Principal-axis Analysis of the Eddington Tensor for the Early Post-bounce Phase of Rotational Core-collapse Supernovae

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Abstract

Using full Boltzmann neutrino transport, we performed 2D core-collapse supernova simulations in axisymmetry for two progenitor models with 11.2 and 15.0 $M_\odot$, both rotational and nonrotational. We employed the results obtained in the early post-bounce phase ($t \lesssim 20$ ms) to assess performance under rapid rotation of some closure relations commonly employed in the truncated moment method. We first made a comparison in 1D under spherical symmetry, though, of the Eddington factor $p$ defined in the fluid rest frame (FR). We confirmed that the maximum entropy closure for the Fermionic distribution (MEFD) performs better than others near the proto-neutron star surface, where $p < 1/3$ occurs, but does not work well even in 1D when the phase-space occupancy satisfies $e < 0.5$ together with $p < 1/3$, the condition known to be not represented by MEFD. For the 2D models with the rapid rotation, we employed the principal-axis analysis of the Eddington tensor. We paid particular attention to the direction of the longest principal axis. We observed in FR that it is aligned neither with the radial direction nor with the neutrino flux in 2D, particularly so in convective and/or rapidly rotating regions, the fact not accommodated in the moment method. We repeated the same analysis in the laboratory frame and found again that the direction of the longest principal axis is not well reproduced by MEFD because the interpolation between the optically thick and thin limits is not very accurate in this frame.

Unified Astronomy Thesaurus concepts: Supernova dynamics (1664); Supernova neutrinos (1666)

1. Introduction

Core-collapse supernovae (CCSNe) have been studied by many researchers over the years (for reviews, see Muller 2016; Janka et al. 2016; Mezzacappa et al. 2020; Burrows & Vartanyan 2021). Although neutrino heating, which plays a crucial role in explosion, should be calculated as accurately as possible, various approximations have been used for multidimensional neutrino transfer to reduce the computational cost (O’Connor & Couch 2018; Vartanyan et al. 2019; Glas et al. 2019; Nagakura et al. 2020; Powell & Müller 2020). One of the most commonly used is the two-moment method. The zeroth and first moments of the transport equation (Lindquist 1966; Castor 1972) are solved with a closure relation, in which the second or third moments are approximately given in terms of the lower-order moments (Shibata et al. 2014). This approach is called the M1 method (Mutschikova et al. 2017) or the algebraic Eddington factor method (Just et al. 2015). The Eddington factor $p$, which is used to interpolate between the optically thick and thin limits, is generally described as a function of the flux factor $f$

Various functional forms of the Eddington factor have been proposed (Smit et al. 2000; Mutschikova et al. 2017). The Wilson closure (Wilson et al. 1975) and Kershaw closure (Kershaw 1976) were presented as flux limiters for the diffusion approximation. The most popular closure is probably the Levermore closure (Levermore 1984). The underlying idea of this closure is the following: if the stress–energy tensor of the isotropic radiation is transformed into other inertial frames by Lorentz transformations, then the relation between $p$ and $f$ is uniquely determined, irrespective of the chosen inertial frames: this relation is imposed as the closure. The maximum entropy closures for Bose–Einstein radiation (MEBE) and for Fermi–Dirac radiation (MEFD) were studied by Cernohorsky & Bludman (1994). Maximizing entropy for the Bose–Einstein or Fermi–Dirac statistics, the resultant angular distribution in momentum space gives the 2D closure $p = p(f, e)$, where $e$ is called the number density or the phase-space occupancy. The well-known Minerbo closure (ME; Minerbo 1978) can be regarded as the maximum entropy closure in the limit of $e \to 0$, in which both statistics give the same relation, while the Levermore–Pomraning closure (Levermore & Pomraning 1981) corresponds to the MEBE closure for $e \to \infty$.

The performance of these closures has been assessed by comparing them with data obtained by Monte Carlo simulations (Janka et al. 1992; Mutschikova et al. 2017; Richers et al. 2017) or by simulations with the discrete ordinate ($S_N$) method (Smit et al. 2000; Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2019, 2020; Iwakami et al. 2020) for matter distributions extracted from core-collapse simulations. Most of the earlier papers employed spherically symmetric backgrounds, either taken from 1D simulations or angle-averaged
for the results of 2D (axisymmetric) simulations. Janka et al. (1992) demonstrated that the Fermi–Dirac form with two parameters that corresponds to the maximum entropy argument above gives a reasonable approximation. Smit et al. (2000) also found for pre- and post-bounce snapshots taken from a 1D simulation that the maximum entropy closure yields results closest to the simulation data obtained with the SN method. Murchikova et al. (2017) employed angle-averaged backgrounds at \( t = 160, 260, \) and 360 ms after bounce in the 2D simulations by Ott et al. (2008) also with the SN method. They found that no single closure works better than the others but that ME and MEFD closures yield better results more often than not. These works commonly pointed out that there are regions of \( \rho < 1/3 \) in the supernova core but only the Wilson and MEFD closures can reproduce such situations; since the former does not satisfy the causality requirement (see Smit et al. 2000), the latter was favored.

More recent investigations (Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2019, 2020; Iwakami et al. 2020) employed the results of multidimensional simulations as they are. For example, Richers et al. (2017) made a detailed comparison between their neutrino transfer simulations with the Monte Carlo method and the discrete ordinate method for the same background model taken from a 2D simulation implemented with the latter scheme and elucidated pros and cons of each method. They also assessed the performance of some closure relations. Nagakura et al. (2018) and Harada et al. (2020), on the other hand, studied the neutrino distributions in momentum space rather in detail, employing the results of their own 2D core-collapse simulations of nonrotating progenitors with the SN method. In these three papers, the stress tensor itself or the Eddington tensor was compared component-wise with the Levermore closure. Harada et al. (2019) extended the analysis of Nagakura et al. (2018) to slowly rotating models. Very recently, Iwakami et al. (2020) performed 3D core-collapse simulations with the discrete ordinate method and compared the resultant neutrino distributions in momentum space with 1D and 2D counterparts. They adopted the principal-axis analysis of the Eddington tensor, the diagnostic first proposed in Harada et al. (2019) and employed also in this paper, to show rather poor performance of the Levermore closure for the hemispheric neutrino distribution.

In this paper, we extend the previous assessments of the closure relations in two directions. In 1D we make a more systematic comparison of different closures, employing the results of our CCSN simulation with the SN method. We confirm that those closures except for the MEFD closure are all in trouble at places with the Eddington factor \( \rho < 1/3 \). This occurs both in the proto–neutron star (PNS; Janka et al. 1992; Smit et al. 2000; Murchikova et al. 2017; Iwakami et al. 2020) and behind the prompt shock wave (Janka et al. 1992; Iwakami et al. 2020) in the early post-bounce phase. The problem is extended even to the MEFD closure when the phase-space occupancy becomes \( \rho < 0.5 \) simultaneously. In 2D, on the other hand, we study in detail the direction of the longest principal axis under rapid rotation. Note that in the previous study, with the principal-axis analysis for slowly rotating models (Harada et al. 2019), only the length of the longest principal axis, the 2D counterpart of the Eddington factor in 1D, was considered; Iwakami et al. (2020) investigated the direction of the principal axis in their 3D simulation, but rotation was ignored. It is obvious, however, that in the genuinely nonspherical situations with rapid rotation systematic misalignment of the longest principal axis from the radial direction becomes significant. Note that in the algebraic closures the principal axis is constrained to be aligned with the flux vector by construction in the fluid rest frame (FR). The validity of this approximation under such rapid rotation has not been addressed so far, and that is the main focus of this paper. We will also pay attention to the subtle difference from the choice of the frames to impose the closure relation.

This paper is organized as follows. The basic equations and numerical setup are described in Section 2, the closures considered in this paper are summarized in Section 3, and the principal-axis analysis of the Eddington tensor is discussed in Section 4. Then, we present the 1D and 2D results in Sections 5 and 6, respectively. Finally, conclusions are given in Section 7. Throughout the paper, the metric signature with \( -+++ \) and the units \( c = G = h = 1 \) are used unless otherwise stated, where \( c, G, \) and \( h \) denote the speed of light, the gravitational constant, and Planck’s constant, respectively. The Greek \((\zeta, \eta, \xi)\) and Latin \((i, j, k)\) indices run over \(0–3\) and \(1–3\), respectively.

### 2. Numerical Modeling

The numerical method used in this study is essentially the same as that used by Nagakura et al. (2018) and Harada et al. (2019, 2020). The Boltzmann radiation hydrodynamics code based on the discrete ordinate SN method (Sumiyoshi & Yamada 2012; Nagakura et al. 2014, 2017, 2019) is employed for core-collapse simulations. The Boltzmann equation for neutrinos with a general metric in the conservative form (Shibata et al. 2014) is

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \left[ n_i \left[ e_{(0)}^{\kappa} + \sum_{j=1}^{3} \ell_{(j)} e_{(j)}^\kappa \right] \right] = \frac{1}{\sqrt{-g}} F
\]

\[
- \frac{1}{e^2} \frac{\partial}{\partial \epsilon} (e^3 F \omega_{(0)}) + \frac{1}{\sin \theta_\nu \partial \theta_\nu} (\sin \theta_\nu F \omega_{(0)}),
\]

\[
+ \frac{1}{\sin^2 \theta_\nu \partial \phi_\nu} (F \omega_{(0)}), = S_{\text{rad}}, \tag{1}
\]

\[
\ell_{(j)} = (\cos \theta_\nu, \sin \theta_\nu, \cos \phi_\nu, \sin \phi_\nu), \tag{2}
\]

\[
\omega_{(0)} = e^{-2p^0 p^\nu} \sum_{j=1}^{3} \omega_{j} \ell_{(j)} \nu_{(0)}, \tag{3}
\]

\[
\omega_{(\nu)} = e^{-2p^0 p^\nu} \sum_{j=1}^{3} \omega_{j} \ell_{(j)} \nu_{(0)}, \tag{4}
\]

\[
\omega_{(\phi)} = e^{-2p^0 p^\nu} \sum_{j=1}^{3} \omega_{j} \ell_{(j)} \nu_{(0)}, \tag{5}
\]

\[
\omega_{(\xi)} = e^{-2p^0 p^\nu} \sum_{j=1}^{3} \omega_{j} \ell_{(j)} \nu_{(0)}, \tag{6}
\]

where \( F, g, \) and \( \nu, \) \( n, p, \) \( S_{\text{rad}}, \) and \( e_{(0)}^\kappa \) are the distribution function, the coordinates, the four-momentum, the coordinates in momentum space, the collision term, and the \( \xi \) component of the tetrad basis \( e_{(0)}, \) respectively. The spacetime metric of the 3 + 1 decomposition is written as

\[
d\xi^2 = g_{\lambda\mu} dx^\lambda dx^\mu \tag{7}
\]

\[
= -\alpha^2 dt^2 + \gamma_{jk}(dx^j + \beta^j dt)(dx^k + \beta^k dt), \tag{8}
\]
where \( t, \gamma_{\nu} = g_{\nu\xi} + n_{\nu} \xi \), \( n^\xi = (1/\alpha, - \beta / \alpha) \), \( \alpha \), and \( \beta \) are time, the spatial metric, the normal vector to the spatial hypersurface with \( t = \text{const} \), the lapse function, and the shift vector, respectively. The zeroth tetrad basis \( e_{(0)}^\xi \) is chosen to agree with \( n^\xi \). In the spherical coordinates \((r, \theta, \phi)\), the other tetrad bases are described as

\[
e^{(1)} = e_r = \frac{\gamma_{rr}}{\sqrt{\gamma_{rr} \gamma_{\theta \theta} - \gamma_{\phi \phi}}} \partial_r,
\]

\[
e^{(2)} = e_\theta = - \frac{\gamma_{r \theta}}{\sqrt{\gamma_{rr} \gamma_{\theta \theta} - \gamma_{\phi \phi}}} \partial_r + \frac{\gamma_{\theta \theta}}{\sqrt{\gamma_{rr} \gamma_{\theta \theta} - \gamma_{\phi \phi}}} \partial_\theta,
\]

\[
e^{(3)} = e_\phi = \frac{\gamma_{r \phi}}{\sqrt{\gamma_{rr} \gamma_{\theta \theta} - \gamma_{\phi \phi}}} \partial_r + \frac{\gamma_{\phi \phi}}{\sqrt{\gamma_{rr} \gamma_{\theta \theta} - \gamma_{\phi \phi}}} \partial_\phi,
\]

where \( \partial_j \) are the coordinate bases of the vector. The flat spacetime moving with the PNS in the velocity of \( \beta \) is considered. Then, \( e_r = \partial_r \), \( e_\theta = r^{-1} \partial_\theta \), and \( e_\phi = (r \sin \theta)^{-1} \partial_\phi \).

The neutrino energy \( \epsilon \) corresponds to the distance from the origin in momentum space, and the neutrino-propagating direction angles \( \theta_\nu \) and \( \phi_\nu \) are defined as the angle from \( e_r \) and the angle from \( e_\theta \) on the \( e_\theta-e_\phi \) plane, respectively, as shown in Figure 1.

Along with the Boltzmann equation for neutrino radiation transport, the Newtonian compressible hydrodynamics equations and the time-evolution equation of electron number density are solved as follows:

\[
\partial_t \mathbf{Q} + \partial_j U^j = W_h + W_i + W_o,
\]

\[
\mathbf{Q} = \sqrt{g} \left( \begin{array}{c}
\rho \\
\rho v_r \\
\rho v_\theta \\
\rho v_\phi \\
\mathcal{E} + \frac{1}{2} \rho v^2 \\
\end{array} \right),
\]

\[
U^j = \sqrt{g} \left( \begin{array}{c}
\rho v_r^j \\
\rho v_r v_j + \mathcal{P} \delta_r^j \\
\rho v_\theta v_j + \mathcal{P} \delta_\theta^j \\
\rho v_\phi v_j + \mathcal{P} \delta_\phi^j \\
\left( \mathcal{E} + \mathcal{P} + \frac{1}{2} \rho v^2 \right) v_j \\
\rho Y_e v_j \\
\end{array} \right),
\]

\[
W_h = \sqrt{g} \left( \begin{array}{c}
0 \\
- \rho \partial_r \Psi + \rho r (v_r)^2 + \rho r \sin^2 \theta (v_\theta)^2 + \frac{2 \mathcal{P}}{r} \\
- \rho \partial_\theta \Psi - \rho v_r \partial_r \Psi \\
- \rho \partial_\phi \Psi \\
0 \\
\end{array} \right),
\]

\[
W_i = \sqrt{g} \left( \begin{array}{c}
0 \\
- G_r \\
- G_\theta \\
- G_\phi \\
- \Gamma \\
\end{array} \right),
\]

\[
W_o = \sqrt{g} \left( \begin{array}{c}
0 \\
- \rho \beta_r \\
- \rho \beta_\theta \\
- \rho \beta_\phi \\
0 \\
\end{array} \right),
\]

where \( \rho, v_r, \mathcal{P}, \mathcal{E}, Y_e, \) and \( \delta_k^j \) are the matter density, velocity, pressure, internal energy density, electron fraction, and Kronecker’s delta, respectively. The Newtonian gravitational potential \( \Psi \) is governed by the Poisson equation:

\[
\Delta \Psi = 4 \pi \rho.
\]
The exchange of momentum and energy between neutrino and matter fields in Equation (17) is described by

\[ G^\nu = \sum_s G^\nu_s, \]  
\[ G^\nu_s = \int p^n e S_{\text{rad}}(\theta, \phi) dV_p, \]  
\[ \Gamma = \Gamma_v - \Gamma_\bar{v}, \]
\[ \Gamma_v = \int e S_{\text{rad}}(\theta, \phi) dV_p, \]

where \( m_n, S_{\text{rad}}(\theta, \phi) \) and \( dV_p \) denote the atomic mass unit, collision term of the neutrino species \( s \), invariant volume in the momentum space, and differential solid angle in the momentum space, respectively. The subscript \( s \) represents three neutrino species: \( \nu_e, \bar{\nu}_e \), and \( \nu_x \). Detailed information of the collision terms is given by Suniymoshi & Yamada (2012).

The equation of state (EOS) based on the liquid drop model of the nuclei and the Skyrme-type interaction with incompressibility \( K = 220 \text{ MeV} \) by Lattimer & Swesty (1991) is used here. The 2D spatial domain in the range \( 0 \leq r \leq 5000 \text{ km} \) and \( 0 \leq \theta \leq \pi \) is divided into \( 384(\theta) \times 64(\theta) \) or \( 128(\theta) \times 128(\theta) \) grid cells, and the momentum space in the range \( 0 \leq \epsilon \leq 300 \text{ MeV} \) over the entire solid angle is discretized into \( 20(\epsilon) \times 10(\theta) \) grid points. The 11.2 and 15.0 \( M_\odot \) progenitor models evolved without rotation (Woosley et al. 2002) are employed in this study. From the onset of the core collapse, 1D simulations are performed for nonrotating models, and 2D low-resolution simulations are done for rotating models, where a rigid rotation of 6 rad s\(^{-1}\) at \( r = 1000 \text{ km} \) is manually added initially. When an entropy gradient becomes negative for the first time soon after the core bounce, the zenith grid number is switched to 128 and 0.1% random velocity perturbations are imposed for both models.

### 3. Closures

The following explains how to calculate the Eddington factor used for the analyses in this paper. An unprojected second moment (Thorne 1981; Shibata et al. 2014) is defined in an arbitrary frame as

\[ M^{\Sigma}(\epsilon_{\text{FR}}) = \int \mathcal{F} \delta \left( \frac{\epsilon_{\text{FR}}}{3} - \frac{(-u_\nu^2 p^\nu e^\nu)^3}{3} \right) p^n p^\nu dV_p, \]  

where \( u_\nu \) and \( \epsilon_{\text{FR}} \) are the four-velocity of medium and the neutrino energy measured in the FR, respectively. The second angular moment can be written as

\[ M^{\Sigma}(\epsilon_{\text{FR}}) = J(\epsilon_{\text{FR}})u_\nu u_\bar{\nu} + H^\nu(\epsilon_{\text{FR}})u^\nu \]
\[ + H_\bar{\nu}(\epsilon_{\text{FR}})u_\bar{\nu} + L^{\Sigma}(\epsilon_{\text{FR}}), \]

where the energy density \( J \), energy flux \( H^\nu \), and radiation pressure tensor \( L^{\Sigma} \) are the variables projected onto the FR.

In 1D, instead of the Eddington tensor but the Eddington factor, i.e., the ratio of the \( rr \)-component of the radiation pressure tensor to the energy density defined as

\[ p = \frac{L_r(\epsilon_{\text{FR}})}{J(\epsilon_{\text{FR}})}. \]

is used. It is straightforward to calculate it in the Boltzmann transport simulation, whereas it is prescribed algebraically in the truncated moment method. The closure relations that are compared in this paper are as follows:

- **MEFD closure** (Cernohorsky & Bludman 1994),
  \[ p_{\text{MEFD}} = \frac{1}{3} + \frac{2}{3} (1 - e)(1 - 2e) \chi \left( \frac{f}{1 - e} \right), \]  
- **ME closure in the classic limit** (Minerbo 1978),
  \[ p_{\text{ME}} (e \rightarrow 0) = \frac{1}{3} + \frac{2f^2}{15} (3 - f + 3f^2); \]  
- **MEBE closure in the limit of** \( e \rightarrow \infty \) (Levermore & Pomraning 1981),
  \[ p_{\text{MEBE}} (e \rightarrow \infty) = f \coth R, \]
  \[ f = \coth R - 1/R; \]
- the Levermore closure (Levermore 1984),
  \[ p_{\text{Levermore}} = \frac{3 + 4f^2}{5 + 2\sqrt{4 - 3f}}. \]

### 4. Principal-axis Analysis of the Eddington Tensor

In 2D, we employ the entire Eddington tensor. This section briefly introduces the principal-axis analysis of the Eddington tensor (see Iwakami et al. 2020, for a detailed explanation). The aforementioned closures are constructed with the assumption of the axisymmetric neutrino distribution around the energy flux vector in the momentum space, which is normally realized for spherically symmetric matter distributions in space. The CCSNe are not spherically symmetric, though, owing to hydrodynamic instabilities or stellar rotation (or magnetic fields). In fact, the neutrino angular distribution in momentum space is complicated in the convective flow in general, with the longest principal axis of the Eddington tensor not always parallel to the flux (Iwakami et al. 2020). Note that, as shown below, the closure relation in the truncated moment method normally employs the flux vector alone to construct the tensor structure in the anisotropic part of the Eddington tensor, thus forcing its longest principal axis either parallel or perpendicular to the flux. The main goal of this paper is to assess this restriction in the closure relations in the genuinely nonspherical

\[ f = \sqrt{\frac{h_{\eta\xi} H^\nu(\epsilon_{\text{FR}})H_\bar{\nu}(\epsilon_{\text{FR}})}{J^2(\epsilon_{\text{FR}})}}, \]  

where \( h_{\eta\xi} = g_{\eta\xi} + u_\eta u_\xi \) is the projection operator onto the FR. In 1D, not the Eddington tensor but the Eddington factor, i.e., the ratio of the \( rr \)-component of the radiation pressure tensor to the energy density defined as

\[ p = \frac{L_r(\epsilon_{\text{FR}})}{J(\epsilon_{\text{FR}})}. \]
settings under rapid rotation. In the following we adopt the MEFD closure as the canonical one, since it performs best according to our comparison in 1D as shown below.

The Eddington tensor calculated in the FR with the Boltzmann transport simulations

\[
\kappa_{\text{Boltz}}(\epsilon_{\text{FR}}) = \frac{L^j(\epsilon_{\text{FR}})}{J(\epsilon_{\text{FR}})},
\]

is compared with the counterpart for the MEFD closure,

\[
\kappa_{\text{MEFD}}(\epsilon_{\text{FR}}) = \frac{L^j_{\text{MEFD}}(\epsilon_{\text{FR}})}{J(\epsilon_{\text{FR}})}.
\]

The radiation pressure tensor for the MEFD closure in the FR is expressed as

\[
L^j_{\text{MEFD}}(\epsilon_{\text{FR}}) = \frac{3}{2}(1 - p_{\text{MEFD}}) L^j_{\text{thick}}(\epsilon_{\text{FR}})
+ \frac{1}{3}(3p_{\text{MEFD}} - 1) L^j_{\text{thin}}(\epsilon_{\text{FR}}),
\]

(36)

\[
L^j_{\text{thick}}(\epsilon_{\text{FR}}) = \frac{1}{3} J(\epsilon_{\text{FR}}) h^j,
\]

(37)

\[
L^j_{\text{thin}}(\epsilon_{\text{FR}}) = J(\epsilon_{\text{FR}}) \frac{H^j(\epsilon_{\text{FR}})}{H(\epsilon_{\text{FR}})^2},
\]

(38)

where \(L^j_{\text{thick}}\) and \(L^j_{\text{thin}}\) are the radiation pressure tensors at the optically thick and thin limits, respectively. Note that these variables are functions of the energy de- 

The Eddington tensors are defined as

\[
k^j_{\text{Boltz}}(\epsilon_{\text{FR}}) = \frac{P^j(\epsilon_{\text{FR}})}{E(\epsilon_{\text{FR}})}
\]

for the Boltzmann transport and as

\[
k^j_{\text{MEFD}}(\epsilon_{\text{FR}}) = \frac{P^j_{\text{MEFD}}(\epsilon_{\text{FR}})}{E(\epsilon_{\text{FR}})}
\]

for the MEFD closure. For the latter the radiation pressure tensor is given by

\[
P^j_{\text{MEFD}}(\epsilon_{\text{FR}}) = \frac{3}{2}(1 - p_{\text{MEFD}}) P^j_{\text{thick}}(\epsilon_{\text{FR}})
+ \frac{1}{3}(3p_{\text{MEFD}} - 1) P^j_{\text{thin}}(\epsilon_{\text{FR}}),
\]

(42)

\[
P^j_{\text{thick}}(\epsilon_{\text{FR}}) = \frac{1}{3} J(\epsilon_{\text{FR}}) \gamma^j + \frac{4}{3} J(\epsilon_{\text{FR}}) v^j v^j
+ (H^j(\epsilon_{\text{FR}}) v^j + \nu^j H(\epsilon_{\text{FR}})),
\]

(43)

\[
P^j_{\text{thin}}(\epsilon_{\text{FR}}) = E(\epsilon_{\text{FR}}) \frac{F^j(\epsilon_{\text{FR}}) F^j(\epsilon_{\text{FR}})}{F(\epsilon_{\text{FR}})^2},
\]

(44)

where \(P^j_{\text{thick}}\) and \(P^j_{\text{thin}}\) are the radiation pressure tensors at the optically thick and thin limits, respectively, in the LB. Equations (42)–(44) are directly evaluated from \(E, F^j, I, J,\) and \(H^j\) obtained by taking the moments in the different frames in this paper. In the simulation with the truncated moment method, one normally knows the energy density and flux in either LB or FR. Then, some iteration procedure is required to obtain the quantities in the other frame in a self-consistent way. However, this iteration procedure is not taken here, as we have the second moment tensor \(M^{ij}\), from which these quantities are obtained in both frames directly. Harada et al. (2020) showed that the two methods give negligible differences.

The eigenvalues and eigenvectors of the Eddington tensor are calculated by the Jacobi method (Iwakami et al. 2020). In the principal-axis analysis, they are visualized as the ellipsoid:

\[
\left(\frac{p^1}{\lambda^1}\right)^2 + \left(\frac{p^2}{\lambda^2}\right)^2 + \left(\frac{p^3}{\lambda^3}\right)^2 = 1,
\]

(45)

where \(\lambda^j\) and \(p^j\) are the \(j\)th eigenvalue and the momentum projected onto the \(j\)th eigenvector, respectively. The surface of the triaxial ellipsoid is parametrically expressed as

\[
\left(\begin{array}{c}
 p^1 \\
 p^2 \\
 p^3
\end{array}\right) = \left(\begin{array}{ccc}
 \nu_1 & \nu_2 & \nu_3 \\
 \nu_2 & \nu_3 & \nu_1 \\
 \nu_3 & \nu_1 & \nu_2
\end{array}\right) \left(\begin{array}{c}
 \lambda^1 \\
 \lambda^2 \\
 \lambda^3
\end{array}\right) \left(\begin{array}{c}
 \cos \theta \cos \phi \\
 \sin \theta \cos \phi \\
 \sin \phi
\end{array}\right)
\]

(46)

where \(a\) and \(b\) are the parameters. The components of the momentum vectors \((p^a, p^b, p^r)\) are the coordinates of a point on the ellipsoidal surface, and \((\nu_1, \nu_2, \nu_3)\) are the \(\theta, \phi,\) and \(r\) components of the \(j\)th eigenvector. We choose \(\lambda^j\) as the largest of these eigenvalues to represent the longest axis of the ellipsoid.

Using Equation (46), an ellipsoid can be drawn as shown in Figure 2. The wireframe shape becomes a sphere, ellipsoid, and line in the optically thick limit, transition region, and optically thin limit, respectively. In the figure, the longest principal axis is represented by \(L\), whereas the neutrino energy flux and matter velocity are denoted by \(F\) and \(V\), respectively. The angle between \(F\) and \(L\) is \(\theta_{FL}\).

5. Principal-axis Analysis in 1D

This section is devoted to the screening of the closure relations commonly used in the literature and listed in Section 3 by using the results of 1D Boltzmann simulations. Such comparisons have been made over the years by other authors in their own contexts (e.g., Janka et al. 1992; Smit et al. 2000; Murchikova et al. 2017), and the results obtained here are consistent with the previous ones. We also apply the principal-axis analysis to the MEFD closure, which is singled out to be
the best and will be adopted also in the 2D analysis later. Particular attention is paid to the shape change of the ellipsoid (Equation (45)) in the MEFD closure, which is shared in 2D and hence will be useful later.

Figure 3 shows the radial profiles of the various quantities for the 11.2 and 15.0 $M_\odot$ models at 10 ms post-bounce, where $S$, $Y_L$, and $\Lambda$ are the entropy, lepton fraction, and mean free path divided by the radius, respectively. The shock wave is located around $r = 70$ km. Since the profiles are not much different from each other for $r \lesssim 100$ km, except for the entropy above the shock wave, only the results for the 11.2 $M_\odot$ model are presented in this section.

The Eddington factor $\rho$ for $\nu_e$ is shown as a function of $f$ (left panel) and $r$ (right panel) in Figure 4. The electron-neutrino energy $\epsilon_{\nu_e}$ is 3.9 MeV in the top panels and 14.5 MeV in the bottom panels. The latter is almost the average energy of $\nu_e$ just behind the shock wave at this time. The Eddington factors $p_{\text{Boltzmann}}$ and $p_{\text{MEFD}}$ are plotted with red and light-blue circles, respectively. In the left panels, the Levermore, MEBE ($e \to \infty$), and ME ($e \to 0$) closures are shown with solid lines, and the MP is indicated by a black line. In the right panels, the green and black lines denote the phase-space occupancy $e$ and flux factor $f$, respectively.

As pointed out in our previous paper (Iwakami et al. 2020), the regions of $p_{\text{Boltzmann}} < 1/3$ appear at $\Lambda \lesssim 1$ (Figures 3(c) and (d)) for both 3.9 and 14.5 MeV (Figure 4). For $\epsilon_{\nu_e} = 3.9$ MeV, this region is overlapped with the region of $p_{\text{MEFD}} < 1/3$, which emerges inside the PNS at 10 km < $r$ < 40 km (Figure 4(b)), where the flux factor is $0 \lesssim f \lesssim 0.2$ (Figure 4(a)). Note that $p_{\text{Levermore}}$, $p_{\text{ME}}$, and $p_{\text{MEBE}}$ increase monotonically with increasing $f$ (Equations (31)–(33)) and are qualitatively different from the result of Boltzmann transport simulation. Similar results were reported by Janka et al. (1992) and Smit et al. (2000). By contrast, at $\epsilon_{\nu_e} = 14.5$ MeV, the region of $p_{\text{Boltzmann}} < 1/3$ is wider, extending from inside the PNS up to behind the shock wave, where the range of the flux factor is $0.05 \lesssim f \lesssim 0.4$ (Figure 4(c)). Note that $p_{\text{MEFD}} > 1/3$ at 50 km < $r$ < 70 km (Figure 4(d)). This is because $e < 0.5$ this time. As a matter of fact, $p_{\text{MEFD}}$ is known to be a monotonically increasing function of $f$ when $e < 0.5$ (Equation (29)). Hence, even the MEFD closure gives qualitatively wrong Eddington factors in such situations.

The Levermore closure seems to be best for $f \geq 0.7$ (Figures 4(c) and (d)). As the Boltzmann transport simulation tends to be underresolved in momentum space (see the Appendix), however, a closure that gives a more forward-peaked distribution is desirable in the optically thin region. As was concluded also by Murchikova et al. (2017), it is confirmed that no single closure can give an accurate estimate to the Eddington factor in the CCSN context.

Figure 5 shows the distribution $\mathcal{F}$ in momentum space for $\nu_e$ in the FR at $r = 10$, 30, 65, and 90 km. In this space, the distance from the origin, the angle from the $P_r$ axis, and the angle from the $P_\theta$ axis on the $P_{\theta} - P_r$ plane correspond to $e$, $\theta_\nu$, and $\phi_\nu$, respectively. In Figure 6, we present $\mathcal{F}$ as a function of $\mu_\nu$ for $\nu_e$ where $\mu_\nu = \cos \theta_\nu$. The neutrinos are uniformly distributed over the entire solid angle for all energy bins at $r = 10$ km in the optically thick region (Figures 5(a) and 6(a)). As the radius increases, more forward-peaked distributions start to emerge in the lower-energy neutrinos (Figures 5(b), (c), and...
is rather meaningless at left panels, the solid lines denote the closures of Levermore tensors obtained in the Boltzmann simulation those given by the MEFD closure the second moment, rendering the Eddington tensor smaller.

\[ \mathbf{r} = \text{lines at uniform distributions denoted by the black, red, and light-blue when the angular distribution is hemispheric, the almost Although the Eddington factor tends to be smaller than 1 the red line for (\( \frac{\epsilon}{\nu} \)) \( \approx \) 3.9 MeV at \( r = 30 \text{ km} \) (Figure 6(b)) and in the red line for \( \epsilon = 14.5 \text{ MeV} \) at \( r = 65 \text{ km} \) (Figure 6(c)). Although the Eddington factor tends to be smaller than 1/3 when the angular distribution is hemispheric, the almost uniform distributions denoted by the black, red, and light-blue lines at \( r = 30 \text{ km} \) (Figure 6(b)) and by the light-blue lines at \( r = 65 \text{ km} \) (Figure 6(c)) also give \( p_{\text{Boltzmann}} \leq 1/3 \). What is essential to produce \( p < 1/3 \) is that neutrinos are abundant at \( \nu \approx 0 \), since they contribute to the zeroth moment but not to the second moment, rendering the Eddington tensor smaller.

Figures 7 and 8 present the ellipsoids of the Eddington tensors obtained in the Boltzmann simulation (top panels) and those given by the MEFD closure (bottom panels) at three radii for \( \epsilon_{\text{FR}} = 3.9 \) and 14.5 MeV, respectively. The Eddington tensors are evaluated in the FR. In the left panels (Figures 7(a), 7(d), 8(a), and 8(d)), on the other hand, an almost-complete sphere is obtained for both Boltzmann and MEFD at \( r = 10 \text{ km} \), where \( \Lambda \lesssim 0.1 \) (Figures 3(e) and (f)). In the right panels (Figures 7(c), 7(f), 8(c), and 8(f)), the vertically elongated ellipsoid, in which the longest axis \( E \) is parallel to \( F \), is drawn for both Boltzmann and MEFD at \( r = 90 \text{ km} \), where \( \Lambda \gtrsim 10 \) (Figures 3(e) and (f)). In the middle panels, the horizontally elongated ellipsoid, where \( L \) is perpendicular to \( F \), is derived for both Boltzmann and MEFD at \( \epsilon_{\text{FR}} = 3.9 \text{ MeV} \) (Figures 7(b) and (e)) and for Boltzmann alone at \( \epsilon_{\text{FR}} = 14.5 \text{ MeV} \) (Figure 8(b)). At this radius \( \Lambda \approx 1 \) (Figures 3(e) and (f)) and matter is neither optically thick nor thin. The difference between Boltzmann and MEFD for average-energy neutrinos is a confirmation that the longest principal axis \( L \) is perpendicular to \( F \) if the flux factor \( p \) is smaller than 1/3 and the MEFD closure fails to reproduce it.

6. Principal-axis Analysis in 2D

Now we move to the main subject of this paper, i.e., the assessment of the closure relation under rapid rotation in 2D. Although most previous works (Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2019, 2020; Iwakami et al. 2020) studied the Levermore closure, we pick up the MEFD closure, since we have found in the preceding section that it performs better in 1D than others including the Levermore closure, at least in the current context. Instead of directly comparing the Eddington tensors as in other works (Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2020), we compare its principal axes in this paper. Since the Eddington tensor is a second-rank symmetric tensor, all of its three eigenvalues are real and the corresponding eigenvectors are orthogonal to one another, characterizing its anisotropy. In fact, one of the remarkable properties that characterize the truncated moment method is that they employ the flux vector alone (in addition to Kronecker’s delta) to build the Eddington tensor in the FR; as a

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8 Since the accuracy of the Jacobi method to obtain eigenvalues and eigenvectors is \( \mathcal{O}(10^{-8}) \) in the principal-axis analysis, the direction of \( L \), the longest principal axis, is rather meaningless at \( \epsilon \lesssim 10 \text{ km} \).
result, the eigenvectors are the flux vector itself and two other mutually orthogonal vectors that are perpendicular to the flux. The latter fact implies in turn that the longest principal axis is always parallel or transverse to the flux vector. This is true in 1D but is not the case in multiple dimensions in general. In this paper, we investigate quantitatively to what extent and in which direction the longest principal axis is misaligned with the flux.

This has never been done before for rapidly rotating models; in fact, Harada et al. (2019) employed a slowly rotating model and paid attention only to the length of the longest principal axis as a function of the radius as in Figure 4, whereas Iwakami et al. (2020) studied in their 3D simulation the effect of convection on the direction of the principal axes of the Eddington tensor in the LB, but rotation is entirely ignored. It should also be mentioned that the frame, in which the closure relation is imposed, matters, since the closure relations in the LB are sometimes not exactly equivalent to those in the FR (see Equations (42)-(44) above). We hence repeat the same exploration in the LB and compare the results with those obtained in the FR. This is also the first-ever attempt of the kind.

6.1. Basic Features in Dynamics and Neutrino Emissions

This section summarizes briefly the basic features of the 2D simulation results for the 11.2 and 15.0 $M_\odot$ progenitor models.

Figures 9 and 10 show color maps of entropy with the matter velocity vectors (top panels) and the number density of $\nu_e$ with its average velocity vectors (bottom panels) at $t = 10$ ms for the 11.2 and 15.0 $M_\odot$ models, respectively. The average neutrino velocity $V_{\nu_\alpha}$ is defined as

$$V_{\nu_\alpha} = \frac{G_i}{N},$$

(47)

where the neutrino number density $N$ and number flux $G_i$ are expressed as

$$N = \int_0^\infty \mathcal{F} (E, \Omega_\nu) \ e^2 E d\Omega_\nu,$$

(48)

$$G_i = \int_0^\pi \hat{\ell}_i \ \mathcal{F}(E, \Omega_\nu) \ e^2 E d\Omega_\nu,$$

(49)

with $\hat{\ell}_i$ being the unit vector in momentum space. Integration over the neutrino energy and solid angle is done in the LB here. The nonrotating models are presented in the left panels, and the rotating ones are shown in the right panels. The shock wave corresponds to the boundary between orange and purple colors in the top panels. The prompt convection grows in the central region (Figures 9(a), 9(b), 10(a), and 10(b)), and neutrinos move in various directions with matter for both nonrotating and rotating models (Figures 9(c), 9(d), 10(c), and 10(d)). The anisotropy of the shock-wave geometry and neutrino propagation in the outer region is weak for the nonrotating models and strong for the rotating models.

Figure 11 shows the radial profiles of the velocity components $V_r$, $V_\theta$, and $V_\phi$ for the $\theta = \pi/2$ and $\pi/4$ directions, where $V_r = v^r$, $V_\theta = r v^\theta$, and $V_\phi = r \sin \theta \ v^\phi$. Figure 12 presents the radial profiles of the density $\rho$ and the mean free path divided by the radius $\Lambda$ for the $\theta = 0$ and $\pi/2$ directions.
The results for the 11.2 and 15.0 $M_\odot$ models are plotted in the left and right panels, respectively. Although the 11.2 $M_\odot$ model experiences stronger prompt convection than the 15.0 $M_\odot$ model, the basic features are very similar between the models at $t = 10$ ms for $r \lesssim 100$ km (Figures 11 and 12). For the nonrotating models, the radial distributions of $\rho$ and $\Lambda$ in the equatorial and polar directions agree well with each other. By contrast, for the rotating models, the shock wave in the equatorial direction propagates faster than in the polar direction, and the resulting radial profiles of $\rho$ and $\Lambda$ in each direction are different (Figure 12). Since such characteristics are common between the models, the results of the analysis for the neutrino transport at $t = 10$ ms are essentially the same between the models. Thus, only the results for the 11.2 $M_\odot$ model are shown in the following sections.

6.2. Principal-axis Analysis in the Fluid Rest Frame

This section presents the results of the principal-axis analysis in the FR for both the nonrotating and rotating progenitor models, the main results of this paper. In their neutrino radiation hydrodynamics codes, Just et al. (2015) and Skinner et al. (2019) solved the two-moment equations in the FR with the closure relation imposed in this frame, whereas O’Connor (2015) employed the moment equations in the LB but applied the closure relations in the FR by consistently transforming the neutrino stress–energy tensor obtained in the LB. Just et al. (2015) compared the results for the Levermore and MEFD closures and found essentially identical results for the time evolutions of the shock wave, neutrino luminosities, and mean energies. Note, however, that neither of the two closures can reproduce $p < 1/3$ at $e < 0.5$. We compare the results for the Boltzmann simulation with those for the MEFD closure, paying particular attention to the misalignment of the longest principal axis with the flux vector in the following.

Figure 13 shows the color maps of $|\mu_\text{FL}|$ in the FR for $\nu_e$ at $t = 10$ ms for $\epsilon_{\text{FR}} = 3.9$ MeV, where $\mu_\text{FL} = \cos \theta_\text{FL}$, with $\theta_\text{FL}$ the angle that the longest principal axis makes with the flux vector (see Figure 2). The dark-blue region is the location where $L$ is perpendicular to $F$ and $p < 1/3$ is commonly observed. Since the central region at $r \lesssim 10$ km is highly isotropic, having anisotropies of $O(10^{-8})$, it is ignored in the following discussion. For $\epsilon_{\text{FR}} = 3.9$ MeV, this blue region is circular for the nonrotating model in both the Boltzmann and MEFD calculations (Figures 13(a) and (c)), and it is oblate for the rotating model in both cases (Figures 13(b) and (d)). The radial position of $\Lambda \approx 1$ is shifted inward near the poles in the rotating model (see the light-blue line in Figure 12(c)). Although the extents of the blue regions are not much different between the Boltzmann and MEFD cases, the color depth tends to be lighter for the Boltzmann case (Figures 13(a) and (b)) than for the MEFD case (Figures 13(c) and (d)). This is because the longest principal axis is perfectly perpendicular to the flux vector in the MEFD case while it is inclined by angles between

![Figure 6](image.png)

**Figure 6.** The angular distribution of $\mathcal{F}$ for $\nu_e$ at $r = 10, 30, 65,$ and 90 km. The symbol $\mu_\text{FL} = (\cos \theta_\text{FL})$ is the cosine of the polar angle in momentum space. The solid lines and circles are the angular distributions corresponding to $p < 1/3$ and $p \geq 1/3$, respectively. The specific values of $p$ are 0.327 (purple), 0.322 (green), 0.324 (yellow), 0.327 (blue), 0.331 (black), 0.333 (red), and 0.333 (light blue) at $r = 30$ km (panel (b)) and 0.311 (red) and 0.331 (light blue) at $r = 65$ km (panel (c)).
0 and π/2 in the Boltzmann case. The convective and/or rotational matter motion makes the neutrino distribution nonaxisymmetric around the flux direction in momentum space, and as a result, the longest principal axis of the Eddington tensor becomes tilted from the perpendicular direction to \( F \).

Figure 14 is the same as the previous figure, except for the neutrino energy \( \epsilon_{\text{FR}} = 3.9 \text{ MeV} \). For the nonrotating model (left panels), two dark-blue regions are observed for Boltzmann (Figure 14(a)), whereas there is only one domain that exists for MEFD. The inner region of the Boltzmann case partly overlaps with the uniform dark-blue region for MEFD, while the outer one just behind the shock wave does not appear for the MEFD closure (Figure 14(c)). More interestingly, in the rotating model the inner dark-blue region is not clearly observed in the Boltzmann case (Figure 14(b)), whereas it is still there, being slightly oblate, in the MEFD case. This is likely because the neutrino distribution, which is otherwise almost isotropic in this region (see the red line in Figure 6(b)), becomes tilted in the azimuthal direction owing to the rotation. The outer dark-blue region also nearly disappears, particularly in the vicinity of the poles (Figure 14(b)). This happens because the hemispheric distribution is disrupted by fast nonradial flows in these regions in the presence of rotation (Figure 9(b)). In the equatorial region, the blue color becomes lighter in the rotating model (Figures 14(b)). This time the hemispheric distribution is deformed to nonaxisymmetric ones because of the azimuthal matter motion and neutrino flux again induced by the rotation.

The distribution function \( \mathcal{F} \) of \( \nu_e \) in momentum space and the corresponding ellipsoid of the Eddington tensor, both in the FR, are presented in Figure 15. The hemispheric distribution, corresponding to red isosurfaces, is observed in momentum space at \( r = 30 \text{ km} \) for the nonrotating model, where the prompt convection grows at \( t = 10 \text{ ms} \) (Figure 15(a)). Note that the hemisphere is slanted. At \( r = 65 \text{ km} \) for the rotating model, the hemispheric distribution, denoted by bluish isosurfaces, also emerges in momentum space (Figure 15(b)). Since the distribution is not completely axisymmetric with respect to \( F \) owing to the rotation, the axis \( L \) is not perfectly perpendicular to \( F \) for Boltzmann (Figures 15(d) and (e)). By contrast, \( L \) is always exactly perpendicular or parallel to \( F \) for MEFD (Figures 15(g) and (h)) by its assumption of axisymmetry in momentum space. In the preshock region, say, at \( r = 90 \text{ km} \), where matter falls almost freely with supersonic speeds even in the rotating model, however, \( L \) is almost completely parallel to \( F \) for both Boltzmann and MEFD. The forward-peaked distribution is simply formed in momentum space (Figures 15(c), (f), and (i)). These results indicate that rapid nonradial matter motions induced either by convection or by rotation can easily disrupt axisymmetry of the neutrino distribution in momentum space and make the longest principal axis neither perpendicular nor parallel to the flux vector. The rapid rotation, in particular, slants the ellipse of the Eddington tensor in the \( \phi \) direction.
6.3. Principal-axis Analysis in the Laboratory Frame

So far we have been working in the FR. The assessment of the closure relations in the LB may also be important, though, since the closure relations are sometimes applied in this frame. In fact, Shibata et al. (2014) gave the closure relation written with the radiation pressure tensor in the LB, and some previous investigations (Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2019, 2020; Iwakami et al. 2020) were conducted in the LB. Picking up a snapshot at 100 ms after bounce obtained by a Boltzmann simulation and comparing $P_{rr}$, $P_{\theta\theta}$, and $P_{r\theta}$ derived in the simulation and those calculated with the Levermore closure, Richers et al. (2017) showed that the relative error in the off-diagonal component is larger than that in the diagonal component in the region, where PNS convection is developed. Nagakura et al. (2018) also pointed out that the difference in $k_{r\theta}$ between the Boltzmann transport and the Levermore closure they observed in the semitransparent region at $t = 15$ ms after bounce is intrinsic and never reduced. Harada et al. (2020), on the other hand, argued that larger $k_{r\theta}$ for the Levermore closure is due to significant lateral motions caused by the prompt convection.

The effect of rotation was considered by Harada et al. (2019), who investigated the Eddington tensor at $t = 12$ ms after bounce for their slowly rotating model. Since the model was a slow rotator, they gave a detailed analysis only to $k_{r\theta}$, although rotation should affect $k_{r\phi}$ more directly. They proceeded then to the analysis of the principal axes of the Eddington tensor. After just mentioning that they do not coincide with the $r$-, $\theta$-, and $\phi$-axes, since all off-diagonal components are nonvanishing, they focused on the radial profiles of the three eigenvalues of the Eddington tensor in their assessment of the closure relations. As we mentioned earlier, the analyses of the eigenvalues alone are insufficient in multiple dimensions, and the corresponding eigenvectors should also be studied, which we have done in the FR in the preceding section. As the closure relations are not always exactly equivalent between the two frames (see Equations (36)–(38) and (42)–(44)), it is nontrivial if the finding in the FR is also true in the LB. Note in passing that the principal-axis analysis, considering both eigenvalues and eigenvectors, was applied already by Iwakami et al. (2020) to their 3D model. It is nonrotating, though. The investigation here is hence complementary to all the previous studies one way or another. We also address the systematic difference between the frames. The individual components of the Eddington tensor are also inspected to understand the effect of convection and rotation on $k_{r\phi}$ and $k_{r\theta}$, respectively.

Figures 16 and 17 show the color maps of $|\mu_{FL}|$ in the LB for $\nu_\alpha$ at $t = 10$ ms for $\epsilon_{FR} = 3.9$ and 14.5 MeV, respectively. For Boltzmann transport, the blue colored region in the LB (Figures 16(a), 16(b), 17(a), and 17(b)) appears at a similar position to the one in the FR (Figures 13(a), 13(b), 14(a), and 14(b)). However, the color depth for the LB is lighter than that for the FR in the regions, where fast nonradial motions exist. Such light-blue regions are sporadic in the nonrotating model.
Figure 9. Color maps of entropy $S$ with the fluid velocity vectors $V$ (top panels) and the number density $N$ of $\nu_e$ with its average velocity vectors $V_\nu$ (bottom panels) for nonrotating (left panels) and rotating (right panels) 11.2 $M_\odot$ models at $t = 10$ ms.

Figure 10. As in Figure 9, but for the progenitor mass 15.0 $M_\odot$. 

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Radial distributions of velocity components for 11.2 and 15.0 \( M_\odot \) progenitor models at \( t = 10 \) ms, where \( V_r, V_\theta \), and \( V_\phi \) are the radial, polar, and azimuthal matter velocities, respectively.

The difference in \( \theta_{FR} \) between the Boltzmann calculation and the MEFD closure approximation for the rotating model comes from the \( k^{r\phi} \) component in the LB. To see this, we plot in the left panels of Figures 19 and 20 the radial distributions of diagonal (top panels) and off-diagonal (bottom panels) components of the Eddington tensor for \( \nu_e \) at \( t = 10 \) ms for the nonrotating and rotating models, respectively. The Eddington tensors \( k_{Boltz}^r \) and \( k_{MEFD}^r \) are calculated directly and in the MEFD closure approximation, respectively, from the numerical data obtained by the Boltzmann simulation. In the nonrotating model, the term \( k_{MEFD}^r \) is smaller than \( k_{Boltz}^r \) for \( \epsilon_{FR} = 3.9 \) MeV at \( r = 30 \) km, while \( k_{Boltz}^{\phi\phi} \) is larger than \( k_{Boltz}^{\theta\theta} \) at the same position (Figure 19(a)). This is why the longest axis of the ellipsoid for MEFD is perpendicular to the \( P_r - P_\theta \) plane at this radius in the LB (Figure 18(g)). The off-diagonal component \( k^{r\phi} \) is nonvanishing (Figure 19(c)) owing to the prompt convection for the nonrotating model and tilts the longest principal axis of the ellipsoid for MEFD from the coordinate axes (Figure 18(d)). The radial profile is qualitatively different between the Boltzmann calculation and the MEFD closure approximation. For the rotating model, on the other hand, \( k^{r\phi} \) grows steeply with radius at \( r \lesssim 90 \) km and then decreases gradually at \( r \gtrsim 90 \) km for \( \epsilon_{FR} = 14.5 \) MeV (Figure 22(c)). The misalignment of \( L \) and \( F \) is related to this growth of \( k^{r\phi} \) (Figures 18(f) and (i)). It should be pointed out that the MEFD closure overestimates the value of \( k^{r\phi} \) by a factor of 2.

The results of the MEFD closure for the LB change from those for the FR but in a different manner from what we have observed above for the Boltzmann transport. In fact, for the nonrotating model, light-blue regions appear around the central dark-blue region for \( \epsilon_{FR} = 3.9 \) MeV (Figures 16(c) and (d)); the central dark-blue region is shrunk for \( \epsilon_{FR} = 14.5 \) MeV (Figures 17(c) and (d)). For the rotating model, on the other hand, greenish regions emerge around the central dark-blue region (Figures 16(d) and 17(d)). In these regions, \( F \) and \( L \) are misaligned with each other in the LB (Figures 18(f) and (i)), while they are well aligned in the FR (Figures 15(f) and (i)). This is understandable from Equation (43): in the LB, the Eddington tensor is constructed not only from the flux vector but also from the matter velocity. It is interesting to point out that the angle between \( F \) and \( L \) for the MEFD closure is larger than that for the Boltzmann transport at \( r = 90 \) km, i.e., in the preshock region, in the LB.
Figure 12. Radial distributions of the density $\rho$ and the mean free path divided by the radius $\Lambda$ for the 11.2 and 15.0 $M_\odot$ progenitor models at $t = 10$ ms.

Figure 13. Color maps of $|\mu_{FL}|$ in the FR for $\epsilon_{FR} = 3.9$ MeV at $t = 10$ ms for the nonrotating (left panels) and rotating (right panels) 11.2 $M_\odot$ models, where $\mu_{FL} = \cos \theta_{FL}$. The results obtained from the Boltzmann transport and MEFD closure are shown in the top and bottom panels, respectively.
Figure 14. As in Figure 13, but for the neutrino energy $\epsilon_{\text{FR}} = 14.5$ MeV.

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(Problem 22(c)). Although it is not shown in this paper, this is also found for the Levermore closure, which is not observed in the previous papers for the slowly rotating model (Harada et al. 2019) and for the 3D nonrotating model (Iwakami et al. 2020).

To identify the origin of the discrepancy between $k^r_{\text{MEFD}}$ and $k^r_{\text{Boltz}}$ in the LB, the contributions of individual terms in Equation (42)–(44) are investigated. The right panels of Figures 19 and 22 present the decompositions of the diagonal (top panels) and off-diagonal (bottom panels) components of the Eddington tensor in the LB for the nonrotating model at $\epsilon_{\text{FR}} = 3.9$ MeV and for the rotating model at $\epsilon_{\text{FR}} = 14.5$ MeV, where “1/3,” “VV,” and “HV,” respectively, denote the first, second, and third terms of Equation (43) multiplied by $3(1 - p_{\text{MEFD}})/2$, and “FF” is the right-hand side of Equation (44) multiplied by $(3p_{\text{MEFD}} - 1)/2$. In the diagonal components of the Eddington tensor, the terms “1/3” and “FF” are the terms common to the FR and the LB. The term “HV,” on the other hand, plays an important role pushing $k^r_{\text{MEFD}}$ close to $k^r_{\text{Boltz}}$ in the region of $\Lambda \lesssim 10$ (Figures 12(c) and 19(b)). In the off-diagonal component, the term “HV” alone reproduces the results for Boltzmann rather well, and the term “FF” seems to be responsible for the discrepancy we observed at $\Lambda \lesssim 10$ (Figure 19(d)). At $\Lambda \gtrsim 10$ (Figure 12(e)), however, the roles of the terms “FF” and “HV” are inverted (Figures 22(b) and 22(d)). This indicates that the interpolation of the optically thick and thin limits employed in Equation (42) does not perform very well in the transition region in the LB. Note that in the derivation of the pressure tensor in the LB, in which the terms “HV” and “VV” obtained, the equations are truncated at the first order of the mean free path (Shibata et al. 2014). Hence, higher-order terms may be needed to better reconstruct the pressure tensor in the transition region in the LB (see also Harada et al. 2019, 2020). Although not presented here, we confirmed that this is a common problem to the Levermore closures in Equation (33).

7. Conclusions

Using the data at the early post-bounce phase derived by the 2D full Boltzmann neutrino transport simulations of CCSNe for 11.2 and 15.0 $M_\odot$ progenitor models with and without rotation, we have assessed the performance of some closure relations that are used commonly in the truncated moment method for neutrino transport in the literature. We have first compared the Eddington factors given by these closure relations with the results of the 1D simulations in spherical symmetry for the nonrotating models. We have then applied the principal-axis analysis to the results of the 2D simulations in axisymmetry for both the nonrotating and rotating models. We have studied in detail the eigenvectors of the Eddington tensor and made a comparison between the simulation results and the maximum entropy closure for Fermi–Dirac radiations (MEFD). The new findings of this study are summarized as follows:

1. The comparison of the Eddington factor obtained directly from the 1D simulation results and the Eddington factors estimated via the closure relations from the energy density and flux in the same simulation results confirmed that MEFD performs best among the closures investigated but that even the MEFD closure fails when $p < 1/3$ happens simultaneously with $e < 0.5$, which occurs, for example, immediately behind the shock wave in the early post-bounce phase.
2. The application of the principal-axis analysis to the Eddington tensors that are obtained either directly from the 2D simulation results or via the MEFD closure in the FR has revealed that the longest principal axis \( L \) in the simulation results is neither parallel nor perpendicular to the flux vector \( F \). This is in sharp contrast to the estimates by the MEFD closure, in which \( L \) is always either parallel or perpendicular to \( F \) because of the axisymmetry assumed in momentum space. The difference is particularly remarkable in the region of rapid convection and/or rotation. In particular, the rapid rotation makes the ellipse of the Eddington tensor inclined in the \( \phi \) direction.

3. The same analysis has been repeated in the LB. Since the closure relations are not always exactly equivalent between the two frames as in Equations (36)–(38) and (42)–(44), this is nontrivial. We have observed that \( L \) is neither parallel nor perpendicular to \( F \) even for the MEFD closure, as it includes explicitly velocity-dependent terms.

Figure 15. Distribution functions \( \mathcal{F} \) of \( \nu_e \) in momentum space (top panels) and the corresponding ellipsoids of the Eddington tensors obtained in the Boltzmann transport (middle panels) and those evaluated in the MEFD closure (bottom panels) for the nonrotating model at \( r = 30 \) km (left panels) and for the rotating model at \( r = 65 \) km (middle panels) and 90 km (right panels). The neutrino energy \( \epsilon_{\text{FR}} \) is 3.9 MeV for the nonrotating model and 14.5 MeV for the rotating model. The reference frame is the FR.
Figure 16. Color maps of $|\mu_{\nu_e}|$ in the LB for $\varepsilon_{FR} = 3.9$ MeV at $t = 10$ ms for the nonrotating (left panels) and rotating (right panels) $11.2 M_\odot$ models. The top and bottom panels show the results for Boltzmann and MEFD, respectively.

Figure 17. As in Figure 16, but for the neutrino energy $\varepsilon_{FR} = 14.5$ MeV.
in the LB. We have demonstrated, however, that the angle that $L$ makes with $F$ is different between the direct calculations and the estimates by the closure relation. In fact, the radial profiles of $k^r \theta$ are qualitatively different between the two cases, while $k^r \phi$ is overestimated by the MEFD closure by a factor of two for the rotating model. We have also found that these discrepancies, which are most remarkable in the regions where matter moves nonradially by convection and/or rotation, are caused by inaccuracy of the interpolation between the optically thick and thin limits. These results are valid not only for the MEFD closure but also for the Levermore closure.

The last point may be the most important, since the closure relation is normally imposed in the LB in the truncated moment methods in the literature. Now that there have been many 3D simulations of CCSNe conducted with the truncated moment methods with some closure relation and their results seem to be converging among different groups, we believe that the subtleties in the closure relations, particularly when it is employed in the LB, should be taken seriously and studied quantitatively.

Of course, we need to extend the analysis to later phases, where the neutrino-driven convection and/or the standing accretion shock instability are in operation in the gain region.
Figure 19. Radial profiles of the diagonal and off-diagonal components (left panels) and the decomposed components (right panels) of the Eddington tensor of $\nu_e$ for the nonrotating model at $\epsilon_{FR} = 3.9$ MeV, where “1/3,” “VV,” and “HV” denote the first, second, and third terms of Equation (43) multiplied by $3(1 - p_{\text{MEFD}})/2$, respectively, and “FF” is the term of Equation (44) multiplied by $(3p_{\text{MEFD}} - 1)/2$. The reference frame is the LB.

Figure 20. As in Figure 19, but for the rotating model at $\epsilon_{FR} = 14.5$ MeV.
and the lepton-driven convection is developing in the PNS. In so doing, higher resolutions should be used for both space and momentum space to reproduce forward-peaked angular distributions accurately. Such simulations will be done on the Fugaku supercomputer in Japan, not to mention that 3D rotating models will also be investigated.

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Appendix
Resolution Test

This appendix shows the results of the resolution tests in this paper. Readers may also refer to earlier publications for more tests (Richers et al. 2017; Nagakura et al. 2018; Harada et al. 2019; Iwakami et al. 2020). Using the data of the 1D resolution test in Iwakami et al. (2020), we show the effect of the angular resolution on the Eddington factor $p$ as a function of the flux factor $f$. In this test, the hydrodynamic simulations with Boltzmann transport start from the onset of core collapse after changing angular resolution. There are some differences in the simulations between the main results and resolution tests in this paper. The Furusawa–Togashi EOS (Furusawa et al. 2017) is used in this test. The neutrino energy mesh is divided into 16 bins in the resolution tests and into 20 bins in the main results. Although the grid number of the polar angle in momentum space is $N_\theta = 10$ in the main results, $N_\theta = 6, 12, 48$ are selected in the resolution tests.

Figures 21 and 22 show $p - f$ plots for $\epsilon_{\text{FR}} = 3.3$ and 12.6 MeV, respectively. The red circles denote $p_{\text{Boltzmann}}$ and the light-blue circles denote $p_{\text{MEFD}}$. The regions of $p < 1/3$ are roughly the same among $N_\theta = 6, 12, 48$. The advantage of MEFD closure is in the region of $p < 1/3$, where $\epsilon > 0.5$ is independent of the angular resolution. However, the difference in $p_{\text{Boltzmann}}$ among $N_\theta = 6, 12, 48$ increases as $f$ increases, especially for $\epsilon_{\text{FR}} = 3.3$ MeV. For Boltzmann transport simulations, the neutrinos tend to have a more forward-peaked distribution with increasing $N_\theta$. Although the convergence is not completely achieved even in $N_\theta = 48$, it was found that the closure matching with the results for Boltzmann transport depends on the flux factor $f$ and the neutrino energy $\epsilon$. Hence, it is difficult to determine the best “algebraic Eddington factor method” proposed so far.

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Figure 22. As in Figure 21, but for $\epsilon_R = 12.6$ MeV, which roughly agrees with the average energy just behind the shock wave.