An Algebraic Approach to the Non-chromatic Adherence of the DP Color Function

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Abstract

DP-coloring (or correspondence coloring) is a generalization of list coloring that has been widely studied since its introduction by Dvořák and Postle in 2015. As the analogue of the chromatic polynomial of a graph $G$, $P(G, m)$, the DP color function of $G$, denoted by $P_{DP}(G, m)$, counts the minimum number of DP-colorings over all possible $m$-fold covers. A function $f$ is chromatic-adherent if for every graph $G$, $f(G, a) = P(G, a)$ for some $a \geq \chi(G)$ implies that $f(G, m) = P(G, m)$ for all $m \geq a$. It is known that the DP color function is not chromatic-adherent, but there are only two known graphs that demonstrate this. Suppose $G$ is an $n$-vertex graph and $\mathcal{H}$ is a 3-fold cover of $G$, in this paper we associate with $\mathcal{H}$ a polynomial $f_{G, \mathcal{H}} \in \mathbb{F}_3[x_1, \ldots, x_n]$ so that the number of non-zeros of $f_{G, \mathcal{H}}$ equals the number of $\mathcal{H}$-colorings of $G$. We then use a well-known result of Alon and Füredi on the number of non-zeros of a polynomial to establish a non-trivial lower bound on $P_{DP}(G, 3)$ when $2n > |E(G)|$. Finally, we use this bound to show that there are infinitely many graphs that demonstrate the non-chromatic-adherence of the DP color function.

Keywords. DP-coloring, correspondence coloring, DP color function, graph polynomial

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1 Introduction

In this paper all graphs are nonempty, finite, undirected loopless multigraphs. For the purposes of this paper, a simple graph is a multigraph without any parallel edges between vertices. Generally speaking we follow West\textsuperscript{[31]} for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. If $G$ is a graph and $S, U \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$, and we use $E_G(S, U)$ for the set consisting of all the edges in $E(G)$ that have one endpoint in $S$ and the other in $U$. When $u, v \in V(G)$ we use $E_G(u, v)$ to denote the set of edges in $E(G)$ with endpoints $u$ and $v$ (note $E_G(u, v) = E_G(v, u)$), and we let $e_G(u, v)$ denote the number

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of elements in $E_G(u,v)$. When $G$ is a multigraph, the \textit{underlying graph} of $G$ is the simple graph formed by deleting all parallel edges of $G$. When $G$ is a simple graph, we can refer to edges by their endpoints; for example, if $u$ and $v$ are adjacent in the simple graph $G$, $uv$ or $vu$ refers to the edge between $u$ and $v$. We say that a multigraph $G$ is $k$-degenerate if every subgraph of $G$ has a vertex of degree at most $k$. If $G$ and $H$ are vertex disjoint multigraphs, we write $G \lor H$ for the join of $G$ and $H$.

### 1.1 DP-Coloring

In classical vertex coloring we wish to color the vertices of a graph $G$ with up to $m$ colors from $[m]$ so that adjacent vertices receive different colors, a so-called \textit{proper} $m$-\textit{coloring}. The smallest $k$ for which a proper $k$-coloring of $G$ exists is called the \textit{chromatic number} of $G$, and it is denoted $\chi(G)$. List coloring is a well-known variation on classical vertex coloring that was introduced independently by Vizing [29] and Erdős, Rubin, and Taylor [13] in the 1970s. For list coloring, we associate a \textit{list assignment} $L$ with a graph $G$ such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$. Then, $G$ is \textit{L-colorable} if there is a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we say $f$ is a \textit{proper} $L$-\textit{coloring} of $G$). A list assignment $L$ is called an $m$-\textit{assignment} for $G$ if $|L(v)| = m$ for each $v \in V(G)$. We say $G$ is $m$-\textit{choosable} if $G$ is $L$-colorable whenever $L$ is an $m$-assignment for $G$.

In 2015, Dvořák and Postle [12] introduced a generalization of list coloring called DP-coloring (they called it correspondence coloring) in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. DP-coloring has been extensively studied over the past 7 years (see e.g., [3, 4, 16, 18, 22, 23, 25]). Intuitively, DP-coloring is a variation on list coloring where each vertex in the graph still gets a list of colors, but identification of which colors are different can change from edge to edge. Due to this property, DP-coloring multigraphs is not as simple as coloring the corresponding underlying graph (see [5]). Following [24], we now give the formal definition. Suppose $G$ is a multigraph. A \textit{cover} of $G$ is a triple $\mathcal{H} = (L, H, M)$ where $L$ is a function that assigns to each $v \in V(G)$ a set $L(v) = \{(v, a) : a \in A_v\}$ where $A_v$ is some nonempty finite set, $H$ is a multigraph with vertex set $\bigcup_{v \in V(G)} L(v)$, and $M$ is a function that assigns to each $e \in E(G)$ a matching $M(e)$ with the property that each edge in $M(e)$ has one endpoint in $L(u)$ and the other endpoint in $L(v)$ where $u$ and $v$ are the endpoints of $e$. Moreover, $L$, $H$, and $M$ satisfy the following conditions:

1. For every $u \in V(G)$, $H[L(u)] = K_{|L(u)|}$;
2. For distinct edges $e_1, e_2 \in E(G)$, $M(e_1) \cap M(e_2) = \emptyset$;
3. For distinct vertices $u, v \in V(G)$, the set of edges between $L(u)$ and $L(v)$ in $H$ is $\bigcup_{e \in E_G(u,v)} M(e)$.

Note that by conditions (2) and (3) in the above definition $H$ may contain parallel edges. Furthermore, note that if $G$ is a simple graph, $H$ must be simple.

Suppose $\mathcal{H} = (L, H, M)$ is a cover of $G$. An $\mathcal{H}$-\textit{coloring} of $G$ is an independent set in $H$ of size $|V(G)|$. It is immediately clear that an independent set $I \subseteq V(H)$ is an $\mathcal{H}$-coloring of $G$ if and only if $|I \cap L(u)| = 1$ for each $u \in V(G)$. We say $\mathcal{H}$ is $m$-\textit{fold} if $|L(u)| = m$ for each $u \in V(G)$. Moreover, we say that $\mathcal{H}$ is a \textit{full} $m$-\textit{fold cover} of $G$ if $|E_H(L(u), L(v))| = e_G(u,v)m$.
whenever \( u \) and \( v \) are distinct vertices of \( G \). The **DP-chromatic number** of \( G \), denoted \( \chi_{DP}(G) \), is the smallest \( k \) such that an \( \mathcal{H} \)-coloring of \( G \) exists whenever \( \mathcal{H} \) is a \( k \)-fold cover of \( G \). Clearly, \( \chi(G) \leq \chi_{DP}(G) \), and if \( G \) is \( d \)-degenerate, then \( \chi_{DP}(G) \leq d + 1 \).

It is easy to demonstrate that DP-coloring is a generalization of list coloring. Suppose that \( K \) is an \( m \)-assignment for the simple graph \( G \). For each \( v \in V(G) \), let \( L(v) = \{(v, j) : j \in K(v)\} \). For each \( uv \in E(G) \), let \( M(uv) = \{(u, j)(v, j) : j \in L(u) \cap L(v)\} \). Finally, let \( H \) be the graph with vertex set \( \bigcup_{v \in V(G)} L(v) \) and edge set that is the union of \( \bigcup_{uv \in E(G)} M(uv) \) and \( \left( \bigcup_{v \in V(G)} \bigcup_{i,j \in K(v), i \neq j} \{(v, i)(v, j)\} \right) \).

Now, let \( \mathcal{H} = (L, H, M) \) and note that \( \mathcal{H} \) is an \( m \)-fold cover of \( G \). Then, if \( \mathcal{L} \) is the set of \( \mathcal{H} \)-colorings of \( G \) and \( \mathcal{C} \) is the set of proper \( L \)-colorings of \( G \), the function \( f: \mathcal{C} \to \mathcal{L} \) given by \( f(c) = \{(v, c(v)) : v \in V(G)\} \) is a bijection.

### 1.2 The DP Color Function

In 1912 Birkhoff introduced the notion of the chromatic polynomial in hopes of using it to make progress on the four color problem. For \( m \in \mathbb{N} \), the **chromatic polynomial** of a graph \( G \), \( P(G, m) \), is the number of proper \( m \)-colorings of \( G \). It can be shown that \( P(G, m) \) is a polynomial in \( m \) of degree \( |V(G)| \) (see \([6]\)). For example, \( P(K_n, m) = \prod_{i=0}^{n-1} (m - i) \) and \( P(T, m) = m(m - 1)^{n-1} \) whenever \( T \) is a tree on \( n \) vertices (see \([31]\)).

The notion of chromatic polynomial was extended to list coloring in the 1990s \([20]\). In particular, if \( L \) is a list assignment for \( G \), we use \( P(G, L) \) to denote the number of proper \( L \)-colorings of \( G \). The **list color function** \( P_L(G, m) \) is the minimum value of \( P(G, L) \) where the minimum is taken over all \( m \)-assignments \( L \) for \( G \). It is clear that \( P_L(G, m) \leq P(G, m) \) for each \( m \in \mathbb{N} \) since we must consider the \( m \)-assignment that assigns \([m]\) to each vertex of \( G \) when considering all possible \( m \)-assignments for \( G \). In general, the list color function can differ significantly from the chromatic polynomial for small values of \( m \). However, for large values of \( m \), Dong and Zhang \([11]\) (improving upon results in \([9, 27, 30]\)) showed that for any graph \( G \) with at least 4 edges, \( P_L(G, m) = P(G, m) \) whenever \( m \geq |E(G)| - 1 \).

With this in mind, we are ready to define a notion that will be important in this paper. Let \( \mathcal{G} \) be the set of all finite multigraphs. We say a function \( f: \mathcal{G} \times \mathbb{N} \to \mathbb{N} \) is **chromatic-adherent** if for every graph \( G \), \( f(G, a) = P(G, a) \) for some \( a \geq \chi(G) \) implies that \( f(G, m) = P(G, m) \) for all \( m \geq a \). It is unknown whether the list color function is chromatic-adherent.

**Question 1** \(([19])\). Is \( P_L \) chromatic-adherent?

In 2019, the second and third author introduced a DP-coloring analogue of the chromatic polynomial called the DP color function in hopes of gaining a better understanding of DP-coloring and using it as a tool for making progress on some open questions related to the list color function \([15]\). Since its introduction in 2019, the DP color function has received some attention in the literature (see e.g., \([2, 10, 14, 17, 21, 26]\)).

Suppose \( \mathcal{H} = (L, H, M) \) is a cover of a multigraph \( G \). Let \( P_{DP}(G, \mathcal{H}) \) be the number of \( \mathcal{H} \)-colorings of \( G \). Then, the **DP color function** of \( G \), \( P_{DP}(G, m) \), is the minimum value of \( P_{DP}(G, \mathcal{H}) \) where the minimum is taken over all full \( m \)-fold covers \( \mathcal{H} \) of \( G \). It is easy to

\(^1\)We take \( \mathbb{N} \) to be the domain of the DP color function of any multigraph. Also, we can restrict our attention to full \( m \)-fold covers since adding edges to a graph can’t increase the number of independent sets of some prescribed size.
show that for any \( m \in \mathbb{N} \),
\[
P_{\text{DP}}(G, m) \leq P_{\ell}(G, m) \leq P(G, m).
\]

Unlike the list color function, it is well known that \( P_{\text{DP}}(G, m) \) does not necessarily equal \( P(G, m) \) for sufficiently large \( m \). Indeed, Dong and Yang recently generalized a result of Kaul and Mudrock \cite{15} and showed the following.

**Theorem 2** (\cite{10}). If \( G \) is a simple graph that contains an edge \( e \) such that the length of a shortest cycle containing \( e \) is even, then there exists an \( N \in \mathbb{N} \) such that \( P_{\text{DP}}(G, m) < P(G, m) \) whenever \( m \geq N \).

It was recently shown that the DP color function is not chromatic-adherent. A **Generalized Theta graph** \( \Theta(l_1, \ldots, l_n) \) consists of a pair of end vertices joined by \( n \) internally disjoint paths of lengths \( l_1, \ldots, l_n \in \mathbb{N} \). It is easy to see that \( \chi_{\text{DP}}(\Theta(l_1, \ldots, l_n)) = 3 \) whenever \( n \geq 2 \).

**Theorem 3** (\cite{8}). If \( G \) is \( \Theta(2, 3, 3, 3, 2) \) or \( \Theta(2, 3, 3, 3, 3, 2, 2) \), then \( P_{\text{DP}}(G, 3) = P(G, 3) \) and there is an \( N \in \mathbb{N} \) such that \( P_{\text{DP}}(G, m) < P(G, m) \) for all \( m \geq N \).

Motivated by Theorem 3, the authors of \cite{8} pose the following question.

**Question 4.** For which graphs \( G \) do there exist, \( a, b \in \mathbb{N} \) with \( \chi(G) \leq a < b \), \( P_{\text{DP}}(G, a) = P(G, a) \), and \( P_{\text{DP}}(G, b) < P(G, b) \)?

The authors of \cite{8} remarked that the two graphs in Theorem 3 are the only examples that they know of that demonstrate that the DP color function is not chromatic-adherent. In this paper we show that there are infinitely many graphs with the property described in Question 4.

### 1.3 Outline of Paper

The proofs of our results are algebraic. We begin by extending an algebraic technique for analyzing full 3-fold covers described in \cite{16} to multigraphs. This technique along with the following well-known result of Alon and Füredi will allow us to establish a non-trivial lower bound on \( P_{\text{DP}}(G, 3) \) for certain graphs \( G \).

**Theorem 5** (\cite{11}). Let \( \mathbb{F} \) be an arbitrary field, let \( A_1, A_2, \ldots, A_n \) be any non-empty subsets of \( \mathbb{F} \), and let \( B = \prod_{i=1}^{n} A_i \). Suppose that \( P \in \mathbb{F}[x_1, \ldots, x_n] \) is a polynomial of degree \( d \) that does not vanish on all of \( B \). Then, the number of points in \( B \) for which \( P \) has a non-zero value is at least \( \min \prod_{i=1}^{n} q_i \) where the minimum is taken over all integers \( q_i \) such that \( 1 \leq q_i \leq |A_i| \) and \( \sum_{i=1}^{n} q_i \geq -d + \sum_{i=1}^{n} |A_i| \).

We prove the following.

**Theorem 6.** Suppose \( G \) is a multigraph with \( \chi_{\text{DP}}(G) \leq 3 \). Also, suppose that \( |V(G)| = n \), \( |E(G)| = l \), and \( 2n \geq l \). Then, \( P_{\text{DP}}(G, 3) \geq 3^{n-1/2} \).
As an immediate application, consider a \( n \)-vertex planar graph \( G \) of girth at least 5. It is known (see \cite{12}) that \( \chi_{DP}(G) \leq 3 \). Since number of edges in \( G \) is at most \( 5n/3 \), Theorem \cite{6} implies \( P_{DP}(G, 3) \geq 3^{n/6} \). Previously, it was only known (see \cite{7}) that the same lower bound holds for \( P_{C}(G, 3) \).

We demonstrate that the bound in Theorem \cite{6} is tight, and in the process, prove the following.

**Theorem 7.** There are infinitely many graphs \( G \) for which \( \chi_{DP}(G) = 3 \), \( P_{DP}(G, 3) = P(G, 3) \), and there is an \( N_{G} \in \mathbb{N} \) such that \( P_{DP}(G, m) < P(G, m) \) whenever \( m \geq N_{G} \).

## 2 Proofs of Results

From this point forward whenever \( G \) is a multigraph with \( n \) vertices, we suppose that \( V(G) = \{ v_{1}, \ldots, v_{n} \} \). We will also suppose, unless otherwise noted, that all addition and multiplication is performed in \( \mathbb{F}_{3} \) where \( \mathbb{F}_{3} \) is the finite field of order 3. Suppose \( \mathcal{H} = (L, H, M) \) is a full 3-fold cover of the multigraph \( G \) on \( n \) vertices. From this point forward, we will always assume under this set up that \( L(v) = \{ (v, j) : j \in \mathbb{F}_{3} \} \). Suppose \( e \) is an arbitrary element of \( E(G) \) with endpoints \( v_{i} \) and \( v_{j} \) where \( i < j \). Also, for each \( k \in \mathbb{F}_{3} \), suppose the edge in \( M(e) \) with endpoint \( (v_{i}, k) \) has \( (v_{j}, c_{k}) \) as its other endpoint. The permutation of \( \mathcal{H} \) associated with \( M(e) \), denoted \( \sigma_{e}^{H} \), is the permutation \( \sigma_{e}^{H} : \mathbb{F}_{3} \rightarrow \mathbb{F}_{3} \) given by \( \sigma_{e}^{H}(k) = c_{k} \). We will now associate a polynomial in \( \mathbb{F}_{3}[x_{1}, \ldots, x_{n}] \) with \( \mathcal{H} \). Before we do this, we need an observation that was specifically used for full 3-fold covers in \cite{16}.

**Observation 8.** Suppose \( \sigma \) is a permutation of \( \mathbb{F}_{3} \). Then, either \( z - \sigma(z) \) is the same for all \( z \in \mathbb{F}_{3} \), or \( z + \sigma(z) \) is the same for all \( z \in \mathbb{F}_{3} \).

Now, let the linear factor of \( \mathcal{H} \) associated with \( M(e) \), denoted \( l_{e}^{H}(x_{1}, \ldots, x_{n}) \), be the polynomial in \( \mathbb{F}_{3}[x_{1}, \ldots, x_{n}] \) given by \( (x_{i} + (-1)^{c}x_{j} - a) \) where \( c \) and \( a \) are chosen so that \( (x_{i} + (-1)^{c}x_{j} - a) \) is zero if and only if \( x_{i}, x_{j} \) are chosen so that \( x_{i} = x_{j} \). Notice that Observation \cite{8} guarantees that such a \( c \) and \( a \) must exist \cite{2}. Finally, we let the graph polynomial of \( G \) associated with \( \mathcal{H} \), denoted \( f_{G,H}(x_{1}, \ldots, x_{n}) \), be the polynomial in \( \mathbb{F}_{3}[x_{1}, \ldots, x_{n}] \) given by

\[
\prod_{E_{G}(v_{i}, v_{j}) \neq \emptyset, i < j} \left( \prod_{e \in E_{G}(v_{i}, v_{j})} l_{e}^{H}(x_{1}, \ldots, x_{n}) \right).
\]

Notice that \( f_{G,H} \) is a polynomial of degree \( |E(G)| \). Also, by construction, the following observation is immediate.

**Observation 9.** Let \( \mathcal{C} = \{ C \subset V(H) : |C \cap L(v)| = 1 \text{ for each } v \in V(G) \} \), and note that all \( \mathcal{H} \)-colorings of \( G \) are contained in \( \mathcal{C} \). Suppose \( C \in \mathcal{C} \) and \( C = \{ (v_{1}, c_{1}), \ldots, (v_{n}, c_{n}) \} \). Then, \( C \) is an \( \mathcal{H} \)-coloring of \( G \) if and only if \( f_{G,H}(c_{1}, \ldots, c_{n}) \neq 0 \). Consequently, if \( B = \prod_{i=1}^{n} \mathbb{F}_{3} \), then the number of points in \( B \) for which \( f_{G,H} \) has a non-zero value is \( P_{DP}(G, \mathcal{H}) \).

It is easier to apply the following corollary of Theorem \cite{5}

\footnote{It is also not too difficult to prove that \( c \) and \( a \) are unique by considering each possible matching.}
Corollary 10 [7]. Let $\mathbb{F}$ be an arbitrary field, let $A_1, A_2, \ldots, A_n$ be any non-empty subsets of $\mathbb{F}$, and let $B = \prod_{i=1}^{n} A_i$. Suppose that $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial of degree $d$ that does not vanish on all of $B$. If $S = \sum_{i=1}^{n} |A_i|$, $t = \max |A_i|$, $S \geq n + d$, and $t \geq 2$, then the number of points in $B$ for which $P$ has a non-zero value is at least $t^{(S-n-d)/(t-1)}$.

We are now ready to complete the proof of Theorem 5. Suppose $V(G) = \{v_1, \ldots, v_n\}$ and $\mathcal{H} = (L, H, M)$ is a full 3-fold cover of $G$ with $P_{DP}(G, \mathcal{H}) = P_{DP}(G, 3)$. Suppose also that the vertices of $H$ are arbitrarily named so that $L(v) = \{(v, j) : j \in \mathbb{F}_3\}$ for each $v \in V(G)$. Let $A_i = \mathbb{F}_3$ for each $i \in [n]$, and let $B = \prod_{i=1}^{n} A_i$. Note that $f_{G, \mathcal{H}}(x_1, \ldots, x_n) \in \mathbb{F}_3[x_1, \ldots, x_n]$ has degree $l$. Moreover, $f_{G, \mathcal{H}}(x_1, \ldots, x_n)$ doesn’t vanish on all of $B$ by the fact that $\chi_{DP}(G) \leq 3$ and Observation 9.

Thus, the number of points in $B$ for which $f_{G, \mathcal{H}}(x_1, \ldots, x_n)$ has a non-zero value is at least $3^{(3n-l)/(3-1)}$ by Corollary 10. The Theorem 5 then follows from Observation 9.

We now mention three simple examples that demonstrate the tightness of Theorem 5. First, notice that if $G_1$ is an edgeless graph on $n$ vertices, then $P_{DP}(G_1, 3) = 3^n$. Second, suppose that $G_2$ is a multigraph on $2$ vertices with $2$ edges. Then, it is easy to see that $P_{DP}(G_2, 3) = 3$ (see Proposition 11 in [24]). Finally, suppose that $k \in \mathbb{N}$ and $G_3 = K_1 \vee C_{2k+2}$. Notice that $|V(G_3)| = 2k + 3$ and $|E(G_3)| = 4k + 4$. It is shown in [2] that $P_{DP}(G_3, 3) = 3$. It is also worth mentioning that Theorem 5 tells us that $P_{DP}(G_3, 3) = 3$ in a manner that is much more elegant than the result used in [2] to demonstrate the same lower bound (see Lemma 16 in [2]).

We now turn our attention to proving Theorem 7. First, we need a definition and a result. A graph $G$ is said to be uniquely $k$-colorable if there is only one partition of its vertex set into $k$ independent sets. It is well known (see [28]) that if $G$ is a uniquely $k$-colorable graph with $n$ vertices, then $|E(G)| \geq (k-1)n - k(k-1)/2$. It is also easy to observe that when $G$ is uniquely $k$-colorable, $\chi(G) = k$ and $P(G, k) = k!$. With this in mind, we have the following lemma.

Lemma 11. Suppose $G$ is a uniquely 3-colorable graph on $n$ vertices with $\chi_{DP}(G) = 3$. If $|E(G)| = 2n - 3$, then $P_{DP}(G, 3) = P(G, 3)$.

Proof. Clearly, $P(G, 3) = 6$ and $P_{DP}(G, 3) \leq P(G, 3)$. Since $\chi_{DP}(G) = 3$, Theorem 5 implies that $P_{DP}(G, 3) \geq 3^{n-(2n-3)/2} = 3^{3/2}$. Since $P_{DP}(G, 3)$ is an integer, this means $P_{DP}(G, 3) \geq 6$.

We are now ready to prove Theorem 7.

Proof. For each $k \in \mathbb{N}$, let $H_k$ be the graph obtained from a copy of $K_1 \vee P_{2k+2}$ by adding a new vertex $z$ and then adding an edge between $z$ and each endpoint of the copy of $P_{2k+2}$. To prove Theorem 7, we will show that for each $k \in \mathbb{N}$, $P_{DP}(H_k, 3) = P(H_k, 3)$, and there is an $N_{H_k} \in \mathbb{N}$ such that $P_{DP}(H_k, m) < P(H_k, m)$ whenever $m \geq N_{H_k}$.

Now, fix a $k \in \mathbb{N}$. It is easy to see that $H_k$ is both 2-degenerate and uniquely 3-colorable. Consequently, $\chi_{DP}(H_k) = 3$. Moreover, $|V(H_k)| = 2k+4$ and $|E(H_k)| = 4k+5 = 2(2k+4)-3$. So, Lemma 11 implies that $P_{DP}(H_k, 3) = P(H_k, 3)$.

Finally, notice that each of the two edges in $H_k$ incident to $z$ have the property that the smallest cycle in $H_k$ containing the edge is of length 4. So, Theorem 2 implies there is an $N_{H_k} \in \mathbb{N}$ such that $P_{DP}(H_k, m) < P(H_k, m)$ whenever $m \geq N_{H_k}$.
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