Coherence and squeezing in superpositions of spin coherent states

Xingshuang Wang

Institute of Physics and Astronomy, Aarhus University, DK-8000, Aarhus C, Denmark, and Quantum information processing group, Institute for Scientific Interchange (ISI) Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy
(November 4, 2018)

We consider the superpositions of spin coherent states and study the coherence properties and spin squeezing in these states. The spin squeezing is examined using a new version of spectroscopic squeezing criteria. The results show that the antibunching effect can be enhanced and spin squeezing can be generated in the superpositions of two spin coherent states.

Keywords: Coherence, spin squeezing and spin coherent state.

PACS numbers: 42.50.Dv

I. INTRODUCTION

There has been much interest in the study of Schrödinger cat states in quantized electromagnetic field [1] and quantized motion of the center of mass of a trapped ion [2]. Two types of cat states, superpositions of the bosonic coherent state $|\alpha\rangle$ and $|-\alpha\rangle$ [3] or superpositions of $|\alpha\rangle$ and $|\alpha^*\rangle$ [4] are studied in the literature. These states exhibit nonclassical properties such as oscillations of photon number distribution, antibunching effects and quadrature squeezing.

Now we replace the bosonic coherent state $|\alpha\rangle$ by the spin coherent state (SCS) [5] $|\eta\rangle$ in the cat states and obtain a superposition of two SCSs. Agarwal et al. [6], Gerry et al. [7] and Recami et al. [8] have introduced and studied the generation scheme and some properties of the superposition state. Here we concentrate on their coherence properties and spin squeezing [6]. There are many ways to characterize spin squeezing [6][7]. Here we use the squeezing criteria given by Wineland et al. [6] and Sørensen et al. [9]. The squeezing parameter is defined as

$$\xi_n^2 = \frac{N(\langle J_{n_i}\rangle)^2 - \langle J_{n_i}^2\rangle}{\langle J_{n_i}^2\rangle + \langle J_{n_i}^3\rangle^2},$$

where $J_{n_i} = i \cdot J$, $n_i (i = 1, 2, 3)$ are orthogonal unit vectors and $J$ is the angular momentum operator. The states with $\xi_n^2 < 1$ are spin squeezed in the direction $n_i$. Under this criteria one remarkable feature is that the squeezing parameters $\xi_1^2 = \xi_2^2 = \xi_3^2 = 1$ [10] for spin coherent states. [6]. Another reason to use this criteria is that the squeezing parameter defined in Eq.(1) is closely related to the entanglement of many qubits [11], i.e., the condition $\xi_n^2 < 1$ also indicates entanglement in multi-qubit systems. It is well-known that quadrature squeezing in bosonic systems can be generated in nonlinear Kerr medium. Similar nonlinear Hamiltonian was proposed by Kitagawa and Ueda [3] to produce spin squeezing. It is also known that the quadrature squeezing exist in the cat states, then it is reasonable to examine the spin squeezing in the superpositions of spin coherent states. So the present works on coherence properties and spin squeezing in spin systems are complementary to the previous works on coherent properties and quadrature squeezing in bosonic systems.

The paper is organized as follows: In sec. II, we introduce the superposition of two SCSs and give its corresponding ladder operator formalism which is useful in the calculation of the expectation value $\langle J_i^2 \rangle$. We examine the coherence properties in Sec. III and spin squeezing in Sec. IV. Finally a conclusion is given in Sec. V.

II. THE SCS AND ITS LADDER OPERATOR FORMALISM

We work in the $(2j + 1)$-dimensional angular momentum Hilbert space $\{|j, m\}; m = -j, \ldots, +j\}$. It is convenient to define the 'number' operator $N = J_z + j$ and 'number states' as

$$|n\rangle = |j, n - j\rangle,$$

$$\hat{N}|n\rangle = n|n\rangle.$$  \hspace{1cm} (2)

The SCS is defined in this Hilbert space and given by [10],

$$|\eta\rangle = (1 + |\eta|^2)^{-j} \sum_{n=0}^{2j} \binom{2j}{n} \eta^n |n\rangle,$$  \hspace{1cm} (3)

where the parameter $\eta$ is complex. The inner product $\langle \eta | - \eta \rangle = \xi^2$, where $\xi = |\eta|^2/(1 + |\eta|^2)$ is a real number. Formally the SCS is exactly of the form of the binomial state [10].

The quantum state, which is of interest here, is the superposition of two SCS $|\pm \eta\rangle$, the SSSC

$$|\eta, \theta\rangle = \frac{1}{\sqrt{2 + 2 \cos \theta \xi^2}} \left(|\eta\rangle + e^{i\theta}| - \eta\rangle\right)$$

$$= \frac{(1 + |\eta|^2)^{-j}}{\sqrt{2 + 2 \cos \theta \xi^2}} \times \sum_{n=0}^{2j} \binom{2j}{n} \eta^n \left[1 + e^{i\theta}(-1)^n\right]|n\rangle,$$ \hspace{1cm} (4)
where $\theta$ is a relative phase.

It is easy to check that a ladder operator formalism of the SSCS is

$$J_-|\eta\rangle = \eta(2j - \mathcal{N})|\eta\rangle,$$

where $J_\pm = J_x \pm i J_y$. From Eqs. (4) and (5), we find that the SSCS satisfies

$$J_-|\eta, \theta\rangle = \eta \sqrt{\frac{1 - \cos \theta \xi^2}{1 + \cos \theta \xi^2}} (2j - \mathcal{N})|\eta, \theta + \pi\rangle$$

and

$$J_-^2|\eta, \theta\rangle = \eta^2 (2j - \mathcal{N})(2j - \mathcal{N} - 1)|\eta, \theta\rangle.$$  

Eq. (7) gives the ladder operator formalism of the SSCS $|\eta, \theta\rangle$.

A specific SSCS $|\eta, \pi/2\rangle$ can be generated in the nonlinear Hamiltonian system $H = \chi \mathcal{N}^2$ since

$$|\eta, \pi/2\rangle = \exp(-i \pi \mathcal{N}^2/2)|\eta\rangle$$

up to a trivial global phase. Next we begin our discussion on the coherence properties and spin squeezing in the SSCS.

### III. COHERENCE PROPERTIES

The coherence properties of a spin state can be characterized by the normalized second-order correlation function

$$g^{(2)} = \frac{\langle J_+ J_- \rangle}{\langle J_- \rangle^2}.$$  

Then we need to know the expectation values $\langle \mathcal{N}^k \rangle (k = 1...4)$, which can be conveniently obtained by the generation function method. The generation function of the SSCS is directly calculated as

$$G(\lambda) = \langle \mathcal{N}, \theta | \mathcal{N} | \eta, \theta \rangle = \frac{(1 + \lambda |\eta|^2) \langle \mathcal{N} \rangle + \cos \theta (1 - \lambda |\eta|^2)^{2j}}{(1 + |\eta|^2)^{2j} + \cos \theta (1 - |\eta|^2)^{2j}}.$$  

The factorial moments follow from the generation function

$$F(k) = \frac{d^k G(\lambda)}{d^k \lambda} \bigg|_{\lambda = 1} = \frac{(2j)!}{(2j - k)!} |\eta|^{2k} \times \frac{(1 + |\eta|^2)^{2j-k} - (-1)^k \cos \theta (1 - |\eta|^2)^{2j-k}}{(1 + |\eta|^2)^{2j} + \cos \theta (1 - |\eta|^2)^{2j}}.$$  

Relations between the expectation values $\langle \mathcal{N}^k \rangle$ and the factorial moments $F(k) (k = 1...4)$ are given by the equation

$$\begin{pmatrix} \langle \mathcal{N} \rangle \\ \langle \mathcal{N}^2 \rangle \\ \langle \mathcal{N}^3 \rangle \\ \langle \mathcal{N}^4 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} F(1) \\ F(2) \\ F(3) \\ F(4) \end{pmatrix}.$$  

The combination of Eqs. (8), (10), and (12) gives the expression for the second-order correlation function.

We first consider the coherent properties in the 'number state' $|n\rangle (n \neq 0)$. From Eq. (9), it is easy to obtain the second-order correlation function

$$g^{(2)} = \frac{(n-1)(2j-n+2)}{n(2j-n+1)},$$  

Then the 'number state' $|j + 1\rangle$ is always coherent as the corresponding $g^{(2)} = 1$, the states $|n\rangle (n > j + 1)$ exhibit bunching, and the states $|n\rangle (n < j + 1)$ antibunching for integer $j$. Particularly $g^{(2)} = 0$ for state $|1\rangle$, irrespective of $j$. Next we numerically calculate $g^{(2)}$ for the SSCS.

Fig. 1 gives the second-order correlation function of the SSCS for different values of $\theta$. We see that there exists a cross point for both the integer and half-integer $j$ when $|\eta| = 1$. The coherence property at this point is independent on the relative phase $\theta$. This is because the two states $|\pm \eta\rangle$ are orthogonal with each other when $|\eta| = 1$. The orthogonality makes the correlation function be independent of $\theta$. For integer $j = 3$, we observe that $g^{(2)}(0) \geq g^{(2)}(\pi/2) \geq g^{(2)}(\pi)$. Here $g^{(2)}(\theta) \equiv g^{(2)}$. However for half-integer $j = 5/2$, $g^{(2)}(0) \geq g^{(2)}(\pi/2) \geq g^{(2)}(\pi)$ when $|\eta| < 1$ and oppositely $g^{(2)}(0) \leq g^{(2)}(\pi/2) \leq g^{(2)}(\pi)$ when $|\eta| > 1$. Obviously, the state $|\eta, \pi\rangle$ exhibits high degree of antibunching. By comparing to the case of the SSCS (the corresponding correlation function is given by $g^{(2)}(\pi/2)$) the antibunching effect is enhanced. In the limit of $|\eta| \rightarrow 0$, the state $|\eta, \pi\rangle$ reduces to the number state $|1\rangle$ and the correlation function is zero, which corresponds highest degree of antibunching. In another limit $|\eta| \rightarrow \infty$ for integer $j$, the limited states of the states $|\eta, 0\rangle$, $|\eta, \pi/2\rangle$ and $|\eta, \pi\rangle$ are $|2j\rangle$, $|2j\rangle$, and $|2j-1\rangle$, respectively. Therefore the corresponding correlation functions are $1 - j/2$, $1 - j/2$, and $3(j-1)/(2j-1)$. For half-integer $j$, the limiting states are $|2j\rangle$, $|2j\rangle$ and $|2j\rangle$ and the limiting values of the corresponding correlation functions are $3(j-1)/(2j-1)$, $1 - j/2$, and $1 - j/2$. These limiting properties can also be seen from the figure.

From Eq. (10) we see that we can measure the second-order correlation function by measuring the quantities $J_\pm^2 (k = 1...4)$. The correlation function is dependent on the absolute value of $\eta$, i.e., it is phase-insensitive. However the spin squeezing discussed below is a phase-sensitive quantity.
Then we need to know expectation values such as (19). Atomic spin states (18), and atomic collisional interactions (9, 15–17), quantum nondemolition measurement of light (8). Later people develop other ways to produce spin j = 0, θ = 0. Maximum squeezing occurs at G = 0.6 and λ = 0.8, 0.2.

IV. SPIN SQUEEZING

In 1993 Kitagawa and Ueda showed that the spin squeezed state can be produced by nonlinear Hamiltonians [8]. Later people develop other ways to produce spin squeezing, such as by interaction of atoms with squeezed light [1, 3, 7], quantum nondemolition measurement of atomic spin states [18], and atomic collisional interactions [19].

In order to study spin squeezing we need to know some expectation values such as (J−) and (J2) et al. From the ladder operator formalism of the cat states [18], the expectation value (J−) is formally expressed as

\[ \langle J_- \rangle = \eta \sqrt{1 - \cos \theta \xi^2} \langle \eta, \theta | (2j - \mathcal{N}) | \eta, \theta + \pi \rangle. \] (14)

Then we need to know \( \langle \eta, \theta | \eta, \theta + \pi \rangle \) and \( \langle \eta, \mathcal{N} | \eta, \theta + \pi \rangle \) to determine \( \langle J_- \rangle \). We define a quantity \( \tilde{G}(\lambda) = \langle \eta, \theta | \mathcal{N} | \eta, \theta + \pi \rangle \), and obviously \( \tilde{G}(1) = \langle \eta, \theta | \eta, \theta + \pi \rangle \) and \( d\tilde{G}(\lambda)/d\lambda |_{\lambda=1} = \langle \eta, \theta | \mathcal{N} | \eta, \theta + \pi \rangle \). From Eq. (4), the quantity \( \tilde{G}(\lambda) \) is obtained as

\[ \tilde{G}(\lambda) = \frac{-i \sin \theta (1 - \lambda |\eta|^2)^2}{\sqrt{1 - \cos^2 \theta \xi^2} (1 + |\eta|^2)^2}, \] (15)

which directly leads to

\[ \langle \eta, \theta | \eta, \theta + \pi \rangle = \frac{-i \sin \theta \xi}{\sqrt{1 - \cos^2 \theta \xi^2}}, \]

\[ \langle \eta, \mathcal{N} | \eta, \theta + \pi \rangle = \frac{i \xi \sin \theta |\eta|^2 \xi}{\sqrt{1 - \cos^2 \theta \xi^2} (1 - |\eta|^2^2)}. \] (16)

Substituting the above equation into Eq. (14) we obtain

\[ \langle J_- \rangle = \frac{-i \xi \sin \theta \xi^2}{(1 + \cos \theta \xi^2) (1 - |\eta|^2)}. \] (17)

From Eqs. (10) and (12), the expectation value of \( J_z \) can be written in terms of the factorial moments as

\[ \langle J_z \rangle = \eta^2 \{ F(2) - 2(2j - 1) [F(1) - j] \}. \] (18)

Having know the expectation values \( \langle \mathcal{N} \rangle, \langle \mathcal{N}^2 \rangle, \langle J_- \rangle \) and \( \langle J_z \rangle \), we can know the expectation values \( J_x, J_y, J_z \) for \( \alpha = x, y, z \) through the relations

\[ J_x = \frac{J_+ - J_-}{2}, \quad J_y = \frac{J_+ + J_-}{2}, \quad J_z = \mathcal{N} - j, \]

\[ J_x = \frac{1}{4} \{ 2j(2j + 1) - 2\mathcal{N}^2 + J_+^2 + J_-^2 \}, \]

\[ J_y = \frac{1}{4} \{ 2j(2j + 1) - 2\mathcal{N}^2 - J_+^2 - J_-^2 \}, \]

\[ J_z = \mathcal{N}^2 - 2j\mathcal{N} + j^2. \] (19)

Then the squeezing parameter \( \xi_x, \xi_y \) and \( \xi_z \) are obtained. We study the spin squeezing in the even SCS \( |\eta, 0 \rangle \) and odd SCS \( |\eta, \pi \rangle \) and assume real parameter \( \eta \). From Eq. (14) we know that the expectation value \( \langle J_- \rangle = 0 \) for \( \theta = 0, \pi \), and then the expectation values \( \langle J_x \rangle = \langle J_y \rangle = 0 \). Therefore the mean spin is along the \( z \) direction. The squeezing parameters, \( \xi_x, \xi_y \), now simplify to

\[ \xi_x = \frac{2j \langle J_z \rangle^2}{\langle J_x \rangle}, \quad \xi_y = \frac{2j \langle J_z \rangle^2}{\langle J_y \rangle}. \] (20)

First we examine the spin squeezing in the even SCS. Before performing numerical calculations, we consider the case \( j = 1 \), which is the smallest \( j \) that can make spin squeezing. From Eqs. (11-12) and (18-20), we obtain

\[ \xi_x = \frac{1 + |\eta|^4}{(1 - |\eta|^2)^2}, \quad \xi_y = \frac{1 + |\eta|^4}{(1 + |\eta|^2)^2}. \] (21)

The above equation shows that the even SCS always exhibits squeezing in the \( y \) direction and no squeezing in the \( x \) direction except two extreme values of \( |\eta|, |\eta| = 0, \infty \). The maximum squeezing occurs at \( |\eta| = 1 \) and the corresponding squeezing parameter \( \xi_y = 1/2 \). These results can also be seen from Fig.2.
Fig. 2 gives a plot of inverse squeezing parameters $\xi^2_x$ and $\xi^2_y$ for different values of $j$. First we see that the even SCS is not squeezed in the $x$ direction, while in the $y$ direction, the state is squeezed. Different from the case $j = 1$, the even SCS with $j > 1$ is not always squeezed. In the initial small range of $|\eta|$, the state is squeezed and becomes not squeezed when $|\eta|$ increases a little. The worst squeezing occurs at $|\eta| = 1$ for $j > 1$, oppositely the best squeezing occurs at the same point for $j = 1$. When $|\eta|$ is large enough, the state with half-integer $j$ is not squeezed, while the state with integer $j > 1$ becomes squeezed again. In the limit of $|\eta| \to \infty$, the squeezing parameter $\xi^2_y$ is equal to 1 for integer $j$ and larger than 1 for half-integer $j$.

Now we examine the squeezing properties in the odd SCS. The numerical results are given by Fig. 3. From the figure we observe that there is no squeezing in the $x$ direction and even no squeezing in the odd SCS with integer $j$ along the $y$ direction. The spin squeezing only exist in the odd SCS with half-integer $j$. The state becomes squeezing after the parameter $|\eta|$ across a critical point $|\eta_c|$. After the critical point the state is squeezed except the limit case $|\eta| \to \infty$. As half-integer $j$ increases we see that the value of $|\eta_c|$ and the maximum value of $\xi^2_y$ increases, and the squeezing range decreases.

V. CONCLUSION

We have studied the coherence properties and spin squeezing in the SSCS. The results show that the antibunching effect can be enhanced and spin squeezing can be generated in the SSCS due to the superpositions of two SCSs. Both the coherence properties and spin squeezing depend sensitively on the parity of $2j$. Particularly the parity determines if the spin squeezing exists or not in the odd SCS.

In this paper we only consider the superpositions of two SCSs $|\pm \eta\rangle$. It will be interesting to consider the spin squeezing in the superpositions of more than two SCSs. As the spin squeezing is closely related the entanglement in multi-qubit systems, it will be helpful to understand entanglement by further investigation of spin squeezing.

ACKNOWLEDGMENTS

The author thanks for many helpful discussions with Klaus Mølmer and Anders Sørensen. This work is supported by the Information Society Technologies Programme IST-1999-11053, EQUIP, action line 6-2-1 and the European Project Q-ACTA.

[1] For a review, see V. Bužek and P. L. Knight, in Progress in Optics XXXIV, edited by E. Wolf (Elsevier, Amsterdam, 1995).
[2] C. Monroe, D. M. Meckhof, B. E. King, and D. J. Wineland, Science 272, 1131 (1996).
[3] V. V. Dodonov, S. Y. Kalmykov and V. I. Man’ko, V. I. Phys. Lett. A199, 123 (1995).
[4] J. M. Radcliffe, J. Phys. A: Gen. Phys. 4, 313 (1971); F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A 6, 2211 (1972); R. Gilmore, C. M. Bowden and L. M. Narducci, Phys. Rev. A 12, 1019 (1975); L. M. Narducci, C. M. Bowden, V. Bluemel, G. P. Garraza, and R. A. Tuft, Phys. Rev. A 11, 973 (1975); G. S. Agarwal, Phys. Rev. A 24, 2889(1981).
[5] G. S. Agarwal, R. R. Puri, and R. P. Singh, Phys. Rev. A 56, 2246 (1997).
[6] C. C. Gerry and R. Grobe, Phys. Rev. A 56, 2390 (1997).
[7] J. Recamier, O. Castanos, R. Jáuregui, and A. Frank, Phys. Rev. A 61, 063808 (2000).
[8] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[9] D. J. Wineland et al., Phys. Rev. A 46, 11 (1992); Phys. Rev. A 46, R6797 (1992); Phys. Rev. A 50, 67 (1994).
[10] D. A. Trifonov, J. Math. Phys. 34, 100 (1993); 35, 2297 (1994); Phys. Lett. A 187, 284 (1994).
[11] A. Sorensen, L.-M. Duan, J. I. Cirac and P. Zoller, Nature 409, 63 (2001).
[12] X Wang, J. Opt. B: Quantum and Semiclassical Optics 3, 93 (2001).
[13] D. Stoler, B. E. A. Saleh, and M. C. Teich, Opt. Acta 32, 345 (1985).
[14] K. Wódkiewicz, Opt. Commun. 51, 198 (1984); Phys. Rev. B 32, 4750 (1985).
[15] A. Kuzmich, K. Mølmer and E. S. Polzik, Phys. Rev. Lett. 79, 4782 (1997); A. Kozhekin, K. Mølmer, and E. S. Polzik, Phys. Rev. A 62, 033809 (2000).
[16] M. D. Lukin, S. F. Yelin, and M. Fleischhauer, Phys. Rev. Lett. 84, 4232 (2000).
[17] L. Vernac, M. Pinard, and E. Giacobino, Phys. Rev. A 62, 063812 (2000).
[18] A. Kuzmich, N. P. Bigelow, and L. Mandel, Europhys. Lett. 43, 481 (1998); A. Kuzmich, L. Mandel, and N. P. Bigelow, Phys. Rev. Lett. 85, 1594 (2000).
[19] A. Sørensen and K. Mølmer, Phys. Rev. Lett. 83, 2274 (1999); X. Wang, A. Sørensen and K. Mølmer, Phys. Rev. A, to appear.