Momentum Distribution of the Hubbard Model

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Abstract
Using the recently amended sea-boson method, we compute the momentum distribution of the one-band Hubbard model in one and two spatial dimensions. We compute the asymptotic features of the momentum distribution explicitly away from half filling for weak coupling in one and two dimensions. While the results are not exact by any means, they provide the exact asymptotics, namely they are able to reproduce the exponents obtained by Shulz in one dimension obtained using Bethe ansatz. The corresponding results in more than one dimension are therefore believable.

1 Introduction
In the present article, we use the recently perfected sea-boson method [1] [2] [3] to compute the momentum distribution of the one-band Hubbard model. The Hubbard model in two dimensions is thought to be important in understanding high-$T_c$ superconductivity. Thus a proper solution of this using the sea-boson method can shed new light on the question of breakdown of Fermi liquid behaviour in two dimensions from which one may decide whether or not the Hubbard model is adequate in describing high-$T_c$ materials. Furthermore, lattice Fermi gases are easy to solve using the sea-boson approach since the hamiltonian remains separable even after invoking the repulsion attraction duality[2][3]. The anomalous exponent is computed for the 1d case and a comparison is made with the exact Bethe ansatz results of Lieb and Wu[4] as elaborated by Shulz[5]. Unfortunately as in the usual bosonization in 1d[7] we are unable to probe the strongly coupled ( large U ) Hubbard model since the velocities of the spinons becomes imaginary and the formalism breaks down. However we take the plausible point of view, supported by earlier works[3] , that says that breakdown of Fermi liquid theory (in any dimension) comes from the infrared divergence of certain integrals in the sea-boson formalism and not from the strength of the coupling. Thus it is sufficient to investigate the weakly coupled regime to
ascertain whether or not Fermi liquid theory has broken down. This point of view is further strengthened by the remarks of Shulz[5].

The Hubbard model is one of the most extensively studied models in Condensed Matter Physics. The collection of reprints in the volume by Korepin and Essler[8] is particularly useful. Also the monumental works by Weng et.al.[11] which is a path integral approach to the 1d Hubbard model cannot be ignored. In more than one dimension, there are many attempts notable among them are by Metzner[9] and Fukuyama [10].

## 2 Hubbard Model in One Dimension

Here we write down the one-band Hubbard model in one dimension in momentum space. To do this we must be careful since there are issues of backward scattering and umklapp process that play an important role the latter especially at half-filling[6]. Consider the model written in the usual manner. We assume that we are in the thermodynamic limit so that we may assume that the number of sites $N_a \gg 1$ is even. Also we set the hopping matrix element $t = 1$ and we work with units such that the lattice spacing $a = 1$. This means $L = N_a$.

$$H = - \sum_{m=-N_a/2}^{N_a/2-1} \sum_{\sigma} (c_{m\sigma}^\dagger c_{m+1\sigma} + c_{m+1\sigma}^\dagger c_{m\sigma}) + U \sum_{m=-N_a/2}^{N_a/2-1} n_{m\uparrow} n_{m\downarrow}$$

(1)

Here $n_{m\sigma} = c_{m\sigma}^\dagger c_{m\sigma}$.

$$c_{m\sigma} = \frac{1}{\sqrt{N_a}} \sum_{k} e^{ikm} c_{k\sigma}$$

(2)

$$c_{k\sigma} = \frac{1}{\sqrt{N_a}} \sum_{m=-N_a/2}^{N_a/2-1} e^{-ikm} c_{m\sigma}$$

(3)

Since we want periodic boundary conditions we must have,

$$c_{m+N_a\sigma} = c_{m\sigma}$$

(4)

This means that $k \cdot N_a = 2\pi \nu$ where $\nu = 0, \pm 1, \pm 2, \ldots$. We can also see that $c_{k+2\pi,\sigma} = c_{k\sigma}$. This means that we may now assume that in the thermodynamic limit, the $k$'s form a continuum and take on values in the interval $k \in (-\pi, \pi) \equiv I$. We may do so without any loss of generality. This periodicity leads to some novel processes such as umklapp processes. To see this more clearly we write down the repulsion term in momentum space. The hopping term is straightforward. To see this we write,

$$H_{kin} = -\frac{1}{N_a} \sum_{\sigma} \sum_{k,k'} e^{ikc_{k'\sigma}} c_{k\sigma} f(k-k') + H.c.$$
where,

\[
f(k - k') = -2i \frac{\sin((k - k')N_a/2)}{1 - e^{i(k-k')}} = 0
\]  

(6)

if \( k - k' \neq 0, \pm 2\pi, \pm 4\pi, \ldots \) and

\[
f(k - k') = N_a
\]  

(7)

if \( k - k' = 0, \pm 2\pi, \pm 4\pi, \ldots \). Since \( k, k' \in (-\pi, \pi) \) there is no chance for \( k - k' = \pm 2\pi \). Thus \( k - k' = 0 \) is the only possibility. However in the repulsion term we find that such umklapp processes are indeed possible. The hopping term is therefore given by,

\[
H_{kin} = \sum_{k \in \mathcal{I}, \sigma} \epsilon_k c_k^\dagger c_k\sigma
\]  

(8)

and \( \epsilon_k = -2 \cos(k) \). The interaction term is given by,

\[
H_U = U \sum_{m=-N_a/2}^{m=N_a/2-1} n_{m\uparrow} n_{m\downarrow}
\]  

(9)

In momentum space it is,

\[
H_U = \frac{U}{N_a} \sum_{k,k',p,p'} c_{k',p}^\dagger c_{k,p}^\dagger c_{p',-p} c_{p,-p'} f(k - k' + p - p')
\]  

(10)

\[
f(k - k' + p - p') = \sum_{m=-N_a/2}^{N_a/2-1} e^{im(k-k'+p-p')}
\]  

(11)

We can see that unless \( p = 0, \pm 2\pi, \pm 4\pi, \ldots \) we must have \( f(p) \equiv 0 \). Thus we may write,

\[
f(k - k' + p - p') = N_a (\delta_{k-k'+p-p',0} + \delta_{k-k'+p-p',2\pi} + \delta_{k-k'+p-p',-2\pi})
\]  

(12)

Since \( k, k', p, p' \in (-\pi, \pi) \), there is no chance for \( k-k'+p-p' = \pm 4\pi \) or anything higher (since \( -4\pi < k-k'+p-p' < 4\pi \)). However, \( k-k'+p-p' = \pm 2\pi \) is perfectly possible, and this is precisely the umklapp process. Therefore we may write,

\[
H_U = \frac{U}{N_a} \sum_{k,k',p,p'} c_{k',p}^\dagger c_{k,p}^\dagger c_{p',-p} c_{p,-p'} (\delta_{k-k'+p-p',0} + \delta_{k-k'+p-p',2\pi} + \delta_{k-k'+p-p',-2\pi})
\]  

(13)

In other words,

\[
H_U = \frac{U}{N_a} \sum_{k,p,q} c_{k+q}^\dagger c_{k}^\dagger c_{p-q} c_{p} + \frac{U}{N_a} \sum_{k,p,q} c_{k+q}^\dagger c_{k}^\dagger c_{p-q} c_{p} - G \sum_{k,p,q} c_{k+q}^\dagger c_{k}^\dagger c_{p-q} c_{p}
\]
\[ + \frac{U}{N_a} \sum_{k,p,q} c^\dagger_{k+q} c^\dagger_k c^\dagger_{p-q+G} c^\downarrow_p \] (14)

The last two terms correspond to umklapp processes. Here \( G = 2\pi \) is the reciprocal lattice vector. Define the sea-displacement operator,

\[ A_{p\sigma} = \frac{n_F(p)(1 - n_F(p'))}{\sqrt{N_{p\sigma}}} c^\dagger_{p\sigma} c^\downarrow_{p'} \] (15)

where \( n_{p\sigma} = c^\dagger_{p\sigma} c^\downarrow_{p\sigma} \) and \( n_F(p) = \theta(\epsilon_F - \epsilon_p) \). Eq. (15) may be formally inverted and a formula for the number conserving product of two Fermi fields may be written down. In the RPA-sense we may regard the object \( A_{p\sigma} \) as being small in the sense that the matrix representation of this operator in a suitably restricted Hilbert space is sparse. Thus we can be content at including only the leading terms. The formula for the off-diagonal product \( c^\dagger_{p\sigma} c^\downarrow_{p'} \sigma' \) for \( p \neq p' \) may be written down as follows.

\[ c^\dagger_{p\sigma} c^\downarrow_{p'} \sigma' \approx A_{p\sigma} p' \sigma' + A_{p\sigma'} \] (16)

If \( p = p' \) we have instead \(^{1}\),

\[ c^\dagger_{p\sigma} c^\downarrow_{p'} \sigma' = \delta_{\sigma,\sigma'} n_F(p) + \sum_{q\sigma_1} A^\dagger_{p-q\sigma_1} A^\sigma_{p-q\sigma_1} - \sum_{q\sigma_1} A^\dagger_{p+q\sigma_1} A^\sigma_{p+q\sigma_1} \] (17)

The interaction term with \( q = 0 \) may be omitted since this leads to a c-number. The terms with \( k = k' \pm q \) are also omitted since they result in a Hamiltonian that is quartic in the sea-bosons (see later for some problems related to this). Since these objects are small in the restricted Hilbert space we may ignore these terms. The full Hamiltonian may be written as the sum of two terms. The first corresponds to holons, the other corresponds to spinons. This is the first hint of the phenomenon of spin charge separation. The full Hamiltonian is \( H = H_c + H_s \) where,

\[ H_c = \sum_{kq} \epsilon_k A^\dagger_{k+q} A^\dagger_{k+q} A_{k-q} A_{k-q} + \sum_{kq} \epsilon_k A^\dagger_{k+q} A^\dagger_{k+q} A_{k-q} A_{k-q} + \sum_{kq} \epsilon_k A^\dagger_{k+q} A^\dagger_{k+q} A_{k-q} A_{k-q} + \frac{U}{N_a} \sum_{k,k',q} (A^\dagger_{k+q} A^\dagger_{k'} (A_{k'-q} + A^\dagger_{k'-q}) + A^\dagger_{k'+q} A^\dagger_{k} (A_{k'+q} + A^\dagger_{k'+q})) + \frac{U}{N_a} \sum_{k,k',q} (A^\dagger_{k+q} + A^\dagger_{k'} (A_{k'-q} + A^\dagger_{k'-q} + A_{k'+q} + A^\dagger_{k'+q})) \]

\(^{1}\) especially if \( n_F(p \uparrow) = n_F(p \downarrow) \) in other words if the number of up-spins is equal to the number of down-spins \( M = M' \).
\[ + \frac{U}{N_a} \sum_{k,k',q} (A_{k+q}^\dagger + A_{k}^\dagger) (A_{k'-q+G}^\dagger + A_{k'}^\dagger) \quad (18) \]

\[ H_s = \sum_{kq} \epsilon_k A_{k-q}^\dagger A_{k-q}^\dagger - \sum_{kq} \epsilon_k A_{k+q}^\dagger A_{k+q}^\dagger \]

\[ + \sum_{kq} \epsilon_k A_{k+q}^\dagger A_{k+q}^\dagger - \sum_{kq} \epsilon_k A_{k+q}^\dagger A_{k+q}^\dagger \]

\[ - \frac{U}{N_a} \sum_{k,k',q} (A_{k+q}^\dagger + A_{k'}^\dagger) (A_{k'-q}^\dagger + A_{k'}^\dagger) \]

\[ - \frac{U}{N_a} \sum_{k,k',q} (A_{k+q}^\dagger + A_{k'}^\dagger) (A_{k'-q-G}^\dagger + A_{k'}^\dagger) \]

\[ - \frac{U}{N_a} \sum_{k,k',q} (A_{k+q}^\dagger + A_{k'}^\dagger) (A_{k'-q+G}^\dagger + A_{k'}^\dagger) \quad (19) \]

\( A_{k' \sigma} \) destroys an electron with some momentum and creates another with a different momentum without changing the spin. This means that this corresponds to charge fluctuations. \( A_{k' \sigma}^\dagger \) destroys an electron with some momentum and creates another with a different momentum, and in addition changes the spin projection by \( \pm 1 \). This means that this corresponds to spin fluctuations. If we want the luxury of interpreting the objects \( A_{k' \sigma} \) as being exact bosons obeying the canonical commutation rules:

\[ [A_{k' \sigma}, A_{k' \sigma}^\dagger] = n_F(k)(1 - n_F(k')) \theta(\pi - |k|) \theta(\pi - |k'|) \quad (20) \]

(And all other commutation rules involving any two of these objects are zero) then we must pay a price in that the momentum distribution is somewhat complicated, namely it is appropriately resummed in order to tame the infrared divergences that are known to be ubiquitous in one dimension. Thus the momentum distribution is given by,

\[ < n_{k\sigma} > = n_F(k) F_1(k\sigma) + (1 - n_F(k)) F_2(k\sigma) \quad (21) \]

\[ F_1(k\sigma) = \frac{1}{2} \left( 1 + e^{-2S_B^0(k\sigma)} \right) \quad (22) \]

\[ F_2(k\sigma) = \frac{1}{2} \left( 1 - e^{-2S_A^0(k\sigma)} \right) \quad (23) \]

\[ S_B^0(k\sigma) = \sum_{q_1\sigma_1} < G_0 | A_{k+q_1\sigma_1} A_{k+q_1\sigma_1}^\dagger | G_0 > \quad (24) \]

\[ S_A^0(k\sigma) = \sum_{q_1\sigma_1} < G_0 | A_{k-q_1\sigma_1} A_{k-q_1\sigma_1}^\dagger | G_0 > \quad (25) \]

Here \( |G_0 > \) is the ground state of \( H = H_c + H_s \). In what follows, we assume that \( M = M' \), that is, the total number of upspins is equal to the total number of downspins.
3 Solution Away From Half-Filling

We focus on the situation away from half-filling where the diagonalisation is simplest since we may ignore umklapp processes. The holon part of the Hamiltonian is given by,

\[ H_c = \sum_{kq} \epsilon_k A_{k-q\uparrow}^\dagger A_{k\uparrow} - \sum_{kq} \epsilon_k A_{k\uparrow}^\dagger A_{k+q\uparrow}^\dagger \]

\[ + \sum_{kq} \epsilon_k A_{k-q\downarrow}^\dagger A_{k\downarrow} - \sum_{kq} \epsilon_k A_{k\downarrow}^\dagger A_{k+q\downarrow} \]

\[ + \frac{U}{N_a} \sum_{k,k',q} (A_{k+q\uparrow}^\dagger + A_{k\uparrow}^\dagger)(A_{k'-q\downarrow}^\dagger + A_{k'\downarrow}^\dagger) \]

and the spinon part is given by,

\[ H_s = \sum_{kq} \epsilon_k A_{k-q\downarrow}^\dagger A_{k\downarrow} - \sum_{kq} \epsilon_k A_{k\downarrow}^\dagger A_{k+q\downarrow}^\dagger \]

\[ + \sum_{kq} \epsilon_k A_{k-q\uparrow} A_{k\uparrow}^\dagger - \sum_{kq} \epsilon_k A_{k\uparrow} A_{k+q\uparrow}^\dagger \]

\[ - \frac{U}{N_a} \sum_{k,k',q} (A_{k+q\downarrow}^\dagger + A_{k\downarrow}^\dagger)(A_{k'-q\uparrow}^\dagger + A_{k'\uparrow}^\dagger) \]

We use the equation of motion method to solve for the boson propagators. Since the details are tedious and the method has already been highlighted in our earlier work[1], we shall merely write down the final answers for the boson occupation.

\[ \langle A_{k-q\sigma}^\dagger(t^+) A_{k-q\sigma}(t) \rangle = \frac{U^2}{N^2} \frac{i}{\beta} \sum_n \frac{P(q, iz_n)}{\epsilon_c(q, iz_n)} \frac{[A_{k-q\sigma}, A_{k-q\sigma}^\dagger]}{(-iz_n - \epsilon_k + \epsilon_{k-q})^2} \]

\[ \langle A_{k-q\sigma}^\dagger(t^+) A_{k-q\sigma}(t) \rangle = -\frac{U}{N^2} \frac{i}{\beta} \sum_n \frac{1}{\epsilon_s(q, iz_n)} \frac{[A_{k-q\sigma}, A_{k-q\sigma}^\dagger]}{(-iz_n - \epsilon_k + \epsilon_{k-q})^2} \]

\[ P(q, iz_n) = \sum_k \frac{[A_{k-q\sigma}, A_{k-q\sigma}^\dagger]}{(-iz_n - \epsilon_k + \epsilon_{k-q})} + \sum_k \frac{[A_{k-q\sigma}, A_{k-q\sigma}^\dagger]}{(iz_n - \epsilon_{k-q} + \epsilon_k)} \]

\[ \epsilon_c(q, iz_n) = 1 - \frac{U^2}{N^2} P^2(q, iz_n) \]

\[ \epsilon_s(q, iz_n) = 1 + \frac{U}{N^a} P(q, iz_n) \]

At absolute zero, the discrete sums become integrals and we have to evaluate the integral by transforming it into a contour integral in the complex plane such
that only zeros of the dielectric function (and the pole of $P$) are included. This means that we have to close the contour such that the zeros of the dielectric function are found at $iz_n > 0$. Thus we may simplify the expressions as follows.

$$
\langle A_{k-q\sigma}^{\dagger} (t^+) A_{k-q\sigma} (t) \rangle = \frac{U^2}{N_a} \frac{1}{\epsilon_c(q, \omega_c)} \left[ A_{k-q\sigma}^{\dagger}, A_{k-q\sigma} \right] \left[ \frac{1}{P^{-1}(q, \omega)} \right] \left[ (-\omega_c(q) - \epsilon_k + \epsilon_{k-q})^2 \right] 
$$

$$
+ \frac{U^2}{N_a} \frac{1}{P^{-1}(q, \omega_c)} \left[ A_{k-q\sigma}^{\dagger}, A_{k-q\sigma} \right] \left[ \frac{1}{\epsilon_c(q, \omega)} \right] \left[ (-\omega_c(q) - \epsilon_k + \epsilon_{k-q})^2 \right]
$$

(33)

Here $\omega_c > 0$ is the zero of $P^{-1}$ and $\omega_c > 0$ is the zero of $\epsilon_c$.

$$
P^{-1}(q, \omega_c) = 0 \quad(34)$$

$$
\epsilon_c(q, \omega_c) = 0 \quad(35)
$$

Similarly we may evaluate the other correlation function as,

$$
\langle A_{k-q\sigma}^{\dagger} (t^+) A_{k-q\sigma} (t) \rangle = \frac{U}{N_a} \frac{1}{\epsilon_s(q, \omega_s)} \left[ A_{k-q\sigma}^{\dagger}, A_{k-q\sigma} \right] \left[ \frac{1}{\epsilon_s(q, \omega_s)} \right] \left[ (-\omega_s(q) - \epsilon_k + \epsilon_{k-q})^2 \right]
$$

(36)

and $\omega_s > 0$ is the zero of $\epsilon_s$.

$$
\epsilon_s(q, \omega_s) = 0 \quad(37)
$$

Since we are interested only in the asymptotics we focus on the small $|q|$ regime. In this regime,

$$
P(q, \omega) \approx \frac{L}{2\pi} \frac{4 \sin(kF) q^2}{(\omega^2 - 4\sin^2(kF)q^2)} \quad(38)
$$

Since $\epsilon_c(q, \omega_c) = \infty$ we may ignore this contribution. Let us now enumerate all the zeros of the dielectric function.

$$
\epsilon_c(q, \omega) = 1 - \frac{U^2}{(2\pi)^2} \frac{(4 \sin(kF) q^2)^2}{(\omega_c^2 - 4\sin^2(kF)q^2)^2} = 0 \quad(39)
$$

$$
\epsilon_s(q, \omega) = 1 + \frac{U}{(2\pi)} \frac{4 \sin(kF) q^2}{(\omega_s^2 - 4\sin^2(kF)q^2)} = 0 \quad(40)
$$

There is only one zero of $\epsilon_s$ namely,

$$
\omega_s(q) = v_F |q| \left( 1 - \frac{U}{\pi v_F} \right) = v_s |q| \quad(41)
$$

where $v_F = 2 \sin(kF)$. Thus we may write the spinon hamiltonian in terms of the elementary excitations of the interacting system as,

$$
H_s = \sum_q \omega_s(q) d_s^\dagger(q) d_s(q) \quad(42)
$$
and for the holons,

\[
\omega_{c,1}(q) = v_F |q| \left( 1 - \frac{U}{\pi v_F} \right)^{\frac{1}{2}} = v_{c,1} |q| \quad (43)
\]

\[
\omega_{c,2}(q) = v_F |q| \left( 1 + \frac{U}{\pi v_F} \right)^{\frac{1}{2}} = v_{c,2} |q| \quad (44)
\]

\[
H_c = \sum_q \omega_{c,1}(q) d_{c,1}^\dagger(q) d_{c,1}(q) + \sum_q \omega_{c,2}(q) d_{c,2}^\dagger(q) d_{c,2}(q) \quad (45)
\]

It appears that these formulas are valid only for weak coupling. For strong coupling we shall have to find some other way. For now we focus only on weak coupling.

\[
\langle A_{k-q\sigma}^{+}(t) A_{k-q\sigma}(t) \rangle = \frac{U^2}{N_a^2} \sum_{j=1,2} P^{-1}(q, \omega_c) \frac{1}{\pi} \frac{[A_{k-q\sigma}^{+}, A_{k-q\sigma}]}{(\omega_c - \epsilon_c(q, \omega)(-\omega_{c,j}(q) - \epsilon_k + \epsilon_{k-q})^2}
\]

\[
\langle A_{k-q\sigma}^{+}(t^+) A_{k-q\sigma}(t) \rangle = \frac{U}{N_a} \left( \frac{|q|}{(2\pi/U)(v_s/v_F)} \right) \left( -v_s |q| - v_F q \text{sgn}(k) \right)^2 \quad (46)
\]

Therefore for \( k \) close to \( +k_F \) we may write (\( k = k_F + x \)),

\[
S_A^0(k\sigma) = \sum_q < A_{k-q\sigma}^{+} A_{k-q\sigma} > + \sum_q < A_{k-q\sigma}^{+} A_{k-q\sigma} >
\]

\[
= \int_{|x|}^\Lambda dq \sum_{j=1,2} \frac{U^2 v_F}{8\pi^2 v_{c,j} (v_{c,j} + v_F)^2} + \int_{|x|}^\Lambda dq \frac{U^2 v_F}{4\pi^2 v_s (v_s + v_F)^2} = \frac{\gamma}{2} \text{Log} \left[ \frac{\Lambda}{|x|} \right] \quad (48)
\]

The anomalous exponent is then given by,

\[
\gamma = \sum_{j=1,2} \frac{U^2 v_F}{4\pi^2 v_{c,j} (v_{c,j} + v_F)^2} + \frac{U^2 v_F}{2\pi^2 v_s (v_s + v_F)^2} \quad (49)
\]

The velocities of spinons and holons are consistent with the ones obtained by traditional bosonization methods. The anomalous exponent is proportional to \( U^2 \) for small \( U \) both in the work by Shulz[5] and our work. The momentum distribution is then given by (\( ||k| - k_F| < \Lambda \)),

\[
n_k = \frac{1}{2} - \frac{1}{2} \text{sgn}(|k| - k_F) \left( \frac{||k| - k_F|}{\Lambda} \right)^\gamma \quad (50)
\]
For a detailed comparison with the results of Shulz[5] we note that Shulz points out that the charge stiffness may be written as a power series in $U$ for small $U$ as $K_p = 1 - U/(πv_F) + ...$. This means that we may read off a formula for the anomalous exponent $γ = (K_p + 1/K_p - 2)/4 ≈ U^2/(2πv_F)^2$. From Eq. (49) we find that since $v_s ≈ v_{c,1} ≈ v_{c,2} ≈ v_F$, we may conclude that $γ ≈ U^2/(4π^2v_F^2)$. This agrees exactly with the Shulz result above. In the large $U$-limit, the velocities of the spinons becomes imaginary and we are unable to make progress. This means that the cubic and quartic terms that the above approach ignores are now relevant. That is, their inclusion alters the value of the exponents. At half-filling again there seem to be some difficulties which we have been unable to overcome. Perhaps future publications will address these issues.

### 4 Momentum Distribution in Two Dimensions

The main advantage of the sea-boson approach is the ease with which one may generalise the above results to more than one dimension by simply promoting the wavenumbers to wave-vectors.

\[
\left\langle A^k_{k-q} (t^+) A^k_{k-q} (t) \right\rangle = U^2 \frac{i}{N^2} \sum_{\mathbf{q}, \mathbf{z}_{\mathbf{n}}} \frac{P(\mathbf{q}, i\mathbf{z}_{\mathbf{n}})}{\epsilon_c(\mathbf{q}, i\mathbf{z}_{\mathbf{n}})} \left[ A^k_{k-q} A^k_{k-q} \right] \tag{51}
\]

\[
\left\langle A^k_{k-q} (t^+) A^k_{k-q} (t) \right\rangle = -\frac{U}{N^2} \sum_{\mathbf{q}, \mathbf{z}_{\mathbf{n}}} \frac{1}{\epsilon_c(\mathbf{q}, i\mathbf{z}_{\mathbf{n}})} \left[ A^k_{k-q} A^k_{k-q} \right] \tag{52}
\]

\[
P(\mathbf{q}, i\mathbf{z}_{\mathbf{n}}) = \sum_{\mathbf{k}} \frac{|A^k_{k-q} A^k_{k-q}|}{(-\epsilon_{c}(\mathbf{k}) - \epsilon_{c}(\mathbf{k} - \mathbf{q}))} + \sum_{\mathbf{k}} \frac{|A^k_{k-q} A^k_{k-q}|}{(i\mathbf{z}_{\mathbf{n}} - \epsilon_{c}(\mathbf{k} - \mathbf{q}) + \epsilon_{c}(\mathbf{k}))} \tag{53}
\]

\[
\epsilon_c(\mathbf{q}, i\mathbf{z}_{\mathbf{n}}) = 1 - \frac{U^2}{N^2} P^2(\mathbf{q}, i\mathbf{z}_{\mathbf{n}}) \tag{54}
\]

\[
\epsilon_s(\mathbf{q}, i\mathbf{z}_{\mathbf{n}}) = 1 + \frac{U}{N^2} P(\mathbf{q}, i\mathbf{z}_{\mathbf{n}}) \tag{55}
\]

For evaluating the low energy, long-wavelength limit of the above expressions, we may ignore the non-zero lattice spacing and instead focus on the continuum limit (which is valid for $k_F ≪ 1$ since we have set $a = 1$) where the expressions for the RPA-polarization have already been derived by Stern[12](here $2m = 1$).

\[
Re[P(\mathbf{q}, \omega)] = -\frac{mk_F A}{2\pi|\mathbf{q}|} \left\{ \left[ \frac{|\mathbf{q}|}{k_F} - \frac{m\omega}{k_F |\mathbf{q}|} \right]^2 - 1 \right\}^{\frac{1}{2}} - C_+ \left\{ \left[ \frac{|\mathbf{q}|}{k_F} + \frac{m\omega}{k_F |\mathbf{q}|} \right]^2 - 1 \right\}^{\frac{1}{2}} \tag{56}
\]
\[ Im[P](q, \omega) = -\frac{mk_F A}{2\pi |q|} \left( D_- \left[ 1 - \left| \frac{q}{2k_F} + \frac{\omega}{k_F |q|} \right|^2 \right]^{\frac{1}{2}} - D_+ \left[ 1 - \left| \frac{q}{2k_F} - \frac{\omega}{k_F |q|} \right|^2 \right]^{\frac{1}{2}} \right) \]

where,

\[ C_\pm = \text{sgn} \left[ \frac{|q|}{2k_F} \pm \frac{\omega}{k_F |q|} \right], \quad D_\pm = 0, \quad \left| \frac{|q|}{2k_F} \pm \frac{\omega}{k_F |q|} \right| > 1 \quad (58) \]

\[ C_\pm = 0, \quad D_\pm = 1, \quad \left| \frac{|q|}{2k_F} \pm \frac{\omega}{k_F |q|} \right| < 1 \quad (59) \]

\[ Re[P](q, \omega) \approx \frac{mA}{2\pi} \left( 1 - \frac{\omega}{\sqrt{\omega^2 - v_F^2 |q|^2}} \right) \quad (60) \]

\[ 0 = 1 + \frac{mU}{2\pi} \left( 1 - \frac{\omega}{\sqrt{\omega^2 - v_F^2 |q|^2}} \right) \quad (61) \]

for \( \omega > v_F |q| \).

\[ v_{eff} = v_F \frac{\left( 1 + \frac{v_F}{\omega} \right)}{\sqrt{\left( 1 + \frac{v_F}{\omega} \right)^2 - 1}} \quad (62) \]

\[ \frac{\partial}{\partial \omega} Re[P](q, \omega) \approx -\frac{Amv_F^2 q^2}{2\pi(\omega^2 - v_F^2 |q|^2)^{3/2}} \quad (63) \]

This means,

\[ \left\langle A_{k-q}^{\dagger \sigma} (t) A_{k}^{\sigma} (t) \right\rangle = \frac{U^2}{N^2} \frac{1}{P^{-1}(q, \omega) \theta(\omega_c(q, \omega))} \frac{[A_{k-q}^{\dagger \sigma}, A_{k}^{\dagger \sigma}]}{(-\omega_c(q) - \epsilon_k + \epsilon_{k-q})^2} \quad (64) \]

\[ \left\langle A_{k-q}^{\dagger \sigma} (t) A_{k-q}^{\sigma} (t) \right\rangle = 0 \quad (65) \]

and \( \omega_c(q) = v_{eff} |q| \). For a triangular lattice the energy dispersion may be written as, \( \epsilon_k = -2 \left[ \cos(k_x) + 2 \cos(k_x/2) \cos(\sqrt{3}k_y/2) \right] \). Let us assume that \( k_F a \ll 1 \) (since we have set \( a = 1 \), we must have \( k_F \ll 1 \) in which case the Fermi surface is a circle,

\[ \left\langle A_{k-q}^{\dagger \sigma} (t) A_{k-q}^{\sigma} (t) \right\rangle = -\frac{1}{2\pi |q| v_{eff}^2 - v_F^2} \frac{[A_{k-q}^{\dagger \sigma}, A_{k-q}^{\dagger \sigma}]}{(-v_{eff} |q| - \epsilon_k + \epsilon_{k-q})^2} \]

\[ = \frac{\pi |q|(v_{eff}^2 - v_F^2)^{3/2}}{Amv_F^2} \frac{[A_{k-q}^{\dagger \sigma}, A_{k-q}^{\dagger \sigma}]}{(-v_{eff} |q| - k^2 + (k - q)^2)^2} \]

\[ \approx \frac{\pi |q|(v_{eff}^2 - v_F^2)^{3/2} \theta(k_F - |q|) \theta(|k| - k_F)}{Amv_F^2} \frac{\theta(k_F - |q|) \theta(|k| - k_F)}{(-v_{eff} |q| - 2k \cdot q + q^2)^2} \]
for $k_F \ll 1$. Also the $U \to +\infty$ result must be taken with a grain of salt since the corresponding 1d case has not worked out. Just as the anomalous exponent saturates to a value of $1/8$ according to Bethe ansatz and also hopefully according to the sea-boson method provided we include the anharmonic terms, we may suspect that the quasiparticle residue also saturates (to a value close to unity) when these anharmonic terms are included. This is completely consistent with the results of Castro-Neto and Fradkin (who predicted a value of $0.78$ I think for the smallest value of the quasiparticle residue). However it is not clear to the author if their results were due to a formalism more powerful than the sea-boson method, or because of a clever use of intuition. In particular one
has to see if the traditional bosonization method can predict the exponent of 1/8 in 1d, without relying on the Bethe ansatz results. The small $U$ results are probably reliable though one has to check this several times more. The quasiparticle residue seems to deviate from unity by a term proportional to $U^3$ rather than $U^2$.

5 Full Propagator

To compute say the density of states, we have to compute the dynamical one-particle Green function. For this we have to complete the construction of the field operator started in the Appendix. In particular we have to find a formula for $R(p\sigma)$.

\begin{align}
\hat{c}_{p,\sigma} & \equiv c\quad \sum_{q\sigma_1} \left( - A_{p\sigma}^{q\sigma_1} + A_{p+q\sigma_1}^{\sigma} \right) T_q(p) R(p\sigma) \\
\hat{c}^{\dagger}_{p,\sigma} & \equiv c\quad \sum_{q\sigma_1} \left( - A_{p\sigma}^{q\sigma_1} + A_{p+q\sigma_1}^{\sigma} \right) T_q(p) R^{*(p\sigma)} \tag{74}
\end{align}

The main problem with these formulas is that it is not possible to use the Baker-Hausdorff theorem since the commutator of the two exponents does not commute with either even at equal times. Therefore it is not possible to systematically deduce $R$ by making contact with the free theory. However we may suspect that perhaps,

$$R(p\sigma) = \varphi_F(p) \ e^{iN^0 \xi(p\sigma)} \tag{76}$$

as usual $\varphi_F(p) = \theta(k_F - |p|)$. Here $\xi(p\sigma)$ is an arbitrary non-constant function (c-number) of the arguments such that,

$$e^{-iN^0 \xi(p\sigma)} e^{iN^0 \xi(p'\sigma')} \approx \delta(p', p) \delta_{\sigma, \sigma'} \tag{77}$$

Therefore,

$$\left\langle \hat{c}^{\dagger}_{p,\sigma'}(t) \hat{c}_{p,\sigma}(0) \right\rangle = \left\langle \quad c\quad \sum_{q\sigma_1} \left( - A_{p',\sigma'}^{q\sigma_1}(t) + A_{p+q\sigma_1}^{\sigma'}(t) \right) T_q(p') \quad \sum_{q\sigma_1} \left( - A_{p\sigma}^{q\sigma_1} + A_{p+q\sigma_1}^{\sigma} \right) \quad T_q(p) \quad R^{*(p'\sigma') \quad R(p\sigma)} \quad \right\rangle \tag{78}$$

For reasons already mentioned this is not possible to simplify further.

6 Conclusions

We have computed the asymptotic features of the momentum distribution of the one-band Hubbard model in one and two spatial dimensions. In the one dimensional case, we obtain a Luttinger liquid with an anomalous exponent identical to the one obtained by Shulz[5] from Bethe ansatz. This is valid for weak coupling only. We are unable to extend the results to strong coupling. In the two dimensional case, we obtain a Landau Fermi liquid with quasiparticle
the commutation rules we proceed as follows. The exponential is interpreted as a power series expansion. In the expansion, the translation operator translates here in Eq.(79) and Eq.(80) and arrive at,

\[ c_{p\sigma}, A_{k,\sigma_1}(q\sigma_2)] \approx \delta_{p+q/2, k} n_F(p)(1 - n_F(p + q)) T_q(p) c_{p\sigma} \delta_{\sigma, \sigma_1} \]  
(79)

\[ c_{p\sigma}, A_{k,\sigma_1}^\dagger(q\sigma_2)] \approx \delta_{p,k+q/2} n_F(p - q)(1 - n_F(p)) T_{-q}(p) c_{p\delta_{\sigma, \sigma_2}} \]  
(80)

where \( T_q(p) \equiv exp(q \nabla_p) \). We may now exponentiate the commutation rules in Eq.(79) and Eq.(80) and arrive at,

\[ c_{p\sigma} \equiv \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_q(p) R(p\sigma) \]  
(81)

\[ c_{p\sigma}^\dagger \equiv \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_{-q}(p) R^*(p\sigma) \]  
(82)

Here \( R(p\sigma) \) is a c-number function that commutes with everything. To verify the commutation rules we proceed as follows. The exponential is interpreted as a power series expansion. In the expansion, the translation operator translates all the \( p \)’s to its right by an amount \( q \) and not just the \( p \) in \( R(p\sigma) \).

\[ [c_{p\sigma}, A_{k,\sigma_1}(q\sigma_2)] = \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_q(p) A_{k,\sigma_1}(q\sigma_2) R(p\sigma) \]

\[ -A_{k,\sigma_1}(q\sigma_2) c_{p\sigma} \equiv \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_q(p) \] 

\[ R(p\sigma) \]

\[ = \sum_{\sigma'} \left[ A_{p+q/2, \sigma}(q\sigma') A_{k,\sigma_1}(q\sigma_2) \right] T_q(p) c_{p\sigma} \]

\[ = \delta_{k,p+q/2,\sigma} n_F(k - q/2)(1 - n_F(k + q/2)) T_q(p) c_{p\sigma} \]  
(83)

\[ [c_{p\sigma}, A_{k,\sigma_1}^\dagger(q\sigma_2)] = \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_q(p) A_{k,\sigma_1}^\dagger(q\sigma_2) R(p\sigma) \]

\[ = \delta_{k,p+q/2,\sigma} n_F(k - q/2)(1 - n_F(k + q/2)) T_q(p) A_{k,\sigma_1}^\dagger(q\sigma_2) R(p\sigma) \]  

\[ = \delta_{k,p+q/2,\sigma} n_F(k - q/2)(1 - n_F(k + q/2)) T_q(p) c_{p\sigma} \]  
(84)

\[ [c_{p\sigma}, A_{k,\sigma_1}^\dagger(q\sigma_2)] = \sum_{q\sigma'} \left( -A_{p+q/2, \sigma}(q\sigma') + A_{p+q/2, \sigma}^\dagger(-q\sigma) \right) T_q(p) A_{k,\sigma_1}^\dagger(q\sigma_2) R(p\sigma) \]

\[ = \delta_{k,p+q/2,\sigma} n_F(k - q/2)(1 - n_F(k + q/2)) T_q(p) c_{p\sigma} \]  
(85)
\[-A_{k\sigma_1}(q\sigma_2)e^{i\sum_{q\sigma'}(-A^\dagger_{\mu+q/2\sigma}(q\sigma')+A_{\mu+q/2\sigma'}(-q\sigma)) T_{q}(p)} R(p\sigma)\]
\[= \sum_{\sigma'} [A_{p-q/2\sigma'}(q\sigma), A^\dagger_{k\sigma_1}(q\sigma_2)] T_{-q}(p) c_{p\sigma}\]
\[= \delta_{k,p-q/2}\delta_{\sigma,\sigma_2} n_F(k - q/2)(1 - n_F(k + q/2)) T_{-q}(p) c_{p\sigma}\] (84)

In the sea-boson method, unlike in the usual bosonization in 1d, the momentum space description is easier than the real space description. Now we have to compute \(R(p\sigma)\) by making contact with the free theory. This is proving to be harder than expected since the commutator of the exponent with the other exponent does not commute with either exponent, hence we may not use the truncated version of the Baker-Hausdorff theorem. If one insists on computing the full dynamical propagator which is needed for obtaining the density of states as a function of the energy, then perhaps a simple route may suffice. It goes by the name ‘serendipitous surmise’.

7.1 A Serendipitous Surmise

Many years ago the author had a conversation with then the student now Prof. A.H. Castro-Neto where the latter suggested that maybe the field operator is simply given by,
\[\psi(x) \approx e^{-i\sum_q e^{i\mathbf{q} \cdot \mathbf{x}} X_q \sqrt{\rho_0}}\] (85)

where by definition \(X_q = i\mathbf{q} \cdot \mathbf{j}(-\mathbf{q})/(\mathbf{q}^2 N^0)\) and \(\rho_0 = N^0/V\). Later he realised that if we choose \([X_q, X_{q'}] = 0\) as is in fact mandatory, then fermion commutation rules are not obeyed. However one may take solace in the fact that at least one commutation rule does come out right namely \([\psi(x), \rho_q] = e^{i\mathbf{q} \cdot \mathbf{x}} \psi(x)\).

A refinement over this ansatz was attempted in our earlier work by introducing an additional phase functional of the density linear in the density and this was also inadequate since by now the author knows that \(\Phi\) there when computed was imaginary when it was postulated to be real. A compromise was also suggested that involved multiplying and dividing by the free propagator and using the exact version in the numerator and the bosonized free propagator in the denominator. This trick though repugnant to most, gives us an anomalous exponent of the Luttinger liquid as we shall see below. However, this anomalous exponent is off by a factor of two from the exact one obtained by Mattis and Lieb. From Eq.(85) we may write,
\[G(x - x') = \langle \psi^\dagger(x') \psi(x) \rangle \approx \left\langle e^{i \sum_q (e^{i\mathbf{q} \cdot \mathbf{x}'} - e^{i\mathbf{q} \cdot \mathbf{x}}) X_q} \right\rangle \rho^0\]
\[= e^{-\sum_q \left(1 - \cos[q,(x - x')]\right)(X_q X_{-q})} \rho^0\] (86)

Since strictly speaking it is the conjugate to \(\rho\) as defined by the line integral of the ratio of the current and density that enters; these commute amongst themselves.
Again using the trick outlined in our earlier work[1] we may write,

\[ G(x - x') = G_0(x - x') e^{- \sum_q \left( 1 - \cos[q.(x-x')] \right) \left( |X_q X_{-q}| - |X_q X_{-q}'| \right) \} \] (87)

Here \( G_0(x - x') \) is the propagator obtained from elementary considerations. In one dimension, we may see that \( \langle X_q X_{-q} \rangle \approx k_F^2 S(q)/(q^2 N^0) \). The structure factor \( S_0(q) = |q|/(2k_F) \) for the interacting case. For the interacting case we have, \( S(q) = (v_F/v_{eff}) S_0(q) \).

\[ G(x - x') = G_0(x - x') e^{- \int_0^\infty dq \frac{1 - \cos[q.(x-x')]}{|q|} \left( \frac{v_F}{v_{eff}} - \frac{1}{2} \right)} \]

\[ \sim G_0(x - x') \left( \frac{1}{|x - x'|} \right)^\gamma \] (88)

where \( \gamma = \frac{v_F}{v_{eff}} - \frac{1}{2} \). This exponent is exactly one half of the exponent obtained by Mattis and Lieb. What is even worse is, we have shown in an earlier preprint that when applied to the X-ray edge problem, we obtain the well-known results of Mahan in one dimension but not in higher dimensions. Thus it would appear that there is something amiss in the expression for the field operator. Nevertheless it may be a quick and easy way of getting the density of states that gives the right qualitative physics if not the right exponents.

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