ON THE GENERIC TRIANGLE GROUP

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Abstract

We introduce the concept of a generic Euclidean triangle $\tau$ and study the group $G_\tau$ generated by the reflection across the edges of $\tau$. In particular, we prove that the subgroup $T_\tau$ consisting of the translations in $G_\tau$ is a free abelian group of infinite rank, and derive from this fact an infinite presentation for $G_\tau$. Moreover, we discuss some examples of non-trivial relations in $T_\tau$ holding for given non-generic triangles $\tau$.

1. Introduction

The term triangle group is generally reserved in the literature to the group $G_\tau$ generated by the reflections $r_1, r_2, r_3$ across the sides of an Euclidean, spherical or hyperbolic triangle $\tau$ with internal angles $\alpha_i = \pi/n_i$, where the specific geometry depends on $1/n_1 + 1/n_2 + 1/n_3$ being $= 1$, $> 1$ or $< 1$, respectively. The structure of these groups is well understood since the seminal works by Fricke and Klein [4] and Coxeter [2] (see also [3]). In particular, based on the fact that the triangle $\tau$ tiles the plane or the sphere, we have the finite presentation $G_\tau = \langle x_1, x_2, x_3 | x_1^n, x_2^n, x_3^n, (x_1x_2)^{n_1}, (x_2x_3)^{n_2}, (x_3x_1)^{n_3} \rangle$ with the symbol $x_i$ corresponding to the reflection $r_i$.

The hyperbolic case (the only non-trivial one) has been widely studied, with a special focus on its complex version, and several notions of generalized triangle groups have been considered, in terms of finite presentation independently on the original geometric setting.

But little seems to be know about the group $G_\tau$ for more general triangles $\tau$, even if the strong hypothesis on the angles $\alpha_i$ is just mildly relaxed to include rational multiples of $\pi$, that is in the case of $\tau$ a rational triangle with $\alpha_i = m_i\pi/n_i$.

On the other hand, in the Euclidean context the structure of the reflection group $G_\tau$ and of its translation subgroup $T_\tau \subset G_\tau$, consisting of all the translations in $G_\tau$, is relevant to the problem of periodic billiard trajectories in $\tau$, when this is considered as an ideal billiard table without any friction.

In a rational triangle the number of directions of a given billiard trajectory is finite. This entails that the billiard flow on the four dimensional phase space (think of the tangent bundle of $\tau$) has another integral of motion besides speed, so that the flow is effectively taking place on surfaces inside the phase space. The major theoretical problem is that of locating closed (or uniformly distributed) orbits on these surfaces (see [7]).

For more general triangles, yet the question of whether there exists at least one periodic billiard trajectory is still open and the problem of classifying billiard trajectories can be more difficult. Here, of special interest are the stable periodic trajectories, namely those that are not destroyed by an arbitrary small deformation of the triangle (see [5]). Their existence as well as their properties appear to be strongly related to the structure of the group $T_\tau$. 
With such motivation, in this note we study the groups $T_\tau \subset G_\tau$ for an arbitrary Euclidean triangle $\tau$, starting from the case of generic triangles. These are introduced in Section 2 as the triangles whose edge lengths are algebraically independent over the rationals (up to a common factor), and can be considered as the opposite to the rational triangles in the spectrum of all Euclidean triangles. In particular, we show that generic triangles include typical triangles, the ones whose angles are linearly independent over the rational, where all the periodic billiard trajectories are stable.

Our main results concerning generic triangles $\tau$ are Theorems 4.1 and 4.2. The first asserts that $T_\tau$ is a free abelian group generated by the translation $t_1 = (r_1 r_2 r_3)^2$ and its conjugates in $G_\tau$, while in the second an infinite presentation of $G_\tau$ is derived.

Finally, in Section 5 we discuss some examples of non-trivial relations in $T_\tau$, holding for continuous families of non-generic but typical triangles $\tau$ and for certain isolated such triangles, respectively.

2. Generic triangles

Given an Euclidean triangle $\tau = A_1 A_2 A_3$, let $\ell_i > 0$ denote the length of the edge $e_i = A_j A_k$ and $\alpha_i > 0$ denote the measure in radians of the (non-oriented) interior angle $A_j A_i A_k$, with $\{i, j, k\} = \{1, 2, 3\}$.

Definition 2.1. We call $\tau$ a generic triangle if for some $k > 0$ (hence for almost every $k \in \mathbb{R}$) the real numbers $k \ell_1, k \ell_2$ and $k \ell_3$ are algebraically independent (over the rationals), namely it does not exist any non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ such that $p(k \ell_1, k \ell_2, k \ell_3) = 0$.

A different formulation of the above condition is that for every non-trivial polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ the polynomial $q(x) = p(x_1, x_2, x_3)$ is non-trivial in $\mathbb{R}[x]$, or equivalently the field $\mathbb{Q}(\ell_1 x, \ell_2 x, \ell_3 x)$ has transcendence degree 3 over $\mathbb{Q}$ (see [8]). Then a straightforward argument on cardinalities shows that generic triangles are dense in the space of all Euclidean triangles (the same argument also explains why “some $k$” could be replaced by “almost every $k$” in the definition).

Next proposition just translates Definition 2.1 in terms of trigonometric functions of the interior angles of the triangle $\tau$.

Proposition 2.2. An Euclidean triangle $\tau$ as above is a generic triangle if and only one of the following equivalent properties holds:

(s) for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k \sin \alpha_1, k \sin \alpha_2$ and $k \sin \alpha_3$ are algebraically independent over the rationals;

(c) for some (hence almost every) $k \in \mathbb{R}$ the real numbers $k \cos \alpha_1, k \cos \alpha_2$ and $k \cos \alpha_3$ are algebraically independent over the rationals.

Proof. The equivalence of the condition in Definition 2.1 with property (s) immediately follows by the law of sines, while some work is needed to verify the equivalence between (s) and (c). Once these properties are reformulated in terms of extensions of $\mathbb{Q}$ involving the indeterminate $x$ as above, we are reduced to proving that $\mathbb{Q}(x \sin \alpha_1, x \sin \alpha_2, x \sin \alpha_3)$ and $\mathbb{Q}(x \cos \alpha_1, x \cos \alpha_2, x \cos \alpha_3)$ have the same transcendence degree over $\mathbb{Q}$. By elementary trigonometry, from $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we get the equations

$$\cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3 + 2 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 = 1,$$

(1)
\[ \sin^4 \alpha_1 + \sin^4 \alpha_2 + \sin^4 \alpha_3 + 4 \sin^2 \alpha_1 \sin^2 \alpha_2 \sin^2 \alpha_3 + 
- 2 \sin^2 \alpha_1 \sin^2 \alpha_2 - 2 \sin^2 \alpha_1 \sin^2 \alpha_3 - 2 \sin^2 \alpha_2 \sin^2 \alpha_3 = 0. \] (2)

Multiplying (1) by \(x^3\), we see that \(x\) is algebraic over \(\mathbb{Q}(x \cos \alpha_1, x \cos \alpha_2, x \cos \alpha_3)\), hence the possibly larger extension \(\mathbb{Q}(x \cos \alpha_1, x \cos \alpha_2, x \cos \alpha_3, x)\) has the same transcendence degree as \(\mathbb{Q}(x \cos \alpha_1, x \cos \alpha_2, x \cos \alpha_3)\) over \(\mathbb{Q}\). Similarly, multiplying (2) by \(x^6\), we see that \(x\) is algebraic over \(\mathbb{Q}(x \sin \alpha_1, x \sin \alpha_2, x \sin \alpha_3)\) as well, hence the transcendence degree of \(\mathbb{Q}(x \sin \alpha_1, x \sin \alpha_2, x \sin \alpha_3, x)\) over \(\mathbb{Q}\) is the same as that of \(\mathbb{Q}(x \sin \alpha_1, x \sin \alpha_2, x \sin \alpha_3)\). Now, the relations \(x^2 \sin^2 \alpha_i + x^2 \cos^2 \alpha_i = x^2\) allow us to conclude that \(\mathbb{Q}(x \cos \alpha_1, x \cos \alpha_2, x \cos \alpha_3, x)\) and \(\mathbb{Q}(x \sin \alpha_1, x \sin \alpha_2, x \sin \alpha_3, x)\) have the same transcendence degree over \(\mathbb{Q}\). \(\square\)

As noticed in the above proof, due to the relation \(\alpha_1 + \alpha_2 + \alpha_3 = \pi\) we cannot have \(k = 1\) (or any rational number) in points (s) and (c) of Proposition 2.2. However, if \(\tau\) is a generic triangle then \(\ell_1, \ell_2, \ell_3\), as well as \(\sin \alpha_1, \sin \alpha_2, \sin \alpha_3\) and \(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3\), form linearly independent triples over the rationals, being linear independence a homogeneous condition where the factor \(r > 0\) can be canceled.

**Lemma 2.3.** If an algebraic relation \(p(\sin \alpha_1, \sin \alpha_2, \sin \alpha_3, \cos \alpha_1, \cos \alpha_2, \cos \alpha_3) = 0\), with \(p(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6]\), holds for a generic triangle, then it holds for every triangle.

**Proof.** By performing in

\[ p(\sin \alpha_1, \sin \alpha_2, \sin \alpha_3, \cos \alpha_1, \cos \alpha_2, \cos \alpha_3) = 0 \] (3)

the replacements

\[ \sin \alpha_i = \frac{\sqrt{\ell_1^2 + \ell_2^2 + \ell_3^2 - 2(\ell_1^2 + \ell_2^2 + \ell_3^2)}}{2\ell_j \ell_k} \quad \text{and} \quad \cos \alpha_i = \frac{\ell_i^2 - \ell_j^2 - \ell_k^2}{2\ell_j \ell_k} \] (4)

given by the laws of sines and cosines, we obtain

\[ q(\ell_1, \ell_2, \ell_3, \sqrt{\ell_1^2 + \ell_2^2 + \ell_3^2 - 2(\ell_1^2 + \ell_2^2 + \ell_3^2)}) = 0, \] (5)

with \(q(x_1, x_2, x_3, x_4) \in \mathbb{Z}[x_1, x_2, x_3, x_4]\).

Considering \(q(x_1, x_2, x_3, x_4)\) as a polynomial in the indeterminate \(x_4\) with coefficients in \(\mathbb{Z}[x_1, x_2, x_3]\) and dividing it by \(x_4^2 - (x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^4 + x_2^4 + x_3^4)\), we get a binomial \(a(x_1, x_2, x_3) x_4 + b(x_1, x_2, x_3)\) as the remainder, with \(a(x_1, x_2, x_3)\) and \(b(x_1, x_2, x_3)\) in \(\mathbb{Z}[x_1, x_2, x_3]\).

Then, equation (5) turns out to be equivalent to

\[ a(\ell_1, \ell_2, \ell_3) \sqrt{\ell_1^2 + \ell_2^2 + \ell_3^2 - 2(\ell_1^2 + \ell_2^2 + \ell_3^2)} = - b(\ell_1, \ell_2, \ell_3), \] (6)

which squared gives the integral algebraic relation

\[ a(\ell_1, \ell_2, \ell_3)^2 [(\ell_1^2 + \ell_2^2 + \ell_3^2 - 2(\ell_1^2 + \ell_2^2 + \ell_3^2))] = b(\ell_1, \ell_2, \ell_3)^2. \] (7)

Taking into account that the replacing expressions in (3) are homogeneous of degree 0 on \(\ell_1, \ell_2\) and \(\ell_3\), we have that also the equation (7) must be homogeneous.
Now, if the relation (3) holds for the angles $\alpha_1, \alpha_2$ and $\alpha_3$ of a generic triangle, then relation (7) holds for the lengths $\ell_1, \ell_2$ and $\ell_3$ of its edges, even if these are scaled by any factor $k > 0$ (by the homogeneity). By definition of a generic triangle, this implies that the polynomial
\[
a(x_1, x_2, x_3)^2[(x_1^2 + x_2^2 + x_3^2)^2 - 2(x_1^4 + x_2^4 + x_3^4)] - b(x_1, x_2, x_3)^2
\]
is trivial in $\mathbb{Z}[x_1, x_2, x_3]$. Then, equations (7) and (5) are identically satisfied and we can conclude that (3) holds for every triangle. \(\square\)

A deeper analysis of the algebraic dependence of $\cos \alpha_1, \cos \alpha_2$ and $\cos \alpha_3$, shows that generic triangles are typical, in the sense of the following definition (see [5]).

**Definition 2.4.** An Euclidean triangle $\tau$ as above is called a typical triangle if the real numbers $\alpha_1, \alpha_2$ and $\alpha_3$ are linearly independent over the rationals.

**Proposition 2.5.** Generic Euclidean triangles are typical.

**Proof.** Let $\tau$ a generic triangle. We want to prove that equation (1) is essentially the only algebraic relation between the cosines of the interior angles of $\tau$, being any polynomial $p(x_1, x_2, x_3) \in \mathbb{Z}[x_1, x_2, x_3]$ such that $p(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3) = 0$ divisible by
\[
x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 - 1.
\]

Assume that the identity $p(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3) = 0$ holds for $\tau$. Then, according to Lemma 2.3, it must hold for any triangle, and using once again the relation $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ we have $p(-\cos(\alpha_2 + \alpha_3), \cos \alpha_2, \cos \alpha_3) = 0$ for every $\alpha_2, \alpha_3 \geq 0$ such that $\alpha_2 + \alpha_3 < \pi$. Therefore, $p(x_1, x_2, x_3)$ is divisible by both the linear binomials $x_1 + x_2x_3 \pm \sqrt{(1-x_2^2)(1-x_3^2)}$ in the indeterminate $x_1$ with coefficients in the quadratic closure of the field of fractions $\mathbb{Q}(x_2, x_3)$, hence it is divisible by $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2x_3 - 1$ in $\mathbb{Z}[x_1, x_2, x_3]$ (notice that the last polynomial is monic with respect to $x_1$).

Now, by contradiction, let $n_1\alpha_1 + n_2\alpha_2 + n_3\alpha_3 = 0$ be a non-trivial vanishing linear combination of the interior angles $\alpha_1, \alpha_2$ and $\alpha_3$ of a generic triangle, with integral coefficients $n_1, n_2$ and $n_3$, which can be assumed coprime without loss of generality. By elementary trigonometry, we have
\[
\cos(n_1\alpha_1) - \cos(n_2\alpha_2)\cos(n_3\alpha_3) = \sin(n_2\alpha_2)\sin(n_3\alpha_3).
\]

Squaring and using the Pythagorean identity, we readily obtain
\[
T_{n_1}^2(\cos \alpha_1) + T_{n_2}^2(\cos \alpha_2) + T_{n_3}^2(\cos \alpha_3) - 2T_{n_1}(\cos \alpha_1)T_{n_2}(\cos \alpha_2)T_{n_3}(\cos \alpha_3) = 1,
\]
where $T_n(x) \in \mathbb{Z}[x]$ denotes the Chebyshev polynomials defined by the identity $T_n(\cos \alpha) = \cos(n\alpha)$. By the above, the polynomial
\[
T_{n_1}^2(x_1) + T_{n_2}^2(x_2) + T_{n_3}^2(x_3) - 2T_{n_1}(x_1)T_{n_2}(x_2)T_{n_3}(x_3) - 1
\]
must be divisible by (9). Since $T_n(1) = 1$ for all $n$, setting $x_2 = x_3 = 1$ both in (9) and (12), we get $(x_1 + 1)^2 = 0$ and $(T_{n_1}(x_1) - 1)^2 = 0$, respectively. Thus, $T_{n_1}(x_1) - 1$ must be divisible by $x_1 + 1$ in $\mathbb{Z}[x_1]$. This implies that $n_1$ is even, because $T_n(x)$ has same parity of $n$. By the symmetry of (9) and (12), the same argument shows that $n_2$ and $n_3$ must be even as well, contradicting the coprimality assumption. \(\square\)

In the light of Proposition 2.5, generic triangles can be somewhat thought of as the opposite end in the spectrum of all Euclidean triangles with respect to the rational ones.
3. The reflection group

For any Euclidean triangle $\tau$ we denote by $G_\tau = \langle r_1, r_2, r_3 \rangle \subset E(2)$ the subgroup of the group $E(2)$ of the Euclidean isometries of the plane generated by the reflections $r_1, r_2$ and $r_3$ on the edges $e_1, e_2$ and $e_3$ of $\tau$, respectively.

The standard exact sequence

\[ 0 \rightarrow \mathbb{R}^2 \overset{\iota}{\rightarrow} E(2) \overset{\lambda}{\rightarrow} O(2) \rightarrow 0, \]  

(13)

where $\iota$ is the inclusion of $\mathbb{R}^2$ in $E(2)$ as the subgroup of translations and $\lambda$ is the linearization homomorphism, induces by restriction the exact sequence

\[ 0 \rightarrow T_\tau \overset{\iota_\tau}{\rightarrow} G_\tau \overset{\lambda_\tau}{\rightarrow} S_\tau \rightarrow 0, \]  

(14)

where $T_\tau \subset G_\tau$ is the subgroup of $G_\tau$ consisting of translations, while $S_\tau = \lambda(G_\tau) = \langle s_1, s_2, s_3 \rangle \subset O(2)$ with $s_i = \lambda(r_i)$ the linearization of $r_i$.

We observe that the structure of the group $G_\tau$ (including the latter exact sequence) is invariant under similarities. Hence, without loss of generality we can assume that the incircle of the triangle $\tau$ is coincides with the unit circle centered at the origin. Under this assumption, we have

\[ (x)r_i = (x)s_i + 2v_i \]  

(15)

for every $x \in \mathbb{R}^2$ (here and in the following we use the right notation for the action of $G_\tau$), where $v_i$ is the unit vector from the origin to the tangency point of the edge $e_i$ and the incircle of $\tau$, as shown in Figure 1.

![Figure 1. The unit vectors $v_1, v_2$ and $v_3$.](image)

We want to determine a minimal presentation of the group $S_\tau$ in the case when $\tau$ is a typical, and hence $S_\tau$ is a dense subgroup of $O(2)$. In order to do that, we first recall from [5] the definition of a stable sequence and the stability criterion for a product of generators of $S_\tau$ to be trivial.

**Definition 3.1.** A sequence $i_1i_2\ldots i_n$ of symbols from $\{1, 2, \ldots, N\}$ is called a **stable sequence** if its terms can be paired into disjoint pairs of identical symbols, one located at an odd and the other at an even position. Differently said, the length $n$ of the sequence is even and the symbolic alternating sum $i_1 - i_2 + \ldots + i_{n-1} - i_n$ vanish (as an algebraic sum of symbols, not of integers).

The motivation for the term “stable” is that a stable sequence as above represents the sequence of sides of a polygonal billiard (whose $N$ sides are arbitrarily numbered) visited by a $n$-periodic trajectory, which is stable in the sense that it survives to small
perturbations of the polygon (see [5]). Actually, also Lemma 3.3 below and its proof are essentially translated from the context of stable trajectories in polygonal billiards, focusing on the case \( N = 3 \).

Before going on, let us give an operational characterization of stability.

**Lemma 3.2.** A sequence \( i_1i_2\ldots i_n \) is stable if and only if one can reduce it to the empty sequence by a finite number of operation of the following types:

(a) transposition of two adjacent subsequences both consisting of two symbols;

(b) deletion of a subsequence consisting of two identical symbols.

**Proof.** First of all, we note that both operations and the inverse of the second one, that is the insertion of two adjacent identical symbols in a sequence, all preserve the parity of the position of each term in the sequence, hence they preserve stability. This immediately gives the “if” part of the statement, being the empty sequence stable.

The “only if” part can be proved by induction on the length of the sequence, starting once again from the empty sequence. For the inductive step, assume we are given any non-empty stable sequence \( i_1i_2\ldots i_n \). The stability implies that \( i_{2k-1} = i_2 \) for some \( 1 \leq k \leq n/2 \). If \( k = 1 \), we can reduce the length of the sequence by deleting the subsequence \( i_1i_2 \). Otherwise, by \( k - 2 \) transpositions of pairs, we get a sequence starting with the four symbols \( i_1i_2i_{2k-1}i_{2k+1} \). Then we can reduce the length of the word by deleting the subsequence \( i_1i_2i_{2k+1} \). □

**Lemma 3.3.** If a sequence \( i_1i_2\ldots i_n \) of symbols from \{1, 2, 3\} is stable, then the product \( s_{i_1}s_{i_2}\ldots s_{i_n} \) is the identity in \( S_\tau \). Moreover, for a typical triangle \( \tau \) the stability of the sequence \( i_1i_2\ldots i_n \) is also necessary in order \( s_{i_1}s_{i_2}\ldots s_{i_n} \) to be the identity.

**Proof.** We proceed in the same spirit as in [5, Sec. 6.B]. We first orient the edges \( e_1, e_2 \) and \( e_3 \) in the counterclockwise way along the boundary of the triangle \( \tau \), and denote by \( \beta_i \) the oriented angle from \( e_1 \) (fixed as reference vector) to \( e_i \). Then, we have \( \beta_1 = 0 \), \( \beta_2 = \pi - \alpha_3 \) and \( \beta_3 = \pi + \alpha_2 \) mod \( 2\pi \). Moreover, any composition \( s_is_k \) gives the linear rotation of angle \( 2(\beta_i - \beta_j) \mod 2\pi \), and hence any product \( s_{i_1}s_{i_2}\ldots s_{i_n} \) with \( n \) even gives the linear rotation of angle

\[
\varphi = 2(\beta_{i_2} - \beta_{i_1}) + \cdots + 2(\beta_{i_n} - \beta_{i_{n-1}}) \mod 2\pi . \quad (16)
\]

Now, the stability of the sequence \( i_1i_2\ldots i_n \) implies that \( \varphi = 0 \mod 2\pi \) and thus \( s_{i_1}s_{i_2}\ldots s_{i_n} \) is the identity in \( S_\tau \).

In the opposite direction, start with a product \( s_{i_1}s_{i_2}\ldots s_{i_n} \) that gives the identity. Then, \( n \) must be even and (16) can be rewritten in terms of the \( \alpha_i \)'s by using the above identities. If the sequence \( i_1i_2\ldots i_n \) is not stable, this yields a non-trivial rational linear relation among the angles \( \alpha_2, \alpha_3 \) and \( \pi \), which implies that \( \tau \) is not typical. □

At this point, we are in position to obtain the wanted presentation of \( S_\tau \).

**Proposition 3.4.** For a typical triangle \( \tau \) the group \( S_\tau \) admits the finite presentation

\[
\langle x_1, x_2, x_3 | x_1^2, x_2^2, x_3^2, (x_1x_2x_3)^2 \rangle ,
\]

with the symbols \( x_1, x_2 \) and \( x_3 \) corresponding to \( s_1, s_2 \) and \( s_3 \), respectively.
Proof. According to Lemma 3.3, all the four relations of the presentation hold in $S$, being the corresponding sequences of indices stable. 

Viceversa, Lemmas 3.2 and 3.3 say that any word $x_{i_1}x_{i_2} \ldots x_{i_n}$ representing the identity in $S$ can be reduced to the empty word by canceling squared terms $x_i^2$ and commuting products $x_i x_j$ and $x_k x_l$. So, to conclude the proof it is enough to show that any commutator $[x_i x_j, x_k x_l]$ is the identity modulo the given four relations. Up to inversions, the only non-trivial cases are $[x_1 x_2, x_1 x_3]$, $[x_1 x_2, x_2 x_3]$ and $[x_1 x_3, x_2 x_3]$. For these we have: 

$$[x_1 x_2, x_1 x_3] = x_1 x_3 (x_1 x_2 x_3)^{-2} x_3 x_1$$ and 

$$[x_1 x_2, x_2 x_3] = [x_1 x_3, x_2 x_3] = x_1(x_1 x_2 x_3)^{-2} x_1.$$ □

4. THE TRANSLATION SUBGROUP

According to the exact sequence (14) and Lemma 3.3 in the previous section, for any Euclidean triangle $\tau$ the product $r_{i_1} r_{i_2} \ldots r_{i_n}$ gives a translation in $T_\tau$ if the sequence $i_1 i_2 \ldots i_n$ is stable. On the other hand, when the triangle $\tau$ is typical we obtain in this way all the translations in $T_\tau$, and it is clear from Proposition 3.4 that a special role is played by the minimal stable product $t_1 = (r_1 r_2 r_3)^2$ and by its conjugation class

$$C(t_1) = \{ (t_1) g = g^{-1} t_1 g \mid g \in G_\tau \} \subset T_\tau.$$ (17)

By identifying translations in $T_\tau$ with the corresponding vectors in $\mathbb{R}^2$ and considering the natural action of $S_\tau \subset O(2)$ on them, for $g = r_{i_1} r_{i_2} \ldots r_{i_n} \in G_\tau$ the conjugate $(t_1) g$ is given by

$$(t_1) r_{i_1} r_{i_2} \ldots r_{i_n} = (t_1) s_{i_1} s_{i_2} \ldots s_{i_n},$$ (18)

according to the notation introduced in equation (15). In the case when $\tau$ is of a typical triangle, the density of the subgroup $S_\tau \subset O(2)$ implies that $C(t_1)$ forms a dense subset of the circle $\rho S^1 \subset R^2$ of radius

$$\rho = ||t_1|| = 4(\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3).$$ (19)

The translation $t_1$ and its conjugates $t_2 = (r_2 r_3 r_1)^2 = (t_1) r_1$ and $t_3 = (r_3 r_1 r_2)^2 = (t_1) r_3$, are represented in Figure 2 (here we assume the same numbering as in the previous Figure 1 for the edges of the triangle $\tau$).

Figure 2. The translations $t_1, t_2$ and $t_3$
Theorem 4.1. For a typical triangle \( \tau \) the subgroup \( T_\tau \subset G_\tau \) is normally generated by the translation \( t_1 = (r_1 r_2 r_3)^2 \). Moreover, if \( \tau \) is generic then \( T_\tau \) is a free abelian group having as a basis the conjugation class \( C(t_1) \).

Proof. Given any \( t \in T_\tau \) with \( \tau \) a typical triangle, we can express it as a product \( r_1, r_2 \ldots, r_n \) of generators of \( G_\tau \). In view of the exact sequence (14), the corresponding product \( s_1 s_2 \ldots s_n \) gives the identity in \( S_\tau \), hence the sequence \( i_1 i_2 \ldots i_n \) is stable by Lemma 3.3. Then, arguing as in the proof of Proposition 3.4, we can rewrite \( Z \) independent over elements of \( r \) by the translation \( t \) having as a basis the conjugation class \( \beta \) equation (16) tells us that the oriented angle from \( t \) with \( m \) cannot be finitely presentable, since it contains a free abelian group of infinite rank. Notice that in the above equations the pairs \( (\alpha, \beta) \) are different from each other, being the conjugates in (20) pairwise distinct. Moreover, possibly after suitable changes of signs in order to have either \( m_j,2 > 0 \) or \( m_j,2 = 0 \) and \( m_j,3 \geq 0 \), we can collect the (at most two) terms corresponding to opposite pairs.

As an immediate consequence of Theorem 4.1, when the triangle \( \tau \) is generic the group \( G_\tau \) cannot be finitely presentable, since it contains a free abelian group of infinite rank. In this case, next theorem provides an essentially minimal presentation of \( G_\tau \). Actually, one could further reduce by symmetry the set of relations, assuming that either \( n > 0 \) or \( n = 0 \) and \( m > 0 \).
Theorem 4.2. For a generic triangle $\tau$ the group $G_{\tau}$ admits the presentation
\[ \langle x_1, x_2, x_3 \mid x_1^2, x_2^2, x_3^2, (x_1x_2x_3)^2, (x_1x_2)^n(x_1x_3)^m(x_2x_3)^{2n} \rangle, \]
with the symbols $x_1, x_2$ and $x_3$ corresponding to $r_1, r_2$ and $r_3$, respectively.

Proof. By a standard argument (see [6, Section 10.2]), a presentation of $G_{\tau}$ can be derived from presentations of the groups $T_{r_i}$ and $S_r$ involved in the exact sequence (14). In view of Proposition 2.5, a presentation of $S_r$ is given by Proposition 3.4. Pulling back the generators $s_i$ of $S_r$ to the generators $r_i$ of $G_{\tau}$, the relations $s_i^2 = 1$ still hold in the same form $r_i^2 = 1$, while the relation $(s_1s_2s_3)^2 = 1$ turns into the identity $(r_1r_2r_3)^2 = t_1$. At this point, to complete the set relations for $G_{\tau}$ it remains to interpret the commutators between all the elements of $C(t_1)$. Actually, it suffices to consider the commutators $[t_1, t]$ with $t \in C(t_1)$.

As we said in the proof of Theorem 4.1, any conjugate $t \in C(t_1)$ can be obtained from $t_1$ by a rotation of $-2m\alpha_2 + 2n\alpha_3$ radians for some (uniquely determined, being $\tau$ typical) integers $m$ and $n$. Then, according to equation (16), we have
\[ t = (t_1)(r_3r_1)^m(r_2r_1)^n. \]
This leads to the specific form of the commutator relations in the presentation of $G_{\tau}$ given in the statement. \qed

5. Examples of non-generic relations

As we have seen in the previous section, apart from the obvious involutive property of the $r_i$’s, the only general relations in $G_{\tau}$, that is the ones holding for $\tau$ a generic triangle or equivalently for every triangle $\tau$, are provided by the commutators of the translations in the free abelian subgroup $T_{r_i}$ generated by the conjugates of $t_1$.

Here, we briefly discuss the existence of extra non-generic relations for the subgroup $T_{r_i}$, and hence for the group $G_{r_i}$, in the case when the triangle $\tau$ is typical but not generic. In this respect, typical triangles are expected to present a rich unexplored structure, in some sense complementary to the one encoded by the relations $(r_1r_2)^{n_2}, (r_2r_3)^{n_1}$ and $(r_3r_1)^{n_3}$ for a rational triangle $\tau$ having angles $m_i\pi/n_i$ with $(m_i, n_i) = 1$, which has been widely considered in the literature after the pioneering work of Coxeter [2].

In Figure 3 two relations of $G_{\tau}$ are represented in terms of the corresponding chain of triangles generated by each next reflection in the word, starting from $\tau$ and ending back to $\tau$. Namely, on the left side there is the generic relation given by the commutator
\[ [t_1, (t_1)r_1r_3] = t_1r_3r_1t_1r_3t_1^{-1}r_3r_1t_1^{-1}r_1r_3 = r_1r_2r_3r_1r_3r_2r_3r_2r_1r_3r_2r_1r_3r_2r_3r_2r_1r_3r_2, \]
where some $r_i^2$ has been canceled in the last expression, while on the right side there is the non-generic relation
\[ t_1 \cdot (t_1^{-1})r_1 \cdot (t_1)r_1r_3r_2 \cdot (t_1^{-1})r_1r_3r_1 \cdot (t_1^{-1})r_3 = r_1r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3r_1r_3r_2r_3, \]
which holds only for the triangles $\tau$ whose angles $\alpha_i$ satisfy a specific condition (in particular for all the triangles such that $2 \cos(2\alpha_2 + 2\alpha_3) - 2 \cos 2\alpha_2 = 1$).
The big vectors superposed to the chains of triangles in the figure correspond to the expression of the relation as a word in the set $C(t_1)$ of generators of the translation subgroup $T_\tau$, while the small vectors indicate the displacement of the incenter of the triangle under the action of each next reflection in the expression of the relation as a word in the generators $r_1, r_2$ and $r_3$ of $G_\tau$. The lengths of these two word representations of a relation, in the generators of $T_\tau$ and $G_\tau$ respectively, provide relatively independent measures of its complexity.

The following table reports the number of cyclically reduced stable words of minimal length from 6 to 24 in the generators $r_1, r_2$ and $r_3$ of $G_\tau$, pairwise distinct up to permutation of the indices, inversion, conjugation and commutation of stable subwords. These have been obtained by a computer procedure in three steps: first, the generation of a complete list of all the cyclically reduced stable words of a given length; then, the elimination of duplicates up to permutation of the indices, inversion and cyclic permutations of the word; finally, the detection of the remaining pair of words (even of different lengths) equivalent up to conjugation and commutation of stable subwords, by comparing their expressions as linear combinations of vectors in $C(t_1)$. The total numbers of words of different minimal lengths in the last column are subdivided in the previous columns, according to the length from 1 to 12 with respect to the set $C(t_1)$ of generators of $T_\tau$.

|     | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | Total |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|-------|
| 6   | 1  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 1     |
| 8   | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 0     |
| 10  | -  | 1  | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 1     |
| 12  | -  | 2  | 1  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 3     |
| 14  | -  | 2  | 2  | 1  | -  | -  | -  | -  | -  | -  | -  | -  | 5     |
| 16  | -  | 2  | 4  | 1  | 1  | -  | -  | -  | -  | -  | -  | -  | 8     |
| 18  | -  | 2  | 8  | 8  | 3  | 3  | 1  | -  | -  | -  | -  | -  | 25    |
| 20  | 2  | 10 | 24 | 11 | 4  | 3  | 1  | -  | -  | -  | -  | -  | 55    |
| 22  | 4  | 22 | 53 | 40 | 23 | 10 | 9  | 4  | 1  | -  | -  | -  | 166   |
| 24  | 2  | 25 | 103| 129| 87 | 49 | 24 | 15 | 9  | 2  | 1  | 444  |

Figure 3. Generic and non-generic relations
Now, in order to determine when a cyclically reduced stable word \( r_{i_1} r_{i_2} \ldots r_{i_n} \) represents the identity in \( T_\tau \), one could directly observe that, according to Lemma 3.3 and equation (15), this happens if and only if
\[
\sum_{j=1}^{n} (v_{ij}) s_{i,j+1} \ldots s_{in} = 0. \tag{28}
\]
Notice that the \( j \)-th term of this summation coincides with half the displacement vector of the incenter of the triangle under the action of the \( j \)-th reflection in \( r_{i_1} r_{i_2} \ldots r_{i_n} \). Taking into account that all these vectors have the same norm, equation (16) could be applied to rewrite equation (33) as a condition on the angles \( \alpha_2 \) and \( \alpha_3 \) of \( \tau \) under which \( r_{i_1} r_{i_2} \ldots r_{i_n} \) is a relation for \( T_\tau \).

A more convenient approach to the same condition on the angles of \( T_\tau \) is provided by the proof of Theorem 4.1. Once the translation vector corresponding to the word \( r_{i_1} r_{i_2} \ldots r_{i_n} \) has been expressed as a linear combination of vectors in \( C(t_1) \), equation (33) can be put in the form
\[
\sum_{j=1}^{m} k_j (t_1) r_{i_1,r_{i_2}} \ldots r_{i_j} = 0. \tag{29}
\]
Then, according to equations (21) and (22), we get the equivalent system
\[
\begin{align*}
\sum_{j=1}^{m} k_j \cos(m_{j,2} \alpha_2 + m_{j,3} \alpha_3) &= 0 \\
\sum_{j=1}^{m} k_j \sin(m_{j,2} \alpha_2 + m_{j,3} \alpha_3) &= 0,
\end{align*} \tag{30}
\]
where \( m_{j,2} \alpha_2 + m_{j,3} \alpha_3 \) is the oriented angle from \( t_1 \) to \( (t_1) r_{i_1} r_{i_2} \ldots r_{i_j} \).

By a systematic computer search among the stable words up to length 24 generated as said above, we found that the shortest words in the \( r_i \)'s giving non-generic relations for some typical triangle have length 18. Up to permutation of the indices, inversion, conjugation and commutation of stable subwords, there are two of such words of length 18. As discussed in the Examples 5.1 and 5.2 below, both the relations hold for a continuous family of triangles, forming a curve in the space of parameters
\[
\mathcal{T} = \{ (\alpha_2, \alpha_3) \mid \alpha_2, \alpha_3 > 0 \text{ and } \alpha_2 + \alpha_3 < \pi \}, \tag{31}
\]
and almost all the triangles in that family are typical. Moreover, the relation presented in Example 5.1 has minimal length also with respect to the set generators \( C(t_1) \), being not difficult to see that, apart from commutators, any extra relation holding in \( T_\tau \) for a typical triangle \( \tau \) must have length at least 5 in terms of conjugates of \( t_1 \).

**Example 5.1.** Consider the stable word of length 18
\[
(r_1 r_2 r_3 r_2 r_3 r_1 r_2 r_1 r_3)^2 = (t_1) r_1 r_3 r_2 r_1 \cdot t_1 \cdot (t_1) r_3 r_1 r_3 r_2 r_3 \cdot (t_1) r_1 r_3 \cdot (t_1) r_3 r_1 r_3. \tag{32}
\]
The corresponding translation vector is
\[
t_1 + (t_1) r_1 r_3 + (t_1) r_1 r_3 r_2 r_1 + (t_1) r_3 r_1 r_3 + (t_1) r_3 r_1 r_3 r_2 r_3 \\
= t_1 + (t_1) r_1 r_3 + (t_1) r_2 r_3 + (t_1) r_1 r_2 r_1 r_3 + (t_1) r_1 r_3 r_1 r_3
\tag{33}
\]
where the simplification is based on the equation (18), the commutativity of the rotations \( s_i s_j \) and \( s_k s_l \), and the identities \((t_1) r_1 r_2 r_3 = t_1 \) and \( s_i^2 = 1 \). Now, the oriented angles from
to the five vectors in (33) are respectively given by \(0, 2(\alpha_2 + \alpha_3), 2\alpha_2, -2(\alpha_2 - \alpha_3)\) and \(4\alpha_2\), and by replacing in (30) we get the system

\[
\begin{align*}
\cos 2(\alpha_2 + \alpha_3) + \cos 2\alpha_2 + \cos 2(\alpha_2 - \alpha_3) + \cos 4\alpha_2 &= -1 \\
\sin 2(\alpha_2 + \alpha_3) + \sin 2\alpha_2 - \sin 2(\alpha_2 - \alpha_3) + \sin 4\alpha_2 &= 0
\end{align*}
\]  

(34)

By standard trigonometric identities, this system is equivalent to

\[
\begin{align*}
(1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3) \cos 2\alpha_2 &= 0 \\
(1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3) \sin 2\alpha_2 &= 0
\end{align*}
\]  

(35)

hence to the equation

\[1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3 = 0.\]  

(36)

The curve solutions of this equation in the space of parameters \(\mathcal{T}\) is plotted on the left side of Figure 4. Since non-typical triangles form a dense countable union of straight lines in \(\mathcal{T}\), it is clear that only countably many triangles along the curve are non-typical. A very special case is represented by the triangle \(d\) in the figure, whose angles are all rational multiples of \(\pi\). Hence, we can conclude that the considered word is a relation in \(T_{\tau}\) for uncountably many typical non-generic triangles \(\tau\), which form a dense subset of the curve. A sample of them is given by the five triangles \(a, b, c, e, f\) depicted in the figure.

\[\text{Figure 4. The non-generic relation of Example 5.1}\]

**Example 5.2.** Arguing as above, we see that the stable word of length 18

\[
(r_1r_2r_3r_2r_3r_1r_3r_1r_2)^2 = (t_1)r_1r_3r_2r_1 \cdot (t_1)r_1r_3r_2r_3 \cdot (t_1)r_1r_3r_1r_3r_1r_2
\]  

(37)

corresponds to the translation vector

\[
t_1 + (t_1)r_3 + (t_1)r_1r_3 + (t_1)r_1r_3r_2r_1 + (t_1)r_1r_3r_2r_3 + (t_1)r_1r_3r_2r_3 + (t_1)r_1r_3r_2r_3 + (t_1)r_1r_3r_2r_3 + (t_1)r_1r_3r_2r_3.
\]  

(38)
This leads to the system
\[
\begin{align*}
(1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3 + 2 \cos 2(\alpha_2 + \alpha_3)) \cos 2\alpha_2 &= 0 \\
(1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3 + 2 \cos 2(\alpha_2 + \alpha_3)) \sin 2\alpha_2 &= 0,
\end{align*}
\]
(39)
hence to the equation
\[
1 + 2 \cos 2\alpha_2 + 2 \cos 2\alpha_3 + 2 \cos 2(\alpha_2 + \alpha_3) = 0.
\]
(40)

Hence, also in this case we can conclude that the considered word is a relation in \(T_\tau\) for uncountably many typical non-generic triangles \(\tau\), which form a dense subset of the curve represented by the equation.

Besides the two relations of length 18 given in the previous examples, our computer search also detected other non-generic relations holding for all the triangles along a curve in the parameter space \(T\), hence for uncountably many typical triangles. Namely, we there are 6 such relations of length 22 and 5 of length 24, but none of length 20.

Moreover, we found a certain number of non-generic relations holding only for isolated typical triangles. One of such relations is discussed in Example 5.3.

Actually, systematic search produced even shorter relations holding in isolated triangles, which present strong evidence of being typical. But we were not able to prove that such triangles are really typical. The shortest one has length 22, and it is the unique that length, up to permutation of the indices, inversion, conjugation and commutation of stable subwords. Up to the same equivalence, there are also 20 similar relations of length 24, some of which have the minimal length 5 with respect to \(C(t_1)\). Such further relations in conjecturally typical triangles are illustrated by Example 5.4.

\textbf{Example 5.3.} Consider the stable word of length 32
\[
\begin{align*}
& r_1r_3r_1r_2r_3r_2r_3r_1r_2r_3r_1r_2r_3r_1r_2r_3r_1r_2r_3r_1r_1r_3
= (t_1)r_3r_1 \cdot (t_1)r_2r_1 \cdot (t_1^2)r_3r_1 \cdot (t_1)r_2r_1 \cdot (t_1)r_3r_2r_1 \cdot t_1 \cdot (t_1)r_1r_3,
\end{align*}
\]
(41)
whose corresponding translation vector is
\[
t_1 + (t_1)r_1r_3 + 2(t_1)r_2r_1 + 3(t_1)r_3r_1 + (t_1)r_2r_3.
\]
(42)

Proceeding as in the previous examples, we see that this word represents the identity in \(T_\tau\) if and only if the angles \(\alpha_2\) and \(\alpha_3\) satisfy the system
\[
\begin{align*}
4 \cos 2\alpha_2 + 2 \cos 2\alpha_3 + \cos 2(\alpha_2 + \alpha_3) &= −1 \\
2 \cos(\alpha_2 + \alpha_3) (\sin(\alpha_2 + \alpha_3) − 2 \sin(\alpha_2 − \alpha_3)) &= 0.
\end{align*}
\]
(43)

Apart from the rational \((\text{mod } \pi)\) solution \(\alpha_2 = \alpha_3 = \pi/4\), the only other acceptable solution is
\[
\begin{align*}
\alpha_2 &= \arctan \sqrt{2} \\
\alpha_3 &= \arctan \sqrt{2}/3.
\end{align*}
\]
(44)

According to Theorem 2 of [1], this solution can be written in the form
\[
\begin{align*}
\alpha_2 &= q\pi ± \langle 3 \rangle_2 \\
\alpha_3 &= q'\pi ± \langle 11 \rangle_2,
\end{align*}
\]
(45)
for certain rational numbers \( q \) and \( q' \), and certain angles \( \langle 3 \rangle_2 \) and \( \langle 11 \rangle_2 \) that are rationally independent together with \( \pi \). This implies that the triangle is typical. The chains of reflections realizing the relation for such triangle is shown in the following Figure 5.

**Example 5.4.** The stable word of length 22

\[
r_1 r_2 r_1 r_2 r_3 r_1 r_3 r_1 r_2 r_3 r_2 r_3 r_1 r_3 r_1 r_2 = (t_1 r_2 r_1 \cdot (t_1) r_1 r_3 r_1 r_2 \cdot (t_1) r_2 r_3 r_1 r_3 \cdot (t_1) r_2 r_3 r_1 r_3 \cdot (t_1) r_3)
\]

represents the identity in \( T_\tau \) if and only if

\[
\begin{align*}
\cos 2\alpha_2 + 2 \cos 2\alpha_3 + \cos 2(\alpha_2 + \alpha_3) + \cos 2(2\alpha_2 + 3\alpha_3) & = -1 \\
\sin 2\alpha_2 + \sin 2(\alpha_2 + \alpha_3) + \sin 2(2\alpha_2 + 3\alpha_3) & = 0
\end{align*}
\]

The only two acceptable approximate solutions of the system are

\[
\begin{align*}
\alpha_2 & = 0.3675592642 \pi \\
\alpha_3 & = 0.1932064551 \pi
\end{align*}
\]

and

\[
\begin{align*}
\alpha_2 & = 0.5971477967 \pi \\
\alpha_3 & = 0.2299624978 \pi
\end{align*}
\]

Analogously, the stable word of length 24

\[
r_1 r_2 r_1 r_2 r_3 r_1 r_3 r_1 r_2 r_3 r_2 r_3 r_1 r_3 r_1 r_2 = (t_1^{-1} r_3 r_1 \cdot (t_1^{-1}) r_2 r_1 r_3 r_1 \cdot (t_1^{-1}) r_3 r_1 r_2 r_1 r_3 r_1 \cdot (t_1) r_3)
\]

represents the identity in \( T_\tau \) if and only if

\[
\begin{align*}
\cos 2\alpha_2 - \cos 2(\alpha_2 + \alpha_3) + \cos 2(\alpha_2 - \alpha_3) + \cos 2(2\alpha_2 - 3\alpha_3) & = 1 \\
\sin 2\alpha_2 + \sin 2(\alpha_2 + \alpha_3) + \sin 2(\alpha_2 - \alpha_3) & = 0
\end{align*}
\]

The only acceptable approximate solution of the system is

\[
\begin{align*}
\alpha_2 & = 0.2961623095 \pi \\
\alpha_3 & = 0.4392394514 \pi
\end{align*}
\]

The chains of reflections realizing both the non-generic relations above are shown in the following Figure 6, on the left side for the two triangles where the former relation holds and on the right side for the unique triangle where the latter holds.
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