ON THE ARAKELOV THEORY OF ELLIPTIC CURVES

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Abstract. This note contains an elementary discussion of the Arakelov intersection theory of elliptic curves. The main new results are a projection formula for elliptic arithmetic surfaces and a formula for the “energy” of an isogeny between Riemann surfaces of genus 1. The latter formula provides an answer to a question originally posed by Szpiro.

1. Introduction

The goal of this note is to present an elementary discussion of the Arakelov intersection theory of elliptic curves. Arakelov intersection theory in general is a theory dealing with curves over number fields, unifying in a subtle way the arithmetic aspects of a curve, present on the reductions of the curve modulo the finite primes of the number field, with its analytic aspects, present on the Riemann surfaces that one obtains by base changing the curve to the complex numbers. The unifying framework is provided by an intersection theory for divisors on an arithmetic surface [1], sharing many formal properties with the traditional intersection theory on proper algebraic surfaces over a field [6]. Although in general working out Arakelov theory is a difficult matter, when we specify to the case of elliptic curves it turns out that a nice, compact and clean theory emerges.

Many results on the Arakelov theory of elliptic curves are already known by the works of Faltings [6] and Szpiro [10], but our approach is different. In particular, we base our discussion on a projection formula for Arakelov’s Green function on Riemann surfaces of genus 1 related by an isogeny. From this formula we derive a projection formula for Arakelov intersections, as well as a formula for the so-called “energy of an isogeny”. Both of these formulas seem new. In fact, the latter formula provides an answer to a question posed by Szpiro in [10].

Using these new results, we give alternative proofs of several of the earlier results. For example, we arrive at explicit formulas for the Arakelov-Green function on an elliptic curve, for the canonical norm in the holomorphic cotangent bundle, and for the Arakelov self-intersection of a point. We also give an elementary proof of a recent result due to Autissier on the average height of the quotients of an elliptic curve by its cyclic subgroups of a fixed order.

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2. Analytic invariants

We start by recalling the main ingredients of the analytic part of Arakelov theory, namely, the Arakelov-Green function $G$ and the canonical metric on the holomorphic cotangent bundle. Our main references are [1] and [6].

Let $X$ be a compact and connected Riemann surface of genus $g > 0$. The space of holomorphic differentials $H^0(X, \Omega^1_X)$ carries then a natural hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2} \int_X \omega \wedge \overline{\eta}$. Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis with respect to this inner product. We have then a fundamental $(1,1)$-form $\mu$ on $X$ given by

$$\mu = \frac{i}{2g} \sum_{k=1}^g \omega_k \wedge \overline{\omega_k}.$$  

It is verified immediately that the form $\mu$ does not depend on the choice of orthonormal basis, and hence it defines a canonical $(1,1)$-form on $X$. Using this form, one defines the canonical Arakelov-Green function on $X$. This function gives the local intersections “at infinity” of two divisors in Arakelov theory.

**Definition 2.1.** The Arakelov-Green function $G$ is the unique function $X \times X \to \mathbb{R}_{\geq 0}$ such that the following three properties hold:

(i) $G(P, Q)^2$ is $C^\infty$ on $X \times X$ and $G(P, Q)$ vanishes only at the diagonal $\Delta_X$, with multiplicity 1;

(ii) for all $P \in X$ we have $\partial_Q \overline{\partial_Q} \log G(P, Q)^2 = 2\pi i \mu(Q)$ for $Q \neq P$;

(iii) for all $P \in X$ we have $\int_X \log G(P, Q) \mu(Q) = 0$.

Properties (i) and (ii) determine $G$ up to a multiplicative constant, which is then fixed by the normalisation condition (iii). By an application of Stokes’ theorem we obtain from (i)–(iii) the symmetry $G(P, Q) = G(Q, P)$ of the function $G$.

Importantly, the Arakelov-Green function gives rise to certain canonical metrics on the line bundles $O_X(D)$, where $D$ is a divisor on $X$. It suffices to consider the case of a point $P \in X$, for the general case follows then by taking tensor products. Let $s$ be the canonical generating section of the line bundle $O_X(P)$. We then define a smooth hermitian metric $\| \cdot \|_{O_X(P)}$ on $O_X(P)$ by putting $\|s\|_{O_X(P)}(Q) := G(P, Q)$ for any $Q \in X$. By property (ii) of the Arakelov-Green function, the curvature form of $O_X(P)$ is equal to $\mu$, and in general, the curvature form of $O_X(D)$ is $\deg(D) \cdot \mu$, with $\deg(D)$ the degree of $D$.

**Definition 2.2.** A line bundle $L$ on $X$ with a smooth hermitian metric $\| \cdot \|$ is called admissible if its curvature form is a multiple of $\mu$. We also call the metric $\| \cdot \|$ itself admissible in this case.

**Proposition 2.3.** Let $\| \cdot \|$ and $\| \cdot \|'$ be admissible metrics on a line bundle $L$. Then the quotient $\| \cdot \|/\| \cdot \|'$ is a constant function on $X$.

**Proof.** The logarithm of the quotient is a smooth harmonic function on $X$, and hence it is constant. \qed
Definition 2.4. The canonical metric ∥ · ∥ Ar on the holomorphic cotangent bundle Ω_X^1 is the unique metric that makes the adjunction isomorphism O_X × X(−Δ_X)|∆_X ↘ Ω_X^1 an isometry. Here the line bundle O_X × X(Δ_X) carries the hermitian metric defined by ∥s∥(P, Q) := G(P, Q), with s the canonical generating section of the line bundle O_X × X(Δ_X).

Proposition 2.5. (Adjunction formula) Let P be a point on X, and let z be a local coordinate about P. Then for the norm ∥dz∥_{Ar} of dz in Ω^1_X the formula ∥dz∥_{Ar} = \lim_{Q → P} |z(P) − z(Q)|/G(P, Q) holds.

Proof. From the definition of the canonical metric on Ω^1_X it follows that dz/z has unit length in Ω^1_X|P. However, this line bundle is isometric to Ω^1_X ⊗ O_X|P, with dz/z corresponding to dz ⊗ z^−1 s with s the canonical generating section of O_X|P. One computes that ∥z^−1 s∥ = \lim_{Q → P} G(P, Q)/|z(P) − z(Q)| and the proposition follows. □

In Sections 3 and 4 we prove some fundamental properties of the Arakelov-Green function and the canonical norm on the holomorphic cotangent bundle in the case that X has genus 1.

3. Analytic projection formula

We start by studying the fundamental (1,1)-form μ with respect to isogenies. Let X and X’ be Riemann surfaces of genus 1, and suppose that f : X → X’ is an isogeny, say of degree N. Let μ_X and μ_X’ be the fundamental (1,1)-forms of X and X’, respectively.

Proposition 3.1. (i) We have f^*μ_X’ = N · μ_X; (ii) the canonical isomorphism f^* : H^0(X’, Ω^1_X’) → H^0(X, Ω^1_X) given by inclusion has norm √N.

Proof. We identify X with a complex torus C/Λ, and obtain X’ as the quotient of C/Λ by a finite subgroup Λ’/Λ. Hence we may identify X’ with C/Λ’. A small computation shows that the differentials ω := dz/√vol(Λ) and ω’ := dz/√vol(Λ’) are orthonormal bases of H^0(X, Ω^1_X) and H^0(X’, Ω^1_X’), respectively. We obtain (ii) by observing that N = vol(Λ)/vol(Λ’). Finally we have μ_X = (i/2) · (dz ∧ dζ)/vol(Λ) and μ_X’ = (i/2) · (dz ∧ dζ)/vol(Λ’) and (i) also follows. □

Proposition 3.1 gives rise to a projection formula for the Arakelov-Green function.

Theorem 3.2. (Analytic projection formula) Let X and X’ be Riemann surfaces of genus 1 and let G_X and G_X’ be the Arakelov-Green functions of X and X’, respectively. Suppose we have an isogeny f : X → X’. Let D be a divisor on X’. Then the canonical isomorphism of line bundles

f^*O_X’(D) → O_X(f^*D)
is an isometry. In particular we have a projection formula: for any \( P \in X \) the formula
\[
G_X(f^*D, P) = G_{X'}(D, f(P))
\]
holds.

**Proof.** Let \( N \) be the degree of \( f \). By Proposition 3.1 we have
\[
\text{curv} f^*O_{X'}(D) = f^*(\text{curv} O_X(D)) = f^*((\deg D) \cdot \mu_{X'}) = N \cdot (\deg D) \cdot \mu_X,
\]
which means that \( f^*O_{X'}(D) \) is an admissible line bundle on \( X \). Hence by Proposition 2.3 we have
\[
\| f^*(s_D) \|_{f^*O_{X'}(D) \cdot \mu_X} = \left( \frac{\text{Im} \tau}{N} \right)^{1/24} \| s_{f^*D} \|_{O_X(f^*D)},
\]
for some constant \( c \) where \( s_D \) and \( s_{f^*D} \) are the canonical sections of \( O_{X'}(D) \) and \( O_X(f^*D) \), respectively. But since
\[
\int_X \log \| f^*(s_D) \|_{f^*O_{X'}(D) \cdot \mu_X} = \frac{1}{N} \int_X \log \| s_D \|_{O_X(f^*D) \cdot f^*\mu_{X'}},
\]
this constant is equal to 1. \( \square \)

4. **Energy of an isogeny**

At this point, we introduce some classical invariants attached to a Riemann surface \( X \) of genus 1.

**Definition 4.1.** Let \( \tau \) be an element of the complex upper half plane, and write \( q = \exp(2\pi i \tau) \). Then we have the eta-function \( \eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k) \) and the modular discriminant \( \Delta(\tau) = \eta(\tau)^{24} = q \prod_{k=1}^{\infty} (1 - q^k)^{24} \). The latter is the unique normalised cusp form of weight 12 on \( \text{SL}(2, \mathbb{Z}) \). Now suppose that we have a Riemann surface \( X \) of genus 1 identified with a complex torus \( \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \). Then we put
\[
\| \eta \|(X) := (\text{Im} \tau)^{1/4} \cdot |\eta(\tau)| \quad \text{and} \quad \| \Delta \|(X) := \| \eta \|(X)^{24} = (\text{Im} \tau)^6 \cdot |\Delta(\tau)|.
\]
These definitions do not depend on the choice of \( \tau \), and hence they define invariants of \( X \).

In [10] Szpiro proves the following statement (cf. Théorème 1): let \( E \) and \( E' \) be semi-stable elliptic curves defined over a number field \( K \), and suppose we have an isogeny \( f : E \to E' \). Then the formula
\[
\sum_{\sigma} \sum_{P_{\sigma} \in \text{Ker}(f_{\sigma})} \log G(0, P_{\sigma}) = \frac{|K : \mathbb{Q}|}{2} \log N + \sum_{\sigma} \log \frac{\| \eta \|(E'_{\sigma})^2}{\| \eta \|(E_{\sigma})^2}
\]
holds, where \( N \) is the degree of \( f \) and where the sum is over the complex embeddings of \( K \). Szpiro then asks whether a similar statement holds without the sum over the complex embeddings. The following theorem gives a positive answer to that question. The terminology “energy of an isogeny” is adopted from [10].
Theorem 4.2. (Energy of an isogeny) Let $X$ and $X'$ be Riemann surfaces of genus 1. Suppose we have an isogeny $f : X \to X'$. Then we have
\[
\prod_{P \in \text{Ker} f, P \neq 0} G(0, P) = \frac{\sqrt{N} \cdot \|\eta\|(X')^2}{\|\eta\|(X)^2},
\]
where $N$ is the degree of $f$.

It is the purpose of the present section to prove Theorem 4.2. En passant we make the Arakelov-Green function and the canonical norm on the holomorphic cotangent bundle explicit, see Propositions 4.7 and 4.8. These formulas are also given in [6], but the proof there relies on a consideration of the eigenvalues and eigenfunctions of the Laplace operator. Our approach is more elementary.

Definition 4.3. Let $X$ be a Riemann surface of genus 1. Let $\omega$ be a holomorphic differential of norm 1 in $H^0(X, \Omega^1_X)$. Then we put $A(X) := \|\omega\|_{\text{Ar}}$ for the norm of $\omega$ in $\Omega^1_X$.

Proposition 4.4. Let $f : X \to X'$ be an isogeny of degree $N$. Then the formula
\[
\prod_{P \in \text{Ker} f, P \neq 0} G(0, P) = \frac{\sqrt{N} \cdot A(X)}{A(X')} \quad \text{holds.}
\]

Proof. Let $\nu$ be the norm of the isomorphism of line bundles $f^*\Omega^1_X \sim \to \Omega^1_{X'}$ given by the usual inclusion. We will compute $\nu$ in two ways. First of all, consider an $\omega' \in H^0(X', \Omega^1_{X'})$ of norm 1, so that $\omega'$ has norm $A(X')$ in $\Omega^1_{X'}$. Then by Proposition 3.1 we have that $f^*(\omega')$ has norm $\sqrt{N}$ in $H^0(X, \Omega^1_X)$, hence it has norm $\sqrt{N} \cdot A(X)$ in $\Omega^1_X$. This gives
\[
\nu = \sqrt{N} \cdot A(X) \quad \text{in } \Omega^1_X.
\]

On the other hand, by Theorem 3.2 the canonical isomorphism $f^*(O_X, (0)) \sim \to O_X(\text{Ker} f)$ is an isometry. Tensoring with the isomorphism $f^*\Omega^1_{X'} \sim \to \Omega^1_X$ gives an isomorphism
\[
f^*(\Omega^1_{X'}, (0)) \sim \to \Omega^1_X (0) \otimes \bigotimes_{P \in \text{Ker} f, P \neq 0} O_X(P)
\]
of norm $\nu$ given in local coordinates by
\[
f^*(\frac{dz}{z}) \mapsto \frac{dz}{z} \otimes s
\]
where $s$ is the canonical section of $\bigotimes_{P \in \text{Ker} f, P \neq 0} O_X(P)$. By the definition of the canonical norm on the holomorphic cotangent bundle, the $dz/z$ have norm 1, so we find
\[
\nu = \prod_{P \in \text{Ker} f, P \neq 0} G(0, P).
\]
Together with the earlier formula for $\nu$ this implies the proposition. \qed
The following corollary seems to be well-known, see for instance [11], Lemme 6.2.

**Corollary 4.5.** Denote by $X[N]$ the kernel of the multiplication-by-$N$ map $X \to X$. Then the formula

$$\prod_{P \in X[N], P \neq 0} G(0, P) = N$$

holds.

**Proof.** Immediate from Proposition 4.4. □

Let $\tau$ be an element of the complex upper half plane. Recall that Riemann’s theta function is given by

$$\vartheta(z; \tau) := \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z)$$
on $\mathbb{C}$. We have the identities

$$(\exp(\pi i \tau/4) \cdot \vartheta(0; \tau) \vartheta(1/2; \tau) \vartheta(\tau/2; \tau))^8 = 2^8 \cdot \Delta(\tau)$$

and

$$(\exp(\pi i \tau/4) \cdot \frac{\partial \vartheta}{\partial z} \left(\frac{1 + \tau}{2}; \tau\right))^8 = (2\pi)^8 \cdot \Delta(\tau),$$

both of which are proved by the fact that the left hand sides are cusp forms on $\text{SL}(2, \mathbb{Z})$ of weight 12.

**Definition 4.6.** (Cf. [6]) Let $\tau$ be in the complex upper half plane. The normalised theta function $\|\vartheta\|$ associated to $\tau$ is defined to be the function

$$\|\vartheta\|(z; \tau) := (\text{Im} \tau)^{1/4} \exp(-\pi (\text{Im} \tau)^{-1} y^2) |\vartheta(z; \tau)|$$
on $\mathbb{C}$ where $y := \text{Im} z$. This function only depends on the class of $z$ modulo $\mathbb{Z} + \tau \mathbb{Z}$.

**Proposition 4.7.** (Faltings [6]) Let $X$ be a Riemann surface of genus 1, and write $X \cong \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$ with $\tau$ in the complex upper half plane. For the Arakelov-Green function $G$ on $X$ the formula

$$G(0, z) = \frac{\|\vartheta\|(z + (1 + \tau)/2; \tau)}{\|\eta\|(X)}$$

holds.

**Proof.** It is not difficult to check that $\|\vartheta\|(z + (1 + \tau)/2)$ vanishes only at $z = 0$, with order 1. Also it is not difficult to check that $\partial_z \text{Log} \|\vartheta\|(z + (1 + \tau)/2)^2 = 2\pi i \mu_X$ for $z \neq 0$. By what we have said in Section 2 we have from this that $G(0, z) = c \cdot \|\vartheta\|(z + (1 + \tau)/2; \tau)$ where $c$ is some constant. It remains to compute this constant. If we apply Corollary 4.5 with $N = 2$ we obtain

$$c^3 \cdot \|\vartheta\|(0; \tau) \|\vartheta\|(1/2; \tau) \|\vartheta\|((\tau/2; \tau) = G(0, 1/2)G(0, \tau/2)G(0, (1 + \tau)/2) = 2.$$

On the other hand we have the formula

$$(\exp(\pi i \tau/4) \cdot \vartheta(0; \tau) \vartheta(1/2; \tau) \vartheta(\tau/2; \tau))^8 = 2^8 \cdot \Delta(\tau)$$

mentioned above. Combining we obtain $c = \|\eta\|(X)^{-1}$. □
Proposition 4.8. (Faltings [6]) For the invariant $A(X)$, the formula

$$A(X) = \frac{1}{(2\pi) \cdot \|\eta\|^2(X)}$$

holds.

Proof. We follow the argument from [6]: writing $X \cong \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$ we can take $\omega = dz/\sqrt{\text{Im}\tau}$ as an orthonormal basis of $H^0(X, \Omega^1_X)$. By Proposition 2.5 we have $\|dz/\sqrt{\text{Im}\tau}\|_{\text{Ar}} = (\sqrt{\text{Im}\tau})^{-1} \cdot \lim_{z \to 0} |z|/G(0, z)$. We obtain the required formula by using the explicit formula for $G(0, z)$ in Proposition 4.7 and the formula

$$\left(\exp\left(\pi i \tau/4\right) \cdot \frac{\partial \vartheta}{\partial z} \left(\frac{1 + \tau}{2}; \tau\right)\right)^8 = (2\pi)^8 \cdot \Delta(\tau)$$

mentioned above. $\square$

Proof of Theorem 4.2. Immediate from Propositions 4.4 and 4.8. $\square$

We conclude this section with a corollary, dealing with the value of the Arakelov-Green function on pairs of 2-torsion points. First we need a classical lemma.

Lemma 4.9. Let $X$ be a Riemann surface of genus 1 and suppose that $y^2 = 4x^3 - px - q =: f(x)$ is a Weierstrass equation for $X$. Write $f(x) = 4(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Let $(\omega_1, \omega_2)$ be the period matrix of the holomorphic differential $dx/y$ on the canonical symplectic basis of homology given by the ordering $\alpha_1, \alpha_2, \alpha_3$ of the roots of $f$ (cf. [3], Chapter IIIa, §5), and put $\tau := \omega_2/\omega_1$. Then we have the formulas

$$\begin{align*}
\omega_1 \sqrt{\alpha_1 - \alpha_3} &= \pi \cdot \vartheta(0; \tau)^2, \\
\omega_1 \sqrt{\alpha_1 - \alpha_2} &= \pi \cdot \vartheta(1/2; \tau)^2, \\
\omega_1 \sqrt{\alpha_2 - \alpha_3} &= \pi \cdot \exp(\pi i \tau/2) \cdot \vartheta(\tau/2; \tau)^2
\end{align*}$$

for appropriate choices of the square roots. Let $D := 16(\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2 = p^3 - 27q^2$ be the discriminant of $f$. Then the formula

$$D = (2\pi)^{12} \cdot \omega_1^{-12} \cdot \Delta(\tau)$$

holds.

Proof. The first set of formulas follows by an application of Thomae’s formula, cf. [3], Chapter IIIa, §5. The other formula follows from the first and from the formula

$$(\exp(\pi i \tau/4) \cdot \vartheta(0; \tau)\vartheta(1/2; \tau)\vartheta(\tau/2; \tau))^8 = 2^8 \cdot \Delta(\tau)$$

mentioned above. $\square$
Proposition 4.10. Let $X$ be a Riemann surface of genus $1$ and suppose that $y^2 = 4x^3 - px - q =: f(x)$ is a Weierstrass equation for $X$. Write $f(x) = 4(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. Let $P_1 = (\alpha_1, 0), P_2 = (\alpha_2, 0)$ and $P_3 = (\alpha_3, 0)$. Then the formulas

\[ G(P_1, P_2)^{12} = \frac{16 \cdot |\alpha_1 - \alpha_2|^2}{|\alpha_1 - \alpha_3| \cdot |\alpha_2 - \alpha_3|}, \]

\[ G(P_1, P_3)^{12} = \frac{16 \cdot |\alpha_1 - \alpha_3|^2}{|\alpha_1 - \alpha_2| \cdot |\alpha_3 - \alpha_2|}, \]

\[ G(P_2, P_3)^{12} = \frac{16 \cdot |\alpha_2 - \alpha_3|^2}{|\alpha_2 - \alpha_1| \cdot |\alpha_3 - \alpha_1|}, \]

hold.

Proof. This follows directly from Lemma 4.9 and the explicit formula for $G(0, z)$ in Proposition 4.7. \qed

We remark that this proposition has been obtained by Szpiro in \cite{10} in the special case that $X$ is the Riemann surface associated to a Frey curve $y^2 = x(x + a)(x - b)$, where $a, b$ are non-zero integers with $2^4 | a$ and $b \equiv -1 \mod 4$ (cf. \cite{10}, Section 1.3).

5. Arakelov projection formula

In this section we prove a projection formula for Arakelov intersections on elliptic arithmetic surfaces. The essential idea is to use the analytic projection formula from Theorem 3.2; the rest of the proof is quite straightforward. We will use the Arakelov projection formula in Section 7.

Let $p : E \to B = \text{Spec}(O_K)$ be an arithmetic surface over the ring of integers $O_K$ of a number field $K$. Here and below we assume that $E$ is a regular scheme. As in \cite{11} we have on $E$ the notion of an Arakelov divisor: this is a formal sum of a Weil divisor $D_{\text{fin}}$ on $E$ and an infinite part $D_{\text{inf}} = \sum_{\sigma} \alpha_{\sigma} \cdot E_{\sigma}$, the sum running over the complex embeddings of $K$, with $\alpha_{\sigma}$ real numbers and with the $E_{\sigma}$ formal symbols corresponding to the Riemann surfaces associated to the curves $E \times_{K, \sigma} \mathbb{C}$. The Arakelov divisors form a group $\widehat{\text{Div}}(E)$. To each non-zero rational function $f \in K(E)$ one associates the corresponding Arakelov principal divisor $(f)$ with $(f)_{\text{fin}}$ the usual principal divisor associated to $f$, and with $\alpha_{\sigma}$ given by $\alpha_{\sigma} = -\int_{E_{\sigma}} \log |f|_{\sigma} \mu_{\sigma}$. Here $\mu_{\sigma}$ is the fundamental (1,1)-form on $E_{\sigma}$. We denote by $\widehat{\text{Cl}}(E)$ the group of Arakelov divisors on $E$ modulo the principal divisors. It was proved in \cite{11} that there exists a natural bilinear symmetric intersection product $(\cdot, \cdot)$ on the group of Arakelov divisors, factoring through the principal divisors to give a natural bilinear symmetric intersection pairing on $\widehat{\text{Cl}}(E)$. The definition of this intersection product is quite straightforward, except for the crucial case of the intersection $(P, Q)$ of two sections $P, Q : B \to E$ of $p$, which consists of a finite contribution $(P, Q)_{\text{fin}}$ given in the usual
way, and an infinite contribution \((P, Q)_{\text{inf}}\) given as a sum \(\sum_{\sigma}(P, Q)_{\sigma}\) with \((P, Q)_{\sigma} = -\log G_{\sigma}(P_{\sigma}, Q_{\sigma})\). Here \(G_{\sigma}\) the Arakelov-Green function on \(E_{\sigma}\).

Our Arakelov projection formula is a projection formula involving pushforwards and pullbacks of Arakelov divisors, which we define as follows.

**Definition 5.1.** Let \(p: E \to B\) and \(p': E' \to B\) be elliptic arithmetic surfaces, and suppose there exists a proper \(B\)-morphism \(f: E \to E'\). Let \(D\) be an Arakelov divisor on \(E\), and write \(D = D_{\text{fin}} + \sum_{\sigma} \alpha_{\sigma} \cdot E_{\sigma}\). The pushforward \(f_* D\) of \(D\) is defined to be the Arakelov divisor \(f_* D := f_* D_{\text{fin}} + d \cdot \sum_{\sigma} \alpha_{\sigma} \cdot E'_{\sigma}\) on \(E'\) where \(f_* D_{\text{fin}}\) is the usual pushforward of the Weil divisor \(D_{\text{fin}}\). Next let \(D'\) be an Arakelov divisor on \(E'\). The pullback \(f^* D'\) of \(D'\) is the Arakelov divisor \(f^* D' := f^* D'_{\text{fin}} + \sum_{\sigma} \alpha'_{\sigma} \cdot E_{\sigma}\) on \(E\) where \(f^* D'_{\text{fin}}\) is the pullback of the Weil divisor \(D'_{\text{fin}}\) on \(E'\), defined in the usual way using Cartier divisors.

Our result is then as follows.

**Theorem 5.2.** (Arakelov projection formula) Let \(E\) and \(E'\) be elliptic curves defined over a number field \(K\), and let \(p: E \to B\) and \(p': E' \to B\) be arithmetic surfaces over the ring of integers of \(K\) with generic fibers isomorphic to \(E\) and \(E'\), respectively. Suppose we have an isogeny \(f: E \to E'\), and suppose that \(f\) extends to a \(B\)-morphism \(f: E \to E'\). Let \(D\) be an Arakelov divisor on \(E\) and let \(D'\) be an Arakelov divisor on \(E'\). Then the equality of intersection products \((f^* D', D) = (D', f_* D)\) holds.

**Proof.** We may restrict to the case where both \(D\) and \(D'\) are Arakelov divisors with trivial contributions “at infinity”. By the moving lemma on \(E'\) (cf. [7], Corollary 9.1.10) we can find a function \(g \in K(E')\) such that \(D'' := D' + (g)_{\text{inf}}\) and \(f_* D\) have no components in common. Obviously \(D'' + (g)_{\text{inf}}\) is Arakelov linearly equivalent to \(D'\), and hence by a computation as in Theorem 9.2 the Arakelov divisor \(f^* D'' + (f^* g)_{\text{inf}}\) is Arakelov linearly equivalent to \(f^* D'\). It is therefore sufficient to prove that \((f^* D'' + (f^* g)_{\text{inf}}, D) = (D'' + (g)_{\text{inf}}, f_* D)\). It is clear that \(((f^* g)_{\text{inf}}, D) = ((g)_{\text{inf}}, f_* D)\), so it remains to prove that \((f^* D'', D) = (D'', f_* D)\). By the traditional projection formula (cf. [7], Theorem 9.2.12 and Remark 9.2.13) we have \((f^* D'', D)_{\text{fin}} = (D'', f_* D)_{\text{fin}}\). For the contributions at infinity we can reduce to the case where \(D\) and \(D''\) are sections of \(E \to B\) and \(E' \to B\), respectively. Let \(\sigma\) be a complex embedding of \(K\). Let \(D_{\sigma}\) and \(D''_{\sigma}\) be the points corresponding to \(D\) and \(D''\) on \(E_{\sigma}\) and \(E'_{\sigma}\). Then for the local intersection at \(\sigma\) we have \((f^* D'', D)_{\sigma} = (D'', f_* D)_{\sigma}\) by the analytic projection formula from Proposition 9.2. The theorem follows.

**Remark 5.3.** In general, an isogeny \(f: E \to E'\) may not extend to a morphism \(E \to E'\). However, if \(E'\) is a minimal arithmetic surface (cf. [7], Section 9.3.2), then it contains the Néron model of \(E'/K\), and hence by the universal property of the Néron model, any isogeny \(f\) extends. In any case we can achieve that \(f\) extends after blowing up finitely many closed points on \(E\).
The following corollary appears in Szpiro’s paper [10].

**Corollary 5.4.** (Szpiro [10]) Let $D_1, D_2$ be Arakelov divisors on $\mathcal{E}'$. Let $N$ be the degree of $f$. Then the formula

$$ (f^*D_1, f^*D_2) = N \cdot (D_1, D_2) $$

holds.

**Proof.** It is not difficult to see (cf. [7], Theorem 7.2.18 and Proposition 9.2.11) that $f_*f^*D_2 = N \cdot D_2$. Theorem 5.2 then gives $(f^*D_1, f^*D_2) = (D_1, f_*f^*D_2) = (D_1, N \cdot D_2)$. □

6. SELF-INTERSECTION OF A POINT

Let $p : \mathcal{E} \to B$ be an elliptic arithmetic surface. The image of a section $P : B \to \mathcal{E}$ gives rise to a divisor on $\mathcal{E}$, which we also denote by $P$. Given the framework of arithmetic intersection theory, it is natural to ask for the self-intersection $(P, P)$ of $P$. The question has been solved in the case that $P$ is the zero section by Szpiro.

**Theorem 6.1.** (Szpiro [10]) Let $E$ be a semi-stable elliptic curve over a number field $K$, and let $p : \mathcal{E} \to B$ be its regular minimal model over the ring of integers of $K$. Let $O : B \to \mathcal{E}$ be the zero section of $p$, and denote by $\Delta(E/K)$ the minimal discriminant ideal of $E/K$. Then the formula

$$ (O, O) = -\frac{1}{12} \log |N_{K/Q}(\Delta(E/K))| $$

holds.

**Proof.** By Proposition 2.3 we need to compute the Arakelov degree $\widehat{\deg} O^*\omega_{\mathcal{E}/B}$, with $\omega_{\mathcal{E}/B}$ the relative dualising sheaf of $p : \mathcal{E} \to B$. It is well-known that there exists a canonical isomorphism $O^*\omega_{\mathcal{E}/B} \sim \pi^*\omega_{\mathcal{E}/B}$. Now the line bundle $(\pi^*\omega_{\mathcal{E}/B})^{\otimes 12}$ contains a canonical global section $\Lambda_{E/B}$ coming from the canonical isomorphism $(\pi_*\omega)^{\otimes 12} \sim O(\Delta)$ on the moduli stack of stable elliptic curves, with $\Delta$ the discriminant locus. Considering then the canonical section $\Lambda_{E/B}$ in $(O^*\omega_{\mathcal{E}/B})^{\otimes 12}$ we compute its norm. First of all, the finite places yield a contribution $\log |N_{K/Q}(\Delta(E/K))|$. Next consider a complex embedding $\sigma$ of $K$. Suppose we have an identification $E_{\sigma} \cong \mathbb{C}/\mathbb{Z} + \tau_{\sigma}\mathbb{Z}$ with $\tau_{\sigma}$ in the complex upper half plane. Let $y^2 = 4x^3 - g_2\sigma x - g_3\sigma$ be the associated Weierstrass equation, where $x = \wp_{\sigma}(z)$ and $y = \wp'_{\sigma}(z)$, with $\wp_{\sigma}$ the Weierstrass $\wp$-function associated to the lattice $\mathbb{Z} + \tau_{\sigma}\mathbb{Z}$. We then have $\Lambda_{\sigma} = D_{\sigma} \cdot (dx/y)^{\otimes 12}$ where $D_{\sigma}$ is the discriminant of the above Weierstrass equation. Moreover $dx/y$ is identified with $dz$. We can now compute $\|\Lambda_{\sigma}\|_{Ar}$ as follows: first by Lemma 1.9 we have $D_{\sigma} = (2\pi)^{12} \cdot \Delta(\tau_{\sigma})$, and second by Proposition 1.8 we have $\|dz\|_{Ar} = \sqrt{\text{Im} \tau/(2\pi)} \cdot \|y\|(E_{\sigma})^2)$. We obtain that $\|\Lambda_{\sigma}\|_{Ar} = 1$ and hence the infinite contributions vanish. This gives the proposition. □
The proof given in [10] is much more involved. The above proof in fact answers a question raised in [10] on the norm $\|\Lambda\|_{Ar}$ of $\Lambda$ in $\Omega^{\otimes 12}$.

The following proposition shows that Theorem 6.1 in fact gives the general answer to our question.

**Proposition 6.2.** Let $E$ be an elliptic curve over a number field $K$, and let $p : E \to B$ be the regular minimal model of $E$ over the ring of integers of $K$. Let $O : B \to E$ be the zero-section. Then for any section $P : B \to E$ of $E \to B$ we have $(P, P) = (O, O)$.

For the proof, we make use of the following lemma.

**Lemma 6.3.** Let $E$ be an elliptic curve over a number field $K$, and let $p : E \to B$ be the regular minimal model of $E$ over the ring of integers of $K$. Let $\omega_{E/B}$ be the relative dualising sheaf of $p : E \to B$. Then we can write $\omega_{E/B} = \sum b \lambda_b E_b + \sum_\sigma \alpha_\sigma F_\sigma$ as Arakelov divisors on $E$, the first sum running over the closed points $b$ of $B$, with $E_b$ denoting the fiber at $b$ and with $\lambda_b$ certain rational numbers; the second sum runs over the complex embeddings of $K$, with $\alpha_\sigma$ certain real numbers.

**Proof.** Since $\omega_{E/B}$ restricts to the trivial sheaf on the generic fiber there exists a vertical divisor $V$ on $\mathcal{E}$ such that $\omega_{E/B} \cong O_E(V)$ as invertible sheaves. Since $\mathcal{E}$ is minimal, the divisor $V$ is numerically effective (cf. [7], Corollary 9.3.26), which implies $(V, C) \geq 0$ for every irreducible component $C$ of a closed fiber. But also by the adjunction formula in the vertical fibers (cf. [7], Section 9.1.3) we have $(V, E_b) = 2p_a(E) - 2 = 0$ for each closed fiber $E_b$ of $E$, so in fact $(V, C) = 0$ for each $C$. Since the kernel of the intersection product is generated by the multiples of the fibers, this implies that $V = \sum b \lambda_b \cdot E_b$, where the $\lambda_b$ are certain rational numbers. The lemma follows immediately from this. □

**Proof of Proposition 6.2.** The adjunction formula Proposition 2.5 shows that we need to prove that $\deg P^*\omega_{E/B} = \deg O^*\omega_{E/B}$. But this is immediate from Lemma 6.3 □

Note that Lemma 6.3 also proves that $(\omega_{E/B}, \omega_{E/B}) = 0$ on a minimal elliptic arithmetic surface $p : E \to B$, a fact observed by Faltings in [6] in the case of a semi-stable elliptic arithmetic surface.

### 7. Average Height of Quotients

In this final section we study the average height of quotients of an elliptic curve by its cyclic subgroups of fixed order. Using our results from the previous sections, we give an alternative proof of a formula due to Autissier [2]. A slightly less general result appears already in [11], and in fact our method is very much in the spirit of this latter paper. The main difference is perhaps that in our approach we do not need to consider the distribution of torsion points on the bad fibers. In fact we do not need any non-trivial
arithmetic information at all; the main ingredients are the Arakelov projection formula from Theorem 5.2, the formula for the “energy of an isogeny” from Theorem 4.2, and the formula for the self-intersection of a point from Theorem 6.1. Amusingly, we shall mention at the end of this section how a purely arithmetic result, namely the injectivity of torsion, follows from our Arakelov-theoretic results.

We start with an explicit formula for $h_F(E)$. This formula is certainly well-known, cf. [9], Proposition 1.1.

**Proposition 7.1.** Let $E$ be a semi-stable elliptic curve over a number field $K$. Let $\Delta(E/K)$ be the minimal discriminant ideal of $E/K$. Then the formula

$$h_F(E) = \frac{1}{[K : \mathbb{Q}]} \left( \frac{1}{12} \log |N_{K/\mathbb{Q}}(\Delta(E/K))| - \frac{1}{12} \sum_{\sigma} \log((2\pi)^{12} \| \Delta \|(E_{\sigma})) \right)$$

holds. Here the sum runs over the complex embeddings of $K$.

**Proof.** As is explained in the proof of Theorem 6.1, the line bundle $(p_* \omega_{E/B})^{\otimes 12}$ contains a canonical section $\Lambda_{E/B}$, which has divisor given by $\Delta(E/K)$ on $B$. This accounts for the finite contribution $|N_{K/\mathbb{Q}}(\Delta(E/K))|$. Next, at a complex embedding $\sigma$ of $K$ we have $\Lambda_{E_{\sigma}} = D_{E_{\sigma}} \cdot (dx/y)^{\otimes 12}$ where $D_{E_{\sigma}}$ is the discriminant of a Weierstrass equation $y^2 = 4x^3 - p_{E_{\sigma}}x - q_{E_{\sigma}}$ associated to $E_{\sigma}$. Let $(\omega_{1\sigma} | \omega_{2\sigma})$ be a period matrix of $dx/y$ on a canonical symplectic basis associated to an ordering of the roots of $f$, and let $\tau_{\sigma} = \omega_{2\sigma}/\omega_{1\sigma}$. By Lemma 4.9 we have $D_{E_{\sigma}} = (2\pi)^{12} \omega_{1\sigma}^{-12} \cdot \Delta(\tau_{\sigma})$, and by Riemann’s second bilinear relations we have $\| dx/y \|_{\sigma}^2 = |\omega_{1\sigma}|^2 \cdot \text{Im} \tau$. Together this yields $\| \Lambda \|_{\sigma} = (2\pi)^{12} \cdot \| \Delta \|(E_{\sigma})$. This gives the infinite contribution to $h_F(E)$. \qed

Now let’s turn to the result of Autissier. First we introduce some notations. Let $N$ be a positive integer. Then we denote by $e_N$ the number of cyclic subgroups of order $N$ on an elliptic curve defined over $\mathbb{C}$, i.e.

$$e_N := N \prod_{p|N} \left( 1 + \frac{1}{p} \right),$$

where the product is over the primes dividing $N$. Further we put

$$\lambda_N := \sum_{p|N} \frac{p^r - 1}{p^{r+1} - 1} \log p,$$

where the notation $p^r|N$ means that $p^r | N$ and $p^{r+1} \nmid N$. For an elliptic curve $E$ and a finite subgroup $C$ of $E$ we denote by $E^C$ the quotient of $E$ by $C$.

In [11] we find the following theorem.

**Theorem 7.2.** (Szpiro-Ullmo, [11]) Let $E$ be a semi-stable elliptic curve defined over a number field $K$. Suppose that $E$ has no complex multiplication over $\overline{K}$ and that the
absolute Galois group $\text{Gal}(\overline{K}/K)$ acts transitively on the points of order $N$ on $E$. Let $C$ be a cyclic subgroup of order $N$ on $E$. Then the formula
\[ h_F(E^C) = h_F(E) + \frac{1}{2} \log N - \lambda_N \]
holds.

One may wonder what one can say without the assumption that $\text{Gal}(\overline{K}/K)$ acts transitively. In [2] we find a proof of the following statement. The price we pay for dropping the assumption is that we can only deal with the average over all $C$.

**Theorem 7.3.** (Autissier [2]) Let $E$ be an elliptic curve defined over a number field $K$. Then the formula
\[ \frac{1}{e_N} \sum_C h_F(E^C) = h_F(E) + \frac{1}{2} \log N - \lambda_N \]
holds, where the sum runs over the cyclic subgroups of $E$ of order $N$.

In fact, this formula was already stated in [11] under the restriction that $N$ is squarefree. Autissier’s proof uses the Hecke correspondence $T_N$ and a generalised intersection theory for higher-dimensional arithmetic varieties. The disadvantage of this approach is that the analytic machinery needed to deal with the contributions at infinity becomes quite complicated. We will give a proof of Theorem 7.3 which is much more elementary. Besides this merit, we also think that the structure of the somewhat strange constant $\lambda_N$ becomes more clear through our approach. It would be interesting to have a generalisation of Theorem 7.3 to abelian varieties of higher dimension.

Theorem 7.3 follows directly from the following two propositions, by using the explicit formula for $h_F$ in Proposition 7.1. The next proposition occurs as Lemme 5.4 in [11].

**Proposition 7.4.** Let $E$ be a semi-stable elliptic curve over a number field $K$ and suppose that all $N$-torsion points are $K$-rational. Then one has
\[ \sum_C (\log |N_{K/Q}(\Delta(E/K))| - \log |N_{K/Q}(\Delta(E^C/K))|) = 0. \]
Here the sum runs over the cyclic subgroups of $E$ of order $N$.

**Proposition 7.5.** Let $X$ be a Riemann surface of genus 1. Then
\[ \frac{1}{e_N} \sum_C \left( \frac{1}{12} \log \|\Delta\|(X) - \frac{1}{12} \log \|\Delta\|(X^C) \right) = \frac{1}{2} \log N - \lambda_N, \]
where the sum runs over the cyclic subgroups of $X$ of order $N$.

Our first step is to reduce these two propositions to the following two:
Proposition 7.6. Let $E$ be a semi-stable elliptic curve over a number field $K$ and suppose that all $N$-torsion points are $K$-rational. Extend all $N$-torsion points of $E$ over the regular minimal model of $E/K$. Then one has

$$
\sum_{C} \sum_{Q \neq O} (Q, O) = 0,
$$

where the first sum runs over the cyclic subgroups of $E$ of order $N$, and the second sum runs over the non-zero points in $C$.

Proposition 7.7. Let $X$ be a Riemann surface of genus 1. Then one has

$$
\frac{1}{e_N} \sum_{C} \sum_{Q \neq O} \log G(Q, 0) = \lambda_N.
$$

Here the first sum runs over the cyclic subgroups of $X$ of order $N$, and the second sum runs over the non-zero points in $C$.

The latter proposition is an improvement of Proposition 6.5 in [11], which gives an analogous statement, but only with the left-hand side summed over the complex embeddings of $K$, and divided by $[K : \mathbb{Q}]$. Our result holds in full generality for an arbitrary Riemann surface of genus 1.

Proof of Proposition 7.6 from Proposition 7.7. Let $C$ be any cyclic subgroup of $E$, and let $O'$ be the zero-section of $E^C$. Extend it over the minimal regular model of $E^C/K$. We then have

$$
\frac{1}{12} \log |N_{K/Q}(\Delta(E/K))| - \frac{1}{12} \log |N_{K/Q}(\Delta(E^C/K))| = (O', O') - (O, O)
$$

by Theorem 6.1. The latter is equal to $\sum_{Q \neq O} (Q, O)$ by Theorem 5.2. Summing over all cyclic subgroups of $E$ of order $N$ and using Proposition 7.6 we find the result. \hfill \Box

Proof of Proposition 7.7 from Proposition 7.5. By Theorem 4.2 we have for any subgroup $C$ of $X$ of order $N$ that

$$
\frac{1}{12} \log \|\Delta\| (X) - \frac{1}{12} \log \|\Delta\| (X^C) = \frac{1}{2} \log N - \sum_{Q \neq 0} \log G(Q, 0).
$$

The statement of Proposition 7.7 is then immediate from Proposition 7.5. \hfill \Box

In order to prove Proposition 7.6 we make use of the following combinatorial lemma.

Lemma 7.8. Let $M$ be a positive integer with $M|N$. Let $E$ be an elliptic curve defined over an algebraically closed field of characteristic zero. Then each cyclic subgroup of $E$ of order $M$ is contained in exactly $e_N/e_M$ cyclic subgroups of order $N$. 

Proof. This follows easily by fixing a basis for the \( N \)-torsion and then considering the induced natural transitive left action of \( \text{SL}(2, \mathbb{Z}) \) on the set of cyclic subgroups of order \( M \) and of order \( N \).
\[ \Box \]

We may argue then as follows.

Proof of Proposition 7.6. Let \( E[M] \) be the set of points of exact order \( M \) on \( E \). By Lemma 7.8 we have
\[
\sum_{C} \sum_{Q \in C \setminus \{O\}} (Q, O) = \sum_{M \mid N} \frac{e_N}{e_M} \sum_{Q \in E[M]} (Q, O).
\]

We claim that for any positive integer \( M \), we have \( \sum_{Q \in E[M]} (Q, O) = 0 \). Indeed, we have
\[
\sum_{Q \in E[M], Q \neq O} (Q, O) = 0
\]
for all \( M \) by Theorem 5.2 and then the claim follows by Möbius inversion.
\[ \Box \]

Also for the proof of Proposition 7.7 we will need a lemma. For a Riemann surface \( X \) of genus 1, and \( M > 1 \) an integer, we put
\[
t(M) := \sum_{Q \in X[M]} \log G(Q, 0),
\]
the sum running over the set \( X[M] \) of points of exact order \( M \) on \( X \).

Part of the following lemma is also given in [11], cf. Lemme 6.2.

Lemma 7.9. We have
\[
t(p^r) = \log p
\]
for any prime integer \( p \) and any positive integer \( r \). Moreover we have \( t(M) = 0 \) for any positive integer \( M \) which is not a prime power.

Proof. By Corollary 4.5 we have
\[
\sum_{Q \in X[M], Q \neq 0} \log G(Q, 0) = \log M.
\]
The lemma follows from this by Möbius inversion.
\[ \Box \]

Proof of Proposition 7.7. For any divisor \( M \mid N \), let \( X[M] \) be the set of points of exact order \( M \) on \( X \) and let \( t(M) = \sum_{Q \in X[M]} \log G(Q, 0) \) as in Lemma 7.9 where it is understood that \( t(1) = 0 \). Then by Lemma 7.8 we can write
\[
\frac{1}{e_N} \sum_{C} \sum_{Q \in C \setminus \{O\}} \log G(Q, 0) = \frac{1}{e_N} \sum_{M \mid N} \frac{e_N}{e_M} \cdot t(M).
\]
Lemma 7.9 gives us that

\[ \frac{1}{e_{p^k}} \sum_{M | N} \frac{e_N}{e_M} \cdot t(M) = \sum_{p \mid N} \left( \frac{1}{e_p} + \cdots + \frac{1}{e_p^r} \right) \log p. \]

Finally note that \( e_{p^k} = p^k(1 + 1/p) \) which gives

\[ \frac{1}{e_p} + \cdots + \frac{1}{e_p^r} = \frac{p^r - 1}{p^r - 1(p^2 - 1)}. \]

From this the result follows. \( \square \)

Remark 7.10. An alternative proof of Proposition 7.5 can be given by classical methods using modular forms identities, see for instance [3], Proposition VII.3.5(b) for the case that \( N \) is a prime, and [2], Lemme 2.2 and Lemme 2.3 for the general case. We preferred to give an argument using Arakelov theory, indicating that Arakelov theory can sometimes be used to derive analytic results on Riemann surfaces in a short and clean manner.

We finish with a corollary from the results above. This corollary gives another interpretation to the constant \( \lambda_N \).

Corollary 7.11. Let \( E \) be a semi-stable elliptic curve over a number field \( K \) and suppose that all \( N \)-torsion points are \( K \)-rational. Extend these torsion points over the minimal regular model of \( E/K \). Then one has

\[ \frac{1}{[K : \mathbb{Q}]} e_N \sum_C \sum_{Q \in C} (Q, O)_{\text{fin}} = \lambda_N, \]

where the first sum runs over the cyclic subgroups of \( E \) of order \( N \), and the second sum runs over the non-zero points in \( C \).

Proof. Let \( C \) be a finite cyclic subgroup of \( E \). Note that by definition of the Arakelov intersection product

\[ \sum_{Q \in C, Q \neq O} (Q, O) = \sum_{Q \in C, Q \neq O} (Q, O)_{\text{fin}} - \sum_{Q \in C, Q \neq O} \log G(Q^\sigma, 0). \]

The corollary follows therefore easily from Proposition 7.9 and Proposition 7.7. \( \square \)

Note that Corollary 7.11 is purely arithmetical in nature. It should also be possible to give a direct proof, but probably this would require a more ad hoc approach, making for instance a case distinction between the supersingular and the non-supersingular primes for \( E/K \). Also note that Corollary 7.11 immediately gives the classical arithmetic result that, for any prime number \( p \), the \( p \)-torsion points are injective on a fiber at a prime of characteristic different from \( p \). Indeed, take \( N = p \) in the formula from Corollary 7.11 then the right hand side is a rational multiple of \( \log p \), and so the same holds for the
left hand side. This means that the local intersections \((Q,O)_{\text{fin}}\), which are always non-negative, are in fact zero at primes of characteristic different from \(p\). Hence, each \(p\)-torsion point \(Q\) stays away from \(O\) on fibers above such primes. Of course the argument can be repeated with \(O\) replaced by any other \(p\)-torsion point.

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