Boundary Korn Inequality and Neumann Problems in Homogenization of Systems of Elasticity

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Abstract

This paper concerns with a family of elliptic systems of linear elasticity with rapidly oscillating periodic coefficients, arising in the theory of homogenization. We establish uniform optimal regularity estimates for solutions of Neumann problems in a bounded Lipschitz domain with $L^2$ boundary data. The proof relies on a boundary Korn inequality for solutions of systems of linear elasticity and uses a large-scale Rellich estimate obtained in [21].

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1 Introduction

This paper concerns with a family of elliptic systems of linear elasticity with rapidly oscillating periodic coefficients, arising in the theory of homogenization. We establish uniform optimal regularity estimates for solutions of Neumann problems in a bounded Lipschitz domain with $L^2$ boundary data. The proof relies on a boundary Korn inequality for solutions of systems of linear elasticity and uses a large-scale Rellich estimate obtained in [21].

More precisely, we consider a family of elasticity operators,

$$\mathcal{L}_\varepsilon = -\text{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i} \left\{ a^{\alpha\beta}_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} \right\}, \quad \varepsilon > 0,$$

(1.1)
where \( A(y) = (a_{ij}^{\alpha \beta}(y)) \) with \( 1 \leq i, j, \alpha, \beta \leq d \). Throughout this paper we will assume that the coefficient matrix (tensor) \( A \) satisfies the elasticity condition,

\[
a_{ij}^{\alpha \beta}(y) = a_{ji}^{\beta \alpha}(y), \quad \kappa_1 |\xi|^2 \leq a_{ij}^{\alpha \beta}(y) \xi^\alpha \xi^\beta \leq \kappa_2 |\xi|^2 \tag{1.2}
\]

for \( y \in \mathbb{R}^d \) and for symmetric matrix \( \xi = (\xi_{ij}^\alpha) \in \mathbb{R}^{d \times d} \), where \( \kappa_1, \kappa_2 > 0 \) (the summation convention is used throughout the paper). We will also assume that \( A(y) \) is 1-periodic,

\[
A(y + z) = A(y) \quad \text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d, \tag{1.3}
\]

and is Hölder continuous,

\[
\|A\|_{C^\sigma(\mathbb{R}^d)} \leq M \tag{1.4}
\]

for some \( \sigma \in (0, 1) \) and \( M > 0 \).

The following is the main result of the paper.

**Theorem 1.1.** Assume that \( A \) satisfies conditions (1.2), (1.3) and (1.4). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Then for any \( g \in L^2_{\mathcal{R}}(\partial \Omega) \), there exists a weak solution \( u_\varepsilon \), unique up to a rigid displacement, to the Neumann problem,

\[
\begin{aligned}
\mathcal{L}_\varepsilon(u_\varepsilon) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} &= g \quad \text{n.t. on } \partial \Omega, \\
(\nabla u_\varepsilon)^* &\in L^2(\partial \Omega),
\end{aligned}
\tag{1.5}
\]

and the solution \( u_\varepsilon \) satisfies the estimate

\[
\|(\nabla u_\varepsilon)^*\|_{L^2(\partial \Omega)} \leq C \|g\|_{L^2(\partial \Omega)}, \tag{1.6}
\]

where \( C \) depends only on \( d, \kappa_1, \kappa_2, (\sigma, M) \) and the Lipschitz character of \( \Omega \).

We introduce the notations used in Theorem 1.1 and hereafter. We use

\[
\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = n \cdot A(x/\varepsilon) \nabla u_\varepsilon
\]

to denote the conormal derivative on \( \partial \Omega \) of \( u_\varepsilon \), associated with the operator \( \mathcal{L}_\varepsilon \), where \( n \) is the outward unit normal to \( \partial \Omega \). The boundary value in (1.5) is taken a.e. in the sense of nontangential (n.t.) convergence. By \( (w)^* \) we mean the nontangential maximal function of \( w \), defined by

\[
(w)^*(z) = \sup \left\{ |w(x)| : x \in \Omega \text{ and } |x - z| < C_0 \delta(x) \right\} \tag{1.7}
\]

for \( z \in \partial \Omega \), where \( \delta(x) = \text{dist}(x, \partial \Omega) \) and \( C_0 > 1 \) is a large constant depending on the Lipschitz character of \( \Omega \). Also,

\[
\mathcal{R} = \left\{ \phi = Bx + b : B \in \mathbb{R}^{d \times d} \text{ is skew-symmetric and } b \in \mathbb{R}^d \right\} \tag{1.8}
\]
is the space of rigid displacements and
\[
L^2_R(\partial \Omega) = \left\{ g \in L^2(\partial \Omega; \mathbb{R}^d) : \int_{\partial \Omega} g \cdot \phi = 0 \text{ for any } \phi \in \mathcal{R} \right\}.
\]  

Boundary value problems in Lipschitz domains with \(L^p\) boundary data have been studied extensively since late 1970’s. We refer the reader to the book [11] for references in this area up to mid-1990’s and to [12, 26, 19, 20, 15, 16, 10, 6] and their references for more recent work. For elliptic systems of linear elasticity, in the case of a homogeneous isotropic body, where the coefficients are constants and given by
\[
a_{ij}^{\alpha \beta} = \mu \delta_{ij} \delta_{\alpha \beta} + \lambda \delta_{i\alpha} \delta_{j\beta} + \mu \delta_{i\beta} \delta_{j\alpha}
\]
with Lamé constants \(\lambda\) and \(\mu\). Theorem [11] as well as the corresponding results for Dirichlet problem was proved in [5] (also see [24, 7, 8, 9, 18]), by the method of layer potentials. The general case of elliptic systems of elasticity with constant coefficients satisfying (1.2) was treated in [25]. Our estimate (1.6) is new for variable coefficients, even in the (local) case \(\varepsilon = 1\).

For elliptic equations and systems with rapidly oscillating periodic coefficients, the Dirichlet and Neumann problems with \(L^p\) boundary data were studied in [2, 1, 3, 15, 16, 14]. In particular, if \(\Omega\) is Lipschitz, the uniform estimate (1.6) in Theorem 1.1 as well as the corresponding estimates (1.11), (1.12) hold for solutions of Dirichlet problem:
\[
L^\varepsilon \left( u^\varepsilon \right) = 0 \text{ in } \Omega \text{ and } u^\varepsilon = f \text{ on } \partial \Omega,
\]
for solutions of Dirichlet problem: \(\mathcal{L}_\varepsilon(u^\varepsilon) = 0\) in \(\Omega\) and \(u^\varepsilon = f\) on \(\partial \Omega\), was established in [16], where it is assumed that \(A = \left( a_{ij}^{\alpha \beta} \right)\), with \(1 \leq i, j \leq d\) and \(1 \leq \alpha, \beta \leq m\), is 1-periodic, symmetric, Hölder continuous, and satisfies the very strong ellipticity condition or the Legendre condition,
\[
\mu |\xi|^2 \leq a_{ij}^{\alpha \beta}(y) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu} |\xi|^2 \text{ for } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d},
\]
where \(\mu > 0\). We mention that for a scalar elliptic equation \((m = 1)\) in a Lipschitz domain, the estimate (1.11) is due to B. Dahlberg (unpublished), while (1.12) and (1.6) were proved in [15]. Also, if \(\Omega\) is a bounded \(C^{1,\alpha}\) domain and \(1 < p < \infty\), the \(L^p\) analogous of estimates (1.11), (1.6) and (1.12) may be found in [2, 1, 14] under the ellipticity condition (1.13).

To describe the main difficulties in the study of the \(L^2\) Neumann problem (1.5) for systems of elasticity and our approach to Theorem 1.1 as well as the structure of this paper, we first note that it is possible to rewrite the system \(\text{div}(\tilde{A}(x/\varepsilon)\nabla u^\varepsilon) = 0\) as \(\text{div}(\tilde{A}(x/\varepsilon)\nabla u^\varepsilon) = 0\) in such a way that the coefficient matrix \(\tilde{A}(y)\) is symmetric and satisfies (1.13). This allows us to use the interior Lipschitz estimates in [11] and the optimal estimates for Dirichlet problem in [16]. As a result, estimates (1.11) and (1.12) hold for solutions of Dirichlet Problem for elliptic systems of elasticity (see Section 2).
We remark that the same technique was used in [5] in the case of a homogeneous isotropic body. Although the rewriting of the system of elasticity $L_\varepsilon(u_\varepsilon) = 0$ does not change Dirichlet problem, it does change the Neumann problem as the conormal derivative $\partial u_\varepsilon/\partial \nu_\varepsilon$ depends on the coefficient matrix. Nevertheless, it makes the method of layer potentials available to $L_\varepsilon$, since the estimates of fundamental solutions and layer potentials only involve the interior estimates. As in the case of constant coefficients [24, 5, 8, 25], to use the method of layer potentials for $L_2$ Neumann problems in Lipschitz domains, the key step is to establish the following Rellich estimate,

$$
\int_{\partial \Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial \Omega} \left| \partial u_\varepsilon / \partial \nu_\varepsilon \right|^2 d\sigma + \frac{C}{r_0^2} \int_{\Omega} |\nabla u_\varepsilon|^2 dx,
$$

(1.14)

for suitable solutions of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, where $r_0 = \text{diam}(\Omega)$. In fact, we will show in Section 3 that for a given pair of $(\Omega, \varepsilon)$, the estimate (1.14) is equivalent to (1.6).

In comparison to elliptic systems with coefficients satisfying (1.13) [16], one of the main difficulties in proving (1.14) is caused by the fact that from the elasticity condition (1.2) one only obtains

$$
\int_{\partial \Omega} A(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon d\sigma \geq c \int_{\partial \Omega} |\nabla u_\varepsilon + (\nabla u_\varepsilon)^T|^2 d\sigma,
$$

(1.15)

where $c > 0$ and $(\nabla u_\varepsilon)^T$ denotes the transpose of the $d \times d$ matrix $\nabla u_\varepsilon$. As a result, to control the full gradient $\nabla u_\varepsilon$ on $\partial \Omega$, we would need some Korn type inequality on the boundary. We remark that the same issue already appears at the small scale, where $\varepsilon = 1$ and $\text{diam}(\Omega) \leq 1$, even in the case of constant coefficients [5, 25]. Also, the techniques developed in [16] for treating the large-scale estimates fail due to the lack of (uniform) Korn inequalities on boundary layers.

Our proof of (1.14) is motivated by the work in [5, 25, 16] and involves several new ideas. We divide the proof into five steps. Note that the first four steps treat the case of small scales, in which the estimates are local and does not use the periodicity assumption.

Step 1. Let $\varepsilon = 1$ and $\Omega$ be a Lipschitz domain with $r_0 = \text{diam}(\Omega) \leq (1/4)$. We establish the boundary Korn inequality

$$
\|\nabla u\|_{L^2(\partial \Omega)} \leq C \|\nabla u + (\nabla u)^T\|_{L^2(\partial \Omega)} + C r_0^{-1/2} \|\nabla u\|_{L^2(\Omega)}
$$

(1.16)

for solutions of $L_1(u) = 0$ in $\Omega$, under the additional assumption:

$$
|\nabla A(x)| \leq M_0[\delta(x)]^{\sigma-1} \quad \text{and} \quad |\nabla^2 A(x)| \leq M_0[\delta(x)]^{\sigma-2}
$$

(1.17)

for any $x \in \Omega$, where $\sigma \in (0, 1)$ and $M_0 > 0$. To do this we rewrite the system $L_1(u) = 0$ in $\Omega$ as

$$
\mu \Delta u^\alpha + \frac{\partial}{\partial x_i} \left( b_{ij}^\alpha \frac{\partial u^\beta}{\partial x_j} \right) = 0 \quad \text{in} \ \Omega,
$$
where $B = (b_{ij}^{\alpha\beta})$ satisfies the elasticity condition (1.2), with (different) constants depending on $\kappa_1$ and $\kappa_2$. We then factor the matrix $B(x)$ so that

$$b_{ij}^{\alpha\beta} = q_i^{\alpha} q_j^{\beta},$$

where $t$ is summed from 1 to $m = d(d+1)/2$. Using the boundary Korn inequality (1.16) for harmonic functions [5] as well as Dahlberg’s bilinear estimate [4, 22], the problem is reduced to the estimates of the nontangential maximal function and the square function of

$$v = (v^t) = \left( q_j^{\beta} \frac{\partial u^\beta}{\partial x_j} \right).$$

To complete the step we observe that $v$ is a solution of an $m \times d$ elliptic system

$$L(v) = F_0 + \text{div}(F_1) \quad \text{in } \Omega,$$

with coefficients satisfying the Legendre condition (1.13). This allows us to use the estimates in [16] to obtain the boundary Korn inequality (1.16). The details of the argument is given in Section 5, while some auxiliary estimates needed for handling terms with $F_0$ and $F_1$ are given in Section 4.

Step 2. Establish the Rellich estimate (1.14) for $\varepsilon = 1$ and Lipschitz domain $\Omega$ with $\text{diam}(\Omega) \leq (1/4)$, under the additional assumption (1.17). The proof uses a Rellich identity and (1.16). See Section 6.

Step 3. Let $\varepsilon = 1$ and $\Omega$ be a Lipschitz domain with $\text{diam}(\Omega) \leq (1/4)$. Prove (1.14) without the condition (1.17). This is done in Section 7, using an approximation scheme taken from [16], with the help of layer potentials.

Step 4. Let $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, where $0 < \varepsilon < \text{diam}(\Omega) < \infty$. It follows from Step 3 by some localization and rescaling techniques that

$$\frac{1}{\varepsilon} \int_{\Omega_r} |\nabla u_\varepsilon|^2 \, dx \leq C \int_{\partial \Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_{\varepsilon}} |\nabla u_\varepsilon|^2 \, dx,$$

(1.18)

where $\Omega_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < r \}$.

Step 5. To control the last term in (1.18), we use the following estimate

$$\frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^2 \, dx \leq C \int_{\partial \Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right|^2 \, d\sigma + \frac{C}{r_0} \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx,$$

(1.19)

for any $0 \leq r < r_0 = \text{diam}(\Omega)$. The estimate (1.19), which is due to homogenization of $\mathcal{L}_\varepsilon$, was proved by the second author in [21] for weak solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ under the conditions (1.2) and (1.3), using a sharp convergence rate in $H^1(\Omega)$ for a two-scale expansion of $u_\varepsilon$. Note that no smoothness condition is needed for (1.19). We remark that estimate (1.19) may be regarded as a large-scale Rellich estimate for two reasons. Firstly, by combining it with the small-scale estimate (1.18), we obtain the full Rellich estimate (1.14) and thus complete the proof of Theorem 1.1 (see Section 8). Secondly, if we are allowed to take the limit $r \to 0$ in (1.19), as in the case of constant coefficients, we would recover the estimate (1.14).
We will use $C$ and $c$ to denote constants that depend at most on $d$, $\kappa_1$, $\kappa_2$, $(\sigma, M)$ in (1.4), and the Lipschitz character of $\Omega$. If a constant also depends on other parameters, it will be pointed out explicitly. Finally, we shall use $\|g\|_2$ to denote the norm of $g$ in $L^2(\partial\Omega)$. The norm in $L^2(\Omega)$ will be denoted by $\|u\|_{L^2(\Omega)}$.

2 Preliminaries

Let $A(y) = (a_{ij}^{\alpha\beta}(y))$, where $a_{ij}^{\alpha\beta}(y)$ satisfies the elasticity condition (1.2). Let

$$\tilde{a}_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + \mu \delta_{i\alpha}\delta_{j\beta} - \mu \delta_{i\beta}\delta_{j\alpha}$$

(2.1)

and

$$b_{ij}^{\alpha\beta}(y) = \tilde{a}_{ij}^{\alpha\beta}(y) - \mu \delta_{ij}\delta_{\alpha\beta},$$

(2.2)

where $\mu = \kappa_1/4 > 0$. Note that

$$\tilde{a}_{ij}^{\alpha\beta} = \tilde{a}_{ji}^{\alpha\beta}, \quad b_{ij}^{\alpha\beta} = b_{ji}^{\alpha\beta} = b_{\alpha\beta}^{ij}$$

for any $1 \leq i, j, \alpha, \beta \leq d$. (2.3)

**Proposition 2.1.** Let $\tilde{a}_{ij}^{\alpha\beta}$ be defined by (2.1). Then

$$\tilde{a}_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} \geq \mu |\xi|^2$$

for any $\xi = (\xi_i^{\alpha}) \in \mathbb{R}^{d \times d}$. (2.4)

**Proof.** Note that by (1.2),

$$a_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} \geq \frac{\kappa_1}{4} |\xi + \xi^T|^2$$

for any $\xi = (\xi_i^{\alpha}) \in \mathbb{R}^{d \times d}$, (2.5)

where $\xi^T$ denotes the transpose of $\xi$. It follows that

$$b_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} = a_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} - \mu |\xi|^2 + \mu \xi_i^{\alpha} \xi_j^{\beta} - \mu \xi_j^{\beta} \xi_i^{\alpha}$$

$$\geq \frac{\kappa_1}{4} |\xi + \xi^T|^2 - \mu |\xi|^2 - \mu \xi_i^{\alpha} \xi_j^{\beta}$$

$$= \frac{\kappa_1}{2} (|\xi|^2 + \xi_i^{\alpha} \xi_j^{\beta} - \mu |\xi|^2 - \mu \xi_j^{\beta} \xi_i^{\alpha})$$

$$= \frac{1}{2} \left( \frac{\kappa_1}{2} - \mu \right) |\xi + \xi^T|^2.$$

Since $\mu = \kappa_1/4$, we obtain

$$b_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} \geq \frac{\kappa_1}{8} |\xi + \xi^T|^2$$

for any $\xi = (\xi_i^{\alpha}) \in \mathbb{R}^{d \times d}$, (2.6)

and

$$\tilde{a}_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} = \mu |\xi|^2 + b_{ij}^{\alpha\beta} \xi_i^{\alpha} \xi_j^{\beta} \geq \mu |\xi|^2$$

for any $\xi = (\xi_i^{\alpha}) \in \mathbb{R}^d$. \qed
Proposition 2.2. Let \( \tilde{A}(y) = (\tilde{a}^{\alpha\beta}_j(y)) \) and \( u_\varepsilon \in H^1_{\text{loc}}(\Omega; \mathbb{R}^d) \). Then
\[
\text{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F \quad \text{in } \Omega \quad \text{if and only if} \quad \text{div}(\tilde{A}(x/\varepsilon)\nabla u_\varepsilon) = F \quad \text{in } \Omega,
\]
where \( F \in (C^\infty(\Omega; \mathbb{R}^d))^' \) is a distribution.

Proof. Let \( u = (u^\alpha) \in C^\infty(\Omega; \mathbb{R}^d) \) and \( \varphi = (\varphi^\alpha) \in C^\infty_0(\Omega; \mathbb{R}^d) \). It follows from integration by parts that
\[
\int_\Omega \left\{ \delta_i\alpha \delta_j\beta - \delta_i\beta \delta_j\alpha \right\} \frac{\partial w^\beta}{\partial x_j} \cdot \frac{\partial \varphi^\alpha}{\partial x_i} = \int_\Omega \text{div}(u) \cdot \text{div}(\varphi) - \int_\Omega \frac{\partial w^\beta}{\partial x_\alpha} \cdot \frac{\partial \varphi^\alpha}{\partial x_\beta} = 0. \tag{2.7}
\]

By a density argument \((2.7)\) continues to hold for any \( u \in H^1_{\text{loc}}(\Omega; \mathbb{R}^d) \) and \( \varphi \in C^\infty_0(\Omega; \mathbb{R}^d) \). Hence,
\[
\int_\Omega A(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi \, dx - \int_\Omega \tilde{A}(x/\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi \, dx = 0
\]
for any \( u_\varepsilon \in H^1_{\text{loc}}(\Omega; \mathbb{R}^d) \) and \( \varphi \in C^\infty_0(\Omega; \mathbb{R}^d) \).

Proposition 2.2 reduces interior estimates for the operator \( L_\varepsilon \) to those for \( \tilde{L}_\varepsilon \), where
\[
\tilde{L}_\varepsilon = -\text{div}(\tilde{A}(x/\varepsilon)\nabla).
\tag{2.8}
\]
Note that by Proposition 2.1, the coefficient matrix \( \tilde{A} \) satisfies the very strong ellipticity condition \( (1.13) \). It follows directly from [1] that if \( L_\varepsilon(u_\varepsilon) = 0 \) in \( B(x_0, r) \), then
\[
|\nabla u_\varepsilon(x_0)| \leq C \left( \int_{B(x_0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2}, \tag{2.9}
\]
where \( C \) depends only on \( \kappa_1, \kappa_2, \) and \( (\sigma, M) \). Proposition 2.2 also shows that Dirichlet problem for \( L_\varepsilon \) is the same as that for \( \tilde{L}_\varepsilon \). Since \( \tilde{A} \) is very strongly elliptic, symmetric, and Hölder continuous, the results in [16] for Dirichlet problems gives the following theorem.

Theorem 2.3. Suppose that \( A \) satisfies conditions \( (1.2), (1.3) \) and \( (1.4) \). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Then for any \( f \in L^2(\partial \Omega; \mathbb{R}^d) \), there exists a unique solution \( u_\varepsilon \) to Dirichlet problem,
\[
\begin{cases}
L_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\
u_\varepsilon = f & \text{n.t. on } \partial \Omega, \\
(u_\varepsilon)^* \in L^2(\partial \Omega),
\end{cases}
\tag{2.10}
\]
and \( u_\varepsilon \) satisfies the estimate \( (1.11) \). Furthermore, if \( f \in W^{1,2}(\partial \Omega; \mathbb{R}^d) \), then the solution satisfies \( (1.12) \). The constants \( C \) in \( (1.11) \) and \( (1.12) \) depend only on \( \kappa_1, \kappa_2, (\sigma, M) \), and the Lipschitz character of \( \Omega \).
We end this section with a Rellich estimate for $L_{\varepsilon}$ in $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$. We will use $\nabla_{\text{tan}} u$ to denote the tangential gradient of $u$ on $\partial \Omega$.

**Theorem 2.4.** Assume that $A$ and $\Omega$ satisfy the same conditions as in Theorem 2.3. Suppose that $L_{\varepsilon}(u_{\varepsilon}) = 0$ in $\Omega_- = \mathbb{R}^d \setminus \Omega$, $\nabla u_{\varepsilon}^* \in L^2(\partial \Omega)$, and $\nabla u_{\varepsilon}$ exists n.t. on $\partial \Omega$. Then
\[
\int_{\partial \Omega} |\nabla u_{\varepsilon}|^2 d\sigma \leq C \int_{\partial \Omega} |\nabla_{\text{tan}} u_{\varepsilon}|^2 d\sigma + \frac{C}{r_0} \int_{\Omega_-} |\nabla u_{\varepsilon}|^2 dx,
\]
where $r_0 = \text{diam}(\Omega)$ and $C$ depends only on $\kappa_1, \kappa_2, (\sigma, M)$, and the Lipschitz character of $\Omega$.

**Proof.** Fix $z \in \partial \Omega$. Since $\widetilde{L}_{\varepsilon}(u_{\varepsilon}) = 0$ in $B(z, r_0) \cap \Omega_-$, it follows from [16] that
\[
\int_{B(z, cr_0) \cap \partial \Omega} |\nabla u_{\varepsilon}|^2 d\sigma \leq C \int_{B(z, 2cr_0) \cap \partial \Omega} |\nabla_{\text{tan}} u_{\varepsilon}|^2 d\sigma + \frac{C}{r_0} \int_{B(z, 2cr_0) \cap \Omega_-} |\nabla u_{\varepsilon}|^2 dx.
\]
Estimate (2.11) follows from this by covering $\partial \Omega$ with a finite number of balls $\{B(z_j, cr_0)\}$ centered on $\partial \Omega$.

## 3 Method of layer potentials

In [16] the $L^2$ Dirichlet and Neumann problems in Lipschitz domains for elliptic systems with rapidly oscillating periodic coefficients satisfying (1.13) are solved by the method of layer potentials. To solve the $L^2$ Neumann problem for the elasticity operator $L_{\varepsilon}$, we will also use the method of layer potentials. Let $\Gamma_{\varepsilon}(x, y) = (\Gamma_{\varepsilon}^{ab}(x, y))$ denote the $d \times d$ matrix of fundamental solutions for $L_{\varepsilon}$ (and $\widetilde{L}_{\varepsilon}$). It follows from the interior Lipschitz estimate (2.9) that
\[
|\Gamma_{\varepsilon}(x, y)| \leq C |x - y|^{2-d},
|\nabla_x \Gamma(x, y)| + |\nabla_y \Gamma_{\varepsilon}(x, y)| \leq C |x - y|^{1-d},
|\nabla_x \nabla_y \Gamma_{\varepsilon}(x, y)| \leq C |x - y|^{-d},
\]
for any $x, y \in \mathbb{R}^d$ and $x \neq y$ (some modifications are needed for the first estimate in the case $d = 2$). Let $u_{\varepsilon} = S_{\varepsilon}(f)$, where $f \in L^2(\partial \Omega; \mathbb{R}^d)$ and
\[
S_{\varepsilon}(f)(x) = \int_{\partial \Omega} \Gamma_{\varepsilon}(x, y) f(y) d\sigma(y)
\]
denotes the single layer potential with density $f$. Then $L_{\varepsilon}(u_{\varepsilon}) = 0$ in $\mathbb{R}^d \setminus \partial \Omega$ and $|u_{\varepsilon}(x)| = O(|x|^{2-d}), |\nabla u_{\varepsilon}(x)| = O(|x|^{-d})$ as $|x| \to \infty$.

We will use $(\nabla u_{\varepsilon})_\pm$ to denote the nontangential limits of $\nabla u_{\varepsilon}$ taken from $\Omega_\pm$ respectively, where $\Omega_+ = \Omega$ and $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$.
Theorem 3.1. Suppose that $A$ satisfies (1.2), (1.3) and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon = S_\varepsilon(f)$, where $f \in L^2(\partial \Omega; \mathbb{R}^d)$. Then
\[ \|(\nabla u_\varepsilon)^*\|_2 \leq C \|f\|_2, \] (3.3)
where $C$ depends only on $\kappa_1$, $\kappa_2$, $(\sigma, M)$, and the Lipschitz character of $\Omega$. Also, $(\nabla u_\varepsilon)_\pm$ exists n.t. on $\partial \Omega$,
\[ (\nabla_{\tan} u_\varepsilon)_+ = (\nabla_{\tan} u_\varepsilon)_- \text{ n.t. on } \partial \Omega, \] (3.4)
and
\[ \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_\pm = \left( \pm \frac{1}{2} I + K_\varepsilon \right) f \text{ n.t. on } \partial \Omega, \] (3.5)
where $K_\varepsilon$ is a bounded linear operator on $L^2(\partial \Omega; \mathbb{R}^d)$.

Proof. Estimates (3.3) and (3.4) follow directly from [16, Section 4], as $L_\varepsilon$ and $\tilde{L}_\varepsilon$ share the same fundamental solutions in $\mathbb{R}^d$. To see (3.5), one uses [16, Theorem 4.4] and also the fact that
\[ a^{\alpha\beta}_{ij}(y) \eta_i \eta_j = \tilde{a}^{\alpha\beta}_{ij}(y) \eta_i \eta_j \]
for any $\eta = (\eta_i) \in \mathbb{R}^d$. \qed

Note that $A(x/\varepsilon) \nabla v = 0$ in $\mathbb{R}^d$ for any $v \in \mathcal{R}$. Thus, if $u_\varepsilon = S_\varepsilon(f)$ and $v \in \mathcal{R}$, then
\[ \int_{\partial \Omega} \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \cdot v d\sigma = \int_{\Omega} A(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla v dx = 0. \]
It follows from this and (3.3) that the operator
\[ (1/2)I + K_\varepsilon : L^2_R(\partial \Omega) \rightarrow L^2_R(\partial \Omega) \]
is bounded uniformly in $\varepsilon > 0$. (3.6)

One of the main goals of this paper is to prove the following theorem, which will allow us to solve the $L^2$ Neumann problem (1.5) for $L_\varepsilon$ with optimal estimates.

Theorem 3.2. Suppose that $A$ satisfies (1.2), (1.3) and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Then $(1/2)I + K_\varepsilon : L^2_R(\partial \Omega) \rightarrow L^2_R(\partial \Omega)$ is invertible and
\[ \|f\|_2 \leq C \|((1/2)I + K_\varepsilon) f\|_2 \text{ for any } f \in L^2_R(\partial \Omega), \] (3.7)
where $C$ depends only on $\kappa_1$, $\kappa_2$, $(\sigma, M)$, and the Lipschitz character of $\Omega$.

We now give the proof of Theorem 1.1, assuming Theorem 3.2.

Theorem 3.2 $\implies$ Theorem 1.1. The uniqueness follows directly from Green’s identity,
\[ \int_{\Omega} A(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx = \int_{\partial \Omega} u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} d\sigma. \] (3.8)
To show the existence as well as the estimate (1.6), for any $g \in L^2_R(\partial \Omega)$, we choose $f \in L^2_R(\partial \Omega)$ such that $g = ((1/2)I + K_\varepsilon) f$ on $\partial \Omega$. Let $u_\varepsilon = S_\varepsilon(f)$. Then $u_\varepsilon$ is a solution to (1.5) and
\[ \|(\nabla u_\varepsilon)^*\|_2 \leq C \|f\|_2 \leq C \|g\|_2, \]
where the last inequality follows from Theorem 3.2. \qed
Remark 3.3. Let $A$ be a matrix satisfying (1.2), (1.3) and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose that the conclusions of Theorem 1.1 hold for $A$ and $\Omega$. Let $u_\varepsilon$ be a weak solution of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ such that $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$. Let $v_\varepsilon$ be the weak solution of $L_\varepsilon(v_\varepsilon) = 0$ in $\Omega$, given by Theorem 1.1 with Neumann data $\partial v_\varepsilon / \partial \nu_\varepsilon = \partial u_\varepsilon / \partial \nu_\varepsilon$ on $\partial\Omega$. Then

$$\|(\nabla v_\varepsilon)^*\|_2 \leq C \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_2. \quad (3.9)$$

Since $u_\varepsilon - v_\varepsilon = \phi \in \mathcal{R}$ in $\Omega$ and $\nabla \phi$ is constant, it follows that

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq 2 \int_{\partial\Omega} |\nabla v_\varepsilon|^2 d\sigma + 2 \int_{\partial\Omega} |\nabla \phi|^2 d\sigma$$

$$\leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma + \frac{C}{r_0^{d+1}} \int_{\Omega} |\nabla \phi|^2$$

$$\leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma + \frac{C}{r_0^{d+1}} \int_{\Omega} |\nabla v_\varepsilon|^2 + \frac{C}{r_0^{d+1}} \int_{\Omega} |\nabla u_\varepsilon|^2,$$

where $r_0 = \text{diam}(\Omega)$. This, together with (3.9), gives

$$\int_{\partial\Omega} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 d\sigma + \frac{C}{r_0^{d+1}} \int_{\Omega} |\nabla u_\varepsilon|^2.$$  \quad (3.10)

In particular, for a given triple $(\varepsilon, A, \Omega)$, the invertibility of $(1/2)I + K_\varepsilon$ on $L^2_\mathcal{R}(\partial\Omega)$ and estimate (3.7) imply the Rellich estimate (3.11).

Next we will show that the Rellich estimate (3.11) implies (3.7). To do this we need some estimates for volume integrals.

Lemma 3.4. Let $u_\varepsilon$ be a weak solution of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$. Suppose that $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$. Then

$$\|\nabla u_\varepsilon + (\nabla u_\varepsilon)^T\|_{L^2(\Omega)} \leq C r_0^{1/2} \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_2, \quad (3.12)$$

where $r_0 = \text{diam}(\Omega)$ and $\Omega$ depends only on $\kappa_1$, $\kappa_2$, and the Lipschitz character of $\Omega$.

Proof. By rescaling we may assume that $r_0 = 1$. It follows by Green’s identity that

$$\int_{\Omega} A(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx = \int_{\partial\Omega} (u_\varepsilon - v) \cdot \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \, d\sigma,$$  \quad (3.13)

for any $v \in \mathcal{R}$. Hence, by (2.5) and the Cauchy inequality,

$$\frac{\kappa_1}{4} \int_{\Omega} |\nabla u_\varepsilon + (\nabla u_\varepsilon)^T|^2 \, dx \leq \|u_\varepsilon - v\|_2 \left\| \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_2.$$  \quad (3.14)
Using the well known trace inequality,
\[\int_{\partial \Omega} |u|^2 d\sigma \leq C \int_{\Omega} |\nabla u|^2 dx + C \int_{\Omega} |u|^2 dx\]  (3.15)
for \(u \in H^1(\Omega)\), and the second Korn inequality [17],
\[\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^2 dx \leq C \int_{\Omega} |\nabla u + (\nabla u)^T|^2 dx,\]  (3.16)
which holds for any \(u \in H^1(\Omega; \mathbb{R}^d)\) such that \(u \perp \mathcal{R}\) in \(L^2(\Omega; \mathbb{R}^d)\), we see that
\[\|u_\varepsilon - v\|_2 \leq C \|\nabla u_\varepsilon + (\nabla u_\varepsilon)^T\|_{L^2(\Omega)},\]  (3.17)
if \(v \in \mathcal{R}\) is chosen so that \(u_\varepsilon - v \perp \mathcal{R}\) in \(L^2(\Omega; \mathbb{R}^d)\). This, together with (3.14), gives (3.12).

**Lemma 3.5.** Let \(u_\varepsilon = S_\varepsilon(f)\), where \(f \in L^2_\mathcal{R}(\partial \Omega)\). Then
\[\int_{\Omega_+} |\tilde{A}(x/\varepsilon) \nabla u_\varepsilon|^2 dx + \int_{\Omega_-} |\nabla u_\varepsilon|^2 dx \leq C \|f\|_2 \left\| \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_+ \right\|_2,\]  (3.18)
where \(r_0 = \text{diam}(\Omega)\) and \(C\) depends only on \(\kappa_1, \kappa_2\), and the Lipschitz character of \(\Omega\).

**Proof.** By rescaling we may assume that \(r_0 = 1\). Let \(\partial u_\varepsilon/\partial \nu_\varepsilon\) denote the conormal derivative on \(\partial \Omega\) of \(u_\varepsilon\) with respect to the operator \(\tilde{L}_\varepsilon\), given by (2.8). Since \(\tilde{L}_\varepsilon(u_\varepsilon) = 0\) in \(\mathbb{R}^d \setminus \partial \Omega\), it follows from integration by parts that
\[\int_{\Omega_+} \tilde{A}(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx = \pm \int_{\partial \Omega} u_\varepsilon \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_\pm d\sigma.\]  (3.19)
This, together with the jump relation
\[f = \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_+ - \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_- \quad \text{on } \partial \Omega,\]  (3.20)
gives
\[\int_{\Omega_+} \tilde{A}(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx + \int_{\Omega_-} \tilde{A}(x/\varepsilon) \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx = \int_{\partial \Omega} u_\varepsilon \cdot f d\sigma.\]  (3.21)
Hence, by the very strong ellipticity of \(\tilde{A}\),
\[\int_{\Omega_+} |\nabla u_\varepsilon|^2 dx + \int_{\Omega_-} |\nabla u_\varepsilon|^2 dx \leq C \int_{\partial \Omega} u_\varepsilon \cdot f d\sigma\]
\[\leq C \|u_\varepsilon - v\|_2 \|f\|_2,\]  (3.22)
for any \(v \in \mathcal{R}\), where we have used the fact \(f \in L^2_\mathcal{R}(\partial \Omega)\). As in the proof of the last proposition, we may choose \(v \in \mathcal{R}\) so that
\[\|u_\varepsilon - v\|_2 \leq C \|\nabla u_\varepsilon + (\nabla u_\varepsilon)^T\|_{L^2(\Omega)} \leq C \left\| \left( \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} \right)_+ \right\|_2,\]
where we have used (3.12) for the last step. This, together with (3.22), yields (3.18). \(\square\)
Remark 3.6. Let $A$ be a matrix satisfying (1.2), (1.3) and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Suppose that for any weak solution $u_\varepsilon$ of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ with the properties that $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$, the Rellich estimate (3.11) holds. We claim that this implies the estimate (3.7) for the operator $(1/2)I + K_\varepsilon$. Indeed, let $u_\varepsilon = S_\varepsilon(f)$ for some $f \in L^2_\kappa(\partial\Omega)$. It follows from (2.11) and (3.11) as well as the jump relation
\[
f = \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ - \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_- \quad \text{on } \partial\Omega,
\]
that
\[
\|f\|_2 \leq \left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ \right\|_2 + \left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_- \right\|_2
\leq \left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ \right\|_2 + C\|\nabla \tan u_\varepsilon\|_2 + Cr_\varepsilon^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_-)}
\leq C\left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ \right\|_2 + Cr_\varepsilon^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_-)}
+C\|\nabla \tan u_\varepsilon\|_{L^2(\Omega_+)} + C r_\varepsilon^{-1/2}\|\nabla u_\varepsilon\|_{L^2(\Omega_+)};
\]
where we also used the fact $(\nabla \tan u_\varepsilon)_+ = (\nabla \tan u_\varepsilon)_-$ on $\partial\Omega$. This, together with Lemma 3.5, gives
\[
\|f\|_2 \leq C\left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ \right\|_2 + C\|f\|_2^{1/2}\left\| \left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)_+ \right\|_2^{1/2},
\]
which, by the Cauchy inequality with an $\varepsilon > 0$, leads to the estimate (3.7).

The remark above reduces Theorem 3.2 to the following.

Theorem 3.7. Assume that $A$ satisfies the conditions (1.2), (1.3) and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $u_\varepsilon$ be a weak solution of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ with the properties that $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$. Then
\[
\int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\sigma \leq C\left(\int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right|^2 \, d\sigma + \frac{C}{r_0} \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx\right),
\]
where $r_0 = \text{diam}(\Omega)$ and $C$ depends only on $\kappa_1$, $\kappa_2$, $(\sigma, M)$ and the Lipschitz character of $\Omega$.

Theorem 3.7 $\implies$ Theorem 3.2. It follows from Remark 3.6 that the Rellich estimate (3.24) implies the inequality (3.7). To show $(1/2)I + K_\varepsilon$ is invertible on $L^2_\kappa(\partial\Omega)$, we apply a continuity argument by considering the matrix
\[
A^t(y) = tA(y) + (1 - t)A^0,
\]
where $t \in [0, 1]$ and $A^0$ is a constant matrix with entries $\delta_{ij}\delta_{i\beta} + \delta_{i\alpha}\delta_{j\beta} + \delta_{ij}\delta_{\alpha\beta}$. Let $L_\varepsilon^t = -\text{div}(A^t(x/\varepsilon)\nabla)$ and $(1/2)I + K_\varepsilon^t$ be the corresponding operator, associated with $L_\varepsilon^t$. Note that $A^t$ satisfies conditions (1.2), (1.3) and (1.4) with constants depending only on $\kappa_1$, $\kappa_2$, $(\sigma, M)$. It follows from Theorem 3.7 that the estimate (3.24) holds for solutions.
of $L^t_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, with constant $C$ depending only on $\kappa_1$, $\kappa_2$, $(\sigma,M)$ and the Lipschitz character of $\Omega$. Consequently, by Remark $3.6$ we obtain

$$\|f\|_2 \leq C \|(1/2)I + K^t_\varepsilon\|f\|_2$$

for any $f \in L^2_R(\partial\Omega)$, (3.25) where $C$ depends only on $\kappa_1$, $\kappa_2$, $(\sigma,M)$ and the Lipschitz character of $\Omega$. Observe that $\|K^t_\varepsilon - K^s_\varepsilon\|_{L^2(\partial\Omega) \to L^2(\partial\Omega)} \to 0$ as $t \to s$ [16]. Since $(1/2)I + K^t_\varepsilon$ is invertible on $L^2_R(\partial\Omega)$ for $t = 0$ [5], it follows from (3.25) that $(1/2)I + K^t_\varepsilon$ is invertible on $L^2_R(\partial\Omega)$ for any $t \in [0,1]$. In particular, the case $t = 1$ gives Theorem $3.2$.\[\square\]

Note that by Remark $3.3$ Theorem $1.1 \implies$ Theorem $3.7$. Thus we have proved that for each $\varepsilon > 0$ and each Lipschitz domain $\Omega$,

$$\text{Theorem } 1.1 \iff \text{Theorem } 3.2 \iff \text{Theorem } 3.7$$

The rest of this paper is devoted to the proof of Theorem $3.7$.

4 Nontangential maximal function and square function estimates

In this section we establish some nontangential maximal function and square function estimates for weak solutions of Dirichlet problem

$$\begin{cases}
-\text{div}(A(x)\nabla u) = F_0 + \text{div}(F_1) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

(4.1)

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. These auxiliary estimates will be used in the next section to prove the boundary Korn inequality. Throughout the section we will assume that $A = (a_{ij}^{\alpha\beta}(x))$ with $1 \leq i,j \leq d$ and $1 \leq \alpha, \beta \leq m$ satisfies the ellipticity condition (1.13) and the Hölder continuity condition (1.4). We also need the symmetry condition $A^* = A$, i.e., $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. The estimates we will prove are local. Thus, without loss of generality, we further assume that $\Omega \subset B(0,1/4)$, $\text{diam}(\Omega) \approx 1$, and $A$ is 1-periodic.

Lemma 4.1. Let $u$ be a weak solution of (4.1) with $F_0 = 0$. Then for any $0 \leq \sigma_1 < \sigma_2 \leq 1$,

$$\int_\Omega |\nabla u|^2 [\delta(x)]^{\sigma_2} \, dx \leq C \int_\Omega |F_1|^2 [\delta(x)]^{\sigma_1} \, dx,$$

(4.2)

where $\delta(x) = \text{dist}(x,\partial\Omega)$ and $C$ depends only on $\mu$, $\sigma_1$, $\sigma_2$, $(\sigma,M)$ in (1.4) and the Lipschitz character of $\Omega$.

Proof. This was proved in [22, pp.373-374] under the assumption that $A$ satisfies (1.13), (1.4), and $A^* = A$.\[\square\]
Lemma 4.2. Let $u$ be a weak solution of (4.1) with $F_1 = 0$. Then for any $0 \leq \sigma_1 < \sigma_2 \leq 1$,
\[
\int_{\Omega} |\nabla u|^2 \left[\delta(x)\right]^{\sigma_2} \, dx \leq C \int_{\Omega} |F_0|^2 \left[\delta(x)\right]^{\sigma_1 + 2} \, dx,
\]
where $C$ depends only on $\mu, \sigma_1, \sigma_2, (\sigma, M)$ in (1.4) and the Lipschitz character of $\Omega$.

Proof. Let $0 \leq \sigma_1 < \sigma_2 \leq 1$. Suppose that $-\text{div}(A \nabla v) = \text{div}(F)$ in $\Omega$ and $v = 0$ on $\partial \Omega$. It follows from Lemma 4.1 by duality that
\[
\int_{\Omega} |\nabla v(x)|^2 \frac{dx}{[\delta(x)]^{\sigma_1}} \leq C \int_{\Omega} |F(x)|^2 \frac{dx}{[\delta(x)]^{\sigma_2}}. \tag{4.4}
\]
This, together with Hardy’s inequality (see e.g. [23]),
\[
\int_{\Omega} |u(x)|^2 \frac{dx}{[\delta(x)]^\alpha} \leq C \alpha \int_{\Omega} |\nabla u(x)|^2 \frac{dx}{[\delta(x)]^{\alpha - 2}}, \tag{4.5}
\]
which holds for $\alpha > 1$ and $u \in H_0^1(\Omega)$, gives
\[
\int_{\Omega} |v(x)|^2 \frac{dx}{[\delta(x)]^{\sigma_1 + 2}} \leq C \int_{\Omega} |F(x)|^2 \frac{dx}{[\delta(x)]^{\sigma_2}}. \tag{4.6}
\]
The estimate (4.3) follows from (4.6) by a duality argument. \qed

By combining Lemmas 4.1 and 4.2 we obtain the following.

Theorem 4.3. Assume that $A$ satisfies (1.13), (1.4), and $A^* = A$. Let $\Omega$ be a bounded Lipschitz domain with $\text{diam}(\Omega) \approx 1$. Let $u$ be a weak solution of (4.1). Then
\[
\int_{\Omega} |\nabla u|^2 \left[\delta(x)\right]^{\sigma_2} \, dx \leq C \int_{\Omega} |F_0|^2 \left[\delta(x)\right]^{\sigma_1 + 2} \, dx + C \int_{\Omega} |F_1|^2 \left[\delta(x)\right]^{\sigma_1} \, dx \tag{4.7}
\]
for any $0 \leq \sigma_1 < \sigma_2 \leq 1$, where $C$ depends only on $\mu, \sigma_1, \sigma_2, (\sigma, M)$ in (1.4) and the Lipschitz character of $\Omega$.

If $\sigma_2 = 1$, estimate (4.7) is a square function estimate. Note that by dilation (4.7) continues to hold for any Lipschitz domain in $\mathbb{R}^d$, with constant $C$ depending also on $\text{diam}(\Omega)$.

The rest of this section is devoted to the estimate of the nontangential maximal function of $u$.

Lemma 4.4. Let $u$ be a weak solution of (4.1). Then
\[
|u(x)| \leq C \int_{B(x,r)} |u| \, dy + C \int_{B(x,r)} \frac{|F_1(y)| \, dy}{|x - y|^{d-1}} + C \int_{B(x,r)} \frac{|F_0(y)| \, dy}{|x - y|^{d-2}} \tag{4.8}
\]
for any $x \in \Omega$, where $r = \delta(x)/4$.\]
Proof. This is a standard interior estimate and may be proved by using the fundamental solution $\Gamma(x, y)$ for the operator $-\text{div}(A\nabla)$ in $\mathbb{R}^d$. Indeed, since we may assume $A$ is 1-periodic, we have $|\Gamma(x, y)| \leq C|x - y|^{2-d}$ and $|\nabla_y \Gamma(x, y)| \leq C|x - y|^{1-d}$.[3] Fix $x_0 \in \Omega$ and let

$$v(x) = -\int_{B(x_0, r_0)} \nabla_y \Gamma(x, y) F_1(y) dy + \int_{B(x_0, r_0)} \Gamma(x, y) F_0(y) dy,$$

where $r_0 = \delta(x_0)/4$. Then $\text{div}(A\nabla(u - v)) = 0$ in $B(x_0, r_0)$. Hence, by the interior $L^\infty$ estimate,

$$|u(x_0) - v(x_0)| \leq C \int_{B(x_0, r_0)} |u - v|,$$

from which the estimate (4.8) follows easily. □

We introduce a modified nontangential maximal function. For $z \in \partial \Omega$, define

$$N(w)(z) = \sup_{x \in \gamma(z) \cap B(x, \delta(x)/4)} |w|, \tag{4.9}$$

where $w$ is a function defined on $\Omega$ and and $\gamma(z) = \{x \in \Omega : |x - z| < C_0 \delta(x)\}$.

Lemma 4.5. Let $N(w)$ be defined by (4.9) and $0 < \alpha < 1$. Then for any $w \in H^1_0(\Omega)$,

$$\int_{\partial \Omega} |N(w)|^2 \, d\sigma \leq C \int_{\Omega} |\nabla w(x)|^2 [\delta(x)]^\alpha \, dx, \tag{4.10}$$

where $C$ depends only on $\alpha$ and the Lipschitz character of $\Omega$.

Proof. Let $z \in \partial \Omega$ and $x \in \gamma(z)$. Note that, if $\delta(x)$ is sufficiently small,

$$\int_{B(x, \delta(x)/4)} |w| \, dy \leq C \int_{B(z, C\delta(x)) \cap \Omega} |w| \, dy \leq C \int_{B(z, C\delta(x)) \cap \partial \Omega} |M_r(w)| \, d\sigma,$$

where $M_r(w)$ denotes the radial maximal function of $w$ (see e.g. [13] for its definition). It follows that

$$N(w)(z) \leq CM_{\partial \Omega}(M_r(w))(z) + C \int_{\Omega} |w|,$$

for any $z \in \partial \Omega$, where $M_{\partial \Omega}$ denotes the Hardy-Littlewood maximal function operator on $\partial \Omega$. Hence, by the $L^2$ boundedness of $M_{\partial \Omega}$,

$$\int_{\partial \Omega} |N(w)|^2 \, d\sigma \leq C \int_{\partial \Omega} |M_r(w)|^2 \, d\sigma + C \int_{\Omega} |w|^2 \, dx$$

$$\leq C \int_{\Omega} |\nabla w(x)|^2 [\delta(x)]^\alpha \, dx + C \int_{\Omega} |w|^2 \, dx,$$

where the last inequality was proved in [13]. This, together with Hardy’s inequality (4.5), gives (4.10). □
To handle the last two terms in the r.h.s. of \((4.8)\), we introduce another nontangential maximal function
\[
\mathcal{M}(w)(z) = \sup \left\{ |w(x)| : x \in \tilde{\gamma}(z) \right\},
\]
(4.11)
where \(z \in \partial \Omega\) and
\[
\tilde{\gamma}(z) = \left\{ x \in \Omega : |x - z| < 10C_0\delta(x) \right\} \supset \gamma(z) = \left\{ x \in \Omega : |x - z| < C_0\delta(x) \right\}.
\]
Observe that if \(x \in \gamma(z)\) and \(y \in B(x, \delta(x)/4)\), then \(y \in \tilde{\gamma}(z)\).

**Theorem 4.6.** Assume that \(A\) satisfies \((1.13)\), \((1.4)\), and \(A^* = A\). Let \(\Omega\) be a bounded Lipschitz domain with \(\text{diam}(\Omega) \approx 1\). Let \(u\) be a weak solution of \((4.1)\) and \(0 < \alpha, \sigma_1 < 1\). Then for any \(0 < t < 1\),
\[
\int_{\partial \Omega} |(u)^*|^2 d\sigma \leq Ct^{2\alpha} \int_{\partial \Omega} |\mathcal{M}(F_1 \delta^{1-\alpha})|^2 d\sigma + Ct^{2\alpha} \int_{\partial \Omega} |\mathcal{M}(F_0 \delta^{2-\alpha})|^2 d\sigma
+ Ct \int_{\Omega} \left| F_1 \right|^2 \left| \delta(x) \right|^\sigma_1^2 dx + Ct \int_{\Omega} \left| F_1 \right|^2 \left| \delta(x) \right|^\alpha dx,
\]
(4.12)
where \(\delta = \delta(x)\) and \(C\) depends only on \(\mu, \sigma_1, \alpha, (\sigma, M)\) and the Lipschitz character of \(\Omega\). The constant \(C_t\) also depends on \(t\).

**Proof.** Fix \(t \in (0, 1)\). Let \(z \in \partial \Omega\) and \(x \in \gamma(z)\). Note that if \(r = \frac{\delta(x)}{4} \leq t\), then
\[
\int_{B(x,r)} \frac{|F_1(y)|}{|x - y|^{d-1}} dy \leq Cr^\alpha \mathcal{M}(F_1 \delta^{1-\alpha})(z) \leq Ct^\alpha \mathcal{M}(F_1 \delta^{1-\alpha})(z).
\]
If \(r \geq t\), we have
\[
\int_{B(x,r)} \frac{|F_1(y)|}{|x - y|^{d-1}} dy = \int_{B(x,t)} \frac{|F_1(y)|}{|x - y|^{d-1}} dy + \int_{B(x,r) \setminus B(x,t)} \frac{|F_1(y)|}{|x - y|^{d-1}} dy
\leq Ct^\alpha \mathcal{M}(F_1 \delta^{1-\alpha})(z) + Ct \int_{\{y \in \Omega : \delta(y) \geq ct\}} |F_1|.
\]
Similarly, we may show that
\[
\int_{B(x,r)} \frac{|F_0(y)|}{|x - y|^{d-2}} dy \leq Ct^\alpha \mathcal{M}(F_0 \delta^{2-\alpha})(z) + Ct \int_{\{y \in \Omega : \delta(y) \geq ct\}} |F_0|.
\]
In view of Lemma 4.4 we obtain
\[
(u)^* \leq CN(u) + C \left\{ \mathcal{M}(F_0 \delta^{2-\alpha}) + \mathcal{M}(F_1 \delta^{1-\alpha}) \right\} + Ct \int_{\{y \in \Omega : \delta(y) \geq ct\}} (|F_0| + |F_1|).
\]
This, together with Lemma 4.5 and Theorem 4.3 gives (4.12). \(\square\)
5 Boundary Korn inequality

Throughout this section we assume that \( A \) satisfies (1.2) and (1.4). Let \( \Omega \) be a bounded Lipschitz domain such that \( \Omega \subset B(0,1/4) \). We further assume that there exists \( M_0 > 0 \) such that

\[
|\nabla A(x)| \leq M_0 [\delta(x)]^{\sigma-1} \quad \text{and} \quad |\nabla^2 A(x)| \leq M_0 [\delta(x)]^{\sigma-2},
\]

for any \( x \in B(0,1/2) \). The goal of this section is to prove the following.

**Theorem 5.1.** Assume that \( A \) and \( \Omega \) satisfy the conditions stated above. Let \( u \) be a weak solution of \( L_1(u) = 0 \) in \( \Omega \) with the properties that \( (\nabla u)^* \in L^2(\partial \Omega) \) and \( \nabla u \) exists n.t. on \( \partial \Omega \). Then

\[
\|\nabla u\|_2 \leq C \|\nabla u + (\nabla u)^T\|_2 + C r_0^{-1/2} \|\nabla u\|_{L^2(\Omega)},
\]

where \( r_0 = \text{diam}(\Omega) \) and \( C \) depends only on \( \kappa_1, \kappa_2, (\sigma, M), M_0, \) and the Lipschitz character of \( \Omega \).

Since \( \Omega \subset B(0,1/4) \), by using a periodic extension of \( A \), we may assume that \( A \) is 1-periodic in \( \mathbb{R}^d \). Also, by approximating \( \Omega \) with a sequence of smooth domains from inside with uniform Lipschitz character \([24]\), we may assume that \( A \in C^2(\Omega) \) and \( u \in C^3(\Omega) \). The assumptions that \( (\nabla u)^* \in L^2(\partial \Omega) \) and \( \nabla u \) exists n.t. on \( \partial \Omega \) allow us to prove (5.2) by a limiting argument.

We begin with a boundary Korn inequality for harmonic functions.

**Lemma 5.2.** Let \( u = (u^1, u^2, \ldots, u^d) \) be a solution of \( \Delta u = 0 \) in \( \Omega \). Suppose that \( (\nabla u)^* \in L^2(\partial \Omega) \). Then the inequality (5.2) holds with a constant \( C \) depending only on the Lipschitz character of \( \Omega \).

**Proof.** This was proved in [5, p.804-805]. Note that for harmonic function \( u \), the condition \( (\nabla u)^* \in L^2(\partial \Omega) \) implies \( \nabla u \) exists n.t. on \( \partial \Omega \).

To utilize Lemma 5.2, we observe that by Proposition 2.2, \( L_1(u) = 0 \) in \( \Omega \) implies \( \text{div}(\tilde{A}(x)\nabla u) = 0 \) in \( \Omega \). It follows that

\[
\mu \Delta u^\alpha + \frac{\partial}{\partial x_i} \left\{ b_{ij}^{\alpha\beta}(x) \frac{\partial u^\beta}{\partial x_j} \right\} = 0 \quad \text{in} \ \Omega,
\]

where \( b_{ij}^{\alpha\beta} \) is given by (2.2). Let

\[
\Gamma(f)(x) = \int_\Omega \Gamma(x-y)f(y) \, dy,
\]

where \( \Gamma(x) \) denotes the fundamental solution for \( \Delta \) in \( \mathbb{R}^d \), with pole at the origin. Then

\[
\Delta \{ \Gamma(f) \} = f \quad \text{in} \ \Omega.
\]

This allows us to rewrite (5.3) as

\[
\Delta \left\{ \mu u^\alpha + \frac{\partial}{\partial x_i} \Gamma \left( b_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \right) \right\} = 0 \quad \text{in} \ \Omega.
\]
Lemma 5.3. Let \( u \in C^3(\overline{\Omega}; \mathbb{R}^d) \) be a solution of \( L_1(u) = 0 \) in \( \Omega \). Then
\[
\| \nabla u \|_2 \leq C \| \nabla u + (\nabla u)^T \|_2 + C \| \nabla^2 \Gamma(B \nabla u) \|_2 + C r_0^{-1/2} \| \nabla u \|_{L^2(\Omega)},
\]
where \( B = (b_{ij}^{\alpha \beta}) \).

Proof. This follows directly from Lemma 5.2 and (5.5). We only need to observe that
\[
\| \nabla^2 \Gamma(B \nabla u) \|_{L^2(\Omega)} \leq C \| \nabla u \|_{L^2(\Omega)},
\]
which is a standard singular integral estimate. \( \square \)

The estimate of the second term in the r.h.s. of (5.6) is much involved. We start with the following.

Lemma 5.4. Let \( \Omega \) be a bounded Lipschitz domain and \( w \in C^1(\overline{\Omega}) \). Then
\[
\| \nabla^2 \Gamma(w) \|_2 \leq C \left\{ \| (w)^* \|_2 + \left( \int_\Omega |\nabla w(x)|^2 \delta(x) \, dx \right)^{1/2} \right\},
\]
where \( C \) depends only on the Lipschitz character of \( \Omega \).

Proof. Let \( g \in L^2(\partial \Omega) \). It follows by Fubini’s Theorem that
\[
\int_{\partial \Omega} \nabla^2 \Gamma(w) \cdot g(y) \, dy = \int_\Omega w(x) \cdot \left\{ \int_{\partial \Omega} \nabla^2 \Gamma(y-x)g(y) \, dy \right\} \, dx = -\int_\Omega w(x) \cdot \left\{ \nabla_x \int_{\partial \Omega} \nabla_y \Gamma(y-x)g(y) \, dy \right\} \, dx.
\]
Since the function \( v(x) = \int_{\partial \Omega} \nabla_y \Gamma(y-x)g(y) \, dy \) is harmonic in \( \Omega \), it follows from Dahlberg’s bilinear estimate [4] that
\[
\left| \int_{\partial \Omega} \nabla^2 \Gamma(w) \cdot g(y) \, dy \right| \leq C \| (v)^* \|_2 \left\{ \| (w)^* \|_2 + \left( \int_\Omega |\nabla w(x)|^2 \delta(x) \, dx \right)^{1/2} \right\}
\]
\[
\leq C \| g \|_2 \left\{ \| (w)^* \|_2 + \left( \int_\Omega |\nabla w(x)|^2 \delta(x) \, dx \right)^{1/2} \right\},
\]
where we have used the well known estimate \( \| (v)^* \|_2 \leq C \| g \|_2 \) [24]. The desired estimate now follows by duality. \( \square \)

Next, we factor the matrix \( B = (b_{ij}^{\alpha \beta}) \). Let \( m = d(d + 1)/2 \).
Lemma 5.5. Let $B(x) = (b_{ij}^\alpha(x))$, where $b_{ij}^\alpha$ is given by (2.2). Then there exists a matrix $Q(x) = (q_{ij}^{t\alpha}(x))$ with $1 \leq \alpha, i \leq d$ and $1 \leq t \leq m$ such that $q_{ij}^{t\alpha} = q_{ji}^t$,

$$b_{ij}^\alpha = q_{ij}^{t\alpha}q_{ji}^t \quad \text{for any } 1 \leq \alpha, \beta, i, j \leq d. \quad (5.8)$$

Moreover, $|Q(x)| \leq M_1$, $|Q(x) - Q(y)| \leq M_1|x - y|^{\sigma}$,

$$|\nabla Q(x)| \leq M_1[\delta(x)]^{\sigma-1} \quad \text{and} \quad |\nabla^2 Q(x)| \leq M_1[\delta(x)]^{\sigma-2} \quad (5.9)$$

for any $x \in B(0, 2)$, where $M_1$ depends only on $\kappa_1, \kappa_2, (\sigma, M)$ and $M_0$ in (5.1).

Proof. We begin by fixing a constant matrix $(h_i^{t\alpha})$ with $1 \leq \alpha, i \leq d$ and $1 \leq t \leq m$ such that $h_i^{ta} = h_i^{ta}$ and

$$(h_i^{t\alpha} \xi_i^\alpha)^{1 \leq t \leq m} = E(\xi) \in \mathbb{R}^m$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times d}$, where $E(\xi)$ is an enumeration of the lower triangular part of matrix $(1/2)(\xi + \xi^T)$,

$$E(\xi) = (\xi_i^\alpha)^{1 \leq t \leq m} = (\xi_1^\alpha, (\xi_2^\alpha + \xi_1^\alpha)/2, \xi_2^\alpha, \ldots, \xi_d^\alpha) = ((\xi_i^\alpha + \xi_j^\alpha)/2)_{1 \leq i,j \leq d, 1 \leq \beta \leq j}.$$

Next we define the $m \times m$ symmetric matrix $G(x) = (g^{ts}(x))$ by the quadratic form

$$\eta^t g^{ts}(x)\eta^s = \xi_i^\alpha b_{ij}^{t\alpha}(x)\xi_j^\beta,$$

where $\eta = (\eta^t) = E(\xi) \in \mathbb{R}^m$ and $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times d}$. Using the fact that $E(\xi) = 0$ if and only if $\xi^T = -\xi$ and that $b_{ij}^\alpha = b_{ji}^{t\alpha} = b_{ij}^{\alpha t}$, it is easy to verify that $G(x)$ is well defined. Moreover, in view of (2.6), we obtain

$$\eta^t g^{ts}(x)\eta^s \geq \frac{\kappa_1}{8} |\xi + \xi^T|^2 \geq \frac{\kappa_1}{8} |\eta|^2$$

for any $\eta \in \mathbb{R}^m$. Thus $G(x)$ is an $m \times m$ symmetric, positive-definite matrix. It follows that there exists a symmetric, positive-definite matrix $P = (p^{ts})$ such that $G = P^2$. Moreover, since $G \geq \kappa_1/8$, we see that $|P(x)| \leq C$, $|P(x) - P(y)| \leq C|x - y|^{\sigma}$, $|\nabla P(x)| \leq C[\delta(x)]^{\sigma-1}$, and $|\nabla^2 P(x)| \leq C[\delta(x)]^{\sigma-2}$, where $C$ depends only on $\kappa_1, \kappa_2, (\sigma, M)$, and $M_0$ in (5.1).

This may be proved by using the formula,

$$P = G^{1/2} = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2}G(\lambda I + G)^{-1} d\lambda$$

and the resolvent identity,

$$(\lambda I + G(x))^{-1} - (\lambda I + G(y))^{-1} = (\lambda I + G(x))^{-1}(G(y) - G(x))(\lambda I + G(y))^{-1}.$$  

Finally, since

$$\xi_i^\alpha b_{ij}^{t\alpha} \xi_j^\beta = h_i^{ta} \xi_i^\alpha g^{ta} h_j^{s\beta} \xi_j^\beta = \xi_i^\alpha h_i^{ta} p^{t \alpha} p^{s \beta} h_j^{s \beta} \xi_j^\beta,$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{d \times d}$, we obtain

$$b_{ij}^{t\alpha} = h_i^{ta} p^{t \alpha} p^{s \beta} h_j^{s \beta} = q_{ij}^{t \alpha} q_{ji}^t,$$

where we have set $g_{ij}^{t \alpha} = q_{ij}^{t \alpha}(x) = h_i^{ta} p^{t \alpha} p^{s \beta} h_j^{s \beta}$. The desired smoothness estimates for $Q = (q_{ij}^{t \alpha})$ follow from the same estimates for $P = (p^{ts})$. \qed
Let \( v = (v^t) \), where \( 1 \leq t \leq m \) and
\[
v^t = q^t_j(x) \frac{\partial u^\beta}{\partial x_j}.
\] (5.10)

Note that since \( q^t_j = q^t_j \), we have \( |v| \leq C|\nabla u + (\nabla u)^T| \).

**Lemma 5.6.** Let \( u \in C^3(\Omega; \mathbb{R}^d) \) be a solution of \( L_1(u) = 0 \) in \( \Omega \). Then
\[
\|\nabla u\|_2 \leq C\|\nabla u + (\nabla u)^T\|_2 + C\|(v)^*\|_2 + C\left( \int_\Omega |\nabla v|^2 \delta(x) \, dx \right)^{1/2}
\]
\[
+ Cr_0^{-1/2}\|\nabla u\|_{L^2(\Omega)},
\] (5.11)
where \( v \) is given by (5.10).

**Proof.** In view of Lemma [5.3](#) it suffices to handle the term \( \|\nabla^2 \Gamma(B\nabla u)\|_2 \). Let \( w = B\nabla u = (w^\alpha_i) \), where
\[
w^\alpha_i = b^{\alpha \beta} \frac{\partial u^\beta}{\partial x_j} = q^{t \alpha}_i q^{t \beta}_j \frac{\partial u^\beta}{\partial x_j} = q^{t \alpha}_i v^t
\]
and we have used Lemma [5.5](#) It follows by Lemma [5.4](#) that
\[
\|\nabla^2 \Gamma(B\nabla u)\|_2 \leq C\|(w)^*\|_2 + C\left( \int_\Omega |\nabla w(x)|^2 \delta(x) \, dx \right)^{1/2}
\]
\[
\leq C\|(w)^*\|_{L^2(\partial \Omega)} + C\left( \int_\Omega |\nabla v|^2 \delta(x) \, dx \right)^{1/2} + C\left( \int_\Omega |v|^2 |\nabla Q|^2 \delta(x) \, dx \right)^{1/2}.
\]

Since \( |\nabla Q(x)| \leq C[\delta(x)]^{\sigma-1} \) for \( x \in \Omega \), we see that \( |\nabla Q(x)|^2 \delta(x) \, dx \) is a Carleson measure on \( \Omega \). It follows that
\[
\left( \int_\Omega |v|^2 |\nabla Q(x)|^2 \delta(x) \, dx \right)^{1/2} \leq C\|(v)^*\|_2
\]
This completes the proof.

We are now in a position to give the proof of Theorem 5.1.

**Proof of Theorem 5.1.** It remains to control the second and third terms in the r.h.s. of (5.11) and show that
\[
\|(v)^*\|_2 + \left( \int_\Omega |\nabla v|^2 \delta(x) \, dx \right)^{1/2} \leq C\|\nabla u + (\nabla u)^T\|_2 + Cr_0^{-1/2}\|\nabla u\|_{L^2(\Omega)},
\] (5.12)
where \( v = Q(x)\nabla u(x) \) and \( Q(x) \) is given by Lemma [5.5](#). To this end we rewrite the equation (5.3) as
\[
\mu \Delta u^\alpha + \frac{\partial}{\partial x_i} \left\{ q^{t \alpha}_i q^{t \beta}_j \frac{\partial u^\beta}{\partial x_j} \right\} = 0 \quad \text{in} \ \Omega.
\] (5.13)
We now differentiate (5.13) in \( x_\ell \) and then multiply the resulting equation by \( q_\ell^{s\alpha} \) to obtain
\[
\mu q_\ell^{s\alpha} \Delta \frac{\partial u_\alpha}{\partial x_\ell} + q_\ell^{s\alpha} \frac{\partial^2}{\partial x_i \partial x_\ell} \left\{ q_i^{t\alpha} v^t \right\} = 0 \quad \text{in } \Omega.
\]

It follows that
\[
\mu \Delta v^s - \mu \left[ \Delta, q_\ell^{s\alpha} \right] \frac{\partial u_\alpha}{\partial x_\ell} + q_\ell^{s\alpha} \frac{\partial}{\partial x_\ell} \left\{ q_i^{t\alpha} \frac{\partial v^t}{\partial x_i} \right\} + q_\ell^{s\alpha} \frac{\partial}{\partial x_\ell} \left\{ q_i^{t\alpha} \frac{\partial v^t}{\partial x_i} \right\} = 0,
\]
where \([S, T] = ST - TS\) denotes the commutator of operators \( S \) and \( T \). This gives
\[
\mu \Delta v^s + \frac{\partial}{\partial x_\ell} \left\{ q_\ell^{s\alpha} q_i^{t\alpha} \frac{\partial v^t}{\partial x_i} \right\} = F^s \quad \text{in } \Omega \quad (5.14)
\]
for \( 1 \leq s \leq m \), where
\[
F^s = \mu \left[ \Delta, q_\ell^{s\alpha} \right] \frac{\partial u_\alpha}{\partial x_\ell} - q_\ell^{s\alpha} \frac{\partial}{\partial x_\ell} \left\{ q_i^{t\alpha} \frac{\partial v^t}{\partial x_i} \right\} + \frac{\partial q_\ell^{s\alpha}}{\partial x_\ell} \cdot q_i^{t\alpha} \frac{\partial v^t}{\partial x_i}.
\]

Using the product rule as well as the symmetry \( q_\ell^{s\alpha} = q_\ell^{s\alpha} \), a computation shows that
\[
F = \text{div}(F_1) + F_0 \quad \text{in } \Omega, \quad (5.15)
\]
where \( F_0 \) and \( F_1 \) satisfy the estimates
\[
|F_0| \leq C \left\| \nabla^2 Q \right\| + \left\| \nabla Q \right\| \left\| \nabla u + (\nabla u)^T \right\|, \quad (5.16)
\]
\[
|F_1| \leq C \left\| \nabla Q \right\| \left\| \nabla u + (\nabla u)^T \right\|. \quad (5.17)
\]

We now decompose \( v = v_1 + v_2 \) in \( \Omega \), where
\[
\begin{cases}
\mu \Delta v_1^s + \frac{\partial}{\partial x_\ell} \left\{ q_\ell^{s\alpha} q_i^{t\alpha} \frac{\partial v_1^t}{\partial x_i} \right\} = (\text{div}(F_1))^s + F_0^s \quad \text{in } \Omega, \\
v_1 = 0 \quad \text{on } \partial \Omega,
\end{cases} \quad (5.18)
\]
and
\[
\begin{cases}
\mu \Delta v_2^s + \frac{\partial}{\partial x_\ell} \left\{ q_\ell^{s\alpha} q_i^{t\alpha} \frac{\partial v_2^t}{\partial x_i} \right\} = 0 \quad \text{in } \Omega, \\
v_2 = v \quad \text{on } \partial \Omega.
\end{cases} \quad (5.19)
\]

To proceed, we point out that the \( m \times d \) system in (5.18) and (5.19) is an elliptic system in divergence form with coefficients
\[
\mu \delta_{st} \delta_{\ell i} + q_\ell^{s\alpha} q_i^{t\alpha},
\]
which are very strongly elliptic, symmetric, and Hölder continuous. As a result, it follows (see e.g. [16]) that the solution \( v_2 \) of Dirichlet problem (5.19) satisfies the estimate
\[
\left\| (v_2)^* \right\|_2 + \left( \int_\Omega \left\| \nabla v_2 \right\|^2 \delta(x) \, dx \right)^{1/2} \leq C \left\| v \right\|_2 \leq C \left\| \nabla u + (\nabla u)^T \right\|_2. \quad (5.20)
\]

21
Finally, to estimate \( v_1 \), we use Theorems 4.3 and 4.6. This gives

\[
\int_{\partial \Omega} |(v_1)|^2 \, d\sigma + \int_{\Omega} |\nabla v_1|^2 \delta(x) \, dx \\
\leq C t \int_{\Omega} |F_0|^2 [\delta(x)]^{\sigma_1 + 2} \, dx + C t \int_{\Omega} |F_1|^2 [\delta(x)]^{\sigma_1} \, dx \\
+ C t^{2a} \int_{\partial \Omega} |\mathcal{M}(F_1 \delta^{1-a})|^2 \, d\sigma + C t^{2a} \int_{\partial \Omega} |\mathcal{M}(F_0 \delta^{2-a})|^2 \, d\sigma,
\]

for any \( t \in (0, 1) \), where \( 0 < \alpha, \sigma_1 < 1 \). Recall that \( |
abla Q(x)| \leq C [\delta(x)]^{\sigma - 1} \) and \( |\nabla^2 Q(x)| \leq C [\delta(x)]^{\sigma - 2} \). In view of (5.17) we obtain

\[
|F_0(x)| \leq C [\delta(x)]^{\sigma - 2} |\nabla u + (\nabla u)^T|,
\]

\[
|F_1(x)| \leq C [\delta(x)]^{\sigma - 1} |\nabla u + (\nabla u)^T|
\]

for any \( x \in \Omega \). By choosing \( \alpha = \sigma \) and \( \sigma_1 = 1 - \sigma \) we may deduce from (5.21) and (5.22) that

\[
\int_{\partial \Omega} |(v_1)|^2 \, d\sigma + \int_{\Omega} |\nabla v_1|^2 \delta(x) \, dx \\
\leq C t \int_{\Omega} |\nabla u + (\nabla u)^T|^2 \, dx + C t^{2a} \int_{\partial \Omega} |\mathcal{M}(\nabla u)|^2 \, d\sigma \\
\leq C t \int_{\Omega} |\nabla u + (\nabla u)^T|^2 \, dx + C t^{2a} \int_{\partial \Omega} |\nabla u|^2 \, d\sigma + C t^{2a} \int_{\partial \Omega} |u|^2 \, d\sigma,
\]

where we have used (1.12) for the last inequality. This, together with (5.20) and Lemma 5.6, gives

\[
||\nabla u||_2 \leq C ||\nabla u + (\nabla u)^T||_2 + C t ||\nabla u||_{L^2(\Omega)} + C t^a ||\nabla u||_2 + C t^a ||u||_2
\]

for any \( t \in (0, 1) \). We now choose \( t \) so small that \( Ct^a \leq (1/2) \). It follows that

\[
||\nabla u||_2 \leq C ||\nabla u + (\nabla u)^T||_2 + C t ||\nabla u||_{L^2(\Omega)} + C ||u||_2 \\
\leq C ||\nabla u + (\nabla u)^T||_2 + C ||\nabla u||_{L^2(\Omega)} + C ||u||_{L^2(\Omega)},
\]

where we have used the trace inequality (3.15) for the last step. The proof is completed by subtracting a constant from \( u \) and using Poincaré inequality.

\[\square\]

### 6 Rellich estimates for small scales

In this section we establish the Rellich estimate

\[
||\nabla u||_2 \leq C ||\frac{\partial u}{\partial y}||_2 + Ct_0^{-1/2} ||\nabla u||_{L^2(\Omega)},
\]

(6.1)
for solutions of \( L(u) = -\text{div}(A\nabla u) = 0 \) in a Lipschitz domain \( \Omega \), and solve the \( L^2 \) Neumann problem in \( \Omega \) by the method layer potentials. This will be done under the assumptions that \( A \) satisfies (1.2), (1.3) (1.4) and (5.1). The extra assumption (5.1) will be eliminated in the next section by an approximation scheme.

Throughout this section \( \Omega \) is a Lipschitz domain with \( r_0 = \text{diam}(\Omega) = 1/4 \).

**Lemma 6.1.** Assume that \( L(u) = 0 \) in \( \Omega \), \( (\nabla u)^* \in L^2(\partial \Omega) \), and \( \nabla u \) exists n.t. on \( \partial \Omega \). Then

\[
\int_{\partial \Omega} |\nabla u + (\nabla u)^T|^2 d\sigma \leq C \int_{\partial \Omega} |\nabla u| \left| \frac{\partial u}{\partial n} \right| d\sigma + C \int_{\Omega} (|\nabla A| + 1)|\nabla u|^2 dx, \tag{6.2}
\]

where \( C \) depends only on \( \kappa_1, \kappa_2 \), and the Lipschitz character of \( \Omega \).

**Proof.** As in the case of constant coefficients \([5]\), the estimate (6.2) follows from the so-called Rellich identity

\[
\int_{\partial \Omega} \langle h, n \rangle a_{ij} \partial u^\alpha \partial u^\beta \partial x_i \partial x_j d\sigma = 2 \int_{\partial \Omega} h \partial u^\alpha \partial x_j \left( \frac{\partial u}{\partial n} \right)^\alpha d\sigma + I, \tag{6.3}
\]

where \( h \in C^1_0(\mathbb{R}^d, \mathbb{R}^d) \),

\[
|I| \leq C \int_{\Omega} \left\{ |\nabla h| + |h||\nabla A| \right\} |\nabla u|^2 dx,
\]

and \( C \) depends only on \( \kappa_1 \) and \( \kappa_2 \). The identity (6.3) is proved by using integration by parts and the symmetry condition \( a_{ij} = a_{ji} \). Since

\[
\frac{\kappa_1}{4} |\nabla u + (\nabla u)^T|^2 \leq a_{ij} \partial u^\alpha \partial x_i \partial u^\beta \partial x_j, \tag{6.4}
\]

the estimate (6.2) follows by choosing \( h \) so that \( \langle h, n \rangle \geq c > 0 \) on \( \partial \Omega \). \( \square \)

**Lemma 6.2.** Assume that \( L(u) = 0 \) in \( \Omega \), \( (\nabla u)^* \in L^2(\partial \Omega) \), and \( \nabla u \) exists n.t. on \( \partial \Omega \). Then for any \( t \in (0, 1) \),

\[
\|\nabla u\|_2 \leq C \left\| \frac{\partial u}{\partial n} \right\|_2 + C t^{\sigma/2} \| (\nabla u)^* \|_2 + C_t \|\nabla u\|_{L^2(\Omega)}, \tag{6.5}
\]

where \( C \) depends only on \( \kappa_1, \kappa_2, (M, \sigma) \), and the Lipschitz character of \( \Omega \). The constant \( C_t \) also depends on \( t \).

**Proof.** It follows by the boundary Korn inequality in Theorem 5.1 and (6.2) that

\[
\|\nabla u\|_2 \leq C \left\| \frac{\partial u}{\partial n} \right\|_2 + C \|\nabla u\|_{L^2(\Omega)} + C \|\nabla A\|^{1/2} \|\nabla u\|_{L^2(\Omega)}, \tag{6.6}
\]
where we have also used the Cauchy inequality with an \( \varepsilon \). To estimate the last term in (6.6), we use the assumption that \( |\nabla A(x)| \leq C[\delta(x)]^{\sigma-1} \) for \( x \in \Omega \). This gives

\[
\int_{\Omega} |\nabla A||\nabla u|^2 dx = \int_{\Omega_t} |\nabla A||\nabla u|^2 dx + \int_{\Omega \setminus \Omega_t} |\nabla A||\nabla u|^2 dx 
\leq C \int_{\Omega_t} [\delta(x)]^{\sigma-1}|\nabla u|^2 dx + C_t \int_{\Omega} |\nabla u|^2 dx 
\leq C t^\sigma \int_{\partial \Omega} |(\nabla u)^\times|^2 d\sigma + C_t \int_{\Omega} |\nabla u|^2 dx,
\]

where \( \Omega_t = \{ x \in \Omega : \delta(x) < t \} \). The estimate (6.5) now follows from (6.6) and (6.7).

Let \((1/2)I + K\) be the operator in (3.5) with \( \varepsilon = 1 \).

**Theorem 6.3.** Suppose that \( A \) satisfies conditions (1.2), (1.3), and (1.4). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \) such that \( \Omega \subset B(0,1) \) and \( \text{diam}(\Omega) = (1/4) \). Also assume that \( A \) satisfies the condition (5.1). Then \((1/2)I + K : L^2(\partial \Omega) \to L^2(\partial \Omega)\) is invertible and

\[
\|f\|_2 \leq C \|((1/2)I + K) f\|_2, \tag{6.8}
\]

where \( C \) depends only on \( \kappa_1, \kappa_2, (M, \sigma), M_0 \) in (5.1), and the Lipschitz character of \( \Omega \).

**Proof.** Let \( u = S(f) \) be the single layer potential for the operator \( L = L_1 \), where \( f \in L^2(\partial \Omega) \). By the jump relation (3.23) it follows that

\[
\|f\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_2 + C \|\nabla u\|_2.
\]

To deal with \( \|\nabla u\|_2 \), we use Theorem 2.4 to obtain

\[
\|\nabla u\|_2 \leq C \|\nabla \tan u\|_2 + C \|\nabla u\|_{L^2(\Omega_-)} 
\leq C \|\nabla u\|_2 + C \|\nabla u\|_{L^2(\Omega_-)},
\]

where we have used the fact (3.4). This, together with (6.5), leads to

\[
\|f\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_2 + C t^{\sigma/2} \| (\nabla u)^\times \|_2 + C_t \|\nabla u\|_{L^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega_-)}
\]

for any \( t \in (0,1) \). Since \( \| (\nabla u)^\times \|_2 \leq C \| f \|_2 \), by choosing \( t \) sufficiently small, we obtain

\[
\|f\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_2 + C\|\nabla u\|_{L^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega_-)} 
\leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_2 + C\|f\|_2^{1/2} \left\| \left( \frac{\partial u}{\partial \nu} \right)_+ \right\|_2^{1/2}, \tag{6.9}
\]

24
where we have used Lemma 3.5 for the last inequality. By using the Cauchy inequality with an $\varepsilon > 0$, this gives

$$\|f\|_2 \leq C \left\| \frac{\partial u}{\partial \nu} \right\|_2 = C \|((1/2)I + \mathcal{K})f\|_2$$

(6.10)

for any $f \in L^2_R(\partial \Omega)$.

Finally, we recall that in the case of an isotropic body, where $a_{ij}^{\alpha \beta} = \lambda \delta_{i\alpha} \delta_{j\beta} + \mu \delta_{ij} \delta_{\alpha \beta} + \mu \delta_{i\beta} \delta_{j\alpha}$, the invertibility of $(1/2)I + \mathcal{K}$ on $L^2_R(\partial \Omega)$ was established in [5]. By a simple continuity argument, this, together with the estimate (6.10), yields the invertibility of $(1/2)I + \mathcal{K}$ on $L^2_R(\partial \Omega)$ under the conditions stated in Theorem 6.3. This completes the proof.

7 $L^2$ Neumann problems for small scales

In this section we use an approximate scheme [16] to get rid of the extra assumption (5.1) in Theorem 6.3. This allows us to solve the $L^2$ Neumann problem (1.5) in the small-scale case where $\varepsilon = 1$ and $\text{diam}(\Omega) \leq 1/4$.

**Theorem 7.1.** Suppose that $A$ satisfies conditions (1.2), (1.3), and (1.4). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ such that $\text{diam}(\Omega) \leq (1/4)$. Then $(1/2)I + \mathcal{K} : L^2_R(\partial \Omega) \rightarrow L^2_R(\partial \Omega)$ is invertible and

$$\|f\|_2 \leq C \|((1/2)I + \mathcal{K})f\|_2 \quad \text{for any } f \in L^2_R(\partial \Omega),$$

(7.1)

where $C$ depends only on $\kappa_1, \kappa_2, (\sigma, M)$, and the Lipschitz character of $\Omega$.

**Remark 7.2.** Theorem 7.1 continues to hold without the periodicity condition (1.3) and for Lipschitz domains $\Omega$ with $r_0 = \text{diam}(\Omega) \geq (1/4)$. However, in this case, the constant $C$ will also depend on $r_0$. This can be seen by a simple rescaling argument. With the periodicity condition (1.3), the estimate (7.1) holds with a constant $C$ independent of $r_0$. This is equivalent to the same estimate for $\mathcal{K}_\varepsilon$ with constant $C$ independent of $\varepsilon$, one of the main results of this paper.

The reduction of Theorem 7.1 from Theorem 6.3 is similar to that in the case of very strong ellipticity condition [16]. We only provide an outline here.

Step 1. By translation and dilation we may assume that $\Omega \subset B(0, 1/4)$ and $\text{diam}(\Omega) = (1/4)$. We construct a coefficient matrix $\overline{A}(x) = (\overline{a}_{ij}^{\alpha \beta}(x))$, with $1 \leq \alpha, \beta, i, j \leq d$, in $\mathbb{R}^d$ with the properties that

(1) $\overline{A} = A$ on $\partial \Omega$;
(2) $\overline{A}$ satisfies the conditions (1.2), (1.3) and (1.4), with possibly different constants depending only on $\kappa_1, \kappa_2, (\sigma, M)$ and the Lipschitz character of $\Omega$;
(3) $\overline{A}$ satisfies the smoothness condition (5.1) with constant $M_0$ depending on $(\sigma, M)$ and the Lipschitz character of $\Omega$. 

25
This is done as follows. In $\Omega$ we let $\bar{\pi}^{\alpha\beta}_{ij}$ be the Poisson extension of $A$, i.e.,

$$\Delta \bar{\pi}^{\alpha\beta}_{ij} = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \bar{\pi}^{\alpha\beta}_{ij} = a^{\alpha\beta}_{ij} \quad \text{on} \quad \partial \Omega.$$ 

On $[-1/2, 1/2]^d \setminus \Omega$, we define $\bar{\pi}^{\alpha\beta}_{ij}$ to be the harmonic function in $(-1/2, 1/2)^d \setminus \Omega$ with boundary data $a^{\alpha\beta}_{ij}$ on $\partial \Omega$ and $\delta_{\alpha\gamma} \delta_{j\beta} + \delta_{i\gamma} \delta_{\alpha\beta} + \delta_{i\beta} \delta_{j\alpha}$ on $\partial [-1/2, 1/2]^d$. We then extend $\bar{A}$ to $\mathbb{R}^d$ by periodicity. The fact that $\bar{A}$ satisfies (1.2), (1.4) and (5.1) follows from the maximum principle and well known estimates for harmonic functions in Lipschitz domains with Hölder continuous data.

Step 2. Let $\theta \in C_0^\infty (-1/2, 1/2)$ such that $0 \leq \theta \leq 1$ and $\theta = 1$ on $(-1/4, 1/4)$. Define

$$A^t(x) = \theta \left( \frac{\delta(x)}{t} \right) A(x) + \left[ 1 - \theta \left( \frac{\delta(x)}{t} \right) \right] \bar{A}(x) \quad (7.2)$$

for $x \in [-1/2, 1/2]^d$, where $t \in (0, 1/8)$ and $\bar{A}(x)$ is the matrix constructed in Step 1. Extend $A^t$ to $\mathbb{R}^d$ by periodicity. Let $\mathcal{K}_{A^t}$ denote the operator on $\partial \Omega$, associated with the conormal derivative of the single layer potential for $-\text{div}(A^t(x) \nabla)$, as in Theorem 3.1.

Then

$$\| \mathcal{K}_{A^t} - \mathcal{K}_\bar{A} \|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)} \leq C t^{\lambda_0}, \quad (7.3)$$

where $\lambda_0 > 0$ depends only on $\sigma$ and the Lipschitz character of $\Omega$. We point out that this estimate was proved in [16] for operators satisfying the very strong ellipticity condition (1.13). Since its proof only involves fundamental solutions, in view of the equivalence of $\mathcal{L}_1$ and $\tilde{\mathcal{L}}_1$, it continues to hold for systems of elasticity.

Write

$$(1/2) I + \mathcal{K}_{A^t} = (1/2) I + \mathcal{K}_\bar{A} + (\mathcal{K}_{A^t} - \mathcal{K}_\bar{A}).$$

Note that by Theorem 6.3

$$\| (1/2) I + \mathcal{K}_{A^t} \|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)} \leq C.$$

It follows by (7.3) that there exist $t \in (0, 1/8)$ and $C > 0$, depending only on $\kappa_1, \kappa_2, (\sigma, M)$ and the Lipschitz character of $\Omega$, such that

$$\| (1/2) I + \mathcal{K}_{A^t} \|_{L^2(\partial \Omega) \rightarrow L^2(\partial \Omega)} \leq C. \quad (7.4)$$

Note that by (7.2),

$$A^t(x) = A(x) \quad \text{if} \quad \delta(x) \leq (t/4). \quad (7.5)$$

Step 3. It follows from (7.4) that the $L^2$ Neumann problem (1.5) in $\Omega$ for the operator $\mathcal{L}^t = -\text{div}(A^t(x) \nabla)$ is solvable and the estimate (1.6) holds. By Remark 3.3 this implies that if $\mathcal{L}^t(w) = 0$ in $\Omega$, $(\nabla w)^* \in L^2(\partial \Omega)$ and $\nabla w$ exists n.t. on $\partial \Omega$, then

$$\int_{\partial \Omega} |\nabla w|^2 d\sigma \leq C \int_{\partial \Omega} \left| \frac{\partial w}{\partial \nu} \right|^2 d\sigma + C \int_{\Omega} |\nabla w dx|^2, \quad (7.6)$$

26
where \( C \) depends only on \( \kappa_1, \kappa_2, (\sigma, M) \) and the Lipschitz character of \( \Omega \). Here we have used the fact that \( A' = A \) on \( \partial \Omega \) and thus the conormal derivative of \( w \) on \( \partial \Omega \) associated with \( \mathcal{L}' \) is the same as that associated with \( \mathcal{L} = -\text{div}(A \nabla) \).

Now let \( u \) be a weak solution of \( \mathcal{L}(u) = 0 \) in \( \Omega \) with the properties that \( (\nabla u)^* \in L^2(\partial \Omega) \) and \( \nabla u \) exists n.t. on \( \partial \Omega \). Let \( \varphi \in C^\infty_0(\mathbb{R}^d) \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi(x) = 1 \) on \( \{ x \in \mathbb{R}^d : \delta(x) \leq (t/8) \} \), and \( \varphi(x) = 0 \) on \( \{ x \in \mathbb{R}^d : \delta(x) \geq (t/6) \} \). Let \( \overline{w} = \varphi(u - E) \), where \( E \) is the \( L^1 \) average of \( u \) over \( \Omega \). Note that by (7.5), we have \( \mathcal{L}'(u) = \mathcal{L}(u) = 0 \) on \( \{ x \in \Omega : \delta(x) < (t/4) \} \). It follows that

\[
\mathcal{L}'( \overline{w}) = -\text{div}(A' \nabla \varphi \cdot (u - E)) - \nabla \varphi \cdot A' \nabla u \quad \text{in } \Omega.
\]

Let

\[
v(x) = \int_{\Omega} \nabla_y \Gamma_t(x, y) A'(y) \nabla \varphi \cdot (u(y) - E) \, dy - \int_{\Omega} \Gamma_t(x, y) \nabla \varphi \cdot A'(y) \nabla u(y) \, dy,
\]

where \( \Gamma_t(x, y) \) denotes the matrix of fundamental solutions for \( \mathcal{L}' \) in \( \mathbb{R}^d \). Then \( \mathcal{L}'(\overline{w} - v) = 0 \) in \( \Omega \) and for any \( x \in \Omega \) with \( \delta(x) \leq (t/16) \),

\[
|\nabla v(x)| + |v(x)| \leq C \left( \int_{\Omega} |u - E| + \int_{\Omega} |\nabla u| \right)^{1/2},
\]

where we have used the observation \( \nabla \varphi = 0 \) on \( \{ x \in \Omega : \delta(x) \leq (t/8) \} \). This allows us to use the Rellich estimate (7.6) for \( w = \overline{w} - v \) and obtain

\[
\int_{\partial \Omega} |\nabla u|^2 \, d\sigma \leq 2 \int_{\partial \Omega} |\nabla w|^2 \, d\sigma + 2 \int_{\partial \Omega} |\nabla v|^2 \, d\sigma
\]

\[
\leq C \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + C \int_{\partial \Omega} |\nabla v|^2 \, d\sigma + C \int_{\Omega} |\nabla u|^2 \, d\sigma + C \int_{\Omega} |\nabla v|^2 \, d\sigma + C \int_{\partial \Omega} |v|^2 \, d\sigma,
\]

which, by (7.7), yields the Rellich estimate for \( u \),

\[
\int_{\partial \Omega} |\nabla u|^2 \, d\sigma \leq C \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + C \int_{\Omega} |\nabla u|^2 \, dx.
\]

Theorem 7.1 now follows from the equivalence of Theorem 3.2 and Theorem 3.7 for \( \varepsilon = 1 \), proved in Section 3.

**Remark 7.3.** Let

\[
D(r, \psi) = \left\{ (x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + 10(C_0 + 1)r \right\},
\]

\[
\Delta(r, \psi) = \left\{ (x', \psi(x')) \in \mathbb{R}^d : |x'| < r \right\},
\]

(7.9)
where $\psi: \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function such that $\psi(0) = 0$ and $\|\nabla \psi\|_\infty \leq C_0$. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in $D(2r, \psi)$, where $0 < r < c_0 \varepsilon$. Let $v(x) = \varepsilon^{-1} u_\varepsilon(\varepsilon x)$. Then

$$\mathcal{L}_1(v) = 0 \quad \text{in} \quad D(2r/\varepsilon, \psi_\varepsilon),$$

where $\psi_\varepsilon(x') = \varepsilon^{-1} \psi(\varepsilon x')$. Under the conditions (1.2) and (1.4), it follows from Theorem 7.1 that

$$\Delta \frac{1}{t} |\nabla u_\varepsilon|^2 \leq C \frac{1}{t} \int_{\partial D(tr/\varepsilon, \psi_\varepsilon)} |\nabla v|^2 d\sigma + C |\partial D(tr/\varepsilon, \psi_\varepsilon)| \frac{1}{t} \int_{\partial \Omega} |\partial u_\varepsilon/\partial \nu_\varepsilon|^2 d\sigma + C \frac{1}{t} \int_{D(2r, \psi_\varepsilon)} |\nabla u_\varepsilon|^2 dx,$$

(7.10)

for $1 \leq t \leq 2$. We point out that since $\psi_\varepsilon(0) = 0$ and $\|\nabla \psi_\varepsilon\|_\infty \leq C_0$, the Lipschitz character of $D(tr/\varepsilon, \psi_\varepsilon)$ depends only on $C_0$. As a result, the constant $C$ in (7.10) depends only on $\kappa_1, \kappa_2, (\sigma, M), c_0$ and $C_0$. By a change of variables we may deduce from (7.10) that

$$\Delta |\nabla u_\varepsilon|^2 \leq C \int_{\Delta(r, \psi)} |\nabla u_\varepsilon|^2 d\sigma + C \frac{1}{r} \int_{\partial D(2r, \psi)} |\nabla u_\varepsilon|^2 d\sigma,$$

(7.11)

We now integrate both sides of (7.11) with respect to $t$ over the interval $(1, 2)$. This gives

$$\int_{\Delta(r, \psi)} |\nabla u_\varepsilon|^2 d\sigma \leq C \int_{\Delta(2r, \psi)} |\nabla u_\varepsilon|^2 d\sigma + C \frac{1}{r} \int_{\partial D(2r, \psi)} |\nabla u_\varepsilon|^2 d\sigma,$$

(7.12)

where $0 < r < c_0 \varepsilon$, $A$ satisfies (1.2) and (1.4), and $C$ depends only on $\kappa_1, \kappa_2, (\sigma, M), c_0$ and $C_0$.

### 8 Proof of Theorems 1.1, 3.2 and 3.7

As these three theorems are equivalent to each other (see Section 3), it suffices to prove Theorem 3.7.

We begin with a large-scale Rellich estimate. Note that the smoothness condition (1.4) is not needed.

**Theorem 8.1.** Assume that $A$ satisfies conditions (1.2) and (1.3). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and $g \in L^2(\partial \Omega)$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ be a weak solution to the Neumann problem,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g \quad \text{on} \quad \partial \Omega.$$

(8.1)

Suppose that $u_\varepsilon \perp \mathcal{R}$ in $L^2(\Omega; \mathbb{R}^d)$. Then for any $\varepsilon \leq r < \text{diam}(\Omega)$,

$$\frac{1}{r} \int_{\Omega_r} |\nabla u_\varepsilon|^2 dx \leq C \int_{\partial \Omega} |g|^2 d\sigma,$$

(8.2)

where $\Omega_r = \{x \in \Omega : \delta(x) < r\}$ and $C$ depends only on $\kappa_1, \kappa_2$, and the Lipschitz character of $\Omega$. 28
Proof. This was proved in [21, Theorem 1.2]. □

**Proof of Theorem 3.7.** Suppose that \( A \) satisfies (1.2), (1.3) and (1.4). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \) and \( u_\varepsilon \in H^1(\Omega; \mathbb{R}^d) \) a weak solution of \( L_\varepsilon(u_\varepsilon) = 0 \) in \( \Omega \). Assume that \( (\nabla u_\varepsilon)^* \in L^2(\partial \Omega) \) and \( \nabla u_\varepsilon \) exists n.t. on \( \partial \Omega \). We need to show that

\[
\int_{\partial \Omega} |\nabla u_\varepsilon|^2 \, d\sigma \leq C \int_{\partial \Omega} |g|^2 \, d\sigma + \frac{C}{r_0} \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx,
\]

(8.3)

where \( r_0 = \text{diam}(\Omega) \) and \( g = \partial u_\varepsilon / \partial \nu_\varepsilon \). We may assume that \( 0 < \varepsilon < c r_0 \). The case \( \varepsilon \geq c r_0 \) follows easily from the local Rellich estimate in Section 7 for \( \varepsilon = 1 \) by a simple rescaling argument.

First, we note that it suffices to prove (8.3) under the additional assumption that \( u_\varepsilon \perp \mathcal{R} \) in \( L^2(\Omega; \mathbb{R}^d) \). Indeed, to handle the general case, we choose \( \phi \in \mathcal{R} \) so that \( u_\varepsilon - \phi \perp \mathcal{R} \) in \( L^2(\Omega; \mathbb{R}^d) \). Then

\[
\int_{\Omega} |\phi|^2 \, dx \leq \int_{\Omega} |u_\varepsilon|^2 \, dx \quad \text{and} \quad \int_{\Omega} u_\varepsilon \, dx = \int_{\Omega} \phi \, dx,
\]

as well as Poincaré inequality, we obtain

\[
\frac{1}{r_0} \int_{\Omega} |\nabla \phi|^2 \, dx \leq \frac{C}{r_0^2} \int_{\Omega} |\phi - \int_{\Omega} \phi| \, dx \leq \frac{C}{r_0^3} \int_{\Omega} |u_\varepsilon - \int_{\Omega} u_\varepsilon| \, dx
\]

\[
\leq \frac{C}{r_0} \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx,
\]

which, together with (8.4), gives (8.3).

Next, fix \( z \in \partial \Omega \). It follows from the local Rellich estimate (7.12) by a change of coordinate systems that

\[
\int_{B(z, c_\varepsilon) \cap \partial \Omega} |\nabla u_\varepsilon|^2 \, d\sigma \leq C \int_{B(z, c_\varepsilon) \cap \partial \Omega} |g|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{B(z, c_\varepsilon) \cap \Omega} |\nabla u_\varepsilon|^2 \, dx,
\]

(8.5)

where \( C \) depends only on \( \kappa_1, \kappa_2, (\sigma, M) \) and the Lipschitz character of \( \Omega \). By covering \( \partial \Omega \) with balls centered on \( \partial \Omega \) with radius \( c_\varepsilon \), we may deduce from (8.5) that

\[
\int_{\partial \Omega} |\nabla u_\varepsilon|^2 \, d\sigma \leq C \int_{\partial \Omega} |g|^2 \, d\sigma + \frac{C}{\varepsilon} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx.
\]

(8.6)

Since \( u_\varepsilon \perp \mathcal{R} \) in \( L^2(\Omega; \mathbb{R}^d) \), it follows from Theorem 8.1 that

\[
\int_{\partial \Omega} |\nabla u_\varepsilon|^2 \, d\sigma \leq C \int_{\partial \Omega} |g|^2 \, d\sigma.
\]

Thus we have proved the Rellich estimate (8.3) and Theorem 3.7.
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