The first fundamental group of Kronecker quaternion group

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Abstract This paper discusses the first fundamental group from the identity graph of the group derived from the Kronecker product of the group quaternion representation. This group is called the Kronecker quaternion group. The purpose of this paper is to show the elements of the first fundamental group of the Kronecker quaternion group. A graph is needed to determine the first fundamental group. In this case, an identity graph is used; that is, graphs obtained from the Kronecker quaternion group. From this identity graph, it’s computed how many directed graphs can be made. The elements of the first fundamental group are the equivalence classes of paths in the directed identity graph that begin and end by a particular vertex. It is shown that the number of equivalence classes of paths for any fundamental group obtained from the identity graph is one plus the number of a completed graph $K_3$ in the identity graph of the Kronecker quaternion group.

1. Introduction
The author in [1] explained the existence of groups obtained from the application of the Kronecker product in the quaternion group representation. The author called this group with the Kronecker quaternion group. This group with multiplication operation and has 32 elements, which are $4 \times 4$ invertible matrices.

In this paper, we discuss the first fundamental group with graphs derived from the Kronecker quaternion group. Study about graphs that are associated with a group has become something interesting at this time. There are many papers on assigning a graph to a finite group and investigating algebraic properties using the associated graph. Some related studies can be seen in [2-5].

The graph from the group in [2] was defined as follows: Let $G$ be a non-abelian finite group and let $Z(G)$ be the center of $G$. A graph $\mathcal{T}$ associated with group $G$ is symbolized by $\mathcal{T}_G$ (called the noncommuting graph of $G$) is graph was described with taking $G \setminus Z(G)$ as the vertices of $\tau_G$ and two distinct vertices $x$ and $y$ are adjacent, whenever $xy = yx$. In this paper, was exploring the graph of subgroup $H$ of $G$, that is, if $\mathcal{T}_G \cong \mathcal{T}_H$ and $G$ is a finite non-abelian group, then $|G| = |H|$. The different definition is given by [3] using centralizer of a group, but it still studies the same problem as [2].

Furthermore, the definition graph of the group in [2], is used by [4] to extend the study to the structure of the graph and determine conditions under which the graph is regular or biregular. It’s investigated this condition for a finite non-abelian group with order $p^n$, $p$ is prime and $n \in \mathbb{N}$. The algebraic properties of this graph associated with the order of elements of the group and order of the group. (noted that, the order of an element of a group is different with an order of the group, see [8]). Another definition of the graph of the group is given by [5], as follows: Let $G$ be a group and $\mathcal{T}_G$ is a graph of $G$; elements of $G$ are elements of $V(\mathcal{T})$ and two distinct vertices $x$ and $y$ are adjacent if $(x, y)$ is cyclic.
In this paper, we use another definition of a graph of a group. This graph is called an identity graph, and the definition will be given in Section 2. This graph is used to determine the fundamental group by first making this graph as a directed graph. Thus there are many possibilities of the first fundamental group that can be made from this directed identity graph. Therefore, in Section 2 it is also explained how many directed identity graphs can be made from identity graphs from the Kronecker quaternion group. In Section 3, it’s present the main result and the definition of the first fundamental group and the example of the first fundamental group are obtained from the directed identity graph of the Kronecker quaternion group.

2. Methods

2.1. Identity graph of the Kronecker quaternion group

We consider the simple graph which is directed, with no loops or multiple edges. For any graph $\mathcal{T}$, we denote the sets of the vertices and edges of $\mathcal{T}$ by $V(\mathcal{T})$ and $E(\mathcal{T})$, respectively. For terminology in graph theory, we refer [6] and for terminology in group theory we refer [7]. For the definition of identity graph of a group we refer [8], and the definition as follows:

Definition 2.1 [8] Let $G$ be a group with identity element $1$. Graph $\mathcal{T}_G$ associated with $G$, is a graph which describes as follows: elements of $V(\mathcal{T})$ are elements of $G$ and elements of $E(\mathcal{T})$ are two elements $x, y$ in $G$ are adjacent or can be joined by an edge if $xy = 1$; every element in $G$ is adjacent with the identity of a group $G$. Graph $\mathcal{T}_G$ is called an identity graph of $G$.

Based on [1], the elements of the Kronecker quaternion group are symbolized by $A_1, A_2, ..., A_{32}$. Identity element is $A_1$. The inverse of each element can be explained as follows: For each $A_k$, $k = 1, 2, ..., 32$, $(A_i)^{-1} = A_{i+1}$ for $i = 3, 5, 7, 9, 17, 25$, $(A_i)^{-1} = A_{i-1}$, for $i = 4, 6, 8, 10, 18, 26$, and other $(A_i)^{-1} = A_i$. For simplicity, in this paper, the writing elements of this group are changed to 1, 2, ..., 32. The following is an identity graph of the Kronecker quaternion group.

![Figure 1. Identity graph of Kronecker quaternion group](image-url)

Based on Figure 2.1, the identity graph of Kronecker quaternion group contains 19 complete graphs $K_2$ and 6 completed graphs $K_3$. Some terms are needed to construct the first fundamental group, i.e the initial vertex, terminal vertex, and directed graph (see [6]).

From Figure 2.1, a number of directed graphs can be determined, namely by giving direction to each edge. Of course, there are many variations of directed graphs obtained. This directed graph is used to
determine the first fundamental group. Since there are many variations of directed graphs from Figure 2.1, then we have the next proposition: (we called the graph in Figure 2.1 as \( \mathcal{T}_G \)).

Proposition 2.1 Consider the identity graph as in Figure 2.1. The number of a directed graph of \( \mathcal{T}_G \) is \( 8^6 \cdot 2^{19} \).

Proof. Graph \( \mathcal{T}_G \) has complete graphs \( K_3 \) and \( K_2 \), 6 and 19 respectively (see Figure 2.1). Consider that each edge of \( \mathcal{T}_G \) has two possible directions. Since \( \mathcal{T}_G \) has 19 complete graphs \( K_2 \), then the possible number of directions is \( 2^{19} \), and \( \mathcal{T}_G \) has 6 complete graphs \( K_3 \), then the possible number of directions is \( 8^{19} \). Thus, the number of a directed graph of \( \mathcal{T}_G \) is \( 8^6 \cdot 2^{19} \). This completes the proof.

3. The main result

In general, this paper is concerned with the first fundamental group. Therefore, the main results of this study are about matters related to it, that is, directed graph and equivalence classes of a closed path.

Let \( \mathcal{G} \) be a directed connected graph, \( v \) be a fixed vertex in \( \mathcal{G} \) and \( \{y\} \) is the equivalent class of closed path \( y \) with \( i(y) = \tau(y) = v \), where \( i(y) \) is an initial vertex of \( y \) and \( \tau(y) \) is the terminal vertex of \( y \), and \( \{\{y\}\} | i(y) = \tau(y) = v \} \) set of classes equivalence of closed path with \( i(y) = \tau(y) = v \). We have the next theorem:

Theorem 3.1 [10] The algebraic system \( \pi_1(\mathcal{G}, v) = \{\{y\}\} | i(y) = \tau(y) = v \} \) with binary operation \( [y_1][y_2] = [y_1y_2] \) form a group named the first fundamental group with base point \( v \). The identity of this group is \([1_v] \) while the inverse of \([y] \) is \([y^{-1}] \) or \([y]^{-1} = [y^{-1}] \).

Next, one of the directed graph from Figure 2:

![Figure 2 Directed graph](image)

Next, it’s explained the first fundamental group with choosing one vertex arbitrary. There are 3 possibilities for determining the first fundamental group for the identity directed graph based on Figure 2.2, that is, the first fundamental group is based on a vertex in \( K_2 \), in \( K_3 \) and vertex which is the identity element in the groups.

Theorem 3.2 Let \( \mathcal{G} \) be a directed identity graph of the Kronecker quaternion group. Then \(|\pi_1(\mathcal{G}, v)| = 1, |\pi_1(\mathcal{G}, v)| = 2, or |\pi_1(\mathcal{G}, v)| = \) number of \( K_3 + 1 \).

Proof. Let \( v \) is a vertex in \( K_2 \) but \( v \) is not an identity element of the group, then \( \pi_1(\mathcal{G}, v) = \{\{y\}\} | i(y) = \tau(y) = v \} = \{[1_v]\}. \) This is caused by closed path arising from \( i(y) = \tau(y) = v \) only the vertex itself. Let \( v \) is a vertex in \( K_3 \) but \( v \) is not the identity element of the group, then \( \pi_1(\mathcal{G}, v) = \)
\[ \{ y \} \ \{ y \} = \tau(y) = v = \{ [1_v], [e_{1,1}, e_{1,4}, e_{1,8,17}] \} \] where \( uv = 1 \), \( e_{1,1} \) is the edge of \( v \) and \( 1, e_{1,4} \) is the edge of \( 1 \) and \( u, e_{1,8,17} \) is the edge of \( u \) and \( v \). Let \( v \) is the identity element of the group, then \( \pi_1(T', v) = \{ [y] \ | \ i(y) = \tau(y) = v = \{ [1_v], [y_i], i = 1, 2, 3, 4, 5, 6 \} \) where \( y_i \) is closed path from \( K_{3i} \).

**Example 3.1**
Consider that Figure 2.2. Based on Theorem 3.2, there are 3 cases for constructing the first fundamental group.

**Case 1.** Let \( v \) is vertex in \( K_2 \) and \( v \) is not the identity element of the group, choose \( v = 2 \), so \( \pi_1(T', 2) = \{ [y] \ | \ i(y) = \tau(y) = 2 \} = \{ [2] \} \). (noted that, there are 19 \( K_2 \) in Figure 2.2).

**Case 2.** Let \( v \) is a vertex in \( K_3 \) but \( v \) is not the identity element of the group, choose \( v = 17 \), then \( \pi_1(T', 17) = \{ [y] \ | \ i(y) = \tau(y) = 17 \} = \{ [17], [e_{17,1}, e_{1,17}, e_{18,17}] \} \) (Noted that, there are 6 \( K_3 \) in Figure 2.2)

**Case 3.** Let \( v \) is the identity element of the group, that is \( v = 1 \) then \( \pi_1(T', 1) = \{ [y] \ | \ i(y) = \tau(y) = 1 \} = \{ [1], [e_{1,17}, e_{17,18}, e_{18,17}], [e_{1,25}, e_{25,26}, e_{26,1}] \}, [e_{1,13}, e_{3,4}, e_{4,1}], [e_{1,5}, e_{5,6}, e_{6,1}], [e_{1,7}, e_{7,10}, e_{8,1}], [e_{1,9}, e_{9,10}, e_{10,1}] \} \). (the sign ‘‘-’’ on the edge indicates the opposite direction)

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