Consistency Problem of the Solutions of the Space Fractional Schrödinger Equation

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Abstract

Recently, consistency of the infinite square well solution of the space fractional Schrödinger equation has been the subject of some controversy. In [J. Math. Phys. 54, 014101 (2013)], Hawkins and Schwarz objected to the way certain integrals are evaluated to show the consistency of the infinite square well solutions of the space fractional Schrödinger equation [J. Math. Phys. 53, 042105 (2012); J. Math. Phys. 53, 084101 (2012)]. Here, we show for general $n$ that as far as the integral representation of the solution in the momentum space is concerned, there is no inconsistency. To pinpoint the source of a possible inconsistency, we also scrutinize the different representations of the Riesz derivative that plays a central role in this controversy and show that they all have the same Fourier transform, when evaluated with consistent assumptions.

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I. Introduction

Fractional calculus is an effective tool in the study of non local and memory effects in physics. Its successful application to anomalous diffusion was immediately followed by other examples in classical physics [1-4]. The first application of fractional calculus to quantum mechanics was given by Laskin in terms of the fractional Riesz derivative as the space fractional Schrödinger equation [5]. Laskin’s space fractional quantum mechanics is intriguing since it follows from the Feynman’s path integral formulation of quantum mechanics over Lévy paths. One of the first solutions of this theory was given by Laskin for the infinite well problem [5-8]. Despite its simplicity, the infinite well problem is very important since it is the prototype of a quantum detector with internal degrees of freedom. In 2010, Jeng et. al. [9] argued that the solutions obtained for the space fractional Schrödinger equation in a piecewise fashion are not valid. Their argument was based on a contradiction they think exists in the ground state wave function of the infinite square well problem. In [10, 11] we have shown that an exact treatment of the integral that lead them to inconsistency proves otherwise. However, in a recent comment, Hawkins and Schwarz point to a possible problem in the proof regarding the analyticity of the relevant integrals [12].

In Sections II and III we present details of the treatment of the relevant integrals and show for general $n$ that there is no inconsistency. Recently, Dong [13] obtained the wave function for the infinite square well problem by using path integrals over Lévy paths and confirmed the solution given by Laskin [5-8].

However, Luchko analyzed the solution in configuration space with a different representation of the Riesz derivative and argued in favor of inconsistency [14]. To pinpoint the source of this controversy and its resolution, in the Section IV we scrutinize the different representations of the Riesz derivative and show that when calculated consistently, they all have the same Fourier transform. The controversy arises when the divergent integrals in the configuration space are evaluated piecewise for the infinite square well problem, thus tampering with the integrity of the Riesz derivative. Finally, Section V is the conclusions.

II. Consistency of the Solutions of the Space Fractional Schrödinger Equation

In one dimension the space fractional Schrödinger equation is written in terms of the quantum Riesz derivative $(-\hbar^2 \Delta)^{\alpha/2}$ [5-8] as

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \Psi(x,t) + V(x)\Psi(x,t),$$  \(1\)

where

$$(-\hbar^2 \Delta)^{\alpha/2} \Psi(x,t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} dp \ e^{ipx/\hbar} |p|^\alpha \Phi(p,t), \ 1 < \alpha \leq 2,$$  \(2\)
and $Φ(p, t)$ is the Fourier transform of the wave function:

$$Φ(p, t) = \int_{-\infty}^{+\infty} dx \Psi(x, t)e^{-ipx/ℏ}.$$  (3)

The restriction on $α$ comes from the requirement of the existence of the first-order moments of the $α$-stable Lévy distribution so that average momentum or position of the quantum particle can be found [5]. For the infinite square well, the potential is given as

$$V(x) = \begin{cases} 0 & ; |x| < a \\ \infty & ; |x| \geq a \end{cases},$$  (4)

where for its separable solutions:

$$Ψ(x, t) = e^{-iEt/ℏ}\psi(x),$$  (5)

$ψ(x)$ satisfies the following eigenvalue problem:

$$D_α (-ℏ^2 \Delta)^{α/2} ψ(x) = Eψ(x), \; ψ(a) = ψ(-a) = 0.$$  (6)

The corresponding energy eigenfunctions and the eigenvalues are obtained as [5-8]:

$$ψ_n(x) = \begin{cases} A \sin \left(\frac{\pi n}{2a}(x + a)\right) & ; |x| < a \\ 0 & ; |x| \geq a \end{cases},$$  (7)

$$E_n = D_α \left(\frac{ℏmπ}{2a}\right)^α, \; n = 1, 2, \ldots .$$

To show the inconsistency of these solutions, Jeng et. al. [9] concentrated on the ground state with $n = 1$:

$$ψ_1(x) = \begin{cases} A \cos \left(\frac{πa}{2a}\right) & ; |x| < a \\ 0 & ; |x| \geq a \end{cases}$$  (8)

and argued that this solution, albeit satisfying the boundary conditions, $ψ_1(-a) = ψ_1(a) = 0$, when substituted back into the space fractional Schrödinger equation leads to a contradiction [9]. Using the Fourier transform of $ψ_1(x)$:

$$φ_1(p) = F \{ψ_1(x)\} = -Aπ \left(\frac{ℏ^2}{a}\right) \frac{\cos (ap/ℏ)}{p^2 - (πℏ/2a)^2} , \; |x| < a,$$  (9)

and the definition of the quantum Riesz derivative [5-8]:

$$(-ℏ^2 Δ)^{α/2} ψ_1(x) = (1/2πℏ) \int_{-∞}^{+∞} dp e^{ipx/ℏ} |p|^α φ_1(p),$$  (10)
in Equation (6), they wrote $\psi_1(x)$ as the integral

$$\psi_1(x) = -\frac{AD_{\alpha}}{2E_1} \left( \frac{\hbar}{a} \right) \int_{-\infty}^{+\infty} dp \left( \frac{2a}{\pi \hbar} \right)^2 |p|^\alpha \cos \left( \frac{ap}{\hbar} \right) e^{ipx/\hbar}, \ |x| < a. \quad (11)$$

Using the substitution $q = \frac{2a}{\pi \hbar} p$, $\psi_1(x)$ becomes

$$\psi_1(x) = -\frac{AD_{\alpha}}{\pi E_1} \left( \frac{\pi \hbar}{2a} \right)^\alpha \int_{-\infty}^{+\infty} dq \frac{|q|^\alpha \cos (\pi q/2)}{q^2 - 1} e^{i\pi qx/2a}. \quad (12)$$

Jeng et. al. [9] argued that the right hand side of the above equation, which they wrote as

$$\psi_1(x) = -\frac{AD_{\alpha}}{\pi E_1} \left( \frac{\pi \hbar}{2a} \right)^\alpha \int_{-\infty}^{+\infty} dq \frac{|q|^\alpha \cos (\pi q/2)}{q^2 - 1} \cos (\pi qx/2a), \quad (13)$$

can not satisfy the boundary conditions that $\psi_1(x)$ satisfies as $x \to \pm a$, thus indicating an inconsistency in the infinite square well solution. However, we have shown that an exact evaluation of the integral in Equation (12) proves otherwise [10, 11]. In the Section III we give the general proof for all $n$.

### III. Proof For All $n$

#### A. The case for odd $n$

For the odd values of $n$, eigenfunctions in Equation (7) become

$$\psi_n(x) = \begin{cases} 
A \cos \frac{n\pi x}{2a} ; & \ |x| < a \\
0 ; & \ |x| \geq a
\end{cases}, \quad (14)$$

$$E_n = D_{\alpha} \left( \frac{\pi n \hbar}{2a} \right)^\alpha, \ n = 1, 3, 5, \ldots.$$

Using the Fourier transform $F\{\psi_n(x)\} = \phi_n(p)$:

$$\phi_n(p) = -\frac{An\pi \hbar^2 \sin(n\pi/2)}{a} \left( \frac{\cos \left( \frac{pa}{\hbar} \right)}{p^2 - (n\pi \hbar/2a)^2} \right), \ n = 1, 3, 5, \ldots, \quad (15)$$

and the definition of the Riesz derivative [Eq. (2)] in the space fractional Schrödinger Equation [Eq. (6)], the corresponding integral expression for $\psi_n(x)$, $n = 1, 3, 5, \ldots$ becomes:

$$\psi_n(x) = -\frac{AD_{\alpha} n\hbar(2a/n\pi \hbar)^2 \sin(n\pi/2)}{2a E_n} \int_{-\infty}^{+\infty} dp \frac{e^{ipx/\hbar} |p|^\alpha \cos \left( \frac{pa}{\hbar} \right)}{(2ap/n\pi \hbar)^2 - 1}. \quad (16)$$

Making the substitution $p = (n\pi \hbar/2a)q$, we write

$$\psi_n(x) = -\frac{AD_{\alpha} \sin(n\pi/2)}{E_n \pi} \left( \frac{n\pi \hbar}{2a} \right)^\alpha \int_{-\infty}^{+\infty} dq \frac{e^{i(n\pi x/2a)q} |q|^\alpha \cos(n\pi q/2)}{(q^2 - 1)}$$

$$= -\frac{AD_{\alpha} \sin(n\pi/2)}{E_n \pi} \left( \frac{n\pi \hbar}{2a} \right)^\alpha I, \quad (17)$$
where $I$ is the integral

$$I = \int_{-\infty}^{+\infty} dq \, e^{i(n\pi x/2\alpha)q} \frac{|q|^\alpha \cos(n\pi q/2)}{(q^2 - 1)}.$$  \hspace{1cm} (18)

Substituting

$$\cos(n\pi q/2) = \frac{1}{2} \left( e^{in\pi q/2} + e^{-in\pi q/2} \right),$$

we can write $I$ as the sum of two integrals:

$$I = I_1 + I_2 = \frac{1}{2} \int_{-\infty}^{+\infty} dq \, \frac{|q|^\alpha e^{i(n\pi x/2\alpha + n\pi/2)q}}{(q + 1)(q - 1)} + \frac{1}{2} \int_{-\infty}^{+\infty} dq \, \frac{|q|^\alpha e^{i(n\pi x/2\alpha - n\pi/2)q}}{(q + 1)(q - 1)},$$  \hspace{1cm} (20)

which can be evaluated by analytic continuation as a Cauchy principal value integral [15 pg. 365]. However, in the above integrals, as it stands, $|q|^\alpha$ cannot be continued analytically. To overcome this difficulty, we resort to the original definition of the Riesz derivative and see where $|q|^\alpha$ comes from.

The Riesz derivative, $R_x^\alpha f(x)$, is defined as [3, 16-18]

$$R_x^\alpha f(x) = -\infty D_x^\alpha f(x) + \infty D_x^\alpha f(x), \quad \alpha > 0, \quad \alpha \neq 1, 3, ...$$  \hspace{1cm} (21)

$$-\infty D_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \int_{-\infty}^{x} (x - x')^{-\alpha-1+n} f^{(n)}(x') dx',$$

$$+\infty D_x^\alpha f(x) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_{x}^{\infty} (x' - x)^{-\alpha-1+n} f^{(n)}(x') dx',$$  \hspace{1cm} (22) (23)

where $n$ is the smallest integer greater than $\alpha$. For the range $1 < \alpha < 2$, $n = 2$. In Equations (22) and (23) we have used the Caputo fractional derivative [Eqs. (A7) and (A8)] since for sufficiently smooth functions:

$$f(x), f'(x), \ldots, f^{(n-1)}(x) \to 0 \text{ as } x \to \pm \infty,$$  \hspace{1cm} (24)

the Caputo and the Riemann-Liouville definitions agree [2, 3, 16-18]. Also note that the quantum Riesz derivative and the Riesz derivative $R_x^\alpha$ are related by [5-8]

$$(-\hbar^2 \Delta)^{\alpha/2} \psi_1(x) = -\hbar^\alpha R_x^\alpha \psi(x).$$  \hspace{1cm} (25)

Using the following Fourier transforms (see Section IV for the detailed derivation):

$$\mathcal{F}\{ -\infty D_x^\alpha f(x) \} = (i\omega)^\alpha g(\omega),$$

$$\mathcal{F}\{ +\infty D_x^\alpha f(x) \} = (-i\omega)^\alpha g(\omega)$$  \hspace{1cm} (26)

where $g(\omega) = \mathcal{F}\{ f(x) \}$ and $\alpha > 0$, we write the Fourier transform of the Riesz derivative as

$$\mathcal{F}\{ R_x^\alpha f(x) \} = -\left( \frac{(i\omega)^\alpha + (-i\omega)^\alpha}{2 \cos \alpha \pi / 2} \right) g(\omega).$$  \hspace{1cm} (27)
When \( \omega \) is restricted to the real axis, this reduces to the familiar expression
\[
\mathcal{F} \{ R_\alpha^x f(x) \} = -|\omega|^\alpha g(\omega),
\]
which is used in Equations (12) and (17). In order to evaluate \( I \) by analytic continuation, we use the above form of the Riesz derivative [Eq. (27)], which allows analytic continuation and write \( I \) [Eq. (20)] as
\[
I = I_1 + I_2 = \frac{1}{2} \int_{-\infty}^{+\infty} dq \left( \frac{(iq)^\alpha + (-iq)^\alpha}{2 \cos \alpha \pi/2} \right) e^{i \left( \frac{\alpha \pi}{2} + \frac{\alpha \pi}{2} \right) q} \frac{q}{(q+1)(q-1)} + \frac{1}{2} \int_{-\infty}^{+\infty} dq \left( \frac{(iq)^\alpha + (-iq)^\alpha}{2 \cos \alpha \pi/2} \right) e^{i \left( \frac{\alpha \pi}{2} - \frac{\alpha \pi}{2} \right) q} \frac{q}{(q+1)(q-1)}.
\]
(28)

Factoring \( q^\alpha \) out, the integrals \( I_1 \) and \( I_2 \):
\[
I_1 = \left( \frac{(i)^\alpha + (-i)^\alpha}{4 \cos \alpha \pi/2} \right) \int_{-\infty}^{+\infty} dq e^{i \left( \frac{\alpha \pi}{2} + \frac{\alpha \pi}{2} \right) q} \frac{q}{(q+1)(q-1)},
\]
(29)
\[
I_2 = \left( \frac{(i)^\alpha + (-i)^\alpha}{4 \cos \alpha \pi/2} \right) \int_{-\infty}^{+\infty} dq e^{i \left( \frac{\alpha \pi}{2} - \frac{\alpha \pi}{2} \right) q} \frac{q}{(q+1)(q-1)},
\]
(30)
can now be evaluated as Cauchy principal value integrals via analytic continuation [15 pg. 365].

In the above integrals, aside from the poles at \( q = \pm 1 \), there is also a branch point and a branch cut at the origin due to the power \( q^\alpha \), \( \alpha > 0 \). For each integral, in contrast to the claims of Hawkins and Schwarz [12], the branch cut can always be chosen away from the region of interest. For the branch values of \( (i)^\alpha \) and \( (-i)^\alpha \), it has to be remembered that the Riesz derivative \( R_\alpha^x \) is defined such that for real \( q \), the Fourier transform of the Riesz derivative corresponds to the logarithm of the characteristic function of the symmetric Lévy probability density function. Therefore, in the definition of the Riesz derivative [Eq. (21)], \( 2 \cos \alpha \pi/2 \) is introduced with the principal branch values of \( (i)^\alpha \) and \( (-i)^\alpha \) in mind, hence \( (i)^\alpha + (-i)^\alpha = 2 \cos \alpha \pi/2 \). This way, along with the minus sign introduced by hand in Equation (21), \( R_\alpha^x \) reproduces the standard derivative \( \frac{d^2}{dx^2} \) for \( \alpha = 2 \) [3, 17]. Other linear combinations of \(-D_\alpha^x f(x)\) and \(D_\alpha^x f(x)\) have also found use in literature as the Feller derivative, which gives an additional degree of freedom in terms of a parameter called the phase or the skewness parameter [3, 17].

1. **Evaluation of \( I_1 \) and \( I_2 \)**

For \( I_1 \) [Eq. (29)] the contour is closed counterclockwise in the upper half complex \( q \)-plane over a semicircular path with radius \( R \), and then the contour detours around the poles on the real axis over semicircular paths of radius \( \delta \) in the upper half \( q \)-plane. Similarly, the contour goes around the branch point at the origin with the branch cut located in the lower half of the \( q \)-plane. Since \( \alpha > 0 \), the integrand vanishes on the contour as the radius of the semicircular
path over $q = 0$ shrinks to zero, hence the integral over the branch point does not contribute to the integral. In the limit as $R \to \infty$, by the Jordan’s lemma, the contribution coming from the large semicircle vanishes, thus allowing the evaluation of this integral as a Cauchy principal value integral in the limit $\delta \to 0$ as [15 pg. 365]

$$PV(I_1) = \left( \frac{i \pi}{2} \right) \frac{\sin n\pi/2}{4 \cos \alpha \pi/2} \left[ (i^\alpha + (-i)^\alpha)(-1 + (-1)^\alpha) \sin (n\pi x/2a) ight. \\
+ (i^\alpha + (-i)^\alpha)(1 + (-1)^\alpha) \cos (n\pi x/2a) \\
= - \left( \frac{\pi \sin n\pi/2}{2} \right) \cos (n\pi x/2a), \ n = 1, 3, \ldots . \tag{31}$$

Note that one also uses the relation $[i^\alpha + (-i)^\alpha] = (-1)^\alpha [i^\alpha + (-i)^\alpha]$.

For $I_2$, the contour is closed counterclockwise in the lower $q$–plane and circles around the poles and the branch point in the lower half $q$–plane. For $I_2$ the branch cut is chosen in the upper half $q$–plane and again since $\alpha > 0$, the integral around the branch point does not contribute to the integral, thus yielding $PV(I_2)$ as

$$PV(I_2) = - \left( \frac{\pi \sin n\pi/2}{2} \right) \cos (n\pi x/2a), \ n = 1, 3, \ldots . \tag{33}$$

which leads to the Cauchy principal value of $I$ as the sum

$$PV(I) = PV(I_1) + PV(I_2)$$

$$= - \pi (\sin n\pi/2) \cos (n\pi x/2a), \ n = 1, 3, \ldots . \tag{34}$$

When this is substituted back into Equation (17) we get

$$\psi_n(x) = - AD_\alpha \left( \frac{\sin(n\pi/2)}{E_n \pi} \right) \left( \frac{n\pi \hbar}{2a} \right)^\alpha PV(I)$$

$$= AD_\alpha \left( \frac{\sin(n\pi/2)}{E_n \pi} \right) \left( \frac{n\pi \hbar}{2a} \right)^\alpha \frac{\cos n\pi x}{2a}. \tag{35}$$

Since $E_n = D_\alpha (\frac{\hbar n \pi}{2a})^\alpha$ and $\sin^2(\frac{n\pi x}{2}) = 1$ for odd $n$, we again obtain the wave function [Eq. (14)] as

$$\psi_n(x) = A \cos \frac{n\pi x}{2a}, \ n = 1, 3, \ldots, \ |x| < a, \tag{37}$$

which on the contrary to Jeng et. al. [9] and Hawkins and Schwarz [12], vanishes at the boundary as $x \to \pm a$, hence there is no inconsistency with the solution outside.
B. The case for even $n$

The proof for the even $n$ values follows along the same lines [10, 11]. We first write the wave function [Eq. (7)] as

$$
\psi_n(x) = \begin{cases} 
A \sin \frac{n\pi x}{2a} & ; \ |x| < a \\
0 & ; \ |x| \geq a
\end{cases}, \quad (38)
$$

and then obtain its Fourier transform as

$$
\phi_n(p) = -iA \frac{\sin(pa/\hbar)}{\alpha} \frac{\sin(n\pi/2)}{\alpha} \sin \left( \frac{n\pi q}{2} \right) \frac{\sin(n\pi q/2)}{(q^2 - 1)}, \quad n = 2, 4, \ldots, \quad (39)
$$

Now the integral representation of $\psi_n(x)$ becomes

$$
\psi_n(x) = - \frac{iA n \pi h^2 (\cos n\pi/2)}{E_n \pi} \left( \frac{n\pi h}{2a} \right)^\alpha \int_{-\infty}^{+\infty} dq \frac{e^{i(n\pi x/2a)q} |q|^\alpha \sin(n\pi q/2)}{(q^2 - 1)}, \quad (40)
$$

where we used the substitution $p = (n\pi h/2a)q$. Finally, using

$$
\sin(n\pi q/2) = \frac{1}{2i} \left( e^{in\pi q/2} - e^{-in\pi q/2} \right), \quad (41)
$$

and the original definition of the Riesz derivative [Eq.(21)], we write

$$
\psi_n(x) = - \frac{AD_\alpha \cos(n\pi/2)}{E_n \pi} \left( \frac{n\pi h}{2a} \right)^\alpha I, \quad (42)
$$

where

$$
I = I_1 - I_2, \quad (43)
$$

$$
I_1 = \left( \frac{(i)^\alpha + (-i)^\alpha}{4 \cos \alpha \pi/2} \right) \int_{-\infty}^{+\infty} dq \frac{q^{\alpha} e^{i\frac{n\pi q}{2a} + \frac{\pi q}{2}}}{(q + 1)(q - 1)}, \quad (44)
$$

$$
I_2 = \left( \frac{(i)^\alpha + (-i)^\alpha}{4 \cos \alpha \pi/2} \right) \int_{-\infty}^{+\infty} dq \frac{q^{\alpha} e^{i\frac{n\pi q}{2a} - \frac{\pi q}{2}}}{(q + 1)(q - 1)}. \quad (45)
$$

The Cauchy principal value of $I$ is now found as

$$
PV(I) = -\pi (\cos n\pi/2) \sin (n\pi x/2a), \quad n = 2, 4, \ldots, \quad (46)
$$

which when substituted into (42) yields the wave function in (38), hence again no inconsistency.

This is not surprising at all. In fact, Equations (14) and (17) and similarly Equations (38) and (40), represent the same wave function, where Equations (17) and (40) are just the integral representations of $\psi_n(x)$ in Equations (14)
and (38) for the odd and the even values of \( n \), respectively. It is true that the Riesz derivative is a non local operator [Eqs. (21-23)] that requires knowledge of the wave function over the entire space. For the infinite square well problem, the system is confined to the region \(|x| < a\) with \( \Psi(x,t) = 0 \) for \(|x| \geq a\). Since the solution for \(|x| < a\) satisfies the boundary conditions as \( x \to \pm a \), the solution inside the well is consistent with the outside.

IV. Scrutinizing the Riesz Derivative

Another source for the proposed inconsistency in the infinite square well solution [Eq. (7)] is that when the Riesz derivative in Equation (21) is directly calculated by evaluating the integrals in Equations (22) and (23), the result does not satisfy the space fractional Schrödinger equation [14]. Note that these integrals are now in the configuration space. This situation is explained by the fact that the Riesz derivative is non local, hence to find the solution outside the well, one also has to consider the solution inside [14]. To shed some light on this problem, we now scrutinize how the different definitions of the Riesz derivative are written and how they are related and calculated.

A. Riesz Fractional Integral

To evaluate the integrals in the definition of \( R_x^\alpha f(x) \) [Eqs. (22) and (23)], we are going to start with the definition of the Riesz fractional integral, which is defined as [Eqs. (A1) and (A2)]

\[
R_x^\alpha f(x) = \frac{-\infty D_x^\alpha f(x) + \infty D_x^\alpha f(x)}{2 \cos \alpha \pi / 2}, \quad \alpha > 0, \quad \alpha \neq 1, 3, \ldots, 
\]

\[
-\infty D_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - x')^{\alpha-1} f(x') dx',
\]

\[
+\infty D_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (x' - x)^{\alpha-1} f(x') dx'.
\]

To evaluate the integral in Equation (48), we define the function

\[
h_+(x) = \begin{cases} 
\frac{x^{\alpha-1}}{\Gamma(\alpha)}, & x > 0 \\
0, & x \leq 0 
\end{cases}
\]

which allows us to write \(-\infty D_x^\alpha f(x)\) as the convolution of \( h_+(x) \) with \( f(x) \):

\[
-\infty D_x^\alpha f(x) = h_+(x) \ast f(x).
\]

It is well known that the Fourier transform of a convolution is equal to the product of the Fourier transforms of the convolved functions, that is,

\[
\mathcal{F}\{ -\infty D_x^\alpha f(x) \} = \mathcal{F}\{ h_+(x) \} \mathcal{F}\{ f(x) \}. 
\]
Using analytic continuation with an appropriate contour, it is straightforward to evaluate the Fourier transform of \(h_+(x)\) as
\[
\mathcal{F}\{h_+(x)\} = \int_{-\infty}^{\infty} x^{\alpha-1} \frac{1}{\Gamma(\alpha)} e^{-i\omega x} dx = (i\omega)^{-\alpha}, \quad \alpha > 0.
\] (53)

Assuming that the Fourier transform of \(f(x)\) exists:
\[
\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = F(\omega),
\] (54)

which only demands an absolutely integrable \(f(x)\), we obtain the Fourier transform
\[
\mathcal{F}\{-\infty D_x^{-\alpha} f(x)\} = (i\omega)^{-\alpha} F(\omega), \quad \alpha > 0.
\] (55)

Following similar steps, we define the function
\[
h_-(x) = \begin{cases} 0, & x \geq 0 \\ \frac{(-x)^{\alpha-1}}{\Gamma(\alpha)}, & x < 0 \end{cases}
\] (56)

with the Fourier transform
\[
\mathcal{F}\{h_-(x)\} = \int_{-\infty}^{0} \frac{(-x)^{\alpha-1}}{\Gamma(\alpha)} e^{-i\omega x} dx = (-i\omega)^{-\alpha}, \quad \alpha > 0.
\] (57)

We can now write \(\infty D_x^{-\alpha} f(x)\) [Eq. (49)] as the convolution
\[
\infty D_x^{-\alpha} f(x) = h_-(x) * f(x),
\] (58)

where its Fourier transform is given as
\[
\mathcal{F}\{\infty D_x^{-\alpha} f(x)\} = \mathcal{F}\{h_-(x)\} \mathcal{F}\{f(x)\} = (-i\omega)^{-\alpha} F(\omega), \quad \alpha > 0.
\] (59)

Using Equations (55) and (60), the Riesz fractional integral, \(R_x^{-\alpha} f(x)\), is defined in terms of its Fourier transform as
\[
\mathcal{F}\{R_x^{-\alpha} f(x)\} = \frac{(i\omega)^{-\alpha} + (-i\omega)^{-\alpha}}{2\cos \alpha \pi/2} F(\omega), \quad \alpha > 0, \quad \alpha \neq 1, 3, \ldots,
\] (61)

\[
= |\omega|^{-\alpha} F(\omega), \quad \text{for real } \omega.
\] (62)

Also note that from Equations (47-49), \(R_x^{-\alpha} f(x)\) is also the integral
\[
R_x^{-\alpha} f(x) = \frac{1}{2\Gamma(\alpha) \cos \alpha \pi/2} \int_{-\infty}^{\infty} |x - x'|^{\alpha-1} f(x') dx', \quad \alpha > 0, \quad \alpha \neq 1, 3, \ldots.
\] (63)
B. Riesz Fractional Derivative

To evaluate the Riesz fractional derivative, we note that in Equations (21-23), the Caputo definition of the fractional derivative is used. Since for sufficiently smooth functions [Eq. (24)]:

\[ f(x), f'(x), \ldots, f^{(n-1)}(x) \to 0 \text{ as } x \to \pm \infty, \]

the Caputo and the Riemann-Liouville definitions agree [Eq. (A9)], we can write Equation (22) as [Eq. (A7) [3, 16-18]]

\[
-\infty D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{x} (x-x')^{-\alpha-1+n} f^{(n)}(x')dx', \quad \alpha > 0, \quad (64)
\]

\[
= -\infty \Gamma_n^{-\alpha} f^{(n)}(x) = -\infty D_x^{\alpha-n} f^{(n)}(x). \quad (65)
\]

Note that we have dropped the abbreviation $R-L$ and $C$ in $R-L D_x^\alpha$ and $C D_x^\alpha$. Since $\alpha - n < 0$, we can use our previous result [Eq. (55)] to obtain [2, 16]

\[
F \{ -\infty D_x^\alpha f(x) \} = F \{ -\infty D_x^{\alpha-n} f^{(n)}(x) \} = (i\omega)^{\alpha-n} F(f^{(n)}(x)) = (i\omega)^{\alpha-n} (i\omega)^n F(\omega) \quad (66)
\]

\[
= (i\omega)^{\alpha-n} F(\omega). \quad (67)
\]

The third step [Eq. (68)], is already assured by the smoothness condition [Eq. (24)]. Similarly, we obtain

\[
F \{ -\infty D_x^\alpha f(x) \} = (-i\omega)^n F(\omega). \quad (69)
\]

Therefore, we can write the Fourier transform of the Riesz derivative [Eq. (21)] as

\[
F \{ R_x^\alpha f(x) \} = -\frac{(i\omega)^n + (-i\omega)^n}{2 \cos \alpha \pi/2} F(\omega), \quad \alpha > 0, \quad \alpha \neq 1, 3, \ldots, \quad (71)
\]

where $F(\omega)$ is the Fourier transform of $f(x)$ [Eq. (54)], which makes use of the values of $f(x)$ over the entire range $x \in (-\infty, \infty)$. For real $\omega$, we can also write this as

\[
F \{ R_x^\alpha f(x) \} = -|\omega|^\alpha F(\omega), \quad (72)
\]

which was used to write Equations (12), (17) and (40). So far, all we have assumed is that $f(x)$ is absolutely integrable, hence its Fourier transform exists and the smoothness condition in Equation (24). Granted that the inverse transform exists, the Riesz derivative is defined as

\[
R_x^\alpha f(x) = \mathcal{F}^{-1} \{ -|\omega|^\alpha F(\omega) \} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^\alpha F(\omega) e^{i\omega x} d\omega. \quad (73)
\]

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Note that our starting point was the integrals in Equations (22−23), hence using (21), \( R^\alpha_x f(x) \) can also be written as

\[
R^\alpha_x f(x) = -\frac{1}{2\Gamma(2-\alpha)\cos \alpha \pi/2} \times \left[ \int_{-\infty}^{x} (x-x')^{-\alpha+1} f^{(2)}(x')dx' + \int_{x}^{\infty} (x'-x)^{-\alpha+1} f^{(2)}(x')dx' \right], \quad 1 < \alpha < 2,
\]

where we have set \( n = 2 \) for \( 1 < \alpha < 2 \).

It is important to note that Equations (74) and (75) correspond to different representations of the Riesz derivative, which have the same Fourier transform. As we have shown, Equation (74) is actually obtained from the Fourier transform of (75). It is not true to say that non local effects are incorporated in (75) but not in (74). In Equation (74), the Fourier transform of \( f(x) \) is obtained by integrating over the entire space as \( F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx. \) In (74), \( R^\alpha_x f(x) \) is given in terms of an integral in the frequency (momentum) space, while in (75), \( R^\alpha_x f(x) \) is given in terms of integrals in the configuration space. In general, the integrals in both of these expressions are singular in their respective spaces. Granted that these singular integrals are treated consistently, they should yield the same result. However, technically, it is easier to work in the momentum space with Equation (74).

### C. Riesz Derivative via the R-L Definition

In Equation (75) we have used the Caputo fractional derivative for \( -\infty D^\alpha_x f(x) \) and \( +\infty D^\alpha_x f(x) \). If we use the Riemann-Liouville definition, The Riesz derivative [Eqs. (21−23)] becomes [Eqs. (A4) and (A6), [3, 16–18]]

\[
R^\alpha_x f(x) = -\frac{-\infty D^\alpha_x f(x) + +\infty D^\alpha_x f(x)}{2 \cos \alpha \pi/2}, \quad \alpha > 0, \quad \alpha \neq 1, 3, ..., \quad (76)
\]

\[
-\infty D^\alpha_x f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^{x} (x-x')^{-\alpha+1} f(x')dx', \quad (77)
\]

\[
+\infty D^\alpha_x f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{x}^{\infty} (x'-x)^{-\alpha+1} f(x')dx', \quad (78)
\]

hence we can also write

\[
R^\alpha_x f(x) = -\frac{1}{2\Gamma(2-\alpha)\cos \alpha \pi/2} \times \left[ \frac{d^2}{dx^2} \int_{-\infty}^{x} (x-x')^{-\alpha+1} f(x')dx' + \frac{d^2}{dx^2} \int_{x}^{\infty} (x'-x)^{-\alpha+1} f(x')dx' \right], \quad (79)
\]

Using the functions \( h_{\pm}(x) \) [Eqs. (50) and (56)] and the convolution theorem, it is straightforward to show that the Fourier transform of (79) is still given by Equation (71), or (72) when \( \omega \) is real.
D. Source of the Controversy

The so-called inconsistency problem of the infinite square well, in the configuration space [14] originates from the piecewise evaluation of the highly singular integrals in Equation (79), which tampers with the integrity of the Riesz derivative, thus affecting its Fourier transform. For example, for a point outside the well, say $x \geq a$, if we write the Riesz derivative [Eq. (79)] as

$$R_\alpha^a \psi_n(x) = \frac{1}{2\Gamma(2-\alpha)\cos \alpha \pi/2} \left\{ \frac{d^2}{dx^2} \int_{-\infty}^{-a} (x-x')^{-\alpha+1} \psi_n(x')dx' + \frac{d^2}{dx^2} \int_{-a}^{a} (x-x')^{-\alpha+1} \psi_n(x')dx' + \frac{d^2}{dx^2} \int_{a}^{x} (x-x')^{-\alpha+1} \psi_n(x')dx' \right\}, 1 < \alpha < 2, \tag{80}$$

and then substitute the square well solution [Eq. (7)], we obtain

$$R_\alpha^a \psi_n(x) = \frac{1}{2\Gamma(2-\alpha)\cos \alpha \pi/2} \left\{ \frac{d^2}{dx^2} \int_{-a}^{a} (x-x')^{-\alpha+1} \psi_n(x')dx' \right\}, x \geq a. \tag{81}$$

The above expression gives the values of the Riesz derivative outside the well, $x \geq a$, in terms of an integral that only makes use of the values of the wave function inside the well. In general, the $R_\alpha^a \psi_n(x)$ given above for $x \geq a$ does not vanish, hence does not satisfy the space fractional Schrödinger equation [Eq. (6)] for $x \geq a$. This implies a potential problem for the infinite square well solution [14]. Note that to write Equation (81), we have used the fact that the wave function outside is zero. Thus, along with the first and the third integrals in Equation (80), we have set the last integral to zero [14]. Even though this procedure looks reasonable, what it essentially does is to set the fractional derivative $\infty D_\alpha^a \psi_n(x)$ to zero for $x \geq a$, that is,

$$\infty D_\alpha^a \psi_n(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{x}^{\infty} (x-x')^{-\alpha+1} \psi_n(x')dx' \tag{82}$$

$$= 0, \quad x \geq a, \tag{83}$$

thus the Fourier transform $\mathcal{F} \{ \infty D_\alpha^a \psi_n(x) \}$ is also set to zero for $x \geq a$. However, in the definition of the Riesz derivative [Eqs. (21–23)], the Fourier transform of $\infty D_\alpha^a \psi_n(x)$, for all $x$, is given as [Eq. (70)]

$$\mathcal{F} \{ \infty D_\alpha^a \psi_n(x) \} = (-i\omega)^\alpha \Phi_n(\omega), \tag{84}$$

where $\Phi_n(\omega)$ is the Fourier transform of the entire solution, $\psi_n(x)$, not just the solution for $x \geq 0$.

Similarly, this procedure also tampers with the Fourier transform of $-\infty D_\alpha^a \psi_n(x)$, thus the Fourier transform of the derivative in (81) is not what it should be,
that is, $\mathcal{F}\{R_x^n f(x)\} = -|\omega|^\alpha F(\omega)$, which is the basic definition of the Riesz derivative used in the space fractional Schrödinger equation.

Similarly, the expressions for $x \leq -a$ and $|x| < a$ can be written as [14]

$$R_x^n \psi_n(x) = -\frac{1}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \left[ \frac{d^2}{dx^2} \int_{-a}^{a} (x' - x)^{-\alpha + 1} \psi_n(x') dx' \right], \quad x \leq -a,$$

$$R_x^n \psi_n(x) = -\frac{1}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \left[ \frac{d^2}{dx^2} \int_{-a}^{a} |x - x'|^{-\alpha + 1} \psi_n(x') dx' \right], \quad |x| < a.$$  

(85)

Note that Equation (81) can also be written as

$$R_x^n \psi_n(x) = -\frac{1}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \int_{-a}^{a} \frac{\psi_n(x')}{(x' - x)^{\alpha + 1}} dx', \quad x \geq a,$$

(87)

$$= -\frac{(-\alpha + 1)(-\alpha)}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \int_{-a}^{a} \frac{\psi_n(x')}{(x' - x)^{\alpha + 1}} dx',$$

(88)

$$= -\frac{1}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \int_{-a}^{a} \frac{\psi_n(x')}{(x - x')^{\alpha + 1}} dx'.$$

(89)

This result was used in [14], which was obtained by using another representation of the Riesz derivative:

$$R_x^n f(x) = \frac{\Gamma(1 + \alpha) \sin \alpha \pi/2}{\pi} \int_{0}^{\infty} \frac{f(x + x') - 2f(x)}{x^{\alpha + 1}} dx',$$

(90)

which is also good for $\alpha = 1$. This representation is obtained by writing $-\infty D_x^\alpha f(x)$ and $\infty D_x^\alpha f(x)$ in Equations (77) and (78) as [3]

$$-\infty D_x^\alpha f(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{f(x) - f(x + x')}{x^{\alpha + 1}} dx',$$

(91)

$$\infty D_x^\alpha f(x) = -\frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{f(x + x') - f(x)}{x^{\alpha + 1}} dx'.$$

(92)

Similarly for $x \leq -a$ and $|x| < a$, we obtain the expressions used in [14] as

$$R_x^n \psi_n(x) = -\frac{1}{2\Gamma(-\alpha) \cos \alpha \pi/2} \left[ \int_{-a}^{a} \frac{\psi_n(x')}{(x' - x)^{\alpha + 1}} dx' \right], \quad x \leq -a,$$

(93)

$$R_x^n \psi_n(x) = -\frac{1}{2\Gamma(2 - \alpha) \cos \alpha \pi/2} \left[ \frac{d^2}{dx^2} \int_{-a}^{a} |x - x'|^{-\alpha + 1} \psi_n(x') dx' \right], \quad |x| < a.$$  

(94)

In summary, the Riesz derivative can be evaluated by using Equation (74), which involves an integration in the frequency (momentum) space. We have shown that for the infinite square well problem, the use of Equation (74) gives consistent results. We can also use the representations in Equations (75) or (79), which involve integrals in configuration space. What is important is that a consistent treatment of all the representations of the Riesz derivative should yield the same Fourier transform, that is, $\mathcal{F}\{R_x^n f(x)\} = -|\omega|^\alpha F(\omega)$.
V. Conclusions

Using the convolution theorem we have demonstrated how the frequency (momentum) space representation of the Riesz derivative [Eq. (74)]:

\[ R_\alpha^x f(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^\alpha F(\omega)e^{i\omega x} d\omega, \]  

(95)
is obtained from the integral representations in the configuration space [Eq. (75)]:

\[ R_\alpha^x f(x) = -\frac{1}{2\Gamma(2-\alpha)} \cos \frac{\alpha\pi}{2} \left[ \int_{-\infty}^{x} (x-x')^{-\alpha+1} f^{(2)}(x')dx' + \int_{x}^{\infty} (x'-x)^{-\alpha+1} f^{(2)}(x')dx' \right], \quad 1 < \alpha < 2, \]

(96)

and similarly from [Eq. (79)]

\[ \frac{d^2}{dx^2} \int_{-\infty}^{x} (x-x')^{-\alpha+1} f^{(2)}(x')dx' + \frac{d^2}{dx^2} \int_{x}^{\infty} (x'-x)^{-\alpha+1} f^{(2)}(x')dx' \right], \quad 1 < \alpha < 2. \]

(97)

 Granted that \( f(x) \) is absolutely integrable and the smoothness condition in Equation (24) is satisfied, all the above representations of the Riesz derivative agree and have the same Fourier transform. The first definition is given in the frequency (momentum) domain while the others are in the configuration space.

For the infinite square well, the controversy proposed in [9, 12] is based on the use of the momentum space definition in Equation (95). In Section II and III, we have shown that if the relevant integrals are evaluated as Cauchy principal value integrals, there is no inconsistency.

As for the inconsistency of the infinite well solution proposed in terms of the configuration space definitions of the Riesz derivative [14], the segmented evaluation of these integrals leads to the Riesz derivative in Equation (81) for \( x \geq a \), (85) for \( x \leq -a \) and (86) for \( |x| < a \). Substituting the eigenfunctions [Eq. (7)] into Equations (89), (93) and (94) we obtain

\[ R_\alpha^x \psi_n(x) = F_1(x) = -\frac{1}{2\Gamma(-\alpha) \cos \alpha \pi/2} \int_{-a}^{a} \frac{\psi_n(x')}{(x-x')^{\alpha+1}} \]  

(98)

\[ = -\frac{A}{2\Gamma(-\alpha) \cos \alpha \pi/2} \int_{-a}^{a} \frac{\sin \frac{\pi}{2a}(x'+a)}{(x-x')^{\alpha+1}}, \quad x \geq a \]  

(99)
and

\[ R_2^\alpha \psi_n(x) = F_2(x) = -\frac{A}{2\Gamma(-\alpha) \cos \alpha \pi/2} \left[ \int_{-a}^{a} \frac{\sin\frac{n\pi}{2} (x' + a)}{(x' - x)^{\alpha+1}}\,dx' \right], \quad x \leq -a, \quad (100) \]

\[ R_3^\alpha \psi_n(x) = F_3(x) = -\frac{A}{2\Gamma(2-\alpha) \cos \alpha \pi/2} \left[ \frac{d^2}{dx^2} \int_{-a}^{a} \frac{\sin\frac{n\pi}{2} (x' + a)}{|x - x'|^{\alpha-1}}\,dx' \right], \quad |x| < a, \quad (101) \]

which are used in [14] to argue for inconsistency. In these expressions, \( F_1(x), \) \( F_2(x), \) and \( F_3(x) \) are functions of \( x, \) in their respective intervals. However, since all the integrands are singular at the end points, none of these functions are well defined, thus the integrals do not exist in the Riemann sense. In this regard, their Fourier transforms do not exist. The segmented evaluation of the integrals destroys the wholeness in the definition of the Riesz derivative, hence does not yield the correct Fourier transform.

In other words, what the above procedure yields in Equations (99-101) is not the Riesz derivative used in the space fractional Schrödinger equation. It does not have the correct Fourier transform. It has to be kept in mind that the Riesz derivative is basically defined in terms of its Fourier transform, which is equal to the logarithm of the characteristic function of the Lévy probability distribution function. This is in keeping with one of the basic premises of the quantum mechanics, which says that the wave functions in position and momentum spaces are related to each other through a Fourier transform. This also shows in the fact that the space fractional Schrödinger equation follows from the Feynman path integral formulation of quantum mechanics over Lévy paths.
Appendix A. Basic Definitions of the Fractional Derivatives and Integrals

The right- and the left-handed Riemann-Liouville integrals, are defined, respectively, as [2, 3, 16-18]

\[ a^+ I^q_x [f(x)] = \frac{1}{\Gamma(q)} \int_a^x (x - \tau)^{q-1} f(\tau) d\tau, \]
\[ b^- I^q_x [f(x)] = \frac{1}{\Gamma(q)} \int_x^b (\tau - x)^{q-1} f(\tau) d\tau, \]

where \( a < x < b \) and \( q > 0 \). In applications we frequently encounter cases with \( a = -\infty \) or \( b = \infty \). Fractional integrals with either the lower or the upper limit is taken as infinity are also called the Weyl fractional integral. Some authors may reverse the definitions of the right- and the left-handed derivatives. Sometimes \( a^+ I^q_x \) and \( b^- I^q_x \) are also called progressive and regressive, respectively.

The right- and the left-handed Riemann-Liouville derivatives of order \( q > 0 \) are defined as [2, 3, 16-18]

\[ a^+ D^q_x f(x) = \frac{d^n}{dx^n} \left( a^+ I^{n-q}_x [f(x)] \right) \]
\[ = \frac{1}{\Gamma(n-q)} \frac{d^n}{dx^n} \int_a^x (x - \tau)^{n-q-1} f(\tau) d\tau, \]
\[ b^- D^q_x f(x) = (-1)^n \frac{d^n}{dx^n} \left( b^- I^{n-q}_x [f(x)] \right) \]
\[ = (-1)^n \frac{d^n}{dx^n} \int_x^b (\tau - x)^{n-q-1} f(\tau) d\tau, \]

where \( a < x < b \) and \( n > q \).

The right-handed Caputo derivative for \( q > 0 \) is defined as

\[ C_{a^+} D^q_x f(x) = a^+ I^{n-q}_x f^{(n)}(x) \]
\[ = \frac{1}{\Gamma(n-q)} \int_a^x \frac{f^{(n)}(\tau) d\tau}{(x - \tau)^{1-n+q}}, \]

where \( n \) is the next integer higher than \( q \).

The left-handed Caputo derivative for \( q > 0 \) is defined as [2, 3, 16-18]

\[ C_{b^-} D^q_x f(x) = (-1)^n b^- I^{n-q}_x f^{(n)}(x) \]
\[ = (-1)^n \frac{1}{\Gamma(n-q)} \int_x^b \frac{f^{(n)}(\tau) d\tau}{(\tau - x)^{1-n+q}}, \]

where \( n \) is again the next integer higher than \( q \) [16-18]. We reserve the letter \( a \) for the lower limit of the integral operators and the letter \( b \) for the upper limit, hence we will ignore the superscripts in \( a^+ \) and \( b^- \).
The two derivatives are related by

\[
\frac{\zeta}{0} D_0^q f(x) = \frac{R}{0} D_0^q f(x) - \sum_{k=0}^{n-1} \frac{x^{k-q}}{\Gamma(k - q + 1)} f^{(k)}(a^+), \quad q > 0, \quad n - 1 < q < n.
\]

(A9)

In other words, the two derivatives are equal when \( f(x) \) and its first \( n - 1 \) derivatives vanish at \( x = a \).
References

1. K. B. Oldham and J. Spanier, *The Fractional Calculus* (Dover, 1974).

2. I. Podlubny, *Fractional Differential Equations* (Academic Press, 1999).

3. R. Herrmann, *Fractional Calculus* (World Scientific, 2011).

4. R. Hilfer R (ed), *Fractional Calculus, Applications in Physics* (World Scientific, 2000).

5. N. Laskin, Phys. Rev. E**62**, 3135 (2000).

6. N. Laskin, Chaos **10**, 780 (2000).

7. N. Laskin, Phys Rev E **66**, 056108 (2002).

8. N. Laskin, [arXiv:1009.5533v1](http://arxiv.org/abs/1009.5533v1) (2010).

9. M. Jeng, S.-L.-Y. Xu, E. Hawkins and J. M. Schwarz, J. Math. Phys. **51**, 062102 (2010).

10. S. S. Bayin, J. Math Phys **53**, 042105 (2012).

11. S. S. Bayin, J. Math. Phys. **53**, 084101 (2012); J. Math. Phys. **54**, 074101 (2013).

12. E. Hawkins and J. M. Schwarz, J. Math. Phys. **54**, 014101 (2013).

13. J. Dong, [arXiv:1301.3009v1](http://arxiv.org/abs/1301.3009v1) [math-ph] (2013).

14. Y. Luchko, J. Math. Phys. **54**, 012111 (2013).

15. S. S. Bayin, *Mathematical Methods in Science and Engineering* (Wiley, 2006).

16. S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach Science Publ., New York-London, 1993).

17. R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, published in *Fractals and Fractional Calculus in Continuum Mechanics*, eds., A. Carpinteri and F. Mainardi (Springer Verlag, Wien, pgs. 223-276, 1997).

18. A. M. A. El-Sayed and M. Gaber, EJTP **3**, 81, (2006).