THE SOFT TORUS II
A Variational Analysis of
Commutator Norms

Ruy Exel†

Abstract. The field of $C^*$-algebras over the interval $[0,2]$ for which the fibers are the Soft Tori is shown to be continuous. This result is applied to show that any pair of non-commuting unitary operators can be perturbed (in a weak sense) in such a way to decrease the commutator norm. Perturbations in norm are also considered and a characterization is given for pairs of unitary operators which are local minimum points for the commutator norm in the finite dimensional case.

1. Introduction.
As in [3], for every $\varepsilon$ in the real interval $[0,2]$ we let $A_\varepsilon$ be the universal unital $C^*$-algebra generated by unitary elements $u_\varepsilon$ and $v_\varepsilon$ subject to the relation

$$||u_\varepsilon v_\varepsilon - v_\varepsilon u_\varepsilon|| \leq \varepsilon.$$  

Clearly, if $\varepsilon_1 \leq \varepsilon_2$ there is a unique homomorphism $A_{\varepsilon_2} \longrightarrow A_{\varepsilon_1}$ sending $u_{\varepsilon_2}$ and $v_{\varepsilon_2}$ respectively to $u_{\varepsilon_1}$ and $v_{\varepsilon_1}$. In case $\varepsilon_2 = 2$ and $\varepsilon_1 = \varepsilon$ we shall denote this map by $\phi_\varepsilon$, i.e.

$$\phi_\varepsilon : A_2 \longrightarrow A_\varepsilon$$

One of the main results of the present work (Theorem 3.4) is the fact that there exists a continuous field of $C^*$-algebras over the interval $[0,2]$ such that $A_\varepsilon$ is the fiber over $\varepsilon$ and moreover such that

$$\varepsilon \in [0,2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

is a continuous section for every $a$ in $A_2$. We refer the reader to [2] for a treatment of the theory of continuous fields of $C^*$-algebras.

The central point in proving our main result is to show that $||\phi_\varepsilon(a)||$ is a continuous function of $\varepsilon$ for all $a$ in $A_2$. If we let

$$J_\varepsilon = \text{Ker}(\phi_\varepsilon)$$

then we have that $||\phi_\varepsilon(a)|| = \text{dist}(a, J_\varepsilon)$ and, since the $J_\varepsilon$'s clearly form a decreasing chain of ideals, $||\phi_\varepsilon(a)||$ is seen to be an increasing function of $\varepsilon$.

Let us denote by $J_{\varepsilon}^+$ the closure of the union of all $J_{\varepsilon'}$ for $\varepsilon' > \varepsilon$ and by $J_{\varepsilon}^-$ the intersection of all $J_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. That is

$$J_{\varepsilon}^+ = \bigcup_{\varepsilon' > \varepsilon} J_{\varepsilon'}$$

† Partially supported by FAPESP, Brazil. On leave from the University of São Paulo.

1991 MR Subject Classification: 46L05, 49R20.
and

\[ J_\varepsilon^- = \bigcap_{\varepsilon' \prec \varepsilon} J_{\varepsilon'}. \]

**1.1 Proposition.** Let \( \varepsilon \) be in \([0,2)\) (resp. \((0,2]\)). A sufficient condition for

\[ \varepsilon' \rightarrow ||\phi_{\varepsilon'}(a)|| \]
to be right (resp. left) continuous at \( \varepsilon \) for all \( a \) in \( A_2 \) is that \( J_\varepsilon = J_\varepsilon^+ \) (resp. \( J_\varepsilon = J_\varepsilon^- \)).

**Proof.** Note that for \( a \) in \( A_2 \) we have

\[ \text{dist}(a, J_{\varepsilon}^+) = \inf_{\varepsilon' > \varepsilon} \text{dist}(a, J_{\varepsilon'}) \]

and that

\[ \text{dist}(a, J_{\varepsilon}^-) = \sup_{\varepsilon' \prec \varepsilon} \text{dist}(a, J_{\varepsilon'}). \]

The first identity follows from trivial Banach space facts. On the other hand, the second one cannot be generalized to Banach spaces so let’s prove it.

Let

\[ \phi : A_2 \rightarrow \prod_{\varepsilon' \prec \varepsilon} A_{\varepsilon'} \]

be given by \( \phi(a) = (\phi_{\varepsilon'}(a))_{\varepsilon' \prec \varepsilon} \) and observe that \( \text{Ker}(\phi) = J_{\varepsilon}^- \) so

\[ \text{dist}(a, J_{\varepsilon}^-) = ||\phi(a)|| = \sup_{\varepsilon' \prec \varepsilon} ||\phi_{\varepsilon'}(a)|| = \sup_{\varepsilon' \prec \varepsilon} \text{dist}(a, J_{\varepsilon'}). \]

The conclusion now follows without much trouble. \( \square \)

En passant, let’s give an example to support our statement that the above fact does not generalize to Banach spaces.

Let \( E = C[0,1] \) and for \( f \) in \( E \) put

\[ ||f|| = |f(0)| + \sup_{t \in [0,1]} |f(t)|. \]

Let

\[ E_n = \{ f \in E : f([1/n,1]) = 0 \} \]

and let \( g \) be the constant function \( g = 1 \). Then

\[ \text{dist}(g, E_n) = 1 \]

for all \( n \) while

\[ \text{dist}(g, \bigcap_{n \geq 1} E_n) = ||g|| = 2. \]
1.2 Proposition. For all $\varepsilon \in [0, 2)$ one has $J_\varepsilon^+ = J_\varepsilon$.

Proof. If we denote by $u_\varepsilon^+$ and $v_\varepsilon^+$ the images of $u_2$ and $v_2$ in $A_2/J_\varepsilon^+$ then it is clear that

$$||u_\varepsilon^+ v_\varepsilon^+ - v_\varepsilon^+ u_\varepsilon^+|| \leq \varepsilon$$

hence by the universal property of $A_\varepsilon$ there is a homomorphism

$$A_\varepsilon = A_2/J_\varepsilon \longrightarrow A_2/J_\varepsilon^+$$

sending $u_\varepsilon$ and $v_\varepsilon$ to $u_\varepsilon^+$ and $v_\varepsilon^+$. Therefore $J_\varepsilon \subseteq J_\varepsilon^+$. But since the reverse inclusion is trivial, our proof is complete. \qed

The main technical result of this work, which shall be proven in the next two sections, is the following.

1.3 Theorem. For all $\varepsilon$ in $(0, 2]$ one has $J_\varepsilon^- = J_\varepsilon$.

Therefore we have

1.4 Corollary. For all $a$ in $A_2$ the function

$$\varepsilon \in [0, 2] \longrightarrow ||\phi_\varepsilon(a)||$$

is continuous.

2. The Case $\varepsilon < 2$.

Throughout this section we shall fix a real number $\varepsilon$ with $0 < \varepsilon < 2$.

2.1 Lemma. Suppose $u_0, u_1, ..., u_n$ is a finite sequence of unitary elements in a $C^*$-algebra $B$ such that $||u_{k-1} - u_k|| \leq \varepsilon$ for all $k$. Then, for every $\delta > 0$, there are unitaries $v_0, v_1, ..., v_n$ in $B$ such that

1. $||v_k - u_k|| \leq \delta$ for $k = 0, ..., n$
2. $||v_{k-1} - v_k|| < \varepsilon$ for $k = 1, ..., n$.

Proof. Since $||u_{k-1} - u_k|| \leq \varepsilon < 2$, it follows that $||u_k u_{k-1}^{-1} - 1|| < 2$ so that $-1$ is not in the spectrum of $u_k u_{k-1}^{-1}$. Therefore we may define

$$h_k = \log(u_k u_{k-1}^{-1})$$

where log is the principal branch of the logarithm. Each $h_k$ is then a skew adjoint element in $B$ and we have

$$u_k = e^{h_k} u_{k-1}$$

for $k = 1, 2, ..., n$. 

3
We shall choose our sequence \( v_0, ..., v_n \) of the form

\[
v_0 = u_0
\]

and

\[
v_k = e^{t_k h_k} u_{k-1} \quad \text{for} \quad k \geq 1
\]

where each \( t_k \) will be a suitably chosen positive real number approaching 1 from below.

A real function which will be useful in our estimates is

\[
d(x) = |1 - e^{ix}| = 2 \sin \left( \frac{x}{2} \right) \quad \text{for} \quad x \in \mathbb{R}.
\]

For instance, observe that if \( h \) is skew adjoint and \( ||h|| \leq \pi \) one has

\[
||1 - e^h|| = d(||h||)
\]

by the spectral theorem. Moreover, for all \( k = 1, ..., n \)

\[
d(||h_k||) = ||1 - e^{h_k}|| = ||u_{k-1} - u_k|| \leq \varepsilon
\]

which implies (since \( d \) is increasing in \([0, \pi]\)) that

\[
||h_k|| \leq \theta
\]

where \( \theta = d^{-1}(\varepsilon) \). Note that \( \theta < \pi \) because \( \varepsilon < 2 \).

In search of the correct choice of the \( t_k \)'s observe that

\[
||v_0 - v_1|| = ||u_0 - e^{t_1 h_1} u_0|| = d(t_1 ||h_1||) \leq d(t_1 \theta)
\]

and that for \( k \geq 1 \) we have

\[
||v_k - v_{k+1}|| \leq ||v_k - u_k|| + ||u_k - v_{k+1}|| =
\]

\[
||e^{t_k h_k} u_{k-1} - e^{h_k} u_{k-1}|| + ||u_k - e^{t_{k+1} h_{k+1}} u_k|| =
\]

\[
d((1 - t_k)||h_k||) + d(t_{k+1} ||h_{k+1}||) \leq
d((1 - t_k) \theta) + d(t_{k+1} \theta).
\]

Note that \( d'(x) = \cos(x/2) \) so for \( x \in [0, \theta] \) we have \( m \leq d'(x) \leq 1 \) where \( m = \cos(\theta/2) \)

is strictly positive since \( \theta < \pi \). By the mean value theorem we then have

\[
m|t - s| \leq |d(t) - d(s)| \leq |t - s|
\]

for all \( t \) and \( s \) in \([0, \theta]\).

If this last fact is used in our previous computations, we obtain

\[
||v_0 - v_1|| \leq d(t_1 \theta) = \varepsilon - (d(\theta) - d(t_1 \theta)) \leq
\]
\[ \varepsilon - m(\theta - t_1\theta) = \varepsilon - m\theta(1 - t_1) \]

while for \( k \geq 1 \)
\[
\|v_k - v_{k+1}\| \leq d((1 - t_k)\theta) + d(t_{k+1}\theta) \leq (1 - t_k)\theta + \varepsilon - (d(\theta) - d(t_{k+1}\theta)) \leq (1 - t_k)\theta + \varepsilon - m(\theta - t_{k+1}\theta) = \varepsilon + (1 - t_k - m(1 - t_{k+1}))\theta.
\]

Therefore the condition that \( \|v_{k-1} - v_k\| < \varepsilon \) will be fulfilled as long as
\[ 1 - t_1 > 0 \]

and
\[ 1 - t_{k+1} > \frac{1 - t_k}{m}. \]

If we thus put \( t_k = 1 - 2^k\sigma/m^k \) for \( \sigma < (m/2)^n \) we have that each \( t_k \) is in \((0,1)\) and \( \|v_{k-1} - v_k\| < \varepsilon \).

Clearly, as \( \sigma \) tends to zero, each \( v_k \) approaches the corresponding \( u_k \) so a suitable choice for \( \sigma \) yields
\[ \|v_k - u_k\| \leq \delta \]
for all \( k = 0, 1, \ldots, n \).

Recall from [3] that \( A_\varepsilon \) is isomorphic to the crossed product
\[ A_\varepsilon \cong B_\varepsilon \times \tau \mathbb{Z} \]
where \( B_\varepsilon \) is the universal \( C^* \)-algebra generated by a sequence \( \{u_n(\varepsilon) : n \in \mathbb{Z}\} \) of unitaries satisfying the relations
\[ \|u_n(\varepsilon) - u_{n+1}(\varepsilon)\| \leq \varepsilon \]
for all \( n \).

Moreover \( \tau \) is the automorphism of \( B_\varepsilon \) given by
\[ \tau(u_n(\varepsilon)) = u_{n+1}(\varepsilon) \quad \text{for} \quad n \in \mathbb{Z}. \]

**2.2 Proposition.** There exists a sequence \( (\psi_n)_{n \in \mathbb{N}} \) of endomorphisms of \( B_\varepsilon \), converging pointwise to the identity map, such that
\[ \sup_{k \in \mathbb{Z}} \|\psi_n(u_k(\varepsilon)) - \psi_n(u_{k+1}(\varepsilon))\| < \varepsilon. \]

**Proof.** By the previous Lemma, let for every \( n \)
\[ v_{-n}, v_{-n+1}, \ldots, v_{0}, v_{1}, \ldots, v_{n} \]
be unitaries in $B_\varepsilon$ such that

$$||v_k^{(n)} - u_k^{(\varepsilon)}|| \leq \frac{1}{n} \quad \text{for} \quad k = -n, \ldots, n$$

and

$$||v_k^{(n)} - v_{k-1}^{(n)}|| < \varepsilon \quad \text{for} \quad k = -n + 1, \ldots, n.$$ 

Define $\psi_n : B_\varepsilon \rightarrow B_\varepsilon$ by

$$\psi_n(u_k^{(\varepsilon)}) = \begin{cases} v_k^{(n)} & \text{if} \quad k < -n \\ v_k^{(n)} & \text{if} \quad -n \leq k \leq n \\ v_n^{(n)} & \text{if} \quad k > n. \end{cases}$$

It is then clear that

$$\lim_{n \rightarrow \infty} \psi_n(u_k^{(\varepsilon)}) = u_k^{(\varepsilon)}$$

which implies that $\psi_n$ converges pointwise to the identity.

The condition that

$$\sup_{k \in \mathbb{Z}} ||\psi_n(u_k^{(\varepsilon)}) - \psi_n(u_{k+1}^{(\varepsilon)})|| < \varepsilon$$

is also clearly satisfied. \hfill \Box

2.3 Definition. Let $K_\varepsilon$ be the ideal in $B_2$ given by the kernel of the canonical map

$$\phi_\varepsilon : B_2 \rightarrow B_\varepsilon.$$

2.4 Theorem. One has $\bigcap_{\varepsilon' < \varepsilon} K_{\varepsilon'} = K_\varepsilon$.

Proof. Let $x$ be in $\bigcap_{\varepsilon' < \varepsilon} K_{\varepsilon'}$ and put $y = \psi_\varepsilon(x)$. We have

$$y = \lim_{n \rightarrow \infty} \psi_n(y) = \lim_{n \rightarrow \infty} \psi_n(\phi_\varepsilon(x)).$$

Now let

$$\varepsilon'_n = \sup_k ||\psi_n \phi_\varepsilon(u_k^{(2)}) - \psi_n \phi_\varepsilon(u_{k+1}^{(2)})|| = \sup_k ||\psi_n(u_k^{(\varepsilon)}) - \psi_n(u_{k+1}^{(\varepsilon)})||$$

which by (2.2) is strictly less than $\varepsilon$. So $\psi_n \phi_\varepsilon$ factors through $B_{\varepsilon'_n}$ and since $x$ is in $K_{\varepsilon'_n}$ we have $\psi_n \phi_\varepsilon(x) = 0$. So $y = 0$ which implies that $x$ is in $K_\varepsilon$. The converse inclusion is trivial. \hfill \Box

2.5 Lemma. Let $\phi : A \rightarrow B$ be a $C^*$-algebra homomorphism and suppose $\phi$ is equivariant with respect to automorphisms $\alpha$ and $\beta$ of $A$ and $B$ respectively.
Let $K = \text{Ker}(\phi)$ and $J = \text{Ker}(\bar{\phi})$ where $\bar{\phi}$ is the canonical extension of $\phi$ to the corresponding crossed products by $\mathbb{Z}$. Then

$$J = \{ x \in A \times_\alpha \mathbb{Z} : E_A(xu^{-n}) \in K \quad \text{for} \quad n \in \mathbb{Z} \}$$

where

$$E_A : A \times_\alpha \mathbb{Z} \to A$$

is the associated conditional expectation [5] and $u$ is the unitary implementing $\alpha$.

**Proof.** It is clear that $\phi \circ E_A = E_B \circ \bar{\phi}$ where $E_B$ is the conditional expectation for $B \times_\beta \mathbb{Z}$.

Let $v$ be the implementing unitary for $B \times_\beta \mathbb{Z}$. Given $x \in A \times_\alpha \mathbb{Z}$ we have that $x$ is in $J$ if and only if $\bar{\phi}(x) = 0$ which is equivalent to the fact that $E_B(\bar{\phi}(x)v^{-n}) = 0$ for all $n$, or that $\phi(E_A(xu^{-n}) = 0$. But this is to say that $E_A(xu^{-n})$ is in $K$. \qed

### 2.6 Theorem.

**For every** $\varepsilon \in (0, 2)$ **one has** $J_{\varepsilon^-} = J_{\varepsilon}$.

**Proof.** Let $E : A_2 \to B_2$ be the conditional expectation induced by the isomorphism $A_2 \simeq B_2 \times_\tau \mathbb{Z}$.

Given $x$ in $J_{\varepsilon^-}$ we have that $E(xu^{-n})$ is in $K_{\varepsilon'}$ for all $n$ and $\varepsilon' < \varepsilon$ so

$$E(xu^{-n}) \in \bigcap_{\varepsilon' < \varepsilon} K_{\varepsilon'} = K_{\varepsilon}$$

for all $n$, which shows that $x \in J_{\varepsilon}$. The converse inclusion is trivial. \qed

### 3. The case $\varepsilon = 2$.

The purpose of this section is to prove that $J_{2^-} = J_2$ or, since $J_2 = (0)$, that

$$\bigcap_{\varepsilon < 2} J_{\varepsilon} = (0).$$

Note that the techniques employed in the previous section do not work here because one couldn’t take logarithms in the proof of (2.1) if $\varepsilon = 2$. A different approach is thus necessary.

#### 3.1 Lemma.

**Let** $w_1$ and $w_2$ be $n \times n$ unitary matrices. **Then** $||w_1 - w_2|| = 2$ **if and only if** $\det(w_1 + w_2) = 0$.

**Proof.** We have that $||w_1 - w_2|| = 2$ if and only if $||w_1w_2^{-1} - 1|| = 2$ which is equivalent to $-1$ being in the spectrum of $w_1w_2^{-1}$ which is to say that $\det(w_1w_2^{-1} + 1) = 0$ or that $\det(w_1 + w_2) = 0$. \qed
3.2 Proposition. Given unitary $n \times n$ matrices $u$ and $v$ such that $\|uv - vu\| = 2$ there is, for every $\delta > 0$, a unitary $u'$ with

$$\|u' - u\| < \delta$$

and

$$\|u'v - vu'\| < 2.$$  

Proof. Write $u = e^h$ for some skew adjoint $h$ and let $u(t) = ue^{-th}$ for all real $t$. Put

$$f(t) = \det (u(t)v + vu(t))$$

and observe that

$$f(1) = \det(2v) \neq 0$$

while

$$f(0) = \det(uv + vu) = 0$$

by (3.1). Therefore $f$ is not a constant function and since it is analytic, its zeros are isolated. So there are arbitrarily small values of $t$ for which $f(t) \neq 0$, which is to say, by (3.1) again, that

$$\|u(t)v - vu(t)\| < 2.$$  

Taking $t$ sufficiently small will also ensure that $\|u(t) - u\| < \delta$. \qed

3.3 Theorem. One has $J_2^- = J_2$.

Proof. Assume by way of contradiction that $a \in J_2^-$ is non-zero.

Note that $A_2$ is isomorphic to the full $C^*$-algebra of the free group on two generators so that by [1] $A_2$ has a separating family of finite dimensional representations. Let therefore

$$\pi : A_2 \longrightarrow M_n(\mathbb{C})$$

be a representation such that $\pi(a) \neq 0$.

Let $u = \pi(u_2)$ and $v = \pi(v_2)$ and write

$$u = \lim_{i \to \infty} u'_i$$

where $\|u'_i v - vu'_i\| < 2$ by (3.2). For each $i$ let $\pi_i$ be the representation of $A_2$ such that

$$\pi_i(u_2) = u'_i \quad \text{and} \quad \pi_i(v_2) = v.$$  

Then, if $\varepsilon_i = \|u'_i v - vu'_i\|$, we have that $\pi_i$ vanishes on $J_{\varepsilon_i}$ and thus also on $J_2^-$ so $\pi_i(a) = 0$.

On the other hand it is clear that $\pi_i$ converges pointwise to $\pi$ so $\pi(a) = 0$ which is a contradiction. \qed
3.4 Theorem. There exists a continuous field of $C^*$-algebras over the interval $[0,2]$ such that $A_\varepsilon$ is the fiber over $\varepsilon$ and such that

$$\varepsilon \in [0,2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

is a continuous section for every $a$ in $A_2$.

Proof. Let $S$ be the set of sections

$$\varepsilon \in [0,2] \mapsto \phi_\varepsilon(a) \in A_\varepsilon$$

for $a$ in $A_2$. According to [2] (Propositions 10.2.3 and 10.3.2) all one needs to check is that $S$ is a *-subalgebra of the algebra of all sections, that the set of all $s(\varepsilon)$ as $s$ runs through $S$ is dense in $A_\varepsilon$ and that $||s(\varepsilon)||$ is continuous as a function of $\varepsilon$ for all $s$ in $S$.

The first two properties are trivial while the last one follows from (1.1), (1.2), (2.6) and (3.3). \hfill \Box

4. Local Minima for Commutator Norms.

As clearly indicated by the results obtained above, the phenomena under consideration is related to the following question

4.1 Question. Given unitary operators $u$ and $v$ which do not commute, when is it possible to perturb $u$ and $v$ in order to obtain a new pair $u'$ and $v'$ such that

$$||u'v' - v'u'|| < ||uv - vu||?$$

In other words (4.1) calls for a characterization of pairs of unitary operators which are not local minimum points for the commutator norm. Proposition (3.2) is clearly a partial answer to (4.1) and it says that when $||uv - vu|| = 2$ in finite dimensions then $(u, v)$ is never such a local minimum point.

In formulating the question above we chose not to specify the precise meaning of “to perturb” in order to allow for different points of view.

The following is a complete answer to (4.1) under quite a loose type of perturbation which we might call *-strong dilated perturbation.

4.2 Theorem. Let $u$ and $v$ be non commuting unitary operators on a Hilbert space $H$. Then there are nets $(u_i)_i$ and $(v_i)_i$ of unitary operators on $H^\infty$ (the direct sum of infinitely many copies of $H$) satisfying

$$||u_iv_i - v_iu_i|| < ||uv - vu||$$

and such that the compressions

$$proj_H \circ u_i|_H \quad \text{and} \quad proj_H \circ v_i|_H$$
of $u_i$ and $v_i$ to $H$ converge *-strongly to $u$ and $v$ respectively.

**Proof.** After Theorems (2.6) and (3.3) this basically becomes a consequence of [4] and some well known results on representation theory. We therefore restrict ourselves to a sketch of the proof.

Let $\varepsilon = ||uv - vu||$ and consider the set $N$ of states on $A_2$ which vanish on some $J_{\varepsilon'}$ for $\varepsilon' < \varepsilon$. Since $\bigcap_{\varepsilon' < \varepsilon} J_{\varepsilon'} = J_\varepsilon$ it can be proved that $N$ is weakly dense in the set of states of $A_2$ that vanish on $J_\varepsilon$.

Given $u$ and $v$ let $\pi$ be the representation of $A_2$ on $H$ such that $\pi(u_2) = u$ and $\pi(v_2) = v$. Assume without loss of generality that $\pi$ is cyclic with cyclic vector $\xi$ and put

$$f : a \in A_2 \longrightarrow \langle \pi(a)\xi, \xi \rangle.$$ 

Since $\pi$ factors through $A_\varepsilon$ we have that $f$ vanish on $J_\varepsilon$ so there exists a net $(f_i)_{i \in I}$ in $N$ converging weakly to $f$. Let, for every $i$, $\pi_i$ be the GNS representation of $A_2$ corresponding to $f_i$. Since each $f_i$ vanish on some $J_{\varepsilon'_i}$ with $\varepsilon'_i < \varepsilon$ the same is true for $\pi_i$ hence

$$||\pi_i(u_2)\pi_i(v_2) - \pi_i(v_2)\pi_i(u_2)|| \leq \varepsilon'_i < \varepsilon.$$ 

Using the methods of [4] one may assume that the space $H_i$ where $\pi_i$ acts is a subspace of $H^\infty$ and that the conclusion holds with

$$u_i = \pi_i(u_2) + 1 - p_i$$
and

$$v_i = \pi_i(u_2) + 1 - p_i$$
where $p_i$ is the projection onto $H_i$. \hfill \Box

We therefore see that there are no local minimum points for the commutator norm other than the commuting pairs, as long as we consider *-strong dilated perturbations.

The situation is quite different if norm perturbations are considered as we shall see in the next Section.

5. Norm Perturbations in Finite Dimensions.

Let us now study question (4.1) for pairs of unitary operators on a finite dimensional Hilbert space. From now on we shall only consider norm perturbations.

For $n \geq 3$ denote by $\Omega_n$ and $S_n$ the $n \times n$ Voiculescu’s unitary matrices (see [6])

$$\Omega_n = \begin{pmatrix} \omega & \omega^2 & \omega^3 & \cdots & \omega^n \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \vdots & \ddots & \ddots \\ 1 & 0 & 1 \end{pmatrix}$$

10
where $\omega = e^{2\pi i/n}$.

**5.1 Theorem.** For $n \geq 3$ there exists a neighborhood $V$ of the pair $(\Omega_n, S_n)$ in $U(n) \times U(n)$ so that

$$||uv - vu|| \geq ||\Omega_n S_n - S_n \Omega_n||$$

for all pairs $(u, v)$ in $V$.

**Proof.** Note that $\Omega_n S_n \Omega_n^{-1} S_n^{-1} = \omega I_n$ so that if $(u, v)$ is close enough to $(\Omega_n, S_n)$ then the spectrum of $uvu^{-1}v^{-1}$ is in a small neighborhood of $\omega$ in the complex plane.

On the other hand, notice that

$$\det(uvu^{-1}v^{-1}) = 1$$

so that if the spectrum of $uvu^{-1}v^{-1}$ is the set $\{e^{i\theta_1}, ..., e^{i\theta_n}\}$ with $-\pi < \theta_i < \pi$ one has that $\sum_{k=1}^{n} \theta_k$ is in $2\pi \mathbb{Z}$. So, by continuity,

$$\sum_{k=1}^{n} \theta_k = 2\pi.$$

Therefore, for some $k_0$ we must have $\theta_{k_0} \geq 2\pi/n$ and it follows that

$$||uv - vu|| \geq |e^{i\theta_{k_0}} - 1| \geq |\omega - 1| = ||\Omega_n S_n - S_n \Omega_n||.$$

\[\square\]

In other words, the pair $(\Omega_n, S_n)$ is a local minimum for the commutator norm. This shows that the situation is quite different from what we saw when we considered *-strong dilated perturbations in Section (4). This result should also be compared with [6].

Clearly the method used in Theorem (5.1) above applies to show that the conclusion is also true for any pair of unitary matrices whose multiplicative commutator is a scalar multiple of the identity, but not equal to $-I$ (see 3.2).

In fact, among irreducible pairs there are no other examples as we shall prove shortly. We say that a pair of unitary operators is irreducible when there is no proper invariant subspace for both elements of the pair.

Denote by $\gamma$ the map

$$\gamma : (u, v) \in U(n) \times U(n) \mapsto uvu^{-1}v^{-1} \in SU(n).$$

**5.2 Lemma.** A point $(u, v) \in U(n) \times U(n)$ is regular for $\gamma$ (in the sense that $\gamma$ is a submersion at $(u, v)$) if and only if $(u, v)$ is an irreducible pair.

**Proof.** If $h, k$ are in the Lie algebra $u(n)$ of $U(n)$, a simple computation shows that

$$d\gamma_{(u, v)}(uh, vk) = uv(v^{-1}hv - h + k - u^{-1}ku)u^{-1}v^{-1}.$$
So $\gamma$ is a submersion at $(u, v)$ if and only if the map

$$L : (h, k) \in u(n) \times u(n) \rightarrow v^{-1}hv - h + k - u^{-1}ku \in su(n)$$

is onto the Lie algebra $su(n)$ of $SU(n)$.

Under the inner product on $su(n)$ defined by

$$\langle x, y \rangle = \text{Trace}(xy^*)$$

the orthogonal space to the image of $L$ can easily be seen to be the set

$$\{ x \in su(n) : xu = ux \text{ and } xv = vx \}.$$ 

Now, by Schur’s lemma, irreducibility of $(u, v)$ can be characterized by the fact that only scalars commute with both $u$ and $v$. Since $su(n)$ contains no scalar matrices the result follows.

\[ \square \]

**5.3 Theorem.** If $(u, v)$ is an irreducible pair in $U(n) \times U(n)$ and at the same time a local minimum for the commutator norm then $uvu^{-1}v^{-1}$ is a scalar.

**Proof.** By the open mapping theorem the image under $\gamma$ of a neighborhood of $(u, v)$ is a neighborhood of $\gamma(u, v)$. But since

$$||uv - vu|| = ||\gamma(u, v) - 1||$$

it follows that $\gamma(u, v)$ is a local minimum for the map

$$w \in SU(n) \rightarrow ||w - 1||$$

and this implies, as a moments thought will reveal, that $\gamma(u, v)$ is a scalar. \[ \square \]

This completes the classification of local minima for irreducible pairs. So let us now consider a reducible pair $(u, v)$ of unitary $n \times n$ matrices. As usual write

$$u = \oplus u_j \quad \text{and} \quad v = \oplus v_j$$

where each pair $(u_j, v_j)$ is irreducible and observe that

$$||uv - vu|| = \max_j ||u_jv_j - v_ju_j||.$$ 

**5.4 Theorem.** Let $u = \oplus u_j$ and $v = \oplus v_j$ be as above and suppose that the pair $(u, v)$ is a local minimum for the commutator norm. Then for some value of $j$ for which

$$||u_jv_j - v_ju_j|| = ||uv - vu||$$

one has that $u_jv_ju_j^{-1}v_j^{-1}$ is a scalar.
Proof. If this is not so then for all such \( j \) the pair \((u_j, v_j)\) admits by (5.3) a small perturbation decreasing the commutator norm. Together, these perturbations yield a contradiction to the hypothesis. \(\square\)

A natural question which one could ask is, of course, whether the converse to Theorem (5.4) is also true. A good test case is given by the pair

\[
(\Omega_n \oplus I_m, S_n \oplus I_m)
\]

that is, the direct sum of Voiculescu’s unitaries with the \( m \times m \) identity matrix.

This pair clearly satisfies the conclusion of (5.4) and so it is natural to ask whether or not it is a local minimum for the commutator norm.

Despite strong favorable evidence given by some partial positive results and a large amount of computer simulation supporting this thesis, we were unable to establish a proof for this fact. In fact we do not even know whether the above pair is a local minimum for \( n = 3 \) and \( m = 1 \). Nevertheless, we conjecture that

5.5 Conjecture. The converse of (5.4) is also true.

6. An Example.

Considering the apparent discrepancy between (4.2) and (5.1) it is perhaps interesting to see a concrete example of nets \((u_i)_i\) and \((v_i)_i\), whose existence is guaranteed by (4.2), in case the given unitaries are taken to be Voiculescu’s unitaries, i.e. \( u = \Omega_n \) and \( v = S_n \).

For that purpose it is enough to find, for all \( \delta > 0 \), unitary operators \( u' \) and \( v' \) on a separable, infinite dimensional Hilbert space \( H \), and an orthonormal set \( \{ \xi_k : k \in \mathbb{Z}/n\mathbb{Z} \} \) of vectors in \( H \) such that

(i) \( \|u'v' - v'u'\| < \|\Omega_n S_n - S_n \Omega_n\| \)
(ii) \( \|u' (\xi_k) - \omega^k \xi_k\| < \delta \) and
(iii) \( \|v' (\xi_k) - \xi_{k+1}\| < \delta \)

for all \( k \) in \( \mathbb{Z}/n\mathbb{Z} \) where \( \omega = e^{2\pi i/n} \).

Let \( H = L_2(S^1) \) and let \( u' \) be the unitary operator on \( H \) defined by

\[
u'(\xi)|_z = z \xi(z) \quad \text{for} \quad \xi \in H, \ z \in S^1\]

and, for \( \theta < 2\pi/n \), let \( v' \) be defined by

\[
v'(\xi)|_z = \xi(e^{-i\theta}z) \quad \text{for} \quad \xi \in H, \ z \in S^1.
\]

Let \( \xi_0 \) be a unit vector in \( H \) represented by a function \( f \) on \( S^1 \) supported in a neighborhood \( V \) of \( z = 1 \) which is small enough so that \( V \) is disjoint from \( e^{ik\theta}V \) for \( k = 1, 2, \ldots, n-1 \). For all such \( k \) let \( \xi_k = u'^k(\xi_0) \). The reader may now check that (i), (ii) and (iii) hold as long as \( \theta \) is close to \( 2\pi/n \) and the diameter of \( V \) is small enough.
References

[1] M. D. Choi, The full C*-algebra of the free group on two generators, Pacific J. Math. 87(1980), 41-48.

[2] J. Dixmier, C*-Algebras, North Holland, 1982.

[3] R. Exel, The Soft Torus and applications to almost commuting matrices, Pacific J. Math, to appear.

[4] J. M. G. Fell, C*-algebras with smooth dual, Illinois J. Math. 4(1960), 221-230.

[5] M. A. Rieffel, Induced representations of C*-algebras, Advances in Math. 13(1974), 176-257.

[6] D. Voiculescu, Asymptotically commuting finite rank unitary operators without commuting approximants, Acta Sci. Math. (Szeged) 45(1983), 429-431.

Current address:
Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131, USA
e-mail: exel@math.unm.edu

Permanent address:
Departamento de Matemática,
Universidade de São Paulo,
Caixa Postal 20570,
01498 São Paulo SP, Brasil