A two-type Bellman–Harris process initiated by a large number of particles

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Abstract

We investigate a two-type critical Bellman–Harris branching process with the following properties: the tail of the life-length distribution of the first type particles is of order $o(t^{-2})$; the tail of the life-length distribution of the second type particles is regularly varying at infinity with index $-\beta$, $\beta \in (0, 1]$; at time $t = 0$ the process starts with a large number $N$ of the second type particles and no particles of the first type. It is shown that the time axis $0 \leq t < \infty$ splits into several regions whose ranges depend on $\beta$ and the ratio $N/t$ within each of which the process at time $t$ exhibits asymptotics (as $N, t \to \infty$) which is different from those in the other regions.

1 Introduction

To some extent, the present paper may be viewed as a continuation of [8] in which the asymptotics of the survival probability of a two-type critical Bellman–Harris branching process $Z(t) = (Z_1(t), Z_2(t))$, $t \geq 0$, was investigated and several conditional limit theorems were proved for the distribution of the number of particles at a distant time $t$ given that the process survives...
up to this time. Since we will use some results obtained in the aforementioned paper we recall some definitions and assertions given in [8].

The model of the two-type Bellman-Harris branching process in focus may be informally described as follows. A particle of type \( i \in \{1, 2\} \) has the life–length distribution \( G_i(t) \), and at the end of her life she produces \( \xi_{i1} \) particles of the first type and \( \xi_{i2} \) particles of the second type in accordance with generating function

\[
F_i(s) = f_i(s_1, s_2) := \mathbb{E}_{s_1, s_2} [s_1^{\xi_{i1}} s_2^{\xi_{i2}}], \quad s := (s_1, s_2) \in [0, 1]^2.
\]

Each particle ever born behaves in a similar manner, and she lives and produces offspring independently of the co-existing particles and the past history of the process.

Using the symbol \(^\dagger\) for the transposition of vectors we introduce two-dimensional vector-columns \( G(t) := (G_1(t), G_2(t))^\dagger \), \( f(s) = (f_1(s), f_2(s))^\dagger \).

Symbols 1 and 0 will be used to denote (depending on the context) either the vector-rows \((1, 1)\) and \((0, 0)\) or the vector-columns \((1, 1)^\dagger\) and \((0, 0)^\dagger\).

Put

\[ m_{ij} := \mathbb{E}_{s_1, s_2} \frac{\partial f_i(s)}{\partial s_j} \bigg|_{s=1} , \quad b_{jk} := \mathbb{E}_{s_1, s_2} \xi_{ij} \xi_{ik} = \frac{\partial^2 f_i(s)}{\partial s_j \partial s_k} \bigg|_{s=1}, \quad i, j, k = 1, 2 \tag{1}\]

and

\[ M(t) := \begin{pmatrix} m_{11} G_1(t) & m_{12} G_1(t) \\ m_{21} G_2(t) & m_{22} G_2(t) \end{pmatrix} , \quad M = M(\infty) := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} . \tag{2}\]

We assume that \( Z(t) \) is an indecomposable, aperiodic and critical process. This means, in particular, that there exists a positive integer \( n_0 \) such that all the elements of the matrix \( M^{n_0} \) are positive, the Perron root of \( M \) is equal to 1 and there exist unique left and right eigenvectors \( v = (v_1, v_2) \) and \( u = (u_1, u_2) \) such that

\[ Mv = uv^\dagger, \quad vM = v, \quad vu^\dagger = 1, \quad u > 0, \quad v > 0, \quad v1 = 1. \tag{3}\]

In addition, we suppose that

\[ B := \frac{1}{2} \sum_{i,j,k=1,2} v_i b_{jk}^i u_j u_k < \infty. \tag{4}\]

Observe that for the critical two-type indecomposable aperiodic process the inequalities \( m_{ii} < 1, \ m_{11} + m_{22} > 0 \) and \( m_{ij} > 0 \) for \( i \neq j \) hold true.
Along with the criticality we impose the following conditions on the tail behavior of the life-length distributions of particles:

\[ 1 - G_1(t) = o(t^{-2}) \] (5)

and

\[ 1 - G_2(t) = \ell(t)t^{-\beta}, \quad \beta \in (0, 1], \] (6)

where \( \ell(t) \) is a function slowly varying at infinity. Here and hereafter all unspecified limit relations are assumed to hold, as \( t \to \infty \) or \( N, t \to \infty \).

Which of the alternatives prevails should be clear from the context.

Let

\[ \mu_i(t) := \int_0^t (1 - G_i(w))dw \quad \text{and} \quad \mu_i := \mu_i(\infty). \] (7)

Clearly, \( \mu_1 < \infty \) and either \( \mu_2 < \infty \) and then \( \beta = 1 \), or \( \mu_2 = \infty \) and then

\[ \mu_2(t) \sim \begin{cases} \frac{t^{1-\beta}\ell(t)}{1-\beta} = \frac{t(1-G_2(t))}{1-\beta}, & \text{if } \beta \in (0, 1), \\ \ell_1(t), & \text{if } \beta = 1, \end{cases} \] (8)

where \( \ell_1(t) := \int_0^t \ell(u)u^{-1}du \to \infty \) is a function slowly varying at infinity and \( \ell(t) = o(\ell_1(t)) \) (see, for example, Proposition 1.5.9a in [1]). Observe that \( \lim_{t \to \infty} \ell_1(t) = \mu_2 \) irrespective of whether \( \mu_2 \) is finite or not, and that \( \mu_i(t) \leq t \). Thus,

\[ 1 \leq R(t) := \frac{t}{\mu_2(t)} \sim \begin{cases} \frac{(1-\beta)t^\beta}{\ell(t)} = \frac{1-\beta}{1-G_2(t)}, & \text{if } \beta \in (0, 1), \\ t/\ell_1(t), & \text{if } \beta = 1. \end{cases} \] (9)

Now we need more notation. For vectors \( s := (s_1, s_2) \in [0, 1]^2 \) and \( z := (z_1, z_2) \in \mathbb{Z}_+^2 := \{0, 1, \ldots\} \times \{0, 1, \ldots\} \) we write \( s^z := s_1^{z_1}s_2^{z_2} \) and for vectors \( x = (x_1, x_2)^\dagger \) and \( y = (y_1, y_2)^\dagger \) we write

\[ x \otimes y := (x_1 y_1, x_2 y_2)^\dagger. \]

Put further

\[ F_i(t; s) = F_i(t; s_1, s_2) := \mathbf{E}_0 s^z(t), \quad F(t; s) := (F_1(t; s), F_2(t; s))^\dagger, \]

where here and hereafter

\[ \mathbf{E}_j \{ \cdot \} := \mathbf{E} \{ \cdot | Z(0) = (\delta_{1j}, \delta_{2j}) \}, \quad \mathbf{P}_j \{ \cdot \} := \mathbf{P} \{ \cdot | Z(0) = (\delta_{1j}, \delta_{2j}) \}, \quad j = 1, 2, \]
and $\delta_{ij}$ is the Kronecker delta.

The vector-function $F(t; s)$ satisfies the following system of integral equations (see, for instance, [8])

$$F(t; s) = s \otimes (1 - G(t)) + \int_0^t f(F(t - w; s)) \otimes dG(w),$$

which, by setting $Q(t; s) := 1 - F(t, s)$, may be rewritten as

$$Q(t; s) = (1 - s) \otimes (1 - G(t)) + \int_0^t (1 - f(F(t - w; s))) \otimes dG(w).$$

Denote $\Phi(s) = (\Phi_1(s), \Phi_2(s))^\dagger := Ms - (1 - f(1 - s))$ and introduce a $2 \times 2$ matrix

$$G_1(t) := (G_i(t)\delta_{ij})_{i,j=1}^2.$$

With this notation at hand a standard renewal argument enables us to rewrite (11) in the form

$$Q(t; s) = (1 - s) \otimes (1 - G(t)) + \int_0^t dM(w)Q(t-w;s)$$

$$- \int_0^t dG_1(w)\Phi(Q(t-w;s)).$$

(12)

One of the main characteristics of any critical branching process is its survival probability. A specialization of Theorem 1 in [6] gives the asymptotic behavior of the survival probability of the two-type Bellman-Harris branching process which satisfies conditions (3) through (6): for any fixed $s := (s_1, s_2) \in [0, 1]^2$

$$Q(t; s) = 1 - F(t; s) \sim u^\dagger \sqrt{\frac{v_2u_2}{B}} (1 - G_2(t)) (1 - s_2).$$

(13)

In particular,

$$Q_i(t) := P_i(Z(t) \neq 0) \sim P_i(Z_2(t) > 0) \sim u_i \sqrt{\frac{v_2u_2}{B}} (1 - G_2(t))$$

(14)

and, moreover,

$$P_i(Z_1(t) > 0) \sim o(Q_i(t)).$$

(15)

The last two asymptotic relations mean that if the two-type population survives up to a distant time $t$, then, with probability close to 1, the
population at that time consists of the second type particles only. We investigate this phenomenon in more detail in the situation when the two-type Bellman-Harris process is initiated at time $t = 0$ by a large number $N$ of the second type particles and no particles of the first type, i.e., $Z(0) = (Z_1(0), Z_2(0)) = (0, N)$, and analyze the distribution of the population size $Z(t) = (Z_1(t), Z_2(t))$, as $t \to \infty$. Note that a similar problem for a single-type critical Bellman-Harris branching process has been investigated in [7]. The critical multitype Sevastyanov branching processes (which are more general than the critical Bellman-Harris branching processes, see [4]) initiated by a large number of particles were (implicitly) considered in [5] under the assumption that the expected life-lengths of particles of all types are finite. In view of (6) our results do not follow (and, in fact, are essentially different) from the results obtained in [5].

Clearly,

$$E\left[ s^{Z(t)} \mid Z(0) = (0, N) \right] = F_N^N(t; s) = e^{-N(1 - F_2(t; s))(1 + o(1))}$$

provided that $\lim_{t \to \infty} (1 - F_2(t; s)) = 0$. Thus, to understand the asymptotic behavior of $Z(t)$ under the present assumptions one has to investigate the behavior of $N(1 - F_2(t; s))$, as $N, t \to \infty$, under a proper scaling of the components of $Z(t)$. If $N$ and $t$ tend to infinity in such a way that $N \sqrt{1 - G_2(t)} \to 0$, then, in view of (13) the population becomes extinct. If, however, $N \sqrt{r_2u_2(1 - G_2(t))/B} \to r \in (0, \infty)$, then

$$\lim_{N, t \to \infty} E\left[ s^{Z(t)} \mid Z(0) = (0, N) \right] = e^{-ru_2 \sqrt{1 - s_2}}.$$ (17)

Thus, despite the indecomposability of the process there is only a finite number of the second type particles and no particles of the first type in the limit. This phenomenon has a natural intuitive explanation: at a distant time $t$ the population only consists of the particles whose life-length distributions have heavy tails (see [7] where a similar effect is discussed for a single-type critical Bellman-Harris process).

Below we list the basic assumptions of the paper.

Hypothesis A. The distribution function $G_1(t)$ satisfies (5) and the distribution function $G_2(t)$ satisfies (6). In addition, if $\beta \in (0, 1/2]$ then there exist positive constants $C$ and $T_0$ such that for $t \geq T_0$ and any fixed $\Delta > 0$

$$G_2(t + \Delta) - G_2(t) \leq C\Delta \ell(t)t^{-\beta - 1}.$$ (18)

Sometimes we will have to deal with ranges of $\beta$ other than $(0, 1]$. In these cases we write, say, that Hypothesis $A(a, b)$ or $A(a, b]$ holds, meaning
that we only consider the range \( \beta \in (a, b) \) or \( \beta \in [a, b] \) and require the validity of Hypothesis A for the indicated range. In the sequel we denote by \( C, C_1, C_2, \ldots \) positive constants whose specific values are of no importance. The values of these constants need not be the same with each usage.

Note that if Hypothesis A holds, then, according to Lemma 12 below,

\[
NP_2(Z_1(t) > 0) = N(1 - F_2(t; 0, 1)) \leq N\mathbb{E}_2[Z_1(t)] \leq CN/\mu_2(t)
\]  (19)

which implies that there are no particles of the first type in the limit if \( \mu_2(t) \gg N \). This means, in particular, that if, given \( \mu_2(t) \gg N \), the limit

\[
\Pi_2(\lambda) := \lim_{N,t \to \infty} N \left( 1 - \mathbb{E}_2 \left[ e^{-\lambda Z_2(t) \psi(t)} \right] \right)
\]

exists for some function \( \psi(t) \) and \( \lambda > 0 \), then for any choice \( s_1 = s_1(t) \in [0, 1] \)

\[
\lim_{N,t \to \infty} N \left( 1 - \mathbb{E}_2 \left[ s_1 Z_1(t) e^{-\lambda Z_2(t) \psi(t)} \right] \right) = \Pi_2(\lambda)
\]  (21)

and vice versa.

Here is a reasonable intuitive explanation of this effect. The branches of the genealogical trees generated by \( N \) initial particles consist of the rays which may be thought of as those generated by renewal processes with increments which (depending on the type of the corresponding particle) have the distribution function \( G_1(t) \) or \( G_2(t) \). As we know by (17), there are only a few surviving branches at a distant time \( t \) such that \( \mu_2(t) \gg N \) and, as a result, not too many rays attain the time-level \( t \). Since the life-length distribution of the first type particles has a light tail \( (o(t^{-2})) \), particles of this type are present in the population at time \( t \) with probability which is negligible in comparison with the survival probability of the whole process up to this time.

It will be shown that there are several natural regions of \( t = t(N) \) which correspond to essentially different limiting distributions, as \( N, t \to \infty \), of the vector \( Z(t) \), properly scaled.

The ranges of these regions depend on the behavior of the product \( N(1 - G_2(t)) \), as \( N, t \to \infty \), or, in view of the relation \( \lim_{t \to \infty} R(t)(1 - G_2(t)) = 1 - \beta \) which holds true whenever \( \beta \in (0, 1) \), on the behavior of the ratio \( R(t)/N \). In the case \( \beta = 1 \) the ranges of the regions will only be described in terms of \( R(t)/N \). This motivates us to formulate the subsequent results in terms of \( R(t)/N \) rather than in terms of a longer expression \( N(1 - G_2(t)) \).
To describe the ranges of the regions in more detail we introduce three functions \( y = g_1(N), \) \( y = g_2(N) \) and \( y = g_3(N) \) which are the inverse functions to

\[
N(y) = (1 - \beta) \frac{y^\beta}{\ell(y)}, \quad N(y) = \frac{y^{1-\beta} \ell(y)}{1 - \beta} \quad \text{and} \quad N(y) = \sqrt{\frac{B}{v_2u_2(1 - G_2(y))}},
\]

respectively, if \( \beta \in (0, 1) \) and

\[
N(y) = \frac{y}{\ell_1(y)}, \quad N(y) = \ell_1(y) \quad \text{and} \quad N(y) = \sqrt{\frac{B}{v_2u_2(1 - G_2(y))}},
\]

respectively, if \( \beta = 1 \). By Theorem 1.8.5 in [1]

\[
g_1(N) = N^{1/\beta} L_1(N), \quad g_2(N) = N^{1/(1-\beta)} L_2(N), \quad g_3(N) = N^{2/\beta} L_3(N)
\]

for some functions \( L_i(\cdot), i = 1, 2, 3, \) slowly varying at infinity, where \( \beta \in (0, 1] \), excluding the case \( \beta = 1 \) for \( g_2 \). Of course, \( g_2(N) \gg N^k \) for all \( k \in \mathbb{N} \), if \( \beta = 1 \).

It can be checked that

\[
g_2(N) \ll g_1(N) \ll g_3(N) \quad \text{for} \quad \beta \in [0, 1/2),
\]

\[
g_1(N) \ll g_2(N) \ll g_3(N) \quad \text{for} \quad \beta \in (1/2, 2/3),
\]

and

\[
g_1(N) \ll g_3(N) \ll g_2(N) \quad \text{for} \quad \beta \in (2/3, 1],
\]

as \( N \to \infty \). We write \( t = t(N) \in (g_i(N), g_j(N)) \) (or \( \in (g_i(N), g_j(N)) \)) if

\[
g_i(N) \ll t(N) \ll g_j(N)
\]

and \( t = t(N) \asymp g_i(N) \) (or \( \asymp g_i(N) \)) if

\[
\lim_{N \to \infty} \frac{t(N)}{g_i(N)} = c \in (0, \infty).
\]

It follows from the definitions above and properties of the functions \( R(t) \) and \( \mu_2(t) \) that

\[
R(t)/N \to 0 \iff t = o(g_1(N)),
\]

\[
R(t)/N \to r \iff t \sim g_1(Nr),
\]

\[
R(t)/N \to \infty \iff t \gg g_1(N)
\]
In the case $\beta \in (0, 1)$ we will call the ranges of $t = t(N)$ satisfying, as $N \to \infty$, either of the conditions

$$N (1 - G_2(t)) \to \infty \quad \text{or} \quad R(t)/N \to 0 \quad \text{or} \quad t = o(g_1(N))$$

the \textit{early evolutionary stages} of the population, the ranges satisfying either of the conditions

$$N (1 - G_2(t)) \to r_1 \in (0, \infty) \quad \text{or} \quad R(t)/N \to r \in (0, \infty) \quad \text{or} \quad t \sim g_1(Nr)$$

the \textit{intermediate evolutionary stages}, and the ranges satisfying either of the conditions

$$N (1 - G_2(t)) \to 0 \quad \text{or} \quad R(t)/N \to \infty \quad \text{or} \quad t \gg g_1(N)$$

the \textit{final evolutionary stages}. In the case $\beta = 1$ we will use the same definition with the first condition (that involves $N (1 - G_2(t))$) omitted.

Two remarks are in order. 1) Although the assumptions arising in the present paper could have been formulated in terms of the time parameter $t = t(N)$ we prefer to state them in terms of $R(t)/N$ and $\mu_2(t)/N$.

2) The ensuing presentation will make it clear that the behavior of the ratio $\mu_2(t)/N$, as $N, t \to \infty$, governs further splitting the evolutionary stages just introduced into subranges which are characterized by different limiting distributions of the vector $Z(t)$, properly scaled.

The remaining part of the paper is structured as follows. While the main results are stated in Section 2, their proofs are given in Section 4 (the early evolutionary stages), Section 5 (the intermediate evolutionary stages) and Section 6 (the final evolutionary stages). In Section 3 we recall some known results taken mainly from [8]. These concern various asymptotic properties of renewal matrices and generating functions, and are extensively used throughout the paper.

2 Main results

2.1 Early evolutionary stages

In this subsection we investigate the asymptotic behavior of the number of particles for the early evolutionary stages.
Denote by \( D = (D_{ij})_{i,j=1}^2 \) a 2 \( \times \) 2 matrix with positive entries
\[
D_{ii} := (1 - m_{ii}) \mu_2 D \quad \text{and} \quad D_{ij} := m_{ij} \mu_2 D, \quad i \neq j,
\]
in the case when \( \mu_2 < \infty \), where \( D := ((1 - m_{22}) \mu_1 + (1 - m_{11}) \mu_2)^{-1} \), and
\[
D_{ii} = \frac{1 - m_{ii}}{1 - m_{11}} \quad \text{and} \quad D_{ij} = \frac{m_{ij}}{1 - m_{11}}, \quad i \neq j,
\]
in the case when \( \mu_2 = \infty \).

Set \( \Gamma_{\beta} = 1 \) for \( \beta = 1 \) and
\[
\Gamma_{\beta} := \frac{\sin \pi \beta}{\pi \beta (1 - \beta)}
\]
for \( \beta \in (0, 1) \).

**Theorem 1** Suppose that Hypothesis A holds and that
\[
\lim_{N,t \to \infty} R(t)N^{-1} = 0 \quad \text{and} \quad \lim_{N,t \to \infty} \mu_2(t)N^{-1} = 0.
\]
If \( \beta = 1/2 \), assume additionally that
\[
\lim_{N,t \to \infty} \frac{\mu_2(t)}{N} \int_0^t \frac{dw}{(1 + \mu_2(w))^2} = 0.
\]
Then, for any \( \lambda_1, \lambda_2 > 0 \),
\[
\lim_{N,t \to \infty} E \left[ \exp \left\{ -\lambda_1 \frac{Z_1(t)\mu_2(t)}{N} - \lambda_2 \frac{Z_2(t)}{N} \right\} \left| Z(0) = (0, N) \right. \right] = \exp \left\{ -\mu_1 \beta \Gamma_{\beta} D_{21} \lambda_1 - D_{22} \lambda_2 \right\}.
\]

**Corollary 2** Suppose that Hypothesis A holds and that \( \lim_{N,t \to \infty} R(t)N^{-1} = 0 \).
Then, for \( \lambda > 0 \),
\[
\lim_{N,t \to \infty} E \left[ \exp \left\{ -\lambda \frac{Z_2(t)}{N} \right\} \left| Z(0) = (0, N) \right. \right] = \exp \left\{ -D_{22} \lambda \right\}.
\]

Note, that this corollary does not require \( \lim_{N,t \to \infty} \mu_2(t)N^{-1} = 0 \), nor condition (25).

Set
\[
O(s) := (O_1(s), O_2(s)) := \beta \Gamma_{\beta} \int_0^\infty D \Phi(Q(w; s, 1))dw.
\]
**Theorem 3** Suppose that Hypothesis $A(0,0.5]$ holds and that
\[
\lim_{N,t \to \infty} R(t)N^{-1} = 0 \quad \text{and} \quad \lim_{N,t \to \infty} \mu_2(t)N^{-1} = r^{-1} \in (0, \infty).
\]
If $\beta = 1/2$, assume additionally that
\[
\Upsilon := \int_0^\infty \frac{dw}{(1 + \mu_2(w))^2} < \infty.
\]
Then, for any $s \in [0,1]$ and $\lambda > 0$,
\[
\lim_{N,t \to \infty} \mathbb{E} \left[ s^{Z_1(t)} \exp \left\{ -\lambda \frac{Z_2(t)}{N} \right\} \mid Z(0) = (0, N) \right] = \exp \left\{ -r \beta \Gamma_{\beta 1} D_{21} (1 - s) + rO_2(s) - D_{22} \lambda \right\}.
\]
Thus, in this case we asymptotically have a few individuals of the first type, while the number of individuals of the second type is still of order $N$. Moreover, $Z_1(t)$ and $Z_2(t)N^{-1}$ are asymptotically independent.

**Remark.** In view of (8) and (9) the assumptions of Theorem 3 hold if either $\beta = 1/2$ and $\lim_{t \to \infty} \ell(t) = \infty$ or $\beta < 1/2$.

**Theorem 4** Suppose that Hypothesis $A(0,0.5]$ holds and that
\[
\lim_{N,t \to \infty} R(t)N^{-1} = 0 \quad \text{and} \quad \lim_{N,t \to \infty} \mu_2(t)N^{-1} = \infty.
\]
Then, for any $\lambda > 0$,
\[
\lim_{N,t \to \infty} \mathbb{E} \left[ \exp \left\{ -\lambda \frac{Z_2(t)}{N} \right\} \mid Z_1(t) = 0 \right] \mid Z(0) = (0, N) = \exp \{-D_{22} \lambda \}. \quad (27)
\]

**Remark.** In view of (8) and (9) the assumptions of Theorem 4 hold for $\beta = 1/2$ only if $\lim_{t \to \infty} \ell(t) = \infty$.

### 2.2 The intermediate evolutionary stages

In this section we formulate theorems which describe the limiting behavior of the population for the intermediate evolutionary ranges, i.e., we assume that the limit of $R(t)/N$ is positive and finite. Unlike the early evolutionary stages the asymptotic results here are affected by genuine properties of branching processes. There are three essentially different intermediate
subranges which are characterized by one of the conditions $\mu_2(t)N^{-1} \to \infty$, $\mu_2(t)N^{-1} \to 0$ or $\mu_2(t)N^{-1} \to r_1 \in (0, \infty)$ which is assumed to hold along with the defining property of the intermediate stages. We only analyze the first and the second subranges. The remaining case $R(t)N^{-1} \to r \in (0, \infty)$ and $\mu_2(t)N^{-1} \to r_2 \in (0, \infty)$ which implies $\beta = 1/2$ will be considered in a separate paper, for it requires much more delicate analysis.

Put

$$N_i(x) := \frac{1}{2} \sum_{j,k=1}^2 b_{j,k} x_j x_k, \quad N(x) := (N_1(x), N_2(x))^\top$$

and observe that by Taylor’s formula

$$\sum_{k=1}^2 m_{ik} (1 - s_k) - (1 - f_i(s)) = \frac{1}{2} \sum_{j,k=1}^2 b_{j,k} (1 - s_j) (1 - s_k) + o\left((1 - s_1)^2 + (1 - s_2)^2\right) = N_i(1 - s) + o\left(\|1 - s\|^2\right)$$

as $s_1, s_2 \uparrow 1$, or, equivalently,

$$\Phi(1 - s) = M(1 - s) - (1 - f(s)) = N(1 - s) + o\left(1 \|1 - s\|^2\right). \quad (29)$$

Letting

$$\bar{b} := \max_{i,j,k} b_{j,k}$$

we infer

$$N(x) \leq \bar{b} \|x\|^2 1 \quad (30)$$

and

$$\|N(x) - N(x^*)\| \leq \bar{b} (\|x\| + \|x^*\|) \|x - x^*\|. \quad (31)$$

In the proof of Theorem 5 it will be shown that the system of equations

$$\Omega(\lambda) = D(0, \lambda) - \Gamma_\beta \int_0^1 DN \left( \frac{\Omega \left(\lambda (1 - w)^\beta\right)}{(1 - w)^{2\beta}} \right) dw^\beta, \quad \lambda > 0, \quad (32)$$

has a unique solution with non-negative components, and we denote this solution by $\Omega(\lambda) := (\Omega_1(\lambda), \Omega_2(\lambda))^\top$.

**Theorem 5** Suppose that Hypothesis A holds and that

$$\lim_{N,t \to \infty} R(t)N^{-1} = 1/r \in (0, \infty). \quad (33)$$

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Then, for any $\lambda > 0$,

$$
\lim_{N,t \to \infty} E \left[ \exp \left\{ -\lambda \frac{rZ_2(t)}{N} \right\} \mid Z(0) = (0, N) \right] = e^{-r \Omega(\lambda)}. \tag{34}
$$

Furthermore, if Hypothesis $A(0,0.5)$ and condition (33) hold, and

$$
\lim_{N,t \to \infty} \mu_2(t)N^{-1} = \infty, \text{ then, for any } \lambda > 0,
$$

$$
\lim_{N,t \to \infty} E \left[ \exp \left\{ -\lambda \frac{rZ_2(t)}{N} \right\} ; Z(0) = (0, N) \right] = e^{-r \Omega(\lambda)}. \tag{35}
$$

Observe that there are no first type particles in the limit under the asymptotic regime $\mu_2(t)N^{-1} \to \infty$. Note also that the assumptions $R(t)N^{-1} \to r^{-1}$ and $\mu_2(t)N^{-1} \to \infty$ entail $\beta = 1/2$ and $\lim_{t \to \infty} \ell(t) = \infty$, or $\beta < 1/2$.

Now we consider the case $R(t)N^{-1} \to r^{-1}$ and $\mu_2(t)N^{-1} \to 0$ which entails $\beta = 1/2$ and $\lim_{t \to \infty} \ell(t) = 0$, or $\beta > 1/2$.

Put

$$
C_\beta := \left( \begin{array}{cc} D_{11} & D_{12} \\ D_{21} & D_{22} \end{array} \right).
$$

In the proof of Theorem 6 it will be shown that the system of equations

$$
H(\theta, \lambda) = C_\beta \left( 1, \lambda \theta^{1-\beta} \right)^\dagger - \Gamma_\beta \theta^{2\beta-1} \int_0^1 \frac{DN(H(\theta(1-y), \lambda))}{(1-y)^{2-2\beta}} dy^{\beta}, \tag{36}
$$

for $\theta, \lambda > 0$, with $\beta \in (1/2, 1]$, has a unique solution with non-negative components, and we denote this solution by $H(\theta, \lambda) = (H_1(\theta, \lambda), H_2(\theta, \lambda))^\dagger$.

**Theorem 6** Suppose that Hypothesis $A(0,0.5)$ holds and that

$$
\lim_{N,t \to \infty} R(t)N^{-1} = r^{-1} \in (0, \infty) \quad \text{and} \quad \lim_{N,t \to \infty} \mu_2(t)N^{-1} = 0.
$$

Then, for $\lambda_1 > 0$ and $\lambda_2 > 0$,

$$
\lim_{N,t \to \infty} E \left[ \exp \left\{ -\lambda_1 \frac{\mu_2(t)Z_1(t)}{\mu_1 R(t)} - \lambda_2 \frac{Z_2(t)}{R(t)} \right\} \mid Z(0) = (0, N) \right]
= \exp \left\{ -r \lambda_1 H_2 \left( \lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)} \right) \right\}.
$$
2.3 Final evolutionary stages

In the first part of Theorem 7 given next we investigate the asymptotic behavior of the number of particles under the conditions

\[ R(t)N^{-1} \to \infty \text{ and } N\sqrt{1-G_2(t)} \to \infty, \]

the first of these being a defining property of the final evolutionary stages. In the second part of that theorem we work under the conditions \((37)\) and \(N/\mu_2(t) \to 0.\)

Relations \(37\) and \(38\) together imply

\[ 0 \leftarrow \frac{N}{\mu_2(t)} = \frac{N(1 - G_2(t))^{1/2}}{\mu_2(t)(1 - G_2(t))^{1/2}} = O\left(\frac{N(1 - G_2(t))^{1/2}}{t(1 - G_2(t))^{3/2}}\right) \]

and thereupon

\[ \lim_{t \to \infty} t(1 - G_2(t))^{3/2} = \infty \]

which means that conditions \((37)\) and \((38)\) may only hold simultaneously if either \(\beta = 2/3\) and \(\lim_{t \to \infty} \ell(t) = \infty\) or \(\beta < 2/3.\)

**Theorem 7** Suppose that Hypothesis \(A(0,1)\) holds, and that

\[ \lim_{N,t \to \infty} \frac{N\psi(t)}{\mu_2(t)} = 0; \]

\[ 1/R(t) = o(\psi(t)), \]

and

\[ \lim_{N,t \to \infty} N\sqrt{\frac{\nu_2 u_2}{B}} \psi(t)(1 - G_2(t)) = r \in (0, \infty) \]

for a non-increasing, regularly varying function \(\psi(t) := t^{-\gamma} \ell_2(t), \gamma \in [0,1),\) such that \(\lim_{t \to \infty} \psi(t) = 0.\) Then, for each \(\lambda > 0,\)

\[ \lim_{N,t \to \infty} \mathbb{E}\left[e^{-\lambda u_2 Z_2(t)}\psi(t) | Z(0) = (0,N) \right] = e^{-ru_2 \sqrt{\lambda}}. \]

Replacing \(41\) by a stronger condition \((38)\) and assuming that all the other assumptions hold we also have

\[ \lim_{N,t \to \infty} \mathbb{E}\left[e^{-\lambda u_2 Z_2(t)}\psi(t) | Z_1(t) = 0, Z(0) = (0,N) \right] = e^{-ru_2 \sqrt{\lambda}}. \]

**Remark.** Recall that relation \((13)\) implies that the population dies out whenever \(N\sqrt{1 - G_2(t)} \to 0,\) whereas the limiting distribution of the number of particles is given by formula \((17)\) whenever \(N\sqrt{1 - G_2(t)} \to r \in (0, \infty).\) In view of the assumption \(\beta < 1\) the latter relation implies \(R(t)/N \sim (1 - \beta)(N(1 - G_2(t)))^{-1} \to \infty.\)
2.4 Summary of the results obtained

In this subsection we give a description of the splitting into the regions in terms of the functions \( g_i(N), i = 1, 2, 3 \).

First we recall that if \( \beta \in (0, 1] \), and \( t, N \to \infty \) in such a way that \( t \sim g_3(N/r) \) or \( t \gg g_3(N) \) (that is, if \( t(N) \) belongs to one of the final evolutionary stages of the process) then the asymptotic behavior of the number of particles is given in (17) with \( r = 0 \) in the second case. Under these conditions the first type particles are absent in the limit. The same is true for the second type particles if \( r = 0 \), while the limiting distribution of the second type particles for \( r > 0 \) is discrete. We summarize this result in Table 0:

| Range of \( t \) | Theorems |
|------------------|----------|
| \( \simeq g_3 \) | \( \simeq g_3 \) |
| \( \gg g_3 \)    | \( \gg g_3 \) |

where the first row shows the set of ranges of \( t = t(N) \) under consideration (with \( g_i \) for \( g_i(N) \)) and "\( \gg \)" in the second row indicates that the statement for the corresponding time interval is given by formula (17).

Given below are three other tables which describe the evolution of the two-type Bellman-Harris branching process in some remaining cases. Tables 1, 2 and 3 concern the cases \( \beta \in (0, 1/2) \), \( \beta \in (1/2, 2/3) \) and \( \beta \in (2/3, 1) \), respectively.

Table 1 case \( \beta \in (0, 1/2) \)

| Range of \( t \) | \( o(g_2) \) | \( \simeq g_2 \) | \( \in (g_2, g_1) \) | \( \simeq g_1 \) | \( \in (g_1, g_3) \) |
|------------------|----------------|----------------|----------------|----------------|----------------|
| Theorems         | \( \Theta_1 \) | \( \Theta_3 \) | \( \Theta_6 \) | \( \Theta_5 \) | \( \Theta_7 \) |

Table 2 case \( \beta \in (1/2, 2/3) \)

| Range of \( t \) | \( o(g_1) \) | \( \simeq g_1 \) | \( \in (g_1, g_3) \) |
|------------------|----------------|----------------|----------------|
| Theorems         | \( \Theta_1 \) | \( \Theta_3 \) | \( \Theta_7 \) |

Here the symbols \( \Theta_1, \Theta_2 \) etc. mean that the result for the corresponding range of \( t = t(N) \) is given in Theorem \( \Theta_1 \), Theorem \( \Theta_2 \) etc. Thus, we have a complete limiting picture for all fixed \( \beta \in (0, 2/3)\setminus\{1/2\} \) as \( t = t(N) \to \infty \) (in some regions \( t(N) \) should be a function regularly varying at \( \infty \)). Moreover, provided that conditions (25) or (26) hold Theorems \( \Theta_1 \) and \( \Theta_3 \) are true for \( \beta = 1/2 \) as well, whereas the case \( \beta = 1/2 \) is still open, otherwise.
Table 3 looks almost the same as Table 2. There are, however, two essential differences. First, Table 2 concerns the case $g_2(N) \ll g_3(N)$, i.e., $\beta < 2(1 - \beta)$, while Table 3 deals with the case $g_3(N) \ll g_2(N)$, i.e., $\beta > 2(1 - \beta)$. Second, in the case $\beta \in (2/3, 1)$ and under certain conditions in the case $\beta = 2/3$ and $g_2(N) \leq t(N) \ll g_3(N)$ we have no results showing that the first type particles are absent within the corresponding time interval (this fact is indicated by the sign “+?” in Table 3). We, however, believe that this is, indeed, the case.

3 Auxiliary results

We stipulate hereafter that, unless otherwise is stated, records like $a(t) = O(b(t))$, $a(t) = o(b(t))$, $a(t) \sim b(t)$ or $\lim a(t) = a$ are assumed to hold, as $t \to \infty$.

Set $I(t) := 1_{\{t \geq 0\}}I$, where $1_{\{A\}}$ denotes the indicator of the event $A$, and $I := (\delta_{ij})_{i,j=1,2}$. We define the convolution $C(t) = A * B(t) = (C_{ij}(t))_{i,j=1}^2$ of two matrices $A(t) = (A_{ij}(t))_{i,j=1}^2$ and $B(t) = (B_{ij}(t))_{i,j=1}^2$ as the matrix with elements

$$C_{ij}(t) = \sum_{k=1}^2 A_{ik} * B_{kj}(t).$$

Put $M^0(t) := I(t)$ and introduce the renewal matrix

$$U(t) = (U_{ij}(t))_{i,j=1}^2 := \sum_{k=0}^{\infty} M^k(t)$$

with the agreement that $U(t)$ is the $2 \times 2$ zero matrix if $t < 0$. Clearly,

$$U(t) = I(t) + M * U(t). \quad (46)$$

We also define the $L_1$ norm $\|\cdot\|$ of matrices and vectors as the sum of the absolute values of all their components, i.e.,

$$\|U(t)\| = \sum_{i,j=1}^2 U_{ij}(t).$$

The following statements concerning various asymptotic properties of the renewal matrix $U(t)$ have been established in [8].
Lemma 8 Under conditions (5) and (6)

\[ U(t) \sim \Gamma_\beta \frac{t}{\mu_2(t)} D = \Gamma_\beta R(t) D. \]

Corollary 9 Suppose that conditions (5) and (6) hold. Then there exists a constant \( C \in (0, \infty) \) such that, for all \( t \geq 0 \),

\[ \|U(t)\| \leq C (R(t) + 1). \]

Lemma 10 Suppose that Hypothesis A holds. Then, for any fixed \( \Delta > 0 \),

\[ U(t + \Delta) - U(t) \sim \Delta \frac{\beta \Gamma_\beta}{\mu_2(t)} D. \]  \hspace{1cm} (47)

Moreover,

\[ U_1(t) := U * G_1(t) \sim U(t) \]  \hspace{1cm} (48)

and

\[ U_1(t + \Delta) - U_1(t) \sim \Delta \frac{\beta \Gamma_\beta}{\mu_2(t)} D. \]  \hspace{1cm} (49)

Remark. In Lemma 8 of [8] relation (49) has only been stated for \( \beta \in (0, 1/2] \). A perusal of the proof of that lemma reveals that (49) holds true for all \( \beta \in (0, 1] \).

Lemma 11 Assume that Hypothesis A holds. Then, for any function \( w(t) \) directly Riemann integrable on \([0, \infty)\)

\[ \int_0^t w(t-u)du(U(u)) \sim \frac{\beta \Gamma_\beta}{\mu_2(t)} \int_0^\infty w(u)duD, \]

and for any function \( W(t) = t^{-\alpha} \ell_W(t) \), where \( \ell_W(t) \) is a function slowly varying at infinity,

\[ \int_0^t W(t-u)du(U(u)) \sim \frac{\beta \Gamma_\beta \Gamma(1-\alpha) \Gamma(\beta) t^{1-\alpha} \ell_W(t)}{\Gamma(1-\alpha+\beta) \mu_2(t)} D, \alpha \in [0, 1), \]

\[ \int_0^t W(t-u)du(U(u)) \sim \frac{\beta \Gamma_\beta}{\mu_2(t)} \int_0^t u^{-1} \ell_W(u)duD, \alpha = 1. \]

Here \( \Gamma(\cdot) \) is the gamma function. In particular,

\[ \int_0^t (1 - G_1(t-u))du(U(u)) \sim \frac{\beta \Gamma_\beta \mu_1}{\mu_2(t)} D, \]

\[ \int_0^t (1 - G_2(t-u))du(U(u)) \sim D. \]

All the previous relations remain valid if we replace \( U(t) \) by \( U_1(t) \).
The results of Lemma 11 concerning $U$ were obtained in Corollaries 7 and 9 in [8]. The results concerning $U_I$ are new and can be derived from Lemma 10 and Lemma 6 in [8].

Let $P(t) := (P_{ij}(t))_{i,j=1,2} = (E_iZ_j(t))_{i,j=1,2}$ denote the mean matrix of the number of particles at time $t$. The asymptotic behavior of $P(t)$ is given by the following lemma.

**Lemma 12** ([see [8]]) Suppose that Hypothesis $A$ holds. Then

$$P(t) \sim \begin{pmatrix} D_{11} \frac{\mu_1 \beta \Gamma_1}{\mu_2(t) + 1} & D_{12} \\ D_{21} \frac{\mu_1 \beta \Gamma_1}{\mu_2(t) + 1} & D_{22} \end{pmatrix} = DJ(t), \quad (50)$$

where

$$J(t) := \begin{pmatrix} \frac{\mu_1 \beta \Gamma_1}{\mu_2(t) + 1} & 0 \\ 0 & 1 \end{pmatrix}. \quad (51)$$

Also, we mention that if

$$P_i(t) := (P_{1i}(t), P_{2i}(t))^\dagger, \quad \delta_i := (\delta_{1i}, \delta_{2i})^\dagger, \quad (52)$$

then (see [8], formula (57) in Section 3)

$$P(t) = (P_1(t), P_2(t)) = U * (I - G_I(\cdot))(t). \quad (53)$$

In particular,

$$P_i(t) = U * (\delta_i \otimes (I - G(\cdot)))(t), \quad i = 1, 2. \quad (54)$$

The section closes with two more technical lemmas of different flavor.

**Lemma 13** Let $H(t)$, $k_1(t)$ and $k_2(t)$ be nonnegative functions defined on $[0, \infty)$ and such that $\lim_{t \to \infty} k_i(t) = 0$, $i = 1, 2$, and $\lim_{t \to \infty} H(t) = \infty$. If

$$\lim_{t \to \infty} H(t)Q(t; 1 - \lambda_1 k_1(t), 1 - \lambda_2 k_2(t)) = h(\lambda_1, \lambda_2), \quad \lambda \geq 0, \lambda_2 \geq 0,$$

for a function $h(\lambda_1, \lambda_2) = (h_1(\lambda_1, \lambda_2), h_2(\lambda_1, \lambda_2))$ with continuous in both arguments components, then

$$\lim_{t \to \infty} H(t)Q(t; e^{-\lambda_1 k_3(t)}, e^{-\lambda_2 k_4(t)}) = h(\lambda_1, \lambda_2)$$

for any functions $k_3(t)$ and $k_4(t)$ such that $\lim_{t \to \infty} k_3(t)/k_1(t) = 1$ and $\lim_{t \to \infty} k_4(t)/k_2(t) = 1$. 17
Proof. The result follows from the monotonicity of \( Q_i(t; s_1, s_2), i = 1, 2, \) in \( s_1 \) and \( s_2 \), the inequality
\[
1 - x \leq e^{-x} \leq 1 - x \left( 1 - \frac{x}{2} \right),
\]
and the continuity of \( h_i(\lambda_1, \lambda_2), i = 1, 2, \) in both arguments.

Lemma 14 Let \( H(t), k_1(t) \) and \( k_2(t) \) be nonnegative functions defined on \([0, \infty)\) and such that \( \lim_{t \to \infty} k_i(t) = 0, i = 1, 2, \) and \( \lim_{t \to \infty} H(t) = \infty. \) If there exist functions \( k_i(t, \lambda_i), i = 1, 2 \) such that \( k_i(t, \lambda_i) \sim \lambda_i k_i(t), t \to \infty, \) for any fixed \( \lambda_i > 0, \) and
\[
\lim_{t \to \infty} H(t)Q(t; 1 - k_1(t, \lambda_1), 1 - k_2(t, \lambda_2)) = h(\lambda_1, \lambda_2), \quad \lambda_1 > 0, \quad \lambda_2 > 0,
\]
for a function \( h(\lambda_1, \lambda_2) = (h_1(\lambda_1, \lambda_2), h_2(\lambda_1, \lambda_2)) \) with continuous in both arguments components, then
\[
\lim_{t \to \infty} H(t)Q(t; e^{-\lambda_1 k_1(t)}, e^{-\lambda_2 k_2(t)}) = h(\lambda_1, \lambda_2), \quad \lambda_1 > 0, \quad \lambda_2 > 0.
\]
If
\[
\lim_{t \to \infty} H(t)Q(t; 1, 1 - k_2(t, \lambda_2)) = h(\lambda_2), \quad \lambda_2 > 0,
\]
for a function \( h(\lambda_2) = (h_1(\lambda_2), h_2(\lambda_2)) \) with continuous components, then
\[
\lim_{t \to \infty} H(t)Q(t; 1, e^{-\lambda_2 k_2(t)}) = h(\lambda_2), \quad \lambda_2 > 0.
\]

Proof. We only prove the first part of the lemma. The proof of the second part is analogous.

For any fixed positive \( \lambda_1, \lambda_2 \) and any \( \varepsilon \in (0, \min\{1, \lambda_1, \lambda_2\}/2) \) there exist \( T_i = T_i(\lambda_i), i = 1, 2 \) such that
\[
\lambda_1 k_i(t) > k_i(t, \lambda_i - \varepsilon), \quad \lambda_1 k_i(t) < k_i(t, \lambda_i + \varepsilon)
\]
for \( t \geq T_i \) which implies that both the upper and the lower limits of \( H(t)Q(t; 1 - \lambda_1 k_1(t), 1 - \lambda_2 k_2(t)) \) are sandwiched between \( h(\lambda_1 + \varepsilon, \lambda_2 + \varepsilon) \) and \( h(\lambda_1 - \varepsilon, \lambda_2 - \varepsilon) \). Using the continuity of the components of \( h \) in \((\lambda_1, \lambda_2)\) yields
\[
\lim_{t \to \infty} H(t)Q(t; 1 - \lambda_1 k_1(t), 1 - \lambda_2 k_2(t)) = h(\lambda_1, \lambda_2),
\]
and an appeal to Lemma 13 completes the proof of the first part.
4 Proofs for the early evolutionary stages

Recalling the definition $U_I(t) = U * G_I(t)$ and viewing equation (12) as a renewal-type equation with respect to $Q(t; s)$ we obtain the representation

$$Q(t; s) = U * ((1 - s) \otimes (1 - G(\cdot))) (t) - \int_0^t dU_I(w) \Phi(Q(t - w; s))$$

which is our main tool in this section.

4.1 Proof of Theorem 1

Invoking Lemma 13 which applies because $\mu_2(t)/N \to 0$ we conclude that it suffices to verify that

$$\lim_{N, t \to \infty} N Q(t; 1 - N^{-1} \lambda_1 \mu_2(t), 1 - N^{-1} \lambda_2) = D(\lambda_1 \mu_1 \beta, \lambda_2)$$

Put $s := (1 - N^{-1} \lambda_1 \mu_2(t), 1 - N^{-1} \lambda_2)$. Our strategy is to show that

$$\lim_{N, t \to \infty} N U * ((1 - s) \otimes (1 - G(\cdot))) = D(\lambda_1 \mu_1 \beta, \lambda_2)$$

and

$$N \int_0^t dU_I(w) \Phi(Q(t - w; s)) = o(1)$$

which entails (58) in view of (57).

Relation (59) is an immediate consequence of

$$U * ((1 - s) \otimes (1 - G(\cdot))) = U * ((I - G_I(\cdot))(t)(1 - s)$$

$$= P(t)(1 - s) = (1 + o(1)) DJ(t)(1 - s),$$

where the second and the third equalities follow from (53) and Lemma 12 respectively.

Left with the proof of (60) we observe that (61) ensures the existence of $c \geq (\lambda_1 + \lambda_2)$ such that

$$Q(w; s) \leq \min \left\{ 1, c \frac{\mu_2(t)}{N \mu_2(w)} \right\}$$
for all pairs $w \leq t$, where $\hat{\mu}_2(t) := \mu_2(t) + 1$. Further we recall that if $\Upsilon$ in (20) is finite then $\beta \in (0, 0.5]$ whereas if $\Upsilon = \infty$ then $\beta \in [0.5, 1]$. Writing

$$\int_0^t dU_1(w) \Phi(Q(t - w; s)) \leq \int_0^t U_1(dw) \Phi \left( \min \left\{ 1, c \frac{\mu_2(t)}{N \hat{\mu}_2(t - w)} \right\} 1 \right)$$

$$\leq C \frac{\mu_2^2(t)}{N^2} \int_0^t \frac{1}{\hat{\mu}_2(t - w)} 1$$

$$= O \left( \frac{1}{N^2} \right) \times \begin{cases} \mu_2^{-1}(t), & \text{if } \Upsilon < \infty, \\ \mu_2^{-1}(t) \int_0^t \frac{1}{\mu_2(u)} du, & \text{if } \Upsilon = \infty, \beta = 0.5, \\ t \mu_2^{-3}(t), & \text{if } \beta \in (0.5, 1] \end{cases}$$

where the first inequality follows from (62), the second is a consequence of (29) and (30), and the third is justified by Lemma 11, and appealing to (25) in the case $\Upsilon = \infty$ and $\beta = 0.5$ and to (24) in the complementary cases we arrive at (60). The proof of Theorem 1 is complete.

4.2 Proof of Corollary 2

Put $s := (1, 1 - \lambda N^{-1})$. A similar argument as in the proof of Theorem 1 shows that it suffices to prove

$$\lim_{N, t \to \infty} NU \ast ((1 - s) \otimes (1 - G(\cdot))) = D(0, \lambda^\dagger)$$

and (60) (with the present $s$).

Using (61) (with the present $s$) proves (63). As for (60), mimicking the argument given in the proof of Theorem 1 leads to $Q(w; s) \leq \min\{1, cN^{-1}\} 1$, $0 \leq w \leq t$ and then to

$$\int_0^t dU_1(w) \Phi(Q(t - w; s)) \leq \int_0^t dU_1(w) \Phi \left( \min\{1, cN^{-1}\} 1 \right)$$

$$= O \left( N^{-2} U_1(t) 1 \right) = o \left( N^{-1} 1 \right),$$

where relation (48), Lemma 8 and the assumption $R(t)/N \to 0$ have been utilized for the last equality. The proof of Corollary 2 is complete.

4.3 Proof of Theorem 3

Put $s := (s, e^{-\lambda/N})$. We intend to prove

$$\lim_{N, t \to \infty} NU \ast ((1 - s) \otimes (1 - G(\cdot))) (t) = D \left( r (1 - s) \mu_1 \beta \Gamma_\beta \right)$$

(64)
and
\[
\lim_{N,t \to \infty} N \int_0^t dU_1(w) \Phi(Q(t - w; s)) = rO(s) \quad (65)
\]
which entails
\[
\lim_{N,t \to \infty} N Q(t; s, e^{-\lambda/N}) = D \left( r(1 - s) \mu_1 \beta \Gamma_\beta \lambda \right) - rO(s)
\]
in view of (67).
Relation (64) is a consequence of (61) and the assumption \(\mu_2(t)/N \to r^{-1}\).
Passing to the proof of (65), we first note that (64) implies
\[
Q_i(t; s, 1) \leq Q_i(t; s) = O(1/N) = O(1/\bar{\mu}_2(t)) \quad (66)
\]
because \(Q(t - w; s, s_2)\) is non-increasing in \(s_2\). Since the function \(1/\bar{\mu}_2(t)\) is directly Riemann integrable on \([0, \infty)\) as a non-increasing Lebesgue integrable function (see (26)), so are \(\Phi_i(Q(t; s, 1)), i = 1, 2\) because these are nonnegative bounded and continuous functions satisfying
\[
\Phi_i(Q(t; s, 1)) \leq C(Q_i^2(t; s, 1)) \leq C_1/\bar{\mu}_2(t),
\]
where the first inequality is a consequence of (29) and (30), and the second follows from (66). With this at hand an application of Lemma 11 yields
\[
\lim_{N,t \to \infty} N \int_0^t dU_1(w) \Phi(Q(t - w; s, 1)) = r\beta \Gamma_\beta \int_0^\infty D\Phi(Q(w; s, 1))dw = rO(s) < \infty. \quad (67)
\]
It remains to check that (67) implies (65). To this end, write
\[
Q(w; s, e^{-\lambda/N}) - Q(w; s, 1) \leq \lambda N^{-1} P_2(w) \leq CN^{-1} 1,
\]
where the last inequality follows from (50), and then
\[
0 \leq \Phi(Q(w; s, e^{-\lambda/N})) - \Phi(Q(w; s, 1)) \\
\leq C \left( Q(w; s, e^{-\lambda/N}) - Q(w; s, 1) \right) \cancel{\|Q(w; s, e^{-\lambda/N})\|} \\
\leq C \frac{\|Q(w; s, e^{-\lambda/N})\|}{N} \frac{1}{\bar{\mu}_2(w) 1}
\]
to infer
\[
0 \leq N \int_0^t dU_1(w) \left[ \Phi(Q(t - w; s, e^{-\lambda/N})) - \Phi(Q(t - w; s, 1)) \right] \\
\leq C \int_0^t dU_1(w) \frac{1}{\bar{\mu}_2(t - w)}.
\]
Using Lemma 11 and the conditions of the theorem we conclude

\[ \int_0^t \frac{dU_1(w)}{\mu_2(t-w)} = O\left( t\mu_2^{-2}(t) \right) = O\left( R(t)/\mu_2(t) \right) = o(1) \]

which completes the proof of Theorem 3.

4.4 Proof of Theorem 4

The second equality in (27) has already been verified in the proof of Corollary 2 under the sole assumption \( R(t)/N \to 0 \). With this at hand the first equality in (27), equivalently,

\[ \lim_{N,t \to \infty} NQ(t; 0, 1 - \lambda/N) = \exp(-D_{22} \lambda) \]

follows from the estimate

\[ 0 \leq N \left( Q(t; 0, 1 - \lambda/N) - Q(t; 1, 1 - \lambda/N) \right) \leq NP_1(t) = O(N/\mu_2(t)) = o(1), \]

where Markov’s inequality has been used for the second inequality and Lemma 12 and the assumption \( \mu_2(t)/N \to \infty \) for the first and the second equalities, respectively. The proof of Theorem 4 is complete.

5 Proofs for the intermediate evolutionary stages

The intermediate evolutionary stages exhibit the most interesting and exotic behavior. Our main technical tool here is the Contraction Principle.

5.1 Proof of Theorem 5

Lemma 15 given below is an important ingredient of the proof of Theorem 5.

**Lemma 15** For each \( \beta \in (0, 1] \) there exists \( \Lambda > 0 \) such that in the domain

\[ \mathcal{K} = \mathcal{K}(\beta, \Lambda) := \{ 0 < \lambda \leq \Lambda \} \subset \mathbb{R} \]

equation (32) has a unique solution in the class of vector-functions with non-negative continuous components.
Proof. We set
\[ \Theta(\lambda) = (\Theta_1, \Theta_2) := \lambda^{-\beta} \Omega \left( \lambda^\beta \right) = \left( \lambda^{-\beta} \Omega_1, \lambda^{-\beta} \Omega_2 \right) \]
and consider the following system which is equivalent to (62):
\[ \Theta(\lambda) = D (0, 1) - \Gamma_\beta \lambda \beta \int_0^1 DN \left( \Theta(\lambda (1 - y)) \right) dy. \] (68)

If a desired solution of (32) exists, then \( \Theta(\lambda) \) is a fixed point of the respective mapping. Thus, it is natural to approximate this fixed point by a sequence of iterates. To this end, starting with \( \Theta(0) \) := \( D (0, 1) \) we define
\[ \Theta^{(n+1)}(\lambda) := D (0, 1) - \Gamma_\beta \lambda \beta \int_0^1 DN \left( \Theta^{(n)}(\lambda (1 - y)) \right) dy \]
for \( n = 0, 1, \ldots \)

The set \( C_+[0, \Lambda] \) of continuous functions \( h : [0, \Lambda] \mapsto [0, \infty) \times [0, \infty) \) equipped with the metric \( \rho(h_1, h_2) = \sup_{\lambda \in [0, \Lambda]} \| h_1(\lambda) - h_2(\lambda) \| \) is a complete metric space. Invoking the Contraction Principle we conclude that it suffices to prove
\[ \Theta^{(n)}(\lambda) \in C_+[0, \Lambda] \] (69)
for \( n = 0, 1, \ldots \) and
\[ \rho(\Theta^{(n+1)}, \Theta^{(n)}) \leq \kappa \rho(\Theta^{(n)}, \Theta^{(n-1)}) \] (70)
for \( n = 1, 2, \ldots \) and appropriate \( \kappa \in (0, 1) \).

In view of
\[ \lambda \beta \int_0^1 DN \left( \Theta^{(0)}(\lambda y) \right) dy \leq C \lambda \beta D \int_0^1 dy \leq C_1 \lambda \beta D (0, 1) \]
where \( \lambda y := \lambda (1 - y) \), there exists a sufficiently small \( \Lambda \in (0, 1) \) such that
\[ \Theta^{(1)}(\lambda) = D (0, 1) - \Gamma_\beta \lambda \beta \int_0^1 DN \left( \Theta^{(0)}(\lambda y) \right) dy \geq 0 \]
for all \( \lambda \in [0, \Lambda] \). Assume now that \( \Theta^{(n)}(\lambda) \geq 0 \) for \( \lambda \in [0, \Lambda] \). Using the estimate
\[ \Theta^{(n)}(\lambda) \leq D (0, 1), \quad \lambda \in [0, \Lambda] \]
and arguing as above we conclude $\Theta^{(n+1)}(\lambda) \geq 0$ for all $\lambda \in [0, \Lambda]$. Continuity of components of $\Theta^{(n)}(\lambda)$, $n = 0, 1, \ldots$ is obvious, and (69) follows.

Further we have

$$
\left\| \Theta^{(n+1)}(\lambda) - \Theta^{(n)}(\lambda) \right\| \leq C \lambda^\beta \int_0^1 \left\| N \left( \Theta^{(n)}(\lambda y) \right) - N \left( \Theta^{(n-1)}(\lambda y) \right) \right\| dy^\beta
$$

for all $\lambda \in [0, \Lambda]$, having utilized (31) and $0 \leq \Theta^{(n)}(\lambda) \leq D (0, 1)^\dagger$, $\lambda \in [0, \Lambda]$, $n = 0, 1, \ldots$ for the second inequality, and thereupon (70) with $\kappa = C_1 \Lambda^\beta$ for $\Lambda > 0$ such that $\kappa < 1$. The proof of Lemma 15 is complete.

**Lemma 16** Suppose that Hypothesis A holds, and

$$
\lim_{N,t \to \infty} R(t)N^{-1} = 1/r \in (0, \infty).
$$

Then, for any $\lambda > 0$,

$$
\lim_{N,t \to \infty} N Q \left( t; 1, e^{-\lambda r/N} \right) = r \Omega(\lambda),
$$

where $\Omega(\lambda) \geq 0$ solves equation (52). Moreover, equation (52) has a unique analytic solution (being a complex-valued vector-function) in the half-plane $\lambda \in \mathbb{C}$, $\Re \lambda > 0$.

**Proof.** For $\lambda > 0$, set

$$
K(t, w; \lambda) = (K_1(t, w; \lambda), K_2(t, w; \lambda)) := R(t/\lambda)Q \left( w; 1, 1 - \frac{1}{R(t/\lambda)} \right)
$$

and

$$
K(t; \lambda) = (K_1(t; \lambda), K_2(t; \lambda)) := K(t, t; \lambda) = R(t/\lambda)Q \left( t; 1, 1 - \frac{1}{R(t/\lambda)} \right).
$$

It suffices to prove the existence of a vector-function $K$ which, in a domain $\mathcal{K}_1 \subseteq \mathcal{K}$ to be specified later, satisfies

$$
K(\lambda) = D (0, 1)^\dagger - \Gamma \lambda^\beta \int_0^1 DN \left( K(\lambda(1 - w)) \right) dw^\beta
$$

(73)
and
\[ \lim_{t \to \infty} K(t; \lambda) = K(\lambda). \] (74)

Indeed, by Lemma 15 equation (68) then has a unique solution \( \Theta(\lambda) = K(\lambda) \) for \( \lambda \in K_1 \) which implies that equation (32) has a unique solution \( \Omega(\lambda) = \lambda \Theta(\lambda^{1/\beta}) \) for \( \lambda \in \mathcal{K}_1 \). Furthermore, (74) entails
\[ \lambda \lim_{t \to \infty} R(t/\lambda^{1/\beta})Q\left(t; 1, 1 - \frac{1}{R(t/\lambda^{1/\beta})}\right) = \Omega(\lambda), \quad \lambda \in \mathcal{K}_1 \]
and thereupon
\[ \lim_{t \to \infty} NQ\left(t; 1, 1 - r\lambda/N\right) = r\Omega(\lambda), \quad \lambda \in \mathcal{K}_1 \]
which implies (71) for \( \lambda \in \mathcal{K}_1 \) in view of Lemma 14. Hence we get
\[ \lim_{N, t \to \infty} F_N^i\left(t; 1, e^{-\lambda r/N}\right) = e^{-r\Omega_i(\lambda)}, \quad \lambda \in \mathcal{K}_1, \ i = 1, 2. \] (75)

Since \( F_N^i\left(t; 1, e^{-\lambda r/N}\right), \ i = 1, 2 \) are the Laplace transforms of nonnegative random variables and the limits exist for \( \lambda \in \mathcal{K}_1 \), it follows by the uniqueness theorem for Laplace transforms that the limits at the left-hand side of (75) exist for all \( \lambda > 0 \). Moreover, the limits are Laplace transforms and, therefore, there exists a function \( \Omega^*(\lambda) = (\Omega^*_1(\lambda), \Omega^*_2(\lambda)), \ \lambda > 0 \) such that
\[ \lim_{N, t \to \infty} NQ\left(t; 1, e^{-\lambda r/N}\right) = r\Omega^*(\lambda), \ \lambda > 0. \]
Also, for \( i = 1, 2 \), the function \( \Omega^*_i(\lambda) \) is analytic for \( \Re \lambda > 0 \) (under an appropriate choice of the branches in the case of singularities) and such that \( \Omega^*_i(\lambda) = \Omega_i(\lambda) \) for \( \lambda \in \mathcal{K}_1 \). Since \( \Omega^*(\lambda) \) solves (32) for \( \lambda \in \mathcal{K}_1 \), it follows from the uniqueness theorem for analytic functions that \( \Omega^*(\lambda) \) solves (32) for all \( \lambda \in \mathbb{C} \) with \( \Re \lambda > 0 \) and, therefore, coincides with \( \Omega(\lambda) \) as desired.

Thus, we concentrate on proving (74) and check that
\[ \lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}_1} \|K(t; \lambda) - \Theta(\lambda)\| = 0 \] (76)
for \( \Theta \) defined in the proof of Lemma 15. To this end, we use (77) to obtain
\[ K(t; \lambda) = R(t/\lambda)U* ((1 - s) \otimes (1 - G(\cdot))) (t) \]
\[ - R(t/\lambda) \int_0^1 dU_1(t_y) \Phi(Q(t_y; s)), \ \lambda > 0, \] (77)
where \( s = (s_1, s_2) := (1, 1 - 1/R(t/\lambda)) \) and \( t_y := t(1 - y) \).

Using (61) and Lemma 12 gives

\[
R(t/\lambda)U * ((1 - s) \otimes (1 - G(\cdot))) (t) = (1 + o(1))R(t/\lambda)DJ(t) (1 - s) = D(0, 1\uparrow) (1 + \delta_0(t; \lambda)),
\]

where \( \lim_{t \to \infty} \delta_0(t; \lambda) = 0 \) uniformly in \( \lambda > 0 \). We have

\[
\Delta_1(t; \varepsilon, \lambda) := R(t/\lambda) \left\| \int_{1-\varepsilon}^1 dU_1(ty) \Phi(Q(ty; s)) \right\|
\leq CR(t/\lambda) \left\| \int_{1-\varepsilon}^1 dU_1(ty) \Phi \left( \frac{1}{R(t/\lambda)} 1 \right) \right\|
\leq \frac{C}{R(t/\lambda)} \|U_1(t) - U_1(t(1 - \varepsilon))\|
\sim C\lambda^\beta (1 - (1 - \varepsilon)^\beta)
\]

for any \( \varepsilon \in (0, 1) \) and all \( \lambda > 0 \) which proves

\[
\lim_{\varepsilon \downarrow 0} \limsup_{t \to \infty} \sup_{\lambda \in K} \Delta_1(t; \varepsilon, \lambda) = 0.
\]

While the fourth line of the last displayed formula follows from (9) and Lemma 8, the second is a consequence of

\[
R(t/\lambda)Q(w; s) \leq R(t/\lambda) (1 - s_2) P_2(w) = P_2(w) \leq C1, \quad w \in [0, t]
\]

which, in its turn, is justified by Lemma 12.

We have, for all \( \lambda \in K \),

\[
R(t/\lambda) \int_{0}^{1-\varepsilon} dU_1(ty) \Phi (Q(ty; s))
= R(t/\lambda) \int_{0}^{1-\varepsilon} dU_1(ty) \cdot \frac{1}{R(t/\lambda)} K(t_y; \lambda_y)
= R(t/\lambda)(1 + \delta_2(t; \lambda)) \int_{0}^{1-\varepsilon} dU_1(ty)N \left( \frac{1}{R(t/\lambda)} K(t_y; \lambda_y) \right)
= \left( \lambda^\beta + \delta_2(t; \lambda) \right) \int_{0}^{1-\varepsilon} \frac{dU_1(ty)}{R(t)} N \left( K(t_y; \lambda_y) \right)
\]

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having utilized (29), (79), (48) and Lemma 8 for the third line. Furthermore,

$$\lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}} |\delta_1(t; \lambda)| = \lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}} |\delta_2(t; \lambda)| = 0$$

because \(\lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}} |R(t)/R(t/\lambda) - \lambda^\beta| = 0\) by Theorem 1.5.2 in [1].

Combining pieces together gives

$$K(t; \lambda) = D(0, 1)^T (1 + \Delta_2(t; \varepsilon, \lambda))$$

$$- \left(\lambda^\beta + \delta_2(t; \lambda) \right) \int_{0}^{1-\varepsilon} \frac{dU_I(ty)}{R(t)} N(K(t; \lambda_y)),$$

(80)

where

$$\lim_{\varepsilon \downarrow 0} \lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}} |\Delta_2(t; \varepsilon, \lambda)| = 0.$$

According to the proof of Lemma 15 each component of the vector-function \(y \mapsto N(\Theta(\lambda (1 - y)))\), \(y \in [0, 1]\), \(\lambda \in \mathcal{K}\) is nonnegative, bounded and continuous. This in combination with Lemma 8 gives

$$\lambda^\beta \Gamma_\beta \int_{0}^{1} dN(\Theta(\lambda_y)) dy^\beta = \lim_{\varepsilon \downarrow 0} \lambda^\beta \int_{0}^{1-\varepsilon} \frac{dU_I(ty)}{R(t)} N(\Theta(\lambda_y))$$

allowing to rewrite equation (80) for \(\lambda \in \mathcal{K}\) as

$$\Theta(\lambda) = D(0, 1)^T (1 + \Delta_3(t; \varepsilon, \lambda)) - \lambda^\beta \int_{0}^{1-\varepsilon} \frac{dU_I(ty)}{R(t)} N(\Theta(\lambda_y)),$$

(81)

where

$$\lim_{\varepsilon \downarrow 0} \lim_{t \to \infty} \sup_{\lambda \in \mathcal{K}} |\Delta_3(t; \varepsilon, \lambda)| = 0.$$

Using (80) and (81) we infer

$$\|K(t; \lambda) - \Theta(\lambda)\|$$

$$\leq \lambda^\beta \left\| \int_{0}^{1-\varepsilon} \frac{dU_I(ty)}{R(t)} (N(K(ty; \lambda_y)) - N(\Theta(\lambda_y))) \right\| + \Delta_4(t; \varepsilon)$$

$$\leq C_1 \lambda^\beta \int_{0}^{1-\varepsilon} \left\| \frac{dU_I(ty)}{R(t)} \right\| \|K(ty; \lambda_y) - \Theta(\lambda_y)\| + \Delta_4(t; \varepsilon)$$

$$\leq C_2 \lambda^\beta \sup_{y \geq t \varepsilon} \theta \in \mathcal{K} \|K(y; \theta) - \Theta(\theta)\| \int_{0}^{1-\varepsilon} \left\| \frac{dU_I(ty)}{R(t)} \right\| + \Delta_4(t; \varepsilon)$$

$$\leq C_3 \lambda^\beta \sup_{w \geq t \varepsilon} \theta \in \mathcal{K} \|K(w; \theta) - \Theta(\theta)\| + \Delta_4(t, \varepsilon)$$

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for \( \lambda \in \mathcal{K} \), where the inequality

\[
\left\| \int_0^{1-\varepsilon} \frac{dU_1(tw)}{R(t)} N\left(K(t_y; \lambda_y)\right) \right\| \leq C \left\| \int_0^{1-\varepsilon} \frac{dU_1(ty)}{R(t)} \right\| \leq C_1
\]

that follows from Lemma 8 and (48) has been utilized for the second line, the third line following from (31) and the last from Lemma 8. Observe that the constants \( C_i, i = 1, 2, 3 \) do not depend on \( \lambda \in \mathcal{K} \) and

\[
\Delta_4(t, \varepsilon) \leq C \sup_{\lambda \in \mathcal{K}} (|\Delta_2(t; \varepsilon, \lambda)| + |\Delta_3(t; \varepsilon, \lambda)| + |\delta_2(t; \lambda)|).
\]

By shrinking, if needed, \( \mathcal{K} \) (which gives \( \mathcal{K}_1 \)) we can and do assume that \( C_3 \sup_{\lambda \in \mathcal{K}_1} \lambda^\beta = C_3 \Lambda^\beta = \kappa < 1 \). Setting

\[
S(t) := \sup_{w \geq t} \sup_{\theta \in \mathcal{K}_1} \| K(w; \theta) - \Theta(\theta) \|
\]

and letting first \( t \to \infty \) and then \( \varepsilon \downarrow 0 \) in the inequality

\[
S(t) \leq \kappa S(t\varepsilon) + \sup_{s \geq t} \Delta_4(s; \varepsilon)
\]

we arrive at (76). The proof of Lemma 16 is herewith finished.

Turning to the proof of Theorem 5 it only takes to observe that (34) is an immediate consequence of Lemma 16 and relation (16), whereas (35) follows from

\[
0 \leq N\left(Q(t; 0, e^{-\lambda r/N}) - Q(t; 1, e^{-\lambda r/N})\right) \leq N\mathcal{P}_1(t) = O(N/\mu_2(t))1 = o(1)1,
\]

where Lemma 12 has been used for the penultimate equality, and the assumption \( \mu_2(t)/N \to \infty \) for the last. The proof of Theorem 5 is complete.

5.2 Proof of Theorem 6

The proof of Theorem 6 rests on the following lemma.

**Lemma 17** For each \( \beta \in (1/2, 1] \) there exists \( \Lambda > 0 \) such that in the domain

\[
\mathcal{K} = \mathcal{K}(\beta, \Lambda, 1) := \{(\theta, \lambda) : 0 < \theta \leq \Lambda, 0 < \lambda \leq 1\} \subset \mathbb{R}^2
\]

equation (36) has a unique solution in the class of vector-functions with non-negative continuous components.
The proof proceeds along the lines of arguments used to demonstrate Lemma 15. Starting with \( H(0)(\theta, \lambda) := C\beta (1, \theta^{1-\beta})^\dagger \) we define

\[
H(n+1)(\theta, \lambda) := C\beta (1, \theta^{1-\beta})^\dagger - \Gamma\beta \theta^{2\beta-1} \int_0^1 \frac{DN(H(n)(\theta, \lambda))}{(1-y)^{2-2\beta}} dy^\beta
\]

for \( n = 0, 1, \ldots \), where \( \theta_y := \theta (1-y), 0 \leq y \leq 1 \).

The set \( C_+ [0, \Lambda] \times [0, 1] \) of functions \( h : [0, \Lambda] \times [0, 1] \to \) \([0, \infty) \times [0, \infty)\) with continuous components equipped with the metric

\[
\rho(h_1, h_2) = \sup_{\theta \in [0, \Lambda], \lambda \in [0, 1]} \|h_1(\theta, \lambda) - h_2(\theta, \lambda)\|
\]

is a complete metric space. Invoking the Contraction Principle we conclude that it suffices to prove

\[
H(n)(\theta, \lambda) \in C_+ [0, \Lambda] \times [0, 1]
\]

for \( n = 0, 1, \ldots \) and

\[
\rho(H^{(n+1)}, H^{(n)}) \leq \kappa \rho(H^{(n)}, H^{(n-1)})
\]

for \( n = 1, 2, \ldots \) and an appropriate \( \kappa \in (0, 1) \).

Note that all the elements of the matrices \( D \) and \( C\beta \) are positive. Therefore, in view of the inequality

\[
\int_0^1 \frac{DN(H(0)(\theta, \lambda))}{(1-y)^{2-2\beta}} dy^\beta \leq C \int_0^1 \frac{\lambda^2 \theta^2 (1-\beta) + \lambda \theta^{1-\beta} + 1}{(1-y)^{2-2\beta}} dy^\beta D1 \leq CD1
\]

which holds for \( \beta \in (1/2, 1], \theta \in [0, 1] \) and \( \lambda \in [0, 1] \), there exists a sufficiently small \( \Lambda > 0 \) such that

\[
H^{(1)}(\theta, \lambda) = C\beta (1, \lambda^{1-\beta})^\dagger - \Gamma\beta \theta^{2\beta-1} \int_0^1 \frac{DN(H(0)(\theta, \lambda))}{(1-y)^{2-2\beta}} dy^\beta \geq 0
\]

for all \( \theta \in [0, \Lambda] \) and \( \lambda \in [0, 1] \).

Assume now that \( H(n)(\theta, \lambda) \geq 0 \) for \( \theta \in [0, \Lambda] \) and \( \lambda \in [0, 1] \). Using the estimate

\[
H^{(n)}(\theta, \lambda) \leq C\beta (1, \lambda^{1-\beta})^\dagger, \ \theta \in [0, \Lambda], \ \lambda \in [0, 1]
\]

for all \( n = 0, 1, \ldots \) and \( \beta \in (1/2, 1] \).
and arguing as above we conclude $H^{(n+1)}(\theta, \lambda) \geq 0$ for all $\theta \in [0, \Lambda]$ and $\lambda \in [0, 1]$. Continuity of components of $H^{(n)}(\theta, \lambda)$, $n = 0, 1, \ldots$ is obvious, and (82) follows.

Further we have

$$\left\| H^{(n+1)}(\theta, \lambda) - H^{(n)}(\theta, \lambda) \right\| \leq C_1 \theta^{2\beta - 1} \int_0^1 \left\| N \left( H^{(n)}(\theta y, \lambda) \right) - N \left( H^{(n)}(\theta y, \lambda) \right) \right\| \frac{dy^\beta}{(1 - y)^{2 - 2\beta}}$$

for all $\theta \in [0, \Lambda]$ and $\lambda \in [0, 1]$, having utilized (31) and (84) for the second inequality, and thereupon (83) with $\kappa = C_2 \Lambda^{2\beta - 1}$ for $\Lambda > 0$ such that $\kappa < 1$.

The proof of Lemma 17 is complete.

Proof of Theorem 6. Although the proof follows the pattern of arguments used to demonstrate Lemma 16, technical details here are more involved. Set

$$K(t; \theta, \lambda) := \mu_2(t) \frac{R(t/\theta)}{\mu_2(t/\theta)} Q \left( t; 1 - \frac{\mu_2(t/\theta)}{\mu_1 R(t/\theta)}, 1 - \frac{\lambda}{R(t/\theta)} \right)$$

for positive $\theta$ and $\lambda$. It suffices to prove the existence of a vector-function $K(\theta, \lambda)$ which, in a domain $K_1 \subset \mathcal{K}(\beta, \Lambda, 1)$ to be specified later, satisfies (85) and such that

$$\lim_{t \to \infty} K(t; \theta, \lambda) = K(\theta, \lambda), \quad (\theta, \lambda) \in K_1.$$  

Indeed, in view of Lemma 17, $K(\theta, \lambda) = H(\theta, \lambda)$ for $(\theta, \lambda) \in K_1$. Furthermore, using (36) with $\theta = \lambda_1^{1/(2\beta - 1)}$ and $\lambda = \lambda_2 \lambda_1^{-\beta/(2\beta - 1)}$ we infer

$$\lim_{t \to \infty} R(t) Q \left( t; \exp \left\{ -\lambda_1 \frac{\mu_2(t)}{\mu_1 R(t)} \right\}, \exp \left\{ -\frac{\lambda_2}{R(t)} \right\} \right) = \lambda_1 H \left( \lambda_1^{1/(2\beta - 1)}, \lambda_2 \lambda_1^{-\beta/(2\beta - 1)} \right), \quad (\theta, \lambda) \in K_1$$

by an appeal to Lemma 14 which applies because the functions $\mu_2(t)/R(t)$ and $1/R(t)$ are regularly varying at $\infty$ with indices $1 - 2\beta < 0$ and $-\beta < 0$, respectively.
Hence we infer for \(\left(\lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)}\right)\) \(\in \mathcal{K}_1\) and \(i = 1, 2\)
\[
\lim_{N,t \to \infty} F_i^{R(t)} \left( t; \exp \left\{ -\lambda_1 \frac{\mu_2(t)}{\mu_1 R(t)} \right\}, \exp \left\{ -\frac{\lambda_2}{R(t)} \right\} \right) = \exp \left\{ -\lambda_1 H_i \left( \lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)} \right) \right\}.
\]

(86)

Since the expressions under the limits in (86) are the Laplace transforms of two-dimensional random vectors with nonnegative components and the limits exist for \(\left(\lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)}\right)\) \(\in \mathcal{K}_1\), where \(\mathcal{K}_1\) contains a ball from \((0, \infty) \times (0, \infty)\), it follows by the uniqueness theorem for Laplace transforms in two variables that the limit in (86) exists for all \(\lambda_1 > 0, \lambda_2 > 0\). Moreover, there exists a vector-function
\[
\Omega^* (\lambda_1, \lambda_2) = (\Omega_1^* (\lambda_1, \lambda_2), \Omega_2^* (\lambda_1, \lambda_2)), \quad \lambda_1 > 0, \quad \lambda_2 > 0,
\]
whose components are analytic in the domain \(\{\Re\lambda_1 > 0, \Re\lambda_2 > 0\}\) (under an appropriate choice of the branches) and such that, for \(i = 1, 2\),
\[
\lim_{N,t \to \infty} F_i^{R(t)} \left( t; \exp \left\{ -\lambda_1 \frac{\mu_2(t)}{\mu_1 R(t)} \right\}, \exp \left\{ -\frac{\lambda_2}{R(t)} \right\} \right) = e^{-\Omega_i^* (\lambda_1, \lambda_2)}
\]

In particular,
\[
\lambda_1 H_i \left( \lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)} \right) = \Omega_i^* (\lambda_1, \lambda_2)
\]

for \(\left(\lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)}\right)\) \(\in \mathcal{K}_1\). Since \(\lambda_i^{-1}\Omega^* (\lambda_1, \lambda_2)\) solves (36) for \(\left(\lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)}\right)\) \(\in \mathcal{K}_1\) it follows from the uniqueness theorem for analytic functions in two variables that \(\lambda_i^{-1}\Omega^* (\lambda_1, \lambda_2)\) solves (36) in the domain \(\{\Re\lambda_1 > 0, \Re\lambda_2 > 0\}\) and, therefore, coincides with \(H \left( \lambda_1^{1/(2\beta-1)}, \lambda_2 \lambda_1^{-\beta/(2\beta-1)} \right)\)
as desired.

The major step towards proving (85) is to check that
\[
\lim_{t \to \infty} \sup_{(\theta, \lambda) \in \mathcal{K}_1} \|K(t; \theta, \lambda) - H(\theta, \lambda)\| = 0. \tag{87}
\]

To this end, we use (87) and the equality
\[
Q \left( t_y; 1 - \frac{\mu_2(t/\theta)}{\mu_1 R(t/\theta)}, 1 - \frac{\lambda}{R(t/\theta)} \right) = \frac{\mu_2(t/\theta)}{R(t/\theta)} \frac{1}{\mu_2(t_y)} K(t_y; \theta_y, \lambda), \quad y \in [0, 1)
\]

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where \( t_y = t(1 - y) \) and \( \theta_y = \theta(1 - y) \), to obtain

\[
K(t; \theta, \lambda) = \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \left( \mathbf{U} \ast ((1 - s) \otimes (1 - \mathbf{G}(\cdot))) (t) \right.
\]

\[
- \int_0^1 d\mathbf{U}_1(ty) \Phi(Q(ty; s)) \left) \right.
\]

\[
= \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \left( \mathbf{U} \ast ((1 - s) \otimes (1 - \mathbf{G}(\cdot))) (t) \right.
\]

\[
- \int_0^{1-\varepsilon} d\mathbf{U}_1(ty) \Phi \left( \frac{\mu_2(t/\theta)}{R(t/\theta)} \frac{1}{\mu_2(t_y)} K(t_y; \theta_y, \lambda) \right) \right) - J_1(t; \varepsilon, \theta)
\]

for positive \( \theta \) and \( \lambda \) and any fixed \( \varepsilon \in (0, 1) \), where

\[
s = (s_1, s_2) := \left( 1 - \frac{\mu_2(t/\theta)}{\mu_1 R(t/\theta)}, 1 - \frac{\lambda}{R(t/\theta)} \right)
\]

and

\[
J_1(t; \varepsilon, \theta) := \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \int_{t(1-\varepsilon)}^t \frac{1}{d\mathbf{U}_1(w)} \Phi(Q(t - w; s)).
\]

Noting that \( \mu_2(t) \) is regularly varying at \( \infty \) with index \( 1 - \beta \) and using (61) and then Lemma 12 give

\[
\frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \mathbf{U} \ast ((1 - s) \otimes (1 - \mathbf{G}(\cdot))) (t) = \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \mathbf{P}(t)(1 - s)
\]

\[
= (1 + \delta_0(t; \theta, \lambda)) \mathbf{C}_\beta \left( 1, \lambda \theta^{1 - \beta} \right)^\dagger
\]

where \( \lim_{t \to \infty} \delta_0(t; \delta, \lambda) = 0 \) uniformly in \( (\delta, \lambda) \in \mathcal{K} \) (while uniformity in \( \theta \) follows from the regular variation of \( \mu_2(t) \) and Theorem 1.5.2 in [1], uniformity in \( \lambda \) is trivial).

To proceed we intend to show that

\[
\int_{t(1-\varepsilon)}^t \frac{d\mathbf{U}_1(y)}{\mu_2(t - y)} \sim \frac{R(t)}{\mu_2(t)} \int_{1-\varepsilon}^1 (1 - y)^{2\beta - 2} y^{\beta - 1} dy \beta \mathbf{D}, \quad t \to \infty
\]

(89)

for any fixed \( \varepsilon \in (0, 1) \), where, as before, \( \hat{\mu}_2(t) = \mu_2(t) + 1 \).

Since

\[
\lim_{t \to \infty} \mathbf{U}_1(ty)/R(t) = \Gamma_\beta y^\beta, \quad y > 0
\]

(90)

by Lemma 9 and \( \lim_{t \to \infty} \hat{\mu}_2(t)/\mu_2(t(1 - y)) = (1 - y)^{2\beta - 2} \) uniformly in \( y \in [0, 1 - \varepsilon] \) we conclude that

\[
\lim_{t \to \infty} \int_{0}^{1-\varepsilon} \frac{\hat{\mu}_2(t)}{\mu_2(t(1 - y))} d\mathbf{U}_1(ty) = \int_{0}^{1-\varepsilon} (1 - y)^{2\beta - 2} y^{\beta - 1} dy \beta \mathbf{D} \quad (91)
\]

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by an application of Lemma A.5 in [3]. A combination of (91) with
\[ \int_0^1 \frac{dU_1(y)}{\hat{\mu}_2'(t-y)} \sim \frac{R(t)}{\mu_2'(t)} \int_0^1 (1-y)^{2\beta-2} y^{\beta-1} dy \Gamma_\beta D, \quad t \to \infty \]
which is just an equivalent form of one part of Lemma 11 proves (89).

As another preparatory result we obtain
\[ Q(w, s) \leq P(w) (1-s) = \frac{1}{R(t/\theta)} P(w) \left( \frac{\mu_2(t/\theta)}{\lambda} \right) \]
\[ \leq \frac{C}{R(t/\theta)} \left( \frac{\mu_2(t/\theta)}{\mu_2(w)} + \lambda \right) 1 \leq C \frac{\mu_2(t/\theta)}{R(t/\theta) \mu_2(w)} 1 \]
(92)
for \( w \leq t \) and positive \( \theta \) and \( \lambda \), where the second inequality follows from
\[ P_1(t) \leq C/\hat{\mu}_2(t), \quad P_2(t) \leq C, \quad t \geq 0 \]
which, in its turn, is a consequence of Lemma 12.

Using (92) and then (89) we have
\[ J_1(t; \epsilon, \theta) \leq \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \int_{t(1-\epsilon)}^t dU_1(w) \Phi \left( C \frac{\mu_2(t/\theta)}{R(t/\theta) \hat{\mu}_2(t-w)} 1 \right) \]
\[ \leq C \mu_2(t) \frac{\mu_2(t/\theta)}{R(t/\theta)} \int_{t(1-\epsilon)}^t \frac{dU_1(w)}{\mu_2'(t-w)} 1 \]
\[ \sim \frac{\mu_2(t/\theta)}{R(t/\theta)} \mu_2(t) \int_{1-\epsilon}^1 (1-y)^{2\beta-2} y^{\beta-1} dy C \beta \Gamma_\beta D 1 \]
for positive \( \theta \) and \( \lambda \), and any fixed \( \epsilon \in (0, 1) \) which implies
\[ \lim_{\epsilon \to 0} \lim_{t \to \infty} \sup_{(\theta, \lambda) \in K} ||J_1(t; \epsilon, \theta)|| = 0 \]
by an application of Theorem 1.5.2 in [1].

We divide the subsequent proof into two parts according to whether \( \Phi(x) = N(x) \) or \( \Phi \) is not restricted in this way.

CASE \( \Phi = N \), i.e., the generating functions of the reproduction laws are polynomials of degree 2. The advantage of this case is that equality (88) takes a simpler form
\[ K(t; \theta, \lambda) = \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) U_* ((1-s) \otimes (1-G(\cdot))(t) - J_1(t; \epsilon, \theta)) \]
\[ - \frac{\mu_2(t/\theta) R(t)}{\mu_2(t) R(t/\theta)} \int_0^{1-\epsilon} dU_1(ty) \frac{\mu_2'(t)}{\mu_2'(ty)} N(K(ty; \theta_y, \lambda)) \]
(93)
which, in view of our findings in the preceding part of the proof, can be represented as follows

\[
K(t; \theta, \lambda) = C_\beta \left( 1, \lambda \theta^{1-\beta} \right) + \delta_1 (t; \varepsilon, \theta, \lambda) 1 \\
- \theta^{2\beta-1} \int_0^{1-\varepsilon} \frac{dU(ty) N(K(ty; \theta, \lambda))}{R(t)} (1 - y)^{-2\beta}
\]

where

\[
\lim_{\varepsilon \downarrow 0} \lim_{t \to \infty} \sup_{(\theta, \lambda) \in K} |\delta_1 (t; \varepsilon, \theta, \lambda)| = 0. \quad (94)
\]

Applying (90) and arguing as in the proof of Lemma 16 we can show that

\[
H(\theta, \lambda) = C_\beta \left( 1, \lambda \theta^{1-\beta} \right) + \delta_2 (t; \varepsilon, \theta, \lambda) 1 \\
- \theta^{2\beta-1} \int_0^{1-\varepsilon} \frac{dU(ty) N(H(\theta, \lambda))}{R(t)} (1 - y)^{-2\beta}
\]

for \((\theta, \lambda) \in K\) and any fixed \(\varepsilon \in (0, 1)\), where

\[
\lim_{\varepsilon \downarrow 0} \lim_{t \to \infty} \sup_{(\theta, \lambda) \in K} |\delta_2 (t; \varepsilon, \theta, \lambda)| = 0. \quad (95)
\]

Put

\[
V(ty; \theta, \lambda) := N(K(ty; \theta, \lambda)) - N(H(\theta, \lambda)), \\
L(t; \theta, \lambda) := \|K(t; \theta, \lambda) - H(\theta, \lambda)\|. 
\]

Using (92) with \(w = ty\) for \(y \in (0, 1 - \varepsilon)\) gives

\[
K(ty; \theta, \lambda) \leq C 1
\]

which together with the boundedness of \(H(\theta, \lambda)\) in \((\theta, \lambda) \in K\) which follows from (36) and (31) implies that

\[
\|V(ty; \theta, \lambda)\| \leq C L(ty; \theta, \lambda).
\]

for \((\theta, \lambda) \in K\). Setting \(\delta(t; \varepsilon, \theta, \lambda) := |\delta_1 (t; \varepsilon, \theta, \lambda)| + |\delta_2 (t; \varepsilon, \theta, \lambda)|\) we get

\[
L(t; \theta, \lambda) \leq C \theta^{2\beta-1} \left( \int_0^{1-\varepsilon} \frac{dU(tw) V(ty; \theta, \lambda)}{R(t)} (1 - y)^{-2\beta} \right)
\]

\[
\leq C \theta^{2\beta-1} \left( \int_0^{1-\varepsilon} \frac{dU(tw) L(ty; \theta, \lambda)}{R(t)} (1 - y)^{-2\beta} \right)
\]

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and thereupon
\[ L(t; \theta, \lambda) \leq C \delta(t; \varepsilon, \theta, \lambda) + C_1 \theta^{2\beta - 1} \sup_{w \geq t; (\theta^*, \lambda^*) \in \mathcal{K}} L(w; \theta^*, \lambda^*) \]
for \((\theta, \lambda) \in \mathcal{K}\). By shrinking, if needed, \(\mathcal{K}\) (which gives \(\mathcal{K}_1\)) we can and do assume that \(\kappa := C_1 \Lambda^{2\beta - 1} < 1\). Setting
\[ S(t) := \sup_{w \geq t; (\theta, \lambda) \in \mathcal{K}_1} L(w; \theta, \lambda) \]
and letting first \(t \to \infty\) and then \(\varepsilon \downarrow 0\) (at which step relations (94) and (95) have to be recalled) in the inequality
\[ S(t) \leq C \sup_{s \geq t; (\theta, \lambda) \in \mathcal{K}_1} \delta(t; \varepsilon, \theta, \lambda) + \kappa S(t \varepsilon) \]
we arrive at (87).

This completes the proof of the theorem under the assumption \(\Phi = N\). **General case** can be treated along the same lines after noting that representation (29) along with relations (92) and (91) implies
\[ \frac{\mu_2(t)}{\mu_2(t/\theta)} R(t/\theta) \int_0^{1-\varepsilon} \|dU_1(ty)\| \left\| \Phi(Q(ty; s)) - \Phi(Q(ty; s)) \right\| = o(1) \]
for any \(\varepsilon \in (0, 1)\). Furthermore, this convergence is uniform in \((\theta, \lambda) \in \mathcal{K}_1\) by Theorem 1.5.2 in [1]. The proof of Theorem 6 is complete.

6 Proofs for the final evolutionary stages

The purpose of this section is to prove Theorem 7 which will be done in a series of lemmas.

The following result was obtained in Lemma 2 in [6].

**Lemma 18** Assume that the Perron root of the aperiodic irreducible matrix \(M\) in (2) is equal to 1. Then the generating vector-function \(F(t; s)\) is non-decreasing in \(t\) for each \(s \in \mathcal{A} := \left\{ s \in [0, 1]^2 : f(s) \geq s \right\} \).
Recalling (3) we set
\[ Q(t; s) := (v, Q(t; s)). \]

The next statement is a natural generalization of Theorem 1 in [2].

**Lemma 19** Let \( Z(t) \) be an irreducible, aperiodic, and critical process satisfying conditions (5) and (6). Then, for each \( \lambda > 0 \) and any non-increasing function \( \psi \) such that \( \lim_{t \to \infty} \psi(t) = 0 \), we have
\[
\lim_{t \to \infty} \frac{1 - F(t; 1 - \lambda \psi(t))}{\Omega(t; 1 - \lambda \psi(t))} = \lim_{t \to \infty} \frac{Q(t; 1 - \lambda \psi(t))}{\Omega(t; 1 - \lambda \psi(t))} = u.
\]

**Proof.** For \( t \) large enough to ensure \( 1 - \lambda \psi(t) \in [0, 1]^2 \) use
\[
1 - f(1 - \lambda \psi(t)) \leq \lambda \psi(t)MU = \lambda \psi(t)u
\]
to infer \( 1 - \lambda \psi(t) \in A \) and further
\[
F(t; 1 - \lambda \psi(t)) \geq F(t - w; 1 - \lambda \psi(t)) \geq F(t - w; \max(1 - \lambda u_1 \psi(t - w), 0), \max(1 - \lambda u_2 \psi(t - w), 0))
\]
for each \( w \in (0, t] \), having utilized Lemma 18 for the first inequality and the monotonicity of \( \psi \) for the second. A minor modification of the proof of Theorem 1 in [2] finishes the proof of the lemma.

**Lemma 20** Suppose that Hypothesis A holds. Then, for each \( \lambda > 0 \) and any non-increasing function \( \psi \) such that \( \lim_{t \to \infty} \psi(t) = 0 \) and \( 1/R(t) = o(\psi(t)) \), we have
\[
\Omega(t; 1 - \lambda \psi(t)) = o(\psi(t)).
\]

**Proof.** In view of (48) and Lemma 8
\[
U_1(t) \sim U(t) \sim \Gamma_\beta R(t) D. \tag{96}
\]
According to Lemma 18 the vector-function \( \Phi(Q(t; s)) \) is non-increasing in \( t \) for each fixed \( s \in A \). This enables us to infer from (57)
\[
U_1(t)\Phi(Q(t; s)) \leq \int_0^t dU_1(w)\Phi(Q(t - w; s)) \leq U_a((1 - s) \otimes (1 - G(\cdot))) (t) \tag{97}
\]
for each fixed \( s \in A \).
Put \( s = s(t) := 1 - \lambda u \psi(t) \) and use \((61)\) and Lemma \([12]\) to obtain

\[
U \ast ((1 - s) \otimes (1 - G(\cdot))) (t) = (1 + o(1)) \lambda \psi(t) D J(t) 1 = O(\psi(t)) 1
\]

which entails

\[
U_1(t) \Phi(Q(t; 1 - \lambda u \psi(t))) = O(\psi(t)) 1
\]

by an appeal to \((97)\) which applies because \( s(t) \in A \) for large enough \( t \) (see the proof of Lemma \([19]\)). Recalling \((96)\) we conclude

\[
\Phi(Q(t; 1 - \lambda u \psi(t))) = O\left(\psi(t)/R(t)\right) 1 = o\left(\psi^2(t)\right) 1
\]

which in view of \((29)\) and \((28)\) ensures

\[
Q(t; 1 - \lambda u \psi(t)) = o(\psi(t)) 1.
\]

The proof of Lemma \([20]\) is complete.

**Lemma 21** Suppose that Hypothesis \( A(0, 1) \) and conditions \((43)\) hold, and that

\[
1/R(t) = o(\psi(t)) \quad \text{for a non-increasing regularly varying function } \psi(t) = t^{-\gamma} \ell_2(t), \quad \gamma \in [0, 1), \quad \text{such that } \lim_{t \to \infty} \ell_2(t) = 0.
\]

Then, for each \( \lambda > 0 \),

\[
Q(t; e^{-\lambda u \psi(t)}) \sim \sqrt{\lambda} \sqrt{\frac{\nu_2 H_2}{B}} \psi(t)(1 - G_2(t)) u^\dagger.
\]

If, in addition, condition \((41)\) holds, then

\[
\lim_{N,t \to \infty} N Q(t; 1, e^{-\lambda u_2 \psi(t)}) = r \sqrt{\lambda} u^\dagger.
\]

**Proof.** Using \((41)\) with \( s = 1 - \lambda u \psi(t) \) for large enough \( t \) to ensure \( s \in A \) we get

\[
Q(t; 1 - \lambda u \psi(t)) = \lambda \psi(t) u \otimes (1 - G(t))
\]

\[
+ \int_0^t (1 - f(F(t - w; 1 - \lambda u \psi(t)))) \otimes dG(w)
\]

\[
\geq \lambda \psi(t) u \otimes (1 - G(t)) + (1 - f(F(t; 1 - \lambda u \psi(t)))) \otimes G(t)
\]

and, therefore,

\[
\Phi(Q(t; 1 - \lambda u \psi(t))) \otimes G(t) \geq \lambda \psi(t) u \otimes (1 - G(t))
\]

\[
+ MQ(t; 1 - \lambda u \psi(t)) \otimes G(t) - Q(t; 1 - \lambda u \psi(t)).
\]
Multiplying both sides of this inequality by \( v \), the left eigenvector of \( M \) corresponding to the Perron root 1, and using the equality \((v, Q(t; s)) = (v, MQ(t; s))\) we obtain

\[
(v, \Phi(Q(t; 1 - \lambda u \psi(t)))) \geq \lambda \psi(t)(v, u \otimes (1 - G(t)))
\]

(100)

which in combination with

\[
(v, MQ(t; 1 - \lambda u \psi(t))) = o(\psi(t))
\]

(see Lemma 20) implies

\[
\liminf_{t \to \infty} \frac{(v, \Phi(Q(t; 1 - \lambda u \psi(t))))}{\psi(t)(v, u \otimes (1 - G(t)))} \geq \lambda.
\]

(101)

Hence

\[
\liminf_{T \to \infty} \frac{\int_0^T t(v, \Phi(Q(t; 1 - \lambda u \psi(t))))dt}{\int_0^T t\psi(t)(v, u \otimes (1 - G(t)))dt} \geq \lambda
\]

(102)

by a variant of l'Hôpital rule which is applicable because

\[
\int_0^\infty t\psi(t)(v, u \otimes (1 - G(t)))dt = \infty.
\]

Now we intend to prove the converse inequality for the upper limit. To this end, we will use the equality

\[
(v, \int_0^t (1 - f(F(t - w; s))) \otimes dG(w) - (v, 1 - f(F(t; s)))
\]

\[
= (v, \Phi(Q(t; s))) - (v, (1 - s) \otimes (1 - G(t)))
\]

(103)

which is a consequence of (11) and \( vM = v \). Denoting

\[
\Delta(T) := \int_0^T t\, dt \int_0^t (1 - f(F(t - w; s))) \otimes dG(w)
\]

\[
- \int_0^T t(t (1 - f(F(t; s(t))))) dt
\]

\[
= \int_0^T dG(w) \otimes \int_0^{T-w} (t + w) (1 - f(F(t; s(t + w)))) dt
\]

\[
- \int_0^T t(t (1 - f(F(t; s(t))))) dt,
\]

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where \( s(t) = 1 - \lambda u \psi(t) \) we observe that (103) with \( s = s(t) \) is equivalent to

\[
(v, \Delta(T)) = \int_0^T t(v, \Phi(Q(t; s(t)))) dt - \lambda \int_0^T t \psi(t)(v, u \otimes (1 - G(t))) dt.
\]

(104)

By the monotonicity of \( \psi(t) \) we have

\[
\int_0^T dG(w) \otimes \int_0^{T-w} t (1 - f(F(t; s(t+w)))) dt \leq \int_0^T t (1 - f(F(t; s(t)))) dt
\]

which implies

\[
\Delta(T) \leq \int_0^T dG(w) \otimes \int_0^{T-w} [(t+w) - t] (1 - f(F(t; s(t+w)))) dt
\]

\[
= \int_0^T w dG(w) \otimes \int_0^{T-w} (1 - f(F(t; s(t+w)))) dt
\]

and further

\[
\Delta(T) \leq \int_0^T w dG(w) \otimes \int_0^{T-w} MQ(t; s(t+w))) dt
\]

\[
\leq \int_0^T w dG(w) \otimes \int_0^T MQ(t; s(t))) dt
\]

\[
= \int_0^T w dG(w) \otimes \int_0^T o(\psi(t)) dt = o\left(1 \int_0^T w \int_0^T \psi(t) dtdG(w)\right),
\]

having utilized \( \Phi(Q(t; s)) = MQ(t; s) - (1 - f(F(t; s))) \geq 0 \) for the first inequality, and the monotonicity of \( \psi \) for the second. The first equality follows from Lemmas 19 and 20, and the second equality is a consequence of \( \int_0^\infty \psi(t) dt = \infty \).

Recalling that \( \psi \) is regularly varying at \( \infty \) with index \( -\gamma, \gamma \in [0,1) \) and invoking Proposition 1.5.8 in \cite{1} we infer

\[
\int_0^T w dG_2(w) \int_0^T \psi(t) dt \sim \frac{\beta}{(1-\beta)(1-\gamma)} T^2 \psi(T)(1 - G_2(T))
\]

and

\[
\int_0^T t \psi(t)(v, u \otimes (1 - G(t))) dt \sim \frac{\nu_2 \nu_2}{2 - \gamma - \beta} T^2 \psi(T)(1 - G_2(T)) \quad (105)
\]

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which implies
\[
\int_0^T w dG(w) \int_0^T \psi(t) dt = O \left( 1 \int_0^T t \psi(t) (v, u \otimes (1 - G(t))) dt \right)
\]
for \(\int_0^\infty w dG_1(w) = \mu_1 < \infty\). Therefore
\[
(v, \Delta(T)) = o \left( \int_0^T t \psi(t) (v, u \otimes (1 - G(t))) dt \right)
\]
which entails
\[
\limsup_{T \to \infty} \frac{\int_0^T t(v, \Phi(Q(t; s(t)))) dt}{\int_0^T t \psi(t) (v, u \otimes (1 - G(t))) dt} \leq \lambda
\]
in view of (104). Combining this with (102) gives
\[
\lambda \int_0^T \psi(t) (v, u \otimes (1 - G(t))) dt \sim \int_0^T \psi(t) (v, \Phi(Q(t; 1 - \lambda \psi(t)))) dt.
\]
Left with the proof of
\[
\lambda \psi(t) (v, u \otimes (1 - G(t))) \sim (v, \Phi(Q(t; 1 - \lambda \psi(t))))
\]
we infer by means of (105)
\[
\int_T^{\delta T} t(v, \Phi(Q(t; 1 - \lambda \psi(t)))) dt \sim \frac{\lambda(\delta^{2-\gamma-\beta}-1)}{2-\gamma-\beta} T^2 \psi(T) (v, u \otimes (1 - G(T)))
\]
for \(\delta > 1\). The function \(t \mapsto (v, \Phi(Q(t; 1 - \lambda \psi(t))))\) is eventually non-increasing whence
\[
\int_T^{\delta T} t(v, \Phi(Q(t; 1 - \lambda \psi(t)))) dt \leq (\delta - 1) \delta T^2 (v, \Phi(Q(T; 1 - \lambda \psi(T))))
\]
for large enough \(T\), and thereupon
\[
\liminf_{t \to \infty} \frac{(v, \Phi(Q(t; 1 - \lambda \psi(t))))}{\psi(t) (v, u \otimes (1 - G(t)))} \geq \frac{\lambda(\delta^{2-\gamma-\beta}-1)}{(\delta - 1) \delta (2 - \gamma - \beta)}.
\]
Letting now \(\delta \downarrow 1\) yields
\[
\liminf_{t \to \infty} \frac{(v, \Phi(Q(t; 1 - \lambda \psi(t))))}{\psi(t) (v, u \otimes (1 - G(t)))} \geq \lambda.
\]
The proof of the converse inequality for the upper limit proceeds similarly, starting with \( \int_{
abla T}^{T} t(\nabla, \Phi(Q(t; 1 - \lambda u \psi(t)))) dt \) for \( \delta \in (0, 1) \).

Using (106), (29) and Lemma 19 we infer
\[
\lambda \psi(t) v_2 u_2 (1 - G_2(t)) \sim B \Omega^2(t, 1 - \lambda u \psi(t))
\]
and then
\[
Q(t; 1 - \lambda u \psi(t)) \sim \sqrt{\lambda} \sqrt{\frac{v_2 u_2}{B}} \psi(t)(1 - G_2(t)) u^{\dagger}
\] (107)
by another appeal to Lemma 19. Since \( \lim_{t \to \infty} \psi(t) = 0 \) and, in view of (13), \( \lim_{t \to \infty} \psi(t)(1 - G_2(t)) = 0 \) Lemma 13 implies that (98) is a consequence of (107).

It remains to prove (99) under additional assumption (11). To this end, we first note that (98) and (13) together imply
\[
\lim_{N,t \to \infty} NQ(t; e^{-\lambda u \psi(t)}) = r \sqrt{\lambda} u^{\dagger}.
\]
Observe further that
\[
0 \leq Q(t; e^{-\lambda u \psi(t)}) - Q(t; 1, e^{-\lambda u_2 \psi(t)})
\]
\[
\leq \lambda u_1 \psi(t) P_1 (t) \sim \lambda \mu_1 \beta \Gamma_{\beta} u_1 \frac{\psi(t)}{\mu_2(t)} (D_{11}, D_{21})^{\dagger},
\]
where the last equivalence follows from Lemma 12 which in combination with (11) yields
\[
NQ(t; e^{-\lambda u \psi(t)}) \sim NQ(t; 1, e^{-\lambda u_2 \psi(t)}).
\]
The proof of Lemma 21 is complete.

Passing to the proof of Theorem 7 we note that (44) follows from (99) and (16), while (45) is a consequence of (98), (99), (16) and
\[
0 \leq Q(t; 0, e^{-\lambda u_2 \psi(t)}) - Q(t; 1, e^{-\lambda u_2 \psi(t)})
\]
\[
\leq P_1 (t) \sim \mu_1 \beta \Gamma_{\beta} \frac{1}{\mu_2(t)} (D_{11}, D_{21})^{\dagger}.
\] (108)
The proof of Theorem 7 is complete.

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