Large deviation induced phase switch in an inertial majority-vote model

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We theoretically study noise-induced phase switch phenomena in an inertial majority-vote (IMV) model introduced in a recent paper [Phys. Rev. E 95, 042304 (2017)]. The IMV model generates a strong hysteresis behavior as the noise intensity f goes forward and backward, a main characteristic of a first-order phase transition, in contrast to a second-order phase transition in the original MV model. Using the Wentzel-Kramers-Brillouin approximation for the master equation, we reduce the problem to finding the zero-energy trajectories in an effective Hamiltonian system, and the mean switching time depends exponentially on the associated action and the number of particles N. Within the hysteresis region, we find that the actions along the optimal forward switching path from ordered phase (OP) to disordered phase (DP) and its backward path, show distinct variation trends with f, and intersect at f = f_c that determines the coexisting line of OP and DP. This results in a nonmonotonic dependence of the mean switching time between two symmetric OPs on f, with a minimum at f_c for sufficiently large N. Finally, the theoretical results are validated by Monte Carlo simulations.

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Noise induced phase switch between coexisting stable phases underlies many important physical, chemical, biological, and social phenomena. Examples include diffusion in solids, switching in nanomagnets and Josephson junctions, nucleation, chemical reactions, protein folding, and epidemics. In this paper, we apply the Wentzel-Kramers-Brillouin approximation to study noise-induced phase switch phenomena in an inertial majority-vote model. The mean switching time is determined by the classical action along the zero-energy trajectories in an effective Hamiltonian system. The results show that the mean switching time between two symmetric ordered phases depends nonmonotonically on the noise intensity. This attributes to the first-order characteristic of phase transition of the model leading to occurrence of a stable disordered phase within the hysteresis region. Our results shed some new understanding for the nontrivial role of noise in social systems.

I. INTRODUCTION

Spin models like the Ising model play a fundamental role in studying phase transitions and critical phenomena in the field of statistical physics and many other disciplines [1]. They have also significant implications for understanding social phenomena where co-ordination dynamics is observed, e.g. in consensus formation and adoption of innovations [2]. The spin orientations can represent the choices made by an agent on the basis of information about its local neighborhood.

One of the simplest nonequilibrium generalizations of the Ising model, called the majority-vote (MV) model, was proposed by Oliveira in 1992 [3]. The model displays an up-down symmetry and a continuous second-order disordered phase transition at a critical value of noise. Studies on regular lattices showed that the critical exponents are the same as those of the Ising model [3,7], in accordance with the conjecture by Grinstein et al. [8]. The MV model has also been extensively studied for various interacting substrates, including random graphs [3,10], small world networks [11,13], scale-free networks [14,16], and some others [17,18]. These studies have shown that the universality classes and the critical exponents depend on the topologies of the underlying interacting substrates.

In a recent paper [19], we have incorporated an inertial effect into the microscopic dynamics of the spin flipping of the MV model, where the spin-flip probability of any individual depends not only on the states of its neighbors, but also on its own state. In contrast to a continuous second-order phase transition in the original MV model, the inertial MV (IMV) model generates a discontinuous first-order phase transition. Such a discontinuous phase transition is manifested by a strong hysteresis behavior as the noise intensity goes forward and backward. Within the hysteresis region, the stochastic fluctuations can induce switches to occur between ordered phase (OP) and disordered phase (DP).
In the present work, we aim to study the switching phenomena based on the Wentzel-Kramers-Brillouin (WKB) approximation for the master equation [20–27]. This method has been used to study large deviation-induced phenomena, e.g., extinction process in a system with an absorbing state (See reviews [28, 29] for two recent reviews). The WKB approximation converts the master equation governing the stochastic spin-flipping processes of the IMV model into an effective Hamiltonian system. This enables us to calculate the mean switching time between different phases given by the classic action along the optimal switching path (zero-energy trajectory). Within the hysteresis region, the coexisting line is determined when the actions along the forward and backward switching path between OP and DP meet, where OP and DP are of equivalent stability. Interestingly, due to the existence of the stable DP, the mean switching time between two OPs shows a nonmonotonic dependence on the noise intensity.

II. MODEL

We consider a system of \(N\) spins, where each spin \(i\) is denoted by either \(\sigma_i = +1\) (up) or \(\sigma_i = -1\) (down). In each time step, we first randomly choose a spin \(i\) and then randomly choose \(q\) other spins as its neighborhood, labeled by \(i_1, \cdots, i_q\), where \(q\) is the number of neighbors. We then try to flip the spin \(i\) with the probability,

\[
w_i(\sigma) = \frac{1}{2} \left[ 1 - (1 - 2f)\sigma_i S(\Theta_i) \right],
\]

where

\[
\Theta_i = (1 - \theta) \sum_{j=1}^{q} \frac{\sigma_{i_j}}{q} + \theta \sigma_i
\]

is the local field of spin \(i\). \(S(x) = sgn(x)\) if \(x \neq 0\) and \(S(0) = 0\). \(f \in [0, 0.5]\) is a noise parameter, and \(\theta \in [0, 0.5]\) is a parameter controlling the weight of the inertia. The larger the value of \(\theta\), the greater is the inertia of the system. For \(\theta = 0\), we recover the original MV model without inertial effect. Since the value of \(q\) does not change qualitatively the results in the present work, \(q = 20\) is fixed throughout the paper.

The MV model does not only play an important role in the study of nonequilibrium phase transitions, but also helps to understand opinion dynamics in social or biological systems. In this model, binary spins can represent two opposite opinions, or competitive language features, and the noise parameter \(f\) plays the role of the temperature in equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbors. Moreover, consideration of the inertial effect is based on the fact that individuals in a social or biological context have a tendency for beliefs to endure once formed. In a recent experiment [30], behavioral inertia was found to be essential for collective turning of starling flocks. A counterintuitive “slower is faster” effect of the inertia on ordering dynamics of the voter model was shown in [31].

III. MEAN-FIELD THEORY

To begin, we proceed our analysis from the deterministic mean-field theory. Let \(x = n/N\) denote the density of up spins, where \(n\) is the number of up spins. The rate equation for the density \(x\) can be written as

\[
\dot{x} = -xw_+(x) + (1 - x)w_-(x),
\]

where \(w_+(x)\) and \(w_-(x)\) are the flipping probabilities of an up spin and a down spin, respectively. According to Eq. (1), \(w_+(x)\) can be written as the sum of three parts,

\[
w_+(x) = \frac{1}{2} P^+_c + \frac{1}{2} P^+_r + (1 - f)P^+_s,
\]

where \(P^+_c\) (\(P^+_r\) and \(P^+_s\) is the probability that the local field of an up spin is larger than zero (equals to zero and is less than zero). Likewise, \(w_-(x)\) can be written as

\[
w_-(x) = (1 - f)P^-_r + \frac{1}{2} P^-_c + fP^-_s,
\]

where \(P^+_c\) (\(P^-_s\) and \(P^-_r\) is the probability that the local field of a down spin is larger than zero (equals to zero and is less than zero). These probabilities can be expressed by binomial distribution

\[
P^+_c = \sum_{k=[q_+]}^{q} C^k_q x^k (1 - x)^{q-k},
\]

\[
P^-_r = \delta_{[q_+]} C^k_q x^k (1 - x)^{q-[q_+]},
\]

\[
P^-_s = \sum_{k=0}^{[q_+]} C^k_q x^k (1 - x)^{q-k},
\]

where \([\cdot]\) (\(\lfloor \cdot \rfloor\)) is the ceiling (floor) function, \(\delta\) is the Kronecker symbol, and \(C^k_q = q!/[k!(q-k)!]\) are the binomial coefficients. \(q_+ = (1 - 2f)q/[2(1 - \theta)]\) and \(q_- = q/[2(1 - \theta)]\) are the number of up-spin neighbors of an up spin and a down spin satisfying \(\Theta = 0\), respectively. It is clear that \(q_+ + q_- = q\) holds for any \(\theta\) and \(P^+_c + P^-_r + P^-_s = 1\) due to probability conservation.

Since \(w_+ = w_-\) at \(x = 1/2\), one can easily check that \(x = 1/2\) is always a stationary solution of Eq. (3). This trivial solution corresponds to a DP. The other possible solutions can be obtained by numerically solving Eq. (4). For \(\theta = 0\), it is well-known that the standard MV model undergoes a continuous second-order phase transition from an OP (there exist two symmetric stable solutions of \(x^\pm \neq 1/2\)) to a DP at \(f = f_c\), as shown in Fig. (1a), where the critical noise \(f_c\) is determined by which the trivial solution \(x = 1/2\) loses its stability. While for a large enough \(\theta\), e.g., \(\theta = 0.35\) as shown in Fig. (1b), the IMV model undergoes a discontinuous first-order phase transition. A hysteresis loop occurs in the range \(f_{cB} < f < f_{cF}\). In detail, for \(f < f_{cB}\) the model has two symmetric stable solutions \(x^\pm\) and an unstable solution \(x = 1/2\), and thus OP is stable in the
region. For $f > f_{cf}$, the DP is the only stable phase. Within the hysteresis region ($f_{co} < f < f_{cf}$), the model is bistable with the coexisting phase (CP) of OP and DP. In this case, there exist three stable solutions. Two of them are symmetric stable solutions $x^+_s$ and $x^-_s$, and the other one is $x = 1/2$. Between $x^+_s$ and $x^-_s$, there is an unstable solution $x_{us}$. The model evolves into either OP or DP depending on the initial value of $x$.

Figure 2 shows the global phase diagram in the $(\theta, f)$ plane. It is separated into three regions by $f_{co}$ and $f_{cf}$: OP, CP, and DP. These three regions collide at a so-called tricritical point, $(\theta^*, f^*) = (0.23, 0.312)$. The phase transition is of second order if $\theta < \theta^*$ and of first order if $\theta > \theta^*$.

FIG. 1: The stationary solution $x_s$ of Eq. 4 as a function of noise parameter $f$ for $\theta = 0$ (a) and for $\theta = 0.35$ (b).

FIG. 2: Phase diagram in the $(\theta, f)$ plane. The phase diagram is divided into three regions: OP, CP, and DP. They collide at a triple point (solid circle). The blue line indicates the coexisting line of OP and DP. Within the CP region, OP is stable and DP is metastable below the coexisting line, but DP is stable and OP is metastable above the coexisting line.

### IV. MASTER EQUATION AND WKB THEORY

However, for a finite size system stochastic fluctuations can induce switches from one phase to another one. In Fig. 3 we show three typical time series of $x$ within the coexisting region for $N = 400$ and $\theta = 0.35$ obtained from Monte Carlo (MC) simulations, where one MC step is defined as each spin is attempted to flip once on average. It can be observed that the switching phenomena between OP and DP rarely occur. As $f$ increases, it seems that the switch from OP to DP happens more frequently, but its backward switch happens more infrequently. Therefore, an interesting question arises: how does one calculate the mean switching time from a theoretical perspective? Since the mean-field treatment ignores the effect of stochastic fluctuations, it fails to account for switching phenomena induced by large deviations. To this end, let $P_n(t)$ be the probability that the number of up spins is $n$ at time $t$. The master equation for $P_n(t)$ reads,

\[
\frac{dP_n(t)}{dt} = W_+(n + 1)P_{n+1}(t) + W_-(n - 1)P_{n-1}(t) - [W_+(n) + W_-(n)] P_n(t).
\]

Here, $W_+(n)$ and $W_-(n)$ are the respective rates of flipping up spins and down spins, which can be written as

\[
W_+(n) = nw_+(x), \quad W_-(n) = (N - n)w_-(x).
\]

By employing the WKB approximation for the probability $P_n$, we write

\[
P_n = e^{NS(x)}.
\]
As usual, we assume that $N$ is large and take the leading order in a $N^{-1}$ expansion, by writing $P(n \pm 1) \approx P_n e^{\mp \partial S/\partial x}$ and $W(n \pm 1) \approx W(n)$. We then arrive at the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \mathcal{H}(x, p) = 0,$$

where $S$ and $\mathcal{H}$ are called the action and Hamiltonian, respectively. As in classical mechanics, $\mathcal{H}$ is a function of the coordinate $x$ and its conjugate momentum $p = \partial S/\partial x$:

$$\mathcal{H}(x, p) = \tilde{w}_-(x)(e^p - 1) + \tilde{w}_+(x)(e^{-p} - 1),$$

where $\tilde{w}_\pm(x) = W_\pm(n)/N$ are the rescaled rates. We then write the canonical equations of motion,

$$\dot{x} = \partial_p \mathcal{H}(x, p) = \tilde{w}_-(x)e^p - \tilde{w}_+(x)e^{-p},$$

$$\dot{p} = -\partial_x \mathcal{H}(x, p) = \tilde{w}'_-(x)(1 - e^p) + \tilde{w}'_+(x)(1 - e^{-p}),$$

where $\tilde{w}'_\pm$ and $\tilde{w}'_\mp$ are the derivative of $\tilde{w}_\pm$ and $\tilde{w}_-\mp$ with respect to $x$, respectively. We focus on the switching trajectory from one phase to another one. This implies that there will be some trajectory along which $S$ is minimized, which represents the maximal probability of such a switching event. This corresponds to the zero-energy ($\mathcal{H} = 0$) trajectory in the phase space $(x, p)$ from one fixed point to another one. According to equation (14), $\mathcal{H} = 0$ implies that

$$p = 0 \quad \text{or} \quad p = \ln \frac{\tilde{w}_+(x)}{\tilde{w}_-(x)}.$$  

In particular, the line $p = 0$ corresponds to the result of the mean-field theory, as Eq. (15) for $p = 0$ recovers to the mean-field equation (13).

Figure 4 depicts three typical zero-energy curves in the $(x, p)$ plane, corresponding to OP (Fig. 4(a)), CP (Fig. 4(b)), and DP (Fig. 4(c)), respectively. These curves determine the topologies of the phase space. All the stable and unstable solutions of $x$ in the mean-field theory become saddle points in phase space. When the model is in OP (Fig. 4(a)), the mean switching time from one OP to another OP is determined by $\langle T \rangle \sim e^{NS}$, where $S = \int x^2 \ln \frac{\tilde{w}_+(x)}{\tilde{w}_-(x)} dx$ is the action from $x_+^-$ to $x_+^-$ or from $x_-^+$ to $x_-^+$ along the zero-energy trajectory. When the model is in CP (Fig. 4(b)), the mean switching time from one OP to DP is determined by $\langle T \rangle \sim e^{NS_1}$, and the mean switching time from DP to one of the OPs is determined by $\langle T \rangle \sim e^{NS_2}$, where $S_1 = \int x^2 \ln \frac{\tilde{w}_+(x)}{\tilde{w}_-(x)} dx$ is the action from $x_+^- (x_+^0)$ to $x = 1/2$ along the zero-energy trajectory, and $S_2 = \int x^2 \ln \frac{\tilde{w}_+(x)}{\tilde{w}_-(x)} dx$ is the action from $x = 1/2$ to $x_+^- (x_+^0)$ along the backward trajectory. Thus, the mean switching time from one OP to another OP is $\langle T \rangle = \langle T_1 \rangle + \langle T_2 \rangle$. When the model is in DP (Fig. 4(c)), the only stable phase is disordered and there is no switching phenomenon.

Figure 5 shows the actions as a function of $f$ for $\theta = 0.35$. In the OP region, $S$ decreases monotonically with $f$. In the CP region, $S_1$ and $S_2$ exhibit distinct variations with $f$, and they intersect at $f = f_c$ where the OP and DP have the same stability. This implies that for $f_{cp} < f < f_c$, the OP is more stable and for $f_c < f < f_{cp}$ the OP is less stable than the DP. We have calculated $f_c$ as a function of $\theta$ (coexisting line), as shown by the blue line in Fig. 5. Since the mean switching times from one OP to DP and then to another OP are both exponentially dependent on the corresponding actions and the number of spins, the mean switching time $\langle T \rangle$ between the two ordered phases is dominated by the larger one of $S_1$ and $S_2$. Therefore, one can expect that $\langle T \rangle$ will change non-monotonically with $f$. A minimum in $\langle T \rangle$ will locate at $f = f_c$ for enough large $N$.

V. NUMERICAL VALIDATION

To obtain the mean switching time, one needs to perform long-time MC simulations to sample enough switching events. However, the phase switch is a rare event that occurs very infrequently, especially for large $N$. 
Thus, the conventional brute-force simulation becomes prohibitively inefficient. To overcome this difficulty, we employ a rare-event sampling method, the forward flux sampling (FFS) [32, 33], combined with MC simulation. The FFS method first defines an order parameter to distinguish between the initial phase $I$ and the final phase $F$, and then uses a series of interfaces to force the system from $I$ to $F$ in a ratchet-like manner. Here, it is convenient to select the number of up spins $n$ as the order parameter. A series of non-intersecting interfaces $n_i$ ($0 < i < N_m$) lie between states $I$ and $F$, such that any path from $I$ to $F$ must cross each interface without reaching $n_{i+1}$ before $n_i$. The algorithm first runs a long-time simulation which gives an estimate of the flux $\Phi_{I,0}$ escaping from the basin of $I$ and generates a collection of configurations corresponding to crossings of interface $n_0$. The next step is to choose a configuration from this collection at random and to use it to initiate a trial run which is continued until it either reaches $n_1$ or returns to $n_0$. If $n_1$ is reached, we store the configuration of the end point of the trial run. We repeat this step, each time choosing a random starting configuration from the collection at $n_0$. The fraction of successful trial runs gives an estimate of of the probability of reaching $n_1$ without going back into $I$, $P(n_1|n_0)$. This process is repeated, step by step, until $n_{N_m}$ is reached, giving the probabilities $P(n_{i+1}|n_i)$ ($i = 1, \cdots, N_m - 1$). Finally, we obtain the mean switching time from state $I$ to state $F$,

$$\langle T \rangle = \frac{1}{\Phi_{I,0}} \prod_{i=0}^{N_m-1} P(n_{i+1}|n_i).$$  

Fig. 6(a) shows $\ln \langle T_1 \rangle / N$ and $\ln \langle T_2 \rangle / N$ as a function of $f$ for different $N$ at $\theta = 0.35$. The lines indicate the actions $S_1$ and $S_2$. (b) $\ln \langle T \rangle / N$ as a function of $f$ for different $N$ at $\theta = 0.35$. The lines indicate the theoretical prediction.

VI. CONCLUSIONS

In the present work, we have studied phase switch phenomena induced by large fluctuations in an inertial MV model with a first-order order-disorder phase transition. By employing the WKB approximation for the master equation, the mean switching time was evaluated by the classical action along the optimal switching path in an effective Hamiltonian system. Within the hysteresis region, the mean switching time from OP to DP and that from DP to OP show opposite variation trends with noise $f$, which leads to a minimal mean switching time between two OPs occurring at the coexisting line $f_c$ where OP and DP are of the same stability. This implies that for $f_c < f < f_{cv}$, noise is unfavorable for the switches between two OPs. As discussed before, behavioral inertia and noise are both essential for social dynamics, and therefore our work may provide a new understanding for their interplay in switch phenomena of social systems, such as the emergence of a consensus and decision making [34, 35], as well as the spontaneous formation of a common language or culture [2, 36]. Since the interacting structures among agents are also vital for dynamics on them, it would be worth studying the phases switches on a networked inertial MV model [32, 35].

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