On concordance indices for models with time-varying risk

A. Gandy
Department of Mathematics, Imperial College London, SW7 2AZ, U.K.
a.gandy@imperial.ac.uk

T. J. Matcham
Department of Mathematics, Imperial College London, SW7 2AZ, U.K.
NIHR ARC Northwest London, SW10 9NH, U.K.
thomas.matcham14@imperial.ac.uk

Abstract
Harrel’s concordance index is a commonly used discrimination metric for survival models, particularly for models where the relative ordering of the risk of individuals is time-independent, such as the proportional hazards model. There are several suggestions, but no consensus, on how it could be extended to models where risk varies over time, e.g. in case of crossing hazard rates. We show that, in the limit, concordance is maximized if and only if the risk score is concordant with the hazard rate, in the sense that for a comparable pair where the first event time is observed, the risk score is concordant with the hazard rate at this first event time. Thus, we suggest using the hazard rate as the risk score when calculating concordance. Through simulations, we demonstrate situations in which other concordance indices can lead to incorrect models being selected over a true model, justifying the use of our suggested risk prediction in both model selection and in loss functions in, e.g., machine learning models.

1 Introduction
Accurate patient prognosis estimation is an important clinical tool with applications including advising patients of their likely disease outcomes, informed selection of patient treatment as well as the design and evaluation of clinical trials. There exist several approaches to quantifying the predictive accuracy of survival models (Harrell Jr et al., 1996), which can in turn be optimized for, in order to improve a given aspect of the predictions. Discrimination metrics focus on a model’s ability to correctly order the predictions of the patient outcomes. This could be important, for example, in deciding the order in which a set of patients should be treated.

The most significant metric of survival model discrimination is Harrel’s concordance index (Harrell et al., 1984), hereafter the C-index, which was first developed as an adaptation of the Kendall-Goodman-Kruskal-Somers type rank correlation index (Goodman and Kruskal, 1979) to right-censored survival data, similar to an adaptation of Kendall’s $\tau$ by Brown Jr et al. (1973) and Schemper (1984).

We use the following setup. Let $(X_i, U_i, Z_i), i \in \mathbb{N}$, be independent and identically distributed with the lifetime $X_i$ and the right censoring time $U_i$ being non-negative random variables. Let the covariate $Z_i$ be an element of some space $Z$. We observe $(T_i, D_i, Z_i), i = 1, \ldots, n$, where $T_i = \min(X_i, U_i)$ is the time at risk and $D_i = 1(X_i \leq U_i)$ is the event indicator.

The C-index estimates the probability that the predicted risk scores of a pair of individuals is concordant with that of their observed survival times. Only for pairs of individuals $(i, j)$, with $i \neq j$,
we use

\[
\pi_{\text{comp}} = P(D_i = 1, T_i < T_j).
\]

Survival predictions are differentiated with functions of the covariate called risk scores. In situations where the relative risk of individual is not changing over time, e.g., in a proportional hazards model with only time-constant covariates, this is sufficient to discriminate between individuals. However, the risk score should arguably be time-dependent in situations where the relative risk of individuals changes over time, e.g., in cases where hazards of risk groups cross (Mantel and Stablein, 1988), where a proportional hazards model has time-dependent covariates, or where the risk prediction is individual over time as in machine learning approaches to survival models (Lee et al., 2018).

Thus, the risk score we use is allowed to depend on the covariate and on time. Specifically, in a paired comparison, we compare the risk scores at the time when the first event occurs. Intuitively, this comparison gives a prediction of who was most at imminent risk of the event, given that they have survived until the first event time. This framework covers previous specific suggestions for dealing with time-varying risks (Antolini et al., 2005; Blanche et al., 2019; Haider et al., 2020). Formally, the risk score is specified through a function \( q : [0, \infty) \times \mathbb{Z} \rightarrow \mathbb{R} \) and for a given pair \((i, j)\), with \(T_i < T_j\), we say that \( i \) has a higher risk score than \( j \) if \( q(T_i | Z_i) > q(T_j | Z_j) \). Higher values of the risk score indicate a propensity towards earlier events.

For a pair \((i, j)\), where we observe that \( i \) has occurred before \( j \), i.e., \( D_i = 1, T_i < T_j \), we say that this pair is concordant if \( q(T_i | Z_i) > q(T_j | Z_j) \). The probability of a pair having observed the event of \( i \) before the event of \( j \) and being concordant is

\[
P[D_i = 1, T_i < T_j, q(T_i | Z_i) > q(T_j | Z_j)].
\]

Defining a concordance index as \( \left(1 \right) \) divided by \( \pi_{\text{comp}} \), would imply the following: a perfect model that could correctly order every pair would have a concordance of 1, a model that simply guesses for each pair would have a concordance of 0.5 on average, and a model that always orders incorrectly would have a concordance of 0. A model that gives the same prediction for each individual would also only get a concordance of 0, which seems undesirable.

To avoid the latter, tied risk scores are often rewarded with a score of 0.5, such that a model with the same risk score for everyone would still score 0.5 (Harrell Jr et al., 1996). Hence, in the C-index

\[
C_q = \frac{\pi_{\text{conc}}}{\pi_{\text{comp}}}
\]

we use

\[
\pi_{\text{conc}} = P[D_i = 1, T_i < T_j, q(T_i | Z_i) > q(T_j | Z_j)] + \frac{1}{2} P[D_i = 1, T_i < T_j, q(T_i | Z_i) = q(T_j | Z_j)].
\]

Given a random sample \((T_i, D_i, Z_i)_{i=1}^n\) we can estimate \( \pi_{\text{conc}} \) and \( \pi_{\text{comp}} \) with:

\[
\hat{\pi}_{\text{conc}} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i} \{I[D_i = 1, T_i < T_j, q(T_i | Z_i) = q(T_j | Z_j)]
\]

\[
+ \frac{1}{2} I[D_i = 1, T_i < T_j, q(T_i | Z_i) > q(T_j | Z_j)]
\]

\[
\hat{\pi}_{\text{comp}} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j=1, j \neq i} I[D_i = 1, T_i < T_j]
\]

and thus estimate the C-index \( C_q^\alpha \) by

\[
C_q^\alpha = \frac{\hat{\pi}_{\text{conc}}}{\hat{\pi}_{\text{comp}}}.
\]

Often, the risk score \( q(t | z) \) being used is not dependent on the first argument \( t \). For example, if a proportional hazards model with covariates \( z \) is used, then often the linear predictor \( q(t | z) = z \hat{\beta} \) is used as risk score, where \( \hat{\beta} \) is an estimate of the regression coefficient.
For more general survival models, where we have access to a survival function $S(t|Z)$ as a function of the covariates $z$, a definition of a risk score is less obvious, as there may not be a clear definition of what constitutes higher risk, for example when the underlying hazard rates of individuals cross. Several methods of computing risk scores in this setting have been considered, for example $q(t|Z) = -S(t_0|Z)$, the negative of the survival function evaluated at some fixed time $t_0 > 0$ (Blanche et al., 2019), or $q(t|Z) = -\inf\{t \text{ s.t. } S(t|Z) \leq 0.5\}$, the negative of the median survival time (Haider et al., 2020). The negative is taken as predicted survival times have the opposite ordering to risk scores. Again, these suggestions do not depend on the first argument of $q$.

In the work of Antolini et al. (2005), a time-dependent concordance index, $C^{td}$, is introduced. This adaptation of the C-index is developed for models with either time-varying covariates or time-varying effects, while supposing the predicted survival function is the 'natural' relative risk predictor. This leads to an event-time dependent risk score

$$q(t|z) = -S(t|z).$$

This index is used widely in deep learning survival models, wherein the survival curves for distinct individuals are prone to crossing (Zhong et al., 2021). A similar concordance index has seen use in loss functions for deep survival models (Lee et al., 2018).

In Section 2 we show through examples that these concordance indices do not always maximally reward correct models, and therefore could lead to selection of inferior predictive models.

The key contribution of this paper (Section 3) is that using the conditional hazard rate $\alpha(t|z)$ as a risk score for an individual with covariates $z$ does not suffer from this problem (as does any risk score that is in a certain sense concordant with the hazard rate). Thus we suggest using $q(t|z) = \alpha(t|z)$ as risk score in concordance rates.

## 2 Main Result

The following theorem shows that, the estimated concordance index $c^n_q$ converges in probability to the concordance $C_q$ for any risk score $q : [0, \infty) \times Z \rightarrow \mathbb{R}$ and that the concordance is maximised if and only if the risk score is concordant with the hazard rate.

We assume that $X_i$ and $U_i$ are independent given $Z_i$, i.e., $X_i \perp \perp U_i \mid Z_i$, that $X_i|Z_i$ has an absolutely continuous distribution, and that there exists $\alpha : [0, \infty) \times Z \rightarrow [0, \infty)$ such that the hazard rate of $X_i$ given $Z_i$ is $\alpha(t|Z_i)$. We also assume that $U_i \leq T$ for some $T \in \mathbb{R}$, i.e. that we have a finite observation window.

**THEOREM 1.** If $\pi_{comp} > 0$ then

$$c^n_q \xrightarrow{p} C_q \quad (n \rightarrow \infty).$$

Furthermore, the following equivalence holds:

$$\forall \tilde{q} : [0, \infty) \times Z \rightarrow \mathbb{R} : \quad C_q \geq C_{\tilde{q}}$$

if and only if

$$E \int_0^{\tau_{j,i}} \{q(s|Z_i) \geq q(s|Z_j), \alpha(s|Z_i) < \alpha(s|Z_j)\} + \{q(s|Z_i) \leq q(s|Z_j), \alpha(s|Z_i) > \alpha(s|Z_j)\} ds = 0. \quad (2)$$

Equation (2) is trivially satisfied if $q = \alpha$, which is why we suggest using the hazard rate as the risk score. More generally, (2) is satisfied if the risk score $q$ and the hazard rate $\alpha$ are concordant in the sense that $\forall s \in [0, \infty), z_1, z_2 \in Z : q(s|z_1) > q(s|z_2) \iff \alpha(s|z_1) > \alpha(s|z_2)$.

To show Theorem 1 we need to introduce some counting process notation.

$$N^{comp}_{ij}(t) = I(T_i \leq t, D_i = 1, T_i < T_j)$$

indicates if the event for $i$ is known to have occurred before the event for $j$ by time $t$. The counting process $N^{comp,1}_{ij}(t)$ indicates if additionally the risk scores are in line with $i$ occurring before $j$, i.e.,

$$N^{comp,1}_{ij}(t) = I[T_i \leq t, D_i = 1, T_i < T_j, q(T_i|Z_i) > q(T_i|Z_j)]$$
and \( N_{ij}^{\text{conc},2}(t) \) indicates if additionally the risk scores for \( i \) and \( j \) are tied, i.e.,
\[
N_{ij}^{\text{conc},2}(t) = \mathbb{I}[T_i \leq t, D_i = 1, T_i < T_j, q(T_i|Z_i) = q(T_j|Z_j)].
\]
\( N_{ij}^{\text{conc}}(t) \) adds these two together, with tied predictions instead contributing 1/2, i.e.,
\[
N_{ij}^{\text{conc}}(t) = N_{ij}^{\text{conc},1}(t) + \frac{1}{2} N_{ij}^{\text{conc},2}(t).
\]
Based on the above, we now define the concordance of \( n \) individuals using information up to time \( t \) as
\[
e_n^c(t) = \frac{\sum_{i=1}^n \sum_{j=1, j \neq i}^n N_{ij}^{\text{conc}}(t)}{\sum_{i=1}^n \sum_{j=1, j \neq i}^n N_{ij}^{\text{comp}}(t)}.
\]
We have \( e_n^c = e_n^c(T) \).

The following lemma derives the compensator of \( N_{ij}^{\text{conc}} \) with respect to the filtration \( (\mathcal{F}_t) \), where \( \mathcal{F}_t = \sigma(Z_i, \mathbb{I}[T_i \leq s], \mathbb{I}[T_i \leq s, D_i = 1], i \in \mathbb{N}, 0 \leq s \leq t \) is the information observed up to time \( t \).

**Lemma 1.** \( N_{ij}^{\text{conc}}(t) \) has a unique decomposition into a martingale \( M_{ij}^{\text{conc}}(t) \) and compensator
\[
\Lambda_{ij}^{\text{conc}}(t) = \int_0^t Y_{ij}(s)\{Q_{ij}^1(s) + \frac{1}{2} Q_{ij}^2(s)\alpha(s|Z_i)\} ds,
\]
where \( Y_{ij}(t) = \mathbb{I}(\tau_{ij} \geq t) \), \( \tau_{ij} = T_i \wedge T_j \), \( Q_{ij}^1(t) = \mathbb{I}[q^{\tau_{ij}}(t|Z_i) > q^{\tau_{ij}}(t|Z_j)], Q_{ij}^2(t) = \mathbb{I}[q^{\tau_{ij}}(t|Z_i) = q^{\tau_{ij}}(t|Z_j)], q^{\tau_{ij}}(t) \cdot \eta = q(t \wedge \tau_{ij}) \).

The proof of this lemma can be found in the Appendix.

**Proof of Theorem 1**
\( \hat{\pi}_{\text{conc}} \) can be written as a U-statistic
\[
\hat{\pi}_{\text{conc}} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n h[(T_i, D_i, Z_i), (T_j, D_j, Z_j)]
\]
with the kernel
\[
h[(T_i, D_i, Z_i), (T_j, D_j, Z_j)] = N_{ij}^{\text{conc}}(T) + N_{ji}^{\text{conc}}(T).
\]

The kernel \( h \) is bounded, implying \( Eh^2[(T_i, D_i, Z_i), (T_j, D_j, Z_j)] < \infty \), and thus Theorem 12.3 of van der Vaart(1998) shows that \( \hat{\pi}_{\text{conc}} \) is asymptotically normal as \( n \to \infty \) with mean \( \frac{1}{2} E[N_{ij}^{\text{conc}}(T) + N_{ji}^{\text{conc}}(T)] = \pi_{\text{conc}} \). Thus, we have \( \hat{\pi}_{\text{conc}} \overset{P}{\to} \pi_{\text{conc}} \) as \( n \to \infty \). Similarly, we can show \( \hat{\pi}_{\text{comp}} \overset{P}{\to} \pi_{\text{comp}} \) as \( n \to \infty \).

Hence, by the assumption \( \pi_{\text{comp}} > 0 \), we have \( e_n^c = \hat{\pi}_{\text{conc}} / \hat{\pi}_{\text{comp}} \overset{P}{\to} \pi_{\text{conc}} / \pi_{\text{comp}} = C_q \) as \( n \to \infty \).

Our choice of \( q \) has no influence on the denominator, so considering the numerator only we find that, using Lemma 1,
\[
2E[N_{ij}^{\text{conc}}(t)] = E[N_{ij}^{\text{conc}}(t) + N_{ji}^{\text{conc}}(t)] = E[M_{ij}^{\text{conc}}(t) + M_{ji}^{\text{conc}}(t)] + E[\Lambda_{ij}^{\text{conc}}(t) + \Lambda_{ji}^{\text{conc}}(t)]
\]
\[
= 0 + E[\Lambda_{ij}^{\text{conc}}(t) + \Lambda_{ji}^{\text{conc}}(t)] = E \int_0^t f_q(s) Y_{ij}(s) ds = E \int_0^{\tau_{ij}} f_q(s) ds,
\]
where
\[
f_q(s) = \alpha(s|Z_i)\mathbb{I}[q(s|Z_i) > q(s|Z_j)] + \alpha(s|Z_j)\mathbb{I}[q(s|Z_i) < q(s|Z_j)]
\]+\(0.5[\alpha(s|Z_i) + \alpha(s|Z_j)]\mathbb{I}[q(s|Z_i) = q(s|Z_j)].
\]

Let \( F_q = E \int_0^{\tau_{ij}} f_q(s) ds \) and let
\[
A_q(s) = \mathbb{I}[q(s|Z_i) \geq q(s|Z_j), \alpha(s|Z_i) < \alpha(s|Z_j)] + \mathbb{I}[q(s|Z_i) \leq q(s|Z_j), \alpha(s|Z_i) > \alpha(s|Z_j)].
\]

Then, for any \( q \),
\[
F_\alpha - F_q = E \int_0^{\tau_{ij}} [f_\alpha(s) - f_q(s)] ds = E \int_0^{\tau_{ij}} [f_\alpha(s) - f_q(s)] A_q(s) ds,
\]
as \( f_q(s) = f_\alpha(s) \) if \( A_q(s) = 0 \). The latter can be seen by going through the three cases \( \alpha(s|Z_i) > \alpha(s|Z_j), \alpha(s|Z_i) < \alpha(s|Z_j) \) and \( \alpha(s|Z_i) = \alpha(s|Z_j) \). Furthermore, \( A_q(s) = 1 \) implies \( f_\alpha(s) > f_q(s) \). Thus, \( F_\alpha \geq F_q \) and \( F_\alpha = F_q \) if and only if \( E \int_0^{\tau_{ij}} A_q(s) ds = 0 \).

We generated 100 different data sets and computed the resulting concordance indices as well as, for use four different risk scores to calculate concordance indices. Our suggestions of the hazard at

We now present an experiment to compare concordance indices produced by different risk scores. As anticipated by Theorem 1, the quantile survival time is denoted by $\tilde{C}_{\alpha}$.

The results in this paper are assuming continuous observations, implying that there are no ties in the observations. In practice, data is recorded to a finite resolution and thus tied observations occur with positive probability. Different approaches in handling of ties in both observation and prediction.

### Table 1: Left: Average concordance scores for each model/risk score. Right: Frequency of model selection by each risk score (all ties in selection counted for all models selected)

| Model | $M_0$ | $M_1$ | $M_2$ | $M_3$ | $C_{\alpha}$ | $C^{bd}$ | $C_{S(0.5)}$ | $C_{S(2.05)}$ | $C_{\mu(0.5)}$ | $C_{\mu(0.75)}$ |
|-------|-------|-------|-------|-------|-------------|----------|--------------|---------------|---------------|---------------|
| $C_{\alpha}$ | 0.57  | 0.57  | 0.55  | 0.53  | 98          | 98       | 2            | 2             | 0             | 0             |
| $C^{bd}$   | 0.53  | 0.57  | 0.57  | 0.52  | 0           | 50       | 50           | 0             | 100           | 100           |
| $C_{S(0.5)}$ | 0.52  | 0.52  | 0.52  | 0.52  | 100         | 100      | 100          | 100           | 100           | 100           |
| $C_{S(2.05)}$ | 0.48  | 0.48  | 0.48  | 0.52  | 4           | 4        | 4            | 4             | 96            | 96            |
| $C_{\mu(0.5)}$ | 0.48  | 0.48  | 0.48  | 0.52  | 4           | 4        | 4            | 4             | 96            | 96            |
| $C_{\mu(0.75)}$ | 0.52  | 0.48  | 0.48  | 0.52  | 96          | 4        | 4            | 96            | 96            | 96            |

### 3 Simulation

We now present an experiment to compare concordance indices produced by different risk scores. The set up is chosen to show that it is possible to favour incorrect models over the true data generating mechanism. Further studies would be needed to show how typical this situation is.

We generate a data set with crossing hazards inspired by the problem discussed by Mantel and Stablein (1988). Let a population of 2000 be divided into two groups, with covariate $Z_i = 0$ for those in group 0 and $Z_i = 1$ for those in group 1. The data generating model $M_0$ is specified by the hazard rates

$$\lambda_{M_0}(t|Z_i = 0) = 0.5, \quad \lambda_{M_0}(t|Z_i = 1) = t$$

There is independent right censoring by an exponential distribution with rate 0.05 as well as censoring for anyone who survives until $t = 1.1$.

Now let there be 3 incorrect models $M_1$, $M_2$, $M_3$, for us to compare to, which are defined by their hazard rates $\lambda_{M_1}$, $\lambda_{M_2}$, $\lambda_{M_3}$ as follows:

$$\lambda_{M_1}(t|Z_i = 0) = 0.5, \quad \lambda_{M_1}(t|Z_i = 1) = \begin{cases} t, & (t \leq 0.5) \\ 10t, & (1 < t) \end{cases}$$

$$\lambda_{M_2}(t|Z_i = 0) = 0.25, \quad \lambda_{M_2}(t|Z_i = 1) = t, \quad \lambda_{M_2}(t|Z_i = 0) = 0.5, \quad \lambda_{M_2}(t|Z_i = 1) = 0.5t.$$  

The hazard and cumulative hazard rates are shown in in Figure[1].

We use four different risk scores to calculate concordance indices. Our suggestions of the hazard at time of first event uses $q(s|Z) = \alpha(s|Z)$ and is denoted by $C_{\alpha}$. Survival at time of first event, the suggestion of Antolini et al. (2005), is denoted by $C^{bd}$ and uses $q(t|z) = -S(t|z)$, where $S(t|z)$ is the survivor function at time $t$ for an individual with covariate $z$. Survival at fixed times 0.5 and 2.05 are denoted by $C_{S(0.5)}$ and $C_{S(1.05)}$ and use $q(t|z) = -S(0.5|z)$ and $q(t|z) = -S(1.05|z)$, respectively. The quantile survival time is denoted by $C_{\mu(s)}$ and uses $q(t|z) = -\inf\{u \text{ s.t } S(u|z) \geq s\}$.

We generated 100 different data sets and computed the resulting concordance indices as well as, for every concordance index, the frequency with which each model achieved the highest concordance index. Results are presented in Table[1].

As anticipated by Theorem[1], $C_{\alpha}$ always selects the correct model, but is unable to distinguish between $M_0$ and $M_1$ as both models have risk scores concordant with the hazard rate of the true model $M_0$. The concordance $C^{bd}$ consistently selects an incorrect model in this situation. $C_{S(0.5)}$ fails to perform any model selection, giving every model an equal score in every experiment. $C_{S(1.05)}$ mostly selects an an incorrect model. Finally, $C_{\mu(0.5)}$ similarly chooses an incorrect model in most iterations, while $C_{\mu(0.75)}$ mostly fails to distinguish between $M_0$ and $M_3$. With each iteration there is a small chance that the randomly generated data will result in concordance calculation orderings that do not match the order of the expected concordances. This has resulted in a small number of deviations in model selection from the general trend.

### 4 Extension to ties in observation

The results in this paper are assuming continuous observations, implying that there are no ties in the observations. In practice, data is recorded to a finite resolution and thus tied observations occur with positive probability. Different approaches in handling of ties in both observation and prediction.
Figure 1: Hazard rates (top row) and cumulative hazard rates (bottom row) of models $M_0, \ldots, M_3$ (left to right).
Group 0: solid lines; group 1: dotted lines.

can be seen with Kendall’s tau-a, the Goodman-Kruskall statistic and Somers’ d $\tau_d$ (Kendall 1938; Goodman and Kruskal 1954; Somers 1962). They respectively handle ties by treating ties as failures, ignoring ties in prediction or observation, and treating ties in observation as incomparable while rewarding ties in prediction 0.5.

Under the existing formulation of $c^{n_q}_{ij}$, ties in observation always contribute 0 to both the concordance count, $N_{conc}^{ij}$, as well as to the the comparable count $N_{comp}^{ij}$. This behaviour could be changed by adding some multiple of $\mathbb{I}(D_i = 1, D_j = 1, T_i = T_j)$ to either, depending on how you would like to handle the ties. In any case, theorem 1 will no longer hold in its current form. We posit that a similar result can be proved, but for points at which equal survival times can be observed, we expect the hazard rate must be substituted by the change in cumulative hazard $\Delta H(s, Z_i)$.

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### Appendix

**Proof of Lemma**[2] Let $i \in \mathbb{N}$. Consider the counting process $N_i(t) = D_i \mathbb{1}(T_i \leq t)$, with a unique decomposition $N_i(t) = \Lambda_i(t) + M_i(t)$ into a compensator $\Lambda_i(t) = \int_0^t \alpha(s|Z_i)Y_i(s)ds$ and a finite variation local martingale $M_i$ with respect to $(\mathcal{F}_t)$. With $j \in \mathbb{N}, j \neq i$, let $N_{ij}^{\tau_{ij}}$ be $N_i$ stopped at $\tau_{ij}$, i.e., $N_{ij}^{\tau_{ij}}(t) = N_i(t \wedge \tau_{ij})$. Since a finite variation local martingale stopped at a stopping time $\tau_{ij}$ is also a finite variation local martingale $M_{ij}^{\tau_{ij}}(t) := M_i(t \wedge \tau_{ij})$ is a finite variation local martingale. By uniqueness of decomposition [Protter, 2010] Theorem III.16), the compensator of $N_{ij}^{\tau_{ij}}(t)$ is therefore

$$
\Lambda_{ij}^{\tau_{ij}}(t) = \int_0^{t \wedge \tau_{ij}} \alpha(s|Z_i)Y_i(s)ds = \int_0^t \alpha(s|Z_i)Y_{ij}(s)ds,
$$

where $Y_{ij}(t) = \mathbb{1}(t \leq \tau_{ij})$. Letting $Q_{ij}^1(s) = \mathbb{E}[q(t \wedge \tau_{ij}|Z_i) > q(t \wedge \tau_{ij}|Z_j)]$, we can write $N_{ij}^{\tau_{ij}}$ as

$$
N_{ij}^{\tau_{ij} \text{ conc.}^1}(t) = \int_0^t Q_{ij}^1(s)dN_i^{\tau_{ij}}(s) = \Lambda_{ij}^{\tau_{ij} \text{ conc.}^1}(t) + M_{ij}^{\tau_{ij} \text{ conc.}^1}(t),
$$

where $\Lambda_{ij}^{\tau_{ij} \text{ conc.}^1}(t) := \int_0^t Q_{ij}^1(s)d\Lambda_i^{\tau_{ij}}(s)$ and $M_{ij}^{\tau_{ij} \text{ conc.}^1}(t) := \int_0^t Q_{ij}^1(s)dM_i^{\tau_{ij}}(s).$
\( M_{ij}^{\text{conc},1}(t) \) is a local martingale with respect to \((\mathcal{F}_t)\) as \( Q_{ij}^1 \) is predictable and bounded \citep[Theorem II.3.1]{Andersen2012}. Thus, again by uniqueness of decomposition, the compensator of \( N_{ij}^{\text{conc},1}(t) \) is given by \( \Lambda_{ij}^{\text{conc},1}(t) \), which can be rewritten as

\[
\Lambda_{ij}^{\text{conc},1}(t) = \int_0^t \alpha(s|Z_i)Q_{ij}^1(s)Y_{ij}(s)ds,
\]

implying that the intensity of \( N_{ij}^{\text{conc},1}(t) \) is \( \lambda_{ij}^{\text{conc},1}(t) = \alpha(t|Z_i)Q_{ij}^1(t)Y_{ij}(t) \). By similar arguments we can show that the process \( N_{ij}^{\text{conc},2}(t) \) has a decomposition into local martingales and compensators with intensity process

\[
\lambda_{ij}^{\text{conc},2}(t) = \alpha(t|Z_i)Q_{ij}^2(t)Y_{ij}(t),
\]

where \( Q_{ij}^2(t) = I[q(t \wedge \tau_{ij}|Z_i) = q(t \wedge \tau_{ij}|Z_i)] \). Now we can decompose \( N_{ij}^{\text{conc}} \) as

\[
N_{ij}^{\text{conc}}(t) = N_{ij}^{\text{conc},1}(t) + N_{ij}^{\text{conc},2}(t)/2 = \Lambda_{ij}^{\text{conc},1}(t) + \lambda_{ij}^{\text{conc},2}(t)/2 + M_{ij}^{\text{conc},1}(t) + M_{ij}^{\text{conc},2}(t)/2.
\]

Since the property of a process being a local martingale is closed under addition and scalar multiplication, the final two terms, which we call \( M_{ij}(t) \), form a local martingale. Therefore, the first two terms are the compensator of \( N_{ij}^{\text{conc}}(t) \), which simplify to

\[
\Lambda_{ij}^{\text{conc}} = \int_0^t \alpha(s|Z_i)[Q_{ij}^1(s) + 0.5Q_{ij}^2(s)]Y_{ij}(s)ds.
\]

To show that \( M_{ij}^{\text{conc}} \) is a martingale, and not just a local martingale, we use Theorem I.51 of \cite{Protter2010}, which requires us to show

\[
E[\sup_{s \leq t} |M_{ij}^{\text{conc}}(s)|] < \infty \quad \forall t \geq 0.
\]

First we write

\[
E[\sup_{s \leq t} |M_{ij}^{\text{conc}}(s)|] \leq E[\sup_{s \leq t} |N_{ij}^{\text{conc}}(s)|] + E[\sup_{s \leq t} |\Lambda_{ij}^{\text{conc}}(t)|] \leq 1 + E[\sup_{s \leq t} |\Lambda_{ij}^{\text{conc}}(t)|],
\]

and then we bound the second term

\[
\sup_{s \leq t} |\Lambda_{ij}^{\text{conc}}(t)| = \int_0^t (\alpha(s|Z_i)Q_{ij}^1(s) + \alpha(s|Z_i)Q_{ij}^2(s)/2)Y_{ij}(s)ds \leq \frac{3}{2} \int_0^t \alpha(s|Z_i)Y_{ij}(s)ds \leq \frac{3}{2} \int_0^X \alpha(s|Z_i)ds = \frac{3}{2} H(X_i|Z_i),
\]

where \( H(t|Z_i) = \int_0^t \alpha(s|Z_i)ds \) is the ith individual’s integrated hazard rate. Suppose \( Y \) is a random variable with integrated hazard rate \( H \) and cumulative distribution function \( F \). Then \( E[H(Y)] = E[-\log(F(Y))] = -\int \log(F(y))dF(y) = -\int_0^1 \log(u)du = 1 \). Hence,

\[
E[\sup_{s \leq t} |M_{ij}^{\text{conc}}(s)|] \leq 1 + \frac{3}{2} E[H(t|Z_i)] = 5/2 < \infty.
\]

\( \square \)