CONGRUENCE LATTICES FORCING NILPOTENCY

ERHARD AICHINGER

Abstract. Given a lattice $L$ and a class $K$ of algebraic structures, we say that $L$ forces nilpotency in $K$ if every algebra $A \in K$ whose congruence lattice $Con(A)$ is isomorphic to $L$ is nilpotent. We describe congruence lattices that force nilpotency, supernilpotency or solvability for some classes of algebras. For this purpose, we investigate which commutator operations can exist on a given congruence lattice.

1. Introduction

We look for structural properties of an algebraic structure that are forced by the shape of its congruence lattice. In particular, we will consider the following properties of an algebra: being abelian, being solvable, being nilpotent, and being supernilpotent; the first three of these properties were first introduced for groups, but they proved meaningful for all algebraic structures. Examples of results in universal algebra [BS81] involving these concepts are that in a congruence modular variety, every abelian algebra is — essentially — a ring module [Her79, Gum83] and that every nilpotent algebra of prime power order has a loop reduct, permutable congruences, and generates a finitely axiomatizable variety [FM87].

For a property $p_A$ of an algebra, we search for a corresponding property $p_L$ of a lattice such that every algebra whose congruence lattice satisfies $p_L$ has the property $p_A$. Since arbitrary algebras can be quite diverse, all our results will be applicable only to restricted classes of algebras, such as the class $D$ of all algebras generating congruence modular varieties.

Definition 1.1. Let $K$ be a class of universal algebras, and let $P$ be the subclass of those algebras in $K$ that fulfil the property $p$. Let $L$ be a lattice. Then $L$ forces $p$ in $K$ if every algebra $A \in K$ such that $Con(A)$ is isomorphic to $L$ lies in $P$.

We will consider this definition first with $K := D$ and $P$ the subclass of solvable algebras in $D$. Then we could pose the following problem:

Date: August 21, 2018.
2010 Mathematics Subject Classification. 08A40,06A30.
Key words and phrases. Commutator theory, modular lattices, nilpotent algebras.
Supported by the Austrian Science Fund (FWF):P24077.
Characterize those finite lattices that force solvability in $D$.

However, among these lattices we also find those finite modular lattices that do not appear as congruence lattices of algebras in $D$. Hence a property $p_L$ characterizing these lattices most hold for all the “forbidden” finite modular lattices that never appear as congruence lattices of an algebra in $D$. This difficulty can be avoided if we only consider those lattices that actually are congruence lattices. To this end, for a class $K$ of algebras, we define the class $L(K)$ by

$$L(K) := \{ L \mid \exists A \in K : L \cong \text{Con}(A) \}$$

as the class of congruence lattices of algebras in $K$. Then in the present note we will

(1) characterize those lattices that force solvability (or supernilpotency) in $D$ among the lattices of finite height in $L(D)$;

(2) characterize those lattices that force nilpotency in $G$ among the lattices in $L(G)$, where $G$ is the class of finite expanded groups;

The properties that characterize these lattices will be rather easy to state, provided that we have some basic notions from lattice theory [MMT87, Grä98] at our disposal. We call $I[\alpha, \beta]$ a prime interval of the lattice $L$, write $\alpha \prec \beta$, and say that $\alpha$ is a subcover of $\beta$ if $\alpha < \beta$ and the interval $I[\alpha, \beta]$ is exactly the set $\{\alpha, \beta\}$. Departing from common usage, we call an element $\eta$ of a complete lattice meet irreducible if $\eta < \bigwedge\{ \beta \mid \eta < \beta \}$, and in this case we abbreviate $\bigwedge\{ \beta \mid \eta < \beta \}$ by $\eta^+$. The set of meet irreducible elements of the complete lattice $L$ is denoted by $M(L)$. For arbitrary $\alpha, \beta, \gamma, \delta \in L$, we write $I[\alpha, \beta] \nsucc I[\gamma, \delta]$ if $\delta = \beta \vee \gamma$ and $\alpha = \beta \wedge \gamma$; projectivity is the smallest equivalence on intervals containing $\nsucc$, and it is denoted by $\sim$. We first state a description of finite lattices that force solvability.

**Theorem 1.2.** Let $L$ be a lattice of finite height that is the congruence lattice of some algebra in a congruence modular variety. Then the following are equivalent:

1. $L$ forces solvability in the class of algebras generating congruence modular varieties.
2. Every algebra $B$ generating a congruence modular variety with $\text{Con}(B) \cong L$ is solvable.
3. The two element lattice $B_2$ is not a homomorphic image of $L$.

We notice that for finite algebras, the implication $[3] \Rightarrow [2]$ is a consequence of [HM88, Theorem 7.7(2)].

For a prime interval $I[\alpha, \beta]$ of the complete lattice $L$, we define the element $\Gamma(\alpha, \beta)$ of $L$ by

$$\Gamma(\alpha, \beta) := \bigvee\{ \eta \in M(L) \mid I[\eta, \eta^+] \sim I[\alpha, \beta] \}.$$

Using these elements $\Gamma(\alpha, \beta)$, we can express a condition forcing nilpotency in finite expanded groups.
Theorem 1.3. Let \( \mathbb{L} \) be a lattice that is the congruence lattice of some finite expanded group. Then the following are equivalent:

1. \( \mathbb{L} \) forces nilpotency in the class of finite expanded groups.
2. Every finite expanded group \( B \) with \( \text{Con}(B) \cong \mathbb{L} \) is nilpotent.
3. For each prime interval \( I[\alpha, \beta] \) of \( \mathbb{L} \), we have \( \Gamma(\alpha, \beta) = 1 \).

The third algebra property for which a lattice property was found is supernilpotency. The following theorem gives a description of congruence lattices that force supernilpotency. We say that a lattice \( \mathbb{L} \) splits if it is the union of two proper subintervals, which is equivalent to saying \( \mathbb{L} \models \exists \delta, \varepsilon : (\delta < 1 \text{ and } \varepsilon > 0 \text{ and } \forall \alpha : (\alpha \leq \delta \text{ or } \alpha \geq \varepsilon) \). A pair \((\delta, \varepsilon) \in (\mathbb{L} \setminus \{1_A\}) \times (\mathbb{L} \setminus \{0_A\})\) with \( \mathbb{L} = I[0_A, \delta] \cup I[\varepsilon, 1_A] \) is also called a splitting pair of \( \mathbb{L} \).

Theorem 1.4. Let \( \mathbb{L} \) be a finite lattice that is the congruence lattice of some algebra in a congruence modular variety. Then the following are equivalent:

1. \( \mathbb{L} \) forces supernilpotency in the class of all algebras that generate a congruence modular variety.
2. Every algebra \( B \) in a congruence modular variety with \( \text{Con}(B) \cong \mathbb{L} \) is supernilpotent.
3. \( \mathbb{L} \) does not split.

The proofs of Theorems 1.2, 1.3, and 1.4 are given in Section 7. Parts of these results will be proved in a purely lattice theoretic setting. To this end, the congruence lattice of an algebra is expanded with the binary operation of taking commutators. One obtains a new algebraic structure called commutator lattice which has been introduced and studied in [Cze08, Cze15]. Section 5 contributes to the structure theory of these commutator lattices.

2. Preliminaries on congruence lattices and commutators

When seeking to describe an algebraic structure \( A = (A, F) \), we can find significant information in the set of its congruence relations. These congruence relations, ordered by \( \subseteq \), are a complete sublattice of the set of equivalence relations on the set \( A \); the set of congruence relations is denoted by \( \text{Con}(A) \). For arbitrary algebras, these congruence relations play the role that ideals play for rings and that normal subgroups play for groups. Commutator theory [FM87] generalizes taking the commutator subgroup of two normal subgroups to arbitrary algebraic structures by associating a new congruence \( \gamma := [\alpha, \beta]_A \) with every pair of congruences \((\alpha, \beta) \) of \( A \). Generalizations of the group commutator can be found, e.g., in [Hig56] and [Sco97], but it was the work of [Smi76, HH79, FM87] that led to the following definition of the term condition commutator, which generalizes at the same time taking the commutator subgroup \([A, B]\) of two normal subgroups of a group, and forming the ideal product \( A \cdot B \) of two ideals of a ring.
Definition 2.1 (cf. [MNT87, Definition 4.150]). Let $A$ be an algebraic structure, and let $\alpha, \beta$ be congruences of $A$. Then the commutator $\gamma := [\alpha, \beta]_A$ is defined as the intersection of all congruence relations $\delta$ of $A$ such that for all $n \in \mathbb{N}$, for all $(n + 1)$-ary term functions $t$ of $A$, and for all $(a, b) \in \alpha$ and $(c_1, d_1), \ldots, (c_n, d_n) \in \beta$ with $(t(a, c_1, \ldots, c_n), t(a, d_1, \ldots, d_n)) \in \delta$ we have $(t(b, c_1, \ldots, c_n), t(b, d_1, \ldots, d_n)) \in \delta$.

Defined for arbitrary algebras, commutators have proved most useful for algebras with a modular congruence lattice, and hence we will restrict ourselves to such algebras, or, in decreasing steps of generality, to algebras in congruence modular varieties, to algebras in congruence permutable varieties, or to expanded groups. In congruence permutable varieties, the term condition commutator admits the following description, which resembles the ideal product defined in [Sco97].

Lemma 2.2 (cf. [AM10, Corollary 6.10]). Let $A$ be an algebra in a congruence permutable variety, and let $\alpha, \beta$ be congruences of $A$. Then the congruence $[\alpha, \beta]_A$ is generated as a congruence of $A$ by $\{(c(a_1, b_1), c(a_2, b_2)) | (a_1, a_2) \in \alpha, (b_1, b_2) \in \beta, c$ is a binary polynomial function of $A$ with $c(a_1, b_1) = c(a_1, b_2) = c(a_2, b_1)\}$. 

From the congruence lattice and the commutator operation of a finite algebra in a congruence modular variety, one can, e.g., determine whether the algebra generates a residually small variety [FM87, Theorem 10.15] or whether every homomorphic image of an algebra in a congruence permutable variety is affine complete [Aic00, Proposition 5.2]. Starting from the commutator operation on congruences, it is possible to define the derived series $(\gamma_n)_{n \in \mathbb{N}}$ and the lower central series $(\lambda_n)_{n \in \mathbb{N}}$ of congruences of the algebra $A$ by $\gamma_1 = \lambda_1 = 1_A$, and the recursion $\gamma_{n+1} = [\gamma_n, \gamma_n]$ and $\lambda_{n+1} = [1_A, \lambda_n]$ for $n \in \mathbb{N}$. An algebra in a congruence modular variety is called solvable (cf. [HM88, Definition 3.6(3)]) if there is $m$ with $\gamma_m = 0_A$, and nilpotent (cf. [FM87, p.69 before Lemma 7.3]) if there is $k$ with $\lambda_k = 0_A$. A. Bulatov [Bul01] introduced a generalization of the binary commutator operation by associating a congruence $[\alpha_1, \ldots, \alpha_n]$ with every finite sequence of congruences; $[\alpha_1, \ldots, \alpha_n]$ is called a higher commutator. In congruence modular varieties, the higher commutator operations enjoy certain properties, such as monotonicity, symmetry, and distributivity with respect to joins; the validity of some of these properties was established only recently in [Moo16]. If an algebra has an $m \in \mathbb{N}$ such that $[\alpha_1, \ldots, \alpha_n] = 0$ whenever $n > m$, then the algebra is called supernilpotent. Every supernilpotent algebra in a congruence modular variety is nilpotent: for congruence permutable varieties, this was proved in [AM10, Corollary 6.15], and for congruence modular varieties, it follows from properties (4) and (8) of higher commutators given in [Moo16], which are called (HC4) and (HC8) in [AM13, p. 860] and in [AM10]. Supernilpotency admits the following combinatorial description: a finite algebra $A$ in a congruence modular variety is supernilpotent if and only if there exists a polynomial $p$
such that the \( n \)-generated free algebra in the variety generated by \( A \) has at most \( 2^{p(n)} \) elements. A self-contained version of this description for the case that \( A \) is an expanded group has been given in Section 4 of [Aic14]; the general result follows from a combination of [HM88, Theorem 9.18 and Lemma 12.4], [EM87, Theorem 6.2, Corollary 7.5, Theorem 14.2], the notion of rank from [Kea99, p. 179], [AM10, Lemma 7.5], the proof of Theorem 1 of [BB87], and the generalization of the properties of higher commutators from congruence permutable to congruence modular varieties in [Moo16]; since we will not make use of the combinatorial description of supernilpotency in this paper, we abstain from a thorough discussion. The definitions of binary commutators, solvability and nilpotency are compatible with the classic use of these notions in group theory (cf. [MMT87, Exercise 4.156(11)]).

Given the congruence lattice of an algebra, it is therefore interesting to know what the possible choices of the commutator operations are. Certain limitations are imposed by the laws \([x, y] \approx [y, x], [x, y] \leq x \land y, x \leq y \to [x, z] \leq [y, z], [x \lor y, z] \approx [x, z] \lor [y, z]\) that are satisfied by every structure \((\text{Con}(A), \land, \lor, [.., ..]_A)\) arising from an algebra \( A \) in a congruence modular variety. It is easy to see that on the five element lattice \( M_3 \), the constant operation \([x, y] = 0\) is the only such operation definable on this lattice; this imposes structural consequences on algebras with such a congruence lattice [MMT87, Lemma 4.153]. Conditions on the higher commutator operations that are imposed by the shape of the congruence lattice are given in [AM13, Lemma 3.3].

Let us now briefly review some properties of the commutator operations in congruence modular varieties. These properties are proved in Chapters 3 and 4 of [FM87].

**Lemma 2.3.** Let \( A \) be an algebra in a congruence modular variety, and let \( \alpha, \alpha_1, \beta, \beta_1 \in \text{Con}(A) \). Then \([\alpha, \beta] = [\beta, \alpha] \leq \alpha \land \beta, [\alpha \lor \alpha_1, \beta] = [\alpha, \beta] \lor [\alpha_1, \beta], \) and if \( \alpha \leq \alpha_1 \) and \( \beta \leq \beta_1 \), then \([\alpha, \beta] \leq [\alpha_1, \beta_1] \). If \((\alpha_i)_{i \in I} \) is a family of congruences of \( A \), we also have \( \bigvee_{i \in I}[\alpha_i, \beta] = [\bigvee_{i \in I} \alpha_i, \beta] \).

The proofs of some of these properties are by no means obvious and require skilful manipulations with Day terms [Day69, FM87]. The proofs become easier when restricting to congruence permutable varieties, and some of these properties have been proposed as exercises in [MMT87]. The introductory chapter of [Aic06b] provides solutions to some of these exercises, as does [AM10].

**Lemma 2.4.** Let \( A = (A, F) \) be an algebra in a congruence modular variety, and let \( B = (A, F \cup G) \) be an expansion of \( A \). Then for all \( \alpha, \beta \in \text{Con}(B) \), we have \([\alpha, \beta]_A \subseteq [\alpha, \beta]_B \). Furthermore, if \( B \) is solvable, then \( A \) is solvable, and if \( B \) is nilpotent, then \( A \) is nilpotent.

**Proof:** Using the definition of the commutator \([\alpha, \beta]_B \) by the term condition, we obtain that \( \alpha \) centralizes \( \beta \) modulo \([\alpha, \beta]_B \) in \( B \) (cf. [MMT87, Definition 4.148]).
Since $\text{Clo}(A) \subseteq \text{Clo}(B)$, $\alpha$ centralizes $\beta$ modulo $[\alpha, \beta]_B$ in $A$. Hence $[\alpha, \beta]_A \leq [\alpha, \beta]_B$. Let $(\gamma^A_n)_{n \in \mathbb{N}}$, $(\gamma^B_n)_{n \in \mathbb{N}}$ be the derived series of $A$ and $B$, resp. Then for each $n \in \mathbb{N}$, we have $\gamma^A_n \leq \gamma^B_n$, which is proved by induction using $\gamma^A_{n+1} = [\gamma^A_n, \gamma^A_n]_A \leq [\gamma^B_n, \gamma^B_n]_A \leq [\gamma^B_n, \gamma^B_n]_B = \gamma^B_{n+1}$ as the induction step. Hence if $B$ is solvable, then so is $A$. The proof for nilpotency is similar.

For an algebra $A$ in a congruence modular variety and $\alpha, \beta \in \text{Con}(A)$, we define $(\alpha : \beta)_A$ as the largest $\gamma \in \text{Con}(A)$ with $[\gamma, \beta]_A \leq \alpha$. We omit the subscript when the algebra is clear from the context. When interpreting commutator theory in group theory, $(\alpha : \beta)$ corresponds to the centralizer $C_G(B/A)$, where $B$ and $A$ are the normal subgroups corresponding to $\beta$ and $\alpha$. Therefore, we will call $(\alpha : \beta)$ the centralizer of $\beta$ over $\alpha$. We note that Proposition 4.2 of [FM87] guarantees that this definition is consistent with [MMT87, Definition 4.150].

Often, we will not use any properties of the binary commutator operation other than its mere definition by the term condition [MMT87, Definition 4.150] and the properties that are stated in Lemma 2.3. Hence it is useful to see what can be derived from these conditions alone; such an investigation was started in [Cze08].

### 3. Preliminaries on commutator lattices

In [Cze08], J. Czelakowski defined commutator lattices. These algebraic structures capture the properties of the structure $(\text{Con}(A), \lor, \land, [\cdot, \cdot])$ that is constructed by expanding the congruence lattice of an algebra $A$ in a congruence modular variety with the binary operation of taking commutators.

**Definition 3.1** ([Cze08, Definition 1.1]). An algebraic structure $L = (L, \lor, \land, [\cdot, \cdot])$ is a commutator lattice if $(L, \lor, \land)$ is a complete lattice, and for all $x, y \in L$ and for all families $(x_i)_{i \in I}$ from $L$, we have $[x, y] = [y, x]$, $[x, y] \leq x \land y$, and $[\lor_{i \in I} x_i, y] = \lor [x_i, y]$. In this case, we call $[\cdot, \cdot]$ a commutator multiplication on the lattice $(L, \lor, \land)$.

The guiding example of this definition comes from congruences and commutators. In fact, we may restate Lemma 2.3 as follows:

**Proposition 3.2.** Let $A$ be an algebra that generates a congruence modular variety, let $(L, \lor, \land) := (\text{Con}(A), \lor, \land)$ be the congruence lattice of $A$, and for $\alpha, \beta \in \text{Con}(A)$, let $[\alpha, \beta]_A$ denote the term condition commutator of $\alpha$ and $\beta$ as defined in [MMT87, Definition 4.150]. Then $(L, \lor, [\cdot, \cdot], A)$ is a commutator lattice.

**Proof:** [FM87, Proposition 4.3].
It is a consequence of the distributivity of [..] with respect to joins that the operation [..] is monotonic with respect to the order of the lattice. An important operation that comes along with a commutator lattice $L$ is that of residuation. For $x, y \in L$, we define

$$ (x : y) := \bigvee \{z \in L \mid [z, y] \leq x\} $$

and call $(., .)$ the residuation operation associated with $L$. We notice that in [Cze08], $(x : y)$ is denoted by $y \to x$; our notation comes from the interpretation of $(x : y)$ as the centralizer of $y$ over $x$ in [FMS7]. In the following lemma, we state some properties of the residuation operation.

**Lemma 3.3.** Let $L = (\mathbb{L}, \lor, \land, [..])$ be a commutator lattice, and let $(., .)$ be the residuation operation associated with $L$. Then for all $x, y, z \in \mathbb{L}$ and for all families $(x_i)_{i \in I}$ from $\mathbb{L}$, we have:

1. $[z, y] \leq x$ if and only if $z \leq (x : y)$,
2. $[(x : y), y] \leq x$,
3. $(\bigwedge_{i \in I} x_i : y) = \bigwedge_{i \in I} (x_i : y)$,
4. $(x : \bigvee_{i \in I} y_i) = \bigwedge_{i \in I} (x : y_i)$,
5. $(x : y) \geq x$,
6. $(x : x) = 1$,
7. $(x : (x : y)) \geq y$.

**Proof:**

1. The “only if”-direction is an immediate consequence of the definition of $(x : y)$. For “if”-direction, we assume $z \leq (x : y)$ and compute $[z, y] \leq [(x : y), y] = \bigvee \{z_1 \in \mathbb{L} \mid [z_1, y] \leq x\}, y = \bigvee \{z_1, y \mid z_1 \in \mathbb{L}, [z_1, y] \leq x\} \leq x$.

2. is a consequence of (1).

3. For $\leq$, we let $j \in I$ and notice that using (2), we have $[(\bigwedge_{i \in I} x_i : y), y] \leq \bigwedge_{i \in I} x_i \leq x_j$, and therefore $(\bigwedge_{i \in I} x_i : y) \leq (x_j : y)$. Hence $(\bigwedge_{i \in I} x_i : y) \leq \bigwedge_{i \in I} (x_i : y)$. For $\geq$, we let $j \in J$ and compute $[\bigwedge_{i \in I} (x_i : y), y] \leq [(x_j : y), y] \leq x_j$. Since $[\bigwedge_{i \in I} x_i : y, y] \leq \bigwedge_{i \in I} x_i$, which implies $\bigwedge_{i \in I} (x_i : y) \leq (\bigwedge_{i \in I} x_i : y)$.

4. For $\leq$, we let $j \in I$ and compute $[(x : \bigvee_{i \in I} y_i), y_i] \leq [(x : \bigvee_{i \in I} y_i), y_i] \leq (x : y_j)$, which implies $(x : \bigvee_{i \in I} y_i) \leq (x : y_j)$, and therefore $(x : \bigvee_{i \in I} y_i) \leq \bigwedge_{i \in I} (x : y_i)$. For $\geq$, we compute $[\bigwedge_{i \in I} (x : y_i), \bigvee_{k \in K} y_k] = \bigvee_{k \in K} [\bigwedge_{i \in I} (x : y_i), y_k] \leq \bigvee_{k \in K} [(x : y_k), y_k] \leq x$, which implies $\bigwedge_{i \in I} (x : y_i) \leq (x : \bigvee_{k \in K} y_k)$.

5. Since $[x, y] \leq x$, we have $x \leq (x : y)$.

6. Since $[1, x] \leq x$, we have $1 \leq (x : x)$.

7. We have $[y, (x : y)] = [(x : y), y] \leq x$, and therefore $y \leq (x : (x : y))$. □

In fact, the properties (3)-(7) in Lemma 3.3 are equivalent to the properties (a)-(e) listed in [Cze08, Theorem 3.1], and therefore provide a different axiomatization
of possible residuation operations of commutator lattices. The following lemma is an abstraction of [FM87, Chapter 9, Exercise 4].

Lemma 3.4. Let \((L, \lor, \land, [\cdot, \cdot])\) be a commutator lattice, and let \((\cdot, \cdot)\) its associated residuation. Let \(\alpha, \beta, \gamma, \delta \in L\) such that \(\alpha \leq \beta, \gamma \leq \delta\), and \(I[\alpha, \beta] \leftrightarrow I[\gamma, \delta]\). Then

\[
(1) \quad (\alpha : \beta) = (\gamma : \delta), \quad \text{and} \\
(2) \quad [\beta, \beta] \leq \alpha \quad \text{if and only if} \quad [\delta, \delta] \leq \gamma.
\]

**Proof:** We assume \(I[\alpha, \beta] \not\supset I[\gamma, \delta]\). Then using Lemma 3.3, we obtain \((\gamma : \delta) = (\gamma : \beta \lor \gamma) = (\gamma : \beta) \land (\gamma : \gamma) = (\gamma : \beta) \land (\beta : \beta) = (\gamma \land \beta : \beta) = (\alpha : \beta)\). For the second item, we first assume that \([\beta, \beta] \leq \alpha\). Then \([\delta, \delta] = [\beta \lor \gamma, \beta \lor \gamma] = [\beta, \beta] \lor [\beta, \gamma] \lor [\gamma, \gamma] \leq \alpha \lor \gamma \lor \gamma = \gamma\). Conversely, if \([\delta, \delta] \leq \gamma\), then \([\beta, \beta] \leq \gamma\), and since \([\beta, \beta] \leq \beta\), we obtain \([\beta, \beta] \leq \gamma \land \beta = \alpha\). \(\square\)

Let \(\alpha, \beta \in L\) with \(\alpha \prec \beta\). The next lemma states that \(\Gamma(\alpha, \beta) = \lor\{\eta \in M(L) \mid I[\eta, \eta^+] \leftrightarrow I[\alpha, \beta]\}\) is a lower bound for the residuum \((\alpha : \beta)\).

Lemma 3.5. Let \(L = (L, \lor, \land, [\cdot, \cdot])\) be a commutator lattice, and let \((\cdot, \cdot)\) be its associated residuation. Let \(\alpha, \beta \in L\) be such that \(\alpha \prec \beta\), and let \(\Gamma(\alpha, \beta) = \lor\{\eta \in M(L) \mid I[\eta, \eta^+] \leftrightarrow I[\alpha, \beta]\}\). Then \(\Gamma(\alpha, \beta) \leq (\alpha : \beta)\).

**Proof:** For every \(\eta \in M(L)\) with \(I[\eta, \eta^+] \leftrightarrow I[\alpha, \beta]\), Lemma 3.3 and Lemma 3.4 yield \(\eta \leq (\eta : \eta^+) = (\alpha : \beta)\). Therefore \(\Gamma(\alpha, \beta) \leq (\alpha : \beta)\). \(\square\)

### 4. Tools from lattice theory

In constructing commutator multiplications on given lattices, we will need some techniques from lattice theory. We will often work in algebraic lattices [MMT87, Definition 2.15], and we call a lattice bialgebraic if the lattice and its dual are both algebraic; for example, every lattice of finite height is bialgebraic. For our purpose, the most important fact in algebraic lattices is that every element is the meet of meet irreducible elements [MMT87, Theorem 2.19]. For any complete lattice \(L\), \(M(L)\) denotes the set of meet irreducible elements of \(L\), and by \(J(L) := \{\rho \in L \mid \rho > \lor\{\alpha \in L \mid \alpha < \rho\}\}\), we denote the set of join irreducible elements of \(L\). The unique subcover of a join irreducible element \(\beta\) is denoted by \(\beta^+\), and \(\beta\) is called a lonesome join irreducible element of \(L\) if \(\{\rho \in J(L) \mid I[\rho^+, \rho] \leftrightarrow I[\beta^+, \beta]\}\) = \(\{\beta\}\); a meet irreducible element \(\eta\) of \(L\) is called a lonesome meet irreducible element if \(\{\varphi \in M(L) \mid I[\varphi^+, \varphi^+] \leftrightarrow I[\eta, \eta^+]\}\) = \(\{\eta\}\). For \(\alpha, \beta, \gamma, \delta \in L\) with \(\alpha \leq \beta\) and \(\gamma \leq \delta\), we say that \(I[\alpha, \beta]\) projects into \(I[\gamma, \delta]\) if there are \(\alpha_1, \beta_1 \in L\) with \(\gamma \leq \alpha_1 \leq \beta_1 \leq \delta\) such that \(I[\alpha, \beta] \leftrightarrow I[\alpha_1, \beta_1]\).

The following proposition collects some well known facts on projectivity.

**Proposition 4.1.** Let \(L\) be a complete lattice.
(1) If \( \mathbb{L} \) is algebraic and \( \alpha, \beta \in \mathbb{L} \) are such that \( \beta \not\leq \alpha \), there is \( \eta \in M(\mathbb{L}) \) such that \( \alpha \leq \eta \) and \( \beta \not\leq \eta \).

(2) If \( \mathbb{L} \) is modular and \( \alpha, \beta \in \mathbb{L} \) are such that \( \alpha \prec \beta \) and \( \eta \in M(\mathbb{L}) \) satisfies \( \alpha \leq \eta \) and \( \beta \not\leq \eta \), then \( I[\alpha, \beta] \not\succ I[\eta, \eta^+] \).

(3) If \( \mathbb{L} \) is modular, and \( \beta \in \mathbb{L} \) and \( \eta \in M(\mathbb{L}) \) are such that \( \eta \not\geq \beta \), then \( I[\eta, \eta^+] \setminus \gamma \subseteq I[\eta \land \beta, \eta^+ \land \beta] \). Dually, if \( \mathbb{L} \) is modular, \( \gamma \in \mathbb{L} \) and \( \rho \in J(\mathbb{L}) \) are such that \( \rho \not\leq \gamma \), then \( I[\rho^-, \rho] \not\succ I[\rho^- \lor \gamma, \rho \lor \gamma] \).

(4) If \( \mathbb{L} \) is algebraic and modular, and \( \beta \in J(\mathbb{L}) \) and \( \gamma \in \mathbb{L} \) are such that \( \beta \not\leq \gamma \), there exists \( \eta \in M(\mathbb{L}) \) are such that \( \gamma \leq \eta \) and \( I[\beta^-, \beta] \not\succ I[\eta, \eta^+] \).

(5) If \( \mathbb{L} \) is modular and \( a, b, x, y \in \mathbb{L} \) are such that \( x \land y \leq a < b \leq x \lor y \), then \( I[a, b] \) projects into \( I[x, x \lor y] \) or into \( I[y, x \lor y] \).

Proof: (1) By [MNT87, Theorem 2.19], \( \alpha = \bigwedge \{ \psi \in M(\mathbb{L}) \mid \psi \geq \alpha \} \). If (1) fails, then \( \{ \psi \in M(\mathbb{L}) \mid \psi \geq \alpha \} \not\subseteq \{ \psi \in M(\mathbb{L}) \mid \psi \geq \beta \} \), and thus \( \alpha = \bigwedge \{ \psi \in M(\mathbb{L}) \mid \psi \geq \beta \} \geq \beta \), contradicting the assumptions.

(2) Since \( \beta > \eta \land \beta \geq \alpha \), we have \( \eta \land \beta = \alpha \), and from \( \eta \lor \beta > \eta \), we obtain \( \eta \lor \beta \geq \eta^+ \). Now suppose \( \eta \lor \beta \not\leq \eta^+ \). Then \( \beta \not\leq \eta^+ \), and therefore \( \beta > \eta^+ \land \beta \geq \alpha \). Hence \( \eta^+ \land \beta = \alpha \), which implies \( \eta = \eta \lor (\beta \land \eta^+) \). By modularity, we have \( \eta \lor (\beta \land \eta^+) = (\eta \lor \beta) \land \eta^+ = \eta^+ \). The contradiction \( \eta = \eta^+ \) completes the proof of \( \eta \lor \beta \leq \eta^+ \), and therefore \( \eta \lor \beta = \eta^+ \).

(3) By modularity, we have \( \eta \lor (\beta \land \eta^+) = (\eta \lor \beta) \land \eta^+ = \eta^+ \), which proves \( I[\beta \land \eta, \beta \land \eta^+] \not\succ I[\eta, \eta^+] \). The statement on join irreducible elements follows from a dual argument.

(4) We first show

\[
(4.1) \quad \beta \not\leq \gamma \lor \beta^-.
\]

Suppose \( \beta \leq \gamma \lor \beta^- \). Then by modularity, \( \beta = \beta \land (\gamma \lor \beta^-) = (\beta \land \gamma) \lor \beta^- \). Since \( \beta \not\leq \gamma \), we have \( \beta \land \gamma \leq \beta^- \) and thus \( (\beta \land \gamma) \lor \beta^- = \beta^- \). The contradiction \( \beta = \beta^- \) establishes (4.1). Using (4.1) and item (1), we find \( \eta \in M(\mathbb{L}) \) such that \( \gamma \lor \beta^- \leq \eta \) and \( \beta \not\leq \eta \). Since \( \beta^- \leq \eta \), item (2) yields \( I[\beta^-, \beta] \not\succ I[\eta, \eta^+] \).

(5) If \( b \lor x = a \lor x \) and \( b \land x = a \land x \), then \( a = a \lor (x \land a) = a \lor (x \land b) = (a \lor x) \land b = (b \lor x) \land b = b \). Hence \( b \lor x > a \lor x \) or \( b \land x > a \land x \). In the case \( b \lor x > a \lor x \), we first observe that then \( a \lor x \not\leq b \). Hence \( a \leq (a \lor x) \land b < b \), and therefore \( (a \lor x) \land b = a \), which implies \( I[a, b] \not\succ I[a \lor x, b \lor x] \), and therefore \( I[a, b] \) projects into \( I[x, x \lor y] \). In the case \( b \land x > a \land x \), we have \( b \land x \not\leq a \), and therefore \( a < a \lor (x \land b) \leq b \), which implies \( a \lor (x \land b) = b \), and therefore \( I[a \land x, b \land x] \not\succ I[a, b] \). Thus \( I[a, b] \) projects into \( I[x \land y, x] \), and therefore by Dedekind’s transposition principle [MNT87, 2.27] into \( I[y, x \land y] \). \( \square \)

Projectivity plays an important role in the description of congruence generation in lattices. In a complete lattice \( \mathbb{L} \), a relation \( \Phi \) on \( \mathbb{L} \) is called a complete
congruence of \( \mathbb{L} \) if \( \Phi \) is an equivalence relation on \( \mathbb{L} \), and for all families \( (x_i)_{i \in I} \) and \( (y_i)_{i \in I} \) from \( \mathbb{L} \), we have \((\forall i \in I : (x_i, y_i) \in \Phi) \Rightarrow ((\bigvee_{i \in I} x_i, \bigvee_{i \in I} y_i) \in \Phi \) and \((\bigwedge_{i \in I} x_i, \bigwedge_{i \in I} y_i) \in \Phi \).

**Proposition 4.2.** Let \( \mathbb{L} \) be a bialgebraic modular lattice, and let \( a, b \in \mathbb{L} \) with \( a \prec b \). Let \( \Phi := \{(x, y) \in \mathbb{L} \times \mathbb{L} \mid I[a, b] \text{ does not project into } I[x \wedge y, x \vee y]\} \). Then \( \Phi \) is a complete congruence on the lattice \( \mathbb{L} \).

Reflexivity and symmetry of \( \Phi \) are obvious. For transitivity, we assume that \( (x, y) \in \Phi \), \( (y, z) \in \Phi \) and \( (x, z) \not\in \Phi \). Since \( (x, z) \not\in \Phi \), \( I[a, b] \) projects into \( I[x \wedge z, x \vee z] \), and hence into \( I[x, x \vee z] \) or into \( I[z, x \vee z] \). We will now distinguish two cases:

1. We assume that \( I[a, b] \) projects into \( I[x, x \vee z] \): Then let \( a_1, b_1 \in \mathbb{L} \) be such that \( x \leq a_1 < b_1 \leq x \vee z \) and \( I[a, b] \hookrightarrow I[a_1, b_1] \). We choose \( \eta \in M(\mathbb{L}) \) such that \( I[a_1, b_1] \not\supset I[\eta, \eta^+] \).
   
   (a) Case \( \eta \not\geq y \): Then \( I[\eta, \eta^+] \not\supset I[\eta \wedge y, \eta^+ \wedge y] \), and therefore \( I[a, b] \) projects into \( I[x \wedge y, y] \), implying \( (x, y) \not\in \Phi \), a contradiction.
   
   (b) Case \( \eta \geq y \): Since \( \eta \geq x \) and \( \eta \not\geq x \vee z \), we have \( \eta \not\geq z \), and therefore \( I[\eta, \eta^+] \not\supset I[\eta \wedge z, \eta^+ \wedge z] \). Hence \( I[a, b] \) projects into \( I[y \wedge z, z] \), implying \( (y, z) \not\in \Phi \), a contradiction.

2. We assume that \( I[a, b] \) projects into \( I[z, x \vee z] \): Then swapping the roles of \( x \) and \( z \) in the previous case, we obtain that \( I[a, b] \) projects into \( I[z \wedge y, y] \) or into \( I[y \wedge x, x] \), again contradicting the assumptions.

Next, we will prove that if \( (x_i)_{i \in I} \) and \( (y_i)_{i \in I} \) are families from \( \mathbb{L} \) such that for all \( i \in I \), \( (x_i, y_i) \in \Phi \), we have \((\bigvee_{i \in I} x_i, \bigvee_{i \in I} y_i) \in \Phi \). Let \( X := \bigvee_{i \in I} x_i \) and \( Y := \bigvee_{i \in I} y_i \). Seeking a contradiction, we assume \((X, Y) \not\in \Phi \). Then \( I[a, b] \) projects into \( I[X \wedge Y, X \vee Y] \), and hence into \( I[X, X \vee Y] \) or into \( I[Y, X \vee Y] \).

In the case that \( I[a, b] \) projects into \( I[X, X \vee Y] \), we choose \( a_1, b_1 \in \mathbb{L} \) with \( X \leq a_1 < b_1 \leq X \vee Y \) and \( I[a, b] \hookrightarrow I[a_1, b_1] \). We pick \( \eta \in M(\mathbb{L}) \) with \( I[a_1, b_1] \not\supset I[\eta, \eta^+] \). Since \( \eta \not\geq b_1 \), we have \( \eta \not\geq X \vee \bigvee_{i \in I} y_i \). Since \( \eta \geq a_1 \geq X \), there is \( j \in I \) such that \( \eta \not\geq y_j \). Then \( I[\eta, \eta^+] \not\supset I[\eta \wedge y_j, \eta^+ \wedge y_j] \), and therefore \( I[a, b] \) projects into \( I[x_j \wedge y_j, y_j] \), implying \( (x_j, y_j) \not\in \Phi \), a contradiction. In the case that \( I[a, b] \) projects into \( I[Y, X \vee Y] \), we swap the roles of \( X \) and \( Y \) and obtain that \( I[a, b] \) projects into some \( I[y_j \wedge x_j, x_j] \). Hence \( \Phi \) is preserved under arbitrary joins.

Now let \( \mathbb{K} \) be the dual of \( \mathbb{L} \), and let \( \Psi := \{(x, y) \in \mathbb{K} \times \mathbb{K} \mid I[b, a] \text{ does not project into } I[x \wedge k, y \wedge k] \text{ in } \mathbb{K}\} \). Since \( \mathbb{L} \) is bialgebraic, so is \( \mathbb{K} \), and hence the previous arguments imply that \( \Psi \) is invariant under arbitrary joins, computed in \( \mathbb{K} \). Hence \( \Psi \) is invariant under arbitrary meets, computed in \( \mathbb{L} \), and since \( \Psi = \Phi \), we obtain that \( \Phi \) is preserved under arbitrary meets.

Hence \( \Phi \) is indeed a complete congruence of the lattice \( \mathbb{L} \). \( \Box \)
We will also need some additional information on lonesome meet irreducible elements.

**Proposition 4.3.** Let $\mathbb{L}$ be an algebraic modular lattice, and let $\eta \in M(\mathbb{L})$. If $\eta$ is not a lonesome meet irreducible element, then there exists $\psi \in M(\mathbb{L})$ with $I[\eta, \eta^+] \nrightarrow I[\psi, \psi^+]$, $\eta \not< \psi$, and $\psi \not< \eta$.

*Proof:* We let $\varphi \in M(\mathbb{L})$ with $\eta \neq \varphi$ such that $I[\eta, \eta^+] \nrightarrow I[\varphi, \varphi^+]$. Since $I[\eta, \eta^+] \nrightarrow I[\varphi, \varphi^+]$, there is a natural number $n$, and there are $\rho_1, \ldots, \rho_{2n-1}, \sigma_1, \ldots, \sigma_{2n-1} \in \mathbb{L}$ such that

$$I[\eta, \eta^+] \not< I[\rho_1, \sigma_1] \not< I[\rho_2, \sigma_2] \not< \cdots \not< I[\rho_{2n-2}, \sigma_{2n-2}] \not< I[\rho_{2n-1}, \sigma_{2n-1}] \nrightarrow I[\varphi, \varphi^+].$$

Now for each $k \in \{1, 2, \ldots, n-1\}$, we pick an element $\eta_k \in M(\mathbb{L})$ with $\eta_k \geq \rho_k$, $\eta_k \not< \sigma_k$. Then by Proposition 4.1(2), $I[\rho_k, \sigma_k] \nrightarrow I[\eta_k, \eta_k^+]$. Since $\nrightarrow$ is transitive, we obtain

$$I[\eta, \eta^+] \not< I[\rho_1, \sigma_1] \not< I[\eta_2, \eta_2^+] \not< \cdots \not< I[\eta_{2n-2}, \eta_{2n-2}^+] \not< I[\rho_{2n-1}, \sigma_{2n-1}] \nrightarrow I[\varphi, \varphi^+].$$

Hence there exists $i \in \{1, \ldots, n\}$ and $\psi$ with $\psi \neq \eta$ and $I[\eta, \eta^+] \not< I[\rho_{2i-1}, \sigma_{2i-1}] \nrightarrow I[\psi, \psi^+]$. If $\eta \leq \psi$, then $\eta^+ \leq \psi$, and therefore $\rho_{2i-1} \equiv \psi \land \sigma_{2i-1} \geq \eta^+ \land \sigma_{2i-1} = \sigma_{2i-1}$, which implies $\psi^+ = \psi \lor \sigma_{2i-1} \leq \psi \lor \rho_{2i-1} = \psi$, a contradiction. Hence $\eta \not< \psi$. Similarly, we obtain $\psi \not< \eta$. \hfill $\square$

**Proposition 4.4.** Let $\mathbb{L}$ be a bialgebraic modular lattice, and let $\beta \in J(\mathbb{L})$ and $\eta \in M(\mathbb{L})$ such that $I[\beta^-, \beta] \nrightarrow I[\eta, \eta^+]$. Then $\beta$ is a lonesome join irreducible element of $\mathbb{L}$ if and only if $\eta$ is a lonesome meet irreducible element of $\mathbb{L}$.

For proving the “only if”-direction, we assume that $\eta$ is a not a lonesome meet irreducible element of $\mathbb{L}$. Let $\eta_1 \in M(\mathbb{L})$ be such that $\eta_1 \geq \beta^-, \eta_1 \not< \beta$. Then $I[\beta^-, \beta] \nrightarrow I[\eta_1, \eta_1^+]$. Since $I[\eta_1, \eta_1^+] \nrightarrow I[\eta, \eta^+]$, $\eta_1$ is not a lonesome meet irreducible element of $\mathbb{L}$, either. Let $\psi$ be a meet irreducible element with $I[\eta_1, \eta_1^+] \nrightarrow I[\psi, \psi^+]$ and $\eta_1 \not< \psi$ as produced in Proposition 4.3. Using that the dual of $\mathbb{L}$ is algebraic and the dual of Proposition 4.3, we can choose $\varepsilon \in J(\mathbb{L})$ such that $\varepsilon \leq \eta_1$ and $I[\varepsilon^-, \varepsilon] \nrightarrow I[\psi, \psi^+]$. Since $\varepsilon \leq \eta_1$ and $\beta \not< \eta_1$, we have $\beta \neq \varepsilon$. Therefore, $\beta$ is not lonesome.

The “if”-direction now follows by applying the direction that has already been proved to the dual of $\mathbb{L}$. \hfill $\square$

We notice that for finite lattices Proposition 4.4 also follows from Corollary 6.2.1 of [Ava58]. To see this, we let $Q$ be the set of all prime intervals in $\mathbb{L}$ that are projective to $I[\beta^-, \beta]$, and use Corollary 6.2.1 to establish that $I[\eta, \eta^+]$ is the only element $I[x, y]$ of $Q$ where $x$ is meet irreducible.
For a prime interval \( I[\alpha, \beta] \) with \( \Gamma(\alpha, \beta) \neq 1_A \) in a bialgebraic modular lattice, we will find a splitting of the congruence lattice. For any complete lattice \( \mathbb{L} \) and \( \alpha, \beta \in \mathbb{L} \), we define
\[
\Delta(\alpha, \beta) := \bigvee \{ \rho \in J(\mathbb{L}) \mid I[\rho^-, \rho] \rightsquigarrow I[\alpha, \beta] \}.
\]

**Proposition 4.5.** Let \( \mathbb{L} \) be an algebraic modular lattice, let \( \alpha, \beta \in \mathbb{L} \) with \( \alpha \prec \beta \).
Then for all \( \varphi \in \mathbb{L} \), we have \( \varphi \leq \Gamma(\alpha, \beta) \) or \( \varphi \geq \Delta(\alpha, \beta) \).

**Proof:** Assume \( \varphi \not\geq \Delta(\alpha, \beta) \). By the definition of \( \Delta(\alpha, \beta) \), there is \( \rho \in J(\mathbb{L}) \) such that \( I[\rho^-, \rho] \rightsquigarrow I[\alpha, \beta] \) and \( \rho \not\leq \varphi \). Using Proposition 4.3, there is \( \psi \in M(\mathbb{L}) \) such that \( \varphi \leq \psi \) and \( I[\rho^-, \rho] \triangleright \bigvee I[\rho, \psi^+] \). By the definition of \( \Gamma(\alpha, \beta) \), we have \( \psi \leq \Gamma(\alpha, \beta) \), and therefore \( \varphi \leq \psi \leq \Gamma(\alpha, \beta) \).

In a bialgebraic modular lattice, we can describe lonesome meet irreducible elements.

**Proposition 4.6.** Let \( \mathbb{L} \) be a bialgebraic modular lattice, let \( \alpha, \beta \in \mathbb{L} \) with \( \alpha \prec \beta \), and let \( \eta \in M(\mathbb{L}) \) with \( I[\alpha, \beta] \rightsquigarrow I[\eta, \eta^+] \). Then the following are equivalent.

1. \( \eta \) is lonesome.
2. \( \eta \) is completely meet prime.
3. There is a complete lattice homomorphism \( h : \mathbb{L} \to \mathbb{B}_2 \) with \( h(\alpha) = h(\eta) = 0 \) and \( h(\beta) = h(\eta^+) = 1 \).

**Proof:** (1) \( \Rightarrow \) (2): We assume that \( \eta \) is lonesome. Then \( \Gamma(\eta, \eta^+) = \eta \). We choose \( \rho \in J(\mathbb{L}) \) with \( \rho \leq \eta^+ \), \( \rho \not\leq \eta \). Then \( I[\rho^-, \rho] \not\triangleright \bigvee I[\eta, \eta^+] \), and therefore \( \Delta(\eta, \eta^+) \geq \rho \). Since \( \eta \not\geq \rho \), this implies \( \eta \not\geq \Delta(\eta, \eta^+) \). Let \( X \subseteq \mathbb{L} \) be such that \( \bigwedge X \leq \eta \). Seeking a contradiction, we assume that for all \( x \in X \), we have \( x \not\leq \eta \). Then by Proposition 4.5 we obtain \( x \geq \Delta(\eta, \eta^+) \) for all \( x \in X \), and therefore \( \eta \geq \bigwedge X \geq \Delta(\eta, \eta^+) \), a contradiction. (2) \( \Rightarrow \) (3): For \( x \in \mathbb{L} \), we define \( h(x) = 0 \) if \( x \not\leq \eta \) and \( h(x) = 1 \) if \( x \not\geq \eta \). Let \( \theta \in \mathbb{L} \) be defined by \( \theta = \bigwedge \{ y \in \mathbb{L} \mid y \not\leq \eta \} \). Since \( \eta \) is completely meet prime, \( \theta \not\leq \eta \). Thus for all \( x \in \mathbb{L} \), we have \( x \not\leq \eta \) if and only if \( x \geq \theta \). Hence \( h(x) = 1 \) if and only if \( x \geq \theta \). Thus if \( h(\bigvee_{i \in I} x_i) = 0 \), then \( \bigvee_{i \in I} x_i \leq \eta \), and therefore for each \( i \in I \), \( h(x_i) = 0 \), implying \( \bigvee_{i \in I} h(x_i) = 0 \), and if \( h(\bigvee_{i \in I} x_i) = 1 \), then \( \bigvee_{i \in I} x_i \not\leq \eta \), hence there is \( j \in I \) with \( x_j \not\leq \eta \), and thus \( \bigvee_{i \in I} h(x_i) \geq h(x_j) = 1 \). Furthermore, if \( \bigwedge_{i \in I} h(x_i) = 1 \), then for all \( i \in I \), we have \( h(x_i) = 1 \) and thus \( x_i \geq \theta \). Hence \( \bigwedge_{i \in I} x_i \geq \theta \), and therefore \( h(\bigwedge_{i \in I} x_i) = 1 \). This is the essential step in proving that \( h \) is also a complete meet homomorphism. Now \( h(\eta) = 0 \) and \( h(\eta^+) = 1 \). Since \( \alpha \leq \beta \), we have \( h(\alpha) \leq h(\beta) \). If \( h(\alpha) = h(\beta) \), then \( (\alpha, \beta) \) lies in the congruence \( \ker(h) \). Since \( I[\eta, \eta^+] \rightsquigarrow I[\alpha, \beta] \), \( (\eta, \eta^+) \) lies in the congruence generated by \( (\alpha, \beta) \), and thus \( (\eta, \eta^+) \in \ker(h) \). This implies \( h(\eta) = h(\eta^+) \), a contradiction. Thus \( h(\alpha) < h(\beta) \), which implies \( h(\alpha) = 0 \) and \( h(\beta) = 1 \). (3) \( \Rightarrow \) (1): If \( \eta \) is not lonesome, then by Proposition 4.3, there is \( \psi \in M(\mathbb{L}) \) such that \( I[\eta, \eta^+] \rightsquigarrow I[\psi, \psi^+] \) and \( \psi \not\leq \eta \).
Since \((\eta, \eta^+) \not\in \ker(h)\), we have \((\psi, \psi^+) \not\in \ker(h)\), and therefore \(h(\psi) = 0\). Thus \(0 = h(\psi) \lor h(\eta) = h(\psi \lor \eta) \geq h(\eta^+) = 1\), a contradiction. \(\square\)

5. Commutator lattices

5.1. Special elements in commutator lattices.

Lemma 5.1. Let \((\mathbb{L}, \lor, \land, [\cdot, \cdot])\) be a commutator lattice and let \((\cdot : \cdot)\) its associated residuation.

1. If \(\eta \in M(\mathbb{L})\) is such that \(\eta = (\eta : \eta^+)\), then \(\eta\) is lonesome.

2. If \(\alpha \in J(\mathbb{L})\) is such that \([\alpha, \alpha] = \alpha\), then \(\alpha\) is lonesome.

Proof: (1) We assume that \(\psi \in M(\mathbb{L})\) is such that \(I[\eta, \eta^+] \rightarrow I[\psi, \psi^+]\). Since \(\eta^+ \not\leq \eta\), \(\eta\) is the \(\eta\) and therefore by Lemma 3.4 \(\psi^+ \not\leq \psi\). Therefore \((\psi : \psi^+) = \psi\). Using Lemma 3.4 again, we obtain \(\eta = (\eta : \eta^+) = (\psi : \psi^+) = \psi\).

(2) We assume that \(\beta \in J(\mathbb{L})\) is such that \(I[\alpha^- : \alpha] \rightarrow I[\beta^- : \beta]\). Since \([\alpha, \alpha] \not\leq \alpha^-\), Lemma 3.4 yields \([\beta, \beta] \not\leq \beta^-\). Therefore \([\beta^- : \beta] = \beta\). Since \((\beta^- : \beta) \not\geq \beta\), Lemma 3.4 yields \((\alpha^- : \alpha) \not\geq \beta\), which implies \([\beta, \alpha] \not\leq \alpha^-\). Thus \([\beta, \alpha] = \alpha\), and therefore \(\alpha \leq \beta\). Exchanging \(\alpha\) and \(\beta\), we obtain \(\beta \leq \alpha\), and therefore \(\alpha = \beta\). \(\square\)

5.2. Constructions of commutator operations. In this section, we will provide three constructions of commutator multiplications on a given lattice \(\mathbb{L}\). For a complete lattice \(\mathbb{K}\), a complete sublattice \(\mathbb{L}\) of \(\mathbb{K}\), and an element \(x \in \mathbb{K}\), we define its \(\mathbb{L}\)-closure \(c_\mathbb{L}(x) := \bigwedge\{y \in \mathbb{L} \mid y \geq x\}\). This operation is \(c_\mathbb{L}\) a monotonic operation from \(\mathbb{K}\) to \(\mathbb{L}\), and \(c_\mathbb{L}(x) \geq x\) for all \(x \in \mathbb{K}\).

Proposition 5.2. Let \(\mathbb{K}\) be a complete lattice, and let \(\mathbb{L}\) be a complete sublattice of \(\mathbb{L}\). Then for all families \((x_i)_{i \in I}\) from \(\mathbb{K}\), we have \(c_\mathbb{L}(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} c_\mathbb{L}(x_i)\).

Proof: For \(\geq\), we let \(j \in I\). Then \(c_\mathbb{L}(\bigvee_{i \in I} x_i) \geq \bigvee_{i \in I} x_i \geq x_j\), and thus \(c_\mathbb{L}(\bigvee_{i \in I} x_i) \geq c_\mathbb{L}(x_j)\). Hence \(c_\mathbb{L}(\bigvee_{i \in I} x_i) \geq \bigvee_{j \in I} c_\mathbb{L}(x_j)\). For \(\leq\), we observe that \(\bigvee_{i \in I} c_\mathbb{L}(x_i) \geq \bigvee_{i \in I} x_i\), and thus \(\bigvee_{i \in I} c_\mathbb{L}(x_i) \geq c_\mathbb{L}(\bigvee_{i \in I} x_i)\). \(\square\)

Lemma 5.3. Let \((\mathbb{K}, \lor, \land, [\cdot, \cdot])\) be a commutator lattice, and let \(\mathbb{L}\) be a complete sublattice of \(\mathbb{K}\). For \(x, y \in \mathbb{L}\), we define \([x, y]_\mathbb{L} := c_\mathbb{L}([x, y])\). Then \((\mathbb{L}, \lor, \land, [\cdot, \cdot])\) is a commutator lattice, and we have \([x, y]_\mathbb{L} \geq [x, y]_\mathbb{K}\) for all \(x, y \in \mathbb{L}\).

Proof: In order to show that \([\cdot, \cdot]_\mathbb{L}\) is a commutator multiplication on \(\mathbb{L}\), we observe that for all \(x, y \in \mathbb{L}\), we have \([x, y]_\mathbb{L} = c_\mathbb{L}([x, y]) = c_\mathbb{L}([y, x]) = [y, x]_\mathbb{L}\). Since \(x \lor y \in \mathbb{L}\) and \([x, y]_\mathbb{L} \leq x \land y\), we also have \(c_\mathbb{L}([x, y]) \leq x \land y\), and therefore \([x, y]_\mathbb{L} \leq x \land y\). Now let \((x_i)_{i \in I}\) be a family from \(\mathbb{L}\). Then Proposition 5.2 yields \(\bigvee_{i \in I} [x_i, y]_\mathbb{L} = c_\mathbb{L}([\bigvee_{i \in I} x_i, y]) = c_\mathbb{L}([\bigvee_{i \in I} x_i, y]) = \bigvee_{i \in I} c_\mathbb{L}([x_i, y]) =
\[ \bigvee_{i \in I} [x_i, y_i]_L. \] Hence \((L, \lor, \land, [.,.],)\) is a commutator lattice. Finally \([x, y]_L = c_L([x, y]) \geq [x, y].\]

**Lemma 5.4.** Let \(L\) be a complete lattice, and let \((K, \lor, \land, [.,.])\) be a commutator lattice. We assume that \(h\) is a complete lattice homomorphism from \(L\) to \(K\). For \(x, y \in L\), we define \([x, y]_L := \{z \in L \mid h(z) \geq [h(x), h(y)]_K\}\). Then \((L, \lor, \land, [.,.])\) is a commutator lattice, and we have \([x, y]_L \geq [h(x), h(y)]_K\) for all \(x, y \in L\).

**Proof:** We fix \(x, y \in L\). For commutativity, we observe that \([x, y]_L = \bigwedge \{z \in L \mid h(z) \geq [h(x), h(y)]_K\} = \bigwedge \{z \in L \mid h(z) \geq [h(y), h(x)]_K\} = [y, x]_L.\) Since \(h(x \land y) = h(x) \land h(y) \geq [h(x), h(y)]_K\), we have \(x \land y \geq [x, y]_L.\) Furthermore, \(h([x, y]_L) = h(\bigwedge \{z \in L \mid h(z) \geq [h(x), h(y)]_K\}) = \bigwedge \{h(z) \mid z \in L, h(z) \geq [h(x), h(y)]_K\} \geq [h(x), h(y)]_K.\) What remains to show is join distributivity. Let \((x_i)_{i \in I}\) be a family from \(L\), and let \(j \in I\). Then \(h(\bigvee_{i \in I} [x_i, y_i]) \geq \bigvee_{i \in I} h([x_i, y_i])_K = \bigvee_{i \in I} [h(x_i), h(y_i)]_K \geq \bigvee_{i \in I} [h(x_j), h(y_j)]_K.\) Now by the definition of \([x_j, y_j]_L\), this inequality implies \([x_j, y_j]_L \geq [x_j, y_j]_L.\) Therefore, \([x_j, y_j]_L \geq \bigvee_{i \in I} [x_j, y_j]_L.\) For the other inequality, we observe that \(h(\bigwedge_{i \in I} [x_i, y_i]) = \bigwedge_{i \in I} h([x_i, y_i])_K \geq \bigwedge_{i \in I} [h(x_i), h(y_i)]_K = \bigwedge_{i \in I} [h(x_i), h(y_i)]_K = [h(\bigwedge_{i \in I} [x_i, y_i]), h(y_j)]_K.\) Using the definition of \([\bigwedge_{i \in I} [x_i, y_i]]_L\), we obtain \([\bigwedge_{i \in I} [x_i, y_i]]_L \geq [\bigwedge_{i \in I} [x_i, y_i]]_L.\)

**Proposition 5.5.** Let \(L\) be a complete lattice, let \(\Theta\) be a complete congruence on \(L\), and let \(s : L \to L\) be the mapping defined by \(s(x) := \bigwedge \{z \in L \mid (z, x) \in \Theta\}.\) Then for all \(x \in L\), we have \((s(x), x) \in \Theta\), and for all families \((x_i)_{i \in I}\) from \(L\), we have \(s(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} s(x_i).\)

**Proof:** The fact that \(\Theta\) is a complete congruence implies that for every \(x \in L\), we have \(\bigwedge \{z \in L \mid (z, x) \in \Theta\} \equiv \Theta x,\) and hence \((s(x), x) \in \Theta.\) We first prove \(s(\bigvee_{i \in I} x_i) \geq \bigvee_{i \in I} s(x_i).\) To this end, let \(j \in I.\) We have \(s(\bigvee_{i \in I} x_i) \equiv \bigvee_{i \in I} s(x_i)\). From the definition of \(s(x_j)\), we obtain \(s(x_j) \leq s(\bigvee_{i \in I} x_i) \land x_j,\) which implies \(s(x_j) \leq s(\bigvee_{i \in I} x_i).\) Thus \(\bigvee_{i \in I} s(x_i) \leq s(\bigvee_{i \in I} x_i).\) For proving \(s(\bigvee_{i \in I} x_i) \leq \bigvee_{i \in I} s(x_i)\), we notice that \(\bigvee_{i \in I} s(x_i) \equiv \Theta \bigvee_{i \in I} x_i.\) Hence from the definition of \(s(\bigvee_{i \in I} x_i)\), we obtain \(s(\bigvee_{i \in I} x_i) \leq \bigvee_{i \in I} s(x_i).\)

**Lemma 5.6.** Let \(L\) be a complete lattice that splits with splitting pair \((\delta, \varepsilon)\). Let \(\Theta\) be a complete congruence of \(L\) with \((\varepsilon, 1) \in \Theta\), and let \(s\) be the complete join homomorphism associated with \(\Theta\) that was defined in Lemma 5.3. For \(x, y \in L\), we define \([x, y] := 0\) if \(x \leq \delta\) and \(y \leq \delta,\) and \([x, y] := s(x \land y) = \bigwedge \{z \in L : (z, x \land y) \in \Theta\}\) otherwise. Then we have:

1. If \(x \not\leq \delta,\) then \([x, y] = [y, x] = s(y).\)
2. \([, , , ]\) is a commutator multiplication on \((L, \lor, \land).\)
Proof: For item (1), we fix $x, y \in L$ with $x \not\leq \delta$. Then $x \geq \varepsilon$, and therefore $(x, 1) \in \Theta$. Then $[x, y] = s(x \land y) = s(1 \land y) = s(y) = s(y \land 1) = s(y \land x) = [y, x]$. For item (2), we observe that commutativity and $[x, y] \leq x \land y$ for all $x, y \in L$ follow immediately from the definition. What remains to be proved is the join distributivity $\bigvee_{i \in I} x_i, y = \bigvee_{i \in I} [x_i, y]$. In the case $y \not\leq \delta$, item (1) yields $\bigvee_{i \in I} x_i, y = s(\bigvee_{i \in I} x_i)$. By Lemma 5.5, this last expression is equal to $\bigvee_{i \in I} s(x_i)$. Applying item (1) again, this last expression is equal to $\bigvee_{i \in I} [x_i, y]$. Again by (1), the first joinand $[x_j, y]$ is equal to $s(y)$. For an arbitrary $i \in I \setminus \{j\}$, $[x_i, y] = 0$ if $x_i \leq \delta$, and $s(y)$ if $x_i \not\leq \delta$. Thus we have $\bigvee_{i \in I \setminus \{j\}} [x_i, y] \leq s(y)$. Hence $\bigvee_{i \in I} [x_i, y] = s(y)$, and so the equation expressing join distributivity also holds in this last case. \hfill \Box

This construction had one origin in the analysis of [IS01, Proposition 16].

5.3 Types of commutator lattices.

Definition 5.7. Let $L = (\mathbb{L}, \lor, \land, [\cdot, \cdot])$ be a commutator lattice. Let $(\gamma_n)_{n \in \mathbb{N}}$ and $(\lambda_n)_{n \in \mathbb{N}}$ be the sequences in $L$ defined by $\gamma_1 = \lambda_1 = 1$ and $\gamma_{n+1} = [\gamma_n, \gamma_n]$ and $\lambda_{n+1} = [1, \lambda_n]$ for $n \in \mathbb{N}$. Then $L$ is of solvable type if there is an $n \in \mathbb{N}$ with $\gamma_n = 0$, of nilpotent type if there is an $n \in \mathbb{N}$ with $\lambda_n = 0$, and of abelian type if $\gamma_2 = 0$.

Lemma 5.8. Let $L = (\mathbb{L}, \lor, \land, [\cdot, \cdot])$ be a commutator lattice of finite height, and let $(\cdot : \cdot)$ its associated residuation. Then we have:

1. $L$ is of solvable type if and only if there is no $\beta \in L$ with $\beta \neq 0$ and $[\beta, \beta] = \beta$.
2. $L$ is of nilpotent type if and only if there is no $\beta \in L$ with $\beta \neq 0$ and $[1_A, \beta] = \beta$.
3. Assume that $L$ is modular. Then $L$ is of nilpotent type if and only if for all $\alpha, \beta \in L$ with $\alpha \prec \beta$, we have $(\alpha : \beta) = 1$.

Proof: (1) If $L$ is not of solvable type, then there will be an $n \in \mathbb{N}$ such that in the derived series of $L$ we have $\gamma_n = \gamma_{n+1}$ and $\gamma_n \neq 0$. Then we set $\beta := \gamma_n$. On the other hand, if $[\beta, \beta] = \beta$, we prove by induction that $\gamma_n \geq \beta$ for all $n \in \mathbb{N}$. The induction step is $\gamma_{n+1} = [\gamma_n, \gamma_n] \geq [\beta, \beta] = \beta$. This proves item (1). Item (2) is proved similarly. For the “if”-direction of (3), we assume that $L$ is not of nilpotent type. By (2), there is $\gamma \in L$ such that $\gamma \neq 0$ and $[1, \gamma] = \gamma$. Let $\delta \prec \gamma$. Then $[1, \gamma] \not\leq \delta$, hence $(\delta : \gamma) \neq 1$. For the “only if”-direction of (3), we assume that there are $\alpha \prec \beta$ in $L$ such that $(\alpha : \beta) < 1$. Let $\gamma$ be minimal with $\gamma \leq \beta$, $\gamma \not\leq \alpha$. Then $\gamma$ is join irreducible and $I[\gamma^-, \gamma] \not\supset I[\alpha, \beta]$. From Lemma 3.4 we
obtain \((\gamma : \gamma) = (\alpha : \beta) < 1\), and therefore \([1, \gamma] \not\subseteq \gamma\). This implies \([1, \gamma] = \gamma\), and hence by item (2), \(L\) is not of nilpotent type.

5.4. The largest commutator operation on a given lattice. Given a lattice \(L\) and \(x, y \in L\), we would like to obtain an upper bound for \([x, y]\) for each commutator multiplication definable on \(L\). From \([\text{Cze08}]\), we know that such a bound is provided by the single largest commutator multiplication on each lattice:

Lemma 5.9 (\([\text{Cze08} \text{ Corollary 1.5}]\)). Let \(L\) be a complete lattice, and let \(([.,.]_i)_{i \in I}\) be the family of all binary operations that turn \(L\) into a commutator lattice. For \(x, y \in L\), we define \([x, y]_L := \bigvee_{i \in I}[x, y]_i\). Then \((L, \lor, \land, [.,.]_L)\) is a commutator lattice.

Czelakowski writes \(\bullet_\Omega\) for the operation \([.,.]_L\) and states that “the characterization of the operation \(\bullet_\Omega\) in modular algebraic lattices is an open and challenging problem” \([\text{Cze08} \text{ p. 114}]\). We will not be able to construct this operation \([.,.]_L\) completely, but we will obtain a description of the associated residuum \((\alpha : \beta)\) if \(\alpha < \beta\).

Definition 5.10. Let \(L\) be a complete lattice. Then \(L\) forces abelian type if \([1, 1]_L = 0\). \(L\) forces nilpotent type if \((L, \lor, \land, [.,.]_L)\) is of nilpotent type, and \(L\) forces solvable type if \((L, \lor, \land, [.,.]_L)\) is of solvable type.

Lemma 5.11. Let \(K\) be a complete lattice, and let \(L\) be a complete sublattice of \(K\). Then for all \(x, y \in L\), we have \([x, y]_L \geq [x, y]_K\).

Proof: We use Lemma 5.3 to construct a multiplication \([.,.]_L\) on \(L\) by \([.,.]_L := c_L([x, y]_K)\) for \(x, y \in L\). Then for all \(x, y \in L\), we have \([x, y]_L \geq [x, y]_L = c_L([x, y]_K) \geq [x, y]_K\).

Lemma 5.12. Let \(L, K\) be complete lattices, and let \(h\) be a complete lattice homomorphism from \(L\) to \(K\). Then for all \(x, y \in L\), we have \(h([x, y]_L) \geq [h(x), h(y)]_K\).

We use Lemma 5.4 to construct a multiplication \([.,.]_L\) on \(L\) by \([.,.]_L := \lor\{z \in L \mid h(z) \geq [h(x), h(y)]_K\}\) for \(x, y \in L\). Now for all \(x, y \in L\), we have \(h([x, y]_L) \geq h([x, y]_L)\). By Lemma 5.4, we have \(h([x, y]_L) \geq [x, y]_K\).

We call a complete sublattice \(L\) of \(K\) a complete \((0, 1)\)-sublattice of \(K\) if \(0_K \in L\) and \(1_K \in L\). In this case, \(K\) is a \((0, 1)\)-extension of \(L\).

Theorem 5.13. Let \(L\) be a complete lattice, and let \(K\) be a complete \((0, 1)\)-extension or a complete \((0, 1)\)-homomorphic image of \(L\). If \(L\) forces abelian, nilpotent, or solvable type, then so does \(K\).

Proof: We use a function \(f\) to treat the lower central and the derived series at once. Let \(f : \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N}\) be a function with \(f(n) < n\) for all \(n \in \mathbb{N}\), and let \((\kappa_n)_{n \in \mathbb{N}}\) be a sequence from \(K\) defined by \(\kappa_1 = 1\) and \(\kappa_n := [\kappa_{f(n)}, \kappa_{n-1}]_K\) for
Let $(\lambda_n)_{n \in \mathbb{N}}$ be the corresponding sequence from $\mathbb{L}$ defined by $\lambda_1 = 1$ and $\lambda_n := [\lambda_{f(n)}, \lambda_{n-1}]_\mathbb{L}$ for $n > 1$.

If $\mathbb{L}$ is a $(0,1)$ sublattice of $\mathbb{K}$, we have $\kappa_n \leq \lambda_n$ for all $n \in \mathbb{N}$. We prove this by induction: for $n > 1$, $\kappa_n = [\kappa_{f(n)}, \kappa_{n-1}]_\mathbb{K}$. By the induction hypothesis and monotonicity, we obtain $[\kappa_{f(n)}, \kappa_{n-1}]_\mathbb{K} \leq [\lambda_{f(n)}, \lambda_{n-1}]_\mathbb{K}$. By Lemma 5.11, we have $[\lambda_{f(n)}, \lambda_{n-1}]_\mathbb{K} \leq [\lambda_{f(n)}, \lambda_{n-1}]_\mathbb{L} = \lambda_n$. Therefore, if for some $k \in \mathbb{N}$, $\lambda_k = 0$, then $\kappa_k = 0$.

If $\mathbb{K}$ is a complete $(0,1)$-homomorphic image of $\mathbb{L}$, we have $\kappa_n \leq h(\lambda_n)$ for all $n \in \mathbb{N}$. Again, we proceed by induction: as the induction basis, we observe that $\kappa_1 = 1_\mathbb{K} = h(1_\mathbb{L}) = h(\lambda_1)$. For the induction step, we let $n > 1$ and compute $\kappa_n = [\kappa_{f(n)}, \kappa_{n-1}]_\mathbb{K}$. By the induction hypothesis and monotonicity, we obtain $[\kappa_{f(n)}, \kappa_{n-1}]_\mathbb{K} \leq [h(\lambda_{f(n)}), h(\lambda_{n-1})]_\mathbb{K}$. By Lemma 5.12, we have $[h(\lambda_{f(n)}), h(\lambda_{n-1})]_\mathbb{K} \leq h([\lambda_{f(n)}, \lambda_{n-1}]_\mathbb{L}) = h(\lambda_n)$. Therefore, if for some $k \in \mathbb{N}$, $\lambda_k = 0$, then $\kappa_k = h(\lambda_k) = 0$.

Now if $\mathbb{L}$ forces abelian type, then $\lambda_2 = 0$, and hence $\kappa_2 = 0$, and therefore $\mathbb{K}$ forces abelian type. If $\mathbb{L}$ forces nilpotent type, we choose $f(n) := 1$ for all $n \in \mathbb{N}$ and observe that there is $k \in \mathbb{N}$ with $\lambda_k = 0$, hence $\kappa_k = 0$, and thus $\mathbb{K}$ forces nilpotent type. For solvable type, the proof is analogous with $f(n) := n - 1$. □

Theorem 5.14. Let $\mathbb{K}$ be a complete lattice. If $\mathbb{K}$ has a complete $(0,1)$-sublattice $\mathbb{L}$ that is algebraic, modular, simple, complemented, and has at least 3 elements, then $\mathbb{K}$ forces abelian type.

Proof: By Theorem 5.13, it is sufficient to prove that $\mathbb{L}$ forces abelian type. Let $T$ be the set of atoms of $\mathbb{L}$. We let $[.,.]$ denote the residuation operation associated with the largest commutator operation $[.,.]_\mathbb{L}$ on $\mathbb{L}$. We show that for all $\alpha \in T$, $[\alpha, \alpha]_\mathbb{L} = 0$. Let $\eta_1$ be a complement of $\alpha$ in $\mathbb{L}$. Then $I[0, \alpha] \succ I[\eta_1, 1]$, and therefore $\eta_1$ is a coatom of $\mathbb{L}$. Since $|\mathbb{L}| \geq 3$, $\eta_1 \neq 0$, and therefore by [MMT87] Lemma 4.83, there is an atom $\beta$ of $\mathbb{L}$ with $\beta \leq \eta_1$. Let $\eta_2$ be the complement of $\beta$ in $\mathbb{L}$. Then $\eta_2$ is a coatom of $\mathbb{L}$ and $\eta_1 \neq \eta_2$. By Lemma 3.3(5), we have $[\eta_1 : 1]_\mathbb{L} \geq \eta_1$. Since $\mathbb{L}$ is simple and modular, Dilworth's congruence generation theorem [MMT87] Theorem 2.66 yields that the intervals $I[\eta_1, 1]$ and $I[\eta_2, 1]$ are projective inside $\mathbb{L}$. Hence by Lemma 3.3, $[\eta_2 : 1]_\mathbb{L} = [\eta_1 : 1]_\mathbb{L} \geq \eta_1$. Since $[\eta_2 : 1]_\mathbb{L} \geq \eta_2$, we obtain $[\eta_2 : 1]_\mathbb{L} \geq \eta_1 \lor \eta_2 = 1$. Thus $\eta_1, \eta_2 \geq 1_\mathbb{L}$, and therefore, since by simplicity all prime intervals of $\mathbb{L}$ are projective, Lemma 3.3 yields $[\alpha, \alpha]_\mathbb{L} = 0$. In an algebraic complemented modular lattice, 1 is the join of atoms [MMT87] Lemma 4.83. Hence $[1, 1]_\mathbb{L} = [\lor T, \lor T]_\mathbb{L} = \lor \{[\alpha, \beta]_\mathbb{L} \mid \alpha, \beta \in T\}$. The joinands of the last expression with $\alpha = \beta = 0$ by the above argument; for the other joinands, we have $[\alpha, \beta]_\mathbb{L} \leq \alpha \land \beta = 0$. This completes the proof that $\mathbb{L}$ forces abelian type; now Theorem 5.13 implies that $\mathbb{K}$ forces abelian type. □
Theorem 5.15. Let $\mathbb{L}$ be a bialgebraic modular lattice, and let $[x : y] := \lor \{ z \in \mathbb{L} \mid [z, y]_\mathbb{L} \leq x \}$ denote the residuation operation associated with $[., .]_\mathbb{L}$. Let $\alpha, \beta \in \mathbb{L}$ be such that $\alpha \prec \beta$. Then $[\alpha : \beta] = \Gamma(\alpha, \beta)$.

Proof: Lemma 5.3 implies $[\alpha : \beta] \geq \Gamma(\alpha, \beta)$. For proving $\leq$, we let $\rho$ be a join irreducible element of $\mathbb{L}$ with $\rho \leq \beta, \rho \not\leq \alpha$. Then $I[\rho^-, \rho] \not\rightarrow I[\alpha, \beta]$, and therefore $\Gamma(\alpha, \beta) = \Gamma(\rho^-, \rho)$, and by Lemma 5.4 $[\alpha : \beta] = [\rho^- : \rho]$. In order to prove $[\rho^- : \rho] \leq \Gamma(\rho^-, \rho)$, we fix $\psi \in \mathbb{L}$ such that such that $[\psi, \rho]_\mathbb{L} \leq \rho^-$, and show that $\psi \leq \Gamma(\rho^-, \rho)$. In the case $\Gamma(\rho^-, \rho) = 1$, this is obviously true, so we assume $\Gamma(\rho^-, \rho) < 1$. We let $\delta := \Gamma(\rho^-, \rho)$ and we define $\varepsilon := \Delta(\rho^-, \rho)$, which is defined as $\lor \{ \sigma \in J(\mathbb{L}) \mid I[\sigma^-, \sigma] \rightarrow I[\rho^-, \rho] \}$. Then $(\delta, \varepsilon)$ is a splitting pair for the lattice $\mathbb{L}$. Let $\Theta$ be the complete congruence of $\mathbb{L}$ that is generated by $(\varepsilon, 1)$. Then we apply Lemma 5.6 to $\Theta$ and the splitting pair $(\delta, \varepsilon)$ and obtain a complete join homomorphism $s$ and a commutator multiplication $[., .]$ on $\mathbb{L}$. Next, we show

$$ (\rho^-, \rho) \not\in \Theta. $$

For this purpose, we construct a complete congruence $\Phi$ of $\mathbb{L}$ such that $(\varepsilon, 1) \in \Phi$ and $(\rho^-, \rho) \not\in \Phi$. Let $a := \rho^-$ and $b := \rho$, let $\Phi$ be the complete congruence of $\mathbb{L}$ produced in Proposition 4.2. Clearly, $(\rho^-, \rho) \not\in \Phi$. Now suppose $(\varepsilon, 1) \not\in \Phi$. Then we have $\rho_1, \rho_2 \in \mathbb{L}$ such that $\varepsilon \leq \rho_1 \prec \rho_2$ and $I[\rho^-, \rho] \rightarrow I[\rho_1, \rho_2]$. From the dual of Proposition 4.1(2), we obtain $\rho_3 \in J(\mathbb{L})$ with $\rho_3 \leq \rho_2, \rho_3 \not\leq \rho_1$. Then by the dual of Proposition 4.1(3), $I[\rho_3^-, \rho_3] \rightarrow I[\rho_1, \rho_2] \rightarrow I[\rho^-, \rho]$, and therefore from the definition of $\varepsilon$ as $\Delta(\rho^-, \rho)$, we obtain $\rho_3 \leq \varepsilon$. Thus $\rho_3 \leq \rho_1$, and therefore $\rho_2 = \rho_1 \lor \rho_3 = \rho_1$, a contradiction. This contradiction proves $(\varepsilon, 1) \in \Phi$. Hence $\Theta \subseteq \Phi$, which completes the proof of (5.1). We will next prove $s(\rho) = \rho$. Suppose $s(\rho) < \rho$. Then $s(\rho) \leq \rho^-$. By Proposition 5.5 we have $(s(\rho), \rho) \in \Theta$, and therefore $(\rho^-, \rho) = (s(\rho) \lor \rho^-, \rho \lor \rho^-) \in \Theta$, contradicting (5.1). Therefore $s(\rho) = \rho$. Since $[\psi, \rho]_\mathbb{L} \leq \rho^-$, we have $[\psi, \rho] \leq \rho^-$. Now if $\psi \not\leq \delta$, then by Lemma 5.6(1), $\rho^- \geq [\psi, \rho] = s(\rho) = \rho$, a contradiction. Therefore $\psi \leq \delta = \Gamma(\rho^-, \rho) = \Gamma(\alpha, \beta)$. \hfill \Box

Theorem 5.16. Let $\mathbb{L}$ be a modular lattice of finite height. Then $\mathbb{L}$ forces nilpotent type if and only if for all $\alpha, \beta \in \mathbb{L}$ with $\alpha \prec \beta$, we have $\Gamma(\alpha, \beta) = 1$.

We let $[x : y] := \lor \{ z \in \mathbb{L} \mid [z, y]_\mathbb{L} \leq x \}$ denote the residuation operation associated with $[., .]_\mathbb{L}$. In order to show that $(\mathbb{L}, \lor, \land, [., .]_\mathbb{L})$ is of nilpotent type, we use Lemma 5.8(3). By this Lemma, $\mathbb{L}$ forces nilpotent type if and only if for all $\alpha, \beta \in \mathbb{L}$ with $\alpha \prec \beta$, we have $[\alpha : \beta] = 1$, which by Theorem 5.15 is equivalent to $\Gamma(\alpha, \beta) = 1$. \hfill \Box

Next, we want to characterize lattices forcing solvable type.
Lemma 5.17. Let $\mathbb{L}$ be a bialgebraic modular lattice, let $[x : y] := \bigvee \{ z \in \mathbb{L} \mid [z, y]_L \leq x \}$ denote the residuation operation associated with $[, ,]_L$, and let $\eta \in M(\mathbb{L})$. Then $[\eta : \eta^+] = \eta$ if and only if $\eta$ is lonesome.

Proof: The “only if”-direction is a consequence of Lemma 5.1. For the “if”-direction, we assume that $\eta$ is lonesome. Then Proposition 4.6 yields a complete lattice homomorphism from $\mathbb{L}$ onto $\mathbb{B}_2$ with $h(\eta) = 0$ and $h(\eta^+) = 1$. Now Lemma 5.12 implies $h([\eta^+, \eta^+]_L) \geq [h(\eta^+), h(\eta^+)]_{\mathbb{B}_2} = [1, 1]_{\mathbb{B}_2}$, which is equal to 1 because $[x, y] := x \wedge y$ is a commutator multiplication on $\mathbb{B}_2$. Therefore $[\eta^+, \eta^+]_L \not\leq \eta$, and then $\eta^+ \not\leq [\eta : \eta^+]$. Since $\eta \leq [\eta : \eta^+]$, we have $[\eta : \eta^+] = \eta$. \qed

Theorem 5.18. Let $\mathbb{L}$ be a modular lattice of finite height. Then $\mathbb{L}$ forces solvable type if and only if the two element lattice $\mathbb{B}_2$ is not a homomorphic image of $\mathbb{L}$.

Proof: For the “only if”-direction, we assume that $\mathbb{L}$ forces solvable type and that $h : \mathbb{L} \to \mathbb{B}_2$ is an epimorphism. Then by Theorem 5.13, $\mathbb{B}_2$ forces solvable type, which contradicts the fact that on $\mathbb{B}_2$, the operation $[x, y] := x \wedge y$ is a commutator multiplication which is not of solvable type. For the “if”-direction, we assume that $\mathbb{L}$ does not force solvable type. Then by Lemma 5.8, there is a $\beta \in \mathbb{L}$ with $\beta > 0$ and $[\beta, \beta]_L = \beta$. Let $\alpha \prec \beta$, and let $\rho$ be minimal with $\rho \leq \beta$, $\rho \not\leq \alpha$. Then $\rho$ is join irreducible and $[\rho, \rho]_L \not\leq \rho^-$, and therefore $[\rho, \rho]_L = \rho$. Now by Lemma 5.1, $\rho$ is lonesome. Taking $\eta \in M(\mathbb{L})$ with $\eta \geq \rho^-$ and $\eta \not\geq \rho$ and using Propositions 4.1 and 4.4, we obtain that $\eta$ is lonesome, and now Proposition 4.6 yields an epimorphism of $\mathbb{L}$ onto $\mathbb{B}_2$. \qed

6. Algebras

6.1. Lattice conditions. The results on commutator multiplications of the previous sections immediately yield the following theorem.

Theorem 6.1. Let $\mathbf{A}$ be an algebra in a congruence modular variety.

1. If $\text{Con}(\mathbf{A})$ has a complete $(0, 1)$-sublattice with at least 3 elements that is algebraic, simple, and complemented, then $\mathbf{A}$ is abelian.
2. If $\text{Con}(\mathbf{A})$ has a finite $(0, 1)$-sublattice $\mathbb{L}$ that does not split, then $\mathbf{A}$ is supernilpotent.
3. If $\text{Con}(\mathbf{A})$ has a $(0, 1)$-sublattice $\mathbb{L}$ of finite height such that for all $\alpha, \beta \in \mathbb{L}$ with $\alpha \not\leq \beta$, we have $\Gamma_\mathbb{L}(\alpha, \beta) = 1$, then $\mathbf{A}$ is nilpotent.
4. If $\text{Con}(\mathbf{A})$ has a $(0, 1)$-sublattice $\mathbb{L}$ of finite height such that $\mathbb{B}_2$ is not a homomorphic image of $\mathbb{L}$, then $\mathbf{A}$ is solvable.

Proof: Let $\mathbb{K}$ be the lattice $(\text{Con}(\mathbf{A}), \lor, \land)$. Since $\mathbf{A}$ lies in a congruence modular variety, Proposition 3.2 tells that $\mathbb{K} := (\text{Con}(\mathbf{A}), \lor, \land, [, ,]_\mathbf{A})$ is a commutator lattice.
From Theorem 5.14, we obtain that $\mathbb{K}$ forces abelian type, and therefore $[1_A, 1_A]_A = 0_A$.

Let $[\alpha_1, \ldots, \alpha_n]_A$ denote the $n$-ary commutator of $\alpha_1, \ldots, \alpha_n \in \text{Con}(A)$. A. Moorhead [Moo16] proved that these higher commutator operations satisfy (among others) the conditions (HC1), (HC3), and (HC7) from [AM13, p. 860]. Let $c_L : \mathbb{K} \to \mathbb{L}$ be the operation defined before Proposition 5.2. Now for every $n \in \mathbb{N}$, we define an operation $f_n^L : \mathbb{L}^n \to \mathbb{L}$ by $f_n^L(x_1, \ldots, x_n) := c_L([x_1, \ldots, x_n]_A)$ for all $x_1, \ldots, x_n \in \mathbb{L}$. Using Proposition 5.2, it is easy to verify that the sequence $(f_n^L)_{n \in \mathbb{N}}$ satisfies the conditions (HC1), (HC3), and (HC7) of [AM13]. Now the proof of [AM13] Lemma 3.3 yields an $n \in \mathbb{N}$ with $f_n^L(1_A, \ldots, 1_A) = 0_A$, and therefore $[1_A, \ldots, 1_A]_A \leq c_L([1_A, \ldots, 1_A]_A) = f_n^L(1_A, \ldots, 1_A) = 0_A$. Hence $A$ is supernilpotent.

From Theorem 5.16 we obtain that $\mathbb{L}$ forces nilpotent type, and hence by Theorem 5.13 $\mathbb{K}$ forces nilpotent type. Hence $\mathbb{K}$ is of nilpotent type, making $A$ nilpotent.

From Theorem 5.18 we obtain that $\mathbb{L}$ forces solvable type, and hence by Theorem 5.13 $\mathbb{K}$ forces solvable type. Thus $\mathbb{K}$ is of solvable type, making $A$ solvable. □

The next sections search for partial converses of these results.

6.2. Nonsolvable and nonnilpotent expansions. We let $\text{Comp}(A)$ be the clone of congruence preserving functions of $A$, and we define $A^c$ as the algebra $(A, \text{Comp}(A))$. Hence $A^c$ is the largest expansion of $A$ that has the same congruence relations as $A$.

Lemma 6.2. Let $A$ be an algebra in a congruence modular variety, and let $\mathbb{L}$ be its congruence lattice. We assume that $\mathbb{L}$ is bialgebraic. Let $\alpha \in J(\mathbb{L})$. Then $[\alpha, \alpha]_{A^c} = \alpha$ if and only if $\alpha$ is lonesome.

Proof: For the “only if”-direction, we assume that $[\alpha, \alpha]_{A^c} = \alpha$. Then by Lemma 5.1 $\alpha$ is lonesome. For the “if”-direction, we assume that $\alpha$ is lonesome. Let $\eta \in M(\mathbb{L})$ be such that $\eta \geq \alpha^-$, $\eta \not\geq \alpha$. Then $I[\alpha^-, \alpha] I[\eta, \eta^+].$ By Proposition 4.3, $\eta$ is lonesome, and we have $\Delta(\alpha^-, \alpha) = \alpha$ and $\Gamma(\alpha^-, \alpha) = \eta$. We choose $(a, b) \in \alpha \setminus \alpha^-$ and define a binary function $f$ by $f(x, y) = b$ if $(x, b) \in \eta$ and $(y, b) \in \eta$, and $f(x, y) = a$ else. By Proposition 4.5 $(\eta, \alpha)$ is a splitting pair of the lattice $\text{Con}(A)$, and the function $f$ is constant on $\eta$-classes and maps into one $\alpha$-class. From this we conclude that $f$ is congruence preserving (an argument is given, e.g., in [AM16, Proposition 3.1]). Thus $f$ is a fundamental operation of $A^c$. We have $f(a, a) = a = f(a, b)$, and therefore $f(b, a) \equiv f(b, b)$ (mod $[\alpha, \alpha]_{A^c}$). Hence $(a, b) \in [\alpha, \alpha]_{A^c}$, and therefore $[\alpha, \alpha]_{A^c} \not\leq \alpha^-$. Thus $[\alpha, \alpha]_{A^c} = \alpha$. □
Theorem 6.3. Let $A$ be an algebra in a congruence modular variety. We assume that $\text{Con}(A)$ is of finite height. Then $A^c$ is solvable if and only if $B_2$ is not a homomorphic image of the lattice $\text{Con}(A)$.

Proof: For the “if”-direction, we use Theorem 5.13 and obtain that $(\text{Con}(A), \vee, \wedge, [., .]_A)$ is a commutator lattice of solvable type, and therefore $A$ is solvable. For the “only if”-direction, we assume that $B_2$ is a homomorphic image of $\text{Con}(A)$. Let $\alpha, \beta \in \mathbb{L}$ be such that $\alpha > \beta$, $h(\alpha_1) = 0$, and $h(\beta_1) = 1$. Take $\eta \in \text{M}(\mathbb{L})$ be such that $I[\alpha_1, \beta_1] \not\sim I[\eta, \eta^+]$. Then $h(\eta) = 0$ and $h(\eta^+) = 1$. Now by Proposition 4.5, $\eta$ is a lonesome meet irreducible element of $\mathbb{L}$. Let $\alpha \in J(\mathbb{L})$ be such that $I[\eta, \eta^+] \not\sim I[\alpha^-, \alpha]$. Then Proposition 4.4 yields that $\alpha$ is a lonesome join irreducible element of $\mathbb{L}$. Now Lemma 6.2 yields $[\alpha, \alpha]_{A^c} = \alpha$, and hence by Lemma 5.8 $A^c$ is not solvable.

For characterizing congruence lattices that force nilpotency, we restrict ourselves to finite expanded groups. For this characterization, we will need to construct congruence preserving functions that destroy nilpotency, similar to the functions destroying solvability produced in the proof of Lemma 6.2. The construction relies on certain unary congruence preserving functions provided by [Aic06a]. We will isolate the arguments that are restricted to expanded groups in the next Lemma.

Lemma 6.4. Let $V$ be a finite expanded group, let $\mathbb{L}$ be its congruence lattice, and let $\alpha \in J(\mathbb{L})$. Then $(\alpha^- : \alpha)_V \leq \Gamma(\alpha^-, \alpha)$.

Proof: We first consider the case that $\alpha$ is a lonesome join irreducible element. Then from Lemma 6.2 we obtain $[\alpha, \alpha]_V = \alpha$. Hence $[\alpha, \alpha]_V \leq \alpha^-$, and therefore $\alpha \not\leq (\alpha^- : \alpha)_V$. Since $\alpha$ is lonesome, we have $\Delta(\alpha, \alpha) = \alpha$. The splitting property from Proposition 4.3 now yields $(\alpha^- : \alpha)_V \leq \Gamma(\alpha^-, \alpha)$.

Let us now consider the case that $\alpha$ is not a lonesome join irreducible element. Let $A := 0/\alpha$, $A^- := 0/\alpha^-$, $C := 0/\Gamma(\alpha^-, \alpha)$, $D := 0/\Delta(\alpha^-, \alpha)$, $E := 0/(\alpha^- : \alpha)_V$. Our goal is to show $E \subseteq C$. To this end, we fix $z \in E$. We first show

\[(6.1) \quad \alpha \leq \Gamma(\alpha^-, \alpha).\]

Since $\alpha$ is not lonesome, we apply Proposition 4.3 to the dual of $\mathbb{L}$ and obtain $\beta \in \text{Con}(V)$ such that $\alpha$ and $\beta$ are not comparable and $I[\alpha -, \alpha] \not\sim I[\beta^-, \beta]$. Then $\Delta^-(\alpha, \alpha) \geq \alpha \vee \beta > \alpha$, and therefore $\alpha \not\leq \Delta^-(\alpha, \alpha)$. Proposition 4.5 now yields (6.1), and thus $A \subseteq C$. Next, we use Proposition 4.3(2) and Theorem 5.1 from [Aic06a] to obtain a unary congruence preserving function $e$ of $V$ with $e(0) = 0$, $e(A) \not\subseteq A^-$ and $e(V) \subseteq D$. From $e$, we define a function $f : V \times V \rightarrow V$ by $f(x, y) := e(z - x + y)$ if $z - x + y \in C$ and $f(x, y) := 0$ otherwise. The range of $f$ is contained in $D$, and the restriction of $f$ to each $\Gamma(\alpha^-, \alpha)$-class, i.e., to each set of the form $(x_1, y_1) + C \times C$, is the restriction of a congruence preserving function of $V$. By Proposition 4.5, $(\Gamma(\alpha^-, \alpha), \Delta(\alpha^-, \alpha))$
is a splitting pair of \( \text{Con}(V) \), and thus from \([ALM16]\) Proposition 3.1 we see that \( f \) is a congruence preserving function of \( V \). We choose \( a \in A \) such that \( e(a) \not\in A^- \). Seeking a contradiction, we suppose that \( z \not\in C \). We have \( f(0,0) = 0 \) because \( z \not\in C \) and \( f(0,a) = 0 \) because \( a \in C \) and thus \( z + a \not\in C \). Hence \((f(z,0), f(z,a)) \in [(\alpha^- : \alpha)_{\text{V}^e}, \alpha]_{\text{V}^e} \). Now \( f(z,0) = e(0) = 0 \) because \( z-z+0 \in C \) and \( f(z,a) = f(z-z+a) = e(a) \) because \( a \in C \). Thus \((0,e(a)) \in \alpha^- \), and therefore \( e(a) \in A^- \), contradicting the choice of \( a \). This contradiction establishes \( z \in C \), which concludes the proof of \( E \subseteq C \). \( \square \)

**Theorem 6.5.** Let \( A \) be a finite expanded group, and let \( \alpha, \beta \in \text{Con}(A) \) be such that \( \alpha < \beta \). Then the centralizer \( (\alpha : \beta)_A \) of \( \beta \) over \( \alpha \) in \( A^c \) is \( \Gamma(\alpha, \beta) \).

Lemma 3.5 yields \( \Gamma(\alpha, \beta) \leq (\alpha : \beta)_{A^c} \). Let \( \alpha_1 \) be minimal in \( \text{Con}(A) \) with \( \alpha_1 \leq \beta, \alpha_1 \not\in \alpha \). Then \( \alpha_1 \) is join irreducible, and \( I[\alpha_1^- : \alpha_1] \not
\subseteq I[\alpha, \beta] \). Hence \( \Gamma(\alpha, \beta) = \Gamma(\alpha_1^- : \alpha_1) \). From Lemma 6.4, we obtain \( \Gamma(\alpha_1^- : \alpha_1) \geq (\alpha_1^- : \alpha_1)_{A^c} \), and the last expression is equal to \( (\alpha : \beta)_{A^c} \) by Lemma 3.4. This establishes the other inclusion. \( \square \)

**Corollary 6.6.** Let \( A \) be a finite expanded group. Then \( A^c \) is nilpotent if and only if for all congruences \( \alpha, \beta \in \text{Con}(A) \) with \( \alpha \leq \beta \), we have \( \Gamma(\alpha, \beta) = 1_A \).

**Proof:** For the “if”-direction, we assume that for all \( \alpha < \beta \), we have \( \Gamma(\alpha, \beta) = 1_A \). Then Lemma 3.5 implies that \( (\alpha : \beta)_{A^c} = 1_A \), and therefore \( A^c \) is nilpotent by Lemma 5.8. For the “only if”-direction, we assume that \( A^c \) is nilpotent and fix \( \alpha < \beta \in \text{Con}(A) \). By Lemma 5.8 we then have \( (\alpha : \beta)_{A^c} = 1_A \), and Theorem 6.5 yields \( \Gamma(\alpha, \beta) = 1_A \). \( \square \)

We now turn to supernilpotency.

**Theorem 6.7.** Let \( A \) be an algebra in a congruence modular variety, and let \( L \) be its congruence lattice. We assume that \( L \) is finite. Then \( A^c \) is supernilpotent if and only if \( L \) does not split.

**Proof:** Assume that \( L \) does not split. From \([Moo16]\), we obtain that the higher commutator operations of \( A \) satisfy \((\text{HC}1)\), \((\text{HC}3)\) and \((\text{HC}7)\) from \([AM13] \) p. 860. Now from the proof of \([AM13] \) Lemma 3.3, we obtain that \( A^c \) is supernilpotent. Conversely, assume that \( (\delta, \varepsilon) \) is a splitting pair of \( L \). Let \( n \in \mathbb{N} \), and let \( (a,b) \in \varepsilon \) with \( a \neq b \). We define an \( n \)-ary operation by \( f(x_1, \ldots, x_n) := a \) if at least one of the \( x_i \) lies in \( a/\delta \), and \( f(x_1, \ldots, x_n) := b \) else. Since \( (\delta, \varepsilon) \) splits \( \text{Con}(A) \), \( f \) is congruence preserving. Now let \( y \in A \setminus (a/\delta) \). Then \( f(y, \ldots, y) = b \). We use the definition of higher commutators from \([Bul01] \) (cf. \([AM10] \) to show that \( [1, \ldots, 1]_A \neq 0 \). To this end, we observe that for all \( x \in \{a, y\}^n \setminus \{(y, \ldots, y)\} \), we have \( f(x) = a \). Hence if \( [1, \ldots, 1]_A = 0 \), \( f(y, \ldots, y, a) = f(y, \ldots, y, y) \), which means \( a = b \), contradicting the choice of \( a \) and \( b \). \( \square \)
7. PROOFS FOR THE THEOREMS FROM SECTION 1

Proof of Theorem 1.2: Item (2) just spells out the definition of forcing solvability, and hence it is equivalent to (1).

(2)⇒(3): Let A be an algebra generating a congruence modular variety with L ∼= Con(A). Since A^c can be seen as an expansion of A, it generates a congruence modular variety, and we have Con(A^c) ∼= L. Thus by the assumptions, A^c is solvable. Now Theorem 6.3 yields that B_2 is not a homomorphic image of L.

(3)⇒(2): Let B be an algebra in a congruence modular variety with Con(B) ∼= L. Then from Theorem 6.1, we obtain that B is solvable.

□

Proof of Theorem 1.3: The items (1) and (2) are equivalent by the definition of forcing nilpotency. If (2) holds and A is a finite expanded group with Con(A) ∼= L, then we also have Con(A^c) ∼= L. Hence from Corollary 6.6 we obtain (3). If (3) holds, then for every finite expanded group with Con(B) ∼= L, Theorem 6.1 yields that B is nilpotent.

□

Proof of Theorem 1.4: The equivalence of items (1) and (2) is immediate.

(2)⇒(3): Let A be an algebra in a congruence modular variety with Con(A) ∼= L. By the assumption (2), A^c is supernilpotent, and thus by Theorem 6.7, L does not split. (3)⇒(2): Theorem 6.1(2).

□

8. OPEN PROBLEMS

We conclude with two questions concerning congruence lattices that make algebras abelian.

Problem 8.1. Characterize those modular lattices of finite height that force abelian type.

Theorem 5.14 provides one source of such lattices. On the algebra side, a corresponding question is to describe those lattices that force abelianity in D among the lattices of finite height in L(D):

Problem 8.2. Among all lattices of finite height that are congruences lattices of some algebra in a congruence modular variety, characterize those L such that every algebra in a congruence modular variety with congruence lattice isomorphic to L is abelian.

ACKNOWLEDGMENTS

The author thanks P. Idziak, K. Kearnes, and T. Vetterlein for valuable discussions.
References

[Aic00] E. Aichinger, *On Hagemann’s and Herrmann’s characterization of strictly affine complete algebras*, Algebra Universalis 44 (2000), 105–121.

[Aic06a] ———, *The near-ring of congruence preserving functions on an expanded group*, Journal of Pure And Applied Algebra 205 (2006), 74–93.

[Aic06b] ———, *The polynomial functions of certain algebras that are simple modulo their center*, Contributions to general algebra. 17, Heyn, Klagenfurt, 2006, pp. 9–24.

[Aic14] ———, *On the Direct Decomposition of Nilpotent Expanded Groups*, Comm. Algebra 42 (2014), no. 6, 2651–2662.

[ALM16] E. Aichinger, M. Lazić, and N. Mudrinski, *Finite generation of congruence preserving functions*, Monatsh. Math. 181 (2016), no. 1, 35–62.

[AM10] E. Aichinger and N. Mudrinski, *Some applications of higher commutators in Mal’cev algebras*, Algebra Universalis 63 (2010), no. 4, 367–403.

[AM13] ———, *Sequences of commutator operations*, Order 30 (2013), no. 3, 859–867.

[Ava58] S. P. Avann, *Dual symmetry of projective sets in a finite modular lattice*, Trans. Amer. Math. Soc. 89 (1958), 541–558.

[BB87] J. Berman and W. J. Blok, *Free spectra of nilpotent varieties*, Algebra Universalis 24 (1987), no. 3, 279–282.

[BS81] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer New York Heidelberg Berlin, 1981.

[Bul01] A. Bulatov, *On the number of finite Mal’tsev algebras*, Contributions to general algebra, 13 (Velké Karlovice, 1999/Dresden, 2000), Heyn, Klagenfurt, 2001, pp. 41–54.

[Cze08] J. Czelakowski, *Additivity of the commutator and residuation*, Rep. Math. Logic 43 (2008), no. 43, 109–132.

[Cze15] ———, *The equationally-defined commutator*, Birkhäuser/Springer, Cham, 2015, A study in equational logic and algebra.

[Day69] A. Day, *A characterization of modularity for congruence lattices of algebras*, Canad. Math. Bull. 12 (1969), 167–173.

[FM87] R. Freese and R. N. McKenzie, *Commutator theory for congruence modular varieties*, London Math. Soc. Lecture Note Ser., vol. 125, Cambridge University Press, 1987.

[Grä98] G. Grätzer, *General lattice theory*, second ed., Birkhäuser Verlag, Basel, 1998, New appendices by the author with B. A. Davey, R. Freese, B. Ganter, M. Greferath, P. Jipsen, H. A. Priestley, H. Rose, E. T. Schmidt, S. E. Schmidt, F. Wehrung and R. Wille.

[Gum83] H. P. Gumm, *Geometrical methods in congruence modular algebras*, vol. 45, Mem. Amer. Math. Soc., no. 286, American Mathematical Society, 1983.

[Her79] C. Herrmann, *Affine algebras in congruence modular varieties*, Acta Sci. Math. (Szeged) 41 (1979), no. 1-2, 119–125.

[HH79] J. Hagemann and C. Herrmann, *A concrete ideal multiplication for algebraic systems and its relations to congruence distributivity*, Arch. Math. (Basel) 32 (1979), 234–245.

[Hig56] P. J. Higgins, *Groups with multiple operators*, Proc. London Math. Soc. (3) 6 (1956), 366–416.

[HM88] D. Hobby and R. McKenzie, *The structure of finite algebras*, Contemporary mathematics, vol. 76, American Mathematical Society, 1988.

[IS01] P. M. Idziak and K. Slomczyński, *Polynomially rich algebras*, J. Pure Appl. Algebra 156 (2001), no. 1, 33–68.
K. A. Kearnes, *Congruence modular varieties with small free spectra*, Algebra Universalis **42** (1999), no. 3, 165–181.

R. N. McKenzie, G. F. McNulty, and W. F. Taylor, *Algebras, lattices, varieties, volume I*, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, California, 1987.

A. Moorhead, *Higher commutator theory for congruence modular varieties*, Talk at the conference *Algebra and Algorithms* (Structure and Complexity Theory; A workshop on constraint satisfaction, structure theory and computation in algebra), University of Colorado, Boulder, May 19-22, 2016, 2016, slides available at http://math.colorado.edu/algebra2016/program.html.

S. D. Scott, *The structure of Ω-groups*, Nearings, nearfields and K-loops (Hamburg, 1995), Kluwer Acad. Publ., Dordrecht, 1997, pp. 47–137.

J. D. H. Smith, *Mal’cev varieties*, Lecture Notes in Math., vol. 554, Springer Verlag Berlin, 1976.

**Erhard Aichinger, Institut für Algebra, Johannes Kepler Universität Linz, 4040 Linz, Austria**

*E-mail address: erhard@algebra.uni-linz.ac.at*

*URL: http://www.jku.at/algebra*