Effects of disorder on two strongly correlated coupled chains

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Abstract

We study the effects of disorder on a system of two coupled chain of strongly correlated fermions (ladder system), using a renormalization group technique. The stability of the phases of the pure system has been investigated as a function of interactions both for fermions with spin and spinless fermions. For spinless fermions the repulsive side is strongly localized whereas the system with attractive interactions is stable with respect to disorder, at variance with the single chain case. For fermions with spins, the repulsive side is also localized, and in particular the d-wave superconducting phase found for the pure system is totally destroyed by an arbitrarily small amount of disorder. On the other hand the attractive side is again remarkably stable with respect to localization. We have also computed the charge stiffness, the localization length and the temperature dependence of the conductivity for the various phases. In the range of parameter where d-wave superconductivity would occur for the pure system the conductivity is found to decrease monotonically with temperature, even at high temperature, and we discuss this surprising result. For a model with one site repulsion and nearest neighbor attraction, the most stable phase is an orbital antiferromagnet. Although this phase has no divergent superconducting fluctuation it can have a divergent conductivity at low temperature. Finally, to make comparison of our results with experimental ladder systems, we treated the interladder coupling in a mean field approximation. We argue based on our results that the superconductivity observed in some of these compounds cannot be a simple stabilization of the d-wave phase found for a pure single ladder. Application of our results to systems such as quantum wires is also discussed. In particular the corrections to conductance in a two channel quantum wire have been obtained as a function of system length, temperature and interactions.

I. INTRODUCTION

Strongly interacting systems constitute nowadays one of the most challenging problem of condensed matter physics. In one dimension a fairly complete solution of the interacting problem can be obtained, and it is well known that one dimensional systems are one of the simplest realizations of non-Fermi liquids, and have generic properties known as Luttinger
liquids. Prompted by a variety of experimental situations ranging from organic conductors to High Tc superconductors, there has been in the recent years, a growing interest in systems of coupled interacting electron chains. Unfortunately, despite the good understanding of purely one dimensional systems, the effects of interchain hopping, allowing to go from one to higher (two or three) dimensions are much less known. Whether non-Fermi liquid properties can be retained even in presence of finite hopping or not is still a highly controversial issue.

Many studies have therefore focused on systems of few coupled chains (two coupled chains being the so-called ladder systems), for which much more controlled analytical or numerical techniques can be applied allowing for a deeper understanding of their physical properties. For commensurate filling, i.e. one electron per site, the system becomes equivalent to coupled spin chains, since the charge degrees of freedom are frozen by a Mott transition. Important differences between ladders with and even and odd number of legs were expected, in a way reminiscent of the Haldane conjecture between one-dimensional systems with integer and half integer spins. In particular ladders with an even number of legs were predicted to have a spin gap. Good experimental realizations of such coupled spin chains like Sr$_{n-1}$Cu$_n$O$_{2n+1}$ and VO$_2$P$_2$O$_7$ compounds have confirmed such behavior. Due to the presence of such a spin gap an even more spectacular effect is expected upon doping. At the opposite of single chains, that exhibit either a spin density wave or charge density wave ground state for repulsive interactions, the ladder system is believed to have a superconducting ground state involving pairing across the chains. That superconducting state has similarities with d-wave paring that has been advocated in some two dimensional models of strongly correlated electrons for High Tc superconductors such as the existence of a spin gap and a sign change of the superconducting order parameter when one moves on the “Fermi Surface”. In the strong coupling limit i.e. the t-J model, the d-wave phase can also be viewed as a RVB state.

However all the studies of ladder systems have been, up to now, restricted to pure systems. Unfortunately (or maybe fortunately) it is well known, that for one dimensional systems disorder has extremely strong effects. For a non-interacting system, it is well known that all states get localized in the presence of an infinitesimal random potential. Interactions can modify this picture, but for one chain system delocalization occurs only for strongly attractive interactions. In particular even normal s-wave superconducting phases are destroyed by non magnetic impurities except for exceedingly attractive interactions (see e.g. Ref. and references therein), and no Anderson’s theorem exists even for weakly coupled one-dimensional systems.

In order to compare the theoretical predictions of d-wave superconductivity in doped ladder systems with experiments, it is therefore of prime importance to understand the effects of disorder on the phase diagram of the pure ladder system. One of the important question is of course the stability of the newly found d-wave superconducting phase since there are no obvious reasons why it would survive the introduction of a small amount of disorder. Such a study is also relevant to the physics of quantum wires with few channels. In quantum wire systems, the situation is however complicated by the occurrence of long range Coulomb forces that can induce a one dimensional analog of the Wigner Crystal, which can drastically modify the response of the system to disorder. However, the presence of charges in the grids of the quantum wire systems can be cleverly used to screen completely the long range interactions, and have an experimental realization of a Luttinger liquid. By
changing the gate voltage it is possible to have more than one band at the Fermi level, in a controlled way. The quantum wire is thus a possible realization of a two (or more) leg ladder. Interband tunneling plays the role of interchain hopping. They provide ideal systems in which to check for the effect of disorder \[40,41\].

Besides the exciting possibility to test the ability of Luttinger liquid models to describe accurately the now available quasi one dimensional experimental systems, investigation of disorder effects in ladders presents in its own right a great theoretical interest. Indeed, the two chain problem is the simplest one to study the effects of interchain hopping onto the Anderson localization in presence of interactions, giving some clues onto this difficult topic in more than one dimension. In particular, one would be interested in obtaining boundaries between localized and delocalized phases and the dependence on localization lengths on disorder. Another question of particular interest is the effect of interactions on physical quantities controlled by disorder such as the conductivity for a macroscopic system, or for a mesoscopic one the persistent currents. In particular, for a one chain system, it was shown that for a system with spin degrees of freedom persistent current were enhanced by repulsive interactions, at variance on what happened for a spinless system. It is therefore important to check whether this striking results still holds in a more two dimensional system.

In this paper, we consider the effects of a weak random potential scattering on systems of coupled fermionic chains both with spin and spinless using bosonization and RG techniques. A short account of some of the results of this paper were presented in Ref. [47]. Besides giving the phase boundaries, the RG also provides us with expressions of the localization lengths, temperature dependence of conductivity, and dependence of persistent currents with system size. The plan of the paper is as follows:

In section II, we discuss the spinless fermions 2 legs ladder problem. We first recall the phase diagram of the pure system \[15\], then consider the effects of disorder. This allows for a detailed comparison of the transport properties of the 2 chain system with the ones of the one chain spinless fermion system and the ones of one chain of fermion with spin. We show that contrarily to naive expectations, the ladder spinless fermions system is very different from the one chain system with spin, and that the effect of interactions on persistent currents is even more violent on a two chain spinless fermions system than it was for a one chain system.

In section III, we discuss the technically more involved case of fermions with spins. Following the same methodology, we first recall the phase diagram of the non-random 2 chain system and then consider the effects of a weak random potential on the phase diagram. As for the spinless problem, we give a detailed discussions of the transport properties in the disordered phases. We compare these results with the ones already known for one chain, and show that the reduction of persistent current by attractive interactions that is observed in one chain of fermions with spin should be almost absent in the two chain system. The d-wave superconductivity of the fermionic 2 chain system is a feature that is not preserved in the presence of a very small amount of disorder. On the other hand for some values of the parameter, and orbital antiferromagnetic phase exists. This phase has an infinite conductivity even in the presence of disorder although it has exponentially decaying superconducting correlations.

In order to compare our results with experiments on doped ladder systems, it is necessary to treat interchain coupling which stabilizes superconductivity at finite temperature in real
systems, and may also reduce the sensitivity of the system to disorder. Thus in section IV we examine the mean field theory for the d-wave superconductor in an array of coupled disordered ladders. We give a criterion for persistence of superconductivity in the presence of disorder and show that d-wave superconductivity remains unstable except in very pure samples or in the presence of a very strong Josephson coupling between ladders.

In section V, we summarize the implications for experimental systems such as doped $SrCuO$ chains and quantum wire with two channels. We claim that in recently synthesized doped ladder systems, the physics of the superconducting phase is more likely to be of two dimensional origin rather than just a stabilization of a ladder d-wave superconductivity. Conclusions can be found in section VI. Finally most of the technicalities can be found in the appendixes.

II. SPINLESS FERMIONS

A. Pure system

Let us consider first two chains of spinless fermions coupled by an interchain hopping $t_\perp$. For simplicity we first consider only nearest neighbor interactions, and interchain interactions. The effect of more complicated interactions will be detailed below. The Hamiltonian for the pure system reads

$$H = -t \sum_{i,p} c_{i,p}^\dagger c_{i+1,p} + h.c. + V \sum_i n_{i,p} n_{i+1,p} + t_\perp \sum_i c_{i,1}^\dagger c_{i,-1} + h.c. + U \sum_i n_{i,1} n_{i,-1} \quad (1)$$

where $p = -1, 1$ is the chain index and $i$ is the site index. To treat the interactions, it is convenient to rewrite the Hamiltonian in term of boson operators. To do so, we linearize the fermions dispersion relation around $k_F$, introduce right (R) and left movers (L) for each chain, and take the continuum limit $c_{n,r,p} \rightarrow \sqrt{\alpha} \psi_{r,p}(x)$ with $r = L, R, p = \pm 1$ the chain index and $\alpha$ the lattice spacing. We use the bonding $\psi_o = \frac{\psi_{1} + \psi_{-1}}{\sqrt{2}}$ and anti-bonding $\psi_\pi = \frac{\psi_{1} - \psi_{-1}}{\sqrt{2}}$ bands base and introduce the densities $\rho_{r,o,\pi}(x) = \psi_{r,o,\pi}^\dagger(x) \psi_{r,o,\pi}(x)$. We then define the canonically conjugate fields $\phi_{\rho,\parallel}$ and $\Pi_{\rho,\parallel}$ via

$$\partial_x \phi_{\rho,\parallel} = -\frac{\pi}{\sqrt{2}}(\rho_{L,o} \pm \rho_{R,o} \pm \rho_{L,\pi} \pm \rho_{R,\pi}) \quad (2)$$
$$\Pi_{\rho,\parallel} = \frac{1}{\sqrt{2}}(\rho_{R,o} \pm \rho_{R,\pi} - \rho_{L,o} \mp \rho_{L,\pi}) \quad (3)$$

and the field $\theta_{\rho,\parallel}(x) = \int_{-\infty}^{x} \Pi_{\rho,\parallel}(x')dx'$. More details on the bosonization technique can be found in Appendix A. In term of these fields the Hamiltonian becomes:

$$H = H_\rho + H_{\parallel}, \quad H_\rho = \int \frac{dx}{2\pi} \left[ u_\rho K_\rho(\pi \Pi_\rho)^2 + \frac{u_\rho}{K_\rho} (\partial_x \phi_\rho)^2 \right] \quad (4)$$
$$H_{\parallel} = \int \frac{dx}{2\pi} \left[ u_{\parallel} K_{\parallel}(\pi \Pi_{\parallel})^2 + \frac{u_{\parallel}}{K_{\parallel}} (\partial_x \phi_{\parallel})^2 \right] + \int dt_\perp \sqrt{\frac{\gamma}{\pi}} \partial_x \phi_{\parallel}$$
The superconducting phases are the standard ones, given on the original model by

and g

with wavevector 2πk⊥ per unit length across the chains. In the orbital antiferromagnet currents go from one chain to the other, with wavevector 2kF, giving currents circulating around plaquettes of length π/kF. In fact (5) describes the most general two chain spinless system. More complicated (i.e. longer range and interchain interactions) lead only to a change in the parameters K, u and g. By adding interchain interactions such as U in formula (5) or longer range interactions one can in particular access the other regimes Kρ > 1 and gF < 0 or Kρ < 1 and gF > 0. The physics of the system is readily seen on (4). The t⊥ term suppresses cos(√8φ∥). Depending on the value of Kρ, the θ∥ can either remain massless or develop a gap. We concentrate here on the case where θ∥ develops a gap and acquires a non-zero expectation value determined by minimizing the ground state energy (see appendix A). This situation always occur for the t-V model. By mapping (4) on a problem of one chain of fermions with spin and spin-anisotropic interactions in a magnetic field, one can obtain the complete phase diagram for the pure case. Since, due to the one dimensional nature of the problem, no true ordered state exists, one has to find the most divergent instability. As for one chain, two main type of instabilities are possible: particle-hole (density, current etc.) instabilities or particle-particle (i.e. superconducting) ones. The operators with the most divergent susceptibilities are in boson form

\[ O_{\text{CDW}} = \psi_{R,1}^\dagger(x)\psi_{L,1}(x) - \psi_{R,-1}^\dagger(x)\psi_{L,-1}(x) \sim e^{i\sqrt{2}\theta_\parallel} \cos(\sqrt{2}\theta_\parallel) \]

\[ O_{\text{SC}} = \psi_{L,o}(x)\psi_{R,o} + \psi_{L,\pi}\psi_{R,\pi} \sim e^{i\sqrt{2}\theta_\parallel} \sin(\sqrt{2}\theta_\parallel) \]

\[ O_{\text{OAF}} = i(\psi_{R,1}^\dagger(x)\psi_{L,-1}(x) - \psi_{R,-1}^\dagger(x)\psi_{L,1}(x)) \sim e^{i\sqrt{2}\theta_\parallel} \sin(\sqrt{2}\theta_\parallel) \]

\[ O_{\text{SC'}} = \psi_{L,o}\psi_{R,\pi} - \psi_{L,\pi}\psi_{R,o} \sim e^{i\sqrt{2}\theta_\parallel} \cos(\sqrt{2}\theta_\parallel) \]

They describe respectively out of phase charge density waves, an orbital antiferromagnetic phase and chain symmetric “s” and antisymmetric “d” type superconductivity. The out of phase charge density has a 2kF modulation of the density along the chain and a change of sign across the chains. In the orbital antiferromagnet currents go from one chain to the other with wavevector 2kF, giving currents circulating around plaquettes of length π/kF. The superconducting phases are the standard ones, given on the original model by
\[ O_{SC^a}(n) = c_{n,1}c_{n,2} \]
\[ O_{SC^b}(n) = c_{n+1,1}c_{n,1} - c_{n+1,2}c_{n,2} \]

(6)

The most stable phase depends on the parameters \( K \) and \( g \). The various cases are given in Table 1 and the phase diagram shown in figure 1. In Ref. 15 the bosonized forms of \( O_{SC^a} \) and \( O_{SC^b} \) are exchanged due to the neglect of anticommuting operators (see appendix A), so that the two superconducting phases have been erroneously exchanged.

**B. Effects of disorder**

Now, we consider the effect of the disorder on (1-4). We introduce a random on-site potential \( \epsilon_{i,p} \) uncorrelated from site to site and from chain to chain:

\[
H_{\text{random}} = \sum_{p=\pm 1} \epsilon_{i,p} c_{i,p} \dagger c_{i,p}
\]

(7)

with \( \epsilon_{i,p} \epsilon_{j,p'} = D \delta_{i,j} \delta_{p,p'} \). In the continuum limit and using the bonding antibonding basis the disorder becomes

\[
H_{\text{random}} = \int dx \left[ \epsilon_s(x) (\psi^\dagger_0(x) \psi_0(x) + \psi^\dagger_\pi(x) \psi_\pi(x)) + \epsilon_a(x) (\psi^\dagger_0(x) \psi_\pi(x) + \psi^\dagger_\pi(x) \psi_0(x)) \right]
\]

(8)

with \( \epsilon_{s,a} = (\epsilon_1 \pm \epsilon_{-1})/2 \) and \( \epsilon_{\alpha,\beta}(x) \epsilon_{\beta,\alpha}(x') = \frac{D \alpha}{2} \delta(x - x') \delta_{\alpha,\beta} \). Using the expression of fermion operators defined in Appendix A and (4) one obtains for the disorder term

\[
H_{\text{random}} = \int dx \left[ \eta_s(x) \frac{\sqrt{2}}{\pi} \partial_x \phi_\rho(x) + \frac{\xi_s(x)}{\pi \alpha} e^{i \sqrt{2} \phi_\rho} \cos(\sqrt{2} \phi_\parallel) + \frac{\xi_s^*(x)}{\pi \alpha} e^{-i \sqrt{2} \phi_\rho} \cos(\sqrt{2} \phi_\parallel) \right]
\]

\[
+ \int dx \left[ \eta_s(x) \frac{\sqrt{2}}{\pi \alpha} \cos(\sqrt{2} \phi_\parallel) \cos(\sqrt{2} \theta_\parallel) + \frac{\xi_s(x)}{\pi \alpha} e^{i \sqrt{2} \phi_\rho} \cos(\sqrt{2} \phi_\parallel) + \frac{\xi_s^*(x)}{\pi \alpha} e^{-i \sqrt{2} \phi_\rho} \cos(\sqrt{2} \phi_\parallel) \right]
\]

(9)

where the disorder has been split in a \( q \sim 0 \) component (\( \eta_{s,a} \)) and a \( q \sim 2k_F \) one (\( \xi_{s,a} \)). As for one chain the \( \eta \) and \( \xi \) are uncorrelated and

\[
\eta_{s,a}(x) \eta_{s,a}(x') = D_{s,a} \delta(x - x')
\]

(10)

\[
\xi_{s,a}(x) \xi_{s,a}(x') = 0
\]

(11)

\[
\xi_{s,a}(x) \xi_{s,a}^*(x') = D_{s,a} \delta(x - x')
\]

(12)

The \( q \sim 0 \) (forward scattering) part of the disorder does not affect the conductivity and cannot lead to localization, but could in principle modify the phase diagram and in particular destroy the gaps of the pure phase. As for one chain, one can eliminate the \( \eta_\rho \partial_x \phi_\rho \) by a transformation \( \phi_\rho \rightarrow \phi_\rho + \frac{\sqrt{8} K}{a_\rho} \int dx \eta_\rho(x) \). The only effect of this term is therefore to give an additional exponential decay in the density-density correlation functions.

Due to the presence of a gap in \( \theta_\parallel \) (see table 1), the \( \eta_\rho(x) \cos(\sqrt{2} \phi_\parallel) \cos(\sqrt{2} \theta_\parallel) \) term is always suppressed at lowest order. It could however generate relevant terms at higher order.
However higher order terms are either identical to backscattering terms already present in the Hamiltonian, or adds random contributions to $g_\perp \cos \sqrt{8} \phi_\parallel$ and $g_f \cos \sqrt{8} \theta_\parallel$. At small disorder these contributions are negligible, and one can completely disregard the forward scattering. We can therefore keep only for the coupling to disorder $H_s + H_a$

$$H_s = \int \frac{dx}{\pi \alpha} \xi_s(x)e^{i\sqrt{2} \phi_\parallel} \cos(\sqrt{2} \phi_\parallel) + \text{h.c.}$$

$$H_a = \int \frac{dx}{\pi \alpha} \xi_a(x)e^{i\sqrt{2} \phi_\parallel} \cos(\sqrt{2} \theta_\parallel) + \text{h.c.}$$

In $H_s$ the symmetric part of the disorder couples to the in-phase charge density wave order parameter $O_{CDW} = \frac{e^{i\sqrt{2} \phi_\parallel}}{\pi \alpha} \cos(\sqrt{2} \phi_\parallel)$, whereas the antisymmetric part involves $O_{CDW}^\ast$. Due to the gap in $\theta_\parallel$, $\phi_\parallel$ has huge quantum fluctuations, and consequently the symmetric part of the disorder $D_s$ is always less relevant than the antisymmetric one $D_a$. We can therefore focus on the latter and forget about the former. The effect of (14) again depends on the values of $g_f$ and $K$.

1. $g_f < 0$

For $g_f < 0$ (i.e. $V > 0$ for the t-V model) we can replace $\cos(\sqrt{2} \theta_\parallel)$ by its (non-zero) mean value and the coupling to disorder (14) reduces to $C \int dx \xi_a(x)e^{i\sqrt{2} \phi_\parallel(x)} + \text{h.c.}$, where $C$ is a constant. The effect of such a term can be determined, as for a single chain, by using a renormalization group (RG) procedure. Upon varying a cutoff $\alpha$, similar to a lattice spacing in the original lattice problem, one find the following renormalization for the disorder

$$\frac{dK_\rho}{dl} = -C_2 D_a$$

$$\frac{dD_a}{dl} = D_a(3 - K_\rho)$$

where $l = \ln(\alpha)$ and $C_2$ a constant. (13) implies a localization-delocalization transition at $K_\rho = 3$. For $K_\rho > 3$ the disorder is irrelevant and the corresponding phase in the pure system is stable. For $K_\rho < 3$ disorder grows. Although the system flows to a strong coupling fixed point, it is natural to interpret this phase as localized by disorder, since the disorder will pin the massless field $\phi_\rho$. As a consequence, the d-wave superconducting phase is unstable in the presence of disorder except for huge attractive interactions. In the case of the t-V model at $V > 0$, we have $K_\rho < 1$ and therefore the $CDW^\ast$ is always pinned by the disorder.

Similarly to the one chain problem the localization length can be computed using the RG. For very weak disorder and far from the transition one can neglect the renormalization of the exponent $K_\rho$ induced by $D_a$. Using that approximation, we obtain:

$$D_a(l) = e^{(3 - K_\rho)l} D_a(0)$$

For $D_a(l) \sim v_F^2/\alpha$ that scheme breaks down and we have a strongly disordered system. For such a system the localization length, i.e. the scale of variation of the phase $\phi_\rho$ is of the order of the (renormalized) lattice spacing $\alpha^\ast$. This occurs for $e^{l^\ast} \sim \left(\frac{v_F^2}{D_a(0)\alpha}\right)^{\frac{3}{3 - K_\rho}}$. Therefore
Let us recall that for a non-interacting system, the localization length is of the order of the mean free path i.e. $L_{\text{loc.}} \sim \frac{v_F}{D\alpha}$.

Using the renormalization equation it is also possible to obtain the temperature dependence of the conductivity for temperatures above the pinning temperature $u/L_{2 \text{ ch.}}$. Below the pinning temperature, the conductivity is expected to decrease as $\exp - \left( \frac{T_{\text{pin.}}}{T} \right)^\mu$, by analogy with non-interacting electrons. A derivation of the temperature dependence of conductivity has been given in Ref. [36]. Another method to derive the temperature (or frequency) dependence of the conductivity is given in appendix B. If one neglects the renormalization of the exponents the conductivity behaves as

$$\sigma(T) \sim T^{2-K_\rho}$$

Therefore, for $K_\rho < 2$, the conductivity decreases, and there is no remnant of any superconducting behavior effect well above the temperature at which the system is effectively pinned $T_{\text{pin.}} \sim u/L_{2 \text{ ch.}}$. Thus the existence of d-wave superconductivity in the pure system affects the transport properties of the disordered system only for quite large attraction. Analogous effects will occur for fermions with spins as will be discussed in section [III].

2. $g_f > 0$

For $g_f > 0$ (i.e. attractive interactions for a t-V model), $\langle \theta_{\parallel} \rangle = \frac{\alpha}{\sqrt{8}}$ and in a first approximation the coupling (14) vanishes. Obviously, this approximation is too crude and one must integrate the fluctuations of $\theta_{\parallel}$ around its mean value to get the effective coupling. This is done in appendix C and gives the following effective action for $\phi_\rho$:

$$S_{\rho} = \int dx d\tau \left[ \frac{(\nabla \phi_\rho)^2}{2\pi K_\rho} + (\xi(x) e^{i\sqrt{8} \phi_\rho(x,\tau)} + \text{h.c.}) \right]$$

with $\xi(x) \xi^*(x') = D \delta(x-x')$ and $D \sim D_0^2$.

The renormalization of the disorder is given by an equation similar to (16):

$$\frac{dD}{dl} = (3 - 4K_\rho)D(l)$$

The disorder is now relevant only for $K_\rho < 3/4$, leading to three different phases for $g_f > 0$: a random orbital antiferromagnet for $K_\rho < 3/4$, an ordered orbital antiferromagnet for $3/4 < K_\rho < 1$ and a s-wave superconducting phase for $K_\rho > 1$. For the t-V model, $K_\rho > 1$, and the “s”-wave superconducting phase is therefore stable with respect to weak disorder, at variance to the single chain problem. For the latter the delocalization only occured for extremely attractive interactions i.e. $K_\rho > 3/2$. For the two chains problem the localization-delocalization transition arises in the immediate vicinity of the non-interacting point. Contrarily to the case of repulsive interactions, interchain hopping now strongly reduces the localization effects.
The localization length in the random orbital antiferromagnet, is now given by

\[
\frac{L_{2\ ch.}}{\alpha} = \left(\frac{1}{D}\right)^{4K\rho} = \left(\frac{v_F^2}{D\alpha}\right)^{\frac{1}{2-4K\rho}}
\]  

(22)

The conductivity behaves both in the OAF and the s-wave phase as

\[
\sigma(T) \sim T^{2-4K\rho}
\]  

(23)

diverges as \( T \to 0 \) since the ground state is superconducting. It is to be noted that although the OAF has no superconducting order parameter, its conductivity can also be divergent for \( K_\rho > 3/4 \) even in the presence of disorder. An expanded discussion of Orbital Antiferromagnet phases can be found in \([\text{III C 1]}\) and appendix \([\text{D}].\)

The resulting phase diagram is summarized on figure 2, together with the single chain phase diagram.

**C. Physical consequences**

The ladder system shows drastically different sensitivity to disorder depending on the sign of \( g_f \): at \( g_f \leq 0 \) localization effects are much stronger than at \( g_f > 0 \). This is obvious both on the phase diagram shown on figure 2 and on the expression (18) and (22) for the localization length. For the case of a pure \( t-V \) model, \( g_f > 0 \) \( K_\rho > 1 \) when \( V < 0 \) (attractive interactions) and as can be seen from figure 2 the system is delocalized. Although our calculation do not allow us to come arbitrarily close to the \( V = 0 \) point for finite disorder, since the disorder has to be smaller than the gaps of the pure system, we see that if we have a very small disorder, the insulator superconductor transition does occur in the vicinity of the non-interacting point. This is remarkable and in marked contrast with the single chain system where the delocalization transition occurs for \( K = 3/2 \) i.e. very strongly attractive interactions even for arbitrarily weak disorder. One could naively think that this effect is simply a manifestation of the delocalization effect seen for non-interacting electrons when one increases the number of channel (or the number of chains). The mechanism is more subtle however, and in in fact controlled by the interactions. Contrarily to the noninteracting case where the localization length is simply proportional to the number of chains, we have here a **complete** delocalization of the attractive region, and the localization length becomes infinite.

For the repulsive case \( V > 0 \) (i.e. \( g_f < 0 \) \( K_\rho < 1 \)) the opposite effect occurs and ladder system is **more** localized than the corresponding one chain system. Indeed for one chain the localization length is given by

\[
\frac{L_{1\ ch.}}{\alpha} \sim \left(\frac{v_F^2}{D\alpha}\right)^{\frac{1}{2-4K\rho}}
\]  

(24)

and is therefore longer than the one of the ladder system shown in (18). For very large repulsion \( (K \to 0) \) these two length give back the standard Fukuyama-Lee pinning length of classical charge density waves. For finite repulsion the localization length of the ladder
system is much shorter than the one of the corresponding one-dimensional system with the same $K$. Close to the non interacting point $K \sim 1$, the localization length of open chain is just the mean free path $L_{1,\text{ch.}} \sim v_F^2/D$ whereas the ladder one is $L_{2,\text{ch.}} \sim \alpha \sqrt{v_F^2/D\alpha}$.

This peculiar behavior of the spinless ladder system is due to the gap of some charge modes, that is different depending on whether the interaction is attractive or repulsive. For the repulsive side $2k_F$ charge fluctuations are still there and the gap just reduces some of the quantum fluctuation and hence reinforce the effects of disorder, whereas for the attractive side the gap kills the dominant charge fluctuation coupled to disorder and helps to delocalize. The sensitivity to disorder is therefore not directly related to the presence or absence of superconducting fluctuations in the pure system, but more on how the density fluctuations behave. The smoother are the density fluctuations, the less localized the system is. These effects will be even more transparent for the system with spins as will be examined in details in section III. As a consequence the transport properties cannot simply be guessed by looking at the phase diagram of the pure system. They even can be opposite to what our intuition based on higher dimensional system could suggest: the more “superconducting” the system is the better is the transport (see e.g. section III C).

D. Persistent currents in the ladder system

In addition to the temperature dependence of the conductivity, one can compute the charge stiffness of the system $D$, which measures the strength of the Drude peak in a macroscopic system $\sigma(\omega) = D \delta(\omega) + \sigma_{\text{reg}}$. The stiffness $D$ can be related to the change of the energy of the ground state of the system in presence of an external flux by

$$ D = \frac{L}{2} \left. \frac{d^2 E_0}{d\phi^2} \right|_{\phi=0} $$

(Eq. 25)

$E_0$ being the ground state energy of a ring in a field. $\phi$ denotes the boundary angle $\phi = 2\pi f/f_0$ where $f$ is the flux threading the ring and $f_0 = \hbar c/e$ is the flux quantum. This quantity is directly related to the persistent currents for a mesoscopic system $J = L \left. \frac{dE_0}{d\phi} \right|_{\phi}$ (Eq. 26)

Therefore the stiffness $D$ provides a measure of the persistent currents for small (or close to a multiple of $2\pi$) flux since $J = 2D\phi$. Although the complete calculation of the persistent currents at finite flux is also possible for a one dimensional interacting system, the calculation is more complicated in the presence of disorder, and the stiffness carries enough information for our present purposes.

The effects of interactions on persistent currents is an extremely difficult question to answer in two or three dimensions. Perturbative calculations suggests that interactions could enhance persistent currents $J$. For a single spinless chain the persistent currents were found to decrease with more repulsive interactions. This effect can naturally be explained using a renormalization group technique, and it was shown that such behavior is
peculiar to the spinless problem and that for a single chain of electrons with spins persistent currents should be enhanced by repulsive interactions. For the ladder system it is therefore very interesting to see if the same effects occur and in particular to check again for the differences between the spinless system and the system with spins. In particular one could imagine that the chain index acts in a similar way than a spin index for a single chain. As we will see this idea is far too naive. We examine the spinless system in this chapter and the system with spins will be investigated in chapter III.

For the ladder system, the conductivity stiffness is obtained using (A13) as $D = 2u_\rho K_\rho$. The factor of two compared to the single chain expression (A13) is due to the fact that there are twice as much degrees of freedom in the 2 chain system. In the following, we consider a finite system, the size $L$ of which is smaller than the localization length.

From the renormalization group equation for $u_\rho, K_\rho$, one can obtain the renormalization group equation for $D$

\[
\frac{dD}{dl} = -D(l)
\]

\[
\text{(27)}
\]

The conductivity stiffness of a disordered system of size $L$, $D(L)$ is then obtained by stopping the RG equation at $\alpha(l) = L$ and taking $D(L) = D(l)$. In the case $g_f < 0$, we have seen that $D(l) = D(0)e^{(3-K_\rho)l}$, at least when $\alpha(l) \ll L_{2\text{ch.}}$. Putting that approximation for $D(l)$ in (27) gives us:

\[
D(L) = D(0) - CD(0) \left[ \left( \frac{L}{\alpha(0)} \right)^{3-K_\rho} - 1 \right]
\]

\[
\text{(28)}
\]

Using the expression for $L_{2\text{ch.}}$, (28) simplifies for length smaller than the localization length into

\[
D_{g_f < 0}(L) = D(0) - C \left[ \left( \frac{L}{L_{2\text{ch.}(g_f < 0)}} \right)^{3-K_\rho} - 1 \right]
\]

\[
\text{(29)}
\]

\[
D_{g_f > 0}(L) = D(0) - C' \left[ \left( \frac{L}{L_{2\text{ch.}(g_f > 0)}} \right)^{3-4K_\rho} - 1 \right]
\]

\[
\text{(30)}
\]

Thus for $g_f > 0$ the reduction of the stiffness is less important than for $g_f < 0$.

Therefore, the length dependence of the conductivity stiffness (and the persistent currents) is extremely sensitive to the attractive or repulsive character of the interactions for the t-V model or any model with intrachain-only interactions. By comparison with the one chain case, we see that the effects of the interactions on the conductivity stiffness are qualitatively the same (i.e. repulsive interactions help in reducing the conductivity stiffness, while attractive interactions reduce the decrease of conductivity stiffness by disorder) but they are much stronger for two chains than for one chain. In fact, for a t-V model, the reduction of conductivity stiffness would be finite for attractive interactions, even in an infinite system since then the disorder is completely irrelevant.

It is noteworthy that the chain index does not act in a similar way as a spin degree of freedom, for which there would be an increase of the persistent currents showing again the important difference between a system with and without spin. The physical reasons for this difference are examined in more details in the next section.
E. Spinless ladder vs. one chain with spin

Naively, one could think that going from one chain to two chains amounts to having one internal degree of freedom that is equivalent to spin, and thus that the results for the system with spin will apply straightforwardly to the ladder system. However, from what we have seen precedingly, this is definitely not the case. In fact, we have properties for the spinless ladder that are just the contrary of the ones of the fermions with spin. Attractive interactions delocalize in the spinless fermions case, whereas they increase localization in the case of fermions with spin. Persistent currents are enhanced for more attractive interactions in the spinless ladder whereas repulsive interactions would enhance the persistent currents in a spin system. The reason for that is that the spinless ladder has no SU(2) symmetry (except for $V = 0$) contrarily to one chain with spin. The minimum of the ground state energy of the spinless ladder corresponds to states that break the SU(2) symmetry because $t_\perp$ plays the role of a magnetic field. Thus such phases cannot be obtained in an isotropic system of fermions with spin.

For attractive interactions, the only way for the symmetric fermions with spin system to preserve SU(2) symmetry is to form singlet phases such as $2k_F$ charge density waves or singlet superconducting state. Coupling the charge density wave fluctuations with a random potential implies strong localization effects. On the other hand, the spinless ladder simply form pairs along the chains and can avoid to form $2k_F$ fluctuations. Translated in the spin language, such phase would be an anisotropic triplet superconductor with a spin gap, and would be forbidden by symmetry. In the same way, for repulsive interactions, preserving SU(2) symmetry prevents the formation of a gap, whereas a gap formation is possible for the spinless ladder giving an out of phase charge density wave. In the spin language, this corresponds to an anisotropic SDW.

Adding random potentials to the spinless ladder results in a rather artificial model of fermions in a random potential and a random field parallel to the z axis. Because of the anisotropy, the system is more sensitive to the random field parallel to the z axis than to the random potential. Thus, for repulsive interactions, the anisotropic system has a very strong coupling to disorder, whereas for repulsive interactions, it is only weakly coupled. On the other hand, the isotropic system is only feeling a random potential. When interactions are attractive, there is a spin gap and CDW fluctuations that can couple to disorder, making the system more localized. When interactions are repulsive, on the other hand, there is no spin gap thus reducing the coupling of the CDW fluctuations with disorder.

We conclude that for interacting systems, contrarily to their non-interacting counterparts, not only the number of available internal degrees of freedom but also the internal symmetries determine the response to random perturbations. Loosing some symmetries allows for a larger variety of ground states, and thus to very different responses to weak perturbations.

III. FERMIONS WITH SPIN
A. Pure system

The pure case has been analyzed in great details both analytically and numerically. A very interesting feature of that model is the existence of a "d-wave" superconducting phase for purely repulsive interactions and the existence of a spin gap. The Hamiltonian is in the extended Hubbard case:

\[ H = -t \sum_{i,\sigma} c_{i+1,\sigma,p}^\dagger c_{i,\sigma,p} + H.\ c. - t_\perp \sum_{i,\sigma,p} c_{i,\sigma,p}^\dagger c_{i,\sigma,-p} + U \sum_{i,p} n_{i,\uparrow,p} n_{i,\downarrow,p} + V \sum_{i,p} n_{i,p} n_{i+1,p} \]  

(31)

with \( p = \pm 1 \) is the chain index and \( \sigma = \uparrow, \downarrow \) labels the spin. In order to treat this Hamiltonian using bosonization one has to separate the bonding and antibonding \( \pi \) bands as was done for spinless fermions. Then, within each band, one can apply the standard bosonization formulas for fermions with spins. As a consequence, the system is described by 4 fields \( \phi_{\rho,\pi},\phi_{\sigma,\rho} \) instead of 2 in the spinning case. For the pure case we follow closely the derivation of Ref. 12. It is convenient in the following to replace the fields \( \phi_{\nu}^{\rho,\pi} (\nu = \rho, \sigma) \) by linear combinations: \( \phi_{\nu,\pm} = \frac{1}{\sqrt{2}} (\phi_{\nu,\rho} \pm \phi_{\nu,\pi}) \) The low energy physics depends on the signs of two constants \( g_1, g_2 \). Physically, \( g_2 \) represents the forward scattering interaction, while \( g_1 \) represents the backward scattering interactions. The Hamiltonian consists of a free part:

\[ H = \sum_{\nu=\rho,\sigma} \int \frac{dx}{2\pi} \left[ u_{\nu,\rho} K_{\nu,\rho} (\pi \Pi_{\nu,\rho})^2 + \frac{u_{\nu,\rho}}{K_{\nu,\rho}} (\partial_x \phi_{\nu,\rho})^2 \right] \]  

(32)

and two sine-Gordon like part, one associated with interband processes induced by intrachain forward scattering:

\[ H_{\text{int},2} = \frac{g_2}{2(\pi \alpha)^2} \int dx \cos 2\theta_{\rho-} (\cos 2\phi_{\sigma-} + \cos 2\theta_{\sigma-}) \]  

(33)

The other associated with the intrachain backward scattering:

\[ H_{\text{int},1} = \frac{2g_1^2}{(2\pi \alpha)^2} \int dx \left[ \cos 2\phi_{\sigma+} (\cos 2\theta_{\rho-} + \cos 2\phi_{\sigma-} + \cos 2\theta_{\sigma-}) - \cos 2\theta_{\rho-} \cos 2\theta_{\sigma-} \right] \]  

(34)

In all cases, only one of the four bosonic fields \( (\phi_{\rho+}) \) is gapless and all physical quantities depend on a parameter \( K_{\rho+} \) of the symmetric charge mode, analogous to the \( K_\rho \) of the spinless problem. In terms of \( g_1, g_2, K_{\rho+} \) is given by:

\[ K_{\rho+} = \left( \frac{2\pi v_F + (g_1 - 2g_2)}{2\pi v_F - (g_1 - 2g_2)} \right)^{1/2} \]  

(35)

That expression is valid for the generic g-ological model. For the extended Hubbard model, we can go further as \( g_1, g_2 \) can be expressed in terms of \( U, V, k_F \) as:

\[ g_1 = U a + 2V a \cos (2k_F a) \]

\[ g_1 - 2g_2 = -(U a + 2V a (2 - \cos (2k_F a))) \]  

(36)
where \( a \) is the lattice spacing. The mean values of the three other fields are determined by minimizing the energy of the ground state. Depending on the interactions one can distinguish four sectors that are summarized in table II.

As for the spinless case one has to consider the various operators with divergent susceptibilities

\[
O_{\text{CDW}}(n) = \sum_{p,\sigma} p c_{n,\sigma,p}^\dagger c_{n,\sigma,p} \tag{37}
\]

\[
O_{\text{OAF}}(n) = \sum_{p,\sigma} p c_{n,\sigma,p}^\dagger c_{n,\sigma,-p} \tag{38}
\]

\[
O_{\text{SC}^s}(n) = \sum_{p} c_{n,\sigma,p} c_{n,-\sigma,p} \tag{39}
\]

\[
O_{\text{SC}^d}(n) = \sum_{p} c_{n,\sigma,p} c_{n,-\sigma,-p} \tag{40}
\]

When taking the continuum limit these expressions become

\[
O_{\text{CDW}} = \sum_{\sigma} (\psi_{L1\sigma}^\dagger \psi_{R1\sigma} - \psi_{L-1\sigma}^\dagger \psi_{R-1\sigma}) \tag{41}
\]

\[
O_{\text{OAF}} = i \sum_{\sigma} (\psi_{L1\sigma}^\dagger \psi_{R-1\sigma} - \psi_{L-1\sigma}^\dagger \psi_{R1\sigma}) \tag{42}
\]

\[
O_{\text{SC}^s} = \sum_{\sigma} (\psi_{L0\sigma}^\dagger \psi_{R0,-\sigma} + \psi_{L\pi\sigma}^\dagger \psi_{R\pi,-\sigma}) \tag{43}
\]

\[
O_{\text{SC}^d} = \sum_{\sigma} (\psi_{L0\sigma}^\dagger \psi_{R0,-\sigma} - \psi_{L\pi\sigma}^\dagger \psi_{R\pi,-\sigma}) \tag{44}
\]

where for the SC operators, one has to retain the \( q \approx 0 \) component, whilst for the OAF and \( \text{CDW}^\pi \) the \( q \approx 2k_F \) component gives the dominant contribution. To get the correct bosonized expression one has to pay extra care to the anticommuting \( U \) operators\( ^{67} \) and one obtains

\[
O_{\text{CDW}^\pi} = \frac{2}{\pi \alpha} e^{i\phi_{\pi}^+} \cos \phi_{\sigma^+} \sin \theta_{\sigma^-} \tag{45}
\]

\[
O_{\text{OAF}} = \frac{2i}{\pi \alpha} e^{i\phi_{\pi}^+} \sin \phi_{\sigma^+} \cos \theta_{\sigma^-} \tag{46}
\]

\[
O_{\text{SC}^s} = \frac{2}{\pi \alpha} e^{-i\theta_{\pi^+}} \cos \phi_{\sigma^+} \cos \phi_{\sigma^-} \tag{47}
\]

\[
O_{\text{SC}^d} = \frac{2}{\pi \alpha} e^{-i\theta_{\pi^+}} \sin \phi_{\sigma^+} \sin \phi_{\sigma^-} \tag{48}
\]

From the bosonized form of these operators (simplified by the fact that \( \langle \theta_{\pi^-} \rangle = 0 \) everywhere and the expressions given in table II) one can deduce that sector I is a \( \text{SC}^d \) phase, sector II an \( \text{OAF} \) phase, sector III a \( \text{SC}^s \) phase and sector IV a \( \text{CDW}^\pi \) phase. The phase diagram of the pure system is summarized in figure 4. Note that for the pure Hubbard model, which corresponds to \( V = 0 \) in (36), one can only have the \( \text{SC}^d \) phase (for \( U > 0 \)) or the \( \text{SC}^s \) phase (for \( U < 0 \)). The other phases could be obtained for a more general model such as the extended Hubbard model. We will come back to that point later.
B. Effects of disorder

Let us now add a weak random on-site potential:

$$H_{\text{random potential}} = \sum_{i,\sigma,p} \epsilon_{i,p} n_{i,\sigma,p}$$  \hfill (49)$$

with $n_{i,p} = c_{i,\uparrow}^\dagger c_{i,\uparrow} + c_{i,\downarrow}^\dagger c_{i,\downarrow}$ and $\epsilon_{i,p} \epsilon_{j,p'} = D \delta_{i,j} \delta_{p,p'}$. We go through the same steps as in the spinless fermions section. We got to the continuum limit, introduce the bonding and antibonding band, and bosonize the resulting coupling to disorder. Let us first consider the $q \sim 0$ part of the coupling to disorder. For the symmetric part of the disorder this coupling is of the form:

$$H_{s,q\sim 0} = \int \eta_s(x) \partial_x \phi_\rho(x) dx$$ \hfill (50)$$

It is clear that this part of the disorder can be eliminated by the transformation $\phi_\rho(x) \rightarrow \phi_\rho(x) + \int x \frac{\pi K_{\alpha\rho}}{n_{\rho}} \eta_s(x') dx'$ For the $q \sim 0$ part of the antisymmetric random potential, we obtain:

$$H_{a,q\sim 0} = \int dx \eta_a(x) \sum_\sigma \left[ \psi_{R,0,\sigma}^\dagger \psi_{R,\pi,\sigma} + \psi_{L,0,\sigma}^\dagger \psi_{L,\pi,\sigma} + \text{ H. c.} \right]$$ \hfill (51)$$

The bosonized form of that operator is the following:

$$H_{a,q\sim 0} = \int dx \frac{\eta_a(x)}{\pi \alpha} \left[ e^{i(\phi_{\rho+} + \theta_{\rho+})} \cos(\phi_{\sigma-} + \theta_{\sigma-}) + e^{i(-\phi_{\rho+} + \theta_{\rho+})} \cos(\phi_{\sigma-} - \theta_{\sigma-}) + \text{ H. c.} \right]$$ \hfill (52)$$

From that equation, we see that the $q \sim 0$ part of the antisymmetric disorder is not coupled to the gapless charge symmetric mode. Moreover, it always contain one term that has exponentially decaying correlations. Therefore, it cannot break any gap by an effect à la Imry Ma and cannot generate any relevant term by a massive mode integration. It will thus be possible to drop it safely in the following. Then, we have to consider the $2k_F$ part of the disorder. We have for the $2k_F$ coupling to disorder two terms:

$$H_a = \int \xi_a(x) O_{\text{CDW}^+}(x) + \xi_a^*(x) O_{\text{CDW}^-}(x) dx$$ \hfill (53)$$

$$H_s = \int \xi_s(x) O_{\text{CDW}^o}(x) + \xi_s^*(x) O_{\text{CDW}^o}(x) dx$$ \hfill (54)$$

Where $\xi_n(x) \xi_{n'}(x')^* = D \delta_{n,n'} \delta(x - x')(n,n' = a,s)$, the $\xi_n$ being random Gaussian distributed potentials. The operators $O_{\text{CDW}^o}$ represents the in-phase charge density wave, and $O_{\text{CDW}^+}$ the out of phase one.

As before we assume that the disorder weak enough not to destroy the gaps in the system. We have already argued that the $q \sim 0$ is irrelevant to our problem. Concerning the $2k_F$ part, we only retain the massless mode. The situation is quite similar to the one of a XXZ spin chain in a random magnetic field. A XXZ spin chain is a Hubbard chain at half filling, and thus has a charge gap. The random magnetic field couples to the spin density that contains (frozen) charge degrees of freedom. However, the random magnetic
field only affects the spin degrees of freedom and does not break the charge gap. By analogy, we expect that even when the random potential gets relevant it will not break the spin gap or the gap in the antisymmetric charge mode. Since the gaps are stable, we can obtain simplified forms for the couplings by replacing the fields by their mean values as we did in the spinless fermions problem.

1. SC$d$ sector

We want to analyze the effect of the weak random potential introduced through (53-54). Making use of the full expressions of $O_{CDW^{o,\pi}}$ and replacing the gapped fields by their mean values (see sector I of table II) we obtain the following simplified forms:

\[ O_{CDW^o} \sim e^{i\phi_{\rho^+}} \sin(\phi_{\rho^-}) \]
\[ O_{CDW^{\pi}} \sim e^{i\phi_{\rho^+}} \sin(\theta_{\sigma^-}) \cos(\phi_{\sigma^+}) \]

These two operators have exponentially decaying correlation functions and no direct coupling with disorder would exist if one just took into account the mean values of the fields $\phi_{\rho^-}$ and $\theta_{\sigma^-}$. As in the spinless case one should integrate over fluctuations to get the effective coupling

\[ S_{\rho^+}^{\text{disorder}} = \int \xi_{\text{eff}}(x)e^{i2\phi_{\rho^+}(x,\tau)}dx d\tau + \text{H. c.} \]  

(57) can be viewed as the coupling of the fermions with the $2(k_{F_o} \pm k_{F_\pi})$ Fourier component of the disordered potential i.e. to a $4k_F$ charge density wave. The origin for such a $4k_F$ charge density wave can be understood in simple terms: at half filling, the strong on site repulsion puts one fermion per site, meaning that there are no $2k_F$ CDW fluctuations. However, the fermion density is maximum on the lattice site and minimums in between giving the $4k_F$ charge density wave fluctuations. In addition due to the spin gap occurring in a ladder with an even number of legs there are no $2k_F$ fluctuations in the spin density as well. As we move away from half filling, the spin gap will survive as well as the absence of $2k_F$ fluctuations. Therefore, a random potential can only couple to the $4k_F$ component of the fermion density even away from half filling. This is to be contrasted to the case of a single chain where the dominant coupling occurs through the $2k_F$ charge fluctuation. One thus expects the disorder effects to be weaker in the ladder system. One can also recover directly the $4k_F$ CDW by looking at higher Fourier components of the density in the bosonization formulas. The physics of the metal insulator transition can be here interpreted as the pinning-depinning transition of this $4k_F$ charge density wave.

Due to the presence of the gaps, the problem has in fact been formally reduced to a problem of one chain of spinless fermions with disorder. Using the results from the one chain problem we find that the localization-delocalization occurs at $K_{\rho^+} = 3/2$. Since purely repulsive interaction imply $K_{\rho^+} < 1$ the $d$-wave phase is therefore unstable to arbitrarily weak disorder. The symmetric (54) and the antisymmetric (53) part of the disorder contribute equally to destroy the $d$-wave superconductivity, in contrast with the spinless case where the antisymmetric part was the most relevant. The localization length of the 2 chain
system with spin and purely repulsive interactions can be obtained by a similar method than for the spinless case and is

$$\frac{L_{2 \text{ ch.}}}{\alpha} \sim \left( \frac{v_F^2}{D\alpha} \right)^{2/(3-2K_{\rho+})}$$  \hspace{1cm} (58)

and therefore longer than the corresponding one for one chain with repulsive interactions\textsuperscript{53,54}

$$\frac{L_{1 \text{ ch.}}}{\alpha} \sim \left( \frac{v_F^2}{D\alpha} \right)^{1/(2-K_{\rho+})}$$  \hspace{1cm} (59)

As for the spinless case (58) is applicable if one is far enough from the noninteracting point so that disorder does not destroy the gaps created by the interactions. In that case one sees from (58) that there is a considerable delocalization in the ladder. Indeed for weakly repulsive interactions $K_{\rho+} \sim 1$, the localization length becomes much longer than the mean free path $l$, since $L_{2 \text{ ch.}} \sim \alpha(l/\alpha)^2$, instead of $L \sim l$ for a single chain. However the more repulsive the interactions become, the more the system localizes (one recovers $L_{2 \text{ ch.}} \sim l$ for $K = 1/2$).

The temperature dependence of the conductivity can be obtained above the pinning temperature $T_{\text{pin.}} = \frac{u_{\rho+}}{L_{2\text{ ch.}}}$ (see appendix B). One gets

$$\sigma(T) \propto T^{2-2K_{\rho+}}$$  \hspace{1cm} (60)

For $K_{\rho+} < 1$, the conductivity decreases as $T \to 0$ even for temperatures much higher than $T_{\text{pin.}}$. There is no remnant of the “superconducting” behavior of the pure system in the whole $SC^d$ sector ($K_{\rho+} < 1$).

2. $SC^s$ sector

For sector III, the $O$ operators take a different simplified form, due to the different gaps in the system

$$O_{CDW^o} \sim e^{i\phi_{\rho+}} \cos(\phi_{\rho-})$$  \hspace{1cm} (61)

$$O_{CDW^e} \sim e^{i\phi_{\rho+}} \sin(\theta_{\sigma-})$$  \hspace{1cm} (62)

By substituting in (53) and (54) and integrating over fluctuations we end with an action of the form (57). This time, $K_{\rho+} > 1$, so the localization-delocalization transition can be reached at $K_{\rho+} = 3/2$. This transition arises for much weaker attraction than in the one dimensional case\textsuperscript{53} where $K_{\rho} = 3$. This critical value of $K$ can be realized for a simple Hubbard model (the maximum $K$ for the Hubbard model is $K = 2\textsuperscript{53,46}$ whereas the one chain Hubbard model is always localized even for very negative $U$\textsuperscript{46}). In addition the localization length is increased

$$\frac{L_{2 \text{ ch.}}}{\alpha} = \left( \frac{v_F^2}{D\alpha} \right)^{\frac{2}{3-2K_{\rho+}}}$$  \hspace{1cm} (63)
whereas in the one chain case

$$\frac{L_{1, \text{ch.}}}{\alpha} = \left(\frac{v_F^2}{D\alpha}\right)^{-1}K_{\rho}$$  \hspace{1cm} (64)$$

Note that here the localization length has the same dependence in disorder on the attractive (63) and the repulsive (58) side, whereas for a single chain the localization length is **reduced** on the attractive side due to the formation of a spin gap (compare (64) and (59)). For the ladder this come from the fact that both in the attractive and repulsive side, three of the modes are always gapped.

The conductivity above the pinning temperature behaves as

$$\sigma(T) \sim T^{2-2K_{\rho+}}$$  \hspace{1cm} (65)$$

with again the same exponent than in the d-wave sector (60). However, since now $K_{\rho+} > 1$ the conductivity now **decreases** with decreasing $T$. There will thus be for $1 < K_{\rho+} < 3/2$ a maximum in the conductivity for $T \sim T_{\text{pin.}}$, and the resistivity will go to zero for high values of $K_{\rho+}$. This maximum can be seen as a remnant of the superconducting behavior of the pure system. For $K_{\rho+} > 3/2$, the system has infinite conductivity for $T \to 0$.

### 3. CDW$^\pi$ sector

Let us now consider sector IV. In that sector, one has strong fluctuations towards a CDW$^\pi$ phase. Such phase is the analog of the CDW$^\pi$ that existed in the spinless fermion problem. We see that the coupling to disorder reduces to (see table [I])

$$\int dx \xi_a(x)e^{i\varphi_{\rho+}} + \text{H. c.}$$  \hspace{1cm} (66)$$

As in the spinless fermion case that antisymmetric Charge density Wave only couples to the antisymmetric disorder. The RG equation for disorder is

$$\frac{dD_a}{dl} = \left(3 - \frac{K_{\rho+}}{2}\right)D_a(l)$$  \hspace{1cm} (67)$$

The antisymmetric disorder is thus relevant for $K_{\rho+} < 6$. Since the CDW$^\pi$ phase only exists at $K_{\rho+} < 1$ the CDW$^\pi$ is always very strongly pinned by disorder. Using (67) we obtain for the localization length in that phase

$$\frac{L_{\text{loc.,CDW}^\pi}}{\alpha} \sim \left(\frac{v_F^2}{D_a\alpha}\right)^{\frac{1}{6-K_{\rho+}}}$$  \hspace{1cm} (68)$$

In the classical limit $K_{\rho+} \to 0$ one recovers again the standard result for the pinning of a classical CDW.

The conductivity of the CDW$^\pi$ above the pinning temperature behaves as

$$\sigma(T) \sim T^{2-K_{\rho+}}$$  \hspace{1cm} (69)$$

showing since $K_{\rho+} < 1$ a very rapid decrease in the conductivity as $T \to 0$. This behavior is a consequence of the very strong pinning of the CDW$^\pi$. This feature of the antisymmetric CDW is similar to the one occurring for the spinless ladder.
4. OAF sector

In the case of the orbital antiferromagnet, the coupling to disorder is made of 2 terms: One term comes from $O_{CDW}$ the other one from $O_{CDW*}$. According to the preceding sections, these terms contain respectively $\cos \phi_{\rho-}$ and $\cos \phi_{\sigma+} \sin \theta_{\sigma-}$ and (see table [I]) therefore have exponentially decaying fluctuations. In order to get nontrivial results, the massive modes have to be integrated out as in the preceding sections. This again leads to an action of the form (57) and the disorder in the OAF phase is relevant for $K_{\rho+} < 3/2$. The OAF phase is therefore as delocalized as the superconducting $SC^s$ phase, although the pure system does not exhibit any obvious superconducting order parameter. The localization length in the disordered OAF is

$$L_{loc} = \left( \frac{v_2^2}{D\alpha} \right)^{4-2K_{\rho+}}$$

For $K_{\rho+} > 3/2$ we have a metallic phase.

The disorder leads to a conductivity of the form

$$\sigma(T) \propto T^{2-2K_{\rho+}}$$

The conductivity in the OAF is therefore identical, as far as the temperature dependence is concerned, as the one in the $SC^s$. It will exhibit in the localized phase $1 < K_{\rho+} < 3/2$ the same maximum in the conductivity for $T \sim T_{pin}$. Once again one sees that the transport properties can hardly be guessed from the phase diagram of the pure system. The OAF is thus also an excellent candidate for a “superconducting” behavior.

Using (36), it is possible to get some hints on the parameter regime of the extended Hubbard model in which the OAF could be achieved. One is in the OAF sector if $g_1 > 0$ and $K_{\rho+} > 1$. In the extended Hubbard language it means

$$2V(2 - \cos(2k_Fa)) < -U < 2V \cos(2k_Fa)$$

Let us assumes a local repulsion $U > 0$ and that one is close to half filling $\cos(2k_Fa) \sim -1$. In that case one reaches the OAF for moderate nearest neighbor attraction $V < -U/6$. Such a situation is likely enough to be realized, specially if additional attractive mechanism such as phonons are taken into account.

5. Differences with the spinless ladder

The spinless ladder and the ladder with spin show some marked physical differences. Some of them are due to the fact that interchain hopping has a different impact on fermions with spin and spinless fermions. In a system of spinless fermions, energy can be gained from hopping only if one site of the rung is occupied and the other one is empty due to the Pauli principle. This induces an enhancement of density fluctuations. On the other hand, a system with spin can gain energy from interchain hopping by having the two sites of the rung occupied by fermions of opposite spins. This leads to spin gap formation.
and a *smoothening* of density fluctuations. This effect is enhanced in the presence of a purely repulsive interaction as it tends to smoothen the density fluctuations in a system with spins, whereas it enhances them in a spinless system\(^{10}\). This has already important consequences in the pure case. In particular, the positions of the \(SC^d\) and OAF phases are different (see figure 1 and figure 3) as the d-wave in the spinless system needs some amount of attraction whereas it is achieved from completely repulsive interactions in the ladder with spin. In the presence of disorder, the d-wave phase of the spinless system can be stabilized by sufficiently attractive interactions, whereas in the system with spin it is always unstable (see figures 2 and 3), being replaced by an s-wave superconducting phase for attractive interactions. Also, in the presence of disorder, the system with spin due to the smoothening of the density, shows delocalization compared to the one chain case both for the attractive and the repulsive side. On the other hand for the spinless system, the reinforcement of the density fluctuations, enhances localization on the repulsive side. The attractive side on the other hand is totally delocalized.

In both case the s-wave phase (occurring for attractive interactions) is very strongly stabilized by the interchain hopping. This can be understood by a picture of tightly bound pairs that behave in both cases as hard core bosons. In that case, the statistics do not influence any more qualitatively the transport properties. Similarly both systems tend to form charge density waves that are extremely well pinned by disorder (usually much more easily than their one chain counterpart). In the case of fermions with spin, this requires some mixing of attractive and repulsive interactions so that pair of fermions of opposite spins are formed in the chains. These pairs then have hard core bosons interactions so that the situation becomes analog to the spinless fermions case. This explains the enhancement of pinning for the antisymmetric charge density wave phase. However, in the system with spin with purely repulsive interactions there is no CDW\(^\pi\) in contrast with the spinless system. Both systems also present an OAF phase that reveals quite stable in the presence of a small disorder. For the spinless ladder the OAF is even stable close to the noninteracting point. Finally, an interesting similarity between the system of fermions with spin and the system of spinless fermions is that pinning on two different CDW phases are possible depending on the interactions: either the antisymmetric \(2k_F\) CDW or a \(4k_F\) CDW. In these two localized phases the behavior of the conductivity at high frequency or high temperature and of the localization length at small disorder are very different (the difference appears in the exponents) the \(4k_F\) being much less well pinned than the \(2k_F\). This is to be contrasted to the one chain case where only one pinned charge density wave phase is realized. Therefore, we may expect to see, for weak disorder, a crossover between two different pinned charge density wave phases in the two chain system when varying the strength of the interactions. Such a crossover needs a more detailed study. Unfortunately it cannot be tackled by the RG since it occurs deep in the localized regime. One interesting question is whether such a transition still occurs for strong disorder.

### C. Transport properties

The ladder with spin shows therefore in presence of disorder transport properties drastically different from the one one could naively expect form the pure phase diagram. In
particular the d-wave phase disappears and does not exhibit any remarkable conductivity. Let us look in more details in the transport properties and compare them to what happens in a single chain for the various sectors.

1. Conductivity

As was mentioned in section II B 2, the ladder s-wave phase is much more stable to disorder than its one chain counterpart (see figures 4 and 5). This effect manifest itself in the location of the superconducting-localized transition, and in the localization length. As for the spinless case, this effect is entirely controlled by the interactions and going from one to two chains affects the power law dependence of the localization length with disorder. It is thus much stronger than the increase of localization length occurring for a noninteracting system (proportional to the number of channels). In presence of interactions the behavior of the localization length cannot be guessed by analogies with the non-interacting system. The case of spinless fermions where repulsive interactions make the two chain system more localized than the one chain system is an excellent counter-example.

The resistivity (see (65)) is also dropping much faster than for one chain for which \( \sigma_{1\text{ch}}(T) \sim T^{2-K_{\rho}} \). The ladder is thus a much better conductor than a single chain both because of the scale of localization and because of the better temperature dependence. In addition even in the localized phase the conductivity will increase for all values of \( K_{\rho+} \) for which the s-wave phase exists in the pure system, till one reaches the localization temperature \( T_{\text{pin}} \). This behavior is qualitatively sketched on figure 6. The SC\textsuperscript{s} phase shows therefore all the “good” characteristics of a “superconducting” phase, and in that respect is much more normal than its one-chain counterpart.

For repulsive interactions a different physical situation occurs. The system is still less localized than the one chain counterpart. The transition occurs for a smaller value of \( K_{\rho+} = 3/2 \) (versus \( K_{\rho} = 2 \) for a single chain), and the localization length is larger than for one chain (see (58)). Contrarily to the single chain where the pinned phase is a random antiferromagnet, here the presence of the spin gap forces the localized phase to become a pinned \( 4k_F \) CDW. However the SC\textsuperscript{d} phase is completely wiped out by the disorder, and what is more surprising, no trace of this “superconducting” phase can be found in the high temperature \( (T > T_{\text{pin}}) \) of the conductivity (see (65)). In particular \( \sigma(T) \) decreases monotonically even at high temperature in stark contrast with the SC\textsuperscript{s} phase as shown on figure 6. This again illustrate the fact that the transport properties are not linked to the behavior of the superconducting order parameter but to the density fluctuations. For a single chain since the density exponent and the superconducting one are related by \( K_{\text{density}} \sim 1/K_{\text{supra}} \) when superconducting fluctuation increases density fluctuations necessarily decrease and the system becomes a better conductor. Or course this is also true in presence of a true superconducting order in higher dimensional systems. For the single chain the fact that superconducting fluctuations do not necessarily imply better transport also appears one the fact that the attractive Hubbard model is more localized than the repulsive one: when the interactions go from repulsive to attractive a spin gap opens and the density fluctuations are suddenly lowered making the system more easy to pin. Similar effect occurs in the d-wave phase of the ladder, in a more dramatic way: the d-wave phase do not look superconducting
at all since it leaves enough room for enough $4k_F$ charge fluctuations. Note that the more repulsive the interactions will be the worse the conductivity, in a similar way than for the single chain where the phase is a spin density wave. The interchain hopping has thus two effects: on the one hand it leads to the appearance of the spin gap that wipes the SDW and replaces it by the $SC^d$ wave and on the other hand it freezes the density fluctuations (in particular the transverse charge modes). Those gaps suppress $2k_F$ CDW fluctuations, and localization happens only through coupling to $4k_F$ CDW fluctuations. Since the mechanism for localization is the same for all sign of the interactions, the transport properties are only weakly dependent of the sign of the interactions. This charge freezing is the dominant effect on transport. The two effects are essentially unrelated.

The most remarkable phase is the OAF which is an illustration of the above. This phase has a localization length and a $\sigma(T)$ as good as a genuine $SC^s$ wave phase! and yet has no genuine superconducting order parameter. In fact the absence of order parameter is here also due to the spin gap since for a single chain the corresponding phase is a triplet superconducting phase. However the fact that density fluctuations are already very small in this phase remains (and is helped by the freezing of transverse charge fluctuations), giving the remarkable transport properties of this phase. This remarkable property is not an artifact of the potential scattering and persists even if coupling to different form of disorder is included. In particular the superconducting-like transport properties of the OAF also exist in the presence of a random hopping along the chains and a random interchain hopping amplitude (see appendix D). Note that this phase has analogies of the so-called flux phase\footnote{\textsuperscript{69}–\textsuperscript{72}}, the size of the plaquette is here fixed by the interparticle distance, and of course this phase could not be reached for a pure Hubbard model (at the opposite of what was claimed in higher dimension). Whether for such a phase, a sort of Meissner effect also exists is of course a very interesting question. The connection between the one dimensional antiferromagnet and its 2d or 3d counterparts clearly deserves further investigations. In particular in two dimensions a phase offering some similarities with the one dimensional OAF, has been proposed for the high-Tc superconductor\textsuperscript{\ref{73}}.

\section*{2. Persistent currents}

In a similar way than for the spinless case one can compute the charge stiffness. For the ladder with attractive interactions one has for $K_{\rho+}, K_{\rho} < 3/2$

$$\mathcal{D}(L) = \mathcal{D}(0) - \left( \frac{L}{L_{\text{loc.,1 ch.}}} \right)^{3-K_{\rho}}$$

whereas for a two chain one, it is

$$\mathcal{D}(L) = \mathcal{D}(0) - \left( \frac{L}{L_{\text{loc.,1 ch.}}} \right)^{3-2K_{\rho+}}$$

These formulas are valid for $\alpha \ll L \ll L_{\text{loc.}}$. It is easy to see that they lead to a smaller reduction of the conductivity stiffness in the 2 chain case, in agreement with the fact that $L_{\text{2ch.}} > L_{\text{1ch.}}$. For repulsive interactions, it is of the form
$$D(L) = D(0) - \left( \frac{L}{L_{\text{loc},1 \text{ ch.}}} \right)^{3-K_\rho}$$  \hspace{1cm} (75)

Whereas for the 2 chain case, it is of the form:

$$D(L) = D(0) - \left( \frac{L}{L_{\text{loc},1 \text{ ch.}}} \right)^{3-2K_\rho}$$  \hspace{1cm} (76)

and the 2 chain system has a smaller reduction of conductivity stiffness than the one chain system. So up to prefactors the reduction in stiffness in the ladder system with spins is identical for repulsive and attractive interactions and the reduction of conductivity stiffness also shows no abrupt change as one goes from attractive to repulsive interactions. By contrast, in the one chain case, attractive interactions induce a spin gap and localization arises from coupling of a single massless mode to $2k_F$ disorder. This gap closes for repulsive interactions and localization arises from the coupling of 2 massless modes with the $2k_F$ random potential. This causes the abrupt change in transport properties and charge stiffness when one goes from attractive to repulsive interactions. This is related to the fact that the localization lengths for attractive and repulsive interactions have the same dependence on disorder, in marked contrast both with the spinless problem and the single chain with spins. The effect of increase of persistent current by repulsive interactions occurring in the single chain is thus either absent or strongly reduced (not an exponent effect any more) in the ladder. It would of course be interesting to investigate in ladder with more than two legs to see if this effect reappears and check for possible difference of behavior between odd and even legs ladders.

**IV. COUPLED LADDERS**

**A. Mean field treatment**

In the preceding sections, we have been considering isolated bichains. To describe realistic compounds, such as SrCuO, and have a finite temperature phase transition, interchain coupling should be taken into account. A realistic coupling is of course single particle hopping between the ladders. However in ladders, due to the existence of single particle gaps (spin and antisymmetric charge mode) for the ladder, single particle hopping is irrelevant, provided that the inter-ladder hopping is much smaller than the gaps of the system. One has therefore to consider only the particle-hole (or particle-particle) coupling generated by the single particle hopping. Such couplings can lead to an ordered phase at a finite temperature. As is very reasonable on physical grounds such interchain couplings stabilize the dominant one-dimensional instability. We focus here on the existence of a stable d-wave superconducting phase. This allows us to keep only the particle-particle (or Josephson) coupling between the ladders. The Hamiltonian for the coupled ladders system is:

$$H = \sum_n \left[ H_{\text{disordered 2 chain system},n} + \frac{J}{2} \int dx (O_{SC,n}^\dagger(x)O_{SC,n+1}(x) + O_{SC,n+1}^\dagger(x)O_{SC,n}(x)) \right]$$  \hspace{1cm} (77)
Where $O_{SC,n}$ is the operator for (d-wave or s-wave) superconductivity for the $n$–th ladder and $J$ is the strength of the Josephson coupling. On can simplify further the Hamiltonian (77) by keeping only massless modes in the ladder. Doing so we assume that the spin gap and the interchain gap of the two chain system are much larger than the disorder and much larger than the Josephson coupling. However, we make no assumption on the relative magnitude of the Josephson coupling and the strength of the random potential. The resulting Hamiltonian is, both for the case where the dominant instability is s-wave or d-wave superconductivity

$$H = \sum_n \left( \int \frac{dx}{2\pi} \left[ u_{\rho+} K_{\rho+} (\pi \Pi_{\rho+}, n)^2 + \frac{u_{\rho+}}{K_{\rho+}} (\partial_x \phi_{\rho+}, n)^2 \right] + \int \frac{dx}{\pi \alpha} \left[ \xi_{\text{eff}, n}(x) e^{i2\phi_{\rho+}, n} + \text{H. c.} \right] + J \int dx \cos(\theta_{\rho+, n} - \theta_{\rho+, n+1}) \right)$$

(78)

To solve (78) we treat the Josephson coupling in mean field assuming the existence a finite superconducting order parameters $\langle \cos(\theta_{\rho+}) \rangle$. By making the replacement $\cos(\theta_{\rho+, n} - \theta_{\rho+, n+1}) \to \langle \cos(\theta_{\rho+}) \rangle \cos(\theta_{\rho+, n})$, the Hamiltonian (78) becomes the one of an isolated ladder system in an external field, the value of which is determined by a self-consistency condition. The Hamiltonian is then

$$H_{MF} = \int \frac{dx}{2\pi} \left[ u_{\rho+} K_{\rho+} (\pi \Pi_{\rho+})^2 + \frac{u_{\rho+}}{K_{\rho+}} (\partial_x \phi_{\rho+})^2 \right] + \int \frac{dx}{\pi \alpha} \left[ \xi_{\text{eff}, n}(x) e^{i2\phi_{\rho+}} + \text{H. c.} \right] - \frac{W}{(2\pi \alpha)^2} \int dx \cos(\theta_{\rho+})$$

(79)

with the self-consistency condition $W = J \langle \cos(\theta_{\rho+}) \rangle$.

The equation determining $T_c$ is

$$\frac{1}{J} = \frac{1}{(2\pi \alpha)^2} \int dx \int_0^{\beta_c} d\tau \langle T, \cos \theta_{\rho+}(x, \tau) \cos \theta(0, 0) \rangle_{H_0}$$

(80)

with $\beta_c = \frac{1}{T_c}$ and $H_0$ is $H_{MF}$ for $W = 0$. To solve (80), one has to compute the finite temperature superconducting response function of a ladder in the presence of disorder. There are presently no methods to do this exactly, but one can get an accurate solution for $T_c$ by making some simplifying approximation. First, one notices that a finite temperature induces a cutoff length $l(T) = u_{\rho+}/T$ beyond which all correlation functions decay exponentially to zero. We make thus the approximation that beyond $l(T)$ all correlation functions are truly zero and below $l(T)$ they are equal to the $T = 0$ correlation functions. This allows us to use the RG equations introduced in section [11]. If we denote by $\chi$ the superconducting response function, when we change the running cutoff $\alpha(l) \to \alpha(l) e^{d \ln(l)}$ we have $\chi \to \chi \exp(-\frac{d l}{2K(l)})$. Thus to compute correlation functions at lengthscale $R$ it is sufficient to integrate the RG equation from the cutoff up to $R$ and follow the renormalization of the response function. Making use of these two approximations, the equation giving $T_c$ simplifies into:

$$\frac{1}{J} = \int_0^{u_{\rho+}/T} \frac{R dR}{2\pi \alpha^2} \exp\left(-\int_0^{\ln(R/\alpha)} \frac{d l}{2K(l)} \right)$$

(81)

The values of $K(l)$ are obtained by numerically solving the RG equations.
\[
\frac{dK}{dl} = -D(l)K(l)^2
\]  
(82)

\[
\frac{dD}{dl} = (3 - 2K(l))D(l)
\]  
(83)

the values of \(T_c\) for \(K = 0.5, 1.2\) and \(J = 0.1\) as a function of \(D\) are shown on figures \(7\) and \(8\) respectively. We note that for \(K = 1.2\) we have an s-wave superconducting phase and for \(K = 0.5\), a d-wave phase. This can be expected since the interchain coupling stabilizes the dominant one dimensional fluctuation (see figure \(5\)). We see that (see fig. \(8\)) as in the case of the single chain mean field theory\(37\) of superconductivity we have an initial linear decrease of the critical temperature with disorder strength. This is to be contrasted with the standard mean field theory of the s-wave superconductor in three dimensions being based on a diffusion approximation that does not include Anderson localization effects and gives \(T_c\) independent of the disorder. This is the well known Anderson theorem. The linear decrease of \(T_c\) with the strength of disorder for s-wave superconductivity in our chain mean field is due to localization effects. This peculiar situation is due to the absence of a diffusive regime in one dimensional disordered systems, which implies that their response functions are always affected by localization effects.

For a d-wave superconductor one expects in mean-field theory a linear decrease of \(T_c\) as a function of \(D\) (see e.g. Ref. \(75\)). For the ladder system however the decrease of \(T_c\) is mainly due to the localization effects, similarly to the s-wave superconductor case. Although it indeed starts linearly for small disorder (see figure \(7\)), localization effects manifest themselves by the sudden drop to \(T_c = 0\) at a critical disorder strength (see fig. \(8\)).

For identical Josephson coupling between the bichains, the critical disorder strength is smaller for the d-wave superconductor than for the s-wave one.

**B. Simplified Treatment:**

Although the mean field theory allows an accurate description of the effects of disorder on \(T_c\) the critical value of disorder above which superconductivity is destroyed can also be obtained by a very simple physical argument. Let \(T_c^{(\text{pure})}(J)\) be the temperature at which the superconducting transition would occur in the array of ladders if there were no impurities. Just above \(T_c^{(\text{pure})}\), the thermal length is \(\frac{u_{\rho+}}{T_c^{(\text{pure})}}\) and beyond that length all phase coherence is lost. Clearly, if the thermal length is smaller than the localization length in a single chain containing impurities \(\xi_{\text{loc}},\) phase coherence is lost before coherent backscattering can build Anderson localization. The system will escape localization due to the building of the (mean-field) superconductivity. Thus, if

\[
\frac{u_{\rho+}}{T_c^{(\text{pure})}} < \xi_{\text{loc}}
\]  
(84)

Anderson localization will not suppress the superconducting transition. Equation \(84\) gives a simplified criterion for the stability of superconductivity. For fixed Josephson coupling \(J, \) \(80\) leads to \(T_c^{(\text{pure})} \sim J\frac{2K_{\rho+}}{2K_{\rho+} - 1}\). Thus, the higher \(K_{\rho+}\) the higher \(T_c^{(\text{pure})}\). From the preceding section, the localization length both in the s-wave and d-wave superconducting
phase is $\xi_{\text{loc.}} \sim (\frac{1}{D})^{1-2K_{\rho^+}}$. Increasing $K_{\rho^+}$ also reduces $\xi_{\text{loc.}}$. Thus, the two effects reinforce each other, and make the $SC^s$ phase, that exists for $K_{\rho^+} > 1$ more stable against Anderson localization than the $SC^d$ phase that exists only at $K_{\rho^+} < 1$.

V. EXPERIMENTAL CONSEQUENCES

The theoretical results obtained in the preceding sections have important consequences for experimental systems that are believed to be well approximated by coupled chains systems, namely the doped ladder systems which present a superconducting transition and the 2 band quantum wire. In the former case, one would like to know if the superconducting transition is related to the divergence of superconducting fluctuations in the strictly one dimensional system that results at the mean field level in a finite $T$ superconducting transition or if the physics of the transition is a two or three dimensional one. We believe that the resilience of superconductivity to disorder is a stringent test of effective dimensionality. In the case of quantum wires, we discuss the experimental consequences of our results for the conductivity and charge stiffness in the interacting system. Measurements of the conductance would allow to check the above theories for the ladders and provide a measurement of the Luttinger liquid parameter in the charge sector, providing some insight on the strength of interactions in these systems.

A. Superconductivity of doped ladder systems

Our study has various experimental consequences for the observation of superconductivity in Ladder systems. First, if the superconductivity is to come from purely repulsive interactions (i.e. to be of the d-wave type), it should be extremely sensitive to disorder as we showed in section V A. In fact, any randomness would induce a conductivity that never increases as temperature decreases (see figure [1]), so that superconductivity would be impossible to probe except in extremely pure samples. Such sensitivity with respect to disorder is certainly consistent with the difficulty in observing any type of superconductivity in the ladder systems $\text{Sr}_{n-1}\text{Cu}_{n+1}\text{O}_{2n+2}$. However superconductivity seems indeed to be observed in $\text{Sr}_{0.4}\text{Ca}_{13.6}\text{Cu}_{22}\text{O}_{41.3}$ under pressure ($\sim 3\text{GPa}$). Whether such superconductivity is of the d-wave type is of course still open. Various experimental facts, however seem to indicate that if it is the case, it is unlikely that such superconducting phase could be described by weakly coupled ladder systems. Indeed one could use the criterion (54) to estimate the localization length. Taking a reasonable value of $10^6\text{ms}^{-1}$ for the Fermi velocity, one obtains from the observed $T_c \sim 10\text{K}$, a minimal localization length of $\xi \sim 10000\text{Å}$. Using (58), this leads to extremely long mean-free paths ($l = \xi$ for $K = 1/2$) when one is in the d-wave sector. So unless the chains are extremely pure, a fact not likely to be true in such doped materials, one expects based on one dimensional physics alone that the superconductivity should be totally suppressed. If the presence of superconductivity is due to an extremely pure system (which is doubtful) then, introducing more disorder in the system (for instance by irradiation) should induce a dramatic decrease of the critical temperature.
Besides the extreme sensitivity of $T_c$ to disorder other arguments are again a simple stabilization of one-dimensional physics in the experimental compound: even if one could be below the critical disorder strength determined by (84) and Figure 7, the physics above $T_c$ should be dominated by the one-dimensional (ladder) effects. In this regime the resistivity goes up with decreasing temperature as described in section III C. The observed resistivity showing a monotonic decrease of the resistivity (roughly with a $T^2$ or $T$ law) is again incompatible with the one-dimensional description. If one is in the purely repulsive sector, the most likely explanation of the main experimental features is that under pressure the interchain hopping between the ladders become strong enough so that the system does not retain its one-dimensional feature, but is more accurately described by two-dimensional physics. Such an interpretation is also compatible with the fact that the system at ambient pressure is insulating. In that case, the coupled bichains treatment becomes extremely questionable, and it is probably better to start from a two-dimensional description, for which disorder effects are probably weaker, but for which the nature of the superconducting phase has yet to be completely elucidated.

Another interesting, but probably more farfetched, possibility could be that the system is in fact in the orbital antiferromagnetic sector. In such a sector the effects of disorder are much more reduced, and even very large localization lengths can lead to reasonable mean-free paths ($\xi/a$ is at worst $(l/a)^2$, and diverges for $K = 3/2$, see (70)). The resistivity decreases with temperature according to (71). Here the difficulty lies more in getting the interactions corresponding to this phase, since one needs local repulsion and a sizeable nearest neighbor attraction. In any case, careful measurements of the temperature dependence of the resistivity above $T_c$ could help to decide if such OAF effects are present. Of course here again, one cannot exclude that the physics is two-dimensional to start with, but at least now the one-dimensional starting point is more consistent with the dominant experimental features.

B. Application to quantum wires

Progress in nanostructure technologies have allowed for measurements of the transport properties of low dimensional electronic systems. In particular, in recent experiments on quantum wires, the conductance of a quasi one dimensional electron gas has been measured at very low temperatures. For the pure system, or extremely weak disorder one finds quantized values of the conductance 41 in good agreement with the theoretical predictions 77, 78 at fractions of $\hbar/e^2$ as a function of width of the quantum wire (i.e. of the number of subbands at the Fermi level). The relation between the number of channels and resistance has been verified 41. Impurities on the other hand induce backward scattering that is known to cause Anderson Localization in a sufficiently long system. In small enough system, it leads to reduction of conductance as the length of the system is increased or the temperature is lowered. Deviations of conductance from $e^2/\hbar$ as a function of temperature have indeed been obtained in experiments 29 as well as deviations as a function of of the length of the wire 1 and can be related to the Luttinger liquid exponent. The correction to the conductance due to impurities 28 is of the form
\[ G(T) = \frac{e^2}{h} - gT^{-\nu} \]  

(85)

where \( \nu = 1 - K_\rho \) is the conductivity exponent. The derivation is similar to the derivation of the T dependence of conductivity in appendix B. For finite size systems \( T \) can be replaced by the lower cutoff \( v_F/L \) in (85). This formula only holds at high enough temperatures or for systems of length \( L \) shorter than the localization length for which the corrections term is small.

If two channels are present in the wire, the system becomes then equivalent to a ladder system. Two band present at the Fermi level are the equivalent of the bonding and antibonding bands of the ladder system. One then expects that the whole physics derived in section III C should apply to these wires. In particular, since one expects reasonably repulsive interactions one should be in the \( SC^d \) of the \( CDW^{4k_F} \) phase. Going from a single chain to the ladder should have observable consequences on the transport properties. First since the localization length increases drastically in the ladder system one would expect the conductance corrections due to disorder, to be much weaker for two channels. This of course assumes that the typical interactions do not vary too much when going from one channel to two channel, a fact which is not certain. Secondly by doing the expansion of the corrections to conductance for the ladder and using the conductivity exponent (III C), one would obtain \( \nu = 2 - 2K_\rho \). A fit of the temperature dependence of the conductance in \( 41 \) could allow to extract the Luttinger liquid exponents for the ladder system, as well as check the above predictions.

VI. CONCLUSION

In this paper we have examined the effects of disorder on a 2-legs ladder system, using RG techniques. We have computed the effects of disorder on the phase diagram as well as the localization length. Disorder has drastic effects on the phase diagram. For spinless fermions, it leads to an extremely strong localization of the charge density wave phase that exists for repulsive interactions. Such localization is even stronger than for a single chain. On the other hand for the ladder system there is a remarkable stability of the s-wave superconducting phase (for attractive interactions), compared to the single chain case. The insulator-superconductor transition occurs in the vicinity of the non interacting point for a pure t-V model whereas in the one chain system it occurs for strongly attractive interactions.

For fermions with spin, the repulsive part of the phase diagram is also strongly localized by disorder. In particular the novel d-wave superconducting phase found for ladder systems is completely suppressed by an arbitrarily small amount of disorder. We emphasize that this is not only a pair breaking effect but a much stronger Anderson localization effect. On the other hand the s-wave superconducting phase occurring for attractive interactions is again much more stable to disorder than its one chain counterpart.

Besides obtaining the phase diagram, we have also investigated the transport properties of the t-V and Hubbard two chain systems. The RG enabled us to compute the localization length and the charge stiffness as a function of disorder (see tables \( III \) and \( IV \)) and the temperature and frequency dependence of the conductivity. Various remarkable fact emerged. First
the behaviors of the spinless ladder and the ladder with spins are very different. In particular the spinless ladder shows the same tendency than the single spinless chain, namely that attractive interactions decrease localization whereas repulsive interactions enhance it. In the two chain case, that effect is even stronger. For attractive interactions there is no localization, whereas for repulsive ones the system is much more localized than its one chain counterpart (the exponent in the dependence of localization length with disorder is changed). On the other hand for the Hubbard ladder there is no such effect: up to a prefactor, the localization length with attractive interaction is the same as for repulsive ones. As a consequence, the corrections to conductivity stiffness are the same for attractive and for repulsive interactions.

The temperature dependence of the conductivity follows a power law of the form $\sigma(T) \sim T^{2-2K_{\rho+}}$, for temperatures above the localization scale $T_{loc}$, where $K_{\rho}$. For the repulsive side, where for the pure system one would have the d-wave superconducting phase, $K_{\rho+} < 1$, and thus the conductivity decreases as a function of the temperature even well above $T_{loc}$. The transport thus shows no remnant of the superconducting behavior one could have naively expected when looking at the pure system. This remarkable fact illustrates that transport is in fact controlled by the density fluctuations of the system and not by the existence of not of slowly decreasing superconducting correlation functions. Ladder system provides evidence of a phase that is genuinely a d-wave superconductor as far as phase diagram is concerned, but would from the transport point of view be closer to an insulator. Of course such an interesting behavior would clearly deserve more studies. In particular it would be interesting to know how the correlation between the density fluctuations and the superconducting one evolves as the number of chain is increased, and how the crossover to the three dimensional situation occurs. Such a study goes of course far beyond the goals of the present paper.

We have applied our results to two types of experimental systems. First our results should be relevant for quantum wires with two channels. Here the prediction for the exponent in the conductivity can be directly checked by the measuring temperature dependence of the conductance of the system. Note that the conductivity/conductance exponent $2 - 2K_{\rho+}$ for the ladder systems is different from the one for a single channel (or a single chain) $1 - K$. Due to the increase of the localization length when going from one channel to two channels one would also expect overall smaller corrections to the conductance for a given strength of the disorder, and roughly constant interactions. Investigation of systems with more than two chains would be useful in order to get a better understanding of the role of internal symmetries and gaps on the transport properties of quasi one dimensional systems. This is of course also useful in connection with experiments on quantum wires. In particular, we expect that the behavior of systems with an even number of legs is dominated by gap formation whereas the behavior of systems with an odd number of legs should be closer to the one of a one chain system.

The other experimental system on which our results could be applied are of course the coupled bichains where superconductivity has recently been obtained under pressure. To compare with this system of coupled ladders, it was necessary to treat the coupling between different ladders, which we did using a mean-field approximation. The results indicate that even in the presence of coupling between ladders the d-wave phase is still much too sensitive to disorder, to be the one experimentally observed. In addition the observed temperature dependence of the conductivity would be incompatible with the one computed here should
these systems be dominated by one-dimensional (ladder) physics. These observations and
the fact that the conductivity occurs under very large pressure tends to indicate that the
mechanism for superconductivity in these systems is very likely to be of a two or three di-
mensional nature and not just the mere stabilization of the ladder superconducting phase.
On the other hand the system without pressure has a resistivity that could be more com-
patible with the localization effects described here. Of course one interesting question would
be whether one can get a one (ladder) to higher dimensional crossover as the pressure is
applied. This of course could only be decided by a more quantitative comparison with exper-
iments as well as further theoretical and experimental work. Adding additional impurities,
for example by irradiation, could allow to distinguish if the system is in a one-dimensional
regime, since one expect much more drastic localization effects in that case.

Finally a ladder system with spins exhibits an extremely interesting orbital antiferro-
magnetic phase. Although such a phase cannot occur in a pure Hubbard system it can in
principle be stabilized if some nearest neighbor attraction is added. Although such a phase
has no superconducting order parameter, it has perfect conductivity in the presence of a
random potential. Moreover that perfect conductivity is also robust in the presence of ran-
dom hopping both along the chains and perpendicular to the chains. As far as transport is
concerned this phase is therefore a one dimensional “superconductor”. Nevertheless, it has
only subdominant (in the spinless case) or exponentially decaying (in the case of fermions
with spin) superconducting correlations, again an illustration that looking at the supercon-
ducting fluctuations is not a good criterion to determine the transport properties. Due to
the peculiar nature of this phase it would be interesting to check whether it survives in lad-
der systems with more than two legs. More generally it also deserves further investigation
in dimensions higher than one, both in relation of flux phases of two dimensional systems
and other orbital phases proposed for the normal state of cuprate superconductors.

The study of the disorder effects could also be extended in various directions. In partic-
ular a more detailed description of the physics inside the localized phase would be suitable.
However such a description is beyond the reach of the simple RG calculation. Going to
strong but diluted disorder is also a challenging problem. In particular understanding the
crossover from the results of our paper to the limit where disorder suppresses gaps altogether
in the system remains yet to be done.

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APPENDIX A: BOSONIZATION TECHNIQUE

In this section, we will give a short review of the bosonization technique in order to fix the
notations. We give the expressions for a single chain of spinless fermions. For more species
of fermions, one can bosonize each specie individually, and the corresponding expressions
are given in the text.
1. Representation of fermion operators in terms of boson ones

Non-interacting one dimensional spinless fermions on a lattice are described by the kinetic energy

\[ H = -t \sum_{n=1}^{N} (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}) = \sum_k \epsilon(k) c_k^\dagger c_k \]  

(A1)

where \( \epsilon(k) = -2t \cos k \) and \( c_n = \frac{1}{\sqrt{N}} \sum_k e^{ikn} c_k \).

To obtain the asymptotic (low energy, long wavelength) properties of the system one can linearize the spectrum near the Fermi “surface” \( \pm k_F \) and take the continuum limit by introducing \( \psi(x) = \frac{1}{\sqrt{N}} \sum_k e^{ikx} c_k \) with \( a \) the lattice spacing and \( x = na \). With our definition the \( \psi \)'s have the commutation relations of continuum fermion operators. We define the \( R \) (resp. \( L \)) (right and left movers) fermions as fermions with momentum close to \( +k_F \) (resp. \( -k_F \)) as

\[
\psi_R(x) = \frac{1}{\sqrt{Na}} \sum_{|k|<\Lambda} e^{ikx} c_{k_F+k}
\]

(A2)

and similarly for \( \psi_L(x) \) with \( k_F \rightarrow -k_F \). \( \Lambda \) is a cut-off needed not to double count fermion states, and imposed by the linearization of the dispersion relation. All asymptotic properties can be expressed in term of \( \psi_{R,L} \). In particular the full fermion operator becomes \( \psi(x) = e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x) \). The Hamiltonian (A1) becomes

\[ H = -i v_F (\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L) \]  

(A3)

with \( v_F = 2ta \sin(k_F a) \).

Due to the separation into two branch of fermions and the linearization of the spectrum, the Fourier components of the fermion density operators

\[ \rho_{R,L}(q) = \sum_k \psi_{R,L,k+q}^\dagger \psi_{R,L,k} \]  

(A4)

have boson commutation relations:

\[
[\rho_R(q), \rho_R(-q')] = -\frac{L}{2\pi} q \delta_{q,q'}
\]

\[
[\rho_L(q), \rho_L(-q')] = \frac{L}{2\pi} q \delta_{q,q'}
\]

\[
[\rho_L(q), \rho_R(-q')] = 0
\]  

(A5)

This allows to rewrite (A3) as

\[ H = \pi v_F \int dx \left[ \rho_R(x)^2 + \rho_L(x)^2 \right] \]  

(A6)

with \( \rho_s(x) = \psi_s^\dagger(x) \psi_s(x) \) for \( s = L, R \). Instead of using the density operators themselves it is more convenient to introduce
\[ \Pi(x) = \rho_R - \rho_L \]

\[ \frac{-1}{\pi} \partial_x \phi = (\rho_R + \rho_L) \]  \hspace{1cm} (A7)

Physically, \( \Pi \) is a momentum density while \( \partial_x \phi \) is proportional to the deviation of the fermion density from its average value. The commutation relations for the \( \rho \)'s imply that \( \left[ \phi(x), \Pi(x') \right] = \imath \delta(x - x') \). Also, the Hamiltonian rewritten in terms of \( \Pi \) and \( \phi \) is:

\[ H = \int dx \frac{v_F}{2\pi} \left[ (\pi \Pi)^2 + (\partial_x \phi)^2 \right] \]  \hspace{1cm} (A8)

which is just the continuum limit of the Hamiltonian of a 1D harmonic chain. Note that the following procedure could have been applied to a more complicated lattice Hamiltonian than \( (A1) \). All that is needed is that the Fermi surface reduces to two points. The effectiveness of bosonization stems from the fact that it is possible to express the fermions operators in terms of \( \Pi(x) \) and \( \phi(x) \). If one introduces \( \theta(x) = \pi \int_{-\infty}^{x} \Pi(x') dx' \) one has the following relations:

\[ \psi_R(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\imath \theta(x) - \phi(x)} U_R \]

\[ \psi_L(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{\imath \theta(x) + \phi(x)} U_L \]  \hspace{1cm} (A9)

\( \alpha \) being a cutoff the presence of which is imposed by the cutoff needed in the linearization of the dispersion relations.

the \( U_R \) and \( U_L \) are anticommuting operators introduced by Haldane that annihilate one fermion at the Fermi level. These operators also anticommute with their hermitian conjugates. It can be verified explicitly that those relations reproduce correctly the commutators of fermion operators. These \( U \) operators give in general corrections vanishing in the thermodynamic limit and can be safely dropped. On the other hand, if there are different species of fermions (such as up and down spin fermions or band degeneracies), one must bosonize separately each fermion species using the formulas for spinless fermions. It is needed to introduce \( U_{L,n}, U_{R,n} \) operators and their complex conjugates \( n \) indexing the internal degrees of freedom such as spin) to enforce proper fermions anticommutation relations. In order to make that bookkeeping less tedious \( \eta \) operators such that:

\[ \eta_\alpha \eta_\beta + \eta_\beta \eta_\alpha = 2 \delta_{\alpha,\beta} \]

\[ \eta_\alpha \dagger = \eta_\alpha \]  \hspace{1cm} (A10)

where \( \alpha = (L,n) \) or \( \alpha = (R,n) \) these operators can therefore introduce minus signs in the various bosonized expressions.

2. handling the interactions with bosonization

Let us consider spinless fermions. Interactions can then be handled straightforwardly: If one adds a density density coupling of the form \( \int dx U(\rho(x))^2 \) The density can be decomposed in a slowly varying part \( \rho_R(x) + \rho_L(x) \) and a \( 2k_F \) part \( e^{2ik_F x} \psi_L(x) \psi_R(x) + \text{H. c.} \).
In the Hamiltonian, one retains only the slowly varying terms (the other term give a zero value when integrated over \( x \)). The \( 2k_F \) always disappear, while the \( 4k_F \) can persist in a half filled lattice system. As a consequence, at a non commensurate filling, the Hamiltonian reduces to:

\[
H = \int \frac{dx}{2\pi} \left[ uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x \phi)^2 \right] \tag{A11}
\]

with \( uK = v_F \) as a consequence of galilean invariance. If one makes the rescaling \( \phi \rightarrow \phi/\sqrt{K} \) and \( \Pi \rightarrow \Pi\sqrt{K} \) one has the same Hamiltonian as in (A8) with the correct commutation relation for \( \phi \) and \( \Pi \). If one computes the physical correlation functions at 0K such as the \( 2k_F \) part of the fermion Green’s function \( G(x - x', t - t') = -i\langle T\psi_R(x,t)\psi_L^\dagger(x',t') \rangle \) it is easily seen that \( K \) controls their power law decay while \( u \) controls the propagation of excitations. \( u \) and \( K \) are also related to physical quantities such as the charge stiffness and the compressibility. More specifically, defining the compressibility by

\[
\chi = -\frac{1}{L}(\frac{\partial P}{\partial L})_T, \quad P = -\left(\frac{\partial F}{\partial L}\right)_T
\]

and taking \( T \rightarrow 0 \), we have:

\[
\chi = \frac{\pi K}{u k_F^2} \tag{A12}
\]

The charge stiffness is defined by

\[
D = L^2 \frac{1}{2} \left(\frac{d^2 E(\phi)}{d\phi^2}\right)_{\phi=0}, \quad \phi \text{ being a flux threading the system.}
\]

From that definition, one obtains:

\[
D = uK \tag{A13}
\]

The case of fermions with internal degrees of freedom is usually more complicated, because some of the interactions cannot be reduced to \((\partial_x \phi)^2\) terms, the most well known example being the backscattering of two fermions with opposite spins. Usually, one finds sine-Gordon Hamiltonians of the form:

\[
H_{SG} = \int \frac{dx}{2\pi} \left[ uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x \phi)^2 \right] + \Delta \int dx \cos(\beta \phi) \tag{A14}
\]

These Hamiltonians can be studied using renormalization group (RG) techniques. The flow equations for \( K \) and \( \Delta \) are of the Kosterlitz-Thouless form. \( \Delta \) has scaling dimension \( 2 - \beta^2 K/4 \). Therefore a small \( \Delta \) is relevant for \( K < 8/\beta^2 \). From the RG equation for \( \Delta \), one sees that there are two regimes: one small \( K \) or large enough \( \Delta \) regime, where \( \Delta \) is relevant and a large \( K \), small enough \( \Delta \) regime where \( \Delta \) is irrelevant. When \( \Delta \) is irrelevant, the correlation functions keep their power law character up to logarithmic corrections. \( \beta \) on the other hand, if \( \Delta \) is relevant, \( \phi \) will acquire a non-zero expectation value that minimizes the ground state energy and a gap will be generated. It can then be shown that there exists a regime where \( \langle f(\phi) \rangle \sim f(\langle \phi \rangle) \) and that \( \langle T e^{ia\theta(x,\tau)} e^{-ia\theta(0,0)} \rangle \sim \exp\left(-\frac{\sqrt{x^2 + (\tau r)^2}}{\xi}\right) \) where \( \xi \) is a correlation length. These results are used extensively in the paper.

**APPENDIX B: MEMORY FUNCTION CALCULATION OF AC AND DC CONDUCTIVITY**

For the sake of clarity, we will explain the technique on the example of one chain of spinless fermions (technically this is the simplest case), and then explain how the calculation
can be extended to more complicated cases. First let us describe the memory function approximation. The conductivity is given by linear response theory as:

$$\sigma(\omega) = -i\chi(0) - \chi(\omega)$$

where $\chi(\omega)$ is the current-current response function. The memory function $M(\omega)$ is defined by:

$$\sigma(\omega) = -i\chi(0) -i\chi(\omega) + M(\omega)$$

This gives the exact formula:

$$M(\omega) = \omega\chi(\omega) - \chi(0)$$

an expansion at high frequency and small impurity concentration gives:

$$M(\omega) = \left(\ll F; F \gg \omega \ll F; F \gg \omega=0\right)/\omega - \chi(0)$$

Where $\ll; \gg$ is a retarded correlator evaluated for the pure system and $F = [J, H]$ $J$ being the total current. To use that formalism in the framework of bosonization, we first need an expression for the current. This can be obtained from the definition of the fermion density $\rho(x) = \rho_0 - \partial_\pi\phi$ and the current conservation equation : $\partial_t \rho + \partial_\pi j = 0$. One obtains $j(x) = \partial_\pi\phi$. Using the Heisenberg equation of motion for $\phi$ and noting that the total current $J = \int dx j(x)$ on finds $J = uK \int \Pi(x)dx$. The coupling to disorder being:

$$H_{\text{imp}} = \int dx \xi(x) e^{i2\phi(x)} + \text{H. c.}$$

We get $F \propto \int dx \frac{\xi(x)}{2\pi\alpha} e^{i2\phi(x)} - \frac{\xi^*(x)}{2\pi\alpha} e^{-i2\phi(x)}$.

This gives:

$$\langle T_F F(\tau) F(0) \rangle \propto \int dx D\delta(x) \left(\frac{1}{x^2 + (u\tau)^2}\right)^K \propto \tau^{-2K}$$

Therefore, $\ll F; F \gg \omega \propto \int d\tau e^{i\omega\tau - 2K} \propto \omega^{2K-1}$. This gives $M(\omega) \propto \omega^{2K-2}$ and for $K > 3/2$ $\sigma(\omega) \propto \omega^{2-2K}$. The formula we have obtained is valid only at high frequency. We can get from it a high temperature formula by using the dimensional equivalence of temperature and frequency ( e. g. $h\omega \sim k_B T$). To generalize the calculation to a more complicated case, we must first note that in the formula for the current, $\phi$ will be replaced by $\phi_\rho$ in the case of two chains of spinless fermions and $\phi_{\rho^+}$ in the case of 2 chains of fermions with spin. The coupling to disorder being some $\int dx \xi(x)e^{in\phi} + \text{H. c.}$, $n$ depending on the problem at hand, we see that in the general case we will just have to make the replacement $2K \to \frac{n^2}{2}K$ in the formulas giving $\sigma(\omega), \sigma(T)$.  

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APPENDIX C: EFFECTIVE RANDOM POTENTIAL IN THE PRESENCE OF GAPS

In that section, we will give a derivation of the RG equation for $D_a$ at $g_f > 0$. We start with the method of \[ \text{C} \]: we compute perturbatively the correlation function: \[ \langle T_\tau e^{i\sqrt{2}\phi_\parallel(x_1,\tau_1)} e^{-i\sqrt{2}\phi_\parallel(x_2,\tau_2)} \rangle \]. In second order in the random potential, since \[ \langle T_\tau \sin(\sqrt{2}\phi_\parallel)(x,\tau) \sin(\sqrt{2}\phi_\parallel)(0,0) \rangle \sim e^{-r/l}, \] there is no singular contribution. Therefore, we must go to fourth order. We will drop the combinatorics since we are only interested in the renormalization of $D$. The fourth order term is of the following form:

\[
D_a^2 \int \frac{dx_1 dx_2 dx_3 dx_4 d\tau_1 d\tau_2 d\tau_3 d\tau_4}{(\pi \alpha)^4} [\delta(x_1 - x_4) \delta(x_2 - x_3) + \delta(x_1 - x_2) \delta(x_3 - x_4)] \\
\langle T_\tau e^{i\sqrt{2}[\phi_\parallel(x,\tau) + \phi_\parallel(x_1,\tau_1) + \phi_\parallel(x_3,\tau_3) - \phi_\parallel(x_2,\tau_2) - \phi_\parallel(x_4,\tau_4) - \phi_\parallel(0,0)]} \rangle \\
\langle T_\tau \sin(\sqrt{2}\phi_\parallel)(x_1,\tau_1) \sin(\sqrt{2}\phi_\parallel)(x_2,\tau_2) \sin(\sqrt{2}\phi_\parallel)(x_3,\tau_3) \sin(\sqrt{2}\phi_\parallel)(x_4,\tau_4) \rangle
\]

(C1)

The $\phi_\parallel$ will be exponentially small except when $|r_1 - r_3| \ll l$ and $|r_2 - r_4| \ll l$ or $|r_1 - r_2| \ll l$ and $|r_3 - r_4| \ll l$ (the other cases are equivalent to these two ones up to a relabeling of dummy integration variables). It is easily seen that the second case is in fact trivial. Therefore, the only interesting contribution comes from the first term. This term reduces to the simple form:

\[
D_a^2 \int dx_1 dx_2 d\tau_2 \delta(x_1 - x_2) \langle T_\tau e^{i\sqrt{2}[\phi_\parallel(x,\tau) + 2\phi_\parallel(x_1,\tau_1) - 2\phi_\parallel(x_2,\tau_2) - \phi_\parallel(0,0)]} \rangle
\]

(C2)

Where $C$ is a constant that depends on the regularization of the continuum model. It can be seen that the term that we obtain can be generated by the following effective coupling:

\[
H_{\text{effective}} = \int dx \xi_{\text{eff}}(x) e^{i\sqrt{2}\phi_\parallel} + \text{H. c.}
\]

(C3)

with $\xi_{\text{eff}}(x) \xi_{\text{eff}}(x') = D \delta(x - x')$ and $D \propto C D_a^2$. It is clear that for couplings of the form $e^{i\phi_\parallel} \cos(\theta_\parallel)$ the same argumentation will be equally valid. Note that using SCHA approximation gives different results; This is due to the fact that normal ordering in SCHA is done without taking the presence of the gaps into account. Therefore standard scaling, irrespective of the presence of the gaps always holds when one uses SCHA.

APPENDIX D: THE ORBITAL ANTIFERROMAGNET IN THE PRESENCE OF RANDOM INTRACHAIN HOPPING AND RANDOM INTERCHAIN HOPPING

We consider the following two types of random hopping: a random hopping along the chains:

\[
H_{\text{intrachain}} = \sum_{i,\sigma} \left[ \delta t_i^1 (c_{i+1,\sigma,1}^\dagger c_{i,\sigma,1} + c_{i,\sigma,1}^\dagger c_{i+1,\sigma,1}) + \delta t_i^2 (c_{i+1,\sigma,2}^\dagger c_{i,\sigma,2} + c_{i,\sigma,2}^\dagger c_{i+1,\sigma,2}) \right]
\]

(D1)

a random interchain hopping amplitude:
\[ H_{\text{interchain}} = \sum_{i, \sigma} \delta t_{\perp, i} \left( c_{i, 1}^\dagger c_{i, 2} + c_{i, 2}^\dagger c_{i, 1} \right) \]  \hspace{1cm} (D2)

Where \( \delta t_{\perp} \) is real. Bosonization of (D1) leads to an expression identical to the one that obtains by bosonizing a random on-site potential. It is then evident that the transport properties of the Orbital Antiferromagnet are the same in the presence of a random potential or random hopping along the chains. Bosonization of equation (D2) gives the following expression:

\[ H_{\text{interchain}} = \int \frac{2dx}{\pi \alpha} t_{\perp}^{2K_F} (x) e^{-i\phi_{\rho+}} [ i \sin \phi_{\rho+} \cos \phi_{\sigma-} \cos \phi_{\sigma+} + \cos \phi_{\rho-} \sin \phi_{\sigma-} \sin \phi_{\sigma+} ] + \text{H. c.} \]  \hspace{1cm} (D3)

It is not difficult to see that such term has exponentially decaying correlations since \( \theta_{\rho-} \) develops a gap. Integration of the massive \( \rho- \) mode leads to a coupling that is identical to the coupling to a random potential. Therefore a random amplitude of the hopping term also does not affect the transport properties of the OAF more severely than a random potential and thus the “superconducting” transport properties of the OAF are not an artifact of restricting to random potentials. All physically admissible random perturbations of the 2 chain system lead to the same limit for localization delocalization \( (K_{\rho+} = 3/2) \) the same behavior for conductivity as a function of frequency and temperature and the same dependence of localization length as a function of disorder.

On the other hand if the random hopping term has a random phase, there is a direct coupling to the OAF order parameter and then the OAF phase is suppressed. Such terms are allowed for instance in a tight binding picture only if the phases on the atoms of the 2 chain system cannot be made real. This could be achieved with a random magnetic flux in each plaquette of the 2 chain system.
TABLE I. The 4 sectors of the pure 2 chain spinless fermions model, as a function of $K_\rho$ and $g_f$. The average value of the massive field $\langle \theta_\parallel \rangle$ are indicated together with the phase with the most divergent susceptibility.

|     | I   | II  | III | IV  |
|-----|-----|-----|-----|-----|
| $g_f$ | +   | +   | -   | -   |
| $K_\rho$ | $\frac{\pi}{\sqrt{8}}$ | $\frac{\pi}{\sqrt{8}}$ | 0   | 0   |
| $\langle \theta_\parallel \rangle$ | $\frac{\pi}{\sqrt{8}}$ | $\frac{\pi}{\sqrt{8}}$ | 0   | 0   |
| phase | OAF | $SC^s$ | $SC^d$ | $CDW^\pi$ |

TABLE II. The 4 sectors of the pure 2 chain Hubbard model, as a function of $K_\rho$ and $g_1$. The average value of the field developing a gap are indicated together with the phase with the most divergent susceptibility.

|     | I   | II  | III | IV  |
|-----|-----|-----|-----|-----|
| $g_1$ | +   | +   | -   | -   |
| $K_\rho$ | $\frac{\pi}{\sqrt{8}}$ | $\frac{\pi}{\sqrt{8}}$ | 0   | 0   |
| $\langle \theta_{\rho-} \rangle$ | $\frac{\pi}{\sqrt{8}}$ | $\frac{\pi}{\sqrt{8}}$ | 0   | 0   |
| $\langle \phi_{\sigma+} \rangle$ | $\frac{\pi}{\sqrt{8}}$ | $\frac{\pi}{\sqrt{8}}$ | 0   | 0   |
| phase | $SC^d$ | OAF | $SC^s$ | $CDW^\pi$ |

TABLE III. The conductivities and localization lengths in the spinless fermions case. The phases are the ones of the non disordered system that are turned into localized ones upon introduction of a small disorder.

| phase | $L_{\text{loc.}}$ | $\sigma(T)$ |
|-------|------------------|-------------|
| $PCDW^{4k_F}$ | $\left( \frac{1}{D_n} \right) \frac{\pi}{3 - 4K_{\rho +}}$ | $T^{2 - 4K_{\rho +}}$ |
| $PCDW^\pi$ | $\left( \frac{1}{D_n} \right) \frac{\pi}{3 - K_{\rho +}}$ | $T^{2 - 4K_{\rho +}}$ |

TABLE IV. The conductivities and localization lengths in the fermions with spin case. The phases are the ones of the non disordered system that are turned into localized ones upon introduction of a small disorder.

| phase | $L_{\text{loc.}}$ | $\sigma(T)$ |
|-------|------------------|-------------|
| OAF, $SC^d, SC^s$ | $\left( \frac{1}{D_n} \right) \frac{\pi}{3 - 2K_{\rho +}}$ | $T^{2 - 2K_{\rho +}}$ |
| $CDW^\pi$ | $\left( \frac{1}{D_n} \right) \frac{\pi}{3 - K_{\rho +}}$ | $T^{2 - \frac{K_{\rho +}}{3}}$ |
FIG. 1. The phase diagram of a generic spinless ladder in terms of $g_f$ and $K_\rho$. $K_\rho > 1$ means attraction in the symmetric charge sector and $K_\rho < 1$ repulsion. The line depicts the phase spanned by the pure t-V ladder, leading to a $CDW^\pi$ phase for $V > 0$ and a superconducting $SC^s$ phase for $V < 0$. 
FIG. 2. The phase diagram of the disordered 2 chain t-V model in terms of $g_f$ and $K_\rho$. For a single chain the system is localized for $K_\rho < 3/2$. Ladder effect therefore delocalize for attractive interactions and enhance localization for repulsive ones.
FIG. 3. The phase diagram of the pure 2 chain Hubbard model in terms of $g_1$ and $K_{\rho+}$. $K_{\rho+} > 1$ and $g_1 < 0$ corresponds to purely attractive interactions. $K_{\rho+} < 1$ and $g_1 > 1$ corresponds to purely attractive interactions. For a Hubbard model, this leads to a $SC_d^d$ phase for $U > 0$ and a $SC^s$ phase for $U < 0$. 
FIG. 4. The phase diagram of the disordered 1 chain Hubbard model in terms of $g_{1\perp}$ and $K_\rho$. Delocalization occurs for $K_\rho > 3$ for $g_{1\perp} < 0$ and for $K_\rho > 2$ for $g_{1\perp} > 0$. 
FIG. 5. The phase diagram of the disordered 2 chain Hubbard model in terms of $g_1$ and $K_{ρ+}$. The $Sc^d$ phase is completely eaten by the $PCDW^{4k_F}$ phase whereas the OAF and the $SC^s$ persist if there is enough attraction. Delocalization occurs for $K > 3/2$ i.e. for less attractive interactions than in the one chain case.
FIG. 6. Behavior of the conductivities of the s-wave (dotted line) and d-wave (solid line) superconductor as a function of temperature. For $T \gg T_{\text{loc}}$, $\sigma(T) \propto T^{2-2\nu'}$. For the d-wave, there is no maximum in the conductivity and therefore no remnant of superconductivity in the localized phase.
FIG. 7. $T_c$ as a function of disorder for the d-wave phase ($K_\rho = 0.5$). $T_c$ drops quickly to zero for $D \approx 0.2$. 
FIG. 8. $T_c$ as a function of disorder for the s-wave phase ($K_\rho = 1.2$). $T_c$ drops to zero for $D \simeq 0.9$. Note the initial linear decay of $T_c$ that shows that Anderson Theorem does not hold in coupled chain system due to strong localization effects and absence of a diffusive regime.
REFERENCES

1. Unité Mixte de Recherche du C. N. R. S.
2. F. D. M. Haldane, Phys. Rev. Lett. 47, 1840 (1981).
3. F. D. M. Haldane, J. Phys. C 14, 2585 (1981).
4. J. Sólyom, Adv. Phys. 28, 209 (1979).
5. V. J. Emery, in Highly Conducting One-Dimensional Solids, edited by J. T. Devreese and et al. (Plenum, New York, 1979), p. 327.
6. D. G. Clarke, S. P. Strong, and P. W. Anderson, Phys. Rev. Lett. 72, 3218 (1994).
7. See e.g. D. Boies, C. Bourbonnais and A.-M. S. Tremblay Phys. Rev. Lett 74 968 (1995) and references therein.
8. E. Dagotto and T. M. Rice, Science 271, 5249 (1996).
9. M. Fabrizio, Phys. Rev. B 48, 15838 (1993).
10. D. V. Kveschenko and T. M. Rice, Phys. Rev. B 50, 252 (1994).
11. A. M. Finkelstein and A. I. Larkin, Phys. Rev. B 47, 10461 (1993).
12. H. J. Schulz, Phys. Rev. B 53, 2959 (1996).
13. L. Balents and M. P. A. Fisher, Phys. Rev. B 53, 12133 (1996).
14. N. Nagaosa, Sol. State Comm. 94, 495 (1995).
15. A. Nersesyan, A. Luther, and F. Kusmartsev, Phys. Lett. A 176, 363 (1993).
16. H. Yoshioka and Y. Suzumura, J. Low Temp. Phys 106, 49 (1997).
17. E. Dagotto, J. Riera, and D. Scalapino, Phys. Rev. B 45, 5744 (1992).
18. R. Noack, S. White, and D. Scalapino, Phys. Rev. Lett. 73, 882 (1994).
19. D. Poilblanc, W. Hanke, and D. Scalapino, 1994, cond-mat/9501093 preprint.
20. D. Poilblanc, H. Tsunetsugu, and T. M. Rice, Phys. Rev. B 50, 6511 (1994).
21. H. Tsunetsugu, M. Troyer, and T. M. Rice, Phys. Rev. B 49, 16078 (1994).
22. M. Takano, Z. Hiroi, M. Azuma, and Y. Bando, J. of Solid State Chem. 95, 230 (1991).
23. M. Takano et al., Phys. Rev. Lett. 73, 3463 (1994).
24. R. S. Eccleston, T. Barnes, J. Brody, and J. W. Johnson, Phys. Rev. Lett. 73, 2626 (1994).
25. D. C. Johnston, J. W. Johnson, D. P. Goshorn, and A. J. Jacobson, Phys. Rev. B 35, 219 (1987).
26. D. J. Scalapino, E. Loh, and J. E. Hirsch, Phys. Rev. B 34, 8190 (1986).
27. R. Bickers, D. J. Scalapino, and R. T. Scalettar, Int. J. Mod. Phys. B 1, 687 (1987).
28. R. Bickers, D. J. Scalapino, and S. R. White, Phys. Rev. Lett. 62, 961 (1989).
29. N. E. Bickers, D. J. Scalapino, and S. R. White, Ann. Phys. 193, 206 (1989).
30. N. E. Bickers, D. J. Scalapino, and S. R. White, Phys. Rev. B 43, 8044 (1991).
31. K. Miyake, S. Schmitt-Rink, and C. M. Varma, Phys. Rev. B 34, 6554 (1986).
32. D. Pines, 1997, see cond-mat preprint 9702187 and references therein.
33. G. Baskaran and P. W. Anderson, Phys. Rev. B 37, 580 (1988).
34. A. A. Abrikosov and J. A. Ryzhkin, Adv. Phys. 27, 147 (1978).
35. V. L. Berezinskii, Sov. Phys. JETP 38, 620 (1974).
36. T. Giamarchi and H. J. Schulz, Phys. Rev. B 37, 325 (1988).
37. Y. Suzumura and T. Giamarchi, J. Phys. Soc. Jpn. 58, 1748 (1989).
38. U. Meirav, M.A. Kastner, and S.J. Wind, Phys. Rev. Lett. 65, 771 (1990).
39. U. Meirav, M. A. Kastner, M. Heilblum, and S. J. Wind, Phys. Rev. B 40, 5871 (1989).
40. A. R. Góñi et al., Phys. Rev. Lett. 67, 3298 (1991).
S. Tarucha, T. Saku, Y. Tokura, and Y. Hirayama, Phys. Rev. B 47, 4064 (1993).

M. A. Kastner, Rev. Mod. Phys. 64, 849 (1992).

L. I. Glazman, I. M. Ruzin, and B. I. Shklovskii, Phys. Rev. B 45, 8454 (1992).

H. J. Schulz, Phys. Rev. Lett. 71, 1864 (1993).

H. Maurey and T. Giamarchi, Phys. Rev. B 51, 10833 (1995).

T. Giamarchi and B. S. Shastry, Phys. Rev. B 51, 10915 (1995).

E. Orignac and T. Giamarchi, Phys. Rev. B 53, 10453 (1996).

M. P. M. den Nijs, Phys. Rev. B 23, 6111 (1981).

H. Maurey and T. Giamarchi, J. Phys. (Paris) 49, 819 (1988).

Y. Suzumura and H. Fukuyama, J. Phys. Soc. Jpn. 52, 2870 (1983).

H. Fukuyama and P. A. Lee, Phys. Rev. B 17, 535 (1978).

W. Kohn, Phys. Rev. 133, A171 (1964).

B. S. Shastry and B. Sutherland, Phys. Rev. Lett. 65, 243 (1990).

D. J. Scalapino, S. R. White, and S. C. Zhang, Phys. Rev. Lett. 68, 2830 (1992).

M. Buttiker, Y. Imry, and R. Landauer, Phys. Lett. A 365, 365 (1983).

H. F. Cheung, Y. Gefen, E. K. Riedel, and W. H. Shih, Phys. Rev. B 37, 6050 (1988).

N. Trivedi and D. A. Browne, Phys. Rev. B 38, 9581 (1988).

H. Bouchiat and G. Montambaux, J. Phys. (Paris) 50, 2695 (1989); G. Montambaux, H. Bouchiat, D. Sigeti and R. Friesner, Phys. Rev. B 42, 7647 (1990).

B. Altshuler, Y. Gefen, and Y. Imry, Phys. Rev. Lett. 66, 88 (1991).

V. Ambegaokar and U. Eckern, Phys. Rev. B 65, 381 (1990).

A. Muller-Groeling, H. A. Weidenmuller, and C. Lewenkopf, Europhys. Lett. 22, 193 (1993).

A. Muller-Groeling, H. A. Weidenmuller, and C. Lewenkopf, Phys. Rev. B 49, 4752 (1994).

M. Ramin, B. Reulet, and H. Bouchiat, Phys. Rev. B 51, 5582 (1995).

G. Bouzerar, D. Poilblanc, and G. Montambaux, Phys. Rev. B 49, 8258 (1994).

G. Bouzerar and D. Poilblanc, 1995, cond-mat preprint 9605188.

R. Berkovits and Y. Avishai, Europhys. Lett. 29, 475 (1995).

H. J. Schulz, 1994, unpublished notes on the 2chain Hubbard model.

N. Kawakami and S. K. Yang, Phys. Rev. B 44, 7844 (1991).

I. Affleck and J. B. Marston, Phys. Rev. B 37, 3774 (1988).

G. Kotliar, Phys. Rev. B 37, 3664 (1988).

P. W. Anderson, B. S. Shastry, and D. Hristopulos, Phys. Rev. B 40, 8939 (1989).

P. Lederer, D. Poilblanc, and T. M. Rice, Phys. Rev. Lett. 63, 1519 (1989).

G. Kotliar and C. M. Varma, cond-mat preprint 9607105 (unpublished).

S. Brazovskii and V. Yakovenko, J. de Phys. (Paris) Lett. 46, L111 (1985).

A. J. Millis, S. Sachdev, and C. M. Varma, Phys. Rev. B 37, 4975 (1988).

M. Uchara et al., J. Phys. Soc. Jpn. 65, 2764 (1996).

I. Safi and H. J. Schulz, Phys. Rev. B 52, R17040 (1995).

D. Maslov and M. Stone, Phys. Rev. B 52, R5539 (1995).

S. Tarucha, T. Honda, and T. Saku, Sol. State Comm. 94, 413 (1995).

C. Kane and M. P. A. Fisher, Phys. Rev. B 46, 15233 (1992).

C. Kane and M. P. A. Fisher, Phys. Rev. Lett. 68, 1220 (1992).

M. Ogata and H. Fukuyama, Phys. Rev. Lett. 73, 468 (1994).

T. Giamarchi and H. Maurey, in Correlated fermions and transport in mesoscopic systems,
There exists a related work on coupled chains systems by Maslov and Sandler (cond-mat/9701153). These authors assume that there are no spin gaps and compute $K_\rho$ for quantum wires. However in experimental systems the disorder is rather weak and thus should only affect the ungapped degrees of freedom. Therefore we think that our assumptions are closer to the experimental situation.

H. J. Schulz, in Correlated fermions and transport in mesoscopic systems, edited by T. Martin, G. Montambaux, and J. Tran Thanh Van (Editions frontières, Gif sur Yvette, France, 1996).

T. Giamarchi and H. J. Schulz, Phys. Rev. B 39, 4620 (1989).

J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).

J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B 16, 1217 (1977).

W. Götzte and P. Wölfle, Phys. Rev. B 6, 1226 (1972).

T. Giamarchi, Phys. Rev. B 44, 2905 (1991).