L-MOMENTS METHOD TO ESTIMATE GAMMA/GOMPERTZ DISTRIBUTION
BASED ON RANKED SET SAMPLING DESIGNS

NURAN M. HASSAN\textsuperscript{1,2,}\textsuperscript{*}, EL-HOUSSAINY A. RADY\textsuperscript{1}, NASR I. RASHWAN\textsuperscript{3}

\textsuperscript{1}Department of Applied Statistics and Econometric, Faculty of Graduate Studies for Statistical Research, Cairo University, Cairo, Egypt
\textsuperscript{2}Department of Basic Science, Faculty of Engineering, Modern Academy, Cairo, Egypt
\textsuperscript{3}Department of of Statistics, Mathematics and Insurance, Faculty of Commerce, Tanta University, Tanta, Egypt

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Abstract. This paper is concerned with the estimation problem using L-moments method of estimation for the unknown parameters of Gamma/Gompertz distribution based ranked set sampling methods. Computer simulation results are given to compare the efficiencies for the Neoteric ranked set sampling (NRSS) as a dependent sampling technique, with ranked set sampling (RSS), median ranked set sampling (MRSS), percentile ranked set sampling (PRSS) as an independent sampling techniques, and simple random sampling (SRS).

Keywords: ranked set sampling; neoteric ranked set sampling; percentile ranked set sampling; median ranked set sampling; Gamma gompertz distribution; L-moments method.

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1. INTRODUCTION

McIntyre \textsuperscript{[13]} submitted a sampling method for the estimation of pasture and forage yields in the field of agriculture it is called RSS. He aim was to maintain the unbiasedness property of the estimators like in SRS estimator with take minimum information about the estimator provided

\textsuperscript{*}Corresponding author

E-mail address: nuru\_medhat@yahoo.com

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by visual inspection or by any inexpensive method. The McIntyre’s technique was used first by Halls and Dell [7]. They found experimentally that RSS was more efficient than SRS. They also stated the name ranked set sampling that is in current use. Dell and Clutter [5] appeared that the mean estimator under imperfect RSS remains an unbiased estimator of a population mean.

RSS method as suggested by McIntyre [13] may be amended to come up with new sampling methods that can be made more efficient than the usual RSS method with small mean square errors (MSE). Muttlak [15] used MRSS to estimate the population mean and he showed that it is more efficient than the usual RSS method. Muttlak [16] introduced PRSS procedure with different value of $0 \leq p \leq 1$ for estimating the population mean. Al-Saleh and Al-Hadrami [2] used different set sizes for RSS technique for estimating the mean of the Normal and Exponential distributions, they explored that this procedure is more useful than RSS for estimating the mean of symmetric distributions. Khamnei and Mayan [12] estimated the parameters of Gumbel distribution based on SRS and RSS with compared the estimators of these two methods. A recently developed modification of RSS, Zamanzade and Al-Omari [21] introduced the NRSS technique and they showed that the efficiency of the estimators based on NRSS are greater than their estimators using RSS and SRS techniques. Esemen and Grler [8] estimated the parameters of Generalized Rayleigh distribution based on RSS and it is some modification.

The Gamma/Gompertz (GG) distribution has been used to estimate customer lifetime and a model of death rate risks. GG distribution was introduced by Bemmaor and Glady [4]. The probability density function (PDF), the cumulative distribution function (CDF), and quantile function (QF) of the GG distribution are, respectively, given by [14]

\[
f(x; b, s, \beta) = \frac{bse^{bx}B^s}{(\beta - 1 + e^{bx})^{s+1}}; \quad 0 \leq x \leq \infty, b > 0, s > 0, \beta > 0,
\]

\[
F(x; b, s, \beta) = 1 - \frac{\beta^s}{(\beta - 1 + e^{bx})^s}; \quad 0 \leq x \leq \infty, b > 0, s > 0, \beta > 0,
\]

and

\[
x(F) = \frac{1}{b} \ln[\beta(1 - F)^{-\frac{1}{s}} - \beta + 1]
\]

where $b$ is the scale parameter, $s$ and $\beta$ are the shape parameters. (see Fig. 1)
In this paper, the performance of NRSS method has been compared with the other ranked set sampling methods using L-moment method of estimation with the usual SRS technique for GG distribution parameters. The remaining part of this paper is organized as follows: In Section 2, introduced ranked set sampling methods. L-moment method of estimation will be discussed in

**Figure 1.** Three parameters Gamma/Gompertz distribution density function.
Section 3. In Section 4, computer simulation results are given to compare the efficiency for the estimators based on SRS with its counterparts RSS, MRSS, PRSS, and NRSS. It is shown that for the GG distributions considered in this study, NRSS has less mean square error and more efficiency than all the other methods. Conclusions are presented in Section 5.

2. Some Ranked Set Sampling Techniques

In this section, different sampling techniques for selection of units in the sample are RSS, MRSS, PRSS and NRSS will be considered with explanation figures for each case for each one of these techniques.

2.1. Ranked set sampling

The original idea of ranked set sampling (RSS) was showed by McIntyre’s [13]. Ranked set sampling is a methodology that can improve the efficiency of techniques such as estimation without taking large a number of substantial observations. In addition, it is designed to decrease the number of measured observations required to achieve the desired precision in making inferences. The RSS scheme can be described as follows:[3]

1. In order to draw a sample of size \( n \), we identify \( n^2 \) units from the target population.
2. Randomly allocate these units to \( n \) sets each of size \( n \). The \( n \) units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.
3. From the first set of \( n \) units, the smallest ranked unit is measured. From the second set of \( n \) units, the second smallest ranked unit is measured. The process is continued until from the \( n^{th} \) set of \( n \) units the largest ranked until is measured. (see Fig. 2)
4. Repeat steps 1 through 3 for \( m \) cycles to obtain a sample of size \( nm \).

Let \( \{X_1, X_2, \ldots, X_n\} \) be a random sample with probability density function \( f(x) \) with mean \( \mu \) and variance \( \sigma^2 \). Let \( \{X_{(1)}_1, X_{(1)}_2, \ldots, X_{(1)}_{n_1}; X_{(2)}_1, X_{(2)}_2, \ldots, X_{(2)}_{n_2}; \ldots, X_{(n)}_1, X_{(n)}_2, \ldots, X_{(n)}_{n_n}\} \) be independent random variables. Let \( X_{(i,n)} \) denotes the \( i^{th} \) order statistic from the \( i^{th} \) sample of size \( n \) where \( (i = 1, 2, \ldots, n) \). The cycle may be repeated \( m \) times to get \( nm \) units. Let \( X_{(i)} \) denotes the \( j^{th} \) cycle of size \( m \) where \( (j = 1, 2, \ldots, m) \). Then PDF and CDF of \( X_{(i)} \) are given by [19]
and

\[ F_n(x_{(ij)}) = \sum_{t=i}^{n} \frac{n!}{t!(n-t)!} \left[ F(x_{(ij)}; \theta) \right]^t \left[ 1 - F(x_{(ij)}; \theta) \right]^{n-t}. \]  

(5)

Figure 2. RSS design in cases of a) \( n \) is an even b) \( n \) is an odd

2.2. Median ranked set sampling

Muttlak \cite{15} introduced Median ranked set sampling (MRSS) scheme for estimating the population mean. MRSS procedure is explained as follows:

(1) In order to draw a sample of size \( n \), we identify \( n^2 \) units from the target population.

(2) Randomly allocate these units to \( n \) sets each of size \( n \). The \( n \) units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.

(3) for even sample size \( n \), identify \( n^2 \) units from the target population. Each sample is ranked in itself as in ranked set sampling design. Select the \( \frac{n}{2} \)th smallest rank from the first \( \frac{n}{2} \) sets and select the \( \frac{n}{2} + \frac{1}{2} \)th smallest rank from the other \( \frac{n}{2} \) sets. Similarly, for odd sample size \( n \), Select the \( \frac{n+1}{2} \)th smallest rank from all sets.

(4) Repeat steps 1 through 3 for \( m \) cycles to draw a sample of size \( nm \). (see Fig. 3)

The PDF of \( X_{(ij)} \) that ranked by MRSS when \( n \) is an even is given by \cite{19}

\[ f_n(x_{(ij)}) = \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} f(x_{(\frac{n}{2})j}; \theta) \left[ F(x_{(\frac{n}{2})j}; \theta) \right]^{\frac{n}{2}} \left[ 1 - F(x_{(\frac{n}{2})j}; \theta) \right]^{\frac{n}{2}} \]

\[ \left[ \frac{n!}{(\frac{n}{2})!(\frac{n}{2})!} f(x_{(\frac{n}{2}+\frac{1}{2})j}; \theta) \left[ F(x_{(\frac{n}{2}+\frac{1}{2})j}; \theta) \right]^{\frac{n}{2}} \left[ 1 - F(x_{(\frac{n}{2}+\frac{1}{2})j}; \theta) \right]^{\frac{n}{2}} \right]. \]  

(6)
Otherwise when \( n \) is an odd is given by

\[
f_n(x_{(i)j}) = \frac{n!}{(\frac{n-1}{2})!(\frac{n-1}{2})!} f(x_{(\frac{n+1}{2})j}; \theta) [F(x_{(\frac{n+1}{2})j}; \theta)]^{\frac{n-1}{2}} [1 - F(x_{(\frac{n+1}{2})j}; \theta)]^{\frac{n-1}{2}}.
\]

\[
\text{Figure 3. MRSS design in cases of a) } n \text{ is an even b) } n \text{ is an odd}
\]

\[\text{2.3. Percentile ranked set sampling}\]

Muttlak \[16\] used percentile ranked set sampling (PRSS) technique with a different values of \( 0 \leq p \leq 1 \) for estimating the population mean. PRSS procedure is explained as follows: \^[18]\]

(1) In order to draw a sample of size \( n \), we identify \( n^2 \) units from the target population.

(2) Randomly allocate these units to \( n \) sets each of size \( n \). The \( n \) units in each sample are ranked visually or by any inexpensive method with respect to the variable of interest.

(3) For even sample size \( n \), select the \((p(n+1))^{th}\) smallest ranked unit from the first \( \frac{n}{2} \) sets and select the \((q(n+1))^{th}\) smallest ranked unit from the other \( \frac{n}{2} \) sets. Similarly, for odd sample size \( n \), select the \((p(n+1))^{th}\) smallest ranked unit from first \( \frac{n-1}{2} \) sets. Select the \((q(n+1))^{th}\) smallest ranked unit from second \( \frac{n-1}{2} \) sets and the median with rank \( \frac{n+1}{2} \) from the remaining set.

(4) Repeat steps 1 through 3 for \( m \) cycles to draw a sample of size \( nm \). (see Fig. 4)

The PDF of \( X_{(i)j} \) that ranked by PRSS when \( n \) is an even is given by

\[
f_n(x_{(i)j}) = \frac{n!}{(p(n+1) - 1)!(n - p(n+1))!} f(x_{(p(n+1))j}; \theta) [F(x_{(p(n+1))j}; \theta)]^{p(n+1)-1} [1 - F(x_{(p(n+1))j}; \theta)]^{n-p(n+1)}
\]
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\[
\begin{align*}
\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x(q(n+1))j; \theta) [F(x(q(n+1))j; \theta)]^{q(n+1)-1} \\
[1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)},
\end{align*}
\]

Otherwise when \( n \) is an odd is given by

\[
\begin{align*}
\frac{n!}{(p(n+1)-1)!(n-p(n+1))!} f(x(p(n+1))j; \theta) [F(x(p(n+1))j; \theta)]^{p(n+1)-1} [1 - F(x(p(n+1))j; \theta)]^{n-p(n+1)} \\
[1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} [1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x(q(n+1))j; \theta) [F(x(q(n+1))j; \theta)]^{q(n+1)-1} [1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
[1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
\frac{n!}{(n+1)!} f(x(n+1)j; \theta) [F(x(n+1)j; \theta)]^{n+1} [1 - F(x(n+1)j; \theta)]^{n+1} \\
[1 - F(x(n+1)j; \theta)]^{n+1} \\
\end{align*}
\]

\[
\begin{align*}
\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x(q(n+1))j; \theta) [F(x(q(n+1))j; \theta)]^{q(n+1)-1} [1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
[1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} [1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
\frac{n!}{(q(n+1)-1)!(n-q(n+1))!} f(x(q(n+1))j; \theta) [F(x(q(n+1))j; \theta)]^{q(n+1)-1} [1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
[1 - F(x(q(n+1))j; \theta)]^{n-q(n+1)} \\
\end{align*}
\]

FIGURE 4. PRSS design in cases of a) \( n \) is an even b) \( n \) is an odd

2.4. Neoteric ranked set sampling

Zamanzade and Al-Omari \[21\] have developed modification of RSS called NRSS. NRSS differs from the original RSS scheme by the composition of a single set of \( n^2 \) units, instead of \( n \) sets of size \( n \). This strategy has been shown to be effective, producing more efficient estimators for the population mean. The NRSS technique can be described as follows:

1. Select a simple random sample of size \( n^2 \) units from the target finite population.
(2) Ranked the $n^2$ selected units in an increasing magnitude based on a visual inspection or any other cost-free method with respect to a variable of interest.

(3) For odd sample size $n$, select the $(\frac{n+1}{2} + (i-1)n)th$ ranked unit for $(i = 1, 2, \ldots n)$. Otherwise for even sample size $n$, if $i$ is an odd, then select the $(\frac{n+2}{2} + (i-1)n)th$ ranked unit. While if $i$ is an even, then select the $(\frac{n}{2} + (i-1)n)th$ ranked unit for $(i = 1, 2, \ldots n)$.

(4) Repeat steps 1 through 3 $m$ cycles if needed to obtain a NRSS of size $nm$.

The PDF of $X_{(i,j)}$ that ranked by NRSS is given by

$$f_n(x_{(i,j)}) = \frac{n^2!}{(k(i) - k(i-1) - 1)!} f(x_{k(i)}; \theta) \left[ F(x_{k(i)}; \theta) - F(x_{k(i-1)}; \theta) \right]^{k(i)-k(i-1)-1},$$

where

$$k(i) = \begin{cases} \frac{n+1}{2} + (i-1)n & \text{n odd} \\ \frac{n+2}{2} + (i-1)n & \text{n even, i odd} \\ \frac{n}{2} + (i-1)n & \text{n even, i even} \end{cases}$$

where $(i = 1, 2, \ldots n)$, $k(0) = 0$ and $x_{k(0)} \equiv -\infty$.

**Figure 5.** NRSS design in cases of a) $n$ is an even b) $n$ is an odd

3. **Estimation Based on L moments**

Hosking [10] formalized methods for the estimation of statistics are measures of higher moments, skewness and kurtosis, are more robust with respect to sample size and the presence of outliers. These measures, called L-moments, are based on expectations of linear combinations
of order statistics. Indeed, Vogel and Fennessey\textsuperscript{[20]} observed that the introduction of the theory of L-moments by Hosking is probably the single most significant recent advance in relating to our understanding of extreme events.

The method of L-moments is a parameter estimation technique that is conceptually the same as the well-known “method of (product) moments”. Specifically, the L-moments is a method of parameter estimation in which the parameters of a distribution are chosen so as to equate the theoretical L-moments of the distribution ($\lambda_r$) to the sample L-moments ($\hat{\lambda}_r$) for $r \leq p$ where $p$ is the number of parameters of a probability distribution.\textsuperscript{[11]}

3.1. Theoretical L-moments

The theoretical L-moments for a real-valued random variable $X$ with a QF of $x(F)$ are defined from the expectations of order statistics. The order statistics of $X$ for a sample of size $n$ are formed by the ascending order $X_{(1;n)} \leq X_{(2;n)} \leq \ldots \leq X_{(l;n)} \leq \ldots \leq X_{(n;n)}$. The theoretical L-moments ($\lambda_r$) are \textsuperscript{[11]}

\begin{equation}
\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E[X_{r-k;r}], \quad r = 1, 2, \ldots
\end{equation}

where $r$ is the order of the L-moment, and $E[X_{r-k;r}]$ is the expectation of the $r-k$ order statistic of a sample of size $r$ and we obtain it as follows:

\begin{equation}
E[X_{r-k;r}] = \frac{6s}{b} \sum_{l=0}^{r-k-1} C_{r-k}^r C_{r-k}^{r-l}(-1)^l \beta^{s(r-l)}(\beta - 1)^{-s(r-l)-1}\Phi_{(s(r-l)+1)}((1 - \beta)^{-1}, 2, 1) \quad \text{for } \beta \neq 1,
\end{equation}

where the Lerch transcendent function is $\Phi_{(s(r-l)+1)}((1 - \beta)^{-1}, 2, 1) = \sum_{h=0}^{\infty} \frac{(s(r-l)+1)_h}{h!} \frac{(1 - \beta)^{-h}}{(h+1)^2}$ for $\beta \neq 1$. By substituting the (12) in (11), the population L-moments of order $r$ for the GG is given by:

\begin{equation}
\lambda_r = \frac{6s}{b} \sum_{k=0}^{r-1} \sum_{l=0}^{r-k-1} \frac{[(r-l)!]^2}{(r-k-1-l)! k! r!} \beta^{s(r-l)}(\beta - 1)^{-s(r-l)-1}\Phi_{(s(r-l)+1)}((1 - \beta)^{-1}, 2, 1).
\end{equation}

If $\beta$ has a special case when equal 1. The $E[X_{r-k;r}]$ is going to be

\begin{equation}
E[X_{r-k;r}] = bs \sum_{l=0}^{r-k-1} C_{r-k}^r C_{r-k}^{r-l}(-1)^l \frac{\Gamma(2)}{(b(s+1)+r-k-1-l)^2}.
\end{equation}
The first three theoretical L-moments can be obtained by taking \( r = 1, 2 \) and 3 in (13) as follows:

\[
\lambda_1 = E[X] = \frac{6s}{b} \beta^s (\beta - 1)^{-s-1} \Phi_{(s+1)}((1 - \beta)^{-1}, 2, 1),
\]

\[
\lambda_2 = \frac{1}{2} E[X_{2:2} - X_{1:2}] = \frac{6s}{b} \left[ -\frac{1}{2} \beta^{2s} (\beta - 1)^{-2s-1} \Phi_{(2s+1)}((1 - \beta)^{-1}, 2, 1) \right. \\
\left. - \frac{1}{2} \beta^s (\beta - 1)^{-s-1} \Phi_{(s+1)}((1 - \beta)^{-1}, 2, 1) \right],
\]

and

\[
\lambda_3 = \frac{1}{3} E[X_{3:3} - 2X_{2:3} + X_{1:3}] = \frac{6s}{b} \left[ \left( -\frac{7}{3} \beta^{3s} \right) (\beta - 1)^{-3s-1} \Phi_{(3s+1)}((1 - \beta)^{-1}, 2, 1) \right.
\]
\[
+ \left( \frac{10}{3} \beta^{2s} \right) (\beta - 1)^{-2s-1} \Phi_{(2s+1)}((1 - \beta)^{-1}, 2, 1) + \frac{1}{3} \beta^s (\beta - 1)^{-s-1} \Phi_{(s+1)}((1 - \beta)^{-1}, 2, 1) \right].
\]

### 3.2. Sample L-moments

The sample L-moments are computed from the sample order statistics. Elamir and Seheult [6] proposed the following formula to calculate sample L-moments:

\[
\hat{\lambda}_r = \frac{1}{r} \sum_{i=0}^{n-r+1} \sum_{k=0}^{r-1} \frac{(-1)^k C_r^{-1} C_{r-k-1} C_k^{n-i}}{C_r^n} x_{i:n}, \quad r = 1, 2, \ldots
\]

L-moment estimators for \( b, s \) and \( \beta \) can be found as follows:

\[
\frac{6s}{b} \beta^s (\beta - 1)^{-s-1} \Phi_{(s+1)}((1 - \beta)^{-1}, 2, 1) = \bar{x},
\]

\[
\frac{6s}{b} \left( -\frac{1}{2} \beta^{2s} \right) (\beta - 1)^{-2s-1} \Phi_{(2s+1)}((1 - \beta)^{-1}, 2, 1) \\
- \frac{1}{2} \beta^s (\beta - 1)^{-s-1} \Phi_{(s+1)}((1 - \beta)^{-1}, 2, 1) \right] = \sum_{i=1}^{n} \frac{(i - 1) - (n - i)}{n(n - 1)} x_{i:n},
\]
4. Simulation Study

Based on 1000 replications, a computer simulation is conducted to study the behavior of the efficiency of the sample mean using RSS, MRSS, PRSS with \( p = 0.2 \), and NRSS with respect to SRS. Random observations are generated from Gamma/Gompertz distribution based on different sample sizes \( n = 15, 20 \) and 25, for cycles \( m = 1 \) and 3, different parameters values \( GG(b, s, beta) = \{ GG(3, 3, 3), GG(2, 2, 4) \} \), were generated by the R package. Nadjafi [17] used 300 replications in his paper because of that the replications were extended to be more than 300. The relative efficiency (RE) of the estimator of any of the RSS methods with respect to the usual estimator SRS is defined as [9]

\[
RE(\hat{\theta}_{SRS}, \hat{\theta}_{RSS}) = \frac{MSE(\hat{\theta}_{SRS})}{MSE(\hat{\theta}_{RSS})},
\]

where \( MSE(\hat{\theta}) = \frac{1}{1000} \sum_{j=1}^{1000} (\hat{\theta} - \theta)^2 \) and \( bias = \hat{\theta} - \theta \).

If \( RE(\hat{\theta}_{SRS}, \hat{\theta}_{RSS}) > 1 \), then \( \hat{\theta}_{RSS} \) is better than \( \hat{\theta}_{SRS} \). The simulation results are summarized in a list of tables and list of figures. The results of the table and these figures show the relative efficiency (the ratio of the mean square errors) of the estimators relative to SRS.
| Distribution | $n$ | $\hat{b}$ | $\hat{s}$ | $\hat{\beta}$ | $\hat{b}$ | $\hat{s}$ | $\hat{\beta}$ |
|--------------|-----|-----------|---------|-------------|-----------|---------|-------------|
| GG(3,3,3)    | 15  | 0.0199    | 0.0170  | 0.0054      | 0.0089    | 0.0049  | 0.0063      | 0.0011    | 0.0010    |
|              | 20  | 0.0273    | 0.0065  | 0.0062      | 0.0198    | 0.0031  | 0.0199      | 0.0028    | 0.0013    |
|              | 25  | 0.0149    | 0.0069  | 0.0067      | 0.0055    | 0.0033  | 0.0034      | 0.0016    | 0.0005    |
|              | 15  | 0.0142    | 0.0095  | 0.0045      | 0.0059    | 0.0028  | 0.0057      | 0.0002    | 0.0003    |
|              | 20  | 0.0187    | 0.0055  | 0.0050      | 0.0035    | 0.0029  | 0.0020      | 0.0039    | 0.0004    |
|              | 25  | 0.0166    | 0.0053  | 0.0053      | 0.0042    | 0.0044  | 0.0030      | 0.0023    | 0.0010    |
| GG(2,2,4)    | 15  | 0.0859    | 0.0685  | 0.0207      | 0.0157    | 0.0079  | 0.0028      | 0.0071    | 0.0014    |
|              | 20  | 0.0031    | 0.0022  | 0.0026      | 0.0012    | 0.0029  | 0.0007      | 0.0016    | 0.0001    |
|              | 25  | 0.0300    | 0.0049  | 0.0100      | 0.0015    | 0.0061  | 0.0023      | 0.0068    | 0.0012    |
|              | 15  | 0.0558    | 0.0085  | 0.0022      | 0.0081    | 0.0034  | 0.0015      | 0.0019    | 0.0010    |
|              | 20  | 0.0221    | 0.0021  | 0.0017      | 0.0051    | 0.0028  | 0.0005      | 0.0015    | 0.0004    |
|              | 25  | 0.0235    | 0.0013  | 0.0005      | 0.0009    | 0.0009  | 0.0038      | 0.0002    | 0.0006    |
|              | 15  | 0.0328    | 0.0086  | 0.0053      | 0.0059    | 0.0039  | 0.0009      | 0.0022    | 0.0057    |
|              | 20  | 0.0820    | 0.0177  | 0.0175      | 0.0102    | 0.0053  | 0.0019      | 0.0044    | 0.0017    |
|              | 25  | 0.0142    | 0.0011  | 0.0060      | 0.0011    | 0.0026  | 0.0075      | 0.0017    | 0.0009    |
| Distribution | n  | SRS    | RSS  | MRSS  | PRSS  | NRSS  |
|--------------|----|--------|------|-------|-------|-------|
|              |    | $m = 1$  | $m = 3$ | $m = 1$ | $m = 3$ | $m = 1$ | $m = 3$ |
| GG(3,3,3)    | 15 | $b$    | $\hat{b}$ | $\hat{b}$ | $\hat{b}$ | $\hat{b}$ | $\hat{b}$ |
|              |    | $\text{Estimator}$ | $\text{Estimator}$ | $\text{Estimator}$ | $\text{Estimator}$ | $\text{Estimator}$ | $\text{Estimator}$ |
|              |    | $\text{bias}$ | $\text{bias}$ | $\text{bias}$ | $\text{bias}$ | $\text{bias}$ | $\text{bias}$ |
|              | 15 | 3.0638 | 0.0206 | 0.0047 | 3.0480 | 3.0380 | 3.0163 | 3.0902 | 3.0211 | 3.0433 |
|              | 20 | 3.0272 | 0.0090 | 0.0064 | 3.0221 | 3.0684 | 3.0105 | 3.0266 | 3.0298 | 3.0471 |
|              | 25 | 3.0631 | 0.0203 | 0.0248 | 3.0313 | 3.1066 | 3.0246 | 3.0642 | 3.0732 | 3.0481 |
|              |    | 0.0630 | 0.0203 | 0.0248 | 0.0313 | 0.1006 | 0.0246 | 0.0642 | 0.0732 | 0.0481 |
|              | 15 | 3.0427 | 0.0103 | 2.9929 | 3.0286 | 3.0176 | 3.0033 | 3.0570 | 3.0081 | 3.0211 |
|              | 20 | 3.0137 | 2.9986 | 3.0175 | 3.0068 | 3.0407 | 3.0015 | 3.0992 | 3.0140 | 3.0239 |
|              | 25 | 3.0384 | 3.0056 | 3.0777 | 3.0159 | 3.0672 | 3.0097 | 3.0367 | 3.0463 | 3.0251 |
|              |    | 0.0384 | 0.0056 | 0.0077 | 0.0159 | 0.0672 | 0.0097 | 0.0367 | 0.0463 | 0.0251 |
|              | 15 | 2.9780 | 3.0067 | 3.0285 | 2.9925 | 3.0100 | 3.0147 | 2.9843 | 3.0130 | 3.0104 |
|              | 20 | 3.0033 | 3.0020 | 3.0175 | 3.0150 | 2.9945 | 3.0120 | 3.0167 | 3.0101 | 3.0085 |
|              | 25 | 2.9891 | 3.0178 | 3.0188 | 3.0039 | 2.9719 | 3.0118 | 2.9992 | 2.9882 | 3.0054 |
|              |    | 0.0109 | 0.0178 | 0.0188 | 0.0039 | 0.0281 | 0.0118 | 0.0008 | 0.0118 | 0.0054 |
|              | 15 | 2.0775 | 2.0131 | 2.0424 | 2.0848 | 2.0232 | 2.0856 | 2.0418 | 2.0307 | 2.0600 |
|              | 20 | 2.0485 | 2.0146 | 2.0756 | 2.0204 | 2.0881 | 2.0605 | 2.0474 | 2.0283 | 2.0470 |
|              | 25 | 2.0643 | 2.0366 | 2.0268 | 2.0668 | 2.0567 | 2.0561 | 2.0325 | 2.0306 | 2.0483 |
|              |    | 0.0643 | 0.0366 | 0.0268 | 0.0668 | 0.0567 | 0.0561 | 0.0325 | 0.0306 | 0.0845 |
|              | 15 | 1.9967 | 1.9996 | 1.9942 | 2.0083 | 1.9740 | 2.0075 | 1.9879 | 2.0089 | 1.9966 |
|              | 20 | 2.0104 | 1.9916 | 2.0088 | 1.9833 | 1.9987 | 1.9975 | 1.9894 | 2.0010 | 1.9899 |
|              | 25 | 2.0050 | 2.0088 | 1.9879 | 2.0189 | 2.0038 | 2.0001 | 1.9879 | 1.9899 | 1.9911 |
|              |    | 0.0050 | 0.0088 | 0.0121 | 0.0189 | 0.0038 | 0.0001 | 0.0121 | 0.0011 | 0.0089 |
|              | 15 | 4.0992 | 4.0468 | 4.0757 | 4.0904 | 4.0905 | 4.0893 | 4.0843 | 4.0583 | 4.0851 |
|              | 20 | 4.0533 | 4.0594 | 4.0791 | 4.0763 | 4.1018 | 4.0839 | 4.0854 | 4.0565 | 4.0853 |
|              | 25 | 4.0764 | 4.0504 | 4.0735 | 4.0582 | 4.0722 | 4.0776 | 4.0771 | 4.0609 | 4.0861 |
From Table 1 and Table 2, it can be observed that:

- In almost all cases, the biases are very small.
- The values of the all estimators for \((b, s and beta)\) is very close to the real values of \(b, s and beta\).
- In almost all cases, MSEs of all estimators for \((b, s and beta)\) based on RSS, MRSS, PRSS and NRSS are smaller than MSEs of the estimators based on SRS.
- In almost all cases, MSEs of all estimators based on RSS, MRSS, PRSS and NRSS decrease as the cycle sizes increase.
- When the values of \(b\) and \(s\) increase, the MSEs of all estimators based on RSS, MRSS, PRSS and NRSS decrease.
- MSEs of the estimators for \((b, s and beta)\) based on NRSS have the smallest MSEs in all cases comparing with other estimators based on RSS, MRSS and PRSS.
From Table 3, it can be observed that:

- Efficiencies of the estimators based on RSS, MRSS, PRSS and NRSS decrease as the values of \((b, s \text{ and } \beta)\) increase.

- In almost all cases, efficiencies of the estimators for \((b, s \text{ and } \beta)\) based on NRSS are greater than the efficiencies based on RSS, MRSS and PRSS, except in some cases.

| Distribution | n  | RSS  | MRSS | PRSS | NRSS |
|--------------|----|------|------|------|------|
|              |    | \(m = 1\) \(r = 3\) | \(m = 1\) \(m = 3\) | \(m = 1\) \(m = 3\) | \(m = 1\) \(m = 3\) |
| GG(3,3,3)    | 15 | 1.1718 | 3.6780 | 2.2389 | 4.0991 | 3.1427 | 13.0538 | 18.8661 | 19.5411 |
|              | 20 | 4.1932 | 4.3836 | 14.6188 | 8.7689 | 14.2698 | 9.7118 | 15.6442 | 20.7295 |
|              | 25 | 2.1648 | 2.2211 | 2.6959 | 4.5559 | 4.4036 | 9.3273 | 28.3575 | 67.9058 |
| GG(2,2,4)    | 15 | 1.4955 | 3.1867 | 2.4200 | 5.1781 | 2.5107 | 68.9798 | 45.9191 | 81.0011 |
|              | 20 | 3.4212 | 3.7668 | 5.3138 | 6.4090 | 9.1615 | 4.8103 | 52.9446 | 9.8355 |
|              | 25 | 3.1413 | 3.1379 | 3.9230 | 3.7967 | 5.5859 | 7.0556 | 16.9299 | 24.1395 |
|              | 15 | 2.2735 | 6.0463 | 2.5305 | 12.2524 | 1.7284 | 26.8113 | 12.5380 | 54.9708 |
|              | 20 | 2.3414 | 2.7351 | 1.7776 | 3.6464 | 4.3516 | 2.0000 | 14.1599 | 3.6610 |
|              | 25 | 4.3420 | 4.1787 | 7.3919 | 2.9307 | 6.0814 | 4.7697 | 8.8934 | 9.6184 |
|              | 15 | 6.1538 | 3.0074 | 20.3993 | 4.9197 | 12.9460 | 4.4248 | 24.0068 | 15.9413 |
|              | 20 | 6.5498 | 25.0192 | 6.8915 | 16.4541 | 38.0996 | 28.8354 | 58.6385 | 49.6355 |
|              | 25 | 10.6288 | 12.9179 | 4.3012 | 7.9081 | 46.1929 | 14.3126 | 50.5101 | 54.7224 |
|              | 15 | 17.7256 | 43.6959 | 26.4348 | 26.3662 | 6.1915 | 134.3402 | 40.4657 | 139.1280 |
|              | 20 | 3.8078 | 6.1727 | 5.5194 | 8.3734 | 35.3623 | 14.9441 | 5.7389 | 16.8289 |
|              | 25 | 4.6395 | 4.6868 | 8.0200 | 15.4738 | 43.6575 | 18.7304 | 48.0546 | 23.8312 |
|              | 15 | 12.5568 | 2.3786 | 12.8752 | 5.4628 | 1.8842 | 8.1582 | 16.5923 | 9.0914 |
FIGURE 6. Efficiencies of the estimators for $GG(3,3,3)$. 

a) For $\hat{b}$ estimator 

b) For $\hat{s}$ estimator 

c) For $\hat{\beta}$ estimator
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From Figure 6 and Figure 7, it can be observed that:

- Efficiencies of the estimators for \((b, s and beta)\) based on NRSS have the largest efficiencies in all cases, except in some cases.
- Efficiencies of the estimators for \((b, s and beta)\) based on PRSS are greater than the efficiencies based on RSS and MRSS, except in some cases.
- Efficiencies of the estimators for \((b, s and beta)\) based on RSS have the smallest efficiencies in all cases, except in some cases.
• When the cycle sizes increase, the efficiencies of all estimators based on RSS, MRSS, PRSS and NRSS increase.

5. SUMMARY AND CONCLUSIONS

Based on the numerical results from the comparative study, the findings may be summarized as follows:

(1) The MSEs based on SRS data has the largest MSEs comparing to RSS and its modifications schemes.

(2) It can be noted that, NRSS technique has superior to the rest of other sampling schemes. It has the smallest MSEs and largest efficiencies.

(3) Also it can be noted that RSS technique has inferior to the rest of other sampling schemes. It has the largest MSEs and smallest efficiencies.

(4) Generally the estimators based on NRSS, PRSS, MRSS and RSS techniques are more efficient than the estimators based on SRS techniques.

(5) L- moments method for estimation is given results very close to the real values of the parameters with very small biases.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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