Small Elliptic Quantum Group $e_{\tau,\gamma}(\mathfrak{sl}_N)$

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Abstract. The small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$, introduced in the paper, is an elliptic dynamical analogue of the universal enveloping algebra $U(\mathfrak{sl}_N)$. We define highest weight modules, Verma modules and contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$, the dynamical Shapovalov form for $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and the contravariant form for highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. We show that any finite-dimensional $\mathfrak{sl}_N$-module and any Verma module over $\mathfrak{sl}_N$ can be lifted to the corresponding $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module on the same vector space. For the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ we construct the evaluation morphism $E_{\tau,\gamma}(\mathfrak{sl}_N) \to e_{\tau,\gamma}(\mathfrak{sl}_N)$, thus making any $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module into an evaluation $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module.

Introduction

The main purpose of this paper is to define a dynamical quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ which is an elliptic dynamical analogue of the universal enveloping algebra $U(\mathfrak{sl}_N)$. We call $e_{\tau,\gamma}(\mathfrak{sl}_N)$ the small elliptic quantum group, comparing it with the elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ introduced in [F]. Our initial motivation to study this object arises from the wish to understand the structure of evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$, which should be analogous to evaluation modules over the Yangian $Y(\mathfrak{sl}_N)$.

Evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_2)$ have been defined in [FV1]. They appear naturally in the description of transition matrices for the trigonometric $q$KZ difference equation [TV1]. They also serve for the definition of the $q$KZB difference equations and occur in the description of its monodromies, see [FTV]. One should expect evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$ for $N > 2$ to play a similar role. Symmetric and exterior powers of the vector representation of $E_{\tau,\gamma}(\mathfrak{sl}_N)$, developed in [FV2], are examples of evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$. In general, evaluation modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$ arise from $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules via the evaluation morphism $E_{\tau,\gamma}(\mathfrak{sl}_N) \to e_{\tau,\gamma}(\mathfrak{sl}_N)$, see Corollary 3.4, analogous to the evaluation homomorphism $Y(\mathfrak{sl}_N) \to U(\mathfrak{sl}_N)$.

In this paper we prove a PBW type theorem for the small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$. We define highest weight modules, Verma modules and contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$. We show that for any finite-dimensional $\mathfrak{sl}_N$-module and any Verma module over $\mathfrak{sl}_N$ one can define the corresponding $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module on the same vector space. Pulling back these $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules through the evaluation morphism we get finite-dimensional evaluation modules and evaluation Verma modules over $E_{\tau,\gamma}(\mathfrak{sl}_N)$. Conjecturally, the same picture takes place for any highest weight $\mathfrak{sl}_N$-module.

We introduce the dynamical Shapovalov form for $e_{\tau,\gamma}(\mathfrak{sl}_N)$, the dynamical Shapovalov pairing and the contravariant form for the highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$. They play an important role in the construction of finite-dimensional highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-modules. From another point of view the contravariant form for $e_{\tau,\gamma}(\mathfrak{sl}_2)$-modules appeared in a disguised form in [TV1, Appendix C].

The small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ admits the trigonometric and rational degenerations. They are closely related to the exchange quantum groups $F_q(SL(N))$ and $F(SL(N))$ introduced in [EV2]. In this paper we consider only the rational dynamical quantum group $e_{\text{rat}}(\mathfrak{sl}_N)$ and its relation to the exchange quantum group $F(SL(N))$. We construct a functor from a certain category of...
\(\mathfrak{sl}_N\)-modules to a category of \(e_{\text{rat}}(\mathfrak{sl}_N)\)-modules, see Theorems 7.5, 7.6, and 9.9. We also establish an equivalence of certain tensor categories of \(e_{\text{rat}}(\mathfrak{sl}_N)\)-modules and rational dynamical representations of \(F(SL(N))\), see Theorem 10.4. In particular, this gives a new construction of highest weight representations for the exchange quantum group \(F(SL(N))\).

Notice that while \(e_{\text{rat}}(\mathfrak{sl}_N)\) and exchange quantum groups have a coproduct structure, no coproduct structure is known for the elliptic quantum group \(e_{\tau, \gamma}(\mathfrak{sl}_N)\). This makes the small elliptic group similar to the Sklyanin algebra \([S], [HW]\).

The paper is organized as follows. After introducing basic notation we recall the definition of the elliptic quantum group \(E_{\tau, \gamma}(\mathfrak{sl}_N)\). The small elliptic quantum group \(e_{\tau, \gamma}(\mathfrak{sl}_N)\) is defined in Section 3. In Section 4 we introduce highest weight modules and Verma modules over \(e_{\tau, \gamma}(\mathfrak{sl}_N)\). The dynamical Shapovalov form for \(e_{\tau, \gamma}(\mathfrak{sl}_N)\) is defined in Section 5. Contragradient modules and the contravariant form for highest weight \(e_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules are defined in Section 6. In Section 7 we study the rational dynamical quantum group \(e_{\text{rat}}(\mathfrak{sl}_N)\). We construct irreducible finite-dimensional \(e_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules in Section 8. A functor from a certain category of \(\mathfrak{sl}_N\)-modules to a category of \(e_{\text{rat}}(\mathfrak{sl}_N)\)-modules is defined in Section 9. Relations between \(e_{\text{rat}}(\mathfrak{sl}_N)\) and \(F(SL(N))\) are studied in Section 10. There are six Appendices in the paper; they contain useful technical information, the \(e_{\tau, \gamma}(\mathfrak{sl}_2)\) example, and some proofs.

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1. Basic notation

Let \(\tau\) be a complex number such that \(\text{Im} \tau > 0\). Let \(\theta(u; \tau)\) be the Jacobi theta function:

\[\theta(u; \tau) = -\sum_{m=-\infty}^{\infty} \exp\left(\pi i \tau (m + 1/2)^2 + 2\pi i (m + 1/2) (u + 1/2)\right)\].

There is a product formula

\[\theta(u; \tau) = 2 e^{\pi i \tau/4} \sin(\pi u) \prod_{s=1}^{\infty} \left(1 - e^{2\pi i s \tau} \right) \left(1 - e^{2\pi i (s \tau + u)} \right) \left(1 - e^{2\pi i (s \tau - u)} \right)\].

The function \(\theta(u; \tau)\) has multipliers \(-1\) and \(-\exp(-2\pi i u - \pi i \tau)\) as \(u \to u + 1\) and \(u \to u + \tau\), respectively. It is an entire function with only simple zeros lying on the lattice \(\mathbb{Z} + \tau \mathbb{Z}\). Usually, we omit the second argument of the theta function, writing \(\theta(u)\) instead of \(\theta(u; \tau)\).

Let \(\mathfrak{h}\) be a finite-dimensional commutative Lie algebra, and let \(\mathfrak{h}^*\) be the dual space. An \(\mathfrak{h}\)-module \(V\) is called diagonalizable if it admits a weight decomposition

\[V = \bigoplus_{\mu \in \mathfrak{h}^*} V[\mu]\],

all weight subspaces \(V[\mu]\) being finite-dimensional and the set \(\{\mu \mid V[\mu] \neq 0\}\) at most countable.

Let \(V_1, \ldots, V_k\) be \(\mathfrak{h}\)-modules. For any function \(f: \mathfrak{h}^* \to \text{End}(V_1 \otimes \ldots \otimes V_k)\) and any \(i = 1, \ldots, k\) we define an operator \(f(h(i)) \in \text{End}(V_1 \otimes \ldots \otimes V_k)\) by the rule:

\[(1.1) \quad f(h(i)) v_1 \otimes \ldots \otimes v_N = f(\mu) v_1 \otimes \ldots \otimes v_N \quad \text{for any } v \in V_1 \otimes \ldots \otimes V_i[\mu] \otimes \ldots \otimes V_k.

For a finite-dimensional vector space \(V\) over \(\mathbb{C}\) denote by \(\text{Fun}(V)\) the space of \(V\)-valued meromorphic functions on \(\mathfrak{h}^*\). If \(V\) is a diagonalizable \(\mathfrak{h}\)-module, set

\[\text{Fun}(V) = \bigoplus_{\mu \in \mathfrak{h}^*} \text{Fun}(V[\mu]).\]

The space \(\text{Fun}(V)\) is a vector space over \(\text{Fun}(\mathbb{C})\). The space \(V\) is naturally embedded in \(\text{Fun}(V)\) as the subspace of constant functions. If \(V\) is an \(\mathfrak{h}\)-module, then \(\text{Fun}(V)\) is an \(\mathfrak{h}\)-module with the natural pointwise action of \(\mathfrak{h}\) and \(\text{Fun}(V)[\mu] = \text{Fun}(V[\mu])\).

Let \(U\) be a diagonalizable \(\mathfrak{h}\)-module and a vector space over \(\text{Fun}(\mathbb{C})\). Suppose that the action of \(\mathfrak{h}\) commutes with multiplication by functions. Then each weight subspace \(U[\mu]\) is a vector space over \(\text{Fun}(\mathbb{C})\). Assume that all the weight subspaces are finite-dimensional over \(\text{Fun}(\mathbb{C})\). Then one can
define a diagonalizable \( h \)-module \( V \), such that \( U = \text{Fun}(V) \) as \( h \)-modules, in the following way. For any \( \mu \) such that \( U[\mu] \neq 0 \), pick up a basis \( f_1, \ldots, f_k \) of \( U[\mu] \) over \( \text{Fun}(\mathbb{C}) \) and set \( V[\mu] = \bigoplus_{i=1}^{k} \mathbb{C} f_i \), otherwise, set \( V[\mu] = 0 \). Then define \( V = \bigoplus_{\mu \in h^*} V[\mu] \) to be the diagonalizable \( h \)-module such that \( V[\mu] \) is a weight subspace of weight \( \mu \).

Let \( V, W \) be diagonalizable \( h \)-modules. The space \( \text{Hom}(V, W) \) has the natural \( h \)-module structure, but in general the weight subspaces are infinite-dimensional. We set

\[
\text{Fun}(\text{Hom}(V, W)) = \text{Hom}(V, \text{Fun}(W)).
\]

A function \( \varphi \in \text{Fun}(\text{Hom}(V, W)) \) induces a linear map \( \text{Fun}(V) \to \text{Fun}(W) \), acting pointwise: \( f(\lambda) \mapsto \varphi(\lambda)f(\lambda) \). This map is usually denoted by the same letter.

Denote by \( D(V) \) the space of difference operators acting in \( \text{Fun}(V) \). It is spanned over \( \mathbb{C} \) by operators of the form \( f(\lambda) \mapsto \varphi(\lambda)f(\lambda + \mu) \) where \( \varphi \in \text{Fun}(\text{End}(V)) \) and \( \mu \in h^* \).

As a rule we do not distinguish a function \( \varphi(\lambda) \in \text{Fun}(\mathbb{C}) \) and the function \( \varphi(\lambda)\text{id} \in \text{Fun}(\text{End}(V)) \).

In this paper we take \( h \) to be the Cartan subalgebra of the Lie algebra \( \mathfrak{sl}_2 \). Fix a basis \( h_1, \ldots, h_{N-1} \) of \( h \). Let \( \omega_1, \ldots, \omega_{N-1} \) be the fundamental weights: \( \langle \omega_a, h_b \rangle = \delta_{ab} \). Let \( \mathbb{P} \subset h^* \) be the weight lattice: \( \mathbb{P} = \bigoplus_{a=1}^{N} \mathbb{Z} \omega_a \). For any \( a = 1, \ldots, N \), set \( \varepsilon_a = \omega_a - \omega_{a-1} \), where by convention \( \omega_0 = \omega_N = 0 \). Let \( \alpha_1, \ldots, \alpha_{N-1} \) be the simple roots: \( \alpha_a = \varepsilon_a - \varepsilon_{a+1} \). For \( \lambda, \mu \in h^* \) say that \( \lambda \leq \mu \) if \( \lambda - \mu \in \bigoplus_{a=1}^{N-1} \mathbb{Z}_{\geq 0} \alpha_a \).

Define a bilinear form \( (, ) \) on \( h^* \) by the rule \( (\alpha_a, \omega_b) = \delta_{ab} \) for any \( a, b = 1, \ldots, N-1 \). For any \( \lambda \in h^* \) set \( \lambda_a = (\lambda, \varepsilon_a) \). It is easy to see that \( \lambda = \sum_{a=1}^{N} \lambda_a \varepsilon_a \), \( \sum_{a=1}^{N} \lambda_a = 0 \) and \( (\lambda, \mu) = \sum_{a=1}^{N} \lambda_a \mu_a \). The Weyl group \( W \) acts on \( h^* \) as the symmetric group \( S_N \) permuting the coordinates \( \lambda_1, \ldots, \lambda_N \).

Let \( \rho = \sum_{a=1}^{N-1} \omega_a = -\sum_{a=1}^{N} a \varepsilon_a \) be the half-sum of positive roots. For any \( w \in W \) and \( \lambda \in h^* \) set \( w \cdot \lambda = w(\lambda + \rho) - \rho \). Notice that \((\lambda, \rho) = -\lambda_1 - 2\lambda_2 - \ldots - N\lambda_N \).

Let \( E_{ab} \in \text{End}(\mathbb{C}^N) \) be the matrix with the only nonzero entry equal to 1 at the intersection of the \( a \)-th row and \( b \)-th column. The assignment \( h_a \mapsto E_{aa} - E_{a+1,a+1} \), \( a = 1, \ldots, N-1 \), makes \( \mathbb{C}^N \) into an \( h \)-module, called the vector representation of \( h \). Henceforth, we always consider \( \mathbb{C}^N \) as the vector representation of \( h \).

Let \( \gamma \) be a nonzero complex number. Introduce functions \( \alpha(u, \xi) \) and \( \beta(u, \xi) \) as follows:

\[
\alpha(u, \xi) = \frac{\theta(u) \theta(\xi + \gamma)}{\theta(u - \gamma) \theta(\xi)}, \quad \beta(u, \xi) = -\frac{\theta(u + \xi) \theta(\gamma)}{\theta(u - \gamma) \theta(\xi)}.
\]

Let \( R(u, \lambda) \) be the elliptic dynamical \( R \)-matrix [F]:

\[
R(u, \lambda) = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a,b=1}^{N} (\alpha(u, \lambda_{ab}) E_{aa} \otimes E_{bb} + \beta(u, \lambda_{ab}) E_{ab} \otimes E_{ba}),
\]

where \( \lambda \in h^* \) and \( \lambda_{ab} = \lambda_a - \lambda_b \). The dynamical \( R \)-matrix has zero weight:

\[
[R(u, \lambda), h^{(1)} + h^{(2)}] = 0,
\]

satisfies the inversion relation:

\[
R(u, \lambda) R^{(21)}(-u, \lambda) = 1,
\]

and the dynamical Yang-Baxter equation:

\[
R^{(12)}(u - v, \lambda - \gamma h^{(3)}) R^{(13)}(u, \lambda) R^{(23)}(v, \lambda - \gamma h^{(1)}) = R^{(23)}(v, \lambda) R^{(13)}(u, \lambda - \gamma h^{(2)}) R^{(12)}(u - v, \lambda).
\]

The last equality holds in \( \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N) \). By standard convention, we assume that \( R^{(ij)}(u, \lambda) \) acts as \( R(u, \lambda) \) on the \( i \)-th and \( j \)-th tensor factors and as the identity operator on the remaining factors.
For instance, in formula (1.6) we have $R^{(12)} = R \otimes \text{id}$ and $R^{(23)} = \text{id} \otimes R$. Notice that for the $R$-matrix (1.3) in addition we have

$$R(u, \lambda - \gamma (h^{(1)} + h^{(2)})) = R(u, \lambda).$$

2. Elliptic quantum group $E_{r, \gamma}(\mathfrak{sl}_N)$

A module over the elliptic quantum group $E_{r, \gamma}(\mathfrak{sl}_N)$ is a diagonalizable $\mathfrak{h}$-module $V$ together with $D(V)$-valued meromorphic functions $T_{ab}(u), \ a, b = 1, \ldots, N$, in a complex variable $u$, subject to relations (2.1) – (2.3). We combine the functions $T_{ab}(u)$ into a matrix $T(u)$ with noncommuting entries:

$$T(u) = \sum_{a,b} E_{ab} \otimes T_{ab}(u).$$

The defining relations are:

$$T_{ab}(u) \varphi(\lambda) = \varphi(\lambda - \gamma \varepsilon_b) T_{ab}(u)$$

for any $\varphi \in \text{Fun}(\mathbb{C})$.

$$(2.2) R^{(12)}(u - v, \lambda - \gamma h^{(3)}) T^{(13)}(u) T^{(23)}(v) = T^{(23)}(v) T^{(13)}(u) R^{(12)}(u - v, \lambda).$$

The last equality holds in $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \text{Fun}(V))$. Here $T^{(13)}(u) = \sum_{a,b} E_{ab} \otimes \text{id} \otimes T_{ab}(u)$ and $T^{(23)}(u) = \sum_{a,b} \text{id} \otimes E_{ab} \otimes T_{ab}(u)$.

Relations (2.1) can be written as $T(u) \varphi(\lambda + \gamma h^{(1)}) = \varphi(\lambda) T(u)$ for any $\varphi \in \text{Fun}(\mathbb{C})$. Formula (2.2) means that for any $\mu \in \mathfrak{h}^*$

$$T_{ab}(u) \text{Fun}(V[\mu]) \subset \text{Fun}(V[\mu - \epsilon_a + \epsilon_b]).$$

Introduce the quantum determinant $\text{Det} T(u)$, cf. [FV1], [FV2], by the rule

$$(2.5) \text{Det} T(u) = \frac{\Theta(\lambda)}{\Theta(\lambda - \gamma h)} \sum_{i \in S_N} \text{sign}(i) T_{i,N}(u + (N - 1)\gamma) \ldots T_{i_2,2}(u + \gamma) T_{i_1,1}(u)$$

where $\Theta(\lambda) = \prod_{1 \leq a < b \leq N} \theta(\lambda_a - \lambda_b)$, the sum is taken over all permutations $i = (i_1, \ldots, i_N)$, and $\text{sign}(i)$ is the sign of the permutation. It is clear that $\text{Det} T(u)$ commutes with multiplication by any function $\varphi(\lambda) \in \text{Fun}(\mathbb{C})$ and with the action of $\mathfrak{h}$. Hence, $\text{Det} T(u)$ acts on $\text{Fun}(V)$ as multiplication by an $\text{End}(V)$-valued meromorphic function of $\mu$ and $\lambda$. We denote this function by $\text{Det} L(u, \lambda)$, cf. (2.6).

**Proposition 2.1.** $[\text{Det} T(u), T_{ab}(v)] = 0$ for any $a, b = 1, \ldots, N$.

The proposition is proved in Appendix B.

According to (2.1), $T_{ab}(u)$ is a difference operator; for any $v \in \text{Fun}(V)$ we have

$$(2.6) (T_{ab}(u) v)(\lambda) = L_{ab}(u, \lambda) v(\lambda - \gamma \varepsilon_b)$$

where $L_{ab}(u, \lambda) \in \text{Fun}(\text{End}(V))$. Set $L(u, \lambda) = \sum_{a,b} E_{ab} \otimes L_{ab}(u, \lambda)$.

**Example.** For any $x \in \mathbb{C}$ the assignment $L(u, \lambda) \mapsto R(u - x, \lambda)$ makes $\mathbb{C}^N$ into an $E_{r, \gamma}(\mathfrak{sl}_N)$-module. This module is called the vector representation of $E_{r, \gamma}(\mathfrak{sl}_N)$ with the evaluation point $x$. The quantum determinant in this $E_{r, \gamma}(\mathfrak{sl}_N)$-module is

$$\text{Det} L(u, \lambda) = \frac{\theta(u - x + (N - 1)\gamma)}{\theta(u - x)}.$$
for any \(a, b = 1, \ldots, N\). Denote by \(\text{Mor}(V, W)\) the space of all morphisms from \(V\) to \(W\). A morphism \(\varphi\) is called an isomorphism if the map \(\varphi(\lambda)\) is bijective for generic \(\lambda\).

An \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-module \(V\) is called irreducible if for any nontrivial morphism \(\varphi \in \text{Mor}(W, V)\) the map \(\varphi(\lambda)\) is surjective, and reducible otherwise.

Let \(V, W\) be \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules. Then the \(\mathfrak{h}\)-module \(V \otimes W\) is made into an \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-module by

\[
L_{ab}(u, \lambda)|_{V \otimes W} = \sum_{c=1}^{N} L_{ac}(u, \lambda - \gamma h^{(2)}) \otimes L_{cb}(u, \lambda),
\]

and \(T_{ab}(u)\) acts on \(\text{Fun}(V \otimes W)\) according to (2.6). Triple tensor products \((U \otimes V) \otimes W\) and \(U \otimes (V \otimes W)\) are canonically isomorphic as \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules. The quantum determinant is group-like, it acts on the \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-module \(V \otimes W\) by

\[
\det L(u, \lambda)|_{V \otimes W} = \det L(u, \lambda - \gamma h^{(2)}) \otimes \det L(u, \lambda).
\]

If \(\varphi_1, \varphi_2\) are morphisms of \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules, \(\varphi_1 \in \text{Mor}(V_1, W_1), \varphi_2 \in \text{Mor}(V_2, W_2)\), then

\[
\varphi_1(\lambda - \gamma h^{(2)}) \otimes \varphi_2(\lambda)
\]

is a morphism of \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules \(V_1 \otimes V_2\) and \(W_1 \otimes W_2\).

**Remark.** Notice that the given definition of modules over the elliptic quantum group \(E_{\tau, \gamma}(\mathfrak{sl}_N)\) is slightly different from the corresponding definition in [FV1]. The definition of morphisms of \(E_{\tau, \gamma}(\mathfrak{sl}_N)\)-modules is also suitably modified. We choose the present version in order to simplify the exposition.

### 3. Small elliptic quantum group \(e_{\tau, \gamma}(\mathfrak{sl}_N)\)

Let \(\text{Fun}^{\otimes 2}(\mathbb{C})\) be the ring of meromorphic functions \(f(\lambda^{(1)}, \lambda^{(2)})\) on \(\mathfrak{h}^* \oplus \mathfrak{h}^*\) such that location of singularities of \(f(\lambda^{(1)}, \lambda^{(2)})\) in \(\lambda^{(1)}\) does not depend on \(\lambda^{(2)}\) and vice versa. For brevity, we write \(f(\lambda^{(1)})\) or \(f(\lambda^{(2)})\) instead of \(f(\lambda^{(1)}, \lambda^{(2)})\) if the function does not depend on the other variable.

Given a diagonalizable \(\mathfrak{h}\)-module \(V\) we define an action of \(\text{Fun}^{\otimes 2}(\mathbb{C})\) in \(\text{Fun}(V)\): for any \(f \in \text{Fun}^{\otimes 2}(\mathbb{C})\) set \(f : v(\lambda) \mapsto f(\lambda, \lambda - \gamma h) v(\lambda)\). For instance, for any \(\varphi \in \text{Fun}(\mathbb{C})\) we have

\[
\varphi(\lambda^{(1)}) : v(\lambda) \mapsto \varphi(v(\lambda)), \quad \varphi(\lambda^{(2)}) : v(\lambda) \mapsto \varphi(\lambda - \gamma h) v(\lambda).
\]

We always assume that \(\text{Fun}^{\otimes 2}(\mathbb{C})\) acts on \(\text{Fun}(V)\) in this way.

The operator algebra \(e_{\tau, \gamma}^{\otimes 2}(\mathfrak{sl}_N)\) is a unital associative algebra over \(\mathbb{C}\) generated by elements \(t_{ab}\), \(a, b = 1, \ldots, N\), and functions \(f \in \text{Fun}^{\otimes 2}(\mathbb{C})\) subject to relations

\[
t_{ab} f(\lambda^{(1)}, \lambda^{(2)}) = f(\lambda^{(1)} - \gamma e_a, \lambda^{(2)} - \gamma e_b) t_{ab}
\]

for any \(f \in \text{Fun}^{\otimes 2}(\mathbb{C})\),

\[
t_{ab} t_{ac} = t_{ac} t_{ab},
\]

\[
t_{ac} t_{bc} = \frac{\theta(\lambda^{(1)}_{ac} + \gamma)}{\theta(\lambda^{(1)}_{ab} - \gamma)} t_{bc} t_{ac}, \quad \text{for} \quad a \neq b,
\]

\[
\frac{\theta(\lambda^{(2)}_{bd} + \gamma)}{\theta(\lambda^{(2)}_{bd})} t_{ab} t_{cd} = \frac{\theta(\lambda^{(1)}_{ac} + \gamma)}{\theta(\lambda^{(1)}_{ac})} t_{cd} t_{ab} = \frac{\theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd}) \theta(\gamma)}{\theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd})} t_{ab} t_{cd},
\]

for \(a \neq c\) and \(b \neq d\). Here \(\lambda^{(i)}_{ab} = \lambda^{(i)}_a - \lambda^{(i)}_b\).

The ring \(\text{Fun}^{\otimes 2}(\mathbb{C})\) is embedded into \(e_{\tau, \gamma}^{\otimes 2}(\mathfrak{sl}_N)\) as a commutative subalgebra. It acts on \(e_{\tau, \gamma}^{\otimes 2}(\mathfrak{sl}_N)\) by left multiplication. In this paper we consider \(e_{\tau, \gamma}^{\otimes 2}(\mathfrak{sl}_N)\) as the corresponding \(\text{Fun}^{\otimes 2}(\mathbb{C})\)-module.
The operator algebras $e_{\tau,\gamma}^\circ (\mathfrak{sl}_N)$, $e_{\tau+1,\gamma}^\circ (\mathfrak{sl}_N)$ and $e_{1/\tau,-\gamma/\tau}^\circ (\mathfrak{sl}_N)$ are isomorphic. The isomorphism $e_{\tau,\gamma}^\circ (\mathfrak{sl}_N) \to e_{\tau+1,\gamma}^\circ (\mathfrak{sl}_N)$ corresponds to the property $\theta_{(u;\tau)} = e^{\pi i/4} \theta(u;\tau + 1)$ of the theta function and is tautological. The isomorphism $e_{\tau,\gamma}^\circ (\mathfrak{sl}_N) \to e_{1/\tau,-\gamma/\tau}^\circ (\mathfrak{sl}_N)$ corresponds to the equality

$$\theta(u; -1/\tau) = (i\tau)^{1/2} \exp(\pi i\tau u^2) \theta(-\tau u; \tau), \quad \text{Im}(i\tau)^{1/2} > 0,$$

and is given by the following formulae

$$f(\lambda^{[1]},\lambda^{[2]}) \mapsto f(-\tau\lambda^{[1]}, -\tau\lambda^{[2]}),$$

$$t_{ab} \mapsto \exp\left(\frac{\pi i\tau}{2N} \left((N + 1) \sum_{c=1}^N (\lambda_{ac}^{[1]})^2 - (N - 1) \sum_{c=1}^N (\lambda_{ac}^{[2]})^2 - 4N\lambda_{a}^{[1]}\lambda_{b}^{[2]}\right)\right) t_{ab}.$$

Introduce a matrix $\mathcal{T}(u)$ with noncommuting entries:

$$\mathcal{T}(u) = \sum_{a,b} E_{ab} \otimes \mathcal{T}_{ab}(u), \quad \mathcal{T}_{ab}(u) = \theta(u - \lambda_a^{[1]} + \lambda_b^{[2]}) t_{ba}. \quad (3.6)$$

**Theorem 3.1.** The commutation relations (3.3)–(3.5) are equivalent to

$$R^{(12)}(u - v, \lambda^{[2]}) \mathcal{T}^{(13)}(u) \mathcal{T}^{(23)}(v) = \mathcal{T}^{(23)}(v) \mathcal{T}^{(13)}(u) R^{(12)}(u - v, \lambda^{[1]}), \quad (3.7)$$

The proof is straightforward and is based on summation formulae for the theta function. Notice that formula (3.7) is similar to (2.3).

Introduce the quantum determinant $\text{Det} \mathcal{T}(u)$ like in (2.5):

$$\text{Det} \mathcal{T}(u) = \frac{\Theta(\lambda^{[1]})}{\Theta(\lambda^{[2]})} \sum_{i \in S_N} \text{sign}(i) \mathcal{T}_{i,N,N}(u + (N - 1)\gamma) \ldots \mathcal{T}_N(2u + (N - 1)\gamma) \mathcal{T}_{N,1}(u) \quad (3.8)$$

where $\Theta(\lambda) = \prod_{1 \leq a < b \leq N} \theta(\lambda_a - \lambda_b)$. By (3.6) and (3.2) we obtain

$$\text{Det} \mathcal{T}(u) = \frac{\Theta(\lambda^{[1]})}{\Theta(\lambda^{[2]})} \sum_{i \in S_N} \text{sign}(i) \prod_{a=1}^N \theta(u + (a - 1)\gamma - \lambda_a^{[1]} + \lambda_b^{[2]}) t_{N,i,N} \ldots t_{1,1}. \quad (3.9)$$

It is clear that $\text{Det} \mathcal{T}(u) f(\lambda^{[1]},\lambda^{[2]}) = f(\lambda^{[1]},\lambda^{[2]}) \text{Det} \mathcal{T}(u)$ for any $f \in \text{Fun}^{\otimes 2}(\mathbb{C})$.

**Proposition 3.2.** $[\text{Det} \mathcal{T}(u), t_{ab}] = 0$ for any $a, b = 1, \ldots, N$.

The proof is similar to the proof of Proposition 2.1.

Let $\tilde{f} = (f_1, \ldots, f_N)$ be a multiplicative cocycle with coefficients in $\text{Fun}(\mathbb{C})$, that is, the functions $f_1, \ldots, f_N \in \text{Fun}(\mathbb{C})$ satisfy the condition

$$f_a(\lambda) f_b(\lambda - \gamma \varepsilon_a) = f_a(\lambda - \gamma \varepsilon_b) f_b(\lambda)$$

for any $a, b = 1, \ldots, N$, cf. Appendix C. Then the assignments

$$t_{ab} \mapsto f_a(\lambda^{[1]}) t_{ab} \quad \text{and} \quad t_{ab} \mapsto f_b(\lambda^{[2]}) t_{ab} \quad (3.10)$$

define two endomorphisms of the operator algebra $e_{\tau,\gamma}^\circ (\mathfrak{sl}_N)$. The endomorphisms are automorphisms if $\tilde{f}$ is nondegenerate, that is, if $f_a \neq 0$ for any $a = 1, \ldots, N$. The automorphisms are inner if $\tilde{f}$ is a multiplicative coboundary: $f_a(\lambda) = \varphi(\lambda - \gamma \varepsilon_a) / \varphi(\lambda)$ for a certain function $\varphi \in \text{Fun}(\mathbb{C})$, see (3.2).

**Remark.** In this paper the inequality $f \neq 0$ for a meromorphic function $f$ means that the function $f$ is not identically zero.
Proposition 3.3. The assignment
\[
(3.11) \quad f(\lambda^{(1)}, \lambda^{(2)}) \mapsto f(-\lambda^{(2)}, -\lambda^{(1)}),
\]
tab \mapsto \prod_{c=1 \atop c \neq b}^{N} \theta(\lambda_{cb}^{(1)}) \prod_{c=1 \atop c \neq a}^{N} (\theta(\lambda_{ca}^{(2)} + \gamma))^{-1} t_{ba}
\]
defines an involutive antiautomorphism of the operator algebra \( e_{\tau,\gamma}^e(sl_N) \).

A module over the small elliptic quantum group \( e_{\tau,\gamma}^e(sl_N) \) is a diagonalizable \( \mathfrak{h} \)-module \( V \) endowed with an action of the operator algebra \( e_{\tau,\gamma}^e(sl_N) \) in the space \( \text{Fun}(V) \). Relations (3.1) and (3.2) mean that \( t_{ab} \) acts on \( \text{Fun}(V) \) as a difference operator:
\[
(3.12) \quad (t_{ab} v)(\lambda) = \ell_{ab}(\lambda) v(\lambda - \gamma \varepsilon_a) \quad \text{for any } v \in \text{Fun}(V)
\]
where \( \ell_{ab}(\lambda) \in \text{Fun} \left( \text{End}(V) \right) \) is a suitable function. In \( D(V) \) relations (3.1), (3.2) are equivalent to
\[
\ell_{ab}(\lambda) = \varphi(\lambda - \gamma \varepsilon_a) t_{ab}
\]
for any \( \varphi \in \text{Fun}(\mathbb{C}) \), and
\[
t_{ab} \text{Fun}(V[\mu]) \subset \text{Fun}(V[\mu + \varepsilon_a - \varepsilon_b])
\]
for any \( \mu \in \mathfrak{h}^* \), which are similar to (2.1) and (2.4). For the quantum determinant we have
\[
\left( \text{Det} T(u) v \right)(\lambda) = D(u, \lambda) v(\lambda)
\]
for any \( v \in \text{Fun}(V) \) where \( D(u, \lambda) \in \text{Fun} \left( \text{End}(V) \right) \) is a suitable function.

Example. The assignment
\[
(3.13) \quad \ell_{aa}(\lambda) \mapsto E_{aa} + \sum_{b=1 \atop b \neq a}^{N} \frac{\theta(\lambda_{ab} + \gamma)}{\theta(\lambda_{ab})} E_{bb},
\]
\[
\ell_{ab}(\lambda) \mapsto \frac{\theta(\gamma)}{\theta(\lambda_{ab})} E_{ab}, \quad a \neq b,
\]
a, b = 1, \ldots, N, makes \( \mathbb{C}^N \) into an \( e_{\tau,\gamma}(sl_N) \)-module. The module is called the vector representation of \( e_{\tau,\gamma}(sl_N) \). In the vector representation
\[
D(u, \lambda) = \theta(u - \gamma) \theta(u + \gamma) \theta(u + 2\gamma) \cdots \theta(u + (N - 1)\gamma).
\]

Corollary 3.4. For any \( e_{\tau,\gamma}(sl_N) \)-module \( V \) and any \( x \in \mathbb{C} \) the rule \( T_{ab}(u) = T_{ab}(u - x)|_V \) makes \( V \) into an \( E_{\tau,\gamma}(sl_N) \)-module called the evaluation module \( V(x) \) over \( E_{\tau,\gamma}(sl_N) \).

By abuse of notation we call the assignment \( T(u) \mapsto T(u) \) the evaluation morphism \( E_{\tau,\gamma}(sl_N) \to e_{\tau,\gamma}(sl_N) \). It is analogous to the evaluation homomorphism from the Yangian \( Y(sl_N) \) to \( U(sl_N) \).

Remark. Corollary 3.4 was our main motivation to discover and study the small elliptic quantum group \( e_{\tau,\gamma}(sl_N) \).

Let \( V, W \) be \( e_{\tau,\gamma}(sl_N) \)-modules. An element \( \varphi \in \text{Fun}(\text{Hom}_\mathfrak{h}(V, W)) \) is a morphism of \( e_{\tau,\gamma}(sl_N) \) modules if the induced map intertwines the corresponding actions of \( e_{\tau,\gamma}(sl_N) \):
\[
\varphi(\lambda) t_{ab} |_{\text{Fun}(V)} = t_{ab} |_{\text{Fun}(W)} \varphi(\lambda)
\]
for any \( a, b = 1, \ldots, N \). Denote by \( \text{Mor}(V, W) \) the space of all morphisms from \( V \) to \( W \). A morphism \( \varphi \) is called an isomorphism if the map \( \varphi(\lambda) \) is bijective for generic \( \lambda \).
An \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V \) is called irreducible if for any nontrivial morphism \( \varphi \in \text{Mor}(W,V) \) the map \( \varphi(\lambda) \) is surjective for generic \( \lambda \), and reducible otherwise.

Say that an \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( W \) is a submodule of \( V \) if there is a morphism \( \varphi \in \text{Mor}(W,V) \) such that the map \( \varphi(\lambda) \) is injective for generic \( \lambda \). The morphism \( \varphi \) is called an embedding. The submodule \( W \) is called proper if \( \varphi \) is not an isomorphism. Any \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V \) has at least two submodules: \( V \) itself and the trivial submodule \( \{0\} \) with obvious embeddings.

Let \( W \) be a submodule of \( V \). Then one can define the quotient \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V/W \) as follows. Fix an embedding \( \varphi \). The subspace \( \varphi(\text{Fun}(W)) \subset \text{Fun}(V) \) is invariant with respect to the action of \( e_{\tau,\gamma}(\mathfrak{sl}_N) \), hence \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) acts on \( \text{Fun}(V)/\varphi(\text{Fun}(W)) \). Let \( \lambda_0 \in \mathfrak{h}^* \) be such that the map \( \varphi(\lambda_0) \) is injective. Take a complement \( U \) of \( \varphi(\lambda_0)W \) in \( V \), that is, \( V = U \oplus \varphi(\lambda_0)W \) as a vector space. Notice that \( V = U \oplus \varphi(\lambda)W \) for generic \( \lambda \) as well. Then \( \text{Fun}(V) = \text{Fun}(U) \oplus \varphi(\text{Fun}(W)) \) and, therefore, \( \text{Fun}(U) = \text{Fun}(V)/\varphi(\text{Fun}(W)) \), which induces an action of \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) on \( \text{Fun}(U) \) and makes \( U \) into an \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module. The constructed \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module does not depend on a choice \( \varphi \), \( \lambda_0 \) and \( U \) up to an isomorphism of \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-modules and is denoted by \( V/W \).

**Lemma 3.5.** An \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V \) is reducible if and only if it has a nontrivial proper submodule.

**Lemma 3.6.** An \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V \) is irreducible if and only if for any nontrivial morphism \( \varphi \in \text{Mor}(V,W) \) the map \( \varphi(\lambda) \) is injective for generic \( \lambda \).

4. **Highest weight modules over \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)**

For any monomial \( t_{a_1b_1} \cdots t_{a_kb_k} \) set \( \text{deg}(t_{a_1b_1} \cdots t_{a_kb_k}) = k \). For \( k = 0 \) we assume that the monomial equals 1. As a \( \text{Fun}^{\otimes 2}(\mathbb{C}) \)-module the operator algebra \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) is generated by all monomials \( t_{a_1b_1} \cdots t_{a_kb_k} \), \( k = 0, 1, \ldots \).

For any function \( \varphi \in \text{Fun}^{\otimes 2}(\mathbb{C}) \) set \( \text{deg}(\varphi) = 0 \). Since relations (3.2)–(3.5) are homogeneous, the algebra \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) is \( \mathbb{Z}_{\geq 0} \)-graded by \( \text{deg} \). Let \( \mathcal{C}_k = \{ x \in e_{\tau,\gamma}(\mathfrak{sl}_N) \mid \text{deg}(x) = k \} \) be the homogeneous subspace of degree \( k \). Each subspace \( \mathcal{C}_k \) is finitely generated over \( \text{Fun}^{\otimes 2}(\mathbb{C}) \).

Consider the normal ordering of generators: \( t_{ab} < t_{cd} \) if \( a - b < c - d \), or \( a - b = c - d \) and \( a < c \). Say that the monomial \( t_{a_1b_1} \cdots t_{a_kb_k} \) is normally ordered if \( t_{a_ib_i} < t_{a_jb_j} \) for any \( i < j \), or \( k = 0 \).

**Theorem 4.1.** For any \( k \in \mathbb{Z}_{\geq 0} \) the normally ordered monomials of degree \( k \) form a basis of \( \mathcal{C}_k \) over \( \text{Fun}^{\otimes 2}(\mathbb{C}) \).

**Proof.** For \( k = 0 \) and \( k = 1 \) the claim is immediate. Let \( k > 1 \). Here we prove that the normally ordered monomials of degree \( k \) span \( \mathcal{C}_k \) over \( \text{Fun}^{\otimes 2}(\mathbb{C}) \). The linear independence of the normally ordered monomials is proved in Appendix D.

It is clear from relations (3.3)–(3.5) that any product \( t_{ab}t_{cd} \) can be written as a linear combination of normally ordered products. Given a monomial \( t_{a_1b_1} \cdots t_{a_kb_k} \) we take any disordered product of adjacent factors and replace it by a suitable sum of normally ordered products, then do the same for each of the obtained monomials. To see that the procedure terminates and, hence, produces a linear combination of normally ordered monomials, introduce auxiliary gradings on monomials by the rule

\[
r(t_{a_1b_1} \cdots t_{a_kb_k}) = \sum_{i=1}^{k} i(a_i - b_i), \quad r'(t_{a_1b_1} \cdots t_{a_kb_k}) = \sum_{i=1}^{k} i b_i,
\]

and observe that at each nontrivial step of the procedure we replace a monomial by a sum of monomials of either less degree \( r \), or the same degree \( r \) and less degree \( r' \).

Introduce modified generators of the algebra \( e_{\tau,\gamma}(\mathfrak{sl}_N) \). For any \( a, b = 1, \ldots, N \) set

\[
\hat{t}_{ab} = \prod_{1 \leq c < a} \theta(\lambda^{(1)}_{cb}) \prod_{1 \leq c < b} (\theta(\lambda^{(2)}_{cb}))^{-1} t_{ab}.
\]

Let \( V \) be an \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module. A nonzero function \( v \in \text{Fun}(V) \) is called a singular vector if \( t_{ab}v = 0 \) for any \( 1 \leq a < b \leq N \). Say that \( v \) is a regular singular vector if, in addition, \( v \) is a weight vector with respect to the action of \( \mathfrak{h} \) and

\[
(t_{aa}v)(\lambda) = Q_a(\lambda) v(\lambda), \quad a = 1, \ldots, N,
\]

(4.2)
for certain functions \( Q_1, \ldots, Q_N \in \text{Fun}(\mathbb{C}) \). We call \( \widehat{Q} = (Q_1, \ldots, Q_N) \) the dynamical weight of \( v \).

Relation (A.2) implies that \( \widehat{Q} \) is a multiplicative cocycle:

\[
Q_a(\lambda) Q_b(\lambda - \gamma \varepsilon_a) = Q_a(\lambda - \gamma \varepsilon_b) Q_b(\lambda)
\]

for any \( a, b = 1, \ldots, N \). If \( f \in \text{Fun}(\mathbb{C}) \), then the function \( \widetilde{v}(\lambda) = f(\lambda) v(\lambda) \) is a regular singular vector of dynamical weight \((\widehat{Q}_1, \ldots, \widehat{Q}_N)\) where

\[
\widehat{Q}_a(\lambda) = Q_a(\lambda) \frac{f(\lambda - \gamma \varepsilon_a)}{f(\lambda)}, \quad a = 1, \ldots, N.
\]

Hence, the subspace \( \text{Fun}(\mathbb{C}) v \) determines the dynamical weight up to a multiplicative coboundary.

Say that \( \widehat{Q} \) and \( v \) are nondegenerate if \( Q_a \neq 0 \) for any \( a = 1, \ldots, N \). Say that \( \widehat{Q} \) and \( v \) are standard if \( Q_a = 1 \) for any \( a = 1, \ldots, N \).

By formula (3.9) the quantum determinant acts on a regular singular vector \( v \) of weight \( \mu \) and dynamical weight \( \widehat{Q} \) as follows:

\[
(\text{Det} \mathcal{T}(u) v)(\lambda) = \prod_{a=1}^{N} \theta(u - \gamma(\mu_a - a + 1)) \prod_{a=1}^{N} Q_a(\lambda - \sum_{a<b\leq N} \gamma \varepsilon_b) v(\lambda).
\]

An \( e_{\tau, \gamma}(\mathfrak{sl}_N) \)-module \( V \) is called a highest weight module with highest weight \( \mu \), dynamical highest weight \( \widehat{Q} \) and highest weight vector \( v \) if \( v \) is a regular singular vector of weight \( \mu \) and dynamical weight \( \widehat{Q} \) generating \( \text{Fun}(V) \) over \( e_{\tau, \gamma}(\mathfrak{sl}_N) \). If \( \widehat{Q} \) is standard (nondegenerate), then \( V \) is called a standard (nondegenerate) \( e_{\tau, \gamma}(\mathfrak{sl}_N) \)-module of highest weight \( \mu \). For example, the vector representation is a standard \( e_{\tau, \gamma}(\mathfrak{sl}_N) \)-module of highest weight \( \omega_1 \).

It is clear that any nondegenerate highest weight module is isomorphic to a pullback of a standard highest weight module of the same highest weight through a suitable automorphism of the form (3.10).

**Proposition 4.2.** Let \( V \) be a highest weight module with highest weight \( \mu \). Then

a) \( V = \bigoplus_{\nu \leq \mu} V[\nu] \) and \( \dim \mathcal{C} V[\mu] = 1 \);

b) \( V \) is reducible if and only if it has a singular vector of weight \( \nu < \mu \).

**Lemma 4.3.** Let \( V \) be a highest weight module of highest weight \( \mu \) and dynamical highest weight \((Q_1, \ldots, Q_N)\). Then for any \( v \in \text{Fun}(V) \)

\[
(\text{Det} \mathcal{T}(u) v)(\lambda) = \prod_{a=1}^{N} \theta(u - \gamma(\mu_a - a + 1)) \prod_{a=1}^{N} Q_a(\lambda - \sum_{a<b\leq N} \gamma \varepsilon_b) v(\lambda).
\]

The statement follows from Proposition 3.2 and formula (4.3).

**Corollary 4.4.** Let \( \gamma \notin \mathbb{Q} + \tau \mathbb{Q} \). Then a nondegenerate highest weight module \( V \) of highest weight \( \mu \) is irreducible unless \( V[w \cdot \mu] \neq 0 \) for some \( w \in W \) such that \( w \cdot \mu < \mu \). Here \( W \) is the Weyl group.

**Proof.** If \( V \) is reducible, then it has a singular vector \( v \) of weight \( \nu < \mu \). Comparing formulae (4.3) and (4.4) for the action of \( \text{Det} \mathcal{T}(u) \) on \( v \) we obtain that \((\nu_1 - 1, \ldots, \nu_N - N) = (\mu_1 - i_1, \ldots, \mu_N - i_N) \) for some permutation \((i_1, \ldots, i_N)\), since \( \gamma \notin \mathbb{Q} + \tau \mathbb{Q} \). That is, \( \nu = w(\mu + \rho) - \rho = w \cdot \mu \) for some \( w \in W \) because \( \rho = -\varepsilon_1 - 2\varepsilon_2 - \ldots - N\varepsilon_N \).

Let \( e(\mathfrak{b}_+ \rangle \) and \( e(\mathfrak{n}_+) \) be the left ideals in \( e_{\tau, \gamma}(\mathfrak{sl}_N) \) generated by the elements \( t_{ab} \) with \( a \leq b \) and \( a < b \), respectively. Let \( B, B', N, N' \) be the following sets of normally ordered monomials

\[
B = \{ t_{a_1b_1} \cdots t_{akbk} \mid k \geq 0, \ a_i \geq b_i, \ i = 1, \ldots, k \},
\]

\[
B' = \{ t_{a_1b_1} \cdots t_{akbk} \mid k > 0, \ a_i \leq b_i, \ i = 1, \ldots, k \},
\]

\[
N = \{ t_{a_1b_1} \cdots t_{akbk} \mid k \geq 0, \ a_i > b_i, \ i = 1, \ldots, k \},
\]

\[
N' = \{ t_{a_1b_1} \cdots t_{akbk} \mid k > 0, \ a_i < b_i, \ i = 1, \ldots, k \}.
\]
Lemma 4.5. The normally ordered monomials of the form \( mm' \) where \( m \in \mathcal{N} \) and \( m' \in \mathcal{B}' \) form a basis of \( e(b_+) \) over \( \text{Fun}^\otimes(\mathbb{C}) \).

Lemma 4.6. The normally ordered monomials of the form \( mm' \) where \( m \in \mathcal{B} \) and \( m' \in \mathcal{N}' \) form a basis of \( e(n_+) \) over \( \text{Fun}^\otimes(\mathbb{C}) \).

The statements easily follows from relations (3.3) – (3.5) and Theorem 4.1.

For any monomial \( t_{a_1b_1} \ldots t_{a_kb_k} \) set \( \text{wt}(t_{a_1b_1} \ldots t_{a_kb_k}) = \sum_{i=1}^{k} (\varepsilon_{a_i} - \varepsilon_{b_i}) \), and for any function \( \varphi \in \text{Fun}^\otimes(\mathbb{C}) \) set \( \text{wt}(\varphi) = 0 \). Since relations (3.2) – (3.5) are homogeneous, the algebra \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \) is \( \mathbb{P} \)-graded by \( \text{wt} \).

Let \( \mu \in \mathfrak{h}^* \) and \( \hat{Q} \) be a multiplicative cocycle. Below we define a Verma module \( M_{\mu,\hat{Q}} \) of highest weight \( \mu \) and dynamical highest weight \( \hat{Q} \) over \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \).

Let \( \mathcal{N}_C = \bigoplus_{m \in \mathcal{N}} \mathbb{C}m \) be a diagonalizable \( \mathfrak{h} \)-module such that a monomial \( m \) has weight \( \text{wt}(m) \), and let \( \mathbb{C}v_{\mu,\hat{Q}} \) be a one-dimensional \( \mathfrak{h} \)-module such that \( v_{\mu,\hat{Q}} \) has weight \( \mu \). Then \( M_{\mu,\hat{Q}} = \mathcal{N}_C \otimes \mathbb{C}v_{\mu,\hat{Q}} \) as an \( \mathfrak{h} \)-module. We define an action of \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \) in \( \text{Fun}(M_{\mu,\hat{Q}}) \) by the rule: \( 1 \otimes v_{\mu,\hat{Q}} \) is a regular singular vector of weight \( \mu \) and dynamical weight \( \hat{Q} \), and

\[
m(1 \otimes v_{\mu,\hat{Q}}) = m \otimes v_{\mu,\hat{Q}}
\]

for any \( m \in \mathcal{N} \). This determines an action on \( v_{\mu,\hat{Q}} \) by any normally ordered monomial and, hence, by any element of \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \), cf. Theorem 4.1. Finally, for any \( x \in e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \) set

\[
x(m \otimes v_{\mu,\hat{Q}}) = (xm)(1 \otimes v_{\mu,\hat{Q}})
\]

where the product \( xm \) should be represented as a linear combination of normally ordered monomials.

Proposition 4.7. \( M_{\mu,\hat{Q}} \) is a well-defined \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module with highest weight \( \mu \), dynamical highest weight \( \hat{Q} \) and highest weight vector \( 1 \otimes v_{\mu,\hat{Q}} \).

The statement follows from Lemmas 4.5, 4.6 and Theorem 4.1.

From now on we suppress the symbol of tensor product in the definition of the Verma module \( M_{\mu,\hat{Q}} \), for instance, we write \( v_{\mu,\hat{Q}} \) instead of \( 1 \otimes v_{\mu,\hat{Q}} \).

If \( \hat{Q} \) is standard, then the \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module \( M_{\mu} = M_{\mu,\hat{Q}} \) is called the standard Verma module with highest weight \( \mu \) and the highest weight vector \( v_{\mu} = v_{\mu,\hat{Q}} \).

Lemma 4.8. Let \( V \) be an \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module. For any \( v \in \text{Fun}(V) \) which is a regular singular vector of weight \( \mu \) and dynamical weight \( \hat{Q} \), there is a unique morphism \( \varphi \in \text{Mor}(M_{\mu,\hat{Q}}, V) \) which sends \( v_{\mu,\hat{Q}} \) to \( v \).

Proposition 4.9. Any highest weight \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module is isomorphic to a suitable quotient of the Verma module over \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \) of the same highest weight and dynamical highest weight.

Proposition 4.10. For any \( \mu \in \mathfrak{h}^* \) and a dynamical weight \( \hat{Q} \) there exists a unique up to an isomorphism irreducible highest weight \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module with highest weight \( \mu \) and dynamical highest weight \( \hat{Q} \).

Proposition 4.11. Let \( \gamma \notin \mathbb{Q} + \tau \mathbb{Q} \). Then a highest weight \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \)-module with highest weight \( \mu \) and nondegenerate dynamical highest weight \( \hat{Q} \) is isomorphic to the Verma module \( M_{\mu,\hat{Q}} \) unless \( w \cdot \mu < \mu \) for some \( w \in W \).

5. Dynamical Shapovalov form

Let \( e(n_-) \) be the right ideal in \( e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \) generated by the elements \( t_{ab} \) with \( a > b \), let \( \mathfrak{d} \) be the \( \text{Fun}^\otimes(\mathbb{C}) \)-submodule generated by normally ordered monomials of the form \( t_{a_1a_1} \ldots t_{a_ka_k} \), and let

\[
\mathfrak{e}[0] = \{ x \in e_{\tau,\gamma}(\mathfrak{s}\mathfrak{l}_N) \mid \text{wt}(x) = 0 \}
\]
be the subalgebra of zero weight elements in \( e_{r,\gamma}^\circ(\mathfrak{sl}_N) \). Consider the quotient
\[
e(\mathfrak{h}) = e_{r,\gamma}^\circ(\mathfrak{sl}_N) / (e(n_+) + e(n_-))
\]

Let \( \eta : e_{r,\gamma}^\circ(\mathfrak{sl}_N) \to e(\mathfrak{h}) \) be the natural projection. By Theorem 4.1 the restriction of \( \eta \) to \( \mathfrak{d} \) is a bijection. Denote by \( \tilde{\eta} : e(\mathfrak{h}) \to \mathfrak{d} \) the inverse map. For any \( x, y \in e(\mathfrak{h}) \) define their product by the rule \( xy = \tilde{\eta}(\eta(x)\tilde{\eta}(y)) \). It is easy to see that this defines an algebra structure on \( e(\mathfrak{h}) \).

**Lemma 5.1.** The restriction of \( \eta \) to \( \mathfrak{c}[0] \) is a homomorphism.

Set \( q_a = \eta(\tilde{\eta}_{aa}) \), \( a = 1, \ldots, N \). It follows from (A.2) that \( e(\mathfrak{h}) \) is generated by functions \( f \in \text{Fun}^{\mathfrak{o}_r}(\mathbb{C}) \) and the pairwise commuting elements \( q_1, \ldots, q_N \) subject to relations
\[
q_a f(\lambda^{(1)}, \lambda^{(2)}) = f(\lambda^{(1)} - \gamma \varepsilon_a, \lambda^{(2)} - \gamma \varepsilon_a) q_a .
\]

The assignment
\[
\varpi : f(\lambda^{(1)}, \lambda^{(2)}) \mapsto f(-\lambda^{(2)}, -\lambda^{(1)}),
\]
\[
\varpi : t_{ab} \mapsto \prod_{1 \leq c \leq b} \theta(\lambda^{(1)}_{cb}) \theta(\lambda^{(1)}_{bc} - \gamma) \prod_{1 \leq c \leq a} (\theta(\lambda^{(2)}_{ca}) \theta(\lambda^{(2)}_{ac} - \gamma))^{-1} t_{ba}
\]
defines an involutive antiautomorphism of the operator algebra \( e_{r,\gamma}^\circ(\mathfrak{sl}_N) \), which differs from the antiautomorphism (3.11) by a suitable automorphism of the form (3.10). We have
\[
\varpi(\tilde{\eta}_{ab}) = \tilde{\eta}_{ba} .
\]

For any \( x \in e(\mathfrak{h}) \) set \( \varpi(x) = \eta(\varpi(\tilde{\eta}(x))) \).

**Lemma 5.2.** For any \( m \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) we have \( \eta(\varpi(m)) = \varpi(\eta(m)) \).

For any \( m_1, m_2 \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) set
\[
S(m_1, m_2) = \eta(\varpi(m_1)m_2) \in e(\mathfrak{h}) .
\]

Since \( \varpi \) is involutive, \( S(m_1, m_2) = \varpi(S(m_2, m_1)) \). Moreover, \( S(m_1m_2, m_3) = S(m_2, \varpi(m_1)m_3) \) for any \( m_1, m_2, m_3 \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \). \( S \) is called the dynamical Shapovalov form on \( e_{r,\gamma}^\circ(\mathfrak{sl}_N) \).

**Example.** Let \( a < b \). Then \( S(\tilde{\eta}_{ba}, \tilde{\eta}_{ba}) = -\frac{\theta(\lambda^{(1)}_{ba} - \lambda^{(2)}_{ab}) \theta(\gamma)}{\theta(\lambda^{(1)}_{ab}) \theta(\lambda^{(2)}_{ab})} q_a q_b .
\]

Let \( \mu \in \mathfrak{h}^* \) and let \( \hat{Q} = (Q_1, \ldots, Q_N) \) be a multiplicative cocycle. Then there is an algebra homomorphism \( \chi_{\mu,\hat{Q}} : e(\mathfrak{h}) \to D(\mathbb{C}) \):
\[
\chi_{\mu,\hat{Q}}(f(\lambda^{(1)}, \lambda^{(2)})) : \varphi(\lambda) \mapsto f(\lambda, \lambda - \gamma \mu) \varphi(\lambda) ,
\]
\[
\chi_{\mu,\hat{Q}}(q_a) : \varphi(\lambda) \mapsto Q_a(\lambda) \varphi(\lambda - \gamma \varepsilon_a) .
\]

Consider the Verma module \( M_{\mu,\hat{Q}} \). For any \( m \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) and \( v \in \text{Fun}(M_{\mu,\hat{Q}}) \) set
\[
S_{\mu,\hat{Q}}(m, v) = \chi_{\mu,\hat{Q}}(S(m, m')) \cdot 1 \in \text{Fun}(\mathbb{C}) .
\]

Here \( m' \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) is determined by \( v = m'v_{\mu,\hat{Q}} \) and \( 1 \in \text{Fun}(\mathbb{C}) \) is the constant function. It is easy to see that \( S_{\mu,\hat{Q}}(m, v) \) does not depend on the choice of \( m' \). We call \( S_{\mu,\hat{Q}} \) the dynamical Shapovalov pairing for \( M_{\mu,\hat{Q}} \).

**Lemma 5.3.** For any \( m_1, m_2 \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) and \( v \in \text{Fun}(M_{\mu,\hat{Q}}) \) we have
\[
S_{\mu,\hat{Q}}(m_1, m_2 v) = S_{\mu,\hat{Q}}(\varpi(m_2)m_1, v) = S_{\mu,\hat{Q}}(1, \varpi(m_1)m_2 v) .
\]

Here \( 1 \in e_{r,\gamma}^\circ(\mathfrak{sl}_N) \) is the identity element.
Lemma 5.4. Let \( m \in e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \) be a wt-homogeneous element, and \( v \in M_{\mu,\widehat{Q}}[v] \). If \( \text{wt}(m) \neq \mu - \nu \), then \( S_{\mu,\widehat{Q}}(m, v) = 0 \). Otherwise, \( \varpi(m) = S_{\mu,\widehat{Q}}(m, v) v_{\mu,\widehat{Q}} \).

Example. \( S_{\mu,\widehat{Q}}(1, v_{\mu,\widehat{Q}}) = 1 \). If \( a < b \), then

\[
S_{\mu,\widehat{Q}}(i_{ba}, i_{ba} v_{\mu,\widehat{Q}}) = -\frac{\theta(\gamma \mu_{ab}) \theta(\gamma)}{\theta(\lambda_{ab}) \theta(\lambda_{ab} - \gamma \mu_{ab})} Q_a(\lambda) Q_b(\lambda - \gamma \varepsilon_a).
\]

Set \( \text{Ker} S_{\mu,\widehat{Q}} = \{ v \in \text{Fun}(M_{\mu,\widehat{Q}}) \mid S_{\mu,\widehat{Q}}(m, v) = 0 \text{ for any } m \in e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \} \).

The subspace \( \text{Ker} S_{\mu,\widehat{Q}} \) is invariant under the action of \( e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \) and defines a proper submodule \( N_{\mu,\widehat{Q}} \) of \( M_{\mu,\widehat{Q}} \).

Proposition 5.5. \( N_{\mu,\widehat{Q}} \) is the maximal proper submodule of \( M_{\mu,\widehat{Q}} \), that is, for any proper submodule \( U \) of \( M_{\mu,\widehat{Q}} \) the embedded image of \( \text{Fun}(U) \) is contained in \( \text{Fun}(N_{\mu,\widehat{Q}}) = \text{Ker} S_{\mu,\widehat{Q}} \).

Proof. Let \( \varphi \in \text{Mor}(U, M_{\mu,\widehat{Q}}) \) be the embedding. \( \varphi(\text{Fun}(U)) \) is a direct sum of its weight components and \( \varphi(\text{Fun}(U))[\mu] = 0 \), since \( U \) is a proper submodule. Therefore, if \( v \in \varphi(\text{Fun}(U))[\nu] \) and \( \text{wt}(m) = \mu - \nu \), then \( \varpi(m)v = 0 \). Hence, by Lemma 5.4 \( S_{\mu,\widehat{Q}}(m, v) = 0 \) for any \( m \in e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \) and \( v \in \varphi(\text{Fun}(U)) \).

Corollary 5.6. The quotient module \( V_{\mu,\widehat{Q}} = M_{\mu,\widehat{Q}}/N_{\mu,\widehat{Q}} \) is the irreducible highest weight \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) module with highest weight \( \mu \) and dynamical highest weight \( \widehat{Q} \).

For any highest weight module \( V \) with highest weight \( \mu \), dynamical highest weight \( \widehat{Q} \) and highest weight vector \( v \) one can define the Shapovalov pairing similarly to (5.4): for any \( m \in e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \) and \( v' \in \text{Fun}(V) \) set

\[
S^V_{\mu,\widehat{Q}}(m, v') = \chi_{\mu,\widehat{Q}}(S(m, m')) \cdot 1 \in \text{Fun}(\mathbb{C})
\]

where \( m' \in e^\circ_{\tau,\gamma}(\mathfrak{sl}_N) \) is determined by \( v' = m'v \). Propositions 4.9 and 5.5 imply that \( S^V_{\mu,\widehat{Q}}(m, v) \) does not depend on the choice of \( m' \).

Proposition 5.7. The module \( V \) is irreducible if and only if \( \text{Ker} S^V_{\mu,\widehat{Q}} \) is trivial. Otherwise, \( \text{Ker} S^V_{\mu,\widehat{Q}} \) defines the maximal proper submodule of \( V \).

Let \( V^\mu_{\widehat{Q}} \) be the irreducible \( \mathfrak{sl}_N \)-module of highest weight \( \mu \). Set \( d_\mu[\nu] = \dim_{\mathbb{C}} V^\mu_{\widehat{Q}}[\nu] \).

Theorem 5.8. Let \( \mu \in \mathfrak{h}^* \) be a dominant integral weight. Then \( \dim_{\mathbb{C}} V^\mu_{\widehat{Q}}[\nu] \leq d_\mu[\nu] \) for any \( \nu \in \mathfrak{h}^* \). In particular, the \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V^\mu_{\widehat{Q}} \) is finite-dimensional.

Theorem 5.9. Let \( \gamma \notin \mathbb{Q} + \tau \mathbb{Q} \). The \( e_{\tau,\gamma}(\mathfrak{sl}_N) \)-module \( V^\mu_{\widehat{Q}} \) for a nondegenerate dynamical weight \( \widehat{Q} \) is finite-dimensional if and only if \( \mu \) is a dominant integral weight. Moreover, \( \dim_{\mathbb{C}} V^\mu_{\widehat{Q}}[\nu] = d_\mu[\nu] \) for any \( \nu \in \mathfrak{h}^* \).

Theorems 5.8 and 5.9 are proved in Section 8.

If \( \widehat{Q} \) is standard, we set \( \chi_{\mu} = \chi_{\mu,\widehat{Q}} \), \( S_\mu = S_{\mu,\widehat{Q}} \), \( N_\mu = N_{\mu,\widehat{Q}} \) and \( V_\mu = V_{\mu,\widehat{Q}} \).

6. Contragradient modules over \( e_{\tau,\gamma}(\mathfrak{sl}_N) \) and contravariant form

For any diagonalizable \( \mathfrak{h} \)-module \( V \) define an involutive linear map \( \psi : \text{Fun}(V) \to \text{Fun}(V) \) by the rule: if \( f \in \text{Fun}(V[\mu]) \), then \( \psi(f)(\lambda) = f(-\lambda + \gamma \mu) \).

Let \( V \) be a diagonalizable \( \mathfrak{h} \)-module and let \( V^* = \bigoplus_{\mu \in \mathfrak{h}^*} (V[\mu])^* \) be its restricted dual space. We consider \( V^* \) as a diagonalizable \( \mathfrak{h} \)-module such that \( (V[\mu])^* \) is a weight subspace of weight \( \mu \). For any \( B \in \text{End}(V) \) we denote by \( B^* \in \text{End}(V^*) \) the dual map. For a difference operator \( A \in D(V) \) we define the dual operator \( A' \in D(V^*) \) by the rule:

\[
\text{if } (Av)(\lambda) = B(\lambda)v(\lambda + \mu), \text{ then } (A'\varphi)(\lambda) = (B(\lambda - \mu))^* \varphi(\lambda - \mu),
\]

and the operator \( A' \in D(V^*) : A' = \psi A' \psi \). The assignment \( A \mapsto A' \) is an involutive antiisomorphism \( D(V) \to D(V^*) \).
Example. Let $V$ be an $e_{\tau, \gamma}(\mathfrak{sl}_N)$-module. Then

$$(t_{ab})^\dagger \varphi(\lambda) = (t_{ab}(-\lambda + (\mu + \varepsilon_b)\gamma))\varphi(\lambda - \gamma \varepsilon_b)$$

for any $\varphi \in \text{Fun}(V^*[\mu])$, cf. (3.12).

Given an $e_{\tau, \gamma}(\mathfrak{sl}_N)$-module $V$ we make the $\mathfrak{h}$-module $V^*$ into an $e_{\tau, \gamma}(\mathfrak{sl}_N)$-module as follows: the action of an element $m \in e_{\tau, \gamma}(\mathfrak{sl}_N)$ in $\text{Fun}(V^*)$ is given by $(\varpi(m))^\dagger$ where $\varpi(m)$ is understood as a difference operator acting in $\text{Fun}(V)$. It is easy to check that the definition is consistent, that is, the ring $\text{Fun}(\otimes^2(C))$ acts on $\text{Fun}(V^*)$ in the prescribed way, cf. (3.1). The obtained $e_{\tau, \gamma}(\mathfrak{sl}_N)$-module $V^*$ is called the contragradient module to the $e_{\tau, \gamma}(\mathfrak{sl}_N)$-module $V$.

Consider the Verma module $M_{\mu, \hat{Q}}$. Recall that $M_{\mu, \hat{Q}}[\mu] = \mathbb{C}v_{\mu, \hat{Q}}$. Fix $v^*_{\mu, \hat{Q}} \in M_{\mu, \hat{Q}}^*[\mu]$ by the rule $\langle v^*_{\mu, \hat{Q}}, v_{\mu, \hat{Q}} \rangle = 1$. For any $a = 1, \ldots, N$ set

$$\tilde{Q}_a(\lambda) = Q_a(-\lambda + (\mu + \varepsilon_a)\gamma).$$

Notice that $Q_a(\lambda) = \tilde{Q}_a(-\lambda + (\mu + \varepsilon_a)\gamma)$ as well.

**Proposition 6.1.** For the contragradient Verma module $M_{\mu, \hat{Q}}^*$ the constant function $v^*_{\mu, \hat{Q}}$ is a regular singular vector of weight $\mu$ and dynamical weight $\tilde{Q} = (\tilde{Q}_1, \ldots, \tilde{Q}_N)$.

**Corollary 6.2.** There is a unique morphism $\pi_{\mu, \hat{Q}} \in \text{Mor}(M_{\mu, \hat{Q}}, M_{\mu, \hat{Q}}^*)$ sending $v_{\mu, \hat{Q}}$ to $v^*_{\mu, \hat{Q}}$.

**Theorem 6.3.** $\ker \pi_{\mu, \hat{Q}} = \ker S_{\mu, \hat{Q}}$.

The morphism $\pi_{\mu, \hat{Q}}$ induces a $\text{Fun}(\mathbb{C})$-bilinear map $B_{\mu, \hat{Q}} : \text{Fun}(M_{\mu, \hat{Q}}) \otimes_{\mathbb{C}} \text{Fun}(M_{\mu, \hat{Q}}) \to \text{Fun}(\mathbb{C})$:

$$B_{\mu, \hat{Q}}(v, \bar{v}) = (\pi_{\mu, \hat{Q}}v, \bar{v}).$$

Define a bilinear map $C_{\mu, \hat{Q}} : \text{Fun}(M_{\mu, \hat{Q}}) \otimes_{\mathbb{C}} \text{Fun}(M_{\mu, \hat{Q}}) \to \text{Fun}(\mathbb{C})$ by the rule

$$(6.1) C_{\mu, \hat{Q}}(v, \bar{v}) = B_{\mu, \hat{Q}}(v, \psi \bar{v}).$$

The map $C_{\mu, \hat{Q}}$ is called the contravariant form.

**Theorem 6.4.** Let $v \in \text{Fun}(M_{\mu, \hat{Q}}[\nu])$ and $\bar{v} \in \text{Fun}(M_{\mu, \hat{Q}}[\bar{\nu}])$. Then $C_{\mu, \hat{Q}}(v, \bar{v}) = 0$ unless $\nu = \bar{\nu}$. Moreover,

$$C_{\mu, \hat{Q}}(v, \bar{v})(\lambda) = C_{\mu, \hat{Q}}(\bar{v}, v)(-\lambda + \gamma \nu).$$

**Theorem 6.5.**

a) $C_{\mu, \hat{Q}}(v, \bar{v}) = 0$ for any $\bar{v} \in \text{Fun}(M_{\mu, \hat{Q}})$ if and only if $v \in \ker S_{\mu, \hat{Q}}$.

b) $C_{\mu, \hat{Q}}(v, \bar{v}) = 0$ for any $v \in \text{Fun}(M_{\mu, \hat{Q}})$ if and only if $\bar{v} \in \ker S_{\mu, \hat{Q}}$.

Theorems 6.3–6.5 are proved at the end of the section.

Let $V$ and $\tilde{V}$ be highest weight $e_{\tau, \gamma}(\mathfrak{sl}_N)$-modules of the same highest weight $\mu$ and dynamical highest weights $\hat{Q}$ and $\tilde{Q}$, respectively. By the last corollary the form $C_{\mu, \hat{Q}}$ descends to a form $\text{Fun}(V) \otimes_{\mathbb{C}} \text{Fun}(\tilde{V}) \to \text{Fun}(\mathbb{C})$, denoted by the same letter. After obvious modification Theorem 6.5 remains true in this case. In particular, for the irreducible highest weight $e_{\tau, \gamma}(\mathfrak{sl}_N)$-modules $V_{\mu, \hat{Q}}$ and $V_{\mu, \tilde{Q}}$ the corresponding function-valued bilinear form $\text{Fun}(V_{\mu, \hat{Q}}) \otimes_{\mathbb{C}} \text{Fun}(V_{\mu, \tilde{Q}}) \to \text{Fun}(\mathbb{C})$ is nondegenerate.

**Example.** $C_{\mu, \hat{Q}}(v_{\mu, \hat{Q}}, v_{\mu, \tilde{Q}}) = 1$. If $a < b$, then

$$C_{\mu, \hat{Q}}(i_{ab}v_{\mu, \hat{Q}}, i_{ba}v_{\mu, \tilde{Q}}) = -\frac{\theta(\gamma \mu_{ab})\theta(\gamma)}{\theta(\lambda_{ab} + \gamma)\theta(\lambda_{ab} - \gamma \mu_{ab} + \gamma)} Q_a(\lambda + \gamma \varepsilon_a)Q_b(\lambda).$$

For a monomial $m = i_{a_1b_1} \ldots i_{a_kb_k}$ set $\zeta'(m) = \sum_{i=1}^k \varepsilon_{a_i}$ and $\zeta''(m) = \sum_{i=1}^k \varepsilon_{b_i}$. Notice that $\text{wt}(m) = \zeta'(m) - \zeta''(m)$.
Lemma 6.6. For any $m_1, m_2 \in e_{r,\gamma}(s|N)$ we have
\[
S_{\mu, \hat{Q}}(m_1, m_2 v_{\mu, \hat{Q}})(\lambda) = S_{\mu, \hat{Q}}(m_2, m_1 v_{\mu, \hat{Q}})(-\lambda + \gamma (\mu + \zeta' (m_1) + \zeta'' (m_2))).
\]

Proof. Without loss of generality we can assume that $\zeta'(m_1) + \zeta''(m_2) = \zeta''(m_1) + \zeta'(m_2)$ because otherwise the expressions on both sides of the formula equal zero by Lemma 5.4.

Let $m_1 = \hat{i}_{a_1 b_1} \cdots \hat{i}_{a_k b_k}$ and $m_2 = \hat{i}_{c_1 d_1} \cdots \hat{i}_{c_l d_l}$. Set $(s_1, \ldots, s_{k+l}) = (a_1, \ldots, a_k, d_1, \ldots, d_l)$. By the definition of the Shapovalov pairing, cf. (5.4), for any multiplicative cocycle $\hat{Q}$ we have
\[
(6.2) \quad S_{\mu, \hat{Q}}(m_1, m_2 v_{\mu, \hat{Q}})(\lambda) = S_{\mu, \hat{Q}}(m_1, m_2 v_{\mu, \hat{Q}})(\lambda) = S_{\mu, \hat{Q}}(m_2, m_1 v_{\mu, \hat{Q}})(\lambda - \sum_{1 \leq j < l} \gamma \varepsilon_{s_j}).
\]

Here $S_{\mu}$ and $v_{\mu}$ correspond to the so-called standard case, cf. Section 5. Since $\hat{Q}$ is a multiplicative cocycle, the product can be written also as $\prod Q_{s_i} (\lambda - \sum_{j > i} \gamma \varepsilon_{s_j})$. By formula (6.2) and the last remark it suffices to verify that
\[
S_{\mu}(m_1, m_2 v_{\mu})(\lambda) = S_{\mu}(m_2, m_1 v_{\mu})(-\lambda + \gamma \mu + \gamma \zeta'(m_1) + \zeta''(m_2))
\]
which follows from the property $S(m_1, m_2) = \varpi(S(m_2, m_1))$, commutation relations (5.1) and formula (5.4). \qed

Proposition 6.7. For any $m_1, m_2 \in N$ we have
\[
(6.3) \quad C_{\mu, \hat{Q}}(m_1 v_{\mu, \hat{Q}}, m_2 v_{\mu, \hat{Q}})(\lambda) = S_{\mu, \hat{Q}}(m_2, m_1 v_{\mu, \hat{Q}})(\lambda + \gamma \zeta''(m_2)).
\]

Proof. Recall that by the definition of Verma modules $m_1 v_{\mu, \hat{Q}}$ and $m_2 v_{\mu, \hat{Q}}$ are constant functions, since $m_1, m_2 \in N$. If $wt(m_1) \neq wt(m_2)$, then the expressions on both sides of formula (6.3) vanish. For $wt(m_1) = wt(m_2)$ the straightforward application of the definition of the contravariant form together with Lemma 5.4 gives
\[
(6.4) \quad C_{\mu, \hat{Q}}(m_1 v_{\mu, \hat{Q}}, m_2 v_{\mu, \hat{Q}})(\lambda) = S_{\mu, \hat{Q}}(m_2, m_1 v_{\mu, \hat{Q}})(-\lambda + \gamma \mu + \gamma \zeta'(m_1)),
\]
and using Lemma 6.6 we complete the proof. \qed

Proof of Theorem 6.4. The first part of the theorem is an easy consequence of the definition of the contravariant form. The second part follows from Proposition 6.7, Lemma 6.6, and the property
\[
(6.5) \quad C_{\mu, \hat{Q}}(f(\lambda) v, g(\lambda) \hat{v}) = f(\lambda) g(-\lambda + \gamma \nu) C_{\mu, \hat{Q}}(v, \hat{v})
\]
for any $f, g \in Fun(\mathbb{C})$, $v \in Fun(M_{\mu, \hat{Q}}[\nu])$ and $\hat{v} \in Fun(M_{\mu, \hat{Q}}[\nu])$, cf. (6.1). \qed

Proof of Theorems 6.3 and 6.5. It is clear that $Ker \pi_{\mu, \hat{Q}} = \{ v \in Fun(M_{\mu, \hat{Q}}) \mid C_{\mu, \hat{Q}}(v, \hat{v}) = 0 \text{ for any } \hat{v} \in Fun(M_{\mu, \hat{Q}}) \}$. So Theorem 6.3 is equivalent to claim a) of Theorem 6.5. Claim b) of the latter follows from claim a) and Theorem 6.4.

Since $\pi_{\mu, \hat{Q}} v_{\mu, \hat{Q}} = v^*_{\mu, \hat{Q}} \neq 0$, the subspace $Ker \pi_{\mu, \hat{Q}} \subset Fun(M_{\mu, \hat{Q}})$ defines a proper submodule of $M_{\mu, \hat{Q}}$, and by Proposition 5.5 we have that $Ker \pi_{\mu, \hat{Q}} \subset Ker S_{\mu, \hat{Q}}$.

Let $v \in Ker S_{\mu, \hat{Q}}$. We write it out as a linear combination of basis vectors: $v = \sum_{m \in N} \varphi_m(\lambda) m v_{\mu, \hat{Q}}$. Then by Proposition 6.7 for any $\bar{m} \in N$ we have
\[
C_{\mu, \hat{Q}}(v, \bar{m} v_{\mu, \hat{Q}})(\lambda) = \sum_{m \in N} \varphi_m(\lambda) S_{\mu, \hat{Q}}(\bar{m} v_{\mu, \hat{Q}}, m v_{\mu, \hat{Q}})(\lambda + \gamma \zeta''(\bar{m})) =
S_{\mu, \hat{Q}}(\bar{m} v_{\mu, \hat{Q}}, v)(\lambda + \gamma \zeta''(\bar{m})) = 0.
\]
Therefore, $C_{\mu, \hat{Q}}(v, \hat{v}) = 0$ for any $\hat{v} \in Fun(M_{\mu, \hat{Q}})$ by property (6.5). \qed
7. Rational dynamical quantum group \( e_{\text{rat}}(\mathfrak{sl}(N)) \)

Introduce the spaces \( \text{Rat}(\mathbb{C}) \), \( \text{Rat}(V) \) and \( \text{Rat} (\text{Hom}(V,W)) \) similar to the spaces \( \text{Fun}(\mathbb{C}) \), \( \text{Fun}(V) \) and \( \text{Fun}(\text{Hom}(V,W)) \), replacing in the definitions meromorphic functions by rational functions. Let \( \text{Rat}^{\otimes 2}(\mathbb{C}) = \text{Rat}(\mathbb{C}) \otimes \varepsilon \text{Rat}(\mathbb{C}) \).

We define the operator algebra \( e_{\text{rat}}^\circ(\mathfrak{sl}(N)) \) and modules over the rational dynamical quantum group \( e_{\text{rat}}(\mathfrak{sl}(N)) \) similar to the elliptic case with the following modification: we replace the spaces of meromorphic functions by rational functions, substitute the theta function \( \theta(u) \) by the linear function \( u \to u \), and set \( \gamma = 1 \). For instance, formulae (3.2), (3.4), (4.1) and (3.12) become

\[
\begin{align*}
t_{ab} f(\lambda^{(1)}, \lambda^{(2)}) &= f(\lambda^{(1)} - \varepsilon_a, \lambda^{(2)} - \varepsilon_b) t_{ab} \\
t_{ac} t_{bc} &= \frac{\lambda_{ab}^{(1)} + 1}{\lambda_{ab}^{(1)} - 1} t_{bc} t_{ac}, \quad \text{for } a \neq b,
\end{align*}
\]

(7.1)

\[
\hat{t}_{ab} = \prod_{1 \leq c < a} \lambda_{ca}^{(1)} \prod_{1 \leq c < b} (\lambda_{cb}^{(2)})^{-1} t_{ab},
\]

(7.2)

\[
(t_{ab} v)(\lambda) = t_{ab}(\lambda) v(\lambda - \varepsilon_a) \quad \text{for any } v \in \text{Rat}(V).
\]

In the last formula \( V \) is an \( e_{\text{rat}}(\mathfrak{sl}(N))-\)module. The definitions of highest weight modules, dynamical weights, etc. can be obviously transferred to the rational case.

The rational case can be considered as a degeneration of the elliptic case obtained by rescaling variables: \( u \to \gamma u \), \( \lambda \to \gamma \lambda \) and taking the limit \( \gamma \to 0 \).

Consider the limit \( \bar{R}(\lambda) \) of the rational version of the \( R \)-matrix (1.3) as \( u \to \infty \):

\[
\bar{R}(\lambda) = \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} + \sum_{a,b=1, a \neq b}^{N} \frac{E_{aa} \otimes E_{bb} - E_{ab} \otimes E_{ba}}{\lambda_{ab}}.
\]

(7.3)

It is a constant solution of the dynamical Yang-Baxter equation:

\[
\bar{R}^{(12)}(\lambda - h^{(3)}) \bar{R}^{(13)}(\lambda) \bar{R}^{(23)}(\lambda - h^{(1)}) = \bar{R}^{(23)}(\lambda) \bar{R}^{(13)}(\lambda - h^{(2)}) \bar{R}^{(12)}(\lambda).
\]

\( \bar{R}(\lambda) \) is the simplest example of the Hecke type dynamical \( R \)-matrix, see [EV1].

Let \( T = \sum_{a,b} E_{ba} \otimes t_{ab} \), where \( t_{ab} \) are the generators of \( e_{\text{rat}}^\circ(\mathfrak{sl}(N)) \) obeying the rational version of commutation relations (3.2) – (3.5). Then one can write relations (3.3) – (3.5) in the \( R \)-matrix form:

\[
\bar{R}^{(12)}(\lambda^{(2)}) T^{(13)} T^{(23)} = T^{(23)} T^{(13)} \bar{R}^{(12)}(\lambda^{(1)}).
\]

(7.4)

Let \( V, W \) be \( e_{\text{rat}}(\mathfrak{sl}(N))-\)modules. Then the \( \mathfrak{h} \)-module \( V \otimes W \) is made into an \( e_{\text{rat}}(\mathfrak{sl}(N))-\)module by the rule

\[
\ell_{ab}(\lambda)|_{V \otimes W} = \sum_{c=1}^{N} \ell_{cb}(\lambda - h^{(2)}) \otimes \ell_{ac}(\lambda),
\]

(7.5)

and \( t_{ab} \) acts on \( \text{Rat}(V \otimes W) \) according to (7.2).

Consider the following element in \( e_{\text{rat}}^\circ(\mathfrak{sl}(N)) \):

\[
t^{\wedge N} = \sum_{i \in S_N} \text{sign}(i) t_{N,i_N} \ldots t_{1,i_1}.
\]

(7.6)

The product \( D = \prod_{1 \leq a < b \leq N} (\lambda^{(1)}_{ab}/\lambda^{(2)}_{ab}) t^{\wedge N} \) coincides with the top coefficient of the rational version of the quantum determinant \( \text{Det} T(u) \), cf. (3.8). Hence, \( D \) is a central element in \( e_{\text{rat}}^\circ(\mathfrak{sl}(N)) \).
Let $V$ be an $e_{rat}(\mathfrak{sl}_N)$-module. The action of $D$ on $\text{Rat}(V)$ commutes with multiplication by any function $\varphi(\lambda) \in \text{Rat}(\mathbb{C})$ and, therefore, is given by multiplication by a certain function $D(\lambda) \in \text{Rat}(\text{End}(V))$. The module $V$ is called nondegenerate if $D(\lambda)$ is invertible for generic $\lambda$, and semistandard if $D(\lambda) = 1$. For instance, any nondegenerate (standard) highest weight $e_{rat}(\mathfrak{sl}_N)$-module is nondegenerate (semistandard) in the new sense.

One can check that the element $D$ is group-like, it acts on the $E_{\tau,\gamma}(\mathfrak{sl}_N)$-module $V \otimes W$ by
\[
D(\lambda) \big|_{V \otimes W} = D(\lambda - \hbar^{(2)}) \otimes D(\lambda).
\]
Therefore, a tensor product of nondegenerate (semistandard) $e_{rat}(\mathfrak{sl}_N)$-modules is nondegenerate (semistandard).

Let $e_{ab}$, $a, b = 1, \ldots, N$, be the standard generators of the Lie algebra $\mathfrak{gl}_N$:
\[
(7.7) \quad \{e_{ab}, e_{cd}\} = \delta_{bc} e_{ad} - \delta_{ad} e_{cb}.
\]
We identify the Lie algebra $\mathfrak{sl}_N$ with the subalgebra of traceless elements in $\mathfrak{gl}_N$:
\[
(7.8) \quad \mathfrak{sl}_N = \left\{ \sum_{a,b} x_{ab} e_{ab} \mid \sum_a x_{aa} = 0 \right\},
\]
and $\mathfrak{h}$ with the subalgebra of diagonal elements in $\mathfrak{sl}_N$: $\mathfrak{h} = \left\{ \sum_{a} x_{aa} e_{aa} \mid \sum_a x_{aa} = 0 \right\}$. The standard basis of $\mathfrak{h}$ is $h_a = e_{aa} - e_{a+1,a+1}$, $a = 1, \ldots, N - 1$. The assignment $e_{ab} \mapsto E_{ab}$, $a, b = 1, \ldots, N$, makes $\mathbb{C}^N$ into the vector representation of $\mathfrak{gl}_N$ and $\mathfrak{sl}_N$.

Let $V$ be an $e_{rat}(\mathfrak{sl}_N)$-module. The elements $\hat{\ell}_{ab}$ act on $\text{Rat}(V)$ as difference operators:
\[
(7.9) \quad (\hat{\ell}_{ab} v)(\lambda) = \hat{\ell}_{ab}(\lambda) v(\lambda - \varepsilon_a),
\]
with coefficients $\hat{\ell}_{ab}(\lambda) \in \text{Rat}(\text{End}(V))$. The module $V$ is called perturbative if these coefficients have the following behaviour as $\lambda$ goes to infinity in a generic direction:

a) for any $a = 1, \ldots, N$ the function $\hat{\ell}_{aa}(\lambda)$ has a limit $\hat{\ell}_{aa}$ which is an invertible operator;

b) for any $a, b = 1, \ldots, N$, $a \neq b$, the function $\lambda_{ab} \hat{\ell}_{ab}(\lambda)$ has a limit $\hat{\ell}_{ab}$.

The operators $\hat{\ell}_{ab}$, $a, b = 1, \ldots, N$, satisfy the following commutation relations:
\[
[x, \hat{\ell}_{bc}] = (x, \varepsilon_b - \varepsilon_c) \hat{\ell}_{bc}, \quad [\hat{\ell}_{aa}, \hat{\ell}_{bc}] = 0,
\]
for any $x \in \mathfrak{h}$ and $a, b, c = 1, \ldots, N$,
\[
[\hat{\ell}_{ab}, \hat{\ell}_{ba}] = (e_{aa} - e_{bb}) \hat{\ell}_{aa} \hat{\ell}_{bb}
\]
for $a \neq b$, and
\[
[\hat{\ell}_{ab}, \hat{\ell}_{bc}] = \hat{\ell}_{ac} \hat{\ell}_{bb}
\]
for pairwise distinct $a, b, c$. Hence, the assignment $e_{ab} \mapsto \hat{\ell}_{ab}^{-1} \hat{\ell}_{ab}$ for $a \neq b$, supplemented by the action of $\mathfrak{h}$, makes $V$ into an $\mathfrak{sl}_N$-module which we denote by $\mathcal{C}(V)$. We say that $V$ is a perturbation of $\mathcal{C}(V)$. It is clear that $V$ coincide with $\mathcal{C}(V)$ as a vector space.

**Lemma 7.1.** Let $V$ be a perturbative $e_{rat}(\mathfrak{sl}_N)$-module. Then $V$ is nondegenerate.

**Proof.** The operator $D(\lambda)$ is invertible for generic $\lambda$ because it has an invertible limit $\hat{\ell}_{11} \ldots \hat{\ell}_{NN}$ as $\lambda$ goes to infinity in a generic direction. \[\Box\]

**Lemma 7.2.** A tensor product of perturbative $e_{rat}(\mathfrak{sl}_N)$-modules is perturbative.

**Example.** The assignment
\[
\hat{\ell}_{aa}(\lambda) \mapsto 1 - \sum_{a < b \leq N} \frac{E_{bb}}{\lambda_{ab}},
\]
\[
\hat{\ell}_{ab}(\lambda) \mapsto \frac{E_{ab}}{\lambda_{ab}}, \quad a \neq b,
\]
If \( a, b = 1, \ldots, N \), makes \( \mathbb{C}^N \) into an \( e_{\text{rat}} (\mathfrak{sl}_N) \)-module \( V \), which is a perturbation of the vector representation of \( \mathfrak{sl}_N \). The module \( V \) is isomorphic to the vector representation \( U \) of \( e_{\text{rat}} (\mathfrak{sl}_N) \), cf. (3.13), by the following isomorphism:

\[
\varphi(\lambda) = \sum_{a=1}^{N} \prod_{1 \leq b < a} \lambda_{ab} E_{aa} \in \operatorname{Mor}(V, U).
\]

**Lemma 7.3.** Let \( V \) be a perturbative \( e_{\text{rat}} (\mathfrak{sl}_N) \)-module. If \( V \) has a weight singular vector, then \( \mathcal{C}(V) \) has a singular vector of the same weight.

**Proof.** Let \( v \in \operatorname{Rat}(V[\mu]) \) be a singular vector. This means that for generic \( \lambda \) the value \( v(\lambda) \) belongs to the subspace \( K_\lambda = \bigcap \ker \ell_{ab}(\lambda + \varepsilon_a)|_{V[\mu]} \subset V \). It is clear that \( \dim K_\lambda \) does not depend on \( \lambda \) for generic \( \lambda \). Moreover, \( K_\lambda \) has a limit \( K_\infty \) as \( \lambda \) goes to infinity in a certain generic direction, and \( \dim K_\infty = \dim K_\lambda \geq 1 \). To complete the proof we observe that the subspace of singular vectors in \( \mathcal{C}(V)[\mu] \) contains \( K_\infty \).

**Lemma 7.4.** Let \( V \) be a perturbative \( e_{\text{rat}} (\mathfrak{sl}_N) \)-module. Then

a) if \( \mathcal{C}(V) \) is irreducible, then \( V \) is irreducible;

b) if \( \mathcal{C}(V) \) is a highest weight module, then \( V \) is a highest weight module of the same highest weight;

c) if \( \mathcal{C}(V) \) is a Verma module, then \( V \) is isomorphic to a Verma module of the same highest weight.

**Proof.** If \( V \) is reducible, then \( \operatorname{Fun}(V) \) has a nontrivial proper invariant \( \operatorname{Fun}(\mathbb{C}) \)-linear subspace \( U \), which is a direct sum of its weight components. For each weight component \( U[\mu] \) consider the subspace \( U_\lambda[\mu] \subset V \) spanned by values of functions from \( U[\mu] \) regular at \( \lambda \). It is clear that \( \dim U_\lambda[\mu] = \dim U_{\mu}[\mu] \) for generic \( \lambda \). Moreover, \( U_\lambda[\mu] \) has a limit \( U_\infty[\mu] \) as \( \lambda \) goes to infinity in a certain generic direction, \( \dim U_\infty[\mu] = \dim U_\lambda[\mu] \), and the direction can be taken the same for all \( \mu \). Then \( U_\infty = \bigoplus U_\mu[\mu] \) is a nontrivial proper invariant subspace in \( \mathcal{C}(V) \), since \( \dim U_\infty[\mu] < \dim V[\mu] \) at least for some \( \mu \). Claim a) is proved.

Let \( v \) be the highest weight vector of \( \mathcal{C}(V) \). Then the constant function \( v \in \operatorname{Rat}(V) \) is a regular singular vector by the weight reason. Any weight subspace \( V[\mu] \) has a basis given by vectors of the form \( e_{a_1 b_1} \cdots e_{a_k b_k} v \). Then the corresponding functions \( t_{a_1 b_1} \cdots t_{a_k b_k} v \) span \( \operatorname{Rat}(V[\mu]) \), which proves claim b). Claim c) follows from claim b) and comparison of dimensions of weight subspaces.

We say that an \( \mathfrak{sl}_N \)-module \( V \) is *admissible* if \( V \) is a diagonalizable \( \mathfrak{h} \)-module. Notice that any highest weight \( \mathfrak{sl}_N \)-module is admissible. In Section 9 we define a functor \( \mathcal{E} \) from the category of admissible \( \mathfrak{sl}_N \)-modules to the category of semistandard \( e_{\text{rat}} (\mathfrak{sl}_N) \)-modules, cf. Theorem 9.9. We summarize the properties of this functor in the next two theorems.

**Theorem 7.5.** Let \( V \) be an admissible \( \mathfrak{sl}_N \)-module. Then

a) \( \mathcal{E}(V) \) coincides with \( V \) as an \( \mathfrak{h} \)-module;

b) \( \mathcal{E}(V) \) is a perturbation of \( V \), moreover, \( \ell_{aa} = 1 \) for any \( a = 1, \ldots, N \);

c) \( \mathcal{E}(V) \) is a semistandard \( e_{\text{rat}} (\mathfrak{sl}_N) \)-module;

d) if \( v \in V \) is a weight singular vector, then \( v \in \operatorname{Fun}(V) \) considered as a constant function, is a standard singular vector of the same weight for \( \mathcal{E}(V) \);

e) if \( V \) is a highest weight module, then \( \mathcal{E}(V) \) is a standard highest weight module with the same highest weight and highest weight vector;

f) if \( V \) is a Verma module, then \( \mathcal{E}(V) \) is isomorphic to the standard Verma module with the same highest weight and highest weight vector.

**Theorem 7.6.** Let \( U, V \) be highest weight \( \mathfrak{sl}_N \)-modules. Then

a) any element of \( \operatorname{Mor}(\mathcal{E}(U), \mathcal{E}(V)) \) is a constant function;

b) the map \( \operatorname{Hom}_{\mathfrak{sl}_N}(U, V) \to \operatorname{Mor}(\mathcal{E}(U), \mathcal{E}(V)) \) defined by \( \mathcal{E} \) is an isomorphism;

c) the above isomorphism coincides with the restriction of the natural embedding of \( \operatorname{Hom}(U, V) \) into \( \operatorname{Fun}(\operatorname{Hom}(U, V)) \).

The theorems are proved in Section 9.

Let \( V \) be a highest weight \( \mathfrak{sl}_N \)-module with highest weight \( \mu \) and highest weight vector \( v \). Let \( S \) be the \( \mathfrak{sl}_N \) Shapovalov form on \( V \), and let \( S_\mu \) be the dynamical Shapovalov pairing for \( \mathcal{E}(V) \).
Proposition 7.7. Let \( S_\mu = \text{Fun}(\text{Ker} \ S) \).

**Proof.** Since \( \text{Ker} \ S \) is an \( \mathfrak{sl}_N \) submodule of \( V \), then \( \mathcal{E}(\text{Ker} \ S) \) is an \( e_{\rho a}(\mathfrak{sl}_N) \) submodule of \( \mathcal{E}(V) \), and \( \text{Fun}(\text{Ker} \ S) \subset \text{Ker} \ S_\mu \) by Proposition 5.7. On the other hand, \( V/\text{Ker} \ S \) is an irreducible \( \mathfrak{sl}_N \)-module, therefore, \( \mathcal{E}(V)/\mathcal{E}(\text{Ker} \ S) = \mathcal{E}(V/\text{Ker} \ S) \) is an irreducible \( e_{\rho a}(\mathfrak{sl}_N) \)-module by Lemma 7.4. Hence, \( \text{Ker} \ S_\mu \subset \text{Fun}(\text{Ker} \ S) \) by Proposition 5.7. \( \square \)

**Lemma 8.7.** Let \( a_i \neq b_i \) for any \( i = 1, \ldots, k \) and let \( c_j \neq d_j \) for any \( j = 1, \ldots, l \). Then

\[
(-1)^k \sum_{i=1}^{k} \lambda_{a_i b_i} \prod_{j=1}^{l} \lambda_{c_j d_j} S_\mu(t_{a_1 b_1} \cdots t_{a_k b_k}, t_{c_1 d_1} \cdots t_{c_l d_l}) \rightarrow \mathcal{E}(e_{a_1 b_1} \cdots e_{a_k b_k} v_\mu, e_{c_1 d_1} \cdots e_{c_l d_l} v_\mu)
\]

as \( \lambda \) goes to infinity in a generic direction.

**Lemma 8.3.** Let \( a < b \). Then \( \hat{t}_{ab} \hat{t}_{ba}^k v = -\frac{\theta((\mu_{ab} - k + 1) \gamma) \theta(k \gamma)}{\theta(\lambda_{ab}) \theta(\mu_{ab} - (2k + 2) \gamma)} \hat{t}_{ba}^{k-1} v \).

**Proof.** Take formula (A.11) for \( a = b \), \( c = d \), and replace \( c \) by \( b \). Since \( \hat{t}_{aa} v = \hat{t}_{bb} v = v \), we have

\[
\hat{t}_{ab} \hat{t}_{ba}^k v = -\frac{\theta(\lambda_{ab}^{(1)}) - \lambda_{ab}^{(2)} + (k - 1) \gamma)}{\theta(\lambda_{ab}^{(1)}), \theta(\lambda_{ab}^{(2)})} \hat{t}_{ba}^{k-1} v.
\]

Now apply convention (3.1) and observe that the vector \( \hat{t}_{ba}^{k-1} v \) has weight \( \mu - (k - 1)(\varepsilon_a - \varepsilon_b) \). Since \( \nu_{ab} = (\nu, \varepsilon_a - \varepsilon_b) \) for any \( \nu \in h^* \) and \( (\varepsilon_a - \varepsilon_b, \varepsilon_b - \varepsilon_b) = 2 \), the lemma is proved. \( \square \)

**Corollary 8.2.** Let \( a < b \) and \( \mu_{ab} \notin \{0, 1, \ldots, k - 1\} \). Then \( \hat{t}_{ba}^k v \neq 0 \).

**Proof.** By Lemma 8.1

\[
\hat{t}_{ab} \hat{t}_{ba}^k v = (-1)^k \sum_{j=0}^{k-1} \frac{\theta((\mu_{ab} - j) \gamma) \theta((j + 1) \gamma)}{\theta(\lambda_{ab} - j \gamma), \theta(\mu_{ab} - (j + 1) \gamma)} v \neq 0.
\]

\( \square \)

**Lemma 8.3.** \( \hat{t}_{cd} \hat{t}_{a+1,a}^k v = 0 \) for any \( c < d \), \( (c, d) \neq (a, a+1) \). \( \hat{t}_{cc} \hat{t}_{a+1,a}^k v = v \) for any \( c \neq a, a+1 \).

**Lemma 8.4.** If \( (\mu, \alpha_a) = k - 1 \) and \( \hat{t}_{a+1,a}^k v \neq 0 \), then \( \prod_{j=1}^{k} \theta(\lambda_{a,a+1} + j \gamma) \hat{t}_{a+1,a}^k v \) is a standard singular vector of weight \( \mu - k \alpha_a \).

**Proof.** By Lemmas 8.1 and 8.3 the function \( \hat{t}_{a+1,a}^k v \) is a singular vector and \( \hat{t}_{cc} \hat{t}_{a+1,a}^k v = v \) for any \( c \neq a, a+1 \). On the other hand, it follows from formulae (A.3) and (A.4) that

\[
\hat{t}_{aa} \hat{t}_{a+1,a}^k v = \frac{\theta(\lambda_{a,a+1} + k \gamma)}{\theta(\lambda_{a,a+1})} \hat{t}_{a+1,a}^k v,
\]

\[
\hat{t}_{a+1,a} \hat{t}_{a+1,a}^k v = \frac{\theta(\lambda_{a,a+1} + k \gamma)}{\theta(\lambda_{a,a+1} - \mu_{a,a+1} + 2k \gamma)} \hat{t}_{a+1,a}^k v = \frac{\theta(\lambda_{a,a+1} + k \gamma)}{\theta(\lambda_{a,a+1} + (k + 1) \gamma)} \hat{t}_{a+1,a}^k v
\]

because \( \mu_{a,a+1} = (\mu, \alpha_a) = k - 1 \). Hence, multiplying \( \hat{t}_{a+1,a}^k v \) by the product \( \prod_{j=1}^{k} \theta(\lambda_{a,a+1} + j \gamma) \) one gets a standard singular vector. \( \square \)
Proposition 8.5. An irreducible standard highest weight $e_{\tau,\gamma}(\mathfrak{sl}_N)$-module with highest weight $\mu$ is infinite-dimensional if $\mu$ is not a dominant integral weight.

Proof. Let $v$ be the highest weight vector. Assume that $\mu$ is not a dominant integral weight, and let $a$ be such that $(\mu, \alpha_a) \not\in \mathbb{Z}_{\geq 0}$. Then the functions $v, \ i_{a+1, a}^1 v, \ i_{a+1, a}^2 v, \ldots$ are linearly independent over $\text{Fun}(\mathbb{C})$ because all of them are nonzero by Corollary 8.2 and they have distinct weights with respect to the action of $\mathfrak{h}$. \hfill \Box

Proposition 8.6. Let $(\mu, \alpha_a) = k - 1$. Then $i_{a+1, a}^k v_\mu$ generates a submodule of the Verma module $M_\mu$ isomorphic to the Verma module $M_{\mu-k\alpha_a}$.

Proof. By Lemma 8.4 there is a nontrivial morphism $\varphi \in \text{Mor}(M_{\mu-k\alpha_a}, M_\mu)$ which sends the highest weight vector $v_{\mu-k\alpha_a} \in M_{\mu-k\alpha_a}$ to $\prod_{j=1}^k \theta(\lambda_{a,a+1}+j\gamma) \ i_{a+1, a}^k v_\mu$, and it remains to show that $\varphi$ is an embedding. In other words, one has to prove that the induced map $\text{Fun}(M_{\mu-k\alpha_a}) \to \text{Fun}(M_\mu)$ is injective.

For any Verma module $M_\nu$ set $\text{Fun}_j(M_\nu) = \{ x v_\nu \mid x \in e_{\tau,\gamma}(\mathfrak{sl}_N), \deg(x) \leq j \}$ and

$$\text{Fun}_\bullet(M_\nu) = \bigoplus_{j=0}^{\infty} \text{Fun}_j(M_\nu)/\text{Fun}_{j-1}(M_\nu).$$

Let $\mathcal{N}$ be given by (4.5). It is clear that the set $\{ x v_\nu \mid x \in \mathcal{N}, \deg(x) \leq j \}$ is a basis of $\text{Fun}_j(M_\nu)$ over $\text{Fun}(\mathbb{C})$, and the set $\{ x v_\nu \mid x \in \mathcal{N}, \deg(x) = j \}$ induces a basis of $\text{Fun}_j(M_\nu)/\text{Fun}_{j-1}(M_\nu)$. We identify $\text{Fun}_\bullet(M_\nu)$ with the space of polynomials in variables $u_{21}, u_{31}, \ldots, u_{N,N-1}$ with coefficients in $\text{Fun}(\mathbb{C})$: for any monomial $t_{b_1 c_1} \cdots t_{b_j c_j} \in \mathcal{N}$ a class of the function $t_{b_1 c_1} \cdots t_{b_j c_j} v_\nu$ in the quotient space $\text{Fun}_j(M_\nu)/\text{Fun}_{j-1}(M_\nu)$ is mapped to $u_{b_1 c_1} \cdots u_{b_j c_j}$.

The map $\varphi : \text{Fun}(M_{\mu-k\alpha_a}) \to \text{Fun}(M_\mu)$ induces a map $\varphi_\bullet : \text{Fun}_\bullet(M_{\mu-k\alpha_a}) \to \text{Fun}_\bullet(M_\mu)$. Consider $\varphi_\bullet$ as a map from $\text{Fun}(\mathbb{C})[u_{21}, \ldots, u_{N,N-1}]$ to itself. By formula (A.10) we find that

$$\varphi_\bullet(u_{b_1 c_1} \cdots u_{b_j c_j}) = f(\lambda - \gamma \sum_{i=1}^j b_i) u_{b_1 c_1} \cdots u_{b_j c_j} u_{a+a+1}^k,$$

where $f(\lambda) = \prod_{j=1}^k \theta(\lambda_{a,a+1}+j\gamma)$. Hence, $\varphi_\bullet$ is injective, and so does $\varphi$. \hfill \Box

From now on till the end of the section fix a dominant integral weight $\mu$ and set $k_a = (\mu, \alpha_a) + 1$, $a = 1, \ldots, N-1$. Notice that $k_a \in \mathbb{Z}_{\geq 0}$ for any $a$. Denote by $Z_\mu$ the subspace in $\text{Fun}(M_\mu)$ generated over $e_{\tau,\gamma}(\mathfrak{sl}_N)$ by functions $i_{a+1, a}^k v_\mu$, $a = 1, \ldots, N-1$. Notice that the functions $i_{a+1, a}^k v_\mu$ are regular singular vectors, cf. Lemma 8.4.

Let $S_\mu$ be the Shapovalov pairing for $M_\mu$, cf. (5.3).

Proposition 8.7. $Z_\mu \subset \text{Ker} S_\mu$.

Proof. Lemma 8.4 implies that $Z_\mu$ is an invariant $\text{Fun}(\mathbb{C})$-linear subspace in $\text{Fun}(M_\mu)$, therefore it defines a submodule of $M_\mu$. Hence, the statement follows from Proposition 5.5. \hfill \Box

Theorem 8.8. $Ker S_\mu = Z_\mu$ for generic $\gamma$.

Proof. Since both $\text{Ker} S_\mu$ and $Z_\mu$ are direct sums of their weight components, we have to prove that $Ker S_\mu)[\nu] = Z_\mu[\nu]$ for any $\nu \leq \mu$. Notice that $\text{Ker} S_\mu)[\mu] = \text{Fun}(\mathbb{C}) v_\mu = Z_\mu[\mu]$.

By Proposition 8.7 it suffices to prove that $Ker S_\mu)[\nu]$ and $Z_\mu[\nu]$ have same dimensions. For doing this we employ the deformation argument.

Consider the Verma module $M^\ast_\mu$ of highest weight $\mu$ over $\mathfrak{sl}_N$ and the Shapovalov form $S^\ast_\mu$ on it. Let $M^\text{rat}_{\mu} = \mathcal{E}(M^\ast_\mu)$, see Theorem 7.5, and let $S^\text{rat}_\mu$ be the corresponding dynamical Shapovalov pairing. Recall that, $M^\text{rat}_\mu$ is isomorphic to the Verma module of highest weight $\mu$ over $e_{\text{rat}}(\mathfrak{sl}_N)$. By abuse of notation we denote generators of $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and $e_{\text{rat}}(\mathfrak{sl}_N)$ by the same letters and write $v_\mu$ for the highest weight vector of each of the modules $M_\mu$, $M^\ast_\mu$ and $M^\text{rat}_\mu$.

Let $Z^\ast_\mu$ be the $\mathfrak{sl}_N$ submodule in $M^\ast_\mu$ generated by singular vectors $e_{\alpha+1, a}^k v_\mu$, $a = 1, \ldots, N-1$, and let $Z^\text{rat}_\mu$ be the subspace in $\text{Rat}(M^\text{rat}_\mu)$ generated over $e_{\text{rat}}(\mathfrak{sl}_N)$ by functions $i_{a+1, a}^k v_\mu$, $a = 1, \ldots, N-1$. It is known that $\text{Ker} S^\ast_\mu = Z^\ast_\mu$. By Theorem 7.5 and Proposition 7.7 we have that $\text{Ker} S^\text{rat}_\mu = \text{Rat}(S^\ast_\mu)$.
Proof. Both vectors $\vec{\phi}_{a+1,a}^k \hat{t}_{a+1,a} v_\mu$ and $\vec{e}_{a+1,a}^k v_\mu$ have weight $\mu - k_a \alpha_a$. Thus they are proportional over $\text{Rat}(C)$ because $\dim C \mathcal{M}_\mu^\text{rat} [\mu - k_a \alpha_a] = 1$. The proportionality coefficient is a constant function, since both of them are standard singular vectors by Theorem 7.5 and the rational version of Lemma 8.4. The constant equals 1 because $\lambda^k_a \hat{t}_{a+1,a} v_\mu \to (-1)^k a^k_{a+1,a} v_\mu$ as $\lambda$ goes to infinity in a generic direction. □

Corollary 8.10. $\text{Ker} S_\mu^\text{rat} = Z_\mu^\text{rat} = \text{Rat}(Z_\mu^\text{rat})$.

Since the elliptic case is a deformation of the rational one we obtain that for generic $\gamma$

$$\dim_{\text{Fun}(C)}(\text{Ker} S_\mu)[\nu] \leq \dim_{\text{Rat}(C)}(\text{Ker} S_\mu^\text{rat})[\nu] = \dim C(\text{Ker} S_\mu^\text{rat})[\nu] = \dim C S_\mu^\text{rat}[\nu] = \dim_{\text{Fun}(C)} Z_\mu[\nu] \leq \dim_{\text{Fun}(C)}(\text{Ker} S_\mu)[\nu].$$

Here the last inequality is due to Proposition 8.7. Therefore, all the dimensions in (8.3) are the same, which proves the theorem. The rest of the proof is a justification of this informal reasoning.

For a pair of sequences $a = (a_1, \ldots, a_j)$ and $b = (b_1, \ldots, b_j)$ let $\hat{t}_{ab} = \hat{t}_{a_1 b_1} \cdots \hat{t}_{a_j b_j}$. Set

$$\text{wt}(a, b) = \text{wt}(\hat{t}_{ab}) = \sum_{i=1}^j (\varepsilon_{a_i} - \varepsilon_{b_i}), \quad \zeta(a, b) = \sum_{i=1}^j \varepsilon_{b_i}.$$ 

Let $\mathcal{M} = \{ (a, b) \mid \hat{t}_{ab} \in \mathcal{N} \}$, the set $\mathcal{N}$ being defined in (4.5). Set

$$\mathcal{M}[\beta] = \{ m \in \mathcal{M} \mid \text{wt}(m) = \beta \} \quad \text{and} \quad \mathcal{M}_\mu[\beta] = \bigcup_{a=1}^{N-1} \mathcal{M}[\beta + k_a \alpha_a].$$

Let $\beta = \nu - \mu$. Introduce a matrix $A^\gamma(\lambda)$ with entries labeled by pairs of elements of $\mathcal{M}[\beta]$:

$$A^\gamma_{m,m'}(\lambda) = S_\mu(\hat{t}_{m}, \hat{t}_{m'} v_\mu)(\lambda + \gamma \zeta(m)),$$

and a matrix $B^\gamma(\lambda)$ with entries labeled by pairs $(m, m') \in \mathcal{M}[\beta] \times \mathcal{M}[\beta]$ and given by the rule:

$$m' \hat{t}_{a+1,a}^k v_\mu = \sum_{m \in \mathcal{M}[\beta]} B^\gamma_{m,m'}(\lambda) \hat{t}_m v_\mu, \quad m' \in \mathcal{M}[\beta + k_a \alpha_a].$$

Both $A^\gamma(\lambda)$ and $B^\gamma(\lambda)$ are meromorphic functions of $\gamma, \lambda$. Moreover, if $\gamma \to 0$, then

$$A^\gamma(\gamma \lambda) \to A^\text{rat}(\lambda), \quad B^\gamma(\gamma \lambda) \to B^\text{rat}(\lambda),$$

where $A^\text{rat}(\lambda)$ and $B^\text{rat}(\lambda)$ are defined in the same way for the rational case (in formula (8.4) for the rational case the argument in the right hand side is $\lambda + \zeta(m)$).

Each matrix naturally defines a linear map: $A^\gamma(\lambda)$ and $A^\text{rat}(\lambda)$ act on $\mathcal{M}_C[\beta] = \bigoplus_{m \in \mathcal{M}[\beta]} \mathbb{C}m$, whilst $B^\gamma(\lambda)$ and $B^\text{rat}(\lambda)$ map $\bigoplus_{m \in \mathcal{M}[\beta]} \mathbb{C}m$ to $\mathcal{M}_C[\beta]$.

Given functions $\varphi_m(\lambda) \in \text{Fun}(C)$, $m \in \mathcal{M}[\beta]$, consider a vector $\tilde{\varphi}(\lambda) = \sum_{m \in \mathcal{M}[\beta]} \varphi_m(\lambda) m \in \mathcal{M}_C[\beta]$. Set $\tilde{\varphi}(\lambda) = \sum_{m \in \mathcal{M}[\beta]} \varphi_m(\lambda) \hat{t}_m v_\mu \in \text{Fun}(M_\mu)$.

Lemma 8.11.

a) $\varphi \in (\text{Ker} S_\mu)[\nu]$ iff $\tilde{\varphi}(\lambda) \in \text{Ker} A^\gamma(\lambda)$ for generic $\lambda$;

b) $\varphi \in (\text{Ker} S_\mu^\text{rat})[\nu]$ iff $\varphi(\lambda) \in \text{Ker} A^\text{rat}(\lambda)$ for generic $\lambda$;

c) $\varphi \in Z_\mu[\nu]$ iff $\tilde{\varphi}(\lambda) \in \text{Im} B^\gamma(\lambda)$ for generic $\lambda$;

d) $\varphi \in Z_\mu^\text{rat}[\nu]$ iff $\tilde{\varphi}(\lambda) \in \text{Im} B^\text{rat}(\lambda)$ for generic $\lambda$.

Proof. Claims a) and b) follow from formulae (5.2)–(5.4) and their rational versions, respectively. Claims c) and d) are straightforward. □
Corollary 8.10 implies that $\text{Ker } A^\gamma(\lambda) = \text{Im } B^\gamma(\lambda)$ for generic $\lambda$. The standard deformation reasoning, cf. (8.5), shows that

$$(8.6) \quad \dim_{\mathbb{C}} \text{Ker } A^\gamma(\lambda) \leq \dim_{\mathbb{C}} \text{Ker } A'^{\text{rat}}(\lambda) = \dim_{\mathbb{C}} \text{Im } B'^{\text{rat}}(\lambda) = \dim_{\mathbb{C}} \text{Im } B^\gamma(\lambda) \leq \dim_{\mathbb{C}} \text{Ker } A^\gamma(\lambda)$$

for generic $\lambda$, provided $\gamma$ is generic. Here the last inequality is due to Proposition 8.7, which implies that $\text{Im } B^\gamma(\lambda) \subseteq \text{Ker } A^\gamma(\lambda)$ for generic $\lambda$. Therefore, all the dimensions in (8.6) are the same. Hence, Ker $A^\gamma(\lambda) = \text{Im } B^\gamma(\lambda)$ for generic $\lambda$, provided $\gamma$ is generic. By Lemma 8.11 we see that $(\text{Ker } S_\mu)[\nu] = Z_\mu[\nu]$ for generic $\gamma$. Theorem 8.8 is proved. \hfill \Box

Let $V_\mu$ be the irreducible standard highest weight $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$-module of highest weight $\mu$. Let $N_\mu$ be the $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$-submodule of $M_\mu$ such that $\text{Fun}(N_\mu) = \text{Ker } S_\mu$. We have $V_\mu = M_\mu/N_\mu$, cf. Corollary 5.6. Let $V_{\mu}^{s\mathfrak{l}}$ be the irreducible $\mathfrak{s\mathfrak{l}}_N$-module of highest weight $\mu$. Set $d_\mu[\nu] = \dim_{\mathbb{C}} V_{\mu}^{s\mathfrak{l}}[\nu]$.

**Proof of Theorem 5.8.** It follows from the proof of Theorem 8.8 that for generic $\gamma$

$$(8.7) \quad \dim_{\mathbb{C}} V_\mu[\nu] = \dim_{\mathbb{C}} M_\mu[\nu] - \dim_{\mathbb{C}} \text{Fun}_{\mathbb{C}}(\text{Ker } S_\mu)[\nu] = \dim_{\mathbb{C}} M_\mu^{s\mathfrak{l}}[\nu] - \dim_{\mathbb{C}} (\text{Ker } S_\mu^{s\mathfrak{l}})[\nu] = \dim_{\mathbb{C}} V_{\mu}^{s\mathfrak{l}}[\nu] = d_\mu[\nu].$$

Since $\dim_{\mathbb{C}} \text{Fun}_{\mathbb{C}}(\text{Ker } S_\mu)[\nu]$ can jump only up at a specific $\gamma$, we have that $\dim_{\mathbb{C}} V_\mu[\nu] \leq d_\mu[\nu]$ for arbitrary $\gamma$. \hfill \Box

**Proof of Theorem 5.9.** We have already proved that $\dim_{\mathbb{C}} V_\mu[\nu] = d_\mu[\nu]$ for any $\nu$, provided $\gamma$ is generic, cf. (8.7). Consider an $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$-module $U_\mu$ defined in the following way. Let $U_\mu = V_\mu$ if $\gamma$ is generic, that is, if $\dim_{\mathbb{C}} V_\mu[\nu] = d_\mu[\nu]$. Otherwise, define $U_\mu$ by analytic continuation from generic $\gamma$. It is not difficult to justify the given definition of $U_\mu$, and to see that $U_\mu$ is a highest weight $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$-module with highest weight $\mu$ and $\dim_{\mathbb{C}} U_\mu[\nu] = d_\mu[\nu]$ for any $\nu$.

If $\mu$ is a dominant integral weight, then $w \cdot \mu < \mu$ and $d_\mu[w \cdot \mu] = 0$ for any nontrivial element $w \in W$. Hence, $U_\mu$ is irreducible by Corollary 4.4 if $\gamma \notin \mathbb{Q} + \tau \mathbb{Q}$, that is, $U_\mu = V_\mu$. The theorem is proved. \hfill \Box

**Remark.** There is another approach to constructing finite-dimensional irreducible representation of $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$. One can start from the vector representation of $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$ and apply the fusion procedure technique developed in the nondynamical case, cf. [C], [N]. If $\gamma \notin \mathbb{Q} + \tau \mathbb{Q}$, then any irreducible finite-dimensional standard highest weight $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$-module can be obtained in this way. The symmetric and exterior powers of the vector representation of $e_{\tau,\gamma}(\mathfrak{s\mathfrak{l}}_N)$ have been constructed by this technique in [FV2]. We will address this approach elsewhere.

**9. Definition of functor $\mathcal{E}$**

In this section we construct a functor from the category of admissible $\mathfrak{s\mathfrak{l}}_N$-modules to the category of standard $e_{\text{rat}}(\mathfrak{s\mathfrak{l}}_N)$-modules. The construction is similar to the construction of the functor from the category of finite-dimensional $\mathfrak{s\mathfrak{l}}_N$-modules to the category of rational representations of the exchange quantum group $F(SL(N))$, developed in [EV2]. In Section 10 we discuss the relation of these two constructions in detail.

Let $n_+ = \{ \sum_{a>b} x_{ab} e_{ab} \}$ and $n_- = \{ \sum_{a>b} x_{ab} e_{ab} \}$ be the standard nilpotent subalgebras in $\mathfrak{s\mathfrak{l}}_N$, and let $\mathfrak{b}_\pm = \mathfrak{h} \oplus n_\pm$. Set

$$\Xi = \frac{1}{2} \sum_{a=1}^{N} e_{aa}^2 + \frac{1}{2N} \left( \sum_{a=1}^{N} e_{aa} \right)^2 \in U(\mathfrak{b}).$$

**Proposition 9.1.** There exists a unique power series $\mathcal{J}(\lambda; z)$ in $z$ with coefficients in $U(\mathfrak{s\mathfrak{l}}_N) \otimes U(\mathfrak{s\mathfrak{l}}_N)$ valued functions of $\lambda$ with the properties:

a) $\mathcal{J}(\lambda; z)$ satisfies the rational $\text{ABRR}$ equation

$$\mathcal{J}(\lambda; z) \left( 1 \otimes (\lambda - z \Xi) \right) = (1 \otimes (\lambda - z \Xi)) + z \sum_{1 \leq a < b \leq N} e_{ab} \otimes e_{ba} \mathcal{J}(\lambda; z);$$

b) the coefficients of the series $(\mathcal{J}(\lambda; z) - 1)$ are $U(\mathfrak{b}_+) n_+ \otimes n_- U(\mathfrak{b}_-) \text{-valued functions of } \lambda$. 21
Proof. Let \( J(\lambda; z) = \sum_{k=0}^{\infty} J_k(\lambda) z^k \). Equation (9.1) is equivalent to certain recurrence relations for coefficients \( J_k(\lambda) \) with the initial condition \( J_0(\lambda) = 1 \). It is straightforward to verify that at each step the recurrence relations uniquely determine \( J_k \) from \( J_0, \ldots, J_{k-1} \). □

It follows from the proof of the last proposition that the coefficients of the series \( J(\lambda; z) \) are rational functions of \( \lambda \), and for any \( x \in \mathfrak{h} \)

\[
(9.2) \quad [J(\lambda; z), x \otimes 1 + 1 \otimes x] = 0.
\]

Denote by \( \Delta : U(\mathfrak{sl}_N) \to U(\mathfrak{sl}_N) \otimes U(\mathfrak{sl}_N) \) the coproduct for \( U(\mathfrak{sl}_N) \).

**Theorem 9.2.** The series \( J(\lambda; z) \) satisfies the equation

\[
(9.3) \quad J^{(1(23))}(\lambda; z) J^{(12)}(\lambda - z h^{(3)}; z) = J^{(1(23))}(\lambda; z) J^{(23)}(\lambda; z).
\]

Here \( J^{(1(23))} = (\Delta \otimes \text{id})(J) \), \( J^{(1(23))} = (\text{id} \otimes \Delta)(J) \), \( J^{(12)} = J \otimes 1 \), \( J^{(23)} = 1 \otimes J \), and the meaning of \( J^{(12)}(\lambda - z h^{(3)}; z) \) is explained below, cf. (9.4).

Proof. The statement is a degeneration of Theorem 3.1 in [ESS], and can be proved in the same way. □

**Remark.** Let \( x_1, \ldots, x_{N-1} \) be a basis of \( \mathfrak{h}^* \), and let \( x_1^*, \ldots, x_{N-1}^* \) be the dual basis of \( \mathfrak{h} \). Write \( \lambda = \lambda_1 x_1^* + \ldots + \lambda_{N-1} x_{N-1}^* \). For a rational function \( f(\lambda) \) we define a series \( f(\lambda - z h^{(3)}) \) by the Taylor series expansion:

\[
(9.4) \quad f(\lambda - z h^{(3)}) = f(\lambda; z) - z \sum_{n=1}^{N-1} \frac{\partial f(\lambda)}{\partial \lambda^n} (x^n)^{(3)} + \ldots,
\]

and extend the definition to series in \( z \) with coefficients in rational functions of \( \lambda \) in the natural way.

**Remark.** The equation (9.3) is usually called the dynamical 2-cocycle condition.

Define a rational exchange matrix \( R(\lambda; z) \) by the rule:

\[
(9.5) \quad R(\lambda; z) = (J(\lambda; z))^{-1} J^{(21)}(\lambda; z).
\]

**Theorem 9.3.** \( R(\lambda) \) satisfies the dynamical Yang-Baxter equation:

\[
R^{(12)}(\lambda - z h^{(3)}; z) R^{(13)}(\lambda; z) R^{(23)}(\lambda - z h^{(1)}; z) = R^{(23)}(\lambda; z) R^{(13)}(\lambda - z h^{(2)}; z) R^{(12)}(\lambda; z).
\]

The statement follows from Theorem 9.2 and cocommutativity of the coproduct \( \Delta \).

Say that an \( n \)-tuple \( V_1, \ldots, V_n \) of \( \mathfrak{sl}_N \)-modules is admissible if for any pairwise distinct \( i_1, \ldots, i_k \) the tensor product \( V_{i_1} \otimes \ldots \otimes V_{i_k} \) is a diagonalizable \( \mathfrak{h} \)-module; that is, \( V_1, \ldots, V_n \) are diagonalizable \( \mathfrak{h} \)-modules and all weight subspaces of any tensor product \( V_{i_1} \otimes \ldots \otimes V_{i_k} \) are finite-dimensional.

Let \( V, W \) be an admissible pair of \( \mathfrak{sl}_N \)-modules. Denote by \( J_{VW}(\lambda; z) \in \text{End}(V \otimes W) \) the action of \( J(\lambda; z) \) in the tensor product \( V \otimes W \). It follows from the explicit form of recurrence relations in the proof of Proposition 9.1 that there is a unique function \( J_{VW}(\lambda) \in \text{Rat}(\text{End}(V \otimes W)) \) such that the series \( J_{VW}(\lambda; z) \) coincides with the expansion of \( J_{VW}(\lambda/ z) \) at \( z = 0 \) . The function \( J_{VW}(\lambda) \) admits the following description.

**Lemma 9.4.** \( J_{VW}(\lambda) \) is the unique solution of the equation

\[
J_{VW}(\lambda)(1 \otimes (\lambda - \Xi))|_{V \otimes W} = (1 \otimes (\lambda - \Xi) + \sum_{1 \leq a < b \leq N} e_{ab} \otimes e_{ba})|_{V \otimes W} J_{VW}(\lambda)
\]

such that \( (J_{VW}(\lambda) - 1)|_{U(\mathfrak{b}_+ n_+ \otimes U(\mathfrak{b}_-))}|_{V \otimes W} \). Moreover, \( J_{VW}(\lambda) \) commutes with the action of \( \mathfrak{h} \) in \( V \otimes W \), cf. (9.2).

**Proposition 9.5.** For any admissible triple of \( \mathfrak{sl}_N \)-modules \( U, V, W \) we have

\[
(9.6) \quad J_{U \otimes V W}(\lambda) (J_{UV}(\lambda - h^{(3)}) \otimes 1) = J_{U, V W}(\lambda) (1 \otimes J_{VW}(\lambda)).
\]
Proof. The function \( (J_{UV}(\lambda - h^{(3)}) \otimes 1) \) is defined by the rule (1.1). To get relation (9.6) from formula (9.3) one needs to verify that the series obtained by expansion of \( (J_{UV}(z^{-1} \lambda - h^{(3)}) \otimes 1) \) at \( z = 0 \) coincides with the action of \( J^{(12)}(\lambda - z h^{(3)}; z) \), defined by (9.4), in \( U \otimes V \otimes W \), which is simple. \( \square \)

For any \( A \in \text{End}(W \otimes V) \) let \( A^{(21)} = P A P^{-1} \in \text{End}(V \otimes W) \) where \( P : W \otimes V \to V \otimes W \) is the permutation map: \( P(x \otimes y) = y \otimes x \). Set

\[
R_{VW}(\lambda) = (J_{VW}(\lambda))^{-1}(J_{WV}(\lambda))^{(21)}.
\]

It is clear that the action of the series \( R(\lambda; z) \) in \( V \otimes W \) coincides with the expansion of \( R_{VW}(\lambda/z) \) at \( z = 0 \). Like in the proof of Proposition 9.5 we get the following assertion from Theorem 9.3.

**Proposition 9.6.** For any admissible triple of \( \mathfrak{sl}_N \)-modules \( U, V, W \) we have

\[
R_{UV}^{(12)}(\lambda - h^{(3)}) R_{UW}^{(13)}(\lambda) R_{VW}^{(22)}(\lambda - h^{(1)}) = R_{UW}^{(23)}(\lambda) R_{UW}^{(13)}(\lambda - h^{(2)}) R_{UV}^{(12)}(\lambda).
\]

Let \( \tilde{J}_{VW}(\lambda) \) be the fusion matrix for \( U(\mathfrak{sl}_N) \) defined in [EV2], and let

\[
\tilde{R}_{VW}(\lambda) = (\tilde{J}_{VW}(\lambda))^{-1}(\tilde{J}_{WV}(\lambda))^{(21)}
\]

be the corresponding dynamical \( \tilde{R} \)-matrix. Let \( X = \sum_{a=1}^{N} (-1)^{a-1} E_{a, N-a+1} \), considered as an element of the group \( \text{SL}(N) \), and let \( w_X \) be the longest element of the Weyl group. We have \( \text{Ad}_{X}(e_{ab}) = e_{N-a+1, N-b+1} \) and \( w_X(\varepsilon_a) = \varepsilon_{N-a+1} \) for any \( a, b = 1, \ldots, N \).

**Lemma 9.7.** For any finite-dimensional \( \mathfrak{sl}_N \)-modules \( V, W \) we have

\[
\tilde{J}_{VW}(\lambda) = (X \otimes X) \tilde{J}_{VW}(w_X(\lambda + \rho))(X \otimes X)^{-1}.
\]

Proof. It is shown in [EV2, Section 9] that \( \tilde{J}_{VW}(\lambda) \) is the only solution of the equation

\[
\tilde{J}_{VW}(\lambda) (1 \otimes (\lambda + \rho - \Xi))|_{V \otimes W} = (1 \otimes (\lambda + \rho - \Xi) + \sum_{1 \leq a < b \leq N} e_{ab} \otimes e_{ab})|_{V \otimes W} \tilde{J}_{VW}(\lambda)
\]

such that \( (\tilde{J}_{VW}(\lambda) - 1) \in (\mathfrak{n}_- \cup \mathfrak{n}_+) \cap \cup (\mathfrak{n}_+) )|_{V \otimes W} \). By Lemma 9.4 the right-hand side of formula (9.9) has the same properties, which proves the claim. \( \square \)

**Corollary 9.8.** \( \tilde{R}_{VW}(\lambda) = (X \otimes X) R_{VW}(w_X(\lambda + \rho))(X \otimes X)^{-1} \).

Henceforward, let \( U \) be the vector representation of \( \mathfrak{sl}_N \). By formula (36) in [EV2] we have

\[
\tilde{R}_{UU}(\lambda) = \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} - \sum_{1 \leq a < b \leq N} \frac{E_{bb} \otimes E_{aa}}{(\lambda_{ab} - a + b)^2} - \sum_{a,b=1}^{N} \frac{E_{ab} \otimes E_{ba}}{\lambda_{ab} - a + b}.
\]

Therefore,

\[
R_{UU}(\lambda) = \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} - \sum_{1 \leq a < b \leq N} \frac{E_{aa} \otimes E_{bb}}{\lambda_{ab}} - \sum_{a,b=1}^{N} \frac{E_{ab} \otimes E_{ba}}{\lambda_{ab}}.
\]

Let \( V \) be an admissible \( \mathfrak{sl}_N \)-module. Introduce functions \( \hat{\ell}_{ab} \in \text{Rat}(\text{End}(V)) \), \( a, b = 1, \ldots, N \), by the equality

\[
R_{UV}(\lambda) = \sum_{a,b=1}^{N} E_{ba} \otimes \hat{\ell}_{ab}(\lambda).
\]

**Theorem 9.9.** Let \( V \) be an admissible \( \mathfrak{sl}_N \)-module. Then the rule \( \hat{\ell}_{ab} v(\lambda) = \hat{\ell}_{ab}(\lambda)v(\lambda - \varepsilon_a) \), for any \( a, b = 1, \ldots, N \) and any \( v \in \text{Rat}(V) \), endows \( V \) with an \( \mathfrak{e}_{\text{rat}}(\mathfrak{sl}_N) \)-module structure. The constructed \( \mathfrak{e}_{\text{rat}}(\mathfrak{sl}_N) \)-module is denoted by \( \mathcal{E}(V) \).

Proof. The statement follows from Proposition 9.6, and formulae (7.1), (7.4), (7.9) and (9.11). \( \square \)
We define the functor $E$ from the category of admissible $\mathfrak{sl}_N$-modules to the category of semistandard $e_{rat}(\mathfrak{sl}_N)$-modules by sending an object $V$ to $E(V)$ and a morphism $\varphi \in \text{Hom}(V,W)$ to the corresponding constant function $\varphi \in \text{Mor}(V,W)$.

**Proof of Theorem 7.5.** Claim a) of the theorem is immediate. Say that $f(\lambda) = O(|\lambda|^k)$ if $f(s\lambda) = O(s^k)$ as $s \to +\infty$ for generic $\lambda$. From Lemma 9.4 and formula (9.7) we have that

$$J_{UV}(\lambda) = 1 - \sum_{1 \leq a < b \leq N} \frac{E_{ba} \otimes e_{ab}}{\lambda_{ab}} + O(|\lambda|^{-2}),$$

$$R_{UV}(\lambda) = 1 + \sum_{a,b=1}^{N} \frac{E_{ba} \otimes e_{ab}}{\lambda_{ab}} + O(|\lambda|^{-2}),$$

which proves claim b). Claim c) follows from Lemma 7.1. Claims e) and f) follow from claims b) and d), and Lemma 7.4.

To prove claim d) one should show that for any singular vector $v \in V$ we have $\tilde{\ell}_{aa}^V(\lambda) v = v$ for any $a$, and $\tilde{\ell}_{ab}^V(\lambda) v = 0$ for any $a < b$. By Lemma 9.4 and formula (9.7) we see that $(\tilde{\ell}_{aa}^\lambda - 1) \in \left( \mathfrak{n}_U(\mathfrak{sl}_N) \mathfrak{n}_e \right)_V$ for any $a$, and $\tilde{\ell}_{ab}^\lambda \in \left( \mathfrak{U}(\mathfrak{sl}_N) \mathfrak{n}_e \right)_V$ for any $a < b$, which implies claim d).

It remains to prove claim c. The element $D$ acts on $\text{Rat}(V)$ as multiplication by $D(\lambda)$. It is clear from the definition of $E(V)$ that there exists some independent of $V$ element in a certain completion of $U(\mathfrak{sl}_N)$ such that its action on $V$ coincides with $D(\lambda)$. Since an element of $U(\mathfrak{sl}_N)$ is uniquely determined by its action in highest weight modules, it suffices to prove claim c) under the assumption that $V$ is a highest weight module. In the last case claim c) follows from claim d).

The proof of Theorem 7.6 is similar to the proof of Theorem 45 in [EV2].

**10. Exchange quantum group $F(SL(N))$**

For any $a = 1, \ldots, N$ let $i^a = (1, \ldots, a-1, a+1, \ldots, N)$. Set

$$t_{ab} = (-1)^{a+b} \sum_{j \in S_{N-1}} \text{sign}(j) \, t_{i_{N-1}^a j_i} \cdots t_{i_1^a j_1} .$$

**Lemma 10.1.** $\sum_{c=1}^{N} t_{ac} \, t_{bc} = \delta_{ab} \, t^{\wedge N}$, where $t^{\wedge N}$ is defined by (7.6).

**Proof.** The formula coincides with the equality of the top coefficients in the rational version of formula (B.6) for $k = 1$ and $T(u) = T(u)$, cf. (3.6). \qed

Consider the exchange quantum group $F(SL(N))$ defined in [EV2]. It admits the following description, see [EV2, Section 5.3]. $F(SL(N))$ is a unital associative algebra over $\mathbb{C}$ generated by functions $f \in \text{Rat} \otimes_2(\mathbb{C})$ and elements $T_{ab}^+, T_{ab}^-$, $a,b = 1, \ldots, N$, subject to relations (10.1)–(10.3) and (10.5).

Let $\tilde{R}(\lambda) = \tilde{R}_{UV}(\lambda)$, cf. (9.10). Set $T^\pm = \sum_{a,b=1}^{N} E_{ab} \otimes T_{ab}^\pm$. The defining relations for $F(SL(N))$ are

$$T^+ T^- = T^- T^+ = \text{id} \otimes 1,$$

$$T_{ab}^+ f(\lambda^{(1)}, \lambda^{(2)}) = f(\lambda^{(1)} - \varepsilon_b, \lambda^{(2)} - \varepsilon_a) \, T_{ab}^+$$

for any $f \in \text{Rat} \otimes_2(\mathbb{C})$,

$$\tilde{R}^{(12)}(\lambda^{(2)}) T^{(13)} T^{(23)} = T^{(23)} T^{(13)} \tilde{R}^{(12)}(\lambda^{(1)})$$

where $T^{(13)} = \sum_{a,b} E_{ab} \otimes \text{id} \otimes T_{ab}^+$ and $T^{(23)} = \sum_{a,b} \text{id} \otimes E_{ab} \otimes T_{ab}^+$, and relation (10.5) below.

**Remark.** In this paper the variables $\lambda^{(1)}, \lambda^{(2)}$ and the generators $T_{ab}^+, T_{ab}^-$ correspond to the variables $\lambda^1, \lambda^2$ and the generators $L_{ab}, L_{ab}^{-1}$ in [EV2].
For any permutation \( i \in S_N \) let \( \Lambda_i(\lambda) = \prod_{1 \leq a < b \leq N \atop i_a < i_b} (1 + \lambda_{ab}^{-1}) \). Set

\[
(10.4) \quad \text{Det} \ T^+ = \frac{1}{\Lambda_{\text{id}}(\lambda(1))} \sum_{i \in S_N} \text{sign}(i) \Lambda_i(\lambda^{[2]}) T^+_{i_{1N},N} \cdots T^+_{i_{11}}
\]

where \( \text{id} = (1, \ldots, N) \). The last defining relation for \( F(SL(N)) \) is

\[
(10.5) \quad \text{Det} \ T^+ = 1.
\]

Remark. The element \( \text{Det} \ T^+ \) corresponds to the element \( D \) in [EV2]. Formula (10.4) can be derived from the definition of \( D \) therein. The complete proof of formula (10.4) will appear elsewhere.

Recall that, given a diagonalizable \( h \)-module \( V \), we assume the following action of \( \text{Rat}^{\otimes 2}(\mathbb{C}) \) on \( V \)-valued functions:

\[
f : v(\lambda) \mapsto f(\lambda, \lambda - \mu) v(\lambda)
\]

for any \( f \in \text{Rat}^{\otimes 2}(\mathbb{C}) \) and any function \( v(\lambda) \) with values in \( V[\mu] \).

A rational dynamical representation of \( F(SL(N)) \) is a diagonalizable \( h \)-module \( V \) endowed with an action of \( F(SL(N)) \) on \( V \)-valued meromorphic functions by difference operators with rational coefficients:

\[
(10.6) \quad (T_{ab}^+ v)(\lambda) = L_{ab}(\lambda) v(\lambda - \varepsilon_b), \quad a, b = 1, \ldots, N,
\]

where \( L_{ab}(\lambda) \in \text{Rat}(\text{End}(V)) \) are suitable functions.

Proposition 10.2. Let \( V \) be a rational dynamical representation of \( F(SL(N)) \). Then the rule

\[
(10.7) \quad (t_{ab} v)(\lambda) = \prod_{1 \leq c < c_a} (\lambda_{ca} + 1)^{-1} \prod_{1 \leq c < c_b} (\lambda_{cb} + \varepsilon_{bb} - \varepsilon_{cc} + 1) L_{ba}(\lambda - \rho) v(\lambda - \varepsilon_a)
\]

for any \( a, b = 1, \ldots, N \) and any \( v \in \text{Rat}(V) \), endows \( V \) with a structure of a semistandard \( \varepsilon_{\text{rat}}(\mathfrak{sl}_N) \)-module.

The proof is straightforward.

Proposition 10.3. Let \( V \) be a semistandard \( \varepsilon_{\text{rat}}(\mathfrak{sl}_N) \)-module. Then formulae (10.6), (10.7) make \( V \) into a rational dynamical representation of \( F(SL(N)) \).

Proof. It is straightforward to verify relations (10.2), (10.3) and (10.5). To complete the proof it remains to find the action of elements \( T_{ab}^+ \) to obey relations (10.1). This can be done using Proposition 10.1, since the \( \varepsilon_{\text{rat}}(\mathfrak{sl}_N) \)-module \( V \) is nondegenerate. \( \square \)

The last two propositions define a functor \( \mathcal{F} \) from the category of rational dynamical representations of \( F(SL(N)) \) to the category of semistandard \( \varepsilon_{\text{rat}}(\mathfrak{sl}_N) \)-modules: an object \( V \) goes to itself and a morphism \( \varphi(\lambda) \in \text{Rat}(\text{Hom}(V, W)) \) goes to \( \varphi(\lambda - \rho) \in \text{Mor}(V, W) \). Furthermore, the propositions imply the following assertion.

Theorem 10.4. The functor \( \mathcal{F} \) is an equivalence of the categories.

For both categories involved in the last theorem the subcategories of finite-dimensional objects are tensor categories, the tensor product of rational dynamical representations of \( F(SL(N)) \) being defined in [EV2]. One can show that the restriction of the functor \( \mathcal{F} \) to these subcategories is an equivalence of tensor categories.

Let \( \mathcal{G} \) be the functor from the category of finite-dimensional \( \mathfrak{sl}_N \)-modules to the category of finite-dimensional dynamical representations of \( F(SL(N)) \) defined in [EV2]: an \( \mathfrak{sl}_N \)-module \( V \) goes to the representation \( \mathcal{G}(V) \) of \( F(SL(N)) \) given by the rule

\[
(10.8) \quad \tilde{R}_{UV}(\lambda) = \sum_{a,b=1}^{N} E_{ab} \otimes L_{ab}(\lambda),
\]

and a morphism \( \varphi \in \text{Hom}(V, W) \) goes to the corresponding constant function \( \varphi \in \text{Rat}(\text{Hom}(V, W)) \).

The composition \( \tilde{\mathcal{E}} = \mathcal{F} \circ \mathcal{G} \) is a functor from the category of finite-dimensional \( \mathfrak{sl}_N \)-modules to the category of semistandard \( \varepsilon_{\text{rat}}(\mathfrak{sl}_N) \)-modules.
Theorem 10.5. The functor $\tilde{E}$ is isomorphic to the restriction of the functor $E$ to the category of finite-dimensional $\mathfrak{sl}_N$-modules.

The theorem is proved in Appendix F.

Remark. Let $V$ be an irreducible finite-dimensional $\mathfrak{sl}_N$-module. Then the $e_{rat}(\mathfrak{sl}_N)$-module $E(V)$ is an irreducible standard highest weight module over the dynamical quantum group $e_{rat}(\mathfrak{sl}_N)$. Such modules have been described in Section 8. Applying the functor inverse to $F$ we get a new description of the dynamical representations of $F(SL(N))$ induced from irreducible finite-dimensional $\mathfrak{sl}_N$-modules. This new description is a new highest weight module theory for the exchange dynamical quantum group $F(SL(N))$.

Appendix A. Commutation relations in $e^\circ_{\tau,\gamma}(\mathfrak{sl}_N)$

In this section we collect useful commutation relations which hold in the operator algebra $e^\circ_{\tau,\gamma}(\mathfrak{sl}_N)$.

In the definition of $e^\circ_{\tau,\gamma}(\mathfrak{sl}_N)$ one can replace relations (3.5) by the following relations:

\begin{equation}
\frac{\theta(\lambda_{ab}^{(1)}+\lambda_{cd}^{(2)})}{\theta(\lambda_{ab}^{(1)})}\hat{t}_{ab}\hat{t}_{cd} - \frac{\theta(\lambda_{ab}^{(1)}+\gamma)}{\theta(\lambda_{ab}^{(1)})}\hat{t}_{cd}\hat{t}_{ab} = \frac{\theta(\lambda_{ab}^{(1)})^2 \theta(\lambda_{ab}^{(2)})}{\theta(\lambda_{cd}^{(1)})^2 \theta(\lambda_{cd}^{(2)})}\hat{t}_{cb}\hat{t}_{ad},
\end{equation}

for $a \neq c$ and $b \neq d$. The last formula implies that

\begin{equation}
\hat{t}_{aa}\hat{t}_{bb} - \hat{t}_{bb}\hat{t}_{aa} = \frac{\theta(\lambda_{ab}^{(1)}) \theta(\lambda_{ab}^{(2)})}{\theta(\lambda_{ab}^{(1)})^2 \theta(\lambda_{ab}^{(2)})}\hat{t}_{ba}\hat{t}_{ab},
\end{equation}

for $a < b$. Under the same assumption we have

\begin{equation}
\frac{\theta(\lambda_{ab}^{(2)} - \gamma)}{\theta(\lambda_{ab}^{(2)})^2}\hat{t}_{ab}\hat{t}_{ba} - \frac{\theta(\gamma)}{\theta(\lambda_{ab}^{(2)})^2}\hat{t}_{ba}\hat{t}_{ab} = - \frac{\theta(\lambda_{ab}^{(1)}) \theta(\lambda_{ab}^{(2)})}{\theta(\lambda_{ab}^{(1)})^2 \theta(\lambda_{ab}^{(2)})}\hat{t}_{ba}\hat{t}_{ab},
\end{equation}

\begin{equation}
\hat{t}_{ab}\hat{t}_{ba} - \frac{\theta(\lambda_{ab}^{(1)} + \lambda_{cd}^{(2)} \gamma)}{\theta(\lambda_{ab}^{(1)}) \theta(\lambda_{ab}^{(2)})}\hat{t}_{ba}\hat{t}_{ab} = - \frac{\theta(\lambda_{ab}^{(1)}) \theta(\lambda_{ab}^{(2)})}{\theta(\lambda_{ab}^{(1)})^2 \theta(\lambda_{ab}^{(2)})}\hat{t}_{ba}\hat{t}_{ab}.
\end{equation}

More general relations are listed below. We assume that $a < c$ and $b < d$ therein.

\begin{equation}
\hat{t}_{ab}\hat{t}_{cb} = \frac{\theta(\lambda_{ab}^{(1)} + k \gamma)}{\theta(\lambda_{ab}^{(1)})}\hat{t}_{cb}\hat{t}_{ab}, \quad \hat{t}_{ab}\hat{t}_{cb} = \frac{\theta(\lambda_{ab}^{(1)} + \gamma)}{\theta(\lambda_{ab}^{(1)}) - (k - 1) \gamma}\hat{t}_{cb}\hat{t}_{ab},
\end{equation}

\begin{equation}
\hat{t}_{ab}\hat{t}_{ad} = \frac{\theta(\lambda_{ab}^{(2)} - \lambda_{cd}^{(2)} \gamma)}{\theta(\lambda_{ab}^{(2)})^2}\hat{t}_{ad}\hat{t}_{ab}, \quad \hat{t}_{ab}\hat{t}_{ad} = \frac{\theta(\gamma)}{\theta(\lambda_{ab}^{(2)})^2}\hat{t}_{ad}\hat{t}_{ab},
\end{equation}

\begin{equation}
\hat{t}_{ab}\hat{t}_{cd} - \hat{t}_{cd}\hat{t}_{ab} = \frac{\theta(\lambda_{ac}^{(1)} + \lambda_{bd}^{(2)} \gamma)}{\theta(\lambda_{ac}^{(1)}) \theta(\lambda_{bd}^{(2)})}\hat{t}_{cb}\hat{t}_{ad},
\end{equation}

\begin{equation}
\hat{t}_{ad}\hat{t}_{cb} = \frac{\theta(\lambda_{ac}^{(1)} + \gamma)}{\theta(\lambda_{ac}^{(1)}) - \lambda_{bd}^{(2)} \gamma}\hat{t}_{cb}\hat{t}_{ad}, \quad \hat{t}_{ad}\hat{t}_{cb} = \frac{\theta(\lambda_{ac}^{(1)} + \lambda_{bd}^{(2)} \gamma)}{\theta(\lambda_{ac}^{(1)}) \theta(\lambda_{bd}^{(2)})}\hat{t}_{cb}\hat{t}_{ad},
\end{equation}

\begin{equation}
\hat{t}_{ad}\hat{t}_{cb} - \hat{t}_{cb}\hat{t}_{ad} = \frac{\theta(\lambda_{ac}^{(1)} + \lambda_{bd}^{(2)} \gamma)}{\theta(\lambda_{ac}^{(1)}) \theta(\lambda_{bd}^{(2)})}\hat{t}_{cb}\hat{t}_{ad},
\end{equation}

\begin{equation}
\hat{t}_{ad}\hat{t}_{cb} - \hat{t}_{cb}\hat{t}_{ad} = \frac{\theta(\lambda_{ac}^{(1)} + \lambda_{bd}^{(2)} \gamma)}{\theta(\lambda_{ac}^{(1)}) \theta(\lambda_{bd}^{(2)})}\hat{t}_{cb}\hat{t}_{ad}.
\end{equation}
All these formulae follow from (3.2) – (3.5) and (A.1), and the summation formulae for the theta function. Combining formula (A.5) with formulae (A.3) and (A.4) for \( k = 1 \) we can obtain the following Serre-type relations:

\[
\begin{align*}
\theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd}) \theta(\lambda^{(1)}_{ac} - \gamma) \theta(\lambda^{(2)}_{bd} - 2\gamma) \theta(\gamma) \hat{t}_{cd}^k \hat{t}_{ab} - \\
- \theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} - \gamma) \theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd} - \gamma) \theta(2^\gamma) \hat{t}_{ab} \hat{t}_{cd} \hat{t}_{cd} + \\
+ \theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} - 2\gamma) \theta(\lambda^{(1)}_{ac} + \gamma) \theta(\lambda^{(2)}_{bd}) \theta(\gamma) \hat{t}_{cd} \hat{t}_{ab}^2 = 0,
\end{align*}
\]

(A.8)

\[
\begin{align*}
\theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} - 2\gamma) \theta(\lambda^{(1)}_{ac} - \gamma) \theta(\gamma) \hat{t}_{ab} \hat{t}_{cd}^2 - \\
- \theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} + \gamma) \theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd}) \theta(2\gamma) \hat{t}_{cd} \hat{t}_{cd} + \\
+ \theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd}) \theta(\lambda^{(1)}_{ac} + 2\gamma) \theta(\lambda^{(2)}_{bd} + \gamma) \theta(\gamma) \hat{t}_{cd}^2 \hat{t}_{ab} = 0.
\end{align*}
\]

(A.9)

**Lemma A.1.** Let \( a < c \) and \( b < d \). Then the following relations hold:

\[
\hat{t}_{ab} \hat{t}_{cd}^k - \hat{t}_{cd}^k \hat{t}_{ab} = \frac{\theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} + (k-1)\gamma) \theta(k\gamma)}{\theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd} + (k-1)\gamma)} \hat{t}_{cb} \hat{t}_{ad}^{k-1} \hat{t}_{cd},
\]

(A.10)

\[
\hat{t}_{ab}^k \hat{t}_{cd} - \hat{t}_{cd}^k \hat{t}_{ab} = \frac{\theta(\lambda^{(1)}_{ac} + \lambda^{(2)}_{bd} - (k-1)\gamma) \theta(k\gamma)}{\theta(\lambda^{(1)}_{ac} - (k-1)\gamma) \theta(\lambda^{(2)}_{bd})} \hat{t}_{cb} \hat{t}_{ad}^{k-1} \hat{t}_{cd},
\]

(A.11)

**Proof.** For \( k = 1 \) these formulae coincide with formulae (A.5) and (A.7), respectively. All the proofs for \( k > 1 \) are similar to each other. So we will prove only formula (A.11).

Multiply formula (A.6) by \( \hat{t}_{cb}^{k-1} \) from the right, and push the factor \( \hat{t}_{cb}^{k-1} \) in the left hand side through all the products from right to left using relations (A.3). Taking into account formula (A.5) we get

\[
\begin{align*}
\hat{t}_{ad} \hat{t}_{cb}^k - \frac{\theta(\lambda^{(1)}_{ac} + k\gamma) \theta(\lambda^{(1)}_{ac} - \gamma) \theta(\lambda^{(2)}_{bd} - (k-1)\gamma)}{\theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd})} \hat{t}_{cb} \hat{t}_{ad} = \\
= - \frac{\theta(\lambda^{(1)}_{ac} - \lambda^{(2)}_{bd} + (k-1)\gamma) \theta(k\gamma)}{\theta(\lambda^{(1)}_{ac}) \theta(\lambda^{(2)}_{bd})} \hat{t}_{cb} \hat{t}_{ad}. \hat{t}_{cd}.
\end{align*}
\]

where

\[
F(u, v) = \frac{\theta(u + (k-1)\gamma) \theta(v + \gamma) \theta(v - k\gamma)}{\theta(u + v) \theta(v) \theta(\gamma)}.
\]

\( F(u, v) + F(-v, -u) \) is a quasiperiodic function of \( u \) with periods 1 and \( \tau \), and it has only simple poles. Thus, it is completely determined by its multipliers and residues. Hence,

\[
F(u, v) + F(-v, -u) = - \frac{\theta(u - v + (k-1)\gamma) \theta(k\gamma)}{\theta(u) \theta(v)}.
\]

□
Appendix B. Quantum determinant

The construction of \( \text{Det} T(u) \) and the proof of Proposition 2.1 can be obtained by extending the standard fusion procedure technique to the dynamical case.

For any \( k = 2, \ldots, N \) let \( A_k \) be the complete antisymmetric in \( (\mathbb{C}^N)^\otimes k \):

\[
A_k(x_1 \otimes \ldots \otimes x_k) = \frac{1}{k!} \sum_{i \in S_k} \text{sign}(i) \left( x_{i_1} \otimes \ldots \otimes x_{i_k} \right).
\]

Set \( A = A_2 \) and let \( S = 1 - A \) be the corresponding symmetrizer. The \( R \)-matrix \( R(u, \lambda) \) has a simple pole at \( u = \gamma \). Denote by \( Q(\lambda) \) the residue of \( R(u, \lambda) \) at \( u = \gamma \).

**Lemma B.1.** \( \text{Ker} Q(\lambda) = \text{Ker} A = \text{Im} S \).

Due to the inversion relation (1.5) we can write relation (2.3) in the following form:

\[
T^{(13)}(u) T^{(23)}(v) R^{(21)}(u - v, \lambda) = R^{(21)}(u - v, \lambda - \gamma h^{(3)}) T^{(23)}(v) T^{(13)}(u).
\]

Then by Lemma B.1 we have that \( A^{(12)} T^{(23)}(u + \gamma) T^{(13)}(u) S^{(12)} = 0 \), which is equivalent to each of the following relations:

\[
A^{(12)} T^{(23)}(u + \gamma) T^{(13)}(u) S^{(12)} = S^{(12)} T^{(23)}(u) T^{(13)}(u) S^{(12)},
\]

\[
A^{(12)} T^{(23)}(u + \gamma) T^{(13)}(u) = A^{(12)} T^{(23)}(u + \gamma) T^{(13)}(u) A^{(12)}.
\]

Formula (B.2) shows that for any \( i = 1, \ldots, k - 1 \) we have

\[
T^{(k,k+1)}(u + (k - 1) \gamma) \ldots T^{(1,k+1)}(u) S^{(i,i+1)} = S^{(i,i+1)} T^{(k,k+1)}(u + (k - 1) \gamma) \ldots T^{(1,k+1)}(u) S^{(i,i+1)},
\]

which implies

\[
A^{(1,\ldots,k)}_k T^{(k,k+1)}(u + (k - 1) \gamma) \ldots T^{(1,k+1)}(u) = A^{(1,\ldots,k)}_k T^{(k,k+1)}(u + (k - 1) \gamma) \ldots T^{(1,k+1)}(u) A^{(1,\ldots,k)}_k
\]

because \( \text{Ker} A_k = \sum_{i=1}^{k-1} \text{Im} S^{(i,i+1)} \) and \( (\mathbb{C}^N)^\otimes k = \text{Im} A_k \oplus \text{Ker} A_k \). We denote the restriction of

\[
A^{(1,\ldots,k)}_k T^{(k,k+1)}(u + (k - 1) \gamma) \ldots T^{(1,k+1)}(u)
\]

to \( \text{Im} A_k \otimes \text{Fun}(V) \) by \( T^{\wedge k}(u) \) and call it the \( k \)-th exterior power of \( T(u) \). Since \( \text{Im} A_N \) is one-dimensional, the top exterior power \( T^{\wedge N}(u) \) can be considered as an element of \( \text{End} \left( \text{Fun}(V) \right) \).

**Lemma B.2.** For any permutation \( j \in S_N \) we have

\[
T^{\wedge N}(u) = \text{sign}(j) \sum_{i \in S_N} \text{sign}(i) T_{i_N,j_N}(u + (N - 1) \gamma) \ldots T_{i_2,j_2}(u + \gamma) T_{i_1,j_1}(u).
\]

**Proof.** Let \( v_1, \ldots, v_N \) be the standard basis of \( \mathbb{C}^N \). Then for any \( j \in S_N \) we have

\[
A_N(v_{j_1} \otimes \ldots \otimes v_{j_N}) = \text{sign}(j) A_N(v_1 \otimes \ldots \otimes v_N).
\]

Let \( v \in \text{Fun}(V) \). By the definition of \( T^{\wedge N}(u) \) and relation (B.3) we get

\[
A_N(v_{j_1} \otimes \ldots \otimes v_{j_N}) \otimes T^{\wedge N}(u) v = \sum_{i \in S_N} A_N(v_{i_1} \otimes \ldots \otimes v_{i_N}) \otimes T_{i_N,j_N}(u + (N - 1) \gamma) \ldots T_{i_2,j_2}(u + \gamma) T_{i_1,j_1}(u) v.
\]

By formula (B.5) the expression in the right hand side equals

\[
\text{sign}(j) A_N(v_{j_1} \otimes \ldots \otimes v_{j_N}) \otimes \sum_{i \in S_N} \text{sign}(i) T_{i_N,j_N}(u + (N - 1) \gamma) \ldots T_{i_2,j_2}(u + \gamma) T_{i_1,j_1}(u) v,
\]

which proves the lemma. \( \square \)
Like in the ordinary linear algebra there is a connection between the complementary exterior powers \( T^\wedge k(u) \) and \( T^\wedge (N-k)(u) \) and the top exterior power \( T^\wedge N(u) \), cf. Theorem B.4. For any two sequences \( a = (a_1, \ldots, a_k) \) and \( b = (b_1, \ldots, b_k) \) set

\[
T^\wedge ab(u) = \sum_{i \in S_k} \text{sign}(i) T_{a_i b_i} (u + (k-1)\gamma) \ldots T_{a_i b_1} (u + \gamma) T_{a_i b_i} (u).
\]

**Lemma B.3.** Let \( i, j \in S_k \), and let \( a' = (a_1, \ldots, a_k) \), \( b' = (a_1, \ldots, a_k) \). Then

\[
T^\wedge a' b'(u) = \text{sign}(i) \text{sign}(j) T^\wedge ab(u).
\]

The proof is similar to the proof of Lemma B.4.

Denote by \( Y_k \) the set of increasing \( k \)-tuples of integers from \( \{1, \ldots, N\} \). For any \( a \in Y_k \) define \( \bar{a} \in Y_{N-k} \) to be the complement of \( a \), that is, \( \{a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_{N-k}\} = \{1, \ldots, N\} \). Denote by \( a\bar{a} \) the permutation \((a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_{N-k}) \in S_N \).

**Theorem B.4.**

\[
(B.6) \quad \text{sign}(a \bar{a}) \sum_{c \in Y_k} \text{sign}(c \bar{a}) T^\wedge (N-k)_{c \bar{a}}(u - (k-1)\gamma) T^\wedge ab(u) = \delta_{ab} T^\wedge N(u).
\]

The proof is similar to the proof of the analogous formula in the ordinary linear algebra.

It is clear from relation (2.1) and formula (B.4) that the difference operator \( T^\wedge N(u) \) commutes with multiplication by scalar functions. So, there exists a function \( L^\wedge N(u, \lambda) \in \text{Fun(End}(V)) \) such that

\[
(T^\wedge N(u) v)(\lambda) = L^\wedge N(u, \lambda) v(\lambda)
\]

for any \( v \in \text{Fun}(V) \). Denote by \( R^\wedge N(u, \lambda) \) the function \( L^\wedge N(u, \lambda) \) for the vector representation of \( E_{\tau, \gamma}(\mathfrak{S}_N) \) with the evaluation point \( x = 0 \).

**Lemma B.5.**

\[
R^\wedge N(u, \lambda) = \frac{\theta(u + (N-1)\gamma)}{\theta(u)} \sum_{a=1}^{N} \prod_{b=1 \atop b \neq a}^{N} \frac{\theta(\lambda_{ab} - \gamma)}{\theta(\lambda_{ab})} E_{aa}.
\]

**Proof.** Recall that in the vector representation

\[
(T_{aa}(u) v)(\lambda) = E_{aa} v(\lambda - \gamma \varepsilon_a) + \sum_{a,b=1 \atop a \neq b}^{N} \alpha(u, \lambda_{ab}) E_{ab} v(\lambda - \gamma \varepsilon_b)
\]

and \( (T_{ab}(u) v)(\lambda) = \beta(u, \lambda_{ab}) E_{ba} v(\lambda - \gamma \varepsilon_b) \), where \( \alpha(u, \xi) \) and \( \beta(u, \xi) \) are given by (1.2). By Lemma (B.2) this implies that \( R^\wedge N(u, \lambda) \) is a linear combination of the matrices \( E_{11}, \ldots, E_{NN} \) with some functional coefficients. Moreover, taking formula (B.4) for the permutation \( j \) such that \( j_1 = a \) we observe that in the sum only the term with \( i = j \) contributes to the coefficient of \( E_{aa} \), which, therefore, can be easily found. \( \Box \)

**Proof of Proposition 2.1.** Following the definition of \( T^\wedge N(u) \) we find from relations (2.1) and (2.3) that

\[
(R^\wedge N(1)(u - v, \lambda - \gamma h^{(2)})) (T^\wedge N(2)(u) T^{(12)}(v)) = T^{(12)}(v) (T^\wedge N(2)(u) (R^\wedge N(1)(u - v, \lambda + \gamma h^{(1)})).
\]

By Lemma B.5 this is equivalent to

\[
\prod_{c=1 \atop c \neq a}^{N} \frac{\theta(\lambda_{ac} - \gamma h_{ac} - \gamma)}{\theta(\lambda_{ac} - \gamma h_{ac})} T^\wedge N(u) T_{ab}(v) = \prod_{c=1 \atop c \neq b}^{N} \frac{\theta(\lambda_{bc} - \gamma)}{\theta(\lambda_{bc})} T_{ab}(v) T^\wedge N(u).
\]

for any \( a, b = 1, \ldots, N \). Since \( \text{Det} T(u) = \frac{\Theta(\lambda)}{\Theta(\lambda - \gamma h)} T^\wedge N(u) \) where \( \Theta(\lambda) = \prod_{1 \leq a < b \leq N} \theta(\lambda_{ab}) \), the proposition is proved. \( \Box \)
Appendix C. Multiplicative forms

In this section we essentially follow [EV1, Section 1.4]. Notice that all over the paper the words\cocycle\ and \coboundary\ mean \1-cocycle\ and \1-coboundary,\ respectively.

Let \( I_k \) be the set of \( k \)-tuples of pairwise distinct integers from \( \{1, \ldots, N\} \). A multiplicative \( k \)-form \( \bar{f} \) is a map \( I_k \to \Fun(\mathbb{C}) \), \( \bar{f}: a \mapsto f_a \), such that for any \( a \in I_k \) and any \( i = 1, \ldots, k \) we have \( f_{a_1, \ldots, a_i(\lambda)} f_{a_1, \ldots, a_{i-1}, a_{i+1}, a_{i+2}, \ldots, a_k(\lambda)} = 1 \)

Let \( \Omega^k \) be the set of all multiplicative \( k \)-forms. If \( \bar{f} \) and \( \bar{g} \) are multiplicative \( k \)-forms, then \( \bar{f} \bar{g}: a \mapsto f_a g_a \) and \( \bar{f} / \bar{g} : a \mapsto f_a / g_a \) are multiplicative \( k \)-forms, which defines an abelian group structure on \( \Omega^k \). The neutral element is the form \( \lambda: I_a(\lambda) = 1 \) for any \( a \in I_k \).

For any nonzero function \( f(\lambda) \) and any \( a = 1, \ldots, N \) set \( (\delta_a f)(\lambda) = f(\lambda - \gamma \varepsilon_a) / f(\lambda) \), and for any \( \bar{f} \in \Omega^k \) define a multiplicative form \( d\bar{f} \in \Omega^{k+1} \) by \( (d\bar{f})_{a_1, \ldots, a_{k+1}(\lambda)} = \sum_{i=1}^{k+1} (\delta_{a_i} f_{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}}(\lambda))(-1)^{i-1} \).

For any multiplicative form \( \bar{f} \) we have \( d^2 \bar{f} = 1 \). The multiplicative form \( \bar{f} \) is called a multiplicative cocycle if \( d\bar{f} = \bar{1} \), and a multiplicative coboundary if \( \bar{f} = d\bar{g} \) for a suitable multiplicative form \( \bar{g} \).

For any meromorphic function \( f(\xi) \) in one variable the multiplicative 1-form \( \bar{F} = (F_1, \ldots, F_N) \):

\[
F_a(\lambda) = \prod_{1 \leq c < a} f(\lambda_{ac}) \prod_{a < c \leq N} (f(\lambda_{ac} - \gamma))^{-1},
\]

is a multiplicative 1-cocycle. If \( f(\xi) = g(\xi + \gamma) / g(\xi) \), then \( \bar{F} \) is a multiplicative 1-coboundary: \( \bar{F} = dG \), where \( G(\lambda) = \prod_{1 \leq b < c \leq N} g(\lambda_{bc}) \).

If \((F_1, \ldots, F_N)\) is a multiplicative 1-cocycle, then so is \((G_1, \ldots, G_N)\): \( G_a(\lambda) = F_a(-\lambda + \gamma \varepsilon_a) \).

Appendix D. Proof of Theorem 4.1

In this section we introduce another ordering on generators of \( e^\mathfrak{g}_{\Gamma, \gamma}(\mathfrak{sl}_N) \), called ordering by rows, and prove the analogue of Theorem 4.1 for the monomials ordered by rows, cf. Theorem D.1. Since the number or ordered by rows monomials of degree \( k \) equals the number of normally ordered monomials of the same degree, and each monomial can be transformed to a linear combination of normally ordered monomials, Theorem D.1 implies Theorem 4.1. We introduce ordered by rows monomials for technical reason because this allows us to reduce the number of cases to be examined at some stage of the proof.

Consider the ordering by rows of generators: \( t_{ab} < t_{cd} \) if \( a < c \), or \( a = c \) and \( b < d \). Say that the monomial \( t_{a_1b_1} \cdots t_{a_kb_k} \) is ordered by rows if \( t_{a_ib_i} < t_{a_jb_j} \) for any \( i < j \), or \( k = 0 \).

**Theorem D.1.** For any \( k \in \mathbb{Z}_{\geq 0} \) the ordered by rows monomials of degree \( k \) form a basis of \( \mathfrak{e}_k \) over \( \Fun^\otimes 2(\mathbb{C}) \).

**Proof.** To save space from now on we write ordered instead of ordered by rows. We call an equality

\[
\text{disordered monomial} = \text{linear combination of ordered monomials}
\]

an ordering rule for the monomial in the left hand side. The commutation relations (3.3)–(3.5) have the important property:

A. Any relation is a linear combination of ordering rules, and the complete list of linear independent ordering rules contains precisely one rule for each disordered product of generators.

For \( k = 0 \) and \( k = 1 \) the claim of Theorem D.1 is immediate. Let \( k > 1 \). First we prove that ordered monomials of degree \( k \) span \( \mathfrak{e}_k \) over \( \Fun^\otimes 2(\mathbb{C}) \). Indeed, one can transform any monomial \( t_{a_1b_1} \cdots t_{a_kb_k} \) to a linear combination of ordered monomials by the following procedure. Pick up any disordered product of adjacent factors and replace it by a sum of ordered products using the ordering rule, then do the same for each of the obtained monomials. To see that the procedure terminates and,
hence, produces a linear combination of ordered monomials, introduce auxiliary gradings on monomials by the rule

\[ r(t_{a_1 b_1} \ldots t_{a_k b_k}) = \sum_{i=1}^{k} i a_i, \quad r'(t_{a_1 b_1} \ldots t_{a_k b_k}) = \sum_{i=1}^{k} i b_i, \]

and observe that at each nontrivial step of the procedure we replace a monomial by a sum of monomials of either less degree \( r \), or the same degree \( r \) and less degree \( r' \). We call the described procedure a regular transformation of the monomial \( t_{a_1 b_1} \ldots t_{a_k b_k} \) to a linear combination of ordered monomials.

If an ordering rule is applied to a product \( t_{a_1 b_1} t_{a_{i+1} b_{i+1}} \) in \( t_{a_1 b_1} \ldots t_{a_k b_k} \), we say that the ordering rule is applied at \( i \)-th place.

By the standard reasoning the property A of the commutation relations (3.3) – (3.5) implies that Theorem D.1 follows from Proposition D.2.

**Proposition D.2.** Any regular transformation of the monomial \( t_{a_1 b_1} \ldots t_{a_k b_k} \) produces the same linear combination of ordered monomials.

**Proof.** For \( k = 2 \) the claim follows from the property A. For \( k = 3 \) the claim can be verified in a straightforward way. We discuss more details of \( k = 3 \) case at the end of the proof.

Let \( k > 3 \). For the proof we use induction with respect to the lexicographical ordering on monomials defined by a pair of degrees \( (r, r') \). The claim of the proposition is obvious for ordered monomials, which provides the base of induction.

Let I and II be two regular transformations of the monomial \( t_{a_1 b_1} \ldots t_{a_k b_k} \) to linear combinations of ordered monomials. If for I and II the first steps coincide, then they produce the same results by the induction assumption. Otherwise, let us construct two additional regular transformations III and IV such that the first steps of I and III coincide, the same holds for I II and IV, and III and IV produce the same results. By the previous remark this proves the proposition.

Assume that for the transformation I an ordering rule at the first step is applied at \( i \)-th place, and for the transformation II at \( j \)-th place. If \(|i - j| > 1\) then we define the transformation III as follows: first apply an ordering rule at \( i \)-th place, next apply an ordering rule at \( j \)-th place for all monomials obtained at the first step, then continue in any possible way. The transformation IV is defined similarly with \( i \) and \( j \) interchanged. By the induction assumption the transformations III and IV produce the same results because after the first two steps of both III and IV one has identical linear combinations of monomials, each of them being lexicographically smaller than the initial monomial \( t_{a_1 b_1} \ldots t_{a_k b_k} \).

The cases \( j = i \pm 1 \) are similar. Assume for example that \( j = i + 1 \). Define the transformation III as follows: apply a regular transformation of the product \( t_{a_{i+1} b_{i+1}} t_{a_{i+2} b_{i+2}} \) to a linear combination of ordered triple products, making the first step at \( i \)-th place, then continue in any possible way. Define the transformation IV similarly, but making the first step at \((i + 1)\)-th place. Then the claim of the proposition for \( k = 3 \) shows that after the first stages of both III and IV one has identical linear combinations of monomials, each of them being lexicographically smaller than the initial monomial \( t_{a_{1} b_{1}} \ldots t_{a_{k} b_{k}} \). Therefore, by the induction assumption the transformations III and IV produce the same results.

It remains to complete the proof for \( k = 3 \). For any monomial \( t_{a_{1} b_{1}} t_{a_{2} b_{2}} t_{a_{3} b_{3}} \) there are at most two regular transformations, and the regular transformation is unique unless \( t_{a_{3} b_{3}} < t_{a_{2} b_{2}} < t_{a_{1} b_{1}} \). The rest of the proof is given by the straightforward calculations. The simplest cases occur if \( a_{1} = a_{2} = a_{3} \) or \( b_{1} = b_{2} = b_{3} \). We present below the most bulky example when \( a_{1} > a_{2} > a_{3} \) and \( b_{1}, b_{2}, b_{3} \) are pairwise distinct.

**Example.** Regular transformations of the monomial \( t_{34} t_{25} t_{16} \). Let functions \( \alpha(u, \xi) \) and \( \beta(u, \xi) \) be given by (1.2). Making the first step at the first place we have

\[
t_{34} t_{25} t_{16} = \alpha(\lambda^{(1)}_{32}, \lambda^{(2)}_{45}) t_{25} t_{34} t_{16} - \beta(\lambda^{(1)}_{32}, \lambda^{(2)}_{45}) t_{24} t_{35} t_{16} = \]

\[
= \alpha(\lambda^{(1)}_{32}, \lambda^{(2)}_{45}) (\alpha(\lambda^{(1)}_{31}, \lambda^{(2)}_{46}) t_{25} t_{16} t_{34} - \beta(\lambda^{(1)}_{31}, \lambda^{(2)}_{46}) t_{25} t_{14} t_{36}) + \\
- \beta(\lambda^{(1)}_{32}, \lambda^{(2)}_{45}) (\alpha(\lambda^{(1)}_{31}, \lambda^{(2)}_{46}) t_{24} t_{16} t_{35} - \beta(\lambda^{(1)}_{31}, \lambda^{(2)}_{46}) t_{24} t_{15} t_{36}) = \]

\[ \]
= \alpha(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) \alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{25} t_{34} - \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{26} t_{34}) + \\
- \alpha(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) \beta(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{25} t_{34} - \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{26} t_{34}) + \\
- \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) \alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{24} t_{35} - \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{26} t_{35}) + \\
+ \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) \beta(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{24} t_{36} - \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{24} t_{36}),

while making the first step at the second place we have

\[ t_{34} t_{25} t_{16} = \alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{34} t_{16} t_{25} - \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{34} t_{15} t_{26} = \]

\[ = \alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{34} t_{25} - \beta(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{36} t_{25}) + \\
- \beta(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{34} t_{26} - \beta(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{35} t_{26}) = \]

\[ = \alpha(\lambda_{21}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{32}^{(1)}, \lambda_{46}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{25} t_{34} - \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{25} t_{34}) + \\
- \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{26} t_{34} - \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{26} t_{34}) + \\
- \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{16} t_{24} t_{35} - \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{24} t_{35}) + \\
+ \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) (\alpha(\lambda_{31}^{(1)}, \lambda_{34}^{(2)}) t_{15} t_{24} t_{36} - \beta(\lambda_{32}^{(1)}, \lambda_{34}^{(2)}) t_{14} t_{24} t_{36}). \]

The coefficients of the monomials \( t_{16} t_{25} t_{34}, t_{15} t_{26} t_{34} \) and \( t_{16} t_{24} t_{35} \) coincide identically. To check that the coefficients for other monomials one should take into account the explicit form of \( \alpha(u, \xi) \) and \( \beta(u, \xi) \) and use summation formulae for the theta function.

**Appendix E. Elliptic quantum group** \( e_{r, \gamma}(sl_2) \)

This section is an illustration of general constructions in the simplest case \( N = 2 \). In this case \( \omega_1 = \varepsilon_1 = -\varepsilon_2 = \rho = a_1/2 \) and \( h^* = \mathbb{C} \omega_1 \). For any \( \lambda \in h^* \) we have \( \lambda_1 = -\lambda_2 = \lambda_1/2 \) and \( \lambda = \lambda_2 \omega_1 \). We identify functions on \( h^* \) with functions of one variable \( \lambda_2 \).

The operator algebra \( e^{\infty}_{r, \gamma}(sl_2) \) is generated over \( \mathbb{C} \) by elements \( t_{11}, t_{12}, t_{21}, t_{22} \) and functions \( f \in Fun^{\otimes 2}(\mathbb{C}) \). According to Theorem 4.1 monomials \( t_{21} t_{11} t_{22} t_{12} \), \( k_{11}, k_{12}, k_{21}, k_{22} \in \mathbb{Z}_{\geq 0} \), form a basis of \( e^{\infty}_{r, \gamma}(sl_2) \) as a \( Fun^{\otimes 2}(\mathbb{C}) \)-module.

Let \( \hat{Q} = (Q_1, Q_2) \) be a multiplicative cocycle, which in this case means that

\[ Q_1(\lambda_1) Q_2(\lambda_2 - \gamma) = Q_1(\lambda_1 + \gamma) Q_2(\lambda_2). \]

A Verma module \( M_{\mu, \hat{Q}} \) of highest weight \( \mu \in h^* \) and dynamical highest weight \( \hat{Q} \) over \( e_{r, \gamma}(sl_2) \) is constructed as follows. As an \( h \)-module \( M_{\mu, \hat{Q}} = \bigoplus_{k=0}^{\infty} \mathbb{C} v_{ \mu, \hat{Q} } [k] \), the vector \( v_{ \mu, \hat{Q} } [k] \) being of weight \( \mu - k \alpha_1 \). Notice that \( v_{\mu, \hat{Q}} = v_{\mu, \hat{Q}} [0] \). The generators \( t_{11}, \hat{t}_{12}, \hat{t}_{21}, \hat{t}_{22} \) act on \( Fun(M_{\mu, \hat{Q}}) \) by the rule

\[ \hat{t}_{21} v_{ \mu, \hat{Q} } [k] = v_{ \mu, \hat{Q} } [k + 1], \]

\[ \hat{t}_{11} v_{ \mu, \hat{Q} } [k] = Q_1(\lambda_2) Q_2(\lambda_2 + \lambda_2 - \gamma) \frac{\theta(\alpha_2 - \beta_2 - \gamma)}{\theta(\alpha_2 + \gamma)} v_{ \mu, \hat{Q} } [k], \]

\[ \hat{t}_{22} v_{ \mu, \hat{Q} } [k] = Q_2(\lambda_2) Q_2(\lambda_2 + \lambda_2 - \gamma) \frac{\theta(\alpha_2 - \beta_2 - \gamma)}{\theta(\alpha_2 + \gamma)} v_{ \mu, \hat{Q} } [k], \]

\[ \hat{t}_{12} v_{ \mu, \hat{Q} } [k] = Q_1(\lambda_2 + \lambda_2 - \gamma) Q_2(\lambda_2 + \lambda_2 - \gamma) \frac{\theta(\alpha_2 - \beta_2 - \gamma)}{\theta(\alpha_2 + \gamma)} v_{ \mu, \hat{Q} } [k - 1], \]
see (8.1), (8.2) and Lemma 8.1. Taking into account relations (4.1) and (3.6), and Corollary 3.4, one
reproduces from these formulae the construction, given in [FV1], of the evaluation Verma module over
$E_{\tau,\gamma}(\mathfrak{sl}_2)$ tensored with a one-dimensional representation of $E_{\tau,\gamma}(\mathfrak{sl}_2)$.

In general, to compute the dynamical Shapovalov form on $e^\mathcal{O}_{\tau,\gamma}(\mathfrak{sl}_2)$ one has to find

\begin{equation}
S(f(\lambda^{(1)}_{12},\lambda^{(2)}_{12})) i^{k_{21}} k_{11}^{k_{22}} k_{12}^{k_{12}}, g(\lambda^{(1)}_{12},\lambda^{(2)}_{12}) i^{m_{21}} m_{11}^{m_{22}} m_{12}^{m_{12}}
\end{equation}

for any $f, g \in \text{Fun}(\mathbb{C})$ and any monomials $i^{k_{21}} k_{11}^{k_{22}} k_{12}^{k_{12}}, i^{m_{21}} m_{11}^{m_{22}} m_{12}^{m_{12}}$. The answer is zero unless $k_{12} = m_{12} = 0$ and $k_{21} = m_{21}$. Moreover, contributions of the functions $f, g$ and factors $t_{aa}$ are easy to calculate, and we find that in the nonzero case expression (E.1) equals

\begin{equation}
f(-\lambda^{(2)}_{12} + (k_{11} - k_{21} - k_{22})\gamma, -\lambda^{(1)}_{12} + (k_{11} + k_{21} - k_{22})\gamma) \times
\end{equation}

\begin{equation}
\times g(\lambda^{(1)}_{12} - (k_{11} + k_{21} - k_{22})\gamma, \lambda^{(2)}_{12} - (k_{11} - k_{21} - k_{22})\gamma) q_{1}^{k_{11}} q_{2}^{k_{22}} S(i^{k_{21}}, i^{k_{21}}) q_{1}^{m_{11}} q_{2}^{m_{22}}
\end{equation}

where

\begin{equation}
S(i^{k_{21}}, i^{k_{21}}) = (-1)^{k} \prod_{j=0}^{k-1} \frac{\theta(\lambda^{(1)}_{12} - \lambda^{(2)}_{12} - j\gamma)}{\theta(\lambda^{(1)}_{12} - j\gamma)\theta(\lambda^{(2)}_{12} + j\gamma)} q_{1}^{k} q_{2}^{k}, \quad k \in \mathbb{Z}_{\geq 0}.
\end{equation}

For the corresponding part of the dynamical Shapovalov pairing for $M_{\mu,\hat{Q}}$ we have

\begin{equation}
S_{\mu,\hat{Q}}(i^{k_{21}}, v_{\mu,\hat{Q}}[k]) = (-1)^{k} \prod_{j=0}^{k-1} \left( Q_{1}(\lambda_{12} + (j + 1)\gamma) Q_{2}(\lambda_{12} + j\gamma) - \frac{\theta(\mu_{12} - j\gamma)\theta((j + 1)\gamma)}{\theta(\lambda_{12} - j\gamma)\theta(\lambda_{12} - (\mu_{12} - j)\gamma)} \right).
\end{equation}

The contravariant form $C_{\mu,\hat{Q}} : \text{Fun}(M_{\mu,\hat{Q}}) \otimes_{\mathbb{C}} \text{Fun}(M_{\mu,\hat{Q}}) \rightarrow \text{Fun}(\mathbb{C})$ is given by

\begin{equation}
C_{\mu,\hat{Q}}(v_{\mu,\hat{Q}}[k], v_{\mu,\hat{Q}}[l]) = \delta_{kl} (-1)^{k} \prod_{j=0}^{k-1} \left( Q_{1}(\lambda_{12} + (j + 1)\gamma) Q_{2}(\lambda_{12} + j\gamma) \times
\end{equation}

\begin{equation}
\times \frac{\theta(\mu_{12} - j\gamma)\theta((j + 1)\gamma)}{\theta(\lambda_{12} + (j + 1)\gamma)\theta(\lambda_{12} - (\mu_{12} - j - k)\gamma)} \right).
\end{equation}

Let $\gamma \notin \mathbb{Q} + \tau\mathbb{Q}$ and assume that $\hat{Q}$ is nondegenerate. Then the module $M_{\mu,\hat{Q}}$ is irreducible provided that $\mu_{12} \notin \mathbb{Z}_{\geq 0}$. If $\mu_{12} \in \mathbb{Z}_{\geq 0}$, then $v_{\mu,\hat{Q}}[\mu_{12} + 1]$ is a regular singular vector generating an irreducible submodule $\mathcal{N}_{\mu,\hat{Q}}$ isomorphic to $M_{-\mu-2\mu,\hat{Q}}$ where $\hat{Q}(\lambda_{12}) = \hat{Q}(\lambda_{12} - \mu_{12} - 2)$. The quotient module $V_{\mu,\hat{Q}} = M_{\mu,\hat{Q}}/\mathcal{N}_{\mu,\hat{Q}}$ is the irreducible highest weight $e_{\tau,\gamma}(\mathfrak{sl}_2)$-module with highest weight $\mu$ and dynamical highest weight $\hat{Q}$, and it has dimension $\mu_{12} + 1$, the same as the irreducible $\mathfrak{sl}_2$-module of highest weight $\mu$.

**Appendix F. Proof of Theorem 10.5**

In [TV2, Section 2.6] for any semisimple Lie algebra $\mathfrak{g}$ we have defined rational functions $B_w(\lambda)$ of $\lambda$ labeled by elements of the Weyl group. The functions takes values in a certain completion of $U(\mathfrak{g})$. Here we need the function $B_w(\lambda - \rho)$ for the particular case of $\mathfrak{g} = \mathfrak{sl}_N$ and $w = w_X$, the longest element of the Weyl group. For brevity we denote this function by $B_X(\lambda)$. We list required properties of $B_X(\lambda)$ below using the notation of the present paper. All of them easily follow from the properties of $B_w(\lambda)$ given in [TV2].

Consider the following series

\begin{equation}
G_{ab}(\lambda) = \sum_{k=0}^{\infty} e_{aa}^{k_{ab}} e_{bb}^{k_{ab}} \prod_{j=1}^{k} \frac{1}{j(\lambda_{ab} - e_{aa} + e_{bb} - j)}.
\end{equation}

Then $B_X(\lambda)$ equals the ordered product $\prod_{1 \leq a < b \leq N} G_{ab}(\lambda)$ where the factor $G_{ab}$ is to the left from the factor $G_{cd}$ if $a < c$, or $a = c$ and $b > d$. For instance, if $N = 3$, then $B_X(\lambda) = G_{12}(\lambda) G_{13}(\lambda) G_{23}(\lambda)$. 
The series \( B_X(\lambda) \) acts on any finite-dimensional \( \mathfrak{sl}_N \)-module \( V \), commuting with the \( h \)-action. The action of \( B_X(\lambda) \) gives an element of \( \text{Rat}(\End(V)) \), which tends to 1 as \( \lambda \) goes to infinity in a generic direction and, therefore, is invertible for generic \( \lambda \).

**Proposition F.1.** For any finite-dimensional \( \mathfrak{sl}_N \)-modules \( V, W \) we have

\[
B_X(\lambda)|_{V \otimes W} (X \otimes X) J_{V,W}(w_X(\lambda))(X \otimes X)^{-1} = J_{V,W}(\lambda) (B_X(\lambda-h^{(2)}) \otimes B_X(\lambda)).
\]

**Corollary F.2.** \( \tilde{R}_{V,W}(\lambda - \rho) = (B_X(\lambda-h^{(2)}) \otimes B_X(\lambda))^{-1}R_{V,W}(\lambda)(B_X(\lambda) \otimes B_X(\lambda-h^{(1)})) \).

The statement follows from Corollary 9.8 and cocommutativity of the coproduct \( \Delta \).

**Lemma F.3.** Let \( U \) be the vector representation of \( \mathfrak{sl}_N \). Then \( B_X(\lambda)|_U = \sum_{a=1}^N E_{aa} \prod_{1 \leq b < a} (1 + \lambda_{ba}^{-1}) \).

**Proof of Theorem 10.5.** By the definition of isomorphic functors the assertion of the theorem means that for any finite-dimensional \( \mathfrak{sl}_N \)-module \( V \) there exists an isomorphism \( \psi_V \in \text{Mor}(\tilde{\mathcal{E}}(V), \mathcal{E}(V)) \), and for any morphism \( \phi : V \to W \) of \( \mathfrak{sl}_N \)-modules one has \( \mathcal{E}(\phi) \circ \psi_V = \psi_W \circ \tilde{\mathcal{E}}(\phi) \).

It follows from formula (10.8), Proposition 10.2, Corollary F.2 and Lemma F.3 that \( B_X(\lambda)|_V \) is a morphism from \( \tilde{\mathcal{E}}(V) \) to \( \mathcal{E}(V) \). It is an isomorphism, since \( B_X(\lambda)|_V \) is invertible for generic \( \lambda \). Moreover, for any morphism \( \phi : V \to W \) of \( \mathfrak{sl}_N \)-modules we have \( \phi B_X(\lambda)|_W = B_X(\lambda)|_W \phi \). Since both \( \mathcal{E} \) and \( \tilde{\mathcal{E}} \) send \( \phi \) to the constant function \( \phi \in \text{Rat}(\text{Hom}(V,W)) \), the theorem is proved. \qed

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Contents

Introduction .......................................................... 1
1. Basic notation ......................................................... 2
2. Elliptic quantum group $E_{\tau,\gamma}(\mathfrak{sl}_N)$ .................. 4
3. Small elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_N)$ ............ 5
4. Highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$ ............... 8
5. Dynamical Shapovalov form ......................................... 10
6. Contragradient modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$ and contravariant form ................. 12
7. Rational dynamical quantum group $e_{\text{rat}}(\mathfrak{sl}_N)$ ........ 15
8. Finite-dimensional highest weight modules over $e_{\tau,\gamma}(\mathfrak{sl}_N)$ .................. 18
9. Definition of functor $E$ ............................................. 21
10. Exchange quantum group $F(SL(N))$ ............................ 24

Appendices

A. Commutation relations in $e_{\tau,\gamma}^O(\mathfrak{sl}_N)$ .................. 26
B. Quantum determinant .................................................. 28
C. Multiplicative forms .................................................... 30
D. Proof of Theorem 4.1 .................................................. 30
E. Elliptic quantum group $e_{\tau,\gamma}(\mathfrak{sl}_2)$ ....................... 32
F. Proof of Theorem 10.5 .................................................. 33

References ............................................................ 34