ON COHOMOLOGY IN SYMMETRIC TENSOR CATEGORIES IN PRIME CHARACTERISTIC

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Abstract

We describe graded commutative Gorenstein algebras \( E_n(p) \) over a field of characteristic \( p \), and we conjecture that \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) \cong E_n(p) \), where \( \text{Ver}_{p+1} \) are the new symmetric tensor categories recently constructed in \([2, 3, 8]\). We investigate the combinatorics of these algebras, and the relationship with Minc’s partition function, as well as possible actions of the Steenrod operations on them.

Evidence for the conjecture includes a large number of computations for small values of \( n \). We also provide some theoretical evidence. Namely, we use a Koszul construction to identify a homogeneous system of parameters in \( E_n(p) \) with a homogeneous system of parameters in \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) \). These parameters have degrees \( 2^i - 1 \) if \( p = 2 \) and \( 2(p^i - 1) \) if \( p \) is odd, for \( 1 \leq i \leq n \). This at least shows that \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) \) is a finitely generated graded commutative algebra with the same Krull dimension as \( E_n(p) \). For \( p = 2 \) we also show that \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) \) has the expected rank \( 2^n(n-1)/2 \) as a module over the subalgebra of parameters.

1. Introduction

In our paper \([2]\), we introduced a nested sequence of incompressible symmetric tensor abelian categories in characteristic two. These were very recently generalised to all primes in our work with Ostrik \([3]\) and simultaneously by Coulembier \([8]\). These categories, \( \text{Ver}_{p^n} \) and \( \text{Ver}^+_{p^n} \), seem to be new fundamental objects deserving further study.

Here, our aim is to state a conjecture describing the ring structure of \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) \).

We have made large numbers of computations using the computer algebra system MAGMA \([5]\), and we conjecture that the answer should be the graded commutative \( k \)-algebra \( E_n(p) \) introduced below, where \( k \) is a field of characteristic \( p \). After defining these algebras, we prove the following.

\textbf{Theorem 1.1.} For \( n \geq 0 \), the algebra \( E_n(p) \) is a graded commutative finitely generated Gorenstein \( k \)-algebra of Krull dimension \( n \). If \( p = 2 \) then it is an integral domain,
while for $p$ odd it has nilpotent elements. The Poincaré series $f(q) = \sum_{d \geq 0} q^d \dim \mathcal{E}_n(p)_d$ is a rational function of $q$ satisfying $f(1/q) = (-q)^n f(q)$.

There are natural inclusion maps $\mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p)$, and in each degree the sequence

$$k = \mathcal{E}_0(p) \to \mathcal{E}_1(p) \to \cdots \to \mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p) \to \cdots$$

stabilises at some finite stage. So it makes sense to examine the colimit $\mathcal{E}_\infty(p) = \lim_{n=0} \mathcal{E}_n(p)$. The Poincaré series of this algebra in the case $p = 2$ is Minc’s partition function [19]. We adapt Andrews’ proof of a Rogers–Ramanujan style formula [1] for the reciprocal of the generating function for this partition function so that it gives us the Poincaré series for $\mathcal{E}_n(p)$ for all $n \geq 0$ and all primes $p$.

**Theorem 1.2.** The dimension of $\mathcal{E}_n(p)_d$ is equal to $\sum_{m=1}^{n} N_p(m, d)$, and the dimension of $\mathcal{E}_\infty(p)_d$ is equal to $\sum_{m=1}^{\infty} N_p(m, d)$, where

$$\sum_{m,d=0}^{\infty} N_p(m, d) t^m q^d = \frac{1}{\sum_{i=0}^{\infty} (-1)^i t^i \ell_{i,p}(q)}$$

and

$$\ell_{i,p}(q) = \begin{cases} \prod_{j=1}^{i} \frac{q^{2^j-1}}{1 - q^{2^j-1}} & p = 2 \\ \prod_{j=1}^{i} \frac{q^{2p^{j-1}(p-1)-1} + q^{2(p^{j-1})}}{1 - q^{2(p^{j-1})}} & p \text{ odd.} \end{cases}$$

The relationship with the symmetric tensor abelian categories constructed in [2] [3] [8] is as follows. Since the subcategory $\text{Ver}^\otimes_n \subset \text{Ver}^{\otimes}_p$ is a direct summand, this inclusion induces an isomorphism $\text{Ext}^\bullet_{\text{Ver}^{\otimes}_n} (1, 1) \cong \text{Ext}^\bullet_{\text{Ver}^{\otimes}_p} (1, 1)$ and so we only consider $\text{Ver}^{\otimes}_p$.

**Conjecture 1.3.** The graded commutative $k$-algebra $\text{Ext}^\bullet_{\text{Ver}^{\otimes}_{p+1}} (1, 1)$ is isomorphic to $\mathcal{E}_n(p)$. The inclusion $\text{Ver}^{\otimes}_p \subset \text{Ver}^{\otimes}_{p+1}$ induces the inclusion map $\mathcal{E}_{n-1}(p) \to \mathcal{E}_n(p)$.

We have the following computational evidence for this conjecture. In all characteristics, this is true for $n \leq 1$. In characteristic two, we have checked both the dimensions and the algebra structure for $n = 2$ in all degrees, for $n = 3$ up to degree 40, and for $n = 4$ up to degree 26. For $p = 3$, $n = 2, 3$, and for $p = 5$, $n = 2$, we have checked the dimensions and algebra structure up to degree 40. These computations were carried out using the computer algebra package MAGMA.

In a symmetric tensor abelian category, the Steenrod operations act on $\text{Ext}^\bullet (1, 1)$ and satisfy the Cartan formula and unstable condition, as well as the homogeneous form of the Adem relations in which it is not assumed that the operation $5q^0$ ($p = 2$), respectively $p^0$ ($p$ odd) acts as the identity (see [13]; the construction there extends to the setting of symmetric tensor categories). We investigate the possibilities for their action on $\mathcal{E}_n(p)$. Our conclusions are clearest when $p = 2$. In that case, we show that the only possible action of the Steenrod operations on $\mathcal{E}_n(2)$ compatible with the inclusions is that all $Sq^1 = 0$ except for the mandatory $Sq^{|x|}(x) = x^2$. This makes the action much more like that on the cohomology of a $p$-restricted Lie algebra than like
that on the cohomology of a finite group. In the case \( p \) odd, the existence of nilpotent elements interferes with the arguments, and we can only prove a weaker statement.

In the final sections, we provide some theoretical evidence for Conjecture 1.3 and some tools that may help prove it. Namely, we first consider the Koszul complex of the generating object \( V \) of \( \text{Ver}_{p^{n+1}} \) and compute its cohomology. Then we use the Koszul complex to express \( \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, X) \) as the cohomology of an explicit complex of vector spaces. While we cannot yet compute this cohomology in general, this construction explains the conjectural shape of the answer and provides upper bounds for dimensions of the individual \( \text{Ext} \) spaces. In particular, it implies the existence of the subalgebra of parameters, \( \mathbb{k}[y_1, \ldots, y_n] \subset \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \), where \( \deg(y_i) = 2^i - 1 \) if \( p = 2 \) and \( 2(p^i - 1) \) if \( p > 2 \). We show that \( \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \) is module-finite over this subalgebra, and for \( p = 2 \) show that the rank of this module is \( 2^n(n-1)/2 \), as predicted by Conjecture 1.3.

More generally, we at least show the following.

**Theorem 1.4.** The graded commutative \( \mathbb{k} \)-algebra \( \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \) is finitely generated, with Krull dimension \( n \). Moreover, for any \( X \in \text{Ver}_{p^{n+1}} \), \( \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, X) \) is a finitely generated module over this algebra.

This confirms Conjecture 2.18 of [14] for the categories \( \text{Ver}_{p^{n+1}} \).

Once the \( \text{Ext} \) algebra is better understood, this will be the starting point for applying support theory to the categories \( \text{Ver}_{p^{n+1}} \), along the lines of the theory for finite groups, developed by Carlson and others [7]. For example, one might hope that \( \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \) stratifies the stable category of \( \text{Ver}_{p^{n+1}} \) as a tensor triangulated category, in the sense of Benson, Iyengar and Krause [4]. This would give a classification of the tensor ideal thick subcategories, as well as the tensor ideal localising subcategories of the stable category of the ind-completion. If Conjecture 1.3 holds, then the inclusion of the subalgebra of parameters \( \mathbb{k}[y_1, \ldots, y_n] \rightarrow \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \) is an inseparable isogeny. This implies that it induces a bijection on homogeneous prime ideals, and so \( \text{Proj} \text{Ext}^*_{\text{Ver}_{p^{n+1}}}(1, 1) \) is a weighted projective space.

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### 2. Graded algebras

For a prime \( p \) let \( \mathbb{Z}[\frac{1}{p}] \) denote the ring of integers with \( p \) inverted. An element of \( \mathbb{Z}[\frac{1}{p}] \) is a rational number \( r \) if \( m/n \) where \( m, n \in \mathbb{Z} \) and 1 is a power of \( p \). We say that such an element \( r \) is even if \( r/2 \) is also in \( \mathbb{Z}[\frac{1}{p}] \) and odd otherwise. So for \( p = 2 \), every element is even. If \( a \in \mathbb{Z}[\frac{1}{p}] \), we write \( (-1)^a \) to denote \( +1 \) if \( a \) is even and \( -1 \) if \( a \) is odd.

We consider \( \mathbb{Z}[\frac{1}{p}] \)-graded algebras \( R \) over a field \( \mathbb{k} \) of characteristic \( p \). If \( x \) is a homogeneous element of \( R \), we write \( |x| \) for the degree of \( x \). We say that such an algebra is graded commutative if it satisfies \( yx = (-1)^{|x||y|}xy \).
If $R$ is a $\mathbb{Z}[\frac{1}{2}]$-graded $k$-algebra, we write $\text{int}(R)$ for the $\mathbb{Z}$-graded algebra derived from $R$ by means of the inclusion of $\mathbb{Z}$ in $\mathbb{Z}[\frac{1}{2}]$. So for $m \in \mathbb{Z}$, the homogeneous part of degree $m$ is given by $\text{int}(R)_m = R_m$.

**Example 2.1.** Let $k$ be a field of characteristic two, and let $k[X^{2^n}]$ be the algebra generated by the elements $X^{2^n}$ with $n \in \mathbb{Z}$, with the obvious relations $(X^{2^n})^2 = X^{2^{n+1}}$. This is a $\mathbb{Z}[\frac{1}{2}]$-graded commutative $k$-algebra, with $|X^{2^n}| = 2^n$. We have $\text{int}(k[X^{2^n}]) = k[X]$.

**Example 2.2.** Let $k$ be a field of odd characteristic $p$, and let $k[X^{p^n}] \otimes \Lambda(Y)$ be the algebra generated by elements $X^{p^n}$ with $n \in \mathbb{Z}$ and $Y$ with the relations $(X^{p^n})^p = X^{p^{n+1}}, Y^2 = 0, XY = YX$. This is a $\mathbb{Z}[\frac{1}{p}]$-graded commutative $k$-algebra, with $|X^{p^n}| = 2p^n$ and $|Y| = 1$. We have $\text{int}(k[X^{p^n}] \otimes \Lambda(Y)) = k[X] \otimes \Lambda(Y)$.

**Definition 2.3.** We define the Reynolds operator $\rho: R \to \text{int}(R)$ to be the map which is the identity on elements of $\text{int}(R)$ and zero on homogeneous elements of $R$ whose degree is not an integer.

**Lemma 2.4.** The map $\rho$ is an $\text{int}(R)$-module homomorphism.

**Proof.** Multiplication by elements of $\text{int}(R)$ preserves whether or not the degree of an element is an integer. \hfill \Box

**Proposition 2.5.** If $R$ is a Cohen–Macaulay $k$-algebra then so is $\text{int}(R)$.

**Proof.** For every element of $R$, some power is an element of $\text{int}(R)$. So $R$ is an integral extension of $\text{int}(R)$. By Lemma 2.3, the Reynolds operator $\rho: R \to \text{int}(R)$ is an $\text{int}(R)$-module homomorphism. The proposition now follows from Proposition 12 of Hochster and Eagon [16]. \hfill \Box

3. The algebra $\mathcal{E}_n(p)$

We treat separately the cases $p = 2$ and $p$ odd.

3.1. The algebra $\mathcal{E}_n(2)$

In this section, we examine the case $p = 2$, and we let $k$ be a field of characteristic two.

**Definition 3.1.** Let $R = R(n, 2)$ be the $\mathbb{Z}[\frac{1}{2}]$-graded commutative polynomial algebra $k[x_1, \ldots, x_n]$ with $|x_i| = \frac{|i|}{2}$, and let $\mathcal{E}_n(2) = \text{int}(R)$.

**Example 3.2.** If $n = 1$, we have $R = k[x_1]$ with $|x_1| = \frac{1}{2}$. The algebra $\text{int}(R)$ is generated by $u = x_1^2$, so $\mathcal{E}_1(2) = k[u]$.

**Example 3.3.** If $n = 2$, we have $R = k[x_1, x_2]$ with $|x_1| = \frac{1}{2}, |x_2| = \frac{3}{2}$. The algebra $\text{int}(R)$ is generated by $u = x_1^2, v = x_1x_2, w = x_2^2$. Then

$$\mathcal{E}_2(2) = \text{int}(R) = k[u, v, w]/(uw + v^2)$$

with $|u| = 1, |v| = 2, |w| = 3$. 
Proof. Macaulay integral domain. So the homogeneous sequence of parameters $y_1 = x_1^3, \ldots, y_3 = x_3^3$, of degrees 1, 3, 7. The quotient by these parameters has the following basis.

| deg | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| elt |   | $x_1 x_2^2$ | $x_1 x_2 x_3^2$ | $x_1 x_3^3$ | $x_2 x_3^3$ | $x_2 x_3^3$ | $x_2 x_3^3$ | $x_1 x_2 x_3^3$ |

The Poincaré series of $E_3(2)$ is therefore given by

$$
\sum_{d \geq 0} q^d \dim E_3(2)_d = \frac{1 + q^2 + q^3 + 2q^4 + q^5 + q^6 + q^8}{(1-q)(1-q^3)(1-q^2)}.
$$

**Theorem 3.5.** The algebra $E_n(2)$ is a Gorenstein integral domain. It has a regular homogeneous sequence of parameters $y_1 = x_1^3, y_2 = x_2^4, y_3 = x_3^5, \ldots, y_n = x_n^{2^n}$ of degrees $1, 3, 7, \ldots, 2^n - 1$. Modulo this regular sequence, we get a graded Frobenius algebra of dimension $2 \binom{n-1}{2}$ with dualising element $\alpha = x_1 x_2^3 x_3^7 \cdots x_{n-1}^{2n-1-1} x_n^{2n-2}$ in degree $2n+1 - 2n - 2$. The Poincaré series $f(q) = \sum_{d \geq 0} q^d \dim E_n(2)_d$ is a rational function of $q$ satisfying $f(1/q) = (-q)^n f(q)$.

**Proof.** It follows from Proposition 2.3 that $E_n(2) = \text{Int}(k[x_1, \ldots, x_n])$ is a Cohen–Macaulay integral domain. So the homogeneous sequence of parameters $y_1, y_2, y_3, \ldots, y_n$ is a regular sequence.

If $x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in $E_n(2)$ then $a_n$ is even. If such a monomial is not divisible by any of the parameters then $a_i \leq 2^i - 1$ for $1 \leq i < n$, and $a_n \leq 2^n - 2$. The monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_{n-1}^{2n-1-a_n-1} x_n^{2n-2-a_n}$ is also a basis element of $E_n(2)$ and the product of this with $x_1^{a_1} \cdots x_n^{a_n}$ is equal to $\alpha$. So $E_n(2)/(x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots x_n^{a_n})$ is a Frobenius algebra with a basis consisting of these monomials, and with dualising element $\alpha$.

It is easy to verify using the Frobenius property that $f(1/q) = (-q)^n f(q)$. It then follows by Theorem 4.4 of Stanley [22] that $E_n(2)$ is a Gorenstein algebra. Alternatively, it is shown in Eisenbud [10] §21.3 that the Gorenstein property holds for a graded Cohen–Macaulay ring if and only if the quotient by a regular sequence of parameters is a Frobenius algebra.

There is a natural inclusion map of algebras $R(n-1, 2) \to R(n, 2)$ given by sending each $x_i$ in $R(n-1, 2)$ to the element with the same name in $R(n, 2)$. It is easy to check that in each degree the sequence

$$
R(1, 2) \to \cdots \to R(n-1, 2) \to R(n, 2) \to \cdots
$$

stabilises at some finite stage. So we take the colimit $R(\infty, 2) = \lim_{n \to \infty} R(n, 2)$. Applying Int, we obtain inclusion maps $E_1(2) \to \cdots \to E_{n-1}(2) \to E_n(2) \to \cdots$ whose colimit we denote $E_\infty(2) = \text{Int}(R(\infty, 2))$.

**3.2. The algebra $E_n(p)$, $p > 2$**

For odd primes, we should double the degrees of the polynomial generators and introduce new exterior generators of degree one smaller.
Let $p$ be an odd prime and let $k$ be a field of characteristic $p$. Let $R = R(n,p)$ be the $\mathbb{Z}[\frac{1}{p}]$-graded commutative algebra $k[x_1, \ldots, x_n] \otimes \Lambda(\xi_1, \ldots, \xi_n)$ with $|x_i| = \frac{2(p-1)}{p^i}$ and $|\xi_i| = |x_i| - 1 = \frac{2}{p^i}$. Note that $|x_i|$ is even and $|\xi_i|$ is odd. We define $\mathcal{E}_n(p) = \text{Int}(R(n,p))$.

Example 3.6. If $n = 1$, we have $R = k[x_1] \otimes \Lambda(\xi_1)$ with $|x_1| = \frac{2(p-1)}{p}$ and $|\xi_1| = \frac{2-2}{p}$. In this case, the algebra $\mathcal{E}_1(p) = \text{Int}(R)$ is generated by the elements $y = x_1^{n}$ and $\eta = x_1^{p-1} \xi_1$ with $|y| = 2p-2$, $|\eta| = 2p-3$, namely, $\mathcal{E}_1(p) = k[y] \otimes \Lambda(\eta)$.

Example 3.7. If $p = 3$ and $n = 2$, we have $R = k[x_1, x_2] \otimes \Lambda(\xi_1, \xi_2)$ with $|x_1| = \frac{4}{9}$, $|x_2| = \frac{10}{9}$, $|\xi_1| = \frac{7}{3}$, $|\xi_2| = \frac{7}{9}$. In this case, the algebra $\mathcal{E}_2(3) = \text{Int}(R)$ is generated by the following elements:

| element | degree |
|---------|--------|
| $x_1^2 \xi_1$ | 3 |
| $x_3$ | 4 |
| $x_1 x_2^2 \xi_1 \xi_2$ | 6 |
| $x_1 x_2^2 \xi_2$ | 7 |
| $x_1 x_2^2 \xi_1 \xi_2$ | 7 |
| $x_1^2 x_2^3 \xi_2$ | 10 |
| $x_1^2 x_2^3 \xi_2$ | 11 |
| $x_1^2 x_2^3 \xi_2$ | 12 |
| $x_1^2 x_2^3 \xi_2$ | 15 |
| $x_1^2 x_2^3 \xi_2$ | 16 |

A regular homogeneous system of parameters is given by $y_1 = x_1^{3}$ and $y_2 = x_2^{3}$, and the quotient by these parameters is a graded Frobenius algebra with dualising element $x_1^2 x_2^3 \xi_1 \xi_2$ in degree 18. We have

$$\sum_{d \geq 0} q^d \dim \mathcal{E}_2(3)_d = \frac{1 + q^3 + q^6 + 2q^7 + q^8 + q^{10} + 2q^{11} + q^{12} + q^{15} + q^{18}}{(1 - q^3)(1 - q^{18})}.$$ 

Theorem 3.8. The ring $\mathcal{E}_n(p)$ is Gorenstein. It has a homogeneous system of parameters $y_1 = x_1^{2p}$, $y_2 = x_2^{2p}$, $\ldots, y_n = x_n^{2p}$ of degrees $2(p - 1)$, $2(p^2 - 1)$, $\ldots, 2(p^n - 1)$.

Modulo this regular sequence, we get a graded Frobenius algebra of dimension $2^np^{\frac{n(n-1)}{2}}$ with dualising element $\alpha = x_1^{2p-1} x_2^{2p-2} \ldots x_n^{2p-n-1} \xi_1 \xi_2 \ldots \xi_n$ in degree $2(\frac{p^n-1}{p-1}) - 3n - 2$. The Poincaré series $f(q) = \sum_{d \geq 0} q^d \dim \mathcal{E}_n(p)_d$ is a rational function of $q$ satisfying $f(1/q) = (-q)^n f(q)$.

Proof. It follows from Proposition 2.5 that $\mathcal{E}_n(p)$ is Cohen–Macaulay. Since $y_1, y_2, \ldots, y_n$ are elements of $\mathcal{E}_n(p)$ which form a regular sequence of parameters in $R(n,p)$, they also form a regular sequence of parameters in $\mathcal{E}_n(p)$. If $x_1^{a_1} \ldots x_n^{a_n} \xi_1^{e_1} \ldots \xi_n^{e_n}$ ($e_i \in \{0,1\}$ for $1 \leq i \leq n$) is a monomial in $\mathcal{E}_n(p)$ which is not divisible by any of the parameters then $a_i \leq p^i - 1$ for $1 \leq i \leq n$. The monomial

$$x_1^{p-1-a_1} x_2^{p-1-a_2} \ldots x_n^{p-1-a_n} \xi_1^{1-e_1} \ldots \xi_n^{1-e_n}$$

is also a basis element of $\mathcal{E}_n(p)$ and its product with $x_1^{a_1} \ldots x_n^{a_n} \xi_1^{e_1} \ldots \xi_n^{e_n}$ is equal to $\alpha$.

Again it is easy to verify using the Frobenius property that $f(1/q) = (-q)^n f(q)$. But this time, we cannot show the Gorenstein property as in the proof of Theorem 3.5 using Theorem 4.4 of [22], because $\mathcal{E}_n(p)$ is not an integral domain. However, the alternative argument using §21.3 of [10] still shows that $\mathcal{E}_n(p)$ is Gorenstein. □
Remark 3.9. Recall that there is an action of the multiplicative group \( \mathbb{G}_m \) on the algebras \( k[x_1, ..., x_n] \) and \( k[x_1, ..., x_n] \otimes \Lambda(\xi_1, ..., \xi_n) \) defined by their \( \mathbb{Z} \)-grading (the fractional degrees multiplied by \( p^n \)). Also we have the semisimple infinitesimal subgroup scheme \( \mu_{p^n} \subset \mathbb{G}_m \) defined by the equation \( a^{p^n} = 1 \) (i.e., \( \mu_{p^n} = (\mathbb{G}_m)_p \), the \( n \)-th Frobenius kernel of \( \mathbb{G}_m \)). For \( p > 2 \) we have \( \mathcal{E}_n(p) = (k[x_1, ..., x_n] \otimes \Lambda(\xi_1, ..., \xi_n))^{\mu_{p^n}} \), the subring of invariants, and for \( p = 2 \) we similarly have \( \mathcal{E}_n(2) = k[x_1, ..., x_{n-1}, x_n]^{\mu_{2^n}} = k[x_1, ..., x_{n-1}, x_n^2]^{\mu_{2^n}} \). Since \( \deg(x_i) - \sum_i \deg(\xi_i) = \deg(x_i) - \sum_i \deg(\xi_i) \) is an integer, the action of \( \mu_{p^n} \) on the super-space spanned by the variables \( x_i, \xi_i \) for \( p > 2 \) has Berezinian equal to 1 (recall that degrees of odd variables should be counted with a minus sign). Similarly, for \( p = 2 \) the action of \( \mu_{2^n} \) on the variables \( x_1, ..., x_{n-1}, x_n^2 \) has determinant equal to 1, as \( \deg(x_i) + 2 \deg(x_n) \) is an integer. This is related to the fact that the ring \( \mathcal{E}_n(p) \) is Gorenstein. For example, for \( p = 2 \) this follows from a group scheme generalization of Watanabe’s theorem: the algebra of invariants for a homogeneous unimodular action of a finite semisimple group scheme on a polynomial algebra is Gorenstein. This is a special case of [17], Theorem 0.1.

4. Generating functions

4.1. Generating functions, \( p = 2 \)

For an integer \( d \geq 0 \), the degree \( d \) part of \( \mathcal{E}_n(2) \) has a basis consisting of the monomials \( x_1^{a_1}x_2^{a_2} \cdots x_n^{a_n} \) such that the \( a_j \) are non-negative integers, and

\[
\sum_{i=1}^n a_i = d = 2^{n-1}a_n.
\]

The smallest integer degree of a term with \( a_j > 0 \) is \( j \), for the monomial \( x_1x_2 \cdots x_{j-1}x_j^2 \). So for \( d \) an integer, we must have \( a_j = 0 \) for \( j > d \). It follows that the maps of vector spaces \( \mathcal{E}_1(2)_d \rightarrow \mathcal{E}_2(2)_d \rightarrow \cdots \) are eventually isomorphisms, and \( \mathcal{E}_\infty(2)_d \) is a finite dimensional vector space. It is spanned by the monomials \( x_1^{a_1}x_2^{a_2} \cdots \) with

\[
\sum_{i=1}^n a_i = d = 2^{n-1}a_n + 2a_2 + \frac{3}{2}a_3 + \cdots.
\]

Such an expression is a partition of \( d \) into parts \( \frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \cdots \). These are enumerated in sequence A002843 of the On-line Encyclopedia of Integer Sequences (which is sequence 405 of Sloane’s Handbook [21]). This sequence has been studied by Mine [19], Andrews [1], and Flajolet and Prodinger [15], see also Nguyen, Schwartz and Tran [20] for a context in algebraic topology. The first few terms are

\[
1, 1, 2, 4, 7, 13, 24, 43, 78, 141, 253, 456, 820, 1472, 2645, 4749, 8523, 15299, 27456, 49267, 88407, 158630, 284622, 510683, 916271, 1643963, 2949570, 5292027, 9494758, \ldots
\]

A few more terms can be found at https://oeis.org/A002843/b002843.txt. This sequence grows like \( C\lambda^n \), where

\[
C := 0.74040259366730734..., \quad \lambda := 1.79414718754168546...
\]

Our analysis of the generating function \( \sum_{d=0}^\infty q^d \dim \mathcal{E}_n(2)_d \) follows Andrews [1]. Since there are many misprints in the relevant section of [1], and we are doing something slightly different, we choose to repeat the argument in our context. The analogous argument for \( p \) odd, which we carry out later, is not dealt with in [1].
Let $N(m, d)$ be the number of monomials of degree $d$ in $x_1, \ldots, x_m$ with $a_m > 0$. Thus the dimension of $E_n(2)_d$ is $\sum_{m=1}^{\infty} N(m, d)$, and the dimension of $E_\infty(2)_d$ is $\sum_{m=1}^{\infty} N(m, d)$.

We can rewrite these monomials in terms of new variables $z_1, z_2, \ldots$ as follows. Set $z_1 = x_1^2$, and $z_i = x_{i-1}^{-1}x_i^2$ for $i \geq 2$. These variables $z_i$ are degree one elements of the larger $\mathbb{Z}[(1/2)]$-graded ring of Laurent polynomials $k[x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots]$. Then we have $x_1^{a_1}x_2^{a_2} \cdots = z_1^{b_1}z_2^{b_2} \cdots$ where $a_i = 2b_i - b_{i+1}$. The constraints $a_i \geq 0$ translate to $2b_i \geq b_{i+1}$ for $i \geq 1$, and since the $b_i$ are eventually zero, they are all non-negative. Thus $N(m, d)$ is the number of sequences $(b_1, \ldots, b_m)$ of nonnegative integers with $\sum_{i=1}^{m} b_i = d$, and $2b_i \geq b_{i+1}$ for $1 \leq i < m$.

Set $\mu_m(q) = \sum_{d=0}^{\infty} N(m, d)q^d$, and $\mu_0(q) = 1$. We would like to compute $\mu_m(q)$.

In fact, we will compute a more general generating function, taking into account the degrees with respect to all $z_i$. Introduce auxiliary variables $q_1, q_2, \ldots$ corresponding to the statistics $b_1, b_2, \ldots$; i.e., we define the multivariate Poincaré series of $E_n(2)$

$$\mu_m(q_1, \ldots, q_m) := \sum_{b_1, \ldots, b_m; 2b_i \geq b_{i+1}} q_1^{b_1} \cdots q_m^{b_m},$$

so that the usual Poincaré series of this algebra is $\mu_m(q) = \mu_m(q_1, \ldots, q)$. Thus we have

$$\mu_m(q_1, \ldots, q_m) = \sum_{b_1=1}^{2b_{m-1}} \cdots \sum_{b_m=1}^{2b_m-1} q_1^{b_1} \cdots q_m^{b_m}. $$

For the last sum we have $\sum_{b_m=1}^{2b_m-1} q_m^{b_m} = \frac{q_m^{2b_m - 1}}{1 - q_m}(1 - q_m^{2b_m - 1})$ and so we obtain

$$\mu_m = \frac{q_m}{1 - q_m} \left( \mu_{m-1} - \sum_{b_1=1}^{2b_1} \cdots \sum_{b_m=1}^{2b_m-1} q_1^{b_1} \cdots q_m^{b_m} \frac{(q_m^{-1}q_1^2)^b}{(q_m^{-1}q_1^2)^{b-1}} \right).$$

Now for the last sum we have $\sum_{b_m=1}^{2b_m-2} (q_m^{-1}q_1^2)^{b_m-1} = \frac{q_m^{-1}q_1^2}{1 - q_m^{-1}q_1^2} (1 - (q_m^{-1}q_1^2)^{2b_m - 2})$ and so we obtain

$$\mu_m = \frac{q_m}{1 - q_m} \left( \mu_{m-1} - \frac{q_m^{-1}q_1^2}{1 - q_m^{-1}q_1^2} \left( \mu_{m-2} - \sum_{b_1=1}^{2b_1} \cdots \sum_{b_m=2}^{2b_m-3} q_1^{b_1} \cdots q_m^{b_m} \frac{(q_m^{-1}q_1^2)^{b_m-2}}{(q_m^{-1}q_1^2)^{b_m-2}} \right) \right).$$

We continue this way, using induction. At the end, we use $\mu_0 = 1$. We obtain

$$\sum_{i=1}^{m} (-1)^i \mu_{m-i} \left( \frac{q_m}{1 - q_m} \left( \frac{q_m^{-1}q_1^2}{1 - q_m^{-1}q_1^2} \right) \frac{q_m^{-2}q_1^4}{1 - q_m^{-2}q_1^4} \right) \cdots \left( \frac{q_m^{-i}q_1^{2i}}{1 - q_m^{-i}q_1^{2i}} \right) = \begin{cases} 0 & m > 0 \\ 1 & m = 0. \end{cases}$$

So we set

$$\ell_m(q_1, \ldots, q_m) = \frac{q_1^3q_2^7 \cdots q_m^{2m-1}}{(1 - q_m)(1 - q_m^{-1}q_1^2) \cdots (1 - q_1^2q_m^{2m-2})},$$

and we have $\sum_{i=0}^{m} (-1)^i \mu_{m-i} \ell_i = \begin{cases} 0 & m > 0 \\ 1 & m = 0. \end{cases}$
Now we introduce another variable $t$, and we have
\[
\sum_{m=0}^{\infty} \sum_{i=0}^{m} t^{m-i} \mu_{m-i} \cdot (-1)^{i} t^{i} \ell_{i} = 1.
\]
Setting $j = m - i$ and
\[
\mu(t, q) := \sum_{m=0}^{\infty} \mu_{m}(q_1, \ldots, q_m) t^{m}, \quad \mu(t, q) := \mu(t, q, q, \ldots) = \sum_{m=0}^{\infty} \mu_{m}(q) t^{m},
\]
we rewrite this as
\[
\mu(t, q) g(t, q) = 1, \quad g(t, q) := \sum_{i=0}^{\infty} (-1)^{i} t^{i} \ell_{i}(q_1, \ldots, q_i).
\] (4.2)
This yields $\mu(t, q) = \frac{1}{g(t, q)}$. In particular, $\mu(t, q) = \frac{1}{g(t, q)}$, where $g(t, q) := g(t, q, q, \ldots)$. Thus we obtain the following result.

**Theorem 4.1.** We have
\[
\mu(t, q) = \left( \sum_{m=0}^{\infty} \frac{(-1)^{m} t^{m} q_{1}^{3} q_{2}^{3} \ldots q_{m}^{3} \ell_{m-1}}{(1 - q_{m})(1 - q_{m-1} q_{m}^{2}) \ldots (1 - q_{1} q_{2}^{2} \ldots q_{m}^{2} \ell_{m-1})} \right)^{-1}.
\]
In particular,
\[
\sum_{m,d=0}^{\infty} N(m, d) t^{m} q^{d} = 1 \left/ \sum_{i=0}^{\infty} \frac{(-1)^{i} t^{i} q^{1+3+7+\ldots+(2^{i}-1)}}{(1-q)(1-q^{2})(1-q^{3}) \ldots (1-q^{2^{i}-1})} \right.
\]
Note that $1 + 3 + 7 + \ldots + (2^{i}-1) = 2^{i+1} - i - 2$.

Expanding this out, the reciprocal of the generating function for $N(m, d)$ is
\[
1 - \frac{1}{1 - q} + \frac{t^{2} q^{4}}{(1-q)(1-q^{3})} - \frac{t^{3} q^{11}}{(1-q)(1-q^{3})(1-q^{7})} + \ldots
\]
which tabulates as follows:

| \( t \) | \( q \) | \( q^{2} \) | \( q^{3} \) | \( q^{4} \) | \( q^{5} \) | \( q^{6} \) | \( q^{7} \) | \( q^{8} \) | \( q^{9} \) | \( q^{10} \) | \( q^{11} \) | \( q^{12} \) | \( q^{13} \) | \( q^{14} \) | \( q^{15} \) | \( q^{16} \) | \( q^{17} \) | \( q^{18} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 4 | 6 | 4 | 12 | 24 | 48 | 96 | 192 | 384 | 768 | 1536 | 3072 | 6144 | 12288 | 24384 | 48768 | 97536 |
| 3 | 3 | 4 | 6 | 9 | 11 | 14 | 18 | 22 | 26 | 31 | 36 | 41 | 47 | 53 | 60 | 67 | 74 | 81 |
| 4 | 1 | 4 | 7 | 11 | 18 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 81 | 89 | 97 | 105 | 113 | 121 |
| 5 | 1 | 5 | 11 | 19 | 31 | 51 | 72 | 102 | 141 | 180 | 219 | 258 | 297 | 336 | 375 | 414 | 453 | 492 |
| 6 | 1 | 6 | 16 | 31 | 57 | 96 | 146 | 216 | 313 | 436 | 595 | 802 | 1056 | 1310 | 1664 | 1918 | 2172 | 2426 |
| 7 | 1 | 7 | 22 | 48 | 94 | 170 | 276 | 432 | 654 | 954 | 1353 | 1853 | 2353 | 2853 | 3353 | 3853 | 4353 | 4853 |
| 8 | 1 | 8 | 29 | 71 | 149 | 287 | 502 | 822 | 1299 | 1979 | 2918 | 3958 | 5098 | 6238 | 7378 | 8518 | 9658 | 10798 |
| 9 | 1 | 9 | 37 | 101 | 228 | 466 | 867 | 1497 | 2470 | 3922 | 5884 | 8846 | 12808 | 17770 | 23742 | 30714 | 37686 | 44658 |
| 10 | 1 | 10 | 46 | 139 | 338 | 732 | 1442 | 2823 | 5646 | 11292 | 22584 | 45168 | 90336 | 180672 | 361344 | 722688 | 1445376 | 2890752 |
| 11 | 1 | 11 | 56 | 186 | 487 | 1117 | 2322 | 4644 | 9288 | 18576 | 37152 | 74304 | 148608 | 297216 | 594432 | 1188864 | 2377728 | 4755456 |
| 12 | 1 | 12 | 67 | 243 | 684 | 1661 | 3322 | 6644 | 13288 | 26576 | 53152 | 106304 | 212608 | 425216 | 850432 | 1700864 | 3401728 | 6803456 |
| 13 | 1 | 13 | 79 | 311 | 939 | 2413 | 4826 | 9652 | 19304 | 38608 | 77216 | 154432 | 308864 | 617728 | 1235456 | 2470912 | 4941824 | 9883648 |
| 14 | 1 | 14 | 92 | 391 | 1263 | 3591 | 7182 | 14364 | 28728 | 57456 | 114912 | 229824 | 459648 | 919296 | 1838592 | 3677184 | 7354368 | 14708736 |
| 15 | 1 | 15 | 106 | 484 | 1621 | 5463 | 10926 | 21852 | 43704 | 87408 | 174816 | 349632 | 699264 | 1398528 | 2797056 | 5594112 | 11188224 | 22376448 |
| 16 | 1 | 16 | 121 | 484 | 1765 | 5895 | 11790 | 23580 | 47160 | 94320 | 188640 | 377280 | 754560 | 1509120 | 3018240 | 6036480 | 12072960 | 24145920 |
| 17 | 1 | 17 | 137 | 517 | 1969 | 6503 | 12906 | 25812 | 51624 | 103248 | 206496 | 412992 | 825984 | 1651968 | 3303936 | 6607872 | 13215744 | 26431488 |
| 18 | 1 | 18 | 154 | 600 | 2292 | 7476 | 14952 | 29904 | 59808 | 119616 | 239232 | 478464 | 956928 | 1913856 | 3827712 | 7655424 | 15310848 | 30621696 |
The coefficients of the Poincaré series for $\mathcal{E}_n(2)$ are given by adding the first $n$ rows of this table, while the coefficients of the Poincaré series for $\mathcal{E}_\infty(2)$ are given by adding all the rows; in other words by setting $t = 1$. Thus, setting $N(d) := \sum_{m \geq 0} N(m, d)$, we get

$$
\sum_{d=0}^{\infty} N(d)q^d = \frac{1}{\phi(q)}, \quad \phi(q) := \sum_{i=0}^{\infty} \frac{(-1)^{i+1}q^{1+3+7+\cdots+(2i-1)}}{(1-q)(1-q^3)(1-q^7)\cdots(1-q^{2i-1})}.
$$

Note that the series $\phi(q)$ defines an analytic function in the disk $|q| < 1$, and that the numbers $C, \lambda$ in (11) are determined as follows: $\lambda = \frac{1}{\alpha}$, where $\alpha$ is the smallest positive zero of $\phi(q)$, while $C = \frac{1}{\phi'(\alpha)}$.

It is easy to see from this computation that the reciprocal of the generating function is much easier to compute than the generating function itself, and has much smaller coefficients. The same will be true for $p$ odd.

Remark 4.2. Recall (2) that the category $\text{Ver}_{2n+1}^+$ is the category of modules in $\text{Ver}_{2n}$ over the algebra $A := \Lambda V$, where $V = X_{n-1}$ is the generating object of $\text{Ver}_{2n}$. Thus the group $G_m$ acts on $A$ by scaling $V$. This action gives rise to an action of $G_m$ on $\text{Ext}_{\text{Ver}_{2n+1}}^*(\mathbb{I}, \mathbb{I})$, i.e., a $\mathbb{Z}$-grading on each cohomology group. We expect that on $\mathcal{E}_n(2)$, this grading is given by the degree with respect to the variable $z_n$. In other words, we expect that the 2-variable Poincaré series of $\mathcal{E}_n(2)$ taking into account this grading is $\mu_m(q, \ldots, q, qv)$.

So let us compute the generating function $\mu(t, q, v) := \sum_{m=0}^{\infty} \mu_m(q, \ldots, q, qv)t^m$. Arguing as above, we get $\mu(t, q, v) = \mu(t, q) + \mu(t, q)g(t, q, v) = 1$, where

$$
g(t, q, v) := \sum_{i=0}^{\infty} \frac{(-1)^{i+1}q^{1+3+7+\cdots+(2i-1)}(1-q^3v^2)(1-q^7v^4)\cdots(1-q^{2i-1}v^{2i-1})}{(1-qv)(1-q^3v^2)(1-q^7v^4)\cdots(1-q^{2i-1}v^{2i-1})}.
$$

Thus, we have

$$
\mu(t, q, v) = 1 + \frac{1 - g(t, q, v)}{g(t, q)}.
$$

4.2. Generating functions, $p > 2$

The details for $p$ odd are similar to those for $p = 2$, but are quite a bit harder to keep straight. So we have chosen to write out the computation again in full.

For an integer $d \geq 0$, the degree $d$ part of $\mathcal{E}_p(n)$ has a basis consisting of the monomials $x_1^{a_1} \cdots x_n^{a_n} \xi_1^{\varepsilon_1} \cdots \xi_n^{\varepsilon_n}$ such that the $a_j$ are non-negative integers, each $\varepsilon_j$ is zero or one, and

$$
d = 2p-2a_1 + 2p^2-2p^2a_2 + \cdots + 2p^{n-2}a_n + \frac{p-2}{p} \xi_1 + \frac{p^2-2}{p} \xi_2 + \cdots + \frac{p^{n-2}}{p} \varepsilon_n.
$$

Let $N_p(m, d)$ be the number of such monomials in degree $d$ with $a_m + \varepsilon_m > 0$. Thus the dimension of $\mathcal{E}_p(n)$ is $\sum_{m=1}^{\infty} N_p(m, d)$, and the dimension of $\mathcal{E}_\infty(p)$ is $\sum_{m=1}^{\infty} N_p(m, d)$.

Set $z_1 = x_1^{p-2}, \xi_1 = x_1^{p-1},$ and $z_i = x_i^{p-1} - x_i, \xi_i = x_i^{p-1} - x_i$ for $i \geq 2$. Then we have $|z_i| = 2p-2, |\xi_i| = 2p-3$ (1 $\leq i \leq n$) and

$$
(x_1^{a_1} x_2^{a_2} \cdots)(\xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \cdots) = (z_1^{b_1} z_2^{b_2} \cdots)(\xi_1^{a_1} \xi_2^{a_2} \cdots)
$$

where $a_i = pb_i + (p-1)\varepsilon_i - b_{i+1} - \varepsilon_{i+1}$. Then the conditions on the $b_i$ and the $\varepsilon_i$
are that $b_i$ are non-negative integers, $\varepsilon_i = 0$ or $1$, and $pb_i + (p-1)\varepsilon_i \geq b_{i+1} + \varepsilon_{i+1}$ for $i \geq 1$.

Set $\mu_m(q) = \sum_{d=0}^{\infty} N_p(m, d)q^d$, and $\mu_0(q) = 1$. Then we have

$$\mu_m(q) = \sum_{b_1 + \varepsilon_1 = 1}^{\infty} \sum_{b_2 + \varepsilon_2 = 1}^{\infty} \cdots \sum_{b_m + \varepsilon_m = 1}^{\infty} q^{(2p-2)(b_1 + \cdots + b_m) + (2p-3)(\varepsilon_1 + \cdots + \varepsilon_m)}.$$  

We would like to compute $\mu_m(q)$. As in the case $p = 2$, we introduce auxiliary variables $q_1, q_2, \ldots, w_1, w_2, \ldots$ corresponding to the statistics $b_1, b_2, \ldots, \varepsilon_1, \varepsilon_2, \ldots$; i.e., we define the multivariate Poincaré series of $E_n(p)$ 

$$\mu_m(q_1, \ldots, q_m; w_1, \ldots, w_m) := \sum_{b_1 + \varepsilon_1 = 1}^{\infty} \sum_{b_2 + \varepsilon_2 = 1}^{\infty} \cdots \sum_{b_m + \varepsilon_m = 1}^{\infty} q_1^{b_1} \cdots q_m^{b_m} w_1^{\varepsilon_1} \cdots w_m^{\varepsilon_m}$$

so that the usual Poincaré series of this algebra is 

$$\mu_m(q) = \mu_m(q^{2p-2}, \ldots, q^{2p-2}, q^{2p-3}, \ldots, q^{2p-3}).$$

We have 

$$\sum_{b + \varepsilon = 1}^{s} q^b w^\varepsilon = \frac{(w + q)(1 - q^s)}{1 - q}.$$  

So, summing over $b_m, \varepsilon_m$, we get 

$$\mu_m = \frac{w_m + q_m}{1 - q_m} \left( \mu_{m-1} - \sum_{b_1 + \varepsilon_1 = 1}^{\infty} \sum_{b_2 + \varepsilon_2 = 1}^{\infty} \cdots \sum_{b_{m-1} + \varepsilon_{m-1} = 1}^{\infty} q_1^{b_1} \cdots q_m^{b_m} w_1^{\varepsilon_1} \cdots w_m^{\varepsilon_m} (q_m - 1 q_m^{-1})^{\varepsilon_m - 1} \right).$$

Thus, summing over $b_m, \varepsilon_m - 1$ and using (4.3) again, we have 

$$\mu_m = \frac{w_m + q_m}{1 - q_m} \left( \mu_{m-1} - \sum_{b_1 + \varepsilon_1 = 1}^{\infty} \sum_{b_2 + \varepsilon_2 = 1}^{\infty} \cdots \sum_{b_{m-2} + \varepsilon_{m-2} = 1}^{\infty} q_1^{b_1} \cdots q_m^{b_m} w_1^{\varepsilon_1} \cdots w_m^{\varepsilon_m} (q_m - 1 q_m^{-1})^{\varepsilon_m - 1} \right).$$

Continuing inductively and using that $\mu_0 = 1$, we obtain 

$$\sum_{i=0}^{m} (-1)^i \mu_{m-i, p} = \begin{cases} 0 & m > 0 \\ 1 & m = 0 \end{cases}$$

where 

$$\mu_{i, p}(q) = \left( \frac{w_m + q_m}{1 - q_m} \right) \left( \frac{w_m - 1 q_m^{p-1} + q_m q_m^p}{1 - q_m q_m^p} \right) \left( \frac{w_m - 2 q_m^{p-1} q_m q_m^p}{1 - q_m q_m^p} \right) \cdots \left( \frac{w_m - i q_m^{p-1} q_m^{p-1} q_m q_m^p}{1 - q_m q_m^p} \right) \cdots \left( \frac{w_m - i q_m^{p-1} q_m^{p-1} q_m q_m^p}{1 - q_m q_m^p} \right).$$
Introducing a new variable $t$, we rewrite this as \( \left( \sum_{j=0}^{\infty} t^j \mu_j \right) \left( \sum_{i=0}^{\infty} (-1)^i t^i \ell_{i,q} \right) = 1 \), so 
\( \mu_j \) can be determined from the generating function
\[
\sum_{j=0}^{\infty} t^j \mu_j = \frac{1}{\sum_{i=0}^{\infty} (-1)^i t^i \ell_{i,q}}.
\]

In particular, setting \( w_i = q^{2i-3} \), \( q_i = q^{2i-2} \), we get
\[
\ell_{i,q}(q) = q^{(2p-2)(p^i-1) - i} \frac{(1 + q)(1 + q^{2p-1}) \cdots (1 + q^{2p^{i-1}-1})}{(1 - q^{2p-2})(1 - q^{2p^{i-2}}) \cdots (1 - q^{2p^{i-2}})}.
\]

Thus we obtain the following result.

**Theorem 4.3.** We have
\[
\sum_{m,d=0}^{\infty} N_p(m,d) t^m = \left( \sum_{i=0}^{\infty} (-1)^i t^i q^{(2p-2)(p^i-1)-i} \frac{(1 + q)(1 + q^{2p-1}) \cdots (1 + q^{2p^{i-1}-1})}{(1 - q^{2p-2})(1 - q^{2p^{i-2}}) \cdots (1 - q^{2p^{i-2}})} \right)^{-1}.
\]

**Remark 4.4.** Recall (\cite{BensonEtAl}, Subsection 4.14) that the principal block of the category \( \text{Ver}_{n,-1} \) (i.e., the block of the unit object) is equivalent to the category of modules in \( \text{Ver}_p \) over the algebra \( A := \Lambda V \), where \( V = T_1 \) is the generating object of \( \text{Ver}_p \).

Thus the group \( G_m \) acts on \( A \) by scaling \( V \). This action gives rise to an action of \( G_m \) on \( \text{Ext}^m_{\text{Ver}_p} (1, 1) \), i.e., a \( \mathbb{Z} \)-grading on each cohomology group. We expect that 

\( \mu_m(q^{2p-2}, ..., q^{2p-2}, (qv)^{2p^i-2}, q^{2p-3}, ..., q^{2p-3}, (qv)^{2p-3}) \).

So let us compute the generating function
\[
\mu(t, q, v) := \sum_{m,d=0}^{\infty} \mu_m(q^{2p-2}, ..., q^{2p-2}, (qv)^{2p^i-2}, q^{2p-3}, ..., q^{2p-3}, (qv)^{2p-3}) t^m.
\]

Arguing as above, we get 
\( g(t, q) := \mu(t, q) - \mu(t, q) g(t, q, v) = 1 \), where

\[
g(t, q, v) := \sum_{i=0}^{\infty} (-1)^i t^i q^{(2p-2)(p^i-1)-i} v^{(2p-2)(p^i-1)-i} \frac{(1 + qv)(1 + q^{2p-1}v^{2p^i-2}) \cdots (1 + q^{2p^{i}-1}v^{(2p-2)p^{i-2}})}{(1 - q^{2p-2}v^{2p^{i-2}})(1 - q^{2p^{i-2}}v^{(2p-2)p^{i-1}}) \cdots (1 - q^{2p^{i-2}}v^{(2p-2)p^{i-1}})}.\]

Thus, we have
\[
\mu(t, q, v) = 1 + \frac{1 - g(t, q, v)}{g(t, q)},
\]

where \( g(t, q) := g(t, q, 1) \).

Here is a table of the coefficients in the reciprocal of the generating function for \( N_p(m,d) \) with \( p = 3 \).

\begin{verbatim}
| i | 1 | q^3 | q^5 | q^7 | q^9 | q^{10} | q^{11} | q^{12} | q^{13} | q^{14} | q^{15} |
|---|---|----|----|----|----|-------|-------|-------|-------|-------|-------|
| t | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| t^2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| t^3 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 |
| t^4 | 1 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 | 4 | 2 |
| t^5 | 1 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
\end{verbatim}

Reciprocating, we obtain the table of coefficients \( N_3(m,d) \). These tables become sparser as the prime increases.
5. Action of the Steenrod operations

In this section, we examine possible actions of the Steenrod operations on the algebra $\mathcal{E}_\infty(p)$.

5.1. Steenrod operations for $p = 2$

We begin with the easier case $p = 2$.

**Theorem 5.1.** There is only one possibility for the action of the Steenrod operations on $\mathcal{E}_\infty(2)$ in such a way that the Cartan formula

$$Sq^n(xy) = \sum_{i+j=n} Sq^i(x) Sq^j(y)$$

and the unstable conditions $Sq^i(x) = x^2$ for $i = |x|$ and $Sq^i(x) = 0$ for $i > |x|$ hold. Namely for $x \in \mathcal{E}_\infty(2)$, we have $Sq^{|x|}(x) = x^2$, and $Sq^i(x) = 0$ for $i \neq |x|$. In particular, if $x$ has degree greater than zero then $Sq^0(x) = 0$.

**Proof.** We begin by examining the elements $x_n^2$ of degree $2^n - 1$, and we show by induction on $n$ that $Sq^i(x_n^2) = 0$ for $i < 2^n - 1$. Let $T = Sq^0 + Sq^1 + Sq^2 + \cdots$ be the total Steenrod operation, which by the Cartan formula is a ring homomorphism. In particular, note that $Sq^i$ of a $2^k$th power vanishes when $i$ is not divisible by $2^n$. Our goal is to show that $T(x_n^2) = (x_n^2)^2$ for all $n \geq 1$.

We begin with $n = 1$. We have $(x_1^2)(x_1^2) = (x_1x_2^2)^2$. Applying $Sq^3$ to this relation, we obtain

$$Sq^0(x_1^2)(x_2^2)^2 + (x_1^2)^2 Sq^2(x_2^2) = Sq^3((x_1x_2^2)^2) = 0.$$ 

Therefore $Sq^0(x_1^2)$ is divisible by $(x_1^2)^2$, and is hence zero, and so $T(x_1^2) = (x_1^2)^2$.

Now for the inductive step. Assume that $T(x_{n-1}^{2^{n-1}}) = (x_{n-1}^{2^{n-1}})^2$. We have the relation

$$(x_{n-1}^{2^{n-1}})^{2^{n-1}-1}(x_{n-1}^{2^{n-1}}) = (x_{n-1}^{2^{n-1}-1})^2 x_{n-1}^{2^{n-1}-1}.$$ 

in $\mathcal{E}_\infty(2)$. Applying $T$, we get

$$(x_{n-1}^{2^{n-1}})^{2^{n-2}} T(x_n^{2^n}) = (T(x_{n-1}^{2^{n-1}})^2 x_n^{2^n}).$$

The right hand side is zero in degrees not divisible by $2^{n-1}$. It follows that $T(x_n^{2^n})$ is zero in degrees not congruent to minus two modulo $2^{n-1}$. So the only possibilities for non-zero Steenrod operations on $x_n^{2^n}$ are $Sq^{2^n-1}$ and $Sq^{2^n-1-1}$.

We also have the relation

$$(x_n^{2^n})(x_n^{2^n+1})^{2^n-1} = (x_n x_n^{2^n+1-2} 2^n)$$ 

in $\mathcal{E}_\infty(2)$. Applying $T$, we get

$$T(x_n^{2^n})(T(x_n^{2^n+1}))^{2^n-1} = (T(x_n x_n^{2^n+1-2}))^{2^n}.$$ 

The right hand side is zero in degrees not divisible by $2^n$. So in particular, examining the term in degree $2^{n+1}(2^n - 1) - 2^{n-1}$, we have

$$(x_n^{2^n})^2 Sq^{(2^n+1)(2^n-1)-2^{n-1}}((x_n^{2^n+1})^{2^n-1}) + (Sq^{2^n-1-1}(x_n^{2^n}))(x_n^{2^n+1})^{2^n+1-2} = 0.$$ 

So $Sq^{2^n-1-1}(x_n^{2^n})$ is divisible by $(x_n^{2^n})^2$, and is hence zero. Hence $T(x_n^{2^n}) = (x_n^{2^n})^2$, and the inductive step is complete.
Finally, given any monomial \(x = x_1^{a_1} \ldots x_n^{a_n} \in \mathcal{E}_\infty(2)\), we raise it to the \(2^n\)th power to obtain an element of the subring generated by \(x_1^2, x_2^2, x_3^2, \ldots\). Then \(T(x^{2^n}) = T(x^{2^n}) = (x^{2^n})^2 = (2^n)^{2^n}\), and since we are in an integral domain of characteristic two, this implies that \(T(x) = x^2\).

5.2. Steenrod operations for \(p > 2\)

Next, we examine possible actions of the Steenrod operations on the algebra \(\mathcal{E}_\infty(p)\) for \(p\) odd. Our conclusions are weaker than in the case \(p = 2\), because of the existence of nilpotent elements.

**Theorem 5.2.** Suppose that the Steenrod operations act on \(\mathcal{E}_\infty(p)\) with \(p\) odd in such a way that the Cartan formula and unstable conditions hold. Then on the subring spanned by the monomials not involving any of the \(\xi_i\), we have \(P^m(x) = x^p\) and \(P^i(x) = 0\) for \(i \neq n\), where \(|x| = 2m\).

**Proof.** Let \(T\) be the total Steenrod operation \(P^0 + P^1 + \ldots\). The argument to show that \(T(x^{a_n}) = (x^{a_n})^p\) for \(p\) odd is similar to the case \(p = 2\), but involves one more induction. We therefore write it out in full.

Our first task is to show that \(T(x_1^2) = (x_1^2)^p\). We begin as before with

\[(x_1^p)(x_2^2)^{p-1} = (x_1x_2^{p(p-1)})^p,
\]
a relation of degree \(2p^2(p-1)\). Applying \(P^{p^2(p-1)-1}\) to this, we get

\[P^{p^2-2}(x_1^p)(x_2^2)^{p(p-1)} + (x_1^p)^p P^{p^2-1(p-1)-1}((x_2^2)^{p-1}) = 0.
\]

Therefore \(P^{p^2-2}(x_1^p)\) is divisible by \((x_1^p)^p\), and hence it is zero. We work downwards in degree by induction. Suppose we have shown that \(P^{p^2-i}(x_1^p), \ldots, P^{p^2-2}(x_1^p)\) are all zero. Then applying \(P^{p^2(p-1)-i}\) to the above relation, we get

\[P^{p^2-i-1}(x_1^p)(x_2^2)^{p(p-1)} + (x_1^p)^p P^{p^2-1(p-1)-1}((x_2^2)^{p-1}) = 0.
\]

Therefore \(P^{p^2-i-1}(x_1^p)\) is divisible by \((x_1^p)^p\), and hence it is zero. Once we reach \(i = p - 1\), we have completed the proof that \(T(x_1^p) = (x_1^p)^p\).

Next, we suppose that we have already shown that \(T(x_n^{a_n-1}) = (x_n^{a_n-1})^p\). We have the relation

\[(x_{n-1}^{p-1})^n - (x_n^{a_n}) = (x_{n-1}^{p-1} - x_n^{a_n})^n \]
in \(\mathcal{E}_\infty(p)\). Applying \(T\), we get

\[(x_{n-1}^{p-1})^n - (x_n^{a_n})^n T(x_n^{a_n}) = (T(x_{n-1}^{p-1} - x_n^{a_n}))^n.
\]

The right hand side is zero in degrees not divisible by \(p^n-1\). So the only possibilities for non-zero Steenrod operations on \(x_n^{a_n}\) are \(P^{p^n-i}P^{p^n-i-1}\) for \(0 \leq i \leq p-1\).

We also have the relation

\[(x_n^{a_n})(x_{n+1}^{a_n+1})^n - (x_n x_{n+1}^{a_n+1-p})^n \]
in \(\mathcal{E}_\infty(p)\). Applying \(T\), we get

\[T(x_n^{a_n})(T(x_{n+1}^{a_n+1}))^n - (T(x_n x_{n+1}^{a_n+1-p}))^n.
\]

The right hand side is zero in degrees not divisible by \(p^n\). We show by induction on \(i\) that \(P^{p^n-i}P^{p^n-i-1}(x_n^{a_n}) = 0\) for \(0 \leq i \leq p-1\). If we have proved this for smaller values
of \( i \), then we get
\[
\mathcal{P}^{p^i - ip^{i-1} - 1}(x_n^{p^i}) (x_{n+1}^{p^{i+1}}) \mathcal{P}^{p^{i+1} - p} + (x_n^{p^i})^p \mathcal{P}^{(p^{i+1} - 1)(p^{i-1} - 1) - ip^{i-1} - 1}((x_{n+1}^{p^{i+1}}) p^{i-1}) = 0.
\]
So \( \mathcal{P}^{p^i - ip^{i-1} - 1}(x_n^{p^i}) \) is divisible by \( (x_n^{p^i})^p \), and is hence zero. This completes the proof that \( T(x_n^{p^i}) = (x_n^{p^i})^p \).

\[\square\]

6. The Koszul complex

We assume that \( p^n > 3 \). We will consider the symmetric tensor categories \( \text{Ver}_{p^n} \) over \( k \) defined in \([13]\). Namely, let \( T_p := \text{Tilt}SL_2(k) \) be the category of tilting modules over \( SL_2(k) \). Let \( T_i \in T_p \) be the tilting module for \( SL_2(k) \) with highest weight \( i \). The module \( T_{i,p-1} \) generates a tensor ideal \( T_i \subset T_{p,n} \) spanned by \( T_i \) for \( i \geq p^n - 1 \). We define \( T_{n,p} \) to be the quotient category \( T_p/T_i \). Then \( \text{Ver}_{p^n} \) is the abelian envelope of \( T_{n,p} \), i.e., the unique abelian symmetric tensor category containing \( T_{n,p} \) such that faithful symmetric monoidal functors out of \( T_{n,p} \) into abelian symmetric tensor categories uniquely factor through \( \text{Ver}_{p^n} \).

More concretely, \( \text{Ver}_{p^n} \) is the category \( R-\text{mod} \) of finite dimensional modules over the algebra \( R := \text{End}(\oplus_{i=p^n-1}^{p^n-2} T_i) \) realized as the homotopy category of projective resolutions \( P^{\bullet} \) in \( R-\text{mod} \) with the usual tensor product. Namely, it turns out that the tensor product of resolutions is a resolution (i.e., acyclic in negative degrees), there is a unit object, and the corresponding tensor category is rigid (with \( T_{n}^{\bullet} \cong T_{n}^{\star} \)) and equipped with a natural faithful symmetric monoidal functor \( \widetilde{T}_{n,p} \rightarrow \text{Ver}_{p^n} \) given by \( P^{\bullet} \mapsto \text{H}^0(P^{\bullet}) \).

Let \( T_i \) be the image of \( T_i \) in \( \text{Ver}_{p^n} \). In particular, we let \( V = T_1 \) be the image of the 2-dimensional irreducible representation \( T_1 \) of \( SL_2(k) \), also denoted by \( V \), (these of course depend on \( n \) but to lighten the notation we do not indicate this explicitly).

Note that in both categories \( \Lambda^2 V \) is the unit object and \( \Lambda^i V = 0 \) for \( i \geq 3 \). Recall \([14]\) that we have the Koszul complex \( K^{\bullet} := S^{\bullet} V \otimes \Lambda V \) in \( \text{Ver}_{p^n} \) (i.e., we use the symmetric power superscript as the cohomological degree). This complex may also be graded by total degree, which is preserved by the differential. So it splits into a direct sum of complexes \( K^{\bullet}_m, m \geq 0: \)

\[
0 \rightarrow S^{m-2} V \rightarrow S^{m-1} V \otimes V \rightarrow S^m V \rightarrow 0
\]

(where we agree that \( S^j V = 0 \) if \( j < 0 \)). The map \( S^{m-1} V \otimes V \rightarrow S^m V \) in this complex is induced by the multiplication map of the algebra \( SV \), so it is surjective when \( m \neq 0 \).

**Proposition 6.1.** If \( 1 \leq m \leq p^n - 2 \) then the complex \( K^{\bullet}_m \) is exact.

**Proof.** It suffices to show that for any \( i \in [p^n-1, p^n-2] \) the sequence

\[\text{It does not matter whether to take endomorphisms in } T_p \text{ or } T_{n,p} - \text{the corresponding natural map of endomorphism rings is an isomorphism.}\]
0 → Hom_{Ver_p}(S^m V, T_i) → Hom_{Ver_p}(S^{m-1} V ⊗ V, T_i) → Hom_{Ver_p}(S^{m-2} V, T_i) → 0

(6.1)
is exact. This sequence can be rewritten as

0 → Hom_{Ver_p}(V^{⊗m}, T_i)^{S_m} → Hom_{Ver_p}(V^{⊗m}, T_i)^{S_{m-1}} → Hom_{Ver_p}(V^{⊗m-2}, T_i)^{S_{m-2}} → 0.

(6.2)

By Theorem 4.2 of [3], sequence (6.2) can be rewritten as

0 → Hom_{τ_{n,p}}(V^{⊗m}, T_i)^{S_m} → Hom_{τ_{n,p}}(V^{⊗m}, T_i)^{S_{m-1}} → Hom_{τ_{n,p}}(V^{⊗m-2}, T_i)^{S_{m-2}} → 0.

(6.3)

Now, if 1 ≤ m ≤ p^n − 2, then by Proposition 3.5 of [3], sequence (6.3) can be rewritten as follows:

0 → Hom_{τ_{p,p}}(V^{⊗m}, T_i)^{S_m} → Hom_{τ_{p,p}}(V^{⊗m}, T_i)^{S_{m-1}} → Hom_{τ_{p,p}}(V^{⊗m-2}, T_i)^{S_{m-2}} → 0,

(6.4)

where V now denotes the 2-dimensional irreducible representation of SL_2(k). The Hom spaces in this sequence are just Homs between representations of SL_2(k). Thus sequence (6.4) can be written as

0 → Hom_{SL_2(k)}(S^m V, T_i) → Hom_{SL_2(k)}(S^{m-1} V ⊗ V, T_i) → Hom_{SL_2(k)}(S^{m-2} V, T_i) → 0.

(6.5)

We will now use the following lemma.

Lemma 6.2. For m ≤ p^n − 1 one has Ext^1_{SL_2(k)}(S^m V, T_i) = 0.

Proof. Since i ≥ p^n − 1, it suffices to show that for any j,

\[ \text{Ext}^1_{SL_2(k)}(S^m V, \text{St}_{n-1} ⊗ T_j) = 0, \]

where \( \text{St}_{n-1} := T_{p^n-1-1} \) is the \((n-1)\)st Steinberg module (note that it is self-dual). We have

\[ \text{Ext}^1_{SL_2(k)}(S^m V, \text{St}_{n-1} ⊗ T_j) = \text{Ext}^1_{SL_2(k)}(S^m V ⊗ \text{St}_{n-1}, T_j). \]

By [3], Lemma 3.3, \( S^m V ⊗ \text{St}_{n-1} \) has a filtration whose successive quotients are tilting modules. Thus, since \( \text{Ext}^1(T_i, T_j) = 0 \), \( S^m V ⊗ \text{St}_{n-1} \) is a direct sum of \( T_i \), i.e., a tilting module. This implies the statement, using again that \( \text{Ext}^1(T_i, T_j) = 0 \).

Now the exactness of (6.5) follows from the fact that the sequence of \( SL_2(k) \)-representations

0 → S^{m-2} V → S^{m-1} V ⊗ V → S^m V → 0

is exact (being a homogeneous part of the ordinary Koszul complex) and Lemma 6.2. This completes the proof of Proposition 6.1.\( \square \)

Let \( q = e^{πi/p^n} \).

Corollary 6.3. (i) For m ≤ p^n − 2 we have

\[ \text{FPdim}(S^m V) = [m + 1]_q := \frac{q^{m+1} - q^{-m-1}}{q - q^{-1}} ∈ \mathbb{R} \]

and \( \dim(S^m V) = m + 1 ∈ \mathbb{R} \).
Remark 6.5. In particular, this implies that

Also we clearly have

Thus the statement follows by induction, using that \( S^0V = 1 \), \( S^1V = V \).

Proof. (i) It follows from Proposition 6.1 that

Thus the \( p > 2 \) and exactly one (the unit) for \( S \).

(ii) This follows from (i), using \( \psi \).

Recall (\ref{prop:FPdim}) that \( \text{Ver}_{p^n} \) has exactly two invertible objects up to isomorphism for \( p > 2 \) and exactly one (the unit) for \( p = 2 \). For \( p > 2 \) let \( \psi \) be the unique non-trivial invertible object of \( \text{Ver}_{p^n} \) (generating the category of supervector spaces). If \( p = 2 \), we agree that \( \psi = 1 \).

Corollary 6.4. (i) \( S^{p^n - 2}V = \psi \).

(ii) \( S^{p^n - 3}V = V \otimes \psi \).

(iii) \( S^1V = 0 \) for all \( j > p^n - 2 \).

Proof. (i) By Corollary 6.3 we have \( \text{FPdim}(S^{p^n - 2}V) = 1 \), which implies that \( S^{p^n - 2}V \) is invertible (see \( \ref{prop:FPdim} \), Ex. 4.5.9). For \( p = 2 \) this implies that \( S^{2^n - 2}V = 1 \), and for \( p > 2 \) that \( S^{p^n - 2}V = \psi \) (as \( S^{p^n - 2}V \in \text{Ver}_{p^n} \), since \( p^n - 2 \) is odd).

(ii) Similarly, by Corollary 6.3 \( \text{FPdim}(S^{p^n - 3}V) = q + q^{-1} < 2 \), so \( S^{p^n - 3}V \) is simple. But by the results of \( \ref{prop:FPdim} \), the only object \( X \in \text{Ver}_{p^n} \) of Frobenius-Perron dimension \( q + q^{-1} \) such that \( \psi \) is a quotient of \( X \otimes V \) is \( X \cong V \otimes \psi \). Thus \( S^{p^n - 3}V \cong V \otimes \psi \).

(iii) The map \( S^{p^n - 3}V \rightarrow S^{p^n - 2}V \otimes V \) corresponds to the surjective map \( S^{p^n - 3}V \otimes V \rightarrow S^{p^n - 2}V \), which is nonzero by (i). Hence by (ii) it is an isomorphism. Thus the morphism \( S^{p^n - 2}V \otimes V \rightarrow S^{p^n - 1}V \) must be 0 (as \( V^* \) is a complex). But this map is surjective, so \( S^{p^n - 1}V = 0 \). This implies the statement.

Remark 6.5. In particular, this implies that

Also we clearly have

Thus the \( p \)-adic dimensions of \( V \) defined in \( \ref{prop:FPdim} \) are as follows:

We also obtain
Corollary 6.6.  (i) The Koszul complex $K^\bullet$ is exact in all degrees except 0 and $p^n - 2$. Moreover $H^0(K^\bullet) = 1$ sitting in total degree 0 and $H^{p^n - 2}(K^\bullet) = \psi$ sitting in total degree $p^n$.

(ii) The algebra $SV$ is $(p^n - 2, 2)$-Koszul and the algebra $\Lambda V$ is $(2, p^n - 2)$-Koszul in the sense of Brenner, Butler and King [6] (see [11], Definition 5.3).

Corollary 6.7. The algebra $SV$ in $\text{Ver}_{p^n}$ is Frobenius.

Proof. Assume the contrary, and let $k$ be the largest integer such that the left kernel of the pairing $S^kV \otimes S^{p^n - 2 - k}V \to S^{p^n - 2}V = \psi$ is nonzero. Denote this kernel by $N$. Then the composite map $N \otimes V \to S^kV \otimes V \to S^{k+1}V$ is zero. Thus the composite map $N \to S^kV \to S^{k+1}V \otimes V$ is zero. But by Proposition 6.1, the map $S^kV \to S^{k+1}V \otimes V$ is injective. Thus $N = 0$, a contradiction.

Remark 6.8. Recall (3, Subsection 4.4) that the category $\text{Ver}_{p^n} = \text{Ver}_{p^n}(K)$ lifts to a semisimple braided (non-symmetric) category $\text{Ver}_{p^n}(\mathbf{K})$ over a field $\mathbf{K}$ of characteristic zero, corresponding to the quantum group $SL_2^q$ where $q$ is a primitive root of unity of order $2p^n$ in $\mathbf{K}$. In $\text{Ver}_{p^n}(\mathbf{K})$ we have the quantum symmetric algebra $S_{-q}V$, which is a lift of $SV$ over $\mathbf{K}$ and is also Frobenius $(p^n - 2, 2)$-almost Koszul (see [11], Subsection 5.5). In particular, we have the quantum Koszul complex $S^\bullet V \otimes \Lambda_{-q}V$ in $\text{Ver}_{p^n}(\mathbf{K})$ which is a flat deformation of the Koszul complex $S^\bullet V \otimes \Lambda V$ and has the cohomology as described in Corollary 6.6.

Corollary 6.6 allows us to construct an injective resolution $Q_\bullet$:

$$Q_0 \to Q_1 \to Q_2 \to \cdots$$

of the augmentation $\Lambda V$-module $1$ by free $\Lambda V$-modules, which is periodic with period $2^n - 1$ for $p = 2$ and antiperiodic with period $p^n - 1$ for $p > 2$ (where antiperiodic means that it multiplies by $\psi$ when shifted by this period; in particular, it is $2(p^n - 1)$-periodic). Namely, for $0 \leq i \leq p^n - 2$ we have $Q_{2r(p^n - 1) + m} = S^mV \otimes \Lambda V$, and $Q_{(2r+1)(p^n - 1) + m} = S^mV \otimes \psi \otimes \Lambda V$.

Remark 6.9. If $p^n = 2$ (i.e., $p = 2, n = 1$) then $V = 0$, so the Koszul complex reduces to $1$ sitting in degree 0 and hence does not fit the above general pattern; but we will not consider this trivial case. If $p^n = 3$ (i.e., $p = 3, n = 1$) then $V = \psi$, so $\Lambda^2 \psi \neq 0$ and hence the Koszul complex $S^\bullet V \otimes \Lambda V$ still does not fit the general pattern (in fact, in this case $\text{Ver}_{p^n} = \text{Sup vec}$, so the Koszul complex is exact except in degree 0). However, now this can be remedied by a slight modification of the definition. Namely, let $\Lambda_1 V$ be the quotient of $\Lambda V$ by $\Lambda^3 V$ (forcing the desired equality $\Lambda^2 V = 0$). Then we have the truncated Koszul complex $K^\bullet_{\psi} := S^\bullet V \otimes \Lambda_1 V$ which is easily shown to have the same properties as the usual Koszul complex $K^\bullet$ for $p^n = 3$. Thus if $p^n = 3$ then, abusing terminology and notation, by $\Lambda V$ we will mean $\Lambda_1 V$, and by the Koszul complex the truncated Koszul complex; then the above results will also apply to this case.

As an application let us compute the multiplicities of the unit object in the symmetric powers of $V$ for $p = 2$.

Proposition 6.10. If $p = 2$ then $[S^mV : 1] = 0$ if $m$ is odd and $[S^mV : 1] = 1$ if $m$ is even. Thus $[SV : 1] = 2^{n-1}$.
Proof. Note that for $X \in \text{Ver}_{2n}$ the multiplicity $[X : 1]$ in $X$ is $\text{Tr}(\text{FPdim}(X))/2^{n-1}$ (the trace of the algebraic number in the field $\mathbb{Q}(q + q^{-1})$) where $q := e^{\pi i/2^n}$: this follows since by \[2\], $\text{TrFPdim}(X) = 0$ for any nontrivial simple $X \in \text{Ver}_{2n+1}$. So we have

$$
\sum_m [S^m V : 1] z^m = \frac{1}{2^{n-1}} \text{Tr} \left( \frac{1 + z^{2^n}}{(1 - qz)(1 - q^{-1}z)} \right).
$$

Thus the result follows from the following lemma.

**Lemma 6.11.** $\frac{1}{2^{n-1}} \text{Tr} \left( \frac{1 + z^{2^n}}{(1 - qz)(1 - q^{-1}z)} \right) = \frac{1 - z^{2^n}}{1 - z^{2}} = \sum_{j=0}^{2^{n-1}-1} z^{2j}$.

Proof. We have

$$
\frac{1}{2^{n-1}} \text{Tr} \left( \frac{1 + z^{2^n}}{(1 - qz)(1 - q^{-1}z)} \right) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} \frac{1 + z^{2^n}}{(1 - q^{2k-1}z)(1 - q^{-2k+1}z)}.
$$

This is the unique polynomial $h(z) \in \mathbb{Q}[z]$ of degree $2^n - 2$ such that $h(q^j) = \frac{2}{1 - q^{2j}}$ for any odd number $j$. But the polynomial $\frac{1 - z^{2^n}}{1 - z^{2}}$ satisfies these conditions, hence the result.

This completes the proof of Proposition \[6.10\].

**Remark 6.12.** Another proof of Proposition \[6.10\] is obtained by applying Proposition 4.16 and Theorem 4.42 of \[3\]. Namely, $[S^i V : 1]$ is an entry of the decomposition matrix of $\text{Ver}_{2n}$, so it is 0 if $i$ is odd and 1 if $i$ is even. This follows since the descendants of the number $2^n - 1$ are exactly all the odd numbers between 1 and $2^n - 1$.

### 7. Ext computations

#### 7.1. Ext computations for $p = 2$

Consider now the case $p = 2$. In this case, we can use the resolution $Q_\bullet$ to give the following recursive procedure of computation of the additive structure of the cohomology $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(1, X)$ (for indecomposable $X$).

We will denote the generating object of $\text{Ver}_{2k+1}$ by $X_k$ and recall that $\text{Ver}_{2k+2}^+$ is the category of $\Lambda X_k$-modules in $\text{Ver}_{2k+1}$. Also the resolution $Q_\bullet$ in $\text{Ver}_{2k+1}$ will be denoted by $S^* X_k [y_k+1] \otimes \Lambda X_k$, where $y_k+1$ is a variable of degree $2^{k+1} - 1$ for $k \geq 0$. This is justified by this resolution being periodic with period $2^{k+1} - 1$. Also if $Y^*, Z^*$ are complexes in an abelian category $A$ then by $\text{Ext}^m(Z^*, Y^*)$ we will mean $\text{Hom}(Z^*[m], Y^*)$ with $\text{Hom}$ taken in the derived category $D(A)$. Recall that $\text{Ver}_{2n+1}^+ = \text{Ver}_{2n+1}^+ \oplus \text{Ver}_{2n+1}^-$. If $X \in \text{Ver}_{2n+1}$, then $\text{Ext}^\bullet_{\text{Ver}_{2n+1}^-}(1, X)$ is zero. Thus, it suffices to compute $\text{Ext}^\bullet_{\text{Ver}_{2n+1}^+}(1, X)$ for $X \in \text{Ver}_{2n+1}^+$. In that case, we
have

$$\operatorname{Ext}^\bullet_{\text{Ver}^{2n+1}}(1, X) \cong \operatorname{Ext}^\bullet_{\Lambda X_{n-1}}(1, X) \cong \operatorname{Ext}^\bullet_{\Lambda X_{n-1}}(1, Q \otimes X)$$

$$\cong \operatorname{Ext}^\bullet_{\Lambda X_{n-1}}(1, S^\bullet X_{n-1}[y_n] \otimes \Lambda X_{n-1} \otimes X)$$

$$\cong \operatorname{Ext}^\bullet_{\text{Ver}^{2n}}(1, S^\bullet X_{n-1}[y_n] \otimes X)$$

$$\cong \operatorname{Ext}^\bullet_{\text{Ver}^{2n}}(1, (S^\bullet X_{n-1}[y_n] \otimes X)^+)$$

where the superscript + means that we are taking the part lying in $\text{Ver}^{2n}_+$, and in the last two expressions $X$ is regarded as an object of $\text{Ver}^{2n}$ using the corresponding forgetful functor $\text{Ver}^{2n}_+ \to \text{Ver}^{2n}$. Here for the penultimate isomorphism we invoked the Shapiro lemma, using that the $\Lambda X_{n-1}$-module $S^k X_{n-1}[y_n] \otimes \Lambda X_{n-1} \otimes X$ is free and therefore coinduced (as $\Lambda X_{n-1}$ is a Frobenius algebra).

Thus we get a recursion expressing of $\operatorname{Ext}^\bullet_{\text{Ver}^{2n}_+}(1, X)$ in terms of $\operatorname{Ext}^\bullet_{\text{Ver}^{2n}_+}(1, X')$.

While this is a good news, unfortunately $X'$ is not an object any more but rather a complex of objects finite in the negative direction. Luckily, the same calculation applies if $X$ is such a complex, i.e., an object of the derived category $D^+(\text{Ver}^{2n}_+)$ of $\text{Ver}^{2n}_+$, which allows us to iterate this construction. Namely, for an object $X \in D^+(\text{Ver}^{2n}_+)$, let

$$E_n(X) := \mathcal{H}om_{\Lambda X_{n-1}}(1, S^\bullet X_{n-1}[y_n] \otimes \Lambda X_{n-1} \otimes X)^+ = (S^\bullet X_{n-1}[y_n] \otimes X)^+$$

(the internal Hom taken in the category $\text{Ver}^{2n}$). This gives an additive functor

$$E_n : D^+(\text{Ver}^{2n}_+) \to D^+(\text{Ver}^{2n}_+).$$

**Lemma 7.1.** If $X \in \text{Ver}^{2n}_+$ (i.e., a trivial $\Lambda X_{n-1}$-module) then the differential in the complex $E_n(X)$ is zero.

**Proof.** It is easy to see that for a finite dimensional vector space $V$ over $k$, the differential on $\mathcal{H}om_{\Lambda V}(k, S^V \otimes \Lambda V) = S^V$ induced by the Koszul differential on $S^V \otimes \Lambda V$ is zero. The lemma is a straightforward generalization of this fact.

**Corollary 7.2.** Suppose $X \in \text{Ver}^{2n}_+$. Then we have an isomorphism

$$\operatorname{Ext}^\bullet_{\text{Ver}^{2n}_+}(1, X) \cong \bigoplus_{i \geq 0} \operatorname{Ext}^\bullet_{\text{Ver}^{2n}_+}(1, S^i X_{n-1}[y_n] \otimes X)$$

$$\cong \bigoplus_{i \geq 0} \operatorname{Ext}^\bullet_{\text{Ver}^{2n}_+}(1, (S^i X_{n-1}[y_n] \otimes X)^+).$$

This isomorphism maps the grading induced by the grading on $\Lambda X_{n-1}$ to the grading defined by $\deg(X_{n-1}) = 1$, $\deg(y_n) = 2^n - 1$ (i.e., it coincides with the cohomological grading).

**Proof.** Follows immediately from Lemma 7.1.

**Remark 7.3.** Corollary 7.2 does not give a recursion to compute the Ext groups, since the object $(S^i X_{n-1} \otimes X)^+$ may not belong to $\text{Ver}^{2n}_+$ (i.e., it may carry a nontrivial action of $\Lambda X_{n-2}$). However, it has some useful consequences given below.
Proposition 7.6. For any 

Note that we have \( 1 ∈ y \) by this proposition.

Proposition 7.7. For any 
The proof is by induction in \( n \).

Corollary 7.5. We have a linear natural isomorphism

\[
\Ext^*_{D^+(\Ver_{2n+1}^+)}(\mathbb{1}, X) ∼ \Ext^*_{D^+(\Ver_{2n}^+)}(\mathbb{1}, E_n(X))
\]

for \( n ≥ 2 \), and

\[
\Ext^*_{D^+(\Ver_{2n}^+)}(\mathbb{1}, X) ∼ \Ext^*_{D^+(\Vec)}(\mathbb{1}, E_1(X)).
\]

This implies the following corollary. Let

\[ E = E_1 ∘ \cdots ∘ E_n : D^+(\Ver_{2n+1}^+) → D^+(\Vec). \]

Corollary 7.5. We have a linear natural isomorphism

\[
\Ext^*_{D^+(\Ver_{2n+1}^+)}(\mathbb{1}, X) = H^*(E(X)).
\]

The complex of vector spaces \( E(X) \) has the following structure:

\[
E(X) = (S^*X_1 ⊕ \cdots ⊕ (S^*X_{n−2} ⊕ (S^*X_{n−1} ⊕ X)^+) \cdots [y_1, y_2, \ldots, y_n]. \quad (7.2)
\]

and it is easy to see that the differential is linear over \( k[y_1, \ldots, y_n] \), since multiplication by \( y_i \) is induced by the shift in the corresponding periodic resolution. Thus we get

Proposition 7.6. For any \( X ∈ \Ver_{2n+1} \), \( \Ext^*_{\Ver_{2n+1}^+}(\mathbb{1}, X) \) is a graded finitely generated module over \( k[y_1, \ldots, y_n] \).

In particular, we get that

\[
\Ext^*_{\Ver_{2n+1}^+}(\mathbb{1}, \mathbb{1}) = \Ext^*_{\Ver_{2n+1}^+}(\mathbb{1}, \mathbb{1}) = H^*(E(\mathbb{1})),
\]

where

\[
E(\mathbb{1}) = (S^*X_1 ⊕ \cdots ⊕ (S^*X_{n−2} ⊕ (S^*X_{n−1})^+) \cdots [y_1, y_2, \ldots, y_n].
\]

Note that we have \( 1 ∈ E(\mathbb{1}) \) and \( d(1) = 0 \), so we obtain a natural linear map

\[
\phi : k[y_1, \ldots, y_n] → \Ext^*_{\Ver_{2n+1}^+}(\mathbb{1}, \mathbb{1}).
\]

Proposition 7.7. For \( 1 ≤ i ≤ n \) multiplication by \( y_i \) on \( \Ext^*_{\Ver_{2n+1}^+}(\mathbb{1}, X) \) coincides with the cup product with \( \phi(y_i) \). In particular, \( \phi \) is an algebra homomorphism.

Proof. The proof is by induction in \( n \). For \( i < n \) the statement follows from the inductive assumption. For \( i = n \), we see that the cup product with \( \phi(y_n) \) can be realised as the Yoneda product (= concatenation) with the Koszul complex \( K^* \), which represents the class \( \phi(y_n) \) in the Yoneda realization of \( \Ext \). This proves the first statement. The second statement then follows since \( \phi(ab) = (ab) · 1 = a · (b · 1) = a · \phi(b) = \phi(a)\phi(b) \).

□
Proposition 7.8. For $X \in \text{Ver}^+_n$ the natural map
\[
\text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, X)[y_n] \to \text{Ext}^\bullet_{\text{Ver}_{2n+1}}(\mathds{1}, X)
\]
is an injective morphism of $k[y_1, \ldots, y_n]$-modules which is also a morphism of algebras for $X = \mathds{1}$.

Proof. This follows from the isomorphism
\[
\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(\mathds{1}, X) \cong \text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, (S^{\text{even}}X_{n-1} \otimes X)[y_n])
\]
since $\mathds{1}$ is a direct summand of $S^{\text{even}}X_{n-1}$.

Proposition 7.9. Let $U \in \text{Ver}_n$ and $X := U \otimes \Lambda X_{n-1} \in \text{Ver}^+_{2n+1}$ be a free $\Lambda X_{n-1}$-module. Then $y_n$ acts on $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(\mathds{1}, X)$ by zero.

Proof. By the Shapiro lemma we have
\[
\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(\mathds{1}, X) \cong \text{Ext}^\bullet_{\Lambda X_{n-1}}(\mathds{1}, X) \cong \text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, U).
\]
Therefore, the group $G_m$ scaling $X_{n-1}$ acts trivially on $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(\mathds{1}, X)$. So the statement follows, as $y_n$ has degree $2^n - 1$ with respect to this action.

Let $S \subset \{1, \ldots, n-1\}$ and $X_S := \bigotimes_{i \in S} X_i$ be the simple object of $\text{Ver}_n$ attached to $S$ in [2].

Proposition 7.10. (i) If $i \in S$ and $Y \in \text{Ver}^+_{2n+1}$ then multiplication by $y_i$ acts by zero on $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(Y, X_S)$. Hence $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(Y, X_S)$ is a torsion module over $k[y_1, \ldots, y_n]$ unless $S = \emptyset$ (i.e., $X_S = \mathds{1}$).

(ii) The annihilator of $\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(X_S, X_S)$ in $k[y_1, \ldots, y_n]$ is generated by $y_i$ with $i \in S$.

Proof. (i) The proof is by induction in $n$. The base is clear, so we just have to justify the induction step. We have
\[
\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(Y, X_S) \cong \text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, (X_S \otimes S^\bullet X_{n-1} \otimes Y^*)^+[y_n]).
\]
If $n - 1 \notin S$ then $X_S \in \text{Ver}^+_2$, so this can be written as
\[
\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(Y, X_S) \cong \text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, (Y \otimes S^\bullet X_{n-1}[y_n]^*)^+, X_S)
\]
and the statement follows from the inductive assumption. On the other hand, if $n - 1 \in S$ then setting $S' = S \setminus \{n - 1\}$, we have $X_S = X_{S'} \otimes X_{n-1}$. So we get
\[
\text{Ext}^\bullet_{\text{Ver}_{2n+1}}(Y, X_S) \cong \text{Ext}^\bullet_{\text{Ver}_n}(\mathds{1}, X_{S'} \otimes X_{n-1} \bigotimes (S^\bullet X_{n-1} \otimes Y^*)^- [y_n]),
\]
where the superscript minus sign means that we are taking the part lying in $\text{Ver}^-_{2n}$. But $(S^\bullet X_{n-1} \otimes Y^*)^- = X_{n-1} \otimes W^\bullet$ for some $W^\bullet \in \text{Ver}^+_2$, and $X_{n-1} \otimes X_{n-1} = \Lambda X_{n-2}$ (and some differential on the tensor product whose exact form is not important for
Remark 7.14. As stated in Conjecture 1.3, we expect that moreover $\Ext_{\Ver_{2n+1}^{\ast}}(Y, X_S) \cong \Ext_{\Ver_{2n}^{\ast}}(1, X_{S'} \otimes X_{n-1} \otimes X_{n-1} \otimes W^\ast[y_n])$

\[ \cong \Ext_{\Lambda X_{n-2}}(1, X_{S'} \otimes \Lambda X_{n-2} \otimes W^\ast[y_n]) \]

\[ \cong \Ext_{\Ver_{2n-1}^{\ast}}(W^\ast[y_n]^\ast, X_{S'}) \].

So by Proposition 7.13, the element $y_{n-1}$ acts on this space by zero, and the statement again follows from the inductive assumption.

(ii) By (i) the annihilator is at least as big as claimed, and we only need to show that it is not bigger. This is shown again by induction in $n$. The base is again easy so we only need to do the induction step. If $n - 1 \notin S$ then by Proposition 7.8 we have an inclusion $\Ext_{\Ver_{2n}^{\ast}}(X_S, X_S)[y_n] \rightarrow \Ext_{\Ver_{2n+1}^{\ast}}(X_S, X_S)$, so the result follows from the inductive assumption for $n - 1$. On the other hand, if $n - 1 \in S$ then $X_S = X_{S'} \otimes X_{n-1}$ so

\[ \Ext_{\Ver_{2n+1}^{\ast}}(X_S, X_S) \cong \Ext_{\Lambda X_{n-1}}(X_{S'}, X_{S'} \otimes \Lambda X_{n-2}) \]

\[ \cong \Ext_{\Lambda X_{n-2}}(X_{S'}, X_{S'} \otimes \Lambda X_{n-2} \otimes S^{\text{even}} X_{n-1}) \]

\[ \cong \Ext_{\Ver_{n}^{\ast}}(X_{S'}, X_{S'} \otimes S^{\text{even}} X_{n-1}) \],

which contains $\Ext_{\Ver_{2n-1}^{\ast}}(X_S, X_S) = \Ext_{\Ver_{n}^{\ast}}(X_S, X_S)$ as a direct summand as $S^{\text{even}} X_{n-1}$ contains $1$ as a direct summand. Thus the result again follows from the inductive assumption (this time for $n - 2$).

Corollary 7.11. The rank $r_n$ of the module $\Ext_{\Ver_{2n+1}^{\ast}}(1, 1)$ over $k[y_1, \ldots, y_n]$ satisfies the equality $r_n = r_{n-1} [SX_{n-1} : 1]$.

Proof. In view of the isomorphism (7.10) applied to $X = 1$, this follows from Proposition 7.10(i). \qed

Corollary 7.12. We have $r_n = 2^{\frac{n(n-1)}{2}}$.

Proof. This follows from Corollary 7.11 and Proposition 6.10 using that $r_1 = 1$. \qed

Recall that the algebra $\Ext_{\Ver_{2n+1}^{\ast}}(1, 1)$ has a $\mathbb{Z}$-grading coming from the grading on $\Lambda X_{n-1}$, where $y_n$ has degree $2^n - 1$. Define the field $F_n := k(y_1, \ldots, y_{n-1})$, and let $r_n(v)$ be the Poincaré polynomial of $\Ext_{\Ver_{2n+1}^{\ast}}(1, 1) \otimes_k [y_1, \ldots, y_{n-1}] F_n$ as a module over the algebra $F_n[y_n]$. Then the above arguments yield

Corollary 7.13. $\Ext_{\Ver_{2n+1}^{\ast}}(1, 1) \otimes_k [y_1, \ldots, y_{n-1}] F_n$ is a free $F_n[y_n]$-module, and for $n \geq 2$

\[ r_n(v) = 2^{|(n-1)(n-2)|/2} \frac{1 - v^{2n}}{1 - v^2} = 2^{(n-1)(n-2)/2} \sum_{j=0}^{2^{n-1}-1} v^{2j} \]

This agrees with Conjecture 1.3. Also the formula $r_n = 2^{\frac{n(n-1)}{2}}$ is now obtained by evaluating $r_n(v)$ at $v = 1$.

Remark 7.14. As stated in Conjecture 1.3, we expect that moreover $\Ext_{\Ver_{2n+1}^{\ast}}(1, 1)$ is a free $k[y_1, \ldots, y_n]$-module (even without localization in $y_1, \ldots, y_{n-1}$).
More generally, for every object $X \in \text{Ver}_{2n+1}^+$ we obtain upper bounds for the Poincaré polynomials of generators of $\text{Ext}_{\text{Ver}_{2n+1}}^*(1, X),$

$$r_n(X, z, v) := \sum_{i,j=0}^{\infty} z^i v^j \dim \left( \text{Ext}_{\text{Ver}_{2n+1}}^*(1, X) / \sum_{k=1}^n \text{Im}(y_k) \right)^{i,j}$$

where $i$ is the cohomological degree and $j$ is the $v$-degree.

**Proposition 7.15.** For $n \geq 2$ we have $r_n(X, z, v) \leq r^*_n(X, z, v),$ in the sense that each coefficient on the left is less than or equal to the corresponding coefficient on the right, where

$$r^*_n(X, z, v) := \frac{1}{2^{n-1}} \text{Tr} \left( \text{FPdim}(X) \frac{1 + (zv)^{2n}}{(1-qzv)(1-q^{-1}zv)} \prod_{j=2}^{n-1} \frac{1 + z^{2^j}}{(1-q^{2^{j-1}}z)(1-q^{-2^{j-1}}z)} \right).$$

In particular, all generators have degree $\leq 2^{n+1} - 2(n + 1)$.

**Proof.** The bound for $r_n(X, z, v)$ follows from the form of $E(X)$ given in (7.2) and formula (6.6) by a direct computation. This implies the bound on the degree of generators, since the degree of $r^*_n$ with respect to $z$ is $2^{n+1} - 2(n + 1)$. \( \square \)

In particular, for $X = 1$ we get

**Corollary 7.16.** $\text{Ext}_{\text{Ver}_{2n+1}}^*(1, 1)$ is a finitely generated module over $k[y_1, \ldots, y_n]$ with Poincaré polynomial of generators $r_n(z, v) \leq r^*_n(z, v),$ where

$$r^*_n(z, v) := \frac{1}{2^{n-1}} \text{Tr} \left( \frac{1 + (zv)^{2n}}{(1-qzv)(1-q^{-1}zv)} \prod_{j=2}^{n-1} \frac{1 + z^{2^j}}{(1-q^{2^{j-1}}z)(1-q^{-2^{j-1}}z)} \right).$$

In particular, all the generators have degree at most $2^{n+1} - 2(n + 1),$ and there is exactly one generator of that degree. Moreover, the Poincaré polynomial of generators is palindromic, i.e., satisfies the equation $P(z) = z^{2^{n+1} - n - 1}P(z^{-1}).$

**Proof.** It only remains to show that the Poincaré polynomial of the generators is palindromic, which follows from the fact that the complex $E(1)$ is self-dual. \( \square \)

**Remark 7.17.** Note that according to Conjecture 1.3 the degree bound of Corollary 7.16 is expected to be sharp: the largest degree of a generator is expected to equal $2^{n+1} - 2(n + 1),$ with exactly one generator in that degree. On the other hand, the bound $r_n \leq r^*_n$ is rather poor: we have $\log_2(r^*_n(1, 1)) \sim n^2$ as $n \to \infty$, while $\log_2(r_n(1, 1)) = \frac{n(n-1)}{2}.$ This is not surprising, as this bound does not take into account the fact that the complex $E(1)$ has a nontrivial differential for $n \geq 3$ and shrinks drastically when we compute its cohomology.

**Example 7.18.** 1. Let $n = 2.$ Then we have $E(1) = (SX_1)^+[y_1, y_2].$ But $S^0X_1 = S^2X_1 = 1,$ $S^1X_1 = X_1,$ and all the other symmetric powers are zero. Thus, $(SX_1)^+ = k \oplus kw,$ where $w$ has cohomological degree 2. Also in this case it is easy to see that the differential in $E(1)$ is zero (so the bound $r^*_2(z, v) = 1 + (zv)^2$ is sharp). Thus $\text{Ext}_{\text{Ver}_{2n}}^*(1, 1)$ is a free $k[y_1, y_2]$ module of rank 2 with generators of degree 0 and 2, which agrees with the result of [2].
2. Let $n = 3$. Let $S^i := S^i X_2$. Then one can show by a direct computation that $S^0 = 1$, $S^1 = X_2$, $S^2 = [1, [1, X]]$, $S^3 = X_1 \otimes X_2$, $S^4 = [X_1, 1]$, $S^5 = X_2$, $S^6 = 1$, and all the other symmetric powers are zero (where $Y = [Y_1, ..., Y_m]$ means that $Y$ is a uniserial object with composition series $Y_1, ..., Y_m$, with head $Y_1$ and socle $Y_m$).

Thus the Poincaré series of $\text{Ext}_{\text{Ver}_2}^\bullet(1, 1)$ is isomorphic to

$$\text{Ext}^\bullet_{\text{Ver}_2}(1, 1)[0] \oplus \text{Ext}^\bullet_{\text{Ver}_2}(1, [1, X])[2] \oplus \text{Ext}^\bullet_{\text{Ver}_2}(1, [X, 1])[4] \oplus \text{Ext}^\bullet_{\text{Ver}_2}(1, 1)[6],$$

where $X = X_1$, and the numbers in square brackets are degree shifts. Now, consider the portion of the long exact sequence

$$\text{Hom}(1, X) \to \text{Ext}^1(1, 1) \to \text{Ext}^1(1, [X, 1]) \to \text{Ext}^1(1, X) \to \text{Ext}^2(1, 1),$$

where $\text{Ext}$ groups are taken in $\text{Ver}_2$. It was shown in [2] that the Poincaré series of $\text{Ext}^\bullet(1, [X, 1])$ is $\frac{1}{1-z}$. Also we have $\dim \text{Ext}^1(1, [X, 1]) \geq 2$ since we have two different nontrivial extensions of $1$ by $[X, 1]$, namely $[1 \otimes X, 1]$ and $[X, 1, X]$ (both indecomposable quotients of the projective cover of $1$ in $\text{Ver}_2$). Thus the dimension of $\text{Ext}^1(1, [X, 1])$ is two, and the sequence (7.3) looks like $0 \to k \to k^2 \to k \to k$.

This implies that the last map in this sequence (the connecting homomorphism $\text{Ext}^1(1, X) \to \text{Ext}^2(1, 1)$) is zero. Since the map $\text{Ext}^\bullet(1, X) \to \text{Ext}^{\bullet+1}(1, 1)$ is linear over $k[y_2]$, we see that this map is zero in all degrees (as $\text{Ext}^\bullet(1, X)$ is a free $k[y_2]$-module on one generator in degree 1). Thus, $\text{Ext}^\bullet(1, [X, 1]) \cong \text{Ext}^\bullet(1, X) \oplus \text{Ext}^\bullet(1, 1)$, so the Poincaré series of $\text{Ext}^\bullet(1, [X, 1])$ is $\frac{1}{1-z}$.

Now, the object $[X, 1, 1, X]$ is the projective cover of $X$. This implies that

$$\text{Ext}^\bullet(1, [1, X]) \cong \text{Ext}^{\bullet+1}(1, [X, 1]).$$

Thus the Poincaré series of $\text{Ext}^\bullet(1, [1, X])$ is $\frac{1}{1-z}$.

Altogether we obtain that the Poincaré series of $\text{Ext}^\bullet_{\text{Ver}_2}(1, 1)$ is given by the formula

$$h(z, v) = \frac{(1 + (vz)^6)(1 + z^2) + (vz)^2(z + z^2) + (vz)^4(1 + z)}{(1 - z)(1 - z^3)(1 - (vz)^2)},$$

$$= \frac{1 + z^2 + v^2z^3 + (v^2 + v^4)z^4 + v^4z^5 + v^6z^6 + v^6z^8}{(1 - z)(1 - z^3)(1 - (vz)^2)}.$$

One can check directly that $\text{Ext}^\bullet_{\text{Ver}_2}(1, 1)$ is a free module over $k[y_1, y_2, y_3]$. Thus the Poincaré polynomial of its generators is

$$r_3(z, v) = 1 + z^2 + v^2z^3 + (v^2 + v^4)z^4 + v^4z^5 + v^6z^6 + v^6z^8.$$

On the other hand, it is easy to compute that

$$r_3^*(z, v) = 1 + (1 + v^2)z^2 + 2v^2z^3 + (v^2 + v^4)z^4 + 2v^4z^5 + (v^4 + v^6)z^6 + v^6z^8,$$

$$= r_3(z, v) + v^2(z^2 + z^3) + v^4(z^5 + z^6).$$

This means that the differential in the complex $E(1)/(y_1, y_2, y_3)$ acts as a rank 1 operator between degrees $2 \to 3$ and $5 \to 6$ and otherwise acts by zero. In other words, when computing the cohomology of this complex, we kill two elements of cohomological degrees 2, 3 in $v$-degree 2 and two elements of cohomological degrees 5, 6 in $v$-degree 4.
It is instructive to write down the complex $E(1)$ explicitly. We have

$$E(1) = M^+[y_1, y_2, y_3], \quad M := S \cdot X_1 \otimes (S \cdot X_2)^+.$$  

The components of $M$ are as follows (with $X := X_1$):

$$M^0 = 1, \quad M^1 = X, \quad M^2 = [1, \cdot X], \quad M^3 = X \otimes [1, \cdot X] = [X \cdot 1, 1],$$

$$M^4 = [1, X] \oplus [X, 1], \quad M^5 = X \otimes [1, X] = [1, 1, X],$$

$$M^6 = 1 \oplus [X, 1], \quad M^7 = X, \quad M^8 = 1.$$  

Thus, $E(1)$ has the following components (as $\Lambda$-modules):

$$E^0 = 1, \quad E^1 = 0, \quad E^2 = 1 \oplus 1, \quad E^3 = [1, 1], \quad E^4 = 1 \oplus 1,$$

$$E^5 = [1, 1], \quad E^6 = 1 \oplus 1, \quad E^7 = 0, \quad E^8 = 1.$$  

The differential maps $E^2 = 1 \oplus 1 \rightarrow E^3 = [1, 1], E^5 = [1, 1] \rightarrow E^6 = 1 \oplus 1$, both by rank 1 operators, and is zero in other degrees.

### 7.2. Ext computations for $p > 2$

In this section we would like to generalise some of the results of the previous section to the case $p > 2$. The constructions and formulas are very similar to the case $p = 2$ but not exactly the same due to presence of the invertible object $\psi$ and some other differences, so we chose to repeat them.

As in the case $p = 2$, we can use the resolution $Q_*$ to give the following recursive procedure for computation of the additive structure of the cohomology $\text{Ext}^*_{\text{Ver}_{p^n+1}}(1, X)$ (for indecomposable $X$).

For $k \geq 1$ we will denote the generating object of $\text{Ver}_{p^k}$ by $X_{k-1}$ and recall (3.14) that the principal block $\text{Ver}_{p^{k+1}}^0$ of $\text{Ver}_{p^{k+1}}$ is naturally equivalent to the category of $\Lambda X_{k-1}$-modules in $\text{Ver}_{p^k}$. Let us denote this equivalence by $F$; i.e., for an object $X \in \text{Ver}_{p^{k+1}}^0$ we denote the corresponding $\Lambda X_{k-1}$-module by $FX$.

In the Yoneda realization of Ext, the Koszul complex $K_* = S^* \cdot X_{k-1} \otimes \Lambda X_{k-1}$ represents a class $\tau_k \in \text{Ext}^{p^k-1}_{\Lambda X_{k-1}}(1, \psi)$, and the class $y_k := \tau_k^p$ of degree $2(p^k - 1)$ is represented by the concatenation of $S^* \cdot X_{k-1} \otimes \Lambda X_{k-1}$ with $S^* \cdot X_{k-1} \otimes \Lambda X_{k-1} \otimes \psi$, which we will denote by $S^* \cdot X_{k-1} \otimes \Lambda X_{k-1} \otimes S^* \cdot \psi_{k-1}$, where $\psi_{k-1}$ is $\psi$ sitting in degree $p^k - 1$. Thus $Q_* = S^* \cdot X_{k-1}[y_k] \otimes \Lambda X_{k-1} \otimes S^* \cdot \psi_{k-1}$.

If $X \in \text{Ver}_{p^n}$ but $X \notin \text{Ver}_{p^n}^0$ then we have $\text{Ext}^*_{\text{Ver}_{p^n}}(1, X) = 0$. So, it suffices to compute $\text{Ext}^*_{\text{Ver}_{p^n+1}}(1, X)$ for $X \in \text{Ver}_{p^n+1}^0$. In that case, we have

$$\text{Ext}^*_{\text{Ver}_{p^n+1}}(1, X) \cong \text{Ext}^*_{\Lambda X_{n-1}}(1, FX) \cong \text{Ext}^*_{\Lambda X_{n-1}}(1, Q_\bullet \otimes FX)$$

$$\cong \text{Ext}^*_{\Lambda X_{n-1}}(1, S^* \cdot X_{n-1}[y_n] \otimes \Lambda X_{n-1} \otimes S^* \cdot \psi_{n-1} \otimes FX)$$

$$\cong \text{Ext}^*_{\text{Ver}_{p^n}}(1, S^* \cdot X_{n-1}[y_n] \otimes S^* \cdot \psi_{n-1} \otimes FX)$$

$$\cong \text{Ext}^*_{\text{Ver}_{p^n}}(1, (S^* \cdot X_{n-1}[y_n] \otimes S^* \cdot \psi_{n-1} \otimes FX)^0),$$

where the superscript zero means that we are taking the part lying in $\text{Ver}_{p^n}^0$, and in the last two expressions $FX$ is regarded as an object of $\text{Ver}_{p^n}$ using the corresponding forgetful functor $\Lambda X_{n-1}\text{-mod} \rightarrow \text{Ver}_{p^n}$ forgetting the structure of a $\Lambda X_{n-1}$-module.
The same calculation applies if $X$ is a complex, i.e., an object of the derived category $D^+(\text{Ver}_p^{0,n+1})$ of $\text{Ver}_p^{0,n+1}$. Namely, for an object $X \in D^+(\text{Ver}_p^{0,n+1})$, let
\[
E_n(X) := \text{Hom}_{\Lambda X_{n-1}}([1, S^*X_{n-1}][y_n] \otimes \Lambda X_{n-1} \otimes S^*\psi_{n-1} \otimes FX)^0 = (S^*X_{n-1}[y_n] \otimes S^*\psi_{n-1} \otimes FX)^0.
\]
This gives an additive functor $E_n : D^+(\text{Ver}_p^{0,n+1}) \to D^+(\text{Ver}_p^{0,n})$.

The following lemma is a straightforward analog of Lemma 7.1.

**Lemma 7.19.** If $X \in \text{Ver}_p^n$ (with trivial action of $\Lambda X_{n-1}$) then the differential in the complex $E_n(X)$ is zero.

**Corollary 7.20.** Suppose $X \in \text{Ver}_p^n$. Then we have an isomorphism
\[
\text{Ext}_{\text{Ver}_p^{0,n+1}}^0([1, X] \cong \bigoplus_{i \geq 0} \text{Ext}_{\text{Ver}_p^n}^i ([1, S^iX_{n-1}[y_n] \otimes S^*\psi_{n-1} \otimes FX)^0.
\]
This isomorphism maps the grading induced by the grading on $\Lambda X_{n-1}$ to the grading defined by $\deg(X_{n-1}) = 1$, $\deg(y_n) = 2p^n - 2$, $\deg(\psi_{n-1}) = p^n - 1$ (i.e., it coincides with the cohomological grading).

As for $p = 2$, Corollary 7.20 does not quite give a recursion to compute the Ext groups, since the object $(S^iX_{n-1} \otimes S^*\psi_{n-1} \otimes FX)^0$ may not belong to $\text{Ver}_p^{0,n-1}$ (i.e., may carry a nontrivial action of $\Lambda X_{n-2}$). However, it has some useful consequences given below.

**Proposition 7.21.** For $n \geq 1$ we have
\[
\text{Ext}_{D^+(\text{Ver}_p^{0,n+1})}^i([1, X] = \text{Ext}_{D^+(\text{Ver}_p^n)}^i ([1, E_n(X)).
\]
This implies the following corollary. Let $E = E_1 \circ \cdots \circ E_n : \text{Ver}_p^{0,n+1} \to \text{Ver}_p^0 = \text{Vec}.$

**Corollary 7.22.** We have a linear isomorphism
\[
\text{Ext}_{D^+(\text{Ver}_p^{0,n+1})}^i([1, X] \cong H^i(E(X)).
\]

The complex of vector spaces $E(X)$ has the following structure:
\[
E(X) = (S^*X_0 \otimes S^*\psi_0 \otimes F(S^*X_1 \otimes S^*\psi_1 \otimes \cdots \cdots \otimes F(S^*X_{n-1} \otimes S^*\psi_{n-1} \otimes FX)^0 \cdots)^0[y_1, y_2, \ldots, y_n],
\]
and it is easy to see as in the case $p = 2$ that the differential is linear over $k[y_1, \ldots, y_n]$. Thus we get

**Proposition 7.23.** For any $X \in \text{Ver}_p^{0,n}$, $\text{Ext}_{\text{Ver}_p^{0,n+1}}^i([1, X]$ is a graded finitely generated module over $k[y_1, \ldots, y_n]$. 

In particular, we get that
\[
\text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}) = \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{0}, \mathbb{1}) = H^\bullet(E(\mathbb{1})),
\]
where
\[
E(\mathbb{1}) = (S^\bullet X_0 \otimes S^\bullet \psi_0 \otimes F(S^\bullet X_1 \otimes S^\bullet \psi_1 \otimes \cdots \otimes F(S^\bullet X_{n-1} \otimes S^\bullet \psi_{n-1}^0 \cdots)^0)^0)[y_1, y_2, \ldots, y_n].
\]
Note that we have \( E(\mathbb{1}) \) and \( d(1) = 0 \), so we obtain a natural linear map
\[
\phi: k[y_1, \ldots, y_n] \rightarrow \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, \mathbb{1}).
\]

**Proposition 7.24.** For \( 1 \leq i \leq n \) multiplication by \( y_i \) on \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, X) \) coincides with the cup product with \( \phi(y_i) \). In particular, \( \phi \) is an algebra homomorphism.

**Proof.** The proof is the same as that of Proposition 7.23, using that \( \phi(y_n) \) can be realised as Yoneda product with the complex \( K^\bullet \otimes S^\bullet \psi_{n-1}^0 \), where \( K^\bullet \) is the Koszul complex. The only difference is the presence of the additional factor \( S^\bullet \psi_{n-1}^0 \).

**Proposition 7.25.** For \( X \in \text{Ver}_p^0 \) the natural map
\[
\text{Ext}^\bullet_{\text{Ver}_p^0}(\mathbb{1}, X)[y_n] \rightarrow \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, X)
\]
is an injective morphism of \( k[y_1, \ldots, y_n] \)-modules which is also a morphism of algebras for \( X = \mathbb{1} \).

**Proof.** This follows from the isomorphism
\[
\text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, X) \cong \text{Ext}^\bullet_{\text{Ver}_p^0}(\mathbb{1}, (S^\bullet X_{n-1} \otimes S^\bullet \psi_{n-1}^0) \otimes X)[y_n]
\]
since \( \mathbb{1} \) is a direct summand of \( (S^\bullet X_{n-1} \otimes S^\bullet \psi_{n-1})^0 \).

8. Some further computations

In order to search for similar patterns for \( \text{Ext}^\bullet_{\text{Ver}_{p+1}}(\mathbb{1}, S) \) with \( S \) simple in the principal block, it makes sense to compute a number of examples. For instance, the simplest case \( n = 1 \) can be computed using the theory of Brauer tree algebras, and the answer is as follows.

Let \( X_0, \ldots, X_{N-1} \) label the simple modules for a chain-shaped Brauer tree algebra of length \( N \), in the order they occur in the Brauer tree (note that in the case of Verlinde categories \( N = p - 1 \) and \( X_0 = \mathbb{1} \)). Then we have

**Proposition 8.1.** (9) The Poincare series of \( \text{Ext}^\bullet(X_i, X_j) \) is given by the formula
\[
\sum_{k=0}^{\infty} t^k \dim \text{Ext}^k(X_i, X_j) = \frac{Q_{ijN}(t) + t^{2N-1}Q_{ijN}(t^{-1})}{1 - t^{2N}},
\]
where \( Q_{ijN}(t) := t^{i-j} + t^{i-j+2} + \ldots + t^{N-1-|N-1-i-j|} \).

**Example 8.2.** If \( i = 0 \), Proposition 8.1 gives
\[
\sum_{k=0}^{\infty} t^k \dim \text{Ext}^k(X_0, X_j) = \frac{t^j + t^{2N-1-j}}{1 - t^{2N}}.
\]
We also computed $\text{Ext}^i_{\text{Ver}_2}(1, S)$ for $S$ simple in the cases $p = 2$ and $p = 3$. For $p = 2$, by the results of [2], we have the following (note that $L_0 = 1$):

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_2}(1, 1) = \frac{1 + t^2}{(1 - t)(1 - t^2)}, \quad \sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_2}(1, L_2) = \frac{t}{1 - t^3}.$$ 

For $p = 3$, the Poincaré series computed using MAGMA agree at least up to degree 100 with the following (again $L_0 = 1$):

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, 1) = \frac{1 + t^3 + t^6 + 2t^7 + t^8 + t^{10} + 2t^{11} + t^{12} + t^{15} + t^{18}}{(1 - t^4)(1 - t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, L_4) = \frac{t + t^4 + t^5 + t^6 + t^8 + 2t^9 + t^{10} + t^{12} + t^{13} + t^{14} + t^{17}}{(1 - t^4)(1 - t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, L_6) = \frac{t^2 + t^{13}}{1 - t^{16}},$$

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, L_{10}) = \frac{t^2 + 2t^3 + t^4 + t^7 + t^8 + t^{10} + t^{11} + t^{14} + 2t^{15} + t^{16}}{(1 - t^4)(1 - t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, L_{12}) = \frac{t + t^2 + t^4 + t^5 + t^6 + 2t^9 + t^{12} + t^{13} + t^{14} + t^{16} + t^{17}}{(1 - t^4)(1 - t^{16})},$$

$$\sum_{i=0}^{\infty} t^i \dim \text{Ext}^i_{\text{Ver}_3}(1, L_{16}) = \frac{t^5 + t^{10}}{1 - t^{16}}.$$ 

References

[1] G. E. Andrews, The Rogers–Ramanujan reciprocal and Minc’s partition function, Pacific J. Math. 95 (1981), no. 2, 251–256.

[2] D. J. Benson and P. Etingof, Symmetric tensor categories in characteristic 2, Adv. in Math. 351 (2019), 967–999.

[3] D. J. Benson, P. Etingof, and V. Ostrik, New incompressible symmetric tensor categories in positive characteristic, arXiv:2003.10499. Preprint, 2020.

[4] D. J. Benson, S. B. Iyengar, and H. Krause, Stratifying triangulated categories, J. Topology 4 (2011), 641–666.

[5] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system, I. The user language, J. Symbolic Comput. 24 (1997), 235–265.

[6] S. Bremer, M. C. R. Butler, and A. D. King, Periodic algebras which are almost Koszul, Algebras and Representation Theory 5 (2002), no. 4, 331–367.

[7] J. F. Carlson, The varieties and the cohomology ring of a module, J. Algebra 85 (1983), 104–143.

[8] K. Coulembier, Monoidal abelian envelopes, arXiv:2003.10105. Preprint, 2020.

[9] O. Dudu, The Ext-algebra of the Brauer tree algebra associated to a line, arXiv:2101.12480.
D. Eisenbud, *Commutative algebra, with a view towards algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, Berlin/New York, 1995.

P. Etingof, *Koszul duality and the PBW theorem in symmetric tensor categories in positive characteristic*, Adv. in Math. **327** (2018), 128–160.

P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor categories*, AMS, Providence, 2015.

P. Etingof, N. Harman, and V. Ostrik, *p-adic dimensions in symmetric tensor categories in characteristic p*, Quantum Topol. **9** (2018), no. 1, 119–140.

P. Etingof and V. Ostrik, *Finite tensor categories*, Moscow Math. J. **4** (2004), 627–654.

P. Flajolet and H. Prodinger, *Level number sequences for trees*, Discrete Math. **65** (1987), 149–156.

M. Hochster and J. A. Eagon, *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.

E. Kirkman, J. Kuzmanovich, and J. J. Zhang, *Gorenstein subrings of invariants under Hopf algebra actions*, J. Algebra **322** (2009), 3640–3669.

J. P. May, *A general algebraic approach to Steenrod operations*, The Steenrod algebra and its applications (F. P. Peterson, ed.), Lecture Notes in Mathematics, vol. 168, Springer-Verlag, Berlin/New York, 1970, pp. 153–231.

H. Minc, *A problem in partitions: Enumeration of elements of a given degree in the free commutative entropic cyclic groupoid*, Proc. Edinb. Math. Soc. **11** (1958/1959), 223–224.

D. H. H. Nguyen, L. Schwartz, and N. N. Tran, *Résolution de certains modules instables et fonction de partition de Minc*, Comptes Rendus Acad. Sci. Paris, Série I **347** (2009), 599–602.

N. J. A. Sloane, *A handbook of integer sequences*, Academic Press, 1973.

R. P. Stanley, *Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), no. 1, 57–83.

D. Tubbenhauer and P. Wedrich, *Quivers of SL(2) tilting modules*, arXiv:1907.11560. Preprint, 2019.

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