Young Graphs: 1089 et al.

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Abstract

This paper deals with those positive integers \( N \) such that, for given integers \( g \) and \( k \) with \( 2 \leq k < g \), the base-\( g \) digits of \( N \) and \( kN \) appear in reverse order. Such \( N \) are called \((g,k)\) reverse multiples. Anne Ludington Young, in 1992, developed a kind of tree reflecting properties of these numbers; N. J. A. Sloane, in 2013, modified these trees into directed graphs and introduced certain combinatoric methods to determine from these graphs the number of reverse multiples for given values of \( g \) and \( k \) with a given number of digits. We extend their work, proving Sloane’s isomorphism conjectures for 1089 graphs and complete graphs, furthering his study of cyclic graphs, and proving a minor result on isomorphism.

1 Introduction

This paper deals with the problem of studying a certain type of positive integer, defined as follows.

Definition 1 (Reverse Multiple). Given \( g,k \in \mathbb{Z}^+ \), with \( 2 \leq k < g \), a \((g,k)\) reverse multiple is a positive integer \( N \) such that the base-\( g \) representations of \( N \) and \( kN \) are reverses of each other. That is, if \((c_{n-1},c_{n-2},\ldots,c_1,c_0)_g \) denotes \( \sum_{i=0}^{n-1} c_i g^i \), then \( N = (a_{n-1},a_{n-2},\ldots,a_1,a_0)_g \) and \( kN = (a_0,a_1,\ldots,a_{n-2},a_{n-1})_g \), for integers \( a_i \) with \( 0 \leq a_i \leq g-1 \) and \( a_{n-1} \) nonzero.

Ironically, the most prominent mention of these numbers may well have curtailed their prominence: G. H. Hardy, in A Mathematician’s Apology, refers to the fact that 1089 and 2178 are the only four-digit \((10,k)\) reverse multiples as one “very suitable for puzzle columns and likely to amuse amateurs,” but “not serious” and “not capable of any significant generalization.” While we abstain from judgment on the first two counts, it is clear, after the intervening decades, that Hardy misjudged regarding the third. A number of works generalize the problem, a list of most of which may be found in the 2013 paper of N. J. A. Sloane, “2178 and All That” [9]; the only works on the topic cited here not found by Sloane are references [3–5]. Hardy’s comment, however, while perhaps self-fulfilling by way of his stature, seems to possess some kernel of truth: Sloane’s list of references, while appreciable, is rather short for a problem given such exposure as a mention in Hardy’s book, albeit a disparaging one, and the problem does, in fact, appear with some frequency in texts on recreational mathematics (e.g. [1]) or mathematical curiosities (e.g. [12]).

We take an approach to the problem, utilizing certain labeled graphs, developed by Anne Ludington Young in 1992 [13] and expanded by Sloane in 2013 [9]. Most serious work takes an approach more algebraic than Young’s, and consists mostly of attempts on the problem for a small, preset number of digits for a general base \( n \). This is largely due to the influence of Alan Sutcliffe, who, with his 1966 paper “Integers That Are Multiplied When Their Digits Are Reversed” [10], initiated what is likely the largest interlinked body of work on the problem, and determined all

[1]The only refuge for his position being in the nebulous qualifier “significant.”
bases $n$ in which there exists a two-digit reverse multiple; Sutcliffe cites no works other than a general number theory reference. In sequence, T. J. Kaczynski\textsuperscript{2} in his 1968 paper “Note on a Problem of Alan Sutcliffe” \cite{Kaczynski68}, proved Sutcliffe’s conjecture on three-digit reverse multiples, and Lara Pudwell, in 2007, with the paper “Digit Reversal Without Apology” \cite{Pudwell07}, extended Kaczynski’s methods for reverse multiples of two or three digits to those of four and five, the behavior of the latter proving to be rather more complicated than that of the former. Pudwell also proved that $(b,b−1,g−b−1,g−b)$ is a $(g,k)$ reverse multiple when $g = b(k+1)$, as did Leonard F. Klosinski and Dennis C. Smolarski \cite{Klosinski69} in 1969, independently of the above material (they cite none of it). Outside of passing mentions in books on recreational mathematics, Sloane found two other treatments of the problem: “On the Trail of Reverse Divisors: 1089 and All That Follow” \cite{Sloane13}, by Roger Webster and Gareth Williams, from 2013, which deals with the case in which $g = 10$, and “Reversible Multiples” \cite{Grimm75}, by C. A. Grimm and D. W. Ballew, from 1975. The latter, after listing all $(10,k)$ reverse multiples less than $10^9$, presents an algorithm that generates three-digit reverse multiples, through a process rather distinct from those of other works, in that it begins with the numbers $r_i$ (defined in Subsection 2.1) and shows how to construct the base $g$, multiplier $k$, and digits $a_i$ of a reverse multiple, while other sources fix $g$, and sometimes $k$, and then construct the digits $a_i$ and numbers $r_i$; Grimm and Ballew \cite{Grimm75} also differ from other sources in that they allow $a_{n−1}$ to be zero. Kaczynski \cite{Kaczynski68} notes that Prasert Na Nagara of Kasetsart University in Thailand arrived at results similar to his independently, but no reference is given, and we could not find this work.

More recently, two papers of Benjamin V. Holt \cite{Holt08,Holt10} present a more general approach to the problem, considering reverse multiples, which he calls “palintiples,”\textsuperscript{3} with any number of digits. In his first and longer paper \cite{Holt10}, after proving several general results on reverse multiples, he first partitions reverse multiples into “symmetric,” “shifted-symmetric,” and “asymmetric” varieties on the basis of patterns in the numbers $r_i$ (defined in Subsection 2.1), and then characterizes all shifted-symmetric reverse multiples and provides partial results on the symmetric. He also extends Pudwell’s study \cite{Pudwell07} of four- and five-digit reverse multiples and of middle-digit truncation, and briefly considers polynomials with coefficients based on the digits of reverse multiples. His second paper \cite{Holt08} is shorter and mostly focuses on answering a number of open questions from his first paper. Some of his results, especially those characterizing symmetric \cite{Holt10} and shifted-symmetric \cite{Holt08} reverse multiples, are related to our results, on which similarities we comment throughout this paper.

Young, with two papers \cite{Young92a,Young92b} in 1992, began studying the problem via a kind of infinite tree, in which edges are labeled with pairs of digits $(a_{n−1−i},a_i)$ and nodes with pairs of numbers $(r_{n−2−i},r_i)$, defined in Section 2 finding a correspondence relating certain kinds of nodes in the tree and reverse multiples. Sloane \cite{Sloane13} extended her work, in 2013, by reformulating these trees into labeled directed graphs, which he calls Young graphs, and applying the transfer matrix method to find the number of $(g,k)$ reverse multiples with a given number of digits (Section 3 in \cite{Sloane13}). He also constructed all Young graphs for $g \leq 40$ and discusses the data thus gained. While Young’s approach can be used to exhaustively list all $(g,k)$ reverse multiples for given $g$ and $k$, Sloane’s approach more obviously raises the questions regarding isomorphism that are dealt with in this paper, and so will be used here.

Young graphs, not reverse multiples, are the predominant focus of this paper. We discuss several types of graphs mentioned in \cite{Sloane13}, proving two of Sloane’s conjectures on isomorphism. We also include minor results determining two points’ images under isomorphism and describing the nodes around $[0,0]$. Section 2 restates the pertinent work of Sloane \cite{Sloane13} and Young \cite{Young92a,Young92b}. Section 3

\textsuperscript{2}“Better known for other work.” \cite{Pudwell07}

\textsuperscript{3}He credits the term to Hoey \cite{Hoey98}. Strictly speaking, “palintiple” refers to a number that is a multiple of its reverse, rather than a divisor, as with reverse multiples.
proves (Theorem 3.1) Sloane’s conjecture on 1089 graphs, Section 4 proves (Theorem 4.2) Sloane’s conjecture on complete graphs, and Section 5 proves a minor result (Theorem 5.1) on cyclic graphs on three nodes and contains several conjectures related to cyclic graphs. Section 4 also contains a result on isomorphism (Theorem 4.1), a result on the predecessors of the node [0, 0] (Corollary 4.2), and a result on two-digit reverse multiples (Proposition 4.2), which is related to the work of Sutcliffe [10]. In Section 6, we discuss possible further research on the topic, including several isomorphism conjectures (Subsection 6.1) made from examination of Young graphs for low values of g, although of a less complete nature than those in [9].

2 Background on Young Graphs

Young [13] developed a method for determining \((g, k)\) reverse multiples for given values of \(g\) and \(k\) by generating a kind of infinite tree; Sloane [9] modified this approach so that the tree became a kind of directed graph. We present Young’s method under Sloane’s modification.

2.1 Equations

Let \(N = (a_{n-1}, a_{n-2}, \ldots, a_1, a_0)_g\) be a \((g, k)\) reverse multiple, with \(0 \leq a_i \leq g - 1\) and \(a_{n-1} \neq 0\). Then \(kN = \text{Reverse}_g(N)\), so there are integers \(r_0, r_1, \ldots, r_{n-2}\) with \(0 \leq r_i < g\) such that

\[
ka_0 = a_{n-1} + r_0 g \\
ka_1 + r_0 = a_{n-2} + r_1 g \\
ka_2 + r_1 = a_{n-3} + r_2 g \\
\vdots \\
ka_n + r_{n-2} = a_1 + r_{n-2} g \\
ka_{n-1} + r_{n-2} = a_0.
\]

These equations are those that arise from the base-\(g\) “columns” multiplication representing the equation \(kN = \text{Reverse}_g(N)\), written

\[
\begin{array}{ccccccc}
(r_{n-2}) & (r_{n-3}) & (r_{n-4}) & \cdots & (r_1) & (r_0) \\
\times & a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & a_0 \\
\hline & a_0 & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\
\end{array}
\]

While the \(r_i\) are defined between 0 and \(g - 1\) inclusive, Lemma 1 in [13] states that \(r_i \leq k - 1\) for all \(i\) and that \(r_0 > 0\). For convenience we may define \(r_{-1} = r_{n-1} = 0\) so that for all \(i, 0 \leq i \leq n-1\), we have

\[
ka_i + r_{i-1} = a_{n-1-i} + r_i g.
\]

**Remark 1.** We use the notation of Young [13, 14] and Sloane [9]. Holt [4, 5] uses a similar setup, but with different notation: \(g, k, n, a_i,\) and \(r_i\) are denoted in his papers by \(b, n, k + 1, d_{k-i},\) and \(c_{i+1},\) respectively.

2.2 Generating the Graph \(H(g, k)\)

\(H(g, k)\) is a kind of directed graph, between a subset of the paths of which and \((g, k)\) reverse multiples Young [13] discovered a correspondence, which we state in Subsection 2.3. The nodes in
this graph will be labeled with pairs \([r_{n-2-i}, r_i]\) of the \(r_i\)s and the edges with pairs \((a_{n-1-i}, a_i)\) of the \(a_i\)s. To construct this graph, we now take the above equations \(1\) in pairs

\[
\begin{align*}
ka_i + r_{i-1} &= a_{n-1-i} + r_i g \\
ka_{n-1-i} + r_{n-2-i} &= a_i + r_{n-1-i} g
\end{align*}
\]

and solve recursively for the pairs \([r_{n-2-i}, r_i]\): if we have a pair \([r_{n-1-i}, r_{i-1}]\) then we can check through all values of \(a_i\) and \(a_{n-1-i}\) (which must be between 0 and \(g-1\) inclusive) for the pairs \((a_{n-1-i}, a_i)\) that give integer values for \(r_{n-2-i}\) and \(r_i\) between 0 and \(k-1\) inclusive. If a pair \((a_{n-1-i}, a_i)\) gives a satisfactory pair \([r_{n-2-i}, r_i]\) then a directed edge, labeled \((a_{n-1-i}, a_i)\), is drawn from the node labeled \([r_{n-1-i}, r_{i-1}]\) to the node labeled \([r_{n-2-i}, r_i]\) (with one exception discussed below). As there are a finite number of potential nodes, this process must terminate.

We begin this process at \([r_{-1}, r_{n-1}]\), which we call the starting node and indicate as \([0,0]\] with double brackets to distinguish it from the node labeled \([0,0]\). No edges are to end at the starting node by definition, and no edges leading from the starting node can have labels with 0s, because this would imply \(a_{n-1} = 0\) or \(a_0 = 0\), which the definition of \(a_{n-1}\) and \(a_0\) forbids. The node \([0,0]\) cannot function as the starting node for this reason, as edges from \([0,0]\) can have 0s, and also for the sake of convenience in Young’s correspondence and Sloane’s enumerations (Section 3 in \([9]\); they are not discussed here); this is why we must have a starting node.

\(H(g,k)\) is the graph thus generated.

### 2.3 Pivot Nodes and Young Graphs

The following definition is made in \([9]\) to condense the statements of Theorems 1 and 2 in \([13]\), Young’s main theorems.

**Definition 2** (Pivot Nodes). An even pivot node is a node of the form \([n, n]\); an odd pivot node is a node of the form \([r, s]\) that is a direct predecessor of the node \([s, r]\) (this includes even pivot nodes with self-loops). The starting node is not considered a pivot node.

Note that the edge label between \([r, s]\) and \([s, r]\) must have the form \((a, a)\) because we can compute, from equation \(2\), that the edge label from \([r, s]\) to \([s, r]\) is \((rg - s)/(k-1), (rg - s)/(k-1)\).

Theorems 1 and 2 in \([13]\) link these nodes of \(H(g,k)\) to \((g,k)\) reverse multiples. The proofs of those theorems are omitted; the results are stated as they are rephrased in \([9]\) for directed graphs.

**Theorem 2.1** (Young’s Theorem). There is a one-to-one correspondence between the \((g,k)\) reverse multiples with an even number of digits and the paths in \(H(g,k)\) from the starting node to even pivot nodes and a one-to-one correspondence between the \((g,k)\) reverse multiples with an odd number of digits and the paths in \(H(g,k)\) from the starting node to odd pivot nodes. The path leading to an even pivot node that consists of the edges labeled \((a_{n-1}, a_0), (a_{n-2}, a_1), \ldots, (a_{n/2}, a_{n/2}-1),\) in that order, corresponds to the reverse multiple \((a_{n-1}, a_{n-2}, \ldots, a_1, a_0)_g\). The path leading to the odd pivot node \([r, s]\), which directly precedes \([s, r]\) by the edge labeled \((a_{(n-1)/2}, a_{(n-1)/2})\), that consists of the edges labeled \((a_{n-1}, a_0), (a_{n-2}, a_1), \ldots, (a_{(n+1)/2}, a_{(n-3)/2})\) corresponds to the reverse multiple \((a_{n-1}, a_{n-2}, \ldots, a_1, a_0)_g\).

Nodes of which no pivot nodes are successors are thus unimportant, if we are interested in Young’s correspondence between paths and reverse multiples. Thus the Young graph is defined as follows.

**Definition 3** (Young Graph). The Young graph for \(g\) and \(k\), denoted \(Y(g,k)\), is the labeled graph obtained by removing from \(H(g,k)\) all the nodes of which no pivot nodes are successors and all the edges starting or ending at these nodes.
Remark 2. Sutcliffe [10], Kaczynski [6], Pudwell [8], and, most generally, Holt [4,5] discuss middle digit truncation of reverse multiples; that is, the question of when removing the middle digit of a \((g,k)\) reverse multiple with an odd number of digits yields another \((g,k)\) reverse multiple. These discussions arose from Sutcliffe’s [10] Theorem 3, which found a correspondence between two-digit reverse multiples and three-digit reverse multiples involving such truncation: namely, that if \((a,b)_g\) is a \((g,k)\) reverse multiple, with \(0 \leq a, b \leq g - 1\), then \((a, a + b, b)_g\) is also a \((g,k)\) reverse multiple. Theorem 10 of [5], which characterizes the occurrences of truncations that yield new reverse multiples, states that a \((g,k)\) reverse multiple \((a_{n-1}, a_{n-2}, \ldots, a_1, a_0)_g\), for odd \(n\), may be truncated by middle digit removal to obtain a new \((g,k)\) reverse multiple if and only if \(r((n-3)/2) = r((n-1)/2)\); while Holt’s proof is algebraic, this result may also be seen as a direct consequence of Young’s theorem and the fact that the edge labels \((a_{n-1-i}, a_i)\) in a Young graph are distinct, as noted in [9], because such reverse multiples correspond to paths to nodes that are both odd and even pivot nodes.

2.3.1 Examples

![Diagram](image1.png)

Figure 1: \(Y(10,9)\). The nodes \([0,0]\) and \([8,8]\) are both even and odd pivot nodes.

In the \((10,9)\) Young graph (Figure 1), \([8,8]\) is an even pivot node, and \(1089\), which has an even number of digits, is the reverse multiple corresponding to the path from the starting node to \([0,8]\) to \([8,8]\), that consists of the edges \((1,9)\) and \((0,8)\). Here \([8,8]\) is also an odd pivot node; so \(1089\) is a reverse multiple, with the same corresponding path as \(1089\) (but now we include in the label from the self-loop). \([0,0]\) is also an even and odd pivot node, and \(10891089\) and \(108901089\), the numbers obtained from the edge labels on paths to \([0,0]\), are also both \((10,9)\) reverse multiples.

In the \((10,4)\) Young graph (Figure 2), \([3,3]\) is an even pivot node; there is a path from \([0,0]\) to \([3,3]\) that consists of the edges labeled \((2,8)\) and \((1,7)\), so \(2178\) should be a \((10,4)\) reverse multiple, and, in fact, \(4 \cdot 2178 = 8712\).
In the (5, 2) Young graph (Figure 3), [1, 1] is an odd pivot node, and there is a path from [[0, 0]] to [1, 1] consisting of the edge labeled (1, 3), and there is also a self-loop labeled (4, 4), and (1, 4, 3)\textsubscript{5} is a (5, 2) reverse multiple.

In the (17, 8) Young graph (Figure 4), the (17, 8) reverse multiple (1, 16, 15, 1, 16, 15, 1, 16, 15)\textsubscript{17} corresponds to the path from [[0, 0]] to [7, 7] to itself to [0, 0] to [7, 7] consisting of the edges (1, 15), (16, 16), (15, 1), (1, 15), and half of (16, 16) — half because [7, 7] is an odd pivot node, and so we only use the 16 of the edge label of its self-loop once in the corresponding reverse multiple.

### 3 1089 Graphs

In this section and the two following it we present our results. This section and Section 4 both prove conjectures from [9] (Theorems 3.1 and 4.2, respectively), while Section 5 discusses another type of graph mentioned in [9]. Section 4 also contains a result (Theorem 4.1) on Young graph isomorphism and a result (Corollary 4.2) on the nodes adjacent to [0, 0].

The results of this section and of Sections 4 and 5 mostly concern isomorphism of Young graphs, which is a somewhat stronger condition than isomorphism in ordinary directed graphs because of the added structure of node labels in Young graphs. The concept was introduced in [9].

**Definition 4** (Isomorphism). Two Young graphs $Y$ and $Y'$ are *isomorphic as Young graphs*, written $Y \simeq Y'$, if there is a function $\phi$ from the nodes of $Y$ to those of $Y'$ such that, if $G$ and $G'$ are respectively the directed graphs\textsuperscript{4} obtained by removing the edge and node labels from $Y$ and $Y'$, $\phi$ acts as an isomorphism of $G$ and $G'$, and furthermore such that $x$ is an even pivot node in $Y$ if and only if $\phi(x)$ is an even pivot node in $Y'$, and $x$ is an odd pivot node in $Y$ if and only if $\phi(x)$ is an odd pivot node in $Y'$ (that is, $\phi$ is a pivot node-preserving isomorphism of the underlying directed graphs).

\textsuperscript{4}These are referred to as “underlying directed graphs” in [9].
Remark 3. Holt [5] classifies reverse multiples based on the “structure of their carries,” that is, the numbers $r_i$. Since it is through the labels $[r_{n-1-i}, r_{i-1}]$ that Young graph isomorphism is determined, studying classes of reverse multiples with similar carry structure should be similar to studying reverse multiples with isomorphic Young graphs; indeed, this relation is seen in Holt’s study of symmetric and shifted-symmetric reverse multiples, as will become evident throughout this paper.

It is noted in Section 3.2 of [9] that Young graphs isomorphic to $Y(10, 9)$ (Figure 1) are fairly common, for example $Y(10, 4)$, $Y(15, 4)$, and $Y(20, 9)$; the following definition is then made.

**Definition 5 (1089 Graph).** A Young graph $Y$ such that $Y \cong Y(10, 9)$ is called a 1089 graph.

The number 1089 is the smallest (10, 9) reverse multiple; thus the name “1089 graph.” Conjecture 3.1 in [9] characterizes the occurrences of 1089 graphs; we prove this conjecture in Theorem 3.1. Holt [4] has also recently provided a proof, approaching the problem more algebraically, without reference to graphs. We require several preliminary results.

**Lemma 3.1.** If a Young graph contains a node $[r, s]$ directly preceding a node $[t, u]$ by an edge $(a, b)$ then it also contains the node $[u, t]$ directly preceding the node $[s, r]$ by the edge $(b, a)$.
We omit the proof; the interested reader is directed to property P3 in Section 2.7 of [9] or Theorem 2 of [14].

**Corollary 3.1.** Every node in a Young graph is a predecessor of [0, 0].

**Proof.** This is a direct consequence of Lemma 3.1 and the fact that all nodes in a Young graph precede some pivot node and succeed the starting node.

Two more lemmas are needed to prove the main theorem; we now prove the first.

**Lemma 3.2.** Let $g = b(k + 1)$, with $b$ a positive integer, and let $m$ be a nonnegative integer. Then, if the $(g,k)$ Young graph contains a node of the form $\left[\frac{s(k-1)}{b^m}, \frac{t(k-1)}{b^{m+1}}\right]$, where $s$ and $t$ are integers, the following are true:

(i) The edges leading from $\left[\frac{s(k-1)}{b^m}, \frac{t(k-1)}{b^{m+1}}\right]$ have the form $\left[\frac{s(k-1)}{b^m} + u, \frac{(t-b^2s)(k-1)}{b^{m+1}} + ku\right]$, where $u$ is some integer.

(ii) The edge $\left[\frac{s(k-1)}{b^m} + u, \frac{(t-b^2s)(k-1)}{b^{m+1}} + ku\right]$ leading from the node $\left[\frac{s(k-1)}{b^m}, \frac{t(k-1)}{b^{m+1}}\right]$ terminates at the node $\left[\frac{(t(k-1))}{b^{m+1}}, \frac{(t-b^2s+b^m+1)u(k-1)}{b^{m+2}}\right]$.

(iii) $t \equiv 0 \mod b$.

**Proof.** First note that $0 \leq s \leq b^m$ and $0 \leq t \leq b^{m+1}$, because $[s(k-1)/b^m, t(k-1)/b^{m+1}]$ is in the Young graph and $0 \leq r_i \leq k-1$ for all $i$. Throughout the following proof let $a = a_i, A = a_{n-1-i}$, $r = r_i$, and $R = r_{n-2-i}$, where $i$ is the number such that $[s(k-1)/b^m, t(k-1)/b^{m+1}] = [r_i, r_{n-1-i}]$.

We will establish the equation $rk = \frac{r(k-1)+t(k-1)}{b^m} - s(k-1)/b^m$, whence we directly relate $a$ and $A$ and the desired conclusions follow easily.

By equation (2), $A = ka + \frac{t(k-1)}{b^{m+1}} - rg$ and $kA + R = a + \frac{s(k-1)}{b^m}g$. These two equations imply $R + (k^2 - 1)a + \frac{tk(k-1)}{b^{m+1}} = g \left(\frac{r(k-1)}{b^m} + rk\right)$. Because $0 \leq R \leq k-1$, we now have

$$(k^2 - 1)a + \frac{tk(k-1)}{b^{m+1}} \leq g \left(\frac{r(k-1)}{b^m} + rk\right) \leq (k^2 - 1)a + \frac{tk(k-1)}{b^{m+1}} + k - 1.$$ 

Since $g = b(k+1)$, multiplying through this inequality by $b^{m+1}/(k+1)$ yields

$$b^{m+2} \left(\frac{s(k-1)}{b^m} + rk\right) \geq b^{m+1}(k-1)a + tk - 2t + \frac{2t}{k+1} + b^{m+1} - \frac{2b^{m+1}}{k+1}.$$ 

The inequalities in (3) give lower and upper bounds on $b^{m+2} \left(\frac{s(k-1)}{b^m} + rk\right)$. Note that the difference between these bounds is $b^{m+1} - \frac{2b^{m+1}}{k+1} < b^{m+1}$, and that $b^{m+2} \left(\frac{s(k-1)}{b^m} + rk\right)$ is a multiple of $b^{m+1}$. This implies that $b^{m+2} \left(\frac{s(k-1)}{b^m} + rk\right)$ is the only multiple of $b^{m+1}$ satisfying the bounds in (3). Now, because $k \geq 2$ and $0 \leq t \leq b^{m+1}$, it may be simply shown that $b^{m+1}(k-1)a + tk - t$ satisfies the bounds in (3), and because $\frac{t(k-1)}{b^{m+1}}$ is a node label $b^{m+1}(k-1)a + tk - t$ must be a multiple of $b^{m+1}$. This establishes $b^{m+1}(k-1)a + tk - t = b^{m+2} \left(\frac{s(k-1)}{b^m} + rk\right)$. 

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It now follows that \( b^{m+2}(\frac{s(k-1)}{b^m} + rk) = b^{m+1}(k-1)a + tk - t \), which implies

\[
rk = \frac{(b^{m+1}a + t)(k-1)}{b^{m+2}} - \frac{s(k-1)}{b^m}.
\] (4)

Equation (2) implies \( kA + rk = k^2a + \frac{kt(k-1)}{b^{m+1}} \). Substituting for \( rk \) via equation (4) and substituting \( b(k+1) \) for \( g \), we obtain

\[
a = kA + \frac{t(k-1)}{b^{m+1}} - \frac{s(k^2-1)}{b^{m-1}}.
\] (5)

Let \( u = A - \frac{s(k-1)}{b^m} \in \mathbb{Z} \). Substituting \( u \) into equation (5) yields \( a = \frac{(t-b^2s)(k-1)}{b^{m+1}} + ku \), establishing (i).

Now equation (2) gives \( A + rg = ka + \frac{t(k-1)}{b^{m+1}} \) and \( a + \frac{s(k-1)}{b^m}g = kA + R \), and substituting in for \( A \) and \( a \) in terms of the parameter \( u \) and for \( g \) in terms of \( b \) and \( k \) yields (iii).

Note that the new node \([t(k-1)/b^{m+1}, (t-b^2s + b^{m+1}u)(k-1)/b^{m+2}] \) has the same form as the directly preceding node \([s(k-1)/b^m, t(k-1)/b^{m+1}] \). Thus all successors of the node \([s(k-1)/b^m, t(k-1)/b^{m+1}] \) have the form \([s'(k-1)/b^{m'}, t'(k-1)/b^{m'+1}] \). Note also that \( t \equiv t - b^2s + b^{m+1}u \mod b \), so that \( t \equiv s' \equiv t' \mod b \) inductively. Now, if \([s(k-1)/b^m, t(k-1)/b^{m+1}] \) is in \( Y(g, k) \), it must precede \([0, 0]\) by Corollary 3.1 so there must be some \( t' = 0 \), which implies (iii).

One further lemma is required before we may characterize all occurrences of the 1089 graph.

**Lemma 3.3.** If, in \( Y(g, k) \), a node \([r, s]\) directly precedes \([0, 0]\) and is not an odd pivot node then \( k+1 \mid g \).

**Proof.** By Lemma 3.1 \([0, 0]\) must directly precede \([s, r]\); let the edge from \([0, 0]\) to \([s, r]\) be labeled \((a, b)\). We then have \( kb = a + rg \) and \( ka + s = b \), by equation (2); adding these equations and rearranging yields \((k-1)(a+b) = rg - s\).

It is also known that there can be no edge from \([r, s]\) to \([s, r]\). If we attempt to find such an edge by solving equation (2) for the edge \((x, y)\) from \([r, s]\) to \([s, r]\), we find \( x = y = \frac{t-b^2s}{k-1} = a + b \), and so \( a + b \) must be an unsatisfactory edge label. It is only required of edge labels that they be integers between 0 and \( g - 1 \) inclusive; \( a + b \) is a nonnegative integer, so \( a + b \geq g \). Now by Theorem 3.4 in [9], we have \( k+1 \mid g \).

We may now prove the main theorem of this section.

**Theorem 3.1** (Conjecture 3.1 in [9]). \( Y(g, k) \) is a 1089 graph if and only if \( k+1 \mid g \). Also, if \( Y(g, k) \) is a 1089 graph, with \( g = b(k+1) \), its labels are those shown in Figure 7.

**Proof.** We divide the proof into its two directions.

**Part 1.** If \( k+1 \mid g \) then \( Y(g, k) \) is a 1089 graph.

Suppose \( g = b(k+1) \). We may now determine \( Y(g, k) \) by repeated application of Lemma 3.2. Consider the starting node: \([0, 0] = [(0(k-1)/b^0, 0(k-1)/b^1)]\), so by Lemma 3.2(ii) the starting node can only directly precede nodes of the form \([0(k-1)/b, bu(k-1)/b^2] = [0(k-1)/b^0, u(k-1)/b^1] \). Note that if \([0(k-1)/b^0, u(k-1)/b^1] \) is a node, we have \( u \equiv 0 \mod b \) by Lemma 3.2(iii). We also have \( 0 \leq u \leq b \) because \( 0 \leq r_i \leq k-1 \) for all \( i \), so \( u = 0 \) or \( u = b \). But the edge leading to this first node has the form \((u, ku)\), by Lemma 3.2(iii) and \( a_0 \neq 0 \) so \( u = b \). Therefore the starting node directly precedes only the node \([0, k-1] \), by the edge \((b, bk)\).
Similar reasoning shows that $[0, k-1]$ only directly precedes $[k-1, k-1]$, by the edge $(b-1, bk-1)$; that $[k-1, k-1]$ only directly precedes itself, by the edge $(bk-1, bk-b-1)$, and $[k-1, 0]$, by the edge $(bk-1, b-1)$; that $[k-1, 0]$ only directly precedes $[0, 0]$, by the edge $(bk, b)$; and that $[0, 0]$ only directly precedes itself, by the edge $(0, 0)$, and $[0, k-1]$, by the edge $(b, bk)$. The graph $Y(g, k)$ thus determined is shown in Figure 5, and is a 1089 graph.

**Part 2.** If $Y(g, k)$ is a 1089 graph then $k + 1 | g$.

We are given that $Y(g, k)$ is isomorphic to $Y(10, 9)$ (Figure 1). The starting node then has a direct successor $[r, s]$, with $r \neq s$, since the node is not a pivot node by the isomorphism. Again by the isomorphism, this node $[r, s]$ has an even pivot node as its only direct successor, which we call $[t, t]$. This node $[t, t]$ has a self-loop and directly precedes only one other node; by Lemma 3.1 this node must be $[s, r]$ – note that $[s, r]$ must be distinct from $[t, t]$ because $s \neq r$. The node $[s, r]$ directly precedes only one other node, and by Lemma 3.1 this must be $[0, 0]$, since $[r, s]$ directly succeeds the starting node. Now $[s, r]$ is not an odd pivot node, because all odd pivot nodes are even pivot nodes in 1089 graphs and $s \neq r$, but it directly precedes $[0, 0]$. Therefore, by Lemma 3.3, we have $k + 1 | g$. By Part 1, the $Y(g, k)$ determined is shown in Figure 5.

**Remark 4.** This theorem is related to Theorem 6 in [5] and Theorems 1 and 3 and Corollary 2 in [4]. These all bear on symmetric reverse multiples (defined in [5]): reverse multiples such that $r_{i-1} = r_{n-2-i}$, as is the case with reverse multiples arising from 1089 graphs. Theorem 6 of [5] states that if $k + 1 | g$ then all $(g, k)$ reverse multiples are symmetric, which is similar to Lemma 3.2(ii) however, the proof of Theorem 3.1 required more specific information about node labels in
graphs where \( k + 1 \mid g \), namely Lemma 3.2(i) and Lemma 3.2(iii). In his more recent paper [4], Holt proves that a \((g, k)\) reverse multiple is symmetric if and only if \( k + 1 \mid g \), and he remarks that Sloane’s conjecture, here Theorem 3.1, follows from his Theorem 3. Holt’s results thus give a further distinction to the 1089 graph, beyond the number theoretical characterization and the corollaries below: it is the graph for which, if \([a, b] \rightarrow [c, d]\) is in the graph, then \( b = c \). Note also that, in finding the edge and node labels in 1089 graphs, Theorem 3.1 also establishes part of Theorem 3.2 in [9], which in addition finds the generating function for the number of reverse multiples with a given number of digits and characterizes the associated reverse multiples for 1089 graphs; Theorem 3 in [4] also finds the digits of these reverse multiples, which he proved to exhaust the class of symmetric reverse multiples.

Several minor results follow immediately from Theorem 3.1.

**Corollary 3.2.** In a Young graph that is not a 1089 graph \((k + 1 \nmid g)\), all direct predecessors of \([0, 0]\) are odd pivot nodes.

**Proof.** This follows from Lemma 3.3 and Theorem 3.1. \(\square\)

**Corollary 3.3.** All Young graphs contain an odd pivot node other than \([0, 0]\).

**Proof.** Corollary 3.1 implies that the node \([0, 0]\) is in all Young graphs \(Y(g, k)\) (also see property P4 in Section 2.7 in [9]). As it is in the Young graph, the node \([0, 0]\) must have some direct predecessor \([r, s] \neq [0, 0]\). If \([r, s]\) is not an odd pivot node then the Young graph is a 1089 graph, by Corollary 3.2 in which case it contains an odd pivot node by definition. \(\square\)

**Corollary 3.4** (Conjecture 3.5 in [9]). If there is a \((g, k)\) reverse multiple with first and last digits that sum to \(g\), \(Y(g, k)\) is a 1089 graph. If \(Y(g, k)\) is a 1089 graph then every \((g, k)\) reverse multiple has first and last digits summing to \(g\).

**Proof.** By Theorem 3.4 in [9], if \(a_0 + a_{n-1} = g\) then we must have \(a_{n-1} = \frac{g}{k+1}\), so that \(Y(g, k)\) is a 1089 graph by Theorem 3.1. If \(Y(g, k)\) is a 1089 graph then, by Theorem 3.1, \(a_0 = \frac{gk}{k+1}\) and \(a_{n-1} = \frac{g}{k+1}\), and these numbers sum to \(g\). \(\square\)

Isomorphism is an equivalence relation on Young graphs, and thus defines a corresponding equivalence relation on pairs of integers \((g, k)\): we take \((g, k) \sim (g', k')\) if and only if \(Y(g, k) \simeq Y(g', k')\). Via these corresponding relations, any equivalence class \(X\) of Young graphs corresponds to an equivalence class \(\tilde{C}\) of pairs \((g, k)\), and so for any equivalence class \(X\) of Young graphs we can also define a relation \(R_X(g, k)\) on \(g\) and \(k\) to be the relation that holds if and only if \((g, k) \in C\); this relation sometimes coincides with a convenient, simply expressed arithmetical condition, as is the case for 1089 graphs and the complete graphs discussed in Section 4. Theorem 3.1 has a somewhat interesting consequence regarding these relations and equivalence classes.

**Corollary 3.5.** The set of 1089 graphs is the only equivalence class \(X\) of Young graphs such that \(R_X(g, k)\) holds if and only if \(f(k) \mid g\), for some fixed function \(f : \mathbb{Z}^+ \rightarrow \mathbb{Z}\).

**Proof.** Suppose there is such an equivalence class \(X\) of Young graphs. Then \(Y(18, f(17), 17)\) is in \(X\), but by Theorem 3.1 is also a 1089 graph, so \(X\) is the set of 1089 graphs. \(\square\)

\(^5\)Note that two degenerate Young graphs – graphs for which \(H(g, k)\) contains no pivot nodes – are isomorphic. The equivalence class of \((g, k)\) pairs corresponding to the equivalence class of degenerate graphs is of great interest, as characterizing this equivalence class is equivalent to characterizing all \(g\) and \(k\) for which reverse multiples exist.
While not related to 1089 graphs, the following result comes from a line of reasoning similar to that of Lemma 3.3 and so we include it in this section.

**Proposition 3.1.** If a node $[r, s]$ directly precedes, by an edge $(a, b)$, a node $[t, t]$ that has a self-loop $(c, c)$, $[r, s]$ is an odd pivot node if and only if $0 \leq a + b - c \leq g - 1$, and if it is an odd pivot node then the edge from $[r, s]$ to $[s, r]$ is labeled $(a + b - c, a + b - c)$.

**Proof.** Taking equation (2) for the given edge $(a, b)$ gives $kb + s = a + tg$ and $ka + t = b + rg$, yielding $(k-1)(a+b) = rg - s + (g-1)t$. Taking equation (2) for the self-loop at $[t, t]$ gives $(g-1)t = (k-1)c$, so that $(rg - s)/(k-1) = a + b - c$. The edge from $[r, s]$ to $[s, r]$ can be found, from equation (2), to be $((rg - s)/(k-1), (rg - s)/(k-1)) = (a + b - c, a + b - c)$. So if $0 \leq a + b - c \leq g - 1$ then there is an edge from $[r, s]$ to $[s, r]$, and it is labeled $(a + b - c, a + b - c)$, and if there is an edge from $[r, s]$ to $[s, r]$ it must be $(a + b - c, a + b - c)$ and so $0 \leq a + b - c \leq g - 1$. □

## 4 Complete Graphs

We now turn to the proof of another of Sloane’s conjectures (Conjecture 3.7 in [9]), concerning complete Young graphs.

**Definition 6** (Complete Graph). A Young graph $Y$ is called a **complete Young graph on $m$ nodes**, denoted $K_m$, when the nodes that are not $[[0, 0]]$ form the complete directed graph on $m$ nodes and there is an edge from $[[0, 0]]$ to every node except for $[0, 0]$.

Examples of complete graphs are shown in Figures 3 and 4. Both are examples of $K_2$. Conjecture 3.7 in [9] characterizes all occurrences of $K_m$; we prove this conjecture in a slightly modified form in Theorem 4.2. Several preliminary results are needed, some of which are proven in [9], and several other interesting digressions from the course of the proof are included.

**Lemma 4.1.** All Young graphs contain the node $[0, 0]$.

**Lemma 4.2.** A Young graph is a complete graph if and only if every node has the form $[r, r]$.

We omit the proofs; the interested reader is directed to property P4 in Section 2.7 (or even Corollary 3.1) for Lemma 4.1 and to Theorem 3.6 for Lemma 4.2 both in [9]. We now begin work towards Theorem 4.2.

**Lemma 4.3.** For given $g$ and $k$, if there is a positive integer $s \leq k - 1$ such that $s \left(\frac{g-k}{k^2-1}\right)$ is an integer then $Y(g, k)$ is a complete Young graph.

**Proof.** We show that all nodes in $Y(g, k)$ have the form $[r, r]$, so that the conclusion follows from Lemma 4.2.

Consider a node $[r, r]$ in $Y(g, k)$. Suppose this node directly precedes some node $[r_{n-2-i}, r_i]$ by an edge $(a_{n-1-i}, a_i)$. We can solve equation (2) for this edge to yield

$$ a_i = \frac{rg - rk + r_i kg - r_{n-2-i}}{k^2 - 1} = r_i \frac{g-k}{k^2-1} + r_i \frac{kg - 1}{k^2 - 1} + \frac{r_i - r_{n-2-i}}{k^2 - 1}. $$

\[\text{In his conjecture, Sloane refers to integers } r \text{ such that } r(g-k)/(k^2-1) \text{ and } r(kg-1)/(k^2-1) < g \text{ are integers; however, } r(kg-1)/(k^2-1) = kr(g-k)/(k^2-1) + r, \text{ so we may drop the second integrality requirement, and } r(kg-1)/(k^2-1 < g) \text{ holds exactly when } r \leq k - 1 \text{ holds.}\]
Therefore, \( s \left( \frac{r_i - r_{n-2-i}}{k^2 - 1} \right) = sa_i - r \left( s \frac{g-k}{k^2 - 1} \right) - r_i \left( ks \frac{g-k}{k^2 - 1} + s \right) \) is an integer. Now
\[
S \left[ \frac{r_i - r_{n-2-i}}{k^2 - 1} \right] \leq (k - 1) \left[ \frac{r_i - r_{n-2-i}}{k^2 - 1} \right] \leq (k - 1) \frac{k - 1}{k^2 - 1} < 1.
\]
Therefore \( S \left[ \frac{r_i - r_{n-2-i}}{k^2 - 1} \right] = 0 \), so \( r_i = r_{n-2-i} \), because \( s > 0 \).

Repeating this argument proves that all successors of a node \([r, r]\) have the form \([r', r']\); since all nodes in a Young graph are successors of \([0, 0]\), all nodes in \(Y(g, k)\) must have the form \([r, r]\), so that \(Y(g, k)\) is complete by Lemma 4.2.

**Remark 5.** This lemma, taken together with Theorem 9 in [5], establishes that if one \((g, k)\) reverse multiple is shifted-symmetric, i.e. \( r_{i-1} = r_{n-1-i} \) (a definition from [5]), then all \((g, k)\) reverse multiples are shifted-symmetric. This is also the conclusion of Corollary 6 in [4], although Holt’s reasoning is more similar to that of Corollary 4.3 and Proposition 4.2; alternatively, this lemma could be seen as a consequence of Corollary 6 in [4] and Theorem 9 in [5]. Note as well that here, as with 1089 graphs, Holt’s partition based on the structure of the carries of reverse multiples aligns nicely with partition by isomorphism, as noted in Remark 3.

While not the full result characterizing the occurrences of complete graphs, the following result is still interesting, and gives several further results, and so we digress from the proof of Theorem 4.2 to include it here.

**Proposition 4.1.** If there are two distinct nodes of the form \([r, r]\) in \(Y(g, k)\) that directly precede the same node, \(Y(g, k)\) is a complete graph.

**Proof.** Suppose there is a node \([x, x]\) which directly precedes the nodes \([r, s]\) by the edge labeled \((a, b)\) and a node \([y, y]\) that directly precedes \([r, s]\) by the edge labeled \((c, d)\), with \(x \neq y\). Then, by equation (2), we must have \(kb + x = a + sg\), \(ka + r = b + gx\), \(kd + y = c + sg\), and \(kc + r = d + gy\). These yield \(y - x = (c - a) - k(d - b)\) and \(g(y - x) = k(c - a) - (d - b)\), which together give \((g - k)(c - a) = (kg - 1)(d - b)\). This rearranges to
\[
d - b = \frac{g - k}{k^2 - 1}((c - a) - k(d - b)) = \frac{g - k}{k^2 - 1}(y - x),
\]
so that \(c - a = \left( \frac{k^2 - 1}{gk - 1} \right) (y - x)\). Therefore \( \left( \frac{g - k}{k^2 - 1} \right) |y - x| = |d - b|\) is a positive integer and \( |y - x| = \left( \frac{k^2 - 1}{gk - 1} \right) |c - a| \leq \left( \frac{k^2 - 1}{gk - 1} \right) g < k\), so, by Lemma 4.3, \(Y(g, k)\) is a complete graph. \(\square\)

Note that this proposition applies to the starting node, as well as all even pivot nodes, so long as the other node is not \([0, 0]\).

**Corollary 4.1.** If, in a Young graph \(Y\), an even pivot node \([s, s]\), with \(s \neq 0\), directly precedes or succeeds \([0, 0]\) (the two are equivalent by Lemma 3.7), \(Y\) is a complete graph.

**Proof.** Under these conditions \([s, s]\) and \([0, 0]\) both directly precede the same node (the node \([0, 0]\)), and are even pivot nodes, so that \(Y\) is complete by Proposition 4.1. \(\square\)

**Corollary 4.2.** In Young graphs that are neither complete graphs nor 1089 graphs, all direct predecessors of \([0, 0]\) are odd, and not even, pivot nodes. The direct successors of \([0, 0]\) are not even pivot nodes.

**Proof.** This follows from Corollaries 3.2 and 4.1. \(\square\)
This line of reasoning culminates in a general result on isomorphism.

**Theorem 4.1.** If $Y$ and $Y'$ are isomorphic as Young graphs by an isomorphism $\phi$ then $\phi([[0,0]]) = [[0,0]]$ and $\phi([0,0]) = [0,0]$.

**Proof.** Clearly $\phi([[0,0]]) = [[0,0]]$, since $[[0,0]]$ is the only node which is the successor of no other node. We must now only show that $\phi([0,0]) = [0,0]$. To this end, suppose $\phi([0,0]) \neq [0,0]$. As a preliminary, note that the direct successors of the starting node are also direct successors of the node $[0,0]$, because $[0,0]$ is in the graph by Lemma 4.1 and the equations given by (2) governing the node and edge labels are the same for the starting node and $[0,0]$.

Because $Y$ exists and is connected, it must include an edge $[[0,0]] \rightarrow [r,s]$, for some node $[r,s]$. Since $\phi$ is an isomorphism, the edge $[[0,0]] \rightarrow \phi([r,s])$ is included in $Y'$, and consequently the edge $[0,0] \rightarrow \phi([r,s])$ is also included in $Y'$. Because $[[0,0]] \rightarrow [r,s]$ is in $Y$, the edge $[0,0] \rightarrow [r,s]$ must also be in $Y$, so $\phi([0,0]) \rightarrow \phi([r,s])$ is included in $Y'$. Thus $[0,0] \rightarrow \phi([r,s])$ and $\phi([0,0]) \rightarrow \phi([r,s])$ are both included in $Y'$, and so $Y$ and $Y'$ must be complete graphs, by Proposition 4.1. But in complete graphs, $[0,0]$ is the only node that does not directly succeed the starting node, so $\phi([0,0]) = [0,0]$, contrary to assumption.

We now return to our proof of Sloane’s conjecture.

**Lemma 4.4.** For all of the nodes $[r,r]$ in a complete graph $Y(g,k)$, $r\left(\frac{g-k}{k-1}\right)$ is an integer and $r \leq k-1$.

**Proof.** Consider a reverse multiple that corresponds, as per Theorem 2.1, to a path through all nodes in $Y(g,k)$. This is a shifted-symmetric reverse multiple, and so the desired conclusion follows by Theorem 9 of \cite{5}.

We may now prove the main result of this section.

**Theorem 4.2** (Conjecture 3.7 in \cite{9}). $Y(g,k)$ is a $K_m$ if and only if there are exactly $m-1$ positive integers $s_1, s_2, \ldots, s_{m-1}$ such that $s_j \left(\frac{g-k}{k-1}\right)$ is an integer and $s_j \leq k-1$ for all $1 \leq j \leq m-1$.

**Proof.** We divide the proof into its two directions.

**Part 1.** If there are exactly $m-1$ positive integers $s_j \leq k-1$ such that $s_j \left(\frac{g-k}{k-1}\right)$ is an integer, then $Y(g,k)$ is a $K_m$.

Since $s_1$ exists, by Lemma 4.3, $Y(g,k)$ must be a $K_l$ for some $l$. By definition $K_l$ contains $l+1$ nodes, of which one is the starting node and one the node $[0,0]$ (by Lemma 4.1). The remaining $l-1$ nodes, by Lemmas 4.2 and 4.4, have the form $[r,r]$ where $r \leq k-1$ is a positive integer such that $r(g-k)/(k^2-1)$. There are only $m-1$ such positive integers, so $l \leq m$. Furthermore, for every positive integer $r \leq k-1$ such that $r(g-k)/(k^2-1)$ is an integer, Theorem 9 of \cite{5} gives a reverse multiple that corresponds via Theorem 2.1 to the path $[[0,0]] \rightarrow [r,r]$. Therefore every node $[r,r]$ is in $Y(g,k)$, so that $m \leq l$. This completes this part of the proof.

**Part 2.** If $Y(g,k)$ is a $K_m$ then there are exactly $m-1$ positive integers $s_j \leq k-1$ such that $s_j \left(\frac{g-k}{k-1}\right)$ is an integer.

\footnote{It is, however, possible to have nodes of the form $[r,0]$ that are direct successors of $[0,0]$ and not of the starting node.}

\bigskip

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By Lemma 4.4, all the nodes \([r, r]\) of \(Y(g, k)\) are such that \(r(g - k)/(k^2 - 1)\) is an integer and \(r \leq k - 1\). In particular, we know that there exist such numbers \(r\). Suppose that exactly \(l - 1\) such numbers \(r\) exist. Then, by Part 1, \(Y(g, k)\) is a \(K_l\). Now, because \(K_l\) and \(K_m\) are isomorphic, they must have the same number of nodes, so \(l = m\).

Note the similarity between the following result, as well as Proposition 4.2, and Theorem 4 of [4], which states that a \((g, k)\) reverse multiple is shifted-symmetric if and only if \((g - k, k^2 - 1) \geq k + 1\).

**Corollary 4.3.** \(Y(g, k)\) is a \(K_m\) if and only if \(\lfloor (g - k, k^2 - 1)/(k + 1) \rfloor = m - 1\).

**Proof.** It is easily shown that there exist \(m - 1\) positive integers \(s \leq k - 1\) such that \(s(g - k)/(k^2 - 1)\) is an integer if and only if \(m - 1 \leq (g - k, k^2 - 1)/(k + 1) < m\).

The earliest paper [10] found in this topic, by Sutcliffe, is largely concerned with reverse multiples with two digits; he concludes (Theorem 2 in [10]) that, for given \(g\), there exists a \(k\) such that there exists a \((g, k)\) reverse multiple with two digits if and only if \(g + 1\) is composite. However, he, and Kaczynski [6] after him, are not particularly concerned with the multiplier \(k\): in their terminology, the \(N\) used throughout this paper is known only as an “n-digit solution for \(g\.” Corollaries 4.1 and 4.3 now imply a result on two-digit reverse multiples with reference to \(k\), a connection hinted at in property P7 of [9].

**Proposition 4.2.** A two-digit \((g, k)\) reverse multiple exists if and only if \(k + 1 \leq (g - k, k^2 - 1)\).

**Proof.** By Theorem 2.1, if there is a two-digit reverse multiple, then there exists a path of length 1 in \(Y(g, k)\) from the starting node to an even pivot node, so that there must be an even pivot node directly succeeding the starting node. This even pivot node will also directly succeed the node \([0, 0]\) as well, because, as mentioned previously, the equations yielded by equation (2) are the same for the starting node and \([0, 0]\). This makes \(Y(g, k)\) complete, by Corollary 4.1. Corollary 4.3 now implies the desired result.

Conversely, suppose \(k + 1 \leq (g - k, k^2 - 1)g\). Now \(Y(g, k)\) is complete by Corollary 4.3 so that there exists a path of length 1 in the graph from the starting node to an even pivot node – namely, from the starting node to any adjacent node – so there exists a \((g, k)\) reverse multiple with two digits, by Theorem 2.1.

This result could also follow by Theorem 4 of [4] and Young’s theorem (Theorem 2.1 here).

Indeed, some simple manipulations show that this result agrees exactly with the characterization of \(g\) by Sutcliffe [10].

### 5 Cyclic Graphs

We now include a minor result and a conjecture on another type of graph mentioned in [9].

**Definition 7** (Cyclic Graph). A Young graph \(Y\) is called a cyclic Young graph on \(n\) nodes, denoted \(Z_n\), if the nodes of \(Y\) that are neither the starting node nor \([0, 0]\) form the cyclic directed graph on \(n\) nodes, and there exist two nodes \(N_1\) and \(N_2 \neq [0, 0]\) such that \(N_1\) directly precedes \(N_2\), the only direct predecessor of \([0, 0]\) (other than itself) is \(N_1\), and the only direct successor of \([0, 0]\) (other than itself) and starting node is \(N_2\).

There is no conjecture made in [9] that is analogous to Theorems 3.1 and 4.2 for cyclic graphs. We also have no such conjecture; however, examination of data for low values of \(g\) suggests the following.
Conjecture 1. For \( t \geq 3 \) and \( n \geq 2 \), the Young graph \( Y(((n^2 + n)t - n^2 + n + 1, n^2t - (n - 1)^2) \) is a \( Z_{2n-1} \).

We now prove a minor result on a similar type of graph.

**Theorem 5.1.** For \( t \geq 3 \) and \( n \geq 2 \), the Young graph \( Y(((n^2 + n)t - 1, n^2t - 1) \) is a \( Z_3 \), with the node and edge labels as shown in Figure 6.

![Figure 6: Y((n^2 + n)t - 1, n^2t - 1) for n \geq 2 and t \geq 3.](image)

**Proof.** Let \( g = (n^2 + n)t - 1 \) and \( k = n^2t - 1 \). Note that \( k^{-1} \equiv -(n + 1) \mod g \); this fact will be used later.

We may compute all direct successors of the starting node. By equation [2], any node \([r'_0, r_0]\) that directly succeeds the starting node by an edge \((a'_0, a_0)\) must satisfy

\[
(n^2t - 1)a_0 = a'_0 + r_0((n^2 + n)t - 1) \\
(n^2t - 1)a'_0 + r'_0 = a_0. \tag{6}
\]

Because \( a_0 \leq g - 1 \) and \( r'_0 \geq 0 \), the second equation of (6) implies \( a'_0 \leq ((n^2 + n)t - 2)/(n^2t - 1) = 1 + 1/n + (1/n - 1)/(n^2t - 1) < 2 \). Recall \( a'_0 \neq 0 \) because it is the last digit of a reverse multiple.
Therefore $a'_0 = 1$. Substituting into the first equation of (6) and taking the equation modulo $g$ gives $(n^2 - 1)a_0 \equiv 1 \mod g$, so that $a_0 \equiv -(n + 1) \mod g$. Since $0 < a_0 \leq g - 1$, this implies $a_0 = (n^2 + n)t - (n + 2)$. Substituting the values $a'_0 = 1$ and $a_0 = (n^2 + n)t - (n + 2)$ into equation (6) gives $r'_0 = nt - (n + 1)$ and $r_0 = n^2t - (n + 1)$. Therefore the only direct successor, in $H(g, k)$, of the starting node is the node $[nt - (n + 1), n^2t - (n + 1)]$, and the corresponding edge is $((1, n^2 + n)(t - (n + 2))$.

Similar considerations for $[nt - (n + 1), n^2t - (n + 1)]$ show that its only direct successor in $H(g, k)$ is $[(n^2 - n)t + 2n - 2, (n^2 - n)t + 2n - 2]$, the corresponding edge being $((n + 1)t - (n + 2), (n^2 - 1)t + 2n - 1)$. In turn, similar considerations for $[(n^2 - n)t + 2n - 2, (n^2 - n)t + 2n - 2]$ show that its only direct successor in $H(g, k)$ is $[n^2t - (n + 1), nt - (n + 1)]$, by the edge $((n^2 - 1)t + 2n - 1, (n + 1)t - (n + 2))$, and similar considerations for $[n^2t - (n + 1), nt - (n + 1)]$ show that its only two direct successors are $[nt - (n + 1), n^2t - (n + 1)]$, by the edge $((n^2 + n)t - (n + 1), (n^2 + n)t - (n + 1))$, and $[0, 0]$, by the edge $((n^2 + n)t - (n + 2), 1)$. It may then be shown that the only direct successors of $[0, 0]$ are $[nt - (n + 1), n^2t - (n + 1)]$ and itself, in reasoning almost identical to that of the computation of the direct successors of the starting node. This fully determines $H(g, k)$. All nodes in the graph precede some pivot node, and so $Y(g, k)$ is the same labeled graph as $H(g, k)$, namely the one depicted in Figure 6.

A similar result seems to hold for cyclic graphs on five nodes.

**Conjecture 2.** For $t \geq 3$ and $n \geq 3$, the Young graph $Y((n^2 + n)t - (n + 2), n^2t - (n + 1))$ is a $Z_5$.

A proof of this could likely proceed along similar lines to the proof of Theorem 5.1.

### 6 Further Research

Given the scarcity of work on the problem, it is not particularly surprising that Young graphs, especially the relation between the associated number theory and graph theory, are not well understood. This paper makes some limited progress, mostly concerning specific equivalence classes of Young graphs, from the work of Young [13,14] and Sloane [9], but still little is known in general. We now discuss some directions for potential further work on the problem, after which discussion we include several conjectures (Subsection 6.1).

Section 5 in Sloane’s paper [9] consists of a list of open problems. He asks what types of Young graphs can occur as underlying directed graphs, which pairs $(g, k)$ have nondegenerate $Y(g, k)$, and whether there are nonisomorphic Young graphs with isomorphic underlying directed graphs. Regarding the last question, our program finds that for $g \leq 336$, there are no pairs of nonisomorphic Young graphs with isomorphic underlying directed graphs; the program could not proceed past this point. Holt [4,5] also lists a number of open problems, mostly concerning bases that only permit symmetric reverse multiples (“strongly symmetric”) or that do not permit asymmetric reverse multiples (“symmetric”). The approach of Grimm and Ballew [2] could also likely be generalized.

Examination of data also suggests the following line of study: given an equivalence class $X$ of Young graphs, let $\text{Freq}_X(x)$ be the number of $Y(g, k) \in X$ with $g \leq x$. We may then examine the growth rates of these functions; for example, if $X$ is the set of 1089 graphs, then $\text{Freq}_X(x)$ grows roughly as $x \ln x$, by Theorem 5.1. The growth rate for the set of $K_m$ graphs is presently unknown, although Corollary 4.3 reduces it to the problem of computing the number of pairs $(g, k)$ with $g \leq x$ and $[(g - k, k^2 - 1)/(k + 1)] = m - 1$. Some graphs pertaining to this issue are included in the URL mentioned at the start of Subsection 6.1.

This paper is largely devoted to characterizing specific equivalence classes. This type of work may be done for other specific classes, for example those featured in Subsection 6.1. The graphs
$Z_n$ present themselves as a natural next step, given the frequency of their occurrence, especially for small values of $n$, and the results of Section 5. While it would be ideal to characterize all equivalence classes, this is likely a rather difficult task given present methods. However, it might further the study of equivalence classes to study of the graphs $H(g, k)$, in addition to the graphs $Y(g, k)$ that are the focus of this paper. It should be easier to study $H$ graphs, because their generation does not involve the additional, potentially complicating step of removing all nodes that do not precede pivot nodes.

Since Young graph isomorphism is a stronger condition than isomorphism of the underlying directed graphs, and since the set of directed graphs that may occur as the underlying directed graphs of Young graphs is a proper subset of the set of all directed graphs, Young graph isomorphism should be an easier condition to test than normal directed graph isomorphism; indeed, Theorem 4.1 demonstrates this. This raises the question of identifying the most efficient test for Young graph isomorphism. Answering this question could significantly speed the computations involved in testing isomorphism, as our current program for this purpose is rather slow. The program, given substantial time, was unable to decide whether $Y(173, 54)$ and $Y(337, 54)$ are isomorphic, whether $Y(458, 431)$ and $Y(459, 436)$ are isomorphic, and whether $Y(471, 45)$ and $Y(159, 45)$ are isomorphic; these might be good cases on which to attempt new methods.

### 6.1 Conjectures

We used a program in Java to generate all graphs with $g \leq 336$ and sort them into the equivalence classes determined by isomorphism. These programs and some data on Young graphs may be found at [https://sites.google.com/site/younggraphs/home](https://sites.google.com/site/younggraphs/home). From the examination of these graphs and their isomorphism classes, we conjecture the following.

**Conjecture 3.** For all $n \geq 0$, the Young graphs $Y(2(n + 2)(n + 3) - 1, 2(n + 2)^2 - 1)$, $Y(23 + 20n, 13 + 12n)$, $Y(23 + 15n, 17 + 10n)$, $Y(26 + 12n, 10 + 4n)$, and $Y(11 + 6n, 7 + 3n)$ are all isomorphic to $Y(11, 7)$.

**Conjecture 4.** For all $n \geq 3$, $Y(n^2 - 1, n^2 - n - 1) \cong Y(8, 5)$.

**Conjecture 5.** For all $t \geq 2$ and $n \geq 2$, $Y((n^2 + n)t + n, n^2t + n - 1) \cong Y(14, 3)$.

**Conjecture 6.** For all $n \geq 2$, $Y(17 + 30n, 14 + 25n) \cong Y(36, 10)$.

Grimm and Ballew [2] found an algorithm for generating three-digit reverse multiples, which goes as follows: take $d_1, d_2 \in \mathbb{Z}^+$ such that $(d_1, d_2) = 1$, and then take some $m \in \mathbb{Z}$ such that $d_2 - d_1 \equiv 0 \pmod{(m - 1)}$, $m \geq d_2 + 1$, $m > d_1 + 1$, and $(m + 1, d_2) = 1$. Then let $k = (d_2^2 - md_1^2)/(m - 1)$, and let the pair $(M, N)$ be an integer solution to $d_1(m + 1)M - d_2N = 1$. Then $(d_2t + kM, d_1(m + 1)t + kN, md_2t + mkM + d_1)_g$ is a $(g, m)$ reverse multiple for $g = (m^2 - 1)t - mNd_1 + d_2M(m + 1)$, for any $t$ such that $g > m$. Note that $d_1$ is $r_1$ and $d_2$ is $r_0$. From examination of some of these multiples, we make Conjecture 7.

**Conjecture 7.** If $g_1$ and $g_2$ are different bases obtained by performing the procedure in [2] for the same $d_1, d_2, m$, and the base, then $Y(g_1, m) \cong Y(g_2, m)$.

**Conjecture 8.** Let $\text{Classes}(x)$ denote the number of equivalence classes that occur in all $Y(g, k)$ for $g \leq x$. Classes($x$) grows roughly proportionately to $x^2$.

Regarding this conjecture, we find that the $r$-value for the set \{(x, Classes($x$)$^{1/2}$ | $x \leq 336$\} is around 0.9996. However, 336 seems to be a rather low upper bound for this type of problem: the functions $\text{Freq}_X(x)$, where $X$ is the set of $K_m$, for $x \leq 400$ are well-approximated by functions of the form $c \cdot x \ln x$, but for $x \leq 10^5$ it becomes apparent that $\text{Freq}_X(x)$ cannot be approximated by such functions. Conjecture 8 is thus made rather hesitantly.
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