Elementary moves on triangulations

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Abstract

It is proved that a triangulation of a polyhedron can always be transformed into any other triangulation of the polyhedron using only elementary moves. One consequence is that an additive function (valuation) defined only on simplices may always be extended to an additive function on all polyhedra.

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An $n$-polyhedron $P$ in $\mathbb{R}^N$, $1 \leq n \leq N$, is a finite union of $n$-dimensional polytopes, where a polytope is the compact convex hull of finitely many points in $\mathbb{R}^N$. A finite set of $n$-simplices $\alpha P$ is a triangulation of $P$ if no pair of simplices intersects in a set of dimension $n$ and their union equals $P$. We shall investigate transformations of triangulations by elementary moves. Here an elementary move applied to $\alpha P$ is one of the two following operations: a simplex $T \in \alpha P$ is dissected into two $n$-simplices $T_1, T_2$ by a hyperplane containing an $(n-2)$-dimensional face of $T$; or the reverse, that is, two simplices $T_1, T_2 \in \alpha P$ are replaced by $T = T_1 \cup T_2$ if $T$ is again a simplex. We say that triangulations $\alpha P$ and $\beta P$ are equivalent by elementary moves, and write $\alpha P \sim \beta P$, if there are finitely many elementary moves that transform $\alpha P$ into $\beta P$. The main object of this note is to show the following result.

Theorem 1. If $\alpha P$ and $\beta P$ are triangulations of the $n$-polyhedron $P$, then $\alpha P \sim \beta P$.

A triangulation $\alpha P$ with the additional property that any pair of simplices intersects in a common face gives rise to a simplicial complex $\hat{\alpha} P$. It is a classical result of algebraic topology due to Alexander [5] and Newman [26, 27] (see also [20]) that a simplicial complex $\hat{\alpha} P$ can always be transformed into any other simplicial complex $\hat{\beta} P$ with the same underlying polyhedron by using only finitely many stellar moves. Here a stellar move is a suitable sequence of elementary moves followed by a simplicial isomorphism, where a simplicial isomorphism between two complexes is a bijection between their vertices that induces a bijection between their $k$-dimensional simplices for $1 \leq k \leq n$. For precise definitions, see [20] and for related results, see [10], [28], [29]. The new feature of Theorem 1 is that simplicial isomorphisms are not allowed. So our theorem belongs to metric geometry whereas the Alexander-Newman theorem is a topological result.

As an application of Theorem 1 we obtain the following results on valuations. Here a function $\mu : S \to \mathbb{R}$ defined on a class $S$ of sets is called a valuation or additive if $\mu(\emptyset) = 0$, where $\emptyset$ is the empty set, and if

$$\mu(S) + \mu(T) = \mu(S \cup T) + \mu(S \cap T),$$

for all $S, T \in S$ such that $S \cup T, S \cap T \in S$ as well.

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Let $T^n$ be the set of simplices of dimension at most $n$ in $\mathbb{R}^N$. Note the following connection between the definition of valuations on simplices and elementary moves. If $S, T \in T^n$, then $S \cup T \in T^n$ implies that $S$ and $T$ can be obtained from $S \cup T$ by elementary moves and the most basic case is when $S \cup T$ is dissected by an elementary move into $S$ and $T$. Let $Q^n$ be the set of polyhedra of dimension at most $n$ in $\mathbb{R}^N$.

**Theorem 2.** Every valuation on $T^n$ has a unique extension to a valuation on $Q^n$.

As a corollary we obtain the analogous theorem for polytopes. Let $P^n$ be the set of polytopes of dimension at most $n$ in $\mathbb{R}^N$.

**Corollary 3.** Every valuation on $T^n$ has a unique extension to a valuation on $P^n$.

A version of Corollary 3 is stated and used in [19]. Note that in the proof of this result in [19], the Alexander-Newman theorem has to be replaced by Theorem 1 of the present paper.

Valuations on polyhedra are classical and they played a critical role in Dehn’s solution of Hilbert’s Third Problem. Results regarding the classification and characterization of invariant valuations are central to convex and integral geometry; see [12], [18], [24], [25]. In recent years, many new results on valuations have been obtained, see, for example, [11], [13], [14], [17], [21], [22], [34]. Also extension questions for general valuations are classical. Volland [37] and Perles and Sallee [30] proved that every valuation on polytopes has a unique extension to a valuation on polyhedra. Their results were generalized by Groemer [11]. Volume is the most basic example of a valuation on polyhedra. The question of how to extend volume from simplices to polyhedra is closely connected with the quest for an elementary definition of volume for polytopes. In his Third Problem, Hilbert [13] asked whether volume on polyhedra can be defined by using only scissors congruences, that is, if two polyhedra $P$ and $Q$ of the same volume can each be cut into a finite number of pieces $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ with $P_i$ congruent to $Q_i$ by a rigid motion for each $i$. This question was answered in the negative by Dehn [8] (see also [7], [31]). However, there are simple definitions for volume of simplices. So the volume of $n$-simplices can be defined as height times the $(n-1)$-dimensional volume of its base divided by $n$. A simple geometric argument (see [35]) shows that this does not depend on the choice of the base and volume defined in this way is a valuation on simplices. Schutonovsky [33] (for $n=3$) and Süss [35] (for general dimensions) proved that there is a unique extension of this valuation from simplices to polyhedra. In that way they obtained an elementary definition of volume for polyhedra. Theorem 2 is the extension of their result to general valuations.

Theorem 1 has already been used by a number of authors. In fact, results more general than Theorem 1 have already been used. For example, Lemma 2.2 in Sah’s book on Hilbert’s Third Problem [31] states that any two triangulations have a common refinement by elementary moves using only dissections. However, this is well-known to be an important open problem in algebraic topology, see [20]. Later, Sah [32] replaced his lemma by Theorem 1 of the present paper. While Sah observes that Theorem 1 suffices for the constructions in his book, he neglects to provide a proof of Theorem 1. For related results, see also [9].

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1 Proof of Theorem 1

A triangulation $\alpha P$ of an $n$-polytope $P$ is called a starring at $a \in P$, if every $n$-simplex in $\alpha P$ has a vertex at $a$. Note that every $n$-polytope $P$ has a starring at every $a \in P$. This can be seen by using induction on $n$. It is trivial for $n = 1$. Suppose that every $(n - 1)$-polytope has a starring. Let $P$ be an $n$-polytope and $a \in P$. For every $(n - 1)$-dimensional face $F_j$ of $P$ with $a \notin F_j$, we choose a starring $\alpha_j F_j$. Then the convex hulls of $a$ and the $(n - 1)$-simplices in $\alpha_j F_j$ are a starring of $P$ at $a$.

A triangulation $\gamma P$ of an $n$-polyhedron $P$ is a refinement of $\alpha P$ if every simplex of $\gamma P$ is contained in a simplex of $\alpha P$. Note that any two triangulations $\alpha P$ and $\beta P$ have a common refinement. To see this, let $P_1, \ldots, P_l$ be the polytopes $S \cap T$, $S \in \alpha P$, $T \in \beta P$, that are $n$-dimensional. For $j = 1, \ldots, l$, we choose a triangulation $\gamma_j P_j$ of $P_j$, for example, we can take a starring of $P_j$ at any point $a_j \in P_j$. Then $\gamma P = \gamma_1 P_1 \cup \cdots \cup \gamma_l P_l$ is a triangulation of $P$. Since every simplex in $\gamma P$ is contained in a suitable $P_j$, $\gamma P$ is a common refinement of $\alpha P$ and $\beta P$.

Let $P$ be an $n$-polyhedron. To show that $\alpha P \sim \beta P$ for any two triangulations $\alpha P$ and $\beta P$ of $P$, we prove that any refinement of a given triangulation can be obtained from the original triangulation by finitely many elementary moves. To show this, it is enough to prove the following proposition.

**Proposition.** If $\alpha T$ is a triangulation of the $n$-simplex $T$, then $\alpha T \sim T$.

Here we write $T$ (instead of $\{T\}$) for the trivial triangulation of $T$.

The rest of this section is devoted to the proof of this proposition. We follow the classical approach of Alexander and Newman as presented by Lickorish [20]. The main new feature of the proof is Lemma 2.

We use induction on the dimension $n$. The case $n = 1$ is trivial. Assume that the proposition is true for dimensions less or equal $n$, that is, for every triangulation $\alpha S$ of a simplex $S$

$$\alpha S \sim S \text{ for } \dim S < n,$$

(1)

where $\dim$ stands for dimension.

Note that, if $S$ is a simplex, then every elementary subdivision of $S$ induces an elementary subdivision of the simplices in the boundary of $S$. The following observation is used several times. Let $P = [Q,v]$ be a pyramid with apex $v$ and base $Q$. Let $\alpha Q = \{S_1, \ldots, S_k\}$ and $\beta Q = \{T_1, \ldots, T_l\}$ be triangulations of the $(n - 1)$-polytope $Q$. Let $[\alpha Q,v] = \{[S_1,v], \ldots, [S_k,v]\}$ and $[\beta Q,v] = \{[T_1,v], \ldots, [T_l,v]\}$. Then $[\alpha Q,v]$ and $[\beta Q,v]$ are triangulations of $P$ and

$$\alpha Q \sim \beta Q \Rightarrow [\alpha Q,v] \sim [\beta Q,v].$$

(2)

**Lemma 1.** If $\alpha T$ is a starring of an $n$-simplex $T$, then $\alpha T \sim T$.

**Proof.** Assume $T \subset \mathbb{R}^n$. First, let $T$ be starred at $a \in \partial T$. Let $S_0, \ldots, S_n$ be the facets of $T$ and let $a \in S_0$. Then

$$\alpha T = [\alpha_1 S_1, a] \cup \cdots \cup [\alpha_n S_n, a],$$

where $\alpha_i S_i$ is a triangulation of $S_i$. The induction hypothesis (1) and (2) imply that $[\alpha_i S_i, a] \sim [S_i, a]$. Thus,

$$\alpha T \sim [S_1, a] \cup \cdots \cup [S_n, a].$$
Let $v$ be the vertex of $T$ opposite to $S_0$. Then there is a starring $\alpha_0S_0$ of $S_0$ at $a$ such that

$$[S_1, a] \cup \cdots \cup [S_n, a] = [\alpha_0S_0, v].$$

The induction hypothesis 1 and 2 imply that $[\alpha_0S_0, v] \sim [S_0, v]$. Consequently,

$$\alpha T \sim T \quad \text{for} \quad a \in \partial T. \quad (3)$$

Now, let $a$ be a point in the interior of $T$. Then

$$\alpha T = [\alpha_0S_0, a] \cup \cdots \cup [\alpha_nS_n, a],$$

where $\alpha_iS_i$ is a triangulation of the facet $S_i$ of $T$. By the induction hypothesis 1 and 2

$$\alpha T \sim [S_0, a] \cup \cdots \cup [S_n, a].$$

We write $aT$ for the starring at $a$ of the $n$-simplex $T$ if every simplex in $aT$ has a facet of $T$ as its base. We dissect $T$ by a hyperplane $H$ through $a$ and an $(n-2)$-dimensional face of $T$ into two simplices $T^+, T^-$. This is an elementary move. Thus by (3)

$$T \sim T^+ \cup T^- \sim aT^+ \cup aT^-.$$  

If a facet $S$ is subdivided by $H$ into $S^+$ and $S^-$, then $[S^+, a] \cup [S^-, a] \sim [S, a]$. Thus

$$aT^+ \cup aT^- \sim aT.$$  

This completes the proof of the lemma.

Note that, for $a \in P$ fixed, 1 and 2 imply that any two starrings at $a$ are equivalent by elementary moves. We write $aP$ for a starring of a polytope $P$ at a point $a \in P$.

**Lemma 2.** For every $n$-polytope $P$ and $a, b \in P$, $aP \sim bP$.

**Proof.** Assume $P \subset \mathbb{R}^n$. We use induction on the number $m$ of vertices of $P$. If $m = n+1$, then $P$ is an $n$-simplex and the statement is true by Lemma 1. Suppose the lemma is true for polytopes with at most $m$ vertices. Write $[A_1, \ldots, A_l]$ for the convex hull of sets $A_1, \ldots, A_l$.

Let $P$ be a polytope with vertices $v_1, \ldots, v_m, v$. Let $a \in P^- = [v_1, \ldots, v_m]$. We say that a facet $F$ of $P^-$ is visible from $v$ if for every $x \in F$, $[v, x] \cap P^- = \{x\}$. By starring $P^-$ at $a$, the facets of $P^-$ that are visible from $v$ are subdivided into $(n-1)$-simplices $S_i$, $i = 1, \ldots, l$. Let $V_a$ be the set of the $n$-simplices $[S_i, v]$, $i = 1, \ldots, l$. Let $aP^-$ contain the simplices $[S_i, a]$, $i = 1, \ldots, l$, that are $n$-dimensional.

The main step is to show that

$$aP \sim aP^- \cup V_a. \quad (4)$$

If (4) holds and if $b \in P^-$, then the induction hypothesis on the number of vertices implies that $aP^- \sim bP^-$. By (1) and (2) we obtain $V_a \sim V_b$. Thus

$$aP \sim aP^- \cup V_a \sim bP^- \cup V_b \sim bP.$$
If there is no vertex \( v \) such that \( a, b \in [v_1, \ldots, v_m] \), then there are vertices \( v, w \) of \( P \) such that \( a \in [v_1, \ldots, v_{m-1}, w] \), \( b \in [v_1, \ldots, v_{m-1}, v] \) and \( P = [v_1, \ldots, v_{m-1}, v, w] \). Choose \( c \in [v_1, \ldots, v_{m-1}, w] \cap [v_1, \ldots, v_{m-1}, v] \) in the interior of \( P \). Then by \( \text{I} \)

\[ aP \sim cP \quad \text{and} \quad cP \sim bP. \]

Thus it is enough to show \( \text{I} \) to prove the lemma.

Let \( H_j, j = 1, \ldots, k \), be the affine hulls of \( S_i, i = 1, \ldots, l \). Denote the intersection point of the hyperplane \( H_j \) with the segment \([a, v]\) by \( x_j \) (these points are not necessarily distinct). Without loss of generality assume that the hyperplanes \( H_j \) are numbered such that \([a, v]\) is dissected into \([a, x_1], [x_1, x_2], \ldots, [x_{k-1}, x_k], [x_k, x_{k+1}] = v\)]. Let \( A_1 \) be the set of those simplices \([S_i, a]\), \( i = 1, \ldots, l \), that are \( n \)-dimensional. Set \( V_1 = V_a \).

First, assume that \( a \neq x_1 \) and \( x_k \neq v \). Starting with \( j = 1 \), we construct \( A_{j+1} \) and \( V_{j+1} \) from \( A_j \) and \( V_j \) in the following way. The simplices in \( A_j \) have \( a \) as a vertex. Let \( A_j(H_j) \subseteq A_j \) be the subset of those simplices that have their \((n - 1)\)-dimensional base in \( H_j \). The simplices in \( V_j \) have \( v \) as a vertex. Let \( V_j(H_j) \subseteq V_j \) be the subset of those simplices that have their \((n - 1)\)-dimensional base in \( H_j \). Note that the set of these \((n - 1)\)-dimensional bases coincides for \( A_j(H_j) \) and \( V_j(H_j) \). These bases form a facet \( F \), say, of \([P^-, x_j]\). By the induction hypothesis, the triangulation of \( F \) by these \((n - 1)\)-dimensional bases is equivalent by elementary moves to a starring of \( F \) at \( x_j \). The moves applied in \( H_j \) induce moves on \( A_j(H_j) \) and \( V_j(H_j) \). Denote the sets obtained by these moves by \( A'_j(H_j) \) and \( V'_j(H_j) \). Note that the set of \((n - 1)\)-dimensional bases coincides for \( A'_j(H_j) \) and \( V'_j(H_j) \). We replace \( A_j(H_j) \) by \( A'_j(H_j) \) and \( V_j(H_j) \) by \( V'_j(H_j) \) and set \( A'_j = (A_j \setminus A_j(H_j)) \cup A'_j(H_j) \) and \( V'_j = (V_j \setminus V_j(H_j)) \cup V'_j(H_j) \).

In the next step, we do the following. If \( x_j = x_{j+1} \), we set \( A_{j+1} = A'_j \) and \( V_{j+1} = V'_j \) and have

\[ A_j \cup V_j \sim A_{j+1} \cup V_{j+1}. \]

If \( x_j \neq x_{j+1} \), denote by \( A'_j(x_{j+1}) \) the subset of those simplices which contain \( x_j \) as a vertex and by \( V'_j(x_{j+1}) \subseteq V'_j \) the subset of those simplices which contain \( x_j \) as a vertex. Each simplex in \( V'_j(x_{j+1}) \) is the convex hull of \( x_j, v \) and an \((n - 2)\)-dimensional face \( B \), say. The convex hull \([B, x_j, v]\) is in \( V'_j(x_{j+1}) \) and \([B, a, x_j]\) in \( A'_j(x_{j+1}) \). We subdivide the simplex \([B, x_j, v]\) into the simplices \([B, x_j, x_{j+1}]\) and \([B, x_{j+1}, v]\). Then we take the union of \([B, a, x_j]\) and \([B, x_j, x_{j+1}] \) and obtain the simplex \([B, a, x_{j+1}] \). These operations are elementary moves. We do this with every simplex with vertex \( x_j \). We obtain the new set \( A'_j(x_{j+1}) \) which contains the simplices \([B, a, x_{j+1}] \) and the new set \( V'_j(x_{j+1}) \) which contains the simplices \([B, x_{j+1}, v]\). Let \( A_{j+1} = (A'_j \setminus A'_j(x_{j+1})) \cup A'_j(x_{j+1}) \) for \( j \leq k \) and \( V_{j+1} = (V'_j \setminus V'_j(x_{j+1})) \cup V'_j(x_{j+1}) \) for \( j < k \) and \( V_{k+1} = \emptyset \). We have

\[ A_j \cup V_j \sim A_{j+1} \cup V_{j+1}. \]

Next, we consider the case \( a = x_1 = \cdots = x_i \). Note that each facet of \( P^- \) that contains \( a \) is already starred at \( a \). Let \( V_1(a) \subseteq V_1 \) be the subset of those simplices which contain \( a \) as a vertex. Each simplex in \( V_1(a) \) is the convex hull of \( a, v \) and an \((n - 2)\)-dimensional face \( B \). We subdivide the simplex \([B, a, v]\) into the simplices \([B, a, x_{i+1}] \) and \([B, x_{i+1}, v]\). These operations are elementary moves. We do this with every simplex with vertex \( a \). We obtain the new set \( A_{i+1} \) which contains the simplices \([B, a, x_{i+1}] \) and the new set \( V_{i+1} \) which contains the simplices \([B, x_{i+1}, v]\) and the simplices in \( V_1 \setminus V_1(a) \). We
have $V_1 \sim A_{i+1} \cup V_{i+1}$. Starting with $j = i + 1$, we construct $A_{j+1}$ and $V_{j+1}$ by the algorithm described above. If $x_i = \cdots = x_k = v$, then the above algorithm implies that $V_i$ contains only lower dimensional simplices. Therefore we set $V_i = \emptyset, \ldots, V_{k+1} = \emptyset$ and $A_1 = A_i, \ldots, A_{k+1} = A_i$.

Thus in all cases

$$A_1 \cup V_1 \sim \cdots \sim A_{k+1} \cup V_{k+1} = A_{k+1}.$$

Since $aP \setminus A_{k+1} = aP \setminus A_1$, this proves (4). □

Let $T$ be an $n$-simplex and let $\alpha T$ be a triangulation of $T$. Assume $T \subset \mathbb{R}^n$. Let $H_1, \ldots, H_l$ be the affine hulls of the facets of the simplices in $\alpha T$. Let $\zeta_j T$ be the dissection into polytopes (in general not simplices) of $T$ by the hyperplanes $H_1, \ldots, H_j$ and let $\zeta_0 T = T$. We dissect each polytope $P$ of $\zeta_j T$ into simplices by starring $P$ at an arbitrary interior point. Let $\beta_j T$ be a triangulation of $T$ obtained from $\zeta_j T$ in this way.

We use induction on the number $k$ of hyperplanes. By Lemma 1, we have $T \sim \beta_0 T$. For $k \geq 1$ assume that $T \sim \beta_j T$ for $j < k$. (5)

Note that we obtain $\zeta_k T$ from $\zeta_{k-1} T$ by cutting by $H_k$. Every cell of $\zeta_{k-1} T$ is either unchanged or it is cut into two pieces. So let $P \in \zeta_{k-1} T$ be cut into pieces $P_1, P_2$. We show that for $a \in P$ and $a_i \in P_i$, $i = 1, 2$,

$$aP \sim a_1 P_1 \cup a_2 P_2. \quad (6)$$

By Lemma 2 $aP \sim bP$ for every $b \in H_k \cap P$. Again by Lemma 2 $a_1 P_1 \sim bP_1$ and $a_2 P_2 \sim bP$. Since any two starings of $P$ at $a \in P$ are equivalent by elementary moves this implies (6). By our definition of $\beta_{k-1} T$ and $\beta_k T$, (6) implies that

$$\beta_{k-1} T \sim \beta_k T.$$

Thus, by induction,

$$T \sim \beta_0 T \sim \cdots \sim \beta_k T. \quad (7)$$

Let $\alpha T$ consist of the simplices $S_1, \ldots, S_m$. Let $\zeta_j S_i$ be the dissection into polytopes of $S_i$ by the hyperplanes $H_1, \ldots, H_j$. Let $\beta_j S_i$ be a triangulation that is obtained by starring each cell of $\beta_j S_i$ at a point in that cell. By Lemma 2 these triangulations for different starring points are equivalent. Thus we obtain as before that

$$S_i \sim \beta_0 S_i \sim \cdots \sim \beta_k S_i.$$

Consequently,

$$\alpha T = (S_1 \cup \cdots \cup S_m) \sim \cdots \sim (\beta_1 S_1 \cup \cdots \cup \beta_1 S_m) = \beta_k T.$$

Combined with (7) this completes the proof of the proposition.
2 Proof of Theorem 2

We use induction on \( n \). The case \( n = 0 \) is straightforward. Suppose that every valuation on \( T^{n-1} \) has a unique extension to a valuation on \( Q^n \). Set \( R^n = \{ T \cup Q \mid T \in T^n, Q \in Q^{n-1} \} \). By the induction hypothesis, it is straightforward that \( \mu \) has a unique extension from \( T^n \) to \( R^n \). We prove that \( \mu \) has a unique extension from \( R^n \) to a valuation on \( Q^n \).

Let \( Q \in Q^n \) and let \( \delta Q = \{ R_1, \ldots, R_k \} \), \( R_i \in R^n \), be a dissection of \( Q \); that is, the intersection of a pair of elements of \( \delta Q \) has dimension less than \( n \) and their union equals \( Q \). For an ordered \( j \)-tuple \( I = \{ i_1, \ldots, i_j \} \), \( 1 \leq i_1 < \cdots < i_j \leq k \), \( 1 \leq j \leq k \), set \( R_I = R_{i_1} \cap \cdots \cap R_{i_j} \) and \( |I| = j \). The set \( Q^n \) is a lattice. Thus, if \( \mu \) can be extended to a valuation on \( Q^n \), then by the inclusion-exclusion principle

\[
\mu(Q) = \sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|-1} \mu(R_I).
\]

We show that, if \( Q \in Q^n \) and if \( \delta Q = \{ R_1, \ldots, R_k \} \), \( R_i \in R^n \), is a dissection of \( Q \), then (8) can be used as a definition of \( \mu(Q) \). Note that \( R_I \in R^n \). Therefore the induction hypothesis implies that the right side of (8) is well defined.

We show that \( \mu(Q) \) as defined by (8) does not depend on the choice of \( \delta Q \). First, let \( Q \) be an \( n \)-polyhedron and let \( \alpha Q = \{ T_1, \ldots, T_k \} \) be a triangulation of \( Q \). By Theorem 1 all triangulations of an \( n \)-polyhedron are equivalent by elementary moves. Thus it is sufficient to show that applying an elementary move to \( \alpha Q \) does not change \( \mu(Q) \). Since an elementary move is either the dissection of a simplex or the reverse, it suffices to show the following. If \( S_i = T_i, i = 1, \ldots, k-1 \), and if \( T_k \) is subdivided into \( S_k \) and \( S_{k+1} \), then

\[
\sum_{I \subseteq \{1, \ldots, k\}} (-1)^{|I|-1} \mu(T_I) = \sum_{J \subseteq \{1, \ldots, k+1\}} (-1)^{|J|-1} \mu(S_J).
\]

Set \( I' = I \setminus \{k\} \). Since \( \mu \) is a valuation on \( R^n \), we have for an ordered \( j \)-tuple \( I \) with \( k \in I \)

\[
\mu(T_I) = \mu(T_I \cap T_k) = \mu(S_I \cap (S_k \cup S_{k+1})) = \mu((S_I \cap S_k) \cup (S_I \cap S_{k+1})) = \mu(S_I) + \mu(S_I \cap S_{k+1}) - \mu(S_I \cap S_k).
\]

It follows that

\[
\sum_J (-1)^{|J|-1} \mu(S_J) = \sum_{k,k+1 \not\in J} (-1)^{|J|-1} \mu(S_J) + \sum_{k \in J, k+1 \not\in J} (-1)^{|J|-1} \mu(S_J)
\]

\[
+ \sum_{J \subseteq I \setminus \{k\}} (-1)^{|J|-1} \mu(S_J) + \sum_{k \in I, k+1 \in J} (-1)^{|J|-1} \mu(S_J)
\]

\[
= \sum_{k \not\in I} (-1)^{|I|-1} \mu(S_I) + \sum_{k \in I} (-1)^{|I|-1} \mu(S_I)
\]

\[
+ \sum_{k \in I} (-1)^{|I|-1} \mu(S_I \cap S_{k+1}) - \sum_{k \in I} (-1)^{|I|-1} \mu(S_I \cap S_k)\]

\[
= \sum_{k \not\in I} (-1)^{|I|-1} \mu(T_I) + \sum_{k \in I} (-1)^{|I|-1} \mu(T_I) = \sum_I (-1)^{|I|-1} \mu(T_I).
\]

Thus (8) holds and there is a unique extension of \( \mu \) to the set of \( n \)-polyhedra.
Next, let $P \in \mathcal{Q}^n$ be decomposed into $Q$ and $R$, where $Q$ is a uniquely defined $n$-polyhedron (or the empty set) and $R \in \mathcal{Q}^{n-1}$. Let $\delta P = \alpha Q \cup R$, where $\alpha Q = \{R_1, \ldots, R_{k-1}\}$ is a triangulation of $Q$, and set $R_k = R$. This is a dissection of $P$. By (8),

$$\mu(P) = \sum_I (-1)^{|I|-1} \mu(R_I) = \sum_{k \in I} (-1)^{|I|-1} \mu(R_I) + \sum_{k \in I} (-1)^{|I|-1} \mu(R_I)$$

$$= \mu(Q) + \mu(R) - \mu(Q \cap R).$$

Since $Q$ is an $n$-polyhedron and since $R, Q \cap R \in \mathcal{Q}^{n-1}$, the terms $\mu(Q), \mu(R), \mu(Q \cap R)$ are well defined. Hence $\mu(P)$ does not depend on $\delta P$ either.

Finally, we show that $\mu$ as defined by (8) is a valuation on $\mathcal{Q}^n$. Let $P, Q \in \mathcal{Q}^n$. We choose a dissection $\{R_1, \ldots, R_m\}$ of $P \cup Q$ such that, for every $R_i$, we have $R_i \subset P$ or $R_i \subset Q$. Then for every ordered $j$-tuple $I$,

$$\mu(P \cap R_I) + \mu(Q \cap R_I) = \mu((P \cup Q) \cap R_I) + \mu((P \cap Q) \cap R_I).$$

By (8), it follows that $\mu$ is a valuation.

3 Proof of Corollary

The proof presented here relies essentially on a theorem of Tverberg [36]. Note that it is also possible to prove the corollary by the extension theorems of Volland [37] or Perles and Sallee [30] or Groemer [11].

The main step is to prove that there is at most one extension. Let $\mu_1$ and $\mu_2$ be valuations on $\mathcal{P}^n$ such that $\mu_1 = \mu_2$ on $\mathcal{T}^n$. Using induction on $n$ we show that $\mu_1 = \mu_2$ also on $\mathcal{P}^n$. The cases $n = 0, 1$ are trivial. Suppose that $\mu_1 = \mu_2$ on $\mathcal{P}^{n-1}$.

A binary space partition is formed by partitioning $\mathbb{R}^N$ by a hyperplane $H$ into two closed halfspaces $H^+, H^-$, and then recursively partitioning each of the two resulting halfspaces; the result is a hierarchical decomposition of space into closed convex cells (cf. [3]). Let $P$ be an $n$-polytope. Since $\mu_1$ and $\mu_2$ are valuations, after the first step of a binary space partition we have for $i = 1, 2$,

$$\mu_i(P) = \mu_i(P \cap H^+) + \mu_i(P \cap H^-) \mu_i(P \cap H).$$

Note that $P \cap H \in \mathcal{P}^{n-1}$ for $P \not\subseteq H$ and thus $\mu_1(P \cap H) = \mu_2(P \cap H)$. Hence to prove $\mu_1(P) = \mu_2(P)$ it suffices to prove $\mu_1(Q) = \mu_2(Q)$ for $Q$ equal to $P \cap H^+$ and $P \cap H^-$. In the next step of the binary space partition the convex polytopes $P \cap H^+$ and $P \cap H^-$ are dissected by suitable hyperplanes and the same argument applies, etc.

Tverberg [36] showed that, given an $n$-polytope $P$, one can find a binary space partition that decomposes $P$ in finitely many steps into $n$-simplices. Since $\mu_1 = \mu_2$ on $\mathcal{T}^n$, this implies $\mu_1(P) = \mu_2(P)$ and the uniqueness is established.

By Theorem 2 there is at least one extension of $\mu$ to a valuation on $\mathcal{P}^n$, the one which can further be extended to $\mathcal{Q}^n$. Combined with the uniqueness result this proves the corollary.
References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. (2) **149** (1999), 977–1005.

[2] S. Alesker, *Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture*, Geom. Funct. Anal. **11** (2001), 244–272.

[3] S. Alesker, *Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations*, J. Differential Geom. **63** (2003), 63–95.

[4] S. Alesker, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26.

[5] J. Alexander, *The combinatorial theory of complexes*, Ann. of Math. (2) **31** (1930), 292–320.

[6] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational geometry*, Springer-Verlag, Berlin, 2000.

[7] V. Boltianskii, *Hilbert’s Third Problem*, John Wiley & Sons, New York, 1978.

[8] M. Dehn, *¨Uber den Rauminhalt*, Math. Ann. **55** (1901), 465–478.

[9] J. L. Dupont, *Algebra of polytopes and homology of flag complexes*, Osaka J. Math. **19** (1982), 599–641.

[10] G. Ewald and G. C. Shephard, *Stellar subdivisions of boundary complexes of convex polytopes*, Math. Ann. **210** (1974), 7–16.

[11] H. Groemer, *On the extension of additive functionals on classes of convex sets*, Pacific J. Math. **75** (1978), 397–410.

[12] H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

[13] D. Hilbert, *Mathematical Problems. Lecture delivered before the International Congress of Mathematicians in Paris, 1900. Translated by M. W. Newson*, Bull. Amer. Math. Soc. **8** (1902), 437-479.

[14] D. A. Klain, *A short proof of Hadwiger’s characterization theorem*, Mathematika **42** (1995), 329–339.

[15] D. A. Klain, *Star valuations and dual mixed volumes*, Adv. Math. **121** (1996), 80–101.

[16] D. A. Klain, *Invariant valuations on star-shaped sets*, Adv. Math. **125** (1997), 95–113.

[17] D. A. Klain, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), 71–93.

[18] D. A. Klain and G.-C. Rota, *Introduction to geometric probability*, Cambridge University Press, Cambridge, 1997.

[19] G. Kuperberg, *A generalization of Filliman duality*, Proc. Amer. Math. Soc. **131** (2003), 3893–3899.

[20] W. B. R. Lickorish, *Simplicial moves on complexes and manifolds*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), Geom. Topol. Monogr., vol. 2, Geom. Topol. Publ., Coventry, 1999, 299–320.

[21] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*, Adv. Math. **170** (2002), 239–256.

[22] M. Ludwig, *Ellipsoids and matrix valued valuations*, Duke Math. J. **119** (2003), 159–188.

[23] M. Ludwig and M. Reitzner, *A characterization of affine surface area*, Adv. Math. **147** (1999), 138–172.

[24] P. McMullen, *Valuations and dissections*, Handbook of Convex Geometry, Vol. B (P.M. Gruber and J.M. Wills, eds.), North-Holland, Amsterdam, 1993, 933–990.

[25] P. McMullen and R. Schneider, *Valuations on convex bodies*, Convexity and its applications (P.M. Gruber and J.M. Wills, eds.), Birkhäuser, 1983, 170–247.

[26] M. H. A. Newman, *On the foundations of combinatorial analysis situs, I,II.*, Proc. Amsterdam **29** (1926), 611–626, 627–641.

[27] M. H. A. Newman, *A theorem in combinatorial topology*, J. Lond. Math. Soc. **6** (1931), 186–192.

[28] U. Pachner, *¨Uber die bistellare Äquivalenz simplizialer Sphären und Polytope*, Math. Z. **176** (1981), 565–576.

[29] U. Pachner, *Shellings of simplicial balls and p.l. manifolds with boundary*, Discrete Math. **81** (1990), 37–47.
[30] M. A. Perles and G. T. Sallee, *Cell complexes, valuations, and the Euler relation*, Canad. J. Math. 22 (1970), 235–241.

[31] C. H. Sah, *Hilbert’s third problem: scissors congruence*, Research Notes in Mathematics, vol. 33, Pitman (Advanced Publishing Program), Boston, Mass., 1979.

[32] C. H. Sah, *Scissors congruences. I. The Gauss-Bonnet map*, Math. Scand. 49 (1981), no. 2, 181–210 (1982).

[33] S. O. Schatunovsky, *Über den Rauminhalt der Polyeder*, Math. Ann. 57 (1903), 496–508.

[34] R. Schneider, *Simple valuations on convex bodies*, Mathematika 43 (1996), 32–39.

[35] W. Süss, *Über das Inhaltsmass bei mehrdimensionalen Polyedern*, Tohoku Math. J. 38 (1933), 252–261.

[36] H. Tverberg, *How to cut a convex polytope into simplices*, Geometriae Dedicata 3 (1974), 239–240.

[37] W. Volland, *Ein Fortsetzungssatz für additive Eipolyederfunktionale im euklidischen Raum*, Arch. Math. 8 (1957), 144–149.

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