1. Introduction

In this paper, we first propose an interpretation of the Kähler-Ricci flow on a manifold $X$ as an exact elliptic equation of Einstein type on a manifold $M$ of which $X$ is one of the (Kähler) symplectic reductions via a (non-trivial) torus action. There are plenty of such manifolds (e.g. any line bundle on $X$ will do).

More precisely, let $M$ be a compact Kähler manifold which admits a Hamiltonian $S^1$-action by holomorphic automorphisms and let $V$ be the vector field generating such an action. Then there is a moment map $\mu : M \to \sqrt{-1}\mathbb{R}$ for this action. Assume that $[0, \bar{\tau}] \subset \mathbb{R}$ consists of regular values of $-\sqrt{-1}\mu$ and for $\tau \in [0, \bar{\tau}]$, $X_\tau = \mu^{-1}(\sqrt{-1}\tau)/S^1$ be the symplectic quotient of $M$ by this action. All these $X_\tau$ are biholomorphic to each other. We consider Kähler metrics which are invariant under the $S^1$-action. As usual, given a Kähler metric $g$, we denote by $\omega_g$ its Kähler form and $\text{Ric}(g)$ its Ricci curvature form.
Our first result (cf. Theorem 3.7, 3.8 and Lemma 3.10) states, loosely speaking, that the normalized Kähler-Ricci flow \( \partial_t \omega_g = -\text{Ric} + \lambda \omega_g \) on \( X_\tau \), where \( \lambda \) is a constant and \( \tau \in \mathbb{R} \), is equivalent to the system of equations on \( M \):

\[
\begin{cases}
\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \left( \log(\|V\|^2_g) + f \right) = \lambda \omega_g \\
\frac{d\tau}{dt} = -\frac{H(\tau)}{4\|V\|^2_g} + \frac{|V|^2_g \partial f}{4} \frac{\partial \tau}{\partial t}
\end{cases}
\]

for some function \( f \) such that \( f = f \cdot (-\sqrt{-1} \mu) \), that is, \( f \) depends only on \( \tau \), satisfying

\[-R(h) + n\lambda - \frac{\partial f}{\partial \tau} < 0.\]

Here \( J \) denotes the complex structure on \( M \) and \( H(\tau) \) denotes the mean curvature of the hypersurface \( Y_\tau := \mu^{-1}(\sqrt{-1} \tau) \subset M \) with respect to the metric \( g \), which we require to be \( S^1 \)-invariant. Also we note that \( R(g) \) is the scalar curvature of \( g \). We will call \( g \) a \( V \)-soliton metric if it is Kähler and satisfies:

\[
\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \left( \log(\|V\|^2_g) + f \right) = \lambda \omega_g. \tag{2}
\]

Such a \( V \)-soliton metric can be regarded as a generalization of Kähler-Einstein metrics or Kähler-Ricci solitons. Similarly to the case of Kähler-Einstein metrics, we can reduce (2) to a scalar equation on Kähler potentials, which is of Monge-Ampere type. To be more explicit, we fix a Kähler metric \( g_0 \) with Kähler form \( \omega_0 \) and write

\[
\omega_g = \omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u.
\]

We can prove that if \( M \) is compact\(^1\), then the above \( V \)-soliton equation is equivalent to the following scalar equation on \( u \):

\[
(\omega_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = |V|^2 g_0 e^{F+f-\lambda u} \omega_0^n, \tag{3}
\]

where \( F \) is determined by \( \text{Ric}(g) - \lambda g = \frac{1}{2} d(JdF) \). We call the equation (3) scalar \( V \)-soliton equation. This equation is of complex Monge-Ampere type.

In the second part of this paper, we prove some preliminary results towards establishing existence of solutions for (3) on a compact Kähler manifold \( M \). In a forthcoming paper [La Nave-Tian], we will establish an existence theorem for (3).

This interpretation can be also extended to any symplectic quotients by more general groups. An holomorphic Hamiltonian action of a Lie group \( G \) on a manifold \( M \) comes with a moment map \( \mu : M \to g^* \): For every coadjoint orbit in the dual Lie algebra of \( G \), \( \tau \subset g^* \), there is a Kähler quotient \( X_\tau := \mu^{-1}(\tau)/G \), as above, we can have an elliptic equation on \( M \) whose solutions can descend to solutions of the Kähler-Ricci flow on \( X_\tau \) (cf. Theorem 3.7).

\(^1\) Even if \( M \) is non-compact, the same conclusion still holds for the solutions of (2) with appropriate asymptotic behaviors.
Our work was inspired by Perelman’s groundbreaking work on Ricci flow. He gave a formal interpretation of the (backwards) Ricci flow on a manifold $M$ in terms of an asymptotically Ricci flat metric on $M \times S^N \times \mathbb{R}_+$, namely, the warped product metric $\tilde{g} = g(\tau) + \tau g_{S^N} + (\frac{N^2}{2\tau} + R) d\tau^2$, where $g(\tau)$ solves the backward Ricci flow and $g_{S^N}$ is the metric on $S^N$ with constant curvature equal to $\frac{1}{2N}$. This allows him to heuristically interpret the monotonicity of the reduced volume purely in terms of an analogue of Bishop-Gromov’s volume comparison theorem for asymptotically Ricci flat metrics. One of the major hurdles for turning his heuristic description into a powerful tool to study the Ricci flow is that the metric on $M \times S^N \times \mathbb{R}_+$ is only asymptotically Ricci-flat in $N$ (the dimension of the sphere). Here we give a precise interpretation in the case of the Kähler-Ricci flow.

One of our main motivations for this interpretation is to study singularity formation of the Kähler-Ricci flow on a manifold with indefinite $c_1(M)$. A singularity can occur when the manifold is forced by the flow to undertake a birational transformation. A large class of birational transformations can be constructed through symplectic quotients. Then our interpretation may reduce studying singularity of the Kähler-Ricci flow on quotients to studying an elliptic problem on $M$ which should be easier. In a subsequent paper, we will discuss how our method can be applied to the study singularity formation along the Kähler-Ricci flow and we will first illustrate it in a concrete example (cf. section 5.2). In fact, our method should be applicable to more general situations than it actually seems at first sight, since there is an associated GIT quotient description for any given flip.

2. GIT versus symplectic quotients

2.1. GIT quotients. From GIT (Geometric Invariant Theory) (cf. [Mumford]), we know that given an action of a reductive group $G$ on a projective manifold $M$ with polarization $L$, one can define the GIT quotient of $M$ via $G$ by considering the Zariski open set $M^{ss}(L) \subset M$ consisting of semistable points of the action, on which $G$ still acts and in fact one can take the quotient $M^{ss}/G$. This depends only on the choice of the polarization $L$, and is denoted by $M/G$. There is a well-defined holomorphic map $\pi : M^{ss} \rightarrow M/G$.

Clearly changing $L$ changes the quotient just in a birational manner, and the change in the GIT quotient as $L$ varies is very well understood (cf. [Thaddeus]). It was shown there that the way the varieties change is by means of birational transformations called flips.

2.2. Symplectic quotients. Let $(M, \omega)$ be a symplectic manifold. Assume there is a group $G$ acting symplectically on $M$. Then there exists a moment map:

$$\mu : M \rightarrow \mathfrak{g}^*,$$

where $\mathfrak{g}$ denotes the Lie algebra of $G$. This is described as follows: if $W \in \mathfrak{g}$, then $\langle \mu(x), W \rangle$ is the Hamiltonian function which generates the flow given by the action of $W$ on $M$.

In these circumstances one can perform the symplectic quotient, defined as $X_\lambda := \mu^{-1}(\lambda)/G$ for some $\lambda$ a $G$-orbit in $\mathfrak{g}^*$. This quotient is in fact a (smooth)
symplectic manifold if $\lambda$ is a regular value for $\mu$, by the Marsden-Weinstein reduction theorem.

If $M$ is a Kähler manifold associated to a (quasi-)projective variety, these quotients coincide with the GIT quotients encountered earlier (cf. [Kirwan]). Furthermore, by the convexity Theorem (cf. [Guillemin-Sternberg1982]), the image of the moment mapping is convex and there is therefore a chamber subdivision according to the critical points of the moment map. When one passes the walls of this chamber subdivision, the symplectic manifold undergoes a symplectic surgery akin to the blowing-up in Algebraic Geometry, as proven by Guillemin and Sternberg in [Guillemin-Sternberg1989].

### 2.3. Kähler quotients and their variations.

Let $G$ be a compact connected Lie group acting symplectically on $(M,g)$ via holomorphic isometries, and let $\mu : M \to \mathfrak{g}^*$ denote the moment map, where $\mathfrak{g} = \text{Lie}(G)$. Denote by $G_e$ the complexification of $G$.

We can think of $\mathfrak{g}$ as a sub-bundle of $TM$ (in fact of $T\mu^{-1}(\tau))$. Let $Q_p(\tau) \subset T_p\mu^{-1}(\tau)$ be the orthogonal complement (with respect to the Kähler metric $g$) of $\mathfrak{g}$. Hence $T_pM = Q_p(\tau) \oplus \mathfrak{g}_p \oplus J\mathfrak{g}_p$ is an orthogonal decomposition. One readily checks that $Q(\tau)$ is $J$-invariant and that $\{Q_p\}_{p \in \mu^{-1}(\tau)}$ is a $G$-invariant distribution. Also observe that if $\pi_\tau : \mu^{-1}(\tau) \to X_\tau$ is the natural projection, then $d\pi_\tau : Q(\tau) \to TX_\tau$ induces an isomorphism.

Recall that the complex structure $J_\tau$ on the Kähler reduction $X_\tau$ is defined by the condition that $d\pi_\tau \circ J = J_\tau \circ d\pi_\tau$ where $\pi_\tau : Y_\tau := \mu^{-1}(\tau) \to X_\tau$ is the natural projection.

The following lemma is well-known.

**Lemma 2.1.** Given any regular value $\tau$ there is a direct sum decomposition of $Q(\tau) \otimes \mathbb{C} = Q(\tau)^{(1,0)} \oplus Q(\tau)^{(0,1)}$ into the $+\sqrt{-1}$ and $-\sqrt{-1}$-eigenspaces respectively. Then $d\pi_\tau$ induces an isomorphism: $Q^{(1,0)}(\tau) \to T^{(1,0)}X_\tau$. Moreover, the induced complex structure $J_\tau$ on $X_\tau$ is integrable.

By imposing that $\pi_\tau : Y_\tau \to X_\tau$ be a Riemannian submersion, we can define a natural Riemannian metric $g_\tau$ on $X_\tau$. Note that

$$g_\tau(d\pi_\tau(W_1), d\pi_\tau(W_2)) = g(W_1, W_2), \ \forall \ W_1, W_2 \in Q(\tau).$$

The metric $g_\tau$ is in fact Hermitian with respect to $J_\tau$. If we denote by $i_\tau : \mu^{-1}(\tau) \to M$ the natural inclusion, we have:

**Lemma 2.2.** The metric $g_\tau$ on $X_\tau$ is Kähler and the corresponding Kähler form $\omega_\tau$ satisfies:

$$\pi_\tau^* \omega_\tau = i_\tau^* \omega.$$ 

Moreover, if $G = S^1$, then for any interval $I \subset \sqrt{-1}\mathbb{R}$ which contains $a$ and consists of regular values of $\mu$, $\mu^{-1}(I)$ is symplectically equivalent to $\mu^{-1}(a) \times I$ (at least in a neighborhood of $Y_a$) endowed with the symplectic structure $\pi_a^* \omega_a + d((\tau-a)\beta)$, where $\beta$ is a connection 1-form on the circle bundle $\pi_a : \mu^{-1}(a) \to X_a$. In particular, the reduced symplectic form on $X_\tau$ is equivalent to $\omega_a + (\tau-a)c_1$, where $c_1$ is the Chern class of the principal bundle $\mu^{-1}(a) \to X_a$. 
Proof. By definition, $\omega_\tau(W, Z) = g_\tau(J_\tau W, Z)$. On the other hand, if $\tilde{W}$ and $\bar{Z}$ are the unique $G$-invariant sections of $Q(\tau)$ such that $d\pi_\tau(\tilde{W}) = W$ and $d\pi_\tau(\bar{Z}) = Z$ respectively, then one has:

\[
\pi_\tau^* \omega_\tau(\tilde{W}, \bar{Z}) = g_\tau(J_\tau d\pi_\tau(\tilde{W}), d\pi_\tau(\bar{Z})) \circ \pi \\
= g_\tau(d\pi_\tau(J\tilde{W}), d\pi_\tau(\bar{Z})) \circ \pi_\tau \\
= g(J\tilde{W}, \bar{Z}) = i_\tau^* \omega(\tilde{W}, \bar{Z})
\]

it is now easy to see that if, say, $\tilde{W}$ is not in $Q(\tau)$, then both sides of the equation amount to zero. This also shows closedness of $\omega_\tau$ since this identity shows that $\pi_\tau^* d\omega_\tau = i_\tau^* d\omega = 0$, and the surjectivity of $\pi_\tau$ implies that $d\omega_\tau = 0$. The statement on the symplectic equivalence follows directly from the uniqueness part of the coisotropic embedding theorem (cf. [Weinstein]), whereas the statement on the nature of $\omega_{X_\tau}$ is a mere consequence of the fact on $Y_\tau = \mu^{-1}(\tau)$, the form $\pi_\tau^* \omega_a + d((\tau - a)\beta)$ restricts to $\pi_\tau^* \omega_a |_{Y_a} + (\tau - a)d\beta$, and clearly $d\beta = \pi_\tau^* c_1$. \hfill \Box

This lemma is of course a special case of a theorem for symplectic quotients (cf. [Guillemin-Sternberg1989]). It is then natural (and essential for our constructions to come) to ask oneself whether such a result carries through to the complex structure of the Kähler quotients. This turns out to be true (cf. [Kirwan]).

Specifically, one can prove that so long as the moment map does not cross critical values, then the complex structure does not change. In order to describe things a little more in depth, we need to introduce some notation:

Let $\Phi_s : M \to M$ represent the gradient flow of the Morse function $||\mu||^2$ (where the norm is in the dual of the Lie algebra $\mathfrak{g}^*), and set (cf. [Kirwan]):

\[
M^{min}(O) := \left\{ x \in M : \lim_{s \to +\infty} \Phi_s(x) \cap \mu^{-1}(O) \neq \emptyset \right\}
\]

and (cf. [Guillemin-Sternberg1982b])

\[
M^s(O) := \left\{ x \in M : G_c x \cap \mu^{-1}(O) \neq \emptyset \right\}
\]

for any coadjoint orbit $O$. Then one can prove:

Lemma 2.3. (cf. [Kirwan], [Guillemin-Sternberg1982b]) $M^{min}(O)$ and $M^s(0)$ are $G_c$-invariant complex submanifolds of $M$. Furthermore, there are natural biholomorphisms between $M^{min}(O)/G_c$ and $\mu^{-1}(O)/G$ and between $M^s(O)/G_c$ and $\mu^{-1}(O)/G$.

For simplicity, we assume that $G = S^1$ and its Lie algebra is identified with $\mathbb{R}$.\footnote{In fact, it was proved by Kempf and Ness that $M^s$ is nothing other than the set of semistable points of the action of $G_c$ on $M$, thereby connecting the GIT quotient with the Kähler reduction.} Then the moment map $\mu$ takes values in $\mathbb{R}$ and

\[
M^{min}(t) := \left\{ x \in M : \lim_{s \to +\infty} \Phi_s(x) \cap \mu^{-1}(t) \neq \emptyset \right\}.
\]

\footnote{All the subsequent discussions go through for a general $G$ which is a maximal compact subgroup of a complex linear group, such as $SL(N, \mathbb{C})$.}
Then $M^\text{min}(t)$ is acted upon by $\mathbb{C}^*$ and if $t$ is a regular value, the natural holomorphic projection: $M^\text{min}(t) \mapsto M^\text{min}(t)/\mathbb{C}^*$ descends to a biholomorphism between $M^\text{min}(t)/\mathbb{C}^* \simeq \mu^{-1}(t)/S^1$. It follows that the complex manifolds $\mu^{-1}(t_1)/S^1$ and $\mu^{-1}(t_2)/S^1$ are biholomorphic to each other whenever $t_1$ and $t_2$ are in an interval which does not contain any critical values of $\mu$. For the readers’ convenience, we will give a direct proof of this fact.

**Proposition 2.4.** If $V$ has no zeros in a neighborhood of $\mu^{-1}([a, a + t_0])$, then the 1-parameter group of diffeomorphisms $\phi_t : M \rightarrow M$ generated by the vector field $U = \frac{JV}{|V|}$ induces biholomorphisms $\tilde{\phi}_t : X_a \rightarrow X_{a+t}$ for $t \in [0, t_0]$.

**Proof.** If we write $\phi(x, t) = \phi_t(x)$, then

$$\begin{cases} \frac{d\phi}{dt} = U(\phi) \\ \phi(x, 0) = x \end{cases}$$

(4)

Since $\nabla \mu = JV$, $U = \frac{\nabla \mu}{|\nabla \mu|}$ and consequently, $\mu(\phi_t(x)) = a+t$ whenever $\mu(x) = a$.

Clearly, through the natural projections $\pi_t : Y_t \rightarrow X_t$, where $Y_t = \mu^{-1}(t')$, $\phi_t$ induce diffeomorphisms $\tilde{\phi}_t : X_a \rightarrow X_{a+t}$.

We want to show that these diffeomorphisms are actually biholomorphic maps. For this purpose, we need to show

$$d\tilde{\phi}_t(J_a \tilde{Z}) = J_{a+t}d\tilde{\phi}_t(\tilde{Z})$$

for any vector field $\tilde{Z}$ of $X_a$. Let $\psi_t : M \rightarrow M$ be an integral curve of the vector field $JV$. They are biholomorphic maps and there is $\lambda : M \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\phi_t(x) = \psi(x, \lambda(x, t))$, since the vector fields $U$ and $JT$ are parallel. In fact, $\lambda(x, t)$ satisfies

$$\begin{cases} \frac{d\lambda}{dt} = \frac{1}{|V|^2} \\ \lambda(x, 0) = 0. \end{cases}$$

(5)

It follows

$$d\phi_t(W)(p) = d\psi_{\lambda(t,p)}(W) + \Lambda(W)JV.$$

Define

$$r_t(x) = \phi(x, t - \mu(x)).$$

Clearly, this defines a retraction $r_t : \mu^{-1}([a, a + t_0]) \rightarrow \mu^{-1}(t)$. In particular, $dr_t(W) = W$ for any $W \in T\mu^{-1}(t)$. Here we think of $T\mu^{-1}(t)$ as a subspace of $TM$. Also, $JV$ lies in the kernel of $dr_t$.

Let $Z = \tilde{Z}^h \in T_p M$ be the horizontal lifting of $\tilde{Z}$, i.e., the unique vector perpendicular to $V$ and $JV$ such that $d\pi_a(Z) = \tilde{Z}$. Then we have
\[ d\tilde{\phi}_t(J_a\tilde{Z}) = d\tilde{\phi}_t(d\pi_a(JZ)) \]
\[ = d\pi_{a+t}(d\phi_t(JZ)) \]
\[ = d\pi_{a+t}(d\psi_{\lambda(p,t)}(JZ) + \Lambda(JZ)JV) \]
\[ = d\pi_{a+t} (dr_{a+t} (d\psi_{\lambda(p,t)}(JZ) + \Lambda(JZ)JV)) \]
\[ = d\pi_{a+t} (dr_{a+t} (d\psi_{\lambda(p,t)}(JZ))) \]
\[ = d\pi_{a+t} (dr_{a+t}(Jd\psi_{\lambda(p,t)}(Z))). \]

Here we used the facts that \( \psi \) is holomorphic and \( \ker(d\pi_{a+t}) = \langle V, JV \rangle \). On the other hand, we have

\[ J_{a+t}d\pi_{a+t}(d\phi_t(Z)) = d\pi_{a+t}([J d\phi_t(Z)]^h) \]
\[ = d\pi_t ([J (d\psi_{\lambda(t,p)}(Z) + \Lambda(Z)JV)]^h) \]
\[ = d\pi_{a+t}dr_{a+t} ([([d\psi_{\lambda(p,t)}(JZ) - \Lambda(Z)V])^h) \]
\[ = d\pi_{a+t} ([([d\psi_{\lambda(p,t)}(JZ) - \Lambda(Z)V)]) \]
\[ = d\pi_{a+t} (dr_{a+t}(d\psi_{\lambda(p,t)}(JZ))) \]

since \( dr_{a+t}(W) = W \) for any \( W \in T\mu^{-1}(a + t) \). This completes the proof. \( \square \)

As a consequence, we conclude that if all \( a + t \), where \( t \in [0, t_0] \), are regular values of \( \mu \) and we denote by \( F_{a+t} : M^{\text{min}}(a) \to X_t \) the above induced holomorphic map, then \( F_{a+t} = \tilde{\phi}_t \cdot F_a \).

3. \( V \)-soliton metrics and Kähler-Ricci flow on symplectic quotients

3.1. The Ricci flow on symplectic quotients. We will start with the case of a complex torus \( T_C = (\mathbb{C}^*)^N \) acting holomorphically on a smooth Kähler manifold \( M \) (in fact \( M \) does not need to be smooth, it could for instance be a Kähler space with canonical singularities)

Note that \( T_C \) is the complexification of the real torus \( T = (S^1)^N \), which acts on \( M \) by Hamiltonian diffeomorphisms. Let \( \mu : M \to \text{Lie}(T)^* = \mathbb{R}^N \) be the associated moment map. We denote by \( g \) an invariant Kähler metric on \( M \).

Let \( z_1, \ldots, z_n \) be holomorphic coordinates on the quotient manifold \( X_a \), and let \( \tau_i \) be "moment map coordinates", i.e., \( \mu = (\tau_1, \ldots, \tau_N) \), sometime, we simply identify \( \mu \) with its value \( \tau = (\tau_1, \ldots, \tau_N) \). If \( N = 1 \), write \( \tau = \tau_1 \). Clearly, we have \( d\tau_k = i_{V_k} \omega \), where \( \{V_k\} \) is a basis of vector fields which generate the Hamiltonian action of \( T \) and correspond to an orthonormal basis of the Lie algebra of \( T \). We can define 1-forms \( \theta_1, \ldots, \theta_N \) by

\[ \theta_k(V_i) = \delta_{kl}, \theta_k(JV_i) = 0, \theta_k|_Q = 0, \]

where \( \nabla \tau_l \) denotes the gradient of \( \tau_l \) with respect to \( g \). By the definition of the moment map, we have \( \nabla \tau_l = JV_l \). In particular, \( \nabla \tau_l \) is tangent to orbits of the action by \( T_C \).
Lemma 3.1. For the above local coordinates, we have \( g(dz_i, d\tau_k) = 0 \) and \( g(dz_i, \theta_k) = 0 \), where \( g \) also denotes the induced metric on the cotangent bundle of \( M \).

Proof. We have
\[
g(dz_i, d\tau_k) = dz_i(\nabla \tau_k) = (\nabla \tau_k)(z_i) = 0.
\]
Clearly, the second follows from the first since \( J(dz_i) = \sqrt{-1}dz_i \). For the third, we have
\[
g(\theta_k, d\tau_l) = \theta_k(\nabla \tau_l) = \theta_k(J\nu_l) = 0.
\]
\[\square\]

It follows from this lemma and a direct computation that in the above local coordinates, we can write the Kähler metric \( g \) on \( M \) as:
\[
g = h_{ij}dz_i d\bar{z}_j + w_{kl}d\tau_k d\tau_l + w^{kl}\theta_k \theta_l \tag{8}
\]
where \( w^{ij} = g(V_i, V_j) \) (this also shows that the \( w^{ij} \)'s are globally defined) and \( \{w^{ij}\} \) is a positive definite matrix and \( \{w_{ij}\} \) is its inverse. Also, in the above proof, we have used the fact that \( d\tau_k(\nabla \tau_i) = \omega(V_k, J\nu_i) = w^{ki} \).

Using \( g(J\nu_i, W) = \omega_g(V_i, W) = d\tau_i(W) \), where \( J \) denotes the complex structure of \( M \), we can deduce \( -J\theta_i = w_{ij}d\tau_j \). We can thus infer that \( w_{ij}d\tau_j - \sqrt{-1}\theta_i \) is of type \((1,0)\), and rewrite \( g \) as
\[
g = h_{ij}dz_i d\bar{z}_j + w^{kl}(w_{kl}d\tau_k - \sqrt{-1}\theta_k)(w_{ij}d\tau_j + \sqrt{-1}\theta_l).
\]
Also, we have the decomposition: \( T^{(1,0)}M = Q^{(1,0)} \oplus \langle w_{ij}d\tau_j - \sqrt{-1}\theta_i \rangle \).

In the sequel we will need the following:

Lemma 3.2. One has:
\[
d\theta_k = \sqrt{-1} \left\{ -\frac{1}{2} \frac{\partial h_{ij}}{\partial \tau_k} dz_i \wedge d\bar{z}_j - \frac{\partial w_{ki}}{\partial \tau_j} d\tau_i \wedge dz_j + \frac{\partial w_{ki}}{\partial \tau_j} d\tau_i \wedge d\bar{z}_j \right\} \tag{9}
\]

Proof. For simplicity, we will assume \( N = 1 \) and write \( \tau = \tau_1 \), the proof for \( N > 1 \) is identical. Observe that the Kähler form of \( g \) is given by
\[
\omega_g = \frac{\sqrt{-1}}{2} h_{ij} dz_i \wedge d\bar{z}_j - d\tau \wedge \theta.
\]
Since this is closed, we get
\[
d\theta = -\frac{\sqrt{-1}}{2} \frac{\partial h_{ij}}{\partial \tau} dz_i \wedge d\bar{z}_j + \beta,
\]
where \( \beta \) is a real 2-form of the form
\[
\beta = \sum_i (q_idz_i \wedge d\tau + \bar{q}_id\bar{z}_i \wedge d\tau) + rd\tau \wedge \theta.
\]
Note that \( r \) is a real function. Since the associated complex structure \( J \) is integrable, the \((0,2)\)-part of \( d(wd\tau - \sqrt{-1}\theta) \) vanishes. This implies
\[
0 = [d(wd\tau - \sqrt{-1}\theta)]^{0,2} = [dw \wedge d\tau - \sqrt{-1}(q_idz_i \wedge d\tau + \bar{q}_i d\bar{z}_i \wedge d\tau)]^{0,2},
\]
consequently,

$$0 = \left[ (\frac{\partial w}{\partial \bar{z}_i} - \sqrt{-1} q_i) d\bar{z}_i \wedge d\tau \right]^{0,2}.$$ 

Hence,

$$q_i = \sqrt{-1} \frac{\partial w}{\partial z_i}.$$

On the other hand, since $\theta$ is invariant under the action, $L_V \theta = 0$, that is, $dV \cdot \theta + i_V d\theta = 0$. But $i_V \theta = 1$, so $d\theta(V, J\theta) = 0$, which implies that $r = 0$ and consequently, the lemma is proved. $\square$

We can now calculate the volume form in a holomorphic frame, namely:

**Lemma 3.3.** There is a holomorphic frame for which the volume form of $\omega_g$ equals to $\det(w)^{-1} \det(h)$.

**Proof.** We first show the following claim: There exists (local) functions $f_{ik}$ such that $\gamma_k = f_{ik} d\bar{z}_i + w_{ik} d\tau_l - \sqrt{-1} \theta_k$ are holomorphic. This is clearly equivalent to showing that there exist smooth functions $f_{ik}$ such that $[d\gamma_i]_{1,1} = 0$, i.e., $d\gamma_i$ is of type $(2, 0)$.

Using the formula for $d\theta_i$ in the above lemma, we have

$$d(w_{ij} d\tau_j - \sqrt{-1} \theta_i) = \frac{\partial w_{ij}}{\partial \tau_k} d\tau_k \wedge d\tau_j + \beta_i^{(2,0)} + \beta_i^{(1,1)},$$

where $\beta_i^{(2,0)}$ and $\beta_i^{(1,1)}$ are of type $(2, 0)$ and $(1, 1)$, respectively. Then

$$\beta_i^{(1,1)} = -\frac{1}{2} \frac{\partial h_{k\bar{j}}}{\partial \tau_i} d\tau_k \wedge d\bar{z}_j + 2 \frac{\partial w_{ij}}{\partial \bar{z}_k} d\bar{z}_j \wedge (d\tau_j + \sqrt{-1} \theta_i).$$

Since each $w_{ij} d\tau_j - \sqrt{-1} \theta_i$ is a $(1, 0)$ form, $d(w_{ij} d\tau_j - \sqrt{-1} \theta_i)$ has vanishing $(0, 2)$-part. It follows that

$$\left[ \frac{\partial w_{ij}}{\partial \tau_k} d\tau_k \wedge d\tau_j \right]^{0,2} = 0.$$

Hence,

$$\frac{\partial w_{ij}}{\partial \tau_k} = \frac{\partial w_{ik}}{\partial \tau_j},$$

consequently, $[d\gamma_i]_{1,1} = 0$ if and only if

$$\begin{cases}
\frac{\partial f_{ik}}{\partial \tau_j} - 2 \frac{\partial w_{ij}}{\partial \bar{z}_k} = 0 \\
\frac{\partial h_{k\bar{j}}}{\partial \tau_i} + 2 \frac{\partial f_{ik}}{\partial \bar{z}_j} = 0
\end{cases} \quad (10)$$

The integrability conditions for this system are

$$\frac{\partial^2 w_{ij}}{\partial \bar{z}_k \partial \tau_l} = \frac{\partial^2 w_{il}}{\partial \bar{z}_k \partial \tau_j}, \quad \frac{\partial^2 h_{ij}}{\partial \tau_k \partial \bar{z}_l} = \frac{\partial^2 h_{il}}{\partial \tau_k \partial \bar{z}_j}, \quad 4 \frac{\partial^2 w_{ij}}{\partial \bar{z}_k \partial \bar{z}_l} = -\frac{\partial^2 h_{k\bar{l}}}{\partial \tau_i \partial \tau_j}. \quad (11)$$

The first identity follows easily from the above symmetry on $\frac{\partial w_{ij}}{\partial \tau_k}$. The second and third follow from $d(d\theta_i) = 0$ and the formula for $d\theta_i$. Hence, we can solve
the equations in (10) for \( f_{ik} \) and our claim is proved. We can therefore infer the existence of a local holomorphic frame \( dz_1, \cdots, dz_n, \gamma_1, \cdots, \gamma_N \). In this local frame, \( \omega_g \) can be written as

\[
\frac{\sqrt{-1}}{2} h_{ij} dz_i \wedge \bar{d}z_j - d\tau_k \wedge \theta_k
\]

\[
= \frac{\sqrt{-1}}{2} \left( h_{ij} dz_i \wedge \bar{d}z_j + w^{ij}(w_{ik}d\tau_k - \sqrt{-1}\theta_i) \wedge (w_{jl}d\tau_l + \sqrt{-1}\theta_j) \right)
\]

\[
= \frac{\sqrt{-1}}{2} \left( (h_{ij} + w^{kl}f_{ki}\bar{f}_{lj}) dz_i \wedge \bar{d}z_j - w^{ij}(f_{ik}d\tau_k \wedge \bar{\gamma}_j + \bar{f}_{jl}\gamma_i \wedge d\bar{z}_l) + w^{ij}\gamma_i \wedge \bar{\gamma}_j \right)
\]

It follows

\[
\omega_g^{n+N} = (n + N)! \left( \frac{\sqrt{-1}}{2} \right)^{n+N} \det(h) \det(w^{ij})^{-1} dz \wedge \bar{d}z \wedge \gamma \wedge \bar{\gamma}.
\]

The lemma is proved. \( \square \)

Next we compute the complex Hessian of any \( T \)-invariant function.

**Lemma 3.4.** For any \( T \)-invariant function \( \phi \in C^2(M) \), we have

\[
\partial \bar{\partial} \phi = \sum \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j + \frac{1}{4} \sum \frac{\partial \phi}{\partial \tau_i} w^{ik} \left( \frac{\partial h_{ij}}{\partial \tau_k} dz_i \wedge d\bar{z}_j \right)
\]

\[
-\frac{1}{2} \sum \frac{\partial}{\partial z_k} \left( \frac{\partial \phi}{\partial \tau_i} w^{ij} \right) dz_k \wedge (w_{jl}d\tau_l + \sqrt{-1}\theta_j)
\]

\[
+ \frac{1}{2} \sum \frac{\partial}{\partial z_k} \left( \frac{\partial \phi}{\partial \tau_i} w^{ij} \right) d\bar{z}_k \wedge (w_{jl}d\tau_l - \sqrt{-1}\theta_j)
\]

\[
+ \frac{\sqrt{-1}}{2} \sum \frac{\partial}{\partial \tau_k} \left( \frac{\partial \phi}{\partial \tau_i} w^{ij} \right) d\tau_k \wedge \theta_j.
\]

**Proof.** First

\[
d\phi = \sum \frac{\partial \phi}{\partial z_i} dz_i + \sum \frac{\partial \phi}{\partial \bar{z}_i} d\bar{z}_i + \sum \frac{\partial \phi}{\partial \tau_i} d\tau_i
\]

Then, using the fact that \( Jd\tau_i = w^{ij}\theta_j \), we get

\[
d(Jd\phi) = \frac{2}{\sqrt{-1}} \sum \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j + \sum \left( d \left( \sum \frac{\partial \phi}{\partial \tau_i} w^{ij} \right) \wedge \theta_j + \left( \frac{\partial \phi}{\partial \tau_i} w^{ij} \right) d\theta_j \right)
\]

\[
+ \sqrt{-1} \sum \left( \frac{\partial^2 \phi}{\partial \tau_j \partial z_i} d\tau_j \wedge dz_i - \frac{\partial^2 \phi}{\partial \tau_j \partial \bar{z}_i} d\tau_j \wedge d\bar{z}_i \right)
\]

Then the lemma follows from the fact that \( d(Jd\phi) = -2\sqrt{-1}\partial \bar{\partial} \phi \) and a direct computation (aided by formula (9)). \( \square \)

We can obtain the following fact about Hamiltonian functions from the Lemma above:
Corollary 3.5. If $T = S^1$ and $\omega_g = \omega_{g_0} - \frac{1}{4} d(J du)$ for some $S^1$-invariant function $u$, and if $\mu : M \rightarrow \mathbb{R}$ is a Hamiltonian function of the $S^1$-action with respect to $\omega_{g_0}$, then $\bar{\mu} := \mu + \frac{1}{4} w_0^{-1} \frac{\partial}{\partial \tau} w_0$ is a Hamiltonian function with respect to $\omega_g$, where $w_0 = g_0(V,V)$ and $V$ is the associated vector field of the $S^1$-action.

Proof. Indeed, using Lemma 3.4, we have

$$\omega_g(V,U) = \omega_{g_0}(V,U) - \frac{1}{4} d(J du)(V,U)$$

$$= d\mu(U) - \frac{1}{4} \left( d\left( \frac{1}{\partial \tau} w_0 \right) \wedge \theta \right)(V,U)$$

$$= U(\mu + \frac{1}{4} \partial u \frac{1}{\partial \tau} w_0) \quad \Box$$

If $X_a$ is smooth, one can compute the curvature of quotient metric $g_a$ in terms of $g$ on $M$ via the Gauss-Codazzi equations and then uses O’Neill’s formula (cf. [O’Neill]) for the Riemannian submersion: $\mu^{-1}(a) \rightarrow X_a$. However, we shall perform our computations by exploring the Kählerian structures.

In order to prove the next theorem, we need the following:

Lemma 3.6. The Hamiltonian functions $\tau_k$ satisfy

$$w^{kl} \frac{\partial \log \det(h)}{\partial \tau_l} = \Delta_g \tau_k - \frac{\partial w^{kl}}{\partial \tau_l} \quad (13)$$

where $\Delta_g$ denotes the Laplacian of $g$ on $M$. Note that the right side of the above is independent of choices of coordinates $z_1, \cdots, z_N$.

Proof. Taking trace of (12), one gets

$$\Delta_g f = h^{ij} \left( 4 \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} + w^{kl} \frac{\partial f}{\partial \tau_k} \frac{\partial h_{ij}}{\partial \tau_l} \right) + \frac{\partial}{\partial \tau_k} \left( \frac{\partial f}{\partial \tau_l} w^{kl} \right). \quad (14)$$

It follows

$$\Delta_g \tau_k = h^{ij} \left( w^{kl} \frac{\partial h_{ij}}{\partial \tau_l} \right) + \frac{\partial w^{kl}}{\partial \tau_l}.$$

Then we get the desired identity by noticing that

$$\frac{\partial \log \det(h)}{\partial \tau_l} = h^{ij} \frac{\partial h_{ij}}{\partial \tau_l}.$$ 

$\Box$

Let $(M,g)$ be a Kähler manifold with a torus $T$-action by holomorphic isometries. Let $V_1, \cdots, V_N$ be a basis of the Killing vector fields generating this $T$-action. As before, we can write the moment map in the form $\mu = (\tau_1, \cdots, \tau_N)$, where $d\tau_k = i_{V_k} \omega_g$, and $w^{ij} = g(V_i, V_j)$. Let $\phi_\tau : X_a \mapsto X_{a+\tau}$ be the biholomorphism defined in Proposition [Proposition 2.4] Fix a unit vector field $V_\tau = \sum_i h_i V_i$, then we get an one-parameter family of metrics on $X_a$: $h(\tau) = \phi_\tau^* g_{a+\tau}$ so long as there
are no critical points of $\mu$ in $\{a + sb \mid 0 \leq s \leq 1\}$, where $\tau = sb$ and $g_{a+\tau}$ is the symplectic reduction of $g$ on $X_{a+\tau}$.

We can now prove:

**Theorem 3.7.** Let $(M, g), h(\tau)$ etc. be as above. Suppose that for some function $f = f(\tau)$ on $M$, $g$ satisfies the following equation:

$$\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \left( \log \det(w^{ij}) + f \right) = \lambda \omega.$$  \hspace{1cm} (15)

Then we have:

1. The function $\Delta_g \tau_k - \frac{\partial w^{kl}}{\partial \tau_l} - w^{kl} \frac{\partial f}{\partial \tau_l}$ is constant along each connected component of $\mu^{-1}(a + \tau)$;

2. Either $\text{Ric}(h(0)) = \lambda \omega_h(0)$, i.e., $h(0)$ is Kähler-Einstein, or $h(\tau) = \phi^* g_{a+\tau}$ is a solution of the normalized Kähler-Ricci flow on $X_a$:

$$\frac{\partial \omega}{\partial t} = -\text{Ric}(\omega) + \lambda \omega,$$  \hspace{1cm} (16)

provided that $\tau_k(t) = c_k (e^{\lambda t} - 1) / \lambda \ (1 \leq k \leq N)$, where

$$c_k = \left( -\frac{1}{4} \Delta_g \tau_k + \frac{1}{4} \frac{\partial w^{kl}}{\partial \tau_l} + \frac{1}{4} w^{kl} \frac{\partial f}{\partial \tau_l} \right) \bigg|_{\mu^{-1}(a)}.$$  \hspace{1cm} (17)

**Proof.** Since $\log \det(g) = \log \det(h) + \log \det(w^{ij})$ in a certain local holomorphic frame, we see that

$$\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \left( \log \det(w^{ij}) + f \right) = \lambda \omega_g$$

is equivalent to:

$$\frac{\sqrt{-1}}{2} \partial \bar{\partial} (-\log \det(h) + f) = \lambda \omega.$$  

In turn, by Lemma 3.4 and the assumption that $f = f(\tau)$, we show that the above equation is equivalent to the following system:

$$\begin{cases}
(i) & 4 \text{Ric}(h)_{\bar{i}i} - w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right) \frac{\partial h_{\bar{i}i}}{\partial \tau_k} = 4 \lambda h_{\bar{i}i} \\
(ii) & \frac{\partial}{\partial z_k} \left( w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right) \right) = 0 \\
(iii) & \frac{\partial}{\partial \bar{z}_k} \left( w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right) \right) = 0 \\
(iv) & \frac{\partial}{\partial \tau_k} \left( w^{lj} \left( \frac{\partial \log \det(h)}{\partial \tau_j} - \frac{\partial f}{\partial \tau_j} \right) \right) = -4 \lambda \delta_{kl} \\
\end{cases}$$  \hspace{1cm} (18)

It follows form (ii) and (iii) in the system (18) that

$$w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right)$$

If $\lambda = 0$, then $\tau_k(t) = c_k t$.\footnote{If $\lambda = 0$, then $\tau_k(t) = c_k t$}
is a function of the Hamiltonian coordinates $\tau_1, \ldots, \tau_N$ only.

On the other hand, from Lemma 3.6 one can infer that:

$$
\frac{1}{4} w^{kl} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right) = \frac{1}{4} \Delta_g \tau_k - \frac{1}{4} \frac{\partial w^{kl}}{\partial \tau_l} \frac{\partial f}{\partial \tau_l}.
$$

This shows (1).

It follows from (iv) in (18), Lemma 3.6 and (17) that

$$
\frac{1}{4} w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right) = -\lambda \tau_k - c_k.
$$

(19)

By our choice of $\tau_k(t)$, this implies

$$
\frac{\partial \tau_k}{\partial t} = -\frac{1}{4} w^{lk} \left( \frac{\partial \log \det(h)}{\partial \tau_l} - \frac{\partial f}{\partial \tau_l} \right).
$$

(20)

Observe that $\tau_k'(t) \neq 0$ for all $t$ whenever $c_k \neq 0$. Hence, if $h(0)$ is not a Kähler-Einstein metric, then $\tau(t)$ is a genuine parameter change of time $t$. Thus we have derived from (i), (ii) and (iii) of (18) the Kähler-Ricci flow

$$
\frac{\partial h}{\partial t} = -\text{Ric}(h) + \lambda h, \text{ on } X_a.
$$

□

Let $(M, g), h$ be in the above theorem. If $h(0)$ is not Kähler-Einstein, then it follows from (20) and a direct computation

$$
R - n\lambda + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} - \frac{\partial \log \det(h)}{\partial t} = \sum_k \left( \frac{\partial f}{\partial \tau_k} - \frac{\partial \log \det(h)}{\partial \tau_k} \right) \frac{d\tau_k}{dt}
$$

$$
= \frac{1}{4} \sum_{k,l} w^{kl} \left( \frac{\partial f}{\partial \tau_k} - \frac{\partial \log \det(h)}{\partial \tau_k} \right) \left( \frac{\partial f}{\partial \tau_l} - \frac{\partial \log \det(h)}{\partial \tau_l} \right).
$$

(21)

Since $w_{ij}$ (and hence $w^{ij}$) is positive definite, i.e., $w^{kl} \xi_l \xi_k > 0$ for every non-zero $(\xi_1, \ldots, \xi_N)$, we have

$$
R(h) - \lambda n + \frac{\partial f}{\partial t} \geq 0
$$

(22)

and the equality holds at some $t$ if and only if $h$ is Kähler-Einstein, where $R(h)$ denotes the scalar curvature of $h$.

There is an integral condition on the descended solution $h(t)$ of the Kähler-Ricci flow from a solution of (15): For simplicity, we assume that $N = 1$, that is, the action group is $S^1$. By (9), we have

$$
d\theta|_{\mu^{-1}(a+\tau)} = -\sqrt{-1} \frac{\partial h(\tau)}{\partial \tau} = \sqrt{-1} \frac{dt}{d\tau} \left( \text{Ric}(h(\tau)) - \lambda \omega_{h(\tau)} \right).
$$
On the other hand, using the Kähler-Ricci flow, we can show
\[-c_1(X_\lambda) + \lambda[\omega_{h(0)}] = e^{\lambda t} \left( -c_1(X_\lambda) + \lambda[\omega_{h(0)}] \right).\]
Here \([\omega]\) denotes the cohomology class represented by \(\omega\). It follows from the above equations
\[
d\theta|_{\mu^{-1}(a+\tau)} = \frac{1}{2c} \left( c_1(X_\lambda) - \lambda[\omega_{h(0)}] \right).
\]
Noticing that \(\theta|_{\mu^{-1}(a+\tau)}\) is a connection of the circle bundle \(\pi : \mu^{-1}(a+\tau) \mapsto X_{a+\tau}\), so its curvature \(d\theta\) represents its first Chern class. Hence, \(\lambda[\omega_{h(0)}]\) must be in \(H^2(X_\lambda, 2\pi\mathbb{Z})\). In particular, if \(\lambda = 0\), the above shows that the associated circle bundle is just the pluri-anti-canonical bundle.

The converse of the above theorem is given in the following:

**Theorem 3.8.** Let \(X\) be a Kähler manifold. If \(\tilde{h}\) is a solution of (16) on \(X \times [t_0, t_1]\) such that \(\lambda[\omega_{h(0)}]\) lies in \(H^2(X, 2\pi\mathbb{Z})\), then there is a unique principal \(S^1\)-bundle over \(X \times [t_0, t_1]\) and a \(S^1\)-invariant metric \(g\) on \(M\) satisfying the equation (15) for some \(f\) and a function \(\tau(t) : [t_0, t_1] \mapsto \mathbb{R}\) such that \(\tau(t_0) = 0\) and \(\tilde{h}(t) = h(\tau(t))\), where \(h(\tau)\) is induced from \(g\) as in last theorem. Moreover, the curvature of the principle bundle \(M\) is given by
\[
\gamma_k := \frac{1}{\sqrt{-1}} \left\{ -\frac{1}{2} \frac{\partial \tilde{h}}{\partial \tau} d\bar{z}_i \wedge d\bar{z}_j - \frac{\partial w}{\partial \bar{z}_j} d\tau \wedge dz_j + \frac{\partial w}{\partial \bar{z}_j} d\tau \wedge d\bar{z}_j \right\}.
\]

**Proof.** First we assume that \(\tilde{h}\) is Kähler-Einstein, i.e., \(\text{Ric}(\tilde{h}) = \lambda\tilde{h}\). Take \(M = X \times \mathbb{C}\) and the vector field \(V = 2 \text{Im}(\bar{z} \frac{\partial}{\partial z})\) is simply the one inducing the standard rotation on \(\mathbb{C}\). The lifting metric \(g\) is of the form
\[
g = \tilde{h} + w\tau^2 + w^{-1}\theta^2,
\]
where \(\theta\) is the dual of \(V\) as we defined before. Then \(\frac{1}{2}|z|^2\) is the associated moment map. Define \(w^{-1} = |z|^2 = 2\tau^2\) and \(f\) as a function of \(\tau\) by
\[
\frac{\partial f}{\partial \tau} = 4\lambda \tau w.
\]
Then one can check directly that \(g\) satisfies
\[
\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \left( \log |V|^2 + f \right) = \lambda \omega_g.
\]
Hence, we get a lifting of \(\tilde{h}\) on \(X \times S^1 \times [\tau_0, \tau_1]\), where \(\tau_0\) and \(\tau_1\) are determined by \(t_0\) and \(t_1\), respectively.

Now we suppose that \(\tilde{h}\) is a non-static solution of the Kähler-Ricci flow:
\[
\frac{\partial \tilde{h}}{\partial t} = -\text{Ric}_X(\tilde{h}) + \lambda \tilde{h}.
\]
We want to find \((M, J)\) and a Kähler metric \(g\) of the form
\[
\tilde{h}_{ij} dz_i dz_j + w\tau^2 + w^{-1}\theta^2,
\]
which satisfies (13):
\[
\text{Ric}(g) + \frac{\sqrt{-1}}{2} \partial \bar{\partial}(-\log w + f) = \lambda \omega_g
\]
for some \( f = f(t) \), i.e., it is constant along \( X \). We will revert the reduction of \( g \) to \( h(\tau) \) in the previous theorem.

As expected, we set
\[
\tau(t) = \frac{c}{\lambda}(e^{\lambda t} - 1),
\]
where \( c \) is a positive constant. Hence,
\[
\frac{d\tau}{dt} = ce^{\lambda t}.
\]
Choose any smooth function \( f = f(\tau) \) such that
\[
-R(\tilde{h}(t)) + n\lambda - \frac{\partial f(\tau(t))}{\partial t} < 0
\]
on \( X \times [t_0, t_1] \), where \( R(\tilde{h}(\tau)) \) is the scalar curvature of \( \tilde{h}(\tau) \). Now we define
\[
w = \frac{e^{-2\lambda t} \left( R(\tilde{h}(t)) - n\lambda + \frac{\partial f(\tau(t))}{\partial t} \right)}{4c^2} > 0.
\]
Since \( \tilde{h} \) is a solution of the Kähler-Ricci flow, we have
\[
\frac{\partial \log \det(\tilde{h})}{\partial t} = -R(\tilde{h}(t)) + n\lambda.
\]
If we regard \( \tilde{h} \) as a function of \( \tau \), we can deduce from the above that:
\[
\frac{\partial \tau}{\partial t} = -\frac{1}{4} w^{-1} \left( \frac{\partial \log \det(\tilde{h})}{\partial \tau} - \frac{\partial f}{\partial \tau} \right). \tag{23}
\]
Define a 2-form as follows:
\[
\gamma := \sqrt{-1} \left\{ -\frac{1}{2} \frac{\partial \tilde{h}_{\bar{j}i}}{\partial \tau} dz_i \wedge d\bar{z}_j - \frac{\partial w}{\partial \bar{z}_j} d\tau \wedge d\bar{z}_j + \frac{\partial w}{\partial z_j} d\tau \wedge dz_j \right\}. \tag{24}
\]
We claim that \( \gamma \) is closed on \( X \times [t_0, t_1] \). The closedness of \( \gamma \) is equivalent to
\[
\frac{\partial^2 \tilde{h}_{\bar{k}i}}{\partial \tau^2} = -4 \frac{\partial^2 w}{\partial z_k \partial \bar{z}_l}. \tag{25}
\]
Using the Ricci flow and the definition of \( \tau(t) \), we see that the left-handed side becomes
\[
\frac{\partial}{\partial \tau} \left( c^{-1} e^{-\lambda t} \left( -\text{Ric}(\tilde{h})_{\bar{k}l} + \lambda \tilde{h}_{\bar{k}l} \right) \right) = -c^{-1} e^{-\lambda t} \frac{\partial^2}{\partial z_k \partial \bar{z}_l} \left( \frac{\partial \log \det(\tilde{h})}{\partial \tau} \right).
\]
On the other hand, using (23), we have
\[
4w = -c^{-1} e^{-\lambda t} \left( \frac{\partial \log \det(\tilde{h})}{\partial \tau} - \frac{\partial f}{\partial \tau} \right).
\]
The claim follows.

Then we can take $M$ to be the unique principal $S^1$-bundle $\pi : M \to X \times [t_0, t_1]$ with the connection 1-form $\theta$ such that

$$d\theta = \pi^* \gamma.$$ 

Here we have used the assumption on the Kähler class of $\tilde{h}(0)$.

We then define the complex structure $J$ on $M$ by imposing that $J\theta = -\sqrt{-1} \theta$ and it restricts to the given one on $X$. Since $T^{1,0} M$ is locally spanned by $dz_1, \cdots, dz_n$ and $wd\tau - \sqrt{-1} \theta$, $J$ is integrable if $d(\sqrt{-1} \theta - w d\tau)$ has no $(0,2)$-components. The latter can be checked directly by using the definition of $\theta$. Hence, $J$ is integrable.

Furthermore, we can endow $(M, J)$ with a Kähler structure: By the definition, we have

$$d\theta = \sqrt{-1} \left\{ -\frac{1}{2} \frac{\partial \tilde{h}_{i\bar{j}}}{\partial \tau} dz_i \wedge d\bar{z}_j - \frac{\partial w}{\partial z_j} d\tau \wedge dz_j + \frac{\partial w}{\partial \bar{z}_j} d\tau \wedge d\bar{z}_j \right\}.$$ 

It follows that $\omega_{\tilde{h}} - d\tau \wedge \theta$ is a closed 2-form. Clearly, this is the Kähler form of the required Kähler metric

$$g = \tilde{h}_{i\bar{j}} dz_i dz_j + wd\tau^2 + w^{-1} \theta^2,$$

that is, $\omega_g = \omega_{\tilde{h}} - d\tau \wedge \theta$.

Let $V$ be the vector field inducing the standard clock-wise rotation on the circle bundle $M$, then $\theta(V) = 1$ and $i_V \omega_g = d\tau$. This means that $\tau$ is a moment map. From the construction, we can easily show that $\tilde{h}$ coincides with $h(\tau)$ from last Theorem.

Remark 3.9. The above lifting is not unique since we do have choices of $f$. If $g$ and $g'$ are such metrics corresponding to $f$ and $f'$, respectively, then we notice that $\gamma$ is independent of $f$ and $f'$, so we have the same circle bundle $M$. Moreover, the symplectic form $\omega_g$ is independent of the choice of $f$.

One may replace (15) by a slightly more general equation: From the above proof, one can see that any solution of $\text{Ric}(g) + \partial \bar{\partial} (\log \det(w^{ij}) + f) = \Omega$ also descends to a solution of the Kähler-Ricci flow, so long as $\Omega$ is a closed $(1,1)$-form such that $(\pi_{a+\tau})_* \Omega = \lambda h(\tau)$ and $\Omega(Z, V_i) = 0$ for every $Z \in Q(a + \tau)$ for every $\tau$. Of course, it holds for $\Omega = \lambda \omega_g$.

In the case in which the action group is just $S^1$, (17) takes a particularly interesting form as it reduces to a (modified) mean curvature flow:

Lemma 3.10. One has that

$$w^{-1} \frac{\partial \log \det(h)}{\partial \tau} = - \frac{H(\tau)}{|V|_g} - \frac{1}{2} \frac{\partial w^{-1}}{\partial \tau}$$

where $H(\tau)$ is the mean curvature of $\mu^{-1}(a + \tau)$ with respect to the unit normal $\sqrt{w} JV$. 
Proof. Given any smooth function \( f \) on a Riemannian manifold \((N, g)\), along its level set \( N_a := \{ x \in M : f(x) = a \} \), we have

\[
Hess_f(Y_1, Y_2) = -\langle \nabla Y_1, \nabla f \rangle = -\langle (B(Y_1, Y_2), \nabla f) \rangle,
\]
where \( Y_1, Y_2 \) are tangent to \( N_a \) and \( B \) denotes the 2nd fundamental form of \( N_a \).

It follows

\[
\Delta_g f |_{N_a} = -\langle H, \nabla f \rangle_g + Hess_f(\nu, \nu),
\]
where \( H \) is the mean curvature of \( N_a \) and \( \nu = \frac{\nabla f}{|\nabla f|} \) is the unit normal.

Now applying (26) to the moment map \( \tau \) regarded as a function on \( M \), we get

\[
\Delta_g \tau = -\langle H, \nabla \tau \rangle_g + Hess_\tau(\nu, \nu).
\]

On the other hand, since \( \nabla \tau = JV \) and \( \nu = \sqrt{w} \nabla \tau \), a straightforward calculation shows

\[
Hess_\tau(\nu, \nu) = \nu(\nu \tau) = \nu(\sqrt{w} \nabla \tau (JV)) = \nu(w^{-1/2}) = w^{-1/2} \frac{\partial w^{-1/2}}{\partial \tau} = \frac{1}{2} \frac{\partial w^{-1}}{\partial \tau}.
\]

Then the claim follows from (13).

It follows from this lemma that the derivative \( \frac{d\tau}{dt} \) in Theorem 3.7 satisfies an evolution equation of mean curvature flow type:

\[
\frac{d\tau}{dt} = \frac{H(\tau)}{4|V|_g} + \frac{1}{8} \frac{\partial w}{\partial \tau} + \frac{1}{4} w^{-1} \frac{\partial f}{\partial \tau}.
\]

4. Scalar V-soliton equation

In this section, we will address the solvability of the following complex Monge-Ampere equation:

\[
(\omega_g + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = \left( |V|_{g_0}^2 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u(V, JV) \right) e^{F - \lambda u} \omega_{g_0}^n,
\]

where \( g_0 \) is a given Kähler metric and \( F \) is a given function satisfying

\[
\int_M (|V|_{g_0}^2 e^F - 1) \omega_{g_0}^n = 0.
\]

We call (42) scalar V-soliton equation. We will assume that both \( g_0 \) and \( F \) are invariant under the \( S^1 \)-action induced by \( V \).

Our main goal here is to develop some preliminary estimates necessary to prove the existence of solutions for this scalar V-soliton equation. Higher order estimates will be done in a forthcoming paper. For simplicity, we assume that \( M \) is compact.

One motivation for studying (42) comes from establishing the existence of V-solitons: Suppose that \( N = 1 \) and \( g \) is a solution of (13), that is, \( g \) is a V-soliton metric. Now we choose \( g_0 \) such that \( c_1(M) \) coincides \( \lambda [\omega_{g_0}] \), then we can write \( g \) as

\[
\omega_g = \omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u.
\]
Define $F = F' + f$ and $F'$ by the equation:
$$ \text{Ric}(g_0) - \lambda \omega_{g_0} = \frac{\sqrt{-1}}{2} \partial \bar{\partial} F'. $$

Such a function $F'$ is only determined up to constants. Then $g$ is a $V$-soliton metric if and only if $u$ satisfies (42) modulo addition of constants. In fact, one can easily see that even if $M$ is not compact, if $u$ is a solution of (42), then $g$ defined as above in terms of $u$ is still a $V$-soliton.

Here we consider (42) only when $\lambda = 0$. The case for $\lambda = -1$ can be done in a similar and simpler way. As usual, the case for positive $\lambda$ is more tricky. We will assume that $u$ is invariant.

**Lemma 4.1.** There is a uniform constant $C = C(g_0)$ such that for any $S^1$-invariant function $u$ with $\omega_{g_0} + \sqrt{-1} \frac{\partial \bar{\partial} u}{2} \geq 0$, we have
$$ w_0^{-1} \left| \frac{\partial u}{\partial \tau} \right| \leq C, $$
where $w_0^{-1} = |V|_{g_0}^2$.

**Proof.** This is a known fact (cf. [Zhu]). For the readers’ convenience, we include a sketched proof here. As before, we denote by $\mu = \tau$ the moment map associated to the $S^1$-action by $V$. Since $M$ is compact, $\mu$ has at least two critical points, so $V$ has at least two zeroes. To estimate $JV(u) = w_0^{-1} |\frac{\partial u}{\partial \tau}|$ at any given $p \in M$, we pick up a trajectory $\gamma$ of the gradient $\nabla \mu$ from one critical point to another. Since $w_0^{-1} = 0$ at critical points of $\mu$, we may assume $p$ is not a critical point. Then $\gamma$ sweeps out a holomorphic sphere $S$ with two punctures by the $S^1$-action. Those two punctures are exactly those critical points which $\gamma$ connects. Using the $S^1$-symmetry, we get
$$ \omega_{g_0}(V, JV) + \sqrt{-1} \frac{\partial \bar{\partial} u}{2}(V, JV) > 0 \text{ on } S. $$
In view of (12), this is the same as
$$ \frac{\partial}{\partial \tau} (JV(u)) = \frac{\partial}{\partial \tau} \left( w_0^{-1} \frac{\partial u}{\partial \tau} \right) > -4. $$
Integrating this along $\gamma$ starting from either $\tau_{\max} = \sup \mu$ or $\tau_{\min} = \inf \mu$, we get
$$ -4(\mu(p) - \tau_{\min}) \leq JV(u)(p) \leq 4(\tau_{\max} - \mu(p)). $$
It follows that $|JV(u)| \leq 4(\tau_{\max} - \tau_{\min})$, so the lemma is proved.

We may use the perturbation method to solve (42). Consider
$$ (\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n = (\epsilon + |V|_{g_0}^2) e^{F+c} \omega_{g_0}^n, $$
where $\epsilon > 0$, $\omega_g = \omega_{g_0} + \sqrt{-1} \frac{\partial \bar{\partial} u}$ and $c$ is chosen such that
$$ \int_M ((\epsilon + |V|_{g_0}^2) e^{F+c} - 1) \omega_{g_0}^n = 0,$$
where \( F_\epsilon = F + c_\epsilon \).

Now let us introduce some notations. Set
\[
C^{k,\alpha}(M, V) := \left\{ u \in C^{k,\alpha}(M) \mid V(u) = 0 \right\},
\]
where \( C^{k,\alpha}(M) \) is the Hölder space of \( C^k \)-smooth functions such that
\[
||u||_{C^{k,\alpha}} := \sum_{i=1}^{k} \sup_{x \in M} |\nabla^i u| + \sup_{x,y \in M, x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{d(x, y)^\alpha} < +\infty,
\]
where \( d(\cdot, \cdot) \) denotes the distance function of any fixed metric \( g \). Clearly, this coincides with the space \( C^{k,\alpha}(M)_{S^1} \) which consists of \( S^1 \)-invariant functions in \( C^{k,\alpha}(M) \).

We further set
\[
C^{k,\alpha}(M; V)_g := \left\{ v \in C^{k,\alpha}(M; V) \mid \int_M ((\epsilon + |V|^2_g)e^u - 1) \omega^n_g = 0 \right\}
\]
and
\[
C^{k,\alpha}_g(M; V) := \left\{ u \in C^{k,\alpha}(M; V) \mid \int_M u \omega^n_g = 0 \right\}.
\]
For \( k \geq 2 \), we also denote by \( P^{k,\alpha}(M, V) \) the set of all \( u \in C^{k,\alpha}(M, V) \) such that \( \omega_u := \omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u > 0 \). Define a differential operator from \( P^{k,\alpha}(M, V) \):
\[
\Phi_\epsilon(u) := \log \left( \frac{(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u)^n}{\omega_{g_0}^n} \right) - \log(\epsilon + |V|^2_u),
\]
where \( |V|^2_u = \omega_u(V, JV) \) is the square norm of \( V \) with respect to the metric given by \( \omega_u \).

Clearly, for \( k \geq 2 \), \( \Phi_\epsilon \) maps into \( C^{k-2,\alpha}(M; V)_{g_0} \). To solve (28), we only need to show that \( \Phi_\epsilon \) is surjective. We will prove that for \( k \) sufficiently large\(^5\)
\[
\Phi_\epsilon(P^{k,\alpha}(M, V) \cap C^{k,\alpha}_g(M; V)) = C^{k-2,\alpha}(M; V)_{g_0}.
\]

The tangent space to \( C^{k-2,\alpha}(M; V)_{g_0} \) at \( \Phi + \epsilon(u) \) is the space: \( C^{k-2,\alpha}_g(M; V) \).

Hence, the differential \( D\Phi_\epsilon|_u \) of \( \Phi_\epsilon \) at \( u \) is a linear map from \( C^{k,\alpha}_g(M; V) \) into \( C^{k-2,\alpha}_g(M; V) \). Furthermore, we have

**Lemma 4.2.** For any \( \epsilon > 0 \), \( \Phi_\epsilon \) is an elliptic operator. Moreover, for any \( u \in P^{k,\alpha}(M, V) \), the differential \( D\Phi_\epsilon|_u \) is surjective with only constant functions in its kernel.

**Proof.** The ellipticity of \( \Phi_\epsilon \) means that for any \( u \in P^{k,\alpha}(M, V) \), \( D\Phi_\epsilon|_u \) is elliptic. A straightforward computation shows:
\[
D\Phi_\epsilon|_u(\bar{u}) = \Delta_{g_0} u - \frac{\sqrt{-1}}{2(\epsilon + |V|^2_u)} \partial \bar{\partial} u(V, JV).
\]

At any given point \( p \in M \), we can choose a basis \( \{e_i\} \) of \( T^1_{p \cdot 0}M \) satisfying:
\[
g_u(e_i, e_j) = \delta_{ij}, \quad \frac{\sqrt{-1}}{2} \partial \bar{\partial} u(e_i, e_j) = a_{ij} \delta_{ij}.
\]

\(^5 k \geq 4 \) should be sufficient.
In terms of this basis, we have

\[ D\Phi_\epsilon|_u(\bar{u})(p) = \sum_i \frac{(\epsilon + \sum_{j \neq i} |v_j|^2) a_i}{\epsilon + \sum_j |v_j|^2}. \]

This shows the ellipticity of \( D\Phi_\epsilon|_u \) at \( p \), and consequently, ellipticity of \( \Phi_\epsilon \).

Moreover, it follows from the above computation and the Maximum principle that \( D\Phi_\epsilon|_u \) is surjective and its kernel consists of only constant functions.

\[ \square \]

**Remark 4.3.** In fact, one can also show that \( D\Phi_\epsilon|_u \) is self-adjoint. Moreover, by the above, we see that \( D\Phi_0|_u \) is also elliptic but degenerate.

As said, we may use the continuity method to solve (28). Fix a large \( k > 0 \).

Choose any path \( F_{\epsilon,s} \) in \( C^{k-2,\alpha}(M;V) \) \((s \in [0,1]) \) with \( F_{\epsilon,0} = -\log(\epsilon + |V|^2_{g_0}) \) and \( F_{\epsilon,1} \) coincides with \( F_\epsilon \) in (28). Consider a family of complex Monge-Ampere equations:

\[ (\omega_{g_0} + \sqrt{-1} \partial\bar{\partial}u)^n = (\epsilon + |V|^2_{g_0}) e^{F_{\epsilon,s} \omega_{g_0}}, \]  

where \( \omega_g = \omega_{g_0} + \sqrt{-1} \partial\bar{\partial}u \). Define

\[ I = \{ s \in [0,1] \mid (29) \text{ has a solution for any } s' \in [0,s] \}. \]

Clearly, \( 0 \in I \) since \( u = 0 \) is a solution. It follows from the above lemma and the Inverse Function Theorem

**Corollary 4.4.** The set \( I \) defined above is open.

Hence, establishing the existence of a solution for (28) is equivalent to proving that \( I \) is closed. For this purpose, we need a priori estimates for solutions of (29).

In view of the \( C^{2,\alpha} \)-estimate due to Evans, Krylov etc. (cf. [Evans82], [Krylov], [Caffarelli], [Trudinger], [Tian84]), it suffices to have a priori \( C^2 \)-estimates for (29).

**4.1. The \( C^0 \)-estimate.** The purpose of this subsection is to derive a \( C^0 \)-estimate by the standard Moser iteration, keeping track of the dependence on \( \epsilon > 0 \). First, we have

**Proposition 4.5.** There is a uniform constant \( C \) which depends only on \((M,g_0), \|F\|_{C^1(M)}, \sup_M |V|_{g_0} \) and \( \sup_M |\text{div}(JV)| \) such that for any solution \( u \) of (29) with \( \int_M u \omega^n_{g_0} = 0 \), we have

\[ \sup_M |u| \leq C. \]

**Proof.** First we assume that \( u \) is a solution of (29) for some \( s \in [0,1] \) such that \( \sup_M u = -1 \). For simplicity, denote \( F_{\epsilon,s} \) by \( \tilde{F} \) and \( u_- = -u \). Note that \( u_- \geq 1 \).
Integrating by parts, we get for $\ell \geq 1$
\[
\begin{align*}
  n \int_M u_\ell^\ell (\omega_n^\ell - \omega_{g_0}^n) \\
  = \frac{n \ell \sqrt{1 - \ell}}{2} \int_M u_\ell^{\ell - 1} \left( \partial u_\ell \land \bar{\partial} u_\ell \land \sum_{j=0}^{n-1} \omega_{g_0}^j \land \omega_n^{n-j-1} \right) \\
  \geq \ell \int_M u_\ell^{\ell - 1} |\nabla_{g_0} u_\ell|_2^2 \omega_{g_0}^n \\
  = \frac{4\ell}{(\ell + 1)^2} \int_M |\nabla_{g_0} u_\ell^{\ell + 1}|_2^2 \omega_{g_0}^n.
\end{align*}
\] (30)

Multiplying $u_\ell$ on both sides of (29) and integrating, we deduce from the above
\[
\int_M |\nabla_{g_0} u_\ell^{\ell + 1}|_2^2 \omega_{g_0}^n \leq \frac{n(n + 1)^2}{4\ell} \int_M u_\ell^\ell (\omega_n^\ell - \omega_{g_0}^n) \\
= \frac{n(n + 1)^2}{4\ell} \int_M u_\ell^\ell \left( (\epsilon + |V|_g^2) e^F - 1 \right) \omega_{g_0}^n. \quad (31)
\]

Using Lemma 3.4 and noticing $w_0 JV = \frac{\partial}{\partial \tau}$, we can compute
\[
\frac{\sqrt{-1}}{2} \partial \bar{\partial} u(V, JV) = \frac{1}{4} w_0^{-1} \frac{\partial}{\partial \tau} \left( w_0^{-1} \frac{\partial \phi}{\partial \tau} \right) = \frac{1}{4} JV(JV(u)).
\]

It follows
\[
\int_M |\nabla_{g_0} u_\ell^{\ell + 1}|_2^2 \omega_{g_0}^n \leq n \ell \int_M u_\ell^\ell \left( e^F (\epsilon + |V|_g^2 + \frac{1}{4} JV(JV(u))) - 1 \right) \omega_{g_0}^n. \quad (32)
\]

Write $W = JV - \sqrt{-1} V$. Then $W$ is a holomorphic vector field. Since $V(u) = 0$, we have $W(u) = JV(u)$ is real-valued and bounded. Recall the identity:
\[
\text{div}(u_\ell^\ell e^F W(u) W) = \text{div}(W) u_\ell^\ell e^F W(u) + W(u_\ell^\ell e^F W(u)),
\]
where the divergence $\text{div}(W)$ is taken with respect to the metric $g_0$. Therefore, there is a constant $C_1$ which depends only on $(M, g_0)$, $|F|_{C^1(M)}$, $\sup_M |V|_{g_0}$ and $\sup_M |\text{div}(JV)|$ such that
\[
u_\ell^\ell e^F W(u) \leq \text{div}(u_\ell^\ell e^F W(u) W) + \ell u_\ell^\ell |W(u)|^2 e^F + C_1 u_\ell^\ell |W(u)|.
\]

Plugging this into (32) and using Lemma 4.1, we obtain
\[
\int_M |\nabla_{g_0} u_\ell^{\ell + 1}|_2^2 \omega_{g_0}^n \leq C_2 \ell \int_M u_\ell^\ell (u_\ell + \ell) \omega_0^n,
\] (33)

where $C_2$ is a uniform constant depending only on $(M, g_0)$, $|F|_{C^1(M)}$, $\sup_M |V|_{g_0}$ and $\sup_M |\text{div}(JV)|$.

Since $u_\ell \geq 1$, it follows
\[
\int_M |\nabla_{g_0} u_\ell^{\ell + 1}|_2^2 \omega_{g_0}^n \leq 2 C_2 \ell^2 \int_M u_\ell^{\ell + 1} \omega_0^n. \quad (34)
\]
Now we can apply the standard Moser iteration scheme: Denote by $C_S$ the Sobolev constant for $g_0$, then for any smooth function $f$ on $M$, we have (cf. Gilbarg-Trudinger Theorem 7.10):

$$||f||_\frac{2n}{n-1} \leq C_S (||\nabla f||_2 + ||f||_2).$$

Applying this to (34) for $f = u_{\ell+1}^{\frac{\ell+1}{n+1}}$, we get

$$(\int_M u_-^{\frac{\ell+1}{n+1}} \omega_{g_0}^n)^{\frac{1}{\ell+1}} \leq \left( C_3 \ell^2 \int_M u_-^{\ell+1} \omega_{g_0}^n \right)^{\frac{1}{\ell+1}},$$

(35)

where $C_3$ is a uniform constant depending only on $(M, g_0)$, $||F||_{C^1(M)}$, sup$_M |V|_{g_0}$ and sup$_M |\text{div}(JV)|$.

Set $\ell_1 = 1$ and $\ell_{i+1} = \frac{2n}{n-i}(\ell_i + 1) - 1$ inductively for $i \geq 1$. Then we have

$$(\int_M u_-^{\ell_{i+1}} \omega_{g_0}^n)^{\frac{1}{\ell_{i+1}}} \leq \prod_{j=1}^{i-1} \left( C_3 (\ell_j + 1)^2 \right)^{\frac{1}{\ell_j + 1}} \left( \int_M u_-^{2} \omega_{g_0}^n \right)^{\frac{1}{2}}.$$

Note that $\ell_j + 1 = 2 \left( \frac{n-1}{2n} \right)^j$, we can deduce from the above

$$\sup_M u_- \leq C_4 \left( \int_M u_-^{2} \omega_{g_0}^n \right)^{\frac{1}{2}},$$

(36)

where $C_4$ is a uniform constant depending only on $(M, g_0)$, $C_S$, $||F||_{C^1(M)}$, sup$_M |V|_{g_0}$ and sup$_M |\text{div}(JV)|$.

Moreover, applying the Poincare inequality to (33) with $\ell = 1$ and noticing $u_- \geq 1$, we get

$$(\int_M u_-^{2} \omega_{g_0}^n)^{\frac{1}{2}} \leq C_5 \int_M u_- \omega_{g_0}^n,$$

(37)

where $C_5$ depends only on $g_0$.

On the other hand, since $n + \Delta_{g_0} u > 0$ on $M$, applying the Green function of $g_0$, we can get

$$\sup_M u \leq \frac{1}{V} \int_M u \omega_{g_0}^n + C_6,$$

(38)

where $V = \int_M \omega_{g_0}^n$ and $C_6$ depends only on $g_0$.

By our assumption on $u$, we get from the above

$$\int_M u \omega_{g_0}^n \leq V (1 + C_6).$$

Combining this with (36) and (37), we obtain an a priori estimate on $||u||_{C^{0}}$ and the proposition is proved in the case that sup$_M u = -1$.

In general, if $u$ is a solution of (29), then $\overline{u} := u - \sup_M u - 1$ is also a solution. Applying the above discussion, we have

$$\sup_M u - \inf_M u \leq C_7,$$
where $C_7$ is a uniform constant. Therefore, if $u$ satisfies $\int_M u \omega_{g_0}^n = 0$, then by (38),
\[ \sup_M u \leq C_6. \]
Hence, we have a uniform estimate on $||u||_{C^0}$ as required by the proposition. \qed

4.2. The higher order estimates. In order to establish the existence of V-solitons, we need higher order estimates for solutions of (29). Based on the known theory on the $C^{2,\alpha}$-estimate for complex Monge-Ampere equations, we only need an a priori $C^2$-estimate.

The following is trivial.

**Lemma 4.6.** Let $u$ be a solution of (29), then $||\bar{\partial} u|| \leq \max\{n + \Delta_{g_0} u, n\}$

Therefore, in order to derive an a priori $C^2$-estimate, we only need to have a $C^0$-estimate for $\Delta_{g_0} u$. This is similar to the second-order estimate in the proof for the Calabi-Yau theorem. However, because of the extra term involving $|V|^2_{u}$, the proof in our case is much more tricky and lengthy. This will be in our forthcoming paper.

4.3. Uniqueness of scalar V-soliton equation. Using the Maximum principle, one can easily show the following:

**Theorem 4.7.** Let $(M, g_0)$ be a compact Kähler manifold with boundary $\partial M$ and a $S^1$-symmetry induced by a Hamiltonian field $V$. Then there is at most one solution of (42) with given boundary value, namely, if $u_1$ and $u_2$ are $S^1$-invariant solutions of (42) with $u_1 = u_2$ along $\partial M$, then $u_1 \equiv u_2$ on $M$.

5. Further directions

In this section, we discuss possible applications of our new correspondence and some further research problems.

5.1. Boundary value problem for V-soliton metrics. First we certainly concern the existence problem of V-soliton metrics. This is amount to solving the scalar V-soliton equation (42). We expect: Given a complete Kähler manifold $(M, g_0)$ with boundary $\partial M$ and finite geometry at $\infty$. Suppose that it admits a $S^1$-symmetry generated by a Hamiltonian field $V$, then for any reasonably ”nice” boundary value $\varphi$ along $\partial M$, there is a unique solution $u$ of (42) on $M$ such that $u|_{\partial M} = \varphi$.

In [La Nave-Tian], we will provide a solution to this existence problem in the case that $M$ is compact or an ALE space. The solution we obtain will be in $C^{1,\alpha}$ in general, but it should be smooth outside the zero set of $V$ as an application of the known regularity theory for complex Monge-Ampere equations. It will be a more challenging problem to study the regularity of such a solution near the zero set of $V$. 
5.2. Finite-time singularities of the Kähler-Ricci flow. Our new correspondence may be applied to studying singularity formation of the Kähler-Ricci flow: Let \((M, g)\) be a Kähler manifold with a \(S^1\)-symmetry generated by a Hamiltonian field \(V\). We further assume that \(g\) is a V-soliton metric (i.e., it satisfies eq. (2)).

Let \(\mu : M \mapsto \mathbb{R}\) be the associated moment map, i.e., the Hamiltonian function of \(V\). Put \(Cr(\mu)\) to be the set of critical values of \(\mu\). Then \(\mathbb{R}\setminus Cr(\mu)\) is a disjoint union of consecutive open intervals \(I_a (a \in \mathbb{Z})\). For each interval \(I_a\), symplectic quotients \(X_\tau\) for \(\tau \in I_a\) are the same complex manifold, but \(X_\tau\) changes when \(\tau\) crosses critical values in \(Cr(\mu)\). Usually, \(X_\tau\) and \(X_\tau'\) are related to each other by so-called flips when \(\tau\) and \(\tau'\) are in two different, but consecutive, intervals. By studying how \(g\) descends to \(X_\tau\) and \(X_\tau'\), we can analyze how the Kähler-Ricci flow transforms under flips. Let us illustrate this by means of an example.

Let \(C^*\) act on \(M := \mathbb{C}^{l+m}\) by:

\[
t(z_1, \ldots, z_{l+m}) = (t^{a_1} z_1, \ldots, t^{a_l} z_l, t^{-a_l+1} z_{l+1}, \ldots, t^{-a_{l+m}} z_{l+m}),
\]

where \(a_1, \ldots, a_l, a_{l+1}, \ldots, a_{l+m} > 0\) are positive integers. This action is Hamiltonian with respect to the standard Kähler structure on \(\mathbb{C}^{l+m}\) with the Hamiltonian \(\mu : \mathbb{C}^{l+m} \mapsto \mathbb{R}\):

\[
\mu(z_1, \ldots, z_{l+m}) = \sum_{i=1}^{l} a_i |z_i|^2 - \sum_{i=l+1}^{l+m} a_i |z_i|^2
\]

One can easily see that \(\tau = 0\) is the only critical value. Therefore, the symplectic quotients \(X_\tau := \mu^{-1}(\tau)/S^1\) are all isomorphic to a fixed variety \(X^-\) for \(\tau < 0\) and to a variety \(X^+\) for \(\tau > 0\). Furthermore, the natural bi-rational map \(\phi : X^- \dashrightarrow X^+\) is a flip for \(l, m \geq 2\) replacing via surgery a neighborhood of \(\mathbb{CP}^{l-1} \subset X^-\) with a neighborhood of \(\mathbb{CP}^{m-1} \subset X^+\). For \(l = 1, m \geq 2\) and \(a_1 = \cdots = a_{l+m} = 1\), \(\phi\) is a blow-down, and for \(l \geq 2, m = 1\) and \(a_1 = \cdots = a_{l+m} = 1\) it is a blow-up. For \(l \geq 2\) and \(a_1 = \cdots = a_{l+m} = 1\) it is a flip or flop (e.g., \(l = m = 2\) gives rise to a flop). Another important special case is when \(l = m = 2, a_1 = 2\) and \(a_2 = \cdots = a_4 = 1\): This is the first non-trivial flip in the Francia series.

Let us consider the simplest case in this context: \(l = 1\) and \(a_1 = \cdots = a_{l+m} = 1\). For \(\tau < 0\), \(X_\tau\) is the \(S^1\)-quotient of

\[
\{ (z_1, z_2, \ldots, z_{m+1}) \mid |z_1|^2 + |\tau| = \sum_{i=2}^{m+1} |z_i|^2 \}
\]

which is the blow-up of \(\mathbb{C}^m\) at \((0, 0)\). On the other hand, one can see easily that \(X_\tau = \mathbb{C}^m\) for \(\tau > 0\). Let us find a special \(V\)-soliton metric \(g\) on \(\mathbb{C}^{n+1}\) of the form

\[
\omega_g = \frac{\sqrt{-1}}{2} \partial \bar{\partial} u, \quad u = (|z_2|^2 + \cdots + |z_{m+1}|^2) h(|z_1|^2).
\]

The holomorphic field whose imaginary part equals \(V\) is given by

\[
W = -z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + \cdots + z_{m+1} \frac{\partial}{\partial z_{m+1}}.
\]
Then the scalar $V$-soliton equation is equivalent to the following:
\[
\det(u_{ij}) = u_{11}|z_1|^2 - \sum_{j=2}^{m+1}(u_{1j}z_1 \bar{z}_j + u_{j1}z_j \bar{z}_1) + \sum_{i,j \geq 2} u_{ij}z_i \bar{z}_j,
\]
where $u_{ij} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$.

If we set $r := |z_1|^2$ and $\rho = |z_2|^2 + \cdots + |z_{m+1}|^2$, one can verify directly that $u = \rho h(r)$ satisfies the V-soliton equation if $h$ satisfies the following ODE:
\[
(rhh'' - r(h')^2 + hh') h^{m-1} = r^2h'' - rh' + h,
\]
where $r \in [0, \infty)$ and $h$ is a function of $r$. Given a solution $h$ of this equation, we obtain a $V$-soliton metric $g$ on $\mathbb{C}^{m+1}$ which descends to a solution of the Kähler-Ricci flow for $\tau < 0$ and converges to a smooth Kähler metric $g_0$ on $\mathbb{C}^m \setminus \{0\}$. One can show easily that $\mathbb{C}^m$ is the metric completion of $\mathbb{C}^n \setminus \{0\}$ by $g_0$. This can be used to verify the first conjecture on finite-time singularity for the Kähler-Ricci flow in [Tian07] and [Song-Tian] in the special case of a blow-up of a smooth manifold. In fact, in order to see how the Kähler-Ricci flow behaves under blowing-down of a $\mathbb{CP}^{m-1}$, it suffices to find a solution of the above ODE near $r = 0$. By using the power series method, one can find a local solution $h$ of the above ODE starting with
\[
h(r) = 1 + r - \frac{m-1}{4}r^2 + O(r^3).
\]
One can use this explicit solution to see that $\mathbb{C}^m$ is the metric completion of $\mathbb{C}^m \setminus \{0\}$ by $g_0$ as claimed above. Similarly, one can find special solutions for the $V$-soliton equation in the general case that both $l, m > 1$.

In fact, this example in the case of $m = 2$ provides the basic picture of finite-time singularity for the Kähler-Ricci flow on complex surfaces. Let us elaborate more on this: Let $X$ be a complex surface and $g(0)$ be a Kähler metric. Then there is a unique solution $g(t)$ of the Kähler-Ricci flow on $[0,T)$, where $T$ is either $\infty$ or the first time when $[\omega_{g(0)}] - tc_1(X)$ fails to be positive. If $T < \infty$ and $([\omega_{g(t)}] - Tc_1(X))^2 > 0$, then as $t \to T$, $g(t)$ converges to $g(T)$ outside finitely many disjoint rational curves $C_1, \ldots, C_k$ of self-intersection number $-1$. For simplicity, assume that $k = 1$. By blowing down $C_1$, we get a new complex surface $\tilde{X}$ with $p$ corresponding to the blow-down $C_1$. We can also extend $g(T)$ to be a solution $g(t)$ of the Kähler-Ricci flow on $\tilde{X}$ for $t \in [T, T + \epsilon]$ for some $\epsilon > 0$. Let $U$ be a small neighborhood of $p$ and $\tilde{U}$ be the blow-up of $U$ at $p$. Then $\tilde{U}$ to be a neighborhood of $C_1$ in $\tilde{X}$, moreover, we can identify $\tilde{U} \times (T - \epsilon, T) \cup U \times [T, T + \epsilon)$ as a quotient of a neighborhood $W \subset \mathbb{C}^3$ of 0 by the $S^1$-action. The solution $g(t)$ lifts to a $V$-soliton metric $\tilde{g}$ on $W \setminus \{0\}$. By solving the boundary value problem for the scalar $V$-soliton equation on $W$, we should be able to extend $\tilde{g}$ on $W$. Then by studying how $\tilde{g}$ descends to $U$, we may prove that $\tilde{X}$ is the metric completion of $\tilde{X} \setminus \{p\}$ by $g(T)$. This verifies the first conjecture on finite-time singularity for the Kähler-Ricci flow in [Tian07] and [Song-Tian] for complex surfaces. Of course, the above discussion just provides a plausible approach. Details remain to be checked.
We believe that this actually provides an effective approach to studying finite-time singularity of the Kähler-Ricci flow in all dimensions, at least for all those flips which can be achieved through variations of symplectic quotients. Indeed, many flips can be achieved in this way. This allows us to carry out a geometric Minimal Model Program using the Ricci flow with "surgeries". The first step in this program is to understand solutions for the $V$-soliton equation on a manifold with boundary. This will be the subject of [La Nave-Tian].

5.3. Kähler-Ricci flow on Fano manifolds. Another possible application of the $V$-soliton equation is to study the Kähler-Ricci flow on Fano manifolds. Let $X$ be a Fano manifold and $g_0$ be a Kähler metric with its Kähler class equal to $c_1(M)$. It is known that the normalized Kähler-Ricci flow

$$\frac{\partial g}{\partial t} = -\text{Ric}(g) + g, \ g(0) = g_0$$

has a global solution $g(t)$ for all $t > 0$. A long-standing problem is on the convergence of $g(t)$ as $t$ goes to $\infty$. The folklore conjecture is that $g(t)$ converges to a Kähler-Ricci soliton (possibly with mild singularity along a subvariety of complex codimension at least 2). Our new correspondence may provide a method of proving this conjecture. By Theorem 3.8, there is a Kähler metric $\bar{g}(\cdot, z)$ on $M = X \times \{z \in \mathbb{C} \mid |z| \geq 1\}$ satisfying:

1. $\tau = e^t - 1$;
2. $\bar{g}$ is invariant under the standard $S^1$-action of $\mathbb{C}$ by rotations;
3. $g(t)$ is the symplectic quotient of $\bar{g}$ on $X$.
4. $\bar{g}$ satisfies the $V$-soliton equation:

$$\text{Ric}(\bar{g}) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} (-\log w + f) = \omega_{\bar{g}},$$

where $w$ is the inverse of the squared norm of $V$ ($w = (|V|_{\bar{g}}^2)^{-1}$) given on $X \times [0, \infty)$ by

$$w = 4^{-1} e^{-2t} \left( R(g(t)) - n + \frac{\partial f(e^t - 1)}{\partial t} \right) > 0.$$ 

This is the same as

$$4^{-1} (1 + \tau)^{-2} \left( R(g(t)) - n + (1 + \tau) \frac{\partial f(\tau)}{\partial \tau} \right) > 0.$$ 

That it is possible to find such an $f$ is insured by a result of Perelman’s to the effect that $R(g(t))$ is bounded (cf. [Sesum-Tian]). Such an $f$ is not unique, so we may choose one that is more convenient to us. For instance, if $c$ is the lower bound of $R(g(t))$, we choose $f = (n - c) \log(1 + \tau) + 2(1 + \tau)^2$. Then

$$w = 4^{-1} (1 + \tau)^{-2} \left( R(g(t)) - n + \frac{\partial f(e^t - 1)}{\partial t} \right) \sim 1.$$ 

It follows that at $\infty$ of $\mathbb{C}$, in polar coordinates $z = (\tau, \varphi)$, we have

$$g \sim \lim_{t \to \infty} g(t) + d\tau^2 + d\varphi^2.$$
If \( g(t) \) converges to a Kähler-Einstein metric \( g_{KE} \) as \( t \) tends to \( \infty \), then \( \bar{g} \) can be extended across \( X \times \{\infty\} \) by adding \( g_{KE} \). Or equivalently, given any sequence \( \{t_i\} \) with \( \lim t_i = \infty \), then \( (X \times \mathbb{C}, \bar{g}(t+t_i)) \) converges to the product of \( g_{KE} \) with the cylinder metric \( d\tau^2 + d\phi^2 \) as \( t_i \) tends to \( \infty \).

In general, it is plausible that the above chosen \( \bar{g} \) can be extended across \( \infty \) of \( \mathbb{C} \) modulo a family of diffeomorphisms of \( X \) or equivalently, \( (X \times \mathbb{C}, \bar{g}(t+t_i)) \) converges to the product of \( g_{KE} \) with the cylinder metric \( d\tau^2 + d\phi^2 \).

Therefore, the above folklore conjecture is closely related to how \( (X \times \mathbb{C}, \bar{g}) \) behaves at the \( \infty \) of \( \mathbb{C} \) and whether or not it can be compactified. We conjecture that \( (X \times \mathbb{C}, \bar{g}(t+t_i)) \) converges to the product of a Kähler-Ricci soliton with \( d\tau^2 + d\phi^2 \) modulo diffeomorphisms. Based on this idea and assuming the analyticity, Arezzo and La Nave (cf. [Arezzo-La Nave]) studied the case that the central fiber of a (non-trivial) special degeneration \( X \to \Delta \) admits a Kähler-Einstein metric. In a forthcoming paper (cf. [Arezzo-La Nave-Tian]), we will discuss this in more details.

5.4. \( V \)-solitons and geodesics in the space of Kähler metrics. On a Kähler manifold \( X \), each \((1,1)\)-form cohomologous to \( \omega \) takes the form \( \omega + \sqrt{-1} \partial \bar{\partial} f \) for some \( f \in C^\infty(X) \). Therefore, the space of all Kähler metrics in the class \([\omega]\) can be identified with

\[
H = \{ \phi \in C^\infty(X) \mid \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi > 0 \}/\sim.
\]

where \( \sim \) if and only if they are different by addition of a constant.

Given \( \phi \in H \), the formal tangent space

\[
T_\phi H = \{ \psi \in C^\infty(X)_0 \mid \int_X \psi \omega_n^{\phi} = 0 \},
\]

where \( \omega_\phi := \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi \). There is an natural metric, introduced by T. Mabuchi, on \( H \) as follows: Let \( \psi_1, \psi_2 \in T_\phi H \), define

\[
<\psi_1, \psi_2>_{\phi} = \frac{1}{n!} \int_X \psi_1 \psi_2 \omega_n^{\phi}.
\]

Given a smooth curve \( \phi(t) : [a, b] \to H \), set \( \phi(x, t) := \phi(t)(x) \). This can be considered as a function on \( X \times S^1 \times [a, b] \) which is \( S^1 \)-invariant. Then the geodesic equation for the above \( L^2 \) metric is equivalent to the following Homogeneous complex Monge-Ampere (in short HCMA) equation on \( M = X \times S^1 \times [a, b] \) (cf. [Semmes], [Donaldson] and [Chen-Tian]):

\[
(\omega + \frac{\sqrt{-1}}{2} \partial_M \bar{\partial}_M \phi)^{n+1} = 0.
\]

On the other hand, consider the Kähler-Ricci flow on the Kähler manifold \( X \times [0, \infty) \):

\[
\frac{\partial g}{\partial t} = -\text{Ric}(g) + g, \quad g(0) = g_0.
\]
By Theorem 3.8, there is a lifting metric $\bar{g}(\cdot, \tau)$ on $M = X \times S^1 \times [0, \infty)$ satisfying the following

1. $\tau = e^t - 1$;
2. $\bar{g}$ is invariant under the obvious $S^1$-action;
3. $g(t)$ is the symplectic quotient of $\bar{g}$ on $X$.
4. For some $f$, $\bar{g}$ satisfies the $V$-soliton equation:

$$\text{Ric}(\bar{g}) + \frac{\sqrt{-1}}{2} \partial \bar{\partial} (\log |V|^2_g + f) = \omega_{\bar{g}},$$

where $V$ is the vector d-field generating the $S^1$-action. As we have shown in the above, the $V$-soliton equation can be reduced to the scalar $V$-soliton equation:

$$(\omega_{g_0} + \frac{\sqrt{-1}}{2} \partial M \bar{\partial} M \phi)^n = |V|^2 g e^{F + f - \phi} \omega_{g_0}^n,$$ (42)

where $F$ is given by

$$\text{Ric}(g_0) - \lambda \omega_{g_0} = \frac{\sqrt{-1}}{2} \partial M \bar{\partial} M F.$$

Since $V$ tends to 0 as $\tau$ goes to $\infty$, we may expect that the solutions to (3) are asymptotic to the solutions of equation (41), more precisely, we expect that the solution $g(t)$ of the Ricci flow is asymptotic to a geodesic ray. This can be a future research topic.

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