ADMISSIBILITY AND FIELDS RELATIONS

DANIEL NEFTIN

Abstract. Let $K$ be a number field. A finite group $G$ is $K$-admissible if there is a $G$-crossed product division $K$-algebra. $K$-admissibility has a necessary condition called $K$-preadmissibility that in many cases is also sufficient (it is known to be insufficient only in very special cases). It is a 20 years old open problem to determine whether two number fields $K$ and $L$ with different degrees over $\mathbb{Q}$ can have the same admissible groups (see [27],[28]). We construct an infinite family of pairs of number fields $(K, M)$ such that $K$ is a proper subfield of $M$ and $K$ and $M$ have the same preadmissible groups. Thus, supplying evidence for a negative answer to the problem. In particular, it will follow from the construction that $K$ and $M$ have the same odd order admissible groups.

1. Introduction

1.1. Admissibility over number fields. Let $G$ be a finite group. A finite dimensional central division algebra $D$ over a field $K$ has the structure of a $G$-crossed product if it has a maximal subfield $L$ that is Galois over $K$ with Galois group $\text{Gal}(L/K) = G$. The following problem arises from [21], given a number field $K$, when is there a $G$-crossed product division algebra. Let us introduce some of the terminology in [21]:

Definition 1.1. Let $L/K$ be a finite extension of fields. The field $L$ is $K$-adequate if there is a division algebra $D$, with center $K$ and a maximal subfield $L$.

Definition 1.2. Let $K$ be a field and let $G$ be a finite group. The group $G$ is $K$-admissible if there exist a $K$-adequate Galois $G$-extension $L/K$ ($\text{Gal}(L/K) \cong G$).

In other words, $G$ is $K$-admissible if there is a $G$-crossed product $K$-division algebra. Over a number field $K$, $K$-admissibility is equivalent to the following realization problem for $G$:

Theorem 1.3. (Schacher, [21]) Let $K$ be a number field and $L/K$ a finite Galois extension. For every $p$, let $p^r$ be the maximal $p$-power that divides $[L : K]$. Then $L$ is $K$-adequate if and only if for every $p|[L : K]$, there are two primes $v_i$ of $K$ for which $p^r|[L_{v_i} : K_{v_i}]$, $i = 1, 2$.

In the second condition $L_{v_i}$ denotes a completion of $L$ at a prime divisor of $v_i$ (since $L/K$ is Galois, the local degree $[L_{v_i} : K_{v_i}]$ does not depend on the choice of the divisor). We will use this notation throughout the text.

This leads to Schacher’s criterion for $K$-admissibility:

Theorem 1.4. (Schacher, [21]) Let $K$ be a number field and let $G$ be a finite group. Then $G$ is $K$-admissible if and only if there exists a $G$-extension $L/K$ such that for every rational prime $p|[G]$, there are two primes $v_i$ of $K$ such that $\text{Gal}(L_{v_i}, K_{v_i})$ contains a $p$-Sylow subgroup of $G$. 

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Note that the property of containing the $p$-Sylow subgroup does not depend on the choice of the prime divisor of $v_i$ in $L$. Schacher’s criterion for $K$-admissibility has several necessary local realization conditions (for more details see [13]):

**Definition 1.5.** Let $K$ be a number field. A finite group $G$ is $K$-preadmissible if there is a set $T = \{v_i(p)|p||G|, i = 1, 2\}$ of primes of $K$ and corresponding subgroups $G^{v_i} \leq G$ for every $v \in T$, such that for every $p||G|$: 
1) $v_1(p) \neq v_2(p)$,
2) $G^{v_i(p)}$ is realizable over $K_{v_i(p)}$ for $i = 1, 2$,
3) $G^{v_i(p)}$ ($i = 1, 2$) contains a $p$-Sylow subgroup of $G$.

Many researches were devoted to the classification of $K$-admissible groups, especially for $K = \mathbb{Q}$. The following conjecture of Schacher suggests a description of the $\mathbb{Q}$-admissible groups. A group is called Sylow metacyclic if its Sylow subgroups are all metacyclic.

**Conjecture 1.6.** (Schacher) A finite group $G$ is $\mathbb{Q}$-admissible if and only if $G$ is Sylow metacyclic.

The conjecture was proved in [27] for solvable groups. Note that a group is $\mathbb{Q}$-preadmissible if and only if it is Sylow metacyclic ([13]). Therefore, Schacher’s conjecture can be stated as follows: a group is $\mathbb{Q}$-admissible if and only if it is $\mathbb{Q}$-preadmissible.

In many cases $K$-preadmissibility is also sufficient for $K$-admissibility. The following theorem which is a corollary to the main embedding Theorem in [15] gives a large set of examples and will also be useful in the sequel.

**Theorem 1.7.** (Neukirch) Let $K$ be a number field with $m(K)$ roots of unity. Let $S$ be a finite set of primes of $K$. Let $G$ be a finite group with order prime to $m(K)$ (it follows that $G$ is of odd order and hence solvable). For every $v \in S$, let $L^{(v)}/K_v$ be a Galois extension whose Galois group is a subgroup of $G$. Then there exist a Galois extension $L/K$ with $\text{Gal}(L/K) \cong G$ for which $L_v = L^{(v)}$ for all $v \in S$.

**Corollary 1.8.** Let $K$ be a number field with $m(K)$ roots of unity and let $G$ be a $K$-preadmissible group satisfying $(|G|, m(K)) = 1$. Then $G$ is $K$-admissible.

For more examples in which preadmissibility implies admissibility and for an example in which this implication fails, see [13].

1.2. **Arithmetic relations.** In [18], Neukirch proved that two number fields $K$ and $L$ with isomorphic absolute Galois groups $G_K \cong G_L$ must have the same $\mathbb{Q}$-normal closure and asked whether $K$ and $L$ must be isomorphic. Uchida ([30]) showed this is indeed the case. In [27], Sonn asked the analogous question for crossed product division algebras or admissibility.

**Definition 1.9.** Two number fields $K, L$ are equivalent by admissibility (resp. preadmissibility) if the set of $K$-admissible (resp. $K$-preadmissible) groups is the same as the set of $L$-admissible (resp. $L$-preadmissible) groups.

It is not known whether two number fields $K, L$ that are equivalent by admissibility are necessarily isomorphic. Moreover the following problem is open ([27],[28]):

**Problem 1.10.** Let $K$ and $L$ be two number fields that are equivalent by admissibility. Does necessarily $[K: \mathbb{Q}] = [L: \mathbb{Q}]$?
However it was proved that equivalence by admissibility determines the $\mathbb{Q}$-normal closure of the field.

**Theorem 1.11.** (Sonn) If $K$ and $L$ are two number fields that are equivalent by admissibility then $K$ and $L$ have the same $\mathbb{Q}$-normal closure.

**Remark 1.12.** The same proof holds if we assume $K$ and $L$ are equivalent by preadmissibility. Indeed any group that is not $L$-admissible (resp. $K$-admissible) in the proof of 1.11 is also not $L$-preadmissible (resp. $K$-preadmissible).

**Remark 1.13.** Two number fields $K$ and $L$ have the same normal closure if and only if the same rational primes split completely in $K$ and $L$. This can be viewed as a weak arithmetical equivalence which we shall call normal equivalence.

Given a relation $\equiv$ on the set of number fields. A number field $K$ is called solitary by $\equiv$ if there is no other number field $L$ for which $K \equiv L$. Theorem 1.11 shows that $\mathbb{Q}$ is solitary by both admissibility and preadmissibility equivalence. Moreover, it shows that if there is a non-solitary number field $K$ which is Galois over $\mathbb{Q}$ then Problem 1.10 has a negative answer. It is important to know that for many pairs of number fields the answer is positive. In particular:

**Theorem 1.14.** (Lochter) Let $K, L$ be two number fields that are equivalent by admissibility with $[K : \mathbb{Q}] = p$ or $[K : \mathbb{Q}] = 4$, where $p$ is prime. Then $[L : \mathbb{Q}] = [K : \mathbb{Q}]$.

We shall be interested in checking the solitariness of Galois extensions $M/\mathbb{Q}$ under the admissibility and preadmissibility equivalence. We shall assume $M/\mathbb{Q}$ is an $l$-extension for some prime $l$ and that $l$ splits completely in $M$ and then study subfields of $M$ which are equivalent to $M$. The following definition is useful for these purposes.

**Definition 1.15.** For a group $G$ and two subgroups $A, B \leq G$ of $G$ a double coset $AxB$ is called split if $|AxB| = |A||B|$.

**Theorem 1.16.** Let $l$ be a prime and $G$ an $l$-group. Let $M/\mathbb{Q}$ be a $G$-extension in which $l$ splits completely. Let $K$ be a subfield of $M$ whose $\mathbb{Q}$-normal closure is $M$ and let $\mathcal{H} = \text{Gal}(M/K)$. Then $K$ is equivalent to $M$ by preadmissibility if and only if for every subgroup $D \leq G$ that appears as a decomposition group, i.e. $D = D(M/\mathbb{Q}, p)$ for some prime $p$ of $M$, there are two distinct split double cosets of the form $Dx\mathcal{H}$, $x \in G$.

Note that there is no number field $K$ over which the set of $K$-admissible groups is known and in many cases the conjecture is that it is exactly the set of $K$-preadmissible groups. Thus, Theorem 1.16 and the following examples provide the closest result to a negative answer for Problem 1.10 that one can obtain using the current knowledge on $K$-admissibility. We can therefore conjecture:

**Conjecture 1.17.** Problem 1.10 has a negative answer. Furthermore, there is a number field $M$ that has a proper subfield $K$ for which $K$ and $M$ have the same admissible groups.

In particular Theorem 1.16 implies that for two fields $K, L$ as above, there are many groups $G$ for which $G$ is $K$-admissible if and only if $G$ is $L$-admissible. For example, by applying Theorem 1.16 and Theorem 1.7, we get:
Corollary 1.18. Let \( l \) be an odd prime, \( \mathcal{G} \) an \( l \)-group and \( M/Q \) a \( \mathcal{G} \)-extension in which \( l \) splits completely. Let \( K \) be a subfield of \( M \) whose \( Q \)-normal closure is \( M \) and \( H := \text{Gal}(M/K) \). Assume that for every subgroup \( D \leq \mathcal{G} \) that appears as a decomposition group, there are two distinct split double cosets of the form \( DxH \), \( x \in \mathcal{G} \). Then every odd order group is \( K \)-admissible if and only if it is \( L \)-admissible, i.e. \( K \) and \( L \) are equivalent by admissibility of odd order groups.

Proof. Any extension of \( Q \) by roots of unity has an even degree. Thus, \( M \) (and hence \( K \)) does not contain any roots of unity except \( \{1, -1\} \). By Theorem 1.17 every odd order group is \( K \)-preadmissible (resp. \( M \)-preadmissible) if and only if it is \( K \)-admissible (resp. \( M \)-admissible). Therefore Theorem 1.16 implies that every odd group \( G \) is \( K \)-admissible if and only if it is \( K \)-preadmissible if and only if it is \( M \)-preadmissible if and only if it is \( M \)-admissible. \( \square \)

In Section 2 we shall use Theorem 1.16 to produce a set of examples, which is described below, of pairs of number fields \( K \) and \( M \) that are equivalent by preadmissibility but have different degrees over \( Q \). Let \( S_n \) denote the symmetric group on \( n \) symbols.

Theorem 1.19. Let \( l \) be an odd prime. Let \((\mathcal{G}_n)_{n \in \mathbb{N}}\) be a sequence of \( l \)-groups for which \( G_i \leq G_{i+1} \) and \( G_i \leq S_k \) for every \( i \in \mathbb{N} \). Let \( d_n \) be the largest cardinality of a metacyclic subgroup of \( G_n \). Assume there is an element \( \alpha \in \mathcal{G} := \bigcup_{n \in \mathbb{N}} G_n \) that has infinitely many conjugates in \( \mathcal{G} \) and that \( \lim_{n \to \infty} \frac{d_n}{|G_n|} = 0 \). Then there is an \( N \) such that for any \( n \geq N \) there is a \( G_n \)-extension \( M/Q \) and a proper subfield \( K \) of \( M \) that is equivalent to \( M \) by preadmissibility and odd order admissibility.

In Section 3, we shall compare equivalence by preadmissibility to other arithmetical equivalences, namely arithmetic equivalence and local isomorphism:

Definition 1.20. 1. Two number fields \( K \) and \( L \) are said to be arithmetically equivalent if their Dedekind Zeta functions satisfy \( \zeta_K(s) = \zeta_L(s) \).

2. The number fields \( K \) and \( L \) are locally isomorphic if there is an isomorphism \( \mathbb{A}_K \cong \mathbb{A}_L \) of rings, where \( \mathbb{A}_K, \mathbb{A}_L \) denote the Adele rings of \( K \) and \( L \), respectively.

We shall see that the following implications diagram holds:

\[
\begin{array}{c}
1 \quad \text{Isomorphism} \\
\downarrow \\
2 \quad \text{Local Isomorphism} \\
\end{array}
\]

\[
\begin{array}{c}
3 \quad \text{Arithmetic Equivalence} \\
\end{array} \quad \begin{array}{c}
4 \quad \text{Preadmissible Equivalence} \\
\end{array}
\]

\[
\begin{array}{c}
5 \quad \text{Normal Equivalence} \\
\end{array}
\]

We shall then prove:

Theorem 1.21. No other implication in the above diagram holds.

For this it is sufficient to show \( 2 \not\rightarrow 1 \), \( 3 \not\rightarrow 4 \), \( 4 \not\rightarrow 3 \). An example for the non-implication \( 2 \not\rightarrow 1 \) appears in [8]. The non-implication \( 2 \not\rightarrow 1 \) also follows from Example 3.9. Example 3.11 shows \( 3 \not\rightarrow 4 \) and Remark 3.5 shows \( 4 \not\rightarrow 3 \).
The paper is based on a work of the author throughout his M.Sc degree under the supervision of Prof. Jack Sonn. The author would like to thank Prof. Sonn for reading several drafts of this paper, suggesting ways to improve it and for introducing the subject and the connections between prime decompositions and double cosets to the author.

2. Equivalence by preadmissibility

2.1. Equivalent subfields. Let $M$ be a Galois extension of $\mathbb{Q}$ with Galois group $\mathcal{G}$ and let $K$ be a subfield of $M$ with $\mathcal{H} = \text{Gal}(M/K)$. Let $p$ be a rational prime and let $v_1, ..., v_k$ be the primes of $K$ lying above $p$. Assume the primes $v_1, ..., v_k$ are ordered so that $[K_{v_i} : \mathbb{Q}_p] \geq [K_{v_j} : \mathbb{Q}_p]$ for $i \geq j$. We shall call the vector $([K_{v_1} : \mathbb{Q}_p], ..., [K_{v_k} : \mathbb{Q}_p])$ the local degree type of $p$ in $K$.

The parallel notion in group theory will be the double coset type. For any two subgroups $A, B$ of $\mathcal{G}$ with double cosets $Ax_1B, ..., Ax_sB$, ordered with decreasing cardinality, i.e. $|Ax_1B| \geq ... \geq |Ax_sB|$, we call the vector $([Ax_1B|, ..., |Ax_sB|)$ the double coset type $(A, B)$. Denote by $(A, B)_k$ the $k$-th entry of the vector $(A, B)$.

Let us describe the connection between local degrees and double cosets. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial that has a root $\alpha$ for which $K = \mathbb{Q}(\alpha)$. So, $M$ is the splitting field of $f$. Let $\alpha_1 := \alpha, \alpha_2, ..., \alpha_d$ be the roots of $f$. Then $\mathcal{G}$ acts transitively on the set $R_f := \{\alpha_1, ..., \alpha_d\}$. This action is equivalent to the action of $\mathcal{G}$ on the set of left cosets $\mathcal{G}/\mathcal{H}$. Let $v_1$ be a prime divisor of $v_1$ in $M$ and $D := D(M/\mathbb{Q}, v_1) \leq \mathcal{G}$. Then $D$ acts on the set $R_f$ which breaks into orbits under this action. Denote these orbits by $O_1, ..., O_s$. These orbits correspond to the orbits of the action of $D$ (as a subgroup of $\mathcal{G}$) on $\mathcal{G}/\mathcal{H}$, that is the set of double cosets $Dx_1\mathcal{H}, ..., Dx_s\mathcal{H}$ where $Dx_1\mathcal{H}$ is treated as the set $\{dx_1\mathcal{H}|d \in D\}$ for any $i = 1, ..., s$. In particular $d = s$. Furthermore, the cardinality of $O_i$ equals the number of elements in the orbit of $x_1\mathcal{H}$ which is $|Dx_1\mathcal{H}|/|\mathcal{H}|$, for any $i = 1, ..., s$. Now, $D$ is isomorphic to the Galois group of $M_{v_1}/\mathbb{Q}_p$ (which is a splitting field of $f$) with an isomorphism that preserves the action on $R_f$. Thus, $f$ factors over $\mathbb{Q}_p$ into $f(x) = f_1(x)...f_s(x)$ where $f_i$ is irreducible over $\mathbb{Q}_p$ and the roots of $f_i$ are the elements of the orbit $O_i$. We therefore have:

$$[K_{v_i} : \mathbb{Q}_p] = \deg(f_i(x)) = |O_i| = \frac{|Dx_i\mathcal{H}|}{|\mathcal{H}|}.$$

We arrive to the following well know description of prime decomposition in $K$:

**Lemma 2.1.** Let $\mathcal{G}$ be a group and $\mathcal{H}$ a subgroup. Let $M$ be a $\mathcal{G}$-extension of $\mathbb{Q}$ and $K = M^\mathcal{H}$. Let $p$ be a rational prime and $p'$ a prime divisor of $p$ in $M$ with decomposition group $D := D(M/\mathbb{Q}, p) \leq \mathcal{G}$. Then the local degree type of $p$ in $K$ equals $\frac{|Dx_i\mathcal{H}|}{|\mathcal{H}|}$.

Note that for a different prime divisor $p'$ of $p$ in $M$ the decomposition group $D' = D(M/\mathbb{Q}, p')$ is a conjugate of $D$ and hence $D$ and $D'$ have the same double coset type $(D', \mathcal{H}) = (D, \mathcal{H})$. Keeping the above notations we have:

$$[K_{v_i} : \mathbb{Q}_p] = \frac{(D, \mathcal{H})_i}{|\mathcal{H}|} = \frac{|Dx_i\mathcal{H}|}{|\mathcal{H}|} = \frac{|x_i^{-1}Dx_i\mathcal{H}|}{|\mathcal{H}|} = \frac{|D|}{|x_i^{-1}Dx_i \cap \mathcal{H}|}$$

for all $i = 1, ..., s$.

We shall use this connection repeatedly through out the text. In particular, Lemma 2.1 will be used to give group interpretations to equivalence by preadmissibility. By Theorem 1.11, two preadmissibly equivalent fields have the same normal closure. So,
let $K$ and $L$ be two number fields that have the same $\mathbb{Q}$-normal closure $M$. Denote by $\mathcal{H} = \text{Gal}(M/K)$ and $\mathcal{H}' = \text{Gal}(M/L)$. The normal closure of $K$ is $M$ if and only if $\text{core}_G(\mathcal{H}) = 1$ and hence $\text{core}_G(\mathcal{H}) = \text{core}_G(\mathcal{H}') = \{1\}$.

**Proposition 2.2.** (Sonn, [27]) Let $p$ be an odd rational prime, $\mathfrak{p}$ a prime divisor of $p$ in $M$ and $D = D(M/\mathbb{Q}, \mathfrak{p})$. If $K$ and $L$ are equivalent by preadmissibility then:

1. $p$ decomposes in $K$ if and only if $p$ decomposes in $L$,

2. if $p$ decomposes in both $K$ and $L$ then $\frac{(D\mathcal{H}_2)}{|H|} = \frac{(D\mathcal{H}'_2)}{|H'|}$, i.e. the second largest double coset $Dx_2H$ in $(D, H)$ and the second largest double coset $Dy_2H'$ in $(D, H')$ satisfy the relation $\frac{|Dx_2H|}{|H|} = \frac{|Dy_2H'|}{|H'|}$.

Most of the proof can be extracted out of the proof of Theorem 1 in [27]. The Proposition also appears without proof in [11]. For the sake of completeness let us give full details. The proof uses the following observation which will be used repeatedly in the sequel:

**Remark 2.3.** (Schacher [21], Liedahl [10], see also [13]) As mentioned above, a group is $\mathbb{Q}$-preadmissible if and only if it is Sylow metacyclic. Moreover, if $p$ does not decompose in $K$ and $G$ is a $K$-preadmissible group then the $p$-Sylow subgroups of $G$ are metacyclic and admit a certain presentation (for details see [10] or [13]). In such case we say $G$ satisfies Liedahl’s condition.

**Proof.** (Proposition 2.2) (1) Assume $p$ does not decompose in $K$ and it has two divisors $v_1, v_2$ in $L$. Then by Remark 2.3 any $K$-admissible $p$-group is metacyclic. The group $C_p \wr C_p$ is not metacyclic and hence not $K$-preadmissible. Let us show $C_p \wr C_p$ is $L$-preadmissible (it is even $L$-admissible). This will show $K$ and $L$ are not equivalent by preadmissibility. To show this, it is enough to show that $C_p \wr C_p$ is realizable over $L_{v_1}$ and $L_{v_2}$. The maximal pro-$p$ extension $M_p$ of $\mathbb{Q}_p$ has Galois group $\text{Gal}(M_p/\mathbb{Q}_p)$ which is the free pro-$p$ group on two generators (see [23], Section 2.5.6). Thus, $C_p \wr C_p$ is realizable over $\mathbb{Q}_p$. By [14], $C_p \wr C_p$ is also realizable over any extension of $\mathbb{Q}_p$ and hence over $L_{v_1}, L_{v_2}$.

Let $v$ be a prime of $K$ dividing $p$. By local class field theory the Galois group $\text{Gal}(K_v(p)/K_v)$, where $K_v(p)$ is the maximal pro-$p$ abelian extension of $K_v$, is isomorphic to the pro-$p$ completion of the group $K_v^*$ and thus has rank $r_v := [K_v : \mathbb{Q}_p] + 1$ if $K_v$ does not contain the $p$-th roots of unity and $r_v := [K_v : \mathbb{Q}_p] + 2$ otherwise (see [24], Chapter 14, Section 6). Thus, the group $C_p^N$ is realizable over all prime divisors $v$ of $p$ in $K$ with $r_v \geq N$. Let $v_2$ (resp. $w_2$) be a prime divisor of $p$ with second largest degree $[K_2 : \mathbb{Q}_p]$ (resp. $[L_{w_2} : \mathbb{Q}_p]$), i.e. $[K_{v_2} : \mathbb{Q}_p]$ (resp. $[L_{w_2} : \mathbb{Q}_p]$) is second in the local degree type of $p$ in $K$ (resp. in $L$).

Assume on the contrary that $\frac{(D\mathcal{H}_2)}{|H|} < \frac{(D\mathcal{H}'_2)}{|H'|}$. By Equation 2.1 $[K_{v_2} : \mathbb{Q}_p] = \frac{(D\mathcal{H}_2)}{|H|}$ and $[L_{w_2} : \mathbb{Q}_p] = \frac{(D\mathcal{H}'_2)}{|H'|}$. Thus, $r_{v_2} \geq [K_{v_2} : \mathbb{Q}_p] + 1 \geq [L_{w_2} : \mathbb{Q}_p] + 2 \geq 3$. The group $G = C_p^{r_{v_2}}$ is therefore not metacyclic and hence realizable only over completions at prime divisors of $p$. The above discussion shows $G$ is realizable over two completions of $K$ but over at most one of $L$. Thus, $G$ is $K$-preadmissible (in fact one can use Grunwald-Wang to show it is $K$-admissible) but not $L$-preadmissible, contradiction. \(\square\)

Let us consider the case $L = M$, i.e. $L$ is Galois over $\mathbb{Q}$. If $L$ is Galois over $\mathbb{Q}$ then all prime divisors $v$ of $p$ in $L$ have the same degree $[L_v : \mathbb{Q}_p] = |D|$, where $D = D(M/\mathbb{Q}, p)$
for a prime divisor \( p \) of \( p \) in \( M \). In such case we shall be able to deduce information on rational primes that decompose in \( M \), i.e. rational primes that have at least two prime divisors in \( M \), and on split double cosets:

**Definition 2.4.** Denote by \( S(A, B) \) the number of distinct split double cosets \( AxB \). Note that \( |AxB| = \frac{|A||B|}{|x^{-1}Ax \cap B|} \) and hence \( AxB \) is a split double coset if and only if \( x^{-1}Ax \cap B = 1 \).

**Corollary 2.5.** Let \( M/Q \) be a \( G \)-extension in which every rational prime \( p \) decomposes. Let \( K \) be a subfield of \( M \) with Galois group \( H = \text{Gal}(M/K) \), that is equivalent by preadmissibility to \( M \). Then for every rational prime \( p \) we have \( S(D, H) > 1 \) where \( D = D(M/Q, p) \) for a prime divisor \( p \) of \( p \) in \( M \).

**Proof.** As all primes decompose in \( M \), and \( K \) is equivalent by preadmissibility to \( L \), Proposition \( \ref{2.2} \) implies \( \frac{|D||H_\ell|}{|H|} = |D| \). We deduce that \( (D, H)_1 = (D, H)_2 = |D||H| \) and hence \( S(D, H) > 1 \). □

**Remark 2.6.** Let \( K \) and \( M \) be as above. As every cyclic subgroup appears as a decomposition group we have that every cyclic subgroup \( C \leq G \) satisfies \( S(C, H) > 1 \).

We shall now aim to show the converse of Corollary \( \ref{2.5} \) when \( G \) is an \( l \)-group and \( l \) splits completely in \( M \). First, let us show that in many cases a group \( G \) that is \( K \)-preadmissible is also \( M \)-preadmissible.

**Proposition 2.7.** Let \( l \) be a prime, \( G \) an \( l \)-group and \( M/Q \) a \( G \)-extension in which \( l \) splits completely and every rational prime decomposes. Let \( K \) be a number field with \( Q \)-normal closure \( M \) and \( H = \text{Gal}(M/K) \). Then every \( K \)-preadmissible group is also \( M \)-preadmissible.

**Remark 2.8.** Proposition \( \ref{2.7} \) implies that under the following assumptions:

(*) \( M/Q \) is a \( G \)-extension in which \( l \) splits completely and \( H \) a subgroup of \( G \) so that for every metacyclic subgroup \( D \leq G \), \( S(D, H) > 1 \),

any group that is \( K := M^H \)-preadmissible is also \( M \)-preadmissible.

Indeed, condition (*) implies that \( G \) is not metacyclic and that any rational prime decomposes in \( M \). As \( M/Q \) is tamely ramified all decomposition groups are metacyclic and hence condition (*) also implies all decomposition groups \( D \) satisfy \( S(D, H) > 1 \).

In order to prove Proposition \( \ref{2.7} \) we shall need the following remark and lemma.

**Remark 2.9.** Let \( G \) be a finite group. Let \( M \) be a number field in which \( l \) splits completely and let \( K \) be a subfield of \( M \) with \( [M : K] = l^r \) and \( M/K \) Galois. Let \( p \) be any rational prime, \( v \) a prime of \( K \) and \( w \) a prime divisor of \( v \) in \( M \). Assume there is a subgroup \( G_1 \leq G \) that contains a \( p \)-Sylow subgroup of \( G \) and is realizable over \( K_v \) and assume either \( p \neq l \) or \( p = l \) and \( v \mid p \). Then there is a subgroup \( G_2 \leq G \) that contains a \( p \)-Sylow subgroup of \( G \) and is realizable over \( M_w \).

**Proof.** (Remark \( \ref{2.9} \)) If \( p = l \): \( p \) splits completely in \( M \) and hence the same groups are realizable over \( Q_p \cong K_v \cong M_w \).

If \( p \neq l \): as \( M/K \) is Galois, \( M_w/K_v \) is an \( l \)-extension and \( ([M_w : K_v], p) = 1 \). Let \( F/K_v \) be a \( G_1 \)-extension. The extension \( FM_w/M_w \) is Galois. Denote its Galois group by \( G_2 \). Then \( G_2 \) is a subgroup of \( G_1 \) for which \( [FM_w : M_w] = [F : F \cap M_w] = |G_2| \). But as \( ([F \cap M_w : K_v], p) = 1 \), \( G_2 \) must also contain a \( p \)-Sylow subgroup of \( G \). □
We shall also use the following Lemma several times to pass from tame realizations to wild realizations.

**Lemma 2.10.** Let $G$ be a metacyclic $p$-group and $k$ a $p$-adic field. Then $G$ is realizable over $k$.

**Proof.** Let $k \neq \mathbb{Q}_2$ be a $p$-adic field, $n := [k : \mathbb{Q}_p]$ and $q$ the number of $p$-power roots of unity in $k$. At first assume $q > 2$. Let $k(p)$ be the maximal pro-$p$ extension of $k$. By [1], the Galois group $G_k(p) := \text{Gal}(k(p)/k)$ has a pro-$p$ presentation (with topological generators):

$$G_k(p) = \langle x_1, \ldots, x_{n+2}|x_1^q[x_1, x_2] \cdots [x_{n+1}, x_{n+2}] \rangle.$$

In such case $n \geq 2$ and $G_k(p)$ has an epimorphism onto $F_p(2) = \langle f_1, f_2 \rangle$ the free pro-$p$ group on the 2 generators $f_1, f_2$ which can be obtained in the following way: send each $x_i$ for every $i \neq 2, 4$ to the trivial element and $x_2 \to f_1$, $x_4 \to f_2$. In particular, $G$ is realizable over $k$.

Now let $q \leq 2$. If $n = 1$ and $p \neq 2$, $k$ does not have any $p$-power roots of unity ($q = 1$) and $G_k(p)$ is the free pro-$p$ group $F_p(n + 1)$. Thus, in this case $G$ is also realizable over $k$.

Let $p = q = 2$ and $n \geq 2$, then $G_k(2)$ has one of the following pro-$p$ presentations:

If $n$ is odd, by [22]:

$$\langle x_1, \ldots, x_{n+2}|x_1^2x_2^4[x_2, x_3] \cdots [x_{n+1}, x_{n+2}] \rangle,$$

and if $n$ is even, by [9], $G$ has one of the following presentations:

$$\langle x_1, \ldots, x_{n+2}|x_1^{2+2^j}[x_1, x_2] \cdots [x_{n+1}, x_{n+2}] \rangle$$

In the cases described in Equations 2.3 and 2.4, $F_2(2) = \langle f_1, f_2 \rangle$ is an epimorphic image of $G_k(2)$ in the following way: send each $x_i$ with $i \neq 2, 4$ to the trivial element and $x_2 \to f_1, x_4 \to f_2$. If $n > 1$ is odd ($n \geq 3$) then $G_k(2)$ is as in Equation 2.2 and by sending $x_i$ for $i \neq 3, 5$ to the trivial element of $F_2(2) = \langle f_1, f_2 \rangle$ and $x_3 \to f_1, x_5 \to f_2$, we achieve an epimorphism onto $F_2(2)$.

We are left with the case $k = \mathbb{Q}_2$, $p = 2$ which is proved in [14].

We can now prove Proposition 2.7.

**Proof.** (Proposition 2.7)

Let $G$ be a $K$-preamissible group and $p || |G|$. There are two primes $v_1(p), v_2(p)$ of $K$ and corresponding subgroups $G^{v_1(p)}, G^{v_2(p)}$ so that $G^{v_i(p)}$ is realizable over $K_{v_i(p)}$ and contains a $p$-Sylow subgroup of $G$. For every $p$, we shall choose two primes $w_1(p), w_2(p)$ of $M$, and corresponding subgroups $G^{w_i(p)} \leq G$ for which:

1. $G^{w_i(p)}$ contains a $p$-Sylow subgroup of $G$,
2. $G^{w_i(p)}$ is realizable over $M^{w_i(p)}$,
3. $w_1(p) \neq w_2(p)$ and $w_i(p) \not| p$.

For any $i = 1, 2$ and $p || |G|$. This implies that $w_i(p) \neq w_j(q)$ for any $i, j \in \{1, 2\}$ and $p \neq q$. Note that such a choice of primes and corresponding subgroups shows $G$ is $M$-preamissible.

Let $p$ be any rational prime. If one of $v_i(p)$, $i = 1, 2$, does not divide $p$ then by [10], $G(p)$ is metacyclic and hence by Lemma 2.10 realizable over any $K_v$ for any $v$ that is a
prime divisor of \( p \). As \( p \) decomposes in \( M \), we can choose both \( w_1(p), w_2(p) \) to be prime divisors of \( p \) and \( G^{u_i(p)} := G(p) \), \( i = 1, 2 \). So, let us next assume \( v_1(p), v_2(p)|p \) and split our proof into two cases: \( p = l \) and \( p \neq l \).

Case \( p = l \): for every prime divisor \( v \) of \( l \) in \( M \) we have \( K_v \cong M_v \cong \mathbb{Q}_p \), and hence \( G \) is realizable over \( K_v \) if and only if it is realizable over \( M_v \). In particular, if \( l \) divides both \( v_1(l), v_2(l) \) then \( G^{v_i(l)} \) is realizable over both \( K_{v_i(l)} \cong M_{v_i(l)}, i = 1, 2 \). Thus, we can choose \( w_i(l) \) to be any prime divisor of \( v_i(l) \) and \( G^{w_i(l)} := G^{v_i(l)} \), for \( i = 1, 2 \).

Case \( p \neq l \): By Remark 2.9 for any \( w_1(p)|v_1(p), w_2(p)|v_2(p) \) primes of \( M \), there are two subgroups \( G^{w_1(p)} \leq G^{v_1(p)}, G^{w_2(p)} \leq G^{v_2(p)} \), each containing a \( p \)-Sylow subgroup of \( G \), so that \( G^{w_i(p)} \) is realizable over \( M_{w_i(p)} \), for \( i = 1, 2 \).

The primes \( w_i(p) \), and the corresponding subgroups \( G^{w_i(p)} \leq G \) for \( i = 1, 2, p||G| \) were chosen so that conditions (1-3) hold and therefore \( G \) is \( M \)-preadmissible.

Since Remark 2.9 uses a rather natural way of extending realizations we deduce the following Corollary:

**Corollary 2.11.** Let \( l \) be a prime and \( G \) an \( l \)-group. Let \( M/\mathbb{Q} \) be a \( G \)-extension in which \( l \) splits completely and \( K \) a subfield of \( M \). Then any group \( G \) that is \( K \)-admissible and has no metacyclic Sylow subgroups is also \( M \)-admissible.

**Proof.** As \( G \) is \( K \)-admissible there is a \( G \)-extension \( L/K \) so that for every \( p||G| \) there are two primes \( v_1(p), v_2(p) \) of \( K \) for which \( G^{v_i(p)} := \text{Gal}(L_{v_i(p)}/K_{v_i(p)}) \) contains a \( p \)-Sylow subgroup of \( G \). We claim \( LM/M \) is an \( M \)-adquate \( G \)-extension. The group \( G^{v_i(p)} \), \( i = 1, 2, p||G| \) is not metacyclic and hence \( v_i(p)|p \). For every \( i = 1, 2, p||G| \), choose a prime \( w_i(p) \) of \( M \) that divides \( v_i(p) \). In our case \( v_i(p)|p \), hence as in Remark 2.9 the subgroup \( G_{w_i(p)} := \text{Gal}(LM_{w_i(p)}/M_{w_i(p)}) \) also contains a \( p \)-Sylow subgroup of \( G \). Therefore \( G \) is \( M \)-admissible.

We are now ready to prove the converse of Corollary 2.5 in case \( G \) is an \( l \)-group.

**Theorem 2.12.** Let \( l \) be a prime and \( G \) an \( l \)-group. Let \( H \) be a subgroup of \( G \) with \( \text{core}_G(H) = 1 \). Let \( M/\mathbb{Q} \) be a \( G \)-extension in which \( l \) splits completely. Assume \( S(D, H) > 1 \) for every subgroup \( D \leq G \) that appears as a decomposition group, i.e. for every \( D = D(M/\mathbb{Q}, p) \) where \( p \) is some prime of \( M \). Then \( M \) and \( K = M^K \) are equivalent by preadmissibility.

**Remark 2.13.** As \( M/\mathbb{Q} \) is a tamely ramified extension, every \( D \) that appears as decomposition group is metacyclic. Therefore, when using Theorem 2.12 it is sufficient to verify \( S(D, H) > 1 \) for every metacyclic subgroup of \( G \).

**Proof.** Let \( G \) be an arbitrary group. Note that as \( S(D, H) > 1 \), there are at least two double cosets in \( (D, H) \) for every decomposition group \( D \) and therefore every rational prime decomposes in \( K \). By Propositions 2.7 if \( G \) is \( K \)-preadmissible then \( G \) is \( M \)-preadmissible. Let us assume \( G \) is \( M \)-preadmissible and show that \( G \) must also be \( K \)-preadmissible. For every \( p||G| \), there are two primes \( w_1(p), w_2(p) \) of \( M \) and corresponding subgroups \( G^{w_1(p)}, G^{w_2(p)} \), so that \( G^{w_1(p)} \) is realizable over \( M_{w_1(p)} \) and contains a \( p \)-Sylow subgroup of \( G \).

First, we claim that \( w_i(p) \) can be chosen so that \( w_i(p)|p \) for every \( i = 1, 2 \) and \( p||G| \). If \( w_i(p) \) does not divide \( p \) and \( F/M_{w_i(p)} \) is a \( G^{w_i(p)} \)-extension then \( F/F^G(p) \) is a tamely ramified extension and therefore \( G(p) \) is metacyclic. In such case, by Lemma 2.10 \( G(p) \)
is realizable over $M_w$ for any prime $w$ that divides $p$. Let us substitute every $w_i(p)$ that is not a divisor of $p$ by a prime divisor $w$ of $p$, that is different from $w_i(p)$, $j = 1, 2$, and set $G^w := G(p)$. We obtain a set of primes $w_i(p)$, $i = 1, 2, p\mid |G|$ and corresponding subgroups $G^{w_i(p)}$ so that for every $i = 1, 2$ and $p\mid |G|$:

1. $w_i(p)|p$,
2. $G^{w_i(p)}$ is realizable over $K_{w_i(p)}$,
3. $G^{w_i(p)}$ contains a $p$-Sylow subgroup of $G$.

Fix a rational prime $p\mid |G|$, a prime divisor $p$ of $M$ and set $D = D(M/Q, p)$. For $i = 1, 2$, $G^{w_i(p)}$ is realizable over $M_{w_i(p)}$ and hence over $M_w$ for any prime $w$ of $M$ that divides $p$. By the correspondence described in the beginning of this section, the existence of two split double cosets in $(D, \mathcal{H})$ implies there are two primes $v_1(p), v_2(p)$ of $K$ for which $[K_{v_i(p)} : \mathbb{Q}_p] = |D|$. Thus, there is a unique prime $w_i$ of $M$ that divide $v_i(p)$. For this prime we have $M_{w_i} \cong K_{v_i(p)}$. Therefore $G^{w_i(p)}$ is realizable over $K_{v_i(p)}$ for $i = 1, 2$ and $p\mid |G|$, which shows $G$ is $K$-preadmissible.

Applying Remark 2.13 and Theorem 1.7 we get:

**Corollary 2.14.** Let $l$ be an odd prime and $G$ an $l$-group that has a subgroup $\mathcal{H} \leq G$ that satisfies $core_G(\mathcal{H}) = 1$ and $S(D, \mathcal{H}) > 1$ for every metacyclic subgroup $D \leq G$. Then there is a $G$-extension $M/Q$ for which $M$ and $K := M^\mathcal{H}$ are equivalent by preadmissibility and odd order admissibility.

### 2.2. Sequences of $l$-groups.

In this section we give examples of infinite families of pairs $(G, \mathcal{H})$ for which $S(D, \mathcal{H}) > 1$ for any metacyclic subgroup $D$ of $G$. Fix a rational prime $l$. Let $(G_n)_{n \in \mathbb{N}}$ denote a sequence of $l$-groups so that $G_i \leq G_{i+1}$ for every $i \in \mathbb{N}$. By setting $G_1 = 1$ and repeating some of the $G_i$'s any such sequence can be refined so that $G_i \leq S_i$. Let $\alpha$ be an element of order $l$ in $G_k$ for some $k \in \mathbb{N}$ and let $\mathcal{H} = \langle \alpha \rangle$.

**Theorem 2.15.** Let $d_n$ denote the maximal order of a metacyclic subgroup of $G_n$, $n \in \mathbb{N}$. Assume the sequence $(G_n)_{n \in \mathbb{N}}$ satisfies:

1. $\lim_{n \to \infty} \frac{|G_n|}{n!} = 0$,
2. the element $\alpha$ has infinitely many conjugates in $\bigcup_{n \in \mathbb{N}} G_n$.

Then there is an $N$ so that for every $n \geq N$ and every metacyclic subgroup $D \leq G_n$ we have $S(D, \mathcal{H}) > 1$.

**Remark 2.16.** Let $c_n$ denote the maximal order of an element in $G_n$. Then $d_n \leq c_n^2$. Therefore the condition $\lim_{n \to \infty} \frac{|G_n|}{c_n} = \infty$ suffices in order for $G$ to satisfy (1). Let $O_{G_n}(\alpha)$ denote the orbit of $\alpha$ under conjugation in $G_n$. Condition (2) can also be stated as $\lim_{n \to \infty} |O_{G_n}(\alpha)| = \infty$.

Letting $l$ be odd, we can apply Theorem 1.7 in order to produce a $G_n$-extension of $\mathbb{Q}$ in which $l$ splits completely. This proves Theorem 1.19.

**Corollary 2.17.** Let $l$ be an odd prime and let $(G_n)_{n=1}^\infty$ and $\mathcal{H}$ be as above. Then there is an $N$ such that for every $n \geq N$ there is a $G_n$-extension $M/Q$ for which $K = M^\mathcal{H}$ is equivalent by preadmissibility and odd order admissibility to $M$.

In order to prove Theorem 2.15 we shall first obtain a bound on the number of occurrences of a given cycle structure (a cycle structure is also often referred as a partition) in an embedding of a metacyclic group in $S_n$. 
Let $S_\infty$ be the group of permutations on $\mathbb{N}$ that fix all elements but a finite set. Let $S_n$ be the subgroup that fixes all elements in $\mathbb{N} \setminus \{1, \ldots, n\}$. Note that $S_\infty$ can also be viewed as $S_\infty = \lim_{n \to \infty} S_n$ or $S_\infty = \bigcup_{n \in \mathbb{N}} S_n$. Any element $\sigma \in S_\infty$ has a cycle structure $p(\sigma)$ which is a vector $(a_1, a_2, \ldots)$ with $a_i \geq a_{i+1}$ that denotes the cycle structure of $\sigma$, i.e. $\sigma$ has an $a_i$-cycle $c_i$ so that $\sigma_j$ and $\sigma_k$ are disjoint whenever $j \neq k$. Let $x$ be the cycle structure of $\sigma$. Denote by $o(x)$ the order of $x$ in $S_\infty$, i.e. $o(x) = \lcm_{i \in \mathbb{N}}(a_i)$. Denote by $l(x)$ the length of $x$: $l(x) := \sum_{a_i \neq 1} a_i$.

**Example 2.18.** Let $x$ be a transposition. A cyclic subgroup of $S_n$ may contain only one transposition as it has only one element of order 2.

This example suggests that the number of occurrences of a given cycle structure depends on the cyclicity level:

**Definition 2.19.** Let $G$ be a solvable group. Then there is a sequence $1 = H_0 \lhd H_1 \lhd \ldots \lhd H_k = G$ so that $H_i$ is normal in $H_{i+1}$ and $H_{i+1}/H_i$ is cyclic. The cyclicity level of $G$ is defined to be the minimal number $k$ for which such a sequence exists.

**Remark 2.20.** When letting $G$ be infinite, we do not necessarily have such a sequence so the requirement that $G$ is solvable does not suffice. One should require $G$ to be polycyclic. In case $G$ is finite the notions polycyclic and solvable coincide.

**Proposition 2.21.** Fix a number $k \in \mathbb{N}$ and some cycle structure $x$ of some element in $S_\infty$. Then there is an $b_x \in \mathbb{N}$ so that for every group $G$ of cyclicity level $k$ and every faithful representation $\phi: G \hookrightarrow S_\infty$ there are at most $b_x$ elements with cycle structure $x$ in $\phi(G)$.

**Proof.** By induction on $k$. The case $k = 0$ is trivial ($b = 1$ always works). Assume by induction that every group of cyclicity level $< k$ has at most $b_y$ element with cycle structure $y$ (in any representation). Fix $\phi$ and identify $\phi(G)$ with $G$. Let $H$ be a normal subgroup of $G$ of cyclicity level $k-1$ such that $C := G/H$ is cyclic and let $\tau \in G$ be an element for which $\langle \tau H \rangle = C := G/H$. So, $H$ has at most $e$ elements with cycle structure $x$. Assume there is an element $u \in G$ with cycle structure $x$. In such case, the order of the coset $uH$ in $C$ divides the order of $u$ which is $o(x)$. As $C$ is cyclic it contains at most $o(x)$ elements of order dividing $o(x)$.

It remains to bound the number of elements with cycle structure $x$ in a single coset. Let $v$ be another element in $uH$ with cycle structure $x$. The element $uv^{-1}$ is in $H$ and has length $l(uv^{-1}) \leq 2l(x)$. For every cycle structure $y$, with length $l(y) \leq 2l(x)$ (clearly there are only finitely many such) there are at most $e_y$ elements with cycle structure $y$ in $H$ and hence $H$ contains at most $\sum_{y: l(y) \leq 2l(x)} e_y$ elements with a cycle structure of length $\leq 2l(x)$. The map $uH \to H$ that sends $v \in uH$ to $u^{-1}v \in H$ is injective and therefore the coset $uH$ contains at most $\sum_{y: l(y) \leq 2l(x)} e_y$ elements with cycle structure $x$. Summing over the cosets of $G/H$ whose order divides $o(x)$ we get:

$$|\{\sigma \in G|p(\sigma) = x\}| \leq b_x := o(x) \sum_{y: l(y) \leq 2l(x)} e_y.$$ 

For $k = 2$, i.e. for a metacyclic group, we have:

**Corollary 2.22.** Let $x$ be any cycle structure. There exists a $b \in \mathbb{N}$ for which every metacyclic subgroup $D \hookrightarrow S_\infty$ contains at most $b$ elements with cycle structure $x$. 
Example 2.23. The maximal number of transpositions in an abelian 2-group of rank r is r. Indeed two distinct transpositions commute if and only if there is no common symbol on which both transpositions act non-trivially.

Example 2.24. The maximal number of transpositions in a metacyclic group is 4. The subgroup \( \langle (123)(45), (12) \rangle \) of \( S_5 \) is a metacyclic group with 4 transpositions, namely \( (12), (23), (13), (45) \). Let us follow one step of the induction in Proposition 2.21 in order to show there can not occur 5 transpositions in \( D \leq S_m \). Let \( C \triangleleft D \) be a cyclic normal subgroup for which \( D/C \) is also cyclic. Then \( C \) contains at most one transposition. Let \( x \not\in C \) be a transposition. Then \( xC \) is a coset of order 2 in \( D/C \) and hence the only coset of order 2. Therefore any transposition of \( D \) is either in \( C \) or in \( xC \). Assume \( y \in xC \) is another transposition. Then \( xy^{-1} \in C \) is either a 3-cycle or an element of cycle structure \((2, 2)\), i.e. a product of two disjoint transpositions.

Let us split the argument into two cases, if \( C \) contains a transposition then it does not contain an element of type \((2, 2)\). Then the fact that \( C \) contains at most two 3-cycles implies \( xC \) contains at most 2 transpositions other than \( y \) and in total \( D \) has at most 4 transpositions. If \( C \) does not contain a transposition then all transpositions are in \( xC \). But the map \( xC \cap T \to C \) that sends \( y \) to \( xy^{-1} \) is injective and its image that consists of 3-cycles and \((2, 2)\) elements is of cardinality at most 4 (which is the maximal amount of such elements in a cyclic group). Thus, in this case \( D \) also has no more than 4 transpositions.

Remark 2.25. Given a cycle structure \( x \) and some \( n \in N \). It is an interesting problem to find a good bound \( b \) on the number of occurrences of \( x \) in a representation of a group with \( n \)-cyclicity level. It is also interesting to understand this number for a given cycle structure and a given group of cyclicity level \( n \).

We can now prove Theorem 2.15.

Proof. As Remark in Definition 2.4, a double coset \( DxH \) splits if and only if \( D \cap xHx^{-1} = 1 \). Let \( n \geq k \) and \( X_n(D, H) = \{|x \in G_n| D \cap H^x = 1 \} \), where \( H^x \) denotes \( xHx^{-1} \). We shall show that there is an \( N \) so that for every \( n \geq N \) and every metacyclic subgroup \( D \) of \( G_n \), \( X_n(D, H) > |D||H| \). In such case the number of elements \( x \in G_n \) for which \( DxH \) splits exceeds \( |D||H| \), which shows there are at least two split double cosets.

Let \( T \) denote the set of all \( l \)-cycles in \( S_n \), then:

\[
X_n(D, H) = |G_n| - |\{x \in G_n| D \cap H^x \neq 1 \}| = |G_n| - |\{x \in G_n| \alpha^x \in D \}| = \\
= |G_n| - \left( \bigcup_{\sigma \in D \cap T} \{x| \alpha^x = \sigma \} \right) = |G_n| - \sum_{\tau \in T \cap D} |\{x| \alpha^x = \tau \}|.
\]

By Corollary 2.22 there is a \( b \) for which every metacyclic subgroup of \( S_n \) (for any \( n \)) contains at most \( b \) element whose cycle structure is \( p(\alpha) \). Thus,

\[
(2.5) \quad X_n(D, H) \geq |G_n| - b \cdot |G_n(\alpha)|.
\]

But by conditions (1) and (2):

\[
(2.6) \quad \lim_{n \to \infty} \frac{|H|d_n + b \cdot |G_n(\alpha)|}{|G_n|} = \lim_{n \to \infty} \frac{|H|d_n}{|G_n|} + \frac{b}{|O_{G_n}(\alpha)|} = 0.
\]
Therefore there is an \( N \) for which \(|G_n| > |H|d_n + b \cdot |N_{G_n}(\alpha)|\) holds for all \( n \geq N \). Using Equation 2.5 we obtain:
\[
X_n(D, H) \geq |G_n| - b \cdot |N_{G_n}(\alpha)| > d_n|H| \geq |D||H|
\]
for every \( n \geq N \) and every metacyclic subgroup \( D \leq G_n \).

2.3. Sylow subgroups of the Symmetric group. We shall devote the rest of the section to constructing a particular sequence of \( l \)-groups \( G_n \) that satisfies the conditions of Theorem 2.15. For this example we shall calculate the smallest constant \( N \) possible in Theorem 2.15.

Example 2.26. Let \( l \) be a prime and \( n \geq 2 \). Our group \( G_n \) will be an \( l \)-Sylow subgroup of \( S_n \). The group \( G_n \) is thus of order \( l^{1+1+...+l^{n-1}} = \frac{l^n - 1}{l - 1} \). Let us construct \( G_n \) explicitly. Let \( \alpha_1 \) be the \( l \)-cycle \((1, 2, ..., l)\), \( \alpha_2 \) be the product of \( l \) \( l \)-cycles:
\[
\alpha_2 = (1, l + 1, 2l + 1, ..., (l - 1)l + 1)(2, l + 2, ..., (l - 1)l + 2)\cdots(l, 2l, 3l, ..., l^2).
\]
Define \( \alpha_r \) to be the product of \( l^{r-1} \) \( l \)-cycles:
\[
\alpha_r = (1, l^{r-1} + 1, ..., (l - 1)l^{r-1} + 1)\cdots(l^{r-1}, 2l^{r-1}, ..., l^r).
\]
Then \( G_n \) is isomorphic to the iterated wreath product \((...((\langle \alpha_1 \rangle \langle \alpha_2 \rangle )\langle \alpha_3 \rangle )\cdots\langle \alpha_n \rangle ))\) or \((...((C_l \langle C_l \rangle C_l)\langle C_l \rangle C_l)\cdots C_l))\) where \( C_l \) appears \( n \) times. The latter is known to be an \( l \)-Sylow subgroup of \( S_n \) \((B2)\). Giving \( G_n \) the structure of an iterated wreath product equips us with generic realizations:

Theorem 2.27. (Saltman, [20]) 1. Let \( l \) be a prime and \( K \) a number field. Then \( C_l \) has a generic extension.

2. For two groups \( G, H \) that have a generic extension over \( K \), \( H \upharpoonright G \) also has a generic extension over \( K \).

This shows \( G_n \) has a generic extension over \( \Q \). It will be useful for us because of the following property:

Theorem 2.28. (Saltman, [20]) Let \( G \) be a group with a generic extension over a number field \( K \). Let \( S \) be a finite set of primes of \( K \) and let \( G^v \leq G \) be a corresponding subgroup that is realizable over \( K_v \), for each \( v \in S \). Then there is a \( G \)-extension \( L/K \) for which \( \text{Gal}(L_v/K_v) \cong G^v \) for every \( v \in S \).

This shows there is a \( G_n \)-extension \( M \) of \( \Q \) in which \( l \) splits completely. Let \( H = \langle \alpha_1 \rangle \) and \( K = M^H \).

By Remark 2.13 in order to show that \( K \) and \( M \) are equivalent by preadmissibility, for large enough \( n \) it suffices to prove the following Lemma:

Lemma 2.29. For every \( n \geq 3 \) and every metacyclic subgroup \( D \leq G_n \), \( S(D, H) > 1 \).

Remark 2.30. We will show that for \( l \geq 5 \) the claim holds for \( n \geq 2 \). For \( l = 2, 3 \) it holds for \( n \geq 3 \). The smallest such example therefore appears when \( l = 2 \) and \( n = 3 \), i.e. \( G := G_n = S_8(2) \) which is of order 128.

Remark 2.31. Let \( l = 3 \) and \( n = 2 \), i.e. \( G_n = S_9(3) = \langle(123), (147)(258)(369)\rangle \) and \( H = \langle(123)\rangle \). Then \( D = \langle(123), (456)\rangle \) is a metacyclic group for which \( S(D, H) = 1 \). For \( l = 2 \) and \( n = 2 \), \( G_n = S_4(2) \) is simply the group of Quaternions which is itself metacyclic. In such case we can choose \( D = G_n \) and then \( S(D, H) = 0 \).
Proof. Fix an $n \geq 2$ and a metacyclic subgroup $D$ of $G_n$. Let $X(D, \mathcal{H}) = |\{x \in D \cap H^x \neq \{1\}\}|$ be as before. We will show that for $l$ and $n$ as in Remark 2.30 we have $X(D, \mathcal{H}) > |D||\mathcal{H}|$.

Let us calculate $X(D, \mathcal{H})$ for a general metacyclic subgroup $D$ of $G_n$. The following $l^{n-1}$ l-cycles are conjugates of $\alpha_1$ in $G_n$:

$$\beta_1 := (1, \ldots, l), \beta_2 := (l + 1, \ldots, 2l), \ldots, \beta_{l^{n-1}} := (l^n - l + 1, \ldots, l^n).$$

Let $T_0 = \{\beta_i | 1 \leq i \leq l^{n-1}\}$.

Remark 2.32. Note that any l-cycle in $G_n$ is of the form $\beta_i^j$ for some $i$ and $j$. Indeed, assume $\gamma \in G_n$ is another l-cycle that is not of this form. Without loss of generality we can assume $\gamma = (b_1 := 1, b_2, \ldots, b_l)$. If one of $b_i$, $i = 2, \ldots, l$, is not in $\{2, \ldots, l\}$ then the subgroup $\langle \alpha_1, \gamma \rangle$ can be embedded in an l-Sylow subgroup of $S_{2l-1}$ (the subgroup is supported by at most $2l - 1$ letters). An l-Sylow subgroup of $S_{2l-1}$ is isomorphic to $C_l$ and hence any two l-cycles in such group are powers of each other. Thus $\gamma = \alpha_1^i = \beta_1^j$ for some $i$, contradiction.

Remark 2.33. Let $G'_n := [G_n, G_n]$. Then $\overline{G}_n := G_n/G'_n$ is isomorphic to $C_n^n$. As all $\beta_i$, $1 \leq i \leq l^{n-1}$ are conjugates they are mapped under $\pi : G_n \to \overline{G}_n$ to the same non-trivial element. This shows that for $1 \leq i \leq l^{n-1}$ and $1 \leq j \leq l - 1$, $\pi(\beta_i^j) = \pi(\alpha_1)$ only if $i = j = 1$. Thus, the only conjugates of $\alpha_1$ in $G_n$ are the elements of $T_0$.

Any metacyclic group can contain at most two elements of $T_0$ and hence:

$$X(D, \mathcal{H}) = |G_n| - |\{x \in G_n| D \cap H^x \neq \{1\}\}| = |G_n| - |\{x \in G_n| \alpha_1^x \in D\}| =$$

$$= |G_n| - \sum_{\sigma \in D \cap T_0} |\{x \in G_n| \alpha_1^x = \sigma\}| = |G_n| - \sum_{\sigma \in T_0 \cap D} |\{x \in G_n| \alpha_1^x = \sigma\}| \geq |G_n| - 2 \cdot |N_{G_n}(\alpha_1)|,$$

where $N_{G_n}(\alpha_1)$ denotes the normalizer of $\alpha_1$ in $G_n$. It is of order $N_{G_n}(\alpha_1) = |G_n|/|G_n(\alpha_1)|$.

As $T_0 = O_{G_n}(\alpha_1)$ is of cardinality $l^{n-1}$, we have $X(D, \mathcal{H}) \geq |G_n|(1 - 2/l^{n-1})$. The maximal order of an element in $S_n(l)$ is $l^n$ and hence the cardinality of $D$ is at most $l^n \cdot l^n = l^{2n}$. So, $|D||\mathcal{H}| \leq l^{2n} \cdot l = l^{2n+1}$. Thus, in order for $X(D, \mathcal{H}) > |D||\mathcal{H}|$ it is sufficient to have:

$$|G_n|(1 - 2/l^{n-1}) = l^{n-1}(1 - 2/l^{n-1}) \geq l^{2n+1}.$$

This inequality holds whenever:

1. $n = 2, l \geq 5$,
2. $n = 3, l \geq 3$ or
3. $n \geq 4$.

This covers all the cases in Remark 2.30 except for $l = 2, n = 3$. In such case

$$G := G_3 = S_8(2) = \langle (12), (13)(24), (15)(26)(37)(48) \rangle.$$  

Note that the only transpositions in $G$ are $T_0 = \{(12), (34), (56), (78)\}$.

Assume on the contrary $D \leq G$ is a metacyclic group for which $S(D, \mathcal{H}) \leq 1$. Then $X(D, \mathcal{H}) \leq |D||\mathcal{H}|$. A metacyclic subgroup $D \leq G$ contains at most 2 transpositions. We shall split the proof into two cases according to whether $D$ contains one or two transpositions.
Assume $D$ contains one transposition. Then $X(D, \mathcal{H}) = \{x | x^{-1}\mathcal{H}x \cap D = 1\} = \frac{3}{4}|G| = 3 \cdot 2^5$. But, if $|D||\mathcal{H}| \geq 3 \cdot 2^5$ then $|D||\mathcal{H}| = 2^7$, i.e. $D\mathcal{H} = G$ and $|D| = 2^6$. Let us show $G$ has no metacyclic subgroup of order $2^6$. Let $\Phi = G_2[\mathcal{G}]$ be the Frattini subgroup of $G$ and let $\pi : G \rightarrow G/\Phi$. Then $G/\Phi = C_2$. If $D$ is metacyclic of order $2^6$ then it maps under $\pi$ onto a subgroup $D'$ of some $C_2$. As $\pi^{-1}(C_2)$ contains $2^6$ elements. We must have $D = \pi^{-1}(C_2)$. But since $D \supseteq [G, \mathcal{G}]$ we get that if $D$ contains one transposition then it must also contain $T_0$. This is a contradiction to $D$ being metacyclic.

Assume now that $D$ contains two transpositions. Then $X(D, \mathcal{H}) = 2^6$ and hence $|D| = 2^6$ or $|D| = 2^5$. We’ve seen $|D| = 2^6$ can not occur. Let us show there is no metacyclic subgroup of $G$ of order $2^5$ that contains two transpositions. Assume without loss of generality $(12) \in D$. There are three cases: $D \supseteq ((12), (34))$, $D \supseteq ((12), (56))$ and $D \supseteq ((12), (78))$.

Case $D \supseteq ((12), (34))$: Let $\beta_i = (2i - 1, 2i), 1 \leq i \leq 4$, $\tau_1 = (13)(24), \tau_2 = (57)(68)$ and $u = \alpha_3 = (15)(26)(37)(48)$. Then
\[ S_6(2) = (\langle \alpha_1 \rangle \cdot \langle \alpha_2 \rangle) \cdot \langle \alpha_3 \rangle = (\langle \beta_1, \beta_2 \rangle \times \tau_1) \times (\langle \beta_3, \beta_4 \rangle \times \tau_2) \times \langle u \rangle. \]

Then any element $x \in S_6(2)$ can be written uniquely in the form $\prod_{i=1}^{4} \beta_i^{t_i(x)} \tau_1^{s_1(x)} \tau_2^{s_2(x)} u^{w(x)}$ for some $t_i(x), s_1(x), s_2(x), w(x) \in \{0, 1\}$ and $i = 1, \ldots, 4$. If there is an element $x \in D$ that has $w(x) = 1$ then $x^{-1}(12)x$ is a transposition that is not $(12)$ nor $(34)$. But $(12), (34)$ are the only transposition in $D$. Thus $D$ can be assumed to be a subgroup of $G' = (\langle \beta_1, \beta_2 \rangle \times \tau_1) \times (\langle \beta_3, \beta_4 \rangle \times \tau_2)$. Let $\Phi'$ be the Frattini subgroup of $G'$ and let $\pi' : G' \rightarrow G'/\Phi' = C_2$. Then $G'/\Phi' = C_2$ and $|\pi^{-1}(C_2)| = 2^4$. Therefore there is no metacyclic subgroup of $G'$ of order $2^5$, contradiction.

Case $D \supseteq ((12), (56))$: Clearly $D \subseteq N_G((12), (56))$ but
\[ N_G((12), (56)) = \langle \beta_1, \beta_2, \beta_3, \beta_4, u \rangle \]
is of cardinality $2^5$ and hence $D = N_G((12), (56))$ which cannot occur as then $D$ will contain all transpositions.

Case $D \supseteq ((12), (78))$, Similarly, $D \subseteq N_G((12), (78))$ but
\[ N_G((12), (78)) = \langle \beta_1, \beta_2, \beta_3, \beta_4, \tau_1 \tau_2 u \rangle \]
is of cardinality $2^5$ and hence $D = N_G((12), (78))$, contradiction.

\[ \square \]

3. Arithmetic equivalences

Let us summarize several interpretations of the arithmetical relations given in Definition [1,20]. The group theoretic analogue of arithmetic equivalence is:

**Definition 3.1.** Two subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathcal{G}$ are said to be Gassmann equivalent if for any $g \in \mathcal{G}$:
\[ |g^G \cap \mathcal{H}| = |g^G \cap \mathcal{H}'|. \]

Here $g^G$ denote the conjugacy class of $g$ in $\mathcal{G}$. Let $D$ be a subgroup of $\mathcal{G}$. Gassmann equivalence can be also defined using the following equivalent condition:

**Theorem 3.2.** (Gassman, [3]) The subgroups $\mathcal{H}$ and $\mathcal{H}'$ are Gassmann equivalent if and only if for any cyclic subgroup $C$ of $\mathcal{G}$, the coset type $(C, \mathcal{H})$ is the same as the coset type $(C, \mathcal{H}')$. 

In [19], it was proved that two number fields that are arithmetically equivalent have the same \( \mathbb{Q} \)-normal closure \( M \). Furthermore:

**Theorem 3.3.** (Gassman [3], Perlis [3]) Let \( K \) and \( L \) be two number fields that have the same \( \mathbb{Q} \)-normal closure \( M \). Denote \( G = \text{Gal}(M/\mathbb{Q}) \), \( H = \text{Gal}(M/K) \), \( H' = \text{Gal}(M/L) \). Then the fields \( K \) and \( L \) are arithmetically equivalent if and only if \( H \) and \( H' \) are Gassmann equivalent.

**Corollary 3.4.** It follows that if \( K \) and \( L \) are arithmetically equivalent then \( |H| = |H'| \) and \([K : \mathbb{Q}] = [L : \mathbb{Q}]\).  

**Remark 3.5.** Example 2.26 thus shows that two number fields which are equivalent by preadmissibility (or even by admissibility of odd order groups) need not be arithmetically equivalent.

**Remark 3.6.** (Gassman [3], Perlis [3]) Let \( K \) and \( L \) be two number fields that have the same \( \mathbb{Q} \)-normal closure \( M \) and let \( H = \text{Gal}(M/K) \) and \( H' = \text{Gal}(M/L) \). By the correspondence in Lemma 2.21, \( K \) and \( L \) have the same splitting type for every prime of \( \mathbb{Q} \) that is unramified in both \( K \) and \( L \) if and only if \( H \) and \( H' \) have the same double coset type for every cyclic subgroup \( C \leq G \). So, \( K \) and \( L \) are arithmetically equivalent if and only if almost (all but a finite number) every rational prime \( p \) has the same splitting type in \( K \) and \( L \).

Thus arithmetical equivalence can also be expressed in terms of the double coset types. However, local isomorphism is a stronger relation that does not have a known group theoretic interpretation.

For a number field \( F \), let \( P(F) \) denote the set of prime ideals of \( F \). By [4], Lemma 7, two fields are locally isomorphic if and only if there is a bijection \( \phi : P(K) \to P(L) \) so that \( K_p \sim L_{\phi(p)} \).

**Remark 3.7.** Two number fields \( K, L \) that are locally isomorphic (with a map \( \phi \) as above) are also arithmetically equivalent (since every \( p \) has the same inertial degree in \( K_p \) and \( L_{\phi(p)} \)) and equivalent by preadmissibility (since a group is realizable over \( K_p \) if and only if it is realizable over \( L_{\phi(p)} \)).

**Remark 3.8.** Let \( k \) and \( l \) be two \( p \)-adic fields. The absolute Galois groups \( G_k \) and \( G_l \) are isomorphic if and only if the local invariants \( n, q, p^* \) introduced in [4] are the same for \( k \) and \( l \). As mentioned in [10] Chapter 12, end of Section 2] \( k \) and \( l \) can be non-isomorphic and have \( G_k \cong G_l \). Two locally isomorphic number fields \( K \) and \( L \) with a bijection \( \phi \) have clearly have the same local invariants for \( K_p \) and \( L_{\phi(p)} \).

In [8], Komatsu gave an example of two locally isomorphic number fields, given explicitly as radical extensions of \( \mathbb{Q} \), that are not isomorphic. In fact, a complete classification of such examples appears in [5]. Such examples are in particular examples of fields that are arithmetically equivalent and equivalent by preadmissibility but not isomorphic. We shall now produce a simple construction that assigns to every two subgroups \( \mathcal{H} \) and \( \mathcal{H}' \) of \( S_n \) which are Gassman equivalent, two number fields \( K \) and \( L \) that are locally isomorphic.

**Example 3.9.** Let \( T/M \) be an unramified \( S_n \)-extension that is defined over \( \mathbb{Q} \), i.e. there is an \( S_n \)-extension \( F/\mathbb{Q} \) for which \( T = MF \) (see [2] Theorem).
Note that given a $\mathcal{G}$-extension $F/Q$, there is a process of creating an extension $M/Q$ (that can be also chosen to be Galois) for which $MF/M$ is an unramified $\mathcal{G}$-extension that is called swallowing ramification or Abhyankar’s Lemma.

Let $\mathcal{H}$ and $\mathcal{H}'$ be two Gassmann equivalent subgroups of $S_n$ that are not conjugate in $S_n$ and have a trivial core, i.e. $\text{core}_{S_n}(\mathcal{H}) = \bigcap_{x \in S_n} x^{-1} \mathcal{H}x = \{1\}$ (resp. $\mathcal{H}'$ = $\{1\}$). As $\text{core}_{S_n}(\mathcal{H}) \triangleleft S_n$ the condition of a trivial core simply means $\mathcal{H}, \mathcal{H}' \not\sim A_n, S_n$. Several methods to construct such $\mathcal{H}$ and $\mathcal{H}'$ are given in [19], Sections 2 and 3.

The Galois extension $T/Q$ has Galois group $\mathcal{G} := \text{Gal}(T/Q) \cong \text{Gal}(T/F) \times \text{Gal}(T/M) \cong \text{Gal}(T/F) \times S_n$. Let us view $\mathcal{H}$ and $\mathcal{H}'$ as subgroups of the latter $S_n$ and let $K = T^H$ and $L = T^{H'}$.

**Proposition 3.10.** The fields $K$ and $L$ are locally isomorphic but $K \not\cong L$.

_PROOF._ First, note that $\text{core}_Q(\mathcal{H}) = \text{core}_Q(\mathcal{H}') = \{1\}$. Indeed, $\text{core}_Q(\mathcal{H}) \subseteq \text{core}_{S_n}(\mathcal{H}) = \{1\}$ (resp. $\text{core}_Q(\mathcal{H}') \subseteq \text{core}_{S_n}(\mathcal{H}') = \{1\}$). This guarantees that the $Q$-normal closure of $K$ (resp. $\mathcal{H}$) is $T$. As $\mathcal{H}$ and $\mathcal{H}'$ were chosen to be non-conjugate in $S_n$ and as $\text{Gal}(T/F)$ commutes with $S_n$ inside $\text{Gal}(T/Q)$, we have that $\mathcal{H}$ and $\mathcal{H}'$ are not conjugate in $\text{Gal}(T/Q)$. This shows the groups $\text{Gal}(Q/K)$ and $\text{Gal}(Q/L)$ are not conjugate in $G_Q$ and hence $K \not\cong L$.

Let $C$ be any cyclic subgroup of $S_n$. Fix a prime $v$ of $M$ whose decomposition group in $T/M$ is (the conjugacy class of) $C$. As in Lemma 2.1 there is a bijection between the primes $v_1, ..., v_r$ (resp. $w_1, ..., w_s$) of $K$ (resp. of $L$) and the double cosets $Cv_1 \mathcal{H}, ..., Cv_r \mathcal{H}$ (resp. $Cy_1 \mathcal{H}', ..., Cy_s \mathcal{H}'$) so that $[K_{v_i} : M_v] = \frac{|Cv_i \mathcal{H}|}{|\mathcal{H}|}$ (resp. $[L_{w_i} : M_w] = \frac{|Cy_i \mathcal{H}'|}{|\mathcal{H}'|}$). As $\mathcal{H}$ and $\mathcal{H}'$ are Gassmann equivalent in $S_n$ (and also in $\mathcal{G}$) the coset type $(C, \mathcal{H})$ is the same as the coset type $(C, \mathcal{H}')$. Thus $r = s$, $|\mathcal{H}| = |\mathcal{H}'|$ and one has:

$$d := [K_{v_i} : M_v] = \frac{|Cv_i \mathcal{H}|}{|\mathcal{H}|} = \frac{|Cv_i \mathcal{H}'|}{|\mathcal{H}'|} = [L_{w_i} : M_w]$$

for $i = 1, 2, ..., r$. But as $T/M$ is unramified and $M_v$ has a unique unramified extension of degree $d$, one has $K_{v_i} \cong L_{w_i}$. We therefore obtain a bijection between primes $v_K$ of $K$ and primes $w_L$ of $L$ so that $K_{v_K} \cong L_{w_L}$. This shows $K$ and $L$ are locally isomorphic. □

We shall devote the second part of this section to give an example of two arithmetically equivalent fields that are not equivalent by preadmissibility. Arithmetic equivalence does not take into consideration any local data except inertial degrees. On the other hand, we shall see in the following example that equivalence by preadmissibility also considers the ramification type and the number of roots of unity that appear at some completions. The following example appears in [7] as an example of arithmetically equivalent fields that are not locally isomorphic. We shall show this example is also an example of two fields that are not equivalent by preadmissibility.

**Example 3.11.** Let $K = \mathbb{Q}(\sqrt[3]{m})$ and $L = \mathbb{Q}(\sqrt[3]{2m})$ where $m \neq \pm 1 \pm 2$ is a square free integer that satisfies $m \equiv 1 \pmod{27}$. In [7] Lemma 4], Komatsu shows that $K$ and $L$ are arithmetically equivalent. Let us show $K$ and $L$ are not equivalent by preadmissibility nor by admissibility. Since $m \equiv 1 \pmod{27}$ there is a unit $u \in \mathbb{Z}_2$ for which $u^{32} = m$. So, the polynomials that define $K$ and $L$ factor over $\mathbb{Q}_2$ into irreducible factors as follows:

$$x^{32} - m = (x - u)(x + u)(x^2 + u^2)(x^4 + u^4)(x^8 + u^8)(x^{16} + u^{16})$$
\[ x^{32} - 2^{16} m = (x^2 - 2u)(x^2 + 2u)(x^2 - 2ux + 2u)(x^2 + 2ux + 2u)(x^8 + 16u^8)(x^{16} + 2^8 u^{16}). \]

Note that by \[7\] Lemma 10 all the above factors are irreducible in \( \mathbb{Q}_2[x] \).

Therefore, there are 6 prime \( p_1, \ldots, p_6 \) that lie above 2 in \( K \) and 6 primes \( p_1', \ldots, p_6' \) lying above 2 in \( L \). Let us assume the primes are ordered so that \([K_{p_i} : \mathbb{Q}_2] \geq [K_{p_{i+1}} : \mathbb{Q}_2]\) and \([L_{p_i} : \mathbb{Q}_2] \geq [L_{p_{i+1}} : \mathbb{Q}_2]\) for all \( i = 1, \ldots, 5 \). Considering the above factorizations we have:

\[
K_{p_1} = \mathbb{Q}_2(\sqrt{2}), \quad K_{p_2} = \mathbb{Q}_2(\mu_{16}), \quad L_{p_i} = \mathbb{Q}_2(\sqrt[p_i]{2}), \quad L_{p_{i+1,2}} = \mathbb{Q}_2(\sqrt[p_i]{2})\quad \text{and} \quad [K_{p_i} : \mathbb{Q}_2] \leq 4, \quad [L_{p_i} : \mathbb{Q}_2] \leq 4 \quad \text{for} \quad i \geq 3.
\]

Now let \( A := C_{16}^0 \). By local class field theory the maximal abelian extension \( k_{ab} \) of a \( p \)-adic field \( k \) has Galois group \( \text{Gal}(k_{ab}/k) \) that is isomorphic to the profinite completion of the group \( k^* \). Therefore the maximal abelian extension of exponent 16 of \( K_{p_2} \) has Galois group \( \text{Gal}(K_{p_2,ab,16}/K_{p_2}) \simeq C_{16}^{10} \). Similarly, \( \text{Gal}(K_{p_1,ab,16}/K_{p_1}) \simeq C_{16}^{18} \), \( \text{Gal}(L_{p_{i,ab,16}}/L_{p_i}) \simeq C_{16}^{17} \times C_2 \), \( \text{Gal}(L_{p_{i+1,2}}/K_{p_i}) \simeq C_{16}^{9} \times C_2 \) \( \text{and} \) \( rk(\text{Gal}(K_{p_1,ab,16}/K_{p_1})), rk(\text{Gal}(L_{p_{i,ab,16}}/L_{p_i})) \leq 6 \) for \( i \geq 3 \). We can now see that \( A \) is realizable over two completions of \( K \) and only one completion of \( L \). Thus, \( A \) is \( K \)-preadmissible but not \( L \)-preadmissible and \( K, L \) are not equivalent by preadmissibility. It follows that \( A \) is not \( L \)-admissible and by Theorem 2.11 in \[13\] \( A \) is \( K \)-admissible (the primes \( p_1, p_2 \) are evenly even). Thus, \( K \) and \( L \) are also not equivalent by admissibility.

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Technion
E-mail address: neftind@tx.technion.ac.il