Zeta functions over zeros of general zeta and $L$-functions

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We describe in detail three distinct families of generalized zeta functions built over the (nontrivial) zeros of a rather general arithmetic zeta or $L$-function, extending the scope of two earlier works that treated the Riemann zeros only. Explicit properties are also displayed more clearly than before. Several tables of formulae cover the simplest concrete cases: $L$-functions for real primitive Dirichlet characters, and Dedekind zeta functions.

1 Generalities

This text is a partial expansion of our oral presentation (which surveyed an earlier paper \[27\] on zeta functions built over the Riemann zeros). Here we fully develop the argument of Sec. 5.5 therein, where we indicated how to change the Riemann zeta function $\zeta(x)$ into a more general “primary function” (i.e., the function providing the zeros on which the newer zeta functions are built, in the language of \[2\]). We also incorporate and extend subsequent work \[28\]. Accordingly, we can now reword the formalism to

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accommodate three distinct kinds of generalized zeta functions built over the (nontrivial) zeros of a fairly arbitrary (number-theoretic) zeta or $L$-function. The resulting explicit special values are presented in (seven) Tables.

Earlier explicit descriptions of such zeta functions, over zeros generalizing those of $\zeta(s)$, hardly exist in the literature. We set apart the zeros of Selberg zeta functions: these zeros correspond to eigenvalues of Laplacians, and zeta functions over them have been analyzed by spectral methods \cite{21, 24, 4, 26} (in the cocompact case, they are particular instances of Minakshisundaram–Pleijel zeta functions). Otherwise, only Dedekind (with Selberg) zeta functions get some mention \cite{19, 14}; already the works on $L$-series \cite{17, 16, 9, 15} do not discuss zeta functions per se over the zeros, but exclusively Cramér functions $V(t) \approx \sum_{\{\text{Im} \rho \geq 0\}} e^{\rho t}$, which are somewhat related but this connection is not covered there either). More general references are listed in greater detail in our previous articles \cite{27, 28}.

As for notations, we basically follow \cite{1, 10, 6}:

\begin{align*}
B_n & : \text{Bernoulli numbers; } B_n(\cdot) : \text{Bernoulli polynomials; } \\
E_n & : \text{Euler numbers; } \gamma : \text{Euler’s constant; } \\
\beta(s) & \overset{\text{def}}{=} \sum_{k=0}^{\infty} (-1)^k(2k+1)^{-s} : \text{a specific Dirichlet } L\text{-function, with } \beta(-n) = \frac{1}{2}E_n \quad (n \in \mathbb{N}) \quad (e.g., \beta(0) = \frac{1}{2}), \quad \beta'(0) = -\frac{3}{2}\log 2 - \log \pi + 2\log \Gamma(\frac{1}{4}); \\
\zeta(s, w) & \overset{\text{def}}{=} \sum_{k=0}^{\infty} (k+w)^{-s} : \text{the Hurwitz zeta function. }
\end{align*}

For fixed $w$, $\zeta(s, w)$ has a single pole at $s = 1$, simple and of residue 1, and the special values (\cite{11}, Sect. 1.10)

\begin{align*}
\zeta(-n, w) &= -B_{n+1}(w)/(n+1) \quad (n \in \mathbb{N}), \quad (e.g., \zeta(0, w) = \frac{1}{2} - w) \\
\text{FP}_{s=1} \zeta(s, w) &= -\Gamma'(w)/\Gamma(w) \quad (\text{FP} \overset{\text{def}}{=} \text{finite part at a pole}) \\
\zeta'(0, w) &= \log [\Gamma(w)/(2\pi)^{1/2}]
\end{align*}

(\text{upon parametric zeta functions as in } \cite{10} \text{ and } \cite{12, 14}, \text{below, } ’ \text{ will always mean differentiation with respect to the first variable: the exponent, } s \text{ or } \sigma).

Our notations (otherwise consistent with \cite{28}) are now generic: objects relative to the primary zeta or $L$-function (to be denoted $L(x)$) will obviously depend on it, usually without mention.
1.1 The primary functions $L(x)$

For the sake of definiteness, we confine interest here to situations still fairly close to the “Riemann case” $L(x) = \zeta(x)$; namely, to primary functions $L(x)$: meromorphic in $\mathbb{C}$ with at most one pole: $x = 1$ (of order $q = 0$ or $1$); (7) nonvanishing in $\{\text{Re } x > 1\}$, with the normalized asymptotic behavior

$$(\log L)^{(n)}(x) = O(x^{-\infty}) \quad \text{for } x \to +\infty \quad (\forall n \in \mathbb{N});$$

(8) obeying a functional equation of the same type as $\zeta(x)$,

$$\Xi(x) \equiv \Xi(1-x), \quad \Xi(x) \overset{\text{def}}{=} G^{-1}(x)(x-1)^q L(x),$$

(9) where both $\Xi(x)$ and $G(x)$ are entire functions of order $\mu_0 = 1$. $G(x)$ is a fully explicit factor (containing inverse Gamma factors) supplying $L(x)$ with explicit (“trivial”) zeros $x_k \in \{\text{Re } x \leq 0\}$. $\Xi(x)$ supplies the remaining (nontrivial) zeros of $L(x)$, which lie in symmetrical pairs within the strip $\{0 < \text{Re } x < 1\}$ and can be labeled as

$$\{\rho = \frac{1}{2} \pm i\tau_k\}_{k=1,2,...}, \quad \text{with } \text{Re } \tau_k > 0 \text{ and non-decreasing}$$

(10) (for simplicity, we exclude the exceptional occurrence of real zeros $\rho$ here). Note: all zeros are systematically counted with multiplicities.

A special notation will be sometimes useful for this Taylor series at $x = 1$,

$$\log [(x - 1)^q L(x)] \equiv \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} g_n^c\{L\} (x - 1)^n.$$  

(11) The coefficients $g_n^c\{L\}$ qualify as generalized Stieltjes cumulants: in the case $L(x) = \zeta(x)$, $q = 1$, then $g_n^c = n\gamma_{n-1}^c$, where $\{\gamma_{n-1}^c\}$ constitutes a cumulant sequence for the Stieltjes constants $\gamma_{n-1}$ (in our notation [27]; cf. also the $\eta_{n-1}$ in [3], Sec. 4; here we switch to a normalization we find more natural).

1.2 Three zeta families

We can then describe three (inequivalent) parametric zeta functions over the nontrivial zeros $\{\rho\}$ of a generic $L$ satisfying the above conditions (7–9):

$$\mathcal{Z}(s, x) \overset{\text{def}}{=} \sum_{\rho} (x - \rho)^{-s} \equiv \sum_{\rho} (\rho + x - 1)^{-s} \quad (\text{Re } s > 1).$$  

(12)
(definable for \((x - \rho) \notin \mathbb{R}^- \ (\forall \rho)\), cf. \([8, 22, 28]\) for the Riemann case);
\[
Z(\sigma, v) \overset{\text{def}}{=} \sum_{k=1}^{\infty} (\tau_k^2 + v)^{-\sigma} \quad (\text{Re } \sigma > \frac{1}{2}) \tag{13}
\]
(definable for \((\tau_k^2 + v) \notin \mathbb{R}^- \ (\forall k)\), cf. \([11, 7, 19, 27]\) for the Riemann case);
\[
Z(\sigma, v) \overset{\text{def}}{=} \sum_{k=1}^{\infty} (\tau_k + y)^{-2\sigma} \quad (\text{Re } \sigma > \frac{1}{2}) \tag{14}
\]
(definable for \((\tau_k + y) \notin \mathbb{R}^- \ (\forall k)\), cf. \([27, 14]\) for the Riemann case),

with shorthand names for interesting special parameter values:
\[
\mathcal{Z}(s) \overset{\text{def}}{=} Z(s, 1), \quad \mathcal{Z}(\sigma) \overset{\text{def}}{=} Z(\sigma, 0) \quad \text{(and, in } \mathcal{Z}(\sigma) \overset{\text{def}}{=} Z(\sigma, \frac{1}{4})). \tag{15}
\]

Each family is a generalized zeta function à la Hurwitz (cf. eq.(3)); its analytic structure mainly interests us in the exponent variable \((s \text{ or } \sigma)\); the translation variable \((x, v, \text{ or } y)\) serves to generate a parametric family in its specified range of values.

The families \(\{\mathcal{Z}\}\) and \(\{\mathcal{Z}\}\) share a single function, through the relation
\[
\mathcal{Z}(\sigma) \overset{(\equiv Z(\sigma, 0))}{=} (2 \cos \pi \sigma)^{-1} \mathcal{Z}(2\sigma, \frac{1}{2}). \tag{16}
\]

The family \(\{\mathcal{Z}\}\) can be generated from the family \(\{\mathcal{Z}\}\) (but not vice-versa), thanks to the identity (for, e.g., \(\text{Re } t > 0\)):
\[
\mathcal{Z}(s, \frac{1}{2} + t) \equiv [e^{\pi s/2} \mathcal{Z}(\frac{1}{2} s, it) + e^{-\pi s/2} \mathcal{Z}(\frac{1}{2} s, -it)]. \tag{17}
\]

The families \(\{\mathcal{Z}\}\) and \(\{\mathcal{Z}\}\) are built by summations over all zeros \((\frac{1}{2} \pm i\tau_k)\) symmetrically; due to resulting cancellations, they will be better behaved overall than the third family \(\{\mathcal{Z}\}\) based on the zeros with only one sign (this type is dubbed “half zeta function” in \([14]\)). Indeed, the first two families are formally expressible by “explicit formulae” à la Weil with suitably chosen test functions \([13]\); however, these formulae strictly diverge outside of clear-cut parameter domains: \(\{\text{Re } x > 1\}\) for \(\{\mathcal{Z}(s, x)\}\), resp. \(\{\text{Re } \sqrt{v} > \frac{1}{2}\}\) for \(\{\mathcal{Z}(\sigma, v)\}\), and that excludes the most interesting special cases for us: \(x = \frac{1}{2}\) and 1, resp. \(v = 0\) and \(\frac{1}{4}\). So, better adapted analytical schemes are still needed. On the basis of all the algorithms used in \([27, 28]\), we see as the most efficient approach to fully describe the family \(\{\mathcal{Z}\}\) first, then to derive the remaining algebraic properties through expansion formulae in the auxiliary
parameter, and the transcendental properties from zeta-regularized factorizations of $L$. Thus, within the same basic framework as before, we will get broader results in fewer steps. Only for the full mathematical justifications must we still invoke [27, 28].

1.3 Range of application, and examples

Some of the restrictions made above are just convenient to keep the paper short and close to concrete cases, and can probably be weakened. For instance, zeta functions over zeros of Selberg zeta functions have yielded results comparable to the Riemann case earlier [21, 22, 10], while they correspond to $\mu_0 = 2$ ($G$ contains a Barnes $G$-function), $q = -1$. Other extensions (e.g., to Hecke $L$-functions, as achieved upon their Cramér functions [15]) are equally conceivable. Currently, the assumptions are meant to closely fit two basic classes of examples (the most explicit properties of their zeta functions over their zeros will be listed in a concluding section).

- Dirichlet $L$-functions for real primitive Dirichlet characters: a Dirichlet $L$-function is associated with a character $\chi$ of a multiplicative group of integers mod $d$ ($d \in \mathbb{N}^*$ is called the modulus or conductor), as [10, 6]

$$L_\chi(x) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \chi(k) k^{-x} \equiv \prod_{\{\text{primes}\}} (1 - \chi(p) p^{-x})^{-1} \quad (\text{Re } x > 1).$$

(18)

Such a character is either even or odd, with a parity index $a$ defined by

$$a = 0 \text{ or } 1, \quad \text{according to } \chi(-1) = (-1)^a.$$  \hspace{1cm} (19)

$L_\chi(x)$ always satisfies conditions (7–8) above. We now restrict to primitive characters [10], and $d > 1$ (to exclude the case $\chi \equiv 1$, $L_\chi(x) \equiv \zeta(x)$, which more readily fits our next class of examples); then, $L_\chi(x)$ is entire, and the following functional equation holds:

$$\Xi_\chi(x) \equiv W_\chi \Xi_\chi(1 - x),$$

(20)

with

$$\Xi_\chi(x) \overset{\text{def}}{=} (d/\pi)^{x/2} \Gamma(\frac{1}{2}(x + a)) \ L_\chi(x),$$

(21)

$$W_\chi = (-i)^a d^{-1/2} \sum_{n \mod d} \chi(n) e^{2\pi in/d}$$

(22)

(the latter sum is called the Gaussian sum for $\chi$). The real ($\chi = \chi$) primitive characters (mod $d$) are given by Kronecker symbols for quadratic number
fields of discriminant \( \pm d \); their Gaussian sums are explicitly known (\[12\] thm 164), implying \( W_\chi \equiv +1 \) always; by way of consequence, the functional equation for their \( L \)-functions reduces to eq.(9), with

\[
q \equiv 0; \quad G(x) \equiv (\pi/d)^{x/2}/\Gamma(\frac{1}{2}(x+a)), \quad a = \begin{cases} 
0 & \text{for } \chi \text{ even} \\
1 & \text{for } \chi \text{ odd} 
\end{cases}
\]

- **Dedekind \( \zeta \)-functions:** for any algebraic number field \( K \), its zeta function is defined as \[12\]

\[
\zeta_K(x) \overset{\text{def}}{=} \sum_a N(a)^{-x} \equiv \prod_p (1 - N(p)^{-x})^{-1} \quad (\text{Re } x > 1)
\]

where \( a \) (resp. \( p \)) runs over all integral (resp. prime) ideals of \( K \) and \( N(a) \) is the norm of \( a \). Then \( L(x) = \zeta_K(x) \) satisfies all conditions \[12\] above, with

\[
q \equiv 1, \quad G(x) \equiv \frac{(4^{r_2} \pi^{n_K}/|d_K|)^{x/2}}{x \Gamma(x/2)^{r_1} \Gamma(x)^{r_2}},
\]

where \( r_1 \) (resp. \( 2r_2 \)) is the number of real (resp. complex) conjugate fields of \( K \), \( n_K (\equiv r_1 + 2r_2) \) the degree of \( K \), and \( d_K (\geq 0) \) its discriminant (\[12\] Sec. 42, \[23\]). For \( K = \mathbb{Q} \), having \( r_1 = 1, r_2 = 0 \) and \( d_K = 1 \), Riemann’s \( \zeta(s) \) and its classic functional equation are recovered, with \( q = 1 \) and \( G(x) \equiv \pi^{x/2}[x \Gamma(x/2)]^{-1} \) in eq.(9).

## 2 The first family \( \{\mathcal{Z}(s, x)\} \)

### 2.1 The zeta function over the trivial zeros

We know \[27\] that a key role must be played by the zeta function wholly analogous to \( \mathcal{Z}(s, x) \) but built on the trivial zeros of \( L(x) \),

\[
\mathcal{Z}(s, x) \overset{\text{def}}{=} \sum_k (x - x_k)^{-s} \quad (\text{Re } s > 1)
\]

(which we call the shadow zeta function of \( \mathcal{Z}(s, x) \)). Here this function (and all quantities referring to it) is to be viewed as fully explicitly known, just as the trivial zeros themselves are (in our concrete examples, \( \mathcal{Z}(s, x) \) will be expressed in terms of the Hurwitz zeta function \[3\]). It is also necessary to
relate $Z(s, x)$ and $G(x)$ as fully as possible. We specially want the results to include formulae for the special values $FP_{s=1}Z(s, x)$ and $FP_{s=1}Z'(s, x)$, resp. $Z'(0, x)$ and $Z''(0, x)$, because comparable formulae for the Hurwitz zeta function are quite important: eqs. $\text{(3)}$, resp. $\text{(4)}$. Then the original normalization of the trivial factor is not fully adequate: it is better to rewrite $L(x)$ as a product of either Weierstrass-like factors (as in $\text{[27]}$) or zeta-regularized factors (as in $\text{[28]}$); we follow the latter course here.

We recall some off-the-shelf rules on zeta-regularization for suitable infinite products of order $\mu_0 = 1$, of the form $\Delta(x) = e^{B_1 x+B_0} \prod_k [(1-x/y_k) e^{x/y_k}]$ ($\text{[28]}$ and refs. therein; the rules will be valid for $G(x)$ and $\Xi(x)$). Actually, eqs. $\text{(27)}$ and $\text{(30)}$ will also serve for $\mu_0 < 1$ in Sec. 3, and (only they) are worded for general $0 < \mu_0 \leq 1$. A key further requirement is an asymptotic (“generalized Stirling” $\text{[16]}$) expansion for $\log \Delta(x)$ ($x \to +\infty$), of the form

$$\log \Delta(x) \sim \tilde{a}_1 x (\log x - 1) + b_1 x + \tilde{a}_0 \log x + b_0 + \sum_{\{\mu_k \} \setminus \{0, 1\}} a_{\mu_k} x^{\mu_k} \quad (27)$$

for some sequence $(1 \geq) \mu_0 > \mu_1 > \cdots > \mu_n \downarrow -\infty$, and indefinitely differentiable term by term. Here, the terms designated by coefficients $a_{\mu}$ or $\tilde{a}_{\mu}$ are those allowed in a zeta-regularized product; any extra terms with a pure $x^1$ or $x^0$ dependence are banned (they read as $b_n x^n$ here). A generalized zeta function is also introduced, as $Z(s, x) = \sum_k (x - y_k)^{-s}$ if $\text{Re } s > \mu_0$. Then:

- the zeta-regularized form $D(x)$ of a product $\Delta(x)$ is explicitly obtained just by removing any “banned” portion present in the large-$x$ expansion of $\log \Delta(x)$: specifically, here,

$$D(x) \overset{\text{def}}{=} e^{-Z'(0, x)} \equiv e^{-(b_1 x + b_0)} \Delta(x); \quad (28)$$

- the logarithmic derivatives of the zeta-regularized product yield

$$\frac{(-1)^{m-1}}{(m-1)!} (\log D)^{(m)}(x) \equiv Z(m, x) \quad \text{for integer } m > \mu_0 \quad (30)$$

(we will also use this result once with $\mu_0 = \frac{1}{2}$, in Sec. 3);

- the results carry over to non-integer $s$ as Mellin-transform formulae:

$$Z(s, x) = \frac{\sin \pi s}{\pi (1-s)} I(s, x), \quad I(s, x) \overset{\text{def}}{=} \int_0^\infty Z(2, x+y) y^{1-s} dy \quad (1 < \text{Re } s < 2); \quad (31)$$
then, \(I(s, x)\) extends to a meromorphic function in the whole \(s\)-plane through repeated integrations by parts, and its polar structure is fully encoded in the \((y \to +\infty)\) expansion of \(Z(2, x + y)\), itself computable (see example next).

We now specialize the above results first to the trivial factor \(G(x)\). Consisting mainly of inverse Gamma factors, \(G(x)\) has a computable Stirling expansion (for \(x \to +\infty\)) which can be reorganized in the form

\[-\log G(x) \sim \tilde{a}_1 x (\log x - 1) + b_1 x + \tilde{a}_0 \log x + b_0 + \sum_{n=1}^{\infty} a_n x^{-n}\]  
(32)

(and which also governs \([\log \Xi(x) - \log(x - 1)]\), by eqs. (8) and (9)). Eq. (28) then implies that the zeta-regularized forms for \(G(x)\) and \(\Xi(x)\) are

\[
D(x) \overset{\text{def}}{=} e^{-Z'(0,x)} \equiv e^{+b_1 x + b_0} G(x),
\]
(33)

\[
\mathcal{D}(x) \overset{\text{def}}{=} e^{-x'(0,x)} \equiv e^{-b_1 x - b_0} \Xi(x),
\]
(34)

which in turn entail a zeta-regularized decomposition of \(L(x)\), as

\[(x - 1)^q L(x) \equiv D(x) \mathcal{D}(x).
\]
(35)

Concretely here (using eq. (33), and \(\mu_0 = 1\)), eqs. (29) and (30) translate to

\[
\begin{align*}
FP_{s=1} & \left(Z(s, x) \equiv (\log G)'(x) + b_1, \right. \\
Z(m, x) & \equiv \frac{(-1)^{m-1}}{(m-1)!} (\log G)^{(m)}(x) \quad \text{for } m = 2, 3, \ldots.
\end{align*}
\]
(36)

The substitution of the Stirling series (32) into eq. (37) with \(m = 2\) leads to the \((y \to +\infty)\) expansion of \(Z(2, x + y)\) in the simple form \(\sum_{n \geq -1} c_{-n}(x) y^{-n-2}\) (the \(c_{-n}(x)\) are polynomials). It follows that eq. (31), written for \(Z = Z\), yields an \(I(s, x)\) with poles at \(s = -n, n = -1, 0, 1, 2, \ldots\), all simple and of residues \(c_{-n}(x)\). As consequences for \(Z(s, x)\), restated in fully explicit form:

- \(Z(s, x)\) extends to a meromorphic function in the whole \(s\)-plane, with

  the single pole \(s = 1\), simple, of residue \(\tilde{a}_1\) (independent of \(x\)).  
  (38)

- the values \(Z(-n, x), n \in \mathbb{N}\) are given by closed polynomial formulae ("trace identities" in a spectral setting), as

\[
Z(-n, x) = -\frac{\tilde{a}_1}{n+1} x^{n+1} - \tilde{a}_0 x^n + n \sum_{j=1}^{n} (-1)^j \binom{n-1}{j-1} a_{-j} x^{n-j},
\]
(39)

e.g., \(Z(0, x) = -\tilde{a}_1 x - \tilde{a}_0\).

Then, the same formulae overall hold with \(D, Z, G\) replaced by the (less explicit) \(\mathcal{D}, \mathcal{Z}', \Xi\) respectively (and with suitably changed coefficients).

8
2.2 The main result

As a generalization of eq.(42) in [28], \( Z(s, x) \) admits an integral representation valid in the half-plane \( \{ \text{Re} s < 1 \} \) and for any eligible value of the parameter \( x \) avoiding the cut \( (-\infty, +1] \):

\[
Z(s, x) = -Z(s, x) + \frac{q}{(x-1)^s} + \frac{\sin \pi s}{\pi} J(s, x), \tag{40}
\]

\[
J(s, x) \overset{\text{def}}{=} \int_0^\infty \frac{L'(y+x)}{L(x+y-1)} y^{-s} \, dy \quad (\text{Re } s < 1); \tag{41}
\]

here, \((x-1)^s\) is given its standard determination for \( x \in \mathbb{C} \setminus (-\infty, +1] \); its discontinuity across the real axis, as well as those of \(-Z(s, x)\), are precisely cancelled through corresponding jumps of \( J(s, x) \) so that only the nontrivial zeros of \( L(x) \) induce genuine \( x \)-plane discontinuities in \( Z(s, x) \) (this can be checked by comparing computations of the right-hand side with small imaginary parts (±i0) added to \( x \)).

Eq.(40) easily takes real forms; a simple one valid for \( x > 0 \) (at least) is

\[
Z(s, x) = -Z(s, x) + \frac{\sin \pi s}{\pi} \int_0^\infty \left[ \frac{L'(y+x)}{L(x+y-1)} + \frac{q}{x+y-1} \right] y^{-s} \, dy \quad (0 < \text{Re } s < 1); \tag{42}
\]

this form only converges in a strip of the \( s \)-plane, but unlike (40), it remains well defined as \( x \to +1 \):

\[
Z(s, 1) = -Z(s, 1) + \frac{\sin \pi s}{\pi} \int_0^\infty \left[ \frac{L'(1+y)}{L} \right] (1+y) + \frac{q}{y} \right] y^{-s} \, dy. \tag{43}
\]

The proof is now a simple application of the general formulae above: the second logarithmic derivative of eq.(35) (or (9)) gives \([Z + Z](2, x) = q(x-1)^{-2} - (L'/L)'(x) \) (using eq.(30) for \( D = G \Xi \) and \( m = 2 \)); upon this and for \( x \notin (-\infty, +1], \) the Mellin formula (31) yields

\[
[Z + Z](s, x) \equiv \frac{\sin \pi s}{\pi(1-s)} \int_0^\infty \left[ \frac{q}{(x+y-1)^2} - \left( \frac{L'}{L} \right)'(x+y) \right] y^{1-s} \, dy. \tag{44}
\]

Now, moreover, this integral is convergent and analytic in \( \{ 0 < \text{Re } s < 2 \} \), the integrand being regular, and \( O(1) \) for \( y \to +0 \), and (using eq.(38)) \( O(y^{-2}) \) for \( y \to +\infty \). Then, splitting the integral in two, the first term evaluates explicitly to \( q(x-1)^{-s} \), and upon restricting to \( 0 < \text{Re } s < 1 \), the second term transforms to \( \pi^{-1} \sin \pi s J(s, x) \) through an integration by parts; hence the result (40). (The same integration by parts upon the whole integral yields eq.(42) instead.)
2.3 Explicit consequences for the family \( \{ \mathcal{F}(s, x) \} \)

The subsequent statements refer to \( s \) as variable, with \( x \) fixed.

Eq. (40) is a true analog for \( \mathcal{F}(s, x) \) of the (Joncqui`ere–Lerch) functional relation for \( \zeta(s, w) \) ([10], Sect. 1.11 eq.(16)). First of all, it gives an explicit one-step analytical continuation of \( \mathcal{F}(s, x) \) to the half-plane \( \{ \text{Re} \, s < 1 \} \). It also implies its meromorphic continuation in \( s \) to all of \( \mathbb{C} \), since a Mellin transform like \( J(s, x) \) has a well understood meromorphic structure: repeated integrations by parts on eq. (41) (invoking the asymptotic form (8)) show that \( J(s, x) \) is meromorphic in the whole \( s \)-plane, and has only simple poles at \( s = 1, 2, \ldots \), with residues

\[
\text{Res}_{s=n} J(s, x) = -(\log |L|)^{(n)}(x)/(n - 1)! \quad (n = 1, 2, \ldots) \tag{45}
\]

It then follows from eq. (40) that \( \mathcal{F}(s, x) \) precisely inherits the polar structure of \( -Z(s, x) \), namely (cf. eq. (48)):

\[
\mathcal{F}(s, x) \text{ has the single pole } s = 1, \text{ simple, of residue } -\tilde{a}_1. \tag{46}
\]

If \( L(x) \) admits an Euler product, as in the examples (18) and (24), then the substitution of its logarithmic derivative into eq. (41), followed by integration term by term, yields an asymptotic \((s \to -\infty)\) expansion for \( J(s, x) \), and thereby for \( \mathcal{F}(s, x) \), just as in the Riemann case ([28], eq.(52)).

Finally, almost all the special values of \( \mathcal{F}(s, x) \) (at integer \( s \)) are explicitly readable off eq. (40) thanks to its vanishing factor \((\sin \pi s)\). On general grounds ([27], Sec. 4), the \( \mathcal{F}(n, x) \) come out algebraically for \( n \in -\mathbb{N} \) (as given in Table 1, upper part), and (together with \( \mathcal{F}'(0, x) \)) transcendentally for \( n \in \mathbb{N}^* \), for instance:

\[
\mathcal{F}'(0, x) = -Z'(0, x) - q \log(x - 1) + J(0, x) \\
= b_1 x + b_0 + \log G(x) - \log [(x - 1)^q L(x)] \tag{47}
\]

\[
\text{FP}_{s=1} \mathcal{F}(s, x) = -\text{FP}_{s=1} Z(s, x) + \frac{q}{x - 1} - \text{Res}_{s=1} J(s, x),
\]

\[
= -b_1 - (\log G)'(x) + \left[ \frac{q}{x - 1} + \frac{L'}{L}(x) \right] \tag{48}
\]

(the reduced forms use eqs. (33), (36), (45)); and likewise for \( \mathcal{F}(n, x) \), \( n \geq 2 \). Terms in brackets stay globally continuous for \( x \to 1 \) (also in eq. (50) below); the limiting special values will be expressed as eq. (56) later.
\[ Z(s, x) = \sum_{\rho} (x - \rho)^{-s} \]

| \( s \) | \( Z(s, x) = \sum_{\rho} (x - \rho)^{-s} \) |
|---|---|
| \( -n \leq 0 \) | \(-Z(-n, x) + q(x - 1)^n \) |
| 0 | \( \tilde{a}_1 x + \tilde{a}_0 + q \) |

\( s \)-derivative at 0

\[ Z'(0, x) = b_1 x + b_0 - \log \Xi(x) \]

finite part at +1

\[ \text{FP}_{s=1} Z(s, x) = -b_1 + (\log \Xi)'(x) \]

\[ +n \geq 1 \]

\[ \left( \frac{-1}{(n-1)!} \right)^{(n)}(x) \]

Table 1: Special values of \( Z(s, x) \) (upper part: algebraic, lower part: transcendental [27]) for a general primary zeta function \( L(x) \) with a pole of order \( q \) (at \( x = 1 \)). Notations: see eqs. ([9]) for \( \Xi(x) \), ([20]) and ([39]) for \( Z(-n, x) \), ([32]) for \( \tilde{a}_j, b_j \); \( n \) is integer.

However, the values \( Z(n, x) \) for \( n \in \mathbb{N}^* \) (also including \( n = 1 \)) emerge more simply by differentiating the logarithm of a symmetrical Hadamard product formula, \( \Xi(x) \propto \prod_{\rho} (1 - x/\rho) \) (just as in the Riemann case [28]), as

\[ Z(n, x) = \left( \frac{-1}{(n-1)!} \right)^{(n)}(x) \]

\[ = -Z(n, x) + \left[ \frac{q}{(x-1)^n} + \left( \frac{-1}{(n-1)!} \right)^{(n)}(\log |L|)^{(n)}(x) \right] \]

This is by far the quickest path to transcendental values, but it altogether misses the pair we declared to be important: \( Z'(0, x) \) and \( \text{FP}_{s=1} Z(s, x) \), determined above by eqs. ([47]) and ([48]) respectively.

For \( n = 1 \), eq. ([50]) cannot hold: \( Z(1, x) \) is finite (eq. ([49]) defines it to be \( \sum_{\rho} (x - \rho)^{-1} \) with the zeros ordered pairwise, as usual), and \( Z(1, x) \) diverges (at the same time, both functions \( Z(s, x) \), \( Z(s, x) \) have a pole at \( s = 1 \)). Instead, the comparison of eq. ([49]) at \( n = 1 \) with eq. ([48]) yields a fixed anomaly, or discrepancy between two natural specifications for a finite value at \( s = 1 \),

\[ Z'(1, x) - \text{FP}_{s=1} Z(s, x) = (\log |\Xi|/\mathcal{D}|)'(x) = b_1 \] (constant).

Table 1 summarizes the special values obtained for \( Z(s, x) \) at general \( x \) (thus extending to general primary functions \( L(x) \) formulae previously restricted to \( L(x) = \zeta(x) \), cf. Table 2 in [28]).
The two sets of linear identities for the values $\mathcal{Z}(n, x)$ in the Riemann case (eqs. (61–62) in [28]), which are purely induced by the symmetry ($\rho \leftrightarrow 1 - \rho$) in eq.(12), naturally persist here:

$$\mathcal{Z}(n, x) = (-1)^n \mathcal{Z}(n, 1 - x) \quad \text{for } n = 1, 2, \ldots; \quad (52)$$

$$\mathcal{Z}(k, x) = -\frac{1}{2} \sum_{\ell=k+1}^{\infty} \binom{\ell-1}{k-1} (2x-1)^{\ell-k} \mathcal{Z}(\ell, x) \quad \text{for each odd } k \geq 1. \quad (53)$$

More explicit formulae can result for exceptional $x$-values such as $x = \frac{1}{2}$ and $x = 1$, which respectively correspond (via the functional equation (9)) to the symmetry center of $\Xi(x)$ and to the origin in the $x$-plane ($\mathcal{Z}(s, x)$ is clearly regular on the real $x$-axis, under our assumption $\{\rho\} \cap \mathbb{R} = \emptyset$). Thus, for $x = \frac{1}{2}$, eq.(49) simplifies to

$$\mathcal{Z}(n, \frac{1}{2}) \equiv 0 \quad \text{for all } n \geq 1 \text{ odd} \quad (54)$$

(and $\text{FP}_{s=1} \mathcal{Z}(s, \frac{1}{2}) = -b_1$ by eq.(51)) \quad (55)

(in combination with eqs.(50), (53), these amount to the explicit specifications $(\log |L|)^{2m+1}(\frac{1}{2}) = (\log G)^{2m+1}(\frac{1}{2}) + 2^{2m+1}q(2m)!$, also directly readable off the functional equation (9)); while for $x = 1$, that same formula brings in the Taylor series (11), to yield

$$\mathcal{Z}(1, 1) = -(\log G)'(1) + g_1^c\{L\},$$

$$\mathcal{Z}(n, 1) = -Z(n, 1) + g_n^c(L)/(n - 1)! \quad (n = 2, 3, \ldots) \quad (56)$$

(also, $\mathcal{Z}'(0, 1) = -Z'(0, 1) + g_0^c(L)$).

Case by case, $\mathcal{Z}(s, x)$ can also be made explicit for $x = \frac{1}{2}$ or 1, just as when $L(x) = \zeta(x)$ (28, Sec. 3.3); for our selected examples (18), (24) it will reduce to combinations of the two fixed Dirichlet series $\zeta(s)$ and $\beta(s)$ (cf. eq.(2)). The accordingly reduced special values of $\mathcal{Z}(s, \frac{1}{2})$ and $\mathcal{Z}(s, 1) (= \mathcal{Z}(s))$ are tabulated in the concluding Sec. 5.

3 The second family \{\mathcal{Z}(\sigma, v)\}

The main starting tool is the relation (16), which allows a 1–1 transfer of the previous results for $\mathcal{Z}(s, \frac{1}{2})$ onto one particular member, $\mathcal{Z}(\sigma) \equiv \mathcal{Z}(\sigma, v = 0)$, of the second family.
3.1 The basic case $v = 0$

The identity (16) shows that $Z(\sigma)$ is meromorphic in all of $\mathbb{C}$ with a double pole at $\sigma = \frac{1}{2}$, simple poles at $\sigma = \frac{1}{2} - m$ ($m = 1, 2, \ldots$), and polar parts:

$$Z\left(\frac{1}{2} + \varepsilon\right) = \frac{-\text{Res}_{s=1}\mathcal{Z}(s, \frac{1}{2})}{4\pi \varepsilon^2} - \text{FP}_{s=1}\mathcal{Z}(s, \frac{1}{2}) \frac{1}{2\pi \varepsilon} + O(1)_{\varepsilon \to 0}$$

$$= \frac{a_1}{4\pi \varepsilon^2} + \frac{b_1}{2\pi \varepsilon} + O(1)_{\varepsilon \to 0}; \quad (57)$$

$$Z\left(\frac{1}{2} - m + \varepsilon\right) = \frac{(-1)^{m+1}}{2\pi \varepsilon} \mathcal{Z}(1 - 2m, \frac{1}{2}) + O(1)_{\varepsilon \to 0} \quad \text{for } m = 1, 2, \ldots,$$

i.e., $R_m \equiv \text{Res}_{\sigma=\frac{1}{2}-m}Z(\sigma) = \frac{(-1)^m}{2\pi} \left[\mathcal{Z}(1 - 2m, \frac{1}{2}) + q 2^{1-2m}\right] \quad (58)$

(eq. (57) follows from eqs. (46), (55), and $R_m$ from using Table 1).

That relation also yields an integral representation for $Z(\sigma)$ (eq.(72) in [27] for the Riemann case), just by specializing eq.(40) to $x = \frac{1}{2} \pm i0$. Finally, it delivers all the special values of $Z(\sigma)$ as

$$Z(m) = \frac{1}{2} (-1)^m \mathcal{Z}(2m, \frac{1}{2}) \quad \text{for } m \in \mathbb{Z}, \quad Z'(0) = \mathcal{Z}'(0, \frac{1}{2}), \quad (59)$$

which become fully explicit using Table 1. The problem is then to extend all those results to general values of the parameter $v$.

3.2 Algebraic results for general $v$

Our best tool here is a straightforward expansion of $Z(\sigma, v)$ around $v = 0$ [27], convergent for $|v| < \min_k \{ |\tau_k|^2 \}$:

$$Z(\sigma, v) = \sum_{k=0}^{\infty} (\tau_k v)^{-\sigma} \left(1 + \frac{v}{\tau_k^2}\right)^{-\sigma} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1 - \sigma)}{\ell! \Gamma(1 - \sigma - \ell)} Z(\sigma + \ell) v^\ell. \quad (60)$$

This first provides the meromorphic continuation of $Z(\sigma, v)$ at fixed $v \neq 0$ to the whole complex $\sigma$-plane, now with double poles at all $\sigma = -m + \frac{1}{2}$, $m \in \mathbb{N}$. More precisely, the polar part of $Z(\sigma, v)$ at each pole only depends on a finite stretch of the series (60),

$$Z(-m + \frac{1}{2} + \varepsilon, v) = \sum_{\ell=0}^{m} \frac{\Gamma\left(\frac{1}{2} + m - \ell\varepsilon\right)}{\ell! \Gamma\left(\frac{1}{2} + m - \ell - \varepsilon\right)} Z(-m + \ell + \frac{1}{2} + \varepsilon) v^\ell + O(1)_{\varepsilon \to 0}; \quad (61)$$
upon importing the polar structure of $\mathcal{Z}(\sigma)$ from eqs.\((57, 58)\), this yields

\[
\mathcal{Z}(-m + \frac{1}{2} + \varepsilon, v) = \frac{\tilde{a}_1}{4\pi} \frac{\Gamma(m + \frac{1}{2})}{m! \Gamma(\frac{1}{2})} v^m \varepsilon^{-2} + \mathcal{R}_m(v) \varepsilon^{-1} + O(1)_{\varepsilon \to 0},
\]

\[
\mathcal{R}_m(v) = -\frac{\Gamma(m + \frac{1}{2})}{m! \Gamma(\frac{1}{2})} \left[ \frac{1}{2\pi} \sum_{j=1}^{m} \frac{1}{2j - 1} - \frac{b_1}{2\pi} \right] v^m + \sum_{j=1}^{m} \frac{\Gamma(\frac{1}{2} + m)}{(m-j)! \Gamma(\frac{1}{2} + j)} \mathcal{R}_j v^{m-j}.
\]

Here, the polar part of order 2 at every $(-m + \frac{1}{2})$ clearly comes from the single double pole of $\mathcal{Z}(\sigma)$ (at $\sigma = \frac{1}{2}$); whereas each residue $\mathcal{R}_m(v)$ has contributions from all the residues of $\mathcal{Z}(\sigma)$ at $\sigma = -j + \frac{1}{2} \geq -m + \frac{1}{2}$ (specified by eqs.\((57)\) for $j = 0$, \((58)\) for $j \geq 1$). For the leading pole $s = \frac{1}{2}$, eq.\((62)\) boils down to eq.\((57)\), i.e., this full polar part is independent of $v$.

For the special values $\mathcal{Z}(-m, v)$, $m \in \mathbb{N}$, the series \((60)\) also terminates:

\[
\mathcal{Z}(-m, v) \equiv \sum_{\ell=0}^{m} \binom{m}{\ell} \mathcal{Z}(-m + \ell) v^{\ell} \quad (m \in \mathbb{N}),
\]

where the values $\mathcal{Z}(-m + \ell)$ are explicit from eq.\((59)\) and Table 1; e.g.,

\[
\mathcal{Z}(0, v) \equiv \frac{1}{2}[-\mathcal{Z}(0, \frac{1}{2}) + q] \equiv \frac{1}{2}(\frac{1}{2} \tilde{a}_1 + \tilde{a}_0 + q) \quad \text{(independent of $v$)}.
\]

In the end, all the polar terms of $\mathcal{Z}(\sigma, v)$, and the special values $\mathcal{Z}(-m, v)$ ($m \in \mathbb{N}$), (listed in Table 2, upper half), are \textit{(computable) polynomials} in $v$.

### 3.3 Transcendental values for general $v$

Now an optimal tool is a variant of the factorization \((35)\), using the alternative zeta-regularized factor

\[
\mathcal{D}(v) \overset{\text{def}}{=} e^{-\mathcal{Z}'(0, v)}
\]

instead of $\mathcal{D}(x)$. The main point here is the replacement of $x$ by $v = (x - \frac{1}{2})^2$ as basic variable: then, in contrast to $\mathcal{D}(x)$ of eq.\((35)\), this zeta-regularization of $\Xi(x)$ now \textit{preserves the symmetry} ($x \leftrightarrow 1 - x$).

Rewritten in the variable $v \to +\infty$, the Stirling formula \((32)\) for $[\log \Xi(x) - q \log(x - 1)]$ becomes of order $\frac{1}{2} < 1$, yielding

\[
\log \Xi(\sqrt{v} + \frac{1}{2}) \sim \frac{1}{2} \tilde{a}_1 v^{\frac{1}{2}} \log v + (b_1 - \tilde{a}_1) v^{\frac{3}{2}} + \frac{1}{2} (\frac{1}{2} \tilde{a}_1 + \tilde{a}_0 + q) \log v + (\frac{1}{2} b_1 + b_0) \left[ + O(v^{-\frac{1}{2}}) \right];
\]
as “banned” terms (cf. Sec. 2.1), only constants ($\propto v^0$) can now occur ($v^\mu \log v$ are allowed if $\mu \notin \mathbb{N}$, they just induce double poles in the zeta functions [27]); here this results in

$$D(v) \equiv e^{-(b_0 + b_1/2)} \Xi \left(\frac{1}{2} + v^{1/2}\right) \equiv e^{b_1v^{1/2}} \xi \left(\frac{1}{2} + v^{1/2}\right)$$

(67)

and in the modified decomposition (cf. eq.(41) in [28] for the Riemann case)

$$(x - 1)^q L(x) \equiv e^{-b_1(x-1/2)} D(x) D(v), \quad v \equiv (x - \frac{1}{2})^2.$$  

(68)

All transcendental special values of $Z(\sigma, v)$ immediately follow, just as in the Riemann case [27]: first, $Z'(0, v) \equiv -\log D(v)$ expresses in terms of $\log \Xi \left(\frac{1}{2} \pm v^{1/2}\right)$ (and thereby, of $\log L \left(\frac{1}{2} \pm v^{1/2}\right)$), using eq.(67); then eq.(30), now applied to $D$ and $Z$ with $v$ as variable hence $\mu_0 = \frac{1}{2}$, yields

$$Z(m, v) = (−1)^{m−1} \frac{d^m}{(m−1)!} \log \Xi \left(\frac{1}{2} \pm v^{1/2}\right), \quad m = 1, 2, \ldots .$$

(69)

By the chain rule, the right-hand side must simplify to a finite linear combination of derivatives ($\log \Xi^{(l)}(x)$ at $x = \frac{1}{2} \pm v^{1/2}$ (and thereby, of $\log L \left(\frac{1}{2} \pm v^{1/2}\right)$), using eq.(67); then eq.(30)], now applied to $D$ and $Z$ with $v$ as variable hence $\mu_0 = \frac{1}{2}$, yields

$$Z(m, v) = (−1)^{m−1} \frac{d^m}{(m−1)!} \log \Xi \left(\frac{1}{2} \pm v^{1/2}\right), \quad m = 1, 2, \ldots .$$

(69)

Now, setting $v \equiv (x - \frac{1}{2})^2$ and $\rho \equiv \frac{1}{2} + i\tau$ throughout here, we may start from the identity

$$\left(1 - \frac{s}{x - \rho}\right) \left(1 - \frac{s}{x - 1 + \rho}\right) = 1 - \frac{s(2x - 1 - s)}{\tau^2 + v},$$

(70)

expand the logarithms of both sides, and identify like powers of $s$ to get a triangular sequence (for $n = 1, 2, \ldots$) of linear identities,

$$\frac{(x - \rho)^{-n} + (x - 1 + \rho)^{-n}}{n} \equiv \sum_{0 \leq \ell \leq n/2} (-1)^{\ell} \binom{n-\ell}{\ell} (2x - 1)^{n-2\ell} \left(\frac{\tau^2 + v}{n - \ell}\right).$$

(71)

Then, on the one hand, summing this over (half) the zeros yields the identities

$$\frac{Z(n, v)}{n} \equiv \sum_{0 \leq \ell \leq n/2} (-1)^{\ell} \binom{n-\ell}{\ell} (2x - 1)^{n-2\ell} \frac{Z(n-\ell, v)}{n - \ell} \text{ for } n = 1, 2, \ldots ,$$

(72)

which clearly generalize eqs.(54) (for $n$ odd) and (59) (for $n$ even) away from $x = \frac{1}{2}$. On the other hand, eqs.(71) invert into finite linear relations of the
\[ Z(\sigma, v) = \sum_{k=1}^{\infty} (\tau_k^2 + v)^{-\sigma} \]

\[
\begin{array}{ll}
-m \leq 0 & \frac{1}{2} \left[ -\frac{m}{2} \left( \sum_{j=0}^{m} \binom{m}{j} (-1)^j Z(-2j, \frac{1}{2})v^{m-j} + q(v - \frac{1}{4})^m \right) \right] \\
0 & \frac{1}{2} (\tilde{a}_1 + \tilde{a}_0 + q) \\
\end{array}
\]

\[
\begin{array}{ll}
\sigma \text{-derivative at 0} & Z'(0, v) = \frac{1}{2} b_1 + b_0 - \log \Xi(\frac{1}{2} \pm v^{1/2}) \\
+m \geq 1 & \frac{1}{2} \left[ -\sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} \left( \frac{\Xi(2v^{1/2}) - m-\ell}{(m-\ell-1)!} \right) (\log \Xi)^{(m-\ell)}(x)|_{x=\frac{1}{2} \pm v^{1/2}} \right] (v \neq 0) \\
& \frac{1}{2} \left[ \frac{(-1)^{m+1}}{2(2m-1)}(\log \Xi)^{(2m)}(\frac{1}{2}) \right] (v = 0) \\
\end{array}
\]

Table 2: Special values of \( Z(\sigma, v) \) (upper half: algebraic, lower half: transcendental [27]) for a general primary zeta function \( L(x) \) with a pole of order \( q \) (at \( x = 1 \)). Notations: see eqs.(9) for \( \Xi(x) \), \( \Xi(1) \) and \( \Xi(2) \) for \( Z(-n, x) \), \( \tilde{a}_j \) and \( b_j \); \( m \) is integer.

Same form (if \( x \neq \frac{1}{2} \)): \( (\tau^2 + v)^{-m} \equiv \sum_{n=1}^{m} V_{m,n}(x)((x-\rho)^{-n} + (x-1+\rho)^{-n}) \). At this point we can identify \( V_{m,n}(x) \): it has to be the coefficient of \( (x-\rho)^{-n} \) in the Laurent series of \( (\tau^2 + v)^{-m} = (x-\rho)^{-m}(2x-1-(x-\rho))^{-m} \) in powers of \( (x-\rho) \), i.e., \( V_{m,n}(x) = \binom{2m-n-1}{m-1} (2x-1)^{n-2m} \). Then, upon summing the above expression for \( (\tau^2 + v)^{-m} \) over half the zeros, we finally get

\[ Z(m, v) \equiv \sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} (2x-1)^{-m-\ell} Z(m-\ell, x) \quad (73) \]

for \( v \equiv (x-\frac{1}{2})^2 \neq 0 \) and \( m = 1, 2, \ldots \),

whereas \( Z(m, 0) \equiv \frac{1}{2} (-1)^m Z(2m, \frac{1}{2}) \), by eq.[59]. Remark: the pair of mutually inverse relations [72] and [73] are clearly similar to the identities [58] and have the same origin. They extend to all primary functions \( L \) and all \( v \)-values previous results written only for the Riemann case and \( v = \frac{1}{4} \) [20, 27].

The resulting values of \( Z(\sigma, v) \) for general \( v \) are listed in Table 2 (lower half).

(These Table improves upon Table 1 of 28 in two independent ways: it is valid for zeros of a general primary function \( L(x) \), not just \( \zeta(x) \), and it specifies \( Z(+m, v) \) more explicitly.)
In the particular case \( v = \frac{1}{4} \), the transcendental special values of \( \mathcal{Z}(\sigma, \frac{1}{4}) \) involve those of \( \mathcal{Z}(s, 1) \) by eq.(73), hence they will likewise end up expressed in terms of the generalized Stieltjes cumulants \([11]\), cf. Tables 3, 4, 6 below.

4 The third family \( \{\mathfrak{Z}(\sigma, y)\} \)

As with the preceding case, a starting point is the knowledge of one particular member of the family, now through the obvious identity \( \mathfrak{Z}(\sigma, 0) \equiv \mathcal{Z}(\sigma, 0) \). All results of Sec. 3.1 then cover this case as well.

The generic \( \mathfrak{Z}(\sigma, y) \) (with \( y \neq 0 \)) is built on a desymmetrized set of zeros, say \( (\frac{1}{2} + i\tau_k) \) only, hence it will be harder to describe explicitly than the other two families. Still, its polar structure can be drawn directly from an expansion (in \( \{|y| < \min_k \{|\tau_k|\}\} \)) similar to eq.(60) for \( \mathcal{Z}(\sigma, v) \) (see also \([14]\)):

\[
\mathfrak{Z}(\sigma, y) = \sum_{k=0}^{\infty} \tau_k^{-2\sigma} \left( 1 + \frac{y}{\tau_k} \right)^{-2\sigma} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-2\sigma)}{\ell!\Gamma(1-2\sigma-\ell)} \mathcal{Z}(\sigma + \frac{1}{2}\ell) y^\ell. \tag{74}
\]

This formula generates a pole for \( \mathfrak{Z}(\sigma, y) \) now at every integer or half-integer \( \frac{1}{2}(1-n), n \in \mathbb{N} \), according to:

\[
\mathfrak{Z}(\frac{1}{2}(1-n)+\varepsilon, y) = \sum_{\ell=0}^{n} \frac{\Gamma(n-2\varepsilon)}{\ell!\Gamma(n-\ell-2\varepsilon)} \mathcal{Z}(\frac{1}{2}(1-n+\ell)+\varepsilon) y^\ell + \left\{ \begin{array}{ll} O(1) & \text{for } n = 0 \\ O(\varepsilon) & \text{for } n = 1, 2, \ldots \end{array} \right. \tag{75}
\]

Concrete differences with eq.(61) arise from the factor \( \Gamma(n-2\varepsilon)/\Gamma(n-\ell-2\varepsilon) \) vanishing whenever \( \ell \geq n > 0 \). Only the polar part at \( \sigma = \frac{1}{2} \) remains the same as for \( \mathcal{Z}(\sigma, v) \) (of order \( r = 2 \) and independent of \( y \), given by eq.(57)); all other poles \( \frac{1}{2}(1-n) \) of \( \mathfrak{Z}(\sigma, y) \) are now simple, of residues

\[
r_n(y) = -\frac{\tilde{a}_1}{2\pi n} y^n + \sum_{0<2m\leq n} \left( \frac{n-1}{2m-1} \right) \mathcal{R}_m y^{n-2m}, \quad n = 1, 2, \ldots \tag{76}
\]

(in terms of the residues \( \mathcal{R}_m \) given by eq.(58)). Moreover, at \( \sigma = 0 \) (only), the \( \varepsilon \)-expansion of eq.(76) captures the finite part too (cf. eqs.(32),(64)):

\[
r_1(y) = \text{Res}_{\sigma=0} \mathfrak{Z}(\sigma, y) = -\frac{\tilde{a}_1}{2\pi} y; \quad \text{FP}_{\sigma=0} \mathfrak{Z}(\sigma, y) = \frac{1}{4}\tilde{a}_1 + \frac{1}{2}(\tilde{a}_0 + q) - \frac{b_1}{n} y. \tag{77}
\]
On the other hand, while previously we could express infinite products over all the zeros such as $D$, resp. $D$, in terms of simpler functions like $D$ and $L$ (through eqs. (35), resp. (68)), now we lack that ability for a similar infinite product but restricted to half the zeros. We thus have no simple formulae for transcendental special values of $Z(\sigma, y)$ (i.e., in the half-plane $\{\Re \sigma > 0\}$). Only a sequence of binary relations results by specializing the identity (17) to $s \in N^*$,

$$[i^m Z(\frac{1}{2}m, it) + i^{-m} Z(\frac{1}{2}m, -it)] \equiv Z(m, \frac{1}{2} + t), \quad m = 1, 2, \ldots, \quad (78)$$

constituting an obvious result except for $m = 1$: then, finite parts are to be taken on the left-hand side only.

5 Concrete examples

We finally illustrate the preceding results upon the two classes of primary zeta functions highlighted in Sec. 1.3. As a rule, it suffices to specialize the general formulae as indicated below (and recalling that the order $\mu_0 = 1$ throughout). To strengthen the practical side of this work, we will further display the final concrete formulae reached when the shift parameters are themselves fixed at specially interesting values. We will mainly show results for $Z(s, 1) \equiv Z(s)$ and $Z(s, \frac{1}{2})$, as Tables 4–7 (generalizing Table 3 in [28], which was specific to $L(x) = \zeta(x)$). The analogous special cases for the second family, $Z(\sigma, 0) \equiv Z(\sigma)$ and $Z(\sigma, \frac{1}{4})$, are easily recovered from the preceding ones by applying the results of Sec. 3, specially as subsumed in Table 3; this stage entails an extension of Table 1 in [27] to general $L(x)$. (We do not further illustrate the third family $\{Z(\sigma, y)\}$, for its lack of explicit special values.)

At those special parameter cases, the relevant values of $Z(s, x)$ will also become more explicit, using

$$\zeta(s, 1) \equiv \zeta(s); \quad \zeta(s, \frac{1}{2}) \equiv (2^s - 1) \zeta(s); \quad 2^{-2s} \zeta(s, \frac{1}{2} \pm \frac{1}{4}) \equiv \frac{1}{2}[(1 - 2^{-s}) \zeta(s) \pm \beta(s)]. \quad (79)$$

(cf. eq. (2)). So, the Dirichlet series $\zeta(s)$ and $\beta(s) \equiv L_{\chi_4}(s)$; $\chi_4$ is the real primitive character for the modulus 4) will both occur ubiquitously in these particular special values, for whatever choice of primary Dirichlet series $L(x)$.

Again, the Riemann case $L(x) = \zeta(x)$, having $q = 1$, is more conveniently treated here as a special Dedekind zeta function only (not an $L$-function).
-m ≤ 0  \ \ \ \ \ \frac{1}{2} (-1)^{m} \mathcal{Z}'(-2m, \frac{1}{2}) \ \ \ \ \ \frac{1}{2} \sum_{j=0}^{m} (-1)^{j} 2^{-(m-j)} \mathcal{Z}'(-2j, \frac{1}{2})

0 \ \ \ \ \ \frac{1}{2} \mathcal{Z}'(0, \frac{1}{2}) \ \ \ \ \ \frac{1}{2} \mathcal{Z}'(0, \frac{1}{2})

+ m ≥ 1 \ \ \ \ \ \frac{1}{2} (-1)^{m} \mathcal{Z}'(2m, \frac{1}{2}) \ \ \ \ \ \frac{1}{2} \sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} \mathcal{Z}'(m-\ell, 1)

Table 3: General formulae for the special values of \( \mathcal{Z}(\sigma, 0) \equiv \mathcal{Z}(\sigma) \) and \( \mathcal{Z}(\sigma, \frac{1}{4}) \), in terms of those for \( \mathcal{Z}(s, \frac{1}{2}) \) and \( \mathcal{Z}(s, 1) \) (such as provided in the subsequent Tables). Notations: see eq.(32) for \( b_{1} \); \( m \) is integer.

5.1 \( L \)-functions for real primitive Dirichlet characters

According to eq.(23), any \( L \)-function \( L_{\chi}(x) \) for such a Dirichlet character \( \chi \) (mod \( d \)) (with \( d > 1 \), to exclude \( \zeta(x) \)) is handled by the choice

\[
q \equiv 0; \quad \mathbf{G}(x) \equiv (\pi/d)^{x/2}/\Gamma(\frac{1}{2}(x+a)), \quad a = \begin{cases} 0 & \text{for } \chi \text{ even} \\ 1 & \text{for } \chi \text{ odd.} \end{cases} \tag{80}
\]

In turn, the other useful quantities specialize as follows:
- the leading coefficients in the Stirling formula (32):
  \[
  \tilde{a}_{1} = \frac{1}{2}, \quad \tilde{a}_{0} = \frac{1}{2}(a-1),
  \]
  \[
  b_{1} = -\frac{1}{2} \log(2\pi/d), \quad b_{0} = \frac{1}{2} \log(2^{2-a}\pi); \tag{81}
  \]
- the shadow zeta function (eq.(26)):
  \[
  \mathcal{Z}(s, x) = 2^{-s}\zeta(s, \frac{1}{2}(x+a)); \tag{82}
  \]
- the lowest generalized Stieltjes cumulants (eq.(11)): \( g_{0} \{ L_{\chi} \} \equiv -\log L_{\chi}(1) \) can always be specified, as well as \( g_{1} \{ L_{\chi} \} \equiv [L_{\chi}' / L_{\chi}](1) \) when \( a = 1 \). First, the general formula

\[
L_{\chi}(x) \equiv d^{-x} \sum_{n=1}^{d} \chi(n) \zeta(x, n/d); \tag{83}
\]
together with the special values (4–6) of the Hurwitz zeta function (also using \( \chi(d) = 0 \) and \( \sum_{n=1}^{d} \chi(n) = 0 \) throughout), yield these special values for \( L_\chi(x) \):

\[
L_\chi(0) = -\frac{1}{d} \sum_{n=1}^{d-1} \chi(n) n \quad \text{(algebraic)} \quad (84)
\]

\[
L_\chi(1) = -\frac{1}{d} \sum_{n=1}^{d-1} \chi(n) \frac{\Gamma'(n/d)}{\Gamma(n/d)} \quad \text{(transcendental)} \quad (85)
\]

\[
L_\chi'(0) = -L_\chi(0) \log d + \sum_{n=1}^{d-1} \chi(n) \log \Gamma(n/d) \quad \text{(transcendental).} \quad (86)
\]

Then, the functional equation (9) also implies (we now suppress the \( \chi \) labels)

if \( a = 1 \):

\[
L(1) = \pi d^{-1/2} L(0), \quad [L'/L](1) = \gamma + \log(2\pi/d) - [L'/L](0); \quad (87)
\]

if \( a = 0 \):

\[
L(1) = 2d^{-1/2} L(0) \quad [L'(1) \text{ involves the unknown } L''(0),... \] \quad (88)
\]

(in the \( a = 0 \) case, moreover, \( L(0) \equiv 0 \), and eq.(86) for \( L'(0) \) simplifies further by the reflection formula for \( \Gamma \), with \( \chi \) being even). The final outcome is:

- when \( a = 1 \), an algebraic explicit formula for \( L(1) \) plus a transcendental one for \([L'/L](1)\) (in terms of Gamma values), amounting to

\[
g_0^c\{L_\chi\} = -\log \left[-\frac{\pi}{d^{3/2}} \sum_{n=1}^{d-1} \chi(n) n\right], \quad g_1^c\{L_\chi\} = \gamma + \log(2\pi/d) + \frac{\sum_{n=1}^{d-1} \chi(n) \log \Gamma\left(\frac{n}{d}\right)}{\sum_{n=1}^{d-1} \chi(n) \frac{n}{d}};
\]

\[
(89)
\]

- when \( a = 0 \), just a formula for \( L(1) \), transcendental but more elementary than (85), and amounting to

\[
g_0^c\{L_\chi\} = -\log \left[-\frac{1}{d^{1/2}} \sum_{n=1}^{d-1} \chi(n) \log \sin \frac{\pi n}{d}\right] \quad [g_1^c\{L_\chi\} \text{ not specified}]. \quad (90)
\]

E.g., for each of \( d = 3 \) and 4 (the lowest possible values of \( d \)), the real primitive character \( \chi_d \) is unique (\( \chi_d(\pm 1 \text{ mod } d) = \pm 1 \), else \( \chi_d(n) = 0 \); in particular, \( L_{\chi_3}(x) \equiv \beta(x) \) as in eq.[2]); they are both odd, giving

\[
d = 3: \quad g_0^c\{L_{\chi_3}\} = -\log(\pi/3^{3/2}), \quad g_1^c\{L_{\chi_3}\} = \log((2\pi)^4/3^{3/2}) + \gamma - 6 \log \Gamma\left(\frac{1}{3}\right);
\]

\[
d = 4: \quad g_0^c\{L_{\chi_4}\} = -\log(\pi/4), \quad g_1^c\{L_{\chi_4}\} = \log(4\pi^3) + \gamma - 4 \log \Gamma\left(\frac{1}{4}\right); \quad (91)
\]

20
\[
\zeta(s) \overset{\text{def}}{=} \zeta(s, 1) \equiv \sum_{\rho} \rho^{-s} \quad [x = 1]
\]

\begin{align*}
\begin{array}{c|c}
 s & \zeta(s) \\
 \hline
 -n < 0 & \left[(a-1)(2^n-1) + a 2^n\right] \frac{B_{n+1}}{n+1} \\
 0 & \frac{1}{2}a \\
 \end{array}
\end{align*}

- derivative at 0: \( \zeta''(0) = \frac{1}{2} \left[(1-a) \log 2 + a \log \pi\right] + g_0^0 \{L_\chi\} \)
- finite part at +1: \( \text{FP}_{s=1} \zeta(s) = (a-\frac{1}{2}) \log 2 - \frac{1}{2} \gamma + g_1^0 \{L_\chi\} \)
- odd +n \geq 1: \( \left[(a-1)(1-2^{-n}) - a 2^{-n}\right] \zeta(n) + \frac{g_0^c \{L_\chi\}}{(n-1)!} \)

\begin{align*}
\begin{array}{c|c}
 s & \zeta(\frac{1}{2}) \\
 \hline
 \text{even } -n \leq 0 & 2^{-n-1}(a-\frac{1}{2}) E_n \\
 \text{odd } -n < 0 & -\frac{1}{2}(1-2^{-n}) \frac{B_{n+1}}{n+1} \\
 0 & \frac{1}{2}(a-\frac{1}{2}) \\
 \end{array}
\end{align*}

- derivative at 0: \( \zeta''(0, \frac{1}{2}) = \left(\frac{3}{4} - a\right) \log 2 + (a-\frac{1}{2}) \log[\Gamma(\frac{1}{2})^2 / \pi] - \log L_\chi(\frac{1}{2}) \)
- finite part at +1: \( \text{FP}_{s=1} \zeta(\frac{1}{2}) = \frac{1}{2} \log(2\pi/d) \)
- odd +n \geq 1: \( 0 \)
- even +n > 1: \( -\frac{1}{2}[2^n - 1] \zeta(n) + (1-2a) 2^n \beta(n)] - \frac{(\log L_\chi)^{(n)}(\frac{1}{2})}{(n-1)!} \)

Table 4: Special values of the zeta function \( \zeta(s, 1) \) over the nontrivial zeros of an \( L \)-function for a real primitive Dirichlet character \( \chi \) of modulus \( d > 1 \) and parity \( a = 0 \) or \( 1 \) (see eqs. (18–19)). For the \( g_n^c \), see eqs. (11), (89–91). Notations: see eqs. (1); \( n \) is integer. In last line, \( \zeta(n) \equiv (2\pi)^n |B_n| / (2n!) \) when \( n \) is even.

Table 5: Same as above, but for the zeta function \( \zeta(\frac{1}{2}) \). In last line, \( n \) being even, \( \zeta(n) \equiv (2\pi)^n |B_n| / (2n!) \) while \( \beta(n) \) (see eq. (2)) remains elusive.
In general, whether \( a = 0 \) or \( 1 \), we cannot specify the \( g^c_n \) any further than stated; we might just relate them, through eq. (83), to special cases of the still more general Laurent coefficients \( \gamma_n(w) \) of \( \zeta(x, w) \) around \( x = 1 \) [2, 18].

The finally resulting special values of \( Z(s, 1) \) and \( Z(s, \frac{1}{2}) \), over zeros of a general real primitive Dirichlet character, are presented in Tables 4 and 5 respectively.

### 5.2 Dedekind zeta functions

Referring back to eq. (25), the Dedekind zeta function \( \zeta_K(x) \) of any algebraic number field \( K \) is handled by the choice

\[
q \equiv 1, \quad G(x) \equiv \frac{(4^{r_2} \pi^{n_K} / |d_K|)^{x/2}}{x \Gamma(x/2)^{r_1} \Gamma(x)^{r_2}}.
\]

Moreover, \( r \overset{\text{def}}{=} \text{Res}_{x=1} \zeta_K(x) \) (the sole residue of \( \zeta_K(x) \)) is strictly positive and computable (in terms of many field invariants) [12 thms 121, 124; [23]).

In turn, the other useful quantities specialize as follows:
- the leading coefficients in the Stirling formula (32):
  \[
  \tilde{a}_1 = \frac{1}{2} n_K, \quad \tilde{a}_0 = 1 - \frac{1}{2} (r_1 + r_2),
  \]
  \[
  b_1 = -\frac{1}{2} \log((2\pi)^{n_K} / |d_K|), \quad b_0 = (r_1 + \frac{1}{2} r_2) \log 2 + \frac{1}{2} (r_1 + r_2) \log \pi;
  \]
- the shadow zeta function (eq. (26)):
  \[
  Z(s, x) = r_1 2^{-s} \zeta(s, \frac{1}{2} x) + r_2 \zeta(s, x) - x^{-s};
  \]
- the lowest generalized Stieltjes cumulants (eq. (11)):
  \[
  g^c_0 \{ \zeta_K \} \equiv - \log r \text{ is fully explicit and, sometimes at least, } g^c_1 \{ \zeta_K \} \equiv r^{-1} \text{FP}_{x=1} \zeta_K(x) \text{ can also be described. First, if } K = \mathbb{Q}; \text{ then } \zeta_K(x) \equiv \zeta(x) \text{ with } r = 1, \text{ hence}
  \]
  \[
  g^c_0 \{ \zeta \} = 0, \quad g^c_1 \{ \zeta \} = \gamma;
  \]

in turn, a general \( g^c_i \) will be some extension of Euler’s constant \( \gamma \). As next example, if \( K \) is a quadratic number field, then \( \zeta_K(x) \equiv \zeta(x) L_\chi(x) \) where \( \chi \) is the real primitive character of modulus \( |d_K| \) given by the Kronecker symbol for the discriminant \( d_K \) [12 Sec. 49]. Now in general, the zeta functions over
\[ \mathcal{Z}(s) \stackrel{\text{def}}{=} \mathcal{Z}(s, 1) \equiv \sum_{\rho} \rho^{-s} \quad [x = 1] \]

| \( s \) | \( \mathcal{Z}(s, 1) \equiv \sum_{\rho} \rho^{-s} \) |
|---|---|
| \(-n < 0\) | \([-r_1(2^n - 1) + r_2] \frac{B_{n+1}}{n+1} + 1\) |
| 0 | \( \frac{1}{2}r_2 + 2\) |

**derivative at 0**
\[ \mathcal{Z}'(0) = \frac{1}{2}[(r_1 + r_2) \log 2 + r_2 \log \pi] + g_n^\mathcal{Z}\{\zeta_K\} \]

**finite part at +1**
\[ \text{FP}_{s=1} \mathcal{Z}(s) = -\frac{1}{2}r_1 \log 2 + 1 - \frac{1}{2}n_K \gamma + g_n^\mathcal{Z}\{\zeta_K\} \]

| \( s \) | \( \mathcal{Z}(s, \frac{1}{2}) \equiv \sum_{\rho} (\rho - \frac{1}{2})^{-s} \) |
|---|---|
| even \(-n \leq 0\) | \( 2^{-n+1}(1 - \frac{1}{8}r_1 E_n) \) |
| odd \(-n < 0\) | \( -\frac{1}{2}n_K(1 - 2^{-n}) \frac{B_{n+1}}{n+1} \) |
| 0 | \( 2 - \frac{1}{4}r_1 \) |

**derivative at 0**
\[ \mathcal{Z}'(0, \frac{1}{2}) = (2 + \frac{3}{4}r_1 + \frac{1}{2}r_2) \log 2 - \frac{1}{2}r_1 \log[\Gamma(\frac{1}{2})^2 / \pi] - \log |\zeta_K|(\frac{1}{2}) \]

**finite part at +1**
\[ \text{FP}_{s=\frac{1}{2}} \mathcal{Z}(s, \frac{1}{2}) = \frac{1}{2}[n_K \log(2\pi) - \log |d_K|] \]

| \( s \) | \( \mathcal{Z}(s, \frac{1}{2}) \equiv \sum_{\rho} (\rho - \frac{1}{2})^{-s} \) |
|---|---|
| even \(-n \leq 0\) | \( 2^{-n+1}(1 - \frac{1}{8}r_1 E_n) \) |
| odd \(-n < 0\) | \( -\frac{1}{2}n_K(1 - 2^{-n}) \frac{B_{n+1}}{n+1} \) |
| 0 | \( 2 - \frac{1}{4}r_1 \) |

Table 6: Special values of the zeta function \( \mathcal{Z}(s, 1) \) over the nontrivial zeros of a Dedekind zeta function for an algebraic number field \( K \) (see eqs.(14). For the \( g_n^\mathcal{Z} \), see eqs.(1), (95–96). Notations: see eqs.(1); \( n \) is integer. In last line, \( \zeta(n) \equiv (2\pi)^n |B_n|/(2n!) \) when \( n \) is even.

Table 7: Same as above, but for the zeta function \( \mathcal{Z}(s, \frac{1}{2}) \). In last line, \( n \) being even, \( \zeta(n) \equiv (2\pi)^n |B_n|/(2n!) \) while \( \beta(n) \) (see eq.(2)) remains elusive.
the zeros (and their linear invariants) obviously add up when their primary functions \( L \) are multiplied. So, for a quadratic number field,

\[
g_0^c \{ \zeta_K \} = g_0^c \{ L_\chi \}, \quad g_1^c \{ \zeta_K \} = \gamma + g_1^c \{ L_\chi \} \tag{96}
\]

referring to the same cumulants for \( L_\chi(x) \) that were precisely described under the previous example by eq. (89) for \( \chi \) odd, or (90) for \( \chi \) even. Two basic examples (both with \( r_1 = 0, r_2 = 1 \)) are: \( K = \mathbb{Q}(i) \) (for which \( d_K = -4 \), \( \chi = \chi_4 \), \( L_\chi(x) \equiv \beta(x) \) as in eq. (2)), and \( K = \mathbb{Q}(\sqrt{-3}) \) (for which \( d_K = -3 \), \( \chi = \chi_3 \)), hence their specific cumulants \( g_0^c, \ g_1^c \) follow from eqs. (91), (96).

The finally resulting special values of \( \mathcal{Z}(s, 1) \) and \( \mathcal{Z}(s, \frac{1}{2}) \), over zeros of a general Dedekind zeta function, are presented in Tables 6 and 7 respectively.

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