REPRESENTATIONS OF C*-CORRESPONDENCES ON PAIRS OF HILBERT SPACES.

ALONSO DELFÍN

Abstract. We study representations of Hilbert bimodules on pairs of Hilbert spaces. If $A$ is a C*-algebra and $X$ is a right Hilbert $A$-module, we use such representations to faithfully represent the C*-algebras $K_A(X)$ and $L_A(X)$. We then extend this theory to define representations of $(A, B)$ C*-correspondences on a pair of Hilbert spaces and show how these can be obtained from any nondegenerate representation of $B$. As an application of such representations, we give necessary and sufficient conditions on an $(A, B)$ C*-correspondences to admit a Hilbert $A$-$B$-bimodule structure. Finally, we show how to represent the interior tensor product of two C*-correspondences.

1. Introduction

Let $A$ and $B$ be C*-algebras. Research on normed $A$-$B$-bimodules usually deals with either Hilbert $A$-$B$-bimodules or $(A, B)$ C*-correspondences. In this paper we study representations of both kinds of bimodules on pairs of Hilbert spaces, which roughly consists of realizing the bimodule as a closed subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, the space of bounded linear maps from a Hilbert space $\mathcal{H}_0$ to a Hilbert space $\mathcal{H}_1$. The Hilbert bimodule case was introduced by R. Exel [5]. We generalize Exel’s definition to the general C*-correspondence setting. Representations of $(A, A)$ C*-correspondences on a Hilbert space, sometimes also called covariant representations, have been studied thoroughly in the past. See for instance [13], [11], and [8]. These representations are the main object used to construct the Cuntz-Pimsner algebra of an $(A, A)$ C*-correspondence. Our definition for representations of $(A, B)$ C*-correspondences on a pair of Hilbert spaces agrees with the other definitions in the literature when $A = B$. We use representations of $(A, B)$ C*-correspondences on pairs of Hilbert spaces as a tool for mainly three purposes:

1. Give representations of adjointable and compact-module maps on right Hilbert modules.
2. Answer when a general $(A, B)$ C*-correspondence can be uniquely given the structure of a Hilbert $A$-$B$-bimodule.
3. Represent the interior tensor product of C*-correspondences as the product of suitable representations of the factors.

Unfortunately, in the early literature, $(A, A)$ C*-correspondences were sometimes also referred to as Hilbert bimodules over $A$. See for instance [13] and [6]. In this paper we follow the current naming conventions. That is, for us both Hilbert $A$-$B$-bimodules and $(A, B)$ C*-correspondences come equipped with a $B$-valued right
inner product which in turn defines a norm on the module. The difference between them lies in the fact that Hilbert $A$-$B$-bimodules come with an $A$-valued left inner product compatible with the $B$-valued one, while for $(A,B)$ $C^*$-correspondences we require $A$ to act as adjointable operators, but an $A$-valued inner product is not assumed.

By a representation of a Hilbert $A$-$B$-bimodule $X$ on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$, we mean a triple of maps $(\lambda_A, \rho_B, \pi_X)$, where $\lambda_A$ is a representation of $A$ on $\mathcal{H}_1$, $\rho_B$ is a representation of $B$ on $\mathcal{H}_0$, and $\pi_X : X \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that $\pi_X(X)$ has the Hilbert $(\lambda_A(A), \rho_B(B))$-bimodule structure where both module actions and both inner products are given by multiplication of operators. See Definition 3.3 for more details. That such representations do exist is shown in Propositions 4.7 and 4.8 of [5]. Since every Hilbert $A$-$B$-bimodule is in particular an $(A,B)$ $C^*$-correspondence, a natural question arises here: can we also represent an $(A,B)$ $C^*$-correspondence on a pair of Hilbert spaces in a way that generalizes representations of Hilbert bimodules? We give a positive answer to this question in Theorem 4.3 by establishing that any $(A,B)$ $C^*$-correspondence can be represented on a particular pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$. In the $(A,B)$ $C^*$-correspondence cases, as in Exel’s result for Hilbert bimodules, the Hilbert space $\mathcal{H}_0$ will also come from a given nondegenerate representation of $B$ and $\mathcal{H}_1$ will be obtained so that $A$ is nondegenerately represented on $\mathcal{H}_1$. The methods we use differ significantly from those used by Exel in [5], Murphy in [10], and Zettl in [16] for their analogous results for Hilbert bimodules and Hilbert modules. In particular, our methods do not rely on the linking algebra of a Hilbert bimodule $X$, nor on the theory of positive definite kernels for Hilbert modules, nor on the dual module $X''$ of a right Hilbert $A$-module $X$, which is a right Hilbert $A^{**}$-module. Nevertheless, our methods can easily be adapted to work in the Hilbert bimodule setting, making our notion of representations of $C^*$-correspondences more general. Indeed, in Theorem 5.4 we adapt our methods from the $C^*$-correspondence case to show the existence of a representation $(\lambda_A, \rho_B, \pi_X)$ for any Hilbert $A$-$B$-bimodule $X$ on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$. In contrast with the analogue result from [5], we give an explicit formula for the map $\pi_X$. All this is used in our proofs for Theorem 4.8 and Corollary 4.11 where we give necessary and sufficient conditions for an $(A,B)$ $C^*$-correspondence to uniquely admit a Hilbert $A$-$B$-bimodule structure.

An important tool when working with $C^*$-correspondences is their interior tensor product. We show, in Theorem 4.11, that representations of $C^*$-correspondences are well behaved with respect to the interior tensor product. By this we mean that it is always possible to find a representation of the tensor product correspondence by looking at suitable representations of the factors. Moreover, if $(X, \varphi_X)$ and $(Y, \varphi_Y)$ are $C^*$-correspondences whose interior tensor product makes sense, we can always think of $X \otimes_{\varphi_Y} Y$ as $\pi_X(X)\pi_Y(Y)$, where $\pi_X$ and $\pi_Y$ come from carefully chosen representations of $(X, \varphi_X)$ and $(Y, \varphi_Y)$. In practice, this makes the tensor product more manageable, as elementary tensors are now replaced by compositions of operators. This can be compared with Theorem 3.2 in [10], where Murphy uses that right Hilbert modules can be concretely represented on pairs of Hilbert spaces to give an elementary construction of the exterior tensor product of right Hilbert modules.

A main advantage of having a right Hilbert module represented as a subspace of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is that, assuming some nondegeneracy conditions, the $C^*$-algebras
of adjointable maps and compact-module maps of the module can be faithfully represented on $H_1$. Indeed, this is shown in Propositions 2.4, 2.8, and 3.10. These representations have not been studied in the current literature. However, they play an important role in current work dealing with modules and correspondences over $L^p$-operator algebras (See Chapters V and VI in [4]).

In fact, our main motivation to study representations of $C^*$-correspondences on a pair of Hilbert spaces is that they naturally give a potential definition of what a correspondence might be when replacing $C^*$-algebras by general Banach algebras. For instance, in Definition 3.7 one can replace Hilbert spaces by what a correspondence might be when replacing $C^*$-algebras by general Banach algebras. Motivated by the results in this paper, the author’s doctoral dissertation, see [4], considers a version of the Cuntz-Pimsner algebras for $L^p$-spaces and define a module over an $L^p$-operator algebra to be a pair of subspaces that satisfy the conditions satisfied by the pair $(\pi_X(X), \pi_X(X)^*)$ in Definition 3.7. Motivated by the results in this paper, the author’s doctoral dissertation, see [4], considers a version of the Cuntz-Pimsner algebras for $L^p$-operator algebras. Indeed, for $p \in (1, \infty)$ and $d \in \mathbb{Z}_{>2}$, it is possible to use a construction similar to the Cuntz-Pimsner construction, see [15] and [8], but on a $L(\ell^1_p)$-module to get $O_p^d$, the $L^p$-version of the Cuntz algebra introduced in [12]. Many results from this paper have motivated definitions for our investigations of modules over $L^p$-operator algebras, $L^p$-correspondences and the $L^p$-operator algebras generated by these. For details see Chapters V and VI in [4].

**Structure of the paper:** In Section 2 we start with the particular case of right Hilbert modules. Roughly speaking, for a pair of Hilbert spaces $(H_0, H_1)$ and a concrete $C^*$-algebra $A \subseteq \mathcal{L}(H_0)$, we analyze the behavior of closed subspaces of $\mathcal{L}(H_0, H_1)$ that have the obvious structure of right Hilbert $A$-modules. We pay close attention to the adjointable maps and compact-module maps of these closed subspaces. We point out in Section 3 that any right Hilbert $A$-module can be represented as an isometric copy of a closed subspace of $\mathcal{L}(H_0, H_1)$ for a pair of Hilbert spaces $(H_0, H_1)$. In fact, we see that this is true for any Hilbert bimodule.

In Section 4 we introduce representations of $(A, B)$ $C^*$-correspondences on a pair of Hilbert spaces. We prove that this is indeed a generalization of Exel’s theory of representations for Hilbert $A$-$B$-bimodules. This yields a framework for representations of general $C^*$-correspondences which, as in the Hilbert bimodule case, shows that any $C^*$-correspondence is isometrically isomorphic to a closed subspace of $\mathcal{L}(H_0, H_1)$ for a pair of Hilbert spaces $(H_0, H_1)$. As an application, we use representations of $C^*$-correspondences to show that an $(A, B)$ $C^*$-correspondence $(X, \varphi)$ admits a unique structure of a Hilbert $A/\ker(\varphi)$-$B$-bimodule if and only if $K_B(X) \subseteq \varphi(A)$. This in turn gives necessary and sufficient conditions for $(X, \varphi)$ to admit a unique structure of a Hilbert $A$-$B$-bimodule. We end the paper by showing that, given any two $C^*$-correspondences that share the middle action, it is always possible to represent their interior tensor product on a pair of Hilbert spaces by using particular representations of the original $C^*$-correspondences.

We end our introduction by establishing some of our notational conventions. Let $a : V_0 \to V_1$ be a linear map between vector spaces. In this case we follow the common convention of suppressing parentheses for linear maps and write $a\xi$ for the action of $a$ on $\xi \in V_0$. However, if $X$ and $Y$ are vector spaces that are also modules over an algebra $A$ and $t : X \to Y$ is a linear module map, then we write $t(x)$ for the action of $t$ on $x \in X$. This is needed to avoid some potential confusion when both $x$ and $t(x)$ happen to also be linear maps between vector spaces, which will occur frequently in this paper.
If $X$ is a subspace of linear maps between vector spaces $V_0$ and $V_1$, the product
$X V_0$ is defined as the linear span of elements in $X$ acting on vectors from $V_0$, that is
$$X V_0 = \text{span}\{x \xi : x \in X \text{ and } \xi \in V_0\} \subseteq V_1.$$ 

**Notation 1.1.** We fix some terminology for Hilbert modules over C*-algebras. The $A$-valued right inner product for a right Hilbert module will be denoted by $\langle -, - \rangle_A$. The map $(x, y) \mapsto \langle x, y \rangle_A$ is assumed to be linear in the second variable and conjugate linear in the first one. Similarly, the $A$-valued left inner product for a left Hilbert module will be denoted by $A \langle -, - \rangle$. The map $(x, y) \mapsto A \langle x, y \rangle$ is assumed linear in the first variable and conjugate linear in the second one. If $X$ is any right Hilbert $A$-module, we use $L_A(X)$ to denote adjointable maps from $X$ to itself. For each $x, y \in X$ we have the “rank one” operator $\theta_{x,y} \in L_A(X)$, given by $\theta_{x,y}(z) = x(y, z)_A$ for any $z \in X$. We write $K_A(X)$ for the compact-module maps from $X$ to itself, which are defined as the closed linear span of the “finite rank” operators. That is,
$$K_A(X) = \text{span}\{\theta_{x,y} : x, y \in X\}.$$ 

Finally, we regard Hilbert spaces as right Hilbert $C^*$-modules. For this reason, our convention for inner products of Hilbert spaces is the physicist’s one: they are linear in the second variable and conjugate linear in the first one.

## 2. Concrete Hilbert Modules

In this section, we describe a concrete example of a right Hilbert module $X$ over a concrete C*-algebra $A \subseteq \mathcal{L}(\mathcal{H}_0)$ for a Hilbert space $\mathcal{H}_0$. This is Example 2.1 below, which was studied by G. J. Murphy in Section 3 of [10], where it was referred to as a Concrete Hilbert C*-Module. We then provide useful representations for the C*-algebras $L_A(X)$ and $K_A(X)$. In Section 8 we will see that any right Hilbert $A$-module can be represented in this fashion.

**Example 2.1.** Let $\mathcal{H}_0, \mathcal{H}_1$ be Hilbert spaces and let $A \subseteq \mathcal{L}(\mathcal{H}_0)$ be a concrete C*-algebra. Suppose that $X \subseteq \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a closed subspace such that $a x \in X$ for all $a \in A$ and $x \in X$, and such that $x^* y \in A$ for all $x, y \in X$. For each $x, y \in X$ we put
$$\langle x, y \rangle_A = x^* y \in A. \tag{2.1}$$

**Proposition 2.2.** Let $X$ be as in Example 2.1. Then, $X$ is a right Hilbert $A$-module with $A$-valued inner product given by equation (2.1) and $\|\langle x, x \rangle_A\|^{1/2} = \|x\|.$

**Proof.** It is clear that $X$ is a right $A$-module. It is easily checked that $\langle x, y \rangle_A$ satisfies all the axioms of an $A$-valued inner product on $X$. We claim that $X$ is complete with the induced norm $\|x\|_A = \|\langle x, x \rangle_A\|^{1/2}$. Indeed, elements of the C*-algebra $\mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H}_1)$ can be written as $2 \times 2$ operator valued matrices and $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is isometrically isomorphic to the lower left corner of $\mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H}_1)$, while $\mathcal{L}(\mathcal{H}_0)$ is isomorphic to the upper left corner. Hence, if $x \in X$, the C*-equation at the second step yields
$$\|x\|^2 = \left\| \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} x^* x & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|x^* x\| = \|x\|_A^2.$$ 

The claim now follows because $X$ is closed in $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$. Thus, $X$ is indeed a right Hilbert $A$-module. 

[10]
Remark 2.3. Thanks to Proposition 2.2 above, when $X$ is as in Example 2.1 we are free to not make any distinction between the norm $x \in X$ has as an element of $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ and the module norm $\|x\|_A$. Thus, from now on we drop the subscript $A$ and simply write $\|x\|$. 

Next, we will show that the compact-module maps and adjointable maps of the Hilbert module in Example 2.1 above can be realized as closed C*-subalgebras of $\mathcal{L}(\mathcal{H}_1)$, provided that some nondegeneracy conditions hold.

**Proposition 2.4.** Let $X$ be the right Hilbert $A$-module described in Example 2.1 above. Suppose that $X\mathcal{H}_0$ is dense in $\mathcal{H}_1$. Then, there is a $*$-isomorphism from $\mathcal{K}_A(X)$ to 
\[ \text{span} \{xy^*: x, y \in X\} \subseteq \mathcal{L}(\mathcal{H}_1) \]
which sends $\theta_{x,y}$ to $xy^*$ for $x, y \in X$.

**Proof.** Define subspaces $K_1 \subseteq \mathcal{L}(\mathcal{H}_1)$ and $K_2 \subseteq \mathcal{K}_A(X)$ by letting 
\[ K_1 = \text{span} \{xy^*: x, y \in X\} , K_2 = \text{span} \{\theta_{x,y}; x, y \in X\}. \]
Recall that $K_2$ is dense in $\mathcal{K}_A(X)$. Let $n \in \mathbb{Z}_{\geq 1}$ and let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Then, for any $z \in X$
\[ \left\| \sum_{j=1}^n \theta_{x_j,y_j}(z) \right\|_X = \left\| \left(\sum_{j=1}^n x_j y_j^*\right) z \right\|_{\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)} \leq \left\| \sum_{j=1}^n x_j y_j^* \right\|_{\mathcal{L}(\mathcal{H}_1)} \|z\|. \]
This implies
\[ \left(\sum_{j=1}^n \theta_{x_j,y_j}\right) \leq \left(\sum_{j=1}^n x_j y_j^*\right). \]
Let $\iota: K_1 \to \mathcal{K}_A(X)$ be the linear extension of the map which sends $xy^*$ to $\theta_{x,y}$ for $x, y \in X$. That is,
\[ \iota \left(\sum_{j=1}^n x_j y_j^*\right) = \sum_{j=1}^n \theta_{x_j,y_j}. \]
That $\iota$ is well defined follows from (2.2). In fact, (2.2) gives $\|\iota(k)\| \leq \|k\|$ for all $k \in K_1$. Thus, we can extend $\iota$ by continuity to a map $\tilde{\iota}: K_1 \to \mathcal{K}_A(X)$ such that $\|\tilde{\iota}(s)\|_{\mathcal{K}_A(X)} \leq \|s\|_{\mathcal{L}(\mathcal{H}_1)}$ for all $s \in K_1$. Our goal is to show that $\tilde{\iota}$ is a $*$-isomorphism from $K_1$ to $\mathcal{K}_A(X)$. Notice that $\tilde{\iota}$ is already a $*$-homomorphism between C*-algebras. We will show that $\tilde{\iota}$ is injective, which in turn will make $\tilde{\iota}$ an isometry. Since $\tilde{\iota}$ maps $K_1$ onto $K_2$, a dense subset of $\mathcal{K}_A(X)$, proving injectivity will automatically show that $\tilde{\iota}$ is a $*$-isomorphism and this will finish the proof.

Take any $s \in K_1$ and fix $x \in X$. We claim that the element $sx \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is actually in $X$ and that it is equal to $\tilde{\iota}(s)(x) \in X$. Indeed, for any $k \in K_1$, the element $kkx \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is an element of $X$ (because $x_1 x_2 x \in \mathcal{X} A \subseteq X$ for all $x_1, x_2 \in X$) and it coincides with $\tilde{\iota}(k)(x) \in X$. Thus, by continuity it follows that $sx = \tilde{\iota}(s)(x)$, as claimed.

Finally, to prove that $\tilde{\iota}$ is injective, let $s \in K_1$ satisfy $\tilde{\iota}(s) = 0 = 0$ in $\mathcal{K}_A(X)$. We have to show that $s = 0$ in $\mathcal{L}(\mathcal{H}_1)$, but since $X\mathcal{H}_0$ is dense in $\mathcal{H}_1$, it is enough to prove that $s(x\xi) = 0$ for all $x \in X$ and $\xi \in \mathcal{H}_0$. Indeed, thanks to our last claim, we have
\[ s(x\xi) = sx(\xi) = [\tilde{\iota}(s)(x)]\xi = 0. \]
This finishes the proof. □
Before characterizing $\mathcal{L}_{A}(X)$, we need to recall a useful lemma and prove a general result about the direct sum of Hilbert modules.

**Lemma 2.5.** Let $A$ be a C*-algebra and let $X$ be any Hilbert $A$-module. Then, for any $t \in \mathcal{L}_{A}(X)$ and any $x \in X$, we have $\langle t(x), t(x) \rangle_{A} \leq \|t\|^2 \langle x, x \rangle_{A}$.

**Proof.** See Proposition 1.2 in [9]. \[ \]

Let $A$ be a C*-algebra, let $X$ be any right Hilbert $A$-module, and let $n \in \mathbb{Z}_{\geq 1}$. The direct sum $X^n$ is usually regarded as a right Hilbert $A$-module in an obvious way. However, $X^n$ can also be identified with $M_{1,n}(X)$, the row vectors with $n$ entries in $X$. This identification makes $X^n$ a right Hilbert $M_n(A)$-module, with action that comes from the formal matrix multiplication $M_{1,n}(X) \times M_n(A) \to M_{1,n}(X)$. That is,

$$
(x_1, \ldots, x_n) \cdot (a_{i,j})_{i,j} = \left( \sum_{i=1}^{n} x_i a_{i,1}, \ldots, \sum_{i=1}^{n} x_i a_{i,n} \right).
$$

The $M_n(A)$-valued right inner product comes from the formal matrix multiplication $M_{n,1}(X) \times M_{1,n}(X) \to M_n(A)$. That is,

$$
\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_{M_n(A)} = \langle (x_i)_A \rangle_{i,j}.
$$

The following result should be well known. We include a complete proof as we couldn’t find one in the current literature.

**Proposition 2.6.** Let $A$ be a C*-algebra, let $X$ be any right Hilbert $A$-module, and let $n \in \mathbb{Z}_{\geq 1}$. For each $t \in \mathcal{L}_{A}(X)$, we define a map $\kappa(t) : X^n \to X^n$ by

$$
\kappa(t)(x_1, \ldots, x_n) = (t(x_1), \ldots, t(x_n)).
$$

Then, $\kappa(t) \in \mathcal{L}_{M_n(A)}(X^n)$, and the map $t \mapsto \kappa(t)$ from $\mathcal{L}_{A}(X)$ to $\mathcal{L}_{M_n(A)}(X^n)$ is a $*$-isomorphism.

**Proof.** Firstly we show that $\kappa(t) \in \mathcal{L}_{M_n(A)}(X^n)$. Indeed, an immediate calculation shows that

$$
\langle (\kappa(t)(x_1, \ldots, x_n), (y_1, \ldots, y_n))_{M_n(A)} = \langle (x_1, \ldots, x_n), \kappa(t^*)(y_1, \ldots, y_n) \rangle_{M_n(A)}.
$$

Therefore, $\kappa(t) \in \mathcal{L}_{M_n(A)}(X^n)$ and $\kappa(t)^* = \kappa(t^*)$. It is now easily checked that $\kappa$ is in fact an injective $*$-homomorphism. Thus, to be done, we only need to show that $\kappa$ is surjective. We establish some notation first. For any $x \in X$ and any $j \in \{1, \ldots, n\}$, we denote by $\delta_j x$ the element of $X^n$ with $x$ in the $j$-th coordinate and zero elsewhere. Thus,

$$
(x_1, \ldots, x_n) = \sum_{j=1}^{n} \delta_j x_j.
$$

Now, take any $s \in \mathcal{L}_{M_n(A)}(X^n)$. We have linear maps $s_1, \ldots, s_n : X^n \to X$ such that

$$
s(x_1, \ldots, x_n) = (s_1(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n)).
$$

For each $i, j \in \{1, \ldots, n\}$, we define a linear map $s_{i,j} : X \to X$ by letting $s_{i,j}(x) = s_i(\delta_j x)$ for any $x \in X$. Therefore,

$$
s(x_1, \ldots, x_n) = \sum_{j=1}^{n} (s_{1,j}(x_j), \ldots, s_{n,j}(x_j)).$$


Notice that since $s$ is adjointable, we have a map $s^*: X^* \to X^{**}$, which in turn gives, for each $i,j \in \{1, \ldots, n\}$, a linear map $(s^*)_{i,j}: X \to X$. The equation

$$
\langle s(x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_{M_n(A)} = \langle (x_1, \ldots, x_n), s^*(y_1, \ldots, y_n) \rangle_{M_n(A)},
$$

becomes

$$
(2.3) \quad \left\langle \sum_{k=1}^{n} s_{i,k}(x_k), y_j \right\rangle_{A_{i,j}} = \left\langle x_i, \sum_{k=1}^{n} (s^*)_{j,k}(y_k) \right\rangle_{A_{i,j}}.
$$

In particular, let $l, m \in \{1, \ldots, n\}$ be such that $l \neq m$. Take any $x, y \in X$ and notice that the $(l, l)$ entry in (2.3), applied to the elements $\delta_l x \in X^n$ and $\delta_l y \in X^n$, becomes the equation

$$
\langle s_{l,m}(x), y \rangle_A = \langle 0, (s^*)_{l,l}(y) \rangle_A = 0.
$$

Thus, for each $l, m \in \{1, \ldots, n\}$ with $l \neq m$, we have shown that $s_{l,m} = 0$. An analogous computation also shows that $(s^*)_{l,m} = 0$ when $l \neq m$. Then, (2.3) implies

$$
\langle s_{i,m}(x_i), y_j \rangle_{A} = \langle x_i, (s^*)_{j,j}(y_j) \rangle_{A}
$$

for all $i,j \in \{1, \ldots, n\}$. It now follows at once that, for all $i \in \{1, \ldots, n\}$, $s_{i,i} \in \mathcal{L}_A(X)$ with $(s^*)_{i,i} = (s^*)_{i,i}^*$. Furthermore, this also proves that $s_{i,i} = s_{j,j}$ for all $i,j \in \{1, \ldots, n\}$. It is now clear that $\kappa(s_{i,i}) = s$ for any $i \in \{1, \ldots, n\}$, which finishes the proof.

\begin{remark}
On page 39 of [14] it is claimed, with no proof, that $\mathcal{L}_A(X^n) \cong \mathcal{L}_{M_n(A)}(X^n)$. Proposition 2.6 shows that the claim is false in general. Indeed, if $A = X = \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 2}$, then it is clear that $\mathcal{L}_C(C^n) \cong M_n(C)$. However, by Proposition 2.6 we have $\mathcal{L}_{M_n(C)}(C^n) \cong \mathcal{L}_C(C) \cong C$.
\end{remark}

\begin{proposition}
Let $X$ be the right Hilbert $A$-module described in Example 2.7 above. Suppose that $XH_0$ is dense in $\mathcal{H}_1$. Define $B \subseteq \mathcal{L}(\mathcal{H}_1)$ by

$$
B = \{ b \in \mathcal{L}(\mathcal{H}_1) : bx, b^*x \in X \text{ for all } x \in X \}.
$$

For each $b \in B$ we get a map $\tau(b): X \to X$, given by $\tau(b)(x) = bx$. Then, $B$ is $\ast$-isomorphic to $\mathcal{L}_A(X)$, via the map that sends $b \in B$ to $\tau(b)$.
\end{proposition}

\begin{proof}
For any $b \in B$ and any $x, y \in X$, we have

$$
\langle bx, y \rangle_A = (bx)^*y = x^*(b^*y) = \langle x, b^*y \rangle_A.
$$

Thus, $\tau(b) \in \mathcal{L}_A(X)$ and $\tau(b)^* = \tau(b^*)$. It is also easily checked that $\tau$ is $\ast$-homomorphism. Furthermore, it follows from density of $XH_0$ in $\mathcal{H}_1$ that $\tau$ is also injective.

We will finish the proof if we show that $\tau$ is surjective. Take any $t \in \mathcal{L}_A(X)$. We have maps $t: X \to X$ and $t^*: X \to X$ satisfying

$$
(2.4) \quad t(x)^*y = x^*t^*(y)
$$

for all $x, y \in X$. Thus, if $n \in \mathbb{Z}_{\geq 1}$, $x_1, \ldots, x_n \in X$, and $\xi_1, \ldots, \xi_n \in \mathcal{H}_0$, then we find, using (2.4) at the final step,

$$
(2.5) \quad \left\| \sum_{j=1}^{n} t(x_j)\xi_j \right\|^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} \langle t(x_j)\xi_j, t(x_i)\xi_i \rangle = \sum_{j=1}^{n} \sum_{i=1}^{n} \langle x_j^*(t^*t)(x_j)\xi_j, \xi_i \rangle.
$$
Recall from Proposition 2.6 that $X^n$ can be viewed as a Hilbert $M_n(A)$-module and that $L_A(X) \cong L_{M_n(A)}(X^n)$ via the map $t \mapsto \kappa(t)$. Applying Lemma 2.5 to $\kappa(t) \in L_{M_n(A)}(X^n)$, we get

\[(2.6) \quad \langle x_i^* (t^* t)(x_j) \rangle_{i,j} = \langle \langle x_i^* (t^* t)(x_j) \rangle \rangle_{i,j} \leq \|\kappa(t)\|^2 \langle \langle x_i x_j \rangle \rangle_{i,j} = \|t\|^2 \langle \langle x_i^* x_j \rangle \rangle_{i,j}.\]

Therefore, if we let $\xi = (\xi_1, \ldots, \xi_n) \in H^n_0$, and consider the obvious action of $M_n(A)$ on $H^n_0$, then we get, using (2.6) at the second step,

\[
\sum_{j=1}^n \sum_{i=1}^n \langle x_i^* (t^* t)(x_j) \rangle_{i,j} \xi_i = \langle \langle x_i^* (t^* t)(x_j) \rangle_{i,j} \rangle_{i,j} \xi,
\]

\[
\leq \|t\|^2 \sum_{j=1}^n \sum_{i=1}^n \langle x_j \xi_j, x_i \xi_i \rangle
\]

\[
= \|t\|^2 \left\| \sum_{j=1}^n x_j \xi_j \right\|^2.
\]

This, together with (2.5), shows that

\[(2.7) \quad \left\| \sum_{j=1}^n t(x_j) \xi_j \right\| \leq \|t\| \left\| \sum_{j=1}^n x_j \xi_j \right\|.
\]

We can now define $b_t : \mathcal{H}_0 \to \mathcal{H}_1$ by letting $b_t(x) = t(x) \xi \in \mathcal{H}_1$, and extending linearly to all of $\mathcal{X}\mathcal{H}_0$. That is,

\[b_t \left( \sum_{j=1}^n x_j \xi_j \right) = \sum_{j=1}^n t(x_j) \xi_j.
\]

Notice that (2.7) shows that $b_t$ is well defined and that $\|b_t(\eta)\| \leq \|t\| \|\eta\|$, for all $\eta \in \mathcal{X}\mathcal{H}_0 = \text{span}\{x \xi : x \in X \text{ and } \xi \in \mathcal{H}_0\}$. Thus, we extend $b_t$ by continuity to all of $\mathcal{H}_1$, and get a well defined map $b_t \in L(\mathcal{H}_1)$ such that $\|b_t(\eta)\| \leq \|t\| \|\eta\|$ for all $\eta \in \mathcal{H}_1$. Let $x \in X$. Since for all $\xi \in \mathcal{H}_0$, we have $b_t(x) \xi = b_t(x) \xi = t(x) \xi$, it follows that $b_t(x) = t(x) \in X$. Similarly, for any $x, y \in X$, we have $x^* t(y) = x^* b_t y = (b_t^* x)^* y$ and therefore

\[\langle t^* (x), y \rangle_A = \langle x, t(y) \rangle_A = x^* t(y) = (b_t^* x)^* y = \langle b_t^* x, y \rangle_A.
\]

Hence, $b_t^* x = t^* (x) \in X$. Thus, $b_t \in B$ and since $\tau(b_t)(x) = b_t x = t(x)$, surjectivity of $\tau$ now follows, finishing the proof.

3. REPRESENTATIONS OF HILBERT BIMODULES AND HILBERT MODULES

The main purpose of this section is to state known results for representations of Hilbert modules and bimodules. Our main contribution here is Proposition 3.10 where we use Proposition 2.6 and Proposition 2.8 above to characterize the ad-jointable and the compact-module maps for a representation of a right Hilbert module. Such representations are guaranteed to exist by Corollary 3.8 below. Roughly, this corollary states that for any right Hilbert $A$-module $X$, there are Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and an isometric linear map $\pi_X : X \to L(\mathcal{H}_0, \mathcal{H}_1)$ such that $\pi_X(X)$ has the right Hilbert module structure from Example 2.4 above. First, we need to recall the concept of Hilbert bimodules and their representations.
**Definition 3.1.** Let $A$ and $B$ be $C^*$-algebras. A Hilbert $A$-$B$-bimodule is a complex vector space $X$ that is a left Hilbert $A$-module and a right Hilbert $B$-module (see Notation 1.1) such that for all $x, y, z \in X$,

$$A \langle x, y \rangle z = x \langle y, z \rangle_B.$$  

(3.1)

**Remark 3.2.** For the definition of Hilbert $A$-$B$-bimodule, some authors also require that $A$ acts on $X$ via $\langle -, - \rangle_B$-adjointable operators and $B$ acts on $X$ via $\langle -, - \rangle_A$-adjointable operators. That is, for all $a \in A$, $b \in B$, and $x, y \in X$ the following holds $\langle ax, y \rangle_B = \langle x, a^* y \rangle_B$ and $A \langle x, yb \rangle = A \langle xb^*, y \rangle$. However, this is redundant as it already follows from (3.1); see comments after Remark 1.9 in [1].

We now turn to representations of Hilbert bimodules. The following comes mostly from Definition 4.5 in [5].

**Definition 3.3.** Let $X$ be a Hilbert $A$-$B$-bimodule. A representation of $X$ on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a triple $(\lambda_A, \rho_B, \pi_X)$, where $\lambda_A$ is a representation of $A$ on $\mathcal{H}_1$, $\rho_B$ is a representation of $B$ on $\mathcal{H}_0$, and $\pi_X : X \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A, b \in B$, and $x, y \in X$, the following compatibility conditions are satisfied.

1. $\pi_X(ax) = \lambda_A(a) \pi_X(x)$,
2. $\pi_X(xb) = \pi_X(x) \rho_B(b)$,
3. $\lambda_A(\langle x, y \rangle) = \pi_X(x) \pi_X(y)^*$,
4. $\rho_B(\langle x, y \rangle_B) = \pi_X(x)^* \pi_X(y)$.

If $\pi_X$ is an isometry, we say the representation $(\lambda_A, \rho_B, \pi_X)$ is isometric.

**Remark 3.4.** The map $\pi_X$ in Definition 3.3 is required to be bounded in Definition 4.5 in [5]. However, since both $\lambda_A$ and $\rho_B$ are $\ast$-homomorphisms, boundedness of $\pi_X$ follows either from compatibility condition 3 or 4. Indeed, for instance, compatibility condition 3 gives

$$\|\pi_X(x)\|^2 = \|\pi_X(x) \pi_X(x)^*\| = \|\lambda_A(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2.$$  

Similarly, Proposition 4.6 in [5] shows that $(\lambda_A, \rho_B, \pi_X)$ is an isometric representation of a Hilbert $A$-$B$-bimodule $X$ whenever either $\lambda_A$ or $\rho_B$ is faithful. Indeed, for example, if $\rho_B$ is isometric, then by the compatibility condition 4 we have

$$\|\pi_X(x)\|^2 = \|\pi_X(x)^* \pi_X(x)\| = \|\rho_B(\langle x, x \rangle_B)\| = \|\langle x, x \rangle_B\| = \|x\|^2.$$  

It is also worth mentioning that conditions 1 and 2 are actually redundant for they respectively follow from conditions 3 and 4. See Remark 2.2.7 in [4] for the details.

The following theorem establishes the existence of representations for any Hilbert $A$-$B$-bimodule.

**Theorem 3.5.** Let $A$ and $B$ be $C^*$-algebras, and let $X$ be a Hilbert $A$-$B$-bimodule. Then, for any nondegenerate representation $\rho_B$ of $B$ on a Hilbert space $\mathcal{H}_0$, there is a nondegenerate representation $\lambda_A : A \to \mathcal{L}(\mathcal{H}_1)$ of $A$ on a Hilbert space $\mathcal{H}_1$ and a bounded linear map $\pi_X : X \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, such that $(\lambda_A, \rho_B, \pi_X)$ is a representation of $X$ on $(\mathcal{H}_0, \mathcal{H}_1)$.

**Proof.** See Proposition 4.7 in [5].
Corollary 3.6. Let $A$ and $B$ be $C^*$-algebras, and let $X$ be a Hilbert $A$-$B$-bimodule. Then there is an isometric representation $(\lambda_A, \rho_B, \pi_X)$ of $X$ on some pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$.

Proof. Let $\rho_B : B \rightarrow \mathcal{L}(\mathcal{H}_0)$ be the universal representation of $B$. Then, $\rho_B$ is faithful and nondegenerate. Hence, this follows at once from Theorem 3.5 and Remark 3.3.

We now present the definition for a representation of a right Hilbert module, which comes from looking at the conditions in Definition 3.7 that only deal with the right action and right inner product.

Definition 3.7. Let $A$ be a $C^*$-algebra and let $X$ be a right Hilbert $A$-module. A representation of $X$ on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a pair $(\rho_A, \pi_X)$ such that $\rho_A$ is a representation of $A$ on $\mathcal{H}_0$, and $\pi_X : X \rightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A$, and all $x, y \in X$, the following compatibility conditions are satisfied.

1. $\pi_X(xa) = \pi_X(x)\rho_A(a)$,
2. $\rho_A(x, y)_A = \pi_X(x)^* \pi_X(y)$.

If $\pi_X$ is an isometry, we say the representation $(\rho_A, \pi_X)$ is isometric.

The map $\pi_X$ in Definition 3.7 is always bounded and this follows exactly as in Remark 3.3. Similarly, faithfulness of $\rho_A$ is sufficient for $(\rho_A, \pi_X)$ to be isometric. Also, as mentioned by the end of Remark 3.4 we point out that condition (1) in Definition 3.7 is actually implied by condition (2).

The following result establishes the existence of (isometric) representations for right Hilbert modules.

Corollary 3.8. Let $A$ be a $C^*$-algebra and let $X$ be a right Hilbert $A$-module. Then, for any nondegenerate representation $\rho_A$ of $A$ on a Hilbert space $\mathcal{H}_0$, there are a Hilbert space $\mathcal{H}_1$ and a linear map $\pi_X : X \rightarrow \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ such that $(\rho_A, \pi_X)$ is a representation of $X$ on $(\mathcal{H}_0, \mathcal{H}_1)$ as in Definition 3.7. Furthermore, if $\rho_A$ is faithful, then $(\rho_A, \pi_X)$ is isometric and in this case $\pi_X(X)$ has the right Hilbert $\rho_A(A)$-module structure from Example 2.2.24.

Proof. It is well known that a right Hilbert $A$-module $X$ is also a Hilbert $K_A(X)$-$A$-bimodule. Hence, the desired result follows at once from Theorem 3.5. The isometric part of the statement follows from Remark 3.4.

Remark 3.9. There are at least two different approaches in the current literature to prove Corollary 3.8 that do not depend on Theorem 3.5. Indeed, the first one is to take $\pi_X$ to be the restriction to $X$ of the map $U$ from Theorem 2.6 in [15]. We are thankful to Julian Kranz for pointing out this reference to us. The second one, uses Murphy’s theory on positive definite kernels for Hilbert modules developed in [10], in which $\pi_X$ comes from a Kolmogorov decomposition of the positive definite map $X \times X \rightarrow \mathcal{L}(\mathcal{H}_0)$ given by $(x, y) \mapsto \rho_A((x, y)_A)$. The details of this approach are in Theorem 3.1 in [10], we are thankful to the referee for pointing out this result.

We end this section by observing that our main results from Section 2 can be stated using the language of Definition 3.7.

Proposition 3.10. Let $A$ be a $C^*$-algebra, let $X$ be any right Hilbert $A$-module, and let $(\rho_A, \pi_X)$ be a representation of $X$ on $(\mathcal{H}_0, \mathcal{H}_1)$, with $\rho_A$ faithful. Suppose
that $\pi_X(\mathcal{H})_0$ is dense in $\mathcal{H}_1$. Then, the $C^*$-algebras $K_A(X)$ and $L_A(X)$ can be represented on $\mathcal{H}_1$ via the maps described below.

1. There is a $*$-isomorphism from $K_A(X)$ to

$$\text{span}\{\pi_X(x)\pi_X(y)^*: x, y \in X\} \subseteq L(\mathcal{H}_1),$$

which sends $\theta_{x,y}$ to $\pi_X(x)\pi_X(y)^*$ for $x, y \in X$.

2. We define

$$B = \{b \in L(\mathcal{H}_1): b\pi_X(x), b^*\pi_X(x) \in \pi_X(X) \text{ for all } x \in X\}.$$

For each $b \in B$ we get a map $\tau(b): \pi_X(X) \to \pi_X(X)$, which sends $\pi_X(x)$ to $\tau(b)(\pi_X(x)) = b\pi_X(x)$. Then, $B$ is $*$-isomorphic to $L_A(X)$, via the map that sends $b \in B$ to $\pi_X^{-1} \circ \tau(b) \circ \pi_X$, where $\pi_X^{-1}$ is interpreted as the inverse of the linear bijection $\pi_X: X \to \pi_X(X)$.

**Proof.** Since $\rho_A$ is faithful, $\pi_X$ is isometric. The result now follows immediately after replacing $A$ with its isometric copy $\rho_A(A)$ and $X$ with its isometric copy $\pi_X(X)$ on Proposition 2.4 for part (1), and on Proposition 2.8 for part (2).

4. REPRESENTATIONS OF $C^*$-CORRESPONDENCES

In this section we define representations of $C^*$-correspondences and present the main result of this paper, Theorem 4.3, which we will see is actually a generalization of Theorem 3.5. We then give two applications of this theorem. The first, contained in Theorem 4.8 and Corollary 4.11, gives necessary and sufficient conditions for a general $(A, B)$ $C^*$-correspondence to admit a Hilbert $A$-$B$-bimodule structure. The second, given in Theorem 4.14, shows that the interior tensor product of correspondences admits a representation as the product of suitable representations of the factors.

4.1. Definitions. We start by recalling the definition of a $C^*$-correspondence.

**Definition 4.1.** Let $A$ and $B$ be $C^*$-algebras. An $(A, B)$ $C^*$-correspondence is a pair $(X, \varphi)$, where $X$ is a right Hilbert $B$-module and $\varphi: A \to L_B(X)$ is a $*$-homomorphism. We say that $A$ acts 	extit{nondegenerately} on $X$ whenever $\varphi(A)X$ is dense in $X$.

We observe that Remark 3.2 implies that any Hilbert $A$-$B$-bimodule, as in Definition 3.1, is in fact an $(A, B)$ $C^*$-correspondence with $\varphi$ given by the left action of the bimodule. In fact, if $X$ is Hilbert $A$-$B$-bimodule, then it is well known that $A$ always acts nondegenerately on $X$. However, not every $C^*$-correspondence is a Hilbert bimodule. Thus, $C^*$-correspondences are a generalization of Hilbert bimodules. Our goal is then to find a general version of Theorem 3.5 that works for the general $C^*$-correspondence setting. For this, we need first to define what we mean by representations of $C^*$-correspondences.

**Definition 4.2.** Let $A$ and $B$ be $C^*$-algebras, and let $(X, \varphi)$ be an $(A, B)$ $C^*$-correspondence. A representation of $(X, \varphi)$ on a pair of Hilbert spaces $(\mathcal{H}_0, \mathcal{H}_1)$ consists of a triple $(\lambda_A, \rho_B, \pi_X)$ where $\lambda_A$ is a representation of $A$ on $\mathcal{H}_1$, $\rho_B$ is a representation of $B$ on $\mathcal{H}_0$, and $\pi_X: X \to L(\mathcal{H}_0, \mathcal{H}_1)$ is a linear map, such that for all $a \in A$, and all $x, y \in X$, the following compatibility conditions are satisfied.

1. $\pi_X(\varphi(a)x) = \lambda_A(a)\pi_X(x),$
2. $\pi_X(xb) = \pi_X(x)\rho_B(b),$
Theorem 4.3. Let degenerate representation of $A, B$ how to produce a representation of any $(X, \varphi)$ on $\rho(X(x)^*\pi_X(y))$.

If $\pi_X$ is an isometry, we say the representation $(\lambda_A, \rho_B, \pi_X)$ is isometric.

As in Remark 3.3 the linear map $\pi_X$ from Definition 3.2 is automatically bounded and faithfulness of $\rho_B$ is sufficient for $(\lambda_A, \rho_B, \pi_X)$ to be isometric. Similarly, condition (2) in Definition 3.2 is automatically implied by condition (3).

We point out that Definition 4.2 agrees with the definitions of representations of $C^*$-correspondences in the literature. Indeed, suppose that $(X, \varphi)$ is an $(A, A)$ $C^*$-correspondence and that $(\lambda_A, \rho_A, \pi_X)$ is a representation of $(X, \varphi)$ as in Definition 4.2 with $H_0 = H_1$ and $\lambda_A = \rho_A$. Then, $(\lambda_A, \pi_X)$ is a representation of $(X, \varphi)$ on $\mathcal{L}(H_0)$ in the sense of Definition 2.1 in [8] and an isometric covariant representation of $(X, \varphi)$ on $H_0$ in the sense of Definition 2.11 in [11]. More generally, suppose that $(X, \varphi)$ is an $(A, B)$ $C^*$-correspondence and that $(\lambda_A, \rho_B, \pi_X)$ is a representation of $(X, \varphi)$ on $(H_0, H_1)$ as in Definition 4.2. Then, letting $C = \mathcal{L}(H_0 \otimes H_1)$, we get obvious maps $\lambda_A: A \to C, \hat{\rho}_B: B \to C$, and $\pi_X: X \to C$ induced by $\lambda_A, \rho_B,$ and $\pi_X$. It is clear that $(\lambda_A, \hat{\rho}_B, \pi_X)$ is a rigged representation of $(X, \varphi)$ on $C$ in the sense of Definition 3.7 in [8].

4.2. Interior tensor product of $C^*$-correspondences. For the rest of this section we will need the interior tensor product of $C^*$-correspondences. This is a well known construction. We only list some of the basic properties that will be needed below. We refer the reader to Proposition 4.5 in [9] and the afterwards discussion for more details. Let $A, B,$ and $C$ be $C^*$-algebras, let $(X, \varphi_X)$ be an $(A, B)$ $C^*$-correspondence and let $(Y, \varphi_Y)$ be a $(B, C)$ $C^*$-correspondence. We consider the algebraic $B$-balanced tensor product of modules $X \otimes_B Y$ which has a $C$-valued right pre-inner product given on elementary tensors by

$$(x_1 \otimes y_1, x_2 \otimes y_2)_C = \langle y_1, \varphi_Y((x_1, x_2)_B)y_2\rangle_C$$

The completion of $X \otimes_B Y$ under the norm induced by the $C$-valued right pre-inner product from equation (4.1) is a right Hilbert $C$-module, which we denote by $X \otimes_{\varphi_Y} Y$. It is useful to keep in mind that, by construction, if $x \in X, b \in B,$ and $y \in Y$, then

$$xb \otimes y = x \otimes \varphi_Y(b)y.$$

Furthermore, $A$ acts on $X \otimes_{\varphi_Y} Y$ via $(-, -)_C$-adjointable operators and the action $\bar{\varphi}_X: A \to \mathcal{L}_C(X \otimes_{\varphi_Y} Y)$ is determined by $\varphi_X$ as follows:

$$\bar{\varphi}_X(a)(x \otimes y) = \varphi_X(a)x \otimes y,$$

for $a \in A$, $x \in X$, and $y \in Y$. All this makes $(X \otimes_{\varphi_Y} Y, \bar{\varphi}_X)$ into an $(A, C)$ $C^*$-correspondence, called the interior tensor product of $(X, \varphi_X)$ with $(Y, \varphi_Y)$.

4.3. Main results. For any two $C^*$-algebras $A$ and $B$, the following result shows how to produce a representation of any $(A, B)$ $C^*$-correspondence out of any non-degenerate representation of $B$.

**Theorem 4.3.** Let $A$ and $B$ be $C^*$-algebras and let $(X, \varphi)$ be an $(A, B)$ $C^*$-correspondence. Then, for any non-degenerate representation $\rho_B$ of $B$ on a Hilbert space $H_0$, there are representations $\lambda_A: A \to \mathcal{L}(H_1)$ of $A$ on a Hilbert space $H_1$ and a bounded linear map $\pi_X: X \to \mathcal{L}(H_0, H_1)$, such that $(\lambda_A, \rho_B, \pi_X)$ is a representation of $(X, \varphi)$ on $(H_0, H_1)$ as in Definition 4.2. If in addition $A$ acts non-degenerately on $X$, then $\lambda_A$ is non-degenerate.
Proof. Notice that \((\mathcal{H}_0, \rho_B)\) is a \((B, \mathbb{C})\) \(C^*\)-correspondence. Let \(\mathcal{H}_1 = X \otimes_{\rho_B} \mathcal{H}_0\) be the interior tensor product of \((X, \varphi)\) with \((\mathcal{H}_0, \rho_B)\), which is in particular a right Hilbert \(C\)-module, that is, a Hilbert space. The representation of \(A\) on \(\mathcal{H}_1\) comes from the left action of \(A\) on \(\mathcal{H}_1\) gotten from equation (4.3) in the interior tensor product construction. Indeed, Proposition 2.66 of \cite{14} gives \(\lambda_A : A \to \mathcal{L}(\mathcal{H}_1)\), a representation of \(A\), such that for each \(a \in A\), each \(x \in X\), and each \(\xi \in \mathcal{H}_0\),

\[\lambda_A(a)(x \otimes \xi) = \varphi(a)x \otimes \xi.\]

Furthermore, it is also shown in Proposition 2.66 of \cite{14} that \(\lambda_A\) is nondegenerate whenever \(A\) acts nondegenerately on \(X\). We now establish the existence of \(\pi_X\). This is motivated by the Fock space construction in \cite{13}. Indeed, for each \(x \in X\), let \(\pi_X(x) : \mathcal{H}_0 \to \mathcal{H}_1\) be the creation operator

\[\pi_X(x)\xi = x \otimes \xi.\]

Then, it is clear that \(x \mapsto \pi_X(x)\) is a linear map from \(X\) to \(\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)\). As in Remark 4.4, boundedness of \(\pi_X\) will follow once we check the compatibility conditions from Definition 3.5, which will in turn prove that \((\lambda_A, \rho_B, \pi_X)\) is indeed a representation of \((X, \varphi)\) on \((\mathcal{H}_0, \mathcal{H}_1)\). First we check condition (1). If \(a \in A, x \in X,\) and \(\xi \in \mathcal{H}_0,\) then

\[\pi_X(\varphi(a)x)\xi = (\varphi(a)x) \otimes \xi = \lambda_A(a)(x \otimes \xi) = \lambda_A(a)\pi_X(x)\xi.\]

That is, \(\pi_X(\varphi(a)x) = \lambda_A(a)\pi_X(x)\) as desired. Now for \(b \in B, x \in X,\) and \(\xi \in \mathcal{H}_0,\) we use equation (4.2) at the second step and find

\[\pi_X(xb)\xi = (xb) \otimes \xi = x \otimes \rho_B(b)\xi = \pi_X(x)(\rho_B(b)\xi).\]

This gives that \(\pi_X(xb) = \pi_X(x)\rho_B(b)\), proving condition (2). Finally, notice that equation (4.1) shows that \(\pi_X(x)^* : \mathcal{H}_1 \to \mathcal{H}_0\) is the annihilation operator satisfying, for any \(z \in X,\) and \(\xi \in \mathcal{H}_0,\)

\[\pi_X(x)^*(z \otimes \xi) = \rho_B(\langle x, z \rangle_B)\xi.\]

Thus, for any \(\xi \in \mathcal{H}_0,\)

\[\pi_X(x)^*\pi_X(y)\xi = \pi_X(x)^*(y \otimes \xi) = \rho_B(\langle x, y \rangle_B)\xi,\]

whence \(\pi_X(x)^*\pi_X(y) = \rho_B(\langle x, y \rangle_B)\), which is compatibility condition (3), so we are done. \hfill \blacksquare

The method we used in the proof of Theorem 4.3 can be easily adapted to produce a different proof of Theorem 3.5. Thus, we present below a restatement of Theorem 3.5 followed by a proof along the lines of the proof of Theorem 4.3.

Theorem 4.4. Let \(A\) and \(B\) be \(C^*\)-algebras, and let \(X\) be a Hilbert \(A\)-\(B\)-bimodule. Then, for any nondegenerate representation \(\rho_B\) of \(B\) on a Hilbert space \(\mathcal{H}_0\), there are a nondegenerate representation \(\lambda_A : A \to \mathcal{L}(\mathcal{H}_1)\) of \(A\) on a Hilbert space \(\mathcal{H}_1\) and a bounded linear map \(\pi_X : X \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)\), such that \((\lambda_A, \rho_B, \pi_X)\) is a representation of \(X\) on \((\mathcal{H}_0, \mathcal{H}_1)\) as in Definition 3.5.

Proof. We get the Hilbert space \(\mathcal{H}_1 = X \otimes_{\rho_B} \mathcal{H}_0\) and the map \(\pi_X : X \to \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)\) exactly as in the proof of Theorem 4.3. Since \(A\) acts on \(X\) via \((-,-)_B\)-adjointable operators (see Remark 3.2), we use Proposition 2.66 of \cite{14} to get \(\lambda_A : A \to \mathcal{L}(\mathcal{H}_1)\), a representation of \(A\), such that for each \(a \in A\), each \(x \in X\), and each \(\xi \in \mathcal{H}_0,\)

\[\lambda_A(a)(x \otimes \xi) = (ax) \otimes \xi.\]
Furthermore, since $X$ is a left Hilbert $A$-module, it follows that $A$ acts nondegenerately on $X$. Thus, Proposition 2.66 of [14] also guarantees that $\lambda_A$ is nondegenerate. Finally, compatibility condition (1) from Definition 3.3 is shown exactly as compatibility condition (1) from Definition 4.2 was shown in the proof of Theorem 4.3. Since compatibility conditions (2) and (4) from Definition 3.3 coincide with compatibility conditions (2) and (3) from Definition 4.2, we only need to make sure that compatibility condition (3) of Definition 3.3 is satisfied. Indeed, for any $x, y, z \in X$, and any $\xi \in H_0$, using equation (4.6) at the first step, equation (4.2) at the second step, and equation (3.1) at the third one, we get

$$\pi_X(x)\pi_X(y)^*(z \otimes \xi) = x \otimes \rho_B(y, z)B\xi$$

$$= x\langle y, z \rangle_B \otimes \xi$$

$$= A\langle x, y \rangle z \otimes \xi$$

$$= \lambda_A(A\langle x, y \rangle)(z \otimes \xi).$$

Thus, $\pi_X(x)\pi_X(y)^* = \lambda_A(A\langle x, y \rangle)$, as wanted. ■

We give three remarks about the last two results. On those we explain how these results are useful to apply the main result of Section 3 and also how the proofs of these theorems compare to those known in the current literature.

Remark 4.5. The construction of the Hilbert space $H_1$ given in the proofs of Theorems 4.3 and 4.4 above clearly implies that $\pi_X(X)H_0$ is dense in $H_1$. This is also true for the Hilbert space $H_1$ obtained from Theorem 3.5. Indeed, according to the proof of Proposition 4.7 in [5], the space $H_1$ is defined as follows. Let $L$ be the linking algebra of $X$ and $\iota: X \to L$ the inclusion of $X$ as the upper right corner of $L$. Then, $H_1$ is defined as the closure of $\pi(\iota(X))H_0$, where $\pi$ is a suitable representation of $L$ on a Hilbert space $H$ that contains $H_0$. For each $x \in X$, the operator $\pi_X(x) \in L(H_0, H_1)$ is then defined as the restriction of $\pi(\iota(x))$ to $H_0$, so it follows that $\pi_X(X)H_0$ is dense in $H_1$. This shows that the map $\pi_X$, no matter from which construction presented so far was obtained, satisfies the nondegeneracy condition on the hypothesis of Proposition 3.10.

Remark 4.6. The proof of Theorem 3.5 given in [5] does not appear to have an obvious modification to make it work for the C*-correspondence case. This is due to the fact that their proof relies on the linking algebra of the bimodule $X$, which does not exist in the general C*-correspondence setting due to the lack of an $A$-valued left inner product. We also believe that the arguments used in [15] to prove our Corollary 3.3 can’t be modified to produce a proof of Theorem 4.9. Finally, we point out that the methods we employed to show Theorem 4.3 differ from those used in [15] and [5]. In particular, we have obtained in equation (4.5) a concise formula for $\pi_X$ that might be useful to produce concrete representations of both Hilbert bimodules and modules.

Remark 4.7. It follows from the definition of $\lambda_A$ in (4.4) that if $\varphi$ is not injective, then $\lambda_A$ is not faithful. The converse holds provided that the representation $\rho_B$ in the hypotheses of Theorem 4.3 (or in Theorem 4.4) is faithful. Indeed, suppose that $\varphi$ is injective and assume, for the sake of a contradiction, that there is a nonzero $a \in A$ such that $\lambda_A(a) = 0$. Since $\varphi$ is injective, there is a nonzero $x \in X$ such that $\varphi(a)x \neq 0$ (for the Hilbert bimodule case we interpret $\varphi(a)x$ as $ax$). We can find
nonzero elements \( y \in X \) and \( b \in B \) such that \( x = yb \) (see for example Proposition 2.31 in [14]). Then, for any \( \xi \in \mathcal{H}_0 \), using (1.2) at the third step,
\[
0 = \lambda_A(a)(x \otimes \xi) = \varphi(a)y b \otimes \xi = \varphi(a)y \otimes \varphi_B(b)\xi.
\]
Since \( \varphi(a)x \neq 0 \), it follows that \( \varphi(a)y \neq 0 \) and therefore \( \varphi_B(b)\xi = 0 \) for all \( \xi \in \mathcal{H}_0 \). Hence, faithfulness of \( \varphi_B \) implies that \( b = 0 \), a contradiction.

We warn the reader that Remark 4.7 above was mistakenly stated in Remark 3.3.6. in [1] for it required \( \varphi \) to be nondegenerate rather than injective. We thank Menevse Eryuzlu for pointing this out to us.

We now present two applications of Theorem 4.3. The first application answers the problem of determining when an \((A, B)\) C*-correspondence can be uniquely given the structure of a Hilbert \(A-B\)-bimodule.

**Theorem 4.8.** Let \( A \) and \( B \) be C*-algebras, let \((X, \varphi)\) be an \((A, B)\) C*-correspondence, let \( A_0 = A/\ker(\varphi) \), and let \( \varphi_0 : A_0 \to \mathcal{L}_B(X) \) be the injective *-homomorphism induced by \( \varphi \), which makes \((X, \varphi_0)\) an \((A_0, B)\) C*-correspondence. Then, there is a unique \( A_0 \)-valued left inner product on \( X \) making it a Hilbert \( A_0-B\)-bimodule if and only if \( \mathcal{K}_B(X) \subseteq \varphi(A) \).

**Proof.** First we establish the uniqueness of the \( A_0 \)-valued left inner product on \( X \). Let \( \rho_B : B \to \mathcal{L}(\mathcal{H}_0) \) be the universal representation of \( B \), which is faithful, and apply Theorem 4.3 to get an isometric representation \((\lambda_{A_0}, \rho_B, \pi_X)\) for the \((A_0, B)\) C*-correspondence \((X, \varphi_0)\). Since \( \varphi_0 \) and \( \rho_B \) are injective, Remark 4.7 shows that \( \lambda_{A_0} \) is injective. Suppose now that \( A_0 \langle \cdot, \cdot \rangle : X \times X \to A_0 \) and \( A_0 \langle \cdot | \cdot \rangle : X \times X \to A_0 \) are two \( A_0 \)-valued left inner products making \( X \) a Hilbert \( A_0-B\)-bimodule. The proof of Theorem 4.4 now shows that, for every \( x, y \in X \),
\[
\lambda_{A_0}(A_0 \langle x, y \rangle) = \pi_X(x)\pi_X(y)^* = \lambda_{A_0}(A_0 \langle x | y \rangle).
\]
Since \( \lambda_{A_0} \) is injective, the above implies that \( A_0 \langle x, y \rangle = A_0 \langle x | y \rangle \), as wanted.

Next, assume that there is \( A_0 \)-valued left inner product on \( X \) making it a Hilbert \( A_0-B\)-bimodule. That is, there is a map \( A_0 \langle \cdot, \cdot \rangle : X \times X \to A_0 \) such that for any \( x, y, z \in X \) we have \( \varphi_0(\langle x, y \rangle)z = \langle y, z \rangle_B \). Since \( \langle x, y \rangle_B = \theta_{x,y}(z) \), this proves \( \theta_{x,y} \in \varphi_0(A_0) = \varphi(A) \) for any \( x, y \in X \), which in turn implies \( \mathcal{K}_B(X) \subseteq \varphi(A) \).

Conversely, assume that \( \mathcal{K}_B(X) \subseteq \varphi(A) \). Then \( \theta_{x,y} \in \varphi(A) = \varphi_0(A_0) \) for each \( x, y \in X \). Thus, we use the *-isomorphism \( \varphi_0^{-1} : \varphi_0(A_0) \to A_0 \) to define
\[
A_0 \langle x, y \rangle = \varphi_0^{-1}(\theta_{x,y}).
\]
It is immediate to check that \( (x, y) \mapsto A_0 \langle x, y \rangle \) is indeed an \( A_0 \)-valued left inner product on \( X \). Furthermore, if \( x \in X \), then
\[
\|A_0 \langle x, x \rangle\| = \|\varphi_0^{-1}(\theta_{x,x})\| = \|\theta_{x,x}\| = \|x\|.
\]
Hence, \( X \) is indeed a left Hilbert \( A_0 \)-module. Finally, by (1.7),
\[
\varphi_0(A_0 \langle x, y \rangle)z = \theta_{x,y}z = \langle y, z \rangle_B,
\]
for all \( x, y, z \in X \), proving equation (3.1) and therefore that \( X \) a Hilbert \( A_0-B\)-bimodule, as wanted.

**Remark 4.9.** If the *-homomorphism \( \varphi : A \to \mathcal{L}_B(X) \) in Theorem 4.8 is injective, then we get at once that \( \mathcal{K}_B(X) \subseteq \varphi(A) \) is a necessary and sufficient condition for the C*-correspondence \((X, \varphi)\) to have a unique structure of a Hilbert \( A-B \)-bimodule.
Remark 4.10. In general, if $(X,\varphi)$ is an $(A,B)$ $C^*$-correspondence where $\varphi$ is not necessarily injective, then $(X,\varphi)$ will have a unique structure of a Hilbert $A$-$B$-bimodule if and only if $A$ has an ideal $J$ such that $J$ is $*$-isomorphic to $K_B(X)$ via $\varphi|_J$. The case $A = B$ was proven by T. Katsura in Lemmas 3.3 and 3.4 from [7]. The general case was later shown by A. Buss and R. Meyer in [2], we are thankful to Ralph Meyer for providing this last reference to us. In Corollary 4.11 we show that the general case (see Lemma 4.2 in [2]) does follow from Theorem 4.8.

Corollary 4.11. Let $A$ and $B$ be $C^*$-algebras and let $(X,\varphi)$ be an $(A,B)$ $C^*$-correspondence. Then, there is a unique $A$-valued left inner product on $X$ making it a Hilbert $A$-$B$-bimodule if and only if $A$ has an ideal $J$ such that $J$ is $*$-isomorphic to $K_B(X)$ via $\varphi|_J$.

Proof. Put $A_0 = A/\ker(\varphi)$. Assume first that $A$ has an ideal $J$ such that $J$ is $*$-isomorphic to $K_B(X)$ via $\varphi|_J$. Then $K_B(X) = \varphi|_J(J) \subseteq \varphi(A)$, so Theorem 4.8 implies that $X$ has a unique left $A_0$-valued inner product, say $A_0(-,-): X \times X \rightarrow A_0$, that makes $X$ into a Hilbert $A_0$-$B$-bimodule. By standard Hilbert module results (see for instance Proposition 1.10 in [1]) the ideal $J_0 = A_0(X,X) \subseteq A_0$, the closed span of the $A_0$-valued inner product, is $*$-isomorphic to $K_B(X) = \varphi|_J(J)$. This implies that the left $J_0$-valued inner product can be thought of as a left $J$-valued inner product, which upgrades $X$ to a Hilbert $A$-$B$-bimodule.

Conversely, if $X$ is a Hilbert $A$-$B$-bimodule, then Proposition 1.10 in [1] shows that $J_A = A(X,X) \subseteq A$, the closed span of the left $A$-valued inner product, is an ideal in $A$ which is $*$-isomorphic to $K_B(X)$ via the restriction of the left action of $A$ to $J_A$.

Finally, the uniqueness of the left $A$-valued inner product is established as in the proof of Theorem 4.8 but this time applying Theorem 4.3 to the $(J,B)$ $C^*$-correspondence $(X,\varphi|_J)$. ■

As an application of Theorem 4.8 we give below an easy proof that a Hilbert space of dimension at least 2, thought of as a $(\mathbb{C},\mathbb{C})$ $C^*$-correspondence, can’t be given the structure of a Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodule.

Example 4.12. Let $\mathcal{H}$ be a Hilbert space with dimension at least 2. Clearly $\mathcal{H}$ is a right Hilbert $\mathbb{C}$-module, $\mathcal{L}_\mathbb{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})$, and $K_\mathbb{C}(\mathcal{H}) = K(\mathcal{H})$. For each $a \in \mathbb{C}$, we define $\varphi(a) = a \cdot \text{id}_\mathcal{H}$. Then, $\varphi: \mathbb{C} \rightarrow \mathcal{L}(\mathcal{H})$ makes $(\mathcal{H},\varphi)$ into a $(\mathbb{C},\mathbb{C})$ $C^*$-correspondence with injective left action. Furthermore, it’s clear that $\varphi(\mathbb{C})\mathcal{H} = \mathcal{H}$, so the left action is nondegenerate. We claim that $K(\mathcal{H}) \not\subseteq \varphi(\mathbb{C})$. Let $(\xi_j)_{j \in J}$ be an orthonormal basis for $\mathcal{H}$. By assumption $\text{card}(J) \geq 2$ and therefore we can find $j,k \in J$ with $j \neq k$. Notice that $\theta_{\xi_j}\xi_k(\xi_k) = \xi_j$, whence $\theta_{\xi_j}\xi_k \neq \varphi(a)$ for all $a \in \mathbb{C}$, proving the claim. Therefore, Theorem 4.8 implies that $(\mathcal{H},\varphi)$ can’t be given the structure of a Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodule.

Remark 4.13. A direct sum of Hilbert $A$-$B$-bimodules is not, in general, a Hilbert bimodule again. However, it is an $(A,B)$ $C^*$-correspondence. It is not hard to see that the $C^*$-correspondence in Example 4.12 is a direct sum of Hilbert $\mathbb{C}$-$\mathbb{C}$-bimodules. We have not investigated which $C^*$-correspondences can be decomposed as a direct sum of Hilbert bimodules, but we believe Theorem 4.8 and Corollary 4.11 might be useful tools to tackle this problem.

As an application of concrete representations of right Hilbert modules (as in Definition 3.7), Murphy gives an easy construction of the exterior tensor product
of right Hilbert modules (see Theorem 3.2 in [14]). In analogy with Murphy’s result, we conclude the paper with a second application of Theorem 4.3 which deals with how to get a representation for the interior tensor product of an \((A, B)\) C*-correspondence \((X, \varphi_X)\) with a \((B, C)\) C*-correspondence \((Y, \varphi_Y)\) using particular representations of the C*-correspondences \((X, \varphi_X)\) and \((Y, \varphi_Y)\). The main point is that we can always make the representation of the C*-algebra \(B\) from the representation of \((X, \varphi_X)\) agree with the representation of \(B\) from the representation of \((Y, \varphi_Y)\).

**Theorem 4.14.** Let \(A, B,\) and \(C\) be C*-algebras, let \((X, \varphi_X)\) be an \((A, B)\) C*-correspondence, let \((Y, \varphi_Y)\) be a \((B, C)\) C*-correspondence such that \(B\) acts nondegenerately on \(Y\), and let \(\rho_C : C \to \mathcal{L}(H_0)\) be any nondegenerate representation of \(C\) on a Hilbert space \(H_0\). Then:

1. There are Hilbert spaces \(H_1, H_2\), maps \(\lambda_A : A \to \mathcal{L}(H_2)\), \(\sigma_B : B \to \mathcal{L}(H_1)\), \(\tau_X : X \to \mathcal{L}(H_1, H_2)\), and \(\pi_Y : Y \to \mathcal{L}(H_0, H_1)\), such that \((\lambda_A, \sigma_B, \tau_X)\) is a representation of \((X, \varphi_X)\) on \((H_1, H_2)\) and \((\sigma_B, \rho_C, \pi_Y)\) is a representation of \((Y, \varphi_Y)\) on \((H_0, H_1)\).

2. For every pair \(((\lambda_A, \sigma_B, \tau_X), (\sigma_B, \rho_C, \pi_Y))\) as in (1), there is a map \(\pi : X \otimes_{\varphi_Y} Y \to \mathcal{L}(H_0, H_2)\) satisfying \(\pi(x \otimes y) = \tau_X(x)\pi_Y(y)\) and such that the triple \((\lambda_A, \rho_C, \pi)\) is a representation of \((X \otimes_{\varphi_Y} Y, \varphi_X)\) on \((H_0, H_2)\).

3. The map \(\pi\) from (2) is an isomorphism from \(X \otimes_{\varphi_Y} Y\) to \(\tau_X(X)\pi_Y(Y)\).

4. If in addition \(\rho_C\) is also faithful then the representation \((\lambda_A, \rho_C, \pi)\) is isometric.

**Proof.** Since \(\rho_C : C \to \mathcal{L}(H_0)\) is nondegenerate and \(B\) acts nondegenerately on \(Y\), Theorem 4.3 gives a Hilbert space \(H_1\), a nondegenerate representation \(\sigma_B : B \to \mathcal{L}(H_1)\), and a bounded linear map \(\pi_Y : Y \to \mathcal{L}(H_0, H_1)\) such that \((\sigma_B, \rho_C, \pi_Y)\) is a representation of \((Y, \varphi_Y)\) on \((H_0, H_1)\). Hence, a second application of Theorem 4.3 gives a Hilbert space \(H_2\), a representation \(\lambda_A : A \to \mathcal{L}(H_2)\), and a bounded linear map \(\tau_X : X \to \mathcal{L}(H_1, H_2)\) such that \((\lambda_A, \sigma_B, \tau_X)\) is a representation of \((X, \varphi_X)\) on \((H_1, H_2)\). This takes care of part 1. For part 2, we first prove the existence of the map \(\pi\). First, fix \(n \in \mathbb{Z}_{\geq 1}\), \(x_1, \ldots, x_n \in X\), and \(y_1, \ldots, y_n \in Y\). Then, using the fact that \((\sigma_B, \rho_C, \pi_Y)\) is a representation at the second and third steps together with equation (11) at the sixth step we find

\[
\left\| \sum_{j=1}^{n} \tau_X(x_j)\pi_Y(y_j) \right\|^2 = \sup_{\|\xi\|=1} \left\| \sum_{j,k=1}^{n} \langle \tau_X(x_k)\pi_Y(y_k)\xi, \tau_X(x_j)\pi_Y(y_j)\xi \rangle \right\|
\]

\[
= \sup_{\|\xi\|=1} \left\| \sum_{j,k=1}^{n} \langle \pi_Y(y_j)\xi, \tau_X(x_j)\pi_Y(y_j)\xi \rangle \right\|
\]

\[
= \sup_{\|\xi\|=1} \left\| \rho_C(\langle y_j, \varphi_Y(\langle x_j, x_k \rangle) y_k \rangle)\xi, \xi \rangle \right\|
\]

\[
\leq \left\| \sum_{j,k=1}^{n} \rho_C(\langle y_j, \varphi_Y(\langle x_j, x_k \rangle) y_k \rangle) \xi \right\|
\]

\[
\leq \left\| \sum_{j,k=1}^{n} \langle y_j, \varphi_Y(\langle x_j, x_k \rangle) y_k \rangle \right\|
\]
Moreover, a direct computation gives 
\[ \tau_X(xb)\pi_Y(y) = \tau_X(x)\pi_Y(\varphi_Y(b)y) \]
for any \( x \in X, y \in Y, \) and \( b \in B. \) Therefore, we can extend the map \( x \otimes y \mapsto \tau_X(x)\pi_Y(y) \) to a well defined bounded linear map \( \pi : X \otimes_{\varphi X} Y \to \mathcal{L}(H_0, H_2). \) To finish part (2), it only remains to check that \( (\lambda_A, \rho_C, \pi) \) is indeed a representation of the correspondence \( (X \otimes_{\varphi X} Y, \varphi_X) \) on \( (H_0, H_2). \) This follows from three immediate computations on elementary tensors using the fact that both \( (\lambda_A, \sigma_B, \tau_X) \) and \( (\sigma_B, \rho_C, \pi_X) \) are representations. Part (3) is now immediate from definition of \( \pi. \) Finally, to check part (4), Remark 3.4 shows that faithfulness of \( \rho_C \) is enough for \( (\lambda_A, \rho_C, \pi) \) to be isometric, so we are done. \[ \square \]

**Acknowledgments:** The author would like to thank N. Christopher Phillips for his advice the past few years and in particular for reading multiple earlier versions of this paper, always giving useful comments that significantly improved the exposition. He is also grateful to Andrey Blinov for stimulating conversations about the topics covered here, some of which led to developing the theory further than originally intended. The author also thanks Matthew Daws for pointing out typos and in particular for sending useful comments that gave rise to Remarks 2.7 and 4.5. Finally, the author thanks the referee their suggestions, including one dealing with an updated notation for representations on pairs of Hilbert spaces, making it more accessible for future applications.

**References**

[1] L. G. Brown, J. A. Mingo, N.-T. Shen, Quasi-Multipliers and embeddings of Hilbert C*-bimodules, *Can. J. Math.*, 46(6)(1994), 1150–1174.

[2] A. Buss, R. Meyer, Inverse Semigroup Actions on Groupoids, *Rocky Mountain J. Math.*, 47(1)(2017), 53–159.

[3] T. M. Carlsen, A. Dor-On, S. Eilers, Shift equivalences through the lens of Cuntz-Krieger algebras, *Anal. PDE*, to appear.

[4] A. Delfín Ares de Parga, *C*-Correspondences, Hilbert Bimodules, and Their \( L^p \) Versions*, PhD Dissertation, University of Oregon: ProQuest Dissertations and Theses, Eugene OR, 2023.

[5] R. Exel, A Fredholm Operator Approach To Morita Equivalence, *K-Theory*, 7(1993), 285–308.

[6] T. Kajiwara, C. Pinzari, Y. Watatani, Ideal Structure and Simplicity of the C*-Algebras generated by Hilbert Bimodules, *J. Funct. Anal.*, 159(1998), 295–322.

[7] T. Katsura, A construction of C*-algebras from C*-correspondences, In *Advances in Quantum Dynamics (South Hadley, MA, 2002)*, AMS, Providence RI, number 335 in *Contemp. Math.*, (2005), 173–182.

[8] T. Katsura, On C*-algebras associated with C*-correspondences, *J. Funct. Anal.*, 217(2004), 366–401.

[9] E. C. Lance, *Hilbert C*-modules: a toolkit for operator algebras*, Cambridge University Press, Cambridge 1995.

[10] G. J. Murphy, Positive definite kernels and Hilbert C*-modules, *Proc. Edinb. Math. Soc.*, 40(2)(1997), 367–374.

[11] P. S. Muhly, B. Solel, Tensor Algebras over C*-Correspondences: Representations, Dilations, and C*-Envelopes, *J. Funct. Anal.*, 158(1998), 389–457.

[12] N. C. Phillips, Analogs of Cuntz algebras on \( L^p \) spaces, arXiv:1201.4196, [math.FA] 2012.
[13] M. V. Pimsner, A class of $C^*$-algebras generalizing both Cuntz-Krieger algebras and crossed products by $Z$, Free probability theory, Fields Inst. Commun., 12 (1997), 189–212.

[14] I. Raeburn, D. P. Williams, Morita equivalence and continuous trace $C^*$ algebras, American Mathematical Society, Providence RI, 1998.

[15] Zettl, H., A Characterization of Ternary Rings of Operators, Adv. Math., 48 (1983) 117–143.

Department of Mathematics, University of Oregon, Eugene OR 97403-1222, USA, Department of Mathematics, University of Colorado, Boulder CO 80309-0395, USA.

Email address: alonsod@uoregon.edu, alonso.delfin@colorado.edu