A Generalized Probabilistic Version of Modus Ponens

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\begin{abstract}
Modus ponens (from \textit{A} and “if \textit{A} then \textit{C}” infer \textit{C}; short: MP) is one of the most basic and important inference rules. The probabilistic MP allows for managing uncertainty by transmitting assigned uncertainties from the premises to the conclusion (i.e., from \(P(A)\) and \(P(C|A)\) infer \(P(C)\)). In this paper, we generalize the probabilistic MP by replacing \textit{A} by the conditional event \(A|H\). The resulting inference rule involves iterated conditionals (formalized by conditional random quantities) and propagates previsions from the premises to the conclusion. Interestingly, the propagation rules for the lower and the upper bounds on the conclusion of the generalized probabilistic MP coincide with the respective bounds on the conclusion for the (non-nested) probabilistic MP.

\textbf{Keywords:} Coherence, Conditional random quantities, Conjoined conditionals, Iterated conditionals, Modus Ponens, Prevision
\end{abstract}

1 Introduction

Modus ponens (from \textit{A} and “if \textit{A} then \textit{C}” infer \textit{C}) is one of the most basic and important inference rules. By instantiating the antecedent of a conditional it allows for detaching the consequent of the conclusion. It is well-known that modus ponens is logically valid (i.e., it is impossible that \textit{A} and \(\textit{A} \lor \textit{C}\) are true while \textit{C} is false, where the event \(\overline{A} \lor C\) denotes the material conditional as defined in classical logic). It is also well-known that there are philosophical arguments \textsuperscript{113} and psychological arguments \textsuperscript{9122} in favor of the hypothesis that a conditional \textit{if \textit{A}, then \textit{C}} is best represented by a suitable conditional probability assertion \(P(C|A)\) and not by a probability of a corresponding material conditional \(P(\overline{A} \lor C)\). Consequently, coherence-based probability logic generalizes the classical modus ponens probabilistically by propagating assigned probabilities from the premises to the conclusion as follows (see, e.g., \textsuperscript{23}2426):

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Probabilistic modus ponens From \( P(A) = x \) (probabilistic categorical premise) and \( P(C|A) = y \) (probabilistic conditional premise) infer \( xy \leq P(C) \leq xy + 1 - y \) (probabilistic conclusion).

In our paper, \( P(C|A) \) is the probability of the conditional event \( C|A \) (see, e.g., \([6,7,12,18,25]\)). The probabilistic modus ponens is \( p \)-valid (i.e., the premise set \( \{A,C|A\} \) \( p \)-entails the conclusion \( C \)) and probabilistically informative \([10,15,12,24]\).

In this paper we generalize the probabilistic modus ponens by replacing the categorical premise (i.e., \( A \)) and the antecedent of the conditional premise (i.e., \( A \) in “if \( A \) then \( C \)”) by the conditional event \( A|H \). The resulting inference rule involves the prevision \( P(C|(A|H)) \) of the iterated conditional \( C|(A|H) \) (formalized by a suitable conditional random quantity, see \([11,13,14,16]\)) and propagates the uncertainty from the premises to the conclusion:

**Generalized probabilistic modus ponens** From \( P(A|H) \) (generalized categorical premise) and \( P(C|(A|H)) \) (generalized conditional premise) infer \( P(C) \) (conclusion).

The conditional event \( A|H \) is interpreted as a conditional random quantity, with \( P(A|H) = P(A|H) \) (see below). As mentioned above, modus ponens instantiates the antecedent of a conditional and governs the detachment of the consequent of the conclusion. In our generalization, we study the case where the unconditional event \( A \) is replaced by the conditional event \( (A|H) \) and the conditional event \( C|A \) is replaced by the iterated conditional \( C|(A|H) \). This corresponds to a common-sense reasoning context where instead of a fact \( A \) a rule \( A|H \) is learned and used for a modus ponens inference.

The outline of the paper is as follows. In Section 2 we first recall basic notions and results on coherence and previsions of conditional random quantities. Then, we illustrate the notions of conjunction between conditional events and of iterated conditional, by recalling some results. In Section 3 we prove a generalized decomposition formula for conditional events, with other results on compounded and iterated conditionals. In Section 4 we propagate the previsions from the premises of the generalized probabilistic modus ponens to the conclusion. We observe that this propagation rule coincides with the probability propagation rule for the (non-nested) probabilistic modus ponens (where \( H = \Omega \)) \([24]\). Section 5 concludes the paper with an outlook for future work.

2 Preliminary notions

In this section we recall some basic notions and results on coherence for conditional prevision assessments. In our approach an event \( A \) represents an uncertain fact described by a (non-ambiguous) logical entity, where \( A \) is two-valued and can be true (\( T \)), or false (\( F \)). The indicator of \( A \), denoted by the same symbol, is a two-valued numerical quantity which is 1, or 0, according to whether \( A \) is true, or false, respectively. The sure event is denoted by \( \Omega \) and the impossible
event is denoted by \( \emptyset \). Moreover, we denote by \( A \land B \), or simply \( AB \), (resp., \( A \lor B \)) the logical conjunction (resp., logical disjunction). The negation of \( A \) is denoted by \( \overline{A} \). Given any events \( A \) and \( B \), we simply write \( A \subseteq B \) to denote that \( A \) logically implies \( B \), that is, \( \overline{A} \overline{B} \) is the impossible event \( \emptyset \). We recall that \( n \) events are logically independent when the number \( m \) of constituents, or possible worlds, generated by them is \( 2^n \) (in general \( m \leq 2^n \)). Given two events \( A \) and \( B \), with \( H \neq \emptyset \), the conditional event \( A|H \) is defined as a three-valued logical entity which is true if \( AH \) is true, false if \( \overline{AH} \) is true, and void if \( H \) is false.

### 2.1 Coherent conditional prevision

We recall below the notion of coherence (see, e.g., [2,3,4,5,12,16,21]). Given a prevision function \( \mathbb{P} \) defined on an arbitrary family \( \mathcal{K} \) of conditional random quantities with finite sets of possible values, consider a finite subfamily \( \mathcal{F}_n = \{ X_i|H_i, i \in J_n \} \subseteq \mathcal{K} \), where \( J_n = \{ 1, \ldots, n \} \), and the vector \( \mathcal{M}_n = (\mu_i, i \in J_n) \), where \( \mu_i = \mathbb{P}(X_i|H_i) \) is the assessed prevision for the c.r.q. \( X_i|H_i \). With the pair \( (\mathcal{F}_n, \mathcal{M}_n) \) we associate the random gain \( G = \sum_{i \in J_n} s_i H_i (x_i - \mu_i) \); moreover, we set \( \mathcal{H}_n = H_1 \lor \cdots \lor H_n \) and we denote by \( \mathcal{G}_{\mathcal{H}_n} \) the set of values of \( G \) restricted to \( \mathcal{H}_n \). Then, using the betting scheme of de Finetti, we stipulate

**Definition 1.** The function \( \mathbb{P} \) defined on \( \mathcal{K} \) is coherent if and only if, \( \forall n \geq 1, \forall \mathcal{F}_n \subseteq \mathcal{K}, \forall s_1, \ldots, s_n \in \mathbb{R} \), it holds that: \( \min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n} \).

Given a family \( \mathcal{F}_n = \{ X_i|H_i, X_n|H_n \} \), for each \( i \in J_n \), we denote by \( (x_1, \ldots, x_{ir_i}) \) the set of possible values for the restriction of \( X_i \) to \( H_i \); then, for each \( i \in J_n \), \( j = 1, \ldots, r_i \), we set \( A_{ij} = (x_i = x_{ij}) \). Of course, for each \( i \in J_n \), the family \( \{ \overline{A}_{1i}, A_{ij}, H_i, j = 1, \ldots, r_i \} \) is a partition of the sure event \( \Omega \), with \( A_{ij} H_i = A_{ij}, \bigvee_{j=1}^{r_i} A_{ij} = H_i \). Then, the constituents generated by the family \( \mathcal{F}_n \) are (the elements of the partition of \( \Omega \)) obtained by expanding the expression \( \bigwedge_{i \in J_n} (A_{1i} \lor \cdots \lor A_{ir_i} \lor \overline{H_i}) \). We set \( C_0 = \overline{\bigwedge_{i \in J_n} A_{1i}} \) (it may be \( C_0 = \emptyset \)); moreover, we denote by \( C_1, \ldots, C_m \) the constituents contained in \( \mathcal{H}_n = H_1 \lor \cdots \lor H_n \). Hence \( \bigwedge_{i \in J_n} (A_{1i} \lor \cdots \lor A_{ir_i} \lor \overline{H_i}) = \bigvee_{h=0}^{m} C_h \). With each \( C_h \), \( h \in J_n \), we associate a vector \( Q_h = (q_{h1}, \ldots, q_{hn}) \), where \( q_{hi} = x_{ij} \) if \( C_h \subseteq A_{ij}, j = 1, \ldots, r_i \), while \( q_{hi} = \mu_i \) if \( C_h \subseteq \overline{H_i} \); \( C_0 \) is associated with \( Q_0 = \mathcal{M}_n = (\mu_1, \ldots, \mu_n) \). Denoting by \( \mathcal{I}_n \) the convex hull of \( Q_1, \ldots, Q_m \), the condition \( \mathcal{M}_n \in \mathcal{I}_n \) amounts to the existence of a vector \( (\lambda_1, \ldots, \lambda_m) \) such that:

\[
\sum_{h \in J_m} \lambda_h q_{hi} = \mathcal{M}_n, \quad \sum_{h \in J_m} \lambda_h = 1, \quad \lambda_h \geq 0, \quad \forall h \in J_m.
\]

\[
(\Sigma) \quad \sum_{h \in J_m} \lambda_h q_{hi} = \mu_i, \quad i \in J_n; \quad \sum_{h \in J_m} \lambda_h = 1; \quad \lambda_h \geq 0, \quad h \in J_m.
\] (1)

Given the assessment \( \mathcal{M}_n = (\mu_1, \ldots, \mu_n) \) on \( \mathcal{F}_n = \{ X_1|H_1, \ldots, X_n|H_n \} \), let \( S \) be the set of solutions \( A = (\lambda_1, \ldots, \lambda_m) \) of system (\( \Sigma \)) defined in (1). Then, the following theorem can be proved (2)

**Theorem 1.** [Characterization of coherence]. Given a family of \( n \) conditional random quantities \( \mathcal{F} = \{ X_1|H_1, \ldots, X_n|H_n \} \) and a vector \( \mathcal{M} = (\mu_1, \ldots, \mu_n) \).
the conditional prevision assessment \( \mathbb{P}(X_i|H_I) = \mu_1, \ldots, \mathbb{P}(X_n|H_n) = \mu_n \) is coherent if and only if, for every subset \( J \subseteq J_n \), defining \( F_J = \{X_i|H_I, i \in J\} \), \( K_J = (\mu_i, i \in J) \), the system \( (\Sigma_J) \) associated with the pair \((F_J, K_J)\) is solvable.

By following the approach given in [13, 14, 16] a conditional random quantity \( X|H \) can be seen as the random quantity \( XH + \mu H \), where \( \mu = \mathbb{P}(X|H) \). In particular a conditional event \( A|H \) can be interpreted as \( AH + x \), where \( x = P(A|H) \).

Moreover, the negation of \( A|H \) is defined as \( A\bar{H} = 1 - A|H = \bar{A}|H \). Coherence can be characterized in terms of proper scoring rules ([5]), which can be related to the notion of entropy in information theory ([19]).

2.2 Conjunction and iterated conditional

**Definition 2.** Given any pair of conditional events \( A|H \) and \( B|K \), with \( P(A|H) = x \), \( P(B|K) = y \), we define their conjunction as the conditional random quantity \( (A|H) \land (B|K) = Z \land (H \lor K) \), where \( Z = \min \{A|H, B|K\} \).

Based on the betting scheme, the compound conditional \((A|H) \land (B|K)\) coincides with \( 1 \cdot AHBK + x \cdot \bar{A}BH + y \cdot A\bar{H}K + z \cdot \bar{A}\bar{H} \), where \( z \) is the prevision of the random quantity \((A|H) \land (B|K)\), denoted by \( \mathbb{P}[(A|H) \land (B|K)] \). Notice that \( z \) represents the amount you agree to pay, with the proviso that you will receive the quantity \((A|H) \land (B|K)\). For examples see [13] and [17]. Notice that this notion of conjunction, with positive probabilities for the conditioning events, has been already proposed in [20]. Now, we recall the notion of iterated conditioning.

**Definition 3 (Iterated conditioning).** Given any pair of conditional events \( A|H \) and \( B|K \), the iterated conditional \((B|K)|(A|H)\) is the conditional random quantity \((B|K)|A|H) = (B|K) \land (A|H) + \mu A|H \), where \( \mu = \mathbb{P}[(B|K)|(A|H)] \).

Notice that, in the context of betting scheme, \( \mu \) represents the amount you agree to pay, with the proviso that you will receive the quantity

\[
(B|K)|(A|H) = \begin{cases} 
1, & \text{if } AHBK \text{ true,} \\
0, & \text{if } A\bar{H}BK \text{ true,} \\
y, & \text{if } AH\bar{K} \text{ true,} \\
\mu, & \text{if } A\bar{H} \text{ true,} \\
x + \mu(1-x), & \text{if } \bar{A}BH \text{ true,} \\
\mu(1-x), & \text{if } \bar{A}\bar{H}K \text{ true,} \\
z + \mu(1-x), & \text{if } \bar{A}\bar{H} \text{ true.} 
\end{cases}
\tag{2}
\]

We recall the following product formula ([14])

**Theorem 2 (Product formula).** Given any assessment \( x = P(A|H), \mu = \mathbb{P}[(B|K)|(A|H)], z = \mathbb{P}[(B|K) \land (A|H)] \), if \((x,\mu,z)\) is coherent, then \( z = \mu \cdot x \).

We recall that coherence requires that \((x,\mu,z) \in [0,1]^3\) (see, e.g., [11]).

**Remark 1.** Given any random quantity \( X \) and any events \( H, K \), with \( H \subseteq K \), \( H \neq \emptyset \), it holds that (see [19] Section 3.3): \((X|H)|K = X|HK = X|H \). In particular, given any events \( A, H, K \), \( H \neq \emptyset \), it holds that: \((A|H)|(H \lor K) = A|H \).
3 Some results on compounded and iterated conditionals

In this section we present a decomposition formula, by also considering a particular case. Then, we give a result on the coherence of a prevision assessment on \( \mathcal{F} = \{A|H, C|(A|H), C|(\overline{A}|H)\} \) which will be used in the next section.

**Proposition 1.** Let \( A|H, B|K \) be two conditional events. Then

\[
B|K = (A|H) \land (B|K) + (\overline{A}|H) \land (B|K).
\]

**Proof.** Let \((x, y, z_1, z_2)\) be a (coherent) prevision on \((A|H, B|K, (A|H) \land (B|K), (\overline{A}|H) \land (B|K))\). Of course, coherence requires that \(P(\overline{A}|H) = 1 - x\). By Definition 2 it holds that

\[
(A|H) \land (B|K) = ABKB + x\overline{A}BK + yKBH + z_1\overline{A}K
\]

and

\[
(\overline{A}|H) \land (B|K) = \overline{A}HBK + (1 - x)\overline{A}BK + y\overline{A}H + z_2\overline{A}K.
\]

Then,

\[
(A|H) \land (B|K) + (\overline{A}|H) \land (B|K) = HKB + \overline{A}BK + y\overline{A}H + z_1\overline{A}K + z_2\overline{A}K =
BK + y\overline{K}H + (z_1 + z_2)\overline{A}K.
\]

Moreover,

\[
B|K = BK + y\overline{K} = BK + y\overline{K}H + y\overline{A}K.
\]

From (1) and (5), when \(H \lor K\) is true, it holds that

\[
(A|H) \land (B|K) + (\overline{A}|H) \land (B|K) = BK + y\overline{K}H = B|K.
\]

Then, the difference \([ (A|H) \land (B|K) + (\overline{A}|H) \land (B|K)] - B|K \) is zero when \(H \lor K\) is true. Thus,

\[
\mathbb{P}([(A|H) \land (B|K) + (\overline{A}|H) \land (B|K) - B|K]|(H \lor K)] =
\mathbb{P}([(A|H) \land (B|K)]|(H \lor K)] + \mathbb{P}([(\overline{A}|H) \land (B|K)]|(H \lor K)] - \mathbb{P}[(B|K)]|(H \lor K)] = 0.
\]

By Remark 4 it holds that \([ (A|H) \land (B|K)]|(H \lor K), [(\overline{A}|H) \land (B|K)]|(H \lor K),\) and \((B|K)]|(H \lor K)\) coincide with \((A|H) \land (B|K), (\overline{A}|H) \land (B|K),\) and \(B|K),\) respectively. Then,

\[
\mathbb{P}([(A|H) \land (B|K) + (\overline{A}|H) \land (B|K) - B|K]|(H \lor K)] =
\mathbb{P}([(A|H) \land (B|K)] + \mathbb{P}[(\overline{A}|H) \land (B|K)] - \mathbb{P}(B|K) = z_1 + z_2 - y = 0.
\]

Therefore, \((A|H) \land (B|K) + (\overline{A}|H) \land (B|K)\) and \(B|K\) also coincide when \(H \lor K\) is false. Thus, \((A|H) \land (B|K) + (\overline{A}|H) \land (B|K) = B|K.\)

**Remark 2.** Notice that Proposition 4 also holds when there are some logical relations among the events \(A, B, H, K,\) provided that \(H \neq \emptyset\) and \(K \neq \emptyset.\) In particular, if \(K = \Omega\) the proof of Proposition 4 is simpler because, by Definition 2

\[
(A|H) \land B = ABHB + x\overline{A}B, \quad (\overline{A}|H) \land B = \overline{A}HB + (1 - x)\overline{A}B,
\]

hence

\[
(A|H) \land B + (\overline{A}|H) \land B = HB + \overline{A}B = B.
\]
Remark 3. Consider a bet on an iterated conditional $C|(A|H)$, with $H \neq \emptyset$, $A \neq \emptyset$, $x = P(A|H)$, and $y = \mathbb{P}[C|(A|H)]$. In this bet, $y$ is the amount that we pay, while $C|(A|H)$ is the amount that we receive. Then, in order to check coherence, the bet on $C|(A|H)$ must be called off when $C|(A|H)$ coincides with its prevision $y$. We distinguish two cases: (i) $x > 0$, (ii) $x = 0$.

Case (i). By applying Definition $\blacksquare$ with $B$ replaced by $C$ and $K = \Omega$, we obtain

$$C|(A|H) = \begin{cases} 1, & \text{if } AHC \text{ true,} \\ 0, & \text{if } A\overline{H} \text{ true,} \\ y, & \text{if } \overline{A}H \text{ true,} \\ x + y(1 - x), & \text{if } \overline{P}C \text{ true,} \\ y(1 - x), & \text{if } \overline{P}\overline{C} \text{ true.} \end{cases} \quad (7)$$

If $x > 0$, $C|(A|H) = y$ when $\overline{A}H$ is true, that is, the bet on $C|(A|H)$ is called off when $\overline{A}H$ is true; then to check coherence we must only consider the constituents contained in $\overline{A}H = AH \lor \overline{P}$.

Case (ii). As $x = 0$, we have

$$C)|(A|H) = \begin{cases} 1, & \text{if } AHC \text{ true,} \\ 0, & \text{if } A\overline{H} \text{ true,} \\ y, & \text{if } \overline{A}H \text{ true,} \\ y, & \text{if } \overline{P}C \text{ true,} \\ y, & \text{if } \overline{P}\overline{C} \text{ true.} \end{cases} \quad (8)$$

Then $C|(A|H) = C|AH$ (see [10] Theorem 4) and to check coherence we must only consider the constituents contained in $AH$.

We denote by $(x > 0)$ an event which is true or false, according to whether $x$ is positive or not. Then, by unifying Case (i) and Case (ii), the constituents such that the bet on $C|(A|H)$ is not called off are those contained in $AH \lor \overline{P}(x > 0)$.

Theorem 3. Let three logically independent events $A, C, H$ be given, with $A \neq \emptyset$, $H \neq \emptyset$. The set of all coherent assessments $\mathcal{M} = \{x, y, z\}$ on $\mathcal{F} = \{A|H, C|(A|H), C|(\overline{A}|H)\}$ is the unit cube $[0, 1]^3$.

Proof. Coherence requires that $x, y$, and $z$ must be in $[0, 1]$ (see, e.g., [11]). Thus, $(x, y, z)$ is not coherent when $(x, y, z) \notin [0, 1]^3$.

Let $\mathcal{M} = \{x, y, z\} \in [0, 1]^3$ be a prevision assessment on $\mathcal{F} = \{A|H, C|(A|H), C|(\overline{A}|H)\}$. Based on Theorem $\blacksquare$ we prove coherence by showing that for each subset $J \subseteq \{1, 2, 3\}$ the system $(\Sigma_J)$ is solvable. By Definition $\blacksquare$

$$C|(A|H) = C \land (A|H) + y\overline{A}|H, \quad C|(\overline{A}|H) = C \land (\overline{A}|H) + zA|H.$$ 

We start with $J = \{1, 2, 3\}$. Based on Remark $\blacksquare$ we observe that $\mathcal{H}_3 = H_1 \lor H_2 \lor H_3 = H \lor (AH \lor \overline{P}(x > 0) \lor (\overline{A}H \lor \overline{P}(x < 1)) \lor H \lor \overline{P} = \Omega$. The constituents $C_h$'s (contained in $\mathcal{H}_3$) and the corresponding points $Q_h$'s associated with $(\mathcal{F}, \mathcal{M})$ are:

$C_1 = AHC$, $C_2 = \overline{A}HC$, $C_3 = AH\overline{C}$, $C_4 = \overline{A}H\overline{C}$, $C_5 = \overline{P}C$, $C_6 = \overline{P}\overline{C}$,

$Q_1 = (1, 1, z)$, $Q_2 = (0, y, 1)$, $Q_3 = (1, 0, z)$, $Q_4 = (0, y, 0)$,

$Q_5 = (x, x + y(1 - x), (1 - x) + xz)$, $Q_6 = (x, y(1 - x), xz)$.
We observe that $Q_5 = xQ_1 + (1 - x)Q_2$ and $Q_6 = xQ_3 + (1 - x)Q_4$, so that for checking the solvability of the system $\Sigma_\lambda$ it is enough to consider the points $Q_1, Q_2, Q_3, Q_4$. The condition $(x, y, z) = \sum_{h=1}^4 \lambda_h Q_h$, with $\lambda_h \geq 0$ and $\sum_{h=1}^4 \lambda_h = 1$, is satisfied for every $(x, y, z) \in [0, 1]^3$. Indeed, the system

$$\begin{cases}
x = \lambda_1 + \lambda_3, \quad y = \lambda_1 + \lambda_2 y + \lambda_4 y, \quad z = \lambda_1 z + \lambda_2 + \lambda_3 z, \\
\sum_{h=1}^4 \lambda_h = 1, \quad \lambda_h \geq 0, \quad h = 1, 2, 3, 4
\end{cases}$$

has the non-negative solution

$$\begin{cases}
\lambda_1 = y(1 - \lambda_2 - \lambda_4) = y(\lambda_1 + \lambda_3) = xy, \quad \lambda_2 = z(1 - x), \\
\lambda_3 = (1 - y)x, \quad \lambda_4 = 1 - z + xz - x = (1 - x)(1 - z).
\end{cases}$$

With $J = \{1, 2\}$ we associate the pair $(F_J, M_J)$, where $F_J = \{A\{H, C|(A\{H)\}$ and $M_J = (x, y)$. By Remark 2 we notice that $H_2 = H_1 \lor H_2 = H \lor (AH \lor \overline{H}(x > 0))$ and then we distinguish two cases: (i) $x > 0$, where $H_2 = \Omega$; (ii) $x = 0$, where $H_2 = H$.

Case (i). The constituents $C_h$’s (contained in $H_2 = \Omega$) and the corresponding points $Q_h$’s are:

- $C_1 = AHC$, $C_2 = \overline{AH}$, $C_3 = AH\overline{C}$, $C_4 = HC$, $C_5 = \overline{HC}$,
- $Q_1 = (1, 1)$, $Q_2 = (0, y)$, $Q_3 = (1, 0)$, $Q_4 = (x, x + y(1 - x))$, $Q_5 = (x, y(1 - x))$.

We observe that $Q_4 = xQ_1 + (1 - x)Q_2$ and $Q_5 = xQ_3 + (1 - x)Q_2$; then we only refer to $Q_1, Q_2, Q_3$. The condition $(x, y) = \sum_{h=1}^3 \lambda_h Q_h$, with $\lambda_h \geq 0$ and $\sum_{h=1}^3 \lambda_h = 1$, is satisfied for every $(x, y) \in [0, 1]^2$. Indeed, the system

$$\begin{cases}
x = \lambda_1 + \lambda_3, \quad y = \lambda_1 + \lambda_2 y, \\
\sum_{h=1}^3 \lambda_h = 1; \quad \lambda_h \geq 0, \quad h = 1, 2, 3
\end{cases}$$

which can be written as

$$\begin{cases}
\lambda_1 = y(1 - \lambda_2) = y(\lambda_1 + \lambda_3) = xy, \\
\lambda_2 = 1 - xy - (1 - y)x = 1 - x, \quad \lambda_3 = (1 - y)x,
\end{cases}$$

is solvable.

Case (ii). As $x = 0$, by Remark 3 it holds that $C\{A\{H) = C\{AH$. The constituents $C_h$’s (contained in $H_2 = H$) and the corresponding points $Q_h$’s are:

- $C_1 = AHC$, $C_2 = \overline{AH}$, $C_3 = AH\overline{C}$, $Q_1 = (1, 1)$, $Q_2 = (0, y)$, $Q_3 = (1, 0)$.

The condition $(0, y) = \sum_{h=1}^3 \lambda_h Q_h$, with $\lambda_h \geq 0$ and $\sum_{h=1}^3 \lambda_h = 1$, is satisfied for every $y \in [0, 1]$. Indeed, the assessment $(0, y)$ coincides with $Q_2$ and the system is solvable, with $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 = 1$.

With $J = \{1, 3\}$ we associate the pair $(F_J, M_J)$, where $F_J = \{A\{H, C|(\overline{A\{H})\}$ and $M_J = (x, z)$. We note that the assessment $F_J$ on $M_J$ is equivalent to the assessment $(1 - x, z)$ on $(\overline{A\{H}, C|(\overline{A\{H})\})$. By the same reasoning as for $J = \{1, 2\}$, the system associated with $(1 - x, z)$ on $\{\overline{A\{H}, C|(\overline{A\{H})\}$ is solvable. Then, the system associated with $(x, z)$ on $\{A\{H, C|(\overline{A\{H})\}$ is solvable too.
With \( J = \{2, 3\} \) we associate the pair \((F_J, M_J)\), where \( F_J = \{C|(A|H), C|(\overline{A}|H)\} \) and \( M_J = (y, z) \); by Remark \(3\) \( H_1 = (AH \lor \overline{H}(x > 0), H_2 = (\overline{A}H \lor H(x < 1), so that \( H_2 = (AH \lor \overline{H}(x > 0) \lor (\overline{A}H \lor H(x < 1)) = \Omega \). The constituents \( C_h \)'s (contained in \( H_2 = \Omega \)) and the corresponding points \( Q_h \)'s are: \( C_1 = AH \overline{C}, C_2 = \overline{A}HC, C_3 = AH \overline{C}, C_4 = \overline{A}HC, C_5 = \overline{H}C, C_6 = \overline{H}C, \) and \( Q_1 = (1, z), Q_2 = (y, 1), Q_3 = (0, z), Q_4 = (y, 0), Q_5 = (x + y(1 - x), (1 - x) + x z), Q_6 = (y(1 - x), x z) \). We observe that \( Q_5 = xQ_1 + (1 - x)Q_2 \) and \( Q_6 = xQ_3 + (1 - x)Q_4 \); then we only refer to the points \( Q_1, Q_2, Q_3, Q_4 \). The condition \((y, z) = \sum_{h=1}^{4} \lambda_h Q_h, \) with \( \lambda_h \geq 0 \) and \( \sum_{h=1}^{4} \lambda_h = 1, \) is satisfied for every \((y, z) \in [0,1]^2\). Indeed, \((y, z) = yQ_1 + (1 - y)Q_3 \).

With \( J = \{1\} \) we associate the pair \((F_J, M_J)\), where \( F_J = \{C|(A|H)\} \) and \( M_J = y \). By Remark \(3\) we notice that \( H_1 = AH \lor \overline{H}(x > 0) \) and then we distinguish two cases: \( (i) \) \( x > 0, \) where \( H_1 = AH \lor \overline{H}; \) \( (ii) \) \( x = AH \).

Case \( (i) \). The constituents \( C_h \)'s (contained in \( H_1 = AH \lor \overline{H} \)) and the corresponding points \( Q_h \)'s are: \( C_1 = AH \overline{C}, C_2 = \overline{A}HC, C_3 = \overline{A}HC, C_4 = \overline{A}HC, \) and \( Q_1 = (1, z), Q_2 = (y, 1), Q_3 = (0, y), Q_4 = (1, 0). \) We observe that \( y = yQ_1 + (1 - y)Q_2, \) then the system \( y = \sum_{h=1}^{4} \lambda_h Q_h, \) with \( \lambda_h \geq 0 \) and \( \sum_{h=1}^{4} \lambda_h = 1, \) is solvable; indeed a solution is \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (y, 1 - y, 0, 0), \) Case \( (ii) \). The constituents \( C_h \)'s (contained in \( H_1 = AH \)) and the corresponding points \( Q_h \)'s are: \( C_1 = AH \overline{C}, C_2 = AH \overline{C}, C_3 = \overline{A}HC, C_4 = \overline{A}HC, \) and \( Q_1 = 1, Q_2 = 0, Q_3 = x + y(1 - x), Q_4 = y(1 - x). \) We observe that \( y = yQ_1 + (1 - y)Q_2, \) then the system \( y = \sum_{h=1}^{4} \lambda_h Q_h, \) with \( \lambda_h \geq 0 \) and \( \sum_{h=1}^{4} \lambda_h = 1, \) is solvable, with the unique solution \((\lambda_1, \lambda_2) = (y, 1 - y), \) With \( J = \{3\} \) we associate the pair \((F_J, M_J)\), where \( F_J = \{C|(\overline{A}|H)\} \) and \( M_J = z. \) In this case the reasoning is the same as for \( J = \{2\}, \) with \( A \) replaced by \( \overline{A}, \) with \( x \) replaced by \( 1 - x, \) and with \( y \) replaced by \( z. \)

In conclusion, the assessment \((x, y, z) \) on \( \{A|H, C|(A|H), C|(\overline{A}|H)\} \) is coherent for every \((x, y, z) \in [0,1]^3. \)

4 Generalized Modus Ponens

We now generalize the Modus Ponens to the case where the first premise \( A \) is replaced by the conditional event \( A|H. \)

Theorem 4. Given any coherent assessment \((x, y) \) on \( \{A|H, C|(A|H)\}, \) with \( A, C, H \) logically independent, with \( A \neq \emptyset \) and \( H \neq \emptyset, \) the extension \( z = P(C) \) is coherent if and only if \( z \in [z', z''] \), where

\[
z' = xy \quad \text{and} \quad z'' = xy + 1 - x.
\]

Proof. We recall that (Theorem 3) the assessment \((x, y) \) on \( \{A|H, C|(A|H)\} \) is coherent for every \((x, y) \in [0,1]^2. \) From (3), by the linearity of prevision, and by Theorem 2 we obtain

\[
z = P(C) = P(A|H) \land C + (\overline{A}|H) \land C] = P(A|H) \land C + P[A|H) \land C] = P(A|H)P[C|(A|H)] + P(\overline{A}|H)P[C|(\overline{A}|H)] = xy + (1 - x)P[C|(\overline{A}|H)].
\]
From Theorem 3, given any coherent assessment \((x, y)\) on \(\{A|H, C|(A|H)\}\), the extension \(t = P[C|(A|H)]\) on \(C|(A|H)\) is coherent for every \(t \in [0, 1]\). Then, as \(z = xy + (1 - x)t\), it follows that \(z' = xy\) and \(z'' = xy + 1 - x\).

Remark 4. We observe that the result given in Theorem 3 also holds when \(H = \Omega\), which is well-known (see, e.g., [24])

We notice that Theorem 4 can be rewritten as

Theorem 4'. Given any logically independent events \(A, C, H\), with \(A \neq \emptyset\) and \(H \neq \emptyset\), the set \(\Pi\) of all coherent assessments \((x, y, z)\) on \(\{A|H, C|(A|H), C\}\) is

\[
\Pi = \{(x, y, z) \in [0, 1]^3 : (x, y) \in [0, 1]^2, z \in [xy, xy + 1 - x]\}. \tag{10}
\]

5 Concluding Remarks

We generalized the probabilistic modus ponens in terms of conditional random quantities in the setting of coherence. Specifically, we replaced the categorical premise \(A\) and the antecedent \(A\) of the conditional premise \(C|A\) by the conditional event \(A|H\). We proved a generalized decomposition formula for conditional events and we gave some results on compound of conditionals and iterated conditionals. We propagated the previsions from the premises of the generalized probabilistic modus ponens to the conclusion. Interestingly, the lower and the upper bounds on the conclusion of the generalized probabilistic modus ponens coincide with the respective bounds on the conclusion for the (non-nested) probabilistic modus ponens. In future work we will focus on similar generalizations of other argument forms like the probabilistic modus tollens. Moreover we will study other instantiations to obtain further generalizations, e.g., by also replacing the consequent \(C\) of the conditional premise \(C|A\) and the conclusion \(C\) by a conditional event \(C|K\).

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References

1. E. W. Adams. A primer of probability logic. CSLI, Stanford, 1998.
2. V. Biazzo, A. Gilio, and G. Sanfilippo. Generalized coherence and connection property of imprecise conditional previsions. In Proc. IPMU 2008, Malaga, Spain, June 22 - 27, pages 907–914, 2008.
3. V. Biazzo, A. Gilio, and G. Sanfilippo. Coherent conditional previsions and proper scoring rules. In Advances in Computational Intelligence. IPMU 2012, volume 300 of CCIS, pages 146–156. Springer Heidelberg, 2012.
4. A. Capotorti, F. Lad, and G. Sanfilippo. Reassessing accuracy rates of median decisions. American Statistician, 61(2):132–138, 2007.
5. G. Coletti and R. Scozzafava. Probabilistic logic in a coherent setting. Kluwer, Dordrecht, 2002.
6. B. de Finetti. The logic of probability. *Philosophical Studies*, 77:181–190, 1936/1995.
7. B. de Finetti. Foresight: Its logical laws, its subjective sources. In H. Jr. Kyburg and H. E. Smokler, editors, *Studies in subjective probability*, pages 55–118. Robert E. Krieger Publishing Company, Huntington, New York, 1937/1980.
8. D. Edgington. Indicative conditionals. Stanford Encyclopedia of Philosophy, 2014.
9. J. St B. T. Evans, S. J. Handley, and D. E. Over. Conditionals and conditional probability. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 29(2):321–355, 2003.
10. A. Gilio, D. E. Over, N. Pfeifer, and G. Sanfilippo. Centering with conjoined and iterated conditionals under coherence. https://arxiv.org/abs/1701.07785.
11. A. Gilio, D. E. Over, N. Pfeifer, and G. Sanfilippo. Centering and compound conditionals under coherence. In *Soft Methods for Data Science*, volume 456 of *AISC*, pages 253–260. Springer, Berlin, Heidelberg, 2017.
12. A. Gilio, N. Pfeifer, and G. Sanfilippo. Transitivity in coherence-based probability logic. *Journal of Applied Logic*, 14:46–64, 2016.
13. A. Gilio and G. Sanfilippo. Conditional random quantities and iterated conditioning in the setting of coherence. In L. C. van der Gaag, editor, *ECSQARU 2013*, volume 7958 of *LNCS*, pages 218–229. Springer, Berlin, Heidelberg, 2013.
14. A. Gilio and G. Sanfilippo. Conjunction, disjunction and iterated conditioning of conditional events. In *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, volume 190 of *AISC*, pages 399–407. Springer, Berlin, 2013.
15. A. Gilio and G. Sanfilippo. Probabilistic entailment in the setting of coherence: The role of quasi conjunction and inclusion relation. *IJAR*, 54(4):513–525, 2013.
16. A. Gilio and G. Sanfilippo. Conditional random quantities and compounds of conditionals. *Studia Logica*, 102(4):709–729, 2014.
17. S. Kaufmann. Conditionals right and left: Probabilities for the whole family. *Journal of Philosophical Logic*, 38:1–53, 2009.
18. F. Lad. *Operational subjective statistical methods: A mathematical, philosophical, and historical introduction*. Wiley, New York, 1996.
19. F. Lad, G. Sanfilippo, and G. Agró. Extropy: complementary dual of entropy. *Statistical Science*, 30(1):40–58, 2015.
20. V. McGee. Conditional probabilities and compounds of conditionals. *Philosophical Review*, 98(4):485–541, 1989.
21. D. Petturiti and B. Vantaggi. Envelopes of conditional probabilities extending a strategy and a prior probability. *IJAR*, 81:160 – 182, 2017.
22. N. Pfeifer. The new psychology of reasoning: A mental probability logical perspective. *Thinking & Reasoning*, 19(3–4):329–345, 2013.
23. N. Pfeifer and G. D. Kleiter. Inference in conditional probability logic. *Kybernetika*, 42:391–404, 2006.
24. N. Pfeifer and G. D. Kleiter. Framing human inference by coherence based probability logic. *Journal of Applied Logic*, 7(2):206–217, 2009.
25. N. Pfeifer and G. Sanfilippo. Square of opposition under coherence. In *Soft Methods for Data Science*, volume 456 of *AISC*, pages 407–414. Springer, Berlin, 2017.
26. C. Wagner. Modus Tollens probabilized. *British Journal of Philosophy of Science*, 55:747–753, 2004.