Estimating Consequences of 3-Body Forces

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Abstract. Classifying the strengths of three-body forces 3BFs with the condition that observables must be cut-off independent, i.e. renormalised at each order, leads to surprising results with relevance for example for thermal neutron capture on the deuteron. Details and a better bibliography in Ref. [1].

Adding 3-Body Forces (3BFs) a posteriori when theory and data disagree is untenable when predictions are required. Effective Field Theories (EFTs), see e.g. [2] for reviews, provide a model-independent way to estimate their typical strength. For systems of three identical particles in which short-range forces produce shallow two-particle bound states, and in particular for the “pion-less” EFT of Nuclear Physics EFT(π/2), consistency arguments from renormalisation lead to a power-counting, namely a recipe to systematically estimate the typical size of 3BFs in all partial waves and orders, including external currents.

\[
\begin{align*}
(l,\lambda) & = K_l + \lambda D K_l (l,\lambda) = q^\lambda \quad (F,\lambda')
\end{align*}
\]

Figure 1. Left: integral equation of nucleon-deuteron scattering. Right: generic loop correction (rectangle) at N\textsuperscript{0}LO. Thick line \((D)\): NN propagator; thin line \((K_l)\): propagator of the exchanged nucleon; ellipse: LO half off-shell amplitude.

We start from the Faddeev equation, Fig. 1 in the \(l\)th partial wave of the spin doublet \((\lambda = 1)\) and quartet \((\lambda = -\frac{1}{2})\) channels of nucleon-deuteron scattering at leading order LO. The \(NN\) amplitude is given by the leading term of the Effective-Range Expansion. The 3-nucleon amplitude converges for large half off-shell momenta \(p\) as \(p^{-s_0-1}\), with \(s_0\) the solution to the algebraic equation

\[
1 = (-1)^l \frac{2^{1-l} \lambda}{\sqrt{3\pi}} \frac{\Gamma \left[ \frac{l+s+1}{2} \right] \Gamma \left[ \frac{l-s+1}{2} \right]}{\Gamma \left[ \frac{2l+3}{2} \right]} 2F_1 \left[ \frac{l+s+1}{2}, \frac{l-s+1}{2}; \frac{2l+3}{2}; \frac{1}{4} \right].
\]

The asymptotics depends thus crucially on \(l\) and the spin-isospin factor \(\lambda\) only.

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As a description in terms of point-like interactions becomes inadequate beyond a maximum momentum, low-energy observables must be insensitive to details of the amplitude at large $p$, i.e. to form and value of the regulator chosen. This is the fundamental tenet of EFT: Include a 3BF if and only if needed to cancel cut-off dependence in low-energy observables, i.e. as counter-term for divergences which can not be absorbed by renormalising 2-body interactions. 3BFs are thus not added out of phenomenological needs but to guarantee that observables are insensitive to off-shell effects. Counting loop momenta in Fig. 1, the superficial degree of divergence of a higher-order correction is non-negative for

$$\text{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geq 0 \ .$$

The order $n$ at which a 3BF is needed is thus determined just by the order at which a correction to the 3-body amplitude with only 2-body interactions starts to depend on unphysical short-distance behaviour.

Let us re-visit the problem in position space. The Schrödinger equation for the wave-function in the hyper-radial deuteron-nucleon distance $r$ at short distances,

$$\left[ -\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{s_l^2(\lambda)}{r^2} - ME \right] F(r) = 0 \ ,$$

looks like the one for a free particle with centrifugal barrier. One would thus expect $s_l = l+1$ (hyper-spherical co-ordinates!). It is however well-known that the centrifugal term is attractive for three bosons ($\lambda = 1$), contrary to expectations. The wave-function collapses to the origin and seems infinitely sensitive to very-short-distance physics. In order to stabilise the system – or, equivalently, remove dependence on details of the cut-off –, a 3BF must be added at LO. This leads to the famous “limit-cycle” with its Efimov and Thomas effects and universal correlations, see e.g. Ref. [2] for reviews.

On the other hand, 3BFs are demoted if $s_l > l + 1$ because the centrifugal barrier provides more repulsion than expected. The wave-function is pushed further out and thus less sensitive to short-distance details. About half of the 3BFs for $l \leq 2$ are weaker, half stronger than one would expect simplistically, see Table 1. The higher partial-waves follow expectation, as the Faddeev equation is then saturated by the Born approximation.

| partial-wave | bosons | fermions | naive dim. analysis | simplistic | typ. size if $Q^n \sim \frac{1}{\pi^n}$ |
|---------------|--------|----------|---------------------|-----------|----------------------------------|
| $^2S-^2S$     | $^2S-^2S$ | LO       | $N^2\text{LO}$ | $N^2\text{LO}$ | promoted 100% (10%) |
| $^2S-^2D$     | $^2S-^2D$ | $N^{1,2}\text{LO}$ | $N^4\text{LO}$ | $N^4\text{LO}$ | promoted 3% (1%) |
| $^2P-^2P$     | $^2P-^2P$ | $N^{1,2}\text{LO}$ | $N^4\text{LO}$ | $N^4\text{LO}$ | demoted 0.2% (1%) |
| $^2P-^2D$     | $^2P-^2D$ | $N^{1,2}\text{LO}$ | $N^4\text{LO}$ | $N^4\text{LO}$ | demoted 2% (1%) |
| $^4S-^4D$     | $^4S-^4D$ | $N^{1,2}\text{LO}$ | $N^4\text{LO}$ | $N^4\text{LO}$ | demoted 0.1% (1%) |
| higher        | $^4S-^4D$ | $N^{1,2}\text{LO}$ | $N^4\text{LO}$ | $N^4\text{LO}$ | demoted 0.4% (1%) |

Table 1. Order of some leading 3BFs in nucleon-deuteron scattering, indicating if actual values from eqs. [12] are stronger (“promoted”) or weaker (“demoted”) than the simplistic estimate. Last column: typical size of 3BF in EFT(}); in parentheses size from the simplistic estimate.
Demotion might seem an academic dis-advantage – to include some unnecessary higher-order corrections does not improve the accuracy of the result. But demotion is pivotal when predicting the experimental precision necessary to disentangle 3BFs in observables, and here the error-estimate of EFTs is crucial.

An example is the cross-section of triton radiative capture $nd \rightarrow t\gamma$ at thermal energies, see Ref. [3] for details and references. Nuclear models give a spread of $[0.49 \ldots 0.66]$ mb, depending on the 2-nucleon potential and inclusion of the $\Delta(1232)$ [4]. Restoring gauge-invariance reduces the spread to $[0.52 \ldots 0.56]$ mb, but the discrepancy to experiment increases when gauge-invariant three-nucleon currents are added [4]. On the other hand, this low-energy process should be insensitive to details of Physics at 300 MeV. Indeed, the power-counting of 3BFs applies equally with external currents. No new 3BFs are needed up to $N^2$LO to render cut-off independence. The result converges order by order,

$$\sigma_{\text{tot}} = [0.485(\text{LO}) + 0.011(\text{NLO}) + 0.007(\text{N}^2\text{LO})] \text{ mb} = [0.503 \pm 0.003] \text{ mb}, \quad (4)$$

is cut-off independent and compares well with experiment, $[0.509 \pm 0.015]$ mb. In contradistinction to earlier potential models, it is manifestly gauge-invariant.

The potential models reproduce the input of EFT(\#): the nucleon magnetic moments, deuteron and triton binding energies, $NN$ and $nd$ scattering lengths, and the thermal cross-section of $np \rightarrow d\gamma$. That their results vary dramatically is at odds with universality, a key aspect of EFTs: Answers from models with the same input should agree with EFT(\#) within the projected accuracy. This conflict poses a puzzle whose resolution is not (yet) clear.

With these findings, EFT(\#) is a self-consistent field theory which contains the minimal number of interactions at each order to be renormalisable. Each 3-body counter-term gives rise to one subtraction-constant, fixed by a 3-body datum. This method is applicable to any EFT with an infinite number of diagrams at LO, e.g. because of shallow bound-states. It leads at each order and to the prescribed level of accuracy to a cut-off independent theory with the smallest number of experimental input-parameters. The power-counting is thus not constructed by educated guesswork but by investigations of the renormalisation-group properties of couplings and observables using rigorous methodology.

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