UNIFORM PERIODIC POINT GROWTH
IN ENTROPY RANK ONE

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Abstract. We show that algebraic dynamical systems with entropy rank one have uniformly exponentially many periodic points in all directions.

1. Introduction

Let $\alpha$ be an action of $\mathbb{Z}^d$ by continuous automorphisms of a compact metrizable abelian group $X$ (such a system is called an algebraic $\mathbb{Z}^d$-action). For a continuous map $\beta : X \to X$ write $h(\beta)$ for the topological entropy and $F(\beta) = \{x \in X \mid \beta x = x\}$ for the set of fixed points. The action $\alpha$ is said to have entropy rank one if, for each $n \in \mathbb{Z}^d$, $h(\alpha^n) < \infty$.

If $\alpha$ is a mixing entropy rank one action and the topological dimension $\dim(X)$ is finite, $F(\alpha^n)$ is finite for all $n \in \mathbb{Z}^d$. Our purpose here is to show that under natural conditions $|F(\alpha^n)|$ exhibits uniform exponential growth. Write $n_j \to \infty$ for a sequence in $\mathbb{Z}^d$ if for any finite set $F \subset \mathbb{Z}^d$ there is some $J$ for which $j > J$ implies that $n_j \notin F$. Equivalently, this means $\|n_j\| \to \infty$ as $j \to \infty$ where $\| \cdot \|$ is the Euclidean norm on $\mathbb{Z}^d$. The Noetherian condition mentioned in Theorem 1.1 is explained in Section 2.

Theorem 1.1. Let $\alpha$ be a mixing algebraic $\mathbb{Z}^d$-action with entropy rank one on a finite-dimensional group $X$. Then there exist constants $C_1, C_2 \geq 0$ such that

$$\limsup_{n \to \infty} \frac{1}{\|n\|} \log |F(\alpha^n)| = C_1 < \infty$$

and

$$\liminf_{n \to \infty} \frac{1}{\|n\|} \log |F(\alpha^n)| = C_2.$$ 

If $\dim(X) > 0$ and the action is Noetherian then $C_2 > 0$. 

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An affirmative answer to Lehmer’s problem (see Remark 2.3) would render the finite dimension assumption in Theorem 1.1 redundant. The assumption is also not required for a mixing Noetherian entropy rank one algebraic action; in that setting it is a consequence of the Noetherian condition. Essentially, the only non-trivial conclusion in the theorem is that $C_2 > 0$.

The action $\alpha$ is expansive if there exists a neighbourhood $U$ of $0_X$ such that

$$\bigcap_{n \in \mathbb{Z}^d} \alpha^n(U) = \{0_X\}.$$  

For $d = 1$ and $\alpha$ expansive, the growth rate of periodic points exists, so $C_1 = C_2$ in Theorem 1.1. The constant coincides with the entropy and is non-zero provided $X$ is infinite. For $d > 1$, the zero-dimensional Example 1.3 is expansive yet has $C_2 = 0$. Expansive actions on connected groups are more indicative of the import of a positive value for $C_2$. In terms of expansive subdynamics (see [3] and [6]), there are sequences $n \to \infty$ converging to non-expansive lines; along such sequences $|F(\alpha^n)|$ is much smaller than the same expression with $n$ of similar Euclidean size and far from non-expansive directions. Nonetheless, there is a uniform exponential growth in all directions.

**Example 1.2.** Consider the $\mathbb{Z}^2$-action $\alpha$ dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto 2r$ and $r \mapsto 3r$ on $\mathbb{Z}^2/\mathbb{Z}$. Figure 1 shows the map $n \mapsto |F(\alpha^n)| \in \mathbb{N}$. Notice that $|F(\alpha^{-n})| = |F(\alpha^n)|$, so only the region $n_2 \geq 0$ is shown, with $\infty$ denoting the lattice point $(0, 0)$ corresponding to the identity map $\alpha^{(0,0)}$.

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211 227 235 239 241 121 485 971 1943 3887 7775
49 65 73 77 79 5 161 323 647 1295 2591
5 11 19 23 25 13 53 107 215 431 863
23 7 1 5 7 1 17 35 71 143 287
29 13 5 1 1 1 5 11 23 47 95
31 5 7 1 1 \infty 1 1 7 5 31
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**Figure 1.** Periodic point counts for $\times 2, \times 3$.

In an expansive direction like $(1, 1)$ we have $|F(\alpha^{(1,1)})| = 6^n - 1$ and the exponential growth rate along the sequence in italics is clear. For lattice points close to the non-expansive line $2^x3^y = 1$ we find (for example) that $|F(\alpha^{(-5,3)})| = 5$, and it is not immediately clear that there is exponential growth along the sequence shown in bold; Theorem 1.1 asserts that there is.
Example 1.3. Consider the $\mathbb{Z}^2$-action $\alpha$ on $X$ dual to the $\mathbb{Z}^2$-action generated by the commuting maps $r \mapsto u_1 r$ and $r \mapsto u_2 r$ on

$$\mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}]/\langle 2, 1 + u_1 + u_2 \rangle$$

(this is an example of the type introduced by Ledrappier [12]; $X$ is a zero-dimensional group). As shown in [3], $|F(\alpha^{(n,0)})| = 2^{n-\text{ord}_2(n)}$. In particular,

$$\lim_{n \to \infty} \frac{1}{\| (2^n, 0) \|} \log |F(\alpha^{(2^n,0)})| = 0,$$

showing that some assumption on the topological dimension on $X$ is needed to have $C_2 > 0$ in Theorem 1.1.

Example 1.4. The Noetherian condition is needed to have $C_2 > 0$ even for $d = 1$. For example, the automorphism $\alpha$ dual to the map $r \mapsto 2r$ on $\mathbb{Q}$ has $|F(\alpha^n)| = 1$ for all $n$. For rings between $\mathbb{Z}$ and $\mathbb{Q}$ a variety of exotic periodic point behavior is possible (see [20] or [21] for the details).

Theorem 1.1 does however apply to Noetherian non-expansive systems. Thus, for example, the genuinely partially hyperbolic systems like that described by Damjanović and Katok [5, Ex. 7.3] satisfy the hypotheses.

2. Proof of Theorem 1.1

The case $d = 1$ is covered by [4] and [16], so we may assume $d \geq 2$. Algebraic $\mathbb{Z}^d$-actions have a convenient description in terms of commutative algebra due to Kitchens and Schmidt [11].

Let $R_d = \mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ be the ring of Laurent polynomials in commuting variables $u_1, \ldots, u_d$ with integer coefficients. If $\alpha$ is an algebraic $\mathbb{Z}^d$-action of a compact abelian group $X$, the character group $\hat{X}$ has the structure of a discrete countable $R_d$-module, obtained by first identifying the dual automorphism $\hat{\alpha}$ with multiplication by the monomial $u^n = u_1^{n_1} \ldots u_d^{n_d}$, and then extending additively to multiplication by polynomials. Conversely, any countable $R_d$-module $M$ has an associated algebraic $\mathbb{Z}^d$-action obtained by dualizing the action induced by multiplying by monomials on $M$. The action $\alpha = \alpha_M$ is described as Noetherian if the module $M$ is Noetherian. A full account of this correspondence and the resulting theory is given in Schmidt’s monograph [19].

Entropy rank one actions are described in [7] and developments concerning their periodic points may be found in the papers [16] and [18]. By a slight abuse of notation, write $h(\cdot)$ for the topological entropy of
maps and for the extension of the entropy function to all of \( \mathbb{R}^d \) in the sense explained below.

**Proposition 2.1.** Let \( \alpha_M \) be a mixing algebraic \( \mathbb{Z}^d \)-action with entropy rank one on a finite-dimensional group. Then

1. The set of associated primes of the associated \( R_d \)-module, \( \text{Asc}(M) \) is finite. For each \( p \in \text{Asc}(M) \), the domain \( R_d/p \) has Krull dimension 1 and its field of fractions \( \mathbb{K}(p) \) is a global field.
2. There exist positive integers \( m(p) \), \( p \in \text{Asc}(M) \), such that for every non-zero \( n \in \mathbb{Z}^d \),
   \[ h(\alpha_M^n) = \sum_{p \in \text{Asc}(M)} m(p)h(\alpha_{R_d/p}^n) \]  
   and
   \[ |F(\alpha_M^n)| \leq \prod_{p \in \text{Asc}(M)} |F(\alpha_{R_d/p}^n)|^{m(p)}, \]
   with equality if \( M \) is Noetherian.
3. For each \( p \in \text{Asc}(M) \), there is a finite set of places \( S(p) \) of \( \mathbb{K}(p) \) such that
   \[ h(\alpha_{R_d/p}^n) = \sum_{v \in S(p)} \max\{l_v \cdot n, 0\} > 0, \]  
   and
   \[ |F(\alpha_{R_d/p}^n)| = \prod_{v \in S(p)} |\xi_v^n - 1|, \]
   where \( \xi_v = (\xi_1, \ldots, \xi_d) \) denotes the image of \( u \) in \( (R_d/p)^d \) and \( l_v = (\log |\xi_1|, \ldots, \log |\xi_d|). \)

**Proof.** Let \( n \in \mathbb{Z}^d \) be non-zero. Since \( \alpha_M \) is mixing, \([19]\) Prop. 6.6\] shows that for each \( p \in \text{Asc}(M) \), \( \alpha_{R_d/p}^n \) is ergodic, so \( h(\alpha_{R_d/p}^n) > 0 \). It follows from \([7]\) that for each \( p \in \text{Asc}(M) \), \( R_d/p \) has Krull dimension 1 and \( \mathbb{K}(p) \) is a global field. If \( \text{char}(R_d/p) > 0 \) then \( h(\alpha_{R_d/p}^n) \geq \log 2 \). Via Yuzvinski\'s formula (see \([19]\) Th. 14.1 or \([23]\)), this implies there are only finitely many such \( p \in \text{Asc}(M) \), as \( h(\alpha_{R_d/p}^n) < \infty \). Also, there can be only finitely many \( p \in \text{Asc}(M) \) with \( \text{char}(R_d/p) = 0 \), as

\[ \dim(X) \geq |\{p \in \text{Asc}(M) : \text{char}(R_d/p) = 0\}|. \]

This establishes (1).

The method of \([7]\) Lem. 8.2\] shows that in any prime filtration of a Noetherian submodule of \( M \), each prime \( p \in \text{Asc}(M) \) appears with a maximum multiplicity

\[ m(p) = \dim_{\mathbb{K}(p)}(M \otimes_{R_d} \mathbb{K}(p)), \]
which is finite by similar reasoning to the proof of (1). By adopting an algorithm for obtaining a prime filtration which selects the associated primes of $M$ first, one obtains a Noetherian submodule $L \subset M$ such that each prime $p \in \text{Asc}(M)$ appears with multiplicity $m(p)$ in a filtration of $L$. Furthermore, if $L \neq M$, each $q \in \text{Asc}(M/L)$ is maximal and $R_d/q$ is a finite field. Hence, Yuzvinskiǐ’s formula shows that $h(\alpha_{M/L}^n) = 0$ and $h(\alpha_M^n) = h(\alpha_L^n)$; the formula (2.1) then follows from [15, Lem. 4.3].

If $M$ is Noetherian, equality in (2.2) is given by [16, Th. 3.2]. If $M$ is not Noetherian, (2.2) follows from [16, Th. 3.2] applied to $L$, together with the inequality

$$|F(\alpha_{M/L}^n)| \leq |F(\alpha_L^n)|,$$

which is established using a similar method to the proof of [17, Lem. 2.6].

Finally, the entropy formula (2.3) follows from [7, Prop. 8.5] and the periodic point counting formula (2.4) is [16, Lem 3.1].

**Proof of Theorem 1.1.** Let $M = \hat{X}$ denote the dual $R_d$-module and let $p \in \text{Asc}(M)$. For a fixed $p \in \text{Asc}(M)$ and any non-zero $n \in \mathbb{Z}^d$, set

$$f(n) = \frac{1}{\|n\|} \log |F(\alpha_{R_d/p}^n)|. \quad (2.5)$$

Let $h : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ denote the directional entropy function for the action $\alpha_{R_d/p}$. This is the function obtained by extending the entropy formula (2.3) to values of $\mathbb{R}^d$ (see [7, Sec. 8] for further details).

For any vector $v \in \mathbb{R}^d \setminus \{0\}$, write $\hat{v} \in S_{d-1}$ for the unique vector of unit length with the property that $v = \lambda \hat{v}$ for some scalar $\lambda > 0$. From (2.3) and (2.4) it follows that

$$f(n) = g(n) + h(\hat{n}) \quad (2.6)$$

where

$$g(n) = \frac{1}{\|n\|} \sum_{v \in S(p)} \log |1 - \phi_v(n)|_v$$

and

$$\phi_v(n) = \begin{cases} \xi_p^{-n} & \text{if } |\xi_p^n|_v > 1; \\ \xi_p^n & \text{if } |\xi_p^n|_v \leq 1. \end{cases}$$

Notice that $\phi_v(n) \neq 1$ if $n \neq 0$ by the assumption that the action is mixing.

To establish the lower bound $C_2 > 0$ in Theorem 1.1 first note that equality in (2.2) gives the expression

$$\frac{1}{\|n\|} \log |F(\alpha_M^n)|$$
as a finite sum of terms of the form (2.5) and – crucially – the assumption that \( \dim(X) > 0 \) means at least one of these terms arises from a prime \( p \) such that \( \mathbb{K}(p) \) is an algebraic number field (rather than a function field of positive characteristic). For the lower bound, it is therefore enough by (2.2) to consider only the asymptotic behavior of \( f \) with the assumption that it arises from such a prime.

We need to know that \( g \) does not make an asymptotic contribution. It is clear that \( g \) cannot be too large, since

\[
\prod_{v \in S(p)} |1 - \phi_v(n)|_v \leq 2^{|S(p)|}.
\]

On the other hand, it cannot be large and negative for the following reason. If \( v \) is an infinite (Archimedean) place, then Baker’s theorem [1] can be used to find constants \( A, B \geq 0 \) such that

\[
|1 - \phi_v(n)|_v \geq \frac{A}{\max\{n_i\}^B}
\]

(see [4] or [9] for similar arguments; roughly speaking, the issue is to bound the proximity of \( \xi_n^p \) and 1 in terms of \( n \)). If \( v \) is a finite place, then Yu’s \( p \)-adic bounds for linear forms in logarithms [22] give a similar lower bound (note that \( d \geq 2 \) by assumption). Since \( S(p) \) is finite, this shows that

\[
g(n) \to 0 \text{ as } n \to \infty. \tag{2.7}
\]

Assume, for a contradiction, that \( C_2 = 0 \). Then there is a sequence \( n_j \to \infty \) as \( j \to \infty \) with the property that

\[
\lim_{j \to \infty} f(n_j) = 0. \tag{2.8}
\]

It follows from (2.6) and (2.7) that

\[
h(n_j) \to 0 \text{ as } j \to \infty. \tag{2.9}
\]

Now the entropy function \( h \) restricted to the unit sphere \( S_{d-1} \) is a continuous function on a compact set, so (2.9) implies that there is some \( v \in S_{d-1} \) for which

\[
h(v) = 0. \tag{2.10}
\]

If the ray through \( v \) happens to meet a point \( m \in \mathbb{Z}^d \) then we have an immediate contradiction: the automorphism \( \alpha_m \) would have zero entropy, contradicting (2.3). In order to show that any point \( v \) with (2.10) gives a contradiction we need to make a quantitative version of that argument.

The explicit formula (2.3) for \( h \) means there is a list of vectors

\[
a_1, \ldots, a_r \in \mathbb{R}^d
\]
with the property that for any $u \in \mathbb{R}^d$, 
\[ h(u) = u \cdot a_k, \]
for some $k$, $1 \leq k \leq r$. It follows that for any $n \in \mathbb{Z}^d$ and $u \in \mathbb{R}^d$, 
\[ |h(n) - h(u)| \leq \max_{1 \leq k \leq r} \max_{1 \leq i \leq d} |a_{k,i}| \|n - u\|, \]
where $a_k = (a_{k,1}, \ldots, a_{k,d})$. In particular, since $h(\lambda v) = 0$ for all $\lambda > 0$ we may find a sequence $(m_j)$ of vectors in $\mathbb{Z}^d$ with $\|m_j - \lambda_j v\| \to 0$ as $j \to \infty$ (for some sequence $(\lambda_j)$ of scalars) and hence have 
\[ h(m_j) \to 0 \text{ as } j \to \infty. \]

(2.11)

On the other hand, the dual group of $\mathbb{R}^d/\mathbb{Z}^d$ is connected and finite-dimensional. Hence, by Yuzvinskii’s formula, for any $n \neq 0$, 
\[ h(n) \geq m(P) > 0 \]
where $m(P)$ denotes the logarithmic Mahler measure of some polynomial $P$ of degree no greater than $\dim_{\mathbb{Q}}(K)$. It follows that there is a constant $C > 0$ (depending only on $\mathbb{p}$) for which 
\[ h(n) > C > 0 \text{ for any } n \neq 0 \]
(the existence of a lower bound for the non-zero logarithmic Mahler measure of polynomials of bounded degree is well-known; see [2] or [8] for the background). This lower bound contradicts (2.11), so (2.8) is impossible. Thus $C_2 > 0$.

The upper bound is clear: the inequality (2.2) together with the explicit formula (2.4) gives a uniform constant $C_1$ with the property that 
\[ |F(\alpha^n)| \leq C_1 \max_{1 \leq i \leq d} (n_i). \]

Remark 2.2. Baker’s theorem provides the key estimates in many dynamical problems; see [4] and [9] for examples. As pointed out by Lind [14], in order to establish the logarithmic growth rate of periodic points for a quasihyperbolic toral automorphism (a typical application), sometimes all that is needed is a weaker and earlier result due to Gelfond [10]. Here we need something closer to the full weight of the theorems of Baker and Yu, because we are in higher rank.

Remark 2.3. Lehmer’s problem [13] asks if there is a uniform lower bound for all positive Mahler measures. As shown by Lind (see [15]) this is equivalent to a uniform lower bound for the topological entropy of any mixing compact group automorphism. If ‘Lehmer’s conjecture’ that there is such a bound and that it is attained by the expected
polynomial, then the topological entropy of any compact group automorphism with positive entropy is at least
\[
0.162 \cdots = \mathbf{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1).
\]
This does not imply a uniform bound for \( C_2 \) in Theorem 1 because \( C_2 \) is influenced by the geometry of the acting group as well as the collection of maps in its image. For example, the \( \mathbb{Z}^2 \) action \( \alpha \) corresponding to the module \( R_2/\langle u_1 - 2, u_1^k u_2 - 3 \rangle \) has the property that \( h(\alpha^{(k,1)}) = \log 3 \), so the corresponding constant \( C_2 \) cannot exceed \( \log 3/\sqrt{1 + k^2} \).

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