Highly Improved Naive and Staggered Fermions

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We present a new action for highly improved staggered fermions. We show that perturbative calculations for the new action are well-behaved where those of the conventional staggered action are badly behaved. We discuss the effects of the new terms in controlling flavor mixing, and discuss the design of operators for the action.

1. INTRODUCTION

Naive and staggered fermion actions (which have very similar properties) have numerous advantages over actions based on four-component Wilson fermions. They have an exact chiral symmetry, allowing a simpler approach to the light-quark limit. They require only even dimension correction operators, simplifying the improvement program. Staggered fermion propagators can be calculated much more rapidly than Wilson fermion propagators. They thus offer the possibility of performing much higher accuracy calculations on current computers than with other methods. However, they have historically suffered some drawbacks which must be addressed before they can be used effectively. Gluons of momentum $\pi$ induce unphysical transitions between different flavors of doubler quarks, leading to large flavor symmetry breaking, especially in the pion sector. The perturbation series, even using improved perturbation theory, is often more poorly behaved than the series for Wilson fermions. With staggered fermions, quarks of different spins live on different sites of the lattice, leading to large $O(a^2)$ errors that are not present for Wilson fermions. Finally, the doubling of the fermion spectrum requires the taking of a root of the determinant in simulations, requiring more care in understanding the locality of the theory.

This paper will show that the improvement program provides an elegant solution to the first three of these problems.

2. IMPROVED STAGGERED FERMIONS

One significant problem with ordinary naive and staggered fermions is the large flavor nondegeneracy. This problem is worst in the pion sector. For the $B$ meson, and other particles with a single light quark, there is no flavor mixing at the valence level, because the heavy quark would be driven far off-shell by a gluon of momentum $\pi$. However, since $B_s$ couple to virtual pions, flavor mixing interactions can still be important.

At tree level, they arise from transitions between doubler quarks of different flavors induced by gluons of momentum $\pi$ \cite{1,2}. In Ref. \cite{3}, it was shown how to turn “fat-link” improvement\cite{4}, which suppresses the coupling of quarks to these gluons, into a tree level $O(a^2)$ improved action by proper tuning of the coefficients and the inclusion of an additional term.

The improved action for naive quarks is given by:

$$S_{\text{imp}} = \sum_x \bar{\psi}(x) \left( \gamma \cdot \Delta' - \frac{a^2}{6} \gamma \cdot \Delta^3 + m \right) \psi(x).$$

(1)

(The connection between naive and staggered quarks is the usual one \cite{5}). $V_\mu$ is a smeared link variable, identical to the ordinary link variable (up to errors of $O(a^2)$) for low gluon momentum, but vanishes when a gluon of momentum $\pi/a$ is extracted. The action has an exact doubling symmetry $\psi(x) \rightarrow B_\zeta(x)\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)B_\zeta(x)$, where $B_\zeta(x) = \prod_\rho (\gamma_\rho y_\rho)^\zeta \exp(i\zeta x_\rho / a)$ for one of fifteen vectors $\zeta$ with one or more non-zero components,
\[ \zeta = (1, 0, 0, 0), (1, 1, 0, 0), \text{etc.} \]

3. HIGHLY IMPROVED STAGGERED FERMIONS

Monte Carlo calculations showed a large reduction in pion flavor nondegeneracy with this action [3]. However, significant flavor breaking still remains, so further improvement is desirable for high precision calculations. In Ref. [3] a nonperturbative determination of gluonic corrections to the tadpole improved tree level \( O(a^2) \) improved staggered action was performed. It found a small improvement in pion flavor breaking was possible, but not the large reduction that is still desirable. Therefore, incorporation of additional operators into the action is needed.

The operators required for full \( O(au a^2) \) improvement are four fermi contact interactions [7]. They arise from transitions in quark–quark scattering between quarks of different doubler flavor induced by two gluon exchange. These contact terms can be transformed into interactions between fermion-bilinears and auxiliary scalar fields. The possible operators are governed by chiral and doubling symmetries.

The required Lagrangian for the quark-scalar field interactions is

\[
\mathcal{L}^{(2)}_q(x) = J_b^\mu(x)\Phi^\mu_b(x) + J_\mu(x)\Phi_\mu(x) + J_{\Phi_\mu}^b(x)\Phi^\mu_{\Phi_b}(x),
\]

where \( b = 1, \ldots, 8 \) is a color index, and where the \( \Phi \) fields are expressed in terms of normalized auxiliary scalar fields \( \Phi \) according to

\[
\Phi^\mu_b(x) = \sum_\zeta c^{(8)}_\zeta(\zeta_b)\mathcal{P}(\zeta)\Phi_{\zeta_b}^\mu(x),
\]

etc. The projection operator \( \mathcal{P}(\zeta) \) is given by

\[
\mathcal{P}(\zeta) = \frac{1}{4} \prod_{\mu=1}^4 \left( \frac{a^2\Delta^{(2)}_\mu}{4} \right)^{\zeta_\mu},
\]

where \( \eta_\mu(x) \) is the usual staggered phase. The color-octet axial-vector current \( J^{\mu}_{\Phi_b} \) for staggered fields is given by

\[
J^{\mu}_{\Phi_b}(x) = \frac{1}{4} \eta_\mu(x) \sum_{\nu = \pm} \left[ \bar{\chi}(x)T^{\nu}_bU_\nu(x)\chi(x + a\nu)
+ \bar{\chi}(x + a\nu)U_{\nu}^\dagger(x)T^{\nu}_b\chi(x) \right],
\]

where the sum over \( \rho \) is over the eight vectors of length \( \sqrt{3a} \) perpendicular to \( \bar{\mu} \)

\[
\rho^2 = 3a^2, \quad \rho \cdot \bar{\mu} = 0,
\]

and

\[
\mathcal{U}(x + \rho) = \left[ U_{\rho_1}(x)U_{\rho_2}(x + a\rho_1)U_{\rho_3}(x + a\rho_1 + a\rho_2) \right]_{\text{symm}},
\]

where the symmetrization is over the six assignments of the components of \( \rho \) to the indices \( \rho_1, \rho_2, \rho_3 \). The symmetrization is done so as to avoid introducing new discretization errors. The phase \( \tau_\mu(x) \) comes from diagonalizing the axial vector current, \( \bar{\psi}(x)\gamma_\mu\gamma_\nu\psi(x + \rho) \).

Finally each of the coefficient functions, is expressed in terms of two scalar functions of \( \zeta^2 \)

\[
c^{(8)}_V(\zeta_b) = \zeta_b c^{(8)}_V(\zeta^2) + (1 - \zeta_b) c^{(8)}_A(\zeta^2),
\]

and analogously for \( c^{(1)}_V, c^{(1)}_A \) and \( c^{(1)}_A \). The results are given in Table 4.

4. OPERATOR DESIGN: NAIVE VS. STAGGERED FERMIONS

Because of the transformation that turns naive fermions into four identical copies of staggered fermions, it is clear that a simulation with naive fermions (taking the sixteenth root of the determinant) and one with staggered fermions (taking the fourth root) would give identical results. Staggered fermions are therefore clearly preferable in simulations since they are a factor of four cheaper. For valence quarks and their operators, on the other hand, we should examine each for advantages and disadvantages. For staggered fermions, different spins live on different lattice sites. Such operators as the \( O(a^2) \) improved four-quark operator used in \( B_K \) are spread
over $4^4$ blocks on the lattice. It is reasonable to expect large $(ap)^2$ discretization errors from such operators, as are indeed observed. For naive quark operators such as $(\bar{\psi}(x)\Gamma \psi(x))^2$, all quarks live on the same site. Hence, at tree-level they have no finite $a$ errors to all orders in $a$.

5. CONVERGENCE OF PERTURBATION THEORY

Table 1
One-loop coefficients for the fermion bilinears in Eq. (3), using the Symanzik-improved gluon action to compute the gluon propagator. The results are in units of $\alpha_V q^\gamma \approx \pi/a$, and are accurate to a few parts in the last digit. Entries with an $i$ are imaginary.

| $c_2$ | $c_4^{(8)}$ | $c_4^{(8)}$ | $c_4^{(1)}$ | $c_4^{(1)}$ |
|-------|-------------|-------------|-------------|-------------|
| 1     | 0.880 i     | 0.500       | 0.643 i     | 0           |
| 2     | 0.335 i     | 0.438       | 0.217 i     | 0           |
| 3     | 0.300 i     | 0           | 0.220 i     | 0           |

| $c_2^2$ | $c_4^{(8)}$ | $c_4^{(8)}$ | $c_4^{(1)}$ | $c_4^{(1)}$ |
|---------|-------------|-------------|-------------|-------------|
| 1       | 0.404 i     | 0.518       | 0.295 i     | 0           |
| 2       | 0.364       | 0.228 i     | 0.166 i     | 0           |
| 3       | 0.198 i     | 0.198       | 0.145 i     | 0           |
| 4       | 0.190       | 0           | 0           | 0           |

The one case in which the improved perturbation theory program of Ref. [8] does not work well is staggered fermions. This was later explained by the existence of fermionic tadpole diagrams in naive and staggered fermions: [9] fermion loops from all doubler quark flavors, created by gluon loops of momentum $\pi$. Since improved staggered fermions suppress precisely the coupling to these gluons, we should expect that these effects are absent here and that perturbation theory is well behaved. Table 2 contains one-loop renormalization coefficients for some of the naive fermion operators advocated in the previous section. It is clear that for the improved case, the coefficients of $\alpha$ are all of order one, as argued, while in the unimproved case they range as high as eight.

6. CONCLUSIONS

We have shown that the improvement program elegantly removes many of the traditional difficulties of naive and staggered fermions. In future work, we will expand on the work presented here, and apply the new action in Monte Carlo calculations.

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