ROMANOV’S THEOREM IN NUMBER FIELDS

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Abstract. Romanov proved that a positive proportion of the integers has a representation as a sum of a prime and a power of an arbitrary fixed integer. We prove the analogous result for number fields. Furthermore we give an explicit lower bound for the lower density of Gaussian integers that have a representation as a sum of a Gaussian prime and a power of $1+i$. Finally, similar to Erdős, we construct an explicit arithmetic progression of Gaussian integers with odd norm which do not have a representation of this type.

1. Introduction and statement of results

The problem of determining the proportion of positive integers which are of the form $p + g^k$ has quite a long history. Especially the case of sums of primes and powers of two received prominent treatment. Mathematicians dating back at least as far as until Euler [3] worked on this problem. In 1934 Romanov [18] proved that the proportion of integers $n \in \mathbb{N}$ of the form $n = p + g^k$ with $p \in \mathbb{P}, k \in \mathbb{N}, 1 < g \in \mathbb{N}$ is positive. For the case $g = 2$ Erdős [2] and van der Corput [23] independently proved a counterpart to Romanov’s theorem stating that a positive proportion of the odd positive integers can not be represented as a sum of a prime and a power of two.

Recently explicit lower bounds for the lower density of integers of the form $p + 2^k$, i.e.

$$\liminf_{x \to \infty} \frac{\# \{ n \leq x : n = p + 2^k, p \in \mathbb{P}, k \in \mathbb{N}_0 \}}{x},$$

were published (see [1], [5], [6] and [11]). Quite recently Shparlinski and Weingartner [20] proved an analogue for Romanov’s Theorem for polynomials over finite fields. Our aim is to find number field analogues to some of the results just mentioned. Before we state those results we need to fix some standard notation.

For the rest of this paper $\mathbb{N}$ and $\mathbb{P}$ will have their usual meaning denoting the set of positive integers and positive primes respectively and $K$ will always denote a number field of degree $n = r_1 + 2r_2$, where the non negative integers $r_1$ and $r_2$, as usual, denote the number of real and pairs of complex conjugated embeddings of $K$. By $\mathcal{O}_K$ we denote the ring of integers of $K$, $\mathcal{P}_K$ is the set of prime elements in $\mathcal{O}_K$ and $\xi \in \mathcal{O}_K$ a fixed non unit. The letter $p$ with or without index in any case denotes a prime ideal of $\mathcal{O}_K$. If $a$ is an ideal of $\mathcal{O}_K$ we write $a \subseteq \mathcal{O}_K$ for short. Implied constants are always allowed to depend on $\xi$ and on $K$. For $\zeta \in K, N(\zeta)$ denotes the field norm of $\zeta$ and $\sigma_i : K \to \mathbb{C}, 1 \leq i \leq n$, are the embeddings of $K$. We write $|\zeta| := \max_{1 \leq i \leq n} |\sigma_i(\zeta)|$ for the house of $\zeta$.

Our main result will be the following number field version of Romanov’s theorem:

**Theorem 1** (Romanov’s Theorem for number fields). Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers, $\mathcal{P}_K$ the set of prime elements in $\mathcal{O}_K$ and $\xi \in \mathcal{O}_K$ a non unit. Then the lower density of algebraic integers $\zeta \in \mathcal{O}_K$ of the form $\zeta = \pi + \xi^k$ for $\pi \in \mathcal{P}_K$ is positive.

Concerning sums of primes and powers of two we want to treat the corresponding case in the ring $\mathbb{Z}[i]$ by computing a lower bound for the density of Gaussian integers $\zeta \in \mathbb{Z}[i]$ with a representation of the form

$$\zeta = \pi + (1 + i)^k.$$
for $\pi \in \mathcal{P}_{Q(i)}$ and $k \in \mathbb{N}$. Note that similar to 2 in $\mathbb{Z}$ the associates of $1 + i$ are exceptional primes in $\mathbb{Z}[i]$: they are the only ramified primes and the only primes whose real- and imaginary parts have the same parity. Furthermore we get the same trivial upper bound for algebraic integers of the form $\pi + (1 + i)^k$ that we get for integers of the form $p + 2^k$. For $k \geq 2$ the real- and imaginary part of $(1 + i)^k$ are even and the real- and imaginary parts of primes in $\mathbb{Z}[i]$, with the exception of the associates of $1 + i$, have different parity hence the norm of sums of prime elements and powers of $1 + i$ is odd. Thus the density of algebraic integers $\zeta \in \mathbb{Z}[i]$ with a representation of the form $\zeta = \pi + (1 + i)^k$ is at most $\frac{1}{2}$. Before we state a theorem concerning the proportion of Gaussian integers we will prove the following lower bound.

**Theorem 2.** In the case of $K = \mathbb{Q}(i)$ and $\xi = 1 + i$ we have

$$\liminf_{x \to \infty} \frac{\sum_{|\zeta| \leq x} \eta_x(\zeta, 1 + i)}{\#\{\zeta \in \mathbb{Z}[i] : |\zeta| \leq x\}} \geq 0.00110183.$$  

Furthermore we prove the following analogue to the result of Erdős and van der Corput.

**Theorem 3.** A positive proportion of the algebraic integers $\zeta \in \mathbb{Z}[i]$ with odd norm $N(\zeta)$ is not of the form $\pi + (1 + i)^k$ for $k \in \mathbb{N}$, $\pi \in \mathcal{P}_{Q(i)}$.

Following the ideas of Romanov [18] we will make use of the Cauchy-Schwarz inequality and a sieve. Applying the Cauchy-Schwarz inequality yields

$$\sum_{|\zeta| \leq x} \eta_x(\zeta, \xi) \geq \left( \sum_{|\zeta| \leq x} r_x(\zeta, \xi) \right)^2 \sum_{|\zeta| \leq x} r_x(\zeta, \xi)^2,$$

and we need to look for a lower bound for the numerator and an upper bound for the denominator of the right hand side of this inequality.

## 2. Romanov’s theorem for number fields

We start with an upper bound for $\sum_{|\zeta| \leq x} r_x(\zeta, \xi)^2$ by using results from sieve theory. Since $r_x(\zeta, \xi)^2$ counts the number of pairs of representations of the algebraic integer $\zeta$ as the sum of a prime and a power of $\xi$ we have

$$\sum_{|\zeta| \leq x} r_x(\zeta, \xi)^2 = |A| + |B|$$

where $A$ corresponds to pairs of equal representations and $B$ to different ones, i.e.

$$A = \left\{ (\pi, k) : \pi \in \mathcal{P}_K, k \in \mathbb{N}, \zeta = \pi + \xi^k, \begin{array}{c} |\zeta| \leq x, |\pi| \leq x, k \leq \frac{\log \sqrt{\frac{\xi}{|\zeta|}}}{\log |\xi|} \end{array} \right\}.$$
and
\[
B = \left\{ (\pi_1, \pi_2, k_1, k_2) : \pi_i \in \mathcal{P}_K, k_i \in \mathbb{N}, \zeta = \pi_1 + \zeta^{k_1}, \pi_1 \neq \pi_2, \right. \\
\left. |\zeta| \leq x, |\pi_i| \leq x, k_i \leq \frac{\log \sqrt{x}}{\log |\zeta|} \right\}.
\]

Finding an upper bound for the size of the set \(B\) for fixed \(k_1\) and \(k_2\) amounts to finding an upper bound for the number of distinct primes \(\pi_1\) and \(\pi_2\) such that
\[
\pi_1 - \pi_2 = \xi^{k_2} - \xi^{k_1}.
\]

Note that with our restriction on the exponents \(k_i\) and with the triangle inequality for the house function we have that \(|\xi^{k_1} - \xi^{k_2}| \leq \sqrt{x}\). An upper bound for the number of solutions in this case is given by the following Theorem:

**Theorem 4.** Let \(K\) be a number field of degree \(n\), \(\mathcal{O}_K\) its ring of integers, \(0 < x \in \mathbb{R}\) and \(\zeta \in \mathcal{O}_K\) such that \(|\zeta| \leq \sqrt{x}\). We denote by \(P(\zeta, x)\) the number of solutions of the equation
\[
\zeta = \pi_1 - \pi_2
\]
where \(\pi_i \in \mathcal{P}_K, |\pi_i| \leq x\). Then there exists a constant \(\kappa\) depending only on \(K\) such that
\[
P(\zeta, x) \leq \kappa x^n \left(1 + o(1)\right).
\]

If \(K = \mathbb{Q}(i)\) then the choice \(\kappa = \frac{1024}{3} \cdot 1.2771\) is admissible.

**Remark 1.** The first part of the proof of Theorem 4 is in large parts the same as Tatuzawa’s proof of [22, Theorem 1]. However, in his proof Tatuzawa makes use of the restriction \(|\zeta| \leq c_K \sqrt{N(\zeta)}\) for a constant \(c_K\) depending only on \(K\) which in our application of this result will not be satisfied in general. Besides of the result of Tatuzawa many of the details appearing in the first part of the proof below can be found in the proof of Lemma 3.2 in Wang’s book [24].

The second part of the proof of Theorem 4 works exactly as the proof of [17, Satz 16] by applying Selberg’s Sieve method for number fields (for a detailed description of the method see [15,17]) instead of the approach of Rademacher [13] using Brun’s sieve that was applied by Tatuzawa in his proof.

We therefore do not give all the details in the proof below, they can be found in the corresponding works of Tatuzawa, Rademacher, Wang and Rieger. The reason why we sketch the proof non the less is because we need to work out an explicit sieve constant for later use. Furthermore our restrictions are slightly different from those in Tatuzawa’s Theorem [22, Theorem 1] as well as Rieger’s Theorems [17, Satz 16 and Satz 17].

**Proof of Theorem 4.** The proof will work in two steps. First we have to count the number of possible pairs \((\zeta, \pi)\). In the second step we will use the Selberg sieve in order to get the desired upper bound.

1. **Counting Lattice Points.** For some ideal \(a \subseteq \mathcal{O}_K\) and some element \(\beta \in \mathcal{O}_K\) we start by finding the asymptotic number of algebraic integers \(\xi \in \mathcal{O}_K\) such that
\[
\xi \equiv \beta \mod a, \quad |\xi| \leq x, \quad |\xi + \zeta| \leq x.
\]

We use the fact that there exists a constant \(c_K\) depending only on \(K\) such that any ideal \(a \subseteq \mathcal{O}_K\) has an integral basis \(\alpha_1, \ldots, \alpha_n\) such that \(|\alpha_j| \leq c_K \sqrt{N(\alpha_j)}\) for all \(1 \leq j \leq n\) (for a proof of this see the first part of the proof of [24, Lemma 3.2]). Using such a basis \((\alpha_j)_{j=1}^n\) and with
\[
u_j = x_1 \alpha_j^{(1)} + \ldots + x_n \alpha_n^{(j)} + \beta^{(j)} \quad \forall 1 \leq j \leq r_1
\]
\[
u_j = x_1 R(\alpha_j^{(j)}) + \ldots + x_n R(\alpha_n^{(j)}) + R(\beta^{(j)}) \quad \forall r_1 + 1 \leq j \leq r_1 + r_2
\]
\[
u_j = x_1 I(\alpha_j^{(j)}) + \ldots + x_n I(\alpha_n^{(j)}) + I(\beta^{(j)}) \quad \forall r_1 + 1 \leq j \leq r_1 + r_2
\]
we may write (4) as
\[
\begin{align*}
|u_j| \leq x, \quad |u_j + \zeta^{(j)}| \leq x, \\
v_j^2 + w_j^2 \leq x^2,
\end{align*}
\]
\((5)\) By enlarging and shrinking the area described by the curves in (5) slightly (for details again see 
(6) \(P(\zeta)\)) we get
\[
\begin{align*}
\forall 1 \leq j \leq r_1

v_j^2 + w_j^2 \leq x^2, \\
(\Re(\zeta^{(j)}))^2 + (\Re(\zeta^{(j)}))^2 \leq x^2
\end{align*}
\]
As in the proof of [24, Lemma 3.2] we use that the Jacobian corresponding to the above change of variables is \(\frac{2\pi^2}{\Delta_K} \) where \(\Delta_K\) is the discriminant of \(K\). We now need to count lattice points in the area enclosed by the curves in (5). We observe that the inequalities concerning the real conjugates describe the intersection of two lines of length \(2x\), and the inequalities concerning the complex conjugates describe the intersection of two circles with radius \(x\) and central distances \(|\zeta^{(j)}|\) for \(r_1 + 1 \leq j \leq r_1 + r_2\).

We start with having a look at the lines described by \(|u_j| \leq x\) and \(|u_j + \zeta^{(j)}| \leq x\). We use that \(\sqrt{v_j^2} \leq \sqrt{x^2}\) and thus get a contribution between \(2x - \sqrt{x}\) and \(2x\) to the volume. All in all we hence get a contribution of \(2x + O(\sqrt{x})\) for the lines. Now we come to the contribution of the intersecting circles described by the inequalities \(v_j^2 + w_j^2 \leq x^2\) and \((\Re(\zeta^{(j)}))^2 + (\Re(\zeta^{(j)}))^2 \leq x^2\). An obvious upper bound for the area enclosed by both of these circles is the area of a full circle with radius \(x\), i.e. \(\pi x^2\). To get a lower bound we again use that \(|\zeta| \leq \sqrt{x}\) and we compute the area enclosed by two circles with radius \(x\) and central distance \(\sqrt{x}\). This is given by
\[
4 \int_0^x \sqrt{x^2 - t^2} dt = 2 \left[ t \sqrt{x^2 - t^2} + x^2 \arctan \left( \frac{t}{\sqrt{x^2 - t^2}} \right) \right]_{t=x} = \pi x^2 + O(x^{\frac{3}{2}}).
\]

To get the error term we used that \(\arctan(x) < x\) for any \(x > 0\) as well as \(\frac{x}{\sqrt{x^2 - t^2}} < \frac{1}{\sqrt{x}}\) for \(x > \frac{1}{2}\). Thus the contribution of all circles is \((\pi x^2 + O(x^{\frac{3}{2}}))^{r_1} = \pi^{r_1} x^{2r_1} + O(x^{2r_2 - \frac{1}{2}})\). Altogether the curves in (5) therefore enclose an area of
\[
2^{r_1} \pi^{r_2} x^n + O(x^{n - \frac{3}{2}}).
\]

By enlarging and shrinking the area described by the curves in (5) slightly (for details again see the proof of [24, Lemma 3.2]) we get
\[
(6) \quad P(a, \zeta, x) = \frac{\pi^{r_2} x^{n-1}}{\mathcal{N}(a) \sqrt{\Delta_K}} x^n + O \left( \frac{x^{n-1}}{\mathcal{N}(a)^{1 - \frac{1}{2}}} \right).
\]

2. Sifting by Prime Ideals

For any details in this part of the proof we refer the reader to the work of Rieger, especially the proof of [17, Satz 16]. Our aim here is just to point out explicit values for the constants appearing in Rieger’s proof.

In his proof Rieger chose the parameter \(z \leq x^{\frac{3}{4}}\). This choice will not work with our error term in (5) and we need to take the slightly worse bound \(z \leq x^{\frac{1}{4}}\). With this choice of \(z\) applying [15, Satz 2] we get an error term of \(O \left( \frac{x^{n - \frac{1}{4}}}{\mathcal{N}(a)} \right)\) in equation (6) and it remains to work out an upper bound for the main term.

The main term is of the form \(\frac{c_1 x^n}{Z}\) where \(c_1 = \frac{\pi^{r_2} x^{n-1} + O(x^{2r_2 - \frac{1}{2}})}{\sqrt{\Delta_K}}\) is the constant form equation (6) and as in [17, p. 86 equation (119)] \(Z\) is bounded from below by
\[
(7) \quad Z \geq \sum_{\mathcal{N}(a) \leq x^{\frac{3}{4}}} \frac{1}{\mathcal{N}(a)} \sum_{\mathcal{N}(b) \leq x^{\frac{1}{4}}} \frac{1}{\mathcal{N}(b)}.
\]
Finding lower bounds for sums of the above type works as in [16, p. 160 equation (40)] where we additionally use that
\[
\prod_{\mathcal{N}(p) \leq \sqrt{z}} \left(1 - \frac{1}{\mathcal{N}(p)}\right) \geq \left(\frac{6}{\pi^2}\right)^n \prod_{\mathcal{N}(p) \leq \sqrt{z}} \left(1 + \frac{1}{\mathcal{N}(p)}\right)^{-1}.
\]

The constant \(\left(\frac{6}{\pi^2}\right)^n\) arises from the inequality
\[
\prod_{\mathcal{N}(p) \leq \sqrt{z}} \left(1 - \frac{1}{\mathcal{N}(p)^2}\right) \geq \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2}\right)^n = \left(\frac{6}{\pi^2}\right)^n.
\]

For the last inequality we used that in the ring of integers of a number field of degree \(n\) there are at most \(n\) prime ideals whose norm is a power of \(p\) for fixed \(p \in \mathbb{P}\). This follows basically from the fact that any prime ideal in \(\mathcal{O}_K\) lies over a prime ideal in \(\mathbb{Z}\) and there are at most \(n\) prime ideals in \(\mathcal{O}_K\) with this property (see for example [9, Proposition 4.2 and Corollary 2 on p. 148]). We note that in the case of the Gaussian integers this bound can be improved. Since we will use it later we also give the improved bound for \(K = \mathbb{Q}(i)\) here:
\[
\prod_{\mathcal{N}(p) \leq \sqrt{z}} \left(1 - \frac{1}{\mathcal{N}(p)^2}\right) \geq \prod_{p \equiv 1 \mod 4} \left(1 - \frac{1}{p^2}\right)^2 \prod_{p \equiv 3 \mod 4} \left(1 - \frac{1}{p^2}\right)^2 
\]
\[
\geq L(\chi_1, 2)^{-1} L(\chi_2, 2)^{-1} \geq 0.88492,
\]
where \(\chi_1\) and \(\chi_2\) are the two Dirichlet Characters mod 4. Another ingredient we will use is Mertens’ formula for number fields (see [7, Theorem 5]) in the form
\[
\prod_{\mathcal{N}(p) \leq x} \left(1 - \frac{1}{\mathcal{N}(p)}\right)^{-1} = e^{\gamma + \log \rho_K} \log x + O\left(\frac{1}{\log^2 x}\right),
\]
where \(\rho_K\) is the residue of the Dedekind zeta function of \(K\) at \(s = 1\). In his proof of a lower bound for \(Z\) in [16, p. 160 equation (40)] Rieger uses that there exists a fixed constant \(c\) independent of \(f\) such that
\[
\sum_{\mathcal{N}(m) \leq x, \frac{m}{f} \in \mathcal{O}_K} \frac{1}{\mathcal{N}(m)} \geq c \sum_{\mathcal{N}(p) \leq x, \frac{p}{f} \in \mathcal{O}_K} \left(1 - \frac{1}{\mathcal{N}(p)}\right)^{-1}
\]
(cf. [16, Hilfssatz 5]). For a proof, Rieger refers to the analogous result in \(Z\) proved in [12, II Lemma 4.1] by using Mertens’ inequality [11] and that there exists a constant \(\alpha\) such that
\[
\sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)} = \alpha \log x + O(1)
\]
(see [16, p. 158 equation (29)]). Since we need the constant \(c\) in inequality (10) for \(K = \mathbb{Q}(i)\) later we give Prachar’s proof of [12, II Lemma 4.1] for this case. To begin with we need to determine the constant \(\alpha\) in equation (11). We note that the left hand side of this equality for the Gaussian integers takes the form
\[
\frac{1}{4} \sum_{m \leq x} \frac{r_2(m)}{m}
\]
where \( r_2(m) \) counts the number of representations of \( m \) as the sum of two squares. A well known result of Sierpiński \(^{21}\) states that
\[
\sum_{m \leq x} \frac{r_2(m)}{m} = \pi \log x + K + O \left( \frac{1}{\sqrt{x}} \right)
\]
where \( K > 0 \) is the Sierpiński constant. The constant \( \alpha \) in equation (11) may therefore be chosen as \( \alpha = \frac{\pi}{4} \). Using that \( \rho_0(i) = \frac{\pi}{4} \) in equation (11) we have that for \( f = \mathbb{Z}[i] \) inequality (11) is satisfied with \( c = e^{-\gamma} \) for sufficiently large \( x \). Following Prachar’s proof we will show that we can keep this constant also in the case \( f \neq \mathbb{Z}[i] \). Take a prime divisor \( q \mid f \) with \( \mathcal{N}(q) \leq x \), then for sufficiently large \( x \) we have
\[
\prod_{\substack{\mathcal{N}(p) \leq x \\ p \neq q}} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)^{-1} = \prod_{\mathcal{N}(p) \leq x} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)^{-1} \left( 1 - \frac{1}{\mathcal{N}(q)} \right) \leq e^{-\gamma} \left( 1 - \frac{1}{\mathcal{N}(q)} \right) \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)}.
\]
Since
\[
\sum_{\substack{\mathcal{N}(m) \leq x \\ q \mid m}} \frac{1}{\mathcal{N}(m)} = \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)} - \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)} \geq \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)} - \frac{1}{\mathcal{N}(q)} \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)}
\]
we conclude that
\[
\prod_{\mathcal{N}(p) \leq x} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)^{-1} \leq e^{-\gamma} \sum_{\mathcal{N}(m) \leq x} \frac{1}{\mathcal{N}(m)}.
\]
Iterating for all of the finitely many prime factors \( q \mid f \) with \( \mathcal{N}(q) \leq x \) gives the desired result. Then with \( c \) as in (11) and \( c = e^{-\gamma} \) if \( K = \mathbb{Q}(i) \) for any ideal \( f \subseteq \mathbb{Z}[i] \) we have
\[
\sum_{\mathcal{N}(m) \leq \sqrt{x}} \frac{1}{\mathcal{N}(m)} \geq c \prod_{\mathcal{N}(m) \leq \sqrt{x}} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)^{-1} = c \prod_{\mathcal{N}(p) \leq \sqrt{x}} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)^{-1} \prod_{\mathcal{N}(p) \leq \sqrt{x}} \left( 1 - \frac{1}{\mathcal{N}(p)} \right)
\]
\[
\geq c \left( \frac{6}{\pi^2} \right)^n \frac{e^{\gamma + \log \rho_K}}{2} \prod_{p \mid f} \left( 1 + \frac{1}{\mathcal{N}(p)} \right)^{-1} \log z \left( 1 + O \left( \frac{1}{\log^2 z} \right) \right)
\]
Using this in inequality (11) we get
\[
Z \geq \log^2(z) c^2 \left( \frac{6}{\pi^2} \right)^n \frac{e^{2(\gamma + \log \rho_K)}}{4} \prod_{p \mid (\zeta)} \left( 1 + \frac{1}{\mathcal{N}(p)} \right)^{-1} \left( 1 + O \left( \frac{1}{\log^2 z} \right) \right)
\]
Altogether we thus get
\[
P(\zeta, x) \leq \kappa \frac{x^n}{\log^2 x} (1 + o(1)),
\]
and in the case \( K = \mathbb{Q}(i) \) we can choose \( \kappa = \frac{1024}{\pi} \cdot 1.2771 \).

Before we start proving Romanov’s theorem for number fields we note that if \( \zeta \in \mathcal{O}_K \) is a non unit then \( \left\lceil \frac{\zeta}{\zeta} \right\rceil > 1 \). Furthermore the following Lemma gives a connection between the norm and the house function:

**Lemma 1.** For a fixed non unit \( \xi \in \mathcal{O}_K \) and \( k \in \mathbb{N} \)
\[
\mathcal{N}(\xi^k - 1) \leq c \left\lceil \frac{\xi}{\xi} \right\rceil^k
\]
for a constant \( c \) depending only on \( K \).

**Proof.** We have
\[
\mathcal{N}(\xi^k - 1) \leq \left\lceil \frac{\xi^k - 1}{\xi^k - 1} \right\rceil \leq \left( \frac{\xi^k}{\xi^k - 1} \right)^n.
\]
The very crude bound \( \binom{n}{k} \leq 2^n \) gives the Lemma with \( c = (n + 1)2^n. \) \( \square \)
Now we have enough tools to bound the number of elements in $A$ and $B$. First we provide a lower bound for the number of elements of the set $A$ defined in (2). We define $L_{\xi} := \frac{\log(2x)}{\log|\xi|}$. Mitsui [8, Main Theorem p. 35] proved that the number of prime elements $\pi \in \mathcal{P}_K$ with $|\pi| \leq x$ is asymptotically of size

$$\frac{\omega x^n}{n h_k 2^m R \log x} =: c_1 \frac{x^n}{\log x}.$$  

Here $\omega$ is the number of roots of unity in $K$, $h_K$ is the class number and $R$ the regulator of $K$.

We thus have

$$\sum_{k \leq L_\xi} r_x(\zeta, \xi) \geq \sum_{k \leq L_\xi} \sum_{\pi \in \mathcal{P}_K, |\pi| \leq x} 1 \sim c_1 \sum_{k \leq L_\xi} \left(\frac{x - |\xi|^k}{\log(x - |\xi|^k)}\right)^n \geq \frac{c_1}{\log x} \sum_{k \leq L_\xi} (x - |\xi|^k)^n \sim \frac{c_1 x^n}{2 \log|\xi|}$$

where we used that

$$\sum_{k \leq L_\xi} (x - |\xi|^k)^n = \frac{x^n \log x}{2 \log|\xi|} \left(1 + O\left(\frac{1}{\log x}\right)\right).$$

On the other hand to get an upper bound we consider

$$\sum_{|\xi| \leq x} r_x(\zeta, \xi) \leq \sum_{k \leq L_\xi} \sum_{|\pi| \leq x} 1 \sim c_1 \sum_{k \leq L_\xi} \frac{x^n \log x}{\log|\xi|} \sim \frac{c_1 x^n}{2 \log|\xi|}$$

Now we count the elements in the set $B$ defined in (3). Following Romanov’s approach we use Theorem 4 to count pairs of primes which sum to $\xi^{k_1} - \xi^{k_2}$ for fixed $k_1, k_2 \in \mathbb{N}$. We thus have

$$\#B \ll \#A + \frac{x^n}{\log^2 x} \prod_{\xi \in \mathcal{P}_K} \left(1 + \frac{1}{\mathcal{N}(p)}\right) \sum_{k_1 < k_2 \leq L_\xi} \xi^{k_2 - k_1 - 1} \prod_{p \in \mathcal{P}_K} \left(1 + \frac{1}{\mathcal{N}(p)}\right)$$

and we need to take care of the sum over $k_1$ and $k_2$. Observing that for the innermost product we have

$$\prod_{\xi^{k_2 - k_1 - 1} \in \mathcal{P}_K} \left(1 + \frac{1}{\mathcal{N}(p)}\right) = \sum_{\xi^{k_1 - k_2 - 1} \in \mathcal{P}_K} \frac{1}{\mathcal{N}(\mu^2(\alpha))}$$

and interchanging summation we end up looking for a bound for

$$\sum_{\mu^2(\alpha) = 1} \frac{1}{\mathcal{N}(\alpha)} \sum_{\xi^{k_1 - k_2 - 1} \in \mathcal{P}_K} \frac{1}{\mathcal{N}(\alpha) \text{ord}_{\xi}(\alpha)} \ll \log^2 x \sum_{\mu^2(\alpha) = 1} \frac{1}{\mathcal{N}(\alpha) \text{ord}_{\xi}(\alpha)}$$

where $\text{ord}_{\xi}(\alpha) := \min\{k \geq 1 : \xi^k - 1 \in \mathcal{P}_K\}$. General versions of the sum appearing on the right hand side of the last inequality were treated by Ram Murty et.al. [14]. Since we later want to work out the constants we will give a detailed proof of the convergence of the above sum by following the lines of the proof of [10, Lemma 7.8] for the case $K = \mathbb{Q}$.

**Lemma 2.** For any non unit $\xi \in \mathcal{O}_K$ the series

$$\sum_{e = 1}^{\infty} \frac{1}{e} \sum_{\mu^2(\alpha) = 1} \frac{1}{\mathcal{N}(\alpha)}$$

converges.

**Proof.** Let $D$ be a product of principal ideals of the form $(\xi^k - 1)$ i.e.

$$D := \prod_{k \leq x} (\xi^k - 1).$$
We consider the sum

\[ E(x) = \sum_{k \leq x} \sum_{\mu^2(a) = 1} \frac{1}{N(a)} \leq \prod_{p \mid D} \left(1 + \frac{1}{N(p)}\right), \]

where the last sum runs over all prime ideals \( p \) dividing \( D \). Using Lemma 1 we get for the number \( \omega(D) \) of distinct prime divisors of \( D \)

\[ 2^{\omega(D)} \leq N(D) = \prod_{k \leq x} N(\xi^k - 1) \leq \prod_{k \leq x} c^{k^2} = \prod_{k \leq x} c_k \leq c^2 \]

where \( c_k = \sqrt[k]{c} \xi \) with \( c \) as in Lemma 1. Hence \( \omega(D) \ll x^2 \). Using that there are at most \( n \) prime ideals in \( K \) whose norm is a power of \( p \in \mathbb{Z} \) we arrive at

\[ E(x) \leq \left(1 + \frac{1}{p_1}\right)^n \leq \prod_{i=1}^{n} \left(1 + \frac{1}{p_i}\right) = \prod_{i=1}^{n} \left(1 - \frac{1}{p_i^2}\right) \ll \log^n x \]

where \( p_i \) is the \( i \)-th prime and we used Mertens’ formula as well as the fact that the first product converges. As in the proof of \([10, \text{Lemma 7.8}]\) we use partial summation to derive the final result:

\[ \sum_{c=1}^{\infty} \frac{1}{c} \sum_{\mu^2(a) = 1} \frac{1}{N(a)} \leq \left[ E(x) \right]_{x=1}^{\infty} + \int_{1}^{\infty} \frac{E(t)}{t^2} dt \ll \int_{1}^{\infty} \frac{(\log t)^n}{t^2} dt = n! \ll 1. \]

\[ \Box \]

\textit{Proof of Theorem 1} By \([12]\) we have that \( \sum_{\xi} r_\chi(\xi) \gg x^n \) and \([14]\) together with Lemma 2 proves that \( \sum_{\xi \leq x} r_\chi(\xi)^2 \ll x^n \). Plugging this in \([14]\) shows that

\[ \sum_{\xi \leq x} \eta_\chi(\xi) \gg x^n. \]

To finish the proof of Theorem 1 we make use of the fact that there are \( O(x^n) \) algebraic integers \( \xi \in \mathcal{O}_K \) such that \( |\xi| \leq x \) (for a proof in the case of counting totally positive algebraic integers see \([24, \text{Lemma 3.2}]\), the proof of the general case works along the same lines as is mentioned after the proof of \([24, \text{Lemma 3.2}]\)). \[ \Box \]

\section{3. The special case of the Gaussian integers}

To get a good lower bound we need to be more careful in estimating Romanov’s sum than we have been in the proof of Lemma 2. In the special case of sums of prime elements and powers of \( 1 + i \) within the Gaussian integers we will apply the following two results.

\textbf{Lemma 3.} The following bound holds:

\[ \sum_{c \leq 200} \frac{1}{c} \sum_{\mu^2(a) = 1} \frac{1}{N(a)} \leq 1.27095. \]

The proof of Lemma 3 is done by explicit calculation using a computer algebra system. The second result is a bound for the remainder term of Romanov’s sum and we will derive it applying the ideas used by Chen and Sun for sums of primes and powers of 2 (cf. \([11, \text{Lemma 4}]\)).
Lemma 4.

\[
\sum_{e>200} \frac{1}{e} \sum_{\mu^2(a)=1 \atop (a,(1+i))=2 \mid \text{ord}_{1+i}(a)=e} \frac{1}{\mathcal{N}(a)} \leq 0.57749.
\]

Proof. As in Lemma 4 we start with a product of the form

\[
D = \prod_{k=\lceil \frac{x}{2} \rceil}^{x} ((1+i)^k - 1).
\]

Here as Chen and Sun \[1\] we note that it suffices to start with \( k = \lceil \frac{x}{2} \rceil \). We want to deduce an upper bound for the largest prime factor \( p_m \) such that \( p_m \mid \mathcal{N}(D) \). Suppose that \( D \) has exactly \( m \) distinct prime factors \( \pi_1, \ldots, \pi_m \) where we note that their norm is odd since \( 1+i \) and its associates do not divide \( D \). Then we have

\[
p_1^2 \cdots p_m^2 \leq \mathcal{N}(\pi_1) \cdots \mathcal{N}(\pi_m) \leq \prod_{k=\lceil \frac{x}{2} \rceil}^{x} ((1+i)^k + 1)^2 \leq \prod_{k=\lceil \frac{x}{2} \rceil}^{x} 2^k,
\]

where the last inequality holds for \( x \geq 6 \) and the \( p_i \) are the first \( \lceil \frac{x}{2} \rceil \) odd primes. Taking the logarithm on both sides of the last inequality yields

\[
2 \sum_{i=1}^{\lceil \frac{x}{2} \rceil} \log p_i \leq \log 2 \left( 1 + \sum_{k=\lceil \frac{x}{2} \rceil}^{x} k \right).
\]

As Chen and Sun \[1\] we will first use computational methods to get a lower bound for the prime number \( p_m \) and apply \[19, Corollary 2*\] to get an explicit upper bound for \( p_m \) altogether. Using a computer algebra system it is easy to verify that for \( x = 200 \) the product \( D \) has exactly 419 different prime factors and that the 209-th odd prime is \( p_{209} = 1291 \). This together with the bound from \[19, Corollary 2*\] implies that

\[
2p_{\lceil \frac{x}{2} \rceil} \left( 1 - \frac{2}{5 \cdot 1291} \right) < \log 2 \left( 1 + \sum_{k=\lceil \frac{x}{2} \rceil}^{x} k \right)
\]

and hence for \( x \geq 200 \)

\[
p_{\lceil \frac{x}{2} \rceil} < \frac{\log 2}{2} \left( 1 - \frac{2}{5 \cdot 1291} \right)^{-1} \frac{15051}{40000} x^2 < 0.130448 x^2.
\]

For \( E(x) \) defined as in \[17\] in the proof of Lemma 2 analogously as in equation \[18\] we get that

\[
E(x) \leq \prod_{i=1}^{\lceil \frac{x}{2} \rceil} \left( 1 - \frac{1}{p_i^2} \right) \prod_{i=1}^{\lfloor \frac{x}{2} \rfloor} \left( 1 - \frac{1}{p_i} \right)^{-2}.
\]

According to \[1\] Lemma 3] we have for \( x \geq 74 \)

\[
\prod_{3 \leq p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} \leq 0.922913686 \log x.
\]

Furthermore with \( \lceil \frac{x}{2} \rceil = 209 \) we have that

\[
\prod_{i=1}^{\lceil \frac{x}{2} \rceil} \left( 1 - \frac{1}{p_i^2} \right) \leq 0.999749
\]

by direct computation. Altogether this implies that

\[
E(x) \leq 3.997993 \log^2 x - 7.503313 \log x + 3.5206
\]
Using a computer algebra system it is easy to verify that $E(200) \geq 3.33018$ and using partial summation as in Lemma \textit{B} we get that

$$
\sum_{e>200} \frac{1}{e} \sum_{\mu^2(a)=1 \atop (a,(1+i))=1 \atop \text{ord}_1+(a)=e} \frac{1}{N(a)} \leq -\frac{E(200)}{200} + 0.999749 \int_{200}^{\infty} \frac{3.997993 \log^2 x - 7.503313 \log x + 3.5206}{x^2} dx
$$

\leq 0.57749.

\hfill \Box

\textbf{Proof of Theorem} \textit{2} We need to work out constants $c_1$, $c_2$ and $c_3$ with

$$
\left( \sum_{|\xi| \leq x} r_x(\zeta,1+i) \right)^2 \geq c_1 x^4, \quad \sum_{|\xi| \leq x} r_x(\zeta,1+i)^2 \leq c_2 x^2 \quad \text{and} \quad \sum_{|\xi| \leq x} 1 \sim c_3 x^n.
$$

Using inequality (1) a lower bound for the lower density is then given by $\frac{c_1}{c_3} x^n$.

Computing the constant $c_3$ reduces to counting integral lattice points within the circle of radius $x$. It is well known that $\sum_{|\xi| \leq x} 1 \sim \pi x^n$ where we note that the asymptotic rate of growth of the error term is linked to the Gauss circle problem which is frequently listed among famous open problems in number theory (cf. for example \cite{4} p. 365 ff). The constant $c_3$ may therefore be chosen as $c_3 = \pi$.

With $\frac{1+i}{1+i} = \sqrt{2}$ equation (12) yields that $c_1 = \frac{4}{\log^2 2}$ (note that there are four roots of unity in $\mathbb{Z}[i]$, the regulator as well as the class number are 1, $r_1 = 0$ and $r_2 = 1$) and it remains to determine the constant $c_2$.

The constant $c_2$ is the sum of the constant in equation (13) which is $\frac{2}{\log 2}$ and the product of the implied constant $c_1 = \frac{1}{\log 2}$ in equation (15), the constant $\tilde{c}_2 = 416.27$ from Theorem \textit{2}, the constant $\tilde{c}_3 = 1.84844$ from Lemma \textit{2} and Lemma \textit{3} and the constant

$$
\tilde{c}_4 = \prod_{1+i \in \mathfrak{p}} \left( 1 + \frac{1}{N(\mathfrak{p})} \right) = \frac{3}{2}.
$$

\hfill \Box

It remains to prove Theorem \textit{3}. To do so we will apply Erdős’ idea (cf. the proof of \textit{2} Theorem 3) of using covering congruences, i.e. a system of $l$ congruences $a_j \mod m_j$ for $a_j, m_j \in \mathbb{N}$ such that

$$
\mathbb{N} \subset \bigcup_{j=1}^{l} \bigcup_{\lambda=0}^{\infty} (a_j + \lambda m_j).
$$

In Nathanson’s book the details of Erdős’ proof were worked out (cf. \cite{10} p. 204 ff) and the reader can find a proof of the following Lemma there (see \cite{10} Lemma 7.11).

\textbf{Lemma B.} \textit{The six congruences} $0 \mod 2$, $0 \mod 3$, $1 \mod 4$, $3 \mod 8$, $7 \mod 12$ \textit{and} $23 \mod 24$ \textit{form a set of covering congruences.}

As Erdős we will use the covering congruences in Lemma \textit{2} to construct an arithmetic progression consisting of algebraic integers in $\mathbb{Z}[i]$ of odd norm not of the form $\pi + (1+i)^k$ for $\pi \in \mathfrak{P}_{\mathbb{Q}(i)}$.

\textbf{Proof of Theorem} \textit{3} We start by choosing prime elements $\pi_j \in \mathbb{Z}[i]$ such that $(1+i)^{m_j} - 1 \in (\pi_j)$ where the $m_j$ are the moduli appearing in the covering congruences in Lemma \textit{2} We will use the following choice of primes $\pi_j$:

$$
(1+i)^2 \equiv 1 \mod (2+i) \quad \text{and} \quad (1+i)^3 \equiv 1 \mod (2+3i) \\
(1+i)^4 \equiv 1 \mod (1+2i) \quad \text{and} \quad (1+i)^8 \equiv 1 \mod (3) \\
(1+i)^{12} \equiv 1 \mod (3+2i) \quad \text{and} \quad (1+i)^{24} \equiv 1 \mod (7).
$$
By the Chinese Remainder theorem the following congruences describe a unique residue class modulo $m = - (1 + i) \prod_{j=1}^{6} \pi_j = 990 + 990i$ in $\mathbb{Z}[i]$:

$$
\begin{align*}
    x &\equiv 1 \mod (2 + i) & x &\equiv 1 \mod (2 + 3i) & x &\equiv 1 + i \mod (1 + 2i) \\
    x &\equiv -2 + 2i \mod (3) & x &\equiv 8 - 8i \mod (3 + 2i) & x &\equiv 2048 - 2048i \mod (7)
\end{align*}
$$

and $x \equiv 1 \mod (1 + i)$. Note that the first 6 of the previous 7 congruences are of the form $(1 + i)\pi_j \mod \pi_j$ where the $a_j$ are the residue classes from Lemma 15. Since the congruences in Lemma 15 are a covering system and by our choice of $x$ whenever we subtract a power of $1 + i$ from any element in our arithmetic progression we get an element divisible by one of the primes $\pi_j$.

Because of $x \equiv 1 \mod (1 + i)$ we have that $N(x)$ is odd. This holds true since $x$ is of the form $1 + (1 + i)(a + ib)$ for $a, b \in \mathbb{Z}$, $(1 + i)(a + ib) = a - b + i(a + b)$ and $a - b$ and $a + b$ are of the same parity.

Thus non of the elements of the residue class $x \mod (m)$ is of the form $\pi + (1 + i)k$ for $\pi \in \mathcal{P}_Q(i)$ with exemptions being algebraic integers of the form $\pi_j + (1 + i)^k$ for $1 \leq i \leq 6$. Since those 6 sets have density 0 in $\mathbb{Z}[i]$ we are done.

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