Frobenius-Perron theory of the bound quiver algebras containing loops

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ABSTRACT
The Frobenius-Perron dimension of a matrix, also known as the spectral radius, is a useful tool for studying linear algebras and plays an important role in the classification of the representation categories of algebras. In this paper, we study the Frobenius-Perron theory of the representation categories of bound quiver algebras containing loops, and find a way to calculate the Frobenius-Perron dimensions of these algebras satisfying the commutativity condition of loops. As an application, we prove that the Frobenius-Perron dimension of the representation category of a modified ADE bounded quiver algebra is equal to the maximal number of loops at each vertex. Finally, we point out that there also exist infinite dimensional algebras whose Frobenius-Perron dimensions is equal to the maximal number of loops by giving an example.

1. Introduction

The Frobenius-Perron dimension (also called the spectral radius) of a matrix is an elementary and extremely useful invariant in linear algebra, combinatorics, topology, probability and statistics. For instance, we can classify all the finite graphs which are simple and connected by applying the spectral radius of their adjacency matrices [5].

The Frobenius-Perron dimension of an object in a semisimple finite tensor (or fusion) category was introduced by Etingof-Nikshych-Ostrik in 2005 [8] (also see [6, 7, 10]). Since then it has become an extremely useful invariant in the study of fusion categories and representations of semisimple (weak and/or quasi-)Hopf algebras.

In 2017, the Frobenius-Perron dimension of an endofunctor of a category was introduced by the authors in [3]. It can be viewed as a generalization of the Frobenius-Perron dimension of an object in a fusion category introduced by Etingof-Nikshych-Ostrik [8]. It was shown in [3, 4, 12] that the Frobenius-Perron dimension has strong connections with the representation type of a category.

To gain a better understanding of the Frobenius-Perron dimension of an endofunctor, Wicks [11] calculated the Frobenius-Perron dimension of the representation category of a modified ADE bounded quiver algebra with arrows in a certain direction. Wicks showed that the Frobenius-Perron dimension of this category is equal to the maximal number of loops at each vertex, and asked what would happen if the directions of the arrows were changed.

In this paper, we study the Frobenius-Perron theory of the representation categories of the bound quiver algebras containing loops. As we know, a bound quiver algebra containing loops is representation-infinite and it is hard to describe completely the homomorphism spaces (or extension spaces) between
objects in the representation category. We focus on these algebras satisfying the commutativity condition of loops (see Definition 3.1). Let \( A \) be a bound quiver algebra satisfying the commutativity condition of loops and \( B = A/J \) be its quotient algebra where \( J \) is the ideal generated by all the loops. Inspired by the work in [2], in which we calculated the Frobenius-Perron dimension of the representation categories of representation-directed algebras, we consider the case that \( B \) is a representation-directed algebra, and prove that the Frobenius-Perron dimension of \( A \) is equal to the maximal number of loops at each vertex. As an application, we confirm that the Frobenius-Perron dimension of the representation category of a modified ADE bound quiver algebra is equal to the maximal number of loops at each vertex no matter what direction of arrows we choose, which presents an explicit answer to the question asked in [11]. We further discuss the case where \( B \) is a canonical algebra of type ADE in this paper, and prove that the Frobenius-Perron dimension of \( A \) falls in an interval with length less than 1. At last, we show that there also exist infinite dimensional algebras whose Frobenius-Perron dimensions are equal to the maximal number of loops by calculating the Frobenius-Perron dimensions of the representation categories of polynomial algebras.

1.1. Conventions

(1) Throughout let \( k \) be an algebraically closed field, and let everything be over the field \( k \).
(2) Usually \( Q \) means a finite connected quiver.
(3) If \( A \) is an algebra over the base field \( k \), then we denote by \( A\text{-mod} \) the category of finite dimensional left \( A \)-modules.

The paper is organized as follows. In Section 1, we introduce the background and summarize the main work of this paper. In Section 2, we review the definition of Frobenius-Perron dimension of a \( k \)-linear category. In Section 3, we study the loop-extended algebras (see Definition 3.1) and describe the properties of the extension spaces over the representation categories of these algebras. In Section 4, we find a way to obtain the Frobenius-Perron dimension of loop-extended algebras of representation-directed algebras which include ADE quiver algebras as special cases. In Section 5, we study the Frobenius-Perron dimension of a tube. In Section 6, we calculate the Frobenius-Perron dimensions of loop-extended algebras of canonical algebras and give some examples. In Section 7, we give the Frobenius-Perron dimensions of the representation categories of the polynomial algebras. The following two theorems are main results of this paper which are proved in Theorems 3.3, 4.1 and 6.3.

**Theorem 1.1.** Let \( A = kQ/I \) be the bound quiver algebra of a finite quiver \( Q \), where \( I \) is an admissible ideal satisfying the commutativity condition of loops. Assume that \( B \) is the loop-reduced algebra of \( A \) (see Definition 3.1). Then the following assertions hold.

1. If \( M, N \) are two \( B \)-modules with \( \text{Hom}_B(M, N) = 0 \), then
   \[
   \text{Ext}^1_A(M, N) \cong \text{Ext}^1_B(M, N).
   \]
2. If \( M \) is a brick \( B \)-module which is not simple, then
   \[
   \text{Ext}^1_A(M, M) \cong \text{Ext}^1_B(M, M).
   \]

**Theorem 1.2.** Keep the notation as in Theorem 1.1. Then the following hold.

1. If \( B \) is representation-directed, then
   \[
   \text{fpd} (A\text{-mod}) = \max_{P \in Q_0} N_P
   \]
   where \( N_P \) is the number of loops at \( P \).
2. If \( B \) is a canonical algebra of type ADE, then
   \[
   \text{fpd} (A\text{-mod}) \in [n_{\text{max}}, n_{\text{max}} + 1)
   \]
   where \( n_{\text{max}} \) is the maximal number of loops at each vertex in the quiver.
2. Preliminaries

2.1. $\mathbb{k}$-linear categories

If $\mathcal{C}$ is a $\mathbb{k}$-linear category, then $\text{Hom}_{\mathcal{C}}(M, N)$ is a $\mathbb{k}$-linear space for all objects $M, N$ in $\mathcal{C}$. If $\mathcal{C}$ is also abelian, then $\text{Ext}^i_{\mathcal{C}}(M, N)$ are $\mathbb{k}$-linear spaces for all $i \geq 0$. Let $\text{dim}$ be the $\mathbb{k}$-vector space dimension.

Throughout the rest of the paper, let $\mathcal{C}$ denote a $\mathbb{k}$-linear category. A functor between two $\mathbb{k}$-linear categories is assumed to preserve the $\mathbb{k}$-linear structure. For simplicity, $\text{dim}(M, N)$ stands for $\text{dim} \text{Hom}_{\mathcal{C}}(M, N)$ for any objects $M$ and $N$ in $\mathcal{C}$.

The set of finite subsets of nonzero objects in $\mathcal{C}$ is denoted by $\Phi$ and the set of subsets of $n$ nonzero objects in $\mathcal{C}$ is denoted by $\Phi_n$ for each $n \geq 1$. It is clear that $\Phi = \bigcup_{n \geq 1} \Phi_n$. We do not consider the empty set as an element of $\Phi$.

**Definition 2.1.** [3, Definition 2.1] Let $\mathcal{C}$ be a $\mathbb{k}$-linear abelian category, and $\phi := \{X_1, X_2, \ldots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$, namely, $\phi \in \Phi_n$.

1. The adjacency matrix of $\phi$ is defined to be $C(\phi) := (a_{ij})_{n \times n}$, where $a_{ij} := \text{dim} \text{Ext}^1_{\mathcal{C}}(X_i, X_j) \ \forall i, j$.
2. An object $M$ in $\mathcal{C}$ is called a brick if $\text{Hom}_{\mathcal{C}}(M, M) = \mathbb{k}$.
3. The set $\phi \in \Phi$ is called a brick set if each $X_i$ is a brick and $\text{dim}(X_i, X_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$. The set of brick $n$-object subsets is denoted by $\Phi_{n, b}$. We write $\Phi_b := \bigcup_{n \geq 1} \Phi_{n, b}$.

Let $\mathcal{C}$ be an $n \times n$-matrix over the complex field $\mathbb{C}$. The spectral radius of $\mathcal{C}$ is defined to be $\rho(\mathcal{C}) := \max\{|r_1|, |r_2|, \ldots, |r_n|\} \in \mathbb{R}$ where $\{r_1, r_2, \ldots, r_n\}$ is the complete multi-set of eigenvalues of $\mathcal{C}$.

**Definition 2.2.** [3, Definition 2.3] Retain the notation as in Definition 2.1, and we use $\Phi_b$ as the testing objects.

1. The $n$th Frobenius-Perron dimension of $\mathcal{C}$ is defined to be $\text{fpd}^n(\mathcal{C}) := \sup_{\phi \in \Phi_{n, b}} \{\rho(\mathcal{C}(\phi))\}$.
   
   If $\Phi_{n, b}$ is empty, then by convention, we write $\text{fpd}^n(\mathcal{C}) = 0$.
2. The Frobenius-Perron dimension of $\mathcal{C}$ is defined to be $\text{fpd}(\mathcal{C}) := \sup_{n} \{\text{fpd}^n(\mathcal{C})\} = \sup_{\phi \in \Phi_b} \{\rho(\mathcal{C}(\phi))\}$.

2.2. Representation of bound quivers

Let $Q$ be a finite connected quiver and $I$ be an admissible ideal of $\mathbb{k}Q$. Then $(Q, I)$ is called a bound quiver and $A = \mathbb{k}Q/\mathbb{k}I$ is called the bound quiver algebra of $Q$ with respect to $I$. A representation of $(Q, I)$ is a tuple $M := (M_a, M_\alpha)_{a \in Q_0, \alpha \in Q_1}$, satisfying

- (R1) To each point $a$, $M_a$ is a finite dimensional $\mathbb{k}$-vector space;
- (R2) To each arrow $\alpha : a \to b$, $M_\alpha$ is a $\mathbb{k}$-linear map from $M_a$ to $M_b$;
- (R3) If $\sum_{i=1}^{m} \lambda_i \alpha_{i,1} \cdots \alpha_{i,n_i} \in I$, where $\lambda_i \in \mathbb{k}$ (not all zero), then it implies $\sum_{i=1}^{m} \lambda_i M_{\alpha_{i,1}} \cdots M_{\alpha_{i,n_i}} = 0$. 


Assume $M = (M_a, M_a)$ and $N = (N_a, N_a)$ are two representations of $(Q, I)$, a morphism $f : M \to N$ from $M$ to $N$ is a tuple $f = (f_a : M_a \to N_a)_{a \in Q_0}$ of $k$-linear maps such that $f_b \circ M_a = N_a \circ f_a$ holds for each arrow $\alpha : a \to b$.

Denote by $\text{rep}(Q, I)$ the category of representations of $(Q, I)$. The following proposition is from [11]. For further application, we present a categorical proof here.

**Proposition 3.2.** [11, Proposition 3.4] The following proposition is from [11]. For further application, we present a categorical proof here.

Let $(Q, I)$ be a bound quiver and $A = kQ/I$ be the bound quiver algebra of $Q$ with respect to $I$. There exists a $k$-linear equivalence of categories

$$F : A\text{-mod} \longrightarrow \text{rep}(Q, I).$$

3. **Representation category of a bound quiver containing loops**

**Definition 3.1.** Let $A = kQ/I$ be the bound quiver algebra of a finite quiver $Q$, where $I$ is an admissible ideal. We say $I$ satisfies the commutativity condition of loops if the following conditions hold.

(a) For a path $\gamma \alpha$, if $\gamma$ is a loop and $\alpha$ is an arrow but not a loop, then $\gamma \alpha$ belongs to $I$;

(b) For a path $\beta \gamma$, if $\gamma$ is a loop and $\beta$ is an arrow but not a loop, then $\beta \gamma$ belongs to $I$;

(c) For any two loops $\gamma_1, \gamma_2$ base at the same vertex, $\gamma_1 \gamma_2 - \gamma_2 \gamma_1$ belongs to $I$.

In this case, let $I$ be the ideal of $A$ generated by all the loops. The quotient algebra $B := A/I$ is called the loop-reduced algebra of $A$ and we call $A$ a loop-extended algebra of $B$.

In this paper, we only consider bound quiver algebras satisfying the commutativity condition of loops. The following proposition is from [11]. For further application, we present a categorical proof here.

**Proposition 3.2.** [11, Proposition 3.4] Let $(Q, I)$ be a bound quiver and $A = kQ/I$ be the bound quiver algebra of $Q$, where $I$ is an admissible ideal of $kQ$ satisfying the commutativity condition of loops. Assume $B$ is the loop-reduced algebra of $A$, then we get a one-to-one correspondence between the isomorphism classes below:

$\{\text{brick } A\text{-modules}\} \leftrightarrow \{\text{brick } B\text{-modules}\}$.

Moreover, for each two brick $B$-modules $M, N$, there exists a isomorphism

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M, N).$$

**Proof.** Since $B$ is a quotient algebra of $A$, we have $\text{Hom}_A(M, M) \cong \text{Hom}_B(M, M) = k$ for any brick $B$-module $M$. So $\{\text{brick } B\text{-modules}\}$ is a subset of $\{\text{brick } A\text{-modules}\}$. Conversely, if there exists a brick $A$-module $N$ not belonging to the set $\{\text{brick } B\text{-modules}\}$, then we can find a vertex $P$ in $Q$ and a loop $\gamma_0$ at $P$ such that $N_{\gamma_0} \neq 0$. Consider the map $f : N \to N$ defined as follow.

$$f_R = \begin{cases} 0, R \in Q_0 \setminus \{P\} \\ N_{\gamma_0}, R = P \end{cases}$$

It is not hard to prove that $f$ is an endomorphism of $N$ because $I$ satisfies the commutativity condition of loops. Since $N_{\gamma_0}$ is nilpotent due to the condition $I$ is admissible, $f$ is linearly independent of $1_N$ which implies $N$ is not a brick. It contradicts the assumption. Therefore, there is a one-to-one correspondence between the isomorphism classes $\{\text{brick } A\text{-modules}\}$ and $\{\text{brick } B\text{-modules}\}$.

Furthermore, by viewing brick $B$-modules $M, N$ as brick $A$-modules, every $B$-module morphism from $M$ to $N$ can be seen as an $A$-modular morphism. It is obvious that an $A$-modular morphism $f : M \to N$ is well-defined as an $B$-modular morphism. Thus, we have a natural isomorphism

$$\text{Hom}_A(M, N) \cong \text{Hom}_B(M, N).$$

** Proposition 3.2 shows that if we want to calculate the Frobenius-Perron dimension of a bound quiver satisfying the commutativity condition, we need only consider the brick sets after removing loops. Therefore, the problem is how the extension spaces change when we remove the loops. Luckily, we get the following important observation.**
Theorem 3.3. Keep the notation as in Proposition 3.2. Then the following hold.
(1) If $M, N$ are two $B$-modules, and $\text{Hom}_B(M, N) = 0$, then we have
$$\text{Ext}_A^1(M, N) \cong \text{Ext}_B^1(M, N).$$
(2) If $M$ is a brick $B$-module and $M$ is not simple, then we have
$$\text{Ext}_A^1(M, M) \cong \text{Ext}_B^1(M, M).$$

Proof. (1) Since $B$-mod is a full subcategory of $A$-mod, each element in $\text{Ext}_B^1(M, N)$ can be viewed as an element in $\text{Ext}_A^1(M, N)$. If $\text{Ext}_A^1(M, N) \not\cong \text{Ext}_B^1(M, N)$, then there exists an element $\eta$ belonging to $\text{Ext}_A^1(M, N)$ but not in $\text{Ext}_B^1(M, N)$. Notice that the element $\eta$ in $\text{Ext}_A^1(M, N)$ can be represented as a short exact sequence
$$\eta : 0 \rightarrow N \xrightarrow{\psi} L \xrightarrow{\psi} M \rightarrow 0$$

where $L$ is in $A$-mod but not in $B$-mod. It means that there exists a loop $\gamma$ such that $L_\gamma$ is nonzero linear map. Assume that $\gamma$ is located at the vertex $P$, the non-loop arrows with the target $P$ are $\beta_i(i = 1, \ldots, n)$, the non-loop arrows with the source $P$ are $\alpha_i(i = 1, \ldots, m)$ and the loops located at $P$ besides $\gamma$ are $\gamma_i(i = 1, \ldots, l)$. We emphasize that the above assumption includes the cases there is no non-loop arrow with the target $P$, there is no non-loop arrow with the source $P$ or there is no loop located at $P$ besides $\gamma$. So the local part at $P$ is described as follows

Therefore, for each $\beta \in \{\beta_1, \beta_2, \ldots, \beta_n\}$, we have the following short exact sequence

$$\begin{align*}
M_\gamma &\xrightarrow{\varphi_P} L_\gamma &\xrightarrow{\psi_P} N_\gamma \\
M_\beta &\xrightarrow{\varphi_T} L_\beta &\xrightarrow{\psi_T} N_\beta \\
M_T &\xrightarrow{\varphi_T} L_T &\xrightarrow{\psi_T} N_T
\end{align*}$$

where $T$ is the target of $\beta$ and $M_\gamma, N_\gamma = 0$. Since there are isomorphisms of $k$-vector spaces $L_P \cong M_P \oplus N_P$ and $L_T \cong M_T \oplus N_T$, we set
$$\varphi_P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_P = (0, 1),$$
$$\varphi_T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \psi_T = (0, 1).$$

Suppose that
$$L_\gamma = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}, L_\beta = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$
Since $L_\gamma \varphi_P = \varphi_P M_\gamma = 0$, we get
\[
\begin{pmatrix}
 f_{11} & f_{12} \\
 f_{21} & f_{22}
\end{pmatrix}
\begin{pmatrix}
 1 \\
 0
\end{pmatrix}
= \begin{pmatrix}
 f_{11} \\
 f_{21}
\end{pmatrix} = 0.
\]
Similarly, we have $f_{22} = 0$. So $L_\gamma = \begin{pmatrix}
 0 & f_{12} \\
 0 & 0
\end{pmatrix}$. Since $L_\beta L_\gamma = 0$, we get
\[
\begin{pmatrix}
 g_{11} & g_{12} \\
 g_{21} & g_{22}
\end{pmatrix}
\begin{pmatrix}
 0 & f_{12} \\
 0 & 0
\end{pmatrix}
= \begin{pmatrix}
 0 & g_{11} f_{12} \\
 0 & g_{21} f_{12}
\end{pmatrix} = 0.
\]
Also by $L_\beta \varphi_P = \varphi_T M_\beta$, we get
\[
\begin{pmatrix}
 g_{11} & g_{12} \\
 g_{21} & g_{22}
\end{pmatrix}
\begin{pmatrix}
 1 \\
 0
\end{pmatrix}
= \begin{pmatrix}
 1 \\
 0
\end{pmatrix} M_\beta.
\]
Thus $g_{11} = M_\beta$ and $g_{21} = 0$. It follows that $M_\beta f_{12} = 0$.

Dually, for each $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, we have the following short exact sequence

\[
\begin{array}{c}
 M_S \xrightarrow{\varphi_S} L_S \xrightarrow{\psi_S} N_S \\
 M_\alpha \xrightarrow{\varphi_P} L_\alpha \xrightarrow{\psi_P} N_\alpha \\
 M_P \xrightarrow{\varphi_P} L_\gamma \xrightarrow{\psi_P} N_\gamma
\end{array}
\]
where $S$ is the source of $\alpha$. By a similar argument as above, we have $f_{12} N_\alpha = 0$.

Now we can construct a nonzero homomorphism $\theta : N \to M$ such that $\theta_P = f_{12}$ and $\theta_{P'} = 0$ for each $P' \in Q_0 \setminus \{P\}$. It contradicts the condition $\text{Hom}_B (M, N) = 0$. Therefore $\text{Ext}_A^1 (M, N) \cong \text{Ext}_B^1 (M, N)$.

(2) If $\text{Ext}_A^1 (M, M) \not\cong \text{Ext}_B^1 (M, M)$, by a similar argument as (1), we can construct a nonzero homomorphism $\theta$ from $M$ to $M$. Since $M$ is not a simple module, there exist at least two vertices $P_1, P_2$ such that $M_{P_1}, M_{P_2} \neq 0$. But there is only one vertex $P_0$ satisfying $\theta_{P_0} \neq 0$. So $\theta$ is not an isomorphism which means $\theta$ and $1_M$ are linearly independent. It follows that $\dim_k \text{Hom}_B (M, M) \geq 2$ and this contradicts the condition that $M$ is a brick. Thus, $\text{Ext}_A^1 (M, M) \cong \text{Ext}_B^1 (M, M)$.

Remark 3.4. For a simple $B$-module $S_P$ with a vertex $P$ in $Q_0$, by [1, Ch.III, Lemma 2.12], the value of $\dim \text{Ext}_A^1 (S_P, S_P)$ is equal to the number of loops at $P$.

4. Loop-extended algebras of representation-directed algebras

In this section, we study the Frobenius-Perron dimension of loop-extended algebras of representation-directed algebras. We recall the definition of representation-directed algebras first. Let $A$ be an algebra. Recall that a path in $A$-mod is a sequence
\[
M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_{t-1}} M_t \xrightarrow{f_t} M_t,
\]
where $t \geq 1$, $M_0, M_1, \ldots, M_t$ are indecomposable $A$-modules and $f_1, \ldots, f_t$ are nonzero non-isomorphism homomorphisms. A path in $A$-mod is called a cycle if its source module $M_0$ is isomorphic to its target module $M_t$. An indecomposable $A$-module that lies on no cycle in $A$-mod is called a directing module. An algebra is called representation-directed if every indecomposable $A$-module is directing. We get the Frobenius-Perron dimension of loop-extended algebras of these algebras as follows.
Theorem 4.1. Let \( A = \mathbb{k}Q/I \) be a bound quiver algebra for some finite quiver \( Q \), where \( I \) is an admissible ideal satisfying the commutativity condition of loops. Assume \( B \) is the loop-reduced algebra of \( A \). If \( B \) is representation-directed, then we have

\[
\text{fpd}(A\text{-mod}) = \max_{P\in Q_0} N_P
\]

where \( N_P \) means the number of loops at the vertex \( P \).

**Proof.** First we prove that for every brick set \( \phi = \{M_i\}_{i=1}^n \) in \( B\text{-mod} \), the adjacency matrix is a strictly upper triangular matrix, that is to say, there exists a permutation \( \sigma \) of \( \{1, \ldots, n\} \) such that for \( i \leq j \),

\[
\text{Hom}_B(M_i, \tau M_j) = 0.
\]

We prove this using induction on \( n \). If \( n = 1 \), there is only one element \( M_1 \) in \( \phi \). If \( \text{Hom}_B(M_1, \tau M_1) \neq 0 \), then there exists a non-zero non-isomorphism homomorphism \( f : M_1 \rightarrow \tau M_1 \). Note that the Auslander-Reiten serie \( \tau M_1 \rightarrow \bigoplus_{i=1}^k N_i \rightarrow M_1 \) gives a path \( \tau M_1 \rightarrow N_1 \rightarrow M_1 \). So we get a cycle \( M_1 \rightarrow \tau M_1 \rightarrow N_1 \rightarrow M_1 \) which contradicts the assumption \( B \) is a representation-directed algebra.

Assume (4.1.1) holds for \( n = k - 1 \). If \( n = k \), then exists a number \( i_0 \), such that \( \text{Hom}_B(M_{i_0}, \tau M_j) = 0 \) for \( j = 1, 2, \ldots, n \). Otherwise, for each \( i \), there exists \( j \), such that \( \text{Hom}_B(M_i, \tau M_j) \neq 0 \). We can get a path

\[
M_{i_1} \rightarrow \tau M_{i_2} \rightarrow N_{i_2} \rightarrow M_{i_2} \rightarrow \tau M_{i_1} \rightarrow \cdots
\]

Since the brick set is finite, there exist \( i_s, i_t \) such that \( i_s = i_t \). Then we get a cycle which contradicts the assumption \( B \) is a representation-directed algebra. Define \( \sigma_1 \) permuting \( i_0 \) with 1 and fix the other numbers. By the induction hypothesis, we can define the permutation \( \sigma_2 \) such that \( \text{Hom}_A(M_{\sigma_2 \sigma_1(i), \tau M_{\sigma_2 \sigma_1(j)}}) = 0 \) for \( 1 < \sigma_1(i) \leq \sigma_1(j) \). Let \( \sigma = \sigma_2 \sigma_1 \); then we get what we need.

Now let us turn to \( A\text{-mod} \). By Corollary 3.2, we have

\[
\{\text{brick } A\text{-modules}\} \leftrightarrow \{\text{brick } B\text{-modules}\}.
\]

For a brick set \( \phi \), by Theorem 3.3 (1), we know the adjacency matrix \( C(\phi_A) \) of \( \phi \) in \( A\text{-mod} \) is the same as the adjacency matrix \( C(\phi_B) \) of \( \phi \) in \( B\text{-mod} \) not considering the diagonal elements. So \( C(\phi_A) \) is a upper triangular matrix. Assume \( M \in \phi \); by Theorem 3.3 (2), we have

\[
\text{Ext}_A(M, M) = 0, \text{ if } M \text{ is not simple},
\]

and

\[
\text{Ext}_A(M, M) = N_P, \text{ for } M = S_P,
\]

where \( S_P \) is the simple module at the vertex \( P \), \( N_P \) is the number of loops at \( P \). Thus the elements in the diagonal of \( C(\phi_A) \) are nonzero means the corresponding module is simple and there are several loops at the corresponding vertex.

Therefore the spectral radius of the adjacency matrix of a brick set in \( A\text{-mod} \) is the maximal number of loops at each vertex, that is,

\[
\text{fpd}(A\text{-mod}) = \max_{P\in Q_0} N_P.
\]

Remark 4.2. For any ADE quiver algebra, whatever direction of arrows we choose, it is a representation-directed algebra. Since all of the modified ADE bounded quiver algebras are loop-extended algebras of representation-directed algebras, by Theorem 4.1, the Frobenius-Perron dimension of representation category of these algebras is equal to the maximal number of loops at each vertex. This answers the question in [11].

We point out that in Theorem 4.1, the condition “\( B \) is representation-directed” is necessary. In fact, if \( B \) is not representation-directed, we have the following counterexample.
Example 4.3. Define quiver $Q$ as follow.

\[
\begin{array}{c}
\alpha & \beta \\
1 & & 4 \\
\gamma & \delta \\
3 & & 2
\end{array}
\]

$Q'$ is the quiver formed from $Q$ by adding $N_i$ loops to each vertex $i$. Let $A = \mathbb{k}Q'/\langle I \cup \{\alpha \beta\} \rangle$, where $I = \{\alpha_1 \alpha_2 | \alpha_1 \text{ is a loop or } \alpha_2 \text{ is a loop}\}$. It is easy to verify that $\langle I \cup \{\alpha \beta\}\rangle$ is an admissible ideal satisfying the conditions (a),(b),(c) in Definition 3.1. Assume $B$ is the loop-reduced algebra of $A$.

We can draw the Auslander-Reiten quiver of $B$ as follows.

It is easy to find a brick set consisting of the two modules at the bottom of the Auslander-Reiten quiver, whose corresponding matrix is not a upper triangular matrix. Explicitly, the corresponding matrix is

\[
\begin{pmatrix}
N_2 & 1 \\
1 & 0
\end{pmatrix},
\]

and the corresponding spectral radius is $\frac{N_2 + \sqrt{N_2^2 + 4}}{2}$.

Notice that the corresponding matrices of all the other brick sets in $A$-mod are upper triangulated matrices. By the same argument as in Theorem 4.1, we have

\[
\text{fpd (}A\text{-mod)} = \max\{\frac{N_2 + \sqrt{N_2^2 + 4}}{2}, N_1, N_3, N_4\}.
\]

5. Some properties of tubes

In the next two sections, we consider the loop-extended algebras of canonical algebras of type ADE and try to calculate the Frobenius-Perron dimension of the corresponding representation category. Notice that there are several tubes in the representation category of canonical algebras. We first give some properties of tubes, compared to [4, Section 2.2].

Recall that in [9], a matrix $C$ is called irreducible if there is no permutation matrix $P$ such that

\[
P^T C P = \begin{pmatrix} C_1 & C_2 \\ 0 & C_3 \end{pmatrix}
\]
where $C_1$ and $C_3$ are nonzero matrices.

**Lemma 5.1.** Let $T$ be a tube, $\phi$ be a brick set in $T$, and $C$ be the adjacency matrix of $\phi$. If $C$ is irreducible, then $C$ is similar to the matrix

$$
\begin{pmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & \ddots & 1 \\
1 & 0
\end{pmatrix}.
$$

**Proof.** Assume $\phi = \{M_1, M_2, \ldots, M_n\}$. For each $M_i$, there exists some $j \in \{1, \ldots, n\}$ such that $\text{Ext}^1_T(M_i, M_j) \neq 0$ since $C$ is irreducible. If there exists $j' (j' \neq j)$ such that $\text{Ext}^1_T(M_i, M_{j'}) \neq 0$, then by Serre duality, we have

$$
\text{Hom}_T(M_{j'}, \tau M_i) \cong \text{Ext}^1_T(M_i, M_{j'}) \neq 0.
$$

Combining it with

$$
\text{Hom}_T(M_{j'}, M_i) = 0,
$$

we find that $M_{j'}$ is restricted on a coray of $T$. Similarly, we have

$$
\text{Hom}_T(M_j, \tau M_i) \cong \text{Ext}^1_T(M_i, M_j) \neq 0
$$

and

$$
\text{Hom}_T(M_j, M_i) = 0.
$$

Then $M_j, M_{j'}$ must be on the same coray of $T$, and it is a contradiction to the condition that $\phi$ is a brick set. Therefore, $M_j$ is the unique object in $\phi$ satisfying $\text{Ext}^1_T(M_i, M_j) \neq 0$. We claim that $\text{Ext}^1_T(M_i, M_j) = k$.

In fact, if $\dim \text{Ext}^1_T(M_i, M_j) \geq 2$, then we have

$$
\dim \text{Hom}_T(M_j, M_i) \geq \dim \text{Hom}_T(M_j, \tau M_i) = \dim \text{Ext}^1_T(M_i, M_j) \geq 2,
$$

which is impossible. By a similar argument, we know that there also exists a unique object $M_k \in \phi$ such that $\text{Ext}^1_T(M_k, M_i) \neq 0$ and we have $\text{Ext}^1_T(M_k, M_i) = k$.

By the condition that $C$ is irreducible, we can find that $C$ is a matrix similar to

$$
\begin{pmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & \ddots & 1 \\
1 & 0
\end{pmatrix}.
$$

\[\square\]

**Corollary 5.2.** Let $T$ be a tube, then we have

$$
\text{fpd}(T) = 1.
$$

**Proof.** Let $\phi$ be a brick set in $T$, and assume $\phi = \phi_1 \cup \cdots \cup \phi_s$ such that the adjacency matrix of $\phi$ is

$$
\begin{pmatrix}
C_1 & * & \cdots & * \\
0 & C_2 & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_s
\end{pmatrix}.
$$


where $C_i$ is the adjacency matrix of $\phi_i$ and $C_i$ is irreducible, $i = 1, \ldots, s$. Then each $C_i$ has the form

$$
\begin{pmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & \ddots & 1 \\
1 & 0
\end{pmatrix}, (0) \text{ or } (1).
$$

So $\rho(C) = \max\{\rho(C_1), \ldots, \rho(C_s)\} \leq 1$, and it follows that fpd($T$) $\leq 1$.

On the other hand, the set consisting of all the simple objects in $T$ is a brick set, and the adjacency matrix is

$$
\begin{pmatrix}
0 & 1 \\
0 & \ddots \\
\vdots & \ddots & 1 \\
1 & 0
\end{pmatrix}
$$

whose spectral radius is 1. Therefore, fpd($T$) $= 1$. \hfill $\square$

**Lemma 5.3.** Let $T$ be a tube. If there exist two different simple objects $S_1, S_2$ in $T$ satisfying that $\text{Ext}^1(S_1, S_2) = 0$, then we can find an object $M$ such that

$$
\text{Ext}^1(S_1, M) = k, \text{Ext}^1(M, S_2) = k,
$$

and $\{S_1, S_2, M\}$ is a brick set.

**Proof.** We have

$$
\text{Hom}(\cdot, \tau S_1) \cong \text{Ext}^1(S_1, \cdot) = k, \text{Hom}(\tau^{-1} S_2, \cdot) \cong \text{Ext}^1(\cdot, -S_2) = k
$$

which give a ray starting at $\tau^{-1} S_2$ and a coray ending at $\tau S_1$ on the tube. Let $M$ be the intersection of the ray and the coray; by Serre duality, we get what we need. \hfill $\square$

### 6. Loop-extended algebras of canonical algebras of type ADE

Before calculating the Frobenius-Perron dimension of this type of algebras, we need some lemmas.

**Lemma 6.1.** Let $f(x) = (x - n_1)^{r_1}(x - n_2)^{r_2} \cdots (x - n_s)^{r_s} \in \mathbb{R}[x]$, where $s \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_s \in \mathbb{Z}_{>0}$ and $n_1, \ldots, n_s \in \mathbb{R}$ satisfying $0 \leq n_1 < n_2 < \cdots < n_{s-1} \leq n_s - 1$. Let $\{x_i\}_{i=1}^{r_1 + \cdots + r_s}$ be the complete set of complex roots of $f(x) - 1$. Denote the value $\max\{|x_i|\}_{i=1}^{r_1 + \cdots + r_s}$ by $\rho(f(x) - 1)$. Then the following hold.

1. $f(x) - 1$ has the unique real root $x_0$ in $(n_s, n_s + 1) \text{ and } \rho(f(x) - 1) = x_0$.
2. If $0 \leq m \leq n_s - 1$, then $\rho((x - m)f(x) - 1) < \rho(f(x) - 1)$.
3. The value $\rho((x - n_s)f(x) - 1) \geq \rho(f(x) - 1)$. Moreover, $\rho((x - n_s)f(x) - 1) = \rho(f(x) - 1)$ if and only if $s = 1$.

**Proof.** (1) Since $f(n_s) - 1 < 0$ and $f(n_s + 1) - 1 \geq 0, f(x) - 1$ has a real root in $(n_s, n_s + 1]$, denote it by $x_0$. The derivative of $f(x)$ is

$$
f'(x) = r_1(x - n_1)^{r_1-1}[(x - n_2)^{r_2} \cdots (x - n_{s-1})^{r_{s-1}} \cdot (x - n_s)^{r_s}]
\quad + \cdots + [(x - n_1)^{r_1}(x - n_2)^{r_2} \cdots (x - n_{s-1})^{r_{s-1}}] \cdot r_s(x - n_s)^{r_s-1}.
$$

It is easy to see that $f'(x) > 0$ in $(n_s, n_s + 1]$. So $x_0$ is the unique real root in $(n_s, n_s + 1]$.

Moreover, if $f(x) - 1$ has a complex root $z_0$ such that $|z_0| > x_0$, we get $|z_0 - n_j| > |x_0 - n_j|$ for $j = 1, \ldots, s$. Therefore, $|f(z_0)| > |f(x_0)| = 1$ which is a contradiction to the condition $f(z_0) = 1$. Hence $\rho(f(x) - 1) = x_0$. 

(2) Denote \((x - m)f(x)\) by \(g(x)\). By (1), \(g(x) - 1\) has the unique real root \(x_0'\) in \((n_2, n_3 + 1]\) and \(\rho(g(x) - 1) = x_0'\). Notice that \(f(x) < g(x)\) always holds in \((n_2, n_3 + 1]\). So \(g(x)\) will meet 1 before \(f(x)\) in \((n_2, n_3 + 1]\), which implies \(x_0' < x_0\). So \(\rho(g(x) - 1) < \rho(f(x) - 1)\).

(3) If \(s = 1\), then it is easy to see that \(\rho((x - n_s)f(x) - 1) = \rho(f(x) - 1)\). So we need only consider the cases of \(s > 1\). Note that the unique root \(x_0\) in \((n_s, n_s + 1]\) of \(f(x) - 1\) is also the unique real root in \((n_s, n_s + 1]\) of \(f(x) - 1\). Let \(h(x) = (x - n_1)^{r_1} (x - n_2)^{r_2} \cdots (x - n_{s-1})^{r_{s-1}}\). Then

\[
f(x)^{1/n_s} = h(x)^{1/n_s} \cdot (x - n_s) \quad \text{and} \quad [(x - n_s)f(x)]^{1/n_s+1} = h(x)^{1/n_s+1} \cdot (x - n_s).
\]

Hence \(h(x)^{1/n_s} \cdot (x - n_s) > h(x)^{1/n_s+1} \cdot (x - n_s)\) in \((n_s, n_s + 1]\), which implies \(h(x)^{1/n_s} \cdot (x - n_s)\) meets 1 before \(h(x)^{1/n_s+1} \cdot (x - n_s)\) in \((n_s, n_s + 1]\). Therefore \(\rho((x - n_s)f(x) - 1) > \rho(f(x) - 1)\).

Following is an example.

**Example 6.2.** Let \(f(x) = x(x - 2) - 1, f_1(x) = x(x - 1)(x - 2) - 1\) and \(f_2(x) = x(x - 2)^2 - 1\). Then \(f_1(x) + 1 = (x - 1)(f(x) + 1)\) and \(f_2(x) + 1 = (x - 2)(f(x) + 1)\). By Lemma 6.1(2), we have \(\rho(f_1(x)) < \rho(f(x))\). By Lemma 6.1(3), we have \(\rho(f_2(x)) > \rho(f(x))\). Therefore we get

\[
\rho(f_1(x)) < \rho(f(x)) < \rho(f_2(x)).
\]

Recall a bound quiver algebra \(B\) is called a canonical algebra of type ADE if \(B\) is one of the following algebras:

(1) \(A(n, m) = \mathbb{k} Q_A\) for \(n \geq 1, m \geq 1\);

\[
Q_A:\begin{array}{c}
(1, 1) \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-1}} (1, n - 1) \\
0 \xleftarrow{\alpha_1}
\end{array}
\]

(2) \(D_f(n) = \mathbb{k} Q_D/I\) for \(n \geq 4\), where \(I\) is the admissible ideal of \(\mathbb{k} Q_D\) generated by \(\alpha_1 \cdots \alpha_{n-2} + \beta_1 \beta_2 + \gamma_1 \gamma_2\);

\[
Q_D:\begin{array}{c}
(1, 1) \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{n-3}} (1, n - 3) \\
0 \xleftarrow{\alpha_1}
\end{array}
\]
Theorem 6.3. Let $A$ be the algebra defined as above, then we have

$$\text{fpd} (A\text{-mod}) \in [n_{\text{max}}, n_{\text{max}} + 1].$$

In addition, the Frobenius-Perron dimension can be calculated precisely in the following cases.

1. If $n_{\text{max}} = \max \{n_0, n_0'\}$ and $n_{\text{max}} > \max \{n_{ij}\}_{ij}$, then we have
   $$\text{fpd} (A\text{-mod}) = n_{\text{max}}.$$

2. If $n_{\text{max}} = n_{i_0j_0}$ for some $(i_0, j_0)$, and $n_{\text{max}} > \max \{n_{ij}\}_{ij}$, $(i, j) \neq (i_0, j_0)$, then we have
   $$\text{fpd} (A\text{-mod}) = \frac{n_{\text{max}} + \sqrt{4 + n_{\text{max}}^2}}{2}.$$

3. If we have $\{n_{ij} = n_{\text{max}}, (i, j) = (i_0, j_0), (i_0, j_0 + 1), \ldots, (i_0, j_0 + s - 1)\}$, $n_{ij} < n_{\text{max}}, \text{others}$, then
   $$\text{fpd} (A\text{-mod}) = \rho(x(x - n_{\text{max}})^s - 1).$$

**Proof.** By Proposition 3.2, we only need to consider the brick set in $B\text{-mod}$. Since each indecomposable object of $B\text{-mod}$ is in $\mathcal{P}, \mathcal{R}$ or $\mathcal{I}$, where $\mathcal{P}, \mathcal{R}, \mathcal{I}$ are the postprojective, regular, preinjective component of $B\text{-mod}$, respectively. There are no non-zero morphisms from $\mathcal{R}$ to $\mathcal{P}$ or $\mathcal{I}$ to $\mathcal{R}$ or $\mathcal{I}$ to $\mathcal{I}$. In other words, $B\text{-mod} = \mathcal{P} \lor \mathcal{R} \lor \mathcal{I}$. Hence $\text{Ext}_A^1(\mathcal{R}, \mathcal{P}) = 0$ and $\text{Ext}_A^1(\mathcal{I}, \mathcal{R}) = 0$, and we have

$$\text{fpd} (A\text{-mod}) = \max \{\text{fpd} (\mathcal{P}), \text{fpd} (\mathcal{R}), \text{fpd} (\mathcal{I})\}.$$

In addition, any adjacency matrix of a brick set in $\mathcal{P}$ or $\mathcal{I}$ is a upper triangular matrix (by a suitable order of objects in the brick set), and the diagonal element is nonzero if and only if its corresponding module is simple and there are loops at the corresponding vertex. By Lemma 5.1, any irreducible adjacency matrix
of a brick set in $\mathcal{R}$ has the following form (by a suitable order of objects in the brick set)
\[
\begin{pmatrix}
  d_1 & 1 & & \\
  & d_2 & \ddots & \\
  & & \ddots & 1 \\
  & & & d_N
\end{pmatrix}
\]
where $d_l$ is the number of loops at the corresponding vertex for $l = 1, \ldots, N$. Since fpd ($\mathcal{R}$) $< \max\{n_{ij}\}_{ij} + 1$, we get $n_{\text{max}} \leq \text{fpd}(\text{mod}(A)) < \max\{\max\{n_{ij}\}_{ij} + 1, n_0 + 1, n'_0 + 1\} = n_{\text{max}} + 1$.

(1) If $n_{\text{max}} = n_0$, which means that $\dim \text{Ext}^1_A(S_0, S_0) = n_{\text{max}}$, where $S_0$ is the simple module at the sink vertex 0. Hence we have fpd ($\mathcal{I}$) $= n_{\text{max}}$. On the other hand, we have fpd ($\mathcal{R}$) $< \max\{n_{ij}\}_{ij} + 1 \leq n_{\text{max}}$, fpd ($\mathcal{P}$) $\leq n_{\text{max}}$. Therefore, if $n_{\text{max}} = n_0$, then fpd ($\text{A-mod}$) $= n_{\text{max}}$.

(2) If $n_{\text{max}} = n_{ij,0}$. First we have fpd ($\mathcal{P}$) $\leq n_{\text{max}}$ and fpd ($\mathcal{I}$) $\leq n_{\text{max}}$. Then we consider the brick set in $\mathcal{R}$.

According to Lemma 5.3, we can find a module $M$ such that $\{M, S_{ij,0}\}$ is a brick set and the adjacency matrix is
\[
\begin{pmatrix}
  0 & 1 & \\
  n_{\text{max}} & & \\
  1 & & \\
\end{pmatrix}
\]
whose spectral radius is $n_{\text{max}} + \sqrt{4 + n_{\text{max}}^2}$. For any other irreducible adjacency matrix of a brick set containing $S_{ij,0}$ in $\mathcal{R}$, the set of the diagonal elements denoted by $\text{Diag}$ must contain $\{0, n_{\text{max}}\}$ and $\max(\text{Diag} - \{n_{\text{max}}, 0\}) \leq n_{\text{max}} - 1$. Therefore, according to Lemma 6.1(2), its spectral radius is less than $n_{\text{max}} + \sqrt{4 + n_{\text{max}}^2}/2$. Hence we have fpd ($\text{A-mod}$) $= n_{\text{max}} + \sqrt{4 + n_{\text{max}}^2}/2$.

(3) According to Lemma 5.3, we can find a brick module $M$ to form a new brick set $\{M, S_{ij,0}, \ldots, S_{ij,0} + s - 1\}$ and the adjacency matrix is
\[
\begin{pmatrix}
  0 & 1 & \cdots & \\
  & n_{\text{max}} & \cdots & \\
  & \cdots & & 1 \\
  & & \cdots & \text{Diag}
\end{pmatrix}
\]
By Lemma 6.1(3), we know
\[
\rho(x(x - n_{\text{max}})^s - 1) > \rho(x(x - n_{\text{max}})^{s-1} - 1) > \cdots > \rho(x(x - n_{\text{max}}) - 1).
\]
It follows that fpd ($\text{A-mod}$) $= \rho(x(x - n_{\text{max}})^s - 1)$. 

For a set $\{n_{ij}\}_{ij}$ not satisfying the cases (1)(2)(3) in Theorem 6.3, there is no obvious relation to the Frobenius-Perron dimension. Sometimes we need to consider $\{n_{ij}\}_{ij}$ other than $n_{\text{max}}$, and sometimes we need not. In the case other than (1)(2)(3) in Theorem 6.3, we have to get the Frobenius-Perron by concrete calculating. We present some examples as follows.

**Example 6.4.** Keep the notation as in Theorem 6.3. Denote the quiver of $B$ by $Q_0$. 

(1) If the following is the quiver of $B$

```
1 ---- 2 ---- 3 ---- 4
    |     |     |
    |     |     |
    v     v     v
5 ---- 6
```

and the corresponding numbers of loops at vertexes 1, 2, 3, 4, 5 are $n_1 = 0$, $n_2 = 2$, $n_3 = 1$, $n_4 = 2$, $n_5 = 0$, then we find that

$$\rho((x - 2)^2(x - 1)x - 1) > \rho((x - 2)x - 1).$$

So we get

$$\text{fpd}(A\text{-mod}) = \rho((x - 2)^2(x - 1)x - 1).$$

In this case, the Frobenius-Perron dimension is determined not only by $n_{\text{max}}$, but also by the numbers of loops at other vertices.

(2) If the following is the quiver of $B$

```
1 ---- 2 ---- 3 ---- 4 ---- 5
    |     |     |
    |     |     |
    v     v     v
6
```

and the corresponding numbers of loops at vertexes 1, 2, 3, 4, 5, 6 are $n_1 = 0$, $n_2 = 3$, $n_3 = 1$, $n_4 = 1$, $n_5 = 3$, $n_6 = 0$, then we find that

$$\rho((x - 3)^2(x - 1)^2x - 1) < \rho((x - 3)x - 1).$$

So we get

$$\text{fpd}(A\text{-mod}) = \rho((x - 3)x - 1).$$

In this case, the Frobenius-Perron dimension is just determined by $n_{\text{max}}$.

(3) If the quiver of $B$ is the same as (2) and the corresponding numbers of loops at vertexes 1, 2, 3, 4, 5, 6 are $n_1 = 0$, $n_2 = 6$, $n_3 = 4$, $n_4 = 5$, $n_5 = 6$, $n_6 = 0$, then we find that

$$\rho((x - 6)^2(x - 5)(x - 4)x - 1) > \rho((x - 6)x - 1).$$

So we get

$$\text{fpd}(A\text{-mod}) = \rho((x - 6)^2(x - 5)(x - 4)x - 1).$$

In this case, we can find that the ratio of the numbers of loops at vertexes 3, 4 to $n_{\text{max}} = n_2 = n_5$ are bigger than (2). Although the quiver of $B$ is the same as (2), the Frobenius-Perron dimension is determined not only by $n_{\text{max}}$, but also by the numbers of loops at other vertexes.

7. Polynomial algebras

In this section, we focus on the polynomial algebras $\mathbb{k}[x_1, x_2, \ldots, x_r]$ for $r \in \mathbb{Z}_{\geq 1}$ which are infinite dimensional algebras. We will calculate the Frobenius-Perron dimensions of the representation categories of these algebras.

As is known that the Auslander-Reiten quiver of the finite dimensional representation category of $\mathbb{k}[x]$ consists of tubes, so the Frobenius-Perron dimension is 1 by Corollary 5.2. Furthermore, we consider
the representation category of \( \mathbb{k}[x_1, x_2] \), which is denoted by \( \mathcal{REP} \) in this section. It is obvious that a representation in \( \mathcal{REP} \) can be written as

\[
\begin{array}{c}
V \\
\downarrow \quad C_1 \\
\downarrow \\
\downarrow \quad C_2
\end{array}
\]

where \( V \) is a \( \mathbb{k} \)-linear space of dimension \( n \), \( C_1, C_2 \) are \( n \times n \) matrices satisfying \( C_1 C_2 = C_2 C_1 \). For simplicity, we denote the representation by \( (V, C_1, C_2) \).

**Lemma 7.1.** A representation in \( \mathcal{REP} \) is a brick if and only if it is of dimension one. In addition, there is no non-zero morphism between two different bricks, so arbitrary many bricks which are different from each other constitute a brick set.

**Proof.** Obviously, a representation of one dimension is a brick. Conversely, let \( M = (V, C_1, C_2) \) be a brick in \( \mathcal{REP} \), \( D \) be an endomorphism of \( M \), i.e.

\[
\begin{array}{c}
V \\
\downarrow \quad C_1 \\
\downarrow \\
\downarrow \quad C_2
\end{array} \xrightarrow{D} \begin{array}{c}
V \\
\downarrow \quad C_1 \\
\downarrow \\
\downarrow \quad C_2
\end{array}
\]

Then we can choose \( D = \text{Id} \). Since \( M \) is a brick, it follows that \( C_1 = \lambda \text{Id} \), \( C_2 = \mu \text{Id} \) for \( \lambda, \mu \in \mathbb{k} \). In this case, any decomposition \( V = V_1 \oplus V_2 \) implies a decomposition \( M = M_1 \oplus M_2 \). Therefore, \( V \) must be of dimension one.

For two bricks \( (\mathbb{k}, \lambda_1, \mu_1), (\mathbb{k}, \lambda_2, \mu_2) \in \mathcal{REP} \), assume there is a non-zero morphism \( v \) as follows

\[
\begin{array}{c}
\mathbb{k} \\
\downarrow \quad \lambda_1 \\
\downarrow \\
\downarrow \quad \mu_1
\end{array} \xrightarrow{v} \begin{array}{c}
\mathbb{k} \\
\downarrow \quad \lambda_2 \\
\downarrow \\
\downarrow \quad \mu_2
\end{array}
\]

Then by the condition, we get \( \lambda_1 = \lambda_2 \) and \( \mu_1 = \mu_2 \).

Now we will try to calculate the extension space between two bricks.

**Lemma 7.2.** For two representations \( (V, C_1, C_2), (W, B_1, B_2) \in \mathcal{REP} \) and \( \lambda, \mu \in \mathbb{k} \), there is an isomorphism of \( \mathbb{k} \)-linear spaces

\[
\text{Ext}^1_{\mathcal{REP}}((V, C_1, C_2), (W, B_1, B_2)) \cong \text{Ext}^1_{\mathcal{REP}}((V, C'_1, C'_2), (W, B'_1, B'_2))
\]

where \( C'_1 = C_1 + \lambda \text{Id}, C'_2 = C_2 + \mu \text{Id}, B'_1 = B_1 + \lambda \text{Id} \) and \( B'_2 = B_2 + \mu \text{Id} \).
Proof. Let $\xi$ be an element in $\text{Ext}^1_{\mathcal{R}E\mathcal{P}}((V, C_1, C_2), (W, B_1, B_2))$. Then $\xi$ is an short exact sequence represented as follow

$$
\xi : 0 \to W \to B_1 \to f \to W \oplus V \to g \to V \to C_1 \to 0
$$
satisfying

$$
B_1 B_2 = B_2 B_1, C_1 C_2 = C_2 C_1, E_1 E_2 = E_2 E_1,
$$

$$
f B_1 = E_1 f, f B_2 = E_2 f,
$$

$$
g E_1 = C_1 g, g E_2 = C_2 g.
$$

It is easy to check that the following short exact sequence $\zeta$ is an element in $\text{Ext}^1_{\mathcal{R}E\mathcal{P}}((V, C'_1, C'_2), (W, B'_1, B'_2))$.

$$
\zeta : 0 \to W \to B_1 + \lambda \text{Id} \to f \to W \oplus V \to E_1 + \lambda \text{Id} \to g \to V \to C_1 + \lambda \text{Id} \to 0
$$

Hence we can construct a corresponding

$$
\psi : \text{Ext}^1_{\mathcal{R}E\mathcal{P}}((V, C_1, C_2), (W, B_1, B_2)) \to \text{Ext}^1_{\mathcal{R}E\mathcal{P}}((V, C'_1, C'_2), (W, B'_1, B'_2)),
$$

$$
\xi \mapsto \zeta.
$$

We will prove that $\psi$ is an isomorphism of $k$-vector spaces.

Firstly, we claim that $\psi$ is well-defined. In $\text{Ext}^1_{\mathcal{R}E\mathcal{P}}((V, C_1, C_2), (W, B_1, B_2))$, the equation $\xi_1 = \xi_2$ means that there exists a isomorphism $h$ such that the following diagram is commutative

$$
\xi_1 : 0 \to W \to B_1 \to f \to W \oplus V \to g \to V \to C_1 \to 0
$$

$$
\xi_2 : 0 \to W \to \hat{B}_1 \to \hat{f} \to W \oplus \hat{V} \to \hat{g} \to \hat{V} \to \hat{C}_1 \to 0
$$

From the commutativity of the diagram above, we have
ψ(ξ₁): \[
\begin{array}{ccc}
W & \xrightarrow{B_1 + \lambda \text{Id}} & W \\
B_2 + \mu \text{Id} & \downarrow f & \downarrow h \\
E_1 + \lambda \text{Id} & \xrightarrow{E_2 + \mu \text{Id}} & V \\
C_1 + \lambda \text{Id} & \xrightarrow{C_2 + \mu \text{Id}} & 0 \\
\end{array}
\]
ψ(ξ₂): \[
\begin{array}{ccc}
W & \xrightarrow{B_1 + \lambda \text{Id}} & W \\
B_2 + \mu \text{Id} & \downarrow f & \downarrow h \\
\hat{E}_1 + \lambda \text{Id} & \xrightarrow{\hat{E}_2 + \mu \text{Id}} & V \\
\hat{C}_2 + \mu \text{Id} & \xrightarrow{\hat{C}_1 + \lambda \text{Id}} & 0 \\
\end{array}
\]
which is also commutative. It follows that ψ(ξ₁) = ψ(ξ₂).

Secondly, ψ preserves scalar multiplication. In fact, for 0 ≠ a ∈ k, we have a · ψ(ξ) = ψ(a · ξ) is the short exact sequence as follow.

Furthermore, ψ preserves addition. In fact, assume ξ₁, ξ₂ are two elements in \( \text{Ext}^1_{\mathcal{R} \mathcal{E} \mathcal{P}}((V, C_1, C_2), (W, B_1, B_2)) \). Then the sum of ξ₁, ξ₂ can be expressed as the Baer sum

\[\xi_1 + \xi_2 = (1, 1)(\xi_1 \oplus \xi_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.\]

which is the pull-back of the push-out (or the push-out of the pull-back). In other words, we have the commutative diagram as follow.

\[\begin{array}{ccc}
\xi_1 \oplus \xi_2: & 0 & \xrightarrow{\text{diag}(B_1, B_1)} \xrightarrow{\text{diag}(f, f)} \xrightarrow{\text{diag}(B_2, B_2)} W \oplus W \\
& \xrightarrow{\text{diag}(g, \hat{g})} \xrightarrow{\text{diag}(E_2, \hat{E}_2)} V \oplus V \\
(1, 1)(\xi_1 \oplus \xi_2): & 0 & \xrightarrow{\text{diag}(C_1, C_1)} \xrightarrow{\text{diag}(C_2, C_2)} \xrightarrow{(1, 1)^T} V \oplus V \\
& \xrightarrow{\text{diag}(C_1, C_1)} \xrightarrow{\text{diag}(C_2, C_2)} \xrightarrow{(1, 1)^T} C_1 \\
& \xrightarrow{\text{diag}(C_2, C_2)} \xrightarrow{(1, 1)^T} C_2 \\
& \xrightarrow{0} \\
\end{array}\]
Notice that the commutativity can be preserved by \( \psi \). We have

\[
(1, 1) \psi (\xi_1 \oplus \xi_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \psi ((1, 1) (\xi_1 \oplus \xi_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}).
\]

Finally, \( \psi \) is invertible. In fact, an element \( \zeta \) in \( \text{Ext}_1^{\mathcal{REP}}((V, C_1', C_2'), (W, B_1', B_2')) \) is an short exact sequence

\[
\zeta: 0 \rightarrow W \xrightarrow{f'} B_1' \oplus V \xrightarrow{g'} E_1' \rightarrow 0.
\]

We can construct a corresponding

\[
\varphi: \text{Ext}_1^{\mathcal{REP}}((V, C_1', C_2'), (W, B_1', B_2')) \rightarrow \text{Ext}_1^{\mathcal{REP}}((V, C_1, C_2), (W, B_1, B_2)),
\]

where \( \xi \) is the short exact sequence represented as follow

\[
\xi: 0 \rightarrow W \xrightarrow{f'} B_1' - \lambda \text{Id} \rightarrow B_2' - \mu \text{Id} \rightarrow E_1' - \lambda \text{Id} \rightarrow E_1' - \mu \text{Id} \rightarrow V \xrightarrow{g'} C_1' - \lambda \text{Id} \rightarrow C_2' - \mu \text{Id} \rightarrow 0.
\]

It is easy to verify that \( \varphi \) is the inverse of \( \psi \). Therefore, \( \psi \) is an isomorphism of \( k \)-linear spaces. \( \square \)

**Lemma 7.3.** For two bricks \((k, \lambda_1, \lambda_2), (k, \mu_1, \mu_2) \in \mathcal{REP}\), there is a nontrivial extension between them if and only if \( \lambda_1 = \mu_1, \lambda_2 = \mu_2 \). In this case, we have

\[
\dim \text{Ext}_1^{\mathcal{REP}}((k, \lambda_1, \lambda_2), (k, \lambda_1, \lambda_2)) = 2.
\]

**Proof.** By Lemma 7.2, we can assume \( \lambda_1 = \lambda_2 = 0 \). If \( \text{Ext}_1^{\mathcal{REP}}((k, 0, 0), (k, \mu_1, \mu_2)) \) is non-zero, then there exists \( 0 \neq \xi \) in \( \text{Ext}_1^{\mathcal{REP}}((k, 0, 0), (k, \mu_1, \mu_2)) \). By choosing a suitable basis of \( k \)-linear spaces, \( \xi \) can be represented as the following commutative diagram

\[
\xi : 0 \rightarrow k \xrightarrow{(1, 0)^T} k^2 \xrightarrow{(0, 1)} k \xrightarrow{0} 0
\]

which implies that there exist \( \mu_3, \mu_4 \in k \) such that \( E_1 = \begin{pmatrix} \mu_1 & \mu_3 \\ 0 & 0 \end{pmatrix} \) and \( E_2 = \begin{pmatrix} \mu_2 & \mu_4 \\ 0 & 0 \end{pmatrix} \). Since \( E_1 E_2 = E_2 E_1 \), we have \( \mu_4 \mu_1 = \mu_2 \mu_3 \).
If $\mu_1 \neq 0$, then $\mu_4 = \mu_2 \mu_3 / \mu_1$ and $\begin{pmatrix} 1 & \mu_3 / \mu_1 \\ 0 & 1 \end{pmatrix} : k^2 \to k^2$ is an isomorphism. We have the following commutative diagram.

It follows that $\xi = 0$, which is a contradiction to $\xi \neq 0$. Hence $\mu_1 = 0$. Similarly, we can prove that $\mu_2 = 0$.

On the other hand, if $\mu_1 = 0$ and $\mu_2 = 0$, then by Lemma 7.2 we have

$$\text{Ext}^1_{\mathcal{REP}}((k, \lambda_1, \lambda_2), (k, \lambda_1, \lambda_2)) \cong \text{Ext}^1_{\mathcal{REP}}((k, 0, 0), (k, 0, 0)) \cong k^2.$$  \hfill \Box

**Theorem 7.4.** We have

$$\text{fpd}(\mathcal{REP}) = 2.$$

**Proof.** The conclusion is directly obtained by Lemmas 7.1 and 7.3. \hfill \Box

**Corollary 7.5.** For the representation category $\mathcal{REP}_r = \text{rep} k[x_1, x_2, \ldots, x_r]$, we have

$$\text{fpd}(\mathcal{REP}_r) = r.$$

**Remark 7.6.** The above result shows that there also exist infinite dimensional algebras whose Frobenius-Perron dimensions are equal to the maximal number of loops. So further study of the characteristics of these infinite dimensional algebras are meaningful and challenging.

**Acknowledgments**

The authors would like to thank the referees for their helpful comments.

**Funding**

This work is supported by the National Natural Science Foundation of China (Grant Nos. 11971398, 12131018, and 12161141001).
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