HEAVY QUARKONIUM SYSTEMS
AND
NONPERTURBATIVE FIELD CORRELATORS

Yu. A. Simonov*

ITEP,
B. Cheremushkinskaya 25,
117259, Moscow, Russia

S. Titard** and F. J. Ynduráin***

Departamento de Física Teórica, C-XI,
Universidad Autónoma de Madrid, Canto Blanco,
E-28049, Madrid, Spain.

Abstract.
Bound states of heavy quarks are considered. Using the path integral formalism we are able to rederive, in a gauge invariant way, the Leutwyler-Voloshin short distance analysis as well as a long distance linear potential. At all distances we describe the states in terms of nonperturbative field correlators, and we include radiative corrections at short and intermediate distances. For intermediate distance states (particularly $b \bar{b}$ with $n = 2$) our results improve, qualitatively and quantitatively, standard analyses, thanks mostly to being able to take into account the finiteness of the correlation time.

Typeset with Plain TeX

* e-mail:simonov@vxitep.itep.ru
** e-mail: stephan@nantes.ft.uam.es
*** e-mail:fjy@delta.ft.uam.es
Some time ago, Leutwyler and Voloshin showed that the gluon condensate \( \langle \alpha_s G^2 \rangle \) controls the leading nonperturbative effects for heavy \( q\bar{q} \) states at short distances. Their analysis was completed in refs. 3,4, in particular by extending it to spin dependent splittings and by including relativistic and one loop radiative corrections, an essential ingredient in the analysis. With these additions it was then shown that a consistent description of states \( n = 1 \) \( b\bar{b} \) and, to a lesser extent, \( b\bar{b} \) states with \( n = 2 \) and \( c\bar{c} \) states with \( n = 1 \) could be obtained.*

As already pointed out in refs. 1,2 the approach fails for large \( n \). The reason is that nonperturbative contributions grow like \( n^6 \), quickly getting out of hand. For example, for the spin-independent spectrum we have,

\[
M(n,l) = 2m \left( 1 - \frac{C_F \tilde{\alpha}_s^2}{8n^2} \right) - \left[ \log \frac{n\mu}{mC_F\alpha_s} + \psi(n+l+1) \right] \frac{C_F \beta_0 \tilde{\alpha}_s \alpha_s^2}{8\pi n^2} + \frac{\pi \epsilon n l n^6 \langle \alpha_s G^2 \rangle}{2(mC_F\alpha_s)^4}.
\]

(1)

Here \( l \) is the angular momentum, \( m \) is the pole mass of the quark, \( \beta_0 = (33 - 2n_f)/3 \), \( C_F = \frac{4}{3} \) and \( \tilde{\alpha}_s \) embodies part of the radiative corrections:

\[
\tilde{\alpha}_s(\mu) = \left\{ 1 + \left( \frac{93 - 10n_f}{36} + \frac{\gamma_E \beta_0}{2} \right) \frac{\alpha_s}{\pi} \right\}.
\]

Here \( \mu \) is the renormalization point. Finally, in the leading nonperturbative approximation, \textit{and assuming a constant gluon condensate density}, \( \langle G_{\mu\nu}(x)G_{\mu\nu}(y) \rangle \simeq \langle G_{\mu\nu}(0)G_{\mu\nu}(0) \rangle \equiv \langle G^2 \rangle \),

the \( \epsilon_{nl} \) are numbers of order unity:

\[
\epsilon_{10} = \frac{624}{425}, \epsilon_{20} = \frac{1051}{663}, \epsilon_{30} = \frac{769456}{463235}, \epsilon_{21} = \frac{9929}{9944}, \ldots
\]

(2)

We will not consider in this note relativistic corrections.

Clearly, the nonperturbative correction in Eq.(1) blows up very quickly for increasing \( n \) making the method totally unsuitable already for \( b\bar{b} \) with \( n = 3 \), and \( c\bar{c} \) with \( n = 2 \). The way out of this difficulty found in most of the literature is to use a phenomenological potential,

\[
\sigma r, \sigma^\frac{1}{2} \simeq 0.45 \text{ MeV}.
\]

(3)

Now, and although a linear potential yields a correct description of long distance \( q\bar{q} \) forces, the methods lack rigour in that, as is well known \(^{[6;1,2]} \), a linear potential is incompatible with known QCD results at short distances, where indeed it does not represent a good approximation of \textit{e.g.}, the nonperturbative part of (1).

* \( n \) is the principal quantum number. We will use standard atomic spectroscopic notation.
Because of this it is desirable to develop a framework which, in suitable limits, implies both the Leutwyler-Voloshin short distance results as well as the long distance description in terms of a linear potential. This framework is an elaboration of that developed in refs. 7, where it was shown from first principles how one can derive a linear potential from first principles. In the present note we explore the short and intermediate distances, where we rederive the Leutwyler-Voloshin description, improved both by getting better agreement with experiment for the states where it is valid, and extending its range of applicability to intermediate distances. The reason for this improvement lies in that our treatment includes the nonlocal character of the gluonic condensate in the form of a finite correlation time, $T_g$. The results of refs. 1 to 4 are then recovered in the limit $T_g \to \infty$.

The nonlocal condensates have been considered previously\cite{9,10}, in particular with the aim of finding this correlation time. In this note we improve upon the treatment of refs. 9,10 first by using a Lorentz and gauge invariant path integral formulation which would allow us, if so wished, to incorporate relativistic and spin effects**. Secondly, we incorporate radiative corrections which are essential to give a meaning to parameters like $\alpha_s(\mu)$ or $m$.

** Description of the method.**

The method uses the path integral formalism. Because in this note we are only interested in nonrelativistic, spin independent splittings, we start directly with the nonrelativistic $q\bar{q}$ Green’s function\cite{7,8}. For large time $T$,

$$G(x, \bar{x}; y, \bar{y}) = 4m^2 e^{-2mT} \int \mathcal{D}z \mathcal{D}\bar{z} e^{-(K_0 + \bar{K}_0)} \langle W(C) \rangle,$$

$$K_0 = \frac{m}{2} \int_0^T dt \dot{z}(t)^2, \quad \bar{K}_0 = \frac{m}{2} \int_0^T dt \dot{\bar{z}}(t)^2.$$  

$W(C)$ is the Wilson loop operator corresponding to the closed contour $C$ which includes the $q, \bar{q}$ paths. $W(C)$ should also include initial and final parallel transporters, $\Phi(x, \bar{x}), \Phi(y, \bar{y})$ with e.g.,

$$\Phi(x, \bar{x}) = P \exp ig \int_{\bar{x}}^x dz_\mu B_\mu(z),$$

$P$ denoting path ordering. Actually, we can omit the parallel transporters, and at the same time avoid problems with the renormalization of the Wilson loop by choosing $x = \bar{x}, y = \bar{y}$, which will prove sufficient for our purposes.

In order to take into account the nonperturbative character of the interaction we split the gluonic field $B$ as

$$B_\mu = b_\mu + a_\mu.$$  

** The methods to accomplish this would be like the ones developed in the fifth paper of ref. 7 and in ref. 8
The separation will be such that, by definition, the vacuum expectation value of Wick ordered products of $a_\mu$ vanishes, so the correlator may be written in terms solely of $b_\mu$:

$$\langle G(x)G(y) \rangle \rightarrow \langle G_b(x)G_b(y) \rangle,$$

(6)

and $G_b$ is constructed with only the $b_\mu$ piece of $B_\mu$. One may expand in powers of the $b_\mu$ and thus write the Wilson loop average as

$$\langle W(C) \rangle = \int \mathcal{D}a \, \mathcal{P} \exp i \int_C \! d\mu a_\mu$$

$$+ \left( \frac{ig}{2!} \right)^2 \int \mathcal{D}a \int \! dz_\mu \int \! dz'_\mu \, \mathcal{P} \Phi_\alpha(z, z') b_\mu(z') \mathcal{P} \Phi_\alpha(z', z) b_\nu(z') + \ldots$$

$$\equiv W_0 + W_2 + \ldots,$$

(7)

where the transporter $\Phi_\alpha$ is given by an equation like (5), but in terms of only the $a_\mu$ piece of $B_\mu$.

Let us discuss the first term in the r.h.s. of Eq. (7). Using the cluster expansion we get

$$W_0 = Z \exp(\phi_2 + \phi_4 + \ldots),$$

$$\phi_2 = -\frac{C_F g^2}{8\pi^2} \int \! dz_\mu \int \! dz'_\mu \frac{1 + z' z}{(z - z')^2},$$

(8)

and regularization (to be absorbed in $Z$) is implied in this integral. It should be noted that $\phi_2$ contains all ladder-type exchanges, and in addition also ”Abelian crossed” diagrams - those where the times of the vertices can be not ordered, but where the color generators $t_{ik}^c$ are always kept in the same order. Because of this, all crossed diagrams (with the exception of the ”Abelian crossed”) are contained in $\phi_4$. It is remarkable that each term $\phi_{2n}$ in Eq.(8) sums up an infinite series of diagrams. In particular, and as we will see below, $\exp \phi_2$ contains all powers of $\alpha_s/v$ ($v$ being the velocity of the quarks) so the calculation is exact in the nonrelativistic limit.

For heavy (and slow) quarks, (8) becomes,

$$\phi_2 = \frac{C_F g^2}{4\pi^2} \int_0^T \! dt \int_0^T \! dt' \frac{1 + z' z}{r^2 + (t - t')^2} = C_F \alpha_s r^{-1} \int_0^T \! dt + O(v^2),$$

(9)

i.e., a singlet one-gluon exchange potential, as expected.

We then turn to $W_2$. When expanding it in powers of $a_\mu$ one gets typical terms like

$$\text{Tr}(t^{c_k}t^{c_{k-1}} \ldots t^{c_1}t^2t^{c_2} \ldots t^{c_k}t^{a_1} \ldots t^{a_n} t^{a} b_\mu t^{a_n} \ldots t^{a_1} b_\nu t^c)$$

$$\rightarrow C_F^k \text{Tr}(t^{a_1} \ldots t^{a_n} t^{a} b_\mu t^{a_n} \ldots t^{a_1} b_\nu t^c \ldots).$$

Because of the equality

$$t^c t^a t^c = -t^a / 2N_C,$$
one obtains, for all exchanges in the time interval between the times of $b_\mu(z)$ and $b_\nu(z')$ a factor $-1/N_C$ instead of the factor $C_F$ that is found for other exchanges. As a result we may write $W_2$ as

$$W_2 = \frac{(ig)^2}{2} \int_C dz_\mu \int_C dz'_\nu \langle b_\mu b_\nu \rangle \times \exp \left[ \int_{t_2}^T dt V_C^{(S)} + \int_{t_1}^{t_2} dt V_C^{(S)} + \int_{t_1}^0 dt V_C^{(8)} \right],$$

with $V_C^{(S)} = C_F \alpha_s / r$, $V_C^{(8)} = -(1/N_C) \alpha_s / r$ the singlet, octet potentials respectively. The relevance of the octet potential was already noted in refs. 1,2. It appears in our derivation in a fully gauge invariant way, in connection with a gauge invariant Green’s function with the gauge invariant quantity $W_2$ as the kernel.

**Evaluation of $W_2$**

One first uses the Fock-Schwinger gauge (see ref. 6 for details, including a modification of this gauge) to write,

$$\int_C dz_\mu \int_C dz'_\nu \langle b_\mu b_\nu \rangle = \int d\sigma_{\mu\rho} d\sigma'_{\nu\lambda} \langle G_{b_\mu\rho}(z) G_{b_\nu\lambda}(z') \rangle,$$

and the $d\sigma$ are surface differentials. Including also parallel transporters, equal to unity in the F-S gauge, we find the gauge-invariant expression,

$$\int_C dz_\mu \int_C dz'_\nu \langle b_\mu b_\nu \rangle = \int d\sigma_{\mu\rho} d\sigma'_{\nu\lambda} \{ \Phi(x_0, w) G_{b_\mu\rho} \Phi(w, x_0)$$

$$\times \Phi(x_0, w') G_{b_\nu\lambda} \Phi(w', x_0) \}.$$  (11)

As was shown in ref. (7) $x_0$ may be chosen between $w$ and $w'$, up to additional contributions of order $b_\mu^4$, that we are neglecting here. Then we divide the total time interval $T$ into three parts (cf. Fig. 1):

(I) $0 \leq t \leq w'_4$

(II) $w'_4 \leq t \leq w_4$

(III) $w_4 \leq t \leq T.$

Separating out the trivial c.m. motion we get, in regions (I), (III), the singlet Coulomb Green’s function,
\[ G^C(r(t_1), r(t_2); t_1 - t_2) = \int \mathcal{D}r(t) \exp \left[ \frac{-m}{2} \int_{t_2}^{t_1} dt \dot{r}^2 + C_F \alpha_s \int_{t_2}^{t_1} dt \frac{\dot{r}}{r(t)} \right], \]

for quarks of equal mass (so that the reduced mass is \( m/2 \)). In region (II), however, we find the octet Green’s function,

\[ G^C(r(w_4), r(w'_4); w_4 - w'_4) = \int \mathcal{D}r(t) \exp \left[ \frac{-m}{2} \int_{w'_4}^{w_4} dt \dot{r}^2 - \frac{\alpha_s}{2N_C} \int_{w'_4}^{w_4} dt \frac{\dot{r}}{r(t)} \right]. \]

Inserting now this into (10) yields the correction to the total Green’s function, \( G = G^{(0)} + \delta G \):

\[ \delta G = \frac{-g^2}{2} \int d^3r(w_4)G_r^{(S)}(r(T), r(w_4); T - w_4) \int d^3r(w'_4)G_r^{(8)}(r(w_4), r(w'_4); w_4 - w'_4) \]

\[ \times \int d\sigma_{\mu\nu}(w) \int d\sigma_{\rho\lambda}(w')(G_{\mu\nu}(w)G_{\rho\lambda}(w'))G_r^{(S)}(r(w'_4), r(0); w'_4). \] (12)

At this point it is convenient to specify the surface inside contour \( C \), which we do by connecting the points \( z(t) \) and \( \bar{z}(t) \) by a straight line. In the c.m. system,

\[ d\sigma_{\mu\nu} = a_{ij} dz_0 d\beta, \]

and, in the nonrelativistic approximation, the \( a_{ij} \) may be neglected. Thus, and as expected, only the chromoelectric piece \( E \) of \( G_{\mu\nu} \) survives in the correlator in Eq. (12). We then expand this correlator in invariants:

\[ \langle g^2 \mathcal{E}_i(x) \mathcal{E}_j(y) \rangle = \frac{1}{12} [\delta_{ij} \Delta(x - y) + h_i h_j \partial D_1 / \partial h^2], \]
\[ h = x - y, \]
and
\[ \Delta(z) = D(z) + D_1(z) + z^2 \frac{\partial D_1}{\partial z^2}. \] (13)

The invariants \( D, D_1 \) are normalized so that
\[ D(0) + D_1(0) = 2\pi \langle \alpha_s G^2 \rangle. \] (15)

Inserting then (13) into (12) gives,
\[ \delta G = -\frac{1}{24} \int d^3r \int d^3r' \int r_i d\beta \int r'_i d\beta' G_C^{(S)}(r(T), r) G_C^{(8)}(r, r') G_C^{(S)}(r', r(0)). \]

We next consider the matrix element of \( \delta G \) between Coulombic states, \( |nl\rangle \). The resulting expression simplifies if we use the spectral decomposition for \( G_C \):
\[ G_C^{(S,8)}(r, r'; t) = \langle r | \exp[-H_C^{(S,8)} t] | r' \rangle \]
\[ = \sum_k \Psi_k^{(S,8)}(r) e^{-E_k^{(S,8)} t} \Psi_k^{(S,8)}(r')^*. \]

Identifying the energy shifts from the relation
\[ G = G^{(S)} + \delta G \simeq G^{(S)}(1 - T \delta E_{nl}) \]
valid for \( T \to \infty \) we get
\[ \delta E_{nl} = \frac{1}{36} \int \frac{d^3p dp_4}{(2\pi)^4} \int \frac{d\beta d\beta'}{4\pi} \tilde{\Delta}(p) \sum_k \langle n| e^{ip(\beta - \frac{i}{2})r} | k(8) \rangle \]
\[ \times \frac{1}{E_k^{(8)} - E_n^{(S)} - ip_4} \langle k(8) | r'_i e^{ip(\beta - \frac{i}{2})r'} | nl \rangle. \] (16)

The states \( |k(8)\rangle \) are eigenstates of the octet Hamiltonian,
\[ H^{(8)} = \frac{p^2}{m} + \alpha_s / 2N_C r \]
with eigenvalues \( E_k^{(8)} \). \( \tilde{\Delta}(p) \) is the Fourier transform of \( \Delta(x) \).

Eq. (16) is our basic equation. The correlator \( \Delta(x) \) depends on \( x \) as
\[ \Delta(x) = f(|x|/T_g) \]
and is expected to decrease exponentially for large \( |x| \), in Euclidean space. The correlation length \( T_g \) may be related to the string tension,
\[ T_g^{-1} \equiv \mu_T = \frac{\pi}{3\sqrt{2}} \frac{(\alpha_s G^2)^{1/2}}{\sigma^{1/2}} \simeq 0.35 \text{ GeV}. \] (17)
(for the derivation of this equation, see below). We have now two regions. For very heavy quarks, and small $n$,

$$\mu_T \ll |E_n^{(S)}| = m(C_F\bar{\alpha}_s)^2/4n^2.$$ 

Then we approximate $\Delta(x) \sim \text{constant}$, and hence $\tilde{\Delta}(p) \sim \delta_4(p)$ so that

$$\delta E_{nl} = \frac{\pi\langle\alpha_s G^2\rangle}{18} \sum_k \frac{\langle nl|ri|k(8)\rangle \langle k(8)|ri|nl\rangle}{E_k^{(8)} - E_n^{(8)}} \approx \frac{\pi\langle\alpha_s G^2\rangle}{18} \langle nl|ri\rangle 1 \frac{1}{H^{(8)} - E_n^{(S)} ri|nl\rangle}. \quad (18)$$

This is the equation obtained in refs. 1,2 and, upon calculating the r.h.s. of (18) one indeed finds the nonperturbative piece of Eq. (1).

The improvement over (18) represented by (16) lies in that it involves the correlator $\langle E_i(x)E_j(y)\rangle$, i.e., one takes account of nonlocality of the correlator. To implement this it is convenient to distinguish two regimes: i) We consider that $\mu_T$ is smaller than $1/a = mC_F\bar{\alpha}_s / 2$, but one has $\mu_T \simeq |E_n|$. ii) $\mu_T \gg |E_n|$. We will first consider case (i). Then one can neglect $|p|$ in the exponents of Eq. (16) but $\mu_T$ should be kept in the denominator there. Using now an exponential form for $\Delta(x)$ we obtain,

$$\tilde{\Delta}(p) = 3(2\pi)^3 \mu_T \frac{H^{(8)} - E_n^{(S)}}{2} \langle\alpha_s G^2\rangle,$$ \quad (19)

Substituting this into (16) we get the energy shifts,

$$\delta E_{nl} = \frac{\pi\langle\alpha_s G^2\rangle}{18} \langle nl|ri\rangle 1 \frac{1}{H^{(8)} - E_n^{(S)} ri|nl\rangle}. \quad (20)$$

This is precisely the approximation postulated in refs. 9,10. Clearly, as $T_g \to \infty (\mu_T \to 0)$, (20) reproduces (18); but, as we will see, (20) represents an important improvement over (18) both from the conceptual and the phenomenological point of view in the intermediate distance region.

Then we turn to the regime (ii), $\mu_T \gg |E_n^{(S)}|$. In this case the velocity tends to zero, the nonlocality of the interaction tends to zero as compared to the quark rotation period (which in the Coulombic approximation would be $T_q = 1/|E_n^{(S)}|$), and the interaction may therefore be described by a local potential. In fact: considering Eq. (16), it now turns out that we can neglect both $E_n^{(S)}$ and the kinetic energy term in $E_k^{(8)}$ (indeed, all of it) as compared to $-i\rho_4$. Then one gets

$$\delta E_{nl} \simeq \frac{1}{36} \langle nl| \int_0^{r_i(w_4)} dw_i \int_0^{r_i(w'_4)} dw'_i \int_0^\infty d(w_4 - w'_4) \Delta(w - w') |nl\rangle$$

and in the limit in which we are now working we can approximate $r_i(w'_4) \simeq r_i(w_4)$. One obtains here the matrix elements of a local potential which may be written in terms of $D, D_1$:[7]

$$U(r) = \frac{1}{36} \{ 2r \int_0^r d\lambda \int_0^\infty d\nu D(\lambda, \nu) \}.$$
\[
+ \int_0^r \lambda \lambda \int_0^\infty d\nu [-2D(\lambda, \nu) + D_1(\lambda, \nu)] \right].
\]

Note that both \(D, D_1\) depend on \(\lambda, \nu\) through the combination \(\lambda^2 + \nu^2\). At large \(r\), and as this equation shows, \(U(r)\) behaves like

\[
U(r) \simeq \sigma r - \text{Constant}, \quad \sigma = \frac{1}{72} \int d1 d2 D(x_1, x_2).
\]

If we now used the ansatz (19), we would obtain the announced relation between \(\mu_T\) and \(\sigma\). Of course in this situation the strategy of treating the effects of the gluon condensate as a perturbation of the Coulombic potential is no more appropriate. One should rather take \(U(r)\), together with the Coulombic potential, as part of the unperturbed Schrödinger equation. We leave the subject here referring to the various existing analyses\(^{[5,11,12]}\) for details. (In particular, refs. 11 are the ones closer in spirit to the work here in what regards the treatment of the nonperturbative effects, while ref. 12 incorporates radiative corrections to a phenomenological long distance potential)

**Phenomenology**

For the phenomenological analysis we will generalize Eq. (1) by writing,

\[
M(n, l) = 2m\{1 - \frac{C_F \bar{\alpha}_s^2}{8n^2}
- \left[\frac{n\mu}{mC_F \bar{\alpha}_s} + \psi(n + l + 1)\right] \frac{C_F^2 \beta_0 \bar{\alpha}_s \alpha_s^2}{8\pi n^2} + \frac{\pi \epsilon_{nl}(\mu_T)n^6 \langle \alpha_s G^2 \rangle}{2(mC_F \bar{\alpha}_s)^4}\}, \tag{21}
\]

where the \(\epsilon_{nl}(\mu_T)\) are obtained solving Eq. (20); for \(\mu_T \to 0\), \(\epsilon_{nl}(0) = \epsilon_{nl}\), this last being the quantities given in Eqs. (1), (2).

Unlike in the case \(\mu_T = 0\), where a closed expression could be found for the \(\epsilon_{nl}\), the \(\epsilon_{nl}(\mu_T)\) may only be computed numerically. However, a fairly precise evaluation may be obtained by neglecting the potential \(\alpha_s / 2N_C\) in (20), then working with \(p\)-space Coulomb functions. To a very tolerable \(\sim 5\%\) accuracy it follows that we may approximate,

\[
\epsilon_{nl}(\mu_T) \simeq \frac{\epsilon_{nl}}{1 + \rho_n \eta_n}, \quad \eta_n \equiv \left(\frac{2n}{C_F \bar{\alpha}_s}\right)^2 \frac{\mu_T}{m}, \tag{22}
\]

where,

\[
\rho_{10} = 0.62, \quad \rho_{20} = 0.76, \quad \rho_{30} \approx 0.9, \quad \rho_{21} = 0.70.
\]

Substituting into Eq. (21) then gives us a very explicit generalization of Eq. (1), valid in the intermediate region \(\mu_T \sim |E_n^{(S)}|\).
For the numerical calculation we proceed as follows. For $\bar{b}b$ we take the optimum values for the renormalization point $\mu$ given in ref. 4 (which were obtained neglecting $\mu_T$). We therefore choose,

$$\mu = 1.5 \text{ GeV, for } n = 1; \mu = 0.95 \text{ GeV, for } n = 2.$$ 

For mixed $n$ we take the value corresponding to the smaller $n$. For $c\bar{c}$ we, somewhat arbitrarily, choose $\mu = 0.95 \text{ GeV}$. For the basic QCD parameters we take

$$\Lambda(n_f = 4, 2 \text{ loops}) = 200 \text{ MeV}, \langle \alpha_s G^2 \rangle = 0.042 \text{ GeV},$$

and thus the corresponding values of $\alpha_s$ are $\alpha_s(1.5 \text{ GeV}) = 0.27$, $\alpha_s(0.95 \text{ GeV}) = 0.35$.

We will not consider varying these quantities. A variation of $\Lambda$ can be largely compensated by a corresponding variation of $\mu$ and likewise, and because the $\epsilon_{nl}(\mu_T)$ depend almost exactly on the ratio $\langle \alpha_s G^2 \rangle / \mu_T$, a variation of the condensate may be balanced by a compensating variation of the correlation time, $T_g = \mu_T^{-1}$.

We now have two possibilities: fit $\mu_T$ to each individual splitting, and compare the results among themselves and with the one coming from the string tension; or take $\mu_T$ from the last, Eq. (17) and then predict the splittings. If we do the first we find

$$\mu_T = 0.40 \text{ GeV (2S }-\text{ 1S)}, \mu_T = 0.76 \text{ GeV (3S }-\text{ 2S)}, \mu_T = 0.59 \text{ GeV (2S }-\text{ 2P}). \tag{23}$$

For the $c\bar{c}$ case we only consider the 2S-1S splitting, where we get

$$\mu_T = 1.23 \text{ GeV.} \tag{24}$$

Clearly, the more reliable calculation is that of the 2S-1S $\bar{b}b$ splitting: not only the radiative corrections are known (unlike for the 2S-2P case) but also it falls inside the conditions of regime ($i$), unlike the 3S-2S splitting and, even more, the 2S-1S $c\bar{c}$ one. It is then gratifying that the value of $\mu_T$ that follows from the $\bar{b}b$ 2S-1S splitting, $\mu_T = 0.4$ GeV, is the one which is in better agreement with the value $\mu_T = 0.32$ GeV obtained with the comparison with the linear potential, Eq. (17).

If we now choose the second possibility, we fix $\mu_T$. The corresponding results are summarized in Table I.

| splitting | $\mu_T = 0$ | $\mu_T = 0.32$ | $\mu_T = 0.40$ | exp. |
|-----------|-------------|-----------------|-----------------|-----|
| 2S − 1S ($\bar{b}b$) | 479a | 590 | 522 | 558 MeV |
| 2S − 2P ($\bar{b}b$) | 181a | 162 | 147 | 123 MeV |
| 3S − 2S ($\bar{b}b$) | 4570 | 748 | 614 | 332 MeV |
| 2S − 1S ($c\bar{c}$) | 9733 | 1930 | 1626 | 670 MeV |

**Table I**- Predicted splittings, and experiment.

(a): Values from ref 4, with $\mu \simeq 0.95$ for both
The \( n = 2 \) \( b \bar{b} \) states are certainly better described than with the approximation \( T_G = \infty \) of refs. 1, 2, 4. Particularly important is the fact that inclusion of the finite correlation time stabilizes the calculation. For example, if we had taken \( \mu = 1.5 \), and \( \mu_T = 0 \), for the 2S-1S splitting for bottomium, we would have obtained the absurd value of 1.944 GeV. In what respects the \( n = 3 \) and the \( n = 2 \) \( c \bar{c} \) states the improvement is marginal, in the sense that the basic assumption, \( \text{viz.} \), that one can treat the nonperturbative effects at leading order fails, as is obvious from the figures in the column \( \mu_T = 0 \) in Table I. Indeed these states fall clearly in regime (ii) and should therefore be better described with a local potential as discussed extensively in the existing literature, of which we, and for illustrative purposes, single out ref. 11, where the nonperturbative effects (including spin-dependent splittings) are treated with methods like ours, but where radiative corrections are ignored; or ref. 12 where radiative corrections are incorporated but the confining potential is introduced phenomenologically.

We would like to end this note with a few words on extensions of this work. An obvious one is to include the treatment of spin effects, and a calculation of the wave functions. Then it would be very desirable to evaluate the radiative corrections to the nonperturbative terms, as this would greatly diminish the dependence of these terms on the renormalization point, \( \mu \), thereby substantially increasing the stability of the calculation. Finally, and to be able to extend the calculation to intermediate distances with success, one should abandon the treatment of nonperturbative effects at first order: an iterative approach should certainly yield better results.

Acknowledgements.-

We are grateful to CICYT, Spain, for partial financial support. One of us (Yu. A. S.) would like to acknowledge the hospitality of the Universidad Autónoma de Madrid, where most of this work was done.
REFERENCES

1.-H. Leutwyler, Phys. Lett., 98B, 447 (1981)
2.-M. B. Voloshin, Nucl. Phys., B154, 365 (1979); Sov. J. Nucl. Phys., 36, 143 (1982)
3.-F. J. Ynduráin, The Theory of Quark and Gluon Interactions, Springer, 1993
4.-S. Titard and F. J. Ynduráin, Phys. Rev. D49, 6007 (1994); FTUAM 94-6, in press in Phys. Rev. FTUAM 94-34.
5.-E. Eichten et al., Phys. Rev., D21, 203 (1980); W. Lucha, F. Schöberl and D. Gromes, Phys. Rep., C200, 128 (1991)
6.-I. I. Balitsky, Nucl. Phys., B254, 166 (1985)
7.-H. G. Dosch, Phys. Lett. B190, 177 (1987); Yu. A. Simonov, Nucl. Phys., B307, 512 (1988) and B324, 56 (1989); H. G. Dosch and Yu. A. Simonov, Phys. Lett., B205, 339 (1988); H. G. Dosch and M. Schiestl, Phys. Lett., B209, 85 (1998). For a review, see Yu. A. Simonov, Sov. J. Nucl. Phys., 54, 192 (1991)
8.-J. A. Tjon and Yu. A. Simonov, Ann. Phys (N.Y.), 228, 1 (1993)
9.-A. Krämer, H. G. Dosch and R. A. Bertlmann, Phys. Lett., B223, 105 (1989); Fort. der Phys., 40, 93 (1992)
10.-M. Campostrini, A. Di Giacomo and S. Olejnik, Z. Phys., C31, 577 (1986)
11.-A. M. Badalian and V. P. Yurov, Yad. Fiz., 51, 1368 (1990); Phys.Rev., D42, 3138 (1990)
12.-See, e.g. S. N. Gupta, S. F. Radford and W. W. Repko, Phys Rev., D26, 3305 (1982)