Quark mass function in Minkowski space

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We investigate the properties of quark mass functions in quantum chromodynamics calculated by the Schwinger–Dyson equation in the strong coupling region, in which the loop integration is performed in Minkowski space. The calculated results are compared with those obtained by integration in Euclidean space.

Subject Index B69

1. Introduction

The Schwinger–Dyson equation (SDE) [1,2] is one of the methods for evaluating nonperturbative phenomena, such as chiral phase transition. So far, much work on chiral symmetry breaking has been done with the SDE in momentum representation, in which a one-loop contribution is integrated over Euclidean space.

Some calculations of fermion mass functions with the SDE have been done in Minkowski space. In Ref. [3], a spectral representation for Green functions is assumed, in which the mass functions are calculated in Lorentz-invariant form. In Ref. [4], explicit one-loop contributions of the mass function have been calculated. However, the mass function is evaluated for only one iteration from a constant initial mass as an input.

It has been pointed out that the spectral functions for the gluon calculated by the lattice simulation in Euclidean space, which is numerically continued to Minkowski space, becomes negative for some range [5]. Similar behavior has been found for generalized perturbative calculations [6].

Analytic continuation from Euclidean space to Minkowski space is valid in perturbative calculation if the pole positions in the complex plane of the energy are known. However, it is not trivial in the nonperturbative region.

So far, the structure of the fermion mass function in the strong coupling region in the entire range of energy and momentum space has not been fully studied in Minkowski space, even at zero temperature.

In the previous paper [7], we formulated the SDE for quantum electrodynamics (QED) in which the momentum integration is performed in Minkowski space without the instantaneous exchange approximation (IEA) [8,9].

In this paper, we apply our previous method to calculate the quark mass function in quantum chromodynamics (QCD) with the SDE in Minkowski space at zero temperature.

In Sect. 2, we present the SDE for QCD in Minkowski space. In Sect. 3, some numerical results are shown and calculated results are compared with those obtained by the SDE in Euclidean space. Section 4 is devoted to a summary and some comments.
2. The SDE for the quark mass function

We calculate a quark self-energy \( \Sigma(P) \) in QCD in four dimensions, which is given by

\[
-i \Sigma(P) = \int \frac{d^4 Q}{(2\pi)^4} \left( i g_s^2 C_F \gamma^\mu \Gamma^\nu \right) iS(Q) \Gamma^\nu iD_{\mu\nu}(K),
\]

where \( S(Q) \) and \( D_{\mu\nu}(K) \) are propagators of a quark with momentum \( Q = (q_0, \mathbf{q}) \) and a gluon with momentum \( K = P - Q = (k_0, \mathbf{k}) \), respectively. Here, \( P = (p_0, \mathbf{p}) \) is an external momentum of the quark. The strong coupling constant and the color factor are denoted by \( g_s \) and \( C_F = 4/3 \), respectively.

The quark propagator is given by

\[
iS(Q) = \frac{iZ}{Q - m_0 - \Sigma(Q) + i\varepsilon} = \frac{i}{A(Q)Q - B(Q) + i\varepsilon},
\]

where \( m_0 \) is a bare quark mass.

In this paper, we calculate a mass function in the Landau gauge, in which the wave-function renormalization constant is \( Z = 1 \) for a one-loop order of perturbation. Therefore, we calculate the self-energy given in Eq. (1) with \( A = 1 \), \( M_M = B = m_0 + \text{Tr}[\Sigma]/4 \), and the quark–gluon vertex with \( \Gamma^\nu = \gamma^\nu \). Here, the gluon propagator is given as

\[
iD_{\mu\nu}(K) = \left(-g_{\mu\nu} + \frac{K_\mu K_\nu}{K^2}\right) \frac{i}{K^2 + i\varepsilon}.
\]

in the Landau gauge. In this paper, we neglect an effective gluon mass. Integrating over the azimuthal angle of the momentum \( \mathbf{q} \), the mass function is given by

\[
M_M(p_0, p) = m_0 - \frac{3iC_F}{2\pi^2} \int_{-\Lambda_0}^{\Lambda_0} dq_0 \int_{-\delta}^{\delta} dq \frac{q}{q^2 - (M^2)^1 - \varepsilon} \alpha_s[M_MIJ](q_0, q),
\]

with \( p = |\mathbf{p}|, q = |\mathbf{q}|, \) and \( \alpha_s = g_s^2/(4\pi) \). The energy \( q_0 \) and the momentum \( q \) are integrated over the ranges \(-\Lambda_0 \leq q_0 \leq \Lambda_0 \) and \( \delta \leq q \leq \Lambda, \) respectively, with cutoff parameters \( \Lambda_0, \Lambda, \) and \( \delta \).

Here, \( I \) and \( J \) are given by

\[
I = \frac{1}{Q^2 - M_M^2 + i\varepsilon} = \frac{Q^2 - (M^2)^1 + i((M^2)^1 - \varepsilon)}{(Q^2 - (M^2)^1 - \varepsilon)^2 + ((M^2)^1 - \varepsilon)^2}
\]

and

\[
J = \int_{-\delta}^{\delta} dk \frac{k}{K^2 + i\varepsilon},
\]

respectively, with \( \eta_{\pm} = |\mathbf{p} \pm \mathbf{q}| \) and \( k = |\mathbf{k}| \). Here, \( (M^2)^{\text{R}} \) and \( (M^2)^{\text{I}} \) denote the real part and the imaginary part of the mass function \( M_M^2 \), respectively.

In Minkowski space, the propagator \( I \) in Eq. (5) rapidly varies near \( Q^2 \simeq (M^2)^{\text{R}} \) if the imaginary part of the mass function \( (M^2)^{\text{I}} \) is small, which is one of the difficulties for numerical calculation in Minkowski space. Therefore, it is necessary to perform integration efficiently. As implemented in the previous work for QED [7], we divide the \( q_0 \) integration into small ranges and integrate the quark propagator over \( q_0^{(l)} \leq q_0 \leq q_0^{(l+1)} \) \( (q_0^{(l)}) = -\Lambda_0, \) \( q_0^{(N)} = \Lambda_0 ) \) as

\[
M_M(p_0, p) \simeq m_0 - \frac{3iC_F}{2\pi^2} \int_{-\delta}^{\delta} dq \frac{q}{q^2 - p^2} \sum_{l=1}^{N-1} \alpha_s([M_MIJ](q)_{\ell} I(q_0^{(l+1)}, q_0^{(l)})
\]

(7)
with
\[ I(q_0^{(l+1)}, q_0^{(l)}) = \int_{q_0^{(l)}}^{q_0^{(l+1)}} dq_0 \frac{d\sigma}{Q^2 - M^2 + i\varepsilon}, \] (8)

where the remaining contributions of the integrand are averaged for the range \( q_0^{(l)} \leq q_0 \leq q_0^{(l+1)} \). Here, \( \langle X \rangle_I \) denotes the average of a function \( X(q_0) \) at \( q_0 = q_0^{(l+1)} \) and \( q_0 = q_0^{(l)} \) as \( \langle X \rangle_I = [X(q_0^{(l+1)}) + X(q_0^{(l)})]/2 \). Convergence of the calculations is significantly improved by our method. The explicit expressions are summarized in Appendix A, in which the improved ladder approximation is implemented.

3. Numerical results

In this section, some numerical results are presented. We solve the SDE presented in Eq. (1) by a recursion method starting from a constant mass.1

For each iteration, we calculate the quark mass function normalized as
\[ M^{(n+1)}_M(p^2) = m(\mu^2) + M^{(n)}_M(p^2) - M^{(n)}_M(\mu^2), \] (9)

where \( n \) denotes the number of iterations. Here, the mass function is normalized by a current quark mass at large \( \mu^2 \), in which perturbative calculations are reliable. In the iteration, the mass function \( M_M(p_0, p) \) in the integrand of Eq. (4) is replaced by the renormalized one obtained by the previous iteration. Here, \( m(\mu^2) \) is a renormalized mass at a renormalization scale \( \mu \).2

Here, \( M_M(p^2) \) is defined as
\[ M_M(p^2) = M_M(p_0, \delta) \theta(|p_0| - \delta) \] (10)
for \( p^2 > 0 \), and
\[ M_M(p^2) = M_M(\delta, p) \theta(p - \delta) \] (11)
for \( p^2 < 0 \), respectively, where \( p^2 = p_0^2 - p^2 \).

The mass function can be written as an absolute value \( |M_M(p^2)| \) and a phase factor \( \exp(i\Phi(p^2)) \) as
\[ M_M(p^2) = |M_M(p^2)| \exp(i\Phi(p^2)). \] (12)

In Fig. 1, the convergence property of the mass function \( |M_M| \) integrated over \( |P^2| \) as
\[ \langle |M_M(p^2)| \rangle = \int_{p^2}^{\Lambda^2} d|P^2||M_M(p^2)| \] (13)
is presented. The solid line and the + symbols represent \( \langle |M_M| \rangle \) integrated over \( P^2 > 0 \) and \( P^2 < 0 \), respectively. The horizontal axis denotes the number of iterations. The dash-dotted line represents the calculated results for \( \langle M_E \rangle \), where the mass function in Euclidean space \( M_E(p^2_E) \) is given by \[10,11\]
\[ M_E(x) = \frac{3C_F}{4\pi} \int_{\delta^2}^{\Lambda^2} dy dy_s \frac{2y}{x + y + \sqrt{(x - y)^2} y + M_E(y),} \] (14)

1 The initial input parameters are \( M_R = \Lambda_{QCD} \) and \( M_I = 0 \), with \( \Lambda_0 = \Lambda = 20\Lambda_{QCD} \) and \( \delta = 0.2\Lambda_{QCD} \), where we define \( M_M = M_R + iM_I \). We set \( \Lambda_{QCD} = 0.5 \text{GeV} \) with \( \varepsilon = 10^{-6} \).

2 We take \( m(\mu^2) = 3 \text{MeV} \) at \( \mu = 20\Lambda_{QCD} \).
Fig. 1. The solid line and the + symbols represent the convergence property of $|M_M|$ integrated over $P^2 > 0$ and $P^2 < 0$ in Minkowski space, respectively. The dash-dotted line denotes the calculated results for $\langle M_E \rangle$ in Euclidean space. The horizontal axis represents the number of iterations.

Fig. 2. The solid line and the + symbols represent the $P^2$ dependence of $|M_M|$ for $P^2 > 0$ and $P^2 < 0$, respectively. The dash-dotted line denotes the result for the $P^2_E$ dependence of $M_E$ in Euclidean space.

which is renormalized as

$$M_E^{(n+1)}(P^2) = m(\mu^2) + M_E^{(n)}(P^2) - M_E^{(n)}(\mu^2).$$  

(15)

As shown in Fig. 1, the mass functions integrated over the momentum for the three cases rapidly converge.

In Fig. 2, the dependencies on $P^2$ for $|M_M|$ in Minkowski space and $M_E$ in Euclidean space are presented. The three cases give similar $|P^2|$ dependencies, though the mass function $|M_M|$ in time-like momentum has an imaginary part.

In Fig. 3, the $P^2$ dependencies of $M_R$ and $M_I$ in Minkowski space are presented. From Eq. (2), we define a spectral function for the quark mass term denoted by $\rho_M(P^2)$ as

$$\frac{1}{4} \text{Tr}[S(P)] = S_M(P^2) = \frac{M_M(P^2)}{P^2 - M_M^2(P^2) + i\epsilon} = \int dQ^2 \frac{\rho_M(Q^2)}{P^2 - Q^2 + i\epsilon},$$  

(16)
Fig. 3. The solid line and the dashed line denote the $P^2$ dependencies of $M_R$ and $M_I$ for $P^2 > 0$, respectively. $M_R$ and $M_I$ for $P^2 < 0$ are represented by the + and the × symbols, respectively. The dotted line denotes the $P^2$ dependence of the spectral function $\rho_M$ defined in Eq. (17) for $P^2 > 0$.

Fig. 4. The mass functions and the spectral function with the analytic coupling $\alpha_s^{(AN)}$. The solid line and the dashed line denote the $P^2$ dependencies of $M_R$ and $M_I$ for $P^2 > 0$, respectively. $M_R$ and $M_I$ for $P^2 < 0$ are represented by the + and the × symbols, respectively. The dotted line denotes the $P^2$ dependence of the spectral function $\rho_M$ defined in Eq. (17) for $P^2 > 0$.

which gives

$$\rho_M(P^2) = -\frac{1}{\pi} \text{Im} S_M(P^2) = -\frac{1}{\pi} \frac{M_I(P^2 + |M_M|^2) - \epsilon M_R}{(P^2 - (M^2)_R)^2 + ((M^2)_I - \epsilon)^2}. \quad (17)$$

As shown in Fig. 3, $M_I$ for $P^2 > 0$ becomes positive in some regions of $P^2$, such as $P^2 > 0.3$ GeV$^2$, in which $\rho_M$ becomes negative. It may be interesting to compare our result with the spectral function for the quark obtained in Ref. [6] in Minkowski space.

In Fig. 4, the $P^2$ dependencies of $M_R$ and $M_I$, as well as $\rho_M(P^2)$ with the analytic coupling $\alpha_s^{(AN)}$ defined in Appendix A, which is valid for a time-like momentum region as well as a space-like one, are presented. The results presented in Fig. 4 show similar behaviors to those in Fig. 3, though the mass functions are smaller than those obtained with the coupling constant defined in Euclidian momentum due to the smaller contribution of the coupling constant $\alpha_s^{(AN)}(P^2, Q^2)$ than $\alpha_s(\bar{P}^2, \bar{Q}^2)$.
Here, $P^2$ and $Q^2$ denote the momenta in Minkowski space, and $\bar{P}^2$ and $\bar{Q}^2$ denote the momenta in Euclidean space, respectively. (See Appendix A.)

4. Summary and comments

In this paper, we studied a quark mass function solved by the Schwinger–Dyson equation in Minkowski space for QCD.

Evaluation of the mass functions in Minkowski space allows us to study the imaginary part of the mass and energy for the massive fermion states.

We examined the properties of the quark mass function in time-like momentum $P^2 > 0$ as well as that in space-like momentum $P^2 < 0$, where $P^2$ denotes the squared four-momentum of the quark.

Furthermore, we also compared our results with the mass function calculated in Euclidean space. We found that the three cases give similar $|P^2|$ dependencies, though the mass function with time-like momentum has an imaginary part.

We also studied the behavior of the spectral function for the quark mass term. We found that there seems to exist a negative spectral function in some momentum regions for $P^2 > 0$ as pointed out in Refs. [5,6], in which the imaginary part of the mass function becomes positive.

Theoretically, we should implement a coupling constant with momenta in Minkowski space. However, coupling constants with nonperturbative contributions in Minkowski space have not yet been studied.

Therefore, we implemented a strong coupling constant with Euclidean momenta, which is a simple model to evaluate the quark mass function in Minkowski space. A different definition of the argument of $\alpha_s$ is a part of the higher-order contributions to the one-loop approximation. Therefore, it is expected that qualitative features of the mass function in Minkowski space may be preserved in our model.

Furthermore, our model gives rather stable numerical results in calculation in Minkowski space.

In order to check our results, we calculated the mass functions as well as the spectral function with an analytic coupling $\alpha_s^{\text{(AN)}}$, which is the perturbative strong coupling constant analytically continued from a space-like momentum region to a time-like one.

We found similar behaviors as those obtained with the coupling constant defined in Euclidean momentum. The results suggest that the qualitative features of the mass function may not be affected by the choice of the momentum in the coupling constant, such as Euclidian or Minkowski momentum, provided that the coupling constant is large enough to break the chiral symmetry of quarks.

It may be expected that the SDE has multiple solutions in numerical calculations. Further studies are needed for solutions of the quark mass function obtained by the SDE in Minkowski space in the strong coupling region.

In this paper, we examined the qualitative features of the quark mass function in Minkowski space. In order to reproduce physical quantities, such as the pion decay constant, or to fit the results obtained by SDE with the numerical data from lattice simulations, we need to fine-tune the parameter $\Lambda_{\text{QCD}}$.

In future works, we shall extend our method to finite temperature and density with the real time formalism.

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Appendix. Approximation formulas

The mass function is given by

\[ M_M(p_0, p) = m_0 - \frac{3iC_F}{2\pi^2} \int_{-\Lambda_0}^{\Lambda_0} dq_0 \int_{-\delta}^{\delta} dq \frac{q}{p} \alpha_s[M_M I](q_0, q), \]

where the propagators \( I \) and \( J \) are defined in Eqs. (5) and (6), respectively.

As shown in Eq. (7), we approximate the integration over \( q_0 \) as

\[ M_M(p_0, p) \approx m_0 - \frac{3iC_F}{2\pi^2} \int_{\delta}^{\Lambda} dq \frac{q}{p} \sum_{I=1}^{N-1} \alpha_s(\bar{P}^2, \bar{Q}^2) \langle [M_M I](q) \rangle_I I(q_0^{(l+1)}, q_0^{(l)}), \]

where \( \langle X \rangle_I \) denotes an average of \( X(q_0^{(l+1)}) \) and \( X(q_0^{(l)}) \) as \( \langle X \rangle_I = [X(q_0^{(l+1)}) + X(q_0^{(l)})]/2. \)

Here, the strong coupling constant \( \alpha_s \) is replaced by the running coupling constant \( \alpha_s(\bar{P}^2, \bar{Q}^2) [12] \), which is defined as

\[ \alpha_s(\bar{P}^2, \bar{Q}^2) = \frac{4\pi}{\beta_0} \times \begin{cases} \frac{1}{\eta} & \text{if } t_F < t < t_C \frac{1}{2\epsilon} + \frac{(t_F - t_C)^2 - (t_C - t)^2}{2(t_F - t_C)} & \text{if } t_C < t < t_F \frac{1}{2\epsilon} \end{cases} \]

with \( \beta_0 = (33 - 2N_f)/3, t = \log((\bar{P}^2 + \bar{Q}^2)/\Lambda_{QCD}^2), t_F = 0.5, \) and \( t_C = -2 \) for \( N_f = 3 \) flavors, where \( \bar{P}^2 = p_0^2 + p^2 \) and \( \bar{Q}^2 = (q_0)^2 + q^2. \)

For \( I, M_M, \) and \( J, \) we separate the real parts, \( I_R, M_R, J_R, \) and the imaginary parts, \( I_I, M_I, J_I, \) respectively.

For \( J \) in Eq. (6), we can integrate over \( k \) as

\[ J_R = -\int_{\eta_-}^{\eta_+} dk \frac{k(k^2 - k_0^2)}{(k^2 - k_0^2)^2 + \epsilon^2} = -\frac{1}{4} \log \left( \frac{\eta_+^2 - k_0^2 + \epsilon^2}{\eta_-^2 - k_0^2 + \epsilon^2} \right) \]

and

\[ J_I = -\int_{\eta_-}^{\eta_+} dk \frac{k\epsilon}{(k^2 - k_0^2)^2 + \epsilon^2} = -\frac{1}{2} \left[ \arctan \frac{\eta_+^2 - k_0^2}{\epsilon} - \arctan \frac{\eta_-^2 - k_0^2}{\epsilon} \right], \]

respectively, with \( \eta_{\pm} = |p \pm q| \) and \( k = |k|. \)

The real and imaginary parts of the quark propagator \( I(q_0^{(l+1)}, q_0^{(l)}) \) defined in Eq. (8) are given by

\[ I_R^{(l)} = \text{Re} I(q_0^{(l+1)}, q_0^{(l)}) = \frac{\epsilon}{2} \frac{\partial(E^2)_R}{\partial q_0} \]

\[ \times \log \left( \frac{(q_0^{(l+1)})^2 - \langle (E^2)_R \rangle_I^2 + \langle (E^2)_I \rangle_I^2}{(q_0^{(l)})^2 - \langle (E^2)_R \rangle_I^2 + \langle (E^2)_I \rangle_I^2} \right) \]

\[ I_I^{(l)} = \text{Im} I(q_0^{(l+1)}, q_0^{(l)}) = \frac{\epsilon}{2} \frac{\partial(E^2)_R}{\partial q_0} \]

\[ \times \log \left( \frac{(q_0^{(l+1)})^2 - \langle (E^2)_R \rangle_I^2 + \langle (E^2)_I \rangle_I^2}{(q_0^{(l)})^2 - \langle (E^2)_R \rangle_I^2 + \langle (E^2)_I \rangle_I^2} \right) \]
and

\[ I_1^{(l)} = \text{Im} \langle q_0^{(l+1)}, q_0^{(l)} \rangle = \frac{\epsilon \left[ \frac{(2q_0 - \frac{\partial (E^2)_R}{\partial q_0})_l}{(2q_0 - \frac{\partial (E^2)_R}{\partial q_0})_l} \right] \epsilon \left( \langle (E^2)_I \rangle \right)}{\left[ 2q_0 - \frac{\partial (E^2)_R}{\partial q_0} \right]_l} \times \left[ \arctan \left( \frac{(q_0^{(l+1)})^2 - \langle (E^2)_R \rangle_1}{\langle (E^2)_I \rangle_1} \right) - \arctan \left( \frac{(q_0^{(l)})^2 - \langle (E^2)_R \rangle_1}{\langle (E^2)_I \rangle_1} \right) \right], \]

respectively, where we define \( \epsilon(z) = \theta(z) - \theta(-z) \) with the step function \( \theta(z) \). Here, the real and imaginary parts of the squared energy denoted by \( (E^2)_R \) and \( (E^2)_I \), respectively, are given as

\[ (E^2)_R = q^2 + (M^2)_R \]

and

\[ (E^2)_I = (M^2)_I - \epsilon, \]

with

\[ (M^2)_R = \text{Re}(M_M^2) = (M_R)^2 - (M_I)^2 \]

and

\[ (M^2)_I = \text{Im}(M_M^2) = 2M_R M_I. \]

In the text, we also implement the coupling constant proposed in Ref. [13], in which the coupling constant in the space-like momentum region is analytically continued to that in the time-like momentum region as

\[ \alpha_s^{(AN)}(P^2, Q^2) = \frac{4\pi}{\beta_0} \times \left\{ \begin{array}{ll}
\frac{1}{\log(-z)} + \frac{1}{1+z} & \text{if } z < 0 \\
\frac{1}{2} - \frac{1}{\pi} \arctan \left( \frac{\log(z)}{\pi} \right) & \text{if } z > 0
\end{array} \right\}, \]

with \( z = (P^2 + Q^2)/\Lambda^2_{\text{QCD}} \), where \( P^2 = p_0^2 - p^2 \) and \( Q^2 = \langle q_0 \rangle^2 - q^2 \), respectively.

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