A $q$-GENERALIZATION OF THE TODA EQUATIONS FOR THE $q$-LAGUERRE/HERMITE ORTHOGONAL POLYNOMIALS

CHUAN-TSUNG CHAN† AND HSIAO-FAN LIU‡

† Department of Applied Physics, Tunghai University
‡ Department of Mathematics, National Tsing Hua University
† ctchan@go.thu.edu.tw, ‡ hfliu@math.nthu.edu.tw

Abstract. Based on the motivation of generalizing the correspondence between the Lax equation for the Toda lattice and the deformation theory of the orthogonal polynomials, we derive a $q$-deformed version of the Toda equations for both $q$-Laguerre/Hermite ensembles, and check the compatibility with the quadratic relation.

1. Introduction

There is an interesting correspondence between the Lax pair formulation of the Toda system [1, 2] and the deformation theory of the orthogonal polynomial systems [3]. In the simplest scenario, Flaschka’s Lax pair equation [4] for the one-dimensional periodic Toda lattice [5, 6]

\[
H := \sum_n \frac{1}{2} P_n^2 + (e^{Q_{n+1}-Q_n} - 1),
\]

\[
a_n := e^{(Q_{n+1}-Q_n)/2}, \quad a_0 := a_N,
\]

\[
b_n := -P_n, \quad b_0 := b_N,
\]

\[
L := \begin{pmatrix}
    b_0 & a_1 & 0 & \cdots & a_N \\
    a_1 & b_1 & a_2 & \cdots & 0 \\
    0 & a_2 & b_2 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    a_N & 0 & 0 & 0 & b_{N-1}
\end{pmatrix} = L^t,
\]

\[
B := \begin{pmatrix}
    0 & a_1 & 0 & \cdots & -a_N \\
    -a_1 & 0 & a_2 & \cdots & 0 \\
    0 & -a_2 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \vdots \\
    a_N & 0 & 0 & -a_{N-1} & 0
\end{pmatrix} = -B^t,
\]

Date: May 3, 2018.

2010 Mathematics Subject Classification. 33D45, 39A45.

Key words and phrases. orthogonal polynomials, Laguerre polynomial, Hermite polynomial, Hankel determinants, Toda equation, matrix model.
\newpage

\section*{2. The Quadratic Relations for the Laguerre/Hermite Orthogonal Polynomials}

\subsection*{2.1. Review of the quadratic relation among recursive coefficients for the orthogonal polynomials associated with the classical Laguerre/Hermite weights.}

Given the classical Laguerre weight defined as

\[ v^{(\alpha)}(x; \kappa) := x^\alpha \exp(-\kappa x), \quad 0 \leq \kappa, \quad -1 < \alpha, \quad 0 \leq x, \quad (2.1) \]
we can compute the orthonormal polynomials \( p_n^{(\alpha)}(x; \kappa) \) as
\[
\int_0^\infty p_m^{(\alpha)}(x; \kappa)p_n^{(\alpha)}(x; \kappa)u^{(\alpha)}(x; \kappa)dx = \delta_{mn} \tag{2.2}
\]
through Gram-Schmidt process.

Similarly, from the classical Hermite weight,
\[
\omega^{(\alpha)}(x; \kappa) := |x|^{2\alpha+1} \exp(-\kappa^2 x^2), \quad 0 \leq \kappa, \quad x \in \mathbb{R}, \tag{2.3}
\]
we obtain associated orthonormal polynomials \( P_n^{(\alpha)}(x, \kappa) \) as
\[
\int_{-\infty}^{\infty} P_m^{(\alpha)}(x; \kappa)P_n^{(\alpha)}(x; \kappa)\omega^{(\alpha)}(x; \kappa)dx = \delta_{mn}. \tag{2.4}
\]

The quadratic relation among these two sets of orthonormal polynomials is based on a simple connection between the Laguerre and Hermite weights. Namely,
\[
\omega^{(\alpha)}(x; \kappa) = |x|^{2\alpha} \nu^{(\alpha)}(x^2; \kappa^2). \tag{2.5}
\]

One immediate consequence of Eq. (2.5) is that, the orthonormal polynomials \( P_n^{(\alpha)}(x; \kappa) \) can be expressed in terms of the orthogonal polynomials \( p_n^{(\alpha)}(x; \kappa) \) as follows,
\[
P_{2n}^{(\alpha)}(x; \kappa) = p_n^{(\alpha)}(x^2; \kappa^2), \quad P_{2n+1}^{(\alpha)}(x; \kappa) = xp_n^{(\alpha+1)}(x^2; \kappa^2). \tag{2.6}
\]

The set of orthonormal polynomials associated with any weight function can be viewed as a complete set of basis for the function space. Hence, it induces a natural realization of the Heisenberg algebra, \([\frac{d}{dx}, x] = 1\). In particular, the matrix elements of the position operator consist of the three-term recursive coefficients among orthonormal polynomials. In the case of the Laguerre weight, it is given as
\[
 xp_n^{(\alpha)}(x; \kappa) = a_{n+1}^{(\alpha)}(\kappa)p_{n+1}^{(\alpha)}(x; \kappa) + b_n^{(\alpha)}(\kappa)p_n^{(\alpha)}(x; \kappa) + a_n^{(\alpha)}(\kappa)p_{n-1}^{(\alpha)}(x; \kappa), \tag{2.7}
\]
and in the case of the Hermite weight, we have
\[
 xp_n^{(\alpha)}(x; \kappa) = A_{n+1}^{(\alpha)}(\kappa)p_{n+1}^{(\alpha)}(x; \kappa) + A_n^{(\alpha)}(\kappa)p_{n-1}^{(\alpha)}(x; \kappa). \tag{2.8}
\]
By computing \( x^2P_n^{(\alpha)}(x, \kappa) \) in two ways (see Theorem 2.1 for details), we obtain the quadratic relation among the two sets of recursive coefficients:
\[
a_n^{(\alpha)}(\kappa^2) = A_{2n}^{(\alpha)}(\kappa)A_{2n-1}^{(\alpha)}(\kappa), \tag{2.9}
\]
\[
b_n^{(\alpha)}(\kappa^2) = \left(A_{2n+1}^{(\alpha)}(\kappa)\right)^2 + \left(A_{2n}^{(\alpha)}(\kappa)\right)^2. \tag{2.10}
\]
2.2. On the quadratic relation between generalized $q$-Laguerre/Hermite ensembles.

In this paper, we take generalized little $q$-Laguerre and $q$-Hermite ensembles \cite{7, 8, 9, 10, 11, 12} as an illustrative example of the quadratic relation. We consider the generalized little $q$-Laguerre weight ($0 \leq \kappa < \frac{1}{q}$)

$$v^{(\alpha)}(x; \kappa, q) := |x|^\alpha (q^{\kappa} x; q)_\infty = |x|^\alpha \prod_{l=0}^{\infty} (1 - q^{l+1} \kappa x), \quad (2.11)$$

and the generalized $q$-Hermite weight

$$\omega^{(\alpha)}(x; \kappa, q) = |x|^{2\alpha+1} (q^{2\kappa} x^2; q^2)_\infty = |x| v^{(\alpha)}(x^2; \kappa^2, q^2). \quad (2.12)$$

Given the orthonormal polynomials of the $q$-Laguerre ensemble $p_n^{(\alpha)}(x; \kappa, q)$, we can express the orthonormal polynomials of the $q$-Hermite ensemble $P_n^{(\alpha)}(x; \kappa, q)$ as follows:

$$P_{2n}^{(\alpha)}(x; \kappa, q) = \sqrt{\frac{1+q}{2}} p_n^{(\alpha)}(x^2; \kappa^2, q^2) \text{ (even, deg = 2n)}, \quad (2.13)$$

$$P_{2n+1}^{(\alpha)}(x; \kappa, q) = \sqrt{\frac{1+q}{2}} x p_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \text{ (odd, deg = 2n + 1)}. \quad (2.14)$$

One can easily check that $P_{2n}^{(\alpha)}(x; \kappa, q)$ and $P_{2n+1}^{(\alpha)}(x; \kappa, q)$ satisfy the orthonormal conditions, for instance,

$$\int_{-1}^{1} P_{2m}^{(\alpha)}(x; \kappa, q) P_{2n}^{(\alpha)}(x; \kappa, q) \omega^{(\alpha)}(x; \kappa, q) d_qx = 2(1-q) \sum_{k=0}^{\infty} P_{2m}^{(\alpha)}(q^k; \kappa, q) P_{2n}^{(\alpha)}(q^k; \kappa, q) \omega^{(\alpha)}(q^k; \kappa, q) q^k$$

$$= 2(1-q) \left( \frac{1+q}{2} \right) \sum_{k=0}^{\infty} p_m^{(\alpha)}(q^{2k}; \kappa^2, q^2) p_n^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k v^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k$$

$$= \int_{0}^{1} p_m^{(\alpha)}(x; \kappa^2, q^2) p_n^{(\alpha)}(x; \kappa^2, q^2) v^{(\alpha)}(x; \kappa^2, q^2) d_qx = \delta_{mn}. \quad (2.15)$$
\[
\int_{-1}^{1} P_{2m+1}^{(\alpha)}(x; \kappa, q) P_{2n+1}^{(\alpha)}(x; \kappa, q) \omega^{(\alpha)}(x; \kappa, q) \, dq \, dx
\]

\[
= 2(1 - q) \sum_{k=0}^{\infty} P_{2m+1}(q^k; \kappa, q) P_{2n+1}(q^k; \kappa, q) \omega^{(\alpha)}(q^k; \kappa, q) q^k
\]

\[
= 2(1 - q) \left( \frac{1 + q}{2} \right) \sum_{k=0}^{\infty} q^k p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) q^k p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) \omega^{(\alpha)}(q^{2k}; \kappa^2, q^2) q^k
\]

\[
= (1 - q^2) \sum_{k=0}^{\infty} p_m^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) p_n^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) \omega^{(\alpha+1)}(q^{2k}; \kappa^2, q^2) q^{2k}
\]

\[
= \int_{0}^{1} p_m^{(\alpha+1)}(x; \kappa^2, q^2) p_n^{(\alpha+1)}(x; \kappa^2, q^2) \omega^{(\alpha+1)}(x; \kappa^2, q^2) \, dq \, dx = \delta_{mn}.
\] (2.16)

Note that the \(q\)-integral (or Jackson integral) is defined in Eq. (A.1). The \(P_{2m}^{(\alpha)} P_{2n+1}^{(\alpha)}\) orthogonality is trivial due to the even parity of the generalized \(q\)-Hermite weight.

Similar to the classical cases Eqs. (2.9), (2.10), there exists a correspondence between recursive coefficients associated with the generalized little \(q\)-Laguerre and \(q\)-Hermite ensembles.

Theorem 2.1. The recursive coefficients associated with \(q\)-generalized Laguerre and Hermite ensembles satisfying the following relations:

\[
a_n^{(\alpha)}(\kappa^2, q^2) = A_{2n}^{(\alpha)}(\kappa, q) A_{2n-1}^{(\alpha)}(\kappa, q),
\] (2.17)

\[
b_n^{(\alpha)}(\kappa^2, q^2) = \left( A_{2n}^{(\alpha)}(\kappa, q) \right)^2 + \left( A_{2n}^{(\alpha)}(\kappa, q) \right)^2,
\] (2.18)

\[
a_n^{(\alpha+1)}(\kappa^2, q^2) = A_{2n+1}^{(\alpha)}(\kappa, q) A_{2n}^{(\alpha)}(\kappa, q),
\] (2.19)

\[
b_n^{(\alpha+1)}(\kappa^2, q^2) = \left[ A_{2n+2}^{(\alpha)}(\kappa, q) \right]^2 + \left[ A_{2n+1}^{(\alpha)}(\kappa, q) \right]^2.
\] (2.20)
Proof.

\[ x^2 P_{2n}^{(\alpha)}(x; \kappa, q) \]

\[ = x \left[ A_{2n+1}^{(\alpha)}(\kappa, q) P_{2n+1}^{(\alpha)}(x; \kappa, q) + A_{2n}^{(\alpha)}(\kappa, q) P_{2n-1}^{(\alpha)}(x; \kappa, q) \right] \]

\[ = A_{2n+1}^{(\alpha)}(\kappa, q) \left[ A_{2n+2}^{(\alpha)}(\kappa, q) P_{2n+2}^{(\alpha)}(x; \kappa, q) + A_{2n+1}^{(\alpha)}(\kappa, q) P_{2n-1}^{(\alpha)}(x; \kappa, q) \right] \]

\[ + A_{2n}^{(\alpha)}(\kappa, q) \left[ A_{2n}^{(\alpha)}(\kappa, q) P_{2n}^{(\alpha)}(x; \kappa, q) + A_{2n-1}^{(\alpha)}(\kappa, q) P_{2n-2}^{(\alpha)}(x; \kappa, q) \right] \]

\[ = \left[ A_{2n+1}^{(\alpha)}(\kappa, q) A_{2n+2}^{(\alpha)}(\kappa, q) \right] P_{2n+2}^{(\alpha)}(x; \kappa, q) + \left[ \left( A_{2n+1}^{(\alpha)}(\kappa, q) \right)^2 + \left( A_{2n}^{(\alpha)}(\kappa, q) \right)^2 \right] P_{2n}^{(\alpha)}(x; \kappa, q) \]

\[ + \left[ A_{2n-1}^{(\alpha)}(\kappa, q) A_{2n}^{(\alpha)}(\kappa, q) \right] P_{2n-2}^{(\alpha)}(x; \kappa, q). \]

On the other hand, using the expression of Eq. (2.13), we have

\[ x^2 P_{2n}^{(\alpha)}(x; \kappa, q) \]

\[ = x^2 \sqrt{\frac{1 + q}{2}} P_n^{(\alpha)}(x^2; \kappa^2, q^2) \]

\[ = \sqrt{\frac{1 + q}{2}} \left[ a_{n+1}^{(\alpha)}(\kappa^2, q^2) P_{n+1}^{(\alpha)}(x^2; \kappa^2, q^2) + b_n^{(\alpha)}(\kappa^2, q^2) P_n^{(\alpha)}(x^2; \kappa^2, q^2) + a_n^{(\alpha)}(\kappa^2, q^2) P_{n-1}^{(\alpha)}(x^2; \kappa^2, q^2) \right] \]

\[ = a_{n+1}^{(\alpha)}(\kappa^2, q^2) P_{2n+2}^{(\alpha)}(x; \kappa, q) + b_n^{(\alpha)}(\kappa^2, q^2) P_{2n}^{(\alpha)}(x; \kappa, q) + a_n^{(\alpha)}(\kappa^2, q^2) P_{2n-2}^{(\alpha)}(x; \kappa, q). \]

By comparing the coefficients on both expressions, we get Eqs. (2.17), (2.18).

If we examine similar calculations for the odd \( q \)-Hermite orthonormal polynomials, we get

\[ x^2 P_{2n+1}^{(\alpha)} = [A_{2n+3}^{(\alpha)} A_{2n+2}^{(\alpha)}] P_{2n+3}^{(\alpha)} \]

\[ + \left[ \left( A_{2n+2}^{(\alpha)} \right)^2 + \left( A_{2n+1}^{(\alpha)} \right)^2 \right] P_{2n+1}^{(\alpha)} + \left[ A_{2n+1}^{(\alpha)} A_{2n}^{(\alpha)} \right] P_{2n-1}^{(\alpha)}. \]

Alternatively,

\[ x^2 P_{2n+1}^{(\alpha)}(x; \kappa, q) \]

\[ = x^2 \sqrt{\frac{1 + q}{2}} x P_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \]

\[ = \sqrt{\frac{1 + q}{2}} \left[ a_{n+1}^{(\alpha+1)}(\kappa^2, q^2) P_{n+1}^{(\alpha+1)}(x^2; \kappa^2, q^2) + b_n^{(\alpha+1)}(\kappa^2, q^2) P_n^{(\alpha+1)}(x^2; \kappa^2, q^2) + a_n^{(\alpha+1)}(\kappa^2, q^2) P_{n-1}^{(\alpha+1)}(x^2; \kappa^2, q^2) \right] \]

\[ = a_{n+1}^{(\alpha+1)}(\kappa^2, q^2) P_{2n+3}^{(\alpha)}(x; \kappa, q) + b_n^{(\alpha+1)}(\kappa^2, q^2) P_{2n+1}^{(\alpha)}(x; \kappa, q) + a_n^{(\alpha+1)}(\kappa^2, q^2) P_{2n-1}^{(\alpha)}(x; \kappa, q). \]

Thus, we have shown Eqs. (2.19), (2.21). \( \square \)
Eliminating the recursive coefficients of the $q$-Laguerre orthonormal polynomials, $a_n^{(\alpha)}, b_n^{(\alpha)}$, in both sets of the equation, we obtain

$$A_{2n}^{(\alpha)}(\kappa, q)A_{2n-2}^{(\alpha)}(\kappa, q) = A_{2n+1}^{(\alpha-1)}(\kappa, q)A_{2n}^{(\alpha-1)}(\kappa, q),$$

(2.21)

and

$$\left( A_{2n+1}^{(\alpha)}(\kappa, q) \right)^2 + \left( A_{2n}^{(\alpha)}(\kappa, q) \right)^2 = \left( A_{2n+2}^{(\alpha-1)}(\kappa, q) \right)^2 + \left( A_{2n+1}^{(\alpha-1)}(\kappa, q) \right)^2.$$ 

(2.22)

For the general $\kappa$ case, we check the compatibility between the quadratic relation and the evolution equations (w.r.t $\kappa$) in Sec. 3.

2.3. On the compatibility of the quadratic relation with deformation.

As mentioned in the introduction, our main calculations are about the derivation of the discrete evolutions of the recursive coefficients with respect to parameter $\kappa$. Before we present the details, it is useful to recall the basic idea in the classical case.

To begin with, we shall study the differential equations associated with the recursive coefficients of the Laguerre/Hermite polynomials under the deformation of the weight.

**Theorem 2.2.** For the weight function of the Laguerre ensembles,

$$v^{(\alpha)}(x; t) := x^\alpha \exp(-tx),$$

(2.23)

the recursive coefficients $a_n^{(\alpha)}(t), b_n^{(\alpha)}(t)$, defined as

$$xp_n^{(\alpha)}(x; t) = a_{n+1}^{(\alpha)}(t)p_{n+1}^{(\alpha)}(x; t) + b_n^{(\alpha)}(t)p_n^{(\alpha)}(x; t) + a_n^{(\alpha)}(t)p_{n-1}^{(\alpha)}(x; t),$$

(2.24)

satisfy the following differential equations

$$\dot{a}_n^{(\alpha)} = \frac{a_n^{(\alpha)}}{2} \left( b_n^{(\alpha)} - b_{n-1}^{(\alpha)} \right),$$

$$\dot{b}_n^{(\alpha)} = \left( a_{n+1}^{(\alpha)} \right)^2 - \left( a_n^{(\alpha)} \right)^2.$$ 

(2.25)

In order to compute the time derivatives of the recursive coefficients $a_n^{(\alpha)}, b_n^{(\alpha)}$, we first compute the Fourier expansion of the time derivatives of the orthogonal polynomials:

$$\frac{\partial}{\partial t} p_n^{(\alpha)}(x; t) = \sum_{k=0}^{n} p_k^{(\alpha)}(x; t)C_{kn}^{(\alpha)}.$$ 

(2.26)

**Theorem 2.3.**

$$\frac{\partial}{\partial t} p_n^{(\alpha)}(x; t) = \frac{b_n^{(\alpha)}}{2} p_n^{(\alpha)}(x; t) + a_n^{(\alpha)} p_{n-1}^{(\alpha)}(x; t).$$ 

(2.27)
Proof. By making the following projections, we can prove that the only non-zero terms in
the expansion (2.26) are \( C^{(\alpha)}_{n-1,n} = a^{(\alpha)}_n \) and \( C^{(\alpha)}_{nn} = \frac{b^{(\alpha)}_n}{2} \). For \( C^{(\alpha)}_{nn} \),

\[
\frac{d}{dt} \int p^{(\alpha)}_n p^{(\alpha)}_m v^{(\alpha)} \, dx = 0
\]  

\( \Rightarrow \)

\[
2 \int p^{(\alpha)}_n \left[ \frac{\partial}{\partial t} p^{(\alpha)}_n \right] v^{(\alpha)} \, dx + \int [p^{(\alpha)}_n]^2 \left[ \frac{\partial}{\partial t} v^{(\alpha)} \right] \, dx = 0. \tag{2.29}
\]

Note that

\[
\frac{\partial}{\partial t} v^{(\alpha)}(x; t) = -xv^{(\alpha)}(x; t), \tag{2.30}
\]

and the first term gives \( 2C^{(\alpha)}_{nn} \). So we conclude that \( C^{(\alpha)}_{nn} = \frac{1}{2} b^{(\alpha)}_n \).

Similarly, for \( C^{(\alpha)}_{n-1,n} \),

\[
\frac{d}{dt} \left[ \int p^{(\alpha)}_n p^{(\alpha)}_m v^{(\alpha)} \, dx \right] = 0, \quad \text{(for } m < n). \tag{2.31}
\]

If \( m = n - 1 \), we derive \( C^{(\alpha)}_{n-1,n} = a^{(\alpha)}_n \), and if \( m < n - 1 \), we obtain \( C^{(\alpha)}_{mn} = 0 \). In conclusion, we get

\[
\frac{\partial}{\partial t} p^{(\alpha)}_n(x; t) = \left( \frac{b^{(\alpha)}_n}{2} - \frac{\dot{a}^{(\alpha)}_n}{2} \right) p^{(\alpha)}_n(x; t) + a^{(\alpha)}_n p^{(\alpha)}_{n-1}(x; t). \tag{2.32}
\]

Having computed the Fourier coefficients of \( \frac{\partial}{\partial t} p^{(\alpha)}_n(x; t) \), we now derive the evolution equations for \( a^{(\alpha)}_n \) and \( b^{(\alpha)}_n \).

Proof of Theorem 2.2. We can compute the time-evolution of \( xp^{(\alpha)}_n(x; t) \) in two ways. Firstly,

\[
\frac{\partial}{\partial t} [xp^{(\alpha)}_n] = \frac{\partial}{\partial t} \left[ a^{(\alpha)}_{n+1} p^{(\alpha)}_{n+1} + b^{(\alpha)}_n p^{(\alpha)}_n + a^{(\alpha)}_n p^{(\alpha)}_{n-1} \right] \tag{2.33}
\]

\[
= \left( \dot{a}^{(\alpha)}_{n+1} + \frac{1}{2} a^{(\alpha)}_{n+1} \dot{b}^{(\alpha)}_{n+1} \right) p^{(\alpha)}_{n+1} + \left( \frac{a^{(\alpha)}_{n+1}}{2} + \frac{1}{2} \dot{a}^{(\alpha)}_{n+1} + \frac{1}{2} \dot{b}^{(\alpha)}_{n+1} \right) p^{(\alpha)}_n
\]

\[
+ \left( \dot{a}^{(\alpha)}_n b^{(\alpha)}_n + \dot{b}^{(\alpha)}_n + \frac{1}{2} a^{(\alpha)}_n \dot{b}^{(\alpha)}_{n-1} \right) p^{(\alpha)}_{n-1} + \left( a^{(\alpha)}_n a^{(\alpha)}_{n-1} \right) p^{(\alpha)}_{n-2}. \tag{2.34}
\]
Next,
\[
\frac{\partial}{\partial t} [xp_n^{(\alpha)}] = x \left[ \frac{\partial}{\partial t} p_n^{(\alpha)} \right]
\]
\[
= x \left[ \frac{b_n^{(\alpha)}}{2} p_n^{(\alpha)} + a_n^{(\alpha)} p_{n-1}^{(\alpha)} \right]
\]
\[
= \left( \frac{a_{n+1}^{(\alpha)} b_n^{(\alpha)}}{2} \right) p_{n+1}^{(\alpha)} + \left[ (a_n^{(\alpha)})^2 + \frac{1}{2} (b_n^{(\alpha)})^2 \right] p_n^{(\alpha)} + \left( a_n^{(\alpha)} b_{n-1}^{(\alpha)} + \frac{1}{2} a_n^{(\alpha)} b_n^{(\alpha)} \right) p_{n-1}^{(\alpha)}
\]
\[
+ \left( a_n^{(\alpha)} a_{n-1}^{(\alpha)} \right) p_{n-2}^{(\alpha)}.
\]
(2.35)

By comparing the two equations, we prove Eq. (2.25). □

Similar calculations can be applied to the case of the Hermite ensembles, so we simply state the results.

**Theorem 2.4.** For Hermite ensembles of orthonormal polynomials defined by the weight function,
\[
w^{(\alpha)}(x; t) := |x|^{2\alpha+1} \exp(-tx^2),
\]

the Fourier expansion of the time-derivative of \( P_n^{(\alpha)} \) is given as
\[
\frac{\partial}{\partial t} P_n^{(\alpha)} = 1 \left[ (A_{n+1}^{(\alpha)})^2 + (A_n^{(\alpha)})^2 \right] P_n^{(\alpha)} + \left[ A_{n-1}^{(\alpha)} A_n^{(\alpha)} \right] P_{n-2}^{(\alpha)}.
\]
(2.37)

**Theorem 2.5.** For Hermite ensembles, the recursive coefficients in the recurrence relation,
\[
x p_n^{(\alpha)}(x; t) = A_n^{(\alpha)}(t) p_{n+1}^{(\alpha)}(x; t) + A_n^{(\alpha)}(t) p_{n-1}^{(\alpha)}(x; t),
\]

satisfy the following differential equation
\[
\dot{A}_n^{(\alpha)} = \frac{1}{2} A_n^{(\alpha)} \left[ (A_{n+1}^{(\alpha)})^2 - (A_{n-1}^{(\alpha)})^2 \right].
\]
(2.39)

Note that the differential equation for the recursive coefficients, Eq. (2.39), is also known as the Volterra equation. See [13, 14, 15] for further elaborations about the Lax pair formulation of this equation.

Having obtained the evolution equations, Eqs. (2.25), (2.39), for the Laguerre and Hermite ensembles, one can check that the quadratic relations, Eqs. (2.9), (2.10), are compatible with the time evolutions. Our aim in this paper is to generalize these computations to a fully \( q \)-discretized evolution of the \( q \)-Laguerre/Hermite orthogonal polynomial systems.

### 3. \( q \)-Generalization of the Toda equations from \( \kappa \)-deformation of the little \( q \)-Laguerre/Hermite ensembles

#### 3.1. \( q \)-Difference equations for the recursive coefficients of the \( q \)-Laguerre polynomials
In this section, we study the $q$-difference equations describing the $\kappa$ dependence of the recursive coefficients $a_n^{(\alpha)}(\kappa), b_n^{(\alpha)}(\kappa)$ associated with the generalized little $q$-Laguerre ensemble. In the classical case, such equations correspond to the Lax equation of the Toda equations \cite{[1]} \cite{[2]}. Hence, our results provide a $q$-generalization of the classical Toda equation. To achieve this, we shall express the Fourier expansion (w.r.t $\kappa$ variable) of the $q$-derivative on the $q$-Laguerre orthonormal polynomials in terms of the recursive coefficients,

$$D_q^{\kappa} p_n^{(\alpha)}(x, \kappa) = \sum_{j=0}^{n} p_j^{(\alpha)}(x, \kappa) \xi_{j,n}^{(\alpha)}(\kappa). \tag{3.1}$$

Following similar discussion as Theorem 2.3, we can express the Fourier coefficients of the $\kappa$-deformation of the orthonormal polynomials in terms of the recursive coefficients, $a_n^{(\alpha)}$ and $b_n^{(\alpha)}$.

**Theorem 3.1.** The Fourier coefficients of the $\kappa$-deformation of the orthonormal polynomials associated with the little $q$-Laguerre weight is given as $(\lambda := (1-q)\kappa)$

$$D_q^{\kappa} p_n^{(\alpha)}(x, \kappa) = p_n^{(\alpha)}(x, \kappa) \xi_n^{(\alpha)}(\kappa) + p_{n-1}^{(\alpha)} \xi_{n-1,n}^{(\alpha)}(\kappa). \tag{3.2}$$

Here

$$\xi_n^{(\alpha)}(\kappa) = \frac{1}{1-q} \left\{ \sqrt{2} - \sqrt{(1-q)\tilde{b}_n^{(\alpha)}} + \sqrt{1 - 2q\kappa \tilde{b}_n^{(\alpha)} + 4(q\kappa)^2} \left[ \left( \tilde{b}_n^{(\alpha)} \right)^2 - \left( \bar{a}_n^{(\alpha)} \right)^2 \right] \right\}, \tag{3.3}$$

and

$$\bar{a}_n^{(\alpha)}(\kappa) := a_n^{(\alpha)}(q\kappa), \quad \bar{b}_n^{(\alpha)}(\kappa) := b_n^{(\alpha)}(q\kappa). \tag{3.4}$$

**Proof.** The main point of this theorem is to find an expression relating $\xi_n^{(\alpha)}$ in terms of the recursive coefficients $a_n^{(\alpha)}, b_n^{(\alpha)}$. By taking $q$-derivative w.r.t $\kappa$ variable on the orthonormal condition, and recalling the $q$-Leibniz rule, Eq. (A.7), we derive a master equation among these Fourier coefficients.

$$D_q^{\kappa} \left[ \int_0^1 p_m^{(\alpha)}(x, \kappa) p_n^{(\alpha)}(x, \kappa) d_q x \right] = 0. \tag{3.6}$$

This implies for $m \leq n$,

$$\left(1 - \lambda \xi_n^{(\alpha)} \right) \xi_n^{(\alpha)} - \lambda \sum_{j=0}^{m-1} \xi_j^{(\alpha)} \xi_{j,n}^{(\alpha)} + \delta_{m,n-1} \left[ \frac{q}{1-q} \bar{a}_n^{(\alpha)} \right] + \delta_{mn} \left[ \frac{q}{1-q} \bar{b}_n^{(\alpha)} \right]. \tag{3.7}$$

From these results, we can extract useful information by specifying the value of $m$:
Case 1: \( m \leq n - 2 \)

We find, by induction, \( \xi^{(a)}_{m,n}(\kappa) = 0 \), if \( m \leq n - 2 \). Consequently, there are only two terms in the \( q \)-derivative (w.r.t \( \kappa \) variable) of the \( q \)-Laguerre orthonormal polynomials,

\[
\mathcal{D}_{q} p^{(a)}_{n}(x, \kappa) = p^{(a)}_{n}(x, \kappa) \xi^{(a)}_{m,n}(\kappa) + p^{(a)}_{n-1} \xi^{(a)}_{n-1,n}(\kappa). \tag{3.8}
\]

Case 2: \( m = n - 1 \)

In this case, we relate the two Fourier coefficients as follows:

\[
\xi^{(a)}_{n-1,n} = \frac{q}{1 - q} a^{(a)}_{n} = \left( \frac{q}{1 - q} \right) \frac{a^{(a)}_{n}}{1 - \lambda \xi^{(a)}_{n-1,n-1}}. \tag{3.9}
\]

Case 3: \( m = n \)

By suitable rearrangements, we derive a recursive equation relating the diagonal Fourier coefficients \( \xi^{(a)}_{n,n} \) to the recursive coefficients \( a^{(a)}_{n}, b^{(a)}_{n} \) as follows:

\[
(1 - \lambda \xi^{(a)}_{n,n})^2 + \frac{(q \kappa a^{(a)}_{n})^2}{(1 - \lambda \xi^{(a)}_{n-1,n-1})^2} = 1 - q \kappa \bar{b}^{(a)}_{n}. \tag{3.10}
\]

By using Eq.(3.22), we can rewrite Eq.(3.10) as a quadratic equation for \( (1 - \lambda \xi^{(a)}_{n,n})^2 \),

\[
(1 - \lambda \xi^{(a)}_{n,n})^2 + \frac{(q \kappa a^{(a)}_{n})^2}{(1 - \lambda \xi^{(a)}_{n,n})^2} = 1 - q \kappa \bar{b}^{(a)}_{n}. \tag{3.11}
\]

From the solution of this equation, we then obtain an expression of \( \xi^{(a)}_{n,n} \) in terms of \( a^{(a)}_{n} \) and \( \bar{b}^{(a)}_{n} \),

\[
\xi^{(a)}_{n,n} = \frac{1}{\sqrt{2(1 - q)\kappa}} \left\{ \sqrt{2} - \sqrt{(1 - q \kappa \bar{b}^{(a)}_{n}) + \sqrt{1 - 2q \kappa \bar{b}^{(a)}_{n} + 4(q \kappa)^2 \left( \bar{b}^{(a)}_{n} \right)^2 - (a^{(a)}_{n})^2}} \right\}. \tag{3.12}
\]

Recalling the definition of the \( q \)-derivative, Eq.(A.3), we can also transform this expansion formula, Eq.(3.2), as a \( q \)-shifting relation \( (\lambda := (1 - q)\kappa) \):

\[
p^{(a)}_{n}(x, q\kappa) = p^{(a)}_{n}(x, \kappa)[1 - \lambda \xi^{(a)}_{n,n}(\kappa)] - \lambda \sum_{j=0}^{n-1} p^{(a)}_{j}(x, \kappa) \xi^{(a)}_{jn}(\kappa). \tag{3.13}
\]
Theorem 3.2. The $\kappa$-deformation of the recursive coefficients associated with the generalized little $q$-Laguerre orthonormal polynomials is given by

$$D_q^\kappa a_n^{(a)}(\kappa) = \frac{\xi_{n-1,n-1}^{(a)} - \xi_{n}^{(a)}(\kappa)}{1 - \lambda \xi_{nn}^{(a)}(\kappa)} a_n^{(a)}(\kappa), \quad (3.14)$$

$$D_q^\kappa b_n^{(a)}(\kappa) = \frac{q}{1 - q} \left[ \left( \frac{a_n^{(a)}(\kappa)}{1 - \lambda \xi_{nn}^{(a)}(\kappa)} \right)^2 - \left( \frac{a_{n+1}^{(a)}(\kappa)}{1 - \lambda \xi_{n+1,n+1}^{(a)}} \right)^2 \right], \quad (3.15)$$

where $\xi_{nn}^{(a)}(\kappa)$ is given in Eq. (3.11).

Proof. We compute the $q$-derivative with respect to the $\kappa$ variable on the action of position operator, $xp_n^{(a)}(x, \kappa)$, in two ways:

We first compute the $q$-derivative, with respect to $\kappa$ of the recursive relation,

$$D_q^\kappa [xp_n^{(a)}(x, \kappa)] = D_q^\kappa [a_{n+1}^{(a)}(\kappa)p_n^{(a)}(x, \kappa) + b_n^{(a)}(\kappa)p_n^{(a)}(x, \kappa) + a_n^{(a)}(\kappa)p_{n-1}(x, \kappa)]$$

$$= [D_q^\kappa a_{n+1}^{(a)} + \xi_{n,n+1}^{(a)} a_{n+1}^{(a)}]p_n^{(a)} + [D_q^\kappa b_n^{(a)} + \xi_{n,n+1}^{(a)} b_n^{(a)}]p_n^{(a)}$$

$$+ [D_q^\kappa a_n^{(a)} + \xi_{n-1,n-1}^{(a)} a_n^{(a)} + \xi_{n-1,n-1}^{(a)} b_n^{(a)}]p_{n-1}^{(a)} + [\xi_{n-2,n-2}^{(a)} a_n^{(a)}]p_{n-2}^{(a)}. \quad (3.16)$$

Here $\xi_{mn}^{(a)}$ are the Fourier coefficients (matrix elements) of Eq. (3.1), and we suppress the dependence on $\kappa$ for simplicity.

On the other hand, since $D_q^\kappa$ commutes with the position operator $x$, we first compute the $q$-derivative (w.r.t $\kappa$ variable) of the orthonormal polynomials and then apply the position operator.

$$xD_q^\kappa [p_n^{(a)}(x, \kappa)]$$

$$= [\xi_{n,n}^{(a)} a_{n+1}^{(a)}]p_{n+1}^{(a)} + [\xi_{n-1,n}^{(a)} a_n^{(a)} + \xi_{n,n}^{(a)} b_n^{(a)}]p_n^{(a)}$$

$$+ [\xi_{n,n}^{(a)} a_n^{(a)} + \xi_{n-1,n}^{(a)} b_n^{(a)}]p_{n-1}^{(a)} + [\xi_{n-1,n}^{(a)} a_n^{(a)}]p_{n-2}^{(a)}. \quad (3.17)$$

By comparing the corresponding coefficients of each orthonormal polynomials as calculated in Eqs. (3.16), (3.17), we get the following set of relations:

$$D_q^\kappa a_n^{(a)} = \xi_{n-1,n-1}^{(a)} a_n^{(a)} - \xi_{n,n}^{(a)} a_n^{(a)}; \quad (3.18)$$

$$D_q^\kappa b_n^{(a)} = \xi_{n,n}^{(a)} (b_n^{(a)} - \bar{b}_n^{(a)}) + \xi_{n-1,n}^{(a)} a_n^{(a)} - \xi_{n,n+1}^{(a)} \bar{a}_{n+1}^{(a)}; \quad (3.19)$$

$$D_q^\kappa a_n^{(a)} = \xi_{n-1,n}^{(a)} (b_n^{(a)} - \bar{b}_n^{(a)}) + [\xi_{n,n}^{(a)} a_n^{(a)} - \xi_{n,n+1}^{(a)} \bar{a}_{n+1}^{(a)}]; \quad (3.20)$$

$$\xi_{n-2,n-1}^{(a)} \bar{a}_n^{(a)} = \xi_{n-1,n}^{(a)} a_n^{(a)} - \xi_{n,n}^{(a)} a_n^{(a)}. \quad (3.21)$$
Note that the last result allows us to replace the rescaled recursive coefficients \( \bar{a}_n^{(\alpha)} \) in terms of the Fourier coefficients and the unscaled recursive coefficients

\[
a_n^{(\alpha)}(q\kappa) = \frac{\xi_{n-1,n}(\kappa)}{\xi_{n-2,n-1}(\kappa)} a_{n-1}^{(\alpha)} = \frac{1 - \lambda \xi_{n-1,n-1}(\kappa)}{1 - \lambda \xi_{n,n}(\kappa)} a_n^{(\alpha)}(\kappa),
\]

where the second equality of the above relation follows from Eq. (3.18). We have also checked that Eqs. (3.18), (3.20) are compatible.

Finally, after some manipulations, we get the coupled \( q \)-difference equations, Eqs. (3.14), (3.15).

\( \square \)

### 3.2. \( q \)-Difference equations for the recursive coefficients of the \( q \)-Hermite orthonormal polynomials.

In this section, we shall derive the \( q \)-difference equation for the recursive coefficients of the \( q \)-Hermite orthonormal polynomials. In order to achieve this, we need to compute the Fourier coefficients of the \( q \)-derivative of the \( q \)-Hermite orthonormal polynomials with respect to parameter \( \kappa \),

\[
D_\kappa q P_n^{(\alpha)}(x;\kappa) = \sum_{j=0}^{n} P^{(\alpha)}_j(x;\kappa) \Xi^{(\alpha)}_{jn}(\kappa), \quad (n - j \text{ is even}).
\]

(3.23)

Note that, by recalling the definition of the \( q \)-derivative, Eq. (A.3), we can also transform the equation above into the Fourier expansion of the \( q \)-evolved \( q \)-Hermite orthonormal polynomials with respect to parameter \( \kappa \),

\[
P_n^{(\alpha)}(x;q\kappa) = [1 - \lambda \Xi_{nn}^{(\alpha)}] P_n^{(\alpha)}(x;\kappa) - \lambda \sum_{j=0}^{n-1} P^{(\alpha)}_j(x;\kappa) \Xi^{(\alpha)}_{jn}(\kappa).
\]

(3.24)

Due to the parity conserving property of the \( q \)-Hermite ensembles, we find that it is easier to firstly present the \( \kappa \)-deformation of the recursive coefficients in terms of \( \Xi_{nn}^{(\alpha)} \) and \( A_n^{(\alpha)} \).

**Theorem 3.3.** The \( \kappa \)-deformation of the recursive coefficients associated with the \( q \)-Hermite orthonormal polynomials is given by

\[
D_\kappa q A_n^{(\alpha)}(\kappa) = \frac{\Xi_{n-1,n-1}^{(\alpha)}(\kappa) - \Xi_{nn}^{(\alpha)}(\kappa)}{1 + \kappa(q-1) \Xi_{nn}^{(\alpha)}(\kappa)} A_n^{(\alpha)}(\kappa).
\]

(3.25)
Proof. We compute the $\mathcal{D}_q^\kappa$ derivative on the product $xP_n^{(\alpha)}(x; \kappa)$ in two ways.

\[
\mathcal{D}_q^\kappa [xP_n^{(\alpha)}(x; \kappa)] = \mathcal{D}_q^\kappa [A_{n+1}^{(\alpha)}(x; \kappa)P_n^{(\alpha)}(x; \kappa) + A_n^{(\alpha)}(x; \kappa)P_{n-1}^{(\alpha)}(x; \kappa)]
\]
\[
= x\sum_{j=0}^{n} P_j^{(\alpha)}(x; \kappa) \Xi_j^{(\alpha)}(\kappa)
\]
\[
= \sum_{j=0}^{n} [A_{j+1}^{(\alpha)}(x; \kappa)P_j^{(\alpha)}(x; \kappa) + A_j^{(\alpha)}(x; \kappa)P_{j-1}^{(\alpha)}(x; \kappa)] \Xi_j^{(\alpha)}(\kappa).
\]  
\noindent(3.26)

By comparing the coefficients of $P_n^{(\alpha)}(x; \kappa)$ of the first and the third lines of the previous equation, we get the following results:

\[
\mathcal{D}_q^\kappa A_{n+1}^{(\alpha)}(\kappa) = \Xi_{mn}^{(\alpha)}(\kappa)A_{n+1}^{(\alpha)}(\kappa) - \Xi_{n+1,n+1}^{(\alpha)}(\kappa)A_{n+1}^{(\alpha)}(q\kappa),
\]  
\noindent(3.27)

which implies

\[
\frac{A_n^{(\alpha)}(q\kappa)}{1 - \lambda \Xi_{n-1,n-1}^{(\alpha)}(\kappa)} = \frac{A_n^{(\alpha)}(\kappa)}{1 - \lambda \Xi_{n,n}^{(\alpha)}(\kappa)},
\]  
\noindent(3.28)

and

\[
\mathcal{D}_q^\kappa A_n^{(\alpha)}(\kappa) = \frac{\Xi_{n-1,n-1}^{(\alpha)}(\kappa) - \Xi_{mn}^{(\alpha)}(\kappa)}{1 + \kappa(q - 1) \Xi_{n,n}^{(\alpha)}(\kappa)} A_n^{(\alpha)}(\kappa).
\]  
\noindent(3.29)

Following similar discussion as Theorem 2.4, we can express the Fourier coefficients of the $\kappa$-deformation of the orthonormal polynomials in terms of the recursive coefficients $A_n^{(\alpha)}(\kappa)$.

**Theorem 3.4.** The Fourier coefficients of the $\kappa$-deformation of the orthonormal polynomials associated with the $q$-Hermite weight is given as ($\lambda := (1 - q)\kappa$)

\[
\mathcal{D}_q^\kappa P_n^{(\alpha)}(x, \kappa) = P_n^{(\alpha)}(x, \kappa)\Xi_{mn}^{(\alpha)}(\kappa) + P_{n-2}(x, \kappa)\Xi_{n-2,n}^{(\alpha)}(\kappa),
\]  
\noindent(3.30)

where,

\[
\Xi_{n-2,n}^{(\alpha)}(\kappa) = \left(\frac{q^2 \kappa}{1 - q}\right) \frac{A_n^{(\alpha)}(\kappa)A_{n-1}^{(\alpha)}(\kappa)}{1 - \lambda \Xi_{n,n}^{(\alpha)}(\kappa)},
\]  
\noindent(3.31)

\[
\Xi_{n,n}^{(\alpha)}(\kappa) = \frac{1}{\sqrt{2}(1 - q)\kappa}\left\{\sqrt{2} - \sqrt{1 - (q\kappa)^2} \bar{A}_n^{(\alpha)}(+) + \sqrt{1 - 2(q\kappa)^2} \bar{A}_n^{(\alpha)}(+) + (q\kappa)^4 \left[\bar{B}_n^{(\alpha)}\right]^2\right\},
\]  
\noindent(3.32)

\[
\bar{A}_n^{(\alpha)}(+) := \left(\bar{A}_{n+1}^{(\alpha)}\right)^2 + \left(\bar{A}_n^{(\alpha)}\right)^2, \text{ and } \bar{B}_n^{(\alpha)} := \left[\bar{A}_{n+1}^{(\alpha)}(+)^2 - 4 \left[A_n^{(\alpha)}A_{n-1}^{(\alpha)}\right]^2\right].
\]  
\noindent(3.33)
Proof. By taking the $q$-derivative (w.r.t $\kappa$) on the orthonormal condition
\[
D_q^\kappa \left( \int_{-1}^{1} P_m^{(\alpha)}(x; \kappa) P_n^{(\alpha)}(x; \kappa) \omega^{(\alpha)}(x; \kappa) d_q x \right) = 0, \tag{3.34}
\]
we get (assuming $m \leq n$)
\[
\int_{-1}^{1} (D_q^\kappa P_m^{(\alpha)}(x; \kappa)) P_n^{(\alpha)}(x; \kappa) \omega^{(\alpha)}(x; \kappa) d_q x
+ \int_{-1}^{1} P_m^{(\alpha)}(x; q\kappa) (D_q^\kappa P_n^{(\alpha)}(x; \kappa)) \omega^{(\alpha)}(x; \kappa) d_q x
+ \int_{-1}^{1} P_m^{(\alpha)}(x; q\kappa) P_n^{(\alpha)}(x; q\kappa) (D_q^\kappa \omega^{(\alpha)}(x; \kappa)) d_q x = 0. \tag{3.35}
\]
Substituting the Fourier expansion, Eqs.(3.23), into the first two terms of (3.35), and using the Pearson relation (in the $\kappa$ variable) for the $q$-Hermite weight, we get
\[
\delta_{mn} \Xi_{mn}^{(\alpha)} + (1 - \lambda \Xi_{mn}^{(\alpha)}) \Xi_{mn}^{(\alpha)} - \lambda \sum_{j=0}^{m-1} \Xi_{jm}^{(\alpha)} \Xi_{jn}^{(\alpha)}
+ \delta_{mn} \left( \frac{q^2 \kappa}{q - 1} \right) \left\{ (\bar{A}_{n+1})^{(\alpha)} \right\}^2 + (\bar{A}_n^{(\alpha)})^2 \right\} + \delta_{m,n-2} \left( \frac{q^2 \kappa}{q - 1} \right) \bar{A}_{n-1}^{(\alpha)} \bar{A}_n^{(\alpha)} = 0. \tag{3.36}
\]
In order to illustrate the content of this equation, we consider the following specializations:

Case 1: $m \leq n - 3$
In this case, the master equation reduces to
\[(1 - \lambda \Xi_{mn}^{(\alpha)}) \Xi_{mn}^{(\alpha)} - \lambda \sum_{j=0}^{m-1} \Xi_{jm}^{(\alpha)} \Xi_{jn}^{(\alpha)} = 0. \]
By the mathematical induction, we show that $\Xi_{mn}^{(\alpha)} = 0$, if $3 \leq n - m$. Consequently, the Fourier expansion of the $q$-derivative (w.r.t $\kappa$ variable) on the $q$-Hermite orthonormal polynomials only consist of two terms:
\[
D_q^\kappa P_n^{(\alpha)}(x; \kappa) = P_n^{(\alpha)}(x; \kappa) \Xi_{mn}^{(\alpha)}(\kappa) + P_{n-2}^{(\alpha)}(x; \kappa) \Xi_{n-2,n}^{(\alpha)}(\kappa). \tag{3.37}
\]
Case 2: $m = n - 2$
In this case, the master equation reduces to
\[(1 - \lambda \Xi_{n-2,n-2}^{(\alpha)}) \Xi_{n-2,n}^{(\alpha)} - \lambda \sum_{j=0}^{n-3} \Xi_{j,n-2}^{(\alpha)} \Xi_{jn}^{(\alpha)} + \frac{q^2 \kappa}{q - 1} \bar{A}_{n-1}^{(\alpha)} \bar{A}_n^{(\alpha)} = 0. \]
Since we have showed that $\Xi^{(\alpha)}_{jn} = 0$ for $j \leq n - 3$ in Case 1, we can use the equation above to express the off-diagonal Fourier coefficient in terms of the diagonal ones:

$$
\Xi^{(\alpha)}_{n-1,n+1} = \left( \frac{q^2 \kappa}{1 - q} \right) \frac{\bar{A}^{(\alpha)}_{n+1} \bar{A}^{(\alpha)}_{n}}{1 - \lambda \Xi^{(\alpha)}_{n-1,n-1}} = \left( \frac{q^2 \kappa}{1 - q} \right) \frac{A^{(\alpha)}_{n+1} A^{(\alpha)}_{n}}{1 - \lambda \Xi^{(\alpha)}_{n+1,n+1}}.
$$

(3.38)

For the second equality of the equation above, we use Eq. (3.28) to replace $\bar{A}^{(\alpha)}_{n}$ in terms of $A^{(\alpha)}_{n}$.

Case 3: $m = n - 1$

Due to the parity preserving property associated with the $q$-Hermite ensemble, $\Xi^{(\alpha)}_{n-1,n} = 0$, we have no constraint in this case.

Case 4: $m = n$

In this case, we have

$$
\Xi^{(\alpha)}_{nn} + (1 - \lambda \Xi^{(\alpha)}_{nn}) \Xi^{(\alpha)}_{n} - \lambda \Xi^{(\alpha)}_{n-2,n} = \left( \frac{q^2 \kappa}{1 - q} \right) \left[ \left( \bar{A}^{(\alpha)}_{n+1} \right)^2 + \left( \bar{A}^{(\alpha)}_{n} \right)^2 \right].
$$

(3.39)

After suitable rearrangement, we get

$$
(1 - \lambda \Xi^{(\alpha)}_{nn})^2 + \frac{(q \kappa \lambda)^2 \left( \bar{A}^{(\alpha)}_{n+1} \right)^2 \left( \bar{A}^{(\alpha)}_{n-1} \right)^2}{(1 - \lambda \Xi^{(\alpha)}_{n-2,n-2})^2} = 1 - (q \kappa)^2 \left\{ \left( \bar{A}^{(\alpha)}_{n+1} \right)^2 + \left( \bar{A}^{(\alpha)}_{n} \right)^2 \right\}.
$$

(3.40)

On the other hand, by replacing $\bar{A}^{(\alpha)}_{n}$ by $A^{(\alpha)}_{n}$, using Eq. (3.38), we derive a quadratic equation for $(1 - \lambda \Xi^{(\alpha)}_{nn})^2$,

$$
(1 - \lambda \Xi^{(\alpha)}_{nn})^2 + \frac{(q \kappa)^4 \left( A^{(\alpha)}_{n+1} \right)^2 \left( A^{(\alpha)}_{n-1} \right)^2}{(1 - \lambda \Xi^{(\alpha)}_{n-2,n-2})^2} = 1 - (q \kappa)^2 \left\{ \left( A^{(\alpha)}_{n+1} \right)^2 + \left( A^{(\alpha)}_{n} \right)^2 \right\}.
$$

(3.41)

Hence, $\Xi^{(\alpha)}_{nn}$ can be solved in terms of $A^{(\alpha)}_{n}$ and $\bar{A}^{(\alpha)}_{n}$.

By substituting the solution of $\Xi^{(\alpha)}_{nn}$ for Eq. (3.41) back to Eq. (3.25), we get closed $q$-difference equations for the recursive coefficients of the generalized $q$-Hermite ensemble.

3.3. On the compatibility of the quadratic relation and $q$-Toda equations.

In this section, we check the compatibility between the quadratic relation Eqs. (2.17), (2.18), (2.19), (2.20) and the $q$-Toda equation Eqs. (3.14), (3.15), (3.25). To see this, we first example the Fourier coefficients of the $q$-derivative of the orthonormal $q$-Laguerre/Hermite polynomials (w.r.t $\kappa$).
Theorem 3.5. The Fourier coefficients of the q-derivative (w.r.t \( \kappa \)) of the orthonormal q-Laguerre/Hermite polynomials (Eqs. (3.1), (3.23)) are related by the quadratic relations

\[
\Xi^{(a)}_{2n,2n}(\kappa, q) = (1 + q)\kappa \xi^{(a)}_{nn}(\kappa^2, q^2),
\]

\[
\Xi^{(a)}_{2n+1,2n+1}(\kappa, q) = (1 + q)\kappa \xi^{(a+1)}_{nn}(\kappa^2, q^2).
\]

Proof. We relate the q-derivative (w.r.t \( \kappa \)) of the orthonormal q-Hermite polynomials Eq. (2.20) to that of the q-Laguerre polynomials in two ways. First of all, for even polynomials

\[
\mathcal{D}_q^n p^{(a)}_{2n}(x; \kappa, q) = \frac{P^{(a)}_{2n}(x; \kappa, q) - P^{(a)}_{2n}(x; q\kappa, q)}{(1 - q)\kappa}
\]

\[
= \frac{p^{(a)}_n(x^2; \kappa^2, q^2) - p^{(a)}_n(x^2; q^2\kappa^2, q^2)}{(1 - q^2)\kappa^2} \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}}
\]

\[
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ \mathcal{D}_q^n p^{(a)}_n(x^2; \kappa^2, q^2) \right]
\]

\[
= \frac{(1 + q)^{\frac{3}{2}}\kappa}{\sqrt{2}} \left[ p^{(a)}_n(x^2; \kappa^2, q^2)\xi^{(a)}_{nn}(\kappa^2, q^2) + p^{(a)}_{n-1}(x^2; \kappa^2, q^2)\xi^{(a)}_{n-1,n}(\kappa^2, q^2) \right].
\]

(3.42)

On the other hand, if we write the Fourier expansion of the q-derivative (w.r.t \( \kappa \)) of the q-Hermite polynomials

\[
\mathcal{D}_q^n p^{(a)}_{2n}(x; \kappa, q) = P^{(a)}_{2n}(x; \kappa, q)\Xi^{(a)}_{2n,2n}(\kappa, q) + P^{(a)}_{2n-2}(x; \kappa, q)\Xi^{(a)}_{2n-2,2n}(\kappa, q)
\]

\[
= \sqrt{\frac{1 + q}{2}} p^{(a)}_n(x^2; \kappa^2, q^2)\Xi^{(a)}_{2n,2n}(\kappa, q) + \sqrt{\frac{1 + q}{2}} p^{(a)}_{n-1}(x^2; \kappa^2, q^2)\Xi^{(a)}_{2n-2,2n}(\kappa, q).
\]

(3.43)

By comparing the two results Eqs. (3.42), (3.43), we obtain

\[
\Xi^{(a)}_{2n,2n}(\kappa, q) = (1 + q)\kappa \xi^{(a)}_{nn}(\kappa^2, q^2),
\]

(3.44)

\[
\Xi^{(a)}_{2n-2,2n}(\kappa, q) = (1 + q)\kappa \xi^{(a)}_{n-1,n}(\kappa^2, q^2).
\]

(3.45)
Next, we compare the odd $q$-Hermite polynomials.

\[
D_q^\kappa P_{2n+1}^{(\alpha)}(x; \kappa, q) \\
= \frac{(1 + q)^{2\kappa}}{\sqrt{2}} x D_q^{\kappa} p_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \\
= \frac{(1 + q)^{2\kappa}}{\sqrt{2}} x \left[ p_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \xi_n^{(\alpha+1)}(\kappa^2, q^2) + p_{n-1}^{(\alpha+1)}(x^2; \kappa^2, q^2) \xi_{n-1,n}^{(\alpha+1)}(\kappa^2, q^2) \right],
\]

and

\[
D_q^\kappa P_{2n+1}^{(\alpha)}(x, \kappa, q) \\
= P_{2n+1}^{(\alpha)}(x; \kappa, q) \Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) + P_{2n-1}^{(\alpha)}(x; \kappa, q) \Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q) \\
= \sqrt{\frac{1 + q}{2}} x p_n^{(\alpha+1)}(x^2; \kappa^2, q^2) \Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) + \sqrt{\frac{1 + q}{2}} x p_{n-1}^{(\alpha+1)}(x^2; \kappa^2, q^2) \Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q).
\]

By comparing the two results, Eqs. (3.46), (3.47), we obtain

\[
\Xi_{2n+1,2n+1}^{(\alpha)}(\kappa, q) = (1 + q)\kappa \xi_n^{(\alpha+1)}(\kappa^2, q^2),
\]

\[
\Xi_{2n-1,2n+1}^{(\alpha)}(\kappa, q) = (1 + q)\kappa \xi_{n-1,n}^{(\alpha+1)}(\kappa^2, q^2).
\]

\[\square\]

In fact, the quadratic relation among the Fourier coefficients of the $q$-Laguerre/Hermite polynomials are equivalent to the quadratic relation among the recursive coefficients of the $q$-Laguerre/Hermite polynomials. To see this, we rewrite the Eq. (3.38) (set $n = m - 1$) as

\[
A_m^{(\alpha)}(\kappa, q)A_{m-1}^{(\alpha)}(\kappa, q) = \frac{1 - q}{q^2} \Xi_{m-2,m}^{(\alpha)}(\kappa, q) \left[ 1 - (1 - q)\kappa \Xi_{n,m}^{(\alpha)}(\kappa, q) \right].
\]

For $m = 2n$, after substituting Eqs. (3.44), (3.45) and using Eq. (3.9), we get

\[
A_{2n}^{(\alpha)}(\kappa, q)A_{2n-1}^{(\alpha)}(\kappa, q) = \frac{1 - q^2}{q^2} \xi_{n-1,n}^{(\alpha)}(\kappa^2, q^2) \left[ 1 - (1 - q^2)\kappa^2 \xi_{n,m}^{(\alpha)}(\kappa^2, q^2) \right] = a_n^{(\alpha)}(\kappa^2, q^2).
\]

For $m = 2n + 1$, after substituting Eqs. (3.48), (3.49) and using Eq. (3.9), we get

\[
A_{2n+1}^{(\alpha)}(\kappa, q)A_{2n}^{(\alpha)}(\kappa, q) = \frac{1 - q^2}{q^2} \xi_{n-1,n}^{(\alpha+1)}(\kappa^2, q^2) \left[ 1 - (1 - q^2)\kappa^2 \xi_{n,m}^{(\alpha+1)}(\kappa^2, q^2) \right] = a_n^{(\alpha+1)}(\kappa^2, q^2).
\]

(3.51)
Similarly, rewriting Eq. (3.40),
\[
(q\kappa)^2 \left[ \left( A_{m+1}^{(\alpha)}(q\kappa, q) \right)^2 + \left( A_m^{(\alpha)}(q\kappa, q) \right)^2 \right]
\]
\[= 1 - \left[ 1 - (1 - q)\kappa \Xi_{mm}(\kappa, q) \right]^2 - \frac{(q\kappa)^4 \left[ A_m^{(\alpha)}(\kappa, q) \right]^2 \left[ A_{m-1}^{(\alpha)}(\kappa, q) \right]^2}{\left[ 1 - (1 - q)\kappa \Xi_{mm}(\kappa, q) \right]^2}, \tag{3.53}
\]
then by substituting Eqs. (2.17), (2.19), (3.10), (3.44), (3.45), for either \( m = 2n \) or \( m = 2n+1 \), we reproduce Eqs. (2.18), (2.20).

4. Summary and Conclusion

In this paper, we study the exponential deformation/evolution of the generalized little \( q \)-Laguerre/Hermite polynomials. Our study serves as a \( q \)-discrete generalization for the correspondence between the Lax equation of the Toda lattice and the exponential deformation of any orthogonal polynomial system. In addition, we also discuss the implications and compatibility of the quadratic relation among \( q \)-Laguerre and \( q \)-Hermite orthogonal system. While it is not clear, at the present stage if we can write down a \( q \)-discrete Lax equation for the systems under study, but our calculation at least provide a possible hint of further explorations. Furthermore, it is of interest to compare with other approaches \cite{16} for this problem, to see if there are simpler expressions for the results obtained in this paper.

Acknowledgement

This research project was initiated in a summer visit (by C. T.) to the Institute of Mathematics at Academia Sinica in 2015. Both authors would like to thank Derchyi Wu, Chueh-Hsin Chang, and Mourad E. H. Ismail for instructive discussions. C.T. would like to thank Derchyi Wu for invitation and hospitality. The research work of C.T. is partially supported by the grant from Academia Sinica for summer visit, and in parts supported by the MOST research grant 104-2112-M-029-001. The research work of H.F. is partially supported by the MOST research grants 104-2115-M-001-001-MY2, 105-2112-M-29-003 and 106-2112-M-29-005.
Appendix A. Some Basic Definitions and Relations for $q$-Analysis ($0 < q < 1$)

In this section, we collect some basic definitions and formulas which are relevant to our discussions.

The $q$-integral (Jackson integral) for a function $f(x)$ over the region $x \in [0, a]$ is defined as

$$F(a) := \int_0^a f(x) d_q x := a(1-q) \sum_{k=0}^{\infty} f(aq^k)q^k,$$

(A.1)

$$F(a,b) := \int_b^a f(x) d_q x := F(a) - F(b).$$

(A.2)

This is compatible with the definition of the $q$-derivative

$$D^x_q f(x) := \frac{f(qx) - f(x)}{qx - x} = \frac{f(x) - f(qx)}{x(1-q)}$$

(A.3)

in the following senses:

1) Fundamental theorem of the $q$-calculus

$$D^a_q F(a) = f(a),$$

(A.4)

$$\int_0^a [D^x_q f(x)] d_q x = f(a) - f(0).$$

(A.5)

2) The linear change of variables can be implemented in $q$-integral:

$$\int_0^a f(cx) d_q x = \frac{1}{c} \int_0^{ca} f(y) d_q y.$$  (A.6)

There are some subtleties associated with the $q$-derivative, in particular, the $q$-Lebinitz rule is given as

$$D^x_q [f(x)g(x)] = \frac{f(qx)g(qx) - f(x)g(x)}{(q-1)x}$$

$$= f(qx) [D^x_q g(x)] + [D^x_q f(x)] g(x)$$

$$= [D^x_q f(x)] g(qx) + f(x) [D^x_q g(x)].$$ (A.7)

References

[1] H. Flaschka, The Toda lattice. II. Existence of integrals, Phys. Rev. B 9, p.1924-1925.
[2] H. Flaschka, On the Toda lattice. II. Inverse-scattering solution, Progr. Theoret. Phys. 51 (1974), 703-716.
[3] Peter A. Clarkson, Recurrence coefficients for discrete orthonormal polynomials and the Painlevé equations, J. Phys. A 46 (2013), no. 18, 185205, 18 pp.
[4] Peter D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. vol. 21 (1968), 467-490.
[5] Morikazu Toda, Vibration of a Chain with Nonlinear Interaction, J. Phys. Soc. Jpn. 22, pp. 431-436 (1967).
[6] Morikazu Toda, Waves in Nonlinear Lattice, Prog. Theor. Phys. Suppl. No. 45 (1970), 174-200.
[7] Mourad E. H. Ismail, *Orthogonal Polynomials, Their Recursions and Functional Equations*, Symmetries and Integrability of Difference Equations, Cambridge Univ. Press, London Math. Soc. Lecture Note Series: 381, 2011.

[8] Walter Van Assche, *Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials*, Difference equations, special functions and orthogonal polynomials, 687-725, World Soc. Publ., Hackensack, NJ, 2007.

[9] Lies Boelen; Walter Van Assche, *Discrete Painlevé equations for recurrence coefficients of semiclassical Laguerre polynomials*, Proc. Amer. Math. Soc. 138 (2010), no. 4, 1317-1331.

[10] R. Simion; D. Stanton, *Specializations of generalized Laguerre polynomials*, SIAM J. Math. Anal. 25 (1994), no. 2, 712-719.

[11] Galina Filipuk; Christophe Smet, *On the Recurrence Coefficients for Generalized q-Laguerre Polynomials*, Journal of Nonlinear Mathematical Physics, 20:sup1, 48-56 (2013).

[12] Yang Chen; James Griffin, *Non linear difference equations arising from a deformation of the q-Laguerre weight*, Indag. Math. (N.S.) 26 (2015), no. 1, 266-279.

[13] M. Kac; Pierre van Moerbeke, *On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices*, Surveys in applied mathematics (Proc. First Los Alamos Sympos. Math. in Natural Sci., Los Alamos, N.M., 1974), pp. 225-234. Academic Press, New York, 1976.

[14] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Advances in Math. 16 (1975), 197-220.

[15] Pantelis A. Damianou, *The Volterra model and its relation to the Toda lattice*, Phys. Lett. A 155 (1991), no. 2-3, 126-132.

[16] Yang Chen; Mourad E. H. Ismail, *Ladder operators for q-orthogonal polynomials*, J. Math. Anal. Appl. 345 (2008), no. 1, 1-10.