VARIATIONAL INTEGRATORS FOR UNDERACTUATED MECHANICAL CONTROL SYSTEMS WITH SYMMETRIES

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Abstract. Optimal control problems for underactuated mechanical systems can be seen as a higher-order optimization problem subject to higher-order constraints. In this paper we discuss the variational formalism for higher-order mechanical systems subject to higher-order constraints (that is, depending on higher-order derivatives as, for example, the acceleration) where the configuration space is a trivial principal bundle.

An interesting family of geometric integrators can be defined using discretizations of the Hamilton’s principle of critical action. This family of geometric integrators is called variational integrators, being one of their main properties the preservation of some geometric features (symplecticity, momentum preservation, good behavior of the energy). We construct variational integrators for higher-order mechanical systems on trivial principal bundles, its extension to the case of systems with higher-order constraints and we devote special attention to a class of controlled mechanical systems, underactuated mechanical systems.

1. Introduction

The construction of variational integrators have deserved a lot of interest in the recent years (from both theoretical and applied points of view). The goal of this paper is to develop variational integrators for optimal control problems of mechanical systems defined on a trivial principal bundle. Our motivation is the optimal control of the class of underactuated mechanical systems. This type of mechanical systems, i.e. the underactuated, is characterized by the fact that there are more degrees of freedom than actuators.

The presence of underactuated mechanical systems is ubiquitous in engineering applications as a result, for instance, of design choices motivated by the search of less cost devices or as a result of a failure regime in fully actuated mechanical systems. Optimal control problems of an underactuated mechanical systems can be seen as an optimization problem involving Lagrangians defined on higher-order tangent bundles subject to higher-order constraints. The purpose is to find an

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optimal curve which solves the controlled equations and minimizes a cost function depending on higher-order derivatives subject to initial and final boundary conditions.

We extend the theory of discrete mechanics and construction of variational integrators, which is based on discrete calculus of variations, to systems which depend on higher-order derivatives and are subject to constraints (also depending of higher-order derivatives). Our discrete setting follows the lines given by Marsden and Wendlandt [31] and Marsden and West [32] for first order systems without constraints.

We shall develop the discrete analogue of higher-order Lagrange-Poincaré equations (introduced by Gay-Balmaz, Holm and Ratiu in [13]) for trivial bundles and its extension to the case of systems subject to higher-order constraints. For the construction of higher-order discrete Lagrange-Poincaré equations with higher-order constraints we use the discrete Hamilton’s principle and the Lagrange multiplier theorem in order to obtain discrete paths that approximately satisfy the dynamics and the constraints. Such formulation gives us the preservation of important geometric properties of the mechanical system, such as momentum, symplecticity, group structure, good behavior of the energy, etc [16].

Our approach is founded on recently developed structure-preserving numeric integrators for optimal control problems (see [6, 9, 10, 11, 19, 21, 22, 25, 29, 33] and references therein) based on solving a discrete optimal control problem as a discrete higher-order variational problem with higher-order constraints (see [3, 10, 11] for the continuous counterpart) which are used for simulating and controlling the dynamics of satellites, spacecrafts, underwater vehicles, mobile robots, helicopters, wheeled vehicles, etc [4, 7, 24].

To be self-contained, we introduce a brief background in discrete mechanics.

1.1. General Background. A discrete Lagrangian is a map $L_d: Q \times Q \to \mathbb{R}$, which may be considered as an approximation of the integral action defined by a continuous Lagrangian $L: TQ \to \mathbb{R}$,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t)) \, dt,$$

where $q(t)$ is the unique solution of the Euler-Lagrange equations for $L; q(0) = q_0$, $q(h) = q_1$, and the time step $h > 0$ is small enough.

We construct the grid $\{t_k = kh \mid k = 0, \ldots, N\}, \, Nh = T$ and define the discrete path space $C_d(Q) := \{q_d : \{t_k\}_{k=0}^N \to Q\}$. We will identify a discrete trajectory $q_d \in C_d(Q)$ with its image $q_d = \{q_k\}_{k=0}^N$ where $q_k := q_d(t_k)$.

Define the action sum $A_d: C_d(Q) \to \mathbb{R}$, corresponding to the Lagrangian $L_d$ by summing the discrete Lagrangian on each adjacent pair and defined by

$$A_d(q_d) := \sum_{k=1}^N L_d(q_{k-1}, q_k),$$

where $q_k \in Q$ for $0 \leq k \leq N$. We would like to point out that the discrete path space is isomorphic to the smooth product manifold which consists on $N+1$ copies of $Q$ and the discrete action inherits the smoothness of the discrete Lagrangian.
The discrete variational principle then requires that $\delta A_d = 0$ where the variations are taken with respect to each point $q_k$, $1 \leq k \leq N - 1$ along the path, and the resulting equations of motion, (a system of difference equations) given fixed endpoints $q_0$ and $q_N$, are

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0,$$

where $D_1$ and $D_2$ denote the derivative to the Lagrangian respect to the first and second arguments, respectively. These equations are usually called \textit{discrete Euler–Lagrange} equations.

If the matrix $(D_{12} L_d(q_k, q_{k+1}))$ is regular, it is possible to define a (local) discrete flow $\Upsilon_{L_d}: Q \times Q \to Q \times Q$, by $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$ from [11]. The discrete algorithm determined by $\Upsilon_{L_d}$ preserves the (pre-)symplectic form on $Q \times Q$, $\omega_d$, i.e., $\Upsilon^*_L \omega_d = \omega_d$ (see [31], [32] and references therein).

Given an action of a Lie group $G$ on $Q$, we can consider the $G-$action on $Q \times Q$ by $g \cdot (q_k, q_{k+1}) := (g \cdot q_k, g \cdot q_{k+1})$. This action is symplectic. Denoting by $\mathfrak{g}$ the Lie algebra of $G$ we can define two discrete momentum maps

$$J^+_d(q_k, q_{k+1}) : \mathfrak{g} \to \mathbb{R}$$

$$\xi \mapsto \{\Theta^+_L(q_k, q_{k+1}), \xi_{Q \times Q}(q_k, q_{k+1})\}$$

for $\xi \in \mathfrak{g}$. Here $\xi_{Q \times Q}(q_k, q_{k+1}) := \{\xi_Q(q_k), \xi_Q(q_{k+1})\}$ where

$$\xi_Q(q) = \frac{d}{dt} \bigg|_{t=0} (exp(t \xi) \cdot q)$$

denotes the fundamental vector field and $\Theta^+_L(q_k, q_{k+1}) := D_2 L_d(q_k, q_{k+1}) dq_{k+1}$, $\Theta^-_L(q_k, q_{k+1}) := -D_1 L_d(q_k, q_{k+1}) dq_k$ are the discrete Poincaré-Cartan 1-forms on $Q \times Q$. If the Lagrangian is $G-$invariant then $J_d := J^+_d = J^-_d$ and $J_d \circ \Upsilon_d = J_d$.

1.2. Organization of the paper. The paper is structured as follows. In §2 we derive second-order Euler-Lagrange (Lagrange-Poincaré) equations for trivial principal bundles from Hamilton’s principle (theorem 2.1) and we study its extension to higher-order Lagrangian systems. The proposed method appears in §3 where we construct, from a discretization of the Lagrangian and through discrete variational calculus, higher-order discrete Euler-Lagrange (Lagrange-Poincaré) equations on trivial principal bundles (theorem 3.1). In §4 we study continuous (proposition 4.1) and discrete higher-order mechanics for systems subject to higher-order constraints. We apply these techniques in §5 to optimal control problems of underactuated mechanical systems with Lie group symmetries and explore two examples: the optimal control of a vehicle whose configuration space is the Lie group $SE(2)$, and the associated optimal control problem of a homogeneous ball rotating on a plate.

2. Higher-order Euler-Lagrange equations on trivial principal bundles

2.1. Higher-order tangent bundles. In this subsection we recall some basic facts of the higher-order tangent bundle theory. At some point, we will particularize this construction to the case when the configuration space is a Lie group $G$. For more details see [20].
Let $M$ be a differentiable manifold of dimension $n$. It is possible to introduce an equivalence relation in the set $C^k(\mathbb{R}, M)$ of $k$-differentiable curves from $\mathbb{R}$ to $M$. By definition, two given curves in $M$, $\gamma_1(t)$ and $\gamma_2(t)$, where $t \in (-a, a)$ with $a \in \mathbb{R}$ have contact of order $k$ at $q_0 = \gamma_1(0) = \gamma_2(0)$ if there is a local chart $(\varphi, U)$ of $M$ such that $q_0 \in U$ and
\[
\left. \frac{d^s}{dt^s} (\varphi \circ \gamma_1(t)) \right|_{t=0} = \left. \frac{d^s}{dt^s} (\varphi \circ \gamma_2(t)) \right|_{t=0},
\]
for all $s = 0, \ldots, k$. This is a well-defined equivalence relation in $C^k(\mathbb{R}, M)$ and the equivalence class of a curve $\gamma$ will be denoted by $[\gamma]_0^{(k)}$. The set of equivalence classes will be denoted by $T^{(k)}M$ and it is not hard to show that it has a natural structure of differentiable manifold. Moreover, $\tau_M^k : T^{(k)}M \to M$ where $\tau_M^k \left[ [\gamma]_0^{(k)} \right] = \gamma(0)$ is a fiber bundle called the tangent bundle of order $k$ of $M$.

Define the left- and right-translation of $G$ on itself
\[
\ell : G \times G \to G, \quad (g, h) \mapsto \ell_g(h) = gh,
\]
\[
r : G \times G \to G, \quad (g, h) \mapsto r_g(h) = hg.
\]
Obviously $\ell_g$ and $r_g$ are diffeomorphisms.

The left-translation allows us to trivialize the tangent bundle $TG$ and the cotangent bundle $T^*G$ as follows
\[
TG \to G \times \mathfrak{g}, \quad (g, \dot{g}) \mapsto (g, g^{-1} \dot{g}) = (g, \xi),
\]
\[
T^*G \to G \times \mathfrak{g}^*, \quad (g, \alpha_g) \mapsto (g, T_e^* \ell_g(\alpha_g)) = (g, \alpha),
\]
where $\mathfrak{g} = T_e G$ is the Lie algebra of $G$ and $e$ is the neutral element of $G$. In the same way, we have the identification $TTG \equiv G \times 3\mathfrak{g}$ where $3\mathfrak{g}$ stands for $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$. Throughout this paper, the notation $nV$, where $V$ is a given space, denotes the cartesian product of $n$ copies of $V$. Therefore, in the case when the manifold $M$ has a Lie group structure, i.e. $M = G$, we can also use the left trivialization to identify the higher-order tangent bundle $T^{(k)}G$ with $G \times k\mathfrak{g}$. That is, if $g : I \to G$ is a curve in $C^{(k)}(\mathbb{R}, G)$:
\[
\Upsilon^{(k)} : T^{(k)}G \to G \times k\mathfrak{g}
\]
\[
[g]_0^{(k)} \mapsto \left( g(0), g^{-1}(0)\dot{g}(0), \frac{d^1}{dt} \big|_{t=0} (g^{-1}(t)\dot{g}(t)), \ldots, \frac{d^{k-1}}{dt^{k-1}} \big|_{t=0} (g^{-1}(t)\dot{g}(t)) \right).
\]
It is clear that $\Upsilon^{(k)}$ is a diffeomorphism.

We will denote $\xi(t) := g^{-1}(t)\dot{g}(t)$, therefore
\[
\Upsilon^{(k)}([g]_0^{(k)}) = (g, \xi, \dot{\xi}, \ldots, \xi^{(k-1)}),
\]
where
\[
\xi^{(l)}(t) = \frac{d^l}{dt^l} (g^{-1}(t)\dot{g}(t)), \quad 0 \leq l \leq k - 1
\]
and $g(0) = g, \xi^{(l)}(0) = \xi^{(l)}, 0 \leq l \leq k - 1$. We will use the following notations without distinction $\xi^{(0)} = \xi, \xi^{(1)} = \dot{\xi}$, when referring to the derivatives.
We may also define the surjective mappings \( \tau_{G}^{(l,k)} : T^{(k)}G \rightarrow T^{(l)}G \), for \( l \leq k \), given by \( \tau_{G}^{(l,k)} ([g]_{(l)}^{(k)}) = [g]_{(l)}^{(l)} \). With the previous identifications we have that

\[
\tau_{G}^{(l,k)}(g(0), \xi(0), \dot{\xi}(0), \ldots, \xi^{(k-1)}(0)) = (g(0), \xi(0), \dot{\xi}(0), \ldots, \xi^{(l-1)}(0)).
\]

It is easy to see that \( T^{(1)}G \equiv G \times \mathfrak{g} \), \( T^{(0)}G \equiv G \) and \( \tau_{G}^{(0,k)} = \tau_{G}^{k} \).

2.2. Euler-Lagrange equations for trivial principal bundles. In this subsection we derive, from a variational point of view, the Euler-Lagrange equations for the trivial principal bundle \( Q = M \times G \) where \( M \) is a \( n \)-dimensional differentiable manifold with coordinates \( (q^{i}) \), \( 1 \leq i \leq n \), and \( G \) is a Lie group.

Let \( L : TQ \rightarrow \mathbb{R} \) be a Lagrangian function. Since \( TQ \) is isomorphic to \( TM \times TG \) and \( TG \) can be identify with \( G \times \mathfrak{g} \) after a left-trivialization, we can consider our Lagrangian function as \( L : TM \times G \times \mathfrak{g} \rightarrow \mathbb{R} \).

The motion of the mechanical system is described by applying the following principle,

\[
\delta A(z(t)) := \delta \int_{0}^{T} L(q(t), \dot{q}(t), g(t), \xi(t))dt = 0, \quad (2)
\]

where \( z(t) \in C^{\infty}(TM \times G \times \mathfrak{g}) \), for all variations \( \delta q(t) \) where \( \delta q(0) = \delta q(T) = 0 \), \( q(t) \in M \) and \( \delta \xi \) verifying \( \delta \xi(t) = \dot{\eta}(t) + [\xi(t), \eta(t)] = \dot{\eta}(t) + ad_{\xi(t)} \eta(t) \), where \( \eta(t) \) is an arbitrary curve on the Lie algebra with \( \eta(0) = \eta(T) = 0 \) and \( \eta = g^{-1}\delta g \) (see [17]). This principle gives rise to the Euler-Lagrange equations on trivial principal bundles

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}, \quad (3a)
\]

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) = ad_{\xi}^{*}\left( \frac{\delta L}{\delta \xi} \right) + \ell_{g}^{*}\frac{\delta L}{\delta g}, \quad (3b)
\]

where \( ad^{*} : \mathfrak{g}^{*} \rightarrow \mathfrak{g}^{*} \) is the coadjoint representation of the Lie algebra \( \mathfrak{g} \). If the Lagrangian \( L \) is left-invariant, that is, \( L \) does not depend of the variable on \( G \), the above equations are rewritten as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}, \quad (4a)
\]

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) = ad_{\xi}^{*}\left( \frac{\delta L}{\delta \xi} \right), \quad (4b)
\]

which shall be considered as the Lagrange-Poincaré equations.

2.3. Second-order (higher-order) Euler-Lagrange equations for trivial principal bundles. In this subsection we deduce, from a variational point of view, the Euler-Lagrange equations for Lagrangians defined on \( T^{(2)}Q \simeq T^{(2)}M \times G \times 2\mathfrak{g} \) from a left-trivization and its extension to higher-order Lagrangian systems.

Let \( L : T^{(2)}M \times G \times 2\mathfrak{g} \rightarrow \mathbb{R} \) be a Lagrangian function. The problem consists on finding the critical curves of the action defined by

\[
\mathcal{A}(c(t)) := \int_{0}^{T} L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi})dt
\]
among all the curves \( c(t) \in C^\infty(T^{(2)}M \times G \times 2g) \), satisfying the boundary conditions for arbitrary variations \( \delta c = (\delta q, \delta q^{(1)}, \delta q^{(2)}, \delta g, \delta \xi, \delta \xi^l) \), where \( \delta q := \frac{d}{dt}_{c=0} g_c, \delta q^{(l)} := \frac{d^l}{dt^l}_{c=0} g_c \). Here, \( \epsilon \mapsto g_\epsilon \) and \( \epsilon \mapsto q_\epsilon \) are smooth curves on \( G \) and \( M \) respectively, for \( \epsilon \in (-a, a) \subset \mathbb{R} \), such that \( g_0 = g \) and \( q_0 = q \). We define, for any \( \epsilon, \xi, \eta := g^{-1}_\epsilon g_\epsilon \). The corresponding variations \( \delta \xi \) induced by \( \delta g \) are \( \delta \xi = \eta + [\xi, \eta] \) where \( \eta := g^{-1}_\epsilon g_\epsilon \). Therefore

\[
\delta A(c(t)) = \delta \int_0^T L(q(t), \dot{q}(t), \ddot{q}(t), g(t), \xi(t), \dot{\xi}(t)) dt
\]

\[
= \frac{d}{d\epsilon} |_{\epsilon=0} \int_0^T L(q_\epsilon(t), \dot{q}_\epsilon(t), \ddot{q}_\epsilon(t), g_\epsilon(t), \xi_\epsilon(t), \dot{\xi}_\epsilon(t)) dt
\]

\[
= \int_0^T \left( \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{d}{dt} \delta q \right\rangle + \left\langle \frac{\partial L}{\partial g}, \delta g \right\rangle + \left\langle \frac{\partial L}{\partial \xi}, \delta \xi \right\rangle \right) dt.
\]

Using integration by parts (twice) and the endpoints condition \( q(0) = q(T) = \dot{q}(0) = \dot{q}(T) = 0 \) and \( \eta(0) = \eta(T) = \dot{\eta}(0) = \dot{\eta}(T) = 0 \), the stationary condition \( \delta A = 0 \) implies

\[
0 = \int_0^T \left\langle \left( \frac{d}{dt} + ad_\xi^* \right) \left( \frac{\partial L}{\partial \dot{\xi}} - \frac{d}{dt} \frac{\partial L}{\partial \xi} \right), \eta \right\rangle dt + \int_0^T \left\langle l^\epsilon_g \left( \frac{\partial L}{\partial g} \right), \eta \right\rangle dt
\]

\[
+ \int_0^T \left\langle \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}} - \frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial \dot{q}}, \delta q \right\rangle dt.
\]

Therefore, \( \delta A(c(t)) = 0 \) if and only if \( c(t) \in C^\infty(T^{(2)}M \times G \times 2g) \) is a solution of the second-order Euler-Lagrange equations for \( L : T^{(2)}M \times G \times 2g \rightarrow \mathbb{R} \),

\[
\left( \frac{d}{dt} - ad_\xi^* \right) \left( \frac{\partial L}{\partial \dot{\xi}} - \frac{d}{dt} \frac{\partial L}{\partial \xi} \right) = l^\epsilon_g \frac{\partial L}{\partial g}, \quad (5a)
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{q}} - \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial \dot{q}}, \quad (5b)
\]

which splits into a \( M \) part (5a) and a \( G \) part (5b). The previous development, reaching equations (5), shall be considered as the proof of the following theorem:

**Theorem 2.1.** Let \( L : T^{(2)}M \times G \times 2g \rightarrow \mathbb{R} \) be a left-trivialized second-order Lagrangian by the left trivialization \( \xi(t) := g^{-1}(t)q(t) \in g \), and \( \eta(t) \) is a curve on \( g \) with fixed endpoints \( \eta(0) = \eta(T) = 0 \). The curve \( c(t) \in C^\infty(T^{(2)}M \times G \times 2g) \) satisfies \( \delta A(c(t)) = 0 \) for the action \( A : C^\infty (T^{(2)}M \times G \times 2g) \rightarrow \mathbb{R} \) given by

\[
A(c(t)) = \int_0^T L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi}) dt,
\]
with respect to the variations $\delta q, \delta q^{(l)} = \frac{d}{de} \delta q$ for $l = 1, 2$ such that $\delta q(0) = \delta q(T) = 0$ and $\delta \dot{q}(0) = \delta \dot{q}(T) = 0$; $\delta g$ and $\delta \xi = \dot{\eta} + \text{ad}_\xi \eta$, if and only if $c(t)$ is a solution of the second-order Euler-Lagrange equations,

$$\ell_g^* \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q} \partial q} + \text{ad}_\xi^* \frac{\partial L}{\partial \dot{q}} - \text{ad}_\xi^* \left( \frac{d}{dt} \frac{\partial L}{\partial q} \right) = 0,$$

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q} \partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q} \partial q} + \frac{\partial L}{\partial q} = 0.$$

Corollary 2.1. If the Lagrangian is left-invariant, that is if $L$ does not depend of $g \in G$, the equations of motion for $L$ are

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \dot{q} \partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q} \partial q} + \frac{\partial L}{\partial q} = 0,$$

which shall be considered the second-order Lagrange-Poincaré equations.

**Higher-order Euler-Lagrange equations on trivial principal bundles:** The previous ideas can be extended to Lagrangians defined on a higher-order trivial principal bundle. We identify the higher-order tangent bundle $T^{(k)}Q$ with $T^{(k)}M \times G \times k\mathfrak{g}$ after a left trivialization.

Let $L$ be a higher-order Lagrangian defined on $T^{(k)}M \times G \times k\mathfrak{g}$ where we have local coordinates $(q, \dot{q}, \ddot{q}, \ldots, q^{(k)}, g, \xi, \dot{\xi}, \dot{\xi}, \ldots, \xi^{(k-1)})$, $\xi = g^{-1} \dot{g}$. Let us denote the variations $

\delta q = \frac{d}{de} \bigg|_{e=0} q_e, \quad \delta q^{(l)} = \left( \frac{d^l}{dt^l} \right) \delta q, \quad \delta \xi^{(j)} = \frac{d^j}{dt^j} \left( \delta \xi \right), \quad \delta g = \frac{d}{de} \bigg|_{e=0} g_e

for $l = 1, \ldots, k$; $j = 1, \ldots, k - 1$ and where the variation $\delta \xi$ is induced by $\delta g$ as $\delta \xi = \dot{\eta} + [\xi, \eta]$ where $\eta$ is a curve on the Lie algebra with fixed endpoints. Therefore, from Hamilton’s principle, integrating $k$ times by parts and using the boundary conditions

$$\begin{cases}
\delta q(0) = \delta q(T) = 0, \\
\delta q^{(l)}(0) = \delta q^{(l)}(T) = 0, \quad l = 1, \ldots, k; \\
\eta(0) = \dot{\eta}(0) = \ldots = \eta^{(k-1)}(0) = 0, \\
\eta(T) = \dot{\eta}(T) = \ldots = \eta^{(k-1)}(T) = 0,
\end{cases}
$$

(and therefore, $\delta \xi^{(j)}(0) = \delta \xi^{(j)}(T) = 0$, for $j = 1, \ldots, k - 1$) the higher-order Euler-Lagrange equations for $L : T^{(k)}M \times G \times k\mathfrak{g} \to \mathbb{R}$ are

$$\sum_{l=0}^{k} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial q^{(l)}} \right) = 0,$$

$$\left( \frac{d}{dt} - \text{ad}_\xi^* \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial \xi^{(l)}} \right) = \ell_g^* \left( \frac{\partial L}{\partial g} \right).$$

As in the previous cases, if the Lagrangian is left-invariant the right-hand side of the second equation vanishes and one obtains the higher-order Lagrange-Poincaré
equations, which coincide with the equations given in [13] for G-invariant Lagrangians.

3. Discrete higher-order Lagrange-Poincaré equations

In this section we will derive, using discrete calculus of variations, discrete Euler-Lagrange equations for Lagrangians defined on a left-trivialized higher-order tangent bundle to $Q = M \times G$, that is, $T^{(k)}M \times G \times k\mathfrak{g}$, where $G$ is a finite dimensional Lie group and $\mathfrak{g}$ its associated Lie algebra.

3.1. Discrete second-order Euler-Lagrange equations on trivial principal bundles. It is well known [2] that a natural discretization of the second order tangent bundle to a manifold $M$ is given by three copies of it. We take $3(M \times G) \equiv 3M \times 3G$ as a natural discretization of $T^{(2)}Q$ since we are considering a discretization of the Lie algebra its associated Lie group.

**Definition 3.1 (Discrete Hamilton’s principle for second-order trivial principal bundles).** Fixed $(q_0, g_0)$, $(q_1, g_1)$, $(q_{N-1}, g_{N-1})$, $(q_N, g_N) \in M \times G$ let define the space of sequences

$$\mathcal{C}^{(N+1)} := \{(q_{0,N}, g_{0,N}) = (q_0, q_1, \ldots, q_N, g_0, g_1, \ldots, g_N) \in (N+1)M \times (N+1)G\}$$

and the discrete action associated with the discrete Lagrangian $L_d : 3(M \times G) \to \mathbb{R}$ as

$$\mathcal{A}_d(q_{0,N}, g_{0,N}) := \sum_{k=0}^{N-2} L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1}),$$

where $W_k := g_k^{-1}g_{k+1} \in G$.

Hamilton’s principle establishes that the sequence $(q_{0,N}, g_{0,N}) \in \mathcal{C}^{(N+1)}$ is a solution of the discrete Lagrangian system determined by $L_d : 3(M \times G) \to \mathbb{R}$ if and only if $(q_{0,N}, g_{0,N})$ is a critical point of $\mathcal{A}_d$.

We now proceed, as in the continuous case, to derive the discrete equations of motion applying Hamilton’s principle of critical action. For it, we consider variations of the discrete action sum, that is,

$$0 = \delta \sum_{k=0}^{N-2} L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1})$$

$$= \sum_{k=0}^{N-2} \left( 3 \sum_{j=1}^{3} (D_j L_d|_k) \delta q_{j+k-1} + (D_4 L_d|_k) \delta g_k + \sum_{j=5}^{6} (D_j L_d|_k) \delta W_{j+k-5} \right),$$

where we use the notation $D_l L_d|_k := D_l L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1})$ and $D_l$ denotes the partial derivative with respect to the $l$-th variable. Taking variations of $W_k$ and considering the Lie algebra element $\Sigma_k := g_k^{-1} \delta g_k$, we obtain

$$\delta W_k := -\Sigma_k W_k + W_k \Sigma_{k+1},$$

where $g_k, W_k \in G$ and $\Sigma_k \in \mathfrak{g}$. 
Theorem 3.1. From these equalities we obtain the following theorem: where the notation \( C \) and \( s \) satisfy \((3)\). Then, the path \( \mathbf{q} \) given in definition \((3.3)\) satisfies the discrete second-order Euler-Lagrange equations \( \delta \mathbf{q} = 0 \) for \( \mathbf{g} \) and \( c \). From these equalities we obtain the following theorem:

**Theorem 3.1.** Let \( L : T^2(Q) \rightarrow \mathbb{R} \) be a regular Lagrangian where \( Q = M \times G \), \( T^2 Q \) is left-trivialized as \( T^2 M \times G \times 2g \) and consider the path \((q_{(0,N)}, g_{(0,N)}) \in C^2(N+1)\). Then, the path \((q_{(0,N)}, g_{(0,N)}) \) satisfies \( \delta \mathbf{A}_d(q_{(0,N)}, g_{(0,N)}) = 0 \) for \( \mathbf{A}_d : C^2(N+1) \rightarrow \mathbb{R} \), given in definition \((3.3)\) with respect to arbitrary variations \( \delta \mathbf{q}, \delta \mathbf{g}, \delta \mathbf{W} \), satisfying \((q_0, \Sigma_0), (q_1, \Sigma_1), (q_{N-1}, \Sigma_{N-1}), (q_N, \Sigma_N) \) fixed and \( \Sigma_k = g^{-1}_k \delta \mathbf{g}_k \); if and only if \((q_{(0,N)}, g_{(0,N)}) \) satisfies the discrete second-order Euler-Lagrange equations

\[
0 = \sum_{k=0}^{N-2} \sum_{j=0}^{3} (D_j L_d[q]) \delta q_{j+k} - 1 = \sum_{k=0}^{N-2} (D_1 L_d[q]) (q_{k-1}, q_k, q_{k+1}) + D_2 L_d[q] (q_{k-2}, q_{k-1}, q_k) \delta q_k,
\]

and

\[
\sum_{k=0}^{N-2} (D_1 L_d[q]) \delta g_k + \sum_{k=0}^{N-2} (D_j L_d[q]) \delta W_{j+k-5} =
\]

\[
\sum_{k=0}^{N-2} D_1 L_d[q] (q_k, W_k, W_{k+1}) \delta g_k + \sum_{k=0}^{N-2} (D_j L_d[q] (q_k, W_k, W_{k+1}) \delta W_{j+k-2} =
\]

\[
\sum_{k=0}^{N-2} D_1 L_d[q] (q_k, W_k, W_{k+1}) (g_k \Sigma_k)
\]

\[
+ \sum_{k=0}^{N-2} (D_j L_d[q] (q_k, W_k, W_{k+1}) (- \Sigma_j + k - 2 W_{j+k-2} + W_{j+k-2} \Sigma_j + k - 1),
\]

where the notation \( L_d[q] \) implies that the \( M \) variables are frozen while, \( L_d[q] \) implies that the \( G \) variables are frozen and where we have used the relations \( \Sigma_k = g^{-1}_k \delta \mathbf{g}_k \) and \((9)\). From these equalities we obtain the following theorem:

**Theorem 3.1.** Let \( L : T^2(Q) \rightarrow \mathbb{R} \) be a regular Lagrangian where \( Q = M \times G \), \( T^2 Q \) is left-trivialized as \( T^2 M \times G \times 2g \) and consider the path \((q_{(0,N)}, g_{(0,N)}) \in C^2(N+1)\). Then, the path \((q_{(0,N)}, g_{(0,N)}) \) satisfies \( \delta \mathbf{A}_d(q_{(0,N)}, g_{(0,N)}) = 0 \) for \( \mathbf{A}_d : C^2(N+1) \rightarrow \mathbb{R} \), given in definition \((3.3)\) with respect to arbitrary variations \( \delta \mathbf{q}, \delta \mathbf{g}, \delta \mathbf{W} \), satisfying \((q_0, \Sigma_0), (q_1, \Sigma_1), (q_{N-1}, \Sigma_{N-1}), (q_N, \Sigma_N) \) fixed and \( \Sigma_k = g^{-1}_k \delta \mathbf{g}_k \); if and only if \((q_{(0,N)}, g_{(0,N)}) \) satisfies the discrete second-order Euler-Lagrange equations

\[
0 = D_1 L_d[q] (q_{k-1}, q_k, q_{k+1}) + D_2 L_d[q] (q_{k-2}, q_{k-1}, q_k) + D_3 L_d[q] (q_{k-3}, q_{k-2}, q_{k-1}),
\]
Corollary 3.2. If the discrete Lagrangian \( L_d \) is \( G \)-invariant, that is, \( L_d \) does not depend on the first entry on \( G \), the equations in theorem 3.1 are rewritten as

\[
0 = D_1 L_d \big|_q (q_k, q_{k+1}, q_{k+2}) + D_2 L_d \big|_q (q_{k-1}, q_k, q_{k+1}) + D_3 L_d \big|_q (q_{k-2}, q_k, q_{k+1}),
\]

\[
0 = \ell^*_W D_1 L_d \big|_q (W_{k-1}, W_k) - r^*_W D_1 L_d \big|_q (W_k, W_{k+1})
- r^*_W D_2 L_d \big|_q (W_{k-1}, W_k) + \ell^*_W D_2 L_d \big|_q (W_{k-2}, W_{k-1}),
\]

\[W_k = g_k^{-1} q_{k+1}, \text{ for } k = 2, \ldots, N - 2.\]

These should be considered as the discrete second-order Lagrange-Poincaré equations.

### 3.2. Discrete higher-order Lagrange-Poincaré equations.

It is easy to extend our techniques for higher-order discrete mechanical systems. Consider a mechanical system determined by a Lagrangian \( L : T^{(k)}(M \times G) \to \mathbb{R} \). As before, we consider the left-trivialized higher-order tangent bundle as \( T^{(k)}M \times G \times k\mathfrak{g} \). The associated discrete problem is found by replacing the left-trivialized higher-order tangent bundle by \((k+1)\) copies of \( M \times G \).

For simplicity, we use the following notation: if \((i, j) \in (\mathbb{N}^*)^2\) with \( i < j \), \( q_{(i,j)} \) denotes the \((j - i) + 1\)-upla \((q_i, q_{i+1}, \ldots, q_{j-1}, q_j)\).

**Definition 3.3 (Discrete Hamilton’s principle for higher-order trivial principal bundles).**

Let \( L_d : (k + 1)(M \times G) \to \mathbb{R} \) be a discrete Lagrangian. Fixed initial and final conditions \((q_{(0,k-1)}, g_{(0,k-1)}, q_{(N-k+1,N)}, g_{(N-k+1,N)}) \in (M \times G)^{2k}\) with \( N > 2k \), we define the set of admissible sequences with boundary conditions \((q_{(0,k-1)}, g_{(0,k-1)})\) and \((q_{(N-k+1,N)}, g_{(N-k+1,N)})\)

\[
\mathcal{C}^{2(N+1)} := \{ (\overline{q}_{(0,N)}, \overline{g}_{(0,N)}) \mid (\overline{q}_{(0,k-1)}, \overline{g}_{(0,k-1)}) = (q_{(0,k-1)}, g_{(0,k-1)}),
(\overline{q}_{(N-k+1,N)}, \overline{g}_{(N-k+1,N)}) = (q_{(N-k+1,N)}, g_{(N-k+1,N)}) \}.
\]

Let define the discrete action over an admissible path as \( \overline{A}_d : \mathcal{C}^{2(N+1)} \to \mathbb{R} \) by

\[
\overline{A}_d(q_{(0,N)}, g_{(0,N)}) := \sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}),
\]

where \( W_i = g_i^{-1} g_{i+1} \in G \).

Hamilton’s principle states that the sequence \((q_{(0,N)}, g_{(0,N)}) \in \mathcal{C}^{2(N+1)}\) is a solution of the discrete Lagrangian system determined by \( L_d : (k + 1)(M \times G) \to \mathbb{R} \) if and only if \((q_{(0,N)}, g_{(0,N)})\) is a critical point of \( \overline{A}_d \).
Taking variations over the discrete action sum we obtain

\[
\delta \sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) = \\
\sum_{i=k}^{N-k} \left[ D_1 L_d \bigg|_q (g_i, W_{(i,i+k)}) (g_i \Sigma_i) + \sum_{j=1}^{k+1} D_j L_d \bigg|_q (q_{(i-j+1,i-j+k+1)}) \delta q_i \\
+ \sum_{j=2}^{k+1} D_j L_d \big|_q (g_i, W_{(i,i+k-1)}) (-\Sigma_{j+i-2} W_{j+i-2} + W_{j+i-2} \Sigma_{j+i-1}) \right],
\]

where we have used the relation (9) and the endpoints condition \(q_{(0,k-1)} = q_{(N-k+1,N)} = 0, \Sigma_{(0,k-1)} = 0, \Sigma_{(N-k+1,N)} = 0 \) for \( \Sigma_i = g_i^{-1} \delta g_i \in \mathfrak{g} \). Therefore, the discrete higher-order Euler-Lagrange equations for \( L_d : (k+1)(M \times G) \to \mathbb{R} \) are

\[
0 = \sum_{j=1}^{k+1} D_j L_d \big|_q (q_{(i-j+1,i-j+1+k)}),
\]

\[
0 = \ell_{W_{i-1}}^* D_1 L_d \big|_q (g_{i-1}, W_{(i-1,i+k-1)}) + \sum_{j=2}^{k+1} \left( \ell_{W_{i-1}}^* \right) D_j L_d \big|_q (g_{i-j+1}, W_{(i-j+1,i-j+k)}) - \sum_{j=2}^{k+1} (r_{W_{i-1}}^*) D_j L_d \big|_q (g_{i-j+2}, W_{(i-j+2,i-j+k+1)}),
\]

\[
W_i = g_i^{-1} g_{i+1}, \text{ where } k \leq i \leq N - k.
\]

Finally, if \( L_d \) is left-invariant the discrete higher-order Lagrange-Poincaré equations on the reduced space \((k+1)M \times kG\) read

\[
0 = \sum_{j=1}^{k+1} D_j L_d \big|_q (q_{(i-j+1,i-j+1+k)}),
\]

\[
0 = \sum_{j=2}^{k+1} \left( \ell_{W_{i-1}}^* \right) D_j L_d \big|_q (W_{(i-j+1,i-j+k)}) - \sum_{j=2}^{k+1} (r_{W_{i-1}}^*) D_j L_d \big|_q (W_{(i-j+2,i-j+k+1)}),
\]

\[
W_i = g_i^{-1} g_{i+1}, \text{ for } k \leq i \leq N - k.
\]

4. Mechanical systems with constraints on higher-order trivial principal bundles

In this section we derive, from a discretization of Hamilton’s principle, using Lagrange’s multiplier theorem, an integrator for higher-order Lagrangian systems with higher-order constraints.

4.1. Mechanical systems defined on higher-order trivial principal bundles subject to higher-order constraints: Consider a higher-order Lagrangian systems determined by \( L : T^{(k)}M \times G \times k\mathfrak{g} \to \mathbb{R} \) with higher-order constraints given
by \( \Phi^\alpha : T^{(k)}M \times G \times k\mathfrak{g} \to \mathbb{R}, \ 1 \leq \alpha \leq m \). We denote by \( \mathcal{M} \) the constraint sub-manifold locally defined by the vanishing of these \( m \) constraint functions. Define the action functional

\[
\mathcal{A}(c(t)) := \int_0^T L(c(t)) dt,
\]

where \( c(t) \) is a smooth curve in \( T^{(k)}M \times G \times k\mathfrak{g} \) which in local coordinates is

\[
c(t) = (q(t), \dot{q}(t), \ldots, q^{(k)}(t), g(t), \xi(t), \dot{\xi}(t), \ldots, \xi^{(k-1)}).
\]

The variational principle is given by

\[
\begin{align*}
\min \mathcal{A}(c(t)), \\
\text{subject to } \Phi^\alpha(c(t)) = 0 \text{ for } 1 \leq \alpha \leq m,
\end{align*}
\]

where we shall consider the boundary conditions \( q(0) = q(T) = q^{(i)}(0) = q^{(i)}(T) = 0, \ l = 1, \ldots, k; \eta^{(j)}(0) = \eta^{(j)}(T) = 0 \) for \( j = 0, \ldots, k - 1 \); and \( \xi = g^{-1}\dot{g} \). Here, \( \eta(t) \) is a curve in the Lie algebra \( \mathfrak{g} \) with fixed endpoints induced by the variations \( \delta \xi = \dot{\eta} + [\xi, \eta] \).

**Definition 4.1.** A curve \( c(t) \in C^\infty(T^{(k)}M \times G \times k\mathfrak{g}) \) will be called a solution of the higher-order variational problem with constraints if \( c \) is a critical point of the functional \( \mathcal{A}|_\mathcal{M} \).

By using the Lagrange multipliers theorem we may characterize the regular critical points of the higher-order problem with constraints as an unconstrained problem for an extended Lagrangian system (See [27] for a detailed proof).

**Proposition 4.1 (Variational problem with higher-order constraints).** A curve \( c \in C^\infty(T^{(k)}M \times G \times k\mathfrak{g}) \) is a critical point of the variational problem with higher-order constraints if and only if \( c \) is a critical point of the functional

\[
\int_0^T \tilde{L}(c(t), \lambda(t)) dt,
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m \) and \( \tilde{L} : T^{(k)}M \times G \times k\mathfrak{g} \times \mathbb{R}^m \to \mathbb{R} \) is defined by

\[
\tilde{L}(c(t), \lambda) = L(c(t)) - \lambda_\alpha \Phi^\alpha(c(t)).
\]

The equations of motion for \( \tilde{L} \) are

\[
0 = \sum_{l=0}^k (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial q^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l)}} \right),
\]

\[
0 = \left( \frac{d}{dt} - \text{ad}^\ast_{\xi} \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left( \frac{\partial L}{\partial \xi^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \xi^{(l)}} \right) - \ell_g \left( \frac{\partial L}{\partial g} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial g} \right),
\]

\[
0 = \Phi^\alpha(c(t)), \quad \dot{g} = g\xi
\]

for \( 1 \leq \alpha \leq m \). These equations are the higher-order Euler-Lagrange equations with higher-order constraints on \( T^{(k)}Q \times \mathbb{R}^m \), where \( Q = M \times G \).
If the Lagrangian is left-invariant (that is, \( L \) does not depend of the variables on \( G \)) these equations are rewritten as the higher-order Lagrange-Poincaré equations with higher-order constraints.

\[
0 = \sum_{i=0}^{k} (-1)^i \frac{d^{i}}{dt^{i}} \left( \frac{\partial L}{\partial q^{(i)}} - \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial q^{(i)}} \right),
\]

\[
0 = \left( \frac{d}{dt} - \text{ad}_x^* \right) \sum_{i=0}^{k-1} (-1)^i \frac{d^{i}}{dt^{i}} \left( \frac{\partial L}{\partial \xi^{(i)}} - \lambda_\alpha \frac{\partial \Phi_\alpha}{\partial \xi^{(i)}} \right),
\]

\[
0 = \Phi_\alpha(c(t)), \quad \dot{g} = g_\xi, \quad \text{for } 1 \leq \alpha \leq m.
\]

### 4.2. Discrete variational problem with constraints on higher-order trivial principal bundles

In this subsection we will get the discretization of the last variational principle to obtain the discrete higher-order Euler-Lagrange equations for systems subject to higher-order constraints.

Let \( L_d : (k+1)(M \times G) \to \mathbb{R} \) and \( \Phi_\alpha^d : (k+1)(M \times G) \to \mathbb{R} \) be the discrete Lagrangian and discrete constraints, respectively, for \( 1 \leq \alpha \leq m \) and denote by \( \mathcal{M}_d \) the constraint submanifold locally determined by the vanishing of these \( m \) discrete constraint functions. As before, let define the discrete action sum by

\[
\mathcal{A}_d(q_{(0,N)}, g_{(0,N-k)}) = \sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}).
\]

Therefore, we can consider the following problem as **higher-order discrete variational problem with constraints**:

\[
\begin{aligned}
\{ & \min \mathcal{A}_d(q_{(0,N)}, g_{(0,N-k)}) \\
\text{subject to} & \Phi_\alpha^d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) = 0,
\end{aligned}
\]

where \( q_{(0,k-1)}, q_{(N-k+1,N)}, g_{(0,k-1)}, g_{(N-k+1,N)} \) are fixed, \( W_i = g_i^{-1}g_{i+1}, \alpha = 1, \ldots, m \) and \( i = 0, \ldots, N - k \).

It is well-known that this optimization problem with higher-order constraints is equivalent to the unconstrained higher-order variational problem for

\[
\tilde{L}_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}, \lambda_i) := L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) + \lambda^i_\alpha \Phi_\alpha^d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)})
\]

defined on \((k+1)(M \times G) \times \mathbb{R}^m\) with \( q_{(i,i+k)} \in (k+1)M, g_i \in G, \lambda_\alpha = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m, \) and \( W_{(i,i+k-1)} \in kG \) for \( i = 0, \ldots, N - k \). Consider the discrete action sum

\[
\tilde{\mathcal{A}}_d(q_{(0,N)}, g_{(0,N-k)}, \lambda^{(0,N-k)}) := \sum_{i=0}^{N-k} \tilde{L}_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}, \lambda^i_\alpha),
\]

where \( \lambda^{(0,N-k)} = (\lambda^0, \ldots, \lambda^{N-k}) \), and \( \lambda^i \) is a vector with components \( \lambda^i_\alpha, 1 \leq \alpha \leq m \). The unconstrained higher-order variational problem is defined as the minimization of \( \tilde{\mathcal{A}}_d \) where \( q_{(0,k-1)}, g_{(0,k-1)}, q_{(N-k+1,N)}, g_{(N-k+1,N)} \) are fixed and \( i = 0, \ldots, N - k \). The critical points of the unconstrained problem, will be those annihilating \( \partial \tilde{\mathcal{A}}_d/\partial q_i \), and \( \partial \tilde{\mathcal{A}}_d/\partial \lambda^i_\alpha \). Thus, the **higher-order discrete Euler-Lagrange equations with constraints** on \((k+1)(M \times G)\) are
0 = \ell_{g_{i-1}}^* \left( D_1 L_{d|q} (g_{i-1}, W_{(i-1,i+k-1)}) + \lambda_{i-1}^* D_1 \Phi^\alpha_{d|q} (g_{i-1}, W_{(i-1,i+k-1)}) \right) \\
+ \sum_{j=2}^{k+1} \left( \ell_{W_{i-1}}^* \right) \left( D_j L_d |_{q} (g_{i-j+1}, W_{(i-j+1,i-j+k)}) \right) \\
+ \lambda_{i-1}^{i-j+1} D_j \Phi^\alpha_{d|q} (g_{i-j+1}, W_{(i-j+1,i-j+k)}) \\
- \sum_{j=2}^{k+1} \left( r_{W_i}^* \right) \left( D_j L_d |_{q} (g_{i-j+2}, W_{(i-j+2,i-j+k+1)}) \right) \\
+ \lambda_{i-1}^{i-j+2} D_j \Phi^\alpha_{d|q} (g_{i-j+2}, W_{(i-j+2,i-j+k+1)}) \\

0 = \sum_{j=1}^{k+1} D_j L_d |_{q} (q_{(i-j+1,i-j+k+1)}) + \lambda_{i-1}^{i-j+1} D_j \Phi^\alpha_{d|q} (q_{(i-j+1,i-j+k+1)}), \\
i = k, \ldots, N - k, \\
0 = \Phi^\alpha_d (q_{(i,i+k)}, g_i, W_{(i,i+k-1)}), \quad W_i = g_i^{-1} g_{i+1} \\
for 0 \leq i \leq N - k, \text{ with } q_{(0,k-1)}, q_{(N-k+1,N)}, g_{(0,k-1)}, g_{(N-k+1,N)} \text{ fixed points on } kM \text{ and } kG \text{ respectively.}

5. APPLICATION TO OPTIMAL CONTROL OF UNDERACTUATED MECHANICAL SYSTEMS

The purpose of this section is to study optimal control problems in the case of underactuated mechanical systems, that is, a Lagrangian control system such that the number of the control inputs is less than the dimension of the configuration space ("superarticulated mechanical system" following the nomenclature given in [11]).

We shall consider the configuration space \(Q\) as a trivial principal bundle. In what follows we assume that all the control systems are controllable, that is, for any two points \(x_0\) and \(x_f\) in the configuration space \(Q\), there exists an admissible control \(u(t)\) defined on some interval \([0, T]\) such that the system with initial condition \(x_0\) reaches the point \(x_f\) at time \(T\) (see [3, 7] for more details).

Define the control manifold \(U \subset \mathbb{R}^n\) where \(u(t) \in U\) is the control parameter. Consider the left-trivialized Lagrangian \(L : TM \times g \to \mathbb{R}\). The equations of motion of the system shall be considered the controlled Euler-Lagrange equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = u_a \mu^a_A (q), \quad (11a)
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \xi^a} \right) - \text{ad}_{\xi^a}^* \left( \frac{\partial L}{\partial \xi^a} \right) = u_a \eta^a_i (q), \quad (11b)
\]

where we denote by \(\mathcal{B}_a := \{ (\mu^a, \eta^a) \}, \mu^a (q) \in T_q^* M, \eta^a (q) \in g^*, a = 1, \ldots, r; \text{ and } A = 1, \ldots, n. \) Here, we are assuming that \(\{ (\mu^a, \eta^a) \}\) are independent elements of
The proposed optimal control problem is equivalent to a second-order variational problem with second-order constraints (see [3] and reference therein), determined by the second-order Lagrangian $\tilde{L} : T^{(2)}M \times 2\mathfrak{g} \to \mathbb{R}$

$$\tilde{L}(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = C \left( q^A, \dot{q}^A, \ddot{q}^A, \xi^i, F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) \right),$$

where

$$F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X^A_a(q) + \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \xi} \right) - \left( \text{ad}^*_{\xi} \frac{\partial L}{\partial \xi} \right) \right) \chi_a(q).$$

subjected to the second-order constraints

$$\Phi^a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X^A_a(q) + \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \xi} \right) - \left( \text{ad}^*_{\xi} \frac{\partial L}{\partial \xi} \right) \right) \chi_a(q).$$

**Remark 5.1.** It is possible to extend our analysis to systems with external forces $f : TM \times \mathfrak{g} \to T^*M \times \mathfrak{g}^*$ given by the following diagram

$$\begin{array}{ccc}
TM \times \mathfrak{g} & \xrightarrow{f} & T^*M \times \mathfrak{g}^* \\
\downarrow & & \downarrow \\
M & & \\
\end{array}$$

just by adding the corresponding terms in the right hand side of (11). These equations are therefore rewritten as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = u_a \mu^a_A(q) + f_A(q, \dot{q}, \xi),$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \xi} \right) - \text{ad}^*_{\xi} \left( \frac{\partial L}{\partial \xi} \right) = u_a \eta^a(q) + \tilde{f}_i(q, \dot{q}, \xi),$$
where
\[
f : \quad TM \times \mathfrak{g} \longrightarrow T^*M \times \mathfrak{g}^*
\]
\[
(q, \dot{q}, \xi) \longmapsto (f(q, \dot{q}, \xi), \bar{f}(q, \dot{q}, \xi)),
\]
such that \(f(q, \dot{q}, \xi) = f_A(q, \dot{q}, \xi) dq^A\) and \(\bar{f}(q, \dot{q}, \xi) \in \mathfrak{g}^*\).

5.1. **Optimal control of an underactuated vehicle.** Consider a rigid body moving in \(SE(2)\) with a thruster to adjust its pose. The configuration of this system is determined by a tuple \((x, y, \theta, \gamma)\), where \((x, y)\) is the position of the center of mass, \(\theta\) is the orientation of the blimp with respect to a fixed basis, and \(\gamma\) the orientation of the thrust with respect to a body basis. Therefore, the configuration manifold is \(Q = SE(2) \times \mathbb{S}^1\) (see [7] and references therein), where \((x, y, \theta)\) are the local coordinates of \(SE(2)\) and \(\gamma\) is the local coordinate of \(\mathbb{S}^1\).

The Lagrangian of the system is given by its kinetic energy
\[
L(x, y, \theta, \gamma, \dot{x}, \dot{y}, \dot{\theta}, \dot{\gamma}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\theta} + \dot{\gamma})^2,
\]
and the input forces are
\[
F^1 = \cos(\theta + \gamma) \, dx + \sin(\theta + \gamma) \, dy - p \sin \gamma d\theta,
\]
\[
F^2 = d\gamma,
\]
where the control forces that we consider are applied to a point on the body with distance \(p > 0\) from the center of mass (\(m\) is the mass of the rigid body), along the body \(x\)-axis. Note this system is an example of underactuated mechanical system when the configuration space is a trivial principal bundle.

The system is invariant under the left multiplication of the Lie group \(G = SE(2)\):
\[
\Phi : \quad SE(2) \times SE(2) \times \mathbb{S}^1 \longrightarrow SE(2) \times \mathbb{S}^1
\]
\[
((a, b, \alpha), (x, y, \theta, \gamma)) \longmapsto (x \cos \alpha - y \sin \alpha + a, x \sin \alpha + y \cos \alpha + b, \theta + \alpha, \gamma).
\]
A basis of the Lie algebra \(\mathfrak{se}(2) \cong \mathbb{R}^3\) of \(SE(2)\) is given by
\[
e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]
from we have that
\[
[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = 0.
\]
Thus, we can write down the structure constants as
\[
\zeta^2_{31} = \zeta^1_{23} = -1, \quad \zeta^2_{13} = \zeta^1_{32} = 1,
\]
and all others zero. An element \(\xi \in \mathfrak{se}(2)\) is of the form \(\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3\); therefore the reduced Lagrangian \(\ell : T\mathbb{S}^1 \times \mathfrak{se}(2) \rightarrow \mathbb{R}\) is given by
\[
\ell(\gamma, \dot{\gamma}, \xi) = \frac{1}{2} m (\xi_1^2 + \xi_2^2) + \frac{J_1 + J_2}{2} \xi_3^2 + J_2 \xi_3 \dot{\gamma} + \frac{J_2}{2} \dot{\gamma}^2.
\]
Then the reduced Euler-Lagrange equations with controls (11) in this case are given by

\[ m\ddot{x}_1 = u_1 \cos \gamma, \]
\[ m\ddot{x}_2 + (J_1 + J_2)x_1 = u_1 \sin \gamma, \]
\[ (J_1 + J_2)x_3 + J_2\dot{\gamma} - m\ddot{x}_2(x_1 + x_3) = -u_1 p \sin \gamma, \]
\[ J_2(\ddot{x}_3 + \dot{\gamma}) = u_2. \]

On the other hand, choosing the adapted basis \( \{B_a, B_o\} \) the modified equations of motion (13) read in this case as

\[
m (\cos \gamma \dot{x}_1 + \sin \gamma (\dot{x}_2 - \xi_1 \dot{\xi}_3)) + (J_1 + J_2)\xi_1 \dot{\xi}_3 \sin \gamma + J_2\dot{\xi}_1 \dot{\xi}_3 \sin \gamma = u_1, \]
\[
m (\cos \gamma (\dot{x}_2 - \xi_1 \dot{\xi}_3) - \sin \gamma \dot{\xi}_1) + \xi_1 \dot{\xi}_3 (J_1 + J_2) \cos \gamma + J_2\dot{\xi}_1 \dot{\xi}_3 \cos \gamma = 0, \]
\[
\frac{J_1 + J_2}{p}(\dot{x}_3 + p\xi_1 \dot{\xi}_3) + \frac{J_2}{p}(\dot{\gamma} + p\xi_1 \dot{\gamma}) + m \left( \dot{\xi}_2 - \xi_1 \dot{\xi}_3 - \frac{\xi_2 \dot{\xi}_1 + \xi_3 \dot{\xi}_2}{p} \right) = 0, \]
\[
J_2(\ddot{x}_3 + \dot{\gamma}) = u_2. \]

Now, we can study the optimal control problem that consists, as mentioned before, on finding a trajectory of state variables and control inputs satisfying the previous equations from given initial and final conditions \((\gamma(0), \dot{\gamma}(0), \xi(0)), (\gamma(T), \dot{\gamma}(T), \xi(T))\) respectively, and extremizing the cost functional

\[
\int_0^T (\rho_1 u_1^2 + \rho_2 u_2^2) \, dt, \]

where \( \rho_1 \) and \( \rho_2 \) are constants.

The related optimal control problem is equivalent to a second-order Lagrangian problem with second-order constraints defined as follows (see \([3]\) for more details). Extremize

\[
\tilde{A} = \int_0^T \tilde{L}(\xi, \dot{\xi}, \gamma, \dot{\gamma}, \tilde{\gamma}) \, dt, \]

subject to the second-order constraints

\[
\Phi^1 = m (\cos \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3) - \sin \gamma \dot{\xi}_1) + \xi_1 \dot{\xi}_3 (J_1 + J_2) \cos \gamma + J_2\dot{\xi}_1 \dot{\xi}_3 \cos \gamma, \quad (15a) \]
\[
\Phi^2 = \frac{J_1 + J_2}{p}(\dot{\xi}_3 + p\xi_1 \dot{\xi}_3) + \frac{J_2}{p}(\dot{\gamma} + p\xi_1 \dot{\gamma}) + m \left( \dot{\xi}_2 - \xi_1 \dot{\xi}_3 - \frac{\xi_2 \dot{\xi}_1 + \xi_3 \dot{\xi}_2}{p} \right). \quad (15b) \]

Here, \( \tilde{L} : T^{(2)}S^1 \times 2S\epsilon(2) \to \mathbb{R} \) is defined by

\[
\tilde{L}(\gamma, \dot{\gamma}, \ddot{\gamma}, \xi, \dot{\xi}) = \rho_2 J_2^2 (\dot{\xi}_3 + \dot{\gamma})^2 + \rho_1 \left( m (\cos \gamma \dot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1 \dot{\xi}_3)) + (J_1 + J_2)\xi_1 \dot{\xi}_3 \sin \gamma + J_2\xi_1 \dot{\xi}_3 \sin \gamma \right)^2. \quad (16) \]

which basically is the cost function \( C = \rho_1 u_1^2 + \rho_2 u_2^2 \) in terms of the new variables.

\textbf{Discrete setting:} Following the prescription in theorem 3.1 and the further conclusion in corollary 3.2 we shall consider a discrete Lagrangian when approaching the discrete associated problem. Moreover, since we are dealing with a constrained problem, we must include the constraints in the variational procedure as shown in 3.2 Therefore, the discrete Lagrangian and constraints read:
The discrete Lagrangian $\tilde{L}_d$ is chosen as a suitable approximation of the action,

$$
\tilde{L}_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, W_k, W_{k+1}) + \lambda^k_\alpha \Phi^\alpha_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, W_k, W_{k+1}) =
$$

$$
h \tilde{L} \left( \frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right) + \lambda^k_\alpha \Phi^\alpha_d \left( \frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right),
$$

where $\tau$ is a general retraction map (see appendix 6.1), $\tilde{L}$ is defined in (16) and $\Phi^\alpha_d$ are defined in (15). Here, $\gamma_k, \gamma_{k+1}, \gamma_{k+2} \in S^1$ while $W_k, W_{k+1} \in SE(2)$. Note that we are taking a symmetric approximation to $\gamma_k$, that is $\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}$. In addition, we are taking the usual discretizations for the first and second derivatives, that is

$$
\dot{\gamma}_k \approx \frac{\gamma_{k+2} - \gamma_k}{2h}, \quad \ddot{\gamma}_k \approx \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \quad \dot{\xi}_k \approx \frac{\xi_{k+1} - \xi_k}{h},
$$

where se(2) $\ni \xi_k = \tau^{-1}(W_k)/h$. Taking advantage of the retraction map, we can define the discrete Lagrangian and the discrete constraints on the Lie algebra, that is, $\tilde{L}_d : 3(S^1) \times 2se(2) \to \mathbb{R}$, $\Phi^\alpha_d : 3(S^1) \times 2se(2) \to \mathbb{R}$, $\alpha = 1, 2$ (with some abuse of notation, we employ the same notation, that is $\tilde{L}_d$ and $\Phi^\alpha_d$ for the Lagrangian and constraints in both spaces). To consider the Lie algebra instead of the Lie group shall be useful since the Lie algebra is a vector space and, moreover, we stay in the space where the original system is defined. Namely

$$
\tilde{L}_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}) + \lambda^k_\alpha \Phi^\alpha_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}) =
$$

$$
h \tilde{L} \left( \frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right) + \lambda^k_\alpha \Phi^\alpha_d \left( \frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right),
$$

where again we take symmetric approximations to $\gamma_k$ and $\xi_k$. Finally, as in [12], applying the usual discrete variational calculus we obtain the discrete algorithm is given by solving equations

$$
0 = D_1 \tilde{L}_d |_{\xi}(\gamma_k, \gamma_{k+1}, \gamma_{k+2}) + \lambda^k_\alpha D_1 \Phi^\alpha_d |_{\xi}(\gamma_k, \gamma_{k+1}, \gamma_{k+2})
$$

$$
+ D_2 \tilde{L}_d |_{\xi}(\gamma_k, \gamma_{k+1}) + \lambda^{k-1}_\alpha D_2 \Phi^\alpha_d |_{\xi}(\gamma_k, \gamma_{k+1})
$$

$$
+ D_3 \tilde{L}_d |_{\xi}(\gamma_{k-2}, \gamma_{k-1}) + \lambda^{k-2}_\alpha D_3 \Phi^\alpha_d |_{\xi}(\gamma_{k-2}, \gamma_{k-1}),
$$

$$
0 = A \Phi^\alpha_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}) + k = 0, \ldots, N - 2; \quad \alpha = 1, 2.
$$
As before, the notation $\tilde{L}_{d,\xi}$, $\Phi^d_0|\xi$ denotes that the variables in $\mathfrak{s}\mathfrak{e}(2)$ are frozen while, correspondingly, $\tilde{L}_{d,\gamma}$, $\Phi^d_0|\gamma$ denotes that the $S^1$ variables are frozen. To derive (18b) the properties of the right-trivialized derivative of the retraction map and its inverse, (see appendix, proposition 6.1) have been used (see [5, 18, 20]). In order to obtain the complete set of unknowns, that is $\gamma(0,N), \xi(0,N), \lambda^{(0,N-2)}_\alpha$, we also have to take into account the reconstruction equation, which in this case has the form

$$g_{k+1} = g_k \tau(h\xi_k), \quad (19)$$

where $g_k \in SE(2)$. Finally, the range of validity of equations (18a) and (18b) is $k = 2, ..., N - 2$.

As was established in §4.2 due to the variational procedure ($\gamma_0, \gamma_1$) and ($\gamma_{N-1}, \gamma_N$) are fixed, which leaves $\gamma_{(2,N-2)}$ as unknowns (i.e. $N - 3$ unknowns). By the same variational procedure ($g_0, g_1$) and ($g_{N-1}, g_N$) are also fixed, which by means of (19) imply that $\xi_0$ and $\xi_{N-1}$ are fixed. Nevertheless, due to the reconstruction discretization $g_{k+1} = g_k \tau(h\xi_k)$, is clear that fixing $\xi_k$ implies constraints in the neighboring points, in this case $g_{k+1}$ and $g_k$. If we allow $\xi_N$, that means constraints at the points $g_{N-1}$ and $g_N$. Since we only consider time points up to $T = Nh$, having a constraint in the beyond-terminal configuration $g_{N+1}$ makes no sense. Hence, to ensure that the effect of the terminal constraint on $\xi$ is correctly accounted for, the set of algebra points must be reduced to $\xi_{(0,N-1)}$. Furthermore, since $\xi_0$ and $\xi_{N-1}$ are also fixed, the final set of algebra unknowns reduces to $\xi_{(1,N-2)}$ (i.e. $3(N - 2)$ unknowns, since $\dim \mathfrak{s}\mathfrak{e}(2) = 3$).

On the other hand, the boundary condition $g(T)$ is enforced by the relation $\tau^{-1}(g_N^{-1}g(T)) = 0$, which basically means that $g_N = g(T)$. It is possible to translate this condition in terms of algebra elements as

$$\tau^{-1}\left(\tau(h\xi_{N-1})^{-1}...\tau(h\xi_0)^{-1}g_0^{-1}g(T)\right) = 0. \quad (20)$$

We have $2(N - 1)$ extra unknowns when adding the Lagrange multipliers $\lambda^{(0,N-2)}_\alpha$ (recall that, in this case $\alpha = 1, 2$). Summing up, we have

$$(N - 3) + 3(N - 2) + 2(N - 1)$$

unknowns (corresponding to $\gamma_{(2,N-2)} + \xi_{(1,N-2)} + \lambda^{(0,N-2)}_\alpha$) for

$$(N - 3) + 3(N - 3) + 3 + 2(N - 1)$$

equations (corresponding to (18a) + (18b) + (20) + (18c)). Consequently, our discrete variational problem (which comes from the original optimal control problem) has become a nonlinear root finding problem. From the set $\xi_{(0,N-1)}$ we can reconstruct the configuration trajectory by means of the reconstruction equation (19). For computational reasons is useful to consider the retraction map $\tau$ as the Cayley map for $SE(2)$ instead of a truncation of the exponential map (see the Appendix for further details).

We would also like to stress that derivation of these discrete equations have a pure variational formulation and as a consequence, the integrators defined in this way are symplectic and Poisson-momentum preserving (see [28]). By using backward error analysis, it is well known that these integrators have a good energy
behavior (see [32]). Similar techniques have been employed in [18] for the discrete optimal control of an underwater vehicle on $SE(3)$.

5.2. Optimal Control of a Homogeneous Ball on a Rotating Plate. We consider the following well-known problem (see [3, 23, 27]), namely the model of a homogeneous ball on a rotating plate. A (homogeneous) ball of radius $r > 0$, mass $m$ and inertia $mk^2$ about any axis rolls without slipping on a horizontal table which rotates with angular velocity $\Omega$ about a vertical axis $x_3$ through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere. Let $(x, y)$ be denote the position of the point of contact of the sphere with the table. The configuration space of the sphere is $Q = \mathbb{R}^2 \times SO(3)$ where may be parametrized $Q$ by $(x, y, g)$, $g \in SO(3)$, all measured with respect to the inertial frame. Let $\omega = (\omega_1, \omega_2, \omega_3)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame. The potential energy is constant, so we may put $V = 0$.

The nonholonomic constraints are given by the non-slipping condition by

$$\dot{x} + \frac{r}{2} Tr(\dot{g}g^T E_2) = -\Omega y, \quad \dot{y} - \frac{r}{2} Tr(\dot{g}g^T E_1) = \Omega x,$$

where $\{E_1, E_2, E_3\}$ is the standard basis of $so(3)$.

The matrix $\dot{g}g^T$ is skew-symmetric therefore we may write

$$\dot{g}g^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $(\omega_1, \omega_2, \omega_3)$ represents the angular velocity vector of the sphere measured with respect to the inertial frame. Then, we may rewrite the constraints in the usual form:

$$\dot{x} + r\omega_2 = -\Omega y, \quad \dot{y} - r\omega_1 = \Omega x.$$

In addition, since we do not consider external forces the Lagrangian of the system corresponds with the kinetic energy

$$K(x, y, g, \dot{x}, \dot{y}, \dot{g}) = \frac{1}{2}(mx^2 + my^2 + mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2)).$$

Observe that the Lagrangian is metric on $Q$ which is bi-invariant on $SO(3)$ as the ball is homogeneous.

Now, it is clear that $Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$-bundle over $\mathbb{R}^2$ with respect the right $SO(3)$—action given by $(x, y, R) \mapsto (x, y, RS)$ for all $S \in SO(3)$ and $(x, y, R) \in \mathbb{R}^2 \times SO(3)$. The action is in the right side since the symmetries are material symmetries.

The bundle projection $\phi : Q \to M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding quotient bundle $E = TQ/SO(3)$ over $M = \mathbb{R}^2$. We will identify the tangent bundle to $SO(3)$ with $so(3) \times SO(3)$ by using right translation. Note that throughout the previous exposition we have employed the left trivialization. However, we would like to point out that the right trivialization just implies minor changes in the derivation of the equations of motion (see [17]).
Under this identification between $T(SO(3))$ and $\mathfrak{so}(3) \times SO(3)$, the tangent action of $SO(3)$ on $T(SO(3)) \cong \mathfrak{so}(3) \times SO(3)$ is the trivial action
\[(\mathfrak{so}(3) \times SO(3)) \times SO(3) \to \mathfrak{so}(3) \times SO(3), \quad ((\omega, g), h) \mapsto (\omega, gh). \quad (21)\]
Thus, the quotient bundle $TQ/SO(3)$ is isomorphic to the product manifold $T^2 \times \mathfrak{so}(3)$, and the vector bundle projection is $\tau_{\mathbb{R}^2} \circ pr_1$, where $pr_1 : T^2 \times \mathfrak{so}(3) \to T^2$ and $\tau_{\mathbb{R}^2} : T^2 \to \mathbb{R}^2$ are the canonical projections.

A section of $E = TQ/SO(3) \cong T^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ is a pair $(X, f)$, where $X$ is a vector field on $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathfrak{so}(3)$ is a smooth map. Therefore, a global basis of sections of $T^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ is
\[e_1 = \left( \frac{\partial}{\partial x}, 0 \right), \quad e_2 = \left( \frac{\partial}{\partial y}, 0 \right), \quad e_3 = (0, E_1), \quad e_4 = (0, E_2), \quad e_5 = (0, E_3).\]

There exists a one-to-one correspondence between the space $\Gamma(E = TQ/SO(3))$ and the $G$-invariant vector fields on $Q$.

If $[\cdot, \cdot]$ is the Lie bracket on the space $\Gamma(E = TQ/SO(3))$, then the only non-zero fundamental Lie brackets are
\[[e_4, e_3] = e_5, \quad [e_5, e_4] = e_3, \quad [e_3, e_5] = e_4.\]
Moreover, it follows that the Lagrangian function $L = K$ and the constraints are $SO(3)$-invariant. Consequently, $L$ induces a Lagrangian function $\ell$ on $E = TQ/SO(3) \cong T^2 \times \mathfrak{so}(3)$.

We have a constrained system on $E = TQ/SO(3) \cong T^2 \times \mathfrak{so}(3)$ and note that in this case the constraints are nonholonomic and affine in the velocities. This kind of systems was analyzed by J. Cortés et al [12] (in particular, this example was carefully studied). The constraints define an affine subbundle of the vector bundle $E \cong T^2 \times \mathfrak{so}(3) \to \mathbb{R}^2$ which is modeled over the vector subbundle $\mathcal{D}$ generated by the sections
\[\mathcal{D} = \text{span}\{e_5; re_1 + e_4; re_2 - e_3\}.\]

Moreover, the angular momentum of the ball about the axis $x_3$ is a conserved quantity since the Lagrangian is invariant under rotations about the axis $x_3$ and the infinitesimal generator for these rotations lies in the distribution $\mathcal{D}$. The conservation law is written as $\omega_3 = c$, where $c$ is a constant or as $\dot{\omega}_3 = 0$. Then by the conservation of the angular momentum the second-order constraints appear.

After some computations the equations of motion for this constrained system are precisely
\[
\begin{align*}
\dot{x} - r \omega_2 &= -\Omega y, \\
\dot{y} + r \omega_1 &= \Omega x, \\
\dot{\omega}_3 &= 0,
\end{align*} \tag{22}
\]

Together with
\[
\ddot{x} + \frac{k^2\Omega}{r^2 + k^2} \dot{y} = 0, \quad \ddot{y} - \frac{k^2\Omega}{r^2 + k^2} \dot{x} = 0.
\]

Now, we pass to an optimization problem. Assume full controls over the motion of the center of the ball (the shape variables). The controlled system can be written as,
\[ \ddot{x} + \frac{k^2 \Omega}{r^2 + k^2} y = u_1, \quad \ddot{y} - \frac{k^2 \Omega}{r^2 + k^2} \dot{x} = u_2, \]

subject to
\begin{align*}
\omega_2 - \frac{1}{r} \dot{x} &= \frac{\Omega y}{r}, \\
\omega_1 + \frac{1}{r} \dot{y} &= \frac{\Omega x}{r}, \\
\dot{\omega}_3 &= 0.
\end{align*}

Next, we consider the optimal control problem for this system following the techniques proposed in this paper.

Let \( C \) be the cost function given by
\[ C = \frac{1}{2} (u_1^2 + u_2^2). \]

Given \( q(0), q(T) \in \mathbb{R}^2 \), \( \dot{q}(0) \in T_{q(0)} \mathbb{R}^2 \), \( \dot{q}(T) \in T_{q(T)} \mathbb{R}^2 \), \( q = (x,y) \in \mathbb{R}^2 \), \( \omega(0), \omega(T) \in so(3) \), we look for an optimal control curve \( (q(t), \omega(t), u(t)) \) on the reduced space that steers the system from \( (q(0), \omega(0)) \) to \( (q(T), \omega(T)) \) minimizing
\[ \int_0^T \frac{1}{2} (u_1^2 + u_2^2) \, dt, \]

subject to the constraints given by equations (23). Note that \( R(0), R(T) \in SO(3) \), the initial and final configurations of the problem, are also fixed. Its dynamics is given by the continuous reconstruction equation \( \dot{R}(t) = R(t) \omega(t) \).

As in the previous example, we define the second order Lagrangian \( \tilde{L} : T^{(2)} \mathbb{R}^2 \times 2so(3) \to \mathbb{R} \) given by
\[ \tilde{L}(x,y,\dot{x},\dot{y},\ddot{x},\ddot{y},\omega_1,\omega_2,\omega_3,\dot{\omega}_1,\dot{\omega}_2,\dot{\omega}_3) = \frac{1}{2} \left( \dddot{x} + \frac{k^2 \Omega}{r^2 + k^2} y \right)^2 + \frac{1}{2} \left( \dddot{y} - \frac{k^2 \Omega}{r^2 + k^2} \dot{x} \right)^2 \]

subject to second-order constraints \( \Phi^\alpha : T^{(2)} \mathbb{R}^2 \times 2so(3) \to \mathbb{R} \), \( \alpha = 1, 2, 3 \),
\begin{align*}
\Phi^1 &= \omega_1 + \frac{1}{r} \dot{y} - \frac{\Omega x}{r}, \\
\Phi^2 &= \omega_2 - \frac{1}{r} \dot{x} - \frac{\Omega y}{r}, \\
\Phi^3 &= \dot{\omega}_3.
\end{align*}

As established in proposition 4.1, as a pure constrained variational problem, the optimal control problem is prescribed by solving the following system of 4-order
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differential equations (ODEs).

\[ 0 = \frac{\lambda_1 \Omega}{r} + \frac{\dot{\lambda}_2}{r} + x^{(iv)} + \frac{2k^2 \Omega \dot{y}}{r^2 + k^2} - \frac{k^4 \Omega^2 \ddot{x}}{(r^2 + k^2)^2}, \]
\[ 0 = \frac{\lambda_2 \Omega}{r} + \frac{\dot{\lambda}_1}{r} + y^{(iv)} - \frac{2k^2 \Omega \dot{x}}{r^2 + k^2} - \frac{k^4 \Omega^2 \ddot{y}}{(r^2 + y^2)^2}, \]
\[ 0 = \dot{\lambda}_1 + \lambda_2 \omega_3 - \lambda_3 \omega_2, \]
\[ 0 = \dot{\lambda}_2 - \lambda_1 \omega_3 + \lambda_3 \omega_1, \]
\[ 0 = \dot{\lambda}_3 + \lambda_1 \omega_2 - \lambda_2 \omega_1, \]
\[ 0 = \omega_1 + \frac{1}{r} \dot{y} - \frac{\Omega(t)x}{r}, \]
\[ 0 = \omega_2 - \frac{1}{r} \dot{x} - \frac{\Omega(t)y}{r}, \]
\[ 0 = \dot{\omega}_3. \]

In addition, the configurations \( R \in SO(3) \) are given by the continuous reconstruction equation \( \dot{R} = R \omega \).

In the particular case when the angular velocity \( \Omega \) depends on the time (see \[3, 22\]), the equations of motion are rewritten as

\[ 0 = \lambda_1 \frac{\Omega(t)}{r} + \frac{\dot{\lambda}_2}{r} + x^{(iv)} + \frac{k^2 \Omega''(t) \dot{y}}{r^2 + k^2} + \frac{2k^2 \Omega'(t) \dot{y}}{r^2 + k^2} + \frac{2k^2 \Omega(t) \ddot{y}}{r^2 + k^2}, \]
\[ + \frac{k^4 \Omega^2(t) \dot{x}}{(r^2 + k^2)^2} - \frac{2k^4 \Omega'(t) \Omega(t)x}{(r^2 + k^2)^2}, \]
\[ 0 = \frac{\lambda_2 \Omega(t)}{r} + \frac{\dot{\lambda}_1}{r} + y^{(iv)} - \frac{k^2 \Omega''(t) \dot{x}}{r^2 + k^2} - \frac{3k^2 \Omega'(t) \dot{x}}{r^2 + k^2} - \frac{2k^2 \Omega(t) \ddot{x}}{r^2 + k^2}, \]
\[ - \frac{k^4 \Omega^2(t) \dot{y}}{(r^2 + y^2)^2} - \frac{2k^4 \Omega(t) \Omega'(t) \dot{y}}{(r^2 + k^2)^2}, \]
\[ 0 = \dot{\lambda}_1 + \lambda_2 \omega_3 - \lambda_3 \omega_2, \]
\[ 0 = \dot{\lambda}_2 - \lambda_1 \omega_3 + \lambda_3 \omega_1, \]
\[ 0 = \dot{\lambda}_3 + \lambda_1 \omega_2 - \lambda_2 \omega_1, \]
\[ 0 = \omega_1 + \frac{1}{r} \dot{y} - \frac{\Omega(t)x}{r}, \]
\[ 0 = \omega_2 - \frac{1}{r} \dot{x} - \frac{\Omega(t)y}{r}, \]
\[ 0 = \dot{\omega}_3. \]

- **Discrete setting:** As in the previous example, we discretize this problem by choosing a discrete Lagrangian \( \widetilde{L}_d \) and discrete constraints \( \Phi^d_2 \). Employing equivalent arguments than in the previous example, we set \( \widetilde{L}_d : 3(\mathbb{R}^2) \times \mathfrak{so}(3) \to \mathbb{R} \).
and \( \Phi^\alpha : 3(\mathbb{R}^2) \times 2\mathfrak{so}(3) \to \mathbb{R}, \alpha = 1, 2, 3 \), as

\[
\tilde{L}_d(q_k, q_{k+1}, q_{k+2}, \omega_k, \omega_{k+1}) + \lambda^k_\alpha \Phi^\alpha_d(q_k, q_{k+1}, q_{k+2}, \omega_k, \omega_{k+1}) = \\
\frac{q_k + q_{k+1} + q_{k+2}}{3} - \frac{q_k - q_{k+1} - q_{k+2}}{2h} - \frac{q_{k+1} - q_k - q_{k+2}}{2h^2} - \frac{\omega_k + \omega_{k+1}}{2} - \frac{\omega_{k+1} - \omega_k}{h} \\
+ \lambda^k_\alpha \Phi^\alpha \left( \frac{q_k + q_{k+1} + q_{k+2}}{3}, \frac{q_k + q_{k+1} + q_k}{2h}, \frac{q_k + q_{k+1} + q_k}{h^2}, \frac{\omega_k + \omega_{k+1}}{2}, \frac{\omega_{k+1} - \omega_k}{h} \right),
\]

We employ the same unknowns-equations counting process than in the previous example to find out that the number of unknowns matches the number of equations. Therefore, our discrete variational problem (which comes from the original optimal control problem) has become again a nonlinear root finding problem. For computational reasons is useful to consider the retraction map \( \tau \) as the Cayley map for \( SO(3) \) instead of a truncation of the exponential map (see the Appendix for further details).

Conclusions

In this paper, we have designed a new class of variational integrators for optimal control problems of underactuated mechanical systems, showing how developments in the theory of discrete mechanics and discrete calculus of variations with constraints can be used to construct numerical optimal control algorithms with certain geometric desirable features.

We construct variational principles for higher-order mechanical systems with higher-order constraints to solve an optimal control problem for systems where its configuration space is a trivial principal bundle. From a discretization of Hamilton’s principle of critical action we derive the discrete version of the problem. We show two concrete applications of our ideas: an underactuated vehicle and a (homogeneous) ball rotating on a plate. It is also possible to use our techniques and the obtained numeric integrators for other interesting problems, for instance, the theory of \( k \)-splines on \( SO(3) \) \[15\].

We would like to point out that a slight modification of the techniques presented in this work would allow to approach the Clebsh-Pontryagin optimal control problem (see \[14, 15\]).

In a future work, we will also extend our construction to the case of non-trivial fiber bundles using a connection to split the reduced space \[8\].

6. Appendix

6.1. Discrete Mechanics on Lie groups. In this appendix we will recall the basics results on discrete mechanics on Lie groups and Hamilton’s principle for the derivation of the discrete Euler-Poincaré equations.

If the configuration space is a finite dimensional Lie group \( G \), then the discrete trajectory is represented numerically using a set of \( N + 1 \) points \( (g_0, g_1, \ldots, g_N) \) with \( g_i \in G, 0 \leq i \leq N \).

The following theorem give us the relation between the discrete Euler-Lagrange equations and the discrete Euler-Poincaré equations.
Theorem 6.1. Let $G$ be a Lie group and $L_d : G \times G \rightarrow \mathbb{R}$ be a discrete Lagrangian function. We suppose that $L_d$ is left-invariant over the diagonal action; that is, $L_d(gk, ggk+1) = L_d(gk, gk+1)$ with $g \in G$. Let $\hat{L}_d : G \rightarrow \mathbb{R}$ be the restriction to the identity, that is, $\hat{L}_d : (G \times G)/G \simeq G \rightarrow \mathbb{R}$, $\hat{L}_d(gk, gk+1) = L_d(gk, gk+1)$. Fixed $g_0$ and $g_N$, for a sequence of points $\{(g_k, g_{k+1})\}_{k=0}^{N-1} \in G \times G$, with $N > 2$, we consider $W_k = g_k^{-1}g_{k+1}$. Then the following statements are equivalent:

1) $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ is a critical point of the discrete action $\sum_{k=0}^{N-1} L_d(g_k, g_{k+1})$ for arbitrary variations $\delta g_k = (d/d\epsilon)|_{\epsilon=0} (g_k^\epsilon)$ where for each $k$, $\epsilon \mapsto g_k^\epsilon$ is a smooth curve on $G$ such that $g_k^0 = g_k$ with fixed end points.

2) $\{(g_k, g_{k+1})\}_{k=0}^{N-1}$ satisfy the Euler-Lagrange equations for $\hat{L}_d$.

3) The discrete Euler-Poincaré equations $r^*_{W_k} \hat{L}'_d(W_k) - \ell^*_{W_{k-1}} \hat{L}'_d(W_{k-1}) = 0$, $k = 1, \ldots, N$ hold, where $'$ denote the partial derivative.

4) $(W_k)_{k=0}^{N}$ extremize $\sum_{k=0}^{N-1} \hat{L}_d(W_k)$ for all variations $\delta W_k = -\eta_k W_k + W_k \eta_{k+1}$ with $\eta_0 = \eta_N = 0$; where $\eta_k \in \mathfrak{g}$ is given by $\eta_k = g_k^{-1}\delta g_k$.

A way to discretize a continuous problem is using a retraction map $\tau : \mathfrak{g} \rightarrow G$ which is an analytic local diffeomorphism. This application maps a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the identity $e \in G$. As a consequence, it is possible to deduce that $\tau(\xi) \tau(-\xi) = e$ for all $\xi \in \mathfrak{g}$.

The retraction map is used to express small discrete changes in the group configuration through unique Lie algebra elements (see [22]), namely $\xi_k = \tau^{-1}(g_k^{-1}g_{k+1})/h$, where $\xi_k \in \mathfrak{g}$. That is, if $\xi_k$ were regarded as an average velocity between $g_k$ and $g_{k+1}$, then $\tau$ is an approximation to the integral flow of the dynamics. The difference $g_k^{-1}g_{k+1} \in G$, which is an element of a nonlinear space, can now be represented by the vector $\xi_k$, in order to enable unconstrained optimization in the linear space $\mathfrak{g}$ for optimal control purposes.

It will be useful in the sequel, mainly in the derivation of the discrete equations of motion, to define the right trivialized tangent retraction map as a function $dT : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$T\tau(\xi) \cdot \eta = T\tau(\xi)d\tau(\eta),$$

where $\eta \in \mathfrak{g}$. Here we use the following notation, $dT^\xi := d\tau(\xi) : \mathfrak{g} \rightarrow \mathfrak{g}$. The function $dT$ is linear in its second argument. From this definition the following identities hold (see [8] for further details)

Proposition 6.1. Given a map $\tau : \mathfrak{g} \rightarrow G$, its right trivialized tangent $dT^\xi : \mathfrak{g} \rightarrow \mathfrak{g}$ and its inverse $d\tau^{-1} : \mathfrak{g} \rightarrow \mathfrak{g}$, are such that for $g = \tau(\xi) \in G$ and $\eta \in \mathfrak{g}$, the following identities hold:

$$\partial_\xi \tau(\xi) \eta = d\tau \eta \tau(\xi) \quad \text{and} \quad \partial_\xi \tau^{-1}(g) \eta = d\tau^{-1}(\eta) \tau(-\xi).$$

The most natural example of retraction map is the exponential map at the identity $e$ of the group $G$, $\exp_e : \mathfrak{g} \rightarrow G$. We recall that, for a finite-dimensional Lie group, $\exp_e$ is locally a diffeomorphism and gives rise a natural chart [30].
Then, there exists a neighborhood $U$ of $e \in G$ such that $\exp^{-1}_e : U \to \exp^{-1}_e(U)$ is a local $\mathcal{C}^\infty$-diffeomorphism. A chart at $g \in G$ is given by $\Psi_g = \exp^{-1}_e \circ \ell_g^{-1}$.

In general, it is not easy to work with the exponential map. In consequence it will be useful to use a different retraction map. More concretely, the Cayley map (see [5, 16] for further details) will provide us a proper framework in the examples shown along the paper.

6.2. The Cayley map. The Cayley map $\text{cay} : g \to G$ is defined by

$$\text{cay}(\xi) = (e - \frac{\xi}{2})^{-1}(e + \frac{\xi}{2})$$

and is valid for a class of quadratic groups (see [16] for example) that include the groups of interest in this paper (e.g. $SO(3)$, $SE(2)$ and $SE(3)$). Its right trivialized derivative and inverse are defined by

$$d\text{cay}_{x}y = (e - \frac{x}{2})^{-1}y(e + \frac{x}{2})^{-1}, \quad d\text{cay}^{-1}_{x}y = (e - \frac{x}{2})y(e + \frac{x}{2}).$$

6.2.1. The Cayley map for $SE(2)$: The coordinates on $SE(2)$ are $(\theta, x, y)$ with matrix representation for $g \in SE(2)$ given by

$$g = \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the isomorphic map $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{se}(3)$ given by

$$\hat{v} = \begin{pmatrix} 0 & -v_1 & v_2 \\ v_1 & 0 & v_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$, the set $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ can be used as a basis for $\mathfrak{se}(3)$, where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. The map $\text{cay} : \mathfrak{se}(3) \to SE(2)$ is given by

$$\text{cay}(\hat{v}) = \begin{pmatrix} 1 \frac{4 - v_1^2}{1 + v_1^2} & -4v_1 & -2v_1v_3 + 4v_2 \\ 4v_1 & 4 - v_1^2 & 2v_1v_2 + 4v_3 \\ 0 & 0 & 1 \end{pmatrix},$$

while the map $d\tau_{\xi}^{-1}$ becomes the $3 \times 3$ matrix

$$d\text{cay}_{\xi}^{-1} = I_3 - \frac{1}{2} \text{ad}_v + \frac{1}{4} (v_1 v_0)_{3 \times 2},$$

where

$$\text{ad}_v = \begin{pmatrix} 0 & 0 & 0 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

and $I_3$ denotes the $3 \times 3$ identity matrix.
6.2.2. **The Cayley map for SO(3):** The group of rigid body rotations is represented by $3 \times 3$ matrices with orthonormal column vectors corresponding to the axes of a right-handed frame attached to the body. On the other hand, the algebra $\mathfrak{so}(3)$ is the set of $3 \times 3$ antisymmetric matrices. A $\mathfrak{so}(3)$ basis can be constructed as \{\(\hat{e}_1, \hat{e}_2, \hat{e}_3\)\}, where \{\(e_1, e_2, e_3\)\} is the standard basis for $\mathbb{R}^3$. Elements $\xi \in \mathfrak{so}(3)$ can be identified with the vector $\omega \in \mathbb{R}^3$ through $\xi = \omega \alpha \hat{e}_\alpha$, or $\xi = \hat{\omega}$. Under such identification the Lie bracket coincides with the standard cross product, i.e., $\text{ad}_{\hat{\omega}} \hat{\rho} = \omega \times \rho$, for some $\rho \in \mathbb{R}^3$. Using this identification we have

$$\text{cay}(\hat{\omega}) = I_3 + \frac{4}{4+\|\omega\|^2} \left( \hat{\omega} + \frac{\hat{\omega}^2}{2} \right),$$

(26)

where $I_3$ is the $3 \times 3$ identity matrix. The linear maps $d\tau_\xi$ and $d\tau^{-1}_\xi$ are expressed as the $3 \times 3$ matrices

$$d\text{cay}_\omega = \frac{2}{4+\|\omega\|^2} (2I_3 + \hat{\omega}), \quad d\text{cay}_\omega^{-1} = I_3 - \frac{\hat{\omega}}{2} + \frac{\omega \omega^T}{4}.$$  

(27)

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