_{[n/2]}},$$

where the second inequality comes from that $x_0$ is non-expandable and the last one from Lemma 3.5. As a consequence, the sum of lengths of the intervals $h^j(\bar{I}_n)$, $j = 0, 1, \ldots, 2n$, is $O(n/S_{[n/2]})$. It is hence smaller than 1 for $n$ large enough, which contradicts the fact that these intervals cover the circle. \hfill \Box

Together with Lemma 3.7, the preceding lemma implies that there exists $n$ such that the maps $f_n, f_{n+1}$ generate a free group and are simultaneously close to the identity on $U_{r,0}^C(x_0)$. By Proposition 2.7, the sequence of their commutators tends to the identity on $U_{r,0}^C(x_0)$. Using Proposition 2.8, we conclude that there exist intervals $J \subset I \subset \mathbb{S}^1$ and a change of variables $\varphi : I \to [-1,2]$ after which the $C^1([0,1],[-1,2])$-closure of the set of restrictions $\{\varphi \circ g_{ij} \circ \varphi^{-1} \mid g \in G\}$ contains all the translations $T_s : y \mapsto y + s$ with $s \in [-1,1]$.

We claim that the last property above yields a contradiction. To see this, first recall that Sacksteder’s theorem is valuable in our context: there exists $f \in G$ with a hyperbolic repelling fixed point $y_0$ (see [8] for a discussion on this). By minimality, $y_0$ can be chosen to belong to any small open subinterval (up to changing $f$ by an appropriate conjugate). Now, the existence of translations in the local $C^1$ closure of the group implies that $x_0$ can be mapped arbitrarily close to $y_0$ keeping the derivative bounded away from zero:

$$\exists C_3 > 0, \ h_n \in G : \ h_n(x_0) \to y_0, \ h'_n(x_0) \geq C_3 \ \forall n.$$

Take $k$ such that $f'(y_0)^k > \frac{1}{C_3}$, and consider the element $g_n := f^k \circ h_n$. We have

$$\limsup_{n \to \infty} g_n'(x_0) = \limsup_{n \to \infty} (f^k \circ h_n)'(x_0) \geq C_3 \cdot f'(y_0)^k > 1,$$
thus obtaining the desired contradiction.

**Remark 3.9.** The final arguments of this proof, together with Remark 2.9, imply that whenever NE ≠ ∅, the group G must be $C^1$-locally discrete.

**Case 2. The stabiliser of $x_0$ is nontrivial.**

In this case, we first need some information on the stabilizer of $x_0$.

**Lemma 3.10.** The stabilizer of $x_0$ in G is infinite cyclic.

*Proof.* By a lemma of I. Nakai (see [25, Section 3]), if this stabilizer was not Abelian, then it would contain a flow in its closure. However, this is impossible by the very same reasons of those at the end of the proof in the preceding case. Hence, the stabilizer of $x_0$ is Abelian, and since G is free, it must be cyclic. 

Let us denote by $\tilde{h}$ the generator of the stabilizer of $x_0$ that is topologically contracting towards $x_0$ on a right neighborhood of it. Notice that $\tilde{h}'(x_0) = 1$, as $x_0$ is non-expandable. Again, for each $n ≥ 1$, let us consider the set $X_n := \{g(x_0) \mid g ∈ B(n)\}$ that is closest to $x_0$ on the right, and let $I_n := [x_0, x_n)$. In what follows, we will assume that $n$ is large enough so that $I_n$ contains no fixed point of $\tilde{h}$ in its interior (this is possible as the orbit of $x_0$ is dense). Although the images $g(I_n)$, with $g ∈ B([n/2])$, are no longer pairwise disjoint, we have the next

**Lemma 3.11.** If $g_1, g_2$ in $B([n/2])$ are such that $g_1(I_n)$ and $g_2(I_n)$ do intersect, then there exists $j$ such that $g_2 = g_1 \tilde{h}^j$ and $|j| ≤ n$.

*Proof.* The arguments of the beginning of the proof of Lemma 3.3 show that $g_2 = g_1 \tilde{f}$ for a certain $\tilde{f}$ in the stabilizer of $x_0$. Writing $\tilde{f}$ as $\tilde{h}^j$ for a certain $j$, we have that $\tilde{h}^j = g_1^{-1}g_2$ belongs to $B(n)$, which forces $|j| ≤ n$. 

**Lemma 3.12.** The length $|I_n|$ decreases to zero as $n$ tends to infinity. More precisely, there exists a constant $C'_1$ such that for all $n ≥ 1$,

$$|I_n| ≤ \frac{C'_1 n}{S[n/2]}.$$  

*Proof.* Let $r'_n > 0$ be such that $r'_n = o(1/n)$ and $n/S[n/2] = o(r'_n)$. (For instance, $r'_n := 1/\sqrt{S[n/2]}$ will do.) Let $I'_n := [x_0, x'_n)$ be the interval to the right of $x_0$ of length exactly equal to $r'_n$. The argument of the proof of Lemma 3.6 yields constants $C'_2, C'_2$ such that each $g ∈ B(n)$ satisfies

$$\kappa(g, I'_n) ≤ C'_2 nr'_n ≤ C'_2.$$  

By the preceding Lemma, the multiplicity of the family of intervals $g(I_n)$, with $g$ ranging over $B([n/2])$, is at most $2n + 1$. Therefore,

$$2n + 1 ≥ \sum_{g ∈ B([n/2])} |g(I_n)|.$$
Assume that $I_n$ is not contained in $I'_n$. Then $g(I'_n) \subset g(I_n)$ for all $g$, which yields

$$2n + 1 \geq \sum_{g \in B([n/2])} |g(I'_n)|,$$

Using (8) we thus obtain

$$2n + 1 \geq \sum_{g \in B([n/2])} e^{-C^2 g'(x_0)} |I'_n| = e^{-C^2} |I'_n| S_{[n/2]}.$$

As a consequence,

$$r'_n = |I'_n| \leq \frac{e^{C^2} (2n + 1)}{S_{[n/2]}},$$

which is impossible for large $n$ due to our assumption (7). Therefore, $I_n$ is contained in $I'_n$ for large $n$, which easily allows to show that its length goes to zero with the claimed speed. 

Starting from this point, the proof proceeds in the very same way as that given for the first case, modulo working on neighborhoods of radius $r'_n$ about $x_0$ instead of radius $r_n$. Moreover, it is useful to point out that, although the elements $f_n \in B(n)$ sending $x_0$ into $x_n$ are no longer uniquely defined, any element in $B(n)$ sending $x_0$ into $x_n$ will actually work. We leave the details to the reader.

**Remark 3.13.** Notice that we could have avoided the second case above just showing by contradiction that (4) still implies property (⋆). However, the more involved argument above allows us to obtain a direct proof for the existence of a Markov partition. Indeed, as we will show in the next Section, the failure of (4) not only leads to property (⋆) with a nonempty set $\text{NE}$, but also to the existence of a Markovian partition for the dynamics. (Compare [9], where the Markovian property is established as a consequence of property (⋆) whenever $\text{NE} \neq \emptyset$.) Moreover, (a careful reading of) the proof also reveals a quite striking fact: any growth rate faster than linear for the sum in (4) still implies property (⋆). Finally, the contradiction obtained in the second case requires only superquadratic growth for the sums $S_n$; due to the next example, it seems that this bound is quite close to the sharp one.

**Example 3.14.** Let us consider the projective minimal (real-analytic) action of $\text{PSL}_2(\mathbb{Z})$ on the circle identified with the projective line (see [7, §5.2] for details). The point $(1 : 0)$ is non-expandable for this action. Moreover, the derivative at this point of an element $[(a \ b)\ ] \in \text{PSL}_2(\mathbb{Z})$ is $1/(a^2 + c^2)$. Now, if such an element belongs to the ball of radius $n$ with respect to the system of generators $[(1\ 0), (1\ 1)]$, then an elementary argument shows that the absolute values of the entries $a, b, c, d$ are smaller than or equal to the $n^{th}$ term $F_n$ of the Fibonacci sequence. Taking into account the action of the stabilizer of $(1 : 0)$, this roughly yields the upper bound

$$n \sum_{|a| \leq F_n, |c| \leq F_n} \frac{1}{1 + a^2 + c^2} \sim n \log(F_n) = O(n^2)$$

for the sum of derivatives of elements in $B(n)$ at this point. Actually, a finer argument shows that this sum is $o(n^2)$: this comes from the fact that the law of elements of
continued fractions (Gauss-Kuzmin distribution; see, e.g., [35] and references therein) has an infinite expectation. On the other hand, it seems quite plausible that the corresponding averages grow at most logarithmically, and thus that the sums $S_n((1 : 0))$ for $\text{PSL}_2(\mathbb{Z})$ are bounded from below by $n^2 / \log^2 n$.

Finally, $\text{PSL}_2(\mathbb{Z})$ contains an index-6 free subgroup (in two generators). For this subgroup, the same arguments apply.

### 3.3 Exponential growth of sums of derivatives along geodesics

This section is devoted to the proof of Proposition 3.2. In fact, we will prove a stronger statement. Namely, for each $\gamma \in G$, consider the function $S_\gamma : \mathbb{S}^1 \to [0, +\infty]$ defined as

$$S_\gamma(y) := \sup \left\{ \sum_{j \geq 0} (\gamma_j \cdots \gamma_1)'(y) \mid \forall j, \gamma_j \neq \gamma_{j+1}, \gamma_1 = \gamma \right\}.$$ 

We will prove that if there exist $\gamma \in G$ and $y \in \mathbb{S}^1$ such that $S_\gamma(y) < \infty$, then $G$ satisfies property ($\ast$).

First, notice that control of distortion arguments (c.f., Proposition 2.4) easily yield that the function $S_\gamma$ is continuous and the set

$$M_\gamma := \{ y \in \mathbb{S}^1 \mid S_\gamma(y) < \infty \}$$

is open, for every $\gamma \in G$. In the sequel, we will actually be mostly concerned with the functions $\widetilde{S}_\gamma(y) := \max_{\tilde{\gamma} \neq \gamma^{-1}} S_{\tilde{\gamma}}(y)$, as well as the sets

$$\widetilde{M}_\gamma := \{ y \in \mathbb{S}^1 \mid \widetilde{S}_\gamma(y) < \infty \} = \bigcap_{\tilde{\gamma} \neq \gamma^{-1}} M_{\tilde{\gamma}}.$$

**Example 3.15.** The free group in two generators may be seen as the Fuchsian group associated to an hyperbolic punctured torus. A fundamental domain of its action on the hyperbolic disc is an absolute quadrangle, and the maps $f$ and $g$ are respectively gluing its opposite sides (see Fig. 2 on the right). The vertices of this quadrangle divide the absolute circle into four arcs.

The dynamics of the action of $\langle f, g \rangle$ on the absolute circle is Shottky-like: each map in $\mathcal{G} := \{ f, g, f^{-1}, g^{-1} \}$ sends three of these arcs into one of them and the other one into the remaining three. It is thus very natural, and not so difficult to see, that these four arcs turn out to be $\widetilde{M}_f, \widetilde{M}_g, \widetilde{M}_{f^{-1}}$ and $\widetilde{M}_{g^{-1}}$. More precisely, the arc into which $\gamma \in \mathcal{G}$ contracts three of them is $M_\gamma$. Indeed, if we apply any geodesic that does not start with $\gamma^{-1}$, we contract the corresponding arc further and further; moreover, one can check (though this is not an immediate computation) that the associated sum of derivatives stay bounded for any point inside the arc.

The idea of the rest of the section is to show that the situation in the general case is quite alike the one that we have just described in the preceding example. More precisely, the sets $\widetilde{M}_\gamma$ will turn out to be unions of finitely many intervals, so that they make a finite partition of the circle, and applying $\gamma^{-1}$ on each $\widetilde{M}_\gamma$ will lead to an expansion-like Markovian dynamics.
Lemma 3.16. The following properties hold:

1) For all $\gamma \in \mathcal{G}$, one has $M_\gamma \cap \widetilde{M}_\gamma = \bigcap_{\gamma \in \mathcal{G}} M_\gamma = \emptyset$; also, for any two different $\gamma_1, \gamma_2$ in $\mathcal{G}$, the sets $\widetilde{M}_{\gamma_1}$ and $\widetilde{M}_{\gamma_2}$ are disjoint.

2) For all $\gamma \in \mathcal{G}$, the image $\gamma(M_\gamma)$ coincides with $\widetilde{M}_\gamma$.

3) For all $\gamma_1, \gamma_2$ in $\mathcal{G}$ such that $\gamma_1 \neq \gamma_2^{-1}$, one has $\gamma_1(M_{\gamma_2}) \subseteq \widetilde{M}_{\gamma_1}$.

4) If at least one of the sets $M_\gamma$ is nonempty, then all sets $\widetilde{M}_\gamma$ are nonempty.

Proof. The first property follows from Proposition 2.5, 2) and 3) are immediate consequences of the definition, and 4) follows as a combination of them. \qed

As we have already mentioned, our idea is to show that the sets $\widetilde{M}_\gamma$ decompose the circle into finitely many intervals that form a Markov partition for the dynamics. In this way, the dynamics of the action of $G$ will turn out to be quite similar to that of the “degenerated” Schottky group $F_2 \subset \text{PSL}_2(\mathbb{Z})$ described in Example 3.15.

Take any connected component $I$ of a set $\widetilde{M}_\gamma$. We will consider the dynamics of $I$ under the action of $G$. We start by studying how the boundedness of $\widetilde{S}_\gamma$ disappears at the endpoints of $I$. To do this, we will consider the images of $I$ until the “first-return”:

Definition 3.17. Let $\gamma \in \mathcal{G}$ and $I$ as above. An element $g = \gamma_n \cdots \gamma_1$ such that $\gamma_1 \neq \gamma^{-1}$ and all the intermediate images $\gamma_k \cdots \gamma_1(I)$, $k = 1, \ldots, n - 1$, are disjoint from $I$ will be said to be $(\gamma, I)$-admissible. A $(\gamma, I)$-admissible element $g$ is a $(\gamma, I)$-first-return if $g(I)$ intersects $I$.

In what follows, the generator $\gamma$ and the interval $I$ will be fixed. Accordingly, we will just speak about admissible elements and first-return maps.

Lemma 3.18. The images of $I$ under any two different admissible elements are pairwise disjoint. Moreover, if $g = \gamma_n \cdots \gamma_1$ is a first-return, then $\gamma_n = \gamma$, and $g(I)$ is a subset of $I$. 

Proof. Assume that the images of $I$ under two admissible elements $g = \gamma_n \cdots \gamma_1$ and $\overline{g} = \overline{\gamma}_m \cdots \overline{\gamma}_1$ intersect. By Lemma 3.16.3), we have $g(I) \subset \overline{M}_{\gamma_n}$ and $\overline{g}(I) \subset \overline{M}_{\overline{\gamma}_m}$. Hence, $\overline{M}_{\gamma_n} \cap \overline{M}_{\overline{\gamma}_m} \neq \emptyset$, which by Lemma 3.16.1) implies that $\gamma_n = \overline{\gamma}_m$. Therefore, we can remove these two letters, thus obtaining shorter admissible elements sending $I$ into non-disjoint intervals. This process stops when one of the words becomes empty. If the other word becomes empty simultaneously, then they are equal, that is, $g = \overline{g}$. Otherwise, a prefix of one of these words is a first-return, and this contradicts the definition of an admissible element.

Now, if $g = \gamma_n \cdots \gamma_1$ is a first-return, then we have $\overline{M}_{\gamma_n} \cap \overline{M}_{\overline{\gamma}_m} \supset g(I) \cap I \neq \emptyset$. By Lemma 3.16.3), we must necessarily have $\gamma_n = \overline{\gamma}_m$. Finally, since the interval $g(I)$ is a subset of $\overline{M}_{\gamma_n}$ and $I$ is a connected component of $\overline{M}_{\gamma_n}$, we must also have $g(I) \subset I$. \hfill $\Box$

We next come up with the key step of the proof.

**Lemma 3.19.** There exist first-returns $g_-, g_+$ that fix respectively the left and right endpoints $x_-, x_+$ of $I$. Moreover, $g_+ \neq g_-$.

Before passing to the proof, let us notice that this is exactly what happens for the Fuchsian group corresponding to a punctured torus:

**Example 3.20.** Let $G = \langle f, g \rangle$ be the group of Example 3.15. Then, the left and right endpoints of $I := \overline{M}_f$ are fixed by $fg^{-1}f^{-1}$ and $fg^{-1}f^{-1}g$, respectively (see Fig. 3).

![Diagram](image)

Figure 3: On the left: (some) of the admissible images and first returns of the interval $I := \overline{M}_f$ for the Fuchsian group considered in Examples 3.15 and 3.20. On the right: the corresponding abstract tree of (disjoint!) admissible images. In both pictures, the composition $g_+$ is shown by bold arrows.

**Proof of Lemma 3.19.** Let us show that there exists a first-return $g_+$ that fixes $x_+$. To do this, first notice that due to Proposition 2.2 and Lemma 3.18 we have a uniform control for the distortion on $I$ of all admissible words: there exists $C_3 > 0$ such that for every admissible word $g$, one has $\kappa(g; I) < C_3$. Moreover, the sum of intermediate
derivatives corresponding to every admissible word at all points of $I$ can be bounded from above by $C_4 := \frac{1}{|I|} e^{C_3}$.

Since the images of $I$ under first-returns are pairwise disjoint, there is only a finite number of first-returns $g$ such that $|g(I)| \geq \frac{1}{2} e^{-C_4}|I|$. Let $g_1, \ldots, g_m$ be the set of all these first-returns. We will show that in fact one of these elements fixes $x_+$. The proof proceeds by contradiction: assuming otherwise, we will show that $\tilde{S}_\gamma(x_+)$ is finite.

From now on, assume that none of $g_1, \ldots, g_m$ fixes $x_+$, and consider the sets

$$U := \bigcup_{j=1}^m g_j(T), \quad \tilde{U} := \bigcup_{j=1}^m g_j(T \setminus U) \subset U.$$ 

Notice that the set $\tilde{U}$ is bounded away from the endpoints of $I$. Indeed, $\tilde{U}$ is bounded away from the right endpoint $x_+$ by our assumption. Next, either for each $j$ we have $g_j(x_-) \neq x_-$, and then the same applies to the left endpoint; or for some $j$ one has $g_j(x_-) = x_-$, and in this case by definition the set $\tilde{U}$ cannot intersect $g_j^2(T) = [x_-, g_j^2(x_+)]$, hence it is also bounded away from $x_-$.

Now, since $\tilde{U}$ is bounded away from the endpoints of $I$, there must exist a finite constant $C_S$ bounding $\tilde{S}_\gamma$ on $\tilde{U}$:

$$C_S := \sup_{x \in \tilde{U}} \tilde{S}_\gamma(x) < \infty.$$ 

We will next prove that for every element $g = \gamma_n \cdots \gamma_1 \notin C_{\gamma^{-1}}$, the sum of the intermediate derivatives at each point of $T \setminus U$ (including $x_+$) does not exceed some constant, thus yielding a contradiction. More precisely, we will show that

$$\sum_{j=0}^{n-1} (\gamma_j \cdots \gamma_1)'(y) \leq 2 \max \{ e^{C_3} \cdot C_S, C_4 \}, \quad \forall y \in T \setminus U, \forall \gamma_1 \neq \gamma^{-1}. \tag{9}$$

(As in Proposition \ref{proposition2.2}, for $j = 0$, the expression $\gamma_j \cdots \gamma_1$ stands for the identity.) To do this, we proceed by induction on the length of $g$. The case $n = 1$ is evident, as well as the case where $g$ is an admissible element. Next, if $g$ is not an admissible element, then it contains a prefix $\overline{g} = \gamma_k \cdots \gamma_1$ that is a first-return. Now notice that

$$\sum_{j=0}^{n-1} (\gamma_j \cdots \gamma_1)'(y) = \overline{g}'(y) \cdot \sum_{j=0}^{n-1} (\gamma_j \cdots \gamma_{k+1})'(\overline{g}(y)) + \sum_{j=0}^{k-1} (\gamma_j \cdots \gamma_1)'(y). \tag{9}$$

Since $\overline{g}$ is a first-return, the second sum in the right-side expression above does not exceed $C_4$. To estimate the first sum, we need to consider two possibilities.

The first possibility is that $\overline{g}$ coincides with one of the $g_j$’s. In this case, we notice that $\overline{g}(y) \in \tilde{U}$, and hence the sum doesn’t exceed $C_S$. We claim that the derivative $\overline{g}'(y)$ doesn’t exceed $e^{C_3}$, and thus the right-side expression of (9) is bounded by $e^{C_3} \cdot C_S + C_4$. Indeed, on the one hand, as $\overline{g}(I) \subset I$, there must be a point in $I$ at which the derivative is less than or equal to $1$. On the other hand, we have $\kappa(\overline{g}, I) \leq C_3$. The claim easily follows from this.

The second possibility is that $\overline{g}$ does not coincide with any of the $g_j$’s. In this case, we have the following uniform upper bound for its derivative:

$$\overline{g}'(x) \leq e^{C_3} \frac{|\overline{g}(I)|}{|I|} \leq \frac{1}{2} \quad \forall x \in I.$$
As a consequence,
\[
\sum_{j=0}^{n-1} (\gamma_j \cdots \gamma_1)'(y) = \overline{g}(y) \cdot \sum_{j=k}^{n-1} (\gamma_j \cdots \gamma_{k+1})'(\overline{g}(y)) + \sum_{j=0}^{k-1} (\gamma_j \cdots \gamma_1)'(y) \leq \frac{1}{2} \sum_{j=k}^{n-1} (\gamma_j \cdots \gamma_{k+1})'(\overline{g}(y)) + C_4. \tag{10}
\]

(As above, for \( j = k \) the expression \( \gamma_j \cdots \gamma_{k+1} \) stands for the identity.) Since \( \overline{g}(y) \in \overline{T} \setminus U \), the induction hypothesis yields that the sum (from \( j = k \) to \( n - 1 \)) in the right-side expression above doesn’t exceed \( \max\{e^{C_3} \cdot C_S, C_4\} \). Thus, we have obtained the desired upper bound:
\[
\max\{e^{C_3} \cdot C_S, C_4\} + C_4 \leq 2 \max\{e^{C_3} \cdot C_S, C_4\}.
\]

This concludes the proof by contradiction, as we have shown that \( \overline{S}_n(x_+) \) is finite. This contradiction came from the assumption that no \( g_j \) fixes \( x_+ \). Thus, there exists a first-return \( g_+ \) among \( g_1, \ldots, g_m \) that fixes \( x_+ \). Similarly, there must exist a first-return \( g_- \) among \( g_1, \ldots, g_m \) that fixes \( x_- \). To show that these are different elements, notice that if we had \( g_+ = g_- = g \), then every fundamental domain \( J \subset I \) for the action of \( g \) on \( I \) would be wandering for the action of \( G \), that is, all its images by elements of \( G \) different from itself would be disjoint (indeed, every element of \( G \) could be represented as the composition of some iterate of either \( g \) or \( g^{-1} \) followed by an admissible word). However, this would clearly contradict the density of the \( G \)-orbits of points of \( I \).

Recall that the general idea is to show that the sets \( \overline{M}_\gamma \) decompose the circle into finitely many intervals that form a Markov partition for the dynamics. Thus, at the same point where a connected component of one of these set ends, a connected component of another (perhaps different) set should start. We have already seen such a behavior in Example 3.15. The following lemma shows that this is always the case.

**Lemma 3.21.** Let \( I, x_+, g_+ \) be as above, and write \( g_+ = \gamma_n \cdots \gamma_1 \). Then \( x_+ \) is the left endpoint of a connected component of some \( \overline{M}_\gamma \), where \( \overline{\gamma} = \gamma \) if \( g_+ \) is topologically contracting on a right neighborhood of \( x_+ \), and \( \overline{\gamma} = \gamma_1^{-1} \) otherwise.

**Proof.** We first claim that in some neighborhood of the point \( x_+ \), we have a uniform upper bound for the sum of derivatives along all geodesics that do not start with neither \( g_+ \) nor \( g_+^{-1} \). To show this, it suffices to establish such an estimate at the point \( x_+ \), as control of distortion will then ensure that we still have a uniform bound for the sum of derivatives on a neighborhood of \( x_+ \) (see Proposition 2.4). We need to consider two cases:

If such a geodesic doesn’t start with \( \gamma^{-1} \): Then this geodesic is either admissible or decomposes as the product of a first-return \( g_j \) different from \( g_+^{\pm 1} \) and a geodesic that (by Lemma 3.18) does not start with \( \gamma^{-1} \). In the former case, the sum of derivatives on \( I \) along the geodesic is bounded from above by \( \frac{1}{M} e^{C_3} \). In the latter case, \( g_j(x_+) \) belongs to \([g_-(x_+), g_+(x_-)]\), and hence is bounded away from the endpoints of \( I \). Thus,
it lies in a region where \( \tilde{S}_\gamma \) is uniformly bounded, so that the sum of derivatives at the point \( x_+ \) along the geodesic is still uniformly bounded.

*If the geodesic starts by \( \gamma^{-1} \), but not with \( g_+^{-1} \):* In this case, let us apply \( g_+ \) before going along this geodesic. This procedure doesn’t change the fact that the sum of derivatives along the geodesic is bounded, due to the fact that \( g_+ \) fixes \( x_+ \). After simplification, the new geodesic doesn’t start with \( \gamma^{-1} \) (because the first element of \( g_+ \) will be not cancelled), hence the sum of derivatives remains uniformly bounded as in the previous case.

Now, assume that \( g_+ \) is contracting in a right neighborhood of \( x_+ \). Our goal is to show that there is a uniform upper bound for the sum of derivatives along geodesics not starting with \( \gamma^{-1} \) at points in a right open neighborhood of \( x_0 \). (Indeed, this will imply that \( \tilde{M}_\gamma \) contains a right open neighborhood of \( x_+ \).) To show this, notice that, since \( \gamma_n = \gamma \), every such geodesic can be decomposed as a non-negative power of \( g_+ \) followed by a word that doesn’t start with neither \( g_+ \) nor \( g_+^{-1} \). If \( g_+ \) does not appear as a prefix, then the arguments of the first part of the proof apply. Assume next that \( g_+ \) does appear. Notice that the sum of the derivatives of any number of iterates of \( g_+ \) at points in a right open neighborhood of \( x_+ \) is uniformly bounded. Indeed, such a neighborhood can be decomposed as the union of the images by iterates of \( g_+ \) of a fundamental domain. Since the interior of these intervals are disjoint, the sum of their lengths is uniformly bounded, and control of distortion ensures the upper bound for the sum of derivatives. Now, after applying this (nontrivial) iterate of \( g_+ \), we are left with a suffix that doesn’t start with neither \( g_+ \) nor \( g_+^{-1} \). Applying to this suffix the estimate from the beginning of the proof provides us the desired uniform bound.

The case where \( g_+^{-1} \) is contracting on a right neighborhood of \( x_+ \) can be treated analogously.

Lemma 3.22. Neither \( g_- \) nor \( g_+ \) have fixed points in the interior of \( I \). Moreover, \( g_-'(x_-) = g_+'(x_+) = 1 \).

*Proof.* Assume first that \( \kappa := g_+'(x_+) < 1 \). We will prove that \( x_+ \) belongs to \( \tilde{M}_\gamma \), thus yielding a contradiction. To show this, we need to consider geodesics that do not start with \( \gamma^{-1} \) and provide an upper bound for the sum of the derivatives at the point \( x_+ \) along it. Now, such a geodesic does not start with \( g_+^{-1} \), hence the arguments of proof of the previous lemma apply except in the case where the geodesic starts with a positive power \( g_n^k \) of \( g_+ \). For this case, we need to provide an alternative argument to bound the sum of derivatives along this iteration (the sum along the suffix remains bounded by the very same argument as before). This is easy: just notice that writing \( g_+ = \gamma_n \cdots \gamma_1 \), the involved sum is

\[
\sum_{j=0}^{k} \sum_{i=1}^{n} (g_+^{j}(x_+))^j (\gamma_i \cdots \gamma_1)^j(x_+) \leq \frac{1}{1 - \kappa} \sum_{i=1}^{n} (\gamma_i \cdots \gamma_1)^j(x_+).
\]

Therefore, \( g_+'(x_+) \geq 1 \).

Now, if we prove that \( g_+ \) has no fixed point inside \( I \), then the fact that \( g_+(I) \) is strictly contained in \( I \) will imply that \( g_+'(x_+) \) cannot be greater than 1, and hence equals 1. Assume for a contradiction that \( g_+(y) = y \in I \). We claim that for every
point \( x \in [y, x_+] \) not fixed by \( g_+ \), the sum of derivatives along all geodesics is finite, which contradicts Proposition 2.5. Indeed, the first arguments of proof of the preceding lemma show that these sums are uniformly bounded for geodesics not starting with neither \( g_+ \) nor \( g_+^{-1} \). Now, to deal with geodesics starting with one of these elements, just notice that the point \( x \) lies in a fundamental domain for the action of both \( g_+ \) and \( g_+^{-1} \) whose whole orbit is contained in \([y, x_+]\). Under such iterations, one can hence control the distortion. If the geodesic has a suffix, then one proceeds as before: either this suffix is an admissible word, and we are done, or it is the composition of a first-return different from \( g_+ \) and an admissible word, and in this case the image point under the first-return lies in a region where \( \tilde{S}_\gamma \) can be uniformly bounded, so we are done as well.

Obviously, similar arguments apply to conclude that \( g_- \) has no fixed point in \( I \), and \( g'_-(x-) = 1 \).

Lemma 3.23. There is only a finite number of connected components of sets \( \tilde{M}_\gamma \).

Proof. For each connected component \( I \) of some \( \tilde{M}_\gamma \), write \( g_+ = \gamma_n \cdots \gamma_1 \), and let \( \Phi_R(I) \) be the connected component of \( \tilde{M}_{\gamma_1} \) that contains \( \gamma_1(I) \). Note that \( \Phi_R(I) \) and \( \gamma_1(I) \) have the same right endpoint \( \gamma_1(x_+) \). Indeed, since \( \gamma_n \cdots \gamma_2(\Phi_R(I)) \subset \tilde{M}_\gamma \), in the case where \( \Phi_R(I) \) contained \( \gamma_1(x_+) \), we would have that \( g_+(x_+) = x_+ \) belongs to \( \tilde{M}_\gamma \), which is absurd.

We next show that the first-return \( \tilde{g}_+ \) fixing the right endpoint of \( \Phi_R(I) \) is exactly the conjugate of \( g_+ \) by \( \gamma_1 \):

\[
\tilde{g}_+ = \gamma_1 \gamma_n \cdots \gamma_2.
\]

Indeed, this conjugate doesn’t start with \( \gamma_1^{-1} \) and fixes the right endpoint of \( \Phi_R(I) \). Hence, it is either \( \tilde{g}_+ \), or a higher power of it. But the latter case would imply that while applying \( \gamma_1 \gamma_n \cdots \gamma_2 \) to \( \gamma_1(x_+) \), we pass through \( \gamma_1(x_+) \) at some intermediate step. Removing the last applied letter (which should be \( \gamma_1 \)), this would imply that \( g_+ \) wasn’t admissible, which is a contradiction.

By the previous discussion, the map \( I \mapsto \Phi_R(I) \) yields a dynamics for which every \( I \) is contained in a cycle. Moreover, the composition of the maps associated to the cycle gives the corresponding element \( g_+ \). Clearly, any two such cycles either coincide or are disjoint.

Similarly, writing \( g_- = \overline{\gamma}_n \cdots \overline{\gamma}_1 \), we may consider the connected component \( \Phi_L(I) \) of \( \tilde{M}_{\overline{\gamma}_1} \) that contains \( \overline{\gamma}_1(I) \). This yields a new dynamics for which every \( I \) belongs to a cycle, and any two such cycles either coincide or are disjoint.

Now, for every connected component \( I \) as above, the intervals \( g_+(I) \) and \( g_-(I) \) are disjoint and contained in \( I \), hence the length of at least one of them doesn’t exceed \( |I|/2 \). Since by the preceding lemma the derivative at the corresponding endpoint equals 1, the distortion of the corresponding first-return is bounded from below by \( \log 2 \). This implies that the sum of the lengths of the image intervals along the composition is bounded from below by \( \log 2 \frac{C}{C_0} \).

All the right cycles are disjoint, and so are all the left cycles. Thus, there is but a finite number of possible right cycles with such a sum of lengths, and there is but a
finite number of possible left cycles. Therefore, there is but a finite number of connected components of the sets $M_j$.  

Lemmas 3.21 and 3.23 imply that the set of endpoints $N = \bigcup_{\gamma} \partial \tilde{M}_{\gamma}$ is finite, and its complement consists of the disjoint union of the connected components of the $\tilde{M}_{\gamma}$'s. Let $\mathcal{J} = \{I_1, \ldots, I_m\}$ be the family of these connected components. Consider the map $R : S^1 \setminus N \to S^1$ given by \[ R|_{\tilde{M}_j} = \gamma^{-1}, \quad \forall \gamma \in \mathcal{J}. \]

Notice that for all $i$, the image $R(I_i)$ is a union of some of the $I_j$'s plus the boundary points of adjacent intervals. Indeed, $R(I_i)$ is an interval, and its endpoints must belong to $N$ due to Lemma 3.16. Thus, it is natural to think of $\mathcal{J}$ as a Markov partition for the map $R$.

Define now $N_j$ as being the set of indeterminacy points for the map $R^j$. More precisely, let $N_0 := N$ and $N_{j+1} := N_j \cup R^{-j}(N)$. Due to the Markov property of the partition $\mathcal{J}$, the map $R^j$ sends each connected component of $S^1 \setminus N_j$ onto one of the intervals $I_i$. We would like now to see $R$ as an expanding dynamics, so that $R^j$ expands on each of the intervals in $\mathcal{J}_j$ for large $j$. Nevertheless, at the end of this iteration, we may fall very close to one of the points of $N$. We are hence forced to compose with another map that allows escaping from such a region. To perform this procedure appropriately (so that we keep control of distortion along the compositions), we will use a technique similar to the one used in [7].

Namely, for each interval $I = (a_-, a_+)$ $\in \mathcal{J}$, consider the corresponding first-returns $g_+$ and $g_-$ fixing the right and left endpoints of $I$, respectively. Let \[ J(I) := (g^2_+(a_+), g^2_-(a_-)), \quad \tilde{N}(I) := \left( \bigcup_{i=2}^{\infty} \{g^i_+(a_+), g^i_-(a_-)\} \right) \cup \{a_-, a_+\}, \]

and denote by $\tilde{R}(I) : I \setminus \tilde{N}(I) \to J(I)$ the "$R$-first-exit to $J(I)$": \[ \tilde{R}(I)(y) = R^{\min\{m|R^m(y) \in J(I)\}}(y), \quad \forall y \in I \setminus \tilde{N}(I). \]

Notice that \[ \tilde{R}(I)(y) = \begin{cases} g^{-1}_-(y), & y \in (g^{i+2}_-(a_+), g^{i+1}_+(a_-)), \\ g^{-1}_+(y), & y \in (g^{i+1}_+(a_-), g^{i+2}_-(a_-)), \\ y, & y \in J(I). \end{cases} \]

A crucial fact is that $\tilde{R}(I)$ locally behaves as an iterate of $R$, with a number of iterations that is constant on each interval of the partition $\mathcal{J}_j$.

Next, let us apply the first-exit map after $j$ iterations of $R$. More precisely, let $\tilde{R}_j : S^1 \setminus \tilde{N}_j \to S^1$ be defined as \[ \forall J \in \mathcal{J}_j, \quad \tilde{R}_j|_J = \tilde{R}(R^j(J)) \circ R^j, \]

where \[ \tilde{N}_j := N_j \cup \bigcup_{J \in \mathcal{J}} R^{-j}(\tilde{N}(J)) \]

and $\mathcal{J}_j$ is the family of connected components of its complement.
Lemma 3.24. For each $j$ and each $J \in \tilde{\mathcal{J}}_j$, the image $\tilde{R}_j(J)$ is one of the finitely many intervals in the family
\[
\mathcal{Q} := \bigcup_{I \in \mathcal{I}} \{(g^-(a_+), g^-(a_-)), (g^-(a_-), g^+(a_-)), (g^+(a_-), g^+(a_+))\}.
\]
Moreover, the distortion coefficients are uniformly bounded: there exists a constant $C_5$ independent of $j$ such that
\[
\kappa(\tilde{R}_j, J) \leq C_5.
\]

Proof. The first claim of the lemma is a direct consequence of the Markov property of $R$ and the definition of $\tilde{R}_j$. For the second, first write the restriction of $\tilde{R}_j$ to $J$ in the form $R^k$, where $k = k_j \geq j$. Next, notice that the sums of the derivatives along the path $R^{-k}$ starting at points in the image interval $\tilde{R}_j(J)$ are uniformly bounded. Indeed, this path corresponds to a geodesic in $C_\gamma$, where the interval $I \in \mathcal{I}$ containing $\tilde{R}_j(J)$ is a connected component of $\tilde{M}_\gamma$. Moreover, the image $\tilde{R}_j(J)$ lies inside an interval of the form $(g^-(a_+), g^-(a_-))$; thus, it is bounded away from the endpoints of $I$, which yields the desired uniform bound for the sum of derivatives. Finally, knowing that this sum is uniformly bounded, Proposition 2.2 together with property (1) guarantee the desired control of distortion.

Lemma 3.25. The diameter of the partition $\tilde{\mathcal{J}}_j$ tends to zero as $j \to \infty$.

Proof. Very close to the right (resp., left) endpoint of an interval $I$ in $\mathcal{I}$, iterations of $R^{-1}$ correspond to following the path given by the corresponding first-return $g_+$ (resp., $g_-$). Since the maps $g_-, g_+$ are different and the images of $I$ under them are disjoint, this implies that each interval in $\mathcal{I}$ is eventually divided into at least two. Actually, according to the construction (e.g., the proof of Lemma 3.19), each of these intervals retain a proportion of the length of $I$ uniformly bounded from below (at least equal to $1/2e^{C_3}$). Together with the uniform control of distortion provided by the preceding lemma, this obviously implies the desired convergence.

By combining these two lemmas, we finally obtain our desired

Corollary 3.26. There exists $j$ such that $(\tilde{R}_j)'(y) > 1$ for all $y \in S^1 \setminus \tilde{N}_j$.

Proof. It suffices to take $j$ such that the diameter of the partition $\tilde{\mathcal{J}}_j$ is less than
\[
\varepsilon_0 := \frac{1}{2} \min_{I \in \mathcal{Q}} |I| \cdot e^{-C_5}.
\]
Indeed, as $\tilde{R}_j(I) \in \mathcal{Q}$, for all $y \in J \in \tilde{\mathcal{J}}_j$ we have
\[
\tilde{R}_j'(y) \geq \frac{|\tilde{R}_j(J)|}{|J|} \cdot e^{-\kappa(\tilde{R}_j, J)} \geq \frac{\min_{I \in \mathcal{Q}} |I|}{\varepsilon_0} \cdot e^{-C_5} \geq 2,
\]
which shows the lemma.
We can finally conclude the proof of Proposition 3.2. Indeed, we have ensured that our procedure of “topological expansion” by iterations of \( R \) indeed generates derivatives greater than 1 everywhere, except on the finite set \( N \) and its preimages under this procedure. Thus, \( \text{NE}(G) \subset G(N) \). Now, every point of \( N \) is a fixed point of the corresponding \( g_+ \), due to Lemma 3.19. Hence, every point of the orbit \( G(N) \) is also a fixed point of the corresponding conjugate of \( g_+ \). We have thus established property (\( \star \)) for the group \( G \).

4 The case of an exceptional minimal set

Most of the previous arguments generalize almost immediately to the case of an action with an exceptional minimal set, provided we mostly work with points therein instead of the whole circle. Below we sketch the involved steps just stressing the points where some significant difference appears.

Assume first that \( G \) is a (finitely-generated, higher-rank) free subgroup of \( \text{Diff}_+^\infty(S^1) \) admitting an exceptional minimal set \( \Lambda \) (the case of a non-necessarily free group will be considered later). Again, the main issue consists in proving an analogous to Theorem 3.1: either \( G \) satisfies property (\( \Lambda \star \)), or there exist positive constants \( c, \lambda \) such that for all \( x \in \Lambda \) and all \( n \geq 1 \),

\[
\sum_{g \in B(n)} g'(x) \geq ce^{\lambda n}. \tag{11}
\]

This will be shown later. For the moment, let us see why the second possibility leads to a contradiction whenever the set \( \text{NE} \cap \Lambda \) is nonempty. Actually, as in the minimal case, such a contradiction will be established under the assumption of super-quadratic growth of sum of derivatives.

Fix \( x_0 \in \text{NE} \cap \Lambda \). As in \( \text{§3.2} \) define \( x_n = f_n(x_0) \neq x_0, f_n \in B(n) \), as being the point in \( X_n := \{g(x_0) \mid g \in B(n)\} \) that is closest to \( x_0 \) on the right if \( x_0 \) is non-isolated in \( \Lambda \) from the right; otherwise, consider the point \( x_n \neq x_0 \) of \( X_n \) that is closest on the left to \( x_0 \). Denote by \( I_n \) the interval of endpoints \( x_0, x_n \). Notice that the length of \( |I_n| \) converges to zero, though we would like to show that this convergence holds at a rate of order \( r'_n := n/S_{n/2} \), where

\[
S_n := \sum_{g \in B(n)} g'(x_0).
\]

To prove this, we notice again that the intervals \( g(I_n), g \in B([n/2]) \), are “almost” disjoint; more precisely, the multiplicity growths linearly with \( n \). (Compare Lemma 3.11) Since \( r'_n = o(1/n) \), the distortion coefficient of such a \( g \) on \( I_n \) is well behaved, which allows establishing the desired rate of convergence. (Compare Lemma 3.12)

Complex control of distortion (c.f., Proposition 2.6) then shows that for every \( g \in B(n) \),

\[
\kappa\left(g, U^{C^1}_{r'_n}(x_0) \right) \leq C_2 nr'_n.
\]

Therefore, the maps \( \tilde{f}_n(y) := \frac{1}{r'_n}(f_n(x_0 + r'_ny) - x_0) \) converge to the identity in the \( C^1 \) topology on \( U^{C^1}_1(0) \). (Compare Lemma 3.7) As in the minimal case (c.f., Lemma 3.8),

\[
{\text{NE} = \text{NE}(K)} \subset G(K(B_n)),
\]

where \( K \) is a subdomain containing \( \text{NE}(G) \) and \( B_n \) is a ball with radius \( r_n \), and \( \text{NE}(K) \) is the set of non-isolated points of \( K \).
passing to a subsequence, we have that \( f_{n_i} \) and \( f_{n_{i+1}} \) generate a free group for each \( i \). This allows performing the commutators procedure, thus finding a local flow in the closure of the group. Nevertheless, the orbit of \( x_0 \) under such a flow is an interval, which is absurd since it must be contained in \( \Lambda \).

We have hence proved that (11) cannot hold for all \( x \in \Lambda \) and all \( n \geq 1 \). Let us next prove that the failure of (11) leads to property \( (\Lambda \star) \). As we will see, the proof is not a direct translation of the arguments given for the minimal case. Several modifications are needed, mainly because we need to concentrate on points of \( \Lambda \), and not of arbitrary points in the circle. We proceed in several steps, invoking the analogue statements for the minimal case at each step.

First of all, the growing trees argument of §3.1 shows that the failure of (11) implies that there must exist \( x \in \Lambda \) and \( \gamma \in G \) such that for every \( n \geq 1 \) and every geodesic \( \gamma_n \cdots \gamma_1 \) starting with \( \gamma_1 = \gamma \),

\[
\sum_{i=1}^{n} (\gamma_i \cdots \gamma_1)'(x) \leq 2.
\]

We may hence consider the (continuous) functions \( S_\gamma \) and \( \tilde{S}_\gamma \) of §3.3, as well as the (open) sets \( M_\gamma \) and \( \tilde{M}_\gamma \). However, it will useful to consider the intersections

\[
M^\Lambda_\gamma := M_\gamma \cap \Lambda, \quad \tilde{M}^\Lambda_\gamma = \tilde{M}_\gamma \cap \Lambda.
\]

These sets satisfy analogous relations to those provided by Lemma 3.16; in particular, they are all nonempty. The fact that the intersection of all the \( \tilde{M}^\Lambda_\gamma \) is empty (hence the disjointness of all \( \tilde{M}^\Lambda_\gamma \)) follows from the next analog of Proposition 2.5.

**Lemma 4.1.** For each \( x \in \Lambda \) there exist geodesics \( g = \gamma_n \cdots \gamma_1 \) with arbitrarily large sum of intermediate derivatives.

**Proof.** Otherwise, as in the proof of Proposition 2.5, the distortion of every group element on a small neighborhood \( U \) of \( x \) would be uniformly bounded. Taking such an element \( f \) having an hyperbolically repelling fixed point inside (this is guaranteed by the classical Sacksteder’s theorem), we would thus conclude that the length of the image \( f^j(U) \) is greater than 1 for \( j \) large enough, which is absurd. \( \square \)

The reason for dealing only with points in \( \Lambda \) is given by the next

**Lemma 4.2.** Let \( J \) be a connected component of \( S^1 \setminus \Lambda \). For all but finitely many points \( x \in J \), one has

\[
\sum_{g \in G} g'(x) < \infty.
\]

Moreover, this sum uniformly converges on compacts subsets of the complement in \( J \) of the finite subset above.

**Proof.** According to a result of Hector [16], the stabilizer of \( J \) in \( G \) is nontrivial and cyclic, say generated by \( h \in G \). Let \( x_1, \ldots, x_k \) be the fixed points of \( h \) inside \( I \). We claim that the sum above is finite for \( x \in J \) different from the \( x_i \)’s. Indeed, each such point belongs to a wandering interval for the action of \( G \), namely a fundamental domain for the action of \( h \). The lemma then follows from control of distortion. \( \square \)
Figure 4: On the left: the interval $I$ (top), the minimal set $\Lambda$ (middle level), and two possible cases in the definition of $\hat{x}_\pm, \hat{x}_\pm^*$. On the right: the interval $I$, the set $\Lambda$, and the three intervals $\hat{I}, \hat{I}_+, \hat{I}_-$. 

While applying the arguments of the previous section in the non-minimal case, one should have in mind that the sets $\tilde{M}_\gamma$ are no longer pairwise disjoint, yet their intersections with $\Lambda$ remain disjoint. At the same time, the endpoints of the sets $\tilde{M}_\gamma$ may not belong to $\Lambda$, creating the problems with applying, for instance, the disjointness arguments for the admissible iterations. In order to handle these problems, for each connected component $I = (x_-, x_+)$ of some $\tilde{M}_\gamma$ that intersects $\Lambda$, let us define $\hat{x}_+ = \hat{x}_+^* = x_+$ if $x_+$ is a point of $\Lambda$ that is non-isolated on both sides. Otherwise, we take $\hat{x}_+$ and $\hat{x}_+^*$ to be respectively the right and the left endpoints of the connected component $J$ of $S^1 \setminus \Lambda$ such that $x_+ \in J$. Similarly, if $x_-$ belongs to such a $J$, we take $\hat{x}_-$ and $\hat{x}_-^*$ to be respectively the right and left endpoints of $J$; otherwise, we let $\hat{x}_- = \hat{x}_-^* = x_-$ (see Fig. 4). Finally, we denote 

$$\hat{I} := (\hat{x}_-, \hat{x}_+), \quad \hat{I}_+ := (\hat{x}_-, x_+), \quad \hat{I}_- := (x_-, \hat{x}_+).$$

Analogously to the minimal case, say that $g = \gamma_n \cdots \gamma_1, \gamma_1 \neq \gamma^{-1}$, is $\hat{I}$-admissible if all intervals $\gamma_k \cdots \gamma_1(I)$, $k = 1, \ldots, n-1$, are disjoint from $\hat{I}$, and that it is a first-return if besides $g(\hat{I})$ intersects $\hat{I}$. The four lemmas below are analogous to Lemmas 3.18, 3.19, 3.21, and 3.22, respectively, that do hold in the minimal case.

**Lemma 4.3.** All the $\hat{I}$-admissible images of $\hat{I}_+$ are pairwise disjoint. The same holds for $\hat{I}$-admissible images of $\hat{I}^-$ (as well as of those of $\hat{I}$ itself). Finally, each first-return $g$ satisfies $g(\hat{I}) \subset \hat{I}$, $g(\hat{I}_+) \subset \hat{I}_+$, and $g(\hat{I}_-) \subset \hat{I}_-$, and $\gamma_n = \gamma$.

**Proof.** Due to the definition, the left endpoint $\hat{x}_-$ of $\hat{I}_+$ is an accumulation point of $\Lambda \cap \hat{I}_+$. If two $\hat{I}$-admissible images of $\hat{I}_+$ under $g = \gamma_n \cdots \gamma_1$ and $\gamma = \gamma_m \cdots \gamma_1$ intersect, then due to the invariance of $\Lambda$, the interval $g(\hat{I}_+) \cap g(\hat{I}_+) \subset M_{\gamma_\gamma} \cap M_{\gamma_\gamma}$ intersects $\Lambda$ near its left endpoint. Hence, if both words $\gamma_n \cdots \gamma_1$ and $\gamma_m \cdots \gamma_1$ were nonempty, we would obtain $\gamma_n = \gamma_m$ and get a contradiction in the same way as in the proof of Lemma 3.18.

Now, if $g$ is a first-return, the above arguments imply that $\gamma_n = \gamma$. Thus, $g(\hat{I}_+) \subset \tilde{M}_\gamma$, and hence $g(\hat{I}_+) \subset I$. Finally, the left endpoint of $g(\hat{I}_+)$ is an accumulation point of $\Lambda \cap g(\hat{I}_+)$, which implies $g(\hat{I}_+) \subset \hat{I}_+$. In the same way we get $g(\hat{I}_-) \subset \hat{I}_-$, and putting together these two inclusions, we obtain $g(\hat{I}) \subset \hat{I}$. 

\[\square\]
Lemma 4.4. There exist first-returns $g_+$ and $g_-$, fixing $x_+$ and $x_-$, respectively. Moreover,
$$g_+(\hat{x}_+) = \hat{x}_+ , \quad g_+(\hat{x}_+^*) = \hat{x}_+^* , \quad g_-(\hat{x}_-) = \hat{x}_- , \quad g_-(\hat{x}_-^*) = \hat{x}_-^* .$$
Finally, $g_- \neq g_+$.

Proof. Literally repeating the arguments of the proof of Lemma 3.19, we may find first-returns $g_+$ fixing the endpoints $x_\pm$ of $I$, and we may check that $g_+ \neq g_-$. Finally, $g_+$ fixes $x_+$, and hence (due to the invariance of $\Lambda$) it also fixes $\hat{x}_+$ and $\hat{x}_+^*$. The same arguments apply for $g_-$. \qed

Lemma 4.5. If $g_+$ writes as $g_+ = \gamma_n \cdots \gamma_1$, then a right neighborhood of $\hat{x}_+^*$ is contained in $\widetilde{M}_\gamma$, where $\gamma := \gamma_n$ if $g_+$ is topologically contracting towards $\hat{x}_+^*$ in such a neighborhood, and $\gamma := \gamma_1^{-1}$ otherwise.

Proof. The same arguments as those of proof of Lemma 3.21 (combined with Lemma 4.2 in case $\hat{x}_+^* \neq x_+$) prove this lemma. \qed

Lemma 4.6. Neither $g_-$ nor $g_+$ have fixed points on $\hat{I}$. Moreover, $g'_-(x_-) \geq 1$, and $g'_+(x_+) \geq 1$.

Proof. In the same way as in Lemma 3.22, we notice that if $g_+$ had a fixed point inside $\hat{I}$, then in a small left neighborhood of $\hat{x}_+$ there would be a fundamental domain for its action intersecting $\Lambda$. At any point of this domain (hence at some points of $\Lambda$), the sum of derivatives along all the geodesics would be uniformly bounded, which in contradiction with Lemma 4.1. The inequality $g'_+(x_+) \geq 1$ follows using the same arguments as in Lemma 3.22; otherwise, we would have $x_+ \in \widetilde{M}_\gamma$. \qed

As we did in the minimal case, we denote by $\Phi_R(I)$ the connected component of $\widetilde{M}_\gamma$ intersecting $\gamma_1(I)$, where the element $g_+$ associated to $I$ writes as $g_+ = \gamma_n \cdots \gamma_1$. Again, we notice that $\Phi_R(I)$ and $\gamma_1(I)$ share the same right endpoint, as otherwise the point $x_+ = (\gamma_n \cdots \gamma_2)(\gamma_1(x_+))$ would belong to $\widetilde{M}_\gamma = \widetilde{M}_\gamma$. Also, the map $\widetilde{g}_+$ corresponding to $\Phi_R(I)$ is $\gamma_1 g_+ \gamma_1^{-1} = \gamma_1 \gamma_n \cdots \gamma_2$.

Similarly, we define the map $\Phi_L$ associated to $g_-$. Finally, we denote by $\mathcal{J}$ the set of connected components $I$ of sets $\widetilde{M}_\gamma$ that intersect $\Lambda$. As in the minimal case, $\mathcal{J}$ decomposes into disjoint cycles corresponding to the application of $\Phi_R$, and the same holds for $\Phi_L$. Moreover, the next analog of Lemma 3.23 holds.

Lemma 4.7. There exists only a finite number of connected components of $\widetilde{M}_\gamma$ that intersect $\Lambda$.

Proof. For each $\hat{I}$ corresponding to an interval $I \in \mathcal{J}$, the images $g_+(\hat{I})$ and $g_-(\hat{I})$ are disjoint. Hence, the derivative of at least one of the maps $g_+, g_-$ attains on $\hat{I}$ a value that does not exceed 1/2. On the other hand, by Lemma 4.6, we have $g'_-(x_-) = g'_+(x_+) \geq 1$. Thus, we have a lower bound for the distortion of at least one of the maps $g_+, g_-$, and hence $I$ belongs to either a left or a right cycle of intervals with sum of lengths bounded from below by $\frac{\log 2}{C_9}$. Since there is only a finite number of such possible cycles, there are only finitely many connected components of each set $\widetilde{M}_\gamma$ that intersect $\Lambda$. \qed
We have seen that the set $\mathcal{J}$ is finite. Moreover, Lemma 4.5 implies that immediately next to the right of a point $\hat{x}^+_{\ast}$ associated to an interval $I \in \mathcal{J}$, there is another interval of the same type. Hence, except for a finite number of points, the set $\Lambda$ is covered by a finite family of intervals $\hat{I}$. Denote this family by $\mathcal{J}_0$, and define $R : \bigcup_{I \in \mathcal{J}_0} \hat{I} \to S^1$ by

$$R|_{\hat{I}} = \gamma^{-1}, \quad \forall \gamma \in \mathcal{G}, \quad \forall \hat{I} \subset \tilde{M}_\gamma.$$

Analogously to the minimal case, this induces a Markovian dynamics on $\Lambda$: the interval $\gamma^{-1}(\hat{I})$ is a union of some intervals from $\mathcal{J}_0$ and connected components of $S^1 \setminus \Lambda$. Then we define the refined partitions $\mathcal{J}_j$ as the set of connected components of all the preimages $R^{-1}(\hat{I})$, where $\hat{I} \in \mathcal{J}_{j-1}$. Analogously to Lemma 3.25, we have

**Lemma 4.8.** The diameter of the partition $\mathcal{J}_j$ tends to zero as $j \to \infty$.

**Proof.** Analogously to the minimal case, consider the first-exit map defined on each $\hat{I} \in \mathcal{J}_0$ as

$$\tilde{R}|_{\hat{I}}(y) = \begin{cases} 
g^{-i}_{-}(y), & y \in (g^{i+2}_{-}(\hat{x}_{+}), g^{i+1}_{-}(\hat{x}_{+})), 
g^{i+1}_{+}(y), & y \in (g^{i+1}_{+}(\hat{x}_{-}), g^{i+2}_{+}(\hat{x}_{-})), 
y, & y \in (g^{2}_{-}(\hat{x}_{+}), g^{2}_{+}(\hat{x}_{-})). \end{cases}$$

In the same way as before, we see that the distortion of the composition $\tilde{R} \circ R^j$ is bounded uniformly in $j$ on any interval of continuity. Thus, we have a uniform control for the distortion of the branches of the inverses of such compositions. The fact that the diameters of the refined partitions $\mathcal{J}_j$ tend to zero follows from this fact as in the minimal case. \qed

Another application of the uniform control of distortion yields that the dynamics of the compositions $\tilde{R} \circ R^j$ becomes uniformly expanding for sufficiently large $j$. (Compare Corollary 3.26.) As a consequence, the points in $\Lambda$ that are never sent by iterations of $R$ into points of the form $\hat{x}_{-}, \hat{x}_{+}$ are expanded under suitable group elements. Since the latter points are fixed by the elements $g_{-}, g_{+}$, this proves property $(\Lambda \ast)$, and hence closes the proof for free-group actions.

Let us now consider consider the general case. Recall that a result of Ghys states that every finitely-generated group of real-analytic circle diffeomorphisms having an exceptional minimal set is virtually free; see [13]. Hence, let $H$ be a finite index free subgroup of $G$, and let $g_1, \ldots, g_k$ be representatives of all left-classes of $H$ in $G$. We claim that $H$ preserves an exceptional minimal set (in particular, it has rank $\geq 2$, because of Denjoy’s theorem). Otherwise, according to [12, 27], the group $H$ would either act minimally or admit a finite orbit $A$. The former case is impossible, as $G$ does not act minimally. The latter case is also impossible, as it would imply that $G$ preserves the finite set $\bigcup_{i=1}^{k} g_i(A)$.

Let then $\hat{\Lambda}$ be the exceptional minimal set for the action of $H$. We claim that the set

$$\bigcup_{i=1}^{k} g_i(\hat{\Lambda})$$

is the exceptional minimal set $\Lambda$ of $G$. Indeed, since $\Lambda$ is $H$-invariant, it contains a minimal $H$-invariant set, which must coincide with $\hat{\Lambda}$ because of the uniqueness of
exceptional minimal sets. As a consequence, the set \( \bigcup_{i=1}^{k} g_i(\bar{\Lambda}) \) is \( G \)-invariant and contained in \( \Lambda \). It must hence coincide with \( \Lambda \), because of the minimality of \( \Lambda \).

Since \( H \) is a (necessarily higher-rank) free group, it must satisfy property (\( \bar{\Lambda}^{\star} \)). Hence, due to Theorem 1.14.3), the set of points of bounded \( H \)-expansivity in \( \bar{\Lambda} \) coincides with \( H(\text{NE}(H)) \). As \( \Lambda = \bigcup_{i=1}^{k} g_i(\bar{\Lambda}) \), and thus any point of \( \Lambda \) can be mapped into \( \bar{\Lambda} \) by one of the \( g_i \)’s, the set of points of \( \Lambda \) of bounded \( G \)-expansivity is a subset of \( G(\text{NE}(H)) \). In particular, any non \( G \)-expandable point belongs to \( G(\text{NE}(H)) \) and hence is fixed by a nontrivial element of \( G \). We conclude that \( G \) satisfies property (\( \Lambda^{\star} \)), thus finishing the proof of the Main Theorem.

**Remark 4.9.** In the proof above, in cases where the set \( \text{NE} \cap \Lambda \) is nonempty, we have established property (\( \Lambda^{\star} \)) simultaneously with the Markovian property for the dynamics. (Strictly speaking, we have done this only for the case of the free group, but passing to finite extensions is straightforward.) This remark is not innocuous, as it shows that the approach of \([3, 4, 24]\) to the question of the Lebesgue measure of exceptional minimal sets, as well as the finiteness of the orbits of the connected components of the complement, pointed in the right direction. (See also \([19]\).) Moreover, this again shows why it was worth considering two different cases in \( \S 3.2 \) instead of reducing the proof to a single one (see Remark 3.13).

We close this work with the

**Proof of Corollary 1.18.** Under the hypothesis, we have established property (\( \Lambda^{\star} \)). Let \( y_1, \ldots, y_m \) be the non-expandable points (we allow the possibility \( m = 0 \)). For each index \( i \), let \( \bar{U}_i \) be a neighborhood of \( y_i \) sufficiently small so that there exists \( g_i \in G \) fixing this point and having no other fixed point therein. By considering smaller neighborhoods \( U_i \subset \bar{U}_i \), we may assume that the first iterate of \( g_i^{\pm 1} \) that sends a point \( x \in U_i \) different from \( y_i \) outside \( \bar{U}_i \) has derivative \( \geq 2 \). Indeed, this follows from control of distortion just by writing \( \bar{U}_i \setminus \{ y_i \} \) as the union of fundamental domains for the action of \( g_i \) on the left and the right neighborhoods of \( y_i \).

Points in the set \( \Lambda \setminus \bigcup U_i \) are expandable, hence we may fix \( \lambda > 0 \) and a covering of this set by open intervals \( V_1, \ldots, V_n \) such that on each \( V_i \), certain element \( h_i \in G \) expands by a factor at least \( e^\lambda \). Now consider the open covering

\[
\Lambda \subset \bigcup_{i=1}^{m} U_i \cup \bigcup_{i=1}^{n} V_i.
\]

There are only finitely many connected components of \( S^1 \setminus \Lambda \) that are not contained in one of these sets \( U_i, V_i \); let us denote them by \( I_1, \ldots, I_k \). We claim that the orbit of every connected component \( I \) of \( S^1 \setminus \Lambda \) coincides with that of one of these intervals. Indeed, if \( I \) does not coincide with none of them, then it is contained in a certain interval \( U_i \) or in one of \( V_i \). In the former case, the action of a suitable power of \( g_i \) expands its length by a factor at least \( 2 \), whereas in the latter, expansion by a factor at least \( e^\lambda \) is performed by \( h_i \). After this first expansion, either we have reached an interval of the form \( I_j \), or the image remains in one of the \( U_i, V_i \), so that we may repeat the procedure. Obviously, this expansion process must stop in finite time. At such a time, we have reached one of the intervals \( I_j \), so that the orbit of this interval coincides with that of \( I \). \( \square \)
Group actions on the circle and codimension-one foliations

Acknowledgments

The authors would like to thank Lawrence Conlon, Étienne Ghys, Steve Hurder, Yulij Ilyashenko, Shigenori Matsumoto, Julio Rebelo, and Dennis Sullivan, for many discussions all along this research project. We would also like to thank the anonymous referee for his careful work and helpful remarks.

B. Deroin’s research was partially supported by ANR-08-JCJC-0130-01, ANR-09-BLAN-0116

V. Kleptsyn acknowledges the support of an RFBR project 13-01-00969-a and of a joint RFBR/CNRS project 10-01-93115-CNRS_a

A. Navas acknowledges the support of the ACT-1103 project DySyRF.

Besides our host institutions, we thank the PUC-Chile, the CRM Barcelona, the Poncelet Laboratory in Moscow, and the MRCC of Bedlewo, for the very nice working conditions provided at several stages of this work.

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