Multiple semiclassical standing waves for fractional nonlinear Schrödinger equations

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Abstract

Via a Lyapunov–Schmidt reduction, we obtain multiple semiclassical solutions to a class of fractional nonlinear Schrödinger equations. To be more precise, we consider

\[ \varepsilon^{2s} (-\Delta)^s u + u + V(x) u = |u|^{p-1} u, \quad u \in H^s(\mathbb{R}^n), \]

where \(0 < s < 1\), \(n > 4 - 4s\), \(1 < p < \frac{n+2s}{n-2}\) (if \(n > 2s\)) and \(1 < p < \infty\) (if \(n \leq 2s\)), and \(V(x)\) is a non-negative potential function. If \(V\) is a sufficiently smooth bounded function with a non-degenerate compact critical manifold \(M\), then, when \(\varepsilon\) is sufficiently small, there exist at least \(l(M)\) semiclassical solutions, where \(l(M)\) is the cup length of \(M\).

Keywords: fractional Laplacian, Schrödinger equation, semiclassical solutions, Lyapunov–Schmidt reduction, concentration phenomena

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1. Introduction

Fractional Schrödinger equations are derived from the path integral over Lévy trajectories. It can be applied, for example, to describe the orbit’s radius for hydrogen-like atoms. (For more details of physical background, see, for example, [29] and the references therein.)

We study the fractional nonlinear Schrödinger equation of the form

\[ i\varepsilon \frac{\partial \psi}{\partial t} = (-\varepsilon^2 \Delta)^s \psi + V(x) \psi - |\psi|^{p-1} \psi \quad \text{in} \; \mathbb{R}^n, \]

where \(\varepsilon\) is a small positive constant which corresponds to the Planck constant, \((-\Delta)^s\), \(0 < s < 1\), is the fractional Laplacian, \(V(x)\) is a potential function, and \(p > 1\).
We shall look for the so-called standing wave solutions which are of the form
\[ \psi(x, t) = e^{i(s/\varepsilon)t}v(x), \]
where \( v \) is a real-valued function depending only on \( x \) and \( E \) is some constant in \( \mathbb{R} \). The function \( \psi \) solves (1.1) provided the standing wave \( v(x) \) satisfies
\[ (-\varepsilon^2 \Delta)^s v + (V(x) + E)v - |v|^{p-1}v = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2} \]

In what follows, we assume that \( E = 1 \) and \( p \) is subcritical. That is, we will study the following equation:
\[ \varepsilon^{2s}(-\Delta)^s u + u + V(x)u = |u|^{p-1}u, \quad u \in H^s(\mathbb{R}^n), \tag{1.3} \]
where \( 0 < s < 1 \), and \( 1 < p < \frac{n+2s}{n-2s} \) for \( n > 2s \), and, \( 1 < p < \infty \) for \( n \leq 2s \).

In quantum mechanics, when \( \varepsilon \) tends to zero, the existence and multiplicity of solutions to (1.3) is of importance. We will find multiple solutions \( u_\varepsilon \) of (1.3) that concentrate near some point \( x_0 \in \mathbb{R}^n \) as \( \varepsilon \to 0 \). By this we mean that, for all \( x \in \mathbb{R}^n \setminus \{x_0\} \), \( u_\varepsilon(x) \to 0 \) as \( \varepsilon \to 0 \). These kinds of solutions are so-called semiclassical standing waves or spike pattern solutions.

When \( s = 1 \), equation (1.3) is a classical nonlinear Schrödinger equation and the existence of semiclassical standing wave solutions was established by Floer and Weinstein [23], and then Oh [32, 33]. There has been a large amount of research carried out on this subject in the past two decades. We refer, for example, to the (far from complete) list of papers [3–7, 9, 13, 16–18, 26–28, 30, 31, 34, 36] and the references therein.

When \( s \in (0, 1) \), the existence of a semiclassical solution to equations (1.3) was obtained by Dávila et al [15], and Chen and Zheng [12]. Using a Lyapunov–Schimdt reduction, [15] proved precisely that if \( V \) is a sufficiently smooth positive function with non-degenerate critical points \( \xi_1, \xi_2, \ldots, \xi_k \) and satisfies some degree conditions around these points, then there exists a solution of (1.3) concentrating on these \( k \) critical points. (See [12] for the case \( k = 1 \) with more technical conditions.) Further, in [20], Fall et al proved that if there exist semiclassical solutions to (1.3) as \( \varepsilon \to 0 \), then the concentration points must be critical points of \( V \).

Moreover, we should mention that the concentration phenomena for fractional Schrödinger equations on a bounded domain with Dirichlet conditions were investigated by Dávila et al [14].

In this paper, we mainly investigate the existence and multiplicity of semiclassical standing wave solutions to equation (1.3) when \( V \) has non-isolated critical points. More precisely, we have the following theorem.

**Theorem 1.1.** Let \( 0 < s < 1 \), \( n > 4 - 4s \). Suppose that \( V \) is a non-negative function in \( C^1_b(\mathbb{R}^n) \) with a non-degenerate smooth compact critical manifold \( M \). Then for small \( \varepsilon > 0 \), equation (1.3) has at least \( l(M) \) solutions concentrating near points of \( M \).

Here \( l(M) \) denotes the cup length of \( M \) (see section 6.1 below) and
\[ C^1_b(\mathbb{R}^n) = \{ v \in C^1(\mathbb{R}^n) \mid \partial^j v \text{ bounded on } \mathbb{R}^n \quad \text{for all } |j| \leq 3 \}. \]

The non-degeneracy of a critical manifold is in the sense of Bott [8]. Precisely, we say that a critical manifold \( M \) of \( V \) is non-degenerate if, for every \( x \in M \), the kernel of \( D^2 f(x) \) equals \( T_x M \).

**Remark 1.2.** When \( s = 1 \), the result of this theorem was obtained by Ambrosetti et al [4].

**Remark 1.3.** Since the unique positive solution (up to translation) to the standard equation decays as \( 1/(1+|x|^{n+2s}) \) (see for example [22, 24] or theorem 2.3 below), we should technically assume that \( 0 < s < 1 \) and \( n > 4 - 4s \) to make some necessary integrals convergent (see the proof of lemma 3.2 below). Based on our observation, this assumption is essential since the decay estimate of the unique standard solution is optimal. We should also note that when \( s \to 1 \), there is no restriction on the dimension \( n \). This is the same as the classical case, \( s = 1 \).
Remark 1.4. Note that the assumption \( \mathcal{V} \geq 0 \) on \( \mathbb{R}^n \) is not essential. In fact, a similar argument to that in section 2.3 below implies that the condition \( \inf (1 + \mathcal{V}) > 0 \) is sufficient. Without loss of generality, in what follows we assume that \( \mathcal{V}(0) = 0 \) for simplicity.

Our proof relies on a singular perturbation argument as in [4]. More precisely, by the change of variable \( x \to \varepsilon x \), equation (1.3) becomes
\[
(-\Delta)^s u + u + V(\varepsilon x)u = |u|^{p-1}u.
\]
(1.4)

Solutions of (1.4) are the critical points \( u \in H^s(\mathbb{R}^n) \) of the functional
\[
f_\varepsilon(u) = f_0(u) + \frac{1}{2} \int_{\mathbb{R}^n} V(\varepsilon x)u^2 \, dx,
\]
(1.5)

where
\[
f_0(u) = \frac{1}{2} \|u\|_s^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \, dx.
\]
(1.6)

Here \( \| \cdot \|_s \) denotes the norm in \( H^s(\mathbb{R}^n) \). We should note that \( f_\varepsilon \in C^2(H^s(\mathbb{R}^n)) \). We will find the solutions of (1.4) near the solutions of
\[
(-\Delta)^s u + u + V(\varepsilon \xi)u = |u|^{p-1}u,
\]
(1.7)

for some \( \xi \in \mathbb{R}^n \) to be fixed. The solutions of (1.7) are critical points of the following functional
\[
F_{\varepsilon,\xi}(u) = f_0(u) + \frac{1}{2} V(\varepsilon \xi) \int_{\mathbb{R}^n} u^2 \, dx.
\]
(1.8)

Since (1.8) has a term of \( V \), \( F_{\varepsilon,\xi} \) inherits the topological features of the critical manifold \( M \) of \( V \). Therefore, if we consider \( f_\varepsilon \) as a perturbation of \( F_{\varepsilon,\xi} \), multiple solutions to (1.4) will be found by a multiplicity theorem from [11] (see theorem 6.1 below). As compared with this perturbation approach, in [15] and [12], the approximate solutions were constructed by the solution to the standard equation (when \( V \equiv 0 \) in (1.4) or see (2.1) below). So, these approximate solutions include less information of the potential function \( V \).

Nevertheless, a direct application of the arguments in [4] to our problem is impossible. There are two reasons which make our proof much more complicated. Firstly, unlike the Laplacian \( -\Delta \), the fractional Laplacian \( (-\Delta)^s \), \( 0 < s < 1 \), is non-local. For this reason, when \( 0 < s < 1 \), the classical local techniques as in the \( s = 1 \) case (see [4]) cannot be used anymore. For instance, instead of using the classical method in [4] which essentially depends on the locality of \( -\Delta \), we employ a functional analysis approach to prove the invertibility of \( D^2 f_\varepsilon \) (see section 4 below). Secondly, the standard solution \( U \) to the unperturbed fractional Schrödinger equation (\( V \equiv 0 \) in equation (1.3)) decays only as \( 1/(1 + |x|^{n+2s}) \) (see section 2.2 below); in particular, it does not decay exponentially as in the \( s = 1 \) case. Therefore, to ensure the necessary functions in certain Sobolev spaces on \( \mathbb{R}^n \) and to recover the estimates for Lyapunov–Schmidt reduction, we need more detailed and involved analysis than the classical case (see sections 3, 4 and 5 below).

Our paper is organized as follows. In section 2, we recall the notations of fractional Sobolev spaces, and some basic properties of the standard equation which is obtained by [22, 24, 25]. Moreover, we formulate the functional corresponding to equation (1.3), and construct the critical manifold of the functional (1.8). In section 3, some useful estimates are shown for further reference. In section 4, we prove the invertibility of linearized operators at the points on the critical manifold of \( F_{\varepsilon,\xi} \). In section 5, we apply the Lyapunov–Schmidt reduction method to our functional. In section 6, we complete the proof of theorem 1.1.
2. Preliminaries

In this section, we recall some results on the fractional Laplacian, fractional Sobolev spaces and some uniqueness, non-degeneracy and decay results for solutions to the standard Schrödinger equations.

2.1. Fractional Laplacian and fractional order Sobolev spaces

For further references, we recall some basic facts involving the fractional Laplacian and fractional order Sobolev spaces. For more details, see, for example, [1, 10, 19, 35].

Mathematically, \((-\Delta)^s\) is defined as

\[
(-\Delta)^s u = C(n, s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy = C(n, s) \lim_{\delta \to 0^+} \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy.
\]

Here P.V. is a commonly used abbreviation for ‘in the principal value sense’ and \(C(n, s) = \pi^{-\frac{(2s+n/2)}{2}} \Gamma\left(\frac{n}{2} - s\right) \Gamma\left(\frac{n}{2} + s\right)^{-1}\). It is well known that \((-\Delta)^s\) on \(\mathbb{R}^n\) with \(s \in (0, 1)\) is a non-local operator.

When \(s \in (0, 1)\), the space \(H^s(\mathbb{R}^n) = W^{s, 2}(\mathbb{R}^n)\) is defined by

\[
H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \frac{|u(x) - u(y)|}{|x-y|^{n+2s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}
\]

and the inner product is

\[
\langle u, v \rangle_s := \int_{\mathbb{R}^n} u v \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy.
\]

Let

\[
[u]_s := [u]_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy \right)^{\frac{1}{2}}
\]

be the Gagliardo (semi) norm of \(u\). The following identity yields the relation between the fractional operator \((-\Delta)^s\) and the fractional Sobolev space \(H^s(\mathbb{R}^n)\),

\[
[u]_{H^s(\mathbb{R}^n)} = C \left( \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = C \left|(-\Delta)^{\frac{s}{2}} u\right|_{L^2(\mathbb{R}^n)}
\]

for a suitable positive constant \(C\) depending only on \(s\) and \(n\).

When \(s > 1\) and it is not an integer we write \(s = m + \sigma\), where \(m\) is an integer and \(\sigma \in (0, 1)\). In this case the space \(H^s(\mathbb{R}^n)\) consists of those equivalence classes of functions \(u \in H^m(\mathbb{R}^n)\) whose distributional derivatives \(D^j u\), with \(|J| = m\), belong to \(H^\sigma(\mathbb{R}^n)\), namely

\[
H^s(\mathbb{R}^n) = \left\{ u \in H^m(\mathbb{R}^n) : D^j u \in H^\sigma(\mathbb{R}^n) \text{ for any } J \text{ with } |J| = m \right\}
\]

and this is a Banach space with respect to the norm

\[
\|u\|_s := \|u\|_{H^s(\mathbb{R}^n)} = \left( \|u\|_{H^m(\mathbb{R}^n)}^2 + \sum_{|J|=m} \|D^j u\|_{H^\sigma(\mathbb{R}^n)}^2 \right)^{\frac{1}{2}}.
\]

Clearly, if \(s = m\) is an integer, the space \(H^s(\mathbb{R}^n)\) coincides with the usual Sobolev space \(H^m(\mathbb{R}^n)\). By this notation, we denote the norm of \(L^2(\mathbb{R}^n)\) by \(\|\cdot\|_0\).

For a general domain \(\Omega\), the space \(H^s(\Omega)\) can be defined similarly.

On the Sobolev inequality and the compactness of embedding, one has
Theorem 2.1. [1] Let $\Omega$ be a domain with smooth boundary in $\mathbb{R}^n$. Let $s > 0$, then
(a) If $n > 2s$, then $H^s(\Omega) \hookrightarrow L^r(\Omega)$ for $2 \leq r \leq 2n/(n - 2s)$,
(b) If $n = 2s$, then $H^s(\Omega) \hookrightarrow L^r(\Omega)$ for $2 \leq r < \infty$.

Theorem 2.2. [35] Let $s > s'$ and $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then the embedding operator
$$
i's : H^s(\Omega) \rightarrow H^{s'}(\Omega)$$
is compact.

2.2. Some results for the standard equation

We recall some basic properties of the solutions to the following equation
$$(−\Delta)u + u - |u|^{p-1}u = 0.$$  \tag{2.1}

The solutions of equation (2.1) are the critical points of $f_0$ given by equation (1.6). The non-degeneracy of the standard solution to equation (2.1) is investigated by many works. For our purpose, we recall the following theorem. (For more results and details on this topic, see, for example, [21, 22, 24, 25] and the references therein.)

Theorem 2.3. There exists a unique solution (up to translation) $U \in H^{2s+1}(\mathbb{R}^n)$ to equation (2.1) such that
$$\frac{C_1}{1 + |x|^{p+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{p+2s}},$$
for some constants $0 < C_1 \leq C_2$. Moreover, the linearized operator $L_0$ at $U$ is non-degenerate, that is, its kernel is given by
$$\ker L_0 = \text{span}\{\partial_{x_1} U, \ldots, \partial_{x_n} U\}.$$

Remark 2.4. By lemma C.2 of [24], $\nabla U$ satisfies
$$|\nabla U(x)| \leq C \frac{1}{1 + |x|^{p+2s}},$$
for some constant $C$.

Remark 2.5. The non-degeneracy of $L_0$ yields the coercivity estimate as follows:
$$\langle L_0 \phi, \phi \rangle_0 \geq C \|\phi\|^2_0$$
for $\phi \perp K$, where $C$ is a positive constant, and $K$ is a suitable chosen $(n + 1)$-dimensional subspace. For example, we can choose $K = \text{span}\{\phi_{-1}, \partial_{x_1} U, \ldots, \partial_{x_n} U\}$ with $\phi_{-1}$ being the linear ground state of $L_0$. For more details, see [24, section 3].

2.3. Critical points of $F_{\varepsilon, \xi}$

Let
$$a = a(\xi) = (1 + V(\xi))^{\frac{1}{p}}$$  \tag{2.2}
and
$$b = b(\xi) = (1 + V(\xi))^{\frac{1}{p+1}}.$$  \tag{2.3}
Then $bU(ax)$ solves equation (1.7). Set
$$z^{\xi} = b(\varepsilon \xi)U(a(\varepsilon \xi)x)$$  \tag{2.4}
and
$$Z^\xi = \{z^{\xi}(x - \xi) \mid \xi \in \mathbb{R}^n\}.$$  \tag{2.5}
Therefore, every point in $Z^\xi$ is a critical point of equation (1.8) or, equivalently, a solution to equation (1.7). For simplicity, we will set $z = z_{\xi} = z_{\varepsilon, \xi} = z^{\xi}(x - \xi)$. 

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3. Some estimates

In this section, we prove some useful estimates for future reference. From now on, \( C \) denotes various constants.

**Lemma 3.1.** Let \( \bar{\rho} > 0 \). For sufficiently small \( \varepsilon \) and \( |\xi| \leq \bar{\rho} \), there holds

\[
\partial_z \zeta^\varepsilon = -\partial_z z^\varepsilon (x - \xi) + O(\varepsilon), \quad \text{in } H^s(\mathbb{R}^n).
\]  

**Proof.** A direct calculation gives

\[
\begin{align*}
\partial_z \zeta^\varepsilon (x - \xi) &= \partial_z [b(\varepsilon \xi)U(a(\varepsilon \xi)(x - \xi))] \\
&= \varepsilon [\partial_z b(\varepsilon \xi)U(a(\varepsilon \xi)(x - \xi)) + \varepsilon b(\varepsilon \xi)[\partial_z a(\varepsilon \xi)]\nabla U(a(\varepsilon \xi)(x - \xi)) \cdot (x - \xi)] \\
&\quad - a(\varepsilon \xi)b(\varepsilon \xi)[\partial_z U(a(\varepsilon \xi)(x - \xi))] := Z_1 + Z_2 + Z_3.
\end{align*}
\]

Note that

\[
Z_3 = -a(\varepsilon \xi)b(\varepsilon \xi)[\partial_z a(\varepsilon \xi)(b(\varepsilon \xi)(x - \xi))] = -\partial_z z^\varepsilon (x - \xi).
\]

By the definition of \( a, b \) and the assumption of \( \mathcal{V} \), we have that

\[
|a(\varepsilon \xi)| \leq C, \quad |b(\varepsilon \xi)| < C, \quad \|\partial_z a(\varepsilon \xi)\| \leq C, \quad \|\partial_z b(\varepsilon \xi)\| < C
\]

for some constant \( C \). From the assumption of \( \mathcal{V} \),

\[
|a(\varepsilon \xi)| \geq 1, \quad |b(\varepsilon \xi)| \geq 1.
\]

Therefore, from \( \partial_z U(\cdot - \xi) \in H^s(\mathbb{R}^n) \), we have \( Z_3 \in H^s(\mathbb{R}^n) \). By \( U \in H^s(\mathbb{R}^n) \), it holds that

\[
\|Z_1\|_s = O(\varepsilon) \|\partial_z [b(\varepsilon \xi)U(a(\varepsilon \xi)(\cdot - \xi))]\|_s = O(\varepsilon).
\]

Since \( \partial_z \zeta^\varepsilon \in H^s(\mathbb{R}^n) \) and \( Z_1, Z_3 \in H^s(\mathbb{R}^n) \), we have that \( Z_2 \) is also in \( H^s(\mathbb{R}^n) \). It follows that \( [\nabla U(a(\varepsilon \xi)(\cdot - \xi)) \cdot (\cdot - \xi)] \in H^s(\mathbb{R}^n) \). So, we obtain that \( [\nabla U(\cdot - \xi) \cdot (\cdot - \xi)] \in H^s(\mathbb{R}^n) \). Again, by the property of \( a \), it holds that

\[
\|Z_2\|_s = O(\varepsilon).
\]

From equations (3.2) and (3.3), we have equation (3.1). This completes the proof. \( \square \)

**Lemma 3.2.** Given \( \bar{\rho} > 0 \) and small \( \bar{\varepsilon} > 0 \), we have that, if \( |\xi| \leq \bar{\rho} \) and \( 0 < \varepsilon < \bar{\varepsilon} \), then

\[
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 z^\varepsilon_2 2 dx \leq C(\varepsilon^2 |\nabla V(\varepsilon \xi)|^2 + \varepsilon^4),
\]

and

\[
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |\partial_z \zeta^\varepsilon_2|^2 2 dx \leq C(\varepsilon^2 |\nabla V(\varepsilon \xi)|^2 + \varepsilon^4).
\]

**Proof.** Since \( V \in C^3_b(\mathbb{R}^n) \) implies that \( |\nabla V(x)| \leq C \) and \( |D^2 V(x)| \leq C \), it holds that

\[
|V(\varepsilon x) - V(\varepsilon \xi)| \leq \varepsilon |\nabla V(\varepsilon \xi)| \cdot |x - \xi| + C \varepsilon^2 |x - \xi|^2.
\]

Therefore,

\[
\begin{align*}
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 z^\varepsilon_2^2 2 dx &\leq C \varepsilon^2 |\nabla V(\varepsilon \xi)|^2 \int_{\mathbb{R}^n} |x - \xi|^2 z^\varepsilon_2^2 (x - \xi) dx \\
&\quad + C \varepsilon^4 \int_{\mathbb{R}^n} |x - \xi|^4 z^\varepsilon_2^2 (x - \xi) dx.
\end{align*}
\]
By the definition of $z_\xi$,
\begin{align*}
\int_{\mathbb{R}^n} |x - \xi|^2 z_\xi^2 (x - \xi) \, dx &= b^2 (\xi) \int_{\mathbb{R}^n} |y|^2 U^2 (a(\xi) y) \, dy \\
&= a^{-\alpha - 2} b^2 \int_{\mathbb{R}^n} |y'|^2 U^2 (y') \, dy'.
\end{align*}

Using theorem 2.3, we obtain
\begin{align*}
\int_{\mathbb{R}^n} |y'|^2 U^2 (y') \, dy' &\leq C_2 \int_{\mathbb{R}^n} \frac{|y|^2}{(1 + |y'|)^{2n+4s}} \leq C.
\end{align*}

Since we assume $n > 4 - 4s$, it follows that
\begin{align*}
\int_{\mathbb{R}^n} |x - \xi|^4 z_\xi^2 (x - \xi) \, dx &\leq C_2 \int_{\mathbb{R}^n} \frac{|x - \xi|^4}{(1 + |x - \xi|)^{2n+4s}} \, dx \\
&\leq C_2 \int_{\mathbb{R}^n} \frac{1}{(1 + |x - \xi|)^{2n+4s-4}} \, dx \leq C.
\end{align*}

Therefore, we get
\begin{equation}
\int_{\mathbb{R}^n} \left| |V(\varepsilon x) - V(\varepsilon \xi)|^2 z_\xi^2 \right| \, dx \leq C (\varepsilon^2 |\nabla V(\varepsilon \xi)|^2 + \varepsilon^4). \tag{3.4}
\end{equation}

For the second estimate, we have
\begin{align*}
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |\partial_r z_\xi|^2 \, dx &\leq C \varepsilon^2 |\nabla V(\varepsilon \xi)|^2 \int_{\mathbb{R}^n} |x - \xi|^2 |\partial_r z_\xi|^2 \, dx \\
&+ C \varepsilon^4 \int_{\mathbb{R}^n} |x - \xi|^4 |\partial_r z_\xi|^2 \, dx.
\end{align*}

From remark 2.4, $|\partial_r z_\xi (x - \xi)| \leq \frac{C}{|1 + |x - \xi|^2|^{1+2s}}$. Then a similar argument as the proof of equation (3.4) gives
\begin{equation}
\int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |\partial_r z_\xi|^2 \, dx \leq C (\varepsilon^2 |\nabla V(\varepsilon \xi)|^2 + \varepsilon^4).
\end{equation}

This completes the proof. \qed

**Lemma 3.3.** Given $\bar{\rho} > 0$ and small $\bar{\varepsilon} > 0$, it holds that, for $|\xi| \leq \bar{\rho}$ and $0 < \varepsilon < \bar{\varepsilon}$,
\begin{equation}
\|DF_f(z_\xi)\|_s \leq C (\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2)), \tag{3.5}
\end{equation}
for some constant $C$.

**Proof.** Rewrite
\begin{equation*}
f_\varepsilon(u) = F^{\varepsilon \xi}(u) + \frac{1}{2} \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) u^2 \, dx.
\end{equation*}

Since $z_\xi$ is a critical point of $F^{\varepsilon \xi}$, we get
\begin{equation*}
\langle DF_f(z_\xi), v \rangle_s = \langle DF^{\varepsilon \xi}(z_\xi), v \rangle_s + \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi v \, dx
= \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi v \, dx.
\end{equation*}

By the Hölder inequality, we have
\begin{align*}
|\langle DF_f(z_\xi), v \rangle_s|^2 &\leq \|v\|^2 \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 z_\xi^2 \, dx \\
&\leq \|v\|^2 \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 z_\xi^2 \, dx.
\end{align*}

Then lemma 3.2 implies equation (3.5). \qed
4. Invertibility

In this section, we will discuss the invertibility of $D^2 f_\varepsilon(z_\xi)$ on $(T_{z_\xi}Z^\varepsilon)^\perp$. Here $T_{z_\xi}Z^\varepsilon$ is the tangent space to $Z^\varepsilon$ at $z_\xi$, and $(T_{z_\xi}Z^\varepsilon)^\perp$ is the orthogonal complement of $T_{z_\xi}Z^\varepsilon$ in $H^s(\mathbb{R}^n)$.

Let

\[ L_{\varepsilon,\xi} : (T_{z_\xi}Z^\varepsilon)^\perp \rightarrow (T_{z_\xi}Z^\varepsilon)^\perp \]

be the tangent operator of $Df_\varepsilon$ restricted on $(T_{z_\xi}Z^\varepsilon)^\perp$, that is, on $(T_{z_\xi}Z^\varepsilon)^\perp$.

\[ \langle L_{\varepsilon,\xi} v, w \rangle_s = D^2 f_\varepsilon(z_\xi)[v, w]. \]

The main aim of this section is to prove the following result which implies that $L_{\varepsilon,\xi}$ is invertible on $(T_{z_\xi}Z^\varepsilon)^\perp$.

**Proposition 4.1.** Given $\tilde{\rho} > 0$, there exists $\tilde{\varepsilon} > 0$ such that, for all $|\xi| \leq \tilde{\rho}$ and $0 < \varepsilon < \tilde{\varepsilon}$, it holds that

\[ \langle L_{\varepsilon,\xi} v, v \rangle_s \geq C \|v\|_{s}^2, \quad \forall v \in (T_{z_\xi}Z^\varepsilon)^\perp, \]

where $C > 0$ is a constant only depending on $\tilde{\xi}$ and $\tilde{\varepsilon}$.

Note that $T_{z_\xi}Z^\varepsilon = \text{span}\{\partial_{\xi_1}z_\xi, \cdots, \partial_{\xi_n}z_\xi\}$.

From lemma 3.1, we know that $\partial_{\xi_i}z_\xi$ is close to $-\partial_{x_i}z_\xi$ in $H^s(\mathbb{R}^n)$ when $\varepsilon \to 0$ and $|\xi| \leq \tilde{\rho}$.

For convenience, we define

\[ K_{\varepsilon,\xi} = \text{span}\{z_\xi, \partial_{x_1}z_\xi, \cdots, \partial_{x_n}z_\xi\}. \tag{4.1} \]

To prove proposition 4.1, we need some lemmas.

**Lemma 4.2.** $z_\xi$ is a critical point of $F^\xi_\varepsilon$ with Morse index one.

**Proof.** Since

\[ D^2 F^\xi_\varepsilon(z_\xi)[z_\xi, z_\xi] = -(p - 1) \int_{\mathbb{R}^n} z_\xi^{p+1} \, dx < 0, \tag{4.2} \]

the operator $D^2 F^\xi_\varepsilon(z_\xi)$ has at least one negative eigenvalue. For the details to prove that the Morse index of $z_\xi$ is one exactly, see section 3 in [24]. \hfill \Box

**Lemma 4.3.** Let $\tilde{\rho} > 0$, there exist $\varepsilon_0 > 0$ and a constant $C_1 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$ and all $|\xi| \leq \tilde{\rho}$, it holds

\[ \langle L_{\varepsilon,\xi} z_\xi, z_\xi \rangle_s \leq -C_1 < 0. \]

**Proof.** Direct calculus yields

\[ \langle L_{\varepsilon,\xi} z_\xi, z_\xi \rangle_s = D^2 F^{\xi_\varepsilon}(z_\xi)[z_\xi, z_\xi] + \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z_\xi^2 \, dx. \tag{4.3} \]

From equations (4.2) and (2.4),

\[ D^2 F^{\xi_\varepsilon}(z_\xi)[z_\xi, z_\xi] = -(p - 1) \int_{\mathbb{R}^n} z_\xi^{p+1} \, dx = -(p - 1) \int_{\mathbb{R}^n} [b(\varepsilon \xi) U(a(\varepsilon \xi)(x - \xi))]^{p+1} \, dx = -(p - 1)[b(\varepsilon \xi)]^{p+1}[a(\varepsilon \xi)]^{-n} \int_{\mathbb{R}^n} U^{p+1}(x) \, dx. \]

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From the definition of $a, b$ (see equations (2.2) and (2.3)) and $V(0) = 0$, we have that, for any fixed $\bar{\rho} > 0$, there exists $\varepsilon_1 > 0$ small enough such that when $|\xi| \leq \bar{\rho}$ and $0 < \varepsilon < \varepsilon_1$, it holds
\[
a(\varepsilon \xi) \in \left[\frac{1}{2}, 2\right] \quad \text{and} \quad b(\varepsilon \xi) \in \left[\frac{1}{2}, 2\right].
\] (4.4)

Since $U$ is the unique solution (up to translation),
\[
\int_{\mathbb{R}^n} U^{\mu+1}(x) \, dx
\]
is a constant. Therefore there is a positive constant $C_0$ such that
\[
D^2 F_{\varepsilon \xi}(z_{\xi}, z_{\xi}) \leq -C_0 < 0.
\] (4.5)

From lemma 3.2, the second term on right-hand side of equation (4.3) satisfies
\[
\left| \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z_{\xi}^2 \, dx \right| \leq \varepsilon \int_{\mathbb{R}^n} |\nabla V(\varepsilon \xi) \cdot (x - \xi)| + \varepsilon^2 |D^2 V(\eta)| \cdot |x - \xi|^2 z_{\xi}^2 \, dx
\]
\[
= \varepsilon \int_{\mathbb{R}^n} |\nabla V(\varepsilon \xi) \cdot (x - \xi)| z_{\xi}^2 \, dx + \varepsilon^2 \int_{\mathbb{R}^n} |D^2 V(\eta)| \cdot |x - \xi|^2 z_{\xi}^2 \, dx.
\]

Here $\eta$ is some point in $\mathbb{R}^n$. Since $V \in C^3_b(\mathbb{R}^n)$, we have that
\[
|\nabla V(\varepsilon \xi) \cdot (x - \xi)| \leq C|x - \xi|,
\]
and
\[
|D^2 V(\eta)| \cdot |x - \xi|^2 \leq C|x - \xi|^2.
\]

Then by the definition of $z_{\xi}$,
\[
\int_{\mathbb{R}^n} |\nabla V(\varepsilon \xi) \cdot (x - \xi)| z_{\xi}^2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^n} |x - \xi| |b(\varepsilon \xi) U(a(\varepsilon \xi)(x - \xi))|^2 \, dx
\]
\[
\leq C[b(\varepsilon \xi)]^2 [a(\varepsilon \xi)]^{-n-1} \int_{\mathbb{R}^n} \frac{|x - \xi|}{(1 + |x - \xi|)^{2n+4\varepsilon}} \, dx.
\]

Taking $|\xi| \leq \rho$ and $\varepsilon < \varepsilon_1$ as in equation (4.4), we obtain that there exists a positive constant $C_2$ such that
\[
\int_{\mathbb{R}^n} |\nabla V(\varepsilon \xi) \cdot (x - \xi)| z_{\xi}^2 \, dx < C_2
\]

A similar argument yields
\[
\int_{\mathbb{R}^n} |D^2 V(\eta)| \cdot |x - \xi|^2 z_{\xi}^2 \, dx \leq C_3 \int_{\mathbb{R}^n} \frac{|x - \xi|^2}{(1 + |x - \xi|)^{2n+4\varepsilon}} \, dx \leq C_4.
\]

Therefore, when $|\xi| \leq \rho$ and $\varepsilon < \varepsilon_1$,
\[
\left| \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi)) z_{\xi}^2 \, dx \right| \leq C_2 \varepsilon + C_3 \varepsilon^2.
\] (4.6)

Then there is an $\varepsilon_0 < \varepsilon_1$ such that when $\varepsilon < \varepsilon_0$,
\[
C_2 \varepsilon + C_3 \varepsilon^2 < \frac{C_0}{2}.
\] (4.7)

From equations (4.5), (4.6) and (4.7), we have
\[
\langle L_{\varepsilon \xi} z_{\xi}, z_{\xi} \rangle_x \leq -\frac{C_0}{2} < 0.
\]

This complete the proof. \(\square\)
Lemma 4.4. Let $\tilde{\rho} > 0$. There exists small $\varepsilon > 0$ such that, for all $0 < \varepsilon < \varepsilon_2$ and $|\bar{\xi}| \leq \tilde{\rho}$, it holds

$$D^2f_0(z_\xi)[\phi, \phi] \geq C_2\|\phi\|_s^2,$$

for $\phi \in K_{\varepsilon, \varepsilon_2}$, where $C_2$ is a positive constant only depending on $\varepsilon_2$ and $\tilde{\rho}$.

If this lemma does not hold, then there exists a sequence of $(\varepsilon_j, \xi_j) \to (0, \tilde{\xi})$ in $\mathbb{R}^n \times B_{\tilde{\rho}} \subset \mathbb{R}^n \times \mathbb{R}^n$ and a sequence $\phi_j \in K_{\varepsilon_j, \varepsilon_2}$ such that

$$\|\phi_j\|_s = 1,$$

(4.8)

and

$$D^2f_0(z_{\xi_j})[\phi_j, \phi_j] \to 0, \quad \text{as} \quad j \to \infty.$$

(4.9)

Since $\{\phi_j\}$ is bounded in $H^s(\mathbb{R}^n)$, we assume (passing to a subsequence) that $\phi_j$ converge weakly to a $\phi_\infty$ in $H^s(\mathbb{R}^n)$.

Lemma 4.5. It holds that

$$\phi_\infty \in K_{0, \varepsilon_2}^{1, 0}.$$

Proof. Rewrite

$$\partial_{\xi_j} z_{\xi} = \partial_{\xi_j} U(x - \bar{\xi}) + \partial_{\xi_j} [b(\varepsilon_j \xi_j)U(a(\varepsilon_j \xi_j)(x - \xi_j)) - U(x - \bar{\xi})]$$

$$:= \partial_{\xi_j} U(x - \bar{\xi}) + \psi_j.$$

By the definition of $a(\xi)$ and $b(\xi)$ (see equations (2.2) and (2.3)), it holds that

$$\|\psi_j\|_s \to 0, \quad \text{as} \quad j \to \infty.$$

From $\phi_j \in K_{\varepsilon_j, \varepsilon_2}^{1, 0}$, it holds that

$$0 = \langle \phi_j, \partial_{\xi_j} z_{\xi} \rangle_s = \langle \phi_j, \partial_{\xi_j} U(\cdot - \bar{\xi}) \rangle_s + \langle \phi_j, \psi_j \rangle_s \to \langle \phi_\infty, \partial_{\xi_j} U(\cdot - \bar{\xi}) \rangle_s.$$

That is, $\phi_\infty \perp \partial_{\xi_j} U(\cdot - \bar{\xi})$. Similarly, we have that $\phi_\infty \perp U(\cdot - \bar{\xi})$. Therefore, we obtain $\phi_\infty \in K_{0, \varepsilon_2}^{1, 0}$. This completes the proof.

Let

$$L_j : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

be the operator given by

$$\langle L_j \phi, \psi \rangle_s = D^2f_0(z_{\xi_j})[\phi, \psi], \quad \text{for} \quad \phi, \psi \in H^s(\mathbb{R}^n),$$

and let

$$L_\infty : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$$

be the operator defined by

$$\langle L_\infty \phi, \psi \rangle_s = D^2f_0(U(\cdot - \bar{\xi}))[\phi, \psi], \quad \text{for} \quad \phi, \psi \in H^s(\mathbb{R}^n).$$

We now have the following lemma.

Lemma 4.6. We have that $\phi_\infty = 0$. 

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Proof. From equations (4.8) and (4.9), we get that
\[ \langle L_j \phi_j, \phi_j \rangle = \| \phi_j \|_2^2 - p \int_{\mathbb{R}^n} z^{p-1}_{\xi_j} \phi_j^2 \, dx \to 0 \]
and then
\[ p \int_{\mathbb{R}^n} z^{p-1}_{\xi_j} \phi_j^2 \, dx \to 1. \]
Hence, from the definition of $z_\xi$ (see section 2.3), we obtain that
\[ p \int_{\mathbb{R}^n} z^{p-1}(x - \bar{\xi}) \phi_j^2 \, dx \to 1. \tag{4.10} \]
Moreover, estimate
\[ \left| \int_{\mathbb{R}^n} U^{p-1}(x - \bar{\xi})(\phi_j^2 - \phi_\infty^2) \, dx \right| \leq \left( \int_{\mathbb{R}^n} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \right)^{\frac{1}{2}} \| \phi_j + \phi_\infty \|_0 \]
\[ \leq C \left( \int_{\mathbb{R}^n} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \right)^{\frac{1}{2}}. \]
Let $B_r(\bar{\xi})$ be the ball centered at $\bar{\xi}$ with radius $r$. Then
\[ \int_{\mathbb{R}^n} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \]
\[ = \left( \int_{B_r(\bar{\xi})} + \int_{\mathbb{R}^n \setminus B_r(\bar{\xi})} \right) U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx. \tag{4.11} \]
For all sufficiently small $\epsilon > 0$, there exists an $r(\epsilon)$ such that if $r > r(\epsilon)$, then, $U^{2(p-1)}(x - \bar{\xi}) < \epsilon$, for all $x \in \mathbb{R}^n \setminus B_r(\bar{\xi})$. Thus,
\[ \left| \int_{\mathbb{R}^n \setminus B_r(\bar{\xi})} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \right| \leq \epsilon \| \phi_j(x) - \phi_\infty(x) \|_0^2. \]
We now estimate the other term in equation (4.11). Let $\chi$ be a smooth function satisfying
\[ \chi(x) = \begin{cases} 1, & \text{for } x \in B_r(\bar{\xi}), \\ 0, & \text{for } x \in \mathbb{R}^n \setminus B_{r+1}(\bar{\xi}). \end{cases} \]
Then $\{\chi \phi_j\}$ is a bounded sequence in $H^s(B_{r+1}(\bar{\xi}))$. Therefore, there exists a function $\eta \in H^s(B_{r+1}(\bar{\xi}))$ such that, up to a subsequence, $\chi \phi_j \rightharpoonup \eta$. Since the embedding $H^s(B_{r+1}(\bar{\xi})) \hookrightarrow L^2(B_{r+1}(\bar{\xi}))$ is compact, we have $\chi \phi_j \to \eta$ in $L^2(B_{r+1}(\bar{\xi}))$. Then
\[ \phi_j|_{B_r(\bar{\xi})} = \chi \phi_j|_{B_r(\bar{\xi})} \rightharpoonup \eta|_{B_r(\bar{\xi})}, \quad \text{in } L^2(B_r(\bar{\xi})). \]
Since $\phi_j \to \phi_\infty$ in $L^2(B_r(\bar{\xi}))$, we obtain that
\[ \phi_j \to \phi_\infty \quad \text{in } L^2(B_r(\bar{\xi})). \tag{4.12} \]
It follows that
\[ \left| \int_{B_r(\bar{\xi})} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \right| \to 0, \quad \text{as } j \to \infty. \]
From the arbitrariness of $\epsilon$, we have that
\[ \int_{\mathbb{R}^n} U^{2(p-1)}(x - \bar{\xi})|\phi_j(x) - \phi_\infty(x)|^2 \, dx \to 0. \]
This yields that
\[ \int_{\mathbb{R}^n} U^{p-1}(x - \bar{\xi})(\phi_j^2 - \phi_{\infty}^2) \, dx \to 0. \] (4.13)

From equations (4.10) and (4.13), we get that
\[ p \int_{\mathbb{R}^n} U^{p-1}(x - \bar{\xi})\phi_j^2 \, dx = 1. \]

However, from \( \phi_j \rightharpoonup \phi_{\infty} \) in \( H^s(\mathbb{R}^n) \), we have that
\[ \langle \phi_{\infty}, \phi_{\infty} \rangle_s - \langle \phi_j, \phi_{\infty} \rangle_s \leq \|\phi_j\|_s \|\phi_{\infty}\|_s = \|\phi_{\infty}\|_s. \]

It follows that
\[ \|\phi_{\infty}\|_s \leq 1. \]

Therefore, we obtain that
\[ \langle L_{\infty}\phi_{\infty}, \phi_{\infty} \rangle_s = \|\phi_{\infty}\|_s^2 - p \int_{\mathbb{R}^n} U^{p-1}(x - \bar{\xi})\phi_{\infty}^2 \, dx \leq 0. \]

By theorem 2.3, remark 2.5 and lemma 4.5, it holds that
\[ \langle L_{\infty}\phi_{\infty}, \phi_{\infty} \rangle_s \geq C \|\phi_{\infty}\|_s^2, \]
where \( C \) is a positive constant. Then we have that
\[ \|\phi_{\infty}\|_s = 0. \]

This completes the proof.

**Proof of lemma 4.4.** Note that \( z_{\xi_j}^{p-1} \) decays uniformly to 0 at infinity as \( 0 < \epsilon_j < \bar{\epsilon} \) and \( |\xi| \leq \bar{\rho} \). Then, for any \( \epsilon > 0 \), there exists a sufficiently large \( r_0 > 0 \) such that, for all \( r > r_0 \), \( |z_{\xi_j}^{p-1}(x)| < \epsilon \) when \( x \in \mathbb{R}^n \setminus B_r \). Therefore, from equation (4.12) and \( \phi_{\infty} = 0 \), we have that
\[ \left| \int_{\mathbb{R}^n} z_{\xi_j}^{p-1} |\phi_j|^2 \, dx \right| \leq C \int_{B_r} |\phi_j|^2 \, dx + \epsilon \int_{\mathbb{R}^n \setminus B_r} |\phi_j|^2 \, dx \to \epsilon, \quad \text{as } j \to \infty. \]

From the arbitrariness of \( \epsilon \), we have that
\[ \left| \int_{\mathbb{R}^n} z_{\xi_j}^{p-1} |\phi_j|^2 \, dx \right| \to 0, \quad \text{as } j \to \infty. \]

Moreover, from equations (4.8) and (4.9), it holds that
\[ 0 \leftarrow D^2 f_0(z_{\xi_j})[\phi_j, \phi_j] = \|\phi_j\|_s^2 - p \int_{\mathbb{R}^n} z_{\xi_j}^{p-1} |\phi_j|^2 \, dx \to 1. \]

It is a contradiction. Thus we have lemma 4.4.

**Lemma 4.7.** Let \( \bar{\rho} > 0 \). There exists small \( \epsilon_3 > 0 \) such that for all \( 0 < \epsilon < \epsilon_3 \) and \( |\xi| \leq \bar{\rho} \), it holds that
\[ D^2 f_\epsilon(z_{\xi})[\phi, \phi] \geq C_3 \|\phi\|_s^2, \quad \text{for } \phi \in K_{\epsilon, \xi}, \]
where \( C_3 \) is a positive constant only depending on \( \epsilon_2 \) and \( \bar{\rho} \).
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Proof. By the non-negativity of $V$ and lemma 4.4, we have that, for all $\phi \in K^{\perp}_{\varepsilon, \xi}$

\[ D^2 f_\varepsilon(z_\xi)[\phi, \phi] = D^2 f_0(z_\xi)[\phi, \phi] + \int_{\mathbb{R}^n} V(\varepsilon x) \phi' \, dx \]

\[ \geq D^2 f_0(z_\xi)[\phi, \phi] + C_0 \|\phi\|_{s}^2. \]

Here $0 < \varepsilon < \varepsilon_2$ and $|\xi| \leq \bar{\rho}$. Letting $\varepsilon_3 = \varepsilon_2$ and $C_3 = C_2$, we obtain the result. \qed

Proof of proposition 4.1. Let $\bar{\varepsilon} = \varepsilon_2$. From lemma 4.3, lemma 4.7, lemma 3.1 and equation (4.1), we have that, for all $|\xi| \leq \bar{\rho}$ and $0 < \varepsilon < \bar{\varepsilon}$,

\[ |\langle L_{\varepsilon, \xi} v, v \rangle_s| \geq C \|v\|_{s}^2, \quad \forall v \in (T_{z_\xi} Z')^{\perp}, \]

where $C > 0$ is a constant only depending on $\bar{\xi}$ and $\bar{\varepsilon}$. This completes the proof. \qed

5. Lyapunov–Schmidt reduction

In this section, we will prove that the existence of critical points of $f_\varepsilon$ can be reduced to find critical points of an auxiliary finite dimensional functional.

5.1. Auxiliary finite dimensional functional

Let $P_{\varepsilon, \xi}$ be the orthogonal projection onto $(T_{z_\xi} Z')^{\perp}$. Our aim is to find a point $w \in (T_{z_\xi} Z')^{\perp}$ satisfying

\[ P_{\varepsilon, \xi} Df_\varepsilon(z_\xi + w) = 0. \] (5.1)

By expansion, we have that

\[ Df_\varepsilon(z_\xi + w) = Df_\varepsilon(z_\xi) + D^2 f_\varepsilon(z_\xi)[w] + \mathcal{R}(z_\xi, w). \]

Here, the map $\mathcal{R}(z_\xi, w)$ is given by

\[ \mathcal{R}(z_\xi, w) : H' \rightarrow \mathbb{R} \]

\[ v \rightarrow \int_{\mathbb{R}^n} R(z_\xi, w)v \, dx, \]

where

\[ R(z_\xi, w) = -(|z_\xi + w|^{p-1}(z_\xi + w) - |z_\xi|^{p-1}z_\xi - p|z_\xi|^{p-1}w). \]

Lemma 5.1. For all $w_1, w_2 \in B_1 \subset H'(\mathbb{R}^n)$, it holds that

\[ \|\mathcal{R}(z_\xi, w_2) - \mathcal{R}(z_\xi, w_1)\|_s \leq C \max\{\|w_1\|_{s}^p, \|w_2\|_{s}^p\}\|w_2 - w_1\|_s, \]

where $\alpha = \min\{1, p - 1\}$, $C$ is a constant independent of $w_1, w_2$. Here $B_1$ is the unit ball in $H'(\mathbb{R}^n)$.

Proof. For all $v \in H'(\mathbb{R}^n)$,

\[ |\langle \mathcal{R}(z_\xi, w_2) - \mathcal{R}(z_\xi, w_1)\rangle(v)| \]

\[ \leq \int_{\mathbb{R}^n} |z_\xi + w_2|^{p-1}(z_\xi + w_2) - |z_\xi + w_1|^{p-1}(z_\xi + w_1) - p|z_\xi|^{p-1}(w_2 - w_1)| \, |v| \, dx \]

\[ \leq p \int_{\mathbb{R}^n} |z_\xi + w_1 + \theta_1(w_2 - w_1)|^{p-1} - |z_\xi|^{p-1}| \, |w_2 - w_1| \, |v| \, dx. \]
Here $\theta_1 \in [0, 1]$. For $1 < p \leq 2$,  
\[
\left| [R(z_\xi, w_2) - R(z_\xi, w_1)](v) \right| \leq p \int_{\mathbb{R}^n} |w_1 + \theta_1(w_2 - w_1)||w_2 - w_1||v| \, dx
\]
\[
\leq C \int_{\mathbb{R}^n} (|w_1| + |w_2|)^{p-1}|w_2 - w_1||v| \, dx
\leq C(\|w_1\|_{L^{\infty}}^{p-1} + \|w_2\|_{L^{\infty}}^{p-1})\|w_2 - w_1\|_{L^p} \|v\|_{L^p}.
\]
By Sobolev imbedding (theorem 2.1), we have that  
\[
H^s(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n).
\]
Therefore, we obtain that  
\[
\left| [R(z_\xi, w_2) - R(z_\xi, w_1)](v) \right| \leq C(\|w_1\|_s^{p-1} + \|w_2\|_s^{p-1})\|w_2 - w_1\|_s \|v\|_s.
\]
For $2 < p < \frac{2n}{n-2}$ (if $2 < \frac{2n}{n-2}$), it holds that  
\[
\left| [R(z_\xi, w_2) - R(z_\xi, w_1)](v) \right| = C \int_{\mathbb{R}^n} \left| z_\xi + \theta_2(w_1 + \theta_1(w_2 - w_1)) \right|^{p-2}|w_2 - w_1|^2|v| \, dx
\]
\[
\leq C \|z_\xi + \theta_2(w_1 + \theta_1(w_2 - w_1))\|_{L^{p^*}}^{p-2}\|w_2 - w_1\|_{L^{p^*}} \|v\|_{L^{p^*}},
\]
where $\theta_2 \in [0, 1]$. Similarly, by Sobolev imbedding, we have that  
\[
\left| [R(z_\xi, w_2) - R(z_\xi, w_1)](v) \right| \leq C(\|z_\xi\|_s + \|w_1\|_s + \|w_2\|_s)^{p-2}\|w_2 - w_1\|_s \|v\|_s.
\]
Therefore, we have  
\[
\|R(z_\xi, w_2) - R(z_\xi, w_1)\|_s \leq C \max(\|w_1\|_s^\sigma, \|w_1\|_s^\sigma)\|w_2 - w_1\|_s,
\]
where $\sigma = \min(1, p - 1)$. This completes the proof. \hfill $\square$

**Corollary 5.2.** It holds that $\|R(z_\xi, w)\|_s = O(\|w\|_s^{1+\sigma})$ where $\sigma = \min(1, p - 1)$.

**Proof.** Choosing $w_1 = 0$ and $w_2 = w$ in lemma 5.1, we find that  
\[
\|R(z_\xi, w)\|_s \leq C(\|w\|_s^{1+\sigma}).
\]
\hfill $\square$

From the definition of $L_{\xi, \tilde{\xi}}$, equation (5.1) becomes  
\[
L_{\xi, \tilde{\xi}} w + P_{\xi, \tilde{\xi}} Df_\xi(z_\xi) + P_{\xi, \tilde{\xi}} R(z_\xi, w) = 0, \quad \text{for } w \in (T_{z_\xi} Z)^{1+}. \tag{5.2}
\]
By proposition 4.1, we know that $L_{\xi, \tilde{\xi}}$ is invertible on $(T_{z_\xi} Z)^{1+}$. Denote the invertible operator by $L_{\xi, \tilde{\xi}}^{-1}$. Then equation (5.2) is equivalent to  
\[
w = N_{\xi, \tilde{\xi}}(w).
\]
Here  
\[
N_{\xi, \tilde{\xi}}(w) = -L_{\xi, \tilde{\xi}}^{-1}(P_{\xi, \tilde{\xi}} Df_\xi(z_\xi) + P_{\xi, \tilde{\xi}} R(z_\xi, w)).
\]

**Lemma 5.3.** There is a small ball $B_{\tilde{\xi}} \subset (T_{z_\xi} Z)^{1+}$ such that $N_{\xi, \tilde{\xi}}$ maps $B_{\tilde{\xi}}$ to itself if $0 < \xi < \tilde{\xi}$ and $|\xi| \leq \tilde{\rho}$. 

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Proof. Using lemma 3.3, we obtain
\[ \| N_{\varepsilon,\xi}(w) \|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2)) + O(\|w\|_s^{1+\sigma}). \] (5.3)
Then there is a small positive constant \( \delta \) such that \( N_{\varepsilon,\xi} \) maps \( B_\delta \subset (T_{z\xi} Z)_{-s} \) to itself if \( 0 < \varepsilon < \bar{\varepsilon} \) and \( |\xi| \leq \bar{\rho} \). □

**Lemma 5.4.** For all \( w_1, w_2 \in B_1 \subset H^s(\mathbb{R}^n) \), we have that
\[ \| N_{\varepsilon,\xi}(w_2) - N_{\varepsilon,\xi}(w_1) \|_s \leq C \max(\|w_1\|_s^p, \|w_2\|_s^p) \|w_2 - w_1\|_s, \]
where \( C \) is a constant independent of \( w_1 \) and \( w_2 \), \( \sigma = \min\{1, p - 1\} \).

**Proof.** Compute
\[ \| N_{\varepsilon,\xi}(w_2) - N_{\varepsilon,\xi}(w_1) \|_s = \| - P_{\varepsilon,\xi}(R(z\xi, w_2) - R(z\xi, w_1)) \|_s \leq C \| R(z\xi, w_2) - R(z\xi, w_1) \|_s. \]
Then by lemma 5.1, we have that
\[ \| N_{\varepsilon,\xi}(w_2) - N_{\varepsilon,\xi}(w_1) \|_s \leq C \max(\|w_1\|_s^p, \|w_1\|_s^p) \|w_2 - w_1\|_s, \]
where \( \sigma = \min\{1, p - 1\} \). This completes the proof. □

**Proposition 5.5.** For \( 0 < \varepsilon < \bar{\varepsilon} \) and \( |\xi| \leq \bar{\rho} \), there exists a unique \( w = w(\varepsilon, \xi) \in (T_{z\xi} Z)_{-s} \) such that \( Df_\varepsilon(z\xi + w) \in T_{z\xi} Z \), and \( w(\varepsilon, \xi) \) is of class \( C^1 \). Moreover, the functional \( \Phi_\varepsilon(\xi) = f_\varepsilon(z\xi + w(\varepsilon, \xi)) \) has the same regularity as \( w \) and satisfies:
\[ \nabla \Phi_\varepsilon(\xi_0) = 0 \Rightarrow Df_\varepsilon(z_0 + w(\varepsilon, \xi_0)) = 0. \]

**Proof.** From lemmas 5.3 and 5.4, the map \( N_{\varepsilon,\xi} \) is a contraction on \( B_\delta \) for \( 0 < \varepsilon < \bar{\varepsilon} \) and \( |\xi| \leq \bar{\rho} \). Then there exists a unique \( w \) such that \( w = N_{\varepsilon,\xi}(w) \). For fixed \( \varepsilon \), define
\[ \Xi_\varepsilon : (\xi, w) \rightarrow P_{\varepsilon,\xi}Df_\varepsilon(z\xi + w). \]
Applying the implicit function theorem to \( \Xi_\varepsilon \), we have that \( w(\varepsilon, \xi) \) is \( C^1 \) with respect to \( \xi \). Then using a standard argument as in [2, 3], we obtain that the critical points of \( \Phi_\varepsilon(\xi) = f_\varepsilon(z\xi + w(\varepsilon, \xi)) \) give rise to critical points of \( f_\varepsilon \). □

In what follows, we use the simple notation \( w \) to denote \( w(\varepsilon, \xi) \) which is obtained in proposition 5.5.

**Remark 5.6.** From equation (5.3), it follows that
\[ \|w\|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + \varepsilon^2), \]
where \( C > 0 \).

**Lemma 5.7.** The following inequality holds:
\[ \|\nabla_\xi w\|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2))^{\sigma}, \]
where \( C > 0 \) and \( \sigma = \min\{1, p - 1\} \).
Proof. From (5.2) and proposition 5.5, we have that, for all \( v \in (T_{z\xi} Z^s)^{\perp} \),
\[
\langle L_{\epsilon,\xi} w, v \rangle_s + \int_{R^n} V(\epsilon x) w v \, dx - p \int_{R^n} z_{\xi}^{p-1} \, dx + \int_{R^n} [V(\epsilon x) - V(\epsilon \xi)] z_{\xi} v \, dx
\]
\[
+ \int_{R^n} R(z_{\xi}, w) \, dx = 0.
\]
(5.4)
Since \( DF_{\epsilon,\xi}(z_{\xi}) = 0 \), equation (5.4) becomes
\[
\langle w, v \rangle_s + \int_{R^n} V(\epsilon x) w v \, dx - p \int_{R^n} z_{\xi}^{p-1} \, dx + \int_{R^n} [V(\epsilon x) - V(\epsilon \xi)] z_{\xi} v \, dx
\]
\[
+ \int_{R^n} R(z_{\xi}, w) \, dx = 0.
\]
Hence
\[
\langle \partial_{\xi_j} w, v \rangle_s + \int_{R^n} V(\epsilon x) (\partial_{\xi_j} w) v \, dx - p(p - 1) \int_{R^n} z_{\xi}^{p-2} (\partial_{\xi_j} z) w v \, dx + \int_{R^n} (V(\epsilon x) - V(\epsilon \xi)) (\partial_{\xi_j} z) v \, dx
\]
\[
- \epsilon (\partial_{\xi_j} V)(\epsilon \xi) \int_{R^n} z_{\xi} v \, dx - \int_{R^n} (R_z \partial_{\xi_j} z + R_w \partial_{\xi_j} w) v \, dx = 0.
\]
Set \( \hat{L} = L_{\epsilon,\xi} - R_w \), where \( \langle R_w v_1, v_2 \rangle = \int_{R^n} R_w v_1 v_2 \, dx \). Since \( R_w \to 0 \) as \( w \to 0 \) and \( L_{\epsilon,\xi} \) is invertible on \((T_{z\xi} Z^s)^{\perp}\), \( \hat{L} \) is also invertible for \( 0 < \epsilon < \bar{\epsilon} \) and \( |\xi| \leq \bar{\rho} \). From equation (5.5), it holds that
\[
\langle \hat{L} \partial_{\xi_j} w, v \rangle = p(p - 1) \int_{R^n} z_{\xi}^{p-2} (\partial_{\xi_j} z) w v \, dx - \int_{R^n} (V(\epsilon x) - V(\epsilon \xi)) (\partial_{\xi_j} z) v \, dx
\]
\[
+ \epsilon (\partial_{\xi_j} V)(\epsilon \xi) \int_{R^n} z_{\xi} v \, dx + \int_{R^n} R_z \partial_{\xi_j} z v \, dx = T_1 + T_2 + T_3 + T_4.
\]
Next, we shall estimate every term on the left of the equation above. By theorem 2.3 and remark 2.4, it holds that, for \( 1 < p \leq 2 \),
\[
|T_1| = p(p - 1) \int_{R^n} z_{\xi}^{p-2} (\partial_{\xi_j} z) w v \, dx
\]
\[
\leq C \int_{R^n} (1 + |x|^{n+2s})^{2-p} \frac{1}{1 + |x|^{n+2s}} |w v| \, dx
\]
\[
\leq C \int_{R^n} \frac{1}{(1 + |x|^{n+2s})^{p-1}} |w v| \, dx
\]
\[
\leq C \int_{R^n} |w v| \, dx \leq C \|w\|_0 \|v\|_0 \leq C \|w\|_s \|v\|_s,
\]
and, for \( 2 < p < \frac{n+2s}{n-2s} \) (if \( 2 < \frac{n+2s}{n-2s} \)),
\[
\int_{R^n} z_{\xi}^{p-2} (\partial_{\xi_j} z) w v \, dx \leq C \int_{R^n} \frac{1}{(1 + |x|^{n+2s})^{p-1}} |w v| \, dx
\]
\[
\leq C \|w\|_s \|v\|_s.
\]
Therefore, we have that
\[
|T_1| \leq C \|w\|_s \|v\|_s.
\]
Since \( 0 < \epsilon < \bar{\epsilon} \) and \( |\xi| \leq \bar{\rho} \), by lemma 3.2 we have
\[
|T_2| = \left| \int_{R^n} (V(\epsilon x) - V(\epsilon \xi)) (\partial_{\xi_j} z) v \, dx \right|
\]
\[
\leq \int_{R^n} |V(\epsilon x) - V(\epsilon \xi)| |\partial_{\xi_j} z| |v| \, dx
\]
\[
(5.6)
\]
\[ \leq \left( \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |\partial_\xi^j z|^2 \, dx \right)^{\frac{1}{2}} \|v\|_0 \]
\[ \leq C\varepsilon \|\nabla V(\varepsilon \xi)\| \|v\|_s. \]

Then we obtain that
\[ |T_2| \leq C\varepsilon \|\nabla V(\varepsilon \xi)\| \|v\|_s. \]

Estimating the third term, we have
\[ |T_3| = \varepsilon \left( \partial_\xi^j V(\varepsilon \xi) \int_{\mathbb{R}^n} z v \, dx \right) \leq \varepsilon \|\nabla V(\varepsilon \xi)\| \|z\|_0 \|v\|_0 \leq \varepsilon \|\nabla V(\varepsilon \xi)\| \|v\|_s. \]

It remains to estimate the final term. A direct computation yields
\[ |T_4| = \left| \int_{\mathbb{R}^n} R_z \partial_\xi^j z v \, dx \right| \leq \varepsilon \left( \|z\|_p \|\partial_\xi^j z\| \|v\| \right) \]
\[ \leq \varepsilon \|z\|_p \|\partial_\xi^j z\| \|v\| \]

Then, for \(1 < p \leq 2\),
\[ \int_{\mathbb{R}^n} |z_\xi + w|^{p-1} - |z_\xi|^{p-1} \cdot |\partial_\xi^j z_\xi| \cdot |v| \, dx \]
\[ \leq C \int_{\mathbb{R}^n} |w|^{p-1} \cdot |\partial_\xi^j z_\xi| \cdot |v| \, dx \]
\[ \leq C \|w\|_{L^{p-1}} \|\partial_\xi^j z_\xi\|_{L^1} \|v\|_{L^1} \leq C \|w\|_{L^1} \|v\|_{L^1}, \]

and, for \(2 < p < \frac{n+2}{n-2}\) (if \(2 < \frac{n+2}{n-2}\)),
\[ \int_{\mathbb{R}^n} |z_\xi + w|^{p-1} - |z_\xi|^{p-1} \cdot |\partial_\xi^j z_\xi| \cdot |v| \, dx \]
\[ \leq \int_{\mathbb{R}^n} (p-1)|z_\xi + \theta_3 w|^{p-2}|w| \cdot |\partial_\xi^j z_\xi| \cdot |v| \, dx \]
\[ \leq C \|z_\xi + \theta_3 w\|_{L^{p-2}} \|\partial_\xi^j z_\xi\|_{L^1} \|w\|_{L^{p-1}} \|v\|_{L^1} \leq C \|w\|_s \|v\|_s. \]

Here \(\theta_3 \in [0, 1]\). Then we have that
\[ \int_{\mathbb{R}^n} |z_\xi + w|^{p-1} - |z_\xi|^{p-1} \cdot |\partial_\xi^j z_\xi| \cdot |v| \, dx \leq C \|w\|_s \|v\|_s, \]

where \(\sigma = \min\{1, p-1\}\). Furthermore, we estimate
\[ \int_{\mathbb{R}^n} |\partial_\xi^j z_\xi| \cdot |w| \cdot |v| \, dx \]
\[ \leq C \int_{\mathbb{R}^n} \left( \frac{1}{(1 + |x - \xi|)^{n+2\sigma}} \right)^{p-1} \cdot |w| \cdot |v| \, dx \]
\[ \leq C \|w\|_0 \|v\|_0 \leq C \|w\|_s \|v\|_s. \]

Therefore, we obtain
\[ |T_4| \leq C \|w\|_s \|v\|_s, \]

where \(\sigma = \min\{1, p-1\}\).
Summarizing the estimates for $T_1, T_2, T_3, T_4$, we get
\[
\| \mathcal{E} \partial_\xi u \|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + \| w \|^p).
\]
Then from remark 5.6, it holds that
\[
\| \mathcal{E} \partial_\xi u \|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2))^p.
\]
Thus, we finally obtain
\[
\| \nabla_\xi w \|_s \leq C(\varepsilon |\nabla V(\varepsilon \xi)| + O(\varepsilon^2))^p.
\]
This completes the proof. \qed

5.2. Analysis of $\Phi_1(\xi)$

In this subsection, we shall expand $\Phi_1(\xi)$. By the definition, we have that
\[
\Phi_1(\xi) = \frac{1}{2} \| z_\xi + w(\varepsilon, \xi) \|^2 + \frac{1}{2} \int_{\mathbb{R}^n} V(\varepsilon x)(z_\xi + w(\varepsilon, \xi))^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^n} |z_\xi + w(\varepsilon, \xi)|^{p+1} \, dx.
\]
Since $(-\Delta)z_\xi + z_\xi + V(\varepsilon \xi)z_\xi = z_\xi^p$, it holds that
\[
\langle z_\xi, w \rangle_s = -V(\varepsilon \xi) \int_{\mathbb{R}^n} z_\xi w \, dx + \int_{\mathbb{R}^n} z_\xi^p w \, dx.
\]
Therefore, we can rewrite
\[
\Phi_1(\xi) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^n} z^{p+1} \, dx + \frac{1}{2} \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi))z^2 \, dx + \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi))z w \, dx + \frac{1}{2} \int_{\mathbb{R}^n} V(\varepsilon x)w^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^n} \left( |z + w|^{p+1} - z^{p+1} - (p + 1)z^p w \right) \, dx.
\]
From the definition of $z(x)$ (see subsection 2.3), $z(x) = b(\varepsilon \xi) U(a(\varepsilon \xi)x)$, where $a(\varepsilon \xi) = (1 + V(\varepsilon \xi))^\frac{2}{n}$ and $b(\varepsilon \xi) = (1 + V(\varepsilon \xi))^{-\theta}$. Then we have that
\[
\int_{\mathbb{R}^n} z^{p+1} \, dx = C_0 (1 + V(\varepsilon \xi))^\theta,
\]
where $C_0 = \int_{\mathbb{R}^n} U^{p+1} \, dx$ and $\theta = \frac{p+1}{p-1} - \frac{n}{2}$. Let $C_1 = \left( \frac{1}{2} - \frac{1}{p+1} \right) C_0$. Then
\[
\Phi_1(\xi) = C_1 (1 + V(\varepsilon \xi))^\theta + \Gamma_\varepsilon(\xi) + \Psi_\varepsilon(\xi),
\]
where
\[
\Gamma_\varepsilon(\xi) = \frac{1}{2} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)]z^2 \, dx + \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)]z w \, dx
\]
and
\[
\Psi_\varepsilon(\xi) = \frac{1}{2} \int_{\mathbb{R}^n} V(\varepsilon x)w^2 \, dx + \frac{1}{2} \| w \|^p_s
\]
\[
- \frac{1}{p+1} \int_{\mathbb{R}^n} \left[ |z + w|^{p+1} - z^{p+1} - (p + 1)z^p w \right] \, dx.
\]
Lemma 5.8. We have the following estimate:
\[ |\nabla \Psi_\varepsilon(\xi) | \leq C \|w\|_s (\|w\|_p^\sigma + \|\nabla_\varepsilon w\|_s). \]

Proof. Direct calculus yields, for \( j = 1, 2, \ldots, n, \)
\[
\left| \partial_{\xi_j} \left( \frac{1}{2} \int_{\mathbb{R}^n} V(\varepsilon x) w^2 \, dx + \frac{1}{2} \|w\|_2^2 \right) \right| = \left| \int_{\mathbb{R}^n} V(\varepsilon x) \partial_{\xi_j} w \, dx + \langle w, \partial_{\xi_j} w \rangle \right|
\leq C (\|w\|_0 \|\partial_{\xi_j} w\|_0 + \|w\|_s \|\partial_{\xi_j} w\|_s) \leq C (\|w\|_s \|\partial_{\xi_j} w\|_s). \tag{5.7}
\]

Estimate
\[
\left| \partial_{\xi_j} \left( \frac{1}{p + 1} \int_{\mathbb{R}^n} (|z + w|^{p+1} - z^{p+1} - (p + 1)z^p w) \, dx \right) \right|
\leq \int_{\mathbb{R}^n} |pw| (|z + \theta_4 w|^{p-1} - z^{p-1} - \theta_4 z^p w) \, dx
\leq C \|\partial_{\xi_j} z\|_{L^{p+1}} \|w\|_{L^{p+1}} \leq C \|w\|_{L^p}.
\]

Here \( \theta_4 \in [0, 1]. \) Then, for \( 1 < p \leq 2, \)
\[
\left| \int_{\mathbb{R}^n} (pw)(|z + \theta_4 w|^{p-1} - z^{p-1}) \partial_{\xi_j} z \, dx \right|
\leq C \|\partial_{\xi_j} z\|_{L^{p+1}} \|w\|_{L^{p+1}} \leq C \|w\|_{L^p}.
\]

and, for \( 2 < p \leq \frac{2^{\frac{1}{p-2}}}{\frac{2}{2-p}} \) (if \( 2 < \frac{2^{\frac{1}{p-2}}}{\frac{2}{2-p}} \)),
\[
\left| \int_{\mathbb{R}^n} (pw)(|z + \theta_4 w|^{p-1} - z^{p-1}) \partial_{\xi_j} z \, dx \right|
\leq C \|\partial_{\xi_j} z\|_{L^{p+1}} \|z + \theta_4 w\|_{L^{p+1}} \|w\|_{L^{p-1}} \leq C \|\partial_{\xi_j} z\|_{L^{p+1}} \|w\|_{L^{p-1}} \leq C \|w\|_{L^p}^2.
\]

Here \( \theta_5 \in [0, 1]. \) Therefore,
\[
\left| \partial_{\xi_j} \left( \frac{1}{p + 1} \int_{\mathbb{R}^n} (|z + w|^{p+1} - z^{p+1} - (p + 1)z^p w) \, dx \right) \right| \leq C \|w\|_{L^\infty}^{1+\sigma}.
\]

Moreover,
\[
\left| \int_{\mathbb{R}^n} pw(|z + \theta_4 w|^{p-1} \partial_{\xi_j} w \, dx \right| \leq C \|z + \theta_4 w\|_{L^{p+1}} \|w\|_{L^{p+1}} \|\partial_{\xi_j} w\|_{L^{p+1}} \leq C \|w\|_s \|\partial_{\xi_j} w\|_s.
\]

Therefore, we have that
\[
|\nabla \Psi_\varepsilon(\xi) | \leq C \|w\|_s (\|w\|_p^\sigma + \|\nabla_\varepsilon w\|_s).
\]

This completes the proof.

Lemma 5.9. It holds that
\[ |\nabla \Gamma_\varepsilon(\xi) | \leq C \varepsilon^{1+\sigma}. \tag{5.8} \]
Proof. Compute
\[
\int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi))z^2 \, dx
\]
\[
= \varepsilon \int_{\mathbb{R}^n} \nabla V(\varepsilon \xi) \cdot (x - \xi)z^2 \, dx
\]
\[
+ \varepsilon^2 \int_{\mathbb{R}^n} D^2 V(\varepsilon \xi + \theta_0(\varepsilon - \xi))[x - \xi, x - \xi]z^2 \, dx
\]
where \(\theta_0 \in [0, 1]\). Since \(V \in C^3_b(\mathbb{R}^n)\), it holds that
\[
\left| \partial_{\xi_j} \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi))z \, dx \right|
\]
\[
= \varepsilon \left| \partial_{\xi_j} \left( \int_{\mathbb{R}^n} D^2 V(\varepsilon \xi + \theta_0(\varepsilon - \xi))[x - \xi, x - \xi]z \, dx \right) \right| \leq C \varepsilon^2.
\]
Estimate
\[
\left| \partial_{\xi_j} \int_{\mathbb{R}^n} [V(\varepsilon x) - V(\varepsilon \xi)]z w \, dx \right|
\]
\[
\leq \varepsilon |\nabla V(\varepsilon \xi)| \int_{\mathbb{R}^n} |z| |w| \, dx + \varepsilon \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)||z| |\partial_{\xi_j} w| \, dx
\]
\[
\leq \varepsilon |\nabla V(\varepsilon \xi)||w||_0 + \left( \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |\partial_{\xi_j} z|^2 \, dx \right)^\frac{1}{2} \|w\|_0
\]
\[
+ \left( \int_{\mathbb{R}^n} |V(\varepsilon x) - V(\varepsilon \xi)|^2 |z|^2 \, dx \right)^\frac{1}{2} \|\partial_{\xi_j} w\|_0.
\]
Thus by lemma 3.2, remark 5.6 and lemma 5.7, we have that
\[
\left| \nabla \left( \int_{\mathbb{R}^n} (V(\varepsilon x) - V(\varepsilon \xi))z \, dx \right) \right|
\]
\[
\leq C \varepsilon (\varepsilon + \|w\|_s + \|\nabla w\|_s) \leq C \varepsilon^{1+\sigma}.
\]
Therefore, from estimates (5.9) and (5.10), equation (5.8) holds.

Let \(\alpha(\varepsilon, \xi) = \theta C_1 (1 + V(\varepsilon \xi))^{\theta-1}, \) where \(\theta = \frac{p+1}{p-1} - \frac{n}{2S}\). Then summarizing all the conclusions above, we get the following proposition.

Proposition 5.10. It holds that
\[
\nabla \Phi_\varepsilon(\xi) = \alpha(\varepsilon \xi) \varepsilon \nabla V(\varepsilon \xi) + \varepsilon^{1+\sigma} \sigma_\varepsilon(\xi),
\]
where \(\sigma_\varepsilon(\xi)\) is a bounded function and \(\sigma = \min\{1, p - 1\}\).

Remark 5.11. Using a similar argument, we can prove that
\[
\Phi_\varepsilon(\xi) = C (1 + V(\varepsilon \xi))^\theta + \gamma_\varepsilon(\xi),
\]
where \(C > 0, \theta = \frac{p+1}{p-1} - \frac{n}{2S}\) and \(|\gamma_\varepsilon(\xi)| \leq C (|\nabla V(\varepsilon \xi)| + \varepsilon^2)\).
6. Proof of the main theorem

In this section, we shall prove the main theorem by a classical perturbation result.

6.1. A multiplicity result by perturbation

Let $M \subset \mathbb{R}^n$ be a non-empty set. We denote by $M_\delta$ its $\delta$-neighbourhood. The cup length $l(M)$ of $M$ is defined by

$$l(M) = 1 + \sup\{k \in \mathbb{N} \mid \exists \alpha_1, \ldots, \alpha_k \in \tilde{H}^*(M) \setminus 1, \alpha_1 \cup \cdots \cup \alpha_k \neq 0\}.$$  

If no such class exists, we set $l(M) = 1$. Here $\tilde{H}^*(M)$ is the Alexander cohomology of $M$ with real coefficients and $\cup$ denotes the cup product.

Assume that $V$ has a smooth manifold of critical points of $M$. According to Bott [8], we say that $M$ is the non-degenerate critical manifold for $V$ if every $x \in M$ is a critical point of $V$ and the nullity of all $x \in M$ equals the dimension of $M$.

Now we recall a classical perturbation result. For more details, see theorem 6.4 of chapter II in [11].

**Theorem 6.1.** Let $h \in C^2(\mathbb{R}^n)$ and $\Sigma \subset \mathbb{R}^n$ be a smooth compact non-degenerate critical manifold of $h$. Let $W$ be a neighbourhood of $\Sigma$ and let $g \in C^1(\mathbb{R}^n)$. Then, if $\|h - g\|_{C^1(W)}$ is sufficiently small, the function $g$ has at least $l(\Sigma)$ critical points in $W$.

6.2. Proof of theorem 1.1

With the preliminary considerations of the sections above, we now prove theorem 1.1 using the abstract perturbation theorem above.

**Proof of theorem 1.1.** Fix $\bar{\rho} > 0$ such that $M \subset B_{\bar{\rho}}$. Since $M$ is a non-degenerate smooth critical manifold of $V$, it is a non-degenerate critical manifold of $C^1(1 + V)\theta$ as well. To use theorem 6.1, we define

$$h(\xi) = C^1(1 + V(\xi))\theta,$$

and

$$g(\xi) = \Phi_x \left( \frac{\xi}{\varepsilon} \right).$$

Set $\Sigma = M$. Fix a $\delta$-neighbourhood $M_\delta$ of $M$ such that $M_\delta \subset B_{\bar{\rho}}$ and the only critical points of $V$ in $M_\delta$ are those of $M$. Let $W = M_\delta$. From proposition 5.10 and remark 5.11, the function $\Phi_x(\cdot/\varepsilon)$ converges to $h(\cdot)$ in $C^1(W)$ as $\varepsilon \to 0$. Then theorem 6.1 yields the existence of at least $l(M)$ critical points of $g$ for sufficiently small $\varepsilon$.

Let $\xi_k \in M_\delta$ be any of those critical points. Then $\xi_k/\varepsilon$ is a critical point of $\Phi_x$ and proposition 5.5 implies that

$$u_{x,\xi_k}(x) = z_{\bar{\delta}} \left( x - \frac{\xi_k}{\varepsilon} \right) + w(\varepsilon, \xi_k)$$

is a critical point of $f_\varepsilon$ and hence a solution of equation (1.4). Thus

$$u_{x,\xi_k}\left( \frac{x}{\varepsilon} \right) \approx z_{\bar{\delta}} \left( x - \frac{\xi_k}{\varepsilon} \right)$$

is a solution of equation (1.3).

Any $\xi_k$ converges to some point $\xi^*_k \in M_\delta$ as $\varepsilon \to 0$. From proposition 5.10, we have that $\xi^*_k$ is a stationary point of $V$. Then the choice of $M_\delta$ implies that $\xi^*_k \in M$. That is, $u_{x,\xi_k}(x/\varepsilon)$ concentrates near a point of $M$. This completes the proof. □
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