Analysis stability of HIV/AIDS epidemic model of different infection stage in closed community

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Abstract. Mathematical modeling can describe about how the epidemic model like virus HIV/AIDS interaction with human, from susceptible individual become individual AIDS. Furthermore, the model is built in the acceleration fraction how fast susceptible individuals can be asymptomatic HIV infected individuals. It can be solved to get the point of free disease and its stability, and also to get endemic point and its stability. The stability for the free disease will get from the Routh’s criterion stability, and for the endemic will be analyzed by Lyapunov function. In the paper, the point of free disease will be asymptotically stable if the basic reproduction of the model less than one, and the point of endemic will stable with Lyapunov if the basic reproduction more than one.

1. Introduction

AIDS (Acquired Immune Deficiency Syndrome), a collection of symptoms that arise due to human immunity is damaged by infection from Human Immunodeficiency Virus (HIV). It is transmitted through direct contact between mucous membranes or bloodstream with bodily fluids that contain HIV, such as blood, semen, vaginal fluid, pre-semenal fluid and breast milk. It can be transmitted to the closed community because it is very easily transmitted through contact body and body fluids [1]. Especially it transmitted when intercourse through the vagina, anal, and oral. By using condom, it still effective to decreasing level of HIV/AIDS in close community, like commercial sex workers and their clients [2]. Other things that can cause transmission of HIV is blood transfusions, contaminated syringes, and need to watch out for more are between mother and baby during pregnancy, childbirth, and breastfeeding.

Scientists generally believe that AIDS originated in Sub-Saharan Africa, and then it has become an epidemic. In 2018, there were 37.8 million people living with HIV, and 1.7 million people become newly infected with HIV. About 79% of all people living with HIV knew their status, and 8.1 million people did not know they were living with HIV. The risk of acquiring HIV is 22 times higher among men who have sex with men, 21 times higher for sex workers, and 12 times higher for transgender people. UNAIDS said that HIV and tuberculosis (TB) remains the leading cause of death among people living. In 2017, an estimated 10 million people developed TB disease approximately 9% were living with HIV [3]. The importance of including the effect of HIV on TB and vice versa for the transmission and progression is studied by [4].

Despite intensive control efforts, the data show that global incidence for TB increasingly, largely due to an association with HIV. TB model can be given a fraction that indicates newly infected
individual is assumed to undergo fast progression directly to the infectious TB class [5]. HIV model can be divided for symptomatic and asymptomatic case, that occurs in individuals infected with HIV [6]. Other, the study about HIV model that is divided into two compartments, there are individuals with non treatment and individuals with treatment [7].

Motivated from above paper, we will investigate about HIV model with symptomatic and asymptomatic case by adding acceleration fraction to infected individuals. For analyze, we were looking for the stability for free disease and the stability for endemic. The global stability for endemic point will use Lyapunov with the method like in [8]. For numerical simulations, we present the dynamics of HIV/AIDS infection.

2. Model formulation
We formulated a model for the transmittal of HIV/AIDS in a human population. Based on epidemiological status, the population is divided into six classes, there are susceptible (S), Exposed (E), asymptomatic infected (I₁), symptomatic infected (I₂) and patients with AIDS (A). We put a fraction \( p \) and \( q \) of newly infected individuals are assumed leading fast progression directly to asymptomatic infected class. The HIV/AIDS model can be explained through the following diagram.

**Figure 1.** Diagram compartment of the HIV/AIDS model

Mathematical modeling for the model can be described in the following equation.

\[
\begin{align*}
\frac{dS}{dt} &= \lambda - \beta_1 S I_1 - \beta_2 S I_2 - \mu S \\
\frac{dE}{dt} &= \beta_1 (1-p) S I_1 + \beta_2 (1-q) S I_2 - \mu E - \omega E \\
\frac{dI_1}{dt} &= p \beta_1 S I_1 + \beta_2 q S I_2 - \mu I_1 - \alpha_1 I_1 + \omega E \\
\frac{dI_2}{dt} &= \alpha_1 I_1 - \mu I_2 - \delta_1 I_2 - \alpha_2 I_2 \\
\frac{dA}{dt} &= \alpha_2 I_2 - \mu A - \delta_2 A
\end{align*}
\]

(1)

The population growth rate for susceptible is \( \lambda \), and natural death rate is \( \mu \). The infection rate of susceptible by asymptomatic patient and symptomatic patient are \( \beta_1 \) and \( \beta_2 \). The population of asymptomatic patients moving to symptomatic patients has the constant rate \( \alpha_1 \). The population of symptomatic patients transformed into AIDS class has the constant rate \( \alpha_2 \). The death rate of the asymptomatic patients and the symptomatic patients are \( \delta_1 \) and \( \delta_2 \), respectively.

3. Model analysis
We will analyze the model to get the basic reproduction, the point of free disease (non endemic) and its stability, and the point of endemic and its stability. The existence of a nontrivial equilibrium in relation to the basic reproduction is guaranteed in [9]. The following subsection will explain about it.

3.1 Basic reproduction

We use the next generation matrix to obtain the basic reproduction for disease. Before we calculate the basic reproduction ($R_0$), we must know about non-endemic equilibrium that is $E_0 = (\frac{\lambda}{\mu}, 0, 0, 0, 0)$. Then, we built in a matrix of subpopulations that can evoke the number of infected individuals. The infected class are $x = [E \quad I_1 \quad I_2]$, so we can calculate $\frac{dx}{dt} = F(x) - V(x)$. From Eq. (1) we get the matrix in the following equation.

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} \beta_1 (1-p)SI_1 + \beta_2 (1-q)SI_2 \\ p\beta_1 SI_1 + \beta_2 qSI_2 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \mu E + \omega E \\ \mu I_1 + \alpha_1 I_1 - \omega E \\ \alpha_1 I_1 - \mu I_2 - \delta_1 I_2 - \alpha_2 I_2 \end{bmatrix}$$

We obtain the Jacobian matrix for $F$ and Jacobian matrix for $V$ at $E_0$, respectively.

$$JF = \begin{bmatrix} 0 & \beta_1 (1-p) \frac{\lambda}{\mu} & \beta_2 (1-q) \frac{\lambda}{\mu} \\ 0 & p\beta_1 \frac{\lambda}{\mu} & \beta_2 q \frac{\lambda}{\mu} \\ 0 & 0 & 0 \end{bmatrix}$$

$$JV = \begin{bmatrix} \mu + \omega & 0 & 0 \\ -\omega & \mu + \alpha_1 & 0 \\ 0 & \alpha_1 & -\mu - \delta_1 - \alpha_2 \end{bmatrix}$$

For inverse of matrix $JV$:

$$JV^{-1} = \begin{bmatrix} \frac{1}{\mu + \omega} & 0 & 0 \\ \omega & \frac{1}{\mu + \alpha_1} & 0 \\ \omega \alpha_1 & \omega \alpha_1 & -1 \end{bmatrix}$$

The next generation matrix can be written as
\[
\begin{bmatrix}
\Phi_1 & \Phi_2 & \Phi_3 \\
\Phi_4 & \Phi_5 & \Phi_6 \\
0 & 0 & 0
\end{bmatrix}
\] (6)

With,
\[
\Phi_i = \frac{\beta_i (1 - p) \lambda \omega}{\mu (\mu + \omega)(\mu + \alpha_i)} + \frac{\beta_i (1 - q) \lambda \omega \alpha_i}{\mu (\mu + \omega)(\mu + \alpha_i)(\mu + \delta_i + \alpha_z)};
\]
\[
\Phi_2 = \frac{\beta_2 (1 - p) \lambda}{\mu (\mu + \alpha_i)} + \frac{\beta_2 (1 - q) \lambda \alpha_i}{\mu (\mu + \alpha_i)(\mu + \delta_i + \alpha_z)};
\]
\[
\Phi_3 = \frac{-\beta_3 (1 - q) \lambda}{\mu (\mu + \delta_i + \alpha_z)};
\]
\[
\Phi_4 = \frac{p b_2 \lambda \omega}{\mu (\mu + \omega)(\mu + \alpha_i)} + \frac{\beta_2 q \lambda \omega \alpha_i}{\mu (\mu + \omega)(\mu + \alpha_i)(\mu + \delta_i + \alpha_z)};
\]
\[
\Phi_5 = \frac{p b_2 \lambda}{\mu (\mu + \alpha_i)} + \frac{\beta_2 q \lambda \alpha_i}{\mu (\mu + \alpha_i)(\mu + \delta_i + \alpha_z)};
\]
\[
\Phi_6 = \frac{-\beta_6 q \lambda}{\mu (\mu + \delta_i + \alpha_z)};
\]

So, we get the basic reproduction for the disease from the dominant of Eigen value Eq. (6). There is
\[
R_0 = \frac{(\omega + p \mu) \lambda \beta_i}{\mu (\mu + \omega)(\mu + \alpha_i)} + \frac{\lambda \beta_i \alpha_i (\mu q + \omega)}{\mu (\mu + \omega)(\mu + \alpha_i)(\mu + \delta_i + \alpha_z)}
\] (7)

3.2 Stability for non-endemic equilibrium
The equilibrium of non endemic is to identify the free infection in the population. We can see from
\[
E_0 (S^0, E^0, I_1^0, I_2^0, A^0) = (\frac{\lambda}{\mu}, 0, 0, 0, 0)
\]

**Theorem 1.** If \( R_0 < 1 \), then the equilibrium state for non endemic \( E_0 \) is locally asymptotically stable and if \( R_0 < 1 \), then the equilibrium is unstable.

**Proof:**
The Jacobian matrix at \( E_0 \) of the system can be written as
\[
J(E_0) =
\]
\[
\begin{bmatrix}
-\mu & 0 & 0 & \frac{-\beta_2 \lambda}{\mu} & \frac{\beta_2 \lambda}{\mu} & 0 \\
0 & -\mu - \omega & 0 & \frac{\beta_2 \lambda}{\mu} & \frac{\beta_2 \lambda}{\mu} & 0 \\
0 & \omega & -\mu - \alpha_i & \frac{-\beta_2 \lambda}{\mu} & \frac{\beta_2 \lambda}{\mu} & 0 \\
0 & 0 & \alpha_i & -\mu - \delta_i - \alpha_z & 0 & 0 \\
0 & 0 & 0 & \alpha_z & -\mu - \delta_z & 0
\end{bmatrix}
\] (8)

The eigenvalues of the Jacobian matrix (8) are \(-\eta_1\), \(-\eta_2\) and other three eigenvalues. It can be analyzed from
\[ \chi^3 + r_2 \chi^2 + r_1 \chi + r_0 = 0 \]

We obtain value of \( r_2 \)

\[
r_2 = -\frac{p_{\beta} \lambda}{\mu} + 3 \mu + \delta_i + \alpha_i + \omega - \frac{p(\mu + \omega)(\mu + \alpha_i)}{\mu p + \omega}(1 - R_0) + \]

\[
\frac{p_{\lambda} \alpha_i (\omega + \mu q) \beta_i}{(\mu + \delta_i + \alpha_i) \mu (\mu p + \omega)} + \frac{1}{\mu p + \omega} (2 \mu^2 p + \mu p \alpha_z + 3 \omega \mu + \mu p \delta_i + \omega^2 + \omega \alpha_z + \omega \alpha_i - \rho \omega \alpha_i + \omega \delta_i) > 0
\]

We obtain value of \( r_1 \)

\[
r_1 = -\frac{\lambda (p \delta_i + p \alpha_z + 2 \mu p + \omega) \beta_i}{\mu} + \frac{1}{\mu} (3 \mu^3 + 2 \mu^2 \delta_i + 2 \mu \alpha_z + 2 \alpha_i \mu - \alpha_i \beta_z q \lambda + \alpha_i \mu \delta_i + \alpha_i \mu \alpha_z + \omega \mu \delta_i + 2 \omega \mu^2 + \omega \alpha_i \mu + \omega \alpha_z) - \frac{(p \delta_i + p \alpha_z + 2 \mu p + \omega) (\mu + \omega) (\mu + \alpha_i)}{\mu p + \omega} (1 - R_0)
\]

\[
\frac{1}{\mu p + \omega} \left( \alpha_i \lambda (q \mu \omega^2 - q \omega \delta_i - q \omega \alpha_z + p \delta_i \omega + p \alpha_z \omega + 2 \mu \omega + \omega^2) \beta_i \right) + \frac{1}{\mu p + \omega} \left( \omega \alpha_i + 2 \omega \mu + \mu \omega^2 + \omega^2 - \omega \alpha_i \mu \right)
\]

We obtain value of \( r_0 \)

\[
r_0 = -\frac{\lambda (\mu + \delta_i + \alpha_z) (\mu p + \omega) \beta_i}{\mu} - \frac{\alpha_i \lambda (\omega + \mu q) \beta_i}{\mu} - (\mu + \delta_i + \alpha_z) (\mu + \alpha_i) (\mu + \omega) (1 - R_0)
\]

By calculated \( r_1 r_2 - r_0 \)

\[
r_1 r_2 - r_0 = -\left[ \frac{\lambda (p \delta_i + p \alpha_z + 2 \mu p + \omega) \beta_i}{\mu} + \frac{1}{\mu} (3 \mu^3 + 2 \mu^2 \delta_i + 2 \mu \alpha_z + 2 \alpha_i \mu - \alpha_i \beta_z q \lambda + \alpha_i \mu \delta_i + \alpha_i \mu \alpha_z + \omega \mu \delta_i + 2 \omega \mu^2 + \omega \alpha_i \mu + \omega \alpha_z) \right] - \frac{\alpha_i \lambda (\omega + \mu q) \beta_i}{\mu} - (\mu + \delta_i + \alpha_z) (\mu + \alpha_i) (\mu + \omega) (1 - R_0)
\]

\[
+ \frac{\lambda (\mu + \delta_i + \alpha_z) (\mu p + \omega) \beta_i}{\mu} - \frac{\alpha_i \lambda (\omega + \mu q) \beta_i}{\mu} - (\mu + \delta_i + \alpha_z) (\mu + \alpha_i) (\mu + \omega) (1 - R_0)
\]

It can be calculated \( r_1 r_2 - r_0 > 0 \), when \( p \geq q \) and \( R_0 < 1 \). All eigenvalues of \( J(E_0) \) are negative, when \( R_0 < 1 \). Therefore, \( E_0 \) is locally asymptotically stable if \( R_0 < 1 \).

3.3 Stability of endemic equilibrium
The endemic equilibrium is identified to identify the spread of infection in the population for the model. We denote the endemic equilibrium is

\[ E_q^* = [S^*, E^*, I_1^*, I_2^*, A^*] \]  

and we calculate

\[ S^* = \frac{\lambda (\mu + \delta_1 + \alpha_2)}{\beta_1 (\mu + \delta_1 + \alpha_2) I_1 + \mu^2 + \mu \delta_1 + \mu \alpha_2 + \beta_2 \alpha_1 I_1} \]  

\[ E^* = \frac{\lambda I_1 ((1 - p)(\mu + \delta_1 + \alpha_2) \beta_1 + \beta_2 \alpha_1 (1 - q))}{((\mu + \delta_1 + \alpha_2)(\mu + \omega) \beta_1 + \beta_2 \alpha_1 (\mu + \omega)) I_1 + \mu (\mu + \delta_1 + \alpha_2)(\mu + \omega)} \]

\[ I_2^* = \frac{\alpha_1 I_1}{\mu + \delta_1 + \alpha_2} ; A^* = \frac{\alpha \alpha_1 I_1}{(\mu + \delta_1 + \alpha_2)(\mu + \delta)} \]

With,

\[ I_1^* = -\frac{\lambda (\mu + \delta_1 + \alpha_2)(p \mu + \omega) \beta_1 - \lambda \alpha_1 (\omega + \mu q) \beta_2 + \mu (\mu + \delta_1 + \alpha_2)(\mu + \alpha_1)(\mu + \omega)}{((\mu + \delta_1 + \alpha_2)(\mu + \alpha_1)(\mu + \omega) \beta_1 + \alpha_1 (\mu + \alpha_1)(\mu + \omega) \beta_2 - \mu (\mu + \delta_1 + \alpha_2)(\mu + \alpha_1)(\mu + \omega)(R_0 - 1) \]

Next, we will explain about the global stability of the endemic equilibrium with Lyapunov.

**Theorem 2.** The endemic equilibrium \( E_q^* = [S^*, E^*, I_1^*, I_2^*, A^*] \) is globally asymptotically stable when \( R_0 > 1 \).

**Proof.**

We consider the Lyapunov function that will use

\[ V(S, E, I_1, I_2, A) = S - S^* \ln S + a_1 \left(E - E^* \ln E\right) + a_2 \left(I_1 - I_1^* \ln I_1\right) + a_3 \left(I_2 - I_2^* \ln I_2\right) + a_4 \left(A - A^* \ln A\right), \]

where \( a_1, a_2, a_3, a_4 > 0 \) will be determined. The derivative of \( V \) respect to \( t \) for the Eq.(1) is given by

\[ \frac{dV}{dt} = \left(1 - \frac{S^*}{S}\right) \frac{dS}{dt} + a_1 \left(1 - \frac{E^*}{E}\right) \frac{dE}{dt} + a_2 \left(1 - \frac{I_1^*}{I_1}\right) \frac{dI_1}{dt} + a_3 \left(1 - \frac{I_2^*}{I_2}\right) \frac{dI_2}{dt} + a_4 \left(1 - \frac{A^*}{A}\right) \frac{dA}{dt} \]

(10)

From the Eq. (1), we can consider respectively

\[ \lambda = \beta_1 SI_1 + \beta_2 SI_2 + \mu S \]

\[ A_1 E = \beta_1 (1 - p) SI_1 + \beta_2 (1 - q) SI_2 \]

\[ A_1 I_1 = p \beta_1 SI_1 + \beta_2 q SI_2 + \omega E \]

\[ A_1 I_2 = \alpha_1 I_1 \]

\[ A_1 A = \alpha_2 I_2 \]

With, \( A_1 = \mu + \alpha; A_2 = \mu + \alpha; A_3 = \mu + \delta_1 + \alpha_2; A_4 = \mu + \delta_2 \). Now, we can substitute Eq.(11) to Eq.(10), so we could get
\[
\frac{dV}{dt} = \left(1 - \frac{S'}{S}\right) \left(\beta_1 S' I_1' + \beta_2 S' I_2' + \mu S' - \beta_1 SI_1 - \beta_2 SI_2 - \mu S\right)
\]
\[
+ a_1 \left(1 - \frac{E'}{E}\right) \left(\beta_1 (1 - p) SI_1 + \beta_2 (1 - q) SI_2\right) - a_4 A_1 E + a_4 A_2 E'
\]
\[
+ a_2 \left(1 - \frac{I_1'}{I_1}\right) \left(p \beta_1 SI_1 + \beta_2 q SI_2 + \omega E\right) - a_2 A_1 I_1 + a_2 A_2 I_1' + a_3 \left(1 - \frac{I_2'}{I_2}\right) \alpha I_1 - a_3 A_1 I_2 + a_3 A_2 I_2'
\]
\[
+ a_4 \left(1 - \frac{A'}{A}\right) \alpha_2 I_2 - a_4 A_1 A + a_4 A_2 A'
\]
\[
= \left(1 - \frac{S'}{S}\right) \left(\beta_1 S' I_1' + \beta_2 S' I_2' + \mu S' - \beta_1 SI_1 - \beta_2 SI_2 - \mu S\right)
\]
\[
+ a_1 \left(1 - \frac{E'}{E}\right) \left(\beta_1 (1 - p) SI_1 + \beta_2 (1 - q) SI_2\right) - a_4 A_1 E + a_4 A_2 E'
\]
\[
+ a_2 \left(1 - \frac{I_1'}{I_1}\right) \left(p \beta_1 SI_1 + \beta_2 q SI_2 + \omega E\right) - a_2 A_1 I_1 + a_2 A_2 I_1' + a_3 \left(1 - \frac{I_2'}{I_2}\right) \alpha I_1 - a_3 A_1 I_2 + a_3 A_2 I_2'
\]
\[
+ a_4 \left(1 - \frac{A'}{A}\right) \alpha_2 I_2 - a_4 A_1 A + a_4 A_2 A'
\]
\[
= -\mu S \left(S - \frac{S'}{S}\right)^2 + \beta_1 S' I_1' \left(1 - \frac{S'}{S}\right) + \beta_2 S' I_2' \left(1 - \frac{S'}{S}\right) + SI_1 \left(-\beta_1 + a_4 \beta_1 (1 - p) + a_4 p \beta_1\right)
\]
\[
+ SI_2 \left(-\beta_2 + a_4 \beta_2 (1 - q) + a_4 \beta_2 q + E(-a_1 A_1 + a_1 \omega) + I_1 \left(\beta, S' - a_2 A_2 + a_2, a_1\right)
\]
\[
+ I_2 \left(\beta, S' - a_2 A_2 + a_2, a_2\right) - a_4 A_4 A + a_4 A_4 A + a_4 \left(1 - p\right) S' I_1' \left(1 - \frac{S}{S'} \frac{I_1}{I_1'} \frac{E}{E'}\right) + a_4 \left(1 - q\right) S' I_2' \left(1 - \frac{S}{S'} \frac{I_2}{I_2'} \frac{E}{E'}\right)
\]
\[
+ a_4 p \beta, S' I_1' \left(1 - \frac{S}{S'} \frac{I_1}{I_1'} \frac{E}{E'}\right) + a_4 q \beta, S' I_2' \left(1 - \frac{S}{S'} \frac{I_2}{I_2'} \frac{E}{E'}\right) + a_4 \omega E \left(1 - \frac{E}{E'} \frac{I_1'}{I_1}\right)
\]
\[
+ a_4 \alpha, I_1' \left(1 - \frac{I_1'}{I_1'} \frac{A'}{A}\right) + a_4 \alpha, I_2' \left(1 - \frac{I_2'}{I_2'} \frac{A'}{A}\right)
\]

The positive constants \(a_1, a_2, a_3, a_4\) are chosen, such that we built the coefficient of \(SI_1, SI_2, E, I_1, I_2\) are equal to zero, respectively, that is,

\[
\begin{cases}
-\beta_1 + a_4 \beta_1 (1 - p) + a_4 p \beta_1 = 0 \\
-\beta_2 + a_4 \beta_2 (1 - q) + a_4 \beta_2 q = 0 \\
-\beta_2 A_2 + a_4 \omega = 0 \\
\beta, S' - a_2 A_2 + a_2, a_2 = 0 \\
\beta, S' - a_2 A_2 + a_2, a_2 = 0
\end{cases}
\]
We can substitute Eq.(12) to the last derive Lyapunov function, that is
\[
\frac{dV}{dt} = -\mu S \left( \frac{S - S^*}{S} \right)^2 + \alpha \beta_1 (1 - p) S^* I_1^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_1^* I_1^* \right) + \alpha \beta_1 p S^* I_1^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_1^* I_1^* \right) + \alpha \beta_2 (1 - q) S^* I_2^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_2^* I_2^* \right) + \alpha \beta_2 q S^* I_2^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_2^* I_2^* \right) + \alpha \omega E^* \left( 1 - \frac{E}{E^*} I_1^* \right) + \alpha \omega E^* \left( 1 - \frac{E}{E^*} I_2^* \right) + \alpha \omega E^* \left( 1 - \frac{E}{E^*} I_1^* \right) - \alpha \omega A^* A^* \frac{A}{A} \right)
\] (13)

From Eq.(11) and Eq.(12), we get the equation above :
\[
\begin{align*}
\alpha \omega A^* A^* \frac{A}{A} &= 0 \\
\alpha \omega A^* A^* \frac{A}{A} &= 0 \\
\alpha \omega A^* A^* \frac{A}{A} &= 0
\end{align*}
\]

Then, we will multiply the above equation with $F_1(u), F_2(u), F_3(u)$ respectively, where 
\[
u = (v, w, x, y, z)^T,
\]
and the function will be determined later. So, we get
\[
\begin{align*}
a \omega A^* A^* \frac{A}{A} &= 0 \\
\alpha \omega A^* A^* \frac{A}{A} &= 0 \\
\alpha \omega A^* A^* \frac{A}{A} &= 0
\end{align*}
\]

If we given \(v = \frac{S}{S^*}; \frac{S}{S^*}; \frac{S}{S^*}; \frac{S}{S^*}; \frac{S}{S^*}; \frac{S}{S^*} \) \(w = \frac{E}{E^*}; \frac{E}{E^*}; \frac{E}{E^*}; \frac{E}{E^*}; \frac{E}{E^*}; \frac{E}{E^*} \) \(x = \frac{I_1}{I_1^*}; \frac{I_1}{I_1^*}; \frac{I_1}{I_1^*}; \frac{I_1}{I_1^*}; \frac{I_1}{I_1^*}; \frac{I_1}{I_1^*} \) \(y = \frac{I_2}{I_2^*}; \frac{I_2}{I_2^*}; \frac{I_2}{I_2^*}; \frac{I_2}{I_2^*}; \frac{I_2}{I_2^*}; \frac{I_2}{I_2^*} \) \(z = \frac{A}{A} \). Then, the last equation will be
\[
\frac{dV}{dt} = -\mu S \left( \frac{S - S^*}{S} \right)^2 + \frac{\alpha \beta_1 (1 - p)}{v} S^* I_1^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_1^* I_1^* \right) + \frac{\alpha \beta_1 p}{w} S^* I_1^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_1^* I_1^* \right) + \frac{\alpha \beta_2 (1 - q)}{x} S^* I_2^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_2^* I_2^* \right) + \frac{\alpha \beta_2 q}{y} S^* I_2^* \left( 2 - \frac{S^*}{S} \frac{S}{S^*} I_2^* I_2^* \right) + \frac{\alpha \omega E^*}{z} \left( 1 - \frac{E}{E^*} I_1^* \right) + \frac{\alpha \omega E^*}{z} \left( 1 - \frac{E}{E^*} I_2^* \right) + \frac{\alpha \omega E^*}{z} \left( 1 - \frac{E}{E^*} I_1^* \right) - \frac{\alpha \omega A^* A^*}{z} \frac{A}{A} \right)
\] (14)

The function $F_1(u), F_2(u), F_3(u)$ are chosen from coefficients of $E^*, I_1^*, I_2^*$ that equal to zero, so we get
\[
F_1(u) = 1 - \frac{y}{z} - z; F_2(u) = 2 - \frac{x}{y} - \frac{y}{z} - z; F_3(u) = 3 - \frac{x}{y} - \frac{y}{z} - z
\]

Finally, the last equation about the derivative Lyapunov function is
\[
V' = \frac{dV}{dt} = \mu S \left( 2 - v - \frac{1}{v} \right) + a_1 \beta_1 (1 - p) S I_1 \left( 3 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} \right) \\
+ a_2 \beta_2 p S I_2 \left( 2 - \frac{1}{v} - v \right) \\
+ a_2 \beta_2 (1 - q) S I_2 \left( 4 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} \frac{x}{y} \right) \\
+ a_3 \beta_2 S I_2 \left( 3 - \frac{1}{v} - \frac{vy}{x} \frac{x}{y} \right).
\]

(15)

Since the arithmetical mean is greater than, or equal to the geometrical mean, then, \(3 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} \leq 0\)
for \(v, w, x > 0\) and \(3 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} = 0\) if only if \(v = w = x = 1\); then \(4 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} \frac{x}{y} \leq 0\) for \(v, w, x, y > 0\) and \(4 - \frac{1}{v} - \frac{vy}{w} \frac{w}{x} \frac{x}{y} = 0\) if only if \(v = w = x = y = 1\); then \(3 - \frac{1}{v} - \frac{vy}{x} \frac{x}{y} \leq 0\) for \(v, x, y > 0\) and \(3 - \frac{1}{v} - \frac{vy}{x} \frac{x}{y} = 0\) if only if \(v = x = y = 1\). Therefore, \(V' \leq 0\) for \(v, w, x, y > 0\) and \(V' = 0\) if only if \(v = w = x = y = 1\), the maximum invariant set of the system (1) on the set \((v, w, x, y)\) : \(V' = 0\) is singleton \((1, 1, 1, 1)\). Thus, the system (1), the endemic equilibrium \(E_q^*\) is globally asymptotically stable if \(R_q > 1\) by LaSalle Invariance Principle [10].

4. Numerical Simulation

In this section, we will give numerical simulation from a numerical data that informing the value of parameter for the model. For the data, we present the analytical result, as follows.

| Parameter | Estimated Value | Data Source |
|-----------|-----------------|-------------|
| \(\lambda\) | 0.55 / year | [11] |
| \(\mu\) | 0.0196 / year | [12] |
| \(\beta_1\) | Variable | |
| \(\beta_2\) | Variable | |
| \(\alpha_1\) | Variable | |
| \(\alpha_2\) | Variable | |
| \(\delta_1\) | 0.009 / year | [12] |
| \(\delta_2\) | 0.00909 / year | [12] |
| \(p\) | Variable | |
| \(q\) | Variable | |
| \(\omega\) | Variable | |

4.1 The equilibrium of non endemic and its stability

We choose \(\alpha_i = 0.021; \alpha_i = 0.0211; \beta_1 = 0.00125; \beta_2 = 0.001; p = q = 0.01\) and \(\omega = 0.12\), numerical simulation gives \(R_q = 0.99531 < 1\). Then, the non endemic equilibrium is globally
asymptotically stable in Fig.2. We can see from Fig.2, in first time, the susceptible is decreasing significantly, but along the time the susceptible will increase that’s going to \( S_t = 28.0612 \). The exposed individual is always decreasing from a value of population size to zero population. On the other hand, the asymptomatic individuals, the symptomatic individuals and patients in AIDS have the same graphical analyze. In the beginning, they are increasing to a peak value, then along the time, they are decreasing until to zero population size.

**Figure 2.** When \( R_0 = 0.99531 < 1 \), the non endemic equilibrium is globally asymptotically stable.

### 4.2 The equilibrium of endemic and its stability

We choose \( \alpha_1 = 0.021; \alpha_2 = 0.0211; \beta_1 = 0.00801; \beta_2 = 0.008; \ p = q = 0.01 \) and \( \omega = 0.12 \) numerical simulation gives \( R_0 = 6.778282203 > 1 \). Then, the endemic equilibrium is globally asymptotically stable in Fig.3. We can see from Fig.3, the susceptible in AIDS is oscillating. In the first time, it is decreasing then increasing again, and along the time, it goes to \( S^* = 4.1399 \). The exposed individual is always increasing. From the first time, it on a value of population size then goes to \( E^* = 3.325 \). For the asymptomatic individuals, it is oscillating with significant increase at the beginning of time, then it will be down sharply then up toward \( I_{t_1}^* = 9.9431 \). Different from the symptomatic individuals and the patient in AIDS, in the first time, they have an increase in population less then the others, but they are decreasing slowly to \( I_{t_2}^* = 4.2013 \) for symptomatic individual and \( A^* = 0.8022 \) for patients in AIDS.

**Figure 3.** When \( R_0 = 6.778282203 > 1 \), the endemic equilibrium is globally asymptotically stable.

### 5. Conclusion

In this study, we constructed HIV/AIDS model that’s has a fraction \( p \) and \( q \) to describe the newly susceptible with fast progression directly to the infectious class, i.e. asymptomatic infected. We obtain the basic reproduction \( (R_0) \) from the next generation matrix method in the non endemic equilibrium. The basic reproduction is used to analyze the stability of equilibrium. From this, the stability of non
endemic equilibrium is globally asymptotically stable if $R_0 < 1$, otherwise, the stability of endemic equilibrium is globally asymptotically stable if $R_0 > 1$. Numerical simulations present about the result of the model, if $R_0 < 1$, then the susceptible is leading to constant depend on the growth population rate of susceptible and the natural death rate. If $R_0 > 1$, then the classes population are leading to the endemic equilibrium.

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