Hamiltonian formulation of the D-brane action and the light-cone Hamiltonian

Julian Lee*
Research Institute for Natural Sciences
Hanyang University
Seoul 133-791, Korea
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Abstract

We present the Hamiltonian formulation of the bosonic Dirichlet $p$-brane action. We rewrite the recently proposed quadratic D-brane action in terms of generalized shift vector and lapse function. The first class and the second class constraints are explicitly separated for the bosonic case. We then impose the gauge conditions in such a way that only time-independent gauge transformations are left. In this gauge we obtain the light-cone Hamiltonian which is quadratic in the field momenta of scalar and vector fields. The constraints are explicitly solved to eliminate part of the canonical variables. The Dirac brackets between the remaining variables are computed and shown to be equal to simple Poisson brackets.

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*e-mail: jlee@dirac.hanyang.ac.kr
I. INTRODUCTION

The action for D-brane which is quadratic in derivatives of $X$ and linear in $F_{ij}$ was introduced in ref. [1] and recently discussed in ref. [2], which takes the form

$$S = -\frac{1}{2} T_p \int d^n \xi \sqrt{-k} \left[ (k^{-1})^{ij} (\partial_i X^\mu \partial_j X^\mu + F_{ij}) - (n - 2) \right]. \quad (1.1)$$

where we are considering a bosonic D-$p$-brane propagating in a $D$ dimensional flat background for simplicity$^1$ with $\mu = 0, \cdots D - 1$, and $T_p$ is the $p$-brane tension, with $p = n - 1$. Here the auxiliary field $k_{ij}$ is introduced so as to remove the square root from the Born-Infeld type action for D-branes $[3-11]$,

$$S_{DBI} = -T_p \int d^n \xi \sqrt{- \det (\partial_i X^\mu \partial_j X^\mu + F_{ij})}, \quad (1.2)$$

Thus the action given by eq. (1.1) is the analogue of the Polyakov action in the case of string $[12,13]$ (or corresponding generalization for higher dimensional branes $[14]$), which is the linearization of the Nambu-Goto action. The only difference from ordinary $p$-brane is that due to the presence of the gauge field strength $F_{ij}$, the auxiliary field $k_{ij}$ must also contain antisymmetric part.

Gauge-fixing of D-brane using the static gauge was discussed in ref. [9], where they started from the Born-Infeld type of action (1.2) and arrived at a gauge-fixed action which is a complicated non-linear action. Also a covariant formalism was studied in ref. [15]. Derivation of the first-order action (1.1) from the Born-Infeld type action (1.2) using a Hamiltonian description was also discussed in ref. [16]. Classical symmetries of the action (1.1) for $n = 2$ (D-string) was investigated in ref. [17].

In this paper, I concentrate my attention to the canonical Hamiltonian formalism. At the classical level, the Hamiltonian analysis has an advantage of exhibiting the relevant physical degrees of freedom more clearly. When one attempts to go to quantum theory, it can be directly connected with the operator formalism via Dirac quantization. Since the action given by (1.2) and (1.1) are classically equivalent, one can apply this formalism to either one, but I chose to use the linear form (1.1) since it is more convenient for manipulations such as separating primarily inexpressible velocities and fixing gauges, and so on. It turns out that when one uses the Born-Infeld type action (1.2), some complicated expressions appear due to the presence of square root. Similar situation arises in case of string or membrane. We then go to the light-cone gauge to obtain the light-cone Hamiltonian which is quadratic in field momenta. This is the generalization of the light-cone quantization of ordinary string $[18]$ and membrane $[19]$.

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$^1$The notation for the target space dimension, $D$, has nothing to do with $D$ in D-$p$-brane, which is just an abbreviation of Dirichlet.
II. THE CONSTRAINTS IN THE HAMILTONIAN FORMALISM

To go to the Hamiltonian formalism, it is convenient to parameterize the auxiliary field $k_{ij}$ in terms of generalized shift vectors $N_a$, $\tilde{N}_a$ and lapse function $N$ as follows,

\begin{align}
  k^{00} &= -N^2 + \gamma_{ab} N^a \tilde{N}^b, \quad k^{0a} = N^b \gamma_{ba} \\
  k^{00} &= -N^{-2}, \quad k^{0a} = N^a N^{-2} \\
  k^{ab} &= \tilde{N}^a N^{-2}, \quad k^{ab} = \gamma^{ab} - \tilde{N}^a N^b N^{-2} \\
  \sqrt{-k} &= N \sqrt{\gamma}
\end{align}

(2.1)

where $\gamma_{ab}$ ($a, b = 1, \ldots, n - 1$) are the spatial components of the auxiliary field and $\gamma^{ab}$ is the inverse, defined by $\gamma^{ab} \gamma_{bc} = \delta^a_c$. In terms of these variables, the action (1.1) is written as,

\begin{align}
  S &= T_p \int d^n \xi \left[ \sqrt{\gamma} \dot{X}^2 - \sqrt{\gamma} (N^a + \tilde{N}^a) \dot{X} \cdot \partial_a X - \frac{\sqrt{\gamma}}{2N}(N^a - \tilde{N}^a) F_{0a} \\
  &- \frac{\sqrt{\gamma}}{2} (\gamma^{ab} N - \tilde{N}^a N^b N^a) \partial_a X \cdot \partial_b X + F_{ab} + \frac{(n - 2)}{2} N \sqrt{\gamma} \right] \\
  &\equiv T_p \int d^n \xi L
\end{align}

(2.2)

The canonical variables are $(X^\mu, N, N^a, \tilde{N}^a, \gamma^{ab}, A_i)$ and their conjugate momenta $(P_\mu, \Pi, \tilde{\Pi}_a, \Pi_{ab}, \pi^i)$, which are given by

\begin{align}
  P_\mu &\equiv \frac{\partial L}{\partial \dot{X}^\mu} = \frac{\sqrt{\gamma}}{N} \dot{X}^\mu - \frac{\sqrt{\gamma}}{2N} (N^a + \tilde{N}^a) \partial_a X^\mu \\
  \Pi &\equiv \frac{\partial L}{\partial \dot{N}} = 0 \\
  \Pi_a &\equiv \frac{\partial L}{\partial \dot{N}^a} = 0 \\
  \tilde{\Pi}_a &\equiv \frac{\partial L}{\partial \dot{\tilde{N}}^a} = 0 \\
  \Pi_{ab} &\equiv \frac{\partial L}{\partial \dot{\gamma}^{ab}} = 0 \\
  \pi^0 &\equiv \frac{\partial L}{\partial \dot{A}_0} = 0 \\
  \pi^a &\equiv \frac{\partial L}{\partial \dot{A}_a} = -\frac{\sqrt{\gamma}}{2N} (N^a - \tilde{N}^a)
\end{align}

(2.3)

The velocity $\dot{X}^\mu$ is a primary expressible velocity, meaning that it can be solved in terms of momenta, whereas the other ones are primary inexpressible and give rise to the primary constraints \[20\] \[22\]
The notation \( \approx 0 \) indicates that these constraints are weakly zero, meaning that they are numerically restricted to be zero but do not identically vanish throughout phase space. This means, in particular, that they have nonzero Poisson brackets with the canonical variables \[20–22\]. The time evolution of the system is generated by the Hamiltonian

\[
H^{(1)} = \int d^{n-1}x [p_\mu \dot{X}_\mu - L + \Sigma \Omega + \Sigma^a \Omega_a + \bar{\varphi}^a \bar{\Omega}_a + \Sigma^{ab} \Omega_{ab} + \lambda^0 \phi_0 + \lambda^a \phi_a] \\
= \int d^{n-1}x \left[ \frac{N}{2\sqrt{\gamma}} p_\mu p_\mu + \frac{N\sqrt{\gamma}}{2} (\gamma^{ab} + \frac{n^a n^b}{4N^2}) \partial_a X^\mu \partial_b X_\mu \\
- \frac{N\sqrt{\gamma}}{2} (n - 2) + \frac{n^a}{2} \partial_a X^\mu P_\mu - \frac{\sqrt{\gamma}}{2N} n^a \partial_a A_0 \\
+ \frac{\sqrt{\gamma}}{2} (N\gamma^{ab} - \frac{N^a N^b}{N}) F_{ab} + \Sigma \Omega + \Sigma^a \Omega_a + \bar{\varphi}^a \bar{\Omega}_a + \Sigma^{ab} \Omega_{ab} + \lambda^0 \phi_0 + \lambda^a \phi_a \right] \\
\equiv H + \int [\Sigma \Omega + \Sigma^a \Omega_a + \bar{\Sigma}^a \bar{\Omega}_a + \Sigma^{ab} \Omega_{ab} + \lambda^0 \phi_0 + \lambda^a \phi_a] \tag{2.5a}
\]

where \( n^a_\pm \equiv N^a \pm \tilde{N}^a \), i.e. for any canonical variable \( \eta \), we have

\[
\dot{\eta} = \{\eta, H^{(1)}\} \tag{2.6}
\]

on the constraint surface, where the Poisson bracket of any two functions \( F(\{\eta\}) \) and \( G(\{\eta\}) \) is defined by

\[
\{F, G\} \equiv \int d^\mu \xi \left[ \frac{\delta F}{\delta X^\mu} \frac{\delta G}{\delta P_\mu} + \frac{\delta F}{\delta N} \frac{\delta G}{\delta \Pi} + \frac{\delta F}{\delta N^a} \frac{\delta G}{\delta \Pi_a} + \frac{\delta F}{\delta N^a} \frac{\delta G}{\delta \Pi_a} + \frac{\delta F}{\delta \gamma^{ab}} \frac{\delta G}{\delta \Pi_{ab}} \\
+ \frac{\delta F}{\delta A^0} \frac{\delta G}{\delta \pi_0} + \frac{\delta F}{\delta A^a} \frac{\delta G}{\delta \pi_a} - (F \leftrightarrow G) \right]. \tag{2.7}
\]

Here, \( \Sigma, \Sigma^a, \tilde{\Sigma}^a, \Sigma^{ab}, \lambda^0, \lambda^a \) are the Lagrange multipliers and they represent the primary inexpressible velocities. We must now require the primary constraints to be maintained in time,

\[
\{\Phi^{(1)}, H^{(1)}\} \approx 0. \tag{2.8}
\]

The relation (2.8) determines the Lagrange multipliers \( \Sigma^a - \tilde{\Sigma}^a, \lambda_a \), and additionally they give rise to second stage constraints,
\[ \phi \equiv \frac{1}{2} P^2 + \frac{\gamma}{2}(\gamma^{cd} + \frac{n^c n^d}{4N^2}) \partial_c X^\mu \partial_d X_\mu - \frac{\gamma}{2}(n - 2) \]
\[ + \frac{\gamma}{2} \gamma^{cd} F_{cd} \]
(2.9a)

\[ \varphi_a \equiv P_\mu \partial_a X^\mu - \sqrt{\gamma} (N^b - \tilde{N}^b) F_{ab} \]
(2.9b)

\[ \varphi_{ab} \equiv \partial_a X^\mu \partial_b X_\mu + F_{ba} - \gamma_{ba} \]
(2.9c)

\[ \bar{\phi} \equiv \partial_a \left[ \frac{\sqrt{\gamma}}{2N} (N^a - \tilde{N}^a) \right] \]
(2.9d)

By requiring these constraints to be maintained in time, i.e. their Poisson brackets with \( H^{(1)} \) vanish weakly, one determines the Lagrange multiplier \( \Sigma_{ab} \), but no new constraint appears. Therefore, there is no third stage constraint and the second stage constraints above are all the secondary constraints we have. Note that the Lagrange multipliers \( \Sigma, \Sigma_a + \tilde{\Sigma}_a, \lambda_0 \) are still undetermined. Since these are primary inexpressible velocities, this means that there are arbitrary functions of time and the time evolution of the system is not unique. Of course, this is a consequence of the gauge symmetry of the system, and any arbitrariness is associated with the unphysical degrees of freedom.

### III. THE FIRST AND SECOND-CLASS CONSTRAINTS

Now, the constraints can be divided into first-class and second-class constraints. A first-class constraint \( \Psi_f \) is the one whose Poisson bracket with any other constraint \( \Phi \) vanishes weakly, \( \{ \Phi, \Psi_f \} \approx 0 \). On the other hand, the second-class constraints \( \chi_l \) are the ones whose Poisson brackets among them form a nonsingular matrix, \( \det \{ \chi_l, \chi_{l'} \} \neq 0 \). For bosonic variables, this matrix is antisymmetric, and therefore the number of second-class constraints should be even [20,21]. After some long and tedious calculations, we find them to be given by:

**First-class constraints**

\[ \bar{\Omega} \equiv \Pi + \frac{(N^a - \tilde{N}^a)}{2N} (\Pi^a - \tilde{\Pi}^a) \]
(3.1a)

\[ \Omega_a^+ \equiv \Pi_a + \tilde{\Pi}_a \]
(3.1b)

\[ \phi^0 \equiv \pi^0 \]
(3.1c)

\[ \phi \equiv \partial_a \phi^a - \bar{\phi} = \partial_a \pi^a \]
(3.1d)

\[ T \equiv \varphi + 2\gamma^{cd} P \cdot \partial_c \partial_d X + P \cdot \partial_c X \partial_d (\Pi^{cd} + \Pi^{dc}) \]

\[ - \frac{\sqrt{\gamma}}{2N} \partial_l X \cdot \partial_m X n^m n^l \phi^d + \left[ \frac{\gamma^{bc}}{2} P \cdot \partial_b \partial_c X n^d \right. \]

\[ + \frac{N P \cdot \partial_b X}{\sqrt{\gamma}} \partial_c \left\{ \frac{\sqrt{\gamma} (\gamma^{bc} + \gamma^{cb}) n^d}{4N} \right\} \]

\[ - \frac{n^l}{8} \partial_l X \cdot \partial_m X \partial_c \left\{ \sqrt{\gamma} (\gamma^{cm} + \gamma^{mc}) n^d \right\} \}

\[ \left. (\Pi^d - \tilde{\Pi}_d) \right\}

\[ + \left[ \frac{\gamma}{2} (\gamma^{cd} - \gamma^{dc}) + \frac{\sqrt{\gamma}}{4N} (\gamma^{bc} + \gamma^{cb}) n^d \right] P \cdot \partial_b X \]
Second-class constraints

\[\Omega_{ab} \equiv \Pi_{ab}\] (3.2a)

\[\varphi_{ab} \equiv \partial_a X^\mu \partial_b X_\mu + F_{ba} - \gamma_{ab} \quad (n \neq 2)\] (3.2b)

\[\Omega_a^- \equiv \Pi_a - \tilde{\Pi}_a\] (3.2c)

\[\phi^a \equiv \pi^a + \frac{\sqrt{\gamma}}{2N}(N^a - \tilde{N}^a)\] (3.2d)

The constraint (3.2b) appears only for \(n \neq 2\). In this case, the constraint (3.2a) is a first-class constraint. Once any set of second-class constraints \(\{\chi_i\}\) (which do not explicitly depend on time) are found, the time evolution of the system is given by

\[\dot{\eta} = \{\eta, H + \tilde{\lambda}_\alpha \tilde{\Phi}_\alpha\}_{D(\chi)}\] (3.3)

where the Dirac bracket with respect to the second-class constraints \(\{\chi_i\}\) is defined as

\[\{F, G\}_{D(\chi)} \equiv \{F, G\} - \{F, \chi_i\}\{\chi_i, \chi_i\}^{-1}\{\chi_i, G\}\] (3.4)

and \(\tilde{\Phi}_\alpha\) and \(\tilde{\lambda}_\alpha\) respectively denote the primary constraints which do not belong to the set \(\{\chi_i\}\) and the corresponding Lagrange multipliers. In particular, when \(\{\chi_i\}\) is the maximal set of second-class constraints in the system, \(\tilde{\Phi}_\alpha\)'s are all necessarily first class. In our case, \(\{\tilde{\Phi}_\alpha\} = \{\Omega, \Omega_a, \tilde{\Omega}_a, \pi^0\}\). When any of the constraint depend on time, this formalism should be modified slightly. We should treat the time \(\tau\) as additional canonical coordinate, and should introduce the conjugate momentum \(\epsilon\) [20]. The Poisson bracket should include the derivative with respect to these variables, and \(H\) should be replaced by \(H + \epsilon\).

IV. COUNTING OF DEGREES OF FREEDOM

The Hamiltonian formalism has an advantage that it exhibits the relevant dynamical degrees of freedom in a clear manner. It is quite well known that by an appropriate canonical transformation, the second class constraints can be identified with pairs of canonically conjugate pairs, \((P_\alpha, Q^\alpha)\), at least for bosonic theory [20,21]. Again we see that the number of second-class constraints for bosonic theory must be even. For each pair of second-class constraints, there is a degree of freedom which is eliminated from the theory. The first-class
constraints are written (after suitable canonical transformation) as a set of momenta $P_\beta$, and the coordinates $Q^\beta$ conjugate to these momenta are completely arbitrary. Since the time evolution of these variables are not determined from the dynamics, we interpret these variables as unphysical. One can in principle put conditions $Q^\beta = 0$ by hand in order to make the first-class constraints into second-class ones. It is the gauge fixing. Therefore we see that for each first-class variables, one degree of freedom is removed. Therefore one has [21]:

$$\text{(Number of physical degrees of freedom)} = \text{(Total number of coordinates)} - \text{(Number of second-class constraints)}/2 - \text{(Number of first-class constraints)} \quad (4.1)$$

In our case, we see that all the components of the auxiliary field $k_{ij}$ are completely removed by the first-class constraints (3.1a), (3.1b) and the second-class constraints (3.2a)-(3.2d). The first-class constraints (3.1c) removes $n$ degrees from $X^\alpha$, and (3.1d) removes 2 degrees from $A_a$, so the number of physical degrees of freedom at each point of worldvolume is given by

$$(D - n) + (n - 2) = D - 2 \quad (4.2)$$

where $D$ is the dimension of the target space. We observe that it is independent of the dimension of the brane. Were it an ordinary $(n - 1)$-brane, we would have the first term $D - n$ only, since there is no gauge field. For a given target space dimensionality, $D - n$ indicates the number of transverse directions. For D-branes, as the dimensionality of the brane grows, the number of transverse directions decreases, however the components of gauge field increases, thus keeping the number of physical degrees constant.

V. GAUGE FIXING

As discussed above, the first-class constraints can be made into the second class by putting additional constraints by hand, which is nothing but gauge fixing. For bosonic variables, the total number of additional constraints should be equal to that of the first-class constraints present if one wants to remove all the unphysical degrees of freedom, without imposing unnecessary restrictions on the physical ones (overfixing). There are several choices one can consider. The static gauge discussed in [3] has an advantage of fixing all the first-class constraints, but since one is identifying worldvolume “time” $\tau$ with a target space-time $X^0$ instead of a light-cone coordinate, one eventually gets some complicated expression for the Hamiltonian which involves square root. On the other hand, in the covariant gauge discussed in [3] the bosonic first-class constraints were not fixed at all, and the gauge-invariance condition was imposed on the physical state vector. Since the primary constraints appearing in $H^{(1)}$ still remains first-class, unless further gauge condition is imposed as constraints at the operator level, no meaningful expression for the Hamiltonian without Lagrange multiplier can be found. In this work, I impose similar conditions as the light-cone gauge for the string [13], (and also for membranes [13]), along with the temporal gauge for the gauge field. I fail to make all the first-class constraints into second class, but we can take advantage
of the fact that as long as we put conditions so as to make all the primary constraints second-class, we can still eliminate all the Lagrange multiplier from $H^{(1)}$ \[20,21\]. Also, leaving just right amount of first-class constraints allow us to simplify Dirac brackets, and the resulting Hamiltonian takes a rather simple form in the light-cone gauge, as we will see below. Because of these advantages, the light-cone gauge has been used not only for string, but also for membrane \[19\]. The conditions are

\[
G_1 \equiv X^+ - P_0^+ \tau \approx 0 \quad (5.1a)
\]

\[
G_2 \equiv \begin{cases} 
P^+ - (P_0^+)^{n-1} & \approx 0 \quad (n \neq 1) \\
P^+ - \frac{p_0^+}{N} & \approx 0 \quad (n = 1)
\end{cases} \quad (5.1b)
\]

\[
G \equiv N - \sqrt{\gamma}(P_0^+)^{2-n} \approx 0 \quad (n \neq 1) \quad (5.1c)
\]

\[
G^a \equiv N^a + \hat{N}^a \approx 0 \quad (5.1d)
\]

\[
F \equiv A_0 \approx 0 \quad (5.1e)
\]

\[
\varphi_{ab} \equiv \partial_a X^a \partial_b X^b + F_{ba} - \gamma_{ab} \approx 0 \quad (n = 2) \quad (5.1f)
\]

where the light-cone index is defined by $V^\pm \equiv \frac{V^0 \pm V^9}{\sqrt{2}}$. Requiring these constraints to be maintained in time does not lead to any new constraint and all the remaining primary inexpressible velocities are determined. This means that there is no ambiguity left in the time evolution of the system. As was discussed above, since the number of gauge-fixing conditions we put in does not match the number of first-class constraints, there are still $n - 1$ local first-class constraints left however. These represent time-independent gauge transformation of the vector field and the volume-preserving diffeomorphisms, and are given by,

\[
\phi = \partial_a \pi^a \quad (5.2a)
\]

\[
\epsilon^{a_1 a_2 \ldots a_{n-1}} \nabla_{a_1} [T_{a_2} + \sqrt{\gamma} \partial_{a_2} \Pi] \quad (5.2b)
\]

It is easily seen that $\phi$ is a first-class constraint by considering its Poisson bracket with other constraints. That (5.2b) is first-class follows from the fact that the only nonvanishing Poisson bracket of $T_a + \sqrt{\gamma} \partial_a \Pi$ with other constraint is given by

\[
\{T_a + \sqrt{\gamma} \partial_a \Pi, P^+ - P_0^+\} \approx P_0^+ \partial_a \delta(x - y) \quad (5.3)
\]

(5.2b) form the divergence-free part of $T_a + \sqrt{\gamma} \partial_a \Pi$ and there are $n - 2$ independent components among them. For D-branes which are not simply connected, one expects that there are also global constraints which remain first class, (for membrane, see ref. \[19\]) but we will discuss only local properties for simplicity. Since we left some of the unphysical degrees of freedom, we have to make sure the physical quantities depend only on physical variables at the classical level. That is, for any physical quantity $F$, we should have

\[
\{\Phi_\alpha, F\}_D(\chi) = 0 \quad (5.4)
\]

where $\Phi_\alpha$ is any one of the remaining first-class constraints \[5.2\], and the Dirac bracket is evaluated using the second-class constraints. As will be shown in the next section, variables other than transverse components of $X^\mu$ and spatial components of $A_i$, respectively, can be eliminated from the theory, and the Dirac bracket reduced to Poisson bracket, so one has
\{\Phi_\alpha, F(X, A)\} = 0 \quad (5.5)

where \(\mathbf{X}, A\) represent transverse and spatial components of \(X^\mu\) and \(A_i\) respectively. At the quantum level, the Poisson bracket above is simply replaced by the commutator. Also, at the quantum level, any physical state vector should satisfy the condition

\[\Phi_\alpha|_{\text{phys}} > = 0, \quad (5.6)\]

That is, the physical wavefunction should depend only on physical variables.

**VI. LIGHT-CONE HAMILTONIAN**

As was discussed in the last section, now we have a set of second-class constraints which are enough fix all the primary inexpressible velocities and the time evolution of the system is governed by \(H\) in (2.5b), which can be rewritten as

\[H = \int d^{n-1}\xi \left[ \frac{N}{\sqrt{\gamma}} \varphi + \frac{N^a + \tilde{N}^a}{2} \varphi_a + A^0 \bar{\phi} \right] \quad (6.1)\]

Since second class constraints can be set strongly equal to zero inside the Dirac bracket, we have \(H = 0\), and the whole dynamics are determined by the dependence of the constraints on the time \(\tau\). One has

\[\dot{\eta} = \{\eta, H + \epsilon\}_{D(\chi)} = \{\eta, \epsilon\}_{D(\chi)} \quad (6.2)\]

where all the Poisson brackets used in defining the Dirac bracket include the derivative with respect to \(\tau\) and \(\epsilon\). One can also make a time-dependent canonical transformation so that all the constraints become time-independent. The new Hamiltonian after this transformation is nonzero, which governs the time evolution of the system. Since we identify the time with one of the light-cone coordinates, this Hamiltonian is called the light-cone Hamiltonian. To be explicit, we make the canonical transformation \(\tilde{\mathbf{X}}^+ = X^+ - P_0^+ \tau\) with other variables unchanged so as to make the time-dependent constraint \(G_1 = X^+ - P_0^+ \tau\) into time-independent one \(\tilde{G}_1 = \tilde{X}^+\). Then the corresponding generating functional \(F\) has the form

\[F = X^\mu P_\mu - \tau P_0^+ P_+ \quad (6.3)\]

Denoting the new Hamiltonian after this transformation by \(H^+\), we have

\[H^+ = \frac{\partial F}{\partial \tau} = -P_0^+ P_+ \quad (6.4)\]

and all the second class constraints can be rewritten in equivalent form which are paired as follows:

\[
\begin{align*}
\left\{ \begin{array}{l}
N - \sqrt{\gamma}(P_0^+)^{2-n} \\
\Pi \\
N^a + \frac{N}{\sqrt{\gamma}}\pi^a \\
\Pi^a
\end{array} \right\} & \quad \left\{ \begin{array}{l}
A_0 \\
\pi^0 \\
\tilde{N}^a - \frac{N}{\sqrt{\gamma}}\pi^a \\
\tilde{\Pi}^a
\end{array} \right\}
\end{align*}
\]

9
\[
\begin{aligned}
\left\{ \gamma_{ab} - \partial_a X^\mu \partial_b X_\mu - F_{ba} \\
\Pi^{ab} \\
\dot{X}^+ = \frac{1}{2P^+_0}[\mathbf{P}^2 + \gamma(\gamma^{cd} + \frac{n^c n^d}{4N^2}) \partial_c X \cdot \partial_d X - \gamma(n - 2) + \gamma \gamma^{cd} F_{cd}] \\
\{ P_+ = \frac{1}{2P^+_0}[\mathbf{P}^2 + \gamma(\gamma^{cd} + \frac{n^c n^d}{4N^2}) \partial_c X \cdot \partial_d X - \gamma(n - 2) + \gamma \gamma^{cd} F_{cd}]
\right.
\end{aligned}
\]

where \( X, P \) indicate the vectors consisting of transverse components \( X_l, P_m \) \((l, m = 1 \cdots 8)\) only. When \( n = 1 \), \( N - \sqrt{\gamma} \) in the first pair is replaced by \( N - \frac{P_0}{P^+_0} \) and the last pair is removed.

As is discussed in Appendix, these constraints are of the special form and the canonical variables \((N, \Pi), (N^a, \Pi^a), (N^a, \Pi^a), (X^+, P_+), (X^-, P_-), (A_0, \pi^0)\) can be solved in terms of the other variables \((X^l, P_m) \) \((l, m = 1 \cdots 8)\), \((A_a, \pi^a)\) using the constraints above and the Dirac brackets between these remaining variables reduces to the Poisson brackets. Therefore, we finally have the light-cone Hamiltonian

\[
H^+ = \frac{1}{2}[\mathbf{P}^2 + \text{det}(\partial_a X \cdot \partial_b X + F_{ba}) + \pi^c \pi^d \partial_c X \partial_d X]
\]

and the equation of motion

\[
\dot{\eta}^* = \{ \eta^*, H^+ \}
\]

where \( \{ \eta^* \} \) are the remaining canonical variables \( X^l, P_m, A_a, \pi^b \).

VII. SUMMARY

In this paper we considered the Hamiltonian formalism for the D-brane action which is written in terms of auxiliary fields. We introduced the generalized shift and lapse functions, which made it very natural to go to the light-cone gauge. Constraint analysis was done in detail for the case of bosonic D-brane. We then derived the light-cone Hamiltonian which is quadratic in field momenta. Although not done in this paper, one might also apply similar analysis for supersymmetric or non-Abelian cases. After this paper was completed and submitted, ref. [23] appeared in the preprint archive, where the light-cone formulation for bosonic D2-brane is also discussed.

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APPENDIX: CONSTRAINTS OF SPECIAL FORM AND SIMPLIFICATION OF DIRAC BRACKETS

Let a complete set of second class constraints be given by χ. Let us solve the equations χ = 0 with respect to some variables ˜η, which are a set of pairs of canonical variables.
We denote the rest of the pairs by \( \bar{\eta} \), and the canonical variables are given by \( \eta = (\tilde{\eta}, \bar{\eta}) \). Since we may use any equivalent set of constraints for evaluating Dirac brackets [20–22], we assume without loss of generality that \( \chi \) takes the form \( \chi = \tilde{\eta} - f(\bar{\eta}) \), that is, \( \chi \) can be used to eliminate variables \( \tilde{\eta} \) in terms of \( \bar{\eta} \). Since the constraints can be set to zero inside the Dirac bracket [20–22], we have
\[
\dot{\bar{\eta}} = \{\bar{\eta}, \bar{H}\} - \{\bar{\eta}, \chi_i\}\{\chi, \chi\}_{ij}^{-1}\{\chi_j, \bar{H}\}
\]
where \( \bar{H} \equiv H(\tilde{\eta}(\bar{\eta}), \bar{\eta}) \), i.e. the Hamiltonian expressed in terms of \( \bar{\eta} \) only. The discussion so far is quite general. However, the equation of motion is considerably simplified when \( \chi \) are of the forms so that the second term on the right-hand side of (A1) vanishes. Such constraints \( \chi \) will be called \textit{constraints of special form} [20]. In particular, we consider the case where the constraints are paired in the form
\[
\tilde{q}^n - \text{constant}, \quad \tilde{p}_n - f^{2n}(\bar{\eta}), \quad \text{or}
\]
\[
\tilde{q}^n - f^{1n}(\bar{\eta}), \quad \tilde{p}_n - \text{constant}.
\] (A2)

Then the matrix \( \{\chi, \chi\} \) are written in a block diagonal form:
\[
\{\chi, \chi\} = \begin{pmatrix}
0 & I & 0 & 0 \\
-I & \{f^{2n}, f^{2n'}\}_{D(\psi)} & \{f^{2n}, f^{1n}\}_{D(\psi)} & 0 \\
0 & \{f^{1n}, f^{2n}\}_{D(\psi)} & \{f^{1n}, f^{1n'}\}_{D(\psi)} & I \\
0 & 0 & -I & 0
\end{pmatrix}
\] (A3)

and the inverse matrix is
\[
(\{\chi, \chi\})^{-1} = \begin{pmatrix}
\{f^{2n}, f^{2n'}\}_{D(\psi)} & -I & 0 & -\{f^{2n}, f^{1n}\}_{D(\psi)} \\
-I & 0 & 0 & 0 \\
0 & 0 & 0 & -I \\
\{f^{1n}, f^{2n}\}_{D(\psi)} & 0 & I & \{f^{1n}, f^{1n'}\}_{D(\psi)}
\end{pmatrix}
\] (A4)

It is easy to see that indeed in this case the second term of the right-hand side of (A1) vanishes and the Dirac bracket simply reduces to the Poisson bracket.