Action scales for quantum decoherence and their relation to structures in phase space.

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Abstract

A characteristic action $\Delta S$ is defined whose magnitude determines some properties of the expectation value of a general quantum displacement operator. These properties are related to the capability of a given environmental ‘monitoring’ system to induce decoherence in quantum systems coupled to it. We show that the scale for effective decoherence is given by $\Delta S \approx \hbar$. We relate this characteristic action with a complementary quantity, $\Delta Z$, and analyse their connection with the main features of the pattern of structures developed by the environmental state in different phase space representations. The relevance of the $\Delta S$-action scale is illustrated using both a model quantum system solved numerically and a set of model quantum systems for which analytical expressions for the time-averaged expectation value of the displacement operator are obtained explicitly.

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I. INTRODUCTION

The superposition principle and the interference terms that generates are the key components of the quantum formalism, and responsible for the main differences between the quantum and classical world. The boundary between these two worlds and the mechanisms that prevent the interference terms from being apparent in the classical realm have been the subjects of many theoretical and experimental studies since the very beginning of the “quantum era”. Significant advances in the analysis and experimentation on the interaction between mesoscopic and microscopic systems are pushing the boundary between the two worlds. An example is the study of measurement processes where the ‘monitoring’ apparatus is represented by a system with an increasingly larger number of degrees of freedom (more classical) and the analysis of the associated disappearance of the non-diagonal terms of the density operator of the microscopic system in some preferred matrix representation [1, 2]. The study of the effectiveness of a given system that plays the role of an environment or of a measurement apparatus to induce decoherence in another system is of fundamental and practical interest. For instance, the advances in the fields of quantum communication and quantum computation depend crucially on our ability to manipulate entanglement [3] and to control the capability of the environment or measurement devices to induce decoherence in our qubit (pointer) system [4, 5].

Many actual interactions between a two-level system $S$, spanned by the pointer states $|+\rangle$ and $|−\rangle$, and a system $E$ playing the role of the environment (for instance as a ‘monitoring’ apparatus), can be described by means of a coupling Hamiltonian of von Neumann’s form [4]. In particular, we will use a generic term $\hat{V}_{SE} = (|+\rangle \langle +| - |−\rangle \langle −|) (c_q \cdot \hat{q} + c_p \cdot \hat{p})$, where $\hat{q} \equiv (\hat{q}_1, \ldots, \hat{q}_f)$ and $\hat{p} \equiv (\hat{p}_1, \ldots, \hat{p}_f)$ are position and momentum operators for an environmental system with $f$ degrees of freedom ($[\hat{q}_j, \hat{p}_j] = i\hbar, j = 1, \ldots, f$). The coefficients $c_q \equiv (c_q^{(1)}, \ldots, c_q^{(f)})$ and $c_p \equiv (c_p^{(1)}, \ldots, c_p^{(f)})$ characterise the strength of the coupling. The reduced density operator describing the state of the system $S$ after its coupling with the environment during a time interval $\delta t$ is given by

$$\hat{\rho}_S = |\alpha|^2 |+\rangle \langle +| + |\beta|^2 |−\rangle \langle −| + (\alpha \beta^* \langle \psi_- | \psi_+ \rangle |−\rangle \langle +| + H.c.) \, ,$$

where $|\psi_\pm\rangle \equiv \hat{D}(\mp c_p \delta t, \mp c_q \delta t) |\psi\rangle$, $\hat{D}(\delta q, \delta p) \equiv \exp\{i(\hat{p} \cdot \delta q + \hat{q} \cdot \delta p)/\hbar\}$, $\delta q$ and $\delta p$ are displacement vectors in $f$-dimensional spaces, and H.c. denotes the Hermitian conjugate of the preceding term in the equation. The states of the environmental and two-level systems
immediately prior to the interaction are \(|\psi\rangle\) and \(|\chi\rangle \equiv \alpha |+\rangle + \beta |-\rangle\) respectively. We have assumed that the coupling strength is large enough so that the evolution induced by each system Hamiltonian (\(\hat{H}_S\) and \(\hat{H}_E\)) can be neglected during the interaction time \(\delta t\). Despite the simplicity of the model considered, it contains the basic elements relevant to our discussion.

Eq. (1) relates the value of the non-diagonal term of the reduced density matrix of \(S\) in the preferred basis \(|{+}\rangle, |{-}\rangle\) to the mean value of a displacement operator over the state \(|\psi\rangle\) of system \(E\) since
\[
\langle \psi | \hat{D}(-2c_p \delta t, -2c_q \delta t) |\psi\rangle = \langle \psi | \hat{D}(-2c_p \delta t, -2c_q \delta t) |\psi\rangle.
\]
Therefore the capability of \(E\) to induce decoherence in \(S\) through the coupling term \(\hat{V}_{SE}\) is characterised by
\[
C_\psi(\delta q, \delta p) \equiv \langle \psi | \hat{D}(\delta q, \delta p) |\psi\rangle = e^{i\delta q \cdot \delta p / 2\hbar} \int d^f q e^{i q \cdot \delta p / \hbar} \psi^*(q) \psi(q + \delta q),
\]
where \(\psi(q) \equiv \langle q |\psi\rangle\), \(\delta q = -2c_p \delta t\), \(\delta p = -2c_q \delta t\), and \(d^f q\) (\(d^f p\)) is the \(f\)-dimensional differential element of volume in positions (momenta). All integrals in this paper run over the entire available volume. Complete decoherence is reached whenever the two states \(|\psi_+\rangle\) and \(|\psi_-\rangle\) are orthogonal to each other; in other words, when \(C_\psi = 0\). At this point, it is important to characterise the scale for which displacements \((\delta q, \delta p)\) in phase space will produce a significant decay of this expectation value of \(\hat{D}\). The main subject of our interest is to find an action scale associated to the effectiveness of system \(E\) to induce decoherence in system \(S\), and to describe its dependence with the particular environmental state.

This question has been previously studied by Zurek by means of the Wigner phase space distribution associated to the state \(|\psi\rangle\),
\[
W_\psi(q, p) = \frac{1}{(2\pi\hbar)^f} \int d^f q' e^{i q' \cdot p / \hbar} \psi(q - q' / 2) \psi^*(q + q' / 2).
\]
In particular, Moyal’s formula was used to analyse the behaviour of the overlap \(|C_\psi|^2\) with \(\delta q\) and \(\delta p\). The choice of the Wigner phase space distribution was motivated by this simple expression for the scalar product between \(|\psi_+\rangle\) and \(|\psi_-\rangle\). In Ref. Zurek showed that for a given time-dependent quantum chaotic system in one dimension \((f = 1)\) confined to a phase space volume characterised by the classical action \(A\), the Wigner distribution associated to the state develops in time a spotty random structure on the scale \(\hbar^2 / A\). Using Eq. (4) he argued that \(|C_\psi|^2 \approx 0\)
for phase space displacements on the scale of the smallest structure of the Wigner distribution $W_\psi(q, p)$. The basis for this result are: (a) Displacements characterised by $\delta q \delta p \approx \hbar^2/A$ produce a significant decrease on the value of the integral in Eq. (1) due to the destructive interference between $W_\psi(q, p)$ and $W_\psi(q + \delta q, p + \delta p)$, and (b) the random distribution of the patches in the structure appearing in the Wigner function associated to such system states prevents the presence of recurrences in the value of the overlap.

Jordan and Srednicki [8] extended the analysis in Ref. [2] to systems with an arbitrary number of degrees of freedom by using

$$C_\psi(\delta q, \delta p) = \int d^f q d^f p e^{i(p \cdot \delta q + q \cdot \delta p)/\hbar} W_\psi(q, p).$$

This equation establishes a relation between the small-scale (large-scale) structure of $W_\psi$ in the variables $(q, p)$ and the large-scale (small-scale) structure of $C_\psi$ in the variables $(\delta p, \delta q)$. Analysing a two-dimensional billiard and a gas of $N$ hard spheres in a three-dimensional box (assuming the Berry-Voros conjecture [9] in both cases) they concluded that for systems with a small number of degrees of freedom, displacements $\delta q_i \approx L_i$ and $\delta p_i \approx P_i$ are needed to avoid oscillations in the overlap, where $L_i$ and $P_i$ are typical classical values of the position $q_i$ and momentum $p_i$ respectively ($i = 1, \ldots, f$). This means that displacements of the order of the size of the state support are needed to guarantee orthogonality in the general case. However, for systems with a large number of degrees of freedom they found that the conclusions in Ref. [2] remain valid, supporting the idea that a larger number of degrees of freedom increases the effectiveness in causing decoherence. Some care must be taken when relating the results in Refs. [2] and [8] since in principle the Berry-Voros conjecture is not valid for the system analysed by Zurek in Ref. [2] and the dependence of the overlap with the displacement could have qualitatively different features.

In this work we characterise the behaviour of $C_\psi$ using a quantity $\Delta S$, with units of action, associated to the displacement $(\delta q, \delta p)$. A formal series expansion of $\hat{D}$ will allow us to identify the scale in the action $\Delta S(\delta q, \delta p)$ for which the overlap decreases significantly for any quantum system, irrespective of the number of degrees of freedom. This scale is manifested in the size of the structures present in the distribution associated to the state in some phase space representations, but they do not necessarily coincide.

The paper is organised as follows. In Sec. II we define the characteristic action $\Delta S$ and determine the scale relevant for the decay of the overlap. In Sec. III we establish the relation
between $C_\psi$ and the structure of the distribution associated to the state in an arbitrary phase space representation. The next two sections are devoted to studying in detail the dependence of $C_\psi$ on $\Delta S$ for states of particular quantum systems. Sec. IV considers a system with a time-dependent Hamiltonian whose classical counterpart exhibits chaos. In Sec. V we analyse the case of non-linear systems with a confining potential and discrete spectrum. In this case the main features of $C_\psi$ can be obtained from time average properties of the state evolution. We will focus on quantum systems with time-independent Hamiltonian for which analytical models are worked out by using the Berry-Voros conjecture [9]. Finally in Sec. VI the main results of this work are discussed.

II. CHARACTERISTIC ACTION SCALES FOR THE DECAY OF $C_\psi(\delta q, \delta p)$

A displacement operator $\hat{D}(\delta q, \delta p)$ acting on the state of an $f$-dimensional quantum system $E$, that describes an environment or a ‘monitoring’ apparatus, can be written as

$$\hat{D}(\delta q, \delta p) = e^{i(\hat{p} \cdot \delta q + \hat{q} \cdot \delta p)/\hbar} \equiv e^{i\hat{S}(\delta q, \delta p)/\hbar}.$$ \hspace{1cm} (6)

The main features of $|C_\psi|^2$ are therefore related to the fluctuation properties of the operator $\hat{S}(\delta q, \delta p)$, since the expectation value of $\hat{D}$ equals the characteristic function of $\hat{S}$ (see Eq. (2)).

A formal expansion of $\hat{D}$ in terms of $\hat{S}$ gives

$$C_\psi(\delta q, \delta p) = 1 + \frac{i}{\hbar} \langle \hat{S} \rangle_\psi - \frac{1}{2\hbar^2} \langle \hat{S}^2 \rangle_\psi + O \left( \frac{s^3 \delta^3}{\hbar^3} \right),$$ \hspace{1cm} (7)

and for the overlap,

$$|C_\psi(\delta q, \delta p)|^2 = 1 - \frac{1}{\hbar^2} \left( \langle \hat{S}^2 \rangle_\psi - \langle \hat{S} \rangle_\psi^2 \right) + O \left( \frac{s^4 \delta^4}{\hbar^4} \right).$$ \hspace{1cm} (8)

We denote by $\delta^n$ general products of $n$ components of the vectors $\delta q$ and $\delta p$, and by $s^n$ terms of the form $\prod_{k=1}^m \langle \hat{O}_k \rangle_\psi$, where $\hat{O}_k$ is the product of $g_k$ operators $\hat{q}$ and $\hat{p}$, with the condition $\sum_{k=1}^m g_k = n$. The characteristic action

$$\Delta S(\delta q, \delta p) \equiv \sqrt{\langle \hat{S}^2 \rangle_\psi - \langle \hat{S} \rangle_\psi^2},$$ \hspace{1cm} (9)

controls the decay of the overlap for sufficiently small values of $\delta q$ and $\delta p$. Eq. (8) suggests that displacements $(\delta q, \delta p)$ for which $\Delta S$ is small compared to $\hbar$ do not lead generally to
an important decay of $|C_\psi|^2$. In other words, displacements leading to $\Delta S$ of the order or larger than $\hbar$ are needed for the states $|\psi_+\rangle$ and $|\psi_-\rangle$ to be orthogonal. Therefore $\Delta S \approx \hbar$ establishes the scale for the action involved in displacements of the environmental state that could induce significant decoherence in system $S$. For the case of Gaussian fluctuations of the operator $\hat{S}$, the only relevant fluctuation is $\Delta S$. In a more general situation higher order fluctuations may play a role in the particular features of the decay of $|C_\psi|^2$, nonetheless the $\Delta S$-action scale is generally expected to be a good measure for the decoherence process. The rest of the paper will provide additional arguments for this interpretation of the scale associated to the quantity $\Delta S$.

To be more specific, let us write $(\Delta S)^2$ in terms of $(\delta q, \delta p)$,

$$
(\Delta S)^2 = \sum_{i=1}^{f} \sum_{j=1}^{f} \left((\langle \hat{q}_i \hat{q}_j \rangle - \langle \hat{q}_i \rangle \langle \hat{q}_j \rangle) \delta p_i \delta p_j + (\langle \hat{p}_i \hat{p}_j \rangle - \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle) \delta q_i \delta q_j
\right.
+ \left( (\langle \hat{q}_i \hat{p}_j \rangle - \langle \hat{q}_i \rangle \langle \hat{p}_j \rangle) \delta p_i \delta q_j + (\langle \hat{p}_i \hat{q}_j \rangle - \langle \hat{p}_i \rangle \langle \hat{q}_j \rangle) \delta q_i \delta p_j \right),
$$

(10)

or

$$
(\Delta S)^2 = \delta p^T \gamma_{qp} \delta p + \delta q^T \gamma_{pq} \delta q + \delta p^T \gamma_{pp} \delta p + \delta q^T \gamma_{qq} \delta q,
$$

(11)

where we have introduced the matrices $\gamma_{ij}^{AB} \equiv \langle \hat{A}_i \hat{B}_j \rangle - \langle \hat{A}_i \rangle \langle \hat{B}_j \rangle$, and $a^T$ denotes the transposed of the vector $a$. To gain some insight into the meaning of this quantity, we will consider $(\Delta S)^2$ for the one-dimensional case,

$$
(\Delta S)^2 = (\sigma_q \delta p)^2 + (\sigma_p \delta q)^2 + (\langle \hat{q} \hat{\hat{p}} + \hat{\hat{p}} \hat{q} \rangle - 2 \langle \hat{q} \rangle \langle \hat{p} \rangle) \delta q \delta p,
$$

(12)

where $\sigma_q$ and $\sigma_p$ are the root-mean-square deviations of $\hat{q}$ and $\hat{p}$ respectively. To continue with our discussion, a rotation in phase space is made, so that the term $(\langle \hat{q} \hat{\hat{p}} + \hat{\hat{p}} \hat{q} \rangle - 2 \langle \hat{q} \rangle \langle \hat{p} \rangle)$ in the previous equation is zero, and $\Delta S$ is given, in terms of the new phase space variables, by

$$
(\Delta S)^2 = (\sigma_q \delta \hat{p})^2 + (\sigma_p \delta \hat{q})^2,
$$

(13)

where $\sigma_{\hat{q}} (\sigma_{\hat{p}})$ gives the support of the state in the variable $\hat{q}$ ($\hat{p}$). A classical action $A \equiv \sigma_{\hat{q}} \sigma_{\hat{p}}$ can be associated to the state of the system. Eq. (13) implies that displacements such that $\sigma_{\hat{q}} \delta \hat{p} \approx \hbar$ or $\sigma_{\hat{p}} \delta \hat{q} \approx \hbar$ give $\Delta S \gtrsim \hbar$, and the main point of our analysis is that they also lead in general to a significant variation of $|C_\psi|^2$, irrespective of the value of action $A$. In this sense values of the order or larger than $\hbar$ of the $\Delta S$-action scale are always needed for this
environmental system to induce decoherence. It is possible to define other relevant quantities with units of action. For instance, values of $\Delta Z \equiv \delta \tilde{q} \delta \tilde{p}$ leading to a significant decrease of the overlap are related to the size of the structure of the distribution associated to the state in some particular phase space representations \[2\]. For the displacements discussed above $\Delta Z \approx \hbar^2/A$, and if $A >> \hbar$ the result that sub-Planck displacements on the $\Delta Z$-action scale are relevant for the decoherence process induced by $\mathcal{E}$ comes naturally. Coming back to the multi-dimensional case, when the dimension of the problem increases more terms will contribute to $(\Delta S)^2$ in Eq. (10), and smaller displacements in each variable are needed to reach the threshold $\Delta S \approx \hbar$, leading to the result that a larger number of degrees of freedom will favour the decoherence process \[8\].

To illustrate the difference between $\Delta Z$- and $\Delta S$-action scales we consider a general Gaussian state in one dimension

$$\psi(q) = \left( \frac{2z_R}{\pi |z|^2} \right)^{1/4} e^{ip_0 q/\hbar} e^{-(q-q_0)^2/\bar{z}},$$

(14)

with $z \equiv z_R + iz_I$, $z_R = (h/\sigma_p)^2$, and $z_I = z_R \sqrt{4\sigma_q^2 \sigma_p^2 - \hbar^2/\hbar}$. Straightforward calculations lead to the exact expression

$$|C_{\psi}(\delta q, \delta p)|^2 = \exp \left[ -(\Delta S)^2/\hbar^2 \right],$$

(15)

where

$$\left( \Delta S \right)^2 = (\sigma_p \delta q)^2 + (\sigma_q \delta p)^2 + \hbar \sqrt{\left( \frac{2\delta p \delta q \sigma_p \sigma_q}{\hbar} \right)^2 - (\delta p \delta q)^2},$$

(16)

in terms of the first two moments of $\hat{S}$, as expected for a Gaussian wavefunction. Eq. (15) shows that values of $\Delta S \gg \hbar$ are needed to obtain a significant decrease of the overlap $|C_{\psi}|^2$. If we now choose, for instance, particular values of the widths $\sigma_q$ and $\sigma_p$ so that the Gaussian state is much narrower in coordinate than in momentum space, say $\sigma_q \approx \sqrt{\hbar}/10$ and $\sigma_p \approx 10\sqrt{\hbar}$ (in arbitrary units), it is clear that a displacement $(\delta q, \delta p) = (\sqrt{\hbar}/2, \sqrt{\hbar}/2)$ will take the shifted Gaussian completely away from the initial one. The different actions associated to that same displacement are $\Delta S \approx 5\hbar$ and $\Delta Z = \hbar/4$, corresponding to over-Planck and sub-Planck values respectively.
III. SUB-PLANCK STRUCTURES IN PHASE SPACE DISTRIBUTIONS.

The behaviour of the overlap $|C_\psi|^2$ with $(\delta q, \delta p)$ can be alternatively studied through the distribution associated to the state in different phase space representations. In this section we will derive the relation between the overlap and the action $\Delta S$ using a wide class of quantum quasi-probability distributions $F(q, p; \chi)$ \cite{10}, the Wigner \cite{6} and Husimi \cite{11} functions being nothing but particular cases. The choice among the $F$ functions associated to the same quantum state of a system, or, equivalently, the selection of a particular representation (given by function $\chi$), is similar to the choice of a convenient set of coordinates \cite{12, 13, 14}. Within this framework, the expectation value of any operator $\hat{G}(\hat{q}, \hat{p})$ is written as the phase space integral

$$\langle \hat{G}(\hat{q}, \hat{p}) \rangle_\psi = \int d\theta d\tau d\mu \, \chi(\theta, \tau) \langle \mu + \frac{i\hbar}{2} \bigg| \psi \bigg| \mu - \frac{i\hbar}{2} \rangle e^{-i[\theta(q-\mu) + \tau(p-\mu)]} , \quad (17)$$

where $F_\psi(q, p; \chi)$ is obtained from the quantum state $|\psi\rangle$ as

$$F_\psi(q, p; \chi) = \frac{1}{(2\pi)^2} \int d\theta d\tau d\mu \, \chi(\theta, \tau) \langle \mu + \frac{i\hbar}{2} \bigg| \psi \bigg| \mu - \frac{i\hbar}{2} \rangle e^{-i[\theta(q-\mu) + \tau(p-\mu)]} . \quad (18)$$

The Wigner and Husimi functions, for instance, are obtained by replacing $\chi(\theta, \tau) = 1$ and $\chi(\theta, \tau) = \exp\{-\frac{\hbar}{4}[(\tau^2 + (\theta^2)]\}$ respectively. The function $g(q, p; \chi)$ is the image of the operator $\hat{G}$ in phase space according to the kernel function $\chi$ \cite{14},

$$g(q, p; \chi) = \left( \frac{\hbar}{2\pi} \right)^f \int d\theta d\tau d\mu \, \frac{1}{\chi(\theta, \tau)} \langle \mu + \frac{i\hbar}{2} \bigg| \hat{G} \bigg| \mu - \frac{i\hbar}{2} \rangle e^{i[\theta(q-\mu) + \tau(p-\mu)]} , \quad (19)$$

and it is not necessarily equal to the classical magnitude. In particular, the expectation value of the displacement operator $\hat{D}$ can be written as the phase space average

$$C_\psi(\delta q, \delta p) = \int d\theta d\tau d\mu \, \chi(\theta, \tau) \langle \mu + \frac{i\hbar}{2} \bigg| \hat{D} \bigg| \mu - \frac{i\hbar}{2} \rangle e^{i[\theta(q-\mu) + \tau(p-\mu)]} \right) , \quad (20)$$

where the function $d(q, p; \chi)$ was obtained by integrating the r.h.s. of Eq. (19) with $\hat{G}$ replaced by $\hat{D}$. Notice that Eq. (13) is a particular case of Eq. (20) for which the Wigner function has been chosen as the distribution associated to the state, $C_\psi(\delta q, \delta p)$ being equal to the Fourier transform of $W_\psi$. Eq. (20) can be used to understand the relation between $C_\psi$ and $\Delta S$ from the point of view of the phase space distribution $F_\psi$. On one hand, if the exponential factor does not vary significantly over the support of $F_\psi(q, p; \chi)$, which
occurs for small enough values of $\delta q$ and $\delta p$, then the overlap will only differ slightly from the normalisation integral of the original distribution, $\int dq dp F_\psi(q,p;\chi) = 1$, leading in general to a small decrease of the function $|C_\psi|^2$. (Notice that $\chi(0,0) = 1$ is needed to guarantee that $F_\psi(q,p;\chi)$ is normalised to one [13].) The condition for these variations not to be significant is equivalent to the condition that the root-mean-square deviation of $\hat{S}$, $\Delta S$, is smaller than $\hbar$. On the other hand, to obtain significant decay of the overlap, rapid oscillations, with $\Delta S$ at least of the order of $\hbar$, are needed. Due to the properties of the Fourier transform, and since the value of $\chi(\delta p/\hbar, \delta q/\hbar)$ is close to one for small enough $\delta q$ and $\delta p$, the initial decay of $|C_\psi|^2$ with the displacement is related to the large scale structure of the distribution $F_\psi$, that depends mainly on the size of the state support in phase space. However, the detailed behaviour of $|C_\psi|^2$ with arbitrary displacements, and, in particular, qualitative features like oscillations, will depend on the state under study.

A different question is how sub-Plank structures emerge in some phase space distributions associated to the state and how they are related to the main features of $C_\psi$. For kernel functions such that $|\chi(\theta,\tau)| = 1$, the corresponding distributions verify [15]

$$|C_\psi(\delta q, \delta p)|^2 = (2\pi\hbar)^f \int dq dp F_\psi(q,p) F_\psi(q+\delta q,p+\delta p).$$  \hspace{1cm} (21)

(Notice that Eq. (4) is a particular case of Eq. (21).) For these representations, the fact that a given displacement leads to $|C_\psi|^2 \approx 0$ is manifested in a complex structure of the distribution $F_\psi$ on the scale of the $\Delta Z$-action for that displacement, as pointed out in Ref. [2] for the Wigner distribution. When displacements with $\Delta Z << \hbar$ lead to small values of $|C_\psi|^2$, the distribution $F_\psi$ will show a complex structure at sub-Planck scales. This result is a consequence of the particular choice of the phase space distribution. For the same state, the Husimi distribution (obtained by smoothing the Wigner function, so eliminating the sub-Planck scale structure) will lead to the same overlap $|C_\psi|^2$. This is not surprising since the overlap depends on the state, and that dependence is manifested in different ways for different phase space representations.

IV. A ONE-DIMENSIONAL TIME-DEPENDENT ENVIRONMENTAL SYSTEM

In this section we will analyse the dependence of the overlap $|C_\psi|^2$ on the action $\Delta S(\delta q, \delta p)$ in the context of a particular one-dimensional model for the environmental system $\mathcal{E}$, de-
scribed by Hamiltonian

\[ \hat{H}_E = \frac{\hat{p}^2}{2m} - \kappa \cos (\hat{q} - l \sin t) + \frac{1}{2} a \hat{q}^2. \]  

(22)

This quantum model system has been previously used in the context of decoherence [2, 16], and describes a particle of mass \( m = 1 \) (arbitrary units are used throughout) confined by a harmonic potential that is perturbed by a spatially and temporally periodic term. For the parameter values used in this work, \( \kappa = 0.36, a = 0.01, l = 3.8, \) and \( \hbar = 0.16, \) the motion in the classical counterpart of this system exhibits a chaotic character [17].

To prepare the state of the environmental system prior to the interaction, we let a given initial state evolve until preparation time \( T \), when it is coupled to the pointer system \( S \). The coupling strength is assumed to be high enough so that the two-systems evolution can be followed as described in the introduction, i.e., by neglecting any contribution coming from the dynamics induced by Hamiltonian (22) during the interaction time. This approach allows us to discuss the values of actions involved in the decoherence process induced on system \( S \) in terms of general displacements in phase space, irrespective of the detailed values of the coupling constants and interaction times [18].

As the initial state \( (T = t = 0) \) for the preparation process we have chosen a coherent state \( |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \) of the harmonic oscillator \( \hat{H}_{OA} = \hat{p}^2/(2m) + a \hat{q}^2/2 \), where \( |n\rangle \) is the eigenstate of \( \hat{H}_{OA} \) with energy \( (n + 1/2)\hbar \sqrt{a/m} \). (We have checked other possible initial states obtaining qualitatively similar results.) Its time propagation under Hamiltonian (22) has been obtained by means of the split-operator method [19].

As time \( T \) increases, the state spreads in coordinate as well as in momentum space through the available phase space as shown in the insets of Fig. 1. To characterise this dynamics the quantity

\[ a \equiv \frac{\hbar^2}{\sigma_q \sigma_p}, \]  

(23)

is used (see Fig. 1). It shows a rapid initial decay (until time \( T \approx 20 \)), followed by a much slower decrease for longer times. The behaviour of \( a \) for small preparation time \( T \) is related to the fast initial increase of the widths \( \sigma_q \) and \( \sigma_p \). The variation of \( a \) for longer times is mainly due to the time-dependent term in the Hamiltonian. Should not be for the presence of this time-dependent term \( a \) would not decrease beyond a certain minimum value related to the maximum position and momentum widths compatible with a fixed mean system energy.
FIG. 1: Action $a \equiv \hbar^2/\left(\sigma_q\sigma_p\right)$ as a function of the preparation time $T$ (solid line) for the initial coherent state with $\alpha = 5 i$. The action $\Delta Z_0$ needed for a displacement in the direction $\delta q \simeq 6.8\delta p$ to reduce the value of $|C_\psi|^2$ to 0.5 is also shown (dashed line). The inset shows the dependence of the widths $\sigma_q$ (left) and $\sigma_p$ (right) with the preparation time. (Notice the different scales in the vertical axis for each case.) Arbitrary units are used.

Fig. 2 shows $|C_\psi|^2$ versus $\Delta S$ for different preparation times and for a given direction in phase space. (The results for any other direction show the same qualitative features.) The different curves, corresponding to different preparation times, decay in the same $\Delta S$-scale. To emphasise this result we represent in the inset the value of $\Delta S$ needed to obtain $|C_\psi|^2 = 0.5$ versus $T$. The $\Delta S$-action values for any preparation time are of the order of $\hbar$, supporting $\Delta S \approx \hbar$ as a relevant scale for the studied decoherence process. (Notice that the apparent convergence of $\Delta S_0$ to a value close to $\hbar$ is only a consequence of the chosen value for $|C_\psi|^2$.)

The values of $\Delta Z$ for which $|C_\psi|^2 = 0.5$ are also shown in Fig. 1 and Fig. 2. After some time $T \approx 5$, the action $a$ sets the scale of the random structure developed in the distribution associated to the states of this system in some phase space representations, for example in the Wigner function [2]. Taking into account the discussion below Eq. (21), the action $\Delta Z$ for displacements producing a significant decrease of $|C_\psi|^2$ will be of the order of the action $a$ after $T \approx 5$, as shown in Fig. 1.
FIG. 2: $|C_\psi|^2$ as a function of the action $\Delta S(\delta q, \delta p)$ in the direction $\delta q \simeq 6.8 \delta p$ in phase space and for different preparation times: $T = 0$ (solid line), $T = 10$ (dashed line), $T = 20$ (dotted line), and $T = 500$ (circles). (Same initial state as in Fig. I.) The inset shows the actions $\Delta S_0$ (upper curve) and $\Delta Z_0$ (lower curve) needed for the displacement to reduce $|C_\psi|^2$ to the value 0.5 versus the preparation time $T$. The straight line shows the action $\hbar$ for reference. Arbitrary units are used.

V. NON-LINEAR CONFINED ENVIRONMENTAL SYSTEMS

For now on, a different model for the environmental system $E$ will be considered, that of a time-independent Hamiltonian, $\hat{H}_{NL}$, with a non-linear confining potential and a discrete energy spectrum. Under certain assumptions, this model will allow us to obtain analytical expressions for the overlap between the states $|\psi_+\rangle$ and $|\psi_-\rangle$. Instead of considering a particular environmental state $|\psi\rangle$, obtained after some fixed preparation time $T$, we will study the dependence of the overlap averaged over the preparation time on an averaged $\Delta S$-action. For non-linear confined systems, the main features of this stationary description can be associated to all the states prepared from a given initial one, provided that their preparation time is long enough. In the first part of this section we will determine the stationary properties of $C_\psi$ relevant to our discussion. We will assume that the states are prepared from a given $|\psi(T = 0)\rangle$, and make use of the Wigner distribution in phase space.
associated to them. Although the procedure and the results are independent of the choice of a particular phase space representation, the use of the Wigner distribution will allows us to extend our analysis afterwards for systems for which the Berry-Voros conjecture is valid.

A. Stationary properties of the overlap

In the basis of eigenstates of the Hamiltonian $\hat{H}_{NL}$, which will be assumed to have, for simplicity, a non-degenerate spectrum, the wave function at preparation time $T$ is given by

$$\psi(q, T) = \sum_n c_n e^{-iE_n T/\hbar} \varphi_n(q),$$

where $\hat{H}\varphi_n(q) = E_n \varphi_n(q)$ and $c_n = \int dq \varphi_n^*(q) \psi(q, 0)$. The Wigner distribution is obtained introducing expansion (24) into Eq. (3). Splitting the result into time-independent and time-dependent terms,

$$W_{\psi}(q, p, T) = \sum_n |c_n|^2 W_{\varphi_n}(q, p)$$

$$+ \sum_{n \neq m} c_n c_m^* e^{-i(E_n - E_m) T/\hbar} \int \frac{d^2 q'}{(2\pi\hbar)^2} e^{i\mathbf{q}' \cdot \mathbf{p}/\hbar} \varphi_n(q - q'/2) \varphi_m^* (q + q'/2),$$

where $W_{\varphi_n}(q, p)$ is the Wigner distribution associated to the energy eigenstate $\varphi_n(q)$. For non-linear systems, it turns out that the Wigner distribution spreads from its initial ($T = 0$) support in phase space until it occupies most of the available phase space volume at some preparation time $T_c$. From time $T_c$ on, the small details of the Wigner distribution will change with time, but in general its long scale structure will remain as a stationary property. To extract that characteristic long scale structure we employ the time-averaged Wigner distribution

$$\overline{W}_{\psi}(q, p) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dT W_{\psi}(q, p, T) = \sum_n |c_n|^2 W_{\varphi_n}(q, p),$$

where we have taken

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dT \sum_{n \neq m} c_n c_m^* e^{-i(E_n - E_m) T/\hbar} \int \frac{d^2 q'}{(2\pi\hbar)^2} e^{i\mathbf{q}' \cdot \mathbf{p}/\hbar} \varphi_n(q - q'/2) \varphi_m^* (q + q'/2) = 0.$$  

Introducing $\overline{W}_{\psi}$ into Eq. (5), we obtain the time-averaged quantity

$$\overline{C}_{\psi}(\delta q, \delta p) = \int d^2 q d^2 p e^{i(p \cdot \delta q + q \cdot \delta p)/\hbar} \overline{W}_{\psi}(q, p),$$
that describes the stationary properties of the overlap between $|\psi_+\rangle$ and $|\psi_-\rangle$. According to Eq. (28), $C_\psi$ can be identified as the generating function of all moments of $S \equiv \mathbf{p} \cdot \delta \mathbf{q} + \mathbf{q} \cdot \delta \mathbf{p}$ with respect to the distribution $\overline{W_\psi}$. Therefore a set of equations similar to Eq. (7) and (8) can be obtained. These equations imply that the initial decay of $C_\psi$ is ruled by the fluctuation properties of $S$ at stationary conditions. The action scale involved in the decay of $C_\psi$ for small displacements $(\delta \mathbf{q}, \delta \mathbf{p})$ can be in general associated to any state with preparation time longer than $T_c$. This result follows from $\overline{W_\psi}$ describing properly the long scale structure for $T > T_c$ and the discussion in Sec. III. In the rest of the section we consider a family of quantum systems for which $C_\psi$ can be obtained analytically.

B. Systems described by the Berry-Voros conjecture

We shall now pay special attention to (1) quantum systems with time-independent Hamiltonians and classical chaotic counterpart and (2) regular quantum systems with particular random components in their potentials [20, 21], for which the relevant quantities are obtained after averaging over the noise. There are both, experimental and numerical evidences, that for these systems the so called Berry-Voros conjecture is valid, namely, that one can approximate the Wigner density associated to an energy eigenstate by a microcanonical density [22, 23, 24],

$$W_{\varphi_n}(\mathbf{q}, \mathbf{p}) \rightarrow \frac{1}{(2\pi \hbar)^d} \frac{\delta(E_n - H(\mathbf{q}, \mathbf{p}))}{\rho(E_n)}, \quad (29)$$

where $\rho(E_n) = \int \frac{d \mathbf{q} d \mathbf{p}}{(2\pi \hbar)^d} \delta(E_n - H(\mathbf{q}, \mathbf{p}))$ is the local average density of states at energy $E_n$, and $H(\mathbf{q}, \mathbf{p})$ is the classical Hamiltonian associated to the quantum one [9, 20, 25, 26]. The Wigner distribution in the semiclassical limit fills the available phase space that corresponds to an energy shell of thickness of the order $\hbar$ and its amplitude fluctuates around the microcanonical density. Furthermore the density function in Eq. (29) is just the leading approximation of a semiclassical expression for $W_{\varphi_n}(\mathbf{q}, \mathbf{p})$. The next to the leading terms depend on the periodic orbits of the classical system and take into account the possible scars [27, 28, 29].

Replacing $W_{\varphi_n}(\mathbf{q}, \mathbf{p})$, implicit in Eq. (28), by the expression in Eq. (29), it follows

$$\overline{C_\psi(\delta \mathbf{q}, \delta \mathbf{p})}^{BV} = \sum_n |c_n|^2 \rho^{-1}(E_n) \left\langle e^{i(\mathbf{p} \cdot \delta \mathbf{q} + \mathbf{q} \cdot \delta \mathbf{p})/\hbar} \right\rangle_{\varphi_n}^{BV}, \quad (30)$$
where
\[ \left\langle e^{i(p\cdot \delta q + q\cdot \delta p)/\hbar} \right\rangle_{\varphi_n}^{BV} \equiv \int \frac{d^dq d^dp}{(2\pi \hbar)^f} e^{i(p\cdot \delta q + q\cdot \delta p)/\hbar} \delta(E_n - H(q, p)) \] (31)

is the microcanonical average of \( e^{iS/\hbar} \). For a Hamiltonian of the form \( H(q, p) = p^2/2M + V(q) \), and after integrating over the momentum variables, one obtains
\[ \left\langle e^{i(p\cdot \delta q + q\cdot \delta p)/\hbar} \right\rangle_{\varphi_n}^{BV} = (2\pi)^{f/2} M \int \frac{dq}{(2\pi \hbar)^f} e^{i(p\cdot \delta q + q\cdot \delta p)/\hbar} \left( \frac{\hbar}{|\delta q|} \sqrt{2M(E_n - V(q))} \right)^{f/2 - 1} \times J_{f/2 - 1} \left( \frac{|\delta q|}{\hbar} \sqrt{2M(E_n - V(q))} \right), \] (32)

where \( J_{f/2 - 1}(z) \) is the Bessel function of order \( f/2 - 1 \). Eqs. (30) and (32) lead to a formal expression of the time-averaged two-point correlation function \( C_{\psi}(\delta q, \delta p)^{BV} \) in terms of the potential \( V(q) \). These equations constitute the main result of this section and are the starting point for the analysis of particular examples. In the following we shall particularise Eq. (30) for systems with a random component in the potential such that the average over the noise is the \( f \)-dimensional harmonic potential.

1. The \( f \)-dimensional harmonic oscillator

The classical Hamiltonian for a generic \( f \)-dimensional harmonic oscillator,
\[ H(\tilde{q}, \tilde{p}) = \sum_{i=1}^{f} \frac{\tilde{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \tilde{q}_i^2, \] (33)

can be rewritten, in terms of the rescaled coordinates and momenta
\[ p_i \equiv \sqrt{\frac{M}{m_i}} \tilde{p}_i, \]
\[ q_i \equiv \sqrt{\frac{m_i \omega_i^2}{M\omega^2}} \tilde{q}_i, \] (34)
as the spherical harmonic oscillator
\[ H(q, p) = \frac{1}{2M}(p_1^2 + p_2^2 + \cdots + p_f^2) + \frac{1}{2} M\omega^2(q_1^2 + q_2^2 + \cdots + q_f^2). \] (35)

The integral in Eq. (31) reads
\[ \left( \prod_{i=1}^{f} \frac{\omega_i}{\omega} \right) \int \frac{d^dq d^dp}{(2\pi \hbar)^f} e^{i(p\cdot \delta q + q\cdot \delta p)/\hbar} \delta(E_n - H(q, p)), \] (36)
\[ \delta q \equiv \left( \sqrt{\frac{m_1}{M}} \delta q_1, \ldots, \sqrt{\frac{m_f}{M}} \delta q_f \right) \]
\[ \delta p \equiv \left( \sqrt{\frac{M \omega^2}{m_1 \omega_f^2}} \delta p_1, \ldots, \sqrt{\frac{M \omega^2}{m_f \omega_f^2}} \delta p_f \right). \] (37)

After some manipulations, it follows that
\[ \left\langle e^{i(p \cdot \delta q + q \cdot \delta p) / \hbar} \right\rangle_{\varphi_n}^{BV} = 2^{f-1} \left( \frac{\omega_f}{\prod_{i=1}^{f} \omega_i} \right) \frac{E_n^{-1} \left( \frac{\hbar}{(\omega_f) f} \sqrt{\frac{1}{2ME_n}} \right) \frac{\hbar}{\sqrt{2E_n}} \frac{\hbar}{\sqrt{2E_n}} \frac{1}{\hbar} \right) \]
\[ \times \int_0^1 \! d \xi \xi^{\frac{f}{2}} \left( \sqrt{1 - \xi^2} \right)^{\frac{f}{2} - 1} J_{\frac{f}{2} - 1} \left( \frac{\left| \delta q \right| \sqrt{2ME_n} \sqrt{1 - \xi^2}}{\hbar} \right) J_{\frac{f}{2} - 1} \left( \frac{\left| \delta p \right| \sqrt{2E_n}}{\hbar} \right), \] (38)
and integrating over variable \( \xi \),
\[ \left\langle e^{i(p \cdot \delta q + q \cdot \delta p) / \hbar} \right\rangle_{\varphi_n}^{BV} = 2^{f-1} \left( \frac{\omega_f}{\prod_{i=1}^{f} \omega_i} \right) \frac{E_n^{-1} \left( \frac{\left| \delta p \right| \sqrt{2E_n}}{\hbar} \right)^2 + \left( \frac{\left| \delta q \right| \sqrt{2ME_n}}{\hbar} \right)^2}{\sqrt{\left( \frac{\left| \delta p \right| \sqrt{2E_n}}{\hbar} \right)^2 + \left( \frac{\left| \delta q \right| \sqrt{2ME_n}}{\hbar} \right)^2} \right)^{f-1}. \] (39)

For an eigenstate of energy \( E_n \), \( \rho(E_n) = E_n^{-1} / (\Gamma(f)(\hbar \omega)^f) \), where \( \Gamma(f) \) denotes the Gamma function of argument \( f \). Besides, \( \sigma_{p,n}^2 = \langle p^2 \rangle_{\varphi_n}^{BV} = ME_n \), \( \sigma_{q,n}^2 = \langle q^2 \rangle_{\varphi_n}^{BV} = E_n/M \omega^2 \) (for this case the mean values of position and of momentum vanish), giving
\[ \left\langle e^{i(p \cdot \delta q + q \cdot \delta p) / \hbar} \right\rangle_{\varphi_n}^{BV} = 2^{f-1} \left( \frac{\omega_f}{\prod_{i=1}^{f} \omega_i} \right) \frac{\Gamma(f)(\hbar E_n)^{\frac{1}{2}}} {\sqrt{\left( \frac{\left| \delta p \right| \sqrt{ME_n}}{\hbar} \right)^2 + \left( \frac{\left| \delta q \right| \sqrt{ME_n}}{\hbar} \right)^2} \right)^{f-1} \]
\[ = 2^{f-1} \left( \frac{\omega_f}{\prod_{i=1}^{f} \omega_i} \right) \frac{\Gamma(f)(\hbar E_n)^{\frac{1}{2}}} {\sqrt{\left( \frac{\left| \delta p \right| \sqrt{ME_n}}{\hbar} \right)^2 + \left( \frac{\left| \delta q \right| \sqrt{ME_n}}{\hbar} \right)^2} \right)^{f-1}, \] (40)

where the characteristic action
\[ \Delta S_n^{BV} = \sqrt{\left( \frac{\left| \delta p \right| \sigma_{p,n}}{\hbar} \right)^2 + \left( \frac{\left| \delta q \right| \sigma_{q,n}}{\hbar} \right)^2} \] (41)
has been introduced. \( \Delta S_n^{BV} \) is nothing but the action \( \Delta S \) introduced in Eq. (10) calculated for the \( n \)th eigenstate using the Berry-Voros conjecture. For the superposition state \[ [\omega] \],
one obtains
\[ C_\psi(\delta q, \delta p)_{\varphi_n}^{BV} = \sum_n 2^{(f-1)/2} |c_n|^2 \frac{\Gamma(f) J_{f-1}(\sqrt{2} \Delta S_n^{BV} / \hbar)} {\Delta S_n^{BV} / \hbar} \frac{1}{f-1}, \] (42)
so that the typical action that controls the decay of the overlap is the one related with the coefficients $c_n$ that contribute more to the initial state. The action $\Delta S_{BV}$, evaluated for the average distribution $\overline{W}_\psi$ under the Berry-Voros conjecture, is related with $(\Delta S_{nBV})$ by

$$\langle \Delta S_{BV} \rangle^2 = \sum_n |c_n|^2 (\Delta S_{nBV})^2.$$  

(43)

For any displacement $(\delta q, \delta p)$ the previous relation can be inverted and used to write $C_{\psi BV}$ in terms of $\Delta S_{BV}$.

To illustrate this result, in Fig. 3 we plot $C_{\psi BV}$ versus $\Delta S_{BV}$ for the one-dimensional case $f = 1$. The $\Delta S$-action scale for the decay of the overlap is dictated by the value $\hbar = 0.16$, as in the case described in the previous section. Note that this result is expected as a power expansion of the Bessel function $J_0$ in Eq. (42) to the second order in $\Delta S_{BV}$ consistently recovers the result in Eq. (8).
VI. DISCUSSION

The results of previous sections show that the relevant $\Delta S$-action scale to the decay of the overlap $|C_\psi|^2$ for small displacements $(\delta q, \delta p)$ is given by $\hbar$. The one-dimensional Gaussian state is a special example for which the dependence of $|C_\psi|^2$ on $\Delta S$ is given explicitly by Eq. (15), and its monotonic exponential decay is independent of particular details of the state, as for instance the widths in position and momentum. (On the contrary, the decay of the overlap with the displacement will depend on $\sigma_q$ and $\sigma_p$ through Eq. (16)). In Figures 2 and 3 the exponential dependence associated to an initial (Gaussian) coherent state is compared to the one corresponding to states at different preparation times. Although all the curves shows a similar initial decay, (dictated by $\Delta S \approx \hbar$), the ulterior behaviour can have qualitatively different features, the presence of oscillations in the overlap for intermediate values of $\Delta S$ being the most relevant one. It is worth noting that these oscillations can never be regarded as true revivals. $|C_\psi|^2$ can be interpreted as the overlap between the states $|\psi\rangle$ and $\hat{D}(\delta q, \delta p)|\psi\rangle$, the second one being obtained by a rigid displacement of $|\psi\rangle$. This implies that $|C_\psi|^2$ can not be equal to one for non-zero displacements since the support of the state in phase-space in finite. However, large amplitude oscillations are possible as shown in Fig. 2 for $T = 10$.

The pattern of oscillations will change in general with the preparation time. In the system described in Sec. IV no oscillations are present for the initial state. For small preparation times some oscillations appear (see Fig. 2 for $T = 10$) but their amplitude decrease when the preparation time increases. For larger preparation times only oscillations with small amplitude are found. This behaviour can be interpreted by using Eq. (21) with the Wigner function, as proposed in Ref. 2. For $T = 0$, the Wigner distribution associated to the coherent initial state is a Gaussian in phase space, and the monotonic decrease of $|C_\psi|^2$ with $\Delta S$ reflects the decrease of the overlapping regions between the states $|\psi_+\rangle$ and $|\psi_-\rangle$ (or, equivalently, between $|\psi\rangle$ and $\hat{D}|\psi\rangle$). For small preparation times, the isolated evolution of the environmental system prior to the coupling generates a regular large scale structure in the distribution (characterised by large values of $\Delta Z_0$ in Fig 2). For this case the coincidence between maxima and minima of that large scale structure in $|\psi_+\rangle$ and $|\psi_-\rangle$ is responsible for the oscillations in the overlap. For longer preparation times, smaller scale structures appear in the distribution (corresponding to smaller values of $\Delta Z_0$), and more importantly,
the randomness of the distribution of the patches in the structure increases (reflected in the similarity of the actions $\Delta Z_0$ and $a$). Then, as $T$ increases the amplitude of the oscillations becomes smaller until they are eventually negligible. This behaviour is expected in general for any non-linear system, with the only difference in the preparation time $T$ needed to develop the small scale structure.

In the light of this discussion, special care must be taken in the interpretation of the results of Sec. V where broad oscillations in the time averaged overlap $|C_{\psi}^{BV}|^2$ could appear for large $\Delta S^{BV}$ (see Fig. 3). As the systems considered are non-linear, the states will develop in general a complex small random structure for long enough preparation times, and only negligible oscillations will be present in the overlap. The broad oscillations in $|C_{\psi}^{BV}|^2$ are the result of the use of the Berry-Voros conjecture, that describes correctly the large scale structure but fails in describing the small scale correlations. Therefore, following the discussion in Sec. III related to Eq. (20), only the initial decay (corresponding to small displacements) for each particular sufficiently long preparation time is well described by $|C_{\psi}^{BV}|^2$.

In the approach used in this work, the effect of the coupled evolution in the environmental system is equivalent to rigid displacements in phase-space of the state $|\psi(T)\rangle$ to give $|\psi_+(T; \delta t)\rangle$ and $|\psi_-(T; \delta t)\rangle$. (The dependence of $|\psi_\pm\rangle$ with the interaction time $\delta t$ is made explicit.) No additional structure in phase-space in the states $|\psi_\pm\rangle$ is generated during the coupling, as the contribution of $\hat{H}_E$ is neglected. The interaction time $\delta t_0$ required to obtain a value $|C_0|^2$ of the overlap is given by the condition $|C_\psi(\delta q_0, \delta p_0)|^2 = |C_0|^2$, where the magnitude of the displacements are $\delta q_0 = -2c_p \delta t_0$ and $\delta p_0 = -2c_q \delta t_0$. Therefore the larger the coupling constants, the smaller the interaction time $\delta t_0$. The condition $\Delta S \approx \hbar$ establishes a lower bound for the value of the displacements and consequently for the interaction time needed to attain effective decoherence. An alternative derivation of the lower bound is pointed out in Ref. [16]. A different aspect is the dependence of this $\delta t_0$ with the environmental state prior to the interaction. The size of the displacements $(\delta q_0, \delta p_0)$ can be described by the action $\Delta Z_0 = \delta q_0 \delta p_0$ for each particular state. As discussed in Sec. III $\Delta Z_0$ is of the order of the action that sets the scale of the structures in the distribution for some phase space representations fulfilling Eq. (21). As a result, $\delta t_0$ decreases as the structure in the distribution associated to the state becomes smaller. For example, in the system analysed in Fig 1 the interaction time $\delta t_0$ is proportional to $\sqrt{a}$ [16], provided the
preparation time is long enough for a to describe properly the small scale structure. A more complex situation appears when the evolution induced by $\hat{H}_E$ is not neglected \[16, 18\]. In that case, besides the displacement, the distribution of the structure in phase space of the states $|\psi_+\rangle$ and $|\psi_-\rangle$ will change during the interaction. For the system studied in Fig 11 two different regimes can be distinguished. For $T \lesssim 20$, a rapid variation of the sizes of the structure with time is found and both mechanisms, the displacement and the development of structure, will determine the interaction time $\delta t_0$. However, for $T \gtrsim 20$, the variation of the sizes of the structure is much slower and $\delta t_0$ is determined by the time required to produce the displacement in phase space. As the displacement is approximately independent of the details of the state, $\delta t_0$ will be weakly dependent on the preparation time for $T \gtrsim 20$ \[16\].

Another important point to discuss is the dependence of the decoherence process with the number of degrees of freedom of the environmental system. As the number of degrees of freedom increases, smaller displacements in each variable are needed to obtain $\Delta S \approx \hbar$, that sets the action scale for the initial decay of the overlap in all cases, and the corresponding interaction time will be smaller too. This is compatible with the observation that the larger the environment the more effective the decoherence process.

Experimental tests of the decoherence process in the context discussed in this work can be in principle realized in the systems described in Refs. \[30\]. The interaction between two oscillators is mediated by a term of the form $\hbar G a^+_S a_S (a_E + a^+_E)$, corresponding to a scattering process in which a quantum of energy of the environmental system $\mathcal{E}$ can be absorbed ($a_E$) or emitted ($a^+_E$) whereas the number of quanta of the pointer system $\mathcal{S}$ remains the same. For these cases, the coherences of the reduced density operator of the pointer system in the basis given by the Fock states are proportional to the overlap between the states $\hat{D}(\alpha = iGn\delta t) |\psi(T)\rangle$ and $\hat{D}(\alpha' = iGn'\delta t) |\psi(T)\rangle$. The operator $\hat{D}(\alpha = iGn\delta t) \equiv \exp\{\alpha a^+_E - \alpha^* a_E\}$ produces a displacement in phase space that depends linearly on the interaction time $\delta t$, the coupling constant $G$, and the index $n$ of one of the Fock states of $\mathcal{S}$ involved in the coherence under consideration.

In summary, the role of $\hbar$ as a boundary between different decoherence regimes has been clarified in the context of a characteristic action $\Delta S$, which depends on the quantum state of the environmental system. We related the action $\Delta S$ with the complementary quantity $\Delta Z$, and described their connection with the pattern of structures developed in phase space.
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