THE UNIVERSAL FUNCTORIAL EQUIVARIANT LEF SCHETZ INVARIANT

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Abstract. We introduce the universal functorial equivariant Lefschetz invariant for endomorphisms of finite proper $G$-CW-complexes, where $G$ is a discrete group. We use $K_0$ of the category of "$\phi$-endomorphisms of finitely generated free $R\Pi(G,X)$-modules". We derive results about fixed points of equivariant endomorphisms of cocompact proper smooth $G$-manifolds.

Introduction

The Lefschetz number is a classical invariant of algebraic topology. Numerous generalizations have been studied, for example the generalized Lefschetz invariant [Rei36, Wec41] and the Lefschetz zeta function [FH94, GN94].

An invariant which maps to all of these generalizations and still has the characteristic properties of the Lefschetz number, namely homotopy invariance and additivity, has been constructed by Lück [Lü99]. He defines a universal functorial Lefschetz invariant $(U,u)$ for endomorphisms of finite CW-complexes. Here, $U$ is a functor which assigns an abelian group $U(X,f)$ to each endomorphism $f: X \rightarrow X$. The function $u$ assigns an element $u(X,f) \in U(X,f)$ to the endomorphism $f$.

The aim of this article is to generalize this construction to the equivariant setting. On the one hand, this is interesting because it gives finer invariants. If there is a $G$-action on a space, this extra information is taken into account in the construction of the invariant. On the other hand, this generalization enlarges the scope of the invariant. If we have an infinite discrete group $G$ acting properly on a finite $G$-CW-complex $X$, the space $X$ seen as a CW-complex is infinite. This situation cannot be treated by the classical theory, but the equivariant version can be applied.

For compact Lie groups $G$, there are equivariant versions of the generalized Lefschetz invariant and related constructions [MP02, Won91]. For discrete groups $G$, equivariant Lefschetz numbers have been defined [LL89, LR03], but none of the generalizations have been studied. We define an equivariant version of the universal invariant, which entails equivariant versions of all intermediate generalizations of the Lefschetz number.

We deal with discrete groups $G$ and finite proper $G$-CW-complexes $X$. To an equivariant endomorphism $f: X \rightarrow X$, we associate an abelian group $U_G(X,f)$ and an element $u_G(X,f) \in U_G(X,f)$. The pair $(U_G,u_G)$ is a functorial equivariant Lefschetz invariant, i.e., it is additive, $G$-homotopy...
invariant and compatible with the induction structure in $G$. We show that it has a universal initial property:

**Theorem 0.1.** The pair $(U_G, u_G)$ is the universal functorial equivariant Lefschetz invariant for equivariant endomorphisms of finite proper $G$-CW-complexes, for discrete groups $G$.

The invariant $(U_G, u_G)$ contains much information, and the abelian group $U_G(X, f)$ can be quite large. We show that $U_G(X, f)$ has a direct sum decomposition into summands corresponding to the fixed point sets $X^H$, for subgroups $H \leq G$. This not only gives structural insight, but is also helpful for concrete calculations. The information contained in $u_G(f)$ splits up into the information given by the restrictions of $f$ to the pairs $(X^H, X^{> H})$, where $X^{> H}$ is the subset of $X^H$ of points with larger isotropy group than $H$.

The decomposition is obtained from a $K$-theoretic splitting theorem, Theorem 4.6. This splitting theorem is valid for all $K$-groups, not only for $K_0$. A more general version for endomorphisms categories of modules over categories can be formulated and applied to the study of $K$-theory of endomorphism categories of modules over group rings.

The study of Lefschetz invariants was motivated by interest in fixed points. As an application of our general constructions, we therefore derive an invariant from $(U_G, u_G)$ extracting information about fixed points, the generalized equivariant Lefschetz invariant $(\Lambda_G, \lambda_G)$. A trace map $\text{tr}_G$ which maps $(U_G, u_G)$ to $(\Lambda_G, \lambda_G)$ is constructed. The use of generalized traces in equivariant fixed point theory already appears in [Mar77], in the context of equivariant $K$-theory.

We prove the refined equivariant Lefschetz fixed point theorem, which shows that the generalized equivariant Lefschetz invariant $\lambda_G(f)$ is equal to the sum of all fixed point contributions. Here, the fixed point data is collected in the generalized local equivariant Lefschetz class $\lambda^\text{loc}_G(f)$.

**Theorem 0.2.** Let $G$ be a discrete group, let $M$ be a cocompact proper smooth $G$-manifold and let $f : M \to M$ be a $G$-equivariant endomorphism such that $\text{Fix}(f) \cap \partial M = \emptyset$ and such that for every $x \in \text{Fix}(f)$ the determinant of the map $(\text{id}_{\mathbb{T}xM} - T_x f)$ is different from zero. Then

$$\lambda_G(f) = \lambda^\text{loc}_G(f).$$

There are further applications to the study of fixed points of equivariant endomorphisms of cocompact proper smooth $G$-manifolds.

Based on the generalized equivariant Lefschetz invariant $\lambda_G(f)$, we can introduce equivariant Nielsen invariants. These give lower bounds for the number of fixed point orbits in the $G$-homotopy class of $f$. Under mild hypotheses, these bounds are sharp. These results generalize results of Wong [Won93] to all discrete groups $G$.

In the same vein, we can define $G$-Jiang spaces and prove a converse to the equivariant Lefschetz fixed point theorem. In particular, under mild hypotheses, a $G$-equivariant endomorphism $f$ is $G$-homotopic to a fixed point free $G$-map if the generalized equivariant Lefschetz invariant $\lambda_G(f)$ is zero.

These applications go beyond the scope of this present article, but they illustrate the interest of the invariants constructed here.
This paper is organized as follows:
1. The Algebraic Approach
2. The Geometric Approach
3. Universality
4. Splitting Results
5. The Generalized Equivariant Lefschetz Invariant
6. The Refined Equivariant Lefschetz Fixed Point Theorem

1. The Algebraic Approach

We start with purely algebraic considerations. Given a commutative ring $R$, we define an abelian group $U(R, \Gamma, \phi)$ for any EI-category $\Gamma$ with endofunctor $\phi$. We proceed to show functoriality of this group in $(\Gamma, \phi)$ and analyze its additive properties.

For us, the most important example of an EI-category is the fundamental category $\Pi(G, X)$ of a topological space $X$ with an action of a discrete group $G$ [L"uc89, Definition 8.15].

**Definition 1.1.** An EI-category is a small category $\Gamma$ such that any endomorphism in $\Gamma$ is an isomorphism. An $R\Gamma$-module is a contravariant functor $M: \Gamma \to R\text{-Mod}$. Given an endofunctor $\phi: \Gamma \to \Gamma$, we define a $\phi$-endomorphism of an $R\Gamma$-module $M$ to be a natural transformation $g: M \to M \circ \phi$.

We assemble all $\phi$-endomorphisms of finite free $R\Gamma$-modules in the category $\phi\text{-end}_{\text{fHRT}}$. Morphisms from $g: M \to M \circ \phi$ to $h: N \to N \circ \phi$ are natural transformations $\tau: M \to N$ such that $h \tau = (\phi^* \tau) g$. This is an exact category whose isomorphism classes of objects form a set.

**Definition 1.2.** We define $U(R, \Gamma, \phi) := K_0(\phi\text{-end}_{\text{fHRT}})$.

Let $H: \Gamma_1 \to \Gamma_2$ be a covariant functor of EI-categories with endofunctors $\phi$ and $\psi$ such that $\psi H = H \phi$. We extend the covariant functor “induction with $H$” [L"uc89, Definition 9.15], defined on $R\Gamma_1$-modules, to $\phi$-endomorphisms.

Induction $\text{ind}_H$ and restriction $\text{res}_H$ are adjoint functors [L"uc89, 9.22], so there are natural transformations $\eta: \text{Id} \to \text{res}_H \text{ind}_H$ and $\varepsilon: \text{ind}_H \text{res}_H \to \text{Id}$. We define $H_*: \phi\text{-end}_{\text{fHRT}_1} \to \psi\text{-end}_{\text{fHRT}_2}$ on objects of $\phi\text{-end}_{\text{fHRT}_1}$, to be the composition $\varepsilon_{\text{res}_H \text{ind}_H} \circ \text{ind}_H \text{res}_H (\eta) \circ \text{ind}_H$. By the triangular identities [Mac71, Theorem IV.1.1] we have $H_*(g) = \text{ind}_H(g)$ if $\phi = \text{Id}$ and $\psi = \text{Id}$. We define $H_*$ on morphisms $\tau$ by setting $H_*(\tau) := \text{ind}_H(\tau)$. The functor $H_*$ is exact and thus induces a group homomorphism $U(R, H) := K_0(H_*): U(R, \Gamma_1, \phi) \to U(R, \Gamma_2, \psi)$.

If we have another covariant functor $K: \Gamma_2 \to \Gamma_3$ to a category $\Gamma_3$ with endofunctor $\vartheta$ such that $\vartheta K = K \psi$, we observe that $H_* G_* = (HG)_*$. We know even more.

**Proposition 1.3.** If $H: \Gamma_1 \rightleftarrows \Gamma_2$ is an equivalence of categories with endofunctors $\phi$ and $\psi$ such that $\psi H = H \phi$, then the natural transformation $H_*: \phi\text{-end}_{\text{fHRT}_1} \to \psi\text{-end}_{\text{fHRT}_2}$ is also an equivalence of categories.

**Proof.** We use $\text{res}_H$ to define a functor $H^*: \psi\text{-end}_{\text{fHRT}_2} \to \phi\text{-end}_{\text{fHRT}_1}$, by setting $H^*(g) := \text{res}_H g$ on objects and $H^*(\tau) := \text{res}_H \tau$ on morphisms.
This is well-defined since $H$ is an equivalence of categories. We now use the triangular identities to show that the adjunction morphisms between $\text{ind}_H$ and $\text{res}_H$ induce natural equivalences $\eta: \text{Id} \to H^*H_*$ and $\varepsilon: H_*H^* \to \text{Id}$.

We obtain a universal additive invariant for the objects of $\phi\text{-end}_{\Gamma R}$ by taking their representatives in $U(R, \Gamma, \phi)$.

**Definition 1.4.** An additive invariant for the category $\phi\text{-end}_{\Gamma R}$ of $\phi$-endomorphisms of finite free $R\Gamma$-chain complexes is a pair $(A, a)$, where $A$ is an abelian group and $a: \text{Ob}(\phi\text{-end}_{\Gamma R}) \to A$, $(g: C \to C \circ \phi) \mapsto a(g)$ is a function, satisfying the following properties:

1. **Additivity**
   For a short exact sequence $0 \to f \to g \to h \to 0$, given by a commutative diagram with exact rows
   \[
   \begin{array}{cccccc}
   0 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & 0 \\
   f & \downarrow & & \downarrow g & & \downarrow h & \\
   0 & \longrightarrow & C \circ \phi & \longrightarrow & D \circ \phi & \longrightarrow & E \circ \phi & \longrightarrow & 0
   \end{array}
   \]
   where $C$, $D$ and $E$ are finite free $R\Gamma$-chain complexes, we have
   \[a(f) - a(g) + a(h) = 0.\]

2. **Homotopy invariance**
   Let $f, g: C \to C \circ \phi$ be $R\Gamma$-chain maps of finite free $R\Gamma$-chain complexes. If $f$ and $g$ are $R\Gamma$-chain-homotopic (i.e., there is an $h: C \to C \circ \phi$ of degree +1 such that $dh + hd = f - g$), then $a(f) = a(g)$.

3. **Elementary chain complexes**
   For a finite free $R\Gamma$-module $F$ and $n \geq 1$ we have
   \[a(0: \text{el}(F, n) \to \text{el}(F, n) \circ \phi) = 0,\]
   where $\text{el}(F, n)$ denotes the elementary chain complex concentrated in dimensions $n$ and $n - 1$ and having $n$-th differential $\text{Id}: F \to F$.

**Definition 1.5.** An additive invariant $(U, u)$ for $\phi\text{-end}_{\Gamma R}$ is called universal if and only if for all additive invariants $(A, a)$ for the category $\phi\text{-end}_{\Gamma R}$ there is exactly one group homomorphism $\xi: U \to A$ such that $\xi(u(g)) = a(g)$ for all $g \in \phi\text{-end}_{\Gamma R}$.

A finite free chain complex of $\phi$-endomorphisms maps into $U(R, \Gamma, \phi)$ by $u: \{\cdots \to g_n \to g_{n-1} \to \cdots\} \mapsto \sum_{n \in \mathbb{Z}} (-1)^n[g_n] \in U(R, \Gamma, \phi)$.

**Theorem 1.6.** The pair $(U(R, \Gamma, \phi), u)$ is the universal additive invariant for $\phi\text{-end}_{\Gamma R}$.

**Proof.** Properties 1 and 3 are clear. In the proof of property 2, we proceed analogously to Lück [Lüc99, Theorem 1.4], using the mapping cone and the suspension of $C$. If $h: C \to C \circ \phi$ is an $R\Gamma$-chain homotopy from $f$ to $g$, we obtain an exact sequence $0 \to f \to k \to \Sigma g \to 0$, where $k: \text{cone}(C) \to \text{cone}(C)$ is given by $k_n := \left( \begin{array}{cc} g_{n-1} & 0 \\ h_{n-1} & f_n \end{array} \right)$. Additivity yields $u(f) - u(g) = u(f) + u(\Sigma g) = u(k)$. The $R\Gamma$-chain complex $\text{cone}(C)$ is contractible. We
finally show that for contractible $C$ and for any $f: C \to C \circ \phi$ we have $u(f: C \to C \circ \phi) = 0 \in U(R, \Gamma, \phi)$.

From the above proof, we can also conclude the following lemma.

**Lemma 1.7.** Let $f: C \to C \circ \phi$ and $g: D \to D \circ \phi$ be $\phi$-endomorphisms of finite free $R\Gamma$-chain complexes. If $h: C \to D$ is a chain homotopy equivalence such that $gh = (\phi^*h)f$, then $u(f) = u(g)$ in $U(R, \Gamma, \phi)$.

**Proof.** We have a short exact sequence $0 \to g \to k \to \Sigma f \to 0$, where $k_n = \left(\begin{array}{cc} -f_{n-1} & 0 \\ h_{n-1} & g_n \end{array}\right)$. We know that cone$(h)$ is contractible if and only if $h$ is a chain homotopy equivalence. So $u(k) = 0 \in U(R, \Gamma, \phi)$, whence $u(g) = u(f)$. \hfill \rlap{$\Box$}

If $R$ is a commutative ring which is not the zero ring, a finite free $R$-module has a well-defined rank. One sees that the invariant $(U(R, \Gamma, \phi), u)$ incorporates the Euler characteristic. For a finite free $R\Gamma$-chain complex $C$, the Euler characteristic is defined to be $\xi(C) := \text{rk}_R([D]) = \sum_{n \in \mathbb{Z}} (-1)^n \text{rk}([C_n]) \in U(\Gamma) := \bigoplus_{k \in \Gamma} \mathbb{Z}$ [Lüc89, Chapter 10]. There is a natural transformation $s: U(\Gamma) \to U(R, \Gamma, \phi)$ mapping an element of $U(\Gamma)$ to the class of the 0-map on the corresponding free module. We have $s(\xi(C)) = u(0: C \to C \circ \phi)$.

**Lemma 1.8.** Let $v: C \to D$ and $f: C \to C \circ \phi$ be $R\Gamma$-chain maps, where $C$ and $D$ are finite free $R\Gamma$-chain complexes. Then

$$u(f \circ v) + s(\xi(D)) = u(\phi^*v \circ f) + s(\xi(C)).$$

**Proof.** We have $\left(\begin{array}{cc} 1 & \phi^*v \\ 0 & 1 \end{array}\right) \left(\begin{array}{cc} 0 & 0 \\ f & f \circ v \end{array}\right) = \left(\begin{array}{cc} \phi^*v \circ f & 0 \\ f & f \circ v \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)$. Since the matrices with 1 on the diagonal are isomorphisms, we obtain from additivity $u\left(\begin{array}{cc} 0 & 0 \\ f & f \circ v \end{array}\right) = u\left(\begin{array}{cc} \phi^*v \circ f & 0 \\ f & f \circ v \end{array}\right)$. Again from additivity we derive

$$u(f \circ v) + u(0: D \to \phi^*D) = u\left(\begin{array}{cc} 0 & 0 \\ f & f \circ v \end{array}\right)$$

and

$$u(\phi^*v \circ f) + u(0: C \to \phi^*C) = u\left(\begin{array}{cc} \phi^*v \circ f & 0 \\ f & f \circ v \end{array}\right).$$ \hfill \rlap{$\Box$}

2. The Geometric Approach

Let $G$ be a discrete group. Let $G\text{-CW}_{fp}$ be the category of finite proper $G$-CW-complexes. Let $\text{End}(G\text{-CW}_{fp})$ be the category of $G$-equivariant endomorphisms of finite proper $G$-CW-complexes.

Given a commutative ring $R$, a finite proper $G$-CW-complex $X$ and a $G$-equivariant cellular endomorphism $f: X \to X$, we want to introduce an abelian group $U^G_{\text{eq}}(X, f)$ and an invariant $u^G_{\text{eq}}(X, f) \in U^G_{\text{eq}}(X, f)$. (The assumption that $f: X \to X$ is cellular can finally be dropped because of homotopy invariance.)

In the non-equivariant case [Lüc99], the construction uses the universal covering space $\tilde{X}$ of the CW-complex $X$. Choosing a basepoint $x$, the fundamental group $\pi_1(X, x)$ acts on $\tilde{X}$. One can lift $f: X \to X$ to a $\phi$-equivariant map $\tilde{f}: \tilde{X} \to \tilde{X}$, where $\phi := c_{x_0} \circ \pi_1(f, x): \pi_1(X, x) \to \pi_1(X, x)$ for a chosen path $w$ from $f(x)$ to $x$. We generalize this non-equivariant construction to $G$-CW-complexes.
The natural generalization of the fundamental group is the fundamental category \( \Pi(G, X) \) [Lüc89, Definition 8.15]. Objects in the fundamental category \( \Pi(G, X) \) are \( G \)-maps \( x: G/H \to X \) for \( H \leq G \), morphisms are pairs \( (\sigma, [w]): x(H) \to y(K) \) consisting of a \( G \)-map \( \sigma: G/H \to G/K \) and a homotopy class \( [w] \) relative \( G/H \times \partial I \) of \( G \)-maps \( w: G/H \times I \to X \) with \( w_1 = x \) and \( w_0 = y \circ \sigma \). This construction is functorial in \( X \) (by composition of maps), in particular the \( G \)-equivariant endomorphism \( f: X \to X \) induces an endofunctor \( \phi := \Pi(G, f): \Pi(G, X) \to \Pi(G, X) \).

There is the contravariant universal covering functor \( \tilde{X}: \Pi(G, X) \to \mathcal{CW} \) at hand [Lüc89, Definition 8.22, Proposition 8.33], generalizing the universal covering space. It maps \( x \in \Pi(G, X) \) to \( \tilde{X}^H(x) \) and \( (\sigma, [w]) \in \text{Mor}(x(H), y(K)) \) to the map \( X^K(y) \to X^H(x) \) induced by composition of morphisms. Here \( X^H(x) \) denotes the connected component of the fixed point set \( X^K \) containing the point \( x(1H) \), for an object \( x: G/H \to X \) of \( \Pi(G, X) \). Composing the universal covering functor \( \tilde{X} \) with the cellular chain complex functor \( C^c: \mathcal{CW} \to \text{R-CH} \) one obtains the cellular \( R\Pi(G, X) \)-chain complex \( C^c(\tilde{X}) \) as a contravariant functor \( C^c \circ \tilde{X} : \Pi(G, X) \to \mathcal{CW} \xrightarrow{C^c} \text{R-CH} \) [Lüc89, Definition 8.37]. The functor \( C^c \circ \tilde{X} \) is a finite free \( R\Pi(G, X) \)-chain complex [Lüc89, 9.18]. The map \( f: X \to X \) induces a \( \phi \)-endomorphism \( C^c(\tilde{f}) \) of the cellular \( R\Pi(G, X) \)-chain complex \( C^c \circ \tilde{X} \).

**Definition 2.1.** We set \( U^R_G(X, f) := U(R, \Pi(G, X), \phi) \). We define the element \( u^R_G(X, f) \in U^R_G(X, f) \) to be \( u(C^c(\tilde{f})) \in U(R, \Pi(G, X), \phi) = U^R_G(X, f) \).

Let \((X, f)\) and \((Y, g)\) be finite proper \( G \)-CW-complexes with \( G \)-equivariant cellular endomorphisms \( f \) and \( g \). Let \( h: X \to Y \) be a \( G \)-equivariant cellular map such that \( gh = hf \). Composition with \( h \) induces a functor \( H := \Pi(G, h): \Pi(G, X) \to \Pi(G, Y) \) between the fundamental categories. Setting \( \phi := \Pi(G, f) \) and \( \psi := \Pi(G, g) \), we have \( \psi H = H \phi \). We obtain \( U^R_G(h) := K_0(H_\ast) : U^R_G(X, f) \to U^R_G(Y, g) \), a group homomorphism. So \( U^R_G \) is a functor from \( \text{End}(G-CW_{fp}) \) to \( \mathbb{Ab} \).

**Lemma 2.2.** If \( h: (X, f) \to (Y, g) \) is a \( G \)-equivariant cellular map between finite proper \( G \)-CW-complexes with endomorphisms such that \( gh = hf \), then \( h \) induces a map \( C^c(\tilde{h}) \) from \( \Pi(G, h)_\ast C^c(\tilde{f}) \) to \( C^c(\tilde{g}) \).

**Idea of proof.** We use adjointness of induction and restriction and the triangular identities. \( \square \)

Let \( X \) be an \( H \)-CW-complex with \( H \)-equivariant endomorphism \( f: X \to X \) and let \( \alpha: H \to G \) be a group homomorphism. Then \( \text{ind}_\alpha X := G \times_H X \) is a \( G \)-CW-complex. It is proper if \( X \) is proper. The map \( \text{ind}_\alpha : X = H \times_H X \xrightarrow{\alpha \times \text{id}} G \times_H X = \text{ind}_\alpha X \) induces a map \( \Pi(\text{ind}_\alpha) \) of the fundamental categories. We obtain a group homomorphism \( \alpha_\ast := K_0(\Pi(\text{ind}_\alpha)_\ast) : U^R_H(X, f) \to U^R_G(\text{ind}_\alpha X, \text{ind}_\alpha f) \).

We defined the invariant \( (U^R_H(X, f), U^R_G(X, f)) \) because it has certain good properties. It is a functorial equivariant Lefschetz invariant, the equivariant generalization of a functorial Lefschetz invariant [Lüc99, Definition 2.3].
Definition 2.3. A functorial equivariant Lefschetz invariant on the family of categories $G$-CW$_{fp}$ of finite proper $G$-CW-complexes for discrete groups $G$ is a pair $(\Theta, \theta)$ that consists of

- A family $\Theta$ of functors $\Theta \colon \text{End}(G\text{-CW}_{fp}) \to \mathcal{A}b$

which is compatible with the induction structure, i.e., for an inclusion $\alpha \colon G \to K$ there is a group homomorphism $\Theta(\alpha) : \Theta_G(X, f) \to \Theta_K(\text{ind}_\alpha X, \text{ind}_\alpha f)$ for every $(X, f) \in \text{End}(G\text{-CW}_{fp})$. We want the equation $\Theta(\alpha) \Theta_G(h) = \Theta_K(\text{ind}_\alpha h) \Theta(\alpha)$ to hold for any morphism $h : (X, f) \to (Y, g)$.

- A family $\theta$ of functions $\theta_G : (X, f) \mapsto \theta_G(X, f) \in \Theta_G(X, f)$.

such that the following holds:

1. **Additivity**
   For a $G$-pushout with $i_2$ a $G$-cofibration
   \[
   (X_0, f_0) \xrightarrow{i_1} (X_1, f_1) \xrightarrow{j_0} (X_2, f_2) \xrightarrow{j_1} (X, f)
   \]
   we obtain in $\Theta_G(X, f)$ that
   \[
   \theta_G(X, f) = \Theta_G(j_1) \theta_G(X_1, f_1) + \Theta_G(j_2) \theta_G(X_2, f_2) - \Theta_G(j_0) \theta_G(X_0, f_0).
   \]

2. **$G$-Homotopy invariance**
   If $h_0, h_1 : (X, f) \to (Y, g)$ are two $G$-maps that are $G$-homotopic in $\text{End}(G\text{-CW}_{fp})$, then
   \[
   \Theta_G(h_0) = \Theta_G(h_1) : \Theta_G(X, f) \to \Theta_G(Y, g).
   \]

3. **Invariance under $G$-homotopy equivalence**
   If $h : (X, f) \to (Y, g)$ is a morphism in $\text{End}(G\text{-CW}_{fp})$ such that $h : X \to Y$ is a $G$-homotopy equivalence, then
   \[
   \Theta_G(h) : \Theta_G(X, f) \xrightarrow{\cong} \Theta_G(Y, g)
   \]
   \[
   \theta_G(X, f) \mapsto \theta_G(Y, g).
   \]

4. **Normalization**: We have $\theta_G(\emptyset, \text{id}_\emptyset) = 0 \in \Theta_G(\emptyset, \text{id}_\emptyset)$.

5. **If $\alpha : G \to K$ is an inclusion of groups, then**
   \[
   \alpha_\ast \theta_G(X, f) = \theta_K(\text{ind}_\alpha X, \text{ind}_\alpha f) \in \Theta_K(\text{ind}_\alpha X, \text{ind}_\alpha f).
   \]

A natural transformation $\tau : (\Theta, \theta) \to (\Xi, \xi)$ of functorial equivariant Lefschetz invariants is a family of natural transformations $\tau_G : \Theta_G \to \Xi_G$ of functors from $\text{End}(G\text{-CW}_{fp})$ to $\mathcal{A}b$ for discrete groups $G$ that preserves all structure.

Proposition 2.4. The invariant $(U_G^R(X, f), u_G^R(X, f))$ is a functorial equivariant Lefschetz invariant on the family of categories $G$-CW$_{fp}$ of finite proper $G$-CW-complexes for discrete groups $G$. 
Proof. 1. Additivity: From [Lüe89, Lemma 13.7] one knows that for the corresponding $G$-pushout of $G$-CW-complexes we obtain a based exact sequence of $R I I (G, X)$-chain complexes

$$0 \to j_0 \ast C^c(\widetilde{X}_0) \xrightarrow{\beta_i} j_1 \ast C^c(\widetilde{X}_1) \oplus j_2 \ast C^c(\widetilde{X}_2)$$

$$C^c(\widetilde{j}_1) \to C^c(\widetilde{f}) \to 0,$$

where $j_0$, $j_1$, and $j_2$ denote induction with $j_0$, $j_1$ and $j_2$ respectively. By Lemma 2.2 we obtain a short exact sequence $0 \to j_0 \ast C^c(\widetilde{f}_0) \to j_1 \ast C^c(\widetilde{f}_1) \oplus j_2 \ast C^c(\widetilde{f}_2) \to C^c(\widetilde{f}) \to 0$. We conclude using the additive properties of the algebraic invariant.

2. $G$-Homotopy invariance: Let $H: X \times I \to Y$ be a $G$-equivariant homotopy between $h_0$ and $h_1$. We want to show that $U^R_G(h_0) = U^R_G(h_1)$. We know that $\Pi(G, h_0) = \Pi(G, H) \circ \Pi(G, i_0)$ and that $\Pi(G, h_1) = \Pi(G, H) \circ \Pi(G, i_1)$, so it suffices to prove that $K_0(\Pi(G, i_0)_\ast(g)) = K_0(\Pi(G, i_1)_\ast(g))$ for all $g \in \phi$-end, where $\Pi(G, i_0)_\ast$ is an equivalence of categories. We can now apply the cellularity property to conclude using the additive properties of the algebraic invariant.

3. Invariance under $G$-homotopy equivalence: If $h : (X, f) \to (Y, g) \in \text{End}(G-CW)$ such that $h : X \to Y$ is a $G$-homotopy equivalence, then $\Pi(G, h)_\ast$ is a bijection on $\phi$-end. By Proposition 1.3 the induced functor $\Pi(G, h)_\ast : \phi$-end $\to \psi$-end is an equivalence of categories. We need to show that $u_G(X, f)$ maps to $u_G(Y, g)$ under this map.

By Lemma 2.2, $h$ induces a map $C^c(\widetilde{h})$ from $\Pi(G, h)_\ast C^c(f)$ to $C^c(\widetilde{g})$. This map is an $R I I (G, X)$-chain homotopy equivalence. By Lemma 1.7 we conclude $u_G(h)_\ast C^c(\widetilde{f}) = u_G(\widetilde{c}(\widetilde{g}))$, and the claim follows.

4. Normalization: We have $\text{id}$-end $\to \text{pt}$ and $K_0(\text{pt}) = \{0\}$.

5. Let $\alpha: G \to K$ be an inclusion of groups. We want to show that $\alpha_* u_G(X, f) = u_K(\text{ind}_\alpha X, \text{ind}_\alpha f)$. By definition we have $\alpha_* u_G(X, f) = u(\Pi(\text{ind}_\alpha)_\ast C^c(\widetilde{f}))$ and $u_K(\text{ind}_\alpha X, \text{ind}_\alpha f) = u(C^c(\text{ind}_\alpha f))$. There is a natural equivalence $T : \text{ind}_\Pi(\text{ind}_\alpha)_\ast \widetilde{X} \to \text{ind}_\alpha X$. We check that $\text{ind}_\alpha fT = (\psi T)\Pi(\text{ind}_\alpha)_\ast \widetilde{f}$. We can now apply the cellular chain complex functor $C^c$ to obtain a natural equivalence which maps $\Pi(\text{ind}_\alpha)_\ast C^c(\widetilde{f})$ to $C^c(\text{ind}_\alpha f)$ and induces the desired equality.

3. Universality

Now we show that the invariant $(U^R_G, u_G)$ has a universal initial property among all functorial equivariant Lefschetz invariants.

**Definition 3.1.** A functorial equivariant Lefschetz invariant $(U_G, u_G)$ is called *universal* if for any functorial equivariant Lefschetz invariant $(\Theta_G, \theta_G)$ there is precisely one family of natural transformations $\tau_G : U_G \to \Theta_G$ such that $\tau_G(\mathcal{X}, f) : U_G(\mathcal{X}, f) \to \Theta_G(\mathcal{X}, f)$ sends $u_G(\mathcal{X}, f)$ to $\theta_G(\mathcal{X}, f)$ for any...
object \((X, f)\) in \(\text{End}(G\text{-CW}_{tp})\), for any discrete group \(G\), and such that the equality \(\tau_K \circ U(\alpha) = \Theta(\alpha) \circ \tau_G\) holds for inclusions \(\alpha: G \to K\).

The goal of this section is the proof of Theorem 0.1.

**Theorem 0.1.** The pair \((\tilde{U}_G^c, \tilde{v}_G^c)\) is the universal functorial equivariant Lefschetz invariant on the family of categories \(\text{End}(G\text{-CW}_{tp})\) for discrete groups \(G\).

The proof is in analogy to Lück [Lüc99, Section 4], of which it is the equivariant generalization. Before starting, we introduce notation.

Let \(X\) be a \(G\)-space. A **retractive \(G\)-space** \(Y\) over \(X\) is a triple \(Y = (Y, i, r)\) which consists of a \(G\)-space \(Y\), a \(G\)-cofibration \(i: X \to Y\) and a \(G\)-map \(r: Y \to X\) satisfying \(r \circ i = \text{id}_X\). Given a retractive \(G\)-space \(Y\) over \(X\), we define retractive \(G\)-spaces \(Y \times_X [0, 1]\) and \(C_X Y\) to be the pushouts

\[
\begin{align*}
X \times [0, 1] & \xrightarrow{\text{pr}_X} X \\
Y \times [0, 1] & \xrightarrow{\text{incl}} Y \times_X [0, 1]
\end{align*}
\]

where \([0, 1]\) is endowed with the trivial \(G\)-action. The inclusions of \(X\) into \(Y \times_X [0, 1]\) and into \(C_X Y\) are the right vertical maps, the retractions \(Y \times_X [0, 1] \to X\) and \(\tilde{r}: C_X Y \to X\) are induced by the retraction \(r: Y \to X\) by the pushout property.

Define the retractive \(G\)-space \(\Sigma_X Y\) to be the pushout of two copies of the inclusion \(\tilde{i}: Y \to C_X Y\) induced by \(Y \times \{0\} \to Y \times [0, 1]\). The retraction is induced by \(\tilde{r}\) by the pushout property. The composition \(\tilde{i} \circ i: X \to C_X Y\) is a \(G\)-homotopy equivalence relative \(X\) with the retraction of \(C_X Y\) onto \(X\) as \(G\)-homotopy inverse relative \(X\).

Given two retractive spaces \(Y\) and \(Z\) over \(X\) and a \(G\)-endomorphism \(f: X \to X\), define \([\langle C_X Y, Y \rangle, \langle C_X Z, Z \rangle]_f^{\tilde{G}}\) to be the set of \(G\)-homotopy classes relative \(X\) of maps of pairs \((\tilde{g}, g): \langle C_X Y, Y \rangle \to \langle C_X Z, Z \rangle\) that induce the given endomorphism \(f\) on \(X\), i.e., such that \(g i_Y = i_Z f\).

We define the \(\text{ZII}(G, X)\)-chain complex \(C^c(\tilde{Y}, \tilde{X})\) by

\[
C^c(\tilde{Y}, \tilde{X}) := \text{coker}(C^c(\tilde{X}) \xrightarrow{C^c(\tilde{i})} \text{res}_{\text{H}(G,i_Y)} C^c(\tilde{Y})).
\]

We call a retractive \(G\)-space \(Y\) over \(X\) a \(d\)-extension if \(Y\) is obtained from \(X\) by attaching finitely many cells in dimension \(d\). If \(Y\) is a \(d\)-extension of \(X\) and \(d \geq 2\), then we have the short exact sequence

\[
0 \to C^c(\tilde{X}) \xrightarrow{C^c(\tilde{i})} \text{res}_{\text{H}(G,i_Y)} C^c(\tilde{Y}) \to C^c(\tilde{Y}, \tilde{X}) \to 0,
\]

where \(C^c(\tilde{X})\) and \(\text{res}_{\text{H}(G,i_Y)} C^c(\tilde{Y})\) are finite free \(\text{ZII}(G, X)\)-chain complexes equipped with the cellular equivalence class of bases [Lüc89, Definition 13.3 and Example 9.18]. Since the above sequence is based split exact, \(C^c(\tilde{Y}, \tilde{X})\) is a chain complex concentrated in degree \(d\), and \(C^c_d(\tilde{Y}, \tilde{X})\) is a finite free \(\text{ZII}(G, X)\)-module with an equivalence class of bases given by the cells of \(Y \setminus X\) [Lüc89, Example 9.18].
If $Y$ is a $d$-extension of $X$ and $d \geq 2$, then $\Pi(G, i_Y) : \Pi(G, X) \to \Pi(G, Y)$ is an equivalence of categories with inverse $\Pi(G, r_Y)$. This implies that $\mathrm{res}_{\Pi(G, i_Y)} C^c(\bar{Y})$ is a finite free $\mathbb{Z}\Pi(G, X)$-module.

For any map $g : Y \to Z$ of $d$-extensions of $X$ ($d \geq 2$) such that $g i_Y = i_Z f$ we obtain a lift $\tilde{g} : \mathrm{res}_{\Pi(G, i_Y)} \bar{Y} \to \mathrm{res}_{\Pi(G, i_Z)} \bar{Z}$ uniquely determined by the fact that it induces $\tilde{f}$ on $X$. We have $C^c(\tilde{g}) C^c(\bar{Y}) = (\phi C^c(\bar{f})) C^c(\bar{f})$ and therefore an induced map $C^c(\tilde{g}, \tilde{f}) : C^c(\bar{Y}, \bar{X}) \to C^c(\bar{Z}, \bar{X}) \circ \phi$ of $\Pi(G, X)$-chain complexes. This leads to a bridge between geometry and algebra, in analogy to Lück [Lüc99, Lemma 4.2].

**Lemma 3.2.** Let $Y$ and $Z$ be $d$-extensions of the $G$-space $X$ with $d \geq 2$. Then there is a bijective map

$$
\eta : [(C_X Y, Y), (C_X Z, Z)]_f^G \to \mathrm{Mor}_{\Pi(G,X)}(C^c_d(\bar{Y}, \bar{X}), C^c_d(\bar{Z}, \bar{X}) \circ \phi)
$$

$$
[(\tilde{g}, g)] \mapsto C^c_d(\tilde{g}, \tilde{f}).
$$

**Proof.** Choose a $G$-pushout

$$
\begin{array}{ccc}
\prod_{i \in I} G/H_i \times S^{d-1} & \xrightarrow{\prod_{i \in I} q_i} & X \\
\downarrow & & \downarrow \ \ \ \ \ \ i_Y \\
\prod_{i \in I} G/H_i \times D^d & \xrightarrow{\prod_{i \in I} Q_i} & Y.
\end{array}
$$

Define a $G$-map $p_i : G/H_i \times S^d \to Y$ by setting $p_i|_{G/H_i \times S^d_s} := Q_i$ and $p_i|_{G/H_i \times S^d_s} := r \circ Q_i$. Then we can see $C_X Y$ as the pushout

$$
\begin{array}{ccc}
\prod_{i \in I} G/H_i \times S^d & \xrightarrow{\prod_{i \in I} p_i} & Y \\
\downarrow & & \downarrow \ \ \ \ \ \ i_Y \\
\prod_{i \in I} G/H_i \times D^{d+1} & \xrightarrow{\prod_{i \in I} P_i} & C_X Y.
\end{array}
$$

Analogously to Lück [Lüc99, Lemma 4.2], we obtain an isomorphism

$$
\mu : [(C_X Y, Y), (C_X Z, Z)]_f^G \xrightarrow{\sim} \prod_{i \in I} \pi_d(Z^{H_i}(f(x_i)), X^{H_i}(f(x_i)), f(x_i))
$$

$$
\xrightarrow{\sim} \prod_{i \in I} \left( C^c_d(\bar{Z}, \bar{X}) \circ \phi \right)(x_i)
$$

$$
\xrightarrow{\sim} \mathrm{Mor}_{\Pi(G,X)}(C^c_d(\bar{Y}, \bar{X}), C^c_d(\bar{Z}, \bar{X}) \circ \phi).
$$

Under this isomorphism,

$$
[(\tilde{g}, g)] \mapsto \left( (g \circ Q_i(1H_i, -), f \circ q_i(1H_i, -)) \right)_{i \in I}
$$

$$
\mapsto \left( (\tilde{g}, \tilde{f}) \left( Q_i(1H_i, [D^d]), \bar{q}_i(1H_i, [S^d]) \right) \right)_{i \in I} = (C^c_d(\tilde{g}, \tilde{f}) (x_i))_{i \in I}
$$

$$
\mapsto C^c_d(\tilde{g}, \tilde{f}).
$$

**Proof of Theorem 0.1.** We have already shown that the pair $(U_G^Z, U_G^Z)$ is a functorial equivariant Lefschetz invariant. It remains to prove universality. For every functorial equivariant Lefschetz invariant $(\Theta_G, \theta_G)$ we need to find a natural transformation $\tau_G : U_G^Z \to \Theta_G$ such that $\theta_G(X, f) = U_G^Z(X, f)$ maps to $\tau_G(X, f)$. This is achieved by constructing $\tau_G$ as the composition of $\mu$ with the inverse of the isomorphism $\eta$. The details of this construction are similar to those in Lück [Lüc99, Lemma 4.2].
\[\theta_G(X, f)\] for all \((X, f)\) in \(\text{End}(G\text{-CW}_\text{ip})\), for discrete groups \(G\), and such that \(\tau_K \circ U(\alpha) = \Theta(\alpha) \circ \tau_G\) for inclusions \(\alpha: G \to K\).

We define \(\tau\) by translating the algebraic data encoded by \(U_G^\mathbb{Z}\) back into geometric information using Lemma 3.2. We proceed in eight steps. We omit details which are completely analogous to [L" uck99, Section 4].

**Step 1:** For any \(d\)-extension \(Y\) of \(X\), with \(d \geq 1\), we define

\[
\tau_Y: \left[\left(\left(\left(C_X Y, Y\right), \left(C_X Y, Y\right)\right)\right)\right] \to \left[\left(\left(\left(C_X \Sigma X Y, \Sigma X Y\right), \left(C_X \Sigma X Z, \Sigma X Z\right)\right)\right)\right] \\
\left(\tilde{g}, \tilde{\gamma}\right) \mapsto \left(\left(\tilde{g} \cup \eta_{\tilde{\gamma}}\right), \hat{\eta}_{\tilde{\gamma}}\right).
\]

The maps \(g\) and \(\tilde{g}\) are only defined up to \(G\)-homotopy (relative \(X\)), but because of the \(G\)-homotopy invariance of \((\Theta_G, \theta_G)\) this does not play a role.

**Step 2:** We define a map \(\tau_d: U_G^\mathbb{Z}(X, f) \to \Theta_G(X, f)\) for \(d \geq 2\). Let \([a: M \to M \circ \phi] \in U_M^\mathbb{Z}(X, f)\). We have \(M \equiv \bigoplus_{i \in I} \text{ZII}(G, X)(?, x_i)\) with \((x_i: G/H_i \to X) \in \text{ObII}(G, X)\) [L" uck89, p. 167] since \(M\) is a finite free \(\text{ZII}(G, X)\)-module. Set \(q_i := x_i \circ \rho_{G/H_i}: G/H_i \times S^{d-1} \to X\) and define \(Y\) to be the pushout defined by the maps \(q_i\), as in diagram 1.

Take an isomorphism \(c: M \to \bigoplus_{i \in I} \text{ZII}(G, X)(?, x_i)\). Setting \([\tilde{g}, \tilde{\gamma}] := \eta^{-1}(\phi^* cac^{-1})\) we have \([a] = [\phi^* cac^{-1}] = [\eta(\tilde{g}, \tilde{\gamma})] = [C_d^c(\tilde{g}, \tilde{\gamma})]\). We define \(\tau_d([a]) := (-1)^d\left(\tau_Y([\tilde{g}, \tilde{\gamma}]) - \theta_G(f)\right)\) and use \(G\)-homotopy invariance to show that this is independent of the choices made.

**Step 3:** We check that \(\tau_d\) is compatible with the additivity relation using the union of \(G\)-spaces over \(X\): For a split short exact sequence \(0 \to a_1 \to a \to a_2 \to 0\) we have \([a_0] + [a_1] = [a]\), and putting \(Y := Y_0 \cup_X Y_1\) we can calculate that \(\tau_d([a_0]) + \tau_d([a_1]) = \tau_d([a])\).

**Step 4:** The map \(\tau_d\) is independent of the integer \(d \geq 2\). Namely, let \([a] \in U_G^\mathbb{Z}(X, f)\) and let \((Y, g)\) be a \(d\)-extension of \(X\) such that \([a] = [C_d^c(\tilde{g}, \tilde{\gamma})]\). In order to show that \(\tau_d([a]) = \tau_{d+1}([a])\), we describe a suspension map

\[
\Sigma_X: \left[\left(\left(\left(C_X Y, Y\right), \left(C_X Z, Z\right)\right)\right)\right] \to \left[\left(\left(\left(C_X \Sigma X Y, \Sigma X Y\right), \left(C_X \Sigma X Z, \Sigma X Z\right)\right)\right)\right] \\
\left(\tilde{g}, \tilde{\gamma}\right) \mapsto \left(\left(\tilde{g} \cup \eta_{\tilde{\gamma}}, \hat{\eta}\right), \hat{\eta}_{\tilde{\gamma}}\right).
\]

in analogy to L" uck [L" uck99, Section 4], with \(\Sigma X g := \tilde{g} \cup g \tilde{g}\). We have

\[
\tau_d([a]) = (-1)^d\left(\tau_Y([\tilde{g}, \tilde{\gamma}]) - \theta_G(f)\right)
\]

and

\[
\tau_{d+1}([a]) = (-1)^{d+1}\left(\tau_{\Sigma X Y}([\Sigma X g, \Sigma X g]) - \theta_G(f)\right).
\]

We obtain the desired equality by calculating that

\[
\tau_{\Sigma X Y}([\Sigma X g, \Sigma X g]) - \theta_G(f) = -\left(\tau_Y([\tilde{g}, \tilde{\gamma}]) - \theta_G(f)\right).
\]

**Step 5:** We show that \(\tau\) is a natural transformation from \(U_G^\mathbb{Z}\) to \(\Theta_G\). Let \(h: X_1 \to X_2\) be a \(G\)-equivariant map between spaces with endomorphisms \(f_1\) and \(f_2\) respectively such that \(f_2 h = h f_1\). Let \([a] \in U_G^\mathbb{Z}(X_1, f_1)\), and let \(Y_1\) be a \(d\)-extension \((d \geq 2)\) of \(X_1\) with endomorphism \(g_1\) extending \(f_1\), i.e., \(g_1|_{X_1} = f_1\), such that \([C_d^c(\tilde{g}_1, \tilde{\gamma}_1)] = [a]\). Define \((Y_2, g_2)\) as the pushout of \((Y_1, g_1) \leftarrow (X_1, f_1) \rightarrow (X_2, f_2)\). The map \(C_d^c(\tilde{g}_2, \tilde{\gamma}_2)\) can be used as a representative of \(U_G^\mathbb{Z}(h)([a])\). This is shown using the fact that \(\Pi(G, i_1)\) and \(\Pi(G, i_2)\) are equivalences of categories, Proposition 1.3 and additivity.
\textbf{Step 6:} The natural transformation \( \tau : U_G^Z \to \Theta_G \) maps \( u_G^Z(X,f) \) to \( \theta_G(X,f) \in \Theta_G(X,f) \). For a finite \( G \)-CW-complex \( X \) with endomorphism \( f \), let \( Y_n \) be the pushout of \( X_n \xrightarrow{k_{n-1}} X_{n-1} \) \xrightarrow{j_{n-1}} X \), where all arrows are canonical inclusions. There is a canonical retraction \( r_n : Y_n \to X \) induced by the inclusions of \( X_{n-1}, X_n \) and \( X \) into \( X \). Defining \( f_k : X_k \to X_k \) as the restriction of \( f \) to \( X_k \), one obtains an endomorphism \( g_n : Y_n \to Y_n \) as \( g_n = f_n \cup f_{n-1} \). Summing and application of \( \Theta_G(r_n) \) yields
\[
\Theta_G(r_n)\theta_G(g_n) = \Theta_G(i_n)\theta_G(f_n) + \theta_G(f) - \Theta_G(i_{n-1})\theta_G(f_{n-1}).
\]
Summing up, we obtain
\[
\sum_{n=0}^{\dim(X)} (\Theta_G(r_n)\theta_G(g_n) - \theta_G(f)) = \theta_G(f) \in \Theta_G(X,f).
\]

The analogous equation holds for \( (U_G^Z, u_G^Z) \). If \( n \geq 2 \), we have
\[
[C^c((g_n, \tilde{f}))] = [\text{res}_{\Pi(G,i_n)} C^c(g_n)] - [C^c(\tilde{f})] = U_G^Z(r_n)u_G^Z(g_n) - u_G^Z(f)
\]
because \( \Pi(G, i_n) \) is an equivalence of categories. We conclude that
\[
\tau_{G(f)}(U_G^Z(r_n)u_G^Z(g_n) - u_G^Z(f)) = \tau_{G(f)}(\Theta_G(\hat{i}_n^i)\theta_G(g_n) - \theta_G(f))
\]
\[
= \Theta_G(r_n)\theta_G(g_n) - \theta_G(f).
\]

The last equation follows because we have strict commutativity \( r_ng_n = fr_n \).

Summing up, we obtain \( \tau_{G(f)}(u_G^Z(f)) = \theta_G(f) \).

If \( n \in \{0, 1\} \), we use the suspension \( \Sigma XY_n \) of \( Y_n \) to get into the range \( d \geq 2 \).

\textbf{Step 7:} We have to show that \( \tau_K \circ U(\alpha) = \Theta(\alpha) \circ \tau_G \) for any inclusion \( \alpha : G \to K \). By definition of the natural transformation \( \tau \) and of \( U(\alpha) \) and using the notation established above, this is equivalent to
\[
(-1)^d(\Theta_K(\text{ind}_\alpha i \circ \text{ind}_\alpha i)^{-1}\Theta_K(\text{ind}_\alpha i)(\theta_K(\text{ind}_\alpha g) - \theta_K(\text{ind}_\alpha f)))
\]
\[
= \Theta(\alpha)((-1)^d(\Theta_G(\hat{i} \circ \hat{i})^{-1}\Theta_G(\hat{i})(\theta_G(g) - \theta_G(f))).
\]
The retractive space constructions commute with induction. Since \( \Theta \) is a family of functors on \( \text{End}(G\text{-CW}_{fp}) \) for discrete groups \( G \) which is compatible with the induction structure, we have \( \Theta(\alpha)\Theta_G(h) = \Theta_K(\text{ind}_\alpha h)\Theta(\alpha) \).

So
\[
\Theta(\alpha)((-1)^d(\Theta_G(\hat{i} \circ \hat{i})^{-1}\Theta_G(\hat{i})(\theta_G(g) - \theta_G(f)))
\]
\[
= (-1)^d(\Theta_K(\text{ind}_\alpha i \circ \text{ind}_\alpha i)^{-1}\Theta_K(\text{ind}_\alpha i)\Theta(\alpha)(\theta_G(g) - \Theta(\alpha)\theta_G(f))
\]

Since \( (\Theta, \theta) \) is a functorial equivariant Lefschetz invariant, the equation \( \Theta(\alpha)(\theta_G(g)) = \theta_K(\text{ind}_\alpha g) \) and the analogous equation for \( f \) hold, so we are finished.

\textbf{Step 8:} The natural transformation \( \tau \) is uniquely determined. Any element \( [a] \) in \( U_G^Z(X,f) \) can be realized by a \( d \)-extension \( Y \) for \( d \geq 2 \) as
\[
[a] = [C^c(g, f)]
\]
\[
= (-1)^d(U_G^Z(\hat{i} \circ \hat{i})^{-1} \circ U_G^Z(\hat{i})(u_G^Z(f) - u_G^Z(f))),
\]
and by \( \tau_{G(f)}(u_G^Z(f)) = \theta_G(f) \) and functoriality it follows that
\[
\tau_{G(f)}([a]) = (-1)^d(\Theta_G(\hat{i} \circ \hat{i})^{-1} \circ \Theta_G(\hat{i})(\theta_G(g) - \theta_G(f))\).
4. Splitting Results

In this section, we derive a direct sum decomposition of the abelian group \( U^G_R(X, f) \), making it more accessible to computations.

For subgroups \( H \leq G \), the fixed point sets \( X^H := \{ x \in X \mid hx = x \text{ for all } h \in H \} \) and the restrictions \( f^H := f|_{X^H} : X^H \to X^H \) come into play. The Weyl group \( WH := N_G H / H \) acts on \( X^H \). We show in Theorem 4.6 that the group \( U^G_R(X, f) \) splits up into summands corresponding to the fixed point sets \( X^H \) for \( (H) \in \text{consub}(G) \), the set of conjugacy classes of subgroups of \( G \).

Let \( X^{>H} := \{ x \in X^H \mid G_x \neq H \} \), where \( G_x \) denotes the isotropy group of \( x \), and \( f^{>H} := f|_{X^{>H}} \). We also show that the element \( u^G_R(X, f) \) maps to the elements given by the relative maps \( (f^H, f^{>H}) \). We even have a finer decomposition corresponding to the orbits of connected components of \( X^H \setminus X^{>H} \) under \( f^H \).

We now start analyzing \( U^G_R(X, f) := K_0(\phi\text{-end}_R(G)) \) for \( \Gamma = \Pi(G, X) \) and \( \phi = \Pi(G, f) \). We restrict ourselves to this geometric case for simplicity. The results hold for more general EI-categories \( \Gamma \) with endofunctors \( \phi \) [Web05].

A partial ordering on the objects of any EI-category \( \Gamma \) is given by \( x \leq y \iff \text{Mor}_\Gamma(x, y) \neq \emptyset \). For an object \( x \): \( G/H \to X \) of \( \Pi(G, X) \), we define the type \( \text{tp}(x) \) to be \( (H) \in \text{consub}(G) \). The partial ordering on \( \Pi(G, X) \) becomes a partial ordering according to the type of \( x \) and the connected component of \( WH \setminus X^H \) which \( x \) maps into. On \( \text{consub}(G) \), we obtain the partial ordering given by \( (H) \leq (K) \iff \text{Mor}(G/H, G/K) \neq \emptyset \). The endofunctor \( \phi \) respects the partial ordering on \( \Pi(G, X) \) since it is given by composition with \( f \).

We work with \( \phi \)-endomorphisms \( g: M \to M \circ \phi \) of finite free \( R\Gamma \)-modules \( M \). Every finite free \( R\Gamma \)-module is isomorphic to one which is of the form \( \bigoplus_{b \in B} R\Gamma(?, \beta(b)) \) with \( B \) a finite set and \( \beta: B \to \text{Ob}\Gamma \) a map.

**Lemma 4.1.** Let \( g: \bigoplus_{b \in B} R\Gamma(?, \beta(b)) \to \bigoplus_{b \in B} R\Gamma(?, \beta(b)) \circ \phi \). Then for \( b_0 \in B \) we have
\[
g(R\Gamma(? , \beta(b_0)))) \subseteq \bigoplus_{b \beta(b) \geq \beta(b_0)} R\Gamma(?, \beta(b)) \circ \phi.
\]

**Proof.** Let \( b_0 \in B_i \), and let \( v \in R\Gamma(?, \beta(b_0)) \). The following diagram commutes (because \( g \) is a natural transformation):

\[
\begin{array}{ccc}
R\Gamma(?, \beta(b_0)) & \xrightarrow{g|_{R\Gamma(?, \beta(b_0))}} & \bigoplus_{b \in B} R\Gamma(\phi(?), \beta(b)) \\
\uparrow_{\phi^*(v^*)} & & \uparrow_{\phi^*(v^*)=(\phi(v))^*} \\
R\Gamma(\beta(b_0), \beta(b_0)) & \xrightarrow{g_{\beta(b_0)}|_{R\Gamma(\beta(b_0), \beta(b_0))}} & \bigoplus_{b \in B} R\Gamma(\phi(\beta(b_0)), \beta(b))
\end{array}
\]

We know that \( R\Gamma(\phi(\beta(b_0)), \beta(b)) = 0 \) unless \( \phi(\beta(b_0)) \leq \beta(b) \). But \( b_0 \in B_i \) implies \( \beta(b_0) \in W_i \) and thus also \( \phi(\beta(b_0)) \in W_i \), so \( \phi(\beta(b_0)) \leq \beta(b) \) is only possible if \( \beta(b) \in W_j \) for \( j \geq i \). So \( R\Gamma(\phi(\beta(b_0)), \beta(b)) = 0 \) if \( b \not\in B_j \) for all \( j \geq i \). The map \( (\phi(v))^* \) respects the direct sum decomposition because it is just precomposition with \( \phi(v) \), and thus \( g_{\beta(b_0)}(v) = (\phi(v))^*(g_{\beta(b_0)}(\text{id}_{\beta(b_0)}) \phi(v))^* \) is nonzero only in components with \( b \in B_j \) for \( j \geq i \). So \( g_{\beta(b_0)}(R\Gamma(? , \beta(b_0)))) \subseteq \bigoplus_{j \geq i} \bigoplus_{b \in B_j} R\Gamma(\phi(?), \beta(b)) \) for all \( b_0 \in B_i \), which implies the statement. \( \square \)
Set $M_H := \bigoplus_{b \text{ with } \text{tp} \beta(b) = (H)} R\Gamma(?, \beta(b))$. Then $M \cong \bigoplus_{(H) \in \text{consub}(G)} M_H$.

**Corollary 4.2.** $g(M_H) \subseteq \bigoplus_{(K) \geq (H)} M_K \circ \phi$.

Analogous results hold for maps $h: M \to M'$ of finite free $R\Gamma$-modules.

We denote by $\Gamma_H$ the full subcategory of $\Gamma$ containing all objects $x$ of type $(H)$. We set $\phi_H := \phi|_{\Gamma_H}: \Gamma_H \to \Gamma_H$. The inclusion map $i_H: \Gamma_H \to \Gamma$ is a map of categories with endofunctors, $\phi i_H = i_H \phi_H$.

We now define functors $S$ and $E$ that induce the desired splitting.

**Definition 4.3.** We define the extension functor $E_H$ by

$$E_H := i_{H*}: \phi \text{-end}_{H(R\Gamma)} \to \phi \text{-end}_{H(R\Gamma)}.$$  

This generalizes the extension functor $E_x$ [Lüc89, Definition 9.28].

We set $S_H M := \text{res}_\Gamma M_H$. A map $g: M \to M \circ \phi$ does not change the type, so it induces a map $g_H: M_H \to M \circ \phi$. We define $S_H(g) := \text{res}_\Gamma g_H: S_H M \to S_H M \circ \phi H$. For a morphism $h: M \to M'$ with $g' = (\phi h)g$ we set $S_H h := \text{res}_\Gamma h_H: S_H M \to S_H M'$. It is easily checked that $S_H(g')S_H h = (\phi h S_H h)S_H(g)$.

**Definition 4.4.** We define the splitting functor $S_H$ by

$$S_H: \phi \text{-end}_{H(R\Gamma)} \to \phi \text{-end}_{H(R\Gamma)}, g \mapsto \text{res}_\Gamma g_H.$$  

This is a variation of the splitting functor $S_x$ [Lüc89, Definition 9.26], for objects $x$ of type $(H)$ we have $\text{res}_x S_H = S_x$. The functors $S_H$ and $E_H$ preserve split exact sequences, so they induce maps on the level of $K$-theory.

We are mostly interested in $K_0$ since that is where our invariant lives. But we can treat all higher $K$-groups since $\phi \text{-end}_{H(R\Gamma)}$ is an exact category. So we let $K$ stand for any $K_n, n \in \mathbb{Z}$, and define the splitting functors for all algebraic $K$-groups simultaneously. We obtain

$$K(S_H): K(\phi \text{-end}_{H(R\Gamma)}) \to K(\phi \text{-end}_{H(R\Gamma)_H})$$

$$K(E_H): K(\phi \text{-end}_{H(R\Gamma)_H}) \to K(\phi \text{-end}_{H(R\Gamma)}).$$

Let another $G$-CW-complex $X'$ with equivariant endomorphism $f'$ be given and set $\Gamma': = \Pi(G, X')$ and $\psi := \Pi(G, f'): \Gamma' \to \Gamma$. A $G$-equivariant map $l: X \to X'$ satisfying $f'l = lf$ induces a functor $L := \Pi(G, l): \Pi(G, X) \to \Pi(G, X')$ such that $\phi L = L \phi'$ and which preserves the type. For every $H \leq G$, the functor $L$ induces a functor $L_{H*}: \phi \text{-end}_{H(R\Gamma)_H} \to \phi' \text{-end}_{H(R\Gamma)'_H}$.

**Definition 4.5.** We define a functor

$$\text{End}(G\text{-CW}_{fp}) \to \mathcal{A}b$$

$$(X, f) \mapsto \bigoplus_{(H) \in \text{consub}(G)} K(\phi \text{-end}_{H(R\Gamma)}(H))$$

$$(l: (X, f) \to (X', f')) \mapsto \bigoplus_{(H) \in \text{consub}(G)} K(L_{H*}).$$

To have compatible notation, we define $K: \text{End}(G\text{-CW}_{fp}) \to \mathcal{A}b$ by $(X, f) \mapsto K(\phi \text{-end}_{H(R\Gamma)}(H), (l: (X, f) \to (X', f')) \mapsto K(L_{H*})$.

For finite subsets $I \subseteq \text{consub}(G)$, set $S_I := \prod_{(H) \in I} K(S_H)$. The map $\text{colim}_{I \subseteq \text{consub}(G)} S_I: K(\phi \text{-end}_{H(R\Gamma)}) \to \bigoplus_{(H) \in \text{consub}(G)} K(\phi \text{-end}_{H(R\Gamma)_H})$ induces
a natural transformation \( S: K \to \text{Split } K \). (Naturality can be checked directly, it also follows from the proof of Theorem 4.6.) The functors \( E_H \) combine to form a natural transformation \( E: \text{Split } K \to K \). Similar to [L"uc89, Theorem 9.34], we prove the following theorem.

**Theorem 4.6.** We have inverse pairs of natural equivalences \( E \) and \( S \) between the functors

\[
K \text{ and } \text{Split } K: \text{End}(G\text{-CW}_{fp}) \to \text{Ab}.
\]

I.e., if \( X \) is a finite proper \( G\text{-CW}\)-complex with equivariant endomorphism \( f: X \to X \), and if \( \Gamma = \Pi(G, X) \) and \( \phi = \Pi(G, f) \), then

\[
K(\phi_{\text{end}_{HFG}}) \cong \bigoplus_{(H) \in \text{consub}(G)} K(\phi_{H\text{-end}_{HFG}}),
\]

where the isomorphism is given by \( S \) with inverse \( E \) and is natural in \((X, f)\).

**Proof.** We have \( S \circ E = \text{colim}_I S_I \circ \text{colim}_I E_I = \text{colim}_I (S_I \circ E_I) \) and \( E \circ S = \text{colim}_I (E_I \circ S_I) \). Hence it suffices to show for any finite \( I \subseteq \text{consub}(G) \) that \( S_I \circ E_I = \text{Id} \) and \( E_I \circ S_I = \text{Id} \) hold.

1) We show that \( S_I \circ E_I = \text{Id} \).

We have \( S_I \circ E_I = \bigoplus_{(H) \in I} K(S_H E_H) \). Thus we need to show that \( K(S_H E_H) = \text{Id} \) for all \((H) \in I \). We know that \( S_H E_H : \text{ff } R\Gamma_H \to \text{ff } R\Gamma_H \) is naturally equivalent to the identity [L"uc89, Lemma 9.31]. Going through the definitions, it is easily checked that this natural equivalence extends to a natural equivalence on \( \phi_{H\text{-end}_{HFG}} \).

2) We show that \( E_I \circ S_I = \text{Id} \).

The proof proceeds inductively over the cardinality of \( I \). The beginning \( I = \emptyset \) is trivial. In the induction step, choose \((H) \) maximal in \( I \) and write \( I' = I \setminus \{(H)\} \). Since we are working with finite free modules, we can restrict ourselves to modules of the form \( M = \bigoplus_{b \in B} R\Gamma(?, \beta(b)) \). Setting \( M_{I'} := \bigoplus_{(K) \in I'} M_K \), we have a split short exact sequence

\[
0 \to M_{I'} \xrightarrow{\text{inc}_{I'}} M \to M_I \to 0.
\]

Since \((H) \) is maximal, by Lemma 4.1 we know that \( g \text{inc}_{I'} = \text{inc}_{I'} g_H \). We obtain an induced map \( g_{I'} := \text{pr}_{M_{I'} \circ \phi} g_{M_I} \) on \( M_{I'} \). So

\[
0 \to g_{I'} \to g \to g_{I'} \to 0
\]

is a short exact sequence in the sense of Definition 1.4.

We call \( \Gamma_{I'} \subseteq \Gamma \) the full subcategory of \( \Gamma \) with objects in \( \bigcup_{(K) \in I'} \Gamma_K \) and set \( \phi_{I'} := \phi_{|\Gamma_{I'}} \). The module \( M_{I'} \) can be restricted to the full subcategory \( \Gamma_{I'} \). Analogously, induction with the inclusion \( \Gamma_{I'} \subseteq \Gamma \) allows us to view an \( R\Gamma_{I'} \)-module \( N \) as an \( R\Gamma \)-module. We extend these assignments to functors

\[
G: \phi_{\text{end}_{HFG}} \to \phi_{\text{end}_{HFG_{I'}}}, g \mapsto \text{res}_{\Gamma_{I'}} g_{I'}, h \mapsto \text{res}_{\Gamma_{I'}}(\text{pr}_{M_{I'}}, h|_{M_{I'}})
\]

\[
F := (\text{inc}_{\Gamma_{I'}})_*: \phi_{\text{end}_{HFG_{I'}}} \to \phi_{\text{end}_{HFG}}
\]

We have \( FG(g) = g_{I'} \) and \( GF = \text{Id} \). We have \( E_H S_H(g) = g_H \). The above sequence induces a cofibration sequence of functors on \( \phi_{\text{end}_{HFG}} \)

\[
0 \to E_H S_H \to \text{Id} \to FG \to 0.
\]
By the additivity theorem [Qui73, p. 103-106], we obtain
\[ K(E_H)K(S_H) + K(F)K(G) = \text{Id}. \]

The following diagram commutes on the outside because of this equation and on the top because of the induction hypothesis \( E_{I'} \circ S_I = \text{Id}. \)

The right triangle commutes if \( (K(F) \oplus K(E_H)) \circ (E_I \oplus \text{Id}) = E \), i.e., if \( K(F) \circ (\oplus_{(K) \in I'} K((E_{I'})_K)) \oplus K(E_H) = \oplus_{(K) \in I} K(E_K) \). This is obviously true: The \( (E_{I'})_K \) on the left land in \( \phi_I^{-\text{end}_{\text{IRR}} \Gamma} \), and the functor \( F \) pushes them forward to \( \phi_I^{-\text{end}_{\text{IRR}} \Gamma} \) where the \( E_K \) on the right hand side land.

The left triangle commutes if \( (K(S_I) \oplus \text{Id}) \circ (K(G) \oplus K(S_H)) = S \), i.e., if \( \prod_{(K) \in I} K((S_{I'})_K \circ G) \oplus K(S_H) = \prod_{(H) \in I} K(S_H) \). This is easily seen: For \( (K) \in I' \), we have \( S_{K} \circ G(g) = S_K(\text{res}_{\phi}(pr_{M_{I'}} \circ g(M_I))) = \text{res}_{\phi} S_K g_K = S_K(g) \). So the left triangle also commutes. We conclude that the bottom triangle commutes and thus that \( E_I \circ S_I = \text{Id} \). Therefore \( E \) and \( S \) are inverse isomorphisms.

It remains to show naturality. An equivariant map \( l: (X_1, f_1) \to (X_2, f_2) \) induces functors \( L := \Pi(G, l)_*: \phi_I^{-\text{end}_{\text{IRR}} \Gamma} \to \phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma} \) and \( L_H := \Pi(G, l|_{X_H^H})_*: \phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma_{1H}} \to \phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma_{2H}} \) for all \( (H) \in \text{consub}(G) \). We have \( SE \cong \text{Id} \) and \( E(\oplus_{(H)} L_H) \cong LE \), so \( SLE \cong SE(\oplus_{(H)} L_H) \cong \oplus_{(H)} L_H \) and the following diagram commutes:

\[
\begin{array}{ccc}
K(\phi_I^{-\text{end}_{\text{IRR}} \Gamma}) & \xrightarrow{K(L_*)} & K(\phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma_1}) \\
\downarrow E & & \downarrow \oplus_{(H)} K(L_H)_* \\
\oplus_{(H)} \phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma_{1H}} & \xrightarrow{\oplus_{(H)} K(L_H)_*} & \oplus_{(H)} \phi_{I'}^{-\text{end}_{\text{IRR}} \Gamma_{2H}}.
\end{array}
\]

Since \( ES \cong \text{Id} \), this implies \( SK(L_*) \cong SK(L_*)ES \cong (\oplus_{(H)} K(L_H)_*)S \), and so \( S \) is a natural transformation.

We can split the groups \( K_0(\phi_H^{-\text{end}_{\text{IRR}} \Gamma_H}) \) up even further, according to the \( WH \)-orbits of connected components \( C \subseteq X_H^H \) and the action of \( f \) on these. We have to distinguish 2 cases:

1) A certain iterate of \( f \) sends \( C \) into \( WH \cdot C \).
2) No iterate of \( f \) sends \( C \) into \( WH \cdot C \).

In case 1, we say that \( C \) is recurring, and we call \( l(C) := \min\{n \geq 1 \mid f^n(C) \subseteq WH \cdot C \} \) the length of \( C \). We denote by \( \Gamma_C \) the full subcategory of \( \Gamma \) with objects isomorphic to \( f^i(x): G/H \to f^i(C) \), for \( x: G/H \to C \) and \( 0 \leq i \leq l - 1 \). We call the set of recurring components \( T \).

In case 2, we say that \( C \) is transient. We call \( ht(C) := \min\{n \in \mathbb{N} \mid f^n(C) \subseteq WH \cdot C' \text{ with } C' \in T \} \) the height of \( C \). We denote by \( \Gamma_C \) the full subcategory of objects isomorphic to \( x: G/H \to C \). (This corresponds to the orbit
Every component $C$ has finite height since we are dealing with finite proper $G$-CW-complexes $X$. Recurring components have height 0.

We choose a set $C$ of representatives, i.e., of connected components $C \subseteq X^H$ such that $\text{Ob}(\Gamma) = \bigsqcup_{C \in C} \text{Ob}(\Gamma_C)$.

**Theorem 4.7.** There is an isomorphism of abelian groups

$$K(\phi_{H-\text{end}}H) \cong \bigoplus_{C \in C \text{ recurrent}} K(\phi_C-\text{end}) \oplus \bigoplus_{C \in C \text{ transient}} K(\text{ff } \Gamma_C).$$

**Proof.** We define a functor

$$A: \bigoplus_{C \in C \text{ recurrent}} \phi_C-\text{end} \oplus \bigoplus_{C \in C \text{ transient}} K(\text{ff } \Gamma_C) \to \phi_{H-\text{end}}H$$

by induction with the inclusion of the relevant subcategories, inserting the 0-map if there is no endomorphism given. Analogously, we define

$$B: \phi_{H-\text{end}} \to \bigoplus_{C \in C \text{ recurrent}} \phi_C-\text{end} \oplus \bigoplus_{C \in C \text{ transient}} K(\text{ff } \Gamma_C)$$

by the corresponding restriction. We proceed as in the proof of Theorem 4.6 to show that $A$ and $B$ are equivalences of categories inverse to each other.

The idea is to split off the transient components, starting from the top. If the largest appearing height is $k$, we call $\Gamma_k$ the full subcategory of $\Gamma_H$ consisting of objects $x$ lying in connected components of height $k$, and we call $\Gamma_{<k}$ the full subcategory consisting of those with smaller height. The inclusion of these subcategories induces induction and restriction maps $\text{ind}_k$, $\text{res}_k$, $\text{ind}_{<k}$ and $\text{res}_{<k}$. In the induction step, the decisive point is that we have a split short exact sequence

$$\begin{array}{ccc}
0 & \to \text{ind}_{<k} \text{ res}_{<k} M & \to M & \to \text{ind}_k \text{ res}_k M & \to 0 \\
\downarrow \text{incl}_{<k} \downarrow \text{res}_{<k} \circ g & & \downarrow g & & \downarrow 0 \\
0 & \to \text{ind}_{<k} \text{ res}_{<k} M \circ \phi & \to M \circ \phi & \to 0 & \to 0
\end{array}$$

to which we apply the additivity theorem. We leave the details to the reader. □

Having established this result, we next replace the groupoids $\Gamma_C$ by corresponding groups. This is analogous to the transition from the fundamental groupoid of a topological space to its fundamental group. Assuming $X$ connected, we choose a basepoint $x \in X$ and look at the fundamental group with respect to this basepoint.

In order for the endomorphism $f: X \to X$ to induce an endomorphism $\phi: \pi_1(X, x) \to \pi_1(X, x)$, we also need to choose a path $v$ from $x$ to $f(x)$. We define $\phi := c_v \circ \pi_1(f)$, where $c_v$ is the conjugation map $c_v: \pi_1(X, f(x)) \to \pi_1(X, x), \gamma \mapsto v\gamma v^{-1}$. Here composition is written from left to right, as is usual for composition of paths. Choosing a path $v$ from $x$ to $f(x)$ corresponds to choosing a morphism $w = (\sigma, [v]) \in \text{Mor}(f(x), x)$.

For transient components $C$, we choose an object $x: G/H \to C$ in $\Gamma_C$. This is identified with the point $x(1H) \in C$. We obtain an equivalence of categories $\text{Aut}(x) \to \Gamma_C$, where the group $\text{Aut}(x)$ is viewed as a category. The choice of $x$ does not play a role, we can use induction and restriction with
this equivalence of categories to identify the groups obtained from different choices.

Now we look at recurrent components $C$. If $l := l(C)$ is the length of $C$, choosing a point $x \in C$ gives a sequence $f(x) \in f(C), \ldots, f^l(x) \in f^l(C)$. There is an element $g \in WH$ such that $gf^l(C) \subseteq C$. We choose a path $v$ from $x$ to $gf^l(C)$. We know that an element $g \in WH$ uniquely determines a map $\sigma_g: G/H \to G/H, g'H \mapsto g'gH$ [tD79, Proposition 1.14]. We set $w = (\sigma_g, [v]) \in \text{Mor}(\phi^l(x), x)$, for $\gamma \in \text{Aut}(\phi^l(x))$, we set $c_w(\gamma) := w\gamma^w^{-1} \in \text{Aut}(x)$. We obtain a group homomorphism

$$\phi_{x,w}: \text{Aut}(\phi^l(x)) \xrightarrow{\phi_{\text{Aut}(\phi^l(x))}} \text{Aut}(\phi^l(x)) \xrightarrow{c_w} \text{Aut}(x).$$

The collection $\Phi := (\phi_{\text{Aut}(\phi^l(x))}, \ldots, \phi_{\text{Aut}(\phi^{l-1}(x))}, \phi_{x,w})$ is an endomorphism of the disjoint union $\bigoplus_{i=0}^{l-1} \text{Aut}(\phi^i(x))$ which on every component is a group homomorphism to the next.

We define a category $\Phi\text{-end}_{\text{HRI}0}\text{Aut}(\phi^i(x))$. An object is a pair of sequences $(\{M_i\}_{0 \leq i \leq l-1}, \{g_i\}_{0 \leq i \leq l-1})$. Here $M_i$ is a finite free $R\text{Aut}(\phi^i(x))$-module. For $0 \leq i < l-1$, the map $g_i: M_i \to M_{i+1} \circ \phi_{\text{Aut}(\phi^i(x))}$ is a $\phi_{\text{Aut}(\phi^i(x))}$-morphism. For $i = l-1$ the map $g_{l-1}: M_{l-1} \to M_0 \circ \phi_{x,w}$ is a $\phi_{x,w}$-morphism. A morphism $h: (M_i)_{0 \leq i \leq l-1} \to (M'_i)_{0 \leq i \leq l-1}$ between these modules is a sequence $(h_0, \ldots, h_{l-1})$ of maps $h_i: M_i \to M'_i$ such that all resulting diagrams commute.

**Theorem 4.8.** There is an equivalence of categories

$$\phi_{C\text{-end}_{\text{HRI}0}\text{Aut}(\phi^i(x))} \sim \Phi\text{-end}_{\text{HRI}0}\text{Aut}(\phi^i(x)).$$

The proof is quite technical, but not very insightful. We state the basic idea and refer the reader to [Web05] for details.

**Idea of proof.** We use induction and restriction with the inclusion of categories $\bigoplus_{i=0}^{l-1} \text{Aut}(\phi^i(x)) \to \Gamma_C$ to define $I_{x,w}: \Phi\text{-end}_{\text{HRI}0}\text{Aut}(\phi^i(x)) \to \phi_{C\text{-end}_{\text{HRI}0}}$ and $R_{x,w}: \phi_{C\text{-end}_{\text{HRI}0}} \to \Phi\text{-end}_{\text{HRI}0}\text{Aut}(\phi^i(x))$. We use the canonical isomorphism $\text{res}_{\phi^l} \circ \text{res}_{\phi(x)}: M_0 \cong \text{res}_{\phi(x)} M_0$ in the definition of $R_{x,w}$ and its inverse in the definition of $I_{x,w}$, alongside with the functors $\eta: \text{Id} \to \text{res ind}$ and $\varepsilon: \text{res ind} \to \text{Id}$ as in Section 1.

We then show that $I_{x,w}$ and $R_{x,w}$ are equivalences of categories inverse to each other, by using the triangular identities to show that the appropriate diagrams commute.

We combine the results of this section in the following statement.

**Theorem 4.9.** Let $G$ be a discrete group, let $X$ be a finite proper $G$-CW-complex and let $f: X \to X$ be a $G$-equivariant endomorphism. Let $(U^G_0(X,f), u^G_0(X,f))$ be the universal functorial Lefschetz invariant of $(X,f)$. For all $H \leq G$ and $C \in \mathcal{C}_H$, the set of representatives of connected components of $X^H$, we choose $x_C: G/H \to C$. If $C$ is recurrent of length $l$, we also choose an element $g_C \in WH$ such that $f^l(C) \subseteq g_C C$ and path $v_C$ from $x_C$ to $g_C f^l(x_C)$. 
Then there is an isomorphism
\[ \zeta: \ U_G^\mathbb{Z}(X,f) \xrightarrow{\sim} \bigoplus_{(H)\in \text{consub}(G)} \left( \bigoplus_{C\in \mathcal{C}_H} K_0(\Phi_{C\text{-end}}\text{fil}_H\text{Aut}(\phi^{f}(x_C))) \right. \]
\[ \left. \oplus K_0(\text{fil } \text{Aut}(x_C)) \right) \]
\[ u_G^\mathbb{Z}(X,f) \mapsto \sum_{(H)\in \text{consub}(G)} \sum_{C\in \mathcal{C}_H} u_G^\mathbb{Z}(X,f)_{x_C}, \]
where for \( C \) recurrent (leaving out the modules) we have
\[ u_G^\mathbb{Z}(X,f)_{x_C} = \left[ (C^c(\tilde{f}|_{\tilde{X}^H(x_C)}), \tilde{f}|_{\tilde{X}^H(x_C)}), \ldots \right. \]
\[ \left. C^c(g_C^{-1}f|_{\tilde{X}^H(f^{-1}(x_C))}, g_C^{-1}f|_{\tilde{X}^H(f^{-1}(x_C))}) \right] \]
and for \( C \) transient we have
\[ u_G^\mathbb{R}(X,f)_{x_C} = \left[ C^c(\tilde{X}^H(x_C), \tilde{X}^H(x_C)) \right]. \]

Proof. The existence of the isomorphism \( \zeta \) was established in Theorems 4.6 and 4.8. It remains to identify the image of \( u_G^\mathbb{Z}(X,f) \).

On the modules, \( \zeta \) is given by \( \text{res}_x S_{\Gamma_G} = S_x \). There is a natural isomorphism \( S_x(C^c(\tilde{X})) \cong C^c(\tilde{X}^H(x), \tilde{X}^H(x)) \) of \( \text{fil } \text{Aut}(x) \)-chain complexes for \((x: G/H \rightarrow X) \in \text{Ob}(\Pi(G,X)) \) [L"uc89, Lemma 9.32]. This gives the result for transient \( C \).

For recurrent \( C \), the modules are
\[ M_i = \text{res}_{f^{i}(x_C)} S_{\Gamma_G}(C^c(\tilde{X})) \cong C^c(\tilde{X}^H(f^i(x_C)), \tilde{X}^H(f^i(x_C))). \]
For \( 0 \leq i \leq l-2 \), the morphisms are
\[ g_i = \text{res}_{f^{i}(x_C)} S_{\Gamma_G}(C^c(\tilde{f})) = C^c(\tilde{f}|_{\tilde{X}^H(f^{i}(x_C))}, \tilde{f}|_{\tilde{X}^H(f^{i}(x_C))}). \]
The canonical isomorphism used in Theorem 4.8 is geometrically the one induced by multiplication with \( g_C^{-1} \),
\[ C^c(g_C^{-1}): C^c(\tilde{X}^H(f^l(x_C)), \tilde{X}^H(f^l(x_C))) \rightarrow C^c(\tilde{X}^H(x_C), \tilde{X}^H(x_C)). \]
So \( g_{l-1} = C^c(g_C^{-1}f|_{\tilde{X}^H(f^{l-1}(x_C))}, g_C^{-1}f|_{\tilde{X}^H(f^{l-1}(x_C))}) \). \( \square \)

5. The Generalized Equivariant Lefschetz Invariant

In this section we develop a generalized equivariant Lefschetz invariant \((\Lambda_G(X,f), \lambda_G(f))\) as the image of the universal functorial equivariant Lefschetz invariant \((U_G^\mathbb{Z}(X,f), u_G^\mathbb{Z}(X,f))\) under a convenient trace map \( \text{tr}_{G(X,f)} \). It is an equivariant analog of the generalized Lefschetz invariant [Rei36, Wec41] and a refinement of the equivariant Lefschetz class [LR03, Definition 3.6].

We start by defining the group \( \Lambda_G(X,f) \) that will be the target group of \( \text{tr}_{G(X,f)} \). We are interested in fixed point information, so only in objects \((x: G/H \rightarrow X) \in \Pi(G,X) \) with \( X^H(f(x)) = X^H(x) \). The splitting
obtained in Theorem 4.9 can be written as
\[ U^H_G(X, f) \cong \bigoplus_{\pi \in \Pi(G,X), \ X^H(\pi) = X^H(\mathfrak{f})} K_0(\phi_x, w\text{-end}_{\text{HAut}(x)}) \oplus \text{other terms.} \]

We design \( \Lambda_G(X, f) \) in the same way. For objects \( x \in \text{Ob}\Pi(G,X) \) with \( X^H(f(x)) = X^H(x) \), we always find morphisms \( w = (\text{id}, [\nu]) \in \text{Mor}(f(x), x) \), and we restrict our attention to morphisms of that form.

**Definition 5.1.** For \( x \in \text{Ob}\Pi(G,X) \) with \( X^H(f(x)) = X^H(x) \) and a morphism \( w = (\text{id}, [\nu]) \in \text{Mor}(f(x), x) \), set
\[ Z\pi_1(X^H(x), x)_{\phi_x, w} := Z\pi_1(X^H(x), x)/\phi_x, w(\gamma)\alpha\gamma^{-1} \sim \alpha, \]
where \( \alpha \in \pi_1(X^H(x), x), \gamma \in \text{Aut}(x) \) and \( \phi_x, w(\gamma) = w\phi(\gamma)w^{-1} \in \text{Aut}(x) \).

We have \( \phi_x, w(\gamma)\alpha\gamma^{-1} \in \pi_1(X^H(x), x) \) for all \( \gamma \in \text{Aut}(x) \) and \( \alpha \in \pi_1(X^H(x), x) \) because the map \( \phi_x, w \) does not change the \( WH_x \)-part of the morphism \( \gamma \) and \( \pi_1(X^H(x), x) \) is normal.

We can move from one basepoint to another in the following way: For a morphism \( (\sigma, [t]) \in \text{Mor}\Pi(G,X)(x(H), y(K)) \), where \( X^H(f(x)) = X^H(x) \) and \( X^K(f(y)) = X^K(y) \), we choose morphisms \( w_x = (\text{id}, [\nu]) \in \text{Mor}(f(x), x) \) and \( w_y = (\text{id}, [\nu]) \in \text{Mor}(f(y), y) \) and set
\[(\sigma, [t])_{w_x, w_y}^* : Z\pi_1(X^K(y), y)_{\phi_y, w_y} \to Z\pi_1(X^H(x), x)_{\phi_x, w_x} \ \\
\alpha \mapsto v_xf(t^{-1})\sigma^* (v_y^{-1})\sigma^* (\alpha)t. \]

One easily checks that this map is well-defined. The next lemma shows that the map \( (\sigma, [t])_{w_x, w_y}^* \) does not depend on the choice of \( (\sigma, [t]) \in \text{Mor}(x, y) \).

**Lemma 5.2.** For \( \overline{\alpha} \in \pi_1(X^K(y), y)_{\phi_y, w_y} \) we have
\[(\sigma, [t])_{w_x, w_y}^* (\overline{\alpha}) = (\tau, [s])_{w_x, w_y}^* (\overline{\alpha}) \in Z\pi_1(X^H(x), x)_{\phi_x, w_x} \]
for all morphisms \( (\sigma, [t]), (\tau, [s]) \in \text{Mor}(x, y) \).

**Proof.** For all \( \gamma = (\sigma\gamma, [\nu]) \in \text{Aut}(x) \), we have
\[(\sigma, [t])_{w_y, w_y}^* (\overline{\alpha}) = v_xf(\gamma)v_x^{-1}(\sigma, [t])_{w_y, w_y}^* (\overline{\alpha})^{-1} \]
\[= v_x\gamma^* f(t^{-1})\gamma^{-1} \sigma^* (v_y^{-1})\gamma^* \sigma^* (\alpha)\gamma^{-1}(tv^{-1}). \]
Setting \( \sigma_{\gamma} = \tau_{\gamma}^{-1} \sigma \) and \( v_{\gamma} = (\tau_{\gamma}^{-1})^* (s^{-1})t \), we obtain
\[(\sigma, [t])_{w_y, w_y}^* (\overline{\alpha}) = v_xf(s^{-1})\tau^* (v_y^{-1})\tau^* (\alpha)s = (\tau, [s])_{w_x, w_y}^* (\overline{\alpha}). \]

We can use these maps to change the morphism \( w \) and the base point \( x \), so the definition is independent of the choices of \( x \) and \( w \). We write \( (\sigma, [t])^* \) from now on.

**Definition 5.3.**
\[ \Lambda_G(X, f) := \bigoplus_{\pi \in \Pi(G,X), \ X^H(\pi) = X^H(\mathfrak{f})} Z\pi_1(X^H(x), x)_{\phi_x, w}. \]
A trace map from $K_0(\phi_{x,w}^{-}\text{end}_{\mathcal{D}(\text{Aut}(x))})$ to $\mathbb{Z}_\pi(X^H(x), x)\phi_{x,w}^{-}$ will now be defined. (The index fp stands for finitely generated projective modules.)

Let $\text{Aut}(x)$ be a group extension $1 \to \pi_1(X^H(x), x) \to \text{Aut}(x) \to WH_x \to 1$, where $WH_x \subseteq WH$ is the subgroup fixing $X^H(x)$. There is a commutative diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(X^H(x), x) & \longrightarrow & \text{Aut}(x) & \longrightarrow & WH_x & \longrightarrow & 1 \\
\phi_{x,w} \downarrow_{\pi_1(X^H(x), x)} & & \phi_{x,w} \downarrow & & \text{id} & & \\
1 & \longrightarrow & \pi_1(X^H(x), x) & \longrightarrow & \text{Aut}(x) & \longrightarrow & WH_x & \longrightarrow & 1.
\end{array}
$$

We combine the trace maps $\text{tr}_{RG} : RG \to R, \sum_{g \in G} r_g \cdot g \mapsto r_1$ [LR03, 1.1 and 1.2], applied to the $WH_x$-part, and $\text{tr}_{(\mathbb{Z}_\pi, \phi)} : \mathbb{Z}_\pi \to \mathbb{Z}_\pi, \sum_{\gamma \in \pi} n_\gamma \cdot \gamma \mapsto \sum_{\gamma \in \pi} n_\gamma \cdot \gamma [\text{Luc99, 3.6}],$ applied to the $\pi_1(X^H(x), x)$-part.

We formulate the definition of the trace map independently of the concrete group $\text{Aut}(x)$ which is given to us geometrically.

**Definition 5.4.** Let $\pi$ and $W$ be discrete groups, and let $G$ be a group extension $1 \to \pi \to G \to W \to 1$. Let an endomorphism $\phi : G \to G$ be given that restricts to an endomorphism $\phi_\pi : \pi \to \pi$ and becomes trivial when the normal subgroup $\pi \leq G$ is divided out. For $R$ be a commutative associative ring with unit, define $R\pi_{\phi'} := R\pi/\sim$, where $\phi(\gamma)\alpha\gamma \sim \alpha$ for $\alpha \in \pi$ and $\gamma \in G$.

Let $\phi^{-}\text{end}_{\mathcal{D}(RG)}$ denote the category of $\phi$-twisted endomorphisms of finitely generated projective $RG$-modules. We define the trace map

$$
\text{tr}_{RG} : \text{Ob}(\phi^{-}\text{end}_{\mathcal{D}(RG)}) \to R\pi_{\phi'}
$$

as follows: On $RG$, we set $\text{tr}_{RG} : RG \to R\pi_{\phi'}, \sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in \pi} r_g \cdot \overline{g}$, where $\overline{\cdot} : \pi \to \pi_{\phi'}, g \mapsto \overline{g}$ denotes the projection. Given a $\phi$-twisted endomorphism $u : P \to \text{res}_\phi P$ of a finitely generated projective $RG$-module, we choose a finitely generated projective $RG$-module $Q$ and an isomorphism $v : P \oplus Q \xrightarrow{\sim} \bigoplus_{i \in I} RG$ for a finite indexing set $I$. Then we have a $\phi$-twisted endomorphism $\phi^*(v) \circ (u \oplus 0) \circ v^{-1} : \bigoplus_{i \in I} RG \to \text{res}_\phi \left( \bigoplus_{i \in I} RG \right)$, to which a matrix $A = (a_{ij})_{i,j \in I}$ is associated. We define

$$
\text{tr}_{RG}(u) := \sum_{i \in I} \text{tr}_{RG}(a_{ii}) \in R\pi_{\phi'}.
$$

Note that $\alpha \in \pi$ implies that $\phi(\gamma)\alpha\gamma^{-1} \in \pi$ since $\text{pr}_W(\phi(\gamma)\alpha\gamma^{-1}) = \text{id}_W(\text{pr}_W(\gamma)) \text{pr}_W(\alpha) \text{pr}_W(\gamma^{-1}) = 1_W$, so $R\pi_{\phi'}$ is well defined. As usual, the definition of the trace map is independent of the choices involved. This trace map has properties generalizing [LR03, Lemma 1.3]:

**Lemma 5.5.** Let $G$ be a discrete group extension $1 \to \pi \to G \to W \to 1$ with endomorphism $\phi : G \to G$ which restricts to $\pi$ and such that $\phi_W = \text{id}_W$.

1. Let $u : P \to Q$ and $v : Q \to \text{res}_\phi P$ be $RG$-maps of finitely generated projective $RG$-modules. Then $\text{tr}_{RG}(v \circ u) = \text{tr}_{RG}(\text{res}_\phi(u) \circ v)$. 


(2) Let $P_1, P_2$ be finitely generated projective $RG$-modules. Given a $\phi$-twisted endomorphism $A = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}: P_1 \oplus P_2 \to P_1 \oplus P_2$ we have $\text{tr}_{RG}(A) = \text{tr}_{RG}(u_{11}) + \text{tr}_{RG}(u_{22})$.

(3) Let $u_1, u_2: P \to \text{res}_G P$ be $\phi$-twisted endomorphisms of a finitely generated projective $RG$-module $P$ and let $r_1, r_2 \in R$. Then

$$\text{tr}_{RG}(r_1 \cdot u_1 + r_2 \cdot u_2) = r_1 \text{tr}_{RG}(u_1) + r_2 \text{tr}_{RG}(u_2).$$

(4) Let $\alpha: G \to K$ be a homomorphism of discrete group extensions with endomorphisms as in Definition 5.4 with $\alpha_W$ injective, lying in a commutative diagram

$$\begin{array}{cccc}
1 & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1 \\
& & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha_W & & \\
1 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & K_2 & \longrightarrow & 1.
\end{array}$$

Let $u: P \to \text{res}_G P$ be a $\phi_G$-twisted endomorphism of a finitely generated projective $RG$-module $P$. Then induction with $\alpha$ yields a $\phi_K$-twisted endomorphism $\alpha_* u$ of a finitely generated projective $RK$-module and

$$\text{tr}_{RK}(\alpha_* u) = \alpha'_* \text{tr}_{RG}(u),$$

where $\alpha'_*: R\pi_{\phi_K} \to R(K_1)_{\phi_K}$ is induced by $\alpha_\pi$.

(5) Let $\alpha: H \to G$ be an inclusion of discrete group extensions with endomorphisms as in Definition 5.4 with finite index $[G: H]$, lying in a commutative diagram

$$\begin{array}{cccc}
1 & \longrightarrow & \pi & \longrightarrow & H & \longrightarrow & H_2 & \longrightarrow & 1 \\
& & \downarrow \text{id} & & \downarrow \alpha & & \downarrow \alpha_{H_2} & & \\
1 & \longrightarrow & \pi & \longrightarrow & G & \longrightarrow & W & \longrightarrow & 1.
\end{array}$$

Let $u: P \to \text{res}_G P$ be a $\phi_G$-twisted endomorphism of a finitely generated projective $RG$-module $P$. Then the restriction to $RH$ with $\alpha$ yields a $\phi_H$-twisted endomorphism $\alpha_* u$ of a finitely generated projective $RH$-module and

$$\text{id}_* \text{tr}_{RH}(\alpha_* u) = [G: H] \cdot \text{tr}_{RG}(u),$$

where $\text{id}_*: R\pi_{\phi_H} \to R\pi_{\phi_G}$ denotes the projection.

(6) Let the subgroup $H \leq G$ be finite such that $|H|$ is invertible in $R$. Let $u: R[G/H] \to \text{res}_H R[G/H]$ be a $\phi$-twisted endomorphism that sends $1H$ to $\sum_{g \in G/H} r_{gH} \cdot gH$. Then $R[G/H]$ is a finitely generated projective $RG$-module and $\text{tr}_{RG}(u) = |H|^{-1} \sum_{g \in \pi} r_{gH} \cdot \bar{g} \in R\pi_{\phi'}$. In particular, $\text{tr}_{RG}(\text{id}_{R[G/H]}) = |H|^{-1} \cdot \left( \sum_{g \in \pi} 1 \cdot \bar{g} \right) \in R\pi_{\phi'}$.

Proof. Check (1) by calculation, using the fact that for $g, g' \in G$ we have $gg' \in U \Leftrightarrow \phi(g')g \in U$. Assertions (2) and (3) are clear by definition, and (4) and (5) are again checked by calculation. In (6), we are given $u: 1H \to \sum_{g \in G/H} r_{gH} \cdot gH = |H|^{-1} \left( \sum_{g \in G} r_{gH} \cdot g \right) H$. So $\text{tr}_{RG}(u) = |H|^{-1} \text{tr}_{RG} \left( \sum_{g \in G} r_{gH} \cdot g \right) = |H|^{-1} \sum_{g \in \pi} r_{gH} \cdot \bar{g}$. \qed
By assertions 1 and 2 of Lemma 5.5, the trace map $\text{tr}_{R\pi}$ is compatible with the relations defining $K_0(\phi\text{-end}_{pRG})$. So we can use its value on any representative and define

$$\text{tr}_{R\pi}: K_0(\phi\text{-end}_{pRG}) \to R\pi[\psi], [u] \mapsto \text{tr}_{R\pi}(u).$$

**Remark 5.6.** The trace map $\text{tr}_{R\pi}$ can be seen as a variation of a trace map between $K$-theory and Hochschild homology with coefficients in the bimodule $M_\phi = RG(\phi(?), ?)$. There is a trace map $\text{tr}_{(K \to HH)}: K(RG; M_\phi) \to HH(RG; M_\phi)$. One has $HH_0(RG; M_\phi) \cong RG_\phi$, where we define $RG_\phi := RG/\phi(\gamma)\beta\gamma^{-1} \sim \beta$ for $\gamma, \beta \in G$. We have an inclusion $R\pi \to RG$ of group rings. This inclusion is respected by the $G$-action given by twisted conjugation since $\beta \in \pi$ implies $\text{pr}_W(\phi(\gamma)\beta\gamma^{-1}) = \text{pr}_W(\gamma) \cdot 1 \cdot \text{pr}_W(\gamma^{-1}) = 1$, so $\phi(\gamma)\beta\gamma^{-1} \in \pi$ for all $\gamma \in G$. It induces an inclusion $R\pi[\psi] \to RG_\phi$ as a direct summand. Denoting the restriction to this summand by $r_\pi$, one can check that $\text{tr}_{R\pi} = r_\pi \circ \text{tr}_{(K \to HH)}$.

We could now return to our original setting, but we keep the more general formulation to make a few observations that will be useful later on.

Let $G$ be a discrete group extension with endomorphism $\phi$ and let $(X, A)$ be a finite proper relative $G$-CW-complex. Let $R$ be a commutative ring such that the order of the isotropy group $|G_x|$ is invertible in $R$ for every $x \in X \setminus A$. Then the cellular $RG$-chain-complex $C^c(X, A)$ is finite projective. Let $(f, f_0): (X, A) \to (X, A)$ be a $\phi$-twisted cellular endomorphism. This induces $C^c(f, f_0): C^c(X, A) \to \text{res}_\phi C^c(X, A)$, a $\phi$-twisted endomorphism of the cellular chain complex.

**Definition 5.7.** With notation as above, we define the **refined equivariant Lefschetz number** of $(f, f_0)$ to be

$$L^{RG}(f, f_0) := \sum_{p \geq 0} (-1)^p \text{tr}_{R\pi}(C^c_p(f, f_0)) \in R\pi[\psi].$$

This generalizes the orbifold Lefschetz number [LR03, Definition 1.4]. Writing $[C^c(f, f_0)] := \sum_{p \geq 0} (-1)^p [C^c_p(f, f_0)] \in K_0(\phi\text{-end}_{pRG})$, this definition becomes $L^{RG}(f, f_0) := \text{tr}_{R\pi}([C^c(f, f_0)]) \in R\pi[\psi]$.

We also have a refinement of the incidence number [LR03, 1.8].

**Definition 5.8.** Let $G$ be a discrete group extension with endomorphism $\phi$, let $(X, A)$ be a finite proper relative $G$-CW-complex, let $e \in I_p(X, A)$ be a $p$-cell and let $(f, f_0): (X, A) \to (X, A)$ be a $\phi$-twisted cellular endomorphism. We define the **refined incidence number** $inc_\phi(f, e) \in \mathbb{Z}\pi[\psi]$ for a $p$-cell $e \in I_p(X, A)$ to be the “degree” of the composition

$$\frac{\pi}{\partial e} \xrightarrow{\text{pr}_\pi} \sum_{e' \in I_p(X, A)} \frac{\overline{\pi}}{\partial e'} \xrightarrow{h} X_p/X_{p-1} \xrightarrow{f} X_p/X_{p-1} \xrightarrow{h^{-1}} \sum_{e' \in I_p(X, A)} \frac{\overline{\pi}}{\partial e'} \xrightarrow{\text{pr}_\pi \cdot \bar{\pi}/\partial e} \pi[\psi] \cdot \overline{\pi}/\partial e.$$

We have $inc_\phi(f, e) = inc_\phi(f, ge)$ for all $g \in G$, this is ensured by using $R\pi[\psi]$. We have the equation $inc_\phi(f, e) = \sum_{\alpha \in \pi} \text{inc}(\alpha^{-1} f, e) \cdot \overline{\pi}$, where the incidence number [LR03, 1.8] appears on the right hand side.
Let $e \in I_p(X, A)$ be a $p$-cell. Then $C^c(X, A)_e$ is the chain complex concentrated in degree $p$ with $C^c_p(X, A)_e = R[G/G_e]$. If

$$C^c_p(f, f_0)|_e = \sum_{g \in G_e \in G/G_e} r_{gG_e} \cdot gg_e : R[G/G_e] \to \text{res}_\phi R[G/G_e],$$

then $\text{inc}_\phi(f, e) = \sum_{g \in \pi} r_{gG_e} \cdot g$, and by assertion 5 of Lemma 5.5 we have

$$\text{tr}_{RG}(C^c_p(f, f_0)|_e) = |G_e|^{-1} \sum_{g \in \pi} r_{gG_e} \cdot g = |G_e|^{-1} \text{inc}_\phi(f, e).$$

This observation helps us prove the following result.

**Lemma 5.9.** Let $G$ be a discrete group extension with endomorphism $\phi$. Let $(X, A)$ be a finite proper relative $G$-CW-complex. Let $R$ be a commutative ring such that the order of the isotropy group $|G_x|$ is invertible in $R$ for every $x \in X \setminus A$. Let $(f, f_0) : (X, A) \to (X, A)$ be a $\phi$-twisted cellular endomorphism. Then

$$L^R_{RG}(f, f_0) = \sum_{p \geq 0} (-1)^p \sum_{G \in G \setminus I_p(X, A)} |G_e|^{-1} \cdot \text{inc}_\phi(f, e) \in R\pi_\phi.$$

**Proof.** We calculate

$$L^R_{RG}(f, f_0) = \sum_{p \geq 0} (-1)^p \sum_{G \in G \setminus I_p(X, A)} \text{tr}_{RG}(C^c_p(f, f_0)|_e)$$

$$= \sum_{p \geq 0} (-1)^p \sum_{G \in G \setminus I_p(X, A)} |G_e|^{-1} \cdot \text{inc}_\phi(f, e). \quad \Box$$

We now return to the original setting. For any $x$ in $\Pi(G, X)$ for which $X^H(f(x)) = X^H(x)$, the group $\text{Aut}(x)$ is a discrete group extension $1 \to \pi_1(X^H(x), x) \to \text{Aut}(x) \to WH_x \to 1$ with endomorphism $\phi_{x, w}$ which restricts to $\pi_1(X^H(x), x)$ as $(\phi_{x, w})|_{\pi_1(X^H(x), x)} = c_w \circ \pi_1(f^H(x), x)$ and such that $(\phi_{x, w})|_w = \text{id}_{WH_x} : WH_x \to WH_x$. We set

$$\text{tr}_{G(X, f), x} := \text{tr}_{\pi_1\text{Aut}(x)} : K_0(\phi_{x, w}-\text{end}) \to \mathbb{Z}\pi_1(X^H(x), x) \rightarrow X_{f^H(x), x}. \phi_{x, w}$$

This definition is independent of the choice of the element $x \in \overline{X}$ and the morphism $w$. The reason is that on the domain as well as on the target space we use morphisms to pass from one choice to another and that these constructions are compatible with the trace map.

**Definition 5.10.** Let $G$ be a discrete group, $X$ a finite proper $G$-CW-complex and $f : X \to X$ a $G$-equivariant endomorphism. We define

$$\text{tr}_{G(X, f), x} := \bigoplus_{x \in 1 \in \Pi(G, X), \pi_1(f(x)) = X^H(x)} \text{tr}_{G(X, f), x} \oplus 0 : U^R_G(X, f) \to \Lambda_G(X, f).$$

The groups $\Lambda_G(X, f)$ combine to form a family of functors we want.

**Lemma 5.11.** The groups $\Lambda_G(X, f)$ are naturally endowed with the structure of a family of functors $\Lambda_G$ from $\text{End}(G\text{-CW}_{fp})$ to $\text{Ab}$, for discrete groups $G$, which is compatible with the induction structure.
Proof. Let \( l: (X, f) \to (Y, g) \) be a \( G \)-equivariant map between \( G \)-CW-complexes with endomorphisms. The map \( l \) induces a functor \( \Pi(G, l) \) and a group homomorphism \( \ell_x := \Pi(G, l)|_{\text{Aut}(x)} : \text{Aut}(x) \to \text{Aut}(\ell(x)) \) for every \( x \in \Pi(G, X) \). We have a map \( \ell_x|_{\pi_1(x_H(x), x)} = \pi_1(l|_{X_H(x), x}) \) which induces a map \((\ell'_x)_* : \mathbb{Z}\pi_1(X_H(x), x) \phi_x' \to \mathbb{Z}\pi_1(Y_H(l(x)), l(x)) \phi'_x \). We set
\[
\Lambda(l)_x := (\ell'_x)_* : \mathbb{Z}\pi_1(X_H(x), x) \phi_x' \to \mathbb{Z}\pi_1(Y_H(l(x)), l(x)) \phi'_x
\]
These maps combine to define \( \Lambda(l) : \Lambda_G(X, f) \to \Lambda_G(Y, g) \).

Given an inclusion \( \alpha : G \to K \), the map \( \alpha|_{\pi_1(X_H(x), x) \phi_x} \) induces a map \( \alpha'_* : \mathbb{Z}\pi_1(X_H(x), x) \phi_G \to \mathbb{Z}\pi_1((\text{ind}_\alpha X)^H(\text{ind}_\alpha x), \text{ind}_\alpha x) \phi'_G \). We set \( \Lambda(\alpha)_x = \alpha'_* \) and obtain \( \alpha_* = \Lambda(\alpha) : \Lambda_G(X, f) \to \Lambda_K(\text{ind}_\alpha X, \text{ind}_\alpha f) \). So \( \Lambda \) is also compatible with the induction structure.

The maps \( \text{tr}_{G(X, f)} \) respect all structure.

**Proposition 5.12.** The collection of maps \( \text{tr}_{G(X, f)} \) is a natural transformation of families of functors from \( \text{End}(G\text{-CW}_{fp}) \) to \( \text{Ab} \), for discrete groups \( G \).

Proof. Let \( l: (X, f) \to (Y, g) \) be a \( G \)-equivariant map between \( G \)-CW-complexes with endomorphisms. Then \( \ell_x := \Pi(G, l)|_{\text{Aut}(x)} : \text{Aut}(x) \to \text{Aut}(\ell(x)) \) lies in the commutative diagram
\[
\begin{array}{ccccccccc}
1 & \to & \pi_1(X_H(x), x) & \to & \text{Aut}(x) & \to & WH_x & \to & 1 \\
& & \downarrow{\pi_1(l|_{X_H(x), x})} & & \downarrow{\ell_x} & & \downarrow{\ell_x=}_{\text{incl}} & & \\
1 & \to & \pi_1(Y_H(l(x)), l(x)) & \to & \text{Aut}(\ell(x)) & \to & WH_{\ell(x)} & \to & 1.
\end{array}
\]
The map \( \ell_x : WH_x \to WH_{\ell(x)} \) is an inclusion since the elements in \( WH \) which fix the connected component \( X_H(x) \) also fix the connected component \( Y_H(l(x)) \), by equivariance and continuity of \( l \). We apply Lemma 5.5, assertion 4 to obtain for all \( u \in \phi_{x,w-\text{end}_{fp}Z\text{Aut}(x)} \)
\[
\text{tr}_{Z\text{Aut}(l(x))}(\ell'_x)_*(u) = (\ell'_x)_* \text{tr}_{Z\text{Aut}(x)}(u).
\]
Taking all induction maps \( K_0((\ell'_x)_*) : U^Z_G(X, f)\mathfrak{m} \to U^Z_G(Y, g)\mathfrak{m} \) together gives \( U^Z_G(l) = K_0(\Pi(G, l)_*) : U^Z_G(X, f) \to U^Z_G(Y, g) \). Combining the above equation for all \( \mathfrak{m} \in \text{Is}\Pi(G, X) \), we arrive at
\[
\text{tr}_{G(Y, g)} \circ U^Z_G(l)(u) = \Lambda_G(l) \circ \text{tr}_{G(X, f)}(u)
\]
for all \( u \in U^Z_G(X, f) \). So the trace map \( \text{tr}_{G(X, f)} \) is a natural transformation of functors from \( \text{End}(G\text{-CW}_{fp}) \) to \( \text{Ab} \).

It remains to show compatibility with the induction structure. Given an inclusion \( \alpha : G \to K \), the functor \( \Pi(\text{ind}_\alpha) : \Pi(G, X) \to \Pi(K, \text{ind}_\alpha X) \) induces a group homomorphism \( \alpha_* : U^Z_G(X, f) \to U^Z_K(\text{ind}_\alpha X, \text{ind}_\alpha f) \). Let \( x : G/H \to X \) be given with \( X^H(f(x)) = X^H(x) \). We set \( W_GH := N_GH/H \).
and $W_K H = N_K H / H$. The map $\alpha : G \to K$ is injective, and so $W_G H \to W_K H$ is injective. If an element of $W_G H$ fixes the component $X^H(x)$, then its image fixes $(\text{ind}_a) X^H((\text{ind}_a) x)$. This implies that $\alpha|_{N_G H} : N_G H \to N_K H$ is injective. So $\Pi(\text{ind}_a)|_W : W_G H_x \to W_K H_{\text{ind}_a(x)}$, induced by $\Pi(\text{ind}_a)|_{\text{Aut}(x)}$, is also injective.

We apply assertion 4 of Lemma 5.5 to obtain $\text{tr}_Z\alpha u = \alpha_0 \text{tr}_G u$ for $u \in \phi_G \text{-end}_{\text{fp}} RG$, where the map $\alpha|_{\pi_1(X^H(x), x)} \phi_G^*$ induces the homomorphism $\alpha^* : \mathbb{Z}\pi_1(X^H(x), x) \phi_G^* \to \mathbb{Z}\pi_1((\text{ind}_a) X^H((\text{ind}_a) x), (\text{ind}_a) x) \phi_K^*$. Since $\Lambda(\alpha)_x$ is defined to be $\alpha_0^*$, these combine to form the map $\alpha^* : \Lambda_G(X, f) \to \Lambda_K((\text{ind}_a) X, (\text{ind}_a) f)$ such that the desired equation $\text{tr}_K((\text{ind}_a) X, (\text{ind}_a) f) \alpha^* = \alpha^* \text{tr}_G(X, f)$ holds on $U^Z_G(X, f)$.

Now we define the invariant which contains the fixed point information we are interested in.

**Definition 5.13.** Let $G$ be a discrete group, let $X$ be a finite proper $G$-CW-complex, and let $f : X \to X$ be a $G$-equivariant cellular endomorphism. We define the **generalized equivariant Lefschetz invariant** of $f$ by

$$\lambda_G(f) := \text{tr}_G(X, f)(u^Z_G(X, f)) \in \Lambda_G(X, f).$$

By Proposition 5.12, the collection of the $\text{tr}_G(X, f)$ is a natural transformation from $(U^Z, u^Z)$ to $(\Lambda, \lambda)$. The pair $(\Lambda, \lambda)$ inherits all structure from $(U^Z, u^Z)$: It is also a functorial equivariant Lefschetz invariant on the family of categories $G$-CW$_{\text{fp}}$ for discrete groups $G$.

**Theorem 5.14.** The pair $(\Lambda, \lambda)$ is a functorial equivariant Lefschetz invariant on the family of categories $G$-CW$_{\text{fp}}$ for discrete groups $G$.

**Proof.** The natural transformation $\text{tr}_G(X, f) : U^Z_G(X, f) \to \Lambda_G(X, f)$ maps $u^Z_G(X, f)$ to $\lambda_G(X, f)$. So $(\Lambda_G(X, f), \lambda_G(X, f))$ is a functorial equivariant Lefschetz invariant.

It therefore has all properties stated in Definition 2.3. Since we can define the universal functorial equivariant Lefschetz invariant for any $G$-equivariant continuous map $f : X \to X$, the same is true for the generalized equivariant Lefschetz invariant.

We can describe the invariant $\lambda_G(f)$ in a more concrete way. We see that $\text{tr}_G(X, f)(u^Z_G(X, f)) = \sum_{x \in H(G, X), f^H(x) = x^H(x)} L^Z_{\text{Aut}(x)}(f^H(x), f^G(x))$. We can use $\mathbb{Z}$ because $\text{Aut}(x)$ operates freely on $X^H(x) \setminus X^{H}(x)$.

**Remark 5.15.** One can obtain any functorial equivariant Lefschetz invariant by applying a suitable natural transformation to $(U^Z_G, u^Z_G)$. For example, the equivariant analog of the Lefschetz number is the equivariant Lefschetz class [LR03, Definition 3.6]. The natural transformation mapping the universal functorial equivariant Lefschetz invariant to the equivariant Lefschetz class is given by the trace map $\text{tr}_G(X, f)$ followed by an augmentation map $s_{G(X, f)}$ induced by the projection $\pi_1(X^H(x), x) \phi_x^* \to \{1\}$, just like in the non-equivariant case.
6. The Refined Equivariant Lefschetz Fixed Point Theorem

In this section, we introduce the generalized local equivariant Lefschetz class $\lambda^G_{loc}(f)$ in terms of fixed point data. It is a refinement of the local equivariant Lefschetz class [LR03, Definition 4.6]. We prove Theorem 0.2 which states that $\lambda_G(f) = \lambda^G_{loc}(f)$ under quite general conditions. So $\lambda^G_{loc}(f)$ gives a concrete geometric description of the fixed point information contained in $\lambda_G(f)$.

We briefly assemble the necessary notation. For a space $X$ with action of a discrete group $G$, we set $U^G(X) := \bigoplus_{[H] \in \text{Irr}(G,X)} \mathbb{Z}$. Let $K$ be a finite group. The Burnside ring $A(K)$ of $K$ is defined to be the Grothendieck ring of finite $K$-sets $S$ with the additive structure induced by disjoint union and the multiplicative structure induced by the Cartesian product. Additively, $A(K) = U^K(\text{pt}) := \bigoplus_{(H) \in \text{consub}(K)} \mathbb{Z}$.

Let $Z$ be a finite $K$-CW-complex and let $\psi : Z \to Z$ be a $K$-equivariant endomorphism. Then the equivariant Lefschetz class with values in the Burnside ring of $\psi$ is defined to be

$$
\Lambda^K_0(\psi) := \sum_{(H) \in \text{consub}(K)} L^{ZKH}(\psi^H, \psi^H) \cdot [K/H] \in A(K) = U^K(*).
$$

We call the injective ring homomorphism

$$
\text{ch}^0_K : A(K) \to \bigoplus_{(H) \in \text{consub}(K)} \mathbb{Z}, \, S \mapsto \{|S^H|\}_{(H) \in \text{consub}(K)}
$$

the character map. Let $V$ be a finite-dimensional $K$-representation and let $\psi : V^c \to V^c$ be a $K$-endomorphism of the one-point compactification $V^c$. Define the equivariant degree of $\psi$ to be

$$
\text{Deg}^K_0(\psi) := (\Lambda^K_0(\psi) - 1) (\Lambda^K_0(\text{id}_{V^c}) - 1) \in A(K) = U^K(*).
$$

Let $G$ be a discrete group, $X$ a $G$-space and $f : X \to X$ an equivariant endomorphism. Let $x \in X$ be a fixed point of $f$. We define

$$
\Lambda_G(x, f) : U^G(G/G_x) \to \Lambda_G(X, f)
$$

$$
1 \cdot [\tau : G/L \to G/G_x] \mapsto \overline{\text{tr}}_{x o \tau} \cdot [x o \tau : G/L \to X],
$$

with $\overline{\text{tr}}_{x o \tau} \in \mathbb{Z} \pi_1(X^L(x o \tau), x o \tau)_\partial_{x o \tau, \text{ext}}$. Choosing $z \cong x o \tau : G/L \to X$ as a fixed representative of the isomorphism class, this is

$$
\Lambda_G(x, f) : U^G(G/G_x) \to \Lambda_G(X, f), 1 \cdot [\tau : G/L \to G/G_x] \mapsto \overline{\text{tr}}_{x o \tau} \cdot [z],
$$

where $\overline{\text{tr}}_{x o \tau} = (\sigma, [t])_w_\text{ext}^{\ast}(\overline{\text{tr}}_{x o \tau}) = v(f(t^{-1})\sigma^{\ast}(\overline{\text{tr}}_{x o \tau})t = v(f(t^{-1})t$ for an isomorphism $(\sigma, [t]) \in \text{Mor}(z, x o \tau)$ and $w = (\text{id}, [v]) \in \text{Mor}(f(z), z)$.

**Definition 6.1.** Let $G$ be a discrete group, let $M$ be a cocompact smooth proper $G$-manifold and let $f : M \to M$ be a $G$-equivariant endomorphism such that $\text{Fix}(f) \cap \partial M = \emptyset$ and such that for every $x \in \text{Fix}(f)$ the determinant of the map $(\text{id}_{T_x M} - T_x f)$ is different from zero. Then $G \setminus \text{Fix}(f)$ is finite. We define the generalized local equivariant Lefschetz class to be

$$
\lambda^G_{loc}(f) := \sum_{Gx \in G \setminus \text{Fix}(f)} \Lambda_G(x, f) \circ \text{ind}_{Gx \leq G} \left( \text{Deg}^G_{0x} \left((\text{id}_{T_x M} - T_x f)^{\ast}\right) \right).
$$
Definition 6.2. We define the character map \( \text{ch}_G(X,f) \) by

\[
\Lambda_G(X,f) \to \bigoplus_{\overline{y} \in \Pi(G,X)} \mathbb{Q} \pi_1(X^K(y), y)_{\phi'_y,w}
\]

\[
\left( \sum_{\overline{y}} n_{\overline{y}} \cdot \overline{\kappa} \right) \cdot \overline{\tau} \to \sum_{y} \sum_{\text{Aut}(y)} \left| \left( \text{Aut}(y) \right)_{(\sigma, [t])} \right|^{-1}(\sigma, [t]) \left( \sum_{\overline{y}} n_{\overline{y}} \cdot \overline{\kappa} \right).
\]

This character map is a generalization of [LR03, Definition 5.1].

Lemma 6.3. The character map \( \text{ch}_G(X,f) \) is injective.

Proof. Let \( u = \sum_{i=1}^{k} a_i \cdot \overline{x_i} \in \Lambda_G(X,f) \), where \( a_i \in \Lambda_G(X,f)_{x_i} \), with \( \text{ch}_G(X,f)(u) = 0 \). Let the \( x_i \) be ordered in accordance with the partial ordering on \( \Pi(G,X) \) given by \( \overline{x} \leq \overline{y} \iff \text{Mor}(x,y) \neq 0 \), so \( x_i \leq x_j \Rightarrow i \leq j \). Suppose without loss of generality that \( a_k \neq 0 \). If \( \text{ch}_G(X,f)(\overline{x_i} \overline{x}) \neq 0 \), then \( \text{Mor}(x_k, x_i) \neq 0 \), which implies \( x_k \leq x_i \) and thus \( x_k = x_i \). Since \( \text{ch}_G(X,f)(a_k \cdot \overline{x_k} \overline{\kappa}) = a_k \), we obtain \( 0 = \text{ch}_G(X,f)(u) = a_k \), a contradiction. \( \square \)

We now calculate the value of \( \text{ch}_G(X,f) \) on the generalized equivariant Lefschetz invariant \( \lambda_G(f) \) and on the generalized local Lefschetz class \( \lambda^{loc}_G(f) \), in analogy to [LR03, Lemma 5.4 and Lemma 5.9]. Using Theorem 6.6, these values will turn out to be equal, proving Theorem 0.2.

Lemma 6.4. Let \( f: X \to X \) be a \( G \)-equivariant endomorphism of a finite proper \( G \)-CW-complex \( X \). Let \( \overline{y} \) be an isomorphism class of objects \( y: G/K \to X \) in \( \Pi(G,X) \). Then

\[
\text{ch}_G(X,f)(\lambda_G(f))_{\overline{y}} = L^\text{Aut}(y)\left( f^K(\overline{y}) \right).
\]

Proof. We first consider the case \( X^K(f(y)) = X^K(y) \). We write the \( p \)-skeleton \( X_p \) as a pushout and call \( x_{p,i}: G/H_i \to X \) for \( 0 \leq i \leq n_p \) the centers of the equivariant \( p \)-cells.

The \( G \)-CW-structure on \( X \) induces an \( \text{Aut}(y) \)-CW-structure on \( f^K(\overline{y}) \). We obtain a pushout diagram of \( \text{Aut}(y) \)-spaces

\[
\begin{array}{ccc}
\prod_{i=1}^{n_p} \text{Mor}(y, x_{p,i}) \times S^{p-1} & \to & X^K(y)_{p-1} \\
\downarrow & & \downarrow \\
\prod_{i=1}^{n_p} \text{Mor}(y, x_{p,i}) \times D^p & \to & X^K(y)_p
\end{array}
\]

(If \( p-1 \leq 1 \), we use the cover corresponding to \( \pi_1(X^K(y),y) \).) We denote the \( p \)-cells of \( X^K(y) \) by \( e_{(\sigma, [t]),p,i} := (\sigma, [t]) \times \tilde{D}^p \), where \( (\sigma, [t]) \in \text{Mor}(y, x_{p,i}) \). The \( \text{Aut}(y) \)-orbit of the cell \( e_{(\sigma, [t]),p,i} \) corresponds to the \( \text{Aut}(y) \)-orbit of \( (\sigma, [t]) \), so we conclude from Lemma 5.9 that

\[
L^\text{Aut}(y)\left( f^K(\overline{y}) \right) = \sum_{p \geq 0} (-1)^p \sum_{i=1}^{n_p} \sum_{\text{Aut}(y)(\sigma, [t]) \in \text{Aut}(y)\backslash \text{Mor}(y, x_{p,i})} \left| \text{Aut}(y)(\sigma, [t]) \right|^{-1} \cdot \text{inc}_{\phi_y,w}(f^K(\overline{y}), e_{(\sigma, [t]),p,i}).
\]
Analogously, we have for any $x: G/H \to X$ a pushout diagram

\[
\begin{array}{ccc}
\prod_{i=1}^{n_p} \text{Mor}(x, x_{p,i}) \times S^{p-1} & \xrightarrow{\sim} & X^H(x)_{p-1} \cup X^>H(x) \\
\downarrow & & \downarrow \\
\prod_{i=1}^{n_p} \text{Mor}(x, x_{p,i}) \times D^p & \xrightarrow{\sim} & X^H(x)_p \cup X^>H(x).
\end{array}
\]

Lemma 5.9 yields

\[
L_{\text{Aut}(x)}(f^H(x), f^>H(x)) = \sum_{p \geq 0} (-1)^p \sum_{i=1}^{n_p} (\tau, [s])^* \text{inc}_{\phi_{x_{p_i}, e}}(f^H(x_{p_i}), e_{p_i}),
\]

for $(\tau, [s]) \in \text{Mor}(x, x_{p,i})$ any morphism. Inserting this formula into the definition of $\text{ch}_G(X, f)(\lambda_G(f))_{\overline{r}}$ proves the claim.

Now we consider the case that $X^K(f(y)) \neq X^K(y)$. This implies that $X^H(f(x)) \neq X^H(x)$ for all $\overline{r}$ with $\text{Mor}(y, x) \neq \emptyset$, so $\lambda_G(f)_{\overline{r}} = 0$ for all $\overline{r}$ with $\text{Mor}(y, x) \neq \emptyset$. Therefore $\text{ch}_G(X, f)(\lambda_G(f))_{\overline{r}} = 0$. \hfill \Box

**Lemma 6.5.** Let $G$ be a discrete group and let $M$ be a cocompact smooth proper $G$-manifold. Let $f: M \to M$ be a smooth $G$-equivariant map. Suppose that $\text{Fix}(f) \cap \partial M = \emptyset$ and that for any $x \in \text{Fix}(f)$ the determinant $\det(\text{id}_{T_xM} - T_xf)$ is different from zero. Then the set $G \setminus \text{Fix}(f)$ is finite. Let $\overline{y}: G/K \to M$ be an object in $\Pi(G, M)$. Then the set $WK_y \setminus \text{Fix}(f|_{M^K(y)})$ is finite and we get

\[
\text{ch}_G(M, f)(\lambda_G^\text{loc}(f))_{\overline{r}} = \sum_{\substack{WK_y, x \in WK_y \setminus \text{Fix}(f|_{M^K(y)})}} |(WK_y)_x|^{-1} \text{deg}\left( (\text{id}_{T_xM^K(y)} - T_x(f|_{M^K(y)}))^c \right) \cdot \overline{a}_x,
\]

where $\overline{a}_x = uf(t^{-1})t \in \pi_1(X^K(y), y)_{\overline{r}}'$ for $(\sigma, [t]) \in \text{Mor}(y, x)$ and $w = (\text{id}, [r]) \in \text{Mor}(f(y), y)$.

**Proof.** The set $G \setminus \text{Fix}(f)$ is finite since $M$ is cocompact and the fixed points are isolated. Analogously, $G \setminus GM^K(y) = WK_y \setminus M^K(y)$ is compact with isolated fixed points, so $WK_y \setminus \text{Fix}(f|_{M^K(y)})$ is finite. Let $x: G/G_x \to M$ be a fixed point of $f$.

We first show that for each $u \in U^{G_x}(\ast)$ we have

\[
\text{ch}_G(M, f)_{\overline{r}} \Lambda_G(x, f) \text{ind}_{G_x \leq G}(u) = \sum_{\substack{\text{Aut}(y)(\sigma, [t]) \in \text{Aut}(y) \setminus \text{Mor}(y, x)}} |\text{Aut}(y)(\sigma, [t])|^{-1} (\sigma, [t])^* \overline{t}_x \text{ch}_G^u((K_x))
\]

where $\overline{t}_x \in \mathbb{Z} \pi_1(X^{G_x}(x), x)_{\phi_{x, e}}$. Here $(K_x) = (g^{-1}_\sigma Kg_\sigma) \in \text{consub}(G_x)$ for $\sigma: G/K \to G/G_x$, $gK \mapsto gg_\sigma G_x$. Let $u = [G_x/L] \in U^{G_x}(\ast)$ be a basis element and $\text{pr}: G/L \to G/G_x$ the projection. Then

\[
\text{ch}_G(M, f)_{\overline{r}} \Lambda_G(x, f) \text{ind}_{G_x \leq G}(G_x/L) = \sum_{\substack{\text{Aut}(y)(\tau, [t]) \in \text{Aut}(y) \setminus \text{Mor}(y, x)_{\text{pr}}}} |\text{Aut}(y)(\tau, [t])|^{-1} (\tau, [t])^* \overline{t}_{x_{\text{pr}}}
\]

where $\overline{t}_{x_{\text{pr}}} \in \mathbb{Z} \pi_1(X^L(x \circ \text{pr}), x \circ \text{pr})_{\phi_{x_{\text{pr}}, e}}$. 
Defining \( q : \text{Mor}(y, x \circ \text{pr}) \to \text{Mor}(y, x) \), \((\tau, [t]) \mapsto (\text{pr} \circ \tau, [t])\) we have
\[
\text{Mor}(y, x \circ \text{pr}) = \coprod_{(\sigma, [t]) \in \text{Mor}(y, x)} q^{-1}(\sigma, [t]) = \prod_{\text{Aut}(y) \setminus \text{Mor}(y, x)} \text{Aut}(y) \times_{\text{Aut}(y) \setminus \text{Mor}(y, x)} q^{-1}(\sigma, [t]).
\]

The \( \text{Aut}(y) \setminus \text{Mor}(y, x) \)-set \( q^{-1}(\sigma, [t]) = \coprod_{i \in I(\sigma, [t])} \text{Aut}(y) / A_i \) is a finite disjoint union of orbits, thus we have a bijection of \( \text{Aut}(y) \)-sets
\[
\text{Mor}(y, x \circ \text{pr}) = \prod_{i \in I(\sigma, [t])} \text{Aut}(y) / A_i.
\]

We know for \((\tau, [t]) \in \text{Mor}(y, x \circ \text{pr})\) that \((\tau, [t])^{\ast} T_{x,y} = (\text{pr} \circ \tau, [t])^{\ast} T_{x,y} \). An orbit \( \text{Aut}(y) / A_i \) corresponds to exactly one orbit \( \text{Aut}(y) \cdot (\tau, [t]) \), where \( i \in I(\tau, [t]) \), so \( |\text{Aut}(y) \cdot (\tau, [t])| = |A_i| \), hence
\[
\sum_{i \in I(\tau, [t])} |\text{Aut}(y) \cdot (\tau, [t])|^{-1} (\tau, [t])^{\ast} T_{x,y} = \sum_{i \in I(\tau, [t])} |A_i|^{-1} (\tau, [t])^{\ast} T_{x,y}.
\]

We have \( |q^{-1}(\sigma, [t])| = |\text{Aut}(y) \setminus \text{Mor}(y, x)| \cdot \sum_{i \in I(\sigma, [t])} |A_i|^{-1} \). Since \( q \) does not change the \([t]\)-part, as in [LR03, Lemma 5.9, Equation 5.14] we have \( |q^{-1}(\sigma, [t])| = |G_{x/L}|^{1/\deg(K_{G_{x/L}})} \).

Inserting these equations into the above formula, we obtain the desired equation for all \( [G_{x/L}] \in U^{G_{x}}(\ast) \), thus for all \( u \in U^{G_{x}}(\ast) \).

We know [LR03, Equations 5.16 and 5.17]
\[
\text{deg'} \left( \text{Deg}_{G_{x}} \left( (\text{id}_{T_{x}M} - T_{x}f)^{\circ} \right) \right) = \text{deg'} \left( (\text{id}_{T_{x}M} - T_{x}f)^{\circ} \right).
\]

We have \( \coprod_{G_{x}\text{Fix}(f) / \text{Mor}(y, x)} \text{WK}_{y} \setminus \text{Fix}(f) \). Under this bijection, \( \text{Aut}(y) \cdot (\sigma, [t]) \mapsto \text{WK}_{y} \cdot t(0) \), where \( t(0) = x \circ \sigma(1K) = x(g_{a}G_{x}) = g_{a}x(1G_{x}) = g_{a}x \) for \( \sigma : G/K \to G/G_{x}, gK \mapsto g_{a}G_{x} \). Since \( |\text{Aut}(y) \setminus \text{Mor}(y, x)| \) is \( |\text{WK}_{y} \cdot g_{a}x| \), inserting the above results into the formula for \( \text{deg'} \) yields the claim.

The final ingredient in the proof of the Theorem 0.2 is a refinement of the orbifold Lefschetz fixed point theorem [LR03, Theorem 2.1].

**Theorem 6.6.** Let \( G \) be a discrete group extension \( 1 \to \pi \to G \to W \to 1 \) with endomorphism \( \phi : G \to G \) such that \( \phi_{W} = \text{id}_{W} \). Let \( M \) be a connected simply connected cocompact proper \( G \)-manifold such that \( \pi \) operates freely on \( M \), and let \( f : M \to M \) be a smooth \( \phi \)-twisted map. Denote by \( \overline{f} : \overline{M} \to \overline{M} \) the \( W \)-equivariant map induced on the manifold \( \overline{M} := \pi \setminus M \) by dividing out the \( \pi \)-action. Suppose that \( \text{Fix}(\overline{f}) \cap \partial \overline{M} = \emptyset \) and that for every \( x \in \text{Fix}(\overline{f}) \) the determinant of the map \( (\text{id}_{T_{x}\overline{M}} - T_{x}\overline{f})^{c} \) is different from zero. Then \( W \setminus \text{Fix}(\overline{f}) \) is finite, and
\[
L^{QG}(\overline{f}) = \sum_{W \cdot x \in W \setminus \text{Fix}(\overline{f})} |W_{x}|^{-1} \cdot \text{deg'}((\text{id}_{T_{x}\overline{M}} - T_{x}\overline{f})^{c} \circ \overline{\alpha}_{x}^{c}).
\]

Here \( (\text{id}_{T_{x}\overline{M}} - T_{x}\overline{f})^{c} : (T_{x}\overline{M})^{c} \to (T_{x}\overline{M})^{c} \) is an endomorphism of the one-point compactification of \( T_{x}\overline{M} \).
The proof is analogous to the proof of the orbifold Lefschetz fixed point theorem [LR03, Theorem 2.1]. We apply the construction from that proof to the W-equivariant map $\overline{f}: \overline{M} \to \overline{M}$. We obtain a W-equivariant map $\overline{f}'$ such that $\text{Fix}(\overline{f}) = \text{Fix}(\overline{f}')$, the map $\overline{f}'$ is of the desired form around the fixed points and agrees with $\overline{f}$ outside a neighborhood of the fixed points. The desired form is that $\exp_{x,\varepsilon}^{-1} \circ \overline{f} \circ \exp_{x,\varepsilon}$ and $\overline{f}'$ agree on $D_\varepsilon T_x \overline{M}$ for some $\varepsilon > 0$ and for all $x \in \text{Fix}(\overline{f})$. Here $\exp_{x,\varepsilon}: D_\varepsilon T_x \overline{M} \sim N_{x,\varepsilon}$ denotes the exponential map.

We lift this construction to $M$ and the $\phi$-twisted endomorphism $f: M \to M$ by extending $\phi$-twisted $G$-equivariantly: Let $z$ be a lift of a fixed point $x$, then $f(z) = \alpha_z \cdot z$ with $\alpha_z \in \pi$, and we set $f'|_{U_z} := \alpha_z \cdot \phi \overline{f'} \phi^{-1}$ on a neighborhood of $z$ that is isomorphic to a neighborhood of $x$ via the isomorphism $\varphi: U_z \sim U_x$ coming from the covering map. Around another lift $\beta \cdot z$, we set $f'|_{U_{\beta z}} := \phi(\beta) \cdot f'|_{U_z} \cdot \beta^{-1}$ on a neighborhood of $\beta z$. (Note that $\alpha_{\beta z} = \phi(\beta) \cdot \alpha_z \cdot \beta^{-1}$.) We obtain a $\phi$-twisted map $f': M \to M$ such that $\text{Fix}(f) = \text{Fix}(f')$, the map $f'$ is of the desired form around the orbits of the fixed points, and it agrees with $f$ outside a neighborhood of the orbits of the fixed points.

Analogously, we lift further constructions such as the $W$-equivariant triangulations $K'(\overline{M})$, $K''(\overline{M})$ and the $W$-homotopy $\overline{h}$ from $\overline{f}$ to $\overline{h}$. The construction of $K''(\overline{M})$ can be done such that there is at most one fixed point $x$ of $\overline{h}$ in each $\overline{e} \in K''(\overline{M})$. We denote $\overline{h}$ again by $\overline{f}$. Then

$$\text{inc}_\phi(f, e) = \begin{cases} \text{inc}(\overline{f}, \overline{e}) \cdot \overline{\alpha_x} & \text{if there is a fixed point } x \in \overline{e} \\ 0 & \text{else.} \end{cases}$$

Here for a basepoint $y$ of $\overline{M}$, a path $v$ from $y$ to $\overline{f}(y)$ and a morphism $(\sigma, [t]): y \to x$ from $y$ to $x$ we set $\overline{\alpha_x} := v f(\overline{f}^{-1}) \overline{t}$. The direction of $v$ corresponds to our usual convention that $w = (\text{id}, [v]) \in \text{Mor}(\overline{f}(y), y)$. Note that $\overline{\alpha_x}$ does not depend on the choice of $(\sigma, [t])$ because of Lemma 5.2. We see that $|G_e| = |W_\overline{e}|$ because $\pi$ operates freely. So using the construction of [LR03, proof of Theorem 2.1] applied to the $W$-equivariant map $\overline{f}$, where $\{x_1, \ldots, x_k\}$ is a complete set of representatives of $W$-orbits of fixed points of $\overline{f}$, we obtain

$$L^QG(f) = L^QG(h_1) = \sum_{p \geq 0} (-1)^p \sum_{G \in G \setminus \text{flip}(K''(\overline{M}))} G_e |^{-1} \text{inc}_\phi(f, e)$$

$$= \sum_{p \geq 0} (-1)^p \sum_{W \in W \setminus \text{flip}(K''(\overline{M}))} |W_\overline{e}|^{-1} \text{inc}(\overline{f}, \overline{e}) \cdot \overline{\alpha_x}$$

$$= \sum_{i=1}^k |W_{x_i}|^{-1} \deg((\text{id} - T_{x_i} \overline{f})^c) \cdot \overline{\alpha_x} = \sum_{W \cdot x \in W \setminus \text{Fix}(\overline{f})} |W_x|^{-1} \cdot \deg((\text{id} - T_x \overline{f})^c) \cdot \overline{\alpha_x}. \quad \square$$

We have assembled all information necessary for the proof of the refined equivariant Lefschetz fixed point theorem.

**Theorem 0.2.** Let $G$ be a discrete group, let $M$ be a cocompact proper smooth $G$-manifold and let $f: M \to M$ be a $G$-equivariant endomorphism
such that $\text{Fix}(f) \cap \partial M = \emptyset$ and such that for every $x \in \text{Fix}(f)$ the determinant of the map $(\text{id}_{T_x M} - T_x f)$ is different from zero. Thus

$$\lambda_G(f) = \lambda_G^{\text{loc}}(f).$$

**Proof.** Using Lemma 6.4, Theorem 6.6 and Lemma 6.5 we have

$$\chi_G(M, f)(\lambda_G(f))_{\mathfrak{p}}
= L_{\mathbb{Q}} \mathbb{A} \mathfrak{t}(y) \left(fK(y)\right)
= \sum_{W_K y, x \in \text{Fix}(f)} |(W_K y)_x|^{-1} \deg \left((\text{id}_{T_x M} M K(y) - T_x f(M K(y)))\right) \cdot \alpha_x
= \chi_G(M, f)(\lambda_G^{\text{loc}}(f))_{\mathfrak{p}}$$

for all $\mathfrak{p} \in \text{Is}(G, X)$. By injectivity of $\chi_G(M, f)$, shown in Lemma 6.3, we obtain the refined equivariant Lefschetz fixed point theorem. \hfill $\square$

From the fact that the units of the Burnside ring $A(K)$ are only $\{1, -1\}$ if $K$ is a finite group of odd order [tD79, Proposition 1.5.1], we obtain in analogy to [LR03, Example 4.7] the following example.

**Example 6.7.** Let $G$ be a discrete group and let $M$ be a cocompact proper smooth $G$-manifold. Suppose that all isotropy groups $G_x$ of points $x \in M$ are of odd order. Let $f: M \to M$ be a smooth $G$-equivariant map such that $\text{Fix}(f) \cap \partial M = \emptyset$ and such that for every $x \in \text{Fix}(f)$ the determinant of the map $(\text{id}_{T_x M} - T_x f)$ is different from zero. Then

$$\lambda_G^{\text{loc}}(f) = \sum_{G_x \in G \setminus \text{Fix}(f)} \frac{\det(1 - T_x f): T_x M \to T_x(M)}{\det(1 - T_x f): T_x M \to T_x(M)} \cdot \alpha_x,$$

where for an isomorphism $(\sigma, [t]) \in \text{Mor}(z, x)$ and a path $w = (\text{id}, [v]) \in \text{Mor}(f(z), z)$ we have $\alpha_x = v f(t^{-1}) t \in \mathbb{Z} \pi_1(X^{G_x}(z), z)_{\phi'}$.

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