Yang–Mills Theory in Three Dimensions as Quantum Gravity Theory

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Abstract
We perform the dual transformation of the Yang–Mills theory in \( d = 3 \) dimensions using the Wilson action on the cubic lattice. The dual lattice is made of tetrahedra triangulating a 3-dimensional curved manifold but embedded into a flat 6-dimensional space (for the \( SU(2) \) gauge group). In the continuum limit the theory can be reformulated in terms of 6-component gauge-invariant scalar fields having the meaning of the external coordinates of the dual lattice sites. These 6-component fields induce a metric and a curvature of the 3-dimensional dual colour space. The Yang–Mills theory can be identically rewritten as a quantum gravity theory with the Einstein–Hilbert action but purely imaginary Newton constant, plus a homogeneous ‘matter’ term. Interestingly, the theory can be formulated in a gauge-invariant and local form without explicit colour degrees of freedom.
1 Lattice partition function

Though our objective is the continuum theory we start by formulating the $SU(N_c)$ gauge theory on a cubic lattice. The partition function can be written as an integral over all link variables being $SU(N_c)$ unitary matrices $U$ with the action being a sum over plaquettes,

$$Z(\beta) = \int \prod_{\text{links}} dU_{\text{link}} \exp \left( \sum_{\text{plaquettes}} \beta \left( \text{Tr} \ U_{\text{plaq}} + \text{c.c.} \right) / 2 \text{Tr} \ 1 \right)$$

(1)

where $\beta$ is the dimensionless inverse coupling. The unitary matrix $U_{\text{plaq}}$ is a product of four link unitary matrices closing a plaquette.

To get to the continuum limit one writes $U_{\text{link}} = \exp(iaA_\mu t^a)$ where $a$ is the lattice spacing and $A_\mu t^a = A_\mu$ is the Yang–Mills gauge potential with $t^a$ being the generators of the gauge group normalized to $\text{Tr} \ t^a t^b = \delta^{ab}/2$, and expands $\text{Tr} \ U_{\text{plaq}}$ in the lattice spacing $a$. As a result one gets for a plaquette lying in the $(12)$ plane:

$$\beta \frac{\text{Tr} \ U_{\text{plaq}} + \text{c.c.}}{2 \text{Tr} \ 1} = \beta \left( 1 - a^4 \frac{\text{Tr} F_{12}^2}{2 \text{Tr} \ 1} + O(a^6) \right),$$

(2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu A_\nu]$ is the Yang–Mills field strength. Summing over all plaquettes one obtains the partition function of the continuum theory,

$$Z_{\text{cont}} = \int DA_\mu \exp \left( -\frac{1}{2g_d^2} \int d^d x \ \text{Tr} \ F_{\mu\nu}^2 \right),$$

(3)

with an obvious relation between the dimensionless lattice coupling $\beta$ and the $SU(N_c)$ gauge coupling constant in $d$ dimensions, $g_d^2$:

$$\beta = \frac{2N_c}{a^4-dg_d^2}. \quad \text{(4)}$$

In this paper we concentrate on the Euclidean $SU(2)$ Yang–Mills theory in $d = 3$ dimensions. In this case eq. (4) reads:

$$\beta = \frac{4}{ag_3^2}. \quad \text{(5)}$$

The continuum limit of the $d = 3$ Yang–Mills theory given by the partition function (4) is obtained as one takes the lattice spacing $a \to 0$ and $\beta \to \infty$ with their product $g_3^2 = 4/(a\beta)$ fixed. This quantity provides the theory with a mass scale. It is widely believed (though not proven so far) that the theory possesses two fundamental properties: 1) the average of a large Wilson loop has an area behaviour with a string tension proportional to $g_3^2$, 2) correlation functions of local operators like $F_{\mu\nu}^2$ decay exponentially at large separations, with a ‘mass gap’ proportional to $g_3^2$.

Our aim will be to rewrite the partition function (4) in dual variables and to study its continuum limit.
2 Dual transformation

The general idea is to integrate over link variables $U_{\text{link}}$ in eq. (1) and to make a Fourier transformation in the plaquette variables $U_{\text{plaq}}$. This will be made in several steps, one for a subsection.

2.1 Inserting a unity into the partition function

First of all, one needs to introduce explicitly integration over unitary matrices ascribed to the plaquettes, $U_{\text{plaq}}$. This is done by inserting a unity for each plaquette into the partition function (1):

$$1 = \prod_{\text{plaquettes}} \int dU_{\text{plaq}} \, \delta(U_{\text{plaq}}, U_1 U_2 U_3 U_4)$$

where $U_{1\ldots4}$ are the link variables closing into a given plaquette. The $\delta$-function is understood with the group-invariant Haar measure. A realization of such a $\delta$-function is given by Wigner $D$-functions:

$$\delta(U, V) = \sum_{J=0, \frac{1}{2}, \frac{3}{2}, \ldots} (2J + 1) D_J^{m_1 m_2} (U^\dagger) D_J^{m_3 m_4} (V).$$

This equation is known as a completeness condition for the $D$-functions [1]. The main properties of the $D$-functions used in this paper are listed in Appendix A.

Eq. (7) should be understood as follows: if one integrates any function of a unitary matrix $U$ with the r.h.s. of eq. (7) over the Haar measure $dU$ one gets the same function but of the argument $V$:

$$\int dU \, f(U) \, \delta(U, V) = f(V).$$

Using the multiplication law for the $D$-functions (see Appendix A, eq. (75)) one can write down the unity to be inserted for each plaquette in the partition function (1) as

$$1 = \int dU_{\text{plaq}} \sum_J (2J + 1) D_J^{m_1 m_2} (U_{\text{plaq}}^\dagger) D_J^{m_3 m_4} (U_1) D_J^{m_4 m_5} (U_2) D_J^{m_5 m_1} (U_3) D_J^{m_1 m_2} (U_4)$$

where $U_{1\ldots4}$ are the corresponding link variables forming the plaquette under consideration.

2.2 Integrating over plaquette variables

Integrating over plaquette unitary matrices $U_{\text{plaq}}$ becomes now very simple. For each plaquette of the lattice one has factorized integrals of the type

$$\int dU_{\text{plaq}} \exp \left( \frac{\beta \Tr U_{\text{plaq}} + \Tr U_{\text{plaq}}^\dagger}{2 \Tr 1} \right) D_J^{m_1 m_2} (U_{\text{plaq}}^\dagger) = \delta_{m_1 m_2} \frac{2}{\beta} I_J (\beta) T_J (\beta),$$

where $T_J (\beta)$ is the ratio of the modified Bessel functions [3],

3
\[ T_J(\beta) = \frac{I_{2J+1}(\beta)}{I_1(\beta)} \xrightarrow{\beta \to \infty} \exp \left[ -\frac{2J(J+1)}{\beta} \right] \] (11)

The quantity \( T_J(\beta) \) is the ‘Fourier transform’ of the Wilson action; since in the lattice formulation the dynamical variables have the meaning of Euler angles and are therefore compact, the Fourier transform depends on discrete values \( J = 0, 1/2, 1, 3/2, \ldots \). However, as one approaches the continuum limit \( (\beta \to \infty) \) the essential values of the plaquette angular momenta increase as \( J \sim \sqrt{\beta} \) and their discreteness becomes less relevant. Strictly speaking, the continuum limit is achieved at plaquette angular momenta \( J \gg 1 \).

We would like to make a side remark on this occasion. The quantity \( T_J(\beta) \) gives the probability that plaquette momenta \( J \) is excited, for given \( \beta \). For a typical value used in lattice simulations \( \beta = 2.6 \) (in 4 dimensions) we find that the probabilities of having plaquette excitations with \( J = 0, 1/2, 1, 3/2 \) and 2 are 56\%, 29\%, 11\%, 3\% and 1\%, respectively. It means that lattice simulations are actually dealing mainly with \( J = 0, 1/2 \) and 1 with a tiny admixture of higher excitations. It would be important to understand why and how continuum physics is reproduced by lattice simulations despite only such small values of plaquette \( J \)'s are involved.

We get, thus, for the partition function:

\[
Z = \left[ \frac{2}{\beta I_1(\beta)} \right] \sum_{J_P} \prod_{\text{plaquettes}} (2J_P + 1) T_{J_P}(\beta) \times \prod_{\text{links}} \int dU \prod_j D_j^{J_P m_j} (U_1) D_j^{J_P m_j} (U_2) D_j^{J_P m_j} (U_3) D_j^{J_P m_j} (U_4) \] (12)

where \( U_{1-4} \) are link variables forming a plaquette with angular momentum \( J_P \).

### 2.3 Integrating over link variables

The difficulty in performing integration over link variables in eq. (12) is due to the fact that any link enters several plaquettes. In \( d = 2 \) dimensions every link is shared by two plaquettes, hence one has to calculate integrals of the type

\[
\int dU D_k^{J_1}(U) D_m^{J_2}(U^\dagger) = \frac{1}{2J_1 + 1} \delta_{J_1 J_2} \delta_k m \] (13)

for all links on the lattice. We shall consider this case later, in section 4.

In \( d = 3 \) dimensions every link is shared by four plaquettes, hence the integral over link variables is of the type

\[
\int dU D_m^{J_1}(U) D_k^{J_2}(U) D_m^{J_3}(U) D_k^{J_4}(U) \] (14)

where \( J_{A,B,C,D} \) are angular momenta associated with four plaquettes intersecting at a given link \( U \), and \( m_{1-8} \) are ‘magnetic’ quantum numbers, to be contracted inside closed plaquettes. In \( d = 4 \) dimensions there will be six plaquettes intersecting at a given link but we shall not consider this case here.
The general strategy in calculating the link integrals (14) is (i) to divide by a certain rule four $D$-functions into two pairs and to decompose the pairs of $D$-functions in terms of single $D$-functions using eq. (82) of Appendix A, (ii) to integrate the resulting two $D$-functions using eq. (13) and, finally, (iii) to contract the ‘magnetic’ indices. Since all ‘magnetic’ indices will be eventually contracted we shall arrive to the partition function written in terms of the invariant $3nj$ symbols.

There are several different tactics how to divide four $D$-functions into two pairs, eventually leading to anything from $6j$ to $18j$ symbols. In this paper we take a route used in refs.[3, 4], leading to a product of many $6j$ symbols, although on this route one looses certain symmetries, and that causes difficulties later on. The gain, however, is that it is more easy to work with $6j$ symbols than with $12j$ or $18j$ symbols. Since important sign factors have been omitted in refs.[3, 4] and only final result has been reported there, we feel it necessary to give a detailed derivation below.

In $d = 3$ dimensions all plaquettes are shared by two adjacent cubes, therefore, it is natural to divide all cubes of the lattice into two classes which we shall call ‘even’ and ‘odd’, and to attribute plaquettes only to even cubes. We shall call the cube even if its left-lower-forward corner is a lattice site with even coordinates, $(-1)^{x+y+z} = +1$. It will be called odd in the opposite case. The even and odd cubes form a 3-dimensional checker board, as illustrated in Fig.1, where only even cubes are drawn explicitly. The even cubes touch each other through a common edge or link, as do the odd ones among themselves. The even and odd cubes have common faces or plaquettes. All plaquettes will be attributed to even cubes only: that is the reason for the division of cubes into two classes.

Let us consider an even cube shown in Fig.2.

$A, B, C, D, E, F$ denote its 6 faces, numbers from 1 to 12 denote its links or edges, $a, b, c, d, e, f, g, h$ denote its 8 vertices or sites. Correspondingly, we shall denote plaquette angular momenta by $J_{A-F}$, link variables by $U_{1-12}$, and the ‘magnetic’ numbers of the $D$-functions will carry indices $a-h$ referring to the sites the $D$-functions are connecting.

One can write the traces of products of four $D$-functions over plaquettes in various ways. To be systematic we shall adhere to the following rule: Link variables in the plaquette are
Figure 2: Elementary even cube

taken in the anti-clock-wise order, as viewed from the center of the even cube to which the given plaquette belongs. If the link goes in the positive direction of the \( x, y, z \) axes we ascribe the \( U \) variable to it; otherwise we ascribe the \( U^\dagger \) variable to it.

With these rules the six plaquettes of the elementary cube shown in Fig.2 bring in the following six traces of the \( D \)-function products:

\[
\text{Cube} = \left[ D_{i\alpha b}^{j_2}(U_1) D_{b\beta c}^{j_2}(U_2) D_{i\alpha d}^{j_2}(U_3) D_{d\gamma e}^{j_2}(U_4) \right] \left[ D_{j\epsilon k}^{j_3}(U_5) D_{k\epsilon f}^{j_3}(U_6) D_{j\gamma f}^{j_3}(U_7) D_{f\epsilon g}^{j_3}(U_8) \right]
\]

\[
\text{Cube} = \left[ D_{k\epsilon b}^{j_2}(U_2) D_{i\alpha k}^{j_2}(U_3) D_{k\epsilon l}^{j_2}(U_4) D_{l\gamma m}^{j_2}(U_5) D_{l\gamma l}^{j_2}(U_6) D_{m\epsilon l}^{j_2}(U_7) D_{l\gamma n}^{j_2}(U_8) \right]
\]

\[
\text{Cube} = \left[ D_{m^\alpha m}^{j_2}(U_1) D_{m^\alpha m}^{j_2}(U_4) D_{m^\alpha m}^{j_2}(U_7) D_{m^\alpha m}^{j_2}(U_8) \right]
\]

\[
\text{Cube} = \sum_{j_1 + j_2}(2j_1 + 1)(2j_2 + 1)
\]

Each link variable \( U_{1-12} \) appears in this product twice: once as \( U \), the other time as \( U^\dagger \). For \( D(U^\dagger) \) we use eq. (13) of the Appendix A to write it in terms of \( D(U) \). After that we can apply the decomposition rule (12) of that Appendix to write down pairs of \( D \)-functions in terms of one \( D \)-function and two \( 3jm \) symbols. The new \( D \)-functions correspond to the links and carry angular momenta which we denote by \( j \)'s. The \( 3jm \) symbols have 'magnetic' indices which get contracted when all indices related to a given corner of the cube are assembled together. Though this exercise is straightforward it is rather lengthy, and we relegate it to Appendix B. As a result we get the following expression which is identically equal to (13):

\[
\text{Cube} = \sum_{j_1 + j_2}(2j_1 + 1)(2j_2 + 1)
\]

\[
D_{i\alpha b}^{j_1}(U_1) D_{\alpha\beta c}^{j_1}(U_2) D_{j\epsilon k}^{j_1}(U_3) D_{\epsilon\gamma f}^{j_1}(U_4) D_{k\epsilon g}^{j_1}(U_5) D_{\epsilon\gamma h}^{j_1}(U_6)
\]

\[
D_{m^\alpha m}^{j_1}(U_1) D_{m^\alpha m}^{j_1}(U_4) D_{m^\alpha m}^{j_1}(U_7) D_{m^\alpha m}^{j_1}(U_8)
\]

\[
\begin{pmatrix}
  j_1 & j_4 & j_2 \\
  o_a & r_a & z_a
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_4 & j_2 \\
  o_b & u_b & p_b
\end{pmatrix}
\]

\[
\begin{pmatrix}
  j_2 & j_3 & j_10 \\
  p_c & q_c & -x_c
\end{pmatrix}
\begin{pmatrix}
  j_4 & j_3 & j_11 \\
  -r_d & q_d & y_d
\end{pmatrix}
\]

\[
\begin{pmatrix}
  j_7 & j_{11} & j_8 \\
  -z_c & v_e & s_c
\end{pmatrix}
\begin{pmatrix}
  j_7 & j_{11} & j_8 \\
  -t_f & s_f & w_f
\end{pmatrix}
\]

\[
\begin{pmatrix}
  j_6 & j_{10} & j_7 \\
  t_g & x_g & u_g
\end{pmatrix}
\begin{pmatrix}
  j_7 & j_{11} & j_8 \\
  -u_h & y_h & v_h
\end{pmatrix}
\]
Figure 3: Several cubes combine to produce $6j$ symbols composed of link momenta $j$. These are the same cubes as in Fig.1.

\[
\times \left\{ \begin{array}{ccc}
  j_7 & j_{11} & j_8 \\
  J_E & J_F & J_D \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_6 & j_{10} & j_7 \\
  J_D & J_F & J_C \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_6 & j_5 & j_9 \\
  J_B & J_C & J_F \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_{12} & j_8 & j_5 \\
  J_F & J_B & J_E \\
\end{array} \right\} \\
\left\{ \begin{array}{ccc}
  j_4 & j_3 & j_{11} \\
  J_D & J_E & J_A \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_2 & j_3 & j_{10} \\
  J_D & J_C & J_A \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_1 & j_9 & j_2 \\
  J_C & J_A & J_B \\
\end{array} \right\} \left\{ \begin{array}{ccc}
  j_1 & j_4 & j_{12} \\
  J_E & J_B & J_A \\
\end{array} \right\}.
\] (16)

Here $j_{1-12}$ are the angular momenta attached to the links of the cube, (...) are $3jm$- and {...} are $6j$-symbols. We see that there is a $6j$ symbol attached to each corner of the even cube; its arguments are three plaquette momenta $J$ and three link momenta $j$ intersecting in a given corner. The $3jm$ symbols involve only link variables $j$.

We have, thus, rewritten all twelve pairs of $D^j$-functions entering a cube as a product of single $D^j$-functions, where $j$’s are the new momenta associated with links. It is understood that this procedure should be applied to all even cubes of the lattice. After that, one has only two $D^j$-functions of the same link variable $U$, for all links of the lattice. It becomes, therefore, straightforward to integrate over link variables, using eq. (13).

It is convenient to integrate simultaneously over six links entering one lattice site, because in that way one gets a full contraction over all ‘magnetic’ numbers. The derivation is, again, straightforward but lengthy: the details are given in Appendix C. The result is that the $3jm$ factors in eq. (16) are contracted with analogous $3jm$ symbols arising from neighbouring even cubes, and produce $6j$ symbols attached to every lattice site and composed of the six link momenta $j$ intersecting at a given lattice site. In notations of Fig.3 we get for the vertices $a, b$:

\[
\begin{align*}
  "a" &= \left\{ \begin{array}{ccc}
    j_1 & j_4 & j_{12} \\
    J_{15} & J_{14} & J_{13} \\
  \end{array} \right\}, \\
  "b" &= \left\{ \begin{array}{ccc}
    j_1 & j_9 & j_2 \\
    J_{17} & J_{18} & J_{16} \\
  \end{array} \right\},
\end{align*}
\] (17)

and similarly for other vertices. A sign factor $(-1)^{2j}$ should be attributed to every link of the lattice. As shown in Appendix C it is actually equivalent to a sign factor $(-1)^{2J}$ attributed to every lattice plaquette.
3 Lattice partition function as a product of 6j symbols

We summarize here the recipe derived in the previous section. One first divides all 3-cubes into two classes, even and odd ones. They form a 3-dimensional checker board depicted in Fig.1. All even cubes are characterized by their plaquette momenta \( J \). The edges of even cubes have link momenta \( j \); each link is shared by two even cubes.

To each of the eight corners of an even cube one attributes a 6j symbol of the type

\[
\begin{array}{c}
\{ j_1 \ j_2 \ j_3 \\
J \ A \ J \ B \ J \ C
\end{array}
\]

(18)

where \( J \)'s are plaquette and \( j \)'s are link momenta intersecting in a given corner of a cube. The rule is that link 1 is perpendicular to plaquette \( A \), link 2 is perpendicular to plaquette \( B \) and link 3 is perpendicular to plaquette \( C \). Four triades, \((j_1J_BJ_C),(j_2J_AJ_C),(j_3J_AJ_B)\) and \((j_1j_2j_3)\) satisfy triangle inequalities.

To each lattice site one attributes a 6j symbol of the type

\[
\begin{array}{c}
j_1 \ j_2 \ j_3 \\
j_4 \ j_5 \ j_6
\end{array}
\]

(19)

where \( j \)'s are the six link momenta entering a given lattice site. The rule is that link 4 is a continuation of link 1 lying in the same direction, link 5 is a continuation of link 2 and link 6 is a continuation of link 3. Four triades, \((j_1j_2j_3),(j_1j_5j_6),(j_2j_4j_6),(j_3j_4j_5)\) satisfy triangle inequalities.

Actually, each lattice site has five 6j symbols ascribed to it: four are originating from the corners of the even cubes adjacent to the site and are of the type (18), and one is of the type (19).

The lattice partition function (1) or (12) can be identically rewritten as a product of the 6j symbols described above. Independent summation over all possible plaquette momenta \( J \) and all possible link momenta \( j \) is understood. We write the partition function in a symbolic form:

\[
Z = \left[ \frac{2}{\beta} I_1(\beta) \right]^{\text{# of plaquettes}} \sum_{J_P, \ j_l} \prod_{\text{plaquettes}} (2J_P + 1) T_{J_P}(\beta) (-1)^{2J_P} \prod_{\text{links}} (2j_l + 1) \\
\times \prod_{\text{even cubes corners}} \begin{array}{c}
j \ J \ j \\
J \ J \ j
\end{array} \prod_{\text{lattice sites}} \begin{array}{c}
j \ j \ j \\
j \ j \ j
\end{array}. \tag{20}
\]

The plaquette weights \( T_J(\beta) \) are given by eq. (11). Apart from the sign factor essentially the same expression was given in refs. [3, 4]. The sign factor is equal to \( \pm 1 \) if the total number of half-integer plaquettes \( J \)'s is even (odd). Since plaquettes with half-integer momenta form closed surfaces it may seem that the sign factor can be omitted. In a general case, however, when one considers vacuum averages of operators this is not so, therefore, it is preferable to keep the sign factor.

\[1\text{We are grateful to P.Pobylitsa who has independently derived eq. (20).}\]
4 Simple example: $d = 2$ Yang–Mills

In a simple exactly soluble case of the 2-dimensional $SU(2)$ theory every link is shared by only two plaquettes. Therefore, the link integration is of the type given by eq. (13): it requires that all plaquettes on the lattice have identical momenta $J$. The partition function thus becomes a single sum over the common $J$:

$$Z = \left[ \frac{2}{\beta} I_1(\beta) \right]^\# \text{ of plaquettes} \sum_J \left[ T_J(\beta) \right]^\# \text{ of plaquettes},$$

the number of plaquettes being equal to $V/a^2$ where $V$ is the full lattice volume (full area in this case) and $a$ is the lattice spacing.

A slightly less trivial exercise is to compute the average of the Wilson loop. Let the Wilson loop be in the representation $j_s$. It means that one inserts $D^{j_s}(U)$ for all links along the loop. One gets therefore integrals of two $D$-functions outside and inside the loop, and integrals of three $D$-functions for links along the loop. The first integral says that all plaquettes outside the loop are equal to a common $J$. The second integral says that all plaquettes inside the loop are equal to a common $J'$. Integrals along the loop require that $J, J'$ and $j_s$ satisfy the triangle inequality. We have thus for the average of the Wilson loop of area $S$:

$$\langle W_{j_s}(S) \rangle = \frac{\sum_J [T_J(\beta)]^{\frac{V}{2}} \sum_{j_s} [T_{j}(\beta) / T_{J}(\beta)]^{\frac{S}{2}}}{\sum_J [T_J(\beta)]^{\frac{V}{2}}}.$$

This is an exact expression for the lattice Wilson loop, however we wish to explore its continuum limit. It implies that $V/a^2 \to \infty$, $S/a^2 \to \infty$ but $S \ll V$; $\beta \to \infty$, $a \to 0$ but $\beta a^2 = 4/g_2^2$ fixed, where $g_2^2$ is the physical coupling constant having the dimension of mass$^2$, see eq. (4).

We take $V/a^2 \to \infty$ first of all, which requires that only the $J = 0$ term contributes to the sum, with $T_0(\beta) \equiv 1$; consequently all momenta inside the loop are that of the source, $J' = j_s$. Taking into account the asymptotics of $T_J(\beta)$ at large $\beta$ we obtain

$$\langle W_{j_s}(S) \rangle = [T_{j_s}(\beta)]^{\frac{S}{2}} = \exp \left[ -\frac{g_2^2}{2} j_s(j_s + 1) S \right]$$

which is, of course, the well-known area behaviour of the Wilson loop with the string tension proportional to the Casimir eigenvalue.

5 Dual lattice: tetrahedra and octahedra

We now turn to the construction of the dual lattice.

Each $6j$ symbol of the exact partition function encodes four triangle inequalities between the plaquette $J$'s and the link $j$'s. It is therefore natural to represent all $6j$ symbols by tetrahedra whose six edges have lengths equal to the six momenta of a given $6j$ symbol. Four faces of a tetrahedron form four triangles, so that the triangle inequalities for the momenta are satisfied automatically.
Figure 4: Tetrahedra corresponding to the $6j$ symbols sitting at vertices $a$ and $b$.

Figure 5: Octahedron dual to the even cube.

Let us first consider the eight $6j$ symbols corresponding to the eight corners of an even cube. These eight $6j$ symbols are given explicitly in eq. (16) with notations shown in Fig.2. Let us represent all of them by tetrahedra of appropriate edge lengths. For example, the tetrahedra corresponding to the corners $a$ and $b$ are shown in Fig.4. We denote the plaquette momenta $J_A, ...$ just by their Latin labels $A, B, ...$ and the link momenta $j_1, j_2, ...$ by their numerical indices 1, 2, ... We notice immediately that the two tetrahedra have a pair of equal faces, in this case it is the triangle $(A, B, 1)$. Therefore, we can glue the two tetrahedra together so that this triangle becomes their common face. The gluing can be done in two ways. To be systematic we shall always glue tetrahedra so that their volumes do not overlap.

In the same way we glue together other tetrahedra. Being glued together the eight tetrahedra of the cube form an octahedron shown in Fig.5. Its center point $O$ is connected with six lines to the vertices denoted as $A - F$; the lengths of these lines are equal to the corresponding plaquette momenta $J_{A-F}$. The external twelve edges of the octahedron have
lengths equal to the link momenta $j_{1-12}$. The eight faces of the octahedron correspond to the eight vertices of the original even cube. One can say that the octahedron is dual to the cube: the faces become vertices and \textit{vice versa}; the edges remain edges.

It is clear that in a case of generic $J$’s and $j$’s the octahedron cannot be placed into a flat 3-dimensional space. Indeed, we have $6+12 = 18$ given momenta, that is fixed lengths, but only 7 points defining the octahedron, including the center one. In three dimensions that gives 21 d.o.f. from which one has to subtract 3+3 to allow for rigid translations and rotations. Therefore, we are left with only 15 d.o.f. instead of the needed 18. [In four dimensions the arithmetic would match: $7 \cdot 4 - 4 - 6 = 18$.]

Each even cube of the original lattice has twelve neighbouring even cubes sharing edges with the first one, and with themselves. If we represent the neighbour even cubes by their own dual octahedra those will also share common edges. Does this network of octahedra cover the space? No, there are holes in between. However, we have not used yet the 6$j$ symbols (19) made solely of the link momenta $j$’s. If we represent these 6$j$ symbols by tetrahedra their triangle faces will coincide with the faces of the octahedra corresponding to the even cubes adjacent to the site. For example, if we consider the 6$j$ symbols corresponding to the site $a$ (see Fig.3 and eq. (17)),

$$\left\{ \begin{array}{ccc} j_1 & j_4 & j_{12} \\ j_{15} & j_{14} & j_{13} \end{array} \right\},$$

it has a common triangle face $(j_1, j_4, j_{12})$ with the octahedron shown in Fig.5. The other faces of this tetrahedron will coincide with the faces of the octahedra corresponding to the even cubes adjacent to the site. For example, if we consider the 6$j$ symbols corresponding to the site $a$ (see Fig.3 and eq. (17)),

Octahedra corresponding to the cubes supplemented by tetrahedra corresponding to the lattice sites cover the space without holes and therefore serve as a simplicial triangulation, see Fig.6.

An equivalent view on the dual lattice has been suggested in ref. [3]. One can connect centers of neighbour cubes (both even and odd) and ascribe plaquette momenta $J$’s to these
lines. The link momenta $j$’s will be then ascribed to diagonal lines connecting only even neighbour sites of that dual lattice, see Fig.7.

The dual lattice can be understood in two senses. On one hand, one can build a regular cubic dual lattice with additional face diagonals like shown in Figs.6 and 7, and ascribe $J$’s and $j$’s to its edges. On the other hand, since variables living on the links of the dual lattice are positive numbers, one can build a lattice with the lengths of edges equal to the appropriate angular momenta. We shall always use the dual lattice in this second sense.

6 Coordinates of the dual lattice as new variables

In the previous section we have already met with a situation when an octahedron dual to a cube did not fit into a 3-dimensional flat space: at least four dimensions were necessary. As one enlarges the triangulated complex more dimensions are needed to match the number of degrees of freedom. In the limiting case of an infinite lattice one needs 6 flat dimensions. This number of dimensions follows from the number of d.o.f. one has to accommodate: at each lattice site there are three plaquette momenta $J$ and three link momenta $j$, and there is a one-to-one correspondence between lattice sites and the cubes.

Therefore, the dual lattice (understood in the second sense, see above) spans a 3-dimensional manifold which can be embedded into a 6-dimensional flat space. Notice that it is the maximal number of flat dimensions needed to embed a general 3-dimensional riemannian manifold; it can be counted from the number of components of the metric tensor, which is 6 in three dimensions. Only very special configurations of $J$’s and $j$’s would be possible to embed into a flat space of less dimensions.

We are primarily interested in the continuum limit of the lattice theory, that is in the small $a$, large $\beta$ case. It implies that large angular momenta $J \sim \sqrt{\beta}$ are involved, and one can pass from summation over $J$’s and $j$’s to integration over these variables in the partition function (20). We replace

$$\sum_{J=0,1/2,1,...} (2J + 1)... \rightarrow 2 \int_0^\infty dJ^2... ,$$

and similarly for the summation over link momenta $j$’s.
The next step is to ascribe a 6-dimensional Lorentz scalar field \( w^\alpha(x), \alpha = 1, \ldots, 6 \) to the centers of all cubes of the original lattice, see Fig. 7. We shall call them coordinates of the dual lattice. They are scalars because in three dimensions the cubes are scalars. The argument of the six-component scalar field is the coordinate of the center of the cube in question, however, we shall consider \( w^\alpha(x) \) as continuous functions. Since six functions depend only on three coordinates there are three relations between \( w^\alpha(x) \) at any point; these relations define a curved 3-dimensional manifold whose triangulation is given by the set of \( J \)'s and \( j \)'s.

We next define six-dimensional angular momenta as differences of \( w^\alpha(x) \) taken at the centers of neighbor cubes:

\[
J^\alpha_x \left( x + \frac{a}{2}, y, z \right) = w^\alpha(x + a, y, z) - w^\alpha(x, y, z) = a\partial_x w^\alpha + \frac{a^2}{2}\partial_x^2 w^\alpha + \ldots ,
\]

\[
j^{\alpha}_{xz} \left( x + \frac{a}{2}, y, z + \frac{a}{2} \right) = w^\alpha(x + a, y, z + a) - w^\alpha(x, y, z + a) = a(\partial_x - \partial_z) w^\alpha + O(a^2),
\]

and so on. The lengths of these 6-vectors are, by construction, the lengths of the edges of the dual lattice.

The six functions \( w^\alpha(x) \) can be called external coordinates of the manifold; they induce a metric tensor of the manifold determined by

\[
g_{ij}(x) = \partial_i w^\alpha \partial_j w^\alpha. \tag{26}
\]

As usual in differential geometry one can define the Christoffel symbol,

\[
\Gamma_{ijk}(x) = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) = \partial_i w^\alpha \partial_j \partial_k w^\alpha \equiv (w_i \cdot w_{jk}), \tag{27}
\]

and the Riemann tensor,

\[
R_{ijkl}(x) = \frac{1}{2}(\partial_j \partial_k g_{il} + \partial_i \partial_k g_{jl} - \partial_l \partial_k g_{ij} - \partial_i \partial_l g_{jk}) + \Gamma_{m,jk}\Gamma^m_{il} - \Gamma_{m,il}\Gamma^m_{jk}
\]

\[
= [(w_{ik} \cdot w_{jl}) - g^{pl}(w_p \cdot w_{ik})(w_q \cdot w_{jl}) - [k \leftrightarrow l]]. \tag{28}
\]

The contravariant tensor is inverse to the covariant one,

\[
g^{ij}g_{jk} = \delta^i_k, \tag{29}
\]

and can be used to rise indices, and for contractions. The determinant of the metric tensor is

\[
g = \det g_{ij} = \frac{1}{3!} \epsilon^{ijk} \epsilon^{lmn} (w_i \cdot w_l)(w_j \cdot w_m)(w_k \cdot w_n), \tag{30}
\]

and the contravariant metric tensor is

\[
g^{ij} = \frac{1}{2g} \epsilon^{ikl} \epsilon^{jmn} (w_k \cdot w_m)(w_l \cdot w_n). \tag{31}
\]

There is a useful identity for the antisymmetrized product of two contravariant tensors, valid in 3 dimensions:
\[ g^{ik}g^{jl} - g^{il}g^{jk} = \epsilon^{ijm} \epsilon^{kln} g_{mn} / g. \]  

(32)

The scalar curvature is obtained as a full contraction:

\[ R = g^{ik}g^{jl} R_{ijkl} = \frac{1}{2}(g^{ik}g^{jl} - g^{il}g^{jk}) R_{ijkl} \]

\[ = \frac{1}{2g^2} \epsilon^{ijkl} \epsilon^{i'j'k'l'} (w_k \cdot w_{k'}) [2g(w_{i'i'} \cdot w_{jj'}) - \epsilon^{plm} \epsilon^{ql'n'} (w_p \cdot w_{i'i'}) (w_q \cdot w_{jj'}) (w_l \cdot w_{m'm'})]. \]  

(33)

Recalling that \( w^\alpha \) is a 6-dimensional vector we can rewrite the scalar curvature in another form:

\[ R = \frac{1}{72g^2} \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta} \epsilon^{\alpha'\beta'\gamma'\delta'\epsilon'\zeta'} \epsilon^{ijkl} \epsilon^{i'j'k'l'} \epsilon^{mnmm'} \epsilon^{\alpha\beta\gamma'\delta'} w^\alpha_i w^\alpha'_i w^\beta_j w^\beta'_j w^\gamma_k w^\gamma'_k w^\delta_l w^\delta'_l w^\epsilon_m w^\zeta_n. \]  

(34)

This form makes it clear that the scalar curvature is zero if \( w^\alpha \) has only three nonzero components, which corresponds to a flat 3-dimensional manifold.

Finally, we would like to point out the Jacobian for the change of integration variables from the set of the lengths of the tetrahedra edges, \( J^2_i \) and \( j^2_i \) given at all lattice sites, to the external coordinates \( w^\alpha \). In the continuum limit this Jacobian is quite simple. It is given by the determinant of a 6 \times 6 matrix composed of the second derivatives:

\[ \prod_x dJ^2_i(x) dj^2_i(x) = \prod_x dw^\alpha(x) \text{Jac}(w), \quad \text{Jac}(w) = \det w^\alpha_{ij}. \]  

(35)

Since \( w^\alpha_{ij} = w^\alpha_{ji} \) there are actually six independent second derivatives. The Jacobian is zero in the degenerate case when the triangulation by tetrahedra can be embedded in less than 6 dimensions.

7 Continuum duality transformation and Bianchi identity

It is instructive at this point to compare the duality transformation on the lattice with that in the continuum theory. The continuum partition function (3) can be written with the help of an additional gaussian integration over the ‘dual field strength’, \( J^a_{ij} \):

\[ Z = \int DA^a_i \exp \int d^3x \left[ -\frac{g^2}{4} J^2_{ij} + \frac{i}{2} J^a_{ij} (\partial_i A^a_j - \partial_j A^a_i + \epsilon^{abc} A^b_i A^c_j) \right]. \]  

(36)

Eq. (36) is usually called the first-order formalism.

In the Abelian case when the \( A_i \) commutator term is absent, integration over \( A_i \) results in the \( \delta \)-function of the Bianchi identity,

\[ \partial_i J_{ij} = 0, \quad \text{or} \quad \epsilon_{ijk} \partial_i J_k = 0, \quad J_k = \frac{1}{2} \epsilon_{ijk} J_{ij}. \]  

(37)

Because of this identity, one can parametrize \( J_k = \partial_k w \), and get for the partition function:
\[ Z_{\text{abel}} = \int D\omega \exp \int d^3x \left[ -\frac{g_3^2}{2}(\partial_k \omega)^2 \right]. \] (38)

It represents a theory of a free massless scalar field \( \omega \). It is in accordance with that in a 3d Abelian theory there is only one physical (transverse) polarization. It is easy to check that gauge-invariant correlation functions of field strengths coincide with those computed in the original formulation.

In the non-Abelian case integration in \( A_{\alpha}^a \) is more complicated, and there is no simple Bianchi identity for \( J_{\alpha}^a_k = (1/2)\epsilon_{ijk}J_{\beta}^a_{ij} \). However, one can formally perform the Gaussian integration over \( A_{\alpha}^a \) resulting in:

\[ Z = \int DJ_{\alpha}^a \det^\frac{2}{3}(J^{-1}) \exp \int d^3x \left[ -\frac{g_3^2}{2}(J_{\alpha}^a)^2 - \frac{i}{2} (\epsilon_{ijm}\partial_j J_{\alpha}^m) (J^{-1})_{ik}^{ab} (\epsilon_{kln}\partial_l J_{\beta}^n) \right] \] (39)

where \( J^{-1} \) is the inverse matrix,

\[ (J^{-1})_{ik}^{ab} \epsilon^{bcd}\epsilon_{klm} J_{m}^d = \delta^{ac}\delta_{il}; \quad \det(J^{-1}) = (\det J_{k}^a)^{-3}. \] (40)

Notice that the second term in the exponent is purely imaginary; the full partition function is real because for each configuration \( J_{\alpha}^a(x) \) there exists a configuration with \( -J_{\alpha}^a(x) \), which adds a complex conjugate expression.

We now turn to the discretized version of the dual theory. As explained above, we need 6 flat dimensions to embed the dual lattice, and we have introduced 6-dimensional momenta \( J^a \), see eq. (25). These momenta apparently satisfy, e.g., the identity (see Fig.7 for notations):

\[ J^a_{\alpha}(x, y, z + \frac{a}{2}) - J^a_{\alpha}(x + \frac{a}{2}, y, z) = w^a(x, y, z + a) - w^a(x + a, y, z) \]

\[ = J^a_{\alpha}(x + a, y, z + \frac{a}{2}) - J^a_{\alpha}(x + \frac{a}{2}, y, z + a), \] (41)

and similarly for other components. This is nothing but a discretized version of the Bianchi identity,

\[ \epsilon_{ijk}\partial_i J^a_{\alpha} = 0, \quad \alpha = 1, ..., 6. \] (42)

Therefore, in 6 dimensions one recovers the simple (flat) form of the Bianchi identity for the dual field strength. One can say that the complicated (nonlinear) form of the usual non-Abelian Bianchi identity is a result of the projection of the flat Bianchi identity onto the curved colour space.

### 8 Wilson loop

In this section we present the Wilson loop in the representation \( j_s \),

\[ W_{j_s} = \frac{1}{2j_s + 1} \text{Tr} P \exp i \oint dx_i A^a_i T^a. \] (43)
in terms of dual variables.

In terms of the original lattice the Wilson loop corresponds to adding a product of the \( D^j(U) \) functions to all links along the loop, with a chain contraction of 'magnetic' indices. Because of these insertions, on links containing the loop one has to integrate over three \( D \)-functions instead of two as for all other links. As a result one gets additional \( 3jm \) symbols along the loop which combine into the new \( 9j \) symbols ascribed to all lattice sites, see Appendix D. For example, the \( 9j \) symbols ascribed to vertices \( a \) and \( b \) are (for notations see Fig.3):

\[
\begin{align*}
\text{"a"} &= \left\{ \begin{array}{ccc}
    j_4 & j_1 & j_{12} \\
    j'_{15} & j_8 & j_{15} \\
    j_{13} & j_1' & j_{14}
\end{array} \right\}, \\
\text{"b"} &= \left\{ \begin{array}{ccc}
    j_2 & j_1 & j_9 \\
    j'_{17} & j_8 & j_{17} \\
    j_{18} & j_1' & j_{16}
\end{array} \right\}.
\end{align*}
\]

The accompanying sign factors are given in Appendix D. Six triades of the \( 9j \) symbol, corresponding to all its rows and columns, satisfy the triangle inequalities.

Contrary to the \( 6j \) symbol the \( 9j \) symbol cannot be represented by a geometrical figure with edges equal to the entries of the \( 9j \) symbol. In addition, the link momenta along the loop split now into pairs: \( j_1 \) and \( j_1' \), \( j_{15} \) and \( j_{15}' \), \( j_{17} \) and \( j_{17}' \), and so on. The 'primed' and 'non-primed' angular momenta satisfy triangle inequalities, with the source \( j_s \) being the third edge of the triangles. If \( j_s \) is an integer, there is always a contribution with \( j_1' = j_1 \) (and so on). If \( j_s \) is a half-integer one has necessarily \( j_1' \neq j_1 \).

Thus, there appears to be a fundamental difference between Wilson loops in integer and half-integer representations. For integer representations one can proceed as in the vacuum case and parametrize the dual lattice sites by the coordinates \( w^\alpha(x) \) related to angular momenta through eq. (25). In the half-integer case one cannot uniquely parametrize the dual lattice by the coordinates \( w^\alpha(x) \). In the presence of the Wilson loop in a half-integer representation the dual space \( w^\alpha \) is not simply connected: there is a infinitely thin cylindrical 'hole' in the dual space along the loop.

### 9 Asymptotics of the 6j symbols

In the continuum limit \( \beta \to \infty, \ J, j \to \infty \) one can replace \( 6j \) symbols by their asymptotics. The asymptotics was ingeniously guessed in a seminal paper by Ponzano and Regge [6] and later on explicitly derived and improved by Schulten and Gordon [7]. The results of these works can be summarized as follows.

First of all one draws a tetrahedron with edges equal to \( j_n + \frac{1}{2} \), where \( j_n \) are the six momenta of a given \( 6j \) symbol. It should be stressed that though four momenta triades satisfy triangle inequalities, the same triades shifted by \( \frac{1}{2} \) need not. In that case the \( 6j \) symbol is said to be 'classically forbidden', and it is exponentially suppressed at large \( j_n \).

If \( j_n \) lie in the 'classically allowed' region, the asymptotics is given by the Ponzano–Regge formula:

\[
\left\{ \begin{array}{ccc}
    j_1 & j_2 & j_3 \\
    j_4 & j_5 & j_6
\end{array} \right\} = \frac{1}{\sqrt{12\pi V(j)}} \cos \left[ \sum_n \left( j_n + \frac{1}{2} \right) \theta_n + \frac{\pi}{4} \right].
\]

(45)
Here \( V(j) \) is the 3-dimensional volume of the tetrahedron and \( \theta_n \) is the dihedral angle in the tetrahedron, corresponding to the edge \( j_n + \frac{1}{2} \). Since we are interested in the large-\( j_n \) limit we shall systematically neglect the shifts by \( \frac{1}{2} \). The tetrahedron volume can be found from the Cayley formula:

\[
V(j)^2 = \frac{1}{288} \begin{vmatrix}
0 & j_2^2 & j_3^2 & j_6^2 & 1 \\
1 & 0 & j_3^2 & j_2^2 & 1 \\
1 & j_6^2 & 0 & j_1^2 & 1 \\
1 & j_2^2 & j_3^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{vmatrix}.
\]

The dihedral angle corresponding, say, to the edge \( j_1 \) can be found from

\[
\cos \theta_1 = \frac{1}{16} \frac{j_1^4 + j_1^2(2j_2^2 - j_3^2 - j_6^2 - j_2^2 - j_3^2) + (j_2^2 - j_3^2)(j_6^2 - j_5^2)}{S(j_1, j_2, j_3) S(j_1, j_5, j_6)},
\]

where

\[
S(j_1, j_2, j_3) = \frac{1}{4} \sqrt{(j_1 + j_2 + j_3)(j_2 + j_3 - j_1)(j_1 - j_2 + j_3)(j_1 + j_2 - j_3)}
\]

is the area of the triangle built on the edges \( j_{1,2,3} \). The dihedral angles are defined such that \( 0 \leq \theta \leq \pi \). Since in section 6 we have defined 6-dimensional angular momenta \( j^\alpha \) whose lengths are the edges of the tetrahedra, we can find the dihedral angles from more simple formulae involving scalar products of momenta in the 6-dimensional space. For example, eq. (47) can be rewritten as

\[
\cos \theta_1 = \frac{(j_1 \cdot j_2)(j_1 \cdot j_6) - j_1^2(j_2 \cdot j_6)}{\sqrt{j_1^2j_2^2 - (j_1 \cdot j_2)^2} \sqrt{j_1^2j_6^2 - (j_1 \cdot j_6)^2}}.
\]

Notice that the angle is defined to be equal to \( \pi \) (not 0!) when the two vectors, \( j_2 \) and \( j_6 \) coincide; it is zero when they point in the opposite directions. We shall use this formula in what follows.

10 Angle defect

The Yang–Mills partition function (20) is a product of many 6\( j \) symbols for each of which we use the asymptotic form (45) in approaching the continuum limit. Each cosine can be written as a half-sum of exponents of imaginary argument. Therefore, we have to consider a sum of a product of many imaginary exponents,

\[
\prod_n^N \cos(\Omega_n) = \frac{1}{2^N} \sum_{\{\epsilon_n = \pm 1\}} \exp \left( i \sum_n \epsilon_n \Omega_n \right),
\]

where \( \Omega_n \) denotes the argument of the cosine in eq. (47), for the \( n^{\text{th}} \) 6\( j \) symbol, and one has to sum over all signs \( \epsilon_n = \pm 1 \).

The expression in the exponent of eq. (50) can be rearranged as follows: We first pick one of the edges of the dual lattice, whose length is a link \( j_l \) or a plaquette \( J_P \), and combine...
all dihedral angles $\theta_n$ related to this edge, as coming from the $n^{th}$ tetrahedron. We then sum over all edges of the dual lattice. Therefore, we can write:

$$\sum_n \epsilon_n \Omega_n = \sum_P J_P \left( \sum_{n=1}^4 \epsilon_n \theta_n(J_P) \right) + \sum_l j_l \left( \sum_{n=1}^6 \epsilon_n \theta_n(j_l) \right), \quad \epsilon_n = \pm 1.$$  \hspace{1cm} (51)

As seen, e.g., from Fig.7, each plaquette $J$ enters four tetrahedra, therefore the corresponding sum over $n$ in eq. (51) goes from 1 to 4. Each link $j$ enters six tetrahedra, therefore in this case the sum is over six dihedral angles $\theta_n(j)$, with appropriate signs $\epsilon_n$.

Let us consider the contribution to eq. (50) when all signs $\epsilon_n = +1$, and let us for a moment assume that the dual lattice spans a 3-dimensional Euclidean manifold. The sum of the dihedral angles about an edge is then equal to $4\pi - 2\pi = 2\pi$ in case of summing over four tetrahedra, and equal to $6\pi - 2\pi = 4\pi$ in case of summing over six tetrahedra. In the first case we get $\exp(2\pi i J) = (-1)^{2J}$; in the second case we get $\exp(4\pi ij) = (-1)^{4j} = 1$. Notice that the sign factor $(-1)^{2J}$ compensates exactly the same factor in the partition function (20). We conclude that, if the configuration of the momenta is ‘flat’, there exists a contribution to the sum (50) that does not oscillate with varying $J$’s and $j$’s. In fact, there are exactly two such contributions corresponding to taking all signs $\epsilon_n = \pm 1$ simultaneously. Contributions of any other choice of the signs are oscillating fast at large $J$’s and $j$’s, and thus die out in the continuum limit.

A generic configuration of momenta cannot be embedded into a flat 3-dimensional space, however. Therefore, the sum of dihedral angles about the edges $J$ and $j$ will, generally, differ from $2\pi$ and $4\pi$, respectively. These differences are sometimes called angle deficiencies or angle defects (we shall use the second term). Let us denote them:

$$\Theta(J) = \sum_{n=1}^4 \theta_n(J) - 2\pi,$$  \hspace{1cm} (52)

$$\Theta(j) = \sum_{n=1}^6 \theta_n(j) - 4\pi.$$  \hspace{1cm} (53)

Our task is to point out contributions to eq. (50) that survive the continuum limit in a general case when the dual lattice is a curved 3-dimensional manifold. To be more precise, we have to consider the sum of all momenta on the lattice times their angle defects, $\Theta$, and to find the contribution of the order of $a^3$ to this exponent, where $a$ is the lattice spacing. The $O(a^3)$ order is needed to compensate for the $1/a^3$ factor arising as one goes from summation over the lattice points to integration over the 3-dimensional space.

In the continuum limit we assume that the momenta are given by the gradients of a 6-component function $w^\alpha(x)$ having the meaning of the 6-dimensional coordinates of the dual lattice sites, see eq. (25). If we restrict ourselves to the first terms in the gradient expansion in eq. (25), the momenta will be expressed only through three vectors, $\partial_x w^\alpha$, $\partial_y w^\alpha$ and $\partial_z w^\alpha$. Three vectors define a flat 3-dimensional space; therefore, the angle defects $\Theta$ are
zero in the first-derivative approximation. To get a non-zero angle defect it is necessary to expand the momenta in eq. (25) up to the second derivatives of $w^\alpha$. We shall see that it is also sufficient in three dimensions.

Since the angle defects $\Theta$’s vanish if $j$’s are taken to the first approximation of the gradient expansion, it means that the expansion of $\Theta$’s starts from terms linear in the lattice spacing $a$. According to eq. (25) the expansion of the momenta also starts from terms linear in $a$. Therefore, one can expect that the expansion of the exponent in eq. (54) starts from the $O(a^2)$ terms. Were that so, the configuration would be too ‘ultraviolet’ and would not survive the continuum limit. Fortunately, there appears to be an exact cancellation of all $O(a^2)$ terms in the sum over several neighbour edges of the dual lattice, so that the exponent in eq. (54) proves to be finite in the continuum limit.

We next embark a rather tedious enterprise of calculating the angle defects about six plaquette $J$’s in a cube (each entering four tetrahedra), and about twelve link $j$’s being edges of that cube (each involved in six tetrahedra, see section 5). Unluckily, it seems that it is the minimal elementary group that is being repeated through the lattice. It means that we have to compute as much as $6 \cdot 4 + 12 \cdot 6 = 96$ dihedral angles, expressing them through the first and second gradients of the 6-component function $w^\alpha$ using eqs. (25, 49). This formidable calculation has been performed by heavily exploiting Mathematica. The intermediate results are very lengthy and we do not present them here. However, the final result is beautiful. From a direct calculation we obtain:

$$
\exp i \left[ \sum_P J_P \Theta (J_P) + \sum_l j_l \Theta (j_l) \right] = \exp i \sum_{ \text{points } x } a^3 \frac{1}{2} \sqrt{g(w)} \ R(w) \\
= \exp \frac{i}{2} \int d^3x \sqrt{g(w)} \ R(w), \quad (55)
$$

where $g$ is the determinant of the induced metric tensor as given by eq. (30), and $R$ is the corresponding scalar curvature given by eq. (33). Actually, we obtain the expression for the l.h.s. of eq. (55) in the form of eq. (33) (written in components, 384 terms!) from where we recognize that we are dealing with the scalar curvature.

In fact this result is a concrete realization of a more general theory developed many years ago by Regge [3, 8]. In these papers it was shown that the l.h.s. of eq. (55) should be equal to its r.h.s. for any simplicial triangulation, provided it has a smooth continuum limit. No relation of the scalar curvature $R$ to any concrete triangulation was given, though. We feel that it is the first time that this ingenious relation has been derived explicitly for a concrete triangulation, and the continuum limit shown to exist.

### 11 Full partition function

Having dealt with the $6j$ symbols of the partition function (20) we now turn to the weight factors $T_J(\beta)$. According to eq. (11) at large $\beta$ and $J$ we have:

$$
\prod_{\text{plaquettes}} T_J(\beta) = \exp \left[ - \sum_{\text{plaquettes}} \frac{2J^2}{\beta} \right]
$$
\[ Z = \int D\omega^\alpha(x) \text{Jac}(\omega) g(\omega)^{-\frac{5}{4}} \exp \int d^3x \left[-\frac{g_3^2}{2} g_{ii} + \frac{i}{2} \sqrt{g} R \right]. \] (57)

The second term is the Einstein–Hilbert action with a purely imaginary Newton constant; it is invariant under global 6-dimensional rotations of the external coordinates \( \omega^\alpha(x) \) and, more important, under local 3-dimensional diffeomorphisms \( \omega^\alpha(x) \to \omega^\alpha(x'(x)) \).

The first term in eq. (57) can be viewed as a ‘matter’ source,

\[ -\frac{g_3^2}{2} \int d^3x g_{ii} = -\frac{g_3^2}{2} \int d^3x \sqrt{g} T^{ij} g_{ij}, \] (58)

with the stress-energy tensor \( T^{ij} \sqrt{g} = \delta^{ij} \) violating the invariance under diffeomorphisms. Since it is homogeneous in space it can be called the ‘ether’.

The functional measure in eq. (57) arises from two sources. One factor is the Jacobian for the change of variables from the tetrahedra edges \( J_i \)'s and \( j_i \)'s to \( \omega^\alpha \), see eq. (35). The other factor arises from the tetrahedra volumes in the asymptotics of the 6j symbols (13). In the continuum limit the tetrahedron volume can be written as \( V(j) \sim \sqrt{g} \), and there are 5 tetrahedra per lattice site, see section 3.

Once the partition function is written in covariant terms one can forget the origin of the external coordinates \( \omega^\alpha \) (as the coordinates of the dual lattice) and consider the metric tensor \( g_{ij} \) as independent dynamical variables over which one integrates in eq. (57). The Jacobian for this change of variables can be easily worked out: in fact it is the inverse of \( \text{Jac}(\omega) \) introduced in eq. (35). As a result we get the integration measure for the partition function (57):

\[ \int Dg_{ij} g^{-\frac{5}{4}}, \] instead of \[ \int Dg_{ij} g^{-2}, \] (59)

which would be the invariant measure in 3d. We shall get an independent check of the power \(-\frac{5}{4}\) in the next section. However, it is anyhow a local counterterm not affecting the physics.

We stress that the partition function written in terms of the metric tensor does not contain explicit colour degrees of freedom. Nevertheless, implicitly the theory does contain three gluons at short distances.

Indeed, let us make a simple dimensional analysis of eq. (57). The dimension of the first term in eq. (57) is \( g_3^2 \partial^2 w^2 \) (we are just counting the number of derivatives and the overall power of \( w \)); the dimension of the second term is \( \partial^3 w^1 \). At short distances where quantum fluctuations of \( w^\alpha(x) \) vary fast, the second term dominates the first one. Meanwhile, the second term is a fast-oscillating functional at nonzero \( R \). Therefore, the leading contribution to the functional integral arises from zero-curvature fluctuations of \( w^\alpha \), that is essentially from the 3-dimensional \( w^\alpha \). Being plugged into the first term, the three components of \( w^\alpha \) describe three massless scalar fields. These fields correspond to three gluons of \( SU(2) \) with one physical (transverse) polarization. It should be paralleled to eq. (38) for free
electrodynamics. This is the correct result for the non-Abelian theory at short distances in three dimensions.

At large distances or at low field momenta the dominant term is, on the contrary, the first one as it has less derivatives. It describes six (instead of three) massless scalar degrees of freedom. It is the correct number of gauge-invariant degrees of freedom in the $SU(2)$ theory. However, the theory remains strongly nonlinear, and it is not clear so far whether massless modes survive in the physical spectrum.

12 Quantum gravity from first-order continuum formalism

In this section we give another derivation of the partition function (57) directly in the continuum theory starting from the first-order formalism, see section 7. We shall show that the two terms in the exponent of eq. (36) are in fact in one-to-one correspondence with the two terms in eq. (57), and that the integration measure coincides with that of eq. (59).

Actually, it has been already derived in the previous section that the first terms of eqs. (36) and (57) are equal:

$$S_1 = -\frac{g^2}{2} \int d^3 x (J_i^a)^2 = -\frac{g^2}{2} \int d^3 x \partial_i \omega^a \partial_i \omega^a = -\frac{g^2}{2} \int d^3 x \ g_{ii}. \quad (60)$$

Let us derive a less trivial relation for the second terms:

$$S_2 = \frac{i}{2} \int d^3 x \ e^{ijk} J_i^a (\partial_j A_k^a - \partial_k A_j^a + \epsilon_{abc} A_c^b A_k^c) = \frac{i}{2} \int d^3 x \sqrt{g} R. \quad (61)$$

This derivation will be done in two steps. We shall first show, following Witten [9], that the l.h.s. of eq. (61) can be presented as a certain Chern–Simons term. Second, we shall show that it is formally equal to the Einstein–Hilbert action. A subtle question about the integration measure will be discussed at the end of the section.

The l.h.s. of eq. (61) is apparently invariant under ordinary gauge transformations:

$$\delta A_i^a = -\partial_i \delta^a \omega^b + \epsilon_{abc} \omega^b A_i^c = -D_i^{ab}(A) \omega^b, \quad \delta J_i^a = \epsilon_{abc} \omega^b J_i^c, \quad (62)$$

where $D_i^{ab}(A) = \partial_i \delta^{ab} + \epsilon_{abc} A_c^b$ is the covariant derivative.

Less evident, it is also invariant under the following local transformation:

$$\delta J_i^a = -\partial_i \rho^a - \epsilon_{abc} \rho^b A_i^c, \quad \delta A_i^a = 0. \quad (63)$$

Indeed, after integrating by parts we obtain the following variation of the action:

$$\delta S_2 = \frac{i}{2} \int d^3 x \rho^a \epsilon_{ijk} D_i^{ab}(A) F_{jk}^b, \quad F_{jk}^b = \partial_j A_k^b - \partial_k A_j^b + \epsilon_{bcd} A_j^c A_k^d. \quad (64)$$

This variation is zero owing to the Bianchi identity, $\epsilon_{ijk} D_i^{ab} F_{jk}^b = 0$.

The two transformations combined form a 6-parameter gauged Poincaré group, called $ISO(3)$. Indeed, let us introduce three ‘momenta’ generators $P_i$ and three ‘angular momenta’ generators $L_i$ satisfying the Poincaré algebra,
\[ [P_a P_b] = 0, \quad [L_a L_b] = i\epsilon_{abc} L_c, \quad [L_a P_b] = i\epsilon_{abc} P_c. \]  

We next introduce a 6-component vector field \( \hat{B}_i \):

\[ \hat{B}_i = J^a_i P_a + A^a_i L_a \equiv B^a_i T^a, \quad T^a = \begin{cases} P_a, & \alpha = a = 1, 2, 3, \\ L_a, & \alpha = 3 + a = 4, 5, 6. \end{cases} \]  

Its gauge transformation has the standard form:

\[ \hat{B}_i \to S^{-1} \hat{B}_i S + iS^{-1} \partial_i S, \quad S = \exp[i\rho^a P_a + i\omega^a L_a]. \]  

Using the Poincare algebra (65) it is easy to check that its infinitesimal form coincides with eqs. (62, 63).

Since the l.h.s. of eq. (61) is invariant under these 6-parameter transformations, it can be rewritten in an explicitly \( ISO(3) \)-invariant form. To that end we notice that the invariant tensor of this group is

\[ M_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  

where “1” is a unit 3 \times 3 matrix. This matrix defines a scalar product, \( B^\alpha M_{\alpha\beta} C^\beta \), which is invariant under global (x-independent) transformations (67). With the help of this invariant tensor we build a local gauge-invariant action having the form of the Chern–Simons term:

\[ S_2 = \frac{i}{2} \int d^3 x \epsilon^{ijk} M_{\alpha\beta} B^\alpha_i \left( \partial_j B^\beta_k + \frac{1}{3} F^\beta_{\gamma\beta} B^\gamma_j B^\delta_k \right), \]  

where \( F^\alpha_{\beta\gamma} = -F^\alpha_{\gamma\beta} \) are the \( ISO(3) \) structure constants. Explicitly,

\[ F^\alpha_{bc} = 0, \quad F^\alpha_{3+b, c} = \epsilon_{abc} \]  

\[ F^{3+a}_{3+b, 3+c} = F^a_{3+b, 3+c} = 0, \quad F^{3+a}_{3+b, 3+c} = \epsilon_{abc}. \]  

Using the definition (66) it is easy to check that eq. (68) coincides with the l.h.s. of eq. (61), however it is explicitly invariant under the 6-parameter gauge transformation (67).

Eq. (69) has the form of the Chern–Simons term in a Yang–Mills theory. Though our derivation above is for the gauge group \( SU(2) \) it is trivially generalized to any Lie group: to that end it is sufficient to replace the \( SU(2) \) structure constants \( \epsilon_{abc} \) by the structure constants \( f_{abc} \) of the gauge group under consideration. We also note in passing that in four dimensions the mixed \( iJ^a_{\mu\nu} F^a_{\mu\nu}(A) \) term of the first-order formalism also possesses an additional local symmetry. To unveil it, it is sufficient to replace the scalar parameter \( \rho^a \) in the transformation (68) by a 4-vector parameter \( \rho^a_{\mu} \): the invariance is again due to the Bianchi identity, this time in four dimensions.

The second step in the derivation is more standard. Introducing the dreibein \( e^a_i \), \( a = 1, 2, 3 \), satisfying the condition \( e^a_i e^{bi} = \delta^{ab} \), so that the metric tensor is \( g_{ij} = e^a_i e^a_j \), and the connection

\[ \omega^a_{i}^{\mu} = \frac{1}{2} e^a_k (\partial_i e^b_k - \partial_k e^b_i) - \frac{1}{2} e^b_k (\partial_i e^a_k - \partial_k e^a_i) - \frac{1}{2} e^a_k e^c_i (\partial_k e^c_i - \partial_i e^c_k), \]  

22
one can identically rewrite $\sqrt{gR}$ as

$$\sqrt{gR} = \frac{1}{2} \epsilon^{ijk} e_i^a \left( \partial_j \omega_k^a - \partial_k \omega_j^a + \epsilon_{abc} \omega_j^b \omega_k^c \right)$$

(72)

where $\omega_{ai} = \omega_i^a = \frac{1}{2} \epsilon_{abc} \omega_j^b \omega_k^c$. Finally, one notices that, if one makes an identification of the dreibein with the dual field strength, $e_i^a = J_i^a$, and of the connection with the Yang–Mills potential, $\omega_i^a = A_i^a$, then eq. (72) takes exactly the form of the l.h.s. of eq. (61). This parallel has been first noticed in ref. [10].

There is a subtle point in this formal derivation, however. The use of the first-order formalism implies that one integrates both over $J_i^a$ and over $A_i^a$ (see eq. (36)) or, equivalently, over the dreibein and over the connection, independently. Meanwhile, the use of the Einstein quantum gravity implies that the connection is rigidly related to the dreibein via eq. (71), moreover, we have explicitly used this relation in the above derivation. In ref. [9] Witten has presented arguments that one can, nevertheless, integrate over the connection as independent variable. However, the arguments rely upon the use of the equations of motion (one of which is the relation (71)), and that might be dangerous in full quantum field theory.

The present paper gives a different kind of argument that the two approaches are in fact equivalent. We start with the Yang–Mills partition function. On one hand it can be presented in the first-order formalism where one integrates independently over $J_i^a$ (the dreibein) and over $A_i^a$ (the connection). On the other hand we have shown that the Yang–Mills theory is equivalent to quantum gravity where one integrates over the external coordinates $w^\alpha$, or over the metric tensor, or over the dreibein only.

Since pure gravity can be rewritten as a Chern–Simons term (69), it is actually a topological field theory [4], with no real propagating particles. It is the ‘ether’ term that violates the invariance under diffeomorphisms and restores the propagation of gluons, as it should be in the Yang–Mills theory, see the end of the previous section.

Finally, we would like to remark that the integration measure (59) could be anticipated from the first-order formalism as well. Indeed, integrating in eq. (39) over $A_i^a$ one gets eq. (44), where the integration measure over the dreibein is $(\det J)^{-3/2} \sim g^{-3/4}$. The Jacobian for the change of variables from the dreibein to the metric tensor is $de_i^a \sim dg_{ij} g^{-1/2}$. Adding the powers we obtain: $-3^3/4 - 1/2 = -5/4$, as in eq. (59).

13 Conclusions and outlook

In this paper, we have studied the dual transformation of the $SU(2)$ Yang–Mills theory in 3 dimensions, both from the continuum and lattice points of view.

On the lattice, one can introduce dual variables being the angular momenta of the plaquettes ($J_i^a$) supplemented by those associated with the links ($j_i^a$). The partition function can be identically rewritten as a product of $6j$ symbols made of those angular momenta. A Wilson loop corresponds to taking a product of $9j$ symbols replacing the $6j$ symbols along the loop. One can construct a dual lattice made of tetrahedra whose edges have the lengths equal to $J_i^a$ and $j_i^a$; the tetrahedra span a 3d curved manifold which can be embedded into a flat 6d space.

In the continuum limit the angular momenta are large, and we have introduced continuum 6d Euclidean external coordinates $w^\alpha(x)$ to describe the curved dual space. The Bianchi
condition for the Yang–Mills field strength has been shown to be trivially soluble in flat six dimensions.

At large angular momenta one can use the asymptotics of the $6j$ symbols, given by Ponzano and Regge. Using a specific simplicial triangulation of the dual space (as dictated by the original lattice) we have shown that the product of the $6j$ symbols does have a smooth continuum limit which appears to be the Einstein–Hilbert action, with the metric tensor $g_{ij}$ and the scalar curvature $R$ expressed through the flat external coordinates $u^a(x)$. This result cannot be considered as particularly new (it is the cornerstone of the Regge’s simplicial gravity), however, to our best knowledge it is the first time that the result has been explicitly derived from a concrete triangulation of the curved space, and the continuum limit shown to exist. We have also found the integration measure for the continuum limit.

The continuum Yang–Mills partition function can be rewritten as a quantum gravity theory but with an ‘ether’ term violating the invariance in respect to general coordinate transformations or diffeomorphisms. This term, however, revives gluons at short distances, in contrast to the topological pure gravity theory where no particles propagate.

The presentation of the Yang–Mills theory in a quantum gravity form is explicitly colour gauge-invariant since the metric tensor of the dual space is colour-neutral. We have, thus, formulated the Yang–Mills theory solely in terms of colourless ‘glueball’ degrees of freedom. It turns out to be an interacting theory of six massless scalar fields. Nevertheless, at small distances it correctly reproduces the propagation of gluons. It is not clear to us at the moment how to proceed best in order to reveal its large-distance behaviour. Let us indicate a few possibilities.

One possibility is to exploit the fact that the pure quantum gravity theory is topological, therefore essentially a free theory. One can try to make a perturbative expansion in $g_3^2$ about it.

Another possibility is to make use of the fact that the Chern-Simons term can be obtained from integrating over heavy fermions, in this case belonging to some $ISO(3)$ representation. The subsequent integration over bosonic fields $A_i, J_i$ is trivial since there is no kinetic energy term for those fields: the result would be a local four-fermion theory with infinitely heavy fermions; it might be soluble, at least in the large-$N_c$ limit.

Probably the most promising possibility is to pursue the analogy with and methods of quantum gravity. One can average eq. (57) over 3d diffeomorphisms: the second term is invariant, the first term is not. Integrating the first term over diffeomorphisms produces diffeomorphism-invariant effective action containing growing powers in the curvature. The effective action may lead to a nonzero v.e.v. of the scalar curvature, and that may yield a mass gap for the diffeomorphism-non-invariant correlation functions, like the correlation functions of $F_{\mu\nu}$.

There are several other tasks for the future, lying on the surface. First, it would be interesting to generalize the present approach to colour groups other than $SU(2)$. In view of the sad fact that the theory of the “$6j$ symbols” for higher Lie groups is not too developed it will be probably difficult to make a straightforward generalization of the lattice formulation. A more promising approach would be to start from the first-order formalism, the more so that the wide local symmetry revealed in section 12 can be directly generalized to any Lie

\[ A \text{ somewhat similar line was developed in ref. [1] for the } 3+1 \text{ dimensional Yang–Mills theory in the Hamiltonian approach; see also ref. [2].} \]
group. Second, it would be interesting to make a transformation similar to that of this paper in \( d = 4 \). The lattice 6\( j \) symbols have been known for a while in this case \[4\] (for the SU(2) colour), however it again seems that the first-order formalism is a more promising start, due to the additional gauge symmetry noticed in section 12.

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**Appendix A. D-functions, 3jm, 6j and 9j symbols**

Wigner D-functions are eigenfunctions of the square of the angular momentum operator (written in terms of, say, three Euler angles \( \alpha, \beta, \gamma \)),

\[
J^2 D_{mn}^J(\alpha, \beta, \gamma) = J(J + 1) D_{mn}^J(\alpha, \beta, \gamma), \quad J = 0, \frac{1}{2}, 1, \frac{3}{2}, ..., \quad -J \leq m, n \leq +J, \quad (73)
\]

and can be said to be eigenfunctions of a spherical top; they are \((2J + 1)^2\)-fold degenerate. The ‘magnetic’ quantum numbers \( m, n \) have the meaning of the projections of the angular momentum of a spherical top on the third axes in the ‘body-fixed’ and ‘lab’ frames. One can parametrize a \( 2 \times 2 \) unitary matrix by Euler angles as

\[
U = \exp(i \alpha \tau^3) \exp(i \beta \tau^2) \exp(i \gamma \tau^3). \quad (74)
\]

It is convenient to use the unitary matrix \( U \) as a formal argument of the D-functions. Their main properties are:

- **Multiplication law:**
  \[
  D^J_{kl}(U_1 U_2) = D^J_{km}(U_1) D^J_{ml}(U_2) \quad \text{(summation over repeated indices understood).} \quad (75)
  \]

- **Unitarity:**
  \[
  D^J_{kl}(U^\dagger) = \left(D^J_{lk}(U)\right)^* \quad \text{("*" denotes complex conjugate).} \quad (76)
  \]

- **Phase condition:**
  \[
  \left(D^J_{lk}(U)\right)^* = (-1)^{l-k} D^{J}_{-l,-k}(U), \quad D^J_{kl}(1) = \delta^J_{kl}. \quad (77)
  \]

- **Orthogonality and normalization:**
  \[
  \int dU D^J_{kl}(U^\dagger) D^J_{mn}(U) = \frac{1}{2J_1 + 1} \delta_{J_1 J_2} \delta_{kn} \delta_{lm}. \quad (78)
  \]
Integration here is over the Haar measure:

\[
\int dU... = \int d(SU)... = \int d(US)...; \quad \int dU = 1.
\]  

(79)

- Completeness (the \(\delta\)-function is understood in the Haar measure sense):

\[
\delta(U, V) = \sum_J (2J + 1) D_{kl}^J(U^\dagger) D_{lk}^J(V).
\]  

(80)

- Matrix element:

\[
\int dU D_{a_1 b_1}^{J_1}(U) D_{a_2 b_2}^{J_2}(U) D_{a_3 b_3}^{J_3}(U) = \begin{pmatrix} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{pmatrix},
\]  

where (...) denote 3jm symbols.

(81)

- Decomposition of a direct product of irreps:

\[
D_{a_1 b_1}^{J_1}(U) D_{a_2 b_2}^{J_2}(U) = \sum_J (2J + 1) \begin{pmatrix} J & J_1 & J_2 \\ -c & a_1 & a_2 \end{pmatrix} \begin{pmatrix} J & J_1 & J_2 \\ -d & b_1 & b_2 \end{pmatrix} (-1)^{d-c} D_{cd}^J(U).
\]  

(82)

The last two factors may be replaced by \(D_{d,-c}^J(U^\dagger)\) using eq. (77).

The 3jm symbols are symmetric under cyclic permutations of the columns. An interchange of two columns gives a sign factor:

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ l & k & m \end{pmatrix}, \quad \text{etc.}
\]  

(83)

If one changes the signs of all ‘magnetic’ quantum numbers or projections, the 3jm symbol also gets a sign factor:

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -k & -l & -m \end{pmatrix}.
\]  

(84)

A “practical” definition of the 6j symbol \{\ldots\} is via a contraction over projections in three 3jm symbols:

\[
\sum_{klm} (-1)^{j_4-k+j_5-l+j_6-m} \begin{pmatrix} j_5 & j_1 & j_6 \\ l & p & -m \end{pmatrix} \begin{pmatrix} j_6 & j_2 & j_1 \\ m & q & -k \end{pmatrix} \begin{pmatrix} j_4 & j_3 & j_5 \\ k & r & -l \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ -p & -q & -r \end{pmatrix} \{\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} \}
\]  

(85)

The summation over projections \(k, l, m\) is such that \(p = m - l, q = k - m\) and \(r = l - k\) are kept fixed.

Another definition of the 6j symbol is via the full contraction of projections in four 3jm symbols:
\[ \sum_{klmnop} (-1)^{j_1+n+j_5+o+j_6+p} \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix} \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & o & m \end{pmatrix} = \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_4 \\ j_6 \\ j_3 \end{array} \middle| \begin{array}{c} j_3 \\ j_2 \\ j_6 \\ j_3 \\ j_3 \end{array} \right\} , \quad (86) \]

Since the three \( j \)'s of any \( 3jm \) symbol satisfy the triangle inequalities, e.g. \( |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \), etc., the following four triades of the \( 6j \) symbols have to satisfy the triangle inequalities: \((j_1,j_2,j_3)\), \((j_1,j_5,j_6)\), \((j_2,j_4,j_6)\) and \((j_3,j_4,j_5)\); otherwise, the \( 6j \) symbol is zero.

The \( 6j \) symbols are symmetric under permutation of any of two columns and under interchange of the upper and lower arguments simultaneously in any two columns, e.g.,

\[ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \end{array} = \begin{array}{c} j_1 \\ j_3 \\ j_2 \\ j_4 \\ j_6 \\ j_5 \end{array} = \begin{array}{c} j_1 \\ j_2 \\ j_6 \\ j_1 \\ j_5 \\ j_3 \end{array} , \quad \text{etc.} \quad (87) \]

A full contraction of six \( 3jm \) symbols yields the \( 9j \) symbol:

\[ \sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix} \times \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix} = \left\{ \begin{array}{c} j_1 \\ j_2 \\ j_3 \\ j_4 \\ j_5 \\ j_6 \\ j_7 \\ j_8 \\ j_9 \end{array} \middle| \begin{array}{c} j_3 \\ j_2 \\ j_6 \\ j_3 \\ j_3 \\ j_3 \\ j_3 \\ j_3 \\ j_3 \end{array} \right\} . \quad (88) \]

\( 9j \) symbol is symmetric under transposition and under even permutations of rows and columns; under odd permutations it acquires a sign factor \((-1)^{j_1+\ldots+j_9}\). As follows from the definition, six momenta triades corresponding to the rows and columns of the \( 9j \) symbol satisfy triangle inequalities.

A convenient reference book on \( D \)-functions, \( 3jm \), \( 6j \) and \( 9j \) symbols is ref. [1] from where we have borrowed the definitions.

**Appendix B. 6j symbols in an ‘even’ cube**

In this Appendix we make the decomposition of two plaquettes \( D^J \)-functions into a sum of single \( D^J \)-functions labelled by link angular momenta \( j \). Then we assemble the arising \( 3jm \) symbols into \( 6j \) symbols attached to the corners of the even cubes. The notations are given in Fig.2.

We find it convenient (though not necessary) to write the decomposition for the pairs containing \( U_{1,4,12,6,7,10} \) (these are links sitting at lower left and upper right corners of the cube) in terms of \( D(U) \), and the rest in terms of \( D(U^+) \).

Exploiting eq. (82) of Appendix A we get:

\[ D_{A_{a}b}^{J_{A}}(U_{1}) \times D_{b,a}^{J_{B}}(U_{1}^+) = (-1)^{j_a-j_b} \sum_{j_1} (2j_1 + 1) \begin{pmatrix} j_1 & j_A & j_B \\ -o_a & i_a & -j_a \end{pmatrix} \begin{pmatrix} j_1 & j_A & j_B \\ -o_b & i_b & -j_b \end{pmatrix} (-1)^{o_b-o_a} D_{a_o b_o}^{j_{a}}(U_{1}) , \]

27
\[-1 \sum_{j_2} (2j_2 + 1) \left( \begin{array}{ccc} j_2 & J_A & J_C \\ -p_b & i_b & -k_b \end{array} \right) \left( \begin{array}{ccc} j_2 & J_A & J_C \\ -p_c & i_c & -k_c \end{array} \right) D_{-p_c,-p_b}(U_2^\dagger), \]

\[-1 \sum_{j_3} (2j_3 + 1) \left( \begin{array}{ccc} j_3 & J_D & J_A \\ -q_d & l_d & -i_d \end{array} \right) \left( \begin{array}{ccc} j_3 & J_D & J_A \\ -q_c & l_c & -i_c \end{array} \right) D_{-q_c,-q_d}(U_3^\dagger), \]

\[-1 \sum_{j_4} (2j_4 + 1) \left( \begin{array}{ccc} j_4 & J_E & J_A \\ -r_a & m_a & -i_a \end{array} \right) \left( \begin{array}{ccc} j_4 & J_E & J_A \\ -r_d & m_d & -i_d \end{array} \right) (-1)^{r_a-r_d} D_{r_a r_d}(U_4), \]

\[-1 \sum_{j_5} (2j_5 + 1) \left( \begin{array}{ccc} j_5 & J_B & J_F \\ -s_e & j_e & -n_e \end{array} \right) \left( \begin{array}{ccc} j_5 & J_B & J_F \\ -s_f & j_f & -n_f \end{array} \right) D_{s_f,-s_e}(U_5), \]

\[-1 \sum_{j_6} (2j_6 + 1) \left( \begin{array}{ccc} j_6 & J_C & J_F \\ -t_f & k_f & -n_f \end{array} \right) \left( \begin{array}{ccc} j_6 & J_C & J_F \\ -t_g & k_g & -n_g \end{array} \right) (-1)^{t_g-t_f} D_{t_f t_g}(U_6), \]

\[-1 \sum_{j_7} (2j_7 + 1) \left( \begin{array}{ccc} j_7 & J_F & J_D \\ -u_h & n_h & -l_h \end{array} \right) \left( \begin{array}{ccc} j_7 & J_F & J_D \\ -u_g & n_g & -l_g \end{array} \right) (-1)^{u_g-u_h} D_{u_h u_g}(U_7), \]

\[-1 \sum_{j_8} (2j_8 + 1) \left( \begin{array}{ccc} j_8 & J_F & J_E \\ -v_e & n_e & -m_e \end{array} \right) \left( \begin{array}{ccc} j_8 & J_F & J_E \\ -v_h & n_h & -m_h \end{array} \right) D_{v_h,v_e}(U_8), \]

\[-1 \sum_{j_9} (2j_9 + 1) \left( \begin{array}{ccc} j_9 & J_C & J_B \\ -w_b & k_b & -j_b \end{array} \right) \left( \begin{array}{ccc} j_9 & J_C & J_B \\ -w_f & k_f & -j_f \end{array} \right) D_{w_f,w_b}(U_9), \]

\[-1 \sum_{j_{10}} (2j_{10} + 1) \left( \begin{array}{ccc} j_{10} & J_D & J_C \\ -x_c & l_c & -k_c \end{array} \right) \left( \begin{array}{ccc} j_{10} & J_D & J_C \\ -x_g & l_g & -k_g \end{array} \right) (-1)^{x_g-x_c} D_{x_g x_c}(U_{10}), \]

\[-1 \sum_{j_{11}} (2j_{11} + 1) \left( \begin{array}{ccc} j_{11} & J_E & J_D \\ -y_d & m_d & -l_d \end{array} \right) \left( \begin{array}{ccc} j_{11} & J_E & J_D \\ -y_h & m_h & -l_h \end{array} \right) (-1)^{y_h-y_d} D_{y_h,y_d}(U_{11}), \]

\[-1 \sum_{j_{12}} (2j_{12} + 1) \left( \begin{array}{ccc} j_{12} & J_F & J_E \\ -y_e & m_e & -l_e \end{array} \right) \left( \begin{array}{ccc} j_{12} & J_F & J_E \\ -y_f & m_f & -l_f \end{array} \right) (-1)^{y_f-y_e} D_{y_f,y_e}(U_{12}), \]
\[= (-1)^{m_a-m_e} \sum_{j_{12}} (2j_{12} + 1) \left( \begin{array}{ccc} j_{12} & J_B & J_E \\ -z_a & j_a & -m_a \end{array} \right) \left( \begin{array}{ccc} j_{12} & J_B & J_E \\ -z_e & j_e & -m_e \end{array} \right) (-1)^{z_e-z_a} D_{z_a z_e}^{j_{12}}(U_{12}). \]  

(89)

We now combine together 3jm symbols related to the same vertices (they are marked by appropriate indices of the projections \(a, b, c, d, r, f, g, h\)), three 3jm symbols for each vertex, together with appropriate sign factors. The three 3jm symbols per vertex combine into 6j symbols, one for each vertex of the cube.

**Vertex a**

Related to vertex a are the factors

\[\sum_{i_a, j_a, m_a} (-1)^{j_a+j_a+m_a-o_a-r_a-z_a} \left( \begin{array}{ccc} j_1 & J_A & J_B \\ -o_a & i_a & -j_a \end{array} \right) \left( \begin{array}{ccc} j_4 & J_E & J_A \\ -r_a & m_a & -i_a \end{array} \right) \left( \begin{array}{ccc} j_{12} & J_B & J_E \\ -z_a & j_a & -m_a \end{array} \right).\]

[we use \(o_a + r_a + z_a = 0\), make cyclic permutations in all 3jm symbols, and change the summation indices \(i, j, m \rightarrow -i, -j, -m\)]

\[= \sum_{i_a, j_a, m_a} (-1)^{-i_a-j_a-m_a} \left( \begin{array}{ccc} J_B & j_1 & J_A \\ j_a & -o_a & -i_a \end{array} \right) \left( \begin{array}{ccc} J_A & j_4 & J_E \\ i_a & -r_a & -m_a \end{array} \right) \left( \begin{array}{ccc} J_E & j_{12} & J_B \\ m_a & -z_a & -j_a \end{array} \right).\]

\[= (-1)^{-J_A-J_B-J_E} \left( \begin{array}{ccc} j_1 & j_4 & j_{12} \\ o_a & r_a & z_a \end{array} \right) \left( \begin{array}{ccc} j_1 & j_4 & j_{12} \\ J_E & J_B & J_A \end{array} \right).\]  

(90)

In the last transformation the definition of the 6j symbol \((85)\) has been used.

**Vertex b**

Related to vertex b are the factors

\[\sum_{i_b, j_b, k_b} (-1)^{o_b+k_b} \bigg|_{o_a=i_b-j_b} \left( \begin{array}{ccc} j_1 & J_A & J_B \\ -o_b & i_b & -j_b \end{array} \right) \left( \begin{array}{ccc} j_2 & J_A & J_C \\ -p_b & i_b & -k_b \end{array} \right) \left( \begin{array}{ccc} j_9 & J_C & J_B \\ -w_b & k_b & -j_b \end{array} \right).\]

[we interchange the first two columns in the first 3jm symbol and change the signs of all its projections; it doesn’t change the sign of the 3jm’s. Also, we make cyclic permutations of the last two 3jm symbols, and change the summation indices \(i, j, k \rightarrow -i, -j, -k\)]

\[= \sum_{i, j, k} (-1)^{-i+j-k} \left[ \text{insert } 1 = (-1)^{2J_B-2j} \right] \]

\[\cdot \left( \begin{array}{ccc} J_A & j_1 & J_B \\ i & o_b & -j \end{array} \right) \left( \begin{array}{ccc} J_B & j_4 & J_C \\ j & -w_b & -k \end{array} \right) \left( \begin{array}{ccc} J_C & j_9 & J_A \\ k & -p_b & -i \end{array} \right).\]

\[= (-1)^{J_B-J_A-J_C} \left( \begin{array}{ccc} j_1 & j_9 & j_2 \\ -o_b & w_b & p_b \end{array} \right) \left( \begin{array}{ccc} j_1 & j_9 & j_2 \\ J_C & J_A & J_B \end{array} \right).\]  

(91)

In each case we combine the three 3jm symbols and the sign factors so that they suit the definition of the 6j symbol given in Appendix A, eq. \((83)\).

An important property of the sign factors is the following: if \(j_1, J_A, J_B\) enter one 3jm symbol, there is an equality:
where all signs are possible. This is because out of three momenta either zero or two moments are half-integer. Another important property is that, if $J$ is the momentum entering a certain $3jm$ symbol, and $m$ is its projection, then $(-1)^{2J\pm2m} = +1$. This is because $J$ and $m$ are either integer or half-integer, but simultaneously.

Below we cite without detailed derivation (which is quite similar to those above) the expressions for other vertices of the cube.

**Vertex c**

$$\begin{align*}
(-1)^{\pm 2j_1 \pm 2J_A \pm 2J_B} &= 1, \\
(92)
\end{align*}$$

$$\begin{align*}
&= (-1)^{j_A+j_D-J_C}
\begin{pmatrix}
\hat{j}_2 & \hat{j}_3 & \hat{j}_{10} \\
p_c & q_c & -x_c
\end{pmatrix}
\begin{pmatrix}
j_2 & j_3 & j_{10} \\
J_D & J_C & J_A
\end{pmatrix}.
(93)
\end{align*}$$

**Vertex d**

$$\begin{align*}
&= (-1)^{j_A-j_D-J_E}
\begin{pmatrix}
\hat{j}_4 & \hat{j}_3 & \hat{j}_{11} \\
r_d & q_d & y_d
\end{pmatrix}
\begin{pmatrix}
j_4 & j_3 & j_{11} \\
J_D & J_E & J_A
\end{pmatrix}.
(94)
\end{align*}$$

**Vertex e**

$$\begin{align*}
&= (-1)^{j_E-j_B-J_F}
\begin{pmatrix}
\hat{j}_{12} & \hat{j}_8 & \hat{j}_5 \\
z_e & v_e & s_e
\end{pmatrix}
\begin{pmatrix}
j_{12} & j_8 & j_5 \\
J_B & J_C & J_E
\end{pmatrix}.
(95)
\end{align*}$$

**Vertex f**

$$\begin{align*}
&= (-1)^{j_B+j_C-J_F}
\begin{pmatrix}
\hat{j}_6 & \hat{j}_5 & \hat{j}_9 \\
t_f & s_f & w_f
\end{pmatrix}
\begin{pmatrix}
j_6 & j_5 & j_9 \\
J_B & J_C & J_F
\end{pmatrix}.
(96)
\end{align*}$$

**Vertex g**

$$\begin{align*}
&= (-1)^{j_C+j_D+J_F}
\begin{pmatrix}
\hat{j}_6 & \hat{j}_{10} & \hat{j}_7 \\
t_g & x_g & u_g
\end{pmatrix}
\begin{pmatrix}
j_6 & j_{10} & j_7 \\
J_D & J_F & J_C
\end{pmatrix}.
(97)
\end{align*}$$

**Vertex h**

$$\begin{align*}
&= (-1)^{j_E+j_F-J_D}
\begin{pmatrix}
\hat{j}_7 & \hat{j}_{11} & \hat{j}_8 \\
u_h & y_h & v_h
\end{pmatrix}
\begin{pmatrix}
j_7 & j_{11} & j_8 \\
J_E & J_F & J_D
\end{pmatrix}.
(98)
\end{align*}$$

Combining all these factors we get eq. (16) corresponding to the cube.
Appendix C. 6j symbols at the lattice sites

In this Appendix we show how integration over link variables in eq. (16) combine, together with the $3jm$ factors, into $6j$ symbols composed of the link momenta $j$, one for each site of the lattice. The notations are given in Fig. 3.

Let us consider integration over link variables $U_{1,4,12,13,14,15}$ entering the vertex $a$ shown in Fig.3. This vertex is an intersection of four even cubes denoted in Fig.3 as $I$, $II$, $III$, and $IV$. Link 1 is common to the cubes $I$ and $II$, link 4 is common to $I$ and $IV$, and so on.

The analytical expression for the cube $I$ is given by eq. (16). The factors relevant to vertex $a$ are

$$D^{j_1}_{o_ar_a}(U_1)D^{j_4}_{r_a y_a}(U_4)D^{j_{12}}_{z_a z_a}(U_{12})\left(\begin{array}{ccc}j_1 & j_4 & j_{12} \\ o_a & r_a & z_a \end{array}\right).$$ (99)

It is not necessary to compute anew corresponding expressions for the cubes $II-IV$. It is sufficient to draw a correspondence between the links and the sites of other cubes with those of the cube $I$. For example, link 1, as seen from the viewpoint of cube $II$, is analogous to link 7 of cube $I$; the vertex $a$ from the viewpoint of cube $II$ is analogous to vertex $h$ of cube $I$, and vertex $b$ is analogous to vertex $g$. In the table below we give the list of the ‘analogs’ of links in cubes $II-IV$ to those of the cube $I$.

|    | II | III | IV |
|----|----|-----|----|
| 1  | 1  | 12  | 4  |
| 13 | 11 | 14  | 13 |
| 14 | 3  | 2   | 5  |
| a  | h  | a   | f  |
| a  | h  | a   | c  |
| a  | f  | a   | c  |

Having this table of correspondence we can immediately read off from eq. (16) the expressions relevant to the vertex $a$, arising from the cubes $II-IV$:

from cube $II$:
$$D^{j_1}_{u_a u_b}(U_1)D^{-y_a,-y_b}_{-y_a,-y_b}(U_{13})D^{j_{13}}_{-u_a,-v_a}(U_{14})D^{j_{14}}_{-v_a,-v_a}(U_{15})\left(\begin{array}{ccc}j_1' & j_{13} & j_{14} \\ -u_a & y_a & v_a \end{array}\right).$$ (100)

from cube $III$:
$$D^{j_{12}}_{x_a x_a}(U_{12})D^{j_4}_{x_a -p_a}(U_{14})D^{j_{15}}_{-q_a,-q_a}(U_{15})\left(\begin{array}{ccc}j_{12}' & j_4 & j_{15} \\ -x_a & p_a & q_a \end{array}\right).$$ (101)

from cube $IV$:
$$D^{j_4}_{u_a u_b}(U_4)D^{j_{15}}_{-s_a,-s_a}(U_{15})D^{j_{13}}_{-t_a,-w_a}(U_{13})\left(\begin{array}{ccc}j_4' & j_{15} & j_{13} \\ -t_a & s_a & w_a \end{array}\right).$$ (102)

Integrating over $U_{1,4,12,13,14,15}$ we get:

$$\int dU_1 D^{j_1}_{o_ar_a}(U_1)D^{j_1'}_{u_a u_b}(U_1) = \frac{\delta_{jj_1'}}{2j_1+1}(-1)^{u_a-u_b}\delta_{o_a-u_a}\delta_{o_b-u_b},$$ (103)
\[ \int dU_4 D_{r a q d}^{j_4} (U_4) D_{a t d}^{j_4} (U_4) = \frac{\delta_{j_4 j_4}}{2j_4 + 1} (-1)^{t_d - t_a} \delta_{r_a, -t_a} \delta_{r_d, -t_d}, \tag{104} \]

\[ \int dU_12 D_{z a x e}^{j_{12}} (U_12) D_{x a x e}^{j_{12}} (U_12) = \frac{\delta_{j_{12} j_{12}}}{2j_{12} + 1} (-1)^{x_e - x_a} \delta_{z_a, -x_a} \delta_{z_e, -x_e}, \tag{105} \]

\[ \int dU_13 D_{-w_a, -w_b}^{j_{13}} (U_13) D_{w_a, -w_b}^{j_{13}} (U_13) = \frac{\delta_{j_{13} j_{13}}}{2j_{13} + 1} (-1)^{w_a - w_b} \delta_{w_a, -y_a} \delta_{w_b, -y_b}, \tag{106} \]

\[ \int dU_14 D_{-p_a, -p_b}^{j_{14}} (U_14) D_{p_a, -p_b}^{j_{14}} (U_14) = \frac{\delta_{j_{14} j_{14}}}{2j_{14} + 1} (-1)^{p_a - p_b} \delta_{p_a, -y_a} \delta_{p_b, -y_b}, \tag{107} \]

\[ \int dU_15 D_{-s_a, -s_b}^{j_{15}} (U_15) D_{s_a, -s_b}^{j_{15}} (U_15) = \frac{\delta_{j_{15} j_{15}}}{2j_{15} + 1} (-1)^{s_a - s_b} \delta_{s_a, -y_a} \delta_{s_b, -y_b}. \tag{108} \]

The four 3jm symbols in eqs. (104-108) get now fully contracted over all indices. This results in a 6j symbol according to eq. (80) of Appendix A. Indeed we have for vertex \( a \):

\[ “a” = \sum_{oqrvyz} (-1)^{o+r+z-q-v-y} \]

\[
\begin{pmatrix}
  j_1 & j_4 & j_{12} \\
  o & r & z
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_{13} & j_{14} \\
  o & y & v
\end{pmatrix}
\begin{pmatrix}
  j_{12} & j_{14} & j_{15} \\
  z & -v & q
\end{pmatrix}
\begin{pmatrix}
  j_4 & j_{15} & j_{13} \\
  r & -q & -y
\end{pmatrix}
\]

\[ = (-1)^{j_1 + j_4 + j_{12} + j_{13} - j_{14} - j_{15}} \sum_{oqrvyz} \]

\[
\begin{pmatrix}
  j_1 & j_4 & j_{12} \\
  o & r & z
\end{pmatrix}
\begin{pmatrix}
  j_1 & j_{14} & j_{13} \\
  o & v & -y
\end{pmatrix}
\begin{pmatrix}
  j_{15} & j_4 & j_{13} \\
  -q & r & y
\end{pmatrix}
\begin{pmatrix}
  j_{15} & j_{14} & j_{12} \\
  q & v & z
\end{pmatrix}
\]

\[ = (-1)^{j_1 + j_4 + j_{12} + j_{13} - j_{14} - j_{15}} \left\{ \begin{array}{c}
  j_1 \\
  j_{14} \\
  j_{13}
\end{array} \right\} \tag{110} \]

[since \( j_{12}, j_{14} \) and \( j_{15} \) came from one 3jm symbol one can use the equation (see eq. (92))

\[ (-1)^{j_{12} - j_{14} - j_{15}} = (-1)^{-j_{12} + j_{14} + j_{15}} \]

\[ = (-1)^{j_1 + j_4 - j_{12} + j_{13} + j_{14} + j_{15}} \left\{ \begin{array}{c}
  j_1 \\
  j_{14} \\
  j_{13}
\end{array} \right\} \tag{111} \]

This is the final result for the vertex \( a \): the six angular momenta ascribed to the six links entering this vertex combine to produce a 6j symbol.

Similarly, one can treat the vertex \( b \), see Fig.3. Links labelled by numbers 1, 2, 9, 16, 17, 18 enter this vertex; they are pair-wise shared by the cubes I, II, V and VI. The correspondence between the links viewed from the viewpoint of the cubes II, V, VI with those of the cube I is given by the following table:
Performing the same steps as in deriving the $6j$ symbol for the vertex $a$ we arrive to the following result for the vertex $b$:

\[
\begin{align*}
\text{"b"} &= (-1)^{j_1+j_2+j_9+j_{16}+j_{17}-j_{18}} \left\{ j_1 & j_9 & j_2 \\
& j_{17} & j_{18} & j_{16} \right\}.
\end{align*}
\] (112)

We notice that vertex $a$ is of the ‘even’ and vertex $b$ is of the ‘odd’ type: all other vertices of the lattice can be considered as either ‘even’ or ‘odd’. Therefore, eqs. (111, 112) give actually the full result. Combining them together we find that a sign factor

\[
(-1)^{2j} = (-1)^{-2j}
\] (113)

should be attributed to all links of the lattice.

Let us prove that this sign factor is equivalent (in the vacuum!) to a sign factor

\[
(-1)^{2J} = (-1)^{-2J}
\] (114)

attributed to all plaquettes of the lattice. We recall that all links are shared by two even cubes whose faces carry plaquette values $J$. We first attribute all links to only one (out of the two possible) cubes, according to some rule. Many such rules can be suggested, the only requirement being that each link is attributed to one and only one even cube. An example is given by the following construction: we choose the edges 12,5,9,2 and 7 (see Fig.2) as ‘belonging’ to the cube shown on that figure. The rest six edges will then ‘belong’ to one of the neighbouring even cubes. For example, the edge 1 will be counted as ‘belonging’ to the cube II (see Fig.3). Indeed, from the cube II point of view that edge will be of the type 7, and so forth. It can be seen that, in these scheme, every link of the full lattice will ‘belong’ to one and only one even cube.

We have, therefore, a sign factor

\[
(-1)^{2j_{12}+2j_5+2j_9+2j_2+2j_7}
\] (115)

attributed to the cube I. Next, we recall that, e.g., $j_{12}$ enters the $3jm$ symbol together with the plaquette angular momenta $J_B$ and $J_E$ (see (90)). Using eq. (92) appropriate to the case we can replace $(-1)^{2j_{12}} = (-1)^{2J_B+2J_E}$. Similarly, $(-1)^{2j_5} = (-1)^{2J_B+2J_E}$, and so on. As a result we get that the sign factor (115) is equal to

\[
(-1)^{2J_A+2J_B+2J_C+2J_D+2J_E+2J_F}.
\] (116)

This procedure can be repeated for all even cubes of the lattice. It proves the above statement that the product of all link sign factors (113) can be replaced by the product of all plaquette sign factors (114). It should be stressed that this proof is valid only for the vacuum, i.e. for the partition function itself but, generally speaking, not for the averages of operators.
Appendix D. $9j$ symbols from the Wilson loop

Let the Wilson loop in the representation $j_s$ go through the links ...,15,1,17,..., see Fig.3 for notations. It means that one has now to integrate three $D$-functions of the link variables $U_{15,1,...}$, instead of two, as it was in eqs. (103, 108) of the previous Appendix, the rest integrations remaining unchanged. We have now

\[
\int dU_1 D_{a_0b_0}(U_1) D_{u_au}(U_1) D_{m_am_b}(U_1) = \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_1' & \hat{j}_s \\
\hat{o}_a & \hat{u}_m & \hat{m}_a
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_1' & \hat{j}_s \\
\hat{o}_b & \hat{u}_m & \hat{m}_b
\end{array} \right),
\]

(117)

\[
\int dU_{15} D_{-q_a,-q_c}(U_{15}) D_{-s_a,-s_c}(U_{15}) D_{m_bm_c}(U_{15}) = (-1)^{m_c-m_a} \left( \begin{array}{ccc}
\hat{j}_{15} & \hat{j}_{15}' & \hat{j}_s \\
\hat{q}_a & \hat{s}_c & \hat{m}_a
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_{15} & \hat{j}_{15}' & \hat{j}_s \\
\hat{q}_a & \hat{s}_c & \hat{m}_a
\end{array} \right).
\]

(118)

Using the other $3jm$ symbols related to the vertex $a$ (see eqs.(99-102)) and the Kronecker symbols from eqs.(105,107), we get for the vertex $a$:

\[
\text{"a"} = \sum (-1)^{r+z+w+p-m} \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_4 & \hat{j}_{12} \\
\hat{o} & \hat{r} & \hat{z}
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_1' & \hat{j}_{13} & \hat{j}_{14} \\
\hat{-u} & \hat{-w} & \hat{-p}
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
\hat{j}_{14} & \hat{j}_{15} & \hat{j}_{12} \\
p & q & z
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_4 & \hat{j}_{15}' & \hat{j}_{13} \\
\hat{r} & \hat{s} & \hat{w}
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_1' & \hat{j}_s \\
\hat{o} & \hat{u} & \hat{m}
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_{15} & \hat{j}_{15}' & \hat{j}_s \\
\hat{q} & \hat{s} & \hat{m}
\end{array} \right)
\]

(119)

[we notice that $r+z=-o$, $w+p=-u$ and that $o+u+m=0$, hence the sign factor is +1; we change the signs of all projections in the second $3jm$ symbol, and permute the columns in other $3jm$ symbols to match the definition of the $9j$ symbols as given by eq. (88)]

\[
= (-1)^{j_1'-j_4+j_{14}+j_{15}+j_s} \left\{ \begin{array}{ccc}
\hat{j}_4 & \hat{j}_1 & \hat{j}_{12} \\
\hat{j}_{15} & \hat{j}_s & \hat{j}_{15} \\
\hat{j}_{13} & \hat{j}_1' & \hat{j}_{14}
\end{array} \right\}.
\]

(120)

To get the final sign factor we have used the relation $(-1)^{\pm 2j_1 \pm 2j_2 \pm 2j_3} = +1$ valid for any $j_{1,2,3}$ originating from one $3jm$ symbol.

Acting in the same fashion we obtain for the vertex $b$:

\[
\text{"b"} = \sum (-1)^{-v-y-t-x+m} \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_9 & \hat{j}_2 \\
\hat{-o} & \hat{-y} & \hat{-v}
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_{16} & \hat{j}_{18} & \hat{j}_1' \\
\hat{t} & \hat{x} & \hat{u}
\end{array} \right)
\]

\[
\left( \begin{array}{ccc}
\hat{j}_{16} & \hat{j}_{17} & \hat{j}_9 \\
t & q & y
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_{18} & \hat{j}_2 & \hat{j}_{17} \\
x & v & s
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_1 & \hat{j}_1' & \hat{j}_s \\
\hat{o} & \hat{u} & \hat{m}
\end{array} \right) \left( \begin{array}{ccc}
\hat{j}_{17} & \hat{j}_{17}' & \hat{j}_s \\
\hat{q} & \hat{s} & \hat{m}
\end{array} \right)
\]

\[
= (-1)^{j_1'-j_2+j_{16}+j_{17}+j_s} \left\{ \begin{array}{ccc}
\hat{j}_2 & \hat{j}_1 & \hat{j}_9 \\
\hat{j}_{17} & \hat{j}_s & \hat{j}_{17} \\
\hat{j}_{18} & \hat{j}_1' & \hat{j}_{16}
\end{array} \right\}.
\]

(121)
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