INITIAL BOUNDARY VALUE PROBLEM FOR ANISOTROPIC FRACTIONAL TYPE DEGENERATE PARABOLIC EQUATION

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Abstract

The aim of the paper is to generalize the author’s previous work [15]. We extend the argument [15] for any uniformly elliptic operator in divergence form $Lu = -\text{div}(A(x)\nabla u)$, more precisely, we study a fractional type degenerate elliptic equation posed in bounded domains with homogeneous boundary conditions

$$\partial_t u = \text{div}(u A(x) \nabla L^{-s} u)$$

where $L^{-s}$ is the inverse $s$-fractional elliptic operator for any $s \in (0, 1)$. This work consists of two part. The first part is devoted to state how the boundary condition will be consider (in the spirit of F. Otto [25]), and to give a formulation for the IBVP. In the second part, It is shown the existence of mass-preserving, non-negative weak solutions satisfying energy estimates for measurable and bounded non-negative initial data.

1 Introduction

The aim of this paper is study the existence of solution of (1.1). More precisely, to state how the boundary conditions will be consider, and to express in a convenient way the concept of solution for the following problem

$$\begin{cases}
\partial_t u = \text{div}(u A(x) \nabla u) & \text{in } \Omega_T, \\
u|_{t=0} = u_0 & \text{in } \Omega, \\
u = 0 & \text{on } (0, T) \times \partial \Omega,
\end{cases}$$

(1.1)

where $\Omega_T := (0, T) \times \Omega$, for any real number $T > 0$, and $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded open set having smooth ($C^2$) boundary $\partial \Omega$. Moreover, the initial data $u_0$ is a measurable, bounded non-negative function in $\Omega$, and considered homogeneous Dirichlet boundary condition, while $K := L^{-s}$, is the inverse of

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the \( s \)-fractional elliptic operator (see Definition 2.1), and the matrix \( A(x) = (a_{ij}(x))_{n \times n} \) satisfy the uniform ellipticity condition.

The nonlocal, possibly degenerate, parabolic type equation is inspired in a non-local Fourier’s law, that is

\[
q := -\kappa(x,u) \nabla K u,
\]

where \( u \) is the temperature, \( q \) is the diffusive flux, and \( \kappa(x,u) \) denotes here the (non-negative definite) thermal conductivity tensor.

Equation (1.1) is motivated in the so-called Caffarelli-Vazquez model of a porous media (degenerate) diffusion model given by a fractional potential pressure law [6]. Under some conditions, they found mass-preserving, nonnegative weak solutions of the equation satisfying energy estimates for the Cauchy problem. Moreover, Caffarelli, Soria and Vazquez establish the Hölder regularity of such weak solutions for the case \( s \neq 1/2 \) in [5] and the case \( s = 1/2 \) has been proved in [7] by Caffarelli and Vazquez.

A similar model was introduced at the same time by Biler, Imbert, Karch and Monneau (see [2] [3] and [16]). A different approach to prove existence based on gradient flows has been developed by Lisini, Mainini and Segatti (see [20]). Then the model has been generalized in [26] [27] [28] [29] [30]. Uniqueness is still open in general, but under some truly restrictive regularity assumption is proven in [31].

On bounded domain, the Caffarelli-Vazquez model was studied by myself and Neves in [15]. The main novelty of this work was to state how the boundary condition is considered. For \( \frac{1}{2} < s < 1 \), the boundary condition is assumed in the sense of trace, and for \( 0 < s \leq \frac{1}{2} \), we inspired in the definition of weak solutions for scalar conservation laws posed in bounded domains as proposed by Otto [25] (see also [21], [22]).

In another context, Nguyen and Vazquez [24] studied a similar model with a different approach in the definition of weak solution. Moreover they proved existence and smoothing effects.

In this paper, we focus in the (simplest) anisotropic degenerate case, that is, \( \kappa(x,u) = u A(x) \), where the coefficients \((a_{ij})\), \( i,j = 1, \cdots, n \) describing the anisotropic, heterogeneous nature of the medium.

The main goal of this work is to state how boundary condition will be considered. In order to treat this part of the boundary, we follow an approach inspired by F. Otto [23]. In method we propose, the boundary conditions, written as limits of integrals on \((0,T) \times \partial \Omega\) of a certain function. To this purpose, it is introduced a function \( \Psi : [0,1] \times \partial \Omega \to \overline{\Omega} \) called \( C^1 \)-admissible deformation (see Section 2.3).

A simple explanation to use the \( C^1 \)-map \( \Psi \) is the following. Consider the equation \( \text{div}(uA(x)\nabla K u) = 0 \) in \( \Omega \), and \( u = 0 \) on \( \partial \Omega \). Multiply it by \( \phi \in C^\infty_c(\mathbb{R}^n) \), integrate by part, and from the boundary condition, we expect that

\[
\int_{\partial \Omega} u(r)A(r)\nabla K u(r) \cdot \nu(r) \phi(r) dr = 0 \quad (1.2)
\]
where $\nu$ is the unit outward normal field on $\partial \Omega$. Notice that the existence of trace for $u$ does not necessarily exist in the sense of traces in $H^s(\Omega)$. Moreover the trace for $u$ and $\nabla K u \cdot \nu$ are mutually exclusive (see Remark 3.1). Then (1.2) is not well defined, to avoid this difficulty, it will be considered a simple modification, as follows

$$\text{ess lim}_{\tau \to 0^+} \int_{\partial \Omega} u(r) A(r) \nabla K u(r) \cdot \nu(r) \phi(\Psi^{-1}(r)) \, dr = 0,$$

where $\Psi(r)$ is a $C^1$-deformation, and $\nu$ is the unit outward normal field on $\partial \Omega = \Psi(\partial \Omega)$ (see Section 2.3).

On the other hand, we also show an equivalent definition of (weak) solutions as given by Definition 3.1, more precisely an integral equivalent definition (see the Equivalence Theorem 3.2).

After introducing the definition of weak solution to above problem, we study of existence of solution in the proposed setting. We prove that the weak solution previously defined can be obtained as the limit of solution of regularized equation (1.1), to prove that we use energy estimates and apply the Aubin-Lions Compacteness Theorem.

On the other hand, an important talk is about the non-homogeneous Dirichlet boundary conditions. First, if a given boundary data $u_b \neq 0$ is smooth enough to be considered as the restriction (in the sense trace in $H^s(\Omega)$) of a function $u_b$ defined in $\Omega_T$, then the strategy developed here follows right way with standard modifications. After that, some forcing terms appears, one of them is

$$\text{div}(u_b A(x) \nabla K u_b), \quad (1.3)$$

thus to make sense (1.3), it is necessary that $u_b \in D(L^{(1-s)/2})$, (see Definition 3.1), but this is not necessary true, since $u_b \neq 0$ on the boundary (see the counterexample in [1]). To avoid this difficulty, it have to use the fractional operators with inhomogeneous boundary conditions as defined in [1].

Finally, we stress that the uniqueness property is not established in this paper. In fact, it seems to be open even if for the Cauchy problem. Somehow, the ideas from scalar conservation laws could be useful, more precise, the doubling of variables of Kružkov [1N].

2 Preliminaries

In this section, we review some results of Dirichlet spectral fractional elliptic (DSFE for short) and admissible deformation. We mainly provide the proofs of the new results, in particular we stress Proposition 2.3. One can refer to [4], [8], and [13] for an introduction.

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. We denote by $\mathcal{H}^\theta$ the $\theta$-dimensional Hausdorff measure, and $(L^2(\Omega))^n$ is the Cartesian product of $L^2(\Omega)$ $n$-times.
2.1 Dirichlet Spectral Fractional Elliptic

Here and subsequently, $\Omega \subset \mathbb{R}^n$ is a bounded open set with $C^2$-boundary $\partial \Omega$. We are mostly interested in fractional powers of a strictly positive self-adjoint operator defined in a domain, which is dense in a (separable) Hilbert space. Therefore, we are going to consider hereupon the operator $L u := -\text{div}(A(x) \nabla u)$ with homogeneous Dirichlet data, where $A(x) = (a_{ij}(x))_{n \times n}$ is a matrix, such that $a_{ij} \in C^\infty(\bar{\Omega})$ ($i,j = 1, \cdots, n$) and satisfy the uniform elliptic condition

$$\Lambda_1 |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda_2 |\xi|^2,$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, for some ellipticity constant $0 < \Lambda_1 \leq \Lambda_2$. Moreover, the coefficients are symmetric $a_{ij}(x) = a_{ji}(x)$, bounded and measurable in $\Omega$.

Due to well-known the elliptic operator $L$ is nonnegative and self-adjoint in $H^1_0(\Omega)$, therefore from spectral theory, there exists a complete orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ of $L^2(\Omega)$, where $\varphi_k \in H^1_0(\Omega)$ are eigenfunction corresponding to eigenvalue $\lambda_k$ for each $k \geq 1$, moreover

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots,$$

Therefor the operator $L$ and its the domain $D(L)$ could be rewrite as follow

$$D(L) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^2 |\langle u, \varphi_k \rangle|^2 < \infty \right\},$$

$$L u = \sum_{k=1}^\infty \lambda_k \langle u, \varphi_k \rangle \varphi_k, \quad \text{for each } u \in D(L).$$

Remark 2.1. Since $\partial \Omega$ is $C^2$, it follows that $\varphi_k \in C^\infty(\Omega) \cap C^2(\overline{\Omega})$, (see [13], p. 214) and $D(L) = H^2(\Omega) \cap H^1_0(\Omega)$ (see [13], p. 186). The former property, that is the regularity of the eigenfunctions $\varphi_k$, help us to study the regularized problem (1.1) and the second property is important in Proposition 2.2.

Now, from functional calculus, we have the following definition

Definition 2.1 (DSFE). Let $\Omega \subset \mathbb{R}^n$ is a bounded open set with $C^2$-boundary $\partial \Omega$. Consider the operator $L u := -\text{div}(A(x) \nabla u)$ with homogeneous Dirichlet data, where $A(x) = (a_{ij}(x))_{n \times n}$ is a symmetric matrix, such that $a_{ij} \in C^\infty(\Omega)$ ($i,j = 1, \cdots, n$) and satisfy the condition (2.1). For each $s \in (0,1)$, the DSFE $L^s : D(L^s) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, is defined as follow

$$L^s u = \sum_{k=1}^\infty \lambda_k^s \langle u, \varphi_k \rangle \varphi_k,$$

$$D(L^s) = \left\{ u \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^{2s} |\langle u, \varphi_k \rangle|^2 < +\infty \right\}.$$
Analogously, we can also define $L^{-s} : D(L^{-s}) \subset L^2(\Omega) \to L^2(\Omega)$ for $s \in (0, 1)$.

The next proposition generalize some properties of the $s$-fractional Laplacian in bounded domain. In particular, we observe that $D(L^{-s}) = L^2(\Omega)$.

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $s \in (0, 1)$, and consider $L^s$, and $L^{-s}$ the operators defined above. Then, we have:

1. $D(L) \subset D(L^s)$, thus $D(L^s)$ is dense in $L^2(\Omega)$.

2. For all $u \in D(L^s)$, there exists $\alpha > 0$ which is the coercivity constant of $L$ and satisfies
   \[
   \langle L^s u, u \rangle \geq \alpha^s \|u\|^2_{L^2(\Omega)}.
   \]  
   Moreover, it follows that $(L^s)^{-1} = L^{-s}$, also $L^s$ and $L^{-s}$ are self-adjoint.

3. $D(L^s)$ endowed with the inner product
   \[
   \langle u, v \rangle_s := \langle u, v \rangle + \int_\Omega L^s u(x) L^s v(x) \, dx
   \]
   is a Hilbert space. In particular the norm $\| \cdot \|_s$ is defined by
   \[
   \|u\|^2_s = \|u\|^2_{L^2(\Omega)} + \|L^s u\|^2_{L^2(\Omega)}.
   \]  

**Proof.** The proof proceed analogously to the proposition 2.1 \[15\] \qed

Now, we state a Poincare’s type inequality for the DSFE, and an equivalent norm for $D(L^s)$.

**Corollary 2.1 (Poincare’s type inequality).** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then for each $s > 0$, we have

\[
\|u\|_{L^2(\Omega)} \leq \lambda_s^{-s} \|L^s u\|_{L^2(\Omega)}, \quad \text{for all } u \in D(L^s).
\]

Moreover, the norm defined in \[2.4\] and

\[
\|u\|^2_s := \int_\Omega |L^s u(x)|^2 \, dx
\]  

are equivalent.

**Remark 2.2.** As a consequence of the above results, we could consider the inner product in $D(L^s)$, as follow

\[
\langle u, v \rangle_s = \int_\Omega L^s u(x) L^s v(x) \, dx.
\]
Now, the aim is to characterize (via interpolation) the space $D(L^s)$. To begin, we consider $u \in D(L)$, then, since $L^{1/2}$ is self-adjoint and from the definition of $L$ we have

$$
\int_{\Omega} |L^{1/2}u(x)|^2 \, dx = \int_{\Omega} L^{1/2}u(x) L^{1/2}u(x) \, dx = \int_{\Omega} L u(x) u(x) \, dx
$$

$$
= \int_{\Omega} A(x) \nabla u(x) \cdot \nabla u(x) \, dx.
$$

On the other hand, using the uniform elliptic condition and choosing $\xi = \nabla u$ in (2.1), and after that integrate over $\Omega$, we obtain

$$
\Lambda_1 \int_{\Omega} |\nabla u(x)|^2 \, dx \leq \int_{\Omega} A(x) \nabla u(x) \cdot \nabla u(x) \, dx \leq \Lambda_2 \int_{\Omega} |\nabla u(x)|^2 \, dx,
$$

hence

$$
\Lambda_1 \|u\|_{H^{1/2}_0(\Omega)} \leq \|L^{1/2}u\|_{L^2(\Omega)} \leq \Lambda_2 \|u\|_{H^1_0(\Omega)},
$$

which mean the norm $\| \cdot \|_{1/2}$ is equivalent to the norm $\| \cdot \|_{H^1_0(\Omega)}$. Consequently, from the density of $D(L)$ in $D(L^{1/2})$, and also in $H^1_0(\Omega)$, it follows that $D(L^{1/2}) = H^1_0(\Omega)$. Similarly, we have the following result:

**Proposition 2.2.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

i) If $s \in (0, 1/2)$, then

$$
D(L^s) = \begin{cases} 
H^{2s}(\Omega), & \text{if } 0 < s < 1/4, \\
H^{1/2}_{00}(\Omega), & \text{if } s = 1/4, \\
H^{s}_{00}(\Omega), & \text{if } 1/4 < s \leq 1/2.
\end{cases}
$$

ii) If $s \in (1/2, 1)$, then

$$
D(L^s) = \left[ H^2(\Omega) \cap H^1_0(\Omega), H^1_0(\Omega) \right]_{1 - \theta},
$$

where $\theta = 2s - 1$. Moreover, $D(L^s) \subset H^{2s}(\Omega) \cap H^1_0(\Omega)$.

**Proof.** The proof follows applying the discrete version of J-Method for interpolation, see [4] and also [14].

Here and subsequently, we denote for each $s \in (0, 1)$ the operators:

$$
K := L^{-s} \quad \text{and} \quad \mathcal{H} = K^{1/2} := L^{-s/2}.
$$

Then, we consider the following

**Proposition 2.3.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.
(1) There exists a constant $C_\Omega > 0$ such that if $u \in H^1_0(\Omega)$, then $\nabla K u \in \left(L^2(\Omega)\right)^n$ and
\[
\int_\Omega |\nabla K u(x)|^2 \, dx \leq C_\Omega \int_\Omega |\nabla u(x)|^2 \, dx. \tag{2.11}
\]
Similarly, for each $u \in H^1_0(\Omega)$, $\nabla H u \in \left(L^2(\Omega)\right)^n$ and
\[
\int_\Omega |\nabla H u(x)|^2 \, dx \leq C_1^{1/2} \int_\Omega |\nabla u(x)|^2 \, dx. \tag{2.12}
\]

(2) If $u \in H^1_0(\Omega)$, then
\[
\Lambda_1 \int_\Omega |\nabla H u|^2 \, dx \leq \int_\Omega A(x) \nabla K u \cdot \nabla u \, dx \leq \Lambda_2 \int_\Omega |\nabla H u|^2 \, dx, \tag{2.13}
\]

Proof. Since $u \in H^1_0(\Omega)$, it is enough to consider $u \in C_\infty(\Omega)$, and then apply a standard density argument.

To show (1), we use the equivalence norm (2.8) or (2.7). Then, we have
\[
\int_\Omega |\nabla K u(x)|^2 \, dx \leq \Lambda_1^{-1} \int_\Omega |\mathcal{L}^{1/2} K u(x)|^2 \, dx = \Lambda_1^{-1} \sum_{k=1}^\infty \lambda_k |\langle K u, \varphi_k \rangle|^2
\]
\[
= \Lambda_1^{-2s} \int_\Omega |\mathcal{L}^{1/2} u(x)|^2 \, dx \leq \Lambda_1^{-1} \sum_{k=1}^\infty \lambda_k |\langle u, \varphi_k \rangle|^2
\]
\[
= \Lambda_1^{-1} \lambda_1^{-2s} \sum_{k=1}^\infty \lambda_k |\langle u, \varphi_k \rangle|^2 \leq \Lambda_1^{-2s} \sum_{k=1}^\infty \lambda_k |\langle u, \varphi_k \rangle|^2
\]
and analogously for $\nabla H u$.

Now, we prove (2). First, we integrate by parts to obtain
\[
\int_\Omega A(x) \nabla K u(x) \cdot \nabla u(x) \, dx = \int_\Omega -\text{div}(A(x) \nabla K u(x)) u(x) \, dx = \int_\Omega \mathcal{L}^{1-s} u(x) u(x) \, dx,
\]
where we have used the definition of $K u$. Due to the $\mathcal{L}^{1-s}$ being self-adjoint (Proposition 2.1(2)), it follows that
\[
\int_\Omega A(x) \nabla K u(x) \cdot \nabla u(x) \, dx = \int_\Omega |\mathcal{L}^{(1-s)/2} u(x)|^2 \, dx.
\]
Therefore, using the equivalence norm (2.8) together with the definition of $H u$, we have
\[
\Lambda_1 \int_\Omega |\nabla H u(x)|^2 \, dx \leq \int_\Omega A(x) \nabla K u(x) \cdot \nabla u(x) \, dx \leq \Lambda_2 \int_\Omega |\nabla H u(x)|^2 \, dx,
\]
\[\square\]
Remark 2.3. Under the above assumptions, and by a similar arguments, we obtain that \( K_u \in H^{1+2s}(\Omega) \cap H^0_0(\Omega) \) and

\[
\Lambda_1 \int_\Omega |\nabla H u(x)|^2 \, dx \leq \int_\Omega A(x) \nabla u(x) \cdot \nabla K u(x) \, dx \leq \Lambda_2 \int_\Omega |\nabla H u(x)|^2 \, dx,
\]

for all \( u \in H^1_0(\Omega) \).

2.2 Heat Semigroup Formula

There are another ways of defining fractional elliptic operator, which turn out to be equivalent to DSFE. Here, we recall the Heat Semigroup formula, and address [8] for a complete description.

First, given a function \( u = \sum_{k=1}^{\infty} u_k \phi_k \) in \( L^2(\Omega) \), the weak solution \( v(t, x) \) of the IBVP

\[
\begin{cases}
v_t + \mathcal{L} v = 0, & \text{in } \Omega \times (0, +\infty), \\
v(x, t) = 0, & \text{on } \partial \Omega \times [0, +\infty), \\
v(x, 0) = u(x), & \text{in } \Omega
\end{cases}
\]

is given by

\[
v(x, t) = e^{-t \mathcal{L}} u(x) = \sum_{k=1}^{\infty} e^{-t \lambda_k} u_k \phi_k(x).
\]

In particular, \( v \in L^2((0, \infty); H^0_0(\Omega)) \cap C([0, \infty); L^2(\Omega)) \) and \( \partial_t v \in L^2((0, \infty); H^{-1}(\Omega)) \).

The following Lemma express in a different way the definition of DSFE.

Lemma 2.1. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set, and \( 0 < s < 1 \).

1. If \( u \in D(\mathcal{L}^s) \), then

\[
\mathcal{L}^s u = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t \mathcal{L}} u - u) \, \frac{dt}{t^{1+s}} \text{ in } L^2(\Omega).
\]

More precisely, if \( w \in L^2(\Omega) \), then

\[
\langle \mathcal{L}^s u, w \rangle_{L^2(\Omega)} = \frac{1}{\Gamma(-s)} \int_0^\infty \left( \langle e^{-t \mathcal{L}} u, w \rangle_{L^2(\Omega)} - \langle u, w \rangle_{L^2(\Omega)} \right) \, \frac{dt}{t^{1+s}}.
\]

2. If \( u \in L^2(\Omega) \), then

\[
\mathcal{L}^{-s} u = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t \mathcal{L}} u \, \frac{dt}{t^{1-s}} \text{ in } L^2(\Omega).
\]

Proof. An excellent reference is the paper by Caffarelli and Stinga [8], see also [15].
The main basic idea of the proof is based on the following observation. For any \( \lambda > 0 \) and \( 0 < s < 1 \) we have
\[
\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-s}}
\]
\[
\lambda^s = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+s}}.
\]
Now, from definition (2.2), and Fubini’s Theorem, the proof follows.

\[\square\]

2.3 Admissible Deformation

Let us fix here some notation and background used in this paper, we first consider the notion of \( C^1 \)-admissible deformations, which is used to give the correct notion of traces. One can refer to [23].

**Definition 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open set. A \( C^1 \)-map \( \Psi : [0,1] \times \partial \Omega \to \overline{\Omega} \) is said a \( C^1 \) admissible deformation, when it satisfies the following conditions:

1. For all \( r \in \partial \Omega \), \( \Psi(0,r) = r \).
2. The derivative of the map \( [0,1] \ni \tau \mapsto \Psi(\tau,r) \) at \( \tau = 0 \) is not orthogonal to \( \nu(r) \), for each \( r \in \partial \Omega \).

Moreover, for each \( \tau \in [0,1] \), we denote: \( \Psi_\tau \) the mapping from \( \partial \Omega \) to \( \Omega \), given by \( \Psi_\tau(r) := \Psi(\tau,r) \); \( \partial \Omega_\tau = \Psi_\tau(\partial \Omega) \); \( \nu_\tau \) the unit outward normal field in \( \partial \Omega_\tau \). In particular, \( \nu_0(x) = \nu(x) \) is the unit outward normal field in \( \partial \Omega \).

**Remark 2.4.** It must be recognized that domains with \( C^2 \) boundaries always have \( C^1 \) admissible deformations. Indeed, it is enough to take \( \Psi(\tau,r) = r - \epsilon \nu(r) \) for sufficiently small \( \epsilon > 0 \). From now on, we say \( C^1 \)-deformations for short.

Now, we state the following Lemma, which will be useful to the define the level set function associated with the \( C^1 \)-deformation \( \Psi \).

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open set with \( C^1 \)-boundary \( \partial \Omega \) and the \( C^1 \) deformation \( \Psi : [0,1] \times \partial \Omega \to \overline{\Omega} \), then there exist \( m \in \mathbb{N} \), \( V_i \subset \mathbb{R}^n \) and \( h_i \in C^1(V_i) \) \( (i = 1, \cdots, m) \), such that
\[
x \in \partial \Omega_\tau \cap V_i \Rightarrow h_i(x) = \tau
\]
for all \( i = 1, \cdots, m \).

**Proof.** Since \( \Omega \subset \mathbb{R}^n \) be an open set with \( C^1 \) boundary. Then, for each \( x \in \partial \Omega \) there exists a neighbourhood \( W \) of \( x \) in \( \mathbb{R}^n \), an open set \( U \subset \mathbb{R}^{n-1} \) and a \( C^1 \) diffeomorphism mapping \( \varsigma : U \to \partial \Omega \cap W \).

On the other hand, we define \( \psi : [0,1] \times U \to \overline{\Omega} \) by
\[
\psi(\tau,y) := \Psi(\tau,\varsigma(y)),
\]
which is a $C^1$ function, due to $\xi$ and $\Psi$ are $C^1$. Moreover from the item (2) of the definition 2.2, we have the Jacobian of $\psi$ in $(0,y)$, satisfies
\[
J\psi(0,y) = J[\xi](y) |\partial_x \Psi(0, \xi(y)) \cdot \nu(\xi(y))| > 0,
\]
for all $y \in U$. Then, applying the Inverse Function Theorem and passing to a smaller neighbourhood if necessary (still denoted by $U$), there exists $\varrho > 0$ such that, the function $\psi : [0,\varrho) \times U \rightarrow \Omega$ is a $C^1$ diffeomorphism onto its image.

At the same time, since $\partial \Omega$ is compact, we can find finitely many points $x_i \in \partial \Omega$, corresponding sets $W_i \subset \mathbb{R}^n$; $U_i \subset \mathbb{R}^{n-1}$ and functions $\gamma_i \in C^1(U_i)$ $(i = 1, \cdots , m)$, such that $\partial \Omega \subset \bigcup_{i=1}^m W_i$ and
\[
\gamma_i : U_i \rightarrow \partial \Omega \cap W_i,
\]
moreover, there exists $\varrho_i > 0$, $(i = 1, \cdots , m)$, such that, $\psi_i : [0,\varrho_i) \times U_i \rightarrow \Omega$ is a $C^1$ diffeomorphism onto its image, where $\psi_i(\tau, y) := \Psi(\tau, \gamma_i(y))$.

Finally, we consider $\varrho = \min\{\varrho_i; i = 1, \cdots , m\}$. Define $V_i := \Psi([0,\varrho) \times \gamma_i(U_i))$ and $h_i : V_i \rightarrow [0,\varrho]$, as follow
\[
h_i(x) := \pi_1 \circ \psi_i^{-1}(x), \quad x \in V_i,
\]
where $\pi_1 : \mathbb{R} \times \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, given by $\pi_1(a,b) = a$. In particular, if $x \in \partial \Omega \cap V_i$, we obtain that $h_i(x) = \tau$.

As a consequence of the above Lemma, we define the level set function associated with the $C^1$-deformation $\Psi$, that is to say, the function
\[
h : \overline{\Omega} \rightarrow [0,\varrho)
\]
by setting $h(x) = h_i(x)$, if $x \in V_i$ and $h(x) = \varrho$, for $x \in \overline{\Omega} \setminus \bigcup_{i=1}^m V_i$, which is clearly a $C^1$ function. Moreover, we have that $\nabla h(x) \neq 0$ for all $x \in \bigcup_{i=1}^m V_i$, and also $\nabla h(r)$ is parallel to $\nu_\tau(r)$ on $\partial \Omega_\tau$.

To follow, we define some auxiliary functions, which are important to show existence of solutions of the IBPV (1.1).

(1) Without loss of generality, we may assume $\varrho = 1$ (define in lemma 2.2), and define
\[
s(x) := \begin{cases} 
  h(x), & \text{if } x \in \Omega, \\
  -h(x), & \text{if } x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

(2) For each $k \in \mathbb{N}$, and all $x \in \mathbb{R}^n$, define $\xi_k$ by
\[
\xi_k(x) := 1 - \exp \left(-k \cdot s(x) \right). \quad (2.14)
\]

**Lemma 2.3.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with $C^2$ boundary. Then, it follows that:
The function \( s(x) \) is Lipschitz continuous in \( \mathbb{R}^n \), and \( C^1 \) on the closure of \( \{ x \in \mathbb{R}^n : |s(x)| < \delta \} \).

The sequence \( \{ \xi_k \} \) satisfies
\[
\lim_{k \to +\infty} \int_{\Omega} |1 - \xi_k|^2 \, dx = 0, \quad \text{and} \quad \lim_{k \to +\infty} \int_{\Omega} |\nabla \xi_k|^2 \, dx = 0. \tag{2.15}
\]

**Proof.** This Lemma is an extension of the result obtained in section 2.8 of Máté, Necas, Rokyta and Ruzicka [21], p. 129.

To finish this section, let us consider the following

(1) Let a non-negative function \( \gamma \in C^1_c(\mathbb{R}) \), with support contained in \([0, 1]\), such that, \( \int \gamma(t) \, dt = 1 \). Then, we consider the sequences \( \{ \delta_j \}_{j \in \mathbb{N}} \), and \( \{ H_j \}_{j \in \mathbb{N}} \), defined by
\[
\delta_j(t) := j \gamma(jt), \quad H_j(t) := \int_0^t \delta_j(s) \, ds.
\]

Thus, for each \( j \geq 1 \), \( H_j'(t) = \delta_j(t) \), and clearly the sequence \( \{ H_j' \} \) converges as \( j \to \infty \) to the Dirac \( \delta \)-measure in \( D'(\mathbb{R}) \), while the sequence \( \{ H_j \} \) converges pointwise to the Heaviside function
\[
H(t) = \begin{cases} 
1, & \text{if } t \geq 0, \\
0, & \text{if } t < 0.
\end{cases}
\]

(2) Let \( \Psi \) a \( C^1 \)-admissible deformation and \( \partial \Omega \) is \( C^2 \). Then for any point \( x \in \partial \Omega \) there exists a neighbourhood \( W \) of \( x \) in \( \mathbb{R}^n \), an open set \( U \subset \mathbb{R}^{n-1} \) and a \( C^2 \) mapping \( \zeta : U \to \partial \Omega \cap W \), which is a \( C^1 \)-diffeomorphism. Moreover, it satisfies
\[
\lim_{\tau \to 0} J[\Psi_\tau \circ \zeta] = J[\zeta] \quad \text{in } C(U),
\]
where \( J[\cdot] \) is the Jacobian. Furthermore \( J[\Psi_\tau] \) defined by
\[
J[\Psi_\tau](r) := \frac{J[\Psi_\tau \circ \zeta^{-1}(r)]}{J[\zeta](\zeta^{-1}(r))}, \tag{2.16}
\]
satisfies \( J[\Psi_\tau] \to 1 \) uniformly as \( \tau \to 0 \).

**3 Initial Boundary Value Problem**

Here we give a definition, which establishes how the boundary condition will be considered for the equation (1.1). We also state an equivalent definition of weak solution ( Equivalent Theorem 3.2 ).
3.1 Definition of weak solution

We seek for a suitable (weak) solution $u(t,x)$ defined in $\Omega_T$, in this way the next definition tells us in which sense $u(t,x)$ is a solution to the IBVP (1.1).

**Definition 3.1.** Given an initial data $u_0 \in L^\infty(\Omega)$ and $0 < s < 1$, a function

$$u \in L^2 \left((0,T);D(\mathcal{L}^{(1-s)/2})\right) \cap L^\infty(\Omega_T)$$

is called a weak solution of the IBVP (1.1), when $u(t,x)$ satisfies:

1. **The integral equation:** For each $\phi \in C_0^\infty(\Omega_T)$

$$\int_{\Omega_T} u(\partial_t \phi - A(x)\nabla K u \cdot \nabla \phi) \, dx \, dt = 0. \quad (3.1)$$

2. **The initial condition:** For all $\zeta \in L^1(\Omega)$

$$\text{ess lim}_{t \to 0^+} \int_{\Omega} u(t,x)\zeta(x) \, dx = \int_{\Omega} u_0(x)\zeta(x) \, dx. \quad (3.2)$$

3. **The boundary condition:** For each $\gamma \in C_0^\infty((0,T) \times \mathbb{R}^n)$ and any $C^1$-deformation $\Psi$

$$\text{ess lim}_{\tau \to 0^+} \int_{\partial\Omega_T} u(t,r)A(r)\nabla K u(t,r) \cdot \nu(r) \gamma(t,\Psi^{-1}(r)) \, d\mathcal{H}^{n-1}(r) \, dt = 0. \quad (3.3)$$

**Remark 3.1.** Given $u \in L^2 \left((0,T);D(\mathcal{L}^{(1-s)/2})\right)$ the limit in the left hand side of (3.3), a priori, does not necessarily exist. Indeed, the existence of trace for $u$ and $\nabla K u \cdot \nu$ are mutually exclusive. For instance, if $0 < s < 1/2$ then from Proposition 2.2, it follows that $u \in L^2 \left((0,T);H^{1-s}_0(\Omega)\right)$, which implies that $u$ has trace on $\partial\Omega$, moreover $u = 0$ on $(0,T) \times \partial\Omega$, contrarily $K u \in L^2 \left((0,T);H^{1+s}(\Omega) \cap H^1_0(\Omega)\right)$, which means that, $\nabla K u \cdot \nu$ does not have trace on $\partial\Omega$. Vice versa result for $1/2 \leq s < 1$.

However, if $u \in L^2 \left((0,T);D(\mathcal{L}^{(1-s)/2})\right) \cap L^\infty(\Omega_T)$ and satisfies (3.1), then the essential limit in (3.3) exist, in particular the boundary condition makes sense. Analogously, the initial condition (3.2).

**Lemma 3.1.** Let $u \in L^2 \left((0,T);D(\mathcal{L}^{(1-s)/2})\right) \cap L^\infty(\Omega_T)$, with $s \in (0,1)$. Then, for each function $\gamma \in C_0^\infty((0,T) \times \mathbb{R}^n)$ and any $C^1$-deformation $\Psi$

$$\int_0^T \int_{\partial\Omega_T} u(t,r)A(r)\nabla K u(t,r) \cdot \nu(r) \gamma(t,\Psi^{-1}(r)) \, d\mathcal{H}^{n-1}(r) \, dt$$

exists for a.e. $\tau > 0$ small enough.
Proof. First, due to $u \in L^2((0,T); D((-\Delta_D)^{(1-s)/2}))$, the integral
\[
\int_0^T \int_{I_m(\Psi)} u(t,x) \nabla K u(t,x) \cdot \nabla h(x) \gamma(t, \Psi^{-1}_h(x)) \, dx dt
\]
exists, where $h$ is the level set function associated with the deformation $\Psi_\tau$, which is defined in Section 2. Hence applying the Coarea Formula for the function $h$, we obtain
\[
\int_0^T \int_{I_m(\Psi)} u(t,x) \nabla K u(t,x) \cdot \nabla h(x) \gamma(t, \Psi^{-1}_h(x)) \, dx dt = \int_0^1 \int_0^T \int_{\partial \Omega} u(t,r) \nabla K u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi^{-1}_\tau(r)) \, d\mathcal{H}^{n-1}(r) \, dt \, d\tau.
\]
(3.4)
Thus, we obtain from (3.4) that
\[
\int_0^T \int_{I_m(\Psi)} u(t,x) \nabla K u(t,x) \cdot \nabla h(x) \gamma(t, \Psi^{-1}_h(x)) \, dx dt
\]
exists for a.e. $\tau \in (0,1)$ and each $\gamma \in C^\infty_c((0,T) \times \mathbb{R}^n)$.

To follow, we define some auxiliary set, which are important to show that the Definition 3.1 makes sense. Let $u \in L^2((0,T); D((-\Delta_D)^{(1-s)/2})) \cap L^\infty(\Omega_T)$ be a function satisfying (3.1), then consider the following sets:

1. Let $E$ be a countable dense subset of $C^1_c(\Omega)$. For each $\zeta \in E$, we define the set of full measure in $(0,T)$ by
   \[
   E_\zeta := \left\{ t \in (0,T) / t \text{ is a Lebesgue point of } I(t) = \int_\Omega u(t,x)\zeta(x) \, dx \right\},
   \]
   and consider
   \[
   E := \bigcap_{\zeta \in E} E_\zeta,
   \]
   which is a set of full measure in $(0,T)$.

2. Let $F$ be a countable dense subset of $C^\infty_c((0,T) \times \mathbb{R}^n)$. For each $\gamma \in F$, we define the set of full measure in $(0,1)$ by
   \[
   F_\gamma = \left\{ \tau \in (0,1) / \tau \text{ is a Lebesgue point of } J(\tau) \right\},
   \]
   where
   \[
   J(\tau) = \int_0^T \int_{\partial \Omega} u(t,r) A(r) \nabla K u(t,r) \cdot \nu_\tau(r) \gamma(t, \Psi^{-1}_\tau(r)) \, d\mathcal{H}^{n-1}(r) \, dt,
   \]
   which makes sense thanks to Lemma 3.1. Moreover, we consider
   \[
   F := \bigcap_{\gamma \in F} F_\gamma,
   \]
   which is also a set of full measure in $(0,1)$. For more details see [15].
The next theorem ensures the existence of the essential limit (3.2) and the boundary condition (3.3).

**Theorem 3.1.** Let \( u \in L^2 \left( (0, T); D \left( \mathcal{L}^{(1-s)/2} \right) \right) \cap L^\infty (\Omega_T) \) and assume that \( u \) satisfies (3.1), then:

1. There exists a function \( \bar{u} \in L^\infty (\Omega) \), such that
   \[
   \text{ess lim}_{t \to 0^+} \int_\Omega u(t, x) \zeta(x) \, dx = \int_\Omega \bar{u}(x) \zeta(x) \, dx, \tag{3.6}
   \]
   for each \( \zeta \in L^1(\Omega) \).

2. For each \( \gamma \in C^\infty_c((0, T) \times \mathbb{R}^n) \), and any \( C^1 \)-deformation \( \Psi \), the
   \[
   \text{ess lim}_{\tau \to 0^+} \int_0^\tau \int_{\partial \Omega} u(t, r) A(r) \nabla K u(t, r) \cdot \nu(r) \gamma(t, \Psi^{-1}(r)) \, d\mathcal{H}^{n-1}(r) \, dt, \tag{3.7}
   \]
   exists.

**Proof.** 1. To prove (3.6), let \( \zeta \in \mathcal{E} \) and consider the set \( E \) defined above. Then, for each \( t \in E \), \( \| u(t, \cdot) \|_\infty \leq C \). Thus we can find a sequence \( \{t_m\} \), \( t_m \in E \), \( m \in \mathbb{N} \), \( t_m \to 0 \) as \( m \to \infty \) and a function \( \bar{u} \in L^\infty (\Omega) \), such that \( u(t_m, \cdot) \to \bar{u}(\cdot) \) weakly-* in \( L^\infty (\Omega) \) as \( m \to \infty \).

   If \( c \in E \), then for large enough \( m \), \( t_m < c \). We fix such \( t_m < c \) and set \( \gamma_j(t) = H_j(t - t_m) - H_j(t - c) \), where the sequence \( H_j(\cdot), j \in \mathbb{N} \) is defined in Section 2. Therefore, taking in (3.1) \( \phi(t, x) = \gamma_j(t) \zeta(x) \), we have
   \[
   \int_{\Omega_T} u(t, x) \gamma_j'(t) \zeta(x) \, dx \, dt = \int_{\Omega_T} u(x) A(x) \nabla K u \cdot \nabla \zeta(x) \gamma_j(t) \, dx \, dt.
   \]
   The above expression may be written as
   \[
   \int_0^T \gamma_j'(t) I(t) \, dt = \int_0^T \gamma_j(t) \int_{\Omega_T} u(x) A(x) \nabla K u \cdot \nabla \zeta(x) \, dx \, dt. \tag{3.8}
   \]

   Then passing to the limit in (3.8) as \( j \to \infty \), and taking into account that, \( t_m, c \) are Lebesgue points of the function \( I(t) \), also that \( \gamma_j(t) \) converges pointwise to the characteristic function of the interval \([t_m, c)\), we obtain
   \[
   I(t_m) - I(c) = \int_{t_m}^c \int_{\Omega} u(x) A(x) \nabla K u \cdot \nabla \zeta(x) \, dx \, dt,
   \]
   which implies in the limit as \( m \to \infty \) that
   \[
   \int_{\Omega} \bar{u}(x) \zeta(x) \, dx - I(c) = \int_0^c \int_{\Omega} u(x) A(x) \nabla K u(x) \cdot \nabla \zeta(x) \, dx \, dt,
   \]
   (14)
for all $c \in E$. Therefore, in view of the density of $\mathcal{E}$ in $L^1(\Omega)$, we have

$$\lim_{E \ni t \to 0} I(t) = \int_\Omega \tilde{u}(x) \zeta(x) \, dx,$$

for each $\zeta \in L^1(\Omega)$, which proves (3.4).

2. Now, we show (3.7). Let $\gamma \in \mathcal{F}$, consider $F$, and define $S := \Psi(F \times \partial \Omega)$. For $\tau_1, \tau_2 \in F$, with $\tau_1 < \tau_2$, define $\zeta_j(\tau) = H_j(\tau - \tau_1) - H_j(\tau - \tau_2)$, $j \in \mathbb{N}$, and take in (3.1) $\phi(t, x)$ defined by

$$\phi(t, x) = \begin{cases} 
\gamma(t, \Psi_{h(x)}^{-1}(x)) \zeta_j(h(x)), & \text{for } x \in S, \\
0, & \text{for } x \in \Omega \setminus S,
\end{cases}$$

where $h(x)$ is the level set associated with the deformation $\Psi$, which is defined in Section 2. Then, we have

$$\int_0^T \int_S u(t, x) \partial_t \zeta_j(t, \Psi_{h(x)}^{-1}(x)) \, \zeta_j(h(x)) \, dx \, dt = \int_0^T \int_S u(t, x) A(x) \nabla K u(t, x) \cdot \nabla \gamma(t, \Psi_{h(x)}^{-1}(x)) \, \zeta_j(h(x)) \, dx \, dt + \int_0^T \int_S u(t, x) A(x) \nabla K u(t, x) \cdot \nabla h(x) \, \zeta_j'(h(x)) \, \gamma(t, \Psi_{h(x)}^{-1}(x)) \, dx \, dt.$$

Consequently, applying the Coarea Formula for the function $h$, we obtain

$$\int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial \Omega_t} u(t, r) \frac{\partial_t \gamma(t, \Psi_{h(r)}^{-1}(r))}{|\nabla h(r)|} \, d\mathcal{H}^{n-1}(r) \, dt \, d\tau = \int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial \Omega_t} u(t, r) A(r) \nabla K u(t, r) \cdot \frac{\nabla \gamma(t, \Psi_{h(r)}^{-1}(r))}{|\nabla h(r)|} \, d\mathcal{H}^{n-1}(r) \, dt \, d\tau + \int_0^1 \zeta_j'(\tau) \int_0^T \int_{\partial \Omega_t} u(t, r) A(r) \nabla K u(t, r) \cdot \nu(r) \gamma(t, \Psi_{h(r)}^{-1}(r)) \, d\mathcal{H}^{n-1}(r) \, dt \, d\tau.$$

Therefore, applying the Dominated Convergence Theorem in the above equation, we get in the limit as $j \to \infty$

$$J(\tau_2) + \int_{\tau_1}^{\tau_2} \Phi(\tau) \, d\tau = J(\tau_1) + \int_{\tau_1}^{\tau_2} \Phi(\tau) \, d\tau,$$

for all $\tau_1, \tau_2 \in F$ and $\gamma \in \mathcal{F}$, where $\Phi(\tau)$ is given by

$$\int_0^T \int_{\partial \Omega_t} u(t, r) \left( \frac{\partial_t \gamma(t, \Psi_{h(r)}^{-1}(r))}{|\nabla h(r)|} - A(r) \nabla K u(t, r) \cdot \frac{\nabla \gamma(t, \Psi_{h(r)}^{-1}(r))}{|\nabla h(r)|} \right) \, d\mathcal{H}^{n-1}(r) \, dt.$$
On the other hand, since $F$ is dense in $C_c^\infty((0,T) \times \mathbb{R}^n)$, we have that (3.12) holds for $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^n)$. Consequently, we obtain

$$\lim_{F \ni \tau \to 0^+} \int_0^T \int_{\partial \Omega} u(t,r) A(r) \nabla K u(t,r) \cdot \nu(r) \gamma(t,\Psi^{-1}(r)) dH_{n-1}(r) dt$$

exists for all $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^n)$.

The following result expresses in convenient way the concept of (weak) solution of the IBVP (1.1) as given by Definition 3.1.

**Theorem 3.2** (Equivalence Theorem). A function $u \in L^2((0,T); D(L^{(1-s)/2})) \cap L^\infty(\Omega_T)$ is a weak solution of the IBVP (1.1) if, and only if, it satisfies

$$\iint_{\Omega_T} u(t,x) \left( \partial_t \phi - A(x) \nabla K u \cdot \nabla \phi \right) dx dt + \int_{\Omega_T} u_0(x) \phi(0,x) dx = 0, \quad (3.10)$$

for each test function $\phi \in C_c^\infty((\mathbb{R},T) \times \mathbb{R}^n)$.

**Proof.** 1. Assume that $u$ satisfies (3.10), then we show that $u$ verifies (3.1)–(3.3).

To show (3.1), it is enough to consider test functions $\phi \in C_c^\infty(\Omega_T)$. In order to show (3.2), let us consider $\phi(t,x) = \gamma_j(t) \zeta(x)$, $\gamma_j(t) = H_j(t + t_0) - H_j(t - t_0)$ for any $t_0 \in E$ (fixed), and $\zeta \in E$. Then, from (3.10) we have

$$\iint_{\Omega_T} u(t,x) \gamma_j'(t) \zeta(x) dx dt + \int_{\Omega} u_0(x) \zeta(x) dx = \iint_{\Omega_T} u(t,x) A(x) \nabla K u(t,x) \cdot \nabla \zeta(x) \gamma_j(t) dx dt.$$

Passing to the limit in the above equation as $j \to \infty$, and taking into account that $t_0$ is Lebesgue point of $I(t)$, we obtain

$$\int_{\Omega} u(t_0,x) \zeta(x) dx = \int_{\Omega} u_0(x) \zeta(x) dx$$

$$- \int_0^{t_0} \int_{\Omega} u(t,x) A(x) \nabla K u(x) \cdot \nabla \zeta(x) dx dt,$$

where we have used the Dominated Convergence Theorem. Since $t_0 \in E$ is arbitrary, and in view of the density of $E$ in $L^1(\Omega)$, it follows from (3.11) that

$$\text{ess lim}_{t \to 0} \int_{\Omega} u(t,x) \zeta(x) dx = \int_{\Omega} u_0(x) \zeta(x) dx$$

for all $\zeta \in L^1(\Omega)$, which shows (3.2).
Finally, let us show (3.3). Similarly to proof in Proposition 3.1 (2), we choose
\[
\phi(t, x) = \begin{cases} 
\gamma(t, \Psi_{h(x)}^{-1}(x)) \zeta_j(h(x)), & \text{for } x \in S, \\
0, & \text{for } x \in \Omega \setminus S,
\end{cases}
\]
where \( \gamma \in \mathcal{F}, \ \zeta_j(\tau) = H_j(\tau + \tau_0) - H_j(\tau - \tau_0), \) with \( \tau_0 \in F, \) and \( S = \Psi(F \times \partial \Omega). \) Therefore, from (3.10) we obtain
\[
\int_0^T \int_S u(t, x) \partial \gamma(t, \Psi_{h_\tau(x)}^{-1}(x)) \zeta_j(h(x)) \ dx dt
= \int_0^T \int_S u(t, x) A(x) \nabla K u(t, x) \cdot \nabla \gamma(t, \Psi_{h_\tau(x)}^{-1}(x)) \zeta_j(h(x)) \ dx dt
+ \int_0^T \int_S u(t, x) A(x) \nabla h(x) \zeta_j'(h(x)) \gamma(t, \Psi_{h_\tau(x)}^{-1}(x)) \ dx dt.
\]
On other hand, applying the Coarea Formula for the function \( h \) in the above equation, we have
\[
\int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial \Omega_r} u(t, r) \frac{\partial \gamma(t, \Psi_{r_\tau}^{-1}(r))}{|\nabla h(r)|} \ d\mathcal{H}^{n-1}(r) dt d\tau
= \int_0^1 \zeta_j(\tau) \int_0^T \int_{\partial \Omega_r} u(t, r) A(r) \nabla K u(t, r) \cdot \frac{\nabla \gamma(t, \Psi_{h_\tau(x)}^{-1}(x)) (r)}{|\nabla h(r)|} \ d\mathcal{H}^{n-1}(r) dt d\tau
+ \int_0^1 \zeta_j'(\tau) \int_0^T \int_{\partial \Omega_r} u(t, r) A(r) \nabla h(r) \cdot \nu_\tau(r) \gamma(t, \Psi_{r_\tau}^{-1}(r)) \ d\mathcal{H}^{n-1}(r) dt d\tau.
\]
Then, passing to the limit in the above equation as \( j \to \infty \) and taking into account that \( \tau_0 \) is a Lebesgue point of \( J(\tau) \), and also that \( \zeta_j(t) \) converges pointwise to the characteristic function of the interval \([-\tau_0, \tau_0]\), we obtain
\[
J(\tau_0) = \int_0^{\tau_0} \Phi(\tau) d\tau, \quad (3.12)
\]
for all \( \tau_0 \in F \) and \( \gamma \in \mathcal{F} \), where \( \Phi(\tau) \) is given by
\[
\int_0^T \int_{\partial \Omega_r} u(t, r) \left( \frac{\partial \gamma(t, \Psi_{r_\tau}^{-1}(r))}{|\nabla h(r)|} - A(r) \nabla K u(t, r) \cdot \frac{\nabla \gamma(t, \Psi_{h_\tau(\cdot)}^{-1}(\cdot))(r)}{|\nabla h(r)|} \right) \ d\mathcal{H}^{n-1}(r) dt.
\]
On the other hand, since \( \mathcal{F} \) is dense in \( C^\infty_c((0, T) \times \mathbb{R}^n) \), we have that (5.12) holds for \( \gamma \in C^\infty_c((0, T) \times \mathbb{R}^n) \). Then, for each \( \tau \in F \) we have
\[
|J(\tau)| \leq C |\Psi((0, \tau) \times \partial \Omega)|,
\]
Finally, we use the Coarea Formula for the function $h$ the above equation, and pass to limit as $j \to \infty$, we obtain

$$\lim_{F \ni \tau \to 0} \int_0^T \int_{\partial \Omega_\tau} u(t,r) \ A(r) \nabla K u(t,r) \cdot \nu_\tau(r) \gamma(t,\Psi_\tau^{-1}(r)) \ d\mathcal{H}^{n-1}(r) dt = 0,$$

for all $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^n)$.

2. Now, let us consider: (3.1)–(3.3) $\Rightarrow$ (3.10). The idea is similar to that one done before; for completeness we give the main points. First, we consider $j \in \mathbb{N}$ sufficiently large and take for any $t_0 \in E$

$$\phi(t,x) = \psi(t,x) H_j(t - t_0),$$

where $\psi \in C_c^\infty((\infty, T) \times \Omega)$, $H_j(t)$ as considered before. Then, from (3.1) we obtain

$$\int_{\Omega_T} u(t,x) \partial_t \psi(t,x) H_j(t - t_0) \ dx dt + \int_{\Omega_T} u(t,x) \ \psi(t,x) \ H'_j(t - t_0) \ dx dt
\begin{equation}
- \int_{\Omega_T} u(t,x) A(x) \nabla K u(t,x) \cdot \nabla \psi(t,x) H_j(t - t_0) \ dx dt = 0. 
\end{equation}

Passing to the limit as $j \to \infty$, and taking into account that $t_0$ is a Lebesgue point of $I(t)$, also that $H_j(\cdot - t_0)$ converges pointwise to the Heaviside function $H(\cdot - t_0)$, after that, taking the limit as $E \ni t_0 \to 0$ and using (3.2), we have

$$\int_{\Omega_T} u(t,x) \partial_t \psi(t,x) \ dx dt + \int_{\Omega} u_0(x) \ \psi(0,x) \ dx
\begin{equation}
- \int_{\Omega_T} u(t,x) A(x) \nabla K u(t,x) \cdot \nabla \psi(t,x) \ dx dt = 0, 
\end{equation}

for all $\psi \in C_c^\infty((\infty, T) \times \Omega)$. In particular, for

$$\psi(t,x) = \phi(t,x)(1 - \zeta_j(h(x))),$$

where $\phi \in C_c^\infty((\infty, T) \times \mathbb{R}^n)$, $h(x)$ as above and we consider the function $\zeta_j(\tau) = H_j(\tau + \tau_0) - H_j(\tau - \tau_0)$, where $\tau_0 \in F$. Then, from (3.13) we obtain

$$\int_{\Omega_T} u(t,x) \partial_t \phi(t,x)(1 - \zeta_j(h(x))) \ dx dt + \int_{\Omega} u_0(x) \ \phi(0,x)(1 - \zeta_j(h(x))) \ dx
\begin{equation}
- \int_{\Omega_T} u(t,x) A(x) \nabla K u(t,x) \cdot \nabla \phi(t,x) \ (1 - \zeta_j(h(x))) \ dx dt
\begin{equation}
+ \int_{\Omega_T} u(t,x) A(x) \nabla K u(t,x) \cdot \nabla h(x) \ \zeta'_j(h(x)) \phi(t,x) \ dx dt dt = 0.
\end{equation}

Finally, we use the Coarea Formula for the function $h$ in the last integral of the above equation, and pass to limit as $j \to \infty$. Therefore, we obtain for all
\[ \phi \in C^\infty_c((\mathbb{R}^n) \times (\mathbb{R}^n)) \]
\[ \int \int_{\Omega} u(t, x)(\partial_t \phi(t, x) - A(x)\nabla u(t, x) \cdot \nabla \phi(t, x)) \, dx \, dt + \int_{\Omega} u_0(x) \phi(0, x) \, dx \, dt = 0, \]
where we have used (3.3).

### 3.2 Solution estimates for the IBVP

Now, we show basic estimates, which are required to show existence of weak solutions to the IBVP (1.1). We perform formal calculations, assuming that \( u \geq 0 \) satisfies the required smoothness and integrability assumptions.

1. **Conservation of mass:** For all \( t \in (0, T) \),
\[ \frac{d}{dt} \int_{\Omega} u(t, x) \, dx = \int_{\Omega} \text{div}(uA(x)\nabla u) \, dx = 0. \]

2. **Conservation of positivity:** If the initial condition \( u_0 \) is non-negative, then the solution \( u \) of (1.1) is non-negative.

Indeed, we assume \( u_0 > 0 \) (without loss of generality). For any \( 0 < t_0 \leq T \) (fixed), let \( x_0 \in \Omega \) be a point where \( u(t_0, \cdot) \) is a minimum, which is to say
\[ u(t_0, x) \geq u(t_0, x_0) \quad \text{for each} \quad x \in \Omega. \]

We claim that \( u(t_0, x_0) \geq 0 \). Note that, since \( t_0 \) is arbitrary, this sentence implies that \( u \) is non-negative. Let us suppose that, \( u(t_0, x_0) < 0 \), and consider for each \( \delta > 0 \),
\[ \varphi_\delta(w) = \begin{cases} (w^2 + \delta^2)^{1/2} - \delta, & \text{for} \quad 0 \leq w, \\ 0, & \text{for} \quad w \leq 0. \end{cases} \quad (3.14) \]

Then, \( \varphi_\delta(w) \) converges to \( w^+ = \max\{w, 0\} \) as \( \delta \to 0^+ \). Now, multiplying the first equation in (1.1) by \( \varphi_\delta'(u) \) and evaluating in \( (t_0, x_0) \), we obtain
\[ \frac{d}{dt} \varphi_\delta(u)(t_0, x_0) = \nabla u(t_0, x_0) \cdot A(x_0)\nabla u(t_0, x_0) \varphi_\delta'(u(t_0, x_0)) \\
+ u(t_0, x_0) \varphi_\delta'(u(t_0, x_0)) \text{div}(A(\cdot)\nabla u)(t_0, x_0). \]

The first term in the right hand side of the above equation is zero, since \( x_0 \) is a point where \( u(t_0, \cdot) \) is a minimum. For the second term, we recall that
\[ -\text{div}(A(\cdot)\nabla u) = \mathcal{L}u = \mathcal{L}^{1-s}u, \]

hence due to Lemma 2.1, it follows that
\[ -\text{div}(A(\cdot)\nabla u)(t_0, x_0) = \frac{1}{\Gamma(s - 1)} \int_0^\infty (e^{-t\mathcal{L}}u(t_0, x_0) - u(t_0, x_0)) \frac{dt}{t^{2-s}}, \quad (3.15) \]
where $\Gamma(s - 1) < 0$ ($s < 1$) and $v(t, x) = e^{-tL}u(t_0, x)$ is the weak solution of the IBVP

\[
\begin{align*}
\frac{\partial v}{\partial t} + Lv &= 0, & \text{in } (0, T) \times \Omega, \\
v(t, x) &= 0, & \text{on } [0, T] \times \partial \Omega, \\
v(0, x) &= u(t_0, x), & \text{in } \Omega.
\end{align*}
\]

Now, applying the (weak) maximum principle, we get

\[
\min_{(t, x) \in \Omega_T} e^{-tL}u(t_0, x) = \min_{(t, x) \in \Gamma_T} e^{-tL}u(t_0, x),
\]

where $\Gamma_T$ is the parabolic boundary of $\Omega_T$, which comprises $\{0\} \times \Omega$ and $[0, T] \times \partial \Omega$. Consequently, we have from (3.16)

\[
\min_{(t, x) \in \Omega_T} e^{-tL}u(t_0, x) = \min \left\{ 0, \min_{x \in \Omega} u(t_0, x) \right\}. \tag{3.16}
\]

Therefore, it follows that $e^{-tL}u(t_0, x) \geq u(t_0, x_0)$, for all $x \in \Omega$. Thus from (3.16) we deduce that, $-\text{div}(A(\cdot)\nabla K u)(t_0, x_0) \geq 0$. Moreover, since $u \varphi_{\delta}(u) \geq 0$, we have at $(t_0, x_0)$ that $\frac{d}{dt}\varphi_{\delta}(u) \geq 0$, and thus

\[
\varphi_{\delta}(u(t_0)) \geq \varphi_{\delta}(u_0). \tag{3.17}
\]

Then, passing to the limit in (3.17) as $\delta \to 0$, we obtain $u^+(t_0) \geq u_0$, which implies that $u(t_0, x_0) > 0$, which is a contradiction, hence $u$ is non-negative.

3. $L^\infty$ estimate: The $L^\infty$ norm of $u$ does not increase in time.

Indeed, for any $0 < t_0 \leq T$ (fixed), let $x_0$ be a point where $u(t_0, \cdot)$ is a maximum, which is to say

\[
u(t_0, x) \leq u(t_0, x_0) \quad \text{for all } x \in \Omega.
\]

Therefore, we have

\[
\frac{du}{dt}(t_0, x_0) = \nabla u(t_0, x_0) \cdot A(x_0)\nabla K u(t_0, x_0) + u(t_0, x_0)\text{div}(A(\cdot)\nabla K u)(t_0, x_0).
\]

The first term in the right hand side of the above equation is zero, since $x_0$ is a point where $u(t_0, \cdot)$ is a maximum. For the second term, we use the same ideas as above, thus $\text{div}(A(\cdot)\nabla K u)(t_0, x_0) \leq 0$. Moreover, since $u \geq 0$, then at $(t_0, x_0)$ we have $\frac{du}{dt} \leq 0$, which implies item 3.

4. First energy estimate: For all $t \in (0, T)$,

\[
\int_{\Omega} u(t, x) \log u(t, x) dx + \Lambda_1 \int_0^t \int_{\Omega} |\nabla H u(t', x)|^2 dxdt' \leq \int_{\Omega} u_0(x) \log u_0(x) dx
\]
Indeed, multiplying the first equation \(1.1\) by \(\log u(t', x)\) and integrate on \(\Omega\). Then after integration by part, we obtain

\[
\frac{\partial}{\partial t} \int_{\Omega} u(t', x) \log u(t', x) dx + \int_{\Omega} A(x) \nabla K u(t', x) \cdot \nabla u(t', x) dx = 0.
\]

On the other hand, from Proposition 2.3 (2), we have

\[
\frac{\partial}{\partial t} \int_{\Omega} u(t', x) \log u(t', x) dx + \Lambda \int_{\Omega} |\nabla \mathcal{H} u(t', x)|^2 dx \leq 0
\]

Then, we integrate over \((0, t)\), for all \(0 < t < T\), to obtain the first energy estimate.

5. **Second energy estimate**: Similar to the above description, is not difficult to show that

\[
\frac{1}{2} \int_{\Omega} |\mathcal{H} u(t_2, x)|^2 dx' + \Lambda \int_{t_1}^{t_2} \int_{\Omega} |\nabla K u(t', x)|^2 dx dt \leq \frac{1}{2} \int_{\Omega} |\mathcal{H} u(t_1, x)|^2 dx,
\]

for \(0 \leq t_1 < t_2 \leq T\).

### 4 Existence of Weak Solutions

The aim of this section is to find a weak solution of \(1.1\). To show that we use the equivalent definition given by the theorem 3.2. The following theorem show the existence of weak solution.

**Theorem 4.1.** Let \(u_0 \in L^\infty(\Omega)\) be a non-negative function. Then, there exists a weak solution \(u \in L^2((0, T); D((1-s)/2)) \cap L^\infty(\Omega_T)\) of the IBVP \(1.1\).

The proof will be divided into two subsections.

### 4.1 Smooth Solution

To show the existence of the solution we use the method of vanishing viscosity and also it will be eliminated the degeneracy by raising the level set \(\{u = 0\}\) in the diffusion coefficient. The basic idea of which is as follows: for \(\delta, \mu \in (0, 1)\) we study the parabolic perturbation of the Cauchy problem \(1.1\) given by

\[
\begin{align*}
\partial_t u_{\mu, \delta} - \delta \text{div}(A(x) \nabla u_{\mu, \delta}) &= \text{div}(q(u_{\mu, \delta}) A(x) \nabla K u_{\mu, \delta}) \quad \text{in } \Omega_T, \quad (4.1) \\
u_{\mu, \delta}(0, \cdot) &= u_{0\delta} \quad \text{in } \Omega, \quad (4.2) \\
u_{\mu, \delta} &= 0 \quad \text{on } (0, T) \times \partial \Omega. \quad (4.3)
\end{align*}
\]

where \(q(\lambda) = \lambda + \mu\), and \(u_{0\delta}\) is a non-negative smooth bounded approximation of the initial data \(u_0 \geq 0\), satisfying \(u_{0\delta} = 0\) on \(\partial \Omega\).

Now, we make use of the well known results of existence, uniqueness and uniform \(L^\infty\) bounds for quasilinear parabolic problems. Therefore, for each
δ, µ > 0, there exists a unique classical solution \( u_{\mu,\delta} \in C^2(\Omega_T) \cap C(\overline{\Omega}_T) \) of the IBVP (4.1)–(4.3), (see [19], p. 449).

The following theorem investigates the properties of the solution \( u_{\mu,\delta} \) to the parabolic perturbation (4.1)–(4.3) for fixed \( \delta, \mu \in (0, 1) \).

**Theorem 4.2.** For each \( \mu, \delta > 0 \), let \( u = u_{\mu,\delta} \in C^2(\Omega_T) \) be the unique classical solution of (4.1)–(4.3) of the parabolic perturbation (4.1)–(4.3). Then, \( u \) satisfies:

1. For all \( \phi \in C_0^\infty((−\infty, T) \times \mathbb{R}^n) \),
   \[
   \int_{\Omega_T} u(t,x)(∂_t \phi(t,x) − \delta \mathcal{L} \phi(t,x)) \, dx \, dt + \int_{\Omega} u_0 \delta(x) \phi(0, x) \, dx = \int_{\Omega_T} q(u(t,x)) A(x) \nabla K u(t,x) \cdot \nabla \phi(t,x) \, dx \, dt.
   \]
   (4.4)

2. For each \( t \in (0, T) \), we have
   \[
   \| u(t) \|_\infty \leq \| u_0 \|_\infty,
   \]
   and the conservation of the “total mass”
   \[
   \int_{\Omega} u(t,x) \, dx = \int_{\Omega} u_0 \delta(x) \, dx \leq \int_{\Omega} u_0(x) \, dx.
   \]
   (4.6)
   Furthermore, for all \( (t, x) \in \Omega_T, 0 \leq u(t, x) \).

3. First energy estimate: For \( \eta(\lambda) := (\lambda + \mu) \log(1 + (\lambda/\mu)) − \lambda, (\lambda \geq 0) \), and all \( t \in (0, T) \),
   \[
   \int_{\Omega} \eta(u(t)) \, dx + \Lambda_1 \delta \int_0^t \int_{\Omega} \frac{|\nabla u|^2}{q(u)} \, dx \, dt + \Lambda_1 \int_0^t \int_{\Omega} |\nabla H u|^2 \, dx \, dt \leq \int_{\Omega} \eta(u_0 \delta) \, dx.
   \]
   (4.7)

4. The second energy estimate: For all \( 0 < t_1 < t_2 < T \),
   \[
   \frac{1}{2} \int_{\Omega} |H u(t_2,x)|^2 \, dx + \Lambda_1 \delta \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2 \, dx \, dt + \Lambda_1 \int_{t_1}^{t_2} \int_{\Omega} q(u) |\nabla K u|^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} |H u(t_1,x)|^2 \, dx.
   \]
   (4.8)

5. For all \( v \in H_0^1(\Omega) \),
   \[
   \int_0^T (\partial_t u(t), v) \, dt = −\delta \int_{\Omega_T} A(x) \nabla u \cdot \nabla v \, dx \, dt + \int_{\Omega_T} q(u) A(x) \nabla K u \cdot \nabla v \, dx \, dt.
   \]
   (4.9)

where \( \langle \cdot, \cdot \rangle \) denote the pairing between \( H^{-1}(\Omega) \) and \( H_0^1(\Omega) \).
Proof. The first part of the theorem (up to (4.6)) is analogous to the theorem 4.2 \cite{15} and therefore we omit the proofs. We will show (4.7)-(4.9).

(1) To get the first energy estimate (4.7), we multiply equation (4.1) by $\eta'(u)$ and integrate on $\Omega$. Then, after integration by parts and taking into account that $\eta'(0) = 0$, we have

$$\frac{\partial}{\partial t} \int_{\Omega} \eta(u) dx = -\delta \int_{\Omega} \frac{1}{q(u)} A(x) \nabla u \cdot \nabla u dx - \int_{\Omega} A(x) \nabla K u \cdot \nabla u dx.$$

Then, we integrate over $(0,t)$, for all $0 < t < T$, to obtain

$$\int_{\Omega} \eta(u(t)) dx + \delta \int_{0}^{t} \int_{\Omega} \frac{1}{q(u)} A(x) \nabla u \cdot \nabla u dx + \int_{0}^{t} \int_{\Omega} A(x) \nabla K u \cdot \nabla u dx = \int_{\Omega} \eta(u_0) dx.$$

On the other hand, due to the uniform ellipticity condition we have an estimate for the second term of the left hand side

$$\Lambda_1 \int_{0}^{t} \int_{\Omega} \frac{1}{A(x)} |\nabla u|^2 dx \leq \int_{0}^{t} \int_{\Omega} \frac{1}{q(u)} A(x) \nabla u \cdot \nabla u dx$$

and for the third term of the left hand side, we use proposition 2.3 item (2), thus we obtain (4.7).

(2) To prove (4.8), we multiply (4.1) by $\xi_k K u$, integrate over $\Omega$ and take into account that $\xi_k = 0$ on $\partial \Omega$. Then, we have

$$\int_{\Omega} \xi_k \frac{\partial u}{\partial t} K u dx + \delta \int_{\Omega} A(x) \nabla u \cdot \nabla (\xi_k K u) dx + \int_{\Omega} q(u) A(x) \nabla K u \cdot \nabla (\xi_k K u) dx = 0.$$

Passing to the limit as $k \to \infty$ and using Lemma 2.3 it follows that

$$\frac{1}{2} \frac{\partial}{\partial t} \int_{\Omega} |Hu(t)|^2 dx + \delta \int_{\Omega} A(x) \nabla u \cdot \nabla K u dx + \int_{\Omega} q(u) A(x) \nabla K u \cdot \nabla K u dx = 0.$$

Then, integrating over $(t_1,t_2)$ we get

$$\frac{1}{2} \int_{t_1}^{t_2} \int_{\Omega} |Hu(t)|^2 dx + \delta \int_{t_1}^{t_2} \int_{\Omega} A(x) \nabla u \cdot \nabla K u dx + \int_{t_1}^{t_2} \int_{\Omega} q(u) A(x) \nabla K u \cdot \nabla K u dx = \frac{1}{2} \int_{t_1}^{t_2} \int_{\Omega} |Hu(t_2, x)|^2 dx.$$

On the other hand, from the uniform ellipticity condition we have and estimate for the third term of the left hand side

$$\Lambda_1 \int_{t_1}^{t_2} \int_{\Omega} q(u) |\nabla K u|^2 dx \leq \int_{t_1}^{t_2} \int_{\Omega} q(u) A(x) \nabla K u \cdot \nabla K u dx$$

and for the second term of the left hand side, we use the remark 2.3 Therefore we get the second energy estimate (4.8), for all $0 < t_1 < t_2 < T$.

(3) It remains to show (4.9), which follows applying the same techniques above, so the proof is omitted. Hence the proof of the Theorem 4.2 is complete. \qed
4.2 Limit transition

Here we pass to the limit in (4.4), as the two parameters $\delta, \mu$ go to zero. To show that we use the first and the second energy estimates together with the Aubin-Lions’ Theorem. After that we apply the Theorem 3.2 to prove the existence of solution.

As a first step, we define $u_\delta := u_{\mu,\delta}$ (fixing $\mu > 0$). Then, we have the following

Proposition 4.1. Let $\{u_\delta\}_{\delta>0}$ be the classical solutions of (4.1)–(4.3). Then, there exists a subsequence of $\{u_\delta\}_{\delta>0}$, which weakly converges to some function $u \in L^2((0,T); D(L^{1-s/2})) \cap L^\infty(\Omega_T)$, satisfying

$$\int\int_{\Omega_T} u(t,x) \partial_t \varphi(t,x) + \int_{\Omega} u_0(x) \varphi(0,x) dx = \int\int_{\Omega_T} q(u(t,x)) A(x) \nabla K u(t,x) \cdot \nabla \varphi(t,x) dx dt.$$  

(4.10)

For all $\varphi \in C^\infty_c((\infty, T) \times \mathbb{R}^n)$

Proof. The idea of the proof of (4.10) is to pass to the limit in (4.4) as $\delta \to 0^+$. Therefore we need to show compactness of the sequence $\{u_\delta\}_{\delta>0}$. From (4.5), it follows that $\{u_\delta\}_{\delta>0}$ is (uniformly) bounded in $L^\infty(\Omega_T)$. Then, it is possible to select a subsequence, still denoted by $\{u_\delta\}$, converging weakly-$\star$ to $u$ in $L^\infty(\Omega_T)$, i.e.

$$\lim_{\delta \to 0^+} \int_{\Omega_T} u_\delta(t,x) \phi(t,x) dt dx = \int_{\Omega_T} u(t,x) \phi(t,x) dt dx,$$

for all $\phi \in L^1(\Omega_T)$, which is enough to pass to the limit in the first integral in the left hand side of (4.4).

Now, we study the convergence of the integral in right hand side of (4.4). First, since $A(x)$ is symmetric, it is sufficient to show $\int_{\Omega_T} q(u_\delta) \nabla K u_\delta$ converges weakly in $(L^2(\Omega_T))^n$. The proof will be divide into two step. First weak convergence of $\nabla K u_\delta$ and strong convergence of $u_\delta$ in $(L^2(\Omega_T))^n$. From (4.8), we have

$$\int\int_{\Omega_T} |\nabla K u_\delta|^2 dx dt \leq \frac{C}{\mu},$$

where $C$ is a positive constant which does not depend on $\delta$. Therefore, the right-hand side is (uniformly) bounded in $L^2(\Omega_T)$ w.r.t. $\delta$. Thus we obtain (along suitable subsequence) that, $\nabla K u_\delta$ converges weakly to $v$ in $(L^2(\Omega_T))^n$.

The next step is to show that $v = \nabla K u$ in $(L^2(\Omega_T))^n$. First we prove the regularity of $u$. From the equivalent norm (2.8) we deduce that

$$\int\int_{\Omega_T} |\mathcal{L}^{(1-s)/2} u_\delta(t,x)|^2 dx dt \leq \Lambda_2 \int\int_{\Omega_T} |\nabla H u_\delta|^2 dx dt.$$
On the other hand, from (4.7), we obtain that $\nabla \mathcal{H}u_\delta$ is (uniformly) bounded in $(L^2(\Omega_T))^n$ w.r.t. $\delta$. Thus $\{u_\delta\}$ is (uniformly) bounded in $L^2((0, T); D(\mathcal{L}^{(1-s)/2}))$. Consequently, it is possible to select a subsequence, still denoted by $\{u_\delta\}$, converging weakly to $u$ in $L^2((0, T); D(\mathcal{L}^{(1-s)/2}))$, where we have used the uniqueness of the limit. Therefore, using again (2.8) and the Poincare’s type inequality (corollary 2.1), we have that

$$\int_{\Omega_T} |\nabla K(t, x)|^2 \, dx \leq 2^{-1}\lambda_1^{-s} \int_{\Omega_T} |\mathcal{L}^{(1-s)/2}u(t, x)|^2 \, dx,$$

where $\lambda_1$ is the first eigenvalue of $\mathcal{L}$. Thus, we obtain that $\nabla K u \in (L^2(\Omega_T))^n$, and hence $\nabla K u_\delta$ converges weakly to $\nabla K u$ in $(L^2(\Omega_T))^n$.

Recall that, we are proving the weak convergence of $q(u_\delta)\nabla K u_\delta$ in $(L^2(\Omega_T))^n$. Now, we prove strong convergence for $\{u_\delta\}_{\delta>0}$ in $L^2(\Omega_T)$, here we apply the Aubin-Lions compactness Theorem. Indeed, from (4.7)–(4.9) and the (uniform) boundedness of $\nabla K u_\delta$ in $(L^2(\Omega_T))^n$, we have

$$\int_0^T \|\partial_t u_\delta\|_{H_{-1}(\Omega)}^2 \, dt \leq C \left( \|u_0\|_\infty + \mu \right).$$

Observe that, at this point $\mu > 0$ is fixed. Thus, the right-hand side of (4.11) is bounded in $L^2((0, T); H^{-1}(\Omega))$ w.r.t. $\delta$. Therefore, exist a subsequence, such that $\partial_t u_\delta$ converges weakly to $\partial_t u$ in $L^2(0, T; H^{-1}(\Omega))$. Then, applying the Aubin-Lions compactness Theorem (see [21], Lemma 2.48) it follows that, $u_\delta$ converges to $u$ (along suitable subsequence) strongly in $L^2(\Omega_T)$ as $\delta$ goes to zero. Consequently, $q(u_\delta)\nabla K u_\delta$ converges weakly to $q(u)\nabla K u$ as $\delta \to 0^+$. Hence, the equality (4.11) holds.

**Corollary 4.1.** Let $u$ the function given by the proposition (4.10), satisfies:

1. For almost all $t \in (0, T)$,

$$\|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{and}$$

$$\int_\Omega u(x, t) \, dx = \int_\Omega u_0(x) \, dx.$$  

Furthermore, $0 \leq u(t, x)$ a.e in $\Omega_T$.

2. First energy estimate: For $\eta(\lambda) := (\lambda + \mu) \log(1 + (\lambda/\mu)) - \lambda$, $(\lambda \geq 0)$, and almost all $t \in (0, T)$,

$$\int_\Omega \eta(u(t)) \, dx + \Lambda_1 \int_0^t \int_\Omega |\nabla \mathcal{H}u| \, dx \, dt \leq \int_\Omega \eta(u_0) \, dx. \quad (4.14)$$

3. Second energy estimate: For almost all $0 < t_1 < t_2 < T$,

$$\frac{1}{2} \int_\Omega |\mathcal{H}u(t_2)|^2 \, dx + \Lambda_1 \int_{t_1}^{t_2} \int_\Omega q(u)|\nabla K u|^2 \, dx \, dt \leq \frac{1}{2} \int_\Omega |\mathcal{H}u(t_1)|^2 \, dx. \quad (4.15)$$
For each \( v \in H^1_0(\Omega) \),
\[
\int_0^T \langle \partial_t u, v \rangle dt = \int_0^T q(u) A(x) \nabla K u \cdot \nabla v \, dx \, dt. \tag{4.16}
\]
where \( \langle \cdot, \cdot \rangle \) denote the pairing between \( H^{-1}(\Omega) \) and \( H^1_0(\Omega) \).

**Proof.** The proof of ((4.12) to (4.16)) is standard, see [15], and therefore we omit the proofs. \( \square \)

**Remark 4.1.** The function \( u \) (obtained in the previous proposition) depends on the fixed parameter \( \mu \). For each \( \mu > 0 \), we write from now on \( u_\mu \) instead of \( u \).

**Proof of Theorem 4.1.** To prove the existence of weak solution of the IBVP (1.1), we consider the sequence \( \{u_\mu\}_{\mu > 0} \), obtained in the proposition 4.1, which satisfies the corollary 4.1 for each \( \mu > 0 \), (4.10)–(4.16).

The idea of the proof is to pass to the limit in (4.10) as \( \mu \to 0^+ \), and obtain the solvability of the IBVP (1.1) applying the Equivalence Theorem 3.2.

From (4.12), we see that \( \{u_\mu\}_{\mu > 0} \) is (uniformly) bounded in \( L^\infty(\Omega_T) \) w.r.t \( \mu \). Hence, it is possible to select a subsequence, still denoted by \( \{u_\mu\} \), converging weakly to \( u \) in \( L^\infty(\Omega_T) \), which is enough to pass to the limit in the first integral in the left hand side of (4.10).

Now, we study the convergence of the integral in right hand side of (4.10). First, since \( A(x) \) is symmetric, it is sufficient to show \( q(u_\delta) \nabla K u_\delta \) converges weakly in \( (L^2(\Omega_T))^n \). On the other hand, we recall that
\[
\eta(\lambda) = (\lambda + \mu) \log(1 + \lambda/\mu) - \lambda,
= \lambda \log(\lambda + \mu) - \lambda \log \mu + \mu \log(1 + \lambda/\mu) - \lambda, \quad (\forall \lambda \geq 0)
\]
Then, from (4.13) and (4.14) we obtain for almost all \( t \in (0, T) \)
\[
\Lambda_1 \int_0^t \int_\Omega |\nabla H u_\mu|^2 \, dx \, dt + \int_\Omega u_\mu(t) (u_\mu(t) + \mu) \, dx \\
\leq \int_\Omega u_0 (\log(u_0 + \mu) + \mu \log(1 + u_0/\mu)) \, dx,
\tag{4.17}
\]
where we have used that \( \mu \int_\Omega \log(1 + u_\mu/\mu) \, dx \geq 0 \) for all \( \mu > 0 \).

Since \( f = f^+ - f^- \), where \( f^\pm = \max\{f, \pm 0\} \), it follows from (4.17) that
\[
\Lambda_1 \int_0^t \int_\Omega |\nabla H u_\mu|^2 \, dx \, dt + \int_\Omega u_\mu(t) (u_\mu(t) + \mu) \, dx \\
\leq \int_\Omega u_0 (\log(u_0 + \mu) + \mu \log(1 + u_0/\mu)) \, dx \\
+ \int_\Omega u_\mu(t) \log^-(u_\mu(t) + \mu) \, dx.
\]
Observe that the right hand side of the above inequality is bounded w.r.t. \( \mu \) (small enough), because \( u_\mu \) is bounded in \( L^\infty(\Omega_T) \) w.r.t. \( \mu \), and

\[
\int_\Omega u_\mu(t) \log^-(u_\mu(t) + \mu) \, dx
\]

is bounded w.r.t. \( \mu \) (small enough). Consequently, we have that \( \nabla H u_\mu \) is (uniformly) bounded in \( L^2(\Omega_T) \).

On the other hand, using (2.8) and the Poincare inequality ( Corollary 2.1 ) we deduce that

\[
\iint_{\Omega_T} |\nabla K u_\mu(t,x)|^2 \, dx \, dt \leq \Lambda_1^{-1} \int_{\Omega_T} |L^{1/2-s} u_\mu(t,x)|^2 \, dx \, dt
\]

\[
\leq \Lambda_1^{-1} \lambda_1^{-s} \int_{\Omega_T} |L^{1/2-s/2} u_\mu(t,x)|^2 \, dx \, dt
\]

\[
\leq \Lambda_1^{-1} \lambda_1^{-s} \Lambda_2 \int_{\Omega_T} |\nabla H u_\mu(t,x)|^2 \, dx \, dt.
\]

Therefore, \( \nabla K u_\mu \) is (uniformly) bounded in \( L^2(\Omega_T) \) w.r.t. \( \mu \), and thus we obtain (along suitable subsequence) that \( \nabla K u_\mu \) converges weakly to \( v \) in \( (L^2(\Omega_T))^n \). It remains to show that \( v = \nabla K u \).

Using the same ideas as in the proof of the proposition 4.1. It is possible to select a subsequence, still denoted by \( \{u_\mu\} \), converging weakly to \( u \) in \( L^2(0,T; D(L^{1-s/2})) \) such that \( v = \nabla K u \) in \( (L^2(\Omega_T))^n \). Hence \( \nabla K u_\delta \) converges weakly to \( \nabla K u \) in \( (L^2(\Omega_T))^n \).

Now, we prove strong convergence for \( \{u_\mu\}_{\mu > 0} \) in \( L^2(\Omega_T) \). To show that, we apply again the Aubin-Lions compactness Theorem. Since the coefficient \( a_{i,j} \) of the matrix \( A(x) \) is in \( C^1(\bar{\Omega}) \), together with the boundedness of \( \nabla K u_\mu \) in \( L^2(\Omega_T) \), and the uniform limitation of \( u_\mu \), we deduce from (4.10) the following we have

\[
\int_0^T \|\partial_t u_\mu\|_{H^{-1}(\Omega)}^2 \, dt \leq C. \tag{4.18}
\]

Passing to a subsequence (still denoted by \( \{u_\mu\} \)), we obtain that

\( \partial_t u_\mu \) converges weakly to \( \partial_t u \) in \( L^2(0,T; H^{-1}(\Omega)) \).

Applying the Aubin-Lions compactness Theorem, it follows that \( u_\mu \) converges strongly to \( u \) (along suitable sequence) in \( L^2(\Omega_T) \). Consequently, we obtain that \( q(u_\mu)\nabla K u_\mu \) converges weakly to \( u \nabla K u \) as \( \mu \to 0^+ \). Then, we are ready to pass to the limit in (4.10) as \( \mu \to 0^+ \) to get

\[
\iint_{\Omega_T} u(t,x)(\partial_t \varphi(t,x) - A(x)\nabla K(u(t,x)) \cdot \nabla \varphi(t,x)) \, dx \, dt + \int_\Omega u_0(x) \varphi(0,x) \, dx = 0,
\]

for all \( \varphi \in C_c^\infty((-\infty,T) \times \mathbb{R}^n) \). According to the Equivalence Theorem 3.2 we obtain the solvability of the IBVP (1.1).
Corollary 4.2. The weak solution $u$ of the IBVP (1.1) given by Theorem 4.1, satisfies:

1. For almost all $t \in (0, T)$, we have
   \[ \|u(t)\|_\infty \leq \|u_0\|_\infty, \quad \text{and} \]
   \[ \int_{\Omega} u(x, t) dx = \int_{\Omega} u_0(x) dx. \]  
   Moreover, $0 \leq u(t, x)$ a.e. in $(0, T) \times \Omega$.

2. First energy estimate: For almost all $t \in (0, T)$,
   \[ \Lambda_1 \int_0^t \int_{\Omega} |\nabla H u|^2 \, dx \, dt' + \int_{\Omega} u(t) \log(u(t)) \, dx \leq \int_{\Omega} u_0 \log(u_0) \, dx. \]  

3. Second energy estimate: For almost all $0 < t_1 < t_2 < T$,
   \[ \frac{1}{2} \int_{\Omega} |H u(t_2)|^2 \, dx + \Lambda_1 \int_{t_1}^{t_2} \int_{\Omega} u \, |\nabla K u|^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} |H u(t_1)|^2 \, dx. \]

Proof. The proof of ((4.19) to (4.22)) is standard, see [15].

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References

[1] H. Antil, J. Pfefferer, S. Rogovs, Fractional operators with inhomogeneous boundary conditions: Analysis, Control, and Discretization, arXiv:1703.05256, 2017.

[2] P. Biler, C. Imbert, G. Karch, The nonlocal porous medium equation: Barenblatt profiles and other weak solution, Arch. Ration. Mech. Anal. 215(2015), 497-529.

[3] P. Biler, G. Karch, R. Monneau, Nonlinear diffusion of dislocation density and self-similar solution, Comm. Math. Phys., 294 (2010), 145-168.

[4] M. Bonforte, Y. Sire, J. L. Vázquez, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains, Manuscript submitted to AIMS’ Journals.
[5] L. CAFFARELLI, F. SORIA, J.L. VAZQUEZ, *Regularity of solution of the fractional porous medium flow* J. Eur. Math. Soc. (JEMS) 155 (2013), 17011746.

[6] L. CAFFARELLI, J.L. VAZQUEZ, *Nonlinear Porous Medium Flow with Fractional Potential Pressure*, Arch. Rational Mech. Anal. 202 (2011), 537–565.

[7] L. CAFFARELLI, J.L. VAZQUEZ, *Regularity of solution of the fractional porous medium flow in the exponent 1/2*, St Petersburg Math Journal 27(2016), no 3, 437460.

[8] L. CAFFARELLI, P.R. STINGA, *Fractional elliptic equations, Caccioppoli estimates and regularity*, Annales de l’Institut Henri Poincaré C, Analyse Non Linéaire, 33 (2016), 767–807.

[9] A. CAPELLA, J. DÁVILA, L. DUPaigne, Y. SIRE, *Regularity of radial extremal solutions for some non local semilinear equations*, Preprint, arXiv:1004.1906.

[10] X. CABRÉ, J. TAN, *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Advances in Mathematics, 224 (2010), 2052–2093.

[11] L.C. EVANS, R.F. GARIepy, *Measure Theory and fine property of function*, CRC Press, Boca Raton, Florida, 1992.

[12] H. FEDERER, *Geometric Measure Theory*, Springer-Verlag, New York, 1969.

[13] D. GILBARG, N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin Heidelberg, 1998.

[14] G. GRUBB, *Regularity of spectral fractional Dirichlet and Neumann problems*, Math. Nachr. 289 (2016) no. 7, 831–844.

[15] G. HUAROTO, W. NEVES, *Initial-boundary value problem for a fractional type degenerate heat equation*, Mathematical Models and Methods in Applied Sciences, Vol. 28, No. 6 (2018) 1199-1231.

[16] C. IMBERT, *Finite speed of propagation for a non-local porous medium equation*, Colloq. Math. 143(2) (2016), 149157.

[17] S. KIM, Ki-Ahm Lee, *Smooth solution for the porous medium equation in a bounded domain*, Journal of Diff. Equations, 347 (2009), 1064-1095.

[18] S. N.,KRUŽKOV , *First-order quasilinear equations in several independent variables*, Math. USSR Sb. 10 (1970), 217-243.

[19] O.A. LADYZHENSKAYA, V.A. SOLONNIKOV, N.N. URAŁ’TSEVA, *Linear and quasilinear equations of parabolic type*, American Mathematical Society, Providence RJ, 1968.
[20] S. Lisini, E. Mainini, A. Segatti, A gradient flow approach to the porous medium equation with fractional pressure, Arch. Rat. Mech. Anal. 227, 567-606.

[21] J. Málek, J. Necas, M. Rokyta, M. Ruzicka, Weak and Measure-valued solutions to evolutionary PDEs, Chapman and Hall, London, 1996.

[22] W. Neves, Scalar multidimensional conservation laws IBVP in noncylindrical Lipschitz domains, Journal of Diff. Equations, 192 (2003), 360–395.

[23] W. Neves, E. Panov, J. Silva, Strong Traces for Conservation Laws with General Nonautonomous Flux SIAM Journal on Mathematical Analysis 50 (6), 6049-6081.

[24] Q. H. Nguyen, J. L. Vazquez, Porous medium equation with nonlocal pressure in a bounded domain, Comm. PDEs, (2018), 1-38. arXiv:1708.00660.

[25] F. Otto Initial-boundary value problem for a scalar conservation law, C.R. Acad. Sci. Paris 322 (1996) 729–734.

[26] D. Stan, F. del Teso, J. L. Vazquez, Finite and infinite speed of propagation for porous medium equation with fractional pressure, C. R. Math. Acad. Sci. Paris, 352(2) (2014), 123-128.

[27] D. Stan, F. del Teso, J. L. Vazquez, Finite and infinite speed of propagation for porous medium equation with nonlocal pressure, J. Diff. Eq., 260 1154-1199, 2016.

[28] D. Stan, F. del Teso, J. L. Vazquez, Existence of weak solution solution for a general porous medium equation with nonlocal pressure, Arch. Rat. Mech. Anal., First online 09 February 2019 in press arXiv: 1609.05139.

[29] D. Stan, F. del Teso, J. L. Vazquez, Porous medium equation with nonlocal pressure, arXiv:1801.04244.

[30] D. Stan, F. del Teso, J. L. Vazquez, Transformations of self-similar solutions for porous medium equations of fractional type, Nolinear Anal., 119(2015), 62-73.

[31] X. Zhou, W. Xiao, J.L. Vazquez, Fractional porous medium and mean field equations in Besov space, Electron. J. Differential Equation (2014) No.199, 14pp.