The closeness of the Ablowitz-Ladik lattice to the Discrete Nonlinear Schrödinger equation

Dirk Hennig and Nikos I. Karachalios
Department of Mathematics, University of Thessaly, 35100, Lamia, Greece

Jesús Cuevas-Maraver
Grupo de Física No Lineal, Departamento de Física Aplicada I,
Universidad de Sevilla. Escuela Politécnica Superior, C/ Virgen de África, 7, 41011-Sevilla, Spain
Instituto de Matemáticas de la Universidad de Sevilla (IMUS). Edificio Celestino Mutis. Avda. Reina Mercedes s/n, 41012-Sevilla, Spain

(Dated: December 14, 2021)

While the Ablowitz-Ladik lattice is integrable, the Discrete Nonlinear Schrödinger equation, which is more significant for physical applications, is not. We prove closeness of the solutions of both systems in the sense of a “continuous dependence” on their initial data in the $l^2$ and $l^\infty$ metrics.

The most striking relevance of the analytical results is that small amplitude solutions of the Ablowitz-Ladik system persist in the Discrete Nonlinear Schrödinger one. It is shown that the closeness results are also valid in higher dimensional lattices as well as for generalised nonlinearities. For illustration of the applicability of the approach, a brief numerical study is included, showing that when the 1-soliton solution of the Ablowitz-Ladik system is initiated in the Discrete Nonlinear Schrödinger system with cubic and saturable nonlinearity, it persists for long-times. Thereby excellent agreement of the numerical findings with the theoretical predictions is obtained.

I. INTRODUCTION

The Discrete Nonlinear Schrödinger equation (DNLS)

$$i\dot{\phi}_n + \kappa \nu (\phi_{n+1} + \phi_{n-1}) + \gamma |\phi_n|^2 \phi_n = 0, \ n \in \mathbb{Z}, \ \gamma > 0,$$

is one of the most important nonlinear lattice systems [1],[2],[3],[4]. It appears as a fundamental, inherently discrete model in a great variety of physical contexts. The study of its dynamics has been an exciting topic of research as it deals with such diverse physical and biological phenomena as wave motion in coupled nonlinear waveguides, dynamics of modulated waves in nonlinear electric lattices, localisation of electromagnetic waves in photonic crystals, energy localisation in discrete condensed matter and biological systems and the dynamics of Bose-Einstein Condensates [5], to mention a few. In Eq. (1.1), $\kappa > 0$ is a discretisation parameter, while the sign of the parameters $\nu$ and $\gamma$ renders the DNLS as focusing (same sign) or defocusing (opposite sign).

A crucial difference from its continuous limit $\kappa \to \infty$ leading to the cubic Nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \nu u_{xx} + \gamma |u|^2 u = 0, \ x \in \mathbb{R},$$

is that the DNLS (1.1) is a non-integrable discretisation of the integrable partial differential equation (1.2). This is not the case for another discretisation of NLS (1.2), known as the Ablowitz-Ladik equation (AL) [6],[7],[8].

$$i\dot{\psi}_n + \kappa \nu (\psi_{n+1} + \psi_{n-1}) + \mu |\psi_n|^2 (\psi_{n+1} + \psi_{n-1}) = 0, \ n \in \mathbb{Z}, \ \mu > 0.$$

Like in (1.1) the sign of the parameters $\nu$ and $\mu$ determines whether the model is of focusing or defocusing kind. Most importantly, the AL lattice is an integrable discretisation of (1.2), as it was shown by the discrete version of the Inverse Scattering Transform [6], and therefore it has an infinite number of conserved quantities. We remark that the AL is one of the few known completely integrable (infinite) lattice systems admitting soliton solutions [9],[10]. On the infinite lattice with vanishing boundary conditions, general solutions can be obtained [7]. For example, for $\kappa = \nu = 1$ and $\mu > 0$, the one-soliton solution reads as

$$\psi_n^s = \frac{\sinh \beta}{\sqrt{\mu}} \text{sech} [\beta(n - v_s t)] \exp(-i(\omega t - \alpha n)),$$

$$\omega = -2 \cos \alpha \cosh \beta,$$

$$v_s = 2 \beta^{-1} \sin \alpha \sinh \beta,$$

(1.4)
with \( \alpha \in [-\pi, \pi] \) and \( \beta \in [0, \infty) \). System (1.3) possesses the following conserved quantity
\[
P_{\mu} = \sum_n \ln(1 + |\mu| \psi_n|^2).
\] (1.5)

In [11], the Inverse Scattering Transform method has been developed for nonvanishing boundary conditions and N-dark soliton solutions of the AL equation have been given in terms of the Casorati determinant in [12]. Yet in similarity with the integrable NLS (1.2), another important class of solutions of the AL is the one of rational solutions which are discrete versions of the Peregrine soliton and the Kuznetsov-Ma breather [13], [14].

Strictly speaking, all the aforementioned analytical solutions exist only for the AL. In the case of the soliton solution (1.4) of the AL this is manifested in the fact that it exhibits continuous translation symmetry and possesses a band of velocities for each \( \beta \), which allow it to travel along the lattice. This is not the case for the DNLS for which a localised state can be pinned due to the Peierls-Nabarro barrier [15], [16], [17].

Therefore, the question of the persistence of solitary wave dynamics and the existence of localised structures in non-integrable lattices, such as the DNLS (1.1), has attracted tremendous interest. As a milestone in this context, we recall the construction of localised in space, time periodic or time-quasiperiodic solutions of lattice dynamical systems, including the DNLS as special case [18], [19], [20], starting from the anti-continuous limit \( \kappa \to 0 \). For key works regarding numerical computations of discrete solitons we refer to [21], [22]. Other seminal results on the existence of nonlinear localised modes of the DNLS equations have been gained by nonlinear analysis methods, in particular variational ones, establishing the existence of localised structures as critical points of suitable functionals [23], [24], [25], [26], [27]. Important extensions concern the existence of more complex structures in higher dimensional set-ups, such as the discrete vortex solutions, see [28] and references therein. The crucial issue of existence and stability of travelling solitons in DNLS lattices has been investigated by a combination of analytical and numerical methods verifying in many cases the robustness of the discrete localised structures under perturbations, [28], [29], [30], [31]. In this context we also refer to the reviews [32], [33], [34].

In the present work, we investigate a persistence/existence problem, by examining in the sense of “continuous dependence”, the closeness of the solutions of the DNLS and the AL. That is, the following question is investigated here: **assuming that the initial data of the DNLS (1.1) and the AL (1.3) are sufficiently close in a suitable metric, do the associated solutions remain close for sufficiently long times?**

We argue that answering this question is important because of the following reasons:

- Whereas, it is natural to expect, at least in some cases of parametric regimes, that sufficiently weak non-integrable perturbations (e.g. stemming from gain/loss or forcing terms or higher order terms) lead to solutions staying close to those of the underlying integrable (core) system dynamics, in the case of the DNLS and AL lattices, there is not such a limiting connection between the systems.

- An affirmative answer to the question above will establish that the already diverse dynamical features of the DNLS itself are even further enriched as then the DNLS closely share such solutions that are provided by the functional form of the analytical solutions of the AL-lattice (at least) for small amplitudes. From this perspective, not only the soliton solutions but also the discrete rational solutions are relevant.

In our aim to answer the above question, we proceed by analytically proving that **at least under certain smallness conditions on the initial data of the DNLS and the AL lattices, the corresponding solutions remain close for all times.**

To be precise, we state the result for the infinite lattice with vanishing boundary conditions
\[
\lim_{|n| \to \infty} \phi_n = \lim_{|n| \to \infty} \psi_n = 0.
\] (1.6)

Hence, the natural phase space for the systems is the Hilbert-space of the square-summable sequences
\[
\ell^2 = \left\{ \phi = (\phi_n)_{n \in \mathbb{Z}} \in \mathbb{C} \mid ||\phi||_{\ell^2} = \left(\sum_n |\phi_n|^2\right)^{1/2} \right\}.
\] (1.7)

Consider then, the initial conditions for the DNLS (1.1) and AL (1.3)
\[
\phi_n(0) = \phi_{n,0}, \quad n \in \mathbb{Z},
\] (1.8)
and
\[
\psi_n(0) = \psi_{n,0}, \quad n \in \mathbb{Z},
\] (1.9)
respectively. The main result of the paper is the following
Theorem I.1. Consider the DNLS equation (1.1). We assume that for every $0 < \epsilon < 1$, the initial conditions $\phi$ of the DNLS (1.1) and the initial conditions $\psi$ of the AL (1.3) satisfy:

$$
\|\phi(0) - \psi(0)\|_2 \leq C_0 \epsilon^3, \quad (1.10)
$$

$$
\|\phi(0)\|_2 \leq C_{\gamma,0} \epsilon, \quad (1.11)
$$

$$
P_\mu(0) = \sum_n \ln(1 + \mu |\psi_n(0)|^2) \leq C_{\mu,0} \epsilon^2 \quad (1.12)
$$

for some constants $C_0, C_{\gamma,0}, C_{\mu,0} > 0$. Then, for arbitrary finite $0 < T_f < \infty$, there exists a constant $C = C(\gamma, \mu, C_{\mu,0}, C_{\gamma,0}, T_f)$, such that the corresponding solutions for every $t \in [0, T_f]$, satisfy the estimate

$$
\|y(t)\|_2 = \|\phi(t) - \psi(t)\|_2 \leq C \epsilon^3. \quad (1.13)
$$

For the proof of Theorem I.1 we use suitable estimates for the solutions of the AL lattice based on its deformed power or norm, and energy arguments for the difference of solutions of the systems. It also makes essential use of the global existence of solutions for both lattices ensured by considering a physically relevant variant of a DNLS system which combines both systems studied first in [35] (the so-called Salerno model, see [35], [38]).

An immediate consequence follows from Theorem I.1 due to embedding $\|y\|_1 \leq \|y\|_2$ which holds for every $y \in L^2$ and can be applied to the estimate (1.13).

Corollary I.1. Under the assumptions (1.10), (1.12), for every $0 < \epsilon < 1$ and $t \in [0, T_f]$, the maximal distance $\|y(t)\|_\infty = \sup_{n \in \mathbb{Z}} |y_n(t)| = \sup_{n \in \mathbb{Z}} |\psi_n(t) - \phi_n(t)|$ between individual units of the systems satisfies the estimate

$$
\|y(t)\|_\infty \leq \tilde{C} \epsilon^3, \quad (1.14)
$$

for some constant $\tilde{C} = \tilde{C}(\gamma, \mu, C_{\mu,0}, C_{\gamma,0}, T_f)$.

Theorem I.1 establishes the closeness of the solutions to the AL and the DNLS as $\epsilon \to 0$, with an explicit expression for the associated constant $C$ in dependence on the parameters of both systems. Its main application is that it rigorously justifies that at least small amplitude localised structures provided by the analytical solutions of the integrable AL-lattice persist in the DNLS lattice. In other words, the DNLS lattice admits small amplitude solutions of the complex Ginzburg-Landau pde and the NLS pde, when the inviscid limit of the former is considered [39], which can even grow exponentially [37].

The closeness result of Theorem I.1 can be extended to other important cases of DNLS systems. These include the DNLS with generic power-nonlinearity $F(z) = |z|^{2\sigma}z$ for $\sigma > 0$ and saturable nonlinearities of the form $F(z) = \frac{z^3}{1 + |z|^2}$ and $F(z) = \frac{z^3}{1 + |z|^2}$. The extensions may consider higher-dimensional lattices $\mathbb{Z}^N, N \geq 1$. Note that for generalisations of the AL lattice in $\mathbb{Z}^2$, analytical localised solutions have been constructed [35], [40], [41], [42].

To corroborate our analytical results we include the results of a numerical study treating the example of the soliton solution (1.4) when launched on DNLS lattice with the cubic nonlinearity (1.1), and the DNLS with saturable nonlinearity. The numerical findings are in excellent agreement with the theoretical results.

The presentation of the paper is as follows: Section II recalls some basic properties of the DNLS and the AL lattices, focusing on their conserved quantities and auxiliary results that will aid the main proofs. In Section III we prove the global existence result for the extended Salerno model of [35], [38]. Section IV contains the proof of the main result Theorem I.1, while section V provides its extensions to higher-dimensional lattices and the saturable nonlinearities. In section VI we present the results of the numerical study. Section VII summarises the findings and provides a brief plan for further relevant studies.
II. PRELIMINARIES

For convenience, we set $\kappa = \nu = 1$, without affecting the generality of the proofs, which are valid in either the focusing or the defocusing case. The AL (1.3) can be derived from the Hamiltonian given by

$$H = \sum_n \overline{\psi}_n (\psi_{n+1} + \psi_{n-1})$$

(2.1)

with the following deformed Poisson bracket

$$\{ \psi_m, \overline{\psi}_n \} = (1 + \mu |\psi_m|^2) \delta_{m,n}, \quad \{ \psi_m, \psi_n \} = \{ \overline{\psi}_m, \overline{\psi}_n \} = 0,$$

(2.2)

yielding the equation of motion as

$$\dot{\psi}_n = \{ H, \psi \}.$$

(2.3)

The DNLS can be derived from the Hamiltonian

$$H = \sum_n \left( \overline{\phi}_n (\phi_{n+1} + \phi_{n-1}) - \frac{\gamma}{2} |\phi|^4 \right),$$

(2.4)

using the standard Poisson bracket and the equation of motion

$$i \partial_t \phi_n = \{ H, \phi \}.$$

(2.5)

For the DNLS the norm

$$P_\gamma = \sum_n |\phi_n|^2,$$

(2.6)

is conserved.

The AL equation is completely integrable [6], whereas its DNLS counterpart (1.1) is known to be nonintegrable [4], [8]. Notice that in (1.3) and (1.1) the nonlinear terms are both of cubic order. However, they are markedly different in the sense that, the nonlinear terms in (1.3) are of nonlocal nature compared to the local terms in (1.1).

In the case of the vanishing boundary conditions (1.6), the functional space setting is based on the spaces of complex summable sequences

$$l^p = \left\{ \phi = (\phi_n)_{n \in \mathbb{Z}} \in \mathbb{C} \mid \|\phi\|_{l^p} = \left( \sum_n |\phi_n|^p \right)^{1/p} \right\}.$$

(2.7)

For any $\phi = (\phi_n)_{n \in \mathbb{Z}}, \psi = (\psi_n)_{n \in \mathbb{Z}} \in l^2$ we consider the inner product

$$(\phi, \psi)_2 = \sum_{n \in \mathbb{Z}} \phi_n \overline{\psi}_n,$$

(2.8)

where $\overline{\psi}$ denotes the conjugate of $\psi_n$. With the associated norm

$$||\phi||_{l^2}^2 = (\phi, \phi),$$

(2.9)

$(l^2, (\cdot, \cdot), || \cdot ||)$ is a complex Hilbert space. We will use the continuous embeddings

$$l^r \subset l^s, \quad ||\phi||_{l^r} \leq ||\phi||_{l^s}, \quad 1 \leq r \leq s \leq \infty.$$

(2.10)

The following auxiliary result will aid the ensuing studies.

**Lemma II.1.** Let $\mu > 0$. Assume that the initial condition (1.9) of the AL-lattice (1.3) is such that

$$P_\mu (0) = \sum_n \ln (1 + \mu |\psi_n(0)|^2) < \infty.$$

(2.11)

Then, the corresponding solution of the AL lattice satisfies the estimate

$$\mu ||\psi(t)||_{l^2}^2 = \sum_n |\psi_n(t)|^2 \leq \exp (P_\mu (0)) - 1, \quad \forall t \geq 0.$$

(2.12)
Proof: Using that the function
\[ f : \mathbb{R}_+ \to \mathbb{R}_+, \ x \mapsto \ln(1 + \mu x), \] (2.13)
is continuous and bijective, we write
\[ P_\mu = \sum_n \ln(1 + \mu |\psi_n|^2) = \sum_n |\lambda_n|^2. \] (2.14)
¿From (2.14), and by using the embedding (2.10) for \( s = 2k \) and \( r = 2 \), we get the estimate:
\[ \mu \sum_n |\psi_n|^2 = \sum_n \left( \exp(|\lambda_n|^2) - 1 \right) = \sum_n \left( \sum_{k=0}^{\infty} \frac{|\lambda_n|^{2k}}{k!} - 1 \right) \]
\[ = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_n |\lambda_n|^2 \leq \sum_{k=1}^{\infty} \frac{P_k}{k!} \] \( = \left( \sum_{k=0}^{\infty} \frac{P_k}{k!} - 1 \right) = \exp(P_\mu) - 1 \). (2.15)
Since \( P_\mu(t) \) is conserved, i.e \( P_\mu(t) = P_\mu(0) \) for all \( t \geq 0 \), it follows that
\[ \mu \sum_n |\psi_n(t)|^2 \leq \exp(P_\mu(0)) - 1, \quad \forall t \geq 0, \] (2.16)
and the proof is finished. □

III. GLOBAL EXISTENCE OF SOLUTIONS FOR THE SALERNO LATTICE

For the current study of existence and uniqueness of a global solution of the AL and DNLS, we combine them in the so called Salerno model introduced first in [38]:
\[ i\frac{d\psi_n}{dt} + (1 + \mu |\psi_n|^2)(\psi_{n+1} + \psi_{n-1}) + \gamma |\psi_n|^2 \psi_n = 0, \quad n \in \mathbb{Z}, \] (3.1)
with \( \psi_n \in \mathbb{C} \) and initial conditions:
\[ \psi_n(0) = \psi_{n,0}, \quad n \in \mathbb{Z}. \] (3.2)
Note that for \( \gamma = 0 \) (\( \mu = 0 \)), the AL (DNLS) results from (3.1). The study of the Salerno model provided information about the intrinsic collapse of localised states in the presence of integrability-breaking terms, in particular, how the reflection symmetry and translational symmetry of the integrable AL are broken by the on-site nonlinearity of the DNLS. Thereby the study of the global existence of solutions of the Salerno system is an essential tool in the present functional analytic set-up for the main closeness results of AL and DNLS.

We start by noticing that for any \( \psi \in l^2 \) the linear operator \( A : l^2 \to l^2, \)
\[ (A\psi)_n = \psi_{n+1} + \psi_{n-1}, \] (3.3)
is continuous, since
\[ ||A\psi||_2^2 \leq 4||\psi||_2^2. \] (3.4)
We formulate the infinite dimensional dynamical system (3.1)-(3.2) as an initial value problem in the Hilbert space \( l^2 \) (see [43]):
\[ \dot{\psi} = F(\psi) \equiv i[(1 + \mu |\psi|^2)A\psi + \gamma |\psi|^2 \psi], \quad t > 0, \] (3.5)
\[ \psi(0) = \psi_0. \] (3.6)
Regarding the global existence of a unique solution to (3.5)-(3.6), we have the following
Proposition III.1. For every \( \psi_0 \in l^2 \), the problem \((3.2)\), \((3.3)\) possesses a unique global solution \( \psi(t) \) on \([0, \infty)\) belonging to \( C^1([0, \infty), l^2) \).

Proof: First, we prove the local existence of a solution: For this aim, the system \((3.1)\)-\((3.2)\) is conveniently expressed as an equivalent system of integral equations
\[
\psi_n(t) = \psi_n(0) + i \int_0^t \left[ (1 + \mu |\psi_n(\tau)|^2) \Delta \psi_n(\tau) + \gamma |\psi_n(\tau)|^2 \psi_n(\tau) \right] d\tau,
\]
with the notation \( \Delta \psi_n = \psi_n+1 + \psi_n-1 \). We consider the set
\[
\mathcal{B} = \{ \phi \in C[0, \tilde{t}], l^2 \ | ||\phi||_2 \leq \kappa \},
\]
which is a Banach space itself, with norm
\[
||\phi||_\mathcal{B} = \sup_{t \in [0, \tilde{t}]} ||\phi||_2.
\]
Next, for \( \phi \in l^2(\mathbb{Z}) \), we define the nonlinear operator
\[
Q_n(\phi(t)) = \phi_n(0) + i \int_0^t \left[ (1 + \mu |\phi_n(\tau)|^2) \Delta \phi_n(\tau) + \gamma |\phi_n(\tau)|^2 \phi_n(\tau) \right] d\tau.
\]
We shall prove that the operator \( Q \) establishes a contraction mapping on \( \mathcal{B} \). Note first, that it satisfies the upper bound
\[
||Q(\phi)||_\mathcal{B} \leq \kappa_0 + \tilde{t} \left[ (1 + \mu \kappa^2) 2\kappa + \gamma \kappa^3 \right].
\]
We may choose \( \kappa_0 < \kappa/2 \) and
\[
\tilde{t} \leq \frac{\kappa}{(1 + \mu \kappa^2) 2\kappa + \gamma \kappa^3},
\]
so that \( Q : \mathcal{B} \to \mathcal{B} \). Now, for every \( \phi, \psi \in \mathcal{B} \), we have
\[
Q_n(\phi(t)) - Q_n(\psi(t)) = \left( i \int_0^t \left[ (1 + \mu |\phi_n(\tau)|^2) \Delta \phi_n(\tau) + \gamma |\phi_n(\tau)|^2 \phi_n(\tau) \right] d\tau - \right]
\[
\left. - \left[ (1 + \mu |\psi_n(\tau)|^2) \Delta \psi_n(\tau) + \gamma |\psi_n(\tau)|^2 \psi_n(\tau) \right] d\tau \right)
\]
\[
= \int_0^t \left[ (\Delta \phi_n - \Delta \psi_n) + \mu (|\phi_n|^2 \Delta \phi_n - |\psi_n|^2 \Delta \psi_n) - \gamma (|\phi_n|^2 \phi_n - |\psi_n|^2 \psi_n) \right] d\tau,
\]
and estimate the norm as
\[
||Q(\phi) - Q(\psi)||_\mathcal{B} \leq \tilde{t} \left[ 2 + (18\mu + 2\gamma)\kappa^2 \right] ||\phi - \psi||_2.
\]
Choosing \( \tilde{t} \) such that
\[
\tilde{t} \leq \min \left\{ \frac{\kappa}{2(1 + \mu \kappa^2)\kappa + \gamma \kappa^3}, \frac{1}{2 + (18\mu + 2\gamma)\kappa^2} \right\},
\]
it is assured that \( Q \) is a contraction mapping on \( \mathcal{B} \). Then by Banach’s fixed point theorem, there exists a unique solution of \((3.1)\) provided by the unique fixed point of \( Q \). To justify that \( \psi(t) \) is \( C^1 \) with respect to \( t \), we see from \((3.1)\) that
\[
\sup_{t \in [0, \tilde{t}]} ||\dot{\psi}(t)||_2 \leq 2(1 + \mu \kappa^2)\kappa + \gamma \kappa^3.
\]
Consequently, the solution belongs to \( C^1([0, \tilde{t}], l^2) \). Then, constructing a maximal solution is achieved by repeating the procedure above with initial conditions \( \psi(t - T_f) \) for some \( 0 < T_f < \tilde{t} \).

To conclude with global existence of solutions, we remark that the Hamiltonian of \((3.1)\) is
\[
H_S = \sum_n (\psi_n \overline{\psi}_{n+1} + \overline{\psi}_n \psi_{n+1}) - \frac{\gamma}{2} \sum_n |\psi_n|^2 - \frac{1}{\mu} \sum_n \ln(1 + \mu |\psi_n|^2).
\]
The deformed Poisson-brackets are
\[
\{ \psi_n, \overline{\psi}_m \} = i(1 + \mu|\psi_n|^2)\delta_{nm},
\]
\[
\{ \psi_n, \psi_m \} = \{ \overline{\psi}_n, \overline{\psi}_m \} = 0,
\]
and the equation of motion (3.1) is obtained as
\[
\psi_t = \{ H_S, \psi \}.
\]

The system (3.1) conserves also the quantity \( P_\mu(t) \) given in (1.5) (see also (38)). Then, for all initial conditions \( \psi_0 \in \ell^2 \), global existence in \( \ell^2 \) follows actually from the conservation of (1.5), the help of the elementary inequality \( \ln(1 + \mu x) \leq \mu x \) for all \( x > 0 \) and Lemma 1.1 providing that \( |\psi(t)|_{\ell^2} < \infty, \forall t \geq 0 \).

Similarly, global existence for the AL (1.3) is ensured by the conservation of (1.5). For the DNLS (1.1), global existence in \( \ell^2 \) is established by the conservation of \( P_\gamma(t) \) given in (2.6). This concludes the proof. \( \square \)

**IV. PROOFS OF CLOSEDNESS OF THE AL AND DNLS SOLUTIONS**

**Proof of Theorem 1.1** Closeness will be proved in the metric \( \text{dist}_{l^2}(\psi, \theta) = ||\psi - \theta||_{l^2}, \forall \psi, \theta \in \ell^2 \). We consider the local distance of the solutions \( y_n = \phi_n - \psi_n \). On the one hand, we have that
\[
\frac{d}{dt}||y(t)||^2_{l^2} = 2||y(t)||_{l^2} \frac{d}{dt}||y(t)||_{l^2}, \tag{4.1}
\]
while, on the other hand, we estimate the derivative of the \( l^2 \)-norm as follows:
\[
\frac{d}{dt}||y||^2_{l^2} = \sum_n \left\{ i \left[ (\overline{\psi}_{n+1} + \overline{\psi}_{n-1})y_n - (y_{n+1} + y_{n-1})\overline{\psi}_n \right] + i\mu|\psi_n|^2 \left[ (\overline{\psi}_{n+1} + \overline{\psi}_{n-1})y_n - (y_{n+1} + y_{n-1})\overline{\psi}_n \right] - i\gamma|\phi_n|^2 \left( \overline{\phi}_n y_n - \phi_n \overline{\psi}_n \right) \right\}
= 2\mu \sum_n |\psi_n|^2 \left[ (\text{Im}\psi_{n+1} + \text{Im}\psi_{n-1})\text{Re}y_n - (\text{Re}\psi_{n+1}\text{Re}\psi_{n-1})\text{Im}y_n \right] \tag{4.2}
+ 2\gamma \sum_n |\phi_n|^2 \left[ \text{Im}y_n \text{Re}\phi_n - \text{Im}\phi_n \text{Re}y_n \right]
\leq 4\mu \sup_n |\psi_n|^2 \sum_n \left[ |\psi_{n+1}| + |\psi_{n-1}| \right] |y_n| + 4\gamma \sup_n |\phi_n|^2 \sum_n |\phi_n||y_n|
\leq 2(4\mu||\psi(t)||^3_{l^2} + 2\gamma||\phi(t)||^3_{l^2})||y(t)||_{l^2}.
\]

For the estimate (4.2), we made use of the Cauchy-Schwarz and the continuous embeddings (2.10). Then, combining (4.1) and (4.2), one has for \( t > 0 \):
\[
\frac{d}{dt}||y(t)||_{l^2} \leq 2(\gamma||\phi(t)||^3_{l^2} + 2\mu||\psi(t)||^3_{l^2}). \tag{4.3}
\]

Note that under the hypotheses (1.10)-(1.12), Proposition 3.1 ensures that the right-hand side of (4.3) is uniformly bounded for all \( t \in [0, \infty) \). Furthermore, due to the conservation of the quantities (1.5) and (2.6) (see also Proposition 3.1), the \( l^2 \) norm of \( \psi \) and \( \phi \) remains of the size \( \varepsilon \) for all times. Integrating the inequality (4.3) in the arbitrary interval \([0, T_f]\), and using the assumption (1.10) on the distance \( ||y(0)||_{l^2} = ||\phi(0) - y(0)||_{l^2} \) of the initial data, we obtain that
\[
||y(t)||_{l^2} \leq 2 \left( \gamma C_{0, \gamma}^3 + 2\mu C_{0, \mu}^3 \right) T_f \varepsilon^3 + ||y(0)||_{l^2} \leq 2 \left( \gamma C_{0, \gamma}^3 + 2\mu C_{0, \mu}^3 \right) T_f \varepsilon^3 + C_0 \varepsilon^3.
\]

Hence, for the constant
\[
C = 2 \left( \gamma C_{0, \gamma}^3 + 2\mu C_{0, \mu}^3 \right) T_f + C_0, \tag{4.4}
\]
we conclude with the claimed estimate \( L^1 \).

The proof of Theorem \([44]\) shows that the distance between the solutions of the AL and the DNLS measured in terms of the \( l^2 \)-metric remains small (bounded above by \( O(\varepsilon^2) \)), compared to the \( l^2 \)-norm of the solutions themselves. Corollary \([44]\) follows immediately from \([43]\) and continuous embedding \( ||y||_\infty \leq ||y||_2 \) and shows features for the \( l^\infty \)-norm (sup norm), determining the maximal distance between individual units, analogous to those of the \( l^2 \)-norm.

V. REMARKS ON EXTENSIONS TO HIGHER DIMENSIONAL LATTICES AND OTHER NONLINEARITIES

In this section, we report on extensions of Theorem \([44]\) in higher dimensional lattices \( \mathbb{Z}^N \), for \( N \geq 2 \) and for generalised nonlinearities. In the first paragraph, we comment on the closeness of the solutions of higher dimensional DNLS lattices with a generalised power nonlinearity to those of the \( N \)-dimensional generalisation of the AL-lattice. In the second paragraph we remark on the validity of Theorem \([44]\) for the case of the DNLS with saturable nonlinearities.

a. DNLS in higher dimensional lattices. The result of Theorem \([44]\) can be extended to higher dimensional DNLS and AL lattices of the form

\[
i\phi_n + (\Delta_d\phi)_n + |\phi_n|^{2\sigma} \phi_n = 0, \quad \sigma > 0,
\]

and

\[
i\psi_n + (\Delta_d\psi)_n + \mu|\psi_n|^2 \sum_{j=1}^N (T_j\psi)_{n+\epsilon n} = 0,
\]

respectively. The \( N \)-dimensional AL \([5,2]\) is motivated by \([44]\), where the case \( N = 2 \) is studied as the specific limit of a 2D generalisation of the Salerno model \([5,1]\). The operator \((\Delta_d\psi)_n\) is the \( N \)-dimensional discrete Laplacian

\[
(\Delta_d\psi)_n = \sum_{m \in \mathcal{N}_n} \psi_m - 2N\psi_n,
\]

where \( \mathcal{N}_n \) denotes the set of 2\( N \) nearest neighbors of the point in \( \mathbb{Z}^N \) with label \( n \). With the linear operator \( T_j \) which is defined for every \( \psi_n, n = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N \), as

\[
(T_j\psi)_{n+\epsilon n} = \psi(n_1, n_2, \ldots, n_j+1, n_{j+1}, \ldots, n_N) + \psi(n_1, n_2, \ldots, n_j-1, n_{j+1}, \ldots, n_N), \quad j = 1, \ldots, N,
\]

the nonlocal nonlinearity in \([5,2]\) generalises the one of \([1,3]\). Analytical solutions of the generalisation of the AL system \([5,2]\) when \( N = 2 \), have been derived in \([39, 40, 41, 42]\).

b. DNLS with saturable nonlinearity. Another important example concerns the DNLS with the saturable nonlinearity

\[
i\phi_n + (\phi_{n+1} - 2\phi_n + \phi_{n-1})_n + \frac{\gamma |\phi_n|^2 \phi_n}{1 + \rho|\phi_n|^2} = 0, \quad \gamma, \rho > 0,
\]

and consequently, its other counterpart

\[
iU_n + (U_{n+1} - 2U_n + U_{n-1})_n - \frac{\Gamma U_n}{1 + |U_n|^2} = 0, \quad \Gamma > 0,
\]

because the models \([5,4]\) and \([5,5]\) are not independent: solutions of the saturable model \([5,4]\) can be mapped to the solutions of the model \([5,5]\) by the invertible transformation

\[
\phi_n(t) = \frac{1}{\sqrt{\rho}} \exp \left( \frac{i\gamma t}{\rho} \right) U_n(t), \quad \Gamma = \frac{\gamma}{\rho},
\]

see \([42]\). For the saturable DNLS, numerous studies have verified the propagation of discrete solitons and the emergence of breathers in the 1D and 2D lattices \([43, 44, 45, 46, 48, 49]\). The conserved quantities of the DNLS model \([5,5]\) are the power \( P_\gamma(t) \) given in \([2,6]\) and the Hamiltonian

\[
\mathcal{H} = \sum_n |U_{n+1} - U_n|^2 + \Gamma \sum_n \ln(1 + |U_n|^2).
\]
FIG. 1: Spatiotemporal evolution of the initial condition (6.1) in the DNLS lattice with $\gamma = 1$ for the cubic nonlinearity (left panel) and the saturable nonlinearity (right panel). Parameters of the initial condition $a = \pi/10$ and $\beta = 0.02, \mu = 1$.

The extension of Theorem I.1 to both saturable models follows from the model (5.4). The corresponding evolution equation for the local distance $y_n = \phi_n - \psi_n$, between the solutions of the DNLS (5.4) and the AL lattice (1.3) is

$$i \dot{y}_n = -(y_{n+1} - 2y_n + y_{n-1}) - \left[ \frac{\gamma|\phi_n|^2 \phi_n}{1 + \rho|\phi_n|^2} - \mu|\psi_n|^2 (\psi_{n+1} + \psi_{n-1}) \right],$$

and the derivative of its $l^2$ norm satisfies

$$\frac{1}{2} \frac{d}{dt} ||y||^2 = -\gamma \text{Im} \sum_n \left| \frac{\phi_n^2 |\phi_n|}{1 + \rho|\phi_n|^2} \right| \overline{y_n} + \mu \text{Im} \sum_n \left[ |\psi_n|^2 (\psi_{n+1} + \psi_{n-1}) \right] \overline{y_n}. \quad (5.6)$$

The first term on the right-hand side of (5.6) is estimated as

$$\left| \text{Im} \sum_n \frac{\phi_n |\phi_n|^2}{1 + \rho|\phi_n|^2} \right| \leq \sum_n |\phi_n| \left| y_n \right| \leq ||\phi||^3 ||y||^2,$$

which can be used to derive exactly the same differential inequality (4.3), yielding Theorem I.1 under the same assumptions and same size of $\epsilon$ for the closeness estimate (1.13). Such an extension is also valid for higher dimensional DNLS saturable models.

c. Comments on global existence and quasi-collapse in higher dimensional lattices. Crucial to our discussion of higher dimensional conservative discrete nonlinear Schrödinger equations is the assurance of the global existence of their solutions. The solutions of the DNLS (5.1) exist unconditionally for any $\sigma > 0$ and any $N \geq 1$. This is a vital difference to its NLS pde counterpart whose solutions may blow-up in finite time when $\sigma > 2/N$. The same unconditional global existence is shared by the higher dimensional AL (5.2) or Salerno models. As explained in [44] for the 2D-lattice, if the value of the deformed power is $P_0$, then the sup-norm $||\phi||_{l^\infty}$ of the solution can never exceed the value $[\exp(\mu P_0) - 1]$. This is actually the argument of Lemma II.1 which can be extended to the case $N \geq 1$, establishing global existence of solutions for the $N$-dimensional AL and Salerno equations. The work [44] provides an analysis of the notion of quasi-collapse which can be observed in discrete systems: Blow-up in finite time for the the NLS pde corresponds to the effect of concentration of all energy in few sites in the conservative discrete NLS counterparts. The quasi-collapse supplies the mechanism for the emergence of very narrow self-trapped states in the lattice from its initial distribution, which however, does not exhibit a finite-time-singularity (unless additional energy gain mechanisms are present [50]).

VI. A NUMERICAL STUDY: PERSISTENCE OF THE AL-SOLITON IN DNLS

To illustrate the applicability of the analytical results we performed a numerical study examining the dynamics of the DNLS lattice for both types of nonlinearity, cubic and saturable. As initial conditions we used the one-soliton
FIG. 2: Top row: Time evolution of $||y(t)||_2$ and $||y(t)||_\infty$, corresponding to the soliton dynamics shown in the upper panel of Figure 1 for the DNLS with cubic and saturable nonlinearity (details in the text of section VI). Bottom row: Space time evolution of the soliton center $X_{CM} = (\sum_n n|\phi_n|^2)/(\sum_n |\phi_n|^2)$ for the cubic and the saturable DNLS.

solution of the AL (1.4):

$$\phi_n(0) = \psi_n^s(0) = \sinh \beta \sqrt{\mu} \text{sech}(\beta n) \exp(i\alpha n), \quad n \in \mathbb{Z},$$

(6.1)

$$||\psi^s(0)||_2 = ||\phi(0)||_2 = \varepsilon,$$

(6.2)

where $\alpha \in [-\pi, \pi]$ and $\beta \in [0, \infty)$. In order to comply with the smallness condition (6.2), we chose the parameter values accordingly so that persistence of the corresponding AL soliton in the DNLS can be expected.

Figure 1 depicts the spatio-temporal evolution of the density $|\phi_n(t)|^2$ of the soliton initial condition when $\alpha = \pi/10$ and $\beta = 0.02$, for the DNLS equation with $\gamma = 1$. The dynamics for the cubic (saturable) DNLS is shown in the left (right) panel. The evolution is presented for the time span $t \in [0, 2500]$ ($T_f = 2500$) and for a chain of $K = 2000$ units with periodic boundary conditions. The evolution in both DNLS systems confirms the persistence of the AL soliton with amplitude of order $O(\varepsilon)$ in both lattices. Notably, persistence lasts for a significant large time interval, in particular with view to that our analysis is a “continuous dependence on the initial data result” where generally, the time interval of such a dependence on the initial data for a given equation might be short. Moreover, we studied the continuous dependence of two different systems, the cubic and the saturable DNLS, respectively. The dynamics of the solitons are almost indistinguishable in both DNLS lattices. Note that for this example of initial condition, the
FIG. 3: Logarithmic scaled plots of the variation of the distance functions $||y(t)||_2$ and $||y(t)||_{\infty}$ as functions of $\varepsilon$, for fixed $T_f = 1000$. Left panel: Cubic nonlinearity. Right panel: Saturable nonlinearity. Details are given in the text.

value of the $l^2$-norm $\epsilon^2$ of the initial condition is $\varepsilon = 0.2$.

Our numerical results confirm convincingly the analytical predictions presented by the Theorem I.1 and Corollary I.1 concerning the distance $y(t) = \phi(t) - \psi(t)$, of the solutions of the DNLS and the AL: First, Figure 2 depicts the time evolution of $||y(t)||_2$ (left panel) and $||y(t)||_{\infty}$ (right panel), corresponding to the dynamics of the cubic DNLS shown in the upper panel of Figure 1. The time evolution for the cubic DNLS is plotted as the blue curve while for the saturable as red. However, the curves are still indistinguishable, in conformity with the dynamics portrayed in Figure 1. The results of Figure 2, provide a first justification that both $||y(t)||_2$ and $||y(t)||_{\infty}$ remain small for significantly long time intervals. Another interesting feature is the preservation of the soliton’s speed as shown in the bottom panel of Figure 2. For the considered value of $\beta$, we have $\sinh(\beta) \approx \beta$, so the soliton’s speed is $v_s = 2\sin(\alpha) = 0.6181$.

To be more precise, we examined the variation of the distances for varying small $\varepsilon$. Figure 3, depicts logarithmic scaled plots of the variation of the distances $||y(t)||_2$ and $||y(t)||_{\infty}$ as functions of $\varepsilon$ for fixed $T_f = 1000$. The left (right) panel illustrates the results of the study for the DNLS with cubic (saturable) nonlinearity. The dashed lines in both panels, correspond to lines of the analytical estimates of Theorem I.1 and Corollary I.1 respectively, of the form $||y||_2 \sim C\varepsilon^a$ and $||y||_{\infty} \sim \tilde{C}\varepsilon^b$. For the cubic case, we have $C = 82.41, \tilde{C} = 26.71$. For the saturable case, $C = 82.69, \tilde{C} = 26.79$. The dots on the full lines correspond to the numerically detected rates of the variations of the distance functions fitted to the lines of the form $||y||_2 \sim Ce^a$ and $||y||_{\infty} \sim \tilde{C}e^b$, for the above values of constants $C$ and $\tilde{C}$: for the case of the cubic nonlinearity we found that $a = 5.00$ and $b = 5.60$, while for the saturable nonlinearity we found that $a = 5.01$ and $b = 5.60$. The numerical results illustrate that the analytical estimates are not only fulfilled, but also that the numerical variation of the distance functions is of significantly lower rate, namely of order $\sim \varepsilon^5$.

VII. CONCLUSIONS

We have proved closeness of solutions of the integrable Ablowitz-Ladik equation and the non-integrable Discrete Nonlinear Schrödinger equation, in the sense of “a continuous dependence of the solutions on their initial data”. For the Discrete Nonlinear Schrödinger equation we have considered the physically important examples of the cubic and the saturable nonlinearity. The analytical results are of relevance in regard to the persistence of small amplitude solutions of the Ablowitz-Ladik equation in non-integrable discrete Nonlinear Schrödinger equations. For an illustration of such persistence we have performed numerical simulations considering the analytical 1-soliton solution of the Ablowitz-Ladik equation. It has turned out that the numerical findings are in excellent agreement with the analytical predictions; thus corroborating that small amplitude solitary waves close to the analytical soliton of the Ablowitz-Ladik equation...
persist in Discrete Nonlinear Schrödinger equations for both types of nonlinearities, cubic as well as saturable. Future plans shall concern studies illustrating the persistence of other localised wave forms, supplied by the analytical solutions of the Ablowitz-Ladik equation, in other (nonintegrable) Discrete Nonlinear Schrödinger models. An important example is that of rational solutions. Particularly these rational solutions are non-trivial for computational studies, as the corresponding parameter values must be suitably chosen such that "small amplitude" waveforms get formed in order to apply our analytical methods appropriate for low-amplitude solutions. It would also be interesting to investigate the extension of the theoretical results to other models with higher order linear and nonlinear coupling operators (such as the discrete p-Laplacian\cite{31}, the discrete biharmonic operator\cite{52}, and even other generalisations of the Ablowitz-Ladik system\cite{53,54,55,56}. Such works are in progress and relevant results will be reported elsewhere\cite{57}.

Acknowledgment

We would like to thank the referee for his/her constructive comments and suggestions. J.C.-M. acknowledges support from the Regional Government of Andalusia and EU (FEDER program) under the projects P18-RT-3480 and US-1380977, and MICINN, AEI and EU (FEDER program) under the projects PID2019-110430GB-C21 and PID2020-112620GB-I00.

[1] D. Hennig and G. P. Tsironis, *Wave transmission in nonlinear lattices*, Phys. Rep. 307 (1999), 333–432.
[2] P. G. Kevrekidis, K. O. Rasmussen and A. R. Bishop, *The discrete nonlinear Schrödinger equation: A survey of recent results*, Int. Journal of Modern Physics B 15 (2001), 2833–2900.
[3] J.C. Eilbeck and M. Johansson, *The discrete nonlinear Schrödinger equation-20 years on in: L. Vázquez, R.S. MacKay, M.P. Zorzano (Eds.), Localization and Energy Transfer in Nonlinear Systems*, World Scientific, Singapore, pp. 44–67 (2005).
[4] P.G. Kevrekidis, *The Nonlinear Discrete Schrödinger Equation: Mathematical Analysis, Numerical Computations, and Physical Perspectives* (Springer-Verlag, Berlin, Heidelberg, 2009).
[5] O. Morsch and M. Oberthaler, *Dynamics of Bose-Einstein condensates in optical lattices*, Rev. Mod. Phys. 78 (2006), 179–215.
[6] M.J. Ablowitz and J.F. Ladik, *Nonlinear differential-difference equations and Fourier analysis*, J. Math. Phys. 17 (1976), 1011–1018.
[7] M.J. Ablowitz and J.F. Ladik, *A nonlinear difference scheme and inverse scattering*, Stud. Appl. Math. 65 (1976), 213–229.
[8] B.M. Herbst and M.J. Ablowitz, *Numerically induced chaos in the nonlinear Schrödinger equation*, Phys. Rev. Lett. 62 (1989), 2065.
[9] M.J. Ablowitz and P.A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge Univ. Press, New York, 1991).
[10] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, 1987).
[11] V. E. Vekslerchik, *Functional representation of the Ablowitz–Ladik hierarchy*, J. Phys. A: Math. Gen. 31 (1998), 1087–1099.
[12] K. Maruno and Y. Ohta, *Casorati Determinant Form of Dark Soliton Solutions of the Discrete Nonlinear Schrödinger Equation*, J. Phys. Soc. Jpn 75 (2006), 054002.
[13] A. Ankiewicz, N. Akhmediev and J. M. Soto-Crespo, *Discrete rogue waves of the Ablowitz-Ladik and Hirota equations*, Phys. Rev. E 82 (2010), 026602.
[14] N. Akhmediev and A. Ankiewicz, *Modulation instability, Fermi-Pasta-Ulam recurrence, rogue waves, nonlinear phase shift, and exact solutions of the Ablowitz-Ladik equation*, Phys. Rev. E 83 (2011), 046603.
[15] R. Peierls, *The size of a dislocation*, Proc. Phys. Soc. 52, 34–37.
[16] R. Hobart, *Peierls-Barrier Minima*, J. Appl. Phys. 36 (1965), 1948–1952.
[17] Y. Kivshar and D. Campbell, *Peierls-Nabarro potential barrier for highly localised nonlinear modes*, Phys. Rev. E 48 (1993), 3077–3081.
[18] S. Aubry, *Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators*, Nonlinearity 7 (1994), 1623–1643.
[19] S. Aubry, *Breathers in nonlinear lattices: Existence, linear stability and quantization*, Physica D 103 (1997), 201–250.
[20] M. Johansson and S. Aubry, *Existence and stability of quasiperiodic breathers in the discrete nonlinear Schrödinger equation*, Nonlinearity 10 (1997), 1151–1178.
[21] J. C. Eilbeck and R. Flesch, *Calculation of families of solitary waves on discrete lattices*, Phys. Lett. A 149 (1990), 200–202.
[22] J. L. Marin and S. Aubry, *Breathers in nonlinear lattices: numerical calculation from the anticontinuous limit*, Nonlinearity 9 (1996), 1501—1528.
[23] M. Weinstein, *Excitation thresholds for nonlinear localised modes on lattices*, Nonlinearity 12 (1999), 673–691.
[24] A. Pankov, *Gap solitons in periodic discrete nonlinear Schrödinger equations*, Nonlinearity 19 (2006), 27–40.
[25] A. Pankov, *Gap solitons in periodic discrete nonlinear Schrödinger equations II: a generalised Nehari manifold approach*, Discrete Contin. Dyn. Syst. 19 (2007), 419–430.
