Degenerate Fermions and Wilson Loop in 1 + 1 Dimensions

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Abstract

We investigate the effect of finite fermion density on symmetry breaking by Wilson loops in (1 + 1) dimensions. We find the breaking and restoration of symmetry at finite density in the models with SU(2) and SU(3) gauge symmetries, in the presence of the adjoint fermions. The transition can occur at a finite density of fermions, regardless of the periodic or antiperiodic boundary condition of the fermion field; this is in contrast to the finite-temperature case examined by Ho and Hosotani (Nucl. Phys. B345 (1990) 445) where the boundary condition of fractional twist is essential to the occurrence of the phase transition.

1 Introduction

The physics of low-dimensional systems is currently studied by many authors. Some recent motivations for the study of low-dimensional physics come from the investigation of string theories [1], i.e., field theory on two-dimensional world sheets, and conformal field theories [2].

More recently, three-dimensional systems have attracted much interest. There are interesting subjects to study, for example, high-$T_c$ superconductors [3], fractional statistics [4], boson-fermion transmutation [5], topological field theories [6], and exactly soluble gravity and the relation to the mathematics of “knots” [7]. In these theories, the peculiarity in low-dimensional space-time is used fully. Note that it seems that some gauge symmetries and “effective” gauge fields play crucial roles.

On the other hand, as is stated in many cases, it is helpful to study models with reduced numbers of degrees of freedom, which depend on the dimensionality of the system. The solution of the models might shed on the more complicated problem in higher dimensions.

In (1+1) dimensional pure Yang-Mills theory there is no dynamical degree of freedom. There is, however, a nontrivial topological quantity if the background space is a circle. Recently, pure Yang-Mills theory applied to a circle has been
investigated by Rajeev [8]. The gauge covariant variable is often called a Wilson loop. It is known that the Wilson loop couples to matter fields and affects their dynamics [9].

Conversely, quantum effects of matter fields determine the vacuum expectation value of the Wilson loop. Symmetry breaking by Wilson loops, not only on a torus space [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] but also on a nonsimply connected lens space [23, 24, 25, 26, 27, 28], has been considered and used in unified models of gauge theories in higher (more than four) dimensions. Other approaches to the mechanism in string theories in novel ways are found in refs. [29, 30, 31, 32].

Recently, Ho and Hosotani [19] examined two-dimensional models with $SU(3)$ gauge symmetry and fermions to investigate phase transitions at finite temperature.

In this paper we consider nonzero fermion density in similar systems. We show the realization of other symmetries of vacuum when fermions are strongly degenerated. We consider Dirac fermions in this paper. Furthermore, we deal with left-right symmetric models; thus, anomalies have nothing to do with the phenomenon that takes place in our models.

## 2 The model and calculation

The action that we consider in this paper is given by

$$I = \int dt \int_0^L dx \left( -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \text{Tr} \bar{\psi} i\gamma^\mu D_\mu \psi \right),$$

where $L$ is circumference of the spatial circle, the field strength is defined as, for example, $F_{01} = \partial_0 A_1 - \partial_1 A_0 + ig[A_0, A_1]$, and the covariant derivative is, for example, $D_1 \psi = \partial_1 \psi + ig[A_1, \psi]$. Here $g$ is a gauge coupling constant. The massless fermion field $\psi$ is in the adjoint representation.

It has been found that the quantum effect of fermions in the adjoint representation or the "faithful, single-valued representation" of the adjoint group [16] can naturally bring about gauge-symmetry breakdown in higher dimensions. On these occasions, the boundary condition on the field is also important in the evaluation of quantum effects. We take the boundary conditions as

$$A_\mu(t, x + L) = A_\mu(t, x),$$
$$\psi(t, x + L) = e^{iB} \psi(x, L),$$

where the notation is the same as in ref. [19] except for the "twist" parameter $B (-\pi \leq B \leq \pi)$. The freedom of the global group transformation has been rotated away in (3).

The physical quantity is the Wilson loop

$$W = P \exp \left( ig \int_0^L dx A_1 \right),$$

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where $P$ means the path-ordered product. In vacuum, we suppose that the vacuum expectation value $\langle F_{01} \rangle = 0$. Nevertheless, $\langle W \rangle$ can be nontrivial.

In two dimensions, there is no dynamical gauge boson; thus, hereafter we call the phenomenon of $\langle W \rangle$ acquiring a nontrivial value as “symmetry breaking.” Moreover, we define symmetry of the vacuum by the subgroup of elements that commute with $\langle W \rangle$.

In the following, we consider nonzero fermion density. For this purpose, we calculate the thermodynamic potential $\Omega$ for the system at one-loop level. It is well known that the thermodynamic potential is expressed formally as [33, 34]

$$\Omega = -\frac{1}{\beta} \ln Z \quad (5)$$

with

$$Z = \text{tr} \left\{ \exp\left[ -\beta(\hat{H} - \mu \hat{N}) \right] \right\}, \quad (6)$$

where $\hat{H}$ is a Hamiltonian operator and $\hat{N}$ is a number operator in the theory. The trace is to be taken on the Hilbert space of states of the theory under consideration. $\beta$ is the inverse temperature, and $\mu$ is the chemical potential associated with $N$. In the present model $N$ is taken as the fermion number:

$$\hat{N} \sim \text{Tr} \int dx \bar{\psi} \gamma^0 \psi. \quad (7)$$

In the field theoretical description at the one-loop level, $\Omega$ is formally written in our model as

$$\Omega = -\frac{1}{2} \text{tr} \ln(i \gamma^\nu D^\nu), \quad (8)$$

where $D_0 = \partial_0 + ig[\langle A_0 \rangle, ]$ and $D_1 = \partial_0 + ig[\langle A_1 \rangle, ]$ with $\langle A_0 \rangle = -i\mu$ [33]. $\text{tr}$ means sum over the oscillator degrees of freedom and the trace over the gamma matrix at the same time. In two dimensions, the contribution of the gauge field to the potential is cancelled by the contribution of the associated ghost fields.

First we consider $SU(2)$ gauge symmetry. By using gauge transformation, we can parametrize a diagonal form of the vacuum gauge field:

$$\langle A_1 \rangle = \frac{1}{gL} \left( \begin{array}{cc} \theta & 0 \\ 0 & -\theta \end{array} \right). \quad (9)$$

The thermodynamic potential or the effective potential in terms of $\theta$ with $\mu \neq 0$ in this background is given by [33, 34]

$$\Omega/L = \frac{2}{\pi L^2} \sum_{l=1}^{\infty} \frac{1}{l^2} \left\{ \cos[l(2\theta - B)] + \cos[l(2\theta + B)] + 1 \right\}$$

$$-\frac{1}{\beta L} \sum_{l=-\infty}^{\infty} \left\{ \ln \left[ \prod_{i=1}^{3} (1 + e^{\beta(\mu - M_i)}) (1 + e^{-\beta(\mu + M_i)}) \right] \right\}, \quad (10)$$

where $M_1 = |2\pi l + 2\theta - B|/L$, $M_2 = |2\pi l - 2\theta - B|/L$ and $M_3 = |2\pi l|/L$. The expression of the vacuum energy in the first part of (10) is obtained after
the regularization in the standard way [34, 13]. Obviously the potential in the region $2\pi \leq 2\theta \leq 4\pi$ has the same form as in the region $0 \leq 2\theta \leq 2\pi$ because of $Z_2$ symmetry [9]. Therefore we restrict ourselves to the examination of the region $0 \leq 2\theta \leq 2\pi$.

At $T = \beta^{-1} = 0$ and $\mu = 0$, performing the sum over $l$, we can obtain the vacuum energy as

$$V_0 = \left( \frac{\Omega}{L} \right)_{\mu=0} = -\left\{ \begin{array}{ll} \frac{(2\theta)^2 + (\pi - B)^2}{(\pi L)^2} & 0 \leq 2\theta \leq B \\ \frac{(2\theta - \pi)^2 + B^2}{(\pi L)^2} & B \leq 2\theta \leq 2\pi - B \\ \frac{(2\theta - 2\pi)^2 + (\pi - B)^2}{(\pi L)^2} & 2\pi - B \leq 2\theta \leq 2\pi \end{array} \right..$$

(11)

The finite density effect becomes important at low temperature. In the limit $T \to 0$ ($\beta \to \infty$), eq. (11) becomes \[13, 34\]

$$\frac{\Omega}{L} = V_0 - \frac{1}{L} \sum_{i=1}^{3} \sum_{M_i \leq \mu} (\mu - M_i).$$

(12)

The sum over $l$ is taken only in the finite range where $M_i \leq \mu$ is satisfied. The quantization of the fermion number is easily read from the expression. (Note: The fermion number is given by $-\partial \Omega/\partial \mu$.)

Here, we pay attention to the case that $B = 0$, i.e., the periodic boundary condition on the fermion field. At high temperatures, no phase transition occurs, as shown by Ho and Hosotani [19]. The minimum of the potential in terms of $\theta$ is located at $2\theta = \pi$ at any finite temperature if $B = 0$. In our case with degenerate fermions, a different aspect comes about.

The shape of the potential $\Omega/L$ is displayed in Fig. 1. The potential is continuous but not smooth everywhere. This feature is found only in two dimensions; in higher dimensions, the potential is smooth everywhere [13].

Figure 1: Schematic view of the thermodynamic potentials at zero temperature including vacuum energy as a function of $2\theta$ for various values of the chemical potential. (a) $\mu L = 0$. (b) $\mu L = \pi/2$. (c) $\mu L = \pi$.

Let us consider the difference in $\Omega/L$ at $2\theta = 0$ and $2\theta = \pi$. It turns out, from eq. (11) and eq. (12), that

$$(\Omega/L)(2\theta = 0) - (\Omega/L)(2\theta = \pi) = -\left\{ \begin{array}{ll} \frac{(\pi - 2\mu L)/L^2}{(-3\pi + 2\mu L)/L^2} & 0 \leq \mu L \leq \pi \\ \frac{(\pi - 2\mu L)/L^2}{(\pi - 2\mu L)/L^2} & \pi \leq \mu L \leq 2\pi \end{array} \right..$$

(13)
Actually, this is a periodic function of $\mu L$. Thus, if $1/4 < (\mu L/2\pi) - [\mu L/2\pi] < 3/4$ (where $[\ ]$ is Gauss’ symbol), the point $2\theta = \pi$ is favoured rather than $2\theta = 0$. This is a new vacuum at finite fermion density.

Next we consider a model with $SU(3)$ symmetry. The form of the action is the same as eq. 1. The vacuum gauge field is parametrized as

$$\langle A_1 \rangle = \frac{1}{gL} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix},$$

with $\theta_1 + \theta_2 + \theta_3 = 0$.

At zero temperature and with vanishing chemical potential, the minima of the effective potential determine the following symmetry of the vacuum:

$$|B| < \pi/3, \quad U(1) \times U(1)$$
$$\pi/3 < |B| < \pi/2, \quad SU(2) \times U(1)$$
$$\pi/2 < |B| < \pi, \quad SU(3).$$

(15)

In the background of the strongly degenerate fermion at zero temperature, one can find the symmetries of the vacua at $B = 0$ as

$$|\mu L| < \pi/3, \quad U(1) \times U(1)$$
$$\pi/3 < |\mu L| < \pi/2, \quad SU(2) \times U(1)$$
$$\pi/2 < |\mu L| < \pi, \quad SU(3).$$

(16)

For general values of $B$ and $\mu L$, the symmetry is exhibited in Fig. 2.

Figure 2: Phase structure of 1 + 1 dimensional $SU(3)$ model with “twisted” adjoint fermions at zero temperature and nonvanishing chemical potential $\mu$. The other region of parameters $B$ and $\mu L$ can be obtained by taking mirror images of this diagram.

In the more general situations that appear in physical problems, we must impose conservation of the fermion number. In other words, $\mu$ is to be determined by $N$. However, we do not pursue further analysis on complicated systems in this paper.
3 Concluding remarks

We wish to emphasize that a new phase exists at finite density even if the boundary condition on the fermion is periodic or antiperiodic. High-temperature phase transitions investigated by Ho and Hosotani [19] are characteristic of the model with matter fields that obey the boundary condition of an intermediate twist angle, i.e., \( B \neq 0 \) or \( \pi \). However, the fractional twist is rather unnatural, especially if we consider higher dimensional theories where we use the symmetry-breaking mechanism of Wilson loops on compact spaces. In the compactified theory, the matter field seems massive even if \( \langle W \rangle = 1 \). Of course, our “phase transition” scenario at finite density can work at unnaturally high densities whose scales are comparable to the Planck scale if we utilize the mechanism in unified theories.

On the other hand, from the low-dimensional field theoretical point of view, the Wilson loop might give an interesting perspective. The boundary conditions of the fields are effectively determined dynamically in some sense. The finite-volume effect on the system may equally be significant in other two- or three-(space-time) dimensional systems. Further non-trivial connections between gauge fields and matter fields will be clarified in the procedure presented in this paper.

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