ON A CONJECTURE OF ALMGREN: AREA-MINIMIZING SURFACES WITH FRACTAL SINGULARITIES

ZHENHUA LIU

Dedicated to Xujing Wei

Abstract. For any integers $n \geq 2$, $0 \leq j \leq n - 2$, any sequence of (not necessarily connected) smooth orientable compact Riemannian manifolds $N_j$ of dimension $j$, and any closed compact subsets $K_j$ in $N_j$, we construct a smooth compact Riemannian manifold $M^{2n-k+1}$, with a smooth $n$-dimensional calibration form $\phi$ and a calibrated irreducible homologically area minimizing surface $T^n$ in $M$. ($k$ is the smallest number with $K_k$ nonempty.) The singular set of $T$ is the disjoint union of $K_0, \cdots, K_{n-2}$, and the $j$-symmetric part of the $j$-th strata in the Almgren stratification of the singular set of $T$ is precisely $K_j$. As a corollary, area-minimizing integral currents can have fractal singular sets. Thus, we settle the Problem 5.4 in [1] by Almgren and the answer is sharp dimension-wise. As a by-product, we also obtain similar conclusions for mod $p$ area minimizing currents and stable minimal surfaces. This also implies the sharpness of the rectifiability results of all non-top strata for stationary varifolds in [23].

Contents

1. Introduction 2

1.1. Sketch of proof 4

Acknowledgements 4

2. Preliminaries and notations 5

2.1. Currents 5

2.2. The basics 5

2.3. Some frequently used concepts 6

2.4. A torus differential form 6

2.5. Smooth functions with controlled zero set 7

2.6. The normal bundle calibration 7

2.7. Transition construction by Zhang 7

3. The model of desingularization 8
1. Introduction

There are many different formulations of area minimizing surfaces. In this paper, area minimizing surfaces refer to area-minimizing integral currents, which roughly speaking are oriented surfaces counted with multiplicity, minimizing the area functional. Calling them surfaces is justified thanks to the Almgren’s Big Theorem \cite{Almgren1984} and De Lellis-Spadaro’s new proof \cite{DeLellis2011} \cite{DeLellis2012} \cite{DeLellis2013}. Their results show that $n$-dimensional area minimizing integral currents are smooth manifolds outside of a singular set of dimension at most $n - 2$. (In the codimension 1 case, the dimension of the singular set can be reduced to $n - 7$ by \cite{DeLellis2014}.)

With the marvelous regularity theorems in mind, a natural next step is to ask what more we can say about the singular set? We know that 2-dimensional area minimizing integral currents are classical branched minimal immersed surfaces by \cite{Huisken1991} \cite{Kim1996} \cite{Kim1997}, and all the tangent cones are unique by \cite{Huisken1993} \cite{Sharp1997}.

It turns out little is known beyond dimension 2. The nature of the singular set is a wide open problem. There is a rectifiability result for every strata in the Almgren stratification of the singular set by \cite{DeLellis2013}. However, to the author’s knowledge, the singularities of all known area minimizing surfaces are (up to diffeomorphisms) subanalytic varieties. There is a huge gap between rectifiable sets and subanalytic varieties.

Professor Frederick Almgren has raised the following question as Problem 5.4 in \cite{Almgren1984}, (to quote him),

"Is it possible for the singular set of an area minimizing integer (real) rectifiable current to be a Cantor type set with possibly non-integer Hausdorff dimension?"

The main theorems of this manuscript settles the question by Almgren completely. The result is sharp dimension-wise for surfaces with codimension larger than 1.

**Theorem 1.1.** For any integer $n \geq 2$ and any nonnegative real number $0 \leq \alpha \leq n - 2$, there exists a smooth compact Riemannian manifold $M^{n+3}$, and an
n-dimensional smooth calibration form \( \phi \) on it, so that \( \phi \) calibrates an irreducible homologically area minimizing surface \( \Sigma^n \) in \( M \), whose singular set is of Hausdorff dimension \( \alpha \).

Theorem 1.1 is a natural corollary of the following more general result.

**Theorem 1.2.** For any integers \( n \geq 2 \), \( 0 \leq j \leq n - 2 \), any sequence of smooth compact (not necessarily connected) Riemannian manifolds \( N_j \) of dimension \( j \), let \( K_j \) be compact subsets of \( N_j \), and \( k \) be the smallest number with \( K_k \) nonempty. There exists a smooth compact Riemannian manifold \( M^{2n-j+1} \), and a smooth calibration \( n \)-form \( \phi \) on it, so that \( \phi \) calibrates an irreducible homologically area minimizing surface \( \Sigma^n \) in \( M \). The singular set of \( \Sigma \) is \( K_0 \cup \cdots \cup K_{n-2} \), and the \( j \)-symmetric part of the \( j \)-th strata of the Almgren stratification of the singular set of \( \Sigma \) is precisely \( K_j \).

**Remark 1.** The singularities all come from non-transverse self-intersections, except for those belonging to the zeroth strata. It is unclear whether under generic perturbations, how the fractal singular sets will change. The generic transversality results of Brian White [28] does not apply here, since there is no guarantee that minimizers in nearby metrics are minimal immersions. However, by construction, there are non-generic ways to dissolve the singularities by perturbing \( f_j \) in Section 3.1.

As a byproduct, we can also prove similar results for mod \( p \) minimizing currents and stable minimal surfaces, which is also sharp dimension-wise when \( p = 2 \), by [14].

**Theorem 1.3.** For any integers \( n \geq 2 \), \( p \geq 2 \), \( 0 \leq j \leq n - 2 \), a smooth compact Riemannian manifolds \( N_j \) of dimension \( j \), and a closed \( K_j \subset N_j \), there exists a smooth compact Riemannian manifold \( M^{2n-j+1} \), and a mod \( p \) area minimizing current \( L \) of dimension \( n \). \( L \) decomposes as the sum of two mod \( p \) area minimizing embeddings \( L_1 \) and \( L_2 \), i.e., \( [L] = [L_1] + [L_2] \). The singular set of \( L \) is precisely \( K_j \), which is realized as the intersection \( L_1 \cap L_2 \). \( K_j \) forms the \( j \)-th strata of \( L \). Thus, any real number \( 0 \leq \gamma \leq n - 2 \) can be realized as the dimension of the singular set of an \( n \)-dimensional mod \( p \) minimizing current.

**Theorem 1.4.** For any integers \( n \geq 2 \), \( 0 \leq j \leq n - 1 \), any smooth Riemannian manifolds \( N_j \) of dimension \( j \), there exists a smooth compact Riemannian manifold \( M^{2n-j+1} \), and two \( n \)-dimensional smoothly embedded boundary-less stable minimal surfaces \( \Sigma_1 \) and \( \Sigma_2 \), so that the intersection \( \Sigma_1 \cap \Sigma_2 \) is \( K_j \). And \( K_j \) forms precisely the \( j \)-th strata of \( \Sigma_1 \cup \Sigma_2 \). Thus, any real number \( 0 \leq \beta \leq n - 1 \) can be realized as the dimension of the singular set of an \( n \)-dimensional stable stationary varifold.

**Remark 2.** By [23], the \( j \)-th strata for stationary varifolds are \( j \)-rectifiable. Theorem 1.1 shows that this cannot be improved to anything much stronger than that, except for the the top dimensional one, which no general regularity result is available beyond the classical Allard’s regularity theorem.

**Remark 3.** Recently, Professor Leon Simon ([26]) has posted the groundbreaking construction of stable minimal hypersurfaces of dimension \( n \) with singular set being any fixed compact subset of \( \mathbb{R}^{n-7} \). Indeed, Professor Simon’s work is one of the main motivations for the author to pursue this problem. Just for the purpose of the peer reviewing process, we comment on the differences between the two. First, it is not known if his examples are area minimizing. Second, the minimal surfaces in his work are based on codimension 1 minimizing cylinders, while ours are of higher
codimension, thus permitting relatively larger singular sets. Third, the singular set of our area minimizing surfaces comes from self-intersection, or more intuitively, of crossing type, while his singular set comes from smoothing of conical points.

Remark 4. In real analytic metrics, immersed minimal surfaces are locally real analytic varieties, so they cannot possess fractal self-intersections. Thus, if we strengthen Almgren’s Problem 5.4 in [2] by imposing the restriction of real analytic metrics, then the stronger statement still remains open, no matter how we adapt our method. In particular, we still do not know even if stationary minimal surfaces can possess fractal singular set in the Euclidean space. (Simon’s work [25] also needs to perturb the metric smoothly and non-analytically.) Moreover, around any singular point, our construction is locally reducible into two smooth pieces. It would be very interesting to get locally non irreducible examples, for example, with true branch points being the singular set. In some sense, the locally non-irreducible singularities are the more genuinely singular points. One can compare the situation with Theorem 1.8 and Conjecture 1.7 in [4].

Remark 5. One might ask why our method cannot give singularities in the top-dimensional strata, i.e., branch/cusp-like points. The idea is that the tangent cones at the singular points of the currents we use, lie in the interior of the moduli space of pairs of minimizing planes. The branch points correspond to the boundary of the moduli space, thus having less degrees of freedom to perturb. The calibrations in this paper are inspired by the work of Frank Morgan ([20],[21]), Gary Lawlor ([18]) and Dana Nance ([24]) on the angle conjecture and intersecting minimal surfaces.

1.1. Sketch of proof. We will first illustrate the main ideas with $n = 3$, $N_1 = \mathbb{R}$, the simplest case. Strictly speaking, we will never use non-compact $N$ in our proof. This is more of a local picture. Consider the standard $\mathbb{C}^3$, with coordinates $z_a = x_a + iy_a$. The model of singularity is $x_1x_2x_3$-plane union $y_1y_2x_3$-plane.

Now along the $x_3$-axis direction, we desingularize the union by lifting the $y_1y_2x_3$-plane smoothly into the $y_3$-direction. The lifting is determined by a smooth function which only vanishes on the singular set. Such a union is point-wise calibrated by a carefully-chosen differential form that calibrates both the $x_1x_2x_3$-plane and the tangent plane of the lifted surface at the same time in a perturbed metric. The calibration comes naturally from summing the volume form in the normal bundle, suitably rescaled.

The picture for general $n \geq 3$ and each $j$-th strata is similar. Finally, for global results and constructing compact examples, one can then apply the gluing techniques by Zhang in [31] and [32] by doing connected sums.

Using roughly the same ideas, similar results can be obtained for stable stationary minimal surfaces and mod p area-minimizing surfaces.

Acknowledgements

The author acknowledges the support of the NSF through the grant FRG-1854147. I cannot thank my advisor Professor Camillo De Lellis enough for his unwavering support, especially when I have been receiving treatments and recovering from recurrent cancer. I would also like to thank him for giving this problem to me and for countlessly many helpful discussions. I feel so lucky that I have Camillo as
my advisor. A special thank goes to Professor Leon Simon, whose groundbreaking work [26] is the ultimate motivation for the author to pursue this problem. Finally, the author would like to thank Professor Max Engelstein, Professor Pei-Ken Hung, Professor Chao Li, Professor Yang Li, Professor Gary Lawlor, Professor Frank Morgan, Professor Leon Simon, and Professor Antoine Song (in alphabetical order of last name) for their interests in this work.

2. Preliminaries and notations

In this section we will collect several notations. The reader can skip this section at the first read, since most notations are standard.

2.1. Currents. When we mention a surface $T$, we mean an integral current $[T]$. For a comprehensive introduction to integral currents, the standard references are [25] and [13]. We will adhere to their notations. Our manuscript mostly focus on the differential geometric side, so the reader can just assume a current to be a sum of chains representing oriented surfaces with singularities. We will use the following definition of irreducibility of currents.

Definition 2.1. A closed integral current $T$ is irreducible in $U$, if we cannot write $T = S + W$, with $\partial S = \partial W = 0$, and $S, W$ nonzero.

Also, we will use the differential geometry convention of completeness. An integral current $T$ is complete, if $\partial T = 0$ and $T$ has compact support. For a smooth manifold $M$, we will use $[W]$ to denote the current corresponding to $M$, i.e., integration over $W$. We will abuse the notation and also use $[W]$ to denote its support, i.e., the oriented manifold $W$.

2.1.1. Mod $p$ area minimizing. The definition of mod $p$ minimizing is the same one as Definition 1.2 in [4].

Definition 2.2. An integral current $T$ on a manifold $M$ is a cycle (or boundary) mod $p$, if for some integral current $S$, $T + pS$ is a cycle (or boundary, respectively) as an integral current. We say a mod $p$ cycle $T$ is area minimizing mod $p$, if for any mod $p$ boundary $S$, we have

$$M(T) \leq M(T + S),$$

with $M$ the mass of currents.

Remark 6. Strictly speaking, the canonical way to define mod $p$ minimizing is to invoke the notion of mod $p$ flat chains and mod $p$ currents. However, by Corollary 1.5 and 1.6 in [30], the above definition is equivalent to the canonical one.

2.2. The basics. When we use $\mathbb{C}^n$, we always equip $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ with the standard complex structure $J$. The coordinates are denoted $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ for $\mathbb{R}^{2n}$, with $z_a = x_a + iy_a$ for the $\mathbb{C}^n$. Moreover, $\partial y_a = J\partial x_a$.

We will frequently use tori of the form $\mathbb{C}^n/\mathbb{Z}^{2n}$. This is formed by taking Riemannian quotient of $\mathbb{C}^n$ with the standard lattice $\mathbb{Z}^{2n}$. In other words we send $x_a$ or $y_a$ to the equivalence classes $x_a(\mod 1)$ or $y_a(\mod 1)$. 
When we use $\mathbb{R}^m$, we will use $(a_1, \cdots, a_m)$ to denote the coordinate system, with axis labels $a_1, \cdots, a_m$.

When we refer to $x_a$-axis, (or $y_a$-axis,) it means the 1 dimensional real vector space spanned by $\partial_{x_a}$ (or $\partial_{y_a}$, respectively.) We will use $[x_1x_2\cdots x_n]$, $[y_1y_2\cdots y_n]$, to denote the current associated with the $x_1x_2\cdots x_n$-plane, and $y_1y_2\cdots y_n$-plane, with orientation precisely the order of the coordinate axis written. When we are in $\mathbb{C}^n/\mathbb{Z}^{2n}$, we will abuse our notation a little and also denote the tori formed from the axis planes by the same notation, i.e., $[x_1 \cdots x_n]$ denoting the quotient torus from $x_1 \cdots x_n$-plane.

For oriented planes $\alpha$ of dimension $l$ in $\mathbb{R}^m$ with the flat metric, we will use $\alpha$ to denote the associated simple $l$ vector and use $\alpha^*$ to denote its dual form. In other words, if $v_1, \cdots, v_l$ is an orthonormal basis of $\alpha$, then $\alpha = v_1 \wedge \cdots \wedge v_l$ and $\alpha^* = v_1^* \wedge \cdots \wedge v_l^*$, with $\sharp$ the musical isomorphism between tangent and cotangent bundle, induced by the flat metric. Direct calculation shows that $\alpha$ and $\alpha^*$ are uniquely defined, and do not depend on the choice of the orthonormal basis.

2.3. Some frequently used concepts. The fundamental theorem of calibrated geometry (Theorem 4.2 and 4.9 in [16]) is frequently used, and we will not cite it explicitly. The reader needs to know that what a calibration form is and for currents are area-minimizing. For the details, [16] is the prime reference. We will use comass/$\phi$ to denote the comass of a form $\phi$.

Also, for the precise definition of the Almgren stratification of minimal surfaces, the work [23] and the references therein are the primary sources. Roughly speaking, the $j$-th strata consists of points, where the tangent cone has at most $j$-dimensional translation invariance.

2.4. A torus differential form. Here we give some basics definitions and lemmas we need.

**Definition 2.3.** We always assume $1 \leq j \leq n-2$. Consider $\mathbb{C}^{n-j} \times \mathbb{R}^m$ for some $m \geq 1$. Let $\alpha, \beta$ be two $l$-dimensional oriented planes in $\mathbb{R}^m$, with $1 \leq l \leq m$. For any positive constants $-1 \leq \lambda, \mu \leq 1$, we call the following $n-j+l$-dimensional differential form $\Psi(\alpha, \beta, \lambda, \mu)$ the torus differential form.

$$\Psi(\alpha, \beta, \lambda, \mu) = \lambda dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \alpha^* + \mu dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \beta^*.$$ 

**Lemma 2.1.** For any $\alpha, \beta$ and $-1 \leq \lambda, \mu \leq 1$, the form $\Psi(\alpha, \beta, \lambda, \mu)$ is a calibration form. If $-1 < \lambda, \mu < 1$, then $\Psi$ calibrates no planes. If $\lambda = 1$, (or $-1$,) then the following plane (or with reverse orientation for $\lambda = -1$) is calibrated by $\phi$,

$$\left[ \partial_{x_1} \wedge \cdots \wedge \partial_{x_{n-j}} \wedge \alpha \right].$$ 

If $\mu = 1$, (or $\mu = -1$,) then the following plane (or with reverse orientation for $\mu = -1$) is calibrated by $\phi$,

$$\left[ \partial_{y_1} \wedge \cdots \wedge \partial_{y_{n-j}} \wedge \beta \right].$$

**Proof.** The claim that $\psi$ is calibration is just a special case of Lemma 2.1 in [12], using the fact that comass of dual forms of planes are 1. Direct calculation verifies that the two planes mentioned are calibrated when $\mu, \lambda = \pm 1$. \hfill $\square$
2.5. Smooth functions with controlled zero set. Let $N$ be a smooth orientable Riemannian manifold. Let $K$ be a compact closed subset of $N$. We need the following lemma.

**Lemma 2.2.** There exists a smooth function $f$ on $N$ with $f^{-1}(0) = K$, and for any fixed $l \geq 0$, the $C^l$ norm of $f$ can be prescribed to be as small as we want.

**Proof.** This is a well-known fact. We only give a sketch here. By Whitney embedding theorem in [29] and the fact that smooth function restricts to smooth function on submanifolds, we reduce to the case of finding a smooth function vanishing only on a compact set $K$ in Euclidean space. By 3.1.13 in [13], there exists a good cover of the complement of $K$ with uniform bound on number of intersections. Then we just use bump functions on the balls in the coverings, with small constant in front of them, so that the sum of $C^k$ norm is absolutely convergent for all $k$. Thus the infinite sum is smooth, and by construction has zero set precisely $K$. After restricting to an embedding of $N$, we can adjust the $C^l$ norm by multiplying with small constants. □

2.6. The normal bundle calibration. Let $(Q^q, g)$ be a smooth Riemannian manifold of dimension $q$, and let $W^w$ be a smooth submanifold of $Q$ of dimension $w$. Suppose the normal bundle $NW$ of $W$ can be exponentiated diffeomorphically into a neighborhood $O_r$ of $W$ in $Q$, up to radius $r$. Let $\pi_0$ be the natural projection from $NW$ to $W$. Define $\pi = \pi_0 \circ \exp_{NW}^{-1}$, i.e., the nearest point projection onto $W$ in $O_r$. Let $\omega$ be the volume form of $W$ on $W$.

We have the following lemma.

**Lemma 2.3.** The comass of the closed form $\pi^* \omega$ in $O_r$ is a smooth function, equal to its Riemannian norm $\|\pi^* \omega\|_g$. Moreover, $\|\pi^* \omega\|_g = 1$ on $W$.

**Proof.** This is just reformulating Remark 3.5 and Lemma 3.4 in [31]. Moreover, as remarked therein, the form $\pi^* \omega$ is a simple form, i.e., dual to multiples of a $w$-vector. Thus, direct calculation gives that its comass coincides with the standard Riemannian norm. □

**Remark 7.** This implies that $W$ is calibrated by $\pi^* \omega$ in the conformal metric $\|\pi^* \omega\|^5 g$.

2.7. Transition construction by Zhang. In the final steps of the proof of the main theorems, we frequently use the following argument by Yongsheng Zhang from Theorem 4.6 in [31].

Suppose we have an $n$-dimensional compactly supported integral current $T$ in a manifold $M^{n+m}$. Suppose there exists two smooth open sets $U_1, U_2$ on $M$, so that $T$ lies in the interior of $U_1 \cup U_2$. Moreover, there exists a smooth (not necessarily connect) compact $n-1$ dimensional submanifold $L^{n-1}$ on the smooth part of $T$, so that $T \setminus L$ decomposes into to two connected components $T_1, T_2$, lying in $U_1$ and $U_2$, respectively. The singular points of $T$ lie in the complement of $U_1 \cap U_2$, and the restriction of $T$ to $U_1 \cap U_2$ is a smooth connected manifold. And $U_1 \cap U_2$ is tubular neighborhood around of $L$.
Lemma 2.4. If $T$ is calibrated by a smooth form $\phi_1$ in $U_1$ and another smooth form $\phi_2$ in $U_2$, in a smooth metric $g$ on $U_1 \cup U_2$, then there exists a smooth metric $\overline{g}$ and a calibration form $\overline{\phi}$ in $\overline{g}$, in $U_1 \cup U_2$. Moreover, $T$ is calibrated by $\overline{\phi}$ in $\overline{g}$.

Proof. If $U_1$ is a smooth ball and $T$ is smooth in $U_2$, with $\phi_2$ the calibration coming from the normal bundle in Lemma 2.3 in a conformal metric, then this is just Theorem 4.6 in [31], the proof of which goes as follows. First, there exists a natural calibration of $T$ in $U_2$, coming from the normal bundle calibration in Lemma 2.3 in a conformal metric. Then since $U_1$ is a ball, all closed forms are exact, so both $\phi$ and the normal bundle calibrations are exact in $U_1 \cap U_2$. We transition between the two forms by taking exterior derivative of the transition between their primitives. Finally, by a theorem of Harvey and Lawson, we can rescale the metric along vertical directions to maintain the calibration. Note that in all the steps, we only use the fact that $U_1$ is a ball, finding primitives to closed $n$-forms. Thus, in our case, if we assume $T$ is smooth in $U_2$, with $\phi_2$ the calibration coming from the normal bundle, then the same proof goes through, since $H_n(U_1 \cap U_2) = H_n(L^{n-1}) = 0$ making closed $n$-forms in $U_1 \cap U_2$ exact. However, this is enough. If $T$ is not smooth in $U_2$ and $\phi_2$ is arbitrary, we can let $\phi_3$ be the normal bundle calibration of $T$ in $U_1 \cap U_2$, and apply the argument above with $U_1 \cap U_2$ replacing $U_2$ to connect between $\phi_3$ and $\phi_1$. Then we apply the argument again above with $U_1 \cap U_2$ replacing $U_2$ and $U_2$ replacing $U_1$ to transit between $\phi_3$ and $\phi_2$. \hfill \Box

This enables us to do connected sums of calibrated currents.

Lemma 2.5. Suppose we have $n$-dimensional integral currents $T_j$ that are calibrated in smooth open sets $U_j$ on manifolds $M^{n+m}$, with $1 \leq j \leq l$. For each $j$ suppose there exists some point $p_j \in T_j$, so that $T_j$ is a multiplicity 1 smooth manifold near $p_j$. Then we can take a simultaneous connected sum of $M = \#_{1 \leq j \leq l} M_j$ and $T = \#_{1 \leq j \leq l} T_j$, so that $T$ is calibrated in a neighborhood containing it in a smooth metric on $M$.

Proof. When taking the connected sum of $M_j \# M_k$, we can take a connected sum $T_j \# T_k$ on the smooth parts by using a neck to connect the smooth parts of $T_j$ near $p_j$. To make $T_j \# T_k$ a current, we might need to reverse the orientation of one of them to make the orientation coherent, and also adding signs in front of the calibrations. Then we can invoke Lemma 2.4 to deduce the existence of a calibration of $T_j \# T_k$ in a neighborhood of it. Finally we just repeat this until we get all the summands. \hfill \Box

3. The model of desingularization

In this section, we will give the model of desingularization that will be used for the local structure of the singular set for each strata $j$.

Fix a dimension $1 \leq j \leq n - 2$, a connected orientable $j$-dimensional Riemannian manifold $(N_j, h_j)$, with $h_j$ the metric, and $K_j \subset N_j$ a closed compact subset of $N_j$. Use Lemma 2.2 to get a smooth function $f_j$ on $N_j$ with $f_j^{-1}(0) = K_j$. 

3.1. The model singularity. From now on we work in $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1$, equipped with the product metric. Here the first factor $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}$ is simply the real torus we get from $\mathbb{C}^{n-j}$ by taking quotient with the standard lattice $\mathbb{Z}^{2(n-j)}$, i.e., Gaussian integer points. In other words sending $x_b$ or $y_b$ to $x_a(\mod 1)$ or $y_b(\mod 1)$. The last $S^1$ is identified with the interval $[-\pi, \pi]$ with two ends identified. We will abuse our notation a little and also denote the tori formed from axis planes by the same notation, i.e., $[x_1 \cdots x_{n-j}]$ denoting the quotient torus from $x_1 \cdots x_{n-j}$-plane.

Let $[N_j]$ denote the current associated with the compact manifold $N_j \times \{0\} \subset N_j \times S^1$. The model singularity comes from the union

$$[x_1 \cdots x_{n-j}] \times [N_j] + [y_1 \cdots y_{n-j}] \times [N_j].$$

The singular set of this union is precisely the $\{0\}^{2(n-j)} \times N_j \times \{0\}$.

Now, consider the graph of $f_j$ in $N_j \times S^1$, i.e., $\{(p, f_j(p)) | p \in N_j\}$, and denote the associated current $[F_j]$. Define

$$[F_j] = [y_1 \cdots y_{n-j}] \times [f_j].$$

Note that $[F_j]$ is a smooth embedding.

Consider the current

$$T = [x_1 \cdots x_{n-j}] \times [N_j] + [F_j].$$

By definition, $T$ is smooth outside the self intersection points of the union, i.e.,

$$[x_1 \cdots x_{n-j}] \times [N_j] \cap [F_j].$$

Thus, the singular set is the self-intersection set, defined by $f_j(p) = 0$, i.e., $\{0\}^{2(n-j)} \times K_j \times \{0\}$. Moreover, the tangent cone to $T$ at any intersection point is the sum $\partial_{x_1} \wedge \cdots \wedge \partial_{x_{n-j}} \wedge TN_j^* + \partial_{y_1} \wedge \cdots \wedge \partial_{y_{n-j}} \wedge TN_j^*$, with $TN_j$ the unit tangent $j$-vector to $[N_j]$. Such tangent cones precisely has $j$-dimensional of translation invariance.

3.2. The calibration form. Now, we can multiply constants in front of $f_j$, to sure that $\|f_j\|_c$ is as small enough. This will ensure that the normal bundle of $[F_j]$ in $N_j \times S^1$ has radius as close to half the circumference of $S^1$ as we want. (The normal exponential maps comes from solutions of second order ODEs with smooth coefficients depending smoothly on the embedding of $[f_j]$ and the metric on $N_j \times S^1$. However, the normal exponential map for $[N_j]$ has differential being identity everywhere, if we identify each leaf $N_j \times \{t\}$ with $N_j \times \{0\}$ by changing the last $t$-coordinate. Thus, with $C^4$ norm small enough for large enough $l$, we can ensure that the normal exponential map of $[F_j]$ is a local diffeomorphism in small balls of uniform size on the normal bundle, excluding nearby points mapped non-injectively. When $f_j$ converges smoothly to 0, the normal exponential map converges smoothly to that of $N_j$, thus excluding non-nearly points mapped non-injectively, provided we stay in a radius a little below that of the normal radius of $N_j$.)

Thus, without loss of generality, suppose that the normal exponential map of $[F_j]$ is injective for radius up to a little above 1 and $[N_j]$ is at most of Hausdorff distance $\frac{1}{10}$ from $[F_j]$.

Now apply Lemma 2.3 with $Q = N_j \times S^1$ and $W$ being $[F_j]$. We get a natural projection $\pi$ defined on a tubular neighborhood $O_1$ of radius 1 around $[F_j]$, and
the volume form of \( \omega_j \) defined on \([F_j]\). Also, let \( \nu_j \) be the volume form of \( N_j \) in \( N_j \times S^1 \), defined by extending constantly over \( S^1 \)-directions. Note that \( \nu_j \) is a calibration form that calibrates \( N_j \).

The following form \( \phi \) defined on \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \) is the calibration form we need.

\[
\phi = dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j + dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j.
\]

Note that \( dx_1 \wedge \cdots \wedge dx_{n-j} \) and \( dy_1 \wedge \cdots \wedge dy_{n-j} \) are smooth and closed, due to construction of the torus \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \). Also, \( \nu_j \) is closed since the extension is constant along \( S^1 \)-directions. \( \pi^* \omega_j = \pi^* d\omega_j = 0 \). Thus, \( \phi \) is closed by construction.

We need to rescale the metric to make it a calibration, i.e., having comass no larger than 1 everywhere. The standard Riemannian metric on \( \mathbb{C}^{n-j} \) is

\[
\delta = \sum_b dx_b^2 + dy_b^2.
\]

The original metric on \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1 \) is

\[
\delta + h_j + dt^2,
\]

with \( dt \) coordinate 1-form on \( S^1 \).

**Lemma 3.1.** For the metric \( g \), defined as

\[
g = \| \pi^* \omega_j \|^2_{h_j + dt^2} \, dx_1^2 + dy_1^2 + \sum_{j=2}^{n-j} (dx_j^2 + dy_j^2) + \| \pi^* \omega_j \|^2_{h_j + dt^2} (h_j + dt^2),
\]

\( \phi \) is a calibration form in \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \) and calibrates \( T \). Moreover, if we add negative signs in front of any of the two summands in \( \phi \), the resulting form is still a calibration.

**Proof.** Consider an arbitrary point \((x_1, \ldots, x_{n-j}, y_1, \ldots, y_{n-j}, p, t)\) in \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \). We can rewrite \( \phi \) as

\[
\phi = (\| \pi^* \omega_j \|^2_{h_j + dt^2} \, dx_1 \wedge \cdots \wedge dx_{n-j} \wedge (\| \pi^* \omega_j \|^2_{h_j + dt^2} \nu_j) + dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j.
\]

In the orthonormal decomposition \( T(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1) = T(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}) \oplus T(N_j \times S^1) \), the following is a set of orthonormal basis of the subspace \( T(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}) \),

\[
e_1 = \| \pi^* \omega_j \|_{h_j + dt^2} \partial x_1, E_1 = \partial y_1, e_2 = \partial x_2, E_2 = \partial y_2, \ldots, e_{n-j} = \partial x_{n-j}, E_{n-j} = \partial y_{n-j},
\]

with dual basis

\[
e_1^* = \| \pi^* \omega_j \|_{h_j + dt^2}^{-1} \, dx_1, E_1^* = dy_1, e_2^* = dx_2, E_2^* = dy_2, \ldots, e_{n-j}^* = dx_{n-j}, E_{n-j}^* = dy_{n-j}.
\]

For conformal change of metric with conformal factor \( \lambda \), the comass of \( j \)-forms changes with factor \( \lambda^{-\frac{j}{2}} \) (Lemma 3.1 in [31]). Thus, we have

\[
\text{comass}_{T(\mathbb{C}^{n-j} \times S^1)} \| \pi^* \omega_j \|_{h_j + dt^2} \nu_j = \| \pi^* \omega_j \|_{h_j + dt^2} \text{comass}_{h_j + dt^2} \| \pi^* \omega_j \|_{h_j + dt^2} \nu_j = \text{comass}_{h_j + dt^2} \nu = 1.
\]

And

\[
\text{comass}_{T(\mathbb{C}^{n-j} \times S^1)} \pi^* \omega_j = \| \pi^* \omega_j \|_{h_j + dt^2}^{-1} \text{comass}_{h_j + dt^2} \pi^* \omega_j = 1.
\]

Also, by construction, \( \| \pi^* \omega_j \|_{h_j + dt^2} \nu \) and \( \pi^* \omega_j \) are both simple \( j \)-forms, i.e., corresponding to multiples of dual forms to \( j \)-dimensional planes in \( g \). Thus, in \( g \), pointwise, \( \phi \) is precisely a torus differential form as in Definition [23], with \( \mu = \nu = 1 \). We
can invoke Lemma 2.4 to conclude that the comass of $\phi$ is 1 in $g$, and it is indeed a calibration. By construction, the two summands in $\phi$ precisely corresponds to dual forms of tangent planes of $[x_1 \cdots x_{n-j}] \times [N_j]$ and $[F_j]$, on the support of $T$. By Lemma 2.4, this implies that $T$ is calibrated by $\phi$.

For the claim about adding negative signs, just apply Lemma 2.4 with $\mu$ or $\nu$ being $-1$.

To sum it up, we have proven the following.

**Lemma 3.2.** There exists a smooth Riemannian metric $g$, and a smooth calibration form $\phi$ on $H = \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1$, where $O_1$ is a tubular neighborhood of $F_j$. The current $T = [x_1 \cdots x_{n-j}] \times [N_j] + [F_j]$ is calibrated by $\phi$ in $g$, and it decomposes as the sum of two smooth embedding of $\mathbb{R}^{n-j}/\mathbb{Z}^{n-j} \times N_j$. The singular set of $T$ is $\{0\}^{2(n-j)} \times K_j \times \{0\}$, with $K_j$ a compact subset of $N_j$.

**4. The proof of the main theorems**

With Lemma 3.2 we are pretty close to getting the main theorems. We will first prove Theorem 1.2 and deduce Theorem 1.1 as a corollary.

**4.1. Proof of Theorem 1.2**

*Proof.* We have to solve several caveats here. First, we have only dealt with one $j$-th strata with $N_j$ having only one connected component. Second, the calibrations are local. Finally, we have not included the 0-th strata.

First, the current we get is decomposable into two parts, so we need to connect them using a connected sum. Take $x = (\frac{1}{2}, \cdots, \frac{1}{2})$ in $x_1 \cdots x_{n-j}$ plane and $y = (\frac{1}{2}, \cdots, \frac{1}{2})$ in $y_1, \cdots, y_{n-j}$ plane. Let $X_r$ and $Y_r$ be small flat disks of radius $r < 1/10$ in these two planes. Let $p$ be a point in $N_j$, and $P_r$ be a small intrinsic ball of radius $r < \text{injrad } N_j$ in $N_j$. We claim there is a smooth homotopy $G_t$ sending $X_r \times P_r$ to $Y_r \times F_j(P_r)$ that never intersects the current $T$ for $0 < t < 1$. Note $X_r$ can be smoothly homotoped into $Y_r$ without intersecting both planes except for end point times, by taking the homotopy $R_t(x_1, \cdots, x_{n-j}) = (\pm e^{\frac{4\pi}{3} ti} x_1, e^{\frac{4\pi}{3} ti} x_2, \cdots, e^{\frac{4\pi}{3} ti} x_{n-j})$, regardless of the orientation, since $e^{\frac{4\pi}{3} ti}$ is never purely imaginary nor real for $0 < t < 1$. $P_r$ can be smoothly homotoped into $F_j(P_r)$ by taking $(p, 0)$ to $(p, tf_j(p))$. The needed homotopy comes from applying these two homotopies together component-wise. Consider the image of the homotopy on $\partial(X_r \times P_r)$. Smoothing out crossings, we can get a smooth neck that connects between the two summands in $T$, and preserves the orientation. Also call this new current $T$. Then we can apply Lemma 2.4 to get $T$ calibrated in a neighborhood of it in a smooth metric.

Next we deal with $N_j$ of several connected components and different $j$. Call the connected components of $N_j$ by $N_{j, 1}, \cdots, N_{j, c_j}$. First, for each component, we can do the connected sum on the current in the previous subsection. Then, by multiplying with $S^1$ and extending calibration forms constantly along the added direction, we can get the ambient manifold to be $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_{j, l} \times S^1 \times S^{j-k}$, for some $l$. Applying Lemma 2.4, we can conclude that $T$ is calibrated by $\phi$.
with \( k \) the smallest number with \( K_k \) nonempty. Call these new ambient manifolds \( L_j, 1, \ldots, L_j, c_j \). This will make all the ambient manifolds of the same dimension.

If \( k = 0 \), instead of using the construction in the previous section, we can use the tori link cones \( M_0 \) in \( \mathbb{C}^n \) in Theorem 3.1 in [19], cut-off in the unit ball. The link in the unit sphere bounds a smooth \( n \)-dimensional solid tori. Now smooth out the crossing of the boundaries of the cone and the solid tori. Name the resulting current \( T_{0,l} \) with \( 1 \leq l \leq c_0 \), and \( c_0 \) the number of points we want in the 0-th strata. Again, we apply Lemma 2.4. Then just take everything into a \( 2n \)-dimensional half sphere and extend the metric smoothly. Embed that into \( S^{2n} \times S^1 \) to give \( L_{0,1}, \ldots, L_{0,c_0} \), all diffeomorphic to \( S^{2n} \times S^1 \).

Finally we take a simultaneous connected sum of all the ambient manifolds \( M = \#_{k \leq j \leq n-2, 1 \leq i \leq c_j} L_{j,l} \), and all the local minimizers \( T = \#_{k \leq j \leq n-2, 1 \leq i \leq c_j} T_{j,l} \). In case \( T \) is homologically trivial in \( M \), we take the simultaneous connected sum with \( S^n \times \{ \text{a point} \} \) in \( S^n \times S^{n-k+1} \). Again, we apply Lemma 2.4 to get \( T \) smoothly calibrated in a neighborhood on \( M \). Finally, as remarked in the last sentence of the proof of Theorem 4.6 in [31], we can invoke Section 3.3 and 3.4 of [31] to produce a smooth metric on \( M \) and a smooth calibration form \( \phi \) defined everywhere that calibrates \( T \). We are done.

\( \square \)

4.2. Proof of Theorem 1.1

Proof. Just take \( k = n-2 \), i.e., we are only prescribing the \( n-2 \)-th strata. Then we just need to show that every nonnegative real Hausdorff dimension \( 0 \leq \alpha \leq n-2 \) can be realized as subsets of Euclidean spaces, and then take \( N_{n-2} = \mathbb{R}^{n-2}/(\rho\mathbb{Z}^{n-2}) \), with \( \rho \) large to enclose this subset. For example, this follows by taking products of Cantor sets and intervals. See the Cantor set construction in Section 4.10 of [19] and Theorem 8.10 in [19]. Also note that the Hausdorff dimension of a compact set on a manifold is independent of the metric. To see this, note that the distance between points on two different metrics are always locally bounded, thus it is straightforward to verify that Hausdorff measure zero and infinity are preserved under all metrics. We are done.

\( \square \)

4.3. Proof of Theorem 1.3

We will first show that the current \( T \) in Lemma 3.2 is area minimizing mod \( p \) in the neighborhood where it is defined. The idea of proving mod \( p \) area minimizing is in the same spirit as the main theorem in [32]. Then we can find a suitable extension of the metric globally by Section 2 in [32].

Roughly speaking, the geometrical idea is to use cuts to unfold the torus \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \), and then compare the integration of the absolute value of each summand of the calibration form with each summand of \( T \) using retractions.

Suppose \( S \) is an \( n \)-dimensional cycle mod \( p \), in \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \), in other words, \( S = pS_0 + \partial S_1 \), for some \( n+1 \)-dimensional integral current \( S_1 \), and \( n \)-dimensional integral current \( S_0 \), all having compact support.

4.3.1. First factor. First, let us do the projection argument for \([x_1 \cdots x_{n-j}] \times [N_j]\). Let \( \psi \) be a smooth \( n \)-dimensional form on \([x_1 \cdots x_{n-j}] \times [N_j]\). Since \( \psi \) is a top dimensional form, we deduce that \( \psi \) must be smooth multiples of the volume form,
i.e.,
\[ \psi(x, p) = f(x, p)dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j. \]
Here \( f \) is a smooth function on \( [x_1 \cdots x_{n-j}] \times [N_j] \), and \( x \) denotes a point in the first factor and \( p \) denotes a point in the second factor.

Thus for any smooth form \( \psi \) on \( [x_1 \cdots x_{n-j}] \times [N_j] \), we can uniquely define a smooth form \( \overline{\psi} \) on \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \) by
\[
\overline{\psi}(x, y, p, t) = f(x, p)dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j.
\]
In this way, every compactly supported \( n \)-dimensional integral current \( S \) on \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \) induces a current \( \overline{S} \) in \( [x_1 \cdots x_{n-j}] \times [N_j] \) by
\[
\overline{S}(\psi) = S(\overline{\psi}).
\]

**Lemma 4.1.** \( \overline{S} \) is also integral, provided that \( S \) is.

**Proof.** Consider the normal exponential map \( \exp_x^\perp \) of \( [x_1 \cdots x_n] \) in the flat metric. For any small positive \( \epsilon \), we can make the fundamental domain on each normal space to be \((y_1, \ldots, y_{n-j}) \in [-\frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon]^{n-j}\). Thus, we can identify \( \mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1 \) with
\[
[x_1 \cdots x_{n-j}] \times \left[ -\frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon \right]^{n-j} \times O_1 \subset \mathbb{R}^{n-j}/\mathbb{Z}^{n-j} \times \mathbb{R}^{n-j} \times O_1,
\]
and identify \( S \) with some current \( S' \) therein. Note that the boundary of \( S' \) has two parts, the boundary of \( S \) and the slicings of \( S \) under the \( y_a \) direction functions for \( 1 \leq a \leq n-j \). By 4.3.6 in [13], we can assume that with suitable choice of \( \epsilon \) small, the slicing of \( S \) is also an integral current. Thus, we can assume \( S' \) is also integral.

Let \( \sigma_x \) be the projection induced by \( \exp_x^\perp \), i.e., \( \sigma_x(x, y, p, t) = (x, 0, p, t) \), and \( \sigma_N \) be the projection onto \( N_j \) by zeroing the \( t \) component. Consider the map \( \sigma_x \times \sigma_N \), i.e., applying the projections on the factors separately. The differential of \( \sigma_x \times \sigma_N \) annihilates the \( y_a \) directions and \( t \) direction. This implies for any differential form \( \psi \) on \( [x_1 \cdots x_{n-j}] \times N_j \), we have \((\sigma_x \times \sigma_N)^* \psi = \psi \). Thus, by definition, we have
\[
(\sigma_x \times \sigma_N)_# S'(\psi) = \overline{S}(\psi).
\]

Note that by construction \( \sigma_x \times \sigma_N \) is smooth, by 27.2(3) in [26], we deduce that \((\sigma_x \times \sigma_N)_# S' \) and thus \( \overline{S} \) are integral currents. □

Direct calculation gives \( [x_1 \cdots x_{n-j}] \times [N_j] = [x_1 \cdots x_{n-j}] \times [N_j] \), and \( [F_j] = 0 \), since the tangent space to \([F_j]\) is always spanned by the direct product of \([y_1 \cdots y_{n-j}]\) with another vector space tangent. Also, note that we always have
\[
d\overline{\psi} = df \wedge dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j = 0.
\]
since the two volume forms are closed, and the way we extend \( f \) constantly does not give any new derivatives.
Note that $dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j$ still have comass 1 in the rescaled metric $g$ by Lemma 3.1. Thus, comass$_g \psi \leq 1$ is equivalent to $|f| \leq 1$. This gives

$$M(T + S) = \sup_{\text{comass}_g \psi \leq 1} \int_{\text{spt}(T + S)} \psi(H + S) d\|T + S\|$$

$$\leq \int_{\text{spt}(T + S)} |dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j(H + S)| \theta_{T + S} dH^n,$$

where we have used the integral representation (27.1 in [26]), with $\theta_{T + S}$ the integer density and $H + S$ the unit oriented tangent $n$-vector in $g$, and $H^n$ the $n$-dimensional Hausdorff measure in $g$.

From $S = pS_0 + \partial S_1$, we have

$$\overline{S}(\psi) = S(\overline{\psi}) = pS_0(\overline{\psi}) + S_1(d\overline{\psi}) = p\overline{S}_0(\psi).$$

Now we consider the mass of $T + S$ in the changed metric. We have

$$M(T + S) = \sup_{\text{comass}_g \psi \leq 1} \int_{\text{spt}(T + S)} \psi T(\overline{\psi}) + S(\overline{\psi})$$

$$= \sup_{\text{comass}_g \psi \leq 1} \int_{\text{spt}(T + S)} \|x_1 \cdots x_{n-j}\| \times [N_j](\psi) + p\overline{S}_0(\psi)$$

$$= M([x_1 \cdots x_{n-j}] \times [N_j] + p\overline{S}_0).$$

On the other hand, we have

$$([x_1 \cdots x_{n-j}] \times [N_j] + p\overline{S}_0)(\psi)$$

$$= \int_{x_1 \cdots x_{n-j} \times N_j} \psi(\xi) dH^n + \int_{\text{spt}S_0} \psi(\pm \xi) p\theta dH^n$$

$$= \int_{x_1 \cdots x_{n-j} \times N_j \setminus \text{spt}S_0} \psi(\xi) dH^n + \int_{\text{spt}S_0} \psi(\pm \frac{\theta}{|\theta| + 1}) |p\theta| + 1 dH^n,$$

where $\xi$ is the oriented unit $n$-vector tangent to $[x_1 \cdots x_{n-j}] \times [N_j]$, the $\pm$ sign is the pointwise orientation of $p\overline{S}_0$, and $\theta$ is the pointwise density of $\overline{S}_0$. Note that $p\theta \pm 1$ is never 0 for $p \geq 2$. Thus, $[x_1 \cdots x_{n-j}] \times [N_j] + p\overline{S}_0$ is supported on all of $x_1 \cdots x_{n-j} \times N_j$, and its density $\theta'$ is always at least 1. This gives

$$M([x_1 \cdots x_{n-j}] \times [N_j] + p\overline{S}_0)$$

$$= \int_{x_1 \cdots x_{n-j} \times N_j} \theta' dH^n$$

$$\geq \int_{x_1 \cdots x_{n-j} \times N_j} 1 dH^n$$

$$= M([x_1 \cdots x_{n-j}] \times [N_j]),$$

where the first equality comes from evaluation of mass by the sentence just after the definition of mass in 4.1.7 of [13].

Combining all the mass estimates in this subsubsection, we deduce that

$$M([x_1 \cdots x_{n-j}] \times [N_j]) \leq M([x_1 \cdots x_{n-j}] \times [N_j] + p\overline{S}_0) = M(T + S)$$

$$\leq \int_{\text{spt}(T + S)} |dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j(H + S)| \theta_{T + S} dH^n.$$
4.3.2. **Second factor.** We carry out the same argument as the last subsubsection. Any smooth form \( \phi \) on \( [F_j] \) can be uniquely written as \( \phi = f(y, q)dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j \) with \( y \) in the \( y_1 \cdots y_{n-j} \) factor and \( q \) in the \( [F_j] \) factor. Again we can get unique global extensions by setting

\[
\tilde{\phi}(x, y, q, s) = f(y, q)dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j,
\]

where \((q, s)\) denotes the point in \( O_1 \) we get from exponentiating the \( s \) times the unit normal to \([F_j]\) at \( q \). This again transforms any \( n \)-dimensional integral current \( S \) on our ambient region into an \( n \)-dimensional current \( \tilde{S} \) on \([F_j]\), defined by

\[
\tilde{S}(\phi) = S(\tilde{\phi}).
\]

**Lemma 4.2.** \( \tilde{S} \) is an integral current if \( S \) is integral.

*Proof.* We can use the same cutting argument as in the previous subsubsection. Again we can make cuts, this time unfolding the \( x_n \)-directions to get integral currents \( S'' \). (Integrality comes from slicing by \( x_n \).) Let \( \sigma_y \) be the projection into \( y_1 \cdots y_{n-j} \). We need to show that \((\sigma_y \times \pi)_y S'' = \tilde{S} \) to deduce integrality. This reduces to showing that \( \tilde{\phi}(x, y, q, s) = (\sigma_y \times \pi)^* \phi \), which follows by definition of \( \tilde{\phi} \).



Again, we always have \( d\tilde{\phi} = 0 \), since \( d\hat{f} \wedge dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j = 0 \). (\( \hat{f} \) has no \( x_n \)-derivatives. Using Fermi-coordinates adapted to \([F_j]\) in the original metric (Section 2 in [15]), one can show that \( \hat{f} \) has no derivatives beyond the directions in \( \pi^* \omega_j \).) Thus, the assumption \( S = pS_0 + \partial S_1 \) again gives

\[
\tilde{S}(\phi) = pS_0(\phi).
\]

Direct calculation shows \( \tilde{T} = [F_j] \). Again, by counting the density of \([F_j] + p\tilde{S}_0 \), we have the inequality

\[
M([F_j]) \leq M([F_j] + p\tilde{S}_0) = M(\tilde{T} + S).
\]

The comass of \( dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j \) is no larger than 1 by Lemma 3.1. This gives

\[
M(\tilde{T} + S) \leq \int_{\text{spt}(T + S)} |dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j(\xi_{T+S})| \theta_{T+S} dH^n.
\]

Combining the two, we have

\[
M([F_j]) \leq \int_{\text{spt}(T+S)} |dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j(\xi_{T+S})| \theta_{T+S} dH^n.
\]

4.3.3. **Mod \( p \) minimizing in the neighborhood.** Summing the estimates in the last two subsections, and use the fact that \( M([x_1 \cdots x_{n-j}] \times [N_j] + [F_j]) = M([x_1 \cdots x_{n-j}] \times [N_j]) + M([F_j]) \), we deduce that

\[
M(T) \leq \int_{\text{spt}(T+S)} \left( |dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j(\xi_{T+S})| + |dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j(\xi_{T+S})| \right) \theta_{T+S} dH^n.
\]

However, by Lemma 3.1 no matter the sign of the summands are, \( \mp dx_1 \wedge \cdots \wedge dx_{n-j} \wedge \nu_j \pm dy_1 \wedge \cdots \wedge dy_{n-j} \wedge \pi^* \omega_j \) always have comass no larger than 1 pointwise. Taking out the absolute value by adding signs, we deduce that

\[
M(T) \leq M(T + S).
\]
This implies that $T$ is area-minimizing mod $p$ in $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times O_1$ in the metric $g$.

Next, we verify that $T$ is a nontrivial homology class in any mod $p$ singular homology. Suppose the contrary, then there exists a integer coefficient $n + 1$-chain $T_1$ and $n$-chain $T_0$ so that $T = pT_0 + \partial T_1$. This gives $p\partial T_0 = 0$ and thus $T_0$ is also a cycle. However, thus, we deduce that $[T] = -p[T_0]$ as homology classes. Now calculate $H_n(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1)$. There are several ways to do this, e.g., by applying the Example 3.B.3 in [17] several times, or using Künneth theorem (Section 3.B in [17]). Note that every integer coefficient singular homology group of $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}$ and $S^1$ are free, giving vanishing tor terms. This implies zero cokernel in Künneth theorem. We deduce that

$$H_n(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1) = H_{n-j}(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}) \otimes H_j(N_j) \bigoplus H_{n-j}(\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)}) \otimes (H_{j-1}(N_j) \otimes H_1(S^1))$$

$$= \mathbb{Z}^{2(n-j)} \bigoplus \cdots$$

Up to isomorphisms, $T$ equals the $(1, 0, \cdots) + (0, 1, 0, \cdots)$ in $\mathbb{Z}^{2(n-j)}$. Thus, we cannot have $[T] = -p[T_0]$ for $p \geq 2$, a contradiction.

Now we can run the argument in Section 2 and 3 of [32] to conformally change the metric while making $T$ a mod $p$ area minimizer on all of $\mathbb{C}^{n-j}/\mathbb{Z}^{2(n-j)} \times N_j \times S^1$ in its mod $p$ homology class. (Section 2 of [32] only uses the fact that the current is a stationary varifold, so it still applies here.) We are done.

### 4.4. Proof of Theorem 1.4

The theorem already holds for the cases with $0 \leq j \leq n - 2$ and $0 \leq \beta \leq n - 2$ by the results of Section 3 and Corollary 4.9 in [31]. We only need to consider $j = n - 1$, and $n - 2 \leq \beta \leq n - 1$.

Essentially the same arguments as in Section 3 works, albeit this time we use two different calibration forms for the two minimal surfaces. The two forms are similar to the two summands of $\psi$ in formula (3.2). Instead of taking $\mathbb{C}^n/\mathbb{Z}^{2n} \times N_j \times S^1$, we take $\mathbb{R}^2/\mathbb{Z}^2 \times N_j \times S^1$, with $dx_1$ replacing $dx_1 \wedge \cdots \wedge dx_{n-j}$ and $dy_1$ replacing $dy_1 \wedge \cdots \wedge dy_{n-j}$. The two calibrated surfaces are the two summands in (3.1) with $[x_1]$ replacing $[x_1 \cdots x_{n-j}]$, tand $[y_1]$ replacing $[y_1 \cdots y_{n-j}]$. Then we change the metric as in Lemma 3.1 to make sure that both are calibrated by their respective forms locally, which yields stability of the union. Finally we just extend the metric smoothly to a global one. Here we don’t need to care about extending calibration, since stability is a local condition near the current. Finally, for the part about realizing any dimension $n - 2 \leq \beta \leq n - 1$, we can use the same reasoning as in Section 12.

### References

[1] Some open problems in geometric measure theory and its applications suggested by participants of the 1984 AMS summer institute. Edited by J. E. Brothers. Proc. Sympos. Pure Math., 44, Geometric measure theory and the calculus of variations (Arcata, Calif., 1984), 441–464, Amer. Math. Soc., Providence, RI, 1986.
Frederick J. Almgren, Jr. Almgren’s big regularity paper. Q-valued functions minimizing Dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension 2. With a preface by Jean E. Taylor and Vladimir Scheffer. World Scientific Monograph Series in Mathematics, 1. World Scientific Publishing Co., Inc.

Sheldon Xu-Dong Chang, Two-dimensional area minimizing integral currents are classical minimal surfaces, J. Amer. Math. Soc. 1 (1988), no. 4, 699–778.

C. De Lellis; G. De Philippis; J. Hirsch; A. Massaccesi, On the boundary behavior of mass-minimizing integral currents, available at https://www.math.ias.edu/delellis/node/148, Regularity of area minimizing currents mod p, to appear in Geometric and Functional Analysis.

C. De Lellis; E. Spadaro, Regularity of area-minimizing currents I: Lp gradient estimates, Geom. Funct. Anal. 24 (2014), no. 6.

C. De Lellis; E. Spadaro, Regularity of area-minimizing currents II: center manifold, Ann. of Math. (2) 183 (2016), no. 2, 499–575.

C. De Lellis; E. Spadaro, Regularity of area-minimizing currents III: blow-up, Ann. of Math. (2) 183 (2016), no. 2, 577–617. Geom. Funct. Anal. 24 (2014), no. 6, 1831–1884.

C. De Lellis; E. Spadaro; L. Spolaor, Uniqueness of tangent cones for 2-dimensional almost minimizing currents, Comm. Pure Appl. Math. 70, 1402-1421.

C. De Lellis; E. Spadaro; L. Spolaor, Regularity theory for 2-dimensional almost minimal currents I: Lipschitz approximation, Trans. Amer. Math. Soc. 370 (2018), no. 3, 1783–1801.

C. De Lellis; E. Spadaro; L. Spolaor, Regularity theory for 2-dimensional almost minimal currents II: branched center manifold, Ann. PDE 3 (2017), no. 2, Art. 18, 85 pp.

C. De Lellis; E. Spadaro; L. Spolaor, Regularity theory for 2-dimensional almost minimal currents III: blowup To appear in Jour. Diff. Geom.

J. Dadok, R. Harvey and F. Morgan, Calibrations on tangent cones, Math. Ann. 261 (1982), no. 1, 225-237

Herbert Federer, Geometric Measure Theory. Springer, New York, 1969.

Herbert Federer, The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension. Bull. Amer. Math. Soc. 76 (1970), 767–771.

Alfred Gray Tubes. Second edition. With a preface by Vicente Miquel. Progress in Mathematics, 221. Birkhäuser Verlag, Basel, 2004.

Reese Harvey; H. Blaine Lawson, Jr. Calibrated geometries. Acta Math. 148 (1982), 41-108

Gary Lawlor, The angle criterion. Invent. Math. 95 (1989), no. 2, 437-446.

Pertti Mattila, Geometry of sets and measures in Euclidean spaces Fractals and rectifiability, Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995. xii+343 pp.

Frank Morgan, On the singular structure of two-dimensional area minimizing surfaces in Rn. Math. Ann. 261 (1982), no. 1, 225-237

Frank Morgan, Examples of unoriented area-minimizing surfaces. Trans. AMS 283 (1984), 41-108

Frank Morgan, Calibrations modulo p. Adv. in Math. 64 (1987), no. 1, 32–50.

Aaron Naber, Daniele Valtorta, The singular structure and regularity of stationary varifolds, J. Eur. Math. Soc., Volume 22, Issue 10, 2020

Dana Mackenzie, Sufficient conditions for a pair of n-planes to be area-minimizing. Math. Ann. 279 (1987), no. 1, 161–164.

Leon Simon, Lectures on Geometric Measure Theory, Proceedings for the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.

Leon Simon, Stable minimal hypersurfaces in Rn+1 with singular set an arbitrary closed K in 0 × R, available at https://arxiv.org/abs/2101.06401

Brian White, Tangent cones to two-dimensional area-minimizing integral currents are unique, Duke Math. J. 50 (1983), no. 1, 143–160.

Brian White, Generic Transversality of Minimal Submanifolds and Generic Regularity of Two-Dimensional Area-Minimizing Integral Currents, preprint available at https://arxiv.org/abs/1901.05514

Yongsheng Zhang, On extending calibration pairs. Adv. Math. 308 (2017), 645–670.

Yongsheng Zhang, On realization of tangent cones of homologically area-minimizing compact singular submanifolds. J. Differential Geom. 109 (2018), no. 1, 177–188.