MARTINGALE APPROXIMATION AND OPTIMALITY OF SOME CONDITIONS FOR THE CENTRAL LIMIT THEOREM

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Abstract. Let \((X_i)\) be a stationary and ergodic Markov chain with kernel \(Q\), \(f\) an \(L^2\) function on its state space. If \(Q\) is a normal operator and \(f = (I - Q)^{1/2}g\) (which is equivalent to the convergence of \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} Q^k f\) in \(L^2\)), we have the central limit theorem (cf. [D-L 1], [G-L 2]). Without assuming normality of \(Q\), the CLT is implied by the convergence of \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|Q^k f\|_2\), in particular by \(\|Q f\|_2 \leq O(\sqrt{n / \log n})\), \(q > 1\) by [M-W u] and [W u-W o] respectively. We shall show that if \(Q\) is not normal and \(f \in (I - Q)^{1/2} L^2\), or if the conditions of Maxwell and Woodroofe or of Wu and Woodroofe are weakened to \(\sum_{n=1}^{\infty} c_n \|Q^k f\|_2 < \infty\) for some sequence \(c_n \to 0\), or by \(\|Q f\|_2 = O(\sqrt{n / \log n})\), the CLT need not hold.

1. Introduction. Let \((S, \mathcal{B}, \nu)\) be a probability space, \((\xi_i)\) a homogeneous and ergodic Markov chain with state space \(S\), transition operator \(Q\), and stationary distribution \(\nu\). For a measurable function \(g\) on \(S\), \((g(\xi_i))\) is then a stationary random process; we shall study the central limit theorem for

\[ S_n(g) = \sum_{i=0}^{n-1} g(\xi_i) \]

where \(g \in L_0^2(\nu)\), i.e. is square integrable and has zero mean. Gordin and Lifšic ([G-L 1]) showed that if \(g\) is a solution of the equation

\[ g = (I - Q)h = h - Qh \]

with \(h \in L^2(\nu)\) then a martingale approximation giving the CLT exists. More precisely, there exists a martingale difference sequence of \(m(\xi_i) = h(\xi_i) - Qh(\xi_{i+1})\) such that \(\|S_n(g - m)\|_2 / \sqrt{n} \to 0\) (as shown in [Vo 1], this condition is equivalent to Gordin’s condition from [G]). The result was extended to normal operators \(Q\) and functions \(g\) satisfying

\[ g = (I - Q)^{1/2} h \]

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with \( h \in L^2 \). The operator \((I - Q)^{1/2}\) is defined using the series of the function \(\sqrt{\sqrt{x} - x}, \ x \in [-1, 1]\) (cf. [D-L 1]). For reversible operators \(Q\) the result was proved by Kipnis and Varadhan in 1986 ([K-V]), for normal operators \(Q\) the result appears in 1981 in [G-Ho 2] with a proof published later in [G-L 3], in 1996 the result was independently proved by Derriennic and Lin in [D-L 1]. Derriennic and Lin formulated the condition (1) in its present form; the other authors used spectral forms of the condition. As noticed by Gordin and Holzmann ([G-Ho]), (1) is equivalent to the convergence of

\[
\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n-1} Q^k g}{n^{3/2}} \quad \text{in} \quad L^2.
\]

They did not present a proof of this statement; for making reader’s homework given in [G-Ho] easier, let us give several arguments.

By [D-L 1] we have that \( f \in \sqrt{T - QL^2} \) iff \( \sum_{j=0}^{\infty} (j+1) a_j Q^j f \) converges, where \( a_j = -(j^{-3/2}) \); by the Stirling formula, \( a_j \sim \frac{j^{-3/2}}{2\sqrt{\pi}} \) with a summable error term.

From the convergence of \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} Q^k f \) and Kronecker’s lemma it follows
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Q^k f \to 0.
\]

By double summation and elementary estimation we can find that the series (2) converges iff the sequence of \( \sum_{k=1}^{n} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n}} \right) Q^k f \) converges. We thus have that \( f \in \sqrt{T - QL^2} \) implies the convergence in (2).

The proof of the converse copies the proof of (a more general) Lemma 4.1 in [Cu 2].

Suppose that the series (2) converges. Denote \( S_k(g) = \sum_{j=0}^{k-1} Q^j g \). We have

\[
\sum_{k=n}^{2n-1} \frac{S_k(g)}{k^{3/2}} = S_n(g) \sum_{k=n}^{2n-1} \frac{1}{k^{3/2}} + Q^n \left( \sum_{k=1}^{n-1} \frac{S_k(g)}{(n+k)^{3/2}} \right) \to 0.
\]

Because the sequence of partial sums of the series (2) is Cauchy and \( Q \) is a Markov operator, \( \| \sum_{k=1}^{n-1} \frac{S_k(g)}{(n+k)^{3/2}} \|_2 \to 0 \) will imply that \( \frac{S_n(g)}{n^{3/2}} \) converges to 0. Let us prove it.

Define \( R_n = \sum_{k=1}^{\infty} \frac{S_k(g)}{k^{3/2}} \). We have

\[
\sum_{k=1}^{n-1} \frac{S_k(g)}{(n+k)^{3/2}} = \sum_{k=1}^{n-1} \frac{R_k - R_{k+1}}{(n+k)^{3/2}} = \sum_{k=2}^{n-1} \frac{k^{3/2}}{(n+k)^{3/2}} - \frac{(k-1)^{3/2}}{(n+k-1)^{3/2}} + \frac{R_1}{(n+1)^{3/2}} - \frac{R_n(n-1)^{3/2}}{(2n-1)^{3/2}}.
\]

By (2), \( \| R_n \|_2 \to 0 \) hence given an \( \epsilon > 0 \) there exists an \( n_0 \) such that \( \| R_n \|_2 < \epsilon \) for \( n \geq n_0 \). We thus have

\[
\| \sum_{k=2}^{n-1} R_k \left( \frac{k^{3/2}}{(n+k)^{3/2}} - \frac{(k-1)^{3/2}}{(n+k-1)^{3/2}} \right) \|_2 \leq \max_{2 \leq k \leq n_0} \| R_k \|_2 \sum_{k=2}^{n_0} \left( \frac{k^{3/2}}{(n+k)^{3/2}} - \frac{(k-1)^{3/2}}{(n+k-1)^{3/2}} \right) + \epsilon.
\]
Therefore \( \| \sum_{k=1}^{n-1} \frac{S_k(g)}{(n+k)^{3/2}} \|_2 \to 0 \), hence \( \frac{S_n(g)}{n^{3/2}} \). The convergence of the series (2) is equivalent to the convergence of \( \sum_{k=1}^{n} \left( \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{n}} \right) Q^k g \), we thus get that \( \sum_{k=1}^{n} \frac{1}{\sqrt{k}} Q^k g \) converges, which is equivalent to \( g \in \sqrt{T - QL^2} \).

As we shall see in Theorem 1, without normality of \( Q \) the condition (2) does not imply the CLT. Maxwell and Woodrooife have shown in [M-Wo] that if

\[
(3) \quad \sum_{n=1}^{\infty} \frac{\| \sum_{k=0}^{n-1} Q^k g \|_2}{n^{3/2}} < \infty
\]

then (without any other assumptions on the Markov operator \( Q \)) the martingale approximation (and the CLT) takes place.

Let \((\Omega, \mathcal{A}, \mu)\) be a probability space with a bijective, bimeasurable and measure preserving transformation \( T \). For a measurable function \( f \) on \( \Omega \), \((f \circ T^i)_i\) is a (strictly) stationary process and reciprocally, using the first canonical process, we get that any (strictly) stationary process can be represented in this way: to a stationary process \((X_i)_i\) defined on a probability space \((\Omega', \mathcal{A}', P)\) we define a mapping \( \psi: \Omega' \to \mathbb{R}^Z \) by \( \psi(\omega) = (X_i(\omega))_i \), and on \( \Omega = \mathbb{R}^Z \) equipped with the product \( \sigma \)-algebra \( \mathcal{A} \) we define the image measure \( \mu = P \circ \psi^{-1} \). By \( T \) we denote the left shift transformation of \( \Omega \) onto itself, \((T \omega)_i = \omega_{i-1} \). If \( Z_i \) is the projection of \( \Omega = \mathbb{R}^Z \) to the \( i \)-th coordinate then the distribution of the process \((Z_i)_i\) is the same as the distribution of \((X_i)_i\) and \( Z_i = Z_0 \circ T^i \).

Any stationary process can be represented by a homogeneous and stationary Markov chain ([Wu-Wo], a similar idea appears already in [R, p.65]). To a process \((X_i)_i\) we associate a Markov chain \((\xi_k)_k\) with \( \mathbb{R}^N \) for the state space, where \( \xi_k = (\ldots, X_{k-1}, X_k) \), the transition operator \( Q \) is given by \( Q(x, B) = \mu(\xi_1 \in B | \xi_0 = x) = \mu(\xi_1 \in B | X_0 = x_0, X_{-1} = x_{-1}, \ldots) \) where \( x = (\ldots, x_{-1}, x_0) \in \mathbb{R}^N \), and a stationary distribution is given by the distribution of the process \((X_i)_{i \leq 0}\). For \( g(x) = x_0, x = (\ldots, x_{-1}, x_0) \in \mathbb{R}^N \), the process \( g(\xi_i) \) has the same distribution as \((X_i)_i\).

For \( g \) integrable we have \( Qg(\xi_i) = E(g(\xi_{i+1})|\xi_i) \). The conditions (2) and (3) thus can be expressed in the following way.

Let \((\Omega, \mathcal{A}, \mu, T)\) be a dynamical system (a probability space with a bimeasurable and measure preserving bijective transformation \( T : \Omega \to \Omega \), \( \mathcal{F}_i \) an increasing filtration with \( T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1} \), \( f \) is a square integrable and zero mean function on \( \Omega \), \( \mathcal{F}_0 \)-measurable. We denote

\[
S_n(f) = \sum_{i=0}^{n-1} f \circ T^i.
\]

The convergence in (2) is then equivalent to the convergence (in \( L^2 \)) of

\[
(2') \quad \sum_{n=1}^{\infty} \frac{E(S_n(f) | \mathcal{F}_0)}{n^{3/2}} < \infty
\]

and (3) becomes

\[
(3') \quad \sum_{n=1}^{\infty} \frac{\| E(S_n(f) | \mathcal{F}_0) \|_2}{n^{3/2}} < \infty.
\]
Remark 1. Notice that in (2') and (3'), the natural filtration need not be used, it is sufficient to suppose that the process \((f \circ T^i)\) is adapted to \((\mathcal{F}_i)\). The natural filtration is the smallest filtration with respect to which the process \((f \circ T^i)\) is adapted, hence the convergence in (2'), (3') for \((\mathcal{F}_i)\) implies the convergence for the natural filtration.

Remark 2. In this article we suppose that the dynamical system \((\Omega, \mathcal{A}, \mu, T)\) is ergodic, i.e. for all sets \(A \in \mathcal{A}\) such that \(A = T^{-1}A\), it is \(\mu(A) = 0\) or \(\mu(A) = 1\). A stationary (here, this always means strictly stationary) process \((X_i)\) is said to be ergodic if there exists an ergodic dynamical system with a process \((f \circ T^i)\), equally distributed as \((X_i)\). Remark that an ergodic process can be represented within a non ergodic dynamical system.

Let \(X_i = f \circ T^i\). Then the mapping \(\psi: \Omega \to \mathbb{R}^Z\) defined by \(\psi(\omega) = (X_i(\omega))_i\) is a factor map of \(\Omega\) onto \(\mathbb{R}^Z\), hence if \((\Omega, \mathcal{A}, \mu, T)\) is ergodic then the dynamical system defined by the first canonical process is ergodic (cf. [C-F-S]). The process \((X_i)\) is thus ergodic if and only if the associated dynamical system defined by the first canonical process is ergodic.

Remark 3. Let \((f \circ T^i)\) be the first canonical process representation for a stationary process \((X_i)\) and let \((g(\xi_i))\) be the Markov chain representation of \((f \circ T^i)\), \(f\) is thus the projection to the zero-th coordinate of \(\Omega = \mathbb{R}^Z\) and \(T\) is the left shift. Ergodicity of the Markov chain \((\xi_i)\) is equivalent to ergodicity of \((X_i)\):

We define \(S = \mathbb{R}^N, \mathcal{B}\) is the product \(\sigma\)-algebra on \(S^Z\). Let us define \(\phi: \Omega = \mathbb{R}^Z \to (\mathbb{R}^N)^Z = S^\phi\) by \(\phi((\omega_k)_{k \in \mathbb{Z}}) = ((\omega_i)_{i \leq k})_{k \in \mathbb{Z}}, \nu = \mu \circ \phi^{-1}\). The measure \(\nu = \mu \circ \phi^{-1}\) is invariant with respect to the left shift \(\tau\) on \(S^Z\), \(\phi\) is a bimeasurable bijection of \(\mathbb{R}^N\) onto \(\phi(\mathbb{R}^N) \in \mathcal{B}\) which commutes with the transformations \(T, \tau\): \(\phi \circ T = \tau \circ \phi\).

The dynamical systems \((\Omega, \mathcal{A}, \mu, T)\) and \((S^Z, \mathcal{B}, \nu, \tau)\) are thus isomorphic.

If \(\xi_i: S^Z \to S\) are the coordinate projections, \((\xi_i)\) is a Markov chain and for \(g: \mathbb{R}^N \to \mathbb{R}, g(x) = x_0, x = (\ldots, x_{-1}, x_0), (g(\xi_i))\) has the same distribution as \((f \circ T^i)\). Because ergodicity is invariant with respect to isomorphism ([CFS]) and the dynamical system \((S^Z, \mathcal{B}, \nu, \tau)\) is the first canonical process for \((\xi_i)\), ergodicity of the Markov chain \((\xi_i)\) is equivalent to ergodicity of \((X_i)\).

Remark 4. In [Vo 2] a nonadapted version of the Maxwell-Woodroofe approximation (3') have been found.

In the present paper we will deal with optimality of the conditions (2') and (3'), hence also of (2) and (3).

Theorem 1. There exists an ergodic process \((f \circ T^i)\) such that the series

\[
(2') \quad \sum_{n=1}^{\infty} \frac{E(S_n(f) | \mathcal{F}_0)}{n^{3/2}}
\]

converges in \(L^2\), but for two different subsequences \((n'_k)\), \((n''_k)\), the distributions of \(S_{n'_k} / \sigma_{n'_k}\) and \(S_{n''_k} / \sigma_{n''_k}\) converge to different limits.

Therefore, if \((\xi_i)\) is a homogeneous ergodic Markov chain \((\xi_i)\) with a transition operator \(Q\), then without normality of \(Q\), the condition \(g \in (I - Q)^{1/2}L^2\) is not sufficient for the CLT.

In the next two theorems we show that in the central limit theorem of Maxwell and Woodroofe, the rate of convergence of \(\|E(S_n(f)|\mathcal{F}_0)\|_2\) towards 0 is practically optimal. We denote \(\sigma_n = \|ES_n(f)\|_2\).
Theorem 2. For any sequence of positive reals \( c_n \to 0 \) there exists an ergodic process \( (f \circ T^n) \) such that

\[
\sum_{n=1}^{\infty} c_n \frac{\|E(S_n(f)|\mathcal{F}_0)\|_2}{n^{3/2}} < \infty
\]

but for two different subsequences \( (n'_k) \), \( (n''_k) \), the distributions of \( S_{n'_k}/\sigma_{n'_k} \) and \( S_{n''_k}/\sigma_{n''_k} \) converge to different limits.

Theorem 3. There exists an ergodic process \( (f \circ T^n) \) such that

\[
\|E(S_n(f)|\mathcal{F}_0)\|_2 = o\left(\frac{\sqrt{n}}{\log n}\right),
\]

but for two different subsequences \( (n'_k) \), \( (n''_k) \), the distributions of \( S_{n'_k}/\sigma_{n'_k} \) and \( S_{n''_k}/\sigma_{n''_k} \) converge to different limits.

Remark 5. In [Pe-U], under the assumption (4) Peligrad and Utev have shown that there exists an \( f \) such the sequence of \( S_n(f)/\sqrt{n} \) is not stochastically bounded.

Remark 6. In [Wu-Wo] M. Woodrooife and W.B. Wu showed that if \( \|E(S_n(f)|\mathcal{F}_0)\|_2 = o(\sigma_n) \) then \( \sigma_n = h(n)/\sqrt{n} \) where \( h(n) \) is a slowly varying function in the sense of Karamata and there exists an array \( D_{n,i} \) of martingale differences such that for each \( n \), the sequence \( (D_{n,i}, i) \) is a strictly stationary martingale difference sequence and \( \|S_n(f) - \sum_{i=0}^{n-1} D_{n,i}\|_2 = o(\sigma_n) \). In particular, under the condition (4) this approximation still takes place.

In the same paper [Wu-Wo], Wu and Woodrooife used the condition

\[
\|E(S_n(f) | \mathcal{F}_0)\|_2 = o\left(\frac{\sqrt{n}}{\log^q n}\right)
\]

where \( q > 1 \). (6) implies (3’) hence a martingale approximation and the CLT. Theorem 3 shows that the result cannot be extended to \( q = 1 \).

Remark 7. The condition (5) which can be written as \( \| \sum_{k=0}^{n-1} Q^k g \|_2 = o\left(\frac{\sqrt{n}}{\log n}\right) \)

implies that \( \sum_{n=1}^{\infty} \| \sum_{k=0}^{n-1} Q^k g \|_2^2 < \infty \). By [Cu 1], Proposition 2.2, for a normal operator \( Q \) this implies \( g \in \sqrt{T - Q}L^2 \) hence a CLT; Theorem 3 shows that the assumption of normality cannot be lifted.

Remark 8. From the construction it follows that in Theorems 1-3, the variances \( \sigma_n^2 \) of \( S_n(f) \) grow faster than linearly. It thus remains an open problem whether with a supplementary assumption \( \sigma_n^2/n \to \text{const.} \) the CLT would hold. As shown by a counter example in [Kl-Vo2], this assumption is not sufficient for \( q \leq 1/2 \) (in [Kl-Vo2], a function \( f \) is found such that (6) holds with \( q = 1/2, \sigma_n^2/n \to \text{const.} \), but the distributions of \( S_n(f)/\sqrt{n} \) do not converge); the only exponents to consider are thus \( 1/2 < q \leq 1 \).

It also remains an open question whether the CLT would hold for \( f \in L^{2+\delta} \) for some \( \delta > 0 \).

2. Proofs. We first define an ergodic dynamical system \((\Omega, \mathcal{A}, \mu, T)\) where the processes \((f \circ T^n)\) which we need can be found.

For \( l = 1, 2, \ldots \) we define \( A_l = \{-1, 0, 1\} \) if \( l \) is odd and \( A_l = \mathbb{R} \) if \( l \) is even, the sets \( A_l \) are equipped with Borel \( \sigma \)-algebras and probability measures \( \nu_l \) such
that \( \nu_i(\{\cdot\}) = 1/(2N_l) = \nu_i(\{1\}) \) and \( \nu_i(\{0\}) = 1 - 1/N_l \) for \( l \) odd, \( \nu_i = N(0,1) \) for \( l \) even. For each \( i \in \mathbb{Z} \) we define \( \Omega_i = \times_{l=1}^{\infty} A_l; \Omega_i \) is equipped with the product measure \( \mu_i = \otimes \nu_i \). On the set \( \Omega = \times_{i \in \mathbb{Z}} \Omega_i \), \( \mu \) is the product measure \( \mu = \otimes \mu_i \) and \( T \) is the left shift transformation. By definition, the dynamical system \( (\Omega, \mathcal{A}, \mu, T) \) is Bernoulli hence ergodic. By \( \mathcal{F}_k, k \in \mathbb{Z} \), we denote the \( \sigma \)-algebra generated by projections of \( \Omega \) onto \( \Omega_i, l \leq k \).

In all of the text, \( \log \) will denote the dyadic logarithm. By \( U \) we denote the unitary operator on \( L^2 \) defined by \( Uf = f \circ T \). \( P_i \) denotes the orthogonal projection on the Hilbert space \( L^2(\mathcal{F}_i) \ominus L^2(\mathcal{F}_{i-1}) \), i.e.

\[
P_i f = E(f|\mathcal{F}_i) - E(f|\mathcal{F}_{i-1}),
\]

\( f \in L^2, i \in \mathbb{Z} \).

For \( k = 1, 2, \ldots \) let \( n_k = 2^k \),

\[
0 \leq a_k \leq 1, \quad a_k/a_{k+1} \to 1, \quad a_{k+1} \leq a_k, \quad \sum_{k=1}^{\infty} a_k/k = \infty;
\]

(\( N_l \), \( l = 1, 2, \ldots \), is an increasing sequence of positive integers such that

\[
2^{2^l} - 1 < \sum_{k \geq 1 : N_l-1 < n_k \leq N_l} \frac{a_k}{k} < 2^{2^l} + 1.
\]

Denote by \( \pi_0 \) the projection of \( \Omega \) onto \( \Omega_0 \) and by \( p_l \) the projection of \( \Omega_0 \) onto \( A_l \). For \( N_{l-1} < n_k \leq N_l \) we then define \( e_k = a_k/k p_l \circ \pi_0 \). Then

\[
\|e_k\|_2 = a_k/k,
\]

and for \( i \neq j \), \( U^i e_k \) and \( U^j e_l \) are independent. Notice that for \( N_{l-1} < n_k \leq N_l \) the random variables \( e_k \) are multiples one of another and are independent of any \( e_j \) with \( n_j < N_{l-1} \) or \( n_j > N_l \). In general, \( e_{k'}, e_{k''} \) are not orthogonal but for all \( 1 \leq k', k'' \) it is \( E(e_{k'} e_{k''}) \geq 0 \) and

\[
\left\| \sum_{k=1}^{n} e_k \right\|_2 \not\rightarrow \infty \quad \text{as} \quad n \rightarrow \infty;
\]

for \( b_k' \geq b_k \geq 0 \) we thus have

\[
\sum_{k=1}^{n} b_k' e_k \geq \left\| \sum_{k=1}^{n} b_k e_k \right\|_2.
\]

Let

\[
f = \sum_{k=1}^{\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} U^{-i} e_k.
\]

We have \( \|f\|_2 \leq \sum_{k=1}^{\infty} \|e_k\|_2/\sqrt{n_k} < \infty \) due to the exponential growth of the \( n_k \)s.
For a positive integer \( N \) we have

\[
S_N(f) = \sum_{k=1}^{\infty} \sum_{j=0}^{N-1} \sum_{i=0}^{n_k-1} \frac{1}{n_k} U^{j-i} e_k = S'_N(f) + S''_N(f)
\]

where

\[
S'_N(f) = S_N(f) - E(S_N(f) | \mathcal{F}_0) = \sum_{k=1}^{\infty} \sum_{j=0}^{N-1} \sum_{i=0}^{n_k-1} \frac{1}{n_k} U^{j-i} e_k
\]

\((j \land n_k = \min\{j, n_k\})\) and

\[
S''_N(f) = E(S_N(f) | \mathcal{F}_0) = \sum_{k=1}^{\infty} \sum_{j=0}^{n_k-1} \sum_{i=0}^{n_k-1} \frac{1}{n_k} U^{j-i} e_k.
\]

We will study the asymptotic behaviour of \( S''_N(f) = E(S_N(f) | \mathcal{F}_0) \) and \( S'_N(f) = S_N(f) - E(S_N(f) | \mathcal{F}_0) \) separately. In Lemmas 1-5 we find estimates for \( E(S_N(f) | \mathcal{F}_0) \) and an approximation of \( S'_N(f) \) by sums of martingale differences. These lemmas do not depend on distributions of the random variables \( e_k \). Then we use the distributions of the random variables \( e_k \) to get a limit behaviour of the distributions of \( S_n(f)/\sigma_n \) which we need.

1. Asymptotics of \( S''_N(f) = E(S_N(f) | \mathcal{F}_0) \).

Lemma 1. The series

\[
(2') \sum_{n=1}^{\infty} \frac{E(S_n(f) | \mathcal{F}_0)}{n^{3/2}}
\]

converges in \( L^2 \).

Proof. For \( N \leq n_k \) we have

\[
\sum_{j=0}^{N-1} \sum_{i=0}^{n_k-1} U^{j-i} e_k = \sum_{i=0}^{n_k-N} N U^{-i} e_k + \sum_{i=n_k-N+1}^{n_k-1} (n_k-i) U^{-i} e_k
\]

and for \( N > n_k \) we have

\[
\sum_{j=0}^{N-1} \sum_{i=0}^{n_k-1} U^{j-i} e_k = \sum_{j=0}^{n_k-1} (n_k-j) U^{-j} e_k,
\]

hence

\[
(8) \quad S''_N(f) = \sum_{k \geq 1: n_k < N} \sum_{j=0}^{n_k-1} \frac{n_k-j}{n_k} U^{-j} e_k +
\]

\[
+ \sum_{k \geq 1: n_k \geq N} \left[ \sum_{j=0}^{n_k-N} \frac{n_k}{n_k} U^{-j} e_k + \sum_{j=n_k-N+1}^{n_k-1} \frac{n_k-j}{n_k} U^{-j} e_k \right].
\]
To prove the lemma it is thus sufficient to show that the sums

\begin{align}
(8a) \quad & \sum_{N=1}^{\infty} \sum_{k \geq 1: n_k < N} \sum_{j=0}^{n_k-1} \frac{1}{N^{3/2}} \frac{n_k-j}{n_k} U^{-j} e_k, \\
(8b) \quad & \sum_{N=1}^{\infty} \sum_{k \geq 1: n_k \geq N} \frac{1}{N^{3/2}} \sum_{j=0}^{n_k-N} \frac{N}{n_k} U^{-j} e_k, \\
(8c) \quad & \sum_{N=1}^{\infty} \sum_{k \geq 1: n_k \geq N} \frac{1}{N^{3/2}} \sum_{j=n_k-N+1}^{n_k-1} \frac{n_k-j}{n_k} U^{-j} e_k,
\end{align}

converge in $L^2$.

Recall that $n_k = 2^k$. We show that the sequence of partial sums for the first series is Cauchy. Let $1 \leq p < q < \infty$.

\[
\left\| \sum_{N=p}^{q} \sum_{k \geq 1: 2^k < N} \sum_{j=0}^{2^k-1} \frac{1}{N^{3/2}} \frac{2^k-j}{2^k} U^{-j} e_k \right\|_2^2 = \]
\[
\left\| \sum_{k=1}^{\infty} \left( \sum_{N=(2^k+1) \lor p}^{q} \sum_{j=0}^{2^k-1} \frac{1}{N^{3/2}} \frac{2^k-j}{2^k} U^{-j} e_k \right) \right\|_2^2 = \]
\[
\left\| \sum_{k=1}^{\infty} \sum_{j=0}^{2^k-1} \frac{2^k-j}{2^{3k/2}} U^{-j} e_k \right\|_2 = \left\| \sum_{k \geq 1 \lor \log(j+1)} \sum_{j=0}^{\infty} \frac{2^k-j}{2^{3k/2}} U^{-j} e_k \right\|_2
\]

where $b_k(p, q) = 2^{k/2} \sum_{N=(2^k+1) \lor p}^{q} \frac{1}{N^{3/2}}$. Notice that $b_k(p, q)$ are uniformly bounded and for each $k$, $\sup_{q \geq p} b_k(p, q) \rightarrow 0$ as $p \rightarrow \infty$. Denote $B = \sup_{k, p, q} b_k(p, q) < \infty$. Using $\|e_k\|_2 \leq 1/k$ we deduce

\[
\left\| \sum_{j=0}^{\infty} \sum_{k \geq 1 \lor \log(j+1)} b_k(p, q) \frac{2^k-j}{2^{3k/2}} U^{-j} e_k \right\|_2^2 \leq \left\| \sum_{j=0}^{\infty} \sum_{k \geq 1 \lor \log(j+1)} b_k(p, q) \frac{2^k-j}{2^{3k/2}} U^{-j} e_k \right\|_2^2 \leq \]
\[
\leq \sum_{j=0}^{\infty} \left( \sum_{k \geq 1 \lor \log(j+1)} b_k(p, q) \right)^2
\]

where

\[
\sum_{k \geq 1 \lor \log(j+1)} b_k(p, q) \frac{2^k-j}{k^{2^{k/2}}} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{B}{[1 \lor \log(j+1)]},
\]

hence

\[
\sum_{j=0}^{\infty} \left( \sum_{k \geq 1 \lor \log(j+1)} b_k(p, q) \right)^2 < \infty,
\]
therefore

\[ \left\| \sum_{j=0}^{\infty} \sum_{k \geq 1 \log(j+1) \leq N} b_k(p,q) \frac{2^k - j}{2^{3k/2}} U^{-j} e_k \right\|^2_2 \leq \lim_{p \to \infty} \sup_{q \geq p} \sum_{j=0}^{\infty} \left( \sum_{k \geq 1 \log(j+1)} b_k(p,q) \right)^2 = 0. \]

For the second sum we define

\[ b_k(p,q) = \frac{1}{2k/2} \sum_{N=p}^{2k/2} \frac{1}{N^{1/2}} \]

and get

\[ \left\| \sum_{N=p}^{q} \sum_{k \geq 1 \log(j+1) \leq N} \frac{1}{N^{3/2}} \sum_{j=0}^{2k-N} \frac{N}{2k} U^{-j} e_k \right\|^2_2 = \left\| \sum_{N=p}^{q} \sum_{j=0}^{\infty} \sum_{k \geq 1 \log(N+j)} \frac{1}{N^{1/2}} \frac{1}{2k} U^{-j} e_k \right\|^2_2 = \left\| \sum_{j=0}^{\infty} \sum_{k \geq 1 \log(j+1)} \left( \sum_{N=p}^{1/2} \frac{1}{N^{1/2}} \right) \frac{1}{2k} U^{-j} e_k \right\|^2_2 \leq \sum_{j=0}^{\infty} \left\| \sum_{k \geq 1 \log(j+1)} \frac{b_k(p,q)}{2k/2} \right\|^2_2 \to 0 \]

as \( p \to \infty \) using similar arguments as in the preceding case.

For the third sum we get

\[ \left\| \sum_{N=p}^{q} \sum_{k \geq 1 \log(j+1) \leq N} \frac{1}{N^{3/2}} \sum_{j=2^k-N+1}^{2k-1} \frac{2^k - j}{2k} U^{-j} e_k \right\|^2_2 = \sum_{j=0}^{\infty} \sum_{k \geq 1 \log(j+1) \leq N=p \vee (2^k-j+1)} q^{2k} \left( \sum_{N=p \vee (2^k-j+1)}^{1/2} \frac{1}{N^{1/2}} \right) \frac{1}{2k} U^{-j} e_k \right\|^2_2 \leq \sum_{j=0}^{\infty} \sum_{k \geq 1 \log(j+1)} q^{2k} \left( \sum_{N=p \vee (2^k-j+1)}^{1/2} \frac{1}{N^{1/2}} \right) \frac{1}{2k} U^{-j} e_k \right\|^2_2 \to 0 \]

as \( p \to \infty \). The sequences of partial sums for (8a), (8b), (8c) are Cauchy hence converge in \( L^2 \). Therefore, the series \( \sum_{N=1}^{\infty} E(S_N(f)|\mathcal{F}_0) \) converges in \( L^2 \).

□

**Lemma 2.** There exists a constant \( 0 < c < \infty \) such that

\[ \left\| E(S_N(f)|\mathcal{F}_0) \right\|^2_2 \leq c \frac{N a^2_{[\log N]}}{\log^2 N} \]

for all \( N \geq 2 \).
Proof. We have
\begin{align*}
(1/\sqrt{6})\|e_k\|_2\sqrt{n_k} & \leq \left\| \sum_{j=0}^{n_k-1} \frac{n_k-j}{n_k} U^{-j}e_k \right\|_2 \leq \|e_k\|_2\sqrt{n_k}, \quad n_k < N,
\end{align*}
(10)
\begin{align*}
\left\| \sum_{j=0}^{n_k-N} \frac{N}{n_k} U^{-j}e_k + \sum_{j=n_k-N+1}^{n_k-1} \frac{n_k-j}{n_k} U^{-j}e_k \right\|_2 & \leq \frac{N}{\sqrt{n_k}}\|e_k\|_2, \quad n_k \geq N.
\end{align*}

Recall that by \([x]\) we denote the integer part of \(x\). Because \(n_k = 2^k\) grows exponentially fast, the norms \(\|e_k\|_2\) are decreasing, and \(\|e_k\|_2/\|e_{k+1}\|_2 \to 1\), there exists a constant \(0 < c < \infty\) not depending on \(N\) such that
\begin{align*}
\sum_{k \geq 1: n_k \geq N} \left(\frac{N}{\sqrt{n_k}}\right)\|e_k\|_2 & \leq c\sqrt{N}\|e_{\lfloor \log N \rfloor}\|_2, \\
\sum_{k \geq 1: n_k < N} \|e_k\|_2\sqrt{n_k} & \leq c\sqrt{N}\|e_{\lfloor \log N \rfloor}\|_2.
\end{align*}

Using (8) and (10) we deduce that for some constant \(c > 0\) we have
\begin{align*}
\|E(S_N(f)|\mathcal{F}_0)\|_2^2 & \leq cN\|e_{\lfloor \log N \rfloor}\|_2^2.
\end{align*}

Because \(\|e_k\|_2 = a_k/k\),
\begin{align*}
\|E(S_N(f)|\mathcal{F}_0)\|_2^2 & \leq c\frac{Na_{\lfloor \log N \rfloor}^2}{\log^2 N}.
\end{align*}

Recall
\begin{align*}
\|e_k\|_2 = a_k/k, \quad 0 \leq a_k \leq 1, \quad \sum_{k=1}^{\infty} a_k/k = \infty.
\end{align*}

As a corollary to Lemma 2 we get
**Lemma 3.**
\begin{align*}
\|E(S_n(f)|\mathcal{F}_0)\|_2 = O\left(\frac{\sqrt{n}}{\log n}\right),
\end{align*}
if \(a_n \to 0\) then
\begin{align*}
\|E(S_n(f)|\mathcal{F}_0)\|_2 = o\left(\frac{\sqrt{n}}{\log n}\right).
\end{align*}

**Lemma 4.** Let \(c_n\) be positive real numbers, \(c_n \downarrow 0\). If
\begin{align*}
\sum_{n=1}^{\infty} \frac{a_n c_n}{n} < \infty \quad \text{then} \quad \sum_{n=1}^{\infty} c_n \frac{\|E(S_n(f)|\mathcal{F}_0)\|_2}{n^{3/2}} < \infty.
\end{align*}

Proof. Recall Lemma 2 and \(a_{k+1} \leq a_k\), denote the constant in (9) by \(C\). We have
\begin{align*}
\sum_{n=2}^{\infty} c_n \frac{\|E(S_n(f)|\mathcal{F}_0)\|_2}{n^{3/2}} & \leq C \sum_{n=2}^{\infty} c_n \frac{a_{\lfloor \log n \rfloor} + 1}{n \lfloor \log n \rfloor} \leq C \sum_{k=1}^{\infty} \sum_{i=0}^{2^k-1} \frac{a_k}{k^{2k + i}} \\
& \leq C \sum_{k=1}^{\infty} c_2 \frac{a_k}{k} \leq C \sum_{k=1}^{\infty} c_k \frac{a_k}{k} < \infty.
\end{align*}
2. Asymptotics of $S'_N(f) = S_N(f) - E(S_N(f)|\mathcal{F}_0)$.

Denote $b(N) = \| \sum_{k \geq 1 : n_k \leq N} e_k \|_2 = \| \sum_{k = 1}^{\lfloor \log N \rfloor} e_k \|_2$; notice that by the assumptions on $e_k$, $b(N) \to \infty$.

**Lemma 5.** We have

\[ \lim_{N \to \infty} \frac{1}{b(N) \sqrt{N}} \left\| S'_N(f) - \sum_{l=0}^{N-1} U^l \sum_{k \geq 1 : n_k \leq N} e_k \right\|_2 = 0. \]

**Proof.** Recall that

\[ S'_N(f) = \sum_{k=1}^{\infty} \sum_{j=0}^{N-1 (j \land n_k) - 1} \frac{1}{n_k} U^{j-i} e_k. \]

For $N \leq n_k$ we have

\[ \sum_{j=0}^{N-1 (j \land n_k) - 1} U^{j-i} e_k = \sum_{j=1}^{N-1} (N-j) U^j e_k \]

and for $N > n_k$ we have

\[ \sum_{j=0}^{N-1 (j \land n_k) - 1} U^{j-i} e_k = \sum_{j=1}^{N-n_k} n_k U^j e_k + \sum_{j=N-n_k+1}^{N-1} (N-j) U^j e_k. \]

For all $k \geq 1$ we have $P_l U^j e_k = 0$ if $j \neq l$, $P_l U^l e_k = U^l e_k$. For $l \geq N$ and $l \leq 0$ we thus have $P_l S_N(f) = 0$ and for $1 \leq l \leq N - 1$ we, using (12) and (13), deduce

\[ P_l S_N(f) = \sum_{k \geq 1: n_k \leq N-l} U^l e_k + \sum_{k \geq 1: n_k \geq N+1-l} N-l \frac{U^l e_k}{n_k}. \]

Recall that $[x]$ denotes the integer part of $x$. We have

\[ S'_N(f) = \sum_{l=1}^{N-1} P_l S_N(f) = \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} P_l S_N(f) + \sum_{l=[N(1-\epsilon)]+1}^{N} P_l S_N(f) \]

and using (14) we get

\[ \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} P_l S_N(f) = \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} U^l \sum_{k \geq 1: n_k \leq N-l} e_k + \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} U^l \sum_{k \geq 1: n_k \geq N+1-l} \frac{N-l}{n_k} e_k = \]

\[ = \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} U^l \sum_{k \geq 1: n_k \leq \epsilon N} e_k + \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} U^l \sum_{k \geq 1: \epsilon N < n_k \leq N-l} e_k + \sum_{l=1}^{\lfloor N(1-\epsilon) \rfloor} U^l \sum_{k \geq 1: n_k \geq N+1-l} \frac{N-l}{n_k} e_k. \]
Because \( n_k = 2^k \),

\[
\left\lVert \sum_{k \geq 1: n_k \geq N + 1 - l} \frac{N - l}{n_k} U^l e_k \right\rVert_2 \leq 2 \left\lVert e_{\lceil \log(N - l) \rceil} \right\rVert_2 \leq 2/\log(N - l),
\]

hence

\[
\left\lVert \sum_{l=1}^{\lceil (N - 1/\epsilon) \rceil} U^l \sum_{k \geq 1: n_k \geq N + 1 - l} \frac{N - l}{n_k} e_k \right\rVert_2 \leq \frac{\sqrt{N}}{\log N + \log \epsilon}
\]

(we can suppose \( N \) big enough to have \( \log N > |\log \epsilon| \)); \( \epsilon N < n_k \leq N \) if and only if \( \log N + \log \epsilon < k \leq \log N \). We thus deduce that for \( \epsilon > 0 \) fixed and \( b(N) = \| \sum_{k=1}^{\log N} e_k \|_2 \to \infty \),

\[
\lim_{N \to \infty} \frac{1}{b(N)} \left\| \sum_{\epsilon N < n_k \leq N} e_k \right\|_2 = 0
\]

and

\[
\lim_{N \to \infty} \frac{1}{b(N) \sqrt{N}} \left\| \sum_{l=\lceil N(1-\epsilon) \rceil + 1}^{\lceil N \rceil} P_l S_N(f) - \sum_{l=\lceil N(1-\epsilon) \rceil + 1}^{\lceil N \rceil} U^l \sum_{k \geq 1: n_k \leq \epsilon N} e_k \right\|_2 = 0.
\]

For all \( \lceil N(1-\epsilon) \rceil + 1 \leq l \leq N - 1 \) we have, by (14) and (15), \( \| P_l S_N(f) \|_2 \leq b(N - l) + 2/\log(N - l) \) hence

\[
\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{b(N) \sqrt{N}} \left\| \sum_{l=\lceil N(1-\epsilon) \rceil + 1}^{\lceil N \rceil} P_l S_N(f) - \sum_{l=\lceil N(1-\epsilon) \rceil + 1}^{\lceil N \rceil} \sum_{k \geq 1: n_k \leq \epsilon N} U^l e_k \right\|_2 = 0
\]

and (11) follows.

\[ \square \]

Recall that

\[
\| e_k \|_2 = a_k/k, \quad 0 \leq a_k \leq 1, \quad \sum_{k=1}^{\infty} a_k/k = \infty
\]

\( N_l, \ l = 1, 2, \ldots, \) is an increasing sequence of positive integers such that

\[
2^l - 1 < \sum_{k \geq 1: N_{l-1} < n_k \leq N_l} \frac{a_k}{k} < 2^l + 1;
\]

for \( N_{l-1} < n_k \leq N_l \) the random variables \( e_k \) are multiples one of another and are independent of any \( e_j \) with \( n_j \leq N_{l-1} \) or \( n_j > N_l \).

**Lemma 6.** Along \( l \) odd the distributions of

\[
\frac{1}{b(N_l) \sqrt{N_l}} \sum_{j=0}^{N_l - 1} U^j \left( \sum_{k \geq 1: N_{l-1} < n_k \leq N_l} e_k \right)
\]

weakly converge to the symmetrised Poisson distribution with parameter \( \lambda = 1/2 \) and for \( l \) even to the standard normal distribution.
Proof. From the definition of the functions $e_k$ it follows that $b^2(N_i) = b^2(N_{i-1}) + \| \sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k \|^2$ and $\| \sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k \|^2 = \sum_{k \geq 1; N_{i-1} < n_k \leq N_i} a_k/k \sim 2^l$, hence $\| \sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k \|^2 \sim b(N_i)$.

For $l$ odd, the random variable $\sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k$ takes values $\approx \pm b(N_i)\sqrt{N_i}$ with probabilities $1/(2N_i)$ and 0 with probability $1 - 1/N_i$.

For $l$ even, $\sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k$ is normally distributed with zero mean and variance $\approx b^2(N_i)$.

By the assumptions, $U^l \sum_{k \geq 1; N_{i-1} < n_k \leq N_i} e_k$ are independent random variables and the statement of the lemma follows.

□

Proof of Theorem 1 and Theorem 3. By Lemma 1 the series $\sum_{n=1}^{\infty} \frac{E(S_n(f) | \mathcal{F}_0)}{n^{3/2}}$ converges in $L^2$. We can have $a_n \searrow 0$, e.g. $a_n = 1/\log n$ for $n \geq 2$; by Lemma 3 then $\|E(S_n(f) | \mathcal{F}_0)\|^2 = O\left(\frac{1}{\log n}\right)$. From Lemma 3 and Lemma 5 it follows

$$\left\| E(S_n(f) - \sum_{l=0}^{N-1} U^l \sum_{k \geq 1; n_k \leq N} e_k \right\|^2 = O\left(\frac{1}{b(N)\sqrt{N}}\right),$$

using Lemma 6 we thus get that along $l$ odd the distributions of $\frac{1}{b(N)\sqrt{N}}S_n(f)$ weakly converge to the symmetrised Poisson distribution with parameter $\lambda = 1/2$ and for $l$ even to the standard normal law.

□

Proof of Theorem 2. Let $c_k > 0$; without loss of generality we can suppose $c_k \searrow 0$. We define $k_0 = 1$ and for $n = 1, 2, \ldots$ let $k_n$ be the first $k$ such that $c_k \leq 1/2^n$ and $k_n - k_{n-1} \geq n$; $a_{k_0} = 1$ and for $n \geq 1$, $a_{k_n}$ is the minimum of $a_{k_{n-1}}$ and $(\sum_{j=k_{n-1}+1}^{k_n} \frac{1}{j})^{-1}$. For $k_{n-1} + 1 \leq j \leq k_n - 1$ we define

$$a_j = a_{k_{n-1}} + \alpha_j (a_{k_n} - a_{k_{n-1}}), \quad \alpha_j = \begin{cases} \frac{j-k_{n-1}}{n} & \text{if } 1 \leq j - k_{n-1} \leq n \\ 1 & \text{if } n+1 \leq j - k_{n-1} \leq k_n - k_{n-1} \end{cases}$$

Then

$$\sum_{n=1}^{\infty} \frac{a_n c_n}{n} < \infty, \quad \sum_{n=1}^{\infty} \frac{a_n}{n} = \infty.$$

To verify the first inequality we notice that

$$\sum_{n=1}^{\infty} \frac{a_n c_n}{n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \left[ a_{k_n} \sum_{j=k_{n-1}+1}^{k_n} \frac{1}{j} + (a_{k_{n-1}} - a_{k_n}) \sum_{j=k_{n-1}+1}^{n} \frac{1 - \alpha_j}{j} \right].$$

By definition, $a_{k_n} \sum_{j=k_{n-1}+1}^{k_n} \frac{1}{j} \leq 1$, and by boundedness of $a_n$ we have $\sum_{n=1}^{\infty} \frac{n a_{k_{n-1}}}{2^n} < \infty$.

To verify the second inequality we notice

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \geq \sum_{n=1}^{\infty} a_{k_n} \sum_{j=k_{n-1}+1}^{k_n} \frac{1}{j}.$$
If $a_k_n = \left(\sum_{j=k_{n-1}+1}^{k_n} \frac{1}{j}\right)^{-1}$ for infinitely many $n$ then the sum is infinite. Otherwise, from some $n_0$ on, the sequence $a_n$ is constant and strictly positive; the series is infinite as well.

A simple calculation shows that the sequence of $a_n$ is decreasing, $0 \leq a_n \leq 1$, and $a_n / a_{n+1} \to 1$.

The inequality $\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty$ makes the assumptions of Theorem 2 satisfied and the equality $\sum_{n=1}^{\infty} \frac{a_n}{n} = \infty$, together with Lemma 6, implies that $f$ can be defined so that along $l$ odd the distributions of $\frac{1}{b(N)\sqrt{N}} S_n(f)$ weakly converge to the symmetrised Poisson distribution with parameter $\lambda = 2$ and for $l$ even to the standard normal law.

□

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References

[B] Billingsley, P., The Lindeberg-Lévy theorem for martingales, Proc. Amer. Math. Soc. 12 (1961), 788-792.
[C-F-S] Cornfeld, I.P., Fomin, S.V., and Sinai, Ya.G., Ergodic Theory, Springer, Berlin, 1982.
[Cu 1] Cuny, Ch., Pointwise ergodic theorems with rate and application to limit theorems for stationary processes, submitted for publication. [arXiv:0904.0185 (2009)].
[Cu 2] Cuny, Ch., Norm convergence of some power-series of operators in $L^p$ with applications in ergodic theory, submitted for publication (2009).
[D-L 1] Derriennic, Y. and Lin, M., Sur le théorème limite central de Kipnis et Varadhan pour les chaînes réversibles ou normales, CRAS 323 (1996), 1053-1057.
[D-L 2] Derriennic, Y and Lin, M., The central limit theorem for Markov chains with normal transition operators, started at a point, Probab. Theory Relat. Fields 119 (2001), 509-528.
[G] Gordin, M.I., A central limit theorem for stationary processes, Soviet Math. Dokl. 10 (1969), 1174-1176.
[G-Ho] Gordin, M.I. and Holzmann, H., The central limit theorem for stationary Markov chains under invariant splittings, Stochastics and Dynamics 4 (2004), 15-30.
[G-L 1] Gordin, M.I. and Lifšic, B.A., Central limit theorem for stationary processes, Soviet Math. Doklady 19 (1978), 392-394.
[G-L 2] Gordin, M.I. and Lifšic, B.A., A remark about a Markov process with normal transition operator, In: Third Vilnius Conference on Probability and Statistics 1 (1981), 147-148.
[G-L 3] Gordin, M.I. and Lifšic, B.A., The central limit theorem for Markov processes with normal transition operator, and a strong form of the central limit theorem, IV.7 and IV.8 in Borodin and Ibragimov, Limit theorems for functionals of random walks, Proc. Steklov Inst. Math. 195(1994), English translation AMS (1995).
[Ha-He] Hall, P. and Heyde, C.C., Martingale Limit Theory and its Application, Academic Press, New York, 1980.
[I] Ibragimov, I.A., A central limit theorem for a class of dependent random variables, Theory Probab. Appl. 8 (1963), 83-89.
[K-V] Kipnis, C. and Varadhan, S.R.S., Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, Comm. Math. Phys. 104 (1986), 1-19.
[Kl-Vo 1] Klicnarová, J. and Volný, D., An invariance principle for non adapted processes, C.R. Acad. Sci. Paris Ser 1 345/5 (2007), 283-287.
[Kl-Vo 2] Klicnarová, J. and Volný, D., Exactness of a Wu-Woodroofe’s approximation with linear growth of variances, Stoch. Proc. and their Appl. 119 (2009), 2158-2165.
[M-Wo] Maxwell, M. and Woodroofe, M., *Central limit theorems for additive functionals of Markov chains*, Ann. Probab. 28 (2000), 713-724.

[P-U] Peligrad, M. and Utev, S., *A new maximal inequality and invariance principle for stationary sequences*, Ann. Probab. 33 (2005), 798-815.

[R] Rosenblatt, M., *Markov Processes: Structure and asymptotic behavior*, Springer, Berlin, 1971.

[Vo 1] Volný, D., *Approximating martingales and the central limit theorem for strictly stationary processes*, Stochastic Processes and their Applications 44 (1993), 41-74.

[Vo 2] Volný, D., *Martingale approximation of non adapted stochastic processes with nonlinear growth of variance*, Dependence in Probability and Statistics Series: Lecture Notes in Statistics, Vol. 187 Bertail, Patrice; Doukhan, Paul; Soulier, Philippe (Eds.) (2006).

[Wo] Woodroofe, M., *A central limit theorem for functions of a Markov chain with applications to shifts*, Stoch. Proc. and their Appl. 41 (1992), 31-42.

[Wu-Wo] Wu, W.B. and Woodroofe, M., *Martingale approximation for sums of stationary processes*, Ann. Probab. 32 (2004), 1674-1690.

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