Generating Higher-Order Lie Algebras by Expanding Maurer Cartan Forms.

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Abstract

By means of a generalization of the Maurer-Cartan expansion method we construct a procedure to obtain expanded higher-order Lie algebras. The expanded higher order Maurer-Cartan equations for the case $\mathcal{G} = V_0 \oplus V_1$ are found.

A dual formulation for the S-expansion multialgebra procedure is also considered. The expanded higher order Maurer Cartan equations are recovered from S-expansion formalism by choosing a special semigroup. This dual method could be useful in finding a generalization to the case of a generalized free differential algebra, which may be relevant for physical applications in, e.g., higher-spin gauge theories.

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I. INTRODUCTION

A Lie algebra $\mathcal{G}$ with basis $\{X_i\}$, may be realized by left-invariant generators $X_i$ on the corresponding group manifold. If $C_{ij}^k$ are the structure constants of $\mathcal{G}$ in the basis $\{X_i\}$, then they satisfy $[X_i, X_j] = C_{ij}^k X_k$. If $\{\omega^i(g)\}$, $i = 1, ..., r = \text{dim} G$, are the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on $G$; then, the Maurer-Cartan equations that characterize $\mathcal{G}$, in a way dual to its Lie bracket description, are given by

$$d\omega^k = -\frac{1}{2} C_{ij}^k \omega^i \wedge \omega^j, \quad i, j, k = 1, ..., r. \quad (1)$$

In direct analogy we can say that a higher-order Lie algebra $(\mathcal{G}, [\cdot, \cdot, \cdots])$ \cite{1}, \cite{2}, \cite{3} with basis $\{X_i\}$, may be realized by left-invariant generators $X_i$ on the corresponding group manifold. If $C_{i_1i_2\cdots i_n}^k$ are the higher order structure constants of $(\mathcal{G}, [\cdot, \cdot, \cdots])$ in the basis $\{X_i\}$, then they satisfy $[X_{i_1}, \ldots, X_{i_n}] = C_{i_1i_2\cdots i_n}^k X_k$, where $[X_{i_1}, \ldots, X_{i_n}]$ are the called higher order Lie bracket or multibracket. If $\{\omega^i\}$, $i = 1, ..., r = \text{dim} G$, are the basis determined by the (dual, left-invariant) Maurer–Cartan one-forms on $G$; then, the generalized Maurer-Cartan equations that characterize $(\mathcal{G}, [\cdot, \cdot, \cdots])$, in a way dual to its higher order Lie bracket description, are given by

$$\tilde{d}\omega^\sigma = \frac{1}{(2m-2)!} \Omega_{i_1, \ldots, i_{2m-2}}^{\sigma} \omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-2}}, \quad (2)$$

where $\tilde{d}_m$ are the so-called higher-order exterior derivations.

As is noted in ref \cite{1}, \cite{3} it could be interesting to find applications of these higher-order Lie algebras to know whether the cohomological restrictions which determine and conditions their existence have a physical significance. Lie algebra cohomology arguments have already been very useful in various physical problems as in the description of anomalies or in the construction of the Wess-Zumino terms required in the actions of extended supersymmetric objects. Other questions may be posed from a purely mathematical point of view. From the discussion in Sect.4 of ref. \cite{1} we know that a representation of a simple Lie algebra may not be a representation for the associated higher-order Lie algebras. Thus, the representation theory of higher-order algebras requires a separate analysis. A very interesting open problem from a structural point of view is the expansions of higher-order Lie algebras, which will take us outside the domain of the simple ones.

The purpose of this paper is to show that the expansion methods developed in ref. \cite{5},...
(see also [7], [8]) can be generalized so that they permits obtaining new higher-order Lie algebras of increasing dimensions from \((G, [\ldots, \ldots])\) by a geometric procedure based on expanding the generalized Maurer Cartan equations.

The paper is organized as follows: In section 2 we shall review some aspects of Generalized Maurer-Cartan equations. The main point of this section is to display the differences between ordinary Maurer-Cartan equations and Generalized Maurer-Cartan equations. In sections 3, 4 we generalize the expansion methods developed in ref. [5], [6] and we give the general structure of the expansion method. Section 4 is devoted to the dual S-expansion of higher-order Lie algebras. We close in section 5 with conclusions and an outlook for future work.

II. GENERALIZED MAURER-CARTAN EQUATIONS

In this section we shall review some aspects of the generalized Maurer-Cartan equations. The main point of this section is to display the differences between ordinary Maurer-Cartan equations and Generalized Maurer-Cartan equations (see [1], [2]).

**Definition 1** Let \(\{X_i\}\) be a basis of \(G\) given in terms of left invariant vector fields on \(G\), and \(\wedge * (G)\) be the exterior algebra of multivectors generated by them \(X_1 \wedge \cdots \wedge X_q \equiv \varepsilon_{i_1 \cdots i_q} X_{i_1} \otimes \cdots \otimes X_{i_q}\). The exterior coderivation \(\partial : \wedge^q \to \wedge^{q-1}\) is given by

\[
\partial (X_1 \wedge \cdots \wedge X_q) = \sum_{l=1}^{q} (-1)^{l+k+1} [X_l, X_k] \wedge X_1 \wedge \cdots \wedge \hat{X}_l \wedge \cdots \wedge \hat{X}_k \cdots \wedge X_q. \quad (3)
\]

This definition is analogous to that of the exterior derivative \(d\), as given by the Palais formula [2] with its first term missing when one considers left-invariant forms (see eq.(2.4) ref. [2]). As \(d\), \(\partial\) is nilpotent, \(\partial^2 = 0\), due to the Jacobi Identity for the commutator. In order to generalize [3], let us note that \(\partial(X_1 \wedge X_2) = [X_1, X_2]\), so that [3] can be interpreted as a formula that gives the action of \(\partial\) on a \(q\)-vector in terms of that on a bivector. For this reason we may write \(\partial_2\) for \(\partial\) above. It is then natural to introduce an operator \(\partial_s\) that on a \(s\)-vector gives the multicommutator of order \(s\).

**Definition 2** The general coderivation \(\partial_s\) of degree \((s-1)\) \((s\ even)\ \partial_s : \wedge^n(G) \to \wedge^{n+2}G\) is given by...
\(^{n-(s-1)}(G)\) is defined by the action on an \(n\)-multivector

\[
\partial_s (X_1 \wedge \cdots \wedge X_n) = \frac{1}{s! (n-s)!} \varepsilon_{i_1 \cdots i_n} \partial_s (X_{i_1} \wedge \cdots \wedge X_{i_s}) \wedge X_{i_{s+1}} \wedge \cdots \wedge X_{i_n},
\]

(4)

\[
\partial_s \wedge^n (G) = 0 \quad \text{for} \ s > n,
\]

(5)

\[
\partial_s (X_1 \wedge \cdots \wedge X_s) = [X_1 \wedge \cdots \wedge X_s],
\]

(6)

\[
\partial_s^2 \equiv 0.
\]

(7)

We may now introduce the corresponding dual higher-order derivations \(\tilde{d}_s\) to provide a generalization of the Maurer-Cartan equations. Since \(\partial_s\) was defined on multivectors that are products of left-invariant vector fields, the dual \(\tilde{d}_s\) will be given for left-invariant forms.

It is easy to introduce dual bases in \(\wedge_n\) and in \(\wedge^n\). With \(\omega^j(X_j) = \delta_{ij}\), a pair of dual bases \(\wedge_n\), \(\wedge^n\) are given by \(\omega^{I_1} \wedge \cdots \wedge \omega^{I_n}\), \(\frac{1}{n!} X_{I_1} \wedge \cdots \wedge X_{I_n}\) \((I_1 < \ldots < I_n)\) since

\[
\left(\varepsilon_{j_1 \cdots j_n} \omega^{j_1} \otimes \cdots \otimes \omega^{j_n}\right) \left(\frac{1}{n!} X_{I_1} \otimes \cdots \otimes X_{I_n}\right) = \varepsilon_{i_1 \cdots i_n} \varepsilon_{I_1 \cdots I_n} \text{ is 1 if all indices coincide and 0 otherwise.}
\]

Nevertheless it is customary to use the non-minimal set \(\omega^{j_1} \wedge \cdots \wedge \omega^{j_n}\) to write \(\alpha = \frac{1}{n!} \alpha_{i_1 \cdots i_n} \omega^{i_1} \wedge \cdots \wedge \omega^{i_n}\). Since \((\omega^{j_1} \wedge \cdots \wedge \omega^{j_n}) (X_{j_1}, \ldots, X_{j_n}) = \varepsilon_{j_1 \cdots j_n} \varepsilon_{i_1 \cdots i_n}\) it is clear that \(\alpha_{i_1 \cdots i_n} = \alpha (X_{i_1}, \ldots, X_{i_n}) = \frac{1}{n!} (X_{i_1} \wedge \cdots \wedge X_{i_n})\).

**Definition 3** The action of \(\tilde{d}_m: \wedge_n \rightarrow \wedge_{n+2m-3}\) (remember that \(s = 2m - 2\)) on \(\alpha \in \wedge_n\) is given by

\[
\tilde{d}_m \alpha (X_{i_1}, \ldots, X_{i_{n+2m-3}}) = \frac{1}{(2m-2)! (n-1)!} \varepsilon^{j_1 \cdots j_{2m-3}} \alpha \left( [X_{j_1}, \ldots, X_{j_{2m-2}}], X_{j_{2m-1}}, \ldots, X_{j_{n+2m-3}} \right),
\]

(8)

\[
\left(\tilde{d}_m \alpha\right)_{i_1 \cdots i_{n+2m-3}} = \frac{1}{(2m-2)! (n-1)!} \varepsilon^{j_1 \cdots j_{2m-3}} \Omega_{j_1 \cdots j_{2m-2}} \rho \alpha_{\rho j_{2m-1} \cdots j_{n+2m-3}}.
\]

(9)

From \((8,9)\) we can see that the coordinates of \(\tilde{d}_m \omega^\sigma\) are given by

\[
\left(\tilde{d}_m \omega^\sigma\right) (X_{i_1}, \ldots, X_{i_{2m-2}}) = \frac{1}{(2m-2)!} \varepsilon^{j_1 \cdots j_{2m-2}} \omega^\sigma \left( [X_{j_1}, \ldots, X_{j_{2m-2}}] \right),
\]

(10)

\[
\left(\tilde{d}_m \omega^\sigma\right) (X_{i_1}, \ldots, X_{i_{2m-2}}) = \omega^\sigma \left( [X_{i_1}, \ldots, X_{i_{2m-2}}] \right) = \omega^\sigma \Omega_{i_1 \cdots i_{2m-2}} \rho X_\rho = \Omega_{i_1 \cdots i_{2m-2}} \omega^\rho,
\]

(11)

from which we conclude that

\[
\tilde{d}_m \omega^\sigma = \frac{1}{(2m-2)!} \Omega_{i_1 \cdots i_{2m-2}} \omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-2}}.
\]

(12)

For \(m = 2\), \(\tilde{d}_2 = -d\), equation \((12)\) reproduces the usual Maurer-Cartan equations. The equation \((12)\) is called "the generalized Maurer-Cartan equation". In the compact notation
that uses the canonical one-form $\theta$, we can say the action of $\tilde{d}_m$ on the canonical form $\theta$ is given by

$$\tilde{d}_m \theta = \frac{1}{(2m-2)!} \left[ \theta, \theta, \cdots, \theta \right],$$

(13)

where the multibracket of form is defined by

$$\left[ \theta, \theta, 2m-2 \cdots \theta, \theta \right] = \omega_{i_1} \wedge \cdots \wedge \omega_{i_{2m-2}} \left[ X_{i_1}, \cdots, X_{i_{2m-2}} \right].$$

(14)

Using Leibniz’s rule for the $\tilde{d}_m$ operator we arrive at

$$\tilde{d}_m^2 \theta = -\frac{1}{(2m-2)!} \frac{1}{(2m-3)!} \left[ \theta, 2m-3 \cdots \theta, \left[ \theta, 2m-2 \cdots \theta \right] \right] = 0,$$

(15)

which again expresses the Generalized Jacobi Identity.

III. EXPANDING HIGHER ORDER LIE ALGEBRAS BY RESCALING SOME COORDINATES OF THE GROUP MANIFOLD

A. The higher-order Lie algebras $(G(N), [[...]])$ generated from $(G, [[...]])$, when $G = V_0 \oplus V_1$.

The generalized Maurer-Cartan equations that characterize the multialgebra $(G, [[...]])$, in a way dual to its higher order Lie bracket description, are given by

$$\tilde{d}_m \omega^k(g) = \frac{1}{(2m-2)!} C^k_{i_1 \cdots i_{2m-2}} \omega^{i_1}(g) \wedge \cdots \wedge \omega^{i_{2m-2}}(g).$$

(16)

Consider the splitting of $G^*$ into the sum of two vector subspaces, $G^* = V_0^* \oplus V_1^*$, where $V_0^*$ and $V_1^*$ are generated by the Maurer-Cartan forms $\omega^{i_0}(g)$ and $\omega^{i_1}(g)$ of $G^*$ with indices corresponding, respectively, to the unmodified and modified parameters,

$$g^{i_0} \rightarrow g^{i_0}, \quad g^{i_1} \rightarrow \lambda g^{i_1}, \quad i_0(i_1) = 1, ..., \dim V_0(V_1).$$

(17)

In general, the series of $\omega^{i_0}(g, \lambda) \in V_0^*$ and $\omega^{i_1}(g, \lambda) \in V_1^*$, will involve all powers of $\lambda$,

$$\omega^{i_p}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_p, \alpha}(g), \quad p = 0, 1.$$

(18)

In terms of the 1-forms $\omega^{i_p}$, the generalized Maurer-Cartan equations (16) take the form
\[ \tilde{d}_m \omega^{k_s} (g) = \frac{1}{(2m-2)!} C_{i_p \cdots i_{2m-2}}^{k_s} \omega^{i_p} (g) \wedge \omega^{i_q} (g) \wedge \omega^{m_r} (g) \cdots \wedge \omega^{n_t} (g), \]  

where, on the right side of equation (16) there is an implicit sum on the \((2m-2)\) indices \(i_p, j_q, m_r, \cdots, n_t = 1, \cdots, \dim V_p(V_q)(V_r) \cdots (V_t)\) and on the \((2m-2)\) indices \(p, q, r, \cdots, t = 0, 1\). Explicitly we have

\[ \tilde{d}_m \omega^{k_s} (g) = \frac{1}{(2m-2)!} \sum_{p,q,r,\ldots,t=0}^{\dim V_p(V_q)(V_r)\cdots(V_t)} \sum_{i_p,j_q,m_r,\ldots,n_t=0}^{\dim V_p(V_q)(V_r)\cdots(V_t)} C_{i_p \cdots i_{2m-2}}^{k_s} \omega^{i_p} (g) \wedge \omega^{i_q} (g) \wedge \omega^{m_r} (g) \cdots \wedge \omega^{n_t} (g). \]  

However, in general, we will consider the sums implicitly. We will denote the set of \((2m-2)\) indices \(i, j, m, ..., n = 0, ..., \dim G, \) by \(i^l\), where \(i^1 = i, i^2 = j, i^3 = k, \cdots, i^{2m-2} = n\) i.e., \(i^l = 0, ..., \dim G; l = 1, ..., 2m-2\). Since \(G = V_0 \oplus V_1\), we have that the set of \((2m-2)\) indices \(p, q, r, ..., t = 0, 1\) is useful to indicate that the forms \(\omega^{i_p}, \omega^{i_q}, \omega^{m_r}, \omega^{n_t}\) belong to the subspaces \(V^*_p, V^*_q, ..., V^*_t\) respectively. This allows to denote the set of indices \(p, q, r, ..., t = 0, 1\) with the index \(p_l\), where the index \(l\) reproduces the \((2m-2)\) indices: \(p_1 = p, p_2 = q, p_3 = r, \cdots, p_{2m-2} = t\).

With this notation, the generalized Maurer-Cartan equations (19) take the form

\[ \tilde{d}_m \omega^{k_s} (g) = \frac{1}{(2m-2)!} C_{i_{p_1} \cdots i_{2p-2}}^{k_s} \omega^{i_{p_1}} (g) \wedge \omega^{2m-2}_{p_{2m-2}} (g), \]  

where we have summed over \(i_{p_l}\) and over \(p_l = 0, 1\) for every \(l = 1, ..., 2m-2\). One might think that the super-index in \(i_{p_l}\) is superfluous. However the super-index \(l\) is really necessary, for example, to distinguish the independent sums existing over the indices \(i_{p_l}\) and \(i_{l+1}\) when \(p_l = p_{l+1}\). In the compact notation that uses the canonical one-form \(\theta = \omega^{k_s} X_k\) [2], the eq. (21) can be written as

\[ \tilde{d}_m \theta = \frac{1}{(2m-2)!} \left[ \theta, \theta, \cdots, \theta \right] , \]  

where the multibracket of forms is defined by

\[ \left[ \theta, \theta, \cdots, \theta \right] = \omega^{i_{p_1}} \wedge \cdots \wedge \omega^{2m-2}_{p_{2m-2}} \left[ X_{i_{p_1}}, \cdots, X_{i_{2m-2}} \right]. \]
Following the procedure of Ref. [5] we now insert the expansions [18] into the Maurer-Cartan equations [16]. After tedious but direct calculation we obtain

$$\bar{d}_m \omega^{k_s,\alpha} = \frac{1}{(2m - 2)!} C^{(k_s,\alpha)}_{(i^1_{p_1},\beta^1),\ldots,(i^{2m-2}_{p_{2m-2}},\beta^{2m-2})} \omega^{i^1_{p_1},\beta^1}_1 \wedge \cdots \wedge \omega^{i^{2m-2}_{p_{2m-2}},\beta^{2m-2}}_N,$$

where on the right side, besides the sums over $i^l_{p_l}$ and $p_l$, a sum exists over $\beta^l = 0, 1, \cdots, \alpha$ for every $l = 0, \ldots, 2m - 2$ and where

$$C^{(k_s,\alpha)}_{(i^1_{p_1},\beta^1),\ldots,(i^{2m-2}_{p_{2m-2}},\beta^{2m-2})} = C^{k_s}_{i^1_{p_1},\ldots,i^{2m-2}_{p_{2m-2}}} \delta^{\alpha}_{\beta^1+\cdots+\beta^{2m-2}}.$$

In the compact notation that now uses the canonical one-form $\theta^{(N)} = \omega^{k_s,\alpha} X_{k_s,\alpha}$ of the expanded multialgebra, $(\mathcal{G}(N), [, ...,])$, the eq. (24) can be written as

$$\bar{d}_m \theta^{(N)} = \frac{1}{(2m - 2)!} \left[ \theta^{(N)}, \theta^{(N)}, 2^{2m-2}, \cdots, \theta^{(N)} \right],$$

where the multibracket of forms is defined by

$$\left[ \theta^{(N)}, \theta^{(N)}, 2^{2m-2}, \cdots, \theta^{(N)} \right] = \omega^{i^1_{p_1},\beta^1}_1 \wedge \cdots \wedge \omega^{i^{2m-2}_{p_{2m-2}},\beta^{2m-2}}_N \left[ X_{i^1_{p_1},\beta^1}, \cdots, X_{i^1_{p_1},\beta^1} \right].$$

Note that $\{X_{k_s,\alpha}\}$ is the basis of $(\mathcal{G}(N), [, ...,])$ while $\{\omega^{k_s,\alpha}\}$ is the dual basis.

The generalized Jacobi identity is obtained the calculation of $\bar{d}_m^2 \omega^{k_s,\alpha}$. The equations (24)-(26) are the direct generalization to the case of higher order Lie algebras of the equations (2.15) of the Ref. [5].

The following theorem generalizes the theorem 1 of Ref. [5] to the case of higher order Lie algebras and it establishes the conditions under which the 1-forms $\omega^{i_0,\alpha_0}, \omega^{i_1,\alpha_1}$ generate new higher order Lie algebras.

**Theorem 4** Let $(\mathcal{G}, [, ...,])$ be a higher order Lie algebra and $\mathcal{G} = V_0 \oplus V_1$ (no higher order Lie sub-algebra conditions are assumed, neither for $V_0$ nor for $V_1$). Let $\{\omega^i\}$, $\{\omega^{i_0}\}$, $\{\omega^{i_1}\}$ ($i = 1, \cdots, \dim \mathcal{G}$, $i_0 = 1, \cdots, \dim V_0$, $i_1 = 1, \cdots, \dim V_1$) be, respectively, the bases of the $\mathcal{G}^*$, $V_0^*$ and $V_1^*$ dual vector spaces. Then, the vector space generated by

$$\{\omega^{i_0,0}, \omega^{i_0,1}, \cdots, \omega^{i_0,N}; \omega^{i_1,0}, \omega^{i_1,1}, \cdots, \omega^{i_1,N}\},$$

(28)

**together with the generalized Maurer-Cartan equations (24)** for the structure constants (26) determine a higher order Lie algebra $\mathcal{G}(N)$ for each expansion order $N \geq 0$ of dimension

$$\dim \mathcal{G}(N) = (N + 1) \dim \mathcal{G}.$$
**Proof.** The generalized Maurer-Cartan equations \( \tilde{d}_m \omega^{k_s,\alpha} = \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}=0}^\alpha C^{k_s}_{i_1^{p_1}, \ldots, i_{2m-2}^{p_{2m-2}}} \delta^{j_1 \beta_1 + \cdots + \beta_{2m-2}}_{\gamma_1} \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}}. \) (29)

Let’s remember that we have sums over \( p_l \) and over \( \beta^l \) such that \( \alpha = \beta^1 + \cdots + \beta^{2m-2} \). We can see that for \( \alpha = N_0 \)

\[
\tilde{d}_m \omega^{k_0,N_0} = \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}=0}^{N_0} C^{k_0}_{i_1^{p_1}, \ldots, i_{2m-2}^{p_{2m-2}}} \delta^{N_0}_{\beta_1 + \cdots + \beta_{2m-2}} \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}}, \quad (30)
\]

appear in the sum terms that contain 1-forms \( \omega^{i_l,N_0} \), whereas

\[
\tilde{d}_m \omega^{k_1,N_1} = \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}=0}^{N_1} C^{k_1}_{i_1^{p_1}, \ldots, i_{2m-2}^{p_{2m-2}}} \delta^{N_0}_{\beta_1 + \cdots + \beta_{2m-2}} \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}}, \quad (31)
\]

appear in the sum terms that contain 1-forms \( \omega^{i_l,N_1} \). Wherefrom we see that the forms \( \omega^{i_l,N_0} \) and \( \omega^{i_l,N_1} \), for any \( l = 1, \ldots, 2m-2 \), are in the base \( \mathcal{G} \), if and only if \( N_0 = N_1 = N \).

This means that the set

\[
\{ \omega^{i_0,0}, \omega^{i_0,1}, \ldots, \omega^{i_0,N}, \omega^{i_1,0}, \omega^{i_1,1}, \ldots, \omega^{i_1,N} \},
\]

generates a higher order Lie algebra of dimension

\[
\dim \mathcal{G}(N) = (N + 1) \dim V_0 + (N + 1) \dim V_1 = (N + 1) \dim \mathcal{G}. \quad (33)
\]

To prove that the generalized Jacobi identity is satisfied we calculate \( \tilde{d}_m^2 \omega^{k_s,\alpha} : \)

\[
\tilde{d}_m^2 \omega^{k_s,\alpha} = \frac{1}{(2m-2)!} C^{(k_s,\alpha)}_{i_1^{p_1}, \beta_1} \cdots (i_{2m-2}^{p_{2m-2}}, \beta_{2m-2}) \tilde{d}_m \left( \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}} \right) \tilde{d}_m \left( \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}} \right),
\]

\[
= \frac{(2m-2)!}{(2m-2)!} C^{(k_s,\alpha)}_{i_1^{p_1}, \beta_1} \cdots (i_{2m-2}^{p_{2m-2}}, \beta_{2m-2}) \tilde{d}_m \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}},
\]

\[
= \frac{1}{(2m-2)! (2m-3)!} C^{(k_s,\alpha)}_{i_1^{p_1}, \beta_1} (i_2^{p_2}, \beta_2) \cdots (i_{2m-2}^{p_{2m-2}}, \beta_{2m-2}) \left( C^{i_1^{p_1}, \beta_1}_{j_1^{p_1}, \gamma_1} \cdots (j_{2m-2}^{p_{2m-2}}, \gamma_{2m-2}) \right) \times \left( \omega^{j_1}_{\gamma_1} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}} \right) \wedge \left( \omega^{j_2}_{\beta_2} \wedge \cdots \wedge \omega^{j_{2m-2}}_{\gamma_{2m-2}} \right) = 0. \quad (34)
\]

Therefore

\[
C^{(k_s,\alpha)}_{i_1^{p_1}, \beta_1} (i_2^{p_2}, \beta_2) \cdots (i_{2m-2}^{p_{2m-2}}, \beta_{2m-2}) C^{(i_1^{p_1}, \beta_1)}_{j_1^{p_1}, \gamma_1} \cdots (j_{2m-2}^{p_{2m-2}}, \gamma_{2m-2}) = 0. \quad (35)
\]

Introducing \( (35) \) into \( (29) \) we find

\[
\delta^{\beta_1 + \cdots + \beta_{2m-2}}_{\gamma_1 + \cdots + \gamma_{2m-2}} C^{k_s}_{i_1^{p_1}, \ldots, i_{2m-2}^{p_{2m-2}}} C^{i_1^{p_1}, \beta_1}_{j_1^{p_1}, \ldots, j_{2m-2}^{p_{2m-2}}} = 0, \quad (36)
\]

which is satisfied identically due to the validity of the generalized Jacobi identity for the original multialgebra \( \mathcal{G}, [\cdot, \cdot, \cdot] \).
B. The case in which \( V_0 \) is a subalgebra of \( \mathcal{G} \) and of a submultialgebra \( (\mathcal{G}, [\ldots]) \)

Let \( (\mathcal{G}, [\ldots]) \) be a Lie algebra and let \( (\mathcal{G}, [\ldots]) \) be a higher order Lie algebra. We will assume that the vector space \( \mathcal{G} = V_0 \oplus V_1 \) is such that \( V_0 \) is a subalgebra of \( (\mathcal{G}, [\ldots]) \) and a submultialgebra of \( (\mathcal{G}, [\ldots]) \). From Ref. [5] it is known that, if \( V_0 \) is a subalgebra, then

\[
\omega^i_p (g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^i_p,\alpha (g),
\]

\[
\omega^i_p,\alpha = 0, \text{ for } \alpha < p.
\]

Introducing (37) into the generalized Maurer-Cartan equations we find that, when \( V_0 \) is a subalgebra, the generalized expanded Maurer-Cartan equations are given by

\[
\tilde{d}_m \omega^{k_s,\alpha_s} = \frac{1}{(2m-2)!} C^{(k_s,\alpha_s)} (i_{1\cdot\cdot\cdot p_l}^1, \beta_{1\cdot\cdot\cdot p_l}^1) \cdot (i_{2m-2,\cdot\cdot\cdot, 2m-2}^1, \beta_{2m-2,\cdot\cdot\cdot, 2m-2}^1) \omega^i_p,^i_p,\beta_{p_l}^1 \wedge \ldots \wedge \omega^i_{p_2,^i_{p_2},\beta_{p_2}^2} \wedge \ldots \wedge \omega^i_{p_{2m-2},^i_{p_{2m-2}},\beta_{p_{2m-2}}^{2m-2}},
\]

where

\[
C^{(k_s,\alpha_s)} (i_{1\cdot\cdot\cdot p_l}^1, \beta_{1\cdot\cdot\cdot p_l}^1) \cdot (i_{2m-2,\cdot\cdot\cdot, 2m-2}^1, \beta_{2m-2,\cdot\cdot\cdot, 2m-2}^1) = C^{(k_s)} (i_{p_1}^1, \beta_{p_1}^1) \cdot (i_{p_2}^2, \beta_{p_2}^2) \cdot \ldots \cdot (i_{2m-2}^m, \beta_{2m-2}^m),
\]

\[
\alpha_s = 0, \ldots, N_s; \beta_{p_l}^l = 0, \ldots, \beta_{p_l}^l = 0, 1; \quad \beta_{p_l}^l = 0, \text{ for } \beta_{p_l}^l < p_l; \quad l = 1, \ldots, 2m-2.
\]

The equations (39)-(42) are a direct generalization to a higher order Lie algebra case of equations (3.13)-(3.14) of Ref. [5]. In the compact notation that uses the canonical one-form \( \theta^{(N)} \), the eq. (39) can be written as

\[
\tilde{d}_m \theta^{(N)} = \frac{1}{(2m-2)!} \left[ \theta^{(N)}, \theta^{(N)}, 2m-2, \ldots, \theta^{(N)} \right],
\]

where

\[
\left[ \theta^{(N)}, \theta^{(N)}, 2m-2, \ldots, \theta^{(N)} \right] = \omega^{i_{p_1},^i_{p_1}, \beta_{p_1}^1} \wedge \ldots \wedge \omega^{i_{p_{2m-2}},^i_{p_{2m-2}}, \beta_{p_{2m-2}}^{2m-2}} \left[ X_{i_{p_1},^i_{p_1}, \beta_{p_1}^1}, \ldots, X_{i_{p_{2m-2}},^i_{p_{2m-2}}, \beta_{p_{2m-2}}^{2m-2}} \right].
\]

Note that the equations (40)-(42) store the structure subspace information of \( \mathcal{G} \) and therefore must be mentioned if we use this free index notation.

The following theorem generalizes theorem 2 of Ref. [5] to the case of higher order Lie algebras and it establishes the conditions under which the 1-forms \( \omega^{i_p,\alpha_p} \) and \( \omega^{i_1,\alpha_1} \) generate new higher order Lie algebras:
Theorem 5 Let \((\mathcal{G}, [\ldots])\) be a higher order Lie algebras with \(\mathcal{G} = V_0 \oplus V_1\), where \(V_0\) is a submultialgebra. Let the coordinates \(g^{i_0}\) of \(G\) be rescaled by \(g^{i_0} \rightarrow g'^{i_0}\), \(g^{i_1} \rightarrow \lambda g^{i_1}\). Then, the coefficient one-forms \(\{\omega^{i_0,0}, \omega^{i_1,0}\}\) of the expansions \((37)\) of the Maurer-Cartan forms of \(\mathcal{G}^*\) determine higher order Lie algebras \((\mathcal{G}, (N_0, N_1) [\ldots])\) when \(N_1 = N_0\) or \(N_1 = N_0 + 1\) of dimension \(\dim \mathcal{G} (N_0, N_1) = (N_0 + 1) \dim V_0 + N_1 \dim V_1\) and with structure constants \((40)\).

Proof. We must prove that the set

\[
\{\omega^{i_0,0}, \omega^{i_1,0}\} \in \{\omega^{i_0,0}, \omega^{i_0,1}, \ldots, \omega^{i_0,N_0}, \omega^{i_1,0}, \omega^{i_1,1}, \ldots, \omega^{i_1,N_1}\},
\]

is closed for the generalized Maurer-Cartan equations \((39)\) and that the Jacobi identity is satisfied. In fact, equation \((39)\) can be written as

\[
\tilde{d}_m \omega^{k_s,\alpha_s} = \frac{1}{(2m - 2)!} \sum_{\beta_p^1, \ldots, \beta_p^{2m-2} = 0}^{\alpha_s} C_{\beta_p^1 \ldots \beta_p^{2m-2}}^{k_s, 2m-2} \delta^{\alpha_s}_{\beta_p^1 \ldots \beta_p^{2m-2}} \omega^{i_1,0} \omega^{i_1,0} \land \ldots \land \omega^{i_1,0} \omega^{i_1,0},
\]

From \((46)\) we have

\[
\tilde{d}_m \omega^{k_0,\alpha_0} = \frac{1}{(2m - 2)!} \sum_{\beta_p^1, \ldots, \beta_p^{2m-2} = 0}^{\alpha_0} C_{\beta_p^1 \ldots \beta_p^{2m-2}}^{k_0, 2m-2} \delta^{\alpha_0}_{\beta_p^1 \ldots \beta_p^{2m-2}} \omega^{i_1,0} \omega^{i_1,0} \land \ldots \land \omega^{i_1,0} \omega^{i_1,0},
\]

\[
\tilde{d}_m \omega^{k_1,\alpha_1} = \frac{1}{(2m - 2)!} \sum_{\beta_p^1, \ldots, \beta_p^{2m-2} = 0}^{\alpha_1} C_{\beta_p^1 \ldots \beta_p^{2m-2}}^{k_1, 2m-2} \delta^{\alpha_1}_{\beta_p^1 \ldots \beta_p^{2m-2}} \omega^{i_1,0} \omega^{i_1,0} \land \ldots \land \omega^{i_1,0} \omega^{i_1,0},
\]

We now consider the forms that contribute to \(\tilde{d}_m \omega^{k_s,\alpha_s}\):

(a) the case \(\alpha = 0\),

\[
\tilde{d}_m \omega^{k_0,0} = \frac{1}{(2m - 2)!} \sum_{\beta_p^1, \ldots, \beta_p^{2m-2} = 0}^{0} C_{\beta_p^1 \ldots \beta_p^{2m-2}}^{k_0, 2m-2} \delta^{0}_{\beta_p^1 \ldots \beta_p^{2m-2}} \bullet \omega^{i_1,0} \omega^{i_1,0} \land \ldots \land \omega^{i_1,0} \omega^{i_1,0},
\]

\[
\tilde{d}_m \omega^{k_0,0} = \frac{1}{(2m - 2)!} C_{i_1}^{k_0, \ldots, i_{2m-2}} \omega^{i_1,0} \land \ldots \land \omega^{i_1,0} \omega^{i_1,0}.
\]
(b) the case $\alpha = 1$

$$\tilde{d}_m \omega^{k_0,1} = \frac{1}{(2m-2)!} \sum_{\beta_1^{p_1}, \ldots, \beta_{2m-2}^{p_{2m-2}} = 0}^1 C_{i_1 \ldots i_{2m-2}}^{k_0} \delta_{\beta_1^{p_1} + \ldots + \beta_{2m-2}^{p_{2m-2}}} \cdot$$

$$\cdot \omega^{i_1^{p_1}} \beta_1^{p_1} \wedge \ldots \wedge \omega^{i_{2m-2}^{p_{2m-2}}} \beta_{2m-2}^{p_{2m-2}}. \quad (52)$$

$$\tilde{d}_m \omega^{k_1,1} = \frac{1}{(2m-2)!} \sum_{\beta_1^{p_1}, \ldots, \beta_{2m-2}^{p_{2m-2}} = 0}^1 C_{i_1 \ldots i_{2m-2}}^{k_1} \delta_{\beta_1^{p_1} + \ldots + \beta_{2m-2}^{p_{2m-2}}} \cdot$$

$$\cdot \omega^{i_1^{p_1}} \beta_1^{p_1} \wedge \ldots \wedge \omega^{i_{2m-2}^{p_{2m-2}}} \beta_{2m-2}^{p_{2m-2}}. \quad (53)$$

(c) the case $\alpha \geq 2$

$$\tilde{d}_m \omega^{k_0,\alpha_0} = \frac{1}{(2m-2)!} \sum_{\beta_1^{p_1}, \ldots, \beta_{2m-2}^{p_{2m-2}} = 0}^{\alpha_0} C_{i_1 \ldots i_{2m-2}}^{k_0} \delta_{\alpha_0}^{\beta_1^{p_1} + \ldots + \beta_{2m-2}^{p_{2m-2}}} \cdot$$

$$\cdot \omega^{i_1^{p_1}} \beta_1^{p_1} \wedge \ldots \wedge \omega^{i_{2m-2}^{p_{2m-2}}} \beta_{2m-2}^{p_{2m-2}}. \quad (54)$$

$$\tilde{d}_m \omega^{k_1,\alpha_1} = \frac{1}{(2m-2)!} \sum_{\beta_1^{p_1}, \ldots, \beta_{2m-2}^{p_{2m-2}} = 0}^{\alpha_1} C_{i_1 \ldots i_{2m-2}}^{k_1} \delta_{\alpha_1}^{\beta_1^{p_1} + \ldots + \beta_{2m-2}^{p_{2m-2}}} \cdot$$

$$\cdot \omega^{i_1^{p_1}} \beta_1^{p_1} \wedge \ldots \wedge \omega^{i_{2m-2}^{p_{2m-2}}} \beta_{2m-2}^{p_{2m-2}}. \quad (55)$$

Therefore:

(1) For $\tilde{d}_m \omega^{k_0,\alpha_0}$, we have

(1a) the forms $\omega^{i_0^{p_0} \beta_0^{p_0}}$ contribute up to the order shown in the following table:

| $\alpha_0$ | Maximum order of $\beta_0^{p_0}$ |
|------------|----------------------------------|
| $0$        | $\beta_0^{p_0} \leq 0$           |
| $1$        | $\beta_0^{p_0} \leq 1$           |
| $\geq 2$   | $\beta_0^{p_0} \leq \alpha_0$   |
(1b) The forms $\omega^{\beta_1^1,\beta_1^1}$ contribute up to the order shown in the following table:

| $\alpha_0$ | Maximum order of $\beta_1^l$ |
|-----------|-------------------------------|
| $\alpha_0 = 0$ | there is no contribution |
| $\alpha_0 = 1$ | $\beta_1^l \leq 1$ |
| $\alpha_0 \geq 2$ | $\beta_1^l \leq \alpha_0$ |

(2) For $\tilde{d}_{m}\omega^{k_1,\alpha_1}$ we have

(2a) with respect to the contribution of the forms $\omega^{\beta_0^l,\beta_0^l}$ we can say

(2ai) for $\alpha_1 = 1$ we have that the maximum order of $\beta_0^l$ can be found by analyzing the equation (53)

$$\tilde{d}_{m}\omega^{k_1,1} = \frac{1}{(2m-2)!} \sum_{\beta_0^{l_1},...,\beta_0^{l_{2m-2}}=0}^{1} C_{l_1,\ldots,l_{2m-2}}^{k_1} \delta_{l_0}^{1} \delta_{l_1}^{p_1+\ldots+p_{2m-2}} \omega^{l_1,\beta_1^1} \wedge \ldots \wedge \omega^{l_{2m-2},\beta_{2m-2}^{l_{2m-2}}}.$$  \hspace{1cm} (56)

The condition of submultialgebra, $C_{i_0,\ldots,i_{2m-2}}^{k_1} = 0$ implies that in the sums on $p_l = \{p_1, \ldots, p_{2m-2}\}$ at least one of them, we say $p_x$, must be equal to 1. So that $p_l = 0$ for $l \neq x$. This means that the condition

$$\beta_0^{l_1} + \cdots + \beta_0^{l_x} + \beta_0^{l_{2m-2}} = 1,$$  \hspace{1cm} (57)

takes the form

$$\beta_0^{l_1} + \cdots + \beta_0^{l_{x-1}} + \beta_0^{l_x} + \beta_0^{l_{x+1}} + \cdots + \beta_0^{l_{2m-2}} = 1.$$  \hspace{1cm} (58)

Since $\omega^{l_i,\beta_0^l} = 0$ for $\beta_0^l < p_l$ we have that to generate a non vanishing element in (53), it is necessary that in the form $\omega^{l_i,\beta_0^l} = \omega^{l_i,\beta_0^l}$ it must be fulfilled that $\beta_0^l = 1$. So $p_l = 0$ and $\beta_0^l = \beta_0^l = 0$ for $l \neq x$. This means that the forms $\omega^{l_0,\beta_0^l}$ contribute to $d_{m}\omega^{k_1,\alpha_1}$ for $\alpha_1 = 1$ only up to the order $\beta_0^l = 0 = \alpha_1 - 1$.

(2aii) Following the same previous procedure we find that, for $\alpha_1 \geq 2$, the forms $\omega^{l_0,\beta_0^l}$ contribute up to the order $\beta_0^l \leq \alpha_1 - 1$. 

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The contribution of the forms $\omega^{l,\beta^l}_0$ to $\tilde{d}_m\omega^{k_1,\alpha_1}$ is shown in the following table:

| $\alpha_1 = 1$ | $\beta^l_0 \leq 0 = \alpha_1 - 1$ |
|-----------------|-------------------------------------|
| $\alpha_1 \geq 2$ | $\beta^l_0 \leq \alpha_1 - 1$ |

(2b) The contribution of the forms $\omega^{l,\beta^l}_1$ to $\tilde{d}_m\omega^{k_1,\alpha_1}$ is given by:

| $\alpha_1 = 1$ | $\beta^l_1 \leq 1 = \alpha_1$ |
|-----------------|--------------------------------|
| $\alpha_1 \geq 2$ | $\beta^l_1 \leq \alpha_1$ |

In the following table are summarized the contributions of the forms $\omega^{l,\beta^l}_s$ to $\tilde{d}_m\omega^{k_s,\alpha_s}$:

| $\alpha_s \geq s$ | $\omega^{l,\beta^l}_0$ | $\omega^{l,\beta^l}_1$ |
|---------------------|-----------------|-----------------|
| $\tilde{d}_m\omega^{k_0,\alpha_0}$ | $\beta^l_0 \leq \alpha_0$ | $\beta^l_0 \leq \alpha_0$ |
| $\tilde{d}_m\omega^{k_1,\alpha_1}$ | $\beta^l_0 \leq \alpha_1 - 1$ | $\beta^l_1 \leq \alpha_1$ |

In order that the generalized Maurer-Cartan equations be satisfied, there must exist in (45) sufficient 1-forms, so that the $(N_0 + 1) \omega^{k_0,\alpha_0}$ and $N_1 \omega^{k_1,\alpha_1}$ (45) must include at least those present in its differential. This means that the above table implies the inverse inequalities shown in the following table:

| $\alpha_s \geq s$ | $\omega^{l,\beta^l}_0$ | $\omega^{l,\beta^l}_1$ |
|---------------------|-----------------|-----------------|
| $\tilde{d}_m\omega^{k_0,\alpha_0}$ | $N_0 \geq N_0$ | $N_1 \geq N_0$ |
| $\tilde{d}_m\omega^{k_1,\alpha_1}$ | $N_0 \geq N_1 - 1$ | $N_1 \geq N_1$ |

The corresponding solutions to the inequations

$$N_1 \geq N_0,$$

$$N_0 \geq N_1 - 1,$$

are

$$N_1 = N_0,$$

or

$$N_0 = N_1 - 1.$$

These equations show the two ways in which the (37) expansions must be truncated.
IV. DUAL FORMULATION OF THE HIGHER-ORDER LIE ALGEBRA S-EXPANSION PROCEDURE

In ref. [9] was constructed an S-expansion procedure which permits obtaining a new higher-order Lie algebra from an original one by choosing an Abelian semigroup \( S \). In the previous sections of the present work we have generalized the expansion procedure of ref. [9] to the higher-order Lie algebra case.

The \( S \)-expansion procedure is defined as the action of a semigroup \( S \) on the generators \( T_A \) of the algebra, and the power series expansion is carried out on the MC forms of the original algebra. On the other hand, the \( S \)-expansion is defined on the algebra \( g \) without referring to the group manifold, whereas the power series expansion is based on a rescaling of the group coordinates.

It is the purpose of this section to study the \( S \)-expansion procedure in the context of the group manifold and then to find the dual formulation of such an \( S \)-expansion procedure.

A. S-expansion of the higher-order Lie algebra

Let's remember that the \( S \)-expansion method is based on combining the structure constants of \((G, [\cdot, \ldots, \cdot])\) with the inner law of a semigroup \( S \) to define the Lie bracket of a new, \( S \)-expanded multialgebra.

Let \( S = \{\lambda_\alpha\} \) be a finite Abelian semigroup endowed with a commutative and associative composition law \( S \times S \to S \), \((\lambda_\alpha, \lambda_\beta) \mapsto \lambda_\alpha \lambda_\beta = K_{\alpha\beta} \gamma \lambda_\gamma \). The direct product \( G = S \otimes G \) is defined as the cartesian product set

\[
G = S \times G = \{T_{(A,\alpha)} = \lambda_\alpha T_A : \lambda_\alpha \in S, T_A \in G\},
\]

with the composition law \([\cdot, \ldots, \cdot]_S : G \times \cdots \times G \to G\), defined by

\[
[T_{(A_1,\alpha_1)}, \ldots, T_{(A_n,\alpha_n)}]_S = \lambda_{\alpha_1} \cdots \lambda_{\alpha_n} [T_{A_1}, \ldots, T_{A_n}],
\]

\[
[T_{(A_1,\alpha_1)}, \ldots, T_{(A_n,\alpha_n)}]_S = K_{\alpha_1 \cdots \alpha_n}^{\gamma} C_{A_1 \cdots A_n}^{\gamma} T_C = C^{(C,\gamma)}_{(A_1,\alpha_1) \cdots (A_n,\alpha_n)} T_{(C,\gamma)},
\]

where \( T_{(A_i,\alpha_i)} \in G, \forall i = 1, \ldots, n \), and \( C^{(C,\gamma)}_{(A_1,\alpha_1) \cdots (A_n,\alpha_n)} = K_{\alpha_1 \cdots \alpha_n}^{\gamma} C_{A_1 \cdots A_n}^{\gamma} \).

**Theorem 6** The set \( G = S \times G \) \((63)\) with the composition law \((65)\) defines a new Lie multialgebra which will be called \( S \)-expanded Lie multialgebra. This algebra is a Lie algebra
structure defined over the vector space obtained by taking $S$ copies of $G$ by means of the structure constant $C^{(C,\gamma)}_{(A_1,\alpha_1)\ldots(A_n,\alpha_n)} = K^\gamma_{\alpha_1\ldots\alpha_n} C^C_{A_1\ldots A_n}$ where $K^\sigma_{\alpha_1\ldots\alpha_{n-1}} K^\gamma_{\alpha_n}$. The structure constants $C^{(C,\gamma)}_{(A_1,\alpha_1)\ldots(A_n,\alpha_n)}$ defined in (65) inherit the symmetry properties of $C^C_{A_1\ldots A_n}$ of $G$ by virtue of the abelian character of the $S$-product.

**Proof.** The proof is direct and may be found in ref. [9]. }

**B. Dual formulation of the S-expansion Procedure**

The above theorem implies that, for every abelian semigroup $S$ and Lie multialgebra $\mathfrak{g}$, the product $\mathfrak{G} = S \times \mathfrak{g}$ is also a Lie multialgebra, with a Lie bracket given by eq. (65). This in turn means that it must be possible to look at this $S$-expanded Lie multialgebra $\mathfrak{G}$ from the dual point of view of the Maurer-Cartan forms [6].

**Theorem 7** If $S = \{\lambda_\alpha, \alpha = 1, \ldots, N\}$ is a finite abelian semigroup and if $\omega^A$ are the Maurer-Cartan forms for a Lie multialgebra $\mathfrak{g}$, then the Maurer-Cartan forms $\omega^{(A,\alpha)}$ associated with the $S$-expanded Lie multialgebra $\mathfrak{G} = S \times \mathfrak{g}$ [cf. Theorem 1] are related to the $\omega^A$ by

$$\omega^A = \sum_{\lambda_\alpha \in S} \lambda_\alpha \omega^{(A,\alpha)}, \quad (66)$$

and satisfy the generalized Maurer Cartan equations

$${\tilde{d}_m}_m \omega^{(A,\alpha)} = \frac{1}{(2m-2)!} C^{(B_1,\beta_1)\ldots (B_{2m-2},\beta_{2m-2})}_{(A,\alpha)} \omega^{(B_1,\beta_1)} \ldots \omega^{(B_{2m-2},\beta_{2m-2})}. \quad (67)$$

**Proof.** Introducing eq. (66) into the generalized Maurer-Cartan equations

$${\tilde{d}_m}_m \omega^A = \frac{1}{(2m-2)!} C^{B_1\ldots B_{2m-2}}_{A} \omega^{B_1} \ldots \omega^{B_{2m-2}}, \quad (68)$$

we obtain

$${\tilde{d}_m}_m \omega^{(A,\alpha)} = \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}} C^{(B_1,\beta_1)\ldots(B_{2m-2},\beta_{2m-2})}_{(A,\alpha)} \omega^{(B_1,\beta_1)} \ldots \omega^{(B_{2m-2},\beta_{2m-2})}, \quad (69)$$

where $\Omega^{(A,\alpha)}_{(B_1,\beta_1)\ldots (B_{2m-2},\beta_{2m-2})} = \Omega^{A}_{B_1\ldots B_{2m-2}} K^\sigma_{\beta_1\ldots\beta_{2m-2}}$. Using the sum convention, equation (69) can be written as

$${\tilde{d}_m}_m \omega^{(A,\alpha)} = \frac{1}{(2m-2)!} C^{(B_1,\beta_1)\ldots(B_{2m-2},\beta_{2m-2})}_{(A,\alpha)} \omega^{(B_1,\beta_1)} \ldots \omega^{(B_{2m-2},\beta_{2m-2})}, \quad (70)$$
This concludes the proof. ■

In the compact notation that uses the canonical one-form \( \theta^{(N)} \omega^{(A,\alpha)} X_{(A,\alpha)} \), the eq. (70) can be written as

\[
\tilde{d}_m \theta^{(S)} = \frac{1}{(2m-2)!} \left[ \theta^{(S)}, \theta^{(S)}, \ldots, \theta^{(S)} \right],
\]

where

\[
\left[ \theta^{(S)}, \theta^{(S)}, \ldots, \theta^{(S)} \right] = \omega^{(B_1,\beta_1)} \wedge \cdots \wedge \omega^{(B_{2m-2},\beta_{2m-2})} \left[ X_{(B_1,\beta_1)}, \cdots, X_{(B_{2m-2},\beta_{2m-2})} \right].
\]

It is perhaps interesting to notice that the relation shown in eq. (66) is analogous to the method of power series expansion developed in Ref. [5] and in the above sections.

C. 0\(_S\)-Reduction of \(S\)-expanded Lie Algebras

Now we present the dual formulation for the 0\(_S\)-reduction of an \(S\)-expanded Lie multialgebra \(G\), formulated in the language of the MC forms.

Let \(S = \{ \lambda_i, i = 1, \ldots, N \} \cup \{ \lambda_{N+1} = 0_S \} \) be an abelian semigroup with zero. The expanded Maurer-Cartan forms \(\omega^{(A,\alpha)}\) are then given by

\[
\omega^A = \sum_{i=1}^{N} \lambda_i \omega^{(A,i)} + 0_S \tilde{\omega}^A,
\]

where \(\tilde{\omega}^A = \omega^{(A,N+1)}\). We shall show that the Maurer Cartan forms \(\omega^{(A,i)}\) by themselves (without including \(\tilde{\omega}^A\)) are those of a Lie multialgebra-the 0\(_S\)-reduced multialgebra \(\mathfrak{G}_R\).

It can be shown [9] that \(C^{(C,k)}_{(A_1,i_1) \ldots (A_n,i_n)} = K^k_{i_1 \ldots i_n} C^C_{A_1 \ldots A_n}\) are the structure constants for the 0\(_S\)-reduced \(S\)-expanded multialgebra \(\mathfrak{G}_R\), which is generated by \(T_{(A,i)}\):

\[
\left[ T_{(A_1,i_1)}, \ldots, T_{(A_n,i_n)} \right]_S = K^k_{i_1 \ldots i_n} C^C_{A_1 \ldots A_n} T_{(C,k)}.
\]

The following Theorem gives the equivalent statement in terms of Maurer-Cartan forms (see [6]):

**Theorem 8** Let \(S = \{ \lambda_i, i = 1, \ldots, N \} \cup \{ \lambda_{N+1} = 0_S \} \) be an abelian semigroup with zero and let \(\{ \omega^{(A,i)}, i = 1, \ldots, N \} \cup \{ \omega^{(A,N+1)} = \tilde{\omega}^A \} \) be the MC forms for the \(S\)-expanded multialgebra \(\mathfrak{G} = S \times g\) of \(g\) by the semigroup \(S\). Then, \(\{ \omega^{(A,i)}, i = 1, \ldots, N \} \) are the Maurer-Cartan forms for the 0\(_S\)-reduced \(S\)-expanded multialgebra \(\mathfrak{G}_R\).
Proof. The Maurer-Cartan forms for the S-expanded multialgebra $\mathfrak{G}$ satisfy the generalized Maurer Cartan equations [cf. eq. (67)]
\[
\tilde{d}_m \omega^{(A,\alpha)} = \frac{1}{(2m-2)!} C(B_1,\beta_1) \cdots (B_{2m-2},\beta_{2m-2}) \omega^{(A,\alpha)}(B_1,\beta_1) \cdots \omega^{(B_{2m-2},\beta_{2m-2})}.
\] (75)
Introducing (73) into (75) we have
\[
\sum_{i=1}^{N} \lambda_i \tilde{d}_m \omega^{(A,i)} + 0_S \tilde{d}_m \omega^{(A,N+1)} = \frac{1}{(2m-2)!} C_{B_1,\ldots,B_{2m-2}} A \left[ \left( \sum_{j_1}^{N} \lambda_{j_1} \omega^{(B_1,j_1)} + 0_S \omega^{(B_1,N+1)} \right) \times \cdots \times \left( \sum_{j_{2m-2}}^{N} \lambda_{j_{2m-2}} \omega^{(B_{2m-2},j_{2m-2})} + 0_S \omega^{(B_{2m-2},N+1)} \right) \right].
\] (76)
On the other hand we can write
\[
\sum_{\alpha=1}^{N+1} \lambda_\alpha \tilde{d}_m \omega^{(A,\alpha)} = \frac{1}{(2m-2)!} C_{B_1,\ldots,B_{2m-2}} A \left( \sum_{\beta_1}^{N+1} \lambda_{\beta_1} \omega^{(B_1,\beta_1)} \right) \cdots \left( \sum_{\beta_{2m-2}}^{N+1} \lambda_{\beta_{2m-2}} \omega^{(B_{2m-2},\beta_{2m-2})} \right),
\] (77)
Since
\[
\sum_{\alpha=1}^{N+1} \lambda_\alpha \tilde{d}_m \omega^{(A,\alpha)} = \sum_{i=1}^{N} \lambda_i \tilde{d}_m \omega^{(A,i)} + 0_S \tilde{d}_m \omega^{(A,N+1)},
\] (78)
we have
\[
\sum_{\alpha=1}^{N+1} \lambda_\alpha \left( \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}}^{N+1} C(B_1,\beta_1) \cdots (B_{2m-2},\beta_{2m-2}) \omega^{(A,\alpha)}(B_1,\beta_1) \cdots \omega^{(B_{2m-2},\beta_{2m-2})} \right)
\]
\[
= \sum_{i=1}^{N} \lambda_i \left( \frac{1}{(2m-2)!} \sum_{i_1,\ldots,i_{2m-2}}^{N} C(B_1,i_1) \cdots (B_{2m-2},i_{2m-2}) \omega^{(A,i)}(B_1,i_1) \cdots \omega^{(B_{2m-2},i_{2m-2})} \right)
\]
\[
+ 0_S \left( \frac{1}{(2m-2)!} \sum_{\beta_1,\ldots,\beta_{2m-2}}^{N+1} C(B_1,\beta_1) \cdots (A,N+1) \omega^{(B_1,\beta_1)} \cdots \omega^{(B_{2m-2},\beta_{2m-2})} \right),
\] (79)
we have that the generalized Maurer-Cartan equations takes the form
\[
\left( \sum_{i=1}^{N} \lambda_\alpha \tilde{d}_m \omega^{(A,\alpha)} + 0_S \tilde{d}_m \omega^{(A,N+1)} \right)
\]
\[ \sum_{i}^{N} \lambda_{i} \left( \frac{1}{(2m-2)!} \sum_{i_{1},...,i_{2m-2}}^{N} C_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2}}^{(A,i)} \omega_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2})} \right) \]

\[ + 0_{S} \left( \frac{1}{(2m-2)!} \sum_{\beta_{1},...,\beta_{2m-2}}^{N+1} C_{(B_{1},\beta_{1}),...,B_{2m-2},\beta_{2m-2}}^{(A,N+1)} \omega_{(B_{1},\beta_{1}),...,B_{2m-2},\beta_{2m-2})} \right). \] (80)

So that,

\[ \tilde{d}_{m} \omega^{(A,i)} = \frac{1}{(2m-2)!} \sum_{i_{1},...,i_{2m-2}}^{N} C_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2}}^{(A,i)} \omega_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2})}, \] (81)

\[ \tilde{d}_{m} \omega^{(A,N+1)} = \frac{1}{(2m-2)!} \sum_{\beta_{1},...,\beta_{2m-2}}^{N+1} C_{(B_{1},\beta_{1}),...,B_{2m-2},\beta_{2m-2}}^{(A,N+1)} \omega_{(B_{1},\beta_{1}),...,B_{2m-2},\beta_{2m-2})}. \] (82)

Applying the so-called 0\_\_reduction we obtain

\[ \tilde{d}_{m} \omega^{(A,i)} = \frac{1}{(2m-2)!} \sum_{i_{1},...,i_{2m-2}}^{N} C_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2}}^{(A,i)} \omega_{(B_{1},i_{1}),...,B_{2m-2},i_{2m-2})}. \] (83)

This concludes the proof. \[ \square \]

In the compact notation that uses the canonical one-form \( \theta^{(S)} = \omega^{(A,i)} X_{(A,i)}, i = 1, ..., N, \) the eq. (83) can be written as

\[ \tilde{d}_{m} \theta^{(S)} = \frac{1}{(2m-2)!} \left[ \theta^{(S)}, \theta^{(S)}, \theta^{(S)}, ..., \theta^{(S)} \right], \] (84)

\[ \left[ \theta^{(S)}, \theta^{(S)}, ..., \theta^{(S)} \right] = \omega^{(B_{1},i_{1})} \land ... \land \omega^{(B_{2m-2},i_{2m-2})} \left[ X_{(B_{1},i_{1})}, ..., X_{(B_{2m-2},i_{2m-2})} \right]. \] (85)

D. Resonant submultialgebras

From ref. \[ 4, 9 \] we known that if \( G = \bigoplus_{p \in I} V_{p} \) is a decomposition of \( G \) into subspaces \( V_{p} \), with a structure described by the subsets \( i_{(p_{1},...,p_{n})} \subset I \) such that

\[ [V_{p_{1}}, ..., V_{p_{n}}] \subset \bigoplus_{r \in i_{(p_{1},...,p_{n})}} V_{r}, \] (86)

and if \( S = \bigcup_{p \in I} S_{p} \) is a subset decomposition of the Abelian semigroup \( S \) such that

\[ S_{p_{1}} \times S_{p_{2}} \times ... \times S_{p_{n}} \subset \bigcap_{r \in i_{(p_{1},...,p_{n})}} S_{r}, \] (87)
then we say that this decomposition is in resonance with the subspace decomposition of $G = \bigoplus_{p \in I} V_p$.

In the same refs. \[4\], \[9\] it was shown that if $G = \bigoplus_{p \in I} V_p$ is a decomposition of $G$ into subspaces $V_p$ with a structure described by $[V_{p_1}, ..., V_{p_n}] \subset \bigoplus_{r \in I(p_1, ..., p_n)} V_r$ and if $S = \bigcup_{p \in I} S_p$ is a subset decomposition of the Abelian semigroup $S$ with the structure given by $S_{p_1} \times S_{p_2} \times \cdots \times S_{p_n} \subset \bigcap_{r \in I(p_1, ..., p_n)} S_r$, then the algebra given by $G_R = \bigoplus_{p \in I} S_p \otimes V_p = \bigoplus_{p \in I} W_p$ is a subalgebra of the $S$-expanded multialgebra called a resonant submultialgebra.

If $\{T_p\}$ denote the basis of $V_p$, $\lambda_{a_q} \in S_q$ and if $T_{(a_p, a_q)} = \lambda_{a_q} T_{a_p}$ then we can write

$$\left[T(a_{p_1}, a_{p_1}), ..., T(a_{p_n}, a_{p_n})\right]_S = C(a_{p_1}, a_{p_1}) ...(a_{p_n}, a_{p_n})^{(c_r, \gamma_r)} T(c_r, \gamma_r),$$

(88)

which means that the structure constants of the resonant submultialgebra are given by

$$C(a_{p_1}, a_{p_1})...(a_{p_n}, a_{p_n})^{(c_r, \gamma_r)} = K^{\gamma_r} C_{a_{p_1} ... a_{p_n}}^{-\gamma_r} C_{a_{p_1} ... a_{p_n}}^{c_r}.$$  

(89)

The following theorem provides the Maurer-Cartan equations for the resonant submultialgebra:

**Theorem 9** Let $\{\omega^{a_p}\}$ be a basis of $V^*_p$ and let $\lambda_{a_q} \in S_q$. Then

$$\omega^{a_p} = \sum_{\lambda_{a_p} \in S_p} \lambda_{a_p} \omega^{(a_p, a_p)},$$

(90)

and the Maurer-Cartan equations for the resonant submultialgebra of the $S$-expanded multialgebra are given by $S$

$$\tilde{d}_m \omega^{(c_r, \gamma_r)} = \frac{1}{(2m - 2)!} C(a_{p_1}, a_{p_1})^{(c_r, \gamma_r)} (a_{p_{2m-2}}, a_{p_{2m-2}}^{2m-2}) \omega(a_{p_1}, a_{p_1})^{1} \cdots \omega(a_{p_{2m-2}}, a_{p_{2m-2}}^{2m-2}),$$

(91)

where

$$C(a_{p_1}, a_{p_1})^{(c_r, \gamma_r)} (a_{p_{2m-2}}, a_{p_{2m-2}}^{2m-2}) = K^{\gamma_s} a_{p_1} ... a_{p_{2m-2}}^{2m-2} C_{a_{p_1} ... a_{p_{2m-2}}^{2m-2}}^{c_r},$$

(92)

with $r, p_i \in I$.

**Proof.** The generalized Maurer-Cartan equations are given by

$$\tilde{d}_m \omega^A(g, \lambda) = \frac{1}{(2m - 2)!} C_{B_1 ... B_{2m-2}} A \omega^B(g, \lambda) \cdots \omega^{B_{2m-2}}(g, \lambda).$$

(93)

Introducing

$$\omega^{a_p} = \sum_{\lambda_{a_p} \in S_p} \lambda_{a_p} \omega^{(a_p, a_p)},$$

(94)
Introducing these results into (95) we have

$$\sum_{\lambda_{\gamma_s} \in S_s} \lambda_{\gamma_s} \bar{d} m \omega^{(c_s, \gamma_s)}$$

$$= \frac{1}{(2m-2)!} C_{a_{P1}^1 \ldots a_{P2m-2}^1}^{c_s} \left( \sum_{a_{P1}^1} \lambda_{\alpha_{P1}^1} \omega^{(a_{P1}^1, a_{P1}^1)} \right) \ldots \left( \sum_{a_{P2m-2}^2} \lambda_{\alpha_{P2m-2}^2} \omega^{(a_{P2m-2}^2, a_{P2m-2}^2)} \right)$$

$$= \frac{1}{(2m-2)!} C_{a_{P1}^1 \ldots a_{P2m-2}^1}^{c_s} \sum_{a_{P1}^1, \ldots, a_{P2m-2}^1} \lambda_{\alpha_{P1}^1} \ldots \lambda_{\alpha_{P2m-2}^2} \omega^{(a_{P1}^1, a_{P1}^1)} \ldots \omega^{(a_{P2m-2}^2, a_{P2m-2}^2)}.$$  \hspace{1cm} (95)

From the generalized resonance condition, we have

$$\lambda_{\alpha_{P1}^1} \ldots \lambda_{\alpha_{P2m-2}^2} = K_{\alpha_{P1}^1 \ldots \alpha_{P2m-2}^2}^{\gamma_s} \text{ donde } \gamma_s \in \tilde{S} (p_1, \ldots, p_{2m-2}) = \bigcap_{t \in \{p_1, \ldots, p_{2m-2}\}} S_t.$$ \hspace{1cm} (96)

Since the condition $C_{a_{P1}^1 \ldots a_{P2m-2}^1}^{c_s} \neq 0$ implies $s \in \{p_1, \ldots, p_{2m-2}\}$, we have

$$\tilde{S} (p_1, \ldots, p_{2m-2}) = \bigcap_{t \in \{p_1, \ldots, p_{2m-2}\}} S_t \subseteq S_s.$$ \hspace{1cm} (97)

This means that if $\tilde{S} \subseteq S_s$ then we can write

$$\lambda_{\alpha_{P1}^1} \ldots \lambda_{\alpha_{P2m-2}^2} = K_{\alpha_{P1}^1 \ldots \alpha_{P2m-2}^2}^{\gamma_s} \lambda_{\gamma_s} \text{ where } \lambda_{\gamma_s} \in S_s.$$ \hspace{1cm} (98)

Introducing these results into (95) we have

$$\sum_{\lambda_{\gamma_s} \in S_s} \lambda_{\gamma_s} \bar{d} m \omega^{(c_s, \gamma_s)}$$

$$= \frac{1}{(2m-2)!} C_{a_{P1}^1 \ldots a_{P2m-2}^1}^{c_s} \sum_{\alpha_{P1}^1, \ldots, \alpha_{P2m-2}^1} \sum_{\lambda_{\gamma_s} \in S_s} K_{\alpha_{P1}^1 \ldots \alpha_{P2m-2}^2}^{\gamma_s} \lambda_{\gamma_s} \omega^{(a_{P1}^1, a_{P1}^1)} \ldots \omega^{(a_{P2m-2}^2, a_{P2m-2}^2)}.$$
Therefore the generalized Maurer-Cartan equations for the resonant submultialgebra are given by

\[
\tilde{d}_m \omega^{(cr, \gamma_r)} = \frac{1}{(2m - 2)!} \sum_{\alpha_{p_1}, \ldots, \alpha_{p_{2m-2}}} C(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}) \omega^{(cr, \gamma_r)}(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}),
\]

(99)

which it can written in the form,

\[
\tilde{d}_m \omega^{(cr, \gamma_r)} = \frac{1}{(2m - 2)!} C(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}) \omega^{(cr, \gamma_r)}(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}).
\]

(100)

In the compact notation we have

\[
\tilde{d}_m \theta^{(S)} = \frac{1}{(2m - 2)!} \left[ \theta^{(S)}, \theta^{(S)}, \ldots, \theta^{(S)} \right],
\]

(101)

where

\[
\theta^{(S)} = \omega^{(cr, \gamma_r)} X^{(cr, \gamma_r)}
\]

(102)

\[
\left[ \theta^{(S)}, \theta^{(S)}, \cdots, \theta^{(S)} \right] = \omega(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}) \left[ X(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}), \ldots, X(a_{p_1}^{\alpha_1} \cdots a_{p_{2m-2}}^{\alpha_{2m-2}}) \right].
\]

(103)

### E. Reduced Multialgebras of a Resonant Submultialgebra

In ref. [9] was shown that, if \( S_p = \hat{S}_p \cup \tilde{S}_p \) is a partition of the subsets \( S_p \subset S \) that satisfy

\[
\tilde{S}_{p_i} \cap \hat{S}_{p_i} = \phi,
\]

(104)

then

\[
\hat{S}_{p_1} \times \hat{S}_{p_2} \times \cdots \times \hat{S}_{p_n} \subset \bigcap_{r \in \{p_1, \ldots, p_n\}} \hat{S}_r.
\]

(105)

The conditions (104) and (105) induce the decomposition \( \mathfrak{g}_R = \hat{\mathfrak{g}}_R \oplus \check{\mathfrak{g}}_R \) on the resonant subalgebra, where

\[
\hat{\mathfrak{g}}_R = \bigoplus_{p \in I} \hat{S}_p \otimes V_p,
\]

(106)

\[
\check{\mathfrak{g}}_R = \bigoplus_{p \in I} \tilde{S}_p \otimes V_p.
\]

(107)

When conditions (104) and (105) hold, then

\[
\left[ \hat{\mathfrak{g}}_R, \check{\mathfrak{g}}_R, \ldots, \check{\mathfrak{g}}_R \right]_S \subset \hat{\mathfrak{g}}_R.
\]

(108)
and therefore $|\mathcal{G}_R|$ corresponds to a reduced algebra of $\mathcal{G}_R$.

The following theorem provides necessary conditions under which a reduced multialgebra can be extracted from a resonant submultialgebra:

**Theorem 10** If $S_p = \hat{S}_p \cup \tilde{S}_p$ is a partition of the subsets $S_p \subset S$ that satisfy

$$\hat{S}_{p_i} \cap \tilde{S}_{p_i} = \phi,$$

$$\hat{S}_{p_1} \times \hat{S}_{p_2} \times \ldots \times \hat{S}_{p_m} \subset \bigcap_{r \in (p_1, \ldots, p_n)} \tilde{S}_r,$$

then the generalized Maurer-Cartan equations for the Reduced Multialgebras of a Resonant Submultialgebra are given by

$$\tilde{d}_m \omega^{(cr, \gamma_r)} = \frac{1}{(2m - 2)!} C^{(cr, \gamma_r)}_{a_{p_1}, \alpha_{p_1}} \ldots a_{p_{2m-2}, \alpha_{p_{2m-2}}} \omega^{(a_{p_1}, \alpha_{p_1})} \ldots \omega^{(a_{p_{2m-2}}, \alpha_{p_{2m-2}})}.$$  \hspace{1cm} (111)

**Proof.**

If $S_p = \hat{S}_p \cup \tilde{S}_p$ is a partition of the subsets $S_p \subset S$ that satisfy

$$\hat{S}_p \cap \tilde{S}_p = \emptyset,$$

$$\hat{S}_p \times \hat{S}_q = \bigcap_{r \in (p, q)} \tilde{S}_r,$$

then

$$\omega^a_p = \sum_{\lambda_a_p \in \hat{S}_p} \lambda_{\hat{a} p} \omega^{(a_p, \hat{a} p)} + \sum_{\lambda_{\hat{a} p} \in \tilde{S}_p} \lambda_{\tilde{a} p} \omega^{(a_p, \tilde{a} p)},$$

where the set of indices $\{a_p\} = \{\hat{a} p, \tilde{a} p\}$ is such that $\lambda_{\hat{a} p} \in \hat{S}_p$ and $\lambda_{\tilde{a} p} \in \tilde{S}_p$. This means that the dual resonant submultialgebra $\hat{\mathcal{G}}_R^*$ is generated by the forms

$$\{\omega^{(a_p, \hat{a} p)}\} = \{\omega^{(a_p, \hat{a} p)}, \omega^{(a_p, \tilde{a} p)}\},$$

so that $\hat{\mathcal{G}}_R^*$ is given by

$$\hat{\mathcal{G}}_R^* = V_0^* \oplus V_1^*,$$

where $V_0^* = \{\omega^{(a_p, \hat{a} p)}\}$, $V_1^* = \{\omega^{(a_p, \tilde{a} p)}\}$. The reduction condition is given by the condition $[V_0, V_1] \subset V_1$ or equivalently $C^{(a_{\hat{a} s}, \hat{a} s)}_{b_{\hat{a} s}, \hat{a} s}(c_{\hat{a} s}, \gamma_{\hat{a} s}) = 0$: Since

$$C^{(a_{\hat{a} s}, \hat{a} s)}_{b_{\hat{a} s}, \hat{a} s}(c_{\hat{a} s}, \gamma_{\hat{a} s}) = K_{\hat{a} s} \gamma_{\hat{a} s} C^{a_{\hat{a} s}}_{b_{\hat{a} s}, \hat{a} s},$$

\hspace{1cm} (117)
and that \([112]\) says to us that \(\lambda_{\beta_r} \in \hat{S}_r, \lambda_{\gamma_i} \in \hat{S}_i\) we have

\[
\lambda_{\beta_r} \lambda_{\gamma_i} = K_{\beta_r \gamma_i}^\alpha \lambda_{\alpha} \in \bigcap_{r \in \{p, q\}} \hat{S}_r,
\]

(118)

which imply \(K_{\beta_r \gamma_i}^\alpha = 0\) and therefore

\[
C^{(\alpha, \alpha)}_{(\beta_r \gamma_i)} = 0.
\]

(119)

This means that the set \(\{\omega^{(\alpha_p, \alpha_p)}\}\) generates the so-called dual reduced multialgebra of a resonant submultialgebra. The corresponding Maurer-Cartan equations for this reduced multialgebra are

\[
\tilde{d}_m \omega^{(\alpha_p, \alpha_p)} = \frac{1}{(2m - 2)!} C^{(\alpha_p, \alpha_p)}_{(\alpha_p, \alpha_p)} \omega^{(\alpha_p, \alpha_p)} \cdot \cdot \cdot \omega^{(\alpha_p, \alpha_p)} \rightarrow 0.
\]

(120)

In the compact notation we have

\[
\tilde{d}_m \theta^{(S)} = \frac{1}{(2m - 2)!} \left[ \theta^{(S)}, \theta^{(S)}, 2m-2, \ldots, \theta^{(S)} \right],
\]

(121)

where

\[
\theta^{(S)} = \omega^{(\alpha_p, \alpha_p)} X_{(\alpha_p, \alpha_p)},
\]

(122)

\[
\left[ \theta^{(S)}, \theta^{(S)}, 2m-2, \ldots, \theta^{(S)} \right] = \omega^{(\alpha_p, \alpha_p)} \cdot \cdot \cdot \omega^{(\alpha_p, \alpha_p)} X_{(\alpha_p, \alpha_p)} \cdot X_{(\alpha_p, \alpha_p)}.
\]

(123)

F. Recovering results of section 3

Now we comment that the expansion method developed in section 3 can be recovered in the S-expansion formalism for a particular election of the semigroup. For example, we will show that the equations \([24]\)

\[
\tilde{d}_m \omega^{(\alpha_p, \alpha_p)} = \frac{1}{(2m - 2)!} C^{(\alpha_p, \alpha_p)}_{(\alpha_p, \alpha_p)} \omega^{(\alpha_p, \alpha_p)} \cdot \cdot \cdot \omega^{(\alpha_p, \alpha_p)} \rightarrow 0,
\]

(124)

\[
C^{(\alpha_p, \alpha_p)}_{(\alpha_p, \alpha_p)} = C^{(\alpha_p, \alpha_p)}_{(\alpha_p, \alpha_p)} \delta_{m-2}^{\alpha_p, \alpha_p, \alpha_p},
\]

(125)

can be recovered in the language of S-expansions.

In fact, let us choose the following semigroup:

\[
S_E^{(N)} = \{\lambda_{\alpha}, \alpha = 0, \ldots, N + 1\},
\]

(126)
with a multiplication rule given by
\[ \lambda_\alpha \lambda_\beta = \lambda_{H_{N+1}(\alpha+\beta)} = \delta_{H_{N+1}(\alpha+\beta)}^\gamma \lambda_\gamma. \]  

The two-selectors for \( S_E^{(N)} \) read
\[ K_{\alpha\beta}^\gamma = \delta_{H_{N+1}(\alpha+\beta)}^\gamma, \]
where \( \delta^\rho_{\sigma} \) is the Kronecker delta. The multiplication rule (127) can be directly generalized to
\[ \lambda_{\alpha_1} \cdots \lambda_{\alpha_n} = \lambda_{H_{N+1}(\alpha_1+\cdots+\alpha_n)} = \delta_{H_{N+1}(\alpha_1+\cdots+\alpha_n)}^\gamma \lambda_\gamma, \]  
\[ K_{\alpha_1\cdots\alpha_n}^\gamma = \delta_{H_{N+1}(\alpha_1+\cdots+\alpha_n)}^\gamma. \]

Consider now a higher order Lie algebra \( (\mathcal{G}, [, , ... ,]) \) of order \( n = 2m - 2 \), whose generalized MC equations are given by
\[ \tilde{d}_m \omega^A = \frac{1}{(2m - 2)!} C_{B_1 \cdots B_{2m-2}} \omega^{B_1} \cdots \omega^{B_{2m-2}}. \]  
Then the generalized MC equations of the \( S_E^{(N)} \)-expanded Lie multialgebra are given by
\[ \tilde{d}_m \omega^{(A,\alpha)} = \frac{1}{(2m - 2)!} C_{(B_1,\beta_1) \cdots (B_{2m-2},\beta_{2m-2})} \omega^{(B_1,\beta_1)} \cdots \omega^{(B_{2m-2},\beta_{2m-2})}. \]  
\[ C_{(B_1,\beta_1) \cdots (B_{2m-2},\beta_{2m-2})}^{(A,\alpha)} = C_{(B_1,\beta_1) \cdots (B_{2m-2},\beta_{2m-2})} A_{B_1 \cdots B_{2m-2}} \delta_{H_{N+1}(\beta_1+\cdots+\beta_{2m-2})}^\gamma, \]
where \( \alpha, \beta_1, \ldots, \beta_{2m-2} = 0, 1, \ldots, N, N + 1 \).

In section 4.3, we used latin indices \( i, j, k \) when we restrict the greek indices (of the semigroup elements) to take values in \( \{0, 1, \ldots, N\} \), following the convention adopted in \[4\], \[6\] and \[9\]. However, in section 3 latin indices were used to label the basis elements of the algebra or multialgebra and their dual forms. This was done so in order to obtain a direct generalization of the expansion method \[5\] to the higher order Lie algebra case. To make a consistent comparison we will not use latin indices to perform the 0-reduction. Instead we continue to use greek indices, but write explicitly that they are restricted to take values in \( \{0, 1, \ldots, N\} \).

Therefore, when the greek indices cannot take the value \( N + 1 \), we have
\[ \delta_{H_{N+1}(\beta_1+\cdots+\beta_{2m-2})}^\gamma = \delta_{\beta_1+\cdots+\beta_{2m-2}}^\gamma. \]
and the 0-reduced multialgebra of the $S_E^{(N)}$-expanded Lie multialgebra is given by the following generalized Maurer-Cartan equations

$$d_m\omega^{(A,\alpha)} = \frac{1}{(2m-2)!} C^{(A,\alpha)}_{(B_1,\beta_1)\ldots(B_{2m-2},\beta_{2m-2})}\omega^{(B_1,\beta_1)}\ldots\omega^{(B_{2m-2},\beta_{2m-2})},$$

(135)

$$C^{(A,\alpha)}_{(B_1,\beta_1)\ldots(B_{2m-2},\beta_{2m-2})} = C^{A\gamma}_{B_1\ldots B_{2m-2} }\delta^\gamma_{\beta_1+\ldots+\beta_{2m-2}},$$

(136)

$$\alpha, \beta_1, \ldots, \beta_{2m-2} = 0, 1, \ldots, N.$$

The equivalence between (124) and (135) is explicit if we consider that the vector space of the original multialgebra is split into a sum of two vector spaces $\mathcal{G} = V_0 \oplus V_1$. Then the dual basis is decomposed as $\{\omega^A\} = \{\omega^{i_0}\} \cup \{\omega^{i_1}\}$ where $\omega^{i_0} \in V_0$ and $\omega^{i_1} \in V_1$.

V. COMMENTS AND POSSIBLE DEVELOPMENTS

We have shown that the successful expansion methods developed in refs. [5], [6] (see also [7], [8]) can be generalized so that they permit obtaining new higher-order Lie algebras of increasing dimensions from $(\mathcal{G}, [\ldots])$ by a geometric procedure based on expanding the generalized Maurer-Cartan equations.

The main results of this paper are: the generalizations of the expansion methods developed in refs. [5], [6] and we give the general structure of the expansion method, as well as to construct the dual S-expansion procedure of higher-order Lie algebras.

The expansion procedures considered here could play an important role in the context of gravity in higher dimensions. In fact, it seems likely that it is possible, in the context of a Chern-Simons action, to construct a theory that describes a consistent coupling of higher-spin fields to a particular form of Lovelock gravity.

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