Asymptotics for the solutions of elliptic systems with fast oscillating coefficients

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Abstract

We consider a singularly perturbed second order elliptic system in the whole space. The coefficients of the systems fast oscillate and depend both of slow and fast variables. We obtain the homogenized operator and in the uniform norm sense we construct the leading terms of the asymptotics expansion for the resolvent of the operator described by the system. The convergence of the spectrum is established. The examples are given.

Introduction

There are many works devoted to the homogenization of the differential operators in bounded domains with fast oscillating coefficients (see, for instance [1]–[6]). Similar questions for the operators in unbounded domains are studied essentially less. At the same time, during last years the case of an unbounded domain is studied intensively. In the series of papers [7]–[15] M.Sh. Birman and T.A. Suslina developed a new original technique which allowed them to prove the convergence theorem, to obtain the precise in order estimates for the rates of convergence, and to construct the first terms in the expansion for the resolvent of a wide class of differential operators in unbounded domains with fast oscillating coefficients. It should be stressed that these results were obtained in the uniform norm sense, while usually the results for the bounded domains were formulated in the sense of strong or weak convergence. The approach of M.Sh. Birman and T.A. Suslina is based on the spectral theory and treats the homogenization as a threshold phenomenon. It is applicable to the operators those can be factorized, and at the same time their coefficients must depend of the fast variable $x/\varepsilon$ only; the dependence of the slow variable $x$ is not allowed. We should also note the paper of V.V. Zhikov [16], where by employing another technique he obtained the precise in order estimates
for the rate of convergence for the resolvent of a scalar operator as well as for the operator of the elasticity theory. It was assumed that the coefficients are periodic and depend of the fast variable only, too.

The one-dimensional scalar operators with the coefficients depending both of fast and slow variables were studied in [17]–[19]. In [17] the Schrödinger operator with fast oscillating compactly supported potential was considered. The object of the study was the phenomenon of new eigenvalue emerging from the threshold of the continuous spectrum. The paper [18] deals with a periodic operator (independent of the small parameter) perturbed by a fast oscillating compactly supported potential with increasing amplitude. Here the structure and the behavior of the spectrum were studied in details. In [19] they studied the Schrödinger operator with a compactly supported potential independent of the small parameter; the perturbation was a fast oscillating periodic potential. The asymptotic behavior of the spectrum was described. We note that the homogenization of the resolvent was not considered in [17]–[19]. At the same time, the technique employed is sufficient to study this question and to obtain the results analogous to [8].

In the present paper we consider a quite general second order elliptic system in the whole space. The first difference to the operators considered in [7]–[15] is the presence of the lower order terms. More precisely, the second order part of our operator is written in the divergence form similar to the cited papers. The lower terms are introduced quite arbitrarily; the only restriction is that the operator is self-adjoint and lower-semibounded uniformly in the small parameter. The certain smoothness for the coefficients is also assumed. One more difference to the [7]–[16] is that in our case the coefficients depend both on slow and fast variables. The dependence of fast variables is periodic. The coefficients are supposed to be uniformly bounded w.r.t. to slow variables; the same is supposed for certain derivatives of the coefficients.

In the work we construct the homogenized operator and obtain the first terms of the asymptotic expansion for the resolvent of the perturbed operator for all values of the spectral parameter separated apriori from the spectrum of the homogenized operator. These asymptotics are obtained for the resolvent treated as an operator in $L_2$ as well as an operator from $L_2$ into $W_2^1$. We borrow the main ideas of [16] to obtain these results. Moreover, we assume the coefficients to be smoother than in [7]–[16] that allows us to simplify certain details in the arguments. In particular, it allows us to avoid the smoothing used in the cited papers. In the end we give examples of some operators to which our results can be applied.
1 Formulation of the problem and the main results

Let $x = (x_1, \ldots, x_d)$ be the Cartesian coordinates in $\mathbb{R}^d$, $d \geq 1$, $B = B(\zeta)$ be a matrix-valued function,

$$B(\zeta) = \sum_{i=1}^{d} B_i \zeta_i,$$

where $\zeta = (\zeta_1, \ldots, \zeta_d)$, $B_i$ are constant complex-valued matrices of the size $m \times n$, and $m \geq n$. Hereafter we assume that rank $B(\zeta) = n$, $\zeta \neq 0$.

Let $Y$ be a Banach space. By $W^k_{\infty}(\mathbb{R}^d; Y)$ we denote the Sobolev space of the functions defined on $\mathbb{R}^d$ and having values in $Y$, and so that

$$\|u\|_{W^k_{\infty}(\mathbb{R}^d; Y)} := \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} \left\| \frac{\partial^{\alpha}u}{\partial x^\alpha} \right\|_Y < \infty.$$

If $k = 0$, we will employ the notation $L^\infty_{\infty}(\mathbb{R}^d; Y)$.

In the space $\mathbb{R}^d$ we select a lattice; its elementary cell is indicated by $\square$. We will employ the symbol $C^\gamma_{\text{per}}(\square)$ to denote the space of $\square$-periodic functions having finite Hölder norm $\| \cdot \|_{C^\gamma(\square)}$. The norm in this space coincides with the norm of $C^\gamma(\square)$.

We will often treat a $\square$-periodic w.r.t. $\xi$ vector-function $f = f(x, \xi)$ as mapping points $x \in \mathbb{R}^d$ into the function depending of $\xi$. The map is defined as $x \mapsto f(x, \cdot)$. It will allow us to speak about the belonging of the function $f(x, \xi)$ to the spaces $W^k_{\infty}(Q; C^\gamma_{\text{per}}(\square))$ and $W^k_{\infty}(Q; C^\gamma_{\text{per}}(\square))$.

Let $A = A(x, \xi)$ be a matrix-valued function of the size $m \times m$. We suppose that the matrix $A$ is hermitian and $\square$-periodic w.r.t. $\xi$, and the uniform in $(x, \xi) \in \mathbb{R}^{2d}$ estimate

$$c_1 E_m \leq A(x, \xi) \leq c_2 E_m,$$  \hspace{1cm} (1.1)

is valid, where $E_m$ is $m \times m$ unit matrix. We also assume that $A \in W^1_{\infty}(\mathbb{R}^d; C^{1+\beta}_{\text{per}}(\square)) \cap W^2_{\infty}(\mathbb{R}^d; C^\beta_{\text{per}}(\square))$ for some $\beta \in (0, 1)$. By $V = V(x, \xi)$, $a_i = a_i(x, \xi)$ we denote $\square$-periodic w.r.t. $\xi$ matrix-valued functions of the size $n \times n$. It is assumed that $a_i \in W^1_{\infty}(\mathbb{R}^d; C^{1+\beta}_{\text{per}}(\square)) \cap W^2_{\infty}(\mathbb{R}^d; C^\beta_{\text{per}}(\square))$, $V \in W^1_{\infty}(\mathbb{R}^d; C^{\beta}_{\text{per}}(\square))$. It is also supposed that the matrix $V$ is hermitian, and the matrices $a_j$ and $B_j$ are complex-valued. Let $b_i = b_i(x) \in W^2_{\infty}(\mathbb{R}^d)$ be complex-matrix-valued functions of the size $n \times n$.

By $\varepsilon$ we denote a small positive parameter. Given a function $f(x, \xi)$, by $f_\varepsilon(x)$ we indicate $f(x, \frac{x}{\varepsilon})$; for instance, $A_\varepsilon(x) := A \left( x, \frac{x}{\varepsilon} \right)$.

The aim of the present work is to study the spectral properties of the operator

$$\mathcal{H}_\varepsilon := B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon,$$  \hspace{1cm} (1.2)

$$a_\varepsilon(x, \partial) := a \left( x, \frac{x}{\varepsilon}, \partial \right), \quad a(x, \xi, \zeta) := \sum_{i=1}^{d} (a_i(x, \xi) \zeta b_i(x) - b_i^*(x) \zeta a_i^*(x, \xi)),$$

where $\partial$ is the vector of the size $d \times 1$. The symbol $\mathcal{H}_\varepsilon$ is defined as $\mathcal{H}_\varepsilon := B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon,$
in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ with $W^2_0(\mathbb{R}^d; \mathbb{C}^n)$ as the domain. Here $\partial = (\partial_1, \ldots, \partial_d)$, $\partial_i$ is the derivative w.r.t. $x_i$, the superscript * indicated conjugation, and

$$B(\partial)^* := - \sum_{i=1}^d B^*_i \partial_i.$$  

We will show that the operator $H$ is self-adjoint and lower-semibounded uniformly in $\varepsilon$ (see Lemma 2.2). Let $\Lambda_0 = \Lambda_0(x, \xi), A_1 = \Lambda_1(x, \xi), i = 0, 1$ be the matrices of the size $n \times n$ and $n \times m$, respectively, being $\Box$-periodic w.r.t. $\xi$ solutions of the equations

$$B(\partial^*) A(x, \xi) B(\partial) \Lambda_0(x, \xi) - \sum_{i=1}^d b_i(x) \frac{\partial a^*_i}{\partial \xi_i}(x, \xi) = 0, \quad (x, \xi) \in \mathbb{R}^{2d}, \quad \text{(1.3)}$$

$$B(\partial^*) A(x, \xi) (B(\partial) \Lambda_1(x, \xi) + E_m) = 0, \quad (x, \xi) \in \mathbb{R}^{2d}, \quad \text{(1.4)}$$

and satisfying the conditions

$$\int \Lambda_i(x, \xi) \, d\xi = 0, \quad x \in \mathbb{R}^d. \quad \text{(1.4)}$$

Here $\partial = \left( \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_d} \right)$. We will show below that the solutions to (1.3), (1.4) exist, are unique, and $\Lambda_i \in W^1_\infty(\mathbb{R}^d; C^{2+\beta}(\Box))$ (see the proof of Lemma 2.3).

Let $H_0$ be an operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ defined as

$$H_0 = B(\partial)^* A_2 B(\partial) + A_1(x, \partial) + A_0, \quad \text{(1.5)}$$

$$A_2(x) := \frac{1}{|\Box|} \int A(x, \xi) \left( B(\partial^*) \Lambda_1(x, \xi) + E_m \right) \, d\xi, \quad \int \Lambda_i(x, \xi) \, d\xi = 0, \quad x \in \mathbb{R}^d. \quad \text{(1.6)}$$

$$A_1(x, \partial) := \frac{1}{|\Box|} B(\partial)^* \int A(x, \xi) B(\partial) \Lambda_0(x, \xi) \, d\xi$$

$$+ \left( \frac{1}{|\Box|} \int \left( B(\partial^*) \Lambda_0(x, \xi) \right) A(x, \xi) \, d\xi \right) B(\partial) + \frac{1}{|\Box|} \int a(x, \xi, \partial) \, d\xi,$$

$$A_0(x) := - \frac{1}{|\Box|} \int \left( B(\partial^*) \Lambda_0(x, \xi) \right) A(x, \xi) B(\partial) \Lambda_0(x, \xi) \, d\xi + \frac{1}{|\Box|} \int V(x, \xi) \, d\xi,$$

on the domain $W^2_0(\mathbb{R}^d; \mathbb{C}^n)$. We will show below (see Lemma 2.4) that this operator is self-adjoint and lower-semibounded, and its coefficients are smooth enough (see Lemma 2.4). By $H_0$ we denote the lower bound of $H_0$.

Let $G = G(x, \xi) \in W^1_{\infty}(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$ be a positive hermitian matrix of the size $n \times n$. We also assume that the inverse matrix $G^{-1}$ is uniformly bounded. By $g_i > 0, i = 1, 2$, we denote constants independent of $\varepsilon, x$ and $\xi$ such that

$$g_1 E_n \lesssim G(x, \xi) \lesssim g_2 E_n. \quad \text{(1.7)}$$
We let

\[ G_0(x) := \frac{1}{|\Box|} \int G(x, \xi) \, d\xi. \]

We introduce an operator

\[ L_\varepsilon := \left( \Lambda_1 \left( x, \frac{x}{\varepsilon} \right) B(\partial) + \Lambda_0 \left( x, \frac{x}{\varepsilon} \right) \right). \] (1.8)

It will be shown that for each \( \varepsilon \) the operator \( L_\varepsilon \) is bounded as one from \( W^{1,2}(\mathbb{R}^d; C^n) \) into \( L^2(\mathbb{R}^d; C^n) \) and from \( W^{2,2}(\mathbb{R}^d; C^n) \) into \( W^{1,2}(\mathbb{R}^d; C^n) \) (see Lemma 3.1).

Our first result describes the approximation for the generalized resolvent of \( H_\varepsilon \).

**Theorem 1.1.** Suppose \( \lambda \in \mathbb{C} \setminus [\mu_0, +\infty) \), \( \mu_0 := \min \left\{ \frac{h_0}{g_1}, \frac{h_0}{g_2} \right\} \). Then for all small \( \varepsilon \) the inequalities

\[ \| (H_\varepsilon - \lambda G_\varepsilon)^{-1} - (H_0 - \lambda G_0)^{-1} \|_{L^2 \to L^2} \leq C\varepsilon, \]
\[ \| (H_\varepsilon - \lambda G_\varepsilon)^{-1} - (I + \varepsilon L_\varepsilon) (H_0 - \lambda G_0)^{-1} \|_{L^2 \to W^1_2} \leq C\varepsilon, \] (1.9)

hold true, where \( I \) is the identical operator, the constants \( C \) are independent of \( \varepsilon \), and the norms are regarded as those for the operators from \( L^2(\mathbb{R}^d; C^n) \) into \( L^2(\mathbb{R}^d; C^n) \) and \( W^1_2(\mathbb{R}^d; C^n) \), respectively.

Hereafter by \( \sigma(\cdot) \) we denote the spectrum.

**Corollary 1.2.** The spectrum of \( H_\varepsilon \) converges to the spectrum of \( H_0 \). Namely, if \( \lambda \notin \sigma(H_0) \), it follows that \( \lambda \notin \sigma(H_\varepsilon) \) for all \( \varepsilon \) small enough, and if \( \lambda \in \sigma(H_0) \), there exists \( \lambda_\varepsilon \in \sigma(H_\varepsilon) \) such that \( \lambda_\varepsilon \to \lambda_0 \) as \( \varepsilon \to +0 \). If \( \alpha_1, \alpha_2 \in \mathbb{R} \setminus \sigma(H_0) \), then the spectral projectors of \( H_\varepsilon \) and \( H_0 \) satisfy the convergence \( P_{(\alpha_1, \alpha_2)}(H_\varepsilon) \to P_{(\alpha_1, \alpha_2)}(H_0), \varepsilon \to +0 \).

We should say that in the papers [12], [15], [16] they considered a particular case of the operator \( H_\varepsilon \) corresponding to the identities \( a_i = 0, b_i = 0, V = 0 \), and also under the assumption \( A = A(\xi) \). In this case the estimates similar to (1.9) were obtained. It should be stressed that in the cited papers the matrices \( A \) and \( G \) were not assumed to be smooth, but bounded only. Moreover, the constants \( C \) in the mentioned estimates depended only of \( L_\infty \)-norm of the matrices \( A, G, G^{-1} \) as well as of the lattice. In our case these constants depend of \( \lambda \), the lattice, and the norms of the coefficients in the spaces the belong to.

## 2 Auxiliary statements

In the present section we prove a series of auxiliary statements required for the proofs of Theorem 1.1 and Corollary 1.2.
Lemma 2.1. For any \( u \in W^1_2(\mathbb{R}^d ; \mathbb{C}^n) \) the uniform in \( \varepsilon \) estimate
\[
C_1 \| \nabla u \|^2_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} \leq \left( A_{\varepsilon} B(\partial) u, B(\partial) u \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} \leq C_2 \| \nabla u \|^2_{L^2(\mathbb{R}^d ; \mathbb{C}^n)}
\]
holds true.

Proof. It follows from (1.1) that
\[
c_1 \| B(\partial) u \|^2_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} \leq \left( A_{\varepsilon} B(\partial) u, B(\partial) u \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} \leq c_2 \| B(\partial) u \|^2_{L^2(\mathbb{R}^d ; \mathbb{C}^n)}.
\]

The desired inequality follows now from [8, Ch. 2, \$1, estimate (1.11)].

Lemma 2.2. The operator \( \mathcal{H}_\varepsilon \) is self-adjoint and lower-semibounded uniformly in \( \varepsilon \).

Proof. The semiboundedness follows easily from the properties of the coefficients of \( \mathcal{H}_\varepsilon \) and the identity
\[
(\mathcal{H}_\varepsilon u, u)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} = h_\varepsilon[u] := \left( A_{\varepsilon} B(\partial) u, B(\partial) u \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} + 2 \text{Re} \sum_{i=1}^d (a_{\varepsilon,i} \partial_i b_i u, u)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} + \left( V_\varepsilon u, u \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)}.
\]

It is clear that the operator \( \mathcal{H}_\varepsilon \) is symmetric; to prove the self-adjointness it is sufficient to check that \( \mathcal{D}(\mathcal{H}_\varepsilon^* \mathcal{H}_\varepsilon) = \mathcal{D}(\mathcal{H}_\varepsilon) \). In its turn, this identity can be established easily, if a generalized solution to the equation
\[
(B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon) u = f, \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}^d ; \mathbb{C}^n), \quad (2.1)
\]
belong to \( W^2_2(\mathbb{R}^d ; \mathbb{C}^n) \). Let us prove it.

It is clear that a generalized solution of (2.1) is also a generalized solution to
\[
B(\partial)^* A_\varepsilon B(\partial) u + u = g, \quad x \in \mathbb{R}^d, \quad (2.2)
\]
where \( g := f - a_\varepsilon(x, \partial) u - V_\varepsilon u + u, \quad \| g \|_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} \leq C \left( \| f \|_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} + \| u \|_{W^2_2(\mathbb{R}^d ; \mathbb{C}^n)} \right) \).

Let \( \delta \neq 0 \) be a small fixed number, \( e_i^{(d)} \), \( i = 1, \ldots, d \) be a standard basis in \( \mathbb{R}^d \). We denote
\[
u_\delta^{(i)}(x) := \frac{1}{\delta} \left( u(x + \delta e_i^{(d)}) - u(x) \right).
\]

This function is a generalized solution to (2.2) with the right hand side
\[
g_\delta^{(i)} = B(\partial)^* A_\varepsilon^{(i)} B(\partial) u(x + \delta e_i^{(d)}),
\]
where \( g_\delta^{(i)}, A_\varepsilon^{(i)} \) are defined via \( g, A_\varepsilon \) similarly to \( u_\delta^{(i)} \). The integral identity corresponding to the equation for \( u_\delta^{(i)} \) reads as follows
\[
\left( A_{\varepsilon} B(\partial) u_\delta^{(i)}, B(\partial) \varphi \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} + \left( u_\delta^{(i)}, \varphi \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} = - \left( g, \varphi_\delta^{(i)} \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)} - \left( A_{\varepsilon}^{(i)} B(\partial) u(\cdot + \delta e_i^{(d)}), B(\partial) \varphi \right)_{L^2(\mathbb{R}^d ; \mathbb{C}^n)},
\]
where \( \varphi \in W^1_2(\mathbb{R}^d; C^n) \). We also observe that inequality (11) in the proof of Item a) of Theorem 3 in [11, Ch. III, Sec. 3.4] implies
\[
\| \varphi^{(i)}_{-\delta} \|_{L_2(\mathbb{R}^d; C^n)} \leq \| \varphi \|_{W^1_2(\mathbb{R}^d; C^n)},
\]
for each \( \varphi \in W^1_2(\mathbb{R}^d; C^n) \). Letting \( \varphi := u^{(i)}_{\delta} \) in two last inequalities and taking into account the smoothness of \( A \) and Lemma 2.1, we arrive at the uniform in \( \delta \) estimate
\[
\| u^{(i)}_{\delta} \|_{W^2_2(\mathbb{R}^d; C^n)} \leq C \left( \| g \|_{L_2(\mathbb{R}^d; C^n)} + \| u \|_{W^2_2(\mathbb{R}^d; C^n)} \right) .
\]
Employing this estimate and repeating the arguments of the proof of Item b) of Theorem 3 in [11, §3.4], one can check easily that there exist second generalized derivations of the function \( u \), and
\[
\| u \|_{W^2_2(\mathbb{R}^d; C^n)} \leq C \left( \| g \|_{L_2(\mathbb{R}^d; C^n)} + \| u \|_{W^2_2(\mathbb{R}^d; C^n)} \right) ,
\]
(2.3)

**Lemma 2.3.** Let \( f(x, \cdot) \in C^\beta_{\text{per}}(\Box) \) for all \( x \in \mathbb{R}^d \). The system
\[
B(\partial_\xi)^* A(x, \xi) B(\partial_\xi) v(x, \xi) = f(x, \xi), \quad \xi \in \mathbb{R}^d,
\]
(2.4)
has \( \Box \)-periodic w.r.t. \( \xi \) solution \( v(x, \cdot) \in C^{2+\beta}_{\text{per}}(\Box) \), if and only if
\[
\int_\Box f(x, \xi) \, d\xi = 0, \quad x \in \mathbb{R}^d.
\]
(2.5)
If the solvability condition holds true, the solution of (2.4) is unique up to a constant (in \( \xi \)) vector. There exists unique solution of (2.4) such that
\[
\int_\Box v(x, \xi) \, d\xi = 0, \quad x \in \mathbb{R}^d.
\]
(2.6)
This solution satisfies the estimate
\[
\| v(x, \cdot) \|_{C^{2+\beta}_{\text{per}}(\Box)} \leq C \| f(x, \cdot) \|_{C^\beta_{\text{per}}(\Box)},
\]
(2.7)
where the constant \( C \) is independent of \( x \in \mathbb{R}^d \) and \( f \). If \( f \in W^k_\infty(\mathbb{R}^d; C^\beta_{\text{per}}(\Box)) \), \( k = 0, 1 \) it follows that \( v \in W^k_\infty(\mathbb{R}^d; C^{2+\beta}_{\text{per}}(\Box)) \), and the estimate
\[
\| v \|_{W^k_\infty(\mathbb{R}^d; C^{2+\beta}_{\text{per}}(\Box))} \leq C \| f \|_{W^k_\infty(\mathbb{R}^d; C^\beta_{\text{per}}(\Box))},
\]
(2.8)
holds true, where the constant \( C \) is independent of \( f \).
Proof. The existence of a $\Box$-periodic generalized solution of (2.4) in $W^1_2(\Box; \mathbb{C}^n)$, the solvability condition (2.5) and the uniqueness of the solution satisfying (2.6) are implied by Theorem 1 in [3, Appendix]. Moreover, it follows from the proof of this theorem that

$$\|v(x, \cdot)\|_{W^1_2(\Box; \mathbb{C}^n)} \leq C\|f(x, \cdot)\|_{L^2(\Box; \mathbb{C}^n)}. \tag{2.9}$$

Throughout the proof by $C$ we indicate the inessential constants independent of $f$ and $x \in \mathbb{R}^d$.

As for the equation (2.4), one can make sure that $v(x, \cdot) \in W^2_{2, loc}(\mathbb{R}^d)$. By theorem 10.7 and Remark 1 in [20, Ch. IV, §10.3] and the periodicity of $f$ and $v$ it implies that

$$\|v(x, \cdot)\|_{C^{2+\beta}(\Box)} \leq C\|f(x, \cdot)\|_{C^\beta(\Box)} + \|v(x, \cdot)\|_{L^1(\Box)}.$$

This estimate and (2.9) yield (2.7).

Assume that $f \in W^k_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$ and let us prove the claimed smoothness of $v$. If $k = 0$ and $f \in L_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$, the belonging $v \in L_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$. The existence of a $\Box$-periodic generalized solution of (2.4) implies that

$$\|v(x, \cdot)\|_{C^{2+\beta}(\Box)} \leq C\|f(x, \cdot)\|_{C^\beta(\Box)} + \|v(x, \cdot)\|_{L^1(\Box)}.$$ 

This right hand side satisfies (2.5) and belongs to $L_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$. It follows from (2.8) for $k = 0$ that

$$\|v^{(i)}(x, \xi) := \frac{1}{\delta}(v(x + \delta e_i^{(d)}, \xi) - v(x, \xi)).$$

This function is the solution to (2.4) at $x$ with the right hand side

$$f^{(i)}_\delta - B(\partial \xi)A^{(i)}_{\varepsilon, \delta}B(\partial \xi)v(x + \delta e_i^{(d)}, \xi),$$

where $f^{(i)}_\delta$ and $A^{(i)}_{\varepsilon, \delta}$ are determined by analogy with $v^{(i)}_\delta$. This right hand side satisfies (2.5) and belongs to $L_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$. It follows now from (2.8) for $k = 0$ that

$$\|v^{(i)}(x, \cdot)\|_{L^\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))} \leq C\|f^{(i)}_\delta - B(\partial \xi)A^{(i)}_{\varepsilon, \delta}B(\partial \xi)v(\cdot + \delta, \cdot)\|_{L^\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))} \leq C\|f^{(i)}_\delta\|_{C^{1}(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))}, \tag{2.10}$$

where the constant $C$ is independent of $f$ and $\delta$.

Let $v^{(i)}_0$ be a solution to (2.4) at $x$ with the right-hand side

$$\frac{\partial f}{\partial x_i}(x, \xi) - B(\partial \xi)\frac{\partial A}{\partial x_i}(x, \xi)B(\partial \xi)v,$$

satisfying (2.6). It is clear that the right hand side satisfies (2.5) and belongs to $L_\infty(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$. The function $v^{(i)}_\delta - v^{(i)}_0$ is the solution to (2.4) at $x$ with the right hand side

$$f^{(i)}_\delta - \frac{\partial f}{\partial x_i} + B(\partial \xi)(\frac{\partial A}{\partial x_i} - A^{(i)}_{\delta})B(\partial \xi)v - \delta B(\partial \xi)^*A^{(i)}_{\delta}B(\partial \xi)v^{(i)}_\delta.$$
Employing now (2.7) and (2.10), we obtain
\[
\|v^{(i)}_\delta - v^{(i)}_0\|_{L_\infty(R^d; C^{2+\beta}_{\text{per}}(\square))} \rightarrow 0.
\]
Hence, the derivative \(\frac{\partial v}{\partial x}\) exists and \(\frac{\partial v}{\partial x} = v^{(i)}_0 \in L_\infty(R^d; C^{2+\beta}_{\text{per}}(\square))\). The inequality (2.4) implies also that the inequality (2.3) is valid with \(k = 1\).

**Lemma 2.4.** The operator \(H_0\) is self-adjoint and lower-semibounded. The matrices \(A_2, A_0\), and the coefficients of \(A_1(x, \partial)\) belong to \(W^1_\infty(R^d)\). The estimate
\[
\|u\|_{L_2^2(R^d; C^n)} \leq C \left( \|H_0 u\|_{L_2(R^d; C^n)} + \|u\|_{L_2(R^d; C^n)} \right) \tag{2.11}
\]
is valid.

**Proof.** We begin with the proof of the solvability of (1.3), (1.4). Let \(\Lambda^{(j)}_0 = \Lambda^{(j)}_0(x, \xi) \in C^n, j = 1, \ldots, n\), \(\Lambda^{(j)}_1 = \Lambda^{(j)}_1(x, \xi) \in C^n, j = 1, \ldots, m\), be \(\square\)-periodic w.r.t. \(\xi\) solutions of
\[
B(\partial_\xi)^* A B(\partial_\xi) \Lambda^{(j)}_0 - \sum_{i=1}^d b_i^{(j)} \frac{\partial a^{(j)}_i}{\partial \xi_i} e_j^{(m)} = 0, \quad (x, \xi) \in R^{2d},
\]
\[
B(\partial_\xi)^* A B(\partial_\xi) \Lambda^{(j)}_1 + e_j^{(m)} = 0, \quad (x, \xi) \in R^{2d},
\]
satisfying (2.6). Here \(A = A(x, \xi), a_i = a_i(x, \xi), e_j^{(m)}, j = 1, \ldots, n\) is the standard basis in \(C^n\). Lemma 2.3 implies that these problems are uniquely solvable and their solutions belong to \(W^1_\infty(R^d; C^{2+\beta}_{\text{per}}(\square))\). The vectors \(\Lambda^{(j)}_0\) are columns of the matrices \(\Lambda_j\). Hence, the matrices \(A_2, A_0\), as well as the coefficients of the operator \(A_1(x, \partial)\) belong to \(W^1_\infty(R^d; C^n)\).

We denote \(X := (B(\partial_\xi) \Lambda_0 + E_m)\). Integrating by parts and taking into account the equations (1.3) we obtain
\[
\int_{\square} \left( B(\partial_\xi) \Lambda_1(x, \xi) \right)^* A(x, \xi) X(x, \xi) \, d\xi
\]
\[
= \int_{\square} \Lambda_1^*(x, \xi) B(\partial_\xi)^* A(x, \xi) \left( B(\partial_\xi) \Lambda_1(x, \xi) + E_m \right) d\xi = 0,
\]
that implies
\[
A_2(x) = \frac{1}{\|\square\|} \int_{\square} X^*(x, \xi) A(x, \xi) X(x, \xi) \, d\xi. \tag{2.12}
\]
This identity and the definition of lower terms of \(H_0\) yield that this operator is symmetric. Let us prove that it is lower-semibounded. In view of the last identity and (1.11) it easy to check that for all \(x \in R^d\) and \(w \in C^n\) the inequalities
\[
(A_2(x) w, w)_{C^n} = (A(x, \cdot) X(x, \cdot) w, X(x, \cdot) w)_{L_2(\square; C^n)}
\]
\[
\geq c_1 \left( \|w\|_{C^n}^2 + 2 \text{Re} \left( B(\partial_\xi) \Lambda_1(x, \cdot) w, w \right)_{L_2(\square; C^n)} + \|B(\partial_\xi) \Lambda_1(x, \cdot) w\|_{L_2(\square; C^n)}^2 \right)
\]
\[
= c_1 \left( \|w\|_{C^n}^2 + \|B(\partial_\xi) \Lambda_1(x, \cdot) w\|_{L_2(\square; C^n)}^2 \right) \geq c_1 \|w\|_{C^n}^2,
\]

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are valid. Together with the inequality (1.11) in [8, Ch. 2, §1] they imply
\[ \| \nabla u \|_{L^2(R^d;C^n)}^2 \leq C \left( A_2 B(\partial u, B(\partial) u) \right)_{L^2(R^d;C^n)}. \]

Employing this estimate by analogy with [7,22] one can check easily that \( H_0 \) is self-adjoint. The estimate (2.11) follows easily from the corresponding analogue of (2.3) and an obvious estimate
\[ \| u \|_{W^1_2(R^d;C^n)} \leq C \left( \| H_0 u \|_{L^2(R^d;C^n)} + \| u \|_{L^2(R^d;C^n)} \right). \]

\[ \square \]

3 Asymptotics for the resolvent

In the section we prove Theorem 1.1 and Corollary 1.2. For the proofs, we require four lemmas.

**Lemma 3.1.** For each value \( \varepsilon > 0 \) the operator \( L_\varepsilon \) defined by (1.8) is bounded as an operator from \( W^2_2(R^d;C^n) \) into \( W^1_2(R^d;C^n) \). The operator \( L_\varepsilon \) is bounded uniformly in \( \varepsilon \) as an operator from \( W^1_2(R^d;C^n) \) into \( L^2(R^d;C^n) \).

The lemma follows from the belonging \( \Lambda_i \in W^1_\infty(R^d;C^{2+\beta}(\Box)) \) proven in Lemma 2.4.

Hereinafter by \( \frac{\partial u}{\partial \xi_i} \) we denote the partial derivatives w.r.t. \( x_i \) for the functions \( u = u(x, \xi) \) treated as ones of independent variables \( x \) and \( \xi = \frac{x}{\varepsilon} \). In the same way we regard the partial derivatives \( \frac{\partial u}{\partial x_i} \). We also remind that we employ the symbols \( \partial_i \) to denote the full derivatives w.r.t. \( x_i \), i.e.,
\[ \partial_i u \left( x, \frac{x}{\varepsilon} \right) = \frac{\partial u}{\partial x_i} \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^{-1} \frac{\partial u}{\partial \xi_i} \left( x, \frac{x}{\varepsilon} \right), \]
and \( \partial = (\partial_1, \ldots, \partial_d) \).

**Lemma 3.2.** Let \( M = M(x, \xi) \) be a \( \Box \)-periodic w.r.t. \( \xi \) matrix of the size \( n \times n \) such that
\[ M \in W^1_\infty(R^d;C^{\beta}(\Box)), \quad \int M(x, \xi) \, d\xi = 0, \quad x \in R^d, \]
and \( u(x) \in W^1_2(R^d;C^n) \) be a vector-function. There exist vector-functions \( v^{(e)}_i(x) \in L^2(R^d;C^n) \), \( i = 0, \ldots, d \), such that the representation and estimates
\[ M \left( x, \frac{x}{\varepsilon} \right) u(x) = \varepsilon \sum_{i=1}^d \partial_i v^{(e)}_i(x) + \varepsilon v^{(e)}_0(x), \quad \| v^{(e)}_i \|_{L^2(R^d;C^n)} \leq C \| u \|_{W^1_2(R^d;C^n)}, \]
are valid, where the constant \( C \) is independent of \( \varepsilon \) and \( u \).
Proof. Let $P = P(x, \xi)$ be a $\Box$-periodic w.r.t. $\xi$ matrix of the size $n \times n$ satisfying the equation
\[ \Delta_\xi P(x, \xi) = M(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}, \]
and the condition (2.6) for all $x \in \mathbb{R}^d$. By Lemma 2.3 this equation is solvable, the matrix $P$ is defined uniquely, and the estimate $\|P\|_{W^{1,\infty}_1(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))} < \infty$ is valid. Using this estimate one can check easily that the lemma is valid for
\[ v^{(e)}_i := \frac{\partial P}{\partial \xi_i} u, \quad v^{(e)}_0 := -\sum_{i=1}^d \frac{\partial^2 P u}{\partial x_i \partial \xi_i}, \quad P = P \left( x, \frac{x}{\varepsilon} \right). \]
\[ \Box \]

We denote
\[ \hat{A}_1(x, \xi) := A(x, \xi)(B(\partial_\xi)\Lambda_1(x, \xi) + E_m) - A_2(x). \quad (3.3) \]

Lemma 3.3. Suppose $u \in W^2_2(\mathbb{R}^d; \mathbb{C}^n)$. There exist vector-functions $v^{(e)}_i \in L^2(\mathbb{R}^d; \mathbb{C}^n)$, $i = 1, \ldots, d$, such that the representation and estimate
\[ B(\partial_x)^* \hat{A}_1 \left( x, \frac{x}{\varepsilon} \right) v(x) = \varepsilon \sum_{i=1}^d \partial_i v^{(e)}_i(x), \quad \|v^{(e)}_i\|_{L^2(\mathbb{R}^d; \mathbb{C}^n)} \leqslant C\|v\|_{W^2_2(\mathbb{R}^d; \mathbb{C}^n)}, \]
are valid, where the constant $C$ is independent of $\varepsilon$ and $u$.

Proof. Let $P^{(i)} = P^{(i)}(x, \xi)$ be $\Box$-periodic w.r.t. $\xi$ matrices of the size $n \times m$ satisfying equations
\[ \Delta_\xi P^{(i)}(x, \xi) = -B_i^* \hat{A}_1(x, \xi), \quad (x, \xi) \in \mathbb{R}^{2d}, \quad (3.4) \]
and the condition (2.6). By Lemma 2.3 these equations are solvable, the matrices $P^{(i)}$ are uniquely defined, and $P^{(i)} \in W^1_1(\mathbb{R}^d; C^{2+\beta}_{per}(\Box))$. The equations (1.3) and the definition of $\hat{A}$ implies that this matrix is $\Box$-periodic w.r.t. $\xi$ and satisfies the equation
\[ B(\partial_\xi)^* \hat{A}_1(x, \xi) = 0, \quad (x, \xi) \in \mathbb{R}^{2d}. \quad (3.5) \]
This equation and (3.4) yield that
\[ \Delta_\xi \sum_{i=1}^d \frac{\partial P^{(i)}}{\partial \xi_i} = 0, \quad (x, \xi) \in \mathbb{R}^d, \quad (3.6) \]
and by the unique solvability of this equation we thus obtain
\[ \sum_{i=1}^d \frac{\partial P^{(i)}}{\partial \xi_i} = 0. \]
Together with (3.5) it follows that

\[- B^*_i \hat{A}_1 = \sum_{j=1}^{d} \frac{\partial M_{ij}}{\partial \xi_j}, \quad M_{ij} := \frac{\partial P^{(i)}}{\partial \xi_j} - \frac{\partial P^{(j)}}{\partial \xi_i}. \tag{3.7}\]

Taking these identities into account we arrive at

\[B(\partial_x)^* \hat{A}_1 v = \sum_{i,j=1}^{d} \frac{\partial^2 M_{ij}}{\partial x_i \partial x_j} \varepsilon^d \sum_{i,j=1}^{d} \frac{\partial M_{ij}}{\partial x_i} - \varepsilon \sum_{i,j=1}^{d} \frac{\partial M_{ij}}{\partial x_i \partial x_j}, \tag{3.8}\]

where \(\hat{A}_1 = \hat{A}_1(x, \xi)\), \(M_{ij} = M_{ij}(x, \xi)\). The second term in the right hand side of the identity obtained is zero due to \(M_{ij} = -M_{ji}\). Now the statement of the lemma follows from the belonging \(P^{(i)} \in W^1_{\infty}(\mathbb{R}^d; C^{2+\beta}_{\text{per}}(\Box))\), if we denote

\[v_i^{(\varepsilon)}(x) := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} M_{ji}(x, \xi) v(x). \]

We denote

\[\hat{A}_0(x, \xi) := A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) - \frac{1}{\Box} \int A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) d\xi. \tag{3.9}\]

**Lemma 3.4.** Let \(v \in W^2_3(\mathbb{R}^d; \mathbb{C}^n)\). There exist vector-functions \(v_i^{(\varepsilon)} \in L^2(\mathbb{R}^d; \mathbb{C}^n)\), \(i = 0, \ldots, d\), such that the representation and the estimate

\[B(\partial_x)^* \hat{A}_0 \left(x, \frac{x}{\varepsilon}\right) v(x) = \varepsilon \sum_{i=1}^{d} \partial v_i^{(\varepsilon)}(x) + \varepsilon v_0^{(\varepsilon)}(x), \quad \|v_i^{(\varepsilon)}\|_{L^2(\mathbb{R}^d; \mathbb{C}^n)} \leq C \|v\|_{W^2_3(\mathbb{R}^d; \mathbb{C}^n)}, \]

hold true, where the constant \(C\) is independent of \(\varepsilon\) and \(u\).

**Proof.** The proof follows the ideas of that of Lemma 3.3; one just need to make some minor corrections. We introduce the matrices \(P^{(i)}\) as the solutions to (3.4), (2.6) with \(\hat{A}_1\) replaced by \(\hat{A}_0\). Then \(P^{(i)} \in W^1_{\infty}(\mathbb{R}^d; C^{2+\beta}_{\text{per}}(\Box))\). By the first equation in (1.3), the analogues of the relations (3.5), (3.6), (3.7) are as follows

\[B(\partial_\xi)^* \hat{A}_0 = \sum_{i=1}^{d} b_i \frac{\partial a_i^*}{\partial \xi_i}, \quad Q := \sum_{i=1}^{d} \frac{\partial P^{(i)}}{\partial \xi_i}, \]

\[- B^*_i \hat{A}_0 = \sum_{j=1}^{d} \frac{\partial M_{ij}}{\partial \xi_j} + \frac{\partial Q}{\partial \xi_i}, \quad \Delta_\xi Q = \sum_{i=1}^{d} b_i \frac{\partial a_i^*}{\partial \xi_i}. \]
Employing these identities, by analogy with (3.8) we obtain

\[
B(\partial)\hat{A}_0 \mathbf{v} = \varepsilon \sum_{i,j=1}^{d} \partial_j \frac{\partial M_{ij}}{\partial x_i} \mathbf{v} + \sum_{i=1}^{d} \frac{\partial^2 Q}{\partial x_i \partial \xi_i} \varepsilon \mathbf{v} = \varepsilon \sum_{i,j=1}^{d} \partial_j \frac{\partial M_{ij}}{\partial x_i} \mathbf{v} + \varepsilon \sum_{i=1}^{d} \frac{\partial Q}{\partial x_i} \mathbf{v} - \varepsilon \sum_{i=1}^{d} \frac{\partial^2 Q}{\partial x_i^2} \mathbf{v}.
\]

Now we let

\[
\mathbf{v}^{(\varepsilon)}(x) := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} M_{ji}(x, \varepsilon) \mathbf{v}(x) + \frac{\partial}{\partial x_i} Q(x, \varepsilon) \mathbf{v}(x),
\]

\[
\mathbf{v}_0^{(\varepsilon)}(x) := - \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} Q(x, \varepsilon) \mathbf{v}(x).
\]

The aforementioned properties of \(P^{(i)}\) yield that the vector-functions \(\mathbf{v}^{(\varepsilon)}\) belong to \(L^2(\mathbb{R}^d; \mathbb{C}^n)\) and satisfy the claimed estimate. To complete the proof, it remains to establish the same fact for \(\mathbf{v}_0^{(\varepsilon)}\). In order to do it, it is sufficient to check that \(Q \in W^{2,\infty}(\mathbb{R}^d; C^\beta_{\text{per}}(\square))\).

Let \(Q^{(i)}\) be the solutions of the equations

\[
\Delta \varepsilon Q^{(i)} = h_i^* \left( a_i^* - \frac{1}{\square} \int_{\square} a_i^*(\cdot, \xi) \, d\xi \right), \quad (x, \xi) \in \mathbb{R}^{2d},
\]

satisfying the condition (2.6). Applying Lemma 2.3 and differentiating these equations w.r.t. \(x_j\), it is easy to check that \(Q^{(i)} \in W^{2,\infty}(\mathbb{R}^d; C^\beta_{\text{per}}(\square))\). Clearly,

\[
Q = \sum_{i=1}^{d} \frac{\partial Q^{(i)}}{\partial \xi_i},
\]

that implies the desired belonging for \(Q\).

Let \(h_\varepsilon\) be the lower bound of \(H_\varepsilon\), \(\mu_\varepsilon := \min \{ h_{\varepsilon}, h_1, h_2 \}\).

**Lemma 3.5.** Suppose that \(\lambda \in \mathbb{C} \setminus [\mu, +\infty)\), where \(\mu_\varepsilon - \mu \geq c > 0\), and the constant \(c\) is independent of \(\varepsilon\). Then the generalized solution \(\mathbf{u} \in W^{1,2}_\varepsilon(\mathbb{R}^d; \mathbb{C}^n)\) of

\[
(B(\partial)^* A, B(\partial)) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \mathbf{u} = \mathbf{f}_0 + \sum_{i=1}^{d} \partial_i \mathbf{f}_i, \quad \mathbf{f}_i \in L^2(\mathbb{R}^d; \mathbb{C}^n),
\]

satisfies the estimate

\[
\| \mathbf{u} \|_{W^{1,2}_\varepsilon(\mathbb{R}^d; \mathbb{C}^n)} \leq C(\lambda) \sum_{i=0}^{d} \| \mathbf{f}_i \|_{L^2(\mathbb{R}^d; \mathbb{C}^n)},
\]

where the constant \(C(\lambda)\) is independent of \(\varepsilon\) and \(\mathbf{f}_i\).
Proof. Basing on Lemma 2.1, the identity

\[ h_\varepsilon[u] - \lambda(G_\varepsilon u, u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = (f_0, u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} - \sum_{i=1}^{d} (f_i, \partial_i u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \]  \tag{3.10}

and (1.7) one can prove that

\[ \|u\|_{W^2_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C(\lambda) \left( \sum_{i=0}^{d} \|f_i\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + \|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \right), \]  \tag{3.11}

where the constant \( C \) is independent of \( \varepsilon \) and \( f_i \). The first term in the left hand side of (3.10) being real, this identity implies that

\[ - \text{Im}(\lambda(G_\varepsilon u, u))_{L_2(\mathbb{R}^d; \mathbb{C}^n)} = \text{Im}(f_0, u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} - \sum_{i=1}^{d} (f_i, \partial_i u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)}. \]

If \( \text{Im} \lambda \neq 0 \), this identity and (1.7) yields

\[ \|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \leq \delta \|u\|_{W^2_2(\mathbb{R}^d; \mathbb{C}^n)}^2 + C(\delta, \lambda) \sum_{i=0}^{d} \|f_i\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2, \]  \tag{3.12}

where the number \( \delta \) can be chosen anyhow small, and the constant \( C \) is independent of \( \varepsilon \) and \( f_i \). If \( \lambda \in (-\infty, \mu) \), the last estimate is valid as well that follows from (3.10) and the inequality

\[ h_\varepsilon[u] - \lambda(G_\varepsilon u, u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \geq h_\varepsilon \|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 - \mu(G_\varepsilon u, u)_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \]

\[ \geq (h_\varepsilon - \mu g)\|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \geq (\mu - \mu) g\|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2 \geq c g\|u\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}^2, \]

where

\[ g = \begin{cases} g_2, & \mu \geq 0, \\ g_1, & \mu < 0. \end{cases} \]

Now the estimates (3.11), (3.12) lead us to the statement of lemma.

Proof of Theorem 1.1 for non-real \( \lambda \). Let \( f \in L_2(\mathbb{R}^d; \mathbb{C}^n) \),

\[ u^{(\varepsilon)} := (\mathcal{H}_\varepsilon - \lambda G_\varepsilon)^{-1} f, \quad u^{(0)} := (\mathcal{H}_0 - \lambda G_0)^{-1} f, \]

\[ u^{(1)}(x, \xi) := (\Lambda_1(x, \xi) B(\partial) + \Lambda_0(x, \xi)) u^{(0)}(x), \quad \tilde{u}^{(\varepsilon)}(x) := u^{(0)}(x) + \varepsilon u^{(1)}(x, \frac{x}{\varepsilon}). \]

It is obvious that

\[ (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) (u^{(\varepsilon)} - \tilde{u}^{(\varepsilon)}) \]

\[ = f - (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \tilde{u}^{(\varepsilon)} \]

\[ = (\mathcal{H}_0 - \lambda G_0) u^{(0)} - (B(\partial)^* A_\varepsilon B(\partial) + a_\varepsilon(x, \partial) + V_\varepsilon - \lambda G_\varepsilon) \tilde{u}^{(\varepsilon)} =: F^{(\varepsilon)}. \]  \tag{3.13}
Let us evaluate the function $F^{(e)}$. Taking into account the identities (3.1) and the second equation in (1.3) we obtain

\[
B(\partial)^* AB(\partial) \hat{u}^{(e)} = B(\partial_x)^* AB(\partial) u^{(0)} + B(\partial_x)^* AB(\partial_x) u^{(1)} \\
+ \varepsilon B(\partial)^* AB(\partial_x) u^{(1)} + \varepsilon^{-1} B(\partial_x)^* AB(\partial) u^{(0)} + \varepsilon^{-1} B(\partial_x)^* AB(\partial_x) u^{(1)} \\
= B(\partial_x)^*(A + B(\partial_\xi) \Lambda_1) B(\partial) u^{(0)} + B(\partial_x)^* AB(\partial_\xi) \Lambda_0 u^{(0)} \\
+ \varepsilon B(\partial)^* AB(\partial_x) u^{(1)} + \varepsilon^{-1} B(\partial_x)^* AB(\partial_\xi) \Lambda_0 u^{(0)},
\]

(3.14)

Here the arguments of all functions except $u^{(0)}(x)$, $u^{(e)}(x)$ and $\hat{u}^{(e)}(x)$ are $(x, \xi)$. Integrating by parts and employing the equations (1.3) one can make sure that

\[
\int (B(\partial_\xi) \Lambda_0)^* A \, d\xi = \int \Lambda_0^* B(\partial_\xi)^* A \, d\xi = - \int \Lambda_0^* B(\partial_\xi)^* AB(\partial_\xi) \Lambda_1 \, d\xi \\
= - \int (B(\partial_\xi)^* AB(\partial_\xi) \Lambda_0)^* \Lambda_1 \, d\xi = - \sum_{i=1}^d \int \frac{\partial a_i}{\partial \xi_i} b_i \Lambda_1 \, d\xi = \sum_{i=1}^d \int a_i b_i \frac{\partial \Lambda_1}{\partial \xi_i} \, d\xi, \\
\int (B(\partial_\xi) \Lambda_0)^* AB(\partial_\xi) \Lambda_0 \, d\xi = \int (AB(\partial_\xi) \Lambda_0)^* B(\partial_\xi) \Lambda_0 \, d\xi \\
= \int (B(\partial_\xi)^* AB(\partial_\xi) \Lambda_0)^* \Lambda_0 \, d\xi = \sum_{i=1}^d \int \frac{\partial a_i}{\partial \xi_i} b_i \Lambda_0 \, d\xi = - \sum_{i=1}^d \int a_i b_i \frac{\partial \Lambda_0}{\partial \xi_i} \, d\xi.
\]

(3.15)

Here the arguments of all matrices are $(x, \xi)$. These identities and (1.6) yield

\[
A_1(x, \partial) := \frac{1}{|\square|} B(\partial)^* \int A(x, \xi) B(\partial_\xi) \Lambda_0(x, \xi) \, d\xi \\
+ \left( \frac{1}{|\square|} \int a_i(x, \xi) b_i(x) \frac{\partial \Lambda_1}{\partial \xi_i}(x, \xi) \, d\xi \right) B(\partial) + \frac{1}{|\square|} \int a(x, \xi, \partial) \, d\xi, \\
A_0(x) := \frac{1}{|\square|} \sum_{i=1}^d \int a_i(x, \xi) b_i(x) \frac{\partial \Lambda_0}{\partial \xi_i}(x, \xi) \, d\xi + \frac{1}{|\square|} \int V(x, \xi) \, d\xi.
\]
Bearing in mind these relations, (3.14), (3.13), and the definition of $u^{(i)}$, we obtain

$$\mathbf{F}^{(e)} = \mathbf{F}_1^{(e)} + \mathbf{F}_2^{(e)} + \mathbf{F}_3^{(e)}, \quad \mathbf{F}_1^{(e)} = -B(\partial_x)^* \widehat{A}_1 B(\partial)u^{(0)} - B(\partial_x)^* \widehat{A}_0 u^{(0)},$$

$$\mathbf{F}_2^{(e)} = \sum_{i=1}^d \left( \frac{1}{|\mathbf{d}|} \int a_i(\cdot, \xi) \, d\xi - a_i \right) \frac{\partial}{\partial x_i} b_i u^{(0)}$$

$$- \sum_{i=1}^d b_i^* \frac{\partial}{\partial x_i} \left( \frac{1}{|\mathbf{d}|} \int a_i^*(\cdot, \xi) \, d\xi - a_i^* \right) u^{(0)}$$

$$+ \sum_{i=1}^d \left( \frac{1}{|\mathbf{d}|} \int a_i(\cdot, \xi) b_i(\cdot) \frac{\partial \Lambda_1(\cdot, \xi) \, d\xi}{\partial \xi_i} - a_i b_i \frac{\partial \Lambda_1(\cdot, \xi) \, d\xi}{\partial \xi_i} \right) B(\partial)u^{(0)}$$

$$+ \sum_{i=1}^d \left( \frac{1}{|\mathbf{d}|} \int a_i(\cdot, \xi) b_i(\cdot) \frac{\partial \Lambda_0(\cdot, \xi) \, d\xi}{\partial \xi_i} - a_i b_i \frac{\partial \Lambda_0(\cdot, \xi) \, d\xi}{\partial \xi_i} \right) u^{(0)}$$

$$+ \left( \frac{1}{|\mathbf{d}|} \int V(\cdot, \xi) \, d\xi - V \right) u^{(0)} + \lambda(G - G_0)u^{(0)};$$

$$\mathbf{F}_3^{(e)} = -\varepsilon B(\partial)^* AB(\partial_x)u^{(1)} - \varepsilon \sum_{i=1}^d \left( a_i \frac{\partial}{\partial x_i} b_i + \frac{\partial b_i^*}{\partial x_i} a_i^* - \partial_i b_i^* a_i^* \right) u^{(1)} - \varepsilon (V - \lambda G)u^{(1)},$$

where the arguments of the functions are $(x, \varepsilon)$. The belonging $\Lambda_i \in W^1_{\infty}(\mathbb{R}^d, C^{2+\beta}(\mathbb{R}))$ and the inequality

$$\|u^{(0)}\|_{W^2_2(\mathbb{R}^d; C^n)} \leq C\|f\|_{L_2(\mathbb{R}^d; C^n)},$$

imply that

$$\left\| B(\partial_x)u^{(1)} \left( x, \frac{x}{\varepsilon} \right) \right\|_{L_2(\mathbb{R}^d; C^n)} \leq C\|f\|_{L_2(\mathbb{R}^d; C^n)}.$$ 

Hereinafter till the end of the proof by $C$ we denote inessential constants independent of $\varepsilon$ and $f$. Taking into account the last estimate we conclude that the function $\mathbf{F}_3^{(e)}$ can be represented as

$$\mathbf{F}_3^{(e)} = \varepsilon \sum_{i=1}^d \frac{\partial F_3^{(e)}}{\partial x_i} + \varepsilon F_3^{(e)}, \quad \|F_3^{(e)}\|_{L_2(\mathbb{R}^d; C^n)} \leq C\|f\|_{L_2(\mathbb{R}^d; C^n)}, \quad i = 0, \ldots, d. \quad (3.17)$$

The formula for $\mathbf{F}_2^{(e)}$ implies immediately that this function is a sum of terms each of them satisfies the hypothesis of Lemma 3.2. Hence by this lemma we have

$$\mathbf{F}_2^{(e)} = \varepsilon \sum_{i=1}^d \frac{\partial F_2^{(e)}}{\partial x_i} + \varepsilon F_2^{(e)}, \quad \|F_2^{(e)}\|_{L_2(\mathbb{R}^d; C^n)} \leq C\|f\|_{L_2(\mathbb{R}^d; C^n)}, \quad i = 0, \ldots, d.$$
where the constant $C$ is independent of $\varepsilon$ and $f$. These identities, Lemma 3.3 and (3.16), (3.17) yield

$$F^{(\varepsilon)} = \varepsilon \sum_{i=1}^{d} \frac{\partial f_i^{(\varepsilon)}}{\partial x_i} + \varepsilon f_0^{(\varepsilon)}, \quad \|f_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\|f\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}, \quad i = 0, \ldots, d.$$

We substitute this representation into (3.13) and by Lemma 3.5 we arrive at the estimate

$$\|u^{(\varepsilon)} - \hat{u}^{(\varepsilon)}\|_{W_1^2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\sum_{i=0}^{d} \|f_i^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\varepsilon \|f\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

This leads us immediately to the latter estimate in (1.9). Employing this estimate and Lemma 3.1 we obtain

$$\|u^{(\varepsilon)} - u^{(0)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\|u^{(\varepsilon)} - \hat{u}^{(\varepsilon)}\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} + C\varepsilon \|L_\varepsilon(H_0 - \lambda G_0)^{-1}f\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)} \leq C\varepsilon \|f\|_{L_2(\mathbb{R}^d; \mathbb{C}^n)}.$$

The former estimate in (1.9) is proven. Corollary 1.2 is implied by the former estimate in (1.9) with $G = G_0 = E_n$ and Theorems VIII.23, VIII.24 in [21, Ch. VIII, §7].

**Proof of Theorem 1.1 for $\lambda \in (-\infty, \mu_0)$.** By Corollary 1.2 the lower bound of $H_\varepsilon$ converges to that of $H_0$. Hence, $\mu_\varepsilon \to \mu_0$ as $\varepsilon \to +0$, and thus, for sufficiently small $\varepsilon$ the number $\lambda \in (-\infty, \mu_0)$ satisfies Lemma 3.5. It follows from (1.7) that this estimate holds true for $G_0$ as well. It is also easy to check that the estimate (3.16) is valid. In view of these facts it is clear that all the arguments in the proof of the theorem for $\text{Im} \lambda \neq 0$ remain valid in the case $\lambda \in (-\infty, \mu_0)$, if $\varepsilon$ is small enough. It proves the estimates (1.3) in the latter case as well. □

### 4 Examples

In the section we give examples of certain operators to which the results of the previous sections can be applied.

Our first example is

$$H_\varepsilon := \sum_{i,j=1}^{d} \left(-\partial_i + a_{i,\varepsilon}^*\right) g_{ij}^{(\varepsilon)} \left(\partial_j + a_{j,\varepsilon}\right) + v_\varepsilon,$$

where

$$g_{ij}^{(\varepsilon)} = g_{ij}(x, \xi) \in W_1^1(\mathbb{R}^d; C^{1+\beta}_{\text{per}}(\square)) \cap C^2(\mathbb{R}^d; C^{\beta}_{\text{per}}(\square)),
\quad a_i = a_i(x, \xi) \in W_1^1(\mathbb{R}^d; C^{1+\beta}_{\text{per}}(\square)) \cap C^2(\mathbb{R}^d; C^{\beta}_{\text{per}}(\square)),
\quad v = v(x, \xi) \in W_1^1(\mathbb{R}^d; C^{\beta}_{\text{per}}(\square))$$
are \( \Box \)-periodic matrices of the size \( n \times n \). Moreover, the identities \( v = v^* \), \( (g_{ij})^* = g_{ji} \) are supposed to be valid as well as

\[
c_1 \sum_{i=1}^{d} \|w_i\|_{C^2}^2 \leq \sum_{i,j=1}^{d} (g_{ij}w_j, w_i) \leq c_2 \sum_{i=1}^{d} \|w_i\|_{C^2}^2
\]

for all \( w_i \in \mathbb{C}^n \), \( (x, \xi) \in \mathbb{R}^{2d} \), where \( c_1, c_2 \) are constants. The operator (1.1) can be written as (1.2); let us indicate the corresponding choice of \( A, a_i, b_i, V \).

We let \( m = nd \) and choose the matrices \( A \) and \( B(\zeta) \) as

\[
B(\zeta) := \begin{pmatrix} \zeta_1 E_n \\ \zeta_2 E_n \\ \vdots \\ \zeta_d E_n \end{pmatrix}, \quad A := \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1d} \\ g_{21} & g_{22} & \cdots & g_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ g_{d1} & g_{d2} & \cdots & g_{dd} \end{pmatrix}.
\]

The matrices \( a_i, b_i \) and \( V \) are introduced as

\[
a_i := \sum_{j=1}^{d} a_{ij}^* g_{ij}, \quad b_i := E_n, \quad V := \nu + \sum_{i,j=1}^{d} a_{ij}^* g_{ij} a_j.
\]

It is easy to check that in this case the operator in (1.2) coincides with the operator in (1.1). Many operators of the mathematical physics are the particular cases of (1.1); let us mention some of them.

If we let \( a_i := 0, g_{ij} := E_n \), the operator (1.1) is a matrix Schrödinger operator. The case \( g_{ij} \neq E_n \) can be considered as the matrix Schrödinger operator with a metric. If, in addition, \( \nu = 0 \), we arrive at the operator of the elasticity theory; one just needs to assume additional symmetry conditions for the coefficients of the matrix \( g_{ij} \) (see, for instance, [1, Ch. 3]).

In the case \( n = 1 \), \( a_i := iA_i \), \( A_i \) are real-valued function, the operator (1.1) describes the magnetic Schrödinger operator. The components of the magnetic potential are the functions \( A_i \); the function \( \nu \) is the electric potential. As above, the functions \( g_{ij} \) correspond to the metric.

One more example is the two- and three-dimensional Pauli operator. We deal with this operator, if \( d = 2 \) or \( d = 3 \), \( n = 2 \), \( a_i := iA_i E_n \), \( A_i \) are real-valued function,

\[
\nu := \sigma_3 B, \quad B = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \quad \text{if} \quad d = 2,
\]

\[
\nu := \sigma_1 B_1 + \sigma_2 B_2 + \sigma_3 B_3, \quad (B_1, B_2, B_3) = \text{rot}(A_1, A_2, A_3), \quad \text{if} \quad d = 3,
\]

\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The case \( g_{ij} = E_n \) corresponds to the usual Pauli operator; if \( g_{ij} \neq E_n \), we obtain the Pauli operator with metric. One can add an additional term to the potential \( \nu \) given above. In this case we have Pauli operator with potential.
In the examples given all the results of Theorem 1.1 and Corollary 1.2 are applicable. The homogenized operator is given by the general formulas (1.5), (1.6).

This is why we will not repeat these formulas for the particular cases described.

The presence of the matrix $G_\varepsilon$ in the estimates (1.9) allow us to widen the class of the examples. In order to do it, we employ the ideas of papers [8], [12], [15].

Let $f = f(x, \xi) \in W^{1, \infty}(\mathbb{R}^d; C^{2+\beta}_{\text{per}}(\Box))$ be a positive matrix of the size $n \times n$ such that the inverse matrix is uniformly bounded. We consider the operator $\tilde{H}_\varepsilon := f^* H_\varepsilon f$, where $H_\varepsilon$ is from (1.2). It is clear that

$$f_\varepsilon (\tilde{H}_\varepsilon - \lambda G_\varepsilon)^{-1} f_\varepsilon^* = (H_\varepsilon - \lambda G_\varepsilon)^{-1}, \quad \tilde{G} = (f^*)^{-1} G f^{-1}.$$  

It allows us to approximate the generalized resolvent of the operator $\tilde{H}_\varepsilon$,

$$\| (\tilde{H}_\varepsilon - \lambda G_\varepsilon)^{-1} - f_\varepsilon^{-1}(H_0 - \lambda G_0)^{-1}(f_\varepsilon^*)^{-1} \|_{L_2 \to L_2} \leq C \varepsilon,$$

$$\| f_\varepsilon (\tilde{H}_\varepsilon - \lambda G_\varepsilon)^{-1} - (I + \varepsilon L_\varepsilon) (H_0 - \lambda G_0)^{-1}(f_\varepsilon^*)^{-1} \|_{L_2 \to W^{1, 2}_2} \leq C \varepsilon.$$  

Let us introduce now the operator $H_\varepsilon$ by (1.1); the corresponding operator $\tilde{H}_\varepsilon$ is determined by the same formula but with the coefficients replaced by

$$\tilde{g}^{ij} := f^* g^{ij} f, \quad \tilde{a}_i := f^{-1} \left( \frac{\partial f}{\partial x_i} + \varepsilon^{-1} \frac{\partial f}{\partial \xi_i} \right) + f^{-1} a_i f, \quad \tilde{v} := f^* v f.$$  

As it follows from these formulas, the coefficients of the operator $\tilde{H}_\varepsilon$ can increase as $\varepsilon \to +0$. It allows us to extend the results of the paper to the certain class of the operators with fast oscillating coefficients increasing as $\varepsilon \to +0$.

In conclusion we also observe that in the case $a_i = 0$, $v = 0$ the operator $\tilde{H}_\varepsilon$ was the main object of the study in [8]–[15]; as it has been already mentioned in the beginning of the paper, under the essentially weaker assumptions for the coefficients. In the cited the authors gave a great number of interesting examples for such operators. Our results can extended to these examples as well. The novelty will be the dependence of the coefficients of slow variable and the asymptotics expansions for the eigenvalues.

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