Revisiting the conformal invariance of the scalar field: from Minkowski space to de Sitter space

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In this article, we clarify the link between the conformal (i.e. Weyl) correspondence from the Minkowski space to the de Sitter space and the conformal (i.e. SO(2, d)) invariance of the conformal scalar field on both spaces. We exhibit the realization on de Sitter space of the massless scalar representation of SO(2, d). It is obtained from the corresponding representation in Minkowski space through an intertwining operator inherited from the Weyl relation between the two spaces. The de Sitter representation is written in a form which allows one to take the point of view of a Minkowskian observer who sees the effect of curvature through additional terms.

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I. INTRODUCTION

In this paper we are interested in the conformally coupled massless scalar field (hereafter conformal scalar field, for simplicity) in both d-dimensional Minkowski and de Sitter spaces for d > 2. It is a known fact that each of these two fields, Minkowskian and de Sitterian, are invariant under the group SO(2, d). It is also well-known that these two spaces are related through a Weyl rescaling. These properties are rarely used at the same time (see however \textsuperscript{1}) and are both termed “conformal invariance” in the literature. In this paper, we show how they hang together. In short, the Weyl rescaling induces a map $\hat{\Xi}_\mu$ between the two Hilbert spaces of solutions of the corresponding conformal equations. This map intertwines the two representations carried by these spaces.

The operator $\hat{\Xi}_\mu$ was introduced, in substance, in a previous article \textsuperscript{2}. In that work we realized Minkowski, de Sitter, and anti-de Sitter spaces on the same underlying set. Since the anti-de Sitter space is not globally hyperbolic we restrict our present investigation to the d-dimensional Minkowski and de Sitter spaces. We used the conformal (Weyl) relation between them to deform Minkowskian objects into de Sitterian objects. In particular, it turns out that the operator $\hat{\Xi}_\mu$ is unitary between the Hilbert spaces of solutions of the conformal scalar field equations on both de Sitter and Minkowski spaces. As remarked, since the respective isometry groups of Minkowski and de Sitter spaces are different, one cannot speak about the covariance of the operator $\hat{\Xi}_\mu$ with respect to these groups. Nevertheless, since these two spaces admit the same conformal group SO(2, d), in which both Poincaré and de Sitter groups are included, the operator $\hat{\Xi}_\mu$ makes explicit the link between the two actions of SO(2, d) on Minkowski and de Sitter spaces. More precisely, this operator intertwines these representations.

Even more, this operator allows an explicit realization of the generators of the so(2, d) algebra for the conformal field on the de Sitter space. This realization is written in a Minkowskian form: in some sense we deal with exact de Sitterian objects (without any approximation) in Minkowski space.

We organize our paper as follows. In Sec. II we comment about the two kinds of conformal invariance. The action of SO(2, d) on Minkowski space is reminded in Sec. III. In Sec. IV the Weyl rescaling between Minkowski and de Sitter space is recovered by using the results of \textsuperscript{2}. This allows to define the operator $\hat{\Xi}_\mu$. This operator is used in Sec. V to move the SO(2, d) representation. Our conventions and some details concerning the so(2, d) algebra are given in the appendix.

II. ABOUT CONFORMAL INVARIANCE

Two notions of conformal invariance are used in this work; let us first make clear the distinction between them. The first one is connected with the conformal group SO(2, d) \textsuperscript{3} \textsuperscript{4}. Let us consider an equation for some field $F$ and write it symbolically as $OF = 0$, where $O$ denotes some linear operator. Such an equation is said to be invariant under SO(2, d), and we will keep this terminology in the sequel, if one can realize the generators $X_{\alpha\beta}$ of the Lie algebra so(2, d) in such a manner that $[O, X_{\alpha\beta}] = \zeta O$, $\zeta$ being some function. The space of solutions of the equation $OF = 0$ is, in this case, invariant under the corresponding action of the group SO(2, d) and thus carries a representation of this group.

The second notion of conformal invariance is related to the so-called Weyl rescaling \textsuperscript{3} \textsuperscript{4}. This transformation consists in a change from the metric $g$ defined on some manifold to a new one $\hat{g} := \omega^2 g$, $\omega$ being a
real function over $M$. Let us consider again the equation $O \phi = 0$; note that $O$ generally depends on $g$. This equation is said to be conformally invariant, if there exists a number $s \in \mathbb{R}$ (the conformal weight of the field) such as $F$ is a solution with $g$ iff $\overline{F} := \omega^s F$ is a solution with $\overline{g}$. In all cases considered here, one has in fact $\overline{O \phi} = \omega^s O \phi$, for some $s' \in \mathbb{R}$. In the sequel, this kind of conformal transformation and invariance will be referred to as “Weyl transformation” and “Weyl invariance”.

It is well known that Minkowski and de Sitter spaces have the same conformal group, namely $SO(2, d)$, whose Poincaré group $SO_0(1, d - 1) \ltimes \mathbb{R}^d$, and de Sitter group $SO_0(1, d)$, are the respective subgroups of isometry. In addition, Minkowski and de Sitter spaces are related by a Weyl transformation. One could ask how the $SO(2, d)$ invariance of the conformal field equation is “transported” under the Weyl transformation? That question is explicitly answered in this work.

III. THE ACTION OF SO(2, d) ON THE MINKOWSKIAN SCALAR FIELD

Now, we want to comment about the scalar representation of $SO(2, d)$ on Minkowski space which realizes the $SO(2, d)$ invariance.

The conformal group of Minkowski space, whose linear representation is $SO(2, d)$, is given by its isometries extended by the dilations and the special conformal transformations. These transformations act, respectively, on the Minkowski coordinates $x^\mu$ as

$$y_D^\mu = \lambda x^\mu, \quad y_{SC}^{\mu} = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2},$$

where $\lambda$ and $b$ being parameters of the transformations. The well-known representation of the $so(2, d)$ algebra follows

$$M_{\mu \nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad P_\mu = \partial_\mu, \quad D = x^\mu \partial_\mu, \quad K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu.$$ 

One could think, and it can be found in some literature, that for any generator $X$ in this representation one has $[\Box, X] = \zeta \Box$. This is not the case. In order to understand this fact, let us consider the usual action $S[\phi]$ for the conformal scalar field in Minkowski space (that is the free massless scalar field). It is of course invariant under Poincaré transformations, but not under the above transformations. In order to restore the action invariance, the field must be changed by an internal transformation a so-called scaling: the change of variables $\phi' = \omega^s \phi$ with $s = -(d - 2)/2$, $\omega_D = \lambda$ and $\omega_{SC} = (1 - 2b \cdot x + b^2 x^2)$ respectively. As a consequence, the action on the conformal scalar field is given by

$$T_D \phi(x) = \omega_D \frac{d^2}{dx^2} \phi(x), \quad T_{SC} \phi(x) = \omega_{SC} \frac{d^2}{dx^2} \phi \left( \frac{x^2 - b^2 x^2}{1 - 2b \cdot x + b^2 x^2} \right).$$

The corresponding representation of the Lie algebra is obtained by differentiation as usual:

$$M_{\mu \nu}^0 = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad P_\mu^0 = \partial_\mu, \quad D^0 = \frac{d^2}{2} \partial_\mu, \quad K_\mu^0 = (d - 2)x_\mu + (2x_\mu x^\nu - x^2 \delta^\mu_\nu) \partial_\nu, \quad \omega_D^0 = \lambda, \quad \omega_{SC}^0 = 4x_\mu \partial_\mu.$$ 

Before closing this section we want to emphasize that the conformal scalar field is not a scalar field in the sense of the differential geometry. Remind that, a complex map $\varphi$ defined on $\mathbb{R}^d$ is a scalar field iff for any diffeomorphism $f$ defined on $\mathbb{R}^d$ one has $(f \cdot \varphi)(f(x)) = \varphi(x)$. In this respect, the conformal scalar field in Minkowski space is not really a scalar field. In fact, it is a scalar field only for Poincaré transformations. The same remark extends to the conformal scalar field in de Sitter space.

IV. THE CONE UP TO THE DILATIONS AND THE WEYL RESCALING

In this section we use the results of our previous work, to which we refer the reader for further details, to define the $\mathbb{H}$ map. This is done by first retrieving the Weyl relation between Minkowski and de Sitter spaces.

Both Minkowski and de Sitter spaces can be obtained as the intersection of the null cone of $\mathbb{R}^{d+2}$ and a moving hyperplane (see the appendix for the notations). Precisely, the manifold

$$X_H = C \cap P_H,$$
where
\[ C = \{ y \in \mathbb{R}^{d+2} : (y^{d+1})^2 + (y^0)^2 - y^2 - (y^d)^2 = 0 \}, \]
and
\[ P_H = \{ y \in \mathbb{R}^{d+2} : (1 + H^2)y^{d+1} + (1 - H^2)y^d = 2 \}, \]
can be shown to be the Minkowski or a de Sitter space for \( H = 0 \) and \( H > 0 \) respectively. In addition, one can show that \( X_H \) can be seen as a subset of the cone up to the dilations \( C' = C' / \sim \), where the relation \( \sim \) is defined through \( u \sim v \) iff there exists \( \lambda > 0 \) such that \( u = \lambda v \). From this point of view \( X_H \) is defined through
\[ X_H = \{ y \in C' : (1 + H^2)y^{d+1} + (1 - H^2)y^d > 0 \}. \quad (4) \]
Let us reemphasize that both Minkowski and de Sitter spaces are realized on the same underlying set.

The line elements \( ds_H \) on \( X_H \) and \( ds \) on \( C' \) are conformally related, i.e.,
\[ ds_H^2 = \Omega_H^2 ds^2. \]
From this relation we deduce
\[ ds_H^2 = \Xi_H^2 ds_0^2, \]
where
\[ \Xi_H := \frac{\Omega_H}{\Omega_0}, \]
which is the Weyl relation between de Sitter and Minkowski spaces.

Owing to this Weyl transformation the solutions of both conformal scalar field equations on Minkowski and de Sitter spaces are obtained from the conformal scalar field equation on \( C' \) which is simpler. Moreover, let \( \mathcal{H}_0 \) and \( \mathcal{H}_H \) be the Hilbert spaces of the solutions, square integrable with respect to the Klein-Gordon scalar product, of the Minkowskian (respectively de Sitterian) conformal scalar field equation, the map
\[ \hat{\Xi}_H : \mathcal{H}_0 \rightarrow \mathcal{H}_H \]
\[ \phi \mapsto \hat{\Xi}_H(\phi) := \Xi_H^{-\frac{d+2}{2}}\phi, \]
is unitary. Now, let us consider some well defined linear operator \( O^0 \) over \( \mathcal{H}_0 \), the unitarity of \( \hat{\Xi}_H \) ensure that the moved operator
\[ O^H := \hat{\Xi}_H O^0 \hat{\Xi}^{-1}_H, \quad (5) \]
is well defined over \( \mathcal{H}_H \).

Since Minkowski and de Sitter spaces are written on the same underlying set one can use a common system of coordinates. A convenient one, for our purpose, is the usual Minkowskian system \( \{ x^\mu \} \) in which
\[ \Xi_H = \frac{1}{1 - \frac{H^2}{4} x^2}. \]

V. THE SO(2, d) COVARIANCE OF THE \( \hat{\Xi}_H \) MAP

Having defined the \( \hat{\Xi}_H \) map, we show how it connects the SO(2, d) invariance of the two conformal scalar fields.

As recalled above, \( \hat{\Xi}_H \) is a unitary operator between the Hilbert spaces of solutions of the conformal scalar field equation on Minkowskian and de Sitter spaces. The group acting on de Sitter and Minkowskian spaces are different; as a consequence, there is no meaning to speak about covariance of \( \hat{\Xi}_H \) with respect to these groups. We thus focus on the conformal group SO(2, d) which is the same for both spaces.

Since these spaces are realized on the cone up to the dilations on which SO(2, d) has a natural action, the geometric action is the same for the two spaces at least locally. As a consequence, the term \( \phi(g^{-1}x) \) in (3) is the same in both Minkowskian and de Sitter spaces. The scaling term \( \omega_g(x) \) will, however, be different, depending on \( H \) and denoted \( \omega^H_g(x) \) in the following. One could compute this term by considering the action \( S[\phi] \) of the field, but it can be more easily recovered by moving the representation from Minkowskian space to de Sitter space using (4). A straightforward computation shows that the de Sitter scaling term \( \omega^H_g(x) \) reads for any \( g \in \text{SO}(2, d) \)
\[ \omega^H_g(x) = (\Xi_H(x))^{-\frac{d+2}{2}} \omega_g(x) (\Xi_H(g^{-1}x))^{\frac{d+2}{2}}. \]

The generators, in Minkowskian coordinates, follow in the same way
\[ M^H_{\mu \nu} = M^0_{\mu \nu}, \quad (6) \]
\[ P^H_\mu = P^0_\mu + \frac{d - 2}{2} (\partial_\mu \ln \Xi_H) \]
\[ = \partial_\mu + \frac{d - 2}{4} H^2 \frac{x_\mu}{1 - \frac{H^2}{4} x^2}, \quad (7) \]
\[ D^H = D^0 + \frac{d - 2}{2} (x^\nu \partial_\nu \ln \Xi_H) \]
\[ = x \cdot \partial + \frac{d - 2}{2} \left( 1 + \frac{H^2}{4} x^2 \right) \frac{x_\nu}{1 - \frac{H^2}{4} x^2}, \quad (8) \]
\[ K^H_\mu = K^0_\mu + \frac{d - 2}{2} \left( 2x_\mu x^\nu - x^2 \delta^\nu_{\mu} \right) \partial_\nu \ln \Xi_H \]
\[ = (2x_\mu x^\nu - x^2 \delta^\nu_{\mu}) \partial_\nu + (d - 2) \frac{x_\mu}{1 - \frac{H^2}{4} x^2}. \quad (9) \]

These generators satisfy by construction the so(2, d) algebra (A1A9). As expected, the operators \( P^0_\mu \) are moved into non-isometric operators; this is confirmed by the presence of the non-derivative term. The de Sitter isometries are identified at the lie algebra level (A10) and then
realized on de Sitter space through (6.12). They read
\[
Y_{\mu}^{\nu} = P_{\mu}^{\nu} - \frac{H^2}{4} K_{\mu}^{\nu} \quad \text{(A8)}
\]
\[
= P_{\mu}^{0} - \frac{H^2}{4}(K_{\mu}^{0} - (d-2)x_{\mu}) \quad \text{(A9)}
\]
\[
= \partial_{\mu} - \frac{H^2}{4}(2x_{\mu}x^{\nu} - x^{2}\delta_{\mu}^{\nu})\partial_{\nu} \quad \text{(A10)}
\]
This can be interpreted as the expression of the infinitesimal translation from the point of view of a Minkowskian observer living in a de Sitter space: the Minkowskian translation plus a term due to the curvature.

Note that the combination \( I_{\mu}^{\mu} := K_{\mu}^{\mu} - x^{2}P_{\mu}^{\mu} \) is invariant under \( \mathfrak{H}_{\mu} \), that is \( [\mathfrak{H}_{\mu}, I_{\mu}^{\mu}] = 0 \) for all values of \( H \).

We finally comment about the expression of the conformal field equation in de Sitter space \( (H > 0) \) which reads \( O\phi^{\mu} = 0 \), with \( O := (\Box_{\mu} + d(d-2)H^2/4) \). One can be tempted to identify the operator \( O \) with the moved \( d'\text{Alembertian} \) \( O' := \mathfrak{H}_{\mu} \mathfrak{H}_{\mu}^{-1} \). In fact a direct calculation shows that
\[
O = \mathfrak{H}_{\mu}^{-\frac{1}{2}} \mathfrak{H}_{\mu}^{\frac{1}{2}} O'.
\]
Of course, both operators equal to zero on \( \mathfrak{H}_{\mu} \).

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APPENDIX A

Here are the conventions about indices:
\[
\begin{align*}
\alpha, \beta, \gamma, \delta, \ldots &= d + 1, 0, \ldots, d, \\
\mu, \nu, \rho, \sigma, \ldots &= 0, \ldots, d - 1, \\
i, j, k, l, \ldots &= 1, \ldots, d - 1.
\end{align*}
\]
The coefficients of the metric of \( \text{diag}(1,1,-1,-1) \) of \( \mathbb{R}^{d+2} \) are denoted \( \eta_{\alpha \beta} \):
\[
\eta_{d+1 \ d+1} = \eta_{00} = 1 = -\eta_{ii} = -\eta_{dd}.
\]

In addition, when no confusion is possible we use the superscript 0 \( (H) \) to denote a Minkowskian (de Sitterian) quantity.

The generators of the algebra \( \text{so}(2,d) \) are \( X_{\alpha \beta} = y_{\alpha} \partial_{\beta} - y_{\beta} \partial_{\alpha} \), they satisfy
\[
[X_{\alpha \beta}, X_{\gamma \delta}] = \eta_{\beta \gamma}X_{\alpha \delta} + \eta_{\alpha \delta}X_{\beta \gamma} - \eta_{\alpha \gamma}X_{\beta \delta} - \eta_{\beta \delta}X_{\alpha \gamma}.
\]
In the basis
\[
M_{\mu \nu} = X_{\mu \nu}, \\
P_{\mu} = \frac{1}{2}(X_{(d+1)\mu} - X_{d\mu}), \\
D = X_{45}, \\
K_{\mu} = 2(X_{(d+1)\mu} + X_{d\mu}),
\]
the \( \text{so}(2,d) \) algebra reads
\[
[D, M_{\mu \nu}] = 0, \quad (A1)
\]
\[
[D, K_{\mu}] = K_{\mu}, \quad (A2)
\]
\[
[P_{\mu}, D] = P_{\mu}, \quad (A3)
\]
\[
[P_{\mu}, K_{\nu}] = 2(\eta_{\mu \nu}D - M_{\mu \nu}), \quad (A4)
\]
\[
[K_{\mu}, M_{\nu \rho}] = \eta_{\mu \nu}K_{\rho} - \eta_{\mu \rho}K_{\nu}, \quad (A5)
\]
\[
[K_{\mu}, K_{\nu}] = 0, \quad (A6)
\]
\[
[M_{\mu \nu}, M_{\rho \sigma}] = \eta_{\nu \rho}M_{\mu \sigma} + \eta_{\nu \sigma}M_{\mu \rho} - \eta_{\mu \rho}M_{\nu \sigma} - \eta_{\mu \sigma}M_{\nu \rho}, \quad (A7)
\]
\[
[P_{\mu}, P_{\nu}] = 0, \quad (A8)
\]
\[
[P_{\rho}, M_{\mu \nu}] = \eta_{\mu \rho}P_{\nu} - \eta_{\nu \rho}P_{\mu}. \quad (A9)
\]

We remind from [2] that the \((d-1)(d-2)/2\) generators \( X_{ij} \) and the \( d - 1 \) generators \( X_{0i} \) of \( \text{so}(2,d) \) leave \( X_{\mu} \) invariant. The \( d \) more generators which also leaves \( X_{\mu} \) invariant are found to be
\[
Y_{\mu} := \frac{1}{2}(1 - H^2)X_{(d+1)\mu} - \frac{1}{2}(1 + H^2)X_{d\mu}. \quad (A10)
\]
A straightforward calculation leads to the following commutations relations:
\[
[Y_{\mu}, Y_{\nu}] = H^2X_{\mu \nu}, \quad (A11)
\]
\[
[Y_{\rho}, X_{\mu \nu}] = \eta_{\mu \rho}Y_{\nu} - \eta_{\nu \rho}Y_{\mu}. \quad (A12)
\]

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