Stress–strain analysis of the antiplane shear problem for an infinite cylindrical inclusion with eigenstrain: an addendum to Arch. Appl. Mech. 2018

1 Introduction

To understand the diffusion process of interstitial atoms (later interstitials) near a defect, it is necessary

- to know the stress field of the defect,
- to know the stress field of the interstitials, which are placed at several distinct sites (e.g., the so-called octahedral or tetrahedral positions in a cubic lattice) and produce according eigenstrain fields. Here we refer to the recent paper [1].

If we consider, e.g., an edge or screw dislocation, interstitials are densely assembled along the dislocation line, forming something like a cylindrical “cloud,” described by a cylinder with the dislocation line as z axis. As already outlined in the paper [2] (to which this paper is an “addendum”), the cylindrical “cloud” is represented as cylindrical inclusion with the radius \( R \) and an eigenstrain state with the components \( \varepsilon_{xx}^{\text{eig}}, \varepsilon_{yy}^{\text{eig}}, \varepsilon_{zz}^{\text{eig}}, \varepsilon_{xy}^{\text{eig}} \). (The coordinates of the cross section are \( x, y \) and \( z \) is the coordinate in the axial direction.) The paper [2] provides the reader with the stress state inside and outside of the inclusion in analytical form. However, it has become...
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recently evident that also eigenstrain terms $\varepsilon_{xz}^{eig}$ and $\varepsilon_{yz}^{eig}$ may occur, which are not treated in [2]. The goal of this addendum is to provide the reader also with the according stress state inside and outside of the cylindrical inclusion.

2 Solution concept

The notation is the same as in [2]. According to Eshelby’s seminal work on inclusions, see the references in [2]; the eigenstress state is spatially constant in the cylindrical inclusion. The \textit{inside} eigenstress state can immediately be calculated in Cartesian coordinates ($x, y, r = \sqrt{x^2 + y^2}$; $z$ axis of rotation), according to [3,4], aspect ratio $\alpha \to \infty$, Eqs. (4.1–2) and (A4), there, with the shear modulus $G = E/(1 + \nu)$, $E$ Young’s modulus, $\nu$ Poisson’s ratio, as

$$0 \leq r \leq R : \quad \sigma_{xz} = -G\varepsilon_{xz}^{eig}, \quad \sigma_{yz} = -G\varepsilon_{yz}^{eig}. \tag{1}$$

As boundary condition at the interface (i.e., the radial position $r = R$), we use the traction vector $t$, which follows with the \textit{inside} stress tensor $\sigma$ (with the only nonzero elements from Eq. (1)) and the normal vector $n$ with the components $(x/R, y/R, 0)$ as $\sigma \cdot n = t$. The only nonzero component of $t$ is

$$r = R, \quad t_z = \sigma_{xz} x/R + \sigma_{yz} y/R = -G \left( \varepsilon_{xz}^{eig} x/R + \varepsilon_{yz}^{eig} y/R \right). \tag{2}$$

To solve the problem for the stress state outside the inclusion, where no eigenstrain is active, cylindrical coordinates, $r$ and $\vartheta$, are used. The only displacement is that in $z$ direction, $w (r, \vartheta)$, which is independent of the $z$ coordinate. As stress components only $\sigma_{rz}$ and $\sigma_{\vartheta z}$ exist. Theory of elasticity, see, e.g., [5], Sect. 4.9.1, teaches the equilibrium equation as

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \sigma_{rz} + \frac{1}{r} \frac{\partial \sigma_{\vartheta z}}{\partial \vartheta} = 0, \tag{3}$$

and [5], Sect. 5.5.3, Hooke’s law as

$$\sigma_{rz} = G \frac{\partial w}{\partial r}, \quad \sigma_{\vartheta z} = G \frac{\partial w}{r \partial \vartheta}. \tag{4}$$

Combining Eqs. (3) and (4) one finds with $w = Cf(r)g(\vartheta)$, $C$ is an integration constant, $f(r)$ has to be derived, $g(\vartheta) = \cos \vartheta$ or $g(\vartheta) = \sin \vartheta$, $g''(\vartheta) = -g(\vartheta)$, the following differential equation

$$f'' + \frac{1}{r} f' - \frac{1}{r^2} f = 0 \tag{5}$$

with the solution

$$r \geq R : \quad w = \frac{C}{r} g(\vartheta). \tag{6}$$

Remark: We denote now first derivatives by $'$ and second derivatives by $''$.

Note that two eigenstrain cases are superposed, see Eq. (1). Therefore, two superposed displacements exist as $w_x$ (subscript $x$ according to $\varepsilon_{xz}^{eig}$) with $C_x = \frac{g_x(\vartheta)}{r}$ and $w_y$ (subscript $y$ according to $\varepsilon_{yz}^{eig}$) with $C_y = \frac{g_y(\vartheta)}{r}$. The total \textit{outside} component $\sigma_{rz}$, Eq. (4), reads with Eq. (6) for both cases as

$$r \geq R : \quad \sigma_{rz} = \frac{G}{r^2} \left( C_x g_x(\vartheta) + C_y g_y(\vartheta) \right). \tag{7}$$

Since a jump of the traction vector $t$ must not exist at the interface $r = R$, $t_z = \sigma_{rz}$, comparison between Eqs. (2) with $\cos \vartheta = x/r$, $\sin \vartheta = y/r$ and (7) yields

$$r \geq R : \quad w_x = \varepsilon_{xz}^{eig} \frac{R^2}{r} \cos \vartheta, \quad w_y = \varepsilon_{yz}^{eig} \frac{R^2}{r} \sin \vartheta. \tag{8}$$

The stress state follows with Eq. (4) as

$$r \geq R : \quad \sigma_{rz} = -G \frac{R^2}{r^2} \left( \varepsilon_{xz}^{eig} \cos \vartheta + \varepsilon_{yz}^{eig} \sin \vartheta \right), \quad \sigma_{\vartheta z} = -G \frac{R^2}{r^2} \left( \varepsilon_{xz}^{eig} \sin \vartheta - \varepsilon_{yz}^{eig} \cos \vartheta \right). \tag{9}$$
It may be of interest to mention that a decay of the stress state proportional to \( R^2/r^2 \) in outside can also be seen in [6], Sect. 4.1.

A final note: If a plane strain configuration (it is, e.g., a disk of infinite extension in the x, y plane with no constraints in the x, y direction and a finite thickness \( h \)) is assumed, with the z axis being the axis of the cylindrical inclusion, \( 0 \leq z \leq h \), then the displacement field \( w = w_x + w_y \),

\[
0 \leq x \leq R : \quad w = (\varepsilon_{xz}^{\text{eig}} \cos \theta + \varepsilon_{yz}^{\text{eig}} \sin \theta) r, \quad r \geq R \text{ see Eq. (9)} \tag{10}
\]

must be suppressed at \( z = h \). However, the average value \( \bar{w} \) of \( w(r, \theta) \) is zero, due to the integration with respect to \( \theta \). Therefore, according to St. Venant’s principle, only a local stress state, i.e., \( \sigma_{zz} \) and corresponding values of \( \sigma_{xx}, \sigma_{yy} \), will develop due to \( w \equiv 0 \) at \( z = h \), which, however, will disappear in certain distance from the surface at \( z = h \). Therefore, we omit, for the sake of simplicity, any estimation of the stress component \( \sigma_{zz} \).

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Appendix: Eigenstress fields in a Cartesian coordinate system

The eigenstress field is expressed in Cartesian coordinates using the abbreviations \( \bar{x} = x/R, \bar{y} = y/R \) and \( \bar{r} = r/R \)

\[
\varepsilon_{xz}^{\text{eig}} \neq 0, \quad \varepsilon_{yz}^{\text{eig}} \neq 0
\]

Inside

\[
\sigma_{xz} = -G \varepsilon_{xz}^{\text{eig}}, \quad \sigma_{yz} = -G \varepsilon_{yz}^{\text{eig}}.
\]

Outside

\[
\sigma_{xz} = -G \frac{\varepsilon_{xz}^{\text{eig}}}{\bar{r}^4} \left[ (\bar{x}^2 - \bar{y}^2) + 2\varepsilon_{yz}^{\text{eig}} \bar{x} \bar{y} \right], \quad \sigma_{yz} = -G \frac{\varepsilon_{yz}^{\text{eig}}}{\bar{r}^4} \left[ 2\varepsilon_{xz}^{\text{eig}} \bar{x} \bar{y} - \varepsilon_{yz}^{\text{eig}} (\bar{x}^2 - \bar{y}^2) \right].
\]

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