Pion-proton scattering and isospin breaking in the $\Delta^0 - \Delta^{++}$ system

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Abstract

We determine the mass and width of the $\Delta^{++}$ ($\Delta^0$) resonance from data on $\pi^+p$ ($\pi^-p$) scattering both, in the pole of the $S$-matrix and conventional Breit-Wigner approaches to the scattering amplitude. We provide a simple formula that relates the two definitions for the parameters of the $\Delta$. Isospin symmetry breaking in the $\Delta^0 - \Delta^{++}$ system depends on the definition of the resonant properties: we find $M_0 - M_{++} = 0.40 \pm 0.57$ MeV, $\Gamma_0 - \Gamma_{++} = 6.89 \pm 0.95$ MeV in the pole approach while $\tilde{M}_0 - \tilde{M}_{++} = 2.25 \pm 0.68$ MeV, $\tilde{\Gamma}_0 - \tilde{\Gamma}_{++} = 8.45 \pm 1.11$ MeV in the conventional approach.

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I. Introduction.

The isospin symmetry of strong interactions is a very good approximation to relate some properties and processes involving hadrons of a given isospin multiplet. The reason for this is that, at the fundamental level, the isospin symmetry is broken only by the electromagnetic interactions and the mass difference of the $u$ and $d$ quarks. However, it is not easy to perform a precise theoretical calculation for isospin breaking effects in hadrons starting from the fundamental theory; for instance, the old problem of the neutron-proton mass difference [1] (which has been measured with an accuracy of 7 parts per million [2]) remain as a challenge for the theory of elementary particles.

In this work we are concerned with the isospin breaking in the masses and widths of the $\Delta^0$, $\Delta^{++}$ members of the $I = J = 3/2$ multiplet of baryon resonances. As is well known, these resonances would have equal masses and widths if isospin symmetry were exact. Actually, the $\Delta$’s undergo strong interaction decays to $N\pi$ final states with branching fractions larger than 99 % [2].

Experimentally, the tests of isospin symmetry in the $\Delta$ system faces the problem that the definition of mass and width for an unstable particle is not unique. In fact, there are two common approaches to extract these resonance parameters from experimental data. In the conventional approach, the transition amplitude is parametrized in terms of a Breit-Wigner containing an energy-dependent width. A partial wave analysis of this amplitude allow to define the mass $\tilde{M}$ as the energy where the phase shift attains 90°. From this, the width is defined as $\tilde{\Gamma}(E = \tilde{M})$. On the other side, the pole approach allows to define the mass $M$ and width $\Gamma$ of the resonance from the real and imaginary parts of the pole position in the $S$-matrix amplitude.

The pole position is believed to be a physical property of the $S$-matrix amplitude [3, 4] and to provide a definition for the mass and width of a
resonance which is independent of the physical process used to extract these parameters. In contrast, in the conventional approach one requires to model the production and decay of the resonance \( i.e. \), the energy dependence of the decay width involved in the amplitude.

In this paper we use the experimental data on \( \pi^\pm p \) scattering \([5]\) to extract the pole parameters of the \( \Delta^0, \Delta^{++} \) baryon resonances. It is found that the isospin splittings in the \( \Delta^0 - \Delta^{++} \) system is different for both definitions of the resonant parameters: the resonant parameters in the conventional approach exhibit a stronger isospin breaking that in the \( S \)-matrix pole approach. Also, a simple formula is provided to relate the resonant parameters defined in the two approaches.

The remaining of this paper is organized as follows. In section II we describe the two approaches for the \( \pi p \) scattering amplitude in the \( \Delta \) resonance region. In sections III and IV we analyse, respectively, the \( \pi^+ p \) and \( \pi^- p \) scattering in order to extract the resonant parameters of the \( \Delta^{++} \) and \( \Delta^0 \). Section V contains a discussion of our results and conclusions and an Appendix is devoted to repeat the analysis of sections III and IV in the case of ‘non-relativistic’ pole scattering amplitudes.

II. Pole and conventional approaches to the \( \Delta \) in \( \pi p \) scattering

In this section we discuss in more detail the two approaches for the description of the \( \Delta \) resonance in \( \pi p \) scattering. We also derive the relations to pass from the resonance parameters in one approach to the other.

The total cross section for \( \pi p \) scattering in the \( \Delta \) resonance region can be written in terms of the partial wave amplitude \( a_{3/2} \) as \([\text{see for example,}\]
p.1293 in Ref. 2]:

$$\sigma_{\frac{3}{2}^{+}}(\pi p) = \frac{8\pi}{k^2} |a_{\frac{3}{2}^{+}}|^2$$  \hspace{1cm} (1)

where $k$ denotes the center of mass momentum of either $\pi$ or $p$.

For elastic scattering, the partial wave amplitude can be written in terms of the corresponding phase shift $\delta_{\frac{3}{2}^{+}}$:

$$a_{\frac{3}{2}^{+}} = \frac{\tan \delta_{\frac{3}{2}^{+}}}{1 - i \tan \delta_{\frac{3}{2}^{+}}}$$  \hspace{1cm} (2)

which automatically satisfies unitarity.

In the conventional approach, the amplitude $a_{\frac{3}{2}^{+}}$ is saturated with the $\Delta$ resonance which is described by an energy dependent width $\tilde{\Gamma}(s)$, where $s$ denotes the squared center of mass energy. If the phase shift is chosen as

$$\tan \delta_{\frac{3}{2}^{+}} = - \frac{\tilde{M}\tilde{\Gamma}(s)}{s - M^2},$$  \hspace{1cm} (3)

we are lead to the usual Breit-Wigner form of the amplitude, namely:

$$a_{\frac{3}{2}^{+}} = - \frac{\tilde{M}\tilde{\Gamma}(s)}{s - M^2 + i\tilde{M}\tilde{\Gamma}(s)}.$$  \hspace{1cm} (4)

Thus, the mass and width of the $\Delta$ in the conventional approach become, respectively, $\tilde{M}$ and $\tilde{\Gamma}(s = \tilde{M}^2)$.

As it was mentionned above, the $S$-matrix approach provides a definition for the parameters of an unstable particle which is independent of the process used to extract them. This happens because, independently of the specific scattering or decay process, the resonance shows up in the amplitude as a physical pole $s$. In this approach, the resonant and background contributions (in the same channel) to the amplitude are explicitly separated according to [3]:

$$a = \frac{R}{s - \bar{s}} + B.$$  \hspace{1cm} (5)
Thus, the mass $M$ and width $\Gamma$ of the resonance in the pole approach are defined as follows [3] (see also the Appendix and Ref. [6] for an alternative definition):

$$\overline{s} \equiv M^2 - iM\Gamma.$$  \hfill (6)

In order to connect the two approaches, let us split the phase shift $\delta_{3\overline{3}}$ into two terms:

$$\delta_{3\overline{3}} = \delta_R + \delta_B$$  \hfill (7)

where $\delta_R$ corresponds to the phase shift due to the $\Delta$ resonance and $\delta_B$ to the background contribution in the $(\frac{3}{2}, \frac{3}{2})$ channel. The choice in Eq. (7), explicitly leads to the scattering amplitude of the form given in Eq. (5) (see Ref. [7] and Eq. (12) below).

Since the background is expected to give a small contribution to the $\pi p$ scattering amplitude in the resonance region, we can choose the following parametrization [7, 8]:

$$\tan \delta_B = x(s)$$  \hfill (8)

where $x(s)$ represents a smooth function of $s$.

If we define

$$\tan \delta_R = -\frac{M\Gamma}{(s - M^2)}$$  \hfill (9)

for the resonance contribution and if we introduce Eqs. (7)-(9) into Eq. (2) we are lead to the following equivalent representations for the amplitude:

\begin{align*}
q_{3\overline{3}} &= -\frac{M\Gamma - x(s)(s - M^2)}{[1 - ix(s)](s - M^2 + iM\Gamma)} \\
&= -\frac{[M\Gamma - x(s)(s - M^2)]}{s - M^2 + x(s)M\Gamma + i[M\Gamma - x(s)(s - M^2)]} \\
&= -\frac{M\Gamma}{s - M^2 + iM\Gamma} \exp(2i\delta_B) + \frac{x(s)}{1 - ix(s)}. \hfill (12)
\end{align*}
If we compare Eqs. (11) and (2) we immediately get the identity:

\[
\tan \delta_{ZZ} = - \frac{M \Gamma - x(s)(s - M^2)}{s - M^2 + x(s)M \Gamma} \tag{13}
\]

\[
= - \frac{\tilde{M}\tilde{\Gamma}(s)}{s - M^2}. \tag{14}
\]

Since the resonant parameters in the conventional approach are defined according to \(\delta_{ZZ} = 90^0\) when \(s = \tilde{M}^2\), from the previous equations we obtain the relations between the resonant parameters in both approaches, namely [7]:

\[
\tilde{M}^2 = M^2 - xM \Gamma \tag{15}
\]

and

\[
\tilde{M}\tilde{\Gamma} = M\Gamma(1 + x^2)/(1 + M\Gamma x') \tag{16}
\]

where \(x\), \(\tilde{\Gamma}\) and \(x' = dx/ds\) are evaluated at \(s = \tilde{M}^2\). Eqs. (15)-(16) will allow us to extract \(\tilde{M}\) and \(\tilde{\Gamma}\) from the fitted values of \(M\), \(\Gamma\) and \(x\) (see sections III and IV).

Defining

\[
M\Gamma(s) \equiv M\Gamma - x(s)(s - M^2),
\]

we get

\[
x(s) = - \frac{M \Gamma \left(\frac{\gamma(s) - 1}{s - M^2}\right)}{s - M^2} \tag{17}
\]

where

\[
\gamma(s) = \frac{\Gamma(s)}{\Gamma} \tag{18}
\]

with \(\gamma(M^2) = 1\). The \(s\)-dependence of the total width \(\Gamma(s)\) (or equivalently \(x(s)\)) will be introduced later\(^1\). Note that \(x(s)\) is a regular function when \(s\) approaches \(M^2\).

\(^1\)We would like to emphasize that various parametrizations for \(x(s)\) were used to fit the \(\pi p\) experimental data (for instance, we used the parametrizations of Ref. [9] for the background contributions to \(e^+ e^- \rightarrow \pi^+ \pi^-\)). As expected, these background parametrizations do not modify the position of the pole.
With the above choice for \( x(s) \), Eq. (10) becomes:

\[
a_{\frac{3}{2}\frac{3}{2}}^{++} = -\frac{M\Gamma(s)}{[1 - ix(s)][s - M^2 + iM\Gamma]} \quad (19)
\]

\[
= -\frac{M\Gamma(s)}{s - M^2 + x(s)M\Gamma + iM\Gamma(s)} \quad (20)
\]

which looks very similar to the usual Breit-Wigner parametrization, Eq. (4), if we define an effective mass \( M_{eff}^2 \approx M^2 - xM\Gamma \) because \( x(s) \) varies smoothly around the resonance.

### III. Analysis of the \( \pi^+p \) scattering.

In this section we perform the fit of the experimental data on \( \pi^+p \) scattering [5] to extract the \( \Delta^{++} \) parameters by using the formalism described in the previous section.

The total cross section for \( \pi^+p \) scattering in the \( (I, J) = (\frac{3}{2}, \frac{3}{2}) \) channel is given by:

\[
\sigma_{\frac{3}{2}\frac{3}{2}}(\pi^+p) = \frac{8\pi}{k^2} |a_{\frac{3}{2}\frac{3}{2}}^{++}|^2. \quad (21)
\]

As discussed in section II, the scattering amplitude \( a_{\frac{3}{2}\frac{3}{2}}^{++} \) can be written as:

\[
a_{\frac{3}{2}\frac{3}{2}}^{++} = -\frac{M_{++}\Gamma_{++} - x_{++}(s)(s - M_{++}^2)}{[1 - ix_{++}(s)][s - M_{++}^2 + iM_{++}\Gamma_{++}]} \quad (22)
\]

where \( x_{++}(s) \) is given by:

\[
x_{++}(s) = -M_{++}\Gamma_{++}\left(\frac{\gamma_{++}(s) - 1}{s - M_{++}^2}\right). \quad (23)
\]

We choose \( \gamma_{++}(s) \) to be the standard parametrization for the energy-dependent width used in the experiments as given, for example, in Ref. [5]:

\[
\gamma_{++}(s) = \left(\frac{k}{k_{++}}\right)^3 \frac{1 + a_{++}(k_{++}/m_{\pi^+})^2}{1 + a_{++}(k/m_{\pi^+})^2} \quad (24)
\]
where $k$ denotes the center of mass momentum of $\pi^+$ and $k_+^+$ the value of $k$ at $\sqrt{s} = M_+^+$. $a_+^+$ is a dimensionless parameter.

Thus, Eq. (22) contains three free parameters to be adjusted from the $\pi^+p$ experimental data: the pole resonance parameters $(M_+^+, \Gamma_+^+)$ and the parameter $a_+^+$. The fitted values for these quantities allow to extract the resonance parameters in the conventional approach by using Eqs. (15) and (16).

In the fit to the experimental data of Ref. [5] we distinguish two cases:

(A) We first take into account the background contributions coming from channels other than $(I, J) = (\frac{3}{2}, \frac{3}{2})$ as given in the last column-Table 1 of Ref. [5].

(B) The same as before but we allow a 10% error for the background contributions of Table 1 in Ref. [5].

The results of the fits are shown in Table 1 and the fit for case (A) is also shown in Fig. 1. The following remarks are in order:

1. The mass and width of the $\Delta^{++}$ in the pole approach are shifted to lower values by around 20 and 12 MeV, respectively, with respect to the resonant parameters in the conventional approach.

2. With the parameters shown in Table 1 and using Eqs. (23)-(24), we can easily check that the variation of $x_+^+(s)$ in the kinematical region $1100 \text{ MeV} < \sqrt{s} < 1300 \text{ MeV}$ is less than 10%.

3. The most important effect of considering case (B) is observed in the parameter $a_+^+$. The pole parameters $M_+^+$, $\Gamma_+^+$ shown in Table 1 are a little bit different from other available determinations which are shown in Table 2 (our results are repeated for comparison).
Similarly, the mass and width values shown in Table 1 for the $\Delta^{++}$ in the conventional approach are very similar to the following results of Ref. [5]:

$$
\begin{align*}
\tilde{M}^{++} &= 1232.1 \pm 0.2 \text{ MeV} \\
\tilde{\Gamma}^{++} &= 109.8 \pm 0.4 \text{ MeV}.
\end{align*}
$$  \hspace{1cm} (25)

IV. The $\Delta^0$ in $\pi^- p$ scattering.

In this section we apply the formalism described in section II to the production of the $\Delta^0$ in $\pi^- p$ scattering. The analysis of $\pi^- p$ scattering is slightly more complicated because both, $\pi^- p$ and $\pi^0 n$, can be reached as final states. Thus, due care of isospin breaking coming from the $\pi^+ - \pi^0$ and $n - p$ mass differences and possible residual isospin breaking effects have to be taken into account.

As in the previous case, the total cross section for $\pi^- p$ scattering in the $(I, J) = (\frac{3}{2}, \frac{3}{2})$ channel is given by:

$$
\sigma_{\frac{3}{2}\frac{3}{2}}(\pi^- p) = \frac{8\pi}{k^2} |a_{\frac{3}{2}\frac{3}{2}}^0|^2.
$$  \hspace{1cm} (26)

In order to incorporate isospin breaking effects we first realize that in the limit of isospin symmetry we would have:

$$
|a_{\frac{3}{2}\frac{3}{2}}^0|^2 = \frac{1}{3} |a_{\frac{3}{2}\frac{3}{2}}^{++}|^2
$$  \hspace{1cm} (27)

and also $M_{\Delta^0} = M_{\Delta^{++}}, \Gamma_{\Delta^0} = \Gamma_{\Delta^{++}}$ (the superindex in $a$ refers to the charge of the $\Delta$, and $M$ and $\Gamma$ are the resonant parameters of the $\Delta$). Observe that, apart from the small radiative decay $\Delta^0 \rightarrow n\gamma (BR(\Delta^0 \rightarrow n\gamma) \sim 0.55 \text{ to } 0.61 \% \ [2])$, the $\Delta$'s undergo strong interaction decays to $\pi N$. 
The isospin breaking can be taken into account by properly modifying Eq. (19), namely by using:

\[ |a_{\frac{3}{2}+}^0|^2 = \frac{M_0^2 \Gamma_{\Delta^0 \rightarrow p\pi^-}(s) \Gamma_0(s)}{(1 + x_0(s))^2 |s - M_0^2 + iM_0 \Gamma_0|^2} \]  

(28)

where,

\[ \Gamma_0(s) = \Gamma_{\Delta^0 \rightarrow p\pi^-}(s) + \Gamma_{\Delta^0 \rightarrow n\pi^0}(s) \]  

(29)

when we neglect the tiny \( \Delta^0 \rightarrow n\gamma \) contribution to the total width of \( \Delta^0 \).

The partial decay widths of the \( \Delta^0 \) can be written as follows:

\[ \Gamma_{\Delta^0 \rightarrow p\pi^-}(s) = \frac{1}{3} (1 + \epsilon) \Gamma_0 \gamma_-(s) \]  

(30)

\[ \Gamma_{\Delta^0 \rightarrow n\pi^0}(s) = \frac{2}{3} (1 - \frac{\epsilon}{2}) \Gamma_0 \gamma_0(s) \]  

(31)

where \((i = -, 0)\),

\[ \gamma_i(s) = \left( \frac{k_i}{k_0} \right)^3 \frac{1 + a_0(k_i^0/m_{\pi^+})^2}{1 + a_0(k_i/m_{\pi^+})^2} \]  

(32)

\( k \) (\( k^0 \)) denotes the center of mass momentum of either one of the final particles coming from the \( \Delta^0 \) at \( \sqrt{s} \) (\( \sqrt{s} = M_0 \)). The small dimensionless parameter \( \epsilon \) takes into account possible residual effects of isospin breaking.

If we neglect second order isospin breaking effects of \( \mathcal{O}(\epsilon[\gamma_-(s) - \gamma_0(s)]) \) in Eq. (29), we obtain the following expression for the total width:

\[ \Gamma_0(s) \approx \Gamma_0 \left\{ \frac{1}{3} \gamma_-(s) + \frac{2}{3} \gamma_0(s) \right\} \]  

(33)

and the expression for the background contribution becomes:

\[ x_0(s) = - \frac{M_0 \Gamma_0}{s - M_0^2} \left( \frac{1}{3} \gamma_-(s) + \frac{2}{3} \gamma_0(s) - 1 \right) \].  

(34)

The set of four free parameters \((M_0, \Gamma_0, a_0, \epsilon)\) can be determined from a fit to the \( \pi^- p \) experimental data of Ref. [5] by using Eqs. (26) and (28–34).
As for $\pi^+p$ scattering, we have considered two cases in the fit:

(C) We have used the experimental data on the $\pi^-p$ cross section and the background contributions coming from channels other than $(\frac{3}{2}, \frac{3}{2})$ as given in Table 1 of Ref. [5].

(D) The same as before but we attribute a $\pm 10\%$ error to the background contributions.

The results of the fits are shown in Table 3 and Fig. 2 (case C). From Table 3 we can draw the following conclusions:

1. The position of the pole remains the same for cases (C) and (D). The most important effect of attributing a $10\%$ error to the background contributions is observed in the dimensionless parameter $a_0$ appearing in the expression for $x_0(s)$.

2. The mass and width of the $\Delta^0$ in the pole approach are shifted to lower values for about 22 and 14 MeV, respectively, respect to the values of those parameters in the conventional approach.

3. The residual isospin breaking parameter $\epsilon$ is of the expected order of magnitude.

The values of the mass and width pole parameters ($M_0$, $\Gamma_0$) can be compared with other determinations of these resonant properties of the $\Delta^0$ as shown in the Table 4.

The values of our fit for the $\Delta^0$ parameters in the conventional approach (see Table 3) as derived from Eqs. (15) and (16) are very similar to the corresponding parameters of Pedroni et. al. [5]:

$$\tilde{M}_0 = 1233.5 \pm 0.2 \text{ MeV} \quad (35)$$
$$\tilde{\Gamma}_0 = 118.4 \pm 0.9 \text{ MeV} \quad (36)$$
V. Discussion of results and conclusions.

In this work we have analysed the experimental data on $\pi p$ scattering [5] in the $\Delta$ resonance region in order to get information about the isospin breaking in the resonant parameters of the $\Delta^{++}$ and $\Delta^{0}$. For these purposes, we have explicitly separated, in the $(\frac{3}{2}, \frac{3}{2})$ channel, the pole and background contributions to the scattering amplitudes and we have obtained simple expressions that relate the resonant properties of the $\Delta$’s in the pole and conventional approaches.

Our main results are summarized in Tables 1 and 3. Our results for the pole parameters of the $\Delta^{++}$ and $\Delta^{0}$ are independent of the precise choice to parametrize the background contribution through the smooth function $x(s)$, as it should be. From these tables we can obtain the isospin breaking in the masses and widths of the $\Delta$’s and we compare our results with other available determinations of these quantities in Table 5 (all entries are given in MeV).

In our analysis we have considered the background contributions as given in Table 1 of Ref. [5] (case I) and we have repeated the analysis by adding a $\pm 10\%$ error to these backgrounds (case II).

Regarding the isospin breaking in the $\Delta^{0} - \Delta^{++}$ system we conclude the following from Table 5:

1). The isospin breaking is larger in the resonant parameters defined in the conventional approach than in the pole approach. The available determinations of isospin breaking in the masses in either of the approaches are rather similar while isospin breaking in the widths spreads over a wider range.

2). The isospin breaking in the pole masses of the $\Delta$’s is consistent with zero ($M_0 \approx M_{++}$). This result differs from the naive expectation based on
rough estimates of mass difference coming from electromagnetic and \( m_d - m_u \) contributions. Indeed, using the expression for the neutron–proton mass difference

\[
m_n - m_p = (\delta m)_{em} + c(m_d - m_u) \tag{37}
\]

and since the quark content of \( \Delta^{++} \) and \( \Delta^0 \) are \( uuu \) and \( ddu \), respectively, we would roughly expect

\[
M_{\Delta^0} - M_{\Delta^{++}} = 2(\delta m)_{em} + 2c(m_d - m_u) = 2(m_n - m_p) \approx 2.6 \text{ MeV}. \tag{38}
\]

In contrast, the isospin breaking in the masses of the \( \Delta \)'s defined in the conventional approach are in agreement with the naive expectation of Eq. (38). Note that the pole mass is the correct way to define a physical mass [3].

3. From Table 5 we observe that our results exhibit an isospin breaking of about 7% in the total widths of the \( \Delta \)'s.

This isospin breaking can also be measured through the background contribution at threshold \( (s_{th} = (m_p + m_{\pi \pm})^2 \approx (m_n + m_{\pi^0})^2) \). More explicitly, since \( M_0 \approx M_{++} \) and \( \gamma(s_{th}) = 0 \) it follows from Eq. (17) and (34) that

\[
x(s_{th}) = M\Gamma/(s_{th} - M^2)
\]

or

\[
\frac{x_0(s_{th})}{x_{++}(s_{th})} \approx \frac{\Gamma_0}{\Gamma_{++}} \approx 1.07 \tag{39}
\]

for the ratio of background contributions.

The numerical value in Eq. (39) follows from \( x_0(s_{th}) = -0.4084 \) and \( x_{++}(s_{th}) = -0.3828 \), which are obtained using the results of Tables 1 (case A) and 3 (case C), respectively. As we have pointed out in the text, \( x(s) \) is
a slowly varying function around the resonance. However it is interesting to observe that it clearly reflects the breaking of isospin symmetry at threshold. As a comparison, let us mention that the corresponding ratio at \( s = \tilde{M}^2 \) gives \( x_0/x_{++} \approx 1.02 \), which exhibits a smaller isospin breaking. Tables 1 and 3 show that isospin symmetry breaking is much smaller in the parameters \( x_i(\tilde{M}^2) \) than in the \( a_i \)'s. Our results are rather insensitive to the exact values of the \( a_i \)'s.

4). As written above, we have neglected the decay \( \Delta^0 \rightarrow n + \gamma \). We have verified that, since \( BR(\Delta^0 \rightarrow n + \gamma) \leq 0.6\% \) [2], to neglect this mode does not affect our results because isospin symmetry breaking in the total widths of the \( \Delta \)'s amounts for 7\%.

**Appendix.**

In this appendix we repeat the analysis of sections III and IV for the case of a ‘non-relativistic’ definition of the pole parameters. As can be concluded by comparing Tables (6) and (7) with Tables (1) and (3), the main conclusions of this paper are not modified by this assumption.

As is well known [6], an alternative definition for the parameters of an unstable particle in the S-matrix approach is obtained by assuming that the phase shift associated to the resonance is given by:

\[
\tan \delta_R = - \frac{\Gamma/2}{\sqrt{s} - M}. \tag{40}
\]

As it will become explicit later (see Eq. (43)), Eq. (40) gives rise to an amplitude with the pole position at

\[
\sqrt{s} = M - \frac{i}{2} \Gamma. \tag{41}
\]
Eq. (40) can be obtained from Eq. (9) by replacing
\[ s - M^2 \rightarrow 2M(\sqrt{s} - M). \] (42)

Note that \( s - M^2 \approx 2M(\sqrt{s} - M) \) is a good approximation for values of \( \sqrt{s} \) close to the resonance. This is the reason for calling Eq. (41) a non-relativistic definition of the pole parameters.

With the above definition for \( \delta_R \), the analogous of Eqs. (22) and (28) become, respectively:
\[
a_{++}^+ = -\frac{\Gamma_{++}/2 - x_{++}(s)(\sqrt{s} - M)}{[1 - ix_{++}(s)](\sqrt{s} - M_{++} + i\Gamma_{++}/2)} \] (43)
and
\[
|a_{0}^0|^2 = \frac{1}{4} \cdot \frac{\Gamma_{0} \rightarrow p\pi^-(s)\Gamma_0(s)}{(1 + x_0^2(s))|\sqrt{s} - M_0 + i\Gamma_0/2|^2}. \] (44)

The relations – Eqs. (15) and (16) – between the resonant parameters in both approaches are also modified to become:
\[
\tilde{M} = M - x\Gamma/2
\] (45)
\[
\tilde{\Gamma} = \Gamma(1 + x^2)/(1 + \frac{\Gamma}{2}x')
\] (46)

where, \( x, \tilde{\Gamma} \) and \( x' = dx/d\sqrt{s} \), are evaluated at \( s = \tilde{M}^2 \).

In order to fit the experimental data of Ref. [5] we have, as in sections III and IV, distinguished two cases: (a) we use the data on \( \pi^+p \) and \( \pi^-p \) scattering by considering also the background contributions as given in Table 1 of Ref. [5] and, (b) the same as before but we attribute a \( \pm 10 \% \) error to the background.

The results of the fits are shown in Table 6 for the \( \Delta^{++} \) and in Table 7 for the \( \Delta^0 \). We observe that the parameters of Tables 6 and 7 agree to a high accuracy with the values in the relativistic definition shown in Tables 1 and 3. In fact we observe that \( M_\Delta(\text{“relativistic”}) \approx M_\Delta(\text{“nonrelativistic”}) - 1 \text{ MeV}, \Gamma_\Delta(\text{“relativistic”}) \approx \Gamma_\Delta(\text{“nonrelativistic”}) + 1 \text{ MeV} \).
From Tables (6) and (7), the isospin breaking in the pole parameters are:

\[ M_0 - M_{++} = 0.70 \pm 0.58 \text{ MeV} \]  \hspace{1cm} (47)
\[ \Gamma_0 - \Gamma_{++} = 6.81 \pm 0.91 \text{ MeV} \]  \hspace{1cm} (48)

for case (a) and

\[ M_0 - M_{++} = 0.80 \pm 0.76 \text{ MeV} \]  \hspace{1cm} (49)
\[ \Gamma_0 - \Gamma_{++} = 6.51 \pm 1.02 \text{ MeV} \]  \hspace{1cm} (50)

for case (b), which are very similar to the results shown in Table 5.

**Note added**

After we have completed this work we became aware of reference [14], where expressions that relate the resonance parameters in the pole and conventional approaches are also provided for the \(N\)'s and \(\Delta\)'s (see Eqs. (3) and (A2) in Ref. [14]). The values quoted for the pole parameters of the *generic* \(\Delta\) resonance using his Eqs. (3) and (A2) are similar to ours. Isospin breaking is not considered in Ref. [14].
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TABLE CAPTIONS
1. Resonant parameters of the $\Delta^{++}$ extracted from $\pi^+p$ scattering. The values of $\tilde{M}_{++}$ and $\tilde{\Gamma}_{++}$ are obtained using Eqs. (15), (16).

2. Comparison of our results for the pole parameters of the $\Delta^{++}$ with other available determinations.

3. Resonant parameters of the $\Delta^0$ extracted from $\pi^-p$ scattering. The values of $\tilde{M}_0$ and $\tilde{\Gamma}_0$ are obtained using Eqs. (15), (16).

4. Comparison of our results for the pole parameters of the $\Delta^0$ with other available determinations.

5. Isospin breaking in the mass and widths of the $\Delta^0 - \Delta^{++}$ baryons. The two cases (I and II) for our results are described in section V. All the quantities are given in MeV units.

6. Resonant parameters of the $\Delta^{++}$ extracted from $\pi^+p$ scattering. The values of $\tilde{M}_{++}$ and $\tilde{\Gamma}_{++}$ are obtained using Eqs. (45), (46).

7. Resonant parameters of the $\Delta^0$ extracted from $\pi^-p$ scattering. The values of $\tilde{M}_0$ and $\tilde{\Gamma}_0$ are obtained using Eqs. (45), (46).

FIGURE CAPTIONS
1. Total cross section for the $\pi^+p$ scattering as a function of kinetic energy in the lab system. The solid line is our fit using the pole parameters given in Table 1 (case A).

2. Total cross section for the $\pi^-p$ scattering as a function of the kinetic energy in the lab system. The solid line is our fit using the pole parameters given in Table 3 (case C).
### Table 1

|                  | Case A               | Case B               |
|------------------|----------------------|----------------------|
| $M_{++}$ (MeV)   | 1212.20 ± 0.23       | 1212.50 ± 0.24       |
| $\Gamma_{++}$ (MeV) | 97.06 ± 0.35       | 97.37 ± 0.42         |
| $a_{++}$         | 0.5978 ± 0.0155      | 0.6256 ± 0.0203      |
| $x_{++}(\tilde{M}^2)$ | −0.4062 ± 0.0015  | −0.4012 ± 0.0017     |
| $\tilde{M}_{++}$ (MeV) | 1231.75 ± 0.27       | 1231.88 ± 0.29       |
| $\tilde{\Gamma}_{++}$ (MeV) | 109.85 ± 0.41     | 109.07 ± 0.48         |

### Table 2

| $M_{++}$ (MeV) | $\Gamma_{++}$ (MeV) | References            |
|----------------|----------------------|-----------------------|
| 1210.9 ± 0.8   | 99.2 ± 1.5           | [10]                  |
| 1210.7 ± 0.16  | 99.21 ± 0.23         | [11]                  |
| 1209.6 ± 0.5   | 100.8 ± 1.0          | [12]                  |
| 1212.20 ± 0.23 | 97.06 ± 0.35         | our results case A    |
| 1213.30 ± 0.23 | 96.17 ± 0.34         | our results case (a)  |
### Table 3

|                | Case C       | Case D       |
|----------------|--------------|--------------|
| $M_0$ (MeV)    | 1212.60 ± 0.52 | 1213.20 ± 0.66 |
| $\Gamma_0$ (MeV) | 103.95 ± 0.88  | 104.10 ± 1.01  |
| $a_0$          | 0.6914 ± 0.0477 | 0.7408 ± 0.0611 |
| $x_0(\tilde{M}^2)$ | −0.4154 ± 0.0035 | −0.4099 ± 0.0040 |
| $\tilde{M}_0$ (MeV) | 1234.00 ± 0.62  | 1234.35 ± 0.75  |
| $\tilde{\Gamma}_0$ (MeV) | 118.30 ± 1.03  | 117.58 ± 1.16  |
| $\epsilon \times 10^{-2}$ | 2.2 ± 0.3       | 2.5 ± 0.4       |

### Table 4

| $M_0$ (MeV)   | $\Gamma_0$ (MeV) | References   |
|---------------|------------------|--------------|
| 1210.9 ± 1.4  | 106.5 ± 3.5     | [10]         |
| 1210.30 ± 0.36 | 108.0 ± 0.52    | [11]         |
| 1210.75 ± 0.60 | 105.6 ± 1.2     | [12]         |
| 1212.60 ± 0.52 | 103.95 ± 0.88   | our results case C |
| 1214.00 ± 0.53 | 102.98 ± 0.85   | our results case (a) |
|                  | $M_0 - M_{++}$ | $\Gamma_0 - \Gamma_{++}$ | $\bar{M}_0 - \bar{M}_{++}$ | $\bar{\Gamma}_0 - \bar{\Gamma}_{++}$ |
|------------------|----------------|--------------------------|-----------------------------|----------------------------------|
| Our results I    | 0.40 ± 0.57    | 6.89 ± 0.95              | 2.25 ± 0.68                 | 8.45 ± 1.11                      |
| Our results II   | 0.70 ± 0.70    | 6.73 ± 1.09              | 2.47 ± 0.80                 | 8.51 ± 1.26                      |
| Pedroni et al [5]| –              | –                        | 1.4 ± 0.3                   | 8.6 ± 1.0                        |
| Koch et al [13]  | –              | –                        | 2.7 ± 0.6                   | 2.0 ± 1.8                        |
| Zidell et al [11]| −0.40 ± 0.39   | 8.79 ± 0.57              | 1.9 ± 0.4                   | 8.1 ± 0.5                        |
| Vasan et al [12] | 1.15 ± 0.78    | 4.8 ± 1.6                | –                          | –                                |
**Table 6**

|                | Case (a)                  | Case (b)                  |
|----------------|---------------------------|---------------------------|
| $M_{++}$ (MeV) | 1213.30 ± 0.23            | 1213.70 ± 0.26            |
| $\Gamma_{++}$ (MeV) | 96.17 ± 0.34               | 96.60 ± 0.43               |
| $a_{++}$       | 0.7175 ± 0.0189           | 0.7725 ± 0.0285           |
| $x_{++}(\tilde{M}^2)$ | −0.3868 ± 0.0014      | −0.3794 ± 0.0017          |
| $M_{++}$ (MeV) | 1231.89 ± 0.25            | 1232.02 ± 0.29            |
| $\Gamma_{++}$ (MeV) | 108.04 ± 0.39             | 107.97 ± 0.50             |

**Table 7**

|                | Case (a)                  | Case (b)                  |
|----------------|---------------------------|---------------------------|
| $M_0$ (MeV)   | 1214.00 ± 0.53            | 1214.50 ± 0.71            |
| $\Gamma_0$ (MeV) | 102.98 ± 0.85             | 103.11 ± 0.93             |
| $a_0$         | 0.8516 ± 0.0623           | 0.9056 ± 0.0826           |
| $x_0(\tilde{M}^2)$ | −0.4033 ± 0.0033      | −0.4012 ± 0.0036          |
| $M_0$ (MeV)   | 1234.76 ± 0.58            | 1235.17 ± 0.78            |
| $\Gamma_0$ (MeV) | 117.88 ± 1.01             | 117.76 ± 1.10             |
| $\epsilon$ ($\times 10^{-2}$) | 2.3 ± 0.3                 | 2.5 ± 0.4                 |
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