ON THE LINEAR WAVE REGIME OF THE GROSS-PITAEVSKII EQUATION

FABRICE BÉTHUEL, RAPHAËL DANCHIN, AND DIDIER SMETS

Abstract. We study a long wave-length asymptotics for the Gross-Pitaevskii equation corresponding to perturbation of a constant state of modulus one. We exhibit lower bounds on the first occurrence of possible zeros (vortices) and compare the solutions with the corresponding solutions to the linear wave equation or variants. The results rely on the use of the Madelung transform, which yields the hydrodynamical form of the Gross-Pitaevskii equation, as well as of an augmented system.

1. Introduction

The dynamics of the Gross-Pitaevskii equation

\begin{equation}
\left.\begin{array}{l}
\frac{i}{\Psi} \frac{\partial \Psi}{\partial t} + \Delta \Psi = \Psi(|\Psi|^2 - 1)
\end{array}\right.\end{equation}

on \( \mathbb{R}^N \times \mathbb{R}, \) for \( N \geq 1, \) with non-trivial limit conditions at infinity, exhibits a remarkable variety of special solutions and regimes. The purpose of this paper is to investigate one of these regimes, namely perturbations of constant maps of modulus one, which are obvious stationary solutions, in a long-wave asymptotics. In particular, we restrict ourselves to solutions \( \Psi \) which do not vanish, so that we may write

\[ \Psi = \rho \exp(i\varphi). \]

In the variables \( (\rho, \varphi), \) \( (GP) \) is turned into the system

\[ \begin{cases} 
\partial_t \rho + 2\nabla \varphi \cdot \nabla \rho + \rho \Delta \varphi = 0, \\
\rho \partial_t \varphi + \rho |\nabla \varphi|^2 - \Delta \rho = \rho(1 - \rho^2). 
\end{cases} \]

Setting \( u = 2\nabla \varphi \) leads to the hydrodynamical form of \( (GP) \)

\begin{equation}
\left.\begin{array}{l}
\partial_t \rho^2 + \text{div}(\rho^2 u) = 0, \\
\partial_t u + u \cdot \nabla u + 2\nabla \rho^2 = 2\nabla \left( \frac{\Delta \rho}{\rho} \right). 
\end{array}\right.\end{equation}

If one neglects the right-hand side of the second equation, which is often referred to as the quantum pressure, system \( (1) \) is similar to the Euler equation for a compressible fluid, with pressure law \( p(\rho) = 2\rho^2. \) In particular, the speed of sound waves near the constant solution \( \Psi = 1, \) that is \( \rho = 1 \) and \( u = 0, \) is given by

\[ c_s = \sqrt{2}. \]
In order to specify the nature of our perturbation as well as of our long-wave asymptotics we introduce a small parameter $\varepsilon > 0$ and set
\begin{equation}
\rho^2(x,t) = 1 + \frac{\varepsilon}{\sqrt{2}}a_\varepsilon(x, \varepsilon t),
\end{equation}
\begin{equation}
u(x,t) = \varepsilon u_\varepsilon(x, \varepsilon t),
\end{equation}
so that system (1) translates into
\begin{equation}
\begin{aligned}
& \frac{\partial_t a_\varepsilon + \sqrt{2} \text{div } u_\varepsilon = -\varepsilon \text{div}(a_\varepsilon u_\varepsilon),} \\
& \frac{\partial_t u_\varepsilon + \sqrt{2} \nabla a_\varepsilon = \varepsilon \left(-u_\varepsilon \cdot \nabla u_\varepsilon + 2\nabla \left(\frac{\Delta \sqrt{2} + \varepsilon a_\varepsilon}{\sqrt{2} + \varepsilon a_\varepsilon}\right)\right).}
\end{aligned}
\end{equation}
The l.h.s. of this system corresponds to the linear wave operator with speed $\sqrt{2}$, whereas the r.h.s. contains terms of higher order derivatives, which correspond to the dispersive nature of the Schrödinger equation (with infinite speed of propagation).

Our first main result provides a lower bound for the first occurrence of a zero of $\Psi$.

**Theorem 1.** Let $s > 1 + \frac{N}{2}$. There exists $C = C(s, N)$ such that for any initial datum $(a_\varepsilon^0, u_\varepsilon^0)$ verifying $(a_\varepsilon^0, u_\varepsilon^0) \in H^{s+1} \times H^s$ and $C\varepsilon \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s} \leq 1$ there exists
\[ T_\varepsilon \geq \frac{1}{C\varepsilon \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}} \]

such that system (3) as a unique solution $(a_\varepsilon(t), u_\varepsilon(t)) \in C^0([0, T_\varepsilon]; H^{s+1} \times H^s)$ satisfying
\[ \|(a_\varepsilon(\cdot, t), u_\varepsilon(\cdot, t))\|_{H^{s+1} \times H^s} \leq C\|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s} \quad \text{and} \quad \frac{1}{2} \leq \rho \left(\cdot, \frac{t}{\varepsilon}\right) \leq 2 \]
whenever $t \in [0, T_\varepsilon]$.

**Remark 1.** i) From the ansatz (2), the time scale of system (3) is accelerated by a factor $\varepsilon$ with respect to the time scale of system (GP). In terms of the Gross-Pitaevskii equation, the lower bound $T_\varepsilon$ given in Theorem 1 translates therefore into the bound
\[ T_\varepsilon = \varepsilon^{-1}T_\varepsilon \geq \frac{1}{C\varepsilon^2 \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}}. \]

ii) A typical initial datum that Theorem 1 allows to handle is
\[ \Psi^0(x) = \sqrt{1 + \frac{\varepsilon}{\sqrt{2}}a^0(\varepsilon x) \exp(i \varphi^0(\varepsilon x))}, \]
where $a^0 \equiv 2\nabla \varphi^0$ and $a^0$ do not depend on $\varepsilon$ and belong to $H^{s+1} \times H^s$. This corresponds to perturbations of the constant map 1 of order $\varepsilon$ for the modulus and of wave-length of order $\varepsilon^{-1}$. In this case, we obtain the lower bound $T_\varepsilon \geq \frac{1}{\varepsilon}$, that is $T_\varepsilon \geq \frac{1}{\varepsilon^2}$.

As a byproduct of Theorem 1, treating the r.h.s of (3) as a perturbation, we deduce the following comparison estimate with loss of three derivatives:

**Theorem 2.** Let $s$, $a_\varepsilon^0$ and $u_\varepsilon^0$ be as in Theorem 1 and let $(a, u)$ denote the solution of the free wave equation
\[ \begin{aligned}
& \frac{\partial_t a + \sqrt{2} \text{div } u = 0} \\
& \frac{\partial_t u + \sqrt{2} \nabla a = 0,}
\end{aligned} \]
with initial datum $(a_\varepsilon^0, u_\varepsilon^0)$. If $\varepsilon \leq 1$ then for $0 \leq t \leq T_\varepsilon$ we have
\[ \|(a_\varepsilon(t), u_\varepsilon(t))\|_{H^{s-2}} \leq C \left[ \varepsilon t \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2 + \varepsilon^2 t \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2 \right] \].
In Theorem 1, the fact that \((a^0_\varepsilon, u^0_\varepsilon) \in H^{s+1} \times H^s\) with \(s \geq 0\) implies in particular that the Ginzburg-Landau energy \(E(\Psi^0)\) of the corresponding function \(\Psi^0\) is finite, where
\[
E(\Psi) = \int_{\mathbb{R}^N} e(\Psi) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla \Psi|^2 + \frac{1}{4} (1 - |\Psi|^2)^2 \right)
\]
is the Hamiltonian for (GP).

Notice that according to [26], the Cauchy problem for (GP) is globally well-posed in the energy space, in dimension \(N = 2, 3\). On the other hand, by means of a basic energy method, it may be easily seen that (GP) is locally well-posed in \(1 + H^s\) in any dimension provided \(s > \frac{N}{2}\). In addition, in both cases, the Ginzburg-Landau energy \(E(\Psi)\) remains conserved during the evolution.

In dimension \(N \geq 2\), in order to handle longer time scales, one may take advantage of the dispersive properties of system (3). As a matter of fact, the linearization about \((0,0)\) of the system (3) does not exactly yield the wave operator, as appearing in Theorem 2, but rather the \(\varepsilon\)-depending operator
\[
L(\varepsilon)(a, u) = \left( \partial_t a + \sqrt{2} \text{div} u, \partial_t u + \sqrt{2} \nabla a - \sqrt{2} \varepsilon^2 \nabla \Delta a \right),
\]
which possesses even better dispersive properties. Indeed, performing a Fourier transform with respect to the space variables, the above operator rewrites for \(\xi \in \mathbb{R}^N\) and \(t \in \mathbb{R}\) as
\[
L(\varepsilon)(a, u)(\xi, t) = \left( \partial_t \hat{a}(\xi, t) + i \left( \sqrt{\frac{2}{\varepsilon^2}} |\xi|^2 + \sqrt{\frac{2}{\varepsilon^2}} |\xi|^2 \right)\xi^T \right) (\hat{a}(\xi, t)).
\]

If we restrict our attention to potential solutions, that is solutions for which \(u\) is a gradient, then the eigenvalues associated to the above system are
\[
\lambda_{\pm} = \pm i \sqrt{2} |\xi| |\sqrt{\varepsilon^2} |\xi|^2 + 1.
\]
Therefore, we expect \(L(\varepsilon)\) to behave as the linear wave operator with velocity \(\sqrt{2}\) for low frequencies \(|\xi| \ll \varepsilon^{-1}\) whereas for high frequencies \(|\xi| \gg \varepsilon^{-1}\), it should resemble the linear Schrödinger equation with small diffusion coefficient equal to \(\sqrt{2} \varepsilon\). We thus expect to glean some additional smallness for the solution to the nonlinear equation (3) by resorting to the dispersive properties of those two linear equations\(^1\). This will enable us to improve the lower bound for \(T(\varepsilon)\) stated in Theorem 1 assuming the dimension \(N\) is larger than or equal to two. More precisely, we prove the following statement.

**Theorem 3.** Under the assumptions of Theorem 1 with \(s > 2 + \frac{N}{2}\) and \(\varepsilon \leq 1\), the time \(T(\varepsilon)\) may be bounded from below by
\[
\begin{align*}
& \frac{c}{\varepsilon^2 \|(a^0_\varepsilon, u^0_\varepsilon)\|_{H^{s+1} \times H^s}^2} \quad \text{if } N \geq 4, \\
& \min \left( \frac{c}{\varepsilon^{1+\alpha} \|(a^0_\varepsilon, u^0_\varepsilon)\|_{H^{s+1} \times H^s}^1 + \alpha}, \frac{1}{\varepsilon^{1+\alpha} \|(a^0_\varepsilon, u^0_\varepsilon)\|_{H^{s+1} \times H^s}^1} \right) \quad \text{if } N = 3 \quad \text{and } 0 < \alpha < 1, \\
& \min \left( \frac{c}{\varepsilon^{q+1} \|(a^0_\varepsilon, u^0_\varepsilon)\|_{H^{s+1} \times H^s}^q}, \frac{1}{\varepsilon^{q+1} \|(a^0_\varepsilon, u^0_\varepsilon)\|_{H^{s+1} \times H^s}^q} \right) \quad \text{if } N = 2 \quad \text{and } 2 > q > \frac{2}{s-2}.
\end{align*}
\]
The constant \(c\) depends only on \(s\) and also on \(N\) if \(N \geq 4\), \(\alpha\) if \(N = 3\) and \(q\) if \(N = 2\).

\(^1\)Note however, that since no dispersion occurs for the wave equation in dimension \(N = 1\), our method does not give any additional information on that case.
Remark 2. With an initial datum as in Remark 1 ii), we obtain, as $\varepsilon \to 0$, $T_\varepsilon \geq C\varepsilon^{-2}$ if $N \geq 4$, $T_\varepsilon \geq C\varepsilon^{-(2-)}$ if $N \geq 3$, and $T_\varepsilon \geq C\varepsilon^{-\frac{4}{3}}$ if $N = 2$.

Remark 3. In dimension 1 and 2, the Gross-Pitaevskii equation is known to have travelling wave solutions $\psi(x,t) = W_\varepsilon(x - c_\varepsilon t)$ which are small amplitude and long wavelength perturbations of the constant 1. They are of the form

$$W_\varepsilon(x) = 1 + \varepsilon^2 w_\varepsilon(\varepsilon x) \quad \text{in dimension 1},$$

and

$$W_\varepsilon(x) = 1 + \varepsilon^2 w_\varepsilon(\varepsilon x_1, \varepsilon^2 x_2) \quad \text{in dimension 2},$$

where the speed $c_\varepsilon$ is given by $c_\varepsilon^2 = 2 - \varepsilon^2$, and where $w_\varepsilon$ remains bounded in strong norms as $\varepsilon \to 0$. For initial data of this form (but not necessarily the travelling waves), the corresponding $a_\varepsilon$ and $u_\varepsilon$ satisfy

$$\|a_\varepsilon^0, u_\varepsilon^0\|_{H^{s+1} \times H^s} = \begin{cases} \mathcal{O}(\varepsilon) & \text{if } N = 1, \\ \mathcal{O}(\varepsilon^{\sqrt{2}}) & \text{if } N = 2. \end{cases}$$

If $N = 1$, Theorem 1 shows that $T_\varepsilon \geq C\varepsilon^{-3}$, and Theorem 3 shows similarly that $T_\varepsilon \geq C\varepsilon^{-3}$ when $N = 2$. In view of Theorem 2, the wave equation is a good approximation on time scales small with respect to $\varepsilon^{-3}$. For times of order $\varepsilon^{-3}$, the wave equation is no longer a good approximation, as can be seen considering the travelling waves. Indeed since the speed of the travelling wave differs from the speed of sound $\sqrt{2}$ by an amount of order $\varepsilon^2$, both solutions are shifted (in the variables for (3)) by an amount of order 1 exactly after a time of order $\varepsilon^{-2}$, which corresponds to a time of order $\varepsilon^{-3}$ in the time variable of (GP).

For such timescales, one is lead to consider nonlinear approximations such as the KdV or the KP equations (see [5, 10]).

Remark 4. It may be worthwhile to compare these existence results with the corresponding ones for the irrotational compressible Euler equation with smooth compactly supported perturbations of size or order of a constant state. In that case, the corresponding $T_\varepsilon$ is known to be $T_\varepsilon = +\infty$ for $N \geq 4$, $T_\varepsilon \geq \exp(\frac{\varepsilon}{\sqrt{2}})$ for $N = 3$, $T_\varepsilon \geq C\varepsilon^{-2}$ for $N = 2$ and $T_\varepsilon \geq C\varepsilon^{-1}$ for $N = 1$ (see e.g. [27, 28, 20] following pioneering ideas by Klainerman [23]).

On the larger time scale given by Theorem 3, equation (3) is better approximated by the linear equation $L_\varepsilon(a, u) = 0$ than by the free wave equation. More precisely, we have

**Theorem 4.** Let $s > 2 + \frac{N}{2}$ and $(a_\varepsilon^0, u_\varepsilon^0)$ be as in Theorem 3, let $(a_\varepsilon, u_\varepsilon)$ be the corresponding maximal solution of (3) and $(a_\varepsilon, u_\varepsilon)$ be the solution to the system

$L_\varepsilon(a, u) = 0$ \quad with initial datum \quad $(a_\varepsilon^0, u_\varepsilon^0)$.

Let $\alpha \in (0, \frac{1}{2})$ (satisfying also $\alpha > 2 - s/2$ if $N = 2$). There exists a constant $C$ depending only on $s, N$ and possibly also on $\alpha$ if $N = 2, 3$ such that for all $t \in [0, T_\varepsilon]$, the difference $(\tilde{a}, \tilde{u}) := (a_\varepsilon - a_\varepsilon, u_\varepsilon - u_\varepsilon)$ satisfies

$$\|(\tilde{a}, \tilde{u})(t)\|_{H^{s-1} + \varepsilon^{-1}\|\tilde{u}_h(t)\|_{H^{s-2}}} \leq C\varepsilon\sqrt{t} \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2$$

if $N \geq 4$,

$$\|(\tilde{a}, \tilde{u})(t)\|_{H^{s-1} + \varepsilon^{-1}\|\tilde{u}_h(t)\|_{H^{s-2}}} \leq C(t^{1-\alpha} + \varepsilon^{\frac{3}{2}}) \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2$$

if $N = 3$,

$$\|(\tilde{a}, \tilde{u})(t)\|_{H^{s-1} + \varepsilon^{-1}\|\tilde{u}_h(t)\|_{H^{s-2}}} \leq C(\varepsilon t^\frac{3}{4} + \varepsilon^{2-\alpha} t^{1-\alpha}) \|(a_\varepsilon^0, u_\varepsilon^0)\|_{H^{s+1} \times H^s}^2$$

if $N = 2$.

Here, $\tilde{u}_h$ and $\tilde{u}_h$ denote respectively the low and high frequency parts of $\tilde{u}$, the threshold between the two being set once more at $\varepsilon^{-1}$ (see the exact definition in (30) below).
In the existing mathematical literature, the Gross-Pitaevskii equation is sometimes considered in its semi-classical form
\begin{equation}
\begin{aligned}
i\varepsilon \frac{\partial \Psi_\varepsilon}{\partial t} + \varepsilon^2 \Delta \Psi_\varepsilon &= \Psi_\varepsilon (|\Psi_\varepsilon|^2 - 1).
\end{aligned}
\end{equation}

One can easily recover the original equation \((GP)\) by mean of the hyperbolic scaling
\begin{equation}
\Psi_\varepsilon (x,t) = \Psi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right).
\end{equation}

In this setting, we have
\begin{equation*}
\begin{cases}
a_\varepsilon &= \frac{\sqrt{2}}{\varepsilon} (|\Psi_\varepsilon|^2 - 1), \\
u_\varepsilon &= 2 \nabla \left( \arg(\Psi_\varepsilon) \right).
\end{cases}
\end{equation*}

In [11], equation (4) is considered on a bounded simply connected domain \(\Omega \subset \mathbb{R}^2\) with Dirichlet boundary condition and initial datum of modulus one (so that \(a_\varepsilon\) vanishes at time zero), independent of \(\varepsilon\) and bounded in \(H^1(\Omega)\). It is proved that \(\Psi_\varepsilon\) converges weakly in \(L^\infty(\mathbb{R}_+, H^1(\Omega))\) and strongly in \(C^0([0,T], L^2(\Omega))\) to \(\Psi_\star\) of modulus one whose phase satisfies the linear wave equation with speed \(\sqrt{2}\). This is consistent with our result. It is stronger in the sense that it allows for rough data, but it is also weaker in the sense that it only provides weak convergence.

Another regime for (4), corresponding to oscillating phases, has been investigated by Grenier in [18], and more recently by Alazard and Carles [1], Lin and Zhang [25], Zhang [30] and Chiron and Rousset [9].

Finally, the Gross-Pitaevskii equation has also been widely considered in a parabolic type scaling, namely
\begin{equation}
\Upsilon_\varepsilon (x,t) = \Psi \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right),
\end{equation}

so that \((GP)\) is turned into
\begin{equation}
\begin{aligned}
i \frac{\partial \Upsilon_\varepsilon}{\partial t} + \Delta \Upsilon_\varepsilon &= \frac{1}{\varepsilon^2} \Upsilon_\varepsilon (|\Upsilon_\varepsilon|^2 - 1),
\end{aligned}
\end{equation}

whose Hamiltonian reads
\begin{equation*}
E_\varepsilon (\Upsilon) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla \Upsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |\Upsilon|^2)^2.
\end{equation*}

Equation (5) is mainly considered in the regime where vortices are present [12, 24, 21, 6] and the energy is essentially reduced to the vortex energy so that no energy is left for wave oscillations as considered here. As long as \(\Psi\) does not vanish, equation \((GP)\) and the system (3) are obviously equivalent. Therefore, Theorem 1 yields a lower bound on the first occurrence of a zero of \(\Psi\) and hence of a vortex. It would be of high interest to combine the two approaches in order to understand the interaction between these two different regimes.

System (1) also enters in the class of capillary fluid equations studied in [4], with capillary coefficient \(K(\rho) = \frac{1}{\rho}\). Indeed, we have
\begin{equation}
\frac{\Delta \rho}{\rho} = K(\rho^2) \Delta \rho^2 + K'(\rho^2) |\nabla \rho^2|^2 \quad \text{with} \quad K(s) = \frac{1}{s}.
\end{equation}

Notice that, if we consider more general nonlinearities for \((GP)\), of the form \(\Psi F(|\Psi|^2)\), the pressure is turned into \(p(\rho) = 2F(\rho^2)\), whereas the capillarity coefficient remains unchanged.
We now come to the main ingredients in the proofs of Theorem 1, 2, 3 and 4. For expository purposes, it is convenient to use the parabolic scaling so as to remove as much as possible the $\varepsilon$–dependence. More precisely, we introduce the new unknowns
\[
\begin{align*}
  b_\varepsilon(x,t) &= a_\varepsilon(x,\frac{t}{\varepsilon}) \\
v_\varepsilon(x,t) &= u_\varepsilon(x,\frac{t}{\varepsilon})
\end{align*}
\]
so that the lower bound that we want to exhibit in Theorem 1 becomes of order 1, for initial data as in Remark 1.

Notice that we have the relation
\[
\Upsilon_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}
\]
with $\rho_\varepsilon^2 := 1 + \frac{\varepsilon}{\sqrt{2}} b_\varepsilon$ and $v_\varepsilon = 2 \nabla \varphi_\varepsilon$,
and that $(b_\varepsilon, v_\varepsilon)$ satisfies the system
\[
\begin{align*}
  \partial_t b_\varepsilon + \frac{\sqrt{2}}{\varepsilon} \text{div} v_\varepsilon &= -\text{div}(b_\varepsilon v_\varepsilon), \\
  \partial_t v_\varepsilon + \frac{\sqrt{2}}{\varepsilon} \nabla b_\varepsilon &= -v_\varepsilon \cdot \nabla v_\varepsilon + 2 \nabla \left( \frac{\Delta \rho_\varepsilon}{\rho_\varepsilon} \right).
\end{align*}
\]

In view of the form of system (7), our aim is to transpose the classical energy estimates for symmetrizable hyperbolic systems. Indeed, in the linear case, the singular terms involving $\sqrt{\frac{\varepsilon}{2}}$ are transparent due to the skewsymmetry, and do not contribute to the final balance. However for the full system, in the computation of the energy estimates, the higher order derivatives are difficult to control, both by themselves and by their interaction with the previously mentioned singular terms. A similar difficulty in a related context was overcome by S. Benzoni-Gavage, the second author and S. Descombes in [4]. The crucial point there, inspired by earlier works by F. Coquel [13], is to consider an augmented system, adding the equation for $\nabla(\log \rho_\varepsilon^2)$.

Therefore, we consider the new $C^N$-valued function
\[
z = v + iw \equiv \nabla(2\varphi - i\log \rho_\varepsilon^2).
\]
We obtain the following system for the functions $z$ and $b$
\[
\begin{align*}
  \partial_t b + \frac{\sqrt{2}}{\varepsilon} \text{div}(\text{Re} z) &= -\text{div}(b \text{Re} z), \\
  \partial_t z + \frac{\sqrt{2}}{\varepsilon} \nabla b &= i \Delta z - \nabla \left( \frac{z \cdot z'}{2} \right).
\end{align*}
\]
Here, for $z, z' \in C^N$, we write $z \cdot z' = \sum_{k=1}^N z_k z'_k$ where the products within the sum are complex multiplications. We first observe that
\[
\nabla \Upsilon_\varepsilon \Upsilon_\varepsilon^{-1} = \frac{i}{2} z \quad \text{and} \quad |\Upsilon_\varepsilon|^2 - 1 = \frac{\varepsilon^2}{2} b.
\]
Therefore
\[
E(\Upsilon_\varepsilon) = \frac{1}{8} \left( \|b\|_{L^2(R^N)}^2 + \|z\|_{L^2(R^N; (1 + \varepsilon b/\sqrt{2}) dx)}^2 \right).
\]

\(^2\)Whenever it does not lead to a confusion, we omit the subscript $\varepsilon$. 

The main ingredient in the proof of Theorem 1 is the following weighted a priori energy estimate involving high-order space derivatives:

**Proposition 1.** Let $s$ be a nonnegative integer and let $\Upsilon_\varepsilon$ be a solution to (5) such that $(b, z) \in C^1([0, T], H^{s+1}(\mathbb{R}^N))$ and $(Db, Dz) \in C^0([0, T]; L^\infty)$ for some $T > 0$. Assume that

$$m := \inf_{x,t} |\Upsilon_\varepsilon(x, t)| > 0.$$  

(10)

Then there exists a constant $C$ depending only on $s$, $m$, $N$, such for any time $t \in [0, T]$ we have for all integer $s' \in \{0, \cdots, s\}$,

$$\frac{d}{dt} \Gamma^{s'}(b, z) \leq C(1 + \varepsilon \|b\|_{L^\infty})(Db, Dz)\|L^\infty \left(\Gamma^{s'}(b, z) + E_\varepsilon(\Upsilon_\varepsilon)\right),$$

where

$$\Gamma^s(b, z) = \|D^s b\|^2_{L^2(\mathbb{R}^N)} + \|D^s z\|^2_{L^2(\mathbb{R}^N; (1 + \varepsilon b/\sqrt{2})dx)}.$$  

**Remark 5.** A generalization of the above proposition to noninteger Sobolev exponents and Besov spaces is given in Section 3.2. Notice that for the case $s' = 0$, we have, in view of the conservation of energy, the identity

$$\frac{d}{dt} \Gamma^0(b, z) = 0.$$

The main idea of the proof of Proposition 1 is that, up to lower order terms which may be bounded with no loss of derivatives provided $Db$ and $Dz$ are in $L^\infty$, the structure for the system satisfied by $(D^k b, D^k z)$ is the same as that of system (9). For the proof of Theorem 1, we perform a time integration in the estimate of Proposition 1 which yields

$$\|(b, z)(t)\|_{H^s} \leq C\|(b^0, z^0)\|_{H^s} \exp\left(\int_0^t \|(Db, Dz)\|_{L^\infty} \, dt\right)$$

whenever $1 + \varepsilon b/\sqrt{2}$ remains bounded and bounded away from zero. In other words, the $H^s$ norms of $(b, z)(t)$ may be bounded in terms of the $H^s$ norms of the initial data provided we have a control over $(Db, Dz)$ in $L^1([0, t]; L^\infty)$. If $s > N/2 + 1$, it follows from the Sobolev embedding that $\|(Db, Dz)\|_{L^\infty}$ may be bounded by $\|(b, z)\|_{H^s}$ so that the above inequality leads to an explicit differential inequality for $\|(b, z)(t)\|_{H^s}$ and it is then straightforward to close the estimate for times of order $\|(b_0, z_0)\|_{H^s}$.

The proof of Theorem 2 is based on elementary energy estimates for the system satisfied by $(a_\varepsilon, u_\varepsilon) - (a, u)$, the source term of which being controlled thanks to Theorem 1.

As mentioned above, the proofs of Theorem 3 and Theorem 4 rely on dispersive properties of the equation. More precisely, we provide in Proposition 4 some Strichartz type estimates (in the spirit of the pioneering work by R. Strichartz in [29] and of the paper by J. Ginibre and G. Velo [17]) tailored for the operator $L_\varepsilon$. Let us emphasize that related estimates have been used by the second author in [14] for the study of slightly compressible fluids and by S. Gustafson, K. Nakanishi and T.P. Tsai in [19] for the Gross-Pitaevskii equation. These estimates allow to improve the control on the term $\|(Db, Dz)\|_{L^\infty}$ appearing in the key inequality of Proposition 1. Indeed, it turns out that in dimension $N \geq 2$, one gets an additional bound for $\varepsilon^{-\frac{1}{2}}\|(Db, Dz)\|_{L^P([0, t]; L^\infty)}$ for some $p \in [2, \infty]$ depending on the dimension.
2. Short time existence and well-posedness for (GP)

This section is devoted to the proof of local well-posedness for (GP) with suitably smooth initial data which bounded away from zero. Since such data do not fit in the standard Sobolev space framework, we introduce, as in [7], the class of maps

\[ V = \{ U \in L^\infty(\mathbb{R}^N, \mathbb{C}), \nabla^k U \in L^2(\mathbb{R}^N), \forall k \geq 2, |U| \in L^2(\mathbb{R}^N), (1 - |U|^2) \in L^2(\mathbb{R}^N) \}. \]

A first short time existence result is given by

**Proposition 2.**

i) Let \( U \in V \) and \( s > \max(1, N/2) \). The Cauchy problem for (GP) is locally well-posed in \( U + H^s(\mathbb{R}^N) \). More precisely, given \( R > 0 \) there exists a time \( T(R) > 0 \) such that if \( \|\Phi^0\|_{H^s} \leq R \) then there exists a unique solution \( t \mapsto \Psi(t) \) in \( C^0([-T(R), T(R)]; U + H^s(\mathbb{R}^N)) \) satisfying the initial time condition

\[ \Psi(0) = U + \Phi^0. \]

ii) The flow map \( \Phi^0 \mapsto \Phi := \Psi - U \) is continuous from the ball \( B(R) \) of \( H^s(\mathbb{R}^N) \) into \( C^0([-T(R), T(R)]; U + H^s(\mathbb{R}^N)) \).

iii) If \( \Psi(0) \in U + H^{s+2}(\mathbb{R}^N) \), then \( t \mapsto \Psi(t) \) belongs to \( C^1([-T(R), T(R)]; U + H^s(\mathbb{R}^N)) \).

iv) If \( E(\Psi(0)) < +\infty \), then

\[ \frac{d}{dt} E(\Psi(t)) = 0, \forall t \in (-T(R), T(R)). \]

v) If \( E(\Psi(0)) < +\infty \), then

\[ \|\Psi(t) - \Psi(0)\|_{L^2(\mathbb{R}^N)} \leq C \exp(C|t|), \forall t \in (-T(R), T(R)), \]

where the constant \( C \) depends only on \( E(\Psi(0)) \).

The proof of Proposition 2 statements i) to iii) is similar to that of [7] Proposition 3, and follows directly from classical semi-group theory with locally lipschitz nonlinearities (see e.g. [8] Section 4.3). For the proof of iv) we invoke the conservation of energy for sufficiently regular solutions (say in \( U + H^{s+2}(\mathbb{R}^N) \)) and then pass to the limit using well-posedness in \( U + H^s(\mathbb{R}^N) \). This only requires \( s > 1 \). For the proof of v), we refer to [7] Lemma 3.

**Remark 6.** In view of Proposition 2, if \( s > 1 + N/2 \) then for \( \Psi^0 \in V + H^{s+1}(\mathbb{R}^N) \) there exists a maximal time of existence \( T^*(\Psi^0) \) and a unique solution \( u \in C([0, T^*(\Psi^0)]; \Psi^0 + H^{s+1}(\mathbb{R}^N)) \) such that \( \Psi(t) = \Psi^0 \). Moreover, either

\[ T^*(\Psi^0) = +\infty \quad \text{or} \quad \limsup_{t \to T^*(\Psi^0)} \|\Psi(t) - \Psi(0)\|_{H^{s+1}(\mathbb{R}^N)} = +\infty, \]

and the map \( \Psi^0 \mapsto T^*(\Psi^0) \) is upper semi continuous for the \( H^{s+1} \) distance.

3. Proof of Proposition 1, and related results

Setting \( X \equiv (b, \text{Re} z, \text{Im} z) \in \mathbb{R}^{2N+1} \), system (9) may be recast in a more abstract form as

\[ \partial_t X = \sum_{j=1}^N A^j X + N_\varepsilon(X) \tag{11} \]

where the \((2N + 1)\)-matrices \( A^j \) are symmetric, and represent the linear one order terms of the r.h.s. of the system, whereas \( N_\varepsilon \) stands for the nonlinear and second order terms. The matrices \( A^j \) are constant, and contain terms which diverge as \( \varepsilon^{-1} \). If the term \( N_\varepsilon \) were not
present in (11), then one would have a linear symmetric hyperbolic system, and therefore conservation of all the \(H^k\) norms of \(X\). Indeed, if

\[
\partial_t Y = \sum_{j=1}^{N} A_j^s \partial_j Y
\]

then,

\[
\frac{1}{2} \frac{d}{dt} \| D^k Y \|_{L^2}^2 = \int_{\mathbb{R}^N} \langle D^k \partial_t Y, D^k Y \rangle = \sum_{j=1}^{N} \int_{\mathbb{R}^N} \langle A_j^s \partial_j D^k Y, D^k Y \rangle,
\]

and, using the symmetry of the matrices,

\[
\sum_{j=1}^{N} \int_{\mathbb{R}^N} \langle A_j^s \partial_j D^k Y, D^k Y \rangle = \sum_{j=1}^{N} \int_{\mathbb{R}^N} \langle \partial_j D^k Y, A_j^s D^k Y \rangle = -\sum_{j=1}^{N} \int_{\mathbb{R}^N} \langle D^k Y, A_j^s \partial_j D^k Y \rangle.
\]

Therefore \(\| D^k Y \|_{L^2}^2\) is time independent.

Owing to the additional term \(N^s(X)\), proving Sobolev estimates (or even energy estimates) for (11) is more involved. The reason why is that the function \(N^s\) contains terms of rather different nature from the “algebraic” point of view:

- semi-linear first order terms, namely \(-\operatorname{Re} z \nabla b - b \operatorname{div}(\operatorname{Re} z)\), and \(-\nabla \left( \frac{\varepsilon}{\sqrt{2}} \right)\);
- the linear second order term \(-i\Delta z\).

It is not clear however that adding this latter terms to (12) would not change the computation in (13). To deal with the semi-linear first order terms, we will have to introduce the quantity \(\Gamma^s(b, z)\) which is different from \(\| D^s X \|_{L^2}\) since the \(z\) part is weighted by the weight \(1 + \frac{\varepsilon}{\sqrt{2}} b\).

This weight plays somehow the role of a symmetrizer. To control its influence (in particular on the second order term), we invoke the relation between the weight and \(z\), namely

\[
-\nabla \left( 1 + \frac{\varepsilon}{\sqrt{2}} b \right) = \left( 1 + \frac{\varepsilon}{\sqrt{2}} b \right) \operatorname{Im} z
\]

which, in some sense, represents a gain of one derivative. When \(s\) is an integer, the computation is a little more explicit. Therefore we present that case first.

3.1. **Proof when \(s\) is an integer.** In this paragraph, we assume that \(s = k\) for some \(k \in \mathbb{N}\). Throughout, it is understood that for \(z_1 \in \mathbb{C}^N\) and \(z_2 \in \mathbb{C}^N\) the notation \(\langle z_1, z_2 \rangle\) stands for the inner product in \(\mathbb{R}^{2N}\) between the vectors \((\operatorname{Re} z_1, \operatorname{Im} z_1)\) and \((\operatorname{Re} z_2, \operatorname{Im} z_2)\). We first compute the time derivative of \(\Gamma^k(b, z)\), namely we have

\[
\frac{d}{dt} \int_{\mathbb{R}^N} \langle 1 + \frac{\varepsilon}{\sqrt{2}} b \rangle \langle D^k z, D^k z \rangle + \langle D^k b, D^k b \rangle
\]

\[
= 2 \int_{\mathbb{R}^N} \langle 1 + \frac{\varepsilon}{\sqrt{2}} b \rangle \langle D^k z, D^k \partial_z z \rangle + \langle D^k b, D^k \partial_z b \rangle + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial_z b \langle D^k z, D^k z \rangle
\]

\[
= I_1 + I_2 + I_3.
\]

**Step 1:** Expansion of \(I_1\) and \(I_2\).

In \(I_1 + I_2\), we replace \(\partial_z z\) and \(\partial_z b\) by their values according to (9), and expand the corresponding expressions. This yields

\[
I_1 = 2(I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5}) \quad \text{and} \quad I_2 = 2(I_{2,1} + I_{2,2})
\]
where

\[ I_{1,1} = \int_{\mathbb{R}^N} \langle D^k z, D^k (\frac{-\sqrt{\varepsilon}}{\varepsilon} \nabla b) \rangle, \quad I_{2,1} = \int_{\mathbb{R}^N} \langle D^k b, D^k (\frac{-\sqrt{\varepsilon}}{\varepsilon} \text{div}(\text{Re} z)) \rangle, \]

\[ I_{1,2} = \int_{\mathbb{R}^N} b \langle D^k z, D^k (-\nabla b) \rangle, \quad I_{2,2} = \int_{\mathbb{R}^N} \langle D^k b, D^k (-\text{div}(b \text{Re} z)) \rangle. \]

\[ I_{1,3} = \int_{\mathbb{R}^N} \langle D^k z, D^k (i \Delta z) \rangle, \]

\[ I_{1,4} = \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} b \langle D^k z, D^k (i \Delta z) \rangle, \]

\[ I_{1,5} = \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle D^k z, D^k (\nabla (\frac{z \cdot \overline{z}}{2})) \rangle. \]

**Step 2:** Both \( I_{1,3} \) and \( I_{1,1} + I_{2,1} \) vanish.

This is a consequence of the properties of the linear part of the equation as explained before. It follows by integration by parts, and, for \( I_{1,1} + I_{2,1} \), from the fact that \( b \) is real valued.

**Step 3:** Estimates for \( I_{1,2} + I_{2,2} \).

Integrating by parts in \( I_{2,2} \) then using Leibniz formula, we obtain

\[ I_{1,2} + I_{2,2} = \int_{\mathbb{R}^N} \langle D^k (\nabla b), D^k (b \text{Re} z) - b D^k z \rangle \]

\[ = \int_{\mathbb{R}^N} \langle D^k (\nabla b), D^k b \text{Re} z \rangle + \sum_{j=1}^{k-1} \int_{\mathbb{R}^N} \langle D^k (\nabla b), D^j b D^{k-j} \text{Re} z \rangle \]

\[ = \int_{\mathbb{R}^N} \langle \nabla |D^k b|^2 / 2, \text{Re} z \rangle - \sum_{j=1}^{k-1} \int_{\mathbb{R}^N} \langle D^k b, \text{div}(D^j b D^{k-j} \text{Re} z) \rangle \]

\[ = - \int_{\mathbb{R}^N} \frac{|D^k b|^2}{2} \text{div}(\text{Re} z) - \sum_{j=1}^{k-1} \int_{\mathbb{R}^N} \langle D^k b, \text{div}(D^j b D^{k-j} \text{Re} z) \rangle. \]

For the first term, we write

\[ \left| \int_{\mathbb{R}^N} |D^k b|^2 \text{div}(\text{Re} z) \right| \leq \||Dz||_{L^\infty} \|b\|_{H^k}^2. \]

In order to bound the second term, one may rely on Lemma 3 in the Appendix which yields, for \( j = 1, \cdots, k - 1, \)

\[ \int_{\mathbb{R}^N} \langle D^k b, \text{div}(D^j b D^{k-j}) \rangle \leq C \| (Db, Dz) \|_{L^\infty} \left( \|b\|_{H^k}^2 + \|z\|_{H^k}^2 \right). \]

Combining the two last inequalities we obtain

\[ |I_{1,2} + I_{2,2}| \leq C \| (Db, Dz) \|_{L^\infty} \left( \|b\|_{H^k}^2 + \|z\|_{H^k}^2 \right). \]

**Step 4:** Estimates for \( 2I_{1,4} + 2I_{1,5} + I_3 \).

The sum of these three terms presents a remarkable compensation. Indeed, integrating by parts in \( I_{1,4} \) we obtain

\[ I_{1,4} = - \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \nabla b \langle D^k z, D^k (i \nabla z) \rangle, \]

where we used the pointwise identity \( \langle D^k (\nabla z), D^k (i \nabla z) \rangle = 0. \)
Using identity (14), we are led to
\[ I_{1,4} = \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}}b)\langle D^k z, D^k(i\nabla z)\rangle \text{Im} z. \]

Next, we turn to \( I_{1,5} \). First, expanding \( \nabla (\frac{z^2}{2}) \), we get
\[
I_{1,5} = - \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}}b)\langle D^k z, D^k (z \cdot \nabla z) \rangle
\]
\[ = - \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}}b)\langle D^k z, D^k (\nabla z) \cdot z \rangle - \sum_{j=0}^{k-1} \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}}b)\langle D^k z, D^j (\nabla z) \cdot D^{k-j} z \rangle
\]
\[ = I'_{1,5} + I''_{1,5}. \]

Relying once more on Lemma 3 of the Appendix, we obtain for \( j = 0, \cdots, k - 1 \),
\[
I''_{1,5} \leq C (1 + \varepsilon \|b\|_{L^\infty}) \|Dz\|_{L^\infty} \|z\|_{H^k}^2.
\]

To estimate the first term \( I'_{1,5} \), we use the algebraic identity
\[
\langle z_1, \zeta z_2 \rangle = \langle z_1, z_2 \rangle \text{Re} \zeta + \langle z_1, iz_2 \rangle \text{Im} \zeta \quad \forall z_1, z_2 \in \mathbb{C}^N, \forall \zeta \in \mathbb{C}.
\]

This yields for all \( j \in \{1, \cdots, N\} \),
\[
(1 + \frac{\varepsilon}{\sqrt{2}}b)\langle D^k z, D^k (\partial_j z) \cdot z^j \rangle = (1 + \frac{\varepsilon}{\sqrt{2}}b)\left[ \langle D^k z, D^k (\partial_j z) \rangle \text{Re} z^j + \langle D^k z, D^k (i\partial_j z) \rangle \text{Im} z^j \right]
\]
\[ = (1 + \frac{\varepsilon}{\sqrt{2}}b)\left[ \text{Re} z^j \partial_j \left( \frac{|D^k z|^2}{2} \right) + \langle D^k z, D^k (i\partial_j z) \rangle \text{Im} z^j \right]
\]
so that, integrating by parts in the first integral,
\[
2(I'_{1,5} + I_{1,4}) + I_3 = - \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}}b)\text{Re} z \cdot \nabla |D^k z|^2 + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial_b (D^k z, D^k z)
\]
\[ = \int_{\mathbb{R}^N} \text{div} \left( (1 + \frac{\varepsilon}{\sqrt{2}}b)\text{Re} z \right) |D^k z|^2 + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial_b |D^k z|^2.
\]
Since system (9) is satisfied, one can now conclude that
\[
2I'_{1,5} + 2I_{1,4} + I_3 = 0.
\]

**Step 5:** Proof of Proposition 1 completed when \( s \) is an integer.

Under condition (10), there exists a constant \( C \) depending only on \( k, m \) and such that
\[
\|(b, z)\|_{H^k}^2 \leq C (E_\varepsilon (Y_\varepsilon) + \Gamma^k (b, z)).
\]

Hence, combining (15), (16), (17) and (18) completes the proof. \( \square \)

3.2. Generalization of Proposition 1. In this section, we extend Proposition 1 to the case of Sobolev spaces with noninteger exponents. The proof that we propose is based on a Littlewood-Paley decomposition and actually covers the case of Besov spaces \( B^{s,r}_2 \), as well.

We first recall the notion of Littlewood-Paley decomposition. Let \( (\chi, \varphi) \) being smooth compactly supported functions such that

1. \( \chi \) is supported in \( B(0, 4/3) \),
2. \( \varphi \) is supported in the annulus \( C(0, 3/4, 8/3) \),
3. \( \forall \xi \in \mathbb{R}^N, \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q} \xi) = 1. \)
We denote\(^3\) \(S_q := \chi(2^{-q}D), \Delta_q := \varphi(2^{-q}D)\) for \(q \in \mathbb{N}\), and \(\Delta_{-1} := S_0 = \chi(D)\). We have \(S_q = \sum_{p=-1}^{q-1} \Delta_p\) and \(u = \sum_{q \geq -1} \Delta_q u\) whenever \(u\) is in \(\mathcal{S}'(\mathbb{R}^N)\). Moreover, we have

\[
|p - q| > 1 \implies \Delta_q \Delta_p u = 0 \quad \text{and} \quad |p - q| > 4 \implies \Delta_q (S_{p-1} u \Delta_p v) = 0.
\]

The Littlewood-Paley decomposition is defined by the identity

\[
u = \sum_{q \geq -1} \Delta_q u
\]

and makes sense for arbitrary tempered distributions. Furthermore, it is not difficult to check that \(H^s(\mathbb{R}^N)\) coincides with the space of tempered distributions \(u\) such that

\[
\left( \sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty
\]

and the left-hand side of this inequality defines a norm on \(H^s(\mathbb{R}^N)\) which is equivalent to the usual one. More generally, one can define the Besov space \(B^s_{2,r}(\mathbb{R}^N)\) as the set of tempered distributions \(u\) such that

\[
\|u\|_{B^s_{2,r}} := \|2^{qs} \|\Delta_q u\|_{L^2}\|_{L^r} < \infty.
\]

For \(r = 2\), we recover the usual Sobolev space since \(H^s(\mathbb{R}^N) = B^s_{2,2}(\mathbb{R}^N)\) with equivalent norms.

The remainder of this section is devoted to the proof of the following proposition.

**Proposition 3.** Let \(s > 0\) and \(r \in [1, \infty]\). Assume that \(\Upsilon_\varepsilon\) is a solution to (5) such that \((b, z) \in C^1([0, T]; B^{s+1}_{2,2}) \cap C^0([0, T]; W^{1,\infty})\) for some \(T > 0\), and that \(m := \inf_{x,t}|\Upsilon_\varepsilon(x,t)| > 0\). There exists a constant \(K\) depending only on \(m, s\) and \(N\) such for any time \(t \in [0, T]\) we have

\[
\frac{d}{dt} \int (1 + \frac{\varepsilon}{\sqrt{2}}) 2^{2qs} |\Delta_q z|^2 + 2^{2qs} |\Delta_q b|^2 \leq K c_q (1 + \varepsilon \|b\|_{L^2}) \|(Db, Dz)\|_{L^2} \|(Db, Dz)\|_{B^{s-1}_{2,1}}
\]

where the sequence \((c_q)_{q \geq -1}\) satisfies \(\|(c_q)\|_{\ell^r} = 1\).

**Remark 7.** Remark that if we assume that

\[
|\Upsilon_\varepsilon(x,t)| \leq M \quad \text{for all} \quad (x, t) \in \mathbb{R}^N \times [0, T],
\]

then a \(\ell^r\) summation and a time integration in (21) implies that we have for some constant \(K\) depending only on \(M, s\) and \(N\),

\[
\|b, z\|_{B^s_{2,r}} \leq K \left( \|(b, z)(0)\|_{B^s_{2,r}} + \int_0^t \|\langle Db, Dz\rangle(\tau)\|_{L^\infty} \|(b, z)(\tau)\|_{B^s_{2,r}} \ d\tau \right).
\]

In particular, taking \(r = 2\) yields

\[
\|b, z\|_{H^s} \leq K \left( \|(b, z)(0)\|_{H^s} + \int_0^t \|\langle Db, Dz\rangle(\tau)\|_{L^\infty} \|(b, z)(\tau)\|_{H^s} \ d\tau \right).
\]

\(^3\)According to a classical convention, \(\psi(D)\) will stand for the Fourier multiplier of symbol \(\psi(\xi)\).
Proof. The proof works follows almost the same lines as the case in the Sobolev case with integer exponents: the main point is to replace the differential operator $D^k$ by the Littlewood-Paley operator $\Delta_q$. Throughout the computation, several commutators will appear, which may be dealt with thanks to Lemma 4. The starting point is the following computation

$$
\frac{d}{dt} \int_{\mathbb{R}^N} \left\{ (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta_q z, \Delta_q z \rangle + |\Delta_q b|^2 \right\} \\
= 2 \int_{\mathbb{R}^N} \left\{ (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta_q z, \Delta_q \partial_t z \rangle + \Delta_q b \Delta_q \partial_t b \right\} + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial_t b \langle \Delta_q z, \Delta_q z \rangle \\
= I_1^q + I_2^q + I_3^q.
$$

As in Section 3.1, we split $I_1^q$ and $I_2^q$ into

$$
I_1^q = 2(I_{1,1}^q + I_{1,2}^q + I_{1,3}^q + I_{1,4}^q + I_{1,5}^q) \quad \text{and} \quad I_2^q = 2(I_{2,1}^q + I_{2,2}^q)
$$

where

$$
I_{1,1}^q = \int_{\mathbb{R}^N} \langle \Delta_q z, \Delta_q (-\sqrt{\frac{\varepsilon}{2}} \nabla b) \rangle, \\
I_{1,2}^q = \int_{\mathbb{R}^N} b \langle \Delta_q z, \Delta_q (-\nabla b) \rangle, \\
I_{1,3}^q = \int_{\mathbb{R}^N} \langle \Delta_q z, \Delta_q (i \Delta z) \rangle, \\
I_{1,4}^q = \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} b \langle \Delta_q z, \Delta_q (i \Delta z) \rangle, \\
I_{1,5}^q = \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta_q z, \Delta_q (-\nabla (\frac{z}{2}) \rangle,
$$

As in Section 3.1, both $I_{1,3}^q$ and $I_{1,1}^q + I_{2,1}^q$ vanish. Next, in order to deal with $I_{1,2}^q + I_{2,2}^q$, one may integrate by parts in $I_{2,2}^q$. We find that

$$
I_{1,2}^q + I_{2,2}^q = \int_{\mathbb{R}^N} \langle \nabla \Delta_q b, \Delta_q (b \text{Re} z) - b \Delta_q \text{Re} z \rangle \\
= \int_{\mathbb{R}^N} \Delta_q b \langle \Delta_q \text{div Re} z - \Delta_q (b \text{div Re} z) \rangle + \int_{\mathbb{R}^N} \Delta_q b (\nabla b \cdot \Delta_q \text{Re} z - \Delta_q (\nabla b \cdot \text{Re} z)) \\
= \int_{\mathbb{R}^N} \Delta_q b [b, \Delta_q] \text{div Re} z + \int_{\mathbb{R}^N} \Delta_q b \nabla b \cdot \Delta_q \text{Re} z \\
- \int_{\mathbb{R}^N} \Delta_q b \Delta_q \nabla b \cdot \text{Re} z + \int_{\mathbb{R}^N} \Delta_q b [\text{Re} z, \Delta_q] \cdot \nabla b \\
= \int_{\mathbb{R}^N} \Delta_q b [b, \Delta_q] \text{div Re} z + \int_{\mathbb{R}^N} \Delta_q b \nabla b \cdot \Delta_q \text{Re} z \\
+ \frac{1}{2} \int_{\mathbb{R}^N} (\Delta_q b)^2 \text{div Re} z + \int_{\mathbb{R}^N} \Delta_q b [\text{Re} z, \Delta_q] \cdot \nabla b.
$$

For the second and third term, we have

$$
\left| \int_{\mathbb{R}^N} \Delta_q b \nabla b \cdot \Delta_q \text{Re} z \right| \leq ||Db||_{L^\infty} ||\Delta_q b||_{L^2} ||\Delta_q z||_{L^2},
$$

$$
\left| \int_{\mathbb{R}^N} (\Delta_q b)^2 \text{div Re} z \right| \leq ||\text{div} z||_{L^\infty} ||\Delta_q b||_{L^2}^2.
$$
The first and last terms may be bounded according to Lemma 4. We find that for some sequence \((c_\ell)_{\ell \geq 1}\) such that \(\|(c_\ell)\|_{\ell^r} = 1\),

\[
\left| \int_{\mathbb{R}^N} \Delta \varphi \left[ b, \Delta \varphi \right] \text{div} \, z \right| \leq C c_\ell 2^{-q_\ell} \left( \|Db\|_{L^\infty} \|\text{div} \, z\|_{B^{r_{\ell-1}}_{2,r}} + \|\text{div} \, z\|_{L^\infty} \|Db\|_{B^{r_{\ell-1}}_{2,r}} \right) \|\Delta \varphi\|_{L^2},
\]

\[
\left| \int_{\mathbb{R}^N} \Delta \varphi \left[ b \text{Re} \, z, \Delta \varphi \right] \nabla b \right| \leq C c_\ell 2^{-q_\ell} \left( \|Dz\|_{L^\infty} \|Db\|_{B^{r_{\ell-1}}_{2,r}} + \|Db\|_{L^\infty} \|Dz\|_{B^{r_{\ell-1}}_{2,r}} \right) \|\Delta \varphi\|_{L^2}.
\]

Combining the previous inequalities, we obtain

\[
\left| I^q_{1,2} + I^q_{2,2} \right| \leq C c_\ell 2^{-q_\ell} \left( \|Dz\|_{L^\infty} \|Db\|_{B^{r_{\ell-1}}_{2,r}} + \|Db\|_{L^\infty} \|Dz\|_{B^{r_{\ell-1}}_{2,r}} \right) \|\Delta \varphi\|_{L^2}.
\]

To finish with, let us prove that

\[
\left| 2I^q_{1,4} + 2I^q_{1,5} + I^q_3 \right| \leq C c_\ell (1 + \varepsilon \|b\|_{L^\infty}) \|Dz\|_{L^\infty} \|Dz\|_{B^{r_{\ell-1}}_{2,r}} \|\Delta q z\|_{L^2}.
\]

Integrating by parts in \(I^q_{1,4}\), and using the pointwise identity \(\langle \Delta q \nabla z, \Delta q (i \nabla z) \rangle = 0\) and (14), we derive the identity

\[
I^q_{1,4} = \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta q z, \Delta q (i \nabla z) \rangle \text{Im} z.
\]

Next, expanding \(\nabla (\frac{\varepsilon z}{2})\), we are led to

\[
I^q_{1,5} = -\int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta q z, \Delta q (\nabla z) \cdot z \rangle + \int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \langle \Delta q z, \Delta q \nabla z \cdot z - \Delta q (z \cdot \nabla z) \rangle
\]

\[
= I^q_{1,5} + I^{mq}_{1,5}.
\]

On the one hand, Lemma 4 ensures that \(I^{mq}_{1,5}\) may be bounded by the right-hand side of (23). On the other hand, mimicking the computations made in Section 3.1, we get

\[
2 \left( I^q_{1,5} + I^q_{1,4} \right) + I^q_3 = -\int_{\mathbb{R}^N} (1 + \frac{\varepsilon}{\sqrt{2}} b) \text{Re} \, z \cdot \nabla |\Delta q z|^2 + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial b \langle \Delta q z, \Delta q z \rangle
\]

\[
= \int_{\mathbb{R}^N} \text{div} \left( (1 + \frac{\varepsilon}{\sqrt{2}} b) \text{Re} \, z \right) |\Delta q z|^2 + \int_{\mathbb{R}^N} \frac{\varepsilon}{\sqrt{2}} \partial b |\Delta q z|^2
\]

so that, since system (9) is satisfied,

\[
2I^q_{1,5} + 2I^q_{1,4} + I^q_3 = 0.
\]

This completes the proof of (23), and thus of (21).

\[\square\]

4. PROOF OF THEOREMS 1 AND 2

We first notice that by Sobolev embedding and the definition of \(a_\varepsilon\) there exists a constant \(C_1(s, N) \geq 1\) independent of \(\varepsilon\) such that if \(C_1(s, N) \varepsilon \|a_\varepsilon\|_{H^{s+1}} \leq 1\) then

\[
\rho_{\varepsilon}^{s+1} \leq 2.
\]

The constant \(C(s, N)\) will be required to satisfy \(C(s, N) > C_1(s, N)\), so that in particular \(\rho_{\varepsilon}^{s+1}(\cdot, 0) < 2\). If we denote by \(\Psi^0\) the corresponding initial datum for \((GP)\) then one may prove that \(\Psi^0 \in \mathcal{V} + H^{s+1}(\mathbb{R}^N)\). In fact, it turns out that for any smooth nonnegative function \(\alpha\) compactly supported in \(\mathbb{R}^N\) and satisfying \(\int \alpha = 1\) the function \(U := \Psi \ast \alpha\) belongs to \(\mathcal{V}\) and \(\Psi^0 - U\) belongs to \(H^{s+1}(\mathbb{R}^N)\) (see e.g. [16]). Therefore, by virtue of Proposition 2, equation \((GP)\) possesses a unique solution \(\Psi\) in \(\Psi^0 + H^{s+1}\) on some time interval \([0, T]\).
Proof of Theorem 1

Step 1: In a first step, we assume that in addition \((a_0^0, u_0^0) \in H^{s+3} \times H^{s+2}\). By Proposition 2 iii) combined with an appropriate change of variable, equation (5) has a unique maximal solution \(Y_\varepsilon\) in \(V + C^1([0, T^*); H^{s+1}(\mathbb{R}^N))\). We introduce the stopping time
\[
t_0 = \sup \left\{ 0 \leq t < T^* \mid \text{s.t. } \rho^\pm_\varepsilon \leq 2 \text{ and } A(\tau) \leq 2A(0), \quad \forall \tau \in [0, t] \right\},
\]
where we have set
\[
A(t) := \Gamma^\varepsilon(b(\cdot, t), z(\cdot, t)) + E_\varepsilon(Y_\varepsilon(\cdot, t)).
\]
By continuity and the fact that \(\rho^\pm_\varepsilon \geq \frac{1}{2}\), we have \(t_0 > 0\). Next, we apply Proposition 1 on the interval \([0, t_0]\), which yields the inequality
\[
\frac{d}{dt} \Gamma^\varepsilon(b, z) \leq C_2(s, N) \| (Db, Dz) \|_{L^\infty} (\Gamma^\varepsilon(b, z) + E_\varepsilon(Y_\varepsilon)).
\]
On the one hand, by conservation of energy, we have on \([0, T]\),
\[
\frac{d}{dt} E_\varepsilon(Y_\varepsilon) = 0.
\]
On the other hand, by Sobolev embedding and (19), we have
\[
\| (Db, Dz) \|_{L^\infty}^2 \leq C_3(s, N) (\Gamma^\varepsilon(b, z) + E_\varepsilon(Y_\varepsilon)).
\]
Therefore, after summation we are led to
\[
\frac{d}{dt} A(t) \leq C_4(s, N) A(t)^{3/2} \quad \text{on } [0, t_0).
\]
Integrating this inequality we obtain
\[
A(t) \leq \frac{A(0)}{1 - C_4(s, N) \sqrt{A(0)/2}} \quad \text{whenever } t < t_1 := \min(t_0, \frac{2}{\sqrt{A(0)/C_4}}).
\]
Notice that, owing to (24) and to the definition of \(\Gamma^\varepsilon\), we have
\[
\frac{1}{2} \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s} \leq \sqrt{A(0)} \leq 2 \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s}.
\]
Therefore, choosing \(C(s, N)\) sufficiently large, we have, for \(t \leq t_* := \frac{1}{C(s, N) \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s}} \cdot \frac{1}{\sqrt{A(0)/C_4}}\),
\[
C_4(s, N) \sqrt{A(0)/2} \leq C_5(s, N) \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s} \cdot \frac{1}{C(s, N) \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s}} \leq \frac{1}{2},
\]
so that
\[
A(t) \leq 2A(0) \quad \text{whenever } t \leq \min(t_0, t_*).
\]

For such \(t\), we then have
\[
\varepsilon \| (a_\varepsilon(\cdot, t_\varepsilon), u_\varepsilon(\cdot, t_\varepsilon)) \|_{H^{s+1} \times H^s} \leq C_6(s, N) \varepsilon \| (a_0^0, u_0^0) \|_{H^{s+1} \times H^s} \leq \frac{C_6(s, N)}{C(s, N)}
\]
so that condition (24) is satisfied provided \(C(s, N)\) is chosen sufficiently large. It follows that \(t_0 > t_*\). The case where \(s \not\in \mathbb{N}\) follows from the same arguments. It suffices to apply Proposition 3 with \(r = 2\) instead of Proposition 1. The conclusion in Theorem 1 therefore holds in the case considered in this step.

Step 2: The general case. In order to prove Theorem 1 in the general case, we mollify the initial datum by an approximation of the identity and then rely on Case 1 and the continuity of the flow map on \(C^0([0, T]; \Psi^0 + H^{s+1})\). The details are standard and left to the reader. \(\Box\)

\footnote{For expository purposes, we assume here that \(s\) is an integer number.}
Proof of Theorem 2

We notice that \((\tilde{a}_\varepsilon, \tilde{u}_\varepsilon) := (a_\varepsilon, u_\varepsilon) - (a, u)\) satisfies the wave equation

\[
\begin{aligned}
\partial_t \tilde{a}_\varepsilon + \sqrt{2} \text{div} \tilde{u}_\varepsilon &= \varepsilon \text{div} f_\varepsilon^1, \\
\partial_t \tilde{u}_\varepsilon + \sqrt{2} \nabla \tilde{u}_\varepsilon &= \varepsilon (g_\varepsilon^1 + 2g_\varepsilon^2)
\end{aligned}
\]

with null initial datum and

\[f_\varepsilon^1 := -a_\varepsilon u_\varepsilon, \quad g_\varepsilon^1 := -|u_\varepsilon|^2 \quad \text{and} \quad g_\varepsilon^2 := \frac{\Delta \sqrt{2 + \varepsilon a_\varepsilon}}{\sqrt{2 + \varepsilon a_\varepsilon}}.\]

Using basic energy estimates for the wave equation, we readily get

\[
\| (\tilde{a}_\varepsilon, \tilde{u}_\varepsilon) (t) \|_{H^{s-2}} \leq \int_0^t \left( \| f_\varepsilon^1 \|_{H^{s-1}} + \| g_\varepsilon^1 \|_{H^{s-1}} + 2 \| g_\varepsilon^2 \|_{H^{s-1}} \right) d\tau.
\]

Now, as \(H^{s-1}\) is an algebra, one can write

\[
\| f_\varepsilon^1 \|_{H^{s-1}} \leq C \| a_\varepsilon \|_{H^{s-1}} \| u_\varepsilon \|_{H^{s-1}} \quad \text{and} \quad \| g_\varepsilon^1 \|_{H^{s-1}} \leq C \| u_\varepsilon \|_{H^{s-1}}^2,
\]

whence, according to Theorem 1,

\[
\| f_\varepsilon^1 \|_{H^{s-1}} + \| g_\varepsilon^1 \|_{H^{s-1}} \leq C \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s}^2 \quad \text{for all} \quad t \in [0, T_\varepsilon).
\]

In order to bound the last term in (26), we notice that, under condition (24), there exist two smooth functions \(K_1\) and \(K_2\) vanishing at 0 and such that

\[
g_\varepsilon^2 = \frac{\varepsilon}{4} \Delta a_\varepsilon + \varepsilon K_1 (a_\varepsilon) \Delta a_\varepsilon + \varepsilon \nabla a_\varepsilon \cdot \nabla K_2 (a_\varepsilon).
\]

Therefore, applying Proposition 6 yields

\[
\| g_\varepsilon^2 \|_{H^{s-1}} \leq C (\varepsilon) \| a_\varepsilon \|_{H^{s+1}} + \varepsilon^2 \| a_\varepsilon \|_{H^{s-1}} \| a_\varepsilon \|_{H^{s+1}} + \varepsilon \| a_\varepsilon \|_{H^s}^2,
\]

so that, using the bounds provided by Theorem 1,

\[
\| g_\varepsilon^2 \|_{H^{s-1}} \leq C (\varepsilon) \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s} + \varepsilon^2 \| (a_\varepsilon^0, u_\varepsilon^0) \|_{H^{s+1} \times H^s}^2 \quad \text{for all} \quad t \in [0, T_\varepsilon).
\]

Using inequalities (27) and (28) in (26), it now easy to complete the proof of Theorem 2. \( \square \)

Remark 8. Using Proposition 3, the results in Theorem 1 and Theorem 2 can be extended to the spaces \(B^{s+1}_{2,r} \times B^s_{2,r}\) instead of \(H^{s+1} \times H^s\) whenever \(B^s_{2,r}\) embeds continuously in \(W^{1,\infty}\). The Besov spaces framework allows to get a result for the critical regularity \(s = 1 + N/2\) since \(B^{1+N/2}_{2,1} \hookrightarrow W^{1,\infty}\) (whereas \(H^{1+N/2} \not\hookrightarrow W^{1,\infty}\)).

5. Proof of Theorems 3 and 4

As mentioned in the introduction, our proofs will be based on the dispersive properties of the linearized system (7) about \((0,0)\), namely

\[
\begin{aligned}
\partial_t b + \sqrt{2} \text{div} v &= f, \\
\partial_t v + \frac{\sqrt{2}}{\varepsilon} \nabla b - \sqrt{2} \nabla \Delta b &= g.
\end{aligned}
\]

More precisely, we shall use the following result, the proof of which is presented in the Appendix.
Proposition 4. Let \((b,v)\) solve system (29) on \([0,T] \times \mathbb{R}^N\). There exists a smooth compactly supported function \(\chi\) with value 1 near the origin such that for all \(\varepsilon > 0\), if we denote
\begin{equation}
(30) \quad b_\varepsilon := \chi(\varepsilon D)b, \quad b := (1 - \chi(\varepsilon D))b, \quad v_\varepsilon := \chi(\varepsilon D)v \quad \text{and} \quad v := (1 - \chi(\varepsilon D))v
\end{equation}
then the following a priori estimates hold true for some constant \(C\) depending only on \(N\):
- if \(N \geq 4\) then \(\| (b,v) \|_{L^4_x(C^{0,1})} \leq C \varepsilon \| (b_\varepsilon, v_\varepsilon) \|_{B_{4,1}^{N+\frac{1}{4}}(\mathbb{R}^N)} + \| (f,g) \|_{L^4_x(B_{4,1}^{N+\frac{1}{4}})}\)
- if \(N = 3\) then for all \(p > 2\), \(\| (b,v) \|_{L^p_x(C^{0,1})} \leq C \varepsilon \| (b_\varepsilon, v_\varepsilon) \|_{B_{2,1}^{N+\frac{5}{4}}(\mathbb{R}^N)} + \| (f,g) \|_{L^p_x(B_{2,1}^{N+\frac{5}{4}})}\)
- if \(N = 2\) then \(\| (b_\varepsilon, v_\varepsilon) \|_{B_{4,1}^{N+\frac{1}{4}}(\mathbb{R}^N)} \leq C \| (b,v) \|_{B_{4,1}^{N+\frac{1}{4}}(\mathbb{R}^N)} + \| (f,g) \|_{L^4_x(B_{4,1}^{N+\frac{1}{4}})}\)
- if \(N = 2\) then for all \(p > 2\),
\(\| (\varepsilon \nabla b, v)_h \|_{L^p_x(C^{0,1})} \leq C \| (\varepsilon \nabla b_\varepsilon, v_\varepsilon)_h \|_{B_{2,1}^{N+\frac{5}{4}}(\mathbb{R}^N)} + \| (\varepsilon \nabla f, g)_h \|_{L^p_x(B_{2,1}^{N+\frac{5}{4}})}\).

Throughout the proof of Theorem 3, we shall use freely the following inequalities which are proved in the Appendix:

Lemma 1. With the notation used in Proposition 4, there exists a constant \(C > 0\) depending only on \(N\) and \(\sigma > 0\) such that, under condition (24),
\begin{equation}
(31) \quad C^{-1} \| (b,z) \|_{B_{2,r}^\sigma} \leq \| (b,v) \|_{B_{2,r}^\sigma} + \| (\varepsilon \nabla b, v)_h \|_{B_{2,r}^\sigma} \leq C \| (b,z) \|_{B_{2,r}^\sigma}, \quad \text{for} \quad r \in [1, \infty],
\end{equation}
\begin{equation}
(32) \quad C^{-1} \| (b,z) \|_{C^{0,1}} \leq \| (b,v) \|_{C^{0,1}} + \| (\varepsilon \nabla b, v)_h \|_{C^{0,1}} \leq C \| (b,z) \|_{C^{0,1}}.
\end{equation}

5.1. Proof of Theorem 3 in the case \(N \geq 4\). According to Proposition 4, the linear system (29) possesses better dispersive properties in high dimension \(N \geq 4\). Therefore, we shall first prove Theorem 3 in this case.

Assume that we are given some map \(\Psi\) solution of \((GP)\) with datum \(\Psi^0\), satisfying \((b,z) \in C^1([0,T];H^s)\) and
\begin{equation}
(33) \quad \frac{1}{2} \leq \rho \leq 2 \quad \text{on} \quad [0,T].
\end{equation}
Integrating the inequality in Proposition 3 in the case \(r = 2\) and taking inequality (33) into account yields for all \(t \in [0,T]\),
\(\| (b,z)(t) \|_{H^s} \leq 2 \| (b_0,z_0) \|_{H^s} + C \int_0^t \| (D_b, Dz) \|_{L^\infty} \| (b,z) \|_{H^s} d\tau\),
whence, according to Cauchy-Schwarz inequality and to inequality (32),
\begin{equation}
(34) \quad \| (b,z) \|_{L^\infty_t(H^s)} \leq 2 \| (b_0,z_0) \|_{H^s} + C T^{\frac{1}{2}} \| (b,v) \|_{B_{4,1}^{N+\frac{1}{4}}(\mathbb{R}^N)} + \| (\varepsilon \nabla b, v)_h \|_{B_{2,1}^{N+\frac{5}{4}}(\mathbb{R}^N)} \| (b,z) \|_{L^\infty_t(H^s)}.
\end{equation}
In order to bound \((b,v)\) and \((\varepsilon \nabla b, v)_h\) in \(L^2([0,T];C^{0,1})\), we shall take advantage of Proposition 4. As \(N \geq 4\) and \((b,v)\) satisfies system (29) with source terms
\(f := -\text{div}(bv)\) and \(g := g_1 + g_2\) with \(g_1 := -\nabla |v|^2\) and \(g_2 := \sqrt{2} \varepsilon \nabla \text{div}(b \text{Im} z)\),
\(^5\)To get the formula for \(g_2\), it suffices to use (7) and the identities (6), (14) which imply that
\[
\frac{\Delta \rho}{\rho} = -\text{div} \text{Im} z \quad \text{and} \quad \frac{\varepsilon}{\sqrt{2}} \Delta b = -\frac{\varepsilon}{\sqrt{2}} \text{div}(b \text{Im} z) - \text{div} \text{Im} z.
\]
Therefore, owing to the support properties of function $\phi$, for all $\varepsilon > 0$, we have
\begin{equation}
\| \phi_h \|_{B^{\frac{N}{2}}_{2,1}} \leq C \varepsilon^\alpha \| \phi_h \|_{B^{\frac{N}{2}+\alpha}_{2,1}}.
\end{equation}
Indeed, owing to the support properties of function $\chi$, one may write for some suitable $\varepsilon_0 > 0$,
\begin{equation}
\| \phi_h \|_{B^{\frac{N}{2}}_{2,1}} = \sum_{2^\nu \geq \varepsilon_0} 2^{\nu \alpha} \| \Delta_q \phi_h \|_{L^2} \leq \left( \frac{\varepsilon}{\varepsilon_0} \right)^\alpha \sum_{2^\nu \geq \varepsilon_0} 2^{\nu (\alpha+\varepsilon)} \| \Delta_q \phi_h \|_{L^2} \leq \left( \frac{\varepsilon}{\varepsilon_0} \right)^\alpha \| \phi_h \|_{B^{\frac{N}{2}+\alpha}_{2,1}}.
\end{equation}
Therefore
\begin{equation}
\varepsilon^\frac{1}{2} \| (b_0, v_0) \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} + \| (\varepsilon Db_0, v_0)h \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} \leq C \varepsilon^\frac{1}{2} \| (b_0, v_0) \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} + \| (\varepsilon Db_0, v_0)h \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}})
\end{equation}
and applying inequality (31) gives (37).

It follows from the previous discussion that the problem reduces to finding suitable bounds for $(f, g) \in L^1([0, T]; B^{\frac{N}{2}+\frac{1}{2}}_{2,1})$, and for $(\varepsilon \nabla f, g)_h$ in $L^1([0, T]; B^{\frac{N}{2}}_{2,1})$. For that purpose, we use standard tame estimates for the product of functions in Besov spaces which are stated in Proposition 5. This yields
\begin{equation}
\| f_{\ell} \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} \leq C \| b \|_{L^\infty} \| v \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} + \| v \|_{L^\infty} \| b \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}},
\end{equation}
\begin{equation}
\| (g_{\ell})_h \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} \leq C \| v \|_{L^\infty} \| v \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}}.
\end{equation}
To deal with the term $(g_2)_\ell$, we notice that for all $\sigma \in \mathbb{R}$ and $\phi \in B^\sigma_{2,1}$, we have
\begin{equation}
\| \varepsilon \nabla \phi \|_{B^\sigma_{2,1}} \leq C \| \phi \|_{B^\sigma_{2,1}}.
\end{equation}
Indeed, owing to the support properties of $\text{Supp} \, \hat{\phi}_\ell$ and Parseval formula, we have for some $\varepsilon_1 > \varepsilon_0$,
\begin{equation}
\| \varepsilon \nabla \phi \|_{B^\sigma_{2,1}} = \sum_{2^\nu \leq \varepsilon_1} 2^{\nu \sigma} \varepsilon \| \nabla \Delta_q \phi \|_{L^2} \leq \sum_{2^\nu \leq \varepsilon_1} (\varepsilon 2^\nu) 2^{\nu \sigma} \| \Delta_q \phi \|_{L^2} \leq \varepsilon \| \phi \|_{B^\sigma_{2,1}}.
\end{equation}
Therefore,
\begin{equation}
\| (g_2)_\ell \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} \leq C \| b \|_{L^\infty} \| \text{Im} \, z \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}},
\end{equation}
\begin{equation}
\leq C \| b \|_{L^\infty} \| \text{Im} \, z \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} + \| \text{Im} \, z \|_{L^\infty} \| b \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}}.
\end{equation}
Summing the inequalities above, we end up with
\begin{equation}
\| (f, g)_\ell \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}} \leq C \| (b, z) \|_{L^\infty} \| (b, z) \|_{B^{\frac{N}{2}+\frac{1}{2}}_{2,1}}.
\end{equation}
To deal with the high-frequency terms, we use Proposition 5 once more. We obtain
\begin{align}
\| \varepsilon \nabla f_h \|_{B^{s}_{2,1}} & \leq C \varepsilon \| b v \|_{B^{s}_{2,1}} \leq C \varepsilon (\| b \|_{L^\infty} \| v \|_{B^{s}_{2,1}} + \| v \|_{L^\infty} \| b \|_{B^{s+2}_{2,1}}), \\
\| (g_2)_h \|_{B^{s}_{2,1}} & \leq C \varepsilon \| b \mathrm{Im} z \|_{B^{s}_{2,1}} \leq C \varepsilon (\| b \|_{L^\infty} \| \mathrm{Im} z \|_{B^{s+2}_{2,1}} + \| \mathrm{Im} z \|_{L^\infty} \| b \|_{B^{s+2}_{2,1}}).
\end{align}

A direct estimate of \((g_1)_h\) would give a term of order 1. To get the factor \(\varepsilon\), one may first take advantage of inequality (38) so as to get,
\[ \| (g_1)_h \|_{B^{s}_{2,1}} \leq C \varepsilon \| g_1 \|_{B^{s+1}_{2,1}}, \]
\[ \leq C \varepsilon \| v \|_{B^{s+2}_{2,1}}, \]
\[ \leq C \varepsilon \| v \|_{L^\infty} \| v \|_{B^{s+2}_{2,1}}. \]

The above inequality together with (41) and (42) implies that
\[ \| (\varepsilon \nabla f, g)_h \|_{B^{s}_{2,1}} \leq C \varepsilon \| (b, z) \|_{L^\infty} \| (b, z) \|_{B^{s+2}_{2,1}}. \]

Finally, inserting inequalities (37), (40), (43) into (35), (36), we end up with
\[ \| (b, v)_t \|_{L^2_t(C^{0,1})} + \| (\varepsilon \nabla b, v)_h \|_{L^2_t(C^{0,1})} \leq C \varepsilon \frac{1}{\varepsilon^2} \left( \| (b_0, z_0) \|_{B^{s+1}_{2,1}} + \sqrt{\varepsilon} \| (b, z) \|_{L^2_t(L^\infty)} \| (b, z) \|_{L^\infty_t(B^{s+2}_{2,1})} \right). \]

Since \(s > N/2 + 2\), we have \(H^s \hookrightarrow B^{s+\frac{1}{2}}_{2,1} \hookrightarrow B^{s+2}_{2,1}\) so that
\[ \| (b, v)_t \|_{L^2_t(C^{0,1})} + \| (\varepsilon \nabla b, v)_h \|_{L^2_t(C^{0,1})} \leq C \varepsilon \frac{1}{\varepsilon^2} \left( \| (b_0, z_0) \|_{H^s} + \sqrt{\varepsilon} \| (b, z) \|_{L^2_t(L^\infty)} \| (b, z) \|_{L^\infty_t(H^s)} \right). \]

Let \(X_0 := \| (b_0, z_0) \|_{H^s}\) and \(X(t) := \| (b, z) \|_{L^\infty_t(H^s)} + c \varepsilon^{-\frac{1}{2}} \| (b, z) \|_{L^2_t(C^{0,1})}\) where \(c = c(S, N)\) is some constant which is assumed to be sufficiently small. We deduce from the previous inequality, (34) and Lemma 1, that, changing possibly the constant \(C\),
\[ X(t) \leq 3X_0 + C\sqrt{\varepsilon} t X^2(t). \]

Therefore, using a stopping time argument similar to the one used in the proof of Theorem 1, we conclude that \(X(t) \leq 4X(0)\) for all \(t \in [0, T]\) whenever \(T\) satisfies
\[ 16C X_0 \sqrt{\varepsilon T} \leq 1. \]

We finally complete the proof of Theorem 3 in the case \(N \geq 4\) as the end of the proof of Theorem 1. The details are left to the reader. \(\square\)

**Remark 9.** The above proof may be easily adapted to the Besov spaces framework, and in particular to the case where the data \((a^0, u^0)\) are in \(B^{s+3}_{2,1} \times B^{s+2}_{2,1}\) and satisfy
\[ C \varepsilon \| (a^0, u^0) \|_{B^{s+3}_{2,1} \times B^{s+2}_{2,1}} \leq 1. \]

As an easy consequence, we discover that under the conditions of Theorem 3 there exists a constant \(c\) independent of \(s\) such that \(|\Psi|\) remains bounded away from zero up to time
\[ \frac{c}{\varepsilon^2 \| (a^0, u^0) \|^2_{B^{s+3}_{2,1} \times B^{s+2}_{2,1}}} \].
5.2. Proof of Theorem 3 in the case \( N = 3 \). The proof Theorem 3 in the three-
dimensional case relies on very similar arguments: however, the endpoint inequality pertaining to \( p = 2 \) in Proposition 4 does not hold for \( N = 3 \) and has to be replaced by slightly more technical arguments.

As above, we assume that we are given some suitably smooth map \( \Psi \), solution of \((GP)\) with datum \( \Psi^0 \) and satisfying \((33)\). Fix some \( \alpha \in (0,1) \), and set \( p := 1 + 1/\alpha \) and \( p' = \alpha + 1 \). Arguing as for the proof \((34)\), we obtain

\[
\| (b, z) \|_{L_t^\infty(H^s)} \leq 2 \| (b_0, z_0) \|_{H^s} + C \left( t^{\frac{1}{p'}} \| (b, v) \|_{L_t^p(C^{0,1})} + t^{\frac{1}{p}} \| (\varepsilon \nabla b, v) \|_{L_t^p(L^2)} \right) \| (b, z) \|_{L_t^\infty(H^s)}.
\]

(45)

It remains to find appropriate bounds for \((b, v)\) in \( L^p([0, T] ; C^{0,1}) \) and \((\varepsilon \nabla b, v) \) in \( L^2([0, T] ; C^{0,1}) \).

For the low-frequency part of the solution, Proposition 4 ensures that

\[
\| (b, v) \|_{L_t^p(C^{0,1})} \leq C \varepsilon^{\frac{1}{p'}} \left( \| (b_0, v_0) \|_{B^{\frac{3}{2}, 1}_2} + \| (f, g) \|_{L_t^1(B^{\frac{3}{2}, 1}_2)} \right).
\]

(46)

As in the case \( N \geq 4 \), the source terms \( f_t \) and \( g_t \) may be easily bounded thanks to Proposition 5. We end up with

\[
\| (f, g) \|_{B^{\frac{3}{2}, 1}_2} \leq C \| (b, z) \|_{L_t^\infty} \| (b, z) \|_{B^{\frac{3}{2}, 1}_2}.
\]

(47)

To deal with the high-frequency terms, we notice that, by virtue of Proposition 4, we have

\[
\| (\varepsilon D b, v)_h \|_{L_t^p(C^{0,1})} \leq C \left( \| (\varepsilon D b_0, v_0)_h \|_{B^{\frac{3}{2}, 1}_2} + \| (\varepsilon D f, g)_h \|_{L_t^1(B^{\frac{3}{2}, 1}_2)} \right).
\]

(48)

Taking advantage of Proposition 5 and inequality \((38)\), we get

\[
\| \varepsilon \nabla h \|_{B^{\frac{3}{2}, 1}_2} \leq C \varepsilon \| (b) \|_{L_t^\infty} \| v \|_{B^{\frac{3}{2}, 1}_2} + \| v \|_{L_t^\infty} \| b \|_{B^{\frac{3}{2}, 1}_2}, \]

\[
\| (g_1)_h \|_{B^{\frac{3}{2}, 1}_2} \leq C \varepsilon \| (g_1)_h \|_{B^{\frac{3}{2}, 1}_2} \leq C \varepsilon \| v \|_{L_t^\infty} \| v \|_{B^{\frac{3}{2}, 1}_2}, \]

\[
\| (g_2)_h \|_{B^{\frac{3}{2}, 1}_2} \leq C \varepsilon \| (b) \|_{L_t^\infty} \| \text{Im } z \|_{B^{\frac{3}{2}, 1}_2} + \| \text{Im } z \|_{L_t^\infty} \| b \|_{B^{\frac{3}{2}, 1}_2}.
\]

Following the lines of the computations leading to inequality \((37)\), it is not difficult to show that

\[
\| (\varepsilon D b_0, v_0)_h \|_{B^{\frac{3}{2}, 1}_2} \leq C \varepsilon \| (b_0, z_0) \|_{B^{\frac{3}{2}, 1}_2}.
\]

It follows that, if \( s > 7/2 \) then inequalities\((46)\) to \((48)\) yield

\[
\varepsilon^{-\frac{1}{p'}} \| (b, v) \|_{L_t^p(C^{0,1})} + \varepsilon^{-1} \| (\varepsilon D b, v)_h \|_{L_t^p(C^{0,1})} \leq C \left( \| (b_0, v_0) \|_{H^s} + \| (b, v) \|_{L_t^p(C^{0,1})} + \| (\varepsilon \nabla b, v)_h \|_{L_t^p(L^2)} \right) \| (b, z) \|_{L_t^\infty(H^s)}.
\]

We introduce, for a constant \( c \) which is assumed to be arbitrarily small, the quantity

\[
X(t) := \| (b, z) \|_{L_t^\infty(H^s)} + c \varepsilon^{-\frac{1}{p'}} \| (b, v) \|_{L_t^p(C^{0,1})} + c \varepsilon^{-1} \| (\varepsilon D b, v)_h \|_{L_t^p(L^2(C^{0,1}))}.
\]

We obtain, in view of the previous estimates

\[
X(t) \leq 3X_0 + C \varepsilon^{\frac{1}{p'}} \varepsilon^{\frac{1}{p'}} + \varepsilon \sqrt{t})X^2.
\]

Using a standard bootstrap argument, one can conclude that \( X(t) \leq 4X(0) \) for all \( t \in [0, T] \) whenever \( T \) satisfies

\[
16C \varepsilon^{\frac{1}{p'}} \varepsilon^{\frac{1}{p'}} + \varepsilon \sqrt{T})X_0 \leq 1.
\]
It is then straightforward to complete the proof of the theorem.

\[ \square \]

**Remark 10.** As a by-product of the above proof, we see that if the data \((a_0^\varepsilon, u_0^\varepsilon)\) are in \(B_{2,1}^2(\mathbb{R}^3) \times B_{2,1}^2(\mathbb{R}^3)\) and satisfy

\[ C\varepsilon\|(a_0^\varepsilon, u_0^\varepsilon)\|_{B_{2,1}^2 \times B_{2,1}^2} \leq 1 \]

then for all \(\alpha \in (0, 1)\), there exists a constant \(c\) such that \(|\Psi|\) remains bounded away from zero up to time

\[ \min\left(\frac{c}{\varepsilon^{1+\alpha\|(a_0^\varepsilon, u_0^\varepsilon)\|_{B_{2,1}^2 \times B_{2,1}^2}^{1+\alpha}}, \frac{c}{\varepsilon^3\|(a_0^\varepsilon, u_0^\varepsilon)\|_{B_{2,1}^2 \times B_{2,1}^2}^2}\right). \]

Let us also point out that, resorting to the logarithmic Strichartz estimate (as in e.g. [3], Th. 8.27), one may replace the factor \(\varepsilon^{1+\alpha}\) by \(\varepsilon^2\sqrt{\log \varepsilon^{-1}}\). However we do not intend to provide proofs in this paper.

### 5.3. Proof of Theorem 3 in the two-dimensional case.

Arguing as in the proof of (34), one may write for all \(p \geq 2\),

\[ \|(b, z)\|_{L^\infty_t(H^s)} \leq \frac{3}{2}\|(b_0, z_0)\|_{H^s} \]

\[ +C\left(\varepsilon^{\frac{1}{2}}\|(b, v)\|_{L^1_t(C^{0,1})} + t^{\frac{1}{p'}}\|(\varepsilon \nabla b, v)\|_{L^p_t(C^{0,1})}\right)\|(b, z)\|_{L^\infty_t(H^s)}. \]

Therefore it remains to bound \((b, v)\) in \(L^4([0, T]; C^{0,1})\) and \((\varepsilon \nabla b, v)\) in \(L^p([0, T]; C^{0,1})\). For the low-frequency part of the solution, Proposition 4 ensures that

\[ \|(b, v)\|_{L^1_t(C^{0,1})} \leq C\varepsilon^{\frac{1}{4}}\|(b_0, v_0)\|_{B_{2,1}^2} + \|(f, g)\|_{L^1_t(B_{2,1}^2)}. \]

The source terms \(f_t\) and \(g_t\) may be easily bounded thanks to Proposition 5: we get

\[ \|(f, g)\|_{L^1_t(B_{2,1}^2)} \leq C\|(b, z)\|_{L^\infty} \|(b, z)\|_{B_{2,1}^2}. \]

Let us now focus on the high-frequency part of the solution. Applying Proposition 4 yields

\[ \|\varepsilon Db\|_{L^1_t(C^{0,1})} \leq C\left(\|(\varepsilon Db_0, v_0)\|_{B_{2,1}^2} + \|\varepsilon Df\|_{L^1_t(B_{2,1}^2)}\right). \]

Taking advantage of Proposition 5 and inequality (38), we get

\[ \|\varepsilon \nabla f\|_{B_{2,1}^{2-\frac{2}{p}}} \leq C\varepsilon\|\|b\|_{L^\infty}\|v\|_{B_{2,1}^{4-\frac{2}{p}}} + \|v\|_{L^\infty}\|b\|_{B_{2,1}^{4-\frac{2}{p}}}, \]

\[ \|\varepsilon \nabla g\|_{B_{2,1}^{2-\frac{2}{p}}} \leq C\varepsilon\|\|v\|_{L^\infty}\|\varepsilon Db\|_{B_{2,1}^{4-\frac{2}{p}}} + \|v\|_{L^\infty}\|\varepsilon Db\|_{B_{2,1}^{4-\frac{2}{p}}}, \]

Following the computations leading to inequality (37), it is not difficult to show that

\[ \|(\varepsilon Db_0, v_0)\|_{B_{2,1}^{2-\frac{2}{p}}} \leq C\varepsilon\|(b_0, z_0)\|_{B_{2,1}^{3-\frac{2}{p}}}. \]

If we assume that \(s > 4 - \frac{2}{p}\) then inequalities (46) to (48) imply that

\[ \varepsilon^{-\frac{2}{p}}\|(b, v)\|_{L^4_t(C^{0,1})} + \varepsilon^{-1}\|(\varepsilon Db, v)\|_{L^4_t(C^{0,1})} \leq C\left(\|(b_0, v_0)\|_{H^s} \right) \]

\[ +\left(t^{\frac{1}{p}}\|(b, v)\|_{L^4_t(C^{0,1})} + t^{\frac{1}{p'}}\|(\varepsilon \nabla b, v)\|_{L^p_t(C^{0,1})}\right)\|(b, z)\|_{L^\infty_t(H^s)}. \]
We introduce as before, for a constant $c$ which is assumed to be sufficiently small, the quantity
\[ X(t) := \| (b, z) \|_{L_t^\infty(H^s)} + c\varepsilon^{-\frac{1}{4}} \| (b, v) \|_{L_t^4(C^{0,1})} + c\varepsilon^{-1} \| (\varepsilon Db, v) \|_{L_t^\infty(C^{0,1})}. \]
we get
\[ X(t) \leq 3X_0 + C(\varepsilon^{\frac{1}{4}}t^\frac{1}{4} + \varepsilon t^\frac{1}{4})X^2. \]
It is now easy to complete the proof of the theorem.

5.4. Proof of Theorem 4. With Theorem 3 at our disposal, we compare the solution $(a_\varepsilon, u_\varepsilon)$ to the hydrodynamical form (3) of the Gross-Pitaevskii equation, to the solution $(a_\varepsilon, u_\varepsilon)$ of the linear system $L_\varepsilon(a_\varepsilon, u_\varepsilon) = 0$ with the same initial datum. We notice that
\[ (\tilde{b}, \tilde{v})(x, t) := (a_\varepsilon - a_\varepsilon, u_\varepsilon - u_\varepsilon)(x, \frac{t}{\varepsilon}) \]
satisfies
\[ \begin{cases} 
\partial_t \tilde{b} + \varepsilon \nabla \tilde{v} = f, \\
\partial_t \tilde{v} + \frac{\sqrt{\varepsilon}}{\varepsilon} \nabla \tilde{b} - \sqrt{2\varepsilon} \Delta \tilde{b} = g
\end{cases} \]
with null initial datum, $f := -\nabla(\varepsilon b v)$ and $g := -\nabla(\varepsilon^2) - \nabla(\sqrt{2\varepsilon} \nabla (b \Im z))$. By standard energy method, it follows that
\[ \| (\tilde{b}, \varepsilon \nabla \tilde{b}, \tilde{v})_\varepsilon(t) \|_{H^{s-1}} \leq \int_0^t \| (f, \varepsilon \nabla f, g)_\varepsilon(\tau) \|_{H^{s-1}} d\tau, \]
\[ \| (\tilde{b}, \varepsilon \nabla \tilde{b}, \tilde{v})_h(t) \|_{H^{s-2}} \leq \int_0^t \| (f, \varepsilon \nabla f, g)_h(\tau) \|_{H^{s-2}} d\tau. \]
Parseval equality entails that
\[ (53) \quad \| (b, \varepsilon \nabla b, v)_\varepsilon \|_{H^{s-1}} \approx \| (b, v) \|_{H^{s-1}} \quad \text{and} \quad \| (b, \varepsilon \nabla b, v)_h \|_{H^{s-2}} \approx \varepsilon \| b_h \|_{H^{s-1}} + \| v_h \|_{H^{s-2}}, \]
and a similar property holds for $(f, g)$. We remark that, thanks to the low frequency cut-off, we have
\[ \| \varepsilon \nabla (b \Im z)_\varepsilon \|_{H^{s-1}} \leq C \| \varepsilon (b \Im z) \|_{H^{s-1}}. \]
Therefore, using Lemma 3, we get
\[ (54) \quad \| (f, g)_\varepsilon \|_{H^{s-1}} \leq C \| (b, z) \|_{L_t^\infty(H^s)} \]
In order to bound $\| (\varepsilon \nabla f, g)_h \|_{H^{s-2}}$, we use the fact that
\[ \| (\nabla |v|^2)_h \|_{H^{s-2}} \leq C\varepsilon \| |v|^2 \|_{H^s} \]
so that we end up with the inequality
\[ (55) \quad \| (\varepsilon \nabla f, g)_h \|_{H^{s-2}} \leq C\varepsilon \| (b, z) \|_{L_t^\infty(H^s)}. \]
Combining inequalities (54) and (55) and making use of (53), we obtain that
\[ (56) \quad \| \tilde{b}(t) \|_{H^{s-1}} + \| \tilde{v}_\varepsilon(t) \|_{H^{s-1}} + \varepsilon^{-1} \| \tilde{v}_h(t) \|_{H^{s-2}} \leq C \int_0^t \| (b, z)(\tau) \|_{L_t^\infty(H^s)} d\tau \]
If we assume that $N \geq 4$ then, according to inequality (44), we have for some constant $C$ depending only on $s$ and on $N$, \[ \| (b, z) \|_{L_t^\infty(H^s)} + \varepsilon^{-\frac{1}{4}} \| (b, z) \|_{L_t^4(C^{0,1})} \leq C \| (b_0, z_0) \|_{H^s} \quad \text{for all} \quad t \in [0, T_\varepsilon]. \]
Inserting the above inequality in (56) directly implies Theorem 4 in the case $N \geq 4$.

The conclusion in the case $N = 2, 3$ follows from similar arguments. The details are left to the reader. \qed
Appendix A. Tame estimates

We recall several Gagliardo-Nirenberg type inequalities, the proof of which may be found in [15], provide the proof to Lemma 1, and finally present a commutation result.

\textbf{Lemma 2.} Let $k \in \mathbb{N}$ and $j \in \{0, \ldots, k\}$. There exists a constant $C_{j,k}$ depending only on $j$ and $k$ and such that the following inequality holds true:
\[
\|D^j v\|_{L^2}^{2k_j} \leq C_{j,k} \|v\|_{L^\infty}^{1+\frac{j}{k}} \|D^k v\|_{L^2}^{\frac{1}{k}}.
\]

The Gagliardo-Nirenberg inequalities stated above will enable us to prove the following tame estimates for the product of two functions:

\textbf{Lemma 3.} Let $k \in \mathbb{N}$ and $j \in \{0, \ldots, k\}$. There exists a constant $C_{k,N}$ depending only on $(k,N)$, such that
\[
\|D^j u D^{k-j} v\|_{L^2} \leq C_{j,k} \left( \|u\|_{L^\infty} \|D^k v\|_{L^2} + \|v\|_{L^\infty} \|D^k u\|_{L^2} \right),
\]
\[
\|uv\|_{H^k} \leq C_k \left( \|u\|_{L^\infty} \|D^k u\|_{H^k} + \|v\|_{L^\infty} \|u\|_{H^k} \right).
\]

\textbf{Proof.} Note that Leibniz formula combined with inequality (57) yields (58). So let us prove the first inequality. According to Hölder inequality, we have
\[
\|D^j u D^{k-j} v\|_{L^2} \leq \|D^j u\|_{L^\infty} \|D^{k-j} v\|_{L^2}.
\]
This yields (57) if $j = 0$ or $k$. Otherwise, using Lemma 2, one can write that
\[
\|D^j u D^{k-j} v\|_{L^2} \leq C_{k,N} \left( \|u\|_{L^\infty} \|D^k v\|_{L^2} + \|v\|_{L^\infty} \|D^k u\|_{L^2} \right),
\]
and Young inequality leads to (57). \hfill \Box

The tame estimates for the product of two functions extend in every $H^s$ with $s \geq 0$ and in the Besov space framework as follows (see the proof in e.g. [3], Chap. 2).

\textbf{Proposition 5.} For any $r \in [1, \infty]$ and $s > 0$ there exists a constant $C$ such that
\[
\|uv\|_{B^s_{2,r}} \leq C (\|u\|_{L^\infty} \|v\|_{B^s_{2,r}} + \|v\|_{L^\infty} \|u\|_{B^s_{2,r}}).
\]

We also recall the following continuity results in Besov spaces for the left-composition (see again e.g. [3], Chap. 2).

\textbf{Proposition 6.} Let $F$ be a smooth function defined on some open interval $I$ containing 0, and such that $F(0) = 0$. For any $r \in [1, \infty]$, $s > 0$ and compact subset $J$ of $I$, there exists a constant $C$ such that for any function $u \in B^s_{2,r}$ valued in $J$ we have
\[
\|F(u)\|_{B^s_{2,r}} \leq C \|u\|_{B^s_{2,r}}.
\]

\textbf{Proof of Lemma 1.} We assume that condition (33) holds. Using the fact that $v = v_\ell + v_h$ and Parseval formula, we easily get
\[
\|v\|_{B^s_{2,r}} \leq \|v_\ell\|_{B^s_{2,r}} + \|v_h\|_{B^s_{2,r}} \leq 2 \|v\|_{B^s_{2,r}}.
\]
Next, because $b = b_\ell + b_h$ and $\epsilon |\xi| \geq \epsilon_0$ for $\xi \in \text{Supp} b_h$, one may write
\[
\|b\|_{B^s_{2,r}} \leq \|b_\ell\|_{B^s_{2,r}} + \epsilon_0^{-1} \|\nabla b_h\|_{B^s_{2,r}} \quad \text{and} \quad \|b_\ell\|_{B^s_{2,r}} \leq \|b\|_{B^s_{2,r}}.
\]
According to (14), we have $\text{Im } z = -\nabla \log(1 + \varepsilon b/\sqrt{2})$. Therefore, condition (33) and Proposition 6 imply that

$$
\|\text{Im } z\|_{B^s_{2,r}} \leq \|\log(1 + \varepsilon b/\sqrt{2})\|_{B^s_{2,r}+1} \leq C\varepsilon\|b\|_{B^s_{2,r}+1}.
$$

Now, using the definition of the $B^s_{2,r}+1$ norm and of $b_\ell, b_h$, one may write for a suitable $\varepsilon_1 > \varepsilon_0$,

$$
\varepsilon\|b\|_{B^s_{2,r}} \leq \sum_{\varepsilon^{2^q} \leq \varepsilon_1} (\varepsilon^{2^q}) 2^q \|\Delta_q b_\ell\|_{L^2} + \sum_{\varepsilon^{2^q} \geq \varepsilon_0} 2^q (\varepsilon^{2^q}\|\Delta_q b_h\|_{L^2}) \leq C(\|b_\ell\|_{B^s_{2,r}} + \|\varepsilon \nabla b_h\|_{B^s_{2,r}}).
$$

Therefore

$$
\|(b, \text{Im } z)\|_{B^s_{2,r}} \leq C(\|b_\ell\|_{B^s_{2,r}} + \|\varepsilon \nabla b_h\|_{B^s_{2,r}}).
$$

In order to complete the proof of (31), we still have to show that

$$
(59) \quad \|\varepsilon \nabla b_h\|_{B^s_{2,r}} \leq C(\|b_\ell\|_{B^s_{2,r}} + \|\text{Im } z\|_{B^s_{2,r}}).
$$

In fact, as $L : z \mapsto \log(1 + z)$ is a smooth diffeomorphism from $(a, b)$ to $L((a, b))$ for any $0 < a < b$, and vanishes at 0, Proposition 6 enables us to write that

$$
\|\varepsilon \nabla b_h\|_{B^s_{2,r}} \leq \|\varepsilon b\|_{B^s_{2,r}+1},
$$

$$
\leq C\|\log(1 + \sqrt{2}b)\|_{B^s_{2,r}+1},
$$

$$
\leq C(\|\log(1 + \sqrt{2}b)\|_{B^s_{2,r}} + \|\nabla \log(1 + \sqrt{2}b)\|_{B^s_{2,r}}),
$$

$$
\leq C(\varepsilon\|b\|_{B^s_{2,r}} + \|\text{Im } z\|_{B^s_{2,r}}).
$$

This completes the proof of (59) thus of (31).

Let us now turn to the proof of inequality (32). Because

$$
z = v - i\frac{\varepsilon}{\sqrt{2}} \frac{\nabla b}{1 + \varepsilon_0^2 b} \quad \text{and} \quad \nabla z = \nabla v - i\frac{\varepsilon}{\sqrt{2}} \frac{\nabla^2 b}{1 + \varepsilon_0^2 b} - i\frac{\varepsilon^2}{2} \frac{|\nabla b|^2}{(1 + \varepsilon_0^2 b)},
$$

condition (33) guarantees that

$$
\|(b, z)\|_{C^{0,1}} \leq \|v\|_{C^{0,1}} + C\varepsilon\|\nabla b\|_{C^{0,1}}.
$$

Let us notice that, whenever $\tilde{\chi} \in C^\infty$ has value 1 on $\text{Supp } \chi$, one may write $b_\ell = \tilde{\chi}(\varepsilon^{-1}D)b_\ell$. Therefore, there exists a $L^1(\mathbb{R}^N; \mathbb{R}^N)$ function $k$ so that

$$
(\varepsilon \nabla b)_\ell = \varepsilon \nabla \tilde{\chi}(\varepsilon D)b_\ell = \varepsilon^{-N}k(\varepsilon^{-1} \cdot) \ast b_\ell.
$$

This ensures that

$$
\varepsilon\|\nabla b\|_{C^{0,1}} \leq \|(\varepsilon \nabla b)_\ell\|_{C^{0,1}} + \|(\varepsilon \nabla b)_h\|_{C^{0,1}} \leq C\|b_\ell\|_{C^{0,1}} + \|\varepsilon\nabla b\|_{C^{0,1}},
$$

so that

$$
\|(b, z)\|_{C^{0,1}} \leq C\|(b, v)_\ell\|_{C^{0,1}} + \|(\varepsilon D b, v)_h\|_{C^{0,1}}.
$$

The reverse inequality follows from similar arguments. The details are left to the reader. \(\square\)

The following commutation lemma is central in the proof of Proposition 3.

**Lemma 4.** Let $s > 0$, $r \in [1, \infty]$ and $\psi$ be a smooth function compactly supported in an annulus $\{ \xi \in \mathbb{R}^N / R_1 \leq |\xi| \leq R_2 \}$. There exists a constant $C$ depending only on $\psi$ and $s$ such that for all $q \in \mathbb{N}$ the following estimate holds true:

$$
(60) \quad \|a, \psi(2^{-q}D)f\|_{L^2} \leq Cc_q2^{-qs}(\|Daf\|_{L^\infty} \|f\|_{B^{s-1}_{2,r}} + \|Da\|_{B^{s-1}_{2,r}} \|f\|_{L^\infty})
$$

for some sequence $(c_q)_{q \in \mathbb{N}}$ with $\|c_q\|_\epsilon = 1$.

A similar estimate is true with $q = 0$ if $\psi$ is only supported in a ball.
Therefore, next, we have
\[ [a, \psi(2^{-q}D)]f = [S_0a, \psi(2^{-q}D)]f + [\bar{a}, \psi(2^{-q}D)]f. \]

Remark that, owing to the support properties of \( \psi \), there exists some integer \( N_0 \) such that
\[ [S_0a, \psi(2^{-q}D)]f = \sum_{|q' - q| \leq N_0} [S_0a, \psi(2^{-q}D)] \Delta_q f. \]

Now, according to Lemma 2.93 in [3], we have
\[
\|[S_0a, \psi(2^{-q}D)] \Delta_q f\|_{L^2} \leq C 2^{-q} \|DS_0a\|_{L^\infty} \|\Delta_q f\|_{L^2},
\]
whence, since \( \|DS_0a\|_{L^\infty} \leq C \|Da\|_{L^\infty} \),
\[
2^{qs} \|[S_0a, \psi(2^{-q}D)]f\|_{L^2} \leq C 2^{-qs} \|Da\|_{L^\infty} \sum_{|q' - q| \leq N_0} 2^{(q-q')(s-1)} (2^{q'(s-1)} \|\Delta_q f\|_{L^2})
\]
so that we find that, for some sequence \( (c_q)_{q \in \mathbb{N}} \) such that \( \|c_q\|_{\ell^1} = 1 \) and
\[
\|[S_0a, \psi(2^{-q}D)]f\|_{L^2} \leq C c_q 2^{-qs} \|Da\|_{L^\infty} \|f\|_{B^{s-1}_{2,1}}.
\]

To deal with the last term in (61), one may take advantage of the paraproductive calculus based on a Littlewood-Paley decomposition, a tool introduced by J.-M. Bony in [2]. The paraproduct of two tempered distributions \( u \) and \( v \) is defined by
\[
T_{uv} := \sum_{q \geq 1} S_{q-1}u \Delta_q v.
\]
and we have the following (formal) Bony’s decomposition for the product of two distributions:
\[
uv = T_{uv} + T'_{uv} \quad \text{with} \quad T'_{uv} := \sum_{q \geq -1} S_{q+2}v \Delta_q u.
\]

This leads us to expand \( [\bar{a}, \psi(2^{-q}D)]f \) into
\[
[\bar{a}, \psi(2^{-q}D)]f = [T_{\bar{a}}, \psi(2^{-q}D)]f + T'_{\bar{a}} \psi(2^{-q}D)f \bar{a} - \psi(2^{-q}D)T'_{\bar{a}}. \]

Taking advantage of the support properties of \( \psi \), one may write for some suitable integer \( N_0 \),
\[
[T_{\bar{a}}, \Delta_q]f = \sum_{|q' - q| \leq N_0} [S_{q'-1} \bar{a}, \psi(2^{-q}D)] \Delta_q f.
\]

Using again Lemma 2.93 in [3], one may write
\[
\|[S_{q'-1} \bar{a}, \psi(2^{-q}D)] \Delta_q f\|_{L^2} \leq C 2^{-q} \|DS_{q'-1} \bar{a}\|_{L^\infty} \|\Delta_q f\|_{L^2},
\]
so that we find
\[
\|[T_{\bar{a}}, \psi(2^{-q}D)]f\|_{L^2} \leq C c_q 2^{-qs} \|Da\|_{L^\infty} \|f\|_{B^{s-1}_{2,1}}.
\]

Next, we have
\[
T'_{\psi(2^{-q}D)f \bar{a}} = \sum_{q \geq q-N_0} S_{q+2} \psi(2^{-q}D)f \Delta_q \bar{a}.
\]
Therefore
\[
\|[T'_{\psi(2^{-q}D)f \bar{a}}]\|_{L^2} \leq \sum_{q \geq q-N_0} \|\psi(2^{-q}D)f\|_{L^2} \|\Delta_q \bar{a}\|_{L^\infty}.
\]
Because $\mathcal{F}a$ is supported away from the origin, Bernstein inequality ensures that $||\Delta_q a||_{L^\infty} \leq C 2^{-q'} ||Da||_{L^\infty}$. Inserting this inequality in (64), we thus get

$$||T'_{\psi(2^{-q}D)} f \tilde{a}||_{L^2} \leq C 2^{-q} ||Da||_{L^\infty} (2^{q(s-1)} ||\psi(2^{-q}D)f||_{L^2}) \sum_{q' \geq q-N_0} 2^{q-q'},$$

whence

$$||T'_{\psi(2^{-q}D)} f \tilde{a}||_{L^2} \leq C ||Da||_{L^\infty} 2^{q(s-1)} ||\psi(2^{-q}D)f||_{L^2}. \tag{65}$$

Note that because $\Delta_q \psi(2^{-q}D) = 0$ for $|q' - q| > N_0$, there exists some sequence $(c_q)_{q \in \mathbb{N}}$ with $||c_q||_{L^r} = 1$ such that (see [3, Section 2.7])

$$2^{q(s-1)} ||\psi(2^{-q}D)f||_{L^2} \leq C c_q ||f||_{B^s_{2,r}}. \tag{66}$$

Finally, standard continuity results for the paraproduct\(^\text{6}\) ensure that

$$||T'_f a||_{B^s_{2,r}} \leq C ||f||_{L^\infty} ||a||_{B^s_{2,r}} \leq C ||f||_{L^\infty} ||Da||_{B^s_{2,r}},$$

so that, because $\Delta_q \psi(2^{-q}D) = 0$ for $|q' - q| > N_0$,

$$||\psi(2^{-q}D)T'_f a||_{L^2} \leq C c_q 2^{-qs} ||f||_{L^\infty} ||Da||_{B^s_{2,r}}. \tag{67}$$

Putting together inequalities (62)--(67) completes the proof.

\(\square\)

APPENDIX B. DISPERSE ESTIMATES

This section is devoted to the proof of Proposition 4. We first symmetrize system (29) by introducing the new functions

$$c = (1 - \varepsilon^2 \Delta)^{\frac{5}{7}} b \quad \text{and} \quad d = (-\Delta)^{-\frac{1}{7}} \text{div} v,$$

and set $F = (1 - \varepsilon^2 \Delta)^{\frac{5}{7}} f$ and $G = (-\Delta)^{-\frac{1}{7}} \text{div} g$. If we restrict ourselves to solutions $(b, v)$ such that $v$ is a gradient, then system (29) translates into

$$\frac{d}{dt} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon^{-1}(1 - \varepsilon^2 \Delta)^{\frac{5}{7}} (1 - \varepsilon^2 \Delta)^{\frac{1}{7}} \\ \varepsilon^{-1}(2\Delta)^{\frac{1}{7}} (1 - \varepsilon^2 \Delta)^{\frac{1}{7}} & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} F \\ G \end{pmatrix}. \tag{68}$$

We recover our original system (29) using the inverse transformation $b = (1 - \varepsilon^2 \Delta)^{\frac{1}{7}} c$ and $v = -\nabla (-\Delta)^{-\frac{1}{7}} d$. For $\varepsilon > 0$, we are hence led to consider the unitary group $(V_\varepsilon(t))_{t \in \mathbb{R}}$ on $L^2(\mathbb{R}^N)$ for the infinitesimal generator $I_\varepsilon = i\varepsilon^{-1}(2\Delta)^{\frac{1}{7}} (1 - \varepsilon^2 \Delta)^{\frac{1}{7}}$. As already suggested in the introduction, when $\varepsilon$ is large $V_\varepsilon$ behaves as the Schrödinger equation, whereas when $\varepsilon$ is small, it behaves as part of the wave system with speed $\sqrt{2}/\varepsilon$. For this reason, we introduce the slowed operator

$$U_\varepsilon(t) = V_\varepsilon(\frac{\varepsilon t}{\sqrt{2}}),$$

which should therefore behave as the wave operator of speed 1.

The main ingredient for the proof of the Strichartz estimates provided in Proposition 4 are the following uniform bounds.

\(^6\)Here we need that $s > 0$, see [3, Section 2.8]
Lemma 5. Let \( R_2 > R_1 > 0 \) and \( a \in L^1(\mathbb{R}^N) \) such that \( \text{Supp} \, \hat{a} \subset \{ \xi \in \mathbb{R}^N / R_1 \leq |\xi| \leq R_2 \} \). There exist two positive constants \( \varepsilon_1 \) and \( C \) depending only on \( N, R_1, R_2 \) and such that for all \( t > 0 \) and \( \varepsilon \geq \varepsilon_1 \), we have

\[
\| V_\varepsilon(t)a \|_{L^\infty} \leq Ct^{\frac{N-1}{2}} \| a \|_{L^1}.
\]

For all \( \varepsilon_0 \) there exists a constant \( C = C(\varepsilon_0, N, R_1, R_2) \) such that for all \( t > 0 \) and \( \varepsilon \in (0, \varepsilon_0] \), we have

\[
\| U_\varepsilon(t)a \|_{L^\infty} \leq Ct^{\frac{1-N}{2}} \| a \|_{L^1}.
\]

Proof. Fix some function \( \phi \in C_0^\infty(\mathbb{R}^N) \) supported in \( \{ R_1/2 \leq |\xi| \leq 2R_2 \} \) and with value 1 on \( \{ R_1 \leq |\xi| \leq R_2 \} \). In view of to the assumption on \( \text{Supp} \, \hat{a} \), we may write

\[
U_\varepsilon(t)a = (2\pi)^{-N} L_\varepsilon(t) \ast a \quad \text{and} \quad V_\varepsilon(t)a = (2\pi)^{-N} H_\varepsilon(\sqrt{2}t) \ast a
\]

where we have set

\[
L_\varepsilon(t,x) := \int_{\mathbb{R}^N} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} \phi(\xi) \, d\xi \quad \text{and} \quad H_\varepsilon(t,x) := \int_{\mathbb{R}^N} e^{i(x \cdot \xi + t|\xi|\sqrt{1+|\xi|^2})} \phi(\xi) \, d\xi.
\]

In order to prove the lemma, it suffices therefore to establish that for all \( \varepsilon_0 > 0 \) there exists a constant \( C \) such that for all \( t > 0 \), we have

\[
\| L_\varepsilon(t) \|_{L^\infty} \leq Ct^{\frac{1-N}{2}} \quad \text{if} \quad \varepsilon \leq \varepsilon_0,
\]

and that there exists \( \varepsilon_1 > 0 \) and a constant \( C' \) such that for all \( t > 0 \),

\[
\| H_\varepsilon(t) \|_{L^\infty} \leq C't^{-\frac{N}{2}} \quad \text{if} \quad \varepsilon \geq \varepsilon_1.
\]

As a matter if fact, inequalities (71) and (72) are derived from the stationary and nonstationary phase theorems. The basic result that we shall invoke (see the proof in e.g. [3], Chap. 8) reads.

Lemma 6. Let \( K \) be a compact subset of \( \mathbb{R}^N \) and \( \psi \) be a smooth function supported in \( K \). Let \( A \) be a real-valued smooth function defined on some neighborhood of \( K \). Set

\[
I(t) := \int_{\mathbb{R}^N} e^{itA(\xi)} \psi(\xi) \, d\xi.
\]

For all couple \( (k,k') \) of positive real numbers, there exists a constant \( C \) depending only on \( k,k' \) and on \( (a \text{ finite number}) \) derivatives of \( A \) and \( \psi \) such that for all \( t > 0 \),

\[
|I(t)| \leq C \left( t^{-k} + \int_{\{\xi \in K / |\nabla A(\xi)| \leq 1\}} (1 + t|\nabla A(\xi)|^2)^{-k'} \, d\xi \right).
\]

Proof of Lemma 5 completed. We first turn to inequality (71). We notice that for any \( x \in \mathbb{R}^N \) and \( t > 0 \), we have

\[
L_\varepsilon(t,x) = \int_{\mathbb{R}^N} e^{i(x \cdot \xi + |\xi|\sqrt{1+|\xi|^2})} \phi(\xi) \, d\xi.
\]

According to Lemma 6, we thus have for some constant \( C \) depending only on \( N \) and \( \phi \),

\[
|L_\varepsilon(t,x)| \leq C \left( t^{\frac{1-N}{2}} + \int_{C_\varepsilon^x} (1 + t|\nabla A^\varepsilon_x(\xi)|^2)^{-N} \, d\xi \right)
\]

where we have set

\[
A^\varepsilon_x(\xi) := x \cdot \xi + |\xi|\sqrt{1+\varepsilon^2|\xi|^2} \quad \text{and} \quad C_\varepsilon^x := \{ \xi \in \text{Supp} \, \phi / |\nabla A^\varepsilon_x(\xi)| \leq 1 \}.
\]
We compute
\[ \nabla A^\varepsilon_\xi(\xi) = x + \left( \frac{1 + 2(\varepsilon|\xi|)^2}{|\xi|\sqrt{1 + (\varepsilon|\xi|)^2}} \right) \xi. \]
We may assume without loss of generality that \( x \neq 0 \), and decompose \( \xi \) into
\[ \xi = \xi_x + \xi'_x \quad \text{where} \quad \xi_x := \left( \xi, \frac{x}{|x|} \right) \frac{x}{|x|}, \]
so that we obtain, for some positive constant \( c \) depending only on \( \varepsilon_0 \),
\[ |\nabla A^\varepsilon_\xi(\xi)| \geq c|\xi'_x| \quad \text{for all} \quad \xi \in \text{Supp} \phi \quad \text{and} \quad \varepsilon \in (0, \varepsilon_0]. \]
Plugging this inequality into (73), one ends up with
\[ |L_\varepsilon(t, tx)| \leq C \left( t^{-\frac{N}{2}} + \int_{-2R_2}^{2R_2} \int_{\mathbb{R}^{N-1}} \left( 1 + t|\xi'_x|^2 \right)^{-N} d\xi'_x d\xi_x \right). \]
The change of variable \( \eta' = \sqrt{t} \xi'_x \) finally yields (71).

For the proof of inequality (72), we use the fact that
\[ H_\varepsilon(t, tx) = \int e^{it(x \cdot \xi + |\xi|^2\sqrt{1 + (\varepsilon|\xi|)^2})} \phi(\xi) \, d\xi. \]
Using Lemma 6 once more, we obtain that
\[ |H_\varepsilon(t, tx)| \leq C \left( t^{-\frac{N}{2}} + \int_{D_x^\varepsilon} \left( 1 + t|\nabla B^\varepsilon_\xi(\xi)|^2 \right)^{-N} d\xi \right) \]
where we have set
\[ B^\varepsilon_\xi(\xi) := x \cdot \xi + |\xi|^2\sqrt{1 + (\varepsilon|\xi|)^2} \quad \text{and} \quad D_x^\varepsilon := \{ \xi \in \text{Supp} \phi / |\nabla B_\xi(\xi)| \leq 1 \}. \]
we write
\[ \nabla B^\varepsilon_\xi(\xi) = x + R_\varepsilon(\xi) \xi \quad \text{where} \quad R_\varepsilon(\xi) := \frac{2 + (\varepsilon|\xi|)^2}{\sqrt{1 + (\varepsilon|\xi|)^2}}. \]
Decomposing \( \xi \) into \( \xi = \xi_x + \xi'_x \) as before, and using the fact that the integration is restricted to the set of \( R_1/2 \leq |\xi| \leq 2R_2 \), Lemma 6 implies that if \( \varepsilon \geq \varepsilon_1 > 0 \) then we have
\[ |H_\varepsilon(t, tx)| \leq C \left( t^{-\frac{N}{2}} + \int_{|\xi_x|<2R_2} \int_{\mathbb{R}^{N-1}} \left( 1 + t(|\xi'_x|^2 + (x + 2\xi_x R_\varepsilon(\xi))^2) \right)^{-N} d\xi'_x d\xi_x \right) \]
for some constant \( C \) depending only on \( \varepsilon_1, N \). If \( \varepsilon_1 \) is assumed to be sufficiently large, then for all \( \varepsilon \geq \varepsilon_1 \) the map
\[ \Phi^\varepsilon_\xi : \xi \longmapsto \sqrt{t}(x + \xi_x R_\varepsilon(\xi) + \xi'_x) \]
is a diffeomorphism from \( \Omega := \{ \xi \in \mathbb{R}^N / R_1/2 < |\xi| < 2R_2 \} \) to \( \Phi^\varepsilon_\xi(\Omega) \) and that the Jacobian of \( \Phi^\varepsilon_\xi \) is bounded by below by \( \alpha t^{N/2} \) for some \( \alpha > 0 \) independent of \( \varepsilon \). Making the change of variable \( \eta = \Phi(\xi) \) in the above integral, we derive inequality (72).

The above lemma will enable us to prove Strichartz estimates for the one-parameter unitary group \((V_\varepsilon(t))_{t \in \mathbb{R}}\). Before we state these estimates, we recall the definition of wave or Schrödinger admissible couples.

**Definition 1.** A couple of numbers \((p, r) \in [2, \infty]^2\) is said to be
- wave admissible if
  \[ \frac{1}{p} + \frac{N - 1}{2r} = \frac{N - 1}{4} \quad \text{and} \quad (p, r, N) \neq (2, \infty, 3), \]
Let \( \varepsilon \) be such that the assumptions of the main result in [22] are met. Since the data are real-valued, we have the identities of \( V(t) \) that the assumptions of the main result in [22] are met. Assume in addition that \( \hat{a}_0 \) and \( \hat{f}(t, \cdot) \) are supported in \( \{ \xi \in \mathbb{R}^N / R_1 \leq |\xi| \leq R_2 \} \).

i) There exists \( \varepsilon_1 = \varepsilon_1(N, R_1, R_2) \) and a constant \( C \) (independent of \( T \)) such that if \( (p, r) \) and \( (p_1, r_1) \) are Schrödinger admissible then for all \( \varepsilon \geq \varepsilon_1 \),

\[
\| V_{\varepsilon}(t) a_0 \|_{L^p_\varepsilon(L^r)} \leq C \| u_0 \|_{L^2},
\]

\[
\left\| \int_0^t V_{\varepsilon}(t-t') f(t') dt' \right\|_{L^p_\varepsilon(L^r)} \leq C \| f \|_{L^1_{p_1}(L^{r_1}_1)}.
\]

ii) For all \( \varepsilon > 0 \) there exists a constant \( C \) (independent of \( T \)) such that if \( (p, r) \) and \( (p_1, r_1) \) are wave admissible then for all \( \varepsilon \in (0, \varepsilon_0) \),

\[
\| V_{\varepsilon}(t) a_0 \|_{L^p_\varepsilon(L^r)} \leq C \| u_0 \|_{L^2},
\]

\[
\left\| \int_0^t V_{\varepsilon}(t-t') f(t') dt' \right\|_{L^p_\varepsilon(L^r)} \leq C \varepsilon^{\frac{1}{p}} \| f \|_{L^p_\varepsilon(L^r)}.
\]

Proof. It follows from Lemma 5 and the fact that \( V_{\varepsilon} \) and \( U_{\varepsilon} \) are unitary operators on \( L^2(\mathbb{R}^N) \) that the assumptions of the main result in [22] are met. The conclusion of [22] yields i) for \( V_{\varepsilon} \) and \( \varepsilon \geq \varepsilon_1 \). For statement ii), it suffices to rephrase the conclusion of [22] for \( U_{\varepsilon} \) in terms of \( V_{\varepsilon} \), since

\[
V_{\varepsilon}(t) a_0 = U_{\varepsilon} \left( \frac{\sqrt{2}}{\varepsilon} t \right) a_0 \quad \text{and} \quad \int_0^t V_{\varepsilon}(t-t') f(t') dt' = \varepsilon \sqrt{2} \int_0^t U_{\varepsilon} \left( \frac{\sqrt{2}}{\varepsilon} t - \tau \right) f \left( \frac{\sqrt{2}}{\varepsilon} \tau \right) d\tau. \quad \square
\]

Lemma 7. Let \((c, d)\) satisfy system (68) with real-valued initial datum \((c_0, a_0)\) and source terms \((F, G)\).

i) For all \( \varepsilon > 0 \) and all wave admissible couples \((p, r)\) and \((p_1, r_1)\) there exists a constant \( C \) such that for all \( q \in \mathbb{Z} \) and \( \varepsilon > 0 \) such that \( 2^q \varepsilon \leq \varepsilon_0 \) we have

\[
2^q \frac{\varepsilon}{p} \| (\Delta_q c, \Delta_q d) \|_{L^p_q(L^r_q)} \leq C \left( \varepsilon^{\frac{1}{p}} 2^{q \left( \frac{N}{2} - \frac{1}{p} \right)} \right) \| (\Delta_q c_0, \Delta_q d_0) \|_{L^2} + \varepsilon^{\frac{1}{p}} \| \Delta_q F, \Delta_q G \|_{L^p_q(L^r_q)}.
\]

ii) There exists a constant \( C \) such that for all \( q \in \mathbb{Z} \) and \( \varepsilon > 0 \) such that \( 2^q \varepsilon \geq \varepsilon_1 \), and all \( \varepsilon > 0 \) such that \( 2^q \varepsilon \geq \varepsilon_1 \), all Schrödinger admissible couples \((p, r)\) and \((p_1, r_1)\), we have

\[
2^q \frac{\varepsilon}{p} \| (\Delta_q c, \Delta_q d) \|_{L^p_q(L^r_q)} \leq C \left( 2^{q \left( \frac{N}{2} - \frac{1}{p} \right)} \| (\Delta_q c_0, \Delta_q d_0) \|_{L^2} + 2^{q \left( \frac{N}{2} - \frac{1}{p} \right)} \| \Delta_q F, \Delta_q G \|_{L^p_q(L^r_q)} \right).
\]

Proof. Since the data are real-valued, we have the identities

\[
c(t) = \text{Re} (V_{\varepsilon}(t) c_0) - \text{Im} (V_{\varepsilon}(t) d_0) + \text{Re} \int_0^t V_{\varepsilon}(t-t') F(t') dt' - \text{Im} \int_0^t V_{\varepsilon}(t-t') G(t') dt',
\]

\[
d(t) = \text{Im} (V_{\varepsilon}(t) c_0) + \text{Re} (V_{\varepsilon}(t) d_0) + \text{Im} \int_0^t V_{\varepsilon}(t-t') F(t') dt' + \text{Re} \int_0^t V_{\varepsilon}(t-t') G(t') dt.
\]

\( \footnote{For the choices \( \sigma = \frac{N}{2} \) for \( V_{\varepsilon} \) and \( \sigma = \frac{N-1}{2} \) for \( U_{\varepsilon} \), \( \sigma \) being a parameter entering in the statement of [22].} \)
Therefore, we introduce the functions
\[(\tilde{c}_q, \tilde{d}_q)(t, x) := (\Delta_q c, \Delta_q d)(2^{-2q}t, 2^{-q}x) \text{ and } (\tilde{F}_q, \tilde{G}_q)(t, x) := 2^{-2q}(\Delta_q F, \Delta_q G)(2^{-2q}t, 2^{-q}x)\]
so that \(\tilde{c}_q, \tilde{d}_q, \tilde{F}_q\) and \(\tilde{G}_q\) are spectrally supported in \(\{3/4 \leq |\xi| \leq 8/3\}\), and we have
\[\tilde{c}_q(t) = \text{Re} (V_{2\varepsilon}(t)\tilde{c}_q(0)) - \text{Im} (V_{2\varepsilon}(t)\tilde{d}_q(0)) + \text{Re} \int_0^t V_{2\varepsilon}(t - t')\tilde{F}_q(t') \, dt' - \text{Im} \int_0^t V_{2\varepsilon}(t - t')\tilde{G}_q(t') \, dt',\]
\[\tilde{d}_q(t) = \text{Im} (V_{2\varepsilon}(t)\tilde{c}_q(0)) + \text{Re} (V_{2\varepsilon}(t)\tilde{d}_q(0)) + \text{Im} \int_0^t V_{2\varepsilon}(t - t')\tilde{F}_q(t') \, dt' + \text{Re} \int_0^t V_{2\varepsilon}(t - t')\tilde{G}_q(t') \, dt'.\]

Next we fix some \(\varepsilon_0 > 0\). Applying the first part of Corollary 1, we derive that for all wave admissible couples \((p, r)\) and \((p_1, r_1)\), and \(\varepsilon \in (0, \varepsilon_0]\), we have
\[\| (\tilde{c}_q, \tilde{d}_q) \|_{L^q_p(L^r)} \leq C \left( (\varepsilon 2^q)^{1/2} \| (\tilde{c}_q(0), \tilde{d}_q(0)) \|_{L^2} + (\varepsilon 2^q)^{1/2} \| (\tilde{F}_q, \tilde{G}_q) \|_{L^q_{p'}(L^{r'})} \right).\]

Going back to the initial variables, we obtain the desired estimate for \((\Delta_q c, \Delta_q d)\).

Proof of Proposition 4 completed. With Lemma 7 at our disposal, we complete the proof of Proposition 4. Indeed, fix some smooth cut-off function \(\chi\) with compact support and value 1 on \(B(0, \frac{4}{3}\varepsilon_1)\) and denote \(z_\ell := \chi(\varepsilon^{-1} D)z\) and \(z_h := z - z_\ell\) for any tempered distribution \(z\). Owing to the spectral properties of \(z_\ell\) and \(z_h\), there exists some \(\varepsilon_0 > \varepsilon_1\) such that
\[\Delta_q z_\ell = 0 \quad \text{for} \quad 2^q \varepsilon > \varepsilon_0 \quad \text{and} \quad \Delta_q z_h = 0 \quad \text{for} \quad 2^q \varepsilon < \varepsilon_1.\]

Let \((b, v)\) satisfy system (29). By virtue of (74) and of Bernstein inequality, one may write for all \(r \in [1, \infty],\)
\[\| (b_\ell, v_\ell) \|_{L^\infty} \leq \sum_{2^q \varepsilon \leq \varepsilon_0} \| (\Delta_q b_\ell, \Delta_q v_\ell) \|_{L^\infty} \leq C \sum_{2^q \varepsilon \leq \varepsilon_0} 2^q \| (\Delta_q b_\ell, \Delta_q v_\ell) \|_{L^r}.\]

Notice that as \(\nabla |D|^{-1}\) and \(|D|^{-1}\text{div}\) are homogeneous multipliers of degree 0, we have (see e.g. Lemma 2.2 in [3])
\[\| \Delta_q v_\ell \|_{L^r} \approx \| \Delta_q b_\ell \|_{L^r}.\]

Next, we have \(b_\ell = (1 - \varepsilon^2 \Delta)^{1/2} \varepsilon \ell\) and it is not difficult to show that \(A_\varepsilon(D) := (1 - \varepsilon^2 \Delta)^{1/2}\)
and its inverse \(A_\varepsilon^{-1}\) are \(S^0\)-multipliers uniformly for \(\varepsilon \leq \varepsilon_0\); for every \(k \in \mathbb{N}\), there exists a constant \(C_k\) such that for every \(\varepsilon \leq \varepsilon_0\) and \(\xi \in \mathbb{R}^N\), we have
\[\| D^k A_\varepsilon^{-1}(\xi) \| \leq C_k (1 + |\xi|^2)^{-k/2}.\]

Therefore, a classical result (see e.g. Lemma 2.2 in [3]) ensures that there exists a constant \(C = C(N)\) such that for all \(q \in \mathbb{Z}, r \in [1, +\infty]\) and tempered distribution \(z\) we have
\[\| \Delta_q (1 - \varepsilon^2 \Delta)^{\pm 1/2} z \|_{L^r} \leq C \| \Delta_q z \|_{L^r} \quad \text{for all} \quad \varepsilon \in [0, \varepsilon_0].\]

Combining these inequalities with (76), we deduce that
\[\| (b_\ell, v_\ell) \|_{L^\infty} \leq C \sum_{2^q \varepsilon \leq \varepsilon_0} 2^q \| (\Delta_q c_\ell, \Delta_q d_\ell) \|_{L^r}.\]
Let us consider first the case \( N \geq 4 \). In this case, we apply the first part of Lemma 7 with the wave admissible couples \((p,r) := (2,2(N-1)/(N-3))\) and \((p_1,r_1) := (\infty,2)\) to deduce that
\[
\| (b_\ell,v_\ell) \|_{L^2_t(L^\infty_x)} \leq C \varepsilon^{\frac{1}{2}} \sum_{2^\ell \leq \varepsilon \leq 0} 2^{\left( \frac{N}{2} - \frac{1}{2} \right)} \left( \left\| \left( \Delta_q c_0, \Delta_q d_0 \right)_\ell \right\|_{L^2} + \left\| \left( \Delta_q F, \Delta_q G \right)_\ell \right\|_{L^1_t(L^2_x)} \right).
\]
In order to bound the r. h. s in terms of the functions \(b_0, v_0 f\) and \(g\), we invoke (77) and (78).

We end up with
\[
\| (b_\ell,v_\ell) \|_{L^2_t(L^\infty_x)} \leq C \varepsilon^{\frac{1}{2}} \sum_{2^\ell \leq \varepsilon \leq 0} 2^{\left( \frac{N}{2} - \frac{1}{2} \right)} \left( \left\| \left( \Delta_q b_0, \Delta_q v_0 \right)_\ell \right\|_{L^2} + \left\| \left( \Delta_q f, \Delta_q g \right)_\ell \right\|_{L^1_t(L^2_x)} \right).
\]
Using similar arguments, we get
\[
\| (\nabla b, \nabla v)_\ell \|_{L^2_t(L^\infty_x)} \leq C \varepsilon^{\frac{1}{2}} \sum_{2^\ell \leq \varepsilon \leq 0} 2^{\left( \frac{N}{2} - \frac{1}{2} \right)} \left( \left\| \left( \Delta_q \nabla b_0, \Delta_q \nabla v_0 \right)_\ell \right\|_{L^2} + \left\| \left( \Delta_q \nabla f, \Delta_q \nabla g \right)_\ell \right\|_{L^1_t(L^2_x)} \right).
\]
Combining this latter inequality with (80) and using Bernstein inequality and the definition of the norm in \( B^{\frac{N}{2} + \frac{1}{2}}_{2,1} \), we conclude that if \( N \geq 4 \) then
\[
\| (b,v)_\ell \|_{L^2_t(L^{\infty}_{0,1})} \leq C \varepsilon^{\frac{1}{2}} \left( \| (b_0,v_0)_\ell \|_{B^{\frac{N}{2} + \frac{1}{2}}_{2,1}} + \| (f,g)_\ell \|_{L^1_t(B^{\frac{N}{2} + \frac{1}{2}}_{2,1})} \right).
\]
If \( N = 3 \), the proof is almost the same except that the endpoint couple \((2,\infty)\) is not admissible. However, we may take any couple \((p,r)\) with \(1/p + 1/r = 1/2\), and (as before) \((p_1,r_1) = (\infty,2)\). Applying Lemma 7, we get after a few computations,
\[
\| (b,v)_\ell \|_{L^2_t(L^{\infty}_{0,1})} \leq C \varepsilon^{\frac{1}{2}} \left( \| (b_0,v_0)_\ell \|_{B^{\frac{N}{2} + \frac{1}{2}}_{2,1}} + \| (f,g)_\ell \|_{L^1_t(B^{\frac{N}{2} + \frac{1}{2}}_{2,1})} \right).
\]
Finally, in the case \( N = 2 \), one can take \((p,r) = (4,\infty)\) and \((p_1,r_1) = (\infty,2)\). We end up with
\[
\| (b,v)_\ell \|_{L^2_t(L^{\infty}_{0,1})} \leq C \varepsilon^{\frac{1}{2}} \left( \| (b_0,v_0)_\ell \|_{B^{\frac{N}{2} + \frac{1}{2}}_{2,1}} + \| (f,g)_\ell \|_{L^1_t(B^{\frac{N}{2} + \frac{1}{2}}_{2,1})} \right).
\]
The part of Proposition 4 pertaining to the high frequencies of the solution may be proved exactly along the same lines. It suffices to apply the second part of Lemma 7. The details are left to the reader.

\[\square\]

**References**

[1] T. Alazard and R. Carles, *WKB analysis for the Gross-Pitaevskii equation with non-trivial boundary conditions at infinity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear.

[2] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*, Ann. Sci. École Norm. Sup. 14(4), (1982), 209–246.

[3] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, to appear.

[4] S. Benzoni-Gavage, R. Danchin and S. Descombes, *On the well-posedness of the Euler-Korteweg model in several space dimensions*, Indiana Univ. Math. J., 56(4), (2007), 1499–1579.

[5] F. Bethuel, P. Gravejat, J.-C. Saut and D. Smets, *On the Korteweg-de Vries long-wave transonic approximation of the Gross-Pitaevskii equation*, in preparation.

[6] F. Bethuel, R.L. Jerrard and D. Smets, *On the NLS dynamics for infinite energy vortex configurations on the plane*, Rev. Mat. Iberoamericana, 24 (2008), 671–702.

[7] F. Bethuel and D. Smets, *A remark on the cauchy problem for the 2D Gross-Pitaevskii equation with nonzero degree at infinity*, Differential Integral Equations, 20 (2007), 325–338.
[8] T. Cazenave and A. Haraux, An introduction to Semilinear Evolution Equations, Oxford University Press, 1998.

[9] D. Chiron and F. Rousset, Geometric optics and boundary layers for nonlinear Schrödinger equations, preprint 2008.

[10] D. Chiron and F. Rousset, work in progress.

[11] T. Colin and A. Soyeur, Some singular limits for evolutionary Ginzburg-Landau equations, Asymptotic Analysis, 13 (1996), no. 4, 361–372.

[12] J.E. Colliander and R.L. Jerrard, Vortex dynamics for the Ginzburg-Landau-Schrödinger equation, Internat. Math. Res. Notices, 7 (1998), 333–358.

[13] F. Coquel, Personal communication.

[14] R. Danchin Zero Mach number limit in critical spaces for compressible Navier-Stokes equations, Ann. Sci. École Norm. Sup. 35(1), (2002), 27–75.

[15] A. Friedman, Partial differential equations, Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.

[16] C. Gallo, The Cauchy problem for defocusing Nonlinear Schrödinger equations with non-vanishing initial data at infinity, preprint.

[17] J. Ginibre and G. Velo, Generalized Strichartz inequalities for the wave equation, Journal of Functional Analysis, 133, (1995), 50–68.

[18] E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, Proc. Amer. Math. Soc., 126 (1998), no. 2, 523–530.

[19] S. Gustafson, K. Nakanishi and T.P. Tsai, Scattering for the Gross-Pitaevskii equation, Math. Res. Lett. 13 (2006), 273–285.

[20] L. Hörmander, Lectures on nonlinear hyperbolic differential equations, Mathématiques & Applications (Berlin), 26. Springer-Verlag, Berlin, 1997.

[21] R.L. Jerrard and D. Spirn, Refined Jacobian estimates and Gross-Pitaevskii vortex dynamics, Arch. Rat. Mech. Anal., to appear.

[22] M. Keel and T. Tao, Endpoint Strichartz estimates, American Journal of Mathematics, 120, (1998), 955–980.

[23] S. Klainerman, Uniform decay estimates and the Lorentz invariance of the classical wave equation, Comm. Pure Appl. Math. 38 (1985), no. 3, 321–332.

[24] F.H. Lin and J.X. Xin, On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation, Comm. Math. Phys., 200 (1999), no. 2, 249–274.

[25] F.H. Lin and P. Zhang, Semiclassical limit of the Gross-Pitaevskii equation in an exterior domain, Arch. Ration. Mech. Anal. 179, (2006), no. 1, 79–107.

[26] P. Gérard, The Cauchy problem for the Gross-Pitaevskii equation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 23, (2006), 765–779.

[27] T. Sideris, The lifespan of smooth solutions to the three-dimensional compressible Euler equations and the incompressible limit, Indiana Univ. Math. J. 40 (1991), no. 2, 535–550.

[28] T. Sideris, Delayed singularity formation in 2D compressible flow, Amer. J. Math. 119 (1997), no. 2, 371–422.

[29] R. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of the wave equations, Duke mathematical Journal, 44, (1977), 705–774.

[30] P. Zhang, Semiclassical limit of nonlinear Schrödinger equation. II, J. Partial Differential Equations, 15 (2002), no. 2, 83–96.

(F. Bethuel) Laboratoire J.-L. Lions UMR 7598, Université Pierre et Marie Curie, 175 rue du Chevaleret, 75013 Paris, France
E-mail address: bethuel@ann.jussieu.fr

(R. Danchin) Université Paris-Est, LAMA, UMR 8050, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France.
E-mail address: danchin@univ-paris12.fr

(D. Smets) Laboratoire J.-L. Lions UMR 7598, Université Pierre et Marie Curie, 175 rue du Chevaleret, 75013 Paris, France
E-mail address: smets@ann.jussieu.fr