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REGULARITY OF PUSH-FORWARD OF MONGE–AMPÈRE MEASURES

by Eleonora DI NEZZA & Charles FAVRE

Dedicated to Jean-Pierre Demailly on the occasion of his 60th birthday

Abstract. — We prove that the image under any dominant meromorphic map of the Monge–Ampère measure of a Hölder continuous quasi-psh function still possesses a Hölder potential. We also discuss the case of lower regularity.

Résumé. — Nous démontrons que l’image par une application méromorphe dominante d’une mesure de Monge–Ampère d’une fonction quasi-psh et hölderienne possède aussi un potentiel hölderien. Nous discutons aussi le cas de régularité plus basse.

1. Introduction

Let $(X,\omega_X)$ be a compact Kähler manifold of dimension $n$ normalized by the volume condition $\int_X \omega^n_X = 1$. We say that a potential $\varphi \in L^1(X)$ is $\omega_X$-plurisubharmonic ($\omega_X$-psh for short) if locally $\varphi$ is the sum of a plurisubharmonic and a smooth function, and $\omega_X + dd^c \varphi \geq 0$ in the weak sense of currents, where $d = \partial + \bar{\partial}$ and $d^c = \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi} \partial \bar{\partial}$.

We denote by $\text{PSH}(X,\omega_X)$ the set of all $\omega_X$-psh functions on $X$. Recall from [13, Section 1] that the non-pluripolar Monge–Ampère measure of a function $\varphi \in \text{PSH}(X,\omega_X)$ is a positive measure defined as the increasing limit

$$(\omega_X + dd^c \varphi)^n = \lim_{j \to +\infty} 1_{\{\varphi > -j\}} (\omega_X + dd^c \max\{\varphi, -j\})^n$$

where the right hand side is defined using Bedford–Taylor intersection theory of bounded psh functions, see [2]. By construction this measure does not charge pluripolar sets.

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One of the main results of [13] states that if $\mu$ is a probability measure on $X$ which does not charge pluripolar sets, then there exists a unique (up to a constant) $\omega_X$-psh function $\varphi$ such that $\int_X (\omega_X + \dd^c \varphi)^n = 1$ and

$$\mu = (\omega_X + \dd^c \varphi)^n.$$  

We denote by $\mathcal{E}(X, \omega_X)$ the set of all $\omega_X$-psh functions whose non-pluripolar Monge–Ampère measure has full mass so that any solution to (1.1) belongs to $\mathcal{E}(X, \omega_X)$.

In the same paper, Guedj and Zeriahi introduced for any $p > 0$ the subset $\mathcal{E}^p(X, \omega_X)$ of $\mathcal{E}(X, \omega_X)$ consisting of all $\omega_X$-psh functions satisfying the integrability condition $\varphi \in L^p((\omega_X + \dd^c \varphi)^n)$. Since $\omega_X$-psh functions are bounded from above it follows that

$$\mathcal{E}^p(X, \omega_X) \subset \mathcal{E}^q(X, \omega_X), \text{ for all } p > q.$$  

Observe also that any $\omega_X$-psh function lying in $L^\infty$ belongs to the intersection of all $\mathcal{E}^p(X, \omega_X)$.

We shall say that a probability measure which does not charge pluripolar sets $\mu = (\omega_X + \dd^c \varphi)^n$ is a Monge–Ampère measure having Hölder, continuous, $L^\infty$ or $\mathcal{E}^p$ potential for some $p > 0$ whenever $\varphi$ is Hölder, continuous, $L^\infty$ or belongs to the energy class $\mathcal{E}^p(X, \omega_X)$ respectively.

Let us now consider any dominant meromorphic map $f : X \to Y$ where $(Y, \omega_Y)$ is also a compact Kähler manifold of volume 1, and denote by $m$ its complex dimension. Let $\Gamma$ be a resolution of singularities of the graph of $f$. We obtain two surjective holomorphic maps $\pi_1 : \Gamma \to X$ and $\pi_2 : \Gamma \to Y$ where $\pi_1$ is bimeromorphic so that $\Gamma$ is a modification of a compact Kähler manifold. By Hironaka’s Chow lemma, see e.g. [17, Theorem 2.8] we may suppose that $\pi_1$ is a composition of blow-ups along smooth centers so that $\Gamma$ is itself a compact Kähler manifold of complex dimension $n$. We fix any Kähler form $\omega_\Gamma$ on it.

One defines the push-forward under $f$ of a measure $\mu$ not charging pluripolar sets as follows. Since $\pi_1$ is a bimeromorphism, there exist two closed analytic subsets $R \subset \Gamma$ and $V \subset X$ such that $\pi_1 : \Gamma \setminus R \to X \setminus V$ is a biholomorphism. One may thus set $\pi_1^* \mu$ to be the trivial extension through $R$ of $(\pi_1)|_{\Gamma \setminus R \mu}$. This measure is again a probability measure which does not charge pluripolar sets.

We then define the probability measure $f_* \mu := (\pi_2)_* \pi_1^* \mu$. We observe that since $f$ is dominant then $\pi_2$ is surjective hence the preimage of a pluripolar set in $Y$ by $\pi_2$ is again pluripolar. By the preceding discussion, there exists $\psi \in \mathcal{E}(Y, \omega_Y)$ such that $f_* \mu = (\omega_Y + \dd^c \psi)^m$.

Our main goal is to discuss the following question.
Problem 1.1. — Suppose $\mu$ is a Monge–Ampère measure having Hölder, continuous, $L^\infty$ or $E^p$ potential. Is it true that $f_\ast \mu$ is also a Monge–Ampère measure of a potential lying in the same class of regularity?

This problem is hard for Monge–Ampère measures having either continuous or $L^\infty$ potentials since there is no known intrinsic characterization of these measures. For these classes of regularity even the case $f$ is the identity map and $X = Y$ is still open (see for example [7, Question 15]).

Problem 1.2. — Suppose $\mu$ is a probability measure on $X$ not charging pluripolar sets and write $\mu = (\omega + dd^c \varphi)^n = (\omega' + dd^c \varphi')^n$ where $\omega, \omega'$ are two Kähler forms of volume 1. Is it true that $\varphi$ is continuous (resp. $L^\infty$) iff $\varphi'$ is?

Remark. — A variant of Problem 1.1 has been recently investigated in [1, 18]. In particular, one can find in these papers a criterion on the singularities of an algebraic map $f : X \to Y$ which ensures that the push-forward of any continuous volume form remains continuous. We refer to these articles for the precise statements and for some far-reaching generalizations of them over any local fields.

Intrinsic characterizations of Monge–Ampère measures of Hölder functions are given by [4] and [9], and in the context of Hermitian compact manifolds by [15]. A characterization of Monge–Ampère measures of functions in the energy class $E^p$ is also obtained in [13, Theorem C] so that Problem 1.2 has a positive answer for these two classes of regularity, see [5, Theorem 4.1]. Problem 1.1 remains though quite subtle. If we restrict our attention to the regularity in the $E^p$ energy classes, then the answer is no in general. Suppose that $\pi : X \to \mathbb{P}^2$ is the blow-up at some point $p \in \mathbb{P}^2$, and let $E = \pi^{-1}(p)$. It was observed by the first author in [6, Proposition B] that there exists a probability measure $\mu = (\omega_X + dd^c \varphi)^2$ with $\varphi \in E^1(X, \omega_X)$ but $\pi_\ast \mu = (\omega_{FS} + dd^c \psi)^2$ with $\psi \notin E^1(\mathbb{P}^2, \omega_{FS})$, where $\omega_{FS}$ denotes the Fubini Study metric on $\mathbb{P}^2$ and $\omega_X$ is a Kähler form.

In this note we answer Problem 1.1 in two situations. We first treat the case $\mu$ is the Monge–Ampère of a Hölder function.

Theorem 1.3. — Let $f : X \dashrightarrow Y$ be any dominant meromorphic map between two compact Kähler manifolds. If $\mu$ is a Monge–Ampère measure having a Hölder potential with Hölder exponent $\alpha$, then $f_\ast \mu$ is a Monge–Ampère measure having a Hölder potential with Hölder exponent bounded by $C \alpha^{\dim(X)}$ for some constant $C > 0$ depending only on $f$. 

We expect that the technics developed in the paper of Kołodziej–Nguyen [15] in the present volume allows one to extend the previous result to arbitrary compact hermitian manifolds.

Next we treat the case the image of the map has dimension 1.

**Theorem 1.4.** — Let $f : X \rightarrow Y$ be any dominant meromorphic map from a compact Kähler manifold to a compact Riemann surface. If $\mu$ is a Monge–Ampère measure having a Hölder, $C^0$, $L^\infty$, $E^p$ potential respectively, then $f_\ast \mu$ is a Monge–Ampère measure having a potential lying in the same regularity class.

Motivations for studying this question come from the analysis of degenerating measures on families of projective manifolds developed in [11]. Let us briefly recall the setting of that paper. Let $X$ be a smooth connected complex manifold of dimension $n + 1$, and $\pi : X \rightarrow \mathbb{D}$ be a flat proper analytic map over the unit disk which is a submerison over the punctured disk and has connected fibers. We assume that $X$ is Kähler so that each fiber $X_t = \pi^{-1}(t)$ is also Kähler.

A tame family of Monge–Ampère measures is by definition a family of positive measures $\{\mu_t\}_{t \in \mathbb{D}}$ each supported on $X_t$ that can be written under the form

$$\mu_t = p_\ast(T|_{X_t}^n),$$

where $T$ is a positive closed $(1, 1)$-current having local Hölder continuous potentials and defined on a complex manifold $X'$ which admits a proper bimeromorphic morphism $p : X' \rightarrow X$ which is an isomorphism over $X := \pi^{-1}(\mathbb{D}^*)$. It follows from [3, Corollary 1.6] that the family of measures $\mu'_t := T|_{X_t}^n$ in $X'$ is continuous so that $\mu'_t$ converges to a positive measure $\mu'_0$ supported on $X'_0$ as $t \rightarrow 0$. It follows that the convergence $\lim_{t \rightarrow 0} \mu_t = \mu_0$ also holds in $X$.

As a corollary of the previous results we show the limiting measure $\mu_0$ is of a very special kind:

**Corollary 1.5.** — Let $\{\mu_t\}_{t \in \mathbb{D}}$ be any tame family of Monge–Ampère measures, so that $\mu_t \rightarrow \mu_0$ as $t \rightarrow 0$.

Then there exist a finite collection of closed subvarieties $\{V_i\}_{i=0,...,N}$ of $X_0$ and for each index $i$ a positive measure $\nu_i$ supported on $V_i$ such that

$$\mu_0 = \sum_{i=1}^{N} \nu_i$$

and $\nu_i$ is a Monge–Ampère measure on $V_i$ having a Hölder potential.
In the previous statement, it may happen that \( V_i \) is singular, in which case it is understood that the pull-back of \( \nu_i \) to a (Kähler) resolution of \( V_i \) is a Monge–Ampère measure having a Hölder continuous potential.

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2. Images of Monge–Ampère measures having a Hölder potential: proof of Theorem 1.3

As already mentioned, a dominant meromorphic map \( f: X \rightarrow Y \) can be decomposed as \( f = \pi_2 \circ \pi_1^{-1} \), where \( \pi_1: \Gamma \rightarrow X \) is holomorphic and bimeromorphic and \( \pi_2: \Gamma \rightarrow Y \) is a surjective holomorphic map. Recall that one can assume \( \Gamma \) to be Kähler, and that \( f^* \mu = (\pi_2)^* \pi_1^* \mu \).

We first claim that if \( \mu \) is the Monge–Ampère of a Hölder continuous function then \( \pi_1^* \mu \) too. Let \( \varphi \in \text{PSH}(X, \omega_X) \) be the Hölder potential such that \( \mu = (\omega_X + dd^c \varphi)^n \). It then follows from Bedford and Taylor theory that \( \pi_1^* \mu = (\pi_1^* \omega_X + dd^c \pi_1^* \varphi)^n \). Since \( \pi_1^* \omega_X \) is a semipositive smooth form, there exists a positive constant \( C > 0 \) such that \( \pi_1^* \mu \leq (C \omega_\Gamma + dd^c \pi^* \varphi)^n \) where \( \omega_\Gamma \) is a Kähler form on \( \Gamma \), and [4, Theorem 4.3] implies that \( \tilde{\mu} := \pi_1^* \mu \) is the Monge–Ampère measure of a Hölder continuous \( C \omega_\Gamma \)-psh function. This proves the claim. We are then left to prove that \( (\pi_2)^* \tilde{\mu} \) is the Monge–Ampère measure of a Hölder potential. This will be done in Lemma 2.4.

We first show that the push-forward of a smooth volume form has density in \( L^{1+\varepsilon} \), for some constant \( \varepsilon > 0 \) depending only on \( f \).

**Proposition 2.1.** — Let \( f: X \rightarrow Y \) be a surjective holomorphic map. Then \( f_* \omega_X^n = g \omega_Y^n \) with \( g \in L^{1+\varepsilon}(\omega_Y^n) \), for some \( \varepsilon > 0 \).

This result is basically [19, Proposition 3.2] (see also [20, Section 2]). We give nevertheless a detailed proof for reader’s convenience. Pick any coherent ideal sheaf \( \mathcal{I} \subset \mathcal{O}_X \), and denote by \( V(\mathcal{I}) = \text{supp}(\mathcal{O}_X/\mathcal{I}) \) the closed analytic subvariety of \( X \) associated to \( \mathcal{I} \). Let \( \{U_i\}_{i=1}^N \) be a finite open covering of \( X \) by balls and \( \{V_i\}_i \) a subcovering such that \( V_i \subset U_i \). The analytic sheaf \( \mathcal{I} \) is globally generated on each \( U_i \) so that we can find holomorphic functions such that \( \mathcal{I}|_{U_i} = (h_1^{(i)}, \ldots, h_k^{(i)}) \cdot \mathcal{O}_{U_i} \). Let \( \{\rho_i\} \) be a partition of unity subordinate to \( V_i \). We then define

\[
\Phi_\mathcal{I} := \sum_{i=1}^N \rho_i \left( \sum_{j=1}^k |h_j^{(i)}|^2 \right).
\]
Then \( \Phi_I : X \to \mathbb{R}_+ \) is a smooth function which vanishes exactly on \( V(I) \).
Observe that if \( \Phi_I \) and \( \Phi'_I \) are defined using two different coverings, then there exists \( C > 0 \) such that
\[
\frac{1}{C} \Phi'_I \leq \Phi_I \leq C \Phi'_I.
\]
In the sequel we shall abuse notation and not write the dependence of \( \Phi_I \) in terms of the local generators of the ideal sheaf. The logarithm of the obtained function is then well-defined up to a bounded function so that all statements in the next Lemma make sense.

**Lemma 2.2.** — Let \( I, J \subset \mathcal{O}_X \) be two coherent ideal sheafs. The followings hold:

1. there exists \( \varepsilon > 0 \) such that \( |\Phi_I|^{-\varepsilon} \in L^1(X) \);
2. if \( I \subset J \) then \( \Phi_J \geq c \Phi_I \) for some positive \( c > 0 \);
3. if \( V(J) \subset V(I) \) then there exists \( c, \theta > 0 \) such that \( \Phi_J \geq c \Phi^\theta_I \);
4. given \( f : X \to Y \) a holomorphic surjective map and a coherent ideal sheaf \( J \subset \mathcal{O}_Y \), then \( \Phi_{f^*J} = \Phi_J \circ f \) (for a suitable choice of local generators of \( J \) and \( f^*J \)).

**Proof.** — Using a resolution of singularities of \( I \), one sees that the statement in (1) reduces to show that \( |z_1|^{-\varepsilon} \) is locally integrable for some \( \varepsilon > 0 \), and this is the case if we choose \( \varepsilon \) small enough. The statements in (2) and (4) follow straightforward from the definition in (2.1). The statement in (3) is a consequence of Łojasiewicz theorem, see e.g. [16, Theorem 7.2].

**Lemma 2.3.** — Let \( f : X \to Y \) be a holomorphic surjective map and let \( I \subset \mathcal{O}_X \) be a coherent ideal sheaf. Then there exists a coherent ideal sheaf \( J \subset \mathcal{O}_Y \), and constants \( c, \theta > 0 \) such that for any \( y \in Y \) we have
\[
\inf_{x \in f^{-1}(y)} \Phi_I \geq c \Phi^\theta_J
\]

**Proof.** — Let \( J \subset \mathcal{O}_Y \) be the coherent ideal sheaf of holomorphic functions vanishing on the set \( f(V(I)) \) which is analytic since \( f \) is proper. Observe that \( V(f^*J) = f^{-1}(V(J)) \supset V(I) \), so that Lemma 2.2(3) and (4) insure that there exist \( c, \theta > 0 \) such that
\[
\Phi_I \geq c \Phi^\theta_{f^*J} = c(\Phi_J \circ f)^\theta.
\]
Hence the conclusion.

**Proof of Proposition 2.1.** — Recall that Sard’s theorem implies the existence of a closed subvariety \( S \subset Y \) such that \( f : X \setminus f^{-1}(S) \to Y \setminus S \) is a submersion.
We first prove that \( f^* \omega^n_X \) is absolutely continuous w.r.t. \( \omega^n_Y \). We need to check that \( \omega_Y^m(E) = 0 \) implies \( f^* \omega^n_X(E) = 0 \) for any Borel subset \( E \subset Y \). As \( S \) and \( f^{-1}(S) \) have volume zero one may assume that \( f \) is a submersion in which case the claim follows from Fubini’s theorem.

Radon–Nikodym theorem now guarantees that \( f^* \omega^n_X = g_{\omega}^m_Y \) for some \( 0 \leq g \in L^1(Y) \). We want to show that the integral
\[
\int_Y g^{1+\varepsilon} \omega_Y^m = \int_Y g^\varepsilon f^* \omega^n_X = \int_X (f^* g)^\varepsilon \omega^n_X
\]
is finite for some \( \varepsilon > 0 \) small enough. Consider the smooth function \( \phi(x) := \frac{f^* \omega^n_Y \wedge \omega_X^{-m}}{\omega_X^n}(x) \), and set \( \phi(y) := \inf_{x \in f^{-1}(y)} \phi(x) \) so that \( \phi \geq f^* \tilde{\phi} \). We claim that for any \( y \in Y \)
\[
(2.2) \quad g(y) \leq \frac{c}{\phi(y)},
\]
for some constant \( c > 0 \). Let \( \chi \) be a test function (i.e. a non negative smooth function) on \( Y \), then
\[
\int_Y \chi g \omega_Y^n = \int_X f^* \chi \omega_X^n = \int_X \frac{\chi}{\phi} f^* \omega^n_Y \wedge \omega_X^{-m} \leq \int_X f^* \left( \frac{\chi}{\phi} \omega_Y^m \right) \wedge \omega_X^{-m} \leq C(f) \int_Y \frac{\chi}{\phi} \omega_Y^m
\]
where \( c := C(f) = \int_{f^{-1}(y)} \omega_X^{-m} \) is the volume of a fiber over a generic point \( y \in Y \). The claim is thus proved. Lemma 2.2(1) and (4) combined with Lemma 2.3 then insure that there exists \( \varepsilon > 0 \) such that \( (f^* g)^\varepsilon \in L^1(\omega_X^n) \).

Theorem 1.3 is reduced to the following result which relies in an essential way on Proposition 2.1.

**Proposition 2.4.** — Suppose \( f : X \to Y \) is a surjective holomorphic map between compact Kähler manifolds. If \( \mu \) is a positive measure on \( X \) with Hölder continuous potentials, then \( f^* \mu \) is a positive measure on \( Y \) with Hölder potentials.

Observe that by multiplying \( \omega_X \) by a suitable positive constant we may assume that \( f^* \omega_Y \leq \omega_X \). The volume normalization is no longer satisfied but a positive multiple of \( \mu \) is still the Monge–Ampère measure of a \( \omega_X \)-psh Hölder continuous function. Write \( f^* \mu = (\omega_Y + \ddc \psi)^m \) with \( \psi \in \text{PSH}(Y, \omega_Y) \).
We claim that there exists $C > 0$, and $\varepsilon > 0$ such that for all $u \in \text{PSH}(Y, \omega_Y)$ with $\int_X u \omega^n_X = 0$

\begin{equation}
\int_Y \exp(-\varepsilon u) \, d(f_*\mu) \leq C.
\end{equation}

Indeed, for any $u \in \text{PSH}(Y, \omega_Y)$ we have that

\begin{equation}
\int_Y e^{-\varepsilon u} \, d(f_*\mu) = \int_X e^{-\varepsilon(u \circ f)} \, d\mu.
\end{equation}

Now the integral $\int_X e^{-\varepsilon(u \circ f)} \, d\mu$ is uniformly bounded by [10, Theorem 1.1] since:

- $\mu$ has Hölder continuous potentials;
- $f^*\omega_Y \leq \omega_X$ hence $u \circ f \in \text{PSH}(X, \omega_X)$;
- and the set of functions in $\text{PSH}(X, \omega_X)$ such that $\int_X u \omega^n_X = 0$ is compact by [12, Proposition 2.6].

This proves our claim. Using the terminology of [9] this means that $f_*\mu$ is moderate. It is worth mentioning that if [7, Question 16] holds true then the conclusion of Proposition 2.4 would follow immediately since any moderate measure would have a Hölder continuous potential. To get around this problem we use the characterization of measures with Hölder potentials given by Dinh and Nguyen.

**Proof of Proposition 2.4.** — By [9, Lemma 3.3], $f_*\mu$ is the Monge–Ampère measure of a Hölder potential if and only if there exist $\tilde{c} > 1$ and $\tilde{\beta} \in (0, 1)$ such that

\begin{equation}
\int_Y |u - v| \, df_*\mu \leq \tilde{c} \max \left( \|u - v\|_{L^1(\omega^n_Y)}, \|u - v\|_{L^1(\omega^n_Y)}^{\tilde{\beta}} \right)
\end{equation}

for all $u, v \in \text{PSH}(Y, \omega_Y)$. By assumption on $\mu$ we know there exist $c > 1$ and $\beta \in (0, 1)$ such that $\int_Y |u - v| \, df_*\mu = \int_X |f^*u - f^*v| \, d\mu$, and

\begin{equation}
\int_X |f^*u - f^*v| \, d\mu \leq c \max \left( \|f^*u - f^*v\|_{L^1(\omega^n_X)}, \|f^*u - f^*v\|_{L^1(\omega^n_X)}^{\beta} \right).
\end{equation}

Also, Proposition 2.1 gives

\begin{equation}
\int_X |f^*u - f^*v| \omega^n_X = \int_Y |u - v| \, g \omega^n_Y \leq \|g\|_{L^{1+\varepsilon}(\omega^n_Y)} \|u - v\|_{L^p(\omega^n_Y)}
\end{equation}

where $p$ is the conjugate exponent of $1 + \varepsilon$. Set $C_g := \|g\|_{L^{1+\varepsilon}(\omega^n_Y)} < +\infty$.

Up to replace $C_g$ with $C_g + 1$ we can assume that $C_g \geq 1$. 

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Denote by $m_u := \int_Y u \omega_X^n$ and observe that $u' := u - m_u$, $v' := v - m_v$ satisfy $\int_X u' \omega_X^n = 0 = \int_X v' \omega_X^n$. Then the triangle inequality gives
\[
\|u - v\|_{L^p(\omega_X^n)} = \left( \int_Y |(u' - v') + (m_u - m_v)|^p \omega_X^n \right)^{1/p}
\leq \|u' - v'\|_{L^p(\omega_X^n)} + |m_u - m_v|
\leq \|u' - v'\|_{L^p(\omega_X^n)} + \|u - v\|_{L^1(\omega_X^n)}.
\]
(2.7)

At this point, we make use of [9, Proposition 3.2] (that holds for normalized potentials) to replace the $L^p$-norm with the $L^1$-norm. We then infer the existence of a constant $c' > 1$ such that
\[
\|u' - v'\|_{L^p(\omega_X^n)} \leq c' \max(1, - \log \|u' - v'\|_{L^1(\omega_X^n)}) \frac{p-1}{p} \|u' - v'\|^\frac{1}{p}_{L^1(\omega_X^n)}.
\]

When $t := \|u' - v'\|_{L^1(\omega_X^n)} \geq 1/e$ we clearly have
\[
\|u' - v'\|_{L^p(\omega_X^n)} \leq c' \|u' - v'\|^\frac{1}{p}_{L^1(\omega_X^n)},
\]
whereas for any integer $N \in \mathbb{N}^*$, there exists a constant $c_N > 0$ such that
\[- \log t \leq c_N t^{-1/N} \text{ when } t \leq 1/e, \text{ hence}
\]
\[
\|u' - v'\|_{L^p(\omega_X^n)} \leq c'' \|u' - v'\|^\frac{1}{p_{L^1(\omega_X^n)}} (1 - \frac{2}{N}).
\]
As $\|u' - v'\|_{L^1(\omega_X^n)} \leq 2\|u - v\|_{L^1(\omega_X^n)}$, combining (2.5), (2.6) and (2.7) we get
\[
\|f^u - f^v\|_{L^1(\mu)} \leq C \max \left( \|u - v\|^\frac{1}{p_{L^1(\omega_X^n)}}, \|u - v\|_{L^1(\omega_X^n)} \right),
\]
with $\hat{\beta} = \frac{\beta}{p} (1 - \frac{2}{N})$. By [9, Lemma 3.3] $f^u \mu = (\omega_Y + dd^c \psi)^n$ where $\psi$ is a Hölder continuous function.

To get a bound on the Hölder regularity of $\psi$, one argues as follows. First if $\mu = (\omega + dd^c \varphi)^n$ with $\varphi$ a $\alpha$-Hölder potential, and $\pi: \Gamma \to X$ is a proper modification, then $\pi^* \mu$ is dominated by a Monge–Ampère measure with $\alpha$-Hölder potential, and [4, Proposition 3.3(ii)] is satisfied with $b = 2\alpha/(\alpha + 2n)$ by [4, Theorem 4.3(iii)]. Hence, following the proof of [4, Theorem], we see that $\pi^* \mu$ is a Monge–Ampère measure of a $\alpha_1$-Hölder continuous potential with $\alpha_1 < b/(n + 1)$ (see Remark below for more details about the latter statement).

By [9, Proposition 4.1], (2.5) holds with $\beta = \alpha^n_1/(2 + \alpha^n_1)$, and (2.4) is then satisfied for any $\hat{\beta} < \beta/p$ so that $f^{u} \mu$ is a Monge–Ampère measure with $\hat{\alpha}$-Hölder potential for any $\hat{\alpha} < 2\hat{\beta}/(m + 1)$, see the discussion on [9, p. 83]. Combining all these estimates we see that any
\[
\hat{\alpha} < \frac{\alpha^n}{p(m + 1)(\alpha/2 + n)^{n}(n + 1)^m}.
\]
works where $p$ is the conjugate of the larger constant $\varepsilon > 0$ for which Proposition 2.1 holds.

\[ \square \]

Remark. — We borrow notations from the proof of [4, Theorem A]. Fix $\alpha_1 < b/(n+1)$ and choose $\varepsilon > 0$ such that $\alpha_1 \leq \alpha \leq \alpha_0 \leq b - \alpha_0(n+\varepsilon)$. By the previous arguments we know that condition $(ii)$ in [4, Proposition 3.3] holds, i.e. for any $\phi \in PSH(\Gamma, \omega_{\Gamma})$, we have $\|\rho_\varepsilon \phi - \phi\|_{L^1(\mu)} = O(\delta^b)$, where $b = 2\alpha/(\alpha + 2n)$.

In particular, this gives $\pi^* \mu(E_0) \leq c_1 \delta^{b-\alpha_0}$.

Let $g \in L^1(\mu)$ be defined as $g = 0$ on $E_0$ and $g = c$ on $\Gamma \setminus E_0$ where $c$ is a positive constant such that $\pi^* \mu(\Gamma) = \int_{\Gamma} g \, d(\mu)$. An easy computation gives that $c = \pi^* \mu(\Gamma)/\pi^* \mu(\Gamma \setminus E_0)$. Let $v \in PSH(\Gamma, \omega_{\Gamma})$ be the bounded solution of the Monge–Ampère equation $(\omega_{\Gamma} + dd^c v)^n = g \cdot \pi^* \mu$. Observe that

$$\|1 - g\|_{L^1(\mu)} = \int_{E_0} d\pi^* \mu + \int_{\Gamma \setminus E_0} |1 - c| \, d\pi^* \mu = 2 \int_{E_0} d\pi^* \mu \leq 2c_1 \delta^{b-\alpha_0}.$$ 

Since $\pi^* \mu = (\omega_{\Gamma} + dd^c \mathring{\varphi})^n$ satisfies the $\mathcal{H}(\infty)$ property we can still apply [8, Theorem 1.1] and get

$$\|\mathring{\varphi} - v\|_{L^\infty} \leq c_3 \delta^{\frac{b-\alpha_0}{n+\varepsilon}}.$$ 

The exact same arguments as in [4, Theorem A] then insure that the Hölder exponent of $\mathring{\varphi}$ is $\alpha_1$.

3. Over a one-dimensional base: proof of Theorem 1.4

In this section we treat Problem 1.1 in the case the base is a Riemann surface.

We start with the case of a surjective holomorphic map $f: X \to Y$ from a Kähler compact manifold to a compact Riemann surface.

Let $\mu = (\omega_X + dd^c \varphi)^n$ be a Monge–Ampère measure of a continuous $\omega_X$-psh function $\varphi$. Suppose $v_k, v$ is a family of $\omega_X$-psh functions such that $v_k \to v$ in $L^1$, then

$$\int_X v_k \, d\mu$$

$$= \int_X v_k (\omega_X + dd^c \varphi)^n$$

$$= \int_X v_k \omega_X^n + \sum_{j=0}^{n-1} \int_X \varphi \, dd^c v_k \wedge \omega_X^j \wedge (\omega_X + dd^c \varphi)^{n-j-1}$$

$$\to \int_X v \, d\mu$$
by [3, Corollary 1.6(a)]. Observe that in the last equality we used the fact that

$$(\omega_X + dd^c \varphi)^n - \omega_X^n = \sum_{j=0}^{n-1} dd^c \varphi \wedge \omega_X^j \wedge (\omega_X + dd^c \varphi)^{n-j-1}$$

and Stokes’ theorem.

Normalize the Kähler form on $Y$ such that $\int \omega_Y = 1$, and pick any sequence $y_k \to y_\infty \in Y$. Let $w_k$ be the solutions of the equations $\Delta w_k = \delta_{y_k} - \omega_Y$ with $sup \ w_k = 0$ so that $w_k(y) - \log |y - y_k|$ is continuous in local coordinates near $y_k$. Write $f_* \mu = \omega_Y + dd^c \psi$ so that

$$\int_Y w_k d(f_* \mu) = \int_Y w_k \omega_Y + \int_Y \psi \Delta w_k = \psi(y_k) + \int_Y (w_k - \psi) \omega_Y.$$ 

Since $w_k \to w_\infty$ in $L^p_{loc}$ for all $p < \infty$, Proposition 2.1 implies that $f^* w_k \to f^* w_\infty$ in the $L^1$ topology, so that the argument above gives $\int_Y w_k d(f_* \mu) = \int_X f^* w_k d\mu$ $\to \int_X f^* w_\infty d\mu = \int_Y w_\infty d(f_* \mu)$ We then conclude that $\psi(y_k) \to \psi(y_\infty)$. Hence $\psi$ is continuous.

Suppose then that $\mu$ is locally the Monge–Ampère of a bounded psh function, and pick any subharmonic function $u$ defined in a neighborhood of a point $y \in Y$. Then $f^* u$ is again psh in a neighborhood of $f^{-1}(y)$, and the standard Chern–Levine–Nirenberg inequalities imply that $f^* u \in L^1(\mu)$ so that $u \in L^1(f_* \mu)$ with a norm depending only on the $L^1$-norm of $u$. It follows that $f_* \mu$ is locally the Monge–Ampère of a bounded subharmonic function.

Finally, assume $\mu = (\omega_X + dd^c \varphi)^n$ for some $\varphi \in \mathcal{E}^p(X, \omega_X)$. By [13, Theorem C] this is equivalent to have that $\mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. Write as usual $f_* \mu = (\omega_Y + dd^c \psi)$ with $\psi \in \mathcal{E}(Y, \omega_Y)$.

We claim that $u \in \mathcal{E}^p(Y, \omega_Y)$ implies $f^* u \in \mathcal{E}^p(X, \omega_X)$. Indeed, without loss of generality we can assume that $\Omega := \omega_X - f^* \omega_Y$ is a Kähler form and by the multilinearity of the non-pluripolar product we have

$$\int_X |f^* u|^p (\omega_X + dd^c f^* u)^n = \int_X |f^* u|^p (f^* \omega_Y + \Omega + dd^c f^* u)^n,$$

where the last identity follows from the fact that $(f^* \omega_Y + dd^c f^* u)^j = 0$ for $j > 1$. The term $\int_X |f^* u|^p \Omega^n$ is bounded thanks to the integrability properties of quasi-plurisubharmonic functions w.r.t. volume forms [14, Theorem 1.47]; while the term

$$\int_X |f^* u|^p (f^* \omega_Y + dd^c f^* u) \wedge \Omega^{n-1} = C(f) \int_Y |u|^p (\omega_Y + dd^c u)$$
is finite since $u \in \mathcal{E}^p(Y, \omega_Y)$. This proves the claim.

Now, given any $u \in \mathcal{E}^p(Y, \omega_Y)$ we have

$$
\int_Y |u|^p \, d(f_* \mu) = \int_X |f^* u|^p \, d\mu < +\infty
$$

since $f^* u \in \mathcal{E}^p(X, \omega_X) \subset L^p(\mu)$. The conclusion follows from [13, Theorem C].

Consider now any dominant meromorphic map $f: X \to Y$ from a Kähler compact manifold to a compact Riemann surface. As above we decompose $f$ such that $f_* \mu = (\pi_2)_* \pi_1^* \mu$ for any positive measure $\mu$ on $X$.

Assume that $\mu$ has continuous potentials. If we write $\mu = (\omega_X + dd^c \varphi)^n$ then $\pi_1^* \mu = (\pi_1^* \omega_X + dd^c \varphi \circ \pi)^n \leq (C \omega_T + dd^c \varphi \circ \pi)^n := \tilde{\mu}$ where $\tilde{\mu}$ has a continuous potential. This implies $f_* \mu \leq (\pi_2)_* \tilde{\mu}$. Observe that by the previous arguments the measure $(\pi_2)_* \tilde{\mu}$ has continuous potential. It follows that locally $f_* \mu = \Delta v \leq \Delta u$ where $u, v$ are subharmonic functions. It follows that $v$ is the sum of a continuous function and the opposite of a subharmonic (hence u.s.c.) function. Since it is also u.s.c we conclude to its continuity.

When $\mu$ has bounded potentials, the same argument applies noting that subharmonic functions are always bounded from above which implies $v$ to be bounded.

Finally, we consider the case where $\mu$ is the Monge–Ampère measure of $\varphi \in \mathcal{E}^p(X, \omega_X)$. We first observe that given $v \in \mathcal{E}^p(\Gamma, \omega_\Gamma)$ we have $(\pi_1)_* v \in \mathcal{E}^p(X, \omega_X)$. Indeed,

$$
\int_X |v \circ \pi^{-1}|^p (\omega_X + dd^c v \circ \pi^{-1})^n = \int_\Gamma |v|^p (\pi_1^* \omega_X + dd^c v)^n
\leq \int_\Gamma |v|^p (C \omega_\Gamma + dd^c v)^n < +\infty.
$$

This and the previous arguments give that if $u \in \mathcal{E}^p(Y, \omega_Y)$ then $f_* u = (\pi_1)_* \pi_2^* u \in \mathcal{E}^p(X, \omega_X)$, hence

$$
\int_Y |u|^p \, df_* \mu = \int_X |u \circ f|^p \, d\mu < +\infty.
$$

It follows from [13, Theorem C] that $f_* \mu$ is the Monge–Ampère measure of a function in $\mathcal{E}^p(Y, \omega_Y)$. 

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4. The case of submersions

In this section we let \((X, \omega_X), (Y, \omega_Y)\) be two compact Kähler manifolds of dimension \(n\) and \(m\), respectively and normalized such that \(\int_X \omega_X^n = \int_Y \omega_Y^m = 1\).

**Proposition 4.1.** — Let \(f: X \to Y\) be a submersion. Then, \(u \in \mathcal{E}^p(Y, \omega_Y)\) implies \(f^*u \in \mathcal{E}^p(X, \omega_X)\). In particular, if a probability measure \(\mu\) is the Monge–Ampère of a function in \(\mathcal{E}^p\) then also \(f_*\mu\) has also a potential in \(\mathcal{E}^p\).

**Proof.** — Since \(f\) is a submersion we can assume that there is a finite number of open neighbourhoods \(U_i\) such that \(X \subset \bigcup_{j=0}^N U_j, f|_{U_j}(z, w) = z\) where \(z = (z_1, \ldots, z_m)\) and \(w = (z_{m+1}, \ldots, z_n)\). Moreover we can assume that on each \(U_j\) we have

\[
\omega_X \leq C_j \left( \frac{i}{2} (dz \wedge d\bar{z} + dw \wedge d\bar{w}) \right), \quad \frac{i}{2} dz \wedge \bar{dz} \leq A_j f^* \omega_Y
\]

where \(A_j, C_j > 1\) and \(dz \wedge d\bar{z}, dw \wedge d\bar{w}\) are short notations for \(\sum_{j=1}^m dz_j \wedge d\bar{z}_j\) and \(\sum_{k=m+1}^n dz_k \wedge d\bar{z}_k\), respectively. We then write

\[
\int_X |f^*u|^p (\omega_X + dd^c f^* u)^n \leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left( C_j \frac{i}{2} dz \wedge \bar{dz} + C_j \frac{i}{2} dw \wedge \bar{dw} + dd^c f^* u \right)^n
\]

\[
\leq \sum_{j=1}^N \int_{U_j} |f^*u|^p \left( A'_j f^* \omega_Y + C_j \frac{i}{2} dw \wedge \bar{dw} + dd^c f^* u \right)^n
\]

\[
= \sum_{j=1}^N \sum_{\ell=0}^n \int_{U_j} |f^*u|^p \left( A'_j f^* \omega_Y + dd^c f^* u \right)^\ell \wedge \left( C_j \frac{i}{2} dw \wedge \bar{dw} \right)^{n-\ell}
\]

\[
= \sum_{j=1}^N \int_{U_j} |f^*u|^p \left( A'_j f^* \omega_Y + dd^c f^* u \right)^m \wedge \left( C_j \frac{i}{2} dw \wedge \bar{dw} \right)^{n-m}.
\]

The above integral is then finite because by assumption \(u \in \mathcal{E}^p(Y, A\omega_Y)\) for any \(A \geq 1\).

The last statement follows from the same arguments in the last part of the proof in the previous section. \(\square\)
5. Tame families of Monge–Ampère measures: proof of Corollary 1.5

Recall the setting from the introduction: $\mathcal{X}$ is a smooth connected complex manifold of dimension $n + 1$, and $\pi: \mathcal{X} \to \mathbb{D}$ is a flat proper analytic map over the unit disk which is a submersion over the punctured disk and has connected fibers. We let $p: \mathcal{X}' \to \mathcal{X}$ be a proper bi-meromorphic map from a smooth complex manifold $\mathcal{X}'$ which is an isomorphism over $\pi^{-1}(\mathbb{D}^*)$.

We let $T$ be any closed positive $(1, 1)$-current on $\mathcal{X}'$ admitting local Hölder continuous potentials. Observe that by e.g. [3, Corollary 1.6] we have

$$\mu'_t = dd^c \log |\pi \circ p - t| \wedge T^n \rightarrow \mu'_0 := dd^c \log |\pi \circ p| \wedge T^n.$$ 

Let us now analyze the structure of the positive measure $\mu_0 := p_\ast \mu'_0$. First observe that $\mu'_0$ can be decomposed as a finite sum of positive measures $\mu'_E := (T|_E)^n$ where the sum is taken over all irreducible components $E$ of $\mathcal{X}'_0$. Each of these measures is locally the Monge–Ampère of a Hölder continuous psh function.

Write $V := (E)$. Since $E$ is irreducible, $V$ is also an irreducible (possibly singular) subvariety of dimension $\ell$. To conclude the proof it remains to show that $p_\ast(\mu'_E)$ is the Monge–Ampère measure of Hölder continuous function that is locally the sum of a smooth and psh function. More precisely, one needs to show that $p_\ast(\mu'_E)$ does not charge any proper algebraic subset of $V$, and given any resolution of singularities $\varpi: V' \to V$ the pull-back measure $\varpi^\ast(p_\ast(\mu'_E))$ can be locally written as $(dd^c u)^\ell$ where $u$ is a Hölder psh function on $V'$.

This follows from Theorem 1.3 applied to any resolution of singularities $V'$ of $V$ and to any $E'$ which admits a birational morphism $E' \to E$ such that the map $E' \to V'$ induced by $p$ is also a morphism.

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