Optimal Hashing in External Memory*

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Abstract

Hash tables are a ubiquitous class of dictionary data structures. However, standard hash table implementations do not translate well into the external memory model, because they do not incorporate locality for insertions.

Iacono and Pătrașcu established an update/query tradeoff curve for external hash tables: a hash table that performs insertions in $O(\lambda/B)$ amortized IOs requires $\Omega(\log N)$ expected IOs for queries, where $N$ is the number of items that can be stored in the data structure, $B$ is the size of a memory transfer, $M$ is the size of memory, and $\lambda$ is a tuning parameter.

They provide a hashing data structure that meets this curve for $\lambda$ that is $\Omega(\log \log M + \log M N)$. Their data structure, which we call an IP hash table, is complicated and, to the best of our knowledge, has not been implemented.

In this paper, we present a new and much simpler optimal external memory hash table, the Bundle of Arrays Hash Table (BOA). BOAs are based on size-tiered LSMs, a well-studied data structure.

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and are almost as easy to implement. The BOA is optimal for a
narrower range of $\lambda$. However, the simplicity of BOAs allows them to
be readily modified to achieve the following results:

- A new external memory data structure, the **Bundle of Trees
  Hash Table** (BOT), that matches the performance of the IP
  hash table, while retaining some of the simplicity of the BOAs.
- The **cache-oblivious Bundle of Trees Hash Table**
  (COBOT), the first cache-oblivious hash table. This data struc-
ture matches the optimality of BOTs and IP hash tables over
the same range of $\lambda$.

1 Introduction

Dictionaries are among the most heavily used data structures. A dictio-
nary maintains a collection of key-value pairs $S \subseteq U \times V$, under operations\(^1\) INSERT($x, v, S$), DELETE($x, S$), and QUERY($x, S$), which returns the value
 corresponding to $x$ when $x \in S$. When data fits in memory, there are many
solutions to the dictionary problem.

When data is too large to fit in memory, comparison-based dictionarie s
 can be quite varied. They include the B$^\varepsilon$-tree [9], the write-optimized skip
  list [7], and the cache-optimized look-ahead array (COLA) [4–6]. All of these
data structures have been deployed extensively, and their best variants are
optimal in the **external-memory comparison model** in that they match
the bound established by Brodal and Fagerberg [9] who showed that for
any dictionary in this model, if insertions can be performed in $O\left(\frac{\lambda \log_\lambda N}{B}\right)$
amortized I/Os, then there exists a query that requires at least $\Omega(\log_\lambda N)$ I/Os.
This trade off has since been extended in several ways [1].

In this paper, we consider the dictionary problem without restriction
to the comparison model, and in particular, we consider external-memory
hashing. This allows for a better insertion/query trade off. Iacono and
Pâtrașcu showed:

**Theorem 1** ([14]). If insertions into an external memory dictionary can
be performed in $O(\lambda/B)$ amortized I/Os, then queries require an expected
$\Omega(\log_\lambda N)$ I/Os.

\(^1\)We do not consider dictionaries that also support the $\text{SUCC}(x, S)$ and $\text{PRED}(x, S)$.
$\text{SUCC}(x, S)$ return $\min\{y | y > x \land y \in S\}$ and $\text{PRED}(x, S)$ is defined symmetrically.
They further describe an external-memory hashing algorithm, which we refer to here as the **IP hash table**, that performs insertions in $O\left(\frac{1}{B} \left(\lambda + \log_\frac{M}{B} N + \log \log N\right)\right)$ IOs and queries in $O(\log_\lambda N)$ IOs w.h.p. Therefore, for $\lambda = \Omega\left(\log_\frac{M}{B} N + \log \log N\right)$, the IP hash table meets the tradeoff curve of Theorem 1 and is thus optimal.

In external memory hashing, we can assume that keys are hashed, that is, that they are uniformly distributed and satisfy some independence properties. The IP hash table and the following results assume that the hash function (and therefore the hashed keys) used is $\Theta(\log N)$-independent.

The base result of this paper is a simple external-memory hashing scheme, the **Bundle of Arrays Hash Table** (BOA), that meets the optimal Theorem 1 trade off curve for large enough $\lambda$. Specifically, we show:

**Theorem 2.** A BOA with growth factor $\lambda$ supports $N$ insertions with amortized per entry insertion cost of $O\left(\frac{\left(\lambda + \log_\frac{M}{B} N + \log_\lambda N\right)}{B}\right)$ IOs and query cost of $O(\log_\lambda N)$ IOs w.h.p.

Thus BOAs are optimal for $\lambda = \Omega(\log_\frac{M}{B} N + \log_\lambda N)$. They are readily modified to provide several variations, the most important of which is the **Bundle of Trees Hash Table** (BOT). BOTs are optimal for the same range of $\lambda$ as the IP hash table:

**Theorem 3.** A BOT with growth factor $\lambda$ supports $N$ insertions with amortized per entry insertion cost of $O\left(\frac{\left(\lambda + \log_\frac{M}{B} N + \log \log M\right)}{B}\right)$ IOs and a query cost of $O(\log_\lambda N)$ IOs w.h.p.

We further introduce the first cache-oblivious hash table, the **Cache-Oblivious Bundle of Trees Hash Table** (COBOT), which matches the IO performance of BOT and IP hash tables.

The remainder of the paper is organized as follows. Section 2.1 describes the properties of the hash function we use and Section 2.2 provides background on size-tiered LSMs. Section 3 introduces the BOAs and Section 4 describes its variant, the BOT. Section 5 adapts the BOT to the cache-oblivious model, resulting in the COBOT.
2 Preliminaries

2.1 Fingerprints and Hashing

In order to achieve our bounds, we need \(\Theta(\log N)\)-wise independent hash functions, which, once again matches IP hash tables. We note that a \(k\)-wise independent hash function is also \(k\)-wise independent on individual bits. Furthermore, the following Chernoff-type bound holds:

**Lemma 1** ([21]). Let \(X_1, X_2, \ldots, X_N\) be \([\mu \delta]\)-wise independent binary random variables, \(X = \sum_{i=1}^{N} X_i\) and \(\mu = E[X]\). Then

\[
\Pr(X > \mu \delta) = O\left(\frac{1}{\delta^3 \mu^3}\right),
\]

for sufficiently large \(\delta\).

In what follows, if the following, we use fingerprint to refer to any key that has been hashed using a \(\Theta(\log N)\)-wise independent hash function. Such hash functions have a compact representation and can be specified using \(\Theta(\log N)\) words. The universe that is hashed into is assumed to have size \(\Theta(N^k)\) for \(k \geq 2\). We ignore collisions, but these can be handled as in [15].

2.2 Log-structured Merge Trees

Log-structured merge trees (LSMs) are (a family of) external-memory dictionary data structures. They come in two varieties: level-tiered LSMs (LT-LSMs) and size-tiered LSMs (ST-LSMs). Both kinds are suboptimal in that they do not meet the optimal insertion-query tradeoff [9], although the COLA [5] is an optimal variant of the LT-LSMs. Both are popular in practice [3, 10, 11, 13, 17–20, 22, 24].

An LSM consists of sets of either B-trees or sorted arrays called runs. In this paper, we describe them in terms of runs, since we use runs below.

An LT-LSM consists of a cascade of levels, where each level consists of at most one run. Each level has a capacity that is \(\lambda\) times greater than the level below it, where \(\lambda\) is called the growth factor.\(^2\) When a level reaches

\(^2\)Sometimes this and related structures are analyzed with a growth factor of \(B^\epsilon\). The two are equivalent. We use \(\lambda\) rather than \(\epsilon\) as the tuning parameter for consistency with the external-memory hashing literature.
capacity, it is merged into the next level (perhaps causing a merge cascade). The amortized IO cost for insertions is small because sequential merging is fast, although each item will participate in $\lambda/2$ merges on average. The IO cost for a query is high because a query must be performed independently on each of $O(\log_\lambda N)$ levels (although Bloom filters [6,8] are used in practice to mitigate this cost).

A ST-LSM further improves insertion IOs at the expense of queries. Each level contains less than $\lambda$ runs. Every run on a given level has the same size, which is $\lambda$ times larger than the runs on the level beneath it. When $\lambda$ runs are present at a level, they are merged into one run and placed at the next level. There are therefore $O(\log_\lambda N)$ levels. Insertions are faster than in LT-LSMs because each item is only merged once on each level. Queries are slower because each query must be perform $O(\lambda)$ times at each level, since the runs on each level are independent.

3 Bundle of Arrays Hashing

A Bundle of Arrays Hash Table (BOA) is an external hash table based on ST-LSMs. As a first step, we show that runs with uniformly distributed keys—for example, hashed keys—can be searched quickly in external memory. This immediately improves the query performance of ST-LSMs.

Interpolation search is not enough to achieve the needed improvement. We show a balls-and-bins analysis that allows for the needed speedup.

**Lemma 2** ([10]). If $N$ balls are thrown into $Q = \Theta(N/\log N)$ bins uniformly and i.i.d., then every bin has $\Theta(N/Q) = \Theta(\log N)$ balls with high probability.

**Lemma 3.** Let $A$ be a sorted array of $N$ uniformly distributed keys in the range $[0, K)$, and assume $B = \Omega(\log N)$. Then $A$ can be written to external memory using $O(N)$ space and $O(N/B)$ IOs so that membership in $A$ can be determined in $O(1)$ IOs with high probability.

**Proof.** Divide the range of keys into $N/B$ uniformly sized buckets; that is, bucket $i$ contains keys in the range $[(i-1)KB/N, iKB/N)$. Because the keys in $A$ are distributed uniformly, and $B = \Omega(\log N)$, every bucket contains $\Theta(B)$ keys with high probability by Lemma 2. Let $F$ be the number of items in the fullest bucket, and write the keys in each bucket to disk in order using $F$ space for each. Because $F = \Theta(B)$, this takes the desired space and IOs.
Now, to find a key, compute which bucket it belongs to. A constant number of IOs will fetch that bucket, whose address is known because all buckets have the same size.

When using a ST-LSM for hashing, we can use Lemma 3 to speed up queries:

**Corollary 1.** If a ST-LSM contains uniformly distributed keys and has growth factor $\lambda$, then by writing the levels as in Lemma 3, queries can be performed in $O(\lambda \log N)$ IOs. The insertion cost is unchanged: $O\left(\frac{1}{B} \left( \log \lambda N + \log_{\lambda} M \right) \right)$ amortized IOs.

While the query performance has improved by a factor of $\log N$, the ST-LSM is still off the optimal tradeoff curve of Theorem 1. In particular, setting $\lambda' = \log \lambda N$ and assuming that $\lambda' = \Omega \log_{\lambda} N$, we see that queries are at least exponentially slower than optimal. The BOA will utilize additional structure in order to reduce this query cost.

### 3.1 Routing Filters

The main result of this section is an auxiliary data structure, the **routing filter** that improves the query cost by a factor of $\lambda$ by further exploiting the hashing of keys. From this point forward, we assume that all keys in the data structure have already been hashed by a pairwise independent hash function, and we refer to these hashed keys as **fingerprints** to make this distinction clear. This combination of techniques will yield a hashing data structure that is optimal for some choices of $\lambda$. In subsequent sections, we show how to achieve optimality for a wider range of $\lambda$.

The purpose of the routing filter is to indicate probabilistically, at each level, which run contains the fingerprint we are looking for. Each level will have its own routing filter, defined as follows. For each level $\ell$, let $h_\ell$ be some number, to be specified below. It will be convenient for a fingerprint $K$ to interpret the bits of $K$ as a string of $\log \lambda$ bit characters, $K = K_0 K_1 K_2 \cdots$. Let $P_\ell(K) = K_0 \cdots K_{h_\ell}$ be the $h_\ell$th prefix, the concatenation of the characters of $K$ up to $K_{h_\ell}$. The routing filter $F_\ell$ for level $\ell$ is a $\lambda^{h_\ell}$ character array, where $F_\ell[i] = j$ if the $j$th run $R_{\ell,j}$ contains a fingerprint $K$ such that the $P_\ell(K) = i$, and no later run $R_{\ell,j'}$ (i.e. with $j' > j$) contains such a fingerprint.

We also modify each run $R_{\ell,j}$ during the merge so that each fingerprint-value pair contains a **previous field** of 1 additional character used to specify
the previous run containing a fingerprint with the same prefix or $j$ to indicate no such run exists. Thus these fingerprint-value pairs now form a singly linked list whose fingerprint share the same prefix, and the routing filter points to the run containing the head.

During a query for a fingerprint $K$, first $F_\ell[P_\ell(K)]$ is checked to find the latest run containing a fingerprint with a matching prefix. Once that fingerprint-value pair is found, its previous field indicates the next run which needs to be checked and so on until all fingerprints with matching prefix in the level are found.

Such routing filters induce a space/cost tradeoff. The greater $h_\ell$ is, the more space the table takes, but the less likely it is that many runs will have fingerprints that collide on their prefixes. The rest of this section shows that when $h_\ell$ is set to be the base $\lambda \log$ of the capacity of level $\ell$, then that yields an optimal external hash table.

Define $\beta$, the routing table ratio, to be the ratio of the number of buckets in the routing filter to the size of a run. The number of entries in a run on level $\ell$ is $B\lambda^{\ell-1}$, so $\beta = \lambda h_\ell / B\lambda^{\ell-1}$. We analyze the per-level insertion and query cost for a given $\beta$ and $\lambda$.

**Lemma 4.** For a BOA with growth factor $\lambda$ and routing table ratio $\beta$, the following hold:

1. Merging a level incurs $\Theta\left(\frac{1}{B} \left(1 + \log \frac{M}{B} \lambda + \beta \log N \lambda\right)\right)$ IOs per fingerprint.

2. Finding a fingerprint in a level takes $\Theta\left(1 + \frac{1}{B} \right)$ IOs in expectation.

**Proof.**

1. Merging a level requires merging together its runs as well as updating the next level’s routing filter. Merging $\lambda$ sorted arrays takes $\Theta\left(\frac{1}{B} \left(1 + \log \frac{M}{B} \lambda\right)\right)$ IOs per fingerprint.

   The routing filter is updated by iterating through it and the new run sequentially. For each fingerprint $K$ appearing in the run, $F_{\ell+1}[P_{h_{\ell+1}}(K)]$ is copied to the previous field in the run, and $F_{\ell+1}[P_{h_{\ell+1}}(K)]$ is set to the number of the current run. Each entry in the routing filter is a character, the routing filter has $\beta$ characters for each new fingerprint, so it requires $\Theta\left(\frac{2^{\beta_B}}{B} \log N \lambda\right)$ IOs per fingerprint to update sequentially.

2. To query for a fingerprint $K$, first the routing filter is checked, which takes $O(1)$ IOs, and then the runs with fingerprints matching the prefix of $K$ are checked.
Given some enumeration of the fingerprints in level \( \ell \), denote the \( i \)th fingerprint by \( K_i \). Let \( X_i \) be the indicator random variable which is 1 if \( P_{h_\ell}(K) = P_{h_\ell}(K_i) \) and 0 otherwise. Because the hash function is pairwise independent, \( K \) and \( K_i \) are uniformly distributed and their bits are pairwise independent. Thus \( E[X_i] \leq \frac{1}{\lambda_\ell} \). The expected number of fingerprints in the level with prefix \( P_{h_\ell}(K) \) is at most \( \sum_{i=1}^{BM} E[X_i] \leq \frac{BM}{\lambda_\ell} = \frac{\lambda}{\beta} \). By Lemma 3, each of these fingerprints can be found and checked in \( O(1) \) IOs. Thus, the expected per-level query cost is \( O \left( 1 + \frac{\lambda}{\beta} \right) \).

**Lemma 5.** A BOA with growth factor \( \lambda \) and routing table ratio \( \beta \) has insertion cost \( O \left( \frac{1}{B} \left( \beta + \log \frac{M}{B} N + \log \lambda N \right) \right) \) and query cost \( O \left( \left( 1 + \frac{\lambda}{\beta} \right) \log \lambda N \right) \).

**Proof.** Because a BOA has \( \log \lambda N \) levels, this follows immediately from Lemma 4. \( \square \)

So for a fixed \( \lambda \), there is no advantage to choosing \( \beta = \omega(\lambda) \). On the other hand, \( \beta = o(\lambda) \) is suboptimal, because then choosing \( \beta' = \lambda' = \beta \) changes a linear factor in the query cost to a logarithmic one. Therefore, it is optimal to choose \( \beta = \Theta(\lambda) \), and in what follows we will fix \( \beta = \lambda \). Now the main theorem follows immediately:

**Theorem 2.** A BOA with growth factor \( \lambda \) supports \( N \) insertions with amortized per entry insertion cost of \( O \left( \left( \lambda + \log \frac{M}{B} N + \log \lambda N \right) / B \right) \) IOs and query cost of \( O(\log \lambda N) \) IOs w.h.p.

Thus, a BOA is optimal for large enough \( \lambda \):

**Corollary 2.** Let \( B \) be a BOA with growth factor \( \lambda \) containing \( N \) entries. If \( \lambda = \Omega \left( \log \frac{M}{B} N + \frac{\log N}{\log \log N} \right) \), then \( B \) is an optimal unsorted dictionary.

Note that the condition that \( \lambda = \Omega(\log M/B N) \) is related to the permutation bound [2]. This is because BOAs and their variations support some form of successor operation for the order of the fingerprints (the hashed order of the keys).
4 Bundle of Trees Hashing

In order for a BOA to be an optimal dictionary, its growth factor $\lambda$ must be $\Omega(\log N / \log \log N)$. Otherwise, the cost of insertion is dominated by the cost of merging, which in slow because it effectively sorts the fingerprints using a $\lambda$-ary merge sort. In this section, we present the Bundle of Trees Hash Table (BOT), which is a BOA-like structure. A BOT stores the fingerprints in a log in the order in which they arrive. Each level of the BOT is like a level of a BOA, where the bundle of arrays on each level is replaced by an search structure on the log (the routing tree) and a data structure needed to merge routing trees (the character queue). The character queue performs a delayed sort on the characters needed at each level, thus increasing the arity of the sort and decreasing the IOs.

A BOT has $s = \lceil \log_\lambda N/B \rceil$ levels, each of which consists of at most one routing tree where the root has degree less than $\lambda$ and all internal nodes have degree $\lambda$. Each node of a routing tree contains a routing filter, which functions similarly to the routing filters in Section 3. In a BOT, the routing filter takes as input a fingerprint and outputs a set of pointers to the children which may contain it, though some of these may be false positives. It also returns some auxiliary information discussed below, which together with the child pointer is referred to as the sketch of the fingerprint.

Each leaf points to a block of $B \log_\lambda N$ fingerprints in the log; the reason for using blocks of this size will be explained in Section 4.2. The deepest level $s$ indexes the beginning of the fingerprint log with a tree of depth $s$, the next level then indexes the next section and so forth. Insertions are appended to the log until they form a block, at which point they are added to the tree in the 1st level of the BOT.

**Querying a BOT.** A query to the BOT for a fingerprint $K$ is performed independently at each level, beginning at the root of each routing tree. At a routing node of height $h$ in a tree, the routing filter is queried. Whereas the routing filter in a BOA returns only the last array containing a fingerprint with a given prefix $P_h(K)$, BOTs use a refined routing filter that returns a full list of all the children with such a fingerprint. The details of the refined routing filter are left for Section 4.1. The query is then passed to each of these children, until it reaches a block of the log, which is then searched in full. In this way queries are “routed” down the tree on each level to the part of log where the fingerprint and its associated value are. In addition, as queries descend the routing tree, they may generate false positives which are
likewise routed down towards the log.

An issue that arises from this querying algorithm is that if a query generates a false positive in a node of height \( h \), that is, there is a fingerprint \( K' \) with \( P_h(K) = P_h(K') \), then in the child containing \( K' \), the shorter prefixes will also match, i.e. \( P_{h-1}(K) = P_{h-1}(K') \). Therefore false positives will always propagate down the routing tree and at each subsequent node may in turn generate more false positives. To prevent this, the routing filter of each node of the routing tree keeps, for each fingerprint \( K \), an additional check character taken from the tail of \( K \). Positive queries must also match this character, and nodes of different heights use different parts of the fingerprint for the check characters so that the probabilities of two fingerprints matching on different levels are independent. Check characters are explained further in Section 4.1.

**Inserting into a BOT.** When a level \( i \) in the BOT fills, its routing tree is merged into the routing tree of level \( i + 1 \), thus increasing the degree of the target routing tree by 1 (and perhaps filling level \( i + 1 \), which triggers a merge of level \( i + 1 \) into \( i + 2 \), and so on). The merge of level \( i \) into level \( i + 1 \) consists of adding the prefix-sketch pairs of the fingerprints from level \( i \) to the routing filter of the root on level \( i + 1 \). The child pointers of these pairs will point to the root of the formerly level-\( i \) routing tree, so it becomes a child of the root of the level \( i + 1 \) routing tree, although it isn’t moved or copied. In this way, a BOT resembles an LT-LSM, described in Section 2.2.

In order to add a fingerprint \( K \) from level \( i \) to the root routing filter on level \( i + 1 \), the prefix \( P_{i+1}(K) \) must be known. However, the root routing filter on level \( i \) only stores the prefix \( P_i(K) \) for each fingerprint \( K \) it contains, so that in particular the last character of \( P_{i+1}(K) \) is missing. As described in Section 4.2, each level has a character queue, which provides this character, as well as the check characters, in order to merge the routing trees efficiently.

By replacing the arrays of a BOA on each level by character queues, the BOT can insert efficiently for a larger range of \( \lambda \). Because the arrays are no longer available to answer queries, BOTs instead use the recursively constructed routing trees, which require some refinement over the routing filters in BOAs. With these in place, however, query performance is optimal, and the BOT becomes an optimal dictionary for a wider range of the parameter \( \lambda \), matching the range of the IP hash table [15].
4.1 BOT Queries

This section covers the routing tree structure in more detail. First, we cover the specifics of check characters, and then we introduce the refined routing filter and prove its performance characteristics.

As described above, in a BOT each false positive in a node of a routing tree queries an additional child. Because the routing filter in the child has shorter prefixes, that false positive will cause an entire path down the tree to be accessed. Moreover, along the way more false positives can be generated. Check characters are used to reduce the probability of false positives in BOTs and short-circuit the paths that do occur.

The $i$th check character $C_i$ of a fingerprint $K$ is the $i$th character from the end of the string representation of $K$. As described in Section 2.1, we assume that the fingerprints are taken from a universe of size at least $N^2$ so that the check characters do not overlap with the characters used in the prefixes of the routing filters.

The routing filter of a node of height $h$ in a routing tree stores $C_h(K)$ in the sketch for each fingerprint $K$ in the filter, and when queried, returns a list of the sketches of prefix-matching fingerprints in the order that they appear in the log.

Now, the query only proceeds on those children whose check characters match the check character $C_h(K)$. Since the characters of the fingerprint are uniformly distributed, the check character of each false positive matches with probability $1/\lambda$. Moreover, the characters of each level are non-overlapping, so for fingerprints $K, K'$ the event that $V_h(K) = V_h(K')$ is independent of the event that $V_{h-1}(K) = V_{h-1}(K')$. Therefore a false positive on a node of height $h$ may still be eliminated in its child (of height $i - 1$), short-circuiting the paths that false positives would otherwise create.

While not strictly necessary, in order to simplify the analysis, we will further arrange it so that false positives may only be created in the root of the tree, and that at each level, only a $1/\lambda$ fraction of the false positives survive in expectation. To prevent new false positives from being generated when a query passes from a parent to a child, we also keep the next character of each fingerprint in its prefix-sketch pair stored in the routing filter. For a fingerprint $K$ in a node of height $h$, the next character is just the next character that follows the prefix, $P_h(K)$, so that its prefix in the parent, $P_{h+1}(K)$, can be obtained. A false positive in the children which is not in the parent will not match this next character and can be eliminated.
When there are multiple prefix-matching fingerprints in both a parent and its child, we would like to be able to align the lists returned by the routing filters so that known false positives in the parent (either from check or next characters) can be eliminated in the child. Otherwise the check character in the child of a known false positive in the parent may match the queried fingerprint, and therefore more than $1/\lambda$ of the false positives may survive in expectation. To this end, we require the routing filter to return the list of sketches of prefix-matching fingerprints in the order they appear in the log. Then after the sketches in the child list whose next characters do not match the parent are eliminated, the remaining phrases will be in the same order as in the parent. In this way, known false positives can also be eliminated in the child.

Now because of the next characters, false positives may only be created in the root of the routing tree. Each false positive in the root corresponds to a fingerprint $K'$ in the level. At each node on the path to $K'$’s location in the log, we use the ordering to determine which returned sketch corresponds to $K'$, so that the false positive corresponding to $K'$ is eliminated with probability $1/\lambda$. Thus the query path for $K'$ causes at most $\sum_{i=1}^{h} 1/\lambda^{i} = O\left(\frac{1}{\lambda}\right)$ node accesses. This is independant of the number of false positives in the root. Since there are $O(1)$ expected false positives in the root, we have shown:

**Lemma 6.** During a query to a routing tree, the expected number of nodes accessed due to false positives is at most $O\left(\frac{1}{\lambda}\right)$.

**Refined Routing Filter.** A BOA routing filter handles prefix collisions by returning only the last run containing the queried fingerprint and then chaining in the runs. Because there are no longer arrays with which to chain, the BOT routing filter, however, must handle prefix collisions itself and return a complete ordered list of sketches for all prefix-matching fingerprints, while having the same performance as in Section 3.1.

The idea behind the refined routing filter is to keep the prefix-sketch pairs in a list, and use a hash table on prefixes to point queries to the appropriate place. Each pointer may require as many as $\Omega(\log N)$ bits, and we require the routing filter to have $O(1)$ characters per fingerprint, so we require the hash table to use shorter prefixes so as to reduce the number of buckets and thus reduce its footprint. In particular, it uses prefixes which are $\log_{\lambda} \log_{\lambda} N$ characters shorter, which we refer to as **pivot prefixes**.

The list delta encodes the prefix $P_h(K)$ for each fingerprint $K$, together with the sketch, $S_h(K)$. This means the difference between $P_h(K)$ and the
preceeding prefix is stored, together with the sketch. In addition, the first entry following each pivot prefix contains the full prefix $P_h(K)$, rather than just the difference. Otherwise, when the hash table routes a query to that place in the list, the full prefix wouldn’t be computable.

We first analyze the space efficiency of a delta encoded list of a collection of prefixes and then analyze the performance characteristics of refined routing filters.

**Lemma 7.** A list of delta-encoded prefixes with density $D$, that is there are $D$ prefixes in the list for every possible prefix, requires $O(-\log\lambda D)$ characters per prefix.

**Proof.** The average difference between consecutive prefixes is $1/D$. Because logarithms are convex, the average number of characters required to represent this difference is therefore $O(-\log\lambda D)$. \hfill \Box

Now we can prove:

**Lemma 8.** A refined routing filter can be updated using $O\left(\frac{\lambda \log \lambda}{B \log N}\right)$ IOs per new entry.

**Proof.** Let $C$ be the capacity of the level. There are $\frac{C}{\log\lambda N}$ pivot prefixes. For each pivot prefix, the hash table stores the bit position in a list with at most $C$ entries, where $C \leq N$. Each entry is at most $\log N$ bits, so this position can be written using $O(\log N)$ bits.

For each fingerprint in the node, the list contains $O(1)$ characters by Lemma 7 or $O(\log \lambda)$ bits. Additionally, each pivot prefix has to an initial entry of length $O(\log N)$ bits, so the list all together uses $O(C \log \lambda + \frac{C}{\log\lambda N} \cdot \log N) = O(C \log \lambda)$ bits.

When the refined routing filter is updated, the old version is read sequentially and the new version is written out sequentially. $C/\lambda$ fingerprints are added at a time, so this incurs $O\left(\frac{\lambda \log \lambda}{B}\right)$ IOs per entry. \hfill \Box

The refined routing filter also still performs constant IO lookups:

**Lemma 9.** A refined routing filter performs lookups in $O(1)$ IOs in expectation.

**Proof.** The pivot bit string of a fingerprint and its successor are accessed from the hash table in $O(1)$ IOs. This return beginning and ending bit positions in
the list. Because the fingerprints are distributed uniformly and are pairwise independent there are $O(\log_3 N)$ fingerprints matching the pivot prefix in expectation. The list has $O(\log \lambda)$ bits per fingerprint, so $O(1)$ words are fetched from the list in expectation, and hence $O(1)$ IOs.

4.2 Character Queue

The purpose of the character queue is to store all the sketches of fingerprints contained in a level $i$ that will be needed during a merge in the future. When level $i$ is merged into level $i+1$, the character queue outputs a sorted list of the delta-encoded prefix-sketch pairs of all the fingerprints, which is used to update the root routing tree. The character queue is then merged into the character queue on level $i+1$.

The character queue effectively performs a merge sort on the sketches. If it were to merge all the sketches as soon as they are available, this would consist of $\lambda$-ary merges. In order to increase the arity of the merges, it defers merging sketches which are not needed immediately. The sketches are collection of series, by which we mean a collection of sorted runs. Each series stores a continuous range of sketches $S_i(K), S_{i+1}(K), \ldots, S_{i+j}(K)$ for each fingerprint $K$, together with the prefix up to the first sketch, $P_{i-1}(K)$. These prefixes are delta encoded in their run. Thus the size of an entry is determined by the number of sketches in the range and the length of the prefix relative to the size of the run (by Lemma 7).

The character queue tradeoff. We are faced with the following tradeoff. If the character queue merges a series frequently, the delta encoding is more efficient, which decreases the cost of the merging. However the arity is lower, which increases it. The character queue uses a merging schedule which balances this tradeoff and thus achieves optimal insertions.

The character queue merging schedule. The character queue on level $i$ contains the sketches $S_{i+1}(K), S_{i+2}(K), \ldots, S_s(K)$ of each fingerprint $K$ in the level. These characters are stored in a collection of series $\{\sigma_{j_q}\}$, where $j_q$ is the smallest multiple of $2^q$ greater than $i$. Series $\sigma_{j_q}$ contains the sketches $S_{j_q}(K), \ldots, S_{j_q+1-1}(K)$. Each series consists of a collection of sorted runs each of which stores the delta encoded prefix of each fingerprint together with its sketches.

When level $i$ fills, the runs in the series $\sigma_{i+1}$ are merged, and the character queue outputs the delta encoded prefix-sketch pairs, $(P_{i+1}(K), S_{i+1}(K))$ to
update the root routing filter on level $i + 1$. If $2^{\rho(i+q)}$ is the greatest power of 2 dividing $i + 1$ ($\rho$ is sometimes referred to as the \textbf{ruler function} \cite{23}), then $\sigma_{i+1}$ also contains the next $2^{\rho(i+1)} - 1$ sketches of each fingerprint. These are batched and delta encoded to become runs in the series $\sigma_j$ for $q = [0, \rho(i+1)]$.

This leads to the following merging pattern: $\sigma_j$ batches $2^{\rho(j)}$ sketches, and has delta encoded prefixes of $2^{\rho(j)}$ characters on average, by Lemma \cite{7}. Therefore,

\textbf{Lemma 10.} A series $\sigma_j$ in a character queue contains $O(2^{\rho(j)})$ characters per fingerprint.

This leads to a merging schedule where the characters per item merged on the $j$th level is $O(2^{\rho(j)})$. Starting from 1 this is $1, 2, 1, 4, 1, 2, 8, 1, 2, 1, 4, 2, 1, 8, 1, 2, 1, 4, 2, 1, 16, \ldots$, which resemble the tick marks of a ruler, hence the name ruler function.

We now analyze the cost of maintaining the character queues.

\textbf{Lemma 11.} The total per-insertion cost to update the character queues in a BOT is $\Theta \left( \frac{1}{B} \left( \log \frac{M}{B} N + \log \log M \right) \right)$.

\textit{Proof.} When $\sigma_j$ is merged, $\lambda^{2^{\rho(j)}}$ runs are merged, which has a cost of $O \left( \frac{2^{\rho(j)}}{B} \left[ \log \frac{M}{B} \left( \lambda^{2^{\rho(j)}} \right) \right] \right)$ characters per fingerprint.

There are $\log \frac{N}{B} = O(\log \lambda N)$ levels, so this leads to the following total cost in terms of characters:

\[ O \left( \sum_{i=1}^{\log \lambda N} 2^{\rho(j)} \left[ \log \frac{M}{B} \left( \lambda^{2^{\rho(j)}} \right) \right] \right) = O \left( \sum_{k=0}^{\log \log \lambda N} \frac{\log \lambda N}{2^k} \cdot 2^k \left[ \log \frac{M}{B} \left( \lambda^{2^k} \right) \right] \right) \]

\[ = O \left( \log \lambda N \left( \log \log M + \sum_{k=\log \log M}^{\log \log \lambda N} 2^k \log \frac{M}{B} \lambda \right) \right) \]

\[ = O \left( \log \lambda N \left( \log \log M + \log \frac{M}{B} N \right) \right), \]

where the last equality is because the RHS sum is dominated by its last term. Because there are $\log \lambda N$ characters in a word, and all reads and writes are performed sequentially in runs of size at lease $B$, the result follows. \hfill $\square$

The character queue is where we require that the blocks of the log have size $B \log \lambda N$, because we want the runs created when the block is added to the first level to be at least size $B$. 

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4.3 Performance of the BOT

We can now prove Theorem 3:

**Theorem 3.** A BOT with growth factor $\lambda$ supports $N$ insertions with amortized per entry insertion cost of $O \left( \left( \frac{\lambda + \log M}{\lambda} N + \log \log M \right)/B \right)$ IOs and a query cost of $O(\log_\lambda N)$ IOs w.h.p.

**Proof.** By Lemma 8, the cost of updating the routing filters is $O \left( \frac{\lambda}{B} \right)$, since there are $O(\log_\lambda N)$ levels. This together with the cost of updating the character queues, given by Lemma 11, is the insertion cost.

By Lemma 4, a query for fingerprint $K$ accesses an average of $O \left( \frac{\log_\lambda N}{\lambda} \right)$ nodes across the routing trees on all level due to false positives. Using Lemma 3 with $\delta = \lambda$, we have that this is $O(\log_\lambda N)$ nodes w.h.p.

If $K$ is contained in the bot, then $O(\log_\lambda N)$ nodes are accessed on its root-to-leaf path.

A block of the log is scanned at most once for a true positive and also whenever a false positive from the level $i$ root survives $i$ times. The expected number of such false positives for level $i$ is $1/\lambda^i$, so the expected number across levels is $O \left( \frac{1}{\lambda} \right)$ Therefore by Lemma 3 the probability that $\omega(1)$ blocks are scanned due to false positives is $O \left( \frac{1}{\lambda} \right)$. Each block can be scanned in $O(\log_\lambda N)$ IOs, so this yields the result. \qed

It follows that:

**Corollary 3.** Let $B$ be a BOT with growth factor $\lambda$ containing $N$ entries. If $\lambda = \Omega \left( \log M N + \log \log M \right)$, then $B$ is an optimal dictionary.

5 Cache-Oblivious BOTs

In this section, we show how to modify a BOT to be cache oblivious. We call the resulting structure a cache-oblivious hash tree (COBOT).

Much of the structure of the BOT translates directly into the cache-oblivious model. However, some changes are necessary. In particular, when the series of character queues are merged, this merge must be performed cache-obliviously using funnels [12], rather than with an (up to) $M/B$-way merge. Also, the log cannot be buffered into sections of size $O(B \log_\lambda N)$, and so instead they are buffered into sections of constant size, items are
immediately added to routing filter, and the extra IOs are eliminated by optimal caching.

When an insertion is made into a CO hash tree, its fingerprint-value pair is appended to the log, and it is immediately inserted into level 1. Thus, the leaves of the routing trees point to single entries in the log.

The series of the character queues must be placed more carefully as well. In particular, for each \(j\), the runs of series \(\sigma_j\) must be laid out back-to-back, so that even when they are short, they may be read efficiently across the level.

The series are merged using a **partial funnelsort**. Funnelsort is a cache-oblivious sorting algorithm that makes use of \(K\)-funnels [12]. A \(K\)-funnel is a CO data structure that merges \(K\) sorted lists of total length \(N\). We make use of the the following lemma.

**Lemma 12** ([12]). A \(K\)-funnel merges \(K\) sorted lists of total length \(N \geq K^3\) in \(O\left(\frac{N}{B} \log_{M/B} \frac{N}{B} + K + \frac{N}{B} \log_K \frac{N}{B}\right)\) IOs, provided the tall cache assumption that \(M = \Omega(B^2)\) holds.

The partial funnelsort used to merge \(K\) runs of a series with total length \(L\) (in words) performs a single merge with a \(K\)-funnel if \(L \geq K^3\) and recursively merges the run in groups of \(K^{1/3}\) runs otherwise.

**Corollary 4.** A partial funnelsort merges \(K\) runs of total word length \(L\) in \(O\left(\frac{L}{B} \log_{M/B} \frac{L}{B} + \frac{1}{B} \log_K \frac{L}{B}\right)\) IOs, provided the tall cache assumption that \(M = \Omega(B^2)\) holds.

**Proof.** The base case of the recursion occurs either when there is only 1 list remaining or the remaining lists fit in memory. In any other case of the recursion, since \(L = \Omega(B^2)\) by the tall cache assumption, the \(K\) term in Lemma [12] is dominated.

The recurrence is dominated by the cost of the funnel merges, which yields the result.

**Theorem 4.** If \(M = \Omega(B^2)\), then a CO hash tree with \(N\) entries and growth factor \(\lambda\) has amortized insertion cost \(\Theta\left(\frac{1}{B} \left(\lambda + \log \log M + \log_{M/B} N/B\right)\right)\) and query cost \(\Theta(\log_\lambda N)\), w.h.p.

**Proof.** We may assume that the caching algorithm sets aside enough memory that the last \(B\) items in the log, together with the subtree rooted at their
least common ancestor, are cached. Thus the log is updated at a per-item cost of $O(1/B)$.

The proof of Theorem 3 now carries over to the CO hash tree. The routing filters are updated the same way, and the cost of updating the character queues is unchanged, by Corollary 4.

Queries are performed as in Section 4.1 except that now the level 1 nodes cover $O(1)$ fingerprints, but the depth of the tree is unchanged, so the cost is the same. □

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