Existence of zeros for operators taking their values in the dual of a Banach space

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Throughout the sequel, $E$ denotes a reflexive real Banach space and $E^*$ its topological dual. We also assume that $E$ is locally uniformly convex. This means that for each $x \in E$, with $\|x\| = 1$, and each $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y \in E$ satisfying $\|y\| = 1$ and $\|x - y\| \geq \epsilon$, one has $\|x + y\| \leq 2(1 - \delta)$. Recall that any reflexive Banach space admits an equivalent norm with which it is locally uniformly convex ([1], p. 289). For $r > 0$, we set $B_r = \{x \in E : \|x\| \leq r\}$.

Moreover, we fix a topology $\tau$ on $E$, weaker than the strong topology and stronger than the weak topology, such that $(E, \tau)$ is a Hausdorff locally convex topological vector space with the property that the $\tau$-closed convex hull of any $\tau$-compact subset of $E$ is still $\tau$-compact and the relativization of $\tau$ to $B_1$ is metrizable by a complete metric. In practice, the most usual choice of $\tau$ is either the strong topology or the weak topology provided $E$ is also separable.

The aim of this short paper is to establish the following result and present some of its consequences:

THEOREM 1. - Let $X$ be a paracompact topological space and $A : X \to E^*$ a weakly continuous operator. Assume that there exist a number $r > 0$, a continuous function $\alpha : X \to \mathbb{R}$ satisfying
\[ |\alpha(x)| \leq r \|A(x)\|_{E^*} \]
for all $x \in X$, a closed set $C \subset X$, and a $\tau$-continuous function $g : C \to B_r$ satisfying
\[ A(x)(g(x)) = \alpha(x) \]
for all $x \in C$, in such a way that, for every $\tau$-continuous function $\psi : X \to B_r$ satisfying $\psi|_C = g$, there exists $x_0 \in X$ such that
\[ A(x_0)(\psi(x_0)) \neq \alpha(x_0) . \]
Then, there exists $x^* \in X$ such that $A(x^*) = 0$.

For the reader’s convenience, we recall that a multifunction $F : S \to 2^V$, between topological spaces, is said to be lower semicontinuous at $s_0 \in S$ if, for every open set $\Omega \subseteq V$ meeting $F(s_0)$, there is a neighbourhood $U$ of $s_0$ such that $F(s) \cap \Omega \neq \emptyset$ for all $s \in U$. $F$ is said to be lower semicontinuous if it is so at each point of $S$.

The following well-known results will be our main tools.

THEOREM A ([3]). - Let $X$ be a paracompact topological space and $F : X \to 2^{B_1}$ a $\tau$-lower semicontinuous multifunction with nonempty $\tau$-closed convex values.
Then, for each closed set \( C \subseteq X \) and each \( \tau \)-continuous function \( g : C \to B_1 \) satisfying \( g(x) \in F(x) \) for all \( x \in C \), there exists a \( \tau \)-continuous function \( \psi : X \to B_1 \) such that \( \psi|_C = g \) and \( \psi(x) \in F(x) \) for all \( x \in X \).

**THEOREM B ([4]).** Let \( X, Y \) be two topological spaces, with \( Y \) connected and locally connected, and let \( f : X \times Y \to \mathbb{R} \) be a function satisfying the following conditions:

(a) for each \( x \in X \), the function \( f(x, \cdot) \) is continuous, changes sign in \( Y \) and is identically zero in no nonempty open subset of \( Y \);

(b) the set \( \{(y, z) \in Y \times Y : \{x \in X : f(x, y) < 0 < f(x, z)\} \text{ is open in } X\} \) is dense in \( Y \times Y \).

Then, the multifunction \( x \mapsto \{y \in Y : f(x, y) = 0 \text{ and } y \text{ is not a local extremum for } f(x, \cdot)\} \) is lower semicontinuous, and its values are nonempty and closed.

**Proof of Theorem 1.** Arguing by contradiction, assume that \( A(x) \neq 0 \) for all \( x \in X \). For each \( x \in X \), \( y \in B_1 \), put

\[
f(x, y) = A(x)(y) - \frac{\alpha(x)}{r}
\]

and

\[
F(x) = \{z \in B_1 : f(x, z) = 0\}.
\]

Also, set

\[
X_0 = \{x \in X : |\alpha(x)| < r\|A(x)\|_{E^*}\}.
\]

Since \( A \) is weakly continuous, the function \( x \mapsto \|A(x)\|_{E^*} \), as supremum of a family of continuous functions, is lower semicontinuous. From this, it follows that the set \( X_0 \) is open. For each \( x \in X_0 \), the function \( f(x, \cdot) \) is continuous and has no local, nonabsolute, extrema, being affine. Moreover, it changes sign in \( B_1 \) since \( A(x)(B_1) = [-\|A(x)\|_{E^*}, \|A(x)\|_{E^*}] \) (recall that \( E \) is reflexive). Since \( f(\cdot, y) \) is continuous for all \( y \in B_1 \), we then realize that the restriction of \( f \) to \( X_0 \times B_1 \) satisfies the hypotheses of Theorem B, \( B_1 \) being considered with the relativization of the strong topology. Hence, the multifunction \( F|_{X_0} \) is lower semicontinuous. Consequently, since \( X_0 \) is open, the multifunction \( F \) is lower semicontinuous at each point of \( X_0 \). Now, fix \( x_0 \in X \setminus X_0 \). So, \( |\alpha(x_0)| = r\|A(x_0)\|_{E^*} \). Let \( y_0 \in F(x_0) \) and \( \epsilon > 0 \). Clearly, since \( y_0 \) is an absolute extremum of \( A(x_0) \) in \( B_1 \), one has \( \|y_0\| = 1 \). Choose \( \delta > 0 \) so that, for each \( y \in E \) satisfying \( \|y\| = 1 \) and \( \|y - y_0\| \geq \epsilon \), one has \( \|y + y_0\| \leq 2(1 - \delta) \). By semicontinuity, the function \( x \mapsto (\|A(x)\|_{E^*})^{-1} \) is bounded in some neighbourhood of \( x_0 \), and so, since the functions \( \alpha \) and \( A(\cdot)(y_0) \) are continuous, it follows that

\[
\lim_{x \to x_0} \left| \frac{A(x)(y_0) - \alpha(x)}{\|A(x)\|_{E^*}} \right| = 0.
\]

So, there is a neighbourhood \( U \) of \( x_0 \) such that

\[
\left| \frac{A(x)(y_0) - \alpha(x)}{\|A(x)\|_{E^*}} \right| < \frac{\epsilon \delta}{2}
\]

(1)
for all $x \in U$. Fix $x \in U$. Pick $z \in E$, with $\|z\| = 1$, in such a way that $|A(x)(z)| = \|A(x)\|_{E^*}$ and

$$\left( A(x)(z) - \frac{\alpha(x)}{r} \right) \left( A(x)(y_0) - \frac{\alpha(x)}{r} \right) \leq 0.$$ 

¿From this choice, it follows, of course, that the segment joining $y_0$ and $z$ meets the hyperplane $(A(x))^{-1}\left( \frac{\alpha(x)}{r} \right)$. In other words, there is $\lambda \in [0, 1]$ such that

$$A(x)(\lambda z + (1 - \lambda)y_0) = \frac{\alpha(x)}{r}. \quad (2)$$

So, if we put $y = \lambda z + (1 - \lambda)y_0$, we have $y \in F(x)$ and

$$\|y - y_0\| = \lambda \|z - y_0\|. \quad (3)$$

We claim that $\|y - y_0\| < \epsilon$. This follows at once from (3) if $\lambda < \frac{\epsilon}{2}$. Thus, assume $\lambda \geq \frac{\epsilon}{2}$. In this case, to prove our claim, it is enough to show that

$$2(1 - \delta) < \|z + y_0\| \quad (4)$$

since (4) implies $\|z - y_0\| < \epsilon$. To this end, note that, by (2), one has

$$\frac{|A(x)(y_0) - \frac{\alpha(x)}{r}|}{\|A(x)\|_{E^*}} = \frac{\lambda |A(x)(z - y_0)|}{\|A(x)\|_{E^*}},$$

and so, from (1), it follows that

$$\frac{|A(x)(z - y_0)|}{\|A(x)\|_{E^*}} < \delta. \quad (5)$$

Suppose $A(x)(z) = \|A(x)\|_{E^*}$. Then, from (5), we get

$$1 - \delta < \frac{A(x)(y_0)}{\|A(x)\|_{E^*}}. \quad (6)$$

On the other hand, we also have

$$1 + \frac{A(x)(y_0)}{\|A(x)\|_{E^*}} = \frac{A(x)(z + y_0)}{\|A(x)\|_{E^*}} \leq \|z + y_0\|. \quad (7)$$

So, (4) follows from (6) and (7). Now, suppose $A(x)(z) = -\|A(x)\|_{E^*}$. Then, from (5), we get

$$1 - \delta < -\frac{A(x)(y_0)}{\|A(x)\|_{E^*}}. \quad (8)$$

On the other hand, we have

$$1 - \frac{A(x)(y_0)}{\|A(x)\|_{E^*}} = -\frac{A(x)(z + y_0)}{\|A(x)\|_{E^*}} \leq \|z + y_0\|. \quad (9)$$
So, in the present case, (4) is a consequence of (8) and (9). In such a manner, we have
proved that $F$ is lower semicontinuous at $x_0$. Hence, it remains proved that $F$ is lower
semicontinuous in $X$ with respect to the strong topology, and so, \textit{a fortiori}, with respect to
$\tau$. Since $F$ is also with nonempty $\tau$-closed convex values, and $\frac{\alpha}{r}$ is a $\tau$-continuous
selection of it over the closed set $C$, by Theorem A, $F$ admits a $\tau$-continuous selection $\omega$ in $X$ such
that $\omega|_C = \frac{\omega}{r}$. At this point, if we put $\psi = r\omega$, it follows that $\psi$ is a $\tau$-continuous function,
from $X$ into $B_r$, such that $\psi|_C = g$ and $A(x)(\psi(x)) = \alpha(x)$ for all $x \in X$, against
the hypotheses. This concludes the proof. \hfill $\triangle$

We now indicate two reasonable ways of application of Theorem 1. The first one is
based on the Tychonoff fixed point theorem.

\textbf{THEOREM 2.} - Assume that $E$ is a separable Hilbert space, with inner product $\langle \cdot, \cdot \rangle$.
Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the weak to the strong
topology. Assume that there exist a weakly continuous function $\alpha : B_r \to \mathbb{R}$ satisfying
$|\alpha(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in B_r$, and a weakly continuous function $g : C \to B_r$ such
\[
\langle A(x), g(x) \rangle = \alpha(x) \quad \text{and} \quad g(x) \neq x
\]
for all $x \in C$, where
\[
C = \{ x \in B_r : \langle A(x), x \rangle = \alpha(x) \} .
\]
Then, there exists $x^* \in B_r$ such that $A(x^*) = 0$.

\textbf{PROOF.} Identifying $E$ with $E^*$, we apply Theorem 1 taking $X = B_r$, with the
relativization of the weak topology of $E$, and taking as $\tau$ the weak topology of $E$. Due to
the kind of continuity we are assuming for $A$, the function $x \to \langle A(x), x \rangle$ turns out to be
weakly continuous (see the proof of Theorem 4), and so the set $C$ is weakly closed. Now, let $\psi : B_r \to B_r$ be any weakly continuous function such that $\psi|_C = g$. By the Tychonoff
fixed point theorem, there is $x_0 \in B_r$ such that $\psi(x_0) = x_0$. Since $g$ ha no fixed points in
$C$, it follows that $x_0 \notin C$, and so
\[
\langle A(x_0), \psi(x_0) \rangle = \langle A(x_0), x_0 \rangle \neq \alpha(x_0) .
\]
Hence, all the assumptions of Theorem 1 are satisfied, and the conclusion follows from it. \hfill $\triangle$

It is worth noticing the following consequence of Theorem 2.

\textbf{THEOREM 3.} - Let $E$ and $A$ be as in Theorem 2. Assume that for each $x \in B_r$, with
$\|A(x)\| > r$, one has
\[
\left\| A \left( \frac{rA(x)}{\|A(x)\|} \right) \right\| \leq r . \quad (10)
\]
Then, the operator $A$ has either a zero or a fixed point.

\textbf{PROOF.} Define the function $\alpha : B_r \to \mathbb{R}$ by
\[
\alpha(x) = \begin{cases} 
\|A(x)\|^2 \quad \text{if} \quad \|A(x)\| \leq r \\
\|A(x)\| \quad \text{if} \quad \|A(x)\| > r .
\end{cases}
\]
Clearly, the function $\alpha$ is weakly continuous and satisfies $|\alpha(x)| \leq r\|A(x)\|$ for all $x \in B_r$. Put

$$C = \{ x \in B_r : \langle A(x), x \rangle = \alpha(x) \}.$$ 

Note that if $x \in C$ then $\|A(x)\| \leq r$. Indeed, otherwise, we would have $\langle A(x), x \rangle = r\|A(x)\|$, and so, necessarily, $x = \frac{rA(x)}{\|A(x)\|}$, against (10). Hence, we have $\langle A(x), A(x) \rangle = \alpha(x)$ for all $x \in C$. At this point, the conclusion follows at once from Theorem 2, taking $g = A|_C$. \triangle

**REMARK 1.** - It would be interesting to know whether Theorem 3 can be improved assuming that $A$ is a continuous operator with relatively compact range.

The second application of Theorem 1 is based on connectedness arguments. For other results of this type we refer to [5] (see also [2]).

**THEOREM 4.** - Let $X$ be a connected paracompact topological space and $A : X \to E^*$ a weakly continuous and locally bounded operator. Assume that there exist $r > 0$, a closed set $C \subset X$, a continuous function $g : C \to B_r$, and an upper semicontinuous function $\beta : X \to \mathbb{R}$, with $|\beta(x)| \leq r\|A(x)\|_{E^*}$ for all $x \in X$, such that $g(C)$ is disconnected,

$$\beta(x) \leq A(x)(g(x))$$

for all $x \in C$ and

$$A(x)(y) < \beta(x)$$

for all $x \in X \setminus C$ and $y \in B_r \setminus g(C)$.

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

**PROOF.** First, note that the function $x \to A(x)(g(x))$ is continuous in $C$. To see this, let $x_1 \in C$ and let $\{x_\gamma\}_{\gamma \in D}$ be any net in $C$ converging to $x_1$. By assumption, there are $M > 0$ and a neighbourhood $U$ of $x_1$ such that $\|A(x)\|_{E^*} \leq M$ for all $x \in U$. Let $\gamma_0 \in D$ be such that $x_\gamma \in U$ for all $\gamma \geq \gamma_0$. Thus, for each $\gamma \geq \gamma_0$, one has

$$|A(x_\gamma)(g(x_\gamma)) - A(x_1)(g(x_1))| \leq M\|g(x_\gamma) - g(x_1)\| + |A(x_\gamma)(g(x_1)) - A(x_1)(g(x_1))|$$

from which, of course, it follows that $\lim_{\gamma} A(x_\gamma)(g(x_\gamma)) = A(x_1)(g(x_1))$. Next, observe that the multifunction $x \to [\beta(x), r\|A(x)\|_{E^*}]$ is lower semicontinuous and that the function $x \to A(x)(g(x))$ is a continuous selection of it in $C$. Hence, by Michael’s theorem, there is a continuous function $\alpha : X \to \mathbb{R}$ such that $\alpha(x) = A(x)(g(x))$ for all $x \in C$ and $\beta(x) \leq \alpha(x) \leq r\|A(x)\|_{E^*}$ for all $x \in X$. Now, let $\psi : X \to B_r$ be any continuous function such that $\psi|_C = g$. Since $X$ is connected, $\psi(X)$ is connected too. But then, since $g(C)$ is disconnected and $g(C) \subset \psi(X)$, there exists $y_0 \in \psi(X) \setminus g(C)$. Let $x_0 \in X \setminus C$ be such that $\psi(x_0) = y_0$. So, by hypothesis, we have

$$A(x_0)(\psi(x_0)) = A(x_0)(y_0) < \beta(x_0) \leq \alpha(x_0).$$

Hence, taking as $\tau$ the strong topology of $E$, all the assumptions of Theorem 1 are satisfied, and the conclusion follows from it. \triangle
REMARK 2. - Observe that when $X$ is first-countable, the local boundedness of $A$ follows automatically from its weak continuity. This follows from the fact that, in a Banach space, any weakly convergent sequence is bounded.

It is worth noticing the corollary of Theorem 4 which comes out taking $X = B_r$, $\beta = 0$ and $g=$identity:

THEOREM 5. - Let $E$ be a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$. Let $r > 0$ and let $A : B_r \to E$ be a continuous operator from the strong to the weak topology. Assume that the set $C = \{ x \in B_r : \langle A(x), x \rangle \geq 0 \}$ is disconnected and that, for each $x, y \in B_r \setminus C$, one has $\langle A(x), y \rangle < 0$.

Then, there exists $x^* \in C$ such that $A(x^*) = 0$.

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