2-subcoloring is NP-complete for planar comparability graphs

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Abstract

A $k$-subcoloring of a graph is a partition of the vertex set into at most $k$ cluster graphs, that is, graphs with no induced $P_3$. 2-subcoloring is known to be NP-complete for comparability graphs and three subclasses of planar graphs, namely triangle-free planar graphs with maximum degree 4, planar perfect graphs with maximum degree 4, and planar graphs with girth 5. We show that 2-subcoloring is also NP-complete for planar comparability graphs with maximum degree 4.

1 Introduction

A $k$-subcoloring of a graph is a partition of the vertex set into at most $k$ cluster graphs, that is, graphs with no induced $P_3$. Unlike $k$-coloring, $k$-subcoloring is already NP-complete for $k = 2$.

Theorem 1. 2-subcoloring is NP-complete for the following classes:

1. $(K_4, bull, house, butterfly, gem, odd-hole)$-free graphs with maximum degree 5 [1],
2. triangle-free planar graphs with maximum degree 4 [2, 3],
3. $(K_{1,3}, K_4, K_4^-, C_4, odd-hole)$ planar graphs [4],
4. planar graphs with girth 5 [7].

A graph $G$ is $(d_1, \ldots, d_k)$-colorable if the vertex set of $G$ can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph induced by the vertices of $V_i$ has maximum degree at most $d_i$ for every $1 \leq i \leq k$. Notice that every $(1,1)$-colorable graph is 2-subcolorable. Moreover, on triangle-free graphs, $(1,1)$-colorable is equivalent to 2-subcolorable. As it is well known, for every $a, b \geq 0$, every graph with maximum degree $a+b+1$ is $(a,b)$-colorable [6]. Thus, every graph with maximum degree 3 is 2-subcolorable, so that the degree bound of 4 in Theorems 1.(2) and 1.(3) is best possible.

Notice that the graphs in Theorem 1.(1) are comparability graphs since they are (bull, house, odd-hole)-free [5]. Our main result restricts the class in Theorem 1.(1) to planar graphs and lowers the maximum degree from 5 to 4.

Theorem 2. Let $\mathcal{G}$ denote the class of $(K_4, bull, house, butterfly, gem, odd-hole)$-free planar graphs with maximum degree 4. 2-subcoloring is NP-complete for $\mathcal{G}$.

2 Main result

The reduction is from 2-subcoloring (or equivalently $(1,1)$-coloring) on triangle-free planar graphs with maximum degree 4, which is NP-complete by Theorem 1.(2). From an instance graph $G$ of this problem, we construct a graph $G'$ in $\mathcal{G}$. Every vertex $v$ of $G$ is replaced by a copy $H_v$ of the vertex gadget $H$ depicted in Figure 1. For every edge $uv$ of $G$, we use two copies of the edge gadget $E$ depicted in Figure 2 to connect $H_u$ and $H_v$ as follows:
We identify the vertex $x_1$ of the first (resp. second) edge gadget with a vertex $a_{2p}$ (resp. $a_{2p+1}$) of $H_u$, with $0 \leq p \leq 3$.

We identify the vertex $x_2$ of the first (resp. second) edge gadget with a vertex $a_i$ (resp. $a_j$) of $H_v$ such that $\min(i, j) \equiv 0 \pmod{2}$, $|j - i| = 1$, and no edge crossing is created.

It is easy to check that $G'$ can be made planar and with maximum degree 4. Moreover, $G'$ is $(K_4, \text{bull, house, gem, butterfly})$-free. By removing the vertices whose neighborhood induces a $P_3$, we obtain a bipartite graph. This shows that $G'$ is odd-hole free. Thus $G'$ belongs to $\mathcal{G}$.

![Figure 1: The vertex gadget $H$.](image1)

![Figure 2: The edge gadget $E$.](image2)

Let us show that $G'$ is 2-subcolorable if and only if $G$ is 2-subcolorable. Given a 2-subcoloring of a graph, we say that a vertex $p$ is saturated if there exists a monochromatic edge $pq$ and is unsaturated otherwise. We will need the following properties of $E$:

1. In every 2-subcoloring of $E \setminus \{x_1, x_2, y_1, y_2\}$, the vertices $z_1$ and $z_2$ get distinct colors and are saturated.
2. In every 2-subcoloring of $E \setminus \{x_1, x_2\}$, the vertices $y_1$ and $y_2$ get distinct colors and are not saturated.
3. There exists a 2-subcoloring of $E$ such that the vertices $x_1$ and $x_2$ get distinct colors and are not saturated.
4. In every 2-subcoloring of $E$ such that the vertices $x_1$ and $x_2$ get the same color, exactly one vertex in $\{x_1, x_2\}$ is saturated.

The six vertices labeled $a_i$ in $H$ are called ports. Using properties (1) and (2), we obtain that every 2-subcoloring of $H$ (in colors red and blue) forces the color of many vertices (see Figure 1). In particular, all the ports in $H$ get the same color. This common color is said to be the color of $H_v$ corresponds to the color of $v$ in a 2-subcoloring of $G$. We also check that in every 2-subcoloring of $H$, at most one of the ports is not saturated.

Suppose that $uv$ is an edge in $G$. Consider the 2-subcolorings of the subgraph of $G'$ induced by $H_u, H_v$, and the two edge gadgets for the edge $uv$. If distinct colors are given to $H_u$ and $H_v$, then
this 2-subcoloring can be extended to the edge gadgets using property (3). Since this extension does not saturate any of the considered ports of $H_u$ and $H_v$, $H_u$ can be connected to any number of vertex gadgets with the color distinct from the color of $H_u$. If the same color is given to $H_u$ and $H_v$, then this 2-subcoloring can be extended using property (4). However, this coloring extension saturates the unique unsaturated port in both $H_u$ and $H_v$. Thus, $H_u$ can be connected to at most one vertex gadget with the same color as $H_u$.

This shows that $G'$ is 2-subcolorable if and only if $G$ is 2-subcolorable.

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