State dependent effective interaction for the hyperspherical formalism with noncentral forces

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Abstract

The recently developed effective interaction method for the hyperspherical harmonic formalism is extended to noncentral forces. Binding energies and radii of three- and four-body nuclei are calculated with AV6 and AV14 NN potentials. Excellent results for the convergence of the expansion are found, particularly for the three-nucleon system. Due to the higher density the convergence rate is a bit slower for the alpha particle. In comparison to central potential models there is only a very slight deterioration of the convergence due the tensor force, while other potential terms have no visible effect on the convergence. The obtained values for binding energy and radii also agree well with the results in the literature obtained with other few-body techniques.

Keywords: Few-body, hyperspherical harmonics, effective interaction
21.45.+v, 21.30.Fe, 31.15.Ja
I. INTRODUCTION

In the last years great progresses have been achieved in microscopic calculations of few-body systems. Quite a few different exact approaches have been developed. Among those the effective interaction method is particularly interesting since it is a typical many-body rather than a few-body approach. In fact effective interactions are used in shell-model calculations of complex nuclei, where the wave functions are expanded in a model space consisting in a truncated single particle harmonic oscillator (HO) basis. The introduction of an effective rather than a bare interaction is necessary in order to compensate for the truncation of the space. A few years ago such an effective interaction method has also been introduced in few-body physics. So called no-core shell-model calculations have been carried out, where one keeps all the nucleons active and where one also removes the spurious center of mass motion finally leading to HO basis functions that depend on the Jacobi coordinates [1].

Recently we have proposed a different effective interaction method [2] using instead of the HO basis an expansion of the wave function in hyperspherical harmonics (HH). In the HH formalism the Jacobi coordinates are replaced by a single length coordinate, the hyperradius, and a set of $3A-4$ hyperangles. The HH are the A-body generalization of the 2-body spherical harmonics, and likewise depend only on the hyperangular (angular) coordinates in the hyperspherical (spherical) decomposition of the A-body (2-body) system. In general, the wave function can be expanded in a series consisting of products of HH basis functions and hyperradial basis functions. The HH basis is widely used in the calculation of few-body wave functions, though the convergence is very often problematic. In order to improve the convergence one may introduce proper correlation functions (see, e.g., [3–5]). The correlation functions, however, lead to various undesirable features in the calculation. One not only loses the orthogonality of the basis functions but also has no unique way to determine the correlation functions. Therefore, as an alternative approach, we have studied the HH effective interaction (EIHH) method to accelerate the slow convergence of the expansion. Employing simple central NN potential models as bare NN interaction for light nuclei in the mass range of $A = 3 – 6$ we could show that the EIHH method leads to an extremely rapid convergence for the investigated observables. One important reason for the excellent convergence behavior is due to the fact that the effective interaction becomes state dependent in a natural way, i.e. the interaction does not only depend on the state of the two interacting particles, but also on the state of the rest system.

Due to the excellent convergence the EIHH method has proven to be a very promising tool in few-body physics. On the other hand, in Ref. [2] only central potential models have been considered while realistic NN interactions are much more complicated and in principle could deteriorate the convergence rate. Therefore it is of great importance to check whether the EIHH method leads to similarly good results also for more realistic NN potentials. In this paper we investigate the method for two noncentral NN interaction models (AV6, AV14 [7]) calculating ground state energies and radii for the $A = 3, 4$ nuclei. To this end the convergence patterns are studied in detail and the final results are compared to results in the literature.

The paper is organized as follows. The EIHH method for noncentral forces is described in Sec. [II]. Numerical results, including a comparison with results obtained with other few-body techniques, are given in Sec. [III] and conclusions are drawn in Sec. [IV].
II. THE HH EFFECTIVE INTERACTION

Before introducing the effective interaction we first give a short summary of the HH formalism. To illustrate the hyperspherical coordinates we start from the definition of the center-of-mass coordinate \( \vec{R} = \frac{1}{A} \sum_{i=1}^{A} \vec{r}_i \) and the normalized reversed order \( N = A - 1 \) Jacobi coordinates

\[
\vec{\eta}_i = \sqrt{\frac{A - i}{A + 1 - i}} \left( \vec{r}_i - \frac{1}{A - i} \sum_{j=i+1}^{A} \vec{r}_j \right). \tag{1}
\]

The Jacobi coordinate \( \vec{\eta}_i \) consists of a radial coordinate \( \eta_i \) and a pair of angular coordinates \( \hat{\eta}_i \equiv (\theta_i, \phi_i) \). These coordinates are then transformed into the hyperangular coordinates \( \alpha_2, \ldots, \alpha_N \) through the relations

\[
\sin \alpha_n = \eta_n / \rho_n, \quad \rho_n^2 = \rho_{n-1}^2 + \eta_n^2 = \sum_{j=1}^{n} \eta_j^2. \tag{2}
\]

Note that the hyperradial coordinate \( \rho \equiv \rho_N \) is symmetric with respect to permutations of the underlying single particle coordinates. The \( 3N = 3(A - 1) \) internal coordinates for the \( A \)-particle system consist of the hyperradial coordinate \( \rho \) and the \( 3N - 1 \) “hyperangular” coordinates \( \Omega_N \equiv \{ \vec{\eta}_1, \vec{\eta}_2, \ldots, \vec{\eta}_N, \alpha_2, \alpha_3, \ldots, \alpha_N \} \).

With the hyperspherical coordinates one can write the Laplace operator for \( n \) Jacobi coordinates \( n = 1 \ldots N \), as a sum of two terms

\[
\Delta_n = \frac{1}{\rho_{n-1}^2 \rho_n^3} \frac{\partial}{\partial \rho_n} \rho_n^3 \frac{\partial}{\partial \rho_n} - \frac{1}{\rho_n^2} \hat{K}_n^2. \tag{3}
\]

The hyperspherical or grand angular momentum operator \( \hat{K}_n^2 \) of the \( n \) Jacobi coordinates can be expressed in terms of the squared angular momentum associated with the \( n \)th Jacobi coordinate, \( \hat{\ell}_n^2 \), and \( \hat{K}_{n-1}^2 \) as follows

\[
\hat{K}_n^2 = -\frac{\partial^2}{\partial \alpha_n^2} + \frac{3n - 6 - (3n - 2) \cos(2\alpha_n)}{\sin(2\alpha_n)} \frac{\partial}{\partial \alpha_n} + \frac{1}{\cos^2 \alpha_n} \hat{K}_{n-1}^2 + \frac{1}{\sin^2 \alpha_n} \hat{\ell}_n^2, \tag{4}
\]

where we define \( \hat{K}_1^2 \equiv \hat{\ell}_1^2 \). The total angular momentum operator associated with \( n \) coordinates is \( \vec{L}_n = \vec{L}_{n-1} + \vec{\ell}_n \). The operators \( \hat{K}_n^2, \hat{\ell}_n^2, \hat{K}_{n-1}^2, \hat{L}_n^2 \) and \( \hat{L}_{nz} \) commute with each other. The hyperspherical harmonic functions \( \mathcal{Y}_{[K_n]} \) are the eigenfunctions of \( \hat{K}_n^2 \) with eigenvalues \( K_n(K_n + 3n - 2) \). The symbol \( [K_n] \) stands for a set of quantum numbers including, e.g., \( \ell_1, \ldots, \ell_n, L_2, \ldots, L_n \) and \( K_2, \ldots, K_n \). For more detailed information on the HH functions see [8,9].

The HH functions do not possess any special properties under particle permutation. Therefore the first step in applying the expansion to the \( A \)-body problem is the symmetrization of the HH basis. In the current work we employ two powerful algorithms [10,11] recently developed for the construction of a HH basis with well defined permutational symmetry.

Employing noncentral potentials it is usually more convenient to work in a \( jj \) coupling scheme. However, constructing symmetrized basis functions in this coupling scheme is more
complicated, roughly by a factor $2^4$, in comparison to the $LS$ scheme. Therefore we work in the $LS$ scheme coupling HH basis functions with definite permutational symmetry with spin-isospin basis functions into anti-symmetric many-body states with total angular momentum $J$.

Now we turn to the question of the effective interaction. In general we would like to use the HH basis functions to solve the A-body Hamiltonian 

$$H = \sum_{i=1}^{A} \frac{\vec{p}_i^2}{2m} + \sum_{i<j}^{A} V_{ij} ,$$

(5)

where $\vec{p}_i$ is the momentum of the $i$th particle and $m$ the nucleon mass, while

$$V_{ij} = V(\vec{r}_{ij}) = \sum_{p} V^{(p)}(r_{ij}) \hat{O}^{(p)}(\vec{r}_{ij}, \vec{\sigma}_i, \vec{\sigma}_j, \vec{\tau}_i, \vec{\tau}_j)$$

(6)

is the NN interaction which also depends on the spin ($\vec{\sigma}_i$) and isospin ($\vec{\tau}_i$) operators of the two interacting particles. As already mentioned one usually introduces correlation functions in order to obtain sufficiently converging results \[3–6\]. In our alternative EIHH method of Ref. \[2\] we use instead of the bare NN interaction an effective interaction inside the model space. Defining $P$ as the projection operator onto the model space and $Q = 1 - P$ as the projection onto the complementary space, the model space Hamiltonian can be written as

$$H_P = P \left[ \sum_{i=1}^{A} \frac{\vec{r}_i^2}{2m} \right] P + P \left[ \sum_{i<j}^{A} V_{ij} \right]_{eff} P .$$

(7)

In the HH formalism the model space can be defined as a product of the hyperradial subspace and the set of HH basis functions with generalized angular momentum quantum number $K \leq K_{max}$.

In general the effective interaction appearing in Eq. (7) is an A-body interaction. If it is determined without any approximation, then the model-space Hamiltonian provides a set of eigenvalues which coincide with a subset of the eigenvalues of the original full-space Hamiltonian, Eq. (5). Since the effective interaction has to contain the whole A-body information of Q-space the calculation of this A-body effective interaction is as difficult as finding the full-space solution. Because of these difficulties we introduced in Ref. \[2\] instead of the exact A-body an approximate two-body effective interaction which, however, satisfies the following properties: (i) $V_{2\text{eff}} \longrightarrow V_{ij}$ as $K_{max} \longrightarrow \infty$ and (ii) the eigenvalues, $E_i(K_{max})$, and eigenvectors of the effective Hamiltonian converge to their limiting values faster than the eigenvalues and eigenvectors of the bare Hamiltonian.

We expressed the HH effective interaction in terms of the matrix element of the $N$th Jacobi coordinate, i.e. the “last” particle pair,

$$(\sum_{i<j}^{A} V_{2\text{eff}}(\vec{r}_{ij})) = \frac{A(A-1)}{2} \langle V_{2\text{eff}}(\vec{r}_{A,A-1}) \rangle .$$

(8)

The relevant hyperspherical degrees of freedom associated with $V_{2\text{eff}}(\vec{r}_{A,A-1})$ are $\hat{n}_N$ and the hyperangle,
\[ \sin \alpha_N = \frac{r_{A,A-1}}{\sqrt{2} \rho}. \]  

For the construction of \( V_{2,\text{eff}}(\vec{r}_{A,A-1}) \) in Ref. [2] we made the following ansatz for the hyperspherical “2-body” Hamiltonian

\[ H_2(\rho) = \frac{1}{2m} \frac{\hat{K}_N^2}{\rho^2} + V(\sqrt{2} \rho \sin \alpha_N \cdot \hat{\eta}_N). \]  

It is a natural choice since \( \frac{\hat{K}_N^2}{\rho^2} \) contains the canonical kinetic energy associated with the two-body variables \( \alpha_N \) and \( \hat{\eta}_N \) (see Eq. (9)). Such an \( H_2 \) is in fact an A-body Hamiltonian. It contains the hyperspherical part of the A-body kinetic energy and thus becomes a function of the collective coordinate \( \rho \). The hyperradial kinetic energy operator has not been included in \( H_2 \). The reason is that we can use a complete basis set for the \( \rho \)-space and therefore we do not need to define an effective interaction for the hyperradial part.

Due to the collective coordinate, \( \rho \), in \( H_2 \) one has automatically a confinement of the 2-body-system: for moderate values of \( \rho \) the relation \( 0 \leq r_{A,A-1} \leq \sqrt{2} \rho \) ensures localization of the 2-body wave function and for large values of \( \rho \) the Hamiltonian coincides with the bare one, since the NN interaction vanishes. Therefore large overlaps between the model space states and the eigenvectors of the 2-body problem are ensured and thus, different from the HO effective interaction, there is no necessity for an additional confining potential.

In order to diagonalize \( H_2 \) we have to recouple the last 2 particles into good spin \( s \), isospin \( t, t_z \) and \( j \) states

\[ \langle T_2 T_3 \ldots T_A T_{A_z} | ([K_{N-1}] L_{N-1}; \ell_N) K_N L_N; S_2 S_3 \ldots S_A) J J Z \rangle \]

\[ H_2 |T_2' T_3' \ldots T'_A T'_{A_z} | ([K'_{N-1}] L'_{N-1}; \ell'_N) K'_N L'_N; S'_2 S'_3 \ldots S'_A) J J Z \rangle \]

\[ = \delta_{[K_{N-1}]} [K'_{N-1}] \delta_{S_2 S'_2} \delta_{S_3 S'_3} \ldots \delta_{S_{A-2} S'_{A-2}} \delta_{t_2 t'_2} \ldots \delta_{t_{A-2} t'_{A-2}} \times \sum_{S' A-2 + S'' A} \sum_{J_{A-2}} \langle 2S_{A-2} + S'' A | (2S_{A-2} + 1)(2S'' A - 1) (2s + 1)(2s' + 1) \]

\[ \times \left\{ \begin{array}{c} S_{A-2} \ \frac{1}{2} \ S_{A-1} \ s \\ \frac{1}{2} \ S'_{A} \ s' \end{array} \right\} \]

\[ \times \sqrt{(2L_{N-1} + 1)(2S_{A-2} + 1)(2J_{A-2} + 1)(2j + 1)} \]

\[ \times \left\{ \begin{array}{c} L_{N-1} \ \ell_N \ L_N \\ S_{A-2} \ s \ S_A \\ J_{A-2} \ j \ J \end{array} \right\} \]

\[ \times \sqrt{(2L'_{N-1} + 1)(2S'' A + 1)(2J_{A-2} + 1)(2j + 1)} \]

\[ \times \left\{ \begin{array}{c} L'_{N-1} \ \ell'_N \ L'_N \\ S'' A \ s' \ S'' A \\ J_{A-2} \ j \ J \end{array} \right\} \]

\[ \times \sum_{2t + 1} (-1)^{2T_{A-2} + T_A + T'_A} \sqrt{(2T_{A-2} + 1)(2T_A + 1)(2t + 1)(2t' + 1)} \]

\[ \times \left\{ \begin{array}{c} T_{A-2} \ \frac{1}{2} \ T_{A-1} \\ T_A \ t \ \frac{1}{2} \ T_A \ t' \end{array} \right\} \]

\[ \times \sum_{t_z} \langle T_{A-2} T_{A-2} T_A T_{A_z} | t t_z \rangle \langle T_{A-2} T_{A-2} T_A T_{A_z} | t' t_z \rangle \]

\[ \times \langle K_N (\ell_N; s) j t t_z | H_2 | K'_N (\ell'_N; s') j t' t_z \rangle \rangle_{K_{N-1}}. \]  

(11)
\[
\langle K_N(l_N; s) j tt_z | H_2 | K_N'(l'_N; s') j t't_z \rangle_{K_{N-1}} = \\
\delta_{K_N, K'_N} \delta_{l_N, l'_N} \delta_{s, s'} \delta_{t, t'} \frac{1}{2m} \frac{K_N(K_N + 3N - 2)}{\rho^2} + V^{K_{N-1}j}_{K_N l_N s t t_z, K'_N l'_N s' t' t_z} (\rho),
\]

where

\[
V^{K_{N-1}j}_{K_N l_N s t t_z, K'_N l'_N s' t' t_z} (\rho) = \sum_{p} W^{(p) K_{N-1}j}_{K_N l_N, K'_N l'_N} (\rho) \hat{O}^{(p) j}_{l_N s t t_z, l'_N s' t' t_z}.
\]

Here

\[
W^{(p) K_{N-1}j}_{K_N l_N, K'_N l'_N} (\rho) = \int d\Omega_N \mathcal{Y}_{K_N}^{\alpha} (\rho) (\sqrt{2} \rho \sin \alpha_N) \mathcal{Y}_{K'_N}^{\beta},
\]

and

\[
\hat{O}^{(p) j}_{l_N s t t_z, l'_N s' t' t_z} = \langle (l_N; s) j tt_z | \hat{O}^{(p)} | (l'_N; s') j t't_z \rangle,
\]

is the usual 2-body matrix element for the operator \( \hat{O}^{(p)} \). One sees that \( H_2 \) is diagonal in the quantum numbers \([K_{N-1}], j, t_z\) and normally, as in case of the potential models considered in this work, also diagonal in \( s \) and \( t \).

Due to the hyperangular integration \( H_2 \) explicitly depends on quantum numbers of the residual system, i.e. \( K_{N-1} \), while it is independent of the other quantum numbers in \([K_{N-1}]\). As a result the below defined HH effective interaction depends on the state of the residual \( A - 2 \) particle subsystem. Such a "medium correction" of the 2-body force is of course a great advantage. On the other hand one has to pay for it with greater numerical effort, since the effective interaction has to be calculated for all the various states and in addition it also depends on the specific \( A \)-body system considered. It is similar to the HO multi-valued effective interaction \([12, 13]\). We solve the hyperradial equation on a grid, where \( H_2 \) is diagonalized for each grid point \( \rho \) and for all the possible values of \( K_{N-1}, j, s, t, t_z \) in our model space.

Having defined \( H_2 \) we employ in Ref. [3] the Lee-Suzuki [4] similarity transformation method to construct the effective interaction. We follow an analogous procedure as that of Barrett and Navrátil [1] for the HO effective interaction. To this end the eigenvectors, \( \{|i\} \) and eigenvalues, \( \{\epsilon_i\} \), of \( H_2(\rho) \), given by Eq. [14], are used to construct the effective interaction. Denoting by \( |\alpha\rangle \) the HH functions that belong to our model space, i.e. the HH function \( |[K_N]\rangle \) with \( K_N \leq K_{\text{max}} \), and by \( |\beta\rangle \) the states that belong to the \( Q \) space, \( Q = \{|[K_N]\rangle; K_N > K_{\text{max}}\} \) the Lee-Suzuki effective Hamiltonian takes the form

\[
P \hat{H}_2 P = PH_2P + PH_2Q_\omega P,
\]

where the transformation operator \( \omega = Q_\omega P \) is given by the equation

\[
\langle \beta | i \rangle = \sum_{\alpha} \langle \beta | \omega | \alpha \rangle \langle \alpha | i \rangle.
\]

If \( n_p \) is the number of model-space HH basis functions that belong to the subspace \([K_{N-1}]\), we may solve Eq. (17) for \( \omega \) by choosing a set, \( A \), of \( n_p \) eigenvectors with the lowest eigenvalues \( \langle i \rangle \) and inverting the matrix \( \langle \alpha | i \rangle \). The resulting effective 2-body Hamiltonian
\begin{equation}
\langle \alpha | \tilde{H}_2(K_{N-1}, \rho) | \alpha' \rangle = \sum_i^{n_F} \left[ \langle \alpha | i \rangle \epsilon_i \langle i | \alpha' \rangle + \sum_{\beta} \langle \alpha | i \rangle \epsilon_i \langle i | \beta \rangle \langle \beta | \omega | \alpha \rangle \right],
\end{equation}

will have the property that \( P|i\rangle, |i\rangle \in \mathcal{A} \), is a right eigenvector of \( \tilde{H}_2 \) with eigenvalue \( \epsilon_i \). The effective Hamiltonian is in general a non-hermitian operator, however it can be hermitized, using the transformation \[13\]

\begin{equation}
H_{2 \text{ eff}} = [P(1 + \omega^\dagger \omega)P]^{1/2} \tilde{H}_2 [P(1 + \omega^\dagger \omega)P]^{-1/2}.
\end{equation}

The effective interaction can be now deduced from \( H_{2 \text{ eff}} \), by subtracting the kinetic energy term,

\begin{equation}
V_{\text{eff}} = H_{2 \text{ eff}} - \frac{1}{2m} \frac{\hat{K}_N^2}{\rho^2}.
\end{equation}

III. NUMERICAL RESULTS

We apply the EIHH method to the three- and four-body nuclei calculating ground state energies (\( E_B \)), matter radii (\( < r^2 >^{1/2} \)), and two-body correlation functions. We have chosen to perform the calculation with two standard noncentral potential models, AV6 and AV14 \[7\]. They are frequently used in other few-body approaches and thus allow to compare to other results in the literature. The model AV14 is a fully realistic potential, while AV6 contains besides spin and isospin dependent central force only the tensor forces in addition.

In Fig. 1 we show the triton binding energy as function of the hyperspherical quantum number \( K \). The convergence pattern looks very similar for both noncentral potential models. One sees that quite precise results are already obtained with rather small \( K \) values. In fact deviations from the converged results amount to less than 0.1 MeV and 0.01 MeV for \( K \geq 8 \) and \( K \geq 14 \), respectively. It is evident from the figure that the additional potential terms of AV14 do not have any important impact on the convergence behavior. In Fig. 1 we illustrate in addition our EIHH results from Ref. \[2\] for the central potential model MTI-III \[16\]. The comparison with the AV6 result shows that the tensor force does not lead to a deterioration of the convergence.

In Fig. 2 the results for the triton radius are shown. One sees that also in this case an extremely good convergence behavior is obtained. For \( K \geq 6 \) one has only very small deviations of less than 0.005 fm from the converged AV6 and AV14 results. As for the binding energy additional potential terms of AV14 do not affect the convergence behavior. Again we show the MTI-III results. Contrary to the case of the binding energy the convergence is slowed down a bit by the tensor force.

Binding energy and radius are typical examples of observables which are not very much affected by high momentum components of the wave function. One has to realize that our effective interaction approach can be considered in some way as a momentum expansion, since with increasing \( K_{max} \) higher and higher momentum components can be incorporated in the wave function. To study this question better we show in Fig. 3 the two-body correlation function
\[ \rho(r) = \langle \Psi | \delta(r - r_{A,A-1}) | \Psi \rangle, \tag{21} \]

where \( \Psi \) is the ground state wave function (\( \int dr \rho(r) = 1 \)). One sees in Fig. 3a that \( \rho(r) \) exhibits a rather broad and asymmetric maximum at about 2 fm. The convergence pattern of Fig. 3b shows that at larger distances convergence is obtained with rather low values of \( K \). Further increase of \( K \) improves the convergence at shorter and shorter distances. In Fig. 3b we also illustrate a result with the bare interaction. For an intermediate distance between about 1 and 3 fm one finds a slight overestimation of \( \rho(r) \), while at smaller and larger \( r \) the correlation function is considerably underestimated.

For the three-nucleon system it is possible to take into account rather high \( K \) values in the calculation, in fact, e.g., for the highest value we considered \( (K = 32) \) one has 374 hyperangular basis functions. Adding just one further nucleon leads to a considerable enhancement of the number of basis functions. Therefore our \( ^4 \text{He} \) calculation is carried out only up to \( K = 18 \), where one already has 1975 hyperangular basis functions.

The results for the \( ^4 \text{He} \) binding energy and radius are depicted in Figs. 4 and 5, respectively. One notes that the convergence behavior is again very good. On the other hand the results do not converge as rapidly as for the three-nucleon case. In fact with \( K = 8 \) one still lacks about 1 MeV binding, while only about one tenth of this value is missing for the triton case at the same \( K \). For the convergence of the radius the differences are smaller between the \( ^3 \text{H} \) and \( ^4 \text{He} \) results. For \( K = 6 \) the \( ^4 \text{He} \) radius is overestimated by 0.015 fm, which is only about three times larger than the corresponding overestimation for the triton case. However, also for \( ^4 \text{He} \) one finds a rather good convergence for the higher \( K \) values.

From Figs. 4 and 5 it is evident that there are no important differences between the convergence patterns of the AV6 and AV14 cases showing again that the additional AV14 potential terms have a rather unimportant effect on the convergence behavior. Also for \( ^4 \text{He} \) we illustrate the results of the central potential model MTI-III in the figures. They exhibit a little better convergence behavior than the potential models with tensor force.

In Fig. 6 we show the two-body correlation function of \( ^4 \text{He} \). The convergence pattern is not as good as for the triton case of Fig. 3. Even at larger distances one needs a higher value of \( K \) in order to reach the same precision. The lower convergence rate can be attributed to the higher density of the alpha particle.

In Table I we compare our results for the three-nucleon system with other results in the literature. Inspecting the table one sees a very good agreement among the various methods. It is evident that the binding energy can be calculated with a precision of about 0.01 MeV in most methods. For the alpha particle we obtain the following results: \( E_B = 25.61(4) \) MeV, \( < r^2 >^{1/2} = 1.4739(4) \) fm (AV6, no Coulomb force) and \( E_B = 24.34(5) \) MeV, \( < r^2 >^{1/2} = 1.5297(3) \) fm (AV14, with Coulomb force). The comparison with other results in the literature is not as good as for the triton case, e.g., comparing with the SVM result \[20\] one finds a difference of 0.2 MeV for the binding energy with AV6. One has similarly large differences comparing our \( E_B \) for AV14 with that of Ref. \[23\]. It shows that further investigations of the various groups \[1,2,8,13,21,24,25\] have to be made in order to reach a better agreement. Such a benchmark study is currently under way.
IV. CONCLUSION

In this work we have applied the recently developed hyperspherical effective interaction method to noncentral potential models including the realistic NN force AV14. In this approach the two-body effective interaction depends on the A-body hyperradius and on the state of the A-2 rest system explicitly. Our results for the ground state properties of three- and four-nucleon systems show that the method leads to an excellent convergence of the hyperspherical expansion. The convergence is extremely fast in case of triton, while it is a bit slower for the alpha particle. The difference is explained by the higher density of the four-nucleon system, since our approach can be interpreted as a kind of momentum expansion.

In view of further applications it will be important to incorporate also three-nucleon forces in the EIHH method. Therefore we would like to mention that, similar to the HO approach \[1,26\], the present formalism can be extended to derive an HH three-body effective interaction.

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FIGURES

FIG. 1. Triton binding energy as function of the hyperangular quantum number $K$ with various NN potentials models: AV14 (a), AV6 (b), and MTI-III (c). The value for the largest $K$ is indicated by a dashed line.

FIG. 2. Triton matter radius as function of the hyperangular quantum number $K$ with various NN potentials models with notations as in Fig. 1.

FIG. 3. EIHH results for the two-body correlation function of triton with the AV14 potential with various $K$ relative to the $K = 26$ result; also shown is the $K = 26$ result with the bare interaction.

FIG. 4. Alpha particle binding energy as function of the hyperangular quantum number $K$ with various NN potentials model with notations as in Fig. 1.

FIG. 5. Alpha particle matter radius as function of the hyperangular quantum number $K$ with various NN potentials model with notations as in Fig. 1.

FIG. 6. EIHH results for the two-body correlation function of $^4\text{He}$ with the AV14 potential with various $K$ relative to the $K = 16$ result.
TABLE I. Comparison of binding energies ($E_B$) in [MeV] and root mean square radii ($<r^2>^{1/2}$) in [fm] obtained with the present effective interaction method in the HH formalism (EIHH) with results of other methods. For EIHH the number in parenthesis indicates the variance with respect to the result obtained with $K = K_{max} - 2$. The quality of the convergence can be inferred from Figs. 1, 2

| Nucleus | Method [Ref.] | AV6 $E_B$ | $<r^2>^{1/2}$ | AV14 $E_B$ | $<r^2>^{1/2}$ |
|---------|---------------|------------|---------------|------------|---------------|
| $^3\text{H}$ | EIHH | 7.1602(3) | 1.77475(5) | 7.6814(3) | 1.77615(3) |
|         | Faddeev [17] | 7.15 | – | 7.670 | – |
|         | GFMC [18] | 7.22(0.12) | 1.75(0.10) | – | – |
|         | VMC [19] | – | – | 7.53 | – |
|         | SVM [20] | 7.15 | 1.76 | – | – |
|         | HH [21] | – | – | 7.684 | – |
|         | CHH [22] | – | – | 7.69 | – |
\[ \rho(r) \text{ [fm}^{-1}\text{]} \]

\[ ^3\text{H} \]

(a)

(b)

- \( K=2 \)
- \( K=8 \)
- \( K=14 \)
- \( K=20 \)
- \( K=24 \)
- \( K=26 \) (AV14 bare)
The graph shows the energy level $E_B$ in MeV as a function of $K$ for three different models:

1. **AV14**: The energy $E_B$ decreases with increasing $K$, starting at a high value and showing a smooth decrease.
2. **AV6**: Similar to AV14, the energy $E_B$ decreases with $K$, though the curve is slightly flatter.
3. **MTI–III**: The energy $E_B$ remains relatively constant across the range of $K$, with minor fluctuations.

Each graph is labeled with the corresponding model name at the top.
