Efficiency Lower Bounds for Distribution-Free Hotelling-Type Two-Sample Tests Based on Optimal Transport

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The Wilcoxon rank-sum test is one of the most popular distribution-free procedures for testing the equality of two univariate probability distributions. One of the main reasons for its popularity can be attributed to the remarkable result of Hodges and Lehmann (1956), which shows that the asymptotic relative efficiency of Wilcoxon’s test with respect to Student’s t-test, under location alternatives, never falls below 0.864, despite the former being exactly distribution-free for all sample sizes. Even more striking is the result of Chernoff and Savage (1958), which shows that the efficiency of a Gaussian score transformed Wilcoxon’s test, against the t-test, is lower bounded by 1.

In this paper we study the two-sample problem in the multivariate setting and propose distribution-free analogues of the Hotelling $T^2$ test (the natural multidimensional counterpart of Student’s t-test) based on optimal transport and obtain extensions of the above celebrated results over various natural families of multivariate distributions. Our proposed tests are consistent against a general class of alternatives and satisfy Hodges-Lehmann and Chernoff-Savage-type efficiency lower bounds, despite being entirely agnostic to the underlying data generating mechanism. In particular, a collection of our proposed tests suffer from no loss in asymptotic efficiency, when compared to Hotelling $T^2$. To the best of our knowledge, these are the first collection of multivariate, nonparametric, exactly distribution-free tests that provably achieve such attractive efficiency lower bounds. We also demonstrate the broader scope of our methods in optimal transport based nonparametric inference by constructing exactly distribution-free multivariate tests for mutual independence, which suffer from no loss in asymptotic efficiency against the classical Wilks’ likelihood ratio test, under Konijn alternatives.

Keywords and phrases: Asymptotic relative efficiency, convergence of optimal transport maps, elliptically symmetric distributions, local contiguous alternatives, multivariate distribution-free testing, score transformed multivariate ranks, Wilks’ likelihood ratio test of independence.

1. Introduction

In this paper we consider the classical multivariate two-sample problem: Given two probability distributions $\mu_1$ and $\mu_2$ on $\mathbb{R}^d$, $d \geq 1$, and independent and identically distributed
(i.i.d.) observations $X_1, \ldots, X_m$ from $\mu_1$ and i.i.d. observations $Y_1, \ldots, Y_n$ from $\mu_2$, the goal is to test

$$H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2.$$  

This is the two-sample equality of distributions testing problem which has been studied extensively over the last hundred years (see, for example, [122, 11, 67, 128, 136, 72, 96] and the references therein). It has numerous applications in different fields, such as pharmaceutical studies [32, 110], causal inference [33, 42], remote sensing [27, 82], econometrics [87, 112], among others. When $d = 1$, some popular testing procedures for problem (1.1) include the Student’s $t$-test [122], the Kolmogorov-Smirnov test [12, 75, 120], the Wald-Wolfowitz runs test [135], and the Wilcoxon rank-sum test/Mann-Whitney $U$-test [84, 137]. When $d > 1$, a plethora of procedures have been proposed for testing (1.1) that include the Hotelling $T^2$ test [68], multivariate Cramér-von Mises test [3], tests based on geometric graphs [19, 35, 91, 108, 113], data depth-based tests [80, 81], kernel maximum mean discrepancy (MMD)/energy distance tests [7, 43, 45, 125], among others.

In the case when $d = 1$, a special feature of some of the tests cited above such as the Wilcoxon rank-sum (among others) is that it is exactly distribution-free under the null, that is, under $H_0$, the distribution of the Wilcoxon rank-sum test statistic is free of the underlying (unknown) data generating distributions, for all sample sizes. Thus, the critical value of this test is universal and one does not need to resort to permutation distributions/asymptotic approximations to carry out the test; thereby yielding uniform level $\alpha$ tests. This rather attractive property arises from the fact that the Wilcoxon rank-sum test is based on the ranks of the pooled sample $X_m \cup Y_n$, where $X_m := \{X_1, \ldots, X_m\}$ and $Y_n := \{Y_1, \ldots, Y_n\}$, instead of the exact values of the individual observations.

In the $d$-dimensional Euclidean space, for $d \geq 2$, due to the absence of a canonical ordering, the existing extensions of concepts of ranks, such as component-wise ranks [11, 104], spatial ranks [18, 85], depth-based ranks [81, 142] and Mahalanobis ranks and interdirections [56, 57, 54, 102, 106], and the corresponding rank-based tests no longer possess exact distribution-freeness. This raises a fundamental question: How do we define multivariate ranks that can lead to distribution-free testing procedures? A major breakthrough in this regard was made very recently in the pioneering work of Marc Hallin and co-authors ([50, 49]) where they propose a notion of multivariate ranks, based on the theory of optimal transport, that possesses many of the desirable properties present in their one-dimensional counterparts. Recently, using this notion of multivariate ranks (defined via optimal transport), [30] proposed a general framework for multivariate distribution-free rank-based testing for problem (1.1).

However, distribution-freeness by itself is not enough. The major reason for the success of univariate rank-based methods in applications over the last 50 years is that rank-based tests, apart from being exactly distribution-free, also possess high asymptotic relative efficiency (ARE) in the Pitman sense (see Definition B.2 in Appendix B.5 in the Appendix for the formal definition). Two remarkable results in this direction are:

1. the celebrated “0.864 result” by Hodges and Lehmann [64] and
2. the efficiency of the Gaussian score transformed Wilcoxon’s test by Chernoff and Savage [21, Theorem 3].

In [64] the authors showed that the efficiency of the Wilcoxon rank-sum test relative to Student’s $t$-test could never fall below $108/125 \approx 0.864$ (when working with contiguous location alternatives); whereas the efficiency can be arbitrarily large (tends to $+\infty$) for
heavy tailed distributions. In [21], the authors showed that surprisingly the efficiency of the Gaussian score transformed Wilcoxon’s test, relative to Student’s $t$-test, never falls below 1. This result shows in particular that there are univariate nonparametric rank-based tests which, in addition to being more robust and consistent beyond location alternatives, unlike the $t$-test, actually do not suffer from any loss in asymptotic efficiency in the Pitman sense relative to the $t$-test. At this point, it is tempting to ask:

*Can we design multivariate nonparametric distribution-free tests for problem (1.1) that enjoy similar AREs when compared to the classical Hotelling $T^2$ test (the natural multivariate counterpart of Student’s $t$-test)?*

In this paper we answer this question in the affirmative by defining a class of distribution-free multivariate analogues of the classical Hotelling $T^2$ test [68], using the aforementioned notion of multivariate ranks based on optimal transport. The proposed tests have a limiting chi-square distribution under the null (Theorem 3.1) and are consistent against a large class of natural alternatives (Theorem 3.2), which includes, among other, the location shift model (Proposition 3.2) and the contamination model (Proposition 3.3). We compute the asymptotic power of these tests for contiguous local alternatives (that is, alternatives shrinking towards null at rate $O(1/\sqrt{N})$, where $N := m + n$), and prove numerous lower bounds on the ARE of our proposed tests with respect to Hotelling $T^2$ for multiple subfamilies of multivariate probability distributions that parallel the univariate efficiency bounds obtained in [64] (the “0.864” result) and [21]. This is the first time that such lower bounds on the ARE (one of the main reasons behind the popularity of univariate rank tests) are being established for multivariate rank tests based on optimal transport.

Our most interesting observation in this regard is that it is possible to get exactly distribution-free nonparametric tests for problem (1.1), when $d \geq 1$, that suffer no loss in efficiency compared to the Hotelling $T^2$ test across multiple subfamilies of multivariate distributions, despite being completely agnostic to the underlying subfamily. Due to this reason, we believe that our proposed rank Hotelling $T^2$ test generalizes the two-sided Wilcoxon rank-sum test and the van der Waerden score test beyond $d = 1$ and is also the proper multivariate rank-based analogue of Hotelling $T^2$. The formal definitions of optimal transport based multivariate ranks and a detailed summary of the results obtained are given in the section below.

### 1.1. Summary of Results

In this section we present a summary of our results. We begin by recalling the notion of multivariate ranks based on optimal transport in Section 1.1.1. The proposed test statistic and its asymptotic properties are described in Section 1.1.2. The ARE results of the proposed tests relative to the Hotelling $T^2$ are given in Section 1.1.3. The broader impacts of our results are briefly discussed in Section 1.1.4.

#### 1.1.1. Multivariate Ranks Defined via Optimal Transport

Denote by $Z_N := X_m \cup Y_n$ the pooled sample, which we enumerate as $Z_N = \{Z_1, \ldots, Z_N\}$, where $N := m + n$. Let $\mathcal{H}_N^d := \{h_1^d, \ldots, h_N^d\}$ denote the set of multivariate ranks — a set of $N$ fixed vectors in $\mathbb{R}^d$ that can be thought of as a “natural” discretization of a prespecified probability distribution $\nu$ on $\mathbb{R}^d$, that is, we assume that the empirical measure on $\mathcal{H}_N^d$ converges weakly to $\nu$. For example, when $d = 1$, $\mathcal{H}_N^d$ is usually chosen as $\{1/N, \ldots, N/N\}$.
and the empirical measure on $\mathcal{H}_N^1$ converges weakly to $\text{Unif}[0,1]$, the uniform distribution on $[0,1]$. Hereafter, $\nu$ will be called the \textit{reference distribution}. Next, let $S_N$ be the set of all permutations of $[N] := \{1,2,\ldots,N\}$ and consider the following optimization problem:

$$\widehat{\sigma} := \arg\min_{\sigma = (\sigma_1,\ldots,\sigma_N) \in S_N} \sum_{i=1}^N \|Z_i - h^d_{\sigma_i}\|^2.$$  

(1.2)

Define the pooled \textit{multivariate ranks} as

$$\widehat{R}_{m,n}(Z_i) := h^d_{\sigma_i}; 1$$  

(1.3)

here $\|\cdot\|$ denotes the standard Euclidean norm. The optimization problem in (1.2) can be viewed as an example of the \textit{assignment problem}, which in turn can be solved using a linear program, for which algorithms of worst case time complexity $O(N^3)$ are available in the literature (see [9, 31, 70, 92]). We also refer the interested reader to [37, 1, 115] and the references therein, for a review of faster approximation algorithms addressing (1.2).

To get a better intuition for (1.2) and (1.3), when $\mathcal{H}_N^1 = \{1/N,2/N,\ldots,N/N\}$ for $d = 1$, then (1.2) and (1.3) reduce to the standard univariate ranks by an application of the rearrangement inequality (see [60, Theorem 368]). In fact, even in multidimension, these empirical ranks preserve a notion of direction, in the sense that the extreme data points get mapped to the corresponding extreme points of the fixed grid $\mathcal{H}_N^d$.

1.1.2. Distribution-Free Hotelling-Type Tests Based on Multivariate Ranks

One of the most celebrated and useful multivariate two-sample tests is based on the Hotelling $T^2$ statistic [68] (the multivariate analogue of Student’s $t$-test), and is given by:

$$T_{m,n} := \frac{mn}{m+n} \left( \bar{X} - \bar{Y} \right)^\top S_{m,n}^{-1} \left( \bar{X} - \bar{Y} \right),$$  

(1.4)

where $\bar{X} := \frac{1}{m} \sum_{i=1}^m X_i$, $\bar{Y} := \frac{1}{n} \sum_{j=1}^n Y_j$,

$$S_{m,n} := \frac{1}{m+n-2} \left\{ \sum_{i=1}^m (X_i - \bar{X})(X_i - \bar{X})^\top + \sum_{j=1}^n (Y_j - \bar{Y})(Y_j - \bar{Y})^\top \right\},$$

and we reject $H_0$ in (1.1) when $T_{m,n}$ exceeds the $(1 - \alpha)$-th quantile of the $\chi^2_d$ distribution. Although the Hotelling $T^2$ statistic is quite classical it still remains the workhorse in many statistical applications (see [16, 62, 139] for a few recent applications).

A prototypical example of the class of multivariate \textit{rank Hotelling $T^2$} statistics proposed in this paper is the following:

$$T_{m,n}^\nu := \frac{mn}{m+n} \left\| \frac{1}{m} \sum_{i=1}^m \widehat{R}_{m,n}(X_i) - \frac{1}{n} \sum_{j=1}^n \widehat{R}_{m,n}(Y_j) \right\|^2.$$  

(1.5)

This statistic can be viewed as a rank version of the Hotelling $T^2$ statistic $T_{m,n}$ in (1.4) in the sense that it is constructed by replacing the observations $\{X_1,\ldots,X_m\}$ and $\{Y_1,\ldots,Y_n\}$ with their empirical (pooled) multivariate ranks $\{\widehat{R}_{m,n}(X_1),\ldots,\widehat{R}_{m,n}(X_m)\}$
and \( \{ \hat{R}_{m,n}(Y_1), \ldots, \hat{R}_{m,n}(Y_n) \} \), respectively. Note that when \( d = 1 \), the statistic (1.5) is equivalent to the two-sided Wilcoxon rank-sum statistic (since the sum of the ranks of the pooled sample is a constant). Hence, \( T_{m,n}^{\nu} \) can also be thought of as a multivariate analogue of the celebrated Wilcoxon rank-sum test. Another interesting thing to note is that, unlike the Hotelling \( T^2 \) statistic, the construction of \( T_{m,n}^{\nu} \) above does not require any covariance matrix estimation. In Section 3 we will consider a class of test statistics which generalizes (1.5) by incorporating score functions (see [130, Chapter 13]).

The distribution-free property of the optimal transport based multivariate ranks, implies that the two-sample test based on \( T_{m,n}^{\nu} \) is distribution-free under the null (see Proposition 3.1 for a formal statement). Consequently, this test is uniformly level \( \alpha \) (see (3.4)). In the following we highlight some of the main properties of \( T_{m,n}^{\nu} \) which we prove later in the paper:

- **Consistency:** The test based on \( T_{m,n}^{\nu} \) is shown to be consistent against a large class of alternatives (see Theorem 3.2). In particular, for \( d = 1 \), this class coincides with the class of alternatives against which the univariate two-sided Wilcoxon rank-sum is consistent (see Remark 3.4). Interestingly, this class also contains the class of location alternatives for \( d \geq 1 \) (see Proposition 3.2). While this inclusion is immediate for \( d = 1 \) due to the canonical ordering of \( \mathbb{R} \), for \( d \geq 2 \), this requires non-trivially exploiting the geometry of population rank maps (see Section 3.1 for more details).

- **Asymptotic distribution under null and contiguous alternatives:** Under \( H_0 \) as in (1.1), the asymptotic distribution of \( T_{m,n}^{\nu} \) is free of \( \mu_1 = \mu_2 \) and converges weakly to a chi-squared distribution with \( d \) degrees of freedom, which can be used to readily compute an asymptotic cutoff for \( T_{m,n}^{\nu} \) (Theorem 3.1). Next, we derive the asymptotic distribution of \( T_{m,n}^{\nu} \) under contiguous alternatives (Theorem 3.3). Here, \( T_{m,n}^{\nu} \) has a limiting non-central chi-squared distribution with \( d \) degrees of freedom. In addition to providing precise expressions for local power, this allows us to explicitly compute the ARE of \( T_{m,n}^{\nu} \), relative to Hotelling \( T^2 \), and optimize it over various classes of multivariate distributions. We discuss the ARE computations and lower bounds in more detail in the next subsection.

- **General reference distributions and scores:** In [21], Chernoff and Savage showed that remarkably, by replacing the univariate ranks \( R_1, \ldots, R_N \) of the pooled sample \( Z_1, \ldots, Z_N \) with the corresponding standard Gaussian quantile functions \( \Phi^{-1}(R_1), \ldots, \Phi^{-1}(R_N) \) in the Wilcoxon statistic, one can design tests which suffer no loss in efficiency against the Student’s \( t \)-test. Since then, it has become a staple in nonparametrics to replace ranks by scores (see, for example, [130, Chapter 13], where general score functions, beyond \( \Phi^{-1}(\cdot) \), are used to construct powerful rank tests). In Section 3 we follow this same principle and further generalize (1.5) by replacing \( \hat{R}_{m,n}(X_i) \) and \( \hat{R}_{m,n}(Y_j) \) with \( J(\hat{R}_{m,n}(X_i)) \) and \( J(\hat{R}_{m,n}(Y_j)) \) for some continuous and invertible score function \( J(\cdot) \) taking values in \( \mathbb{R}^d \) (see (3.1)). This comes in addition to our use of a general reference distribution \( \nu \), which is itself unexplored in the optimal transport based nonparametric testing literature (see, for example, [30, 51, 117, 14]).

All our results on consistency against fixed alternatives, asymptotic distribution under null and contiguous alternatives are actually developed in this general score and reference distribution based framework. It is important to stress that this is not just a technical contribution, this generalization indeed offers practical gains by facilitating the construction of tests with provably higher ARE against Hotelling \( T^2 \) as discussed
below (see Theorems 3.4 and 3.5; also see Appendix D in the Appendix).

1.1.3. ARE against Hotelling $T^2$

In this subsection, we highlight our results which show that the rank Hotelling $T^2$ test in (1.5), has a high efficiency compared to the Hotelling $T^2$ test in (1.4). We begin by recalling the notion of asymptotic relative efficiency (ARE) of two level $\alpha \in (0,1)$ (consistent) tests $T_1$ and $T_2$ for problem (1.1). This entails studying the limiting power of the tests as the alternative hypothesis comes closer to the null with increasing sample size. More concretely, fixing a target power level, say $\beta \in (\alpha,1)$, the asymptotic relative efficiency (ARE) of the test $T_1$ with respect to $T_2$ can be described informally as follows (see Definition B.2 in Appendix B.5 in the Appendix for the formal definition):

The ARE of $T_1$ relative to $T_2$ is the limiting ratio of the number of samples needed to attain a power of $\beta$ when using the test $T_2$ compared to the same for test $T_1$, where the limit is taken as $\mu_2$ “converges” to $\mu_1$.

For example, if the ARE of $T_1$ with respect to $T_2$ is 0.9, intuitively it means that $T_2$ takes 10% fewer samples than $T_1$ to attain the power level $\beta$. Quite often (especially when the corresponding test statistics have asymptotic normal distributions) the ARE does not depend on $\alpha$, $\beta$, and the way the alternative hypothesis comes closer to the null. Moreover, compared to other notions of asymptotic efficiency, such as the Bahadur efficiency [5], the ARE (in the sense of Pitman [103]) is reputed to be a fairly good approximation for moderate sample sizes in many testing problems (see [47]; also see Appendix D in the Appendix). Due to these reasons, coupled with its easy interpretability, the study of ARE and power functions of two-sample tests under local contiguous alternatives has been a popular research area for over half a century (see [10, 21, 23, 55, 56, 54, 64, 74, 48, 59] and the references therein).

In this paper we compute the ARE of $T^\nu_{m,n}$ (as in (1.5)) with respect to $T_{m,n}$ under location shift alternatives, which, in this section is denoted by $\text{ARE}(T^\nu, T)$, for different choices of the reference distribution (see Section 1.1.1). Our candidate reference distributions are:

- the spherical uniform which has been heavily advocated in [49, 50, 116, 117, 52, 51],
- the Unif[0,1]$^d$ distribution (see [30]), and
- the standard Gaussian distribution. We will see that this proposal enjoys distinct, provable advantages in terms of the ARE with respect to the Hotelling’s $T^2$ test.

In Section 3.3 we compute $\text{ARE}(T^\nu, T)$, for different reference distributions $\nu$, and optimize it over many natural classes of multivariate distributions. As a consequence, we derive multivariate analogues of the classical results of Hodges and Lehmann [64] and Chernoff and Savage [21]. The following is a summary of the results obtained.

- **Gaussian location problem**: To begin with, consider the Gaussian location problem with unknown covariance. Even in this simple case, the ARE framework offers an interesting distinction between the candidate reference distributions mentioned above. In particular, in Proposition 3.4 we show that when the reference distribution $\nu$ is Gaussian, then

$$\text{ARE}(T^\nu, T) = 1,$$

and when the “effective” reference distribution (see Definition 3.1) is Unif[0,1]$^d$ (obtained by taking a standard multivariate Gaussian reference distribution, and score
function \( J(x) = (\Phi(x_1), \ldots, \Phi(x_d)) \) where \( \Phi(\cdot) \) is the standard normal distribution function), then

\[
\text{ARE}(T^{\nu}, T) \approx 0.95.
\]

Note that in both cases, the ARE is free of the underlying dimension. On the other hand, the ARE depends on dimension when the reference distribution is the spherical uniform (see Proposition 3.4 for the precise expression). One of the interesting things that emerge from this computation is that the above ARE is smaller than 0.95, for \( d \geq 5 \). This shows that in this popular multivariate example, for \( d \geq 5 \), the Unif[0, 1]^d “effective” reference distribution has better efficiency than the spherical uniform, when compared to the Hotelling’s \( T^2 \)-test. Of course, the Gaussian reference distribution has the best ARE which is to be expected as in this case, both the data distribution and the reference distribution belong to the same family. It is useful to note that although \( \Sigma \) is unknown, (1.5) attains these efficiencies without having to estimate the unknown \( \Sigma \).

- **Data with independent components:** Product distributions are perhaps the most natural ways to construct multivariate distributions from univariate ones. Note that two of our candidate reference distributions, namely the standard Gaussian and the Unif[0, 1]^d also have independent components. In Theorem 3.4, we establish lower bounds on \( \text{ARE}(T^{\nu}, T) \) for these two reference distributions uniformly over the class \( \mathcal{F}_{\text{ind}} \) of multivariate product distributions. In particular, we show that when the reference distribution is Unif[0, 1]^d, then

\[
\inf_{\mathcal{F}_{\text{ind}}} \text{ARE}(T^{\nu}, T) = 0.864,
\]

and if a standard Gaussian reference distribution is chosen, then

\[
\inf_{\mathcal{F}_{\text{ind}}} \text{ARE}(T^{\nu}, T) = 1.
\]

These uniform lower bounds have far reaching consequences. The first lower bound shows that the efficiency of \( T_{m,n}^{\nu} \) never falls below 0.864, the classical Hodges-Lehmann lower bound (see [64]), uniformly over \( \mathcal{F}_{\text{ind}} \) and for any dimension \( d \). On the other hand, the Gaussian reference distribution based \( T_{m,n}^{\nu} \) suffers from no loss in efficiency compared to Hotelling \( T^2 \); akin to the univariate Chernoff and Savage result (see [21]).

- **Elliptically symmetric distributions:** Next, we optimize \( \text{ARE}(T^{\nu}, T) \) over the class \( \mathcal{F}_{\text{ell}} \) of elliptically symmetric distributions (formally defined in Section 3.3). Elliptically symmetric distributions form a rich class of multivariate probability measures which includes the spherical uniform, multivariate Gaussian, \( t \), and logistic distributions among others. This class of distributions has attracted a lot of attention in statistical theory (see [24, 34] and the references therein for a review) and applications such as in graphical modeling [134], mathematical finance [65]. In the same vein as before, we show in Theorem 3.5 that

\[
\inf_{d} \inf_{\mathcal{F}_{\text{ell}}} \text{ARE}(T^{\nu}, T) = 0.648
\]

when the spherical uniform reference distribution is used, and on the other hand,

\[
\inf_{\mathcal{F}_{\text{ell}}} \text{ARE}(T^{\nu}, T) = 1
\]
when the standard Gaussian reference distribution is used. Once again, the Gaussian reference distribution based \( T_{m,n}^{\nu} \) suffers from no loss in efficiency for any \( d \). For the spherical uniform distribution, this worst case lower bound actually depends on and decreases with \( d \), that is, for smaller values of \( d \), the worst case lower bound is higher than 0.648 (see Theorem 3.5 and Figure 1). These uniform lower bounds match those for tests proposed in [55, 97, 99]. However, the construction of the test statistics in [55, 97, 99] specifically assumes the knowledge of elliptical symmetry of the underlying distribution. On the other hand, \( T_{m,n}^{\nu} \) assumes no such knowledge on the underlying distribution, and yet successfully attains the same ARE lower bounds.

- **Generative model for blind source separation:** Finally, we optimize \( \text{ARE}(T^{\nu}, T) \) over the class \( \mathcal{F}_{\text{gen}} \) of generative models often used in blind source separation; see [26, 118, 17]. These include the class of distributions which can be written as:

\[
X_{d \times 1} = A_{d \times d} W_{d \times 1} \tag{1.6}
\]

where \( W_{d \times 1} \) has independent components and \( A_{d \times d} \) is an orthogonal matrix, that is, \( AA^\top = A^\top A = I_d \) (the \( d \times d \) identity matrix). In the statistics literature, the above model is often used in the classical independent component analysis (ICA) problem; see [20, 111, 4, 69]. In the same vein as before, we show in Theorem 3.6 that

\[
\inf_{\mathcal{F}_{\text{gen}}} \text{ARE}(T^{\nu}, T) = 1,
\]

for any \( d \geq 1 \), when the standard Gaussian reference distribution is used. Here again, we observe the benefits (in terms of ARE) of using \( T_{m,n}^{\nu} \) with the Gaussian reference distribution \( \nu \), compared to the Hotelling \( T^2 \) test.

The discussions above show that working under a general class of reference distributions indeed offers provable gains in terms of ARE. In particular, the test based on \( T_{m,n}^{\nu} \) with the Gaussian reference distribution is at least as efficient as the Hotelling \( T^2 \) in both the independent components and elliptically symmetric cases discussed above, for any dimension \( d \), despite being completely agnostic to the true data generating distribution and also being exactly distribution-free under \( H_0 \) for all sample sizes. Also the 0.864 lower bound across all dimensions for product measures using a \( \text{Unif}[0,1]^d \) reference distribution matches the classical Hodges and Lehmann lower bound in [64]. Given these factors, we believe that our tests are the most appropriate multivariate extensions to the Wilcoxon rank-sum test/rank-based versions of the Hotelling \( T^2 \) test to have been proposed thus far in the literature.

1.1.4. Broader Scope

Our general strategy as described at the start of Section 1.1.2 is not just confined to the two-sample problem in (1.1), but is useful in other nonparametric testing problems for obtaining efficient, distribution-free optimal transport based analogues of classical univariate rank tests. We illustrate this in Section 4.1 where we construct a class of multivariate analogues of the classical Spearman’s rank correlation (see [121]) and use it for testing independence, that is, given \((X_1, Y_1), \ldots, (X_n, Y_n)\) i.i.d. \( \mu \) supported on some subset of \( \mathbb{R}^{d_1 + d_2} \), we test the null hypothesis \( H_0: X_1 \) and \( Y_1 \) are independent. In Theorem 4.2, we show that, once again, under the Gaussian reference distribution, our proposed test of independence suffers from no loss in efficiency over different classes of multivariate probability distributions compared to the Wilks’ test [138] (the natural multivariate analogue of Pearson’s correlation [101]).
In Section 4.2, we demonstrate that our techniques are useful towards proving consistencies of other nonparametric testing procedures based on optimal transport. One of the main technical tools in this regard is Theorem 2.1, which provides a general convergence result for continuous functions of empirical rank maps to their population counterparts. In particular, we use Theorem 2.1 to answer an open question regarding the consistency of the Gaussian score transformed rank distance covariance test for independence, as laid out in [117] (see Proposition 4.3). In fact, the same theorem can be used to prove consistencies of other tests in [30, 52, 51, 116].

Very recently, a statistic closely related to (1.5) was presented in passing in [51, Page 25] for the special case when the reference distribution is spherical uniform and for a specific choice of the set \( \{h_d^1, \ldots, h_d^N\} \). However, none of its theoretical properties, pertaining to consistency or ARE, were derived. In Section 4.2 we show that this statistic is a special case of our general framework introduced in Section 3, and our results readily imply its asymptotic properties, such as consistency and ARE lower bounds (see Proposition 4.2 for details).

1.2. Related Work

Nonparametric multivariate two-sample testing has been studied in great detail over the years. In this context, rank and data-depth based methods have mostly been restricted to testing against location-scale alternatives [25, 63, 107, 90, 95]. Asymptotically distribution-free depth-based tests which are consistent if restricted to the above class of alternatives are discussed in [79, 109]. Multivariate generalizations of the Wilcoxon rank-sum test based on data depth were studied in [80, 81, 141], which are also asymptotically distribution-free. However, these tests are not exactly distribution-free and are difficult to compute when the dimension is large because computation of depth-functions generally require time that scales exponentially with the dimension.

An alternative route for testing against general alternatives is through geometric graphs. This includes the celebrated Friedman-Rafsky test based on the minimum spanning tree (MST) [35], the tests based on nearest-neighbor graphs [61, 113], and their recent generalizations [19], and the cross-match test of Rosenbaum [108] based on minimum non-bipartite matching. These tests are asymptotically distribution-free (apart from the cross-match test, which is exactly distribution-free in finite samples), computationally efficient, and universally consistent, but have no power against \( O(1/\sqrt{N}) \) alternatives, that is, they have zero asymptotic Pitman efficiency [10].

Yet another approach is to compare the sum of pairwise-distances between and within the samples. This is the celebrated energy distance test [123, 125, 124, 7] which is a special case of the kernel MMD [43, 44]. Due to its simplicity, the kernel MMD has been studied extensively over the past decade. These tests are consistent against general fixed alternatives, but are not distribution-free, even asymptotically. So, its rejection thresholds have to be approximated by generating a number of permutation samples, which can often be computationally cumbersome for large sample sizes. In forthcoming work [29], we consider rank-analogues of the kernel MMD, which gives a class of distribution-free two-sample tests that are universally consistent, computationally efficient, and have non-trivial asymptotic efficiency.

Over the past few years, a new line of work has emerged with the use of optimal transport to design distribution-free testing procedures. This area, initiated by a new notion of
multivariate ranks in [49, 50] has since found applications in the two-sample testing problem [14, 30], independence testing problem [30, 116, 117], linear regression [51, 52]), etc. We will discuss more on the connections between our work and these references in Section 4.

1.3. Organization

The rest of the paper is organized as follows: In Section 2 we provide some background on population multivariate ranks defined via optimal transport and present a general convergence result. The proposed class of test statistics and its asymptotic properties (consistency, null distribution, power against local alternatives, and ARE computations) are described in Section 3. Connections to other testing problems and the broader applicability of our techniques are discussed in Section 4. Proofs of our results, numerical experiments depicting the finite sample performance of our proposed tests, and other additional technical details are given in Appendices B-E in the Appendix.

2. Background

In this section, we collect some important background material. In Section 2.1 we formally define the population counterpart of the empirical rank map defined in (1.3). We then present a general convergence result which gives conditions under which the empirical and population ranks are asymptotically “close” (Section 2.2).

2.1. Population Rank Map

Let \( P(\mathbb{R}^d) \) and \( P_{ac}(\mathbb{R}^d) \) denote the space of probability measures and the space of Lebesgue absolutely continuous probability measures on \( \mathbb{R}^d \), respectively. Given measures \( \mu, \nu \in P(\mathbb{R}^d) \), consider the following optimization problem:

\[
\inf_{F: \mathbb{R}^d \to \mathbb{R}^d} \int \| x - F(x) \|^2 \, d\mu(x) \quad \text{subject to} \quad F\#\mu = \nu; \tag{2.1}
\]

where \( F\#\mu = \nu \) means that \( F(X) \sim \nu \), where \( X \sim \mu \). This optimization problem is often referred to as the Monge’s problem (see [89]) and a minimizer of (2.1), if it exists, is referred to as an optimal transport map. One of the most powerful results in this field was proved by Robert McCann in 1995, where he took a geometric approach to (2.1). His result is the defining tool we will need to make sense of the definitions in this section. Let us state McCann’s theorem in a form which will be useful to us (see [133, Theorem 2.12 and Corollary 2.30]).

**Proposition 2.1** (McCann’s theorem [88]). Suppose that \( \mu, \nu \in P_{ac}(\mathbb{R}^d) \). Then there exists functions \( R(\cdot) \) and \( Q(\cdot) \) (hereafter referred to as “transport maps”), both of which are gradients of (extended) real-valued \( d \)-variate convex functions, such that \( R\#\mu = \nu, Q\#\nu = \mu \), \( R \) and \( Q \) are unique (\( \mu \) and \( \nu \) a.e., respectively), \( R \circ Q(y) = y \) (\( \nu \) a.e.) and \( Q \circ R(x) = x \) (\( \mu \) a.e.). Moreover, if \( \mu \) and \( \nu \) have finite second moments, \( R(\cdot) \) is also the solution to the problem in (2.1).

In Proposition 2.1, by “gradient of a convex function” we essentially mean a function from \( \mathbb{R}^d \) to \( \mathbb{R}^d \) which is \( \mu \) (or \( \nu \)) a.e. equal to the gradient of some convex function. We are now in a position to define our population multivariate rank map.
**Definition 2.1** (Population multivariate rank map). Given a pre-specified reference distribution \( \nu \in \mathcal{P}_{ac}(\mathbb{R}^d) \) and a measure \( \mu \in \mathcal{P}_{ac}(\mathbb{R}^d) \), the population rank map for the measure \( \mu \) with reference distribution \( \nu \) is the function \( R(\cdot) \) as in Proposition 2.1. Note that \( R(\cdot) \) is unique up to measure zero sets with respect to \( \mu \).

**Remark 2.1** (Univariate case). When \( d = 1 \), a natural choice for the reference distribution \( \nu \) is Unif[0, 1]. In this case, by Proposition 2.1, it is easy to check that the population rank map is the cumulative distribution function associated with the probability measure \( \mu \).

Hereafter, we will fix a pre-specified reference distribution \( \nu \) on \( \mathbb{R}^d \) and assume that \( \mu_1, \mu_2, \nu \in \mathcal{P}_{ac}(\mathbb{R}^d) \). The densities of \( \mu_1 \) and \( \mu_2 \) with respect to the Lebesgue measure on \( \mathbb{R}^d \) will be denoted by \( f_1(\cdot) \) and \( f_2(\cdot) \). The hypothesis testing problem (1.1) can then be reformulated as:

\[
H_0 : f_1 = f_2 \quad \text{versus} \quad H_1 : f_1 \neq f_2. \tag{2.2}
\]

Throughout the paper, we will work in the usual asymptotic regime where \( N = m + n \to \infty \) such that

\[
m/N \to \lambda \in (0, 1). \tag{2.3}
\]

We will also denote by \( R_{H_1}^\nu(\cdot) \) the population rank map, associated with the measure \( \lambda \mu_1 + (1 - \lambda)\mu_2 \) and reference distribution \( \nu \) (as in Definition 2.1), that is, \( R_{H_1}^\nu \#[\lambda \mu_1 + (1 - \lambda)\mu_2] = \nu \). Similarly, \( R_{H_0}^\nu(\cdot) \) will denote the rank map associated with \( \mu_1 \), that is, \( R_{H_0}^\nu \# \mu_1 = \nu \). Note that by the absolute continuity assumptions, these rank maps are well-defined by Definition 2.1.

### 2.2. Convergence of Empirical Rank Maps

In this section we will address the question as to how the empirical rank map defined in (1.2) and (1.3) estimates its population version (see Definition 2.1 above). To this end, define \( [N] := \{1, 2, \ldots, N\} \), for any \( N \geq 1 \). Also, recall that \( Z_N = \{Z_1, \ldots, Z_N\} \) denotes the pooled sample \( X_m \cup Y_n \).

**Theorem 2.1.** Suppose \( \mu_1, \mu_2, \nu \in \mathcal{P}_{ac}(\mathbb{R}^d) \) and

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{h_i^{\nu}} \xrightarrow{w} \nu, \tag{2.4}
\]

where \( \xrightarrow{w} \) denotes weak convergence. Fix \( p, q, r \in \mathbb{N} \) and assume that \( \mathcal{F}(\cdot) : (\mathbb{R}^d)^p \to \mathbb{R}^q \), \( J : \mathbb{R}^d \to \mathbb{R}^d \) are continuous Lebesgue a.e., and \( r \in [p] \). Suppose that

\[
\limsup_{N \to \infty} \frac{1}{N^p} \mathbb{E} \left[ \sum_{(i_1, \ldots, i_p) \in [N]^p} \left\| \mathcal{F}(J(\hat{R}_{m, n}(Z_{i_1})), \ldots, J(\hat{R}_{m, n}(Z_{i_r})), J(R_{H_1}^\nu(Z_{i_{r+1}})), \ldots, J(R_{H_1}^\nu(Z_{i_p}))) \right\| \right] \leq \int \left\| \mathcal{F}(J(z_1), \ldots, J(z_p)) \right\| \, d\nu(z_1) \ldots d\nu(z_p) < \infty
\]

(2.5)

for all \( 0 \leq r \leq p \). Then the following conclusion holds:

\[
\frac{1}{N^p} \sum_{(i_1, \ldots, i_p) \in [N]^p} \left\| \mathcal{F}(J(\hat{R}_{m, n}(Z_{i_1})), \ldots, J(\hat{R}_{m, n}(Z_{i_r})), J(R_{H_1}^\nu(Z_{i_{r+1}})), \ldots, J(R_{H_1}^\nu(Z_{i_p}))) \right\|
\]
conditionally on the set $\mathcal{H}_N^d = \{ h_1^d, \ldots, h_N^d \}$. Further, if $\mathcal{F}(\cdot)$ and $J(\cdot)$ are Lipschitz, $\text{supp}(\nu)$ is compact, and $\{ h_1^d, h_2^d, \ldots, h_N^d \} \subseteq \text{supp}(\nu)$, then the above convergence holds a.s. Also, the same conclusions hold if $R_{H_1}^\nu(\cdot)$ is replaced with $R_{H_0}^\nu(\cdot)$ provided $\mu_1 = \mu_2$.

Theorem 2.1 (see Appendix B.1 of the Appendix for its proof) shows how a function $\mathcal{F}(\cdot) : (\mathbb{R}^d)^p \rightarrow \mathbb{R}^d$, where $1 \leq r \leq p$ of its arguments are evaluated at the empirical rank map and the remaining $p - r$ coordinates are evaluated at the population rank map, can be approximated by its population counterpart in the asymptotic limit. In particular, by choosing $r = p = 1$, $q = d$, $\mathcal{F}(x) = x$ and $J(x) = x$ in Theorem 2.1 gives,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N \left\| \hat{R}_{m,n}(Z_i) - R_{H_1}^\nu(Z_i) \right\| \overset{P}{\rightarrow} 0,$$

whenever (2.4) holds. This establishes the convergence of the empirical ranks to the population ranks in probability in the $L_2$ sense, with respect to the empirical distribution of the pooled sample. Note that the above conclusion does not require $\nu$ to be compactly supported. On the other hand, it yields convergence in probability instead of almost sure convergence, which in turn requires more stringent assumptions on $\mathcal{F}(\cdot)$, $J(\cdot)$ and $\nu$. We will invoke Theorem 2.1 in its generality in the subsequent sections while deriving the asymptotic properties of the proposed tests. In fact, we expect Theorem 2.1 to be of independent interest, because it can be more widely applicable to other nonparametric two-sample and independence tests based on multivariate ranks (see Proposition 4.3 in Section 4).

**Remark 2.2** (Verifying (2.5)). A natural way to verify (2.5) is by showing that

$$\limsup_{N \to \infty} \frac{1}{N^p} \mathbb{E} \left[ \sum_{(i_1, \ldots, i_p) \in [N]^p} \left( \left\| \mathcal{F}(J(\hat{R}_{m,n}(Z_{i_1})), \ldots, J(\hat{R}_{m,n}(Z_{i_r})), J(R_{H_1}^\nu(Z_{i_{r+1}})), \ldots, J(R_{H_1}^\nu(Z_{i_p})) \right\|^{1+\delta} \right)^{1+\delta} \right] < \infty$$

for some $\delta > 0$. The above condition can be easily verified for many natural choices of $\mathcal{F}(\cdot)$, $J(\cdot)$, and $\nu$ (see Appendix E in the Appendix for some examples). In fact, the above condition will imply that (2.5) is satisfied with $\overset{=}{\text{instead of } \leq}$.

**Remark 2.3** (Choice of $\mathcal{H}_N^d$). Note that Theorem 2.1 is flexible on the choice of the elements of $\mathcal{H}_N^d$. One choice includes drawing a random sample of size $N$ from $\nu$. Otherwise one can also choose deterministic sequences that approximate $\nu$. When $\nu = \text{Unif}[0,1]^d$, some popular examples of such sequences can be found in the quasi-Monte Carlo literature (see [105] and the references therein). The use of deterministic sequences to define multivariate ranks has been advocated in [30, 49, 117].

**Remark 2.4** (On the notion of convergence). If $h_1^d, h_2^d, \ldots, h_N^d$ are chosen using a random sample from $\nu$, then assumption (2.4) is to be interpreted as weak convergence a.s. (which follows using the Varadarajan theorem, see [132]). In this case, crucially, the convergence results hold conditionally on $\{ h_1^d, \ldots, h_N^d \}$.

$^1$Here $\text{supp}(\nu)$ denotes the support of the distribution $\nu$, that is, the smallest closed set $C$ such that $\nu(C) = 1$. 

\[ -\mathcal{F}\left(J(R_{H_1}^\nu(Z_{i_1})), \ldots, J(R_{H_1}^\nu(Z_{i_p}))\right) \overset{P}{\rightarrow} 0, \]
3. Score Transformed Hotelling-Type Tests Based on Optimal Transport

In this section we introduce a class of distribution-free test statistics based on multivariate ranks (defined via optimal transport) that generalizes the classical Wilcoxon rank-sum test to higher dimensions and provides distribution-free analogues of the Hotelling $T^2$ test. A basic version of such a test statistic has already been presented in (1.5). Here, we generalize it further by incorporating score functions, which are injective functions $J(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ that are continuous Lebesgue a.e. in $\mathbb{R}^d$. A natural example of a score function is $J(\cdot) := (F_1^{-1}(\cdot), \ldots, F_d^{-1}(\cdot))$, where $F_1(\cdot), F_2(\cdot), \ldots, F_d(\cdot)$ are univariate distribution functions. When $d = 1$ and $F(\cdot) = \Phi(\cdot)$ is the standard Gaussian distribution function, then the corresponding score function $J(\cdot)$ is called the van der Waerden score function (see [131]).

We begin with the notion of an effective reference distribution which is obtained by combining a score function with a reference measure as follows:

**Definition 3.1.** Given a reference distribution $\nu$ and a score function $J(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, the effective reference distribution, hereby abbreviated as ERD, is the push-forward measure $J^\# \nu$. For example, in the classical univariate Chernoff-Savage framework (see [21, 38]), the ERD is the $\mathcal{N}(0,1)$ distribution, which can be obtained by choosing the reference distribution $\nu = \text{Unif}[0,1]$ and the score function $J(\cdot) = \Phi^{-1}(\cdot)$, the standard Gaussian quantile function. The analogous example for dimension $d > 1$ would be to set the standard $d$-variate Gaussian distribution as the ERD, with $\nu = \text{Unif}[0,1]^d$ and $J(x) := (\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_d))$, where $x = (x_1, \ldots, x_d)$.

Hereafter, we will assume that the reference distribution and the score function is chosen such that the ERD is non-degenerate in the following sense:

**Assumption 3.1.** The effective reference distribution (ERD) has a well-defined finite and positive definite covariance matrix $\Sigma_{\text{ERD}}$.

Under the above assumption, we define the rank Hotelling $T^2$ statistic with reference distribution $\nu$ and score function $J$ as:

$$T_{\nu, J}^{m,n} := \frac{mn}{m+n} (\Delta_{\nu,J}^{m,n})^\top \Sigma_{\text{ERD}}^{-1} \Delta_{\nu,J}^{m,n},$$

where

$$\Delta_{\nu,J}^{m,n} := \frac{1}{m} \sum_{i=1}^m J \left( \hat{R}_{m,n}(X_i) \right) - \frac{1}{n} \sum_{j=1}^n J \left( \hat{R}_{m,n}(Y_j) \right).$$

is the difference of the means of the score transformed ranks of the two samples.

**Remark 3.1.** Note that $T_{\nu,J}^{m,n}$ is equivalent to statistic $T_{\nu,J}^{m,n}$ introduced (1.5) when the score function is $J(x) = x$ and the reference distribution $\nu$ has uncorrelated components with the same marginal distributions, each of which have finite second moments. This is because, in this case, $\Sigma_{\text{ERD}} = \sigma^2 I_d$, where $\sigma > 0$.

The class of statistics in (3.1) leads to a new family of two-sample tests which are distribution-free and consistent for a general collection of alternatives. We begin with the distribution-free property. This is formalized in the following proposition, which is an immediate consequence of [50, Proposition 2.5] and [30, Proposition 2.2].
Proposition 3.1 (Distribution-free). Assume that $H_0$ is true and $\mu_1 = \mu_2 \in \mathcal{P}_{ac}(\mathbb{R}^d)$. Then the distribution of $T^{\nu,J}_{m,n}$ is universal, that is, it is free of $\mu_1 = \mu_2$, for all $m, n \geq 1$.

Using the above result we can readily obtain a finite sample distribution-free two-sample test that uniformly controls the Type I error. To this end, fix a level $\alpha \in (0, 1)$ and let $c_{m,n}$ denote the upper $\alpha$ quantile of the universal distribution in Proposition 3.1. Consider the test function:

$$\phi_{m,n}^{\nu,J} := 1 \{ T^{\nu,J}_{m,n} \geq c_{m,n} \}. \quad (3.3)$$

This test is exactly distribution-free for all $m, n \geq 1$ and uniformly level$^2$ $\alpha$ under $H_0$, that is,

$$\sup_{\mu_1 = \mu_2 \in \mathcal{P}_{ac}(\mathbb{R}^d)} \mathbb{E} [ \phi_{m,n}^{\nu,J} ] = \alpha. \quad (3.4)$$

Remark 3.2. In addition to being distribution-free, the statistic (3.1) has the advantage that the matrix $\Sigma_{ERD}$ is deterministic. In fact, it can be computed once the reference distribution $\nu$ and score function $J$ are specified, and does not depend on the observed samples or the unknown distribution $\mu_1 = \mu_2$. This makes the implementation of the test (3.3) particularly convenient, because no covariance matrix estimation is required for computing $T^{\nu,J}_{m,n}$, unlike other nonparametric tests of location such as the interdirections-based tests in [54, 102, 106] and the tests based on data-depth in [81].

3.1. Asymptotic Null Distribution and Consistency

In this section we discuss the consistency of the test $\phi_{m,n}^{\nu,J}$ and the asymptotic null distribution of the test statistic $T^{\nu,J}_{m,n}$. Throughout we will assume that $X \sim \mu_1$ and $Y \sim \mu_2$. We first describe the asymptotic null distribution of the statistic $T^{\nu,J}_{m,n}$. This is formalized in the following theorem, which is proved in Appendix B.3 in the Appendix.

Theorem 3.1 (Limiting null distribution). Suppose the condition in (2.4) and Assumption 3.1 hold. Further, assume that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \| J(h^i_d) \|^2 \leq \int \| J(z) \|^2 \, d\nu(z). \quad (3.5)$$

Then, under $H_0$ as in (2.2), in the usual asymptotic regime (2.3),

$$T^{\nu,J}_{m,n} \xrightarrow{w} \chi^2_d. \quad (3.6)$$

On account of having a simple limiting null distribution, the result above can be used to calibrate the statistic $T^{\nu,J}_{m,n}$ to obtain an asymptotic level $\alpha$ test. To this end, denote by $\chi^2_{d,1-\alpha}$ the $(1 - \alpha)$-th quantile of the $\chi^2_d$ distribution. Then the test which rejects when $\{ T^{\nu,J}_{m,n} > \chi^2_{d,1-\alpha} \}$ satisfies (3.4) asymptotically. One might also use the finite sample distribution-free test $\phi_{m,n}^{\nu,J}$ as in (3.3).

The proof of Theorem 3.1 is given in Appendix B.3 in the Appendix. It proceeds in two steps: First, we prove a Hájek representation type result which shows that the error incurred

$^2$Strictly speaking, to guarantee exact level $\alpha$, we have to randomize $\phi_{m,n}^{\nu,J}$, as the exact distribution of $T^{\nu,J}_{m,n}$ is discrete. But unless $m, n$ are extremely small, this makes no practical difference.
in replacing the empirical rank maps in (3.1) with their population counterparts (that is, $R_{H_0}^\nu(\cdot)$) is asymptotically negligible under $H_0$. The asymptotic distribution in (3.6) then follows from the asymptotic normality of

$$
\Delta_{m,n}^\nu := \frac{1}{m} \sum_{i=1}^m J(R_{H_0}^\nu(X_i)) - \frac{1}{n} \sum_{j=1}^n J(R_{H_0}^\nu(Y_j))
$$

(3.7)

and the continuous mapping theorem. Note that $\Delta_{m,n}^\nu$ is the oracle counterpart of $\Delta_{m,n}^\nu$ obtained by replacing the empirical rank maps in (3.2) with their population analogues.

**Remark 3.3** (On assumption (3.5)). A simple case where assumption (3.5) can be verified is when $h_1^d, h_2^d, \ldots, h_N^d$ are i.i.d. samples from the reference distribution $\nu$. In that case, equality holds in (3.5) almost surely by the strong law of large numbers and Theorem 3.1 holds conditionally on $\{h_1^d, \ldots, h_N^d\}$. There has however been a push towards opting towards deterministic choices for $\{h_1^d, \ldots, h_N^d\}$ in recent works such as [49, 30, 116] (see also [30, Table 12] for some potential benefits of deterministic choices). We will discuss how to verify (3.5) in some such cases in Appendix E in the Appendix.

We now proceed to show that the test $\phi_{m,n}^\nu J$ is consistent against a large class of alternatives in (2.2). This is formalized in the following theorem.

**Theorem 3.2** (Consistency). Suppose the conditions in (2.4), (3.5) and Assumption 3.1 hold. Then, for problem (2.2) in the usual asymptotic regime (2.3),

$$
\lim_{m,n \to \infty} \mathbb{E}_{H_1} [\phi_{m,n}^\nu J] = 1
$$

provided $\mathbb{E} J(R_{H_1}^\nu(X))$ and $\mathbb{E} J(R_{H_1}^\nu(Y))$ are finite and $\mathbb{E} J(R_{H_1}^\nu(X)) \neq \mathbb{E} J(R_{H_1}^\nu(Y))$.

The proof of this theorem is given in Appendix B.2 in the Appendix. It follows from Theorem 3.1 and the condition $\mathbb{E} J(R_{H_1}^\nu(X)) \neq \mathbb{E} J(R_{H_1}^\nu(Y))$: these imply that $T_{m,n}^\nu J$ converges to zero under $H_0$ and a positive number under the alternative, thus, implying consistency.

**Remark 3.4** (Connection to Wilcoxon’s test). For $d = 1$, if $J(\cdot)$ is nondecreasing (as is the case with quantile functions) then it is easy to see that whenever $Y$ is stochastically larger (respectively, smaller) than $X$, we have $\mathbb{E} J(R_{H_1}^\nu(Y)) > \mathbb{E} J(R_{H_1}^\nu(X))$ (respectively, $\mathbb{E} J(R_{H_1}^\nu(Y)) < \mathbb{E} J(R_{H_1}^\nu(X))$). This is the exact condition for the consistency of Wilcoxon’s rank sum test (see [140] for details).

While Theorem 3.2 gives the general condition under which the test $\phi_{m,n}^\nu J$ is consistent, it is not directly apparent how it applies to the location alternatives where $f_2(\cdot) = f_1(\cdot - \Delta)$, for some $\Delta \in \mathbb{R}^d \setminus \{0\}$. In this case, the two-sample testing problem (2.2) becomes:

$$
H_0 : \Delta = 0 \quad \text{versus} \quad H_1 : \Delta \neq 0.
$$

(3.8)

The following proposition shows the consistency $\phi_{m,n}^\nu J$ for the above problem.

**Proposition 3.2** (Consistency under location alternatives). Suppose the conditions in (2.4), (3.5) and Assumption 3.1 hold with $J(x) = x$. Recall from Proposition 2.1 that

\[\text{for relevant definitions and properties of stochastic ordering see [114, Equation (1.A.1)].}\]
the population rank map $R_{H_1}^\nu (\cdot)$ is the gradient of a convex function, say $u_{H_1}^\nu (\cdot)$. Assume that $u_{H_1}^\nu (\cdot)$ is strictly convex\(^4\) on an open set of positive measure with respect to $\mu_1$. Then, for problem (3.8) in the usual asymptotic regime (2.3),

$$\lim_{m,n \to \infty} \mathbb{E}_\Delta \left[ \phi_{m,n}^\nu J \right] = 1$$

for any $\Delta \in \mathbb{R}^d \setminus \{0\}$.

The proof of Proposition 3.2 is given in Appendix B.2. The main ingredient of the proof is the cyclical monotonicity\(^5\) property of the population multivariate rank maps (recall Definition 2.1) which is used to show that the consistency condition $\mathbb{E} J(R_{H_1}^\nu (X)) \neq \mathbb{E} J(R_{H_1}^\nu (Y))$ in Theorem 3.2 holds whenever $\Delta \neq 0$.

Another common class of nonparametric alternatives is the contamination model where

$$f_2(\cdot) = (1 - \delta)f_1(\cdot) + \delta g(\cdot), \quad (3.9)$$

where $\delta \in [0, 1)$ and $g \neq f_1$ is a probability density function with respect to the Lebesgue measure in $\mathbb{R}^d$. In this case, the testing problem (2.2) simplifies to:

$$H_0 : \delta = 0 \quad \text{versus} \quad H_1 : \delta \neq 0. \quad (3.10)$$

In the same spirit as Proposition 3.2, we can obtain a consistency result for $\phi_{m,n}^\nu J$ in the contamination model, which simplifies nicely when the contamination density $g(\cdot)$ is itself a location shift of $f_1(\cdot)$ as was studied in the seminal Hodges and Lehmann [64] paper.

**Proposition 3.3** (Consistency under contamination alternatives). Suppose the condition in (2.4), (3.5) and Assumption 3.1 hold with $J(x) = x$. Then, for the testing problem (3.10) in the usual asymptotic regime (2.3),

$$\lim_{m,n \to \infty} \mathbb{E}_\delta \left[ \phi_{m,n}^\nu J \right] = 1, \quad (3.11)$$

for any $\delta \in (0, 1)$, provided $\mathbb{E} R_{H_1}^\nu (X) \neq \mathbb{E} R_{H_1}^\nu (W)$, where $W$ has density $g(\cdot)$. In particular, (3.11) holds if $g(\cdot) = f_1(\cdot - \Delta)$, for some $\Delta \neq 0$, provided $R_{H_1}^\nu (\cdot)$ satisfies the same assumption as in Proposition 3.2.

The proof of Proposition 3.3 follows along the same lines as the proof of Proposition 3.2. We provide a sketch in Appendix B.2 in the Appendix. We conclude this section with the following crucial observation.

**Remark 3.5** (No moment assumptions required). Note that the moment assumptions in Theorem 3.2, Proposition 3.2, and Proposition 3.3 are not impositions on $\mu_1$ or $\mu_2$, but on the reference distribution $\nu$ and score function $J(\cdot)$. For instance, if $J(x) = x$ is the identity function and $\nu = \text{Unif}[0,1]^d$, then the moment assumptions are always satisfied, even if the distributions $\mu_1, \mu_2$ do not have finite moments. In contrast, the Hotelling $T^2$ test requires finite second moments of $\mu_1$ and $\mu_2$ for consistency.

\(^4\)A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be strictly convex on an open set $U$, if, given any $x, y \in U$ and any $\alpha \in (0, 1)$, $f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y)$.

\(^5\)For $S \subseteq \mathbb{R}^d$, a function $f : S \to \mathbb{R}^d$ is said to be cyclically monotone if given any $k \geq 1$, $x_1, \ldots, x_k \in S$, $x_{k+1} \equiv x_1$, the following hold: $\sum_{i=1}^k (x_{i+1} - x_i, f(x_i)) \geq 0$. It is easy to see that, if $f(\cdot)$ is the gradient of a convex function, then it is necessarily cyclically monotone.
3.2. Local Asymptotic Power

In this section we derive the asymptotic power of the test $\phi_{\nu,J}^{\nu,J}$ against local contiguous alternatives. To quantify the notion of local alternatives, we will adopt the standard smooth parametric model assumptions from the theory of local asymptotic normality (LAN) (see, for example, [130, Chapter 7]). To this end, let $\Theta \subseteq \mathbb{R}^p$ ($p$ fixed, may or may not be equal to $d$) and $\{P_{\theta}\}_{\theta \in \Theta}$ be a parametric family of distributions in $\mathbb{R}^d$ with density $f(\cdot|\theta)$, with respect to Lebesgue measure, indexed by a $p$-dimensional parameter $\theta \in \Theta$. We will assume the following standard regularity conditions on this parametric family:

- The family $\{P_{\theta}\}_{\theta \in \Theta}$ is quadratic mean differentiable (QMD) at $\theta = \theta_0 \in \Theta$ (see [78, Definition 12.2.1] for related definitions). The QMD assumption implies differentiability in norm of the square root of the density, which holds for most standard families of distributions, including exponential families in natural form.

- For $X \sim P_{\theta_0}$, $\mathbb{E}_{\theta_0}(\|\eta(X,\theta_0)\|^2) < \infty$, where $\eta(\cdot,\theta) := \nabla_{\theta} f(\cdot|\theta)$ is the score function. Also, suppose that the Fisher information exists at $\theta_0$, that is, $I(\theta_0) := \mathbb{E}_{\theta_0}(\eta(X,\theta_0)\eta(X,\theta_0)^\top)$ exists and is invertible.

Under these regularity assumptions, we will consider the following sequence of hypotheses:

$$H_0 : f_1(\cdot) = f_2(\cdot) = f(\cdot|\theta_0) \quad \text{versus} \quad H_1 : f_1(\cdot) = f(\cdot|\theta_0), \quad f_2 = f\left(\cdot|\theta_0 + h/\sqrt{N}\right)$$

(3.12)

for some $h \neq 0$ (more generally, one can define local perturbations along a general direction $h \in \mathbb{R}^p$ as $\theta_0 + h/\sqrt{N}$. While all the results in the sequel easily extend to general directions, we have chosen $h$ as the direction $1$ for simplicity). It is worth noting that under these contiguous alternatives, no test can be consistent, that is, no test can have power converging to $1$ (see Appendix C in the Appendix; cf. [117, Theorem 5.4]). Therefore it is interesting to look at the power curves of tests along such contiguous alternatives as a way to draw direct non-trivial comparisons between them.

Another commonly used model for local alternatives is to consider local perturbations of the mixing proportion in the contamination model (3.9). For this, suppose $f_2(\cdot) = (1 - \delta)f_1(\cdot) + \delta g(\cdot)$ as in (3.9) such that the following hold:

- The support of $g$ is contained in that of $f_1(\cdot)$.
- $0 < \mathbb{E}_{\mu_1}\left[(g(X)/f_1(X) - 1)^2\right] < \infty$, for $X \sim \mu_1$.

Under these assumptions, we will consider the following sequence of hypotheses:

$$H_0 : \delta = 0 \quad \text{versus} \quad H_1 : \delta = h/\sqrt{N},$$

(3.13)

for some $h \neq 0$.

In the following theorem we derive the asymptotic distribution of $T_{\nu,J}^{\nu,J}$ under the local alternatives (3.12) and (3.13) described above. In addition to providing precise expressions for the asymptotic local power, this result will be key in deriving bounds on the ARE of $T_{\nu,J}^{\nu,J}$ with respect to Hotelling $T^2$ test in Section 3.3 below.

**Theorem 3.3** (Asymptotics under local alternatives). Suppose the condition in (2.4) and Assumption 3.1 hold. Then in the usual asymptotic regime (2.3) the following hold:
Recall that $\eta(\cdot, \theta) = \frac{\nabla \theta f(\cdot| \theta)}{f(\cdot| \theta)}$. Then, under $H_1$ from (3.12),

$$T_{m,n}^{\nu, J} \overset{w}{\longrightarrow} \left\| h \sqrt{(1-\lambda)} \Sigma_{\text{ERD}} \mathbb{E}_{\theta_0} \left[ J(R_{H_0}^\nu(X)) \eta(X, \theta) \right] + G \right\|^2,$$

where $X \sim P_{\theta_0}$ and $G \sim \mathcal{N}(0, I_d)$. 

(2) Under $H_1$ from (3.13),

$$T_{m,n}^{\nu, J} \overset{w}{\longrightarrow} \left\| h \sqrt{(1-\lambda)} \Sigma_{\text{ERD}} \mathbb{E}_{\theta_0} \left[ J(R_{H_0}^\nu(X)) \left( \frac{g(X)}{f_1(X)} - 1 \right) \right] + G \right\|^2,$$

where $X \sim f_1$ and $G \sim \mathcal{N}(0, I_d)$.

The proof of the above result is given in Appendix B.4 in the Appendix. For this we first derive the joint limiting distribution of $\Delta_{m,n}^{J, \nu}$ (recall (3.2)) and the likelihood ratio (for the testing problems (3.12) and (3.13)) under $H_0$. The limiting distribution of $\Delta_{m,n}^{J, \nu}$ under $H_1$ can then be obtained by invoking Le Cam’s third lemma [78, Theorem 12.3.2], from which the result follows by an application of the continuous mapping theorem.

Remark 3.6 (On the role of $J(\cdot)$ and $\nu$ in ERD). Given an ERD $\nu_1$, there may be different choices, say $(J(\cdot), \nu)$ and $(\tilde{J}(\cdot), \tilde{\nu})$ such that $J(R_{H_0}^\nu(X)) \sim \nu_1$ and $J(R_{H_0}^{\tilde{\nu}}(X)) \sim \nu_1$.

However, the test statistics $T_{m,n}^{\nu, J}$ and $T_{m,n}^{\tilde{\nu}, \tilde{J}}$ are clearly different. Further Theorem 3.3 shows that the corresponding limiting distributions under local alternatives can also be different. This is because the corresponding limiting distributions are driven by the joint distributions $(J(R_{H_0}^\nu(X)), \eta(X, \theta))$ and $(\tilde{J}(R_{H_0}^{\tilde{\nu}}(X)), \eta(X, \theta))$. These two distributions may not be the same unless the maps $J(R_{H_0}^\nu(\cdot))$ and $\tilde{J}(R_{H_0}^{\tilde{\nu}}(\cdot))$ are the same, given $\nu_1$ (characterizing situations when these functions may be the same is an active research area in optimal transport theory; for more on this, see [100, Section 4.4.2] and [15, Section 4.1]). This is why we incorporate both a score function $J(\cdot)$ and a reference distribution $\nu$ separately, instead of working with the ERD by itself (also see Remark 3.8 for a more analytic advantage of using score functions).

Based on Theorem 3.3, we can now obtain explicit expressions of the local power for $\phi_{m,n}^{\nu, J}$ and compute its efficiency relative to the Hotelling $T^2$, as described in the following section.

3.3. ARE against Hotelling $T^2$

We will now compute the ARE of $T_{m,n}^{\nu, J}$ against the Hotelling $T^2$ test for a variety of settings in location shift alternatives. As is evident from Theorem 3.3, the asymptotic efficiency of $T_{m,n}^{\nu, J}$ will depend on the choices of $J(\cdot)$ and $\nu$ (collectively on the ERD; recall Definition 3.1) and also on the data generating distribution. This makes it important and interesting to study how the relative efficiencies depend on the aforementioned variables. We begin by describing three popular choices of ERDs:

1) Uniform on the hypercube: In this case, the ERD is $\text{Unif}[0, 1]^d$, the uniform distribution on the hypercube $[0, 1]^d$. This choice has appeared in [22, 30, 39], and is a very natural generalization of the univariate case, where the $\text{Unif}[0, 1]$ distribution is the most
popular reference distribution for rank-based tests. In this paper, we will obtain this ERD by setting \( J(x) = (\Phi(x_1), \ldots, \Phi(x_d)) \) for \( x = (x_1, \ldots, x_d) \) and \( \nu = \mathcal{N}(0, I_d) \). Note that this ERD has independent components.

(2) **Spherical uniform**: Let \( U_1 \sim \text{Unif}[0, 1] \) and \( U_2 \) be drawn independently and uniformly from the unit sphere \( \{ x \in \mathbb{R}^d : \|x\| = 1 \} \). Then the spherical uniform ERD is the distribution of the random variable \( U_1 U_2 \). It is easy to see that the spherical uniform distribution is spherically symmetric, that is, its density at \( x \) is only a function of \( \|x\| \). The spherical uniform reference distribution has been used in [50, 116, 117] while defining multivariate ranks. In our applications, unless mentioned otherwise, for obtaining this ERD we will set \( J(x) = x \) and \( \nu = \mathcal{N}(0, I_d) \).

(3) **Standard multivariate normal**: Here, the ERD is \( \mathcal{N}(0, I_d) \), the standard \( d \)-variate Gaussian distribution with mean vector \( 0 \) and covariance matrix \( I_d \). Note that this is the only multivariate distribution which is both spherically symmetric and has independent components [28]. As in the previous case, unless mentioned otherwise, for obtaining this ERD we will set \( J(x) = x \) and \( \nu = \mathcal{N}(0, I_d) \).

**Remark 3.7.** We observe that for all 3 ERDs described above \( \Sigma_{\text{ERD}} = \sigma^2 I_d \), for some \( \sigma > 0 \). In particular, for the uniform distribution on the hypercube \( \sigma^2 = 1/12 \), for the spherical uniform distribution \( \sigma^2 = (3d)^{-1} \), and for the standard \( d \)-variate normal distribution \( \sigma^2 = 1 \).

Following the seminal paper of Hodges and Lehmann [64], we will compute the ARE of \( T_{m,n}^{\nu,J} \) against the Hotelling \( T^2 \) test under local perturbations in location shift models. This corresponds to \( f_2(\cdot) = f_1(\cdot - \Delta) \), where \( f_1 \equiv f_1(\cdot | \theta_0) \) satisfies the regularity assumptions of Theorem 3.3-(1), and considering the following hypothesis:

\[
H_0 : \Delta = 0 \quad \text{versus} \quad H_1 : \Delta = h \sqrt{\frac{1}{N}} \quad (3.14)
\]

for some \( h \neq 0 \). Hereafter, we will abbreviate the ARE of \( T_{m,n}^{\nu,J} \) against the Hotelling \( T^2 \) test as \( \text{ARE}(T_{\nu,J}^{\nu}, T) \). Recall that the ARE is essentially the inverse ratio of the number of samples required by the respective tests (individually calibrated as asymptotically level \( \alpha \) tests) to achieve power \( \beta \) when the null shrinks to the alternative at rate \( O(1/\sqrt{N}) \) (refer to Definition B.2 in the Appendix for the formal definition of the ARE between two tests). As a consequence, the ARE of two tests in general depends on the level parameter \( \alpha \) and the power parameter \( \beta \). However, interestingly, for tests which have asymptotically non-central \( \chi^2 \) distributions with the same degrees of freedom under contiguous alternatives (such as in (3.14)), the ARE is simply the ratio of the corresponding non-centrality parameters (see [55, Proposition 5]). This implies, \( \text{ARE}(T_{\nu,J}^{\nu}, T) \) is completely free of parameters \( \alpha \) and \( \beta \) and, hence, can be used to compare the two testing procedures uniformly over the level and power parameters.

We begin by computing \( \text{ARE}(T_{\nu,J}^{\nu}, T) \) in the simple case of the Gaussian location family for the different ERDs described above. The proof of the following result is given in Appendix B.5 in the Appendix.

**Proposition 3.4** (Gaussian location problem). Suppose \( \mathcal{P}_\theta \) is the \( d \)-variate normal distribution \( \mathcal{N}(\theta, \Sigma) \), for some unknown positive definite covariance matrix \( \Sigma \). Then, under condition (2.4), Assumption (3.5), the following conclusions hold for the testing problem (3.14) in the usual asymptotic regime (2.3):

1. If the ERD is \( \text{Unif}[0, 1]^d \), obtained by choosing \( \nu = \mathcal{N}(0, I_d) \) and \( J(x) = \)
(Φ(x₁), . . . , Φ(xₙ)), then

\[ \text{ARE}(T^{νJ}, T) = 3/π \approx 0.95. \]

(2) When the ERD is spherical uniform with \( J(x) = x \) and \( ν = \) spherical uniform,

\[ \text{ARE}(T^{νJ}, T) = \kappa_d := \frac{3}{d} \left\{ \frac{1}{2^d-1} \cdot \frac{Γ(d-0.5)}{(Γ(d/2))^2} + \omega_d \right\}^2, \]

with

\[ \omega_d := \frac{\sqrt{2π}(d-1)}{2^{d/2}Γ(d/2)} E[|Z|^{d-2}] - \frac{\sqrt{2π}(2d-2)!!}{2^{d-1}(Γ(d/2))^2} F_1(d - \frac{1}{2}, \frac{d}{2}, \frac{d}{2} + \frac{1}{2}, -1), \]

where \( Z \sim \mathcal{N}(0, 1) \), \( (2d-2)!! := 1 \times 3 \times . . . \times (2d-3) \) and \( F_1(·, ·; ·; ·) \) is the hypergeometric function (see [98]).

(3) When the ERD is \( \mathcal{N}(0, I_d) \) with \( J(x) = x \) and \( ν = \mathcal{N}(0, I_d) \), \( \text{ARE}(T^{νJ}, T) = 1. \)

The result above shows that how even in this simple case of the Gaussian location, the ARE depends crucially on the choice of the ERD. In particular, when the ERD is chosen to be \( \text{Unif}[0, 1]^d \) or \( \mathcal{N}(0, I_d) \), the ARE stays constant over dimension. On the other hand, when the ERD is the spherical uniform distribution, the ARE varies with dimension.

Figure 1 shows the plots of the efficiencies in Proposition 3.4 as a function of the dimension. Here, the hypergeometric function in (3.16) is computed using the \texttt{hypergeo} package in R. One of the surprising things that emerge from this plot is that \( \kappa_d \) (recall (3.15)) falls below 0.95 when \( d \geq 5 \), that is, in the normal family for \( d \geq 5 \), the rank Hotelling statistic with cubic uniform ERD (with \( J(·) \) and \( ν \) chosen as in Proposition 3.4, part (1)) is more efficient than the spherical uniform (in terms of ARE with respect to the Hotelling’s \( T^2 \)). It is also worth noting that although the ARE decreases with increasing dimension when the ERD is the spherical uniform distribution, it stabilizes to a non-zero limit. In fact, from Theorem 3.5, it is easy to see that \( \lim_{d \to \infty} \kappa_d \geq 0.648 \).

**Remark 3.8 (Advantages of incorporating score functions).** In Proposition 3.4 (1) we see the theoretical benefits of incorporating score functions. For instance, if we had simply set \( J(x) = x \) and \( ν = \text{Unif}[0, 1]^d \), then by Theorem 3.3, we would have to work with the population rank map from \( \mathcal{N}(θ₀, Σ) \) to \( \text{Unif}[0, 1]^d \), which is harder to obtain in a closed form. On the other hand, when we set \( ν = \mathcal{N}(0, I_d) \), then the population rank map from \( \mathcal{N}(θ₀, Σ) \) to \( ν = \mathcal{N}(0, I_d) \) has the simple closed form expression \( R^*_{θ₀}(X) = Σ^{-\frac{1}{2}}(X - θ₀) \). This implies, \( J(R^*_{θ₀}(X)) = Φ(Σ^{-\frac{1}{2}}(X - θ₀)) \), where the standard normal distribution function \( Φ \) function is applied coordinate-wise, which is again a simple function to analyze. It is interesting to note that using this approach, we are able to recover the efficiency of \( 3/π \) which is the same as the ARE of the Wilcoxon’s test against the \( t \)-test (see [64]).

We now move on to the independent components case. To this end, denote by \( \mathcal{F}_{\text{ind}} \) the class of \( d \)-dimensional product distributions where the distribution of each component belongs to a location family which is absolutely continuous with respect to the Lebesgue measure. More precisely, \( \{P_θ\}_{θ ∈ Θ} = \{P_θ(z) = \prod_{i=1}^d f_i(z_i - θ_i)\}_{θ ∈ Θ} \), where \( f_1, f_2, . . . , f_d \) are univariate densities which are absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \), \( z = (z_1, z_2, . . . , z_d) \), and \( θ = (θ_1, . . . , θ_d) ∈ Θ ⊆ \mathbb{R}^d \).
Theorem 3.4 (Independent components case). Suppose condition (2.4) and Assumption (3.5) hold. Then the following conclusions hold for testing the hypothesis (3.14) in the usual asymptotic regime (2.3):

1. If the ERD is $\text{Unif}[0,1]^d$ obtained by choosing $\nu = N(0, I_d)$ and $J(x) = (\Phi(x_1), \ldots, \Phi(x_d))$ for $x = (x_1, \ldots, x_d)$, we have

$$\inf_{\mathcal{F}_{\text{ind}}} \text{ARE}(T^\nu \cdot J, T) \geq \frac{108}{125} \approx 0.864.$$  

Further, equality holds if and only if the $i$-th component of $X$ has density of the form:

$$f_i(x) = \frac{3}{20 \sqrt{5} \sigma_i^3} \left(5\sigma_i^2 - (x - \theta_i)^2\right) 1\{|x - \theta_i| \leq \sqrt{5}\sigma_i\},$$  

for all $i \in [d]$, where $\sigma_1, \ldots, \sigma_d > 0$.

2. When the ERD is $N(0, I_d)$ with $\nu = N(0, I_d)$ and $J(x) = x$,

$$\inf_{\mathcal{F}_{\text{ind}}} \text{ARE}(T^\nu \cdot J, T) \geq 1.$$  

Further, equality holds in the above display if and only if $X \sim N(\theta, D)$, where $D$ is a diagonal matrix with strictly positive entries.

The proof of this result is given in Appendix B.5 in the Appendix. From the proof, it will immediately be evident that the same conclusion as in Theorem 3.4, part (1) also holds if we use $\nu = \text{Unif}[0,1]^d$ and $J(x) = x$ to obtain the Unif[0,1]$^d$ ERD. Theorem 3.4 provides worst case lower bounds for the efficiency of $\phi_{\nu, J}^{m,n}$ against Hotelling $T^2$ over the class of multivariate distributions with independent components for the two different choices of ERD with independent components. These can be interpreted as multivariate analogues of the celebrated results of Hodges and Lehmann [64] and Chernoff and Savage [21]:

Figure 1: The left panel shows the ARE when $X$ follows a multivariate Gaussian distribution for our three candidate ERDs. The black and blue lines correspond to the AREs for the standard Gaussian and the Unif[0,1]$^d$ ERD. Note that these lines are constant over dimension at 1 and $3/\pi$, respectively. For the spherical uniform ERD, the ARE decreases for $d \geq 2$ and is lower than that of the Unif[0,1]$^d$ ERD for $d \geq 5$. In the right panel, we plot the lower bound for the ARE of $T^\nu$, $J_{m,n}$ with respect to the Hotelling $T^2$ test under elliptical symmetry when the spherical uniform ERD is used (see Theorem 3.5). Once again, this ARE decreases for $d \geq 2$. 
• **Multivariate Hodges-Lehmann phenomenon:** The result in Theorem 3.4-(1) shows that, in the worst case, $\phi_{m,n}^{\nu,J}$ needs only around 15% more samples than Hotelling $T^2$ to yield the same power, if Unif$[0,1]^d$ is used as the ERD. Note that the bound in this case does not depend on the dimension $d$ and matches the univariate Hodges-Lehmann bound [64].

• **Multivariate Chernoff-Savage phenomenon:** Theorem 3.4-(2) shows that if $N(0, I_d)$ is used as the ERD, then $\phi_{m,n}^{\nu,J}$ attains at least as large an asymptotic power as Hotelling $T^2$ with as many samples, thereby extending the Chernoff-Savage bound [21] for the univariate Wilcoxon rank-sum test. This, in particular, implies the following remarkable fact: *If the distribution $\mu_1$ has independent components but is not Gaussian, then $\phi_{m,n}^{\nu,J}$ will attain the same asymptotic power as Hotelling $T^2$ with fewer samples.*

Next, we consider the class of multivariate elliptically symmetric distributions. To be precise, $X \sim \text{Elliptically symmetric distribution}$ if there exists $\theta \in \mathbb{R}^d$, a positive definite $d \times d$ matrix $\Sigma$, and a function $f(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that the density $f_1$ of $X$ satisfies:

$\begin{align*}
f_1(x) &\propto (\det(\Sigma))^{-\frac{1}{2}} \int \left( (x - \theta)^\top \Sigma^{-1}(x - \theta) \right) \, \text{for all } x \in \mathbb{R}^d.
\end{align*}$

(3.18)

In the following, we will denote by $\mathcal{F}_{\text{ell}}$ the class of $d$-dimensional elliptically symmetric distributions satisfying some regularity conditions (see Appendix B.5.3 in the Appendix for details) with location parameter $\theta$ and a positive definite matrix $\Sigma$, both unknown.

**Theorem 3.5 (Elliptically symmetric case).** Suppose condition (2.4) and Assumption (3.5) hold. Then the following conclusions hold for the hypothesis testing problem (3.14) in the usual asymptotic regime (2.3):

1. When the ERD is the spherical uniform with $J(x) = x$ and $\nu = \text{spherical uniform}$,

   $$\inf_{\mathcal{F}_{\text{ell}}} \text{ARE}(T^{\nu,J}, T) \geq 81 \cdot \frac{1}{500} \cdot \frac{(\sqrt{2d-1} + 1)^5}{d^2(\sqrt{2d-1} + 5)} \geq 0.648.$$  

   The explicit form of the radial density for which the inequality is attained is given in Appendix B.5.3 in the Appendix.

2. When the ERD is $N(0, I_d)$ with $J(x) = x$ and $\nu = N(0, I_d)$,

   $$\inf_{\mathcal{F}_{\text{ell}}} \text{ARE}(T^{\nu,J}, T) \geq 1.$$  

   Also, equality holds if and only if $X \sim N(\theta, \Sigma)$ for some positive definite $\Sigma$ and $\theta \in \mathbb{R}^d$.

The proof of this result is given in Appendix B.5 in the Appendix. Again we have the following multivariate counterparts of the classical univariate results [21, 64] for the class of symmetric elliptical distributions, which includes, among others, the multivariate Gaussian and the multivariate $t$-distributions.

• **Multivariate Hodges-Lehmann phenomenon:** For the spherical uniform ERD, the lower bound on $\text{ARE}(T^{\nu,J}, T)$ depends on the dimension $d$, which is always bounded below by 0.648. Note that when $d = 1$ this lower bound equals 0.864 as one would expect in light of Theorem 3.4, because for $d = 1$ the spherical uniform is simply Unif$[-1,1]$, a location-scale shift of Unif$[0,1]$. For $d \geq 2$, the lower bound is decreasing in $d$ and converges to $81/125 = 0.648$, as $d \rightarrow \infty$. The plot of the lower bound as a function of $d$ is shown in Figure 1.
Next, we look at the class $F_{\text{gen}}$ described in (1.6) — a class of generative models used in blind source separation [118, 17, 26] and in independent component analysis (ICA) [20, 111, 4, 69]. To be precise, $F_{\text{gen}}$ includes the class of distributions which can be written as $X_{d \times 1} = A_{d \times d} W_{d \times 1}$, where $A_{d \times d}$ is an orthogonal matrix (unknown) and $W$ has independent components with Lebesgue density $\tilde{f}(w) = \prod_{i=1}^{d} \tilde{f}_i(w_i)$, where $w = (w_1, \ldots, w_d)$ and $\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_d$ are univariate densities. It is easy to check that the density of $X$ is given by:

$$f_1(x) = \prod_{i=1}^{d} \tilde{f}_i \left( \sum_{j=1}^{d} a_{ji} x_j \right), \quad \text{for all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

where $A_{d \times d} = (a_{ij})_{1 \leq i, j \leq d}$. In the following, we will obtain a Chernoff-Savage type result under the assumption that the family of densities $\{f_1(x - \theta)\}_{\theta \in \mathbb{R}^d}$ is a smooth parametric model in the sense of Section 3.2. Note that the matrix $A_{d \times d}$ is unknown.

**Theorem 3.6 (Generative model for blind source mixing).** Suppose condition (2.4) and Assumption (3.5) hold. Then the following conclusion holds for the hypothesis testing problem (3.14) in the usual asymptotic regime (2.3):

$$\inf_{F_{\text{gen}}} \text{ARE}(T^{\nu, J}, T) \geq 1,$$

when $J(x) = x$ and $\nu = N(0, I_d)$. Further, equality holds if and only if $X_{d \times 1} = A_{d \times d} W_{d \times 1}$ where $A_{d \times d}$ is orthogonal, $W \sim N(\theta, \Sigma)$ for some diagonal matrix with positive diagonal entries.

The proof of the above result can be found in Appendix B.5 in the Appendix. Theorem 3.6 can be viewed as a Chernoff-Savage type result (see [21]) for the class of distributions $F_{\text{gen}}$. Here again, $T^{\nu, J}_{m, n}$ with the Gaussian reference distribution has an uniformly higher ARE against the Hotelling $T^2$ test.

The results above clearly make a strong case for using $\phi_{m, n}^{\nu, J}$ with Gaussian ERD compared to Hotelling $T^2$. We elaborate more on this in the following remark.

**Remark 3.9 (Benefits of Gaussian ERD).** Proposition 3.4, Theorems 3.4, and 3.5 reveal that $\phi_{m, n}^{\nu, J}$, when the ERD is Gaussian (with $J(x) = x$ and $\nu = N(0, I_d)$), automatically adapts to the underlying family $\{P_{\theta}\}_{\theta \in \Theta}$ provided the family is Gaussian with unknown covariance, has independent components, or is elliptically symmetric, respectively. Therefore, $\phi_{m, n}^{\nu, J}$ with the Gaussian ERD yields a test which is always as efficient as the parametric Hotelling $T^2$ test, for the aforementioned families. The fact that the standard Gaussian is both spherically symmetric (has a density of the form (3.18) with $\theta = 0$ and $\Sigma = I_d$) and has independent components, plays a crucial role in the proof. This shows the benefits (in terms of the ARE) of using the Gaussian ERD when the performance is compared with respect to the Hotelling’s $T^2$ test. Interestingly, the advantage of using a Gaussian reference distribution was also observed in simulations in the related problem of mutual independence testing in the recent paper [117]. Our paper corroborates this theme for the two-sample problem using the ARE framework, which provides a theoretical foundation for Multivariate Chernoff-Savage phenomenon: For the Gaussian ERD, as in the independent components case, $\text{ARE}(T^{\nu, J}, T)$ is lower bounded by 1 in the worst case, irrespective of the dimension. Once again this showcases the strength of the proposed statistic in detecting location changes and the benefits of the Gaussian ERD.
making an informed choice about the underlying reference distribution. Analogous results for mutual independence testing are presented in Section 4.1.

Having discussed ARE lower bounds for the testing problem in (3.14), we now focus our attention on the contamination model discussed in (3.13). Unfortunately, under this model, as in the univariate case (see [64]), such ARE lower bounds do not exist. To see this we first need the following proposition, which can be proved using the same argument as in the proof of Proposition 3.4.

**Proposition 3.5.** Suppose \( X \) has a well-defined positive definite covariance matrix, and \( W \sim g \) (where \( g(\cdot) \) is defined as in (3.9)) such that \( EW \) is finite. Then, under Assumption 3.1, the following conclusion holds in the usual asymptotic regime:

\[
\text{ARE}(T^\nu J, T) = \left\| \Sigma^{-\frac{1}{2}} \mathbb{E}_{H_0} \left[ J(R^\nu_{H_0}(X)) \left( \frac{g(X)}{f_1(X)} - 1 \right) \right] \right\|^2 \left/ \left\| \Sigma^{-\frac{1}{2}} (EW - EX) \right\|_2^2 \right. 
\]

To see that the above expression of the ARE cannot have a non-trivial lower bound in general, suppose, for simplicity that the ERD is compactly supported on \([-M, M]^d\) for some \( M > 0 \). Observe that,

\[
\left\| \Sigma^{-\frac{1}{2}} \mathbb{E}_{H_0} \left[ J(R^\nu_{H_0}(X)) \left( \frac{g(X)}{f_1(X)} - 1 \right) \right] \right\|^2 \leq M^2 1^\top \Sigma_{ERD}^{-1} \mathbb{E}_{H_0} \left| \frac{g(X)}{f_1(X)} - 1 \right| \leq 2M^2 1^\top \Sigma_{ERD}^{-1} 1.
\]

On the other hand,

\[
\left\| \Sigma^{-\frac{1}{2}} (EW - EX) \right\|^2 \gtrsim \left\| EW - EX \right\|^2,
\]

where the hidden constant above depend only on the maximum eigenvalue of \( \Sigma_{ERD} \). Therefore, by making \( \left\| EW - EX \right\|^2 \) arbitrarily large, there is no hope of getting any lower bounds on ARE\((T^\nu J, T)\) in Proposition 3.5.

The same phenomenon happens in \( d = 1 \) while comparing the Wilcoxon rank-sum test with Student’s \( t \), as was duly noted in Hodges and Lehmann [64]. In fact, the authors in [64] point out that the lack of sensitivity of rank-based procedures to extreme contamination is, in many cases, a blessing. This is because, contamination is often due to gross errors in observation resulting in outliers. In such cases, a small proportion of the observed data is expected to come from a very different distribution with potentially larger means than that of the signal distribution \( \mu_1 \). As this difference in mean between the signal and the outlier distributions grow larger and larger, the Hotelling \( T^2 \) becomes more and more efficient (see the ARE expression above) compared to \( T^\nu_{m,n} \), even when the proportion of contamination is fixed. This shows that the Hotelling \( T^2 \) test is more sensitive to outlier distribution than \( T^\nu_{m,n} \) even when the proportion of outliers is small. This greater sensitivity to outliers for Hotelling \( T^2 \) makes it less robust as an inference procedure compared to \( T^\nu_{m,n} \).

4. Broader Scope

In this section, we will illustrate the broader scope of our techniques to other nonparametric testing problems, by establishing analogous results in the context of mutual independence testing and discussing connections and refinements to related methods in recent literature. In particular, in Section 4.1 we construct a class of distribution-free nonparametric tests of
independence which are natural multivariate analogues of Spearman’s rank correlation [121] and enjoy favorable ARE properties similar to $T_m^J$. In Section 4.2 we discuss how our techniques provide direct improvements of the results in some related papers such as [117, 51, 30, 52, 53], including the resolution of an open question from [117].

4.1. Applications to Independence Testing

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. observations from $\mu \in \mathcal{P}(\mathbb{R}^{d_1+d_2})$ with absolutely continuous marginals $\mu_1$ and $\mu_2$ (note that $\mu$ need not be absolutely continuous). We are interested in the following test of independence problem:

$$H_0 : X_1 \perp \perp Y_1 \quad \text{(i.e., } \mu = \mu_1 \otimes \mu_2 \text{)} \quad \text{versus} \quad H_1 : X_1 \not\perp \not\perp Y_1 \quad \text{(i.e., } \mu \neq \mu_1 \otimes \mu_2 \text{)}. \quad (4.1)$$

This is the classical multivariate mutual independence testing problem which has received enormous attention in the past hundred years (see [67, Chapters 1 and 8] and the references therein) with applications in finance [83], statistical genetics [79], survival analysis [86], etc.

When $d_1 = d_2 = 1$, the earliest attempt at problem (4.1) is the Pearson’s correlation [101] which can only detect linear association between two variables. This was soon extended through the classical Spearman’s rank correlation coefficient [121] which can detect any monotonic association between the variables and has the additional benefit of being exactly distribution-free under $H_0$ for all sample sizes. Since then rank-based distribution-free correlation measures in the univariate case have received a lot of attention (see, for example, [73, 66, 13, 8] and the references therein).

In the multivariate setting, the earliest test for problem (4.1) is probably due to Wilks [138] which is constructed using the Gaussian likelihood ratio test statistic, and reduces to Pearson’s correlation for $d_1 = d_2 = 1$. In this sense, Wilks’ test is the natural multivariate analogue of Pearson’s correlation. Since the advent of Wilks’ test, a number of other multivariate tests have been proposed including the Friedman-Rafsky test based on geometric graphs [36] and the celebrated distance covariance test [126] (see [71] for a comprehensive survey of other procedures). However, none of these proposals are exactly distribution-free under $H_0$, the chief hurdle once again, being the lack of a canonical ordering in $\mathbb{R}^d$, for $d \geq 2$. This gap in the literature was recently bridged in a series of works [30, 116, 117] where multivariate ranks based on optimal transport were used (as we did in Section 1.1) to construct multivariate, nonparametric, exactly distribution-free tests for (4.1). However, none of these papers provide any explicit expressions for the ARE of their tests against natural counterparts, and consequently do not guarantee any ARE lower bounds.

The main goal of this section is to overcome the aforementioned gap in the optimal transport based independence testing literature by constructing a multivariate version of Spearman’s rank correlation coefficient that is exactly distribution-free and has high ARE compared to the Wilk’s test, the natural multivariate analogue of Pearson’s correlation coefficient.

4.1.1. Multivariate Spearman’s Correlation

Before defining our statistic, let us fix some notation. Let $\nu_1 \in \mathcal{P}_{ac}(\mathbb{R}^{d_1})$ and $\nu_2 \in \mathcal{P}_{ac}(\mathbb{R}^{d_2})$ be two reference distributions. Let $\hat{R}_1(X_1), \ldots, \hat{R}_1(X_n)$ denote the multivariate ranks of
$X_1, \ldots, X_n$ constructed as in (1.2) and (1.3) with a fixed grid \( \{ h_{i,1} \}_{i \in [n]} \) satisfying (2.4) with \( \nu = \nu_1 \). Construct \( \tilde{R}_2(Y_1), \ldots, \tilde{R}_2(Y_n) \) using \( \{ h_{i,2} \}_{i \in [n]} \) analogously, where \( \{ h_{i,2} \}_{i \in [n]} \) is a discretization of \( \nu_2 \). Also, given any \( d_1 \times d_2 \) matrix \( H \), let \( \text{vec}(H) \) be the vector obtained by unlisting the entries of \( H \) row-wise. For example,

\[
H = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \implies \text{vec}(H) = (1 \ 3 \ 5 \ 2 \ 4 \ 6).
\]

Next, for any two matrices \( H_1 \) and \( H_2 \), we will use \( H_1 \otimes H_2 \) to denote the standard Kronecker product. Finally, let \( J_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}, J_2 : \mathbb{R}^{d_2} \to \mathbb{R}^{d_2} \) be two injective, continuous score functions and assume that the ERD-s, \( J_1 \# \nu_1 \) and \( J_2 \# \nu_2 \) both satisfy Assumption 3.1 with positive definite covariance matrices \( \Sigma^{(1)}_{\text{ERD}} \) and \( \Sigma^{(2)}_{\text{ERD}} \).

Based on the above notation, setting \( \nu := (\nu_1, \nu_2) \) and \( J := (J_1, J_2) \), our version of multivariate Spearman’s correlation is given as:

\[
R_{n,sc}^{\text{rank}} := \left\| \left( \Sigma^{(1)}_{\text{ERD}} \otimes \Sigma^{(2)}_{\text{ERD}} \right)^{-\frac{1}{2}} \text{vec} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (J_1(\tilde{R}_1(X_i)) - \bar{J}_1)(J_2(\tilde{R}_2(Y_i)) - \bar{J}_2) \right) \right\|^2,
\]

where \( \bar{J}_1 := \frac{1}{n} \sum_{i=1}^{n} \tilde{R}_1(X_i) \) and \( \bar{J}_2 := \frac{1}{n} \sum_{i=1}^{n} \tilde{R}_2(Y_i) \).

**Remark 4.1** (Extension of Spearman’s rank correlation). When \( d_1 = d_2 = 1 \) and the fixed grids \( \{ h_{1,i} \}_{i \in [n]}, \{ h_{2,i} \}_{i \in [n]} \) are both chosen as \( \{ i/n \}_{i \in [n]} \), then \( R_{n,sc}^{\text{rank}} \) is the same as the squared Spearman’s rank correlation coefficient. Moreover, \( R_{n,sc}^{\text{rank}} \) is exactly distribution-free under \( H_0 \) as shown in Proposition 4.1 below. In this sense, \( R_{n,sc}^{\text{rank}} \) is a multivariate extension of the classical Spearman’s rank correlation coefficient.

Our first result shows that \( R_{n,sc}^{\text{rank}} \) is distribution-free under \( H_0 \). This follows directly from [30, Proposition 2.2] and is formalized in the following proposition:

**Proposition 4.1.** Under \( H_0 \) as specified in (4.1), \( R_{n,sc}^{\text{rank}} \) is distribution-free for all \( n \geq 1 \), that is, its distribution is free of \( \mu_1 \) and \( \mu_2 \).

Using the above result we can readily obtain a finite sample distribution-free independence test. Fix a level \( \alpha \in (0, 1) \) and let \( c_n \) denote the upper \( \alpha \) quantile of the universal distribution in Proposition 4.1. Consider the test function:

\[
\phi_n^\nu J := 1 \left( R_{n,sc}^{\text{rank}} \geq c_n \right).
\]

This test is exactly distribution-free for all \( n \geq 1 \) and uniformly level \( \alpha \) under \( H_0 \), in the sense of (3.4). From Proposition 4.1, it is clear that the asymptotic null distribution of \( R_{n,sc}^{\text{rank}} \) should be free of \( \mu_1 \) and \( \mu_2 \). In the following theorem, we make this explicit.

**Theorem 4.1.** Suppose the fixed grids \( \{ h_{1,i} \}_{i \in [n]}, \{ h_{2,i} \}_{i \in [n]} \) satisfy (3.5) with score functions \( J_1(\cdot), J_2(\cdot) \) and reference distributions \( \nu_1, \nu_2 \), respectively. Then under \( H_0 \) as in (4.1),

\[
R_{n,sc}^{\text{rank}} \overset{w}{\to} \chi^2_{d_1 d_2}.
\]

The proof of Theorem 4.1 is given in Appendix B.6 in the Appendix. The simple limiting null distribution of \( R_{n,sc}^{\text{rank}} \) as presented in Theorem 4.1 can be used to calibrate the statistic \( R_{n,sc}^{\text{rank}} \) to obtain an asymptotically level \( \alpha \) test.
4.1.2. ARE against Wilks’ Test [138]

In this section, we will derive lower bounds on the ARE of $R_{n,sc}^{\text{rank}}$ against Wilks’ statistic over various classes of multivariate distributions. To begin with, we recall the Wilks’ test statistic:

$$R_n := n \log \left( \frac{\det(Q_{1,1}) \cdot \det(Q_{2,2})}{\det(Q)} \right), \quad Q := \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} := \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} X_i - \bar{X} \\ Y_i - \bar{Y} \end{pmatrix} \begin{pmatrix} X_i - \bar{X} \\ Y_i - \bar{Y} \end{pmatrix}^\top,$$

(4.4)

$$\bar{X} := \frac{1}{n} \sum_{i=1}^{n} X_i \text{ and } \bar{Y} := \frac{1}{n} \sum_{i=1}^{n} Y_i.$$ Under $H_0$ as in (4.1), it is well-known that $R_n \xrightarrow{w} \chi^2_{d_1 \times d_2}$, which is the same as the limiting distribution obtained in Theorem 4.1.

To compare the local power of $R_{n,sc}^{\text{rank}}$ against $R_n$, we need to fix a notion of local alternatives as in Section 3.2. One of the most popular choices of local alternatives in mutual independence testing is the sequence of Konijn alternatives (see [76, 127, 58, 41]) defined below.

**Definition 4.1** (Konijn alternatives). Suppose $X'_1$ and $Y'_1$ are independent random vectors with Lebesgue absolutely continuous distributions $\mu_1$ and $\mu_2$. Define,

$$\begin{pmatrix} X'_1 \\ Y'_1 \end{pmatrix} := \begin{pmatrix} (1 - \delta n^{-\frac{2}{3}})I_{d_1} & \delta n^{-\frac{1}{2}}M_{d_1 \times d_2} \\ \delta n^{-\frac{1}{2}}M_{d_1 \times d_2}^\top & (1 - \delta n^{-\frac{2}{3}})I_{d_2} \end{pmatrix} \begin{pmatrix} X'_1 \\ Y'_1 \end{pmatrix},$$

(4.5)

where $\delta > 0$ and $M_{d_1 \times d_2}$ is a $d_1 \times d_2$ dimensional matrix. Note that if $\delta = 0$, then $X'_1$ and $Y'_1$ are independent. Therefore, problem (4.1) can be restated in this framework as:

$$H_0 : \delta = 0 \text{ versus } H_1 : \delta \neq 0.$$

(4.6)

We will further assume [117, Assumption 5.1]. This assumption ensures that the probability measures under $H_0$ and $H_1$ as in (4.6) are contiguous to each other.

In the sequel, we will write $\text{ARE}(R^{\nu,J}, R)$ to denote the ARE of $R_{n,sc}^{\text{rank}}$ with respect to $R_n$ under the Konijn alternatives defined above, that is, when $(X_1, Y_1)$ are generated according to (4.5). In order to obtain lower bounds for $\text{ARE}(R^{\nu,J}, R)$, we can safely assume that both $X'_1$ and $Y'_1$ have finite variances, as otherwise $\text{ARE}(R^{\nu,J}, R)$ is trivially $\infty$. We also note that problem (4.6) is affine invariant, in the following sense: If we replace $X'_1, Y'_1$ by $A(X'_1 - a)$ and $B(Y'_1 - b)$ in (4.5), where $A, B$ are invertible matrices of dimensions $d_1 \times d_1$ and $d_2 \times d_2$, respectively, and $a \in \mathbb{R}^{d_1}, b \in \mathbb{R}^{d_2}$, then $X'_1$ and $Y'_1$ so obtained are still independent if and only if $\delta = 0$. Therefore, without loss of generality we can assume:

**Assumption 4.1.** $E X'_1 = 0_{d_1}, E Y'_1 = 0_{d_2}$ and $\text{Var}(X'_1) = I_{d_1}$ and $\text{Var}(Y'_1) = I_{d_2}$.

We now present our main theorem of this section in which we provide Chernoff-Savage [21] type lower bounds for problem (4.6). It shows that, even in the worst case, $R_{n,sc}^{\text{rank}}$ is at least as efficient as the Wilks’ statistic $R_n$ in (4.4), for all fixed dimensions $d_1$ and $d_2$.

**Theorem 4.2** (Lower bounds on $\text{ARE}(R^{\nu,J}, R)$ with Gaussian ERD). Suppose the conditions in Theorem 4.1 and Assumption 4.1 hold. Then, with $\nu_1 = \mathcal{N}(0_{d_1}, I_{d_1}), \nu_2 = \mathcal{N}(0_{d_2}, I_{d_2}), J_1(x) = x$, and $J_2(y) = y$ (both ERDs are standard Gaussians of appropriate dimensions), the following holds:

$$\inf_{\mu_1, \mu_2 \in \mathcal{F}_{\text{ind}} \cup \mathcal{F}_{\text{el}} \cup} \text{ARE}(R^{\nu,J}, R) \geq 1,$$

(4.7)
where $\mathcal{F}_{\text{ind}}$ and $\mathcal{F}_{\text{ell}}$ are defined as in Section 3.3. Furthermore, equality holds in (4.7) if and only if both $\mu_1$ and $\mu_2$ are standard Gaussians of appropriate dimensions.

The proof of Theorem 4.2 can be found in Appendix B.6 in the Appendix. Theorem 4.2 shows the benefits of using $R_{\nu,sc}^{\text{rank}}$ with Gaussian ERDs for problem (4.6). Note that the bound in (4.7) is free of the dimensions $d_1$, $d_2$ and also free of the matrix $M$ in Definition 4.1. The benefits of using a Gaussian ERD were also noted in our analysis of $T_{m,n}^{\nu, J}$ (see Remark 3.9). In fact, we believe that advantages of using a Gaussian ERD are ubiquitous and should extend to other natural multivariate rank-based procedures for testing symmetry, significance of regression coefficients, etc. We would also like to point out that while Theorem 4.2 provides Chernoff-Savage type lower bounds, it is also possible to have Hodges-Lehmann type lower bounds in this setting. In fact, the Hodges-Lehmann type bounds match those obtained in [58, Proposition 2]. These bounds have a complicated form and they are strictly smaller than 1 for all fixed dimensions $d_1$ and $d_2$. We omit those results for brevity.

4.2. Implications of our results to existing literature

In this section, we discuss the implications of our results to other existing papers that deal with nonparametric testing problems using optimal transport. As stated in the Introduction, our goals are quite different from the existing papers, but nevertheless our results have interesting consequences that help resolve open problems in existing literature. We illustrate this by using two examples — Hallin et al. [51] and Shi et al. [117], although similar comments apply to [30, 116, 52].

Comparison with Hallin et al. [51]: In the recent work [51], a basic version of the rank Hotelling $T^2$ statistic $T_{m,n}^{\nu, J}$ was presented in passing in [51, Page 25] for the special case when the reference distribution is spherical uniform and for a specific choice of the set $\{h_1^d, \ldots, h_N^d\}$, such that $\sum_{i=1}^N h_i^d = 0$ (a slightly stringent requirement that may be hard to satisfy for generic sequences). However, the authors did not study its theoretical properties, such as consistency and asymptotic efficiency. Our results, when applied to the special case proposed in [51], imply the consistency of their corresponding test (see Theorem 3.2, Propositions 3.2 and 3.3), and can be used to derive explicit ARE expressions (see Theorem 3.3) and, most importantly, lower bounds for the AREs (see Theorems 3.4 and 3.5).

In this section, we will focus on the ARE results that can be obtained for the special case presented in [51] and highlight some of the additional benefits to be gained by adopting our general framework. In fact, for the particular case proposed in [51], our arguments directly imply the lower bound in Theorem 3.5-(1). The conclusion in Theorem 3.5-(2) also follows if the statistic in [51] is transformed using the van der Waerden score, given below:

$$J(x) := F_{\chi_d}^{-1}(|x|) \frac{x}{||x||} 1(x \neq 0).$$

Here $F_{\chi_d}(-)$ is the distribution function of a $\chi_d^2$ random variable. The corresponding conclusion is formalized in the following proposition.

**Proposition 4.2.** Consider the same assumptions as in Theorem 3.5 and recall the definition of $\mathcal{F}_{\text{ell}}$. Suppose the reference distribution $\nu$ is the spherical uniform and $J(\cdot)$ is the
van der Waerden score function in (4.8) (same combination as discussed in [51]). Then,
\[
\inf_{\mathcal{F}_{\text{all}}} \text{ARE}(T^{\nu,J}, T) \geq 1
\]
where equality holds if and only if \( X \sim \mathcal{N}(\theta, \Sigma) \) for some \( \theta \in \mathbb{R}^d \) and positive definite \( \Sigma \).

On observing that the function \( J(\mathcal{P}_{\text{Ha}}(\cdot)) \) (with \( J(\cdot) \) and \( \nu \) as above) is in fact the optimal transport map from \( X \) to a standard normal, the proof of Proposition 4.2 follows immediately from the proof of Theorem 3.5-(2). While the statistic in [51] (that is, \( T_{\text{in}}^{\nu,J} \) with spherical uniform reference distribution and a specific choice of \( \{h_i^d\}_{i \in [n]} \)) attains the same ARE lower bound as in Theorem 3.5 over the class of elliptically symmetric distributions, as shown above, when it comes to distributions with independent components (as in Theorem 3.4), this special case proposed in [51] falls short. This is because the optimal transport map from distributions with independent components to the spherical uniform (in the sense of Proposition 2.1) is not explicit and consequently not analytically tractable. In fact, we believe that with the spherical uniform reference distribution, the same lower bounds as in Theorem 3.4 are no longer true for \( d > 1 \). On the other hand, optimal transport maps from distributions with independent components to the standard Gaussian or Unif\([0,1]^d\) distributions are tractable which we are able to exploit in our lower bound computations in Theorem 3.4, thanks to our general framework. In fact, it is this flexibility that allows us to show that the standard Gaussian reference distribution is at least as efficient as Hotelling \( T^2 \) uniformly over both the class of distributions with independent components and those having an elliptically symmetric density.

**Comparison with Shi et al. [117]:** We now discuss the implications of our results for the paper [117], where the authors consider the independence testing problem as introduced in (4.1). Recall the notation for multivariate ranks used in the beginning of Section 4.1.1 and let the fixed grids \( \{h^d_{1,i_1}\}_{i_1 \in [n]} \) and \( \{h^d_{2,i_2}\}_{i_2 \in [n]} \) be chosen as in [117, Page 9]. Using this notation, a prototypical example of a test statistic considered in [117] would be the rank distance covariance given as:
\[
\text{RdCov}_n^2 := \frac{1}{n^2} \sum_{i,j} \Delta_{i,j}^{(1)} \Delta_{i,j}^{(2)} + \frac{1}{n^4} \left( \sum_{i,j} \Delta_{i,j}^{(1)} \right)^2 - \frac{2}{n^3} \sum_{i,j,k} \Delta_{i,j}^{(1)} \Delta_{i,j}^{(2)}, \tag{4.9}
\]
where \( \Delta_{i,j}^{(1)} := \|J_1(\hat{R}_1(X_i)) - J_1(\hat{R}_1(X_j))\| \) and \( \Delta_{i,j}^{(2)} := \|J_2(\hat{R}_2(Y_i)) - J_2(\hat{R}_2(Y_j))\| \) for score functions \( J_1(\cdot) \) and \( J_2(\cdot) \). In [117], the authors conjecture that the test based on the above statistic with van der Waerden score function (see (4.8)) is consistent, but a proof was not provided. Here, we answer this question in the affirmative by using Theorem 2.1 and techniques as in the proof of Theorem 3.2. This is formalized in the following proposition (see Appendix B.7 in the Appendix for a proof).

**Proposition 4.3.** Assume that \( R_1(\cdot), R_2(\cdot) \) are the optimal transport maps from \( \mu_1 \) and \( \mu_2 \) (both are Lebesgue absolutely continuous) to reference distributions \( \nu_1 \) and \( \nu_2 \). Also, suppose \( \mathbb{E}\|J_1(R_1(X_1))\|^2 < \infty \) and \( \mathbb{E}\|J_2(R_2(Y_1))\|^2 < \infty \). Then provided both \( \{h^d_{1,i_1}\}_{i_1 \in [n]} \) and \( \{h^d_{2,i_2}\}_{i_2 \in [n]} \) satisfy (2.4), (3.5) with \( \nu_1 \) and \( \nu_2 \), respectively, the following hold as \( n \to \infty \):
\[
\text{RdCov}_n^2 \xrightarrow{P} \mathbb{E}\left[\Delta_{1,2}^{(1),\text{or}} \Delta_{1,2}^{(2),\text{or}}\right] + \mathbb{E}\left[\Delta_{1,2}^{(1),\text{or}}\right] \mathbb{E}\left[\Delta_{1,2}^{(2),\text{or}}\right] - 2\mathbb{E}\left[\Delta_{1,2}^{(1),\text{or}} \Delta_{1,3}^{(2),\text{or}}\right], \tag{4.10}
\]
where $\Delta^{(k)}_{i,j} := \|J_k(R_k(X_i)) - J_k(R_k(X_j))\|$ for $1 \leq i, j \leq n$, $k = 1, 2$. Moreover, the right hand side of the display above equals 0 if and only if $\mu = \mu_1 \otimes \mu_2$.

In fact, the same technique can be used to establish the consistency of the more general class of tests considered in [117], the details of which we omit for brevity. It is worth emphasizing that Theorem 2.1 is not restricted to any particular statistic, instead it establishes consistency for continuous functions of empirical rank maps. Consequently, it can be applied to establish the consistency of a wide variety of statistics such as those in [30, 52, 51].

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Appendix

The Appendix Section is organized as follows:

- Appendix B contains the proofs of our main results,
- Appendix C contains some asymptotic minimax type lower bounds for the two sample testing problem (2.2) from the main paper,
- Appendix D contains detailed simulation studies that support our theoretical results, and
- Appendix E contains some simple examples where (3.5) from the main paper, holds.

Appendix B: Proofs of the Main Results

In this section we present the proofs of our main results from the main paper. The section is organized as follows: We begin with the proof of Theorem 2.1 in Appendix B.1. The proofs of the consistency results from Section 3.1 are given in Appendix B.2. In Appendix B.3, we provide the proof of Theorem 3.1. The proof of Theorem 3.3 is presented in Appendix B.4. Further, in Appendix B.5 we prove the results from Section 3.3. Finally, the proof of the results from Section 4 have been provided in Appendix B.6.

Hereafter, given two positive sequences \( \{a_N\}_{N \geq 1} \) and \( \{b_N\}_{N \geq 1} \), we will write \( a_N \lesssim b_N \) to denote \( a_N \leq C b_N \), for some constant \( C > 0 \) not depending on \( N \).

B.1. Proof of Theorem 2.1

Throughout this proof, we will assume \( r = p \) for notational simplicity. As will be evident from the arguments below, the proof for other values of \( r \) will follow similarly.

We begin our proof by recalling the following well-known result from convex analysis.

**Lemma B.1** (Alexandroff’s Theorem, Alexandroff [2]). Let \( f : U \to \mathbb{R} \) be a convex function, where \( U \) is an open convex subset of \( \mathbb{R}^n \). Then \( f \) has a second derivative Lebesgue a.e. in \( U \).

Hereafter, we assume that all the random variables are defined on the same probability space. Then in the usual asymptotic regime (2.3) in the main paper,

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{Z_i} \xrightarrow{w} \lambda \mu_1 + (1 - \lambda) \mu_2 =: \mu \quad \text{a.s.} \tag{B.1}
\]

Let \( \{(C_{N,1}, D_{N,1}),(C_{N,2}, D_{N,2}), \ldots, (C_{N,p}, D_{N,p})\} \) denote \( p \) i.i.d. draws from the following distribution

\[
\frac{1}{N} \sum_{i=1}^{N} \delta_{(Z_i, \hat{R}_{m,n}(Z_i))},
\]

which is the empirical distribution on the (random) set \( \{(Z_j, \hat{R}_{m,n}(Z_j))\}_{j \in [N]} \). Also, let \( \nu^* \) be the law induced by the random variable \( (Z, R_{H_1}(Z)) \), \( Z \sim \lambda \mu_1 + (1 - \lambda) \mu_2 \). Note that, by (2.4) (see the main paper) and (B.1), \( (C_{N,1}, D_{N,1}) \) is asymptotically tight almost surely.
as both the variables converge weakly marginally. Consequently, by using the same sequence of steps\(^6\) as in [30, Theorem 2.1], we get
\[
((C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p})) \xrightarrow{w} \nu^* \otimes \ldots \otimes \nu^*, \quad \text{a.s.} \quad (B.2)
\]
Next, observe that \(R_{H_1}^\nu(\cdot)\) is continuous a.e. in the interior of the support of \(\mu\), by Alexanderoff’s theorem (see Lemma B.1) and the a.e. continuity of \(\mathcal{F}(\cdot)\) and \(J(\cdot)\) (by the assumptions in the theorem). Therefore, the map:
\[
g : ((y_1, z_1), \ldots, (y_p, z_p)) \mapsto \left\| \mathcal{F}(J(R_{H_1}^\nu(y_1)), \ldots, J(R_{H_1}^\nu(y_p))) - \mathcal{F}(J(z_1), \ldots, J(z_p)) \right\|
\]
is continuous a.e. with respect to the \(p\)-fold product measure \(\nu^* \otimes \ldots \otimes \nu^*\). Suppose \((Z_1, R_{H_1}^\nu(Z_1)), \ldots, (Z_p, R_{H_1}^\nu(Z_p)) \sim \nu^* \otimes \ldots \otimes \nu^*\). Then observe that
\[
g\left( (Z_1, R_{H_1}^\nu(Z_1)), \ldots, (Z_p, R_{H_1}^\nu(Z_p)) \right) = 0.
\]
Therefore, by a direct application of the continuous mapping theorem,
\[
g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \xrightarrow{w} 0 \quad \text{a.s.}
\]
Now, since weak convergence to a degenerate measure implies convergence in probability, given any \(\varepsilon > 0\), the following holds:
\[
\mathbb{P}\left[ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) > \varepsilon \mid \{Z_1, \ldots, Z_N\} \right] \rightarrow 0 \quad \text{a.s.}
\]
and consequently by the bounded convergence theorem,
\[
\mathbb{P}\left[ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) > \varepsilon \right] \rightarrow 0. \quad (B.3)
\]
To complete the proof, define
\[
V_{m,n} := \sum_{(i_1, \ldots, i_p) \in [N]^p} \left\| \mathcal{F}(J(\hat{R}_{m,n}(Z_{i_1})), \ldots, J(\hat{R}_{m,n}(Z_{i_p})) - \mathcal{F}(J(R_{H_1}^\nu(Z_{i_1})), \ldots, J(R_{H_1}^\nu(Z_{i_p}))) \right\|.
\]
This implies, by recalling the definition of the function \(g\),
\[
\mathbb{P}\left( \frac{1}{N^p} V_{m,n} > \varepsilon \right) = \mathbb{P}\left( \mathbb{E} \left[ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \mid Z_1, \ldots, Z_N \right] > \varepsilon \right)
\]
\[
\leq \varepsilon^{-1} \mathbb{E} \left[ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \right], \quad (B.4)
\]
where the last line uses Markov’s inequality. We next show that
\[
\left\{ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \right\}_{N \geq 1} \quad \text{is uniformly integrable.} \quad (B.5)
\]
---

\(^6\)First, by using Prokhorov’s Theorem, one can show that given any subsequence, there exists a further subsequence such that \(((C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}))\) converges weakly to some distribution on \((\mathbb{R}^{2d})^p\), almost surely. By using [88, Corollary 14 and Lemma 2], one can then show that the limiting distribution above is free of the subsequence thereby concluding the proof.
Note that if we establish (B.5), then by (B.3), we will have

\[ \mathbb{E}\left[ g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \right] \to 0 \text{ as } N \to \infty. \]

Combining this observation with (B.4) would then imply \( \frac{1}{N^p} V_{m,n} \) converges in probability to zero, as required.

**Proving (B.5):** Observe that

\[
g\left( (C_{N,1}, D_{N,1}), \ldots, (C_{N,p}, D_{N,p}) \right) \leq \left\| \mathcal{F}(J(\hat{R}_{H_1}^\nu(C_{N,1}))), \ldots, J(\hat{R}_{H_1}^\nu(C_{N,p})) \right\| + \left\| \mathcal{F}(J(D_{N,1}), \ldots, J(D_{N,p})) \right\| =: \mathcal{G}_N + \mathcal{H}_N.
\]

By using the above display, (B.5) would follow if \( \{\mathcal{G}_N\}_{N \geq 1} \) and \( \{\mathcal{H}_N\}_{N \geq 1} \) are uniformly integrable. Observe that \( \mathcal{G}_N \xrightarrow{u} \mathcal{F}(J(Z_1), \ldots, J(Z_p)) \), where \( Z_1, \ldots, Z_p \overset{i.i.d.}{\sim} \nu \). The same conclusion also holds for \( \mathcal{H}_N \). Now observe that:

\[
\limsup_{N \to \infty} \mathbb{E}\left[ \mathcal{G}_N \right] = \limsup_{N \to \infty} \frac{1}{N^p} \mathbb{E}\left[ \sum_{(i_1, \ldots, i_p) \in [N]^p} \left\| \mathcal{F}(J(\hat{R}_{m,n}^{z_1}(Z_{i_1})), \ldots, J(\hat{R}_{m,n}^{z_p}(Z_{i_p}))) \right\| \right] \leq \int \left\| \mathcal{F}(J(z_1), \ldots, J(z_p)) \right\| \, d\nu(z_1) \ldots d\nu(z_p) < \infty.
\]

The same conclusion also holds with \( \mathcal{G}_N \) replaced with \( \mathcal{H}_N \). Using the above display, (B.5) then follows by Vitali’s convergence theorem (see [119, Theorem 5.5]).

To establish the almost sure convergence, recall that in this case \( \mathcal{F}(\cdot) \) and \( J(\cdot) \) are both assumed to be Lipschitz. Consequently,

\[
\frac{1}{N^p} V_{m,n} \lesssim \frac{1}{N} \sum_{i=1}^{N} \| \hat{R}_{m,n}(Z_i) - R_{H_1}^\nu(Z_i) \|.
\]

The conclusion then follows in the same manner as the proof of [30, Theorem 2.1], once again by noting that \( (C_{N,1}, D_{N,1}) \) is asymptotically tight almost surely by (2.4) (see the main paper and (B.1)).

**B.2. Proofs of Theorem 3.2, Proposition 3.2, and Proposition 3.3**

In this section we present the proofs of the consistency results of the test \( \phi_{m,n}^{\nu,J} \). We begin with the proof of Theorem 3.2 from the main paper.

**Proof of Theorem 3.2.** Throughout this proof all expectations are taken under \( H_1 \) and we will assume \( X \sim \mu_1 \) and \( Y \sim \mu_2 \). With this in mind, we first show the following:

\[
\frac{T_{m,n}^{\nu,J}}{N\lambda(1-\lambda)} \overset{P}{\to} \left( \mathbb{E}J(R_{H_1}^\nu(X)) - \mathbb{E}J(R_{H_1}^\nu(Y)) \right)^\top \Sigma^{-1}_{ERD} \left( \mathbb{E}J(R_{H_1}^\nu(X)) - \mathbb{E}J(R_{H_1}^\nu(Y)) \right). \tag{B.6}
\]
Towards proving this, using Theorem 2.1 (see the main paper, with \( p = 1, \ r = 1, \ q = d, \) and \( \mathcal{F}(x) = x, \) note that
\[
\frac{1}{m} \sum_{i=1}^{m} \| J(\hat{R}_{m,n}(X_i)) - J(R_{H_1}^\nu(X_i)) \| + \frac{1}{n} \sum_{i=1}^{n} \| J(\hat{R}_{m,n}(Y_i)) - J(R_{H_1}^\nu(Y_i)) \| = o_p(1)
\]
This implies,
\[
\left\| \frac{1}{m} \sum_{i=1}^{m} J(\hat{R}_{m,n}(X_i)) - \frac{1}{n} \sum_{i=1}^{n} J(\hat{R}_{m,n}(Y_i)) \right\| - \frac{1}{m} \sum_{i=1}^{m} J(R_{H_1}^\nu(X_i)) + \frac{1}{n} \sum_{i=1}^{n} J(R_{H_1}^\nu(Y_i)) = o_p(1).
\]
Next, by using the weak law of large numbers together with Slutsky’s theorem gives,
\[
\frac{1}{m} \sum_{i=1}^{m} J(\hat{R}_{m,n}(X_i)) - \frac{1}{n} \sum_{i=1}^{n} J(\hat{R}_{m,n}(Y_i)) \xrightarrow{P} \mathbb{E}J(R_{H_1}^\nu(X)) - \mathbb{E}J(R_{H_1}^\nu(Y)).
\]
An application of the continuous mapping theorem then completes the proof of (B.6).

Now, to complete the proof of Theorem 3.2, note that whenever \( \mathbb{E}J(R_{H_1}^\nu(X)) - \mathbb{E}J(R_{H_1}^\nu(Y)) \neq 0, \) (B.6) implies that \( T_{m,n}^\nu \xrightarrow{P} \infty. \) Also, under \( H_0, \) \( T_{m,n}^\nu \) is \( O_p(1) \) (by Theorem 3.1 in the main paper, and consequently \( c_{m,n} \) in (3.3) is \( O(1). \) Combining these observations immediately yields consistency.

Proof of Proposition 3.2. Let \( B \subset \mathbb{R}^d \) be an open set such that \( \mu_1(B) > 0 \) and \( u_{H_1}^\nu(\cdot) \) is strictly convex on \( B. \) Fix \( y \in B. \) Let \( \tilde{y} \) be an element in the support of \( \mu_1. \) We now claim that
\[
\langle R_{H_1}^\nu(\tilde{y}) - R_{H_1}^\nu(y), \tilde{y} - y \rangle > 0.
\]
Note that the LHS above is always \( \geq 0, \) as \( R_{H_1}^\nu(\cdot) \) is the gradient of a convex function. To show strict inequality, firstly note that (B.7) is immediate if \( \tilde{y} \in B \setminus \{y\} \) as \( u_{H_1}^\nu(\cdot) \) is strictly convex on \( B. \) Now suppose \( \tilde{y} \notin B. \) As \( B \) is open, there exists \( \alpha \in (0,1) \) small enough such that the following holds:
\[
\langle R_{H_1}^\nu(y) - R_{H_1}^\nu(\alpha y + (1 - \alpha)\tilde{y}), (1 - \alpha)(y - \tilde{y}) \rangle > 0,
\]
where we use the fact that \( R_{H_1}^\nu(\cdot) \) is the gradient of a convex function (see Proposition 2.1 in the main paper and consequently satisfies cyclical monotonicity.

Next, observe that
\[
\langle R_{H_1}^\nu(y) - R_{H_1}^\nu(\tilde{y}), \tilde{y} - y \rangle = \langle R_{H_1}^\nu(y) - R_{H_1}^\nu(\alpha y + (1 - \alpha)\tilde{y}), \tilde{y} - y \rangle + \langle R_{H_1}^\nu(\alpha y + (1 - \alpha)\tilde{y}), y - \tilde{y} \rangle \\
\geq \langle R_{H_1}^\nu(y) - R_{H_1}^\nu(\alpha y + (1 - \alpha)\tilde{y}), y - \tilde{y} \rangle > 0,
\]
where the last line follows from the fact that \( R_{H_1}^\nu(\cdot) \) is the gradient of a convex function and (B.8).

Now, it suffices to prove that \( \Delta \neq 0 \) implies \( \mathbb{E}R_{H_1}^\nu(X) \neq \mathbb{E}R_{H_1}^\nu(Y). \) We will prove this by contradiction. Towards this direction, suppose that \( \Delta \neq 0. \) Observe that the condition
\( \mathbb{E} R_{H_1}^\nu(X) = \mathbb{E} R_{H_1}^\nu(Y) \) can be written as: \( \int (R_{H_1}^\nu(x + \Delta) - R_{H_1}^\nu(x)) \, d\mu_1(x) = 0 \), which implies,
\[
\int (R_{H_1}^\nu(x + \Delta) - R_{H_1}^\nu(x), \Delta) \, d\mu_1(x) = 0. \tag{B.9}
\]
As \( (R_{H_1}^\nu(x + \Delta) - R_{H_1}^\nu(x), \Delta) \geq 0 \), (B.9) implies that \( (R_{H_1}^\nu(x + \Delta) - R_{H_1}^\nu(x), \Delta) = 0 \), \( \mu_1 \)-almost everywhere \( x \). However, (B.7) implies, \( (R_{H_1}^\nu(x + \Delta) - R_{H_1}^\nu(x), \Delta) > 0 \) for all \( x \in B \), where \( \mu_1(B) > 0 \). This is a contradiction, thereby completing the proof.

**Proof of Proposition 3.3.** Recall that \( W \) has density \( g(\cdot) \). Observe that
\[
\mathbb{E} R_{H_1}^\nu(Y) = (1 - \delta) \mathbb{E} R_{H_1}^\nu(X) + \delta \mathbb{E} R_{H_1}^\nu(W)
\]
and consequently, \( \mathbb{E} R_{H_1}^\nu(X) \neq \mathbb{E} R_{H_1}^\nu(Y) \) if and only if \( \mathbb{E} R_{H_1}^\nu(X) \neq \mathbb{E} R_{H_1}^\nu(W) \). This observation combined with Theorem 3.2 proves the first part of the proposition. The second part then follows from Proposition 3.2 in the main paper.

**B.3. Proof of Theorem 3.1**

Recall the definition of \( \Delta_{\nu}^{or} \) from (3.7) in the main paper. Now, define
\[
T_{m,n,sc}^{\nu,or} := \frac{m}{N} \sum_{k=1}^{\nu} \left( \Delta_{m,n}^{\nu,J^{or}} \right) \Sigma_{ERD}^{-1} \left( \Delta_{m,n}^{\nu,J^{or}} \right)
\]
Throughout we will take limit as \( N \to \infty \) such that (2.3) from the main paper holds. The main step in the proof of Theorem 3.1 is to show that
\[
|T_{m,n}^{\nu,J} - T_{m,n,sc}^{\nu,or}| \xrightarrow{P} 0 \tag{B.10}
\]
under \( H_0 \). The proof of (B.10) is deferred. It can be used to complete the proof of Theorem 3.1 as follows: Note that by a direct application of the multivariate central limit theorem,
\[
\sqrt{\frac{mn}{N}} \Delta_{m,n}^{\nu,J^{or}} \xrightarrow{w} \mathcal{N}(0, \Sigma_{ERD}),
\]

since \( \{J(R_{H_0}^\nu(X_1)), J(R_{H_0}^\nu(X_2)), \ldots, J(R_{H_0}^\nu(X_m))\}, \{J(R_{H_0}^\nu(Y_1)), J(R_{H_0}^\nu(Y_2)), \ldots, J(R_{H_0}^\nu(Y_n))\} \) are independent and identically distributed random variables under \( H_0 \). This implies, by the continuous mapping theorem,
\[
T_{m,n,sc}^{\nu,or} \xrightarrow{w} \chi^2_d,
\]
under \( H_0 \). Combining this with (B.10) and the Slutsky’s theorem, gives
\[
T_{m,n}^{\nu,J} \xrightarrow{w} \chi^2_d,
\]
which completes the proof of Theorem 3.1.

**Proof of (B.10):** Note that it suffices to show that:
\[
\lim_{N \to \infty} \frac{mn}{N} \mathbb{E} \left\| \Delta_{m,n}^{\nu,J} - \Delta_{m,n}^{\nu,J^{or}} \right\|^2 = 0, \tag{B.11}
\]
where \( \Delta_{m,n} \) is defined as in (3.2) from the main paper. For the proof of claim (B.11), we need the notion of permutation distributions as defined below:
Definition B.1 (Permutation distribution). Recall that \( \mathcal{Z}_N = \{Z_1, Z_2, \ldots, Z_N\} \) denotes the pooled sample \( \mathcal{X}_m \cup \mathcal{Y}_n \). For \( 1 \leq i \leq N \), define

\[
L_i = \begin{cases} 
1 & \text{if } Z_i \in \mathcal{X}_m, \\
2 & \text{if } Z_i \in \mathcal{Y}_n.
\end{cases}
\]

Note that if \( \mu_1 = \mu_2 \), then

\[
P(L_i = 1|\mathcal{Z}_N) = \frac{m}{N} = 1 - P(L_i = 2|\mathcal{Z}_N). \tag{B.12}
\]

(Observe that \( L_1, L_2, \ldots, L_N \) are identically distributed, but they are not independent.) In particular, the distribution of \((T_{\nu,J}^{\nu,J}, T_{\nu,J}^{\nu,J,or})\) is completely determined by the joint distribution of \((L_1, \ldots, L_N)\) conditional on \(\{Z_1, \ldots, Z_N\}\). We will refer to this distribution as the permutation distribution.

Let \( E_{\mathcal{Z}_N} \) denote the conditional expectation with respect to \( \{Z_1, \ldots, Z_N\} \). Define

\[
\hat{S}_i := J(\hat{R}_{m,n}(Z_i)), \quad S_i := J(R_{m,n}^{\nu,J}(Z_i)).
\]

Observe that the left hand side of (B.11) can be written as:

\[
E\left\|\Delta_{\nu,J}^{\nu,J} - \Delta_{\nu,J,or}^{\nu,J}\right\|^2 = \frac{mn}{N} \left\| \sum_{i=1}^{N} \hat{S}_i \left\{ \frac{1}{m} \mathbf{1}\{L_i = 1\} - \frac{1}{n} \mathbf{1}\{L_i = 2\} \right\} - \sum_{i=1}^{N} S_i \left\{ \frac{1}{m} \mathbf{1}\{L_i = 1\} - \frac{1}{n} \mathbf{1}\{L_i = 2\} \right\} \right\|^2
\]

\[
= \frac{mn}{N} \left\| \sum_{i=1}^{N} \hat{S}_i \ell_i - \sum_{i=1}^{N} S_i \ell_i \right\|^2, \tag{B.13}
\]

where \( \ell_i := \frac{1}{m} \mathbf{1}\{L_i = 1\} - \frac{1}{n} \mathbf{1}\{L_i = 2\} \), for \( i \in [N] \). Now, (B.13) can be written as

\[
E\left\|\Delta_{\nu,J}^{\nu,J} - \Delta_{\nu,J,or}^{\nu,J}\right\|^2 = T_1 + T_2 - T_3, \tag{B.14}
\]

where

\[
T_1 = \frac{mn}{N} \left\| \sum_{i=1}^{N} \hat{S}_i \ell_i \right\|^2, \quad T_2 = \frac{mn}{N} \left\| \sum_{i=1}^{N} S_i \ell_i \right\|^2, \quad T_3 = \frac{2mn}{N} \left\{ \sum_{1 \leq i,j \leq N} \hat{S}_i^T S_j \ell_i \ell_j \right\}. \tag{B.15}
\]

We will now show that each of the three terms in (B.13) converges to the same limit as \( N \to \infty \). We begin with \( T_1 \). Note that \( \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_N \) are measurable with respect to the sigma field induced by \( \mathcal{Z}_N \). Therefore the conditional expectation \( E_{\mathcal{Z}_N} \) only operates on the indicator variables above to yield the corresponding probabilities. In particular, recall (B.12) and note that, for \( i \neq j \),

\[
P(L_i = 1, L_j = 1|\mathcal{Z}_N) = \frac{m(m-1)}{N(N-1)}, \quad P(L_i = 2, L_j = 2|\mathcal{Z}_N) = \frac{n(n-1)}{N(N-1)},
\]

and,

\[
P(L_i = 1, L_j = 2|\mathcal{Z}_N) = \frac{mn}{N(N-1)}.
\]
The above identities imply \( \mathbb{E}_{Z_N} \ell_i = 0 \), for all \( i \in [N] \), and, for \( i \neq j \),

\[
\frac{mn}{N} \mathbb{E}_{Z_N} [\ell_i \ell_j] = -\frac{1}{N(N-1)}.
\] (B.16)

Now, recalling the definition of \( T_1 \) from (B.15) and taking iterated expectation, first with respect to the permutation distribution (conditional on \( Z_N \)) and then with respect to the randomness of \( Z_N \), gives,

\[
T_1 = \frac{mn}{N} \mathbb{E} \mathbb{E}_{Z_N} \left[ \sum_{i=1}^{N} \tilde{S}_i \ell_i \right]^2
= \frac{mn}{N} \mathbb{E} \left[ \sum_{i=1}^{N} \| \tilde{S}_i \|^2 \right] \left\{ \frac{P(L_i = 1|Z_N)}{m^2} + \frac{P(L_i = 2|Z_N)}{n^2} \right\} + \sum_{1 \leq i \neq j \leq N} \tilde{S}_i^\top \tilde{S}_j \mathbb{E}_{Z_N} [\ell_i \ell_j]
= \frac{1}{N} \mathbb{E} \left[ \sum_{i=1}^{N} \| \tilde{S}_i \|^2 \right] - \frac{1}{N(N-1)} \mathbb{E} \left[ \sum_{1 \leq i \neq j \leq N} \tilde{S}_i^\top \tilde{S}_j \right],
\] (B.17)

where the last step uses (B.12) and (B.16). Now, invoking Theorem 2.1 from the main paper, with \( p = q = r = 1, \mathcal{F}(x) = \|x\|^2 \) and Theorem 2.1, gives

\[
\frac{1}{N} \sum_{i=1}^{N} \| \tilde{S}_i \|^2 - \| S_i \|^2 \xrightarrow{P} 0.
\] (B.18)

By using assumption (3.5) coupled with Vitali’s convergence theorem (see [119, Theorem 5.5]), the above convergence happens in \( L_1 \). Similarly, using Theorem 2.1, with \( p = 2, r = 2, q = 1, \mathcal{F}(x, y) = x^\top y \), together with Vitali’s theorem gives,

\[
\frac{1}{N(N-1)} \mathbb{E} \left[ \sum_{1 \leq i \neq j \leq N} \tilde{S}_i^\top \tilde{S}_j - S_i^\top S_j \right] \xrightarrow{} 0.
\] (B.19)

Finally, using the weak law of large numbers in (B.18) and (B.19), it follows that

\[
\lim_{N \to \infty} \mathbb{E} \left[ \frac{1}{N} \sum_{i=1}^{N} \| \tilde{S}_i \|^2 \right] = \int \| J(x) \|^2 d\nu(x),
\] (B.20)

and

\[
\lim_{N \to \infty} \frac{1}{N(N-1)} \mathbb{E} \left[ \sum_{1 \leq i \neq j \leq N} \tilde{S}_i^\top \tilde{S}_j \right] = \int J(x)^\top J(y) d\nu(x) d\nu(y).
\] (B.21)

Combining (B.17), (B.20) and (B.21), gives

\[
\lim_{N \to \infty} T_1 = \eta_J := \int \| J(x) \|^2 d\nu(x) - \int J(x)^\top J(y) d\nu(x) d\nu(y).
\]

Similar arguments can be applied to the other two terms in (B.16) to show that \( \lim_{N \to \infty} T_2 = \eta_J \) and \( \lim_{N \to \infty} T_3 = 2\eta_J \). This proves (B.11).
B.4. Proof of Theorem 3.3-(1)

Recall the setup of (3.12) and its corresponding assumptions from Section 3.2, both from the main paper. Define the likelihood ratio statistic as:

\[ V_N := \sum_{j=1}^{n} \log \frac{f(Y_j|\theta_0 + h\frac{1}{\sqrt{N}})}{f(Y_j|\theta_0)}. \]

By using the local asymptotic normality under (3.12), \( V_N \) can be written as:

\[ V_N = \dot{V}_N - h^2 (1 - \lambda) \frac{1}{2} \mathbf{1}^\top \mathbf{1} + o_P(1), \]

where \( \dot{V}_N := \frac{h}{\sqrt{N}} \sum_{j=1}^{n} \frac{1}{f(Y_j|\theta_0)} \frac{\nabla_{\theta} f(Y_j|\theta)|_{\theta_0}}{f(Y_j|\theta_0)}. \)  

Denote

\[ T_1 := \frac{1}{m} \sum_{i=1}^{m} \left\{ J(R_{H_0}^\nu(X_i)) - \mathbb{E}_{H_0} J(R_{H_0}^\nu(X_1)) \right\}, \]

\[ T_2 := \frac{1}{n} \sum_{j=1}^{n} \left\{ J(R_{H_0}^\nu(Y_j)) - \mathbb{E}_{H_0} J(R_{H_0}^\nu(Y_1)) \right\}. \]

Consequently, by the multivariate central limit theorem, the following result holds:

\[ \left( \sqrt{\frac{m}{n} T_1 \frac{V_N}{n}} \right) \xrightarrow{w} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} (1 - \lambda)\Sigma_{ERD} & 0 \\ 0 & \lambda \Sigma_{ERD} \end{pmatrix} \right). \]

under \( H_0 \), where \( \mathbf{0} \) denotes the vector of zeros of length \( d \), \( \mathbf{O} \) the \( d \times d \) matrix of zeros, \( c_1 := h^2 (1 - \lambda) \frac{1}{2} \frac{1}{\sqrt{N}} \mathbf{1}^\top \mathbf{I}(\theta_0) \mathbf{1} \), and

\[ \gamma_1 := h \sqrt{\lambda (1 - \lambda)} \mathbb{E}_{H_0} \left[ J(R_{H_0}^\nu(Y)) \frac{1}{f(Y|\theta_0)} \frac{\nabla_{\theta} f(Y|\theta)|_{\theta_0}}{f(Y|\theta_0)} \right]. \]

Note that under \( H_0 \), \( \Delta_{m,n}^{\nu,J,\text{or}} = T_1 - T_2 \) (recall (3.7) from the main paper). Therefore, by (B.11) and (B.24), under \( H_0 \),

\[ \left( \sqrt{\frac{m}{n} \Delta_{m,n} \frac{V_N}{n}} \right) \xrightarrow{w} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -c_1 \gamma_1 \\ \gamma_1 \end{pmatrix} \right). \]

Then by Le Cam’s third lemma [78, Corollary 12.3.2] and the continuous mapping theorem, we have:

\[ T_{m,n,\text{sc}}^{\nu,J,\text{or}} \xrightarrow{w} \left\| - \Sigma_{ERD}^{-\frac{1}{2}} \gamma_1 + \mathbf{G} \right\|_2^2, \]

under \( H_1 \), where \( \mathbf{G} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d) \). Next note that in (B.10), we showed that \( |T_{m,n}^{\nu,J} - T_{m,n,\text{sc}}^{\nu,J,\text{or}}| \xrightarrow{P} 0 \) under \( H_0 \). By contiguity, the same conclusion also holds under \( H_1 \). Therefore, by an application of Slutsky’s theorem, we get:

\[ T_{m,n}^{\nu,J} \xrightarrow{w} \left\| - \Sigma_{ERD}^{-\frac{1}{2}} \gamma_1 + \mathbf{G} \right\|_2^2. \]  

(B.25)
This completes the proof of part (1).

Proof of Theorem 3.3-(2): Recall the setup of (3.13) and its corresponding assumptions from Section 3.2, both in the main paper. In this case the likelihood ratio $V_N$ is defined by:

$$V_N := \sum_{j=1}^{N} \log \left[ \frac{\left(1 - \frac{h}{\sqrt{N}}\right)f(Y_j) + \frac{h}{\sqrt{N}}g(Y_j)}{f(Y_j)} \right].$$

Once again, by using local asymptotic normality, $V_N$ can be written as:

$$V_N = \hat{V}_N - \frac{1 - \lambda}{2} h^2 \int_{A} \left( \frac{g(x)}{f_1(x)} - 1 \right)^2 f_1(x) \, dx + o_P(1),$$

where

$$\hat{V}_N = \frac{h}{\sqrt{N}} \sum_{j=1}^{n} \left( \frac{g(Y_j)}{f_1(Y_j)} - 1 \right).$$

Now, recalling (B.23) and the multivariate central limit theorem, the following result holds:

$$\left(\sqrt{\frac{mN}{n}}T_1 \quad \sqrt{\frac{mN}{n}}T_2 \quad \sqrt{\frac{mN}{n}}V_N \right) \xrightarrow{w} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \\ -c_2^{2} \end{pmatrix}, \begin{pmatrix} (1 - \lambda)\Sigma_{\text{ERD}} & O & O \\ O & \lambda\Sigma_{\text{ERD}} & \gamma_2 \\ O^T & \gamma_2 & c_2 \end{pmatrix} \right)$$

under $H_0$, where $c_2 := (1 - \lambda)h^2A$ and

$$\gamma_2 := h\sqrt{\lambda(1 - \lambda)}\mathbb{E}_{H_0} \left[ J(R_{H_0}^{\nu}(Y)) \left( \frac{g(Y)}{f_1(Y)} - 1 \right) \right].$$

The conclusion in part (2) then follows from Le Cam’s third lemma [78, Corollary 12.3.2] and the continuous mapping theorem.

### B.5. Proofs from Section 3.3

In this section we will present the proofs of the results from Section 3.3 in the main paper. We begin with the formal definition of the asymptotic (Pitman) relative efficiency of two tests (see [103, 94, 130]).

**Definition B.2** (Asymptotic (Pitman) relative efficiency). Consider the sequence of testing problems

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_{1,K},$$

for $\theta_0, \theta_{1,K} \in \mathbb{R}$ and all $K \geq 1$, where $\theta_{1,K} \to \theta_0$, as $K \to \infty$. Suppose $\{T_{1,K}\}_{K \geq 1}$ and $\{T_{2,K}\}_{K \geq 1}$ are two sequences of level $\alpha \in (0, 1)$ tests for the problem (B.27), with associated test functions $\{\phi_{1,K}\}_{K \geq 1}$ and $\{\phi_{2,K}\}_{K \geq 1}$. Fixing a power level $\beta \in [\alpha, 1)$, define

$$N_1(\alpha, \beta, \theta_1) := \min\{K \geq 1 : \mathbb{E}_{\theta_1} \phi_{1,K'} \geq \beta, \quad \text{for all} \ K' \geq K\},$$
and $N_2(\alpha, \beta, \theta_1)$ similarly as above with $\phi_{1,K'}$ replaced by $\phi_{2,K'}$. Then the asymptotic (Pitman) relative efficiency (ARE) of the sequence of tests $\phi_{1,K}$ with respect to $\phi_{2,K}$, along a sequence $\theta_{1,K} \to \theta_0$, as $K \to \infty$, is given by

$$\text{ARE}(\{T_{1,K}\}_{K \geq 1}, \{T_{2,K}\}_{K \geq 1}) = \lim_{\theta_{1,K} \to \theta_0} \frac{N_2(\alpha, \beta, \theta_{1,K})}{N_1(\alpha, \beta, \theta_{1,K})}$$

provided the limit exists.

As is evident from the above definition, the ARE of two tests will in general depend on $\alpha$ and $\beta$. However, when the tests have asymptotically non-central chi-squared distributions with the same degrees of freedom (as is the case with the Hotelling $T^2$ and the $T_{m,n}^{\nu,J}$ statistics), the ARE is simply the ratio of the corresponding non-centrality parameters (see [55, Proposition 5] and [130, Theorem 14.19]). Equipped with this fact, we begin the proofs of our results from Section 3.3 in the main paper.

### B.5.1. Proof of Proposition 3.4

Recall the definition of the Hotelling $T^2$ statistic $T_{m,n}$ from (1.4) in the main paper. Note that if $\text{Var}_{H_0} X$ exists then under the regularity assumptions of Theorem 3.3 in the main paper, it is easy to check that:

$$T_{m,n} \xrightarrow{w} \|h \sqrt{\lambda(1 - \lambda)} \Sigma^{-\frac{1}{2}} \text{E}_{H_0} \left[ X 1^T \nabla f(X|\theta) |_{\theta_0} \right] + G \|^2,$$

where $G \sim N(0, I_d)$ and

$$\tilde{\Sigma} := \text{E}(X - \text{E}X)(X - \text{E}X)^\top.$$

Recall from Section 3.3 (see the main paper that $\text{ARE}(T_{\nu,J}^{\nu,J}, T)$ denotes the ARE of $T_{m,n}^{\nu,J}$ with respect to $T_{m,n}$. Then, using the non-centrality parameters of $T_{m,n}^{\nu,J}$ and $T_{m,n}$ from (B.25) and (B.28), respectively, and invoking [130, Theorem 14.19] gives,

$$\text{ARE}(T_{\nu,J}^{\nu,J}, T) = \frac{\left\| \Sigma^{-\frac{1}{2}} \text{E}_{H_0} \left[ J(P_{H_0}^{\nu,J}(X)) 1^T \nabla f(X|\theta)|_{\theta_0} \right] \right\|^2}{\left\| \tilde{\Sigma}^{-\frac{1}{2}} \text{E}_{H_0} \left[ X 1^T \nabla f(X|\theta)|_{\theta_0} \right] \right\|^2}.$$

**Proof of Proposition 3.4 (1):** Let us define $A := \Sigma^{-\frac{1}{2}} := ((a_{ij}))_{1 \leq i, j \leq d}$, where $a_{ij}$ denotes the $(i, j)$-th element of the matrix $A$. In this case, $\tilde{\Sigma} = \Sigma$. Also note that $A(X - \theta_0) \equiv G \sim N(0, I_d)$. Using

$$\frac{\nabla f(X|\theta)|_{\theta_0}}{f(X|\theta_0)} = \Sigma^{-1}(X - \theta_0) = AG,$$

and $\text{E}_{H_0} \left[ \frac{\nabla f(X|\theta)|_{\theta_0}}{f(X|\theta_0)} \right] = 0$, observe that

$$\left\| \Sigma^{-\frac{1}{2}} \text{E}_{H_0} \left[ X 1^T \nabla f(X|\theta)|_{\theta_0} \right] \right\|^2 = \text{E}_{H_0} \left[ G : 1^T AG \right]^2 = \sum_{j=1}^{d} \left( \sum_{i=1}^{d} a_{ij} \right)^2.$$
Moreover, with Remark 3.7 in the main paper and the integration by parts formula gives, 

\[ J(R_{H_0}^\nu(X)) = \left( \Phi(e_1^T A(X - \theta_0)), \ldots, \Phi(e_d^T A(X - \theta_0)) \right), \]

where \( e_i \) is the \( i \)-th vector of the canonical basis in \( \mathbb{R}^d \). Combining the above display with Remark 3.7 in the main paper and the integration by parts formula gives,

\[
\left\| \Sigma_{ERD}^{-\frac{1}{2}} E_{H_0} \left[ J(R_{H_0}^\nu(X)) \frac{1}{f(X | \theta_0)} \right] \right\|^2 \\
= 12 \sum_{j=1}^{d} \left\{ \left( \sum_{i=1}^{d} E \left[ \frac{\partial}{\partial X_i} \Phi(e_j^T A(X - \theta_0)) \right] \right)^2 \right\} \\
= 12 \sum_{j=1}^{d} \left\{ \left( \sum_{i=1}^{d} a_{ij} \right)^2 \left( E \phi(e_j^T A(X - \theta_0)) \right)^2 \right\},
\]

where \( X = (X_1, \ldots, X_d)^T \). Now, for \( j \in [d] \), define \( d_j := A e_j \) and note that by the standard block determinant formula,

\[ \det(\Sigma^{-1} + d_j d_j^T) = \det(\Sigma^{-1}) (1 + d_j^T \Sigma d_j) = 2(\det(\Sigma))^{-1}. \]

Using the above display, we get the following chain of equalities, for \( j \in [d] \):

\[
E \phi(e_j^T A(X - \theta_0)) = E \phi(d_j^T (X - \theta_0)) \\
= \frac{\sqrt{\det(\Sigma^{-1})}}{(\sqrt{2\pi})^{d+1}} \int \exp \left( -\frac{1}{2} (x - \theta_0)^T (d_j d_j^T + \Sigma^{-1}) (x - \theta_0) \right) \, dx \\
= \left( \frac{\det(\Sigma^{-1})}{2\pi \cdot \det(\Sigma^{-1} + d_j d_j^T)} \right)^\frac{1}{2} \\
= \frac{1}{2\sqrt{\pi}}.
\]

Using (B.32) in (B.33) gives,

\[
\left\| \Sigma_{ERD}^{-\frac{1}{2}} E_{H_0} \left[ J(R_{H_0}^\nu(X)) \frac{1}{f(X | \theta_0)} \right] \right\|^2 = \frac{3}{\pi} \sum_{j=1}^{d} \left( \sum_{i=1}^{d} a_{ij} \right)^2 = \frac{3}{\pi} \cdot 1^T \Sigma^{-1} 1.
\]

This implies, by (B.30) and (B.31), \( \text{ARE}(T^\nu, J) = \frac{3}{\pi} \), which completes the proof of part (1).

**Proof of Proposition 3.4 (2):** Let \( H_d(\cdot) \) be the cumulative distribution function of a \( \sqrt{\chi_d^2} \) distribution and \( h_d(\cdot) \) be the associated probability density. It is easy to check that

\[
h_d(r) = \frac{1}{2^{d/2-1} \Gamma(d/2)} r^{d-1} e^{-r^2/2},
\]

(B.34)
for \( r \geq 0 \). Next, define
\[
R_{H_0}^r(X) := \frac{A(X - \theta_0)}{\sqrt{(X - \theta_0) \Sigma (X - \theta_0)}} : H_d \left( \sqrt{(X - \theta_0)^\top A^2(X - \theta_0)} \right). \tag{B.35}
\]

We will first show that \( R_{H_0}^r(X) \) defined above is the optimal transport map from \( X \) to the spherical uniform distribution. Towards this direction, consider the following standard lemma which we state and prove for completeness.

**Lemma B.2.** Suppose \( X_1' \) has an elliptically symmetric distribution as in (B.54) with parameters \( \theta, \Sigma \) and \( X_2' \) has an elliptically symmetric distribution with parameters \( \mathbf{0}_d \) and \( I_d \) (that is, the distribution of \( X_2' \) is spherically symmetric). Let \( H_1(\cdot) \) be the distribution function of \( \|\Sigma^{-\frac{1}{2}}(X_1' - \theta)\| \) and \( H_2(\cdot) \) be the distribution function of \( \|X_2\| \). Then the optimal transport map from the distribution of \( X_1' \) to that of \( X_2' \) is given by:
\[
R(X_1') := \frac{\Sigma^{-\frac{1}{2}}(X_1' - \theta)}{\|\Sigma^{-\frac{1}{2}}(X_1' - \theta)\|} H_2^{-1} \left( H_1 \left( \|\Sigma^{-\frac{1}{2}}(X_1' - \theta)\| \right) \right). \tag{B.36}
\]

**Proof.** \( R(\cdot) \) as defined in (B.36) clearly satisfies \( R(X_1') \overset{d}{=} X_2' \). Further note that \( R(\cdot) \) is the gradient of the function:
\[
\int_0^{\|\Sigma^{-\frac{1}{2}}(X_1' - \theta)\|} H_2^{-1}(H_1(r)) \, dr,
\]
which is a convex function due to the monotonicity of \( H_2^{-1} \circ H_1(\cdot) \). Therefore, by Proposition 2.1 in the main paper, \( R(\cdot) \) as defined in (B.36) is the optimal transport map in this case. \( \square \)

It follows from Lemma B.2 with \( X_1' \overset{d}{=} X_1 \) and \( X_2' \) as the spherical uniform distribution, that \( R_{H_0}^r(\cdot) \) as defined in (B.35) above is the optimal transport map in this case.

Next, let us write:
\[
R_{H_0}^r(X) = (r_1(X), \ldots, r_d(X)),
\]
where \( r_j(X) \) denotes the \( j \)-th coordinate of the rank vector \( R_{H_0}^r(X) \), for \( j \in [d] \). Now, write \( G = A(X - \theta_0) \) and note \( G \overset{w}{=} N(0, I_d) \). For any \( i, j \in [d] \), observe that:
\[
\frac{\partial}{\partial X_i} r_j(X) = \frac{a_{ij} h_d(\|G\|)}{\|G\|} + \frac{(e_j^\top G) h_d(\|G\|)}{\|G\|^2} \cdot e_i \top AG - \frac{(e_j^\top G) h_d(\|G\|)}{\|G\|^3} \cdot e_i \top AG. \tag{B.37}
\]

By using the spherical symmetry of \( G \), we further get:
\[
\mathbb{E} \left[ \frac{(e_j^\top G) h_d(\|G\|)}{\|G\|^2} \cdot e_i \top AG \right] = \mathbb{E} \left[ \frac{a_{ij} G_j^2 h_d(\|G\|)}{\|G\|^2} \right]
= \frac{a_{ij}}{d} \mathbb{E} [h_d(\|G\|)] \tag{B.38}
= \frac{a_{ij}}{d} \cdot \frac{1}{2^{d-2} \Gamma(d/2)^2} \int_0^\infty e^{-r^2} r^{2d-2} \, dr \tag{B.39}
= \frac{a_{ij}}{d} \cdot \frac{1}{2^{d-1} \Gamma(d - 0.5)}. \tag{B.40}
\]
Here, (B.38) follows by using that conditional on $\|G\|$, $G_1, \ldots, G_d$ have the same marginal distribution, (B.39) uses (B.34), and (B.40) is a simple integration exercise using the properties of the Gamma integral.

Similarly,

$$
E \left[ \left( e_j^\top G \right) H_d(\|G\|) \cdot e_i^\top A G \right] 
$$

$$
= E \left[ \left( a_{ij} G_j^2 \right) H_d(\|G\|) \right] 
$$

$$
= \frac{a_{ij}}{d} \cdot \frac{1}{2d-2(\Gamma(d/2)^2)} \int_0^\infty \int_0^t e^{-\frac{x^2}{2}+\frac{t^2}{2}} x^{-1} t^{d-2} \, dt \, dx
$$

$$
= \frac{a_{ij}}{d} \cdot 2d/2 \Gamma(d/2) E|G_1|^{d-2} - \frac{a_{ij}}{d} \cdot 2d/2 - 1(\Gamma(d/2))^2 \int_0^\infty \Gamma(d/2, t^2/2) e^{-\frac{t^2}{2}} t^{d-2} \, dt
$$

where

$$
\Gamma(a, b) := \int_b^\infty \exp(-x)x^{a-1} \, dx
$$

for $a, b > 0$ is popularly called the upper incomplete Gamma function. By [77], the incomplete Gamma function in (B.42) can alternatively be written as:

$$
\Gamma(d/2, t^2/2) = e^{-t^2/2} \frac{t^d}{2d/2} \int_0^\infty e^{-u^2/2} (1 + u)^{d/2 - 1/2} \, du.
$$

Using the above identity gives,

$$
\int_0^\infty \Gamma(d/2, t^2/2) e^{-\frac{t^2}{2}} t^{d-2} \, dt
$$

$$
= \frac{\sqrt{2\pi}(2d - 2)!!}{2d+1} \int_1^\infty \frac{u^{d/2-1}}{(1 + u)^{d-0.5}} \, du
$$

$$
= \frac{\sqrt{2\pi}(2d - 2)!!}{2d^2} 2F_1(d - 0.5, d/2 - 0.5; d/2 + 0.5; -1) := C_d
$$

where the last line follows from Kummer’s identity (see [6, Section 2.3]).

Next, by using the integration by parts formula, we have:

$$
\left\| \Sigma^{-\frac{1}{2}} \text{E}_{R_{H_0}} \left[ J(R_{H_0}^\nu(X)) \right. \right. 
\left. \left. \frac{1}{f(X|\theta_0)} \nabla_{\theta} f(X|\theta_0) \right] \right\|^2
$$

$$
= 3d \sum_{j=1}^d \sum_{i=1}^d \left[ \frac{\partial}{\partial X_i} r_j(X) \right]^2
$$

$$
= 3 \Sigma^{-1} \cdot \frac{1}{d} \left[ \frac{1}{2d-1} \cdot \Gamma(d - 0.5) + \frac{\sqrt{2\pi}(d - 1)}{2d^2/2d} \Gamma(d/2) \cdot E[G_1|^{d-2}] - C_d \right]^2
$$

Here (B.45) follows by plugging the expressions obtained in (B.40), (B.42), and (B.43) in (B.37).
Using (B.45) with (B.31) and (B.30) completes the proof of Proposition 3.4 (2) from the main paper.

Proof of Proposition 3.4 (3): Note that in this case the optimal transport map is the same as the function \( \mathbf{R}_{\mathcal{H}_0}^\nu(\cdot) \) defined in part (1). Once again using Remark 3.7 gives,

\[
\left\| \Sigma_{\text{ERD}} \mathbb{E}_{\mathcal{H}_0}[\mathbf{J}(\mathbf{R}_{\mathcal{H}_0}(\mathbf{X})) \frac{\mathbf{1}^T \nabla_{\theta} f(\mathbf{X}|\theta_0)}{f(\mathbf{X}|\theta_0)}] \right\|^2 = \left\| \mathbb{E}_{\mathcal{H}_0}[ \Sigma_\nu^{-\frac{1}{2}}(\mathbf{X} - \theta_0) \mathbf{1}^T \mathbf{A} \Sigma_\nu^{-\frac{1}{2}}(\mathbf{X} - \theta_0) ] \right\|^2,
\]

which is exactly the same as (B.31). Hence, \( \text{ARE}(\mathbf{T}^\nu, \mathbf{T}) = 1. \) \( \square \)

B.5.2. Proof of Theorem 3.4

Part (1): It is easy to see that \( \text{ARE}(\mathbf{T}^\nu, \mathbf{T}) = \infty \) if \( \text{Var}(X_i) = \infty, \) for some \( 1 \leq i \leq d. \) Therefore, we will assume here that \( \text{Var}(\mathbf{X}) = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2), \) where \( 0 < \sigma_i^2 < \infty, \) for \( i \in [d]. \) Write \( \theta_0 := (\theta_{0,1}, \ldots, \theta_{0,d}). \) Then for \( f(\cdot|\theta_0) \in \mathcal{F}_{\text{ind}}, \)

\[
\frac{\mathbf{1}^T \nabla_{\theta} f(\mathbf{x}|\theta_0)}{f(\mathbf{x}|\theta_0)} = - \sum_{i=1}^{d} \frac{d}{f_i(x_i - \theta_{0,i})} \frac{d}{f_i(x_i - \theta_{0,i})}, \tag{B.46}
\]
since \( f(\mathbf{x}|\theta_0) = \prod_{i=1}^{d} f_i(x_i - \theta_{0,i}). \)

To begin with we consider the non-centrality parameter of the Hotelling \( T^2 \) statistic \( T_{m,n} \) under the contiguous alternative. For this, recalling (B.28) and using (B.46), note that,

\[
\left\| \Sigma_\nu^{-\frac{1}{2}} \mathbb{E}_{\mathcal{H}_0}[\mathbf{X} \frac{\mathbf{1}^T \nabla_{\theta} f(\mathbf{X}|\theta_0)}{f(\mathbf{X}|\theta_0)}] \right\|^2 = \sum_{i=1}^{d} \frac{1}{\sigma_i^2} \left( \int x_i \frac{d}{d x_i} f_i(x_i - \theta_{0,i}) \mathbf{d} x_i \right)^2
\]

\[
= \sum_{i=1}^{d} \frac{1}{\sigma_i^2}. \tag{B.47}
\]

Let \( F_i(\cdot - \theta_{0,i}) \) be the cumulative distribution function associated with \( f_i(\cdot - \theta_{0,i}) \) and \( \Phi(\cdot) \) be the cumulative distribution function of a Gaussian random variable. Now, define

\[
\mathbf{R}_{\mathcal{H}_0}^\nu(\mathbf{X}) := (\Phi^{-1} \circ F_1(X_1 - \theta_{0,1}), \ldots, \Phi^{-1} \circ F_d(X_d - \theta_{0,d})).
\]

To see that this is indeed the optimal transport map in this case, note that \( \mathbf{R}_{\mathcal{H}_0}^\nu(\mathbf{X}) \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_d) \) and \( \mathbf{R}_{\mathcal{H}_0}^\nu(\mathbf{X}) \) as defined is the gradient of the following function:

\[
\sum_{i=1}^{d} \int_{-\infty}^{X_i} \Phi^{-1} \circ F_i(t_i - \theta_{i}) \mathbf{d} t_i,
\]

which due to the monotonicity of \( \Phi^{-1} \circ F_i(\cdot - \theta_{i}) \) is a convex function. Therefore, by applying Proposition 2.1 in the main paper, \( \mathbf{R}_{\mathcal{H}_0}^\nu(\mathbf{X}) \) is the required optimal transport map in this case. This implies that

\[
\mathbf{J}(\mathbf{R}_{\mathcal{H}_0}^\nu(\mathbf{X})) = (F_1(X_1 - \theta_{0,1}), \ldots, F_d(X_d - \theta_{0,d})),
\]
as \( \mathbf{J}(\mathbf{x}) = (\Phi(x_1), \ldots, \Phi(x_d)) \) where \( \mathbf{x} = (x_1, \ldots, x_d). \)
Next, we consider the non-centrality parameter of the statistic $T_{m,n}^\nu$ under the contiguous alternative. Here, Remark 3.7 from the main paper, and (B.46) gives,

$$\left\| \Sigma_{\text{ERD}} \mathbb{E}_{H_0} \left[ J(R^\nu_{H_0}(X)) \frac{1}{f(X|\theta_0)} \frac{\nabla_{\theta} f(X|\theta)}{f(X|\theta_0)} \right] \right\|^2 = 12 \sum_{i=1}^d \left( \int F_i(x_i - \theta_{0,i}) \frac{d}{dx_i} f_i(x_i - \theta_{0,i}) \, dx_i \right)^2$$

$$= 12 \sum_{i=1}^d \left( \int f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2,$$

where the last step uses the integration by parts formula.

Now, combining (B.30), (B.47), and (B.48) gives,

$$\inf_{T_{\text{ind}}} \text{ARE}(T^\nu, T) = \inf_{T_{\text{ind}}} \left\{ \sum_{i=1}^d \left( \int f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2 \right\}. \tag{B.49}$$

Next, consider the following optimization problem:

$$\inf_{f_i} \sigma_i^2 \left( \int f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2,$$ \tag{B.50}

such that $\int f_i(x_i - \theta_{0,i}) \, dx_i = 1$. Here $\int f_i(x_i - \theta_{0,i})^2 f_i(x_i - \theta_{0,i}) \, dx_i = \sigma_i^2$. This is precisely the optimization problem that arises in the 1-dimensional case while minimizing the ARE of the Wilcoxon’s test with respect to Student’s $t$-test over location families. In particular, [64, Theorem 1] shows that the infimum in (B.50) is attained by the class of densities in (3.17) (see the main paper) and the minimum value is 9/125. Plugging this in (B.49), we get:

$$\inf_{T_{\text{ind}}} \text{ARE}(T^\nu, T) \geq \inf_{T_{\text{ind}}} \frac{108}{125} \left( \sum_{i=1}^d \left( \frac{1}{\sigma_i^2} \right) \right)^{-1} \left( \sum_{i=1}^d \frac{1}{\sigma_i^2} \right) = 0.864.$$ 

This proves the result in part (1).

**Proof of Theorem 3.4 (2):** Recall that $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and $\phi(\cdot)$ denotes the standard normal density. Similar to part (1), in this case it can be checked that

$$R^\nu_{H_0}(X) = (\Phi^{-1}(F_1(X_1 - \theta_{0,1})), \ldots, \Phi^{-1}(F_d(X_d - \theta_{0,d})))$$

is the optimal transport map. This implies,

$$\left\| \Sigma_{\text{ERD}} \mathbb{E}_{H_0} \left[ J(R^\nu_{H_0}(X)) \frac{1}{f(X|\theta_0)} \frac{\nabla_{\theta} f(X|\theta)}{f(X|\theta_0)} \right] \right\|^2$$

$$= \sum_{i=1}^d \left( \int \Phi^{-1}(F_i(x_i - \theta_{0,i})) \frac{d}{dx_i} f_i(x_i - \theta_{0,i}) \, dx_i \right)^2$$

$$= \sum_{i=1}^d \left( \left( \phi(\Phi^{-1}(F_i(x_i - \theta_{0,i}))) \right)^{-1} f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2, \tag{B.51}$$
where the last step uses integration by parts. Now, as in part (1) combining (B.30), (B.47), and (B.51) gives,

$$\inf_{\mathcal{F}_{\text{ind}}} \text{ARE}(T^\nu, J, T)$$

$$= \inf_{\mathcal{F}_{\text{ind}}} \left( \sum_{i=1}^d \frac{1}{\sigma_i^2} \left( \sum_{i=1}^d \left( \frac{1}{\phi^{-1}((F_i(x_i - \theta_{0,i})))} \right) f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2 \right)$$

$$= 1,$$  \hfill (B.52)

Here (B.52) follows by considering the following optimization problem:

$$\inf_{\sigma_i^2} \sigma_i^2 \left( \phi^{-1}((F_i(x_i - \theta_{0,i})))^{-1} \int f_i^2(x_i - \theta_{0,i}) \, dx_i \right)^2,$$ \hfill (B.53)

under the same constraints as in part (1). This is precisely the optimization problem that arises in the 1-dimensional case [38, Theorem 2.1], where the minimum value is 1 and the minimizing density is that of $\mathcal{N}(0, \sigma^2)$ for any $\sigma > 0$. This completes the proof. \hfill \Box

B.5.3. Proof of Theorem 3.5

We begin by formally defining the class $\mathcal{F}_{\text{ell}}$. Recall that $X$ is said to have an elliptically symmetric distribution if there exists $\theta \in \mathbb{R}^d$, a positive definite $d \times d$ matrix $\Sigma$, and a radial density function $f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that the density $f_1$ of $X$ satisfies:

$$f_1(x) \propto (\det(\Sigma))^{-\frac{1}{2}} f \left( (x - \theta)^\top \Sigma^{-1} (x - \theta) \right).$$ \hfill (B.54)

We denote by $\mathcal{F}_{\text{ell}}$ the class of $d$-dimensional elliptically symmetric distributions satisfying the following standard regularity conditions on the function $f$ (see, for example, [49]):

- $\int_{\mathbb{R}^+} r^{d+1} f(r) \, dr < \infty$,
- $\sqrt{f(\cdot)}$ admits a weak derivative, which is denoted by $(\sqrt{f})'(\cdot)$. This means that

$$\int_{\mathbb{R}^+} \sqrt{f(r)} \psi'(r) \, dr = - \int_{\mathbb{R}^+} (\sqrt{f})'(r) \psi(r) \, dr,$$

for all $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which are compactly supported and infinitely differentiable.
- $\int_{\mathbb{R}^+} r^{d-1} \left[ (\sqrt{f})'(r) \right]^2 \, dr < \infty$.

Proof of Theorem 3.5 (1): Define $\overline{X} := \Sigma^{-\frac{1}{2}}(X - \theta_0)$. It is easy to check that, in this case,

$$\overline{\Sigma} = \mathbb{E}(X - \theta_0)(X - \theta_0)^\top = \left( \frac{1}{d} \mathbb{E} \|X\|^2 \right) \left( \frac{1}{1^\top \Sigma^{-1} 1} \right).$$

Therefore, using the same computation as in (B.31), we get that:

$$\text{ARE}(T^\nu, J, T) \leq \frac{\mathbb{E} \|\overline{X}\|^2}{d} \left\| \sum_{i=1}^d \left[ J(R_{H_0}(\theta)) \frac{1^\top \nabla f(X|\theta_0)}{f(X|\theta_0)} \right] \right\|^2.$$ \hfill (B.55)
By the Lagrange formula, the optimal solution is an element \( \mathbf{v} \) that satisfies the constraints:

\[
\frac{\partial F}{\partial v} - \frac{\partial}{\partial r} \frac{\partial F}{\partial \dot{v}} = 0, \quad \int_0^\infty \dot{v} \, dr = 1, \quad \text{and} \quad \int_0^\infty r^2 \dot{v} \, dr = 1.
\]

Let \( \lambda_1 \) and \( \lambda_2 \) be the Lagrange multipliers associated with the two constraints in (B.58). Define,

\[
F(r, v, \dot{v}) := \left( \dot{v} + \frac{(p-1)v}{r} + \frac{\lambda_1}{r} + \frac{\lambda_2}{r^2} \right) \dot{v}.
\]

By the Lagrange formula, the optimal solution is an element \( v \) that satisfies the constraints:

\[
\frac{\partial F}{\partial v} - \frac{\partial}{\partial r} \frac{\partial F}{\partial \dot{v}} = 0, \quad \int_0^\infty \dot{v} \, dr = 1, \quad \text{and} \quad \int_0^\infty r^2 \dot{v} \, dr = 1.
\]
It is easy to check that \(\frac{\partial F}{\partial v} = \frac{(p-1)\hat{v}}{r}\) and

\[
\frac{\partial}{\partial r} \frac{\partial F}{\partial \hat{v}} = 2\hat{v} + \frac{(p-1)\hat{v}}{r} - \frac{(p-1)v}{r^2} + 2\lambda_2 r.
\]

Therefore the Euler-Lagrange equation becomes:

\[
r^2\hat{v} - \frac{(p-1)v}{2} = -\lambda_2 r^3.
\]

This is the standard non-homogeneous Cauchy-Euler differential equation. In fact, precisely the same equation appears in [54, Equation 21]. Consequently, using the arguments in [54, Page 29-30], the lower bound in Theorem 3.5 (1) follows, with the minimizing radial density function given as:

\[
f(r) = \frac{1}{\sigma} \left( \frac{9\sqrt{3}}{5\sqrt{5}} \left( \frac{\sqrt{2d-1} + 1}{2d-1} \right)^{5/2} \cdot \left( \frac{r}{\sigma} \right)^2 - \frac{3(\sqrt{2d-1} + 1)}{\sqrt{2d-1} - 5} \cdot \left( \frac{3(\sqrt{2d-1} + 1)}{5(\sqrt{2d-1} + 5)} \right)^{(\frac{\sqrt{2d-1}+1}{2})} \cdot \left( \frac{r}{\sigma} \right)^{(\frac{\sqrt{2d-1}-1}{2})} \right) \mathbb{1}_{0 < r < \sigma} \left( \frac{5(\sqrt{2d-1} + 5)}{3(\sqrt{2d-1} + 1)} \right)^{1/2}
\]

for \(d \neq 13\), and

\[
f(r) = \frac{243}{125\sigma^3} \left( \ln \frac{5}{3} - \ln \frac{r}{\sigma} \right) \cdot r^2 \cdot \mathbb{1}_{0 < r < \frac{5\sigma}{3}}
\]

for \(d = 13\), and some \(\sigma > 0\).

**Proof of Theorem 3.5 (2):** By using Lemma B.2, it follows that:

\[
R_{H_0}^\nu(X) = \frac{\Sigma^{-\frac{1}{2}}(X - \theta_0)}{\|\Sigma^{-\frac{1}{2}}(X - \theta_0)\|} \cdot H_d^{-1} \circ \Pi \left( \|\Sigma^{-\frac{1}{2}}(X - \theta_0)\| \right)
\]

is the required optimal transport map. Recall that \(H_d(\cdot)\) is the distribution function of a \(\chi_d^2\) distribution and \(\Pi(\cdot)\) is the distribution function of \(\|\Sigma^{-\frac{1}{2}}(X - \theta_0)\|\). Once again, we write \(R_{H_0}^\nu(X) = (r_1(X), \ldots, r_2(X))\) and set \(\overline{X} = \Sigma^{-\frac{1}{2}}(X - \theta_0)\). Note that (B.37) holds with \(H_d(\cdot)\) replaced with \(H_d^{-1} \circ \Pi(\cdot)\) and \(h_d(\cdot)\) replaced with,

\[
\frac{d}{dr} H_d^{-1}(\Pi(r)) = \frac{\overline{h}(r)}{h_d(H_d^{-1}(\Pi(r)))}.
\]

Using the above observation in (B.37) and (B.44), we get:

\[
\left\| \Sigma^{-\frac{1}{2}} \mathbb{E}_{H_0} \left[ J(R_{H_0}^\nu(X)) \frac{\mathbf{1}^\top \nabla_\theta f(X|\theta) |\theta_0}{f(X | \theta_0)} \right] \right\|^2 = \frac{1}{d^2} \left( \mathbb{E} \left[ \overline{h}(\|\overline{X}\|) \right] + (d - 1) \mathbb{E} \left[ \frac{\Pi(\|\overline{X}\|)}{\|\overline{X}\|} \right] \right)^2.
\]

Plugging the above observation in (B.55) gives,

\[
\inf_{T \in \mathcal{F}} \text{ARE}(T^{\nu,J}, T)
\]
By using the same computation as in (B.31) coupled with (B.61), we get that:

\[
\text{Proof of Theorem 3.6.}
\]

Let \( R \) to get:

\[
\inf_{\mathcal{F}_{\text{erd}}} \frac{\mathbb{E}\|X\|^2}{d^3} \left\{ \mathbb{E} \left[ \frac{\overline{h}(\|X\|)}{h_d(H_d^{-1}(\mathbb{H}(\|X\|)))} \right] + (d-1)\mathbb{E} \left[ \frac{H_d^{-1} \circ \mathbb{H}(\|X\|)}{\|X\|} \right]^2 \right\}.
\] (B.59)

Now, by [99, Theorem 1] (also see [58, Lemma 1]), the following holds:

\[
\left( \mathbb{E} \left[ \frac{\overline{h}(\|X\|)}{h_d(H_d^{-1}(\mathbb{H}(\|X\|)))} \right] + (d-1)\mathbb{E} \left[ \frac{H_d^{-1} \circ \mathbb{H}(\|X\|)}{\|X\|} \right]^2 \right) \geq d^4 (\mathbb{E} [\|X\| H_d^{-1}(\mathbb{H}(\|X\|))]^{-2}).
\] (B.60)

By the Cauchy Schwartz inequality, \( (\mathbb{E} [\|X\| H_d^{-1}(\mathbb{H}(\|X\|))]^{-2}) \leq d\mathbb{E}\|X\|^2 \). Using this observation in (B.60) yields:

\[
\left( \mathbb{E} \left[ \frac{\overline{h}(\|X\|)}{h_d(H_d^{-1}(\mathbb{H}(\|X\|)))} \right] + (d-1)\mathbb{E} \left[ \frac{H_d^{-1} \circ \mathbb{H}(\|X\|)}{\|X\|} \right]^2 \right) \geq d^4 \frac{\mathbb{E}\|X\|^2}{d\mathbb{E}\|X\|^2} = \frac{d^3}{\mathbb{E}\|X\|^2}.
\]

Plugging this observation in (B.59) completes the proof of Theorem 3.5 (2) from the main paper. \( \square \)

\textbf{Proof of Theorem 3.6.} Let \( \mathbb{E}[(W - \mathbb{E}W)(W - \mathbb{E}W)^\top] = \Sigma \). Note that \( \Sigma \) is a diagonal matrix, since \( W \) has independent components. We write \( \Sigma = \text{diag}(\sigma^2_1, \ldots, \sigma^2_d) \). Recall the definition of \( \tilde{\Sigma} \) from (B.29). Note that

\[
\tilde{\Sigma} = A\mathbb{E}[(W - \mathbb{E}W)(W - \mathbb{E}W)^\top]A^\top = A\Sigma A^\top.
\] (B.61)

Define \( R_i := \sum_{j=1}^d a_{ji} \). Observe that

\[
f(x|\theta) := f_1(x - \theta) = \prod_{i=1}^d \tilde{f}_i \left( \sum_{j=1}^d a_{ji} (x_j - \theta_j) \right), \quad \text{for all } x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]

By using the same computation as in (B.31) coupled with (B.61), we get that:

\[
\text{ARE}(T^{\nu,J}, T) = \frac{\left\| \sum_{i=1}^d R_i^{-\frac{3}{2}} \mathbb{E}_{H_0} \left[ J(R_{H_0}^\nu(X)) \frac{1^\top \nabla \theta f(X|\theta)|_{\theta=0}}{f(X|\theta=0)} \right] \right\|^2}{\sum_{i=1}^d R_i^2 / \sigma_i^2}.
\] (B.62)

With \( J(x) = x, \nu \) as the standard Gaussian distribution, it is easy to see that \( \Sigma_{\text{erd}} = I_d \).

Next observe that

\[
1^\top \nabla \theta f(X|\theta)|_{\theta=0} = \sum_{i=1}^d R_i \tilde{f}_i (\sum_{j=1}^d a_{ji} x_j) / \tilde{f}_i (\sum_{j=1}^d a_{ji} x_j).
\] (B.63)

We now compute \( R_{H_0}^\nu(X) \). Let \( \tilde{F}_i(\cdot) \) be the distribution function of \( W_i \), for \( 1 \leq i \leq d \). As \( (W_1, \ldots, W_d) \) has independent components, \( A \) is orthogonal, we can use [39, Lemma A.8] to get:

\[
R_{H_0}^\nu(X) = A v(X),
\] (B.64)
where \( v(X) := (v_1(X), \ldots, v_d(X)) \), with \( v_j(X) := \Phi^{-1} \circ \tilde{F}_j \left( \sum_{k=1}^{d} a_{jk} X_j \right) \) and \( \Phi(\cdot) \) the standard normal cumulative distribution function.

Next, by using (B.63) and (B.64), we have:

\[
\left\| \sum_{\text{ERD}} \mathbb{E}_{H_0} \left[ J(R_{H_0}^\nu(X)) \frac{1^T \nabla \theta f(X|\theta)|_{\theta=0}}{f(X|\theta=0)} \right] \right\|^2 \\
= \sum_{i=1}^{d} \left( \mathbb{E} \left[ \left( \sum_{j=1}^{d} R_{ij} \frac{\tilde{f}_j(W_j)}{f_j(W_j)} \right) \cdot \left( \sum_{j=1}^{d} a_{ij} \Phi^{-1} \circ \tilde{F}_j(W_j) \right) \right] \right)^2 \\
= \sum_{i=1}^{d} \left( \sum_{j=1}^{d} a_{ij} R_{ij} \Phi^{-1} \circ \tilde{F}_j(W_j) \frac{\tilde{f}_j(W_j)}{f_j(W_j)} \right)^2 \\
= \sum_{j=1}^{d} R_{jj}^2 \left( \mathbb{E} \left[ \Phi^{-1} \circ \tilde{F}_j(W_j) \frac{\tilde{f}_j(W_j)}{f_j(W_j)} \right] \right)^2 .
\]

By using the same argument as in (B.52) and right after (B.53), we further have:

\[
\sum_{j=1}^{d} R_{jj}^2 \left( \mathbb{E} \left[ \Phi^{-1} \circ \tilde{F}_j(W_j) \frac{\tilde{f}_j(W_j)}{f_j(W_j)} \right] \right)^2 \geq \sum_{j=1}^{d} \frac{R_{jj}^2}{\sigma_j^2}.
\]

Plugging the above observation into (B.62) completes the proof. □

### B.6. Proofs from Section 4.1

**Proof of Theorem 4.1.** Let \( R_1(\cdot) \) and \( R_2(\cdot) \) be the optimal transport maps from the distribution of \( X_1 \) and \( Y_1 \) to \( v_1 \) and \( v_2 \), respectively. Define \( \hat{J}_{1\text{or}} := \frac{1}{n} \sum_{i=1}^{n} J_1(R_1(X_i)) \), \( \hat{J}_{2\text{or}} := \frac{1}{n} \sum_{i=1}^{n} J_2(R_2(Y_i)) \), and

\[
R_{n,\text{sc}}^{\text{rank,or}} := \left\| \left( \Sigma_{\text{ERD}\otimes \Sigma_{\text{ERD}}} \right)^{\frac{1}{2}} \text{vec} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (J_1(R_1(X_i)) - \hat{J}_{1\text{or}})(J_2(\hat{R}_2(Y_i)) - \hat{J}_{2\text{or}}) \right)^\top \right\|^2 .
\] (B.65)

Note that \( R_{n,\text{sc}}^{\text{rank,or}} \) is obtained by replacing the empirical rank maps \( \hat{R}_1(\cdot) \) and \( \hat{R}_2(\cdot) \) in (4.2) (see the main paper with their population counterparts \( R_1(\cdot) \) and \( R_2(\cdot) \). The proof of Theorem 4.1 in the main paper now proceeds in two steps:

- In the first step we will show that, under \( H_0 \),

\[
| R_{n,\text{sc}}^{\text{rank}} - R_{n,\text{sc}}^{\text{rank,or}} | \xrightarrow{P} 0. \quad (B.66)
\]

- Next, we will show that, under \( H_0 \),

\[
R_{n,\text{sc}}^{\text{rank,or}} \xrightarrow{w} \chi_{d_1 d_2}^2 . \quad (B.67)
\]

Combining (B.66) and (B.67) with Slutsky’s theorem completes the proof of Theorem 4.1.

We begin with the proof of (B.67). For this, let \( m_1 := \mathbb{E} J_1(R_1(X_1)) \), \( m_2 := \mathbb{E} J_2(R_2(Y_1)) \) and observe that,

\[
R_{n,\text{sc}}^{\text{rank,or}} = \left\| \left( \Sigma_{\text{ERD}\otimes \Sigma_{\text{ERD}}} \right)^{\frac{1}{2}} \text{vec} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (J_1(R_1(X_i)) - m_1)(J_2(\hat{R}_2(Y_i)) - m_2) \right)^\top \right\|^2 .
\]
Note that the first term in the RHS of the above display converges weakly to $\chi^2_{d_1d_2}$ by combining the multivariate central limit theorem with the continuous mapping theorem. This completes the proof of (B.67).

To prove (B.66), note that it suffices to prove that:

$$E \left\| \operatorname{vec} \left( \sum_{i=1}^{n} \hat{S}_X, \hat{S}_Y^\top - \sum_{i=1}^{n} S_X, S_Y^\top \right) \right\|^2 = o(n), \tag{B.69}$$

where, for $i \in [n],$

$$\hat{S}_X := J_1(\hat{R}_1(X_i)) - \bar{J}_1, \quad \hat{S}_Y := J_2(\hat{R}_2(Y_i)) - \bar{J}_2,$$

and

$$S_X := J_1(R_1(X_i)) - \bar{J}_1, \quad S_Y := J_2(R_2(Y_i)) - \bar{J}_2.$$

To prove (B.69), let $S_n$ denote the set of all permutations of the set $\{1, 2, \ldots, n\}$. Also, suppose $\sigma$ is a random permutation sampled uniformly over $S_n$ and independently of $(X_1, Y_1), \ldots, (X_n, Y_n)$. It is easy to see that, under $H_0$, we have:

$$(X_1, Y_1), \ldots, (X_n, Y_n) \overset{D}{=} (X_1, Y_{\sigma(1)}), \ldots, (X_n, Y_{\sigma(n)}).$$

Let $X_n := \{X_1, \ldots, X_n\}$ and $Y_n := \{Y_1, \ldots, Y_n\}$ be the unordered sets of observations. Denote by $E_{Z_n}$ the expectation conditional on $Z_n := (X_n, Y_n)$. Based on this notation, the LHS of (B.69) can be written as:

$$E \left\| \operatorname{vec} \left( \sum_{i=1}^{n} \hat{S}_X, \hat{S}_Y^\top - \sum_{i=1}^{n} S_X, S_Y^\top \right) \right\|^2 = T_1 + T_2 - 2T_3, \tag{B.70}$$

where

$$T_1 := \mathbb{E} \operatorname{Tr} \left( E_{Z_n} \left[ \sum_{1 \leq i, j \leq n} \hat{S}_X, \hat{S}_Y^\top, \hat{S}_Y, \hat{S}_X^\top \right] \right),$$

$$T_2 := \mathbb{E} \operatorname{Tr} \left( E_{Z_n} \left[ \sum_{1 \leq i, j \leq n} S_X, S_Y^\top, S_Y, S_X^\top \right] \right),$$

$$T_3 := \mathbb{E} \operatorname{Tr} \left( E_{Z_n} \left[ \sum_{1 \leq i, j \leq n} \hat{S}_X, \hat{S}_Y^\top, S_Y, S_X^\top \right] \right),$$

with $\operatorname{Tr}(\cdot)$ denoting the trace of a matrix.

We will focus on $T_3$. The analysis for the other two terms will follow along similar lines. Towards this direction, note that:

$$\mathbb{E}_{Z_n} \left[ \sum_{1 \leq i, j \leq n} \hat{S}_X, \hat{S}_Y^\top, S_Y, S_X^\top \right]$$
Proof of Theorem 4.2. Consider the testing problem (4.6) under the Konijn alternatives as in Definition 4.1 (both from the main paper). Also, let the Lebesgue densities associated with \( \mu_1 \) and \( \mu_2 \) be \( f_1(\cdot) \) and \( f_2(\cdot) \), respectively. In this setting, by [40, Lemma 3.2.1], the sequence of joint distributions of \( (X_1, Y_1), \ldots, (X_n, Y_n) \) under \( H_1 \) and \( H_0 \) are contiguous to each other. In fact, by defining the joint density of \( (X_1, Y_1) \) under \( H_1 \) as \( f_{n, \delta}(\cdot, \cdot) \), it follows from the same lemma that,

\[
L_{n, \delta} := \sum_{i=1}^{n} \log \frac{f_{n, \delta}(X_i, Y_i)}{f_{n, 0, \delta}(X_i, Y_i)} = \frac{\delta}{\sqrt{n}} \sum_{i=1}^{n} \ell(X_i, Y_i) - \frac{\delta^2}{2} \text{Var}(\ell(X_1, Y_1)) + o_P(1),
\]

(B.71)

where

\[
\ell(x, y) := d_1 + d_2 - \frac{x^\top \nabla f_1(x)}{f_1(x)} - \frac{y^\top \nabla f_2(y)}{f_2(y)} + \frac{x^\top M \nabla f_2(y)}{f_2(y)} + \frac{y^\top M \nabla f_1(x)}{f_1(x)}.
\]

Let \( R_1(\cdot) \) and \( R_2(\cdot) \) denote the optimal transport maps (in the sense of Proposition 2.1 from the main paper) from \( \mu_1 \) and \( \mu_2 \) (the marginal distributions of \( X'_1 \) and \( Y'_1 \) to the reference
distributions \( \nu_1 \) and \( \nu_2 \). Then, by a standard application of the multivariate central limit theorem, we have: under \( H_0 \),
\[
\left( \vec{\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (J_1(R_1(X_i)) \right.} - \bar{J}_1, (J_2(R_2(Y_i)) - \bar{J}_2) \right) \right) \xrightarrow{w} N(\kappa, \Gamma), \tag{B.72}
\]
where
\[
\kappa := \left( \frac{d_1^2}{2} \text{Var}(\ell(X_1, Y_1)) \right) \quad \text{and} \quad \Gamma := \left( \begin{array}{cc} \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} & \delta^2 \gamma_{d_1d_2} \\ \delta \gamma_{d_1d_2} & \delta^2 \text{Var}(\ell(X_1, Y_1)) \end{array} \right),
\]
\[
\gamma_{d_1d_2} := g(X_1', Y_1') \cdot \vec{\left( \text{Var}(J_1(R_1(X_1')) - m_1) (J_2(R_2(Y_1')) - m_2) \right)} \tag{B.73},
\]
with
\[
g(X_1', Y_1') := \frac{(X_1')\top M\gamma f_2(Y_1')}{f_2(Y_1')} + \frac{(Y_1')\top M\gamma f_1(X_1')}{f_1(X_1')},
\]
\[
m_1 := \mathbb{E}[J_1(R_1(X_1'))], \quad \text{and} \quad m_2 := \mathbb{E}[J_2(R_2(Y_1'))].
\]
Now, recall the definition of \( R_{n,sc}^{\text{rank},or} \) from (B.65). Then, by using (B.72) with Le Cam’s third lemma [78], under \( H_1 \) as in (4.6) from the main paper, the following holds:
\[
R_{n,sc}^{\text{rank},or} \xrightarrow{w} \chi^2_{d_1d_2} \left( \delta^2 \gamma_{d_1d_2} \left( \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} \right)^{-1} \gamma_{d_1d_2} \right). \tag{B.74}
\]
Note that RHS of (B.74) denotes the \( \chi^2 \) distribution with \( d_1d_2 \) degrees of freedom and non-centrality parameter
\[
\delta^2 \gamma_{d_1d_2} \left( \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} \right)^{-1} \gamma_{d_1d_2}.
\]
This implies, using (B.66), (B.74) and contiguity, that
\[
R_{n,sc}^{\text{rank}} \xrightarrow{w} \chi^2_{d_1d_2} \left( \delta^2 \gamma_{d_1d_2} \left( \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} \right)^{-1} \gamma_{d_1d_2} \right), \tag{B.75}
\]
under \( H_1 \).

Next, recall the Wilks’ test defined in (4.4) (see the main paper) using the variable \( R_n \). From [127], we have,
\[
R_n \xrightarrow{w} \chi^2_{d_1d_2} \left( 4\delta^2 \parallel \text{vec}(M) \parallel^2 \right), \tag{B.76}
\]
under \( H_1 \). Therefore, using (B.75) and (B.76),
\[
\text{ARE}(R^{\mu,J}, R) = \frac{\gamma_{d_1d_2} \left( \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} \right)^{-1} \gamma_{d_1d_2}}{4\parallel \text{vec}(M) \parallel^2}. \tag{B.77}
\]
Note that the expression of \( \text{ARE}(R^{\mu,J}, R) \) in (B.77) holds in general without restricting to \( \mu_1, \mu_2 \in \mathcal{F}_{\text{ind}} \) or \( \mathcal{F}_{\text{ell}} \). We now provide lower bounds to \( \text{ARE}(R^{\mu,J}, R) \) obtained above. Recall that the theorem specifies both our ERDs to be standard Gaussian. Therefore, \( \Sigma_{ERD}^{(1)} = I_{d_1}, \Sigma_{ERD}^{(2)} = I_{d_2} \) which implies \( \Sigma_{ERD}^{(1)} \otimes \Sigma_{ERD}^{(2)} = I_{d_1d_2}, m_1 = 0_{d_1}, \text{ and } m_2 = 0_{d_2}. \) By symmetry,
it suffices to consider the following three cases: (1) \( \mu_1, \mu_2 \in F_{\text{ind}} \), (2) \( \mu_1, \mu_2 \in F_{\text{ell}} \), and (3) \( \mu_1 \in F_{\text{ind}}, \mu_2 \in F_{\text{ell}} \).

**Case (1):** Denote by marginal distribution functions of \( X'_1 \) and \( Y'_1 \) by \( F(\cdot) \) and \( G(\cdot) \), respectively. As \( \mu_1 \) has independent components, \( F(\cdot) \) has the following form:

\[
F(x_1, \ldots, x_{d_1}) = \prod_{i=1}^{d_1} F_i(x_i),
\]

where \( F_i(\cdot), F_1(\cdot), \ldots, F_d(\cdot) \) are univariate cumulative distribution functions. Let \( f_i(\cdot) \) denote the probability density function associated with \( F_i(\cdot) \), for \( i \in [d] \). Similarly, let \( G_1(\cdot), G_2(\cdot), \ldots, G_d(\cdot) \) and \( g_1(\cdot), g_2(\cdot), \ldots, g_d(\cdot) \) be the distribution and density functions associated with the components of \( Y'_1 \), respectively. (Note that \( Y'_1 \) also has independent components by assumption.) Write \( X'_1 = (X'_{1,1}, \ldots, X'_{1,d_1}) \) and \( Y'_1 = (Y'_{1,1}, \ldots, Y'_{1,d_1}) \). Using this notation and the same argument as in the proof of Theorem 3.4 from the main paper, shows that

\[
R_1(X'_1) = (\Phi^{-1}(F_1(X'_{1,1})), \ldots, \Phi^{-1}(F_{d_1}(X'_{1,d_1}))),
\]

\[
R_2(Y'_1) = (\Phi^{-1}(G_1(Y'_{1,1})), \ldots, \Phi^{-1}(G_{d_2}(Y'_{2,d_2}))),
\]

(B.78)

are the required optimal transport maps, where \( \Phi(\cdot) \) is the standard Gaussian cumulative distribution function. Next, recalling (B.73) from the main paper and noting that in this case \( m_1 = 0_{d_1}, m_2 = 0_{d_2} \) gives,

\[
\gamma_{d_1,d_2} = \text{vec} \left( \mathbb{E} \left( R_1(X'_1) R_2(Y'_1) \right) \right) = \Phi^{-1}(F_{k}(X'_{1,k})) \dagger \mathbb{E} \left( \nabla f_1(X'_1) \Phi^{-1}(F_k(X'_{1,k})) \right) = e_k \int (\Phi^{-1}(F_k(x))) f_1^2(x) \, dx.
\]

(B.79)

Now, for \( k \in [d_1] \), note that by the integration by parts formula,

\[
\mathbb{E} \left[ \nabla f_1(X'_1) \Phi^{-1}(F_k(X'_{1,k})) \right] = e_k \int (\Phi^{-1}(F_k(x))) f_1^2(x) \, dx.
\]

(B.80)

Define,

\[
C_{1,k} := \mathbb{E} \left( \Phi^{-1}(F_k(X'_{1,k})) (X'_{1,k}) \right), \quad D_{1,k} := e_k \int (\Phi^{-1}(F_k(x))) f_1^2(x) \, dx,
\]

(B.81)

and, similarly, \( C_{2,\ell}, D_{2,\ell} \), for \( \ell \in [d_2] \) with the \( k \)-th entry of \( X'_1 \) replaced by the \( \ell \)-th entry of \( Y'_1 \). Note that under Assumption 4.1 in the main paper, for \( k \in [d_1] \) and \( \ell \in [d_2] \),

\[
\mathbb{E} \left( \Phi^{-1}(F_k(X'_{1,k}))(X'_{1,k}) \right) = C_{1,k} e_k^{\top} \quad \text{and} \quad \mathbb{E} \left( \Phi^{-1}(G_\ell(Y'_{1,\ell}))(Y'_{1,\ell}) \right) = D_{1,\ell} e_\ell^{\top}.
\]

(B.82)

Denoting the matrix \( M := ((m_{k\ell}))_{k\in[d_1],\ell\in[d_2]} \) and combining (B.78), (B.79), (B.80), (B.81), and (B.82) gives,

\[
\|\gamma_{d_1,d_2}\|^2 = \sum_{k=1}^{d_1} \sum_{\ell=1}^{d_2} m_{k\ell}^2 (C_{1,k} D_{2,\ell} + D_{1,k} C_{2,\ell})^2 \\
\geq 4 \sum_{k=1}^{d_1} \sum_{\ell=1}^{d_2} m_{k\ell}^2 (C_{1,k} D_{1,k} + C_{2,\ell} D_{2,\ell}),
\]

(B.83)
where the last step uses using the elementary inequality \((a + b)^2 \geq 4ab\), for \(a, b \in \mathbb{R}\).

Observe that the lower bound in (B.83) has effectively decoupled as the product of two quantities, the first of which only depends on the distribution of the \(k\)-th component of \(X'_1\) while the second one only depends on the distribution of the \(\ell\)-th component of \(Y'_1\). Now, following the proof of [38, Theorem 2.1] shows, \(C_{1,k}D_{1,k} \geq 1\), for any distribution \(F_k\) (the cumulative distribution function of \(X'_{1,k}\)), and similarly, \(C_{2,\ell}D_{2,\ell} \geq 1\) for any distribution \(G_\ell\) (the cumulative distribution function of \(Y'_{1,\ell}\)). Moreover, equality holds if and only if both \(X'_{1,k}\) and \(Y'_{1,\ell}\) have standard normal distribution. This observation together with (B.83) in (B.77), shows that

\[
\text{ARE}(R'^{\nu}, R) = \frac{4\sum_{k=1}^{d_1} \sum_{\ell=1}^{d_2} m_{k\ell}^2}{4\|\text{vec}(M')\|^2} = 1.
\]

This completes the proof for case (1).

**Case 2:** Recall that \(f_1\) and \(f_2\) denote the probability density functions of \(X'_1\) and \(Y'_1\), respectively. Since, \(\mu_1, \mu_2 \in \mathcal{F}_\text{ell}\) in this case and Assumption 4.1 (see the main paper) is satisfied, we can assume without loss of generality that \(f_1\) is proportional to \(f_1'\) and \(f_2\) is proportional to \(f_2'\), for some radial density functions \(f_1\) and \(f_2\). Also, recall that \(H_{d_1}()\) and \(H_{d_2}()\) are the cumulative distribution functions of \(\sqrt{\chi^2_d}\) and \(\sqrt{\chi^2_{d_2}}\) distributions, respectively (as defined in the proof of Proposition 3.4 from the main paper).

Now, using the same argument as in Lemma B.2 we get,

\[
R_1(X'_1) = \frac{X'_1}{\|X'_1\|}H_{d_1}^{-1}(\tilde{\Psi}_1(\|X'_1\|)) \quad \text{and} \quad R_2(Y'_1) = \frac{Y'_1}{\|Y'_1\|}H_{d_2}^{-1}(\tilde{\Psi}_1(\|Y'_1\|)),
\]

where \(\tilde{\Psi}_1()\) and \(\tilde{\Psi}_2()\) are the distribution functions of \(\|X'_1\|\) and \(\|Y'_1\|\), respectively. For \(k \in [d_1]\), define

\[
C_{1,k} := \mathbb{E}[(R_1(X'_1))_k(X'_1)] = \mathbb{E}\left[\frac{\|X'_1\|H_{d_1}^{-1}(\tilde{\Psi}_1(\|X'_1\|))}{C_1}\right] e_k \quad \text{and} \quad d_1
\]

and

\[
D_{1,k} := \mathbb{E}\left[\frac{\nabla f_1(X'_1)}{f_1(X'_1)}(R_1(X'_1))_k\right] = \mathbb{E}\left[\frac{f'_1}{f_1}(\|X'_1\|)H_{d_1}^{-1}(\tilde{\Psi}_1(\|X'_1\|))\right] e_k \quad \text{and} \quad d_2.
\]

Similarly, for \(\ell \in [d_2]\), define \(C_{2,\ell}, D_{2,\ell}, C_2\) and \(D_2\) with the \(k\)-th element of \(X'_1\) replaced by the \(\ell\)-th element of \(Y'_1\). Now, using the same steps as in (B.83), we get:

\[
\|d_{d_1d_2}\|^2 = \sum_{k,\ell}(C_{1,k}MD_{2,\ell} + D_{1,k}MC_{2,\ell})^2 \geq \frac{4}{d_1d_2}(C_1D_1)(C_2D_2) \sum_{k=1}^{d_1} \sum_{\ell=1}^{d_2} m_{k\ell}^2. \quad \text{(B.84)}
\]

Note that the once again the lower bound in (B.84) has decoupled into two separate problems, one involving the distribution of \(X'_1\) and the other involving the distribution of \(Y'_1\).
Now, by [99, Theorem 1], \( C_1D_1 \geq d_1^2 \) and \( C_2D_2 \geq d_2^2 \), where equality holds if and only if both \( X'_1 \) and \( Y'_1 \) have standard normal distributions of appropriate dimensions. Using this observation and (B.84) in (B.77) gives,
\[
\text{ARE}(R^{\text{or}J}, R) = \frac{\|\gamma_{d_1}d_2\|}{4\|\text{vec}(M)\|^2} \geq \frac{4d_1^2d_2^2}{4d_1^2d_2^2\|\text{vec}(M)\|^2} = 1,
\]
which completes the proof for case (2).

**Case (3):** When \( \mu_1 \in \mathcal{F}_{\text{ind}} \) and \( \mu_2 \in \mathcal{F}_{\text{ell}} \), the proof proceeds exactly similar to the above two cases. In particular, we can get a similar lower bound to those obtained in (B.83) and (B.84), which will again decouple into two separate problems, one involving \( \mu_1 \) and the other involving \( \mu_2 \). We can then separately optimize over \( \mu_1 \in \mathcal{F}_{\text{ind}} \) as we did in case (1) and \( \mu_2 \in \mathcal{F}_{\text{ell}} \) as we did in case (2), to complete the proof. The details are omitted. \( \square \)

**B.7. Proof of Proposition 4.3**

Note that, an application of the triangle inequality followed by the Cauchy-Schwarz inequality gives,
\[
\frac{1}{n^2} \left| \sum_{i,j} \left( \Delta_{i,j}^{(1)} \Delta_{i,j}^{(2)} - \Delta_{i,j}^{(1),\text{or}} \Delta_{i,j}^{(2),\text{or}} \right) \right| \leq \sqrt{Q_1T_2} + \sqrt{Q_2T_1} \tag{B.85}
\]
where
\[
Q_1 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(1)} - \Delta_{i,j}^{(1),\text{or}})^2, \quad Q_2 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(2)} - \Delta_{i,j}^{(2),\text{or}})^2, \quad T_1 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(1),\text{or}})^2, \quad T_2 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(2)})^2.
\]
and
\( T_1 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(1),\text{or}})^2 \), \( T_2 := \frac{1}{n^2} \sum_{i,j} (\Delta_{i,j}^{(2)})^2 \). Note that \( T_1 = O_P(1) \) and \( T_2 = O_P(1) \) by Assumption 3.1. Moreover, by using Theorem 2.1 from the main paper with \( p = r = 2 \), \( q = 1 \), \( \mathcal{F}(x_1, x_2) = ||x_1 - x_2|| \), we get \( Q_1 \xrightarrow{P} 0 \) and \( Q_2 \xrightarrow{P} 0 \). This implies, the LHS of (B.85) converges to 0 in probability. Consequently, the weak law of large numbers for V-statistics gives,
\[
\frac{1}{n^2} \sum_{i,j} \Delta_{i,j}^{(1)} \Delta_{i,j}^{(2)} \xrightarrow{P} \mathbb{E} \left[ \Delta_{1,2}^{(1),\text{or}} \Delta_{1,2}^{(2),\text{or}} \right].
\]
Using similar computations we can find the weak limits of the other two terms in the definition of RdCov2 in (4.9) to establish (4.10) (both equations are from the main paper).

By using [126, Theorem 3(i)], the right hand side of (4.10) converges to 0 in probability if and only if \( J_1(R_1(X_1)) \) and \( J_2(R_2(Y_1)) \) are independent. Further, as \( J_1(\cdot), J_2(\cdot), R_1(\cdot), R_2(\cdot) \) are injective (in the sense of Proposition 2.1), \( J_1(R_1(X_1)) \) and \( J_2(R_2(Y_1)) \) are independent if and only if \( X_1 \) and \( Y_1 \) are independent.

**Appendix C: Lower Bounds**

This section is devoted to proving lower bounds for the testing problems (3.12) and (3.13) described in Section 3.2 (all from the main paper). In other words, we show that for both these problems, the power of any level \( \alpha \) test function is upper bounded by \( \alpha + \varepsilon \), for any given \( \varepsilon > 0 \) and for all large enough \( m, n \). This is formalized in the following proposition for the hypothesis (3.12). The proof for (3.13) is similar.
**Proposition C.1** (Lower bound in testing). Fix any $\varepsilon > 0$ and let $T_{m,n}^\alpha$ be the set of all level $\alpha$ test functions based on $Z_N$. Then, provided $\alpha + \varepsilon < 1$, there exists $h_\varepsilon > 0$ such that, for all $m, n$ large enough,

$$\inf_{T_\alpha \in T_{m,n}^\alpha} \sup_{|h| \geq h_\varepsilon} \mathbb{P}_{H_1}(T_\alpha = 0) \geq 1 - \alpha - \varepsilon,$$

where $H_1$ is specified as in (3.12) from the main paper, with $h = h_\varepsilon$.

**Remark C.1** (Rate-optimality of $T_{m,n}^{\nu, J}$). Recall that in Theorem 3.3 (see the main paper) we show, for the testing problems in (3.12) and (3.13), $T_{m,n}^{\nu, J}$ has a non-trivial power $\in (\alpha, 1)$ under $H_1$. This combined with Proposition C.1 above, shows the rate-optimality of the test based on $T_{m,n}^{\nu, J}$.

**Proof of Proposition C.1.** The proof is a standard application of the connection between minimax lower bounds and total variation distance between probability measures (see, for example, [46, Chapter 6]). Towards this, set $P^{(N)} := \mu_1^{(N)}$ and $Q^{(N)} := \mu_2^{(m)} \otimes \mu_2^{(n)}$, where $\mu_2$ is as specified under $H_1$ in (3.14) (see the main paper). Let $TV(P^{(N)}, Q^{(N)})$ denote the total variation distance between $P^{(N)}$ and $Q^{(N)}$, and $HD(P^{(N)}, Q^{(N)})$ denote the Hellinger distance between $P^{(N)}$ and $Q^{(N)}$. It suffices to show that given any $\varepsilon > 0$, $TV(P^{(N)}, Q^{(N)}) \leq \varepsilon$, for all large enough $m, n$. In fact, since [129, Equation 2.20], $TV(P^{(N)}, Q^{(N)}) \leq HD(P^{(N)}, Q^{(N)})$, it suffices to show $HD(P^{(N)}, Q^{(N)}) \leq \varepsilon$, for all large enough $m, n$. To this end, following [129, Page 83], we have

$$1 - \frac{HD^2(P^{(N)}, Q^{(N)})}{2} = \mathbb{E}_{\theta_0} \left[ \prod_{i=1}^{n} \sqrt{\frac{f(X_i | \theta_0) + h \mathbf{1}/\sqrt{N}}{f(X_i | \theta_0)}} \right]. \quad (C.1)$$

Now, using (B.22) gives,

$$\prod_{i=1}^{n} \sqrt{\frac{f(X_i | \theta_0) + h \mathbf{1}/\sqrt{N}}{f(X_i | \theta_0)}} \xrightarrow{w} \exp \left( -\frac{h^2 (1 - \lambda)}{4} \cdot \mathbf{1}^\top \mathbf{1} (\theta_0) \mathbf{1} + \frac{h \sqrt{(1 - \lambda)} \mathbf{1}^\top \mathbf{1} (\theta_0) \mathbf{1}}{2} G \right),$$

where $G \sim \mathcal{N}(0, 1)$. Observe that

$$\mathbb{E}_{\theta_0} \left[ \left( \prod_{i=1}^{n} \sqrt{\frac{f(X_i | \theta_0) + h \mathbf{1}/\sqrt{N}}{f(X_i | \theta_0)}} \right)^2 \right] = 1.$$

Therefore, using uniform integrability and (C.1) gives,

$$1 - \frac{HD^2(P^{(N)}, Q^{(N)})}{2} \rightarrow \exp \left( -\frac{h^2 (1 - \lambda)}{8} \cdot \mathbf{1}^\top \mathbf{1} (\theta_0) \mathbf{1} \right).$$

Hence, one can choose $h_\varepsilon > 0$ such that when $h \geq h_\varepsilon$, then $HD(P^{(N)}, Q^{(N)}) \leq \varepsilon$, for all large enough $m, n$. \(\square\)
Appendix D: Simulations

In this section, we will illustrate our theoretical findings through numerical experiments. The section is organized as follows: In Appendix D.1 we use numerical experiments to demonstrate the multivariate Hodges-Lehmann phenomenon and the multivariate Chernoff-Savage phenomenon which we discussed after Theorem 3.4 in the main paper. In Appendix D.2 we compare the finite sample power of $T_{m,n}^{\nu,J}$ to Hotelling $T^2$.

D.1. Numerical illustration of Hodges-Lehmann and Chernoff-Savage type results

![Graphs showing empirical power with adjusted sample size for different distributions and test types](image)

**Figure 2:** In the left panel, we sample $X_1$ and $Y_1$ according to setting (H1). We plot the power curve for $T_{m,n}^{\nu,J}$ with $\nu = \nu_U$ with sample sizes $m = n$ varying in [100, 1100] (in blue), the power curve for Hotelling $T^2$ with the number of samples $m, n$ replaced by $\approx 0.864m, 0.864n$ (in black), and the power curve for $T_{m,n}^{\nu,J}$ with $\nu = \nu_G$ with the same number of samples as used for Hotelling $T^2$ (in red). In the right panel, we do the same but with $X_1, Y_1$ sampled according to setting (H2).

In this section, we use numerical experiments to demonstrate the Hodges-Lehmann and Chernoff-Savage type behavior theoretically observed in Theorem 3.4 in the main paper. We do so with the following simulation settings:

(H1) $X_1$ follows a bivariate Epanechnikov distribution (see Theorem 3.4) with independent components and location parameter $0$, and $Y_1$ has the same distribution with location parameter $0.1 \cdot 1_2$.

(H2) $X_1$ follows a bivariate standard normal distribution with location parameter $0$, and $Y_1$ has the same distribution with location parameter $0.1 \cdot 1_2$.

For each of these settings, we compare the power curves of the Hotelling $T^2$ test versus tests based on $T_{m,n}^{\nu,J}$, where we use $\nu \equiv \nu_U \equiv \text{Unif}[0,1]^d$ and $J(x) = x$, or $\nu \equiv \nu_G \equiv N_d(0,I_d)$ and $J(x) = x$. These will be referred to as the RankUniform and RankGaussian versions of the test based on $T_{m,n}^{\nu,J}$, respectively. In the sequel, we will not repeat the choice of the score function as it is set to the identity map in both cases. Each of the tests are carried
out at level 0.05 and the power curves are plotted as the sample size $m = n$ varies in the interval [100, 1100]. To obtain the power curves, we used 500 independent replications.

To understand how Figure 2 should be interpreted in the aforementioned settings, it is instructive to recall the informal/intuitive understanding of ARE which we presented in Section 1.1.3 in the main paper, namely:

The ARE of $T_1$ relative to $T_2$ is the ratio of the number of samples needed to attain the same power when using the test $T_2$ compared to the same for test $T_1$.

In other words, if the ARE of $T_1$ with respect to $T_2$ is 0.9, intuitively it means that the power of $T_1$ with $n$ samples and that of $T_2$ with $\approx 0.9n$ should be similar, at least for large $n$.

We now look at Figure 2 in light of the above discussion. In the left panel of Figure 2, we focus on setting (H1). Note that, by Theorem 3.4 in the main paper, the ARE of $T_{m,n}^{\nu_U, J}$ with $\nu = \nu_U$ against Hotelling $T^2$ is 0.864 under setting (H1). Therefore we plot the power curve for $T_{m,n}^{\nu_U, J}$ with $\nu = \nu_U$ with $m = n \in [100, 1100]$ samples and the power curve for Hotelling $T^2$ with the number of samples $m,n$ replaced by $\approx 0.864m, 0.864n$. As per the aforementioned heuristic understanding of ARE, these two power curves should be fairly close, specially for large $n$. This is exactly what we observe in the left panel of Figure 2. The black and blue lines (for Hotelling $T^2$ and $T_{m,n}^{\nu_U, J}$ respectively) are very close for the entire spectrum of sample sizes considered in the left panel of Figure 2. In the same plot, we also show the power curve of $T_{m,n}^{\nu_G, J}$ with $\nu = \nu_G$ where the sample sizes $m, n$ in this case were chosen in the same way as that for Hotelling $T^2$ test. By Theorem 3.4 in the main paper, the ARE against Hotelling $T^2$ in this case is larger than 1. Therefore one would expect $T_{m,n}^{\nu_G, J}$ with $\nu = \nu_G$ to have better power in this setting which is exactly what we observe in the left panel of Figure 2. The power curve of $T_{m,n}^{\nu_G, J}$ with $\nu = \nu_G$ is significantly higher than that of Hotelling $T^2$ or $T_{m,n}^{\nu_U, J}$ with $\nu = \nu_U$.

A similar observation is made in the right panel of Figure 2 under setting (H2) - the standard Gaussian location model. In this case, by Proposition 3.4 (see the main paper), we have $\text{ARE}(T_{m,n}^{\nu_G, J}, T) = 1$ and $\text{ARE}(T_{m,n}^{\nu_U, J}, T) = 0.95$. We therefore allot $m, n \in [100, 1100]$ samples for computing the power curve for $T_{m,n}^{\nu_U, J}$ with $\nu = \nu_U$ and sample sizes of $\approx 0.95m, 0.95n$ while computing power curves for both Hotelling $T^2$ and $T_{m,n}^{\nu_G, J}$ with $\nu = \nu_G$. As the theory predicts, all the power curves are virtually identical in the right panel of Figure 2. The resemblance can be seen through the entire spectrum of sample sizes considered.

### D.2. Power comparisons

In this subsection, we perform numerical experiments for 4 settings. In each of these settings, we choose the dimension $d = \{2, 4\}$ and the sample sizes as $m = n = 300$.

(A1) $X_1 \sim \mathcal{N}(0_d, I_d)$ and $Y_1 \sim \mathcal{N}(\theta \cdot 1_d, I_d)$ where $\theta \in \mathbb{R}$ varies in the interval $[0.01, 0.20]$.

(A2) $X_1$ has a logistic distribution with location parameter $0_d$ and scale parameter $I_d$, and $Y_1$ has a logistic distribution with the same scale parameter but with location parameter $\theta \cdot 1_d$. Once again, we choose $\theta \in \mathbb{R}$ with $\theta \in [0.01, 0.20]$.

(A3) $X_1$ has a Laplace distribution with location parameter $0_d$ and scale parameter $0.5 \times I_d + 0.5 \times I_d^\top$, and $Y_1$ has a Laplace distribution with the same scale parameter but with location parameter $\theta \cdot 1$, $\theta \in [0.01, 0.5]$.

(A4) $X_1, Y_1$ both belong to a log-normal family of distributions. In particular, $\log X_1 \sim \mathcal{N}(0_d, I_d)$ and $\log Y_1 \sim \mathcal{N}(\theta \cdot 1_d, I_d)$ where $\theta \in \mathbb{R}$ varies in the interval $[-0.25, -0.01]$. 

As before, for each of the settings (A1)-(A4), we compare the power curves of the Hotelling \( T^2 \) test versus tests based on \( T_{m,n}^{\nu,J} \), where we use \( \nu \equiv \nu_U \equiv \text{Unif}[0,1]^d \) and \( J(x) = x \), or \( \nu \equiv \nu_G \equiv \mathcal{N}_d(0,I_d) \) and \( J(x) = x \). Each of the tests are carried out at level 0.05 and the power curves are plotted as the parameter \( \theta \) varies over the aforementioned ranges in settings (A1)-(A4). To obtain the power curves, we used 500 independent replications. The plots can be found in Figures 3-6 respectively. We now discuss our principle findings from these simulations.

Let us begin with the Gaussian setting (A1) (see Figure 3). In this case, for \( d = 2 \) (left panel of Figure 3), the performance of Hotelling \( T^2 \) and \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \), \( J(x) = x \) are almost identical, whereas the performance of \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_U \), \( J(x) = x \) is slightly worse. This is in alignment with Proposition 3.4 in the main paper where we show that the ARE of \( T_{m,n}^{\nu,J} \) based on \( \nu = \nu_G \) with respect to Hotelling \( T^2 \) is 1; and the same for \( T_{m,n}^{\nu,J} \) based on \( \nu = \nu_U \) is 0.95. Therefore, we would expect the performances of \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \) and Hotelling \( T^2 \) to be close, and that of \( \nu = \nu_U \) to be slightly worse, asymptotically. It is interesting to see a similar behavior manifest itself for a moderate sample size of \( m = n = 300 \). For \( d = 4 \) (right panel of Figure 3), the performance of Hotelling \( T^2 \) is slightly better than the performance of \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \), whose performance is in turn slightly better than \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_U \). As expected, the agreement with the aforementioned AREs is slightly weaker in the \( d = 4 \) case than in the \( d = 2 \) case. However, we do believe that for \( d = 4 \), the power curves are still reasonably close so as to justify the theoretical AREs of 1 and 0.95 as mentioned above.

Note that, when working with \( T_{m,n}^{\nu,J} \), \( \nu = \nu_G \), the standard Gaussian example in the preceding paragraph is in a way, the “worst case” distribution as per Proposition 3.4, and Theorems 3.4, 3.5, and 3.6 (all from the main paper). The aforementioned results show that \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \) has ARE 1 against Hotelling \( T^2 \) and an ARE larger than 1 for any distribution belonging to the families of distributions covered in Theorems 3.4-3.6. We illustrate this using settings (A2) and (A3), featuring a standard Laplace distribution (independent components, see Theorem 3.4) and a correlated logistic distribution (elliptically symmetric, see Theorem 3.5 in the main paper). In both cases, the power curves in Figures 4 and 5 show that \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \) has higher power than Hotelling \( T^2 \). This difference in the power curves is quite pronounced in Figure 4 (standard Laplace with \( d = 2, 4 \)) and the left hand panel of Figure 5 (correlated logistic with \( d = 2 \)), while this difference is only marginal in the right hand panel of Figure 5 (correlated logistic with \( d = 4 \)). Further \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_U \) has the highest power in Figure 4 (standard Laplace with \( d = 2, 4 \)). This too, is justified by our theoretical results, in particular Theorem 3.3 from the main paper, using which it is easy to check that the following approximations hold (upto errors due to numerical integration):

\[
\text{ARE}(T_{m,n}^{\nu,J}, T) = \frac{3}{2} > \frac{32}{25} \approx \text{ARE}(T_{m,n}^{\nu_G,J}, T).
\]

For the correlated logistic setting (A3), see Figure 5, the power curve of \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_U \) is very similar to \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_G \) and none of these two power curves seem to be uniformly better than the other. Note that, particularly for the \( d = 2 \) case, \( T_{m,n}^{\nu,J} \) with \( \nu = \nu_U \) performs significantly better than Hotelling \( T^2 \). For \( d = 4 \), it has similar performance compared to Hotelling \( T^2 \); once again none of the power curves uniformly dominate the other.

In the next simulation setting, we explore the performance of the 3 candidate tests in a heavy-tailed log-normal setting (A4). The log-normal distribution is heavy tailed in the
sense that it has all moments finite but has an infinite exponential moment. Traditionally, rank-based procedures perform better in such cases than their non-rank-based counterparts. Both our proposed tests $T_{\nu,J}^{m,n}$ with $\nu = \nu_G$ and $\nu = \nu_U$ manifest this behavior by significantly outperforming the Hotelling $T^2$ test both for $d = 2$ and 4 (see Figure 6). The difference in the power curves between the rank-based procedures and the Hotelling $T^2$ test is the largest in this setting compared to the other settings ((A1)-(A3)) considered in this section. A similar observation was also made in [30] where the authors show that other (optimal transport based) multivariate rank tests for the two-sample problem outperform their non-rank-based counterparts in different heavy-tailed settings. We believe that theoretically investigating the robustness properties of these multivariate rank tests would be an interesting future research direction.

![Figure 3](image)

**Figure 3:** In the left panel, we sample $X_1, Y_1$ according to setting (A1) with $d = 2$. The power curves are plotted as $\theta \in [0.01, 0.2]$. The red line represents the power curve for $T_{\nu,J}^{m,n}$ with $\nu = \nu_G$, the blue line represents the same for $T_{\nu,J}^{m,n}$ with $\nu = \nu_U$, and the black line represents the same for Hotelling $T^2$. In the right panel, we plot the same with $d = 4$.

**Appendix E: Assumption (3.5) for Deterministic Sequences**

In this section, we discuss how to verify (3.5) in the main paper for some popular deterministic sequences and score function combinations. The case when $h_d^1, \ldots, h_d^N$ are sampled randomly has already been discussed in Remark 3.3 in the main paper.

**Example E.1** (When ERD is uniform on the unit cube or spherical uniform). When the ERD is the Unif$[0,1]^d$ distribution or the spherical uniform distribution, the natural choice in the literature is to choose $J(x) = x$ and $h_d^1, h_d^2, \ldots, h_d^N$ in $[0,1]^d$ such that

$$\frac{1}{N} \sum_{i=1}^{N} \delta h_d^i \overset{w}{\rightarrow} \nu$$

where, depending on the case considered, $\nu$ is either Unif$[0,1]^d$ or the spherical uniform distribution. For the Unif$[0,1]^d$ distribution, popular choices of the $\{h_d^i\}_{i \in [N]}$ include the
Figure 4: Power curves under setting (A2) with $d = 2$ (left panel) and $d = 4$ (right panel). The rest of the description of this figure is similar to that of Figure 3.

regular grid or quasi-Monte Carlo sequences such as the Halton sequence (see [30, Section D.3] for a discussion). For the spherical uniform distribution, a suitable choice was constructed explicitly in [50] which has since been used in [116, 117]. For these choices, \{J(h^d_i)\}_{i \in [N]} and \{K(J(h^d_i), J(h^d_j))\}_{i \in [N]} are uniformly bounded (assuming $K(\cdot, \cdot)$ is continuous on $[0, 1]^d$). Therefore, the assumption (3.5) in the main paper follows directly using the dominated convergence theorem.

Example E.2 (When ERD is standard Gaussian). A natural way to obtain the Gaussian ERD would be to start with $\nu = \text{Unif}[0, 1]^d$ or $\nu$ equals to the spherical uniform distribution and choose $J(\cdot)$ appropriately such that $J \# \nu$ is standard Gaussian.

1. When $\nu = \text{Unif}[0, 1]^d$: Suppose we choose \{h^d_i\}_{i \in [N]} as the standard Halton sequence (see [93] for details on its construction). For $x = (x_1, \ldots, x_d)$, set $J(x) := (\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_d))$. Also, write $h^d_i = (h^d_{i,1}, \ldots, h^d_{i,d})$ for $i \in [N]$. Then by condition (2.4) and the dominated convergence theorem, (3.5) from the main paper follows if we show that:

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (\Phi^{-1}(h^d_{i,j}))^4 < \infty$$

for all $j \in [d]$. Towards this direction, let $p_j$ denote the $j$-th prime number. Without loss of generality, suppose $h^d_{1,j} < h^d_{2,j} < \ldots < h^d_{N,j}$. Then by construction of Halton sequences,

$$h^d_{i,j} - h^d_{i-1,j} \geq (N p_j)^{-1}.$$ 

Also, as $\Phi^{-1}(\cdot)$ is increasing, using the above lower bound yields that,

$$\frac{1}{N} \sum_{i=1}^{N} (\Phi^{-1}(h^d_{i,j}))^4 \leq p_j^{-1} \int_0^1 (\Phi^{-1}(u))^4 \, du < \infty,$$

which establishes (E.1) and consequently (3.5), both from the main paper.
The same idea can be used to establish (E.1) and (3.5) (see the main paper) for other quasi-Monte Carlo sequences or even the uniform \(d\)-dimensional grid. Moreover, as the arguments above show, it is easy to replace \(\Phi^{-1}(\cdot)\) with other quantile functions under appropriate moment assumptions.

2. When \(\nu\) is the spherical uniform: In this case, the popular choice is to choose \(\{h_i\}_{i \in [N]}\) as Hallin’s discrete grid (see [50]) and \(J(\cdot)\) as the van der Waerden score given in (4.8) in the main paper. Then condition (3.5) from the main paper can be verified in the same way as in [117, Proof of Theorem 5.1].
Figure 6: Power curves under setting (A4) with $d = 2$ (left panel) and $d = 4$ (right panel). The rest of the description of this figure is similar to that of Figure 3.