Deformation quantization: genesis, developments and metamorphoses

Giuseppe Dito and Daniel Sternheimer

Abstract

We start with a short exposition of developments in physics and mathematics that preceded, formed the basis for, or accompanied, the birth of deformation quantization in the seventies. We indicate how the latter is at least a viable alternative, autonomous and conceptually more satisfactory, to conventional quantum mechanics and mention related questions, including covariance and star representations of Lie groups. We sketch Fedosov’s geometric presentation, based on ideas coming from index theorems, which provided a beautiful frame for developing existence and classification of star-products on symplectic manifolds. We present Kontsevich’s formality, a major metamorphosis of deformation quantization, which implies existence and classification of star-products on general Poisson manifolds and has numerous ramifications. Its alternate proof using operads gave a new metamorphosis which in particular showed that the proper context is that of deformations of algebras over operads, while still another is provided by the extension from differential to algebraic geometry. In this panorama some important aspects are highlighted by a more detailed account.

1 Introduction: the background and around

The passage from commutative to noncommutative structures is a staple in frontier physics and mathematics. The advent of quantum mechanics is of course an important example but it is now becoming increasingly clear that the appearance, 25 years ago [FLS0, BFFLS], of deformation quantization, followed by numerous developments and metamorphoses, has been a major factor in the present trend. The epistemological background can be traced back [FL, SD] to the deformation philosophy which Moshé Flato developed in the early seventies, motivated by deep (decades old) physical ideas and (then recent) mathematical developments. Since that time, inspiring many, he...
has consistently pursued and promoted physical and mathematical consequences of that idea in several directions, of which deformation quantization is presently the most widely recognized.

Sir Michael Atiyah said recently, after Oscar Wilde, that mathematics and physics are two communities separated by a common language. Sir Michael added that the two communities managed to communicate rather well until the beginning of the twentieth century, then became so separated that, half a century ago, Eugene Wigner marveled at what he called the unreasonable effectiveness of mathematics in physics. In the past decade however, in spite of the ever increasing “Babel tower effect” in Science, some form of communication was developed, with a converse phenomenon: what may seem an unreasonable effectiveness of physics in mathematics, including in such abstract fields as algebraic geometry. Now, if we remember that mathematics arose as an abstraction of our understanding of the physical world, neither effectiveness should be unreasonable. We shall see here that deformation quantization is a perfect example of both of them.

Physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. concerned. But the passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way. Rather, experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified. Then the attached structures (symmetries, observables, states, etc.) deform the initial structure. Namely, we have a new structure which in the limit, when the new parameter goes to zero, coincides with the previous formalism. The only question is, in which category do we seek for deformations? Usually physics is rather conservative and if we start e.g. with the category of associative or Lie algebras, we tend to deform in the same category, but there are important examples of generalizations of this principle (e.g. quantum groups are deformations of Hopf algebras).

The discovery of the non-flat nature of Earth may be the first example of this phenomenon. Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity: there the Galilean geometrical symmetry group of Newtonian mechanics is deformed to the Poincaré group, the new fundamental constant being $c^{-1}$ where $c$ is the velocity of light in vacuum. At around the same time the Riemann surface theory can be considered as a first mathematical example of deformations, even if deformations became systematically studied in the mathematical literature only at the end of the fifties with the profound works of Kodaira and Spencer [KS] on deformations of complex analytic structures.

Now, when one has an action on a geometrical structure, it is natural to try and “linearize” it by inducing from it an action on an algebra of functions on that structure. This is implicitly what Gerstenhaber did in 1963 [Ge] with his definition and thorough study of deformations of rings and algebras. It is in the Gerstenhaber sense that the Galileo group is deformed to the Poincaré group; that operation is the inverse of the notion of group contraction introduced ten years before, empirically, by İnönü and Wigner [VI]. This fact triggered strong interest for deformation theory in France among a number of theoretical physicists, including Flato who had just arrived from the Racah school and knew well the effectiveness of symmetry in physical problems.

In 1900, as a last resort to explain the black body radiation, Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency.
He wrote $h$ for the proportionality constant which bears his name. This paradoxical situation got a beginning of a theoretical basis when, again in 1905, Einstein came with the theory of the photoelectric effect. Around 1920, an “agréé d’histoire”, Prince Louis de Broglie, was introduced to the photoelectric effect, together with the Planck–Einstein relations and the theory of relativity, in the laboratory of his much older brother, Maurice duc de Broglie. This led him, in 1923, to his discovery of the duality of waves and particles, which he described in his celebrated Thesis published in 1925, and to what he called ‘mécanique ondulatoire’. German and Austrian physicists, in particular, Hermann Weyl, Werner Heisenberg and Erwin Schrödinger, transformed it into the quantum mechanics that we know, where the observables are operators in Hilbert spaces of wave functions.

Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = \frac{h}{2\pi}$ goes to zero. But how can this be realized when in classical mechanics the observables are functions over phase space (a Poisson manifold) and not operators? The deformation philosophy promoted by Flato showed the way: one has to look for deformations of algebras of functions over Poisson manifolds endowed with the Poisson bracket and realize, in an autonomous manner, quantum mechanics there. This required, as a preliminary, a detailed study of the corresponding cohomology spaces. As a first step, in 1974, the cochains were assumed [FLS] to be 1-differentiable (given by bidifferential operators of order $(1, 1)$). This fell short of the solution but inspired Vey [Ve] who was able in 1975 (for symplectic manifolds with vanishing third Betti number) to show the existence of such differentiable deformations. Doing so, he rediscovered a formula for the deformed bracket (the sine of the Poisson bracket) that he did not know had been obtained in an entirely different context by Moyal [Mo] in 1949. The technical obstacle (a solution of which, in an algebraic context, could later be traced back to a result hidden in a fundamental paper [HKR] from 1962) was lifted and deformation quantization could be developed [FLSc, BFFLS] in 1976–78. In this approach, as we shall see in the next section, quantization is a deformation of the associative (and commutative) product of classical observables (functions on phase space) driven by the Poisson bracket, namely a star-product.

In a context which then seemed unrelated to deformation theory, pseudodifferential operators were introduced also at the end of the fifties and became a very hot subject in mathematics thanks to the publication in 1963 of the first index theorems by Atiyah and Singer [AS], which express an analytically defined index in topological terms. The composition of symbols of pseudodifferential operators, an important ingredient in the proof, is a nontrivial example of a star-product, but this fact was noticed only years later, after deformation quantization was introduced.

There have been many generalizations of the original index theorems, including algebraic versions developed in particular by Connes [Co] in the context of noncommutative geometry, a natural continuation of his important works of the seventies on operator algebras that were motivated by physical problems. It was developed shortly after the appearance of deformation quantization, using cyclic rather than plain Hochschild cohomology. If one adds to star-products the notion of trace (see Section 3.2.2), which in this case comes from integration over the manifold on which the functions are defined, one gets closed star-products [CFS] (classified by cyclic cohomology) that provide [Co] other examples of algebras falling in the framework of noncommutative geometry.
Now the Gelfand isomorphism provides a realization of a commutative algebra as an algebra of functions over a manifold, i.e., its spectrum. A natural but difficult question is then to develop a theory of noncommutative manifolds. Truly nontrivial examples (if we exclude the group case, cf. below) are appearing only recently (see e.g. [Ma, DV]).

Simple Lie groups and algebras are rigid for the Gerstenhaber notion of deformation but if one goes to the category of Hopf algebras, they can be deformed. This is what Drinfeld [Dr] realized with a notion to which he gave the spectacular (albeit somewhat misleading) name of quantum group, by considering star-products deforming the product in a Hopf algebra of functions on Lie groups having a compatible Poisson structure. The dual approach of deforming the coproduct in a closure of enveloping algebras, generalizing the $su(2)$ example discovered empirically [KR] in 1981 in relation with quantum inverse scattering problems, was also taken independently by Jimbo [Ji]. The domain has since known an extensive development, at the beginning mostly in Russia [FK]. Numerous applications to physics were developed in many fields and several metamorphoses occurred both in the mathematical methods used and in the concept itself. The subject now covers thousands of references, a fraction of which can be found in [ES, Maj, SS], which are recent complementary expositions ([SS] has 1264 references!) We shall not attempt to develop further this important avatar of deformation quantization.

2 Genesis and first developments

2.1 A few preliminaries

In this part we recall some very classical notions, definitions and properties, for the sake of completeness and the benefit of theoretical physicists who might not be familiar with them but would be willing to take up the mathematical jargon required without having to read an extensive literature. It may be skipped by many readers. As for all of this paper, the interested reader is strongly encouraged to study the references, and the references quoted in references (etc.), at least in the topics that are closest to his interests: the paper is meant to be mostly a starting point for newcomers in the domains covered.

2.1.1 Hochschild and Chevalley–Eilenberg cohomologies. When $A$ is an associative algebra (over some commutative ring $\mathbb{K}$), we can consider it as a module over itself with the adjoint action (algebra multiplication) and we shall here define cohomology in that context. The generalization to cohomology valued in a general module is straightforward. A Hochschild $p$-cochain is a $p$-linear map $C$ from $A^p$ to $A$ and its coboundary $bC$ is a $(p + 1)$-cochain given by

$$bC(u_0, \ldots, u_p) = u_0 C(u_1, \ldots, u_p) - C(u_0 u_1, u_2, \ldots, u_p) + \cdots + (-1)^p C(u_0, u_1, \ldots, u_{p-1} u_p) + (-1)^{p+1} C(u_0, \ldots, u_{p-1}) u_p.$$  \hspace{1cm} (2.1)

One checks that we have here what is called a complex, i.e. $b^2 = 0$. We say that a $p$-cochain $C$ is a $p$-cocycle if $bC = 0$. We denote by $Z^p(A, A)$ the space of $p$-cocycles and by $B^p(A, A)$ the space of those $p$-cocycles which are coboundaries (of a $(p - 1)$-cochain). The $p$th Hochschild cohomology
space (of \(A\) valued in \(A\) seen as a bimodule) is defined as \(H^p(A,A) = Z^p(A,A)/B^p(A,A)\). Cyclic cohomology is defined using a bicomplex which includes the Hochschild complex. We refer to [Cô] for a detailed treatment.

For **Lie algebras** (with bracket \(\{\cdot,\cdot\}\)) one has a similar definition, due to Chevalley and Eilenberg [CE]. The (ad-valued) \(p\)-cochains are here skew-symmetric, i.e., linear maps \(B : \wedge^p A \to A\), and the Chevalley coboundary operator \(\partial\) is defined on a \(p\)-cochain \(B\) by (where \(\hat{u}_j\) means that \(u_j\) has to be omitted):

\[
\partial C(u_0, \ldots, u_p) = \sum_{j=0}^{p} (-1)^j \{u_j, C(u_0, \ldots, \hat{u}_j, \ldots, u_p)\} + \sum_{i<j} (-1)^{i+j} C(\{u_i, u_j\}, u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_p).
\]

(2.2)

Again one has a complex (\(\partial^2 = 0\)), cocycles and coboundaries spaces \(Z^p\) and \(B^p\) (resp.) and by quotient the Chevalley cohomology spaces \(H^p(A,A)\), or in short \(H^p(A)\); the collection of all cohomology spaces will be denoted here \(H^\bullet\), or \(H^\bullet\) for the Hochschild cohomology.

### 2.1.2 Gerstenhaber theory of deformations of algebras

Let \(A\) be an algebra. By this we mean an **associative**, **Lie** or **Hopf** algebra, or a **bialgebra** [an associative algebra \(A\) where one has in addition a coproduct \(\Delta : A \to A \otimes A\) and the obvious compatibility relations, see e.g. [ES] for a precise definition]. Whenever needed we assume it is also a **topological** algebra, i.e., endowed with a locally convex topology for which all needed algebraic laws are continuous. For simplicity we may think that the base (commutative) ring \(\mathbb{K}\) is the field of complex numbers \(\mathbb{C}\) or that of the real numbers \(\mathbb{R}\). Extending it to the ring \(\mathbb{K}[[\lambda]]\) of formal series in some parameter \(\lambda\) gives the module \(\tilde{A} = A[[\lambda]]\), on which we can consider the preceding various algebraic (and topological) structures. In a number of instances we also need to look at \(A[\lambda^{-1}, \lambda]]\), formal series in \(\lambda\) and polynomials in \(\lambda^{-1}\) (considered at first as an independent parameter).

#### 2.1.2.1. Deformations and cohomologies

A concise formulation of a Gerstenhaber deformation of an algebra (which we shall call in short a DrG-deformation whenever a confusion may arise with more general deformations) is [Ge, GS, BFGP]:

**Definition 2.1.** A deformation of such an algebra \(A\) is a \(\mathbb{K}[[\lambda]]\)-algebra \(\tilde{A}\) such that \(\tilde{A}/\lambda\tilde{A} \approx A\). Two deformations \(\tilde{A}\) and \(\tilde{A}'\) are said equivalent if they are isomorphic over \(\mathbb{K}[[\lambda]]\) and \(\tilde{A}\) is said trivial if it is isomorphic to the original algebra \(A\) considered by base field extension as a \(\mathbb{K}[[\lambda]]\)-algebra.

Whenever we consider a topology on \(A\), \(\tilde{A}\) is supposed to be topologically free. For associative (resp. Lie) algebras, Definition [2.1] tells us that there exists a new product \(\ast\) (resp. bracket \(\{\cdot,\cdot\}\)) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. \(\{\cdot,\cdot\}\)) this means that, for \(u, v \in A\) we have:

\[
u \ast v = uv + \sum_{r=1}^{\infty} \lambda^r C_r(u, v)
\]

(2.3)

\[
[u, v] = \{u, v\} + \sum_{r=1}^{\infty} \lambda^r B_r(u, v)
\]

(2.4)
the $C_r$ being Hochschild 2-cochains and the $B_r$ (skew-symmetric) Chevalley 2-cochains, such that for $u,v,w \in A$ we have $(u \ast v) \ast w = u \ast (v \ast w)$ and $\mathcal{S}[u,v],w] = 0$, where $\mathcal{S}$ denotes summation over cyclic permutations (we extend (2.3) and (2.4) to $A[[\lambda]]$ by $\mathbb{K}[[\lambda]]$-linearity). At each level $r$ we therefore need to fulfill the equations $(j,k \geq 1)$:

$$D_r(u,v,w) \equiv \sum_{j+k=r} (C_j(C_k(u,v),w) - C_j(u,C_k(v,w))) = bC_r(u,v,w)$$  \hspace{1cm} (2.5)

$$E_r(u,v,w) \equiv \sum_{j+k=r} \mathcal{S}B_j(B_k(u,v),w) = \partial B_r(u,v,w)$$  \hspace{1cm} (2.6)

where $b$ and $\partial$ denote (respectively) the Hochschild and Chevalley coboundary operators. In particular we see that for $r = 1$ the driver $C_1$ (resp. $B_1$) must be a 2-cocycle. Furthermore, assuming one has shown that (2.5) or (2.6) are satisfied up to some order $r = t$, a simple calculation shows that the left-hand sides for $r = t + 1$ are then 3-cocycles, depending only on the cochains $C_k$ (resp. $B_k$) of order $k \leq t$. If we want to extend the deformation up to order $r = t + 1$ (i.e. to find the required 2-cochains $C_{t+1}$ or $B_{t+1}$), this cocycle has to be a coboundary (the coboundary of the required cochain): The obstructions to extend a deformation from one step to the next lie in the 3-cohomology. In particular (and this was Vey’s trick) if one can manage to pass always through the null class in the 3-cohomology, a cocycle can be the driver of a full-fledged (formal) deformation.

For a (topological) bialgebra, denoting by $\otimes_\lambda$ the tensor product of $\mathbb{K}[[\lambda]]$-modules, we can identify $\hat{A} \otimes_\lambda \hat{A}$ with $(A \hat{\otimes} A)[[\lambda]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some operator topology (e.g. projective for Fréchet nuclear topology), we similarly have a deformed coproduct $\hat{\Delta} = \Delta + \sum_1^\infty \lambda^r D_r$, $D_r \in \mathcal{L}(A,A \hat{\otimes} A)$ and in this context appropriate cohomologies can be introduced. As we have said we shall not elaborate on these, nor on the additional requirements for Hopf algebras, referring for more details to original papers and books.

2.1.2.2. Equivalence means that there is an isomorphism $T_\lambda = I + \sum_{r=1}^\infty \lambda^r T_r$, $T_r \in \mathcal{L}(A,A)$ so that $T_\lambda(u \ast' v) = (T_\lambda u \ast T_\lambda v)$ in the associative case, denoting by $\ast$ (resp. $\ast'$) the deformed laws in $\hat{A}$ (resp. $\hat{A}'$); and similarly in the Lie case. In particular we see (for $r = 1$) that a deformation is trivial at order 1 if it starts with a 2-cocycle which is a 2-coboundary. More generally, exactly as above, we can show that if two deformations are equivalent up to some order $t$, the condition to extend the equivalence one step further is that a 2-cocycle (defined using the $T_k$, $k \leq t$) is the coboundary of the required $T_{t+1}$ and therefore the obstructions to equivalence lie in the 2-cohomology. In particular, if that space is null, all deformations are trivial.

An important property is that a deformation of an associative algebra with unit (what is called a unital algebra) is again unital, and equivalent to a deformation with the same unit. This follows from a more general result of Gerstenhaber (for deformations leaving unchanged a subalgebra); a proof can be found in [GS].

In the case of (topological) bialgebras or Hopf algebras, equivalence of deformations has to be understood as an isomorphism of (topological) $\mathbb{K}[[\lambda]]$-algebras, the isomorphism starting with the identity for the degree 0 in $\lambda$. A deformation is again said trivial if it is equivalent to that obtained by base field extension. For Hopf algebras the deformed algebras may be taken (by equivalence) to have the same unit and counit, but in general not the same antipode.
2.1.2.3. Homotopy of deformations. Recently Gerstenhaber has considered (for reasons that are related to the so-called Donald–Flanigan conjecture, see \([GG, GGS]\)) the question of (formal) compatibility of deformations, a kind of homotopy in the variety of algebras between two deformations \(N\) with parameters \(\lambda\) and \(\lambda'\) and cochains \(C_r\) and \(C'_r\). By this he means a 2-parameter deformation of the form

\[
u \star v = uv + \lambda C_1(u,v) + \lambda' C'_1(u,v) + \sum_{r=2}^{\infty} \Phi_r(u,v;\lambda,\lambda')\tag{2.7}
\]

where each \(\Phi_r\) is a polynomial of total degree \(r\) in \(\lambda\) and \(\lambda'\), which reduces to the first one-parameter deformation when \(\lambda' = 0\) and to the second when \(\lambda = 0\). At the first order the condition for this to hold (e.g. for associative algebras) is that the Gerstenhaber bracket \([C_1, C'_1]_G\) is a 3-coboundary, and here also there are higher obstructions. As an example, it follows from \([HKR]\) that the Weyl algebra and the quantum plane are formally (but non analytically \([GZ]\)) compatible nonequivalent deformations of the polynomial algebra \(\mathbb{C}[x,y]\). In (2.2.3.4) below we shall see another appearance of such a 2-parameter deformation in a physical context.

2.1.3 The differentiable case, Poisson manifolds. Consider the algebra \(N = C^\infty(X)\) of functions on a differentiable manifold \(X\). When we look at it as an associative algebra acting on itself by pointwise multiplication, we can define the corresponding Hochschild cohomologies. Now let \(\Lambda\) be a skew-symmetric contravariant two-tensor (possibly degenerate) defined on \(X\), satisfying \([\Lambda, \Lambda]_{SN} = 0\) in the sense of the Schouten–Nijenhuis bracket (a definition of which, both intrinsic and in terms of local coordinates, can be found in \([BFFLS, FLS]\); see also (4.3) below).

Then the inner product of \(\Lambda\) with the 2-form \(du \wedge dv, P(u,v) = i(\Lambda)(du \wedge dv), u, v \in N\), defines a Poisson bracket \(P\): it is obviously skew-symmetric, satisfies the Jacobi identity because \([\Lambda, \Lambda]_{SN} = 0\) and the Leibniz rule \(P(uv,w) = P(u,v)w + uP(v,w)\). It is a bidifferential 2-cocycle for the (general or differentiable) Hochschild cohomology of \(N\), skewsymmetric of order \((1,1)\), therefore nontrivial \([BFFLS]\) and thus defines an infinitesimal deformation of the pointwise product on \(N\). \((X, P)\) is called a Poisson manifold \([BFFLS, LiP]\).

When \(\Lambda\) is everywhere nondegenerate \((X\) is then necessarily of even dimension \(2\ell)\), its inverse \(\omega\) is a closed everywhere nondegenerate 2-form \((d\omega = 0)\) which is equivalent to \([\Lambda, \Lambda]_{SN} = 0\) and we say that \((X, \omega)\) is symplectic; \(\omega^\ell\) is a volume element on \(X\). Then one can in a consistent manner work with differentiable cocycles \([BFFLS, FLS]\) and the differentiable Hochschild \(p\)-cohomology space \(H^p(N)\) is that of all skew-symmetric contravariant \(p\)-tensor fields \([HKR, Ve]\), therefore is infinite-dimensional. Thus, except when \(X\) is of dimension 2 (because then necessarily \(H^3(N) = 0\)), the obstructions belong to an infinite-dimensional space where they may be difficult to trace. On the other hand, when \(2\ell = 2\), any 2-cocycle can be the driver of a deformation of the associative algebra \(N\): “anything goes” in this case; some examples for \(\mathbb{R}^2\) can be found already in \([Ve]\).

Now endow \(N\) with a Poisson bracket: we get a Lie algebra and can look at its Chevalley cohomology spaces. Note that \(P\) is bidifferential of order \((1,1)\) so it is important to check whether the Gerstenhaber theory is consistent when restricted to differentiable cochains (in both cases, of arbitrary order and order at most 1). The answer is positive; in brief, if a coboundary is differen-
tiable, it is the coboundary of a differentiable cochain. Later it was found [33, 24] that assuming only continuity gives the same type of results.

Since \( P \) is of order \((1,1)\), it was natural to study first the 1-differentiable cohomologies. When the cochains are restricted to be of order \((1,1)\) with no constant term (then they annihilate constant functions, which we write “n.c.” for “null on constants”) it was found [11] that the Chevalley cohomology \( \mathcal{H}_{1-diff,n.c.}^p (N) \) of the Lie algebra \( N \) (acting on itself with the adjoint representation) is exactly the de Rham cohomology \( \mathcal{H}_{diff}^p(X) \). Thus \( \dim \mathcal{H}_{1-diff,n.c.}^p (N) = b_p(X) \), the \( p \)th Betti number of the manifold \( X \). Without the n.c. condition one gets a slightly more complicated formula [31]: in particular if \( X \) is symplectic with an exact 2-form \( \omega = d\alpha \), one has here \( \mathcal{H}_{1-diff}^p (N) = H^p_{dR}(X) \oplus H^{p-1}_{dR}(X) \).

Then the “three musketeers” [10] could, in 1974, study 1-differentiable deformations of the Poisson bracket Lie algebra \( N \) and develop some applications at the level of classical mechanics. In particular it was noticed that the “pure” order \((1,1)\) deformations correspond to a deformation of the 2-tensor \( \Lambda \); allowing constant terms and taking the deformed bracket in Hamilton equations instead of the original Poisson bracket gave a kind of friction term.

Shortly afterwards, triggered by that work, a “fourth musketeer” J. Vey [26] noticed that in 1974, study 1-differentiable deformations of the Poisson bracket Lie algebra \( N \) and develop some applications at the level of classical mechanics. He could then study more general differentiable deformations, rediscovering in the \( \mathbb{R}^{2\ell} \) case the Moyal bracket. The latter was at that time rather “exotic” and few authors (except for a number of physicists, see e.g. [40]) paid any attention to it. In Mathematical Reviews this bracket, for which [24] is nowadays often quoted, is not even mentioned in the review! Vey’s work permitted to tackle the problem with differentiable deformations, and deformation quantization was born [10].

### 2.2 The founding papers and some follow-up

#### 2.2.1 Classical mechanics and quantization.

Classical mechanics, in Lagrangean or Hamiltonian form, assumed at first implicitly a “flat” phase space \( \mathbb{R}^{2\ell} \), or at least considered only an open connected set thereof. Eventually more general configurations were needed and so the mathematical notion of manifold, on which mechanics imposed some structure, was used. This has lead in particular to using the notions of symplectic and later of Poisson manifolds, which were introduced also for purely mathematical reasons. One of these reasons has to do with families of infinite-dimensional Lie algebras, which date back to works by Élie Cartan at the beginning of last century and regained a lot of popularity (including in physics) in the past 30 years.

In 1927, Weyl came out with his quantization rule [37]. If we start with a classical observable \( u(p,q) \), some function on (flat) phase space \( \mathbb{R}^{2\ell} \) (with \( p,q \in \mathbb{R}^{\ell} \)), one can associate to it an operator (the corresponding quantum observable) \( \Omega(u) \) in the Hilbert space \( L^2(\mathbb{R}^{\ell}) \) by the following general recipe:

\[
 u \mapsto \Omega_w(u) = \int_{\mathbb{R}^{2\ell}} \tilde{u}(\xi,\eta) \exp(i(P.\xi + Q.\eta)/\hbar) w(\xi,\eta) \ d^\ell\xi d^\ell\eta
\]  

(2.8)

where \( \tilde{u} \) is the inverse Fourier transform of \( u \), \( P_\alpha \) and \( Q_\alpha \) are operators satisfying the canonical commutation relations \( [P_\alpha, Q_\beta] = i\hbar \delta_{\alpha\beta} \) (\( \alpha, \beta = 1, \ldots, \ell \)), \( w \) is a weight function and the integral is
taken in the weak operator topology. What is now called normal ordering corresponds to choosing
the weight \( w(\xi, \eta) = \exp\left(-\frac{1}{4}(\xi^2 + \eta^2)\right) \), standard ordering (the case of the usual pseudodifferential
operators in mathematics) to the weight \( w(\xi, \eta) = \exp\left(-\frac{1}{4}\xi\eta\right) \) and the original Weyl (symmetric)
ordering to \( w = 1 \). An inverse formula was found shortly afterwards by Eugene Wigner \([W]\) and maps an operator into what mathematicians call its symbol by a kind of trace formula.
For example \( \Omega_1 \) defines an isomorphism of Hilbert spaces between \( L^2(\mathbb{R}^2\ell) \) and Hilbert–Schmidt
operators on \( L^2(\mathbb{R}\ell) \) with inverse given by
\[
 u = (2\pi\hbar)^{-\ell} \text{Tr}\left[\Omega_1(u) \exp((\xi, \ell P + \eta, Q)/\hbar)\right]
\]
and if \( \Omega_1(u) \) is of trace class one has \( \text{Tr}(\Omega_1(u)) = (2\pi\hbar)^{-\ell} \int u \omega^\ell \equiv \text{Tr}_M(u) \), the “Moyal trace”,
where \( \omega^\ell \) is the (symplectic) volume form \( dx \) on \( \mathbb{R}^{2\ell} \). Numerous developments followed in the direction
of phase-space methods, many of which can be found described in \([AW]\). Of particular interest to
us here is the question of finding a physical interpretation to the classical function \( u \), symbol of the
quantum operator \( \Omega_1(u) \); this was the problem posed (around 15 years after \([Wi]\)) by Blackett to
his student Moyal. The (somewhat naive) idea to interpret it as a probability density had of course
no reason to be positive) but, looking for a direct expression for the
symbol of a quantum commutator, Moyal found \([Mo]\) what is now called the Moyal bracket:
\[
 M(u, v) = \lambda^{-1} \sinh(\lambda P)(u, v) = P(u, v) + \sum_{r=1}^{\infty} \lambda^{2r} P^{2r+1}(u, v) \tag{2.10}
\]
where \( 2\lambda = i\hbar \), \( P^r(u, v) = \Lambda^{ij_1} \ldots \Lambda^{ij_r} (\partial_{j_1} \ldots \partial_{j_k}) u(\partial_{j_1} \ldots \partial_{j_k}) v \) is the \( r \)-th power \((r \geq 1)\) of the Poisson
bracket bidifferential operator \( P, i_k, j_k = 1, \ldots, 2\ell, k = 1, \ldots, r \) and \( \Lambda = (\delta_{ij} - \eta_{ij}) \). To fix ideas we may
assume here \( u, v \in C^\infty(\mathbb{R}^{2\ell}) \) and the sum taken as a formal series (the definition and convergence
for various families of functions \( u \) and \( v \) was also studied, including in \([BFFLS]\)). A similar
formula for the symbol of a product \( \Omega_1(u) \Omega_1(v) \) had been found a little earlier \([Gr]\) and can now
be written more clearly as a (Moyal) star-product:
\[
 u \star_M v = \exp(\lambda P)(u, v) = uv + \sum_{r=1}^{\infty} \lambda^r P^r(u, v). \tag{2.11}
\]
Several integral formulas for the star-product have been introduced (we shall come back later
to this question). In great part after deformation quantization was developed, the Wigner image
of various families of operators (including all bounded operators on \( L^2(\mathbb{R}^\ell) \)) was studied and an
adaptation to Weyl ordering of the mathematical notion of pseudodifferential operators (ordered,
like differential operators, “first \( q \), then \( p \)” had been developed. We shall not give here more
details on those aspects, but for completeness mention a related development. Starting from field
theory, where normal (Wick) ordering is essential (the role of \( q \) and \( p \) above is played by \( q \pm ip \)),
Berezin \([Bc, BS]\) developed in the mid-seventies an extensive study of what he called “quantization”,
based on the correspondence principle and Wick symbols. It is essentially based on Kähler
manifolds and related to pseudodifferential operators in the complex domain \([BC]\). However in
his approach, as in the studies of various orderings \([AW]\), the important concepts of deformation
and autonomous formulation of quantum mechanics in general phase space are absent.
Quantization involving more general phase spaces was treated, in a somewhat systematic manner, only with Dirac constraints \([D]\): second class Dirac constraints restrict phase space from some \(\mathbb{R}^{2l}\) to a symplectic manifold \(W\) imbedded in it (with induced symplectic form), while first class constraints further restrict to a Poisson manifold with symplectic foliation (see e.g. \([FLSq]\)). The question of quantization on such manifolds was certainly treated by many authors (including \([D]\)) but did not go beyond giving some (often useful) recipes and hoping for the best.

A first systematic attempt started around 1970 with what was called soon afterwards geometric quantization \([Ko]\), a by-product of Lie group representations theory where it gave significant results. It turns out that it is geometric all right, but its scope as far as quantization is concerned has been rather limited since few classical observables could be quantized, except in situations which amount essentially to the Weyl case considered above. In a nutshell one considers phase-spaces \(W\) which are coadjoint orbits of some Lie groups (the Weyl case corresponds to the Heisenberg group with the canonical commutation relations and \(\hbar\) as Lie algebra); there one defines a “pre-quantization” on the Hilbert space \(L^2(W)\) and tries to halve the number of degrees of freedom by using polarizations (often complex ones, which is not an innocent operation as far as physics is concerned) to get a Lagrangean submanifold \(L\) of dimension half of that of \(W\) and quantized observables as operators in \(L^2(L)\). A recent exposition can be found in \([Wo]\).

2.2.2 Star-products. Star-products as a deformation of the usual product of functions on a phase-space meant for an understanding of quantum mechanics were introduced in \([FLSq]\) and their relevance examplified by the founding papers \([BFFLS]\), which included significant applications. Their general definition is as follows.

Let \(X\) be a differentiable manifold (of finite, or possibly infinite, dimension). We assume given on \(X\) a Poisson structure \((\text{a Poisson bracket } P)\).

**Definition 2.2.** A star-product on \(X\) is a deformation of the associative algebra of functions \(N = C^\infty(X)\) of the form \(\star = \sum_{n=0}^{\infty} \lambda^n C_n\) where \(C_0(u,v) = uv, C_1(u,v) - C_1(v,u) = 2P(u,v), u, v \in N,\) and the \(C_n\) are bidifferential operators (locally of finite order). We say a star-product is strongly closed if \(\int_X (u \star v - v \star u) dx = 0\) where \(dx\) is a volume element on \(X\).

The parameter \(\lambda\) of the deformation is taken to be \(\lambda = \frac{i}{2\hbar}\) in physical applications.

2.2.2.1. \(a.\) Using equivalence one may take \(C_1 = P\). That is the case of Moyal, but other orderings like standard or normal do not verify this condition (only the skew-symmetric part of \(C_1\) is \(P\)).

Again by equivalence, in view of Gerstenhaber’s result on the unit (cf. 2.1.2.2), we may take cochains \(C_r\) which are without constant term (what we called n.c. or null on constants). In fact, in the original paper \([BFFLS]\), only this case was considered and the accent was put on “Vey products”, for which the cochains \(C_r\) have the same parity as \(r\) and have \(P'\) for principal symbol in any Darboux chart, with \(X\) symplectic.

\(b.\) It is also possible to consider star-products for which the cochains \(C_n\) are allowed to be slightly more general. Allowing them to be local \((C_n(u,v) = 0\) on any open set where \(u\) or \(v\) vanish\) gives nothing new. Note that this is not the same as requiring the whole associative product to be local (in fact the latter condition is very restrictive and, like true pseudodifferential operators, a star-product is a nonlocal operation). In some cases (e.g. for star representations of Lie groups)
it may be practical to consider pseudodifferential cochains. As far as the cohomologies are concerned, as long as one requires at least continuity for the cochains, the theory is the same as in the differentiable case [ACMP].

Also, due to formulas like (2.9) and the relation with Lie algebras (see below), it is sometimes convenient to take \( \mathbb{K}[\lambda^{-1}, \lambda] \) (Laurent series in \( \lambda \), polynomial in \( \lambda^{-1} \) and formal series in \( \lambda \)) for the ring on which the deformation is defined. Again, this will not change the theory.

c. By taking the corresponding commutator \([u, v]_\lambda = (2\lambda)^{-1}(u \ast v - v \ast u)\), since the skew-symmetric part of \( C_1 \) is \( P \), we get a deformation of the Poisson bracket Lie algebra \((N, P)\). This is a crucial point because (at least in the symplectic case) we know the needed Chevalley cohomologies and (in contradistinction with the Hochschild cohomologies) they are small [Ve, DLG]. The interplay between both structures gives existence and classification; in addition it will explain why (in the symplectic case) the classification of star-products is based on the 1-differentiable cohomologies, hence ultimately on the de Rham cohomology of the manifold.

2.2.2.2. Invariance and covariance. The Poisson bracket \( P \) is (by definition) invariant under all symplectomorphisms, i.e. transformations of the manifold \( X \) generated by the flows \( x_\mu = i(\Lambda)(du) \) defined by Hamiltonians \( u \in N \). But already on \( \mathbb{R}^{2\ell} \) one sees easily that its powers \( P^p \), hence also the Moyal bracket (2.10), are invariant only under flows generated by Hamiltonians \( u \) which are polynomials of maximal order 2, forming the “affine” symplectic Lie algebra \( \mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot h_c \). For other orderings the invariance is even smaller (only \( h_c \) remains). For general Vey products the first terms of a star-product are [BFFLS] \( C_2 = P_1^2 + bH \) and \( C_3 = S_1^3 + T + 3\partial H \). Here \( H \) is a differential operator of maximal order 2, \( T \) a 2-tensor corresponding to a closed 2-form, \( \partial \) the Chevalley coboundary operator. \( P_1^2 \) is given (in canonical coordinates) by an expression similar to \( P^2 \) in which usual derivatives are replaced by covariant derivatives with respect to a given symplectic connection \( \Gamma \) (a torsionless connection with totally skew-symmetric components when all indices are lowered using \( \Lambda \)). \( S_1^3 \) is a very special cochain given by an expression similar to \( P^3 \) in which the derivatives are replaced by the relevant components of the Lie derivative of \( \Gamma \) in the direction of the vector field associated to the function \((u \lor v)\). Fedosov’s algorithmic construction [FC] shows that the symplectic connection \( \Gamma \) plays a role at all orders. Therefore the invariance group of a star-product is a subgroup of the finite-dimensional group of symplectomorphisms preserving a connection. Its Lie algebra is \( g_0 = \{ a \in N; [a, u]_\lambda = P(a, u) \forall u \in N \} \), the elements of which are preferred observables, i.e., Hamiltonians for which the classical and quantum evolutions coincide. We are thus lead to look for a weaker notion and shall call a star-product covariant under a Lie algebra \( g \) of functions if \([a, b]_\lambda = P(a, b) \forall a, b \in g\). It can be shown [ACMP] that \( \ast \) is \( g \)-covariant iff there exists a representation \( \tau \) of the Lie group \( G \) whose Lie algebra is \( g \) into \( \text{Aut}(N[\lambda]); \ast \) such that \( \tau_g u = (Id_N + \sum_{\ell=1}^\infty \lambda^\ell \tau_\ell)(g, u) \) where \( g \in G, u \in N, G \) acts on \( N \) by the natural action induced by the vector fields associated with \( g \), \( (g \cdot u)(x) = u(g^{-1}x) \), and where the \( \tau_\ell \) are differential operators on \( W \). Invariance of course means that the geometric action preserves the star-product: \( g \cdot u \ast g \cdot v = g \cdot (u \ast v) \). This is the basis for the theory of star representations.
which we shall briefly present below, and of the relevance of star-products for important problems in group theory.

2.2.3 Quantum mechanics. Let us start with a phase space $X$, a symplectic (or Poisson) manifold and $N$ an algebra of classical observables (functions, possibly including distributions if proper care is taken for the product). We shall call star quantization a star-product on $X$ invariant (or sometimes only covariant) under some Lie algebra $g_0$ of “preferred observables”. Invariance of the star-product ensures that the classical and quantum evolutions of observables under a Hamiltonian $H \in g_0$ will coincide [BFFLS]. The typical example is the Moyal product on $W = \mathbb{R}^{2\ell}$.

2.2.3.1 Spectrality. Physicists want to get numbers matching experimental results, e.g. for energy levels of a system. That is usually achieved by describing the spectrum of a given Hamiltonian $\hat{H}$ supposed to be a self-adjoint operator so as to get a real spectrum and so that the evolution operator (the exponential of $i\hat{H}$) is unitary (thus preserves probability amplitude). A similar spectral theory can be done here, in an autonomous manner. The most efficient way to achieve it is to consider [BFFLS] the star exponential (corresponding to the evolution operator)

$$\exp(\hat{H}t) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{\hbar} \right)^n (H \ast)^n$$

where $(H \ast)^n$ means the $n^{th}$ star power of the Hamiltonian $H \in N$ (or $N[[\lambda]]$). Then one writes its Fourier–Stieltjes transform $d\mu$ in the distribution sense as $\exp(\hat{H}t) = \int e^{\lambda t/\hbar} d\mu(\lambda)$ and defines the spectrum of $(H/\hbar)$ as the support $S$ of $d\mu$ (incidentally this is the definition given by L. Schwartz for the spectrum of a distribution, out of motivations coming from Fourier analysis). In the particular case when $H$ has discrete spectrum, the integral can be written as a sum (see the top equation in (2.14) below for a typical example): the distribution $d\mu$ is a sum of “delta functions” supported at the points of $S$ multiplied by the symbols of the corresponding eigenprojectors.

In different orderings with various weight functions $w$ in (2.8) one gets in general different operators for the same classical observable $H$, thus different spectra. For $X = \mathbb{R}^{2\ell}$ all orderings are mathematically equivalent (to Moyal under the Fourier transform $T_\w$ of the weight function $w$). This means that every observable $H$ will have the same spectrum under Moyal ordering as $T_\w H$ under the equivalent ordering. But this does not imply physical equivalence, i.e. the fact that $H$ will have the same spectrum under both orderings. In fact the opposite is true: if two equivalent star-products are isospectral (give the same spectrum for a large family of observables and all $\hbar$), they are identical [CFG].

It is worth mentioning that our definition of spectrum permits to define a spectrum even for symbols of non-spectrable operators, such as the derivative on a half-line which has different deficiency indices; this corresponds to an infinite potential barrier (see also [UU] for detailed studies of similar questions). That is one of the many advantages of our autonomous approach to quantization.

2.2.3.2 Applications. In quantum mechanics it is preferable to work (for $X = \mathbb{R}^{2\ell}$) with the star-product that has maximal symmetry, i.e. $\mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot \mathfrak{h}_\ell$ as algebra of preferred observables: the Moyal product. One indeed finds [BFFLS] that the star exponential of these observables
Calculations for angular momentum to be (as they should, with not be considered as perturbation series.

Quantization, the results of "conventional" quantum mechanics in those typical examples (many more integrals appearing in the various expressions being convergent as distributions, both in phase-space variables and in t or λ.)

\[
\text{Exp}(Ht) = \begin{cases} 
(\cos \delta t)^{-1} \exp((H/\hbar) \tan(\delta t)) & \text{for } d > 0 \\
\exp(Ht/\hbar) & \text{for } d = 0 \\
(\cosh \delta t)^{-1} \exp((H/\hbar) \tanh(\delta t)) & \text{for } d < 0
\end{cases}
\]

\[
\text{Exp}(Ht) = \begin{cases} 
\sum_{n=0}^{\infty} \Pi_{n}(\ell) e^{(n+\frac{\ell}{2})\hbar} & \text{for } d > 0 \\
\int_{-\infty}^{\infty} e^{\lambda t/\hbar} \Pi(\lambda, H) d\lambda & \text{for } d < 0
\end{cases}
\]

We get the discrete spectrum \((n + \frac{\ell}{2})\hbar\) of the harmonic oscillator and the continuous spectrum \(\mathbb{R}\) for the dilation generator \(pq\). The eigenprojectors \(\Pi_{n}(\ell)\) and \(\Pi(\lambda, H)\) are given \([BFFLS]\) by known special functions on phase-space (generalized Laguerre and hypergeometric, multiplied by some exponential). Formulas \((2.13)\) and \((2.14)\) can, by analytic continuation, be extended outside singularities and (as distributions) for singular values of \(t\).

Other examples can be brought to this case by functional manipulations \([BFFLS]\). For instance the Casimir element \(C\) of \(so(\ell)\) representing angular momentum, which can be written \(C = \frac{1}{2} pq - (\ell(\ell - 1))\hbar^{2}\) for spectrum. For the hydrogen atom, with Hamiltonian \(H = \frac{1}{2} p^{2} - |q|^{-1}\), the Moyal product on \(\mathbb{R}^{2\ell+2}\) \((\ell = 3\) in the physical case) induces a star-product on \(X = \mathbb{T}^{\ast} \mathbb{S}^{d}\); the energy levels, solutions of \((H - E) \star \phi = 0\), are found from \((2.14)\) and the preceding calculations for angular momentum to be (as they should, with \(\ell = 3\)) \(E = \frac{1}{2} (n + 1)^{-2} \hbar^{-2}\) for the discrete spectrum, and \(E \in \mathbb{R}^{\ast}\) for the continuous spectrum.

We thus have recovered, in a completely autonomous manner entirely within deformation quantization, the results of “conventional” quantum mechanics in those typical examples (many more can be treated similarly). It is worth noting that the term \(\frac{\ell}{2}\) in the harmonic oscillator spectrum, obvious source of divergences in the infinite-dimensional case, disappears if the normal star-product is used instead of Moyal – which is one of the reasons it is preferred in field theory.

2.2.3.3. Remark on convergence and the physical meaning of deformation quantization. We have always considered star-products as formal series and looked for convergence only in specific examples, generally in the sense of distributions. The same applies to star exponentials, as long as each coefficient in the formal series is well defined. In the case of the harmonic oscillator or more generally the preferred observables \(H\) in Weyl ordering, the study is facilitated by the fact that the powers \((H \star \phi)^{n}\) are polynomials in \(H\).

Integral formulas for star-products and star exponentials can be used to give a meaning to these expressions beyond the domains where the sums are convergent or summable. The sums must not be considered as perturbation series, rather taken as a whole, whether expressed as integral formulas or formal series.
We stress that deformation quantization should not be seen as a mere reformulation of quantum mechanics or quantum theories in general. At the conceptual level, it is the true mathematical formulation of physical reality whenever quantum effects have to be taken into account. The above examples show that one can indeed perform important quantum mechanical calculations, in an autonomous manner, entirely within deformation quantization—and get the results obtained in conventional quantum mechanics. Whether one uses an operatorial formulation or some form of deformation quantization formulation is thus basically a practical question, which formulation is the most effective, at least in the cases where a Weyl or Wigner map exists. When such a map does not exist, a satisfactory operatorial formulation will be very hard to find, except locally on phase space, and deformation quantization is the solution.

One can of course (and should in practical examples, as we have done here, and also for algebraic varieties) look for small domains (in $N$) where one has convergence. We can then speak of “strict” deformation quantization. In particular we can look for domains where pointwise convergence can be proved; this was done e.g. for Hermitian symmetric spaces $[CGR]$. But it should be clearly understood that one can consider wider classes of observables—in fact, the latter tend to be physically more interesting—than those that fit in a strict $C^*$-algebraic approach.

2.2.3.4. A 2-parameter star-product and statistical mechanics. A relation similar to (2.7) had been considered for star-products in $[BFLS]$, in connection with statistical mechanics. Indeed, in view of our philosophy on deformations, a natural question to ask is their stability: Can deformations be further deformed, or does “the buck stops there”? As we indicated at the beginning, the answer to that question may depend on the context. Quantum groups are an example, when dealing with Hopf algebras; properties of Harrison cohomology require going to noncontinuous cochains if one wants nontrivial abelian deformations (see (4.3.3.3) and $[Fr]$). Here is a simpler example.

If one looks for deformations of the Poisson bracket Lie algebra $(N, P)$ one finds (under a mild assumption) that a further deformation of the Moyal bracket, with another deformation parameter $\rho$, is again a Moyal bracket for a $\rho$-deformed Poisson structure; in particular, for $X = \mathbb{R}^{2\ell}$, quantum mechanics viewed as a deformation is unique and stable.

Now, for the associative algebra $N$, the only local associative composition laws are of the form $(u, v) \mapsto ufv$ for some $f \in N$. If we take $f = f_\beta \in N[[\beta]]$ we get a 0-differentiable deformation (with parameter $\beta$) of the usual product. We were thus lead $[BFLS]$ to look, starting from a product $\star_\lambda$, for a new composition law $(u, v) \mapsto u^{\star_\lambda, \beta} v = u \star_\lambda f_{\lambda, \beta} \star_\lambda v$ with $f_{\lambda, \beta} = \sum_{\tau=0}^\infty \lambda^{2\tau} f_{2\tau, \beta} \in N[[\lambda^2], [\beta]]$, where $f_{0, \beta} \equiv f_\beta \neq 0$ and $f_0 = 1$. The transformation $u \mapsto T_{\lambda, \beta} u = f_{\lambda, \beta} \star_\lambda u$ intertwines $\star_\lambda$ and $\star_{\lambda, \beta}$ but it is not an equivalence of star-products because $\star_{\lambda, \beta}$ is not a star-product: it is a $(\lambda, \beta)$-deformation of the usual product with at first order in $\lambda$ the driver given by $P_{\beta}(u, v) = f_{\beta} P(u, v) + u P(f_\beta, v) - P(f_\beta, u)v$, a conformal Poisson bracket associated with a conformal symplectic structure given by the 2-tensor $\Lambda_{\beta} = f_{\beta} \Lambda$ and the vector $E_{\beta} = [\Lambda, f_{\beta}]_{SN}$.

We then start with a star-product, denoted $\star$, on some algebra $\mathcal{A}$ of observables, and take $f_{\lambda, \beta} = \exp_\star (c \beta H)$ with $c = -\frac{1}{2}$. The star exponential $\exp(Ht)$ defines an automorphism $u \mapsto \alpha_t(u) = \exp(-Ht) \star u \star \exp(Ht)$. A KMS state $\sigma$ on $\mathcal{A}$ is a state (linear functional) satisfying, $\forall a, b \in \mathcal{A}$, the Kubo–Martin–Schwinger condition $\sigma(\alpha_t(a) \star b) = \sigma(b \star \alpha_{t+i\beta}(a))$. Then the
can recover known features of statistical mechanics by introducing a new deformation parameter $\beta = (kT)^{-1}$ and the related conformal symplectic structure. This procedure commutes with usual deformation quantization. The question was recently treated from a more conventional point of view in deformation quantization (see e.g. [BRW, Wm]) using the notion of formal trace [NT].

2.2.4 Star representations, a short overview. Let $G$ be a Lie group (connected and simply connected), acting by symplectomorphisms on a symplectic manifold $X$ (e.g. coadjoint orbits in the dual of the Lie algebra $\mathfrak{g}$ of $G$). The elements $x, y \in \mathfrak{g}$ will be supposed to be realized by functions $u_x, u_y$ in $\mathcal{N}$ so that their Lie bracket $[x, y]_\mathfrak{g}$ is realized by $P(u_x, u_y)$. Now take a $G$-covariant star-product $\ast$, that is $P(u_x, u_y) = [u_x, u_y] \equiv (u \ast v - v \ast u) / 2 \lambda$, which shows that the map $\mathfrak{g} \ni x \mapsto (2\lambda)^{-1} u_x \in \mathcal{N}$ is a Lie algebra morphism. The appearance of $\lambda^{-1}$ here and in the trace (see 2.2.1) cannot be avoided and explains why we have often to take into account both $\lambda$ and $\lambda^{-1}$. We can now define the star exponential

$$E(e^x) = \operatorname{Exp}(x) \equiv \sum_{n=0}^\infty (n!)^{-1}(u_x / 2\lambda)^n$$

(2.15)

where $x \in \mathfrak{g}$, $e^x \in G$ and the power $\ast n$ denotes the $n$th star-power of the corresponding function.

By the Campbell–Hausdorff formula one can extend $E$ to a group homomorphism $E : G \to (\mathcal{P}[[\lambda^{-1}]], \ast)$ where, in the formal series, $\lambda$ and $\lambda^{-1}$ are treated as independent parameters for the time being. Alternatively, the values of $E$ can be taken in the algebra $(\mathcal{P}[[\lambda^{-1}]], \ast)$, where $\mathcal{P}$ is the algebra generated by $\mathfrak{g}$ with the $\ast$-product (it is a representation of the enveloping algebra). A star representation [BFFLS] of $G$ is a distribution $\mathcal{E}$ (valued in $\text{Im}E$) on $X$ defined by

$$D \ni f \mapsto \mathcal{E}(f) = \int_G f(g) E(g^{-1}) dg$$

where $D$ is some space of test-functions on $G$. The corresponding character $\chi$ is the (scalar-valued) distribution defined by $D \ni f \mapsto \chi(f) = \int_X \mathcal{E}(f) d\mu$, $d\mu$ being a quasi-invariant measure on $X$.

The character is one of the tools which permit a comparison with usual representation theory. For semi-simple groups it is singular at the origin in irreducible representations, which may require caution in computing the star exponential (2.13). In the case of the harmonic oscillator that difficulty was masked by the fact that the corresponding representation of $\mathfrak{sl}(2)$ generated by $(p^2, q^2, pq)$ is integrable to a double covering of $SL(2, \mathbb{R})$ and decomposes into a sum $D(\frac{1}{2}) \oplus D(\frac{3}{2})$: the singularities at the origin cancel each other for the two components.

This theory is now very developed, and parallels in many ways the usual (operatorial) representation theory. A detailed account of all the results would take us too far, but among the most notable one may quote:

i) An exhaustive treatment of nilpotent or solvable exponential [AC] and even general solvable Lie groups [ACL]. The coadjoint orbits are there symplectomorphic to $\mathbb{R}^{2l}$ and one can
lift the Moyal product to the orbits in a way that is adapted to the Plancherel formula. Polarizations are not required, and “star-polarizations” can always be introduced to compare with usual theory. Wavelets, important in signal analysis, are manifestations of star-products on the (2-dimensional solvable) affine group of \( \mathbb{R} \) or on a similar 3-dimensional solvable group [BB].

ii) For semi-simple Lie groups an array of results is already available, including [ACG, Mr] a complete treatment of the holomorphic discrete series (this includes the case of compact Lie groups) using a kind of Berezin dequantization, and scattered results for specific examples. Similar techniques have also been used [CGR, Ka] to find invariant star-products on Kähler and Hermitian symmetric spaces (convergent for an appropriate dense subalgebra). Note however, as shown by recent developments of unitary representations theory (see e.g. [SW]), that for semi-simple groups the coadjoint orbits alone are no more sufficient for the unitary dual and one needs far more elaborate constructions.

iii) For semi-direct products, and in particular for the Poincaré and Euclidean groups, an autonomous theory has also been developed (see e.g. [ACM]).

Comparison with the usual results of “operatorial” theory of Lie group representations can be performed in several ways, in particular by constructing an invariant Weyl transform generalizing (2.8), finding “star-polarizations” that always exist, in contradistinction with the geometric quantization approach (where at best one can find complex polarizations), study of spectra (of elements in the center of the enveloping algebra and of compact generators) in the sense of (2.2.3.1), comparison of characters, etc. Note also in this context that the pseudodifferential analysis and (non autonomous) connection with quantization developed extensively by Unterberger, first in the case of \( \mathbb{R}^2 \), has been extended to the above invariant context [U2, UU]. But our main insistence is that the theory of star representations is an autonomous one that can be formulated completely within this framework, based on coadjoint orbits (and some additional ingredients when required).

3 Existence and classification: geometric approach

The first proof of the existence of star-products on any (real) symplectic manifold was given by De Wilde and Lecomte [DL] and is based on the idea of gluing local Moyal star-products defined on Darboux charts. Their proof is algebraic and cohomological in essence. A more geometrical proof was provided by the construction of Omori, Maeda, and Yoshioka [OMY1]. They have introduced the notion of Weyl bundle as a locally trivial fiber bundle whose fibers are Weyl algebras. A Weyl algebra is a unital \( \mathbb{C} \)-algebra generated by \( 1, \lambda, X_1, \ldots, X_n, Y_1, \ldots, Y_n \) satisfying the commutation relations \([X_i, Y_j] = -\delta_{ij}\lambda\) and all other commutators are trivial. The main step was to show the existence of a Weyl bundle \( W \) over any symplectic manifold \( M \). This was done by defining a special class of sections (the Weyl functions), first locally and then globally by showing that the class of Weyl functions is stable under different choices of local trivialization. Then, the composition of Weyl sections induces a star-product on \( M \). The construction of a Weyl bundle
shares some features with the De Wilde–Lecomte approach: in both cases, derivations play an important role and the basic idea is still to glue together local objects defined on Darboux charts by using cohomological arguments.

In 1985, motivated by index theory and independently of the previous proofs, Fedosov [Fe1] announced a purely geometrical construction, also based on Weyl algebras, of star-products on a symplectic manifold. Unfortunately, the announcement appeared in a local volume with a summary in Doklady and has stayed almost unknown for several years until a complete version was published in an international journal [Fe]. The beautiful construction of Fedosov does provide a new proof of existence, but above all it gives a much better understanding of deformation quantization that allowed many major developments that will be discussed below. Let us first briefly review the Fedosov construction (see also e.g. [WB]). For a comprehensive treatment and much more, we recommend Fedosov’s book [Fe3].

3.1 The construction of Fedosov

Let \((M, \omega)\) be a \(2m\)-dimensional symplectic manifold. Let \(\nabla\) be a symplectic connection, i.e., a linear connection without torsion and preserving the symplectic form \(\omega\). At each point \(x \in M\), the symplectic form endows the tangent space \(T_xM\) with a (constant) symplectic structure and we can define the Moyal star-product \(\star_\lambda\) associated to \(\omega\) on \(T_xM\). The space of formal series in \(\lambda\) with polynomial coefficients on \(T_xM\), endowed with \(\star_\lambda\), gives us the Weyl algebra \(W_x\) associated to \(T_xM\). The algebras \(W_x, x \in M\), can be smoothly patched and we get a fiber bundle \(W = \bigcup_{x \in M} W_x\) on \(M\), called the Weyl bundle over \(M\). The fiberwise Moyal star-product endows naturally the space of sections \(\Gamma(W)\) with a structure of unital associative algebra. The center of \(\Gamma(W)\) can be identified as a vector space with \(C^\infty(M)[[\lambda]]\). The basic idea of Fedosov’s construction is to build from \(\nabla\) a flat connection \(D\) on \(W\) such that the algebra of horizontal sections for \(D\) induces a star-product on \(M\). This “infinitesimal” approach bypasses the need of gluing together star-products defined on (large) Darboux charts.

In a local Darboux chart at \(x \in M\), denote by \((y^1, \ldots, y^{2m})\) the coordinates on \(T_xM\). A section of \(W\) can be locally written as

\[
a(x, y) = \sum_{k \geq 0, |\alpha| \geq 0} \lambda^k a_{k, \alpha}(x) y^\alpha,
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_{2m})\) is a multi-index. We shall need the bundle

\[
\Gamma(W) \otimes \Lambda = \bigoplus_{0 \leq q \leq 2m} \Gamma(W) \otimes \Lambda^q
\]

of differential forms on \(M\) taking their values in \(W\). A section of \(\Gamma(W) \otimes \Lambda\) can be expressed locally as

\[
a(x, y, dx) = \sum_{k \geq 0, |\alpha|, |\beta| \geq 0} \lambda^k a_{k, \alpha, \beta}(x) y^\alpha(dx)^\beta,
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_{2m})\) is a multi-index.
where \((dx)^\beta \equiv dx^{\beta_1} \land \cdots \land dx^{\beta_k}\). The natural extension of the product in \(\Gamma(W)\) obtained by taking the Moyal star-product \(\ast_\lambda\) between the \(y^i\)'s and the wedge product between the \(dx^j\)'s, defines a product \(\circ\) which makes \(\Gamma(W) \otimes \Lambda\) an algebra. There is also a structure of graded Lie algebra on \(\Gamma(W) \otimes \Lambda\) given by the bracket \([a, b] = a \circ b - (-1)^{|a||b|} b \circ a\) for \(a \in \Gamma(W) \otimes \Lambda^p\) and \(b \in \Gamma(W) \otimes \Lambda^q\). It just the natural extension of the bracket in \(\Gamma(W)\) for the Moyal star-product \(\ast_\lambda\) to the differential forms taking values in \(W\).

The algebra \(\Gamma(W) \otimes \Lambda\) is filtered in the following way. Define the degrees of \(\lambda\) and \(y^j\) by \(\deg(\lambda) = 2\) and \(\deg(y^j) = 1\). Then \(\Gamma(W) \otimes \Lambda\) has a filtration with respect to the total degree \(2k + |\alpha|\) of the terms appearing in \((3.2)\): \(\Gamma(W) \otimes \Lambda \supset \Gamma_1(W) \otimes \Lambda \supset \Gamma_2(W) \otimes \Lambda \supset \cdots\). Notice that the center of \(\Gamma(W) \otimes \Lambda\) for the bracket \([\ , \ ]\) is the set of sections independent of the \(y^i\)'s. We shall denote by \(\sigma\) the projection of \(\Gamma(W) \otimes \Lambda\) onto its center.

The exterior differential \(d\) on \(M\) extends to an antiderivation \(\delta\) of the algebra \(\Gamma(W) \otimes \Lambda\) by the formula \(\delta(a) = \sum_j dx^j \land \partial_j a\), where \(\partial_j = \partial / \partial y^j\). The antiderivation \(\delta\) reduces by one unit the degree of the filtration. Fedosov has introduced another operation \(\delta^*\) on \(\Gamma(W) \otimes \Lambda\). It is defined by the formula \(\delta^*(a) = \sum_j y^j (\partial_j)(a)\), where \(i\) denotes the interior product. One can check that \(\delta^2 = \delta^* = 0\) and \(\delta \delta^* + \delta^* \delta = (p + q)ld\) on homogeneous elements of degrees \(p\) in \(y^i\) and \(q\) in \(dx^j\). There is a Hodge-like decomposition for the sections \((3.2)\). Denote by \(a_{00}\) the term of the section \(a\) in \((3.2)\) which is independent of the \(y^i\)'s and the \(dx^j\)'s. By defining \(\delta^{-1}\) on \((p, q)\)-homogeneous elements by \(\delta^{-1} = 0\) if \(p + q = 0\) and \(\delta^{-1} = \delta^*/(p + q)\) if \(p + q > 0\), every section \(a\) admits the decomposition \(a = \delta \delta^{-1}(a) + \delta^{-1} \delta(a) + a_{00}\).

The next step is to introduce another connection on \(W\). First, the symplectic connection \(\nabla\) defines on \(W\) a connection \(\hat{\nabla}: \Gamma(W) \otimes \Lambda^q \to \Gamma(W) \otimes \Lambda^{q+1}\), by \(\hat{\nabla}_i = \sum dx^j \land \nabla_i \). Fedosov introduces the new connection on \(W\) as follows: for an element \(\gamma \in \Gamma(W) \otimes \Lambda^1\) such that \(\sigma(\gamma) = 0\), define the connection \(D = \hat{\nabla} - [\gamma, \cdot] / i \lambda\). The curvature of \(D\) is \(\Omega = R + \hat{\nabla} \gamma - \gamma^2 / i \lambda\), where \(R\) is the curvature of \(\nabla\). \(\Omega\) is called the Weyl curvature of \(D\). The usual relation \(D^2 = -[\Omega, \cdot] / i \lambda\) holds,

Here comes the essential ingredient of Fedosov's construction. The idea is to make quantum corrections to \(D\) in order to make it flat, i.e., \(D^2 = 0\). In other words, it consists to make the Weyl curvature \(\Omega\) central (Abelian connection). Fedosov showed the existence of a flat Abelian connection on \(W\), normalized by the condition \(\sigma(r) = 0\), of the form \(D = -\delta + \hat{\nabla} - [r, \cdot] / i \lambda\), with \(r \in \Gamma_3(W) \otimes \Lambda^1\). The strategy used is to solve iteratively the equation \(\delta(r) = R + \hat{\nabla}r - r^2 / i \lambda\) with respect to the filtration of \(\Gamma(W) \otimes \Lambda\). It can be shown that the preceding equation has a unique solution if the initial condition \(\delta^{-1}(r) = 0\) is satisfied. It turns out that for such specific solution \(r\) the Weyl curvature is nothing else than \(\Omega = -\omega\). Such a flat Abelian connection is called a Fedosov connection on \(W\).

Given a Fedosov connection \(D\) on \(W\), let us see how it induces a star-product on \(M\). The set of horizontal sections, i.e., sections \(a\) such that \(Da = 0\), is an algebra for the product \(\circ\). We shall denote this algebra by \(\Gamma_\sigma(W)\). The restriction of the projection \(\sigma: \Gamma(W) \to C^\infty(M)[[\lambda]]\) to \(\Gamma_\sigma(W)\) is a bijective map. To any \(f \in C^\infty(M)[[\lambda]]\) we can then associate a horizontal section denoted by \(\sigma^{-1}(f)\). Then \(f \ast g = \sigma(\sigma^{-1}(f) \circ \sigma^{-1}(g))\) defines a star-product on \(M\).
The method of Fedosov extends to regular Poisson manifolds. An earlier proof along the lines of De Wilde and Lecomte and Omori, Maeda, and Yoshioka was published in 1992 by Massoudi [Mas] where the existence of tangential star-products is established.

3.2 Classification, characteristic classes, and closed star-products

3.2.1 Classification. There are several occurrences in the literature of the second de Rham cohomology space $H^2_{dR}(M)$ in relation with the question of equivalence of star-products and deformed Poisson brackets.

For formal 1-differentiable deformations of the Poisson bracket (i.e., formal Poisson bracket) on a symplectic manifold, the first of these occurrences goes back to [BFFLS], where it is shown that the obstructions to equivalence (at each step in the power of the deformation parameter) are in $H^2_{dR}(M)$. By taking advantage of graded Lie structures related to deformations, Lecomte [Le] showed that equivalence classes of formal Poisson brackets are in one-to-one correspondence with $H^2_{dR}(M)[[\lambda]]$.

There are similar occurrences for star-products. For example, it is implicit in the work of Gutt [Gu1] and in [Gu2, Gth] where the second and third Chevalley–Eilenberg cohomology spaces have been computed. Also, as a consequence of their proof of existence when $b_3(M) = 0$, Nersesov and Vlasov [NV] found that the classification for this class of star-products was given by sequences in $H^2_{dR}(M)$.

The Fedosov construction provides a more geometric point of view on the classification problem. Fedosov’s method allows to canonically construct a star-product on $(M, \omega)$ whenever a symplectic connection $\nabla$ is chosen. Any star-product obtained in such a way is called a Fedosov star-product. The iterative construction reviewed in Section 3.1 can be generalized to the case where one starts with a formal symplectic form on $M$, $\omega_{\lambda} = \omega + \sum_{k \geq 1} \lambda^k \omega_k$, the $\omega_k$’s being closed 2-forms on $M$. Hence we have a canonical method to construct a star-product $\star$ from the object $(M, \omega_{\lambda}, \nabla)$. Then the Weyl curvature is equal to $-\omega_{\lambda}$ and Fedosov has established that two Fedosov star-products $\star$ and $\star'$ are equivalent if and only if their Weyl curvatures belong to the same cohomology class in $H^2_{dR}(M)[[\lambda]]$.

Shortly after Fedosov’s paper, Nest and Tsygan [NT] showed that any differentiable star-product on a symplectic manifold is equivalent to a Fedosov star-product. This fact allowed them to define the characteristic class of a star-product $\star$ to be the class of the Weyl curvature in $H^2_{dR}(M)[[\lambda]]$ of any Fedosov star-product equivalent to $\star$. As a consequence, one gets a parametrization of the equivalence classes of star-products on $(M, \omega)$ by elements in $H^2_{dR}(M)[[\lambda]]$.

The parametrization of equivalence classes of star-products has also been made explicit by Bertelson, Cahen and Gutt [BCC] (using older results in Gutt’s thesis [Gth]) and by Xu [Xu]; the relation between classes of star-products and classes of 1-differentiable deformations of a Poisson bracket was described in [Bon]. Deligne [D] obtained the same classification result, without any reference to a particular method of construction of deformations, by using cohomological Čech
techniques and gerbes. Deligne’s approach has been studied further by Gutt and Rawnsley [Gu, GR].

3.2.2 Trace and closed star products. The prominent role played by traces of operators in index theory has its counterpart in deformation quantization. One can naturally define a trace on the deformed algebra of smooth functions having compact support $(\mathcal{C}_c^\infty(M)[[\lambda]], \star)$. It is a $\mathbb{C}[[\lambda]]$-linear map $\text{Tr}: (\mathcal{C}_c^\infty(M)[[\lambda]] \to \mathbb{C}[\lambda^{-1}, \lambda])$ such that $\text{Tr}(f \star g) = \text{Tr}(g \star f)$. The negative powers of $\lambda$ appear already in the classical examples and $\text{Tr}$ is allowed to take its values not only in formal series in $\lambda$ but in Laurent series in $\lambda^{-1}$ as well. Such a trace always exists and is essentially unique [NT, Fe3]. Around 1995, new algebraic versions of the (generalized) Atiyah-Singer theorem appeared: one is due to Nest and Tsygan [NT] where Gelfand-Fuks techniques play an important role, the other, by Fedosov, uses Thom isomorphism. Recently, Fedosov [Fe4] published a new proof based on the topological invariants of symplectic connections of Tamarkin [Ta].

3.2.2.1 Closed star-products. The trace allows to distinguish a particular class of star-products on a symplectic manifold $M$. These are the so-called closed star-products introduced in [CFS]. A star-product $\star$ is closed if the map

$$T: f \mapsto \frac{1}{(2\pi \lambda)^m} \int f_m \omega^m \frac{\text{d}}{m!}$$

is a trace ($2m$ is the dimension of $M$ and $f_m$ is the coefficient of $\lambda^m$ in $f \in \mathcal{C}_c^\infty(M)[[\lambda]]$). Closed star-products exist on any symplectic manifold [OMY2] and any star-product is equivalent to a closed star-product. Cyclic cohomology replaces the Hochschild one for closed star-products: obstructions to existence and equivalence live in the third and second cyclic cohomology spaces, correspondingly. Another important feature of closed star-products is the existence of an invariant $\phi$ defined in [CFS]. It is a cocycle for the differential of the cyclic bicomplex of $M$, i.e., Hochschild in one direction and cyclic in the other. Let $\theta(f, g) = f \star g - fg$; for an integer $k$ define the map

$$\phi_{2k}: \mathcal{C}_c^\infty(M)[[\lambda]]^{\otimes 2k+1} \to \mathbb{R}, \ (f_0, \ldots, f_{2k}) \mapsto \tau(f_0 \star \theta(f_1, f_2) \star \cdots \star \theta(f_{2k-1}, f_{2k}))$$

where $\tau = (2\pi \lambda)^m T$ does not depend on $\lambda$. It turns out that $\phi_{2k} = 0$ for $k > m$ and the cocycle is defined by $\phi = \sum_k \phi_{2k}/k!$. The computation of $\phi$, called the CFS-invariant, has been performed for the case where $M$ is a cotangent bundle of a compact Riemannian manifold [CFS]. Notice that in that case, associative deformations correspond to the standard calculus of pseudo-differential operators, and the CFS-invariant is nothing else than the Todd class of $M$. This result has been generalized by Halbout [Ha] to the case of any symplectic manifold.

4 Metamorphoses

4.1 Poisson manifolds and Kontsevich’s formality

The problem of deformation quantization of general Poisson differentiable manifolds has been left open for a long time. For the nonregular Poisson case, first examples of star-products appeared
in [BFILS] in relation with the quantization of angular momentum. They were defined on the dual of \( \mathfrak{so}(n) \) endowed with its natural Kirillov–Poisson structure. The case for any Lie algebra follows from the construction given by Gutt [Gu3] of a star-product on the cotangent bundle of a Lie group \( G \). This star-product restricts to a star-product on the dual of the Lie algebra of \( G \). It translates the associative structure of the enveloping algebra in terms of functions on the dual of the Lie algebra of \( G \). Omori, Maeda and Yoshika [DMY] have constructed quantization for a class of quadratic Poisson structures.

There is no equivalent of Darboux theorem for Poisson structures and this fact is at the origin of the main difficulty to construct (even local) star-products in the Poisson case. The situation changed drastically in 1997, when Kontsevich [Ko2] proved his Formality Conjecture [Ko1]. The existence of star-products on any smooth Poisson manifold appears as a consequence of the Formality Theorem.

We shall review the construction of a star-product on \( \mathbb{R}^d \) given in [Ko2] and then present the essential steps of the Formality Theorem.

### 4.1.1 Kontsevich star-product

Consider \( \mathbb{R}^d \) endowed with a Poisson bracket \( \alpha \). We denote by \( (x^1, \ldots, x^d) \) the coordinate system on \( \mathbb{R}^d \); the Poisson bracket of two smooth functions \( f, g \) is given by \( \alpha(f, g) = \sum_{1 \leq i,j \leq n} \alpha^{ij} \partial_i f \partial_j g \), where \( \partial_k \) denotes the partial derivative with respect to \( x^k \). Instead of the whole of \( \mathbb{R}^d \), one may consider an open subset of it.

The formula for the Kontsevich star-product is conveniently defined by considering, for each \( n \geq 0 \), a family of oriented graphs \( G_n \). To a graph \( \Gamma \in G_n \) is associated a bidifferential operator \( B^\Gamma \) and a weight \( w(\Gamma) \in \mathbb{R} \). The sum \( \sum_{\Gamma \in G_n} w(\Gamma) B^\Gamma \) gives the coefficient of \( \lambda^n \), i.e., the cochain \( C_n \) of the star-product.

For later use, we shall reproduce the general definitions of [Ko2]. They are more general than the ones needed in this section for the star-product.

- **An oriented graph** \( \Gamma \) belongs to \( G_{n,m} \), with \( n, m \geq 0 \), \( 2n - m + 2 \geq 0 \), if:

  i) The set of vertices \( V_\Gamma \) of \( \Gamma \) has \( n + m \) elements labeled \( \{1, \ldots, n; \bar{1}, \ldots, \bar{m}\} \). The vertices \( \{1, \ldots, n\} \) are said of the first type and \( \{\bar{1}, \ldots, \bar{m}\} \) are said of the second type.

  ii) \( E_\Gamma \) is the set of oriented edges of \( \Gamma \). \( E_\Gamma \) has \( 2n - m + 2 \) edges. There is no edge starting at a vertex of the second type. The set of edges starting at a vertex of the first type \( k \) is denoted by \( \text{Star}(k) \) and its cardinality by \( |\text{Star}(k)| \), hence \( \sum_{1 \leq k \leq n} |\text{Star}(k)| = 2n - m - 2 \). The edges of \( \Gamma \) starting at vertex \( k \) will be denoted by \( \{e^1_k, \ldots, e^\#(k)_k\} \). When it is needed to make explicit the starting and ending vertices of some edge \( v \), we shall write \( v = (s(v), e(v)) \), where \( s(v) \in \{1, \ldots, n\} \) is its starting vertex and \( e(v) \in \{1, \ldots, n; \bar{1}, \ldots, \bar{m}\} \) is its ending vertex.

  iii) \( \Gamma \) has no loop (edge starting at some vertex and ending at that vertex) and no parallel multiple edges (edges sharing the same starting and ending vertices).

Let us specialize the preceding definition to the case of star-products. Here \( m = 2 \) and only the family of graphs \( G_{n,2} \), \( n \geq 0 \), will be of interest. Moreover there are exactly 2 edges starting at each vertex of the first type, i.e., \( \# k = 2 \), \( \forall k \in \{1, \ldots, n\} \). Hence \( E_\Gamma \) has \( 2n \) edges for \( \Gamma \in G_{n,2} \). For
The configuration space of Section 4.1.2. Let \( \phi(z_1, z_2) \) be the function (angle function):

\[
\phi(z_1, z_2) = \frac{1}{2\sqrt{-1}} \log \left( \frac{(z_2 - z_1)(\bar{z}_2 - \bar{z}_1)}{(z_2 - \bar{z}_1)(\bar{z}_2 - z_1)} \right),
\]

\( \phi(z_1, z_2) \) is extended by continuity for \( z_1, z_2 \in \mathbb{R}, z_1 \neq z_2 \). The function \( \phi \) measures the angle at \( z_1 \) between the geodesic passing by \( z_1 \) and the point at infinity, and the geodesic passing by \( z_1 \) and \( z_2 \) for the hyperbolic metric.
Theorem 4.1 (Kontsevich [Ko2]). The map \( \star : C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d)[[\lambda]] \) defined by
\[
(f, g) \mapsto f \star g = \sum_{n \geq 0} \sum_{\Gamma \in G_{n,2}} w(\Gamma) B_1(f, g)
\]
defines a differential star-product on \((\mathbb{R}^d, \alpha)\). The classes of equivalence of star-products are in a one-to-one correspondence with the classes of equivalence of formal Poisson brackets.

The first cochains of the Kontsevich star-product can be computed by elementary means. Of course we have \( C_0(f, g) = fg \). At first order, \( G_{1,2} \) has 2 graphs: one has weight \( \frac{1}{2} \) and as bidifferential operator the Poisson bracket \( \{f, g\}_\alpha = \alpha(df \wedge dg) \), the other graph has weight \( -\frac{1}{2} \) and bidifferential operator \( \{g, f\}_\alpha \). The sum gives \( C_1(f, g) = \{f, g\}_\alpha \). All the graphs contributing to \( C_2 \), up to symmetries, are shown in Fig. 2, and the second order term is given by:
\[
C_2(f, g) = \frac{1}{2} \alpha^{i_1j_1} \alpha^{j_1j_2} \partial_{i_1} f \partial_{j_1j_2} g \\
+ \frac{1}{3} \alpha^{i_1j_1} \partial_{i_1} \alpha^{j_1j_2} (\partial_{j_1} f \partial_{i_2} g + \partial_{i_2} f \partial_{j_1} g) \\
- \frac{1}{6} \partial_{i_2} \alpha^{i_1j_1} \partial_{i_1} \alpha^{j_1j_2} \partial_{i_2} f \partial_{j_1j_2} g
\]
algebras are “equivalent” in a sense we now make precise. Roughly speaking, the Formality Theorem states that these two differential graded Lie structures constructed on the associative algebra \( \mathcal{A} \) resulted in what is now called Formality Theorem. Given a smooth manifold \( (M, \omega) \), the Kontsevich star-product induces a convolution on the space of distributions supported at 0 in the Lie algebra \( \mathfrak{g} \). This has allowed [MRS] to prove the conjecture on invariant distributions on a Lie group formulated by Kashiwara and Vergne in 1978.

The linear Poisson case allows to quantize the class of quadratic Poisson brackets that are in the image of a map introduced by Drinfeld which associates a quadratic Poisson to a linear Poisson bracket; not all quadratic brackets arise in this way [MRS].

### 4.1.2 Formality Theorem

Theorem 4.1 is a particular consequence of a much more general result called Formality Theorem. Given a smooth manifold \( M \) one may consider two algebraic structures constructed on the associative algebra \( A \equiv C^\infty(M) \), namely, the Hochschild complex \( C^*(A, A) \) and its cohomology \( H^*(A, A) \). They both have a structure of differential graded Lie algebra. Roughly speaking, the Formality Theorem states that these two differential graded Lie algebras are “equivalent” in a sense we now make precise.

#### 4.1.2.1. Differential graded Lie algebras

Let \( \mathbb{K} \) be a field of characteristic zero. All vector spaces are over \( \mathbb{K} \). Let \( V \) be a \( \mathbb{Z} \)-graded vector space: \( V = \bigoplus_{k \in \mathbb{Z}} V^k \). An element \( x \in V \) is called homogeneous of degree \( k \) if \( x \in V^k \). The degree of a homogeneous element \( x \) is denoted \( |x| \).

A differential graded Lie algebra over \( \mathbb{K} \) (DGLA for short) is a graded vector space \( \mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k \) endowed with Lie superbracket \( [\cdot, \cdot] \) and a differential \( d \) of degree 1. It means that \( (\mathfrak{g}, [\cdot, \cdot], d) \) fulfills \( (a, b, c) \) are homogeneous elements:

1. \( [\cdot, \cdot] \) is a bilinear map, \( [\cdot, \cdot] : \mathfrak{g}^i \times \mathfrak{g}^j \to \mathfrak{g}^{i+j} \), satisfying:
   \[
   [a, b] = -(-1)^{|a||b|} [b, a],
   \]
   \[
   (-1)^{|a||c|} [a, [b, c]] + (-1)^{|b||a|} [b, [c, a]] + (-1)^{|c||b|} [c, [a, b]] = 0.
   \]

2. \( d : \mathfrak{g}^i \to \mathfrak{g}^{i+1} \) is a linear map satisfying \( d^2 = 0 \) and:
   \[
   d[a, b] = [d a, b] + (-1)^{|a|} [a, d b].
   \]

The Hochschild complex \( C^*(A, A) = \bigoplus_{k \geq 0} C^k(A, A) \) consists of polydifferential operators on \( M \), \( C^k(A, A) = \{ D : A^\otimes k \to A \} \). In the following, we will detail the ingredients that make the Hochschild complex a DGLA. We shall look at the Hochschild complex as a graded vector space \( D^*_\text{poly}(M) = \bigoplus_{k \in \mathbb{Z}} D^k_{\text{poly}}(M) \), where \( D^k_{\text{poly}}(M) \) is equal to \( C^{k+1}(A, A) \) for \( k \geq -1 \) and equal to \( \{0\} \). Where summation over repeated indices is understood. For higher cochains, there is no systematic way to compute weights of graphs, especially in the presence of cycles (e.g. graph \( \Gamma_4 \) in Fig. 2), but we should mention that Polyak [PR] has provided an interpretation of weights in term of degree of maps which allowed him to "easily" compute a large class of graphs and all of them for the linear Poisson structure case.

The Kontsevich star-product has been extensively studied for linear Poisson structures, i.e., on the dual of a Lie algebra. An important result in that context is due to Shoikhet [Sho] who showed the vanishing of the weight of all even wheel graphs. As a consequence, on the dual of a Lie algebra, the Kontsevich star-product coincides with the one given by the Duflo isomorphism. Moreover, the Kontsevich star-product induces a convolution on the space of distributions supported at 0 in the Lie algebra \( \mathfrak{g} \). This has allowed [ADS] to prove the conjecture on invariant distributions on a Lie group formulated by Kashiwara and Vergne in 1978.
otherwise. Notice the shift by one unit in the grading of $D^\bullet_{\text{poly}}(M)$ with respect to that of $C^\bullet(A,A)$: $|D| = k$ if $D \in C^{k+1}(A,A)$.

The Gerstenhaber bracket $[\cdot,\cdot]_G$ on $D^\bullet_{\text{poly}}(M)$ is defined for $D_i \in D^k_{\text{poly}}(M)$ by:

$$[D_1,D_2]_G = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1,$$

where $\circ: D^k_{\text{poly}}(M) \times D^{k_2}_{\text{poly}}(M) \to D^{k_1+k_2}_{\text{poly}}(M)$ is a nonassociative composition law for polydifferential operators:

$$(D_1 \circ D_2)(f_0,\ldots,f_{k_1+k_2}) = \sum_{0 \leq j \leq k_1} (-1)^{j k_2} D_1(f_0,\ldots,f_{j-1},D_2(f_j,\ldots,f_{j+k_2}),f_{j+k_2+1},\ldots f_{k_1+k_2}).$$

Finally, we have the usual Hochschild differential $b$ defined by (2.1). It can be expressed in terms of the Gerstenhaber bracket. If $m: A \otimes A \to A$ denotes the product of $A$ (i.e., usual product of functions on $M$), then one can verify that $b = -[\cdot,m]_G$. The usual Hochschild differential is not a differential of the Gerstenhaber bracket, but $\delta = [m,\cdot]_G$ is a differential of degree 1 (for $D \in D^k_{\text{poly}}(M)$, we have $bD = (-1)^k \delta D$). It is an important fact that $(D^\bullet_{\text{poly}}(M),[\cdot,\cdot]_G,\delta)$ is a DG$A$.

The cohomology $H^\bullet(A,A)$ of the complex $C^\bullet(A,A)$ (with $b$ or $\delta$ as differential) is the space of polyvectors on $M$: $\Gamma(\wedge^\bullet TM)$. We denote by $T^\bullet_{\text{poly}}(M)$ the graded vector space $T^\bullet_{\text{poly}}(M) = \bigoplus_{k \geq 0} T^k_{\text{poly}}(M)$, where $T^k_{\text{poly}}(M) = \Gamma(\wedge^{k+1} TM)$ for $k \geq -1$ and is equal to $\{0\}$ otherwise. On $T^\bullet_{\text{poly}}(M)$ one has the Schouten–Nijenhuis bracket $[\cdot,\cdot]_{\text{SN}}$ which is the natural extension of the bracket of vector fields. On decomposable tensors it is defined by:

$$[\xi_0 \wedge \cdots \wedge \xi_l, \eta_0 \wedge \cdots \wedge \eta_l]_{\text{SN}} = \sum_{0 \leq i \leq l} \sum_{0 \leq j \leq l} (-1)^i j [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_l \wedge \eta_0 \wedge \cdots \wedge \hat{\eta}_j \wedge \cdots \wedge \eta_l. \quad (4.5)$$

The Schouten–Nijenhuis bracket makes $(T^\bullet_{\text{poly}}(M),[\cdot,\cdot]_{\text{SN}})$ into a graded Lie algebra. On it, take as the differential 0, then $(T^\bullet_{\text{poly}}(M),[\cdot,\cdot]_{\text{SN}},0)$ is trivially a DG$A$.

4.1.2.2. $L_\infty$-morphism. We recall some definitions needed to state the Formality Theorem. A graded coalgebra $c$ is a graded vector space with a coproduct $\Delta: c_i \to \bigoplus_{m+n=i} c_m \otimes c_n$. To any graded vector space $g$ is associated a graded coalgebra $\mathcal{C}(g) = \bigotimes_{k \geq 1}(\wedge^k g)[k]$, where $[k]$ indicates the shift to the left by $k$ units in the grading.

An $L_\infty$-algebra $(g,Q)$ is a graded vector space $g$ with a codifferential $Q$ satisfying $Q^2 = 0$ on $\mathcal{C}(g)$. $L_\infty$-algebras are also known as strong(h) homotopy Lie algebras (see e.g. [Sta]). The differential $Q$ defines a sequence $\{Q_k\}$ of morphisms $Q_k: \wedge^k g \to g[2-k]$, $k \geq 1$ and the relation $Q^2 = 0$ puts quadratic constraints on the $Q_k$’s. These constraints imply that $Q_1: g \to g[1]$ is a differential of degree 1, hence $(g,Q_1)$ is a complex, and $Q_2: \wedge^2 g \to g$ defines a superbacket on $g$ for which $Q_1$ is compatible. Hence any $L_\infty$-algebra $(g,Q)$ such that $Q_k = 0$ for $k \geq 3$ is a DG$A$.

An $L_\infty$-morphism $\mathcal{F}$ between two $L_\infty$-algebras $(g_1,Q^{(1)})$ and $(g_2,Q^{(2)})$ is a morphism between the associated coalgebras such that $\mathcal{F} Q^{(1)} = Q^{(2)} \mathcal{F}$. We have similar constraints on the sequence
of morphisms \( \{ \mathcal{F}_k \} : \mathcal{F}_k : \wedge^k (g_1) \to g_2[1-k], \ k \geq 1. \) At first order, they imply that the formalism \( \mathcal{F} \) is a morphism of complexes \( (g_1, Q_1^{(1)}) \to (g_2, Q_1^{(2)}) \), but the constraint at order 2 shows that \( \mathcal{F}_2 \) is not a homomorphism of DGLA. Finally, an \( L_\infty \)-morphism is a quasi-morphism if the morphism of complexes \( \mathcal{F}_1 : (g_1, Q_1^{(1)}) \to (g_2, Q_1^{(2)}) \) induces an isomorphism between the corresponding cohomology spaces (i.e., quasi-isomorphism of complexes in the usual sense).

The two DGLA \( T_{\text{poly}}^\bullet (M), [\cdot, \cdot]_{\text{SN}}, 0 \) and \( (D_{\text{poly}}^\bullet (M), [\cdot, \cdot]_G, \delta) \) are \( L_\infty \)-algebras by definition. The natural injection \( \mathcal{J} : T_{\text{poly}}^\bullet (M) \to D_{\text{poly}}^\bullet (M) \) defined by

\[
\mathcal{J} (\xi_1 \wedge \cdots \wedge \xi_k) : (f_1, \ldots, f_k) = \frac{1}{k!} \det (\xi_j (f_k)),
\]

is a quasi-isomorphism of complexes, but is not a homomorphism of DGLA since it is incompatible with the brackets. But Kontsevich has shown that:

**Theorem 4.2 (Formality).** There is a quasi-isomorphism

\[
\mathcal{K} : (T_{\text{poly}}^\bullet (M), [\cdot, \cdot]_{\text{SN}}, 0) \to (D_{\text{poly}}^\bullet (M), [\cdot, \cdot]_G, \delta)
\]

such that \( \mathcal{K}_1 \) is equal to \( \mathcal{J} \).

The main part of the proof consists in writing down explicitly a quasi-isomorphism for the case \( M = \mathbb{R}^d \). The quasi-isomorphism \( \mathcal{K} \) is expressed in terms of graphs and weights and it generalizes the formula for the star-product given in Section 4.1.1.

**4.1.2.3. Construction of \( \mathcal{K} \) for \( M = \mathbb{R}^d \).** We have already defined the family of graphs \( G_{n,m} \). Here we need a generalization of the bidifferential operators \( B_I \) of the star-product formula. For a graph \( \Gamma \) in \( G_{n,m} \), take \( (\alpha_1, \ldots, \alpha_n) \) \( n \) polyvectors such that \( \alpha_k \in T_{\text{poly}}^{(k)} (M) \). By definition of \( G_{n,m} \) we have \( \sum_{1 \leq k \leq n} \| \kappa (k) \| = 2n + m - 2 \), and consequently \( \alpha_1 \wedge \cdots \wedge \alpha_n \) is in \( T_{\text{poly}}^{n+m-2} (M) \). Define a polydifferential operator \( B_I (\alpha_1 \wedge \cdots \wedge \alpha_n) \) belonging to \( T_{\text{poly}}^{m-1} (M) \) by a formula similar to (4.1). For \( m \) smooth functions \( (f_1, \ldots, f_m) \) on \( \mathbb{R}^d \), we have:

\[
B_I (\alpha_1 \wedge \cdots \wedge \alpha_n) : (f_1, \ldots, f_m) =
\sum_I \left( \left( \prod_{k=1}^n \prod_{k=1}^n \partial_i (\kappa (k)) \right) \left( \prod_{i=1}^m \prod_{j=1}^n \partial_i (j) f_i \right) \right),
\]

where the sum is on all maps \( I : E_\Gamma \to \{ 1, \ldots, d \} \).

The weight of a graph is defined by integration on compactified configuration spaces that we now make precise. For \( 2n + m - 2 \geq 0 \), let

\[
\text{Conf}_{n,m} = \{ (p_1, \ldots, p_n; q_1, \ldots, q_m) \in \mathcal{K}^n \times \mathbb{R}^m \ | \ p_i \neq p_j \text{ for } i \neq j; q_i \neq q_j \text{ for } i \neq j \}.
\]

\( \text{Conf}_{n,m}^+ \) will denote the connected component of \( \text{Conf}_{n,m} \) for which \( q_1 < \cdots < q_m \). \( \text{Conf}_{n,m} \) is acted upon by the group \( G \) of transformations \( z \mapsto a z + b, \ a > 0, \ b \in \mathbb{R} \). Define \( C_{n,m} = \text{Conf}_{n,m} / G \): It inherits a natural orientation. \( C_{n,m}^+ \) will denote the connected component corresponding to \( \text{Conf}_{n,m}^+ \).
The compactified spaces $\bar{C}_{n,m}$ are manifolds with corners. The weight of a graph $\Gamma \in G_{n,m}$ is defined by:

$$w(\Gamma) = \prod_{1 \leq k \leq n} \frac{1}{(\pi)^{2n+m-2}} \int_{\bar{C}_{n,m}} \wedge d\Gamma$$

Notice that the order in the set of edges $E_{\Gamma}$ gives the order in the wedge product. We can now write down the sequence of morphisms $\{K_n\}$ of the quasi-isomorphism $K$:

$$K_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} w(\Gamma)B_{\Gamma}, \quad n \geq 1.$$ 

The verification that $K$ is a quasi-isomorphism amounts to check an infinite number of quadratic equations for the weights, which are essentially a consequence of Stokes theorem applied to the weights. One has to keep track of all the signs mixed by the gradings and the orientations of the configuration spaces. A detailed description of these computations can be found in [AMM].

### 4.2 Field theory and path integral formulas

#### 4.2.1 Star-products in infinite dimension.

The deformation quantization of a given classical field theory consists in giving a proper definition for a star-product on the infinite-dimensional manifold of initial data for the classical field equation and constructing with it, as rigorously as possible, whatever physical expressions are needed. As in other approaches to field theory, here also one faces serious divergence difficulties as soon as one is considering interacting fields theory, and even at the free field level if one wants a mathematically rigorous theory. But the philosophy in dealing with the divergences is significantly different and one is in position to take advantage of the cohomological features of deformation theory to perform what can be called cohomological renormalization [Dth].

Poisson structures are known on infinite-dimensional manifolds since a long time and there is an extensive literature on this subject. A typical structure, for our purpose, is a weak symplectic structure such as that defined by Segal [Se] (see also [Ko74]) on the space of solutions of a classical field equation like $\square \Phi = F(\Phi)$, where $\square$ is the d’Alembertian. Now if one considers scalar-valued functionals $\Psi$ over such a space of solutions, i.e., over the phase space of initial conditions $\phi(x) = \Phi(x,0)$ and $\pi(x) = \frac{\delta}{\delta \phi} \Phi(x,0)$, one can consider a Poisson bracket defined by

$$P(\Psi_1, \Psi_2) = \int \left( \frac{\delta \Psi_1}{\delta \phi} \frac{\delta \Psi_2}{\delta \pi} - \frac{\delta \Psi_1}{\delta \pi} \frac{\delta \Psi_2}{\delta \phi} \right) dx$$

where $\delta$ denotes the functional derivative. The problem is that while it is possible to give a precise mathematical meaning to (4.8) by specifying an appropriate algebra of functionals, the formal extension to powers of $P$, needed to define the Moyal bracket, is highly divergent, already for $P^2$. This should not be so surprising for physicists who know from experience that the correct approach to field theory starts with normal ordering, and that there are infinitely many inequivalent representations of the canonical commutation relations (as opposed to the von Neumann uniqueness
in the finite-dimensional case, for projective representations), even if in recent physical literature some are working formally with Moyal product.

Starting with some star-product \( \star \) (e.g., an infinite-dimensional version of a Moyal-type product or, better, a star-product similar to the normal star-product (4.9)) on the manifold of initial data, one would interpret various divergences appearing in the theory in terms of coboundaries (or cocycles) for the relevant Hochschild cohomology. Suppose that we are suspecting that a term in a cochain of the product \( \star \) is responsible for the appearance of divergences. Applying an iterative procedure of equivalence, we can try to eliminate it, or at least get a lesser divergence, by subtracting at the relevant order a divergent coboundary; we would then get a better theory with a new star-product, “equivalent” to the original one. Furthermore, since in this case we can expect to have at each order an infinity of non equivalent star-products, we can try to subtract a cocycle and then pass to a nonequivalent star-product whose lower order cochains are identical to those of the original one. We would then make an analysis of the divergences up to order \( \hbar' \), identify a divergent cocycle, remove it, and continue the procedure (at the same or hopefully a higher order). Along the way one should preserve the usual properties of a quantum field theory (Poincaré covariance, locality, etc.) and the construction of adapted star-products should be done accordingly. The complete implementation of this program should lead to a cohomological approach to renormalization theory. It would be interesting to formulate the Connes–Kreimer [CK] rigorous renormalization procedure in a way that would fit in this pattern.

A very good test for this approach would be to start from classical electrodynamics, where (among others) the existence of global solutions and a study of infrared divergencies were recently rigorously performed [FST], and go towards mathematically rigorous QED. Physicists may think that spending so much effort in trying to give complete mathematical sense to recipes that work so well is a waste of time, but the fact it has for so long resisted a rigorous formulation is a good indication that the new mathematical tools needed will prove very efficient.

### 4.2.1.1. Normal star-product

In the case of free fields, one can write down an explicit expression for a star-product corresponding to normal ordering. Consider a (classical) free massive scalar field \( \Phi \) with initial data \( (\phi, \pi) \) in the Schwartz space \( \mathcal{S} \). The initial data \( (\phi, \pi) \) can advantageously be replaced by their Fourier modes \( (\bar{a}, a) \), which for a massive theory are also in \( \mathcal{S} \) seen as a real vector space. After quantization \( (\hat{a}, a) \) become the usual creation and annihilation operators, respectively. The normal star-product \( \star_N \) is formally equivalent to the Moyal product and an integral representation for \( \star_N \) is given by [Di]:

\[
(F \star_N G)(\hat{a}, a) = \int_{\mathcal{S}' \oplus \mathcal{S}'} d\mu(\vec{\xi}, \xi) F(\hat{a} + \vec{\xi}, a + \xi) G(\hat{a} + \vec{\xi}, a),
\]

where, by Bochner–Minlos theorem, \( \mu \) is the Gaussian measure on \( \mathcal{S}' \oplus \mathcal{S}' \) defined by the characteristic function \( \exp\left(-\frac{1}{4} \int dk \, \bar{a}(k)a(k)\right) \) and \( F, G \) are holomorphic functions with semi-regular kernels. The reader may object that creation and annihilation operators are operator-valued distributions and one should consider \( (\hat{a}, a) \) in \( \mathcal{S}' \oplus \mathcal{S}' \) instead. In fact the distribution aspect is present in the very definition of the cochains of the star-product through (4.9).
Likewise, Fermionic fields can be cast in that framework by considering functions valued in some infinite dimensional Grassmann algebra and super-Poisson brackets (for the deformation quantization of the latter see e.g. \[Bm\]).

For the normal product (4.9) one can formally consider interacting fields. It turns out that the star-exponential of the Hamiltonian is, up to a multiplicative well-defined function, equal to Feynman’s path integral. For free fields, we have a mathematical meaningful equality between the star-exponential and the path integrals as both of them are defined by a Gaussian measure, and hence well-defined. In the interacting fields case, giving a rigorous meaning to either of them would give a meaning to the other.

The interested reader will find in \[Di\] calculations performing some steps in the above direction, for free scalar fields and the Klein–Gordon equation, and an example of cancellation of some infinities in \[\lambda \phi^4\]-theory via a \[\lambda\]-dependent star-product formally equivalent to a normal star-product.

Recently many field theorists became interested in deformation quantization in relation with M-theory and string theory. The main hope seems to be that deformation quantization may give a good framework for noncommutative field theory. We refer the reader to the thorough review by Douglas and Nekrasov \[DN\] for more details.

### 4.2.2 Path integral formula of Cattaneo and Felder

Kontsevich \[Ko2\] suggested that his formula comes from a perturbation theory for a related bidimensional topological field theory. The physical origin of the weights and graphs has been elucidated by Cattaneo and Felder \[CF\]. They have constructed a topological field theory on a disc whose perturbation series makes Kontsevich graphs and weights appear explicitly after a finite renormalization. The field model in question is the so-called Poisson sigma model \[Ik\]; it is made up of the following ingredients:

a) the closed unit disc \(\Sigma\) in \(\mathbb{R}^2\),

b) a \(d\)-dimensional Poisson real manifold \((M, \alpha)\),

c) a map \(C^1\) \(X: \Sigma \rightarrow M\),

d) a 1-form \(\eta\) on \(\Sigma\) taking values in the space of 1-forms on \(M\), i.e., \(\eta \in \Gamma(\Sigma, T^*\Sigma \otimes T^*M)\).

In physics, \(\Sigma\) is called the worldsheet, and \(M\) the target space.

The action functional for the Poisson sigma model is given by the integration of the sum of two 2-forms on \(\Sigma\). Let \(V_1\) and \(V_2\) be two vector fields on \(\Sigma\), then \(\langle \eta, V_i \rangle, i = 1, 2\), is a 1-form on \(M\). We can define a 2-form \(\beta\) on \(\Sigma\) as follows:

\[
\langle \beta, V_1 \wedge V_2 \rangle = \frac{1}{2} (\langle \langle \eta, V_1 \rangle, DX.V_2 \rangle - \langle \langle \eta, V_2 \rangle, DX.V_1 \rangle),
\]

where \(DX: T\Sigma \rightarrow TM\) is the tangent map of \(X\). The 2-form \(\beta\) is also denoted \(\langle \eta, DX \rangle\). A second 2-form \(\gamma\) on \(\Sigma\) involves the Poisson tensor \(\alpha\), and is given by:

\[
\langle \gamma, V_1 \wedge V_2 \rangle = \frac{1}{2} \langle \alpha \circ X, \langle \eta, V_1 \rangle \wedge \langle \eta, V_2 \rangle \rangle,
\]
and we shall write $\gamma = \frac{1}{2} (\alpha \circ X, \eta \wedge \eta)$. If $(u^1, u^2)$ is a chart at $s \in \Sigma$, and $(x^1, \ldots, x^d)$ is a chart at $x = X(s)$, the local expressions for $\beta$ and $\gamma$ read:

$$\beta(s) = \sum_{i, \mu, \nu} \eta_{\mu i}(s) \frac{\partial X^i}{\partial u^\nu}(s) \, du^\mu \wedge du^\nu$$

$$\gamma(s) = \frac{1}{2} \sum_{i, j, \mu, \nu} \alpha^i j(x) \eta_{\mu i}(s) \eta_{\nu j}(s) \, du^\mu \wedge du^\nu$$

Finally, the action integral is defined by:

$$S[X, \eta] = \int_\Sigma \left( \langle \eta, DX \rangle + \frac{1}{2} \langle \alpha \circ X, \eta \wedge \eta \rangle \right). \quad (4.10)$$

The 1-form $\eta$ is required to vanish on $T(\partial \Sigma)$ (tangent vectors of $\Sigma$ tangential to the boundary $\partial \Sigma = S^1$).

The dynamics of $S[X, \eta]$ is one of a constrained Hamiltonian system governing the propagation of an open string. Let $\alpha^*: T^* M \to TM$ be the canonical homomorphism of vector bundles induced by $\alpha$. The variation of (4.10) under the constraint $\eta_{|\partial \Sigma} = 0$ gives us the critical point $(X, \eta)$ which is a solution of (here $x = X(s)$):

$$DX(s) + \langle \alpha^* (x), \eta(s) \rangle = 0,$$

$$d_s \eta(s) + \frac{1}{2} d_s \langle \alpha(x), \eta(s) \wedge \eta(s) \rangle = 0,$$

where $d_s$ (resp. $d_x$) is the de Rham differential on $\Sigma$ (resp. $M$). Equations (4.11) admit the trivial solution: $X$ is a constant map equal to $x$, and $\eta = 0$.

The integral formula of Cattaneo and Felder is defined through path integration over $(X, \eta)$ subject to the following conditions: Take three cyclicly counterclockwise ordered points $a, b, c$ on the unit circle, the boundary of $\Sigma$. $X(c) = x$ a fixed point in $M$, $\eta_{|\partial \Sigma} = 0$; then the functional integral

$$(f \ast_k g)(x) = \int D(X) D(\eta) f(X(a)) g(X(b)) e^{i S[X, \eta]}.$$
We conclude this section by remarking that the methods used by Cattaneo and Felder proved to be very efficient. For example, they have provided a simple way to understand the globalization of Kontsevich star-products in more geometrical terms \cite{CI}. The proof goes much along the lines of Fedosov’s construction.
4.3 Recent metamorphoses: operadic approach, algebraic varieties

This section will deal mostly with the metamorphoses that occurred in deformation quantization during the past three years. About one month after the ICM98 Congress in Berlin, Tamarkin [Tam] found a new short derivation (for $X = \mathbb{R}^n$) of the Formality Theorem, based on a very general result concerning all associative algebras: For any algebra $A$ (over a field of characteristic 0), its cohomological Hochschild complex $C^*(A, A)$ and its Hochschild cohomology $B \equiv H^*(A, A)$ (this graded space being considered as a complex with zero differential) are algebras over the same operad (up to homotopy). However abstruse this statement may seem to a physicist, when Kontsevich saw Tamarkin’s result he thought to himself, like Commissaire Maigret in Simenon’s detective stories, “Bon sang, mais c’est bien sûr . . .”. In fact, he had been close to such a result in some works of his in the early nineties but somehow missed the point at that time. He then devoted a lot of his time and efforts to understand fully Tamarkin’s result and generalize it, which he did to a considerable extent. That is probably the main reason why he did not finalize in print his fundamental 1997 e-print [Ko2], nor write his ICM98 lecture on the relations between deformations, motives (in particular, the motivic Galois group) and the Grothendieck–Teichmüller group: all of this became much more clear when expressed in the new language. Instead he pushed a lot further the various ramifications of the new metamorphosis in two fundamental papers [Ko3, KoS], the latter with Soibelman with whom he is working on what will certainly be a very seminal book on these subjects (and many more).

Recently he accomplished still another leap forward [Ko4] by going from the $C^\infty$ situation to the algebro-geometric setting, performing what can be called ‘noncommutative algebraic geometry’. In the previous case, operads and deformations of algebras over operads were essential notions. In the new setting he had to use sheaves of algebroids, a slightly more complicated structure. We shall not attempt here to summarize in a couple of pages the very dense content of several long and complicated papers, rather to try and convey a few touches of (what we understand of) the flavor that emerges from them, giving only very basic definitions. Both developments have a wide array of ramifications and implications in a variety of frontier mathematics.

4.3.1 Operadic approach. To fix ideas we shall give a short definition of an operad and of related notions as in [Ko3]. More abstract (and general) definitions can be found in [KoS]. The language of operads, convenient for descriptions and constructions of various algebraic structures, became recently quite popular in theoretical physics because of the emergence of many new types of algebras related with quantum field theories.

Definition 4.3. An operad (of vector spaces) consists of the following:

$O_1$ a collection of vector spaces $P(n), n \geq 0$, 

$O_2$ an action of the symmetric group $S_n$ on $P(n)$ for every $n$, 

$O_3$ an identity element $id_p \in P(1)$, 

$O_4$ compositions $m_{(n_1, \ldots, n_k)}$:

$$P(k) \otimes (P(n_1) \otimes P(n_2) \otimes \cdots \otimes P(n_k)) \rightarrow P(n_1 + \cdots + n_k)$$

(4.12)

for every $k \geq 0$ and $n_1, \ldots, n_k \geq 0$ satisfying a natural list of axioms.
If we replace vector spaces by topological spaces, $\otimes$ by $\times$, and require continuity of all maps, we get a topological operad.

An algebra over an operad $P$ consists of a vector space $A$ and a collection of polylinear maps $f_n : P(n) \otimes A^\otimes n \to A$ for all $n \geq 0$ satisfying the following list of axioms:

A.1 for any $n \geq 0$ the map $f_n$ is $S_n$-equivariant,

A.2 for any $a \in A$ we have $f_1(id_P \otimes a) = a$,

A.3 all compositions in $P$ map to compositions of polylinear operations on $A$.

The little discs operad $C_d$ is a topological operad such that $C_d(0) = \emptyset$, $C_d(1) = \text{point} = \{\text{id}_{C_d}\}$, and for $n \geq 2$ the space $C_d(n)$ is the space of configurations of $n$ disjoint discs $(D_i)_{1 \leq i \leq n}$ inside the standard disc $D_0$. The composition $C_d(k) \times C_d(n_1) \times \cdots \times C_d(n_k) \to C_d(n_1 + \cdots + n_k)$ is obtained by applying the affine transformations in $AF(\mathbb{R}^d)$ associated with discs $(D_i)_{1 \leq i \leq k}$ in the configuration in $C_d(k)$ (the translations and dilations transforming $D_0$ to $D_i$) to configurations in all $C_d(n_i)$, $i = 1, \ldots, k$ and putting the resulting configurations together. The action of the symmetric group $S_n$ on $C_d(n)$ is given by renumberings of indices of discs $(D_i)_{1 \leq i \leq n}$.

An associative algebra is an algebra over some operad (denoted Assoc in [Ko3]). This explains the universality of associative algebras and suggests how to develop the deformation theory of algebras over operads, which is the right context for the deformation theory of many algebraic structures.

Now, for a topological space $X$, denote by Chains$(X)$ the complex concentrated in $\mathbb{Z}_{\leq 0}$ whose $(-k)$-th component for $k = 0, 1, \ldots$ consists of the formal finite additive combinations $\sum_{i=1}^N n_i \cdot f_i$, $n_i \in \mathbb{Z}$, $N \in \mathbb{Z}_{\geq 0}$ of continuous maps $f_i : [0,1]^k \to X$ (singular cubes in $X$), modulo the following relations: $f \circ \sigma = \text{sign} (\sigma) f$ for any $\sigma \in S_k$ acting on the standard cube $[0,1]^k$ by permutations of coordinates, and $f' \circ pr_{k\to (k-1)} = 0$ where $pr_{k\to (k-1)} : [0,1]^k \to [0,1]^{k-1}$ is the projection onto the first $(k-1)$ coordinates, and $f' : [0,1]^{k-1} \to X$ is a continuous map. The boundary operator on cubical chains is defined in the usual way. A $d$-algebra is an algebra over the operad Chains$(C_d)$.

If $P$ is a topological operad then the collection of complexes $(\text{Chains}(P(n)))_{n \geq 0}$ has a natural operad structure in the category of complexes of Abelian groups. The compositions in Chians$(P)$ are defined using the external tensor product of cubical chains. Passing from complexes to their cohomology we obtain an operad $H_*(P)$ of $\mathbb{Z}$-graded Abelian groups (complexes with zero differential), the homology operad of $P$.

A remarkable fact is the relationship of the deformation theory of associative algebras to the geometry of configuration spaces of points on surfaces. One of its incarnations is the so-called Deligne conjecture. A formulation of the conjecture is that there is a natural action of $\text{Chains}(C_2)$, the chain operad of the little discs operad $C_d$, on the Hochschild complex $C^*(A,A)$ for an arbitrary associative algebra $A$ (we refer to [Ko3, Ko5] for a more precise formulation). The conjecture was formulated in 1993, deemed proved in 1994 until in the Spring of 1998 Tamarkin found a very subtle flaw in these proofs (he had used the result in an earlier version of his new proof of the Formality Theorem). New proofs can be found in [Ko5] and other papers published around that time. The power of those notions can be grasped if we realize that the Formality Theorem in (4.1.2) becomes a relatively easy consequence of the formality of $\text{Chains}(C_2)$, which shows that
the Hochschild complex of an associative algebra is a homotopic Gerstenhaber algebra. A general result is \[ [Ko3] \]:

**Theorem 4.4.** The operad \( \text{Chains}(C_d) \otimes \mathbb{R} \) of complexes of real vector spaces is quasi-isomorphic to its cohomology operad endowed with the zero differential.

Then Tamarkin’s result (which implies the Formality theorem; see \[ TaT \] for further developments) can be formulated as

**Theorem 4.5.** Let \( A := \mathbb{R}[x_1, \ldots, x_n] \) be the algebra of polynomials considered merely as an associative algebra. Then the Hochschild complex \( \text{Hoch}(A) \) is quasi-isomorphic as 2-algebra to its cohomology \( B := H^*(\text{Hoch}(A)) \), the space of polynomial polyvector fields on \( \mathbb{R}^n \), considered as a Gerstenhaber algebra, hence a 2-algebra.

Pursuing further in the directions hinted at in \[ Ko3 \], he could show in \[ KoS \], among other, that the Grothendieck–Teichmüller group acts homotopically on the moduli space of structures on 2-algebras on the Hochschild complex. The weights \( w(\Gamma) \) that appear in Kontsevich’s quasi-isomorphism are examples of some very special numbers, the periods \( [K\text{per}] \) of rational algebraic varieties, a countable set of numbers lying between rational algebraic numbers and all complex numbers. At the end of \[ KoS \] there is a theory of singular chains, suitable for working with manifolds with corners, where one sees appearing a notion of piecewise algebraic spaces.

### 4.3.2 Algebraic varieties

There have been a number of studies of star-products in the complex domain, starting with e.g. \[ BG \] (even if the authors had not realized at that time that this was in fact what they were studying). A more recent development, with a number of interesting examples, can be found in \[ Bo \]. However the most systematic study appeared very recently with Kontsevich’s paper \[ Ko4 \] devoted to peculiarities of the deformation quantization in the algebro-geometric context.

A direct application of the Formality Theorem to an algebraic Poisson manifold gives a canonical sheaf of categories deforming coherent sheaves. The global category is very degenerate in general. For a general algebraic Poisson manifold one gets a canonical presheaf of algebroids and eventually an Abelian category, that can be quite degenerate. Thus, he introduces a new notion of a semi-formal deformation, a replacement in algebraic geometry of a weakened notion of strict deformation quantization (versus a formal one), which is quite natural here because of the context. To give a flavor of the kind of metamorphosis undergone here by deformation quantization, we reproduce the precise definition of \[ Ko4 \]:

**Definition 4.6.** Let \( R \) be a complete pro-Artin local ring with residue field \( k \). A deformation over \( \text{Spec}(R) \) of a \( k \)-algebra \( A \) is an algebra \( \hat{A} \) over \( R \), topologically free as \( R \)-module, considered together with an identification of \( k \)-algebras \( A \cong \hat{A} \otimes_R k \). A semi-formal deformation over \( R \) of a finitely generated associative algebra \( A/k \) is an algebra \( \hat{A}_{\text{finite}} \) over \( R \) endowed with an identification of \( k \)-algebras \( A \cong \hat{A}_{\text{finite}} \otimes_R k \) such that there exists an exhaustive increasing filtration of \( \hat{A}_{\text{finite}} \) by finitely generated free \( R \)-modules \( \hat{A}_{\leq n} \), compatible with the product, admitting a splitting as a filtration of \( R \)-modules, and such that the Rees ring of the induced filtration of \( A \) is finitely generated over \( k \).
Deformed algebras obtained by semi-formal deformations are Noetherian and have polynomial growth. Kontsevich gives constructions of semi-formal quantizations (noncommutative deformations over the ring of formal power series) of projective and affine algebraic Poisson manifolds satisfying certain natural geometric conditions. Projective symplectic manifolds (e.g., K3 surfaces and abelian varieties) do not satisfy those conditions (in contradistinction with the $C^\infty$ situation where quantum tori are important examples), but projective spaces with quadratic Poisson brackets and Poisson–Lie groups can be semi-formally quantized. In other words, in that framework, semi-formal deformation quantization is either canonical or impossible.

4.3.3 Generalized deformations, singularities. We shall end this (long) review by various attempts, motivated by physical problems, to go beyond DrG (continuous) deformations. The first two involve deformations that are formally different from the latter, while the last one (mentioned only for completeness) involves going to noncontinuous cochains in a context which has a flavor of the preceding Section 4.3.2.

4.3.3.1. Nambu mechanics and its “Fock–Zariski” quantization. We mention this aspect here mainly as an example of generalized deformation. Details can be found in [DFST], so we shall just briefly indicate a few highlights.

In 1973 Nambu published some calculations which he had made a dozen years before: with quarks in the back of his mind he started with a kind of “Hamilton equations” on $\mathbb{R}^3$ with two “Hamiltonians” $g, h$ functions of $r$. In this new mechanics the evolution of a function $f$ on $\mathbb{R}^3$ is $\frac{df}{dt} = \frac{\partial (f, g, h)}{\partial (x, y, z)}$, a 3-bracket, where the right-hand side is the Jacobian of the mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by $(x, y, z) \mapsto (f, g, h)$. That expression was easily generalized to $n$ functions $f_i, i = 1, \ldots, n$. One introduces an $n$-tuple of functions on $\mathbb{R}^n$ with composition law given by their Jacobian, linear canonical transformations $\text{SL}(n, \mathbb{R})$ and a corresponding $(n - 1)$-form which is the analogue of the Poincaré–Cartan integral invariant. The Jacobian has to be interpreted as a generalized Poisson bracket: It is skew-symmetric with respect to the $f_i$’s, satisfies an identity (called the fundamental identity) which is an analogue of the Jacobi identity and is a derivation of the algebra of smooth functions on $\mathbb{R}^n$ (i.e., the Leibniz rule is verified in each argument). There is a complete analogy with the Poisson bracket formulation of Hamilton equations, including the important fact that the components of the $(n - 1)$-tuple of “Hamiltonians” $(f_2, \ldots, f_n)$ are constants of motion. Shortly afterwards it was shown that Nambu mechanics could be seen as a coming from constrained Hamiltonian mechanics; e.g., for $\mathbb{R}^3$ one starts with $\mathbb{R}^6$ and an identically vanishing Hamiltonian, takes a pair of second class constraints to reduce it to some $\mathbb{R}^4$ and one more first-class Dirac constraint, together with time rescaling, will give the reduction. This “chilled” the domain for almost 20 years – and gives a physical explanation to the fact that Nambu could not go beyond Heisenberg quantization.

In order to quantize the Nambu bracket, a natural idea is to replace, in the definition of the Jacobian, the pointwise product of functions by a deformed product. For this to make sense, the deformed product should be Abelian, so we are lead to consider commutative DrG-deformations of an associative and commutative product. Looking first at polynomials we are lead to the commutative part of Hochschild cohomology called Harrison cohomology, which is trivial.
with polynomials, a natural idea is to factorize them and take symmetrized star-products of the irreducible factors. More precisely we introduce an operation $\alpha$ which maps a product of factors into a symmetrized tensor product (in a kind of Fock space) and an evaluation map $T$ which replaces tensor product by star-product. Associativity will be satisfied if $\alpha$ annihilates the deformation parameter $\lambda$ (there are still $\lambda$-dependent terms in a product due to the last action of $T$); intuitively one can think of a deformation parameter which is $\lambda$ times a Dirac $\gamma$ matrix. This fact brought us to generalized deformations, but even that was not enough. Dealing with distributivity of the product with respect to addition, and with derivatives, posed difficult problems. In the end we took for observables Taylor developments of elements of the algebra of the semi-group generated by irreducible polynomials (“polynomials over polynomials”, inspired by second quantization techniques) and were then able to perform a meaningful quantization of these Nambu–Poisson brackets (cf. [DFS] for more details and subsequent developments). We call this approach “Fock–Zariski” to underline the decomposition of polynomials into irreducible factors and the role of these factors, similar to one-particle states in Fock quantization. Related cohomologies were studied [Ga].

4.3.3.2. Generalized deformations. The fact that in the Fock–Zariski quantization, the deformation parameter behaves almost as if it was nilpotent, has very recently induced Pinczon [Pi] and Nadaud [NaD] to generalize the Gerstenhaber theory to the case of a deformation parameter which does not commute with the algebra. For instance one can have $\tilde{a} = \sum_n a_n \lambda^n$, $a_n \in A$, a left multiplication by $\lambda$ of the form $\lambda \cdot \tilde{a} = \sum_n \sigma(a_n) \lambda^{n+1}$ where $\sigma$ is an endomorphism of $A$. A similar theory can be done in this case, with appropriate cohomologies. While that theory does not yet reproduce the above mentioned Nambu quantization, it gives new and interesting results. In particular [Pi], while the Weyl algebra $W_1$ (generated by the Heisenberg Lie algebra $h_1$) is known to be DrG-rigid, it can be nontrivially deformed in such a supersymmetric deformation theory to the supersymmetry enveloping algebra $U(osp(1,2))$. More recently [NaD], on the polynomial algebra $C[x,y]$ in 2 variables, Moyal-like products of a new type were discovered; a more general situation was studied, where the relevant Hochschild cohomology is still valued in the algebra but with “twists” on both sides for the action of the deformation parameter on the algebra. Though this more balanced generalization of deformations also does not (yet) give Nambu mechanics quantization, it opens a whole new direction of research for deformation theory. This is another example of a physically motivated study which goes beyond a generally accepted framework and opens new perspectives.

4.3.3.3. Singularities. We have already mentioned the case of manifolds with boundaries or with corners, where deformation quantization can be extended [NT], as was the pseudodifferential calculus by the Melrose $b$-calculus [Mc]. More general situations are being studied. An interesting new idea, where a nontrivial Harrison cohomology can be expected, is to try and use noncontinuous cochains. When the manifold has singularities (like a cone, a most elementary example), this is a reasonable thing to do [Fr].

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References

[AW] Agarwal G.S. and Wolf E. “Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics I, II, III.” Phys. Rev. D2 (1970), 2161–2186, 2187–2205, 2206–2225.

[ADS] Andler M., Dvorsky A. and Sahi S. “Kontsevich quantization and invariant distributions on Lie groups,” math.QA/9910104 (1999).

[ACG] Arnal A., Cahen M. and Gutt S. “Representations of compact Lie groups and quantization by deformation,” Bull. Acad. Royale Belg. 74 (1988), 123–141; “Star exponential and holomorphic discrete series,” Bull. Soc. Math. Belg. 41 (1989), 207–227.

[AC] Arnal D. and Cortet J-C. “Nilpotent Fourier transform and applications,” Lett. Math. Phys. 9 (1984), 483–494. “Deformation theory and quantization I, II,” Ann. Phys. (NY) 111 (1977), 521–530. “Deformation theory and quantization I, II,” Ann. Phys. (NY) 111 (1977), 61–110, 111–151.

[Be] Berezn F.A. “General concept of quantization,” Comm. Math. Phys. 40 (1975), 153–174; “Quantization,” Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175; “Quantization in complex symmetric spaces,” Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 363–402, 472. [English translations: Math. USSR-Izv. 38 (1975) 1109–1165 and 39 (1976) 341–379].

[BS] Berezn F.A. and Šubin M.A. “Symbols of operators and quantization,” pp. 21–52 in: Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), Colloq. Math. Soc. Janos Bolyai 5, North-Holland, Amsterdam 1972.

[BG] Boutet de Monvel L. and Guillemin V. The spectral theory of Toeplitz operators, Annals of Mathematics Studies 99, Princeton University Press 1981.

[CFG] Cahen M., Flato M., Gutt S. and Sternheimer D. “Do different deformations lead to the same spectrum?” J. Geom. Phys. 2 (1985), 35–48.
[CGR] Cahen M., Gutt S. and Rawnsley J. “Quantization of Kähler manifolds IV,” *Lett. Math. Phys.* **34** (1995), 159–168.

[CF] Cattaneo A.S. and Felder G. “A path integral approach to the Kontsevich quantization formula,” *Comm. Math. Phys.* **212** (2000) 591–611; “On the Globalization of Kontsevich’s Star Product and the Perturbative Poisson Sigma Model,” hep-th/0111028.

[CFT] Cattaneo A.S., Felder G. and Tomassini L. “From local to global deformation quantization of Poisson manifolds,” math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.

[Co] Connes A. *Noncommutative Geometry*, Academic Press, San Diego 1994; “Noncommutative differential geometry,” Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360.

[CDV] Connes A. and Dubois-Violette M. “Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples”. math.QA/0012228 (2000); “Fedosov connections on jet bundles and deformation quantization,” in *Deformation quantization* (G. Halbout ed.), IRMA Lectures in Math. Theoret. Phys. **1**, Walter de Gruyter, Berlin 2000 (math.QA/0111290).

[CE] Chevalley C. and Eilenberg E. “Cohomology theory of Lie groups and algebras,” *Trans. Amer. Math. Soc.* **63** (1948), 85–124.
Deformation Quantization: Genesis, Developments and Metamorphoses

[Fe] Fedosov B.V. “A simple geometrical construction of deformation quantization,” J. Diff. Geom. 40 (1994), 213–238.

[Fe3] Fedosov B.V. Deformation quantization and index theory. Mathematical Topics 9, Akademie Verlag, Berlin 1996.

[Fe4] Fedosov B.V. “The Atiyah-Bott-Patodi method in deformation quantization,” Comm. Math. Phys. 209 (2000), 691–728.

[Fl] Flato M. “Deformation view of physical theories,” Czechoslovak J. Phys. B32 (1982), 472–475. “Two disjoint aspects of the deformation programme: quantizing Nambu mechanics; singleton physics,” pp. 49–52 in: J. Rembieliński (ed.), Particles, Fields, and Gravitation (Lodz 1998), AIP Press, New York 1998 (hep-th/9809073).

[FLS1] Flato M., Lichnerowicz A. and Sternheimer D. “Déformations 1-differentiables d’algèbres de Lie attachées à une variété symplectique ou de contact,” C.R. Acad. Paris Sér. A 279 (1974), 877–881; Compositio Mathematica, 31 (1975), 47–82.

[FLSq] Flato M., Lichnerowicz A. and Sternheimer D. “Crochets de Moyal-Vey et quantification,” C.R. Acad. Sci. Paris Sér. A 283 (1976), 19–24.

[FST] Flato M., Simon J. C. H. and Taflin E. The Maxwell-Dirac equations: the Cauchy problem, asymptotic completeness and the infrared problem, Mem. Amer. Math. Soc., 127 (number 606) 1997.

[Fr] Frønsdal C. “Harrison cohomology and Abelian Deformation Quantization on Algebraic Varieties,” pp. 151–163 in Deformation quantization (G.Halbout ed.), IRMA lectures in Math. Theoret. Phys. 1, Walter de Gruyter, Berlin 2000 (hep-th/0109001).

[Ga] Gautheron P. “Some remarks concerning Nambu mechanics”. Lett. Math. Phys. 37 (1996), 103–116. “Simple facts concerning Nambu algebras,” Comm. Math. Phys. 195 (1998), 417–434.

[Ge] Gerstenhaber M. “The cohomology structure of an associative ring,” Ann. Math. 78 (1963), 267–288. “On the deformation of rings and algebras,” ibid. 79 (1964), 59–103; and (IV), ibid. 99 (1974), 257–276.

[GG] Gerstenhaber M. and Giaquinto A. “Compatible deformations,” pp. 159–168 in: E.L. Green and B. Huisgen-Zimmermann (eds.), Trends in the Representation Theory of Finite Dimensional Algebras, Contemporary Mathematics 229, American Mathematical Society, Providence 1998.

[GGS] Gerstenhaber M., Giaquinto A. and Schaps M.E. “The Donald-Flanigan Problem for Finite Reflection Groups,” Lett. Math. Phys. 56 (2001), 41–72.

[GS] Gerstenhaber M. and Schack S.D. “Algebraic cohomology and deformation theory,” pp. 11-264 in: M. Hazewinkel and M. Gerstenhaber (eds.), Deformation Theory of Algebras and Structures and Applications, NATO ASI Ser. C 247, Kluwer Acad. Publ., Dordrecht 1988.

[GZ] Giaquinto A. and Zhang J.J. “Quantum Weyl Algebras,” J. of Algebra 176 (1995), 861–881.

[Gu1] Gutt S. “Equivalence of deformations and associated ∗-products,” Lett. Math. Phys. 3 (1979), 297–309.

[Gth] Gutt S. Déformations formelles de l’algèbre des fonctions différentiables sur une variété symplectique, Thesis, Université Libre de Bruxelles 1980.

[Gu2] Gutt S. “Second et troisième espaces de cohomologie différentiable de l’algèbre de Lie de Poisson d’une variété symplectique,” Ann. Inst. H. Poincaré Sect. A (N.S.) 33 (1980), 1–31.

[Gu3] Gutt S. “An explicit ∗-product on the cotangent bundle of a Lie group,” Lett. Math. Phys. 7 (1983), 249–258.

[Gu] Gutt S. “Variations on deformation quantization,” pp. 217–254 in: G. Dito and D. Sternheimer (eds.), Conference Moshé Flato 1999, Math. Phys. Stud. 21, Kluwer Acad. Publ., Dordrecht 2000.

[GR] Gutt S. and Rawnsley J. “Equivalence of star-products on a symplectic manifold: an introduction to Deligne’s Čech cohomology class,” J. Geom. Phys. 29 (1999), 347–392.

[Gr] Groenewold A. “On the principles of elementary quantum mechanics,” Physica 12 (1946), 405–460.

[Ha] Halbout G. “Calcul d’un invariant de star produit fermé sur une variété symplectique,” Comm. Math. Phys. 205 (1999), 53–67.

[HKR] Hochschild G., Kostant B. and Rosenberg A. “Differential forms on regular affine algebras,” Trans. Am. Math. Soc. 102 (1962), 383–406.

[Ik] Ikeda N. “Two-dimensional gravity and nonlinear gauge theory,” Ann. Phys. 235 (1994), 435–464.

[WI] Inönü E. and Wigner E.P. “On the contraction of groups and their representations, Proc. Nat. Acad. Sci. U. S. A. 39 (1953), 510–524.
Jimbo M. “A $q$-difference algebra of $U_q(\mathfrak{g})$ and the Yang-Baxter equation,” Lett. Math. Phys. 10 (1985), 63–69.

Karabegov A.V. “Cohomological classification of deformation quantizations with separation of variables,” Lett. Math. Phys. 43 (1998), 347–357; “Berezin’s quantization on flag manifolds and spherical modules,” Trans. Amer. Math. Soc. 350 (1998), 1467–1479.

Kodaira K. and Spencer D.C. “On deformations of complex analytic structures,” Ann. Math. 67 (1958), 328–466.

Kontsevich M. “Formality conjecture,” pp. 139–156 in: D. Sternheimer, J. Rawnsley and S. Gutt (eds.) Deformation theory and symplectic geometry (Ascona, 1996), Math. Phys. Stud. 20, Kluwer Acad. Publ., Dordrecht 1997.

Kontsevich M. “Deformation quantization of Poisson manifolds I”, q-alg/9709040 (1997).

Kontsevich M. “Deformation quantization of algebraic varieties,” Lett. Math. Phys. 56 (2001), 271–294.

Kostant B. “Symplectic spinors” pp. 139–152 in: Symposia Mathematica 14, Academic Press, London, 1974.

Kulish P.P. and Reshetikhin N.Yu. “Quantum linear problem for the sine-Gordon equation and higher representations”. Zap. Nauch. Sem. LOMI 101 (1981), 101–110 (English translation in Jour. Sov. Math. 23 (1983), 24–35).

Lee P.B.A. “Application of the cohomology of graded Lie algebras to formal deformations of Lie algebras,” Lett. Math. Phys. 13 (1987), 157–166.

Lichnerowicz A. “Cohomologie 1-differentiable des algèbres de Lie attachées à une variété symplectique ou de contact,” J. Math. Pures Appl. 53 (1974), 459–483.

Lichnerowicz A. “Les variétés de Poisson et leurs algèbres de Lie associées,” J. Diff. Geom. 12 (1977), 253–300; “Variétés de Poisson et feuilletages,” Ann. Fac. Sci. Toulouse Math. (5) 4 (1983), 195–262.

Majid S. Foundations of quantum group theory, Cambridge University Press, Cambridge 1995.

Manchon D., Masmoudi M. and Roux A. “On quantization of quadratic Poisson structures,” Comm. Math. Phys. 225 (2002), 121–130 (math.QA/0105068).

Manin Yu. “Theta functions, quantum tori and Heisenberg groups,” Lett. Math. Phys. 56 (2001), 295–320.

Masmoudi M. “Tangential formal deformations of the Poisson bracket and tangential star products on a regular Poisson manifold,” J. Geom. Phys. 9 (1992), 155–171.

Melrose R. The Atiyah-Patodi-Singer index theorem. Research Notes in Mathematics 4, A.K. Peters Ltd., Wellesley MA 1993.

Moreno C. “Invariant star products and representations of compact semi-simple Lie groups,” Lett. Math. Phys. 12 (1986), 217–229.

Moyal J.E. “Quantum mechanics as a statistical theory,” Proc. Cambridge Phil. Soc. 45 (1949), 99–124.

Nazaikinskii F. “Generalized deformations, Koszul resolutions, Moyal Products,” Reviews Math. Phys. 10 (1998), 685–704; Thèse, Dijon (janvier 2000); “Generalized Deformations and Hochschild Cohomology,” Lett. Math. Phys. 58 (2001), 41–55.

Nazaikinskii F. “On continuous and differentiable Hochschild cohomology,” Lett. Math. Phys. 47 (1999), 85–95.

Nest R. and Tsygan R. “Algebraic index theorem,” Comm. Math. Phys. 172 (1995), 223–262; “Algebraic index theorem for families,” Adv. Math. 113 (1995), 151–205; “Formal deformations of symplectic manifolds with boundary,” J. Reine Angew. Math. 481 (1996), 27–54.
[NV] Neroslavsky O. M. and Vlasov A. T. “Sur les déformations de l’algèbre des fonctions d’une variété symplectique,” *C. R. Acad. Sc. Paris Sér. I* 292 (1981), 71–76.

[OMY1] Omori H., Maeda Y. and Yoshioka A. “Weyl manifolds and deformation quantization,” *Adv. Math.* 85 (1991), 225–255.

[OMY2] Omori H., Maeda Y. and Yoshioka A. “Existence of a closed star product,” *Lett. Math. Phys.* 26 (1992), 285–294.

[OMY3] Omori H., Maeda Y. and Yoshioka A. “Deformation quantizations of Poisson algebras,” *Contemp. Math.* 179 (1994), 213–240.

[Pi] Pinczon G. “Non commutative deformation theory,” *Lett. Math. Phys.* 41 (1997), 101–117.

[Po] Polya M. “Quantization of linear Poisson structures and degrees of maps,” To appear in *Lett. Math. Phys.* (2002).

[SB] Schmid W. “Character formulas and localization of integrals,” pp. 259–270 in *Deformation theory and symplectic geometry, Proceedings of Ascona meeting, June 1996* (D. Sternheimer, J. Rawnsley and S. Gutt, Eds.), Math. Physics Studies 20, Kluwer Acad. Publ., Dordrecht 1997.

[Se] Segal I. E. “Symplectic structures and the quantization problem for wave equations” pp. 79-117 in: *Symposia Mathematica* 14, Academic Press, London 1974.

[SS] Shnider S. and Sternberg S. *Quantum Groups*, Graduate Texts in Mathematical Physics vol. II, International Press, Boston & Hong-Kong 1993.

[Sho] Shoikhet B. “Vanishing of the Kontsevich Integrals of the Wheels,” *Lett. Math. Phys.* 56 (2001), 141–149.

[Sta] Staškevičius J. “Deformation theory and the Batalin-Vilkovisky master equation,” pp. 271–284 in: D. Sternheimer, J. Rawnsley and S. Gutt (eds.) *Deformation theory and symplectic geometry* (Ascona, 1996), Math. Phys. Stud. 20, Kluwer Acad. Publ., Dordrecht 1997.

[SD] Sternheimer D. “Deformation quantization: Twenty years after,” pp. 107–145 in: J. Rembieliński, (ed.), *Particles, Fields, and Gravitation (Łódź 1998)*, AIP Press, New York 1998 (math. QA/9809056). “In retrospect: A personal view on Moshé Flato’s scientific legacy,” pp. 31–41 in: *Conférence Moshé Flato 1999*, Math. Phys. Studies 21, Kluwer Acad. Publ., Dordrecht 2000.

[Ta] Tamarkin D.E. “Topological invariants of connections on symplectic manifolds,” *Funct. Anal. Appl.*, 29 (1996), 258–267.

[Tam] Tamarkin D.E. “Another proof of M. Kontsevich formality theorem,” math.QA/9803026 (1998).

[TaT] Tamarkin D.E. and Tsygan B. “Cyclic Formality and Index Theorems,” *Lett. Math. Phys.* 56 (2001), 85–97.

[U2] Unterberger A. & J. “Quantification et analyse pseudo-différentielle,” and “La série discrète de SL(2, ℝ) et les opérateurs pseudo-différentiels sur une demi-droite,” *Ann. Sci. Éc. Norm. Sup.* (4) 21 (1988), 133–158 and 17 (1984), 83–116.

[UU] Unterberger A. and Upmeier H. *Pseudodifferential analysis on symmetric cones*, Studies in Advanced Mathematics, CRC Press, Boca Raton FL 1996; “The Berezin transform and invariant differential operators,” *Comm. Math. Phys.* 164 (1994), 563-597.

[Ve] Vey J. “Déformation du crochet de Poisson sur une variété symplectique,” *Comment. Math. Helv.* 50 (1975), 421–454.

[Wb] Weinstein A. “Deformation quantization,” Séminaire Bourbaki, exposé 789 (juin 1994), *Astérisque* 227, 389–409.

[Wm] Weinstein A. “The modular automorphism group of a Poisson manifold,” *J. Geom. Phys.* 23 (1997), 379–394.

[We] Weyl H. *The theory of groups and quantum mechanics*, Dover, New-York 1931, translated from *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig 1928; “Quantenmechanik und Gruppentheorie,” *Z. Physik* 46 (1927), 1–46.

[Wi] Wigner E.P. “Quantum corrections for thermodynamic equilibrium,” *Phys. Rev.* 40 (1932), 749–759.

[Wo] Woodhouse N.M.J. *Geometric quantization* (2nd edition). Oxford Science Publications, Oxford Mathematical Monographs. Oxford University Press, New York 1992.

[Xu] Xu P. “Fedosov *-Products and Quantum Momentum Maps,” *Comm. Math. Phys.* 197 (1998) 167–197.