Besov-weak-Herz spaces and global solutions for Navier-Stokes equations

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Abstract

We consider the incompressible Navier-Stokes equations (NS) in $\mathbb{R}^n$ for $n \geq 2$. Global well-posedness is proved in critical Besov-weak-Herz spaces (BWH-spaces) that consist in Besov spaces based on weak-Herz spaces. These spaces are larger than some critical spaces considered in previous works for (NS). For our purposes, we need to develop a basic theory for BWH-spaces containing properties and estimates such as heat semigroup estimates, embedding theorems, interpolation properties, among others. In particular, it is proved a characterization of Besov-weak-Herz spaces as interpolation of Sobolev-weak-Herz ones, which is key in our arguments. Self-similarity and asymptotic behavior of solutions are also discussed. Our class of spaces and its properties developed here could also be employed to study other PDEs of elliptic, parabolic and conservation-law type.

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Key: Navier-Stokes equations; Well-posedness; Besov-weak-Herz spaces; Interpolation; Heat semigroup estimates; Self-similarity

1 Introduction

This paper is concerned with the incompressible Navier-Stokes equations

$$
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \rho = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
\nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\
u(0) = u_0 & \text{in } \mathbb{R}^n
\end{cases}
$$

(1.1)

where $n \geq 2$, $\rho$ is the pressure, $u = (u_j)_{j=1}^n$ is the velocity field and $u_0$ is a given initial velocity satisfying $\nabla \cdot u_0 = 0$.

After applying the Leray-Hopf projector $\mathbb{P}$ and using Duhamel’s principle, the Cauchy problem (1.1) can be reduced to the integral formulation

$$
u(t) = G(t) u_0 - \int_0^t G(t - \tau) \mathbb{P} \text{div} (u \otimes u) (\tau) d\tau := G(t) u_0 + B(u, u)(t),
$$

(1.2)

where $u \otimes v := (u_i v_j)_{1 \leq i, j \leq n}$ is a matrix-valued function and $G(t) = e^{t\Delta}$ is the heat semigroup. The operator $\mathbb{P}$ can be expressed as $\mathbb{P} = (\mathbb{P}_{i,j})_{n \times n}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + \mathcal{R}_i \mathcal{R}_j$, $\delta_{i,j}$ is the Kronecker delta and $\mathcal{R}_i = (-\Delta)^{-1/2} \partial_i$ is the $i$-th Riesz transform. Divergence-free solutions for (1.2) are called mild solutions for (1.1). Note that if $u$ is a smooth solution for (1.1) (or (1.2)), then

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t)
$$

(1.3)

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is also a solution with initial data

\[ (u_0)_\lambda(x) = \lambda u_0(\lambda x). \tag{1.4} \]

Recall that given a Banach space \( Y \) we say that it has scaling degree equal to \( k \in \mathbb{R} \) if \( \| f(\lambda x) \|_Y \approx \lambda^k \| f \|_Y \) for all \( \lambda > 0 \) and \( f \in Y \). Motivated by (1.4), a Banach space \( Y \) is called critical for (1.1) if it has scaling degree equal to \(-1\), that is, if \( \| f \|_Y \approx \| \lambda^1 f(\lambda x) \|_Y \) for all \( \lambda > 0 \) and \( f \in Y \). In turn, a solution of (1.1) which is invariant by the scaling (1.3), i.e., \( u = u_\lambda \), is called a self-similar solution of (1.1). Note that in order to obtain self-similar solutions, the initial data should be homogeneous of degree \(-1\).

Over the years, global-in-time well-posedness of small solutions for (1.1) in critical spaces has attracted the interest of a number of authors. Without making a complete list, we mention the works in homogeneous Sobolev \( \dot{H}^{1/2}(\mathbb{R}^3) \) [8], Lebesgue \( L^n(\mathbb{R}^n) \) [14], Marcinkiewicz \( L^{n,\infty}(\mathbb{R}^n) \) [1, 27], Morrey \( \mathcal{M}_q^n(\mathbb{R}^n) \) [9, 15, 24], weak-Morrey \( \mathcal{M}_q^n(\mathbb{R}^n) \) [23, 21, 7], \( PM^{n-1} \)-spaces [5], Fourier-Besov \( \dot{B}_p^{\frac{n}{p-1}} \) [12, 17], homogeneous weak-Herz spaces \( \dot{W}_p^{\frac{n}{p},\infty}(\mathbb{R}^n) \) [25], Fourier-Herz \( B_{r}^{-1} = \dot{F}_B^{-1,r} \) with \( r \in [1,2] \) [6, 12, 19], homogeneous Besov-Morrey \( \mathcal{N}_{r,q}^{-1,1} \) with \( r > n \) [18, 22], and \( BMO^{-1} \) [16]. The reader can find other examples in the nice review [20]. Up until now, we know that \( BMO^{-1} \) and \( \mathcal{N}_{r,1}^{-1,\infty} \) are maximal critical spaces for (1.1) in the sense that it is not known a larger critical space in which small solutions of (1.1) are globally well-posed.

The propose of this paper is to provide a new critical Besov type class for global well-posedness of solutions for (1.1) by assuming a smallness condition on initial data norms. Here we consider homogeneous Besov-weak-Herz spaces \( \dot{B} W^{\alpha,a}_{p,q,r} \) that are a type of Besov space based on homogeneous weak-Herz spaces \( \dot{W}^{\alpha}_{p,q,r} \). They are a natural extension of the spaces \( B \dot{K}^{\alpha,a}_{p,q} \) introduced in [26] (see Definition 2.5 in subsection 2.2). The Herz space \( \dot{K}_{p,q}^{\alpha} \) was introduced by Herz in [11] but his definition is not appropriate for our purposes. Later, Johnson [13] obtained a characterization of the \( \dot{K}_{p,q}^{\alpha} \)-norm in terms of \( L^p \)-norms over annuli which is the base for the definition of the spaces \( \dot{W}^{\alpha}_{p,q} \) in [25] and is the same one that we use in the present paper.

In order to achieve our aims, we need to develop properties for \( \dot{W}^{\alpha}_{p,q} \) and \( \dot{B} W^{\alpha,a,s}_{p,q,r} \)-spaces such as Hölder inequality, estimates for convolution operators, embedding theorems, interpolation properties, among others (see Section 2). In particular, it is proved a characterization of Besov-weak-Herz spaces in terms of interpolation of Sobolev-weak-Herz ones, which is key in our arguments (see Lemma 2.14). Moreover, we prove estimates for the heat semigroup, as well as for the bilinear term \( B(u,v) \) in (1.2), in the context of \( \dot{B} W^{\alpha,a,s}_{p,q,r} \)-spaces. We also point out that these spaces and their basic theory developed here could be employed to study other PDEs of elliptic, parabolic and conservation-law type. It is worthy to observe that some arguments in this paper are inspired by some of those in [18] that analyzed (1.1) in Besov-Morrey spaces.

In what follows, we state our global well-posedness result.

**Theorem 1.1.** Let \( 1 \leq q \leq \infty, \frac{n}{2} < p < \infty \) and \( 0 \leq \alpha < \min\{1 - \frac{n}{2p}, \frac{n}{2p}\} \). There exist \( \epsilon > 0 \) and \( \delta > 0 \) such that if \( u_0 \in \dot{B} W^{\alpha,a+\frac{n}{p},1}_{p,q} \) with \( \nabla \cdot u_0 = 0 \) and \( \| u_0 \|_{\dot{B} W^{\alpha,a+\frac{n}{p},1}_{p,q}} \leq \delta \), then there exists a unique mild solution \( u \in L^\infty((0,\infty); \dot{B} W^{\alpha,a+\frac{n}{p},1}_{p,q}) \) for (1.1) such that

\[
\| u \|_{X^\epsilon} := \| u \|_{L^\infty((0,\infty); \dot{B} W^{\alpha,a+\frac{n}{p},1}_{p,q})} + \sup_{t > 0} t^{-\frac{\alpha}{p}} \| u \|_{\dot{B} W^{\alpha,a,n}_{p,q,2}} \leq 2\epsilon.
\]

Moreover, \( u(t) \xrightarrow{\lambda} u_0 \) in \( \dot{B}^{1-\epsilon,\infty}_{\infty,\infty} \) as \( t \to 0^+ \), and solutions depend continuously on initial data.

We have the continuous inclusions \( L^n \subset L^{n,\infty} \subset W^{0,0}_{n,\infty} \subset \dot{B} W^{0,0}_{n,\infty} \) (see Lemmas 2.7 and 2.12) and

\[
\dot{H}^{\frac{n}{p}-1} \subset L^n \subset \dot{B}^{\frac{n}{p}-1}_{p,\infty} \subset \dot{B} W^{0,\frac{n}{p}-1}_{p,\infty}, \quad \text{for } p \geq n
\]

So our initial data class extends those of some previous works; for instance, the ones in [8, 14, 1, 4, 27, 25].
Notice that the parameter $s$ corresponds to the regularity index of the Besov type space $\dot{B}W^{s,p,q,r}$. Considering the family $\{\dot{B}W^{0,\frac{n}{p},1}_{p,\infty}\}_{p>n/2}$, in the positive regularity range $n/2 < p < n$ we are dealing with spaces smaller than those with $p > n$, because of the Sobolev embedding $\dot{B}W^{0,\frac{n}{p},1}_{p,\infty} \subset \dot{B}W^{0,\frac{n}{p},1}_{p,\infty}$ when $p_2 < p_1$ (see Lemma 2.13). For $p = 2$, $n = 3$ and $s = 1/2$ (positive regularity), one can show $\dot{B}W^{0,1/2}_{2,\infty} \subset BMO^{-1}$ by using duality and $A_\infty$-atom decomposition. However, for $p > n$ (negative regularity and larger spaces) it is not clear for us whether there are inclusion relations between $\dot{B}W^{0,\frac{n}{p},1}_{p,\infty}$ and $BMO^{-1}$ or between $\dot{B}W^{0,\frac{n}{p},1}_{p,\infty}$ and $\mathcal{N}^{\alpha,1}_{r,1,\infty}$ with $r > n$. In this sense, our result seems to give a new critical initial data class for existence of small global mild solutions for (1.1). In any case, it would be suitable to recall that well-posedness involves more properties than only existence of solutions, namely existence, uniqueness, persistence, and continuous dependence on initial data, which together characterize a good behavior of the Navier-Stokes flow in the considered space.

We finish with some comments about self-similarity and asymptotic behavior of solutions. It is not difficult to see that for $n \leq p < \infty$ the function $f(x) = |x|^{-1}$ belongs to $\dot{B}W^{0,\frac{n}{p},1}_{p,\infty}$. So, the homogeneous Besov-weak-Herz spaces (at least some of them) contain homogeneous functions of degree $-1$. Thus, if one assumes further that the initial data $u_0$ is a homogeneous vector field of degree $-1$, then a standard procedure involving a Picard type sequence gives that the solution obtained in Theorem 1.1 is in fact self-similar. Moreover, following some estimates and arguments in the proof of Theorem 1.1, with an extra effort, it is possible to prove that if we have $u_0$ and $v_0$ satisfying $\lim_{t \to \infty} \|G(t)(u_0 - v_0)\|_{\dot{B}W^{0,\alpha,\alpha+\frac{n}{p}+1}_{p,q,\infty}} = 0$, then

$$\lim_{t \to \infty} \|u(\cdot, t) - v(\cdot, t)\|_{\dot{B}W^{0,\alpha,\alpha+\frac{n}{p}+1}_{p,q,\infty}} = 0,$$

where $u$ and $v$ are the solutions obtained in Theorem 1.1 with initial data $u_0$ and $v_0$, respectively.

The plan of this paper is as follows. Section 2 is devoted to function spaces where Herz and Sobolev-Herz spaces are considered in subsection 2.1 while Sobolev-weak-Herz and Besov-weak-Herz spaces are addressed in subsection 2.2. The proof of Theorem 1.1 is performed in the final section through three subsections, namely 3.1, 3.2 and 3.3. In the first we provide linear estimates for the heat semigroup. The second is devoted to bilinear estimates for $B(\cdot, \cdot)$ in our setting. After obtaining the needed estimates, the proof is concluded in subsection 3.3 by means of a contraction argument.

## 2 Function spaces

In this section we recall some definitions and properties about function spaces that will be considered throughout this paper.

### 2.1 Weak-Herz and Sobolev weak-Herz spaces

For an integer $k \in \mathbb{Z}$, we define the set $A_k$ as

$$A_k = \left\{ x \in \mathbb{R}^n; 2^{k-1} \leq |x| < 2^k \right\},$$

and observe that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k \in \mathbb{Z}} A_k$. Taking $x \in A_k$ we have that

- $y \in A_m$ and $m \leq k \Rightarrow 2^{k-1} - 2^m \leq |x - y| < 2^k + 2^m$,
- $y \in A_m$ and $m \geq k \Rightarrow 2^{m-1} - 2^k \leq |x - y| < 2^m + 2^k$.

Consider also the sets
\[ C_{m,k} = \{ \xi; 2^{k-1} - 2^m \leq |\xi| < 2^k + 2^m \}, \]
\[ \bar{C}_{m,k} = \{ \xi; 2^{m-1} - 2^k \leq |\xi| < 2^m + 2^k \}. \tag{2.2} \]

Now we are able to define the weak-Herz spaces.

**Definition 2.1.** Let \( 1 < p \leq \infty, 1 \leq q \leq \infty \) and \( \alpha \in \mathbb{R} \). The Homogeneous weak-Herz space \( \dot{W}^{\alpha}_{p,q}(\mathbb{R}^n) \) is defined as the set of all measurable functions such that the following quantity is finite

\[
\|f\|_{\dot{W}^{\alpha}_{p,q}} := \left\{ \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|f\|_{L^p,\infty(A_k)}^{q} \right)^{1/q} \right\} if \ q < \infty, \]
\[
\|f\|_{\dot{W}^{\alpha}_{p,q}} := \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|f\|_{L^p,\infty(A_k)} \quad if \ q = \infty. \tag{2.3} \]

For \( \alpha \in \mathbb{R}, 1 < p \leq \infty \) and \( 1 \leq q \leq \infty \), the quantity \( \|\cdot\|_{\dot{W}^{\alpha}_{p,q}} \) defines a norm in \( \dot{W}^{\alpha}_{p,q} \) and the pair \( (\dot{W}^{\alpha}_{p,q}, \|\cdot\|_{\dot{W}^{\alpha}_{p,q}}) \) is a Banach space (see e.g. \([10, 25]\)).

Hölder inequality holds in the setting of homogeneous Weak-Herz spaces (see \([25]\)). To be more precise, if \( 1 < p, p_1, p_2 \leq \infty, 1 \leq q, q_1, q_2 \leq \infty \) and \( \alpha, \alpha_1, \alpha_2 \in \mathbb{R} \) are such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \) and \( \alpha = \alpha_1 + \alpha_2 \), then

\[ \|fg\|_{\dot{W}^{\alpha}_{p,q}} \leq C \|f\|_{\dot{W}^{\alpha_1}_{p_1,q_1}} \|g\|_{\dot{W}^{\alpha_2}_{p_2,q_2}}, \tag{2.4} \]

where \( C > 0 \) is an universal constant. In fact, for all \( k \in \mathbb{Z} \), we have

\[ \|fg\|_{L^{p,\infty}(A_k)} \leq C \|f\|_{L^{p_1,\infty}(A_k)} \|g\|_{L^{p_2,\infty}(A_k)}, \]

and therefore

\[
\|fg\|_{\dot{W}^{\alpha}_{p,q}} = \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha q} \|fg\|_{L^{p,\infty}(A_k)}^{q} \right)^{1/q} \leq \]
\[ \leq C \left( \sum_{k \in \mathbb{Z}} 2^{k\alpha_1 q} \|f\|_{L^{p_1,\infty}(A_k)}^{q} 2^{k\alpha_2 q} \|g\|_{L^{p_2,\infty}(A_k)}^{q} \right)^{1/q} \leq C \|f\|_{\dot{W}^{\alpha_1}_{p_1,q_1}} \|g\|_{\dot{W}^{\alpha_2}_{p_2,q_2}}. \tag{2.5} \]

Taking in particular \( (\alpha_1, p_1, q_1) = (0, \infty, \infty) \) in (2.5), we obtain

\[ \|fg\|_{\dot{W}^{\alpha}_{p,q}} \leq C \|f\|_{L^{\infty}(\mathbb{R}^n)} \|g\|_{\dot{W}^{\alpha}_{p,q}}. \tag{2.6} \]

More below we will need to estimate some convolution operators, particularly the heat semigroup, in weak-Herz and Besov-weak-Herz spaces. The following lemma will be useful for that propose.

**Lemma 2.2.** (Convolution) Let \( 1 \leq p_1 < \infty \) and \( 1 < r, p_2 < \infty \) be such that \( 1 + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} \). Also, let \( 1 \leq q \leq \infty, -\frac{n}{p} < \alpha < n \left( 1 - \frac{1}{p_2} \right), \) and \( \theta \in L^{p_1}(\mathbb{R}^n) \) be such that \( \theta |\cdot|^{\alpha/p_1} \in L^{\infty}(\mathbb{R}^n) \). Then, there exists a positive constant \( C \) independent of \( \theta \) such that

\[ \|\theta * f\|_{\dot{W}^{\alpha}_{p,r}} \leq C \max \left\{ \|\theta\|_{L^{p_1}}, \left\| \theta |\cdot|^{\alpha/p_1} \right\|_{L^{\infty}} \right\} \|f\|_{\dot{W}^{\alpha}_{p_2,q}}, \tag{2.7} \]

for all \( f \in \dot{W}^{\alpha}_{p_2,q} \).
Proof. Denote $f_m = f|_{A_m}$. Recalling the decomposition (2.1), for $k \in \mathbb{Z}$ we can estimate

$$2^{k\alpha} \|\theta * f\|_{L^r(A_k)} \leq 2^{k\alpha} \left\{ \left\| \sum_{m \leq k-2} \theta * f_m \right\|_{L^r(A_k)} + \left\| \sum_{m=k-1}^{k+1} \theta * f_m \right\|_{L^r(A_k)} + \left\| \sum_{m \geq k+2} \theta * f_m \right\|_{L^r(A_k)} \right\} $$

$$:= I^1_k + I^2_k + I^3_k. \quad (2.8)$$

Using the notations in (2.2) and the change of variable $z = k - m$, we handle the term $I^3_k$ as follows

$$I^3_k \leq 2^{k\alpha} \left\| \sum_{m \geq k+2} \theta * f_m \right\|_{L^r(A_k)} \leq 2^{k\alpha} \left\| \sum_{m \geq k+2} \theta * f_m \right\|_{L^r(A_k)}$$

$$\leq 2^{k\alpha} \left( \int_{A_k} \left\| \sum_{m \geq k+2} \int_{R^n} \theta(x-y)f_m(y)\,dy \right\|_r \, dx \right)^{1/r}$$

$$= 2^{k\alpha} \left( \int_{A_k} \left\| \sum_{m \geq k+2} \int_{R^n} \theta(x-y)X_{C_m,k}(x-y)f_m(y)\,dy \right\|_r \, dx \right)^{1/r}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} 2^{k\alpha} \left( \int_{A_k} \left\| \sum_{m \geq k+2} \int_{R^n} |x-y|^{-n/p_1}X_{C_m,k}(x-y)|f_m(y)|\,dy \right\|_r \, dx \right)^{1/r}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} 2^{k\alpha} \left( \int_{A_k} \left\| \sum_{m \geq k+2} 2^{-nm/p_1} \|f\|_{L^1(A_m)} \right\|_r \, dx \right)^{1/r}. \quad (2.9)$$

Recalling the inclusion $L^{p_2,\infty}(A_m) \hookrightarrow L^1(A_m)$, we can continue to estimate the right-hand side of (2.9) in order to obtain

R.H.S. of (2.9)

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} 2^{k\alpha} \left( \int_{A_k} \left\| \sum_{m \geq k+2} 2^{-nm/p_1} 2^{-nm(1-1/p_2)} \|f\|_{L^{p_2,\infty}(A_m)} \right\|_r \, dx \right)^{1/r}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} 2^{k\alpha} 2^{k\varphi} \sum_{m \geq k+2} 2^{-m\varphi} \|f\|_{L^{p_2,\infty}(A_m)}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} \sum_{-2 \geq z} 2^{k(\alpha + \frac{\varphi}{r})} 2^{z(k-z)\frac{\varphi}{r}} \|f\|_{L^{p_2,\infty}(A_{k-z})}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} \sum_{-2 \geq z} 2^{k\alpha} 2^{z(k-z)\alpha} 2^{(k-z)\alpha} \|f\|_{L^{p_2,\infty}(A_{k-z})}$$

$$\leq C \left\| |z|^{-n/p_1} \right\|_{L^\infty} \sum_{-2 \geq z} 2^{z(\frac{\varphi}{r} + \alpha)} 2^{(k-z)\alpha} \|f\|_{L^{p_2,\infty}(A_{k-z})}. \quad (2.10)$$

The above estimates and Minkowski inequality lead us to (with usual modification in the case $q = \infty$)
\[
\left( \sum_{k \in \mathbb{Z}} \left( I_k^q \right)^q \right)^{1/q} \leq CM_\theta \| f \|_{W K^{\alpha}_{p_2,q}}.
\]

For the parcel \( I_2^k \), we estimate
\[
I_2^k \leq 2^{k\alpha} \sum_{m=k-1}^{k+1} \| \theta \ast f_m \|_{L^{r,\infty}(A_k)} \leq 2^{k\alpha} \sum_{m=k-1}^{k+1} \| \theta \ast f_m \|_{L^{r,\infty}(\mathbb{R}^n)}
\]
\[
\leq 2^{k\alpha} \sum_{m=k-1}^{k+1} \| \theta \|_{L^{p_1,\infty}} \| f_m \|_{L^{p_2,\infty}} \leq C \| \theta \|_{L^{p_1}} \sum_{l=-1}^{1} 2^{(k+l)\alpha} \| f \|_{L^{p_2,\infty}(A_{k+l})},
\]
which implies
\[
\left( \sum_{k \in \mathbb{Z}} \left( I_2^k \right)^q \right)^{1/q} \leq CM_\theta \| f \|_{W K^{\alpha}_{p_2,q}}.
\]

Proceeding similarly to the estimates (2.9)-(2.10) but considering \( C_{m,k} \) in place of \( \tilde{C}_{m,k} \), the parcel \( I_1^k \) can be estimated as
\[
I_1^k \leq C \left\| |n|/p \theta \right\|_{L^\infty} 2^{k\alpha} \left( \int_{A_k} \left( \sum_{m \leq k-2} \int_{\mathbb{R}^n} |x-y|^{-n/p_1} \chi_{C_{m,k}}(x-y) |f_m(y)| \, dy \right)^r \, dx \right)^{1/r}
\]
\[
\leq C \left\| |n|/p \theta \right\|_{L^\infty} 2^{k\alpha} \left( \int_{A_k} \left( \sum_{m \leq k-2} 2^{-kn/p_1} \| f \|_{L^1(A_m)} \right)^r \, dx \right)^{1/r}
\]
\[
\leq C \left\| |n|/p \theta \right\|_{L^\infty} \sum_{m \leq k-2} 2^{k \left( \alpha - \frac{n}{p_1} \right)} \| f \|_{L^1(A_m)}
\]
\[
\leq C \left\| |n|/p \theta \right\|_{L^\infty} \sum_{2 \leq z} 2^{k \left( \alpha - \frac{n}{p_1} \right) 2^{n(z-1)/p_2}} \| f \|_{L^{p_2,\infty}(A_{k-2})}
\]
\[
\leq C \left\| |n|/p \theta \right\|_{L^\infty} \sum_{2 \leq z} 2^{k \left( \alpha - \frac{n}{p_1} \right) 2^{(k-2)\alpha}} \| f \|_{L^{p_2,\infty}(A_{k-2})},
\]
(2.11)

It follows from (2.11) that
\[
\left( \sum_{k \in \mathbb{Z}} \left( I_1^k \right)^q \right)^{1/q} \leq CM_\theta \| f \|_{W K^{\alpha}_{p_2,q}}.
\]

Finally, the desired estimate is obtained after recalling the norm (2.3) and using the above estimates for \( I_j^k \) in (2.8).

\[\diamond\]

Let \( \varphi \in C_c^\infty (\mathbb{R}^n \setminus \{0\}) \) be radially symmetric and such that

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Finally, Bony's decomposition (see [3]) gives

$$\text{supp } \varphi \subset \left\{ x; \frac{3}{4} \leq |x| \leq \frac{8}{3} \right\}$$

and

$$\sum_{j \in \mathbb{N}} \varphi_j(\xi) = 1, \ \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\varphi_j(\xi) := \varphi(\xi 2^{-j})$. Now we can define the well-known localization operators $\Delta_j$ and $S_j$

$$\Delta_j f = \varphi_j(D)f = (F^{-1}\varphi_j) * f,$$

$$S_k f = \sum_{j \leq k} \Delta_j f.$$

It is easy to see that we have the identities

$$\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \text{ and } \Delta_j (S_{k-2} \Delta_k f) = 0 \text{ if } |j - k| \geq 5.$$

Finally, Bony’s decomposition (see [3]) gives

$$fg = T fg + T g f + R(fg),$$

where

$$T fg = \sum_{j \in \mathbb{Z}} S_{j-2} \Delta_j g, \ R(fg) = \sum_{j \in \mathbb{Z}} \Delta_j f \tilde{\Delta}_j g \text{ and } \tilde{\Delta}_j g = \sum_{|j-j'| \leq 1} \Delta_{j'} g.$$

The next lemma will be useful in order to estimate some multiplier operators in Besov-Weak-Herz spaces.

**Lemma 2.3.** Let $1 < p < \infty, 1 \leq q \leq \infty, -\frac{n}{p} < \alpha < n \left(1 - \frac{1}{p}\right)$, $m \in \mathbb{R}$ and $D_j = \{ x; \frac{3}{4}2^j \leq |x| \leq \frac{8}{3}2^j \}$ for $j \in \mathbb{Z}$. Let $P$ be a $C^\infty$-function on $\check{D}_j := D_{j-1} \cup D_j \cup D_{j+1}$ such that $|\partial^{\beta}_{\xi} P(\xi)| \leq C 2^{(m-|\beta|)j}$ for all $\xi \in \check{D}_j$ and multi-index $\beta$ satisfying $|\beta| \leq [n/2] + 1$. Then, we have that

$$\left\| \left( \check{P} \hat{f} \right)^\vee \right\|_{W^{K}_{p,q}} \leq C^2 j^m \| f \|_{W^{K}_{p,q}},$$

for all $f \in W^{K}_{p,q}$ such that $\text{supp } \hat{f} \subset D_j$.

**Proof.** We start by defining $\check{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and $K(x) = (P \check{\varphi}_j)^\vee$. Since $\text{supp } \hat{f} \subset D_j$ we have that $P(\xi) \hat{f}(\xi) = P(\xi) \check{\varphi}_j(\xi) \hat{f}(\xi)$, and therefore $\left( \check{P} \hat{f} \right)^\vee = (P \check{\varphi}_j \hat{f})^\vee = K * f$.

Using Lemma 2.2 we get

$$\left\| \left( \check{P} \hat{f} \right)^\vee \right\|_{W^{K}_{p,q}} \leq C \max \{ \| K \|_{L^1}, \| \cdot \|_n K \|_{L^\infty} \} \| f \|_{W^{K}_{p,q}}.$$

It remains to show that $\max \{ \| K \|_{L^1}, \| \cdot \|_n K \|_{L^\infty} \} \leq C 2^{mj}$. For that, let $N \in \mathbb{N}$ be such that $\frac{n}{2} < N \leq n$ and proceed as follows.
\[ \| K \|_{L^1} = \int_{B(0,2^{-j})} K(y) + \int_{|y| \geq 2^{-j}} K(y) \leq \left( \int_{B(0,2^{-j})} 1 \right)^{1/2} \left( \int_{B(0,2^{-j})} |K(y)|^2 \right)^{1/2} + \left( \int_{|y| \geq 2^{-j}} |y|^{-2N} \right)^{1/2} \left( \int_{|y| \geq 2^{-j}} |y|^{2N} |K(y)|^2 \right)^{1/2} \leq C 2^{-j \frac{p}{2}} \| P\tilde{\varphi}_j \|_{L^2} + C 2^{-j \left( -N + \frac{p}{2} \right)} \sum_{|\beta| = N} \| \partial^\beta (P\tilde{\varphi}_j) \|_{L^2} \leq C 2^{-j \frac{p}{2}} C 2^{m_j} 2^{j \frac{p}{2}} + C 2^{-j \left( -N + \frac{p}{2} \right)} C 2^{j(m-N)\frac{p}{2}} \leq C 2^{m_j}. \]

For the norm \( \| \cdot \|_{L^\infty} \), we have that
\[
\| |n| K \|_{L^\infty} \leq C \sum_{|\beta| = n} \| \partial^\beta (P\tilde{\varphi}_j) \|_{L^1} \leq C \sum_{|\beta| = n} 2^{j(m-n)\frac{p}{2}} \leq C 2^{m_j},
\]
as required.

\section*{2.2 Sobolev-weak-Herz spaces and Besov-weak-Herz spaces}

In this section we introduce the homogeneous Sobolev-weak-Herz spaces and Besov-weak-Herz spaces. We also shall prove a number of properties about these spaces that will be useful in our study of the Navier-Stokes equations. These spaces are a generalization of Sobolev-Herz and Besov-Herz spaces found in [26].

\begin{definition}
Let \( 1 < p \leq \infty, 1 \leq q \leq \infty \) and \( a, s \in \mathbb{R} \). Recall the Riesz operator \( \hat{I}^s \hat{f} = |\xi|^s \hat{f} \). The homogeneous Sobolev-weak-Herz spaces \( W\dot{K}^{a,s}_{p,q} = W\dot{K}^{a,s}_{p,q}(\mathbb{R}^n) \) are defined as
\[ W\dot{K}^{a,s}_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \; \| I^s f \|_{W\dot{K}^{a,s}_{p,q}} < \infty \right\}. \tag{2.12} \]
\end{definition}

\begin{definition}
Let \( 1 < p \leq \infty, 1 \leq q, r \leq \infty \) and \( a, s \in \mathbb{R} \). The homogeneous Besov-weak-Herz spaces \( \dot{B}W\dot{K}^{a,s}_{p,q,r} = \dot{B}W\dot{K}^{a,s}_{p,q,r}(\mathbb{R}^n) \) are defined as
\[ \dot{B}W\dot{K}^{a,s}_{p,q,r} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}; \; \| f \|_{\dot{B}W\dot{K}^{a,s}_{p,q,r}} < \infty \right\}, \]
where
\[
\| f \|_{\dot{B}W\dot{K}^{a,s}_{p,q,r}} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \| \Delta_j f \|_{W\dot{K}^{a,s}_{p,q}}^r \right)^{1/r} & \text{if } r < \infty \\ \sup_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_{W\dot{K}^{a,s}_{p,q}} & \text{if } r = \infty. \end{cases} \tag{2.13} \]
\end{definition}

\begin{remark}
(i) The spaces \( W\dot{K}^{a,s}_{p,q} \) and \( \dot{B}W\dot{K}^{a,s}_{p,q,r} \) are Banach spaces endowed with the norms \( \| \cdot \|_{W\dot{K}^{a,s}_{p,q}} \) and \( \| \cdot \|_{\dot{B}W\dot{K}^{a,s}_{p,q,r}} \), respectively.
\end{remark}
(ii) The continuous inclusion $\dot{B}^s_{p,r}(\mathbb{R}^n) \subset \dot{B}W^{0,s,r}_{p,\infty}(\mathbb{R}^n)$ holds for all $s \in \mathbb{R}$, $1 < p \leq \infty$, and $1 \leq r \leq \infty$, where $\dot{B}^s_{p,r}$ stands for homogeneous Besov spaces. For that, it is sufficient to recall the definition of Besov spaces (see [2, pg.146]) and (2.13) and to use the inclusion $L^p \subset W\dot{K}^0_{p,\infty}$ that is going to be showed in the lemma below.

The next lemma contains relations between weak-$L^p$, weak-Herz and Morrey spaces. For the definition and some properties about Morrey spaces we refer the reader to [18] (see also [15] for an equivalent definition and further properties).

**Lemma 2.7.** For $1 < p < \infty$, we have the continuous inclusion

$$L^p \subset L^{p,\infty} \subset W\dot{K}^0_{p,\infty}.$$  \hspace{1cm} (2.14)

Moreover, let $\mathcal{M}^r_q$ stand for homogeneous Morrey spaces, $1 \leq q \leq r < \infty$ and $n/r \neq \alpha + n/p$ when $q < p$. Then

$$W\dot{K}^\alpha_{p,\infty} \subset \mathcal{M}^r_q.$$  \hspace{1cm} (2.15)

**Proof.** The first inclusion in (2.14) is well-known, so we only prove the second one. For that, it is sufficient to note that $\|f\|_{L^{p,\infty}(A_k)} \leq \|f\|_{L^{p,\infty}(\mathbb{R}^n)}$ $\forall k \in \mathbb{Z}$ and after to take the supremum over $k$. In order to see the strictness of the inclusion, take $x_k = \frac{3}{2}2^{-k+1}e_1$ and $h(x) := \sum_{k=1}^{\infty} |x - x_k|^{-\frac{\alpha}{r}} \chi_{B(0,1/2)}(x - x_k)$. It is clear that $h \in W\dot{K}^\alpha_{p,\infty}$ but not to $L^{p,\infty}(\mathbb{R}^n)$.

Now we turn to (2.15). For $f(x) = |x|^{-\frac{n}{r}}$, we have that $f \in L^{p,\infty} \subset W\dot{K}^0_{p,\infty}$. On the other hand, for any $q \geq p$ note that $\|f\|_{L^n(B(0,R))} = \infty$, and then $f \notin \mathcal{M}^r_q$ for any $r$. Finally, if $n/r \neq \alpha + n/p$ then $W\dot{K}^\alpha_{p,\infty} \subset \mathcal{M}^r_q$ (and the reverse) never could hold. This follows from an easy scaling analysis of the space norms; in fact, the scaling of $\mathcal{M}^r_q$ is $-n/r$ and the one of $W\dot{K}^0_{p,\infty}$ is $-\alpha - n/p$.

\[ \Box \]

In the next remark, we recall some inclusion and non-inclusion relations involving Herz, weak-Herz, Besov and $bmo^{-1}$ spaces that can be found in [25].

**Remark 2.8.**

i) For $1 < p < \sigma < \infty$ and $0 < \alpha < n (1 - 1/p)$, we have

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha+n(1/p-1/\sigma)}_{p,\sigma}, \dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha}_{p,\infty} \text{ and } W\dot{K}^0_{p,\sigma} \hookrightarrow \dot{B}^{-n(1/p-1/\sigma)}_{p,\sigma}.$$  

ii) For $1 < p < \infty$ and $0 \leq \alpha < n (1 - 1/p)$, we have $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha+n/p}_{\infty}$.  

iii) For $0 \leq \alpha < n$, we have $\dot{K}^\alpha_{\infty,\infty} \hookrightarrow \dot{B}^{-\alpha}_{\infty,\infty}$.  

iv) For $1 < p \leq \infty$ and $0 \leq \alpha \leq n (1 - 1/p)$, we have $W\dot{K}^\alpha_{p,1} \hookrightarrow \dot{B}^{-\alpha+n/p}_{\infty}$.  

v) We have $L^1 = \dot{K}^0_{1,1} \hookrightarrow \dot{B}^{-n}_{\infty,\infty}$. For $n < p \leq \infty$ and $0 \leq \alpha < 1 - n/p$, the inclusion $W\dot{K}^\alpha_{p,\infty} \hookrightarrow bmo^{-1}$ holds.  

vi) For $1 < p < \sigma < \infty$ and $-n (1/p - 1/\sigma) < \alpha \leq 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha+n(1/p-1/\sigma)}_{\sigma,\infty}$ does not hold.  

vii) For $1 < p < \infty$ and $-n/p < \alpha < 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha+n/p}_{\infty}$ does not hold.
Remark 2.9. Using the interpolation properties of homogeneous Besov spaces and homogeneous Besov-
weak-Herz spaces (see Lemma 2.14 below) and item (ii) of Remark 2.8, for $1 < p < \infty$ and $0 \leq \alpha < n \left(1 - \frac{1}{p}\right)$ we can obtain
\[ \dot{B}W^{\alpha,s}_{p,\infty,r} \hookrightarrow \dot{B}^{s-(\alpha+n/p)}_{\infty,r}. \] (2.16)

In particular, $\dot{B}W^{\alpha,\alpha+n/p-1}_{p,\infty,\infty} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$ and
\[ \dot{B}W^{\alpha,0}_{p,\infty,1} \hookrightarrow \dot{B}^{0}_{0,1} \hookrightarrow L^{\infty}. \] (2.17)

Moreover, from Remark 2.8 (vi) and Lemma 2.12 below, it follows that the inclusion
\[ \dot{B}W^{0,s}_{p,\infty,\infty} \hookrightarrow \dot{B}^{s-n(1/p-1/\sigma)}_{\infty,\infty} \]
does not hold for any $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq \sigma < \infty$.

Remark 2.10. Note that for $s - (\alpha + n/p) < 0$ and $r > 1$, or $s - (\alpha + n/p) \leq 0$ and $r = 1$, the inclusion (2.16) implies that for $f \in \dot{B}W^{\alpha,\alpha+n/p}_{p,\infty,r}$ the series $\sum_{j=\infty}^{\infty} \Delta_j f$ converges in $S'$ to a representative of $f$ in $S'/P$ (see e.g [20]). So, in these cases the space $\dot{B}W^{\alpha,\alpha+n/p}_{p,\infty,r}$ can be regarded as a subspace of $S'$. Hereafter, we say that $f \in S'$ belongs to $\dot{B}W^{\alpha,\alpha+n/p}_{p,\infty,r}$ with $s - (\alpha + n/p) < 0$ and $r > 1$, or $s - (\alpha + n/p) \leq 0$ and $r = 1$, if $f$ is the canonical representative of the class in $S'/P$, namely $f = \sum_{j=\infty}^{\infty} \Delta_j f$ in $S'$.

A multiplier theorem of Hörmander-Mihlin type will be needed in our setting. This is the subject of the next lemma. In fact, the main part of the proof has already been done in Lemma 2.3.

Lemma 2.11. Let $1 < p < \infty$, $1 \leq q, r \leq \infty$, $-\frac{n}{p} < \alpha < n \left(1 - \frac{1}{p}\right)$ and $m, s \in \mathbb{R}$. Let $P \in C^m \left(\mathbb{R}^n \setminus \{0\}\right)$ be a function such that $|\partial^\beta P(\xi)| \leq C |\xi|^{(m-|\beta|)}$ for all multi-index $\beta$ satisfying $|\beta| \leq n$. Then
\[ \|P(D) f\|_{\dot{B}W^{\alpha,s-m}_{p,q,r}} \leq C \|f\|_{\dot{B}W^{\alpha,s}_{p,q,r}}. \]

Proof. Note that for each $j \in \mathbb{Z}$ we have that $|\xi|^{m-|\beta|} \leq C 2^{j(m-|\beta|)}$ for all $\xi \in \tilde{D}_j$, and therefore $|\partial^\beta P(\xi)| \leq C 2^{j(m-|\beta|)}$. On the other hand, since $\text{supp} \hat{\Delta}_j f \subset D_j$ we can use Lemma 2.3 in order to get
\[ \|\Delta_j (P(D) f)\|_{W^{\alpha,q}_{p,q}} = \|P(D) (\Delta_j f)\|_{W^{\alpha,q}_{p,q}} \leq C 2^{jm} \|\Delta_j f\|_{W^{\alpha,q}_{p,q}}. \] (2.18)

Now the result follows by multiplying (2.18) by $2^{j(s-m)}$ and after taking the $l^r$-norm.

\[ \Diamond \]

In what follows we present some inclusions involving Sobolev-weak-Herz and Besov-weak-Herz spaces.

Lemma 2.12. Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $-\frac{n}{p} < \alpha < n \left(1 - \frac{1}{p}\right)$. We have the following continuous inclusions
\[ \dot{B}W^{\alpha,0}_{p,q,1} \subset W^{\alpha}_{p,q} \subset \dot{B}W^{\alpha,0}_{p,q,\infty} \] (2.19)
\[ \dot{B}W^{\alpha,s}_{p,q,1} \subset W^{\alpha,s}_{p,q} \subset \dot{B}W^{\alpha,s}_{p,q,\infty} \] (2.20)
Proof. For \( f \in \dot{BW}K_{p,q,1}^{\alpha,0} \), we can employ the decomposition \( f = \sum_{j \in \mathbb{Z}} \Delta_j f \) in order to estimate

\[
\|f\|_{L^p,\infty(A_k)} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^p,\infty(A_k)}.
\]

Thus, using Minkowski inequality, we arrive at (with usual modification in the case \( q = \infty \))

\[
\|f\|_{W^{k,q}_p} \leq \left[ \sum_{k \in \mathbb{Z}} 2^{kq} \|f\|_{L^p,\infty(A_k)}^q \right]^{\frac{1}{q}} \leq \left[ \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^p,\infty(A_k)} \right)^q \right]^{\frac{1}{q}} \leq \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{W^{k,q}_p} = \|f\|_{\dot{BW}K_{p,q,1}^{\alpha,0}},
\]

which implies the first inclusion in (2.19). Now, let \( f \in W^{k,q}_p \) and note that in fact we have that \( f \in S'/P \). Moreover, using Lemma 2.2 we get

\[
\|f\|_{\dot{BW}K_{p,q,\infty}^{\alpha,0}} = \sup_{j \in \mathbb{Z}} \|\Delta_j f\|_{W^{k,q}_p} \leq C \sup_{j \in \mathbb{Z}} \|f\|_{W^{k,q}_p} = C \|f\|_{W^{k,q}_p},
\]

and then the second inclusion in (2.19) holds.

For (2.20), we can use Lemma 2.3 in order to estimate

\[
\|f\|_{W^{k,q}_p} = \|I^s f\|_{W^{k,q}_p} \leq \|I^s f\|_{\dot{BW}K_{p,q}^{\alpha,0}} = \sum_{j \in \mathbb{Z}} \|\Delta_j I^s f\|_{W^{k,q}_p} \leq C \sum_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{W^{k,q}_p} = C \|f\|_{\dot{BW}K_{p,q}^{\alpha,s}}.
\]

Moreover, Lemma 2.3 also can be used to obtain

\[
\|f\|_{\dot{BW}K_{p,q,\infty}^{\alpha,s}} = \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{W^{k,q}_p} = \sup_{j \in \mathbb{Z}} 2^{js} \|I^{-s} \Delta_j I^s f\|_{W^{k,q}_p} \leq C \sup_{j \in \mathbb{Z}} \|\Delta_j I^s f\|_{W^{k,q}_p} \leq C \sup_{j \in \mathbb{Z}} \|I^s f\|_{W^{k,q}_p} = C \|f\|_{W^{k,q}_p}
\]

for all \( f \in W^{k,q}_p \), as required.

Now we present an embedding theorem of Sobolev type.

Lemma 2.13. Let \( s \in \mathbb{R}, 1 < p < \infty, 1 \leq q, r \leq \infty, p \leq p_1 < \infty, 1 < p_2 \leq p_1 \) and \(-\frac{n}{p} < \alpha < n \left(1 + \frac{1}{p_1} - \frac{1}{p_2} - \frac{1}{p}\right)\). Then

\[
\|f\|_{BW^{k,q,\infty}_p} \leq C \|f\|_{BW^{k,q,\infty}_p} \leq C \|f\|_{BW^{k,q,\infty}_p},
\]

(2.21)
In particular, for $\frac{2}{n} < p < \infty$ and $0 < \alpha < \min \left\{ 1 - \frac{m}{2p}, \frac{n}{2p} \right\}$, it follows that
\[
\| f \|_{\dot{B}W^{\alpha,s}_{2p,q,r}} \leq C \| f \|_{\dot{B}W^{\alpha,s}_{p,q,r}} .
\] (2.22)

**Proof.** Using Hölder inequality, it follows that
\[
\| \Delta_j f \|_{W^{\alpha+n} \left( \frac{1}{p} - \frac{1}{q} \right)} \leq C \max \left\{ \| (\varphi_j)^* \|_{L^p}, \| | \Delta_j \varphi_j |^{\frac{1}{p}} \|_{L^\infty} \right\} \| \Delta_j f \|_{W^{\alpha+n} \left( \frac{1}{p} - \frac{1}{q} \right)},
\]
where $1 + \frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$. It is easy to check that $\max \left\{ \| (\varphi_j)^* \|_{L^p}, \| | \Delta_j \varphi_j |^{\frac{1}{p}} \|_{L^\infty} \right\} \leq C 2^{jn} \left( \frac{1}{p_2} - \frac{1}{p} \right)$, and then
\[
\| \Delta_j f \|_{W^{\alpha+n} \left( \frac{1}{p} - \frac{1}{q} \right)} \leq C 2^{jn} \left( \frac{1}{p_2} - \frac{1}{p} \right) \| \Delta_j f \|_{W^{\alpha+n} \left( \frac{1}{p} - \frac{1}{q} \right)},
\]
which gives (2.21). We conclude the proof by noting that for $0 \leq \alpha < n/2p$ there exists $p_1$ such that $p_1 \geq 2p$ and $\alpha = n \left( \frac{1}{p_1} - \frac{1}{p} \right)$. Moreover, $\alpha < n + \frac{n}{p_1} - \frac{1}{p} - \frac{2}{2p}$ because $\alpha < 1 - \frac{p}{2p} \leq \frac{n}{2} - \frac{2}{2p}$. So, (2.22) follows from (2.21) by choosing this value of $p_1$.

We finish this section with a result that provides a characterization of homogeneous Besov-weak-Herz spaces as interpolation of two homogeneous Sobolev-weak-Herz ones.

**Lemma 2.14.** Let $s_0, s_1, s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $-\frac{n}{p} < \alpha < n \left( 1 - \frac{1}{p} \right)$. If $s_0 \neq s_1$ and $s = (1 - \theta)s_0 + \theta s_1$ with $\theta \in (0, 1)$, then
\[
\left( \dot{W}^{\alpha,s_0}_{p,q}, \dot{W}^{\alpha,s_1}_{p,q} \right)_{\theta,r} = \dot{B}W^{\alpha,s}_{p,q,r}.
\]

**Proof.** Let $f = f_0 + f_1$ with $f_i \in \dot{W}^{\alpha,si}_{p,q}$ $i = 0, 1$. By using the Lemma 2.3 we get
\[
\| \Delta_j f \|_{\dot{W}^{\alpha}_{p,q}} \leq \| \Delta_j f_0 \|_{\dot{W}^{\alpha,s_0}_{p,q}} + \| \Delta_j f_1 \|_{\dot{W}^{\alpha,s_1}_{p,q}} 
\leq C \left( 2^{-s_0j} \| I^{s_0} \Delta_j f_0 \|_{\dot{W}^{\alpha,s_0}_{p,q}} + 2^{-s_1j} \| I^{s_1} \Delta_j f_1 \|_{\dot{W}^{\alpha,s_1}_{p,q}} \right) 
\leq C \left( 2^{-s_0j} \| I^{s_0} f_0 \|_{\dot{W}^{\alpha,s_0}_{p,q}} + 2^{-s_1j} \| I^{s_1} f_1 \|_{\dot{W}^{\alpha,s_1}_{p,q}} \right) 
\leq C 2^{-s_0j} \left( \| f_0 \|_{\dot{W}^{\alpha,s_0}_{p,q}} + 2^{(s_0-s_1)} \| f_1 \|_{\dot{W}^{\alpha,s_1}_{p,q}} \right). 
\] (2.23)

It follows from (2.23) that
\[
\| \Delta_j f \|_{\dot{W}^{\alpha}_{p,q}} \leq C 2^{-s_0j} K \left( 2^{(s_0-s_1)} \| f, \dot{W}^{\alpha,s_0}_{p,q}, \dot{W}^{\alpha,s_1}_{p,q} \right). 
\]
Noting that \( s - s_0 = -\theta (s_0 - s_1) \) and multiplying by \( 2^j \) the previous inequality, we arrive at
\[
2^{sj} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_0}} \leq C \left( 2^{(s_0 - s_j)j} \right)^{\theta} K \left( 2^{(s_0 - s_1)j} f, W\dot{K}_{p,q}^{\alpha,s_0}, W\dot{K}_{p,q}^{\alpha,s_1} \right),
\]
and then (see [2, Lemma 3.1.3]) we can conclude that
\[
\| f \|_{\dot{B}W\dot{K}_{p,q}^{\alpha,s_0}} \leq C \| f \|_{(W\dot{K}_{p,q}^{\alpha,s_0}, W\dot{K}_{p,q}^{\alpha,s_1})_{q,r}}.
\]

In order to prove the reverse inequality, note that by using again Lemma 2.3 we have
\[
2^{(s-s_0)j} J \left( 2^{(s_0 - s_j)j} \Delta_j f, W\dot{K}_{p,q}^{\alpha,s_0}, W\dot{K}_{p,q}^{\alpha,s_1} \right) = 2^{(s-s_0)j} \max \left\{ \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_0}}, 2^{(s_0 - s_1)j} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_1}} \right\} \leq 2^{(s-s_0)j} \max \left\{ 2^{s_0j} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_0}}, 2^{s_0j} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}} \right\} \leq 2^{sj} \max \left\{ \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_0}}, \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}} \right\} = 2^{sj} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}}.
\]

Now the Equivalence Theorem (see [2, Lemma 3.2.3]) leads us to
\[
\| f \|_{(W\dot{K}_{p,q}^{\alpha,s_0}, W\dot{K}_{p,q}^{\alpha,s_1})_{q,r}} \leq C \| f \|_{\dot{B}W\dot{K}_{p,q}^{\alpha,s}}.
\]

The remainder of the proof is to show that in fact \( f \in \dot{B}W\dot{K}_{p,q}^{\alpha,s} \) implies that \( f \in W\dot{K}_{p,q}^{\alpha,s_0} + W\dot{K}_{p,q}^{\alpha,s_1} \).

Suppose that \( s_0 > s_1 \) (without loss of generality). Using the decomposition \( f = \sum_{j<0} \Delta_j f + \sum_{j \geq 0} \Delta_j f = f_0 + f_1 \) and Lemma 2.3, we obtain
\[
\| f_0 \|_{W\dot{K}_{p,q}^{\alpha,s_0}} \leq \sum_{j<0} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_0}} \leq \sum_{j<0} 2^{j(s_0 - s)} 2^{sj} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}} \leq C \left( \sum_{j<0} 2^{j(s_0 - s)r} \right)^{1/r} \left( \sum_{j<0} 2^{jsr} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}}^r \right)^{1/r} \leq C \| f \|_{\dot{B}W\dot{K}_{p,q}^{\alpha,s}}.
\]

Similarly, one has
\[
\| f_1 \|_{W\dot{K}_{p,q}^{\alpha,s_1}} \leq \sum_{j \geq 0} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha,s_1}} \leq \sum_{j \geq 0} 2^{j(s_1 - s)} 2^{sj} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}} \leq C \left( \sum_{j \geq 0} 2^{j(s_1 - s)r'} \right)^{1/r'} \left( \sum_{j \geq 0} 2^{jsr} \| \Delta_j f \|_{W\dot{K}_{p,q}^{\alpha}}^r \right)^{1/r} \leq C \| f \|_{\dot{B}W\dot{K}_{p,q}^{\alpha,s}}.
\]

and then we are done.
3 Proof of Theorem 1.1

In the previous sections, we have derived key properties about homogeneous Besov-weak-Herz spaces. With these results in hands, we prove Theorem 1.1 in the present section.

3.1 Heat kernel estimates

We start by providing estimates for the heat semigroup \( \{G(t)\}_{t \geq 0} \) in Besov-weak-Herz spaces. Recall that in the whole space \( \mathbb{R}^n \) this semigroup can be defined as \( G(t)f = \left( \exp \left( -t|\xi|^2 \right) \hat{f} \right)^\# \) for all \( f \in S' \) and \( t \geq 0 \).

Lemma 3.1. Let \( s, \sigma \in \mathbb{R}, s \leq \sigma, 1 < p < \infty, 1 \leq q, r \leq \infty \) and \(-\frac{n}{p} < \alpha < n \left( 1 - \frac{1}{p} \right)\). Then, there is \( C > 0 \) (independent of \( f \)) such that

\[
\|G(t)f\|_{\dot{B}W^{\alpha,\sigma}_{p,q,r}} \leq C t^{(s-\sigma)/2} \|f\|_{\dot{B}W^{\alpha,s}_{p,q,r}},
\]

for all \( t > 0 \). Moreover, if \( s < \sigma \), then we have the estimate

\[
\|G(t)f\|_{\dot{B}W^{\alpha,\sigma}_{p,q,1}} \leq C t^{(s-\sigma)/2} \|f\|_{\dot{B}W^{\alpha,s}_{p,q,\infty}},
\]

for all \( t > 0 \).

Proof. Firstly, observe that for each multi-index \( \beta \) there is a polynomial \( p_\beta(\cdot) \) of degree \|\beta\| such that

\[
\partial^{\beta}_\xi \left( \exp(-t|\xi|^2) \right) = t^{|\beta|/2} p_\beta(\sqrt{t}\xi) \exp \left( -t|\xi|^2 \right).
\]

Therefore, for some \( C > 0 \) it follows that

\[
\left| \partial^{\beta}_\xi \left( \exp(-t|\xi|^2) \right) \right| \leq C t^{-m/2} |\xi|^{-m-|\beta|}.
\]

By employing Lemma 2.11, we obtain

\[
\|G(t)f\|_{\dot{B}W^{\alpha,\sigma-m}_{p,q,r}} \leq C t^{-m/2} \|f\|_{\dot{B}W^{\alpha,s}_{p,q,r}}.
\]

Taking now \( m = s - \sigma \) we arrive at the inequality (3.1).

Next we turn to (3.2) and let \( s < \sigma \). From (3.1) with \( r = \infty \) we get

\[
\|G(t)f\|_{\dot{B}W^{\alpha,2\sigma-s}_{p,q,\infty}} \leq C t^{s-\sigma} \|f\|_{\dot{B}W^{\alpha,s}_{p,q,\infty}}
\]

and

\[
\|G(t)f\|_{\dot{B}W^{\alpha,s}_{p,q,\infty}} \leq C \|f\|_{\dot{B}W^{\alpha,s}_{p,q,\infty}}.
\]

By using Lemma 2.14 and the Reiteration Theorem (see [2, Theorem 3.5.3 and its remark]) we conclude that

\[
G(t) : \dot{B}W^{\alpha,s}_{p,q,\infty} \rightarrow \left( \dot{B}W^{2\alpha-s}_{p,q,\infty}, \dot{B}W^{\alpha,s}_{p,q,\infty} \right)^{\frac{1}{2},1} = \dot{B}W^{\alpha,s}_{p,q,1},
\]

with \( \|G(t)\|_{\dot{B}W^{\alpha,s}_{p,q,\infty} \rightarrow \dot{B}W^{\alpha,s}_{p,q,1}} \leq C t^{(s-\sigma)/2} \), which gives (3.2).

\( \diamond \)
3.2 Bilinear estimate

Let us define the space $X$ as

$$X = \left\{ u : (0, \infty) \rightarrow \mathcal{B}WK^{\alpha, \alpha + \frac{n}{2p} - 1}_{p, q, 0} \cap W^{2p, 2q}_{0, \alpha} \right\},$$

where

$$\|u\|_X := \|u\|_{L^\infty((0, \infty); \mathcal{B}W^{\alpha, \alpha + \frac{n}{2p} - 1}_{p, q, 0})} + \sup_{t > 0} t^{\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|u\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}}. \quad (3.3)$$

We are going to prove the bilinear estimate

$$\|B(u, v)\|_X \leq K \|u\|_X \|v\|_X. \quad (3.4)$$

We start by estimating the second part of the norm (3.3). For that matter, we use (2.19), (2.22), (3.2) and Lemma 2.11 in order to get

$$\|B(u, v)(t)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} \leq \|B(u, v)(t)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} \leq \|B(u, v)(t)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}}$$

$$\leq C \int_0^t \|G(t - \tau) \text{div} (u \otimes v)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|\text{div} (u \otimes v)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|u \otimes v\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|u\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|v\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} d\tau$$

$$\leq C \int_0^t (t - \tau)^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \tau^{-2 \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} d\tau \|u\|_X \|v\|_X$$

$$\leq C t^{-\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \mathcal{B}\left( \alpha + \frac{n}{2p}, \frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right) \right) \|u\|_X \|v\|_X,$$

where $\mathcal{B}(\cdot, \cdot)$ denotes the beta function. The previous estimate leads us to

$$\sup_{t > 0} t^{\frac{1}{2} - \left( \frac{\alpha}{2} + \frac{n}{4p} \right)} \|B(u, v)(t)\|_{\mathcal{B}W^{2p, 2q}_{0, \alpha}} \leq C \|u\|_X \|v\|_X. \quad (3.5)$$

Moreover, for the first part of the norm (3.3), we have
In other words, we have obtained the estimate

\[ \|B(u, v)(t)\|_{BW_{(p, q)}^{\alpha+\frac{n}{2p}-1}} \leq \int_0^t \|G(t - \tau)\|_{\mathbb{P} \text{div } [u \otimes v]}_{BW_{(p, q)}^{\alpha+\frac{n}{2p}-1}} d\tau \]

Moreover, using (3.8) we obtain

\[ \sup_{t>0} t^{\frac{1}{2} - \left(\frac{\alpha}{2} + \frac{n}{2p}\right)} \|G(t)u_0\|_{BW_{p, 2q}^{\alpha}} \leq C \sup_{t>0} t^{\frac{1}{2} - \left(\frac{\alpha}{2} + \frac{n}{2p}\right)} \|G(t)u_0\|_{BW_{p, 2q, \infty}^{\alpha+\frac{n}{2p}-1}} \]

Moreover, using (3.1) we obtain

\[ \|G(t)u_0\|_{BW_{p, q}^{\alpha+\frac{n}{2p}-1}} \leq C \|u_0\|_{BW_{p, q, \infty}^{\alpha+\frac{n}{2p}-1}}. \]

From the last two estimates, we get

\[ \|G(t)u_0\|_X \leq C \|u_0\|_{BW_{p, q, \infty}^{\alpha+\frac{n}{2p}-1}}. \] (3.8)

Take \( 0 < \epsilon < 1/4K \) and \( 0 < \delta < \epsilon/C \) where \( C \) is as in (3.8). It follows from (3.8) and (3.4) that

\[ \|\Psi(u)\|_X \leq \|G(t)u_0\|_X + \|B(u, u)\|_X \]

Finally, notice that the estimates (3.5) and (3.6) together give (3.4).

### 3.3 Proof of Theorem 1.1

**Existence and Uniqueness.** For \( \epsilon > 0 \) (to be chosen later) let \( \overline{B}(0, \epsilon) \) denote the closed ball in \( X \) and define the operator \( \Psi : \overline{B}(0, 2\epsilon) \to \overline{B}(0, 2\epsilon) \) as

\[ \Psi(u) = G(t)u_0 + B(u, u). \]

First, note that by using (2.19), (3.2), \( \alpha + \frac{n}{2p} - 1 < 0 \) and (2.21) it follows that

\[ \|B(u, v)(t)\|_{BW_{(p, q)}^{\alpha+\frac{n}{2p}-1}} \leq C \|u\|_X \|v\|_X. \] (3.6)

From the last two estimates, we get

\[ \|G(t)u_0\|_X \leq C \|u_0\|_{BW_{p, q, \infty}^{\alpha+\frac{n}{2p}-1}}. \] (3.8)
So, $\Psi$ is well-defined, moreover for $u, v \in \tilde{B}(0, 2\epsilon)$ we have that

$$\|\Psi(u) - \Psi(v)\|_X = \|B(u - v, u) + B(v, u - v)\|_X$$

$$\leq K \|u - v\|_X \|u\|_X + K \|v\|_X \|u - v\|_X$$

$$\leq 4K\epsilon \|u - v\|_X.$$  \hspace{1cm} (3.9)

Since $4K\epsilon < 1$, we get that $\Psi$ is a contraction and then this part is concluded by the Banach fixed-point theorem. Notice that the continuous dependence with respect to the initial data $u_0$ follows from estimates (3.8) and (3.9).

**Time-weak continuity at $t = 0$.** The proof of the weak-* convergence follows from the two following lemmas.

The first one is due to Kozono and Yamazaki [18, pg. 989].

**Lemma 3.2.** For every real number $s$ and $u_0 \in \dot{B}^s_{\infty, \infty}$, we have $G(t)u_0 \overset{s}{\rightharpoonup} u_0$ in $\dot{B}^s_{\infty, \infty}$ as $t \to 0^+$.

The second one is concerned with the weak-convergence of the bilinear term $B(u, u)$ and it concludes the proof.

**Lemma 3.3.** Let $v \in X$. We have that $B(v, v)(t)$ converges to 0 in the weak-* topology of $\dot{B}^{-1}_{\infty, \infty}$ as $t \to 0^+$.

**Proof.** Let $\tilde{\phi} \in \dot{B}^1_{1,1}$ and $\epsilon > 0$ an arbitrary number. We can choose $\tilde{\phi} \in \mathcal{S}$ such that $\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} < \epsilon$.

Then we have that

$$\left| \left\langle B(v, v)(t), \phi \right\rangle \right| \leq \|B(v, v)(t)\|_{\dot{B}^{-1}_{\infty, \infty}} \left\| \phi - \tilde{\phi} \right\|_{\dot{B}^1_{1,1}}$$

$$\leq C \|B(v, v)(t)\|_{\dot{B}^{1,0}_{\alpha, \alpha + n/p - 1}} \left\| \phi - \tilde{\phi} \right\|_{\dot{B}^1_{1,1}} \leq K \|v\|_X^2 \epsilon \leq C\epsilon. \hspace{1cm} (3.10)$$

On the other hand,

$$\left| \left\langle B(v, v)(t), \tilde{\phi} \right\rangle \right| \leq \int_0^t \left| \left\langle G(t - \tau)\text{div} [v \otimes v] (\tau), \tilde{\phi} \right\rangle \right| d\tau \leq \int_0^t \left| \left\langle \text{div} [v \otimes v] (\tau), G(t - \tau)\tilde{\phi} \right\rangle \right| d\tau$$

$$\leq \int_0^t \|\text{div} [v \otimes v] (\tau)\|_{\dot{B}^{-1-2\alpha - n/p}_{\infty, \infty}} \|G(t - \tau)\tilde{\phi}\|_{\dot{B}^{1+2\alpha + n/p}_{1,1}} d\tau$$

$$\leq C\tilde{\phi} \int_0^t \|v \otimes v\|_{\dot{B}^{-2\alpha - n/p}_{\infty, \infty}} d\tau \leq C\tilde{\phi} \int_0^t \|v \otimes v\|_{W^{\alpha}_{2,2}} d\tau$$

$$\leq C\tilde{\phi} \int_0^t \|v\|_X^2 \int_0^\tau \tau^{-1 - \alpha + \frac{n}{2p}} d\tau \leq C\tilde{\phi} \|v\|_X^{\alpha + \frac{n}{2p}}. \hspace{1cm} (3.11)$$

From (3.10) and (3.11), we obtain

$$0 \leq \limsup_{t \to 0^+} \left\langle |B(v, v)(t), \phi| \right\rangle \leq \limsup_{t \to 0^+} \left| \left\langle B(v, v)(t), \phi \right\rangle \right| + \limsup_{t \to 0^+} \left| \left\langle B(v, v)(t), \tilde{\phi} \right\rangle \right|$$

$$\leq C\epsilon + 0.$$
Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{t \to 0^+} |\langle B(v,v)(t), \phi \rangle| = 0$. Now, using that $\phi \in \dot{B}^1_{1,1}$ is arbitrary, we get the desired convergence.

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