Ferrimagnetism in a system of two antiferromagnetically coupled Heisenberg models

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Abstract. We phenomenologically investigate the phase diagram of some ferrimagnetic materials on the basis of two bilinearly coupled Heisenberg models sitting on two interpenetrating simple magnetic cubic lattices. Computations were performed with the help of the Landau free energy obtained through applying the Hubbard-Stratonovich transformation to the initial microscopic Heisenberg Hamiltonian. The transitions to the magnetically ordered phases within the model are found to be of second order with the occurrence of a compensation point at lower temperatures for some values of the system’s parameters. It is found that the main stable phase is a two-sublattice collinear ferrimagnet in addition to a metastable non-collinear phase. The numerical results give a detailed description of temperature dependence of magnetization on the strength of intersublattice interaction and the difference between the effective exchanges of two ferromagnetically ordered sublattices. Our results are in qualitative agreement with the experimental results on Y₃Fe₅O₁₂ in the strong coupling limit and the compounds ErFe₂ and GdCo₁₂B₆ for weak couplings.

1. Introduction
Ferrimagnets are substances made up of various components possessing different magnetic properties. The difference in magnetic moments leads to geometric frustration that may arise because either different elements occupy the lattice sites or the same element occupies nonequivalent crystallographic sites surrounded by a different number or type of non-magnetic ions, which effectively results in different magnetic properties. For complex alloys a combination of both may take place (For an extensive review see reference [1] and references therein). Within the mean-field approach it is generally accepted that ferrimagnets can be modeled with the help of several interpenetrating sublattices each ordered ferromagnetically with an effective antiferromagnetic coupling between them. In the pioneering works of Néel on ferrimagnetism within the molecular field approach (see e.g. [2]), a two sublattice model is used to compute the thermal magnetization behaviour of ferrimagnets, and six possible magnetization curves were derived. There, special attention was paid to iron garnets, where the spontaneous magnetization in comparison with experiment can be interpreted by applying a three-sublattice model. In view of experimental study of magnetocaloric effect of rare-earth based ferrimagnets [3] which has great potential for technological applications in environmentally-friendly refrigeration, the theoretical mean-field description of such alloys with three sublattices, is further elaborated [4]. Such studies are based on considering microscopic classical Heisenberg models with different exchange and spin-orbit interactions depending on the crystal structure, and chemical composition of the particular alloy under study. There is another theoretical mean field approach based on considering mixed spin Ising model for description of ferrimagnets, see for example [5–7], where a very detailed review of the literature on this approach is presented.
In the present paper we consider ferrimagnets that can be described by different magnetic ions sitting on two interpenetrating simple cubic sublattices arranged in a body-centred-cubic structure. The magnetic interaction between ions on each sublattice is supposed ferromagnetic, while ions on the different sublattices are coupled antiferromagnetically. The magnetic properties will be investigated on the basis of bilinearly coupled Heisenberg classical model in a mean-field approximation which is performed using the Hubbard-Stratonovich transformation for obtaining the respective Landau free energy and its analysis.

We aim at clarifying two aspects, namely, the influence of the magnitude of inter-sublattice interaction on the behaviour of sublattice magnetizations as a function of the temperature, as well as the dependence of the sublattice magnetizations on the difference between the magnitudes of exchange interactions in the sublattices. For this reason we will not consider any anisotropy effects and will adhere to high-symmetry crystal structure, for which such anisotropic effects can be neglected as small. Let us anticipate that despite the simplifications introduced in the model considered here our theoretical results compare qualitatively well with experimental measurements on the compounds Y_{3}Fe_{2}O_{12} [4] in the strong interaction limit, as well as, ErFe_{2} [8] and GdCo_{12}B_{6} [9] in the weak interaction limit.

The rest of the paper is organized as follows: in Section 2 we describe in detail the derivation of the expression of the Landau free energy, within the mean field approach, from the classical Heisenberg model with competing interactions following closely the approach of Ref. [10]. In Section 3 the solutions of the equations of state obtained after minimization of the Landau free energy derived in Section 2 are analyzed. Section 4 discusses the analytical and the numerical results depending on the model’s parameters both in strong- and weak-coupling limits for the ferrimagnetic substances under consideration. Section 5 generalizes the conclusions and possible further development of our study.

2. The model and derivation of Landau free energy

The microscopic Heisenberg Hamiltonian that describes two bilinearly coupled magnetic sublattices reads

$$H = -\frac{1}{2} \sum_{ij} \left( J_{ij}^{(1)} S_{i}^{(1)} \cdot S_{j}^{(1)} + J_{ij}^{(2)} S_{i}^{(2)} \cdot S_{j}^{(2)} + 2 \mathcal{K}_{ij} S_{i}^{(1)} \cdot S_{j}^{(2)} \right).$$

(1)

Here $S_{i}^{(1,2)}$, are $n$-component classical Heisenberg spins whose magnitude is normalized on the unit sphere in spin space through the condition $|S_{i}^{(1,2)}| = 1$. The exchange parameters $J_{ij}^{(1,2)}$, $\mathcal{K}_{ij}$ in the general case are $N \times N$ symmetric matrices with $N$ - the number of lattice sites, assumed equal for both subsystems. This condition simplifies the consideration as it makes the system symmetric with respect to the interchange of the subsystems. The exchange matrices $J_{ij}^{(1,2)}$ denote the interaction between ions on the same sublattice and $\mathcal{K}_{ij}$ - between magnetic ions on the different sublattices.

Hamiltonian (1) may be applied to the description of magnetic systems which consist of two distinct magnetic materials. Regardless of the microscopic origin of mechanisms underlying the magnetic properties such systems can be effectively described by different exchange interactions within the two subsystems. There may be other situations when the substance is made only of one type of magnetic ions that occupy two different crystallographic positions in the Bravais lattice, separated by some non-magnetic atoms. Such substance may also be considered as built of two magnetic subsystems with different exchange interactions within them.

In order to analyse the behaviour of the magnetization and the phase behaviour in systems that can be described by the microscopic Hamiltonian (1) we have to find the associated mean-field free energy. This is achieved by calculating the partition function which in this case is represented by functional integral in $n$-dimensional spin space, where $n$ - is the number of spin components. To this end we apply the Hubbard-Stratonovich transformation; see e.g. ref. [11] and references therein. This approach was described in detail and used in reference [10] to explore the magnetic properties of a
system of two ferromagnetically coupled subsystems. Here we will just outline the main steps for the derivation of the Landau free energy, especially in respect to the antiferromagnetic coupling between the subsystems.

We consider the two interacting magnetic subsystems sitting on a body-centered crystal lattice, for which the corners of elementary cube are occupied by one kind of magnetic atoms, and at the centre of the cube the atoms of the other kind. Thus, the nearest neighbours belong to different magnetic subsystems and the next-nearest neighbours to same subsystem. Hence, the system may be described as two interpenetrating sublattices, consisting of different magnetic atoms and we assume that the interaction within the sublattices $J_{ij}$ is ferromagnetic and between them, $K_{ij}$, it is antiferromagnetic.

The Hubbard-Stratonovich transformation expresses the initial microscopic Hamiltonian in new $n$-component variables $\Psi_i^{(1)}, \Psi_i^{(2)}$ defined in real space, directly connected with the initial spins (see [10]), namely:

$$
\mathcal{H} = \frac{1}{2} \sum_{ij} \left( J_{ij}^{(1)} \Psi_i^{(1)} \cdot \Psi_j^{(1)} + J_{ij}^{(2)} \Psi_i^{(2)} \cdot \Psi_j^{(2)} + 2K_{ij} \Psi_i^{(1)} \cdot \Psi_j^{(2)} \right) - \ln \left[ \sum_x \left( \frac{\lambda(x)}{2} \right) ^{-\frac{x}{4}} \Gamma \left( \frac{N}{2} \right) \right].
$$

$$
\text{(2)}
$$

Here $I_{\frac{N}{2}-1} \left( \frac{x}{2} \frac{1}{2} \right)$ is the modified Bessel function of the first kind, and $\Gamma \left( \frac{N}{2} \right)$ is the Gamma function. In the above expression the exchange parameters $J_{ij}^{(1,2)}$ and $K_{ij}$ are connected to those in the initial Hamiltonian (1) through the relations:

$$
J_{ij}^{(1,2)} = \frac{\mathcal{H}_{ij}^{(1,2)}}{T}, \quad K_{ij} = \frac{\mathcal{H}_{ij}}{T},
$$

$$
\text{(3)}
$$

with $T$ - the temperature. We have denoted by $x_i^{(1)}$ and $x_i^{(2)}$ in (2) the following expressions:

$$
x_i^{(1)} = \left| J_{ij}^{(1)} \Psi_j^{(1)} + K_{ij} \Psi_j^{(2)} \right|, \quad x_i^{(2)} = \left| J_{ij}^{(2)} \Psi_j^{(2)} + K_{ij} \Psi_j^{(1)} \right|.
$$

The terms in (2) containing the Bessel functions will be further used only in the form of an expansion with respect to $x_i^{(1,2)}$ up to forth order by using the relation:

$$
\Gamma \left( \frac{N}{2} \right) \left( \frac{2}{\lambda} \right) ^{\frac{N}{2}-1} I_{\frac{N}{2}-1} (\lambda) = 1 + \sum_{k=1}^{\infty} \frac{(\frac{2}{\lambda})^{2k}}{k!(\frac{N}{2}+1)(\frac{N}{2}+2)\cdots\frac{N}{2}}.
$$

The next step is to perform a Fourier transformation to the $k$-space, and pass to the continuum limit in $k$ as the finite size effects will not be considered at this stage. The quadratic part of the obtained Hamiltonian again contains a bilinear term with respect to $\Psi^{(1,2)}(k)$ and we have to diagonalise it. This is achieved with the help of the unitary matrix $\hat{S}$:

$$
\hat{S} = \begin{pmatrix}
S_0(k) & -S_1(k) \\
S_1^*(k) & S_0(k)
\end{pmatrix}.
$$

$$
\text{(4)}
$$

The eigenvalues of the matrix $\hat{S}$ read:

$$
\lambda_{1,2}(k) = \frac{1}{2} \left[ J_1(k) + J_2(k) \pm \sqrt{(J_1(k) - J_2(k))^2 + 4K(k)^2} \right],
$$

$$
\text{(5)}
$$
where \( J_{1,2}(k) \) and \( K(k) \) are the Fourier transforms of \( f_{i,j}^{(1,2)} \) and \( K_{ij} \) from (3), respectively.

In order to compute the integral in the partition function we use the steepest-descent method, i.e. the integration contour is taken around the maxima of the eigenvalues (5). For the bcc structure, if we take the nearest neighbour interaction between atoms of the same sort and the nearest neighbour interaction between the atoms of different sorts, the Fourier transforms of exchange interactions read:

\[
J_{1,2}(k) = \frac{J_{1,2}}{3} \sum_{i=1}^{3} \cos k_i a
\]

and

\[
|K(k)| = \frac{K^2}{8} \left[ 1 + \sum_{i=1}^{3} \cos k_i a + \sum_{i<j}^{3} \cos k_i a \cos k_j a + \cos k_1 a \cos k_2 a \cos k_3 a \right];
\]

here \( J_{1,2} \) are the exchange interactions between the nearest neighbour atoms of the same sort on sublattices 1 and 2, respectively, and \( K \) is the exchange interaction between nearest neighbour atoms of different sort; \( a \) is the lattice constant. The calculation using the above expressions shows that the eigenvalues \( \lambda_{1,2}(k) \) have a maximum in the centre of the Brillouin zone that gives ferromagnetic ordering for the sublattices with antiferromagnetic \( K < 0 \) interaction between them, and this is the focus of our consideration below. We should note that \( \lambda_{1,2}(k) \) have maxima also at the border of the Brillouin zone \( k = \pi / a \) which presupposes antiferromagnetically ordered sublattices. There may be also some local maxima inside the Brillouin zone, which give some incommensurate ordering within the sublattices, but this case is beyond the scope of the present study.

After performing the inverse Fourier transform to real space, we obtain the dimensionless Landau free energy in the following form:

\[
f = \frac{F}{T} = \frac{t_1}{2} \overline{\psi}_1^2 + \frac{t_2}{2} \overline{\psi}_2^2 + \frac{g}{4} \left[ (\overline{\psi}_1^2)^2 + (\overline{\psi}_2^2)^2 \right] + \frac{b_1}{2} \overline{\psi}_1 \overline{\psi}_2 + b_2 \overline{\psi}_1 \overline{\psi}_2 + \overline{\psi}_2 \overline{\psi}_1 (\overline{\psi}_1 \cdot \overline{\psi}_2),
\]

where the coefficient

\[
t_{1,2} = \frac{1}{\lambda_{1,2}} - \frac{1}{n}; \quad g = \frac{\mu}{2} (S_0^4 + S_1^4), \quad b = n^2 (n+2) S_1^2 S_0^2, \quad w = \frac{\mu}{2} S_0 S_1 (S_0^2 - S_1^2).
\]

are expressed in terms of the elements of the matrix (4) and its eigenvalues (5) with \( k = 0 \), i.e.

\[
S_0 = \frac{1}{D} \left( J_1 - J_2 + \sqrt{(J_1 - J_2)^2 + 4K^2} \right), \quad S_1 = \frac{2K}{D},
\]

where \( D \) is introduced to satisfy the condition \( \| \bar{S} \| = 1 \), namely \( S_0^2 + S_1^2 = 1 \):

\[
D = \sqrt{2} \left[ (J_1 - J_2)^2 + 4K^2 \right]^{1/4} \left[ J_1 - J_2 + \sqrt{(J_1 - J_2)^2 + 4K^2} \right]^{1/2}.
\]

The real vector fields \( \overline{\psi}_1 \) and \( \overline{\psi}_2 \) in the Landau free energy (6) play the role of two coupled order parameters, and the averaged sublattice magnetizations are obtained through the equations

\[
\overline{m}_1 = \frac{S_0}{\lambda_1} \overline{\psi}_1 - \frac{S_1}{\lambda_2} \overline{\psi}_2; \quad \overline{m}_2 = \frac{S_1}{\lambda_1} \overline{\psi}_1 + \frac{S_0}{\lambda_2} \overline{\psi}_2.
\]
3. Solving the mean-field equations

The initial microscopic Hamiltonian is symmetric with respect to the rotation of all spins through the same angle. The application of Hubbard-Stratonovich transformation for the derivation of Landau free energy, presented in the previous section, preserves the symmetry of the initial Hamiltonian with respect to the auxiliary field variables $\Psi_{1,2}$, as well, which means that we can find the magnitude and the mutual orientation between the variables $\Psi_{1,2}$ but not their orientation with respect to crystallographic axes. This may be achieved for a particular magnetic substance by including in the initial microscopic Hamiltonian (1) terms accounting for the magnetic anisotropy. For pure exchange interactions we can introduce the following notations \[ (10): \Psi_{1,i} = |\Psi_{1}| \beta_{i} \text{ and } \Psi_{2,i} = |\Psi_{2}| \delta_{i}, \]
where $|\Psi_{1}| = \psi_{1}$, $|\Psi_{2}| = \psi_{2}$ are the magnitudes of the vector fields, and $\beta_{i}$, $\delta_{i}$ are the respective direction cosines, which fulfill the condition:

\[ \sum_{i=1}^{3} \beta_{i}^2 = 1 \quad \text{and} \quad \sum_{i=1}^{3} \delta_{i}^2 = 1. \tag{11} \]

Then, the equations of state will be

\[ \frac{\partial f}{\partial X_{i}} = 0, \quad \text{where} \quad X_{i} = (\psi_{1}, \psi_{2}, \beta_{i}, \delta_{i}). \tag{12} \]

Solving the above equations with respect to direction cosines $\beta_{i}$, $\delta_{i}$ renders two possible orientations between the vector fields $\Psi_{1}$, $\Psi_{2}$ for $K < 0$:

(i) The collinear phase with $\sum_{i} \beta_{i} \delta_{i} = -1$, that is, $\Psi_{1}$ and $\Psi_{2}$ are antiparallel, and

(ii) The non-collinear phase with $\sum_{i} \beta_{i} \delta_{i} = 0$, that is, $\Psi_{1}$ and $\Psi_{2}$ are perpendicular.

Hereafter, we will discuss in detail the non-collinear phase 2. The general form of angle obtained between $\Psi_{1}$ and $\Psi_{2}$ is:

\[ \sum_{i} \beta_{i} \delta_{i} = - \frac{w(\psi_{2}^2 - \psi_{1}^2)}{2b\psi_{1}\psi_{2}} \tag{12} \]

and is defined only when $\psi_{1} \neq 0$ and $\psi_{2} \neq 0$. For $K < 0$, the analysis shows that the non-collinear phase exists only when the field variables $\psi_{1}$ and $\psi_{2}$ are of equal magnitudes and mutually perpendicular. Substituting the above expression for the angle of non-collinear phase in (12) we find for the magnitude $\psi = \psi_{1} = \psi_{2}$ of non-collinear phase:

\[ \psi^2 = - \frac{f_{1} + f_{2}}{u}. \tag{13} \]

Then, the sublattice magnetization magnitudes calculated using (10) for the non-collinear phase are

\[ |\mathbf{m}_{1}| = \psi \sqrt{\frac{S_{0}}{\lambda_{1}^2} + \frac{S_{1}^2}{\lambda_{2}^2}}, \quad |\mathbf{m}_{2}| = \psi \sqrt{\frac{S_{1}^2}{\lambda_{1}^2} + \frac{S_{0}}{\lambda_{2}^2}}. \tag{14} \]

Note that the sublattice magnetizations are not perpendicular, but form an angle $\gamma = \zeta(\mathbf{m}_{1}, \mathbf{m}_{2})$ with each other, expressed by

\[ \cos(\gamma) = \frac{S_{0} S_{1} (\lambda_{1}^2 - \lambda_{2}^2)}{\sqrt{(S_{0}^2 \lambda_{2}^2 + S_{1}^2 \lambda_{1}) (S_{1}^2 \lambda_{2}^2 + S_{0}^2 \lambda_{1})}}. \]

A thorough analysis of the Landau free energy shows that this non-collinear phase has no domain of stability, for $K < 0$, within the exchange approximation used here. We should mention that the free energy (6) is very sensitive to the sign of the inter-sublattices interaction $K$. When $K > 0$, i.e.
the interaction between the sublattices is ferromagnetic then there is small domain in which the respective non-collinear phase is stable [12].

For antiparallel \( \psi_1 \) and \( \psi_2 \) it is obvious that the sublattice magnetizations (10) will be also antiparallel. In this case, we may write the resulting equations, that minimize the free energy, for the magnitudes of the field variables \( \psi_1 \) and \( \psi_2 \) of the collinear phase and \( K < 0 \) in the following form:

\[
t_1 \psi_1 + g \psi_1^3 + 3b \psi_1 \psi_2^2 - w \psi_2 (\psi_2^2 - 3\psi_1^2) = 0, \\
t_2 \psi_2 + g \psi_2^3 + 3b \psi_1^2 \psi_2 - w \psi_1 (3\psi_2^2 - \psi_1^2) = 0,
\]

with the stability conditions given by:

\[
t_1 + 3g \psi_1^2 + 3b \psi_2^2 + 3w \psi_1 \psi_2 > 0, \\
(t_1 + 3g \psi_1^2 + 3b \psi_2^2 + 6w \psi_1 \psi_2)(t_2 + 3g \psi_1^2 + 3b \psi_2^2 - 6w \psi_1 \psi_2) - 9[w(\psi_1^2 - \psi_2^2) + 2b \psi_1 \psi_2]^2 \geq 0.
\]

We will make some remarks on the dependence of solutions of (15) on the magnitude of the exchange parameters \( J_1, J_2 \) and \( K \). When \( J_1 < |K| \) and \( J_2 < |K| \), the leading interaction is determined by the antiferromagnetic coupling between the two sublattices. This may be dubbed a strong coupling limit for which the eigenvalue

\[
\lambda_2 = \frac{1}{2} \left[ J_1 + J_2 - \sqrt{(J_1 - J_2)^2 + 4K^2} \right],
\]

turns negative. This is equivalent to the inequality \( K^2 - J_1 J_2 > 0 \). The coefficient \( t_2 \) in front of \( \psi_2^2 \) is positive; see (7), and the field \( \psi_2 \) becomes redundant and the Landau free energy (6) reads

\[
\left( \frac{F}{T} \right)_s = f_s = \frac{t_1}{2} \psi_1^2 + \frac{g}{4} (\psi_1^2)^2.
\]

The minimization of the above expression with respect to \( \psi_1 \) yields the solution

\[
(\psi_1^2)^2 = -\frac{t_1}{g},
\]

which exists and is stable for \( t_1 < 0 \).

The sublattice magnetizations:

\[
\vec{m}_1 = \frac{S_0}{A_1} \psi_1 \quad \text{and} \quad \vec{m}_2 = \frac{S_1}{A_2} \psi_1
\]

are antiparallel as \( S_1 \sim K/D \) and \( K < 0 \). The phase described by the above equations corresponds to two antiparallel sublattices with different magnitudes of sublattice magnetizations.

In the weak coupling limit for antiparallel configuration, i.e., when \( J_1 > |K| \) and \( J_2 > |K| \), or equivalently \( J_1 J_2 > K^2 \), the system of equations (15), together with the stability conditions (16) is hard to solve analytically. To gain insights into the phase behaviour of the considered system we will perform a numerical analysis and the results are presented in the next Section.
4. Results and discussion

The analytical result for sublattice magnetizations (20) in the case of strong antiferromagnetic coupling gives for the magnitude of total magnetization \( |\mathbf{M}| = |\mathbf{m}_1 + \mathbf{m}_2| \) the following expression:

\[
|\mathbf{M}| = \frac{S_0}{\lambda_1} \left| 1 + \frac{S_1}{S_0} \right| |\psi_1|
\]

with \( |\psi_1|^2 \), from (19). It is found that regardless of the values of the model’s parameter, the phase transition is of second order. The total magnetization behaviour with temperature is smooth resembling the one of Weiss ferromagnet with the exception that no saturation is reached at \( T = 0 \).

According to the Néel’s classification of ferrimagnets, see [2], the change of magnetization with temperature in the strong coupling limit falls within R-type curve. It is worth noticing that a qualitatively similar curve is obtained theoretically and compared with the respective experimental measurements for \( Y_3\text{Fe}_5\text{O}_{12} \) [4] where two sublattice model with strong antiferromagnetic coupling is considered.

When \( J_1 = J_2 = J \) the relation \( S_0 = -S_1 = 1/\sqrt{2} \) holds and an antiferromagnetic ordering with \( \mathbf{m}_1 = -\mathbf{m}_2 \) sets in, only if \( \psi_2 \equiv 0 \) and \( |\psi_1|^2 = -2t_1/u \). The transition temperature to antiferromagnetic ordering is given by \( T_c^0 = \frac{1}{\beta}(J + |K|) \).

Further we present the numerical results for the temperature dependence of sublattice magnetizations and the total magnetization of the system in the weak-coupling limit which we define here in the following way: \( \mathcal{J}_1 > |\mathcal{K}| \) and \( \mathcal{J}_2 > |\mathcal{K}| \). Such a situation is present, for example in some ferrimagnetic compounds like \( \text{GdCo}_{12}\text{B}_6 \) [9]. It is experimentally found that the exchange constants within sublattices are ferromagnetic and larger than the antiferromagnetic coupling between the sublattices; moreover there the contribution of the magnetic anisotropy may be disregarded.

Experiments for some R-T compounds where R is a rare earth element and T is a transition one, show that the exchange in the transition metal sublattice is leading in magnitude, while the exchange in the rare-earth ion sublattice can be safely ignored and considered as negligible. The intersublattice interaction is also small for the compounds \( \text{DyFe}_5\text{Al}_7 \) [13], \( \text{ErFe}_{11}\text{Th} \) [14], \( \text{RCO}_2 \) (R = Tb and Gd and R = Er, Ho, and Dy) [15]. In our notations the relation between the exchange integrals in this case will be \( \mathcal{J}_1 > |\mathcal{K}| \gg \mathcal{J}_2 \). This does not fall into our assumption of weak coupling and will not be considered here.

In order to solve numerically the equations of state (15) for weak coupling between sublattices we introduce the following dimensionless parameters.

\[
t = \frac{T}{\mathcal{J}_1 + \mathcal{J}_2}, \quad \alpha = \frac{\mathcal{J}_1 - \mathcal{J}_2}{\mathcal{J}_1 + \mathcal{J}_2}, \quad \beta = \frac{\mathcal{K}}{\mathcal{J}_1 + \mathcal{J}_2}, \quad \tag{21}
\]

with \( t \) – the dimensionless temperature. In the above expression we have supposed that \( \mathcal{J}_1 > \mathcal{J}_2 \) which in view of symmetry in interchanging the sublattices does not limit the consideration; then \( \alpha > 0 \) and \( \beta < 0 \) as \( \mathcal{K} < 0 \).

The weak coupling between the sublattices, namely \( \mathcal{J}_1 > |\mathcal{K}| \) and \( \mathcal{J}_1 > |\mathcal{K}| \) may be expressed by the parameters from (21) with the relation:

\[
\alpha^2 + \beta^2 < 1.
\]

The parameter \( \alpha \) is a measure for the difference in exchange parameters of the two sublattices and by its definition \( 0 < \alpha < 1 \). The total magnetization of the system is the sum of the two sublattice magnetizations (10):

\[
\mathbf{\bar{M}} = \mathbf{\bar{m}}_1 + \mathbf{\bar{m}}_2 = \frac{S_0 + S_1}{\lambda_1} \psi_1 + \frac{S_0 - S_1}{\lambda_2} \psi_2. \quad \tag{22}
\]
Hereafter we will use the following notations for the magnitudes of the sublattice magnetizations and the total magnetization both in the text and in figures:

\[ m_1 = |\vec{m}_1|, \quad m_2 = |\vec{m}_2| \quad \text{and} \quad M = |\vec{M}| \]

The phase transition to the ordered ferrimagnetic state for weak antiferromagnetic coupling occurs at the temperature:

\[ t_c = \frac{1}{6} \left( 1 + \sqrt{\alpha^2 + \beta^2} \right) \]

which increases when either the difference between the exchange interactions in the sublattices grows, or when the antiferromagnetic coupling becomes larger, or both. The phase transition from the disordered to the ordered phase is of second order.

Notice that within the exchange approximation for the regime of weak coupling defined above with the decrease of temperature a compensation point sets in no matter how small is the difference of the exchange interaction between the sublattices. At the compensation temperature \( t_{\text{comp}} \), the sublattice magnetizations \( \vec{m}_1 \) and \( \vec{m}_2 \) are equal in magnitude and antiparallel, thus \( M \) vanishes. The relation between the field variables magnitudes at \( t_{\text{comp}} \) is given by:

\[ \psi_1 = \frac{\lambda_1 (S_0 - S_1)}{\lambda_2 (S_0 + S_1)} \psi_2. \]

As the calculations show \( \psi_2 < \psi_1 \) for all values of \( \alpha \) and \( \beta \), but \( \psi_1 \) grows with the decrease of temperature in a monotonic way, while \( \psi_2 \) grows more rapidly. The quantity

\[ \frac{\lambda_1 (S_0 - S_1)}{\lambda_2 (S_0 + S_1)} = \frac{(6t_c)^2}{1 - \alpha^2 - \beta^2} \left( \frac{\sqrt{\alpha^2 + \beta^2} - \beta}{\alpha} \right) \]

is always > 1 as \( \beta < 0 \sim K \) and \( \alpha > 0 \) so at some temperature \( t_{\text{comp}} < t_c \), and respective values of \( \psi_1 \), \( \psi_2 \) the condition (24) is fulfilled. In figure 1, we depict the change of the net magnetization magnitude \( M \) as a function of the reduced temperature \( t \) for \( \alpha = 0.1 \), i.e., \( J_2 = 0.83 J_1 \) and different values of \( \beta \). One can see that the increase of antiferromagnetic coupling between the sublattices slightly shifts the compensation temperature to higher values, and \( M \) grows more rapidly below the compensation temperature as the temperature decreases. We suppose that within the exchange

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**Figure 1.** The dependence of net magnetization \( M \) on reduced temperature \( t \) for fixed \( \alpha \) and different values of the antiferromagnetic coupling \( \beta \).
approximation and in weak coupling limit the key factor for the compensation point to appear is the weakness of antiferromagnetic exchange between the sublattices compared to the ferromagnetic exchange of sublattices 1 and 2, respectively.

Hereafter we turn our attention to the influence of the difference between the magnitudes of exchange interactions within the sublattices, represented by the parameter $\alpha$ on the magnetization $M$ as a function of $t$ and the sublattice magnetizations $m_1$, and $m_2$. For $\alpha = 0.08$, i.e., $J_2 = 0.85 J_1$, the behaviours of $M$, $m_1$, and $m_2$ are shown in figure 2. At $t_c$ the transition is of second order and when lowering the temperature a compensation point $t_{\text{comp}}$ shows up closer to $t_c$ than $t = 0$. The sublattice magnetizations vary with temperature in a monotonic way, and in the temperature interval $t_{\text{comp}} < t < t_c$, the relation between the sublattice magnetizations is $m_1 > m_2$, as expected, since we have $J_1 > J_2$. Below $t_{\text{comp}}$, the magnetization of the weaker sublattice $m_2$ becomes larger than $m_1$. For intermediate values of $\alpha = 0.4$ or $J_2 = 0.43 J_1$, see figure 3, by lowering the temperature below the compensation point the total magnetization rapidly grows in non-monotonic way as $t \to 0$, similar to the V-curve according to Néel’s classification. The behaviour of the sublattice magnetizations with
temperature is quite different; $m_1$, which is the sublattice magnetization with stronger exchange interaction $J_1$ grows in smooth way with decrease of temperature, while $m_2$ for weaker sublattice interaction $J_2$, below the compensation point grows drastically in a non-monotonic way and in the temperature interval $0 < t < t_{comp}$, $m_2 \gg m_1$.

Such behaviour is described in detail in [16] for ferrimagnets with a compensation point for many experimentally investigated substances. The behaviour of $M$ below the compensation point is interpreted by introducing the notion of weak sublattice where depending on the particular substance considered different mechanisms for the explanation of this effect are pointed out. Within our exchange model the effect of weak sublattice is readily seen and mainly depends on the difference $J_1 - J_2 \sim \alpha$. When $\alpha$ further grows the compensation point is shifted to lower temperatures and the magnetization $m_1$ of the stronger sublattice decreases for $t \to 0$. This is illustrated in figure 4 for $J_2 = 0.19 J_1$.

![Figure 4](image)

**Figure 4.** Illustration of the dependence of the net magnetization $M$ and the sublattice magnetizations $m_1$ and $m_2$ on the reduced temperature $t$ when the exchange in sublattice 2 is very small compared to that in sublattice 1.

A qualitatively similar behaviour of the total magnetization is observed in experiments with ErFe$_2$ [8] and GdCo$_{12}$B$_6$ [9], although direct comparison with the experimental curves can hardly
be made, as these substances have crystallographic and magnetic structure that differs from the one assumed within our model.

To shed light on the influence of the parameter \( \alpha \) on the temperature behaviour of net magnetization we show it in figure 5 for small value of antiferromagnetic coupling \( \beta = 0.08 \) and different values of the parameter \( \alpha \). As the difference between the magnitudes of sublattice exchange interactions grows the transition temperature is shifted to higher values as expected, see (23) and the compensation temperature is lowered. Another effect is the change of net magnetization behaviour below \( t_{\text{comp}} \) from monotonic to nearly exponential when \( \alpha \) increases.

5. Conclusions
We performed a mean-field analysis of two antiferromagnetically coupled Heisenberg models (1) with in-sublattice ferromagnetic interaction on a bcc lattice. The magnetic structure of the collinear ferrimagnet is described by two interpenetrating simple cubic magnetic sublattices shifted with respect to one another along the cube diagonal by \( a(1/2, 1/2, 1/2) \) with \( a \) - the lattice constant which interact antiferromagnetically. Our study shows that the behaviour of the net magnetization of the two-sublattice ferrimagnet depends essentially on the difference in the magnetic interactions between the sublattices within the weak coupling limit. The weak antiferromagnetic interaction as defined above between the ferromagnetically ordered sublattices is responsible for the appearance of compensation point when lowering the temperature. The model may be generalized to include the expansion of the effective Hamiltonian up to sixth-order terms in the field vectors on the two sublattices in order to consider the possibility of a first order phase transition to ferrimagnetic state due to the influence of external parameters, such as pressure and change of concentration.

Within our approach the parameters of the Landau free energy can be directly related to the averaged microscopic exchange interactions as described in section 2, and this relation is quite simple in the case of high symmetry crystal structure as the one considered here. It will be of interest also to include the influence of an external magnetic field and to perform calculations for ferrimagnets, for which the relation between exchange interactions fulfills the condition \( J_1 > |K| \gg J_2 \), that is, when one of the sublattices is very weak.

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