On the P-representable subset of all bipartite Gaussian separable states

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P-representability is a necessary and sufficient condition for separability of bipartite Gaussian states only for the special subset of states whose covariance matrix are $Sp(2,R) \otimes Sp(2,R)$ locally invariant. Although this special class of states can be reached by a convenient $Sp(2,R) \otimes Sp(2,R)$ transformation over an arbitrary covariance matrix, it represents a loss of generality, avoiding inference of many general aspects of separability of bipartite Gaussian states.

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I. INTRODUCTION

In the recent years the question to whether a given quantum state is separable or entangled has become central to the quantum information and to the quantum optics communities. Mostly because fault-tolerant quantum information protocols, such as quantum computation and quantum teleportation are completely dependent on the ability to prepare pure (or close to pure) entangled states. Peres [2,3], whose efficiency rests on the ability to generate entangled states of systems with infinite dimensional Hilbert spaces. Bipartite systems with finite Hilbert space have been exhaustively investigated in order to achieve a precise quantification of entanglement. Peres [4] and Horodecki [5] demonstrated that a necessary and sufficient condition for separability of bipartite systems with Hilbert spaces of dimension $\leq 2 \otimes 3$ is the positivity of the partial transpose of the system density matrix. On the other hand, algebraic similarities between bipartite states with Hilbert space of dimension $2 \otimes 2$ and bipartite Gaussian states (described by $4 \times 4$ covariance matrices) allows the extension of the positivity criterion to those special continuous variable states as firstly developed in Refs. [6,7], and considered afterwards in discussions on entanglement of Gaussian bipartite states (e.g. [8–18]).

Of particular importance is the connection between Glauber P-representability of a bipartite quantum state and separability [10]. A P-representable state is the one that is represented by a positive Glauber P distribution function $P(\alpha,\beta)$, which is less (or equally)-singular than the delta distribution, such as

$$
\rho = \int d\alpha^2 d\beta^2 P(\alpha,\beta) |\alpha,\beta\rangle \langle \alpha,\beta|.
$$

Under this condition $P(\alpha,\beta)$ assumes the structure of a legitimate probability distribution function over an ensemble of states, allowing the connection between separability and classicality. However, although any P-representable bipartite state is separable, as can be immediately seen by the P representation definition, the inverse is not necessarily true. P-representability and separability are completely equivalent only for Gaussian states with locally $Sp(2,R) \otimes Sp(2,R)$ invariant covariance matrices. Since any covariance matrix can be brought to this invariant form under appropriate $Sp(2,R) \otimes Sp(2,R)$ transform, P-representability and separability have been misleadingly accepted as one-to-one equivalent properties of bipartite Gaussian states.

The purpose of the present paper is to give a complete classification of the set of all bipartite Gaussian separable states (BGSS). Particularly we show that P-representable Gaussian bipartite states form a subset of BGSS with locally $Sp(2,R) \otimes Sp(2,R)$ invariant form. In Section II we begin by a revision of some necessary properties of bipartite Gaussian states and in Sec. III we give the necessary and sufficient conditions for the state to be separable. In Sec. IV we discuss the P-representability of those states and show that they actually form a subset of the separable states. In Sec. V we describe the the unitary $Sp(2,R) \otimes Sp(2,R)$ map that connects the two sets and finally in Sec. VI a conclusion encloses the paper.

II. BIPARTITE GAUSSIAN STATES

Any bipartite quantum state $\rho$ is Gaussian (see e.g. [17,19]) if its symmetric characteristic function is given by

$$
C(\eta) = Tr[D(\eta)\rho] = e^{-\frac{i}{2} \eta^\dagger V \eta},
$$

where $D(\eta) = e^{-\eta^\dagger \eta}$ is a displacement operator in the parameter four-vector $\eta$-space, with

$$
\eta^\dagger = (\eta_1^*, \eta_1, \eta_2^*, \eta_2), \quad V^\dagger = \begin{pmatrix} a_1^\dagger, a_1, a_2^\dagger, a_2 \end{pmatrix},
$$

and

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where $a_1$ ($a_1^\dagger$) and $a_2$ ($a_2^\dagger$) are annihilation (creation) operators for party 1 and 2, respectively. \( V \) is the Hermitian 4 \times 4 covariance matrix with elements $V_{ij} = (-1)^{i+j} \langle \{v_i, a_j^\dagger\} \rangle / 2$, which can be decomposed in four block 2 \times 2 matrices,

\[
V = \begin{pmatrix} V_1 & C \\ C^\dagger & V_2 \end{pmatrix},
\]

(5)

where $V_1$ and $V_2$ are Hermitian matrices containing only local elements while $C$ is the correlation between the two parties. Any covariance matrix must be positive semidefinite ($V \geq 0$), furthermore the generalized uncertainty principle

\[
V + \frac{1}{2}E \geq 0,
\]

(6)

must also be applied. Those general positivity criteria can be decomposed into block using matrix positivity properties. There are many ways to check a matrix positivity, such as above, embodies a manifestation of a physical operator for dimension 2 to bipartite systems of infinite dimension, Simon [6] has discovered an elegant geometrical interpretation of separability in terms of the Wigner distribution function for the density operator. The Peres-Horodecki separability criterion in the Simon framework reads: if a bipartite density operator is separable, then its Wigner distribution necessarily goes over into a Wigner distribution under a phase space mirror reflection. The separability criterion can be understood as a valid Wigner-class-conservative quantum map under local time reversal. Following [6] a necessary and sufficient condition for a Gaussian quantum state to be separable,

\[
\rho = \sum_k p_k \rho_k^A \otimes \rho_k^B
\]

(17)

is that its covariance matrix must satisfy

\[
\overline{V} + \frac{1}{2}E \geq 0,
\]

(18)

under a partial phase space mirror reflection (partial Hermitian conjugation) $\overline{V} = TVT$ :

\[
Tv = v_T = \begin{pmatrix} a_1 \\ a_1^\dagger \\ a_2 \\ a_2^\dagger \end{pmatrix},
\]

(19)

with

\[
T = \begin{pmatrix} I & 0 & 0 & X \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(20)
otherwise the state is entangled. Similarly to the generalized uncertainty (6) decomposition, the separability condition (18) is satisfied if and only if (9) and

\[
\left( XV_2X + \frac{1}{2}Z \right) - XC^\dagger \left( V_1 + \frac{1}{2}Z \right)^{-1} CX \geq 0, \tag{21}
\]

are both satisfied, which explicitly implies in (12) and

\[
n_2 \geq \frac{s}{d} + \sqrt{\frac{1}{4} \left[ \frac{|m_c|^2 - |m_s|^2}{d} + 1 \right]^2 + |m_2 - c|^2}, \tag{22}
\]

respectively. We call the set of states \( \rho \) that fall inside (12) and (22) the set BGSS of all bipartite Gaussian separable states \( S \). Any state that does not follow those inequalities is entangled, being it pure or not. Remark also that purity is only reached when the equalities in (12) and (13) hold.

**IV. P-REPRESENTABILITY OF GAUSSIAN STATES**

The very definition of a separable state (17) can be written in a coherent state representation through the Glauber P-function (1), but it is not obvious that \( P(\alpha, \beta) \) is a legitimate probability distribution function. That is only reached if the state is P-representable, i.e., if the P-function is non-negative and less (or equal) singular than the delta distribution. In terms of the covariance matrix, a quantum state is P-representable [19] if

\[
V - \frac{1}{2}I \succeq 0, \tag{23}
\]

which in terms of the upper left block matrix and its Schur complement writes as

\[
V_1 - \frac{1}{2}I \succeq 0, \tag{24}
\]

and

\[
\left( V_2 - \frac{1}{2}I \right) - C^\dagger \left( V_1 - \frac{1}{2}I \right)^{-1} C \succeq 0. \tag{25}
\]

Thus

\[
n_1 \geq |m_1| + \frac{1}{2}, \tag{26}
\]

and

\[
n_2 \geq \frac{s'}{d'} + \frac{|m_2 - c'|}{d'} + \frac{1}{2}, \tag{27}
\]

with

\[
s' = (n_1 - \frac{1}{2})(|m_c|^2 + |m_s|^2) - m_c m_s m_1^*, \tag{28}
\]

\[
c' = 2(n_1 - \frac{1}{2})m_c m_s - m_c^2 m_1^* - (m_s^*)^2 m_1, \tag{29}
\]

\[
d' = (n_1 - \frac{1}{2})^2 - |m_1|^2. \tag{30}
\]

States that follow (26) and (27) form the set of all bipartite P-representable Gaussian states \( \mathbb{P} \).

From an operator formalism for the density matrix, Englert and Wódkiewicz [10] have recently stated that P-representability is equivalent to the separability condition, for the specific symmetric situation where \( m_1 = m_2 = m_3 = 0, n_1 = n_2 = n, \) and \( m_c = m \), which indeed set \( \mathbb{P} = S \) as we see below. The generality of their statement is justified only if \( Sp(2, R) \otimes Sp(2, R) \) local operations are used to bring those parameters to the special symmetric class described above (see also [19]). However, this particular situation does not represents a total equivalence between \( S \) and the set of all P-representable states. In general the P-representability conditions, (26) and (27), are more restrictive than the separability ones, (12) and (22), respectively, as we now investigate.

Firstly observe that (12) is less restrictive than (26), equaling only for \( |m_1| = 0 \) or \( |m_1| \to \infty \), being enough to check if (27) dominates over (22) for the simplest \( |m_1| = 0 \) situation. For that we make use of the knowledge that (13) is always stronger than (27), including the situation where \( d = 0 \), i.e., \( n_1 = 1/2 \). In such a case, the comparisons of the (27) lower bound to (22) and to (13) are equivalent and thus if (22) is violated so is (13). These inequalities must satisfy

\[
(|m_2| + |m_c|^2)(|m_2| + |m_s|^2) \geq 0, \tag{31}
\]

and since the quantities involved are always strictly positive the criterion (31) is always satisfied. The equality however occurs only if \( |m_2| = |m_c|^2 = 0 \) or \( |m_2| = |m_s|^2 = 0 \), which then set the equivalence \( \mathbb{P} = S \) for the two following special \( Sp(2, R) \otimes Sp(2, R) \) invariant forms for \( V \):

**Invariant form 1:**

\[
\begin{pmatrix}
  n_1 & 0 & 0 & m_c \\
  0 & n_1 & m_c^* & 0 \\
  0 & m_c & n_2 & 0 \\
  m_c^* & 0 & 0 & n_2
\end{pmatrix}, \tag{32}
\]

**Invariant form 2:**

\[
\begin{pmatrix}
  n_1 & 0 & m_s & 0 \\
  0 & n_1 & m_s^* & 0 \\
  m_s^* & 0 & n_2 & 0 \\
  0 & m_s & 0 & n_2
\end{pmatrix}. \tag{33}
\]

Special forms 1 and 2 are locally \( Sp(2, R) \otimes Sp(2, R) \) invariant covariance matrices that form the \( \mathbb{P} = S \) subset. The separability and thus P-representability criterion is then reduced to
for the special form 1, and to
\[
\left(n_1 - \frac{1}{2}\right) \left(n_2 - \frac{1}{2}\right) \geq |m_c|^2, \tag{34}
\]
for the special form 2. The physical condition of existence of a general bipartite Gaussian state of the form 1 or 2 writes
\[
\left(n_1 - \frac{1}{2}\right) \left(n_2 + \frac{1}{2}\right) \geq |m_c|^2, \tag{35}
\]
or
\[
\left(n_1 - \frac{1}{2}\right) \left(n_2 + \frac{1}{2}\right) \geq |m_s|^2, \tag{36}
\]
respectively.

Remark 1: There are P-representable Gaussian operators that violate (13), which however do not represent any valid positive definite quantum state. As an example in Fig. 1 through comparison of the limiting bounds (22) and (27), assuming all real coefficients and setting \(m_1 = 0.5\) and \(m_2 = 1\). Only those states that lay in or above the separable class are valid P-representable separable states.

Remark 2: The special symmetric situation depicted in Ref. [10,13] for the two-mode thermal squeezed state, where \(m_1 = m_2 = m_s = 0\), \(n_1 = n_2 = n\), and \(m_c = m\), is a particular example of the specific form 1, and thus a separable state in this case is always P-representable.

V. \(S \subset P\) MAPPING

Any general covariance matrix can be mapped into one of those invariant forms under appropriate \(Sp(2,R) \otimes Sp(2,R)\) transform. In other words, it is possible to map \(S\) into \(P\) such that
\[
\rho_{SP} = U_L \rho_G U_L^{-1}, \tag{38}
\]
be the state obtained by the local unitary transform \(U_L = U_1 \otimes U_2\) over a general bipartite Gaussian density operator \(\rho_G\) assuming
\[
U_L \rho U_L^{-1} = S_L \rho, \quad S_L = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}, \tag{39}
\]
with the condition \(S_L^{-1} = ES_L|E\). The new symmetric characteristic function writes
\[
C_{SP}(\eta) = Tr[D(\eta)\rho_{SP}] = Tr[U_L^{-1}D(\eta)U_L \rho_G] = e^{-\frac{1}{2} \eta^T V_{SP} \eta}, \tag{40}
\]
with
\[
V_{SP} = S_L^1 V S_L. \tag{41}
\]
The transformed covariance matrix writes as (5), but with new block elements
\[
V'_i = S_i^1 V_i S_i, \quad C' = S_1^1 C S_2. \tag{42}
\]
Assuming a local \(Sp(2,R)\) transform as
\[
S_i \equiv \begin{pmatrix} e^{i \phi_i} \cosh \theta_i & e^{i \varphi_i} \sinh \theta_i \\ e^{-i \varphi_i} \sinh \theta_i & e^{-i \phi_i} \cosh \theta_i \end{pmatrix}, \tag{43}
\]
the condition to bring \(V\) to the invariant form 1:
\[
\begin{pmatrix} \nu_1 & 0 & 0 & \mu_c \\ 0 & \nu_1 & \mu_c^* & 0 \\ 0 & \mu_c & \nu_2 & 0 \\ \mu_c^* & 0 & 0 & \nu_2 \end{pmatrix}, \tag{44}
\]
is obtained by setting \(\phi_i + \varphi_i = -\mu_i + \pi\) and \(\tanh 2 \theta_i = |m_i|/|n_i| = |m_c|/|m_s|\), for \(i = 1, 2\), respectively, where \(e^{-i \varphi_i} = m_i/|m_i|\), and assuming (for \(|m_c| \geq |m_s|\)).

Now the condition to bring \(V\) to the invariant form 2:
\[
\begin{pmatrix} \nu_1 & 0 & 0 & \mu_s \\ 0 & \nu_1 & \mu_s^* & 0 \\ 0 & \mu_s & \nu_2 & 0 \\ \mu_s^* & 0 & 0 & \nu_2 \end{pmatrix}, \tag{45}
\]
is immediately attained if \(\phi_i + \varphi_i = -\mu_i + \pi\) also, but now with \(\tanh 2 \theta_i = |m_i|/|n_i| = |m_s|/|m_c|\) (assuming \(|m_c| \leq |m_s|\)). Since both \(V_1\) and \(V_1^1\) are proportional to the identity, they do not change under unitary local rotations and the two invariant forms are then connected through those operations. As such, the last two conditions on \(|m_c|\) and \(|m_s|\) can be waved by appropriate rotations.

The new transformed elements are
\[
\nu_i = \sqrt{n_i^2 - |m_i|^2}, \tag{46}
\]
\[
\mu_s = e^{-i (\phi_1 - \phi_2)} \frac{m_s}{|m_s|} \sqrt{|m_s|^2 - |m_c|^2}, \tag{47}
\]
(for $|m_c| \leq |m_s|$), and
\[
\mu_c = e^{-i(\phi_1 + \phi_2)} \frac{m_c}{|m_c|} \sqrt{|m_c|^2 - |m_s|^2},
\]
(48)
(for $|m_c| \geq |m_s|$), which then turn explicit the four invariants of the $Sp(2,R) \otimes Sp(2,R)$ group: $I_1 = \det V_1^\dagger$, $I_2 = \det V_2^\dagger$, $I_3 = \det C'$, and $I_4 = Tr [V_1^\dagger ZC'ZV_2^\dagger Z(C')^\dagger Z]$. The general $Sp(2,R) \otimes Sp(2,R)$ transformation \( (41) \) of the \( (43) \) form is reached through the squeezing operation $U_L = U_1 \otimes U_2$:
\[
U_1 = e^{i \frac{\theta_i}{2}} (\kappa_i a_i^\dagger e^{\phi_i} - \kappa_i^* a_i^* e^{-i\phi_i}),
\]
(49)
over the bipartite Gaussian state $\rho_G$, with $|\kappa_i| t = \theta_i \equiv 2r_i$, the squeezing parameter associated with the transformation on the mode $i$ and $e^{i\phi_i} = \kappa_i/|\kappa_i|$. An important result is that while all the BGSS set can be mapped into the $P$-representable set by suitable $Sp(2,R) \otimes Sp(2,R)$ transforms, it is not possible to restore the original matrices $V_1$ and $V_2$ with unitary rotations. That is only reached applying over the squeezing operation. This is immediate from the two invariant forms. Since both covariances reduced matrices $V_1'$ and $V_2'$ are proportional to the identity, unitary rotations transform the invariant forms among themselves.

Remark 3: Through the $\mathbb{S} \rightrightarrows \mathbb{P}$ mapping, we have reached the special subset of locally $Sp(2,R) \otimes Sp(2,R)$ invariant forms. However the general separability condition can possibly be set equivalent to the special $P$-representable subset under an appropriate nonlocal operation forming the mapping $\mathbb{S} \rightarrow \mathbb{P}$. Let us consider again the condition for separability \( (18) \). It can be equivalently written as
\[
V + \frac{1}{2} \text{TET} \geq 0.
\]
(50)
Now let $U_{NL}$ be a nonlocal operation:
\[
U_{NL} VU_{NL}^\dagger = M v,
\]
(51)
where $M$ is a general transformation matrix: $M \in Sp(4,R)$. Such a general $M$, when acting on \( (50) \) must leave $V$ invariant in form \( (V') \), while $M' \text{TETM}$ must go necessarily to $-I$, such that \( (50) \) writes as
\[
V' - \frac{1}{2} I \geq 0,
\]
(52)
which is the transformed $P$-representable subset condition. The Stone - von Neumann theorem provides that if $M$ exists it must be unitarily implementable [23]. Nonetheless, finding the corresponding $U_{NL}$ operator may not be a simple exercise and we leave this point for future research.

VI. CONCLUSION

In conclusion, we have derived a complete description of bipartite Gaussian separable states, and have proved that $P$-representable states form a subset of the set of all bipartite Gaussian separable states, existent only under special symmetry of the covariance matrix. We can state that for positive definite bipartite Gaussian operators, which describe physical quantum states, $P$-representability is a necessary and sufficient condition for separability only for the subset of locally $Sp(2,R) \otimes Sp(2,R)$ invariant Gaussian states [24]. In General $P \subset S$.

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[1] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge Univ. Press, UK, 2000).
[2] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. 80, 869 (1998).
[3] A. Furusawa et al., Science 282, 706 (1998).
[4] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[5] P. Horodecki, Phys. Lett. A 232, 333 (1997).
[6] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).
[7] L.-M. Duan et al., Phys. Rev. Lett. 84, 2722 (2000).
[8] P. Marian, T.A. Marian, and H. Scutaru, J. Phys. A: Math. Gen. 34, 6969 (2001).
[9] V.V. Dodonov, A.S.M. de Castro, and S.S. Mizrahi, Phys. Lett. A 296, 73 (2002).
[10] B.-G. Englert and K. Wódkiewicz, Phys. Rev. A 65, 054303 (2002).
[11] L. Mišta Jr., R. Filip, and J. Fiurášek, Phys. Rev. A 65, 062315 (2002).
[12] E. Santos, Eur. Phys. J. D 22, 423 (2003).
[13] S. Daffer, K. Wódkiewicz, and J.K. McIver, Phys. rev. A 68, 012104 (2003).
[14] M.S. Kim, J. Lee, and W.J. Munro, Phys. Rev. A 66, 030301(R) (2002).
[15] P. Marian, T.A. Marian, and H. Scutaru, Phys. Rev. A 68, 062309 (2003).
[16] G. Adesso, A. Serafini, and F. Illuminati, quant-ph/0310150.
[17] J. Eisert and M.B. Plenio, quant-ph/0312071.
[18] M.C. de Oliveira, quant-ph/0401055.
[19] B.-G. Englert and K. Wódkiewicz, Int. J. Quant. Inf. 1, 153 (2003).
[20] N.J. Higham, Math. Comp. 67, 1591 (1998).
[21] J. Eisert, S. Scheel, and M. B. Plenio, Phys. Rev. Lett. 89, 137903 (2002).
[22] J. Fiurášek, Phys. Rev. Lett. 89, 137904 (2002).
[23] Arvind, B. Dutta, N. Mukunda, and R. Simon, quant-ph/9509002.
[24] It is interesting to note that the author of Ref. [12] have arrived to similar conclusions from a somewhat different approach on classicality and Q-representation.