This article concerns topologically non-trivial interface Hamiltonians that find many applications in materials science and geophysical fluid flows. The non-trivial topology manifests itself in the existence of topologically protected, asymmetric edge states at the interface between two two-dimensional half spaces. The asymmetric transport is characterized by a quantized interface conductivity. The objective of this article is to compute such a conductivity and show its stability under perturbations. We present two methods. The first one computes the conductivity using the winding number of branches of absolutely continuous spectrum of the interface Hamiltonian. This calculation is independent of any bulk properties but requires a sufficient understanding of the spectral decomposition of the Hamiltonian. In the fluid flow setting, it also applies in cases where the so-called bulk-interface correspondence fails. The second method establishes a bulk-interface correspondence between the interface conductivity and a so-called bulk-difference invariant. We introduce the bulk-difference invariants characterizing pairs of half spaces. We then relate the interface conductivity to the bulk-difference invariant by means of a Fedosov–Hörmander formula, which computes the index of a related Fredholm operator and is obtained using semiclassical calculus. The two methods are used to compute invariants for representative 2 × 2 and 3 × 3 systems of equations that appear in the applications.

1. Introduction

Topological invariants offer a useful description of phenomena that appear to be immune to large classes of perturbations. The macroscopic transport properties of topological insulators are for instance known to be dictated to a large extent by a topological invariant characterizing their phase [1–3]. More recently, the behavior of equatorial waves and that of other wave phenomena was also shown to afford a topological description [4–7]. Its main practical consequence is the stability of macroscopic transport properties in the presence of defects. This article addresses the computation of these invariants and their relations to quantized physical observables for continuous two-dimensional models that appear in the analysis of topological insulators and topological waves.

We consider Hamiltonians generically denoted by $H[\mu]$ with $\mu$ an order parameter taking values in $\mathbb{R}$ in the examples of this article. A typical Hamiltonian that appears in the analysis of topological insulators with $\mu = m$ a mass term is the $2 \times 2$ Dirac system...
\[ H[m(y)] = D \cdot \sigma + m(y)\sigma_3, \quad H[m] = k \cdot \sigma + m\sigma_3 = \begin{pmatrix} m & \xi - i\zeta \\ \zeta + i\zeta & -m \end{pmatrix}, \] (1)

in the physical and Fourier variables (when \( m \) is constant), respectively. Here, \((x, y)\) are the spatial variables, \(k = (\xi, \zeta)\) the dual Fourier variables, \(D = (D_x, D_y)\) with \(D_x = \frac{1}{i} \partial_x\) and \(D_y = \frac{1}{i} \partial_y\) while \(\sigma = (\sigma_1, \sigma_2)\) and \(\sigma_{1,2,3}\) are the standard Pauli matrices. The above system as well as more general, and more physical, block-diagonal direct sums of such elementary blocks (in which case \(m(y)\) may be a vector-valued order parameter) appear naturally in the analysis of topological insulators \([8–12]\).

For equatorial waves, \(\mu = f\) is the Coriolis force (with as a possible second order parameter an odd viscosity term \(\epsilon\) in which case \(f\) below is replaced by \(f + \epsilon \Delta [4, 6, 7]\)). The typical \(3 \times 3\) system we consider is

\[ H[f(y)] = (D_x, D_y, -f(y)) \cdot \Gamma, \quad H[f] = (\xi, \zeta, -f) \cdot \Gamma = \begin{pmatrix} 0 & \xi & \zeta \\ \zeta & 0 & i\xi \\ -i\xi & 0 & 0 \end{pmatrix}, \] (2)

where \(\Gamma = (\gamma_1, \gamma_4, \gamma_2)\) involves some of the Gell–Mann matrices used in the representation of the Lie algebra \(su(3)\); see \([4, 6]\) for derivations of such a fluid model and its applications.\(^1\) The above system applies to a vector field \((\eta, u, v)\) with \(\eta\) the (deviation of the equilibrium) thickness of the two dimensional fluid and \((u, v)\) its velocity components.

We systematically use the notation \(H[\mu]\) for a bulk Hamiltonian with fixed value of the order parameter \(\mu\) and \(H[\mu(y)]\) for an interface Hamiltonian with varying order parameter \(\mu(y)\).

Physically interesting phenomena appear when insulating domains with different bulk topologies are brought next to each other. Consider a material modeled by a bulk Hamiltonian \(H[\mu]\). It is said to be insulating in the energy range \([E_1, E_2]\) when the latter interval and the spectrum of \(H[\mu]\) are disjoint; that is, \([E_1, E_2]\) is in a bulk band gap of \(H[\mu]\). A bulk topology may then formally be assigned to \(H[\mu]\) \([1–3, 8]\).

Consider now two materials modeled by \(H[\mu_\pm]\) in the half spaces \(\pm y > 1\) with bulk topologies characterized by integers \(I_\pm = I(\mu_\pm)\). The interface between the two bulk materials is modeled by a varying order parameter \(\mu = \mu(y)\) and a corresponding interface Hamiltonian \(H[\mu(y)]\). At the interface \(\mu(y) \sim 0\), the bulk invariants are not defined and transport along the interface is allowed. Quite surprisingly, this transport is typically asymmetric, with more modes moving in one direction than in the other one, and quantized.

Bounded domain walls \(\mu(y)\) and the construction of spatial truncations and of density of states within the bulk band gap \([E_1, E_2]\) are all conveniently represented as switch functions \(f \in \mathcal{G}[f_-, f_+, x_-, x_+]\) which we define as the family of bounded functions from \(\mathbb{R}\) to \(\mathbb{R}\) such that

\(^1\)The physical system \(H[f(y)]\) is the same as in \([15, 43, 44]\) while its Fourier domain representation \(H[f]\) is not. Our convention for plane wave representations is \(e^{i(k \cdot x - \xi t)}\) (rather than \(e^{i(\xi t - k_1 x)}\)) so that \(k = (\xi, \zeta)\) is the Fourier multiplier of \(D = \nabla\), a non-negotiable convention in semiclassical analysis. This choice affects the numerical values of our topological invariants. The dispersion relations \(E = E(k)\) and group velocities \(\nabla_k E(k)\) are, however, the same in both representations.
\[ f(x) = f_- \quad \forall x < x_- \quad \text{and} \quad f(x) = f_+ \quad \forall x \geq x_+. \] (3)

We also denote by \( \mathcal{S}[f_-, f_+] \) the union of the above switch functions over all \( x_- \leq x_+ \). A smooth switch function is a switch function in \( C^\infty(\mathbb{R}) \). Bounded domain walls \( \mu(y) \) will then often assumed to be elements in \( \mathcal{S}[\mu_-, \mu_+] \). We will also consider unbounded, linear domain walls \( \mu(y) = \lambda y \) for \( \mathbb{R} \ni \lambda \neq 0 \).

The topologically protected asymmetric transport is characterized by what we call an interface conductivity in analogy to the linear response electric fields in condensed matter physics [1–3]. This conductivity is defined for a Hamiltonian \( H = H[\mu(y)] \) as

\[ \sigma_I = \sigma_I[H] := \text{Tr} i[H, P] \phi'(H). \] (4)

Here, \( \text{Tr} \) is operator trace, \( [A, B] = AB - BA \) is the usual commutator, \( P = P(x) \) is either a smooth switch function from \( \mathbb{R} \) to \([0, 1]\) in \( \mathcal{S}[0, 1, x_0 - \beta, x_0 + \beta] \) for some \( \beta > 0 \) or the Heaviside function \( P(x) = H(x - x_0) = \chi_{(x \geq x_0)} \), and \( \phi(E) \) is a smooth switch function in \( \mathcal{S}[0, 1, E_1, E_2] \) such that \( \phi'(E) \geq 0 \) while \( \phi'(H) \) is then constructed by spectral calculus. The interval \([E_1, E_2] \) is chosen to lie in a bulk band gap of \( H[\mu_{\pm}] \).

The interpretation of \( \sigma_I \) is as follows. \( \phi'(H) \) may be interpreted as a (nonnegative) density of states (integrating to 1). Next, \( P \) may be seen as an observable with \( \langle \psi(t)|P|\psi(t) \rangle \) for \( \psi(t) = e^{-itH}\psi \) quantifying an amount of signal to the right of \( x_0 \). Its evolution is given by

\[ \frac{d}{dt} \langle \psi(t)|P|\psi(t) \rangle = \langle \psi(t)|i[H, P]|\psi(t) \rangle = \text{Tr} i[H, P] \psi(t) \otimes \psi(t). \]

We then formally replace \( \psi(t) \otimes \psi(t) \) by the stationary density \( \phi'(H) \). The above (operator) trace then has a clear physical interpretation (if defined) as a rate of signal moving from the left of \( x_0 \) to the right of \( x_0 \). This conductivity has appeared in different contexts to analyze the topological protection of edge states [3, 8, 13–15].

It turns out that \( \sigma_I \) is often quantized \((2\pi\sigma_I \in \mathbb{Z})\) and equal to an invariant that is immune to perturbations in the Hamiltonian \( H \). Moreover, the asymmetry typically follows a general principle, the bulk-interface correspondence, which states that the excess of modes going in one direction versus the other is given by \( I(\mu_+) - I(\mu_-) \).

While appealing, this correspondence does not (quite) always apply in the case of equatorial waves [4]. In particular, the number of edge states depends on the structure of the Coriolis force term \( f(y) \) with a number of asymmetric interface modes equal to 2 when \( f(y) = \lambda y \) and equal to only one when \( f(y) = f_0 \text{ sign}(y) \). The main heuristic reason for such difficulties comes from the fact that the topological change occurs when \( f(y) \) changes signs. For such a value, the gap closing of the Hamiltonian \( H[f] \) coincides with infinitely degenerate (essential) spectrum at \( 0 \in \text{Sp}(H) \) (modes in geostrophic balance), whose behavior is influenced by the profile of the variation \( f(y) \).

This article introduces two methodologies to compute the interface conductivity \( \sigma_I \). The first one, presented in section 2, extends the work in [8] on the system (1) to more general Hamiltonians including the one in (2). The method is based on mapping \( H[\mu(y)] \) to an appropriate Fredholm operators as in [3, 8]. The index of the operator is then related to a sequence of winding numbers associated to the Hamiltonian’s continuous spectrum. Such invariants are defined independently of any bulk invariant. Moreover, the method allows us to compute \( \sigma_I \) for \( H \) defined in (2) both when \( f(y) = \lambda y \) and when \( f(y) = f_0 \text{ sign}(y) \). The main drawback of the method is that it requires
explicit knowledge of the structure of the branches of absolutely continuous spectrum. Once these invariants are computed, one shows as in [8] that they are stable with respect to appropriate classes of perturbations $H_V := H[\mu(y)] + V$.

The second methodology is based on a new bulk-interface correspondence relating the above interface conductivity to the symbol of an appropriate Fredholm operator whose index is computed directly from its symbol in a Fedosov–Hörmander formula. This correspondence may be interpreted as a topological charge conservation [16, 17]. A reinterpretation of the formula then relates the conductivity to several equivalent expressions of a quantity we call a bulk-difference invariant, which generalizes the difference of bulk invariants we mentioned above.

The bulk-difference invariant is introduced in section 3. We present in section 4 a semi-classical analysis of the interface conductivity and show how to relate it to the bulk-difference invariant and to an index theory similar to that of Fedosov–Hörmander [18, Chapter 19].

The correspondence shares some similarities with the bulk-interface correspondence based on spectral flows and spectral asymmetry in, for example, [16, 17, 19].

As an application to the system (2), we obtain that the bulk-interface correspondence applies provided that $f(y)$ varies sufficiently slowly. This is consistent with the explicit computations obtained when $f(y) = ky$ and tends to indicate that the bulk-interface correspondence does not apply when $f(y) = f_0 \text{sign}(y)$ because the latter is too singular.

### 2. Spectral computation of interface invariants

This section computes the interface conductivity $\sigma_I$ defined in (4) for interface Hamiltonians $H = H[\mu(y)]$ in terms of the winding number of their branches of absolutely continuous spectrum. This extends work in [8] on the system (1).

The order parameter $\mu = \mu(y) \in \mathbb{S}[\mu_-, \mu_+]$ generates a domain wall in the topologically non-trivial case $\mu_+ \mu_- < 0$. The non-vanishing values of $\mu$ act as insulators while the interface in the vicinity of $y = 0$ where $\mu$ changes signs becomes conducting. This is the physical origin for the topologically protected asymmetric transport.

The results in [8] show that we can relate $\sigma_I$ to the index of a Fredholm operator and that both remain quantized when $H$ is replaced by $H + V$ for $V$ in an appropriate class of perturbations. The computation of $\sigma_I$ based on branches of absolutely continuous spectrum is independent of any definition of bulk invariants and hence of any bulk-interface correspondence. However, it is useful in practice only when the branches can be analyzed in sufficient detail. The analysis carried out in [8] for the system (1) applies to general mass terms $m(y)$. However, the algebra for the $3 \times 3$ system in (2) is more complex and carried out only in specific situations such as when $f(y) = ky$ and $f(y) = f_0 \text{sign}(y)$. The details of these calculations are presented in Appendix B.

We consider an interface Hamiltonian $H = H[\mu(y)]$ as an unbounded operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$ with coefficients $\mu = \mu(y)$ independent of $x$ so that the following decomposition holds:

$$H = \mathcal{F}^{-1}_{\xi \rightarrow x} \hat{H}(\xi) \mathcal{F}_{x \rightarrow \xi}. \quad (5)$$

Here $\mathcal{F}$ is the one dimensional Fourier transform in the $x$-variable and $\xi \rightarrow \hat{H}(\xi)$ is a family of unbounded operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^n$. We make the following assumptions on $H$ and $\hat{H}(\xi)$:
[H-AC]: (i) $H$ is an unbounded self-adjoint operator $D(H) \subset L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$ to $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$; (ii) for each $\xi \in \mathbb{R}$, $\hat{H}(\xi)$ is self adjoint as an unbounded self-adjoint operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^n$ and when restricted to the interval $[E_1, E_2]$ has finite rank and vanishes for $\xi$ outside of a compact set; and (iii) the existence of smooth (in $\xi$) corresponding eigenvalues $E_j(\xi)$ and rank-one projectors $\Pi_j(\xi)$ (with Schwartz kernel $\psi_j(y, \xi)\psi_j^*(y', \xi)$ for normalized eigenvectors $\int_{\mathbb{R}} |\psi_j(y, \xi)|^2dy = 1$) parametrizing a finite number $J$ of branches of absolutely continuous spectrum.

The domain $D(H)$ may be defined as $(H \pm i)^{-1}(L^2(\mathbb{R}^2) \otimes \mathbb{C}^n)$ independent of $\pm$ [20, Theorem 1.2.7]. We verify in Appendix B that the above assumption holds for systems (1) and (2). Since $H$ is self-adjoint, we have access to the corresponding spectral calculus; see [20] and Appendix D. The above branches $E_j(\xi)$ are so far defined on an implicit compact domain $\Xi_j \ni \xi$. They are extended by continuity to $\xi \in \mathbb{R}$ as the unique continuous branches still called $E_j(\xi)$ such that $E_j(\xi) \in \{E_1, E_2\}$ for $\xi \notin \Xi_j$.

We now construct a unitary operator from the restriction of $H$ to the interval $[E_1, E_2]$. Let $\varphi(E)$ be a smooth switch function in $\mathbb{G}[0, 1]$ $E_1, E_2]$. We then define the unitary operator

$$U(H) = e^{i2\pi\varphi(H)}, \quad W(H) = U(H) - I$$

by spectral calculus [21]. Note that $W(E)$ is compactly supported in $(E_1, E_2)$. By assumption, we have the decomposition

$$W(H) = \mathcal{F}^{-1}\sum_{j=1}^J W(E_j(\xi))\Pi_j(\xi) \mathcal{F}.$$ 

Let $P$ be the spatial projector onto $x \geq x_0$ for $x_0 \in \mathbb{R}$, that is, point-wise multiplication by $H(x - x_0)$ with $H$ the Heaviside function. Then, as an extension of the results [8], we have

**Theorem 2.1.** Let $P$ and $U(H)$ be defined as above for $H$ satisfying the hypothesis [H-AC(I)-(III)]. Then $PU(H)P$ is a Fredholm operator on the range of $P$ in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$. Moreover,

$$I[H] := \text{Index}(PUP) = -\text{Tr}[P, U]U^* = \sum_{j=1}^J \mathcal{W}_1(e^{i2\pi\varphi E_j})$$

with $\mathcal{W}_1(f)$ the winding number of a unimodular complex function $f$ with compactly supported gradient, given explicitly by

$$\mathcal{W}_1(f) = \frac{1}{2\pi i} \int_{\mathbb{R}} \partial f(\xi)f^*(\xi).$$

The proof of this theorem is given in Appendix A. Applying the above theorem yields

$$I[H] = \text{Tr}[U, P]U^* = N_+ - N_-,$$

where $N_+$ is the number of branches $E_j$ such that $E_j(\xi) \in \mathbb{G}[E_1, E_2]$ while $N_-$ is the number of branches $E_j$ such that $E_j(\xi) \in \mathbb{G}[E_2, E_1]$. Branches such that $E_j(\xi)$ belongs to $\mathbb{G}[E_1, E_1]$ or $\mathbb{G}[E_2, E_2]$ do not contribute to $I[H]$. 

2.1. Stability and interface conductivity

One of the main reasons to develop the above apparatus borrowed from non-commutative geometry (see [3, 8] and their references) is that it allows us to include spatial perturbations in the Hamiltonians. Let $H$ be as above and $H_V = H + V$ a perturbed Hamiltonian.

We recall results from [8] for completeness. Stability of the interface conductivity $\sigma_I$ is guaranteed when $V$ is relatively compact with respect to $H$, which means that $V(H + i)^{-1}$ is a compact operator in the $L^2$ sense. For the $2 \times 2$ problem (1), any $V = V(x, y)$ an operator of point-wise multiplication by a bounded function that decays to 0 at infinity satisfies such hypotheses since the symbol of $(H + i)^{-1}$ decays to 0 as $z \to \infty$; this is essentially an application of Rellich’s compactness criterion [22]. In other words, the topology is fixed by our assumptions ‘at infinity’ but $V$ can be arbitrarily large (bounded) so long as it decays at infinity. For the $3 \times 3$ system (2), the symbol of $(H + i)^{-1}$ does not converge to 0 as $z \to \infty$ and it is more difficult to characterize operators $V$ for which the above criterion holds.

Let us define $U[H_V] = e^{2\pi\phi(H_V)}$ spectrally. Then we have [8]

**Theorem 2.2.** $PU[H_V]P$ on the range of $P$ is a Fredholm operator and

$$I[H_V] := \text{Index}(PU[H_V]P) = I[H].$$

The interface index is thus independent of any relatively compact perturbation.

Still following [8], we now relate the above index to the interface conductivity in (4). The above stability results show that $P$ may be replaced by a smooth switch function $P \in \mathcal{S}[0,1]$, which we now assume for the rest of the section. The conductivity $\sigma_I$ is also stable with respect to perturbations $V$ although it is less stable than the above index $I[H_V]$. Let us now impose that $V$ is of the form $V = V_1 V_2$ with

$$\|V_j\| \leq C \quad \text{and} \quad \|(z - H)^{-1} V_j\|_{HS} \leq C |\Im z|^{-1} \quad j = 1, 2,$$

where $HS$ is the Hilbert-Schmidt norm. It is then shown in [8] (see also Proposition 4.3 below) that $[H_V, P] \phi'(H_V)$ is indeed a trace-class operator and we have (for $P$ smooth):

**Theorem 2.3.** Let $H$ be as above and $V$ a perturbation satisfying (9). Then the operator $i[H_V, P] \phi'(H_V)$ is trace-class and

$$\sigma_I := i\text{Tr}[H_V, P] \phi'(H_V) = \frac{1}{2\pi} I[H_V].$$

This provides a means to compute the topologically protected conductivity $\sigma_I$ by spectral analysis of the unperturbed Hamiltonian $H[\mu(y)]$ using Theorem 2.1 above.

That the interface conductivity is quantized is consistent with the numerical simulations presented in, for example, [4, 6]. The calculations of interface conductivities for the systems (1) and (2) are presented in Appendix B. A summary for system (1) is that $2\pi \sigma_I = - \text{sign}(m_+)$ when $m(y)$ is a domain wall between $m_-$ and $m_+$ when $m_+ m_- < 0$ (with $\sigma_I = 0$ when $m_+ m_- > 0$) while $2\pi \sigma_I = - \text{sign}(\lambda)$ when $m(y) = \lambda y$. For system (2), we find that $2\pi \sigma_I$ is given by $\text{sign}(f_0)$ when $f(y) = f_0 \text{ sign}(y)$ while it is given by $2 \text{ sign}(\lambda)$ when $f(y) = \lambda y$. 
This discrepancy in the invariant for different profiles of \( f \) is reminiscent of correction terms in Levinson’s theorem caused by the presence of resonances for the Hamiltonian at energy 0 \([23, 24]\).}

### 3. Bulk-difference invariant

Bulk invariants are defined when the coefficients \( \mu \) such as \( m \) or \( f \) above are constant. They are of interest in their own right and may be extended to operators with spatially varying coefficients as in \([8]\) and in a variety of other settings \([3]\). They are also instrumental in bulk-boundary correspondences \([3, 10, 14, 25–31]\), which relate them to the interface conductivity \( \sigma_i \).

With \( \mathcal{F} = \mathcal{F}_{(x, y) \rightarrow k=(\xi, \zeta)} \) the two-dimensional Fourier transform, we may then write the decomposition

\[
H = \mathcal{F}^{-1} \hat{H}(k) \mathcal{F},
\]

where \( \hat{H}(k) \) is \( M_n(\mathbb{C}) \)-valued (square matrices of dimension \( n \)) with \( n=2 \) in \((1)\) and \( n=3 \) in \((2)\). We then have the spectral decomposition

\[
\hat{H}(k) = \sum_{i=1}^{n} h_i(k) \Pi_i(k)
\]

with eigenvalues \( h_i(k) \in \mathbb{R} \) for \( k \in \mathbb{R}^2 \) ordered with increasing values and rank-one orthogonal projectors \( \Pi_i(k) = \psi_i(k)\psi_i^*(k) \) for \( 1 \leq i \leq n \). We make the following assumption:

\[\text{([H-Bulk]): the eigen-energies } k \rightarrow h_i(k) \text{ and eigen-spaces } k \rightarrow \Pi_i(k) \text{ are } C^2 \text{ functions.}\]

This holds for instance for \( \hat{H}(k) \) smooth and in the presence of (local) spectral gaps separating the energies \( i \rightarrow h_i(k) \). For the \( 2 \times 2 \) problem in \((1)\), we even have global spectral gaps since \( h_{1,2}(k) = \mp \sqrt{|k|^2 + m^2} \) with one spectral gap given by \((-|m|, |m|)\)

while for the \( 3 \times 3 \) problem in \((2)\), \( h_{1,2,3}(k) = \left(-\sqrt{|k|^2 + f^2}, 0, \sqrt{|k|^2 + f^2}\right) \) with two global spectral gaps given by \((-|f|, 0)\) and \((0, |f|)\). The (smooth) rank-one projectors are also well known; see \((41)\) below for the \( 3 \times 3 \) system as well as \([2, 4, 6, 7]\). In both cases, \( \Pi_1 \equiv \Pi_- \) corresponds to the projection of the Hamiltonian onto its (strictly) negative spectrum.

Let \( \Pi(k) \) be any sum \( \Pi(k) = \sum_{j \in J} \Pi_i(k) \) for \( J \) a subset of \( \{1, ..., n\} \) independent of \( k \). Then \( k \mapsto \text{Ran} \Pi(k) \) defines a vector bundle over the base manifold \( \mathbb{R}^2 \) \([2, 32]\). When \( \mathbb{R}^2 \) is replaced by a compact manifold \( M \), then vector bundles over \( M \) admit topological classifications based on their Chern classes. For the above projectors, living in spaces of matrices over \( M \), such classes integrated over the base manifold give rise to integer-valued objects called Chern numbers:

\[
\tilde{c}[\Pi] = \frac{i}{2\pi} \int_M \text{tr} \Pi d\Pi \wedge d\Pi = \frac{i}{2\pi} \int_M \text{tr} \Pi [\partial_1 \Pi, \partial_2 \Pi] d^2 k \in \mathbb{Z}.
\]

Here, \([A, B] = AB - BA\) and \( \text{tr} \) refers to the standard matrix trace.

When \( M \) is not compact, for instance \( \mathbb{R}^n \), the domain of interest in this article, the above integrals are often still defined but are no longer guaranteed to be integer-valued....
[2, 4, 6–8]. For instance, for the $2 \times 2$ problem, $\tilde{c}[\Pi_1] = \frac{1}{2} \text{sign}(m)$. A possible solution to this issue is to regularize the Hamiltonian and its projectors in such a way that they take a unique value as $|k| \to \infty$. This allows one to compactify the plane around the unit sphere (one point compactification) mapping $\infty$ to the south pole, say, and still obtain a continuous family of projectors on the sphere. The above integral, which is manifestly invariant by change of variables from its $PdP \wedge dP$ form, may then be computed for $M = S^2$ and shown to be integral. A typical regularization consists in replacing $m$ by $m - \eta|k|^2$ for $\alpha > 1$ and $\eta \neq 0$. We may then show that $\tilde{c}[\Pi_1] = \frac{1}{2} (\text{sign}(m) + \text{sign}(\eta)) \in \{-1, 0, 1\} \subset \mathbb{Z}$ [8].

A similar regularization, based on replacing $f$ by $f - \eta|k|^2$ in the $3 \times 3$ model (2), also allows one to define a topological invariant on the sphere $S^2$. One then finds that $\tilde{c}[\Pi_3] = -\tilde{c}[\Pi_1] = (\text{sign}(f) + \text{sign}(\eta)) \in \{-2, 0, 2\} \subset \mathbb{Z}$ [6, 7].

The bulk invariants of such regularized operators therefore depend on the sign of the regularization. The main advantage of this regularization is that it still applies in the setting where $H$ is no longer translationally invariant [8]. We now propose a different method to bypass this somewhat artificial regularization term and define topological invariants in the specific situation of interest here, namely the analysis of interface Hamiltonians. Let us assume the existence of two families of Hamiltonians

$$
\hat{H}^\pm(k) = \sum_{i=1}^n h_i^\pm(k) \Pi_i^\pm(k)
$$

with smooth projectors $k \to \Pi_i^\pm(k)$ in the sense of $[(H\text{-Bulk})]$. The structure of the energies $h_i^\pm(k)$ is irrelevant beyond the existence of well-defined gaps and each $h_i(k)$ may be continuously modified (homotopically transformed) to a single value $h_i$ (flat band) that depends on neither $k$ nor $\pm$. In the applications of interest here, the two Hamiltonians are identical except for the value of their mass terms $\mu_\pm$.

### 3.1. Construction of the bulk-difference invariants

Let $\Pi^\pm(k) = \sum_{i \in J} \Pi_i^\pm(k)$ for $J \subset \{1, \ldots, n\}$ independent of $k \in \mathbb{R}^2$ be two smooth families of projectors. Defining $k = |k|\theta$ in polar coordinates, we assume the continuous matching (gluing condition) of the projectors in all directions at infinity:

$$
\lim_{|k| \to \infty} \Pi^\pm(|k|\theta) = \lim_{|k| \to \infty} \Pi^-(|k|\theta) \quad \text{for all} \ \theta \in S^1.
$$

We assume that these limits exist and are continuous in $\theta$. In our applications, the projectors at $\infty$ do not depend on the mass terms $m$ or $f$ and thus satisfy the above hypothesis.

We then define a new projector $\Pi(k)$ for $k$ an element in the union of two planes $P_\pm \simeq \mathbb{R}^2$ that are wrapped around the unit sphere $S^2 \simeq (P_+ \cup P_-)/\sim$ by radial compactification so that the circles at infinity are glued (identified by $\sim$) along the sphere’s equator. For $k \in P_\pm$, we define $\Pi(k) = \Pi^\pm(k)$. For a point $\phi$ on the sphere, a form of stereographic projection $\pi$ maps $\phi$ in the upper half sphere to $k \in P_+$ and $\phi$ in the lower half sphere to $k \in P_-$. More precisely, with $\phi \in S^2$ parametrized by $(x, y, z)$, we have
\[ (x, y) = \frac{k}{\sqrt{1 + |k|^2}}, \quad z = \frac{\pm 1}{\sqrt{1 + |k|^2}}, \quad k \in \mathbb{P}_\pm, \]

with \( \pi \) the inverse map, that is, \( k = \pi(\phi) \). We then define \( \pi^* \Pi(\phi) = \Pi(\pi(\phi)) \) the pull-back by \( \pi \) (still called \( \Pi(\phi) \approx \pi^* \Pi(\phi) \) to simplify notation) a projector that is now continuous on \( S^2 \) thanks to the continuity assumption (12). We may therefore define the Chern numbers as integrals over the sphere, a compact cycle, which written on the sphere and then pushed by \( \pi \) to the planar variables, are given by

\[
c[\Pi] = \frac{i}{2\pi} \int_{S^2} \text{tr} \Pi d\Pi \wedge d\Pi = \frac{i}{2\pi} \int_{\mathbb{R}^2} \text{tr} (\Pi^+ [\partial_1 \Pi^-, \partial_2 \Pi^-] - \Pi^+ [\partial_1 \Pi^+, \partial_2 \Pi^+]) dk,
\]

where the \(-\) sign above is necessary to ensure that \( S^2 \) has a given orientation, here inherited from that of the lower plane \( P_- \) and opposite that of the upper plane \( P_+ \). This ensures that \( S^2 \) also inherits its orientation from the \( dx \wedge dy \wedge dz > 0 \) positive orientation of \( \mathbb{R}^3 \) it is embedded in.

**Definition 3.1** (Bulk-difference invariant): Let \( \hat{H}^+ \) be decomposed as in (11) and let \( \Pi^\pm(k) = \sum_{i \in J} \Pi^\pm_i(k) \) for \( J \subset \{1, \ldots, n\} \) satisfying the gluing condition (12). Then the above integrals \( c[\Pi] \) are well defined integers we call the bulk-difference invariants. We define \( c_i = c[\Pi_i] \) when \( J = \{i\} \).

These invariants are by construction immune to any (continuous) perturbation of \( \hat{H}(k) \) that maintains the spectral gaps and the gluing assumptions as \( |k| \to \infty \) and satisfy the general additivity property of Chern numbers [33] resulting from the additivity property of Chern classes [32]:

\[
c[\Pi_i + \Pi_{i+1}] = c[\Pi_i] + c[\Pi_{i+1}].
\]

When the two Hamiltonians satisfy \( H_+ = H_- \), then the above integral vanishes.

However, for the system \( H_\pm = k \cdot \sigma + m_\pm \sigma_3 \), which satisfies all the above assumptions, we find, following calculations as in, for example [8] that

\[
c_- := c[\Pi_1] = -c_+ := -c[\Pi_2] = \frac{1}{2} \text{sign}(m_-) - \frac{1}{2} \text{sign}(m_+) \in \mathbb{Z}
\]

which is a bona fide invariant even in the absence of regularization (\( \eta = 0 \) above). For the system (2), which also satisfies the above assumptions, we find that

\[
c_+ := c[\Pi_3] = c_0 + c_+ := c[\Pi_3 + \Pi_2] = -c_- := -c[\Pi_1] = \text{sign}(f_-) - \text{sign}(f_+) \in \mathbb{Z}.
\]

The calculations are given in some detail in Appendix C; see (41) and the following computations. For the above problem, only \( c_+ \) needs to be computed since \( c_- = -c_+ \) by symmetry and \( c_+ + c_0 + c_- = 0 \).

Explicit calculations of Chern numbers may be obtained in several ways. The first one is to compute the Berry curvature \( \text{itr} \Pi_i d\Pi_i \wedge d\Pi_i \) directly (from \( \psi_i(k) \)) and integrate it over the plane(s) \( \mathbb{R}^2 \), as done for instance in [4, 6, 8]. The second method consists in directly looking at the line bundle generated by \( \psi_i(k) \) and how charts covering the sphere need to be glued by an appropriate transition of connections to respect the twists of the eigenvectors [1, 2, 7]. The third method, which is the most versatile when it applies, recasts the
3.2. Green’s function invariant

We now consider a different form of the bulk-difference invariant based on the notion of resolvent or Green’s function [16, 17] and sharing similarities with the Kubo formula [1]. It is given by

\[ G = G_x(\omega, k) = (z - \hat{H}(k))^{-1} = \sum_{j=1}^{n} (z - h_j(k))^{-1} \Pi_j(k) \]

for \( k \in \mathbb{R}^2 \) and \( z = \alpha + i\omega \) for \( \omega \in \mathbb{R} \). We assume that \( \alpha \) is a fixed real number in a global spectral gap, that is, \( \alpha \neq h_j(k) \) for all \( 1 \leq j \leq n \) and \( k \in K \). Thus, \( G \) and \( G^{-1} \) are well-defined with obviously \( G^{-1}(k, \omega) = z - \hat{H}(k) \).

Consider two operators \( \hat{H}(k) \) as in (11) and the corresponding Green’s functions \( \mathbb{R}^3 \ni (\omega, K) \rightarrow G^\pm_x(\omega, K) \). As above, we assume the following gluing conditions

\[ \lim_{|[k, \omega]| \to \infty} G^+_{x} (|(k, \omega)| \theta) = \lim_{|[k, \omega]| \to \infty} G^-_{x} (|(k, \omega)| \theta) \quad \text{for all } \theta \in S^2. \]  

(16)

We observe that this condition holds again for the Dirac 2 \times 2 problem in (1) and the wave 3 \times 3 problem in (2). Note that both limits vanish when \( \theta = (0, 0, 1) \) and that the projectors satisfy the matching condition by assumption (12) when \( \theta \neq (0, 0, 1) \).

As we did earlier to define \( \Pi(k) \) on a sphere, we now define \( G_x(\omega, k) \) as a continuous map on a three-sphere \( M = S^3 \simeq (\mathbb{R}^3 \sqcup \mathbb{R}^3) / \sim \) by radial compactification and gluing of the spheres at infinity. Then, \( G_x(\omega, k) \) takes values in \( GL_n(\mathbb{C}) \), the space of \( n \)–dimensional invertible matrices. We may now define the standard winding number [34] of such maps

\[ W_x = \frac{1}{24\pi^2} \int_M \text{tr}(dG^{-1}G)^3 = \frac{1}{8\pi^2} \int_M \text{tr}\partial_\omega G^{-1}G[\partial_1 G^{-1}G, \partial_2 G^{-1}G] d\omega d^2k, \]

(17)

where, with a slight abuse of notation, the above right-hand side should be interpreted as an integration over \( \tilde{M} = \mathbb{R} \times \mathbb{R}^2 \cup \mathbb{R} \times \mathbb{R}^2 \), the product of \( \omega \in \mathbb{R} \) with the two half planes \( k \in P_\pm \) with appropriate orientation as in (13).

This winding number is related to the Chern numbers previously defined [3, 16, 17] by:

**Lemma 3.2.** For \( \alpha \) in a global spectral gap, let \( G = G_x \) be constructed from \( H^\pm_x(k) \) as described above. Let \( W_x \) and \( c_i = c[\Pi_i] \) be defined in (17) and (13), respectively. Then we have the relation

\[ W_x = -\sum_{h_i < \alpha} c_i = \sum_{h_i > \alpha} c_i. \]

(18)

The proof of the lemma is given in Appendix C.

Note that the Chern numbers \( c[\Pi_i] \) have been defined for sufficiently smooth projectors without conditions on the energies \( h_i(k) \) besides local gaps ensuring the regularity of \( k \rightarrow \Pi_i(k) \). In contrast, \( G_x \) is defined only for \( \alpha \) in a global spectral gap and requires
the more constraining (16). The global spectral gap condition is satisfied in the consid-
ered applications, with a global gap $\alpha \in (-|m|, |m|)$ for (1) and two global gaps $\alpha \in (-|f|, 0) \cup (0, |f|)$ for (2).

The form of the invariant (17) appears naturally in the subsequent analysis of the
bulk-interface correspondence. It also offers a convenient computational tool, in par-
ticular (40) below that appears in the proof of the above lemma. The formula is used to
compute the bulk-difference invariant for the system (2) in Appendix C.

4. Bulk-interface correspondence

We turn to the computation of the interface invariant $\sigma_I$ using the bulk-difference
invariant.

The bulk-interface correspondence stipulates that the number of topologically pro-
tected edge states characterizing the asymmetric current is given by the bulk-difference
invariant. For the system (1), the difference therefore equals $\epsilon_- = 2\pi \sigma_I = - \text{sign}(m)$ in
the topologically nontrivial case, whereas for the system (2), it is given by $\epsilon_- = 2\pi \sigma_I = 2 \text{sign}(f)$. In both cases, $\mu(y)$ is a smooth switch function in $S[\mp \mu] + \mu$.

Many techniques have been developed to prove or at least build intuition on the cor-
respondence between bulk invariants and the number of edge or interface modes. The
edge problem considers bulk Hamiltonians in the plane and their restriction to a half
plane with appropriate boundary conditions along the edge. The relation between the
bulk invariant and the number of topological edge states is then referred to as a bulk-
boundary correspondence [3, 14, 25–31].

This article considers a similar problem, the domain wall problem, where the order
parameter $\mu(y)$ transitions, smoothly or not, from one bulk topology to another. The
relation between the two bulk invariants, and more precisely the bulk-difference as
established in the preceding section, and the number of protected interface modes is the
bulk-interface correspondence [17, 19, 35].

The main advantage of deriving a bulk-interface correspondence is that it avoids the
explicit spectral decomposition and the search for topologically non-trivial branches of
continuous spectrum that we carried out in section 2. The computation of $\sigma_I$ is directly
obtained from the often significantly simpler calculation of bulk invariants. For an
application of such a result, see, for example, [36].

The main result of this section is to recast the interface conductivity as an appropri-
ate integral of the symbol of the Hamiltonian $H$ that is familiar in index theory [18, 37,
38]. This integral is in the form of a Fedosov–Hörmander formula and in a general set-
ing [18, Chapter 19], computes the index of a Fredholm operator naturally related to
$H$. By an application of the Stokes’ theorem, it is also related to the bulk-difference
invariant in (17). This correspondence essentially realizes the topological charge conser-
vation of [17, 35].

The chain of relations leading to the correspondence, for $P(x) = H(x - x_0)$ projection
onto the half space $x \geq x_0$ and $P_1$ a smooth switch function in $S[0,1]$, is:

$$2\pi \sigma_I = 2\pi \text{Tr} i[H, P_1] \varphi' = \text{Tr}[U, P] U^* = \text{Tr}[U, P] U^* = \text{Index} PU(H) P_{\text{Ran} P} =: I[H].$$

We recall that $\varphi$ is a smooth function increasing from 0 to 1 with $\varphi' \geq 0$ supported
in a bulk spectral gap about 0 while $U(H) = e^{i2\pi \phi(H)}$. The main objective of the section 4.1
is to find sufficient conditions on the symbol of $H$ such that (19) holds. When the bulk gap is about an energy $E = x$ for $x \in \mathbb{R}$ (and $x \neq 0$ for the system (2)), we apply the above relation to the operator $H - x$.

Once $2\pi \sigma_I$ is guaranteed to be integer-valued and constant over continuous deformations of $H$, we evaluate the trace defining $\sigma_I$ by applying semiclassical calculus tools similar to those used to derive the index of elliptic pseudo-differential operators [18, Chapter 19] in section 4.2.

Extensions of the proposed approach under hypothesis (H2) below to more general Hamiltonians are considered in [39]. Generalizations to higher dimensions for operators with unbounded domains walls are presented in [40].

### 4.1. Interface conductivity and pseudo-differential calculus

The terminology used in the rest of the article on pseudo-differential operator ($\Psi$DO) and semiclassical pseudo-differential operators ($h\Psi$DO) is borrowed from [21]; see also [41] for the extension of results to matrix-valued symbols. Appendix D summarizes the notation and results on $\Psi$DO and $h\Psi$DO we need in this article. Throughout, we use the notation $\langle X \rangle = \sqrt{1 + |X|^2}$ for $X \in \mathbb{R}^N$ for $p \geq 1$ and $\langle X \rangle^{-\infty}$ as a quantity bounded by $C_N(X)^{-N}$ for every $N \in \mathbb{N}$.

We first identify sufficient criteria ensuring that $[H, P] \phi'(H)$ and $[U(H), P] U^*(H)$ are trace-class operators. We separate the cases where $P$ is a smooth switch function in $\mathcal{S}(0,1]$ and $P$ is a Heaviside function $P(x) = H(x - x_0)$.

Let $H = \text{Op}^w(\sigma)$ be a self-adjoint $\Psi$DO with symbol $\sigma = \sigma(x, y, \xi, \zeta)$ in $\mathcal{S}(m_r)$ with $m_r(x, y, \xi, \zeta) := (y, \xi, \zeta)^T$ for $r \in \mathbb{N}$. Define $\tau$ as the symbol of $H^2$, that is, $H^2 = \text{Op}^w(\tau)$. The sufficient criterion is based on assuming the existence of $G = \text{Op}^w(\tilde{\tau})$ a self-adjoint operator such that $G$ has no spectrum in $(-\infty, E_0^2)$ for $E_0 > 0$ fixed and such that either:

- (H1) $\tilde{\tau} = \tau$ for $y^2 + \xi^2 + \zeta^2 > R^2$; or:
- (H2) $\tau - \tilde{\tau}$ in $\mathcal{S}(m)$ with $m = (y)^{-\infty}(\xi, \zeta)^{2\tau}$ and $(I + H^2)^{-1}$ has Weyl symbol in $\mathcal{S}(m)$ with $m = (\xi, \zeta)^{-s\tau}$ for $s > 0$.

The favorable (H2) is verified for operators with symbols converging to infinity when the dual variables $(\xi, \zeta)$ converge to infinity. It applies to (1) but not to (2). In the latter case, the presence of essential spectrum at 0 forces us to consider another theory. We will see that (H1) applies and allows us to obtain a bulk-interface correspondence in some cases.

**Lemma 4.1.** Let $H = \text{Op}^w(\sigma)$ and $H^2 = \text{Op}^w(\tau)$ as above and $E_0 > 0$. Let $G = \text{Op}^w(\tilde{\tau})$ with $G > E_0^2$ satisfy either (H1) or (H2). Let $\phi$ be a smooth function with compact support in $(-E_0, E_0)$ and $P$ a smooth switch function in $\mathcal{S}(0,1]$.

Then $\phi(H) \in \text{Op}^w S((y, \xi, \zeta)^{-\infty})$ and $[\phi(H), P]$ and $H^q[H^q, P] \phi(H)$ for $p, q \geq 0$ are trace-class operators with symbols in $\mathcal{S}((x, y, \xi, \zeta)^{-\infty})$. Moreover, $\text{Tr}[\phi(H), P] = 0$.

**Proof.** By hypothesis, $G$ has no spectrum in $(-\infty, E_0^2 + \delta)$ for some $\delta > 0$ and we can thus construct a smooth function compactly supported $\psi : [0, \infty) \to \mathbb{R}$ such that...
\[ \psi(G) = 0 \text{ and } \psi(E) = 1 \text{ for } 0 \leq E \leq E_0. \] This implies that \[ \psi(j^2)\phi(\lambda) = \phi(\lambda) \] by assumption on \( \phi \).

By spectral calculus and following a similar construction in [42], we have
\[ \phi(H) = \psi(H^2)\phi(H) = (\psi(H^2) - \psi(G))\phi(H). \]

Therefore, \( \mathcal{H}^q[H^p, P]\phi(H) \) is trace-class when \( \mathcal{H}^q[H^p, P]\psi(H^2) \) is since \( \phi(H) \) is bounded. Taking a commutator with \( P \) provides localization in \( x \) since
\[ [A, P] = (1 - \chi(x))A\chi(x) - \chi(x)A(1 - \chi(x)), \]
with \( \chi(x) \) a multiplication operator with symbol bounded by \( \langle x \rangle^{-\infty} \) for \( x < 1 \) and \( 1 - \chi(x) \) an operator with symbol bounded by \( \langle x \rangle^{-\infty} \) for \( x > -1 \). For \( A \) a \( \Psi \)DO with symbol in \( S(m_A) \), then \([A, P]\) has a symbol in \( S(m_A \langle x \rangle^{-\infty}) \) by composition of three \( \Psi \)DO; see [21, Chapter 7] and [13].

Let us first prove the results under hypothesis (H1). The Helffer-Sjöstrand formula (47) (see also [20, 21] and [21, Eq. (9.11)]) gives the expression:
\[ \psi(H^2) = \psi(H^2) - \psi(G) = -\frac{1}{\pi} \int_C \tilde{\partial}\psi(z - H^2)^{-1}(H^2 - G)(z - G)^{-1}d^2z, \]

where \( d^2z = d\lambda d\omega \) for \( z = \lambda + i\omega \), and \( \tilde{\psi} \) is an almost analytic extension of \( \psi \) as in (47); see Appendix D. The operators \( (z - H^2)^{-1} \) and \( (z - G)^{-1} \) have symbols in \( S(1) \) bounded by \( C|\text{Im}z|^{-1} \) uniformly on the compact support of \( \tilde{\partial}\psi \), itself satisfying \( |\tilde{\partial}\psi| \leq C_N|\text{Im}z|^N \) for any \( N \geq 1 \), while the symbol of \( (H^2 - G) \) is by assumption in \( S((y, \xi, \zeta)^{-\infty}) \cdot \) By composition calculus, \( \psi(H^2) \) is also a \( \Psi \)DO with symbol in \( S((y, \xi, \zeta)^{-\infty}) \cdot As a consequence, \( \mathcal{H}^q[H^p, P]\psi(H^2) \) has a symbol in \( S((x, y, \xi, \zeta)^{-\infty}) \cdot \) The operator \( \phi(H) = \psi(H^2) \) also has a symbol in \( S((y, \xi, \zeta)^{-\infty}) \cdot \) Therefore, \( [\phi(H), P] \) has symbol in \( S((x, y, \xi, \zeta)^{-\infty}) \cdot \)

The above operators are thus trace-class by [21, Theorems 9.3 and 9.4] with traces computed by integrating the symbols in \( (x, y, \xi, \zeta) \) or equivalently the Schwartz kernels along the diagonal. If \( k(x, y, x', y') \) is the Schwartz kernel of \( \phi(H) \), then we find that
\[ \text{Tr}[\phi(H), P] = \int_{\mathbb{R}^2} (p(x) - p(x))k(x, y, x, y)dxdy = 0. \]

Let us now assume hypothesis (H2), which applies in the slightly more general setting where \( H \in \text{Op}^s\mathcal{S}(\langle x, y, \xi, \zeta \rangle^\tau) \cdot \) Then, for \( t > 0, \)
\[ \phi(H) = \phi(H)(I + H^2)^t(I + H^2)^{-t}\psi(H^2) - \psi(G) \]
with \( \phi(H)(I + H^2)^t \) a bounded operator and by the Helffer-Sjöstrand formula
\[ (I + H^2)^{-t}\psi(H^2) - \psi(G) = -\frac{1}{\pi} \int_C \tilde{\partial}\psi(z - H^2)^{-1}(H^2 - G)(z - G)^{-1}dz. \]

On the compact support of \( \tilde{\partial}\psi \), the symbols of \( (z - H^2)^{-1} \) and \( (z - G)^{-1} \) and \( \tilde{\partial}\psi \) are bounded as described above and the symbol of \( (I + H^2)^{-q}(H^2 - G) \) is in \( S((y)^{-\infty}(\xi, \zeta)^{2r-st}) \cdot \) The symbol of \( [\phi(H), P] \) is therefore in \( S((x, y)^{-\infty}(\xi, \zeta)^{2r-st}) \) for any \( t > 0 \) and since \( s > 0 \), we obtain our result. The same result holds for \( \mathcal{H}^q[H^p, P]\phi(H) \) by composition calculus. [21, Theorems 9.3 and 9.4] apply again so that the operators are also trace-class and \( \text{Tr}[\phi(H), P] = 0 \) for the same reasons as above. \( \square \)
The above proof shows that $\phi(H) \in \text{Op}^m S(\langle y, \xi, \zeta \rangle^{-\infty})$ provided that $G > E^2_0$ satisfies either (H1) or (H2) and $\phi$ has compact support in $[-E_0, E_0]$. To link the conductivity to the index of a Fredholm operator, we need to extend the above result to $P(x) = H(x - x_0)$ as follows.

**Lemma 4.2.** Let $\phi \in \text{Op}^m S(\langle y, \xi, \zeta \rangle^{-\infty})$ and $P(x)$ a switch function in $\mathcal{S}[0, 1]$, possibly a Heaviside function $H(x - x_0)$. Let $B$ be a bounded operator.

Then $[\phi, P]$ is a trace-class operator with vanishing trace and $[\phi, P]B$ is a trace-class operator with trace given as the integral along of the diagonal of its Schwartz kernel.

Let $P_1$ be another switch function (smooth or not). Then $(P - P_1)\phi$ and $\phi(P - P_1)$ are trace-class operators.

**Proof.** We write the Schwartz kernel of $\phi$ in the variables $w(\frac{1}{2} (x + x'), x - x', y, y')$. By assumption, it is smooth in all variables with rapid decay in the last three variables. We first show that $[\phi, P]$ is trace-class with vanishing trace.

We first decompose $w(x_1, x_2, y, y') = \sum_{m,n} w_{mn}(x_1, x_2) h_m(y) h_n(y')$ with $h_m$ an orthonormal basis of $L^2(\mathbb{R})$, for instance the Hermite functions. Similarly $\phi = \sum_{m,n} \phi_{mn}$. By assumption $\sum_{m,n} m^n \phi_{mn}$ is bounded uniformly in $(x_1, x_2)$. We thus need to show that $[\phi, P] := [\phi_{mn}, P]$ (dropping the indices $(m, n)$) is trace-class with trace-norms summable in $\langle m, n \rangle$ since $w_{mn}$ decays more rapidly than $\langle m \rangle^{-2} \langle n \rangle^{-2}$, say.

Now $w(\frac{1}{2} (x + x'), x - x')$ is smooth in both variables but rapidly decaying only in $x - x'$. Let $k(x, x')$ be the Schwartz kernel of the corresponding $[\phi, P]$.

We introduce a partition of unity $1 = \sum_{j \in \mathbb{Z}} \chi_j(x)$ such that $\chi_0(x)$ has support in $(-2 - \eta, 2 + \eta)$ and equals 1 on $(-2 + \eta, 2 - \eta)$ for $0 < \eta < 1/2$; $\chi_j(x)$ has support in $(1 + j - \eta, 2 + j + \eta)$ and equals 1 on $(1 + j + \eta, 2 + j - \eta)$ while $\chi_{-j}(x) = \chi_j(-x)$ for $j \geq 1$.

The Schwartz kernel of $[\phi, P]$ is therefore given by

$$k(x, x') = \sum_{i,j \in \mathbb{Z}} k_{i,j}(x, x'), \quad k_{i,j}(x, x') = (p(x') - P(x)) w\left(\frac{x + x'}{2}, x - x'\right) \chi_i(x) \chi_j(x').$$

We wish to show that each $k_{i,j}(x, x')$ is the kernel of a trace-class operator with trace-class norm summable in $(i, j)$. This will prove the first result.

By assumption on $p$, we observe that the latter kernel vanishes when $0 < i, j$ and when $i, j < 0$. By symmetry, it remains to consider the cases $\{i = 0 \text{ and } j \geq 0\}$ as well as $\{i \leq -1 \text{ and } j \geq 1\}$. Assume first $i = 0$ and $0 \leq j \leq 5$. These contributions are of the form

$$(p(x') - P(x)) w\left(\frac{x + x'}{2}, x - x'\right) \phi_1(x) \phi_2(x')$$

with $\phi_p$ compactly supported for $p = 1, 2$. Following [21, Section 9], this is decomposed as

$$\int_{\mathbb{R}^2} \hat{w}(\xi, \zeta) e^{\frac{i}{2} (\xi - \zeta)} e^{i(x-x')\xi} d\xi d\zeta (p(x') - P(x)) \phi_1(x) \phi_2(x').$$

This may be seen as a superposition in $(\xi, \zeta)$ of rank-one operators with traces uniformly bounded in $(\xi, \zeta)$ since $\phi_j$ and $p$ are bounded. By regularity assumptions on $w$, then $\hat{w}(\xi, \zeta) \in L^1(\mathbb{R}^2) \otimes \mathbb{C}^{n \times n}$. For instance by decomposing $\hat{w}$ in a basis of Hermite...
functions, the above trace is well approximated by that of finite rank operators so that all traces are computed as integrals of Schwartz kernels along the diagonal $x = x'$.

Let us now consider the cases $i = 0$ and $j \geq 6$ or $i \leq -1$ and $j \geq 1$. We observe for $x \neq x'$ that $e^{i(x-x') \xi} = \frac{1}{|i(x-x')|^n} \partial^n_x e^{i(x-x') \xi}$. We then find after integrations by parts that

$$k_{ij}(x, x') = \int_{\mathbb{R}^2} \partial_\xi^2 \hat{w}(\xi, \zeta) e^{i(x-x') \xi} \frac{e^{i(x-x') \zeta} d\xi d\zeta}{(x-x')^3}.$$  

By assumption on $w$, we have that $\partial_\xi^2 \hat{w}(\xi, \zeta)$ is also integrable.

In all cases, we observe that on the support of $\chi_i(x)\chi_j(x')$, we have $3 < x' - x < 5$. Therefore $x - x' = q + \tilde{x} - \tilde{x}'$ for $q \geq 3$ an integer and $|\tilde{x} - \tilde{x}'| \leq 2$. Thus

$$\frac{1}{(x-x')^3} = \frac{1}{(q + \tilde{x} - \tilde{x}')^3} = \frac{1}{q^3} \sum_{m_1, m_2, m_3 \geq 0} \left( \frac{\tilde{x}' - \tilde{x}}{q} \right)^{m_1 + m_2 + m_3}.$$  

This is an absolutely convergent sum that is well-approximated by finite sums, all of which give rise to finite-rank operators (degenerate Schwartz kernels). This shows, using the regularity of $w$, that $k_{ij}$ is the kernel of an operator given as a limit in trace-class norm of trace-class operators with a trace norm bounded by $C(|i| + j)^{-3}$ when $i < 0 < j$ and bounded by $j^{-3}$ when $i = 0$ and $j \geq 6$. Since the latter bound is summable in $(i, j)$, this shows that $[\phi, P]$ is trace-class as limit of finite rank trace-class operators. Since the traces of the latter are given by the integral of their kernels along the diagonal, this property also holds for $[\phi, P]$. Since the kernel of that operator vanishes along the diagonal, we find that $\text{Tr}[\phi, P] = 0$.

Let us now consider the case $[U, P]B$. We obtained above that $[U, P]$ was the limit of finite rank trace-class operators, and hence as a sum of rank-one operators. Let $\phi_1(x) \phi_2(x')$ be the kernel of such an operator $K_{12}$. Then $K_{12}B$ may be written as $\phi_1 \otimes \phi_2 B = \phi_1 \otimes (B^* \phi_2) = \phi_1 \otimes \phi_3$ and is also rank-one with trace norm increased by at most $\|B\|$. This shows that $[U, P]B$ is also a limit of uniformly trace-class finite rank operators and hence the result.

It remains to consider the operators $(P - P_1)\phi$ and $\phi(P - P_1)$. Since $P - P_1$ is compactly supported in $x$, we can define the above partition of unity such that $\chi_0$ equals 1 on the support of $P - P_1$. Then only the operator with kernel $k_{00}$ contributes and it is clearly trace-class using the same decomposition in $(\xi, \zeta)$. This concludes the proof of the lemma.

We now prove the first main result of the section:

**Proposition 4.3.** Let $E_0 > 0$ and $G > E_0^2$ such that (H1) or (H2) holds. Let $\varphi(H) \in \mathcal{G}[0, 1, -E_0, E_0]$ and $U(H) = e^{i2\pi \varphi(H)}$. Let $P \in \mathcal{G}[0, 1]$ such that $P^2 = P$ and $P_1$ a smooth switch function in $\mathcal{G}[0, 1]$. Then all terms in (19) are defined and equal.

**Proof.**

(i) The first equality in (19) is a definition. We know from Lemma 4.1 that $[H, P_1]\varphi'(H)$ is trace-class.

(ii) The second inequality has been shown to hold in a variety of contexts [3, 8, 13, 14]. We first show that for $g = g(H)$ smooth compactly supported with support in $(-E_0, E_0)$,
\[ \text{Tr}[W, P_1]g = \text{Tr}[H, P_1]W'g, \quad W(H) = U(H) - I. \quad (20) \]

Let \( \chi \) be smooth, compactly supported and equal to 1 on the support of \( W \) and \( g \) also with support in \((-E_0, E_0)\). Let \( W_p \) be a sequence of polynomials chosen such that \( \chi(W - W_p) \) and \( \chi(W' - W_p') \) uniformly on \( \mathbb{R} \) decrease to 0 as \( p \to \infty \). We find, with \( \delta W_p := W - W_p \)

\[
[\delta W_p, P_1]g \chi = \delta W_p[P_1, g] \chi + \delta W_p g[P_1, \chi] + [\delta W_p g \chi, P_1].
\]

We deduce from Lemma 4.1 that \( \text{Tr}[\delta W_p g \chi, P_1] = 0 \) and that \( \delta W_p[P_1, g] \) is trace-class since \([P_1, g] \in \text{Op}^w S(x, y, z, \xi), \xi^{-\infty} \). Since \( \text{Tr}AB = \text{Tr}BA \) when \( A \) is trace-class and \( B \) bounded, we find that \( \text{Tr}\delta W_p[P_1, g] \chi = \text{Tr}\chi \delta W_p[P_1, g] = \text{Tr}[P_1, g] \chi \delta W_p \).

Therefore,

\[
\text{Tr}[W - W_p, P_1]g = \text{Tr}[P_1, g](W - W_p) \chi + \text{Tr}(W - W_p)g[P_1, \chi] \to 0 \quad \text{as } p \to \infty.
\]

It remains to analyze \( \text{Tr}[W_p, P_1]g \). We now verify that

\[
\text{Tr}[H^n, P_1]g = \text{Tr}[n[H, P_1]H^{n-1}g, \quad \text{Tr}[W_p, P_1]g = \text{Tr}[H, P_1]W_p g.
\]

Indeed, from \([AB, C] = A[B, C] + [A, C]B\),

\[
\text{Tr}[H^{n+1}, P_1]g = \text{Tr}[H^n[H, P_1]g + [H^n, P_1]Hg = \text{Tr}H^n[H, P_1]g \chi + [H^n, P_1]Hg
\]

\[= \text{Tr}[H, P_1]H^n g \chi + [H^n, P_1]Hg = \text{Tr}[H, P_1]H^n g + [H^n, P_1]Hg
\]

\[= \text{Tr}[H, P_1](n + 1)H^n g
\]

using that \( H^n[H, P_1]g \) is trace-class so that \( \text{Tr}H^n[H, P_1]g \chi = \text{Tr}H^n[H, P_1]g \) and for the last equality an induction in \( n \geq 1 \). This proves the result for \( W_p \) as well.

It remains to realize that \( (W_p' - W')g \) is uniformly small as \( p \to \infty \) to obtain (20). We next compute

\[
\text{Tr}[U, P_1]U^* = \text{Tr}[W, P_1] + \text{Tr}[W, P_1]W^* = \text{Tr}[H, P_1]W'W^*
\]

\[= \text{Tr}[H, P_1]W'U^* - \text{Tr}[H, P_1]W'.
\]

It thus remains to show that \( 0 = \text{Tr}[H, P_1]W' \). Let \( 1 = \psi_1^2 + \psi_2^2 \) a partition of unity with \( 0 \leq \psi_j \leq 1 \) for \( j = 1, 2 \) and such that \( \psi_j \in C^\infty_c(\mathbb{R}) \) has support in \((-E_0, E_0)\) and equals 1 on the support of \( W \). Then, using (20), with \( \psi_j = \psi_j(H),
\]

\[
\text{Tr}[H, P_1]W' = \text{Tr}[H, P_1]W'\psi_1^2 = \text{Tr}[W(H), P_1]\psi_1^2.
\]

Now, since \( [W(H), P_1] \) is trace-class,

\[
\text{Tr}[W, P_1]\psi_2^2 = \text{Tr}\psi_2[W, P_1]\psi_2 = 0
\]

since \( W(H)\psi_2(H) = 0 \). Since \( \text{Tr}[W, P_1] = 0 \), we have \( \text{Tr}[H, P_1]W' = \text{Tr}[W, P_1]\psi_1^2 = \text{Tr}[W, P_1] = 0 \). This proves that \( \text{Tr}[U, P_1]U^* = 2\pi i \text{Tr}[H, P_1]\phi'(H) \) since \( W'U^* = U'U^* = 2\pi i \phi \).

(iii) For the third equality, we deduce from Lemmas 4.1 and 4.2 that \( (P - P_1)W^* \) is trace-class with \( W = U - I \) and that the trace of \([P - P_1, W] \) vanishes. Thus
\[ \text{Tr}[U, P - P_1]U^* = \text{Tr}[W, P - P_1]U^* = \text{Tr}[W, P - P_1]W^* = \text{Tr}(P - P_1)(W^*W - WW^*) = 0 \]

since \( \text{Tr}W(P - P_1)W^* = \text{Tr}(P - P_1)W^*W \) as \((P - P_1)W^*\) is trace-class and \(WW^* = W^*W\).

(iv) The fourth equality, which holds when \( P \) is a projection, is also a (non-trivial) consequence of the trace-class nature of \([P, U(H)]\) as shown in [43, Theorems 4.1 and 5.3].

The previous result shows that (19) holds when \( H^2 \) is well approximated by \( G \) such that \( \|G\| > E_0^2 \). In particular, \( 2\pi \sigma t \in \mathbb{Z} \) is quantized. We will find sufficient criteria ensuring either (H1) or (H2) for the systems (1) and (2). Before doing so, we prove several results showing that the conductivity \( \sigma_t \) is stable against changes in \( P, \phi', \) and \( H \). Note that the index of \( PU(H)P_{\text{Ran}P} \) in (19) is defined only for \( P \) a projection, where it is also the index of the operator \( PU(H)P + I - P \) defined on \( L^2(\mathbb{R}^2) \otimes \mathbb{C}^n \). When \( P \) is a smooth switch function, we can show that the latter, or better yet \( PU(H)P + I - P^2 = I + PW(H)P \) with \( W(H) = U(H) - I \), is still a Fredholm operator under (H1) or (H2) using the Fedosov formula [8, 43]. We do not pursue this further here as we do not use such a result.

**Proposition 4.4.** Let \( P_j \) for \( j = 1, 2 \) be smooth switch functions in \( \mathcal{S}[0, 1] \), and \( \phi_j \) for \( j = 1, 2 \) be smooth switch functions in \( \mathcal{S}[0, 1, -E_0, E_0] \) for \( E_0 > 0 \). Assume that \( H \) satisfies the hypotheses of Proposition 4.3. Then we have

\[ \sigma_t = \text{Tr} i[H, P_1]\phi'_1(H) = \text{Tr} i[H, P_2]\phi'_2(H). \]

**Proof.** Both traces are defined thanks to Lemma 4.1 and (19) applies thanks to Proposition 4.3. This directly gives that \( \sigma_t \) is independent of \( P_j \) for a fixed \( \phi'(H) \).

Let us now assume that \( P_1 = P_2 \) and let \( P = H(x) \) a Heaviside function. Then Proposition 4.3 applies and the above traces are quantized and related to the indices of \( PU_t(H)P \). We then construct a continuous family \([1, 2] \ni t \mapsto \phi_t \) of smooth switch functions in \( \mathcal{S}[0, 1, -E_0, E_0] \) and obtain a continuous map in operator norm \( t \mapsto U_t(H) = e^{i\sigma_t(H)} \) for instance by invoking the Helffer–Sjöstrand formula (47). Using (19), we know that \( 2\pi \sigma_t(t) \) is the index of \( PU_t(H)P \) and is therefore an integer for all \( t \in [1, 2] \). Thus, \( t \mapsto PU_t(H)P \) is continuous in operator norm and the index of these Fredholm operators is therefore independent of \( t \) [18, Chapter 19]. Appealing to (19) concludes the proof of the proposition.

Note that the proof of Proposition 4.3 shows that \( \text{Tr}[H, P_1]\phi'(\psi_1^2 + \psi_2^2) = \text{Tr}[\phi(H), P_1](\psi_1^2 + \psi_2^2) = 0 \) when \( \phi(H) \) has compact support in \((-E_0, E_0)\). It remains to apply this identity to \( \phi = \phi_2 - \phi_1 \), which has compact support in \((-E_0, E_0)\), to obtain another proof of the proposition.

We now extend the above results to families of operators \( H = H(t) \) for \( t \in [0, 1] \) such that the invariants in (19), defined for \( P(x) = H(x - x_0) \), are independent of \( t \). This allows us to include the presence of perturbations \( V(x, y) \) in the Hamiltonian and to vary semiclassical parameters in the next section. We first recall the following result:
Lemma 4.5. Let $A$ be trace-class operator on $L^2(\mathbb{R}^n)$ and $\Lambda$ a linear invertible transform in $\text{GL}(n, \mathbb{R})$. Then $\Lambda^{-1}AA$ is also trace-class and $\text{Tr} A = \text{Tr} \Lambda^{-1}AA$.

The proof is a direct consequence of the cyclicity of the trace. The only such linear transform we use in this article is the scaling $Y \mapsto y = hY$ for $h > 0$ so that $hD_y$ and $\mu(y)$ are pulled back to $D_Y$ and $\mu(hY)$. Recall the notation $I[H] := \text{Index} \text{PU}(H)P$. Let $H = \text{Op}^w(a(x, y, \xi, \zeta))$ and $H_h = \text{Op}^w(a(x, hy, \xi, h^{-1}\zeta))$. Then the lemma and identities such as $U(\Lambda^{-1}HA) = \Lambda^{-1}U(H)A$ imply that $I[H] = I[H_h]$ when either one is defined. What we need is a different equality. Let $H_h = \text{Op}^w(a(x, y, \xi, h\zeta))$ the semiclassical rescaling (in the variable $y$). We want to show that $I[H_h] = I[H]$. The above lemma states that $I[H_h] = I[H]$ with $H = \text{Op}^w(a(x, hy, \xi, \zeta))$. It thus remains to show that transforming a coefficient $\mu(y)$ to $\mu(hy)$ does not modify the invariants in (19). Similarly, let $H_t = H + tV$. We want to show that $I[H_t]$ is independent of $t$. A sufficient condition is as follows.

Proposition 4.6. Let $H_t$ be as in Proposition 4.3 for all $t$ in a connected interval $I \subset \mathbb{R}$. Let us assume that $H(t) - H(s) = (t - s)B(s, t)$ for $B(s, t)$ a bounded operator in uniform norm on $L^2(\mathbb{R}^2; \mathbb{M}_n)$ uniformly in $(s, t) \in I^2$. Then $I[H_t] = \text{Index} \text{PU}(H_t)P$ is independent of $t \in I$.

Proof. Let $H_t$ be as above and consider the unitary $U(H_t)$. By the Helffer–Sjöstrand formula (47), we have

$$U(H_t) - U(H_s) = W(H_t) - W(H_s) = -\frac{1}{\pi} \int_{\mathbb{C}^2} \partial \overline{\partial} U(z - H_t)^{-1}(H(t) - H(s))(z - H_s)^{-1} \, d^2z.$$ 

By hypothesis, $H(t) - H(s) = (t - s)B(t, s)$ with $B(t, s)$ bounded. Since $(z - H_t)$ is bounded by $|\text{Im} z|^{-1}$ uniformly in $t$ (at least locally) and $|\partial \overline{\partial} U| \leq C_N |\text{Im} z|^N$ for a well-chosen almost analytic extension $\tilde{U}$, we find that $U(H_t) - U(H_s)$ is bounded in the operator norm by a constant times $(t - s)$. This ensures that $t \mapsto U(H_t)$ is continuous in the operator norm and hence so is the Fredholm operator $\text{PU}(H_t)P$. By continuity of the index of Fredholm operators [18, Chapter 19], the index is independent of $t$. \hfill \square

The above result addresses the question of stability under perturbation $H_V = H + V$. Assuming that $V$ is bounded and that the index $I[H + tV]$ is defined for $0 \leq t \leq 1$, the above proposition directly implies that $I[H] = I[H + V]$. The above result also allows us to address stability in the semiclassical regime. Let $H_h = H_0 + \mu(hy)$ with $\mu$ multiplication by a bounded (matrix-valued) function and $h > 0$. Let us assume that $\mu(y)$ takes constant values for $y$ large and $-y$ large. Then $\mu(hy) - \mu(ly) = (h - l)y\mu'(hy)$ for $h$ between $h$ and $l$. By assumption on $\mu$, $y\mu'(hy)$ is bounded and Proposition 4.6 applies. We collect the above results as:

Corollary 4.7. Let $H_{h,t} = H_0 + \mu(hy) + tV$ be an operator with $H_0$ independent of $y$, $V$ a bounded multiplication operator, and $\mu(y)$ a smooth domain wall taking constant values outside of a compact interval. Assume that $H_{h,t}$ is as in Proposition 4.3 for all $h_0 \leq h \leq h_1$ and $0 \leq t \leq 1$. Then the index $I[H_{h,t}]$ is an integer independent of $h$ and $t$ in these ranges.
The above two results imply the stability of the index $I[H_{h,t}]$ but not necessarily of the corresponding traces in (19). Indeed, the latter holds when $[P, U(H_V)]$ is trace-class according to [8, Lemma 4.7]. That $[P, U(H_0)]$ is trace class for $\mu(h\nu)$ holds independently of $h$. For perturbations in $V$, it holds when $U(H_V) - U(H) = W(H_V) - W(H)$ is itself trace-class. Sufficient conditions ensuring the latter result are given in [8, Proposition 4.4]. We can for instance show that Hermitian, compactly supported perturbations $V \in L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ generate trace-class perturbations for the system (1). Note that $H + V$ is then no longer necessarily a pseudo-differential operator (with smooth symbol).

The main difficulty in applying Proposition 4.3 is therefore to show the existence of a positive definite operator $G$ satisfying either (H1) or (H2). If one such operator $G$ may be found for each $H_{h,t}$ in Corollary 4.7, then $I[H_{h,t}]$ is an integer independent of $h$ and $t$. It thus remains to construct such an operator $G$. The construction is reasonably explicit in the favorable situation when (H2) is satisfied.

**Proposition 4.8.** Let $H$ be a self-adjoint unbounded differential operator on $L^2(\mathbb{R}^2; \mathcal{M}_d)$ such that $(I + H^2)^{-1}$ is a VDO with symbol in $S(m)$ with $m = \langle \xi, \zeta \rangle^{-s}$ for $s > 0$ and such that $H$ has constant coefficients when $y \geq R$ and when $y \leq -R$ for some $R \geq 0$. Let $H \pm$ be the constant coefficient operators such that $H\pm = H\pm f_\pm$ for any test function $f_\pm$ supported on $y \geq R$ and $f_-$ supported on $y < -R$. We assume that $H\pm$ have a spectral gap $[-m, m]$ for some $m > 0$. Let now $\phi(H)$ be a smooth switch function in $\mathcal{G}[0, 1, -m, m]$. Then there is a differential operator $G$ with no spectrum in $(-\infty, m^2]$ and such that hypothesis (H2) holds. As a consequence, (19) holds.

**Proof.** By assumption, $(I + H^2)^{-1}$ has a (Weyl) symbol with the required decaying assumption. It remains to construct $G$. Let $(\phi_1(y), \phi_2(y))$ be a partition of unity such that $\phi_1^2(y) + \phi_2^2(y) = 1$ and such that $\phi_1(y)$ is compactly supported and equal to 1 on $(-R, R)$. We then construct

$$G = \phi_2 H^2 \phi_2 + m^2 \phi_1^2.$$

The (Weyl) symbols of $H^2$ and $G$ agree for $|y|$ sufficiently large (outside of the support of $\phi_1$). The hypothesis on $\tau - \tilde{\tau}$ is therefore satisfied. Moreover, we find for any function $f \in L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$ that

$$(f, Gf) = (f \phi_2, H^2 f \phi_2) + m^2 (f \phi_1, f \phi_1) \geq m^2 \|f\|^2$$

since $H$ is given by $H\pm$ on the support of $f \phi_2$ and these operators have a spectral gap in $[-m, m]$. Here, $(\cdot, \cdot)$ and $\| \cdot \|$ are the inner product and norm on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$. This shows the existence of $G$ with the required properties. We deduce from Proposition 4.3 that (19) holds.

It is straightforward to apply the above result to the operator $H - \alpha$ with $H$ given by the $2 \times 2$ system (1) assuming that $m(y) = \pm m_0$ away from a compact domain and $-|m_0| < \alpha < |m_0|$ so that $E_0 = |m_0| - |\alpha|$. The result also holds with $D_y$ replaced by $hD_y$ for any $h > 0$. It continues to hold if $H$ is replaced by $H + V$ for any Hermitian compactly supported (smooth) $\mathcal{M}_2(\mathbb{C})$ - valued function $V(x, y)$. Indeed, for any of these operators, we find that $(I + (H - \alpha)^2)^{-1}$ has symbol in $S(\langle \xi, \zeta \rangle^{-2})$. 


The $3 \times 3$ system in (2) is more challenging. The reason is the presence of the flat band $E_0(\xi) = 0$ observed when $f$ is constant. There is therefore no reason for $(I + (H - ax)^2)^{-1}$ to be a negative-order PDO even for $x$ in a bulk gap. Only when the coefficients are sufficiently slowly varying can one expect to leverage the presence of a spectral gap.

The construction of $G$ under hypothesis (H1) is done microlocally and proceeds as follows.

**Proposition 4.9.** Let $H = \text{Op}^w(\sigma(x, y, \xi, \zeta))$ be a differential operator and assume that $H^2 = \text{Op}^w(\tau(x, y, \xi, \zeta))$ is such that as a symmetric matrix, $\tau(x, y, \xi, \zeta) \geq E_0^2 > 0$ uniformly in $|(y, \xi, \zeta)| \geq R$ for some $R > 0$ and in $x \in \mathbb{R}$.

Then for any $\delta > 0$, there exists $G = \text{Op}^w(\tilde{\tau}(x, y, \xi, \zeta))$ with $\tilde{\tau}(x, y, \xi, \zeta) \geq (E_0 - \delta)^2$ uniformly in $(x, y, \xi, \zeta)$ and $\tilde{\tau} = \tau$ when $|(y, \xi, \zeta)| \geq R$.

Let $h > 0$ and define $H_h = \text{Op}^w(\sigma(x, y, h\xi, h\zeta))$ as well as $G_h = \text{Op}^w(\tilde{\tau}(x, y, h\xi, h\zeta))$. Then there is $h_0 > 0$ (sufficiently small) and a constant $C$ independent of $0 < h \leq h_0$ such that $\tilde{\tau}(x, y, h\xi, h\zeta) = \tau(x, y, h\xi, h\zeta)$ when $|(y, \xi, \zeta)| \geq R/h$ and $G_h \geq F^2 := (E_0 - \delta - Ch)^2 > 0$.

Let $\varphi \in \mathcal{S}[0, 1, -F, F]$ and define $\tilde{H}_h = \text{Op}^w(\sigma(hx, hy, \xi, \zeta))$ for $0 < h \leq h_0$. Then hypothesis (H1) holds for $H_h$ and Lemma 4.5 implies that (19) holds for $H$ replaced by $\tilde{H}_h$.

This result thus states that (19) holds for Hamiltonians $\tilde{H}_h = \text{Op}^w(\sigma(hx, hy, \xi, \zeta))$ with spatial coefficients of the form $c(hx, hy)$ for $h$ sufficiently small.

**Proof.** By hypothesis, we can construct a matrix-valued symbol $\tilde{\tau} = \tau \geq E_0^2$ when $|(y, \xi, \zeta)| \geq R$. This symbol can then be extended to $\mathbb{R}^4$ smoothly such that $\tilde{\tau} \geq (E_0 - \delta)^2$. The positivity of the symbol is not sufficient to ensure that the operator $G = \text{Op}^w(\tilde{\tau}(x, y, \xi, \zeta))$ itself is positive. Replacing $H$ by $H_h$, with $H_h^2 = \text{Op}^w(\tau(x, y, h\xi, h\zeta))$, we observe that $\tilde{\tau}(x, y, h\xi, h\zeta) = \tau(x, y, h\xi, h\zeta)$ when $|(y, h\xi, h\zeta)| \geq R$ and hence certainly when $|(y, \xi, \zeta)| \geq R/h$. Moreover, we still have that $\tilde{\tau}(x, y, h\xi, h\zeta) \geq (E_0 - \delta)^2$. The Gårding inequality (46) allows us to conclude that $G_h \geq (E_0 - \delta - Ch)^2$ and hence is bounded below by a positive constant when $h$ is sufficiently small.

We therefore obtain that (H1) holds for $H_h$ and hence that the index $I[\text{Op}^w(\sigma(x, y, h\xi, h\zeta))]$ is defined by Proposition 4.3. We invoke Lemma 4.5 to observe that $I[\text{Op}^w(\sigma(x, y, h\xi, h\zeta))] = I[\text{Op}^w(\sigma(hx, hy, \xi, \zeta))]$, which proves that (19) holds for $H$ replaced by $\tilde{H}_h$.

In the remainder of this section, we apply the above result to the system (2)

$$H = D_x y_1 + D_y y_4 - f(y) y_7 + V(x, y)$$

with $f(y)$ a smooth switch function in $\mathcal{S}[-f_0, f_0]$ and $h > 0$. We assume that $V(x, y)$ is multiplication by a smooth compactly supported function (still called $V(x, y)$). Let $\tau_x$ be the symbol of $(H - ax)^2$ for $x \in (-f_0, 0) \cup (0, f_0)$. $H - ax$ has a bulk gap given by $E_0 = \min(|x|, f_0 - |x|)$. Our objective is to find constraints on $f$ and $V$ such that the matrix $\tau_x(x, y, \xi, \zeta)$ is bounded uniformly below by a positive constant when $|(y, \xi, \zeta)| \geq R$. The construction of the operator $G_x$ associated to $H - ax$ then proceeds as in the above proposition.
Let \((y, \xi, \zeta) = \rho(y_0, \xi_0, \zeta_0)\) with \(|(y_0, \xi_0, \zeta_0)| = 1\) and the scaling factor \(\rho \in \mathbb{R}^+\). Consider \(|y_0|^2 \geq \frac{1}{2}\) first. Then for \(\rho\) sufficiently large, the symbol of \((H_{i\hbar}, V - \alpha)^2\) becomes constant with eigenvalues bounded below by \(\min(1/|x|, f_0 - |x|)\) as expected.

The region \(|y_0|^2 \leq \frac{1}{2}\) is more challenging. We have \((\xi, \zeta) \to \infty\) as \(\rho \to \infty\) in that sector. However, only two out of three eigenvalues of the symbol of \((H - \alpha)^2\) diverge to infinity as \((\xi, \zeta) \to \infty\). The third eigenvalue remains bounded and large fluctuations in \(V(x, y)\) or \(f(y)\) may force it to vanish or become arbitrarily small, whereby closing the spectral gap and destroying any hope of “topological protection.” In such a case, we cannot construct a positive definite approximation of \((H - \alpha)^2\). Constraints on \(V\) and \(f\) thus need to be imposed. The Hermitian operator \((H - \alpha)^2\) is given explicitly by

\[
\begin{pmatrix}
W_{11}^2 + \delta_x^2 + \delta_y^2 & \delta_x W_{21} + \delta_y W_{22} + \delta_y g^* & \delta_x W_{31} + \delta_y W_{32} + \delta_y W_{33} \\
\delta_x^* \delta_x + W_{22}^2 + |g|^2 & \delta_y^* \delta_y + W_{32}^2 + g W_{33} \\
\delta_x \delta_y^* + W_{31}^2 & \delta_y \delta_y^* + g W_{33}^2 & W_{33}^2
\end{pmatrix},
\]

where we have defined \(W = V - \alpha\) for \(V = (V_{ij})_{1 \leq i, j \leq 3}\), \(\delta_x = D_x + V_{12}\), \(\delta_y = D_y + V_{13}\), and \(g = i f_h + V_{23}\). The full Weyl symbol of \((H - \alpha)^2\) is obtained by using identities implying that the symbol of \(D_y g(y)\) is \(\zeta g(y) - \frac{1}{2} \partial_y g(y)\) and that of \(g(y) D_y\) is \(\zeta g(y) + \frac{1}{2} \partial_y g(y)\). We wish to show that the eigenvalues \(\lambda_{1,2,3}\) of the above symbol are positive independently of the scaling factor \(\rho \geq R\) for \((y, \xi, \zeta) = \rho(y_0, \xi_0, \zeta_0)\) and \(|(\xi_0, \zeta_0)|^2 \geq \frac{1}{2}\).

Let us assume that \(V_{12}\) and \(V_{13}\) are real-valued and \(V_{23}\) is purely imaginary. Assuming constant coefficients (independent of \((x, y)\)), the symbol of \((H - \alpha)^2\) is the square of

\[
\begin{pmatrix}
W_1 & \tilde{\xi} & \tilde{\zeta} \\
\tilde{\xi} & W_2 & g \\
\tilde{\zeta} & -g & W_3
\end{pmatrix},
\]

where \(\tilde{\xi}\) and \(\tilde{\zeta}\) are the symbols of \(\delta_x\) and \(\delta_y\), respectively, and \(W_j\) stands for \(W_{jj}\). Therefore, \(|(\tilde{\xi}, \tilde{\zeta})| \to \infty\) as \(\rho \to \infty\). The invariants of the above matrix satisfy

\[
\lambda_1 \lambda_2 \lambda_3 = -W_3 \zeta^2 - W_2 \tilde{\zeta}^2 + O(1),
\]

\[
\sum_{i \neq j} \lambda_i \lambda_j = -(\zeta^2 + \tilde{\zeta}^2) + O(1), \quad \lambda_1 + \lambda_2 + \lambda_3 = O(1),
\]

where \(O(1)\) means independent of \(\rho\). This implies that two eigenvalues are asymptotically equal to \(\pm \rho\) while the third eigenvalue is asymptotically given by \(\frac{W_3 \zeta^2 + W_2 \tilde{\zeta}^2}{\rho^2}\). When \(V_{22} = V_{33} = 0\), the latter term equals \(-\alpha \neq 0\). In order for that term not to vanish, we need \(V_{22}\) and \(V_{33}\) to be sufficiently small compared to \(\alpha\). Large variations in \(V_{22}\) and \(V_{33}\) prevent the approximation of \((H - \alpha)^2\) by a positive definite operator. Note that we do not have any constraint on \(V_{11}\). Similar calculations show that the imaginary part of \(V_{12}\) and \(V_{13}\) and the real part of \(V_{23}\) also need to be small for the above eigenvalues to remain bounded away from 0.

The full symbol of \((H - \alpha)^2\) is given by the square of the above matrix plus contributions coming from differentiations of the coefficient such as \(\partial_x g\) or \(\partial_x V_{i,j}\). For instance, the term \(\partial_x g\), which involves \(f'(y)\), modifies the determinant and the sum of
squares of eigenvalues by a quantity of order \((\xi^2 + \zeta^2)\varepsilon^2\), which is leading-order in \(\rho^4\).
Such terms have to be sufficiently small for the symbol of \((H - \varepsilon)^2\) to have positive eigenvalues uniformly.

Let \(\varepsilon > 0\) and assume the coefficients in (21) are of the form \(f := f(\varepsilon y)\) scalar-valued and \(V := V(\varepsilon x, \varepsilon y)\) Hermitian-valued. Let \(\sigma_e\) be the Weyl symbol of \(H = \text{Op}^w(\sigma_e)\). We introduce the following hypothesis:

(H3 \(\times \) 3): The coefficients \(V_{22}, V_{33}, \text{Im}V_{12}, \text{Im}V_{13}\) and \(\text{Re}V_{23}\) are sufficiently small in \(S(1)\) and \(\varepsilon\) is sufficiently small.

Under (H3 \(\times \) 3), the above calculations show that the symbol of \((H + V - \varepsilon)^2\) has eigenvalues bounded away from 0 uniformly in \(|(y, \xi, \zeta)| > R\) for \(R\) sufficiently large. Therefore, \(H = \text{Op}^w(\sigma_e)\) satisfies the hypotheses of Proposition 4.9. Let \(F = E_0 - \delta\) for \(\delta > 0\) and consider a density of states built on \(\phi \in S[0, 1, x-F, x+F]\). Then Proposition 4.9 implies that (19) holds provided that \(\varepsilon\) is further reduced to \(h_0\varepsilon\).

Relabeling \(\varepsilon\) the small term \(\varepsilon h_0\), we thus deduce that (19) holds for any Hamiltonian \(H = D_x \gamma_1 + D_y \gamma_4 - f(\varepsilon y)\gamma_7 + V(\varepsilon x, \varepsilon y)\) satisfying (H3 \(\times \) 3).

This shows in particular that the interface conductivity in (19) is well defined provided that \(f(y)\) is a sufficiently slowly varying domain wall. We saw in section 2 that (19) also held for \(f(y) = f_0\ \text{sign}(y)\). However, this result came from a sufficiently explicit knowledge of the spectral decomposition of \(H[\mu(y)]\). The result we just established applies to any smooth \(f(y)\) that has sufficiently slow variations.

Numerical simulations in [39] confirm the above theoretical results and show that the value of \(\sigma_I\) is indeed quite stable against smooth perturbations in \(V_{11}, V_{12}, V_{13}\) and \(\text{Im}V_{23}\) while even small variations in the other components of \(V\) rapidly destabilize it.

### 4.2. Fedosov–Hörmander formula and bulk-interface correspondence

This section presents a general bulk-interface correspondence (BIC) for the family of differential operators

\[
H_h = D_x \gamma_1 + \gamma_2(hD_y, y) \tag{22}
\]

for \(\gamma_1\) a Hermitian matrix in \(\mathbb{M}(\mathbb{C}^n)\) and \(\gamma_2(D_y, y)\) a differential operator taking values in \(\mathbb{M}(\mathbb{C}^n)\). We also define \(H_V = H_h + V\) a perturbed operator and set \(H := H_1\). In the applications considered in this article, \(\gamma_2(hD_y, y) = hD_y \gamma_2 + \mu(y)\gamma_3\) although the BIC holds in the above more general setting. This class of models finds applications outside of the systems considered here, for instance for the \(n-\) replica models that appear in the analysis of Floquet topological insulators [36]. The separation of the component \(D_x \gamma_1\) simplifies the presentation of the result but is not essential either. Generalizations to a larger class of models mixing the \(x\) and \(y\) derivatives and involving higher-order derivatives are worked out in [39]. A generalized bulk-interface correspondence in arbitrary dimension for operators with unbounded domain walls is given in [40].

We assume that \(\sigma_I(H_h)\) is quantized, independent of \(h\), and that it can be computed as the trace of a trace-class operator with rapidly decaying symbol. This holds thanks to Proposition 4.8 for the system (1) and thanks to Proposition 4.9 for the system (2) provided that \(\mu(y) = f(\varepsilon y)\) with \(\varepsilon\) sufficiently small. The preceding section also indicated which perturbations \(V\) were allowed for the two systems (1) and (2) to ensure that \(\sigma_I(H_h + V) = \sigma_I(H_h)\). In the rest of the section, we thus assume \(V = 0\).
Let $x \in \mathbb{R}$ fixed. Assuming $x$ is in a bulk band-gap, we assigned two invariants so far. One is the conductivity $\sigma_i$ in (19). The other one is the bulk-difference invariant $W_x$ and the related Chern numbers in (18). We now introduce a third invariant as the index of the operator $H_h - x - i\nu(x)$ for well-chosen domain walls $\nu(x)$. The objective of the bulk-interface correspondence is to show that these invariants are all equal and different calculations of the same topological charge.

The operator $H_h = \text{Op}_h^w(\sigma)$ with $S^0(m) \ni \sigma = \tilde{\zeta}\gamma_1 + \gamma_2(y,z)$ for the order $m$ that makes $H_h$ self-adjoint as an unbounded operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$. We define $\tilde{H}_h(\zeta)$ in (5) as the Fourier transform $x \rightarrow \zeta$ of $H_h$, which is invariant by translation along the $x$ axis. We collect our assumptions as follows:

(h1) $2\pi \sigma_1$ is an integer independent of $0 < h \leq 1$ and $\zeta \mapsto \gamma_1 \varphi'(\tilde{H}_h(\zeta))$ is an operator with semiclassical symbol $a(y, \zeta, \xi; h) \in S^0(1)$ such that for $N$ sufficiently large, we have $|\partial^\alpha a(y, \zeta, \xi; h)| \leq C(y, \zeta, \xi)^{-N}$ for $\zeta = C(\zeta)$ and all $|x| \leq N$. Moreover, $i[H_h, P] \varphi'(H_h)$ is a trace-class operator with trace given by the diagonal integral of its Schwartz kernel.

Here, $\varphi$ is a smooth switch function in $\mathcal{G}[0,1, x - E_0, x + E_0]$ for $x \in \mathbb{R}$ and $E_0 > 0$ and $P$ is pointwise multiplication by a smooth function $\chi \in \mathcal{G}[0,1, x_0 - \beta, x_0 + \beta]$ for $\beta > 0$. Then $[H_h, P] = -i\chi'(x)\gamma_1$ with $\chi'(x)$ compactly supported.

(h2) Let $\zeta = x + i\omega$ and $\sigma_z(y, \zeta, \xi)$ be the semiclassical symbol of $z - \tilde{H}_h(\zeta) = \text{Op}_h(\sigma_z)$ in the variables $(y, \zeta)$ with $(x, \omega, \xi)$ seen as parameters. Let $\sigma_z(\omega, y, \zeta, \xi) = (\omega, y, \zeta, \xi)$ be the Weyl symbol at $h = 1$ of $x = \omega - h$. We assume that $\sigma_z$ is invertible and $\sigma_z^{-1}d_{\omega, y', \zeta', \xi'}(\sigma_z)$ is uniformly bounded (one-form with bounded coefficients) for $|\omega|^2 + |y|^2 + |\zeta|^2 + |\xi|^2 \geq R^2$ sufficiently large. Finally, we assume that $\sigma_z^{-1}d_{\omega, y', \zeta', \xi'}(\sigma_z) \rightarrow 0$ as $|\zeta| \rightarrow \infty$.

**Remark 4.10.** Let $H = \text{Op}_h^w(\sigma)$ and $H_h = \text{Op}_h^w(h\zeta)$ with all other variables fixed. Define $M_1 = (y, \zeta, \xi)$. The results obtained in Lemma 4.1 show that $\varphi'(H_h(\zeta)) = \text{Op}_h^w(a(h)$ for some symbol in $S(M_1^{-\infty})$ and hence the above bound for each fixed $h$ with a bound uniform for $h \in [h_1, 1]$ with $h_1 > 0$. For $0 < h < h_0$ with $h_0$ sufficiently small, the results in [21, Chapter 8] show that $\varphi'(H_h)$ also belongs to $S^0(M_1^{-\infty})$ with an asymptotic expansion in powers of $h$ and $a$ bound on $|\partial^\alpha a(y, \zeta, \xi; h)\gamma(y, \zeta, \xi)|$ that is independent of $0 < h < h_0$.

**Remark 4.11.**

(i) Condition (h2) is very similar to what is necessary to construct bulk-difference invariants. Indeed by the same diagonalization as in (10), we find

$$\sigma_z = x + iy - \sum_{j=1}^n h_j(y, \zeta)\Pi_j(\zeta, y, \zeta),$$

$$\sigma_z^{-1} = \sum_{j=1}^n (x + iy - h_j(\zeta, y, \zeta))^{-1}\Pi_j(\zeta, y, \zeta)$$

where we see that $\sigma_z$ is invertible if all $x + iy - h_j(\zeta, y, \zeta) \neq 0$ for $(x, \zeta, y, \xi)$ outside of a compact set. The choice $x = 0$ implies that $x$ lives within the bulk gap of the system.

(ii) In this article, we considered two forms of mass terms $\mu(y)$: either bounded domain walls or $\lambda y$. The same applies to the confining term $x$ above, which may be replaced by any domain wall $\nu(x) \in \mathcal{G}[0,1] - \delta, |x| + \delta]$ for $\delta > 0$ or a term of the form $\nu(x) = |x|\arctan x$ with a gap sufficiently large to include the energy
level $\alpha$. The above invertibility condition then takes the form $\alpha + i\nu(x) - h_j(\xi, y, \zeta) \neq 0$ outside of a compact set.

(iii) Let $\nu(x) = x$ or $\nu(x) \in \mathcal{S}[-|x| - \delta, |x| + \delta]$ for $\delta > 0$. For system (1), condition (h1) is $\alpha + i\nu(x) \neq \pm \sqrt{\xi^2 + \zeta^2 + m^2(y)}$. For $m(y) \in \mathcal{S}[-|m_0|, |m_0|]$, this condition is satisfied when $|x| < |m_0|$ (but not for $|x| > |m_0|$). For the geophysical system (2), the same conditions hold with $m(y)$ replaced by $f(y)$ with the additional constraint in (h1) that $\alpha + i\nu(x) \neq 0$, that is, $\alpha \neq 0$ when $\nu(x) = 0$. We verify that these conditions are satisfied for $\alpha$ in a bulk band gap, that is, $\alpha \in (-|f_0|, 0) \cup (0, |f_0|)$ for smooth $f(y) \in \mathcal{S}[-f_0, f_0]$ and $\alpha \neq 0$ for $f(y) = \lambda y$, $\lambda \neq 0$.

While the main objective of the bulk interface correspondence in Theorem 4.13 below is to relate the interface conductivity $\alpha_I$ to the bulk-difference invariant $W_\alpha$, it turns out that both invariants are naturally related to the index of the Fredholm operator:

$$F_h = H_h - \alpha - i\nu(x) \quad (23)$$

as alluded to in the above remark with some choices for $\nu(x)$ given there. We set $F := F_1$.

We assume $\nu(x)$ bounded to simplify the presentation and define $F_h$ as an unbounded operator from $\mathcal{D}(H)$ to $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$ where the domain of $H_h$ (which makes it self-adjoint as an unbounded operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$) is independent of $h$. There is in fact no reason for any operator of the form (23) to be Fredholm, that is an operator that admits left and right inverses modulo compact operators. In the preceding section, we introduced two hypotheses (H1) and (H2) on the symbol of $H$. We state corresponding hypotheses on that of $F_h$ when $\nu(x) \in \mathcal{S}[-\delta - |x|, |x| + \delta]$ to simplify; similar results hold for $\nu(x) = x$ and unbounded domain walls $\mu(y)$ after appropriate changes of the domain $\mathcal{D}(F)$, which we do not consider here in detail. Note that we can always choose $\nu \in \mathcal{S}[-\delta - |x|, |x| + \delta]$ such that $\nu(x) = x$ for $|x| \leq R$ so that the Weyl symbol of $F_h$ is given by $-\sigma(x, y, \xi, \bar{\xi})$ defined in (h2) when restricted to the vicinity of the sphere $x^2 + y^2 + \xi^2 + \bar{\xi}^2 = R^2$. Only those values appear in the Fedosov–Hörmander formula (24) below. The hypotheses are:

(H1’) Let $F_h$ be as in (23) and assume that (h2) holds and that $0 < h \leq h_0$ for $h_0$ sufficiently small.

(H2’) Assume that (h2) holds and that $(I + F^*F)^{-1} = \text{Op}^n(a)$ with $a \in S(m)$ for $m = (1 + \xi^2 + \bar{\xi}^2)^{-1}$ with $s > 0$.

Hypothesis (h2) may be seen as an ellipticity condition at infinity. In the favorable case (H2’) of a regularizing operator, then (h2) is sufficient to obtain a Fredholm operator. For systems such as (2), where regularization does not occur for all components, a smallness condition in (H1’) is necessary to ensure the Fredholm structure, in parallel to (H1) in section 4.1.

**Proposition 4.12.** Let $F_h$ be the differential operator defined in (23). Let us assume that:

either (H1’) holds with $h_0$ sufficiently small; or (H2’) holds. Then $F_h$ is a Fredholm operator from $\mathcal{D}(H)$ to $L^2(\mathbb{R}^2) \otimes \mathbb{C}^n$.

**Proof.** Assume (H2’). By assumption, the differential operator $F^*F$ is a constant-coefficient operator outside of a compact domain $\Omega$, which is invertible. Let $G = F^*F +
\[ \chi^2(x,y) \] with \( \chi^2 \in C_0^\infty(\mathbb{R}^2) \) with \( \chi^2 = 1 \) on \( \Omega \). Then \( G \) is positive definite and hence invertible with \( G^{-1} \in S(m) \) by assumption. We thus have \( G^{-1}F^*F = I + G^{-1}\chi^2 \) and we find that \( G^{-1}\chi^2 \) is compact since \( \chi^2 \) is compactly supported in space and \( G^{-1} \) is smoothing. We have thus constructed a left inverse up to a compact operator. We can similarly construct a right inverse and conclude that \( F \) is Fredholm \([18, \text{Corollary 19.1.9}]\).

Let us now consider the case \((H1')\). We can no longer invoke a compactness argument alone and thus need a smallness condition to obtain the Fredholm property. Let \( F_h = \text{Op}_h^n(f) \). Let \( \chi + \psi = 1 \) with \( \psi(x,y,\xi,\zeta) \in C_0^\infty(\mathbb{R}^4) \) compactly supported and equal to 1 on a sufficiently large domain that \( g = \chi^{-1} \in S(m^{-1}) \) is defined. Let \( G = \text{Op}_h^n(g) \). By semiclassical calculus \([21, \text{Chapters 7 and 8}]\), we find that \( GF = \text{Op}_h^n(\chi) + h\text{Op}_h^n(r_h) \) with \( r_h \in S^0(1) \). Therefore, for \( 0 \leq h \leq h_0 \), \( I + h\text{Op}_h^n(r_h) \) is invertible and \( (I + h\text{Op}_h^n(r_h))^{-1}GF = I - (I + h\text{Op}_h^n(r_h))^{-1}\text{Op}_h^n(\psi) \).

However, \( \text{Op}_h^n(\psi) \) is a compact operator so that \( F \) admits a left inverse up to a compact perturbation. We construct a right-inverse similarly and conclude that \( F_h \) is Fredholm for \( 0 < h \leq h_0 \).

It remains to show that \((H1')\) applies to \( (2) \) when the coefficients (including appropriate perturbations \( V \)) vary sufficiently slowly and that \((H2')\) applies to \( (1) \). This is done along the lines of the derivation in the preceding section. We leave the details to the reader. That \( F \) is a Fredholm operator extends to unbounded domain walls as in \([18, \text{Chapter 19}]\) as well as operators with coefficients whose derivatives vanish at infinity; see for instance \([44]\).

After these preliminary results and hypotheses, we state the main result of this section:

**Theorem 4.13** (Bulk-interface correspondence.). Let \( H \) be given by \((22)\) with \( h = 1 \). Assume \((h1)-(h2)\) above for \( x \in \mathbb{R} \). Then

\[
2\pi \sigma_I = -W_x = \frac{1}{24\pi^2} \int_\Sigma \text{tr}(\sigma_z^{-1}d\sigma_z)^3, \tag{24}
\]

where \( \Sigma = \{(\omega, y, \xi, \zeta) \in \mathbb{R}^4, \omega^2 + y^2 + \xi^2 + \zeta^2 = R^2 \} \) with the orientation on \( \mathbb{R}^4 \) given by \( d\omega \wedge d\xi \wedge dy \wedge d\zeta > 0 \). When \( F \) satisfies the hypotheses stated in \([18, \text{Chapter 19.3}]\), then the above quantity also equals \( \text{Index}(F) \).

We recall that \( \sigma_I \) is the interface invariant while \( W_x \) is the bulk-difference invariant. The above formula also provides the \( L^2 \)-index of the unbounded Fredholm operator \( F = H - x - iv(x) \) on \( L^2(\mathbb{R}^2) \otimes \mathbb{C}^n \) with (Weyl) symbol at \( h = 1 \) given by \( -\sigma_z(\nu(x), y, \xi, \omega) \) whenever applicable \([18, \text{Chapter 19.3}]\). The invariance properties of the integral in \((35)\) show that it takes the same values for \( \sigma_z(\nu(x), y, \xi, \omega) \) and \( \sigma_z(x, y, \xi, \omega) \). While it is likely \( 2\pi \sigma_I = \text{Index}(F) \) when the bulk-interface correspondence \( 2\pi \sigma_I = -W_x \) holds, we do not make such a claim and do not need it to establish the BIC.

The rest of this section is devoted to a proof of this result and some corollaries. We start with two lemmas summarizing some properties of the symbol of (semiclassical) resolvents and approximations in the computation of traces of trace-class operators.

**Lemma 4.14.** Let \( H_h = \text{Op}_h^n(a) \) with \( a \in S^0(m) \). Let \( z = \lambda + io \in \mathbb{C} \) with \( \omega \neq 0 \). Then \( (z - H_h)^{-1} \) is a bounded operator and there exists an analytic function \( z \to r_z = r_z(y, \zeta; h) \) such that \( (z - H_h)^{-1} = \text{Op}_h^n(r_z) \). The coefficient \( r_z \in S^0(1) \) for a fixed \( z \) satisfies
\[ |\partial^{\beta} r_z| \leq C_{\beta}|\omega|^{-1-|\beta|}\left(1 + h^{|\beta|}\right)^{-|\beta|} \]

for all multi-indices \( \beta = (\beta_y, \beta_z) \) and a constant \( C_{\beta} \) independent of \( z \in Z \) a compact set in \( \mathbb{C} \) and of \( 0 < h \leq 1 \).

**Proof.** We follow [21, Chapters 7 and 8] and [41] for the extension to matrix-valued operators. \( z - H_h = \text{Op}_h(z - a) \) with \( \omega \not\equiv 0 \) so that \( z - a \) invertible. The Beals’ criterion applied in [21, (8.10)] shows that \( (z - H_h)^{-1} = \text{Op}_h(r_z) \) for \( r_z \in S^0(1) \). This bound is uniform on any compact domain away from \( \omega = 0 \). [21, Proposition 8.6] shows that \( |\partial^{\beta} r| \) is bounded by the maximum of \( |\omega|^{-1-|\beta|} \) and \( \frac{1}{h^{2|\beta|}}|\omega|^{-2d-2|\beta|} \) where dimension \( d = 1 \) in our applications. Finally, \( r_z \) may be written as an appropriate transform of the Schwartz kernel of \( (z - H_h)^{-1} \) [21, Chapters 7], which is analytic since \( \partial(z - H_h)^{-1} = 0 \). This implies the analyticity of \( z \rightarrow r_z \) on \( Z \).

**Lemma 4.15.** Let \( \mathbb{R} \ni \xi \mapsto \gamma_1\phi'(\hat{H}_h(\xi)) \) be a family of operators with semiclassical symbol \( a(\xi, y, \zeta; h) \in S^0(1) \) for \( 0 < h \leq h_0 \) and \( (y, \zeta) \in \mathbb{R}^2 \) such that for any \( N \geq 1 \) (\( N \) large enough is sufficient), there is a constant \( C_N \) with

\[ |\partial^{\beta} a(\xi, y, \zeta; h)| \leq C_N((\xi, y, \zeta))^{-N}, \quad \beta = (\beta_y, \beta_z), \quad |\beta| \leq 3. \quad (25) \]

Then \( \gamma_1\phi'(\hat{H}_h(\xi)) \) is trace-class for each \( \xi \) and moreover,

\[ I := \int_{\mathbb{R}} \text{Tr}_\gamma \phi'(\hat{H}_h(\xi))d\xi = \frac{1}{2\pi h} \int_{\mathbb{R}^3} a(\xi, y, \zeta; h)dR, \quad dR := d\xi dyd\zeta. \quad (26) \]

Consider the domain \( R_h = [-R_h, R_h]^3 \) and \( R_h = h^{-\kappa}R \). We assume \( 0 < \kappa < 2(1 + \kappa^{-1}) \) so that \( \eta := \frac{h}{2}N - 1 > 0 \). We define the approximation

\[ I_{R_h} := \frac{1}{2\pi h} \int_{R_h} a(\xi, y, \zeta; h)dR. \quad (27) \]

Then \( |I - I_{R_h}| \leq Ch^{N}R^{1-\frac{\kappa}{2}} \) and \( R_h \) is a domain of volume \( CR^3h^{-\kappa} \).

**Proof.** That the operators \( \xi \mapsto \gamma_1\phi'(\hat{H}_h(\xi)) \) are trace-class and \( I \) above is defined as a Lebesgue integral is a consequence of (25) and [21, Theorem 9.4]. We verify that \( \eta > 0 \) for the above choice of \( N \).

Thanks to (25) with \( \beta = 0 \), the integral over \( \mathbb{R}^3 - R_h \) is bounded by a constant times

\[ \int_{R_h} h^{-1}r^{-N}r^2dr \leq CR^3h^{3-N}h^{-1} = Ch^{3(N-3)-1}R^{3-N} \leq Ch^{n}R^{-\frac{\kappa}{2}}. \]

This shows the convergence error. We also verify that \( R_h \) is a domain of volume \( 2^3R^3h^{-\kappa} \).

For \( \mathbb{C} \ni z = \lambda + i\omega \) with \( \omega \not\equiv 0 \), we define \( r_z \) such that

\[ (z - \hat{H}_h(\xi))^{-1} = \text{Op}_h(r_z) \quad (28) \]

with \( r_z = r_z(y, \zeta; h) \in S^0(1) \) a symbol in the variables \((y, \zeta)\) with \((z, \xi)\) as parameters. The bulk of the proof of the above theorem is to relate \( 2\pi \sigma_1 \) to the symbols \( \sigma_z \) and \( \sigma_x \) defined in (h2) as follows.
Proposition 4.16. Under hypotheses (h1)-(h2), we have

\[
2\pi\sigma_I = \frac{i}{8\pi^2} \lim_{R \to \infty} \int_{\mathbb{R} \times \partial R} \epsilon_{ijk} \text{tr}(\sigma^{-1}_x \partial_i \sigma^{-1}_z \partial_j \sigma_\nu_k) d\nu d\sigma,
\]

where the variables labeled 1, 2, 3 are identified with \(\xi, \zeta, \gamma\); \(\nu_k\) is the \(k\)th component of the outward unit normal to the boundary of the rectangle \(R = \prod_{i=1}^{3} (-R, R)\); and \(d\nu\) is the Euclidean measure on that boundary. Also, \(\epsilon_{ijk}\) is the totally antisymmetric tensor such that \(\epsilon_{111} = 1\).

By hypothesis (h2), the matrix \(\sigma^{-1}_x\) is indeed defined when \(R\) is sufficiently large. The above result shows that the conductivity, and hence all quantities appearing in (19) may be written in terms of the symbol \(-\sigma_x\) of \(H - \alpha - i\omega\).

**Proof.** Recall that \([H_h, P] = -i\phi'(x)\gamma_1\). By hypothesis (h1), traces may be computed as (Lebesgue) integrals along diagonals. Integrating in \(x\) and denoting by \(\text{Tr}_y\) the matrix trace and integration of the Schwartz kernel along the diagonal in the variable \(y\), we have (by Fubini)

\[
2\pi\sigma_I = 2\pi \text{Tr} \phi'(x)\gamma_1 \phi'(H_h) = 2\pi \int_{\partial} \text{Tr}_y \phi'(x)\gamma_1(\phi'(H_h))(x, x) dx.
\]

Since \(\phi'(H_h)\) is invariant by translation, then the Schwartz kernel satisfies \(\phi'(H_h)(x, x) = \phi'(H_h)(0, 0)\) and up to \(2\pi\) equals the integral over the dual variable \(\xi\) so that (dropping \(y\) in \(\text{Tr} \equiv \text{Tr}_y\) moving forward),

\[
2\pi\sigma_I = \text{Tr} \int_{\partial} \gamma_1 \phi'(\hat{H}_h(\xi)) d\xi = \int_{\partial} \text{Tr}_y \gamma_1 \phi'(\hat{H}_h(\xi)) d\xi.
\]

Note that \(\gamma_1 = \partial_\xi \hat{H}_h(\xi)\) and the above integrand is formally given by \(\partial_\xi \phi'(\hat{H}_h(\xi)) d\xi = d\phi(\hat{H}_h(\xi))\) so that \(2\pi\sigma_I\) depends only on the behavior of \(\hat{H}_h(\xi)\) for \(|\xi|\) large. However, \(\phi\), unlike \(\phi'\), is not compactly supported and the above expression is more suitable for the sequel.

We now use the Helffer–Sjöstrand formula and an almost analytic extension \(\hat{\phi}'(z)\) of \(\phi'(\lambda)\) to write \(\phi'(\hat{H}_h(\xi))\) as a superposition of resolvent operators of \(\hat{H}_h(\xi)\); see (47) and [20,21]

\[
2\pi\sigma_I = \text{Tr} -\frac{1}{\pi} \int_{\partial} \gamma_1 \partial \hat{\phi}'(z)(z - \hat{H}_h(\xi))^{-1} d^2 z d\xi = \frac{1}{\pi} \int_{\partial} \partial \hat{\phi}'(z) \gamma_1 \text{Op}_h(r_z) d^2 z d\xi.
\]

Here, \(d^2 z = d\lambda d\omega\) for \(z = \lambda + i\omega\) and \(\partial = \frac{1}{2} \partial_\lambda + \frac{i}{2} \partial_\omega\). The resolvent has a semiclassical symbol \(r_z \in S^0(1)\) defined in (28) with properties collected in Lemma 4.14.

We use hypothesis (h1) and Lemma 4.15 to write

\[
2\pi\sigma_I = \lim_{R \to \infty} \frac{-1}{2\pi^2 h} \int \partial \hat{\phi}'(z) \text{tr} \gamma_1 r_z(y, \xi, \zeta, h) h^{2} d^2 z dR.
\]

Here, tr is matrix trace, \(dR := d\xi d\zeta d\gamma\) on \(R_h = \prod_{i=1}^{3} (-R_h, R_h)\) with \(R_h = h^{-\frac{3}{2}} R\) for \(0 < \kappa\), and where we identify the variables \((1, 2, 3)\) with \((\xi, \zeta, \gamma)\). We then obtain that the above integral over \(R_h\), a domain of volume of order \(R^3 h^{-\kappa}\), gives the integral \(2\pi\sigma_I\) up to a negligible error of order \(R^{-\frac{7}{2}} h^n\) with \(\eta > 0\).
By construction of the almost analytic extension $\tilde{\phi}'(z)$, the above integration in $z$ is performed over a bounded domain $Z$. We recall that $|\partial^\beta_r z|$ is bounded by $C(|\omega|^{-3} + h^2|\omega|^{-6})$ uniformly in $h$ and $0 \leq |\beta| \leq 2$. We choose $\tilde{\phi}$ such that $|\partial^\phi(\phi')(z)| \leq C(|\omega|^N$ and $\delta(h) = h^2$. Then, for $|\omega| \leq \delta$, we find $h^{-1}|\partial^\phi(\phi')(z)r_z(y, z; \zeta; h)| \leq Ch^{-1}\delta^{-N-1}$ so that its integral over $R_h \times \{|\omega| < \delta\}$ is bounded by $C\delta^3 h^{-1-\kappa}\delta^{-N-4}$. We choose $N > 4 + \frac{1 + \kappa}{2}$ so that $C\delta^3 h^{-1-\kappa}\delta^{-N-4} = o(1)$ as $h \to 0$.

Let us call $Z_\delta = Z \cap \{|\omega| \geq \delta = h^2\}$. We wish to estimate

$$I_{\delta, R_h} = \frac{-1}{2\pi^2 h^2} \int_{Z_\delta \times R_h} \partial^\phi_{\phi'}(z) \text{tr}_{r_z}(y, z; \zeta; h) d^2z dR,$$

(31)

since the above considerations and Lemma 4.15 show that $2\pi \sigma_I - I_{\delta, R_h} = o(1)$.

After the above truncation $|\omega| \geq \delta$, we observe that $h^{3+i} \partial^\phi_r z \in S^0(1)$ uniformly in $z \in Z_\delta, (\zeta, \zeta, y) \in R_h$, and $|\beta| \leq 2$ and that the integral of $\partial^\phi_r z$ over $Z_\delta \times R_h$ for $|\beta| \leq 2$ is thus bounded by $h^{-\kappa-3\gamma}$. In what follows, we assume that $\kappa + 3\gamma < 1$ to control asymptotic expansions in $h$ integrated over $R_h$. We collect the above definitions and assumptions for $\gamma, \kappa > 0$ as

$$R_h = (-R_h, R_h)^3, \quad R_h = h^{-4k}R, \quad \delta(h) = h^2, \quad Z_\delta = Z \cap \{|\omega| \geq \delta\}, \quad 3\gamma + \kappa < 1.$$

The function $z \to r_z$ is analytic away from $\omega = 0$ so that $\partial_z r + i\partial_\omega r = 0$. Moreover $r_z \to 0$ as $|\omega| \to \infty$ since $\|(z - H_h(\zeta))^{-1}\| \leq |\omega|^{-\gamma}$. We can therefore write for $\omega > 0$

$$r(z) = -i \int_0^\infty \partial_\omega r_{z+\alpha} dt = i \int_0^\infty \partial_z r_{z+\alpha} dt.$$

(32)

The domain $\omega < 0$ is treated similarly by observing that $r(z) = {\tilde{r}(z)}$. We now focus on $\partial_z r_z(y, z; \zeta; h)$ for $|\omega| \geq \delta$. Since $h^{3+i} \partial^\phi_{r_z+\alpha} \in S^0(1)$ uniformly for $t \geq 0$ for $|\beta| \leq 2$, this allows us to obtain asymptotic expansions in $h$ that hold uniformly in $|\omega| \geq \delta$. We then perform an integration in $(\zeta, y, \zeta, \zeta)$ to display the topological properties of the resolvent and obtain a function that no longer has any singularity at $\omega = 0$ thanks to the assumption that $\sigma_z$ is invertible outside of a large sphere in $(\omega, y, \zeta, \zeta)$.

By analyticity, $\partial_z r_z = \partial_z r_z$. We wish to perform an expansion of the integral in $(\zeta, y, \zeta)$ of $\gamma_1 \partial_z r_z$. At the operator level, we find $\partial_z (z - H_h)^{-1} = -(z - H_h)^{-2}$, which at the level of symbols translates into $\partial_z r_z = -r_z^{\#} h r_z$. We also deduce from $\gamma_1 = \partial_z H_h(z)$ that $\partial_\zeta (z - H_h)^{-1} = (z - H_h)^{-1}\partial_\zeta H_h(z - H_h)^{-1} = (z - H_h)^{-1}\gamma_1 (z - H_h)^{-1}$ that at the level of symbols, $\partial_\zeta r_z = r_z^{\#} h \gamma_1 r_z - \partial_\zeta r_z$. We thus find

$$\gamma_1 \partial_z r_z = -\gamma_1 r_z^{\#} h r_z + r_z^{\#} h \gamma_1 r_z - \partial_\zeta r_z.$$

We are therefore interested in the quantity for $\omega > 0$

$$\Phi(R_h) := \frac{1}{h} \int_{R_h} \text{tr}_{\gamma_1} \partial_z r_z dR = \frac{1}{h} \int_{R_h} \text{tr}[-\gamma_1 r_z^{\#} h r_z + r_z^{\#} h \gamma_1 r_z - \partial_\zeta r_z]dR.$$

We want to show that the above trace is asymptotically in divergence form and thus an integral over $\partial R$. For two symbols $a$ and $b$ in $S^0(1)$, we find [21, 45]

$$a h b = ab + h \frac{1}{2} \{a, b\} + O(h^2)$$

with $\{a, b\} = \partial_z a \partial_\zeta b - \partial_\zeta a \partial_z b$. By cyclicity of the trace, $\text{tr}ab = \text{tr}ba$ and $\text{tr}\{a, b\} = -\text{tr}\{b, a\}$ so that $\text{tr}a h b - \text{tr}b h a = -i h \text{tr}\{a, b\} + O(h^2)$. Here and below, all terms
O(h^k) are so in the sense of \( S^0(1) \) symbols [21]. This implies, since \( h^3y_0 y_\beta y\tau \in S^0(1) \) for \( 0 \leq |\beta| \leq 2 \), that expansions up to second order provide the estimate

\[
\text{tr}(-\gamma_1 r_z \frac{\partial}{\partial h} r_z + r_z \frac{\partial}{\partial h} \gamma_1 r_z) = ih \text{tr} \{ \gamma_1 r_z, r_z \} + O(h^{2-3\gamma}).
\]

The integral \( I_{\delta, R_h} \) involves a multiplicative factor \( h^{-1} \) and a volume \( Ch^{-\kappa} \) with \( 3\gamma + \kappa < 1 \) so that only \( \text{itr} \{ \gamma_1 r_z, r_z \} \) survives in the limit \( h \to 0 \). It thus remains to integrate \( \text{tr}[ih \{ \gamma_1 r_z, r_z \} - \partial_n r_z] \) over \( R_h \).

We now verify that \( 2 \{ a, b \} = \partial_n (a \partial_n b - \partial_n ab) - \partial_n (a \partial_n b - \partial_n ab) \) in divergence form in these variables. We thus find that

\[
\text{tr} \{ \gamma_1 r_z, r_z \} = \partial_n \tilde{a}_n - \partial_n \tilde{a}_\pi, \quad \tilde{a}_n = \frac{1}{2} [\gamma_1 r_z, \partial_n r_z], \quad \pi = y, \zeta.
\]

This means that the integral of \( \text{tr}[ih \{ \gamma_1 r_z, r_z \} - \partial_n r_z] \) over \( R_h \) may be written as

\[
\Phi_{R_h}(z) = \frac{1}{h} \text{tr} \int_{\partial R_h} [ih (\tilde{a}_n \nu_n - \tilde{a}_\pi \nu_\pi) + r_z \nu_\pi] d\Sigma + O(h^{1-3\gamma-\kappa}),
\]

with \( \nu_\tau \) the \( \tau \)th component of the normal unit vector to \( R_h \) at the boundary and \( d\Sigma \) the surface measure there. Provided that \( R \) is large enough, \( \sigma^{-1}_z \) is defined by assumption (h2) for all \( \lambda \) sufficiently close to \( \tau \) by continuity. That we may only consider values of \( \lambda \) close to \( \tau \) in \( Z \) is ensured by possibly shrinking the support of \( \phi' \) and replacing it by \( \phi_\rho' (\lambda) = \frac{1}{\rho} \phi' (\frac{1}{\rho} (\lambda - \tau) + \tau) \). We know from Proposition 4.4 that \( \sigma_1 \) is independent of \( 0 < \rho \leq 1 \).

The inverse \( \sigma^{-1}_z \) now exists uniformly in \( \omega \), even when \( |\omega| \leq \delta \), still by assumption. This allows us to approximate \( r_z \) using \( \text{Op}_h^w (r_z) \text{Op}_h^w (\sigma_z) = I \) so that \( I = r_z \frac{\partial}{\partial h} \sigma_z \). By semi-classical calculus of Weyl operators this implies that

\[
I = r_z \sigma_z + \frac{h}{2i} \{ r_z, \sigma_z \} + O(h^{2-3\gamma}).
\]

From this, we deduce that in the \( S^0(1) \) sense

\[
r_z = \sigma^{-1}_z + \frac{ih}{2} \{ r_z, \sigma_z \} \sigma^{-1}_z + O(h^{2-3\gamma}) = \sigma^{-1}_z + \frac{ih}{2} \{ \sigma^{-1}_z, \sigma_z \} \sigma^{-1}_z + O(h^{2-3\gamma}). \tag{33}
\]

To leading order, we may therefore replace \( r_z \) by \( \sigma^{-1}_z \) in the terms \( \tilde{a}_n \) above, defining the corresponding terms \( a_\pi \) and obtain

\[
\Phi_{R_h}(z) = \text{tr} \int_{\partial R_h} \left[ i (a_n \nu_n - a_\pi \nu_\pi) + \frac{i}{2} \{ \sigma^{-1}_z, \sigma_z \} \sigma^{-1}_z \nu_\pi + \frac{1}{h} \sigma^{-1}_z \nu_\pi \right] d\Sigma + O(h^{1-3\gamma-\kappa}).
\]

We now write, using that \( \gamma_1 = -\partial_\pi \sigma_z \) is a constant matrix,

\[
2 \text{tr} \{ \sigma^{-1}_z, \gamma_1 \sigma^{-1}_z \} = \text{tr} \left[ \partial_\pi \left( \left( \frac{\partial}{\partial \pi} \sigma^{-1}_z, \partial_\pi \sigma_z \right) \sigma^{-1}_z \right) \right] - \text{tr} \left[ \partial_\pi \left( \partial_\pi \sigma^{-1}_z, \partial_\pi \sigma_z \sigma^{-1}_z \right) \right].
\]

The last term equals \( \partial_\pi (\partial_\pi \sigma^{-1}_z, \partial_\pi \sigma_z \sigma^{-1}_z) \). We also use the identity \( \partial_\pi \sigma^{-1}_z = -\sigma^{-1}_z \partial_\pi \sigma_z \sigma^{-1}_z \) to find eventually that
\[ \Phi_{R_\alpha}(z) = \Phi_{R_\alpha,0}(z) + O(h^{1-\kappa}), \quad \Phi_{R_\alpha,0}(z) = \frac{i}{2} \text{tr} \left[ \frac{1}{h} \sigma_{z+\theta} \nu_z d\Omega \right] \]

We thus obtain, with an integration in \( t \) over \( R_+ \) and over the interval \([0,R]\) that
\[ \Psi_{R_\alpha}(z) := \frac{1}{h} \text{tr} \left[ \psi_{j} r_{1} d\Omega = \Psi_{R_\alpha,1}(z) + \Psi_{R_\alpha,0}(z), \quad \Psi_{R_\alpha,0}(z) := -i \int_{0}^{\infty} \Phi_{R_\alpha,j}(z + it) dt \]

with here \( o(1) \) negligible as \( R \to \infty \) and \( h \to 0 \). We have thus shown that
\[ I_{\delta,R_\alpha} = -\frac{1}{2\pi^2} \int_{Z_\delta} \partial \phi'(z) \Psi_{R_\alpha,0}(z) dz + o(1) + \frac{1}{2\pi^2} \int_{Z_\delta} \partial \phi'(z) \Psi_{R_\alpha,1}(z) dz \]

The term \( I_{\delta,R_\alpha} \) is close to an integer as \( h \to 0 \) and \( R \to \infty \) since \( 2\pi \sigma_{j} \) is known to be an integer independent of \( h \). We will show below that the first term on the above right-hand side (let us call it \( I_{0,h} \)) is also close to an integer (all we need is that it converges to a real number independent of the choice of \((\gamma, \kappa)\)). Anticipating this result, let us call \( I_{\delta,R_\alpha} \) the term formally proportional to \( h^{-1} \). It must therefore also be close to an integer \( n \) as \( h \to 0 \). We now show that \( n = 0 \). We observe that \( I_{\delta,R_\alpha} = n + o(1) \) is \( h^{-1} \) times the integral of an \( h \)-independent integrand over the domain \( X_{h} := Z_\delta \times R_\alpha \times [0,\infty) \).

Let us now consider two different values \( 0 < \kappa_1 \neq \kappa_2 \) and \( h_{1}^{\kappa_1} = h_{2}^{\kappa_2} \) so that \( R_{h_1} = R_{h_2} = h^{-\kappa_1/\kappa_2} R \). We also choose \( \gamma_{1,2} \) such that \( h_{1}^{\gamma_1} = h_{2}^{\gamma_2} \) (i.e., \( \kappa_2 \gamma_1 = \kappa_1 \gamma_2 \)) so that \( \delta_1 = \delta_2 = h_{1}^{\gamma_1-j} \). We assume that \( \kappa_j + 3\gamma_j < 1 \) for \( j = 1, 2 \), which is possible. Applying the above derivation to these choices of \( \delta(h) \), we obtain that \( I_{\delta,R_\alpha} = n + o(1) \) with \( n \) independent of \( j \). Since \( X_{h_1} = X_{h_2} \), the above integral is independent of \( j = 1, 2 \). Since \( h^{-1} \) times that integral needs to be close to \( j \)-independent integer as \( h \to 0 \) while \( h_{1} = h_{2}^{\kappa_2/\kappa_1} \) for \( \kappa_2 \neq \kappa_1 \), the latter integer has to vanish.

We thus obtained that \( \Psi_{R_\alpha}(z) = \Psi_{R_\alpha,0}(z) + o(1) \) and proceed to compute that contribution. Since \( \Psi_{R_\alpha}(z) \) is a smooth function in \( z \) close to \( x \) even when \( \omega \) is small, then
\[ I_{\delta,R_\alpha} = -\frac{1}{2\pi^2} \int_{Z} \partial \phi'(z) \Psi_{R_\alpha}(z) dz + o(1) \]

Since all terms involved (from the beginning) are analytic in \( z \), then is \( \Psi_{R_\alpha}(z) \) away from the real axis \( \omega = 0 \). Let \( Z = \overline{Z} \cup \overline{Z} \), where \( Z_{\pm} = Z \cap \{ \pm \omega > 0 \} \). Using \( \partial \phi'(z) \Psi_{R_\alpha}(z) = \partial(\phi'(z) \Psi_{R_\alpha}(z)) \), we have by the Stokes formula
\[ \int_{Z_{\pm}} \partial \phi'(z) \Psi_{R_\alpha}(z) d^2z = -\frac{i}{2} \int_{\partial Z_{\pm}} \phi'(z) \Psi_{R_\alpha}(z) dz, \]

where \( dz = \pm d\lambda \) on the part of \( \partial Z_{\pm} \) on the real axis, \( \phi'(z) = \phi'(\lambda) \) while \( \Psi_{R_\alpha}(z) \) is defined as the limit from \( \omega > 0 \) while \( \Psi_{R_\alpha}(z) = \Psi_{R_\alpha}(z) \) for \( \omega < 0 \). Overall, we therefore find
\[
\sigma(1) + I_R = \frac{i}{4\pi^2} \int_{\mathbb{R}} \varphi'(\lambda) \left[ \Psi_{R_0}(\lambda + i0) - \Psi_{R_0}(\lambda - i0) \right] d\lambda.
\]

Each of the \( \Psi_{R_0} \) terms is an integral over a half line in \( \omega \) while the difference involves a line integral in \( \omega \).

As mentioned earlier, this term is independent of \( \varphi'_{\rho}(\lambda) = \frac{1}{\rho} \varphi'_{\rho}(\frac{1}{\rho} (\lambda - x) + x) \). As \( \rho \) to \( 0 \), and using the smoothness of \( \lambda \to \Psi_{R_0}(\lambda + i0) \), we evaluate the above at \( \lambda = x \) with \( \int_\mathbb{R} \varphi'_{\rho}(\lambda)d\lambda = 1 \) to deduce in the limit \( R \to \infty \) that \( l_{0,h} \) converge as \( h \to 0 \), and hence the equality of integers:

\[
2\pi \sigma_I = \lim_{R \to \infty} \frac{i}{8\pi^2} \int_{\partial\mathbb{R} \times \mathbb{R}} \text{tr} \left[ \epsilon_{ijk} \sigma^{-1}_x \partial_i \sigma_x \sigma^{-1}_x \partial_j \sigma_x \sigma^{-1}_x \nu_k \right] d\Sigma d\omega.
\]

This concludes the proof of the proposition.

We now deduce several corollaries from the preceding result. The most natural result is the following relation between the interface conductivity and the Fedosov-Hörmander formula for the index of the operator \( i\omega - H \).

**Corollary 4.17** (Spectral flow Index). Let \( \Sigma \) be a surface in the variables \((\xi, \zeta, y, \omega)\) such that \( \sigma^{-1}_x \) is defined on and outside of the surface. Then

\[
2\pi \sigma_I = \frac{1}{24\pi^2} \int_{\Sigma} \text{tr}(\sigma^{-1}_x d\sigma_x)^3.
\]

(34)

Here and below, the orientation of \( \mathbb{R}^4 \) is \( d\xi \wedge d\zeta \wedge dy \wedge d\omega = d\omega \wedge d\xi \wedge dy \wedge d\zeta > 0 \).

**Corollary 4.18** (Bulk Interface Correspondence). We find that

\[
2\pi \sigma_I = \lim_{y \to \infty} \frac{i}{8\pi^2} \int_{\mathbb{R}^3} \text{tr} \left[ \sigma^{-1}_x \partial_i \sigma_x, \sigma^{-1}_x \partial_j \sigma_x, \sigma^{-1}_x \partial_k \sigma_x \right] \sigma^{-1}_x \nu_{ij} d\xi d\zeta d\omega = -W_x = \sum_{\xi < x} c_i.
\]

(35)

This is the bulk-interface correspondence, stating that the interface conductivity equals the bulk-difference invariant. We use the notation \( X|_{-y}^{y} = X(y) - X(-y) \). We also have the relation to the spectral asymmetry:

**Corollary 4.19** (Spectral Asymmetry). Let us assume that the 3-form \( \text{tr}(\sigma^{-1}_x d\sigma_x)^3 \) integrated over bounded domain in \((\omega, \xi, \zeta)\) at fixed values of \( y \) converges to 0 as \( |y| \to \infty \). Then we find that:

\[
2\pi \sigma_I = \lim_{\xi \to \infty} \frac{i}{8\pi^2} \int_{\mathbb{R}^3} \text{tr} \left[ \sigma^{-1}_x \partial_i \sigma_x, \sigma^{-1}_x \partial_j \sigma_x, \sigma^{-1}_x \partial_k \sigma_x \right] \sigma^{-1}_x \nu_{ij} d\xi dy d\zeta d\omega.
\]

(36)

This is the three-dimensional winding number of the Green’s function used in [17, 35].

**Proof.** We first note that \( i = \partial_x \sigma_x \) with \( \sigma_x = \sigma_x(\omega, y, \xi, \zeta) \). The result of Proposition 4.16 states that \( \sigma_I \) is up to a constant the limit as \( R \to \infty \) of the integral of \( \text{tr}(\sigma^{-1}_x d\sigma_x)^3 \) over the cylinder \( \mathbb{R} \times \mathbb{R} \). A direct calculation (as in the proof of the index theorem in [18, Chapter 19.3]) shows that

\[
\text{tr}(\sigma^{-1}_x d\sigma_x)^4 = 0,
\]

which implies that \( \text{tr}(\sigma^{-1}_x d\sigma_x)^3 \) is a closed form since \( d \text{tr}(\sigma^{-1}_x d\sigma_x)^3 = -3 \text{tr}(\sigma^{-1}_x d\sigma_x)^4 = 0 \).
Note that the latter form equals 3 times the form written using commutators as in the proposition; this explains the factor $24 = 8 \times 3$ in the first corollary. For $M_i$ sufficiently large, let $\Omega$ be the volume with boundary $\Sigma \cup \partial R_4$, where $R_4 = \prod_{i=1}^{4}(-M_i,M_i)$ this time also including the fourth variable identified with $\omega$. Then the Stokes theorem implies that the integral of $\text{tr}(\sigma_x^{-1}d\sigma_x)^3$ over $\Sigma$ and that over $\partial R_4$ (with the same orientation) are equal.

To prove Corollary 4.18, we need to approximate the integral over the cylinder $\mathbb{R} \times \partial R_3$ where $R_3 = \prod_{i=1}^{3}(-M_i,M_i)$ by an integral over the boundary of a four-dimensional rectangle $R_4 = \prod_{i=1}^{4}(-M_i,M_i)$. This means that, as $M_4 = M_\omega \to \infty$, we want the integral of $(\sigma_x^{-1}d\sigma_x)^3$ over the rectangle $R_3$ for a fixed value $\omega_0$ of $\omega$ to be small as $\omega_0 \to \infty$. From (h2), $\sigma_x^{-1}d\sigma_x$ is bounded (uniformly as a matrix-valued one form) and $\partial_\xi \sigma_x = -\gamma_1$ bounded while $\sigma_x^{-1}$ goes to 0 (uniformly in $(y,\xi,\zeta)$ as a $N \times N$ matrix) as $\omega_0 \to \infty$. Passing to the limit in $R \to \infty$ gives the spectral flow index in Corollary 4.18. Note that the orientation used in the computation of $W_x$ in (17) is $0 < \omega d\sigma_x \wedge d\zeta \wedge dy = -\omega d\sigma_x \wedge d\zeta \wedge dy \wedge d\zeta$ and hence the sign difference in (35).

Note that the above derivation shows that the limit as $R \to \infty$ is replaced by the integral of the trace of the matrix-valued three-form over any closed (three-dimensional) surface $\Sigma$.

The other corollaries are obtained similarly by appropriately deforming $\Sigma$ to two surfaces since $\text{tr}(\sigma_x^{-1}d\sigma_x)^3$ is a closed form. To prove Corollary 4.19, which aims to compute the invariant by integrals at fixed values of $\xi$, we need to ensure that the integral of $(\sigma_x^{-1}d\sigma_x)^3$ over any (lateral) rectangle (involving an integration in $\xi$) goes to 0 as $(\omega,y,\xi) \to \infty$. To prove Corollary 4.17, which aims to compute the invariant by integrals at fixed values of $\gamma$, we need to ensure that the integral of $(\sigma_x^{-1}d\sigma_x)^3$ over any (lateral) rectangle (involving an integration in $\gamma$) goes to 0 as $(\omega,\xi,\zeta) \to \infty$.

We already know that the integrals over (lateral) rectangles with constant values of $\omega$ converge to 0 as $|\omega| \to \infty$. The same argument shows the same result for integrals over (lateral) rectangles at constant values of $|\xi| \to \infty$. By hypothesis (h2) on the behavior of $\sigma_x$ in $\zeta$, we also deduce the result for integrals over (lateral) rectangles at constant values of $|\zeta| \to \infty$. This proves Corollary 4.18. To prove Corollary 4.19, it therefore remains to obtain that the integral over a fixed rectangle in the variables $(\omega,\xi,\zeta)$ goes to 0 as $|\gamma| \to \infty$. This is in fact incorrect in general and this is why it was added as an assumption in the corollary.

The results (34) and (35) conclude the proof of Theorem 4.13. It remains to address the equality involving the index of the Fredholm operator $F$. Under conditions (H1') or (H2'), we proved in proposition 4.12 that $F$ was indeed Fredholm. That its index is given by the formula in the theorem can be found or proved as in [18, Theorem 19.3.1]. This concludes the proof of Theorem 4.13.

Proposition 4.20. For the systems (1) and (2), the assumptions in Corollary 4.19 hold when $m(y)$ and $f(y)$, respectively, converge to $\infty$ at $\infty$.

Proof. In both cases, at fixed values of $y$ the integral of $\text{tr}[\sigma_x^{-1}\partial_y \sigma_x, \sigma_x^{-1}\partial_y \sigma_x, \sigma_x^{-1}]$ over a bounded domain at fixed values of $y$ involves an integrand that goes to 0 when $|\mu(y)|$ converges to $\infty$. 

\qed
The above result states that while Corollaries 4.17 and 4.18 always hold, the regularized spectral asymmetry in [16, 35] holds only when the range of the domain wall is the whole of \( \mathbb{R} \) (in the topologically non-trivial case). A calculation for system (1) shows that (36) does not hold when \( m(y) \) takes values between \(-m_0 < 0 \) and \( m_0, \) say. Indeed, the bulk-difference invariant \( c_+ \) is given by [8]

\[
\frac{2}{4\pi} \int_{\mathbb{R}^2} \frac{m}{(\xi^2 + \zeta^2 + m^2)^{3/2}} d\xi d\zeta = \text{sign}(m),
\]

whereas the spectral asymmetry ‘invariant’ (36) would be given by

\[
-\frac{2}{4\pi} \int_{\mathbb{R}^2} \frac{m'(y)\zeta}{(\xi^2 + \zeta^2 + m^2)^{3/2}} d\xi dy = -\frac{2}{4\pi} \int_{\mathbb{R}^2} \frac{\text{sign}(m_0)\zeta}{(\xi^2 + \zeta^2 + m^2)^{3/2}} d\zeta dm.
\]

Calculated for \( \xi > 0 \) large enough, it is equals to \( c_+ \) only in the limit \( m_0 \to +\infty \) and in fact converges to 0 as \( \xi \to \infty. \)

For (2), we thus find that for smooth and sufficiently slowly varying profiles of the Coriolis force \( f(y), \) the bulk-interface correspondence (35) always holds. This was one of the main motivations for the theory presented in this section.

5. Conclusions

This article proposes two methods to compute the topologically quantized interface conductivity \( 2\pi \sigma_I \) of Hamiltonians of the form \( H[\mu(y)] + V \) where \( \mu(y) \in \mathfrak{S}[\mu_-, \mu_+] \) implements a domain wall and \( V \) models a large class of perturbations.

The first method presented in section 2 is a direct application of the heuristics of spectral flows. It requires a diagonalization of the domain wall problem and shows that the conductivity is given by the sum of the winding numbers of the branches of absolutely continuous spectrum that cross a spectral region of interest. The mathematical backbone of the derivation borrows from the machinery of non-commutative geometry to construct a Fredholm operator of the form \( PU(H)P \) whose index is given by the formula \( \text{Tr}[U, P]U^* \) and as a topological object is immune to large classes of perturbations \( V. \) This immunity is the main reason for the practical interest of such theories and gives a quantitative meaning to the perceived topological protection of transport phenomena observed numerically or experimentally in several areas of materials science and geophysical fluid flows.

The main drawback of the method is that it requires in practice a sufficiently explicit expression of the spectral decomposition of \( H[\mu(y)] \). Its main advantage is that it applies even when \( \mu(y) \) displays discontinuities. It allows one to justify the surprising result for the system (2) that the interface conductivity may take different quantized values for different profiles in the same topological class of domain walls \( \mathfrak{S}[\mu_-, \mu_+] \).

The second method is based on computing a bulk invariant involving only the constant-coefficient Hamiltonians \( H[\mu_-] \) and relating it to the interface conductivity \( 2\pi \sigma_I \). We introduced a notion of bulk-difference invariant in section 3 that appropriately combines the projectors associated to both bulk Hamiltonians \( H[\mu_-] \) and is computed either by means of Chern numbers on the sphere or three-dimensional winding numbers of associated Green’s functions. These invariants are typically easier to compute.
than the ones introduced in section 2; see [36] for an application of Floquet topological insulators, where the computations of these bulk invariants is possible while the spectral decomposition of the domain walls seems intractable.

Once these bulk invariants are computed, they are related to the interface conductivity by a general principle called here a bulk-interface correspondence. While the latter does not always hold as we just mentioned, we show in section 4 that it applies to a large class of differential operators when their coefficients are sufficiently smooth, and for the system (2) sufficiently slowly varying.

The correspondence is proved by relating both indices to a third one given as the index of a Fredholm differential operator constructed from $H[\mu(y)]$ by adding a second domain wall $i\nu(x)$ (and hence enforcing localization) in the direction of propagation along the interface. We first introduced some sufficient (pseudo-)differential conditions ensuring that the conductivity was quantized. We then related the conductivity to an integral involving only the symbol of the Hamiltonian $H[\mu(y)]$ and identified this integral with the Fedosov–Hörmander formula for the index of the differential operator.

Our class of operators was restricted to the form (22) that includes generalizations of models such as (1) and (2) or the Floquet model in [36]. The method extends to a larger class of topologically nontrivial differential operators such as those appearing in superconductor theory [17]; see [39]. For extensions to larger dimensions, see [40].

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A. Index of interface Hamiltonian

The proof of Theorem 2.1 is similar to those derived in [8] to compute the index of interface Hamiltonians. We highlight the main differences. It is shown in [8] that under the assumptions on the spectral decomposition of $U(H)$, then $PU(H)P_{\text{ran}P}$ is a Fredholm operator and

$$-\text{Index}(PUPU) = \text{Tr}[P, U^*U] = \text{Tr}[P, W](I + W^*) = \text{Tr}[P, W]W^* + \text{Tr}[P, W]$$

since $[P, W]$ is trace-class (see also the proof of Lemma 4.1). Moreover, as in [8, 46], the above traces and given by the integrals along the diagonals of their Schwartz kernels. The trace of $[P, W]$ is seen to vanish since $(p(x) - p(y))|_{y=x} = 0$. The Schwartz kernel of $W$ is given by

$$w(x - x', y, y') = \int_{\mathbb{R}} \sum_j W(E_j(\xi))\psi_j(y, \xi)\psi^*_j(y', \xi)\frac{\delta(x - x')\delta(y - y')}{{2\pi}}d\xi.$$

The kernel of $W^*$ is given by $w^*(x' - x; y', y)$ with $w^*_{ij} = \bar{w}_{ji}$. The kernel of $[P, W]W^*$ is thus given by

$$t(x, x'; y, y') = \int_{\mathbb{R}^{d+1}} (\chi(x) - \chi(x'))w(x - x', y, y')w^*(x' - x, y, y')dx'dx,$$

where $P$ corresponds to multiplication by $\chi(x)$. Therefore, $T := \text{Tr}[P, W]W^*$ is computed by $T = \int_{\mathbb{R}^{d+1}} t(x, x; y, y')dx'dy$. The change of variables $(x, x') \rightarrow (z, x'') = (x - x', x'')$ with $dxdx'' = dzdx''$, and computing $\int_{\mathbb{R}} (\chi(x'') + z) - \chi(x''))dz = z$, we obtain

$$T = \text{tr} \int_{\mathbb{R}^{d+1}} zw(z; y, y')w^*(z; y, y')dzdy'dy'.$$

Using the Fourier transform from $z$ to $\xi$ yields by Parseval

$$T = \frac{-\text{tr}}{2\pi i} \int_{\mathbb{R}^{d+1}} \partial_{\xi}\hat{w}(\xi; y, y')\hat{w}^*(\xi; y, y')d\xi dy'dy'.$$
where \( \hat{w}(\xi; \cdot) \) is the component-wise Fourier transform of \( w(x; \cdot) \) given by

\[
\hat{w}(\xi; y, y') = \sum_j W(E_j(\xi)) \psi_j(y, \xi) \psi_j^*(y', \xi).
\]

The derivative \( \partial_\xi \) applies to \( W \circ E_j \) and to \( \psi_j(y, \xi) \psi_j^*(y', \xi) \). At a fixed \( \xi \), consider the latter contribution, which is given by

\[
\tau(\xi) := \int \sum_{j,k} \text{tr} \partial_\xi \left[ \psi_j(y, \xi) \psi_j^*(y', \xi) \right] \psi_k(y', \xi) \psi_k^*(y, \xi) dy dy'.
\]

We show that \( \tau(\xi) = 0 \). Indeed, we distribute \( \partial_\xi \) over the product, exchange \( y \) and \( y' \) in the second contribution to get (dropping the \( \xi \)-dependence to simplify notation)

\[
\text{tr} \int \sum_{j,k} \left[ \partial_\xi \psi_j(y) \psi_k(y) \psi_j^*(y') \psi_k^*(y) + \partial_\xi \psi_j(y) \psi_k(y) \psi_j^*(y') \psi_k^*(y') \right] dy dy'.
\]

Applying traces to these products of rank-one matrices yields (with \( \psi \) as a column vector)

\[
\tau(\xi) = \int \sum_{j,k} \left[ \partial_\xi \psi_j(y) \cdot \psi_k(y) \psi_j^*(y') + \partial_\xi \psi_j(y) \cdot \psi_k(y) \psi_j^*(y') + \partial_\xi \psi_j(y) \cdot \psi_k(y) \psi_j^*(y') \right] dy dy'.
\]

By orthogonality of the eigenvectors, only the terms \( j = k \) survive the integration and then \( \tau(\xi) = \int \sum_j |\partial_\xi |\psi_j(y)|^2 dy = \int \sum_j |\partial_\xi |\psi_j(y)|^2 dy = 0 \). As a consequence,

\[
T = -\frac{\text{tr}}{2\pi i} \sum_{j,k} \int \partial_\xi W \circ E_j(\xi) \psi_j(y, \xi) \psi_j^*(y', \xi) W^* \circ E_j(\xi) \psi_k(y', \xi) \psi_k^*(y, \xi) dy dy' d\xi.
\]

Taking traces again and integrating in \( y \) and \( y' \) yields

\[
T = -\frac{1}{2\pi i} \sum_j \int \partial_\xi W \circ E_j(\xi) W^* \circ E_j(\xi) d\xi = -\sum_j W_1(W \circ E_j).
\]

Here, \( W_1(f) \) is the winding number of a (compactly supported) function \( f \). Since the integral of \( W' \) vanishes, the winding number of \( W \circ E_j \) and that of \( U \circ E_j \) are the same. This concludes the proof of Theorem 2.1.

### B. Calculations of conductivities in systems (1) and (2)

We show that the corresponding operators are unbounded self-adjoint operators on \( L^2(\mathbb{R}^2) \otimes \mathbb{C}^n \) and compute the interface conductivity in some tractable cases.

**The 2 × 2 system (1).** We briefly recall and slightly extend results computed in [8] for this system so that they can be contrasted with the more challenging 3 × 3 system. The domain of definition of \( H \) making it self-adjoint is given by \( \mathcal{D}(H) := (H + \epsilon i)^{-1} \mathcal{H} \) with \( \mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2 \). Here \( \epsilon = \pm 1 \). Following [21, Chapters 4 and 8], we verify that the above domain is independent of \( \epsilon \). Indeed the source problem, with \( \epsilon = 1 \), may be recast as

\[
(m + i) \psi_1 + (D_x - iD_y) \psi_2 = s_1, \quad (D_x + iD_y) \psi_3 + (i - m) \psi_2 = s_2
\]

which after elimination of one variable yields, for instance,

\[
((m + i) + (D_x - iD_y) \frac{-1}{i - m} (D_x + iD_y)) \psi_1 = s_1 - (D_x - iD_y) \frac{1}{i - m} s_2.
\]

This is an elliptic problem that implicitly defines a \( m(y) \)-dependent domain of definition. The same domain of definition applies to \( \psi_2 \).
Now, for any bounded (multiplication or not) operator $V$, the problem $(H + V + ei)\psi = 0$ implies that $\psi = 0$ since $((H + V)\psi, \psi)$ is real and $(ei\psi, \psi)$ is purely imaginary. This shows that for such perturbations, $H_V = H + V$ is self-adjoint and the spectral calculus applies.

Spectral decomposition of system (1). Let us first consider $m(y)$ either of the form $m(y) = \lambda y$ or $m(y)$ a switch function in $\mathcal{S}[m_-, m_+]$ with $m_-m_+ < 0$. Detailed calculations are in [8,9]. We summarize the main relevant results.

Let us consider $\hat{H}_h(\xi)$, the partial Fourier transform from $x$ to $\xi$, which we represent [8,9] as $D_x \sigma_1 + m(y) \sigma_2 + \xi \sigma_3$ after a global unitary transform permuting $\sigma_{1,2,3}$ to $\sigma_{3,1,2}$. In the case considered in (i), $\hat{H}_h(\xi)$ has compact resolvent, while in case (ii), its restriction to the interval $(-m_0, m_0)$ with $m_0 = \min(\{|m_-|, |m_+|\})$, is also compact. As $\xi$ varies, this provides branches of absolutely continuous spectrum for the operator $H[m(y)]$.

In either case, we observe that solving $(\hat{H}_h(\xi) - E)\psi = 0$, that is, finding the edge states, amounts to solving

$$(\xi - E)\psi_1 - i a \psi_2 = 0, \quad i a^* \psi_1 - (\xi + E)\psi_2 = 0, \quad a = \partial_y + m(y), \quad a^* = -\partial_y + m(y),$$

which leads to trivial branches of spectrum corresponding to the shared (strictly) positive eigenvalues of $a a^*$ and $a^* a$, and one special branch of spectrum corresponding to the nontrivial kernel of $a$ when $m_- < 0 < m_+$ (the case we now consider). The branch is given by $a \psi_2 = 0$, $\psi_1 = 0$, and $E(\xi) = -\xi$, with group velocity $E'(\xi) = -1$ corresponding to a mode moving toward negative values along the $x$ axis. In other words, the operator $H[m(y)]$ admits the branch of continuous spectrum (parametrized by $E$) $e^{-M(y)} e^{-iEx} (0, 0, 1)$ at least for values $|E| < m_0$ in case (ii), where $M$ is the antiderivative of $m$ with $M(0) = 0$.

The above calculations imply that $2\pi \sigma_i = -\text{sign}(m_+)$ when $m(y) \in \mathcal{S}[m_-, m_+]$ and $m_- < 0$ (with $\sigma_i = 0$ when $m_+ m_- > 0$) while $2\pi \sigma_i = -\text{sign}(\lambda)$ when $m(y) = \lambda y$ (and would vanish for $m(y) = \lambda |y|$).

The 3 × 3 system (2). We start by analyzing its Hermitian structure. We consider the $3 \times 3$ system with perturbation given by $H_V = H + V$. We define the domain of such an operator as $\mathcal{D}(H) = (H + ei)^{-1}\mathcal{H} \subset \mathcal{H}$ with $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^3$ with $\epsilon = \pm 1$ and then show that $H + V + ei$ is still surjective from $\mathcal{D}(H)$ to $\mathcal{H}$, or equivalently that the kernel of $H + V + ei$ on that domain is trivial. The problem $(H + i)\psi = S$, with $\epsilon = 1$ to simplify may be recast as

$$
\begin{pmatrix}
 i + D_x \sigma_1 & B \\
 B^* & i
\end{pmatrix}
\begin{pmatrix}
w \\
v
\end{pmatrix}
= S
\begin{pmatrix}
w \\
v
\end{pmatrix}
$$

where we have defined $w = (\eta, u)^t$ and $B = (D_y, if(y))^t$. We find $(i + D_x \sigma_1)^{-1} = -(1 + D_x^2)^{-1}(i - D_x \sigma_1)$ so that the above system is equivalent to

$$(B^*(1 + D_x^2)^{-1}(i - D_x \sigma_1)B + i)v = (1 + D_x^2)^{-1}(i - D_x \sigma_1)S_w,$$

along with $w = (1 + D_x^2)^{-1}(S_w + (i - D_x \sigma_1)B)v$. The equation for $v$ is therefore $(B^*(i - D_x \sigma_1)B + i(1 + D_x^2))v = (1 + D_x^2)S_w + (i - D_x \sigma_1)S_w$ or equivalently

$$(i(D_y^2 + f^2) + f'(y)D_x + i(1 + D_x^2))v = (1 + D_x^2)S_w + (i - D_x \sigma_1)S_w$$

This is an elliptic problem with a unique solution, which provides a domain of definition for $v$ given by $(D_x^2 + D_y^2 + (f^2)^{-1}(1 + D_x^2)L^2(\mathbb{R}^2))$. This propagates to a domain of definition for $(\eta, u)$ coming from $w = (1 + D_x^2)^{-1}(S_w + (i - D_x \sigma_1)B)v$. We observe that the spaces are independent of $\epsilon = \pm 1$.

For any bounded Hermitian perturbation $V$, the same spaces provide that $(H + V + ei)\psi = 0$ implies that $\psi = 0$. Indeed on that domain $((H + V)\psi, \psi)$ is real while $(\psi, ei\psi)$ is purely imaginary. This shows that $H_V = H + V$ is a self-adjoint operator on $\mathcal{D}(H)$ and the spectral calculus applies.
Spectral decomposition of system (2). Consider now the Hamiltonian $H[f(y)]$, which is invariant by translation along the $x-$axis. One difficulty not present in the $2 \times 2$ system is the presence of essential spectrum at energy 0. The bulk-interface correspondence predicts a number of protected interface modes equal to 2, which is 'often' the case, as we demonstrate in section 4, but not always. We now compute the winding number of branches of continuous spectrum for several profiles $f(y)$. Similar calculations have been worked out in [4, 6, 7, 47] without explicitly looking to identify spectral branches.

We are left with verifying that $H_h(\xi)\psi = E(\xi)\psi$ for $\psi = (\eta, u, v)^T$, which is given by the following system:

$$\begin{align*}
\xi u + Dv &= E\eta, \\
\xi \eta + iv &= Eu, \quad D\eta - ifu = Ev,
\end{align*}$$

with $D = \frac{i}{\xi} \partial_y$. We first make general remarks on such systems and then consider specific examples that can be solved explicitly. For concreteness, we consider the setting where $f'(y) \geq 0$. We first look for solutions with $v = 0$. We find from the other equations that $|E| = |\xi|$ and $u = \pm \eta$ when $E \neq 0$ with $\partial_y \eta + fu = 0$. This shows that $\eta = u$ provides a solution $u = e^{-F(y)}$ with $F'(y) = f(y)$ and $F(0) = 0$, say, which is unique up to normalization. Note that $au = 0$ with $a = \partial_y + f$ an operator with non-trivial kernel and trivial co-kernel (and hence a Fredholm operator in appropriate topologies with index equal to 1). Note that $E = E(\xi) = \xi$ provides a branch of continuous spectrum. Moreover, $F'(\xi) = 1$ corresponds to waves propagating toward positive values of $x$; these are the eastward propagating Kelvin waves in the geophysical application.

Still with $v = 0$, it remains to look at the case $E = \xi = 0$ where we find that $D\eta = ifu$ as the only constraint. There is an infinite number of solutions and $E = \xi = 0$ corresponds to essential spectrum that needs to be avoided. We thus assume $E \neq 0$.

Let us now look for solutions with $v \neq 0$. Eliminating $\eta$, assuming $\xi \neq 0$, yields

$$(\xi D + ifE)v = (E^2 - \xi^2)u, \quad (ED - if\xi)u = (E\xi + iDf)v,$$

and further eliminating $u$ provides the following equation for $v$:

$$\left( D^2 + f^2 + \frac{\xi}{E} f' \right) v = \left( -\partial_y^2 + f^2 + \frac{\xi}{E} f' \right) v = (E^2 - \xi^2) v.$$ 

We verify from (37) that the above equation still holds when $\xi = 0$.

Let us assume that $E = \xi^2$. Then $\xi = -E$ implies that the above equation for $v$ is $a^*^{-1}v = 0$ whereas $\xi = E$ implies that $aa^* v = 0$. The latter admits $v = 0$ as a unique solution while the former admits the solution $v = e^{-F(y)}$. When $E = -\xi$, we then verify that

$$(\partial_y - f)u = -a^* u = \frac{i}{\xi} (\xi^2 + f^2 - f^2) v, \quad -a^* \xi = \frac{i}{\xi} (\xi^2 - f^2) v.$$ 

These equations admit solution if and only if the right hand sides are orthogonal to the solutions in the kernel of $a$ by the Fredholm alternative. As a consequence, we observe that $\int R(\zeta^2 + f^2 - f^2)^2 dy = \int R(\zeta^2 - f^2)^2 dy = 0$. Only for specific $f$ are such constraints satisfied (they are for instance when $f = f_0$ sign($y$)). But in any case, they are satisfied for two values of $\xi$ at most and therefore only generate discrete spectrum in $H[f(y)]$ at most. Such discrete spectrum is irrelevant in our pursuits of branches of continuous spectrum and does not modify the value of Fredholm indices.

Therefore, $|E| \neq |\xi|$ when $v \neq 0$. Once $v$ is found, then $u$ and $\eta$ are given by

$$u = \frac{1}{E^2 - \xi^2} (\xi D + ifE)v, \quad \eta = \frac{1}{E} (\xi u + Dv).$$ 

We are left with verifying that $u$ and $\eta$ thus defined are normalizable when $v$ is normalizable, which may depend on $f(y)$ but holds in the cases considered below.

We thus look for non-trivial solutions of the equation $\left( -\partial_y^2 + f^2 + \frac{\xi}{E} f' \right) v = (E^2 - \xi^2) v$. It does not seem possible to solve such an equation in closed form for arbitrary functions $f(y)$. We consider cases that admit closed-form expressions.
We first consider the case $f(y) = f_0 \, \text{sign}(y)$. The solution $v$ is even in $y$ and given by $e^{-\mu y}$ for $y > 0$ with $\mu > 0$ such that $-\mu^2 + f_0^2 = E^2 - \xi^2$. The jump conditions at $y=0$ read $v'(0-) - v'(0+) + \frac{E}{\bar{E}} 2f_0 = 0$ or equivalently $\mu = -\frac{\bar{E}}{\bar{E}} f_0$. Plugging this into the above equation yields two solutions $E^2 = f_0^2$ or $E^2 = \xi^2$. We have already ruled out the latter. The former gives $E = -f_0$ sign($\xi$) so that $\mu > 0$. As a consequence, we observe that $E = \mp f_0$ is infinitely degenerate. The effect of the interaction of the bulk 0 sheet and the varying $f$ is the presence of stationary waves since $E(\xi) = 0$. Moreover, we observe a spectral gap in $(-f_0, f_0)$ for the part of the spectrum corresponding to $v \neq 0$. For energies $|E| < f_0$, we thus obtain only one branch of continuous spectrum given by the Kelvin waves. The number of interface modes is equal to 1, not 2 as (wrongly) predicted by the bulk interface correspondence. We refer the reader to [7, 47] for other explicit calculations, obtained for instance in the presence of an odd viscosity contribution $\varepsilon$ so that $f$ in the equations is replaced by $f + \varepsilon \Delta$.

We now consider the practically relevant case $f(y) = \lambda y$ with $\lambda > 0$ for concreteness. Since $f' = \lambda$ is constant, we can relate the spectrum of the problem of interest to that of the quantum harmonic oscillator. We find that $\alpha^* a v = (E^2 - \xi^2 - \lambda (1 + \frac{E}{\bar{E}})) v$. The spectrum of $\alpha^* a$ is given by $2n\lambda$ so that

$$E_n^2 - \xi^2 - \lambda \left(1 + \frac{E}{\bar{E}}\right) = (2n + 1) \lambda,$$

$$E_n^3 = (\lambda(2n + 1) + \xi^2)E_n + \lambda \xi.$$

We look for all solutions to the above equation, except for the case $E^2 = \xi^2$, which we ruled out and occurs only when $n = 0$. These are all cubic equations, which can be solved explicitly. The case $n = 0$ provides two solutions, one with positive energy and the other one with negative energy. These are called Yanai waves. We verify that $E_0(\xi) > 0$ for such waves and they therefore also propagate eastward.

The rest of the solutions are composed of Rossby waves and Poincaré waves coming in continuous branches of spectrum crossing an energy level different from 0 a finite and even number of times so that they are topologically trivial (with a vanishing winding number); see [4, 7] as well.

As a summary of the above calculations, we therefore find that $2\pi \sigma_\ell$ is given by $+1$ when $f(y) = f_0 \, \text{sign}(y)$ with $f_0 > 0$ while it is given by $+2$ when $f(y) = \lambda y$ with $\lambda > 0$.

**Coriolis force taking two values on half lines.** We finally consider the case with $f(y) = f_+ \, \text{for } y > 0$ and $f(y) = f_- \, \text{for } y < 0$. Then $v$, assuming $E \neq 0$, solves the following system:

$$-\partial_y^2 v + f^2 v = (E^2 - \xi^2)v, \quad y > 0; \quad -\partial_y^2 v + f^2 v = (E^2 - \xi^2)v - v'(0+) - v'(0-) + \frac{E}{\bar{E}} (f_+ - f_-) v(0) = 0.$$

The solutions are given by $v(y) = e^{-\mu y}$ for $y > 0$ and $v(y) = e^{\mu y}$ for $y < 0$ with $\mu_0 > 0$ as a necessary condition for normalization. We thus obtain the system of equations for $(E, \mu_+, \mu_-)$ knowing $(\xi, f_+, f_-)$:

$$E^2 - \xi^2 = \mu_+^2 - \mu_-^2 = f_+^2 - f_-^2, \quad \mu_+ + \mu_- + \frac{\xi}{\bar{E}} (f_+ - f_-) = 0.$$

We want solutions such that $\mu_+ > 0$, $f_+ - f_- > 0$ implies that $\frac{\xi}{\bar{E}} < 0$. We introduce:

$$f_0 = \frac{1}{2} (f_+ + f_-), \quad f_0 = \frac{1}{2} (f_+ - f_-) > 0.$$

From $\mu_+^2 - \mu_-^2 = (\mu_+ + \mu_-)(\mu_+ - \mu_-) = f_+^2 - f_-^2$ we deduce that $\mu_+ - \mu_- + \frac{\xi}{\bar{E}} (f_+ + f_-) = 0$. Let us define $\nu = \frac{\xi}{\bar{E}}$. Then,

$$\mu_+ + \nu^{-1} f_0 + \nu f_0 = 0 \quad \text{so that} \quad \mu_+^2 = \nu^{-2} f_0^2 + f_0 f_+ + \nu^2 f_0^2 = f_+^2 + \xi^2 (1 - \nu^2).$$

This gives an equation for $\nu$, or equivalently $E$, using $f_+^2 - 2f_0 f_+ = f_0^2 + f_+^2$:

$$(f_+^2 + \xi^2) \nu^4 - (f_0^2 + f_+^2 + \xi^2) \nu^2 + f_0^2 = 0 \quad \text{or} \quad (f_+^2 + \xi^2) (\nu^2 - 1) \left(\nu^2 - \frac{f_0^2}{f_+^2 + \xi^2}\right) = 0.$$
We know the existence of a branch \( E(\xi) = \xi \) (Kelvin waves with \( v = 0 \)) and ruled out \( E(\xi) = -\xi \). With \(|E| \neq |\xi|\), we know that \( \xi/E < 0 \) so that, since \( f_o > 0 \), the only admissible solution is
\[
E = -\frac{\xi f_o}{\sqrt{f_o^2 + \xi^2}}.
\]
(38)

For such a branch, we verify that \( \partial_\xi E < 0 \). The only valid solutions of interest satisfy that \( \mu_+ = -\nu f_o + \nu^{-1} f_o > 0 \), or equivalently,
\[
f_o > \nu^2 |\xi| \quad \text{or} \quad \xi^2 \geq |\xi| (|f_o - |\xi||),
\]
where we used the above expression for \( \nu \). An admissible branch of continuous spectrum inside the bulk gaps implies \(|E| < M := \min(|f_-|, |f_+|) = |f_o - |\xi||\). We thus find the constraints
\[
\xi^2 \geq |\xi| (|f_o - |\xi||), \quad \text{and} \quad (2f_o - |\xi|) \xi^2 < |\xi| (|f_o - |\xi||)^2.
\]
We then consider three cases. (i) \( |\xi| \geq 2f_o \) in which case both constraints are satisfied; (ii) \( 2f_o > |\xi| \geq f_o \) in which case the first constraint is always satisfied and the second one provides
\[
\xi^2 < |\xi| (|f_o - |\xi||)^2.
\]
We verify that \( f_o < 0 < f_+ \) belongs to case (iii). Only the Kelvin waves \( E(\xi) = \xi \), as in the case \( f_- = -f_+ \), cross the bulk band gap. Surprisingly, as soon as \( |\xi| \geq f_o \), which corresponds to \( f_- < f_+ \leq 0 \) or \( 0 \leq f_- < f_+ \), then a branch of absolutely continuous spectrum crosses the intervals \((-M, 0)\) and \((0, M)\) with \( M = |f_o - f_0| \). Since \( \partial_\xi E < 0 \), these branches have a winding number equal to \(-1\).

We have thus considered the three different configurations: (i) when \( f(y) = |\xi|y \), then \( I[H] = 2 \) as predicted by the bulk interface correspondence; in the piece-wise constant case \( f = f_-, x < 0 + f_+, x > 0 \) (ii) when \( |\xi| < f_o \), then \( I[H] = 1 \); and (iii) when \( |\xi| > f_o \), then \( I[H] = 1 - 1 = 0 \) as predicted by the bulk-interface correspondence, since \( f_+ \) and \( f_- \) have the same sign.

C. Equivalence of bulk-difference invariants

Lemma 3.2. The objective is to recast \( W_s = \frac{1}{8\pi} \int T d^2 k \) in terms of the projectors \( \Pi_\rho \) where we use that \( \partial_\omega G^{-1} = i \) and
\[
T = T(k) := \int_{\mathbb{R}} \text{tr} G \left[ \partial_\omega G^{-1} G, \partial_\omega G^{-1} G \right] d\omega.
\]
For \( \hat{H} = \sum_i h_i \Pi_i \) so that \( G = \sum_i (z - h_i)^{-1} \Pi_i \) we thus need to estimate
\[
\sum_{i,j,k} \text{tr} \left[ \Pi_i \partial_\omega G \Pi_j \partial_\omega H - \partial_\omega G \Pi_i \partial_\omega H \right] \Pi_k d\omega = \sum_{i,j} \text{tr} \int \frac{\Pi_i \partial_\omega G \Pi_j \partial_\omega H - \partial_\omega G \Pi_i \partial_\omega H \Pi_k}{(z - h_i)(z - h_j)(z - h_k)} d\omega
\]
since \( \Pi_i \Pi_k = \delta_{ik} \Pi_i \) and the trace is cyclic. We now evaluate the integrals over \( \omega \), which have a different form depending on whether \( i = j \) or not. For the latter case, we have
\[
\frac{1}{(z - h_i)^2} = \frac{b_1}{z - h_i + i\omega} + \frac{b_2}{(z - h_i + i\omega)^2} + \frac{b_3}{z - h_i + i\omega}
\]
with \( b_3 = -b_1 = (h_i - h_j)^{-1} \) and \( b_2 = (h_i - h_j)^{-1} \). It remains to evaluate the integrals using that
\[
\int (\omega + i\omega)^{-1} d\omega = \pi \text{ sign}(\omega) \text{ to obtain}
\]
\[
\pi \frac{1}{(h_i - h_j)} \left( \text{ sign}(z - h_j) - \text{ sign}(z - h_i) \right)
\]
since \( \int (x + i\omega)^{-k} d\omega = 0 \) for \( k \geq 2 \) integer by residue calculation. When \( i = j \), we similarly obtain a vanishing contribution. Thus,

\[
T = \sum_{i \neq j} \pi(\text{sign}(x - h_i) - \text{sign}(x - h_j)) \frac{\Pi_i(\partial_1 H \Pi_i \partial_2 H - \partial_2 H \Pi_i \partial_1 H)}{(h_i - h_j)^2}.
\]

By cyclicity of the trace, we obtain

\[
T = \sum_{i < j} 2\pi(\text{sign}(x - h_j) - \text{sign}(x - h_i)) \frac{\Pi_i(\partial_1 H \Pi_i \partial_2 H - \partial_2 H \Pi_i \partial_1 H)}{(h_i - h_j)^2}.
\]

(39)

The above formula is convenient for explicit computations of the invariants. When the projectors \( \Pi_i = \psi_i \otimes \psi_i \) are rank one, using that \( \langle \psi_i, \gamma \psi_i \rangle = \langle \psi_i, \psi_i \rangle \) for any Hermitian matrix \( \gamma \), we observe that

\[
T = 2\pi \sum_{i < j} \frac{\text{sign}(x - h_j) - \text{sign}(x - h_i)}{(h_i - h_j)^2} 2i\mathcal{I}(\langle \psi_i, \partial_1 H \psi_j \rangle \langle \psi_j, \partial_2 H \psi_i \rangle).
\]

(40)

From the above expression for \( \hat{H} \), we deduce that \( \partial_k \hat{H} = \sum_l \partial_k h_l \Pi_l + h_l \partial_k \Pi_l \). For \( i \neq j \), all terms involving \( \partial_k h_l \) cancel as \( \Pi_i \Pi_j \Pi_k = 0 \) so that

\[
\sum_{i < j} 2\pi(\text{sign}(x - h_j) - \text{sign}(x - h_i)) \frac{h_k h_l}{(h_i - h_j)^2} \text{tr} \Pi_i(\partial_1 h_l \Pi_i \partial_2 h_l - \partial_2 h_l \Pi_i \partial_1 h_l).
\]

\[
\Pi_m \Pi_n = \delta_{mn} \Pi_n \text{ implies } \partial_m \Pi_k \Pi_l + \Pi_k \partial_m \Pi_l = \delta_{kl} \partial_m \Pi_k \text{ so that } (k, l) \text{ in the above formula have to equal } i \text{ or } j.
\]

The sum over \( (k, l) \) is thus

\[
\frac{h_k^2 \Pi_i \partial_1 h_l \Pi_i \partial_2 h_l + h_l h_j (\Pi_i \partial_1 h_l \Pi_i \partial_2 h_l + \Pi_i \partial_1 h_l \Pi_i \partial_2 h_l) + h_j^2 \Pi_i \partial_1 h_l \Pi_i \partial_2 h_l}{(h_i - h_j)^2}
\]

minus the contribution exchanging the order of the derivatives. The above term is

\[
\text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i) = \text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i) = \text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i).
\]

Therefore,

\[
T = -4\pi \sum_{h_i < x} \sum_{h_j > x} \text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i), \text{ or using } \text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i) = \text{tr} \Pi_i(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i).
\]

This is also, for \( i_0 \) the index so that \( h_{i_0} < x < h_{i_0 + 1} \),

\[
T = 4\pi \sum_{h_i < x} \text{tr}(\sum_{h_j > x} \Pi_j)(\partial_1 \Pi_i \partial_2 \Pi_i - \partial_2 \Pi_i \partial_1 \Pi_i).
\]

This shows that for \( x \) in the \( i_0 \)th gap, the winding number is given by the sum of the first \( i_0 \) Chern numbers as expected. \( \square \)

**Application to bulk invariant calculations for the 3 \times 3 system (2)**: We now use (40) to compute the bulk-difference \( W_x \) invariant. We recast the system (2) as \( \hat{H} = \xi_4 \gamma_1 + \xi_4^* \gamma_4 - f \gamma_7 \). This is diagonalized as follows. We find \( I = \Pi_- + \Pi_0 + \Pi_+ \) for \( \hat{H} = \kappa(\Pi_+ - \Pi_-) \) and \( \Pi_j = \psi_j \psi_j^\dagger \) rank one projectors given by
\[ \psi_0 = \frac{1}{\kappa} \begin{pmatrix} \text{if} \\
\zeta \end{pmatrix}, \quad \psi_\pm = \frac{1}{\rho} \begin{pmatrix} \text{if} \zeta \mp \kappa \\
\zeta^2 + f^2 \end{pmatrix} \] (41)

and \((\zeta, \zeta)\)-dependent eigenvalues given by

\[ h_0 = 0, \quad h_\pm = \pm \kappa, \quad \text{for} \quad \kappa = \sqrt{\zeta^2 + f^2} \quad \text{and} \quad \rho = \kappa \sqrt{2(f^2 + \zeta^2)}. \]

We find \( \partial_z \tilde{H} = -\partial_{\zeta} G^{-1} = \gamma_1 \) and \( \partial_x \tilde{H} = -\partial_{\xi} G^{-1} = \gamma_4 \). As a consequence, (40) gives \( W_x = \frac{i}{8\pi}\int_{\mathbb{R}^2} T d\zeta d\xi \) with

\[ T = 2\pi \sum_{1 \leq i < j \leq 3} \frac{\text{sign}(x - h_i) - \text{sign}(x - h_j)}{(h_i - h_j)^2} 2i\Im(\langle \psi_i, \gamma_1 \psi_j \rangle \langle \psi_j, \gamma_4 \psi_i \rangle). \]

For \( x > 0 \), the only differences of signs contributing non trivial terms are \((i, j)\) with \( i < j \) in \( \{(+, +), (0, +)\} \) with then \( \text{sign}(x - h_j) - \text{sign}(x - h_i) = -2 \). We find

\[ \langle \psi_+, \gamma_4 \psi_- \rangle = 0, \quad \langle \psi_+, \gamma_4 \psi_0 \rangle = \frac{1}{\rho}(\text{if} \kappa - \zeta), \quad \langle \psi_0, \gamma_1 \psi_+ \rangle = \frac{1}{\rho}(\zeta^2 + f^2). \]

Thus, \( 2i\Im(\langle \psi_0, \gamma_1 \psi_+ \rangle \langle \psi_+, \gamma_4 \psi_0 \rangle) = \frac{\kappa}{\rho} \). Note that \( h_+ - h_0 = \kappa \). The bulk difference is defined for \( f_\pm = \pm f \) on the top and bottom planes, which yields

\[ -W_x = 2 \text{sign}(f) \frac{i}{8\pi^2} \int_{\mathbb{R}^2} \frac{\text{sign}(f)}{\kappa^2} \frac{d\zeta d\xi}{\pi} = \frac{\text{sign}(f)}{\pi} \int_{\mathbb{R}^2} \frac{\text{sign}(f)}{\kappa^2} \frac{d\zeta d\xi}{(\zeta^2 + f^2)^2} = 2 \text{sign}(f), \]

as a standard integration for two-dimensional invariants. This was obtained for \( x > 0 \) and \( f_- = -f_+ \). All other expressions are obtained similarly. In particular, we observe that \( W_x \) after some algebra takes the same value for \( x < 0 \) so that \( c_+ = c_+ + c_0 = -c_- = -2 \text{ sign}(f) \), which yields (15). See [6,7] and in particular [23, Appendix A] for alternative methods to compute the above invariants.

**D. Semiclassical calculus and Helffer–Sjöstrand formula**

We collect here notation and results on pseudo-differential, semiclassical, and spectral calculus used in this article following [20,21, 41, 45], to which we refer for details.

Let \( V = \mathbb{R}^d \) and \( V' \simeq \mathbb{R}^d \) its dual. A bounded matrix-valued operator \( A \) from \( \mathcal{S}(V) \otimes \mathbb{C}^n \) the Schwartz space to its dual \( \mathcal{S}'(V) \otimes \mathbb{C}^n \) admits a (Schwartz) distribution kernel \( K_A \in \mathcal{S}'(V \times V) \otimes M_n(\mathbb{C}) \) and can be represented for \( 0 < h \leq 1 \) as

\[ A = \text{Op}_h^w(a) := \text{Op}_h(a), \quad \text{Op}_h(a)\psi(x) = \frac{1}{(2\pi h)^d} \int_{V' \times V} e^{i\frac{x}{2h} \cdot \frac{A}{2h}\xi} \psi(y)dyd\xi, \] (42)

where \( a(x, \xi) \in \mathcal{S}'(V \times V') \otimes M_n(\mathbb{C}) \) is the Weyl symbol of \( A \) defined as

\[ a(x, \xi) = \int_V e^{-i\frac{\xi}{h} K_A(x + \frac{y}{2}, x - \frac{y}{2})} dy. \]

The notation \( a''(x, hD) := \text{Op}_h(a) \) is also used to define the semiclassical operator \((h\Psi DO)\) for the Weyl quantization of \( a \) with \( D = \frac{i}{h} \nabla \). Finally, we use the notation \( \text{Op}_h(a) := \text{Op}_h(a) := \text{Op}_h(a) \) for the Weyl quantization of pseudo-differential operators defined at the scale \( h = 1 \).

To define operators with smoother kernels that can be composed with each other, we define the space of \textit{order functions} \( m(x, \xi) \) from \( V \times V' \to [0, \infty) \) satisfying the growth condition:

\[ m(x, \xi) \leq C(1 + |x - y| + |\xi - \zeta|)^N m(y, \zeta) \] (43)
for some constants $C = C(m)$ and $N = N(m)$. Examples of interest are $(1 + |x|^2 + |\xi|^2)^s$ for $s \in \mathbb{R}$ as well as $\max(0, \pm x_1)$. We then denote by $S(m)$ the Fréchet space of symbols $a(x, \xi) \in C^\infty(V \times V') \otimes \mathbb{M}_n$ for the semi-norns

$$\|a(x, \xi)\|_{S(m)} = \sup_{|x| + |\xi| = k_0(x, \xi) \in V \times V'} \frac{1}{m(x, \xi)} \|\partial_x^a \partial_\xi^b a(x, \xi)\|_{\mathbb{M}_n},$$

(44)

implying the existence of constants $C_{a, \beta}$ such that $\|\partial_x^a \partial_\xi^b a(x, \xi)\|_{\mathbb{M}_n} \leq C_{a, \beta} m(x, \xi)$. We use the notation $Op^w S(m)$ for the class of operators with symbols in $S(m)$. For $h-$ dependent operators $a(\cdot; h)$, we say that $a \in S^0(m)$ when $a(\cdot; h) \in S(m)$ uniformly in $0 < h \leq 1$.

For two operators $a^w$ and $b^w$ with symbols $a \in S^0(m_1)$ and $b \in S^0(m_2)$, we then define the composition $c^w = a^w \circ b^w$ with symbol $c \in S^0(m_1 m_2)$ given by the Moyal product

$$c(x, \xi) = (a^w b)(x, \xi) := (e^{i\bar{x}\partial_x - i\bar{\xi}\partial_\xi}) a(x, \xi) b(y, \bar{\xi}) |_{y = x; \xi = \bar{\xi}}.$$  

(45)

We also define $\sharp = \sharp_1$ when $h = 1$.

For $a \in S^0(1)$, we obtain ([21, Th. 7.11], [41, Prop. 1.4]) that $Op_h(a)$ is bounded as an operator in $L(L^2(V) \otimes \mathbb{C}^n)$ with bound uniform in $0 < h \leq 1$ so that $I - h Op_h(a)$ is invertible on that space when $h$ is sufficiently small.

An operator is said to be (semiclassically) elliptic when the symbol $a = a(x, \xi; h) \in S^0(m)$ is invertible in $\mathbb{M}_n$ for all $(x, \xi) \in V \times V'$ with $a^{-1} \in S^0(m^{-1})$. For $\sigma$ a Hermitian-valued symbol and $H_0 = Op_h(\sigma)$, we then obtain that $z - H_0$ is elliptic for $z = \lambda + i\omega$ with $\omega \neq 0$. Moreover, $(z - H_0)^{-1}$ is a symbol $r \in S^0(1)$ uniformly in $0 < h \leq h_0$ sufficiently small, $|\lambda|$ bounded and $|\omega|$ bounded away from 0; see [21, Prop. 8.6], which extends to the matrix-valued case, and Lemma 4.14 for a more precise bound for $|\omega|$ small.

We next recall the sharp semiclassical Gårding inequality [21, Theorem 7.12] (which extends to the vectorial case without modification) stating that for $a$ a Hermitian-valued symbol in $S^0(1)$ with eigenvalues greater than or equal to $\beta$ in $\mathbb{R}$ for all $(x, \xi) \in V \times V'$, then

$$(Op_h(a) \psi, \psi) \geq (\beta - Ch) \|\psi\|^2$$

(46)

for all $\psi \in L^2(V) \otimes \mathbb{C}^n$ and for $C$ a constant independent of $0 < h \leq h_0$ sufficiently small. Thus, for $\beta > 0$ and $h$ sufficiently small, the operator $Op_h(a)$ has spectrum bounded away from 0.

Finally, we recall some results on spectral calculus and the Helffer-Sjöstrand formula following [20,21]; see also [41] for the vectorial case. For any self-adjoint operator $H$ from its domain $D(H)$ to $L^2(V) \otimes \mathbb{C}^n$ and any bounded continuous function $\phi$ on $\mathbb{R}$, then $\phi(H)$ is uniquely defined as a bounded operator on $L^2(V) \otimes \mathbb{C}^n$ [21, Chapter 4]. Moreover, for $\phi$ compactly supported, we have the following representation

$$\phi(H) = -\frac{1}{\pi} \int_{\mathbb{C}} \partial \bar{\phi}(z) (z - H)^{-1} d^2z,$$

(47)

where, for $z = \lambda + i\omega$, $d^2z := dl d\omega$, $\partial = \frac{1}{2} \partial_\lambda + \frac{1}{2} \partial_\omega$, and where $\bar{\phi}(z)$ is an almost analytic extension of $\phi$. The extension $\bar{\phi}$ is compactly supported in $\mathbb{C}$. Moreover, $\bar{\phi}(\lambda + i0) = \phi(\lambda)$ and $\partial \bar{\phi}(\lambda + i0) = 0$, whence the name of almost analytic extension. We can choose the almost analytic extension such that $|\partial \bar{\phi}| \leq C_N |\omega|^N$ for any $N \in \mathbb{N}$ in the vicinity of the real axis uniformly in $(\lambda, \omega)$ on compact sets. Several explicit expressions, which we do not need here, for such extensions are available in [20,21].