On symplectic automorphisms of elliptic surfaces acting on $CH_0$

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Abstract Let $S$ be a complex smooth projective surface of Kodaira dimension one. We show that the group $\text{Aut}_s(S)$ of symplectic automorphisms acts trivially on the Albanese kernel $\text{CH}_0^{\text{alb}}(S)$ of the 0-th Chow group $\text{CH}_0(S)$, unless possibly if the geometric genus and the irregularity satisfy $p_g(S) = q(S) \in \{1, 2\}$. In the exceptional cases, the image of the homomorphism $\text{Aut}_s(S) \to \text{Aut}(\text{CH}_0(S)_{\text{alb}})$ has the order at most 3. Our arguments actually take care of the group $\text{Aut}_f(S)$ of fibration-preserving automorphisms of elliptic surfaces $f: S \to B$. We prove that if $\sigma \in \text{Aut}_f(S)$ induces the trivial action on $H^i_f(S, \mathbb{Q})$ for $i > 0$, then it induces the trivial action on $\text{CH}_0(S)_{\text{alb}}$. As a by-product we obtain that if $S$ is an elliptic K3 surface, then $\text{Aut}_f(S) \cap \text{Aut}_s(S)$ acts trivially on $\text{CH}_0(S)_{\text{alb}}$.

Keywords symplectic automorphism, elliptic surface, Chow group

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1 Introduction

We work over the complex numbers $\mathbb{C}$ in this paper.

For a complex smooth projective variety $X$ of dimension $d$, its $k$-th Chow group $\text{CH}_k(X)$ is defined to be $Z_k(X)/\sim_{\text{rat}}$, where $Z_k(X)$ is the free abelian group on the $k$-dimensional closed subvarieties of $X$ and $\sim_{\text{rat}}$ denotes the rational equivalence. We have $\text{CH}_d(X) = \mathbb{Z}[X] \cong \mathbb{Z}$ and $\text{CH}_{d-1}(X) = \text{Pic}(X)$. However, $\text{CH}_k(X)$ becomes very hard to compute for $0 \leq k \leq d - 2$. For example, for a smooth projective surface $S$ with $p_g(S) > 0$, the degree zero part $\text{CH}_0(S)_{\text{hom}}$ of $\text{CH}_0(S)$ is infinite-dimensional [18] \(^1\).

The Bloch-Beilinson conjecture predicts the existence of a finite decreasing filtration on each Chow group $\text{CH}_k(X)_{\mathbb{Q}}$ with rational coefficients whose graded pieces are, in terms of correspondence between smooth projective varieties, controlled by the Hodge decomposition of the cohomology groups (see [24, Subsection 11.2.2] for precise statements). The philosophy behind this conjecture is that the topology (Hodge theory) determines the algebraic geometry of cycles.

\(^1\) This means that the natural map $\text{Sym}^n(S) \times \text{Sym}^n(S) \to \text{CH}_0(S)_{\text{hom}}, (A,B) \mapsto [A-B]$ is not surjective for any natural number $n$, where $\text{Sym}^n(S)$ denotes the $n$-th symmetric product of $S$.\n
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Specifically for the 0-th Chow group, one defines the kernel of the degree map

$$\text{CH}_0(X)_{\text{hom}} := \ker(\text{CH}_0(X) \xrightarrow{\text{deg}} \mathbb{Z})$$

and in turn the kernel of the Albanese map

$$\text{CH}_0(X)_{\text{alb}} := \ker(\text{CH}_0(X)_{\text{hom}} \xrightarrow{\text{alb}} \text{Alb}(X)).$$

Then $\text{CH}_0(X) \supset \text{CH}_0(X)_{\text{hom}} \supset \text{CH}_0(X)_{\text{alb}}$ are expected to be the first three terms of the Bloch-Berline filtration for $\text{CH}_0(X)$, and if $X = S$ is a surface, then this should be the full filtration. As a consequence, one expects the following results.

**Conjecture 1.1.** Let $S$ be a smooth projective surface. Let $X$ be a smooth projective variety, and $\Gamma \in \text{CH}^2(X \times S)$ be a cycle of codimension 2. If $[\Gamma]^*: \text{H}^{2,0}(S) \rightarrow \text{H}^{2,0}(X)$ vanishes, then $\Gamma_*: \text{CH}_0(X)_{\text{alb}} \rightarrow \text{CH}_0(S)_{\text{alb}}$ also vanishes.

Taking $X = S$ and $\Gamma = \Gamma_\sigma - \Delta_S$, where $\Gamma_\sigma$ is the graph of an automorphism $\sigma \in \text{Aut}(S)$ and $\Delta_S \subset S \times S$ is the diagonal, we obtain the following special and more tractable case of Conjecture 1.1.

**Conjecture 1.2.** Let $S$ be a smooth projective surface. Then any symplectic automorphism acts trivially on $\text{CH}_0(S)_{\text{alb}}$.

Here, an automorphism $\sigma \in \text{Aut}(S)$ is called symplectic if the induced map $\sigma^*: \text{H}^{2,0}(S) \rightarrow \text{H}^{2,0}(S)$ is the identity. We use $\text{Aut}_s(S)$ to define the group of symplectic automorphisms of $S$.

Surfaces with $\kappa(S) \leq 1$ and $p_g(S) = 0$ have trivial $\text{CH}_0(S)_{\text{alb}}$ by Bloch et al. [9], so Conjecture 1.2 is automatically true in this case. The Bloch conjecture [8, Conjecture 1.8 and Proposition 1.11], which is again a consequence of Conjecture 1.1 (take $\Gamma = \Delta_S$), asserts that surfaces of general type with $p_g(S) = 0$ also have trivial $\text{CH}_0(S)_{\text{alb}}$. This has been verified in some special cases by various authors: surfaces with “enough automorphisms” such as Godeaux surfaces, Burniat-Inoue surfaces and Campedelli surfaces [1,3,4,6,10,13], surfaces with “finite-dimensional Chow motives” such as surfaces rationally dominated by a product of curves [15], surfaces with “nice moduli spaces” such as Catanese surfaces, Barlow surfaces [26] and some numerical Campedelli surfaces [17], and so on. Since the complete classification for surfaces of general type with $p_g = 0$ is still unknown and the effective methods avoiding the classification results of such surfaces are not established, the Bloch conjecture is open by now [5].

For surfaces with $p_g(S) > 0$, the Albanese kernel $\text{CH}_0(S)_{\text{alb}}$ is huge [18]. Nevertheless, Conjecture 1.2 has been confirmed for abelian surfaces [9,19] as well as symplectic automorphisms of the finite order of K3 surfaces [12,25]. For Kummer K3 surfaces, infinite order symplectic automorphisms coming from the covering abelian surface are treated in [19].

The main result of this paper is the following theorem.

**Theorem 1.3.** Conjecture 1.2 holds for surfaces $S$ with Kodaira dimension $\kappa(S) = 1$, unless possibly $q(S) = p_g(S) \in \{1,2\}$. In these cases, the image of the homomorphism $\text{Aut}_s(S) \rightarrow \text{Aut}(\text{CH}_0(S)_{\text{alb}})$ has the order at most 3.

Strengthening the condition of Conjecture 1.2 (compare with [25, Conjecture 1.1]), we have the following theorem.

**Theorem 1.4.** Let $S$ be a smooth projective surface with $\kappa(S) = 1$. If an automorphism $\sigma \in \text{Aut}(S)$ induces the trivial action on $\text{H}^{i,0}(S)$ for $i > 0$, then it induces the trivial action on $\text{CH}_0(S)_{\text{alb}}$.

The results are based on the following elementary observation about the zero cycles of an elliptic surface $f: S \rightarrow B$: the Albanese kernel $\text{CH}_0(S)_{\text{alb}}$ is contained in the so-called $f$-kernel

$$\text{CH}_0(S)_f := \ker(\text{CH}_0(S) \xrightarrow{f_*} \text{CH}_0(B)).$$

For any cycle class $\alpha \in \text{CH}_0(S)_f$, one can find a positive integer $d$ and finitely many $\alpha_i \in \text{CH}_0(S)_{\text{hom}}$ such that

$$d\alpha = \sum_i \alpha_i \in \text{CH}_0(S),$$

(1.1)
and the support supp(α_i) lies on a single fiber for each i (see Lemma 2.6).

Let Aut_f(S) be the subgroup of automorphisms of S preserving the fibration structure f; for the precise definition, see Section 2. What we are dealing with in this paper is in fact the group Aut_f(S) ∩ Aut_s(S).

Note that, if κ(S) = 1, then there is a unique elliptic fibration structure on S and hence Aut_f(S) = Aut(S); it follows that Aut_f(S) ∩ Aut_s(S) is the whole Aut_s(S).

In any case, we have an exact sequence

$$1 \to \text{Aut}_B(S) \to \text{Aut}_f(S) \to \text{Aut}(B),$$

where Aut_B(S) := \{σ ∈ Aut_f(S) | f ⋄ σ = f\}. By the decomposition (1.1), it is clear that σ ∈ Aut_B(S) induces the trivial action on α ∈ CH_0(S)_{alb} if the restriction σ|_F to a general fiber F of f is a translation. The latter property is guaranteed if p_g(S) > 0 (see Lemma 2.9). On the other hand, if p_g(S) = 0, then CH_0(S)_{alb} = 0 by [9] and there is nothing to prove. The conclusion is that Aut_B(S) ∩ Aut_s(S) acts trivially on CH_0(S)_{alb} (see Proposition 2.5).

Note that Aut_B(S) ∩ Aut_s(S) also acts trivially on the Jacobian j: J → B. Using this fact, we can reduce the problem to the one for finite order symplectic automorphisms of J that fix the distinguished section. This is carried out in Section 4.

Replacing the elliptic fibration f: S → B with its Jacobian j: J → B and σ ∈ Aut_f(S) ∩ Aut_s(S) with its induced automorphism σ_J ∈ Aut_j(J) ∩ Aut_s(J), we can assume that S has a section, and then work only with those fibration-preserving symplectic automorphisms σ of the finite order such that |σ| = |σ_B|, where σ_B ∈ Aut(B) is induced by σ.

Here, we observe that in the case p_g(S) > 0, the canonical map φ_S: S → P^{p_g-1} factors through f: S → B by the canonical bundle formula for elliptic fibrations. Since σ acts trivially on H^0(S, K_S), φ_S also factors through the quotient map π: B → B/⟨σ_B⟩, where σ_B ∈ Aut(B) is the automorphism induced by σ. Therefore, we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{φ_S} & B/⟨σ_B⟩ \\
\downarrow{f} & & \downarrow{π} \\
B & \xrightarrow{φ_B} & \mathbb{P}^{p_g-1},
\end{array}
\]

where φ_B is the morphism associated with the linear system |K_B + L| with L being a line bundle of degree χ(O_S) on B. If p_g(S) ≥ 2, we obtain |σ_B| ≤ deg φ_B.

If χ(O_S) ≥ 3, then deg(K_B + L) = 2g(B) − 2 + χ(O_S) ≥ 2g(B) + 1 and thus φ_B is an embedding. Therefore, σ_B = id_B, and it follows that σ is also the identity (see Theorem 3.1).

In the case χ(O_S) ∈ {1, 2}, we cannot conclude that σ_B = id_B, but a similar consideration yields strong restrictions on what σ_B can be. In fact, if we impose the additional condition that σ acts trivially also on H^1(S, O_S), then σ_B = id_B holds, which implies Theorem 1.4 in this case. In most cases, we can show that |σ| = |σ_B| ∈ {1, 2, 3, 4, 6}. Using the method of “enough automorphisms” [13], we see that either σ^2 or σ^3 has the order at most 2, and thus acts trivially on CH_0(S)_{alb} (see Lemma 5.2).

In the case χ(O_S) = 0, the relatively minimal model of the surface S over B is an elliptic quasi-bundle and hence is the quotient of a product of two curves. We can thus draw on the finite-dimensionality of the Chow motives of such surfaces [14, 15].

Our arguments have a byproduct concerning (possibly infinite order) symplectic automorphisms of elliptic K3 surfaces.

**Theorem 1.5** (Corollary 5.5). Let f: S → B be an elliptic K3 surface. Then Aut_f(S) ∩ Aut_s(S) acts trivially on CH_0(S)_{alb}.

### 2 Fibration-preserving automorphisms

Let S be a smooth projective surface and f: S → B be a fibration, i.e., a morphism onto the smooth projective curve B with connected fibers. We define the following subgroups of Aut(S):

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**Du J B et al. Sci China Math March 2023 Vol. 66 No. 3 445**
The subgroup of fibration-preserving automorphisms

\[ \text{Aut}_f(S) := \{ \sigma \in \text{Aut}(S) \mid \exists \sigma_B \in \text{Aut}(B) \text{ such that } \sigma_B \circ f = f \circ \sigma \}. \]

These automorphisms may permute the fibers of \( f \).

The subgroup of fiber-preserving automorphisms

\[ \text{Aut}_B(S) := \{ \sigma \in \text{Aut}_f(S) \mid f = f \circ \sigma \}. \]

These automorphisms preserve each fiber of \( f \).

There is an obvious exact sequence of groups

\[ 1 \to \text{Aut}_B(S) \to \text{Aut}_f(S) \xrightarrow{r} \text{Aut}(B), \tag{2.1} \]

where \( r \) sends an automorphism \( \sigma \in \text{Aut}_f(S) \) to \( \sigma_B \in \text{Aut}(B) \) such that \( \sigma_B \circ f = f \circ \sigma \).

**Lemma 2.1.** Let \( S \) be a smooth projective surface and \( f : S \to B \) be a fibration. Suppose that one of the following conditions holds:

1. \( f \) is not isotrivial, i.e., not all the smooth fibers of \( f \) are isomorphic to each other;
2. \( g(B) \geq 2 \);
3. \( f \) has at least three singular fibers (resp. one singular fiber) if \( g(B) = 0 \) (resp. \( g(B) = 1 \)).

Then the image \( \text{Im}(r) \) of the homomorphism \( r \) in (2.1) is finite.

**Proof.**

(1) If \( f \) is not isotrivial, then the rational map from \( \lambda : B \dashrightarrow M_g \) to the moduli space of curves of the genus \( g \) is generically finite, where \( g \geq 1 \) is the genus of the general fibers of \( f \). On the other hand, for any \( b \in B \), the fibers over the points in \( \text{Orbit}(b) := \{ \sigma_B(b) \mid \sigma_B \in \text{Im}(r) \} \) are isomorphic, so \( \text{Orbit}(b) \) are mapped to the same point by \( \lambda \). Thus \( \text{Orbit}(b) \) is finite for all \( b \), and it follows that \( \text{Im}(r) \) is finite.

(2) If \( g(B) \geq 2 \), then \( \text{Aut}(B) \) is finite, so \( \text{Im}(r) \subset \text{Aut}(B) \) is automatically finite.

(3) Let \( \Sigma = \{ b \in B \mid f^*b \text{ is singular} \} \), which is a finite set. Then there is a natural homomorphism \( \text{Im}(r) \to \text{Perm}(\Sigma) \) into the (finite) permutation group of \( \Sigma \); the kernel of this homomorphism is finite if either \( g(B) = 0 \) and \( \# \Sigma > 2 \) or \( g(B) = 1 \) and \( \# \Sigma > 0 \).

The proof of the lemma is completed. \( \square \)

We recall some facts about isotrivial fibrations (see [23, Section 1]). Let \( S \) be a smooth projective surface and \( f : S \to B \) be an isotrivial fibration whose smooth fibers are isomorphic to a fixed curve \( F \) with \( g(F) \geq 1 \). Birationally \( f \) becomes a trivial fibration after a base change. More precisely, there exist a smooth projective curve \( \tilde{B} \), a finite group \( G \) acting on \( \tilde{B} \) and \( F \) such that \( B \cong \tilde{B}/G \), and the following diagram is commutative:

\[
\begin{array}{ccc}
S & \to & (\tilde{B} \times F)/G \\
\downarrow f & & \downarrow \pi \\
\tilde{B} & \cong & \tilde{B}/G,
\end{array}
\tag{2.2}
\]

where the horizontal dashed arrow is a birational map, \( G \) acts diagonally on \( \tilde{B} \times F \), and \( \pi \) is induced by the projection \( \tilde{B} \times F \to \tilde{B} \). It is easy to check that if \( b \in B \) is a branch point of the quotient map \( \tilde{B} \to \tilde{B}/G \cong B \), then the fiber \( f^*b \) of \( f \) over \( b \) is singular.

**Lemma 2.2.** Let \( S \) be a smooth projective surface with Kodaira dimension \( \kappa(S) \geq 0 \), and \( f : S \to B \) be a fibration. If the image \( \text{Im}(r) \) of the homomorphism in (2.1) is an infinite group, then \( g(B) = 1 \) and \( f \) is a fiber bundle.

**Proof.** Suppose that \( \text{Im}(r) \) is infinite. By Lemma 2.1, \( f \) is isotrivial and \( g(B) \leq 1 \). Moreover, if \( g(B) = 1 \) then \( f \) has no singular fibers, so it is necessarily a fiber bundle.

Now suppose that \( g(B) = 0 \). We draw a contradiction by showing that \( \text{Im}(r) \) is finite in this case, and thus complete the proof. Since \( f \) is isotrivial, we have a commutative diagram as in (2.2). Since \( \kappa(\tilde{B} \times F) \geq \kappa(S) \geq 0 \), one has \( g(\tilde{B}) \geq 1 \). By the Riemann-Hurwitz formula, the quotient map \( \tilde{B} \to \tilde{B}/G \cong B \) has at least three branch points. The fibers of \( f \) over these branch points are necessarily singular, so \( \text{Im}(r) \) is finite by Lemma 2.1(3). \( \square \)
Corollary 2.3. Let $S$ be a smooth projective surface with Kodaira dimension $\kappa(S) \in \{0, 1\}$, and $f: S \to B$ be an elliptic fibration. Then the image $\text{Im}(r)$ of the homomorphism in (2.1) is finite unless possibly if $S$ is an abelian surface or a bi-elliptic surface.

Proof. If $\text{Im}(r)$ is infinite, then $f$ is an elliptic bundle over an elliptic curve by Lemma 2.2. According to [2, Subsection V.5], this can only happen when $S$ is an abelian surface or a bi-elliptic surface. □

Restricting (2.1) to the group of symplectic automorphisms, we obtain another exact sequence

$$1 \to \text{Aut}_B(S) \cap \text{Aut}_s(S) \to \text{Aut}_f(S) \cap \text{Aut}_s(S) \to \text{Aut}(B).$$

Suppose that $S$ is a surface with Kodaira dimension $\kappa(S) = 1$. Then the Iitaka fibration $f: S \to B$, defined by the pluri-canonical systems, is the unique elliptic fibration on $S$. Therefore, any automorphism of $S$ preserves $f$, i.e., $\text{Aut}(S) = \text{Aut}_f(S)$. In this case, the exact sequences (2.1) and (2.3) can be rewritten as

$$1 \to \text{Aut}_B(S) \to \text{Aut}(S) \to \text{Aut}(B)$$

and

$$1 \to \text{Aut}_B(S) \cap \text{Aut}_s(S) \to \text{Aut}_s(S) \to \text{Aut}(B).$$

The image of $\text{Aut}(S) \to \text{Aut}(B)$ is finite by Corollary 2.3 (see also [20, Proposition 1.2]).

The following fact about elliptic fibrations will be used later on.

Lemma 2.4. Let $f: S \to B$ be a relatively minimal elliptic fibration. Then one has

$q(S) \leq g(B) + 1,$

and the equality holds if and only if there exist a smooth projective curve $\tilde{B}$, an elliptic curve $F$ and a finite group $G$ acting on $\tilde{B}$ and $F$ such that $S \cong (\tilde{B} \times F)/G$, $B \cong \tilde{B}/G$, the action of $G$ on $F$ is by translations, and the diagram

$$
\begin{array}{ccc}
S & \cong & (\tilde{B} \times F)/G \\
\downarrow f & & \downarrow \\
B & \cong & \tilde{B}/G
\end{array}
$$

commutes, where the right vertical arrow is the natural projection.

Proof. By [22, Lemma 1.6] (see also [7, Lemme]), one has the inequality $q(S) \leq g(B) + 1$. Moreover, in the equality case, one has $\chi(O_S) = 0$ and thus $f$ is an elliptic quasi-bundle, i.e., the possible singular fibers of $f$ are multiples of smooth elliptic curves [22, Lemma 1.5]. Then the existence of the commutative diagram (2.6) follows (see [23]). We remark that the action of $G$ on $F$ is necessarily by translations, since $g(B) + 1 = q(S) = g(B) + g(F/G)$ and hence $g(F/G) = 1 = g(F)$. □

Now we state the main result of this section.

Proposition 2.5. Let $f: S \to B$ be an elliptic fibration. Then the group $\text{Aut}_B(S) \cap \text{Aut}_s(S)$ of fiber-preserving symplectic automorphisms acts trivially on $\text{CH}_0(S)_\text{alb}$.

We need some preparations before giving the proof of Proposition 2.5 at the end of this section.

First, define

$$\text{CH}_0(S)_f := \ker(\text{CH}_0(S) \to \text{CH}_0(B)),$$

and call it the $f$-kernel of $\text{CH}_0(S)$. The following elementary observation about $\text{CH}_0(S)_f$ is the basis of further arguments.

Lemma 2.6 (See [9]). For any $\alpha \in \text{CH}_0(S)_f$, there is a positive integer $d$ such that

$$d\alpha = \sum_i \alpha_i \in \text{CH}_0(S),$$

where $\deg \alpha_i = 0$ and $\text{supp}(\alpha_i)$ is contained in a single smooth fiber of $f$ for each $i$.  

Proof. By [25, Fact 3.3], we can assume that supp(α) is contained in some union of smooth fibers of \( f \).
Take an ample smooth curve \( C \subset S \), and denote by \( d \) the degree of \( f \mid_C : C \to B \). Write \( \alpha = \sum_i n_i[p_i] \).
Since \( \alpha \in \ker(f_*), \) we have
\[
f_*\alpha = \sum_i n_i[f(p_i)] = 0 \in \text{CH}_0(B).
\]
It follows that
\[
\sum_i n_i F_{p_i} = f^* \left( \sum_i n_i[f(p_i)] \right) = 0 \in \text{Pic}(S),
\]
and hence
\[
\left( \sum_i n_i F_{p_i} \right) \cdot C = 0 \in \text{CH}_0(S),
\]
where \( F_{p_i} \) denotes the fiber containing \( p_i \). Now we can write
\[
d\alpha = \sum_i n_i d[p_i] - \left( \sum_i n_i F_{p_i} \right) \cdot C = \sum_i n_i (d[p_i] - [F_{p_i} \cdot C]).
\]
Taking \( \alpha_i = n_i (d[p_i] - [F_{p_i} \cdot C]) \), we have deg\( \alpha_i = 0 \) and supp\( (\alpha_i) \subset F_{p_i} \). The proof of the lemma is completed. \( \square \)

By the universal property of Albanese maps, we have a commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0(S)_{\text{hom}} & \xrightarrow{f_*} & \text{CH}_0(B)_{\text{hom}} \\
\text{alb}_S & \downarrow & \text{alb}_B \\
\text{Alb}(S) & \xrightarrow{\text{alb}} & \text{Alb}(B).
\end{array}
\]

By the Abel-Jacobi theorem, al\( b_B : \text{CH}_0(B)_{\text{hom}} \to \text{Alb}(B) \) is an isomorphism. It follows from (2.7) that
\[
\text{CH}_0(S)_{\text{alb}} = \ker(\text{alb}_S) \subset \ker(f_*) = \text{CH}_0(S)_f,
\]
and \( \text{CH}_0(S)_{\text{alb}} = \text{CH}_0(S)_f \) if and only if the induced map \( \text{Alb}(S) \to \text{Alb}(B) \) is an isomorphism.

We recall how the translations of a smooth elliptic fiber \( F \) of \( f \) act on its cycle classes and holomorphic one-forms. The universal cover of the (complex) elliptic curve \( F \) is \( \mathbb{C} \), and \( F \cong \mathbb{C}/\Gamma \) for some lattice \( \Gamma \subset \mathbb{C} \). Any \( \bar{c} \in F \) determines an automorphism \( \tau_{\bar{c}} : \bar{z} \mapsto \bar{z} + \bar{c} \) of \( F \), called the translation by \( \bar{c} \); it is descended from the usual translation \( \tau_{\bar{c}} : z \mapsto z + c \) on \( \mathbb{C} \).

Note that the translation \( \tau_{\bar{c}} \) induces the trivial action on \( H^0(F,K_F) \). In fact, a basis element \( \xi \) of the one-dimensional vector space \( H^0(F,K_F) \) is descended from the one form \( dz \) on the universal cover \( \mathbb{C} \). Since \( \tau_{\bar{c}}^* dz = d(z + c) = dz \) on \( \mathbb{C} \), we also have \( \tau_{\bar{c}}^* \xi = \xi \) on \( F \).

Fixing a point \( \bar{e} \in F \) as the origin, one has the identifications
\[
F \cong \text{Pic}^0(F) \cong \text{CH}_0(F)_{\text{hom}}
\]
by sending \( \bar{z} \in F \) to \( \mathcal{O}_F([\bar{z} - [\bar{e}]) \in \text{Pic}^0(F) \) and to \( [\bar{z}] - [\bar{e}] \in \text{CH}_0(F)_{\text{hom}}, \) respectively. In \( \text{CH}_0(F)_{\text{hom}}, \) we have by the Abel-Jacobi theorem
\[
\tau_{e*}([\bar{z}] - [\bar{e}]) = [\bar{z} + e] - [\bar{e} + e] = [\bar{z}] - [\bar{e}],
\]
so \( \tau_{\bar{c}} \) induces the trivial action on \( \text{CH}_0(F)_{\text{hom}}. \)

**Lemma 2.7.** Let \( f : S \to B \) be an elliptic fibration. Then for any \( \sigma \in \text{Aut}_B(S) \) such that its restriction \( \sigma_f \) to a general fiber \( F \) is a translation, the induced homomorphism \( \sigma_* : \text{CH}_0(S)_{f,Q} \to \text{CH}_0(S)_{f,Q} \) is the identity.
Proof. Take any $\alpha \in \mathrm{CH}(S)_f$. By Lemma 2.6, we can write $d\alpha = \sum \alpha_i \in \mathrm{CH}_0(S)$, where $d$ is a positive integer, and for each $i$, $\deg \alpha_i = 0$ and $\text{supp}(\alpha_i)$ is contained in a single smooth fiber, say $F_i$. Since $\sigma_F$ is a translation of $F_i$ and $\deg \alpha_i = 0$, we have $\sigma_{F,*}(\alpha_i) = \alpha_i$, viewed as elements in $\mathrm{CH}_0(F_i)$. Pushing the equality to $S$ by the inclusion map $F_i \hookrightarrow S$, we obtain $\sigma_*(\alpha_i) = \alpha_i \in \mathrm{CH}_0(S)$. Therefore, we have

$$\sigma_*(d\alpha) = \sum_i \sigma_*(\alpha_i) = \sum_i \alpha_i = d\alpha.$$ 

The proof is completed. \hfill $\square$

Corollary 2.8. Let $f : S \to B$ be an elliptic fibration. Then for any $\sigma \in \text{Aut}_B(S)$ that induces a translation on a general fiber $F$, its action on the Albanese kernel $\sigma_* : \mathrm{CH}_0(S)_{\text{alb}} \to \mathrm{CH}_0(S)_{\text{alb}}$ is the identity.

Proof. By Lemma 2.7, $\sigma_* : \mathrm{CH}_0(S)_{\text{alb},Q} \to \mathrm{CH}_0(S)_{\text{alb},Q}$ is the identity. Since $\mathrm{CH}(S)_{\text{alb}}$ is torsion free by [21], we infer that $\sigma_*$ is the identity on $\mathrm{CH}_0(S)_{\text{alb}}$. \hfill $\square$

Let $f : S \to B$ be a relatively minimal elliptic fibration. In order to describe the fiber-preserving symplectic automorphisms of $S$, we take a closer look at the canonical bundle formula (see [2, Chapter V, Theorem 12.1] and its proof, and ultimately [16, Theorem 12.1])

$$\omega_S = f^*(f_*\omega_{S/B} \otimes \omega_B) \otimes \mathcal{O}_S\left(\sum_i (m_i - 1)F_i\right). \tag{2.8}$$

where $\omega_S = \mathcal{O}_S(K_S)$ and $\omega_B = \mathcal{O}_B(K_B)$ are the canonical sheaves of $S$ and $B$, respectively, $\omega_{S/B} = \omega_S \otimes f^*\omega_B^{-1}$ is the relative canonical sheaf of $f$, and $m_i F_i$’s are the multiple fibers of $f$. There is a natural inclusion of invertible sheaves

$$f^*(f_*\omega_{S/B} \otimes \omega_B) \hookrightarrow \omega_S,$$

which is an isomorphism over $B^0 := \{ b \in B \mid f^*b \text{ is smooth} \}$ and which induces an isomorphism of global sections

$$f^* : H^0(B, f_*\omega_{S/B} \otimes \omega_B) \cong H^0(S, \omega_S). \tag{2.9}$$

Analytically locally around each $b \in B^0$, there is a small coordinate disk $b \in \Delta \cong \{ t \in \mathbb{C} \mid |t| < \epsilon \}$ such that the sections of $f_*\omega_{S/B} \otimes \omega_B$ have the form

$$\xi_t \otimes h(t)dt,$$

where $h(t)$ is a holomorphic function on $\Delta$ and $\xi_t$ is a basis of $H^0(F_t, K_{F_t})$, varying holomorphically in $t \in \Delta$. Pulling it back to $S$, we obtain a description, which is local in $B$, of the global sections $\omega \in H^0(S, \omega_S)$:

$$\omega = h(t)\xi_t \wedge dt. \tag{2.10}$$

This is used in the proof of the next lemma.

Lemma 2.9. Let $f : S \to B$ be an elliptic fibration with $p_g(S) > 0$. Then a fiber-preserving automorphism $\sigma \in \text{Aut}_B(S)$ is symplectic if and only if it induces translations on the smooth fibers.

Proof. Let $\sigma \in \text{Aut}_B(S)$ be a nontrivial fiber-preserving automorphism and $\sigma_F := \sigma|_F$ be its restriction to a smooth fiber $F$ of $f$.

(i) Suppose that $\sigma \in \text{Aut}_B(S)$ is symplectic. We want to show that $\sigma|_F$ is a translation, i.e., it acts freely on $F$. Suppose on the contrary that $\sigma|_F$ has a fixed point $p$. Then $\sigma|_F$ is of the finite order, and $F/\langle \sigma_F \rangle$ has the genus 0 by the Riemann-Hurwitz formula. It follows that $\sigma$ is of the finite order, and a resolution $X$ of the quotient surface $S/\langle \sigma \rangle$ is a $\mathbb{P}^1$-fibration over $B$. But then

$$H^0(S, K_S) = H^0(S, K_S)^{\sigma} \cong H^0(X, K_X) = 0.$$

This contradicts the assumption that $p_g(S) > 0$. 

Du J B et al. Sci China Math March 2023 Vol. 66 No. 3 449

1. [2] Liu J. et al. Sci China Math March 2023 Vol. 66 No. 3 449
(ii) Now suppose that \( \sigma_F \in \text{Aut}(F) \) is a translation. Then \( \sigma_F^* \xi = \xi \) for \( \xi \in H^0(F,K_F) \) (see the discussion above Lemma 2.7). Taking a local coordinate \( t \) around \( f(F) \) in \( B \), we can write locally \( \omega = h(t)\xi_t \wedge dt \) as in (2.10), where \( \xi_t \in H^0(F_t,K_{F_t}) \) is a basis element, so
\[
\sigma^* \omega = \text{id}_B^* h(t)\sigma_F^* \xi_t \wedge \text{id}_B^* dt = h(t)\xi_t \wedge dt = \omega.
\]
It follows that \( \sigma \) is a symplectic automorphism of \( S \).

**Lemma 2.10.** Let \( f: S \to B \) be an elliptic surface. Suppose that a fiber-preserving automorphism \( \sigma \in \text{Aut}_B(S) \) induces translations on smooth fibers. Then it induces the trivial action on \( H^1_B(S) = H^0(S,\Omega^1_B) \).

**Proof.** Since \( \sigma \in \text{Aut}_B(S) \) descends to a fiber-preserving automorphism of the relatively minimal model of \( S \), we can assume without loss of generality that \( f \) is already relatively minimal.

By Lemma 2.4, we have \( q(S) \leq g(B) + 1 \). If \( q(S) = g(B) \), then
\[
H^0(S,\Omega^1_B) = f^* H^0(B,K_B).
\]

Since \( \sigma \) induces the trivial action on \( B \), it induces the trivial action on \( H^0(B,K_B) \) and hence also on \( H^0(S,\Omega^1_B) \).

If \( q(S) = g(B) + 1 \), then \( S \equiv (\tilde{B} \times F)/G \) as in (2.6). Since \( \sigma \) induces \( \text{id}_B \) on \( B \) and a translation on the fiber \( F \), one sees that it induces the trivial action on \( H^0(S,\Omega^1_B) \).

Finally, we give the proof of Proposition 2.5.

**Proof of Proposition 2.5.** If \( p_g(S) = 0 \), then \( \text{CH}_0(S)_{\text{alb}} = 0 \) by [9] and there is nothing to prove. In the case \( p_g(S) > 0 \), it suffices to combine Lemma 2.9 and Corollary 2.8.

3 Elliptic surfaces with \( \chi(\mathcal{O}_S) \geq 3 \)

In this section, we prove the following theorem.

**Theorem 3.1.** Let \( S \) be a smooth projective surface with \( \kappa(S) = 1 \) and \( \chi(\mathcal{O}_S) \geq 3 \). Then \( \text{Aut}_s(S) \) acts trivially on \( \text{CH}_0(S)_{\text{alb}} \).

**Proof.** Since \( \kappa(S) = 1 \), there is a unique elliptic fibration \( f: S \to B \), and any automorphism of \( S \) preserves the fibration \( f \).

By Proposition 2.5, it suffices to show that \( \text{Aut}_s(S) = \text{Aut}_B(S) \cap \text{Aut}_s(S) \), i.e., any symplectic automorphism of \( S \) preserves the fibers of \( f \).

By the canonical bundle formula (2.8), we have
\[
|K_S| = f^*|K_B + L| + \sum_i (m_i - 1)F_i.
\]

Thus the canonical map \( \varphi_S \) of \( S \), induced by the linear system \( |K_S| \), factors as
\[
\varphi_S: S \xrightarrow{f} B \xrightarrow{\varphi_B} \mathbb{P}^{p_g-1},
\]
where \( \varphi_B \) is the map defined by the linear system \( |K_B + L| \).

Now, since \( \chi(\mathcal{O}_S) \geq 3 \), we have
\[
\deg(K_B + L) = 2g(B) - 2 + \deg L = 2g(B) - 2 + \chi(\mathcal{O}_S) \geq 2g(B) + 1.
\]

It follows that \( K_B + L \) is very ample and hence \( \varphi_B \) is an embedding. A symplectic automorphism \( \sigma \) induces an automorphism \( \sigma_B \in \text{Aut}(B) \) and the identity on \( \mathbb{P}^{p_g-1} \), and they act equivariantly on the respective varieties. Since \( \varphi_B \) is an embedding, it can only happen that \( \sigma_B = \text{id}_B \). In other words, \( \sigma \) preserves each fiber of \( f \), which is what we wanted to prove.
4 Reduction to the Jacobian fibration

Given an elliptic fibration \( f: S \to B \), a natural idea is to reduce the problem at hand to the Jacobian fibration \( j: J \to B \). The following construction has been used by [9] in proving the vanishing of \( \text{CH}(S)_{\text{alg}} \) for surfaces with \( p_g(S) = 0 \) and \( \kappa(S) \leq 1 \). We apply it to deal with the fibration-preserving automorphisms of \( S \).

For any irreducible curve \( C \subset S \), horizontal with respect to \( f \), one can define a rational dominant map \( \phi_C: S \to J \) as follows: to a point \( p \) on a smooth fiber \( F_b \) over \( b \in B \), we associate

\[
\phi_C(p) := d[p] - C|_{F_b} \in j^*b = \text{Pic}^0(F_b),
\]

where \( d \) is the degree of the finite morphism \( f|_C: C \to B \). It is clear that \( \deg \phi_C = d^2 \).

**Lemma 4.1** (See [9, p.138] and [24, Proof of Theorem 11.10]). For a smooth ample curve \( C \subset S \), the induced homomorphism

\[
\phi_{C*}: \text{CH}_0(S)_{f,Q} \to \text{CH}_0(J)_{f,Q}
\]

is an isomorphism, which restricts to an isomorphism between the Albanese kernels \( \phi_{C*}: \text{CH}_0(S)_{\text{alg},Q} \to \text{CH}_0(J)_{\text{alg},Q} \).

**Proof.** We define a homomorphism \( \lambda: \text{CH}_0(J)_{f,Q} \to \text{CH}_0(S)_{f,Q} \) as follows: for any \( \gamma \in \text{CH}_0(J)_{f,Q} \), we can assume that \( \text{supp} (\gamma) \) is on a smooth fiber \( j^*b \) of \( j \) and \( \gamma = [\gamma] - [\alpha_0] \), where \( \alpha_0 \) denotes the origin of \( j^*b = \text{Pic}^0(f^*b) \). Then we set

\[
\lambda(\gamma) = \frac{1}{d^2} ([p_1'] + \cdots + [p'_d]) - ([p_1] + \cdots + [p_d]),
\]

where \([p_1] + \cdots + [p_d] = C|_{J^*b}\), and for each \( 1 \leq i \leq d \), \( p'_i \) is the unique point of \( f^*b \) such that \([p'_i] - [p_i] = \gamma' \in \text{Pic}^0(f^*b) \). Then it is straightforward to check that \( \lambda \) is the inverse of \( \phi_{C*} \).

Note that the irregularities of \( S \) and \( J \) are the same by the following Lemma 4.2, and thus \( \phi_C \) induces an isomorphism \( \phi_{C*}: \text{Alb}(S)_{Q} \cong \text{Alb}(J)_{Q} \). In view of the following commutative diagram, where the rows are exact, we infer that the left vertical map \( \phi_{C*}: \text{CH}_0(S)_{\text{alg},Q} \to \text{CH}_0(J)_{\text{alg},Q} \) is an isomorphism

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{CH}_0(S)_{\text{alg},Q} \longrightarrow \text{CH}_0(S)_{f,Q} \longrightarrow \text{Alb}(S)_{Q} \\
\phi_{C*} & \cong & \phi_{C*} \\
0 & \longrightarrow & \text{CH}_0(J)_{\text{alg},Q} \longrightarrow \text{CH}_0(J)_{f,Q} \longrightarrow \text{Alb}(J)_{Q}.
\end{array}
\]

The proof is completed. \( \square \)

Many of the numerical invariants of an elliptic fibration and its Jacobian fibration turn out to be the same. We give a proof of this fact for lack of an adequate reference.

**Lemma 4.2** (See [11, Proposition 5.3.6, p.308, and Corollaries 5.3.4 and 5.3.5, p.310]). The following equalities hold:

\[
\chi(O_S) = \chi(O_J), \quad p_g(S) = p_g(J), \quad q(S) = q(J).
\]

If \( f \) is relatively minimal, then for each \( b \in B \), \( e(f^*b) = e(j^*b) \), where \( e(\cdot) \) denotes the Euler characteristic of a topological space.

**Proof.** Replacing \( f: S \to B \) with the relatively minimal elliptic fibration does not change the invariants \( \chi(O_S), p_g(S) \) and \( q(S) \), as well as its Jacobian fibration. Thus we can assume that \( f \) is relatively minimal. Then we have \( K_S^2 = 0 \), and hence by the Noether formula,

\[
12\chi(O_S) = e(S). \tag{4.1}
\]

On the other hand, \( j: J \to B \) is a relatively minimal fibration such that for each \( b \in B \), one has \( e(f^*b) = e(j^*b) \). It follows that

\[
e(J) = e(S) \quad \text{and} \quad 12\chi(O_J) = e(J). \tag{4.2}
\]
Combining (4.1) and (4.2), we obtain \( \chi(O_S) = \chi(O_J) \).

Since \( \chi(O_J) = 1 - q(J) + p_g(J) \), it remains to show \( q(S) = q(J) \). First, we have the easy inequalities

\[
g(B) \leq q(J) \leq q(S) \leq g(B) + 1,
\]

where the second inequality holds because of the existence of dominant maps such as \( \phi_C \) from \( S \) to \( J \), and the last inequality is given by Lemma 2.4.

Thus, if \( q(S) = g(B) \), then \( q(J) = g(S) = g(B) \).

Now suppose \( q(S) = g(B) + 1 \). Then \( S \cong (\tilde{B} \times F)/G \) as in (2.6), and it is straightforward to check that \( J = B \times F \). Therefore,

\[
q(J) = g(B) + g(F) = g(B) + 1 = q(S).
\]

The proof is completed. \( \square \)

The induced map \( \phi_{C,s} \) of zero cycles and holomorphic forms actually does not depend on the choice of the curve \( C \), as the following lemma shows.

**Lemma 4.3.** Let \( f: S \to B \) be an elliptic surface, and \( C \) and \( C' \) be two smooth ample curves on \( S \). Suppose that \( \deg f|_{C'} = \deg f|_{C} \). Then the rational maps \( \phi_C \) and \( \phi_{C'} \) from \( S \) to \( J \) induce the same maps between the \( 0 \)-th Chow groups and the spaces of holomorphic forms, i.e.,

1. \( \phi_{C,s} = \phi_{C',s}: CH_0(S)_{f,Q} \to CH_0(J)_{f,Q} \), and
2. \( \phi_{C}^* = \phi_{C'}^* : H^1(J) \to H^1(J) \) for any \( i \).

**Proof.** Let \( d \) be the degree \( \deg f|_{C'} = \deg f|_{C} \). Then

\[
\phi_{C'}(p) = d[p] - C'|_{F_b} = d[p] - C|_{F_b} + (C - C')|_{F_b} = \phi_C(p) + (C - C')|_{F_b} \in j^*b,
\]

so \( \phi_{C'}(p) \) and \( \phi_C(p) \) differ by a translation of \( F_b \) by \( (C - C')|_{F_b} \). These translations along the fibers glue to an automorphism of \( J \) over \( B \), which we denote by \( \phi_{C-C'} \). In other words, we have a commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\phi_C} & J \\
\downarrow{\phi_{C'}} & & \downarrow{\phi_{C-C'}} \\
J
\end{array}
\]

(4.3)

where \( \phi_{C-C'} \in Aut_B(J) \) induces translations on general fibers of \( j: J \to B \).

By Lemmas 2.7, 2.9 and 2.10, \( \phi_{C-C'} \) induces the identity map on \( CH_0(J)_{f,Q} \) as well as on \( H^1(J) \). In view of (4.3), the desired equalities \( \phi_{C,s} = \phi_{C',s} \) and \( \phi_{C}^* = \phi_{C'}^* \) follow. \( \square \)

By the universal property of \( J \), any automorphism \( \sigma \in \text{Aut}(J) \) preserving the fibration \( f \) induces an automorphism \( \sigma_J \in \text{Aut}_J(J) \) such that they induce the same automorphism \( \sigma_B \in \text{Aut}(B) \) on the base curve \( B \) and the following diagram is commutative:

\[
\begin{array}{ccc}
S & \xrightarrow{\phi_C} & J \\
\downarrow{\phi_{\sigma(C)}} & & \downarrow{\phi_{\sigma(J)}} \\
S & \xrightarrow{\sigma_J} & J
\end{array}
\]

(4.4)

This defines a group homomorphism \( \text{Aut}_f(S) \to \text{Aut}_J(J) \), \( \sigma \mapsto \sigma_J \). For a point \( b \in B \) such that the fiber \( F_b := f^*(b) \) is smooth, we have \( j^*b = \text{Pic}^0(F_b) \), and for any \( \alpha \in \text{Pic}^0(F_b) \),

\[
\sigma_J(\alpha) = (\sigma^{-1})^*(\alpha) = \text{Pic}^0(\sigma(F_b)) = j^*(\sigma_B(b)).
\]

**Lemma 4.4.** Let \( f: S \to B \) be an elliptic fibration and let \( j: J \to B \) be the Jacobian of \( f \). Then an automorphism \( \sigma \in \text{Aut}_f(S) \) acts as the identity on \( CH_0(S)_{f, Q} \) (resp. \( CH_0(S)_{\text{alb}, Q} \), \( H^{2,0}(S), H^{1,0}(S) \)) if and only if so does the induced automorphism \( \sigma_J \in \text{Aut}_J(J) \) on \( CH_0(J)_{f, Q} \) (resp. \( CH_0(J)_{\text{alb}, Q} \), \( H^{2,0}(J), H^{1,0}(J) \)).
Proof. It follows from Lemma 4.3 that $\phi_C$ and $\phi_{\sigma(C)}$ in (4.4) induce the same maps on the Chow groups as well as on the spaces $H^i_0$, and they are all the isomorphisms by Lemmas 4.1 and 4.2. In view of (4.4), the assertion of the lemma follows.

Corollary 4.5. Let $f: S \to B$ be an elliptic fibration and let $j: J \to B$ be the Jacobian of $f$. Then $\sigma \in \text{Aut}_f(S)$ acts trivially on $\text{CH}_0(S)_{\text{ab}}$ if and only if $\sigma J \in \text{Aut}_j(J)$ acts trivially on $\text{CH}_0(J)_{\text{ab}}$.

Proof. The natural maps $\text{CH}_0(S)_{\text{ab}} \to \text{CH}_0(S)_{\text{ab}, J}$ and $\text{CH}_0(J)_{\text{ab}} \to \text{CH}_0(J)_{\text{ab}, J}$ are injective by [21], and hence the corollary follows from Lemma 4.4.

The following lemma on the orders of the induced automorphisms $\sigma J$ and $\sigma B$ will be used in Section 5.

Lemma 4.6. Let $f: S \to B$ be an elliptic fibration such that $p_g(S) > 0$, and let $j: J \to B$ be the Jacobian of $f$. Let $\sigma \in \text{Aut}_f(S) \cap \text{Aut}_s(S)$ be a symplectic fibration-preserving automorphism. Then the induced automorphisms $\sigma J \in \text{Aut}_s(J)$ and $\sigma B \in \text{Aut}_B(B)$ have the same order.

Proof. For any integer $n$, we have

$$j \circ \sigma_j^n = \sigma_B^n \circ j. \quad (4.5)$$

If the order $|\sigma_B|$ is infinite, then $\sigma J$ necessarily has the infinite order.

Now suppose that $\sigma B$ has the finite order $m$. Then $\sigma^m B$ lies in $\text{Aut}_B(S) \cap \text{Aut}_s(S)$. By Lemma 2.9, $\sigma^m$ induces translations on the smooth fibers $F_s$. It follows that $\sigma^m J$ induces the identity on $j^* b = \text{Pic}^0(F_s)$ and is itself the identity. Therefore, $|\sigma J|$ is finite, with the order dividing $m = |\sigma_B|$. On the other hand, $m$ divides $|\sigma J|$ by (4.5). We infer that $|\sigma J| = |\sigma B|$.

5 Elliptic surfaces with $\chi(\mathcal{O}_S) \leq 2$

In this section, we deal with elliptic surfaces $f: S \to B$ with $\chi(\mathcal{O}_S) \leq 2$.

We need a lemma for the action of the Klein group $(\mathbb{Z}/2\mathbb{Z})^2$ on $\text{CH}_0(S)$. It is based on the idea of “enough automorphisms” of [13].

Lemma 5.1. Let $S$ be a smooth projective surface. Let $G = \langle \sigma, \tau \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ be a subgroup of $\text{Aut}(S)$ such that the smooth models of the quotient surfaces $S/\langle \tau \rangle$ and $S/\langle \sigma \tau \rangle$ are not of general type and have vanishing geometric genera. Then $\sigma$ induces the identity on the $\text{CH}_0(S)_{\text{ab}}$.

Proof. Let $X_1$ and $X_2$ be the smooth models of $S/\langle \tau \rangle$ and $S/\langle \sigma \tau \rangle$, respectively. Then by [9], we have

$$\text{CH}_0(S)_{\text{ab}, \mathbb{Q}} = \text{CH}_0(X_1)_{\text{ab}, \mathbb{Q}} = 0, \quad \text{CH}_0(S)_{\text{ab}, \mathbb{Q}}^\sigma = \text{CH}_0(X_2)_{\text{ab}, \mathbb{Q}} = 0.$$

Since $\tau$ and $\sigma \tau$ are both involutions, they act as $-\text{id}$ on $\text{CH}_0(S)_{\text{ab}, \mathbb{Q}}$. It follows that $\sigma = (\sigma \tau) \tau$ acts trivially on $\text{CH}_0(S)_{\text{ab}, \mathbb{Q}}$. Since $\text{CH}_0(S)_{\text{ab}}$ has no torsion by [21], the lemma follows.

Lemma 5.2. Let $f: S \to B$ be a smooth projective elliptic surface. Then any symplectic involution of $S$ acts trivially on $\text{CH}_0(S)_{\text{ab}}$.

Proof. If $\kappa(S) \leq 0$, this is a consequence of [9,25].

In the following, we assume that $\kappa(S) = 1$. Let $\sigma \in \text{Aut}_s(S)$ be a symplectic involution. By Lemma 4.4, it suffices to prove that the induced automorphism $\sigma J \in \text{Aut}_s(J)$ acts trivially on $\text{CH}_0(J)_{\text{ab}}$, where $j: J \to B$ is the Jacobian fibration of $f$. Note that $|\sigma J| \leq |\sigma| = 2$. If $\sigma J = \text{id}_J$, then there is nothing to prove. We can thus assume that $\sigma J$ is also an involution.

Note that the $a$-section of $j$ is preserved by $\sigma J$. Let $\tau \in \text{Aut}_B(J)$ be the involution that restricts to $-\text{id}_F$ on a general fiber $F$ of $j$. Then the subgroup $G = \langle \sigma J, \tau \rangle \subset \text{Aut}_B(J)$, generated by $\sigma J$ and $\tau$, is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. It is easy to see that the smooth models of the quotient surfaces $J/\langle \tau \rangle$ and $J/\langle \tau \sigma J \rangle$ have vanishing geometric genera. By Lemma 5.1, $\sigma J$ acts as the identity on $\text{CH}_0(J)_{\text{ab}}$.

Theorem 5.3. Let $f: S \to B$ be an elliptic fibration with $\chi(\mathcal{O}_S) = 2$. Then $\text{Aut}_f(S) \cap \text{Aut}_s(S)$ acts trivially on $\text{CH}_0(S)_{f, \mathbb{Q}}$.

As a consequence, $\text{Aut}_f(S) \cap \text{Aut}_s(S)$ acts trivially on $\text{CH}_0(S)_{\text{ab}}$.
Proof. Since \( \chi(\mathcal{O}_S) = 2 \), one has \( p_g(S) = q(S) + 1 \geq 1 \). Let \( j : J \rightarrow B \) be the Jacobian fibration of \( f \). By Lemma 4.4, it suffices to show that \( \sigma_j \) acts trivially on \( \text{CH}_1(J,\mathbb{Q}) \), where \( \sigma_j \in \text{Aut}_J(J) \cap \text{Aut}_s(J) \) is the automorphism induced by \( \sigma \).

First, we assume that \( q(J) = 0 \). In this case, \( J \) is an elliptic K3 surface. Recall that the induced automorphisms \( \sigma_j \in \text{Aut}(J) \) and \( \sigma_B \in \text{Aut}(B) \) have the same order by Lemma 4.6, which is finite by Corollary 2.3.

Now the triviality of the action of \( \sigma_J \) on \( \text{CH}_0(J,\mathbb{Q}) \) follows from the results of Voisin [25] and Huybrechts [12].

Next, we can assume that \( q(J) > 0 \). By the canonical bundle formula, we have \( |K_J| = j^*|K_B + L| \), where \( L \) is a line bundle of degree \( \chi(\mathcal{O}_J) = \chi(\mathcal{O}_S) = 2 \): Since \( \deg(K_B + L) = 2q(B) \), the linear system \( |K_B + L| \) is base point free and the map \( \varphi_B \) defined by \( |K_B + L| \) is a morphism. It follows that the canonical map \( \varphi_J \) of \( J \) is a morphism which factors as

\[
\varphi_J : J \xrightarrow{j} B \xrightarrow{\varphi_B} \mathbb{P}^{p_g - 1},
\]

where \( p_g(J) = p_g(S) \).

On the one hand, \( \deg(K_B + L) = 2g(B) = 2(p_g - 1) \) by the Riemann-Roch theorem. On the other hand, we know that \( \deg(K_B + L) = \deg(\varphi_B) \cdot \deg(\text{Im}(\varphi_B)) \) and \( \deg(\text{Im}(\varphi_B)) \geq \deg(\varphi_B) \leq 2 \).

Since \( \sigma_j \) acts trivially on \( H^0(J,\mathcal{O}_J) \), the morphism \( \varphi_B \) factors through the quotient map \( B \rightarrow B/\sigma_B \).

Therefore, one has \( |\sigma_J| = |\sigma_B| \leq \deg(\varphi_B) \leq 2 \). We are done by Lemma 5.2.

We have the following two immediate corollaries.

**Corollary 5.4.** Let \( S \) be a smooth projective surface with \( \kappa(S) = 1 \) and \( \chi(\mathcal{O}_S) = 2 \). Then \( \text{Aut}_s(S) \) acts trivially on \( \text{CH}_0(S)_{\text{ab}} \).

**Corollary 5.5.** Let \( f : S \rightarrow B \) be an elliptic K3 surface. Then \( \text{Aut}_J(S) \cap \text{Aut}_s(S) \) acts trivially on \( \text{CH}_0(S)_{\text{ab}} \).

Next, we treat the case \( \chi(\mathcal{O}_S) = 1 \).

**Theorem 5.6.** Let \( f : S \rightarrow B \) be a smooth projective elliptic surface with \( \chi(\mathcal{O}_S) = 1 \). Then

\[
\text{Aut}_s(S) \cap \text{Aut}_f(S) \text{ acts trivially on } \text{CH}_0(S)_{\text{ab}} \text{ if } p_g(S) = q(S) \notin \{1, 2\}. \text{ Otherwise, the image of the homomorphism }
\]

\[
\text{Aut}_s(S) \cap \text{Aut}_f(S) \rightarrow \text{Aut}(\text{CH}_0(S)_{\text{ab}})
\]

has the order at most 3.

**Proof.** If \( p_g(S) = q(S) = 0 \), then \( \text{CH}_0(J)_{\text{ab}} = 0 \) by [9], and there is nothing to prove.

So we may assume that \( p_g(S) = q(S) > 0 \). Let \( j : J \rightarrow B \) be the Jacobian fibration of \( f \). Let \( \sigma_j \in \text{Aut}_J(J) \cap \text{Aut}_s(J) \) and \( \sigma_B \in \text{Aut}(B) \) be the automorphisms induced by \( \sigma \). By Corollary 4.5, it suffices to show that \( \sigma_j \) acts trivially on \( \text{CH}_0(J)_{\text{ab}} \).

Since \( \chi(\mathcal{O}_S) = 1 \), the surface cannot be abelian or bielliptic, so \( \sigma_B \) is of the finite order by Corollary 2.3.

Also, \( \sigma_J \) has the same finite order as \( \sigma_B \) by Lemma 4.6.

By Lemma 4.2, \( \chi(\mathcal{O}_J) = \chi(\mathcal{O}_S) = 1 \) and thus \( q(J) = p_g(J) \). Since \( \chi(\mathcal{O}_J) > 0 \), the fibration \( j : J \rightarrow B \) cannot be an elliptic quasi-bundle and it follows that \( q(B) = q(J) \).

By the canonical bundle formula, we have \( |K_J| = j^*|K_B + L| \), where \( L \) is an invertible sheaf on \( B \) of degree \( \chi(\mathcal{O}_J) = 1 \). Then the canonical map of \( J \) factors through \( j \) followed by the map \( \varphi_B \) induced by the linear system \( |K_B + L| \) as in the proof of Theorem 5.3.

If \( g(B) = q(J) \geq 3 \), then

\[
|\sigma_J| = |\sigma_B| \leq \deg(\varphi_B) \leq 2.
\]

In this case, \( \sigma_J \) acts trivially on \( \text{CH}_0(J)_{\text{ab}} \) by Lemma 5.2.

If \( g(B) = q(J) = 2 \), then

\[
|\sigma_J| \leq \deg(\varphi_B) \leq 3.
\]
If \( g(B) = q(J) = 1 \), then \( \deg(K_B + L) = 1 \) and \( |K_B + L| \) consists of a unique element, say \( p \in B \), which is necessarily fixed by \( \sigma_B \). In these last two cases, the order \( |\sigma_j| \) is one of \( \{1, 2, 3, 4, 6\} \). It follows that either \( \sigma_2^j \) or \( \sigma_3^j \) has the order at most 2, and thus acts trivially on \( \text{CH}_0(J)_{\text{alb}} \) by Lemma 5.2. This completes the proof.

Now we deal with the case \( \chi(O_S) = 0 \).

**Theorem 5.7.** Let \( S \) be a smooth projective surface with \( \chi(O_S) = 0 \). Then \( \text{Aut}_s(S) \) acts trivially on \( \text{CH}_0(S)_{\text{alb}} \).

**Proof.** Since \( \chi(O_S) = 0 \), we have \( \kappa(S) \leq 1 \). If \( \kappa(S) \leq 0 \) or \( p_g(S) = 0 \), then the assertion follows from [9].

Therefore, we can assume that \( \kappa(S) = 1 \) and \( p_g(S) > 0 \). We can also assume that \( S \) is minimal. Let \( f: S \to B \) be the Iitaka fibration of \( S \) and \( j: J \to B \) be its Jacobian. For any \( \sigma \in \text{Aut}_s(S) \), the induced automorphism \( \sigma_j \in \text{Aut}_s(J) \cap \text{Aut}_j(J) \) has the finite order by Lemma 4.6 and Corollary 2.3.

By Lemma 4.4, it suffices to show that \( \sigma_j \) induces the trivial action on \( \text{CH}_0(J)_{\text{alb}, \mathbb{Q}} \). Since \( \chi(O_J) = \chi(O_S) = 0 \), \( j \) is a quasi-bundle and hence \( J \) is isogenous to a product of curves. It follows that the Chow motive \( h(J) \) is finite-dimensional in the sense of Kimura [15], and the assertion follows from Lemma 5.9. □

**Remark 5.8.** It would be interesting to give a direct proof of Theorem 5.7 without involving the theory of Chow motives.

The following lemma should be well known to experts. We write down a proof for lack of an adequate reference.

**Lemma 5.9.** Let \( S \) be a smooth projective surface with \( p_g(S) > 0 \). Assume that the Chow motive \( h(S) \) of \( S \) is finite-dimensional in the sense of Kimura [15]. Then any symplectic automorphism \( \sigma \in \text{Aut}_s(S) \) of the finite order acts as the identity on \( \text{CH}_0(S)_{\text{alb}} \).

**Proof.** The Chow motive of \( S \) has a Chow-K"unneth decomposition (see [14, Proposition 7.2.1])

\[
h(S) = h_0(S) \oplus h_1(S) \oplus h_2(S) \oplus h_3(S) \oplus h_4(S)
\]

in the category of Chow motives with rational coefficients. There is a further decomposition for \( h_2(S) \) (see [14, Proposition 7.2.3]), i.e.,

\[
h_2(S) = h_2^{\text{alg}}(S) \oplus t_2(S),
\]

where \( h_2^{\text{alg}}(S) \) denotes the algebraic part and \( t_2(S) \) denotes the transcendental part. We have

\[
\text{CH}_0(t_2(S)) = \text{CH}_0(S)_{\text{alb}, \mathbb{Q}} \quad \text{and} \quad H^2(t_2(S)) = H^2_0(S, \mathbb{Q}),
\]

where \( H^2_0(S, \mathbb{Q}) \) denotes the transcendental part of \( H^2(S, \mathbb{Q}) \).

Since the motive \( h(S) \) is finite-dimensional in the sense of Kimura [15], its direct summand \( t_2(S) \) is also finite-dimensional. For any symplectic automorphism \( \sigma \in \text{Aut}_s(S) \), it acts trivially on \( H^2_0(S, \mathbb{Q}) = H^2(t_2(S)) \). Therefore, \( (\Gamma_\sigma - \Delta_S)_j: t_2(S) \to t_2(S) \) is a numerically trivial morphism, where \( \Gamma_\sigma \) is the graph of \( \sigma \) and \( \Delta_S \subset S \times S \) is the diagonal. Then by Kimura’s nilpotence theorem [15, Proposition 7.5], \( (\Gamma_\sigma - \Delta_S)_i \), is nilpotent as an endomorphism of \( t_2(S) \).

It follows that the action of \( \sigma \) on \( \text{CH}_0(t_2(S)) = \text{CH}_0(S)_{\text{alb}, \mathbb{Q}} \) is unipotent. Since \( \sigma \) is of the finite order, we infer that \( \sigma \) acts as the identity on \( \text{CH}_0(S)_{\text{alb}, \mathbb{Q}} \). It also acts trivially on \( \text{CH}_0(S)_{\text{alb}} \), because \( \text{CH}_0(S)_{\text{alb}} \) has no torsion by [21]. □

Finally, strengthening the hypothesis in Conjecture 1.2, we obtain the following theorem.

**Theorem 5.10.** Let \( S \) be a smooth projective surface with \( \kappa(S) = 1 \). If an automorphism \( \sigma \in \text{Aut}(S) \) induces the trivial action on \( H^{i,0}(S) \) for \( i > 0 \), then it induces the trivial action on \( \text{CH}_0(S)_{\text{alb}} \).

**Proof.** Let \( f: S \to B \) be the Iitaka fibration of \( S \) and \( \sigma \) be an automorphism of \( S \) acting trivially on \( H^{i,0}(S) \) for \( i > 0 \). By Theorems 3.1, 5.3, 5.6 and 5.7, it is enough to show the result for surfaces with \( q(S) = p_g(S) \in \{1, 2\} \).
Let $\sigma_B \in \text{Aut}(B)$ be the automorphism induced by $\sigma$. It suffices to show that $\sigma_B = \text{id}_B$, since $\sigma \in \text{Aut}_{\mathbb{S}}(B) \cap \text{Aut}_S(B)$ and we can conclude by Proposition 2.5.

Observe that since $\sigma$ acts trivially on $H^{1,0}(S)$ and $f^*: H^{1,0}(B) \to H^{1,0}(S)$ is injective, $\sigma_B$ acts trivially on $H^{1,0}(B)$. It follows that $\sigma_B = \text{id}_B$ if $g(B) = 2$. In the case $g(B) = 1$, the automorphism $\sigma_B$ is necessarily a translation. Since $\chi(O_S) > 0$, by the holomorphic Lefschetz fixed point formula, the fixed locus $S^\sigma$ is non-empty. Thus the translation $\sigma_B$ fixes a nonempty subset $f(S^\sigma) \subset B$, and we infer that $\sigma_B = \text{id}_B$.

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