On the Ballistic Flow of Two-Dimensional Electrons in a Magnetic Field

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Abstract—In conductors with a very low defect density, electrons at low temperatures collide mainly with the sample edges; therefore, the ballistic transport of charge and heat is implemented. An applied perpendicular magnetic field significantly modifies ballistic transport. For the case of two-dimensional electrons, in magnetic fields at which the diameter of cyclotron trajectories is smaller than the sample width, the hydrodynamic transport regime forms. In this regime, the flow is mainly controlled by rare electron-electron collisions, which determine the viscosity effect. In this work, we study the ballistic flow of two-dimensional electrons in long samples in magnetic fields up to the critical field of the transition to the hydrodynamic regime. By solving the kinetic equation, we obtain analytical formulas for the current density and the Hall electric field far and close to the ballistic-hydrodynamic transition, as well as for the longitudinal and Hall resistances in these ranges. Our theoretical results apparently describe the observed longitudinal resistance of pure graphene samples in the magnetic-field range below the ballistic-hydrodynamic transition.

Keywords: two-dimensional electrons, high mobility, ballistic transport, magnetoresistance, Hall effect

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1. INTRODUCTION

In rather small pure two- and three-dimensional conductor samples, electrons at very low temperatures most often collide with the edges of a sample. In this case, electron transport is ballistic. As the temperature increases, electron-electron collisions can lead to the formation of a viscous electron fluid and the implementation of hydrodynamic transport. Although the theory of a viscous electron fluid has been intensively developed for a long time [1–3], undoubted experimental evidence for the formation of such a fluid was obtained only recently in high-quality graphene, Weyl semimetals, and GaAs quantum wells [4–25]. The conclusion regarding the presence of a hydrodynamic flow in these experiments was drawn on the basis of the observed specific dependence of the average resistance of a sample on its width [4], nonlocal negative resistance [5–7], giant negative magnetoresistance [12–20], and magnetic resonance at the double cyclotron frequency [21–25]. The experimental implementation of hydrodynamic transport has led to the development of its theory in new directions (see, for example, [26–33]).

In recent works [10, 11], the spatial distribution of the current density and Hall electric field in a flow of two-dimensional (2D) electrons in graphene strips was measured. In [10], the observed evolution of the Hall-field profile curvature served as evidence for a transition between the ballistic and hydrodynamic transport regimes. In [11], the current density of hydrodynamic and Ohmic flows in a narrow strip was determined by measuring the distribution of a local magnetic field induced by the current in the strip.

In [34, 35], the flow of interacting 2D electrons in a narrow ballistic sample was theoretically investigated in the limit of weak magnetic fields using the analytical solution of a simplified kinetic equation. It was shown that, at low temperatures, the Hall electric field in almost the entire sample, except the very edge vicinities, is half of its usual value in macroscopic ohmic samples (the exception of the edge vicinities was established in [38]). In addition, it was demonstrated in [34, 35] that interparticle collisions, first, control the maximum ballistic trajectory length, which determines the ballistic current and the Hall electric field and, second, lead to small hydrodynamic corrections, which are precursors of the formation of a viscous flow.

In [36], the transition between the ballistic and hydrodynamic transport regimes for the Poiseuille flow of 2D electrons in a perpendicular magnetic field was theoretically investigated using numerical solution of the kinetic equation. In the magnetic field \( B = B_c \), in which the diameter of the electron cyclotron orbit \( 2R_c \) becomes equal to the sample width \( W \) and some electrons start to make a complete revolution without scattering at the edges, the longitudinal and Hall resis-
The aim of this study is to theoretically investigate the ballistic transport of interacting 2D electrons in high-quality long samples in magnetic fields. Using the general analytical solution of the kinetic equation in the ballistic regime from [37, 38], we show that, in the field range of $B_1 < B < B_2$ (more precisely, $B \gg B_1$ and $B_2 - B_1 \gg B - B_2$), the electron flow at low temperature is mainly determined by electron scattering at the sample edges and the cyclotron effect of a magnetic field. Weak electron–electron collisions with the intensity $\gamma$ determine the $B_1$ and $B_2$ values ($B_1 \propto \gamma^2$ and $B_2 - B_1 \propto \gamma$) and only lead to minor corrections to all the characteristics of the flow in this range.

We calculate the current density and Hall field, as well as the longitudinal and Hall resistances $\rho_{xx}(B)$ and $\rho_{xy}(B)$. With an increase in the magnetic field from $B_1$ to $B_2$, the current density profile $j(y)$ evolves from almost flat to a deformed semicircle. The obtained Hall-field profiles $E_H(y)$, both in low fields ($B_1 \ll B \ll B_2$) and near the transition ($B_2 - B_1 \ll B - B_1$), are nonplanar and diverge at the sample edges: $E_H(y) \neq \text{const}$ and $E_H(y) \to \infty$ as $y \to \pm W/2$. The Hall-field amplitude becomes independent of magnetic field $B$ in the region of moderately low fields ($B_1 \ll B \ll B_2$); therefore, the $\rho_{xy}(B)$ value in this region is plateau-like. This $\rho_{xy}(B)$ behavior is apparently a form of the ballistic anomaly of magnetoresistance, which was previously observed experimentally and obtained by numerical simulation for 2D samples with four contacts [39].

The magnetic-field dependences of the resistances $\rho_{xx}(B)$ and $\rho_{xy}(B)$ and their derivatives with respect to $B$ over the entire range of $B_1 < B < B_2$ exhibit a nontrivial nonmonotonic behavior and agree with the results of numerical simulation [36]. The theoretical result for $\rho_{xx}(B)$ apparently corresponds to the dependences of the resistance of high-quality graphene samples and GaAs quantum wells experimentally observed in [10, 11, 18].

2. MODEL

We consider a flow of 2D electrons in a high-quality long sample with the length $L$ and rough edges at low temperature. The scattering of electrons at the edges is assumed to be diffusive: the momentum of reflected electrons is isotropically distributed regardless of the momentum of incident electrons. In the bulk of the sample, electrons are rarely scattered by each other and/or weak disorder (Fig. 1a).

Our approach allows one to consider systems that are mixtures of two limiting cases: (i) there are no defects inside a sample and electrons in the bulk are only scattered at each other with momentum conservation and (ii) there are no interparticle collisions and electrons are scattered by disorder in the bulk of the sample, which leads to weak relaxation of the momentum. We assume the rate of any scattering in the bulk of the sample to be low: $W \ll l$, where $l = \nu_F/\gamma$ is the electron mean free path with respect to all scattering mechanisms in the sample volume, $\gamma$ is the total scat-
tering rate, and \( v_F \) is the electron velocity at the Fermi level.

In weak magnetic fields \((B < B_c)\), when the diameter of the cyclotron orbit is larger than the sample width \((2R_c > W)\), each electron is mainly scattered at the sample edges. Consequently, in the main order with respect to \( \gamma \) in such magnetic fields, the electron-scattering regime, generally speaking, is ballistic. In this case, scattering in the bulk can limit the time spent by electrons on ballistic trajectories \([36–38]\) and to small corrections to the parameters of a purely ballistic flow, which completely disregard electron–electron collisions \([38]\).

In strong magnetic fields \((B > B_c)\) corresponding to \(W > 2R_c\), electrons are divided into two groups according to their dynamics: “edge” electrons, which mainly move along the “jumping” trajectories and only scatter at one of the sample edges and “central” electrons with trajectories that do not touch the edges \([36–38]\).

Near the transition field \((B - B_c \ll B_c)\), the centers of the “central” electron trajectories lie in a narrow range of coordinates \(y\) at the sample center, \(|y| < W/2\) therefore, their fraction is much smaller than the fraction of “edge” electrons, so they are scattered mainly at “edge” electrons and/or the bulk disorder. The occurrence of the “central” electron phase is the initial stage of the formation of the phase of bulk (2D) electrons, which is responsible for hydrodynamic/Ohmic transport at \(W \gg R_c\) \([38]\).

We are searching for the linear response of 2D electrons on a uniform electric field \(E_0 \parallel x\) in an external magnetic field \(B\) perpendicular to the sample plane (Fig. 1a). The corresponding 2D electron distribution function acquires a nonequilibrium part \(\delta f(y, p) = -f_0(\epsilon)f(y, \varphi, \epsilon) \approx E_0\), where \(f_0(\epsilon)\) is the Fermi distribution function, \(\epsilon\) is the electron energy, \(\varphi\) is the angle between the electron velocity \(v = v(\epsilon)\sin \varphi, \cos \varphi\) and the normal to the left sample edge, \(p = mv\) is the electron momentum, and \(m\) is the electron mass (Fig. 1a). There is no dependence of \(\delta f\) on coordinate \(x\) along the sample, since \(L \gg W\). In addition, we ignore the dependence of the electron velocity \(v(\epsilon) = |v(\epsilon)|\) and the nonequilibrium part of the distribution function \(\delta f(y, p)\) on the electron energy. Such a simplification is allowed for a degenerate electron distribution. Below, we use units in which the characteristic electron velocity \(v(\epsilon) \equiv v_F\) and the elementary charge \(e\) are set to unity. In the selected units, the coordinate, time, and reverse field \(1/E_0\) have the same dimensionality.

The kinetic equation for the nonequilibrium distribution function \(f(y, \varphi, \epsilon)\) takes the form

\[
\cos \varphi \frac{\partial f}{\partial y} - \sin \varphi E_0 - \cos \varphi E_{1y} - \omega_0 \frac{\partial f}{\partial \varphi} = St[f], \tag{1}
\]

where \(\omega_0 = eB/mc\) is the cyclotron part, \(E_{1y} = E_0(y)\) is the Hall electric field induced by the redistribution of electrons in a magnetic field, and the collision integral \(St[f]\) describes both the momentum-conserving electron–electron collisions and scattering at bulk disorder, which leads to momentum relaxation.

In this work, we use a simplified form of the integrals of electron–electron collisions and scattering at disorder, which allows us to obtain an asymptotically accurate (with respect to \(\gamma W \ll 1\)) analytical solution of the kinetic equation at \(B < B_c\)

\[
St[f] = -\gamma_f + \gamma_{ee} \hat{P}_e[f] + \gamma_d \hat{P}_d[f], \tag{2}
\]

where \(\gamma_{ee}\) and \(\gamma_d\) are the rates of electron–electron scattering and scattering at disorder, \(\gamma = \gamma_{ee} + \gamma_d\) is the total scattering rate, and \(\hat{P}_e\) and \(\hat{P}_d\) are the projection operators for the \(f(\varphi)\) functions in the subspaces \([1, e^{\varphi \varphi}]\) and \([1]\), respectively. Such a collision integral preserves the distribution function perturbations corresponding to the nonequilibrium density. Integral (2) at \(\gamma_d = 0\) describes also momentum conservation in interparticle collisions. This \(St\) form was used, in particular, in \([34, 35]\) to study the ballistic transport of interacting 2D electrons at \(B \to 0\) and in \([38]\) to construct the mean field theory for describing the phase transition between the ballistic and hydrodynamic-transport regimes near \(B = B_c\).

We assume that the longitudinal edges of the sample are completely rough. Thus, the electron scattering at the edges is of the diffusion type and the boundary conditions for the distribution function have the form \([35, 38, 39]\) (see also Figs. 1a, 1b):

\[
f(-W/2, \varphi) = c_t \text{ at angles within } \pm \pi/2 < \varphi < \pi/2 \text{ and } f(W/2, \varphi) = c_r \text{ at angles within } \pi/2 < \varphi < 3\pi/2,
\]

where the quantities \(c_t = c_t[f]\) and \(c_r = c_r[f]\) are proportional to the \(y\) components of the incident electron flow on the left \((y = -W/2)\) and right \((y = W/2)\) sample edges.

\[
\begin{align*}
    c_t &= \frac{1}{2} \int_{\pi/2}^{3\pi/2} d\varphi' \cos \varphi' f(-W/2, \varphi' ), \\
    c_r &= \frac{1}{2} \int_{\pi/2}^{3\pi/2} d\varphi' \cos \varphi' f(W/2, \varphi' ). \tag{3}
\end{align*}
\]

Boundary conditions with coefficients (3) mean that the probability of electron reflection at a rough edge is independent of the scattering angle \(\varphi\) and the transverse component of the electron flow \(j(x) = [n_0/(\pi m \gamma)] \int_{2\pi}^{2\pi} d\varphi' \cos \varphi' f(y, \varphi')\) vanishes at the edges:

\[
    j(x = \pm W/2) = 0 \text{ (therefore, everywhere in the sample, due to the continuity equation div } j = dj/dy \text{ = 0).}
\]

The current density along the sample \(j(y) \equiv j_x(y)\) is calculated by the formula

\[
    j(y) = \frac{n_0}{\pi m} \int_{-\pi/y/2}^{\pi/y/2} d\varphi' \sin \varphi' f(y, \varphi'). \tag{4}
\]

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If an electric current flows through the sample in an external magnetic field, then, under the action of the Lorentz force, the charge-density perturbation and the Hall electric field arise. Both effects are described by the zero \((m = 0)\) angular harmonic of the distribution function

\[
f_{m=0}(y) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi f(y, \varphi). \tag{5}\]

Figure 1b shows the regions on the \((y, \varphi)\) plane that correspond to the ballistic motion of electrons reflected at the right and left sample edges. In the narrow samples \((W \ll R_c)\), the shape of the “left” and “right” ballistic regions is close to rectangular: \([-\pi/2, \pi/2] \times [-W/2, W/2]\) and \([\pi/2, 3\pi/2] \times [-W/2, W/2]\). In wider samples \((W \sim R_c, W < 2R_c)\), the boundaries of the “left” and “right” regions \{curves \(\varphi_+(y)\) and \(\pm \pi + \varphi_-(y)\), \(\varphi_+(y) = \arcsin[1 - \varphi_0(W/2 \pm y)]\}\}, become dependent on the \(y\) coordinate (see Fig. 1b). The boundary lines coincide with the trajectories of electrons falling tangentially to the edges. Such electrons on the boundary trajectories are not described by the above boundary conditions. Consequently, the distribution function \(f(y, \varphi)\) is not defined at the points \(\varphi_+(y)\) and \(\pm \pi + \varphi_-(y)\) and can therefore have a discontinuity or another singularity at these points.

It is convenient to rewrite kinetic equation (1) as

\[
\left[ \cos \varphi \frac{\partial}{\partial y} + \gamma \right] \tilde{f} - \sin \varphi E_0 = \gamma_c \frac{\partial \tilde{f}}{\partial \varphi}, \tag{6}\]

where the function \(\tilde{f}(y, \varphi) = f(y, \varphi) + \tilde{f}(y, \varphi)\) is introduced and the departure and arrival terms of the collision integral were transferred from the left to the right. Here, \(\tilde{f}\) is the electrostatic potential of the Hall electric field \(E_H = -\tilde{f}\). Indeed, it follows from Eq. (1) that the Hall potential \(\tilde{f}(y)\) plays the same role in the transport equation as the zero harmonic of the distribution function, which is proportional to the electron density and related to the Hall potential by electrostatic equations. Thus, it is convenient to introduce the function \(\tilde{f}(y, \varphi) = f(y, \varphi) + \tilde{f}(y)\) in uniform way.

The zero harmonic \(f_{m=0}(y)\) of the generalized distribution function \(\tilde{f}\) has the form \(f_{m=0}(y) = \tilde{f}(y) + \phi(y)\), where \(\tilde{f}(y)\) is the perturbation of the chemical potential of electrons. For stationary (and rather slow) flows, the quantities \(\tilde{f}(y)\) and \(\phi(y)\) are connected by electrostatic relations [25]. For the investigated case of a one-component 2D electron gas, the electrostatic potential \(\phi\) is usually significantly larger than the corresponding chemical-potential perturbation \(\tilde{f}\) [25].

Therefore, the Hall electric field is calculated using the simple formula \(E_H(y) = -df_{m=0}(y)/dy\).

For brevity, we hereinafter omit the tilde in the function \(\tilde{f}\) and use simply \(f \equiv \tilde{f}\).

### 3. General Solution

#### IN THE BALLISTIC REGIME

As was shown in [37, 38], in the presence of weak electron–electron scattering, in almost all magnetic fields below the critical one \((B < B_c)\), specifically, at \(2 - \omega_c W \gg \gamma W\), the electron flow is ballistic. In this case, in the main order of the scattering rate \(\gamma\), such flows are described by kinetic equation (6) with an omitted arrival term [37, 38]

\[
\left[ \cos \varphi \frac{\partial}{\partial y} + \gamma \right] d - \sin \varphi E_0 = \omega_c \frac{\partial f}{\partial \varphi}. \tag{7}\]

The analysis of Eq. (7) showed that the ballistic regime is divided into three sub-regimes [38]. The transition between them is accompanied by the evolution of the current density \(j(y)\) and the Hall electric field \(E_H(y)\) profiles and the change in the type of magnetic-field dependences of the longitudinal \((\rho_{xx}(B))\) and Hall \((\rho_{xy}(B))\) resistances.

These three sub-regimes are as follows.

(i) \(\omega_c \ll \gamma W\). The length of the maximum cyclotron orbit segment that can be adjusted in a strip \(l_b^{(2)} = R/W\) is larger than the average electron mean free path relative to bulk scattering: \(l_b^{(2)} \gg l = 1/\gamma\). Therefore, bulk scattering determines the effective ballistic-trajectory length for most electrons: \(l_b^{(3)} \sim l\).

(ii) \(\gamma^2 W \ll \omega_c \ll 1/W\). In this sub-regime, on the contrary, we have \(l_b^{(2)} \ll l\), while the magnetic parameter \(\omega_c W\) is small. The maximum length of the ballistic trajectory is determined by the geometry of the trajectories and therefore is equal to \(l_b^{(2)}\).

(iii) \(\omega_c \sim 1/W\) under the condition \(2 - \omega_c W \gg \gamma W\).

In this case, we also have \(l_b^{(2)} \ll l\), but the magnetic parameter \(\omega_c W\) is about 1, \(\omega_c W \sim 1\), or close to the critical value of \(\omega_c W = 2\). The average ballistic trajectory length \(l_b^{(3)}\) is also determined by the geometry of electron trajectories and has the same order of magnitude as \(W\).

An analytical solution to kinetic equation (7), which describes all three sub-regimes (i)–(iii) was obtained in [37, 38] by the method of characteristics for ordinary differential equations. The obtained solution \(f(y, \varphi)\) is a discontinuous function with continuity domains shown in Fig. 1b. For electrons reflected from the left and the right sample edges, the trajec-
ries of which lie within \(-\pi + \phi_\gamma(y) < \phi < \phi_\gamma(y)\) and \(\phi_\gamma(y) < n + \phi < \phi_\gamma(y)\), respectively, we use the designation \(f(y, \phi) = f_\omega(y, \phi)\) (see Fig. 1b). The two components \(f_\omega(y, \phi)\) of function \(f\) have the form

\[
f_\omega(y, \phi) = \frac{E_0}{\gamma + \omega_\phi^2} \times [\omega_\phi \cos \phi + \gamma \sin \phi + e^{i\omega_\phi y} Z_\omega(\sin \phi + \omega_\phi y)],
\]

where the contributions \(\omega_\phi \cos \phi\) and \(\gamma \sin \phi\) independent of coordinate \(y\) are particular solutions of Eq. (6) corresponding to the Drude formulas (when the scattering rate \(\gamma\) corresponds to scattering at disorder only). The contribution \(e^{i\omega_\phi y} Z_\omega(X)\) is the general solution of kinetic equation (7) without the field term \(\sin \phi E_0\). It allows to satisfy the correct boundary conditions with nonzero parameters \(c_{l, r}\) (3).

Substituting Eq. (8) into the boundary conditions \(f(y = \pm W/2) = c_{l, r}\), we obtain the explicit form of \(Z_\omega(X) [38]

\[
Z_\omega(X) = e^{-\frac{\arcsin(X + \omega_\phi W/2)}{\omega_\phi}} \times [c_l - \gamma(X + \omega_\phi W/2) - \omega_\phi \sqrt{1 - (X + \omega_\phi W/2)^2}]
\]

and

\[
Z_{-\omega}(X) = e^{-\frac{\pi - \arcsin(X - \omega_\phi W/2)}{\omega_\phi}} \times [c_r - \gamma(X - \omega_\phi W/2) + \omega_\phi \sqrt{1 - (X - \omega_\phi W/2)^2}],
\]

where the coefficients \(c_l\) and \(c_r\) are determined from the balance boundary conditions \(j_\gamma(y = \mp W/2) = 0\), which take the form

\[
\begin{pmatrix}
I_{ll} & I_{lr} \\
I_{rl} & I_{rr}
\end{pmatrix}
\begin{pmatrix}
c_l \\
c_r
\end{pmatrix}
= -\begin{pmatrix}
I_l \\
I_r
\end{pmatrix}.
\]

In this equation, the coefficients in the first row of the matrix are expressed in the form of the integrals

\[
I_{ll} = 2 + \int_{-\pi/2}^{\pi/2} d\phi \cos \phi e^{\frac{\pi + 2\phi}{\omega_\phi}}
\]

and

\[
I_{rl} = \int_{\pi/2}^{\pi + \phi_0} d\phi \cos \phi e^{\frac{\phi - \arcsin(\phi + \omega_\phi W)}{\omega_\phi}},
\]

while the first component on the right-hand side is

\[
I_l = \frac{\pi \omega_\phi}{2} + \int_{-\pi/2}^{\pi/2} d\phi \cos \phi e^{\frac{2\phi - \pi}{\omega_\phi}} (\omega_\phi \cos \phi - \gamma \sin \phi)
\]

\[
- \int_{\pi/2}^{\arcsin(\phi + \omega_\phi W)} d\phi \cos \phi e^{\frac{\phi - \arcsin(\phi + \omega_\phi W)}{\omega_\phi}}
\]

\[
\times [\omega_\phi \sqrt{1 - (\sin \phi + \omega_\phi W)^2} + \gamma(\sin \phi + \omega_\phi W)].
\]

The remaining coefficients in Eq. (11), i.e., \(L_{ll}, L_{lr}\) and \(L_{rl}\), are related to \(I_{ll}, I_{lr}\) and \(I_{rl}\) as \(I_{ll} = -I_{rl}\), \(I_{lr} = -I_{ll}\), and \(I_{rl} = I_{lr}\). In Eqs. (12)–(14), the designation \(\phi_\gamma = \arcsin(1 - \omega_\phi W)\) is introduced.

For an arbitrary value \(\omega_\phi W\), the integrals (12)–(14) can be determined only numerically. The explicit expressions for them and, consequently, for the quantities \(c_{l, r}\) and \(j(y)\), \(E_{\text{int}}(y)\) can be obtained analytically in the following limiting cases: in subdomain (i) of the ballistic regime, \(\omega_\phi \ll \gamma W\) (see [35, 38]; in subdomain (ii) of the ballistic regime, \(\gamma W \ll \omega_\phi \ll 1/W\) (see [37, 38] and Section 4 of this paper); and in the right-hand singular part of subdomain (iii) of the ballistic regime, \(\gamma W \approx 2 - \omega_\phi W \ll 1\) (see [38] and Section 4 of this paper).

Sub-regime (i) was studied in detail in [34, 35, 38] by solving kinetic equation (7) using the method of successive approximations and by analyzing the exact solution of (8). In this sub-regime, the external magnetic field introduces only small corrections to the central part of the flow in the region \(|W/2 - |y|| \gg \omega_\phi/\gamma^2\), but, at the same time, it leads to the solution for electrons at the near-edge regions \(|W/2 - |y|| \leq \omega_\phi/\gamma^2\), which is not described by perturbation theory.

Sub-regimes (ii) and (iii) were investigated in part in [37, 38]. Further, using the general solution of (8), we obtain new results for the flows in the ranges (ii) and (iii), specifically, we analytically calculate the \(j(y)\) and \(E_{\text{int}}(y)\) profiles and the magnetoresistances \(\rho_\phi(\omega_\phi)\) and \(\rho_{\text{int}}(\omega_\phi)\).

4. PURELY BALLISTIC TRANSPORT IN MODERATE MAGNETIC FIELDS

In the second and third ballistic sub-regimes, where

\[
\omega_\phi \gg \gamma^2 W, \quad 2 - \omega_\phi W \gg \gamma W,
\]

generally speaking, the electron dynamics in the framework of Eqs. (3) and (7), even in the main order by the small parameter \(\gamma W\), could be controlled by applied fields and scattering at the edges, as well as by scattering in the bulk of the sample. Indeed, in classically weak fields (\(\omega_\phi \ll \gamma\)) included in interval (15), the rate of the momentum redistribution due to cyclotron rotation \(\omega_\phi\). This is reflected, in particular, in the fact that the first term of formula (8) dominates in the denominator \(\gamma^2 + \omega_\phi^2\). However, our performed analysis of Eqs. (8)–(14) shows that, in both subdomains \(\gamma^2 W \ll \omega_\phi \ll \gamma\) and \(\gamma \ll \omega_\phi \ll 1/W\) of interval (15), the electron distribution function in the main order for \(\gamma\) is determined by scattering at the edges and the effect of applied fields,
while the electron—electron collisions only lead to minor corrections.

This analysis is based on asymptotic expansion of the argument of the exponent in functions $f_\gamma (8)$—(10) in all $\gamma$ ranges, both for the small differences $(\pi / 2 - |\phi| \leq \omega_c W)$ and for the large ones $(\pi / 2 - |\phi| \gg \omega_c W)$. The form of these expansions depends on the ratios $\gamma / \omega_c$ and $\gamma^2 W / \omega_c$. When the latter quantity is small, the exponent argument is small over the entire angular range. Due to the preservation of only one (at $\omega_c \gg \gamma$) or two (at $\gamma^2 W \ll \omega_c \ll \gamma$) terms in the expansion of the exponent in a series, distribution function $f_\gamma (8)$ in the main order for the parameter $\gamma^2 W / \omega_c \ll 1$ (and $\omega_c W \ll 1$ at $\omega_c \ll \gamma$) becomes independent of the $\gamma$ value at any angles $\phi$.

The negligible role of bulk scattering in the main order for $\gamma$ in both ranges $\gamma^2 W \ll \omega_c \ll \gamma$ and $\gamma \ll \omega_c \ll 1 / W$ can be qualitatively explained as follows. As we noted in the previous section, at $\omega_c \gg \gamma$ the maximum size of the ballistics trajectory is limited not by the length $l = 1 / \gamma$ related to bulk scattering, but by the maximum length $l_{bc}^{(2)} = \sqrt{R W}$ of the cyclotron orbit segment that can be fit to the sample (see Fig. 1a). Consequently, for both these $\omega_c$ ranges, its subsequent reflection from the same or opposite edge is more probable than scattering in the bulk at other electrons or disorder. Such dynamics of electrons determines the current density along the $x$ axis and the Hall electric field, which is obtained from the requirement for the absence of current and average acceleration along the $y$ axis. Therefore, the current density and the Hall field in the main order for the parameter $l_{bc}^{(2)} / l \ll 1$ (and $W / l_{bc}^{(2)} \ll 1$ at $\omega_c \ll \gamma$) are described by the purely ballistic formulas that do not take into account scattering in the bulk.

Thus, over the entire range (15), Eq. (8) at $\gamma = 0$ yields the desired distribution function in the main order in the bulk-scattering rate $\gamma$ for all $y$ and $\phi$:

$$f_\gamma(y, \phi) = \bar{c}_{ir} + \frac{E_0}{\omega_c} \times \{\cos \phi \pm \sqrt{1 - [\sin \phi + \omega_c(y \pm W / 2)]^2}\},$$

where $\bar{c}_{ir} = E_0 \bar{c}_{ir} / \omega_c^2$. We note that, at $\gamma^2 W \ll \omega_c \ll \gamma$, the literal transition to the limit $\gamma \to 0$ in function (8) is impossible and the coincidence of the final form of the function $f_\gamma$ at $\gamma^2 W \ll \omega_c \ll \gamma$ with Eq. (16) in the main order for the parameter $\omega_c W \ll 1$ is the result of analysis of the first two terms of the asymptotics of general expression (8) by the parameter $\gamma^2 W / \omega_c \ll 1$ under the condition $\omega_c W \ll 1$.

Any solution of Eq. (6) at $\gamma = 0$ can be found accurate to a constant, which corresponds to the absence of relaxation of electron-density perturbations. Consequently, the system of algebraic equations for coefficients $c_{ir}$ (11) is degenerate. Imposing the symmetry condition $c_i + c_r = 0$, we find from system (11)

$$\bar{c}_{ir} = \frac{E_0}{2\omega} \frac{U - V}{2\omega - 2\omega},$$

where $U = \arccos(1 - \omega_c W)$ and $V = (1 - \omega_c W)\sqrt{\omega_c W \times 2 - \omega_c W}$. We note that the form of solution (16) was obtained recently in [37, 38], but its range of applicability was not fully analyzed there.

As can be seen from Eq. (16), for intermediate magnetic fields at $\omega_c \sim 1 / W$ (the left-hand side of subregime (iii)), the current density $j(y)$ is estimated as

$$j(y) \sim j_0,$$

where $j_0 = n_0 E_0 W / m$ is the characteristic density of the purely ballistic current. Equation (18) follows from the fact that the typical ballistic trajectories, which make the main contribution to the current $j$ at $\omega_c \sim 1 / W$, have the length $\sim W$. In the same fields, $\omega_c \sim 1 / W$, the distribution function (16) also allows us to estimate the Hall field:

$$E_H(y) \sim E_0.$$
Next, we obtain analytical expressions for the flow characteristics within the limits $\omega cW \to 0$ and $\omega cW \to 2$, i.e., in sub-regime (ii) and the right-hand singular part of sub-regime (iii).

According to distribution function (16), in the weak magnetic-field limit $\omega cW \ll 1$, the main contribution to the transport characteristics is made by electrons moving along the edges of the sample with angles $\varphi \approx \pm \pi/2$. The asymptotic form of Eq. (16) at angles $|\varphi| \ll |\pi/2 - |\varphi|| \ll 1$ is

$$f_\pm(y, \varphi) = \hat{c}_{trl} + E_0 \left[ \left( y \pm \frac{W}{2} \right) \sin \varphi \cos \varphi \right] + \omega_c \left( \frac{y \pm W/2}{2} \right)^2 \left( \frac{1}{\cos \varphi} + \frac{\sin^2 \varphi}{\cos^3 \varphi} \right),$$  

where

$$\hat{c}_{trl} = \mp \frac{\sqrt{3}E_0}{2} \sqrt{\omega cW^3}. \quad (21)$$

At the angles $|\pi/2 - |\varphi|| \lesssim \sqrt{\omega cW}$, it is necessary to use the complete expression (16), in which $\sin \approx \pm 1 - (|\varphi| - \pi/2)^2/2$ and $\cos \varphi \approx ||\varphi| - \pi/2|$.

The current density $j(y)$ (Eq. (4)) corresponding to distribution function (20) consists of the main part independent of the $y$ coordinate and a small correction to the logarithmic parameter $L_B = \ln[1/(\omega cW)] \gg 1$ dependent on $y$

$$j(y) = j_B + \Delta j(y), \quad \frac{j_B}{j_0} = \frac{1}{\pi} \ln \left( \frac{1}{\omega cW} \right),$$

where

$$\Delta j(y) = f_1(y) + f_2(y) + f_3(y). \quad (22)$$

The $f_1(y), f_2(y)$, and $f_3(y)$ functions were calculated analytically. They have similar profiles with a divergent coordinate derivative at the sample edges $y = \pm W/2$

$$f_1(y) = \frac{1}{\pi} \left( \frac{1}{\sqrt{2} \frac{W}{y}} + \frac{1}{\sqrt{2} \frac{y}{W}} \right). \quad (23)$$

Fig. 2. (a) Total current, (b) Hall electric field, and (c) longitudinal and (d) Hall resistance for a sample without bulk scattering ($\gamma = 0$) as functions of the magnetic field in the second ($0 < \omega cW \ll 1$) and the third ($1 \lesssim \omega cW < 2$) ballistic sub-regimes. The resistances are given in units of $\rho_0 = E_0/j_0 = m/(n_0W)$. 
The similarity of the profiles can be seen in Fig. 3a, which shows the $f_1(y)$, $f_2(y)$, and $f_3(y)$ dependences.

The Hall-field potential (5) calculated from the distribution function (20) takes the form

\[
\phi(y) = E_0 [g_1(y) + g_2(y)],
\]

where

\[
g_1(y) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{y/W}} - \frac{1}{\sqrt{1 + y/W}} \right),
\]

and

\[
g_2(y) = \frac{1}{2\pi} \left( \frac{1}{\sqrt{1 + y/W}} + \frac{1}{\sqrt{1 - y/W}} \right) \ln \left( \frac{\sqrt{1 + y/W}}{\sqrt{1 - y/W}} \right).
\]

These formulas describe the Hall field in the main order for the parameter $\omega_c W \ll 1$. Both functions, $g_1(y)$ and $g_2(y)$, have derivatives divergent near the sample edges, which leads to divergence of the Hall field $E_{H}(y) = -\phi(y)$ (see Fig. 3b).

In Fig. 4, we compare the $j(y)$ and $E_{H}(y)$ profiles numerically calculated using exact distribution function (16) and analytical expressions (22) and (26). It can be seen that analytical curves (22) and (26) correctly reproduce the results of the numerical calculation using (16) for the current and the Hall field in the limit $\omega_c W \ll 1$. The current density $j(y)$ calculated accurately contains a correction of $\sim 1$ independent of $y$, which is not taken into account in formulas (22)–(25); the numerically obtained Hall field is accurately reproduced by Eqs. (22)–(26).

Using Eqs. (22)–(28) in the main order for $\omega_c W$, we obtain the following results for the total current $I$ and the Hall voltage $U_{H} = E_{H}/(W/2) - E_{H}/(-W/2)$:

\[
I = \frac{L_B}{\pi} j_0 W, \quad U_{H} = \frac{1}{\pi} E_0 W.
\]

Consequently, the longitudinal and Hall resistances are

\[
\rho_{xx} = \frac{\rho_0 L_B}{\pi}, \quad \rho_{xy} = \frac{\rho_0 L_B}{\pi},
\]

where $\rho_0 = E_0/j_0 = m/(\hbar n_0 W)$. It can be seen that both these quantities have a weak singularity due to the presence of the factor $1/L_B(B) \sim 1/\ln(1/B)$.

Figure 5 shows the dependences of the total current, the average Hall field, and the corresponding resistances in the second ballistic sub-regime $\omega_c W \ll 1$ on the parameter $\omega_c W$, which characterizes the magnetic-field value. We compare the results of numerical calculation of all the quantities using distribution function (16) with the analytical results of (29).
and (30). It can be seen that the latter describe well the numerical calculation. In addition, it can be seen that the singularities in the resistances \( \rho_{xx} \) and \( \rho_{xy} \) at \( B \to 0 \) are very weak and, thus, are looked like the dependences with some finite limits.

Near the transition, at \( \gamma W \ll 2 - \omega_c W \ll 1 \) (the right-hand side of the third ballistic sub-regime), coefficients \( \tilde{c}_{l, r} \), (17) rapidly diverge. In the main order by the parameter \( u = 2 - \omega_c W \ll 1 \), they take the form

\[
\tilde{c}_{l, r} = \pm (\pi E_0/2 \omega_c (2 - \omega_c W)) \quad \text{and become larger than}\quad \text{the other terms of Eq. (16).}
\]

Then, the main part of the distribution function is

\[
f_{\pm}(y, \varphi) = \pm \frac{\pi E_0}{2 \omega_c u}.
\]

The terms omitted in this formula are of the order of magnitude \( E_0/\omega_c \).

Distribution function (31) describes the disbalance between the densities of excess electrons reflected from the left and right edges, which is needed to compensate the nonequilibrium current \( j_y \) induced by the direct action of the fields \( E_0 \) and \( B \) on electrons (the first two terms in formula (8)).

Calculation by formulas (4) and (5) with distribution function (31) yields the current density

\[
j(y) = \frac{E_0}{\omega_c u} \left[ \sqrt{1 - \left( \frac{\omega_c y - \mu}{2} \right)^2} + \sqrt{1 - \left( \frac{\omega_c y + \mu}{2} \right)^2} \right] \quad (32)
\]

and the electrostatic potential of the Hall field

\[
\phi_H(y) = \frac{E_0}{2 \omega_c u} \left[ \arcsin \left( \frac{\omega_c y - \mu}{2} \right) + \arcsin \left( \frac{\omega_c y + \mu}{2} \right) \right].
\]

Both quantities (32) and (33), as well as distribution function (31), diverge as \( \sim 1/u \) in magnetic fields close to the transition field \( \gamma W \ll 2 - \omega_c W \ll 1 \). It can be seen that the current-density profile is a distorted semicircle with very large derivatives at the sample edges \( y = \pm W/2 \), while the Hall potential profile has the form of a distorted arccosine. These quantities for the sample without bulk scattering \( (\gamma = 0) \) are shown in Fig. 6 for the transition point \( u = 0 \) \( (B = B_c) \), and for the magnetic field near the transition from the weaker-field side \( u \ll 1 \) \( (B_c - B \ll B_c) \).

For the total current and Hall voltage in the main and subsequent orders for the parameter \( \sqrt{u} \), from formulas (32) and (33), we obtain

\[
I = \frac{\pi j_0 W}{4 u}, \quad U_H = \frac{\pi E_0 W}{2 u} \left( 1 - \frac{\sqrt{2u}}{\pi} \right). \quad (34)
\]

The corresponding resistances are

\[
\rho_{xx} = \frac{4 u}{\pi} \rho_0, \quad \rho_{xy} = \left( 1 - \frac{\sqrt{2u}}{\pi} \right) 2 \rho_0. \quad (35)
\]

The longitudinal resistance \( \rho_{xx} \) as a function of \( u \) decreases linearly to zero at \( u \to 0 \), while the Hall
resistance $\rho_{xy}$ shows the root divergence at $u \to 0$. This behavior of the resistances was also obtained by us using direct numerical calculation by distribution function (16) (see Fig. 2).

5. DISUSSION OF THE RESULTS

The comparison of the calculated current and Hall field in relatively low ($\gamma^2 W^2 \ll \omega_c W \ll 1$), relatively large near-critical ($\gamma W \ll 2 - \omega_c W \ll 1$), and intermediate ($\omega_c W \sim 1$) magnetic fields shows that, with an increase in the magnetic field, the current density profile becomes increasingly convex and, for the Hall-field profiles, the width of the diverging features at the sample edges increases (the profiles for the first and second cases are shown in Figs. 4 and 6 and, for the third case, the shapes of the $j(y)$ and $E_H(y)$ curves are intermediate between the corresponding curves in Figs. 4 and 6 and therefore are not shown). For the current density, the ratio between the uniform $j(y)$ part and the nonuniform one decreases with an increase in $\omega_c$ in the range of $\gamma^2 W^2 \ll \omega_c W \ll 1$ as a logarithm, $\propto \ln \left[ 1/(\omega_c W) \right]$ (see formulas (22)). The last formula, as well as numerical calculation of the current using distribution function (16), show that the uniform and nonuniform $j(y)$ portions are of the same order of magnitude as $\omega_c W \sim 1$.

We note the nontrivial features of the obtained magnetic-field dependences of the longitudinal and Hall resistances $\rho_{xx}$ and $\rho_{xy}$.

First, both these functions are nonanalytical in the limit $\omega_c W \ll 1$: $\rho_{xx}, \rho_{xy} \propto 1/\ln \left[ 1/(\omega_c W) \right]$. A strongly singular behavior of this type is not encountered in the ohmic- and hydrodynamic-transport modes and is characteristic of ballistic transport, when the main contribution to the current can be made by a group of trajectories of selected geometry. In the investigated system, in the limit $\omega_c W \ll 1$, such a group of trajecto-
ries are segments of cyclotron circles with velocity angles close to the sample direction: $|\phi| \approx \pi/2$.

Second, it is noteworthy that, in the lower vicinity of the transition, the Hall resistance $\rho_{xy}$ takes exactly the nominal ballistic value $\rho_0$ and the longitudinal resistance $\rho_{xx}$ tends to zero linearly with respect to the difference $B_c - B$ (for the case $\gamma = 0$). The vanishing of $\rho_{xx}$ at $B \to B_c$ is consistent with the fact that 2D electrons in a perpendicular magnetic field in a strip with a width equal to or greater than the critical value ($W \geq 2R_c$) in the complete absence of disorder and electron–electron interaction ($\gamma = 0$) do not have a finite resistance. In such a system, at $W \geq 2R_c$, the applied external field $E_0 \parallel x$, at a certain instant of time $t = 0$, leads to unlimited growth of all the quantities in the current system $j(y, t) = j_c(y, t)$ along the external field with time, as well as the current along the normal to the sample edge $j_y(y, t)$ and, consequently, the Hall field $E_H(y, t)$.

The calculated dependences of the resistances $\rho_{xx}(B)$ and $\rho_{yx}(B)$ in the range of $\gamma W^2 \ll \omega_c W < 2$ (Figs. 2c, 2d; formulas (30) and (35)) are in good agreement with the behavior of the resistances obtained by the numerical solution of kinetic equation (1) for the case of weak interparticle scattering $\gamma W \ll 1$ (see Figs. 1a and 2a in [36]).

The coincidence of the features of the obtained theoretical field dependence of the longitudinal resistance $\rho_{xx}(B)$ (the maximum in the region of $\omega_c W \approx 1$ and much smaller values at $\omega_c W \ll 1$ and $\omega_c W \approx 2$) with the features of the resistance observed at low temperatures on high-quality long graphene and GaAs quantum-well samples (see Fig. 1 in [10], Fig. S21 in the appendix to [11], and Fig. 2 in [19]) apparently indicates that our theory and the numerical theory from [36] describe the experiments reported in [10, 11, 18].

6. CONCLUSIONS

The ballistic flow of two-dimensional electrons in a magnetic field in long samples with rough edges was studied. It was shown that, in a wide magnetic-field range, up to the critical field of the transition to the hydrodynamic regime, the flow is mainly determined by the scattering of electrons at the strip edges and their acceleration by magnetic and electric fields. The current density and Hall electric field distributions over the sample cross section were calculated and the longitudinal and Hall resistances as functions of the magnetic field were determined. The obtained dependence of the longitudinal resistance apparently agrees with the dependences observed experimentally in [10, 11, 18] on pure graphene and GaAs quantum-well samples in magnetic fields below the transition between the ballistic- and hydrodynamic-transport regimes. It seems important to compare in detail the obtained ballistic profiles of the current density and Hall field with recent measurements of these quantities on graphene samples from [10, 11].

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CONFLICT OF INTEREST
The authors declare that they have no conflict of interest.

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