A Link Invariant from Quantum Dilogarithm

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Abstract

The link invariant, arising from the cyclic quantum dilogarithm via the particular $R$-matrix construction, is proved to coincide with the invariant of triangulated links in $S^3$ introduced in [14]. The obtained invariant, like Alexander-Conway polynomial, vanishes on disjoint union of links. The $R$-matrix can be considered as the cyclic analog of the universal $R$-matrix associated with $U_q(sl(2))$ algebra.

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Introduction

Solutions to the Yang-Baxter equation (YBE) \([20, 2]\) play important role both in the theory of solvable two-dimensional models of statistical mechanics and field theory (for a review see \([3, 10]\)), and in the theory of braid groups and links \([12]\). Algebraically YBE is deeply related with quantum groups \([8]\).

Recently, Rogers' dilogarithm identity \([18]\) attracted much attention in connection with solvable models and 3d mathematics, see \([9]\) and references therein. In \([11]\) quantum generalization of the dilogarithm function has been suggested. Remarkably, quantized dilogarithm identity, like YBE, is also connected with quantum groups \([14, 15]\).

In \([14]\) the invariant of triangulated links in three-manifolds has been defined. It was conjectured that the choice of the triangulation is inessential. In this paper we prove this conjecture for the case of three-sphere. Namely, using particular \(R\)-matrix, we define in the standard way the link invariant, and then show that it coincides with the three-dimensional construction of \([14]\).

In Sect.\(3\) the \(R\)-matrix is defined. The corresponding link invariant is constructed in Sect.\(4\). In Sect.\(4\) the three-dimensional description is developed and the equivalence with construction of \([14]\) is established.

2 The \(R\)-Matrix

First introduce the necessary notations. Let \(\omega\) be some primitive \(N\)-th root of unity for some \(N \geq 2\). For any non-negative integer \(n < N\) define the symbol:

\[
(\omega)_n = \begin{cases} 
1, & \text{if } n = 0; \\
\prod_{j=1}^{n}(1 - \omega^j), & 0 < n < N, 
\end{cases}
\]  

(2.1)

Under complex conjugation we have the following properties:

\[
(\omega)_n^* = (\omega^*)_n, \quad \omega^* \omega = 1.
\]  

(2.2)

It will be convenient to use the “theta”-function, defined on integers:

\[
\theta(n) = \begin{cases} 
1, & 0 \leq n < N; \\
0, & \text{otherwise, } n \in \mathbb{Z}. 
\end{cases}
\]  

(2.3)

For any \(n \in \mathbb{Z}\) denote by \([n]\) the corresponding integer residue modulo \(N\), lying in the interval \(\{0, \ldots, N - 1\}\):

\[
[n] = n \quad \text{(mod } N\}, \quad 0 \leq [n] < N, \quad n, [n] \in \mathbb{Z}.
\]  

(2.4)

The evident property of this function is its’ periodicity:

\[
[n + N] = [n], \quad n \in \mathbb{Z}.
\]  

(2.5)

It should be stressed however that the “bracketing” is a map from \(\mathbb{Z}\) to \(\mathbb{Z}\) rather than to \(\mathbb{Z}_N\), the ring of residues modulo \(N\). Besides, it does not preserve the addition operation:

\[
[m] + [n] \geq N \Rightarrow [m] + [n] \neq [m + n].
\]  

(2.6)
Define a function with four integer arguments:

\[ W(k, l, m, n) = N \omega^{(l+m)(m+n)} \frac{\theta([k] + [m]) \theta([l] + [n])}{(\omega)_{[k]}(\omega^*)_{[l]}(\omega)_{[m]}(\omega^*)_{[n]}}, \quad k, l, m, n \in \mathbb{Z}. \]  

(2.7)

Using these definitions consider the following indexed set of numbers:

\[ R(i, j, k, l|a, b, c, d) = \omega^{a-k-j} W(j - i - a, i - l - d, l - k - c, k - j - b), \]  

(2.8)

where all indices are integers, and \(a, b, c, d\) being constrained to have the unit sum modulo \(N\):

\[ a + b + c + d = 1 \pmod{N}. \]  

(2.9)

There is a symmetry group of these symbols generated by relations:

\[ R(i, j, k, l|a, b, c, d) = R(k, l, i, j|c, d, a, b), \]  

(2.10)

and

\[ R(i, j, k, l|a, b, c, d) = R(-j, -i, -l, -k|a, d, c, b) \omega^{-i-j-k-l}, \]  

(2.11)

which have a natural geometrical interpretation. Indeed, consider the diagram given by a projection of two non-coplanar segments in \(R^3\) to some plane, the images of the segments having an intersection point. Place indices \(j\) and \(l\) on two ends of the overcrossing segment, and indices \(i\) and \(k\), on the ends of the other segment, choosing among two possibilities that, where indices \(i, j, k, l\) go round the intersection point in the counter-clockwise direction. Place indices \(a, b, c, d\) on four sectors also in the counter-clockwise order, starting from the sector bounded by segments with the end labels \(i\) and \(j\). Associate with this configuration the “Boltzmann” weight (2.8). The above mentioned symmetry group acquires now an interpretation of the symmetry group of the diagram. Namely, relation (2.10) corresponds to the rotation by \(\pi\) aroung the intersection point, while (2.11) corresponds to looking at the projection image from the opposite side of the plane.

The next important property of the \(R\)-symbols is the Yang-Baxter identity, which they satisfy. Using periodicity property (2.5), confine the range, where the indices take their values, to the finite set \(\{0, \ldots, N - 1\}\). Consider an \(N^2\)-by-\(N^2\) \(R\)-matrix, described by the matrix elements:

\[ \langle i, j|R|k, l \rangle = R(i, j, k, l|1, 0, 0, 0) \omega^{k+l}. \]  

(2.12)

Then, the following YBE holds:

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \]  

(2.13)

where the standard matrix notations are used:

\[ R_{12}|i, j, k \rangle = \sum_{i', j' = 0}^{N-1} |i', j', k \rangle \langle i', j'|R|i, j \rangle, \]  

(2.14)
and similarly for the other matrices. Later we will argue that the $R$-matrix (2.12) is in fact a cyclic analog of the universal $R$-matrix associated with the quantum group $U_q(sl(2))$ (see the remark in the end of Sect.4).

The inverse $R$-matrix is given in terms of the same $R$-symbols:

$$\langle i,l|R|k,j\rangle = R(i,j,k,l,0,0,0,1)\omega^{j+k}. \quad (2.15)$$

So, in matrix notations we have

$$R_{12}R_{12} = 1. \quad (2.16)$$

Thus, taking into account above diagrammatic interpretation as well as the symmetry property (2.10), we have already realizations of two Reidemeister moves, corresponding to the regular isotopy of links. The following relation provides us with the third Reidemeister move, necessary for the ambient isotopy:

$$\sum_{j=0}^{N-1} \omega^j R(i,j,j,l|0,1,0,0) = \omega^{-i}\delta_{i,l}. \quad (2.17)$$

In the next section, using these results, we define the “state sum”, leading to an ambient isotopy link invariant.

3 Invariants of Links in Three-Sphere

Let link $L$ in $S^3$ is given by a two-strand tangle. Consider some non-singular (i.e. with only simple intersection points) planar projection of this tangle. Denote by $V,E,F$ the sets of vertices, edges, and faces, respectively, in the planar graph. We will write $v \in f$, for some vertex $v$ and face $f$ if $v$ lies in the boundary of $f$. There are two distinguished elements $e_1$ and $e_2$ in $E$, corresponding to the two strands of the tangle, and one distinguished element $f_0$ in $F$, corresponding to the outer region in the plane. Define the following two maps:

$$j:E \to Z_N, \quad a:V \times F \to Z_N, \quad (3.1)$$

where we identified $Z_N$ with the set $\{0,\ldots,N-1\}$. Impose the restrictions on $j$:

$$j(e_1) = j(e_2) = 0, \quad (3.2)$$

and on $a$:

$$a(v,f_0) = 0; \quad v \notin f \Rightarrow a(v,f) = 0, \quad v \in V; f \in F, \quad (3.3)$$

$$\sum_{f \in F} a(v,f) = \sum_{v \in V} a(v,f) = 1, \quad f \neq f_0. \quad (3.4)$$

Using these maps, associate with each vertex the Boltzmann weight according to Sect.2. This gives us one more map:

$$r_{j,a}:V \to C. \quad (3.5)$$

Consider now the “partition” function:

$$\langle L \rangle = \sum_j \prod_{v \in V} r_{j,a}(v) \prod_{e \in E} \omega^{j(e)}. \quad (3.6)$$
Theorem 1  (1) The quantity $\langle L \rangle^N$ is an ambient isotopy invariant of the link $L$ in three-sphere; (2) for odd $N$ the following equality holds:

$$\langle L \rangle^N = Q(S^3, L),$$  \hspace{1cm} (3.7)

where $Q(M, L)$ is the invariant of triangulated link $L$ in three-manifold $M$ defined in [14].

The first part of the theorem is a consequence of the identities, satisfied by the $R$-symbols (see Sect. 2). There is a freedom in choosing the second map in (3.1). In fact, different choices lead to one and the same partition function (3.4) up to $N$-th roots of unity. Note that the face indices are nothing else but the angle dependent data, corresponding to “enhanced” vertex models [4, 13, 19]. This becomes evident after Fourier transformation over the edge indices. The face index - angle correspondence is described by the formula:

$$\phi = \pi - 2\pi a,$$ \hspace{1cm} (3.8)

where $\phi$ is the angle between the adjoining straight edges in the diagram, corresponding to $R$-symbol (see Sect. 2), and $a$ is the face index associated with the sector bounded by these edges.

The proof of the second part of the theorem is given in the next section.

Note that the two strand tangle representation of links has been used in [16] to construct the state sum for Alexander-Conway polynomial [1, 7], because the latter is zero for disjoint union of links. This is also true for the invariant (3.6):

$$\langle L_1 \sqcup L_2 \rangle = 0.$$ \hspace{1cm} (3.9)

In the language of quantum group theory, property (3.9) reflects the vanishing quantum dimension. In the three-dimensional definition given in [14] (see also the next section) no tangle description is needed.

4 Three-Dimensional Description

In this section we establish the connection of the link invariant constructed in Sect. 3 with the invariant of triangulated links of [14], related in turn to cyclic quantum dilogarithm [11]. First, remind the necessary formulas from [14] in slightly different notations.

We will assume that the integer $N$ is odd, and the square root of $\omega$ will be thought to be taken as an $N$-th root of unity:

$$\omega^{1/2} = \omega^{(N+1)/2}.$$ \hspace{1cm} (4.1)

For complex $x, y, z$, satisfying the Fermat equation

$$x^N + y^N = z^N,$$ \hspace{1cm} (4.2)

and integers $m, n$ define the function

$$w(x, y, z|m, n) = w(x, y, z|m - n)\omega^{n^2/2},$$ \hspace{1cm} (4.3)
where
\[ w(x, y, z | n) = \prod_{j=1}^{n} \frac{y}{z - x \omega^j}. \] (4.4)

The Fermat equation (4.2) ensures the periodicity relation:
\[ w(x, y, z | n + N) = w(x, y, z | n). \] (4.5)

This latter property enables us to consider the finite set \( Z_N \) instead of \( Z \).

For any non-coinciding complex numbers \( z_0, z_1, z_2, z_3, \) and \( \alpha, \beta, \gamma, \delta, a, c \in Z_N \) define two functions:
\[
T_{0123}^{\alpha, \beta}(a, c) = \rho_{0123} \omega^{\alpha \delta - ac/2} w(x_{03}x_{12}, x_{01}x_{23}, x_{02}x_{13} | \gamma - a, \alpha) \delta_{\beta, \gamma + \delta},
\] (4.6)
and
\[
T_{0123}^{\alpha, \beta}(a, c) = \rho_{0123} \omega^{\alpha \delta - ac/2} \delta_{\beta, \gamma + \delta},
\] (4.7)
where
\[ x_{ij} = (z_i - z_j)^{1/N}, \quad i, j = 0, 1, 2, 3, \] (4.8)
with \( N \)-th root being chosen to be real for real difference \( z_i - z_j \); scalar functions \( \rho_{0123} \) and \( \rho_{0123} \) depend on only \( z \)'s and their explicit form can be found in [14]. These symbols can be associated with tetrahedrons in \( R^3 \) as is described below.

In what follows for a polyhedron \( X \), considered as a collection of vertices, edges, triangles, and tetrahedrons, we will denote by \( \Lambda_i(X) \), \( i = 0, 1, 2, 3 \) the sets of simplices of corresponding dimension. Consider a topological \( 1 \) tetrahedron \( t \) in \( R^3 \). Order the vertices by fixing a bijective map
\[ u: \{0, 1, 2, 3\} \ni i \mapsto u_i \in \Lambda_0(t). \] (4.9)
Put an arrow on each edge, pointing from a “larger” vertex (with respect to above ordering) to a “smaller” one. Each face also gets its’ own orientation. Let \( u_0 \) be the top of the tetrahedron. Let us look from it down at the vertices \( u_i, u_j, u_k \), where
\[ \{i, j, k\} = \{0, 1, 2, 3\} \setminus \{l\} \quad i < j < k. \] (4.10)
We will see two possible views: either \( u_i, u_j, u_k \), in the order which they are written, go round in the counter-clockwise direction (the “right” orientation) or, in the clockwise one (the “left” orientation). The tetrahedron itself has two possible orientations in the following sense. Call the tetrahedron “right (left)”-oriented if the face \( u_0u_1u_2 \) has the right (left) orientation in the above sense.

Introduce three maps:
\[ s: \Lambda_0(t) \to C, \quad c: \Lambda_1(t) \to Z_N, \quad \alpha: \Lambda_2(t) \to Z_N. \] (4.11)

\(^1\) the term “topological” here means that edges and faces of the tetrahedron can be curved
where $s$ is injective, and $c$ satisfies the following relations:

$$\sum_{e \ni v} c(e) = 1/2, \quad v \in \Lambda_0(t). \quad (4.12)$$

Let $c_{ij} = c(u_i u_j)$ ($u_i u_j$ is the edge having ends $u_i$ and $u_j$), $\alpha_i = \alpha(u_i u_j u_i)$, $\{j, k, l\} = \{0, 1, 2, 3\} \setminus \{i\}$, and $z_i = s(u_i)$. Define the symbol associated with the tetrahedron $t$:

$$t_u(s, c, \alpha) = \begin{cases} T^{0123}_{c_{01}}, c_{12}, c_3, c_0, \alpha_1, \alpha_2, \alpha_3, \alpha_0, & \text{right orientation;} \\ T^{0123}_{c_{01}}, c_{12}, c_3, c_0, \alpha_1, \alpha_2, \alpha_3, \alpha_0, & \text{left orientation.} \end{cases} \quad (4.13)$$

The variables associated with the second map in (4.11) will be called $Z_N$-charges. Using symbols (4.13), one can associate a “partition” function to any finitely triangulated three-manifold with a triangulated tangle passing through the all interior 0-simplices. The construction goes as follows.

Let $M$ be a finite triangulation of an oriented 3-dimensional manifold with boundary. Fix a subset of 1-simplices $T \subset \Lambda_1(M \setminus \partial M)$ in such a way that any interior 0-simplex belongs to exactly two elements from $T$, so the latter is some triangulated tangle in $M$, passing through the all interior vertices. Denote $I = \{0, 1, \ldots, K - 1\}$, where $K$ is the number of vertices in $M$, and fix the following maps:

$$u: I \to \Lambda_0(M), \quad s: \Lambda_0(M) \to C, \quad c_T: \Lambda_3(M) \times \Lambda_1(M) \to Z_N, \quad \alpha: \Lambda_2(M) \to Z_N,$$

where $u$ is bijective, $s$, injective, and $c_T$ satisfies the restrictions:

$$e \notin \Lambda_1(t) \Rightarrow c_T(t, e) = 0; \quad \sum_{e \ni v} c_T(t, e) = 1/2, \quad v \in t \in \Lambda_3(M); \quad (4.14)$$

$$\sum_{t \in \Lambda_3(M)} c_T(t, e) = \begin{cases} 0, & e \in T; \\ 1, & e \in \Lambda_1(M \setminus \partial M) \setminus T. \end{cases} \quad (4.15)$$

Denote the total $Z_N$-charges on $\partial_1(\partial M)$ by $c_{\text{tot}}: \Lambda_1(\partial M) \ni e \mapsto \sum_{t \in \Lambda_3(M)} c_T(t, e) \in Z_N,$

and also the restrictions of the remained maps in (4.14) by

$$\partial u: \Lambda_0(\partial M) \to I, \quad \partial s: \Lambda_0(\partial M) \to C, \quad \partial \alpha: \Lambda_2(\partial M) \to Z_N.$$ \quad (4.16)

Then, the partition function reads:

$$M_{\partial u}(T, \partial s, \partial c_{\text{tot}}, \partial \alpha) = N^{-K_0} \sum_{\alpha, \partial \alpha = \text{fixed}} \prod_{t \in \Lambda_3(M)} t_u(s, c_T, \alpha) \prod_{e \in \Lambda_1(M \setminus \partial M) \setminus T} \langle s(\partial e) \rangle^{-1},$$

where

$$\langle s(\partial e) \rangle = (s(u_j) - s(u_i))^{(N-1)/N}, \quad e = u_i u_j; \quad (4.20)$$

and $K_0$ being the number of interior 0-simplices. As is reflected in the left hand side of (4.19), the partition function depends (up to $N$-th roots of unity) only
on the boundary data as well as on the topological class of the triangulated tangle (see [14] for the definition). This is the consequence of the relations the symbols (4.13) satisfy, which are described in [14]. When \( \partial M = \emptyset \) partition function (4.19) multiplied by \( N^2 \) reduces to the invariant of triangulated links [14].

To make contact with the results of Sect. 3, we have just to specify the triangulated 3-manifold with tangle, corresponding to \( R \)-symbol (2.8). The most economic way is to consider singular triangulated manifolds [17].

Let \( u_1, u_2, u_3, u_4 \) be four points in \( \mathbb{R}^3 \), specifying the right-oriented tetrahedron \( u_1 u_2 u_3 u_4 \). Glue to four faces of the latter four other tetrahedrons

\[
\begin{align*}
&u_1 u_2 u_4 u_5, & &u_2 u_3 u_4 u_5', & &u_0 u_1 u_2 u_3, & &u_0' u_1 u_3 u_4.
\end{align*}
\]

We have added four new vertices \( u_0, u_0', u_5, u_5' \). Among the latter identify the “primed” vertices with the corresponding “non-primed” ones:

\[
u_0 = u_0', \quad u_5 = u_5'.
\]

(4.21)

While doing this, we glue also the added tetrahedrons pairwise along the corresponding faces. As a result we get a triangulated octahedron \( u_0 u_1 u_2 u_3 u_4 u_5 \). The next step consists in attaching four more tetrahedrons, \( u_0 u_1 u_2 u_5, u_0 u_2 u_3 u_5, u_0 u_3 u_4 u_5, u_0 u_1 u_4 u_5 \), to the octahedron, each one being glued along two faces. For example, tetrahedron \( u_0 u_1 u_2 u_5 \) is attached along faces \( u_0 u_1 u_2 \) and \( u_1 u_2 u_5 \), and similarly the others. Thus we obtain singular triangulated three-ball \( B \), where among others four different 1-simplices have coinciding ends \( u_0 \) and \( u_5 \). We will distinguish them by specifying the tetrahedron to which they belong. For example,

\[
[u_0 u_5]_{12} \in \Lambda_1(u_0 u_1 u_2 u_5),
\]

(4.22)

and similarly for the others. Besides, \( \Lambda_2(\partial B) \) consists of four pairs of 2-simplices, each being given by one and the same triple of vertices:

\[
\Lambda_2(\partial B) = \{[u_0 u_i u_5]^\pm, \quad i = 1, 2, 3, 4\},
\]

(4.23)

where we distinguish elements within each pair by their orientation \((+ (−) \) corresponds to the right (left) orientation). To write down the partition function, we have to specify the tangle. Let us choose it as follows:

\[
T = \{u_1 u_3, u_2 u_4\}.
\]

(4.24)

Now, providing \( B \) with additional data (4.14), one can calculate the partition function (4.19). The result reads:

\[
B_{\partial B}(T, \partial s, \partial c_{\text{tot}}, \partial \alpha) = (s_{05})^{N-1} \sum_{i_1, i_2, i_3, i_4 = 0}^{N-1} R(i_1, i_2, i_3, i_4|c_{12}, c_{23}, c_{34}, c_{14}) \prod_{m=1}^4 \psi^{\alpha_m - \alpha_m^*}(4.25)
\]

\[
\]
where
\[ \alpha_m^\pm = \alpha([u_0 u_m u_5]^\pm), \quad s_{ij} = (s(u_i) - s(u_j))^{1/N} \] (4.26)
\[ c_m = \partial c_{\text{tot}}(u_m u_5), \quad c_{ij} = \partial c_{\text{tot}}([u_0 u_5]_{ij}), \] (4.27)
and
\[ \psi_{m,j}^k = \omega^k s_{0m}^{[k-1]} s_{m5}^{-[k]}, \] (4.28)
see also (2.4). The role of \( \psi \)-functions is to intertwine the indices of different nature. Note that the \( Z_N \)-charges are related with the angle-dependent data for the enhanced \( R \)-matrix.

Formula (4.25) enables us to prove the second part of Theorem 1. Indeed, let we are given a link, represented by a two-strand tangle. Then, using correspondence (4.25), we associate to the latter partition function of a singular triangulated three-ball with four boundary 0-simplices, the tangle by its strands being attached to two of them. To get the invariant in \( S^3 \), we have to glue the ball with another (empty) one, making the tangle to pass through the other two vertices and converting it into the link under consideration. This last process is equivalent to choosing boundary conditions (3.2).

One can prove also the YBE (2.13) via consideration the triangulated three-balls, corresponding to two sides of YBE, and transforming them to each other by a sequence of elementary moves defined in [14].

A remark is in order. In [15] it has been shown that the universal \( R \)-matrix in Drinfeld double can be represented as a product of solutions to the constant pentagon relation, the canonical element in Heisenberg double being such solution. This construction can be extended also to the case of non-constant solutions to the pentagon relation. In particular, using cyclic quantum dilogarithm of [11] and specializing the parameters, one can derive \( R \)-matrix (2.12). The derivation given in this section is essentially equivalent to the latter. On the other hand, the cyclic quantum dilogarithm can be obtained from the quantum dilogarithm with \( q \) non-root of unity \( 2 \) in the limit, where \( q \) approaches a root of unity \( 3 \). Thus, it is natural to consider \( R \)-matrix (2.12) as a cyclic counterpart of the universal \( R \)-matrix in \( U_q(\mathfrak{sl}(2)) \).

5 Summary

According to [14], cyclic quantum dilogarithm leads to invariants of triangulated links in triangulated three-manifolds. It was conjectured that these invariants are independent of triangulations used. The \( R \)-matrix derivation, presented in this paper (Sect.3), proves the conjecture for the case of three-sphere. The essential part of the proof is the realization of the \( R \)-matrix (2.12) as a three-dimensional partition function (4.13), (4.25) for the triangulated three-ball with properly chosen tangle in it.

The \( R \)-matrix (2.12) can be considered as a cyclic counterpart of the universal \( R \)-matrix for the algebra \( U_q(\mathfrak{sl}(2)) \). The factorized structure of the matrix

\[ \text{in this case the quantum dilogarithm is precisely the canonical element in Heisenberg double of } BU_q(\mathfrak{sl}(2)), \text{ the Borel sub-algebra of } U_q(\mathfrak{sl}(2)). \] [14]
elements \((2.5)\), \((2.7)\) probably reflects the fact that this \(R\)-matrix can be obtained in some limit from the chiral Potts model \(R\)-matrix \([5]\).

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