Abstract

Four years ago the Extended Scale Relativity (ESR) theory in C-spaces (Clifford manifolds) was proposed as the plausible physical foundations of string theory. In such theory the speed of light and the minimum Planck scale are the two universal invariants. All the dimensions of a C-space can be treated on equal footing by implementing the holographic principle associated with a nested family of p-loops of various dimensionalities. This is achieved by using polyvector valued coordinates in C-spaces that encode in one stroke points, lines, areas, volumes,... We review the derivation of the minimal length/time string/brane uncertainty relations and the maximum Planck temperature thermodynamical uncertainty relation. The Weyl-Heisenberg algebra in C-spaces is constructed which induces a Noncommutative Geometric structure in the \( X^A \) coordinates. Hence quantization in C-spaces involves in a natural fashion a Noncommutative Quantum Mechanics and Field Theory rather than being postulated ad-hoc. A QFT in C-spaces may very likely involve (Braided Hopf) Quantum Clifford algebras and generalized Moyal-like star products associated with multisymplectic geometry.

Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein’s general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime resolutions was developed long ago by Nottale where the Planck scale was postulated as the minimum observer independent invariant resolution \([1]\) in Nature. Since “points” cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution-dependent, one must also include a theory with all topologies as well. It is our belief that this may lead to the proper formulation of string and M theory.

In \([2]\) we applied this Extended Scale Relativity principle to the quantum mechanics of \( p \)-branes which led to the construction of C-space (a dimension
category) where all \( p \)-branes were taken to be on the same footing; i.e. transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of polydimensional transformations [3, 4, 5].

Clifford algebras have been a very useful tool for a description of geometry and physics [4, 5, 6, 7, 8]. In [3,5] it was proposed that every physical quantity is in fact a **polyvector**, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left or right minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of supersymmetries, i.e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of relativistic particle [4] can be embedded in the Clifford algebra of spacetime [3]. Many important aspects of Clifford algebra are described in [3,5,6,7,8].

Using these methods the bosonic \( p \)-brane propagator, in the quenched min-superspace approximation, was constructed in [9]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [12]; the action for higher derivative gravity with torsion from the geometry of C-spaces and how the Conformal algebra of spacetime emerges from the Clifford algebra was performed in [11]; the resolution of the ordering ambiguities of QFT in curved spaces was resolved by [3].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra valued objects, polyvectors. Such a manifold has been called a pan-dimensional continuum [5] or C-space [2]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes..., called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc.. It is a sort of a **dimension** category, where the role of functorial maps is played by C-space transformations which reshuffles a \( p \)-brane history for a \( p' \)-brane history or a mixture of all of them, for example. The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed \( p \)-branes. Technically those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space. In that sense, the C-space is roughly speaking a sort of generalized Penrose-Twistor space from which the ordinary spacetime is a **derived** concept.

In [2] we derived the minimal length uncertainty relations as well as the full blown uncertainty relations due to the contributions of all branes of every dimensionality, ranging from \( p = 0 \) all the way to \( p = \infty \). In [14] we extended this derivation to include the maximum Planck Temperature condition which recently has been recast into a maximum temperature thermodynamical uncertainty relation involving the internal Energy, temperature and the Boltzmann constant [19].

In section I we will review the Extended Relativity in C-spaces and the explicit derivation, from first principles, of all the generalized minimal length-time (and maximum temperature) uncertainty relations based on the effective-running Planck “constant” \( \hbar \) (energy dependent) emerging from the geometry.
of C-spaces.

In section II we proceed with the canonical quantization in C-spaces and construct the Weyl-Heisenberg algebra. We show rigorously how the Noncommutative Geometry of the C-space coordinates \( X \) is a direct consequence of the Weyl-Heisenberg algebra in C-spaces in contradistinction with the ordinary phase space commutators \( [x^i, p^j] = i\hbar \delta^{ij} \) which imply that \( [x^i, x^j] = 0 \) as a result of the Jacobi identities.

In C-space this will no longer be the case due to the highly nontrivial Weyl-Heisenberg algebra. This is the main result of this work: Canonical Quantization in C-spaces automatically yields a Noncommutative Geometric structure for the coordinates \( X \) and hence it involves a Noncommutative QM. It is unnecessary to put in by hand the noncommutativity of the coordinates in terms of a length scale (like the Planck scale) like Snyder did long ago [22]. The mere quantization in C-spaces induces the Noncommutativity of coordinates. We believe this is an important result deeply ingrained in the Extended Scale Relativistic structure of C-spaces.

We finalize by discussing how one can extract an effective \( \hbar \), which is a function of the Lorentz invariant quantity \( p^2 = p_\mu p^\mu \), from the Weyl-Heisenberg algebra in C-space. Effective \( \hbar \) of this sort are the ones which furnish the minimal length/time string/brane uncertainty relations. We conclude with some remarks pertaining multisymplectic geometry and QFT in C-spaces based on Quantum Clifford Algebras and Braided Hopf Quantum Clifford algebras to study q-deformations of C-spaces.

A Moyal star product construction deserves further study as well. Since C-spaces involves the physics of all \( p \)-loops it is warranted to use methods of multisymplectic mechanics since phase spaces in C-spaces involve antisymmetric tensors of arbitrary rank. Nambu-Poison QM seems to be the most appropriate one to study C-space QM. In particular the use of the Zariski star product deformations vs the Moyal one will be welcome [24].

I. Extending Relativity from Minkowski spacetime to \( C \)-space

We embark into the extended relativity theory in C-spaces by a natural generalization of the notion of a space-time interval in Minkowski space to C-space:

\[
dX^2 = d\Omega^2 + dx_\mu dx^\mu + dx_\mu dx_\nu + ... 
\]

The Clifford valued poly-vector:

\[
X = \Omega I + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... 
\]

denotes the position in a manifold, called Clifford space or C-space. The coordinates \( x^{\mu\nu} \), ..., are the holographic areas, volumes, ...shadows or projections of the nested family of \( p \)-loops onto the embedding spacetime coordinate planes/hyperplanes. Since the Planck scale is given by: \( \Lambda = (G_N)^{1/(D-2)} \) in \( D \) dimensions, in units of \( \hbar = c = 1 \), and since we wish to have a universal value
for the minimum distance in all dimensions, we will set $\Lambda = G_N = 1$, which is the consistent value in all dimensions ranging from $D = 2$ all the way to $D = \infty$. The ESR theory admits naturally this system of units, of setting all fundamental constants to unity, including Boltzmann constant and the Planck Temperature.

If we take differential $dX$ of $X$ and compute the scalar product $dX \cdot dX$ we obtain:

$$d\Sigma^2 = (d\Omega)^2 + \Lambda^{2D-2}dx_\mu dx^\mu + \Lambda^{2D-4}dx_{\mu\nu}dx^{\mu\nu} + ..$$

(3)

Here we have introduced the Planck scale $\Lambda$ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops, ..., $p$-loops. Einstein introduced the speed of light as a universal absolute invariant in order to “unite” space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2dt^2 - dx_i dx^i.$$  

(4)

A similar unification is needed here to “unite” objects of different dimensions, such as $x^\nu$, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [2].

To continue along the same path, we consider the analog of Lorentz transformations in C-spaces which transform a poly-vector $X$ into another poly-vector $X'$ given by $X' = RXR^{-1}$ with

$$R = \exp \left[ i(\theta I + \theta^{\mu\nu}\gamma_{\mu\nu} + \theta^{\mu_1\mu_2\gamma_{\mu_1\mu_2}} \land \gamma_{\mu_2} \ldots) \right].$$

(5)

and

$$R^{-1} = \exp \left[ -i(\theta I + \theta^{\nu\gamma_{\nu}} + \theta^{\nu_1\nu_2\gamma_{\nu_1\nu_2}} \land \gamma_{\nu_2} \ldots) \right].$$

(6)

where the theta parameters:

$$\theta; \theta^{\mu\nu}; \ldots$$

(7)

are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: $||X||^2 = Trace X^2$

Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$Trace X'^2 = Trace [RX^2R^{-1}] = Trace [RR^{-1}X^2] = Trace X^2.$$  

(8)

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$.

1.2 Planck scale as the minimum invariant in Extended Scale Relativity

Long time ago L.Nottale proposed to view the Planck scale as the absolute minimum invariant (observer independent) scale in Nature in his formulation of
scale relativity [1]. We can apply this idea to C-spaces by choosing the correct analog of the Minkowski signature:

\[ ||\text{d}X||^2 = (\text{d}\Omega)^2[1 - \Lambda^2 D - 2 \left( \frac{\text{d}x_\mu}{\text{d}\Omega} \right)^2 - \Lambda^2 D - 4 \left( \frac{\text{d}x_{\mu\nu}}{\text{d}\Omega} \right)^2 - \Lambda^2 D - 6 \left( \frac{\text{d}x_{\mu\nu\rho}}{\text{d}\Omega} \right)^2 - \ldots] \]

where the sequence of variable scales \( \lambda_1, \lambda_2, \lambda_3, \ldots \) are related to the generalized (holographic) velocities defined as follows:

\[ \frac{(\text{d}x_\mu)^2}{(\text{d}\Omega)^2} \equiv (V_1)^2 = \left( \frac{1}{\lambda_1} \right)^{2D-2}. \]
\[ \frac{(\text{d}x_{\mu\nu})^2}{(\text{d}\Omega)^2} \equiv (V_2)^2 = \left( \frac{1}{\lambda_2} \right)^{2D-4}. \]
\[ \frac{(\text{d}x_{\mu\nu\rho})^2}{(\text{d}\Omega)^2} \equiv (V_3)^2 = \left( \frac{1}{\lambda_3} \right)^{2D-6}. \]

It is clear now that if \( ||\text{d}X||^2 \geq 0 \) then the sequence of variable lengths \( \lambda_n \) cannot be smaller than the Planck scale \( \Lambda \). This is analogous to a situation with the Minkowski interval:

\[ ds^2 = c^2 dt^2 \left[ 1 - \frac{v^2}{c^2} \right]. \]

when it is \( \geq 0 \) if, and only if, the velocity \( v \) does not exceed the speed of light. If any of the \( \lambda_n \) were smaller than the Planck scale the C-space interval will become tachyonic-like \( d\Sigma^2 < 0 \). Photons in C-space are tensionless branes/loops. Quite analogously one can interpret the Planck scale as the postulated minimum universal distance in nature, not unlike the postulate about the speed of light as the upper limit on the speed of signal propagation.

What seems remarkable in this scheme of things is the nature of the singularities and the emergence of two times. One of the latter is the local mode, a clock, represented by \( t \) and the other mode is a “global” one represented by the volume of the space-time filling brane \( \Omega \). For more details related to this Fock-Stückelberg-type parameter see [3]. We must emphasize that one must not confuse these global and local time modes with the two modes of time in other branches of science [13].

Another immediate application of this theory is that one may consider “strings” and “branes” in C-spaces as a unifying description of all branes of different dimensionality. As we have already indicated, since spinors are left/right ideals of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world volume and target space...
supersymmetry simultaneously [17]. We hope that the C-space “strings” and “branes” may lead us towards discovering the physical foundations of string and M-theory.

1.3 The Generalized String/Brane Uncertainty Relations

Below we will review how the minimal length string uncertainty relations can be obtained from C-spaces [2]. The norm of a momentum poly-vector was defined:

\[ P^2 = \pi^2 + p_\mu p^\mu + p_{\mu\nu} p^{\mu\nu} + p_{\mu\nu\rho} p^{\mu\nu\rho} + \ldots = M^2 \]  

Nottale has given convincing arguments why the notion of dimension is resolution dependent, and at the Planck scale, the minimum attainable distance, the dimension becomes singular, that is blows-up. If we take the dimension at the Planck scale to be infinity, then the norm \( P^2 \) will involve an infinite number of terms since the degree of a Clifford algebra in \( D \)-dim is \( 2^D \). It is precisely this infinite series expansion which will reproduce all the different forms of the Casimir invariant masses appearing in kappa-deformed Poincare algebras [11,12].

It was discussed recently why there is an infinity of possible values of the Casimirs invariant \( M^2 \) due to an infinite choice of possible bases. The parameter \( \kappa \) is taken to be equal to the inverse of the Planck scale. The classical Poincare algebra is retrieved when \( \Lambda = 0 \). The kappa-deformed Poincare algebra does not act in classical Minkowski spacetime. It acts in a quantum-deformed spacetime. We conjecture that the natural deformation of Minkowski spacetime is given by C-space.

The way to generate all the different forms of the Casimirs \( M^2 \) is by “projecting down” from the \( 2^D \)-dim Clifford algebra to \( D \)-dim. One simply “slices” the \( 2^D \)-dim mass-shell hyper-surface in C-space by a \( D \)-dim one. This is achieved by imposing the following constraints on the holographic components of the polyvector-momentum. In doing so one is explicitly breaking the poly-dimensional covariance and for this reason one can obtain an infinity of possible choices for the Casimirs \( M^2 \).

To demonstrate this, we impose the following constraints:

\[ p_{\mu\nu} p^{\mu\nu} = a_2 (p_\mu p^\mu)^2 = a_2 p^4. \quad p_{\mu\nu\rho} p^{\mu\nu\rho} = a_3 (p_\mu p^\mu)^3 = a_3 p^6. \quad \ldots \]  

Upon doing so the norm of the poly-momentum becomes:

\[ P^2 = \sum_n a_n p^{2n} = M^2(1, a_2, a_3, \ldots, a_n, \ldots) \]  

Therefore, by a judicious choice of the coefficients \( a_n \), and by reinserting the suitable powers of the Planck scale, which have to be there in order to combine objects of different dimensions, one can reproduce all the possible Casimirs in the form:
\[ M^2 = m^2 |f(\Lambda m/\hbar)|^2, \quad m^2 \equiv p_\mu p^\mu = p^2. \]  

(15)

where the functions \( f(\Lambda m/\hbar) \) are the scaling functions with the property that when \( \Lambda = 0 \) then \( f \to 1 \).

To illustrate the relevance of poly-vectors, we will summarize our derivation of the minimal length string uncertainty relations [2]. Because of the existence of the extra holographic variables one cannot naively impose \([x, p] = i\hbar \) due to the effects of the other components. The units of \([x_{\mu\nu}, p^{\mu\nu}] \) are of \( \hbar^2 \) and of higher powers of \( \hbar \) for the other commutators. To achieve covariance in C-space which reshuffles objects of different dimensionality, the effective Planck constant in C-space should be given by a sum of powers of \( \hbar \).

This is not surprising. Classical C-space contains the Planck scale, which itself depends on \( \hbar \). This implies that already at the classical level, C-space contains the seeds of the quantum space. At the next level of quantization, we have an effective \( \hbar \) that comprises all the powers of \( \hbar \) induced by the commutators involving all the holographic variables. In general one must write down the commutation relations in terms of polyvector-valued quantities. In particular, the Planck constant will now be a Clifford number, a polyvector with multiple components. This will be the subject of section \( \text{II} \).

The simplest way to infer the effects of the holographic coordinates of C-space on the commutation relations is by working with the effective \( \hbar_{\text{eff}} \) that appears in the nonlinear de Broglie dispersion relation. The mass-shell condition in C-space, after imposing the constraints among the holographic components, yields an effective mass \( M = m f(\Lambda m/\hbar) \). The generalized De Broglie relations, which are no longer linear, are [2]:

\[ |P_{\text{effective}}| = |p| f(\Lambda m/\hbar) = \hbar_{\text{effective}} |k|, \quad \hbar_{\text{effective}} = \hbar f(\Lambda m/\hbar) = \hbar \sum a_n (\Lambda m/\hbar)^{2n}. \quad m^2 = p^2 = p_\mu p^\mu = (\hbar k)^2. \]  

(16)

Using the effective \( \hbar_{\text{eff}} \), the well known relation based on the Schwartz inequality and the fact that \(|z| \geq |Imz|\) we obtain:

\[ \Delta x^i \Delta p^j \geq \frac{1}{2} |[x^i, p^j]| \geq |[x^i, p^j]| = i\hbar_{\text{eff}} \delta^{ij}. \]  

(17)

Using the relations

\[ \langle p^2 \rangle \geq (\Delta p)^2, \quad \langle p^4 \rangle \geq (\Delta p)^4 \ldots \]  

(18)

and the series expansion of the effective \( \hbar_{\text{eff}} \), we get for each component (we omit indices for simplicity):

\[ \Delta x \Delta p \geq \frac{1}{2} \hbar + \frac{n \Lambda^2}{2\hbar} (\Delta p)^2 + \ldots \]  

(19)

This yields the minimal length string uncertainty relations:
\[ \Delta x \geq \frac{\hbar}{2\Delta p} + \frac{a\Lambda^2}{2\hbar} \Delta p \ldots \] (20)

By replacing lengths by times and momenta by energy one reproduces the minimal Planck time uncertainty relations. One could include all the terms in the series expansion and derive a generalized string/brane uncertainty relation which still retains the minimal length condition, of the order of the Planck scale [2]. For example, if one chooses the same value for all the coefficients in the Taylor expansion, an isotropy condition in C-spaces is selected where all directions have equal weight, the full blown uncertainty relation due to all branes is given by [2]:

\[ \Delta x \geq \sqrt{2\Lambda} \frac{e^{(\Delta z)^2/4}}{(\Delta z)^2} \sqrt{\sinh \left[ (\Delta z)^2/2 \right]} \] (21)

where \( \Delta z = \Lambda \Delta k \) and \( k = \sqrt{k_{\mu}k^\mu} \) and we took all the coefficients of the Taylor expansion to be equal to unity. This relation also obeys the minimal length condition [2] of the order 1.2426 \( \Lambda \). Uncertainty relations for a particular p-brane (for a specific value of \( p \)) has been given by [20]. Relations given by eq-(21) are due to the contribution of all values of \( p \). In the limit that \( \Lambda \) goes to zero we recover the standard Heisenberg uncertainty relations.

The physical interpretation of these uncertainty relations follows from the extended relativity principle. As we boost the string to higher transPlanckian energies part of the energy will always be invested into the string’s potential energy, increasing its length in bits of Planck scale sizes so that the original string will decompose into two, three, four...strings of Planck sizes carrying units of Planck momentum; i.e. the notion of a single particle/string loses its meaning beyond that point. This reminds one of ordinary relativity, where boosting a massive particle to higher energy increases its speed while a part of the energy is also invested into increasing its mass. In this process the speed of light remains the maximum attainable speed (it takes an infinite energy to do so) and in our scheme the Planck scale is never surpassed. The effects of minimal length can be clearly seen in Finsler geometries having both a maximum four acceleration \( c^2/\Lambda \) (maximum tidal forces) and a maximum speed [21]. The Riemannian limit is reached when the maximum four acceleration goes to infinity, i.e. the Finsler geometry “collapses” to a Riemannian one.

It is straightforward now to derive the maximum Planck temperature condition [14] and the Thermodynamic Uncertainty relations [19]. Based on the old known results of Euclidean QFT, we simply identify the inverse temperature \( 1/T \) with the period of the Euclideanized temporal coordinate \( x_0 \). By using the simple correspondence in eqs-(20):

\[
x_0 \rightarrow 1/T, \quad \hbar \rightarrow k_B, \quad \Delta E \rightarrow \Delta U
\]

we will recover the maximum Planck temperature Uncertainty relations [19]:

\[ \ldots \]
\[ \Delta(1/T) \geq \frac{k_B}{2\Delta U} + \frac{a}{2k_B T_P^2} \Delta U \]  \hspace{1cm} (22) \]

in terms of the temperature \( T \), the internal energy \( U \) and the Boltzmann constant \( k_B \). The Planck temperature \( T_P \) is defined by \( M_P/k_B \) in units of \( c = 1 \).

II. Weyl-Heisenberg Algebra in C-spaces

A straightforward procedure to visualize the C-space algebraic-geometric structure can be achieved by recalling that a realization of the basis elements \( E_A \) exists in terms of Dirac matrices, and their suitable antisymmetrized products of matrices, until saturating the dimensionality of spacetime. A Clifford algebra in \( D \) dimension has \( 2^D \) basis elements including the unit element. For \( D = 2n \) one has a basis of \( 2^D \) matrix-elements where each matrix-element is given by a \( 2^n \times 2^n \) Dirac matrix, for example.

Due to the noncommutative nature of the basis vectors of the Clifford algebra one has:

\[ [E_A, E_B] = F^M_{AB} E_M = F_{ABM} E^M, \quad E_A = \{I; \gamma_\mu; \gamma_\mu \wedge \gamma_\nu; \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho; \ldots\}. \]  \hspace{1cm} (23a) \]

the quantities \( F^M_{AB} \) play a similar role as the structure constants in ordinary Lie algebras. A commutator of two matrices is itself a matrix, which in turn, can be expanded in a suitable matrix basis due to the Clifford algebraic (vector space) structure inherent in C-spaces. The commutator algebra obeys the Jacobi identities, the Liebnitz rule of derivations and the antisymmetry properties.

The Clifford geometric product of two basis elements is defined in terms of a “scalar” (symmetric) and “outer” product (antisymmetric) respectively:

\[ E_A E_B = \frac{1}{2} \{E_A, E_B\} + \frac{1}{2} [E_A, E_B]. \]  \hspace{1cm} (23b) \]

In general, the geometric product of two multivectors \( E^A, E^B \) of ranks \( r, s \), respectively, is given by an aggregate of multivectors of the form:

\[ E^A E^B = \langle E^A E^B \rangle_{r+s}, \quad \langle E^A E^B \rangle_{r+s-2}, \quad \langle E^A E^B \rangle_{r+s-4}, \ldots, \langle E^A E^B \rangle_{|r-s|} \]

The first term of rank \( r + s \) is the wedge product \( E^A \wedge E^B \) and the last term of rank \( |r - s| \) is the dot product \( E^A E^B \) which is obtained by a contraction of indices and must not be confused with the scalar part of \( E^A E^B \) unless \( r = s \). In general, the scalar product among two equal-rank multivectors \( r = s \) cannot longer be written in terms of the anticommutator \( \{E^A, E^B\} \) except in the case when \( r = s = 1 \): \( \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1. \) However, for equal-rank multivectors, the scalar part \( \langle E_A E_B \rangle_{0} = G_{AB}I \) where \( I \) is the unit element of the Clifford algebra and \( G_{AB} \) is the C-space metric.

To begin with we will write for the putative Weyl-Heisenberg algebra:
\[ [X^A, P^B] = H^0 G^{AB} - H^C F^{AB}_C. \]  

(24)

where \( H^C, H^0 \) are themselves the components of a polyvector-valued C-space generalization of Planck’s constant \( H = H^C E_C \). The \( C \) multi-index runs over all the basis elements except the unit element \( H^0 \). Later on we will see that this relation needs to be modified by the addition of crucial terms involving the \( X, P \) variables.

First of all we must keep track of the correct units. We will treat the \( P \) and \( X \) exactly on the same footing. For this reason we will scale all the basis elements by judicious powers of \( \sqrt{\hbar} \):

\[
\gamma^\mu \rightarrow \sqrt{\hbar} \gamma^\mu. \quad \gamma^\mu \wedge \gamma^\nu \rightarrow \hbar^{-1/2} \gamma^\mu \wedge \gamma^\nu; \quad \ldots.
\]

\[ (25a) \]

and

\[
\gamma^\mu \rightarrow \hbar^{1/2} \gamma^\mu. \quad \gamma^\mu \wedge \gamma^\nu \rightarrow \hbar^{-1} \gamma^\mu \wedge \gamma^\nu; \quad \ldots.
\]

\[ (25b) \]

The units of \( X^A \) and \( P^B \) are taken to be \( \hbar^{r_A/2} \) and \( \hbar^{r_B/2} \) respectively where \( r_A, r_B \) are the ranks of the antisymmetric tensor components \( X^A, P^B \) of the polyvectors \( X, P \) respectively.

The scaling of the commutator is:

\[
[E_A, E_B] = F^{M}_{AB} E_M \rightarrow \hbar^{-r_A/2 - r_B/2} [E_A, E_B] = (\hbar^{-r_A/2 - r_B/2} F^{M}_{AB}) E_M. 
\]

(26)

hence, we will absorb the powers of \( \hbar \) appropriately in the structure constants as indicated above. Upper indices carry positive powers of \( \sqrt{\hbar} \) whereas lower indices carry negative powers. Notice that there are no powers of \( \sqrt{\hbar} \) associated with the index \( M \) above, only w.r.t the two \( AB \) indices.

Since we are scaling the basis vectors \( E_A \) this means that we are choosing the \( H^C, H^0 \) to be dimensionless. In this fashion we automatically obtain quantities with the correct units. For example: \([x^\mu, p^\nu]\) will contain a single power of \( \hbar \) due to the two factors of \( \hbar^{1/2} \) appearing in the \( G^{\mu\nu} \) as a result of the scaling of the \( \gamma^\mu \) and \( \gamma^\nu \). Their anticommutator yields the \( G^{\mu\nu} \) (after the saling takes place). One obtains identical results with the other holographic components of the polyvectors. The commutator of \([x^\mu\nu, p^{\rho\tau}]\) will automatically have the correct \( \hbar^2 \) power, etc…… Identical results follow for the \( H^C F^{AB}_C \) terms as well.

If one uses these putative Weyl-Heisenberg algebra relations in the Jacobi identities for the set of variables \( X^A, P^B, X^C \) ordinary commutativity of the coordinates will be maintained, \([X^A, X^C]\) = 0. However this is not longer the case in the full-fledged algebra as we shall see next.

A direct evaluation of the commutator of two polyvectors in terms of the Clifford-valued Planck constant \( H = H_M E_M \) is:

\[
[X, P] = H = [X_A E^A, P_B E^B] = H = H^C E_C = [X_A, P_B]E^A E^B + P^M X^N [E_M, E_N] =
\]
the $E^A E^B$ terms in the r.h.s of (27) can be reshuffled to the l.h.s by means of writing the inverse of the geometric product as:

$$(E_A E_B)^{-1} = E_B^{-1} E_A^{-1} = E^B E^A.$$  

and this allows us to write the Weyl-Heisenberg algebra in C-spaces in terms of the scalar part of the triple geometric product $< E^C E^B E^A >$ as:

$$[X_A, P_B] = (H^C + P^M X^N F^C_{MN}) < E^C E^B E^A >_0.$$  

where $< E^C E^B E^A >_0 \equiv \Omega_{CBA}$ is the scalar part of the geometric triple product. Eq.(29) is the fundamental result of this work.

Inspired on this result (29), if one wishes to write the Weyl-Heisenberg algebras in terms of $G^{AB}$ and the structure constants $F^{AB}_C, K^{AB}_C$ of the commutators and anticommutators, respectively, $[E^A, E^B] = F^{AB}_C E^C, \{E^A, E^B\} = K^{AB}_C E^C$, the Weyl-Heisenberg algebra reads:

$$[X_A, P_B] = H_0 G^{AB} + H^C [F^C_{BA} + K^C_{AB}] +$$

$$P_M X_N F^M_N [F^C_{BA} + K^C_{AB}].$$

Once again the $\hat{C}$ index in (27b) runs over all multi-indices of the Clifford algebra except the unit element.

The Weyl-Heisenberg algebra can be written compactly in the 'spin' plus 'orbital' angular momentum form:

$$[X_A, P_B] = H_{AB} + J_{AB}.$$  

with the standard Planck constant-like terms of the form:

$$H_{AB} = H_0 G^{AB} + H^C [F^C_{BA} + K^C_{AB}].$$  

Notice the mixed symmetry of this expression, a symmetric plus antisymmetric piece in $A, B$. Had one had commuting basis elements, like in ordinary spacetime, and a scalar component for $H = H_0$ one would have had the standard Weyl-Heisenberg algebra $[X^A, P^B] = H^0 G^{AB}$. Since the powers of $\hbar$ are absorbed by the metric $G^{AB}$ this implies that $H^0 = i$. The extra term in (30) is the analog of the orbital angular momentum in C-spaces given by:

$$J_{AB} = P_M X_N F^M_N < E^C E^B E^A >_0 \equiv J_C < E^C E^B E^A >_0 = J_C \Omega_{ABC}.$$  

it also has mixed symmetry in the indices $A, B$.

The elements $H^{AB}$ involving the components of the polyvector-valued Planck constant resemble the familiar quaternionic and octonionic expansions of a quaternion and octonion in terms of their components. This indicates that
QM in C-spaces may be intrinsically linked with Quaternionic and Octonionic QM [23].

An immediate consequence of the C-space Weyl-Heisenberg algebra is that it induces automatically a Noncommutative Geometric structure in the $X^A$ coordinates. To satisfy the Bianchi identities among the triples $X^A, P^B, X^C$ and $X^A, X^B, X^C$ it is fairly clear that the coordinates cannot commute due to the explicit $X, P$ terms in the modified Weyl-Heisenberg algebra. Hence, the Jacobi identities require:

$$[X^A, X^B] = \Sigma^{AB}, \quad [X^C, \Sigma^{AB}] = 0. \quad (33)$$

where $\Sigma^{AB}$ is a tensor-like operator-valued object in C-space that does not destroy C-space Lorentz invariance and which is implicitly defined by the Jacobi identities. Suitable powers of the Planck scale are absorbed in the defining relations for $\Sigma^{AB}$ in order to match units. Since the Planck scale is a C-space invariant one will maintain C-space Lorentz invariance. To evaluate explicitly the expression for $\Sigma^{AB}$ will be the subject of future investigation. It is nontrivial even if we set in flat C-spaces: $[P^A, P^B] = 0$.

Another important consequence is that we cannot represent naively the operators by the old QM prescriptions:

$$X^A \rightarrow (H^{AB} + J^{AB}) \frac{\partial}{\partial P^B}. \quad (34a)$$

$$P^A \rightarrow (H^{AB} + J^{AB}) \frac{\partial}{\partial X^B}. \quad (34b)$$

These naive representations of the $X, P$ operators in the Weyl-Heisenberg algebra do not longer hold due to the explicit $X, P$ dependence of the C-space angular momentum $J^{AB}$ in the Weyl-Heisenberg algebra.

Using the effective $\hbar(p^2)$, where $p^2 = p_\mu p^\mu$, [2] we could still represent the position operator in terms of the momentum variables:

$$x_i \rightarrow i\hbar_{effective}(p^2) \frac{\partial}{\partial p^i}. \quad (35a)$$

but no longer we may write that:

$$p_i \rightarrow -i\hbar_{effective}(p^2) \frac{\partial}{\partial x^i}. \quad (35b)$$

otherwise one would not have been able to satisfy the Weyl-Heisenberg relation:

$$[x^i, p^j] = i\hbar_{effective}(p^2)\delta^{ij}. \quad (36)$$

assuming a flat spacetime $[p^i, p^j] = 0$. Hence, the symmetry between $x, p$ is broken already in these cases where one works with a modified Weyl-Heisenberg algebra using an effective $\hbar(p^2)$ with $p^2 = p_\mu p^\mu$.

Using other effective matrix valued $\hbar_{ij}$ that depend on $\vec{p} = p^i$ and on products like $p^i p^j$ happen to break Lorentz invariance explicitly despite maintaining
rotational symmetry [21, 25]. The fact that Lorentz invariance is broken is not surprising since these models are based on kappa-deformed Poincare symmetries [11, 12]. We have shown how one can break C-space Lorentz invariance to obtain a kappa-deformed effective Lorentz transformations which leave the Planck scale invariant [2, 16].

A length scale was introduced by hand by Snyder [22] when he wrote down the commutation relations for the four spacetime coordinates based on a 5-dimensional spacetime, whose fifth dimension was discrete in units of length $l$:

$$[x^\mu, x^\nu] = l^2 [J^{5\mu}, J^{5\nu}] = i l^2 J^{\mu \nu}$$

where $J^{\mu \nu}$ is an angular momentum in four dimensions and the four spacetime coordinates $x^\mu$ (divided by $l$) are identified with the components of the angular momentum which contain the fifth direction. Lorentz invariance is maintained in the four-dimensions by construction. The origin of the scale $l$ is due to the discrete fifth dimension. The authors in [11, 12, 21] have related this scale to the deformation parameter of kappa-deformed Poincare algebras $l = 1/\kappa = \Lambda$, with the fundamental difference that the four dim Lorentz invariance is broken, only rotational symmetry is conserved.

The advantage of C-spaces is that one does not need to introduce ad hoc this angular momentum type commutators for the four spacetime coordinates, by recurring to an extra discrete dimension of size $l$. The Weyl-Heisenberg and $X^A$ coordinate algebras in C-space are indeed compatible with the C-space Lorentz invariance without introducing extra dimensions. The Planck scale is a true invariant of C-space. In [13] we have shown that the Conformal algebra in four dimensions $SO(4, 2)$ does not require the six-dimensional interpretation associated with the Anti de Sitter group. Instead, it can be obtained directly from the Clifford algebra of four dim spacetime.

It is fairly clear why C-space QM differs from the ordinary QM in many aspects. To start it is already a Noncommutative QM since it involves a Noncommutative Geometric structure for the $X^A$ coordinates. The main task in the near future will be to construct a QFT in C-spaces, in particular, to use Quantum Clifford Algebras and Braided Hopf Quantum Clifford algebras to study q-deformations of C-spaces [15]. A Moyal-like star product construction deserves further study as well. Since C-spaces involves the physics of all $p$-loops it is warranted to use methods of multisymplectic geometry (mechanics) since phase spaces in C-spaces involve antisymmetric tensors of arbitrary rank. Nambu-Poison QM seems to be the most appropriate one to study C-space QM. In particular the use of the Zariski star product deformations vz the Moyal one [24] will be welcome.

To finalize we discuss how one would take contractions of the Weyl-Heisenberg algebra to obtain an effective matrix valued $\hat{h}_{ij}$, or for that matter, to generate a single effective Planck constant in the form of $\hat{h}\delta_{ij}$. The most natural candidate is to take the norm-squared as an effective $\hbar$:
\[ H^{AB}H_{AB} + J^{AB}J_{AB} + H^{AB}J_{AB} + J^{AB}H_{AB} . \]  

(37)

C-space polydimensional invariance can be broken by imposing similar type of constraints like we had in section 1, relating the holographic norms of polyvectors to the powers of ordinary vector norms:

\[ J^{AB}J_{AB} = \sum_n a_n (J/j)^{2n} \]

(38)

and in this way the norm-squared (37) reduces to:

\[ (\hbar_{eff}/\hbar)^2 = H^{AB}H_{AB} + H^{AB}J_{AB} + J^{AB}H_{AB} + \sum_n a_n (J/j)^{2n} \]  

(40)

Eq-(40) for the effective \( \hbar(p^2) \) is far more general than the ones discussed in the previous section. In particular, the last terms of (40) do contain the required terms \( \sum p^{2n} \) in the effective \( \hbar \). This can be understood after relating the angular momentum \( J \) to the \( m^2 = p^2 \) of the center of mass coordinates. In the ordinary mechanics of a rigid top there are two Casimirs, the angular momentum \( J^2 \) and the Energy given in terms of the angular momentum, in the case of a symmetric top, as \( E = J^2/2I \) where \( I \) is the moment of inertia of the symmetric top.

If one assumes a similar relationship among Energy and angular momentum in the relativistic case, one will have the familiar Regge-type of relation associated with the on-shell string spectrum: \( J \sim (\alpha)^2 m^2 \), where on shell, \( m^2 = p^2 \) and \( (\alpha)^2 \) is the inverse string tension, of the order of \( \Lambda^2 \). In this case one will be able to match the results of section 1, involving an effective \( \hbar_{eff} (p^2) \), for on-shell values \( p^2 = m^2 \), with the last terms of (40) after breaking the C-space polydimensional invariance in (38) and using the desired Regge relation.

Concluding: Quantization in C-spaces contains a very rich Noncommutative structure from which many old results can be derived after breaking the C-space Lorentz invariance. There is no need to introduce ad-hoc nontrivial commutation relations for the spacetime coordinates. These are induced from the mere quantization process. No extra discrete fifth dimension is required to introduce a length scale. C-space Relativity already has a natural invariant minimum Planck scale by definition.

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