Algebras of Quantum Variables for Loop Quantum Gravity

II. A new formulation of the Weyl $C^*$-algebra

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Abstract

In this article a new formulation of the Weyl $C^*$-algebra, which has been invented by Fleischhack [7], in terms of $C^*$-dynamical systems is presented. The quantum configuration variables are given by the holonomies along paths in a graph. Functions depending on these quantum variables form the analytic holonomy $C^*$-algebra. Each classical flux variable is quantised as an element of a flux group associated to a certain surface set and a graph. The quantised spatial diffeomorphisms are elements of the group of bisections of a finite graph system. Then different actions of the flux group associated to surfaces and the group of bisections on the analytic holonomy $C^*$-algebra are studied. The Weyl $C^*$-algebra for surfaces is generated by unitary operators, which implements the group-valued quantum flux operators, and certain functions depending on holonomies along paths that satisfy canonical commutation relations. Furthermore there is a unique pure state on the commutative Weyl $C^*$-algebra for surfaces, which is a path- or graph-diffeomorphism invariant.

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1 Introduction

The LQG-viewpoint

In LQG the algebra of holonomy variables has been introduced by Ashtekar and Lewandowski [1]. The analytic holonomy algebra is given by a commutative $C^*$-algebra $C(\tilde{A})$, which is an inductive limit of a family $\{C(\tilde{A}_\Gamma)\}$ of commutative $C^*$-algebras associated to graphs. The inductive limit of $C^*$-algebras corresponds to a projective limit of a family $\{A_\Gamma\}$ of configuration spaces. Due to the Tychonov-theorem the inductive limit space $\tilde{A}$, which is constructed from the compact Hausdorff spaces, is a compact Hausdorff space, too. Each configuration space $\tilde{A}_\Gamma$ is identified with $G^{[\Gamma]}$ where $G$ is the structure group of a principal fibre bundle. Usually this group is chosen to be a compact Lie group. On the configuration space $A_\Gamma$ associated to each graph $\Gamma$ there exists a Haar measure $\mu_\Gamma$. The consistent family $\{\mu_\Gamma\}$ of measures defines a measure $\mu_{AL}$ on $\tilde{A}$. Homeomorphisms on the compact Hausdorff space $\tilde{A}$ leaving the measure $\mu_{AL}$ invariant corresponds one-to-one to unitary operators $U(g)$ for elements $g$, which are contained in the structure group $G$. These unitary operators implement the fluxes associated to a surface $S$. In particular, measure preserving transformations associated to a graph $\Gamma$ define $G^{[\Gamma]}$-invariant states on the $C^*$-algebra $C(\tilde{A}_\Gamma)$. For a detailed investigation of this construction in the context of compact Hausdorff spaces and measures refer to Marolf and Mourão [23] for graphs containing only analytic loops and Fleischhack [6] for general index sets. A study of the interplay of the projective structure of the configuration space and the inductive structure of the $C^*$-algebra is given in the article of Ashtekar and Lewandowski [2], Fleischhack [8], Velhinho [30, 31]. Although several other diffeomorphism invariant states on $C(\tilde{A})$ are available due to the work of Baez [4, 5], or, Ashtekar and Lewandowski [2, 3], only states which are $G^{[\Gamma]}$-invariant will allow to extend the quantum algebra by the flux operators for a surface $S$. This question has been analysed for example by Sahlmann in [24]. In the context of Weyl algebras constructed from holonomies and quantum flux operators, which are exponentiated Lie algebra-valued operators, the first attempts are due to Sahlmann and Thiemann in [26]. The Weyl algebra of holonomies and (exponentiated) quantum fluxes, which are introduced by particular pull-backs of homeomorphisms on the configuration space, has been constructed by Fleischhack in [7]. The development of the Weyl algebra can be related to transformation groups associated to a flux group and the configuration space. First attempts in this direction has been presented by Velhinho in [32]. The irreducibility of the Weyl $C^*$-algebra has been studied first by Sahlmann and Thiemann in [26]. Fleischhack has proved in [4] irreducibility and under some technical assumptions that there is only one irreducible and diffeomorphism-invariant representation of his Weyl $C^*$-algebra on the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$. For a short overview refer to Fleischhack [9]. In comparison with the Weyl algebra presented in the project $AQV$, Fleischhack has considered more general stratified objects, instead of $D − 1$-dimensional surfaces in a $D$-dimensional manifold only, for the construction of his Weyl $C^*$-algebra.

The operator-algebraic viewpoint

The quantum configuration variables: holonomies along paths

The fundamental geometric objects for a theory of Loop Quantum Gravity are (semi-) analytic paths and loops that form graphs. In section 2 the basic quantum variables derived from these geometric objects will be introduced. A short overview will be presented in this section.

A graph contains a finite set of independent edges. A set of edges is called independent if the edges only intersect each other in the source or target vertices. A finite groupoid is a finite set of paths equipped with a groupoid structure. The finite graph system associated to a graph $\Gamma$ is given by all subgraphs of $\Gamma$. A finite path groupoid associated to the graph $\Gamma$ is generated by all compositions of elements or their inverse elements of the set of edges that defines the graph $\Gamma$. Note that an element of a finite path groupoid is not necessarily an independent path. Clearly, for all these objects there exists an ordering such that

(i) an inductive family of graphs

(ii) an inductive family of finite path groupoids and

(iii) an inductive family of finite graph systems can be studied.
Furthermore, a holonomy map is a groupoid morphism from the path groupoid to the compact structure group $G$. If a graph is considered, then the holonomy map maps each edge of the graph to an element of the structure group $G$. For generality it is assumed that $G$ is a compact group. In 2.2 two ways of an identification of the holonomy map evaluated for a subgraph of $\Gamma$ with elements in $G^{|\Gamma|}$ will be presented. One distinguishes between the natural or the non-standard identification of the configuration space $\mathcal{A}_\Gamma$ with $G^{|\Gamma|}$. Recall that a subgraph of $\Gamma$ is a set of independent paths, which are generated by the edges of the graph $\Gamma$. In the natural identification these paths are decomposed into the edges, which define the graph $\Gamma$. In the non-standard identification only graphs that contain only non-composable paths are considered. In both cases the holonomy maps evaluated on a subgraph $\Gamma$ of $\Gamma$. In the non-standard identification only graphs that contain only non-composable paths are considered. In both cases the holonomy maps evaluated on a subgraph $\Gamma$ are elements of $G^{|\Gamma|}$, where $M$ is the number of paths in $\Gamma'$, $\Gamma'$ is a set of independent paths and $\Gamma$ is the number of edges in $\Gamma$. One obtains a product group $G^M$ for $M \leq |\Gamma|$, and which is embedded into $G^{|\Gamma|}$ by $G^M \times \{e_G\} \times \ldots \times \{e_G\}$. Hence, in both cases the holonomy evaluated on a subgraph of a graph $\Gamma$ is an element of $G^{|\Gamma|}$. In LQG 1, 2, 29 a holonomy map evaluated at the graph $\Gamma$ is an element of $G^{|\Gamma|}$, too.

The analytic holonomy $C^*$-algebra restricted to a finite graph system associated to a graph is given by the commutative unital $C^*$-algebra $C(\mathcal{A}_\Gamma)$ of continuous functions on the configuration space $\mathcal{A}_\Gamma$ vanishing at infinity and supremum norm.

In the project $AQV$ the inductive limit $C^*$-algebra is constructed from an inductive family of $C^*$-algebras, which depend on finite graph systems. The reason is the following: Consider graph-diffeomorphisms of the finite graph system associated to a graph $\Gamma$. These objects are pairs of maps and will be presented in more detail in section 2.2. For short such a pair consists of a bijective map from vertices to vertices, which are situated in the manifold $\Sigma$, and a map that maps subgraphs to subgraphs of $\Gamma$. Then there are actions of these graph-diffeomorphisms on the analytic holonomy $C^*$-algebra restricted to a finite graph system associated to the graph $\Gamma$. There is no well-defined action of these graph-diffeomorphisms on the analytic holonomy $C^*$-algebra restricted to a fixed graph in general. This can be verified as follows. Assume that $\Gamma := \{\gamma_1, \gamma_2, \gamma_3\}$ is a graph and $\Gamma' := \{\gamma_1\}$, $\Gamma'' := \{\gamma_1 \circ \gamma_3\}$ are subgraphs of $\Gamma$. Then consider a graph-diffeomorphism $(\varphi, \Phi)$ such that $\Phi(\Gamma') = \Gamma''$. Now the action $\zeta_{(\varphi, \Phi)}$ on the analytic holonomy $C^*$-algebra restricted to the graph $\Gamma$, which is defined by

$$\left(\zeta_{(\varphi, \Phi)} f\right)(h\gamma) = f_{\Phi(\gamma)}(h\Phi(\gamma))(\Phi(\gamma)) = f_{\gamma''}(h\gamma''(\gamma'''))$$

whenever $\Phi(\Gamma') = \Gamma''$ is not well-defined. The reason is: $\Gamma''$ is not a graph. If $\Phi(\Gamma')$ is a subgraph of $\Gamma'$, then in particular $f_{\Phi(\Gamma')}$ is an element of the analytic holonomy $C^*$-algebra restricted to the subgraph $\Phi(\Gamma')$. The analytic holonomy $C^*$-algebra restricted to every subgraph of $\Gamma$ is a $C^*$-subalgebra of the analytic holonomy $C^*$-algebra restricted to the graph $\Gamma$. Hence, the last $C^*$-algebra is in particular a $C^*$-algebra, which is characterised by the finite graph system associated to $\Gamma$. An action of graph-diffeomorphisms is an automorphism of the analytic holonomy $C^*$-algebra restricted to finite graph system associated to $\Gamma$. Summarising, the concepts of the limit of $C^*$-algebras restricted to finite graph systems, and actions of graph-diffeomorphisms on the holonomy $C^*$-algebra restricted to finite graph systems engage with each other.

Finally note that the inductive limit $C^*$-algebra of the inductive family of $C^*$-algebras $\{C(\mathcal{A}_\Gamma), \beta_{\Gamma, \Gamma'}\}$ defines the projective limit configuration space $\bar{\mathcal{A}}$. The inductive limit $C^*$-algebra $C(\bar{\mathcal{A}})$ is called the analytic holonomy $C^*$-algebra in the project $AQV$.

The idea of using families of graph systems is influenced by the work of Giesel and Thiemann 10 in the LQG framework. They use particular cubic graphs instead of sets of paths in a groupoid and their inductive limit is constructed from families of cubic graph systems. In the project $AQV$ the inductive limit Hilbert space $\mathcal{H}_\infty$ will be derived from the natural or non-standard identified configuration spaces, the Haar measure on the structure group $G$ and an inductive limit of finite graph systems. It will be assumed that the inductive limit graph system only contains a countable set of subgraphs of an inductive limit graph $\Gamma_\infty$. This is contrary to the Hilbert space used in LQG literature 29, which is the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL}$. The Hilbert space $\mathcal{H}_{AL}$ is manifestly non-separable, since the limit is taken over all sets of paths in $\Sigma$ and, hence over an infinite and uncountable set of all graphs. Clearly, the Hilbert space $\mathcal{H}_\infty$ is constructed by using certain identification of the configuration space and the countable set of subgraph. In this simplified formulation some important aspects of the theory can be studied. It is possible to generalise partly the results for the Ashtekar-Lewandowski Hilbert space.

The classical configuration space in the context of LQG and Ashtekar variables is the space of smooth connections $\hat{\mathcal{A}}$, on an arbitrary principal fibre bundle $P(\Sigma, G)$. In this project the quantum operator $Q(A)$ of the infinitesimal connection $A$ is given by the holonomy $h$ along a path $\gamma$. The operator $Q(\mathcal{A})$ is represented as a multiplication operator on the inductive limit Hilbert space $\mathcal{H}_\infty$. 

3
The quantum momentum variables: group-valued flux operators

In the project AQV the quantum operator $Q(E^i)$ of the classical fluxes $E^i$ is either a group- or Lie algebra-valued operator, which depend on a surface $S$ and a path $\gamma$ or a graph $\Gamma$. The idea of this definition is the following: Consider a surface $S$ and a path $\gamma$ that intersects each other in the source vertex of $\gamma$ and the path lies below the orientated surface $S$. Let $G$ be compact group. The group-valued quantum flux operator $\rho_S(\gamma)$ is given by the value of a map $\rho_S : P\Sigma \rightarrow G$ evaluated for a path $\gamma$ in the set $P\Sigma$ of paths in $\Sigma$. This definition does not coincide with the usual definition presented in LQG literature completely.

In general the idea is to obtain algebras, which are generated by

(i) the group-valued quantum flux operators and the holonomies along paths in a graph, or

(ii) the group-valued quantum flux operators and certain functions depending on holonomies along paths in a graph, or

In this work an algebra derived from the operators given in (ii) satisfying some canonical commutator relations will be presented in section 3.

Until now, a suitable set of surfaces in $\Sigma$ and a path $\gamma$ in the finite path groupoid $P\Gamma\Sigma$ are fixed. For a general situation the following maps are studied in section 2.3 and 2.4.

(i) a certain map $\rho_S : P\Gamma\Sigma \rightarrow G$

(ii) a certain map $\rho_S : P\Gamma\Sigma \rightarrow \mathbb{Z}$, where $\mathbb{Z}$ denotes the center of the group $G$, and

(iii) a certain map $\varrho : P\Gamma\Sigma \rightarrow G$ and this map $\varrho$ is called admissible in analogy to Fleischhack [7].

Then the maps $\rho_S$ given by (i) (or (ii)) define a group, which depend on the fixed path $\gamma$ and a suitable fixed surface set $\tilde{S}$. Note that the surface set always contains at least one surface in $\Sigma$. This group is called flux group $G_{S,\gamma}$ associated to a surface set $\tilde{S}$ and a path $\gamma$. Clearly, for each suitable surface set there exist a flux group associated to this surface set. The maps of the form $\varrho$ given by (iii) are used to define a more complicated structure. Furthermore, this concept generalises to holonomies of a graph $\Gamma$, which are maps from graphs to products of the structure group $G$. Then for example the flux group $G_{\tilde{S},\Gamma}$ associated to a surface set and a graph exists.

Now, for the group-valued quantum flux operators different actions on the configuration space will be explicitly considered in section 3.1. In particular the left, right and inner actions are studied independently from each other and are denoted by $L,R$ or $I$. Furthermore, only the maps (ii) and (iii) define groupoid morphisms by composition of the action $L$ (or $R$, or $I$) and the holonomy map. For an overview about which maps define groupoid morphisms consider [15] table 11.2. Note that using admissible maps (maps of the form (iii)) particular morphisms are defined. These morphisms are called equivalent groupoid morphisms in analogy to Mackenzie [22] and are related to gauge transformations on the configuration space. The flux groups constructed from the maps (i) and (ii) the analytic holonomy $C^*$-algebra $C(A_i)$ and the actions $L$, $R$ or $I$ define $C^*$-dynamical systems. If admissible maps are taken into account, the $C^*$-dynamical systems are very complicated.

In general the parameter group of automorphism, which is defined from arbitrary group-valued quantum flux operators $\rho_S(\gamma)$ for every surface $S$ and a fixed path $\gamma$ to the group of automorphisms, i.e. $\rho_S(\gamma) \mapsto \alpha(\rho_S(\gamma)) \in \text{Aut}(C(A_i))$, does not define a group homomorphism to the group of automorphisms in $C(A_i)$. This is only true for certain group-valued quantum flux operators, which form a flux group associated to a certain surface set. Therefore, $C^*$-dynamical systems are defined by the analytic holonomy $C^*$-algebra $C(A_i)$ restricted to the finite graph system and actions of the flux group associated to a certain surface set and the graph $\Gamma$ on this $C^*$-algebra.

Furthermore, the analytic holonomy $C^*$-algebra can be restricted to certain subgraphs of a graph $\Gamma$. Therefore, the following object is important. A finite orientation preserved graph system is a set of certain subgraphs of a graph $\Gamma$ such that all paths in a subgraph are generated by compositions of the edges that generate the graph $\Gamma$. Note that in this definition the composition of edges and inverses of this edges are excluded. Then clearly there is an action of the flux group associated to the graph $\Gamma$ and a surface set on the analytic holonomy $C^*$-algebra restricted to the finite orientation preserved graph system $P_G^\Sigma$. Furthermore, there is an action of the flux group associated to every subgraph of the finite orientation preserved graph
system $\mathcal{P}_G$ and a surface set on the analytic holonomy $C^*$-algebra restricted to a finite orientation preserved graph system. There is a set of exceptional $C^*$-dynamical systems, which is defined by these automorphisms of the flux groups associated to suitable surface sets and graphs on the analytic holonomy algebras restricted to finite orientation preserved graph systems. The restriction to orientation preserved subgraphs is necessary to obtain either a purely left or right action of the flux group associated to a fixed surface set and subgraphs of a particular graph system on the holonomy $C^*$-algebra restricted to suitable graph systems.

The Gelfand-Naimark theorem implies that there is an isomorphism between commutative $C^*$-algebras and continuous function algebras on configuration spaces. If other in particular non-abelian $C^*$-algebras are studied, then automorphisms of the algebras do not correspond to certain homeomorphisms on the configuration spaces. More generally, covariant representations of the $C^*$-dynamical systems replace the construction of Fleischhack. A covariant representation is a pair of maps, which is given by a representation of the $C^*$-algebra on the Hilbert space and a unitary representation of the flux group, and these maps satisfy a certain canonical commutator relation. In this project the Weyl $C^*$-algebra for surfaces is constructed from all $C^*$-dynamical systems, which contains all actions of the flux groups associated to all different surface sets on the analytic holonomy $C^*$-algebra. In particular an element of the Weyl algebra of a surface set $\mathcal{S}$ restricted to a finite graph system $\mathcal{G}$ is for example of the form

$$\sum_{l=1}^{L} l_{\mathcal{G}} U_{S_{l}}(\rho_{S_{l}}(\Gamma)) + \sum_{k=1}^{K} \sum_{j=1}^{M} f_{k}^{j} U_{S_{k}}(\rho_{S_{k}}(\Gamma)) + \sum_{k=1}^{K} \sum_{i=1}^{M} U_{S_{i}}(\rho_{S_{i}}(\Gamma)) f_{k}^{i} U_{S_{i}}(\rho_{S_{i}}(\Gamma))^* + \sum_{p=1}^{P} f_{p}^{*}$$

whenever $f_{k}^{j}, f_{k}^{i}, f_{p}^{*} \in C(\bar{\mathcal{A}}_{\Gamma})$, $U_{S_{k}} \in \text{Rep}(\bar{G}_{S_{k}}, \mathcal{K}(\mathcal{H}_{\Gamma}))$. The notion $U_{S_{k}} \in \text{Rep}(\bar{G}_{S_{k}}, \mathcal{K}(\mathcal{H}_{\Gamma}))$ means that the unitary operators are represented on the $C^*$-algebra $\mathcal{K}(\mathcal{H}_{\Gamma})$ of compact operators on the Hilbert space $\mathcal{H}_{\Gamma}$. Furthermore, the unitaries and products of these unitaries, which satisfy the canonical commutator relation, are called Weyl elements in this project.

The quantum spatial diffeomorphisms

In the project of Algebras of Quantum Variables in LQG the classical spatial diffeomorphisms are replaced by new quantum diffeomorphisms. The classical diffeomorphisms are certain diffeomorphisms in the spatial hypersurface $\Sigma$. In Mackenzie \[22\] a concept of translations in a general Lie groupoid has been presented. The ideas are used in section \[2.2.3\] for a redefinition of the classical diffeomorphisms. The new operators are called bisections. The idea of the definition of a bisection is presented in the next paragraph.

In the theory of groupoids the following object is often used: the groupoid isomorphism in a path groupoid, which consists of the classical diffeomorphism in $\Sigma$ and an additional bijective map from paths to paths in the path groupoid over $\Sigma$. This pair of maps is called the path-diffeomorphisms of a path groupoid. The path-diffeomorphisms extend the notion of graphomorphisms, which have been introduced by Fleischhack \[7\]. There is only a slight difference between these objects: A graphomorphism is a map from $\Sigma$ to $\Sigma$ that preserves additionally the path groupoid structure, whereas a path-diffeomorphism is a pair of maps. In particular finite path-diffeomorphisms are given by a pair of maps, which contains a map that maps paths to paths in a finite path groupoid $\mathcal{P}_{\mathcal{G}} \Sigma$ and a bijective map that maps vertices to vertices of the vertex set of the graph $\Gamma$. Moreover, since graph systems are used in this project, a pair of maps that contains a map, which maps subgraphs to subgraphs, plays a fundamental role and is called finite graph-diffeomorphism. Graphomorphisms define in particular groupoid isomorphisms and, hence, they transform non-trivial paths to non-trivial paths. To define maps that transform a trivial path at a vertex in $\Sigma$ to a non-trivial path other objects have to be considered.Translations in a finite path groupoid are naturally given by adding or deleting edges, which generate the graph $\Gamma$. One distinguishes between three translations in a path groupoid. One is given by adding a path $\gamma$ to a path $\theta$ at the source vertex $s(\theta)$ of the path $\theta$. The other case is given by composition of a path $\gamma$ to a path $\theta$ at the target vertex $t(\theta)$ of the path $\theta$. Finally, two paths can be composed with a path at the source and target vertices simultaneously. Hence, there is a natural map from the set of vertices to the set of paths, which is called a bisection of a finite path groupoid. For such a bisection $\sigma$ the map $t \circ \sigma$ is assumed to be bijective, where $t$ denotes the target map of the finite path groupoid. In the definition of a bisection of a path groupoid the map $t \circ \sigma$ is required to be a diffeomorphism from $\Sigma$ to $\Sigma$ and the map $\sigma$ maps vertices to paths in a path groupoid. Furthermore a right-translation $R_{t}$ of a bisection $\sigma$ is a map that composes a path $\gamma$ with the path $\sigma(t(\gamma))$, i.e. $R_{t}(\gamma) = \gamma \circ \sigma(t(\gamma))$. Furthermore a left-translation $L_{t}$ and an inner-translation $I_{t}$ of a bisection $\sigma$ can be defined similarly. The pair consisting of the composition $t \circ \sigma$ of the bisection and the target map and the right translation $R_{t}$ define in general no groupoid isomorphism. Nevertheless there are particular translations of suitable bisections that define
path-diffeomorphisms. There is no doubt that the notion of a bisection can be generalised to a bisection of a path groupoid or a bisection of a finite graph system. Moreover, the bisections of a path groupoid form a group and there is a group homomorphism between this group and the group of diffeomorphisms in Σ. Moreover, the bisections of a finite path groupoid or a finite graph system equipped with a sophisticated group multiplication form groups, too. Finally, a quantum diffeomorphism is assumed to be an element of the group of bisections of a path groupoid, a finite path groupoid or a finite graph system.

Now, actions of the group of bisections on the analytic holonomy C*-algebra restricted to a finite graph system will be used in section 3.2 to construct C*-dynamical systems. If the group \( \mathcal{B}(\mathcal{P}_\Gamma) \) of bisections of a finite graph system \( \mathcal{P}_\Gamma \) is considered, then the right-, left- or inner-translation of the bisections define three different C*-dynamical systems. For example, there is a C*-dynamical system \( (C(\mathcal{A}_\Gamma), \mathcal{B}(\mathcal{P}_\Gamma), \zeta) \), where the action \( \zeta \) is defined by the right-translation of the bisections. For each C*-dynamical system there exists a covariant representation on the Hilbert space \( \mathcal{H}_\Gamma \). Hence, the right-, left- or inner-translation of the bisections define unitary operators on the Hilbert space \( \mathcal{H}_\Gamma \) associated to a graph. The main advantage of these maps from a graph to a graph is that they define graph changing operators. In particular these maps transform subgraphs into subgraphs of a graph \( \Gamma \) such that the number of edges of the subgraphs can change.

Both actions, which are the action of the group of bisections of a finite graph system and the action of the flux group on the configuration space, lead to automorphisms on the analytic holonomy C*-algebra. A comparison of the actions can be found in [15, table 11.2]. Similarly to actions of the flux group, the actions of the group of bisections composed with holonomy maps do not define groupoid morphisms in general. This causes no problems, since the configuration space restricted to a finite graph system \( \mathcal{P}_\Gamma \) is identified (naturally or in non-standard way) with \( \mathcal{G} \mathcal{P}_\Gamma \). Finally, notice that only actions of certain bisections preserve the flux operators associated to a surface \( S \). For example consider the bisection \( \sigma \) of a path groupoid and recall the diffeomorphism \( t \circ \sigma \). Then for example the diffeomorphism \( t \circ \sigma \) is required to preserve the surface \( S \). This particular bisection is called the surface-preserving bisection of a path groupoid. There exists a similar description for a surface-preserving bisection for a finite path groupoid or a finite graph system. Then the concept can be extended to bisections of a finite graph system that map surfaces to surfaces in a certain surface set and preserve the orientation of the surfaces with respect to the transformed subgraph. In this situation the bisections are called surface-orientation-preserving bisections for a finite graph system and they form a subgroup of the group of bisection of a finite graph system. Finally both actions on the analytic holonomy C*-algebra restricted to a finite graph system:

(i) the action of the group of surface-orientation-preserving bisections for a finite graph system and

(ii) the action of the center of the flux group associated to a surface set

commute. In analogy to the surface-orientation-preserving bisections of a finite graph system the surface-orientation-preserving graph-diffeomorphisms can be constructed.

Finally there is an action of bisections of the path groupoid \( \mathcal{P} \) over \( \Sigma \) or the inductive limit graph system \( \mathcal{P}_{\Gamma,\infty} \) on the analytic holonomy C*-algebra \( C(\mathcal{A}) \). This action is not point-norm continuous. Consequently, the infinitesimal diffeomorphism constraint is not implemented as a Hilbert space operator.

**Representations for the Weyl C*-algebras for surfaces on a Hilbert space**

The main objects, which are introduced in this project \( \text{AQV} \), are given by

- the flux groups or the Lie flux algebras of Lie flux groups associated to surface sets,
- the analytic holonomy C*-algebra, which is given by the inductive limit C*-algebra of an inductive family of analytic holonomy C*-algebras restricted to finite graph systems.

In the previous subsection the construction of the Weyl algebra has been introduced briefly. The Weyl algebra of Quantum Geometry [21] has been constructed from the analytic holonomy C*-algebra and unitary operators, which are defined by weakly continuous one-parameter unitary groups of \( \mathbb{R} \) on the Hilbert space \( \mathcal{H}_\mathcal{AL} \). The unitaries have been called Weyl operators by Fleischhack. The Weyl C*-algebra for a surface set and restricted to a finite graph system is generated by the analytic holonomy C*-algebra restricted to a finite
graph system and Weyl elements. Assume for a moment that \( G \) is a compact connected Lie group. Then consider a strongly continuous one-parameter unitary group of \( \mathbb{R} \), which is given by \( \mathbb{R} \ni t \mapsto U(\exp(tE_S(\gamma))) \), on the Hilbert space \( \mathcal{H}_\infty \). Then each unitary \( U(\exp(tE_S(\gamma))) \) defines a Weyl element, too.

To obtain a uniqueness result of a representation of a \( C^* \)-algebra the following general facts will be used. Since irreducible representations of a \( C^* \)-algebra on a Hilbert space correspond one-to-one to pure states on the \( C^* \)-algebra, the uniqueness of a particular representation of the \( C^* \)-algebra on a Hilbert space corresponds to a unique state. The inductive limit of an inductive family of \( C^* \)-algebras corresponds one-to-one to a projective limit on the projective family of state spaces of the \( C^* \)-algebras. The GNS-representation associated to a state of a \( C^* \)-algebra consists of a cyclic vector \( \Omega \) on a Hilbert space and a representation of the \( C^* \)-algebra on the Hilbert space.

The uniqueness of a finite surface-orientation-preserving graph-diffeomorphism invariant pure state of the commutative Weyl \( C^* \)-algebra for surfaces is obtained in Theorem [108] by several steps. The commutative Weyl \( C^* \)-algebra for surfaces is constructed similarly to the Weyl \( C^* \)-algebra for surfaces with the difference that the group \( G \) is replaced by the center of the group \( G \). Then graph-diffeomorphism invariant states of the commutative Weyl algebra for surfaces restricted to a graph system \( \mathcal{P}_\Gamma \) are analysed. It turns out that a difference occur, if either the natural or if the non-standard identification of the configuration space \( \mathcal{A}_\Gamma \) is taken into account. In particular, for the natural identification, the state is a sum over states, which are indexed by bisections. For the commutative Weyl algebra for surfaces the difference disappears. There exists a pure and unique state, which is invariant under finite graph-diffeomorphisms. This result is similar to the uniqueness of the representation of the Weyl algebra of Quantum Geometry and it is obtained in a complete new operator algebraic formulation.

There is a problem of finding other representations of the Weyl \( C^* \)-algebra for surfaces. For example for the Weyl \( C^* \)-algebra for surfaces the important fact is that the flux group associated to a surface set is a commutative \( C^* \)-algebra on a Hilbert space corresponds one-to-one to \( C^* \)-algebras. The GNS-representation of the \( C^* \)-algebra for surfaces restricted to a graph system \( \mathcal{P}_\Gamma \) are analysed. It turns out that a difference occur, if either the natural or if the non-standard identification of the configuration space \( \mathcal{A}_\Gamma \) is taken into account. In particular, for the natural identification, the state is a sum over states, which are indexed by bisections. For the commutative Weyl algebra for surfaces the difference disappears. There exists a pure and unique state, which is invariant under finite graph-diffeomorphisms. This result is similar to the uniqueness of the representation of the Weyl algebra of Quantum Geometry and it is obtained in a complete new operator algebraic formulation.

2 The basic quantum operators

2.1 Finite path groupoids and graph systems

Let \( c : [0,1] \to \Sigma \) be continuous curve in the domain \([0,1]\), which is (piecewise) \( C^k \)-differentiable (\( 1 \leq k \leq \infty \)), analytic (\( k = \omega \)) or semi-analytic (\( k = \omega \)) in \([0,1]\) and oriented such that the source vertex is \( c(0) = s(c) \) and the target vertex is \( c(1) = t(c) \). Moreover assume that, the range of each subinterval of the curve \( c \) is a submanifold, which can be embedded in \( \Sigma \). An edge is given by a curve of class (piecewise) \( C^k \). The maps \( s_\Sigma, t_\Sigma : P\Sigma \to \Sigma \) where \( P\Sigma \) is the path space are surjective maps and are called the source or target map.

A set of edges \( \{e_i\}_{i=1,...,N} \) is called independent, if the only intersections points of the edges are source \( s_\Sigma(e_i) \) or target \( t_\Sigma(e_i) \) target points. Composed edges are called paths. An initial segment of a path \( \gamma \) is a path \( \gamma_1 \) such that there exists another path \( \gamma_2 \) and \( \gamma = \gamma_1 \circ \gamma_2 \). The second element \( \gamma_2 \) is also called a final segment of the path \( \gamma \).

Definition 1. A graph \( \Gamma \) is a union of finitely many independent edges \( \{e_i\}_{i=1,...,N} \) for \( N \in \mathbb{N} \). The set \( \{e_1,...,e_N\} \) is called the generating set for \( \Gamma \). The number of edges of a graph is denoted by \( |\Gamma| \). The elements of the set \( \mathcal{P}_\Gamma \Sigma := \{s_\Sigma(e_k),t_\Sigma(e_k)\}_{k=1,...,N} \) of source and target points are called vertices.

A graph generates a finite path groupoid in the sense that, the set \( \mathcal{P}_\Gamma \Sigma \) contains all independent edges, their inverses and all possible compositions of edges. All the elements of \( \mathcal{P}_\Gamma \Sigma \) are called paths associated to a graph. Furthermore the surjective source and target maps \( s_\Sigma \) and \( t_\Sigma \) are restricted to the maps \( s, t : \mathcal{P}_\Gamma \Sigma \to V_\Gamma \), which are required to be surjective.

Definition 2. Let \( \Gamma \) be a graph. Then a finite path groupoid \( \mathcal{P}_\Gamma \Sigma \) over \( V_\Gamma \) is a pair \( (\mathcal{P}_\Gamma \Sigma, V_\Gamma) \) of finite sets equipped with the following structures:

(i) two surjective maps \( s, t : \mathcal{P}_\Gamma \Sigma \to V_\Gamma \), which are called the source and target map,
(ii) the set \( P \Sigma \Sigma := \{(\gamma_i, \gamma_j) \in P \Sigma \times P \Sigma : t(\gamma_i) = s(\gamma_j)\} \) of finitely many composable pairs of paths,

(iii) the composition \( \circ : P \Sigma \Sigma \to P \Sigma \), where \((\gamma_i, \gamma_j) \mapsto \gamma_i \circ \gamma_j\),

(iv) the inversion map \( \gamma_i \mapsto \gamma_i^{-1} \) of a path,

(v) the object inclusion map \( i : V \Gamma \to P \Gamma \Sigma \)

(vi) \( P \Gamma \Sigma \) is defined by the set \( P \Gamma \Sigma \) modulo the algebraic equivalence relations generated by

\[
\gamma_i^{-1} \circ \gamma_i \simeq \mathbb{I}_{s(\gamma_i)} \text{ and } \gamma_i \circ \gamma_i^{-1} \simeq \mathbb{I}_{t(\gamma_i)}
\]

(1)

Shortly write \( P \Gamma \Sigma \xrightarrow{\equiv} V \Gamma \).

Clearly, a graph \( \Gamma \) generates freely the paths in \( P \Gamma \Sigma \). Moreover the map \( s \times t : P \Gamma \Sigma \to V \Gamma \times V \Gamma \) defined by \((s \times t)(\gamma) = (s(\gamma), t(\gamma))\) for all \( \gamma \in P \Gamma \Sigma \) is assumed to be surjective (\( P \Gamma \Sigma \) over \( V \Gamma \) is a transitive groupoid), too.

A general groupoid \( G \) over \( G^0 \) defines a small category where the set of morphisms is denoted in general by \( G \) and the set of objects is denoted by \( G^0 \). Hence in particular the path groupoid can be viewed as a category, since,

\[ \begin{align*}
& \cdot \text{ the set of morphisms is identified with } P \Gamma \Sigma,
& \cdot \text{ the set of objects is given by } V \Gamma \text{ (the units)}
\end{align*} \]

From the condition (1) it follows that, the path groupoid satisfies additionally

(i) \( s(\gamma_i \circ \gamma_j) = s(\gamma_i) \) and \( t(\gamma_i \circ \gamma_j) = t(\gamma_j) \) for every \((\gamma_i, \gamma_j) \in P \Sigma \Sigma\)

(ii) \( s(v) = v = t(v) \) for every \( v \in V \Gamma \)

(iii) \( \gamma \circ \mathbb{I}_{s(\gamma)} = \gamma = \mathbb{I}_{t(\gamma)} \circ \gamma \) for every \( \gamma \in P \Gamma \Sigma \) and

(iv) \( \gamma \circ (\gamma_i \circ \gamma_j) = (\gamma \circ \gamma_i) \circ \gamma_j \)

(v) \( \gamma \circ (\gamma_i^{-1} \circ \gamma_j) = \gamma_i \circ (\gamma_i \circ \gamma) \circ \gamma^{-1} \)

The condition (iii) implies that the vertices are units of the groupoid.

**Definition 3.** Denote the set of all finitely generated paths by

\[ P \Gamma \Sigma^{(n)} := \{(\gamma_1, ..., \gamma_n) \in P \Gamma \times ... P \Gamma : (\gamma_i, \gamma_{i+1}) \in P^{(2)}, 1 \leq i \leq n - 1\} \]

The set of paths with source point \( v \in V \Gamma \) is given by

\[ P \Gamma \Sigma^v := s^{-1}\{v\} \]

The set of paths with target point \( v \in V \Gamma \) is defined by

\[ P \Gamma \Sigma^t := t^{-1}\{v\} \]

The set of paths with source point \( v \in V \Gamma \) and target point \( u \in V \Gamma \) is

\[ P \Gamma \Sigma^u := P \Gamma \Sigma^v \cap P \Gamma \Sigma^u \]

A graph \( \Gamma \) is said to be disconnected if it contains only mutually pairs \((\gamma_i, \gamma_j)\) of non-composable independent paths \( \gamma_i \) and \( \gamma_j \) for \( i \neq j \) and \( i, j = 1, ..., N \). In other words for all \( 1 \leq i, l \leq N \) it is true that \( s(\gamma_i) \neq t(\gamma_l) \) and \( t(\gamma_i) \neq s(\gamma_l) \) where \( i \neq l \) and \( \gamma_i, \gamma_l \in \Gamma \).

**Definition 4.** Let \( \Gamma \) be a graph. A subgraph \( \Gamma' \) of \( \Gamma \) is a given by a finite set of independent paths in \( P \Gamma \Sigma \).
For example let $\Gamma := \{\gamma_1, ..., \gamma_N\}$ then $\Gamma' := \{\gamma_1 \circ \gamma_2, \gamma_3^\top, \gamma_4\}$ where $\gamma_1 \circ \gamma_2, \gamma_3^\top, \gamma_4 \in P_\Sigma$ is a subgraph of $\Gamma$, whereas the set $\{\gamma_1, \gamma_1 \circ \gamma_2\}$ is not a subgraph of $\Gamma$. Notice if additionally $(\gamma_2, \gamma_4) \in P_\Sigma^{(2)}$ holds, then $\{\gamma_1, \gamma_3^\top, \gamma_2 \circ \gamma_4\}$ is a subgraph of $\Gamma$, too. Moreover for $\Gamma := \{\gamma\}$ the graph $\Gamma^{-1} := \{\gamma^{-1}\}$ is a subgraph of $\Gamma$. As well the graph $\Gamma$ is a subgraph of $\Gamma^{-1}$. A subgraph of $\Gamma$ that is generated by compositions of some paths, which are not reversed in their orientation, of the set $\{\gamma_1, ..., \gamma_N\}$ is called an orientation preserved subgraph of a graph. For example for $\Gamma := \{\gamma_1, ..., \gamma_N\}$ orientation preserved subgraphs are given by $\{\gamma_1 \circ \gamma_2\}, \{\gamma_1, \gamma_2, \gamma_N\}$ or $\{\gamma_{N-2} \circ \gamma_{N-1}\}$ if $(\gamma_1, \gamma_2) \in P_\Sigma^{(2)}$ and $(\gamma_{N-2}, \gamma_{N-1}) \in P_\Sigma^{(2)}$.

**Definition 5.** A finite graph system $P_\Gamma$ for $\Gamma$ is a finite set of subgraphs of a graph $\Gamma$. A finite graph system $P_\Gamma'$ for $\Gamma'$ is a finite graph subsystem of $P_\Gamma$ for $\Gamma$ if the set $P_\Gamma'$ is a subset of $P_\Gamma$ and $\Gamma'$ is a subgraph of $\Gamma$. Shortly write $\mathcal{P}_\Gamma \leq P_\Gamma$.

A finite orientation preserved graph system $P_\Gamma^o$ for $\Gamma$ is a finite set of orientation preserved subgraphs of a graph $\Gamma$.

Recall that, a finite path groupoid is constructed from a graph $\Gamma$, but a set of elements of the path groupoid need not be a graph again. For example let $\Gamma := \{\gamma_1 \circ \gamma_2\}$ and $\Gamma' = \{\gamma_1 \circ \gamma_3\}$, then $\Gamma'' = \Gamma \cup \Gamma'$ is not a graph, since this set is not independent. Hence only appropriate unions of paths, which are elements of a fixed finite path groupoid, define graphs. The idea is to define a suitable action on elements of the path groupoid, which corresponds to an action of diffeomorphisms on the manifold $\Sigma$. The action has to be transferred to graph systems. But the action of bisection, which is defined by the use of the groupoid multiplication, cannot easily generalised for graph systems.

**Problem 2.1:** Let $\hat{\Gamma} := \{\Gamma_i\}_{i=1, ..., N}$ be a finite set such that each $\Gamma_i$ is a set of not necessarily independent paths such that

(i) the set contains no loops and

(ii) each pair of paths satisfies one of the following conditions

- the paths intersect each other only in one vertex,
- the paths do not intersect each other or
- one path of the pair is a segment of the other path.

Then there is a map $\circ : \hat{\Gamma} \times \hat{\Gamma} \rightarrow \hat{\Gamma}$ of two elements $\Gamma_1$ and $\Gamma_2$ defined by

$$\{\gamma_1, ..., \gamma_M\} \circ \{\tilde{\gamma}_1, ..., \tilde{\gamma}_M\} := \\{\gamma_i \circ \tilde{\gamma}_j : t(\gamma_i) = s(\tilde{\gamma}_j)\}_{1 \leq i, j \leq M}$$

for $\Gamma_1 := \{\gamma_1, ..., \gamma_M\}, \Gamma_2 := \{\tilde{\gamma}_1, ..., \tilde{\gamma}_M\}$. Moreover define a map $^{-1} : \hat{\Gamma} \rightarrow \hat{\Gamma}$ by

$$\{\gamma_1, ..., \gamma_M\}^{-1} := \{\gamma_1^{-1}, ..., \gamma_M^{-1}\}$$

Then the following is derived

$$\{\gamma_1, ..., \gamma_M\} \circ \{\gamma_1^{-1}, ..., \gamma_M^{-1}\} = \\{\gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j)\}_{1 \leq i, j \leq M}$$

$$= \\{\gamma_i \circ \gamma_j^{-1} : t(\gamma_i) = t(\gamma_j) \text{ and } i \neq j\}_{1 \leq i, j \leq M}$$

$$\cup \\{\mathbb{1}_{\gamma_i}\}_{1 \leq i \leq M} \neq \cup \\{\mathbb{1}_{\gamma_i}\}_{1 \leq j \leq M}$$

The equality is true, if the set $\hat{\Gamma}$ contains only graphs such that all paths are mutually non-composable. Consequently this does not define a well-defined multiplication map. Notice that, the same is discovered if a similar map and inversion operation are defined for a finite graph system $P_\Gamma$.

Consequently the property of paths being independent need not be dropped for the definition of a suitable multiplication and inversion operation. In fact the independence property is a necessary condition for the construction of the holonomy algebra for analytic paths. Only under this circumstance each analytic path is decomposed into a finite product of independent piecewise analytic paths again.
Definition 6. A finite path groupoid \( P_G \Sigma \) over \( V_G \) is a finite path subgroupoid of \( P_G \Sigma \) over \( V_G \) if the set \( V_G \) is contained in \( V_G \) and the set \( P_G \Sigma \) is a subset of \( P_G \Sigma \). Shortly write \( P_G \Sigma \leq P_G \Sigma \).

Clearly for a subgraph \( \Gamma_1 \) of a graph \( \Gamma_2 \), the associated path groupoid \( P_{\Gamma_1} \Sigma \) over \( V_{\Gamma_1} \) is a subgroupoid of \( P_{\Gamma_2} \Sigma \) over \( V_{\Gamma_2} \). This is a consequence of the fact that, each path in \( P_{\Gamma_1} \Sigma \) is a composition of paths or their inverses in \( P_{\Gamma_2} \Sigma \).

Definition 7. A family of finite path groupoids \( \{ P_{\Gamma_i} \Sigma \}_{i=1,\ldots,\infty} \), which is a set of finite path groupoids \( P_{\Gamma_i} \Sigma \) over \( V_{\Gamma_i} \), is said to be inductive if for any \( P_{\Gamma_i} \Sigma , P_{\Gamma_j} \Sigma \) exists a \( P_{\Gamma_k} \Sigma \) such that \( P_{\Gamma_i} \Sigma , P_{\Gamma_j} \Sigma \leq P_{\Gamma_k} \Sigma \).

A family of graph systems \( \{ P_{\Gamma_i} \Sigma \}_{i=1,\ldots,\infty} \), which is a set of finite path systems \( P_{\Gamma_i} \) for \( \Gamma_i \), is said to be inductive if for any \( P_{\Gamma_i} , P_{\Gamma_j} \) exists a \( P_{\Gamma_k} \) such that \( P_{\Gamma_i} , P_{\Gamma_j} \leq P_{\Gamma_k} \).

Definition 8. Let \( \{ P_{\Gamma_i} \Sigma \}_{i=1,\ldots,\infty} \) be an inductive family of path groupoids and \( \{ P_{\Gamma_i} \Sigma \}_{i=1,\ldots,\infty} \) be an inductive family of graph systems.

The inductive limit path groupoid \( \Gamma \) over \( \Sigma \) of an inductive family of finite path groupoids such that
\[
\Gamma := \lim_{i \to \infty} P_{\Gamma_i} \Sigma
\]

Moreover there exists an inductive limit graph \( \Gamma_\infty \) of an inductive family of graphs such that \( \Gamma_\infty := \lim_{i \to \infty} \Gamma_i \).

The inductive limit graph system \( \Gamma_{\infty} \) of an inductive family of graph systems such that \( \Gamma_{\infty} := \lim_{i \to \infty} \Gamma_i \).

Assume that, the inductive limit \( \Gamma_\infty \) of a inductive family of graphs is a graph, which consists of an infinite countable number of independent paths. The inductive limit \( \Gamma_{\infty} \) of a inductive family \( \{ P_{\Gamma_i} \} \) of finite graph systems contains an infinite countable number of subgraphs of \( \Gamma_\infty \) and each subgraph is a finite set of arbitrary independent paths in \( \Sigma \).

2.2 Holonomy maps for finite path groupoids, graph systems and transformations

In section 2.1 the concept of finite path groupoids for analytic paths has been given. Now the holonomy maps are introduced for finite path groupoids and finite graph systems. The ideas are familiar with those presented by Thiemann [29]. But for example the finite graph systems have not been studied before. Ashtekar and Lewandowski [1] have defined the analytic holonomy \( C^* \)-algebra, which they have based on a finite set of independent hoops. The hoops are generalised for path groupoids and the independence requirement is implemented by the concept of finite graph systems.

2.2.1 Holonomy maps for finite path groupoids

Groupoid morphisms for finite path groupoids

Let \( G_1 \xrightarrow{s_1} t_1 \) \( G^0_1 \), \( G_2 \xrightarrow{s_2} t_2 \) \( G^0_2 \) be two arbitrary groupoids.

Definition 9. A groupoid morphism between two groupoids \( G_1 \) and \( G_2 \) consists of two maps \( h : G_1 \to G_2 \) and \( h : G_1^0 \to G_2^0 \) such that
\[
(G1) \quad h(\gamma \circ \gamma') = h(\gamma) h(\gamma') \quad \text{for all } (\gamma, \gamma') \in G_1^{(2)}
\]
\[
(G2) \quad s_2(h(\gamma)) = h(s_1(\gamma)), \quad t_2(h(\gamma)) = h(t_1(\gamma))
\]

A strong groupoid morphism between two groupoids \( G_1 \) and \( G_2 \) additionally satisfies
\[
(SG) \quad \text{for every pair } (h(\gamma), h(\gamma')) \in G_2^{(2)} \text{ it follows that } (\gamma, \gamma') \in G_1^{(2)}
\]
Let $G$ be a Lie group. Then $G$ over $e_G$ is a groupoid, where the group multiplication $\cdot : G^2 \to G$ is defined for all elements $g_1, g_2, g \in G$ such that $g_1 \cdot g_2 = g$. A groupoid morphism between a finite path groupoid $\mathcal{P}_T\Sigma$ to $G$ is given by the maps

$$ h_T : \mathcal{P}_T\Sigma \to G, \quad h_T : V_T \to e_G $$

Clearly

$$ h_T(\gamma \circ \gamma') = h_T(\gamma)h_T(\gamma') \quad \text{for all } (\gamma, \gamma') \in \mathcal{P}_T\Sigma^{(2)} $$

$$ s_G(h_T(\gamma)) = h_T(s_{\mathcal{P}_T\Sigma}(\gamma)), \quad t_G(h_T(\gamma)) = h_T(t_{\mathcal{P}_T\Sigma}(\gamma)) $$

But for an arbitrary pair $(h_T(\gamma_1), h_T(\gamma_2)) =: (g_1, g_2) \in G^{(2)}$ it does not follows that, $(\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)}$ is true. Hence $h_T$ is not a strong groupoid morphism.

**Definition 10.** Let $\mathcal{P}_T\Sigma \supseteq V_T$ be a finite path groupoid.

Two paths $\gamma$ and $\gamma'$ in $\mathcal{P}_T\Sigma$ have the **same-holonomy for all connections** iff

$$ h_T(\gamma) = h_T(\gamma') \quad \text{for all } (h_T, h_T) \quad \text{groupoid morphisms} $$

$$ h_T : \mathcal{P}_T\Sigma \to G, \quad h : V_T \to \{e_G\} $$

Denote the relation by $\sim_{s.hol.}$.

**Lemma 11.** The same-holonomy for all connections relation is an equivalence relation.

Notice that, the quotient of the finite path groupoid and the same-holonomy relation for all connections replace the hoop group, which has been used in [1].

**Definition 12.** Let $\mathcal{P}_T\Sigma \supseteq V_T$ be a finite path groupoid modulo same-holonomy for all connections equivalence.

A **holonomy map for a finite path groupoid** $\mathcal{P}_T\Sigma$ over $V_T$ is a groupoid morphism consisting of the maps $(h_T, h_T)$, where $h_T : \mathcal{P}_T\Sigma \to G, h_T : V_T \to \{e_G\}$. The set of all holonomy maps is abbreviated by $\text{Hom}(\mathcal{P}_T\Sigma, G)$.

For a short notation observe the following. In further sections it is always assumed that, the finite path groupoid $\mathcal{P}_T\Sigma \supseteq V_T$ is considered modulo same-holonomy for all connections equivalence although it is not stated explicitly.

**Admissable maps and equivalent groupoid morphisms**

Now consider a finite path groupoid morphism $(h_T, h_T)$ from a finite path groupoid $\mathcal{P}_T\Sigma$ over $V_T$ to the groupoid $G$ over $\{e_G\}$, which is contained in $\text{Hom}(\mathcal{P}_T\Sigma, G)$.

Consider an arbitrary map $g_T : \mathcal{P}_T\Sigma \to G$. Then there is a groupoid morphism defined by

$$ \mathcal{G}_T(\gamma) := g_T(\gamma)h_T(\gamma)g_T(\gamma^{-1})^{-1} \quad \text{for all } \gamma \in \mathcal{P}_T\Sigma $$

if and only if

$$ \mathcal{G}_T(\gamma_1 \circ \gamma_2) = \mathcal{G}_T(\gamma_1)\mathcal{G}_T(\gamma_2) \quad \text{for all } (\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)} $$

holds. Then $\mathcal{G}_T \in \text{Hom}(\mathcal{P}_T\Sigma, G)$.

Hence for all $(\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)}$ it is necessary that

$$ \mathcal{G}_T(\gamma_1 \circ \gamma_2) = \mathcal{G}_T(\gamma_1 \circ \gamma_2)h_T(\gamma_1 \circ \gamma_2)g_T(\gamma_1 \circ \gamma_2^{-1} \circ \gamma_1^{-1})^{-1} $$

$$ = \mathcal{G}_T(\gamma_1 \circ \gamma_2)h_T(\gamma_1)h_T(\gamma_2)g_T(\gamma_1^{-1} \circ \gamma_1^{-1})^{-1} $$

$$ = \mathcal{G}_T(\gamma_1)\mathcal{G}_T(\gamma_1^{-1})^{-1}\mathcal{G}_T(\gamma_2)h_T(\gamma_2)g_T(\gamma_2^{-1})^{-1} $$

is satisfied. Therefore the map is required to fulfill

$$ g_T(\gamma_1) = g_T(\gamma_1 \circ \gamma_2), \quad g_T(\gamma_2^{-1}) = g_T((\gamma_1 \circ \gamma_2)^{-1}) $$

and

$$ g_T(\gamma_1^{-1})^{-1}g_T(\gamma_2) = e_G \quad \text{for all } (\gamma_1, \gamma_2) \in \mathcal{P}_T\Sigma^{(2)} \quad \text{in particular,} $$

$$ g_T(\gamma^{-1})^{-1}g_T(\gamma) = e_G \quad \text{for all } (\gamma^{-1}, \gamma) \in \mathcal{P}_T\Sigma^{(2)} $$

for every refinement $\gamma_1 \circ \gamma_2$ of each $\gamma$ in $\mathcal{P}_T\Sigma$ and $\gamma_1$ being an initial segment of $\gamma_1 \circ \gamma_2$ and $\gamma_2^{-1}$ an final segment of $(\gamma_1 \circ \gamma_2)^{-1}$. In comparison with Fleischhack’s definition in [7] Def. 3.7] such maps are called admissible.
Consider a map \( g_Γ : V_Γ \to G \) such that
\[
(g_Γ, h_Γ) \in \text{Map}(V_Γ, G) \times \text{Hom}(P_Γ Σ, G)
\]
which is also called a local gauge map. Then the map \( \tilde{G}_Γ \) defined by
\[
\tilde{G}_Γ(γ) := g_Γ(s(γ))h_Γ(γ)g_Γ(s(γ^{-1}))^{-1} \text{ for all } γ ∈ P_Γ Σ
\]
is a groupoid morphism. This is a result of the computation:
\[
\tilde{G}_Γ(γ_1γ_2) = g_Γ(s(γ_1))h_Γ(γ_1γ_2)g_Γ(t(γ_2))^{-1} = g_Γ(s(γ_1))h_Γ(γ_1)g_Γ(t(γ_1))^{-1}g_Γ(s(γ_2))h_Γ(γ_2)g_Γ(t(γ_2))^{-1}
\]
since \( t(γ_1) = s(γ_2) \).

**Definition 13.** The set of maps \( g_Γ : P_Γ Σ \to G \) satisfying (1) for all pairs of decomposable paths in \( P_Γ (2) Σ \) is called the set of admissible maps and is denoted by \( \text{Map}^A(P_Γ Σ, G) \).

2.2.2 Holonomy maps for finite graph systems

Ashtekar and Lewandowski [1] have presented the loop decomposition into a finite set of independent hoops (in the analytic category). This structure is replaced by a graph, since a graph is a set of independent edges. Notice that, the set of hoops that is generated by a finite set of independent hoops, is generalised to the set of finite graph systems. A finite path groupoid is generated by the set of edges, which defines a graph \( Γ \), but a set of elements of the path groupoid need not be a graph again. The appropriate notion for graphs constructed from sets of paths is the finite graph system, which is defined in section 2.2.1. Now the concept of holonomy maps is generalised for finite graph systems. Since the set, which is generated by a finite number of independent edges, contains paths that are composable, there are two possibilities to identify the image of the holonomy map for a finite graph system on a fixed graph with a subgroup of \( G^{[Γ]} \). One way is to use the generating set of independent edges of a graph, which has been also used in [1]. On the other hand, it is also possible to identify each graph with a disconnected subgraph of a fixed graph, which is generated by a set of independent edges. Notice that, the author implements two situations. One case is given by a set of paths that can be composed further and the other case is related to paths that are not composable. This is necessary for the definition of an action of the flux operators. Precisely the identification of the image of the holonomy maps along these paths is necessary to define a well-defined action of a flux element on the configuration space. This issue is studied in remark 2.2.1 in section 4.1.

First of all consider a graph \( Γ \) that is generated by the set \( \{γ_1, ..., γ_N\} \) of edges. Then each subgraph of a graph \( Γ \) contain paths that are composition of edges in \( \{γ_1, ..., γ_N\} \) or inverse edges. For example the following set \( Γ' := \{γ_1 \circ γ_2 \circ γ_3, γ_4\} \) defines a subgraph of \( Γ := \{γ_1, γ_2, γ_3, γ_4\} \). Hence there is a natural identification available.

**Definition 15.** A subgraph \( Γ' \) of a graph \( Γ \) is always generated by a subset \( \{γ_1, ..., γ_M\} \) of the generating set \( \{γ_1, ..., γ_N\} \) of independent edges that generates the graph \( Γ \). Hence each subgraph is identified with a subset of \( \{γ_1, ..., γ_M\} \). This is called the natural identification of subgraphs.

**Example 2.1:** For example consider a subgraph \( Γ' := \{γ_1 \circ γ_2, γ_3 \circ γ_4, ..., γ_{M-1} \circ γ_M\} \), which is identified naturally with a set \( \{γ_1, ..., γ_M\} \). The set \( \{γ_1, ..., γ_M\} \) is a subset of \( \{γ_1, ..., γ_N\} \) where \( N = |Γ| \) and \( M \leq N \).

Another example is given by the graph \( Γ'' := \{γ_1, γ_2\} \) such that \( γ_2 = γ_1 \circ γ_2', \) then \( Γ'' \) is identified naturally with \( \{γ_1, γ_1', γ_2\} \). This set is a subset of \( \{γ_1, γ_1', γ_2, γ_3, ..., γ_{N-1}\} \).
Definition 16. Let \( \Gamma \) be a graph, \( \mathcal{P}_\Gamma \) be the finite graph system. Let \( \Gamma' := \{ \gamma_1, ..., \gamma_M \} \) be a subgraph of \( \Gamma \).

A **holonomy map for a finite graph system** \( \mathcal{P}_\Gamma \) is a given by a pair of maps \((h_\Gamma, h_G)\) such that there exists a holonomy map \( h_\Gamma : \mathcal{P}_\Gamma \rightarrow G^{[\Gamma]} \), \( h_\Gamma(\{ \gamma_1, ..., \gamma_M \}) = (h_\Gamma(\gamma_1), ..., h_\Gamma(\gamma_M), e_G, ..., e_G) \)

\( h_G : V_\Gamma \rightarrow \{ e_G \} \)

The set of all holonomy maps for the finite graph system is denoted by \( \text{Hom}(\mathcal{P}_\Gamma, G^{[\Gamma]}) \).

The image of a map \( h_\Gamma \) on each subgraph \( \Gamma' \) of the graph \( \Gamma \) is given by

\[
(h_\Gamma(\gamma_1), ..., h_\Gamma(\gamma_M), e_G, ..., e_G)
\]

is an element of \( G^{[\Gamma]} \). The set of all images of maps on subgraphs of \( \Gamma \) is denoted by \( \mathcal{A}_\Gamma \).

The idea is now to study two different restrictions of the set \( \mathcal{P}_\Gamma \) of subgraphs. For a short notation of a "set of holonomy maps for a certain restricted set of subgraphs of a graph" in this article the following notions are introduced.

Definition 17. **If the subset of all disconnected subgraphs of the finite graph system** \( \mathcal{P}_\Gamma \) **is considered**, then the restriction of \( \mathcal{A}_\Gamma \), which is identified with \( G^{[\Gamma]} \) appropriately, is called the **non-standard identification of the configuration space**. **If the subset of all natural identified subgraphs of the finite graph system** \( \mathcal{P}_\Gamma \) **is considered**, then the restriction of \( \mathcal{A}_\Gamma \), which is identified with \( G^{[\Gamma]} \) appropriately, is called the **natural identification of the configuration space**.

A comment on the non-standard identification of \( \mathcal{A}_\Gamma \) is the following. If \( \Gamma' := \{ \gamma_1 \circ \gamma_2 \} \) and \( \Gamma'' := \{ \gamma_2 \} \) are two subgraphs of \( \Gamma := \{ \gamma_1, \gamma_2, \gamma_3 \} \). The graph \( \Gamma' \) is a subgraph of \( \Gamma \). Then evaluation of a map \( h_\Gamma \) on a subgraph \( \Gamma' \) is given by

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2), h_\Gamma(s(\gamma_2)), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), e_G, e_G) \in G^3
\]

and the holonomy map of the subgraph \( \Gamma'' \) of \( \Gamma' \) is evaluated by

\[
h_\Gamma(\Gamma'') = (h_\Gamma(s(\gamma_1)), h_\Gamma(s(\gamma_2)), h_\Gamma(s(\gamma_3))) = (h_\Gamma(\gamma_2), e_G, e_G) \in G^3
\]

Example 2.2: Recall example 2.2[1] For example for a subgraph \( \Gamma' := \{ \gamma_1 \circ \gamma_2 \circ \gamma_3 \circ \gamma_4, ..., \gamma_M \circ \gamma_1 \circ \gamma_i \circ \gamma_j \} \), which is naturally identified with a set \( \{ \gamma_1, ..., \gamma_M \} \). Then the holonomy map is evaluated at \( \Gamma' \) such that

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), h_\Gamma(\gamma_3), e_G, ..., e_G) \in G^N
\]

where \( N = |\Gamma'| \). For example, let \( \Gamma' := \{ \gamma_1, \gamma_2 \} \) such that \( \gamma_2 = \gamma_1 \circ \gamma_2' \) and which is naturally identified with \( \{ \gamma_1, \gamma_1', \gamma_2 \} \). Hence

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1), h_\Gamma(\gamma_1'), h_\Gamma(\gamma_2'), e_G, ..., e_G) \in G^N
\]

is true.

Another example is given by the disconnected graph \( \Gamma' := \{ \gamma_1 \circ \gamma_2 \circ \gamma_3, \gamma_4 \} \), which is a subgraph of \( \Gamma := \{ \gamma_1, \gamma_2, \gamma_3, \gamma_4 \} \). Then the non-standard identification is given by

\[
h_\Gamma(\Gamma') = (h_\Gamma(\gamma_1 \circ \gamma_2 \circ \gamma_3), h_\Gamma(\gamma_4), e_G, e_G) \in G^4
\]

If the natural identification is used, then \( h_\Gamma(\Gamma') \) is identified with

\[
(h_\Gamma(\gamma_1), h_\Gamma(\gamma_2), h_\Gamma(\gamma_3), h_\Gamma(\gamma_4)) \in G^4
\]

Consider the following example. Let \( \Gamma''' := \{ \gamma_1, \alpha, \gamma_2, \gamma_3 \} \) be a graph such that

\[\text{In the work the holonomy map for a finite graph system and the holonomy map for a finite path groupoid is denoted by the same pair \( (h_\Gamma, h_G) \).}\]
Then notice the sets $\Gamma_1 := \{\gamma_1 \circ \alpha, \gamma_3\}$ and $\Gamma_2 := \{\gamma_1 \circ \alpha^{-1}, \gamma_3\}$. In the non-standard identification of the configuration space $A_{\Gamma''}$ it is true that,

$$h_{\Gamma''}(\Gamma_1) = (h_{\Gamma''}(\gamma_1 \circ \alpha), h_{\Gamma''}(\gamma_3), e_G, e_G) \in G^4,$$

$$h_{\Gamma''}(\Gamma_2) = (h_{\Gamma''}(\gamma_1 \circ \alpha^{-1}), h_{\Gamma''}(\gamma_3), e_G, e_G) \in G^4$$

holds. Whereas in the natural identification of $A_{\Gamma''}$

$$h_{\Gamma''}(\Gamma_1) = (h_{\Gamma''}(\gamma_1 \circ \alpha), h_{\Gamma''}(\gamma_3), e_G) \in G^4,$$

$$h_{\Gamma''}(\Gamma_2) = (h_{\Gamma''}(\gamma_1 \circ \alpha^{-1}), h_{\Gamma''}(\gamma_3), e_G) \in G^4$$

yields.

The equivalence class of similar or equivalent groupoid morphisms defined in definition 14 allows to define the following object. The set of images of all holonomy maps of a finite graph system modulo the similar or equivalent groupoid morphisms equivalence relation is denoted by $A_{\Gamma}/\tilde{\Phi}_T$.

2.2.3 Transformations in finite path groupoids and finite graph systems

The aim of this section is to clarify the graph changing operators in LQG framework and the role of finite diffeomorphisms in $\Sigma$. Therefore operations, which add, delete or transform paths, are introduced. In particular translations in a finite path graph groupoid and in the groupoid $G$ over $\{e_G\}$ are studied.

Transformations in finite path groupoid

**Definition 18.** Let $\varphi$ be a $C^k$-diffeomorphism on $\Sigma$, which maps surfaces into surfaces.

Then let $(\Phi_T, \varphi_T)$ be a pair of bijective maps, where $\varphi|_{V_T} = \varphi_T$ and

$$\Phi_T : P_T \Sigma \to P_T \Sigma \text{ and } \varphi_T : V_T \to V_T$$

such that

$$(s \circ \Phi_T)(\gamma) = (\varphi_T \circ s)(\gamma), \quad (t \circ \Phi_T)(\gamma) = (\varphi_T \circ t)(\gamma) \text{ for all } \gamma \in P_T \Sigma$$

holds such that $(\Phi_T, \varphi_T)$ defines a groupoid morphism.

Call the pair $(\Phi_T, \varphi_T)$ a **path-diffeomorphism of a finite path groupoid** $P_T \Sigma$ over $V_T$. Denote the set of finite path-diffeomorphisms by $\text{Diff}(P_T \Sigma)$.

Notice that, for $(\gamma, \gamma') \in P_T \Sigma^{(2)}$ it is true that

$$\Phi_T(\gamma \circ \gamma') = \Phi_T(\gamma) \circ \Phi_T(\gamma')$$

requires that

$$(t \circ \Phi_T)(\gamma) = (s \circ \Phi_T)(\gamma')$$

Hence from (8) and (9) it follows that, $\Phi_T(1_v) = 1_{\varphi_T(v)}$ is true.
A path-diffeomorphism \((\Phi_\Gamma, \varphi_\Gamma)\) is lifted to \(\text{Hom}(P_\Gamma \Sigma, G)\).

The pair \((h_\Gamma \circ \Phi_\Gamma, h_\Gamma \circ \varphi_\Gamma)\) defined by

\[
\begin{align*}
  h_\Gamma \circ \Phi_\Gamma : P_\Gamma \Sigma &\to G, \quad \gamma \mapsto (h_\Gamma \circ \Phi_\Gamma)(\gamma) \\
  h_\Gamma \circ \varphi_\Gamma : V_\Gamma &\to \{e_G\}, \quad (h_\Gamma \circ \varphi_\Gamma)(v) = e_G
\end{align*}
\]

such that

\[
\begin{align*}
  s_{\text{Hol}}((h_\Gamma \circ \Phi_\Gamma)(\gamma)) &= (h_\Gamma \circ \varphi_\Gamma)(s(\gamma)) = e_G, \\
  t_{\text{Hol}}(h_\Gamma \circ \Phi_\Gamma)(\gamma) &= (h_\Gamma \circ \varphi_\Gamma)(t(\gamma)) = e_G \text{ for all } \gamma \in P_\Gamma \Sigma
\end{align*}
\]

whenever \((h_\Gamma, h_\Gamma)\) is \(\text{Hom}(P_\Gamma \Sigma, G)\) and \((\Phi_\Gamma, \varphi_\Gamma)\) is a path-diffeomorphism, is a holonomy map for a finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\).

**Definition 19.** A left-translation in the finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) at a vertex \(v\) is a map defined by

\[
L_\theta : P_\Gamma \Sigma^u \to P_\Gamma \Sigma^w, \quad \gamma \mapsto L_\theta(\gamma) := \theta \circ \gamma
\]

for some \(\theta \in P_\Gamma \Sigma^u\) and all \(\gamma \in P_\Gamma \Sigma^u\).

In analogy a right-translation \(R_\theta\) and an inner-translation \(I_{\theta, \theta'}\) in the finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) at a vertex \(v\) can be defined.

**Remark 20.** Let \((\Phi_\Gamma, \varphi_\Gamma)\) be a path-diffeomorphism on a finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\). Then a left-translation in the finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) at a vertex \(v\) is defined by a path-diffeomorphism \((\Theta_\Gamma, \varphi_\Gamma)\) and the following object

\[
L_{\Phi_\Gamma} : P_\Gamma \Sigma^u \to P_\Gamma \Sigma^{\varphi(\gamma)}, \quad \gamma \mapsto L_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma) \text{ for } \gamma \in P_\Gamma \Sigma^u
\]

Furthermore a right-translation in the finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) at a vertex \(v\) is defined by a path-diffeomorphism \((\Phi_\Gamma, \varphi_\Gamma)\) and the following object

\[
R_{\Phi_\Gamma} : P_\Gamma \Sigma^v \to P_\Gamma \Sigma^{\varphi(\gamma)}, \quad \gamma \mapsto R_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma) \text{ for } \gamma \in P_\Gamma \Sigma^v
\]

Finally an inner-translation in the finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) at the vertices \(v\) and \(w\) is defined by

\[
I_{\Phi_\Gamma} : P_\Gamma \Sigma^u \to P_\Gamma \Sigma^{\varphi(\gamma)}, \quad \gamma \mapsto I_{\Phi_\Gamma}(\gamma) := \Phi_\Gamma(\gamma) \text{ for } \gamma \in P_\Gamma \Sigma^w
\]

where \((s \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(v)\) and \((t \circ \Phi_\Gamma)(\gamma) = \varphi_\Gamma(w)\).

In the following discussions the right-translation in a finite path groupoid is focused, but there is a generalisation to left-translations and inner-translations.

**Definition 21.** A bisection of a finite path groupoid \(P_\Gamma \Sigma\) over \(V_\Gamma\) is a map \(s : P_\Gamma \Sigma \to V_\Gamma\) (i.o.w. \(s \circ \sigma = \text{id}_{V_\Gamma}\)) and such that \(t \circ \sigma : V_\Gamma \to V_\Gamma\) is a bijective map. The set of bisections on \(P_\Gamma \Sigma\) over \(V_\Gamma\) is denoted \(B(P_\Gamma \Sigma)\).

**Remark 22.** Discover that, a bisection \(\sigma \in B(P_\Gamma \Sigma)\) defines a path-diffeomorphism \((\varphi_\Gamma, \Phi_\Gamma) \in \text{Diff}(P_\Gamma \Sigma)\), where \(\varphi_\Gamma = t \circ \sigma\) and \(\Phi_\Gamma\) is given by the right-translation \(R_{\sigma(v)} : P_\Gamma \Sigma_v \to P_\Gamma \Sigma_{\varphi_\Gamma(v)}\) in \(P_\Gamma \Sigma \overset{\simeq}{\to} V_\Gamma\), where \(R_{\sigma(v)}(\gamma) = \Phi_\Gamma(\gamma)\) for all \(\gamma \in P_\Gamma \Sigma_v\) and for a fixed \(v \in V_\Gamma\). The right-translation is defined by

\[
R_{\sigma(v)}(\gamma) := \begin{cases} 
  \gamma \circ \sigma(v) & v = t(\gamma) \\
  \gamma \circ \tau(\gamma) & v \neq t(\gamma)
\end{cases}
\]

whenever \(t(\gamma)\) is the target vertex of a non-trivial path \(\gamma\) in \(\Gamma\). For a trivial path \(\mathbb{I}_v\) the right-translation is defined by \(R_{\sigma(v)}(\mathbb{I}_v) = \mathbb{I}_{(t \circ \sigma)(v)}\) and \(R_{\sigma(v)}(\mathbb{I}_w) = \mathbb{I}_w\) whenever \(v \neq w\). The right-translation \(R_{\sigma(v)}\) is required to be bijective. Before this result is proven in lemma 23 notice the following considerations.

\(^2\text{Note that in the infinite case of path groupoids an additional condition for the map } t \circ \sigma : \Sigma \to \Sigma \text{ has to be required. The map has to be a diffeomorphism. Observe that, the map } t \circ \sigma \text{ defines the finite diffeomorphism } \varphi_\Gamma : V_\Gamma \to V_\Gamma.\)
Note that, \((R_{\sigma(v)}, t \circ \sigma)\) transfers to the holonomy map such that
\[
(\mathfrak{h}_\Gamma \circ R_{\sigma(t(\gamma'))}(\gamma \circ \gamma') = \mathfrak{h}_\Gamma(\gamma \circ \gamma' \circ \sigma(t(\gamma'))))
= \mathfrak{h}_\Gamma(\gamma) \mathfrak{h}_\Gamma(\gamma' \circ \sigma(t(\gamma')))
\]
is true. There is a bijective map between a right-translation \(R_{\sigma(v)} : \mathcal{P}_T \Sigma_v \to \mathcal{P}_T \Sigma_{(t \circ \sigma)(v)}\) and a path-diffeomorphism \((\varphi_T, \Phi_T)\). In particular observe that, \(\sigma \in \mathbb{B}(\mathcal{P}_T \Sigma_v)\) and \((\varphi_T, \Phi_T) \in \text{Diff}(\mathcal{P}_T \Sigma_v)\). Simply speaking the path-diffeomorphism does not change the source and target vertex at the same time. The path-diffeomorphism changes the target vertex by a (finite) diffeomorphism and, therefore, the path is transformed.

Bisections \(\sigma\) in a finite path groupoid can be transferred, likewise path-diffeomorphisms, to holonomy maps. The pair \((\mathfrak{h}_\Gamma \circ \Phi_T, h_t \circ \varphi_T)\) of the maps defines a pair of maps \((\mathfrak{h}_\Gamma \circ \Phi_T, h_T \circ \varphi_T)\) by
\[
\mathfrak{h}_\Gamma \circ \Phi_T : \mathcal{P}_T \Sigma_v \to G \text{ and } h_T \circ \varphi_T : V_T \to \{ e_G \}
\]
which is a holonomy map for a finite path groupoid \(\mathcal{P}_T \Sigma\) over \(V_T\).

**Lemma 23.** The set \(\mathbb{B}(\mathcal{P}_T \Sigma)\) of bisections on the finite path groupoid \(\mathcal{P}_T \Sigma\) over \(V_T\) forms a group.

**Proof:** The group multiplication is given by
\[
(\sigma \ast \sigma')(v) = \sigma'(v) \circ \sigma(t(\sigma'(v))) \text{ for } v \in V_T
\]
whenever \(\sigma'(v) \in \mathcal{P}_T \Sigma_{v(\varphi_T(v))}\) and \(\sigma(t(\sigma'(v))) \in \mathcal{P}_T \Sigma_{(t \circ \sigma)(v)}\).

Clearly the group multiplication is associative. The unit \(id\) is equivalent to the object inclusion \(v \mapsto \mathbb{I}_v\) of the groupoid \(\mathcal{P}_T \Sigma \rightrightarrows V_T\), where \(\mathbb{I}_v\) is the constant loop at \(v\), and the inversion is given by
\[
\sigma^{-1}(v) = \sigma((t \circ \sigma)^{-1}(v))^{-1} \text{ for } v \in V_T
\]

The group property of bisections \(\mathbb{B}(\mathcal{P}_T \Sigma)\) carries over to holonomy maps. Using the group multiplication \(\ast\) of \(G\) conclude that
\[
(\mathfrak{h}_\Gamma \circ R_{(\sigma \ast \sigma')(v)}(\mathbb{I}_v) = \mathfrak{h}_\Gamma \circ (R_{\sigma'(v)} \circ R_{\sigma(t(\sigma'(v)))})(\mathbb{I}_v) = \mathfrak{h}_\Gamma(\sigma'(v)) \cdot \mathfrak{h}_\Gamma(\sigma(t(\sigma'(v)))) \text{ for } v \in V_T
\]
is true.

**Remark 24.** Moreover right-translations define path-diffeomorphisms, i.e. \(R_{(\sigma \ast \sigma')}(v) = \Phi_T\) and \(\varphi_T = t \circ \sigma\) whenever \(v \in V_T\). But for two bisections \(\sigma_T, \tilde{\sigma}_T \in \mathbb{B}(\mathcal{P}_T \Sigma)\) the object \(\sigma_T(v) \circ \tilde{\sigma}_T(v)\) is not comparable with \((\sigma_T \ast \tilde{\sigma}_T)(v)\). Then for the composition \(\Phi_1(\gamma) \circ \Phi_2(\gamma)\), there exists no path-diffeomorphism \(\Phi \) such that \(\Phi_1(\gamma) \circ \Phi_2(\gamma) = \Phi(\gamma)\) yields in general. Moreover generally the object \(\Phi_1(\gamma) \circ \Phi_2(\gamma') = \Phi(\gamma \circ \gamma')\) is not well-defined.

But the following is defined
\[
R_{(\sigma \ast \sigma')(v)}(\gamma) = \Phi_T'(\gamma) \circ \Phi_T(\mathbb{I}_v) =: (\Phi_T' \ast \Phi_T)(\gamma)
\]
whenever \(\gamma \in \mathcal{P}_T \Sigma_v\), \((\varphi_T, \Phi_T) \in \text{Diff}(\mathcal{P}_T \Sigma_v)\) and \((\varphi_T', \Phi_T') \in \text{Diff}(\mathcal{P}_T \Sigma_{v(v)})\) are path-diffeomorphisms such that \(\varphi_T = t \circ \sigma\), \(\Phi_T = R_{\sigma(\varphi_T(v))}\) and \(\varphi_T' = t \circ \sigma'\), \(\Phi_T' = R_{\sigma'(v)}\).

Moreover for \((\gamma, \gamma') \in \mathcal{P}_T \Sigma_v^{(2)}\) and \(\gamma' \in \mathcal{P}_T \Sigma_v\) it is true that
\[
(\Phi_T' \ast \Phi_T)(\gamma \circ \gamma') = \Phi_T'(\gamma \circ \gamma') \circ \Phi_T(\mathbb{I}_v) = \Phi_T'(\gamma) \circ \Phi_T'(\gamma') \circ \Phi_T(\mathbb{I}_v) \circ \Phi_T(\mathbb{I}_v) = \Phi_T'(\gamma) \circ (\Phi_T' \ast \Phi_T)(\gamma')
\]
holds.

Then the following lemma easily follows.

**Lemma 25.** Let \(\sigma\) be a bisection contained in \(\mathbb{B}(\mathcal{P}_T \Sigma)\) and \(v \in V_T\).

The pair \((R_{\sigma(v)}, t \circ \sigma)\) of maps such that
\[
R_{\sigma(v)} : \mathcal{P}_T \Sigma_v \to \mathcal{P}_T \Sigma_{(t \circ \sigma)(v)}, \quad s \circ R_{\sigma(v)} = (t \circ \sigma) \circ s
\]
\(t \circ \sigma : V_T \to V_T, \quad t \circ R_{\sigma(v)} = (t \circ \sigma) \circ t\)
defined in remark 23 is a path-diffeomorphism in \(\mathcal{P}_T \Sigma \rightrightarrows V_T\).
$R$ is defined by the bijective maps $P$-path groupoid

The inverse map satisfies

In general a right-translation ($\sigma$:

Moreover derive

Moreover derive

for all $\gamma^{' \prime} \in \mathcal{P}_1\Sigma_v$ and a fixed bisection $\sigma \in \mathcal{B}(\mathcal{P}_1\Sigma)$.

Notice that, $L_{\sigma(v)}$ and $I_{\sigma(v)}$ similarly to the pair $(R_{\sigma(v)}, t \circ \sigma)$ can be defined. Summarising the pairs $(R_{\sigma(v)}, t \circ \sigma)$, $(I_{\sigma(v)}, t \circ \sigma)$ and $(I_{\sigma(v)}, t \circ \sigma)$ for a bisection $\sigma \in \mathcal{B}(\mathcal{P}_1\Sigma)$ are path-diffeomorphisms of a finite path groupoid $\mathcal{P}_1\Sigma \rightrightarrows V_1$.

In general a right-translation $(R_{\sigma}, t \circ \sigma)$ in the finite path groupoid $\mathcal{P}_1\Sigma$ over $\Sigma$ for a bisection $\sigma \in \mathcal{B}(\mathcal{P}_1\Sigma)$ is defined by the bijective maps $R_{\sigma}$ and $t \circ \sigma$, which are given by

For example for a fixed suitable bisection $\sigma$ the right-translation is $R_{\sigma}(1_v) = \gamma$, then $R_{\sigma}^{-1}(\gamma) = \gamma \circ \gamma^{-1} = 1_v$ for $v = s(\gamma)$. Clearly the right-translation $(R_{\sigma}, t \circ \sigma)$ is not a groupoid morphism in general.

**Definition 26.** Define for a given bisection $\sigma \in \mathcal{B}(\mathcal{P}_1\Sigma)$, the right-translation in the groupoid $G$ over $\{e_G\}$ through

Furthermore for a fixed $\sigma \in \mathcal{B}(\mathcal{P}_1\Sigma)$ define the left-translation in the groupoid $G$ over $\{e_G\}$ by

and the inner-translation in the groupoid $G$ over $\{e_G\}$

The pairs $(R_{\sigma}, t \circ \sigma)$ and $(L_{\sigma}, t \circ \sigma)$ are not groupoid morphisms. Whereas the pair $(I_{\sigma}, t \circ \sigma)$ is a groupoid morphism, since for all pairs $(\gamma, \gamma') \in \mathcal{P}_1\Sigma(2)$ such that $t(\gamma) = s(\gamma')$ it is true that $\sigma(t(\gamma)) \circ \sigma((t \circ \sigma)^{-1}(t(\gamma)))^{-1} = 1_{s(\gamma)}$ holds. Notice that, in this situation $\sigma(t(\gamma)) = \sigma(t(\gamma) \circ \gamma')$ is satisfied.
Proposition 27. The map $\sigma \mapsto R_\sigma$ is a group isomorphism, i.e. $R_{\sigma\circ\sigma'} = R_\sigma \circ R_{\sigma'}$, and where $\sigma \mapsto t \circ \sigma$ is a group isomorphism from $\mathcal{B}(P_T \Sigma)$ to the group of finite diffeomorphisms $\text{Diff}(V_T)$ in a finite subset $V_T$ of $\Sigma$. The maps $\sigma \mapsto L_\sigma$ and $\sigma \mapsto I_\sigma$ are group isomorphisms.

There is a generalisation of path-diffeomorphisms in the finite path groupoid, which coincide with the graphomorphism presented by Fleischhack in [7]. In this approach the diffeomorphism $\varphi : \Sigma \to \Sigma$ changes the source and target vertex of a path $\gamma$. Consequently the path-diffeomorphism $(\Phi, \varphi)$, which implements the inner-translations $I_\sigma$ in the path groupoid $P\Sigma \equiv \Sigma$, is a graphomorphism in the context of Fleischhack. Some element of the set of graphomorphisms is directly related to a right-translation $R_\sigma$ in the path groupoid. Precisely for every $v \in \Sigma$ and $\sigma \in \mathcal{B}(P\Sigma)$ the pairs $(R_{\sigma(v)}, t \circ \sigma)$, $(L_{\sigma(v)}, t \circ \sigma)$ and $(I_{\sigma(v)}, t \circ \sigma)$ define graphomorphism. Furthermore the right-translation $R_{\sigma(v)}$, the left-translation $L_{\sigma(v)}$ and the inner-translation $I_{\sigma(v)}$ are required to be bijective maps, and hence the maps cannot map non-trivial paths to trivial paths. This property restricts the set of all graphomorphisms, which is generated by these translations. In particular in this article graph changing operations, which change the number of edges of a graph, are studied. Hence the left- or right-tranlation shows that composed paths arise, too. The maps $R_{\sigma(v)}$, $L_{\sigma(v)}$, and $I_{\sigma(v)}$ are constructed such that (13) generalises. But note that, in the infinite case considered by Fleischhack the action of the bisections $\mathcal{B}(P\Sigma)$ of a bisection $\tilde{\sigma}(\gamma_1, \gamma_2)$ is disconnected and the bisection $\tilde{\sigma}_2$ given by the elements $\tilde{\sigma}(\gamma_1, \gamma_2)$. In this case the graph-diffeomorphism acts on all vertices in the set $V_{\tilde{\sigma}}$, which is indeed a graph. Set $\Gamma := \{\gamma_1, \gamma_2\}$ and a holonomy map for the path groupoid $P\Sigma$ of diffeomorphisms in $\Sigma$ and a point norm continuous action of the group. Consequently there is an action of the group of diffeomorphisms in $\Sigma$ on the finite path groupoid, which is used to define an action of the group of diffemorphisms in $\Sigma$ on the analytic holonomy $C^*$-algebra.

Transformations in finite graph systems

To proceed it is necessary to transfer the notion of bisections and right-translatings to finite graph systems. A right-translation $R_{\sigma}$ is a mapping that maps graphs to graphs. Each graph is a finite union of independent edges. This causes problems. Since the definition of right-translation in a finite graph system $P_T$ is often not well-defined for all bisections in the finite graph system and all graphs. For example if the graph $\Gamma := \{\gamma_1, \gamma_2\}$ is disconnected and the bisection $\tilde{\sigma}$ in the finite path groupoid $P_T \Sigma$ over $V_T$ is defined by $\tilde{\sigma}(\gamma_1) = \gamma_1$, $\tilde{\sigma}(\gamma_2) = \gamma_2$, $\tilde{\sigma}(t(\gamma_1)) = \gamma_1^{-1}$ and $\tilde{\sigma}(t(\gamma_2)) = \gamma_2^{-1}$ where $V_T := \{s(\gamma_1), t(\gamma_1), s(\gamma_2), t(\gamma_2)\}$. Let $\mathbb{I}_T$ be the set given by the elements $\mathbb{I}_{s(\gamma_1)}, \mathbb{I}_{s(\gamma_2)}$, $\mathbb{I}_{t(\gamma_1)}$ and $\mathbb{I}_{t(\gamma_2)}$. Then notice that, a bisection $\sigma$, which maps a set of vertices in $V_T$ to a set of paths in $P_T \Sigma$, is given for example by $\sigma \in P_T \Sigma$ and hence not a graph. Loosely speaking the graph-diffeomorphism acts on all vertices in the set $V_T$ and hence implements four new edges. But a bisection $\sigma$, which maps a subset $V := \{s(\gamma_1), s(\gamma_2)\}$ of $V_T$ to a set of paths, leads to a translation $R_{\sigma v}(V) = \{\mathbb{I}_{s(\gamma_1)}, \mathbb{I}_{s(\gamma_2)}\} = \{\gamma_1, \gamma_2\}$, which is indeed a graph. Set $\Gamma' := \{\gamma_1\}$ and $V' := \{s(\gamma_1)\}$. Then observe that, for a restricted bisection, which maps a set $V'$ of vertices in $V_T$ to a set of paths in $P_T \Sigma$, the right-translation become $R_{\sigma v}(V') = \{\mathbb{I}_{s(\gamma_1)}\} = \{\gamma_1\}$, which defines a graph, too. Notice that $\mathbb{I}_{s(\gamma_1)}$ is a subgraph of $\Gamma'$. Hence in the simplest case new edges are emerging. The next definition of the right-tranlation shows that composed paths arise, too.

Definition 28. Let $\Gamma$ be a graph, $P_T \Sigma \equiv V_T$ be a finite path groupoid and let $P_T$ be a finite graph system. Moreover the set $V_T$ is given by $\{v_1, \ldots, v_{2N}\}$.

A bisection of a finite graph system $P_T$ is a map $\sigma : V_T \to P_T$ such that there exists a bisection $\tilde{\sigma} \in \mathcal{B}(P_T \Sigma)$ such that $\sigma T(v) = \{\tilde{\sigma}(v) : v \in V\}$ whenever $V$ is a subset of $V_T$.

Define a restriction $\sigma' : V_T \to P_{T'}$ of a bisection $\sigma$ in $P_T$ by

$$\sigma'(V) := \{\tilde{\sigma}(w) : w \in V\}$$

for each subgraph $\Gamma'$ of $\Gamma$ and $V \subseteq V_T$.

A right-translation in the finite graph system $P_T$ is a map $R_{\sigma v} : P_{T'} \to P_{T'}$, which is given by a
bisection $\sigma_\Gamma : V_\Gamma \to P_\Gamma$, such that
\[
R_{\sigma_\Gamma}(\Gamma') = R_{\sigma_\Gamma}(\{\gamma_1''', \ldots, \gamma_M''', \|_{w_i} : w_i \in \{s(\gamma_1'), \ldots, s(\gamma_k') \in V_{\Gamma'} : s(\gamma_i') \neq s(\gamma_j') \forall i \neq j \} \setminus V_{\Gamma''})
\]
\[
= \begin{cases}
\gamma_1'', \ldots, \gamma_j'', \gamma_{j+1}'' \circ \tilde{\sigma}(t(\gamma_j + 1)), \ldots, \gamma_{M}'' \circ \tilde{\sigma}(t(\gamma_M)), \|_{w_i} \circ \tilde{\sigma}(w_i) : w_i \in \{s(\gamma_1'), \ldots, s(\gamma_k') \in V_{\Gamma'} : s(\gamma_i') \neq s(\gamma_j') \forall i \neq j \} \setminus V_{\Gamma''},
\end{cases}
\]
where $\tilde{\sigma} \in B(P_\Gamma \Sigma)$, $K := |\Gamma'|$ and $M := |\Gamma''|$, $V_{\Gamma'}^s$ is the set of all source vertices of $\Gamma'$ and such that $\Gamma'' := \{\gamma_1'', \ldots, \gamma_M''\}$ is a subgraph of $\Gamma' := \{\gamma_1', \ldots, \gamma_K'\}$ and $\Gamma''_\rho$ is a subgraph of $\Gamma'$. Derive that, for $\tilde{\sigma}(t(\gamma_i)) = \gamma_i^\rho$ it is true that $(t \circ \tilde{\sigma})(s(\gamma_i^\rho)) = s(\gamma_i) = (t \circ \tilde{\sigma})(t(\gamma_i))$ holds.

Example 2.3: Let $\Gamma$ be a disconnected graph. Then for a bisection $\tilde{\sigma} \in B(P_\Gamma \Sigma)$ such that $\sigma(t(\gamma_i)) = \gamma_i^{-1}$ for all $1 \leq i \leq |\Gamma|$ it is true that
\[
R_{\sigma_\Gamma}(\Gamma) = \{\gamma_1 \circ \tilde{\sigma}(t(\gamma_1)), \ldots, \gamma_N \circ \tilde{\sigma}(t(\gamma_N)), \|_{s(\gamma_1)}, \ldots, \|_{s(\gamma_N)} \circ \tilde{\sigma}(s(\gamma_N))\}
\]
yields. Set $\Gamma' := \{\gamma_1', \ldots, \gamma_M'\}$, then derive
\[
R_{\sigma_\Gamma}(\Gamma') = \{\gamma_1' \circ \tilde{\sigma}(t(\gamma_1')), \ldots, \gamma_M' \circ \tilde{\sigma}(t(\gamma_M')), \|_{s(\gamma_1')}, \ldots, \|_{s(\gamma_N-M)} \circ \tilde{\sigma}(s(\gamma_N-M))\}
\]
if $\Gamma = \Gamma' \cup \{\gamma_1, \ldots, \gamma_{N-M}\}$.

To understand the definition of the right-translation notice the following problem.

Problem 2.1: Consider a subgraph $\Gamma$ of $\tilde{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, a map $\tilde{\sigma} : V_{\tilde{\Gamma}} \to P_{\tilde{\Gamma}} \Sigma$. Then the map
\[
R_{\sigma_{\tilde{\Gamma}}}(\Gamma) = \{\gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ \|_{t(\gamma_2)}, \gamma_3 \circ \|_{t(\gamma_3)}, \|_{s(\gamma_1)} \circ \|_{t(\gamma_4)}\} =: \Gamma_{\sigma}
\]
is not a right-translation. This follows from the following fact. Notice that, the map $\sigma$ maps $t(\gamma_1) \mapsto s(\gamma_1)$, $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $s(\gamma_1) \mapsto t(\gamma_4)$. Then the map $\tilde{\sigma}$ is not a bisection in the finite path groupoid $P_{\tilde{\Gamma}} \Sigma$ over $V_{\tilde{\Gamma}}$ and does not define a right-translation $R_{\sigma_{\tilde{\Gamma}}}$ in the finite graph system $P_{\tilde{\Gamma}}$.

This is a general problem. For every bisection $\sigma$ in a finite path groupoid such that a graph $\Gamma := \{\gamma\}$ is translated to $\{\gamma \circ \tilde{\sigma}(t(\gamma), \tilde{\sigma}(s(\gamma))\}$. Hence either such translations in the graph system are excluded or the definition of the bisections has to be restricted to maps such that the map $t \circ \tilde{\sigma}$ is not bijective. Clearly, the restriction of the right-translation such that $\tilde{\Gamma}$ is mapped to $\{\gamma \circ \tilde{\sigma}(t(\gamma), \|_{s(\gamma)})\}$ implies that a simple path orientation transformation is not implemented by a right-translation.

Furthermore there is an ambiguity for graph containing to paths $\gamma_1$ and $\gamma_2$ such that $t(\gamma_1) = t(\gamma_2)$. Since in this case a bisection $\sigma$, which maps $t(\gamma_1)$ to $t(\gamma_3)$, the right-translation is $\{\gamma_1 \circ \gamma_3, \gamma_2 \circ \gamma_3\}$, is not a graph anymore.
Example 2.4: Otherwise there is for example a subgraph $\Gamma'$ of $\Gamma := \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and a bisection $\tilde{\sigma}_\Gamma$ such that

$$\Gamma'_\sigma := \{\gamma_1 \circ \gamma_1^{-1}, \gamma_2 \circ s(\gamma_2), \gamma_3 \circ s(\gamma_3)\}$$

Notice that, $t(\gamma_1) \mapsto s(\gamma_1)$, $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $t(\gamma_4) \mapsto t(\gamma_4)$. Hence the the map $\tilde{\sigma}_\Gamma : V_\Gamma \to \mathcal{P}_\Gamma \Sigma$ is bijective map and consequently a bisection. The bisection $\sigma_\Gamma$ in the graph system $\mathcal{P}_\Gamma$ defines a right-translation $R_{\sigma_\Gamma}$ in $\mathcal{P}_\Gamma$.

Moreover for a subgraph $\Gamma'' := \{\gamma_2, \gamma_3\}$ of the graph $\tilde{\Gamma} := \{\gamma_1, \gamma_2, \gamma_3\}$ there exists a map $\sigma_t : V_\Gamma \to \mathcal{P}_\Gamma$ such that

$$R_{\sigma_t}(\Gamma'') = \{\gamma_2, \gamma_3, s(s(\gamma_1))\} = \{\gamma_2, \gamma_3, \gamma_1\}$$

where $t(\gamma_2) \mapsto t(\gamma_2)$, $t(\gamma_3) \mapsto t(\gamma_3)$ and $s(\gamma_1) \mapsto t(\gamma_1)$. Consequently in this example the map $\tilde{\sigma}_\Gamma$ is a bisection, which defines a right-translation in $\mathcal{P}_\Gamma$.

Note that, for a graph $\Gamma$ such that $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ are subgraphs the bisection $\sigma_\Gamma$ extends to a bisection $\sigma$ in $\mathcal{P}_\Gamma$ and $\sigma_\Gamma$ extends to a bisection $\tilde{\sigma}$ in $\mathcal{P}_\Gamma$.

Moreover the bisections of a finite graph system are transfered, analogously, to bisections of a finite path groupoid $\mathcal{P}_\Gamma \Sigma \equiv V_\Gamma$ to the group $G^{[\Gamma]}$. Let $\sigma \in \mathcal{B}(\mathcal{P}_\Gamma)$ and $(\eta_\Gamma, h_\Gamma) \in \text{Hom}(\mathcal{P}_\Gamma, G^{[\Gamma]})$. Thus there are two maps

$$h_\Gamma \circ R_{\sigma} : \mathcal{P}_\Gamma \to G^{[\Gamma]} \quad \text{and} \quad h_\Gamma \circ (t \circ \sigma) : V_\Gamma \to \{e_G\}$$

which defines a holonomy map for a finite graph system if $\sigma$ is suitable.

Now, a similar right-translation in a finite graph system in comparison to the right-translation $R_{\sigma(v)}$ in a finite path groupoid is studied. Let $\sigma_{\Gamma'} : V_{\Gamma'} \to \mathcal{P}_{\Gamma'}$ be a restriction of $\sigma_{\Gamma} \in \mathcal{B}(\mathcal{P}_{\Gamma})$. Moreover let $V$ be a subset of $V_{\Gamma'}$, let $\Gamma'''$ be a subgraph of $\Gamma''$ and $\Gamma'''$ be a subgraph of $\Gamma''$. Then a right-translation is given by

$$R_{\sigma_{\Gamma'}(V)}(\Gamma''')$$

$$\begin{cases} R_{\sigma_{\Gamma'}}(\{\gamma_1', \ldots, \gamma_M, w_i : w_i \in \{s(\gamma_1), \ldots, s(\gamma_M)\} \in V_{\Gamma'} : s(\gamma_i) \neq s(\gamma_j) \forall i \neq j\} \setminus V_{\Gamma''}) : V_{\Gamma''} \subset V \\ R_{\sigma_{\Gamma'}}(\{\gamma_1', \ldots, \gamma_M, w_i : w_i \in \{s(\gamma_1'), \ldots, s(\gamma_M')\} \in V_{\Gamma''} : s(\gamma_i') \neq s(\gamma_j') \forall i \neq j\} \setminus V_{\Gamma''}) \cup \{x_i : x_i \in V \setminus V_{\Gamma''} \} \cup \{\Gamma'' : \Gamma'' \setminus V_{\Gamma'''} \} : V_{\Gamma''} \not\subset V, V_{\Gamma'''} \subset V \end{cases}$$

Loosely speaking, the action of a path-diffeomorphism is somehow localised on a fixed vertex set $V$.

For example note that for a subgraph $\Gamma' := \{\gamma \circ \gamma'\}$ of $\Gamma := \{\gamma, \gamma'\}$ and a subset $V := \{t(\gamma')\}$ of $V_{\Gamma'}$, it is true that

$$(h_\Gamma \circ R_{\sigma_{\Gamma'}(V)})(\gamma \circ \gamma') = (h_\Gamma \circ R_{\sigma_{\Gamma'}(V)})(\gamma) \cdot (h_\Gamma \circ R_{\sigma_{\Gamma'}(V)})(\gamma') = h_\Gamma(\gamma) \cdot (h_\Gamma \circ R_{\sigma_{\Gamma'}(V)})(\gamma') = h_\Gamma(\gamma \circ \gamma' \circ \sigma(t(\gamma'))$$

20
yields whenever \( \sigma_T \in \mathcal{B}(\mathcal{P}_T \Sigma) \). For a special bisection \( \tilde{\sigma} \) it is true that,

\[
(\mathfrak{h}_T \circ R_{\sigma_T})(\gamma) = \mathfrak{h}_T(\gamma \circ \gamma') = (\mathfrak{h}_T \circ R_{\sigma_T})(\gamma) \cdot (\mathfrak{h}_T \circ R_{\sigma_T})(\gamma')
\]

holds whenever \( \tilde{\sigma} \in \mathcal{B}(\mathcal{P}_T \Sigma) \), \( \tilde{\sigma}_T(t(\gamma')) = \mathbb{1}_{t(\gamma')} \) and \( \tilde{\sigma}_T(t(\gamma)) = \gamma' \). Let \( \tilde{\sigma} \) be the bisection in the finite path groupoid \( \mathcal{P}_T \Sigma \) that defines the bisection \( \sigma \in \mathcal{P}_T \mathcal{P}_T \). Then the last statement is true, since \( R_{\sigma_T}(\gamma') = \gamma' \circ \gamma'^{-1} \) requires \( \tilde{\sigma}_T : t(\gamma') \mapsto s(\gamma') \) and \( R_{\sigma_T}(\gamma) = \gamma \circ \gamma' \) needs \( \tilde{\sigma}_T : t(\gamma) \mapsto t(\gamma') \), where \( s(\gamma') = t(\gamma) \). Then \( R_{\sigma_T}(\gamma) \) and \( R_{\sigma_T}(t(\gamma''))(\gamma) \) coincide if \( \tilde{\sigma}_T(t(\gamma)) = \sigma_T(t(\gamma)) \) and \( \tilde{\sigma}_T(t(\gamma')) = \mathbb{1}_t(\gamma) \) holds.

**Proposition 3.2:** Let \( \Gamma' \) be a subgraph of the graph \( \Gamma \), \( \sigma_T \) be a bisection in \( \mathcal{P}_T \), \( \sigma_T : V_T \rightarrow \mathcal{P}_T \) be a restriction of \( \sigma_T \in \mathcal{B}(\mathcal{P}_T) \). Moreover let \( V' \) be a subset of \( V_T \), let \( \Gamma'' := \{ \gamma \circ \gamma' \} \) be a subgraph of \( \Gamma' \). Let \( \langle \gamma, \gamma' \rangle \in \mathcal{P}_T \Sigma \).

Then even for a suitable bisection \( \sigma \) in \( \mathcal{P}_T \) it follows that,

\[
R_{\sigma_T(v)}(\gamma \circ \gamma') \neq R_{\sigma_T(v)}(\gamma) \circ R_{\sigma_T(v)}(\gamma') \quad (17)
\]

yields. This is a general problem. In comparison with problem 2.1 the multiplication map \( \circ \) is not well-defined and hence

\[
R_{\sigma_T(v)}(\gamma) \circ R_{\sigma_T(v)}(\gamma')
\]

is not well-defined. Recognize that, \( R_{\sigma_T(v)} : \mathcal{P}_T \rightarrow \mathcal{P}_T \).

Consequently in general it is not true that,

\[
(h_T \circ R_{\sigma_T(v)})(\gamma \circ \gamma') = h_T(R_{\sigma_T(v)}(\gamma) \circ R_{\sigma_T(v)}(\gamma')) = (h_T \circ R_{\sigma_T(v)})(\gamma) \cdot (h_T \circ R_{\sigma_T(v)})(\gamma') \quad (18)
\]

yields.

With no doubt the left-translation \( L_{\sigma_T} \) and the inner automorphisms \( I_{\sigma_T} \) in a finite graph system \( \mathcal{P}_T \) for every \( \Gamma' \in \mathcal{P}_T \) are defined similarly.

**Definition 29.** Let \( \sigma_T \in \mathcal{B}(\mathcal{P}_T) \) be a bisection in the finite graph system \( \mathcal{P}_T \). Let \( R_{\sigma_T(v)} \) be a right-translation, where \( V \) is a subset of \( V_T \).

Then the pair \( (\Phi_T, \varphi_T) \) defined by \( \Phi_T = R_{\sigma_T(v)} \) (or, respectively, \( \Phi_T = L_{\sigma_T(v)} \), or \( \Phi_T = I_{\sigma_T(v)} \)) for a subset \( V \subseteq V_T \) and \( \varphi_T = t \circ \sigma_T \) is called a **graph-diffeomorphism of a finite graph system**. Denote the set of finite graph-diffeomorphisms by \( \text{Diff}(\mathcal{P}_V) \).

Let \( \Gamma' \) be a subgraph of \( \Gamma \) and \( \sigma_{T'} \) be a restriction of bisection \( \sigma_T \) in \( \mathcal{P}_T \). Then for example another graph-diffeomorphism \( (\Phi_{T'}, \varphi_{T'}) \) in \( \text{Diff}(\mathcal{P}_T) \) is defined by \( \Phi_{T'} = R_{\sigma_{T'}(V)} \) for a subset \( V \subseteq V_T \) and \( \varphi_{T'} = t \circ \sigma_{T'} \).

Remembering that the set of bisections of a finite path groupoid forms a group (refer 23) one may ask if the bisections of a finite graph system form a group, too.

**Proposition 30.** The set of bisections \( \mathcal{B}(\mathcal{P}_T) \) in a finite graph system \( \mathcal{P}_T \) forms a group.

**Proof:** Let \( \Gamma \) be a graph and let \( V_T \) be equivalent to the set \( \{ v_1, ..., v_{2N} \} \).

First two different multiplication operations are studied. The studies are comparable with the results of the definition of a right-translation in a finite graph system. The easiest multiplication operation is given by \( \ast_1 \), which is defined by

\[
(\sigma \ast \sigma')(V_T) := \{(\tilde{\sigma} \circ \tilde{\sigma}')(v_1),..., (\tilde{\sigma} \circ \tilde{\sigma}')(v_{2N}) : v_i \in V_T\}
\]

where \( \circ \) denotes the multiplication of bisections on the finite path groupoid \( \mathcal{P}_T \Sigma \Rightarrow V_T \). Notice that, this operation is not well-defined in general. In comparison with the definition of the right-translation in a finite graph system one has to take care. First the set of vertices doesn’t contain any vertices twice, the map \( \sigma \) in the finite path system is bijective, the mapping \( \sigma \) maps each set to a set of vertices containing no vertices twice and the situation in problem 2.14 has to be avoided.

Fix a bisection \( \tilde{\sigma} \) in a finite path groupoid \( \mathcal{P}_T \Sigma \Rightarrow V_T \). Let \( V_{\sigma_T} \) be a subset of \( V_T \) where \( \Gamma := \{ \gamma_1, ..., \gamma_N \} \) and for each \( v_i \) in \( V_{\sigma_T} \) it is true that \( v_i \neq v_j \) and \( v_i \neq (t \circ \tilde{\sigma})(v_j) \) for all \( i \neq j \). Define the set \( V_{\sigma, \sigma'} \) to be equal
to a subset of the set of all vertices \( \{v_k \in V_{\sigma'} : 1 \leq k \leq 2N \} \) such that each pair \((v_i, v_j)\) of vertices in \( V_{\sigma, \sigma'} \) satisfies \((t \circ (\hat{\sigma} \ast \hat{\sigma}'))(v_i) \neq (t \circ \hat{\sigma}')(v_j) \) and \((t \circ \hat{\sigma}')(v_i) \neq (t \circ \hat{\sigma}')(v_j)\) for all \( i \neq j \). Define 

\[
W_{\sigma, \sigma'} := \{ w_i \in V_{\sigma'} \setminus V_{\sigma, \sigma'} : (t \circ \hat{\sigma})(w_j) \neq (t \circ \hat{\sigma}')(w_i) \quad \forall i \neq j, \quad 1 \leq i, j \leq l \}
\]

The set \( W_{\sigma, \sigma'} \) is a subset of all vertices \( \{v_k \in V_{\sigma, \sigma'} : 1 \leq k \leq 2N \} \) such that each pair \((v_i, v_j)\) of vertices in \( V_{\sigma, \sigma'} \) satisfies \((t \circ (\hat{\sigma} \ast \hat{\sigma}'))(v_i) \neq (t \circ (\hat{\sigma} \ast \hat{\sigma}'))(v_j) \) and \((t \circ (\hat{\sigma} \ast \hat{\sigma}'))(v_i) \neq (t \circ (\hat{\sigma} \ast \hat{\sigma}'))(v_j)\) for all \( i \neq j \).

Consequently define a second multiplication on \( \mathcal{B}(P_T) \) similarly to the operation \( *_1 \). This is done by the following definition. Set

\[
(\sigma \ast_2 \sigma')(V_{\sigma'}) := \{ (\hat{\sigma} \ast \hat{\sigma}')(v_1), ..., (\hat{\sigma} \ast \hat{\sigma}')(v_k) : v_1, ..., v_k \in V_{\sigma, \sigma'}, 1 \leq k \leq 2N \}
\]

\[
\cup \{ \hat{\sigma}(w_1), \hat{\sigma}(w_1), ..., \hat{\sigma}(w_1), \hat{\sigma}'(w_1) : w_1, ..., w_l \in W_{\sigma, \sigma'}, 1 \leq l \leq 2N \}
\]

\[
\cup \{ \mathbb{I}_{p_1}, ..., \mathbb{I}_{p_n} : p_1, ..., p_n \in V_{\sigma'} \setminus \{ V_{\sigma, \sigma'} \cup W_{\sigma, \sigma'} \}, 1 \leq n \leq 2N \}
\]

Hence the inverse is supposed to be \( \sigma^{-1}(V_T) = \sigma((t \circ \sigma^{-1})(V_T))^{-1} \) such that

\[
(\sigma \ast_2 \sigma^{-1})(V_{\sigma^{-1}}) = \{ (\hat{\sigma} \ast \hat{\sigma}^{-1})(v_1), ..., (\hat{\sigma} \ast \hat{\sigma}^{-1})(v_{2N}) : v_i \in V_{\sigma, \sigma^{-1}} \}
\]

\[
\cup \{ \hat{\sigma}(w_1), \hat{\sigma}(w_1), ..., \hat{\sigma}(w_1), \hat{\sigma}'(w_1) : w_1, ..., w_l \in \mathcal{W}_{\sigma, \sigma^{-1}}, 1 \leq l \leq 2N \}
\]

\[
\cup \{ \mathbb{I}_{p_1}, ..., \mathbb{I}_{p_n} : p_1, ..., p_n \in V_{\sigma'} \setminus \{ V_{\sigma, \sigma^{-1}} \cup W_{\sigma, \sigma^{-1}} \}, 1 \leq n \leq 2N \}
\]

Notice that, the problem \( 22 \) is solved by a multiplication operation \( \circ_2 \), which is defined similarly to \( *_2 \). Hence the equality of \( 17 \) is available and consequently \( 18 \) is true. Furthermore a similar remark to \( 15 \) can be done.

**Example 2.5:** Now consider the following example. Set \( \Gamma' := \{ \gamma_1, \gamma_3 \} \), let \( \Gamma := \{ \gamma_1, \gamma_2, \gamma_3 \} \) and \( V_{\Gamma} := \{ (\gamma_1, t)(\gamma_1), s(\gamma_2), t(\gamma_2), s(\gamma_3), t(\gamma_3) : s(\gamma_i) \neq s(\gamma_j), t(\gamma_i) \neq t(\gamma_j) \forall i \neq j \} \). Set \( V \) be equal to \( \{ (\gamma_1, \gamma_3), s(\gamma_2), s(\gamma_3) \} \). Take two maps \( \sigma \) and \( \sigma' \) such that \( \sigma'(V) = \{ \gamma_1, \gamma_3 \} \), \( \sigma(V) = \{ \gamma_2 \} \), where \( t \circ \sigma(\gamma_3) = t(\gamma_3) \), \( \sigma'(s(\gamma_3)) = \gamma_3 \), \( \sigma'(s(\gamma_1)) = \gamma_1 \) and \( \sigma(t(\gamma_3)) = \gamma_2 \). Then \( s(\gamma_3) \in V_{\sigma, \sigma'} \), \( t(\gamma_3) \in V_{\sigma', \sigma'} \), and \( s(\gamma_1) \in W_{\sigma, \sigma'} \). Derive

\[
(\sigma \ast_1 \sigma')(V) = \{ \gamma_3 \circ \gamma_2, \gamma_1 \}
\]

Then conclude that,

\[
(\sigma \ast_2 \sigma')(V_T) = \{ \gamma_3 \circ \gamma_2, \gamma_1 \}
\]

holds. Notice that

\[
(\sigma \ast_2 \sigma')(V) \neq (\sigma' \ast_2 \sigma)(V) = \{ \gamma_2, \gamma_1, \gamma_3 \}
\]

is true. Finally obtain

\[
(\sigma \ast_2 \sigma^{-1})(V_T) = \{ \gamma_3 \circ \gamma_3^{-1}, \gamma_1 \circ \gamma_1^{-1} \} = \{ \mathbb{I}_s(\gamma_3), \mathbb{I}_s(\gamma_1) \}
\]

Let \( \sigma'(V_T) = \{ \gamma_1, \gamma_3 \} \) and \( \delta(V_T) = \{ \gamma_2, \gamma_4 \} \). Then notice that,

\[
(\hat{\sigma} \ast_1 \sigma')(V_T) = \{ \gamma_3 \circ \gamma_2, \gamma_1 \}
\]

and

\[
(\hat{\sigma} \ast_2 \sigma')(V_T) = \{ \gamma_3 \circ \gamma_2, \gamma_1, \gamma_4 \}
\]

yields.

Furthermore assume supplementary that \( t(\gamma_3) = t(\gamma_1) \) holds. Then calculate the product of the maps \( \sigma \) and \( \sigma' \):

\[
(\sigma \ast_2 \sigma')(V) = \{ \gamma_3 \circ \gamma_2, \gamma_1 \} \notin \mathcal{P}_T
\]

and

\[
(\sigma \ast_2 \sigma')(V_T) = \{ \mathbb{I}_t(\gamma_1), \mathbb{I}_t(\gamma_3) \} \in \mathcal{P}_T
\]
The group structure of \( \mathcal{B}(\mathcal{P}_r) \) transfers to \( G \). Let \( \tilde{\sigma} \) be a bisection in the finite path groupoid \( \mathcal{P}_r \Sigma \xrightarrow{\tilde{\sigma}} V \), which defines a bisection \( \sigma \) in \( \mathcal{P}_r \) and let \( \tilde{\sigma}' \) be a bisection in \( \mathcal{P}_r \Sigma \xrightarrow{\tilde{\sigma}'} V \), which defines another bisection \( \sigma' \) in \( \mathcal{P}_r \). Let \( V_{\sigma,\sigma'} \) be equal to \( V_r \), then derive

\[
\begin{align*}
\mathfrak{h}_r (\sigma * \sigma') & = \{ \mathfrak{h}_r ((\tilde{\sigma} * \tilde{\sigma}')(v_1)), \ldots, \mathfrak{h}_r ((\tilde{\sigma} * \tilde{\sigma}')(v_2)) \} \\
& = \{ \mathfrak{h}_r(\tilde{\sigma}' (v_1) \circ \tilde{\sigma} (t(\tilde{\sigma}')(v_1))), \ldots, \mathfrak{h}_r(\tilde{\sigma}' (v_2) \circ \tilde{\sigma} (t(\tilde{\sigma}')(v_2))) \}
\end{align*}
\]

Consequently the right-translation in the finite product \( G^{[r]} \) is definable.

**Definition 31.** Let \( \sigma_{\Gamma'} \) be in \( \mathcal{B}(\mathcal{P}_r) \), \( \Gamma' \) a subgraph of \( \Gamma \), \( \Gamma'' \) a subgraph of \( \Gamma' \) and \( R_{\sigma_{\Gamma'}} \) a right-translation, \( L_{\sigma_{\Gamma'}} \), a left-translation and \( I_{\sigma_{\Gamma'}} \), an inner-translation in \( \mathcal{P}_r \).

Then the **right-translation in the finite product** \( G^{[r]} \) is given by

\[
\mathfrak{h}_r \circ R_{\sigma_{\Gamma'}} : \mathcal{P}_r \to G^{[r]}, \quad \Gamma'' \mapsto (\mathfrak{h}_r \circ R_{\sigma_{\Gamma'}})(\Gamma'')
\]

Furthermore define the **left-translation in the finite product** \( G^{[r]} \) by

\[
\mathfrak{h}_r \circ L_{\sigma_{\Gamma'}} : \mathcal{P}_r \to G^{[r]}, \quad \Gamma'' \mapsto (\mathfrak{h}_r \circ L_{\sigma_{\Gamma'}})(\Gamma'')
\]

and the **inner-translation in the finite product** \( G^{[r]} \)

\[
\mathfrak{h}_r \circ I_{\sigma_{\Gamma'}} : \mathcal{P}_r \to G^{[r]}, \quad \Gamma'' \mapsto (\mathfrak{h}_r \circ I_{\sigma_{\Gamma'}})(\Gamma'')
\]

such that \( I_{\sigma_{\Gamma'}} = L_{\sigma_{\Gamma'}} \circ R_{\sigma_{\Gamma'}} \).

**Lemma 32.** It is true that \( R_{\sigma_{\Gamma''} \circ \sigma_{\Gamma'}} = R_{\sigma_{\Gamma'}} \circ R_{\sigma_{\Gamma''}} \), \( L_{\sigma_{\Gamma''} \circ \sigma_{\Gamma'}} = L_{\sigma_{\Gamma'}} \circ L_{\sigma_{\Gamma''}} \) and \( I_{\sigma_{\Gamma''} \circ \sigma_{\Gamma'}} = I_{\sigma_{\Gamma'}} \circ I_{\sigma_{\Gamma''}} \) for all bisections \( \sigma_{\Gamma'} \) and \( \sigma_{\Gamma''} \) in \( \mathcal{B}(\mathcal{P}_r) \).

There is an action of \( \mathcal{B}(\mathcal{P}_r) \) on \( G^{[r]} \) by

\[
(\zeta_{\sigma_{\Gamma'}} \circ \mathfrak{h}_r)(\Gamma'') \equiv (\mathfrak{h}_r \circ R_{\sigma_{\Gamma'}})(\Gamma'')
\]

whenever \( \sigma_{\Gamma'} \in \mathcal{B}(\mathcal{P}_r) \), \( \Gamma'' \in \mathcal{P}_r \), and \( \Gamma' \in \mathcal{P}_r \). Then for another \( \tilde{\sigma} \in \mathcal{B}(\mathcal{P}_r) \) it is true that,

\[
((\zeta_{\sigma_{\Gamma'}} \circ \zeta_{\sigma_{\Gamma'}}) \circ \mathfrak{h}_r)(\Gamma'') = (\mathfrak{h}_r \circ R_{\sigma_{\Gamma''} \circ \sigma_{\Gamma'}})(\Gamma'') = (\zeta_{\sigma_{\Gamma''} \circ \sigma_{\Gamma'}} \circ \mathfrak{h}_r)(\Gamma'')
\]

yields.

Recall that, the map \( \tilde{\sigma} \mapsto t \circ \tilde{\sigma} \) is a group isomorphism between the group of bisections \( \mathcal{B}(\mathcal{P}_r \Sigma) \) and the group \( \text{Diff}(V_r) \) of finite diffeomorphisms in \( V_r \). Therefore if the graphs \( \Gamma' = \Gamma'' \) contain only the path \( \gamma \), then the action \( \zeta_{\sigma_{\Gamma'}} \circ \mathfrak{h}_r \) is equivalent to an action of the finite diffeomorphism group \( \text{Diff}(V_r) \). Loosely speaking, the graph-diffeomorphisms \( (R_{\sigma_{\Gamma'}})(t \circ \sigma_{\Gamma'}) \) on a subgraph \( \Gamma'' \) of \( \Gamma' \) transform graphs and respect the graph structure of \( \Gamma' \). The diffeomorphism \( t \circ \tilde{\sigma} \) in the finite path groupoid only implements the finite diffeomorphism in \( \Sigma \), but it doesn’t adopt any path groupoid or graph preserving structure. Summarising the bisections of a finite graph system respect the graph structure and implement the finite diffeomorphisms in \( \Sigma \). There is another reason why the group of bisections is more fundamental than the path- or graph-diffeomorphism group.

In section \( \S 2.6 \) the concept of \( C^* \)-dynamical systems is studied. It turns out that, there are three different \( C^* \)-dynamical systems, each is build from the analytic holonomy \( C^* \)-algebra and a point-norm continuous action of the group of bisections of a finite graph system. The actions are implemented by one of the three translations, i.e. the left-, right- or inner-translation in the finite product \( G^{[r]} \). Furthermore the actions are related to each other by an unitary 1-cocycle. Hence the automorphisms are exterior equivalent \( \S 3.3 \) Def.:2.66]. In this case the full information is contained in the \( C^* \)-dynamical systems constructed from the right-translation in a finite graph system. Moreover the path- (or graph-) diffeomorphisms define particular right-, left- or inner-translations associated to suitable bisections. This is why in the introduction and later in section \( \S 3 \), the path- (or graph-) diffeomorphism group are often indentified with the group of bisections in a finite path groupoid (or graph system).

Finally the left or right-translations in a finite path groupoid can be studied in the context of natural or non-standard identification of the configuration space. This new concept leads to two different notions of diffeomorphism-invariant states. The actions of path- and graph-diffeomorphism and the concepts of natural or non-standard identification of the configuration space was not used in the context of LQG before.
2.3 The group-valued quantum flux operators associated to surfaces and graphs

Let $G$ be the structure group of a principal fibre bundle $P(\Sigma, G, \pi)$. Then the quantum flux operators, which are associated to a fixed surface $S$, are $G$-valued operators. For the construction of the quantum flux operator $\rho_S(\gamma)$ different maps from a graph $\Gamma$ to a direct product $G \times G$ are considered. This is related to the fact that, one distinguishes between paths that are ingoing and paths that are outgoing with respect to the surface orientation of $S$. If there are no intersection points of the surface $S$ and the source or target vertex of a path $\gamma_i$ of a graph $\Gamma$, then the map maps the path $\gamma_i$ to zero in both entries. For different surfaces or for a fixed surface different maps refer to different quantum flux operators.

**Definition 33.** Let $\mathcal{S}$ be a finite set $\{S_i\}$ of surfaces in $\Sigma$, which is closed under a flip of orientation of the surfaces. Let $\Gamma$ be a graph such that each path in $\Gamma$ satisfies one of the following conditions

- the path intersects each surface in $\mathcal{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\mathcal{S}$,
- the path intersects each surface in $\mathcal{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\mathcal{S}$,
- the path intersects each surface in $\mathcal{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\mathcal{S}$,
- the path does not intersect any surface $S$ contained in $\mathcal{S}$.

Finally let $\mathcal{P}_\Gamma$ denotes the finite graph system associated to $\Gamma$.

Then define the intersection functions $\iota_L : \mathcal{S} \times \Gamma \to \{\pm 1, 0\}$ such that

$$
\iota_L(S, \gamma) := \begin{cases} 
1 & \text{for a path } \gamma \text{ lying above and outgoing w.r.t. } S \\
-1 & \text{for a path } \gamma \text{ lying below and outgoing w.r.t. } S \\
0 & \text{the path } \gamma \text{ is not outgoing w.r.t. } S
\end{cases}
$$

and the intersection functions $\iota_R : \mathcal{S} \times \Gamma \to \{\pm 1, 0\}$ such that

$$
\iota_R(S, \gamma) := \begin{cases} 
-1 & \text{for a path } \gamma' \text{ lying above and ingoing w.r.t. } S \\
1 & \text{for a path } \gamma' \text{ lying below and ingoing w.r.t. } S \\
0 & \text{the path } \gamma' \text{ is not ingoing w.r.t. } S
\end{cases}
$$

whenever $S \in \mathcal{S}$ and $\gamma \in \Gamma$.

Define a map $o_L : \mathcal{S} \to G$ such that

$$
o_L(S) = o_L(S^{-1})
$$

whenever $S \in \mathcal{S}$ and $S^{-1}$ is the surface $S$ with reversed orientation. Denote the set of such maps by $\delta_L$. Respectively the map $o_R : \mathcal{S} \to G$ such that

$$
o_R(S) = o_R(S^{-1})
$$

whenever $S \in \mathcal{S}$. Denote the set of such maps by $\delta_R$. Moreover there is a map $o_L \times o_R : \mathcal{S} \to G \times G$ such that

$$
(o_L, o_R)(S) = (o_L(S), o_R(S^{-1}))
$$

whenever $S \in \mathcal{S}$. Denote the set of such maps by $\delta$.

Then define the **group-valued quantum flux set for paths**

$$
\mathcal{G}_{\mathcal{S}, \Gamma} := \bigcup_{o_L \times o_R \in \delta} \bigcup_{S \in \mathcal{S}} \left\{ (\rho_L, \rho_R) \in \text{Map}(\Gamma, G \times G) : (\rho_L, \rho_R)(\gamma) = (o_L(S)^{\iota_L(S, \gamma)}, o_R(S)^{\iota_R(S, \gamma)}) \right\}
$$

where $\text{Map}(\Gamma, G \times G)$ denotes the set of all maps from the graph $\Gamma$ to the direct product $G \times G$. 

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Define the set of group-valued quantum fluxes for graphs

\[ G_{\tilde{S},\Gamma} := \bigcup_{o_L \times o_R \in \delta} \bigcup_{S \in \tilde{S}} \left\{ \rho_{S,\Gamma} \in \text{Map}(\mathcal{P}_\Gamma^S, G^{[\Gamma]} \times G^{[\Gamma]} ) : \rho_{S,\Gamma} := \rho_S \times \ldots \times \rho_S \right\} \]

where \( \rho_S(\gamma) := (o_L(S)^{\gamma,L}(S), o_R(S)^{\gamma,R}(S)) \),

\[ \rho_S \in G_{\tilde{S},\Gamma}, S \in \tilde{S}, \gamma \in \Gamma \}

Notice if \( H \) is a closed subgroup of \( G \), then \( H_{\tilde{S},\Gamma} \) can be defined in analogy to \( G_{\tilde{S},\Gamma} \). In particular if the group \( H \) is replaced by the center \( Z(G) \) of the group \( G \), then the set \( G_{\tilde{S},\Gamma} \) is replaced by \( Z(G_{\tilde{S},\Gamma}) \) and \( G_{\tilde{S},\Gamma} \) is changed to \( Z_{\tilde{S},\Gamma} \).

Furthermore observe that, \((\iota_L \times \iota_R)(S^{-1}, \gamma) = (-\iota_L \times -\iota_R)(S, \gamma)\) for every \( \gamma \in \Gamma \) holds. Remark that, the condition \( \rho^L(\gamma) = \rho^R(\gamma^{-1}) \) is not required.

**Example 2.6:** For example the following example can be analysed. Consider a graph \( \Gamma \) and two disjoint surface sets \( \tilde{S} \) and \( \tilde{T} \).

Then the elements of \( G_{\tilde{S},\Gamma} \) are for example the maps \( \rho_i^L \times \rho_i^R \) for \( i = 1, 2 \) such that

\[ \rho_1(\gamma) := (\rho_1^L, \rho_1^R)(\gamma) = (\sigma_L(S_1)^{\gamma,L}(S_1, \gamma), \sigma_R(S_1)^{\gamma,R}(S_1, \gamma)) = (g_1, 0) \]

\[ \rho_2(\gamma) := (\rho_2^L, \rho_2^R)(\gamma) = (\sigma_L(S_2)^{\gamma,L}(S_2, \gamma), \sigma_R(S_2)^{\gamma,R}(S_2, \gamma)) = (g_2, 0) \]

This example shows that, the surfaces \{\( S_1, S_2 \)\} are similar, whereas the surfaces \{\( T_1, T_2 \)\} produce different signatures for different paths. Moreover the set of surfaces are chosen such that one component of the direct sum is always zero.

For a particular surface set \( \tilde{S} \), the following set is defined

\[ \bigcup_{\sigma_L \times \sigma_R \in \delta} \bigcup_{S \in \tilde{S}} \left\{ (\rho^L, \rho^R) \in \text{Map}(\Gamma, G \times G) : (\rho^L, \rho^R)(\gamma) := (\sigma_L(S)^{\gamma,L}(S, \gamma), 0) \right\} \]

can be identified with

\[ \bigcup_{\sigma_L \in \delta} \bigcup_{S \in \tilde{S}} \left\{ \rho \in \text{Map}(\Gamma, G) : \rho(\gamma) := \sigma_L(S)^{\gamma,L}(S, \gamma) \right\} \]
The same is observed for another surface set $\mathcal{T}$ and the set $\mathbb{G}_{\mathcal{T},\Gamma}$ is identifiable with
\[
\bigcup_{R \in \mathcal{T}} \bigcup_{\sigma \in \sigma_{R}} \{ \rho \in \text{Map}(\Gamma, G) : \rho(\gamma) := \sigma_{R}(T_{R}(\gamma)) \}
\]

The intersection behaviour of paths and surfaces plays a fundamental role in the definition of the flux operator. There are exceptional configurations of surfaces and paths in a graph. One of them is the following.

**Definition 34.** A surface $S$ has the **surface intersection property for a graph** $\Gamma$, if the surface intersects each path of $\Gamma$ once in the source or target vertex of the path and there are no other intersection points of $S$ and the path.

This is for example the case for the surface $S_{1}$ or the surface $S_{3}$, which are presented in example 2.6. Notice that in general, for the surface $S$ there are $N$ intersection points with $N$ paths of the graph. In the example the evaluated map $\rho_{1}(\gamma) = (g_{1}, 0) = \rho_{1}(\tilde{\gamma})$ for $\gamma, \tilde{\gamma} \in \Gamma$ if the surface $S_{1}$ is considered.

The property of a path lying above or below is not important for the definition of the surface intersection property for a surface. This indicates that the surface $S_{4}$ in the example 2.6 has the surface intersection property, too.

Let a surface $S$ does not have the surface intersection property for a graph $\Gamma$, which contains only one path $\gamma$. Then for example the path $\gamma$ intersects the surface $S$ in the source and target vertices such that the path lies above the surface $S$. Then the map $\rho_{\mathcal{T}} \times \rho_{\mathcal{R}}$ is evaluated for the path $\gamma$ by
\[
(\rho_{\mathcal{T}} \times \rho_{\mathcal{R}})(\gamma) = (g, h^{-1})
\]
Hence simply speaking the surface intersection property reduces the components of the map $\rho_{\mathcal{T}} \times \rho_{\mathcal{R}}$, but for different paths to different components.

Now, consider a bunch of sets of surfaces such that for each surface there is only one intersection point.

**Definition 35.** A set $\mathcal{S}$ of $N$ surfaces has the **surface intersection property for a graph** $\Gamma$ with $N$ independent edges, if it contains only surfaces, for which each path $\gamma_{i}$ of a graph $\Gamma$ intersects each surface $S_{i}$ only once in the same source or target vertex of the path $\gamma_{i}$, there are no other intersection points of each path $\gamma_{i}$ and each surface in $\mathcal{S}$ and there is no other path $\gamma_{j}$ that intersects the surface $S_{i}$ for $i \neq j$ where $1 \leq i, j \leq N$.

Then for example consider the following configuration.

**Example 2.7:**

The sets $\{S_{6}, S_{7}\}$ or $\{S_{5}, S_{8}\}$ have the surface intersection property for the graph $\Gamma$. The images of a map $E$ is
\[
\rho_{5}(\tilde{\gamma}) = (g_{5}, 0), \quad \rho_{8}(\gamma) = (0, h_{8})
\]
Note that simply speaking the property indicates that each map reduces to a component of $\rho^L \times \rho^R$.

A set of surfaces that has the surface intersection property for a graph is further specialised by restricting the choice to paths lying ingoing and below with respect to the surface orientations.

**Definition 36.** A set $S$ of $N$ surfaces has the **simple surface intersection property for a graph** $\Gamma$ with $N$ independent edges, if it contains only surfaces, for which each path $\gamma_i$ of a graph $\Gamma$ intersects only one surface $S_i$ only once in the target vertex of the path $\gamma_i$, the path $\gamma_i$ lies above and there are no other intersection points of each path $\gamma_i$ and each surface in $\hat{S}$.

**Example 2.8:** Consider the following example.

The sets $\{S_9, S_{11}\}$ or $\{S_{10}, S_{12}\}$ have the simple surface intersection property for the graph $\Gamma$. Calculate $\rho_9(\hat{\gamma}) = (0, h_9^{-1})$, $\rho_{11}(\gamma) = (0, h_{11}^{-1})$.

In this case the set $G_{S,\Gamma}$ reduces to

$$\bigcup_{\sigma \in \sigma_n} \bigcup_{S \in \hat{S}} \{\rho \in \text{Map}(\Gamma, g) : \rho(\gamma) := \sigma_R(S)^{-1} \text{ for } \gamma \cap S = t(\gamma)\}$$

Notice that, the set $\Gamma \cap \hat{S} = \{t(\gamma_i)\}$ for a surface $S_i \in \hat{S}$ and $\gamma_i \cap S_j \cap S_i = \{\emptyset\}$ for a path $\gamma_i$ in $\Gamma$ and $i \neq j$.

On the other hand, there exists a set of surfaces such that each path of a graph intersects all surfaces of the set in the same vertex. This contradicts the assumption that each path of a graph intersects only one surface once.

**Definition 37.** Let $\Gamma$ be a graph that contains no loops.

A set $\hat{S}$ of surfaces has the **same surface intersection property for a graph** $\Gamma$ iff each path $\gamma_i$ in $\Gamma$ intersects with all surfaces of $\hat{S}$ in the same source vertex $v_i \in V_{\Gamma}$ ($i = 1, ..., N$), all paths are outgoing and lie below each surface $S \in \hat{S}$ and there are no other intersection points of each path $\gamma_i$ and each surface in $\hat{S}$.

A surface set $\hat{S}$ has the **same right surface intersection property for a graph** $\Gamma$ iff each path $\gamma_i$ in $\Gamma$ intersects with all surfaces of $\hat{S}$ in the same target vertex $v_i \in V_{\Gamma}$ ($i = 1, ..., N$), all paths are ingoing and lie above each surface $S \in \hat{S}$ and there are no other intersection points of each path $\gamma_i$ and each surface in $\hat{S}$.

Recall the example 2.8. Then the set $\{S_1, S_2\}$ has the same surface intersection property for the graph $\Gamma$.

Then the set $G_{S,\Gamma}$ reduces to

$$\bigcup_{\sigma \in \sigma_n} \bigcup_{S \in \hat{S}} \{\rho \in \text{Map}(\Gamma, g) : \rho(\gamma) := \sigma_L(S)^{-1} \text{ for } \gamma \cap S = s(\gamma)\}$$

Notice that, $\gamma \cap S_1 \cap ... \cap S_N = s(\gamma)$ for a path $\gamma$ in $\Gamma$ whereas $\Gamma \cap \hat{S} = \{s(\gamma_i)\}_{1 \leq i \leq N}$. Clearly $\Gamma \cap S_i = s(\gamma_i)$ for a surface $S_i$ in $\hat{S}$ holds. Simply speaking the physical intuition behind that is given by fluxes associated to different surfaces that should act on the same path.

A very special configuration is the following.
Definition 38. A set $\tilde{S}$ of surfaces has the same surface intersection property for a graph $\Gamma$ containing only loops iff each loop $\gamma_i$ in $\Gamma$ intersects with all surfaces of $\tilde{S}$ in the same vertices $s(\gamma_i) = t(\gamma_i)$ in $V_\Gamma$ ($i = 1, \ldots, N$), all loops lie below each surface $S \in \tilde{S}$ and there are no other intersection points of each loop in $\Gamma$ and each surface in $\tilde{S}$.

Notice that, both properties can be restated for other surface and path configurations. Hence a surface set have the simple or same surface intersection property for paths that are outgoing and lie above (or ingoing and below, or outgoing and below). The important fact is related to the question if the intersection vertices are the same for all surfaces or not.

Finally for the definition of the quantum flux operators notice the following objects.

Definition 39. The set of all images of maps in $G_{\tilde{S}, \Gamma}$ for a fixed surface set $\tilde{S}$ and a fixed path $\gamma$ in $\Gamma$ is denoted by $G_{\tilde{S}, \gamma}$.

The set of all finite products of images of maps in $G_{\tilde{S}, \Gamma}$ for a fixed surface set $\tilde{S}$ and a fixed graph $\Gamma$ is denoted by $G_{\tilde{S}, \Gamma}$.

The product · on $G_{\tilde{S}, \Gamma}$ is given by

$$
\rho_{S_1, \Gamma}(\gamma) \cdot \rho_{S_2, \Gamma}(\gamma) = (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), \ldots, \rho_{S_1}(\gamma_N) \cdot \rho_{S_2}(\gamma_N))
$$

$$
= (o_L(S_1)^{-1}o_L(S_2)^{-1}, \ldots, o_L(S_2)^{-1}o_L(S_1)^{-1})
$$

$$
= ((o_L(S_2)o_L(S_1))^{-1}, \ldots, (o_L(S_2)o_L(S_1))^{-1})
$$

$$
= \rho_{S_3, \Gamma}(\Gamma)
$$

Definition 40. Let $S$ be a surface and $\Gamma$ be a graph such that the only intersections of the graph and the surface in $S$ are contained in the vertex set $V_\Gamma$. Moreover let $P_{\Gamma \Sigma} 
\Rightarrow V_\Gamma$ be a finite path groupoid associated to $\Gamma$.

Then define the set for a fixed surface $S$ by

$$
\text{Maps}_S(P_{\Gamma \Sigma}, G \times G) := \bigcup_{o_L \times o_R \in \tilde{S}} \bigcup_{S \in \tilde{S}} \{ (\rho^L, \rho^R) \in \text{Map}(P_{\Gamma \Sigma}, G \times G) : (\rho^L, \rho^R)(\gamma) := (o_L(S)^{e_L(S, \gamma)}, o_R(S)^{e_R(S, \gamma)}) \}
$$

Then the quantum flux operators are elements of the following group.

Proposition 41. Let $\tilde{S}$ a set of surfaces and $\Gamma$ be a fixed graph, which contains no loops, such that the set $\tilde{S}$ has the same surface intersection property for the graph $\Gamma$.

The set $G_{\tilde{S}, \gamma}$ has the structure of a group.

The group $G_{\tilde{S}, \gamma}$ is called the flux group associated a path and a finite set of surfaces.

Proof: This follows easily from the observation that in this case $G_{\tilde{S}, \gamma}$ reduces to

$$
\bigcup_{o_L \in \tilde{S}} \bigcup_{S \in \tilde{S}} \{ \rho^L \in \text{Map}(\Gamma, G) : \rho^L(\gamma) := o_L(S)^{-1} \text{ for } \gamma \cap S = s(\gamma) \}
$$

There always exists a map $\rho^L_{\tilde{S}, \beta} \in G_{\tilde{S}, \gamma}$ such that the following equation defines a multiplication operation

$$
\rho^L_{\tilde{S}, 1}(\gamma) \cdot \rho^L_{\tilde{S}, 2}(\gamma) = g_1g_2 := \rho^L_{\tilde{S}, 3}(\gamma) \in G_{\tilde{S}, \gamma}
$$

with inverse $(\rho^L_{\tilde{S}, \gamma})^{-1}$ such that

$$
\rho^L_{\tilde{S}, \gamma}(\gamma) \cdot (\rho^L_{\tilde{S}, \gamma})^{-1} = (\rho^L_{\tilde{S}, \gamma})^{-1} \cdot \rho^L_{\tilde{S}, \gamma}(\gamma) = e_G \quad \forall \gamma \in \Gamma
$$
Notice that for a loop \( \alpha \) an element \( \rho_S(\alpha) \in G_{S,\gamma} \) is defined by
\[
\rho_S(\alpha) := (\rho_S^1 \times \rho_S^2)(\alpha) = (g, h) \in G^2
\]
In the case of a path \( \gamma' \) that intersects a surface \( S \) in the source and target vertex there is also an element \( \rho_S(\gamma') \in G_{S,\gamma} \) defined by
\[
\rho_S(\gamma') := (\rho_S^1 \times \rho_S^2)(\gamma') = (g, h) \in G^2
\]

**Proposition 42.** Let \( \mathcal{S} \) be a set of surfaces and \( \Gamma \) be a fixed graph, which contains no loops, such that the set \( S \) has the same surface intersection property for the graph \( \Gamma \). Let \( \mathcal{P}_\Gamma^o \) be a finite orientation preserved graph system such that the set \( \mathcal{S} \).

The set \( G_{\mathcal{S},\Gamma} \) has the structure of a group.

The set \( G_{\mathcal{S},\Gamma} \) is called the **flux group associated a graph and a finite set of surfaces**.

**Proof:** This follows from the observation that the set \( G_{\mathcal{S},\Gamma} \) is identified with
\[
\bigcup \bigcup \left\{ \rho_{S,\Gamma} \in \text{Map}(\mathcal{P}_\Gamma^o, G^{(|\mathcal{E}_\Gamma|)}) : \rho_{S,\Gamma} := \rho_S \times \ldots \times \rho_S \\
\text{where } \rho_S(\gamma) := o_L(S)^{-1}, \rho_S \in G_{\mathcal{S},\Gamma}, S \in \mathcal{S}, \gamma \in \Gamma \right\}
\]
Let \( \mathcal{S} \) be a surface set having the same intersection property for a fixed graph \( \Gamma := \{\gamma_1, \ldots, \gamma_N\} \). Then for two surfaces \( S_1, S_2 \) contained in \( \mathcal{S} \) define
\[
\rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma) = (\rho_{S_1}(\gamma_1) \cdot \rho_{S_2}(\gamma_1), \ldots, \rho_{S_1}(\gamma_N) \cdot \rho_{S_2}(\gamma_N))
\]
\[
= (g_{S_1}, \ldots, g_{S_1}) \cdot (g_{S_2}, \ldots, g_{S_2}) = (g_{S_1}g_{S_2}, \ldots, g_{S_1}g_{S_2})
\]
where \( \Gamma = \{\gamma_1, \ldots, \gamma_N\} \). Note that, since the maps \( o_L \) are arbitrary maps from \( \mathcal{S} \) to \( G \), it is assumed that the maps satisfy \( o_L(S_1) := g_{S_1}^{-1} \in G \) for \( i = 1, 2 \). Clearly this is related to in this particular case of the graph \( \Gamma \) and can be generalised.

The inverse operation is given by
\[
(\rho_{S,\Gamma}(\Gamma))^{-1} = ((\rho_S(\gamma_1))^{-1}, \ldots, (\rho_S(\gamma_N))^{-1})
\]
where \( N = |\Gamma| \) and \( \rho_S \in G_{S,\gamma} \) for \( S \in \mathcal{S} \). Since it is true that
\[
\rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma)^{-1} = (g_{S_1}, \ldots, g_{S_1}) \cdot (g_{S_1}^{-1}, \ldots, g_{S_1}^{-1})
\]
\[
= (\rho_S(\gamma_1) \cdot \rho_S(\gamma_1)^{-1}, \ldots, \rho_S(\gamma_N) \cdot \rho_S(\gamma_N)^{-1})
\]
\[
= (g_{S_1}g_{S_1}^{-1}, \ldots, g_{S_1}g_{S_1}^{-1}) = (e_G, \ldots, e_G)
\]
yields.

\[
\rho_{S_1,\Gamma}(\Gamma) \cdot \rho_{S_2,\Gamma}(\Gamma) = (o_L(S_1)^{-1}o_L(S_1)^{-1}, \ldots, o_L(S_2)^{-1}o_L(S_2)^{-1}) = ((o_L(S_1)o_L(S_2))^{-1}, \ldots, (o_L(S_1)o_L(S_2))^{-1})
\]
is true. Moreover observe that, if all subgraphs of a finite orientation preserved graph system are **naturally identified**, then \( G_{\mathcal{S},\Gamma} \) is a subgroup of \( G_{\mathcal{S},\Gamma} \) for all subgraphs \( \Gamma' \) in \( \mathcal{P}_\Gamma \). If \( G \) is assumed to be a compact Lie group, then the flux group \( G_{\mathcal{S},\Gamma} \) is called the Lie flux group.

There is another group, if another surface set is considered.
Proposition 43. Let $\tilde{T}$ be a set of surfaces and $\Gamma$ be a fixed graph such that the set $\tilde{T}$ has the simple surface intersection property for the graph $\Gamma$. Let $\mathcal{P}_T^\widetilde{\Gamma}$ be a finite orientation preserved graph system.

The set $\mathcal{G}_{\tilde{T}, \Gamma}$ has the structure of a group.

The same arguments using the identification of $\mathcal{G}_{\tilde{T}, \Gamma}$ with

$$\bigcup_{\sigma_n \in \sigma_n} \left\{ \rho_{\tilde{T}, \Gamma} \in \text{Map}(\mathcal{P}_T^\widetilde{\Gamma}, G^{[\tilde{\Gamma}]}) : \rho_{\tilde{T}, \Gamma} := \rho_{\tilde{T}_1} \times ... \times \rho_{\tilde{T}_N} \right\}$$

where $\rho_{\tilde{T}_i} := o_R(T_i)^{-1}, \sigma_i \in \mathcal{G}_{\tilde{T}, \Gamma}, T_i \in \tilde{T}, \gamma \in \Gamma$.

which is given by

$$\rho_{\tilde{T}_1, \Gamma} \cdot ... \cdot \rho_{\tilde{T}_n, \Gamma} = (\rho_{\tilde{T}_1}(\gamma_1), e_G, e_G, ..., e_G) \cdot (e_G, \rho_{\tilde{T}_2}(\gamma_2), e_G, ..., e_G) \cdot ... \cdot (e_G, ..., e_G, \rho_{\tilde{T}_N}(\gamma_N), e_G)$$

$$= (\rho_{\tilde{T}_1}(\gamma_1), ..., \rho_{\tilde{T}_N}(\gamma_N)) = (g_1, ..., g_N) \in G^N$$

$$=: \rho_{\tilde{T}, \Gamma}(\Gamma)$$

Then the multiplication operation is presented by

$$\rho_{\tilde{T}_1, \Gamma}^1(\Gamma) \cdot \rho_{\tilde{T}_2, \Gamma}^2(\Gamma) = (\rho_{\tilde{T}_1}(\gamma_1)^1, \rho_{\tilde{T}_2}(\gamma_1)^2, ..., \rho_{\tilde{T}_N}(\gamma_N)^1, \rho_{\tilde{T}_N}(\gamma_N)^2)$$

$$= (g_{1,1}, ..., g_{1,N}) \cdot (g_{2,1}, ..., g_{2,N}) = (g_{1,1}g_{2,1}, ..., g_{1,N}g_{2,N}) \in G^N$$

where $\Gamma = \{\gamma_1, ..., \gamma_N\}$.

It is also possible that, the fluxes are located only in a vertex and do not depend on ingoing or outgoing, above or below orientation properties.

Definition 44. Let $\mathcal{P}_T$ be a finite graph groupoid associated to a graph $\Gamma$ and let $N$ be the number of edges of the graph $\Gamma$.

Define the set of maps

$$G_{\Gamma}^{\text{loc}} := \left\{ g_T \in \text{Map}(\mathcal{P}_T, G^{[\Gamma]}), g_T := g_{1}^1 \circ s \times ... \times g_{N}^N \circ s \right\}$$

$$g_{i}^j \in \text{Map}(\Gamma, G) \right\}$$

Then $G_{\Gamma}^{\text{loc}}$ is the set of all images of maps in $G_{\Gamma}^{\text{loc}}$ for all graphs in $\mathcal{P}_T$ and $G_{\Gamma}^{\text{loc}}$ is called the local flux group associated a finite graph system.

2.4 The group-valued quantum flux operators associated to surfaces and finite path groupoids

Recall the set of admissible maps $\text{Map}^A(\mathcal{P}_T \Sigma, G)$ presented in definition 13.

Definition 45. Let $\tilde{S}$ be a finite set of surfaces which is closed under a flip of orientation of the surfaces. Let $\mathcal{P}_T \Sigma \vDash V_T$ be a finite path groupoid associated to a graph $\Gamma$ such that each path in $\mathcal{P}_T \Sigma$ satisfies one of the following conditions

- the path intersects each surface in $\tilde{S}$ in the source vertex of the path and there are no other intersection points of the path and any surface contained in $\tilde{S}$,
- the path intersects each surface in $\tilde{S}$ in the target vertex of the path and there are no other intersection points of the path and any surface contained in $\tilde{S}$,
- the path intersects each surface in $\tilde{S}$ in the source and target vertex of the path and there are no other intersection points of the path and any surface contained in $\tilde{S}$,
- the path does not intersect any surface $S$ contained in $\tilde{S}$.
Finally, let $\mathcal{P}_\Gamma$ denote the finite graph system associated to $\Gamma$.

Then the set of admissible maps associated to a graph and surfaces $\bar{S}$ are defined by

$$G^A_{\bar{S},\Gamma} := \bigcup_{aL \times aR \in \partial S \in S} \{(g^L, g^R) \in \text{Map}^A(\mathcal{P}_\Gamma \Sigma, G \times G) : (g^L, g^R)(\gamma) := (o_L(S)^{\bar{S}}(S, \gamma), o_R(S)^{\bar{S}}(S, \gamma))\}$$

Define the set of admissible maps associated to a finite graph system and surfaces $\bar{S}$ is presented by

$$G^A_{\bar{S},\Gamma} := \bigcup_{aL \times aR \in \partial S \in S} \{g_{S,\Gamma} \in \text{Map}^A(\mathcal{P}_\Gamma, G^{[\Gamma]} \times G^{[\Gamma]}) : g_{S,\Gamma} := g_S \times \ldots \times g_S$$

where $g_{S}(\gamma_i) = (o_L(S)^{\bar{S}}(S, \gamma_i), o_R(S)^{\bar{S}}(S, \gamma_i))$

and $g_S \in G^A_{\bar{S},\Gamma}, S \in \bar{S}, \gamma_i \in \Gamma, \Gamma' \in \mathcal{P}_\Gamma\}$. 

Observe that, these maps have the following properties. For all elements of $\mathcal{P}_\Sigma \Sigma_0$ (or $\mathcal{P}_\Sigma \Sigma_0$) that intersect the surface $S$ only in their target (or source) vertex $v$ the maps $g^L_S$ (or $g^R_S$) in $G^A_{\bar{S},\Gamma}$ satisfies

$$g^L_S(\gamma) = g^L_S(\gamma \circ \gamma') = g^L_S(\gamma') = g_{S,L} \quad \forall \gamma, \gamma \circ \gamma', \gamma'' \in \mathcal{P}_\Sigma \Sigma_0$$

and $v = s(\gamma) = S \cap \gamma$

$$g^L_S(\gamma)^{-1} = g^L_S((\gamma \circ \gamma')^{-1}) = k_{S,L} \quad \forall \gamma^{-1}, (\gamma \circ \gamma')^{-1} \in \mathcal{P}_\Sigma \Sigma_0$$

Furthermore for paths $\gamma$ and $\gamma'$ that compose and intersect $S$ in the common vertex $t(\gamma) = s(\gamma')$ it is true that

$$(g^L_S(\gamma)^{-1})^{-1} g^L_S(\gamma') = e_G$$

and $$(g^R_S(\gamma)) g^L_S(\gamma)^{-1} = e_G$$

whenever $(\gamma, \gamma') \in \mathcal{P}_\Sigma \Sigma^{(2)}$ and for all maps $(g^L_S, g^R_S) \in G^A_{\bar{S},\Gamma}$. 

In both definitions of the sets $G^A_{\bar{S},\Gamma}$ or $G^A_{\bar{S},\Gamma}$ of maps, there is a mapping $\rho_S$ or, respectively, $\bar{g}_S$, which maps all paths in $\mathcal{P}_\Sigma \Sigma_0$ to one element $X_S$, i.e. $\rho_S(\gamma) = ps(\gamma \circ \gamma')$ for all $\gamma, \gamma \circ \gamma' \in \mathcal{P}_\Sigma \Sigma_0$ where $v = s(\gamma)$. But the equalities (21) are required only for maps in $G^A_{\bar{S},\Gamma}$.

Notice that, if the group $G$ is replaced by the center $Z(G)$ of the group $G$, then the set $G^A_{\bar{S},\Gamma}$ is replaced by $Z(G_{\bar{S},\Sigma})^{\bar{S},\Gamma}$ and $G^A_{\bar{S},\Gamma}$ is changed to $Z^A_{\bar{S},\Gamma}$.

3 The analytic holonomy $C^*$-algebra and Weyl $C^*$-algebra

3.1 Dynamical systems of actions of the flux group on the analytic holonomy $C^*$-algebra

The analytic holonomy $C^*$-algebra for finite graph systems

In this article the analytic holonomy algebra $C(\bar{A}_\Gamma)$ restricted to a graph system $\mathcal{P}_\Gamma$ is given by the set $C(\bar{A}_\Gamma)$ of continuous functions on $\bar{A}_\Gamma$, pointwise multiplication, complex conjugation and the completion is taken with respect to the sup-norm. The analytic holonomy $C^*$-algebra $C(\bar{A})$ is given by the inductive limit of the family of unital commutative $C^*$-algebras $\{(C(\bar{A}_\Gamma), \beta_{\Gamma',\Gamma}) : \mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma}, i, j \in \mathbb{N}\}$ for an inductive family $\{\mathcal{P}_{\Gamma}\}$ of finite graph systems, where $\beta_{\Gamma',\Gamma}$ is a unit-preserving injective $^*$-homomorphism from the analytic holonomy algebra $C(\bar{A}_\Gamma)$ to $C(\bar{A}_{\Gamma'})$. The maps $\beta_{\Gamma',\Gamma}$ satisfy the consistency conditions

$$\beta_{\Gamma',\Gamma''} = \beta_{\Gamma',\Gamma} \circ \beta_{\Gamma,\Gamma''}$$

whenever $\mathcal{P}_{\Gamma} \leq \mathcal{P}_{\Gamma'} \leq \mathcal{P}_{\Gamma''}$. The configuration space $\bar{A}_\Gamma$ associated to a graph $\Gamma_i$ is derived from the set of all holonomy maps from the finite graph system $\mathcal{P}_{\Gamma_i}$ to the product group $G^{[\Gamma_i]}$. Recall the notion of natural or non-standard identification of the configuration space $\bar{A}_\Gamma$, which is presented in subsection 2.2.2. The elements of $\bar{A}_\Gamma$ are identified naturally or in a non-standard way with $G^{[\Gamma]}$ by the evaluation of the holonomy map for a subset of the finite graph system $\mathcal{P}_{\Gamma}$. Simply speaking the choice of the identification is
a matter of the labeling of the configuration variables. In this article the identifications are needed for the definition of graph changing automorphisms acting on the analytic holonomy $C^*$-algebra.

An element of the analytic holonomy $C^*$-algebra $C(\mathring{A})$ is of the form

$$ f = f_{\Gamma_i} \circ \pi_{\Gamma_i} = \beta_{\Gamma_i} \circ f_{\Gamma_i} $$

where $f \in C(\mathring{A})$, $\pi_{\Gamma_i} : \mathring{A} \to \mathring{A}_{\Gamma_i}$, $f_{\Gamma_i} \in C(G^{[\Gamma_i]})$ and the map $\beta_{\Gamma_i} : C(\mathring{A}_{\Gamma_i}) \to C(\mathring{A})$ is an unit-preserving injective $^*$-homomorphisms. Furthermore the maps $\beta_{\Gamma} : C(\mathring{A}_{\mathring{T}}) \to C(\mathring{A})$ are isometries, since

$$ \|f\| = \|\beta_{\Gamma_i}f_{\Gamma_i}\| = \sup |f_{\Gamma_i}| $$

yields whenever $f \in C(\mathring{A})$ and $f_{\Gamma_i} \in C(\mathring{A}_{\Gamma_i})$ for all graphs $\Gamma_i$.

The idea is to define actions of groups on the $C^*$-algebra $C(\mathring{A}_{\mathring{T}})$ of continuous functions on the compact Hausdorff space $\mathring{A}_{\mathring{T}}$ associated to a graph $\Gamma$, which can be extended to actions on the inductive limit algebra $C(\mathring{A})$.

**Group actions on the configuration space**

Let $\Gamma$ be a graph, $\mathring{P}_\mathring{T}$ be the associated finite graph system. Assume that the subgraphs in a finite graph system $\mathring{P}_\mathring{T}$ are identified naturally and hence the configuration space $\mathring{A}_{\mathring{T}}$ is identified in the natural way with $G^{[\Gamma]}$.

Then there is a group action

$$ G^N \times \mathring{A}_{\mathring{T}} \ni (g_1, ..., g_N, (b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N))) \mapsto (g_1 b_{\Gamma}(\gamma_1), ..., g_N b_{\Gamma}(\gamma_N)) \in \mathring{A}_{\mathring{T}} $$

of a finite product of a compact group $G$ on the compact Hausdorff space $\mathring{A}_{\mathring{T}}$ where $N := |\Gamma|$. For each $g := (g_1, ..., g_N) \in G^{[\Gamma]}$ the map $L(g)$ given by

$$ \mathring{A}_{\mathring{T}} \ni (b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) \mapsto L(g)(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) := (g_1 b_{\Gamma}(\gamma_1), ..., g_N b_{\Gamma}(\gamma_N)) \in \mathring{A}_{\mathring{T}} $$

is a homeomorphism $L(g) : \mathring{A}_{\mathring{T}} \to \mathring{A}_{\mathring{T}}$. Moreover

$$ L(g)L(h)(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) = (L(gh))(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) $$

for all $g, h \in G^{[\Gamma]}$ and $(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) \in \mathring{A}_{\mathring{T}}$ yields. Clearly there is a right action presented by the map $R(g)$, which is defined by

$$ \mathring{A}_{\mathring{T}} \ni (b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) \mapsto R(g)(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) := (b_{\Gamma}(\gamma_1)g_1^{-1}, ..., b_{\Gamma}(\gamma_N)g_N^{-1}) \in \mathring{A}_{\mathring{T}} $$

such that

$$ R(gh)(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) = (b_{\Gamma}(\gamma_1)g_1h_1^{-1}, ..., b_{\Gamma}(\gamma_N)g_Nh_N^{-1}) $$

$$ = (b_{\Gamma}(\gamma_1)h_1^{-1}g_1^{-1}, ..., b_{\Gamma}(\gamma_N)h_N^{-1}g_N^{-1}) $$

$$ = R(g)(R(h)(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N))) $$

Consider a **finite orientation preserved graph system** $\mathring{P}_\mathring{T}^\circ$ associated to a graph $\Gamma$ and a finite set of surfaces $\mathring{S}$ such that the set $\mathring{S}$ has the same surface intersection property for the graph $\Gamma$. Then the flux group $G_{\mathring{S},\mathring{T}}$ is a subgroup of $G^{[\Gamma]}$. Since each subgraph $\mathring{T}'$ of $\mathring{T}$, for example, consists only paths that intersect each surface in $\mathring{S}$ in the source vertex of the path and lie above. The evaluation of a map $\rho_{\mathring{S},\mathring{T}} \in G_{\mathring{S},\mathring{T}}$ for a subgraph $\mathring{T}'$ in $\mathring{P}_\mathring{T}^\circ$ is given by $\rho_{\mathring{S},\mathring{T}}(\mathring{T}') = (\rho_{\mathring{S}}(\gamma_1), ..., \rho_{\mathring{S}}(\gamma_M))$. The element $\rho_{\mathring{S},\mathring{T}}(\mathring{T}')$ is contained in $G_{\mathring{S},\mathring{T}}$. Furthermore the element $(\rho_{\mathring{S}}(\gamma_1), ..., \rho_{\mathring{S}}(\gamma_M), e_G, ..., e_G)$ is contained in $G_{\mathring{S},\mathring{T}}$.

Consider a graph $\Gamma := \{\gamma_1, ..., \gamma_N\}$ and a subgraph $\mathring{T}' := \{\gamma_1, ..., \gamma_M\}$ of $\mathring{T}$, a finite graph system $\mathring{P}_\mathring{T}$ and a finite orientation preserved graph system $\mathring{P}_\mathring{T}^\circ$ exists. Then there is a surfaces set $\mathring{S}$, which has the same surface intersection property for $\Gamma$. Moreover assume that $\mathring{T}' \in \mathring{P}_\mathring{T}^\circ$. Then for a map $\rho_{\mathring{S},\mathring{T}} \in G_{\mathring{S},\mathring{T}}$ there exists a left action $L : G_{\mathring{S},\mathring{T}} \to \mathring{A}_{\mathring{T}}$, which is given by

$$ L(\rho_{\mathring{S},\mathring{T}}(\mathring{T}))(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) = L(\rho_{\mathring{S}}(\gamma_1), ..., \rho_{\mathring{S}}(\gamma_N))(b_{\Gamma}(\gamma_1), ..., b_{\Gamma}(\gamma_N)) $$

$$ := (\rho_{\mathring{S}}(\gamma_1)b_{\Gamma}(\gamma_1), ..., \rho_{\mathring{S}}(\gamma_N)b_{\Gamma}(\gamma_N)) $$

(22)
and which defines a homeomorphism on $\tilde{A}_\Gamma$. Certainly, if the surface set $\tilde{S}$ has the same surface intersection property for $\Gamma$, then there is a right action $R$ of $G_{\tilde{S}, \Gamma}$ on $\tilde{A}_\Gamma$. This action $R$ is of the form

$$R(p_{\tilde{S}}(\Gamma')(\delta(\gamma_1), ..., \delta(\gamma_N)) = R(p_{\tilde{S}}(\gamma_1), ..., p_{\tilde{S}}(\gamma_M))(\delta(\gamma_1), ..., \delta(\gamma_N))$$

$$:= (\delta(\gamma_1)p_{\tilde{S}}(\gamma_1)^{-1}, ..., \delta(\gamma_M)p_{\tilde{S}}(\gamma_M)^{-1}, \delta(\gamma_{M+1}), ..., \delta(\gamma_N))$$

(23)

for $p_{\tilde{S}}(\Gamma') \in G_{\tilde{S}, \Gamma}$. This action defines a homeomorphism of $\tilde{A}_\Gamma$, too. Notice that, the flux operator given by $p_{\tilde{S}}(\Gamma')$ is for example restricted to a subgraph $\Gamma'$, whereas the holonomies are computed on the whole graph $\Gamma$. Mathematically, this is well-defined. Physically, the flux operators are somehow localised on a subgraph.

In subsection 2.3 the flux operators are constructed from maps, which map a graph $\Gamma$ to the structure $Z$ and $\rho$ for $\rho(p)$ shown by the computation for $\rho$ homomorphism $h$.

Due to the specific structure of the groupoid homomorphisms $h_{\Gamma} \colon \mathcal{P}_{\Gamma} \to G$ there is in general no groupoid morphism $\tilde{s}$ defined by

$$\text{Hom}(\mathcal{P}_{\Gamma}, G) \ni h_{\Gamma} \mapsto \tilde{s}_{\Gamma}(\gamma) := p_{\tilde{S}}(\gamma)h_{\Gamma}(\gamma) \notin \text{Hom}(\mathcal{P}_{\Gamma}, G)$$

for $p_{\tilde{S}} \in \text{Hom}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, G)$. Let $\Gamma = \{ \gamma, \gamma' \}$, then $\gamma, \gamma' \in \mathcal{P}_{\Gamma}$ and assume that $(\gamma, \gamma') \in \mathcal{P}_{\Gamma}(2)$. Then this is shown by the computation

$$\tilde{s}_{\Gamma}(\gamma \circ \gamma') = p_{\tilde{S}}(\gamma \circ \gamma')h_{\Gamma}(\gamma \circ \gamma') = p_{\tilde{S}}(\gamma)p_{\tilde{S}}(\gamma'h_{\Gamma}(\gamma'h_{\Gamma}(\gamma'))$$

$$\neq p_{\tilde{S}}(\gamma)h_{\Gamma}(\gamma)p_{\tilde{S}}(\gamma'h_{\Gamma}(\gamma'))$$

$$= \tilde{s}_{\Gamma}(\gamma)\tilde{s}_{\Gamma}(\gamma')$$

(24)

for $p_{\tilde{S}} \in \text{Hom}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, G)$.

The equality holds for all $p_{\tilde{S}} \in \text{Map}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, Z(G))$, where $Z(G)$ is the center of the group $G$. Hence $L(p_{\tilde{S}}) \circ h_{\Gamma} \in \text{Hom}(\mathcal{P}_{\Gamma}, G)$ for $(L(p_{\tilde{S}}) \circ h_{\Gamma})(\gamma) := p_{\tilde{S}}(\gamma)h_{\Gamma}(\gamma)$ and $p_{\tilde{S}} \in \text{Map}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, Z(G))$ and $\gamma \in \mathcal{P}_{\Gamma}$. This indicate that for $p_{\tilde{S}} \in \text{Hom}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, Z(G))$ the following properties are true

(i) $p_{\tilde{S}}(\gamma')h_{\Gamma}(\gamma) = h_{\Gamma}(\gamma)p_{\tilde{S}}(\gamma')$ for all paths $\gamma$ and $\gamma'$ contained in $\mathcal{P}_{\Gamma}$, 
(ii) $p_{\tilde{S}}(\gamma^{-1}) = p_{\tilde{S}}(\gamma)^{-1}$ and $p_{\tilde{S}}(\gamma \circ \gamma') = p_{\tilde{S}}(\gamma)p_{\tilde{S}}(\gamma')$ for all paths $\gamma, \gamma' \in \mathcal{P}_{\Gamma}$, 
(iii) $p_{\tilde{S}}(\gamma) = p_{\tilde{S}}(\gamma'n)$ for all $\gamma, \gamma'' \in \mathcal{P}_{\Gamma}$ where $v = s(\gamma) = s(\gamma'')$, 
(iv) $p_{\tilde{S}}(\gamma \circ \gamma') = e_G$ for each path $\gamma$ in $\mathcal{P}_{\Gamma}$ and 
(v) $p_{\tilde{S}}(\gamma) = e_G$ if $\gamma$ in $\mathcal{P}_{\Gamma}$ does not intersect $S$ in the source or target vertex.

Moreover consider $Z_{\tilde{S}, \Gamma}$ to be the the set

$$Z_{\tilde{S}, \Gamma} := \{ p_{\tilde{S}, \Gamma} \in \text{Hom}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, Z(G)) : \exists \ p_{\tilde{S}} \in \text{Hom}_{\mathcal{P}_{\Gamma}, G}(\mathcal{P}_{\Gamma}, Z(G)) \ s.t. \ p_{\tilde{S}, \Gamma}(\Gamma') = (p_{\tilde{S}}(\gamma_1), ..., p_{\tilde{S}}(\gamma_M))$$

$$\forall \Gamma' \in \mathcal{P}_{\Gamma} \}$$

and $Z_{\tilde{S}, \Gamma} := \times_{S \in \tilde{S}} Z_{\tilde{S}, \Gamma}$.

Remark 47. Otherwise, in subsection 2.4 the set of maps $G^A_{\tilde{S}, \Gamma}$ associated to a set of surfaces $\tilde{S}$ is presented. For a pair of maps $\varphi_S^k$ and $\varphi_S^R$ in $G^A_{\tilde{S}, \Gamma}$ and for each surface $S$ in $\tilde{S}$ it is true that
\((\text{vi})\) \(S \cap \gamma = \{s(\gamma), t(\gamma)\}\) for all \(\gamma \in \mathcal{P}_T \Sigma,\)

\((\text{vii})\) \(\varrho_S^2(\gamma \circ \gamma') = \varrho_S^2(\gamma), \varrho_S^3(\gamma \circ \gamma') = \varrho_S^3(\gamma)\) for all \(\gamma, \gamma \circ \gamma' \in \mathcal{P}_T \Sigma^v\) and \(S \cap \gamma \circ \gamma' = \{s(\gamma)\},\)

\((\text{viii})\) \(\varrho_S^2((\gamma \circ \gamma)^{-1}) = \varrho_S^2(\gamma^{-1}), \varrho_S^3((\gamma \circ \gamma)^{-1}) = \varrho_S^3(\gamma^{-1})\) for all \(\gamma^{-1} \circ \gamma^{-1} \in \mathcal{P}_T \Sigma^v\) and \(S \cap \gamma^{-1} \circ \gamma^{-1} = \{t(\gamma^{-1})\},\)

\((\text{ix})\) \(\varrho_S^2(\gamma)^{-1} = \varrho_S^2(\gamma^{-1}), \varrho_S^3(\gamma)^{-1} = \varrho_S^3(\gamma^{-1})\) for \(\gamma \in \mathcal{P}_T \Sigma,\)

\((\text{x})\) \(\varrho_S^2(\gamma) = \varrho_S^2(\gamma'), \varrho_S^3(\gamma) = \varrho_S^3(\gamma')\) for all \(\gamma, \gamma' \in \mathcal{P}_T \Sigma^v\) where \(v = s(\gamma) = s(\gamma'),\)

\((\text{xi})\) \(\varrho_S^2(\gamma^{-1})^{-1} \varrho_S^2(\gamma) = e_G\) for all \((\gamma, \gamma') \in \mathcal{P}_T \Sigma^2\)

\((\text{xii})\) \(\varrho_S^3(\gamma^{-1})^{-1} \varrho_S^3(\gamma) = e_G, \varrho_S^3(\gamma \circ \gamma^{-1}) = e_G\) for a path \(\gamma\) in \(\mathcal{P}_T \Sigma\) and

\((\text{xiii})\) \(\varrho_S^2(\gamma) = e_G, \varrho_S^3(\gamma) = e_G\) if \(\gamma\) in \(\mathcal{P}_T \Sigma\) does not intersect \(S\) in the source or target vertex.

Then for example, for a path \(\gamma\) that intersects a fixed surface \(S\) in the source vertex \(s(\gamma)\) the map \(\vartheta_{T}^{S}\) defined by

\[
\text{Hom}(\mathcal{P}_T \Sigma, G) \ni \vartheta_{T}(\gamma) \mapsto \vartheta_{T}^{S}(\gamma) := \varrho_S^2(\gamma)\vartheta_{T}(\gamma) \notin \text{Hom}(\mathcal{P}_T \Sigma, G)
\]

is not a groupoid morphism. This is verified by

\[
\vartheta_{T}^{S}(\gamma \circ \gamma') = \varrho_S^2(\gamma \circ \gamma')\vartheta_{T}(\gamma \circ \gamma') = \varrho_S^2(\gamma)\varrho_S^3(\gamma)\varrho_S^2(\gamma')\varrho_S^3(\gamma')\vartheta_{T}(\gamma)\vartheta_{T}(\gamma')
\]

\[
\neq \vartheta_{T}^{S}(\gamma)\vartheta_{T}^{S}(\gamma') = \varrho_S^2(\gamma)\varrho_S^3(\gamma)\varrho_S^2(\gamma')\varrho_S^3(\gamma')\vartheta_{T}(\gamma)\vartheta_{T}(\gamma')
\]

Otherwise, if the path \(\gamma\) intersects the surface \(S\) in the target vertex \(t(\gamma)\) then the map \(\vartheta_{T}^{R}\) given by

\[
\vartheta_{T}(\gamma) \mapsto \vartheta_{T}^{R}(\gamma) := \varrho_S^2(\gamma)\vartheta_{T}(\gamma)\varrho_S^3(\gamma)^{-1}
\]

is not a groupoid morphism, too.

Recall in definition \(14\) a groupoid morphism \(\vartheta_{T}\) has been defined for a path \(\gamma\) that intersects the surface \(S\) in the source vertex \(s(\gamma)\) and the target \(t(\gamma)\) by

\[
\vartheta_{T}(\gamma) \mapsto \vartheta_{T}^{S}(\gamma) := \varrho_S^2(\gamma)\vartheta_{T}(\gamma)\varrho_S^3(\gamma)^{-1}
\]

Notice that, there is a difference between \(\text{Hom}_S(\mathcal{P}_T \Sigma, Z(G))\) and \(\mathcal{G}^{A}_{S,T} \). For example in condition \(\text{(i)}\) which solve the problem \(\text{(24)}\) of the groupoid multiplication by using the center of \(G\) and condition \(\text{(ii)}\) for maps in \(\text{Hom}_S(\mathcal{P}_T \Sigma, Z(G))\). Otherwise the maps in \(\mathcal{G}^{A}_{S,T}\) satisfy in particularly condition \(\text{(xiii)}\) for composable paths.

A dynamical system of actions of the flux group on the analytic holonomy algebra for a fixed finite graph system and surfaces

Let \(\mathcal{S}\) be a suitable surface set for a finite orientation preserved graph system \(\mathcal{P}_T^\mathcal{G}\) and the object \(\hat{G}_{S,T}\) is the flux group. Then equivalently to a group action of \(\hat{G}_{S,T}\) on the configuration space \(A_T\) an action \(\alpha\) of \(\hat{G}_{S,T}\) on the analytic holonomy \(C^*\)-algebra \(C(\hat{A}_T)\) associated to a graph \(\Gamma\) is studied as follows. The action is for example of the form

\[
(\alpha(\rho_{S,T}(\Gamma))(f_T))(\vartheta_{T}(\Gamma)) := f_T(L(\rho_{S,T}(\Gamma))(\vartheta_{T}(\Gamma)))
\]

where \(\rho_{S,T} \in G_{S,T}\) and \(f_T \in C(\hat{A}_T)\).

Notice that, there is a state on \(C(\hat{A}_T)\), which is \(\hat{G}_{S,T}\)-invariant. In this subsection a bunch of different actions of this form is constructed.

Before the investigations start the following remark on the nature of the definition of the flux operators has to be done. In subsection \(2.3\) the flux operators are constructed from maps, which map a graph \(\Gamma\) to the structure group \(G\). If the flux operators would be defined by groupoid morphisms between the finite path groupoid \(\mathcal{P}_T \Sigma \equiv V_T\) and the groupoid \(G\) over \(\{e_G\}\) then difficulties arise, which have been studied in \(\text{[15]}\) Sect.: 6.1.
First restrict the surface and graph configuration to the simplest case. Consider a surface \( S \), which has the same surface intersection property for a graph \( \Gamma \). This equivalent to consider a surface \( S \) that intersects each path of \( \Gamma \) in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of \( S \). Furthermore there are no other intersection points of the surface \( S \) with paths of the graph \( \Gamma \).

In this subsection representations and actions of the flux group \( \mathcal{G}_{S,\Gamma} \) in the \( C^* \)-algebra \( C(\mathcal{A}_\Gamma) \) are studied instead of analysing group actions on the configuration space or transformation groups. These representations are maps from the flux group \( \mathcal{G}_{S,\Gamma} \) to the multiplier algebra of the \( C^* \)-algebra having several properties presented in [15, Appendix 12.2]. In the following different actions of the flux group \( \mathcal{G}_{S,\Gamma} \) on \( C^* \)-algebra \( C(\mathcal{A}_\Gamma) \) are investigated first.

Assume that, \( G \) is a compact group. Therefore the configuration space \( \mathcal{A}_\Gamma \) for a finite graph system \( \mathcal{P}_\Gamma \) associated to a graph \( \Gamma \) is a compact Hausdorff space. Let \( \mathfrak{A}_\Gamma \) be the quantum algebra generated by the configuration variables, which is isomorphic to \( C(\mathcal{A}_\Gamma) \). Notice that, the elements of \( \mathcal{A}_\Gamma \) are identified with \( G^{[\Gamma]} \) by the natural identification. The evaluation of the holonomy map \( \mathfrak{h}_\Gamma \) for a finite graph system \( \mathcal{P}_\Gamma \) on a subgraph \( \Gamma' \) of \( \Gamma \) is \( \mathfrak{h}_\Gamma(\Gamma') = (\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_M),e_G,...,e_G) \) an element in \( G^{[\Gamma]} \).

An important property of actions on commutative \( C^* \)-algebras is the following.

**Definition 48.** Let \( \mathfrak{A} \) be an unital commutative \( C^* \)-algebra isometrically isomorphic to \( C(X) \) where \( X \) is a compact space, \( G \) be an arbitrary group and \( \alpha \) be an automorphism of \( \mathfrak{A} \).

Then the action \( \alpha \) of \( G \) on \( \mathfrak{A} \) is **automorphic** if the following conditions are satisfied

(i) \( \alpha(gh)(f) = \alpha(g)(\alpha(h)(f)) \) for any \( f \in \mathfrak{A}, \, g,h \in G \)

(ii) \( \alpha(g)(f_1f_2) = \alpha(g)(f_1)\alpha(g)(f_2) \) for any \( f_1,f_2 \in \mathfrak{A}, \, g \in G \)

(iii) \( \alpha(g)(f^*) = \alpha(g)(f)^* \) for any \( f \in \mathfrak{A}, \, g \in G \)

In this article the flux operators w.r.t. a surface \( S \) are implemented as group actions of \( \mathcal{G}_{S,\Gamma} \) on the configuration space \( \mathcal{A}_\Gamma \) or equivalently as group actions on \( C^* \)-algebras. Consequently for each surface set and graph system configuration an action on the holonomy algebra for a suitable finite graph system is defined. Investigate the following actions on the holonomy \( C^* \)-algebra \( C(\mathcal{A}_\Gamma) \), where the configuration space \( \mathcal{A}_\Gamma \) is identified with \( G^{[\Gamma]} \) naturally.

**Lemma 49.** Let \( \Gamma \) be a graph and \( \mathcal{P}_\Gamma^\gamma \) be the finite orientation preserved graph system associated to \( \Gamma \). Furthermore let \( S \) be a fixed surface in \( \Sigma \) such that \( S \) intersects each path of \( \Gamma \) in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of \( S \). There are no other intersection points of the surface \( S \) with paths of the graph \( \Gamma \).

Let \( \mathcal{G}_{S,\Gamma} \) denotes the flux group and \( \mathcal{A}_\Gamma \) denotes the configuration space for the finite orientation preserved graph system \( \mathcal{P}_\Gamma^\gamma \), where all elements of \( \mathcal{P}_\Gamma^\gamma \) are identified in the natural way with a subset of the set of generators of \( \Gamma \).

Then there is an action \( \alpha \) of \( \mathcal{G}_{S,\Gamma} \) on \( C(\mathcal{A}_\Gamma) \) defined by

\[
(\alpha(\rho_{S,\Gamma}(\Gamma))f_{\Gamma})(\mathfrak{h}_\Gamma(\Gamma)) := f_{\Gamma}(\rho_{S}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_{S}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))
\]

for \( \rho_{S,\Gamma} \in \mathcal{G}_{S,\Gamma} \) and \( \rho_{S} \in \mathcal{G}_{S,\Gamma} \), which is automorphic.

**Proof:** Observe

\[
(\alpha(\rho_{S,\Gamma}(\Gamma))\rho_{S,\Gamma}(\Gamma))f_{\Gamma})(\mathfrak{h}_\Gamma(\Gamma)) = f_{\Gamma}(\rho_{S}(\gamma_1)\rho_{S}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_{S}(\gamma_N)\rho_{S}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N)) = (\alpha(\rho_{S,\Gamma}(\Gamma))(\alpha(\rho_{S,\Gamma}(\Gamma))f_{\Gamma}))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\]

The multiplication between two functions in \( C(\mathcal{A}_\Gamma) \) is pointwise:

\[
\alpha(\rho_{S,\Gamma}(\Gamma))(f_{\Gamma})(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)) = (\alpha(\rho_{S,\Gamma}(\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)))
\]

\[
\alpha(\rho_{S,\Gamma}(\Gamma))(f_{\Gamma})(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\]

\[
= (\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))
\]

\[
= (f_{\Gamma}(\rho_{S}(\gamma_1)\mathfrak{h}_\Gamma(\gamma_1),...,\rho_{S}(\gamma_N)\mathfrak{h}_\Gamma(\gamma_N))(\alpha(\rho_{S,\Gamma}(\Gamma))(f_{\Gamma}))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)))
\]

\[
= (\alpha(\rho_{S,\Gamma}(\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N))(\alpha(\rho_{S,\Gamma}(\Gamma))(\mathfrak{h}_\Gamma(\gamma_1),...,\mathfrak{h}_\Gamma(\gamma_N)))
\]

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Finally, recover
\[
(\alpha(\rho_{S,T}(\Gamma)))f_T^\ast)(h_T(\gamma_1),...,h_T(\gamma_N)) = f_T(\rho_S(\gamma_1)h_T(\gamma_1),...,\rho_S(\gamma_N)h_T(\gamma_N))
= (\alpha(\rho_{S,T}(\Gamma))f_t)(h_T(\gamma_1),...,h_T(\gamma_N))^\ast
\]

In particular there is a map \( \tilde{J} : G_{S,T} \rightarrow G_{S,T} \), \( J : \rho_S(\gamma) \mapsto \rho_{S^{-1}}(\gamma) \). Then for \( \rho_S(\gamma) = g \), where the surface \( S \) and the path \( \gamma \) are suitable and \( \tilde{S} := \{S, S^{-1}\} \), the map satisfies \( (\tilde{J}(\rho_S))((\gamma)) = \rho_{S^{-1}}(\gamma) = g^{-1} \).

With no doubt there is a general map \( J : G_{S,T} \rightarrow G_{S,T} \), \( J : \rho_S(\Gamma) \mapsto \rho_{S^{-1,T}}(\Gamma) \). Then derive
\[
(\alpha(\rho_{S,T}(\Gamma))(J(\rho_S(\Gamma))(\Gamma))f_T)(h_T(\gamma_1),...,h_T(\gamma_N)) = f_T(\rho_S(\gamma_1)\rho_{S^{-1}}(\gamma_1)h_T(\gamma_1),...,\rho_S(\gamma_N)\rho_{S^{-1}}(\gamma_N)h_T(\gamma_N)) = f_T(gg^{-1}h_T(\gamma_1),...,gg^{-1}h_T(\gamma_N)) = f_T(h_T(\gamma_1),...,h_T(\gamma_N)) \tag{26}
\]

It is necessary that, the action is defined for a finite orientation preserved graph system, since the following observation is made. Let \( \Gamma \) be equivalent to a path \( \gamma \). Then it is true that \( h_T(\gamma^{-1}) = (h_T(\gamma))^{-1} \) and consequently
\[
(\alpha(\rho_{S,T}(\Gamma))f_T)((h_T(\gamma)^{-1}) = f_T(\rho_S^2(\gamma)((h_T(\gamma)^{-1}) = f_T(\rho_S^2(\gamma)h_T(\gamma)^{-1})
\]
holds for \( \rho_S \in G_{S,T} \) and \( \rho_S^2 \in G_{S,T} \). Although the holonomy is evaluated for the path \( \gamma^{-1} \) the action is defined by the left action \( L \). Later it is analysed, how a left action \( L \) can be transferred to a right action \( R \). The right action is given by
\[
(\alpha(\rho_{S,T^{-1}}(\Gamma)^{-1}))f_T)((h_T(\gamma^{-1}) = f_T(R(\rho_S^R(\gamma)^{-1}))(h_T(\gamma^{-1})) = f_T(h_T(\gamma^{-1})\rho_S^R(\gamma)^{-1})^{-1}
\]
for \( \rho_S \in G_{S,T^{-1}} \) and \( \rho_S^R \in G_{S,T^{-1}} \).

Furthermore an automorphic action has another interesting property, which allows to speak about \( C^\ast \)-dynamical systems.

**Corollary 50.** Let \( \Gamma \) be a graph and \( P^G_{\Gamma} \) be the finite orientation preserved graph system associated to \( \Gamma \). Furthermore let \( S \) be a fixed surface in \( \Sigma \) such that \( S \) intersects each path of \( \Gamma \) in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of \( S \). There are no other intersection points of the surface \( S \) with paths of the graph \( \Gamma \).

Let \( G_{S,T} \) denotes the flux group and \( A_T \) denotes the configuration space for the finite orientation preserved graph system \( P^G_{\Gamma} \), where all elements of \( P^G_{\Gamma} \) are identified in the natural way with a subset of the set of generators of \( \Gamma \), where all elements of \( P^G_{\Gamma} \) are identified in the natural way with a subset of the set of generators of \( \Gamma \).

The triple \( (G_{S,T}, C(A_T)\alpha) \) consisting of a compact group \( G_{S,T} \), a \( C^\ast \)-algebra \( C(A_T) \) and an automorphic action \( \alpha \) of \( G_{S,T} \) on \( C(A_T) \) such that for each \( f_T \in C(A_T) \) the function \( G_{S,T} \ni \rho_S(\Gamma) \mapsto \|\alpha(\rho_S(\Gamma))(f_T)\| \) is continuous is a \( C^\ast \)-dynamical system for a finite orientation preserved graph system associated to a graph \( \Gamma \).

**Proof:** Let \( f_T \in C(A_T) \) and \( \Gamma := \{\gamma\} \) then for a fixed suitable surface \( S \) it is true that
\[
\lim_{\rho_S(\gamma) \rightarrow \text{id}_{S,T}(\gamma)} \|\alpha(\rho_S(\Gamma))(f_T) - f_T\| = \lim_{\rho_S(\gamma) \rightarrow \text{id}_{S,T}(\gamma)} \|f_T(L(\rho_S(\gamma))(h_T(\gamma)))(f_T) - f_T(h_T(\gamma))\|
= 0
\]
yields for \( \rho_S, \text{id}_S \in G_{S,T} \) and \( \rho_S, \text{id}_S \in G_{S,T} \), if \( \rho_S(\gamma) = g \in G \) and \( \text{id}_S(\gamma) = e_G \) for all \( \gamma \in \Gamma \).
Let \( \mathfrak{A}_\Gamma \) be a \( C^* \)-subalgebra of the \( C^* \)-algebra \( \mathcal{L}(\mathcal{H}_\Gamma) \) of bounded operators on a Hilbert space \( \mathcal{H}_\Gamma \), then \( \mathfrak{A}_\Gamma \) is non-degenerately represented on \( \mathcal{H}_\Gamma \) if the inclusion map of \( \mathfrak{A}_\Gamma \) into \( \mathcal{L}(\mathcal{H}_\Gamma) \) is a non-degenerate representation of \( \mathfrak{A}_\Gamma \). Set \( \mathcal{H}_\Gamma \) be equal to \( L^2(\mathcal{A}_\Gamma, \mu_\Gamma) \). Then the multiplication representation \( \Phi_M \) of \( \mathcal{C}(\mathcal{A}_\Gamma) \) in \( \mathcal{H}_\Gamma \) defined by

\[
\Phi_M(f)\psi_T = f_T \cdot \psi_T \quad \text{for} \quad \psi_T \in \mathcal{H}_\Gamma \quad \text{and} \quad f_T \in \mathfrak{A}_\Gamma
\]  

is non-degenerate.

**Definition 51.** Let \( S \) be a fixed surface in \( \Sigma \), \( \Gamma \) be a graph, \( \mathcal{P}_\text{fin}^\oplus \) be the finite orientation preserved graph system associated to \( \Gamma \) such that the surface \( S \) has the surface intersection property for a finite orientation preserved graph system \( \mathcal{P}_\text{fin}^\oplus \). Let \( G_{\mathcal{S},\Gamma} \) denotes the flux group and \( \mathcal{A}_\Gamma \) denotes the configuration space for the finite orientation preserved graph system \( \mathcal{P}_\text{fin}^\oplus \), where all elements of \( \mathcal{P}_\text{fin}^\oplus \) are identified in the natural way with a subset of the set of generators of \( \Gamma \).

A **covariant representation** of the \( C^* \)-dynamical system \((G_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma), \alpha)\) in a \( C^* \)-algebra \( \mathcal{L}(\mathcal{H}_\Gamma) \) consists of a pair \((\Phi_M, U)\) where \( \Phi_M \) is a non-degenerate representation of \( C(\mathcal{A}_\Gamma) \) on a Hilbert space \( \mathcal{H}_\Gamma \) (i.e. \( \Phi_M \in \text{Mor}(C(\mathcal{A}_\Gamma), \mathcal{L}(\mathcal{H}_\Gamma)) \)) and \( U \) is a (strongly continuous) unitary representation of \( G_{\mathcal{S},\Gamma} \) on \( \mathcal{H}_\Gamma \) such that for all \( \rho_{\mathcal{S},\Gamma}(\Gamma) \in G_{\mathcal{S},\Gamma} \) and \( f_T \in C(\mathcal{A}_\Gamma) \) it is true that

\[
U(\rho_{\mathcal{S},\Gamma}(\Gamma))\Phi_M(f_T)U^*(\rho_{\mathcal{S},\Gamma}(\Gamma)) = \Phi_M(\alpha(\rho_{\mathcal{S},\Gamma}(\Gamma))(f_T)) \tag{28}
\]

holds. The pair \((\Phi_M, U)\) is also called a **covariant Hilbert space representation of the \( C^* \)-dynamical system** \((G_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma), \alpha)\).

There is a covariant representation of \((G_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma), \alpha)\) with respect to the automorphic action \( \alpha \) of \( G_{\mathcal{S},\Gamma} \) on \( C(\mathcal{A}_\Gamma) \) given by \( M \) and the unitary operator \( U \), which is a map \( U : G_{\mathcal{S},\Gamma} \to \mathcal{U}(\mathcal{H}_\Gamma) \), where \( \mathcal{U}(\mathcal{H}_\Gamma) \) is the unitary group of \( \mathcal{L}(\mathcal{H}_\Gamma) \), since

\[
U(\rho_{\mathcal{S},\Gamma}(\Gamma))U(\rho_{\mathcal{S},\Gamma}(\Gamma))^* = U(\rho_{\mathcal{S},\Gamma}(\Gamma)^{-1})(U^*(\rho_{\mathcal{S},\Gamma}(\Gamma)^{-1}))(U(\rho_{\mathcal{S},\Gamma}(\Gamma))) \tag{29}
\]

holds for \( \psi_T \in \mathcal{H}_\Gamma \) and which satisfies \( (28) \). Notice that, \( U(\rho_{\mathcal{S},\Gamma}(\Gamma))^* = U(\rho_{\mathcal{S},-1,\Gamma}(\Gamma)) \).

In the following it is often assumed that, a strongly continuous unitary representation of the flux group on a Hilbert space is a representation of the group on the \( C^* \)-algebra of compact operators on the Hilbert space. Therefore this relation is explicitly given in the following lemma.

**Lemma 52.** Let \((\Phi_M, U)\) a covariant representation \( U \) of a \( C^* \)-dynamical system \((G_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma), \alpha)\). Then the strongly continuous unitary representation of \( G_{\mathcal{S},\Gamma} \) on \( \mathcal{H}_\Gamma \) is a representation of the group \( \mathcal{G}_{\mathcal{S},\Gamma} \) on the \( C^* \)-algebra of compact operators on the Hilbert space \( \mathcal{H}_\Gamma \), i.e. \( U \in \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma)) \).

**Proposition 53.** Let \( \Gamma \) be a graph and \( \mathcal{P}_\text{fin}^\oplus \) be the finite orientation preserved graph system associated to \( \Gamma \). Furthermore let \( S \) be a fixed surface in \( \Sigma \) such that \( S \) intersects each path of \( \Gamma \) in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of \( S \). There are no other intersection points of the surface \( S \) with paths of the graph \( \Gamma \).

Let \( \mathcal{G}_{\mathcal{S},\Gamma} \) denotes the flux group and \( \mathcal{A}_\Gamma \) denotes the configuration space for a identified in the natural way finite orientation preserved graph system \( \mathcal{P}_\text{fin}^\oplus \). Moreover let \((\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{C}(\mathcal{A}_\Gamma), \alpha)\) a \( C^* \)-dynamical system and \( (M, U) \) a covariant representation of \((\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{C}(\mathcal{A}_\Gamma), \alpha)\) on a Hilbert space \( \mathcal{H}_\Gamma \).

Then there exists a GNS-triple \((\mathcal{H}_\Gamma, \Phi_M, \Omega_\Gamma)\), where \( \Omega_\Gamma \) is the cyclic vector for \( \Phi_M \) on \( \mathcal{H}_\Gamma \). Moreover the associated GNS-state \( \omega_M^\Gamma \) on \( \mathcal{C}(\mathcal{A}_\Gamma) \) is \( G_{\mathcal{S},\Gamma} \)-invariant, i.e.

\[
\omega_M^\Gamma(\alpha(\rho_{\mathcal{S},\Gamma}(\Gamma))(f_T)) = \omega_M^\Gamma(f_T) := \langle \Omega_\Gamma, \Phi_M(f_T)\Omega_\Gamma \rangle \tag{27}
\]

holds for all \( \rho_{\mathcal{S},\Gamma} \in \mathcal{G}_{\mathcal{S},\Gamma} \) and \( f_T \in C(\mathcal{A}_\Gamma) \).

In general automorphic actions on \( C^* \)-algebras define \( C^* \)-dynamical systems, since they are related to covariant representations of groups on \( C^* \)-algebras (refer to [15], Appendix]). This is connected to the definition of inner automorphisms.
Definition 54. Let $G$ be group and $\mathfrak{A}$ be a $C^*$-algebra.

An automorphic action $\alpha$ of a group $G$ on $\mathfrak{A}$ is called inner, if there is a representation of the group $G$ on $\mathfrak{A}$, i.e. $U \in \text{Rep}(G, \mathfrak{A})$ such that
\[
\alpha_g(A) = U(g)AU(g)^* \quad \text{whenver} \quad A \in \mathfrak{A} \quad \text{and} \quad g \in G.
\]
Otherwise $\alpha$ is called outer.

But since the holonomy $C^*$-algebra $C(\bar{A}_\Gamma)$ for a finite graph system is commutative, there is only one inner automorphic action of $\bar{G}_{S,\Gamma}$ on $C(\bar{A}_\Gamma)$ given by the trivial one.

Due to the different intersection behavior of surfaces and paths, there are a lot of different automorphic actions on the holonomy $C^*$-algebra for finite graph systems.

**Dynamical systems of the analytic holonomy algebra and actions of the flux group w.r.t. different graph and surface configurations**

A graph is a set of paths, each path and a surface $S$ has a specific intersection behavior. In general a graph does not contain only paths that are ingoing and lie above w.r.t. the surface orientation of $S$. In this subsection different actions for general graphs are studied. In the interesting configurations the corresponding actions on the analytic holonomy algebra turn out to be automorphic and point-norm continuous. There are only few which have to be excluded.

**Purely left or right actions of the flux group**

In the construction of dynamical systems the following surface and graph configurations play a particular role. This implies that, particular actions, which are for example purely left or right actions of a group on a $C^*$-algebra, are analysed.

During the whole subsection the set $\bar{G}_{S,\Gamma}$ and the multiplication operation
\[
(\rho^S_1(\gamma_1),...,\rho^S_N(\gamma_N)) \cdot (\tilde{\rho}^S_1(\gamma_1),...,\tilde{\rho}^S_N(\gamma_N)) = (\rho^S_1(\gamma_1)\tilde{\rho}^S_1(\gamma_1),...,\rho^S_N(\gamma_N)\tilde{\rho}^S_N(\gamma_N))
\]
where $x$ is equal to $L$ or $R$ and $\tilde{S} = \{S_i\}_{1 \leq i \leq |\Gamma|}$, is considered. If $\tilde{S}$ contains for example only one surface, then $S_i = S$ for all $1 \leq i \leq |\Gamma|$. The right group multiplication is explicitly assumed in the context. If both left and right multiplication would be used, the set $G_{A_{\Gamma}}$ does not form a group and consequently there are problems in the definition of automorphic actions, which are stated in the problem 3.1\[1\]

For a summary recall the definition of the last subsection.

**Lemma 55.** Let $\Gamma$ be a graph and $\mathcal{P}^*_\Gamma$ be the finite orientation preserved graph system associated to $\Gamma$. Furthermore let $S$ be a fixed surface in $\Sigma$ such that $S$ intersects each path of $\Gamma$ in the source vertex of the path such that each path lies below the surface and is outgoing w.r.t. the surface orientation of $S$. There are no other intersection points of the surface $S$ with paths of the graph $\Gamma$.

Then redefine the action
\[
(\alpha_{\Gamma}^{\bar{G}}(\rho^S_1)_{\bar{f}_\Gamma})(\text{hr}(\Gamma)) := \text{fr}(\rho_S(\gamma_1)\text{hr}(\gamma_1),...,\rho_S(\gamma_N)\text{hr}(\gamma_N))
\]
\[
= \text{fr}(\text{gs}hr(\gamma_1),...,\text{gs}hr(\gamma_N)) \quad \text{(30)}
\]
for $\rho^S_1 = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (\text{gs},...,\text{gs})$, $\rho^S_1 \in \bar{G}_{S,\Gamma}$ such that $\rho_S \in \bar{G}_{S,N}$ and $f_\Gamma \in C(\bar{A}_\Gamma)$.

Then the action $\alpha_{\Gamma}^{\bar{G}}$ of $\bar{G}_{S,\Gamma}$ on $C(\bar{A}_\Gamma)$ is automorphic and point-norm continuous.

For a simplification of the following considerations always assume that, there exists a finite orientation preserved graph system $\mathcal{P}^*_\Gamma$ associated to $\Gamma$. Furthermore there are no other intersection points of the surface $S$ with paths of the graph $\Gamma$ except the intersections, which are required in the different lemmata. The proofs can be found in [15, Section 6.1].
Lemma 56. Let only the path $\gamma_N$ in $\Gamma$ intersect in (source) vertex of the set $V_\Gamma$ with a surface $S$ such that $\gamma_N$ is outgoing and lies above the surface $S$.

Then define an action

$$
(a_{L}^{1,1}(\rho_{S,\Gamma})_{fr})(b_{fr}(\gamma_1),...,b_{fr}(\gamma_N)) := f_r(\rho_S(\gamma_1)b_{fr}(\gamma_1),...,\rho_S(\gamma_N)b_{fr}(\gamma_N))
$$

$$
= f_r(b_{fr}(\gamma_1),...,g_S^{-1}b_{fr}(\gamma_N))
$$

(31)

for $\rho_{S,\Gamma}^{1,1} \in G_{S,\Gamma}$ and $\rho_S \in G_{S,\gamma}$ and $\rho_{S,\Gamma}^{1,1} = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (e_G,...,e_G,g_S^{-1})$.

Then the action $a_{L}^{1,1}$ of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ is automorphic and point-norm continuous.

Lemma 57. Let $\Gamma$ be a graph given by $\{\gamma_1,...,\gamma_N\}$. Moreover let only the paths $\gamma_1,...,\gamma_{N-1}$ intersect in (source) vertices of the set $V_\Gamma$ with a surface $S$ such that all paths are outgoing and lie below.

Then define the action

$$
(a_{L}^{1,N-1}(\rho_{S,\Gamma}^{1,N-1})_{fr})(b_{fr}(\gamma_1),...,b_{fr}(\gamma_N)) := f_r(\rho_S(\gamma_1)b_{fr}(\gamma_1),...,\rho_S(\gamma_N)b_{fr}(\gamma_N))
$$

$$
= f_r(g_Sb_{fr}(\gamma_1),...,g_{S}^{-1}b_{fr}(\gamma_N))
$$

(32)

for $\rho_{S,\Gamma}^{1,N-1} \in G_{S,\Gamma}$ and $\rho_S \in G_{S,\gamma}$ and $\rho_{S,\Gamma}^{1,N-1} = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (g_S,...,g_S^{-1})$.

The action $a_{L}^{1,N-1}$ of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ is automorphic and point-norm continuous.

In the following an action is defined, which does not lead to a point-norm continuous automorphic action on the analytic holonomy algebra.

Lemma 58. Let $\Gamma$ be a graph given by $\{\gamma_1,...,\gamma_N\}$. Moreover let all paths intersect in (source) vertices of the set $V_\Gamma$ with a surface $S$ such that all paths $\gamma_1,...,\gamma_{N-1}$ are outgoing and lie below, $\gamma_N$ is outgoing and lies above the surface $S$. There are no other intersection point of paths and the surface $S$.

Then define the action

$$
(a_L^{1}(\rho_{S,\Gamma})_{fr})(b_{fr}(\gamma_1),...,b_{fr}(\gamma_N)) := f_r(\rho_S(\gamma_1)b_{fr}(\gamma_1),...,\rho_S(\gamma_N)b_{fr}(\gamma_N))
$$

$$
= f_r(g_Sb_{fr}(\gamma_1),...,g_{S}^{-1}b_{fr}(\gamma_N))
$$

(33)

for $\rho_{S,\Gamma}^{1} \in G_{S,\Gamma}$ and $\rho_S \in G_{S,\gamma}$ and $\rho_{S,\Gamma}^{1} = (\rho_S(\gamma_1),...,\rho_S(\gamma_N)) = (g_S,...,g_S^{-1})$.

The action $a_{L}^{1}$ of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ is automorphic and the action is not point-norm continuous.

Lemma 59. Let $\Gamma := \{\gamma_1,...,\gamma_N\}$ be a graph. The paths $\{\gamma_1,...,\gamma_{N-1}\}$ intersect in their source vertices with a surface $S$ such that the paths are outgoing and lie below the surface $S$. The path $\gamma_N$ in $\Gamma$ intersects in the source vertex with the surface $S'$ such that $\gamma_N$ is outgoing and lies above the surface $S'$.

There are no other intersection point of paths and the surface $S$ and $S'$.

Then the action defined by

$$
(a_L^{2}(\rho_{S,\Gamma}(\Gamma),\rho_{S',\Gamma}(\Gamma))_{fr})(b_{fr}(\gamma_1),...,b_{fr}(\gamma_N)) := f_r(\rho_S(\gamma_1)b_{fr}(\gamma_1),...,\rho_{S'}(\gamma_N)b_{fr}(\gamma_N))
$$

$$
= f_r(g_Sb_{fr}(\gamma_1),...,g_{S'}^{-1}b_{fr}(\gamma_N))
$$

(34)

for $\rho_{S,\Gamma}^{1} \in G_{S,\Gamma}$ and $\rho_S,\rho_{S'} \in G_{S,\gamma}$ and $\rho_{S,\Gamma}^{1} = (\rho_S(\gamma_1),...,\rho_{S'}(\gamma_N)) = (g_S,...,g_{S'}^{-1})$ where $g_S \neq g_{S'}$.

The action $a_{L}^{2}$ of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ is automorphic and point-norm continuous.

Lemma 60. Let all paths in $\Gamma := \{\gamma_1,...,\gamma_N\}$ intersect in vertices of the set $V_\Gamma$ with a surface $S_N$ such that $\gamma_1,...,\gamma_N$ are outgoing and lie below the surface $S_N$. Let the paths $\gamma_1,...,\gamma_{N-1}$ in $\Gamma$ intersect in vertices of the set $V_\Gamma$ with a surface $S_{N-1}$ such that $\gamma_1,...,\gamma_{N-1}$ are outgoing and lie below the surface $S_{N-1}$. The same is true for a surface $S_{N-2}$ and paths $\{\gamma_1,...,\gamma_{N-2}\}$, and so on, til $S_1$ and $\{\gamma_1\}$. There are no other intersections between the paths and surfaces $S_1,...,S_{N-1}$ and $S_N$. Moreover let each path $\gamma_i$ in $\Gamma$ intersects in vertices of the set $V_\Gamma$ with a surface $S_i$, such that $\gamma_i$ is outgoing and lies below the surface $S_i$ for $i = 1,...,N$. There are no other intersections between the paths in $\Gamma$ and surfaces $S_{1,1},...,S_{1,N}$. The surfaces $S_N$ and $S_{1,N}$ coincide. The set $\mathcal{S} := \{S_{i,j}\}_{1\leq i\leq N}$ has the simple surface intersection property for $\Gamma$. 

39
The action for two different maps \( \rho_{S^{-1}, \Gamma} \) in \( G_{S^{-1}, \Gamma} \) and \( \tilde{\rho}_{S, \Gamma} \) in \( G_{S, \Gamma} \), such that there is an action of \( G_{S^{-1}, \Gamma} \times G_{S, \Gamma} \) on \( C(\hat{A}_\Gamma) \) given by

\[
(\alpha^2_{\Gamma} (\rho_{S^{-1}, \Gamma}, \tilde{\rho}_{S, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S^{-1}, \Gamma}((\gamma_1), \ldots, \gamma_N) \tilde{\rho}_{S, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

(35)

The action for \( (N-1) \)-different maps \( \rho_{S^{-1}, \Gamma} \) in \( G_{S^{-1}, \Gamma} \) \( \tilde{\rho}_{S_i, \Gamma} \) in \( G_{S_i, \Gamma} \) till \( \tilde{\rho}_{S_{N-2}, \Gamma} \) in \( G_{S_{N-2}, \Gamma} \), such that there is an action of \( G_{S_1, \Gamma} \times G_{S_3, \Gamma} \times \cdots \times G_{S_{N-2}, \Gamma} \) on \( C(\hat{A}_\Gamma) \) given by

\[
(\alpha^{N-1}_{\Gamma} (\rho_{S_{1}, \Gamma}, \ldots, \tilde{\rho}_{S_{N-2}, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S_1, \Gamma}((\gamma_1), \ldots, \gamma_N) \rho_{S_{N-1}, \Gamma}((\gamma_1), \ldots, \gamma_N) \tilde{\rho}_{S_{N-2}, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

(36)

whenever \( N > 5 \).

Respectively the action of \( N \)-different maps is equivalent to an action of \( G_{S_1, \Gamma} \times \cdots \times G_{S_{N}, \Gamma} \) on \( C(\hat{A}_\Gamma) \), which is defined by

\[
(\alpha^{N}_{\Gamma} (\rho_{S_1, \Gamma}, \ldots, \tilde{\rho}_{S_{N}, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S_1, \Gamma}((\gamma_1), \ldots, \gamma_N) \rho_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

(37)

Then the actions \( \alpha^2_{\Gamma}, \ldots, \alpha^{N-1}_{\Gamma} \) and \( \alpha^{N}_{\Gamma} \) of \( G_{S, \Gamma} \) on \( C(\hat{A}_\Gamma) \) are automorphic and point-norm continuous actions.

**Lemma 61.** Let all paths of a graph \( \Gamma \) intersect in vertices of the set \( V_\Gamma \) with a surface \( S \) such that all paths are ingoing and lie above the surface. Moreover let the set \( \tilde{S} := \{ S_i \}_{1 \leq i \leq N} \) has the simple surface intersection property for \( \Gamma \).

Then there is an action such that

\[
(\alpha^{\tilde{S}}_{\Gamma} (\rho_{S_{1}, \Gamma}, \ldots, \tilde{\rho}_{S_{N}, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S_1, \Gamma}((\gamma_1), \ldots, \gamma_N) \rho_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N) \tilde{\rho}_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

This action is changed such that

\[
(\alpha^{\tilde{S}}_{N} (\rho_{S_{1}, \Gamma}, \ldots, \tilde{\rho}_{S_{N}, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S_1, \Gamma}((\gamma_1), \ldots, \gamma_N) \rho_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N) \tilde{\rho}_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

(39)

Then the actions \( \alpha^{\tilde{S}}_{1} \) of \( G_{S, \Gamma} \) on \( C(\hat{A}_\Gamma) \), \ldots and \( \alpha^{\tilde{S}}_{N} \) of \( G_{S, \Gamma} \) on \( C(\hat{A}_\Gamma) \) are automorphic and point-norm continuous actions.

**Lemma 62.** Let all paths intersect in their target vertices contained in the set \( V_\Gamma \) with a surface \( S \) such that all paths are ingoing and lie below the surface \( S \).

Then there is an action such that

\[
(\alpha^{\tilde{S}}_{1} (\rho_{S_{1}, \Gamma}, \ldots, \tilde{\rho}_{S_{N}, \Gamma})) f_r ((\gamma_1), \ldots, \gamma_N)) = f_r (\rho_{S_1, \Gamma}((\gamma_1), \ldots, \gamma_N) \rho_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N) \tilde{\rho}_{S_{N}, \Gamma}((\gamma_1), \ldots, \gamma_N))
\]

(40)

holds. Then the action \( \alpha^{\tilde{S}}_{1} \) of \( G_{S, \Gamma} \) on \( C(\hat{A}_\Gamma) \) is an automorphic and point-norm continuous action.
The actions in the last paragraphs are constructed such that there is a always decomposition of left and right structures. For example, if graphs are considered such that all paths have the same intersection behavior w.r.t a fixed surface set $\tilde{S}$, then for elements of a finite orientation preserved graph system $\mathcal{P}_\Gamma^\mathcal{P}$ an action is defined. On the other hand for every graph $\Gamma$ in $\mathcal{P}_\Gamma^\mathcal{P}$ there always exists a graph $\Gamma^{-1}$, which refers to the set \{\$1^{-1},...,$\gamma^{-1}_M$\}, which is obviously not an element of $\mathcal{P}_\Gamma$. But this graph of reversed path orientations forms a second finite orientation preserved graph system $\mathcal{P}_\Gamma^{-1}$. Moreover there is an action of the corresponding flux group $\mathcal{G}_{S,\Gamma^{-1}}$ on $C(\mathcal{A}_\Gamma)$, where the configuration space is constructed from the finite graph groupoid $\mathcal{P}_\Gamma$. Recall that, $h_{\Gamma}(\gamma^{-1}) = h_{\Gamma}(\gamma)^{-1}$ yields for an arbitrary $\gamma \in \mathcal{P}_\Gamma \Sigma$. Hence it is easy to verify that,

\[
(\alpha_{\Gamma}^{L}(\rho_{S,\Gamma}^{1})f_{\Gamma})(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N}))
\]
\[
= fr(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1),...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N}))
\]
\[
(\alpha_{\Gamma}^{R}(\rho_{S,\Gamma}^{1}))f_{\Gamma})(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N}))
\]
\[
= fr(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})
\]
\[
= fr(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})
\]
\[
= fr(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})
\]

is fulfilled, whenever $\rho_{S,\Gamma}^{2}$ and $\rho_{S,\Gamma}^{2}$ denote the maps in $\mathcal{G}_{S,\Gamma}$.}

**Definition 63.** Define the map $I : C(\mathcal{A}_\Gamma) \rightarrow C(\mathcal{A}_\Gamma)$

$I : f_{\Gamma} \mapsto \tilde{f}_{\Gamma}$, where $\tilde{f}_{\Gamma}(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N})) := fr(\gamma_{1},...h_{\Gamma}(\gamma_{N})^{-1})$

such that $I^2 = \mathbb{1}$ where $\mathbb{1}$ is the identical automorphism on $C(\mathcal{A}_\Gamma)$.

Then deduce

\[
I(\alpha_{\Gamma}^{L}(\rho_{S,\Gamma}^{1})f_{\Gamma})(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N})^{-1})
\]
\[
= (f_{\Gamma}(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})^{-1}
\]
\[
= (f_{\Gamma}(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})^{-1}
\]
\[
= (f_{\Gamma}(\rho_{S,\Gamma}^{2}(\gamma_1)h_{\Gamma}(\gamma_1)^{-1},...\rho_{S,\Gamma}^{2}(\gamma_{N})h_{\Gamma}(\gamma_{N})^{-1})^{-1}
\]
\[
= (\alpha_{\Gamma}^{R}(\rho_{S,\Gamma}^{1}))f_{\Gamma})(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N})^{-1})
\]

if $\rho_{S,\Gamma}^{2}(\gamma_i) = \rho_{S,\Gamma}^{2}(\gamma_i)^{-1}$ for all $i = 1,...,N$, $\rho_{S,\Gamma}^{2}$ and $\rho_{S,\Gamma}^{2}$ are maps in $\mathcal{G}_{S,\Gamma}$. Consequently the actions satisfy

\[
I\alpha_{\Gamma}^{L}(\rho_{S,\Gamma}^{1})I \Gamma = \alpha_{\Gamma}^{L}(\rho_{S,\Gamma}^{1})I \Gamma
\]
\[
I\alpha_{\Gamma}^{R}(\rho_{S,\Gamma}^{1})I \Gamma = \alpha_{\Gamma}^{R}(\rho_{S,\Gamma}^{1})I \Gamma
\]

Notice that, if $\rho_{S,\Gamma}^{2}(\gamma_i)^{-1} = \rho_{S,\Gamma}^{2}(\gamma_i)^{-1}$ for all $i = 1,...,N$, $\rho_{S,\Gamma}^{2}$ is satisfied and $\rho_{S,\Gamma}^{2}$ are maps in $\mathcal{G}_{S,\Gamma}$ then

\[
I\alpha_{\Gamma}^{L}(J(\rho_{S,\Gamma}^{1}))I \Gamma = \alpha_{\Gamma}^{R}(\rho_{S,\Gamma}^{1})I \Gamma
\]

yields, whenever $J$ is the map $J : \mathcal{G}_{\tilde{S},\Gamma} \rightarrow \mathcal{G}_{\tilde{S},\Gamma}, J : \rho_{S,\Gamma}(\Gamma) \mapsto \rho_{S,\Gamma^{-1}}(\Gamma)$ where $\tilde{S} := \{S, S^{-1}\}$.

**Lemma 64.** Let $S_1,...,S_N$ form a set $\tilde{S}$, where the set $\tilde{S}$ has the simple surface intersection property for a graph $\Gamma$, let $\mathcal{P}_\Gamma^\mathcal{P}$ be a finite orientation preserved graph system for $\Gamma$.

Then there is an action of $\mathcal{G}_{\tilde{S},\Gamma}$ on $C(\mathcal{A}_\Gamma)$ given by

\[
\alpha_{\Gamma}^{R}(\rho_{S,\Gamma}(\Gamma'))(h_{\Gamma}(\gamma_1),...h_{\Gamma}(\gamma_{N})) := fr(\gamma_{1},...h_{\Gamma}(\gamma_{N})^{-1})
\]
\[
= fr(\gamma_{1},...h_{\Gamma}(\gamma_{N})^{-1})
\]

\[
\gamma := \{\gamma_1,...,\gamma_M\} \ \text{is an element of} \ \mathcal{P}_\Gamma^\mathcal{P} \ \text{for all} \ \rho_{S,\Gamma} \in \mathcal{G}_{\tilde{S},\Gamma}. \ \text{This action is point-norm continuous and automorphic.}
\]

Notice that, in this case of a suitbale surface set $\tilde{S}$ instead of $\mathcal{G}_{\tilde{S},\Gamma}$ one can use $\times_{i=1}^{N} G_{S_i,\Gamma}$ equivalently.
Left and right actions of the flux group

Left and right actions are defined on the same level for some configurations of the surfaces and paths. Therefore recall the maps contained in the set $G_{S,G}$ with left multiplication operation, which decomposes into $\rho^L \times \rho^R$.

**Lemma 65.** Let all paths in $\Gamma$ intersect in vertices of the set $V_\Gamma$ with a surface $S$ such that $\gamma_1, \ldots, \gamma_{1,N-1}$ are ingoing paths and lie above the surface $S$, whereas $\gamma_N$ is a outgoing path lying below w.r.t. the surface orientation of $S$.

Then the action defined by

$$\begin{align*}
(\alpha_{\overline{\rho}^L,1}(\rho^L_{S,\Gamma})f_r)(b_r(\gamma_1), \ldots, b_r(\gamma_N)) := & f_r(b_r(\gamma_1)\rho^L_S(\gamma_1)^{-1}, \ldots, \rho^L_S(\gamma_N)b_r(\gamma_N)) \\
= & f_r(b_r(\gamma_1)g_{S,1}, \ldots, g_{S,N}b_r(\gamma_N))
\end{align*}
$$

(46)

for $\rho^L_S = (\rho^L_S(\gamma_1), \ldots, \rho^L_S(\gamma_N)) = (g_{S,1}, \ldots, g_{S,N})$, $\rho^L_{S,\Gamma} \in \tilde{G}_{S,\Gamma}$ and $f_r \in C(\overline{\mathbb{A}_r})$.

Then the action $\alpha_{\overline{\rho}^L,1}$ of $\tilde{G}_{S,\Gamma}$ on $C(\overline{\mathbb{A}_r})$ is automorphic and point-norm continuous.

In Section 6.1 similar actions build from left and right actions have been presented. For example, consider a set $\{S_i\}_{1 \leq i \leq N} = \tilde{S}$ of surfaces. Furthermore let each path $\gamma_i$ in $\Gamma$ intersects in one vertex of the set $V_\Gamma$ with a surface $S_i$ and there are no other intersections with any other surface. In particular for $i = 1, \ldots, N-1$ each the path $\gamma_i$ is an ingoing path and lies above the surface $S_i$, whereas $\gamma_N$ is an outgoing path lying below w.r.t. the surface orientation of $S_N$.

Then the action of $\tilde{G}_{S,\Gamma}$ on $C(\overline{\mathbb{A}_r})$ is

$$\begin{align*}
(\alpha_{\overline{\rho}^L,1}(\rho^L_{S,\Gamma})f_r)(b_r(\gamma_1), \ldots, b_r(\gamma_N)) := & f_r(b_r(\gamma_1)\rho^L_S(\gamma_1)^{-1}, \ldots, \rho^L_{S,N}(\gamma_N)b_r(\gamma_N)) \\
= & f_r(b_r(\gamma_1)g_{S,1}, \ldots, g_{S,N}b_r(\gamma_N))
\end{align*}
$$

(47)

holds for $\rho^L_{S,\Gamma} = (\rho^L_S(\gamma_1), \ldots, \rho^L_{S,N}(\gamma_N)) = (g_{S,1}, \ldots, g_{S,N})$, $\rho^L_{S,\Gamma} \in \tilde{G}_{S,\Gamma}$ and $f_r \in C(\overline{\mathbb{A}_r})$. Then the action $\alpha_{\overline{\rho}^L,1}$ is an automorphic and point-norm continuous action of $\tilde{G}_{S,\Gamma}$ on $C(\overline{\mathbb{A}_r})$.

Clearly in the same way the actions $\alpha_{\overline{\rho}^L,1}, \alpha_{\rho^L,1}, \alpha_{\rho^L,1}$ and so on are automorphic and point-norm continuous actions of the flux group associated to a suitable surface set on $C(\overline{\mathbb{A}_r})$.

Furthermore the actions defined above can be easily generalised to finite orientation preserved graph systems.

**Problem 3.1:** Observe that there is a problem if the following actions\footnote{Notice that, there is a difference between a left action of a group on a space and a left action of a group on a $C^*$-algebra.} on $\overline{\mathbb{A}_r}$ for a surface $S$ are defined. Assume that, the set $\tilde{G}_{S,\Gamma}$ equipped with a left and right multiplication on the same time. This could be the case if a surface $S$ is considered such that $S$ intersects the paths $\gamma_1, \ldots, \gamma_{1,N-1}$ in the source vertices, hence the paths are outgoing and lie below. Furthermore $S$ intersects the path $\gamma_N$ such that the path lies outgoing and above. Then a left action, which is defined by the set $\tilde{G}_{S,\Gamma}$ with a left multiplication for the paths $\gamma_1, \ldots, \gamma_{1,N-1}$ and a right multiplication for the path $\gamma_N$, is not automorphic. This is verified by the following computation:

$$\begin{align*}
(\alpha_{\overline{\rho}^L,1}(\rho^L_{S,\Gamma})f_r)(b_r(\gamma_1), \ldots, b_r(\gamma_N)) & = f_r(\rho_S(\gamma_1)\rho_S(\gamma_1)b_r(\gamma_1), \ldots, \rho_S(\gamma_N)\rho_S(\gamma_N)b_r(\gamma_N)) \\
& = f_r(g_{S,1}g_{S,1}b_r(\gamma_1), \ldots, g_{S,N}g_{S,N}b_r(\gamma_N)) \\
& \neq (\alpha_{\overline{\rho}^L,1}(\rho^L_{S,\Gamma})(\alpha_{\overline{\rho}^L,1}(\rho^L_{S,\Gamma})f_r(1)))(\gamma_1, \ldots, b_r(\gamma_N))
\end{align*}
$$

(48)

This is the reason why all paths that lie outgoing w.r.t. a surface $S$ are defined by left actions of $\tilde{G}_{S,\Gamma}$ with a left multiplication on $\overline{\mathbb{A}_r}$. Hence this problem is absent by definition of the actions, which are stated before. A left action of $\tilde{G}_{S,\Gamma}$ with a left multiplication on $\overline{\mathbb{A}_r}$ is called the **left action** of $\tilde{G}_{S,\Gamma}$ on $\overline{\mathbb{A}_r}$. Whereas a left action of $\tilde{G}_{S,\Gamma}$ with a right multiplication on $\overline{\mathbb{A}_r}$ is called the **inverse left action** of $\tilde{G}_{S,\Gamma}$ on $\overline{\mathbb{A}_r}$. A right action of $\tilde{G}_{S,\Gamma}$ with a left multiplication on $\overline{\mathbb{A}_r}$ is called the **right action** of $\tilde{G}_{S,\Gamma}$ on $\overline{\mathbb{A}_r}$. Whereas a right
action of $\tilde{G}_{S,T}$ with a right multiplication on $A_T$ is called the *inverse right action* of $G_{S,T}$ on $A_T$. Hence the action defined in equation (48) corresponds to a left action of $G_{S,T}$ on $A_T$ for the paths $\gamma_1, ..., \gamma_{N-1}$ and a left inverse action of $G_{S,T}$ on $A_T$ for the path $\gamma_N$.

Recognize that, there is no homeomorphism $H$ on $A_T$ corresponding to a group action of $\tilde{G}_{S,T}$ with a left and right multiplication on $A_T$ defined in (48), i.o.w.

$$H(gs\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N)) = (gs\bar{g}h_T(\gamma_1), ..., \bar{g}^{-1}g^{-1}h_T(\gamma_N))$$

(49)

$$H(gs)(H(\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N))) = (gs\bar{g}h_T(\gamma_1), ..., g^{-1}g^{-1}h_T(\gamma_N))$$

In the same way the problem occurs if there is a inverse right action of $\tilde{G}_{S,T}$ on $A_T$ for the paths $\gamma_1, ..., \gamma_{N-1}$ intersecting a surface $S$ such that they are ingoing and lie above, and a right action of $\tilde{G}_{S,T}$ on $A_T$ for the path $\gamma_N$ intersecting $S$ such that the path is ingoing and lies below, is studied. Then it is true that

$$H(\bar{g}s\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N)) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., \bar{g}s^{-1}g^{-1}h_T(\gamma_N))$$

(50)

$$H(\bar{g}^{-1}g^{-1})(H(\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N))) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., g^{-1}g^{-1}h_T(\gamma_N))$$

yields.

There is also a problem if there is a left action of $G_{S,T}$ on $A_T$ for the paths $\gamma_1, ..., \gamma_{N-1}$, which intersect a surface $S$ such that they are outgoing and lie above, and a inverse right group action on $A_T$ for the path $\gamma_N$ lying ingoing and below, is considered. Then derive

$$H(\bar{g}^{-1}g^{-1})(\bar{g}s\bar{g}(h_T(\gamma_1), ..., h_T(\gamma_N))) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., \bar{g}s^{-1}g^{-1}h_T(\gamma_N))$$

(51)

$$H(\bar{g}^{-1}g^{-1})(H(\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N))) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., g^{-1}g^{-1}h_T(\gamma_N))$$

Finally there is also a problem if there is a left inverse action of $G_{S,T}$ on $A_T$ for the paths $\gamma_1, ..., \gamma_{N-1}$ lying outgoing and below, and a right action of $G_{S,T}$ on $A_T$ for the path $\gamma_N$ lying ingoing and above is considered. In this case,

$$H(\bar{g}s\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N)) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., \bar{g}s^{-1}g^{-1}h_T(\gamma_N))$$

(52)

$$H(\bar{g}^{-1}g^{-1})(H(\bar{g})(h_T(\gamma_1), ..., h_T(\gamma_N))) = (\bar{g}s\bar{g}h_T(\gamma_1), ..., g^{-1}g^{-1}h_T(\gamma_N))$$

holds. All these problems are excluded by the definition of the actions. Since for example there is an action defined by $\tilde{α}_L^{1,1}$ instead of the action $\tilde{α}_L^{1,1}$ given in (52). Respectively there is an action defined by $\tilde{α}_L^{1,1}$ instead of the action $\tilde{α}_L^{1,1}$ given in (51).

Non-standard identification of the configuration space

At the beginning of this considerations one assumes that the subgraphs of the finite graph system are identified naturally. If instead the non-standard identification of the configuration space is used, the following observation is made.
If the set $\text{Hom}(\mathcal{P}_\Gamma, G^{(\Gamma)})$ of holonomy maps for a finite graph system $\mathcal{P}_\Gamma$ is considered and the configuration space is identified with $G^N$ by the non-standard way. Then there exists a situation such that $h \Gamma(\gamma \circ \gamma') = h \Gamma(\gamma)h \Gamma(\gamma')$ for arbitrary $(\gamma, \gamma') \in \mathcal{P}_\Gamma \Sigma^{(2)}$ holds. Despite the property there is an action of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ derivable.

First observe that, for all paths of $\Gamma$ intersecting $S$ in their source vertices and all paths lie below one concludes that

$$
(\alpha \Gamma_L(\rho_S,\Gamma)) \circ \alpha \Gamma_L(\rho_S,\Gamma^{-1}))_{\Gamma^t}(h \Gamma(\gamma_1)h \Gamma(\gamma_1^{-1}), \ldots, h \Gamma(\gamma_N)h \Gamma(\gamma_N^{-1}))
$$

$$
= f_{\Gamma}(h \Gamma(\gamma_1)h \Gamma(\gamma_1^{-1}), \ldots, h \Gamma(\gamma_N)h \Gamma(\gamma_N^{-1}))
$$

$$
= f_{\Gamma}(g_S h \Gamma(\gamma_1)h \Gamma(\gamma_1^{-1})g_S^{-1}, \ldots, g_S h \Gamma(\gamma_N)h \Gamma(\gamma_N^{-1})g_S^{-1})
$$

$$
= f_{\Gamma}(e_G, \ldots, e_G)
$$

holds whenever $\rho_S, \Gamma \in G_{S,\Gamma}$, $\rho_S, \Gamma^{-1} \in G_{S,\Gamma^{-1}}$ and $f_{\Gamma} \in C(\mathcal{A}_\Gamma)$, where $\Gamma^{-1}$ refers to the set $\{\gamma_1^{-1}, \ldots, \gamma_N^{-1}\}$ and if it is assumed that $\rho_S(\gamma_1) = \rho_S(\gamma_1^{-1}) = g_S$ for all $\gamma_i \in \Gamma$ is fulfilled. In this case there is an action $\alpha \Gamma_L$ of $\tilde{G}_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ defined by

$$
(\alpha \Gamma_L(\rho_S,\Gamma))_{\Gamma^t} = (\alpha \Gamma_L(\rho_{S,\Gamma^{-1}}))_{\Gamma^t})_{\Gamma^t} = f_{\Gamma}(h \Gamma(\gamma_1)h \Gamma(\gamma_1^{-1}), \ldots, h \Gamma(\gamma_N)h \Gamma(\gamma_N^{-1}))
$$

Moreover for a graph $\Gamma$ consisting of two paths $\gamma$ and $\gamma'$ such that $\gamma$ and $S$ intersect in the target vertex $t(\gamma)$ of $\gamma$ and the path lies above and respectively $\gamma'$ and $S$ intersect in $s(\gamma')$ such that $\gamma'$ lies above and is outgoing. Then $(\gamma, \gamma') \in P_{\mathcal{A}}^{(2)}$. Set $\Gamma' = \{\gamma, \gamma'\}$, $\Gamma'' := \{\gamma \circ \gamma'\}$ and assume $S \cap \{\gamma, \gamma'\} = \{t(\gamma)\}$. Then there is an action of $G_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$ given by

$$
(\alpha \Gamma_L(\rho_S,\Gamma))_{\Gamma^t} = f_{\Gamma}(h \Gamma(\gamma), h \Gamma(\gamma'))
$$

$$
= f_{\Gamma}(h \Gamma(\gamma)g_S^{-1}, g_S^{-1} h \Gamma(\gamma'))
$$

where it is assumed that $\rho_S(\gamma) = \rho_S(\gamma') = g_S$ for all $(\gamma, \gamma') \in P_{\mathcal{A}}^{(2)}$. For the definition of an action of $\tilde{G}_{S,\Gamma}$ on $C(\mathcal{A}_\Gamma)$, whenever the configuration space is identified with $G^N$ in the non-standard way, it is necessary to define a the following map.

Let $D_S : C(\mathcal{A}_\Gamma) \rightarrow C(\mathcal{A}_\Gamma)$ be a map such that

$$
(D_S f_{\Gamma})(h \Gamma(\gamma), h h \Gamma(\gamma')) = f_{\Gamma}(h \Gamma(\gamma)g h h \Gamma(\gamma'))
$$

whenever $g, h \in G$ and if $(\gamma, \gamma') \in P_{\mathcal{A}}^{(2)}$ and $S \cap \{\gamma, \gamma'\} = \{t(\gamma)\}$. There is an ambiguity in the definition of the inverse of $D_S$, since it is possible that

$$
(D_S^{-1} f_{\Gamma})(h \Gamma(\gamma)g h h \Gamma(\gamma')) = f_{\Gamma}(h \Gamma(\gamma), g h h \Gamma(\gamma'))
$$

for $g \in G$

or

$$
(D_S^{-1} f_{\Gamma})(h \Gamma(\gamma)g h h \Gamma(\gamma')) = f_{\Gamma}(h \Gamma(\gamma), g h h \Gamma(\gamma'))
$$

or

$$
(D_S^{-1} f_{\Gamma})(h \Gamma(\gamma)g h h \Gamma(\gamma')) = f_{\Gamma}(h \Gamma(\gamma), g h h \Gamma(\gamma'))
$$

holds whenever $g, h \in G$. Recall that, $h \Gamma(\gamma) = h$ is an element of $G$. Let $g \in Z(G)$ and $h \Gamma(\gamma) \neq h \Gamma(\gamma')$. Then there always exists a groupoid morphism $h \Gamma$ such that for a pair $(\gamma, \gamma') \in P_{\mathcal{A}}^{(2)}$ it is true that, $h \Gamma(\gamma \circ \gamma') = h \Gamma(\gamma)h \Gamma(\gamma')$ and $h \Gamma(\gamma) = h \Gamma(\gamma')$ yields. Then

$$
(D_S^{-1} f_{\Gamma})(h \Gamma(\gamma)g h h \Gamma(\gamma')) = f_{\Gamma}(h \Gamma(\gamma)g h h \Gamma(\gamma'))
$$

is well-defined. The reason is given by the following property. Since for example for $h \Gamma(\gamma) = h$ and $g \in Z(G)$, it is true that $h g(h g) = h g(h g) = (h g)^2$.

More generally consider all $g \in G$ such that there exists a $k \in G$ and $h \Gamma(\gamma)g g h \Gamma(\gamma') = k^2$ for all $h \Gamma(\gamma), h \Gamma(\gamma') \in \mathcal{A}_\Gamma$ for a fixed pair $(\gamma, \gamma') \in P_{\mathcal{A}}^{(2)}$. 

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Definition 66. Let $S$ be a surface and $\Gamma$ be a graph, which consists of two paths $\gamma$ and $\gamma'$ such that $\gamma$ and $S$ intersect in the target vertex $t(\gamma)$ of $\gamma$ and lies above and respectively the path $\gamma'$ and $S$ intersect in $s(\gamma')$ such that the path $\gamma'$ lies above and outgoing.

Then define the action of $\mathcal{Z}_{S,\Gamma}$ on $C(\tilde{A}_\Gamma)$ by

\[
(D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\Gamma')) := f_r(\mathfrak{h}_r(\gamma)\rho_S^{-1}(\gamma')\mathfrak{h}_r(\gamma))
\]

whenever $\rho_{S,\Gamma}(\Gamma) \in \mathcal{Z}_{S,\Gamma}$ and it is assumed that

\[
\rho_S^L(\gamma) = g_S^{-1} \quad \text{for} \quad \rho_S^L, \rho_S^R \in \mathcal{Z}(G_{S,\gamma})_{S,\Gamma},
\]

\[
\mathfrak{h}_r(\gamma) = \mathfrak{h}_r(\gamma') \quad \text{for} \quad \mathfrak{h}_r \in \text{Hom}(\mathcal{P}_r, G_{r}^{(\Gamma)})
\]

holds for the pair $(\gamma, \gamma') \in \mathcal{P}_r\Sigma^{(2)}$ such that $\Gamma' := \{\gamma \circ \gamma'\}$. The action is redefined by

\[
(D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\Gamma')) := (\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma'))f_r)(\mathfrak{h}_r(\Gamma'))
\]

Notice that, the action is computed in the following way

\[
(D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\Gamma')) = (D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\gamma))
\]

\[
= (D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))f_r)(\mathfrak{h}_r(\gamma), \mathfrak{h}_r(\gamma'))
\]

\[
= (D_Sf_r)(\mathfrak{h}_r(\gamma)\rho_S^{-1}(\gamma'), \rho_S^L(\gamma)\mathfrak{h}_r(\gamma'))
\]

\[
= (D_Sf_r)(\mathfrak{h}_r(\gamma)g_S^{-1}g_S^{-1}\mathfrak{h}_r(\gamma'))
\]

whenever $\rho_{S,\Gamma}(\Gamma) \in \mathcal{Z}_{S,\Gamma}$ and it is assumed that $\rho_S^L(\gamma) = g_S^{-1}$ holds for the pair $(\gamma, \gamma') \in \mathcal{P}_r\Sigma^{(2)}$ such that $\Gamma' := \{\gamma \circ \gamma'\}$.

Finally derive that, for this action it is true that

\[
(\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma'))\rho_{S,\Gamma}(\Gamma'))f_r)(\mathfrak{h}_r(\Gamma')) = (\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma'))\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma'))f_r)(\mathfrak{h}_r(\Gamma'))
\]

holds whenever $\rho_{S,\Gamma}, \rho_{S,\Gamma} \in \mathcal{Z}_{S,\Gamma}$. Hence the action $\alpha_{\tilde{L}}^{-1}$ is automorphic. One can show that the action $\alpha_{\tilde{L}}^{-1}$ is point-norm continuous.

If the graph is changed only slightly, then recognize the following.

Definition 67. Let $S$ be a surface and $\Gamma'$ be a graph, which is given by the composed path of a path $\gamma$ and $\gamma'$ such that $\gamma$ and $S$ intersects in the target vertex $t(\gamma)$ of $\gamma$ and lies below and respectively $\gamma'$ and $S$ intersect in $s(\gamma')$ such that $\gamma'$ lies above and outgoing. Then $(\gamma, \gamma') \in \mathcal{P}_r\Sigma$.

The action of $\tilde{G}_{S,\Gamma}$ on $C(\tilde{A}_\Gamma)$ in this case is presented by

\[
(D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\Gamma')) := f_r(\mathfrak{h}_r(\gamma)\rho_S^{-1}(\gamma')\mathfrak{h}_r(\gamma))
\]

\[
= f_r(\mathfrak{h}_r(\gamma)g_S^{-1}g_S^{-1}\mathfrak{h}_r(\gamma'))
\]

whenever $f_r \in C(\tilde{A}_\Gamma)$, $\rho_{S,\Gamma} \in G_{S,\Gamma}$ and it is assumed that

\[
\rho_S^L(\gamma) = g_S^{-1} \quad \text{for} \quad \rho_S^L, \rho_S^R \in \mathcal{Z}(G_{S,\gamma})_{S,\Gamma},
\]

\[
\mathfrak{h}_r(\gamma) = \mathfrak{h}_r(\gamma') \quad \text{for} \quad \mathfrak{h}_r \in \text{Hom}(\mathcal{P}_r, G_{r}^{(\Gamma)})
\]

for the pair $(\gamma, \gamma') \in \mathcal{P}_r\Sigma^{(2)}$ such that $\Gamma' := \{\gamma \circ \gamma'\}$. Set

\[
(D\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma))D_S^{-1}f_r)(\mathfrak{h}_r(\Gamma')) := (\alpha_{\tilde{L}}^{-1}(\rho_{S,\Gamma}(\Gamma'))f_r)(\mathfrak{h}_r(\Gamma'))
\]

yields.
Since in this case
\[(D^{-1}_S f_R)(h_R(\gamma)gg^{-1}h_R(\gamma')) = f_R(h_R(\gamma), h_R(\gamma')) \quad \text{for} \quad g \in G\]
is well-defined. Clearly this action can be restricted to those maps that are elements of \(Z_{S,R}\). The action \(\alpha^{R,1}_{L,R}\) is automorphic and point-norm continuous.

Notice that, there are two actions \(\alpha^{R,1}_{L,R}\) and \(\alpha^{L,1}_{R,R}\) of \(Z_{S,R}\) are restricted by the requirement \((56)\) or \((58)\). The actions depend on the orientation of both paths of the pair \((\gamma, \gamma') \in P_{T}\Sigma^{(2)}\) w.r.t. the surface orientation of \(S\). Clearly the actions can be generalised to graphs containing a set of paths \{\((\gamma, \gamma')\) of paths in \(P_{T}\Sigma^{(2)}\).

In [13] Section 6.1] actions of fluxes, which are constructed from admissible maps associated to surfaces, have been introduced. These actions on the analytic holonomy \(C^*\)-algebra restricted to a finite graph system are more complicated. A summary of the results is given by the following. If the non-standard identification of the configuration space is used, then the holonomy maps are defined on arbitrary elements of the finite graph groupoid. In particular, a graph \(\Gamma'\), which is a subgraph of \(\Gamma\) and contains a path \(\gamma \circ \gamma'\). Then consider a surface \(S\) that intersects only the paths \(\gamma\) and \(\gamma'\) of the graph \(\Gamma\) in the vertex \(t(\gamma)\). Then different actions of an element \(g_{S,R}(\Gamma')\) on a function in \(C(\mathcal{A}_R)\) are considered. There are two different kinds of actions. One refers to a translation of the center of the flux group \(Z_{S,R}\). The other is related to a translation of the fluxes related to admissible maps. Moreover each actions depend on the orientation of the paths \(\gamma\) and \(\gamma'\) w.r.t. the surface \(S\).

The set of actions of \(\mathcal{G}_{S,R}\) on \(C(\mathcal{A}_R)\)

Assume that, the configuration space \(\mathcal{A}_R\) of generalised connections is identified in the natural way with \(C^{[1]}\). Certainly there are a lot of different actions on \(C(\mathcal{A}_R)\) corresponding to different surfaces and graph configurations, which are build from left and right actions of the group \(\mathcal{G}_{S,R}\) on the \(C^*\)-algebra. In general there is an exceptional set of all well-defined point-norm continuous automorphic actions.

**Definition 68.** Denote the set of all point-norm continuous automorphic actions of \(\mathcal{G}_{S,R}\) on \(C(\mathcal{A}_R)\) by \(\text{Act}(\mathcal{G}_{S,R}, C(\mathcal{A}_R))\) for every suitable set \(\tilde{S}\) of surfaces, a graph \(\Gamma\) and a finite graph system \(P_{T}\).

Let \(\Phi_M\) be the multiplication representation of \(C(\mathcal{A}_R)\) on \(\mathcal{H}_R\). For all automorphic and point-norm continuous actions presented above there are unitary representations \(U\) of \(\mathcal{G}_{S,R}\) or \(\mathcal{G}_{S,R}\) on the Hilbert space \(\mathcal{H}_R\), which satisfy for a suitable surface \(S\) or surface set \(\tilde{S}\) and graph configuration one of the following or some equivalent Weyl relations

\[U_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))\Phi_M(f_R)U_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))^{-1} = \Phi_M(\alpha^p_L(\rho_{S_1,\Gamma}(\Gamma), ..., \rho_{S_p,\Gamma}(\Gamma))f_R)\]

\[\forall p \in \mathbb{N},\]

\[U^k_L(\rho_{S_1,\Gamma}^k)\Phi_M(f_R)U^k_L(\rho_{S_1,\Gamma}^k)^{-1} = \Phi_M(\alpha^k_L(\rho_{S_1,\Gamma}^k)f_R) \quad \forall k \in \mathbb{N},\]

\[U^k_R(\rho_{S_1,\Gamma}^k)\Phi_M(f_R)U^k_R(\rho_{S_1,\Gamma}^k)^{-1} = \Phi_M(\alpha^k_R(\rho_{S_1,\Gamma}^k)f_R) \quad \forall k \in \mathbb{N},\]

\[(59)\]

\[U^k_{p,k}(\rho_{S_1,\Gamma}^k)\Phi_M(f_R)U^k_{p,k}(\rho_{S_1,\Gamma}^k)^{-1} = \Phi_M(\alpha^k_{p,k}(\rho_{S_1,\Gamma}^k)f_R) \quad \text{for} \quad p \leq k \leq N - 1 \quad \text{or} \]

\[U^k_{L,R}(\rho_{S_1,\Gamma}^k)\Phi_M(f_R)U^k_{L,R}(\rho_{S_1,\Gamma}^k)^{-1}U^k_{L,R}(\rho_{S_1,\Gamma}^k)^{-1} = \Phi_M(\alpha^k_{L,R}(\rho_{S_1,\Gamma}^k)f_R) \quad \forall k \in \mathbb{N},\]

and so on for all \(f_R \in C(\mathcal{A}_R), \rho_{S_1,\Gamma}^k \in \mathcal{G}_{S,R}, \rho_{S_1,\Gamma}^k \in \mathcal{G}_{S,R}\) and \(\rho_{S_1,\Gamma} \in \mathcal{G}_{S,R}\) for \(i = 1, ..., k\) where \(\tilde{S} := \{S_i\}_{1 \leq i \leq p}\) is suitable.

Observe that, for each unitary \(U\) defined above the pair \((U, \Phi_M)\) consisting of \(U \in \text{Rep}(\mathcal{G}_{S,R}, \mathcal{K}(\mathcal{H}_R))\) and \(\Phi_M \in \text{Mor}(\mathcal{A}_R, \mathcal{L}(\mathcal{H}_R))\) is a covariant pair of the dynamical \(C^*\)-system \((C(\mathcal{A}_R), \mathcal{G}_{S,R}, \alpha)\) for an action \(\alpha \in \text{Act}(\mathcal{G}_{S,R}, C(\mathcal{A}_R))\).

**Problem 3.2.** There are no unitary operators, which satisfy the Weyl relations for the actions

\[\alpha^1_{L,R}(\rho_{S,\Gamma}), \quad \alpha^1_{L,R}(\rho_{S,\Gamma}),...\]
which are presented in problem 3.1.0.1 and which are not automorphic actions of \( G_{S,\Gamma} \) on \( C(\tilde{\mathcal{A}}_\Gamma) \). This is true, since there are unitary operators such that

\[
U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})(U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))^* = U_{\mathcal{L},\mathcal{L}}^1(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1),...,\tilde{\rho}_S(\gamma_1)\rho_S(\gamma_1))U_{\mathcal{L},\mathcal{L}}^1(\tilde{\rho}_S(\gamma_1)^{-1}\rho_S(\gamma_1)^{-1},...,\rho_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)^{-1})
\]

\[
= U_{\mathcal{L},\mathcal{L}}^1(\rho_S(\gamma_1)\tilde{\rho}_S(\gamma_1)^{-1}\rho_S(\gamma_1)^{-1},...,\rho_S(\gamma_1)^{-1}\tilde{\rho}_S(\gamma_1)^{-1})^* = I
\]

holds. But \( U_{\mathcal{L},\mathcal{L}}^1(\tilde{G}_{S,\Gamma}) \) does not form a group. Consequently

\[
U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma})U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma})\Phi_M(f_\Gamma)(U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma})U_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))^* \neq \Phi_M(a_{\mathcal{L},\mathcal{L}}^1(\rho^1_{S,\Gamma}\tilde{\rho}^1_{S,\Gamma}))
\]

holds.

**Definition 69.** Let \( \tilde{S} \) be an arbitrary set of surfaces. The Hilbert space \( \mathcal{H}_\Gamma \) is identified with \( L^2(\tilde{\mathcal{A}}_\Gamma, \mu_\Gamma) \).

Each unitary \( U(\rho_S(\Gamma)) \) for \( U \in \text{Rep}(\tilde{G}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma)) \) and \( \rho_S(\Gamma) \in \tilde{G}_{S,\Gamma} \) is called a **Weyl element**. The set of all linearly independent Weyl elements is denoted by \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \). The vector space of all finite complex linear combinations of Weyl elements, which are unitary operators satisfying the Weyl relations, is generated by the constant function \( \Phi_M(\tilde{G}_{S,\Gamma}) \).

Each unitary \( U(\rho_S(\Gamma)) \) for \( U \in \text{Rep}(\tilde{Z}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_\Gamma)) \) is called the **commutative Weyl element**. The set of all linearly independent commutative Weyl elements is denoted by \( \mathcal{W}(\tilde{Z}_{S,\Gamma}) \). The vector space \( \mathcal{W}(\tilde{Z}_{S,\Gamma}) \) is given by the set of all finite complex linear combinations of commutative Weyl elements such that the linear combinations are unitary operators and satisfy the Weyl relations.

In general finite linear combinations of unitary elements (elements such that \( UU^* = U^*U = I \)) form a unital *-algebra. Otherwise, for each fixed suitable surface set and \( \tilde{G}_{S,\Gamma} \), the set \( U(\tilde{G}_{S,\Gamma}) \) form a group with the usual left multiplication operation. The group \( U(\tilde{G}_{S,\Gamma}) \) is a subalgebra of the group \( U(\mathcal{H}_\Gamma) \) of unitaries on a Hilbert space \( \mathcal{H}_\Gamma \). Notice that, the Weyl algebra associated to surfaces and a graph, which is introduced in the next subsection, is generated by the constant function \( 1 \), the elements of the analytic holonomy \( C^* \)-algebra \( C(\tilde{\mathcal{A}}_\Gamma) \) and the unitaries associated to surfaces. For example, an element of the Weyl algebra is of the form

\[
\sum_{i=1}^L a_\Gamma(\rho^i_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K} \sum_{i=1}^M f^k_\Gamma(b_\Gamma(\Gamma))U(\rho^i_{S,\Gamma}(\Gamma)) + \sum_{k=1}^{K} \sum_{i=1}^M U(\rho^i_{S,\Gamma}(\Gamma))f^k_\Gamma(b_\Gamma(\Gamma))U(\rho^i_{S,\Gamma}(\Gamma))^* + \sum_{p=1}^P f^p_\Gamma(\theta_\Gamma(\Gamma))
\]

whenever \( f^k_\Gamma, f^p_\Gamma \in \mathcal{C}(\tilde{\mathcal{A}}_\Gamma) \) and \( \rho^i_{S,\Gamma}, \tilde{\rho}^i_{S,\Gamma} \in \tilde{G}_{S,\Gamma} \). Notice that, for a compact group \( G \) the analytic holonomy \( C^* \)-algebra is unital.

In this context a reformulation of the Weyl \( C^* \)-algebra is given as follows.

**Lemma 70.** Let \( \mathcal{H}_\Gamma \) be the Hilbert space \( L^2(\tilde{\mathcal{A}}_\Gamma, \mu_\Gamma) \) with norm \( \| \cdot \|_2 \).

With the involution * and the natural product of unitaries the vector space \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \) is a unital *-algebra, where \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \) stands for the *-algebra of Weyl elements.

The *-algebra \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \) of Weyl elements completed w.r.t. the strong operator norm is a \( C^* \)-algebra. Denote this algebra by \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \).

Notice that, \( \mathcal{W}(\tilde{G}_{S,\Gamma}) \) is a \( C^* \)-subalgebra of the \( C^* \)-algebra \( \mathcal{L}(\mathcal{H}_\Gamma) \) of bounded operators on the Hilbert space \( \mathcal{H}_\Gamma \), which is equal to \( L^2(\tilde{\mathcal{A}}_\Gamma, \mu_\Gamma) \).

Recall that, \( G \) is assumed to be a compact group and \( \tilde{\mathcal{A}}_\Gamma \) is identified in the natural way with \( G^{[\Gamma]} \). Then remember the action \( \beta^{[\Gamma]}_L \), which defines a \( C^* \)-dynamical system \( (\mathcal{A}_\Gamma, \tilde{G}_{S,\Gamma}, \beta^{[\Gamma]}_L) \) for a fixed graph and a suitable surface set \( \tilde{S} \).
Proposition 71. Let \( \tilde{S}' \) and \( \tilde{S} \) be two suitable surface sets, let \( \Gamma \) be a graph and let \( \mathcal{P}_\Gamma \) be the finite graph system associated to \( \Gamma \).

Furthermore let \((U_{\mathcal{R}}^{1}, \Phi_M)\) be a covariant pair of a dynamical \( C^* \)-system
\((C(\mathcal{A}_\Gamma), \mathcal{G}_{\mathcal{S},\Gamma}, \beta_{\mathcal{R}}^{1})\) associated to the surface set \( \tilde{S}' \) and the orientation-preserved finite graph system \( \mathcal{P}^\mathcal{R}_\Gamma \) associated to the graph \( \Gamma \).

Denote a general covariant pair by \((U, \Phi_M)\) of a dynamical \( C^* \)-system \((C(\mathcal{A}_\Gamma), \mathcal{G}_{\mathcal{S},\Gamma}, \alpha)\) for an action \( \alpha \in \text{Act}(\mathcal{G}_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma)) \) for the surface set \( \tilde{S} \) and the finite graph system \( \mathcal{P}_\Gamma \). The set \( S \) of all suitable surface sets for \( \Gamma \) contains all surface sets such that there exists an action in \( \text{Act}(\mathcal{G}_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma)) \).

Then there exists a GNS-triple \((\mathcal{H}_\Gamma, \Phi_M, \Omega_\Gamma)\) where \( \Omega_\Gamma \) is the cyclic vector for \( \Phi_M \) on \( \mathcal{H}_\Gamma \). Moreover the associated GNS-state \( \omega_M^\mathcal{R} \) on \( C(\mathcal{A}_\Gamma) \) is \( \mathcal{G}_{\mathcal{S},\Gamma} \)- and \( \mathcal{G}_{\mathcal{S}',\Gamma} \)-invariant, i.e.

\[
\omega_M^\mathcal{R}(\beta_{\mathcal{R}}^{1}(\rho_{\mathcal{S}',\Gamma})(f)) = \omega_M^\mathcal{R}(f) := (\Omega_\Gamma, \Phi_M(f)\Omega_\Gamma)_\Gamma \\
= \omega_M^\mathcal{R}(\alpha(\rho_{\mathcal{S},\Gamma}(\Gamma))(f))
\]

for \( \alpha, \beta_{\mathcal{R}}^{1} \in \text{Act}(\mathcal{G}_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma)), \rho_{\mathcal{S},\Gamma}, \rho_{\mathcal{S}',\Gamma}(\Gamma) \in \mathcal{G}_{\mathcal{S},\Gamma} \) and \( f \in C(\mathcal{A}_\Gamma) \).

Moreover the set \( \mathcal{M} := \Phi_M(C(\mathcal{A}_\Gamma)) \cup \{U(\mathcal{G}_{\mathcal{S},\Gamma}) : U \in \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{A}_\Gamma)\} \) is irreducible on \( \mathcal{H}_\Gamma \).

Remark that, the state \( \omega_M^\mathcal{R} \) on \( C(\mathcal{A}_\Gamma) \) is \( \mathcal{G}_{\mathcal{S},\Gamma} \)-invariant for many different suitable surface sets. The surface sets are required to intersect the graph \( \Gamma \) only in vertices and hence such that there exists an action in \( \text{Act}(\mathcal{G}_{\mathcal{S},\Gamma}, C(\mathcal{A}_\Gamma)) \).

Proof: The GNS-triple is constructed on the Hilbert space \( \mathcal{H}_\Gamma \), which is given by \( L^2(\mathcal{A}_\Gamma, \mu_\Gamma) \). The unitaries \( U \) of the set \( \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma)) \) for every suitable surface set \( \tilde{S} \) in \( S \) and the representation \( \Phi_M \) are given on the same Hilbert space \( \mathcal{H}_\Gamma \). Consequently a state \( \omega_M^\mathcal{R} \) exists.

The crucial property is the irreducibility of the set \( \Phi_M(C(\mathcal{A}_\Gamma)) \cup \{U(\mathcal{G}_{\mathcal{S},\Gamma}) : U \in \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))\} \).
Notice that, \( \mathcal{M}' = \Phi_M(C(\mathcal{A}_\Gamma))' \cap \{U(\mathcal{G}_{\mathcal{S},\Gamma}) : U \in \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma))\}' \). First notice that \( \Phi_M(C(\mathcal{A}_\Gamma)) \subset \Phi_M(C(\mathcal{A}_\Gamma))' \). Then it is true that \( U_L^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma}) \subset U_L^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma})' \) and \( U_R^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma}) \subset U_R^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma})' \) whenever \( U_L^\mathcal{E}, U_R^\mathcal{E} \in \text{Rep}(\mathcal{G}_{\mathcal{S},\Gamma}, \mathcal{L}(\mathcal{H}_\Gamma)) \) for \( 1 \leq k \leq N - 1 \). Clearly \( U_R^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma}) \notin U_R^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma})' \) is satisfied. Then observe that, 
\( U_L^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma})' \cap U_R^\mathcal{E}(\mathcal{G}_{\mathcal{S},\Gamma})' = \{\lambda \cdot 1 : \lambda \in \mathbb{R}\} \) yields. Hence \( \mathcal{M}' \) is given by \( \{\lambda \cdot 1 : \lambda \in \mathbb{R}\} \).

The proposition 71 is reformulated for the non-standard identification of the configuration space \( \mathcal{A}_\Gamma \) if \( \mathcal{G}_{\mathcal{S},\Gamma} \) is replaced by \( \mathcal{Z}_{\mathcal{S},\Gamma} \).

Corollary 72. Let \( \mathcal{P}_\Gamma \) be a finite graph system associated to the graph \( \Gamma \). Let \( \mathcal{A}_\Gamma \) be the set of generalised connections, which is identified in the natural way with \( \mathcal{G}^N \).

There is a unique measure on \( \mathcal{A}_\Gamma \) given by the Haar measure \( \mu_\Gamma \) on the product \( \mathcal{G}^N \) of a compact group \( \mathcal{G} \).

If \( \mathcal{G}_{\mathcal{S},\Gamma} \) is identified with \( \mathcal{G}^N \), then there is a unique state \( \omega_M^\mathcal{R} \) on \( C(\mathcal{A}_\Gamma) \), which is \( \mathcal{G}^N \)-invariant and which is given by

\[
\omega_M^\mathcal{R}(f) = \int_\mathcal{G}^N d\mu_N(h_\Gamma(\Gamma))f_\Gamma(h_\Gamma(\Gamma)) = \int_\mathcal{G}^N d\mu_N(h_\Gamma(\Gamma))f_\Gamma(R(g)(h_\Gamma(\Gamma))) \\
= \int_\mathcal{G}^N d\mu_N(h_\Gamma(\Gamma))f_\Gamma(L(g)(h_\Gamma(\Gamma)))
\]

for all \( f_\Gamma \in C(\mathcal{A}_\Gamma) \) and \( g \in \mathcal{G}^N \).

The same result is obtained for the non-standard identification of the finite graph system and the configuration space.

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Remark 73. Let \( \mathcal{P}_G \) be a finite graph system associated to the graph \( \Gamma \)'s. Let \( \tilde{A}_G \) be the space of generalised connections identified in the natural way with \( G^{[\Gamma]} \) and \( G \) be compact group.

Let \( f \) be a function in the convolution *-algebra \( C(\tilde{A}_G) \), which is given by the algebra \( C_c(\tilde{A}_G) \) of compactly supported functions on \( \tilde{A}_G \) equipped with the convolution as the multiplication operation. For example for two continuous compactly supported function \( f, k \) on \( \tilde{A}_G \) the convolution product is illustrated by

\[
(f * k)(h_G(\gamma), h_G(\gamma')) = \int_{\tilde{A}_G} d\mu_G(h_G(\Gamma')) f(h_G(\gamma)) h_G(\gamma)^{-1} h_G(\gamma') h_G(\gamma')^{-1} k(h_G(\gamma), h_G(\gamma'))
\]

for \( \Gamma := \{ \gamma, \gamma' \} \). The involution is given by

\[
f(h_G(\Gamma'))^* = f(h_G(\gamma), h_G(\gamma'))^* := f(h_G(\gamma)^{-1}, h_G(\gamma')^{-1})
\]

Notice that, \( h_G(\Gamma) \) and \( h_G(\Gamma') \) are elements of \( G^N \) and hence the convolution *-algebra \( C(\tilde{A}_G) \) is identified with \( C(G^N) \).

Then there exists a state on \( C(\tilde{A}_G) \) given by

\[
\omega^\Gamma_{M,f}(f_G) = \int_{G^N} d\mu_G(h_G(\Gamma')) f_G(h_G(\Gamma')) f(h_G(\Gamma'))
\]

where \( f_G \in C(\tilde{A}_G) \) and \( f \in C(\tilde{A}_G) \). Derive from

\[
\omega^\Gamma_{M,f}(\alpha(k)(f_G)) = \int_{G^N} d\mu_G(h_G(\Gamma')) f_G(R(k^{-1})h_G(\Gamma')) f(h_G(\Gamma'))
\]

\[
= \int_{G^N} d\mu_G(h_G(\Gamma')) f_G(h_G(\Gamma')) f(R(k)h_G(\Gamma'))
\]

\[
= \int_{G^N} d\mu_G(h_G(\Gamma')) f_G(h_G(\Gamma')) f(h_G(\Gamma'))
\]

that \( \omega^\Gamma_{M,f} \) is \( G^N \)-invariant iff

\[
f(R(k)h_G(\Gamma')) = f(h_G(\Gamma'))
\]

for any \( k \in G^N \). If \( C(\tilde{G}_{S,G}) \) is identified with \( C(G^N) \), then for \( f_S \in C(G^N) \) and \( f_S(gk^{-1}) = f_S(g) \) for all \( k \in G^N \) there is another state on \( C(G^N) \) defined by

\[
\omega^\Gamma_{M,f_S}(f_G) = \int_{G^N} d\mu_G(g) f_G(g) f_S(g)
\]

whenever \( f_G \in C(G^N) \). But both states will be not finite path-and graph-diffeomorphism invariant. This is shown in problem 3.2.1. Clearly the states \( \omega^\Gamma_{M,f} \) on \( C(\tilde{A}_G) \) are not invariant under all actions \( \alpha \in \text{Act}(\tilde{G}_{S,G}, C(\tilde{A}_G)) \).

Notice the function \( f \) is also an element of \( C(\tilde{G}_{S,G}) \), but in this case the state \( \omega^\Gamma_{M,f} \) is obviously not finite graph-diffeomorphism invariant.

Proof of corollary 72

After the natural identification of \( \tilde{A}_G \) with \( G^N \) there is a unique Haar measure \( \mu_G \) on \( G^{[\Gamma]} \). The dual of \( C(\tilde{A}_G) \) is given by the Banach space of all bounded complex Baire measures on \( \tilde{A}_G \). Furthermore there exists an extension of each Baire measure to a regular Borel measure on \( \tilde{A}_G \). The linear space of regular Borel measures equipped with the convolution operation form a Banach *-algebra \( M(G^N) \). The algebra decomposes into norm closed subspaces consisting of measures absolutely continuous with respect to the Haar measure of \( G^N \), continuous measures singular with respect to the Haar measure and discrete measures (refer to [14] chap.: 19]). The Banach *-algebra generated by Dirac point measures is excluded in the following considerations, a closer look on this structure will be presented in [17] and [15] Section 7.1. The subspace \( M_c(G^N) \) of all continuous measures singular with respect to the Haar measure is not a subalgebra of \( M(G^N) \), in general. Consequently the space \( M_c(G^N) \) is not considered. Notice the space \( M_c(G^N) \) consisting of measures absolutely continuous with respect to the Haar measure of \( G^N \) is identified with \( L^1(G^N, \mu_N) \).

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The norm-closed subspace of all regular Borel measures on $\bar{A}_G$, which are absolutely continuous to the uniquely defined Haar measure $\mu_G$ is given by $L^1(\bar{A}_G, \mu_G)$. Hence a state on $C(\bar{A}_G)$ is given by

$$\omega^G_M(f) = \int_{G^N} d\mu_G(h)(G') f_G(h)(G')$$

for all $f \in L^1(\bar{A}_G, \mu_G)$ and $f_G \in C(\bar{A}_G)$. The Banach $*$-algebra $L^1(\bar{A}_G, \mu_G)$ is the completion of $C(\bar{A}_G)$ w.r.t. the $\|\cdot\|_1$-norm.

But there is only one state on $C(\bar{A}_G)$ given by

$$\omega^G_M(f) = \int_{G^N} d\mu_G(h)(G') f_G(h)(G')$$

which is invariant under all actions $\alpha \in \text{Act}(\bar{G}_S, C(\bar{A}_G))$ for any arbitrary set $\bar{S}$ of suitable surfaces. The invariance of the state under different actions of $\bar{G}_S$ is derived from the fact that the product Haar measure of the product of the compact group $G$ is left and right invariant.

**Remark 74.** Let $\bar{A}_G$ be the space of generalised connections identified in the natural way with $G^N$. Let $H$ be a closed subgroup of the compact group $G$.

Let $f$ be a function in $C(\bar{A}_G)$ such that $f(h_G(G')k^{-1}) = f(h_G(G'))$ for any $k \in H^N$ and consider the state

$$\omega^G_H(f) = \int_{G^N} d\mu_G(h)(G') f_G(h)(G')$$

where $f_G \in C(\bar{A}_G)$. Then this state $\omega^G_H$ is $H$-invariant. But this state will be not path- or graph-diffeomorphism invariant in general.

### 3.2 Dynamical systems of actions of the group of bisectons on two $C^*$-algebras

#### Actions of the group of bisectons of the analytic holonomy algebra for finite graph systems

In this subsection the new concept for graph changing operations is introduced. On the level of finite path groupoids the bisectons of the path groupoid $P_T \Sigma$ over $V_T$ implement path-diffeomorphisms. A path-diffeomorphism is a pair of maps such that one bijective mapping maps vertices to vertices and the second bijective mapping maps non-trivial paths to non-trivial paths. A fixed set of independent paths is a graph, on the other hand, each path of a graph is an element of a path groupoid. Hence there is a concept of bisections of finite graph systems. This concepts are presented in subsection 2.2. The action of a bisection of finite path groupoids changes paths by adding or deleting segments of paths. In particular an action maps a non-trivial path to a trivial one or conversely. Hence an action of a bisection of a finite graph system transforms graphs to graphs. Actions of bisectons of a finite path groupoid are either right-, left- or inner-translations in the finite path groupoid. Hence actions of bisectons of a finite graph system are defined by right-, left- or inner-translations in the finite graph system. For a detail analysis refer to definition of a right translation in a finite graph system. Recall the lemma which states that the set $\mathcal{B}(P_T)$ of bisectons form a group w.r.t. a multiplication operation $*$ and an inverse $^{-1}$. Furthermore each bisection $\sigma$ defines a right translation $R_\sigma$ on a finite graph system $P_T$.

In the following investigations the non-standard identification of the configuration space is used, but it is also possible to derive results for the natural identification.

**Proposition 75.** Let $P_T$ be a finite graph system associated to a graph $\Gamma$ and let $C(\bar{A}_G)$ be the analytic holonomy $C^*$-algebra associated to $\Gamma$.

There is an action $\zeta$ of the group $\mathcal{B}(P_T)$ of bisectons equipped with $*$ and an inverse $^{-1}$ on $C(\bar{A}_G)$ defined by

$$(\zeta_\sigma f_T)(h_G(G')) := f_T((h_G \circ R_\sigma)(G')) = f_T(h_G(G'))$$

whenever $f_T \in C(\bar{A}_G)$, $\sigma \in \mathcal{B}(P_T)$ and for the subgraphs $\Gamma', \Gamma_\sigma$ of $\Gamma$. The inverse action is given by

$$(\zeta_\sigma^{-1} f_T)(h_G(G')) := f_T((h_G \circ R_{\sigma^{-1}})(G')) = f_T(h_G(G_{\sigma^{-1}}))$$

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Proposition 76. The
whenever $f_T \in C(\bar{A}_T)$, $\sigma \in \mathfrak{B}(\mathcal{P}_T)$ and for the subgraphs $\Gamma', \Gamma_{\sigma^{-1}}$ of $\Gamma$.

This action $\zeta$ is point-norm continuous and automorphic.

Hence $(\mathfrak{B}(\mathcal{P}_T), C(\bar{A}_T), \zeta_{\sigma})$ is a $C^*$-dynamical system.

Proof: First it is proved that $\zeta$ defines an automorphic action. Moreover for simplicity reasons the arguments are verified for one particular graph. The proof generalises to arbitrary graphs. Assume that $\Gamma$ is a graph, $V_{\bar{\Gamma}} = \{v_1, v_2, v_3, v_4, w_1, w_2\}$ and $\Gamma' := \{\gamma_1\}$ is a subgraph of $\bar{\Gamma}$. Let $\sigma$ and $\sigma'$ be two bisections of $\mathcal{P}_T$ such that $\sigma'(z) = z$ if $z \neq v_1$ and $z \neq w_1$ for $z \in V_{\bar{\Gamma}}$. Moreover let $v_i' = s(\gamma_i)$, $v_i = t(\gamma_i)$ and

\[
\|f_T(1)\|_{\bar{\Gamma}} = \lim_{\sigma \to \text{id}} \|f_T(h_{\bar{\Gamma}}(\zeta_{\sigma} f_T) h_{\bar{\Gamma}}(\bar{\Gamma}))\|_{\bar{\Gamma}} = 0
\]

for a graph $\Gamma = \{\gamma\}$ and $\Gamma_{\sigma} = \{\gamma \circ \sigma(v)\}$ being subgraphs of $\bar{\Gamma}$, and if for the bisection $\sigma \in \mathfrak{B}(\mathcal{P}_T)$ the equality $\sigma(w) = \text{id}_w$ holds for any $w \in V_{\bar{\Gamma}} \setminus \{v\}$ where $v = t(\gamma)$. Moreover the map $\text{id}(w) = \text{id}_w$ for all $w \in V_{\bar{\Gamma}}$, in particular for $v = t(\gamma)$, is the identity bisection. Deduce the properties for arbitrary subgraphs $\Gamma$ of an arbitrary graph $\bar{\Gamma}$.

Clearly there are also $C^*$-dynamical systems $(\mathfrak{B}(\mathcal{P}_T), C(\bar{A}_T), \zeta_{\sigma})$ and $(\mathfrak{B}(\mathcal{P}_T), C(\bar{A}_T), \zeta_{\sigma}')$ for the actions

\[
(\zeta_{\sigma} f_T)(h_{\bar{\Gamma}}(\Gamma')) := f_T((h_{\bar{\Gamma}} \circ L_\sigma)(\Gamma'))
\]

and

\[
(\zeta_{\sigma'} f_T)(h_{\bar{\Gamma}}(\Gamma')) := f_T((h_{\bar{\Gamma}} \circ L_{\sigma'})(\Gamma'))
\]

whenever $L_\sigma$ and $L_{\sigma'}$ are translations in $\mathcal{P}_T$. Refer to subsection 2.2.3 for the definition of these objects.

Proposition 76. The $C^*$-dynamical systems $(\mathfrak{B}(\mathcal{P}_T), C(\bar{A}_T), \zeta_{\sigma})$ and $(\mathfrak{B}(\mathcal{P}_T), C(\bar{A}_T), \zeta_{\sigma}')$ for the actions

\[
\zeta_{\sigma} f_T \cdot k_T = f_T \circ R_{\sigma} \cdot k_T = f_T \circ L_{\sigma^{-1}} \circ L_\sigma \cdot k_T \circ R_{\sigma^{-1}} = u_\sigma^* \zeta_{\sigma'}(f_T) u_\sigma
\]

holds whenever $f_T, k_T \in C(\bar{A}_T)$, $u_\sigma f_T := f_T \circ L_{\sigma^{-1}}$, $u_\sigma^* f_T := f_T \circ R_{\sigma^{-1}}$ and $\cdot$ is the multiplication in $C(\bar{A}_T)$. Notice that, $u_\sigma u_\sigma^* = L_{\sigma^{-1}} \circ R_{\sigma^{-1}} = \text{id}$. 

\[51\]
For every graph-diffeomorphism in $\text{Diff}(\mathcal{P}_\Gamma)$ there exists a bisection $\sigma \in \mathcal{B}(\mathcal{P}_\Gamma)$ and either a left-, or right- or inner-translation such that $\Phi_\Gamma = X_{\sigma}$, where $X$ is equivalent to $L$, or $R$ or $I$, and $\varphi_\Gamma = t \circ \sigma$. The set $\text{Diff}(\mathcal{P}_\Gamma)$ does not form a group in general. If one of the actions is fixed, then loosely speaking, the group of graph-diffeomorphisms is the set of graph-diffeomorphism in $\text{Diff}(\mathcal{P}_\Gamma)$, which are defined by a bisection $\sigma \in \mathcal{B}(\mathcal{P}_\Gamma)$ and a left- translation such that $\Phi_\Gamma = L_\sigma$ and $\varphi_\Gamma = t \circ \sigma$. Clearly the left-translation can be replaced by right- or inner-translation.

Note that, the $C^*$-dynamical systems $(\mathcal{B}(\mathcal{P}_\Gamma),\mathcal{C}(\mathcal{A}_\Gamma),\zeta_\sigma)$ and $(\mathcal{G}_\mathcal{S}_\Gamma,\mathcal{C}(\mathcal{A}_\Gamma),\alpha)$ are not exterior or equivariantly isomorphic [33 Def.: 2.64].

Groups of surface or surface-orientation-preserving bisections for finite graph systems

Up to now only actions of bisections of the holonomies are considered. Therefore in the next investigations the behavior of an action of bisections on the flux operators is analysed. Clearly this action is required to preserve the structure of the group-valued quantum flux operators.

Proposition 77. Let $\tilde{S}$ be a set of surfaces and $\Gamma$ be a graph such that $\tilde{S} \cap \Gamma \subset V_\Gamma$. Let $\varphi$ be a diffeomorphism in $\Sigma$, which maps each surface $S \in \tilde{S}$ to a surface $S_\sigma \in \tilde{S}$.

A surface-preserving bisection $\sigma$ of a finite path groupoid and a set of surfaces $\tilde{S}$ is defined by a bisection $\sigma : V_\Gamma \rightarrow \mathcal{P}_\Gamma \Sigma$ in $\mathcal{P}_\Gamma$ such that

- the map $\varphi_\Gamma : V_\Gamma \rightarrow V_\Gamma$, which is given by $\varphi_\Gamma = t \circ \sigma$, is bijective and $(t \circ \sigma)(v) = v$ whenever $v \in S \cap V_\Gamma$ and each $S \in \tilde{S}$ yields,
- for each path $\gamma \in \mathcal{P}_\Gamma \Sigma$ that intersects a surface $S$ and, such that $\gamma \cap S = \{s(\gamma)\}$ holds, the non-trivial transformed path is presented by $\gamma \circ \sigma(t(\gamma))$ for $S \in \tilde{S},$
- for each path $\gamma \in \mathcal{P}_\Gamma \Sigma$ that intersects a surface $S$ and, such that $\gamma \cap S = \{t(\gamma)\}$ is satisfied, then $\sigma(t(\gamma)) = 1_{t(\gamma)}$ and $\gamma \circ \sigma(t(\gamma)) = \gamma$ yields for a surface $S \in \tilde{S},$
- the bisection $\sigma : V \rightarrow \mathcal{P}_\Gamma \Sigma$, where $V = V_\Gamma \setminus S$ and $V_\tilde{S} = V_\Gamma \cap \{S_i : S_i \in \tilde{S}\}$, is such that $(t \circ \sigma)(v) \in V$ for all $v \in V$ holds. Then for each $\gamma \in \mathcal{P}_\Gamma \Sigma$ such that $\gamma \cap S = \emptyset$, the transformed path is given by $\gamma \circ \sigma(t(\gamma))$.

The set of all surface-preserving bisections for a path groupoid forms a group equipped with $*$ and $^{-1}$ and is called the group $\mathcal{B}_{S_{\text{surf}}}(\mathcal{P}_\Gamma \Sigma)$ of surface-preserving bisections of a finite path groupoid.

Observe for a certain bisection $\sigma \in \mathcal{B}_{S_{\text{surf}}}(\mathcal{P}_\Gamma \Sigma)$ the right translation $R_\sigma(v) = \{\sigma(v) = \sigma(w) : w \in V_\Gamma \setminus V_S\}$ for $v \in V_\Gamma$ defines a path-diffeomorphism $\Phi_\Gamma(\gamma)$ such that $(\varphi_\Gamma, \Phi_\Gamma) \in \text{Diff}(\mathcal{P}_\Gamma \Sigma)$. Consequently similar to surface-preserving bisections of a finite groupoid the corresponding surface-preserving path-diffeomorphisms is defined. Refer to [33 Section 6.2] for a detailed investigation.

In general it follows that, $R_\sigma(1_v) = \sigma(v)$ for $v \in V$, where $V = V_\Gamma \setminus V_S$ and $V_S = V_\Gamma \cap S$, and $R_\sigma(1_w) = 1_w$ for $w \in V_\tilde{S}$ for each $S \in \tilde{S}$ is satisfied. Since $1_v$ and $1_w$ for $v \in V$ and $w \in V_\tilde{S}$ are elements of $\mathcal{P}_\Gamma \Sigma$.

Proposition 78. A surface-preserving bisection $\sigma$ of a finite graph system is defined as a bisection $\sigma : V_\Gamma \rightarrow \mathcal{P}_\Gamma$ in $\mathcal{P}_\Gamma$ such that there is a surface-preserving bisection $\tilde{\sigma}$ of a finite path groupoid $\mathcal{P}_\Gamma \Sigma \equiv V_\Gamma$ and $\sigma_\Gamma(V) = \{\sigma(v) : v_i \in V\}$ whenever $V$ is a subset of $V_\Gamma$.

The set surface-preserving bisections for a finite graph system forms a group equipped with $*_2$ and $^{-1}$ and is called the group $\mathcal{B}_{S_{\text{surf}}}(\mathcal{P}_\Gamma)$ of surface-preserving bisections of a finite graph system and a surface set $\tilde{S}$.

A right-translation $R_\sigma$ in the finite graph system $\mathcal{P}_\Gamma$ is defined for a bisection $\sigma \in \mathcal{B}_{S_{\text{surf}}}(\mathcal{P}_\Gamma)$ in definition 28.

For example for a path $\gamma$ that intersect a surface $S$ in $t(\gamma)$, then $\gamma_\sigma = \gamma \circ \sigma(t(\gamma)) = \gamma$ is satisfied. For a path $\gamma'$ that intersects $S'$ in $s(\gamma')$ the transformed path is $\gamma_\sigma = \gamma \circ \sigma(t(\gamma'))$. In both cases the surfaces satisfy $S = S_\sigma$ and $S' = S'_\sigma$.  

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Definition 79. Let \( bisection in \Sigma \) and the surfaces such that there is no interchange of the left and right unitary representation of \( \bar{S} \) a surface \( S \) modified path-diffeomorphisms are maps such that paths, which intersect a surface \( \sigma \) surfaces are fixed, whereas the graph \( \Gamma \) is changed to \( \bar{\Gamma} \) intersections between surfaces in \( \bar{\Sigma} \).

Now the question arise whether another dissimilar situation is possible. The idea is to implement bisections of such a way that the orientation of the transformed surface with respect to path obtained by the bisection is preserved and paths that are ingoing \( \text{w.r.t.} \) the orientation of the unchanged surface are ingoing \( \text{w.r.t.} \) the transformed surface. The same is true for paths that are outgoing \( \text{w.r.t.} \) the orientation of the unchanged surface.

Now consider a diffeomorphism or actions of bisections, which maps a surface \( S \) into another \( S_\sigma \). Then the modified path-diffeomorphisms are maps such that paths, which intersect a surface \( S \) and are ingoing \( \text{w.r.t.} \) a surface \( S_\sigma \), to paths that are ingoing \( \text{w.r.t.} \) the transformed surface \( S_\sigma \). It is possible that the path \( \gamma \) lies above the surface \( S \), whereas the transformed path \( \Phi_\gamma(\gamma) \) lies below the surface \( \varphi(S) \). This case is needed to be excluded. The new action of bisections are required to preserve the orientation properties of the paths and the surfaces such that there is no interchange of the left and right unitary representation of \( G_{S,\bar{\Gamma}} \) on \( \bar{H}_\Gamma \).

**Definition 79.** Let \( \bar{S} \) be a set of surfaces and \( \Gamma \) be a graph such that \( \bar{S} \cap \Gamma \subset V_\Gamma \). Let \( \varphi \) be a diffeomorphism in \( \Sigma \), which maps each surface \( S \in \bar{S} \) to a surface \( S_\sigma \in \bar{S} \).

* A map \( \sigma \) is called **surface-orientation-preserving bisection for a finite path groupoid**, if \( \sigma \) is a bisection in \( \mathcal{B}(P_\Gamma \Sigma) \) such that

- the map \( \varphi_\Gamma : V_\Gamma \to V_\Gamma \), which is given by \( \varphi_\Gamma = t \circ \sigma \), is bijective and \( \varphi_\Gamma = \varphi | V_\Gamma \), and
- for each \( \gamma \in \mathcal{P}_\Gamma \Sigma \) that intersects a surface \( S \) and, such that \( \gamma \cap S = \{ t(\gamma) \} \) holds, the non-trivial transformed path is given by \( \gamma \cap S = \{ t(\gamma) \} \) \( \Rightarrow \gamma_S \) for a surface \( S \in \bar{S} \) yields. Moreover if \( \gamma \) lie above (or below) the surface \( S \) and \( \gamma_S \) is non-trivial, then \( \gamma_S \) lie above (or below) the surface \( S_\sigma \). Except of a vertex \( s(\gamma) \) such that \( s(\gamma) \cap S_\sigma = \{ s(\gamma) \} \), the vertex \( t(\gamma) \) is the only intersection vertex of \( S_\sigma \) and \( \gamma_S \).
- For each \( \gamma \in \mathcal{P}_\Gamma \Sigma \) that intersects a surface \( S \), and such that \( \gamma \cap S = \{ s(\gamma) \} \), it is true that \( \sigma(s(\gamma)) = 1_{s(\gamma)} \), \( (t \circ \sigma)(s(\gamma)) = s(\gamma) \) and hence \( \gamma \circ \sigma(s(\gamma)) = \gamma \circ \Gamma(s(\gamma)) = \gamma \) for a surface \( S \in \bar{S} \). Furthermore if \( \gamma \) is located above \( S \), then \( \gamma \) is located above the surface \( S_\sigma \).
- The map \( \sigma : V \to \mathcal{P}_\Gamma \Sigma \), where \( V = V_\Gamma \setminus V_\Sigma \) and \( V_\Sigma = V_\Gamma \cap \{ S_i : S_i \in \bar{S} \} \), is such that \( (t \circ \sigma)(v) \in V \) \( \forall v \in V \) yields. Then for each \( \gamma \in \mathcal{P}_\Gamma \Sigma \) such that \( \gamma \cap S = \{ \emptyset \} \), the transformed path is given by \( \gamma \circ \sigma(t(\gamma)) \).

Clearly this concept can be generalised to surface-orientation-preserving bisections of a finite graph system. Moreover similar to surface-orientation-preserving bisections of a finite graph system the corresponding surface-preserving-orientation graph-diffeomorphisms can be defined ([15] Section 6.2]).

**Corollary 80.** The set \( \mathcal{B}_{S,\bar{\sigma}}(P_\Gamma \Sigma) \) of all surface-orientation-preserving bisections of a finite path groupoid equipped with multiplication \( * \) and inversion \( ^{-1} \) forms a group and it is called the **group of surface-orientation-preserving bisections of a finite path groupoid associated to surfaces**.

The set \( \mathcal{B}_{S,\bar{\sigma}}(P_\Gamma) \) of all surface-orientation-preserving bisections of a finite graph system equipped with multiplication \( * \) and inversion \( ^{-1} \) forms a group and is called the **group of surface-orientation-preserving bisections associated to graphs and surfaces**.

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For example, consider a graph \( \Gamma = \{ \gamma, \gamma', \gamma'' \} \) and the surfaces \( \tilde{S} := \{ S, S', S'' \} \), which are presented in the picture below. Then let \( (\varphi_T, \Phi_T) \) be a surface-orientation-preserving path-diffeomorphism for \( \tilde{S} \) defined by a bisection \( \sigma \) such that \( S \) is mapped to \( S_\sigma \) and so on. Let the path \( \gamma \) intersects the surface \( S \) in \( t(\gamma) \) such that \( \gamma \) lies below \( S \) and the path \( \gamma_\sigma \) intersects \( S_\sigma \) in \( t(\gamma_\sigma) \) such that \( \gamma_\sigma \) lies below \( S_\sigma \). Moreover let \( \gamma' \) intersect \( S' \) in \( t(\gamma') \) such that \( \gamma' \) lies below \( S' \) and the path \( \gamma'_\sigma := \gamma' \circ \sigma(t(\gamma')) \) intersects \( S'_\sigma \) in \( t(\gamma'_\sigma) \) and \( \gamma'_\sigma \) lies below \( S'_\sigma \). Finally for a path \( \gamma'' \) intersecting \( S'' \) in \( s(\gamma'') \) and the path \( \gamma'' \) lies above, then the transformed path \( \tilde{\gamma}'_\sigma \) is equivalent to \( \gamma'' \) and \( \gamma'' \) is outgoing and lies above the surface \( S''_\sigma \). Summarising, there is a map such that \( \Phi_T(\Gamma) = \{ \gamma_\sigma, \gamma'_\sigma, \gamma''_\sigma \} \).

There is a problem if the bisection \( \sigma \) maps \( s(\tilde{\gamma}) \) to a path \( \tilde{\gamma}_\sigma \), which is not equivalent to \( \tilde{\gamma} \). Since the resulting path \( \tilde{\gamma}_\sigma \) is ingoing and lies above the surface \( S_\sigma \) whereas \( \tilde{\gamma} \) is outgoing and above \( S \).

The situation is restricted to the case that the surface set \( \tilde{S} \) is chosen such that each path \( \gamma_i \) in \( \mathcal{P}_T \Sigma \), which intersects only the surfaces \( S_j \), which is contained in \( \tilde{S} := \{ S_j \}_{1 \leq j \leq K} \), and lies above and ingoing, (or below and ingoing, or above and outgoing, or below and outgoing) w.r.t. \( S_j \) and there are no other intersection vertices of this path with any other surface \( S_k \) for \( k \neq j \). Then it is required that, the path \( \gamma_i \circ \sigma(t(\gamma_i)) \) (or \( \sigma(s(\gamma_i)) \)) lies above and ingoing (or below and ingoing, or above and outgoing, or below and outgoing) w.r.t. each \( \varphi(S_j) \). Hence all actions of these bisections preserve the quantum flux operators associated to different surface sets and graphs presented in subsection 2.3 can be treated.

**Actions of the group of surface-preserving bisections of the \( C^* \)-algebra \( W(\tilde{G}_{S, \Gamma}) \) of Weyl elements**

The next question is related to different actions of group of bisections of the algebra of Weyl elements. First consider the action of a surface-preserving group \( \mathfrak{B}_{S, \text{surf}}(\mathcal{P}_T) \) of bisections of a finite graph system on the \( C^* \)-algebra \( W(\tilde{G}_{S, \Gamma}) \).

**Lemma 81.** The action is trivial, i.e. for \( \Gamma' \in \mathcal{P}_T \)

\[
(\zeta_\sigma U)(\rho_{S, \Gamma}(\Gamma')) = U(\rho_{S, \Gamma}(\Gamma'_\sigma)) = U(\rho_{S, \Gamma}(\Gamma'))
\]

yields for \( \rho_{S, \Gamma} \in G_{S, \Gamma} \).

**Proof:** This is true, since \( \rho_{S}(\gamma) = e_G \) holds if the path \( \gamma \in \Gamma' \) does not intersect with a surface \( S \) in \( \tilde{S} \) in a source or target vertex. For a subgraph \( \Gamma' \) of \( \Gamma = \{ \gamma_1, \ldots, \gamma_N \} \), where each path \( \gamma_i \) in \( \Gamma \) intersect a surface in \( \tilde{S} \) in a vertex, the action is given by \( (\zeta_\sigma U)(\rho_{S, \Gamma}(\Gamma')) = U(R_\sigma(\rho_{S, \Gamma}(\Gamma'))) \) where \( R_\sigma(\rho_{S, \Gamma}(\Gamma')) = \rho_{\varphi(S), \Gamma}(R_\sigma(\Gamma')) = \rho_{\varphi(S), \Gamma}(\Gamma'_\sigma) \) for all \( S \in \tilde{S} \) and \( \varphi(S) = S \), \( (t \circ \sigma)(v) = v \) and hence \( \sigma(v) = 1_v \) for all \( v \in V_T \cap \tilde{S} \), then finally deduce \( \Gamma'_\sigma = \Gamma' \).
Action of the group of surface-orientation-preserving bisectons of the $C^*$-algebra $W(\hat{G}_{S,\Gamma})$ of Weyl elements

**Proposition 82.** Let $\tilde{S}$ be a finite set of surfaces.

The action of a surface-orientation-preserving group $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ of bisectons of a finite graph system on the $C^*$-algebra $W(\hat{G}_{S,\Gamma})$ is presented by

$$((\zeta_U)(\rho_{S,T}(\Gamma')) = U(\rho_{\varphi(S)}(\Gamma'))$$

for all $U \in \text{Rep}(\hat{G}_{S,\Gamma}, \mathcal{K}(\mathcal{H}_T))$

satisfies the following properties:

(i) The action $\zeta$ of $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ in $W(\hat{G}_{S,\Gamma})$, which is a $C^*$-subalgebra of $\mathcal{L}(\mathcal{H}_T)$ is automorphic,

(ii) The action $\zeta$ of $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ in $W(\hat{G}_{S,\Gamma})$ is point-norm continuous.

(iii) The automorphic action $\alpha$ the group of bisection $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ on $W(\hat{G}_{S,\Gamma})$ is inner such that there exists an unitary representation $V$ of $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ on the $C^*$-algebra $W(\hat{G}_{S,\Gamma})$, i.e. $V \in \text{Rep}(\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T), W(\hat{G}_{S,\Gamma}))$, which satisfy

$$V_{\sigma}U(\rho_{S,T}(\Gamma'))V_{\sigma}^* = (\zeta_{\sigma}(U))(\rho_{S,T}(\Gamma')) \quad \forall U \in W(\hat{G}_{S,\Gamma}), \sigma \in \mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$$

For each $\sigma \in \mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ the unitary $V_{\sigma}$ are called the **unitary bisectons of a finite graph system and surfaces in $\tilde{S}$**.

For example, for an appropriate surface set $\tilde{S}$, a graph $\Gamma = \{\gamma, \gamma_1, ..., \gamma_M\}$ and a subgraph $\Gamma' = \{\gamma\}$ it is true that

$$((\zeta_{\sigma^{-1}}U_{\tilde{\Gamma}})(\rho_{S}(\gamma)) = U_{\tilde{\Gamma}}(\rho_{\varphi'(S)}(\gamma \circ \sigma(v)))$$

whenever $v = t(\gamma)$ and $\Gamma' = \{\gamma \circ \sigma(v)\}$. Certainly, there is also an action of the group of bisection $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ on $W(\hat{G}_{S,\Gamma})$ given by a left translation or a inner automorphism of the path groupoid $\mathcal{P}_T \Sigma$ over $V_T$. Each action is inner and hence for each action there is a unitary representation of $\mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ on $W(\hat{G}_{S,\Gamma})$.

**Proof:** For the proof of the action being automorphic (property [1]) derive the following. Set $\varphi_{\gamma} = \varphi_{T} = t \circ \sigma$ and $\varphi'_{\gamma} = \varphi'_{T} = t \circ \sigma'$ for two bisectons $\sigma, \sigma' \in S_{\text{or}}(\mathcal{P}_T)$. For simplicity assume that, $\Gamma' = \{\gamma\}$ such that $\rho_{S,T}(\Gamma') = \rho_S(\gamma)$ and $v = t(\gamma)$, $w = t(\sigma'(v))$, then derive

$$((\zeta_{\varphi(S)}U_{\Gamma})\rho_S(\gamma))
= U_{\Gamma}(\rho_{\varphi'(S)}(\gamma \circ \sigma'(v) \circ \sigma(w)))
= (\zeta_{\varphi(S)}U_{\Gamma})\rho_S(\gamma))$$

which proves condition [1] of definition [48].

Condition [3] of definition [48] is shown by the observation that

$$(\zeta_{\varphi(S)}U^*)\rho_S(\gamma)) = U(\rho_{\varphi(S)}^{-1}(\gamma \circ \sigma(v))) = (\zeta_{\varphi(S)}U^*)\rho_S(\gamma))$$

where $v = t(\gamma)$ and $\varphi(S)^{-1} = \varphi(S^{-1})$ holds.

The action satisfies property [3] of the proposition, since it is true that,

$$\lim_{\sigma \to \text{id}} \|\zeta_{\sigma}(U(\rho_{S,T}(\Gamma'))) - U(\rho_{S,T}(\Gamma'))\| = \lim_{\sigma \to \text{id}} \|U(\rho_{\varphi'(S)}'(\Gamma'_\sigma)) - U(\rho_{S,T}(\Gamma'))\| = 0$$

yields for a graph $\Gamma' = \{\gamma\}$ and $\Gamma'_\sigma = \{\gamma \circ \sigma(v)\}$ being subgraphs of $\Gamma$, $v = t(\gamma)$, $\sigma \in \mathfrak{B}_{S,\text{or}}(\mathcal{P}_T)$ and where $\text{id}(v) = v$ for $v = t(\gamma)$ is the identity bisection.
The action is indeed inner (property (iii) of the proposition), since there is a unitary representation $V : \mathcal{B}_{S,\text{or}}(P_T) \to M(\mathcal{W}(G_{S,T}))$, where $M(\mathcal{W}(G_{S,T}))$ is the multiplier algebra of $\mathcal{W}(G_{S,T})$, such that

$$(\zeta_* U)(\rho_S(\gamma)) = V_\sigma U(\rho_S(\gamma))V_\sigma^*$$

is satisfied.

Observe that, $\rho_S(\mathbb{1}_v) = \epsilon_G$ for any $v \in V_T$. Set $(V_\sigma U)(\rho_S(\Gamma')) = U(\rho_S(\Gamma'\sigma))$ and $(V_\sigma^* U)(\rho_S(\Gamma')) = U(\rho_S(\Gamma'\sigma^{-1}))$, in example

$$(V_\sigma U)(\rho_S(\gamma)) = U(\rho_S(\gamma \circ \sigma(v)))$$
$$(V_\sigma^* U)(\rho_S(\gamma)) = U(\rho_S(\gamma \circ (\sigma \ast \sigma^{-1})(v)))$$

is satisfied.

Observe that, $\rho_S(\mathbb{1}_v) = \epsilon_G$ for any $v \in V_T$. Set $(V_\sigma U)(\rho_S(\Gamma')) = U(\rho_S(\Gamma'\sigma))$ and $(V_\sigma^* U)(\rho_S(\Gamma')) = U(\rho_S(\Gamma'\sigma^{-1}))$, in example

$$V_\sigma V_{\sigma^{-1}} = \text{id} = V_{\sigma^{-1}} V_\sigma$$

yields whenever $w = (t \circ \sigma)^{-1}(v), v \in t(\gamma), \Gamma' = \{\gamma, \mathbb{1}_w, \gamma_1, \ldots, \gamma_M\}$, $\Gamma' := \{\gamma \circ \sigma(v), \mathbb{1}_w, \gamma_1, \ldots, \gamma_M\}$ and $\Gamma'_{\sigma^{-1}} := \{\gamma, \mathbb{1}_w \circ \sigma^{-1}(v), \gamma_1, \ldots, \gamma_M\}$.

To show that, the $^\ast$-representation $V$ of $\mathcal{B}_{S,\text{or}}(P_T)$ on the Hilbert space $\mathcal{H}_T$ is really unitary consider the following example. Let $\gamma_1$ and $\gamma_2$ be two disjoint paths defining two disjoint graphs, then derive

$$(V_\sigma V_\sigma^* \psi_{\gamma_1})(h_{\gamma_1}(\gamma_1)) = \psi_{\gamma_1}(h_{\gamma_1}(\gamma_1))$$
$$(V_\sigma^* V_\sigma \psi_{\gamma_2})(h_{\gamma_2}(\gamma_2)) = \psi_{\gamma_2}(h_{\gamma_2}(\gamma_2))$$

and, consequently,

$$V_\sigma V_\sigma^* \left(\psi_{\gamma_1}(h_{\gamma_1}(\gamma_1)), \psi_{\gamma_2}(h_{\gamma_2}(\gamma_2))\right) = \left(\psi_{\gamma_1}(h_{\gamma_1}(\gamma_1)), \psi_{\gamma_2}(h_{\gamma_2}(\gamma_2))\right)$$

$$= (V_\sigma^* V_\sigma) \left(\psi_{\gamma_1}(h_{\gamma_1}(\gamma_1)), \psi_{\gamma_2}(h_{\gamma_2}(\gamma_2))\right)$$

for $v = t(\gamma_2), t(\gamma_1) = (t \circ \sigma)^{-1}(v), \psi_{\gamma_1}(h_{\gamma_1}(\gamma_1)) \in \mathcal{H}_{\gamma_1}, \psi_{\gamma_2}(h_{\gamma_2}(\gamma_2)) \in \mathcal{H}_{\gamma_2}$ where $\Gamma = \{\gamma_1, \gamma_2\}$ and $\mathcal{H}_T = \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2}$.

Then collect the following facts to conclude that $V$ is a unitary representation of $\mathcal{B}_{S,\text{or}}(P_T)$ in $\mathcal{W}(G_{S,T})$:

(i) $V_\sigma$ is unitary for any $\sigma \in \mathcal{B}_{S,\text{or}}(P_T)$

(ii) $V_\sigma V_{\sigma'} = V(\sigma \ast \sigma')$ for all $\sigma, \sigma' \in \mathcal{B}_{S,\text{or}}(P_T)$

(iii) $V_\sigma$ is point-norm continuous, since the associated action $\zeta_\sigma$ is.

Dynamical systems of an action of the group of surface-preserving bisections of the $C^*$-algebra $\mathcal{W}(G_{S,T})$ of Weyl elements and states on $C(\mathcal{A}_T)$

Then the last proposition implies the following.

**Proposition 83.** Let $\mathcal{S}$ be a set of surfaces.

The triple $(\mathcal{B}_{S,\text{or}}(P_T), \mathcal{W}(G_{S,T}), \zeta)$ of a surface-orientation-preserving group $\mathcal{B}_{S,\text{or}}(P_T)$ of bisections, a $C^*$-algebra $\mathcal{W}(G_{S,T})$ w.r.t. a set $\mathcal{S}$ of surfaces and a graph $\Gamma$ and the action $\zeta$ is a $C^*$-dynamical system.

**Lemma 84.** Let $\Phi : \mathcal{W}(G_{S,T}) \to \mathcal{L}(\mathcal{H}_T)$ be the natural $^\ast$-homomorphism.

Then the set $\{\Phi(W)B : W \in \mathcal{W}(G_{S,T}), B \in \mathcal{L}(\mathcal{H}_T)\}$ is dense in $\mathcal{L}(\mathcal{H}_T)$. Consequently $\Phi$ is a morphism of $C^*$-algebras $\mathcal{W}(G_{S,T})$ and $\mathcal{L}(\mathcal{H}_T)$.

Hence it is obvious to consider the following covariant pair.
Proposition 85. The pair $(\Phi, V)$ consisting of a morphism $\Phi \in \text{Mor}(\mathcal{W}(\mathcal{G}_{\Sigma_1}), \mathcal{L}(H_T))$ and a unitary representation $V$ of $\mathfrak{B}_{S,\sigma^R}(P_T)$ on the Hilbert space $H_T$, i.e. $V_\sigma \in \text{Rep}(\mathfrak{B}_{S,\sigma^R}(P_T), \mathcal{K}(H_T))$ such that

$$\Psi(\zeta W) = V_\sigma \Psi(W) V_\sigma^*$$

whenever $W \in \mathcal{W}(\mathcal{G}_{\Sigma_1})$ and $\sigma \in \mathfrak{B}_{S,\sigma^R}(P_T)$, is a covariant representation of the $C^*$-dynamical system $(\mathfrak{B}_{S,\sigma^R}(P_T), \mathcal{W}(\mathcal{G}_{\Sigma_1}), \zeta)$ in $\mathcal{L}(H_T)$.

Proof: Conclude for a subgraph $\Gamma' = \{\gamma\}$ and $\Gamma'_\sigma = \{\gamma \circ \sigma(v)\}$ of $\Gamma$, $v = t(\gamma)$, a surface $S$ such that $\gamma$ is outgoing and lies below and $\psi \in H_T$ that

$$(V_\sigma U(\rho_{S,T}(\Gamma')) V_\sigma^* \psi_T(\gamma)) = (V_\sigma U(\rho_{S,T}(\Gamma'_\sigma)) \psi_T(\gamma')) = (V_\psi_T(\rho_{S,T}(\Gamma_\sigma)) \psi_T(\gamma')) = (\zeta(\rho_{S,T}(\Gamma'_\sigma)) \psi_T(\gamma'))$$

holds for $U \in \mathcal{W}(\mathcal{G}_{\Sigma_1})$. Hence the proposition is true.

Covariant pairs are constructed from the multiplication representation $\Phi_M$ of the holonomy algebra $C(\mathcal{A}_T)$ for a finite graph system $P_T$. The next proposition is valid for both identifications of the configuration space $\mathcal{A}_T$.

Proposition 86. Let $\Phi_M$ be the multiplication representation of $C(\mathcal{A}_T)$ on $H_T$.

Then there is an unitary representation $V_\sigma$ of $\mathfrak{B}(P_T)$ on $H_T$ such that

$$V_\sigma \Phi_M(f_T) V_\sigma^* = \Phi_M(\zeta \sigma f_T)$$

whenever $f_T \in C(\mathcal{A}_T)$, $\sigma \in \mathfrak{B}(P_T)$ and $(V, \Phi_M)$ is a covariant pair of the $C^*$-dynamical system $(\mathfrak{B}(P_T), C(\mathcal{A}_T), \zeta)$.

Then there is a $\mathfrak{B}(P_T)$-invariant state $\Omega_{\mathfrak{B}_2}^\Gamma$ on $C(\mathcal{A}_T)$ such that

$$\omega^\Gamma_{\mathfrak{B}_2}(\zeta \sigma f_T) = \Omega_{\mathfrak{B}_2}^\Gamma(f_T) := \langle \Omega_{\mathfrak{B}_2}^\Gamma, \Phi_M(f_T) \Omega_{\mathfrak{B}_2}^\Gamma \rangle$$

where $\Omega_{\mathfrak{B}_2}^\Gamma$ is a cyclic vector in $H_T$ for the GNS-triple $(H_T, \Phi_M, \Omega_{\mathfrak{B}_2}^\Gamma)$.

Notice that, the state $\omega_{\mathfrak{B}_2}^\Gamma$ is also a $\mathfrak{B}_{S,\sigma^R}(P_T)$-and $\mathfrak{B}_{S,\sigma}(P_T)$-invariant state on $C(\mathcal{A}_T)$. Hence

$$V_\sigma \Omega_{\mathfrak{B}_2}^\Gamma = \Omega_{\mathfrak{B}_2}^\Gamma$$

for all $\sigma \in \mathfrak{B}_{S,\sigma^R}(P_T)$

is true.

Recall the $\mathcal{G}_{\Sigma_1}$-invariant state $\omega^\Gamma_M$ of $C(\mathcal{A}_T)$ defined in proposition 71.

Problem 3.1: For a path-diffeomorphism $(\varphi_T, \Phi_T) \in \text{Diff}(P_T)$ such that for a graph

$$\Gamma := \{\gamma_1, \gamma_1', ..., \gamma_M, \gamma_M', ..., \gamma_M''\}$$

where $|\Gamma| := N = 3M$ and $\Phi_T(\Gamma') = (\gamma_1' \circ \gamma_1'', ..., \gamma_M' \circ \gamma_M'')$ the natural identification is used, i.e.

$$\Phi_T(\Gamma') = (\gamma_1', \gamma_1'', ..., \gamma_M', \gamma_M'')$$

where $\Gamma' := (\gamma_1', ..., \gamma_M')$ and $\Gamma'' := (\gamma_1'', ..., \gamma_M'')$ are subgraphs of $\Gamma$, then

$$\omega^\Gamma_M(\zeta(\varphi_T, \Phi_T) f_T) = \int_{G_{2^M}} d\mu_T(\varphi_T(\Phi_T(\Gamma')) f_T(\varphi_T(\Phi_T(\Gamma'))))$$

where $\Gamma' := (\gamma_1', ..., \gamma_M')$ and $\Gamma'' := (\gamma_1'', ..., \gamma_M'')$ are subgraphs of $\Gamma$, then

$$\omega^\Gamma_M(\zeta(\varphi_T, \Phi_T) f_T) = \int_{G_{2^M}} d\mu_T(\varphi_T(\Phi_T(\Gamma')) f_T(\varphi_T(\Phi_T(\Gamma'))))$$

$$= \int_{G_{2^M}} d\mu_T(\varphi_T(\Phi_T(\Gamma')) f_T(\varphi_T(\Phi_T(\Gamma'))))$$

$$= \int_{G_{2^M}} d\mu_T(\varphi_T(\Phi_T(\Gamma')) f_T(\varphi_T(\Phi_T(\Gamma'))))$$

$$= W \int_{G_{2^M}} d\mu_T(\varphi_T(\Phi_T(\Gamma')) f_T(\varphi_T(\Phi_T(\Gamma'))))$$

$$\neq \omega^\Gamma_M(f_T)$$

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where $W$ is a suitable constant. Clearly the state $\omega_M^\Gamma$ is not graph-diffeomorphism invariant.

On the other hand, if instead of the natural identification the non-standard identification is taken into account, then this state is graph-diffeomorphism invariant, since for a graph-diffeomorphism $(\varphi, \Phi_T) \in \text{Diff}(\mathcal{P}_T)$ such that for $\Gamma := \{\gamma'_1, \gamma'_2, \ldots, \gamma'_M, \gamma'_1', \ldots, \gamma'_N'\}$ and $N = 3M$,

$$\Phi_T(\Gamma') = (\gamma'_1 \circ \gamma''_1, \ldots, \gamma'_M \circ \gamma''_M)$$

Then derive

$$\omega_M^\Gamma(\zeta(\varphi, \Phi_T)(f_T)) = \int_{G^{2M}} d\mu_T(h_T(\Phi_T(\Gamma')) f_T(h_T(\Phi_T(\Gamma')))
= \int_{G^M} d\mu_T(h_T(\Phi_T(\Gamma')) f_T(h_T(\gamma'_1), \ldots, h_T(\gamma'_M) h_T(\gamma''_M))
= \int_{G^M} d\mu_T(h_T(\Phi_T(\Gamma')) f_T(h_T(\gamma'_1), \ldots, h_T(\gamma'_M))
= \omega_M^\Gamma(\Phi_T(\Gamma'))$$

Furthermore recall that, the state $\omega_M^\Gamma f$ defined in remark 73. Then this state is not path-diffeomorphism invariant, since for a path-diffeomorphism $(\varphi, \Phi_T) \in \text{Diff}(\mathcal{P}_T)$ such that for $\Gamma := \{\gamma'_1, \gamma'_2, \ldots, \gamma'_M, \gamma''_1', \ldots, 
\gamma''_N'\}$ and $N = 3M$,

$$\Phi_T(\Gamma') = (\gamma'_1 \circ \gamma''_1, \ldots, \gamma'_M \circ \gamma''_M)$$

it is true that for $f \in C(\bar{\mathcal{A}}_T)$ such that $f(h_T(\Gamma')) = f(h_T(\Gamma'))$ for all $k \in G^N$ the state satisfies

$$\omega_M^\Gamma(\zeta(\varphi, \Phi_T)(f_T)) = \int_{G^N} d\mu_T(h_T(\Phi_T(\Gamma')) f_T(h_T(\Phi_T(\Gamma')))
= \int_{G^N} d\mu_T(h_T(\Phi_T(\Gamma')) f_T(h_T(\gamma'_1), \ldots, h_T(\gamma'_M)) f_T(h_T(\gamma''_1), \ldots, h_T(\gamma''_M))
= \omega_M^\Gamma(f_T)$$

Consequently if the function $f$ satisfies additionally

$$f(h_T(\Gamma')) = f(h_T(\gamma'_1), \ldots, h_T(\gamma'_M)) = f(h_T(\Phi_T(\Gamma')) = f(h_T(\gamma'_1), \ldots, h_T(\gamma'_M))$$

for all $(\varphi, \Phi_T) \in \text{Diff}(\mathcal{P}_T)$ then the state $\omega_M^\Gamma f$ is Diff($\mathcal{P}_T$)-invariant. But the only function, which satisfies (66) for all graph-diffeomorphism for a finite graph groupoid, is the constant function.

Restrict the state $\omega_M^\Gamma f$ presented in remark 74, which is $H^N$-invariant, to functions in $f \in C(\bar{\mathcal{A}}_T)$ such that

$$f(R(k)(h_T(\Gamma')) = f(h_T(\Gamma')) = f(L(k)(h_T(\Gamma')))$$

for all $k \in H^N$ and

$$f(h_T(\Gamma')) = f(h_T(\Phi_T(\Gamma')))$$

for all $\Gamma' \in \mathcal{P}_T$ and $(\varphi, \Phi_T) \in \text{Diff}(\mathcal{P}_T)$

Then the state $\omega_M^\Gamma f$ is $H^N$ - and Diff($\mathcal{P}_T$)-invariant. Observe that, this would give a new state for the holonomy-flux $C^*$-algebra, but the flux operators would be implemented by maps $H_{S, \Gamma}$ instead of $G_{S, \Gamma}$.

Recognize that, the elements of $\bar{\mathcal{A}}_T$ are of the form $h_T(\Gamma')$ where $\Gamma'$ is a subgraph of $\Gamma$ and the natural or the non-standard identification is applied to identify $\bar{\mathcal{A}}_T$ with $G^N$. Hence there exists, additionally to the actions in $\text{Act}(G_{S, \Gamma}, C(\bar{\mathcal{A}}_T))$, the actions of $G_{S, \Gamma}$ or $Z_{S, \Gamma}$ on $C(\bar{\mathcal{A}}_T)$, which are defined in remark 16.

Furthermore due to the fact that the number of subgraphs of $\Gamma$ generated by the edges of $\Gamma$ is finite, there exists a finite set $\mathcal{B}_{S, \Gamma}(\mathcal{P}_T)$ of bisections such that each of bisection is a map from the set $V_T$ to a distinct subgraph of $\Gamma$ such that all elements of $\mathcal{P}_T$ are constructed from the finite set $\mathcal{B}_{S, \Gamma}(\mathcal{P}_T)$. Call such a set of bisections a generating system of bisections for a graph $\Gamma$.
**Proposition 87.** Let $\mathcal{B}_{S,\text{surf}}^\Gamma(\mathcal{P}_\Gamma) := \{\sigma_t \in \mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)\}_{1 \leq t \leq k}$ be a subset of $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ that forms a generating system of bisections for the graph $\Gamma$.

Then there is a state $\hat{\omega}_\mathcal{B}^\Gamma$ on $C(\mathcal{A}_\Gamma)$ given by

$$\hat{\omega}_\mathcal{B}^\Gamma(f) := \frac{1}{k} \sum_{i=1}^{k} \omega^\Gamma_i(\sigma_i(f)) \quad \text{for } \sigma_i \in \mathcal{B}_{S,\text{surf}}^\Gamma(\mathcal{P}_\Gamma)$$

which is $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant and where $k$ is the maximal number of subgraphs, which are generated by all edges and their compositions of the graph $\Gamma$.

Consequently the state $\hat{\omega}_\mathcal{B}^\Gamma$ on $C(\mathcal{A}_\Gamma)$ is $\hat{\mathcal{Z}}_{S,\Gamma}$, $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$- and $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant, i.e. $\hat{\omega}_\mathcal{B}^\Gamma$ is contained in the set $\mathcal{S}^{S,\text{surf,or}}(C(\mathcal{A}_\Gamma))$ of all $\mathcal{Z}_{S,\Gamma}$, $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$- and $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant states on $C(\mathcal{A}_\Gamma)$.

The actions $\alpha \in \text{Act}(\hat{\mathcal{Z}}_{S,\Gamma}, C(\mathcal{A}_\Gamma))$ and $\zeta \in \text{Act}(\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma), C(\mathcal{A}_\Gamma))$ commute, i.e.

$$(\alpha(\rho_{S,\Gamma}(\Gamma')) \circ \zeta)(f) = (\zeta \circ \alpha(\rho_{S,\Gamma}(\Gamma')))(f) \quad \forall f \in \mathcal{A}_\Gamma$$

where $S = S, \rho_{S,\Gamma}(\gamma) \circ \sigma(t(\gamma_i)) = \rho_{S,\Gamma}(\gamma_i)$ for $i = 1, ..., M$ yields for $\sigma \in \mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$, $\gamma \in S = \{t(\gamma_i)\}$. Clearly equality holds for every $\rho_{S,\Gamma}(\Gamma') \in \hat{\mathcal{Z}}_{S,\Gamma}$. Hence the action of $\hat{G}_{S,\Gamma}$ and the action of $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ on the analytic holonomy $C^*$-algebra do not commute.

In particular observe that, if the natural identification of $\mathcal{A}_\Gamma$ is assumed, then there exists no map $D$ such that

$$(D(\zeta \circ \alpha(\rho_{S,\Gamma}(\Gamma')))(f))((\zeta \circ \alpha(\rho_{S,\Gamma}(\Gamma'))))$$

holds for every element $\rho_{S,\Gamma} \in G_{S,\Gamma}$, $\rho_{S,\Gamma} \in G_{S,\Gamma}$ and whenever $\sigma(t(\gamma_i))$ and $S_\sigma$ do not intersect each other for every $i = 1, ..., M$.

But if $\rho_{S,\Gamma} \in \mathcal{Z}_{S,\Gamma}$, then there is a map $D$ such that

$$(D(\zeta \circ \alpha(\rho_{S,\Gamma}(\Gamma')))(f))((\zeta \circ \alpha(\rho_{S,\Gamma}(\Gamma'))))$$

is fulfilled.

Consequently the actions of the flux group and the group of bisections are treated simultaneously only in the case of the commutative flux group $\hat{\mathcal{Z}}_{S,\Gamma}$.
Proof: First observe that, the state $\tilde{\omega}_B^G$ defined in the proposition is well-defined in both cases of a natural or non-standard identification of $\tilde{A}_G$ and $G^N$. To conclude that, (67) yields for an action of $\tilde{Z}_{S,G}$ on $C(\tilde{A}_G)$, investigate the computation

$$\tilde{\omega}_B^G((\zeta_\sigma \circ \alpha(\rho_{S,G}(\Gamma')))(f_\Gamma)) = \frac{1}{k} \sum_{i=1}^{k} \omega_M^G(\zeta_{\sigma_{i+1}}(\alpha(\rho_{S,G}(f_\Gamma))))$$

$$= \frac{1}{k} \sum_{i=1}^{k} \omega_M^G(\alpha(\rho_{S,G}(\Gamma'))(\zeta_{\sigma_{i+1}}(f_\Gamma)))$$

$$= \frac{1}{k} \sum_{i=1}^{k} \omega_M^G(\zeta_{\sigma_{i+1}}(f_\Gamma))$$

$$= \tilde{\omega}_B^G(f_\Gamma) = \tilde{\omega}_B^G((\alpha(\rho_{S,G}(\Gamma')) \circ \zeta_\sigma)(f_\Gamma))$$

for $\rho_{S,G}(\Gamma'), \rho_{S,G}(\Gamma') \in \tilde{Z}_{S,G}$, $\sigma \in B_{S,surf}(P_G)$ or $\sigma \in B_{S,os}(P_G)$.

It is possible to construct a state $\tilde{\omega}_B^G$ on $C(\tilde{A}_G)$, which is $\tilde{G}_{S,G}^\omega$-invariant. But this need more technical details, which are not concerned in this article.

An action of the local flux group on the holonomy algebra for finite graph systems

Let $\Gamma' := \{\gamma_1, \ldots, \gamma_M\}$ be a subgraph of $\Gamma$. Recall the maps $G^\text{loc}_\Gamma$ presented in definition 44. For an element $g_\Gamma \in G^\text{loc}_\Gamma$ it is true that, $g_\Gamma(\Gamma')$ is identified with the element $(g_\Gamma(s(\gamma_1)), \ldots, g_\Gamma(s(\gamma_M)))$ in $G^{\Gamma'}$. Notice that, it is not necessary to focus on natural identified graphs in a finite graph system. Then there is an action $\alpha_{\text{loc}}$ of $G^\text{loc}_\Gamma$ on $C(\tilde{A}_G)$ given by $$(\alpha_{\text{loc}}(g_\Gamma(\Gamma'))f_\Gamma)(h_\Gamma(\Gamma')) := f_\Gamma(g_\Gamma(s(\gamma_1))h_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}, \ldots, g_\Gamma(s(\gamma_M))h_\Gamma(\gamma_M)g_\Gamma(t(\gamma_M))^{-1})$$

Consider the example, which is given by a graph $\Gamma := \{\gamma_1, \gamma_2\}$ and a subgraph $\Gamma' := \{\gamma_1 \circ \gamma_2\}$. Then calculate

$$(\alpha_{\text{loc}}(g_\Gamma(\Gamma'))f_\Gamma)(h_\Gamma(\Gamma')) = (D_S \alpha_{\text{loc}}(g_\Gamma(\Gamma'))D_S^{-1}f_\Gamma)(h_\Gamma(\gamma_1 \circ \gamma_2))$$

$$= (D_S \alpha_{\text{loc}}(g_\Gamma(\Gamma'))f_\Gamma)(h_\Gamma(\gamma_1), h_\Gamma(\gamma_2))$$

$$= (D_S f_\Gamma)(g_\Gamma(s(\gamma_1))h_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1}, g_\Gamma(s(\gamma_2))h_\Gamma(\gamma_2)g_\Gamma(t(\gamma_2))^{-1})$$

$$= f_\Gamma(g_\Gamma(s(\gamma_1))h_\Gamma(\gamma_1)g_\Gamma(t(\gamma_1))^{-1} g_\Gamma(s(\gamma_2))h_\Gamma(\gamma_2)g_\Gamma(t(\gamma_2))^{-1})$$

$$= f_\Gamma(g_\Gamma(s(\gamma_1))h_\Gamma(\gamma_1 \circ \gamma_2)g_\Gamma(t(\gamma_2))^{-1})$$

Definition 88. The $G^\text{loc}_\Gamma$-fixed point subalgebra of $C(\tilde{A}_G) := A_G$ is given by

$$A^\text{loc}_G := \{f_\Gamma \in A_G : \alpha_{\text{loc}}(g_\Gamma(\Gamma'))(f_\Gamma) = f_\Gamma \quad \forall g_\Gamma(\Gamma') \in G^\text{loc}_\Gamma\}$$

and this algebra is called the $C^*$-algebra of gauge invariant holonomies restricted to finite graph systems.

Notice that, there is a isomorphism between the $C^*$-algebras $A^\text{loc}_G$ and $C(\tilde{A}_G/\tilde{\mathcal{G}}_\Gamma)$.

3.3 Weyl $C^*$-algebras associated to surfaces and inductive limits of finite graph systems

Weyl $C^*$-algebras associated to surfaces and finite graph systems

Definition 89. Let $\Gamma$ be a graph and $P_G$ be a finite graph system associated to $\Gamma$ and $S$ a surface set. Let $S$ be the set of all suitable surface sets for $\Gamma$.

The algebra generated by all elements of $C(\tilde{A}_G)$ and $\mathcal{W}(\tilde{G}_{S,G})$, which satisfy the canonical commutator relations, form an abstract Weyl $*$-algebra $\mathcal{W}(S, \Gamma)$ for a surface set and a finite graph system associated to a graph $\Gamma$. The algebra generated by all elements of $C(\tilde{A}_G)$ and $\mathcal{W}(\tilde{G}_{S,G})$ forms an abstract Weyl $*$-algebra $\mathcal{W}(S, \Gamma)$ for surfaces and a finite graph system associated to a graph $\Gamma$. 60
Due to the fact that all unitaries $U$ (or $V$) define a homomorphism of $\tilde{O}_{S,\Gamma}$ (or $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$) into a unitary group of $C(H_{\Gamma})$, the abstract Weyl $*$-algebra $\mathcal{W}(\tilde{S},\Gamma)$ is completed to a $C^*$-algebra.

Summarising the Weyl $C^*$-algebra of Loop Quantum Gravity is generated by continuous functions depending on holonomies along paths and the (strongly) continuous unitary flux operators.

**Proposition 90.** Let $H_{\Gamma}$ be the Hilbert space $L^2(\tilde{A}_{\Gamma},\mu_{\Gamma})$ with norm $\|\cdot\|_2$.

The $*$-algebra generated by all elements of $C(\tilde{A}_{\Gamma})$ and $\mathcal{W}(G_{S,\Gamma})$ for every surface set $\tilde{S}$ in $\mathbb{S}$, which satisfy the canonical commutator relations (59), completed w.r.t. the $\|\cdot\|_2^*-\text{norm}$ is a $C^*$-algebra. This $C^*$-algebra is the **Weyl $C^*$-algebra for surfaces and a finite graph system**. Denote this $C^*$-algebra by $\mathcal{W}\text{y}(\tilde{S},\Gamma)$.

The set $\mathbb{S}$ of surface sets, which is used to define the $C^*$-algebra $\mathcal{W}\text{y}(\mathbb{S},\Gamma)$, and the set $\mathbb{S}_{\mathcal{Z}}$, which defines $\mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$, are distinguished from each other. The set $\mathbb{S}_{\mathcal{Z}}$ contains the set $\mathbb{S}$.

**Proposition 91.** The $*$-algebra generated by all elements of $C(\tilde{A}_{\Gamma})$ and $\mathcal{W}(G_{S,\Gamma})$ for every surface set $\tilde{S}$ in $\mathbb{S}$, which satisfy the canonical commutator relations (59), completed w.r.t. the norm

$$\|W\| := \text{sup}\{\|\pi_r(W)\|_r : \pi_r \text{ a unital } *\text{-representation of } \mathcal{W}(\tilde{S},\Gamma) \text{ on } H_r \text{ for all } \tilde{S} \in \mathbb{S}\}$$

is a $C^*$-algebra. This $C^*$-algebra is called the **universal Weyl $C^*$-algebra for surfaces and a finite graph system** and will be denoted by $\mathcal{W}\text{y}(\mathbb{S},\Gamma)$.

The Weyl algebra for surfaces

**Proposition 92.** Define the action of $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ (or $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$) on $\mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ by

$$\zeta_\sigma(U(\rho_{S,\Gamma}(\Gamma'))f_r) := (\zeta_\sigma(U))(\rho_{S,\Gamma}(\Gamma'))\zeta_\sigma(f_r)$$

whenever $\sigma \in \mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ and $U(\rho_{S,\Gamma}(\Gamma'))f_r \in \mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ for every surface $S$ in the set $\tilde{S}$, which is contained in $\mathbb{S}_{\mathcal{Z}}$. This action is automorphic and point-norm continuous.

Let $\tilde{S}$ and $\tilde{S}'$ be two disjoint surface sets in $\mathbb{S}_{\mathcal{Z}}$. Then the action of $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ on $\mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ satisfies

$$\left(\zeta_\sigma(U)\right)(\rho_{S,\Gamma}(\Gamma')) := 1_{\Gamma}$$

for $\sigma \in \mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$ and $U(\rho_{S,\Gamma}(\Gamma')) \in \mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ and $S \in \tilde{S}'$.

**Proposition 93.**

(i) The state $\omega^G_{\mathcal{Z}}$ on $C(\tilde{A}_{\Gamma})$, which is defined in 71 and which is $\tilde{O}_{S,\Gamma}$-invariant for every surface set $\tilde{S}$ in $\mathbb{S}$, extends to a state $\tilde{\omega}^G_{\mathcal{Z}}$ on $\mathcal{W}\text{y}(\mathbb{S},\Gamma)$. The state $\tilde{\omega}^G_{\mathcal{Z}}$ is pure and unique.

(ii) The state $\omega^G_{\mathcal{Z}}$ on $C(\tilde{A}_{\Gamma})$, which is defined in 87 and which is $\tilde{Z}_{S,\Gamma}$, $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$- and $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant for every surface set $\tilde{S}$ in $\mathbb{S}_{\mathcal{Z}}$, extends to a state on $\mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$. The state $\omega^G_{\mathcal{Z}}$ is $\tilde{Z}_{S,\Gamma}$, $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$- and $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant.

**Definition 94.** The set of all not necessarily pure states on $\mathcal{W}\text{y}(\mathbb{S}_{\mathcal{Z}},\Gamma)$ that are $\tilde{Z}_{S,\Gamma}$, $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$- and $\mathcal{B}_{S,\text{surf}}(\mathcal{P}_\Gamma)$-invariant are denoted by $\mathcal{W}\text{y,}\mathcal{S}\text{surf,}\mathcal{P}\text{surf}(\mathbb{S}_{\mathcal{Z}},\Gamma)$.

**Proof of proposition 93.**

The part (i) of the proposition follows from the corollary 72 and the proposition 71. The part (ii) of the proposition follows from the following derivations. First fix a surface set $\tilde{S}$ in $\mathbb{S}_{\mathcal{Z}}$. 

---

$\mathbb{Z}$: modulo the two-sided self-adjoint ideal of the $*-$algebra defined by $I = \{W : \|W\|_2 = 0\}$

$\mathbb{Z}$: modulo the two-sided self-adjoint ideal of the $*-$algebra defined by $I = \{W : \|W\|_2 = 0\}$

$\mathbb{Z}$: modulo the two-sided self-adjoint ideal of $\mathcal{W}\text{y}(\tilde{S},\Gamma)$ defined by $I = \{W : \|W\|_2 = 0\}$ for all $\tilde{S} \in \mathbb{S}$
Step 1:
For the covariant pair \((\Phi_M, V)\) of \((\mathfrak{B}_{S,\text{or}}(P_T), C(\tilde{A}_T), \zeta)\) or \((\mathfrak{B}_{S,\text{surf}}(P_T), C(\tilde{A}_T), \zeta)\) on \(H_T\) there exists an invariant state. In proposition [87] this state is defined and satisfies
\[
\hat{\omega}_M^\Gamma (\zeta_\sigma (f_T)) = \hat{\omega}_B(f_T)
\]
for all \(f_T \in C(\tilde{A}_T)\) and for arbitrary \(\sigma \in \mathfrak{B}_{S,\text{surf}}(P_T)\) or \(\sigma \in \mathfrak{B}_{S,\text{or}}(P_T)\). Recall that, the state \(\omega_B^\Gamma\) on \(C(\tilde{A}_T)\) is required to be \(\tilde{Z}_{S,\Gamma}\)-invariant, too. Hence the state satisfies
\[
\hat{\omega}_M^\Gamma (\alpha(\rho_{S,\Gamma}(\Gamma))(f_T)) = \hat{\omega}_B(f_T)
\]
for all \(f_T \in C(\tilde{A}_T)\) and \(\rho_{S,\Gamma}(\Gamma) \in G_{S,\Gamma}\). While \(\alpha(\rho_{S,\Gamma}(\Gamma))(f_T) \in C(\tilde{A}_T)\) and the actions \(\zeta\) and \(\alpha\) commute, the state \(\hat{\omega}_B^\Gamma\) fulfill
\[
\hat{\omega}_M^\Gamma (\alpha(\rho_{S,\Gamma}(\Gamma))(f_T)) = \hat{\omega}_B^\Gamma (\zeta_\sigma (\alpha(\rho_{S,\Gamma}(\Gamma))(f_T))) = \hat{\omega}_B^\Gamma (\alpha(\rho_{S,\Gamma}(\Gamma)) (\zeta_\sigma (f_T)))
\]
Clearly there is a morphism \(\Phi \in \text{Mor}(C(\tilde{A}_T), \text{Weyl}_Z(S, \Gamma))\).

Step 2:
On the other hand, there are covariant representations \((\Psi, V)\) of the \(C^\ast\)-dynamical systems
\((\mathfrak{B}_{S,\text{or}}(P_T), W(G_{S,\Gamma}), \zeta)\) and \((\mathfrak{B}_{S,\text{surf}}(P_T), W(G_{S,\Gamma}), \zeta)\) in \(L(H_T)\). There is a \(G_{S,\Gamma}\)-invariant, \(\mathfrak{B}_{S,\text{surf}}(P_T)\)-invariant and \(\mathfrak{B}_{S,\text{or}}(P_T)\)-invariant state \(\omega_{M,2B}^\Gamma\) on \(W(G_{S,\Gamma})\).

Step 3:
There are covariant representations \((\Phi_T, V)\) of the \(C^\ast\)-dynamical systems
\((\mathfrak{B}_{S,\text{or}}(P_T), \text{Weyl}_Z(S, \Gamma), \zeta)\) and \((\mathfrak{B}_{S,\text{surf}}(P_T), \text{Weyl}_Z(S, \Gamma), \zeta)\) in \(L(H_T)\), where
\(\Phi_T(W) = \Psi(W)\) for \(W = U \in W(Z_{S,\Gamma})\) or \(\Phi_T(W) = \Phi_M(W)\) for \(W = f_T \in C(\tilde{A}_T)\). Consequently there exists a \(\tilde{Z}_{S,\Gamma}\)-invariant, \(\mathfrak{B}_{S,\text{surf}}(P_T)\)-invariant and \(\mathfrak{B}_{S,\text{or}}(P_T)\)-invariant state \(\omega_{M,2B}^\Gamma\) on \(\text{Weyl}_Z(S, \Gamma)\). This state is an extension of the state \(\omega_B^\Gamma\) on \(C(\tilde{A}_T)\) by Hahn-Banach theorem.

Then the state restricted to \(C(\tilde{A}_T)\) is given by
\[
\omega_{M,2B}^\Gamma(f_T) = \hat{\omega}_B^\Gamma(f_T) \quad \forall f_T \in C(\tilde{A}_T)
\]
and restricted to \(W(\tilde{Z}_{S,\Gamma})\) it is given by
\[
\omega_{M,2B}^\Gamma(W) = \hat{\omega}_{M,2B}^\Gamma(1_T) \quad \forall W \in W(\tilde{Z}_{S,\Gamma})
\]
such that \(\hat{\omega}_{M,2B}^\Gamma(W^*W) = 1\) holds. Then
\[
\omega_{M,2B}^\Gamma(W f_T) = \omega_{M,2B}^\Gamma(f_T) = \omega_{M,2B}^\Gamma(f_T W)
\]
yields for all \(W \in W(\tilde{Z}_{S,\Gamma})\) and \(f_T \in C(\tilde{A}_T)\).

Observe that,
\[
\omega_{M,2B}^\Gamma(V^*e_V) = 1 \quad \forall V \in \text{Rep}(\mathfrak{B}_{S,\text{or}}(P_T), K(H_T)) \text{ or } V \in \text{Rep}(\mathfrak{B}_{S,\text{surf}}(P_T), K(H_T))
\]
holds.

Finally all steps are true for every surface set \(\tilde{S}\) and hence for all surface sets in \(S_Z\).

\[\blacksquare\]

Now, in the next investigations the focus lies on actions of fluxes and diffeomorphisms on the inductive limit algebra \(C(\tilde{A})\).

**Definition 95.** Let \(\Gamma_\infty\) be the inductive limit of a inductive family \(\{\Gamma_i\}\) of graphs. Then \(\mathcal{P}_{\Gamma_\infty}\) denotes the inductive limit of a inductive family of finite graph systems \(\mathcal{P}_{\Gamma_i}\). Moreover let \(G^\infty\) be the projective limit of the family of groups \(\{G^N_i\}\) if \(G^N_i = G \times \ldots \times G\) and \(G\) is a compact group.
Let \((\varphi, \Phi) \in \text{Diff}(\mathcal{P})\) be a path-diffeomorphism of a path groupoid \(\mathcal{P} \equiv \Sigma\) such that \(\varphi : \Sigma \to \Sigma\) and \(\Phi : \mathcal{P} \to \mathcal{P}\). Then a graph-diffeomorphism for a limit graph system \(\mathcal{P}_{\Gamma_\infty}\) is given by the pair \((\varphi_\Sigma, \Phi_\infty)\) of maps such that \(\varphi_\Sigma : \Sigma \to \Sigma\), \(\Phi_\infty : \mathcal{P}_{\Gamma_\infty} \to \mathcal{P}_{\Gamma_\infty}\) and

\[
\Phi_\infty(\Gamma) = (\Phi(\gamma_1), \ldots, \Phi(\gamma_N)) = \Gamma_\Phi
\]

for \(\Gamma := \{\gamma_1, \ldots, \gamma_N\}\) and \(\Gamma_\Phi\) being two subgraphs of \(\mathcal{P}_{\Gamma_\infty}\). The set of such graph-diffeomorphisms for a limit graph system \(\mathcal{P}_{\Gamma_\infty}\) is denoted by \(\text{Diff}(\mathcal{P}_{\Gamma_\infty})\).

Then there is an action of graph-diffeomorphisms for a limit graph system \(\mathcal{P}_{\Gamma_\infty}\) on the analytic holonomy \(C^*\)-algebra \(C(\tilde{\mathcal{A}})\) defined by

\[
(\theta_{(\varphi_\Sigma, \Phi_\infty)} f) (b(\Gamma)) := (f)(b(\Phi_\infty(\Gamma))) = (f)(b(\Gamma_\Phi))
\]

whenever \(\Gamma, \Gamma_\Phi \in \mathcal{P}_{\Gamma_\infty}\), \((\varphi_\Sigma, \Phi_\infty) \in \text{Diff}(\mathcal{P}_{\Gamma_\infty})\) and for

\[
f(b(\Gamma)) = (f_\Gamma \circ \pi_\Gamma)(b(\Gamma)) = (\beta_\Gamma f_\Gamma)(b(\Gamma))
\]

where \(\pi_\Gamma : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_\Gamma\) is a surjective projection and \(\beta_\Gamma : C(\tilde{\mathcal{A}}_\Gamma) \to C(\tilde{\mathcal{A}})\) are injective unit-preserving \(*\)-homomorphisms satisfying consistency conditions.

The group of bisections \(\mathcal{B}(\mathcal{P})\) is defined to be the set of all smooth maps \(\sigma\) from \(\Sigma\) to the path groupoid \(\mathcal{P} \equiv \Sigma\) such that \(s \circ \sigma = \text{id}_\Sigma\) and \(t \circ \sigma : \Sigma \to \Sigma\) is a diffeomorphism. Therefore due to the group morphism \(\mathcal{B}(\mathcal{P}) \supset \sigma \mapsto t \circ \sigma \in \text{Diff}(\Sigma)\) there exists also an action of the bisections of the \(C^*\)-algebra \(C(\tilde{\mathcal{A}})\). Recognize that, it is possible to rewrite

\[
\Phi_\infty(\Gamma) =: \Gamma_{\sigma_\Sigma}\text{ for a bisection }\sigma_\Sigma \in \mathcal{B}(\mathcal{P}_{\Gamma_\infty})\text{ on the limit graph system }\mathcal{P}_{\Gamma_\infty}.
\]

Recall the action \(\zeta\) of the group \(\mathcal{B}(\mathcal{P}_{\Gamma})\) of bisections for a finite graph system \(\mathcal{P}_{\Gamma}\) on \(C(\tilde{\mathcal{A}}_{\Gamma})\), which is given in proposition [75] by

\[
(\zeta_{\sigma_{\Sigma}} f)(b(\Gamma)) = (f_\Gamma \circ \pi_{\Gamma})(b(\Gamma)) = f_\Gamma(b(\Gamma_{\sigma_{\Sigma}}))
\]

whenever \(\sigma \in \mathcal{B}(\mathcal{P}_{\Gamma})\), \(f_\Gamma \in C(\tilde{\mathcal{A}}_{\Gamma})\) and \(\mathcal{P}_{\sigma} \leq \mathcal{P}_{\Gamma}\).

**Definition 96.** There is an action of the group of global bisections \(\mathcal{B}(\mathcal{P}_{\Gamma_\infty})\) on the algebra \(C(\tilde{\mathcal{A}})\) given by

\[
(\zeta_{\sigma_\Sigma} f)(b(\Gamma)) := f(R_{\sigma_\Sigma}(b(\Gamma))) = f(b(\Gamma_{\sigma_\Sigma}))
\]

\[
= \left(\left(\beta_{\Gamma} \circ \beta_{\Gamma_{\sigma_\Sigma}}\right) f \Gamma_{\sigma_\Sigma}\right) (b(\Gamma_{\sigma_\Sigma}))
\]

\[
= \left(\beta_{\Gamma} f_{\Gamma}\right)\left(\beta_{\Gamma}(\Gamma_{\sigma_\Sigma})\right) = \left(\beta_{\Gamma}(\zeta_{\sigma_{\Sigma}} f)(b(\Gamma))\right)
\]

\[
= \left(\beta_{\Gamma} f_{\Gamma}\right)\left(\beta_{\Gamma_{\sigma_\Sigma}}(b(\Gamma))\right)
\]

whenever \(\mathcal{P}_{\Gamma} \leq \mathcal{P}_{\Gamma_{\sigma_\Sigma}} \leq \mathcal{P}_{\Gamma}\), for a function \(f \in C(\tilde{\mathcal{A}})\), where \(\beta_{\Gamma} : C(\tilde{\mathcal{A}}_{\Gamma}) \to C(\tilde{\mathcal{A}}), \beta_{\Gamma_{\sigma_\Sigma}} f_{\Gamma} : C(\tilde{\mathcal{A}}_{\Gamma_{\sigma_\Sigma}}) \to C(\tilde{\mathcal{A}}_{\Gamma})\) are unit-preserving injective \(*\)-homomorphisms satisfying consistency conditions and for a global bisections \(\sigma_\Sigma \in \mathcal{B}(\mathcal{P}_{\Gamma_\infty})\) such that for a bisection \(\sigma \in \mathcal{B}(\mathcal{P}_{\Gamma})\) on a finite graph system \(\mathcal{P}_{\Gamma}\) it is true that \(\sigma_{\Sigma_{\Gamma_{\sigma_\Sigma}}} = \sigma(V_{\Gamma})\).

The limit Hilbert space \(\mathcal{H}_\infty\) with norm \(\| \cdot \|_\infty\) is constructed from the inductive family of Hilbert spaces \(\mathcal{H}_{\Gamma}\).

But these actions related to the limit graph system \(\mathcal{P}_{\Gamma_\infty}\) are not norm-point continuous. This is proved by the following argument. Since from \(b_{\Gamma}\) is not a continuous groupoid morphism between \(\mathcal{P} \equiv \Sigma\) to \(G\) over \(\{e_G\}\) it follows that,

\[
\lim_{\sigma_{\Sigma}(\Sigma) \to \text{id}(\Sigma)} \| \zeta_{\sigma_{\Sigma}} f \|_\infty = \lim_{\sigma_{\Sigma}(\Sigma) \to \text{id}(\Sigma)} \| \beta_{\Gamma}(\zeta_{\sigma_{\Sigma}} f_{\Gamma}) - f \|_\infty
\]

\[
= \lim_{\sigma_{\Sigma}(\Sigma) \to \text{id}(\Sigma)} \| (\beta_{\Gamma} f_{\Gamma})(b_{\Gamma}(\Gamma))(\sigma(V'))(b_{\Gamma}(\sigma(V))) \|_\infty
\]

yields for a function \(f \in C(\tilde{\mathcal{A}})\), a subgraph \(\Gamma := \{\gamma_1, \ldots, \gamma_N\}\) of \(\tilde{\Gamma}\), a subset \(V_{\tilde{\Gamma}} := V' \cup V_{\tilde{\Gamma}}\) where \(V' := \{t(\gamma_1), \ldots, t(\gamma_N)\}\) and \(N = |\Gamma|\), a global bisection \(\sigma_{\Sigma} \in \mathcal{B}(\mathcal{P}_{\Gamma_\infty})\) such that \(\sigma_{\Sigma}(V_{\tilde{\Gamma}}) = (\tilde{\sigma}_{\Sigma}(v_1), \ldots, \tilde{\sigma}_{\Sigma}(v_{2N}))\)
where $\bar{\sigma}_C \in \mathcal{B}(P)$ and there is a bijection $\sigma \in \mathcal{B}(P_{\Gamma})$ such that $\sigma(V_{\Gamma}) = \sigma_C(V_{\Gamma})$ yields. Since there is an group morphism between $\mathcal{B}(P)$ and the group of diffeomorphisms $\text{Diff}(\Sigma)$ on the spatial manifold $\Sigma$, the diffeomorphism cannot be implemented as strongly or weakly continuous representations on the limit Hilbert space $\mathcal{H}_\infty$. Nevertheless the action $\zeta$ of $\mathcal{B}(P)_{\Gamma_{\infty}}$ on $C(\bar{A})$ is automorphic. Denote the set of automorphic actions of a group $\mathcal{B}(P_{\Gamma})$ on the commutative $C^*$-algebra $C(\bar{A})$ by $\text{Act}_0(\mathcal{B}(P_{\Gamma}), C(\bar{A})).$

Despite the discontinuity of the action on the inductive limit of $C^*$-algebras $C(\bar{A})$, there are injective $^*$-homomorphisms $\beta_{\Gamma,\Gamma'}$, such that

$$\begin{align*}
f(b(\Gamma')) &= \left((\beta_{\Gamma} \circ \beta_{\Gamma,\Gamma'})(b_{\Gamma'})\right)(b_{\Gamma'}(\Gamma'))
\end{align*}$$

yields for any graphs such that $P_{\Gamma'} \leq P_{\Gamma} \leq P_\Gamma$.

In LQG literature the set of surfaces is not restricted, the set $\bar{S}$ is an infinite set of surfaces and the inductive limit of a family of graph systems is constructed from a limit of a family of graphs. In this article the infinite set of surfaces is decomposed into several finite sets. To implement an action of $G_{\bar{S},\Gamma}$ the inductive limit structure of graphs has to preserve the particular sort of the action for a fixed suitable surface set $\bar{S}$.

**Proposition 97.** Let $\Gamma_{\infty}$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that the set $\bar{S}$ of surfaces has the same surface intersection property for each graph $\Gamma_i$ of the family. Let $\bar{S}$ be a suitable surface set with same right surface intersection property for each graph $\Gamma_i$ of the family. Then $P_{\Gamma_{\infty}}$ is the inductive limit of a inductive family $\{P_{\Gamma_i}\}$ of finite orientation preserved graph systems.

Then there is an action of $G_{\bar{S},\Gamma_{\infty}}$ on $C(\bar{A})$ given by

$$\begin{align*}
(\alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))f)(b(\Gamma)) &= f(L(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))(b(\Gamma))) \\
&= (\beta_{\Gamma} \circ \alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma)) f)(b_{\Gamma}(\Gamma)) = (\beta_{\Gamma} f_{\Gamma})(L(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))(b_{\Gamma}(\Gamma)))
\end{align*}$$

for $P_{\Gamma_{\infty}} \leq P_{\Gamma} \leq P_{\Gamma_{\infty}}$, injective unit-preserving $^*$-homomorphism $\beta_{\Gamma} : C(\bar{A}) \longrightarrow C(\bar{A})$ satisfying consistency conditions, elements $\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma_{\infty}}$ and there are elements $\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma}$ such that $\rho_{\bar{S},\Gamma}(\Gamma) = \rho_{\bar{S},\Gamma_{\infty}}(\Gamma)$ for all $\Gamma \in P_{\Gamma}$ and every surface $S_i \in \bar{S}$.

There is another action of $G_{\bar{S},\Gamma_{\infty}}$ on $C(\bar{A})$ defined by

$$\begin{align*}
(\alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))f)(b(\Gamma)) &= f(R(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))(b(\Gamma))) \\
&= (\beta_{\Gamma} \circ \alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma)) f)(b_{\Gamma}(\Gamma)) = (\beta_{\Gamma} f_{\Gamma})(R(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))(b_{\Gamma}(\Gamma)))
\end{align*}$$

for $P_{\Gamma_{\infty}} \leq P_{\Gamma} \leq P_{\Gamma_{\infty}}$, injective unit-preserving $^*$-homomorphism $\beta_{\Gamma} : C(\bar{A}) \longrightarrow C(\bar{A})$ satisfying consistency conditions, elements $\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma_{\infty}}$ and there are elements $\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma}$ such that $\rho_{\bar{S},\Gamma}(\Gamma) = \rho_{\bar{S},\Gamma_{\infty}}(\Gamma)$ for all $\Gamma \in P_{\Gamma}$ and every surface $S_i \in \bar{S}$.

These actions of the flux group $G_{\bar{S},\Gamma_{\infty}}$ for the surface set $\bar{S}$ and $G_{\bar{S},\Gamma_{\infty}}$ for surfaces in $\bar{S}$ on $C(\bar{A})$ are automorphic and point-norm continuous.

**Proof:** The point-norm continuity follows from the observation that

$$\begin{align*}
\lim_{\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \rightarrow \text{id}_{\bar{S},\Gamma_{\infty}}(\Gamma)} \| \alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma))(f) - f \|_{\text{sup}} &= 0 \\
\lim_{\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \rightarrow \text{id}_{\bar{S},\Gamma_{\infty}}(\Gamma)} \| (\beta_{\Gamma}(\alpha(\rho_{\bar{S},\Gamma_{\infty}}(\Gamma)) f_{\Gamma})) - \beta_{\Gamma}(f_{\Gamma}) \|_{\text{sup}} &= 0
\end{align*}$$

holds whenever $P_{\Gamma} \leq P_{\Gamma}^\prime$, $\rho_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma_{\infty}}$ and $\text{id}_{\bar{S},\Gamma_{\infty}}(\Gamma) \in G_{\bar{S},\Gamma_{\infty}}$, which is defined by $\text{id}_{\bar{S},\Gamma_{\infty}}(\Gamma) = (\text{id}_{\bar{S}(\gamma_1)}, \ldots, \text{id}_{\bar{S}(\gamma_N)}) = (e_G, \ldots, e_G)$ for a graph $\Gamma := \{\gamma_1, \ldots, \gamma_N\}$.

**Definition 98.** Let $\Gamma_{\infty}$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that the set $\bar{S}$ of surfaces has the surface intersection property for each graph $\Gamma_i$ of the family. Then $P_{\Gamma_{\infty}}$ is the inductive limit of a inductive family $\{P_{\Gamma_i}\}$ of finite graph systems.

Let $(\varphi, \Phi) \in \text{Diff}(P)$ be a path-diffeomorphism of a path groupoid $P \to \Sigma$ such that
The set of surface-preserving graph-diffeomorphism for a limit graph system

\( \mathcal{P} \in \mathbb{C} \), which leave each surface in \( \tilde{S} \) and a suitable neighborhood of each surface in \( \tilde{S} \) invariant

and \( \Phi : \mathcal{P} \to \mathcal{P} \);

- if a path \( \gamma \) in \( \mathcal{P} \) does not intersect all surfaces, then \( \Phi(\gamma) \) does not intersect all surfaces and

- the number of all generators \( \{ \gamma_i \} \) of \( \Gamma_i \) and the number of all transformed paths \( \{ \Phi(\gamma_i) \} \) that intersect each surface in \( \tilde{S} \) in their target vertices are constant and equal

is called a surface-preserving path-diffeomorphism for a path groupoid \( \mathcal{P} \rightrightarrows \Sigma \) and a surface set \( \tilde{S} \).

Then a surface-preserving graph-diffeomorphism for a limit graph system \( \mathcal{P}_{\Gamma_\infty} \) is given by the pair \((\varphi_\Sigma, \Phi_\infty)\) of maps such that

- \( \varphi_\Sigma : \Sigma \to \Sigma \), \( \Phi_\infty : \mathcal{P}_{\Gamma_\infty} \to \mathcal{P}_{\Gamma_\infty} \) and

\[ \Phi_\infty(\Gamma) = (\Phi(\gamma_1), ..., \Phi(\gamma_N)) = \Gamma_\Phi \]

for \( \Gamma := \{ \gamma_1, ..., \gamma_N \} \) and \( \Gamma_\Phi \) being two subgraphs of \( \mathcal{P}_{\Gamma_\infty} \) and

- \( (\varphi_\Sigma, \Phi) \) is a surface-preserving path-diffeomorphism for a path groupoid \( \mathcal{P} \rightrightarrows \Sigma \) and a surface set \( \tilde{S} \).

The set of surface-preserving graph-diffeomorphism for a limit graph system \( \mathcal{P}_{\Gamma_\infty} \) is denoted by \( \text{Diff}_{\text{surf}}(\mathcal{P}_{\Gamma_\infty}) \).

With no doubt the group \( \mathbb{B}_{\text{surf}}(\mathcal{P}_{\Gamma_\infty}) \) of surface-preserving bisections of a limit graph system \( \mathcal{P}_{\Gamma_\infty} \) can be defined, too.

**Definition 99.** Let \( \Gamma_\infty \) be the inductive limit of a family of graphs \( \{ \Gamma_i \} \) such that the set \( \tilde{S} \) of surfaces has the simple surface intersection property for each graph \( \Gamma_i \) of the family. Then \( \mathcal{P}_{\Gamma_\infty} \) is the inductive limit of a inductive family \( \{ \mathcal{P}_{\Gamma_i} \} \) of finite orientation preserved graph systems.

Let \( (\varphi, \Phi) \in \text{Diff}(\mathcal{P}) \) be a path-diffeomorphism of a path groupoid \( \mathcal{P} \rightrightarrows \Sigma \) such that

- \( \varphi : \Sigma \to \Sigma \) such that each surface \( S \) in \( \tilde{S} \) is mapped to another surface \( S_\varphi \) in \( \tilde{S} \) and \( \Phi : \mathcal{P} \to \mathcal{P} \);

- if a path \( \gamma \) in \( \mathcal{P} \) does not intersect a surface in \( \tilde{S} \), then \( \Phi(\gamma) \) does not intersect a surface in \( \tilde{S} \) and

- if a path intersects a surface \( S \), lies below and is outgoing (or above and outgoing, above and ingoing, below and ingoing) and the transformed path \( \Phi(\gamma) \) is non-trivial, then \( \Phi(\gamma) \) intersects the transformed surface \( S_\varphi \), lies below and is outgoing (or above and outgoing, below and ingoing, above and ingoing), too,

is called a surface-orientation-preserving path-diffeomorphism for a path groupoid \( \mathcal{P} \rightrightarrows \Sigma \) and a surface set \( \tilde{S} \).

Then a surface-orientation-preserving graph-diffeomorphism for a limit orientation preserved graph system \( \mathcal{P}_{\Gamma_\infty}^o \) is given by the pair \((\varphi_\Sigma, \Phi_\infty)\) of maps such that

- \( \varphi_\Sigma : \Sigma \to \Sigma \), \( \Phi_\infty : \mathcal{P}_{\Gamma_\infty}^o \to \mathcal{P}_{\Gamma_\infty}^o \) and

\[ \Phi_\infty(\Gamma) = (\Phi(\gamma_1), ..., \Phi(\gamma_N)) = \Gamma_\Phi \]

for \( \Gamma := \{ \gamma_1, ..., \gamma_N \} \) and \( \Gamma_\Phi \) being two subgraphs of \( \mathcal{P}_{\Gamma_\infty}^o \) and

- \( (\varphi_\Sigma, \Phi) \) is a surface-orientation-preserving path-diffeomorphism for a path groupoid \( \mathcal{P} \rightrightarrows \Sigma \) and a surface set \( \tilde{S} \).

The set of surface-orientation-preserving graph-diffeomorphism for a orientation preserved limit graph system \( \mathcal{P}_{\Gamma_\infty}^o \) is denoted by \( \text{Diff}_{\text{or}}(\mathcal{P}_{\Gamma_\infty}^o) \).

With no doubt the group \( \mathbb{B}_{\text{or}}(\mathcal{P}_{\Gamma_\infty}^o) \) of surface-orientation-preserving bisections of a limit orientation preserved graph system \( \mathcal{P}_{\Gamma_\infty}^o \) can be defined.

Apart from the problems of defining a point-norm continuous action of the group of global bisections of the inductive limit holonomy \( C^* \)-algebra, the Weyl algebra can be realized as inductive limit \( C^* \)-algebra, too.
Definition 100. Let $\hat{S}$ be a finite set of surfaces in $\Sigma$. Moreover let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the surface intersection property for the set $\hat{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of a inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.

The Weyl $C^*$-algebra $\mathfrak{Weyl}(\hat{S})$ for a surface set is generated by the inductive limit $C(\hat{A})$ of the family of $C^*$-algebras $\{(C(\hat{A}_i), \beta_{\Gamma_i, \Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i, j \in \mathbb{N}\}$ and the Weyl elements $\mathcal{W}(G_{\hat{S}_n})$, which satisfy the canonical commutator relations $[59]$, completed w.r.t. the $\|\|_\infty$-norm defined by the Hilbert space $\mathcal{H}_\infty$, which is given as the limit of the Hilbert spaces $\mathcal{H}_\Gamma$.

Definition 101. Let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ and let $\mathcal{P}_{\Gamma_\infty}$ be the inductive limit of a inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems associated to the family of graphs $\{\Gamma_i\}$. Let $\mathfrak{S}$ and $\mathfrak{S}_Z$ be two sets of suitable surfaces for each graph $\Gamma_i$ of the family of graphs $\{\Gamma_i\}$.

Moreover $\mathfrak{P}_{\Gamma_\infty}$ is the inductive limit of a inductive family $\{\mathfrak{P}_{\Gamma_i}\}$ of finite orientation preserved graph systems.

The Weyl $C^*$-algebra $\mathfrak{Weyl}(\mathfrak{S})$ for surfaces is generated by the inductive limit $C(\hat{A})$ of the family of $C^*$-algebras $\{(C(\hat{A}_i), \beta_{\Gamma_i, \Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i, j \in \mathbb{N}\}$ and the Weyl elements $\mathcal{W}(G_{\hat{S}_n})$ for each surface set $\hat{S}$ in $\mathfrak{S}$, which satisfies the canonical commutator relations $[59]$, completed w.r.t. the $\|\|_\infty$-norm defined by the Hilbert space $\mathcal{H}_\infty$, which is given as the limit of the Hilbert spaces $\mathcal{H}_\Gamma$.

The commutative Weyl $C^*$-algebra $\mathfrak{Weyl}_Z(\mathfrak{S}_Z)$ for surfaces is generated by the inductive limit of the family of $C^*$-algebras $\{(C(\hat{A}_i), \beta_{\Gamma_i, \Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i, j \in \mathbb{N}\}$ and the Weyl elements $\mathcal{W}(G_{\hat{S}_n})$ for each surface set $\hat{S}$ in $\mathfrak{S}_Z$, which satisfies the canonical commutator relations $[59]$, completed w.r.t. the $\|\|_\infty$-norm defined by the Hilbert space $\mathcal{H}_\infty$.

3.4 Flux and graph-diffeomorphism group-invariant states of the Weyl $C^*$-algebra for surfaces

Consider the inductive limit algebra $C(\hat{A})$ of the family of $C^*$-algebras $\{(C(\hat{A}_i), \beta_{\Gamma_i, \Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i, j \in \mathbb{N}\}$, where $C(\hat{A}_i)$ is isomorphic to $C(G^N)$ by the natural or non-standard identification.

Proposition 102. Let $\hat{S}$ be a finite set of surfaces in $\Sigma$. Moreover let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the surface intersection property for the set $\hat{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of a inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems. Let $\hat{A}_{\Gamma_i}$ be identified naturally with $G_{\Gamma_i}^N$.

The limit $\hat{\omega}_{\mathfrak{B}_Z}$ on $C(\hat{A})$ is defined by

$$\hat{\omega}_{\mathfrak{B}_Z}(f) := \lim_{\Gamma_i \rightarrow \Gamma_\infty} \frac{1}{k_{\Gamma_i}} \sum_{i=1}^{k_{\Gamma_i}} \omega_{\mathfrak{B}_Z}^\Gamma(\zeta_{\sigma_i}(f_{\Gamma_i})) \text{ for } \sigma_i \in \mathfrak{B}_{\mathfrak{S}_Z, \text{surf}}(\mathcal{P}_{\Gamma_i})$$

for $f \in C(\hat{A})$ and where $k_{\Gamma_i}$ is the maximal number of subgraphs in $\mathcal{P}_{\Gamma_i}$, which are generated by all edges and their compositions of the graph $\Gamma_\infty$. The limit $\hat{\omega}_{\mathfrak{B}_Z}$ is $\mathfrak{B}_{\mathfrak{S}_Z, \text{surf}}(\mathcal{P}_{\Gamma_\infty})$-invariant and does not converge in weak *-topology.

Proof: There are two disjoint families of graph $\{\Gamma_i\}$ and $\{\Gamma_i\}$ such that the union converges to $\Gamma_\infty$ and such that for a suitable constants $k_{\Gamma_i}$ it is true that

$$\lim_{\Gamma_i \rightarrow \Gamma_\infty} \left| \hat{\omega}_{\mathfrak{B}_Z}(f) - \frac{1}{k_{\Gamma_i}} \sum_{i=1}^{k_{\Gamma_i}} \omega_{\mathfrak{B}_Z}^\Gamma(\zeta_{\sigma_i}(f_{\Gamma_i})) \right| > \lim_{\Gamma_i \rightarrow \Gamma_\infty} \left| \frac{1}{k_{\Gamma_i}} \sum_{i=1}^{k_{\Gamma_i}} \omega_{\mathfrak{B}_Z}^\Gamma(\zeta_{\sigma_i}(f_{\Gamma_i})) \right| > 0$$

If there would be a asymptotic condition for the state such that for the limit to the infinite graph the state does not depend on the action of $\zeta$ anymore, then the state defined above would be weakly converging.

Consequently if the natural identification of $\hat{A}_\Gamma$ with $G^N$ is used, then there is no state, which is $\mathfrak{Z}_{\mathfrak{S}_Z, \text{inf}}, \mathfrak{B}_{\mathfrak{S}_Z}, \text{surf}(\mathcal{P}_{\Gamma_\infty})$-invariant on $C(\hat{A})$.

Let each finite graph system $\mathcal{P}_{\Gamma_i}$ be identified naturally or in the non-standard way, then the following observations are made.
**Proposition 103.** Let $\bar{S}$ be a finite set of surfaces in $\Sigma$. Moreover let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the surface intersection property for the set $\bar{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of a inductive family $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems.

There is a $G_{\bar{S},\Gamma_\infty}$-invariant state $\omega_M$ on $C(\bar{A})$ presented by

$$
\omega_M(f) = \int_{\mathcal{G}^N} f_{\Gamma_i}(b_{\Gamma_i}(\Gamma_i')) \, d\mu_{\Gamma_i}(b_{\Gamma_i}(\Gamma_i'))
$$

for all $f \in C(\bar{A})$, $\mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}$, which satisfies

$$
\omega_M \circ \beta_{\Gamma_i} = \omega_M^{\Gamma_i}
$$

where $\beta_{\Gamma_i} : C(\bar{A}_{\Gamma_i}) \to C(\bar{A})$ is an injective $\ast$-homomorphism.

Moreover there is a state on $C(\bar{A})$ given by

$$
\omega_S(f) = (\omega_M \circ \beta_{\Gamma_i})(f) := \frac{1}{k_{\Gamma_i}} \sum_{i=1}^{k_{\Gamma_i}} \omega_M^{\Gamma_i}(\zeta_{\Gamma_i}(f_{\Gamma_i}))
$$

for $\sigma_i \in \mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i})$ and $f_{\Gamma_i} \in C(\bar{A}_{\Gamma_i})$, which is invariant under the automorphic actions of the groups $\text{Diff}_{S,\infty}(\mathcal{P}_{\Gamma_i})$, $\mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i})$ for a fixed graph $\Gamma_i$ and $\bar{Z}_S$ for a suitable set $\bar{S}$ of surfaces in $S_\Sigma$.

In other words,

$$
\omega_S(\theta_{(\varphi_{\Gamma_i}, \varphi_{\Gamma_i})} f) = \omega_S(f), \quad \omega_S(\zeta_{\sigma} f) = \omega_S(f),
$$

yield for all $f \in C(\bar{A})$, $(\varphi_{\Gamma_i}, \varphi_{\Gamma_i}) \in \text{Diff}_{S,\infty}(\mathcal{P}_{\Gamma_i})$, $\theta \in \text{Act}_0(\text{Diff}_{S,\infty}(\mathcal{P}_{\Gamma_i}), C(\bar{A}))$, $\sigma \in \mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i})$, $\zeta \in \text{Act}(\mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i}), C(\bar{A}))$, $\rho_{S,\Gamma_i}(\Gamma_i') \in \bar{Z}_S$, $\alpha \in \text{Act}(\bar{Z}_S, C(\bar{A}))$ for any surface set $\bar{S}$ in $S_\Sigma$.

Furthermore the state $\omega_S$ and the actions $\alpha \in \text{Act}(\bar{Z}_S, C(\bar{A}))$ and $\zeta \in \text{Act}(\mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i}), C(\bar{A}))$ satisfy

$$
\omega_S^{\Gamma_i} \circ \alpha(\rho_{S,\Gamma_i}(\Gamma_i')) \circ \zeta_{\sigma} = \omega_S^{\Gamma_i} \circ \zeta_{\sigma} \circ \alpha(\rho_{S,\Gamma_i}(\Gamma_i'))
$$

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Note that, the state $\omega_S$ is even invariant under a graph-diffeomorphism $(\varphi_{\Gamma_i}, \varphi_{\Gamma_i})$ such that there exists a diffeomorphism $\varphi : \Sigma \to \Sigma$ that maps surfaces into surfaces in $\bar{S}$ and $\varphi(v) = \varphi_{\Gamma_i}(v)$ for all $v \in V_{\Gamma_i}$. In the following only graph-diffeomorphisms in $\text{Diff}(\mathcal{P}_{\Gamma_i})$ and consequently also the induced bisections of $\mathcal{B}(\mathcal{P}_{\Gamma_i})$, which satisfy this requirement are considered. Therefore the restricted sets are denoted by $\text{Diff}_{S}(\mathcal{P}_{\Gamma_i})$ and $\mathcal{B}_{S}(\mathcal{P}_{\Gamma_i})$.

**Proof:** Recall the inductive limit $C(\bar{A})$ of the $C^*$-algebras $\{C(\bar{A}_{\Gamma_i}), \beta_{\Gamma_i, \Gamma_j} : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}, i, j \in \mathbb{N}\}$ where $\beta_{\Gamma_i, \Gamma_j}$ is an injective $\ast$-homomorphism satisfying $\beta_{\Gamma_i, \Gamma_j} = \beta_{\Gamma_i, \Gamma_k} \circ \beta_{\Gamma_k, \Gamma_j}$ whenever $\mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_k} \leq \mathcal{P}_{\Gamma_j}$ for all $i, j, k \in \mathbb{N}$.

There is a state $\omega_M^{\Gamma_i}$ on $C(\bar{A}_{\Gamma_i})$, which is $G_{\bar{S},\Gamma_i}$-invariant, and a state $\omega_M^{\bar{S}}$, which is $Z_{\bar{S},\Gamma_i}$- and $\mathcal{B}_{S,\infty}(\mathcal{P}_{\Gamma_i})$-invariant due to proposition 5.7 and 8.2. Every inductive limit of $C^*$-algebras corresponds to a projective limit of the state space of the $C^*$-algebras. Hence there are conjugate maps $\beta_{\Gamma_i, \Gamma_j} : C(\bar{A}_{\Gamma_i}) \to C(\bar{A}_{\Gamma_j})$ such that $\omega_M^{\Gamma_i} = \beta_{\Gamma_i, \Gamma_j} \omega_M^{\Gamma_j}$ and $\omega_M^{\bar{S}} = \beta_{\bar{S}, \Gamma_i} \omega_M^{\bar{S}}$. Denote the projective limit state of $\{(\omega_M^{\Gamma_i}, \beta_{\Gamma_i, \Gamma_j}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}\}$ by $\omega_M$ on $C(\bar{A})$ respectively $\{(\omega_M^{\bar{S}}, \beta_{\bar{S}, \Gamma_i}) : \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_j}\}$ by $\omega_{\bar{S}}$ on $C(\bar{A})$.

Then the state on $C(\bar{A})$ satisfies

$$
\omega_M(f) = \beta_{\Gamma_i}^{\ast}(\omega_M^{\Gamma_i}(f_{\Gamma_i})) = (\beta_{\Gamma_i}^{\ast} \circ \beta_{\Gamma_i, \Gamma_j})(\omega_M^{\Gamma_j}(f_{\Gamma_j})) = \beta_{\Gamma_i}^{\ast}(\omega_M^{\Gamma_j}(f_{\Gamma_j}))
$$

$$
= \int_{\mathcal{G}^N} f_{\Gamma_j}(b_{\Gamma_i}(\Gamma)) \, d\mu_{\Gamma_i}(b_{\Gamma_i}(\Gamma))
$$

$$
= \int_{\mathcal{G}^N} f_{\Gamma_j}(L(\rho_{S,\Gamma_i}(\Gamma'))(b_{\Gamma_i}(\Gamma')) \, d\mu_{\Gamma_i}(b_{\Gamma_i}(\Gamma'))
$$

$$
= \int_{\mathcal{G}^N} f_{\Gamma_j}(R(\rho_{S,\Gamma_i}(\Gamma''))(b_{\Gamma_i}(\Gamma'')) \, d\mu_{\Gamma_i}(b_{\Gamma_i}(\Gamma''))
$$

for suitable surface $S$ and $S'$, graphs $\Gamma, \Gamma'$ and $\Gamma''$, maps $\rho_{S,\Gamma_i} \in G_{\bar{S},\Gamma_i}$ and $\rho_{S',\Gamma''} \in G_{\bar{S},\Gamma''}$. 

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Notice that, the state \( \omega_\mathcal{S} \) is only invariant under the group \( \mathcal{Z}_\mathcal{S} \). This follows from the fact that, the action \( \zeta \) for \( \mathfrak{B}_{\mathcal{S},or}(\mathcal{P}_\mathcal{R}) \) and the action \( \alpha \) for \( \check{G}_{\mathcal{S},\Gamma} \) do not commute.

**Corollary 104.** Let \( \check{S} \) be a finite set of surfaces in \( \Sigma \). Moreover let \( \Gamma_\infty \) be the inductive limit of a family of graphs \( \{ \Gamma_i \} \). Then \( \mathcal{P}_{\Gamma_\infty} \) is the inductive limit of a inductive family \( \{ \mathcal{P}_{\Gamma_i} \} \) of finite graph systems. Let \( \check{\mathcal{A}}_\Gamma \) be identified in the non-standard identification with \( \check{G}^{[\Gamma]} \).

Then the state \( \omega_M \) defined by

\[
\omega_M(f) = \int_{\mathcal{G}^{\infty}} f_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i')) d\mu_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i'))
\]

for all \( f \in C(\check{\mathcal{A}}) \), \( \mathcal{P}_{\Gamma_i} \leq \mathcal{P}_{\Gamma_i'} \), which satisfies

\[
\omega_M \circ \beta_{\Gamma_i} = \omega_{M,\Gamma_i}
\]

is the unique state on \( C(\check{\mathcal{A}}) \), which is invariant under the automorphic actions of the groups \( \text{Diff}_\mathcal{S}(\mathcal{P}_{\Gamma_i}) \), \( \mathfrak{B}_{\mathcal{S}}(\mathcal{P}_{\Gamma_i}) \) for each graph \( \Gamma_i \) and \( \check{\mathcal{Z}}_\mathcal{S} \).

Let each finite graph system \( \mathcal{P}_{\Gamma_i} \) be identified naturally or in the non-standard way.

**Proposition 105.** Let \( \check{S} \) be a finite set of surfaces in \( \Sigma \). Moreover let \( \Gamma_\infty \) be the inductive limit of a family of graphs \( \{ \Gamma_i \} \) such that each graph \( \Gamma_i \) of the family has the surface intersection property for the set \( \check{S} \) of surface \( \check{S} \).

Then \( \mathcal{P}_{\Gamma_\infty} \) is the inductive limit of a inductive family \( \{ \mathcal{P}_{\Gamma_i} \} \) of finite graph systems.

There is a GNS-representation \( (\mathcal{H}_\infty, \Phi, \Omega_M) \) of \( \mathfrak{Weyl}(\check{S}) \) on \( \mathcal{H}_\infty \) of the pure and unique state \( \check{\omega}_M \) on \( \mathfrak{Weyl}(\check{S}) \), which is given by

\[
\check{\omega}_M(f) = (\omega_M^{\Gamma_i} \circ \beta_{\Gamma_i})(f) = \langle \Omega_M, \Phi(f)\Omega_M \rangle
\]

\[
= \int_{\mathcal{G}^{\infty}} f_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i')) d\mu_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i'))
\]

\[
\check{\omega}_M(U^*(\rho_{\mathcal{S},\Gamma}(\Gamma))U(\rho_{\mathcal{S},\Gamma}(\Gamma))) = \langle \Omega_M, \Phi(U^*(\rho_{\mathcal{S},\Gamma}(\Gamma))U(\rho_{\mathcal{S},\Gamma}(\Gamma)))\Omega_M \rangle = \check{\omega}_M(\mathbb{1}) = 1
\]

where \( \mathbb{1} \) is the identity on \( \mathcal{H}_\infty \), for \( f \in C(\check{\mathcal{A}}) \) and \( U(\rho_{\mathcal{S},\Gamma}(\Gamma)) \in \mathcal{W}(\check{G}_{\mathcal{S},\Gamma_\infty}) \) whenever \( \Phi \in \text{Mor}(\mathfrak{Weyl}(\check{S}), \mathcal{L}(\mathcal{H}_\infty)) \) and \( \Phi_M := \Phi|_{C(\check{\mathcal{A}})} \). The state \( \check{\omega}_M \) is invariant under the automorphic actions of the flux group \( \check{G}_{\mathcal{S},\Gamma_\infty} \).

**Proof.** First use the proposition \[13\] for finite graph systems and the uniqueness of the construction of the limit of states on the inductive limit of \( C^*-\)algebras. Equivalently it can be shown that, the representation \( \Phi_M \in \text{Rep}(C(\check{\mathcal{A}}), \mathcal{L}(\mathcal{H}_\infty)) \) extends uniquely to a representation \( \Phi \in \text{Mor}(\mathfrak{Weyl}(\check{S}), \mathcal{L}(\mathcal{H}_\infty)) \). \( \square \)

Furthermore this proposition extends to sets of surface sets.

**Proposition 106.** Let \( \Gamma_\infty \) be the inductive limit of a family of graphs \( \{ \Gamma_i \} \) and let \( \mathcal{P}_{\Gamma_\infty} \) be the inductive limit of a inductive family \( \{ \mathcal{P}_{\Gamma_i} \} \) of finite graph systems associated to the family of graphs \( \{ \Gamma_i \} \). Let \( \mathcal{S} \) be a set of suitable surfaces for each graph \( \Gamma_i \) of the family of graphs \( \{ \Gamma_i \} \).

There is a GNS-representation \( (\mathcal{H}_\infty, \Phi, \Omega_M) \) of \( \mathfrak{Weyl}(\check{S}) \) on \( \mathcal{H}_\infty \) of the pure and unique state \( \check{\omega}_M \) on \( \mathfrak{Weyl}(\mathcal{S}) \), which is given by

\[
\check{\omega}_M(f) = (\omega_M^{\Gamma_i} \circ \beta_{\Gamma_i})(f) = \langle \Omega_M, \Phi(f)\Omega_M \rangle
\]

\[
= \int_{\mathcal{G}^{\infty}} f_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i')) d\mu_{\Gamma_i}(h_{\Gamma_i}(\Gamma_i'))
\]

\[
\check{\omega}_M(U^*(\rho_{\mathcal{S},\Gamma}(\Gamma))U(\rho_{\mathcal{S},\Gamma}(\Gamma))) = \langle \Omega_M, \Phi(U^*(\rho_{\mathcal{S},\Gamma}(\Gamma))U(\rho_{\mathcal{S},\Gamma}(\Gamma)))\Omega_M \rangle = \check{\omega}_M(\mathbb{1}) = 1
\]

where \( \mathbb{1} \) is the identity on \( \mathcal{H}_\infty \), for \( f \in C(\check{\mathcal{A}}) \) and \( U(\rho_{\mathcal{S},\Gamma}(\Gamma)) \in \mathcal{W}(\check{G}_{\mathcal{S},\Gamma_\infty}) \) for every suitable surface set \( \check{S} \) in \( \mathcal{S} \) whenever \( \Phi \in \text{Mor}(\mathfrak{Weyl}(\check{S}), \mathcal{L}(\mathcal{H}_\infty)) \) and \( \Phi_M := \Phi|_{C(\check{\mathcal{A}})} \) and which is invariant under the automorphic actions of the flux group \( \check{G}_{\mathcal{S},\Gamma_\infty} \) for each suitable surface set \( \check{S} \) in \( \mathcal{S} \).

*This condition is necessary, since otherwise \( \check{G}_{\mathcal{S},\Gamma_i} \) does not form a group.
Now the focus lies on graph-diffeomorphism invariant states on a Weyl $C^*$-algebra. Then the following theorem is stated.

**Theorem 107.** Let $\hat{S}$ be a finite set of surfaces in $\Sigma$. Moreover let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$ such that each graph $\Gamma_i$ of the family has the surface intersection property for the set $\hat{S}$ of surfaces. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of a family of graph systems $\{\mathcal{P}_{\Gamma_i}\}$ of finite graph systems and each $\mathcal{P}_{\Gamma_i}$ is identified in the non-standard way.

The state $\hat{\omega}_M$ on $\mathcal{Weyl}_\Sigma(\hat{S})$ given in proposition 105 is the unique state, which is invariant under the automorphic actions of the groups $\text{Diff}_S(\mathcal{P}_{\Gamma_i})$, $\mathcal{B}_S(\mathcal{P}_{\Gamma_i})$ for each graph $\Gamma_i$ and $\hat{Z}_S$ and pure.

Furthermore the state $\hat{\omega}_M$ and the actions $\alpha \in \text{Act}(\hat{Z}_S, \mathcal{Weyl}_\Sigma(\hat{S}))$ and $\zeta \in \text{Act}(\mathcal{B}_S(\mathcal{P}_{\Gamma_i}), \mathcal{Weyl}_\Sigma(\hat{S}))$ satisfy

$$
(\beta^*_\Gamma, \hat{\omega}_M) \circ \alpha(\rho_{\Gamma,S}(\Gamma')) \circ \zeta = (\beta^*_\Gamma, \hat{\omega}_M) \circ \zeta \circ \alpha(\rho_{\Gamma,S}(\Gamma'))
$$

(72)

Notice that, this theorem generalises to the non-standard identification and for a suitable set $S_\Sigma$ of surface sets.

**Proof:** This follows from corollary 104 and proposition 105.

For the natural or non-standard and natural identification the theorem follows.

**Theorem 108.** Let $\hat{S}$ be a finite set of surfaces in $\Sigma$. Moreover let $\Gamma_\infty$ be the inductive limit of a family of graphs $\{\Gamma_i\}$. Then $\mathcal{P}_{\Gamma_\infty}$ is the inductive limit of a family of graph systems.

There is a GNS-representation $(\mathcal{H}_\infty, \Phi, \Omega_{M,\mathcal{B}})$ of $\mathcal{Weyl}_\Sigma(\hat{S})$ on $\mathcal{H}_\infty$ of the state $\omega_{M,\mathcal{B}}$ on $\mathcal{Weyl}_\Sigma(\hat{S})$, which is given by

$$
\omega_{M,\mathcal{B}}(f_{\Gamma_i}) := \frac{1}{k_{\Gamma_i}} \sum_{l=1}^{k_{\Gamma_i}} \omega_M(\zeta_{\sigma_l}(f_{\Gamma_i}))
$$

$$
= (\Omega_{M,\mathcal{B}}, \Phi(f_{\Gamma_i}))\Omega_{M,\mathcal{B}}
$$

$$
\omega_{M,\mathcal{B}}(f) = \omega_M(f)
$$

$$
\omega_{M,\mathcal{B}}(U^*(\rho_{\Gamma,S}(\Gamma'))U(\rho_{\Gamma,S}(\Gamma'))) = (\Omega_{M,\mathcal{B}}, \Phi(U^*(\rho_{\Gamma,S}(\Gamma'))U(\rho_{\Gamma,S}(\Gamma'')))\Omega_{M,\mathcal{B}} = \omega_{M,\mathcal{B}}(1)
$$

where $1$ is the identity on $\mathcal{H}_\infty$, for $\sigma_l \in \mathcal{B}_{S,surf}(\mathcal{P}_{\Gamma_i})$, $f_{\Gamma_i} \in C(\hat{\mathcal{A}}_{\Gamma_i})$, $f \in C(\hat{\mathcal{A}})$, $U(\rho_{\Gamma,S}(\Gamma')) \in \mathcal{W}(\hat{Z}_{S,\Gamma_\infty})$ whenever $\Phi \in \text{Mor}(\mathcal{Weyl}_\Sigma(\hat{S}), \mathcal{L}(\mathcal{H}_\infty))$ and $\Phi := \Phi|_{C(\hat{\mathcal{A}})}$.

The state $\omega_{M,\mathcal{B}}$ on $\mathcal{Weyl}_\Sigma(\hat{S})$ is $\text{Diff}_{S,surf}(\mathcal{P}_{\Gamma_i})$, $\text{Diff}_{S,or}(\mathcal{P}_{\Gamma_i})$, $\mathcal{B}_{S,surf}(\mathcal{P}_{\Gamma_i})$, $\mathcal{B}_{S,or}(\mathcal{P}_{\Gamma_i})$-invariant for a fixed graph $\Gamma_i$ and $\hat{Z}_{S}$-invariant.

In other words, it is true that

$$
\omega_{M,\mathcal{B}}(\theta(\rho_{\Gamma_i}, \Phi_{\Gamma_i})(W)) = \omega_{M,\mathcal{B}}(W), \quad \omega_{M,\mathcal{B}}(\zeta_\sigma(W)) = \omega_{M,\mathcal{B}}(W),
$$

$$
\omega_{M,\mathcal{B}}(\alpha(\rho_{\Gamma_i,S}(\Gamma'))(W)) = \omega_{M,\mathcal{B}}(W)
$$

holds for all $W \in \mathcal{Weyl}_\Sigma(\hat{S})$ and $\theta \in \text{Act}_0(\text{Diff}(\mathcal{P}_{\Gamma_i}), \mathcal{Weyl}_\Sigma(\hat{S}))$, $
\zeta \in \text{Act}_0(\mathcal{B}_{S,surf}(\mathcal{P}_{\Gamma_i}), \mathcal{Weyl}_\Sigma(\hat{S}))$ or $\zeta \in \text{Act}_0(\mathcal{B}_{S,or}(\mathcal{P}_{\Gamma_i}), \mathcal{Weyl}_\Sigma(\hat{S}))$ for each fixed graph $\Gamma_i$,

$\alpha \in \text{Act}(\hat{Z}_{S,\Gamma_\infty}, \mathcal{Weyl}_\Sigma(\hat{S}))$ for any surface $S \in \hat{S}$.

Finally the state $\omega_{M,\mathcal{B}}$ and the actions $\alpha \in \text{Act}(\hat{Z}_S, \mathcal{Weyl}_\Sigma(\hat{S}))$ and $\zeta \in \text{Act}(\mathcal{B}_{S,or}(\mathcal{P}_{\Gamma_i}), \mathcal{Weyl}_\Sigma(\hat{S}))$ satisfy

$$
(\beta^*_\Gamma, \hat{\omega}_M) \circ \alpha(\rho_{\Gamma,S}(\Gamma')) \circ \zeta = (\beta^*_\Gamma, \hat{\omega}_M) \circ \zeta \circ \alpha(\rho_{\Gamma,S}(\Gamma'))
$$

(73)

This theorem can be generalised to $\mathcal{Weyl}_\Sigma(S_\Sigma)$, since this theorem is true for all surface sets in $S_\Sigma$.
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