ON THE ALGEBRAIC K-THEORY
OF THE COMPLEX K-THEORY SPECTRUM

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Abstract. Let \( p \geq 5 \) be a prime, let \( ku \) be the connective complex \( K \)-theory spectrum, and let \( K(ku) \) be the algebraic \( K \)-theory spectrum of \( ku \). In this paper we study the \( p \)-primary homotopy type of the spectrum \( K(ku) \) by computing its mod \( (p, v_1) \) homotopy groups. We show that up to a finite summand, these groups form a finitely generated free module over the polynomial algebra \( \mathbb{F}_p[b] \), where \( b \) is a class of degree \( 2p + 2 \) defined as a “higher Bott element”.

1. Introduction

The algebraic \( K \)-theory of a local or global number field \( F \), with suitable finite coefficients, is known to satisfy a form of Bott periodicity. Bott periodicity refers here to the periodicity of topological complex \( K \)-theory, and is an example of \( v_1 \)-periodicity in the sense of stable homotopy theory. For example, if \( p \) is an odd prime and if \( F \) contains a primitive \( p \)-th root of unity, then the mod \( (p) \) algebraic \( K \)-theory \( K_*(F; \mathbb{Z}/p) \) of \( F \) contains a non-nilpotent Bott element \( \beta \) of degree 2, with

\[
\beta^{p-1} = v_1.
\]

In one of its reformulations \[19, 41\], the Lichtenbaum-Quillen Conjecture asserts that the localization

\[
K_*(F; \mathbb{Z}/p) \to K_*(F; \mathbb{Z}/p)[\beta^{-1}]
\]

away from \( \beta \) is an isomorphism in positive degrees. In particular, \( K_*(F; \mathbb{Z}/p) \) is periodic of period 2 in positive degrees. In the local case, this follows from \[23\] Theorem D.

The \( p \)-local stable homotopy category also features higher forms of periodicity \[25\], one for each integer \( n \geq 0 \), referred to as \( v_n \)-periodicity. It is detected for example by the \( n \)th Morava \( K \)-theory \( K(n) \), having coefficients \( K(0)_* = \mathbb{Q} \) and \( K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}] \) with \( |v_n| = 2p^n - 2 \) if \( n \geq 1 \). The study of \( v_2 \)-periodicity is at the focus of current research in algebraic topology, as illustrated for example by the efforts to define the elliptic cohomology theory known as topological modular forms \[24\].

Waldhausen \[41\] extended the definition of algebraic \( K \)-theory to include specific “rings up to homotopy” called structured ring spectra, like \( E_\infty \) ring spectra \[30\], \( S \)-algebras \[20\], or symmetric ring-spectra \[26\]. The chromatic red-shift conjecture \[4\] of John Rognes predicts that the algebraic \( K \)-theory of a suitable \( v_n \)-periodic structured ring-spectrum is essentially \( v_{n+1} \)-periodic, as illustrated above in the case of number fields (which are \( v_0 \)-periodic). For an example with the next level of periodicity, we consider the algebraic \( K \)-theory of topological \( K \)-theory.
Let $p \geq 5$ be a prime, and let $ku_p$ denote the $p$-completed connective complex $K$-theory spectrum with coefficients $ku_{p,*} = \mathbb{Z}_p[u], |u| = 2$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. Let $\ell_p$ be the Adams summand of $ku_p$ with coefficients $\ell_{p,*} = \mathbb{Z}_p[v_1]$ and $v_1 = u^{p-1}$. In joint work with John Rognes [3], we have computed the mod $(p,v_1)$ algebraic $K$-theory of the $S$-algebra $\ell_p$, denoted $V(1)_s K(\ell_p)$, and we have shown that it is essentially $v_2$-periodic. This computation provides a first example of red-shift for non-ordinary rings.

In this paper, following the discussion in [1, Section 10], we interpret $ku_p$ as a tamely ramified extension of $\ell_p$ of degree $p - 1$, and we compute $V(1)_s K(ku_p)$. As expected, the result is again essentially periodic. However, $V(1)_s K(ku_p)$ has a shorter period: its periodicity is given by multiplication with a higher Bott element $b \in V(1)_s K(ku_p)$, of degree $2p + 2$. We defer a definition of $b$ to Section 3 below, and summarize our main result in the following statement.

**Theorem 1.1.** Let $p \geq 5$ be a prime. The higher Bott element $b \in V(1)_{2p+2} K(ku_p)$ is non-nilpotent and satisfies the relation

$$b^{p-1} = -v_2.$$  

Let $P(b)$ denote the polynomial $\mathbb{F}_p$-sub-algebra of $V(1)_s K(ku_p)$ generated by $b$. Then there is a short exact sequence of graded $P(b)$-modules

$$0 \to \Sigma^{2p-3} \mathbb{F}_p \to V(1)_s K(ku_p) \to F \to 0,$$

where $\Sigma^{2p-3} \mathbb{F}_p$ is the sub-module of $b$-torsion elements and $F$ is a free $P(b)$-module on $8 + 4(p - 1)$ generators.

A detailed description of the free $P(b)$-module $F$ is given in Theorem 8.1. The proof is based on evaluating the cyclotomic trace map [11]  

$$\text{trc} : K(ku_p) \to T(C(ku_p))$$

to topological cyclic homology. We emphasize that the higher Bott element $b$ is not the reduction of a class in the mod $(p)$ or integral homotopy of $K(ku_p)$.

The cyclic subgroup $\Delta \subset \mathbb{Z}_p$ of order $p - 1$ acts on $ku_p$ by $p$-adic Adams operations. The Adams summand is defined as the homotopy fixed-point spectrum $\ell_p = ku_p^{h\Delta}$, and $\Delta$ qualifies as the Galois group of the tamely ramified extension $\ell_p \to ku_p$ of commutative $S$-algebras given by the inclusion of homotopy fixed-points. We proved in [1, Theorem 10.2] that the induced map $K(\ell_p) \to K(ku_p)$ factors through a weak equivalence

$$K(\ell_p) \xrightarrow{\sim} K(ku_p)^{h\Delta}$$

after $p$-completion. The mod $(p,v_1)$ homotopy groups of $K(\ell_p)$ and $K(ku_p)$ are related as follows.

**Proposition 1.2.** Let $i_* : V(1)_s K(\ell_p) \to V(1)_s K(ku_p)$ be the homomorphism induced by the extension of $S$-algebras $\ell_p \to ku_p$.

(a) The homomorphism $i_*$ factors through an isomorphism

$$V(1)_s K(\ell_p) \cong (V(1)_s K(ku_p))^\Delta \subset V(1)_s K(ku_p)$$

onto the classes fixed by the Galois group. The higher Bott element $b$ is not fixed under the action of $\Delta$, but $b^{p-1} = -v_2$ is, accounting for the $v_2$-periodicity of $V(1)_s K(\ell_p)$. 


(b) The homomorphism
\[ \mu : P(b) \otimes_{P(v_2)} V(1)_s K(\ell_p) \to V(1)_s K(\text{ku}_p) \]
induced by \( i_s \) and the \( P(b) \)-action has finite kernel and cokernel, and is an isomorphism in degrees larger than \( 2p^2 - 4 \). By localizing away from \( b \), we obtain an isomorphism of \( P(b, b^{-1}) \)-modules
\[ P(b, b^{-1}) \otimes_{P(v_2)} V(1)_s K(\ell_p) \xrightarrow{\cong} V(1)_s K(\text{ku}_p)[b^{-1}] \, . \]

In particular, the \( P(b) \)-module \( V(1)_s K(\text{ku}_p) \) is almost the module obtained from the \( P(v_2) \)-module \( V(1)_s K(\ell_p) \) by the extension \( P(v_2) \subset P(b) \) of scalars. The kernel of \( \mu \) consists of \( b \)-multiples of the \( v_2 \)-torsion elements, and we have a non-trivial cokernel because some of the \( P(v_2) \)-module generators of \( V(1)_s K(\ell_p) \) are multiples of \( b \) in \( V(1)_s K(\text{ku}_p) \), see Corollary \( 8.2 \).

Notice that for the cyclotomic extension \( \mathbb{Z}_p \to \mathbb{Z}_p[\zeta_p] \) of complete discrete valuation rings with Galois group \( \Delta \) (where \( \zeta_p \) is a primitive \( p \)th root of unity), we have corresponding results in mod \( (p) \) algebraic \( K \)-theory. In effect, the natural homomorphism \( K_*(\mathbb{Z}_p; \mathbb{Z}/p) \to K_*(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p) \) factors through an isomorphism onto the \( \Delta \)-fixed classes. The Bott class \( \beta \in K_2(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p) \) is not fixed under \( \Delta \), but \( \beta^{p-1} \) is. This accounts for the fact that \( K_*(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p) \) has a shorter period than \( K_*(\mathbb{Z}_p; \mathbb{Z}/p) \). Moreover, the \( P(\beta) \)-module \( K_*(\mathbb{Z}_p[\zeta_p]; \mathbb{Z}/p) \) is essentially obtained from the \( P(v_1) \)-module \( K_*(\mathbb{Z}_p; \mathbb{Z}/p) \) by the extension \( P(v_1) \subset P(\beta) \) of scalars. These facts are extracted from computations by Hesselholt and Madsen \( [23, \text{Theorem D}] \). We therefore interpret Proposition \( 1.2 \) as follows: up to a chromatic shift of one in the sense of stable homotopy theory, the algebraic \( K \)-theory spectra of the tamely ramified extensions

\[ \begin{array}{ccc}
\mathbb{Z}_p[\zeta_p] & \xrightarrow{} & \text{ku}_p \\
\Delta & & \Delta \\
\mathbb{Z}_p & \xrightarrow{} & \ell_p
\end{array} \]

have a comparable formal structure.

This example of red-shift provides evidence that structural results for the algebraic \( K \)-theory of ordinary rings might well be generalized to provide more conceptual descriptions of the algebraic \( K \)-theory of \( S \)-algebras. See Remarks \( 3.5 \) and \( 8.4 \) for a discussion of the results we have in mind here.

We now turn to the algebraic \( K \)-theory \( K(\text{ku}) \) of the (non \( p \)-completed) connective complex \( K \)-theory spectrum \( \text{ku} \), with coefficients \( \text{ku}_* = \mathbb{Z}[u], |u| = 2 \). The \( p \)-completion \( \text{ku} \to \text{ku}_p \) induces a map
\[ \kappa : K(\text{ku}) \to K(\text{ku}_p) \, , \]
and the higher Bott element \( b \in V(1)_{2p+2} K(\text{ku}_p) \) is in fact defined as the image of a class with same name in \( V(1)_{2p+2} K(\text{ku}) \). The difference between \( K(\text{ku}) \) and \( K(\text{ku}_p) \) can be measured by means of the homotopy Cartesian square after \( p \)-completion
\[ \begin{array}{ccc}
K(\text{ku}) & \xrightarrow{\pi} & K(\mathbb{Z}) \\
\kappa & & \kappa \\
K(\text{ku}_p) & \xrightarrow{\pi} & K(\mathbb{Z}_p)
\end{array} \]
of Dundas [18, page 224]. Here π denotes the map induced in K-theory by the zeroth Postnikov sections ku → HZ and ku_p → HZ_p, where HR is the Eilenberg-Mac Lane spectrum of the ring R. The homotopy type of the p-completion of K(Z_p) has been computed by Bökstedt, Hesselholt and Madsen [22, 13]. The Lichtenbaum-Quillen Conjecture for K(Z) (see for example [34, §6]) implies that the homotopy fiber of K(Z) → K(Z_p) has finite V(1)-homotopy groups, which are concentrated in degrees smaller than 2p − 1. This implies the result below. In fact there seems to be some consensus that work of Vladimir Voevodsky and Markus Rost should imply the Lichtenbaum-Quillen Conjecture, but to our knowledge this has not appeared in written form. We therefore keep it as an assumption in the following results.

**Proposition 1.3.** Let p ≥ 5 be a prime, and assume that the Lichtenbaum-Quillen Conjecture for K(Z) holds at p. Then the homomorphism of P(b)-modules

\[ \kappa_* : V(1)_*K(ku) \to V(1)_*K(ku_p) \]

is an isomorphism in degrees larger than 2p − 1. Localizing the V(1)-homotopy groups away from b, we obtain an isomorphism

\[ V(1)_*K(ku)[b^{-1}] \cong V(1)_*K(ku_p)[b^{-1}] \]

of P(b, b^{-1})-algebras.

This result is of interest beyond algebraic K-theory. Baas, Dundas and Rognes have proposed a geometric definition of a cohomology theory derived from a suitable notion of bundles of complex two-vector spaces [7]. These are a two-categorical analogue of the ordinary complex vector bundles which enter in the geometric definition of topological K-theory. They conjectured in [7, 5.1] that the spectrum representing this new theory is weakly homotopy equivalent to K(ku), and this was proved by these authors and Birgit Richter in [6]. The next statement follows from Theorem 1.3 and Proposition 1.3.

**Proposition 1.4.** If the Lichtenbaum-Quillen Conjecture for K(Z) holds, then at any prime p ≥ 5 the spectrum K(ku) is of telescopic complexity two in the sense of [7, 6.1].

This result was anticipated in [7, §6], and ensures that the cohomology theory derived from two-vector bundles is, from the viewpoint of stable homotopy theory, a legitimate candidate for elliptic cohomology.

The computations presented in this paper fail at the primes 2 and 3, because of the non-existence of the ring-spectrum V(1). Theoretically, computations in mod (p) homotopy or in integral homotopy could also be carried out, but the algebra seems quite intractable. Another approach [16, 28] is via homology computations. There are ongoing projects in this direction by Robert Bruner, Sverre Lunøe-Nielsen and John Rognes.

Up to degree three, the integral homotopy groups of K(ku) can be computed essentially by using the map \( \pi : K(ku) \to K(Z) \) introduced above. The map \( \pi_* : K_*(ku) \to K_*(Z) \) is 3-connected, so that

\[ K_0(ku) \cong \mathbb{Z}, \ K_1(ku) \cong \mathbb{Z}/2 \text{ and } K_2(ku) \cong \mathbb{Z}/2. \]

Here \( K_1(ku) \) and \( K_2(ku) \) are generated by the image of \( \eta \in \pi_1S \) and \( \eta^2 \in \pi_2S \), respectively, under the unit \( S \to K(ku) \). Let \( w : BBU \to \Omega^\infty K(ku) \) be the map induced by the inclusion of units, see [33, 34]. There is a non-split extension

\[ 0 \to \pi_3(BBU) \xrightarrow{\pi_*} K_3(ku) \xrightarrow{w} K_3(Z) \to 0 \]
with \( \pi_3(BBU_\otimes) \cong \mathbb{Z}\{\mu\} \), \( K_3(ku) \cong \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\} \) and \( K_3(\mathbb{Z}) \cong \mathbb{Z}/48\{\lambda\} \), where \( \nu \) is the image of the Hopf class \( \nu \), which generates \( \pi_3 S \cong \mathbb{Z}/24 \). We have \( w_*(\mu) = 2\varsigma - \nu \) and \( \pi_*(\varsigma) = \lambda \). See [2] for details. This indicates that the integral homotopy groups \( K_*(ku) \) contain intriguing non-trivial extensions from subgroups in \( \pi_* S \), \( \pi_* BBU_\otimes \) and \( K_*(\mathbb{Z}) \).

The rational algebraic \( K \)-groups of \( ku \) are well understood. In joint work with John Rognes [5], we have proved that after rationalization, the sequence

\[
BBU_\otimes \xrightarrow{w} \Omega^\infty K(ku) \xrightarrow{\pi} \Omega^\infty K(\mathbb{Z})
\]

is a split homotopy fibre-sequence. A rational splitting of \( w \) is provided by a rational determinant map \( \Omega^\infty K(ku) \to (BBU_\otimes)_\mathbb{Q} \). In particular, by Borel’s computation [13] of \( K_*(\mathbb{Z}) \otimes \mathbb{Q} \), there is a rational equivalence

\[
\Omega^\infty K(ku) \cong_{\mathbb{Q}} SU \times (SU/ SO) \times \mathbb{Z}.
\]

All but finitely many of the non-torsion classes in the integral homotopy groups \( \pi_* K(ku) \) detected by this equivalence reduce mod \( (p) \) to multiples of \( v_1 \), and hence reduce to zero in \( V(1)_* K(ku) \).

We briefly discuss the contents of this paper. In Section 2, we study the \( V(1) \)-homotopy of the Eilenberg-Mac Lane space \( K(\mathbb{Z}, 3) \), which is a subspace of the space of units of \( ku \). In Section 3, we define low-dimensional classes in \( V(1)_* K(ku) \) corresponding to units of \( ku \), and in particular we introduce the higher Bott element. We prove in Section 4 that the images of the Hopf class \( \nu \) to topological Hochschild homology. In Section 5, we compute the determinant map \( \Omega^\infty \) to topological cyclic homology. In Section 6 we compute the various homotopy fixed points of \( THH(ku_\otimes) \) under the action of the cyclic groups \( C_p \) and the circle, which are the ingredients for the computation of \( V(1)_* TC(ku_\otimes) \) in Section 7. In Section 8 we prove Theorem [14] on the structure of \( V(1)_* K(ku_\otimes) \) stated above. We also give a computation of \( V(1)_* K(KU_\otimes) \) for \( KU_\otimes \) the \( p \)-completed periodic \( K \)-theory spectrum, up to some indeterminacy.

**Notations and conventions.** Throughout the paper, unless stated otherwise, \( p \) will be a fixed prime with \( p \geq 5 \), and \( \mathbb{Z}_p \) will denote the \( p \)-adic integers. For an \( \mathbb{F}_p \)-vector space \( V \), let \( E(V) \), \( P(V) \) and \( \Gamma(V) \) be the exterior algebra, polynomial algebra and divided power algebra on \( V \), respectively. If \( V \) has a basis \( \{x_1, \ldots, x_n\} \), we write \( V = \mathbb{F}_p\{x_1, \ldots, x_n\} \) and \( E(x_1, \ldots, x_n) \), \( P(x_1, \ldots, x_n) \) and \( \Gamma(x_1, \ldots, x_n) \) for these algebras. By definition, \( \Gamma(x) \) is the \( \mathbb{F}_p \)-vector space \( \mathbb{F}_p\{\gamma_k x \mid k \geq 0\} \) with product given by \( \gamma_i x \cdot \gamma_j x = \binom{i+j}{i} \gamma_{i+j} x \), where \( \gamma_0 x = 1 \) and \( \gamma_1 x = x \). Let \( P_h(x) = P(x)/(x^h) \) be the truncated polynomial algebra of height \( h \). For an algebra \( A \), we denote by \( A\{x_1, \ldots, x_n\} \) the free \( A \)-module generated by \( x_1, \ldots, x_n \).

If \( Y \) is a space and \( E_* \) is a homology theory, such as mod \( (p) \) homology, \( V(1) \)-homotopy or Morava \( K \)-theory \( K(2)_* \), we denote by \( E_* (Y) \) the unreduced \( E_* \)-homology of \( Y \), which we identify with the \( E_* \)-homology of the suspension spectrum \( \Sigma^\infty(Y_+) \), where \( Y_+ \) denotes \( Y \) with a disjoint base-point added. We usually write \( \Sigma^\infty_+ Y \) instead of \( \Sigma^\infty(Y_+) \).
The reduced $E_*$-homology of a pointed space $X$ is denoted $\tilde{E}_*(X)$. We denote $\pi_* X$ the (unstable) homotopy groups of $X$, and $\pi_* \Sigma^\infty X$ its stable homotopy groups.

If $f : A \to B$ is a map of $S$-algebras, we also denote by $f$ its image under various functors like $\text{THH}$, $TC$ or $K$.

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In our computations with spectral sequences, we often determine a differential $d$ only up to multiplication by a unit. We use the notation $d(x) \equiv y$ to indicate that the equation $d(x) = \alpha y$ holds for some unit $\alpha \in \mathbb{F}_p$. Classes surviving to the $E^r$-term of a spectral sequence, for $r \geq 3$, are often given as a product of classes in the $E^2$-term. To improve the readability, we denote the product of two classes $x, y$ in $E_r$ by $x \cdot y$.

2. On the $V(1)$-homotopy of $K(\mathbb{Z}, 3)$

If $G$ is a topological monoid, let us denote by $BG$ its classifying space, obtained by realization of the bar construction, see for example [36, §1]. If $G$ is an Abelian topological group, then so is $BG$. The space $BG$ is equipped with the bar filtration

$$\{e\} = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_m \subset \cdots BG,$$

with filtration quotients $B_m/B_m-1 \cong \Sigma^n(G^m)$. In particular, we have a map

$$s : \Sigma G = B_1 \subset BG,$$

which in any homology theory $E_*$ induces a map

$$\sigma : E_* G \to E_{*+1}BG$$

called the suspension. If $E_*$ is a multiplicative homology theory satisfying the Künneth isomorphism, we have the bar spectral sequence [36, §2]

$$E^1_{s,t}(G) = \tilde{E}_s(G)^{\otimes E_t},$$

$$E^2_{s,t}(G) = \text{Tor}^{E_*}_{s,t}(E_*, E_*) \Rightarrow E_{s+t}(BG)$$

associated to the bar filtration (2.1).

Let $K(\mathbb{Z}, 0)$ be equal to $\mathbb{Z}$ as a discrete topological group, and for $m \geq 1$, we define recursively the Eilenberg-Mac Lane space $K(\mathbb{Z}, m)$ as the Abelian topological group $BK(\mathbb{Z}, m-1)$. We recall Cartan’s computation of the algebra $H_*(K(\mathbb{Z}, m); \mathbb{F}_p)$ for $p$ an odd prime and $m = 2, 3$. The generators are constructed explicitly from the unit $1 \in H_*(K(\mathbb{Z}, 0); \mathbb{F}_p)$ by means of the suspension $\sigma$ and two further operators

$$\varphi : H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p) \to H_{2pq+2}(K(\mathbb{Z}, m+1); \mathbb{F}_p)$$

and

$$\gamma_p : H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p) \to H_{2pq}(K(\mathbb{Z}, m); \mathbb{F}_p),$$

called the transposition [17, page 6-06] and the $p$-th divided power [17, page 7-07], respectively. The transposition is an additive homomorphism since $p$ is odd. For $x \in H_{2q}(K(\mathbb{Z}, m); \mathbb{F}_p)$, the class $\varphi(x)$ is represented, for example, by

$$x^{p-1} \otimes x \in E^1_{2,2pq}(K(\mathbb{Z}, m))$$

in the bar spectral sequence. The algebra $H_*(K(\mathbb{Z}, m); \mathbb{F}_p)$ has the structure of an algebra with divided powers, which are uniquely determined by $\gamma_p$.

Theorem 2.1 (Cartan). Let $p$ be an odd prime. There are isomorphisms of $\mathbb{F}_p$-algebras with divided powers

$$\Gamma(y) \xrightarrow{\cong} H_*(K(\mathbb{Z}, 2); \mathbb{F}_p)$$

given by \( y \mapsto \sigma \sigma(1) \), with \(|y| = 2\), and
\[
\bigotimes_{k \geq 0} E(e_k) \otimes \Gamma(f_k) \xrightarrow{\cong} H_*(K(\mathbb{Z}, 3); \mathbb{F}_p),
\]
given by \( e_k \mapsto \sigma \gamma_p^k \sigma(1) \) and \( f_k \mapsto \varphi \gamma_p^k \sigma(1) \), with degrees \(|e_k| = 2p^k + 1\) and \(|f_k| = 2p^{k+1} + 2\). For \( k \geq 0 \), the generators \( f_k \) and \( e_{k+1} \) are related by a primary mod \( (p) \) homology Bockstein
\[
\beta(f_k) = e_{k+1}.
\]

**Proof.** The computation of \( H_*(K(\mathbb{Z}, m); \mathbb{F}_p) \) as an algebra is given in [17, Théorème fondamental, p. 9-03]. The Bockstein relation \( \beta(f_k) = e_{k+1} \) is established in [17 page 8-04].

Ravenel and Wilson [36] make use of the bar spectral sequence to compute the Morava \( K \)-theory \( K(n)_* K(\pi, m) \) as an algebra when \( \pi = \mathbb{Z} \) or \( \mathbb{Z}/p^j \). All generators can be defined explicitly, starting with the unit \( 1 \in K(n)_* K(\pi, 0) \) and using the suspension, divided powers, transpotence and the Hopf-ring structure on \( K(n)_* K(\pi, *) \). We refer to [36, 5.6 and 12.1] for the following result, and for the definition of the generators \( \beta(k) \) and \( b(2k,1) \).

**Theorem 2.2** (Ravenel-Wilson). Let \( p \geq 3 \) be a prime and let \( K(2) \) be the Morava \( K \)-theory spectrum with coefficients \( K(2)_* = \mathbb{F}_p[v_2, v_2^{-1}] \). There are isomorphisms of \( K(2)_* \)-algebras
\[
K(2)_* K(\mathbb{Z}, 2) \cong K(2)_*[\beta(k) \mid k \geq 0] / (\beta_0^p, \beta_{(k+1)}^p - v_2^k \beta(k) \mid k \geq 0)
\]
where \(|\beta(k)| = 2p^k\), and
\[
K(2)_* K(\mathbb{Z}, 3) \cong K(2)_*[b(2k,1) \mid k \geq 0] / (b_{(2k,1)}^p + v_2^k b(2k,1) \mid k \geq 0)
\]
where \(|b(2k,1)| = 2p^k(p+1)\). The class \( \beta(0) \in K(2)_2 K(\mathbb{Z}, 2) \) is equal to \( \sigma \sigma(1) \), and the class \( b(0,1) \in K(2)_2 p+2 K(\mathbb{Z}, 3) \) is the transpotence of \( \beta(0) \).

We now turn to \( V(1) \)-homotopy. For an integer \( n \geq 0 \), we denote by \( V(n) \) the Smith-Toda complex [42], with mod \( (p) \) homology given by
\[
H_*(V(n); \mathbb{F}_p) \cong E(\tau_0, \ldots, \tau_n)
\]
as a left sub-comodule of the dual Steenrod algebra. In particular, \( V(0) = S/p \) is the mod \( (p) \) Moore spectrum, and the spectra \( V(0) \) and \( V(1) \) fit in cofibre sequences
\[
S \xrightarrow{p} S \xrightarrow{i_0} V(0) \xrightarrow{j_0} \Sigma S
\]
and
\[
\Sigma^{2p-2} V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1} V(0),
\]
where \( v_1 \) is a periodic map. For \( n = 0, 1 \) and \( p \geq 5 \), the spectrum \( V(n) \) is a commutative ring spectrum [35], and its ring of coefficients \( V(n)_* \), is an \( \mathbb{F}_p \)-algebra which contains a non-nilpotent class \( v_{n+1} \), of degree \( 2p^{n+1} - 2 \). We call “\( V(n) \)-homotopy” the homology theory associated to the spectrum \( V(n) \). In other words, the \( V(n)_* \)-homotopy groups of a spectrum \( X \) are defined by
\[
V(n)_* X = \pi_*(V(n) \wedge X).
\]
Notice that $V(0)_sX$ is denoted $\pi_*(X; \mathbb{Z}/p)$ by some authors, and called the mod $(p)$ homotopy groups of $X$. By analogy, we sometimes call $V(1)_sX$ the mod $(p, v_1)$ homotopy groups of $X$. If $Y$ is a space, then $V(n)_sY$ is defined as $V(n)_s\Sigma^n Y$.

The primary mod $(p)$ homotopy Bockstein $\beta_{0,1} : V(0)_sX \rightarrow V(0)_{s-1}X$ is the homomorphism induced by $(\Sigma i_0)_j$, and the primary mod $(v_1)$ homotopy Bockstein $\beta_{1,1} : V(1)_sX \rightarrow V(1)_{s-2p+1}X$ is the homomorphism induced by $(\Sigma^{2p-1}i_1)_j$. The homomorphisms $i_0 : \pi_*(X) \rightarrow V(0)_sX$ and $i_1 : V(0)_sX \rightarrow V(1)_sX$ are called the mod $(p)$ reduction and the mod $(v_1)$ reduction, respectively.

Let $HF_p$ be the Eilenberg-Mac Lane spectrum of $\mathbb{F}_p$. The unit map $S \rightarrow HF_p$ factors through a map of ring spectra $h : V(1) \rightarrow HF_p$, which induces an injective homomorphism in mod $(p)$ homology. Identifying the homology of $V(1)$ with its image in the dual Steenrod algebra $A_*$, we obtain the isomorphism

$$H_*(V(1); \mathbb{F}_p) \cong E(\tau_0, \tau_1)$$

of left $A_*$-comodule algebras mentioned above. Toda [22, Theorem 5.2] computed $V(1)_s$ in a range of degrees for which the Adams spectral sequence collapses. Up to some renaming of the classes, we deduce from his theorem that for $p \geq 5$ there is an isomorphism of $P(v_2) \otimes P(\beta_1)$-modules

$$P(v_2) \otimes P(\beta_1) \otimes \mathbb{F}_p\{\alpha_1, \beta_1, (\alpha_1\beta_1)^2\} \rightarrow V(1)_s$$

(2.3) in degrees $* < 4p^2 - 2p - 4$. The classes $\alpha_1$ and $\beta_1$ are the mod $(p, v_1)$ reduction of the classes with same name in $\pi_*(S)$, of degrees $2p - 3$ and $2p^2 - 2p - 2$, respectively. The class $\beta'_1$ is the mod $(v_1)$ reduction of the class with same name in $V(0)_s$ that supports a primary mod $(p)$ homotopy Bockstein $\beta_{0,1}(\beta'_1) = \beta_1$, and is of degree $2p^2 - 2p - 1$. The classes $v_2$ and $(\alpha_1\beta_1)^2$, of degree $2p^2 - 2$ and $2p^2 + 2p - 6$ respectively, support a primary mod $(v_1)$ homotopy Bockstein, given by $\beta_{1,1}(v_2) = \beta'_1$ and $\beta_{1,1}((\alpha_1\beta_1)^2) = \alpha_1\beta_1$. The class $v_2$ is non-nilpotent. The lowest-degree class in $V(1)_s$ that is not in the image of (2.3) is the mod $(p, v_1)$ reduction of the class $\beta_2$ in $\pi_*(S)$, of degree $4p^2 - 2p - 4$.

If $X$ is a connective spectrum of finite type, the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t} = H_s(X; \mathbb{F}_p) \otimes V(1)_t \Rightarrow V(1)_{s+t}X$$

(2.4) converges strongly, and we can use it to compute $V(1)_sX$ in low degrees. The first non-trivial Postnikov invariant of $V(1)$ is Steenrod’s reduced power operation $P^1$, corresponding to the first possibly non-trivial differential of the spectral sequence on the zeroth line, see Remark 2.4. This operation detects the class $\alpha_1$, which belongs to the kernel of the Hurewicz homomorphism $V(1)_s \rightarrow H_*(V(1); \mathbb{F}_p)$. In some more details, we have a commutative diagram

$$\begin{array}{c}
V(1) \\
\downarrow \rho
\end{array} \xrightarrow{g} \begin{array}{c}
\Sigma^{2p-3}HF_p \\
\downarrow \rho
\end{array} \begin{array}{c}
V(1)[2p-3] \\
\downarrow \rho
\end{array} \xrightarrow{h} \begin{array}{c}
HF_p \\
\downarrow \rho
\end{array} \begin{array}{c}
\Sigma^{2p-2}HF_p \\
\downarrow \rho
\end{array},
$$

(2.5)

where $\rho$ is the $(2p - 3)$th-Postnikov section, and the horizontal sequence is a cofibre sequence. Notice that by (2.3) the map $\rho$ is $(2p^2 - 2p - 2)$-connected, so that under our assumptions on $X$ we have a well defined homomorphism

$$\alpha = (\rho_*)^{-1}g_* : H_{n-2p+3}(X; \mathbb{F}_p) \rightarrow V(1)_nX$$

for $n \leq 2p^2 - 2p - 3$. 
Lemma 2.3. Let $X$ be a connective spectrum of finite type, and let $p \geq 3$ be a prime. For $n \leq 2p^2 - 2p - 3$, the group $V(1)_n X$ fits in an exact sequence

$$H_{n+1}(X; \mathbb{F}_p) (P^1)^* \rightarrow H_{n-2p+3}(X; \mathbb{F}_p) \xrightarrow{\alpha} V(1)_n X \xrightarrow{h^*} H_n(X; \mathbb{F}_p).$$

Here $(P^1)^*$ denotes the homology operation dual to $P^1$. If $X$ is a ring spectrum then $\alpha$ sends the unit $1 \in H_0(X; \mathbb{F}_p)$ to $\alpha_1$. Moreover, for any $X$ and any $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccc}
V(1)_n X & \xrightarrow{h^*} & H_n(X; \mathbb{F}_p) \\
\downarrow{\beta, 1} & & \downarrow{Q_1^*}
\end{array}$$

relating the primary mod $(v_1)$ homotopy Bockstein $\beta_{1,1}$ to the homology operation $Q_1^*$ dual to Milnor’s primitive $Q_1 = P^1(\delta - \delta P^1) \in A$.

Proof. This exact sequence is the sequence associated to the cofibre sequence in (2.5), where we have replaced $V(1)[2p - 3]_n X$ by $V(1)_n X$ via $\rho_*$, which is an isomorphism for these values of $n$, by strong convergence of the Atiyah-Hirzebruch spectral sequence. The assertion on $\alpha_1$ is true if $X = S$, and follows by naturality for $X$ an arbitrary ring spectrum.

The self-map $f = (\Sigma^{2p-1} i_1) j_1$ of $V(1)$, which induces $\beta_{1,1}$, is given in mod $(p)$ homology by the homomorphism $f_* : E(\tau_0, \tau_1) \rightarrow E(\tau_0, \tau_1)$ of degree $1 - 2p$ with $f_*(1) = f_*(\tau_0) = 0$, $f_*(\tau_1) = 1$ and $f_*(\tau_0 \tau_1) = \tau_0$. We have a commutative diagram

$$\begin{array}{ccc}
V(1)_* X & \xrightarrow{\beta_{1,1}} & V(1)_* X \\
\downarrow{g_*} & & \downarrow{g_*} \\
E(\tau_0, \tau_1) \otimes H_*(X; \mathbb{F}_p) & \xrightarrow{f_* \otimes 1} & E(\tau_0, \tau_1) \otimes H_*(X; \mathbb{F}_p) \\
\downarrow{e_*} & & \downarrow{\mu} \\
A_* \otimes H_*(X; \mathbb{F}_p) & \xrightarrow{\tau_1^* \otimes 1} & H_*(X; \mathbb{F}_p). 
\end{array}$$

The horizontal arrows are of degree $1 - 2p$, and $\tau_1^* : A_* \rightarrow \mathbb{F}_p$ is the dual of $\tau_1$ with respect to the standard basis $\{\tau(E) \xi(R)\}$ of $A_*$ given in [33, §6]. The homomorphism $g_*$ is induced in homotopy by the smash product of the unit $S \rightarrow H\mathbb{F}_p$ with the identity of $V(1) \wedge X$, $\mu$ is induced by the right homotopy action $H\mathbb{F}_p \wedge V(1) \rightarrow H\mathbb{F}_p$, and $e_*$ is induced by $1 \wedge h \wedge 1 : H\mathbb{F}_p \wedge V(1) \wedge X \rightarrow H\mathbb{F}_p \wedge H\mathbb{F}_p \wedge X$. We have $\mu g_* = h_*$ and $e_*(g_*) = \nu_* h_*$, where $\nu_*$ is the left $A_*$-coaction on $H_*(X; \mathbb{F}_p)$. This completes the proof since $((\tau_1^* \otimes 1) \nu_*) = Q_1^*$ by definition of $Q_1$, see [33, page 163].

Remark 2.4. For $X$ connective, the Atiyah-Hirzebruch spectral sequence (2.4) has only two non-trivial lines in internal degrees $t \leq 2p^2 - 2p - 3$, corresponding to 1 and $\alpha_1$ in $V(1)_*$, see (2.3). The argument above shows that there is a differential

$$d^{2p-2}(z) = (P^1)^*(z) \alpha_1$$

for $z \in E^2_{*0}$. In total degrees less than $2p^2 - 2p - 3$ this is the only possibly non-trivial differential.
Lemma 2.5. The map
\[ \mathbb{F}_p\{\alpha_1\} \oplus P_\mu(x) \to V(1)_*K(\mathbb{Z}, 2) \]
given by \( x \mapsto \sigma \sigma(1) \) with \( |x| = 2 \) is an isomorphism in degrees less than \( 4p - 3 \).

Proof. This follows from Theorem 2.1, Lemma 2.3 and the relation
\[ (P^1)^*(\gamma_{k+p-1}(y)) = k\gamma_k(y) \]  
\[ (2.6) \]
in \( H_*(K(\mathbb{Z}, 2); \mathbb{F}_p) \cong \Gamma(y) \).

Consider the cofibration
\[ B_1 = \Sigma K(\mathbb{Z}, 2) \xrightarrow{i} B_2 \xrightarrow{j} \Sigma^2 K(\mathbb{Z}, 2) \]
extracted from the bar filtration \( (2.1) \) of \( K(\mathbb{Z}, 3) \). It induces an exact sequence
\[ V(1)_*\Sigma K(\mathbb{Z}, 2) \xrightarrow{i_*} V(1)_*K(\mathbb{Z}, 2) \xrightarrow{j_*} \Sigma V(1)_*K(\mathbb{Z}, 2) \]
\[ \quad \quad \quad \quad \quad \quad \xrightarrow{\Sigma \mu_*} V(1)_*\Sigma^2 K(\mathbb{Z}, 2) \]
where \( \mu_* \) is induced by the product on \( K(\mathbb{Z}, 2) \). We know that \( V(1)_{2p+1}K(\mathbb{Z}, 2) = 0 \), by Lemma 2.5, which implies that the homomorphism
\[ V(1)_{2p+2}B_2 \xrightarrow{j_*} \Sigma V(1)_{2p}K(\mathbb{Z}, 2) \]
is injective. We know as well that the composition
\[ \Sigma V(1)_*K(\mathbb{Z}, 2) \xrightarrow{k} V(1)_*K(\mathbb{Z}, 2) \xrightarrow{\Sigma \mu_*} V(1)_*\Sigma V(1)_*K(\mathbb{Z}, 2) \]
sends the class \( x^{p-1} \otimes x \) to zero. In particular, the class \( k(x^{p-1} \otimes x) \) is in the image of \( j_* \).
Let \( \hat{b}' \in V(1)_{2p+2}B_2 \) be the unique class which satisfies the equation
\[ j_*(\hat{b}') = \Sigma^2 k(x^{p-1} \otimes x) \].

Definition 2.6. We define the fundamental class \( e'_0 \in V(1)_3K(\mathbb{Z}, 3) \) as the image of the unit \( 1 \in V(1)_0K(\mathbb{Z}, 0) \) under the iterated suspension \( \sigma^3 \). We define
\[ b' \in V(1)_{2p+2}K(\mathbb{Z}, 3) \]
as \( b' = l_{2*}(\hat{b}') \), where \( l_2 : B_2 \to K(\mathbb{Z}, 3) \) is the inclusion of the second subspace in the bar filtration.

Notice that the definition of \( b' \) in \( V(1)_* \)-homotopy, using \( x^{p-1} \otimes x \) as above, lifts the definition of the transpotence in the homology of the bar construction. We use this fact in the proof of the following proposition.

Proposition 2.7. The class \( b' \in V(1)_*K(\mathbb{Z}, 3) \) is non-nilpotent, and satisfies the relation
\[ b'^p = -v_2b' \].

There is a primary mod \( (v_1) \) homotopy Bockstein
\[ \beta_{1, 1}(b') = e'_0 \].

Proof. First, we notice that the \( \mathbb{F}_p \)-vector space \( V(1)_{2p^2+2p}K(\mathbb{Z}, 3) \), which contains \( b'^p \), is of rank at most one. Indeed, consider the Atiyah-Hirzebruch spectral sequence
\[ E^2_{s,t} \cong H_s(K(\mathbb{Z}, 3); \mathbb{F}_p) \otimes V(1)_t \Rightarrow V(1)_{s+t}K(\mathbb{Z}, 3) \].

From Theorem 2.1 and the formula (2.3) for \( V(1)_* \) in low degrees we deduce that \( E^2_{s,t} \) consists of \( \mathbb{F}_p\{f_0 \cdot v_2, e_0 \cdot f_0 \cdot \alpha_1 \cdot \beta_1\} \) in total degree \( 2p^2 + 2p \). Suspending the relation (2.6) for \( k = 1 \) we get a relation
\[ (P^1)^*(e_1) = e_0 \].
\[ (2.7) \]
Taking the vertical fiber over 1 implies the claim on $V(1)_{2p^2 + 2p} K(Z, 3)$. The unit map $S \to K(2)$ factors through a map of ring spectra $V(1) \to K(2)$. The induced ring homomorphism

$$V(1)_* K(Z, 2) \to K(2)_* K(Z, 2)$$

maps $x$ to $\beta(0)$, since these classes are defined as the double suspension of the unit in $V(1)_0 K(Z, 0)$, respectively $K(2)_0 K(Z, 0)$. By construction, the class $b'$ maps to the transpotence of $\beta(0)$, which is $b(0,1)$. We deduce that the sub-$V(1)_*$-algebra of $V(1)_* K(Z, 3)$ generated by $b'$ maps surjectively onto the subalgebra

$$P(v_2, b(0,1))/(b^p_{(0,1)} + v_2 b(0,1))$$

of $K(2)_* K(Z, 3)$ generated by $v_2$ and $b(0,1)$. In particular $b'$ is non-nilpotent. Thus $V(1)_{2p^2 + 2p} K(Z, 3)$ is of rank one, and injects into $K(2)_{2p^2 + 2p} K(Z, 3)$. This implies the identity $b^p = -v_2 b'$.

To prove the Bockstein relation, we map to homology. The Hurewicz homomorphism $h_* : V(1)_* K(Z, 3) \to H_*(K(Z, 3); \mathbb{F}_p)$ is an isomorphism in degrees 3 and $2p + 2$, mapping $e_0$ to $e_0$ and $b'$ to the transpotence $\varphi(y) = f_0$ of $y$. We have a primary homology Bockstein $\beta(f_0) = e_1$ by Theorem \[2.4\] and combining with \[2.7\] we obtain $(P^1)^* \beta(f_0) = e_0$. We also have $\beta(P^1)^* (f_0) = 0$ for degree reasons. Finally,

$$Q_1'(f_0) = ((P^1)^* \beta - \beta(P^1)^*) (f_0) = e_0,$$

so by Lemma \[2.3\] the relation $\beta_{1,1}(b') = e'_0$ holds. □

3. The units of $ku$ and the higher Bott element

The aim of this section is to define low-dimensional classes in $V(1)_* K(ku)$ by using the inclusion of units.

We recall from [30] or [31, Definition 7.6] that the space of units $GL_1(A)$ of an $E_\infty$-ring spectrum $A$ is defined by the following pull-back square of spaces

$$
\begin{array}{ccc}
GL_1(A) & \longrightarrow & \Omega^\infty A \\
\downarrow & & \downarrow \pi_0 \\
GL_1(\pi_0 A) & \longrightarrow & \pi_0 A.
\end{array}
$$

Taking the vertical fiber over $1 \in GL_1(\pi_0 A)$, we obtain a fiber sequence of group-like $E_\infty$-spaces or infinite loop spaces

$$SL_1(A) \to GL_1(A) \to GL_1(\pi_0 A),$$

with products given by the multiplicative structure of $A$. Here we can assume that we have a model of $GL_1(A)$ and of $SL_1(A)$ which is actually a topological monoid, see for example [39, §2.3]. The functor $GL_1$ from $E_\infty$-ring spectra to infinite loop spaces is right adjoint, up to homotopy, to the suspension functor $\Sigma_+^\infty$. This follows from [31, Lemma 9.6].
In the case of $ku$, the space $SL_1(ku)$ is commonly denoted $BU\otimes$. This notation refers to the product of the underlying $H$-space of $BU\otimes$, which represents the tensor product of virtual line bundles.

The first Postnikov section $\pi : BU\otimes \to K(Z, 2)$, with homotopy fiber denoted by $BSU\otimes$, admits a section $j : K(Z, 2) \simeq BU(1) \to BU\otimes$. Here the map $j$ represents viewing a line bundle as a virtual line bundle. Both $\pi$ and $j$ are infinite loop maps, and we have a splitting of infinite loop-spaces

$$BU\otimes \simeq K(Z, 2) \times BSU\otimes,$$

see [30, V.3.1]. We denote by $Bj : K(Z, 3) \to BBU\otimes$ a first delooping of $j$, fitting in a homotopy commutative diagram

$$
\begin{array}{ccc}
K(Z, 2) & \xrightarrow{j} & BU\otimes \\
\downarrow s & & \downarrow \tilde{s} \\
\Omega K(Z, 3) & \xrightarrow{\Omega Bj} & \Omega BBU\otimes,
\end{array}
$$

where $\tilde{s}$ denotes the homotopy equivalence which is right adjoint to the suspension $s$ as in (2.2). We name $y_1 \in \pi_2 K(Z, 2) \cong \mathbb{Z}$ the generator that maps to $y \in H_2(K(Z, 2); \mathbb{F}_p)$ by the Hurewicz homomorphism. We have maps of based spaces

$$K(Z, 2) \xrightarrow{j} BU\otimes \xrightarrow{c_0} BU \times \{0\} \subset BU \times \mathbb{Z},$$

where $c_0$ is the inclusion in $BU \times \mathbb{Z}$ followed by the translation of the component of 1 to that of 0 in the $H$-group $BU \times \mathbb{Z}$. The map $c_0j$ is a $\pi_2$-isomorphism, and we define

$$u = c_0j_*(y_1) \in \pi_2(BU \times \mathbb{Z}).$$

We call $u$ the Bott class. We have an isomorphism of rings

$$\pi_*(BU \times \mathbb{Z}) = \pi_*(ku) \cong \mathbb{Z}[u]$$

given by Bott periodicity. The map $c_{0*} : \pi_*(BU\otimes) \to \pi_*(BU \times \mathbb{Z})$ is an isomorphism in positive degrees, and we define $y_n \in \pi_{2n}(BU\otimes)$ by requiring $c_{0*}(y_n) = u^n$. Finally, we define

$$\sigma'_n \in V(1)_{2n+1}BBU\otimes$$

as the image of $y_n$ under the composition

$$\pi_{2n}BU\otimes \xrightarrow{h_*} V(1)_{2n}BU\otimes \xrightarrow{\sigma} V(1)_{2n+1}BBU\otimes.$$ 

Here the first map is the Hurewicz homomorphism from (unstable) homotopy to $V(1)$-homotopy, and $\sigma$ is the suspension induced by the map $s : \Sigma BU\otimes \to BBU\otimes$.

**Lemma 3.1.** Consider the homomorphism

$$Bj_* : V(1)_3 K(Z, 3) \to V(1)_3 BBU\otimes$$

induced by the map defined above. We have $\sigma'_1 = (Bj)_*(e'_0)$, where $e'_0 = \sigma^3(1) \in V(1)_3 K(Z, 3)$, as given in Definition 2.6

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
\pi_2 K(Z, 2) & \xrightarrow{h_*} & V(1)_2 K(Z, 2) & \xrightarrow{\sigma} & V(1)_3 K(Z, 3) \\
\downarrow j_* & & \downarrow j_* & & \downarrow Bj_* \\
\pi_2 BU\otimes & \xrightarrow{h_*} & V(1)_2 BU\otimes & \xrightarrow{\sigma} & V(1)_3 BBU\otimes.
\end{array}
$$
The right-hand square is induced in $V(1)$-homotopy from the square left adjoint to the square $(3.1)$. The class $y_1 \in \pi_2 K(\mathbb{Z}, 2)$ was chosen so that $h_*(y_1) = \sigma^2(1)$ in $V(1)_2 K(\mathbb{Z}, 2) \cong H_2(K(\mathbb{Z}, 2); \mathbb{F}_p)$. The lemma follows, since
\[
\sigma_1' = \sigma h_* j_*(y_1) = (Bj)_* \sigma h_*(y_1) = (Bj)_* \sigma^3(1) = (Bj)_*(e'_0).
\]

The space $\Omega^\infty K(ku)$ is defined as the group completion of the topological monoid $\prod_n BGL_n(ku)$, with product modelling the block-sum of matrices, see for instance [20 VI.7]. The composition
\[
w : BBU_\otimes \to BGL_1(ku) \to \prod_n BGL_n(ku) \to \Omega^\infty K(ku)
\]
factors through an infinite loop map $BBU_\otimes \to SL_1 K(ku)$, which is right adjoint to a map
\[
\omega : \Sigma^\infty_+ BBU_\otimes \to K(ku)
\]
of commutative $S$-algebras. We consider also the map of commutative $S$-algebras
\[
\phi : \Sigma^\infty_+ K(\mathbb{Z}, 3) \to K(ku)
\]
defined as the composition of the suspension of $Bj : K(\mathbb{Z}, 3) \to BBU_\otimes$ with the map $\omega$.

**Definition 3.2.** For $n \geq 1$, we define
\[
\sigma_n = \omega_*(\sigma'_n) \in V(1)_{2n+1} K(ku),
\]
where $\sigma'_n$ is the class given in $(3.2)$. We define the “higher Bott element” as
\[
b = \phi_*(b') \in V(1)_{2p+2} K(ku),
\]
where $b' \in V(1)_{2p+2} K(\mathbb{Z}, 3)$ is the class given in Definition $(2.6)$.

**Remark 3.3.** Notice that by Proposition $(2.7)$ the classes $b$ and $\sigma_1$ are related by a primary mod $(v_1)$ homotopy Bockstein $\beta_{1,1}(b) = \sigma_1$.

**Remark 3.4.** Assume that $p$ is an odd prime. If $R$ is a number ring containing a primitive $p$-th root of unity $\zeta_p$, for example $R = \mathbb{Z}[\zeta_p]$, then the mod $(p)$ algebraic $K$-theory of $R$ contains a non-nilpotent class
\[
\beta \in V(0)_2 K(R),
\]
called the Bott element, which we referred to in the introduction. It was defined by Browder [15] using the composition
\[
BC_p \to BGL_1 R \to \Omega^\infty K(R)
\]
analogous to $(3.3)$, and its adjoint
\[
\phi : \Sigma^\infty_+ BC_p \to K(R).
\]
Here $C_p$ denotes the cyclic subgroup of order $p$ of $GL_1(R)$ generated by $\zeta_p$. By inspection, the class $x = \zeta_p - 1$ satisfies $x^p = 0$ in the group-ring $\mathbb{F}_p[C_p] = V(0)_0 C_p$, and has a well defined “transpotence” $\beta' \in V(0)_2 BC_p$, supporting a primary mod $(p)$ homotopy Bockstein $\beta_{0,1}(\beta') = \sigma(1) \in V(0)_1 BC_p$. The classical Bott element can then be defined as
\[
\beta = \phi_*(\beta') \in V(0)_2 K(R).
\]
An embedding of rings $R \subset \mathbb{C}^{\text{top}}$, where $\mathbb{C}^{\text{top}}$ has the Euclidean topology, induces a map of commutative $S$-algebras $\iota : K(R) \to K(\mathbb{C}^{\text{top}}) = ku$ in algebraic $K$-theory. Browder’s
Proposition [15 2.2] implies that \( \iota \phi_s(\beta') = u \), where \( u \) is the Bott class in \( V(0)_* ku \cong P(u) \). This proves that \( \beta \) is non-nilpotent and is related to the Bott periodicity of topological \( K \)-theory. Snaith showed [40] that the relation \( \beta^p = v_1 \beta' \) in \( V(0)_* BC_p \) promotes to the relation \( \beta^{p-1} = v_1 \) in \( V(0)_* K(R) \).

The remark above makes it clear that our construction of \( b \in V(1)_{2p+2} K(ku) \) is inspired from the classical Bott element, and that these classes share interesting properties. This provides some justification for calling \( b \) a higher Bott element. Here higher refers to the fact that \( b \) lives one chromatic step higher than \( \beta \), in the sense that it is defined only in algebraic \( K \)-theory modulo \( (p, v_1) \) and that it is related to \( v_2 \)-periodicity. Indeed, recall from Theorem 1.1 and Proposition 1.3 that \( b \) is non-nilpotent and that the relation \( b^p = -v_2 b' \) in \( V(1)_* K(Z, 3) \) promotes to the relation \( b^{p-1} = -v_2 \) in \( V(1)_* K(ku) \). Our proof of these assertions relies on the computation of the cyclotomic trace for \( ku \), and is much more technical than in the number ring case: unfortunately, in the present situation we don’t have an analogue of the map \( K(R) \to K(\mathbb{C}^{top}) \), but see the remark below for a possible candidate.

**Remark 3.5.** John Rognes conjectured [4] that if \( \Omega_1 \) is a separably closed \( K(1) \)-local pro-Galois extension of \( ku \), in the sense of [38], then there is a weak equivalence

\[
L_{K(2)} K(\Omega_1) \simeq E_2,
\]

where \( L_{K(2)} \) is the Bousfield localization functor with respect to the Morava \( K \)-theory \( K(2) \), and where \( E_2 \) is the second Morava \( E \)-theory spectrum [21] with coefficients

\[
(E_2)_* = W(\mathbb{F}_p)[[u]][u, u^{-1}].
\]

This would provide a map

\[
\iota : K(ku) \to L_{K(2)} K(\Omega_1) \simeq E_2
\]

that might play the role, at this chromatic level, of the map \( K(R) \to K(\mathbb{C}^{top}) \) mentioned in Remark 3.4. Since \( V(1)_* E_2 \cong \mathbb{F}_p[u, u^{-1}] \) with \( u^p - 1 = v_2 \), we presume that the class \( b \) would be detected by the non-nilpotent class

\[
\iota_*(b) = \alpha^p u^{p+1} \in V(1)_* E_2
\]

for some \( \alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p \) with \( \alpha^{p-1} = -1 \). More generally, we expect that a periodic higher Bott element can be defined in \( V(1)_* K(A) \) if \( A \) is an commutative \( S \)-algebra with an \( S \)-algebra map \( A \to \Omega_1 \) and a suitable \( (p - 1) \)th-root of \( v_1 \) in \( V(0)_* A \).

### 4. The Trace Map

In this section, we consider the Bökstedt trace map [11]

\[
\text{tr} : K(ku) \to THH(ku)
\]

to topological Hochschild homology. This is a map of commutative \( S \)-algebras, and it induces a homomorphism of graded-commutative algebras in \( V(1) \)-homotopy, which we just call the trace. Our aim here is to prove that for \( n \leq p - 2 \) the classes \( \sigma_n \) and \( b \) defined above are non-zero in \( V(1)_* K(ku) \), as well as some of their products, see Proposition 4.6.
We achieve this by showing that these classes have a non-zero trace in $V(1)_*THH(ku)$. To this end, we briefly recall the computation of $V(1)_*THH(ku)$ given in [11, 9.15].

The topological Hochschild homology spectrum $THH(ku)$ is a $ku$-algebra, and its $V(1)$-homotopy groups form an algebra over the truncated polynomial algebra $V(1)_*ku = P_{p-1}(u)$, where we also denote by $u$ the mod $(p, v_1)$ reduction of the Bott class $u \in \pi_2 ku$. There is a free $\mathbb{F}_p$-sub-algebra $E(\lambda_1) \otimes P(\mu)$ in $V(1)_*THH(ku)$, and there is an isomorphism of $E(\lambda_1) \otimes P(\mu) \otimes P_{p-1}(u)$-modules

$$V(1)_*THH(ku) \cong E(\lambda_1) \otimes P(\mu) \otimes Q_*,$$

where $Q_*$ is the $P_{p-1}(u)$-module given by

$$Q_* = P_{p-1}(u) \oplus P_{p-2}(u)\{a_0, b_1, a_1, b_2, \ldots, a_{p-2}, b_{p-1}\} \oplus P_{p-1}(u)\{a_{p-1}\}.$$

The degree of these generators is given by $|\lambda_1| = 2p - 1$, $|\mu| = 2p^2$, $|a_i| = 2pi + 3$ and $|b_j| = 2pj + 2$. The isomorphism (4.1) is an isomorphism of $P_{p-1}(u)$-algebras if the product on the $P_{p-1}(u)$-module generators of $Q_*$ is given by the relations

$$\begin{align*}
b_ib_j &= ub_{i+j} & i + j &\leq p - 1, \\
b_ib_j &= ub_{i+j-p} & i + j &> p, \\
a_ib_j &= ua_{i+j} & i + j &\leq p - 1, \\
a_ib_j &= ua_{i+j-p} & i + j &> p, \\
a_ia_j &= 0 & 0 &\leq i, j \leq p - 1.
\end{align*}$$

Here by convention $b_0 = u$. For example we have a product

$$(u^ka_i)(u^jb_j) = u^{p-2}a_{p-1}$$

if $k + l = p - 3$ and $i + j = p - 1$.

**Remark 4.1.** The class $\mu$ is called $\mu_2$ in [11], but we adopt here the notation of [31].

The classes $u^{n-1}a_0 \in V(1)_{2n+1}THH(ku)$ for $1 \leq n \leq p - 2$ are constructed as follows. The circle action $S^1_+ \wedge THH(ku) \rightarrow THH(ku)$ restricts in the homotopy category to a map $d : \Sigma THH(ku) \rightarrow THH(ku)$, which in any homology theory $E_*$ induces Connes' operator

$$d : E_*THH(ku) \rightarrow E_{*+1}THH(ku).$$

We have an $S$-algebra map $l : ku \rightarrow THH(ku)$ given by the inclusion of zero-simplices. Composing the induced map in $E_*$-homology with $d$ yields a suspension homomorphism

$$dl_* : E_*ku \rightarrow E_{*+1}THH(ku),$$

see [32, 3.2] (it is often denoted $\sigma$). For $1 \leq n \leq p - 2$, we define the class $u^{n-1}a_0$ as the image

$$u^{n-1}a_0 = dl_*(u^n)$$

of $u^n \in V(1)_*ku$. Mapping to homology, we can show that these classes are non-zero. By Lemma [23, 22] the Hurewicz homomorphism

$$h_* : V(1)_*THH(ku) \rightarrow H_*(THH(ku); \mathbb{F}_p)$$

is an isomorphism in degrees $* \leq 2p - 3$ (notice that $\alpha_1 = 0$ in $V(1)_*THH(ku)$ since $THH(ku)$ is a $ku$-algebra). Let $x = h_*(u) \in H_2(ku; \mathbb{F}_p)$ be the image of $u \in V(1)_2ku$. 

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We then have \( h_*(u^{n-1}a_0) = dl_*(x^n) \) in \( H_{2n+1}(\text{THH}(ku); \mathbb{F}_p) \), and this class represents the permanent cycle \( 1 \otimes x^n \) in \( E_1^{1,2n}(ku) \) in the Bökstedt spectral sequence
\[
E_1^{s,t}(ku) = H_s(ku; \mathbb{F}_p)^{\otimes (s+1)}, \\
E_2^{s,t}(ku) = HHH^p_s(H_s(ku; \mathbb{F}_p)) \Rightarrow H_{s+t}(\text{THH}(ku); \mathbb{F}_p).
\]
This proves that the classes \( h_*(u^{n-1}a_0) \) are non-zero for these values of \( n \). We refer to \([11] \S 9\) for more details.

**Lemma 4.2.** If \( 1 \leq n \leq p - 2 \), the class \( \sigma'_n \) of \((3.2)\) maps to the class \( u^{n-1}a_0 \) under the composition
\[
V(1)_*BBU \xrightarrow{w} V(1)_*K(ku) \xrightarrow{\text{tr}} V(1)_*\text{THH}(ku).
\]

**Proof.** As mentioned above, \( h_* : V(1)_{2n+1}\text{THH}(ku) \to H_{2n+1}(\text{THH}(ku); \mathbb{F}_p) \) is an isomorphism for \( n \leq p - 2 \) and maps \( u^{n-1}a_0 \) to \( dl_*(x^n) \). Thus, passing to homology and using the definition of \( \sigma'_n \) in \((3.2)\), it suffices to prove that the composition
\[
H_{2n}(BU_\otimes; \mathbb{F}_p) \xrightarrow{\sigma} H_{2n+1}(BU_\otimes; \mathbb{F}_p) \xrightarrow{\text{tr}_{\omega_0}} H_{2n+1}(\text{THH}(ku); \mathbb{F}_p)
\]
maps \( z_n = h_*(y_n) \in H_{2n}(BU_\otimes; \mathbb{F}_p) \) to \( dl_*(x^n) \). Here we also denoted by \( h_* \) the Hurewicz homomorphism \( \pi_{2n}BU \to H_{2n}(BU_\otimes; \mathbb{F}_p) \). First, we need some information on the trace map. We will use the following commutative diagram of spaces

\[
\begin{array}{ccc}
BBU_\otimes & \xrightarrow{i} & B^\gamma BU_\otimes \\
\downarrow{w} & & \downarrow{\tau} \\
\Omega^\infty K(ku) & \xrightarrow{\Omega^\infty w} & \Omega^\infty \text{THH}(ku) \xrightarrow{i} \Omega^\infty ku,
\end{array}
\]

which is assembled from \([39] \S 4\). Here the space \( B^\gamma BU_\otimes \) is the realization of the cyclic nerve of the topological monoid \( BU_\otimes \) and, as \( \Omega^\infty \text{THH}(ku) \), is equipped with a canonical \( S^1 \)-action. The map \( \tau \) is the realization of a morphism of cyclic spaces, and is therefore \( S^1 \)-equivariant. The maps \( l \) are given by the inclusion of 0-simplices, while \( c_1 \) is the inclusion of the component of 1. There is a homotopy fibration \([39] \text{Proposition } 3.1\]
\[
BU_\otimes \xrightarrow{l} B^\gamma BU_\otimes \xrightarrow{p} BBU_\otimes,
\]
and the map \( p \) admits a section up to homotopy \( i : BBU_\otimes \to B^\gamma BU_\otimes \).

Let \( d \) be Connes’ operator on \( H_*(B^\gamma BU_\otimes; \mathbb{F}_p) \) and \( H_*(\Omega^\infty \text{THH}(ku); \mathbb{F}_p) \). It commutes with \( \tau_* : H_*(B^\gamma BU_\otimes; \mathbb{F}_p) \to H_*(\Omega^\infty \text{THH}(ku); \mathbb{F}_p) \) since \( \tau \) is equivariant. In the next lemma, we prove that
\[
dl_*(z_n) = i_*\sigma(z_n)
\]
holds in \( H_{2n+1}(B^\gamma BU_\otimes; \mathbb{F}_p) \). Using \((4.3)\), we deduce
\[
(\Omega^\infty \text{tr})_w^* \sigma(z_n) = \tau_* l_* \sigma(z_n) = d\tau_* l_*(z_n) = dl_* c_{1*}(z_n).
\]
Finally, composing with the stabilization map
\[
st : H_*(\Omega^\infty \text{THH}(ku); \mathbb{F}_p) \to H_*(\text{THH}(ku); \mathbb{F}_p)
\]
to spectrum homology, we obtain
\[
\text{tr}_{\omega_0} \sigma(z_n) = st(\Omega^\infty \text{tr})_w^* \sigma(z_n) = stdl_*(c_{1*}(z_n)) = dl_*(x^n).
\]
For the last equality, we used that stabilization commutes with \( dl_* \), and that \( stc_{1*}(z_n) = x^n \) for \( 1 \leq n \leq p - 2 \). \( \square \)
Lemma 4.3. The equality \( dl_*(z_n) = i_*\sigma(z_n) \) holds in \( H_{2n+1}(B^{cy}BU_\otimes; \mathbb{F}_p) \).

Proof. We consider the homotopy fibration \((4.5)\). Since \( H_*(BU_\otimes; \mathbb{F}_p) \) is concentrated in even degrees, the map \( p_* : H_*(BU_\otimes; \mathbb{F}_p) \to H_*(BBU_\otimes; \mathbb{F}_p) \) restricts to an isomorphism

\[
p_* : \text{Prim}(H_{2n+1}(B^{cy}BU_\otimes; \mathbb{F}_p)) \to \text{Prim}(H_{2n+1}(BBU_\otimes; \mathbb{F}_p))
\]

of the subgroups of primitive elements in degree \( 2n + 1 \), with the restriction of \( i_* \) as inverse. The class \( l_*(z_n) \) is spherical, hence primitive, and it follows from \( d(1) = 0 \) that \( dl_*(z_n) \) is also primitive.

Next, we consider the diagram

\[
\begin{array}{ccc}
S^1 \times BU_\otimes & \xrightarrow{1 \times l} & S^1 \times B^{cy}BU_\otimes \\
\downarrow & & \downarrow \\
S^1 \wedge BU_\otimes & \xrightarrow{s} & BBU_\otimes,
\end{array}
\]

where \( \mu \) denotes the \( S^1 \)-action on \( B^{cy}BU_\otimes \) and \( s \) the suspension map \((2.2)\). This diagram is commutative, as can be checked at simplicial level by using the definition of \( \mu \), see for example \([27, 7.1.9]\). Therefore \( p_* dl_*(z_n) = \sigma(z_n) \), and since \( dl_*(z_n) \) is primitive, we have

\[
dl_*(z_n) = i_*p_*dl_*(z_n) = i_*\sigma(z_n).
\]

\[\square\]

Lemma 4.4. The class \( b' \) maps to the class \( b_1 \) under the composition

\[
V(1)_*K(\mathbb{Z}, 3) \xrightarrow{\phi_*} V(1)_*K(ku) \xrightarrow{tr_*} V(1)_*THH(ku).
\]

Proof. We know from Lemma \([11,4]\) and Lemma \([11,2]\) that \( e'_0 \in V(1)_3K(\mathbb{Z}, 3) \) maps to the class \( a_0 \) in \( V(1)_3THH(ku) \). We have primary mod \((v_1)\) homotopy Bockstein

\[
\beta_{1,1}(b') = e'_0 \quad \text{and} \quad \beta_{1,1}(b_1) = a_0
\]

in \( V(1)_*K(\mathbb{Z}, 3) \) and \( V(1)_*THH(ku) \) respectively, see Proposition \([27, 9.19]\). Moreover \( V(1)_2THH(ku) = \mathbb{F}_p\{b_1\} \), so that \( \beta_{1,1} \) is injective on this group. The result follows, since

\[
\beta_{1,1}tr_*\phi_*(b') = tr_*\phi_*(\beta_{1,1}(b')) = tr_*\phi_*(e'_0) = a_0.
\]

\[\square\]

Let \( \kappa : ku \to ku_p \) be the completion at \( p \). It induces the inclusion \( \mathbb{Z}[u] \to \mathbb{Z}_p[u] \) of coefficients rings.

Definition 4.5. We also denote by

\[
\sigma_n \in V(1)_{2n+1}K(ku_p) \quad \text{and} \quad b \in V(1)_{2p+2}K(ku_p)
\]

the image under \( \kappa_* : V(1)_*K(ku) \to V(1)_*K(ku_p) \) of the classes \( \sigma_n \) and \( b \) defined in \([3.2]\).

Proposition 4.6. The classes

\[
\begin{cases}
   b^k & \text{for } 0 \leq k \leq p - 2, \text{ and} \\
   \sigma_n b^l & \text{for } 1 \leq n \leq p - 2 \text{ and } 0 \leq l \leq p - 2 - n
\end{cases}
\]

are non-zero in \( V(1)_*K(ku) \) and in \( V(1)_*K(ku_p) \).
Proof. For $V(1)_*K(ku)$, it follows from Lemma 4.2 and the structure of $V(1)_*THH(ku)$ given in (4.2). In more detail, we have $\text{tr}_*(b^p) = b^p_1 \neq 0$ for $k \leq p - 2$ and $\text{tr}_*(\sigma_nb^n) = u^{n-l}a_0b_1^n = u^{n+l-1}a_l \neq 0$ for $l \leq p - 3$ and $n + l - 1 \leq p - 3$. Notice that we have a commutative diagram

$$
\begin{array}{ccc}
V(1)_*K(ku) & \xrightarrow{\text{tr}_*} & V(1)_*THH(ku) \\
\kappa_* & & \kappa_* \\
V(1)_*K(ku_p) & \xrightarrow{\text{tr}_*} & V(1)_*THH(ku_p).
\end{array}
$$

The map $\kappa : THH(ku) \to THH(ku_p)$ is a weak equivalence after $p$-completion, so in this diagram the right-hand $\kappa_*$ is an isomorphism. This proves that the result also holds for $V(1)_*K(ku_p)$. \qed

Remark 4.7. We claimed in Theorem 1.1 and Proposition 1.3 that $b$ is non-nilpotent in $V(1)_*K(ku)$. However, we have

$$
\text{tr}_*(b^{p-1}) = \text{tr}_*(b)^{p-1} = b_1^{p-1} = u^{p-2}b_{p-1} = 0
$$

in $V(1)_*THH(ku)$, so that the Bökstedt trace is not sufficient for proving this assertion. This is of course also predicted by our other claim that $b^{p-1} = -v_2$ holds in $V(1)_*K(ku)$. Indeed, $v_2$ maps to zero in $V(1)_*THH(ku)$ since $THH(ku)$ is a $ku$-algebra.

5. Algebraic $K$-theory in low degrees

In this section, we compute the groups $V(1)_*K(ku_p)$ in degrees $* \leq 2p - 2$. This complements the computations presented in the next sections, which are based on evaluating the fixed points of $THH(ku)$ and which are valid only in degrees larger than $2p - 2$, see Proposition 6.7.

Consider the Adams summand

$$
\ell_p = ku_p^{h\Delta}
$$

of $ku_p$, where $\Delta \cong \mathbb{Z}/(p - 1)$ is the finite subgroup of the $p$-adic units, acting on $ku_p$ by $p$-adic Adams operations, and where $(-)^{h\Delta}$ denotes the homotopy fixed points. By Theorem 10.2 of [1], the natural map $V(1)_*K(\ell_p) \to V(1)_*K(ku_p)$ factors through an isomorphism

$$
V(1)_*K(\ell_p) \cong (V(1)_*K(ku_p))^\Delta \subset V(1)_*K(ku_p)
$$

onto the elements of $V(1)_*K(ku_p)$ fixed under the induced action of $\Delta$. In the sequel, we identify $V(1)_*K(\ell_p)$ with its image in $V(1)_*K(ku_p)$.

The $V(1)_*$-homotopy of $K(\ell_p)$ is computed in [3]. In the degrees we are concerned with here, namely $* \leq 2p - 2$, $V(1)_*K(\ell_p)$ is generated as an $\mathbb{F}_p$-vector space by the classes listed in

$$
\{1, \lambda_1t^d, s, \partial\lambda_1 \mid 0 < d < p\},
$$

of degree $|\lambda_1t^d| = 2p - 2d - 1$, $|s| = 2p - 3$ and $|\partial\lambda_1| = 2p - 2$, see [3, 9.1] (where the sporadic $v_2$-torsion class $s$ was denoted $a$). The zeroth Postnikov section $\ell_p \to HZ_p$ is a $(2p - 2)$-connected map, so that the induced map $K(\ell_p) \to K(Z_p)$ is $(2p - 1)$-connected [12, Proposition 10.9]. All the classes listed in (5.2) map to classes with same name in $V(1)_*K(Z_p)$, which is given by the formula

$$
V(1)_*K(Z_p) \cong E(\lambda_1) \oplus \mathbb{F}_p\{s, \partial\lambda_1\} \oplus \mathbb{F}_p\{\lambda_1t^d \mid 0 < d < p\}.
$$
The name of the classes in this formula refers to permanent cycles in the $S^1$ homotopy fixed-point spectral sequence used in the computation of $V(1)_*K(\mathbb{Z}_p)$ by traces, compare with Theorem 7.9. If desired, these classes could be given a more memorable name by means of the inclusion

$$V(0)_*K(\mathbb{Z}_p) \to V(0)_*K(\mathbb{Q}_p(\zeta_p)),$$

in the target of which they can be decomposed as a product of a unit and a power of the Bott element $\beta \in V(0)_2K(\mathbb{Q}_p(\zeta_p))$.

Using the inclusion given in (5.1), we view the classes listed in (5.2) as elements of $V(1)_*K(ku_p)$. The following lemma implies that these classes are linearly independent of the classes in $V(1)_*K(ku_p)$ constructed in the previous section.

Lemma 5.1. The non-zero classes $b^k$ and $\sigma_n b^l$ in $V(1)_*K(ku_p)$ given in Proposition 4.6 are not fixed under the action of $\Delta$.

Proof. All these classes map into $V(1)_*THH(ku)$ to classes which do not lie in the image of $V(1)_*THH(ku)$, and hence which are not fixed under the action of $\Delta$, see Proposition 10.1 of [1].

Proposition 5.2. The inclusion

$$\mathbb{F}_p\{1, \sigma_n, \lambda_1 t^d, s, \partial \lambda_1 \mid 1 \leq n \leq p-2, \ 0 < d < p\} \subset V(1)_*K(ku_p)$$

of graded $\mathbb{F}_p$-vector spaces is an isomorphism in degrees $\leq 2p - 2$.

Proof. We have constructed all the classes listed above and have argued that they are linearly independent. It suffices therefore to compute the dimension of $V(1)_nK(ku_p)$ as an $\mathbb{F}_p$-vector space for all $0 \leq n \leq 2p - 2$.

Consider a double loop map $\Omega S^3 \to BU_\otimes$ such that the composition

$$S^2 \to \Omega S^3 \to BU_\otimes,$$

where $S^2 \to \Omega S^3$ is the adjunction unit, represents the class $y_1 \in \pi_2 BU_\otimes$ defined in Section 3. By adjunction we have a map of $E_2$-ring spectra

$$S[\Omega S^3] \to ku,$$

where $S[\Omega S^3]$ is another notation for the suspension spectrum $\Sigma^\infty_+ \Omega S^3$. We refer to [5, Proposition 2.2] for some more details on the construction of this map. After $p$-completion this map is $(2p - 3)$-connected, and induces a $(2p - 2)$-connected map $K(S[\Omega S^3])_p \to K(ku_p)$. The dimension of the $\mathbb{F}_p$-vector space $V(1)_nK(S[\Omega S^3])_p$ for $n \leq 2p - 2$ is computed in the following lemma, and this completes the proof of this proposition. Notice that a priori

$$V(1)_{2p-2}K(S[\Omega S^3])_p \to V(1)_{2p-2}K(ku_p)$$

is only surjective, but luckily $V(1)_{2p-2}K(S[\Omega S^3])_p$ is of rank one. Since we know that the rank of $V(1)_{2p-2}K(ku_p)$ is at least one, we also have an isomorphism in this degree.

Lemma 5.3. The dimension of $V(1)_nK(S[\Omega S^3])_p$ as an $\mathbb{F}_p$-vector space is

$$\begin{cases} 
1 & \text{if } n = 0, 1, 2p - 2, \\
2 & \text{if } n \text{ is odd with } 3 \leq n \leq 2p - 5, \\
3 & \text{if } n = 2p - 3, \\
0 & \text{for other values of } n \leq 2p - 2.
\end{cases}$$
Proof. We compute \( V(1)_*K(S[\Omega S^3]_p) \) in degrees less than \( 2p - 1 \) by using the cyclotomic trace map to topological cyclic homology \([11]\), which sits in a cofibre sequence \([22]\)

\[
K(S[\Omega S^3]_p)_p \xrightarrow{\text{tc}} TC(S[\Omega S^3]_p) \rightarrow \Sigma^{-1} \mathbb{H}Z_p \rightarrow \Sigma K(S[\Omega S^3]_p)_p.
\]

Here \( TC(X) = TC(X; p) \) denotes the \((p\text{-completed})\) topological cyclic homology spectrum of a spectrum \( X \). By inspection, it suffices to prove that we have

\[
\dim_{\mathbb{F}_p} V(1)_* TC(S[\Omega S^3]_p) = \begin{cases} 
1 & \text{if } n = -1, 0, 1, 2p - 2, \\
2 & \text{if } n \text{ is odd with } 3 \leq n \leq 2p - 3, \\
0 & \text{for other values of } n \leq 2p - 2.
\end{cases}
\]

(5.3)

Indeed, \( V(1)_* \Sigma^{-1} \mathbb{H}Z_p \) consists of a copy of \( \mathbb{F}_p \) in degrees \(-1\) and \( 2p - 2 \), and is zero in other degrees. We have an isomorphism \( V(1)_{-1} TC(S[\Omega S^3]_p) \rightarrow V(1)_{-1} \Sigma^{-1} \mathbb{H}Z_p \), and the sporadic class \( s \) is in the image of the connecting homomorphism

\[
V(1)_{2p-3} \Sigma^{-1} \mathbb{H}Z_p \rightarrow V(1)_{2p-3} K(S[\Omega S^3]_p),
\]

by naturality with respect to \( S[\Omega S^3]_p \rightarrow \mathbb{H}Z_p \), see for example \([3, \text{Proof of 9.1}]\).

The reduced topological cyclic homology spectrum \( \tilde{TC}(S[\Omega S^3]_p) \) is the homotopy fibre of the map \( c : TC(S[\Omega S^3]_p) \rightarrow TC(S_p) \) induced by the map \( S^3 \rightarrow \ast \) to a one-point space. The maps \( c \) admits a splitting, and we have a decomposition

\[
TC(S[\Omega S^3]_p) \simeq TC(S_p) \lor \tilde{TC}(S[\Omega S^3]_p).
\]

The spectrum \( TC(S_p) \) decomposes as

\[
TC(S_p) \simeq S_p \lor \Sigma \mathbb{C}P^\infty_1,
\]

where \( \mathbb{C}P^\infty_1 \) is the \((p\text{-completed})\) Thom spectrum of minus the canonical line bundle on \( \mathbb{C}P^\infty \), see \([29]\). The homology of \( \Sigma \mathbb{C}P^\infty_1 \) is given by

\[
H_*(\Sigma \mathbb{C}P^\infty_1; \mathbb{F}_p) \cong \mathbb{F}_p\{x_i \mid i \geq -1\}
\]

with \( |x_i| = 2i + 1 \). Moreover these classes can be chosen so that the relations

\[
(P^1)^*(x_{p-2}) = x_{-1} \quad \text{and} \quad (P^1)^*(x_{p-1}) = 0
\]

hold. It follows from Lemma \([23]\) that we have an inclusion

\[
\mathbb{F}_p\{c_i \mid -1 \leq i \leq p - 3\} \cup \mathbb{F}_p\{\alpha(x_0)\} \subset V(1)_*(\Sigma \mathbb{C}P^\infty_1),
\]

which is an isomorphism in degrees \(* \leq 2p - 2 \), with \( h_*(c_i) = x_i \). These classes have degree \( |c_i| = 2i + 1 \) and \( |\alpha(x_0)| = 2p - 2 \).

By \([10, \text{3.9}]\), we have a decomposition

\[
\tilde{TC}(S[\Omega S^3]_p) \simeq \Sigma^\infty S^3_p \lor \tilde{V},
\]

where \( \tilde{V} \) is the \((p\text{-completed})\) homotopy fiber of the composition

\[
\Sigma^\infty \Sigma(ES^1_+ \wedge S^3 \overset{\text{trf}}{\longrightarrow} \Sigma^\infty LS^3 \overset{\epsilon_1}{\longrightarrow} \Sigma^\infty S^3).
\]

Here \( \text{trf} \) is the dimension-shifting \( S^1\text{-transfer} \) on the free loop space \( LS^3 \) of \( S^3 \), and \( \epsilon_1 \) is the evaluation at \( 1 \in S^1 \), see \([29]\). We consider the Serre spectral sequence

\[
E^2_{*,*} = H_*(BS^1; H_*(LS^3, \mathbb{F}_p)) \Rightarrow H_*(ES^1 \times S^1 LS^3; \mathbb{F}_p).
\]
We have isomorphisms
\[ H_\ast(BS^1; \mathbb{F}_p) \cong H_\ast(K(\mathbb{Z}, 2); \mathbb{F}_p) = \Gamma(y) \quad \text{and} \]
\[ H_\ast(LS^3; \mathbb{F}_p) \cong P(z) \otimes E(dz). \]

Here \( z \in H_2(\Omega S^3; \mathbb{F}_p) \subset H_2(LS^3; \mathbb{F}_p) \) and \( dz \in H_3(LS^3; \mathbb{F}_p) \) is the suspension of \( z \) associated to the circle action on \( LS^3 \). In particular, we have a non-zero \( d^2 \)-differential
\[ d^2(yz) = dz. \]

For degree reasons no further non-zero differential involves the classes in total degree less than \( 2p \), and we have an inclusion
\[ P_p(y) \oplus \mathbb{F}_p\{z^j | 1 \leq j \leq p - 1\} \subset H_\ast(ES^1 \times \mathbb{S}^1; LS^3; \mathbb{F}_p) \]
which is an isomorphism in degrees less than \( 2p \). We deduce that the inclusion
\[ \Sigma \mathbb{F}_p(z^j | 1 \leq j \leq p - 1\} \subset H_\ast(\Sigma^\infty(ES^1_+ \wedge \mathbb{S}^1; LS^3); \mathbb{F}_p) \]
is an isomorphism in degrees less the \( 2p - 1 \). The homomorphism
\[ (e_1 \text{trf})_\ast : H_\ast(\Sigma^\infty(ES^1_+ \wedge \mathbb{S}^1; LS^3); \mathbb{F}_p) \rightarrow H_\ast(S^3; \mathbb{F}_p) = E(e) \]
maps \( \Sigma z \) to a generator \( e \) of \( H_3(S^3; \mathbb{F}_p) \) since the restriction of \( \text{trf} \) to \( \Sigma^\infty(\mathbb{S}^1_+ \wedge \mathbb{S}^1; LS^3) \) is induced by the circle action. This implies that we have an inclusion
\[ \mathbb{F}_p\{e, \Sigma z^j | 2 \leq j \leq p - 2\} \subset H_\ast(\Sigma^\infty S^3_p \vee \tilde{V}; \mathbb{F}_p) \cong H_\ast(\tilde{T}C(S[\Omega S^3_3], \mathbb{F}_p)) \]
which is an isomorphism in degrees smaller than \( 2p - 1 \). By Lemma 2.3, \[ \mathbb{F}_p\{e, \Sigma z^j | 2 \leq j \leq p - 2\} \subset V(1)_*TC(S[\Omega S^3_3]) \]
is also an isomorphism in degrees less than \( 2p - 1 \). In summary, we have
\[ V(1)_*TC(S[\Omega S^3_3]) \cong V(1)_* \oplus V(1)_*\Sigma\mathbb{CP}^\infty_1 \oplus V(1)_*\tilde{T}C(S[\Omega S^3_3]) \]
which is isomorphic to
\[ \mathbb{F}_p\{1, \alpha_1, c, \alpha(x_0), e, \Sigma z^j | 1 \leq i \leq p - 3, 2 \leq j \leq p - 2\} \]
in degrees smaller than \( 2p - 1 \). This proves that formula (5.3) for the rank of the \( \mathbb{F}_p \)-vector space \( V(1)_*TC(S[\Omega S^3_3]) \) is correct. \( \square \)

Remark 5.4. In an earlier proof of this lemma we used the space \( BU(1) \) and the map \( \theta : \Sigma^\infty BU(1) \rightarrow ku \) of commutative \( S \)-algebras. I thank John Rognes for noticing that using \( \Omega S^3 \) instead simplifies the computation. The maps
\[ S[\Omega S^3] \rightarrow \Sigma^\infty BU(1) \rightarrow ku \]
are \( \pi_0 \)-isomorphisms and rational equivalences. We use this in [5] to determine the rational homotopy type of \( K(ku) \).
6. The fixed points

In this section we compute the $V(1)$-homotopy groups of the homotopy limit

$$TF(ku_p) = \operatorname{holim}_{n,F} THH(ku_p)^{C_p^n},$$

where $F: THH(ku_p)^{C_p^{n+1}} \to THH(ku_p)^{C_p^n}$ is the Frobenius map. This will be used in the next section to compute the topological cyclic homology of $ku_p$. The strategy to perform such computations was developed in [12, 22, 43], but we will closely follow the exposition and adopt the notations of [3, §3, §5 and §6], with an exception: the $G$ Tate construction on an equivariant spectrum $X$ will be denoted by $X^{tG}$ instead of $\hat{\mathbb{H}}(G, X)$. We refer the reader to [3, §3] for a brief review of the homotopy commutative norm-restriction diagram

$$\begin{array}{ccc}
K(ku_p) & \xrightarrow{\operatorname{tr}_F} & THH(ku_p)^{h_{C_p^n}} \\
\downarrow \operatorname{tr}_n & & \downarrow \operatorname{tr}_{n-1} \\
\Sigma THH(ku_p)^{h_{C_p^n}} & \xrightarrow{N} & THH(ku_p)^{C_p^n-1} \\
\downarrow \Gamma_n & & \downarrow \Gamma_n \\
THH(ku_p)^{h_{C_p^n}} & \xrightarrow{N^h} & THH(ku_p)^{h_{C_p^n}} \\
\downarrow \tilde{\Gamma}_n & & \downarrow \tilde{\Gamma}_n \\
THH(ku_p)^{h_{C_p^n}} & \xrightarrow{R^h} & THH(ku_p)^{h_{C_p^n}} \\
& & \downarrow \\
& & \Sigma THH(ku_p)^{h_{C_p^n}}
\end{array}$$

for any $n \geq 1$, which is our essential tool. By passage to homotopy limits over the Frobenius maps, we obtain the homotopy commutative diagram

$$\begin{array}{ccc}
K(ku_p) & \xrightarrow{\operatorname{tr}_F} & TF(ku_p) \\
\downarrow \operatorname{tr}_F & & \downarrow \operatorname{tr}_F \\
\Sigma THH(ku_p)^{h_{S^1}} & \xrightarrow{N} & TF(ku_p) \\
\downarrow \Gamma & & \downarrow \Gamma \\
\Sigma THH(ku_p)^{h_{S^1}} & \xrightarrow{N^h} & THH(ku_p)^{h_{S^1}} \\
\downarrow \tilde{\Gamma} & & \downarrow \tilde{\Gamma} \\
\Sigma THH(ku_p)^{h_{S^1}} & \xrightarrow{R^h} & THH(ku_p)^{t_{S^1}} \\
& & \downarrow \\
& & \Sigma^2 THH(ku_p)^{h_{S^1}}
\end{array}$$

The map $i_*: V(1)_* THH(\ell_p) \to THH(ku_p)$ factors through an isomorphism onto the $\Delta$-fixed elements of $V(1)_* THH(ku_p)$,

$$i_*: V(1)_* THH(\ell_p) \xrightarrow{\cong} (V(1)_* THH(ku_p))^\Delta \subset V(1)_* THH(ku_p), \quad (6.1)$$

see [1, 10.1]. The corresponding results hold also for the $C_{p^n}$ or $S^1$ homotopy fixed points of $THH$, for the $C_{p^n}$ or $S^1$ Tate construction on $THH$, and for $TC$ and $K$, see [1, 10.2]. In the sequel, we identify $V(1)_* THH(\ell_p)$, $V(1)_* TC(\ell_p)$, etc. with their image under $i_*$. We have a similar statement for the various spectral sequences computing the $V(1)$-homotopy of these spectra.

**Lemma 6.1.** Let $G = S^1$ or $G = C_{p^n}$, and let $E^*(G, \ell_p)$ and $E^*(G, ku_p)$ be the $G$ homotopy fixed-point spectral sequences converging strongly to $V(1)_* THH(\ell_p)^{hG}$ and to $V(1)_* THH(ku_p)^{hG}$, respectively. Then the morphism of spectral sequences induced by the map $\ell_p \to ku_p$ is equal to the inclusion of the $\Delta$ fixed points

$$E^*(G, \ell_p) = (E^*(G, ku_p))^\Delta \subset E^*(G, ku_p).$$
This follows from (6.1).

\[ \hat{E}^*(G, \ell_p) = (\hat{E}^*(G, ku_p))^\Delta \subset \hat{E}^*(G, ku_p). \]

**Proof.** The group \( \Delta \) acts on \( ku_p \) by \( S \)-algebra maps, and it acts \( S^1 \)-equivariantly on \( THH(ku_p) \). In particular \( \Delta \) acts by morphisms of spectral sequences on \( E^*(G, ku_p) \) and \( \hat{E}^*(G, ku_p) \), and hence it suffices to prove that the claims hold at the level of the \( E^2 \)-terms. This follows from (6.1).

From now on, we will omit \( ku_p \) from the notation and just write \( E^*(G) \) and \( \hat{E}^*(G) \) for the \( G \) homotopy fixed-point and Tate spectral sequences converging to \( V(1)_*, THH(ku_p)^{BG} \) and \( V(1)_*, THH(ku_p)^{BG} \), respectively.

At this point, we recall the notion of \( \delta \)-weight introduced in \([11] \). We fix a generator \( \delta \) of the group \( \Delta \) acting on \( ku_p, K(ku_p), THH(ku_p), TC(ku_p) \), etc. The self-map \( \delta_\ast \) of \( V(1)_* ku_p = P_{p-1}(u) \) maps \( u \) to \( \alpha u \) for some generator \( \alpha \) of \( \mathbb{F}_p^\times \). We say that a class \( v \in V(1)_* K(ku_p) \) has \( \delta \)-weight \( i \in \mathbb{Z}/(p - 1) \) if \( \delta_\ast(v) = \alpha^i v \). The same convention holds for classes in \( V(1)_* THH(ku_p), V(1)_* TC(ku_p) \), etc. For example, the generators \( a_i \) and \( b_j \) of \( V(1)_* THH(ku_p) \) given in \([11] \) all have \( \delta \)-weight 1, see \([11] \). Similarly, it follows from its definition that \( b \in V(1)_* K(ku_p) \) has \( \delta \)-weight 1. Since \( \delta_\ast \) is diagonalizable, we can reinterpret Lemma \((6.1) \) by saying that each of these spectral sequences for \( ku_p \) has an extra \( \mathbb{Z}/(p - 1) \)-grading given by the \( \delta \)-weight, and that its homogeneous summand of \( \delta \)-weight 0 consists of the corresponding spectral sequence for \( \ell_p \). Together with the internal and filtration degrees, the \( \delta \)-weight endows the \( E^r \)-terms of these spectral sequences with a tri-grading that we will refer to in the computations below.

By a computation of McClure and Staffeldt \([11], [3], 2.6 \), we have an isomorphism of \( \mathbb{F}_p \)-algebras

\[ V(1)_* THH(\ell_p) \cong E(\lambda_1, \lambda_2) \otimes P(\mu). \]

The induced map \( V(1)_* THH(\ell_p) \to V(1)_* THH(ku_p) \) sends \( \lambda_1 \) and \( \mu \) to the classes with same name, and \( \lambda_2 \) to the class \( a_1 b_1^{p-2} \).

**Remark 6.2.** In the sequel, we will frequently denote by \( \lambda_2 \) the class \( a_1 b_1^{p-2} \).

The \( C_p \)-Tate spectral sequence

\[ \hat{E}(C_p)^2 = \hat{H}^s(C_p, V(1)_* THH(ku_p)) \Rightarrow V(1)_{* + i} THH(ku_p)^{C_p} \]

has an \( E_2 \)-term given by

\[ \hat{E}(C_p)^2 = P(t, t^{-1}) \otimes E(u_1) \otimes V(1)_* THH(ku_p) \]

with \( t \) in bidegree \((-2, 0)\), \( u_1 \) in bidegree \((-1, 0)\), and \( w \in V(1)_* THH(ku_p) \) in bidegree \((0, t)\). Recall the description of \( V(1)_* THH(ku_p) \) given in \([41] \).

**Lemma 6.3.** In the \( C_p \) Tate spectral sequence \( \hat{E}^*(C_p) \) the classes \( \lambda_1, \lambda_2, b_1 \) and \( t \mu \) are infinite cycles. There are non-zero differentials

\[ d^2(b_1) = (1 - i)a_i t \]

\[ d^{2p}(t^{1-p}) = \lambda_1 \cdot t \]

\[ d^{2p^2}(t^{p^2-p^2}) = \lambda_2 \cdot p^p \]

\[ d^{2p^2+1}(u_1 \cdot t^{p^2-p^2}) = t \mu \]
with \( 0 \leq i \leq p - 1 \). The spectral sequence collapses at the \( \hat{E}^{2p^2+2} \)-term, leaving
\[
\hat{E}^\infty(C_p) = P(t^{\pm p}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}.
\]

**Remark 6.4.** Beware that in the lemma above, the index \( j \) appearing as a power of \( t \) runs over all integers, positive or negative, with specified \( p \)-adic valuation. The same remark holds for the Lemmas 6.10 and 6.12 below, and also for the power \( j \) of \( \mu \) in Lemmas 6.11 and 6.13 below.

**Proof.** We know from [3] Proposition 4.8 that \( t\mu \) is an infinite cycle. The classes \( \lambda_1, \lambda_2 \) and \( b_1 \) are also infinite cycles, see the argument given at the top of [3] page 21.

Let \( d \) be Connes’ operator \([1,3]\) on \( V(1)_* \text{THH}(ku_p) \), and recall from above the notation \( b_0 = u \). We have
\[
d(b_0) = a_0,
\]
and this relation is detected via the Hurewicz homomorphism in mod \((p)\) homology, see [1] §9. It follows from \([37, 3.3]\) that in the \( S^1 \) homotopy fixed-point spectral sequence
\[
E^2(S^1) = P(t) \otimes V(1)_* \text{THH}(ku_p) \Rightarrow V(1)_* \text{THH}(ku_p)^{hS^1}
\]
we have a \( d^2 \)-differential
\[
d^2(b_0) = a_0 t.
\]
Since \( E^2(S^1) \) injects into \( \hat{E}^2(C_p) \) via \( R^hF \), this differential is also present in \( \hat{E}^2(C_p) \). The differentials \( d^2(b_i) = (1 - i)a_1 t \) for \( i \neq 0 \) follow easily from the case \( i = 0 \) and the multiplicative structure. Indeed \( d^2(\mu) = 0 \) for degree reasons, and hence \( d^2(u^{2}\mu) = 2u\mu a_0 t \). From the relation \( b_i b_{p-i} = u^2 \mu \) we deduce that \( d^2(b_i) = \alpha_i a_1 t \) for some \( \alpha_i \in \mathbb{F}_p \), because in \( V(1)_* \text{THH}(ku_p) \) the equation \( xb_{p-i} = u\mu a_0 x = a_i \) as unique (homogeneous) solution. First, notice that \( 0 = d^2(b_i^{p-1}) = (p-1)\alpha_i \lambda_2 \), so we have \( \alpha_1 = 0 \). Next, the relation \( b_i b_{p-i} = u^2 \mu \) implies that \( \alpha_{p-1} = 2 \), while \( b_i b_i = ub_{i+1} \) for \( i < p-2 \) implies that \( \alpha_i = 1 + \alpha_{i+1} \). We deduce that \( \alpha_i = 1 - i \), proving the claim on the \( d^2 \)-differential, which leaves
\[
\hat{E}^3(C_p) = P(t^{\pm 1}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1).
\]
Lemma 6.1 determines the given next three non-zero differentials, by comparison with the case of the \( \ell_p \) treated in [3] 5.5], and this takes care of the summand of \( \delta \)-weight zero.

The only algebra generators of \( \hat{E}^3(C_p) \) of non-zero \( \delta \)-weight are \( a_1 \) and \( b_1 \). We know that \( b_1 \) is an infinite cycle. In the \( S^1 \) Tate spectral sequence, using the known differentials, the tri-grading and the product, it is easy to see that \( a_1 \) survives to the \( E^{2p^2+2} \)-term. Therefore \( a_1 \) also survives to the \( E^{2p^2+2} \)-term in \( E^*(C_p) \), via the morphism of spectral sequences induced by \( F \). The \( d^{2p} \) differential leaves
\[
\hat{E}^{2p+1}(C_p) = P(t^{\pm p}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1),
\]
and the \( d^{2p^2} \) differential leaves
\[
\hat{E}^{2p^2+1}(C_p) = P(t^{\pm p^2}, t\mu) \otimes E(u_1, \lambda_1, a_1) \otimes P_{p-1}(b_1) \\
\oplus E(u_1, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\},
\]
as can be computed using the relation \( a_1 \cdot b^{p-2} = \lambda_2 \). Finally, \( d^{2p^2+1} \) leaves
\[
\hat{E}^{2p^2+2}(C_p) = P(t^{\pm p^2}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\},
\]
Lemma 6.10. In the \( r \)-adic Tate spectral sequence, the claims for \( \Gamma \) and \( \hat{\Gamma} \) follow by passage to homotopy limits.

Definition 6.6. We call a homomorphism of graded groups \( k \)-coconnected if it is an isomorphism in all dimensions greater than \( k \) and injective in dimension \( k \).

Proposition 6.7. The algebra map

\[
(\hat{\Gamma}_1)_*: V(1)_* \text{THH}(ku_p) \to V(1)_* \text{THH}(ku_p)^{\text{C}_p}
\]

factorizes as the localization away from \( \mu \), followed by an isomorphism

\[
V(1)_* \text{THH}(ku_p)[\mu^{-1}] \to V(1)_* \text{THH}(ku_p)^{\text{C}_p}
\]

given by

\[
\lambda_1 \mapsto \lambda_1, \quad \mu \mapsto t^{-p^2}, \quad b_i \mapsto t^{(1-i)p}b_1, \quad \text{and} \quad a_i \mapsto t^{(1-i)p}a_1,
\]

for \( 0 \leq i \leq p - 1 \), up to some non-zero scalar multiples. In particular the map \( (\hat{\Gamma}_1)_* \) is \((2p - 2)\)-coconnected.

Proof. By naturality with respect to \( \ell_p \to ku_p \) and by the computation of \( (\hat{\Gamma}_1)_* \) for \( \ell_p \) given in [3, Theorem 5.5], we know that the map \( (\hat{\Gamma}_1)_* \) for \( ku_p \) satisfies

\[
\lambda_1 \mapsto \lambda_1, \quad \lambda_2 \mapsto \lambda_2 \quad \text{and} \quad \mu \mapsto t^{-p^2}.
\]

In \( V(1)_* \text{THH}(ku_p) \) we have multiplicative relations \( w^{p-3}a_ib_j = \lambda_2 \) for \( i + j = p - 1 \), from which we deduce that \( (\hat{\Gamma}_1)_*(a^k_b) \neq 0 \) and \( (\hat{\Gamma}_1)_*(a^kb) \neq 0 \) for any \( 0 \leq k \leq p - 3 \) and any \( 0 \leq i \leq p - 1 \). For degree reasons, this forces

\[
(\hat{\Gamma}_1)_*(a_i) = t^{(1-i)p}a_1 \quad \text{and} \quad (\hat{\Gamma}_1)_*(b_i) = t^{(1-i)p}b_1
\]

up to some non-zero scalar multiples. \( \square \)

Corollary 6.8. The canonical maps

\[
\Gamma_n : \text{THH}(ku_p)^{\text{C}_p^n} \to \text{THH}(ku_p)^{\text{h}\text{C}_p^n}, \quad \hat{\Gamma}_n : \text{THH}(ku_p)^{\text{C}_p^{n-1}} \to \text{THH}(ku_p)^{\text{C}_p^n},
\]

\[
\Gamma : \text{TF}(ku_p) \to \text{THH}(ku_p)^{\text{h}\text{S}^1}, \quad \hat{\Gamma} : \text{TF}(ku_p) \to \text{THH}(ku_p)^{\text{S}^1},
\]

for \( n \geq 1 \) all induce \((2p - 2)\)-coconnected maps in \( V(1)_* \)-homotopy.

Proof. The claims for \( \Gamma_n \) and \( \hat{\Gamma}_n \) follow from Proposition 6.7 and the generalization of a theorem of Tsalidis [13] given in [9]. The claims for \( \Gamma \) and \( \hat{\Gamma} \) follow by passage to homotopy limits. \( \square \)

Definition 6.9. Let \( r(n) = 0 \) for all \( n \leq 0 \), and let \( r(n) = p^n + r(n - 2) \) for all \( n \geq 1 \). Thus \( r(2n - 1) = p^{2n-1} + \cdots + p \) (odd powers) and \( r(2n) = p^{2n} + \cdots + p^2 \) (even powers).

Lemma 6.10. In the \( \text{C}_p^n \) Tate spectral sequence \( \hat{E}^*(\text{C}_p^n) \) the classes \( \lambda_1, \lambda_2, b_1 \) and \( t\mu \) are infinite cycles. There are non-zero differentials

\[
d^2(b_i) = (1 - i)a_i t, \quad d^{2p}(t^{1-p}) = \lambda_1 \cdot t, \quad d^{2p^2}(t^{p-p^2}) = \lambda_2 \cdot t^p.
\]
with \( 0 \leq i \leq p - 1 \), leaving
\[
\hat{E}^{2p^i+1}(C_{p^n}) = P(t^{\pm p^i}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \\
\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes F_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 1\}.
\]
If \( n \geq 2 \), then for each \( 1 \leq k \leq n - 1 \) there is a triple of non-zero differentials
\[
d^{2r(2k)+2} (b_1 t^j) = a_1 t^j \cdot t^{p^k} \cdot (t\mu)^{r(2k-2)+1} \\
d^{2r(2k+1)} (t^{p^k} - t^{2k+1}) = \lambda_1 \cdot t^{p^k} \cdot (t\mu)^{r(2k-1)} \\
d^{2r(2k+2)} (t^{p^k+1} - t^{2k+2}) = \lambda_2 \cdot t^{p^k} \cdot (t\mu)^{r(2k)}
\]
with \( v_p(j) = 2k - 1 \), leaving
\[
\hat{E}^{2r(2k+2)+1}(C_{p^n}) = P(t^{\pm p^{2k+2}}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \\
\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes F_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2k+1\} \\
\oplus \bigoplus_{1 \leq m \leq k} \hat{T}_m(C_{p^n}),
\]
where
\[
\hat{T}_m(C_{p^n}) = E(u_n, \lambda_1) \otimes P_r(2m)(t\mu) \otimes F_p\{t^j \mid v_p(j) = 2m+1\} \\
\oplus E(u_n, a_1) \otimes P_{p-1}(b_1) \otimes P_r(2m-1)(t\mu) \otimes F_p\{t^j \mid v_p(j) = 2m\} \\
\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P_r(2m-2)+1(t\mu) \otimes F_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2m-1\}.
\]
For \( n \geq 1 \), there is a last non-zero differential
\[
d^{2r(2n)+1} (u_n \cdot t^{-p^{2n}}) = (t\mu)^{r(2n-2)+1}
\]
after which the spectral sequence collapses, leaving
\[
\hat{E}^\infty(C_{p^n}) = P(t^{\pm p^{2n}}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P_r(2n-2)+1(t\mu) \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_r(2n-2)+1(t\mu) \otimes F_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2n-1\} \\
\oplus \bigoplus_{1 \leq m \leq n-1} \hat{T}_m(C_{p^n}).
\]
Next, we describe the \( C_{p^n} \)-homotopy fixed-point spectral sequence \( E^*(C_{p^n}) \). It is algebraically easier to describe the \( E^* \)-terms of the \( C_{p^n} \)-homotopy fixed-point spectral sequence for \( THH(ku_p)^{hC_p} \), which we denote abusively by
\[
\mu^{-1} E^*(C_{p^n}) \Rightarrow V(1)_*(THH(ku_p)^{hC_p})^{hC_{p^n}},
\]
compare with [3 page 23]. We know from Proposition 6.7 that the map
\[
\hat{\Gamma}_1^{hC_{p^n}} : THH(ku_p)^{hC_{p^n}} \rightarrow (THH(ku_p)^{tC_p})^{hC_{p^n}}
\]
induces a morphism of spectral sequences
\[
E^*(C_{p^n}) \rightarrow \mu^{-1} E^*(C_{p^n})
\]
which on \( E^2 \)-terms (but not on higher terms) indeed corresponds to inverting \( \mu \). By the same Proposition and by strong convergence of the spectral sequences, the map \( \hat{\Gamma}_1^{hC_{p^n}} \) induces a \((2p-2)\)-coconnected homomorphism in \( V(1) \)-homotopy.
Lemma 6.11. In the $C_{p^n}$ homotopy fixed-point spectral sequence $\mu^{-1}E^*(C_{p^n})$ the classes $\lambda_1$, $\lambda_2$, $b_1$ and $t\mu$ are infinite cycles. There are non-zero differentials

$$d^2(b_1) = (1 - i)a_1t$$
$$d^{2p}(\mu^{p-1}) = \lambda_1 \cdot \mu^{-1} \cdot (t\mu)^p$$
$$d^{2p^2}(\mu^{p^2-p}) = \lambda_2 \cdot \mu^{-p} \cdot (t\mu)^{p^2}$$

with $0 \leq i \leq p - 1$, leaving

$$\mu^{-1}E^{2p^2+1}(C_{p^n}) = P(\mu^{p^2}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu)$$
$$\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 1\}$$
$$\oplus T_1(C_{p^n}),$$

where

$$T_1(C_{p^n}) = E(u_n, \lambda_1) \otimes P_{p^2}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^j \mid v_p(j) = 1\}$$
$$\oplus E(u_n, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{\lambda_1\mu^j \mid v_p(j) = 0\}$$
$$\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(\mu^{p+1}) \otimes \mathbb{F}_p\{a_1 \mid 0 \leq i \leq p - 1, \ i \neq 1\}.$$

If $n \geq 2$, then for each $2 \leq k \leq n$ there is a triple of non-zero differentials

$$d^{2r(2k)+2}(b_1\mu^j) = a_1\mu^j \cdot \mu^{-2k-2} \cdot (t\mu)^{r(2k-2)+1}$$
$$d^{2r(2k-1)}(\mu^{2k-1-\mu^{2k-2}}) = \lambda_1 \cdot \mu^{-2k-2} \cdot (t\mu)^{r(2k-1)}$$
$$d^{2r(2k)}(\mu^{p^2-2\mu^{2k-1}}) = \lambda_2 \cdot \mu^{-2k-1} \cdot (t\mu)^{r(2k)}$$

with $v_p(j) = 2k - 3$, leaving

$$\mu^{-1}E^{2r(2k)+1}(C_{p^n}) = P(\mu^{p^2}) \otimes E(u_n, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu)$$
$$\oplus E(u_n, \lambda_1) \otimes P_{p-2}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 2k - 1\}$$
$$\oplus \bigoplus_{1 \leq m \leq k} T_m(C_{p^n}),$$

where for $m \geq 2$ we have

$$T_m(C_{p^n}) = E(u_n, \lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2\mu^j \mid v_p(j) = 2m - 1\}$$
$$\oplus E(u_n, a_1) \otimes P_{r-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1\mu^j \mid v_p(j) = 2m - 2\}$$
$$\oplus E(u_n, \lambda_1) \otimes P_{r-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \otimes \mathbb{F}_p\{a_1\mu^j \mid v_p(j) = 2m - 3\}.$$

For $n \geq 1$, there is a last non-zero differential

$$d^{2r(2n)+1}(u_n \cdot \mu^{p^2n}) = (t\mu)^{r(2n)+1}$$

after which the spectral sequence collapses, leaving

$$\mu^{-1}E^\infty(C_{p^n}) = P(\mu^{p^2}) \otimes E(\lambda_1, a_1) \otimes P_{r(2n)+1}(b_1) \otimes P(t\mu)$$
$$\oplus E(\lambda_1) \otimes P_{r-2}(b_1) \otimes P_{r(2n)+1}(t\mu) \otimes \mathbb{F}_p\{a_1\mu^j, b_1\mu^j \mid v_p(j) = 2n - 1\}$$
$$\oplus \bigoplus_{1 \leq m \leq n} T_m(C_{p^n}).$$
Proof. We prove these two lemmas by induction on $n$, showing that Lemma 6.10 for $C_p^n$ implies Lemma 6.11 for $C_p^{n+1}$, which in turn implies Lemma 6.10 for $C_p^{n+1}$. The induction starts with Lemma 6.10 for $C_p$, which is the content of Lemma 6.3. Let us therefore assume given $n \geq 1$ such that Lemma 6.10 holds for $C_p^n$. The homotopy restriction map

$$R^h : \text{THH}(ku_p)^{bC_p^n} \to \text{THH}(ku_p)^{1C_p^n}$$

induces a morphism of spectral sequences $$(R^h)^* : E^* (C_p^n) \to \hat{E}^* (C_p^n),$$

which at the $E^2$-terms corresponds to inverting the class $t \in E^2_{2,0}(C_p^n)$,

$$(R^h)^2 : E^2(C_{p^n}) \subset E^2(C_{p^n})[t^{-1}] \cong \hat{E}^2(C_{p^n}),$$

(6.2)

and can be pictured as the inclusion of the second quadrant into the upper-half plane. As we will see below, although $(R^h)^r$ is not injective for $r \geq 3$, it detects all the non-trivial differentials of $E^r(C_{p^n})$. Taking into account the multiplicative structure and the fact that $\lambda_1, \lambda_2, b_1$ and $t\mu$ are infinite cycles, we claim that these differential are given by

$$d^2(b_i) = (1 - i)a_it$$
$$d^{2p}(t) = \lambda_1 \cdot t^{1+p^2}$$
$$d^{2p^2}(t^p) = \lambda_2 \cdot t^{p^2+p^2}$$

with $0 \leq i \leq p - 1$,

$$d^{2r(2k)+2} (b_i t^j) = a_1 t^j \cdot \mu^{2k} \cdot (t\mu)^{r(2k-2)+1}$$
$$d^{2r(2k+1)} (t^{2k}) = \lambda_1 \cdot t^{2k+1+p^{2k+1}} \cdot (t\mu)^{r(2k-1)}$$
$$d^{2r(2k+2)} (t^{2k+1}) = \lambda_2 \cdot t^{2k+1+p^{2k+2}} \cdot (t\mu)^{r(2k)}$$

if $n \geq 2$, $1 \leq k \leq n - 1$ and $v_p(j) = 2k - 1$ with $i \geq 0$, and finally

$$d^{2r(2n)+1} (\mu^n) = (t\mu)^{r(2n-2)+1} \cdot \mu^{2n}. $$

To prove this claim, we assume that some $r \geq 2$ is given, and that $E^r(C_{p^n})$ has been computed using the differentials $d^r$ above with $r' < r$. The class $t\mu$ is an infinite cycle, and $E^r(C_{p^n})$ is a $P(t\mu)$-module. Our choice of generators induces a decomposition $E^r(C_{p^n}) \cong F^r(C_{p^n}) \oplus T^r(C_{p^n})$, where $F^r(C_{p^n})$ is a free $P(t\mu)$-module and $T^r(C_{p^n})$ is a $t\mu$-torsion module. By inspection, the non-zero elements of $T^r(C_{p^n})$ are concentrated in filtration degrees $s$ with $-r < s \leq 0$, so they cannot be boundaries. They cannot support non-zero differentials either since a $t\mu$-torsion class cannot map to a non-torsion class. Thus the differential $d^r$ maps $F^r(C_{p^n})$ to itself and $T^r(C_{p^n})$ to zero. The morphism $(R^h)^r$ maps $F^r(C_{p^n})$ injectively into $E^r(C_{p^n})$, and it therefore detects the non-zero differentials of $E^r(C_{p^n})$ as the non-zero differential of $E^r(C_{p^n})$ which lie in the second quadrant. These are precisely the differentials given above. By induction on $r$, this determines all the non-trivial differentials of $E^r(C_{p^n})$. In the $\mu$-inverted homotopy fixed-point spectral sequence $\mu^{-1} E^*(C_{p^n})$, these can be rewritten as the claimed differentials. This proves Lemma 6.11 for $C_p^n$.

We now turn to the proof of Lemma 6.10 for $C_{p^{n+1}}$. In the Tate spectral sequence $\hat{E}^*(C_{p^n})$ the first non-zero differential of odd length originating from a column of odd $s$-filtration is $d^{2r(2n)+1}$. By [3 Lemma 5.2] the spectral sequences $\hat{E}^*(C_{p^n})$ and $\hat{E}^*(C_{p^{n+1}})$ are abstractly isomorphic up to the $E^{2r(2n)+1}$-term included. The Frobenius map

$$F : \text{THH}(ku_p)^{1C_{p^{n+1}}} \to \text{THH}(ku_p)^{1C_{p^n}}$$
induces a morphism of the corresponding Tate spectral sequences, which on $E^r$-terms with $2 \leq r \leq 2r(2n) + 1$ maps the columns of even $s$-filtration isomorphically. This detects all the claimed differentials of $\hat{E}^r(C_{p^{n+1}})$ for $2 \leq r \leq 2r(2n)$, and leaves

$$\hat{E}^{2r(2n)+1}(C_{p^{n+1}}) = \hat{F}^{2r(2n)+1}(C_{p^{n+1}}) \oplus \bigoplus_{m=1}^{n-1} \tilde{T}_m(C_{p^{n+1}}),$$

where $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$ is the $t\mu$-torsion free summand

$$\hat{F}^{2r(2n)+1}(C_{p^{n+1}}) = P(t^{p2^n}) \otimes E(u_{n+1}, \lambda_1) \otimes P(t\mu)$$

$$\otimes \left( P_{p-1}(b_1) \otimes E(a_1) \oplus P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1t^{-ip2^{n-1}}, b_1t^{-ip2^{n-1}} | 0 < i < p \} \right).$$

The non-zero $t\mu$-torsion elements of $\hat{E}^{2r(2n)+1}(C_{p^{n+1}})$ are concentrated in internal degrees $t$ with $0 \leq t < 2r(2n)$. In particular these elements cannot be boundaries, and they cannot map to non-$t\mu$-torsion elements. As in the case of the homotopy fixed-point spectral sequence above, we deduce that for $r \geq 2r(2n) + 1$ the differential $d^r$ can only affect the summand $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$. By Lemma 6.1 the summand of $\delta$-weight $0$ of $\hat{E}^*(C_{p^{n+1}})$ is equal to the image of the injective morphism of spectral sequences

$$\hat{E}^*(C_{p^{n+1}}, \ell_p) \to \hat{E}^*(C_{p^{n+1}}, ku_p) = \hat{E}^*(C_{p^{n+1}})$$

induced by the map $\ell_p \to ku_p$. Therefore, by [3, Theorem 6.1], the differentials affecting the summand of $\delta$-weight $0$ of $\hat{F}^{2r(2n)+1}(C_{p^{n+1}})$ at a later stage are given by

$$d^{2r(2n)+1}(t^{p2^n-2^{n+1}}) \cong \lambda_1 \cdot t^{p2^n} \cdot (t\mu)^{r(2n)-1}$$

$$d^{2r}(t^{p2^n+2}) \cong \lambda_2 \cdot t^{p2^n+1} \cdot (t\mu)^{r(2n)}$$

$$d^{2r(2n)+2}(u_{n+1} \cdot t^{-p2^n+2}) \cong (t\mu)^{r(2n)+1},$$

(6.3)

$$d^{2r(2n)+2}(b_1t^j) \cong a_1t^j \cdot t^{p2^n} \cdot (t\mu)^{r(2n)-2}+1$$

(6.4)

for $v_p(j) = 2n - 1$. First, notice that for tri-degree reasons $d^{2r(2n)+1} = 0$, so that $\hat{F}^{2r(2n)+2}(C_{p^{n+1}}) = \hat{F}^{2r(2n)+1}(C_{p^{n+1}})$. To detect the differential (6.3) we make use of the $(2p - 2)$-cocompact map

$$(\hat{\Gamma}_{n+1})_* : V(1)_* \text{THH}(ku_p)C_{p^n} \to V(1)_* \text{THH}(ku_p)^tC_{p^{n+1}},$$

and argue as in [3, proof of 6.1]. There is a commutative diagram

$$\text{THH}(ku_p)^{hC_{p^n}} \xrightarrow{\hat{\Gamma}_n} \text{THH}(ku_p)^{C_{p^n}} \xrightarrow{\hat{\Gamma}_{n+1}} \text{THH}(ku_p)^{tC_{p^{n+1}}},$$

where the vertical arrows are the $n$-fold Frobenius maps. The left-hand Frobenius is given in $V(1)$-homotopy on the associated graded by the edge homomorphism

$$E_{\infty,s}(C_{p^n}) \to E_{0,s}(C_{p^n}) \subset E_{0,s}^2(C_{p^n}) = V(1)_* \text{THH}(ku_p),$$

and the right-hand Frobenius is given in $\text{THH}(ku_p)^{tC_{p^{n+1}}}$.
which is known by induction hypothesis. For each $0 < \ell < p$ there is a direct summand

$$P_{r(2n-2)+1}(t^\mu)\{a_1\mu^{\ell p^{2n-3}}\} \subset E^{\infty}_{s,*}(C_{p^n}),$$

and $a_1\mu^{\ell p^{2n-3}}$ maps by $F^n_*$ to the class with same name in $V(1)_*THH(ku_p)$. Since $(\Gamma_n)_*$ is $(2p-2)$-coconnected, there is a class $x_\ell \in V(1)_*THH(ku_p)C_{p^n}$ with $F^n_*(x_\ell) = a_1\mu^{\ell p^{2n-3}}$ in $V(1)_*THH(ku_p)$. In $E^{\infty}(C_{p^n})$ we have no non-zero class of same total degree, same $\delta$-weight and lower $s$-filtration than

$$(t^\mu)^{r(2n-2)+1} \cdot a_1\mu^{\ell p^{2n-3}},$$

which forces $v_2^{r(2n-2)+1} x_\ell = 0$ in $V(1)_*THH(ku_p)C_{p^n}$. By Proposition 6.7, the class $(\hat{\Gamma}F^n)_*(x_\ell)$ is represented by $a_1t^{-\ell p^{2n-1}} \in \hat{E}^\infty(C_p)$, and therefore $(\hat{\Gamma}_n)_*(x_\ell)$ must be detected in $s$-filtration $2\ell p^{2n-1}$ or higher. The only suitable class in $\hat{E}^{2r(2n)+2}(C_{p^{n+1}})$ is $a_1t^{-\ell p^{2n-1}}$, which therefore is a permanent cycle representing $(\hat{\Gamma}_n)_*(x_\ell)$. Notice for later use that the same argument shows that

$$a_1 \in \hat{E}^{2r(2n)+2}_{0,2p+3}(C_{p^{n+1}})$$

is a permanent cycle. The map $(\hat{\Gamma}_n+1)_*$ is an isomorphism in degrees larger than $2p-2$, and the relation $v_2^{r(2n-2)+1} (\hat{\Gamma}_n+1)_*(x_\ell) = 0$ implies that the infinite cycle $(t^\mu)^{r(2n-2)+1} \cdot a_1t^{-\ell p^{2n-1}}$, of total degree $2p^{2n} + 2\ell p^{2n-1} + 2p + 1$ and of $\delta$-weight 1, is a boundary. On the other hand, the component of $\hat{E}^{2r(2n)+1}(C_{p^{n+1}})$ of total degree $2p^{2n} + 2\ell p^{2n-1} + 2p + 2$, of $\delta$-weight 1 and of $s$-filtration degree exceeding by at least $2r(2n) + 2$ the $s$-filtration degree of $(t^\mu)^{r(2n-2)+1} \cdot a_1t^{-\ell p^{2n-1}}$ reduces to

$$\mathbb{F}_p \{ b_1 t^{-\ell p^{2n-1}} \cdot t^{p^{2n}} \}.$$

This proves the existence of a non-zero differential

$$d^{2r(2n)+2}(b_1 t^{-\ell p^{2n-1}} \cdot t^{p^{2n}}) \simeq (t^\mu)^{r(2n-2)+1} \cdot a_1t^{-\ell p^{2n-1}}$$

for $0 < \ell < p$. Since $t^{p^{2n}}$ is a unit and a cycle we obtain the claimed differentials (5.4). This leaves

$$\hat{E}^{2r(2n)+3}(C_{p^{n+1}}) = \hat{E}^{2r(2n)+3}(C_{p^{n+1}}) \oplus E(u_{n+1}, \lambda_1) \otimes P_{r-2}(b_1) \otimes P_{r(2n-2)+1}(t^\mu) \otimes \mathbb{F}_p \{ a_1 t^j \mid v_p(j) = 2n - 1 \} \oplus \bigoplus_{m=1}^{n-1} \hat{T}_m(C_{p^{n+1}}),$$

with a $t^\mu$-torsion free summand

$$F^{2r(2n)+3}(C_{p^{n+1}}) = P(t^{\pm p^{2n}}) \otimes E(u_{n+1}, \lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t^\mu).$$

Again, further differentials can only affect the summand $F^{2r(2n)+3}(C_{p^{n+1}})$. Since $b_1$ and $a_1$ are infinite cycles, the next non-zero differentials are $d^{2r(2n+1)}$ and $d^{2r(2n+2)}$, as given in (6.3), leaving

$$\hat{E}^{2r(2n+2)+1}(C_{p^{n+1}}) = \hat{E}^{2r(2n+2)+1}(C_{p^{n+1}}) \oplus \bigoplus_{m=1}^{n} \hat{T}_m(C_{p^{n+1}}),$$

where $\hat{T}_m(C_{p^{n+1}})$ are finite dimensional.
with
\[
\hat{E}^{2r(2n+2)+1}(C_{p^{n+1}}) = P(t^{2p^{2n+2}}) \otimes E(u_{n+1}, \lambda_1) \otimes P(t\mu) \\
\otimes \left( P_{p-1}(b_1) \otimes E(a_1) \oplus P_{p-2}(b_1) \otimes \mathbb{F}_p\{a_1 t^{-i p^{2n+1}}, b_1 t^{-i p^{2n+1}} \mid 0 < i < p\} \right).
\]

Notice that for tri-degree reasons, the classes \(a_1 t^{-i p^{2n+1}}\) and \(b_1 t^{-i p^{2n+1}}\) are cycles at the \(E^{2r(2n+2)+1}\)-stage. The third differential of (6.3) remains, after which the spectral sequence collapses for bidegree reasons, leaving
\[
\hat{E}^\infty(C_{p^{n+1}}) = P(t^{2p^{2(n+1)}}) \otimes E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P_{r(2n)+1}(t\mu) \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2n-1)+1}(t\mu) \otimes \mathbb{F}_p\{a_1 t^j, b_1 t^j \mid v_p(j) = 2n + 1\} \\
\oplus \bigoplus_{1 \leq m \leq n} \hat{T}_m(C_{p^{n+1}}),
\]
as claimed. This completes the induction step and the proof of Lemmas 6.10 and 6.11. 

Taking the limit over the Frobenius maps we obtain the following two lemmas.

**Lemma 6.12.** The associated graded \(\hat{E}^\infty(S^1)\) of \(V(1)_*\text{THH}(ku_p)^{hS^1}\) is given by
\[
\hat{E}^\infty(S^1) = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \oplus \bigoplus_{m \geq 1} \hat{T}_m(S^1),
\]
where
\[
\hat{T}_m(S^1) = E(\lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 t^j \mid v_p(j) = 2m + 1\} \\
\oplus E(\lambda_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 t^j \mid v_p(j) = 2m\} \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \otimes \mathbb{F}_p\{a_1 t^j \mid v_p(j) = 2m - 1\}.
\]

**Lemma 6.13.** The associated graded \(E^\infty(S^1)\) of \(V(1)_*\text{THH}(ku_p)^{hS^1}\) is mapped by a \((2p - 2)\)-coconnected homomorphism to
\[
\mu^{-1} E^\infty(S^1) = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \oplus \bigoplus_{m \geq 1} T_m(S^1),
\]
where
\[
T_1(S^1) = E(\lambda_1) \otimes P_{p-1}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 1\} \\
\oplus E(\lambda_1) \otimes P_{p-1}(b_1) \otimes P(t\mu) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 0\} \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P(\mu^{p+1}) \otimes \mathbb{F}_p\{a_i \mid 0 \leq i \leq p-1, i \neq 1\}
\]
and, for \(m \geq 2,
\[
T_m(S^1) = E(\lambda_1) \otimes P_{r(2m)}(t\mu) \otimes \mathbb{F}_p\{\lambda_2 \mu^j \mid v_p(j) = 2m - 1\} \\
\oplus E(\lambda_1) \otimes P_{p-1}(b_1) \otimes P_{r(2m-1)}(t\mu) \otimes \mathbb{F}_p\{\lambda_1 \mu^j \mid v_p(j) = 2m - 2\} \\
\oplus E(\lambda_1) \otimes P_{p-2}(b_1) \otimes P_{r(2m-2)+1}(t\mu) \otimes \mathbb{F}_p\{a_1 \mu^j \mid v_p(j) = 2m - 3\}.
\]
7. Topological cyclic homology

We now evaluate the restriction map $R : TF(ku_p) \to TF(ku_p)$ in $V(1)$-homotopy. Consider the homotopy commutative diagram

$$
\begin{array}{ccc}
TF(ku_p) & \xrightarrow{R} & TF(ku_p) \\
\Gamma \downarrow & & \Gamma \downarrow \\
THH(ku_p)^{hS^1} & \xrightarrow{G} & (THH(ku_p)^{hS^1})_R
\end{array}
$$
displayed in [3, page 27], and with $G$ a $V(1)$-equivalence. By the argument in [3 Lemma 7.5], we know that on $V(1)_*TF(ku_p)$ the profinite topology coincides with the topology induced by the spectral sequence filtration of $V(1)_*THH(ku_p)^{hS^1}$ via $\Gamma_*$, and that the restriction map

$$
R_* : V(1)_*TF(ku_p) \to V(1)_*TF(ku_p)
$$

is continuous in degrees larger than $2p - 2$. In this range of degrees, we will identify $V(1)_*TF(ku_p)$ with $V(1)_*THH(ku_p)^{hS^1}$ via the homeomorphism $\Gamma_*$. Under this identification $R_*$ corresponds to $(\Gamma_* \hat{\Gamma}_-^1)R_*^h$, and we first describe $R_*^h$ and $\Gamma_* \hat{\Gamma}_-^1$ separately.

**Lemma 7.1.** In total degrees larger than $2p - 2$, the morphism

$$(R^h)^* : E^\infty(S^1) \to \hat{E}^\infty(S^1)$$

has the following properties.

1. It maps $E(\lambda_1, a_1) \otimes P_{p-1}(b) \otimes P(t\mu)$ isomorphically to the summand with same name;
2. It maps $E(\lambda_1) \otimes P_{(k)}(t\mu) \otimes F_p\{\lambda_2 t^{-d_p^{k-1}}\}$ onto

$$E(\lambda_1) \otimes P_{(k-2)}(t\mu) \otimes F_p\{\lambda_2 t^{-d_p^{k-1}}\}$$

and $E(\lambda_1) \otimes P_{p-2}(b) \otimes P_{(k)+1}(t\mu) \otimes F_p\{a_1 t^{-d_p^{k-1}}\}$ onto

$$E(\lambda_1) \otimes P_{p-2}(b) \otimes P_{(k-2)+1}(t\mu) \otimes F_p\{a_1 t^{-d_p^{k-1}}\}$$

for $k \geq 2$ even and $0 < d < p$;
3. It maps $E(a_1) \otimes P_{p-1}(b) \otimes P_{(k)}(t\mu) \otimes F_p\{\lambda_1 t^{-d_p^{k-1}}\}$ onto

$$E(a_1) \otimes P_{p-1}(b) \otimes P_{(k-2)}(t\mu) \otimes F_p\{\lambda_1 t^{-d_p^{k-1}}\}$$

for $k \geq 3$ odd and $0 < d < p$;
4. It maps the remaining summands to zero.

**Proof.** This follows from the description of $(R^h)^2$, see (5.2).

**Lemma 7.2.** In degrees larger then $2p - 2$, the homomorphism $\Gamma_* \hat{\Gamma}_-^1$ maps

1. the classes in $V(1)_*THH(ku_p)^{hS^1}$ represented in $\hat{E}^\infty(S^1)$ by

$$\lambda_1^\epsilon_1 \alpha_1^\epsilon_2 b_k(t\mu)^m t^i$$

for $\nu_0(i) \neq 1$, $\epsilon_1$ and $\epsilon_2 \in \{0, 1\}$, $0 \leq k \leq p - 2$ and $m \geq 0$, to classes in $V(1)_*THH(ku_p)^{hS^1}$ represented in $E^\infty(S^1)$ by

$$\lambda_1^\epsilon_1 \alpha_1^\epsilon_2 b_k(t\mu)^m \mu^j$$

with $i + p^2 j = 0$, up to multiplication with a unit in $F_p$;
(b) the classes in \(V(1)_*\text{THH}(ku_p)^{tS^1}\) represented in \(\hat{E}(S^1)\) by
\[
\lambda_1 t^p a_1 t^i
\]
for \(v_p(i) = 1, \epsilon_1 \in \{0, 1\}\) and \(0 \leq k \leq p - 3\), to classes in \(V(1)_*\text{THH}(ku_p)^{hS^1}\) represented in \(E^\infty(S^1)\) by
\[
\lambda_1 t^p a_1 t^i
\]
with \(i = (1 - j)p - lp^2\) for \(0 \leq j \leq p - 1\) such that \(j \neq 1\), up to multiplication with a unit in \(\mathbb{F}_p\).

**Proof.** The proof is similar to the proof of [3 Proposition 7.4], and we omit it. \(\square\)

**Definition 7.3.** We recall from [3 Theorem 9.1] that there are classes \(\lambda_1 t^{p-1}, \lambda_1, \lambda_2\) in \(V(1)_*K(\ell_p) \subset V(1)_*K(ku_p)\), of degree 1, \(2p - 1\) and \(2p^2 - 1\), respectively. We denote by
\[
\tilde{\lambda}_1 t^{p-1}, \tilde{\lambda}_1 \text{ and } \tilde{\lambda}_2
\]
their image in \(V(1)_*TF(ku_p)\) under \(\text{tr}_{F_*}\). The latter classes are represented by
\[
\lambda_1 t^{p-1} = (t\mu)^{p-1} \cdot \lambda_1 t^{1-p}, \lambda_1 \text{ and } \lambda_2
\]
in \(E^\infty(S^1)\), respectively, see [3 Theorem 8.4]. We further denote by \(b\) and \(v_2\) the image in \(V(1)_*TF(ku_p)\) under \(\text{tr}_{F_*}\) of the classes with same name in \(V(1)_*K(ku_p)\). These classes are represented by \(b_1 \) and \(t\mu\) in \(E^\infty(S^1)\), respectively, see Lemma 4.4 and [3 Proposition 4.8].

**Lemma 7.4.** There exists a unique class \(\tilde{a}_1 \in V(1)_{2p+3}TF(ku_p)\) with the following two properties:

(a) \(\tilde{a}_1\) has \(\delta\)-weight \(1\) and \(b^{p-2}\tilde{a}_1 = \tilde{\lambda}_2\),

(b) \(R_*(\tilde{a}_1) = \tilde{a}_1\).

Moreover, this class \(\tilde{a}_1\) is represented by \(a_1\) in \(E^\infty(S^1)\).

**Proof.** For \(i = 0\) or \(1\), let us denote by \(T^{(i)}_*\) and \(\text{ker}(R - 1)^{t(1)}_*\) the summand of \(\delta\)-weight \(i\) of \(V(1)_*TF(ku_p)\) and \(\text{ker}(R - 1)_* \subset V(1)_*TF(ku_p)\), respectively. We make the following claims:

(1) The homomorphism given by multiplication with \(b^{p-2}\) on \(T^{(1)}_{2p+3}\) fits in a short exact sequence
\[
0 \rightarrow \mathbb{F}_p \{z\} \rightarrow T^{(1)}_{2p+3} \xrightarrow{b^{p-2}} T^{(0)}_{2p^2 - 1} \rightarrow 0,
\]
where the class \(z\) is represented by \(b_1 \cdot (t\mu)^{p-1} \cdot \lambda_1 t^{1-p}\) in \(E^\infty(S^1)\);

(2) The class \(z\) does not belong to \(\text{ker}(R - 1)_*\).

Using these claims, it is easy to deduce that multiplication with \(b^{p-2}\) restricts to an isomorphism
\[
\text{ker}(R - 1)^{t(1)}_{2p+3} \xrightarrow{\cong} \text{ker}(R - 1)^{t(0)}_{2p^2 - 1}.
\]
We have \(\tilde{\lambda}_2 \in \text{ker}(R - 1)^{t(0)}_{2p^2 - 1}\) since \(\tilde{\lambda}_2\) has \(\delta\)-weight \(0\) and is in the image of \(\text{tr}_{F_*}\). Therefore, there is a unique pre-image \(\tilde{a}_1 \in \text{ker}(R - 1)^{t(1)}_{2p+3}\) of \(\tilde{\lambda}_2 \in \text{ker}(R - 1)^{t(0)}_{2p^2 - 1}\), or, in other words, there is a unique class \(\tilde{a}_1 \in V(1)_{2p+3}TF(ku_p)\) with properties (a) and (b).

Moreover, \(\tilde{\lambda}_2\) is represented in \(E^\infty(S^1)\) in filtration zero by \(\lambda_2 = b_1^{p-2}a_1\), and we deduce that \(\tilde{a}_1\) must be represented in filtration zero by \(a_1\). Thus this lemma follows from claims (1) and (2), which we now prove.
First, notice that the group $T_*^{(1)}$ inherits via $\Gamma_*$ the spectral sequence filtration of $V(1)_* \text{THH}(ku_p)^{hS^1}$. Denoting by $E^\infty(S^1)_*^{(1)}$ its associated graded, we know from Lemma 6.13 that

$$E^\infty(S^1)_*^{(1)} = E_{p+3}^\infty \{ a_1, b_1 \cdot x_n \mid n \geq 0 \} \quad \text{and} \quad E^\infty(S^1)_*^{(0)} = E_{p+1}^\infty \{ \lambda_2, t_\mu \cdot x_n \mid n \geq 1 \},$$

where $x_n = (t_\mu)^c_{(2n+1)} - r(2n-1) \cdot \lambda_1 \mu^{(1-p)n^2}$.

Next, the relation $b^p + v_2b' = 0$ in $V(1)_*K(Z, 3)$, established in Proposition 2.7, maps under $\text{tr}_{F_*}\phi_*$ to the relation $b^p + v_2b = 0$ in $T_*^{(1)}$. The class $v_2b$ in $T_*^{(1)}$ is represented by the non-zero class $t_\mu \cdot b_1$ in $E^\infty(S^1)$ in filtration $-2$, and we deduce that $b^{p-1} \in T_*^{(0)}$ must be represented by $-t_\mu$ in $E^\infty(S^1)$. It follows that if a class $x \in T_*^{(1)}$ is represented by $b_1 \cdot x_n$, then $b^{p-2}x$ is represented by $-t_\mu \cdot x_n$ in filtration degree lower. Using a coarser filtration that ignores this shift, and considering our formulas for $E^\infty(S^1)_*^{(1)}$ and $E^\infty(S^1)_*^{(0)}$, we deduce claim (1) from the corresponding claim for the associated graded, with $z$ represented by $b_1 \cdot x_0$.

To prove claim (2), we notice that if a class $y \in T_*^{(1)}$ is represented by $b_1 \cdot x_n$, with $n \geq 1$, then $R_*(y)$ will be represented by $b_1 \cdot x_{n-1}$ in higher filtration, up to some non-zero scalar multiple: this follows directly from Lemmas 7.1 and 7.2. In particular, $R_*(y) \neq y$. This implies the following claim:

(3) The group $\ker(R - 1)_*^{(1)}$ contains at most one class represented by $b_1 \cdot x_0$.

Now consider the class $\tilde{x}_0 = \lambda_1 t_\mu b^{-1} \in T_*^{(0)}$ given in Definition 7.3. By definition, this class lies in $\ker(R - 1)_*^{(0)}$ and is represented by $x_0$. We also claim that

(4) The class $b\tilde{x}_0 \in \ker(R - 1)_*^{(1)}$ is not annihilated by $b^{p-2}$.

Since $b\tilde{x}_0$ is represented by $b_1 \cdot x_0$, claim (2) follows from claims (3) and (4).

Finally, to prove claim (4), we recall from [3, Theorem 8.2] that the class $v_2\tilde{x}_0 \in \ker(R - 1)_*^{(0)}$ is non-zero, and must be represented, in filtration degree lower than $-2p + 2$, by a class in

$$F_p \{ t_\mu \cdot x_n \mid n \geq 1 \}.$$

None of these classes is annihilated by $b_1$. Therefore $bv_2\tilde{x}_0 = -b^p\tilde{x}_0$ is non-zero, and we deduce that $b\tilde{x}_0 \in \ker(R - 1)_*^{(1)}$ is not annihilated by $b^{p-2}$. \qed

Remark 7.5. The lemma above implies that the class $a_1 \in V(1)_* \text{THH}(ku_p)$ has a lift $a_1 \in V(1)_*K(ku_p)$ under the trace, with $b^{p-2}a_1 = \lambda_2$, see Theorem 8.1. It would be nice to have a more direct construction of such a lift. In fact, we conjecture that $a_1 \in V(1)_*K(ku_p)$ decomposes as $bd$, where $d \in V(1)_*K(U_p)$ is a unit class, when mapped into $V(1)_*K(KU_p)$, see the discussion preceding Theorem 8.3 below.
Definition 7.6. We consider the following subgroups of $E^\infty(S^1)$:

\[
A = E(\lambda_1, a_1) \otimes P_{p-1}(b_1) \otimes P(t\mu),
\]

\[
B_0 = E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \mathbb{F}_p\{\mu^{-1}a_i, a_0 \mid 2 \leq i \leq p - 1\},
\]

\[
B_k = \left(E(\lambda_1) \otimes P_{p-2}(b_1) \otimes \bigoplus_{0 < d < p} P_{r(k)-dp^{k-1}+1}(t\mu) \otimes \mathbb{F}_p\{a_1t^{dp^{k-1}}\}\right)
\]

\[
\quad \oplus \left(E(\lambda_1) \otimes \bigoplus_{0 < d < p} P_{r(k)-dp^{k-1}}(t\mu) \otimes \mathbb{F}_p\{\lambda_2t^{dp^{k-1}}\}\right) \quad \text{for } k \geq 2 \text{ even},
\]

\[
B_k = E(a_1) \otimes P_{p-1}(b_1) \otimes \bigoplus_{0 < d < p} P_{r(k)-dp^{k-1}}(t\mu) \otimes \mathbb{F}_p\{\lambda_1t^{dp^{k-1}}\} \quad \text{for } k \geq 1 \text{ odd},
\]

and we let $C$ be the span of the remaining monomials in $E^\infty(S^1)$. We then have a direct sum decomposition $E^\infty(S^1) = A \oplus B \oplus C$, with $B = \bigoplus_{k \geq 0} B_k$.

Lemma 7.7. In dimensions larger than $2p - 2$ there are closed subgroups $\tilde{A}$, $\tilde{B}_k$ and $\tilde{C}$ in $V(1)_*TF(ku_p)$, represented by $A$, $B_k$ and $C$ in $E^\infty(S^1)$ respectively, such that

(a) $R_*$ restricts to the identity on $\tilde{A}$,

(b) $R_*$ maps $\tilde{B}_{k+2}$ onto $\tilde{B}_k$ for $k \geq 0$,

(c) $R_*$ maps $B_0$, $B_1$ and $C$ to zero.

In these degrees $V(1)_*TF(ku_p) \cong \tilde{A} \oplus \tilde{B} \oplus \tilde{C}$, where $\tilde{B} = \prod_{k \geq 0} \tilde{B}_k$.

Proof. On the associated graded $E^\infty(S^1)$, the homomorphism $(\Gamma_1\hat{\Gamma}^{-1}_*)R_*^h$ has been described in Lemmas [3.1] and [3.2], and maps $A$ isomorphically to itself, $B_{k+2}$ onto $\tilde{B}_k$ for $k \geq 0$, and $B_0$, $B_1$ and $C$ to zero. It remains to find closed lifts of these groups in $V(1)_*TF(ku_p)$ with desired properties. We take $\tilde{A}$ to be the (closed) subalgebra of $V(1)_*TF(ku_p)$ generated by $\tilde{\lambda}_1$, $\tilde{a}_1$, $b$ and $v_2$. Then $\tilde{A}$ lifts $A$, by definition of its algebra generators and by the fact, proved above, that $b^{p-1}$ is represented by $-t\mu$ in $E^\infty(S^1)$. Also, $\tilde{\lambda}_1$, $b$ and $v_2$ are fixed under $R_*$, since they are in the image of $\text{tr}_{\mathcal{F}}$, and $\tilde{a}_1$ is fixed by definition. To construct $\tilde{B}_k$ for $k \geq 0$ and $\tilde{C}$, we follow the procedure given in [3, Theorem 7.7].

Definition 7.8. We denote $b \in V(1)_{2p+2}TC(ku_p)$ the image of the higher Bott element $b$, defined in [3.2], under the cyclotomic trace map

\[
(\text{tr}_c)_* : V(1)_*K(ku_p) \to V(1)_*TC(ku_p).
\]

Theorem 7.9. The class $b \in V(1)_{2p+2}TC(ku_p)$ satisfies the relation

\[
b^{p-1} = -v_2.
\]

There is an isomorphism of $P(b)$-modules

\[
V(1)_*TC(ku_p) \cong P(b) \otimes E(\partial, \lambda_1, a_1)
\]

\[
\quad \oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\}
\]

\[
\quad \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{u^ia_0, t^{p-2-p}\lambda_2 \mid 0 \leq i < p - 2\},
\]

where the degree of the classes is $|\partial| = -1$, $|\lambda_1| = 2p - 1$, $|a_1| = 2p + 3$, $|u^ia_0| = 2i + 3$, $|\lambda_2| = 2p^2 - 1$ and $|t| = -2$. 

Proof. Recall that $TC(ku_p)$ is defined as the homotopy fiber of the map
$$R - 1 : TF(ku_p) \to TF(ku_p).$$
In $V(1)$-homotopy, it gives a short exact sequence of $P(v_2)$-modules
$$0 \to \Sigma^{-1} \text{cok}(R - 1)_* \to V(1)_* TC(ku_p) \to \ker(R - 1)_* \to 0. \quad (7.1)$$
We have isomorphisms of $P(v_2)$-modules
$$\Sigma^{-1} \text{cok}(R - 1)_* \cong \Sigma^{-1}\hat{A}$$
$$\ker(R - 1)_* \cong \hat{A} \oplus \lim_{k \geq 0 \text{ even}} \hat{B}_k \oplus \lim_{k \geq 1 \text{ odd}} \hat{B}_k. \quad (7.2)$$
Indeed, $R_*$ maps each factor of the decomposition $V(1)_* TF(ku_p) \cong \hat{A} \oplus \hat{B} \oplus \hat{C}$ to itself. It restricts to zero on $\hat{A}$ and to the identity on $\hat{C}$. We have a short exact sequence
$$0 \to \lim_{k \geq 0 \text{ even}} \hat{B}_k \to \prod_{k \geq 0 \text{ even}} \hat{B}_k \underset{R_* - 1}{\to} \prod_{k \geq 0 \text{ even}} \hat{B}_k \to \lim_{k \geq 0 \text{ even}} \hat{B}_k \to 0,$$
and similarly for the $\hat{B}_k$ with $k$ odd. Here the limits are taken over the sequential system of maps $R_* : \hat{B}_{k+2} \to \hat{B}_k$ for $k \geq 0$ even or $k \geq 1$ odd. Since these maps are surjective, the $lim^1$-terms are trivial. This proves our claims on $\Sigma^{-1} \text{cok}(R - 1)_*$ and $\ker(R - 1)_*$ in (7.2).

For $k \geq 1$ odd, the group $\hat{B}_k$ is isomorphic as a $P(v_2)$-module to a sum of $2(p - 1)^2$ cyclic $P(v_2)$-modules
$$\hat{B}^k \cong E(a_1) \otimes P_{r(k)}(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{\lambda_1^{t^{dpk-1}} \mid 0 < d < p\}.$$

The map $R_*$ respects this decomposition into cyclic $P(v_2)$-modules. Since the height of these modules grows to infinity with $k$, we deduce from the surjectivity of $R_*$ that $\lim_{k \geq 1 \text{ odd}} \hat{B}_k$ is a sum of $2(p - 1)^2$ free cyclic $P(v_2)$-modules, given by an isomorphism
$$\lim_{k \geq 1 \text{ odd}} \hat{B}_k \cong E(a_1) \otimes P(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{\lambda_1^{t^d} \mid 0 < d < p\}.$$

Similarly, for $k \geq 2$ even, $\hat{B}_k$ is isomorphic to a sum of $2(p - 1)^2$ cyclic $P(v_2)$-modules of height growing with $k$, and passing to the limit we have an isomorphism of $P(v_2)$-modules
$$\lim_{k \geq 0 \text{ even}} \hat{B}_k \cong E(\lambda_1) \otimes P(v_2) \otimes P_{p-1}(b) \otimes \mathbb{F}_p\{a_1^{t^{dp}} \mid 0 < d < p\}.$$

Thus $\ker(R - 1)_*$ is a free $P(v_2)$-module, and the exact sequence (7.1) splits. We have an isomorphism of $P(v_2)$-modules
$$V(1)_* TC(ku_p) \cong P(v_2) \otimes P_{p-1}(b) \otimes E(\partial, \lambda_1, a_1)$$
$$\oplus P(v_2) \otimes P_{p-1}(b) \otimes E(a_1) \otimes \mathbb{F}_p\{\lambda_1^{t^d} \mid 0 < d < p\} \quad (7.3)$$
$$\oplus P(v_2) \otimes P_{p-1}(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{a_1^{t^{dp}} \mid 0 < d < p\}$$
in degrees larger than $2p - 2$, where the summand
$$P(v_2) \otimes P_{p-1}(b) \otimes E(\lambda_1, a_1) \otimes \mathbb{F}_p\{\partial\}$$
is the group $\text{cok}(R - 1)_* \cong \Sigma^{-1}\hat{A}$.

We now show that the relation
$$b^{p-1} = -v_2$$
holds in $V(1)_*TC(ku_p)$. We recall from Proposition 2.7 that the class $b^{p-1} + v_2$ in $V(1)_{2p^2-2}K(\mathbb{Z},3)$ is annihilated by $b$. This class maps by $\text{trc}_s\phi_s$ to the class

$$b^{p-1} + v_2 \in V(1)_{2p^2-2}TC(ku_p),$$

which is therefore annihilated by $b$. Thus it suffices to show that zero is the only class in $V(1)_{2p^2-2}TC(ku_p)$ that is annihilated by $b$. We consider the short exact sequence

$$0 \to \text{cok}(R - 1)_{2p^2-1} \to V(1)_{2p^2-2}TC(ku_p) \to \ker(R - 1)_{2p^2-2} \to 0$$

given in (7.1) above. Here

$$\ker(R - 1)_* \subset V(1)_*TF(ku_p)$$

inherits via $\Gamma_*$ the spectral sequence filtration of $V(1)_*THH(ku_p)^{hS^1}$. By (7.3), this filtration gives the short exact sequence

$$0 \to \mathbb{F}_p \{b^{p-2} \cdot \lambda_1 \cdot a_1^{p} \} \to \ker(R - 1)_{2p^2-2} \to \mathbb{F}_p \{v_2 \} \to 0$$

in dimension $2p^2 - 2$, while in dimension $2p^2 + 2$ it gives the short exact sequence

$$0 \to \mathbb{F}_p \{v_2 \cdot \lambda_1 \cdot a_1^{p} \} \to \ker(R - 1)_{2p^2+2} \to \mathbb{F}_p \{b \cdot v_2 \} \to 0.$$ 

Here $v_2$ and $b \cdot v_2$ are represented by $t_\mu$ and $b_1 \cdot t_\mu$ in $E^\infty(S^1)$, respectively. Multiplication with $b$ is compatible with the filtration, and maps the former sequence to the latter one. First, notice that the class $v_2$ maps to a non-zero class in $\mathbb{F}_p \{b \cdot v_2\}$, since $b \cdot v_2$ is represented by $b_1 \cdot t_\mu$ in $E^\infty(S^1)$. Next, the relation $b^{p} = -bv_2$ in $\ker(R - 1)_*$ implies

$$b^{p} \cdot \lambda_1 \cdot a_1^{p} = -v_2 \cdot b \cdot \lambda_1 \cdot a_1^{p},$$

which is non-zero by (7.3). A fortiori $b^{p-1} \cdot \lambda_1 \cdot a_1^{p} \in \mathbb{F}_p \{v \cdot \lambda_1 \cdot a_1^{p} \}$ is not zero either. Thus $\ker(R - 1)_{2p^2-2}$ contains no non-zero class annihilated by $b$, and we deduce that

$$b^{p-1} + v_2 \in \partial(\text{cok}(R - 1)_{2p^2-1}) = \mathbb{F}_p \{b^{p-2} \cdot a_1 \cdot \partial \}.$$ 

However the class $b^{p-2} \cdot a_1 \cdot \partial$ is not annihilated by $b$, since by (7.3) we know that $b^{p} \cdot a_1 \cdot \partial = -v_2 \cdot b \cdot a_1 \cdot \partial$ is non-zero. This proves that $b^{p-1} + v_2$ must be zero.

In particular $b$ is not a nilpotent class, and we have an isomorphism of $P(b)$-modules

$$V(1)_*TC(ku_p) \cong P(b) \otimes E(\partial, \lambda_1, a_1)$$

$$\oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p \{t^d \lambda_1 \mid 0 < d < p \}$$

$$\oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p \{a_1 t^{pd} \mid 0 < d < p \}$$

in degrees larger than $2p - 2$. This proves that our formula for $V(1)_*TC(ku_p)$ is correct in dimensions greater than $2p - 2$. Let us define $M$ and $N$ as

$$M = \bigoplus_{-1 \leq n \leq 2p-2} V(1)_n TC(ku_p) \text{ and } N = \bigoplus_{n \geq 2p-1} V(1)_n TC(ku_p).$$

We just argued that $N$ is a free $P(b)$-module. We know by (5.3) that there is an isomorphism

$$M \cong \mathbb{F}_p \{\partial, 1, u^i a_0, \lambda_1 t^i, \partial \lambda_1 \mid 0 \leq i \leq p - 3, 1 \leq d \leq p - 1 \}$$

of $\mathbb{F}_p$-modules. This proves that the formula for $V(1)_*TC(ku_p)$ in Theorem 7.9 holds as an isomorphism of $\mathbb{F}_p$-modules. It only remains to show that for any non-zero class $m \in M$, we have $bm \neq 0$ in $V(1)_*TC(ku_p)$. By comparison with $V(1)_*TC(t_p)$ or with $V(1)_*THH(ku_p)^{hS^1}$, we know that either $m \lambda_1$ or $mv_2$ is non-zero. These products lie in $N$ for degree reasons, so are not $b$-torsion classes. Therefore $m$ is not a $b$-torsion class either. 

\qed
8. Algebraic K-theory

Theorem 8.1. There is an isomorphism of $P(b)$-modules

$$V(1)_* K(ku_p) \cong P(b) \otimes E(\lambda_1, a_1) \oplus P(b) \otimes \mathbb{F}_p \{ \partial \lambda_1, \partial b, \partial a_1, \partial \lambda_1 a_1 \}
+ P(b) \otimes E(a_1) \otimes \mathbb{F}_p \{ t^d \lambda_1 | 0 < d < p \}
+ P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p \{ \sigma_n, \lambda_2 b^{p^2-p} | 1 \leq n \leq p-2 \}
+ \mathbb{F}_p \{ s \},$$

with $b^{p-1} = -v_2$. The degree of the generators is given by $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|a_1| = 2p+3$, $|\sigma_n| = 2n+1$, $|t| = -2$, $|\lambda_2| = 2p^2-1$ and $|s| = 2p-3$. The classes $1$, $\sigma_n$, $\lambda_1$, $b$ and $a_1$ map under the trace to $1$, $u^{a-1} a_0$, $\lambda_1$, $b_1$ and $a_1$ in $V(1)_* \text{THH}(ku_p)$, respectively, and the other given $P(b)$-module generators map to zero.

Proof. There is a cofibre sequence of spectra $[22]$

$$K(ku_p) \to TC(ku_p) \to \Sigma^{-1} HZ_p \to \Sigma K(ku_p).$$

We have an isomorphism $V(1)_* \Sigma^{-1} HZ_p \cong \mathbb{F}_p \{ \partial, \epsilon \}$ with a primary $v_1$ Bockstein $\beta_{1,1} (\epsilon) = \partial$. Here $\partial$ is the image of the class $\partial \in V(1)_* TC(ku_p)$, while $\epsilon$ maps by the connecting homomorphism to a class $s \in V(1)_{2p-3} K(ku_p)$. These facts, together with Theorem 7.9, allow us to establish our formula for $V(1)_* K(ku_p)$. The statement on the trace follows from the definition of the given $P(b)$-module generators.

The following corollary is a restatement of Proposition [12] part (b) of the introduction.

Corollary 8.2. There is a short exact sequence of $P(b)$-modules

$$0 \to K \to P(b) \otimes_{P(b^2)} V(1)_* K(\ell_p) \to K(ku_p) \to Q \to 0$$

where $K$ and $Q$ are finite (and hence torsion) $P(b)$-modules given by

$$K = \mathbb{F}_p \{ b^k a | 1 \leq k \leq p-2 \}, \text{ and}$$

$$Q = P_{p-2}(b) \otimes \mathbb{F}_p \{ \partial b, \partial a_1, \partial a_1, \partial \lambda_1 a_1, \lambda_1 a_1 \}
+ P_{p-2}(b) \otimes \mathbb{F}_p \{ a_1 \lambda_1 t^d | 0 < d < p \}
+ E(\lambda_1) \otimes \mathbb{F}_p \{ \sigma_n b^{p^2-n} | 1 \leq n \leq p-2, 0 \leq i_n \leq p-2 - n \}.$$ Here $a \in V(1)_{2p-3} K(\ell_p)$ is the class annihilated by $v_2$ and mapping to $s$. In particular we have an isomorphism $P(b, b^{-1})$-algebras

$$P(b, b^{-1}) \otimes_{P(b^2)} V(1)_* K(\ell_p) \cong V(1)_* K(ku_p)[b^{-1}].$$

Proof. This follows from the formulas for $V(1)_* K(\ell_p)$ and for $V(1)_* K(ku_p)$ given in [3, Theorem 9.1] and Theorem 8.1 and the fact that $V(1)_* K(\ell_p)$ includes as the summand of $\delta$-weight zero in $V(1)_* K(ku_p)$, see [11, Theorem 10.2]. Notice that for $\frac{1}{2} \leq d \leq p-2$ the class $\lambda_2 b^{p-1-d}$ maps to $s$ under the connecting homomorphism, up to a non-zero scalar multiple. □

Blumberg and Mandell [8] have proved a conjecture of John Rognes that there is a localization cofibre sequence

$$K(\mathbb{Z}_p) \xrightarrow{\tau} K(ku_p) \xrightarrow{j} K(KU_p) \to \Sigma K(\mathbb{Z}_p),$$

relating the algebraic $K$-theory of $ku_p$, of its localization $KU_p = ku_p[u^{-1}]$ (i.e. periodic $K$-theory), and of its mod ($u$) reduction $HZ_p$. The $V(1)$-homotopy of $K(\mathbb{Z}_p)$ and $K(ku_p)$ is known, but we need to compute also the transfer map $\tau_*$ and solve a $P(b)$-module
extension if we seek a decent description of $V(1)_*K(KU_p)$. Let us therefore assume that this localization sequence maps via trace maps to a corresponding localization sequence in topological Hochschild homology, building a homotopy commutative diagram of horizontal fibre sequences

$$
\begin{align*}
K(\mathbb{Z}_p) & \xrightarrow{\tau} K(ku_p) \xrightarrow{j} K(KU_p) \xrightarrow{\Sigma} \Sigma K(\mathbb{Z}_p) \\
\text{tr} & \quad \text{tr} & \quad \text{tr} & \quad \Sigma \text{tr}
\end{align*}
\tag{8.1}
$$

$$
\begin{align*}
\text{THH} (\mathbb{Z}_p) & \xrightarrow{\tau} \text{THH}(ku_p) \xrightarrow{j} \text{THH}(ku_p|KU_p) \xrightarrow{\Sigma} \Sigma \text{THH}(\mathbb{Z}_p)
\end{align*}
$$

as conjectured by Lars Hesselholt, compare with Remark 8.4 below. The $V(1)$-homotopy of the bottom line was described in [10, §10]. The $V(1)$-homotopy groups of $K(\mathbb{Z}_p)$ are given by an isomorphism [22]

$$
V(1)_*K(\mathbb{Z}_p) \cong E(\lambda_1) \oplus F_p\{\partial v_1, \partial \lambda_1\} \oplus F_p\{\lambda_1 t^d \mid 0 < d < p\}.
$$

The class $\partial v_1$ maps to $s$ in $V(1)_*K(ku_p)$ via $\tau_s$. The class $1 \in V(1)_0 K(\mathbb{Z}_p)$ is in the kernel of $\tau_s$, because it is $v_2$-torsion and there is no torsion class in $V(1)_0 K(ku_p)$. Let $d \in V(1)_1 K(KU_p)$ be the class mapping to $1 \in V(1)_0 K(\mathbb{Z}_p)$ via the connecting homomorphism. Presumably $d$ corresponds to the added unit or the self-equivalence

$$
KU_p \xrightarrow{u} \Sigma^{-2} KU_p \xrightarrow{\cong} KU_p,
$$

where $u$ denotes multiplication by the Bott class and the second map is the Bott equivalence. The class $d$ maps in $V(1)_1 \text{THH}(ku_p|KU_p)$ to a class with the same name. In [10, §10] we establish an (additive) isomorphism

$$
V(1)_* \text{THH}(ku_p|KU_p) \cong P_{p-1}(u) \otimes E(d, \lambda_1) \otimes P(\mu_1).
$$

If this is an isomorphism of algebras, then $j_*(b_1) d = j_*(a_1)$ holds in $V(1)_* \text{THH}(ku_p|KU_p)$, and lifts to the relation $j_*(b_1) d = j_*(a_1)$ in $V(1)_* K(KU_p)$. By inspection this determines the structure of $V(1)_* K(KU_p)$ as a $P(b)$-module.

**Theorem 8.3.** Under the hypothesis that there exists a commutative diagram of localization sequences (8.1), and that the isomorphism (8.2) is one of algebras, we have an isomorphism of $P(b)$-modules

$$
V(1)_* K(KU_p) \cong P(b) \otimes E(\lambda_1, d) \oplus P(b) \otimes F_p\{\partial \lambda_1, \partial b, \partial a_1, \partial \lambda_1 d\} \\
\oplus P(b) \otimes E(d) \otimes F_p\{t^d \lambda_1 \mid 0 < d < p\} \\
\oplus P(b) \otimes E(\lambda_1) \otimes F_p\{\sigma_n, \lambda_2 t^{p-2} \mid 1 \leq n \leq p - 2\}.
$$

The class $d$ has degree 1, and the other classes have the degree given in Theorem 8.1.

**Remark 8.4.** Consider a complete discrete valuation field $K$ of characteristic zero with perfect residue field $k$ of characteristic $p \geq 3$, and let $A$ be its valuation ring. Hesselholt and Madsen [23] compute the $V(0)$-homotopy of $K(A)$ and $K(K)$ by means of the cyclotomic trace. They introduce a relative version of topological cyclic homology, denoted $TC(A|K)$, that sits in a localization cofibre sequence

$$
TC(k) \xrightarrow{} TC(A) \xrightarrow{} TC(A|K) \xrightarrow{} \Sigma TC(k).
$$

The computation of $V(0)_*TC(A|K)$ is achieved by using the rich algebraic structure on the $V(0)$-homotopy groups of the tower $TR^*(A|K)$, and described in terms of the
De Rham-Witt complex with log poles

\[ W_{\omega}(A, A \cap K^n) \],

see [23] Th. C. Then \( V(0)_*TC(A) \) can be evaluated by means of the localization sequence. This approach has, in particular, the advantage of avoiding a computation of \( V(0)_*TR^n(A) \), which seems quite intractable.

Continuing the discussion in [1, §10] on a relative trace for \( ku_p \), and following Lars Hesselholt, one could speculate on the existence of a relative term \( TC(ku_p|KU_p) \) fitting in a localization sequence

\[ TC(HZ_p) \to TC(ku_p) \to TC(ku_p|KU_p) \to \Sigma TC(HZ_p) \],

through which the trace of diagram (8.1) factorizes. By analogy with the case of complete discrete valuation fields, we expect that a computation of \( V(1)_*TR^n(ku_p|KU_p) \) should be easier to handle than the computation of \( V(1)_*TR^n(ku_p) \) presented in this paper. In fact, the advantage of such an approach is already apparent when comparing

\[ V(1)_*TR^1(ku_p|KU_p) = V(1)_*THH(ku_p|KU_p) \]

in (8.2) with \( V(1)_*THH(ku_p) \) in (4.1), and is also confirmed by partial, hypothetical computations of \( V(1)_*TR^n(\ell_p|L_p) \) and \( V(1)_*TR^n(ku_p|KU_p) \) by Lars Hesselholt (private communication) and the author.

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