Abstract. In this paper we study the initial boundary value problem for the system \( \text{div}(\sigma(u)\nabla \varphi) = 0, \quad u_t - \Delta u = \sigma(u)|\nabla \varphi|^2 \) in two space dimensions. This problem is also known as the thermistor problem which models the electrical heating of conductors. Our assumptions on \( \sigma(u) \) leave open the possibility that \( \liminf_{u \to \infty} \sigma(u) = 0 \), while \( \limsup_{u \to \infty} \sigma(u) \) is large. This means that \( \sigma(u) \) can oscillate wildly between 0 and a large positive number as \( u \to \infty \). Thus our singularity and degeneracy are fundamentally different from those that are present in porous medium type of equations. In spite of this, we are still able to obtain a solution \((u, \varphi)\) with \( |\nabla \varphi|, |\nabla u| \in L^\infty \). This result is a little bit surprising in view of the fact that classical regularity theory for elliptic equations like the first one in our system requires that the elliptic coefficient \( \sigma(u) \) be an \( A_2 \) weight, which implies that \( \ln \sigma(u) \) is “nearly bounded”.

1. Introduction

Electrical heating of a conductor is a ubiquitous phenomenon. A mathematical description of this is given by the so-called thermistor problem. It states that the temperature \( u \) and electrical potential \( \varphi \) of a conductor, which is represented by a bounded domain \( \Omega \) in \( \mathbb{R}^N \), are governed by the following system of partial differential equations

\begin{align*}
(1.1) \quad u_t - \Delta u &= \sigma(u)|\nabla \varphi|^2 \quad \text{in } \Omega_T, \\
(1.2) \quad \text{div}(\sigma(u)\nabla \varphi) &= 0 \quad \text{in } \Omega_T,
\end{align*}

where \( \Omega_T = \Omega \times (0, T) \) and \( T \) is any positive number. The heat source is the Joule heating \( \sigma(u)\nabla \varphi \cdot \nabla \varphi \), where \( \sigma(u) \) is the temperature-dependent electrical conductivity. We have taken the thermal conductivity to be 1. Precise assumptions on \( \sigma(u) \) will be made later. The system is coupled with the initial boundary conditions

\begin{align*}
(1.3) \quad u &= u_0 \quad \text{on } \partial_p \Omega_T, \\
(1.4) \quad \varphi &= \varphi_0 \quad \text{on } \Sigma_T,
\end{align*}

where

\begin{align*}
(1.5) \quad \Sigma_T &= \partial \Omega \times (0, T), \quad \text{the lateral boundary of } \Omega_T, \\
(1.6) \quad \partial_p \Omega_T &= \Sigma_T \cup \Omega \times \{0\}, \quad \text{the parabolic boundary of } \Omega_T.
\end{align*}

Without loss of generality, assume that

\begin{align*}
(1.7) \quad u &\geq 0 \quad \text{on } \Omega_T.
\end{align*}

This can easily be achieved by assuming

\begin{align*}
(1.8) \quad u_0 |_{\partial_p \Omega_T} &\geq 0
\end{align*}

because (1.1) satisfies the minimum principle. Our main result is:
Theorem 1.1 (Main Theorem). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with \( C^3 \) boundary \( \partial \Omega \). Assume that

(H1) the function \( \sigma \) is continuously differentiable on the interval \([0, \infty)\). Furthermore,

\[
(1.9) \quad c_0 e^{-\beta s} \leq \sigma(s) \leq c_1 \text{ on } [0, \infty) \text{ for some } c_0, c_1, \beta \in (0, \infty),
\]

\[
(1.10) \quad |\sigma'(s)| \leq c_2 e^{\gamma s} \text{ on } [0, \infty) \text{ for some } c_2, \gamma \in (0, \infty);
\]

(H2) \( \nabla u_0, \nabla \varphi_0 \in L^\infty(\Omega_T) \) and \( \Delta \varphi_0, \partial_t u_0 - \Delta u_0 \in L^\infty(0, \infty; L^2(\Omega)) \) for each \( s > 1 \).

Then there is a weak solution \((u, \varphi)\) to (1.1)-(1.4) with \( \nabla u, \nabla \varphi \in L^\infty(\Omega_T) \).

Under our assumptions, the function \( \sigma(s) \) can oscillate wildly between 0 and a positive number as \( s \to \infty \). For example, we can take \( \sigma(s) \) to be a function of the form

\[
(1.11) \quad \sigma(s) = c_3 (1 + \sin e^{\gamma s}) + c_0 e^{-\beta s}, \quad c_3 > 0.
\]

That is, we allow the possibility that

\[
(1.12) \quad \limsup_{s \to \infty} \sigma(s) = 2c_3 \quad \text{and} \quad \liminf_{s \to \infty} \sigma(s) = 0
\]

hold simultaneously. This is certainly not conducive to the regularity of solutions [17]. In fact, the classical regularity theory [6] for degenerate and/or singular elliptic equations of the type (1.2) requires that \( \sigma(u) \) be an \( A_2 \)-weight. That is, there is a positive number \( c \) such that

\[
(1.13) \quad \int_{B_r(y)} \frac{\sigma(u)}{\sigma(u)} dx \leq c \quad \text{for all } y \in \Omega, r > 0 \text{ such that } B_r(y) \subset \Omega.
\]

It was shown in [15] that \( \sigma(u) \) is an \( A_2 \) weight when \( \sigma(u) \) is roughly a constant multiple of the function \( e^{-cu} \) for some \( c > 0 \). A function \( f \) is an \( A_2 \) weight if and only if \( \ln f \) belongs to \( \text{BMO} \), which asserts that over any ball, the average oscillation of \( f \) must be bounded ([11], p.141). Thus our situation here seems to lie outside the scope of [6].

There is a large body of literature devoted to the study of (1.1)-(1.4) under various assumptions on \( \sigma(s) \). See, e.g., [17] and the references therein. In a series of three papers ([12]-[14]), the author obtained the boundedness of \( u \) under the assumptions that the given function \( \sigma(s) \) has the properties:

(C1) \( \sigma(s) \) is continuous, positive, and bounded above;
(C2) \( \lim_{s \to \infty} \sigma(s) = 0 \); and
(C3) \( \lim_{\tau \to 0^+} \frac{\sigma(s+\tau)}{\sigma(s)} = 1 \) uniformly on \([0, \infty)\).

In particular, condition (C2) is essential to the argument there. We have managed to remove this condition here, thereby allowing oscillation in \( \sigma \). A result in [14] asserts that (C3) implies that \( \sigma(u) \) is bounded below by an exponential function. Thus we have also weaken (C3) substantially.

The trade-off for us is that we have to assume that \( N = 2 \) and \( \sigma \) is continuously differentiable.

Recall that solutions to the initial boundary value problem for the equation \( u_t - \Delta u = \sigma(u) \) can blow up in finite time when \( \sigma(u) \) is superlinear, i.e.,

\[
(1.14) \quad \lim_{u \to \infty} \frac{\sigma(u)}{u} = \infty, \quad \int_0^\infty \frac{1}{\sigma(u)} du < \infty.
\]

See, for example, [3]. It would be interesting to know if we can allow \( \sigma(u) \) to be bounded above by a linear function. The difficulty we are facing here is that the exponential integrability \( u \) becomes an issue.

Our approach has two components. One is that the \( L^\infty \) norm of \( u \) can be bounded by the logarithm of the \( L^p \)-norm of the term on the right-hand side of (1.1). This idea is based upon a recent result of the author in [16]. However, we must point out that the claim about the case \( N = 2 \)
there is not quite accurate. In fact, the following problem is still open: Let $b$ be a weak solution of the problem
\begin{align}
  b_t - \Delta b &= g &\text{in } \Omega_T, \\
  b &= 0 &\text{on } \partial_p \Omega_T.
\end{align}

Is there $\alpha > 0$ such that $\sup_{0 \leq t \leq T} \int_{\Omega} e^{\alpha b} dx < \infty$ if we only assume that $g \in L^\infty(0, T; L^2(\Omega)) \equiv L^\infty(0, T; L^2(\Omega))$? Of course, the answer is yes if $N > 2$. It can also be shown that the answer is positive if we can extend $g$ so that $g \in L^\infty(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$. One just needs to apply Corollary 1.84 in [10] suitably. Unfortunately, under our assumptions it does not seem likely we can have $\sigma(u)|\nabla \varphi|^2 \in L^\infty(0, T; H^1_{\text{loc}}(\mathbb{R}^2))$. This means that obtaining the exponential integrability of $u$ via the inequality
\begin{equation}
  \int_{\Omega} \sigma(u)|\nabla \varphi|^2 v^2 dx \leq c \int_{\Omega} |\nabla v|^2 dx \quad \text{for all } v \in W^{1,2}_0(\Omega)
\end{equation}
is not available to us. We derive this using a different method. The second is that an equation for $\sigma(u)|\nabla \varphi|^2$, $j \geq 1$, can be derived based upon a method in [1, 18]. Even though this equation is both singular and degenerate we are still able to apply the De Giorgi iteration method to obtain the boundedness of solutions. Observe that condition (C3) is assumed to ensure the a priori bound for $\nabla \varphi$ in $(L^2(\Omega_T))^N$ [13]. Without this condition, any a priori bound for $\nabla \varphi$ becomes an issue. We overcome this by treating $\sigma(u)|\nabla \varphi|^2$ as a single unit.

This work is organized as follows. Section 2 is largely preparatory. We collect some relevant known results. The proof of the main theorem is contained in Section 3.

We follow the well-established notation convention whenever possible. Therefore, throughout this paper, the letter $c$ will be used to denote a positive number that depends only on the given data. The dot product of two column vectors $\mathbf{F}, \mathbf{G}$ is denoted by $\mathbf{F} \cdot \mathbf{G}$, and so on.

\section{2. Preliminaries}

In this section we collect some known results for later use. The first part deals with differentiation formulas. In these formulas capital letters represent matrix-valued functions, bold face letters are vector-valued functions, and lower case letters are scalar functions.

The following identities will be frequently used
\begin{align}
  \nabla (\mathbf{F} \cdot \mathbf{G}) &= \nabla \mathbf{F} \mathbf{G} + \mathbf{G} \nabla \mathbf{F}, \\
  \text{div} (A \mathbf{F}) &= A : \nabla \mathbf{F} + \text{div} A \mathbf{F}, \\
  \nabla (A \mathbf{F}) &= \nabla A \mathbf{F} + (A_{x_1} \mathbf{F}, A_{x_2} \mathbf{F})^T, \\
  \text{div}(uA) &= u\text{div} A + (\nabla u)^T A, \\
  \nabla |\nabla u|^2 &= 2\nabla^2 u \nabla u.
\end{align}

We also need the interpolation inequality
\begin{equation}
  \|u\|_q \leq \epsilon \|u\|_r + \epsilon^{-\mu} \|u\|_{\ell},
\end{equation}
where $1 \leq \ell \leq q \leq r$ with $\mu = \left(\frac{1}{q} - \frac{1}{r}\right) / \left(\frac{1}{\ell} - \frac{1}{q}\right)$.

The next lemma deals with sequences of nonnegative numbers which satisfy certain recursive inequalities.

\begin{lemma}
  Let $\{y_n\}, n = 0, 1, 2, \cdots$, be a sequence of positive numbers satisfying the recursive inequalities
  \begin{equation}
    y_{n+1} \leq cb^n y_n^{1+\alpha} \quad \text{for some } b > 1, c, \alpha \in (0, \infty).
  \end{equation}
\end{lemma}
If
\[ y_0 \leq c^{-1} b^{-\frac{1}{\sigma^2}}, \]
then \( \lim_{n \to \infty} y_n = 0. \)

This lemma can be found in ([4], p.12).

Let \( \varphi \) be a solution of the equation
\[ A : \nabla^2 \varphi = a_{11} \varphi_{x_1 x_1} + 2a_{12} \varphi_{x_1 x_2} + a_{22} \varphi_{x_2 x_2} = w. \]

Introduce the following functions:
\[
\begin{align*}
v &= A \nabla \varphi \cdot \nabla \varphi, \\
G &= v^{-1} \left( A_{x_1} \nabla \varphi \cdot \nabla \varphi \right), \\
A_1 &= \left( \begin{array}{cc} a_{11}(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2}) & a_{12}(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2}) \\ a_{11}(a_{12} \varphi_{x_1} + a_{22} \varphi_{x_2}) & a_{22}(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2}) \end{array} \right), \\
A_2 &= \left( \begin{array}{cc} a_{11}(a_{12} \varphi_{x_1} + a_{22} \varphi_{x_2}) & a_{22}(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2}) \\ -(a_{22}a_{11} - 2a_{12}^2) \varphi_{x_1} + a_{12}a_{22} \varphi_{x_2} & a_{22}(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2}) \end{array} \right), \\
A_3 &= \left( \begin{array}{cc} -(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2})^2 & -(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2})(a_{12} \varphi_{x_1} + a_{22} \varphi_{x_2}) \\ -(a_{11} \varphi_{x_1} + a_{12} \varphi_{x_2})(a_{12} \varphi_{x_1} + a_{22} \varphi_{x_2}) & -(a_{12} \varphi_{x_1} + a_{22} \varphi_{x_2})^2 \end{array} \right), \\
H &= \frac{1}{\det(A)v} \left( \begin{array}{c} (\nabla \varphi)^T A \end{array} \right)_{A_1} \nabla \det(A) - AG, \\
K &= AG + 2v^{-1} w A \nabla \varphi, \\
h &= \left( 2v^{-1} w A \nabla \varphi - \frac{1}{\det(A)v} \left( \begin{array}{c} (\nabla \varphi)^T A \end{array} \right)_{A_1} \nabla \det(A) + AG \right) \cdot G \\
&\quad + \frac{2w}{\det(A)v^2} A_3 \nabla \varphi \cdot \nabla \det(A).
\end{align*}
\]

Our study is based upon the following result from [18].

**Theorem 2.2.** For each \( j \geq 1 \) the function \( \psi = v^j \) satisfies the equation
\[ \text{div} \left( \frac{1}{\psi} A \nabla \psi \right) = \frac{1}{\psi} \nabla \psi \cdot \nabla \psi + jh + j\text{div}K \text{ in } \{|\nabla \varphi| > 0\}. \]

The proof of this theorem involves tons of calculations [18]. To gain some insights into this theorem, here we offer a proof in the case where
\[ A = I. \]

**Proof of (2.8) under assumption (2.9).** We begin by computing
\[ \nabla \psi = j |\nabla \varphi|^2 (j^{-1}) \nabla |\nabla \varphi|^2. \]

Thus
\[ \frac{1}{\psi} \nabla \psi = j |\nabla \varphi|^{-2} \nabla |\nabla \varphi|^2. \]

This yields
\[ \text{div} \left( \frac{1}{\psi} \nabla \psi \right) = -j |\nabla \varphi|^{-4} |\nabla |\nabla \varphi|^2|^2 + j |\nabla \varphi|^{-2} \Delta |\nabla \varphi|^2 \]
\[ \quad + j |\nabla \varphi|^{-2} (\nabla |\nabla \varphi|^2)^2 + \Delta |\nabla \varphi|^2). \]
We calculate the two terms between the preceding two parentheses. For the first term we have from (2.5) that
\[
-|\nabla \varphi|^2 |\nabla |\nabla \varphi|^2|^2 = -|\nabla \varphi|^2 |2\nabla^2 \varphi \nabla \varphi|^2 = -4|\nabla \varphi|^2 (\nabla^2 \varphi)^2 \nabla \varphi.
\]
(2.13)

Observe that
\[
(\nabla^2 \varphi)^2 = \begin{pmatrix}
\varphi_{x_1 x_1} + \varphi_{x_1 x_2}^2 & \varphi_{x_1 x_1} \varphi_{x_1 x_2} + \varphi_{x_2 x_2} \varphi_{x_1 x_2} \\
\varphi_{x_1 x_1} \varphi_{x_1 x_2} + \varphi_{x_2 x_2} \varphi_{x_1 x_2} & \varphi_{x_2 x_2} + \varphi_{x_1 x_1} \varphi_{x_2 x_2}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\varphi_{x_1 x_1} + \varphi_{x_1 x_2} - \det(\nabla^2 \varphi) & \varphi_{x_2 x_2} \Delta \varphi \\
\varphi_{x_1 x_2} \Delta \varphi & \varphi_{x_2 x_2} \Delta \varphi - \det(\nabla^2 \varphi)
\end{pmatrix}
\]
(2.14)
\[
= \Delta \varphi \nabla^2 \varphi - \det(\nabla^2 \varphi) = w \nabla^2 \varphi - \det(\nabla^2 \varphi) I.
\]

Use this in (2.13) to obtain
\[
-|\nabla \varphi|^2 |\nabla |\nabla \varphi|^2|^2 = -4|\nabla \varphi|^2 (w(\nabla \varphi)^T \nabla^2 \varphi \nabla \varphi - \det(\nabla^2 \varphi) |\nabla \varphi|^2)
\]
(2.15)
\[
= -2w|\nabla \varphi|^2 \nabla \varphi + 4\det(\nabla^2 \varphi).
\]

As for the second term we have
\[
\Delta |\nabla \varphi|^2 = \text{div}(2\nabla^2 \varphi \nabla \varphi)
\]
(2.16)
\[
= 2|\nabla^2 \varphi|^2 + 2\text{div} \nabla^2 \varphi \nabla \varphi
\]

Note that
\[
|\nabla^2 \varphi|^2 = \varphi_{x_1 x_1}^2 + 2\varphi_{x_1 x_2}^2 + \varphi_{x_2 x_2}^2
\]
\[
= \varphi_{x_1 x_1}^2 + 2(\varphi_{x_1 x_1} \varphi_{x_2 x_2} - \det(\nabla^2 \varphi)) + \varphi_{x_2 x_2}^2
\]
\[
= (\Delta \varphi)^2 - 2\det(\nabla^2 \varphi)
\]
(2.17)
\[
= w^2 - 2\det(\nabla^2 \varphi),
\]

and
\[
\text{div} \nabla^2 \varphi \nabla \varphi = (\varphi_{x_1 x_1} \varphi_{x_1 x_2} + \varphi_{x_1 x_1} \varphi_{x_1 x_2} + \varphi_{x_2 x_2} \varphi_{x_1 x_2}) \nabla \varphi
\]
\[
= (w_{x_1}, w_{x_2}) \nabla \varphi
\]
(2.18)
\[
= \text{div} (w \nabla \varphi) - w^2.
\]

Plug the preceding two formulas into (2.16) to obtain
\[
\Delta |\nabla \varphi|^2 = -4\det(\nabla^2 \varphi) + 2\text{div} (w \nabla \varphi).
\]
(2.19)

Combining this with (2.12) and (2.15) yields
\[
\text{div} \left( \frac{1}{\psi} \nabla \psi \right) = j|\nabla \varphi|^2 \left( -2w|\nabla \varphi|^2 \nabla \varphi + 2\text{div} (w \nabla \varphi) \right)
\]
(2.20)
\[
= 2j \left( -w|\nabla \varphi|^2 \nabla \varphi + |\nabla \varphi|^2 \text{div} (w \nabla \varphi) \right)
\]

We would like to remark that the preceding argument only works for the two-dimensional case. It is not possible to represent $|\nabla^2 \varphi|^2$ in terms of $\det(\nabla^2 \varphi)$ if the space dimensions are bigger than or equal to three.
3. Proof of the Main Result

In this section we first establish the exponential integrability of \( u \). This results in a logarithmic upper bound for \( \|u\|_{\infty,T} \). These results put us in a position to use a result in \([18]\) to prove the main theorem.

**Lemma 3.1.** For each \( m \in (0, \frac{1}{c_1\|\varphi_0\|_{\infty,\Omega}}) \) there is a positive number \( c \) such that

\[
(3.1) \quad \sup_{0 \leq t \leq T} \int_{\Omega} e^{mu}dx + \int_{\Omega_T} e^{mu}|\nabla u|^2 + \sigma(u)e^{mu}|\nabla \varphi|^2 \, dxdt \leq c.
\]

**Proof.** The weak maximum principle asserts that

\[
(3.2) \quad \|\varphi\|_{\infty,\Omega} \leq \|\varphi_0\|_{\infty,\Omega}.
\]

We use \( \varphi - \varphi_0 \) as a test function in \((1.2)\) to obtain

\[
(3.3) \quad \int_{\Omega} \sigma(u)|\nabla \varphi|^2 \, dx \leq \int_{\Omega} \sigma(u)|\nabla \varphi_0|^2 \, dx \leq c.
\]

For each \( L > 0 \) define

\[
(3.4) \quad \delta_L(s) = \begin{cases} 
L & \text{if } s > L, \\
\phantom{=}s & \text{if } s \in [-L, L], \\
\phantom{=}L & \text{if } s < -L.
\end{cases}
\]

Multiply through \((3.16)\) by \( \delta_L(\phi) \) and integrate the resulting equation over \( \Omega \) to get

\[
(3.5) \quad \frac{d}{dt} \int_{\Omega} \int_0^\phi \delta_L(s)dsdx + \int_{\Omega} |\nabla \delta_L(\phi)|^2 \, dx = \int_{\Omega} \sigma(u)|\nabla \varphi|^2 \delta_L(\phi) \, dx + \int_{\Omega} f \delta_L(\phi) \, dx \leq cL,
\]

from whence follows

\[
(3.6) \quad \sup_{0 \leq t \leq T} \int_{\Omega} \int_0^\phi \delta_L(s)dsdx + \int_0^t \int_{\Omega} |\nabla \delta_L(\phi)|^2 \, dx \, d\tau \leq c.
\]

Fix

\[
(3.7) \quad K \geq \|u_0\|_{\infty,\Omega_T}.
\]

For any \( C^1 \) function \( f \) on \( \mathbb{R} \) with

\[
(3.8) \quad f > 0 \quad \text{and} \quad f' > 0
\]

we use \((f(u) - f(K))^\dagger \) as a test function in \((1.1)\) to obtain

\[
(3.9) \quad \frac{d}{dt} \int_{\Omega} \int_0^u (f(s) - f(K))^\dagger dsdx + \int_{\{u \geq K\}} (f'(u)|\nabla u|^2 + \sigma(u)\varphi\nabla f'(u)\nabla u) \, dx = 0.
\]

On the other hand, use \((f(u) - f(K))^\dagger \varphi \) as a test function in \((1.1)\) to yield

\[
(3.10) \quad \int_{\{u \geq K\}} (f(u)\sigma(u)|\nabla \varphi|^2 + \sigma(u)\varphi\nabla \varphi f'(u)\nabla u) \, dx = f(K) \int_{\{u \geq K\}} \sigma(u)|\nabla \varphi|^2 \, dx \leq cf(K).
\]

Combining the preceding two equations, we arrive at

\[
(3.11) \quad \frac{d}{dt} \int_{\Omega} \int_0^u (f(s) - f(K))^\dagger dsdx + \int_{\{u \geq K\}} (f'(u)|\nabla u|^2 + f(u)\sigma(u)|\nabla \varphi|^2) \, dx
\]

\[
+ \int_{\{u \geq K\}} ((1 - \varepsilon)f'(u)|\nabla u|^2 + 2\sigma(u)\varphi\nabla \varphi f'(u)\nabla u + (1 - \varepsilon)f(u)\sigma(u)|\nabla \varphi|^2) \, dx \leq cf(K),
\]

where \( \varepsilon \in (0, 1) \). The last integrand in the above inequality is non-negative if \( f \) is so chosen that

\[
(3.12) \quad \frac{f'(u)}{f(u)} \leq \frac{(1 - \varepsilon)^2}{c_1\|\varphi_0\|^2_{\infty,\Omega}} \leq \frac{(1 - \varepsilon)^2}{\sigma(u)\varphi^2}.
\]
We take
\[ f(s) = e^{mu}. \]
For (3.12) to hold for \( \varepsilon \) sufficiently small, it is enough to take
\[ m < \frac{1}{c_1 \| \varphi u \|_{\infty, \Omega}}. \]
Use this in (3.11), integrate, and keep in mind (3.6) to derive the desired result. The proof is complete. \( \square \)

Now let
\[ \phi = u - u_0. \]
Then \( \phi \) satisfies the problem
\[ \begin{align*}
\phi_t - \Delta \phi &= \sigma(u)|\nabla \varphi|^2 + f \quad \text{in } \Omega_T, \\
\phi &= 0 \quad \text{on } \partial_p \Omega_T,
\end{align*} \]
where
\[ f = -(u_0)_t + \Delta u_0. \]

**Lemma 3.2.** Let \( \phi \) be given as in (3.16) and (3.17). For each \( \ell \in (1, 2) \) there is a positive number \( c \) such that
\[ \| \phi \|_{\infty, \Omega_T} \leq c \ln \left( \| \sigma(u)|\nabla \varphi|^2 \|_{\frac{1}{\ell-1}, \Omega_T} + 1 \right) + c. \]

**Proof.** Without loss of generality, assume
\[ \max_{\Omega_T} \phi = \| \phi \|_{\infty, \Omega_T}. \]
Otherwise, we consider \(-\phi\). Define
\[ a = e^{\varepsilon \phi}, \quad \varepsilon \in (0, 1). \]
A simple calculation shows that \( a \) satisfies the equation
\[ a_t - \Delta a + \frac{1}{a}|\nabla a|^2 = \varepsilon ag \leq a|g|, \]
where
\[ g = \sigma(u)|\nabla \varphi|^2 + f. \]
Note that
\[ a - 1 |\partial_p \Omega_T| = 0. \]
Let
\[ k \geq 2. \]
be selected as below. Set
\[ k_n = k - \frac{k}{2^{n+1}}, \quad n = 0, 1, \ldots \]
Then we have
\[ (a - k_n)^+ |\partial_p \Omega_T| = 0. \]
Use \((a - k_{n+1})^+\) as a test function in (3.22) to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} [(a - k_{n+1})^+]^2 dx + \int_{\Omega} |\nabla (a - k_{n+1})^+|^2 dx \\
\leq \int_{\Omega} |g|a(a - k_{n+1})^+ dx
\]
(3.28)
\[
\leq c\|g\|_{\ell-\Omega} \left( \int_{\Omega} (a(a - k_{n+1})^+)^\ell dx \right)^{\frac{1}{\ell}}, \quad \ell > 1.
\]
Integrate to get
\[
\max_{0 \leq t \leq T} \int_{\Omega} [(a - k_{n+1})^+]^2 dx + \int_{\Omega_T} |\nabla (a - k_{n+1})^+|^2 dx dt
\]
(3.29)
\[
\leq c\|g\|_{\ell-\Omega_T} \left( \int_{\Omega_T} (a(a - k_{n+1})^+)^\ell dx dt \right)^{\frac{1}{\ell}}.
\]
Now we further require
\[
\ell < 2.
\]
Let
\[
y_n = \int_{\Omega_T} [(a - k_n)^+]^{2\ell} dx dt.
\]
We estimate from the Sobolev embedding theorem that
\[
y_{n+1} = \int_{0}^{T} \int_{\Omega} [(a - k_{n+1})^+]^{\ell+\ell} dx dt \\
\leq \int_{0}^{T} \left( \int_{\Omega} [(a - k_{n+1})^+]^{2} dx \right)^{\frac{\ell}{2}} \left( \int_{\Omega} [(a - k_{n+1})^+]^{2\ell-2} dx \right)^{\frac{1}{2}} dt \\
\leq c \left( \max_{0 \leq t \leq T} \int_{\Omega} [(a - k_{n+1})^+]^{2} dx \right)^{\frac{\ell}{2}} \int_{0}^{T} \int_{\Omega} |\nabla (a - k_{n+1})^+|^\ell dx dt
\]
(3.32)
\[
\leq c \left( \max_{0 \leq t \leq T} \int_{\Omega} [(a - k_{n+1})^+]^{2} dx \right)^{\frac{\ell}{2}} \left( \int_{\Omega_T} |\nabla (a - k_{n+1})^+|^2 dx dt \right)^{\frac{1}{2}} |A_{n+1}|^{1-\frac{\ell}{2}},
\]
where
\[
A_{n+1} = \{a \geq k_{n+1}\}.
\]
This combined with (3.28) gives
\[
y_{n+1} \leq c\|g\|_{\ell-\Omega_T} \int_{\Omega_T} (a(a - k_{n+1})^+)^\ell dx dt |A_{n+1}|^{1-\frac{\ell}{2}}
\]
(3.34)
On the other hand, we have
\[
y_n \geq \int_{A_{n+1}} [(a - k_n)^+]^{2\ell} dx dt \\
\geq \int_{A_{n+1}} a^\ell [(a - k_n)^+]^\ell \left( 1 - \frac{k_n}{a} \right)^\ell dx dt \\
\geq \int_{A_{n+1}} a^\ell [(a - k_n)^+]^\ell \left( 1 - \frac{k_n}{k_{n+1}} \right)^\ell dx dt
\]
(3.35)
\[
\geq \frac{1}{2(n+2)^{\ell}} \int_{A_{n+1}} a^\ell [(a - k_n)^+]^\ell dx dt.
\]
Furthermore,

\begin{equation}
(3.36) \quad y_n \geq (k_{n+1} - k_n)^2 |A_{n+1}| = \frac{k_n^{2\ell}}{2^{(n+2)\ell}} |A_{n+1}|.
\end{equation}

Finally, we arrive at

\begin{equation}
(3.37) \quad y_{n+1} \leq c2^{(n+2)\ell} \|g\|_{L_T^\ell, \Omega_T} |A_{n+1}|^{1 - \frac{k}{2}} \|g\|_{L_T^\ell, \Omega_T} y_n^{1 + \frac{(2 - \ell)}{2}}.
\end{equation}

Thus if we take \(k\) so that

\begin{equation}
(3.38) \quad y_0 \leq c \left( \frac{k^{(2 - \ell)}}{\|g\|_{L_T^\ell, \Omega_T}} \right)^{\frac{2}{(2 - \ell)}},
\end{equation}

then

\begin{equation}
(3.39) \quad a \leq k.
\end{equation}

This together with (3.25) implies

\begin{equation}
(3.40) \quad a \leq cy_0^{\frac{1}{2}} \|g\|_{L_T^\ell, \Omega_T} + 2e\|u_0\|_{\infty, \Omega_T}.
\end{equation}

Choose \(\epsilon\) suitably small so that

\begin{equation}
(3.41) \quad 2\ell \epsilon < \frac{1}{c_1 \|\varphi_0\|_{\infty, \Omega}}.
\end{equation}

By Lemma 3.1, we have

\begin{equation}
(3.42) \quad y_0 \leq \int_\Omega e^{2\ell \epsilon u} dx \leq c.
\end{equation}

Note from (3.18) that

\begin{equation}
(3.43) \quad \|g\|_{L_T^\ell, \Omega_T} \leq \|\sigma(u)|\nabla \varphi|^2\|_{L_T^\ell, \Omega_T} + \|\varphi_0 + (u_0)t + \Delta u_0\|_{L^1, \Omega_T}.
\end{equation}

This concludes the proof. \(\square\)

**Lemma 3.3.** We have

\begin{equation}
(3.44) \quad \|\nabla u\|_{\infty, \Omega_T} \leq c\|\sigma(u)|\nabla \varphi|^2\|_{\infty, \Omega_T} + c.
\end{equation}

**Proof.** Define

\begin{equation}
(3.45) \quad g(x, t) = \begin{cases} 
\sigma(u)|\nabla \varphi|^2 & \text{if } (x, t) \in \Omega_T, \\
0 & \text{if } (x, t) \text{ lies outside } \Omega_T.
\end{cases}
\end{equation}

Consider the function

\begin{equation}
(3.46) \quad G = \frac{1}{4\pi} \int_0^t \frac{1}{t - \tau} \int_{\mathbb{R}^2} \exp \left( -\frac{|x - y|^2}{4(t - \tau)} \right) g(y, \tau) dy d\tau.
\end{equation}

We see from ([7], Chapter IV) that \(G\) satisfies

\begin{equation}
(3.47) \quad G_t - \Delta G = g \quad \text{in } \mathbb{R}^2 \times (0, \infty),
\end{equation}

\begin{equation}
(3.48) \quad G(x, 0) = 0 \quad \text{on } \mathbb{R}^2.
\end{equation}

Furthermore, for each \(s > 1\) there is a positive number \(c\) such that

\begin{equation}
(3.49) \quad \|G_t\|_{s, \Omega_T} + \|G\|_{L^s(0, T; W^{2,s}(\Omega))} \leq c\|g\|_{s, \Omega_T}.
\end{equation}
Set
\[ l = \frac{|x - y|}{2\sqrt{t - \tau}}. \]  

For each \( \delta \in (2, 3) \) we estimate
\[
|\nabla G| = \left| \frac{1}{16\pi} \int_0^t \frac{1}{(t - \tau)^2} \int_{\mathbb{R}^2} (x - y) \exp \left( -l^2 \right) g(y, \tau) dy d\tau \right| 
\leq c \int_0^t \frac{1}{(t - \tau)^2} \int_{\mathbb{R}^2} \frac{(2\alpha \sqrt{t - \tau})^\delta}{|x - y|^{\delta - 1}} l^\delta \exp \left( -l^2 \right) |g(y, \tau)| dy d\tau 
\leq c \|\sigma(u) \nabla \varphi\|^2 \|\infty, \Omega_T \int_0^t \frac{1}{(t - \tau)^{2 - \frac{\delta}{2}}} \int_{\mathbb{R}^2} \frac{\chi_{\Omega_T}}{|x - y|^{\delta - 1}} dy d\tau 
\leq c \|\sigma(u) |\nabla \varphi|^2\|_{\infty, \Omega_T}. \]

Obviously, \( F \equiv u - G \) satisfies the problem
\[
(3.52) \quad F_t - \Delta F = 0 \text{ in } \Omega_T, \\
(3.53) \quad F = u_0 - G \text{ on } \partial_p \Omega_T. 
\]

We can easily conclude from (3.49) and the classical regularity theory for the heat equation ([7], Chapter IV) that \( \|\nabla F\|_{\infty, \Omega_T} \leq c \|\sigma(u) |\nabla \varphi|^2\|_{\infty, \Omega_T} + c. \) Hence we have
\[
(3.54) \quad \|\nabla u\|_{\infty, \Omega_T} \leq c \|\sigma(u) |\nabla \varphi|^2\|_{\infty, \Omega_T} + c. 
\]

We derive from (1.2) that
\[
(3.55) \quad \Delta \varphi = -\frac{\sigma'(u)}{\sigma(u)} \nabla u \cdot \nabla \varphi \equiv w. 
\]

Thus if we take \( A = I, v = |\nabla \varphi|^2 \) then equation for \( \psi = v^j \) becomes
\[
(3.56) \quad \text{div} \left( \frac{1}{\psi} \nabla \psi \right) = 2j \text{div} (|\nabla \varphi|^{-2} w \nabla \varphi) = -2j \text{div} \left( |\nabla \varphi|^{-2} \frac{\sigma'(u)}{\sigma(u)} \nabla u \cdot \nabla \varphi \nabla \varphi \right). 
\]

Unfortunately, we cannot take this path. This is due to the fact that we are not able to prove \( \sup_{0 \leq t \leq T} \int_{\Omega} |\nabla \varphi|^2 dx \leq c \) because we allow \( \sigma(u) \) to oscillate between 0 and \( \infty \). We are forced to work with (1.2) and take
\[
(3.57) \quad A = \sigma(u)I, \quad w = 0, \quad v = \sigma(u) |\nabla \varphi|^2. 
\]
Subsequently,

\[
A_1 = \begin{pmatrix}
    a_{11}(a_{11}\varphi_{x_1} + a_{12}\varphi_{x_2}) & a_{12}(a_{11}\varphi_{x_1} + a_{12}\varphi_{x_2}) \\
    a_{11}(a_{12}\varphi_{x_1} + a_{22}\varphi_{x_2}) & a_{22}(a_{11}\varphi_{x_1} + a_{12}\varphi_{x_2})
\end{pmatrix},
\]

\[
= \begin{pmatrix}
    \sigma^2(u)\varphi_{x_1} & 0 \\
    \sigma^2(u)\varphi_{x_2} & \sigma^2(u)\varphi_{x_1}
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
    a_{11}(a_{12}\varphi_{x_1} + a_{22}\varphi_{x_2}) & a_{22}(a_{11}\varphi_{x_1} + a_{12}\varphi_{x_2}) \\
    -(a_{12}a_{11} - 2a_{12}^2)\varphi_{x_1} + a_{12}a_{22}\varphi_{x_2} & a_{12}(a_{11}\varphi_{x_1} + a_{12}\varphi_{x_2})
\end{pmatrix},
\]

\[
= \begin{pmatrix}
    \sigma^2(u)\varphi_{x_2} & \sigma^2(u)\varphi_{x_1} \\
    -\sigma^2(u)\varphi_{x_1} & \sigma^2(u)\varphi_{x_2}
\end{pmatrix},
\]

\[
(\nabla \varphi)^T A_1 = (\nabla \varphi)^T \begin{pmatrix}
    \sigma^2(u)\varphi_{x_1} & 0 \\
    \sigma^2(u)\varphi_{x_2} & \sigma^2(u)\varphi_{x_1}
\end{pmatrix} = \sigma^2(u) |\nabla \varphi|^2 \varphi_{x_2}\varphi_{x_1},
\]

\[
(\nabla \varphi)^T A_2 = (\nabla \varphi)^T \begin{pmatrix}
    \sigma^2(u)\varphi_{x_2} & -\sigma^2(u)\varphi_{x_1} \\
    \sigma^2(u)\varphi_{x_1} & \sigma^2(u)\varphi_{x_2}
\end{pmatrix} = \sigma^2(u) |\nabla \varphi|^2.
\]

\[
G = v^{-1} \begin{pmatrix}
    A_{x_1} \nabla \varphi \cdot \nabla \varphi \\
    A_{x_2} \nabla \varphi \cdot \nabla \varphi
\end{pmatrix} = \frac{1}{\sigma(u)} \nabla \sigma(u),
\]

\[
H = \frac{1}{\det(A)^v} \begin{pmatrix}
    (\nabla p)^T A_1 \\
    (\nabla p)^T A_2
\end{pmatrix} \nabla \det(A) - AG,
\]

\[
= \frac{2}{|\nabla \varphi|^2} \begin{pmatrix}
    |\nabla \varphi|^2 \varphi_{x_1}\varphi_{x_2} \\
    0 \quad |\nabla \varphi|^2
\end{pmatrix} \nabla \sigma(u) - \nabla \sigma(u)
\]

\[
K = AG + 2v^{-1}wA\nabla \varphi = \nabla \sigma(u),
\]

\[
h = -\frac{2}{\sigma(u)|\nabla \varphi|^2} \begin{pmatrix}
    |\nabla \varphi|^2 & 0 \\
    \varphi_{x_1}\varphi_{x_2} & |\nabla \varphi|^2
\end{pmatrix} \nabla \sigma(u) \cdot \nabla \sigma(u) + \frac{1}{\sigma(u)} |\nabla \sigma(u)|^2.
\]

Equation (2.8) becomes

\[
(3.58) \quad \text{div} \left( \frac{\sigma(u)}{\psi} \nabla \psi \right) = \frac{1}{\psi} H \cdot \nabla \psi + jh + j \text{div} K \quad \text{in} \quad \{|\nabla \varphi| > 0\}.
\]

We are ready to prove the main theorem.

**Proof of the main theorem.** Fix a point \(x_0 \in \Omega\). Then pick a number \(R\) from \((0, \text{dist}(x_0, \partial \Omega))\). Define a sequence of concentric balls \(B_{R_n}(x_0)\) in \(\Omega\) as follows:

\[
(3.59) \quad B_{R_n}(x_0) = \{x : |x - x_0| < R_n\},
\]

where

\[
(3.60) \quad R_n = \frac{R}{2} + \frac{R}{2^{n+1}}, \quad n = 0, 1, 2, \ldots.
\]

Choose a sequence of smooth functions \(\theta_n\) so that

\[
\theta_n(x) = 1 \quad \text{in} \quad B_{R_n}(x_0),
\]

\[
\theta_n(x) = 0 \quad \text{outside} \quad B_{R_{n-1}}(x_0),
\]

\[
|\nabla \theta_n(x)| \leq \frac{c2^n}{R} \quad \text{for each} \quad x \in \mathbb{R}^2, \quad \text{and}
\]

\[
0 \leq \theta_n(x) \leq 1 \quad \text{in} \quad \mathbb{R}^2.
\]

Select

\[
(3.61) \quad K \geq 2
\]
as below. Set

$$K_n = K - \frac{K}{2^{n+1}}, \quad n = 0, 1, 2, \ldots.$$  \hspace{1cm} (3.62)

Hence,

$$K_n \geq 1 \quad \text{for each } n.$$  \hspace{1cm} (3.63)

Note that (3.58) holds on the set \{\psi > 1\}. We use \(\theta_{n+1}^2(\psi - K_{n+1})^+\) as a test function in (3.58) to obtain

\[
\int_{\Omega} \frac{1}{\psi} \sigma(u) \nabla \psi \cdot \nabla (\psi - K_{n+1})^+ \theta_{n+1}^2 dx \\
= -2 \int_{\Omega} \frac{1}{\psi} \sigma(u) \nabla \psi \cdot \nabla \theta_{n+1}(\psi - K_{n+1})^+ \theta_{n+1} dx \\
- \int_{\Omega} \frac{1}{\psi} H \nabla \psi \theta_{n+1}^2 (\psi - K_{n+1})^+ dx - j \int_{\Omega} h \theta_{n+1}^2 (\psi - K_{n+1})^+ dx \\
+ j \int_{\Omega} K \cdot \nabla (\psi - K_{n+1})^+ \theta_{n+1}^2 dx \\
+ 2j \int_{\Omega} K \cdot \nabla \theta_{n+1}(\psi - K_{n+1})^+ \theta_{n+1} dx.
\]  \hspace{1cm} (3.64)

Note that

\[
\nabla \psi = \nabla (\psi - K_{n+1})^+ \quad \text{on } S_{n+1}(t),
\]

where

$$S_{n+1}(t) = \{x \in B_n(x_0) : \psi(x, t) \geq K_{n+1}\}.$$  \hspace{1cm} (3.66)

We obtain from (3.64) that

\[
\int_{\Omega} \frac{\sigma(u)}{\psi} |\nabla (\psi - K_{n+1})^+|^2 \theta_{n+1}^2 dx \\
\leq \frac{c^4}{R^2} \int_{S_{n+1}(t)} \frac{\sigma(u)}{\psi} \left[(\psi - K_{n+1})^+\right]^2 dx \\
+ \int_{\Omega} \frac{c}{\psi \sigma(u)} |H|^2 \theta_{n+1}^2 \left[(\psi - K_{n+1})^+\right]^2 dx + j \int_{\Omega} |h| \theta_{n+1}^2 (\psi - K_{n+1})^+ dx \\
+ \int_{S_{n+1}(t)} c^2 \frac{\psi}{\sigma(u)} |K|^2 \theta_{n+1}^2 dx + \frac{c^2}{R} \int_{\Omega} |K|(\psi - K_{n+1})^+ \theta_{n+1} dx.
\]  \hspace{1cm} (3.67)

The last term in (3.67) can be estimated as follows:

\[
\frac{2^n}{R} \int_{\Omega} |K|(\psi - K_{n+1})^+ \theta_{n+1} dx \leq \frac{c^4}{R^2} \int_{S_{n+1}(t)} \frac{\sigma(u)}{\psi} \left[(\psi - K_{n+1})^+\right]^2 dx \\
+ \int_{S_{n+1}(t)} c^2 \frac{\psi}{\sigma(u)} |K|^2 \theta_{n+1}^2 dx.
\]  \hspace{1cm} (3.68)
Observe that
\[ \frac{1}{\psi} |\nabla (\psi - K_{n+1})^+|^2 = 4|\nabla (\sqrt{\psi} - \sqrt{K_{n+1}})^+|^2, \]
\[ \frac{1}{\psi} [(\psi - K_{n+1})^+]^2 = \frac{1}{\psi} \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 \left( \sqrt{\psi} + \sqrt{K_{n+1}} \right)^2 
= \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 \left( 1 + \frac{\sqrt{K_{n+1}}}{\sqrt{\psi}} \right)^2 \]
(3.69)\]
\[ \leq 4 \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2. \]
(3.70)

Notice that
\[ \sqrt{K_{n+1}} - \sqrt{K_n} \sqrt{K_{n+1}} = \sqrt{1 - \frac{1}{2^{n+2}}} - \sqrt{1 - \frac{1}{2^{n+1}}} \]
\[ \leq \frac{1}{2^{n+2}} \left( \sqrt{1 - \frac{1}{2^{n+2}}} + \sqrt{1 - \frac{1}{2^{n+1}}} \right) \sqrt{1 - \frac{1}{2^{n+2}}} \]
(3.71)
\[ \geq \frac{1}{2^{n+3}}. \]

With this in mind, we estimate
\[ \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \geq \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \chi_{S_{n+1}(t)} 
= \frac{1}{2} (\sqrt{\psi} - \sqrt{K_n})^+ (\sqrt{\psi} + \sqrt{\psi}) \left( 1 - \frac{\sqrt{K_n}}{\sqrt{\psi}} \right) \chi_{S_{n+1}(t)} 
\geq \frac{1}{2} (\sqrt{\psi} - \sqrt{K_n})^+ (\sqrt{\psi} + \sqrt{K_{n+1}}) \left( 1 - \frac{\sqrt{K_n}}{\sqrt{K_{n+1}}} \right) \chi_{S_{n+1}(t)} \]
(3.72)
\[ \geq \frac{1}{2^{n+4}} (\psi - K_{n+1})^+. \]

Here \( \chi_{S_{n+1}(t)} \) is the indicator function of the set \( S_{n+1}(t) \). Similarly,
\[ \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \geq \psi \left[ \left( 1 - \frac{\sqrt{K_n}}{\sqrt{\psi}} \right)^+ \right]^2 \chi_{S_{n+1}(t)} \geq \frac{1}{2^{2(n+3)}} \psi \chi_{S_{n+1}(t)}. \]
(3.73)

Plugging the preceding results into (3.67), we obtain
\[ \int_{\Omega} \sigma(u)|\nabla (\sqrt{\psi} - \sqrt{K_{n+1}})^+|^2 \theta_{n+1}^2 dx \]
\[ \leq \frac{c4^n}{R^2} \int_{S_{n+1}(t)} \sigma(u) \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 dx 
+ c \int_{\Omega} \frac{||H||^2}{\sigma(u)} \left[ (\sqrt{\psi} - \sqrt{K_{n+1}})^+ \right]^2 \theta_{n+1}^2 dx + c2^n \int_{\Omega} |h| \theta_{n+1}^2 \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 dx 
(3.74) \]
\[ + c2^{2n} \int_{S_{n+1}(t)} \frac{||K||^2}{\sigma(u)} \left[ (\sqrt{\psi} - \sqrt{K_n})^+ \right]^2 \theta_{n+1}^2 dx. \]
We pick a number \( r \) from the interval \((1, \infty)\). Define
\[
(3.75) \quad y_n = \left( \int_{B_{R_n}(x_0)} \left( \sqrt{\psi} - \sqrt{K_n} \right)^{2r} \, dx \right)^{\frac{1}{r}}.
\]
We conclude from (3.74) that
\[
\int_{\Omega} \sigma(u) |\nabla (\sqrt{\psi} - \sqrt{K_{n+1}})^{+} \cdot \theta_{n+1}^2 \, dx \\
\leq \frac{c4^n}{R^2} \|\sigma(u)\|_{\tau^{-1}, S_{1}(t)} y_n + c \frac{\|H\|^2}{\|\sigma(u)\|} y_n + c2^n \|h\|_{\tau^{-1}, S_{1}(t)} y_n \\
+ c2^n \frac{\|K\|^2}{\|\sigma(u)\|} y_n
\]
(3.76) \leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n,
where
\[
\Gamma_{S_{1}(t)} = \|\sigma(u)\|_{\tau^{-1}, S_{1}(t)} + \frac{\|H\|^2}{\|\sigma(u)\|} + \frac{\|h\|_{\tau^{-1}, S_{1}(t)}}{\|\sigma(u)\|} + \frac{\|K\|^2}{\|\sigma(u)\|}.
\]
By Poincaré’s inequality, we have
\[
y_{n+1} \leq \left( \int_{\Omega} \left( \sqrt{\psi} - \sqrt{K_{n+1}}^{+} \theta_{n+1}^{2r} \right) \, dx \right)^{\frac{1}{r}} \\
\leq c \left( \int_{\Omega} [\sigma(u)]^{-\frac{1}{r}} \, dx \right)^{\frac{1}{r}} \left( \int_{\Omega} \left| \nabla \left( \sqrt{\psi} - \sqrt{K_{n+1}}^{+} \theta_{n+1}^{2r} \right) \right|^2 \, dx \right)^{\frac{1}{r}} \\
\leq c \left( \int_{\Omega} \left| \nabla \left( \sqrt{\psi} - \sqrt{K_{n+1}}^{+} \theta_{n+1}^{2r} \right) \right|^2 \, dx \right)^{\frac{1}{r}} \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq c \left( \int_{\Omega} \left| \nabla \left( \sqrt{\psi} - \sqrt{K_{n+1}}^{+} \theta_{n+1}^{2r} \right) \right|^2 \, dx \right)^{\frac{1}{r}} \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
+ \frac{c4^n}{R^2} \left( \int_{B_{R_n}(x_0)} \left| \nabla \left( \sqrt{\psi} - \sqrt{K_{n+1}}^{+} \theta_{n+1}^{2r} \right) \right|^2 \, dx \right)^{\frac{1}{r}} \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}} \\
\leq \frac{c4^n}{R^2} \Gamma_{S_{1}(t)} y_n \left( \int_{S_{n+1}(t)} [\sigma(u)]^{-r} \, dx \right)^{\frac{1}{r}}
\]
(3.78) for each \( s > r \).
We easily see that
\[
y_n \geq \left( \int_{S_{n+1}(t)} (\sqrt{K_{n+1}} - \sqrt{K_n})^2 \, dx \right)^{\frac{1}{r}} \Rightarrow \frac{K}{2^n(s+1)} \left| S_{n+1}(t) \right|^{\frac{1}{r}}.
\]
(3.79)
Substituting this into (3.78) yields

\begin{equation}
 y_{n+1} \leq \frac{cb^n}{R^2 K^{r+\frac{1}{r}}} \left( \int_{S_1(t)} [\sigma(u)]^{-s} \right)^{\frac{1}{s}} \Gamma_{S_1(t)} y_n^{1+\frac{s-r}{s}},
\end{equation}

where \( b = \left[ \max\{4, 4^{\frac{1}{s}}\} \right]^2 \). In view of Lemma 2.1 and (3.61), it is enough for us to take

\begin{equation}
 K = \frac{c}{R^{2r}} y_0 \Gamma^{\frac{s}{s-r}}_{S_1(t)} \left( \int_{S_1(t)} [\sigma(u)]^{-s} \right)^{\frac{1}{s}} + 2
\end{equation}

to obtain

\begin{equation}
 \sup_{B_R(x_0)} \psi \leq K.
\end{equation}

Note that

\begin{equation}
 y_0 = \left( \int_{B_R(x_0)} \left[ \left( \sqrt{\psi} - \sqrt{\frac{K}{2}} \right)^{2r} + 2^r \right] dx \right)^{\frac{1}{r}} \leq \|v\|_{j, B_R(x_0)}^j.
\end{equation}

Collecting the preceding estimates in (3.82) and taking the \( j \)th root of the resulting inequality, we arrive at

\begin{equation}
 \sup_{B_R(x_0)} v \leq \frac{c}{R^{2r}} \|v\|_{j, B_R(x_0)}^{\frac{s}{s-r}} \left( \int_{S_1(t)} [\sigma(u)]^{-s} \right)^{\frac{1}{s}} + c.
\end{equation}

By an argument in ([5], p. 303), we can extend the above estimate to the whole \( \Omega \). That is, we have

\begin{equation}
 \sup_\Omega v \leq c\|v\|_{j, \Omega}^{\frac{s}{s-r}} \left( \int_{\Omega} [\sigma(u)]^{-s} \right)^{\frac{1}{s}} + c,
\end{equation}

where \( \Omega_1 = \{ v \geq 1 \} \). The idea here is that one can turn a boundary point into an interior point by introducing a suitable change of variables. A description on how this is done can also be found in [13]. However, when we do boundary estimates, \( w \) in (3.57) will not be zero. As a result, \( K \) and \( h \) in (2.8) will have more terms. But they are of the same type as those already there, and so the same argument still carries through.

On account of (2.6) and (3.3), we have

\begin{equation}
 \|v\|_{j, \Omega} \leq \varepsilon \|v\|_{\infty, \Omega} + \frac{1}{\varepsilon^{1-r}} \|v\|_{1, \Omega} + \frac{c}{\varepsilon^{1-r}}, \quad \varepsilon > 0.
\end{equation}

By choosing \( \varepsilon \) suitably, we can derive from (3.85) that

\begin{equation}
 \|v\|_{\infty, \Omega} \leq c \Gamma^{\frac{s}{s-r}}_{\Omega_1} \left( \int_{\Omega_1} [\sigma(u)]^{-s} \right)^{\frac{1}{s}} + c.
\end{equation}

By (H1), we have

\begin{equation}
 \|\sigma(u)\|_{\frac{r}{r-1}, \Omega_1} \leq c.
\end{equation}
It is easy to see from (H1) that
\[\chi_{\Omega_1}|\mathbf{H}| \leq c|\sigma'(u)\nabla u| \leq ce^{\gamma u}|\nabla u|,
\chi_{\Omega_1}|\mathbf{K}| \leq ce^{\gamma u}|\nabla u|,
\chi_{\Omega_1}|h| \leq ce^{(2\gamma+\beta)u}|\nabla u|^2.\]

We estimate from (H1), (3.19), and Lemma 3.3 that
\[\Gamma_{\Omega_1} = \|\sigma(u)\|_{r^{-1},\Omega_1} + \left\|\frac{|\mathbf{H}|^2}{\sigma(u)}\right\|_{r^{-1},\Omega_1} + \left\|h\right\|_{r^{-1},\Omega_1} + \left\|\frac{|\mathbf{K}|^2}{\sigma(u)}\right\|_{r^{-1},\Omega_1}
\leq c + ce^{(2\gamma+\beta)u}|\nabla u|_{\infty,\Omega_1}^2 \|
\leq c + ce^{(2\gamma+\beta)u}\|u\|_{\infty,\Omega_T}^2 \|
\leq c\|v\|_{\infty,\Omega_T}^c + c.
\]

Similarly,
\[\int_{\Omega} [\sigma(u)]^{-s} dx \leq ce^{\beta s}\|u\|_{\infty,\Omega_T} \leq c\|v\|_{\infty,\Omega_T}^c + c.
\]

Substitute the preceding two into (3.87) to obtain
\[\|v\|_{\infty,\Omega} \leq c\left(\frac{\|v\|_{\infty,\Omega}^c + 1}{j} \right)^{1 + \frac{1}{jr}} + c.
\]

Pick \(j\) so large that the exponent in (3.91)
\[\frac{2c}{j} \left(1 + \frac{1}{jr - 1}\right) < 1.
\]

This implies that
\[\|v\|_{\infty,\Omega} \leq c.
\]

This together with Lemma (3.3) yields
\[\|\nabla u\|_{\infty,\Omega} \leq c.
\]

from whence follows that \(u\) is bounded. This gives
\[\|\nabla \varphi\|_{\infty,\Omega} \leq c.
\]

This completes the proof of the main theorem. \(\square\)

It is important to note that our argument has worked because we can choose \(j\) big enough. Obviously, we can also deduce higher regularity for solutions from the main theorem. We shall not elaborate.

Finally, we remark that what we are doing here is to turn qualitative assumption into quantitative estimates. That is, we assume that our problem has a “classical” solution. Then the solution must be bounded by given data in a certain sense. But this issue can be addressed by constructing a sequence of smooth approximate solutions. This can be done easily in our context. For example, we can approximate \(\sigma\) by bounded and twice differentiable functions suitably and \(\varphi_0\) by functions in \(W^{3,2}(\Omega)\). We shall omit the details.
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