Chiral gauge theories on the lattice with exact gauge invariance

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A recently proposed formulation of chiral lattice gauge theories is reviewed, in which the locality and gauge invariance of the theory can be preserved if the fermion representation of the gauge group is anomaly-free.

1. INTRODUCTION

At the quantum level chiral gauge theories have always been relatively difficult to treat, because of the gauge anomaly and the related fact that there seemed to be no way to regularize these theories without breaking the gauge symmetry. The famous no-go theorem of Nielsen and Ninomiya [1,2] was often taken as a proof of this and many people working on the subject suspected that the problem might reflect a fundamental limitation of quantum field theory. As a consequence it remained unclear whether anomaly-free chiral gauge theories can be defined consistently beyond perturbation theory. The existence of global anomalies [3,4] in fact suggests that the answer might be “no” in some cases.

In this review we consider left-handed fermions, $P^- \psi = \psi$, $\overline{\psi} P^+ = \overline{\psi}$, $P_{\pm} \equiv \frac{1}{2} (1 \pm \gamma_5)$, transforming according to some unitary representation $R$ of the gauge group. The action density of the continuum theory is assumed to be of the standard form

$$\mathcal{L}(x) = \frac{1}{4g^2} F_{\mu \nu}^a(x) F_{\mu \nu}^a(x) + \overline{\psi}(x) \gamma_\mu D_\mu \psi(x),$$

where $F_{\mu \nu}^a$ denotes the gauge field tensor and $D_\mu$ the appropriate covariant derivative. The gauge anomaly in this theory is proportional to

$$d^{abc}_R \equiv 2i \text{tr} \{ R(T^a) R(T^b) R(T^c) + (b \leftrightarrow c) \}$$

with $R(T^a)$ the anti-hermitian generators of the fermion representation of the gauge group. If this tensor vanishes (i.e. if one has chosen an anomaly-free fermion multiplet), the uniqueness and gauge invariance of the theory can be proved to all orders of perturbation theory.

On the lattice one would like to formulate these theories in such a way that

(a) one has the right number and type of fermions from the beginning,
(b) locality is preserved,
(c) gauge invariance is unbroken,
(d) the perturbation expansion has the conventional form.

The locality of the lattice theory is essential, since the universality of the continuum limit depends on it. Exact gauge invariance is also very important as otherwise the gauge degrees of freedom are not guaranteed to decouple from the physical sector of the theory. Many attempts to put chiral gauge theories on the lattice in fact failed for this reason [5]. Requirement (d) is included in the list, because perturbation theory is currently the only framework where one has analytical control over the continuum limit [6,7].

Until recently it seemed to be impossible to fulfil all these conditions simultaneously. The situation has now changed completely as a result of the developments following the rediscovery of the Ginsparg–Wilson relation [8],

$$\gamma_5 D + D \gamma_5 = aD \gamma_5 D,$$

and the construction of gauge-covariant lattice Dirac operators $D$ satisfying this equation [9,10] (here and below $a$ denotes the lattice spacing).
It is striking that two totally different lines of research produced such Dirac operators at about the same time. One of them goes under the heading of the perfect-action approach to asymptotically free lattice field theories [11] (see ref. [12] for a review and further references). The Dirac operators that are obtained in this framework provide particular solutions of eq. (4), with good localization properties and small cutoff dependence [9].

The other theoretical development started with Kaplan’s observation [13] that the chiral nature of fermions bound to a 4-dimensional defect in 4+1 dimensions is preserved on the lattice. The overlap representation of the fermion determinant [14,15] derives from this and also the domain-wall fermion formulation of lattice QCD [16,17]. In both cases an effective Dirac operator can be extracted [10,18], which turns out to satisfy the Ginsparg–Wilson relation.

The significance of eq. (4) has only been fully appreciated last year. A key step has been to note that the zero-modes of any such Dirac operator are chiral and that the associated index is a topological invariant which represents the Chern character on the lattice [19]. Shortly after this the relation was shown to imply an exact chiral symmetry [20] with the correct flavour-singlet anomaly [19,21–27]. Left- and right-handed fermions are then easily introduced [28–31] and (after a lot more work) this has now led to a formulation of abelian chiral gauge theories on the lattice, which complies with all the basic requirements including exact gauge invariance [31]. The construction is completely general and extends to any gauge group [32], but there are still a few loose ends in the non-abelian case.

In the following we focus on the most recent advances in this field. Particular attention will be paid to the cancellation of the gauge anomaly on the lattice. Local cohomology plays an important rôle here and the well-known results on the structure of the anomaly in continuum chiral gauge theories [33–38] turn out to be very useful at this point (for a review and an extensive list of references see refs. [39,40]). There are also intriguing links [41,42] to the earlier work of Alvarez-Gaumé et al. [43–45] and of Ball and Osborn [46,47] relating the chiral determinant to the \( \eta \)-invariant of the Dirac operator in 4+1 dimensions, but this topic will not be addressed here.

### 2. WEYL FERMIONS

Once it had been understood how to preserve chiral symmetry on the lattice without having to compromise in other ways, the projection to chiral fermions turned out to be straightforward [28–31]. One begins by choosing any gauge-covariant lattice Dirac operator \( D \) which

(a) satisfies the Ginsparg–Wilson relation and the hermiticity condition \( D^\dagger = \gamma_5 D \gamma_5 \),

(b) respects the lattice symmetries,

(c) has the correct behaviour in the free fermion limit,

(d) is local and smoothly dependent on the gauge field.

Neuberger’s operator [10] (with the link variables \( U(x, \mu) \) replaced by \( R[U(x, \mu)] \) to ensure the correct gauge transformation behaviour) provides an example of such a lattice Dirac operator. In this case property (d) is rigorously guaranteed if one assumes that the gauge field satisfies

\[
||1 - R[U(p)]|| < \epsilon \tag{5}
\]

for all plaquettes \( p \), where \( U(p) \) denotes the product of the link variables around \( p \) and \( \epsilon \) any fixed positive number smaller than \( \frac{1}{30} \). Usually not all statistically relevant fields are of this type, but this can be enforced through a simple modification of the Wilson plaquette action [31]. As far as the continuum limit in the weak coupling phase is concerned, such actions are perfectly acceptable, because the bound \( \frac{1}{\lambda} \) constrains the gauge field fluctuations at the scale of the cutoff only and does not violate the locality or the gauge invariance of the theory.

Chiral fields may now be defined as follows. One first observes that the operator

\[
\hat{\gamma}_5 = \gamma_5 (1 - aD) \tag{6}
\]

satisfies

\[
(\hat{\gamma}_5)^\dagger = \hat{\gamma}_5, \quad (\hat{\gamma}_5)^2 = 1, \quad D\hat{\gamma}_5 = -\gamma_5 D. \tag{7}
\]
The fermion action
\[ S_F = a^4 \sum_x \bar{\psi}(x) D\psi(x) \] (8)

thus splits into left- and right-handed parts if the chiral projectors for fermion and antifermion fields are defined through
\[ \hat{P}_\pm = \frac{1}{2}(1 \pm \gamma_5), \quad P_\pm = \frac{1}{2}(1 \pm \gamma_5), \] (9)
respectively. In particular, by imposing the constraints
\[ \hat{P}_- \psi = \psi, \quad \bar{\psi} P_\pm = \bar{\psi}, \] (10)
the right-handed components are eliminated and one obtains a classical lattice theory where a multiplet of left-handed Weyl fermions couples to the gauge field in a consistent way.

3. FERMION MEASURE

To define the theory beyond the classical level one also needs to specify the functional integration measure. For the gauge field one can take the standard measure, but the definition of the measure for left-handed fermions turns out to be non-trivial, because the projector \( \hat{P}_- \) depends on the gauge field.

To make this clear, let us suppose that \( v_j(x), \) \( j = 1, 2, 3, \ldots, \) is a basis of complex-valued lattice Dirac fields such that
\[ \hat{P}_- v_j = v_j, \quad (v_k, v_j) = \delta_{kj}, \] (11)
the bracket being the obvious scalar product for such fields. The quantum field may then be expanded according to
\[ \psi(x) = \sum_j v_j(x) c_j, \] (12)
where the coefficients \( c_j \) generate a Grassmann algebra. They represent the independent degrees of freedom of the field and an integration measure for left-handed fermion fields is thus given by
\[ D[\psi] = \prod_j dc_j. \] (13)

Evidently if we pass to a different basis
\[ \tilde{v}_j(x) = \sum_i v_i(x)(Q^{-1})_ij, \quad \tilde{c}_j = \sum_i Q_{ij}c_i, \] (14)
the measure changes by the factor \( \det Q \), which is a pure phase factor since \( Q \) is unitary.

It follows from this that any two bases \( v_j \) and \( \tilde{v}_j \), which are related to each other by a unimodular transformation, define the same measure. Specifying an integration measure for left-handed fermions thus amounts to choosing a basis \( v_j \) modulo such transformations. Moreover, once a particular choice has been made, any other measure is obtained by multiplication with a phase factor. An important point to understand here is that the measure implicitly depends on the gauge field through the basis vectors \( v_j \). In general the phase ambiguity is hence gauge-field-dependent too and does not cancel in expectation values.

In the case of the antifermion fields the subspace of left-handed fields is independent of the gauge field and one can take the same orthonormal basis \( \bar{v}_k(x) \) for all gauge fields. The ambiguity in the integration measure
\[ D[\bar{\psi}] = \prod_k d\bar{c}_k, \quad \bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x), \] (15)
is then only a constant phase factor.

The fermion integral of any product \( \mathcal{O} \) of the basic fields may now be defined through
\[ \langle \mathcal{O} \rangle_F = \int D[\psi] D[\bar{\psi}] \mathcal{O} e^{-S_F}. \] (16)

In particular, if \( D \) has no zero-modes, the fermion two-point function is given by
\[ \langle \psi(x) \bar{\psi}(y) \rangle_F = \langle 1 \rangle_F \times \hat{P}_- S(x, y) P_+, \] (17)
where \( S(x, y) \) denotes the Green function of \( D \). The integral of any other product of fields is then obtained by applying Wick’s theorem. Note that only the left-handed components of the fermion field propagate (as it should be).

To obtain the full correlation functions one finally has to integrate over the gauge field variables as usual. The definition of the theory is then complete apart from the fact that the phase of the fermion measure has not been fixed. In the following we shall almost exclusively be concerned with this problem. We first determine which properties the measure must have in order to comply with the basic principles and then address the question of whether such measures exist.
4. LOCALITY & GAUGE INVARIANCE

The locality properties of a euclidean field theory are usually obvious from the classical action. Although the action and the chiral projectors are local, the situation is more complicated here, because the fermion measure is not a simple product of local measures.

To gain some insight into this problem, let us consider the fermion partition function

$$\langle 1 \rangle_F = \det M, \quad M_{kj} = a^4 \sum_x \bar{v}_k(x) Dv_j(x), \quad (18)$$

in the vacuum sector. If we vary the link variables $U(x, \mu)$ in some direction $\eta_\mu(x)$,

$$\delta_\eta U(x, \mu) = a \eta_\mu(x) U(x, \mu), \quad (19)$$

the effective action changes according to

$$\delta_\eta \ln \det M = \text{Tr} \left\{ \delta_\eta \hat{P}_- D^{-1} P_+ \right\} - i \mathfrak{L}_\eta. \quad (20)$$

The first term in this equation is the naively expected one while the second,

$$\mathfrak{L}_\eta = i \sum_j (v_j, \delta_\eta v_j), \quad (21)$$

arises from the fermion integration measure. $\mathfrak{L}_\eta$ is linear in the variation $\eta_\mu(x)$ and a current $j_\mu(x)$ may thus be introduced through

$$\mathfrak{L}_\eta = a^4 \sum_x \eta_\mu^a(x) j_\mu^a(x). \quad (22)$$

Note that $j_\mu(x)$ is a well-defined expression in the gauge field once the fermion integration measure has been specified.

On the level of the effective action, the implicit dependence of the measure on the gauge field thus generates additional gauge field vertices. The current $j_\mu(x)$ also appears in the classical action of the fermion field as a second term besides the current derived from the classical action. To preserve the locality of the theory the fermion measure should hence be chosen so that the associated current $j_\mu(x)$ is a local field.

In eq. (20) the measure term then assumes the form of a local counterterm and the field equations become relations between local operator insertions as usual.

For gauge variations $\eta_\mu(x) = -\nabla_\mu \omega(x)$, the change of the effective action is given by

$$\delta_\eta \ln \det M =$$

$$ia^4 \sum_x \omega^a(x) \left\{ A^a(x) - [\nabla_\mu j_\mu]^a(x) \right\}, \quad (23)$$

$$A^a(x) = \frac{ia}{2} \text{tr} \left\{ \gamma_5 R(T^a) D(x, y) \right\}. \quad (24)$$

In these equations $\nabla_\mu$ and $\nabla^*_\mu$ denote the gauge-covariant forward and backward lattice derivatives and $D(x, y)$ the kernel of the Dirac operator in position space.

From the definition (24) and the properties of the Dirac operator it is obvious that $A(x)$ is a gauge-covariant local expression in the gauge field. It can easily be worked out in the classical continuum limit, where the link variables are assumed to be given by a smooth background gauge potential $A_\mu(x)$ through the usual path-ordered exponentials. As a result one obtains

$$A^a(x) = c_1 a^{abc} \epsilon_{\mu\nu\rho\sigma} F^b_{\mu\nu}(x) F^c_{\rho\sigma}(x) + O(a), \quad (25)$$

with $c_1 = -1/128 \pi^2 \langle 1 \rangle_F \langle 2 \rangle_F \langle 3 \rangle_F$, and $A(x)$ thus represents the covariant anomaly on the lattice.

Now if we require that the gauge symmetry be unbroken, the fermion determinant should be gauge-invariant and from the above we then conclude that

$$[\nabla_\mu j_\mu]^a(x) = A^a(x). \quad (26)$$

Moreover, recalling eq. (24), it is obvious that the current $j_\mu(x)$ has to be a gauge-covariant expression in the gauge field. The converse is also true, i.e. any fermion integration measure that yields a current with these properties preserves the gauge symmetry $\langle 1 \rangle_F \langle 2 \rangle_F \langle 3 \rangle_F$.

As is well known one cannot have both locality and gauge invariance, unless the anomaly cancellation condition

$$d^{abc}_R = 0 \quad (27)$$

is fulfilled. In the present framework this may be shown by noting that eq. (24) has no solution in the classical continuum limit if the anomaly does not vanish at $a = 0$. 

5. INTEGRABILITY CONDITION

So far we have assumed that the current $j_\mu(x)$ is obtained from a given fermion integration measure through eqs. (\ref{eq:21}), (\ref{eq:23}). The question may now be asked whether one could also start with a prescribed current and derive the measure from it. As explained below this is indeed the case if the current satisfies a certain integrability condition. The construction of the lattice theory is thus simplified, because the measure (which is a relatively complicated object) no longer needs to be specified explicitly.

To derive the integrability condition, let us consider a smooth curve

$$U_t(x, \mu), \quad 0 \leq t \leq 1,$$

in field space. The change of the effective action along this curve is given by

$$\partial_t \ln \det M = \text{Tr} \left\{ \partial_t \hat{D} \hat{P} \cdot D^{-1} \hat{P} \right\} - i \mathcal{L}_\eta,$$

where

$$a \eta_\mu(x) = \partial_t U_t(x, \mu) U_t(x, \mu)^{-1}$$

(30)

(for simplicity the $t$-dependence of $\eta_\mu(x)$ is suppressed). At $t = 1$ the solution of this equation yields

$$\det M \det M_0^\dagger = \det \{1 - P_+ + P_+ DQ_1 D_0^\dagger \} W^{-1},$$

(31)

where

$$W = \exp \left\{ i \int_0^1 dt \mathcal{L}_\eta \right\}$$

(32)

is the total change of phase of the fermion measure and the operator $Q_t$ is defined through

$$\partial_t Q_t = [\partial_t P_t, P_t] Q_t, \quad P_t = \hat{P}_0 U_t,$$

(33)

with initial value $Q_0 = 1$. $Q_t$ is unitary and satisfies $P_t Q_t = Q_t P_0$, i.e. it maps left-handed fields at $U = U_0$ to left-handed fields at $U = U_t$.

We now note that the left-hand side of eq. (32) only depends on the end-points of the chosen path in field space. On the other side of the equation, a closer examination shows this to be a consequence of the fact that

$$W = \det \{1 - P_0 + P_0 Q_1\}$$

(34)

for all closed curves. The important point to understand here is that this identity holds whenever the current $j_\mu(x)$ is obtained from an underlying fermion measure in the way we have described. In other words, if we start from an arbitrary current and define the phase factors $W$ through eqs. (\ref{eq:22}), (\ref{eq:30}), (\ref{eq:32}), the validity of eq. (34) is a necessary condition for the current to be associated with a fermion measure.

As it turns out, this is in fact also a sufficient condition \cite{22}. Moreover the current determines the measure uniquely up to a constant phase factor in each topological sector. The construction of the lattice theory is thus reduced to the task of finding a current $j_\mu(x)$ that

(a) is a gauge-covariant local expression in the gauge field,

(b) transforms like an axial vector current under the lattice symmetries,

(c) satisfies the anomalous conservation law (\ref{eq:26}),

(d) fulfills the integrability condition (34).

Once this is achieved, the lattice theory is completely specified and guaranteed to comply with the basic principles listed earlier.

At first sight the integrability condition looks quite inaccessible, since it involves the determinant of a complicated operator. In its differential form,

$$\delta_\eta \mathcal{L}_\zeta - \delta_\zeta \mathcal{L}_\eta + a \mathcal{L}_{[\eta, \zeta]} =$$

$$i \text{Tr} \left\{ \hat{P}_- [\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\},$$

(35)

the equation is, however, much more tractable. It can be shown, for example, that the right-hand side of eq. (35) is proportional to

$$\int d^4x \, d^3y \, \epsilon_{\mu \nu \rho \sigma} \eta^\mu_\alpha(x) \zeta^\nu_\beta(x) F^\rho_\sigma(x)$$

(36)

in the classical continuum limit. This expression vanishes if the fermion multiplet is anomaly-free, and setting $j_\mu(x) = 0$ in this limit then fulfills both the requirement of gauge invariance and the integrability condition in its differential form. As far
as the classical continuum limit goes, this completes the definition of the lattice theory. In particular, in the formula (21) for the fermion determinant, the phase factor $W$ is equal to 1 up to terms vanishing proportionally to $a$.

A last comment which should be made here is that our discussion of the fermion determinant is in many respects similar to Leutwyler’s approach in the continuum theory [19], where one first applies a finite-part prescription to the variation of the determinant and then adds a local counter-term to restore the integrability of the expression. The measure term $\mathcal{L}_0$ effectively plays the rôle of this counterterm and eq. (35) may actually be derived directly from eq. (21) by noting that the right-hand side has to be integrable.

6. U(1) THEORIES

At this point we still need to prove that there exists a current $J_\mu(x)$ with the required properties if the fermion multiplet is anomaly-free. It suffices to construct the current on the subset of all gauge fields satisfying the bound (3), since only these contribute to the functional integral. While this is technically helpful, a definitive answer to the question has so far only been given for abelian gauge groups [31].

The simplest case to consider is a theory with $N$ left-handed fermions coupled to a U(1) gauge field. Without loss the fermion representation of the gauge group may be taken to be diagonal,

$$R[A]_{\alpha\beta} = \Lambda^{c_\alpha} \delta_{\alpha\beta}, \quad A \in U(1),$$

where the indices label the fermion flavours and the integers $c_\alpha$ denote their charges. The anomaly cancellation condition then reads

$$\sum_{\alpha=1}^N c_\alpha^3 = 0$$

and an interesting example of an anomaly-free charge assignment is thus

$$e_1 = \ldots = e_8 = 1, \quad e_9 = -2.$$  \hspace{1cm} (38)

The main theorem established in ref. [23] applies to the theory in infinite volume and asserts that a current satisfying all conditions (a)–(d) listed above exists if eq. (38) holds. The proof of the theorem is constructive and one ends up with a complicated but well-defined expression for the current.

In finite volume with periodic boundary conditions, the space of gauge orbits divides into topological sectors with non-contractible closed loops. Global obstructions can then arise, but for most charge assignments, including the multiplet (8) and all representations with only even charges, the theorem extends to all topological sectors in finite volume [31].

A more detailed description of this result and its derivation is beyond the scope of this review. Instead we now briefly discuss how the anomaly cancellation works out on the lattice. In the U(1) theories considered in this section, the anomaly (24) assumes the form

$$\mathcal{A}(x) = -\frac{1}{2} a \text{tr}\{\gamma_5 T D(x, x)\},$$

where $T$ is the charge matrix $T_{\alpha\beta} = e_\alpha \delta_{\alpha\beta}$. The anomaly is thus a gauge-invariant local expression in the gauge field. A less obvious property is that

$$a^4 \sum_x \delta \mathcal{A}(x) = 0$$

for any local variation $\delta U(x, \mu)$ of the gauge field. To prove this one first notes that the left-hand side of eq. (31) is proportional to $\text{Tr}\{T \delta \hat{\gamma}_5\}$. Using the identities

$$\hat{\gamma}_5^2 = 1, \quad \{\hat{\gamma}_5, \delta \hat{\gamma}_5\} = 0, \quad \{\hat{\gamma}_5, T\} = 0$$

the trace is then easily seen to vanish.

Equation (11) says that the abelian anomaly is a topological field, i.e. it has all the characteristic properties of a topological density. Modulo divergence terms (which are topologically uninteresting) there are usually not many fields of this type and the form of the anomaly is thus strongly constrained. To make this explicit, we first note that the Chern polynomial

$$c(x) = \alpha + \beta_{\mu\nu} F_{\mu\nu}(x) + \gamma \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma} (x + a\hat{\mu} + a\hat{\nu})$$

(43)

provides a simple example of a non-trivial topological field. The field tensor $F_{\mu\nu}(x)$, which appears here, is defined as usual from the plaquette...
loops and $\hat{\mu}$ denotes the unit vector in direction $\mu$ (the displacement of the coordinates in the last term accounts for the fact that the Leibniz rule is modified on the lattice). The crucial observation is now that up to divergence terms there are actually no further topological fields [50,51]. In other words, any topological field $q(x)$ constructed from a U(1) lattice gauge field is of the form

$$q(x) = c(x) + \partial^\mu k_\mu(x),$$  \hspace{1cm} (44)

where $k_\mu(x)$ is a gauge-invariant local current. In particular, the anomaly $A(x)$ can be written in this way, with some constants $\alpha$, $\beta_{\mu\nu}$ and $\gamma$.

It is easy to see that $\alpha$ and $\beta_{\mu\nu}$ have to vanish, because the anomaly transforms as a pseudo-scalar under the lattice symmetries. Concerning the coefficient $\gamma$ we note that the anomaly is a sum of terms, one for each fermion flavour. Since the field tensor scales with the charge and since there is another power of the charge coming from the charge matrix $T$ in eq. (31), we have

$$\gamma \propto \sum_{\alpha=1}^N e^3_\alpha. \hspace{1cm} (45)$$

The topologically non-trivial part of the anomaly thus cancels if the fermion multiplet is anomaly-free and the condition for exact gauge invariance, eq. (29), then reduces to

$$\partial^\mu j_\mu(x) = \partial^\mu k_\mu(x). \hspace{1cm} (46)$$

Although more work is required to actually show this [31], it should now be quite plausible that this relation can be satisfied. At least the topological obstruction represented by the anomaly has completely disappeared at this stage.

7. NON-ABELIAN THEORIES

If the gauge group is not abelian, the anomaly has a more complicated structure and does not seem to have any obvious topological properties. In the continuum limit the anomaly is, however, known to be closely related to the Chern character in 4+2 dimensions. Algebraically the link is provided by the descent equations [32,33] and there is also an associated topological interpretation of the anomaly [34]. Somewhat surprisingly the relation persists on the lattice and the exact cancellation of the anomaly is then again mapped to the problem of classifying topological fields, although this time in 4+2 dimensions [35].

We now describe this result in outline and begin by considering gauge fields

$$U(z, \mu), \ A_t(z), \ A_s(z), \ \ z = (x, t, s), \hspace{1cm} (47)$$

on Lattice $\times \mathbb{R}^2$, the continuous extra coordinates being $t$ and $s$. The gauge potentials in these directions take values in the Lie algebra of the gauge group as usual. They allow one to define gauge-covariant derivatives such as

$$D^A_t \hat{P}_- = \partial_t \hat{P}_- + [R(A_t), \hat{P}_-]. \hspace{1cm} (48)$$

In particular, the field

$$q(z) = \frac{1}{2} \text{Im tr} \left\{ [\gamma_5 D^A_t \hat{P}_-, D^A_s \hat{P}_-] + R(F_{ts}) \gamma_5 \right\}(x, x) \hspace{1cm} (49)$$

(where $\ldots(x, y)$ denotes the kernel in position space representing the operator enclosed in the square bracket) is invariant under arbitrary gauge transformations in 4+2 dimensions. $q(z)$ is also a local field and it can be shown to be topological in the sense explained above, viz.

$$a^4 \sum_x \int dt \ ds \ \delta q(z) = 0 \hspace{1cm} (50)$$

for all local deformations of the gauge field.

The importance of all this is now made clear by the following

**Theorem.** The field $q(z)$ is topologically trivial if and only if there exists a gauge-covariant local current $j_\mu(x)$ satisfying the integrability condition in its differential form, eq. (35), and the anomalous conservation law [24].

To establish the exact cancellation of the anomaly on the lattice we thus need to show that $q(z)$ is topologically trivial. Presumably there are just a few non-trivial topological fields on Lattice $\times \mathbb{R}^2$. The field $q(z)$ is a linear combination of these plus a divergence term and one then has to prove that the coefficients of the non-trivial fields vanish if the fermion multiplet is anomaly-free.
The classification of topological fields modulo divergence terms is a particular case of a local cohomology problem, a subject that has received a lot of attention in continuum field theory. In particular, using the descent equations \[33\] [37], a general theorem has been established, for any gauge group and in any dimension, which states that in the absence of matter fields the Chern monomials are the only non-trivial topological fields \[32\] [34].

Although this result is for the continuum theory, it allows us to prove the anomaly cancellation on the lattice to all orders of an expansion in powers of \(a\) around the classical continuum limit. As explained before, the limit is approached by assuming the link variables \(U(z, \mu)\) to be given by a smooth background gauge potential \(A_\mu(z)\). For \(a \to 0\) the asymptotic expansion

\[
q(z) = \sum_{k=0}^{\infty} a^{k-6} O_k(z) \tag{51}
\]

is then obtained, \(O_k(z)\) being polynomials of dimension \(k\) in the potentials \(A_\mu(z), A_t(z), A_s(z)\) and their derivatives. The first few terms are not difficult to work out and one finds that

\[
q(z) = \frac{1}{\pi} c_1 \epsilon^{abc} \epsilon_{\mu_1...\mu_6} \times F_{\mu_1\mu_2}(z) F_{\mu_3\mu_4}(z) F_{\mu_5\mu_6}(z) + O(a), \tag{52}
\]

where the obvious notations are being used, with space-time indices running from 0 to 5. In the continuum limit \(q(z)\) is thus proportional to the Chern character in 4+2 dimensions \[53\]. The connection between the anomaly and the Chern character is thus made explicit and the equation also shows that all fields \(O_k(z)\) with dimension \(k \leq 6\) vanish if the fermion multiplet is anomaly-free.

We now note that the higher-order terms have to be gauge-invariant and topological, because \(q(z)\) has these properties. The classification theorem quoted above then implies that they are all equal to a sum of Chern monomials plus a divergence term. Since there are no Chern monomials with scale dimension greater than the space-time dimension, we thus conclude that \(q(z)\) is topologically trivial to all orders in \(a\) if the anomaly cancels at \(a = 0\).

Evidently it is not certain that this result extends to any fixed value of the lattice spacing, but it would be a surprise if it did not. What is lacking at present is a classification theorem for topological fields on Lattice \(\times \mathbb{R}^2\). Presumably the non-trivial classes match with the non-trivial classes in the continuum theory, and the exact cancellation of the anomaly on the lattice would then be obvious from our discussion above.

8. CONCLUDING REMARKS

The starting point in this review has been the Ginsparg–Wilson relation, which implies an exact chiral symmetry of the fermion action and thus allows one to introduce Weyl fermions in a sensible way. Another consequence of this relation is that the gauge anomaly on the lattice descends from a topological field in 4+2 dimensions. For the exact cancellation of the anomaly, this property is essential as otherwise there is no reason for the lattice corrections to the anomaly to cancel if the fermion representation is anomaly-free.

An important topic, which we did not discuss, are global anomalies \[44\]. Such anomalies can arise if there are non-contractible loops in field space, because the integrability condition \[44\] is then a slightly stronger constraint than its differential form. The latter implies

\[
W = h \det \{1 - P_0 + P_0 Q_1\} \tag{53}
\]

for all closed curves in field space, where \(h\) is a constant phase factor depending on the homotopy class of the curve. Evidently \(h = 1\) for all contractible loops, but in general this need not be so.

For illustration let us consider a Weyl fermion coupled to an SU(2) gauge field in the fundamental representation. The anomaly and the right-hand side of eq. \[53\] vanish in this case and setting \(j_\mu(x) = 0\) thus appears to fulfill all conditions. As has recently been shown by Bär and Campos \[66\], there are, however, closed loops in field space for which

\[
\det \{1 - P_0 + P_0 Q_1\} = -1. \tag{54}
\]

Moreover they explain in detail how this relates to Witten’s original derivation of the anomaly and
the conclusion is then that there is no acceptable fermion measure in this theory.

While this leaves little doubt that the known global anomalies can all be reproduced on the lattice, it is not obvious that such anomalies are absent in those theories where they are not expected to occur. So far this has only been shown for abelian gauge groups [31].

Although only purely left-handed models have been considered here, no additional difficulties are expected to arise when more general multiplets of Weyl fermions and Higgs fields are included. In particular, one may now envision to study non-leptonic weak decays using an exactly gauge-invariant lattice formulation of the Standard Model. To check the $\Delta I = 1/2$ rule, for example, an interesting option in such an approach is to set the mass of the $W$ boson to unphysically low values. The operator product expansion is then not needed and the hard renormalization problems that go with it are thus avoided (for a recent discussion of this issue and a related proposal see ref. [57]).

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