Symmetries on the lattice of $k$-bounded partitions.*†

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Abstract

We generalize the symmetry on Young’s lattice, found by Suter [Su1], to a symmetry on the $k$-bounded partition lattice of Lapointe, Lascoux and Morse [LLM].

1 Introduction

In [Su1], Suter found a dihedral symmetry which exists in Young’s lattice, by taking all partitions whose bounding rectangle is contained within the staircase $(k, k-1, k-2, \ldots, 2, 1)$. He recognized that these partitions would have the same symmetries as the affine Dynkin diagram of type $A_k$.

While studying $k$-Schur functions, we noticed that the rectangles which Suter uses are the same rectangles that appear in Morse and Lapointe’s

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paper [LM3]. The rectangles in this picture correspond to special elements of the homology of the affine Grassmannian [L1, L2, L3]. For this reason, the lattice of $k$-bounded partitions related to the algebra of $k$-Schur functions is a natural place to view a generalization of the symmetry observed by Suter.

Recent results of Berg, Bergeron, Thomas and Zabrocki [BBTZ] developed some geometric properties of the affine hyperplane arrangement. We use this geometric picture to generalize the symmetry that Suter found to the $k$-bounded partition lattice of Lapointe, Lascoux and Morse [LLM]. We do this by recognizing that the $k$-bounded partitions which are contained in a concatenation of $r$ rectangles with a $k$ hook is isomorphic to an $r$-dilation in the geometric picture.

1.1 From root systems in type $A_k$ to the affine Grassmannian

Let $\alpha_1, \ldots, \alpha_k$ denote the simple roots of type $A_k$, which form a basis for a vector space $V$. $V$ has a symmetric bilinear form given by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{else.} \end{cases}$$

and we let $\{\Lambda_i\}_{1 \leq i \leq k}$ denote the basis dual to $\{\alpha_i\}_{1 \leq i \leq k}$ under this bilinear form. The $\mathbb{Z}$ span of the $\{\Lambda_i\}_{1 \leq i \leq k}$ will be called the weights.

For $v \in V$, we let $H_v$ denote the hyperplane through the origin, perpendicular to $v$. We write $H_i$ for $H_{\alpha_i}$ and $H_{v,p}$ for the points $x$ satisfying $\langle v, x \rangle = p$.

Let $s_i$ represent the reflection of a vector $v$ through the hyperplane $H_i$ so that the set of reflections $s_1, \ldots, s_k$ corresponding to the roots $\alpha_1, \ldots, \alpha_k$ generate a reflection group $W_0$ which is isomorphic to the symmetric group $S_{k+1}$. The corresponding (finite) root system is $\Phi_0$ is the closure of the set of vectors $\{\alpha_i\}_{1 \leq i \leq k}$ under the action of $W_0$. The element $\phi = \alpha_1 + \cdots + \alpha_k$ is known as the highest root of the root system.

The affine arrangement is the collection of all hyperplanes $H_{\alpha,p}$ for $\alpha \in \Phi_0$ and $p \in \mathbb{Z}$.

The dominant chamber is the (closed) collection of points in $V$ which are bounded by the hyperplanes $H_{\alpha_i,0}$. We denote it by $C$. A weight is called dominant if it lies in the dominant chamber.
The fundamental alcove is bounded by the walls of the dominant chamber, together with the hyperplane \( H_{\phi,1} \). We denote it by \( A_{\emptyset} \).

The affine reflection group, \( W \), has an additional generator \( s_0 \), which acts as reflection in \( H_{\phi,1} \). The generators \( s_0, s_1, \ldots, s_k \) satisfy the affine type A Coxeter relations:

\[
\begin{align*}
    s_i^2 &= 1 \text{ for } i \in \{0, 1, \ldots, k\} \\
    s_is_j &= s_js_i \text{ if } i - j \neq \pm 1 \\
    s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \text{ for } i \in \{0, 1, \ldots, k\}
\end{align*}
\]

where \( i - j \) and \( i + 1 \) are understood to be taken modulo \( k + 1 \).

There is an action of \( W \) on \( V \) defined by \( s_i \) reflecting across the hyperplane \( H_i \) for \( i \in \{1, 2, \ldots, k\} \) and \( s_0 \) reflecting across the hyperplane \( H_{\phi,1} \).

We let \( A_w := w^{-1}A \). The collection of \( A_w \) are called the alcoves of the affine arrangement. The hyperplanes \( H_{\alpha_i,n} \) will intersect with \( A_w \) either in the empty set, at a single weight, or in a facet of the alcove (the convex hull of \( k \) of the vertices of \( A_w \)). An alcove \( A_w \subset C \) if and only if \( w \) is a minimal length coset representative of \( W/W_0 \). The set of minimal length coset representatives is denoted \( W_0 \). A permutation \( w \in W^0 \) is called an affine Grassmannian permutation.

A partition \( \lambda \) is called a \((k + 1)\)-core if \( \lambda \) has no removable \((k + 1)\)-rim hook. Define the size of a \((k + 1)\)-core, \( |\lambda| \), to be the number of cells \((i, j)\) with hook smaller than \( k + 1 \) where the hook of a cell is \( \lambda_i + \lambda_j' - i - j + 1 \).

Let \( C^{(k+1)} \) denote the set of all \((k + 1)\)-cores.

\( W \) has an action on \( C^{(k+1)} \). Let the content of a cell \((i, j)\) in the Young diagram of \( \lambda \) be the integer \( j - i \mod k + 1 \). If \( \lambda \) is a \((k + 1)\)-core then \( s_i\lambda \) is \( \lambda \) union all addable cells of content \( i \), if \( \lambda \) has such an addable cell, \( s_i\lambda \) is \( \lambda \) minus all removable boxes of content \( i \) from \( \lambda \) if \( \lambda \) has such a removable box (a \((k + 1)\)-core cannot have both a removable box and an addable position of the same content), and \( s_i\lambda = \lambda \) otherwise.

**Proposition 1.1.** [Lascoux] There is a bijection between affine Grassmannian permutations of length \( r \) and the set of \((k + 1)\)-cores of size \( r \) by sending \( w \in W^0 \) to \( w\emptyset \).

## 2 Background: Suter symmetry

For a fixed positive integer \( k \), let \( R_1 = (1^k), R_2 = (2^{k-1}), \ldots, R_k = (k) \) denote the rectangular partitions which have largest hook length equal to \( k \). Let \( Y^k \)
denote the (finite) sublattice of Young’s lattice which contains everything smaller than \( R_1, R_2, \ldots, R_k \), i.e. \( Y^k = \{ \lambda : \lambda \subset R_i \text{ for some } i \} \).

Suter [Su1] noticed that \( Y^k \) had a dihedral symmetry, coming from the usual symmetry of partition transposition, as well as a \( k \)-fold rotational symmetry, as pictured in Figure 1.

Suter defined a cyclic action on \( Y^k \) of order \( k+1 \), described on a Young diagram of a partition. We will not present this here; our generalization comes from a different description of this cyclic action which we now introduce.

### 2.1 Suter symmetry on alcoves

Since every partition in \( Y^k \) is a \((k+1)\)-core, we can associate each partition \( \lambda \in Y^k \) with some affine Grassmannian permutation, or equivalently, to an alcove \( A_w \) in the dominant chamber. It was noticed by Suter in [Su2] that all partitions whose hook is smaller than or equal to \( k \) are in bijection with the alcoves in the fundamental chamber bounded by \( H_{\phi, 2} \). The elements of \( Y^k \), viewed as alcoves, now form a 2 fold dilation of the fundamental alcove. The fundamental alcove has a \((k+1)\) cyclic symmetry (cycling the vertices of the dialated alcove) and so the elements of \( Y^k \) also have this symmetry. We will generalize this version of Suter symmetry in Section 4.

### 3 Combinatorics of \( k \)-bounded partitions

Lapointe and Morse [LM2] introduced a bijection between \((k+1)\)-cores and \( k \)-bounded partitions (a partition is \( k \)-bounded if all of it’s parts are less than or equal to \( k \)). The bijection is sends a \((k+1)\)-core \( \mu \) to the \( k \)-bounded partition \( \lambda \) whose \( i^{th} \) part is equal to the number of cells \((i, j)\) in \( \mu \) with hook less than \( k+1 \). For a \((k+1)\)-core \( \mu \), we let \( p(\mu) \) denote the corresponding \( k \)-bounded partition, and we will let \( c \) denote the inverse map (so \( c(p(\mu)) = \mu \)).

Lapointe, Lascoux and Morse [LLM] introduced a \( k \)-version of Young’s lattice. It is a sublattice of Young’s lattice whose vertices are labeled by \( k \)-bounded partitions. It is the lattice generated by the covering relation \( \lambda \lessdot \mu \) if \( |\lambda| + 1 = |\mu| \) and \( s_i c(\lambda) = c(\mu) \) for some \( i = 0, 1, \ldots, k \).

The rectangles \( R_1, \ldots, R_k \) described above play an important role in the study of \( k \)-Schur functions. \( k \)-Schur functions, first introduced by Lapointe, Lascoux and Morse [LLM], were motivated in the study of Macdonald polynomials, but have since appeared in other contexts (see, in particular,
Figure 1: Three examples of the $k + 1$ dihedral symmetry of $Y^k$ for $k \in \{2, 3, 4\}$.
Each $k$-Schur function $s^{(k)}_\lambda$ is indexed by a $k$-bounded partition $\lambda$ (or equivalently a $(k+1)$-core, or an affine Grassmannian permutation).

An important open problem in the study of $k$-Schur functions is to understand their multiplication rule. One special case is very explicitly understood, due to the following theorem of Lapointe and Morse. For two partitions $\lambda$ and $\mu$, let $\lambda \cup \mu$ denote the partition obtained by combining the parts of $\lambda$ and $\mu$ and placing them into non-increasing order.

**Theorem 3.1** (Lapointe, Morse [LM3]). $s^{(k)}_\lambda s^{(k)}_\mu = s^{(k)}_{\lambda \cup \mu}$ for a rectangle $R = R_1, \ldots, R_k$.

### 4 Generalized Suter symmetry

We now fix an integer $m > 1$. With Theorem 3.1 in mind, we will study all partitions contained in a product of $m$ rectangles. Let $Y^k_m$ denote the subposet of the $k$-Young’s lattice which contains all partitions contained in a stack of $m - 1$ of the $k$-rectangles (so $\lambda \in Y^k_m$ if $\lambda \subseteq R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-1}}$ for some $i_1, \ldots, i_{m-1}$). By this definition, $Y^k_2 = Y^k$ from the beginning of Section 2. As exhibited in Figure 4, the set $Y^k_m$ also has a $k + 1$ cyclic symmetry. We will prove this by appealing to the geometric description of Suter symmetry.
Figure 3: The poset $Y_4^2$ labeled by cores which exhibits a dihedral 3-fold symmetry. A reflection in this symmetry is realized through conjugation of the 3-cores. The red indicates the cells added to the core are content 0 ($mod$ 3), blue at the cells are content 1 ($mod$ 3), green the cells are content 2 ($mod$ 3)

The collection of alcoves in the dominant chamber which are bounded by the affine hyperplane $H_{\phi,m}$ again inherits the cyclic $k+1$ symmetry of the fundamental alcove, thus proving that a cyclic $k+1$ symmetry exists on the alcoves. It remains to be shown that the alcoves in the dominant chamber bounded by the hyperplane $H_{\phi,m}$ correspond to the partitions which are contained in a product of $m-1$ rectangles. Once we have shown this, our main theorem, that $Y_m^k$ has a cyclic $k+1$ action, will follow.
Figure 4: The poset $Y_3^3$ exhibits a cyclic 4 symmetry. The vertices are labelled by 4-cores, and corresponding 3-bounded partitions are obtained by deleting shaded boxes and left justifying the partition. The edge colors correspond to the integer modulo 4 of the content of the cells being added; red is 0, blue is 1, yellow is 2 and green is 3.
5 The affine Nil-Coxeter algebra and rectangle $k$-Schur functions

The affine nilCoxeter algebra $A$ is the algebra generated by $u_i$ for $i \in \{0, 1, \ldots, k\}$, subject to the relations (see for instance [L1]):

\[ u_i^2 = 0 \text{ for } i \in \{0, 1, \ldots, k\} \]
\[ u_iu_j = u_ju_i \text{ if } i - j \neq \pm 1 \]
\[ u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1} \text{ for } i \in \{0, 1, \ldots, k\} \]

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

If $s_{i_1} \ldots s_{i_m}$ is a reduced word for an element $w \in W$, we let $u(w) = u_{i_1} \ldots u_{i_m}$, then $U := \{u(w) : w \in W\}$ is a basis of $A$.

The affine nilCoxeter algebra has an action on the free abelian group with basis the $(k + 1)$-cores. Let $\nu \in C^{(k+1)}$ and then define $u_i\nu$ to be the $(k + 1)$-core formed by adding all addable boxes of content $i$ if $\nu$ has at least one such addable box, and $u_i\nu$ is 0 otherwise.

Within the affine nilCoxeter algebra, Lam [L1] found elements $h_i$ for $1 \leq i \leq k$ which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions $h_1, \ldots, h_k$.

Definition 5.1. An element $u = u_{i_1}u_{i_2}\ldots u_{i_m} \in U$ is said to be cyclically decreasing if each of $i_1, \ldots, i_m$ are distinct, and whenever $j = i_s$ and $j + 1 = i_t$ then $t < s$ (here $j + 1$ is taken modulo $k + 1$). To a strict subset $D \subset \{0, 1, \ldots, k\}$, we let $u_D$ denote the unique element of $U$ which is cyclically decreasing and is a product of the generators $u_m$ for $m \in D$.

Lam then defines elements $h_i := \sum_{|D|=i} u_D \in A$ for $i \in \{0, 1, \ldots, k\}$.

Theorem 5.2 (Lam [L1] Corollary 14). The $h_i$ for $i \in \{1, 2, \ldots k\}$ generate a subalgebra isomorphic to the ring generated by the complete homogenous symmetric functions $h_i$ for $i \in \{1, 2, \ldots k\}$. The isomorphism identifies $h_i$ and $h_i$.

One can then define the $k$-Schur functions.

Definition 5.3. Let $\lambda$ be a $k$-bounded partition. Then we define $s^{(k)}_\lambda$ to be the unique elements of the subring generated by the $h_i$ which satisfy the
following rule, known as the \( k \)-Pieri rule:

\[
h_i s^{(k)}_\lambda = \sum_{\mu} s^{(k)}_\mu ; \quad s^{(k)}_\emptyset = 1.
\]

where \( \mu = u(y) \lambda \) and \( y \) is a cyclically decreasing word of length \( i \).

**Remark 5.4.** In general, expanding \( s^{(k)}_\lambda = \sum_w c_w u(w) \) is an open problem, and has been shown to be equivalent to understanding the structure coefficients of \( k \)-Schur functions (called the \( k \)-Littlewood Richardson coefficients).

### 5.1 Expression of rectangle \( k \)-Schur functions as pseudo-translations

In [BBTZ], the authors introduced the notion of a pseudo-translation in order to describe the expansion of \( k \)-Schur functions corresponding to \( R_1, \ldots, R_k \) in \( \mathbb{A} \). Pseudo-translations have since been realized by Lam and Shimozono as being translations of the extended affine Weyl group (see [LS2]).

**Definition 5.5.** Let \( \eta \) be a weight. We say \( y \in W \) is a pseudo-translation of \( A_w \) in direction \( \eta \) if \( A_yw = A_w + \eta \).

For a weight \( \gamma \) we let \( z_\gamma \) denote the pseudo-translation of the fundamental alcove \( A_\emptyset \) in direction \( \gamma \).

**Theorem 5.6** (Berg, Bergeron, Thomas, Zabrocki [BBTZ]). *Inside \( \mathbb{A} \),*

\[
s^{(k)}_{R_i} = \sum_{\gamma \in W_0 \Lambda_i} u(z_\gamma).
\]

### 5.2 Alcoves with a facet on the hyperplane \( H_{\phi,m} \)

Let \( R \) be the \( k \)-bounded partition \( R_i \cup \cdots \cup R_{i_{m-1}} \) with \( i_j \in \{1, 2, \ldots, k\} \). Then \( s^{(k)}_R = s^{(k)}_{R_{i_{m-1}}} \cdots s^{(k)}_{R_{i_1}} \). By Theorem 5.6, the \( k \)-bounded partition \( R \) corresponds to the alcove \( A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}) \).

**Lemma 5.7.** There are \( \binom{m-1+k-1}{k-1} \) distinct \( k \)-bounded partitions of the form \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \).
Proof.  The partition \( R \) will be the union some number (possibly 0) of each of the different rectangles \( R_1, R_2, \ldots, R_k \). Hence the number of such rectangles is the number of ways to pick a set of \( m - 1 \) objects from a set of \( k \) elements with repetition.

Lemma 5.8. The alcove \( A \) corresponding to the \( k \)-bounded partition \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \) shares a facet with the wall \( H_{\phi,m} \).

Proof. The fundamental alcove \( A_\emptyset \) shares a facet with the hyperplane \( H_{\phi,1} \). The fundamental weights \( \Lambda_i \) all satisfy \( \langle \Lambda_i, \phi \rangle = 1 \) and are the coordinates of the vertices of this facet. Since \( A \) is a translate of the fundamental alcove, \( A = A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}) \), the vertices of \( A \) which are not translates of the origin will have weight \( v_d = \Lambda_d + \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}} \). They will satisfy

\[
\langle v_d, \phi \rangle = 1 + \sum_{j=1}^{m-1} \langle \Lambda_j, \phi \rangle = m,
\]

and so will lie on the wall \( H_{\phi,m} \).

Lemma 5.9. The number of vertices on the hyperplane \( H_{\phi,m-1} \) which are in the fundamental chamber is \( \binom{m-1+k-1}{k-1} \). There is a bijection between the alcoves corresponding to products of rectangles and these vertices; we identify an alcove with its unique vertex on \( H_{\phi,m-1} \).

Proof. Each vertex on \( H_{\phi,m-1} \) has the form \( \sum_i a_i \Lambda_i \) with \( a_i \) all non-negative integers and \( \sum_i a_i = m - 1 \).

We conclude then that the vertices are then in bijection with non-negative integer solutions \( (a_i \geq 0) \) to the equation \( \sum_{i=1}^{k} a_i = m - 1 \) and this is well known to be \( \binom{m-1+k-1}{k-1} \).

For the last statement it is sufficient to remark that each alcove corresponding to partition \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \) contains the vertex \( \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}} \) which lies on \( H_{\phi,m-1} \) and by Lemma 5.7 these sets have the same number of elements.

Lemma 5.10. Let \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \) and let \( R \) be the \( (k+1) \)-core which corresponds to the \( k \)-bounded partition \( R \). Then \( R \) has only one addable residue, that is there exists a unique \( i \) for which \( u_i R \neq 0 \).

Proof. The only residue which is addable is \( i = i_1 + \cdots + i_{m-1} \). The core \( R \) is obtained by appending rectangles ordered by their widths in a skew
fashion, stacking the rectangles so that adjacent rectangles share neither row nor column. Cells which are on the opposite sides of a rectangle in the core will have the same residue because they are separated by a hook of \( k \) therefore only one residue is addable. The length of the first row of \( R \) will be \( i = i_1 + \cdots + i_{m-1} \) and so it is also the residue of the addable corner.

**Corollary 5.11.** Let \( \lambda \) be a \( k \)-bounded partition and suppose that \( \lambda \) corresponds to an alcove \( A_w \) in the fundamental chamber which is bounded by \( H_{\phi,m} \). Then there exists an \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \) such that \( \lambda \subseteq R \).

**Proof.** The proof is by induction on \( m \). When \( m = 1 \) the statement is trivial; the only dominant alcove bounded by \( H_{\phi,1} \) is the fundamental alcove, which corresponds to the empty partition \( \emptyset \), which is contained in an empty product of rectangles.

Now we fix \( m \). If \( A \) is bounded by \( H_{\phi,m-1} \) then the statement follows by induction, and know that there is a \( R' = R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-2}} \) and then \( \lambda \subseteq R \cup R_{i_{m-1}} \) for any other \( i_{m-1} \in \{ 1, 2, \ldots, k \} \). So we assume that \( A \) is between \( H_{\phi,m-1} \) and \( H_{\phi,m} \) and so has at least one vertex on \( H_{\phi,m-1} \). Let that vertex be \( \Lambda = \sum_{j=1}^{m-1} \Lambda_{i_j} \). Let \( R = R_{i_1} \cup \cdots \cup R_{i_{m-1}} \) as in Lemma 5.9, then we claim that \( \lambda \subseteq R \).

Let \( B \) denote the alcove corresponding to \( R \). By Lemma 5.10, \( B \) has a unique addable residue, which we shall denote \( r \). This residue corresponds to crossing the hyperplane \( H_{\phi,m} \), since crossing the hyperplane will increase the length of the corresponding core and we know there is only one reflection which will add box to \( R \), by Lemma 5.10. Applications of all other generators \( s_i \) for \( i \neq r \) must therefore decrease the size of the partition. Since \( B \) shares a vertex with \( A_w \), there is an element \( s_{a_1} s_{a_2} \cdots s_{a_x} \) of \( W_r \) which takes \( A_w \) to \( B \) (i.e. \( A_{s_{a_1} s_{a_2} \cdots s_{a_x} w} = B \) for some \( a_j \neq r \)). Therefore \( \lambda \subseteq R \), since \( \lambda = s_{a_x} \cdots s_{a_1} R \).

As a consequence of Corollary 5.11 we have the following results.

**Theorem 5.12.** The set \( Y^k_m \) has a cyclic \( k+1 \) action.

**Proof.** By Corollary 5.11, \( Y^k_m \) corresponds precisely with alcoves in the dominant chamber which are bounded by \( H_{\phi,m} \). The region in the dominant chamber bounded by \( H_{\phi,m} \) has the same shape as the fundamental alcove; the lengths of the edges of the fundamental alcove have been multiplied by \( m \) in \( H_{\phi,m} \). Since the fundamental alcove has a cyclic \( k+1 \) action which is inherited from the affine Dynkin diagram, the collection of alcoves in this region inherits the cyclic \( k+1 \) action. 

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As a corollary we also have as a consequence an enumeration of the elements in $Y^k_m$.

**Proposition 5.13.** The number of partitions in $Y^k_m$ is $m^k$.

**Proof.** As noted in the previous result, $Y^k_m$ is in bijection with the alcoves which lie inside of an $m$-dilation of the fundamental alcove. In a $k$ dimensional space the volume of a region diluted by $m$ on a side will be $m^k$ times the original, hence there are $m^k$ alcoves within this region. \(\Box\)

Others (e.g. Sommers [Som, Theorem 5.7]) have considered this dilated alcove for reasons other than the connection with $k$-bounded partitions and $(k + 1)$-cores and so this lattice may have unexpected algebraic applications.

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7 Apendix

Using sage [sage], sage-combinat [sage-combinat] and graphviz [graphviz] software, we were able to generate some pictures which exhibit this symmetry. The cases when \( m = 2 \) are the symmetries that Suter found in \([Su1]\).

Figure 5: Suter symmetry of type \( k = 3 \) and \( m = 2 \)

Figure 6: Suter symmetry of type \( k = 4 \) and \( m = 2 \)
Figure 7: Suter symmetry of type $k = 5$ and $m = 2$
Figure 8: Suter symmetry of type $k = 3$ and $m = 3$
Figure 9: Suter symmetry of type $k = 4$ and $m = 3$
Figure 10: Suter symmetry of type $k = 5$ and $m = 3$
Figure 11: Suter symmetry of type $k = 3$ and $m = 4$
Figure 12: Suter symmetry of type $k = 4$ and $m = 4$