Online Best Reply Algorithms for Resource Allocation Problems

Max Klimm \(^{1}\), Daniel Schmand \(^{1,2}\), and Andreas Tönnis \(^{1,3}\)

\(^1\)Operations Research, HU Berlin, Germany  
\(^2\)Chair of Management Science, RWTH Aachen University, Germany  
\(^3\)Department of Computer Science, University of Bonn, Germany

May 8, 2018

Abstract

We study online resource allocation problems with a diseconomy of scale. In these problems, there are certain requests, each demanding a set of resources, that arrive in an online manner. The cost of each resource is semi-convex and grows superlinearly in the total load on the resource. An irrevocable allocation decision has to be made directly after the arrival of each request with the goal to minimize the total cost on the resources. We focus on two simple greedy online policies that provide very fast and easy approximation algorithms.

The first policy is to minimize the individual cost of the current online request with respect to all previous requests that have been allocated before. The second policy is to minimize the marginal total cost over all requests that have arrived up to this point. In the literature, these type of algorithms is also considered as one-round walks in congestion games starting from the empty state.

We consider the weighted and unweighted version of the problem. In the weighted variant, and for cost functions that are polynomials with maximal degree \(d\) and positive coefficients, we proof a tight competitive ratio of \(\left(\frac{\sqrt{2}}{2} - 1\right)^{-\frac{1}{d+1}}\) for the marginal total cost policy. This interestingly exactly matches the approximation factor for the corresponding multiple-round walk algorithm. Our work indicates that one-round walks that start in an empty starting state are exactly as efficient as multiple-round walks. We also show that this does not carry over to the unweighted version of the problem. For unweighted instances, we provide lower bounds for both policies that are significantly larger than the corresponding multiple-round walks. We complement our results with an upper and lower bound on the solution quality of the personal cost policy for weighted and unweighted instances.

1 Introduction

We consider the online variant of weighted and unweighted resource allocation problems where a set of requests jointly uses a set of resources. The set of feasible allocations for each request is a set of subsets of the resources. Each resource is endowed with a cost function that is semi-convex and non-decreasing in the total load of that resource and thus models a diseconomy of scale.

In the online variant, considered in this paper, the requests arrive one after the other and upon arrival of a request, this request reveals its weight, and its set of feasible allocations. At this point in time, we immediately have to irrevocably allocate the request to a set of resources without any knowledge about requests that subsequently arrive. The objective is to find an allocation for each request such that the total cost over all requests is minimized.

\[^{1}\text{max.klimm@hu-berlin.de}^{1}\text{daniel.schmand@oms.rwth-aachen.de}^{1}\text{atoennis@uni-bonn.de} \text{ Partially supported by CONICYT grant PCI PII 20150140 and ERC Starting Grant 306465 (BeyondWorstCase).} \]
We consider both weighted and unweighted problems. In the weighted case, the non-negative, non-decreasing and semi-convex cost functions on the resources depend on the total load on that resource. The total cost for some chosen allocation vector is separable over resources and is defined as the sum of the resource costs. The cost on a resource is given by the weight on that resource times the cost function evaluated for this weight.

This resource allocation problems naturally arise in the context of congestion minimization in road networks. A widely used theoretical model to study congestion effects are congestion games (Rosenthal [23]). In a congestion game, each request is identified with a player that selfishly minimizes her private cost that she incurs due to her selected allocation. In contrast to many recent works on the existence and inefficiency of equilibria and their computational tractability, we focus on the online setting. This setting occurs naturally when requests to a supplier of connected automotive navigation systems appear online and the supplier wishes to route the requests in a way that minimizes overall congestion. Online variants of congestion games have been analyzed in the context of one-round walks when starting in the empty state. Here, we are interested in the competitive ratio of our algorithms, which is defined by the worst possible factor of the solution quality of the calculated solution compared to the offline optimal solution. Note, that this online procedure immediately translates into an efficient, decentralized approximation algorithm with the approximation guarantee of the derived competitive ratio.

In this work, we analyze the following two very basic greedy procedures. In the first algorithm which we term PERSONALCOSTWALK requests are allocated to the set of resources that minimize their personal cost. In the second algorithm termed SOCIALCOSTWALK requests are allocated to the set of resources that minimize the total cost with respect to the current state. Although the setting of online variants of congestion games and resource allocation problems is well studied in the literature, the two most basic greedy approaches have only been analyzed for the case of linear cost functions. In this domain for unweighted requests, minimizing personal cost is provably better than minimizing social cost by a constant factor [8, 13]. We extend this work to weighted requests and arbitrary non-negative, semi-convex cost functions. We derive explicit bounds on the solution quality for polynomial cost functions with positive coefficients and maximal degree $d$ that do only depend on $d$. We show that the constant gap for linear cost functions grows exponentially in $d$. Furthermore, all our bounds are independent of the number of requests.

As we discuss in more detail in Section 1.1, online problems have been studied for a related problem where the cost of a resource is the integral of the cost function from zero to the load rather than the product of the load and the cost as in our setting [16, 19, 18]. As remarked by Farzad [16], for polynomial cost function, these models are actually equivalent in the sense that for each instance in one model one can construct an equivalent instance in the other model. As a consequence, the lower bound obtained by Farzad et al. [16] and the upper bound obtained by Harks et al. [19] translate to our setting.

### 1.1 Related Work

Already the offline version of the resource allocation problem studied in this paper are very challenging. Roughgarden [24] showed that there is a constant $\beta > 0$ such that even the unweighted version of the offline problem cannot be approximated in polynomial time by a factor better than $(\beta d)^{d/2}$ when all cost functions are polynomials of maximum degree $d$ with non-negative coefficients. The currently best-known algorithm is due to Makarychev and Srividenko [20] and uses a convex programming relaxation. They showed that randomly rounding an optimal fractional solution gives an $O((\frac{2.792d}{d+1})^d)$ approximate solution. This approach is highly centralized and relies on the fact all requests are initially known, which both might be unrealistic assumptions for large-scale problems.

Offline and decentralized algorithms for this problem, that have been studied in the literature, are local search algorithms and multi-round best-response dynamics. For best-responses with respect to personal cost, local search algorithms are closely related to the studies of Nash equilibria. Nash equilibria are typically good approximations to the total cost. The notion of the price of stability serves as a lower bound on the approximation factor, where the price of anarchy is an upper bound. For polynomial congestion games that admit a Nash equilibrium, Aland et al. [1] proved tight results on the price on anarchy for
both weighted and unweighted congestion games by solving an optimization problem of the form

\[
\min_{\lambda > 0, \mu \in [0,1)} \left\{ \frac{\lambda}{1 - \mu} : \text{c}(x + y) \leq \lambda \text{c}(x) + \mu \text{c}(y), \forall x, y \in \mathbb{N}, c \in \mathcal{C}_d \right\},
\]

where \( \mathcal{C}_d \) is the set of cost functions in the game. When \( \mathcal{C}_d \) is the set of polynomial functions with maximum degree \( d \) and positive coefficients, this gives a bound on the price of anarchy of \( \Phi_d^{d+1} \) where \( \Phi_d \in \Theta\left(\frac{d}{\ln(d)}\right) \) is the solution to \((x + 1)^d = x^{d+1}\). The price of stability as a lower bound on the solution quality was not well understood for a long time for weighted congestion games. Very recently, Christodoulou et al. [11] showed that the price of stability is at least \( \left(\Phi_d^{d+1}/2\right) \) for large \( d \). For unweighted congestion games, Christodoulou and Gairing [10] showed a tight bound on the price of stability in the order of \( \Theta(d) \). Unfortunately, the deterministic best-response walk towards such a solution can take exponential time [14] in unweighted congestion games, or might even cycle in weighted instances. Though random walks [17] or walks using approximate best-response steps [3] converge to approximate Nash equilibria in polynomial time.

Mirrokni and Vetta [21] are the first to study best-response dynamics with respect to social cost. Bjelde, Klimm and Schmand [7] analyzed the solution quality of local minima of the social cost function both for weighted and unweighted resource allocation problems, see Table 1. By a result of Orlin et al. [22], this admits a PTAS in the sense that an \((1 + \epsilon)\)-approximate local optimal solution can be computed in polynomial time via local improvement steps. This local search algorithm is highly related to our work since this can be seen as a multiple round social cost work.

In contrast to this, one-round walks touch every request only once at the point in time when it arrives. Fanelli et al. have shown a linear lower bound even for linear cost functions if the requests are restricted to make one best-response starting from a bad initial configuration [15]. This lower bound does not hold for the empty state. There is a set of results for one-round works with respect to personal cost [13, 6, 8] and social cost for linear cost functions [4, 8, 26], for details see Table 1.

Closest to our work are the works of Harks et al. [19, 18] and Farzad et al. [15] who studied very related algorithms for online routing and even a generalization where requests are only present during certain time windows. However, they measure cost slightly differently as they define the cost of a resource as the integral of its cost function from 0 to the current load. This different cost measure leads also to a different notion of the private cost of a request. However, as remarked by Farzad et al. [15], the models are equivalent when only polynomial costs are considered. Farzad et al. provided a lower bound of \((d + 1)^{d+1}\) on the competitive ratio of the personal cost one-round walk for polynomial cost functions of degree \( d \) and positive coefficients. This lower bound also applies to our setting and we could strengthen this bound to \( \Omega\left(\left(\frac{d}{\ln(d)}\right)^{d+1}\right) \) for the weighted setting. Harks et al. [19] also consider personal cost one-round walks and show that it is \( O(1.77^d d^{d+1}) \)-competitive. We are able to slightly refine and improve this bound as we show that the algorithm is \( (\frac{d}{\ln(d)})^{d+1} \) where \( W \) is the product logarithm. In contrast to the bound of Harks et al., our bound holds for all \( d \), asymptotically it approaches \( (\frac{d}{\ln(d)})^{d+1} \approx (1.763d)^{d+1} \).
1.2 Our Contribution

We show upper and lower bounds for the online variant of weighted and unweighted resource allocation problems with semi-convex cost functions. We analyze two different very simple greedy techniques in one-round walks starting in the empty state. We use best-responses with respect to a request’s personal cost or the increase of the social cost incurred by a request, and we call the algorithms PersonalCostWalk and SocialCostWalk, respectively.

Let $C_d$ be the class of polynomial functions with maximal degree $d$ and positive coefficients. For weighted resource allocation problems, we give a tight bound of $\left(\frac{d+1}{\sqrt{2}} - 1\right)^{-d+1} \in O\left(\left(\frac{d}{\sqrt{2}}\right)^{d+1}\right)$ for SocialCostWalk. This is exactly the locality gap of the social cost function. It shows that the very easy and fast one-round walk algorithm establishes the same approximation guarantee as the PTAS local search procedure described in Bjelde et al. [7]. Our algorithm even works in an online setting and does only need $n$ best-response computations. For PersonalCostWalk, we get an upper bound of $\Psi_d^{d+1} \in O\left(\left(\frac{d}{W(d+1)}\right)^{d+1}\right)$. Here $\Psi_d$ is the unique solution to $(d+1)(x+1)^d = x^{d+1}$ and $W$ is the product logarithm function on $\mathbb{R}_{>0}$. This upper bound is significantly larger than the one for the price of anarchy. However, note that due to the absence of a potential function, deterministic best response dynamics do not converge to a Nash equilibrium in weighted congestion games.

We complement these results with a matching lower bound for social cost. Due to the special structure of the lower bound instance, it also holds for the case of personal cost. This improves the previously known bound by Farzad et al. [10]. Please note that Bjelde et al. also presented an instance in [7], which can be shown to be a matching lower bound for social cost. Due to the special structure of the lower bound instance, it also holds for the case of personal cost, but cannot be transferred to the case of personal cost easily.

For unweighted resource allocation problems, we separate the asymptotic performance guarantee of both considered one-round walks from the local search procedure. We provide a lower bound of $A_d \in \Omega\left(\left(\frac{d+1}{\sqrt{2}}\right)^{d+1}\right)$ for SocialCostWalk. Here $A_d$ is the $d$th ordered Bell number [5]. The lower bound of $(d+1)^{d+1}$ by Farzad et al. applies to PersonalCostWalk. Please note that both lower bounds are of the form $\Omega\left(\left(\frac{d}{\sqrt{2}}\right)^{d+1}\right)$ for some constant $c$. This indicates that, in contrast to the weighted setting, the one-round or online variant of the problem significantly increases the approximation ratio compared to the local search algorithm for the unweighted case.

We complete our analysis of the two one-round walks by an upper bound for personal cost. We separate the unweighted case from the weighted case and show an upper bound of $O\left((\Xi d)^{d+1}\right)$ for sufficiently large $d$, where $\Xi$ is the solution to the equation $2\Xi e^{\Xi} + \Xi^2 = e^{\frac{1}{2}} + e^{\frac{1}{2}}$, i.e. $\Xi \approx 1.523$.

2 Preliminaries

We consider online algorithms for unsplittable resource allocation problems. Let $R$ be a finite set of resources $r$ each endowed with a non-negative cost function $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. There is a set $N = \{1, \ldots, n\}$ of $n$ players. At time step $i$, player $i$ reveals its existence, its weight $w_i \in \mathbb{R}_{\geq 0}$ and a set $S_i \subseteq 2^R$ of possible allocations. If $w_i = 1$ for all $i \in N$, we call the instance unweighted. Upon arrival of player $i$, an allocation $S_i \in S_i$ of player $i$ has to be fixed irrevocably by an online algorithm.

For $i \in N$, let $S_{\leq i} = S_1 \times \cdots \times S_{i-1} \times S_i$ be the set of all allocation vectors up to player $i$. The cost of an allocation vector $S_{\leq i} = (S_1, \ldots, S_{i-1}, S_i) \in S_{\leq i}$ is defined as

$$C(S_{\leq i}) = \sum_{j=1}^{i} \sum_{r \in S_j} w_j c_r(w_r(S_{\leq i})), $$

where $w_i(S_{\leq i}) = \sum_{j \in N: r \in S_j} w_j$ denotes the load of resource $r$ under allocation vector $S_{\leq i}$. In the following, we write $S$ and $\mathbf{S}$ instead of $S_{\leq n}$ and $S_{\leq n}$. Given a sequence $R = (w_1, S_1), \ldots, (w_n, S_n)$ of requests, the offline optimum solution is be denoted by $\text{OPT}(R) = \min_{S \in S} C(S)$. As a convention, system optimal strategy profiles are denoted by $S^*$. For a sequence of requests $R$ and $i \in N$, denote by $R_{\leq i} = (w_1, S_1), \ldots, (w_{i-1}, S_{i-1}), (w_i, S_i)$ the
subsequence of requests up to player \(i\). An online algorithm \(\text{ALG}\) is a family of functions \(f_i: \mathcal{R}_{\leq i} \rightarrow \mathcal{S}_i\), mapping partial requests up to player \(i\) to a respective allocation for player \(i\). For a sequence of requests \(\mathcal{R}\), the cost of an online algorithm \(\text{ALG}\) with a family of functions \((f_i)_{i \in \mathbb{N}}\) is given by \(\text{ALG}(\mathcal{R}) = C(\mathcal{S})\) where \(\mathcal{S} = S_1 \times \cdots \times S_n\) and \(S_i = f_i(\mathcal{R}_{\leq i})\).

We measure the performance of an online algorithm by its competitive ratio which is \(\rho = \sup_{\mathcal{R}} \text{ALG}(\mathcal{R})/\text{OPT}(\mathcal{R})\) where the supremum is taken over all finite sequences of requests for which \(\text{OPT}(\mathcal{R}) > 0\). Typically, the response steps used by our online algorithms are tracktable and therefore \(\rho\) is also the approximation factor for the corresponding distributed approximation algorithm. When the sequence of requests \(\mathcal{R}\) is clear from context, we write \(\text{ALG}(\mathcal{R})\) and \(\text{OPT}(\mathcal{R})\) instead of \(\text{ALG}(\mathcal{R}_{\leq i})\) and \(\text{OPT}(\mathcal{R}_{\leq i})\).

We consider two greedy online algorithms. The algorithm \(\text{PERSONALCOSTWALK}\) greedily minimizes the cost of the current request. Let again denote \(S_{\leq i}\) the allocation vector of the algorithm before the \(i\)-th request is revealed. Then, \(w_i c_r(w_r(S_{\leq i})))\) is the per request cost at the arrival of request \(i\) on resource \(r\). Upon arrival of request \(i\), the greedy allocation algorithm chooses an allocation \(S_i \in \mathcal{S}_i\) that minimizes the cost of that request, i.e. we choose some allocation \(S_i\) such that,

\[
\sum_{r \in S_i} w_i c_r(w_r((S_{\leq i}))) \leq \sum_{r \in S'_i} w_i c_r(w_r(S_{\leq i})) .
\]  

The algorithm \(\text{SOCIALCOSTWALK}\) minimizes the marginal increase of the total cost in each step. Let \(S_{\leq i} = (f_1(\mathcal{R}_{\leq i}), \ldots, f_{i-1}(\mathcal{R}_{\leq i-1}))\) denote the allocation vector before the \(i\)-th request is revealed. Then, \(c_r(w_r(S_{\leq i})))\) is the cost on resource \(r\) at this step of the algorithm. We denote the corresponding total cost on resource \(r\) by \(C_r(w_r(S_{\leq i})) = w_r(S_{\leq i}) c_r(w_r(S_{\leq i}))\). Upon arrival of request \(i\), we allocate a set of resources to request \(i\), such that the marginal increase in the total social cost is minimized. Denoting the choice of the algorithm by \(S_i = f_i(\mathcal{R}_{\leq i})\), we have

\[
\sum_{r \in S_i} C_r(w_r(S_{\leq i}) + w_i) - C_r(w_r(S_{\leq i})) \leq \sum_{r \in S'_i} C_r(w_r(S_{\leq i}) + w_i) - C_r(w_r(S_{\leq i})) ,
\]  

for all other feasible allocations \(S'_i \in \mathcal{S}_i\).

|                      | linear \(c_e\) | \(c_e\) of max. degree \(d\) |
|----------------------|---------------|-------------------------------|
|                      | lower upper   | lower bnd. upper bnd.         |
| inefficiency of Nash eq. | 1.57 2.5     | \(d + 1\) \(\Theta\left(\frac{d}{\ln d}\right)^{d+1}\) |
| local search soc. cost | 3 3          | \(\Omega\left(\frac{d}{\ln d}\right)^{d+1}\) |
| one-r. pers. cost    | 4.24 4.24    | \(d + 1\) \(O\left(\frac{d}{\ln d}\right)^{d+1}\) |
| one-r. social cost   | 5.66 5.66    | \(\Omega\left(\frac{d+1}{e \ln 2}\right)^{d+1}\) \(\Theta\left(\frac{d}{\ln d}\right)^{d+1}\) |

Table 1: Overview over our and related results. Result obtained in this paper are marked in blue. The lower (upper) bound on the inefficiency of Nash equilibria is the price of anarchy (price of stability). Here \(\Xi \approx 1.523\), and \(W\) denotes the product logarithm function on \(\mathbb{R}_{\geq 0}\).
3 Weighted Resource Allocation Problems

In this section, we give upper and lower bounds on the competitive ratio for PersonalCostWalk and SocialCostWalk in weighted resource allocation problems. The study of these algorithms is motivated by the work of [13] and [8]. They showed that one-round walks with respect to the cost per request result in a provably better competitive ratio than walks with respect to the social cost for congestion games with unweighted requests and linear cost functions. We show that, for \( d > 1 \), this also holds for walks on weighted instances and that the multiplicative gap between the two algorithms increases exponentially.

For the upper bounds, we use the smoothness framework which has also been used before by [1] and [25]. We show that we can derive an optimization problem such that the solution to the optimization problem is an upper bound on the competitive ratio. We show that for SocialCostWalk this optimization problem is exactly the same as the one that appeared in [7], thus our upper bound here is identical. For PersonalCostWalk, the optimization problem is different. For the case that all cost functions are in the class \( C_d \), we achieve a bound of \( O\left(\frac{d}{W(\frac{d}{e})}\right)^{d+1} \) where \( W \) is the product logarithm function defined on \( \mathbb{R}_{\geq 0} \).

We give a lower bound construction on which both algorithms, PersonalCostWalk and SocialCostWalk, behave identically and show a bound of \( \Omega\left(\frac{d}{W(\frac{d}{e})}\right)^{d+1} \). Furthermore, we show that the bound on the locality gap by Bjelde et al. [7] also applies to SocialCostWalk. Unfortunately, it cannot be easily extended to PersonalCostWalk.

3.1 Personal Cost Upper Bound

In order to show an upper bound on the competitive ratio of PersonalCostWalk, we use Inequality [1] that captures the local decision policy. We derive a new \((\lambda, \mu)\)-optimization problem and show that there is an upper bound on the optimal solution value of \( \Psi_{d+1} \), where \( \Psi_d \) is the unique solution to the equality \((d+1)(x+1)^d = x^{d+1} \). The analysis is in large parts similar to that of the price of anarchy given in Aland et al. [1].

Lemma 1. Let \( C \) be a set of semi-convex and non-decreasing cost functions and let \( \beta = \sup_{c \in C, x \in \mathbb{R}_{\geq 0}} \frac{c(x)+c'(x)x}{c(x)}. \) Further, let \( \lambda > 0 \) and \( \mu \in [0, \beta) \) be such that

\[
xc(x+y) \leq \lambda xc(x) + \mu yc(y) \text{ for all } x, y \in \mathbb{R}_{\geq 0} \text{ and } c \in C.
\]

Then PersonalCostWalk is \( \frac{\beta \lambda}{1-\beta \mu} \)-competitive.

Proof. Let \( S_{<i} = (f_1(R_{<1}), \ldots, f_{i-1}(R_{<i-1})) \) denote the allocation vector before the \( i \)-th request is revealed and \( c_r(w_r(S_{<i})) \) denote the correspond cost on resource \( r \). Denoting the choice of the algorithm by \( S_i = f_i(R_{<i}) \) and an optimal allocation by \( S^* \), we obtain the Inequality [1]

\[
\sum_{r \in S_i} c_r(w_r(S_{<i}) + w_i) \leq \sum_{r \in S^*_i} c_r(w_r(S_{<i}) + w_i).
\] (3)

The total cost after the \( i \)-th request is revealed is

\[
C(S_{<i}) = \sum_{r \in R} w_r(S_{<i})c_r(w_r(S_{<i}))
\]

\[
= \sum_{r \in S_i} (w_r(S_{<i-1}) + w_i)c_r(w_r(S_{<i-1}) + w_i) + \sum_{r \notin S_i} w_r(S_{<i-1})c_r(w_r(S_{<i-1}))
\]

\[
\leq C(S_{<i-1}) + \sum_{r \in S_i} (w_r(S_{<i-1}) + w_i)c_r(w_r(S_{<i-1}) + w_i) - w_r(S_{<i-1})c_r(w_r(S_{<i-1}))
\]

\[
\leq C(S_{<i-1}) + \beta \sum_{r \in S_i} w_i \left( c_r(w_r(S_{<i})) + c'_r(w_r(S_{<i})w_r(S_{<i})) \right) \leq C(S_{<i-1}) + \beta \sum_{r \in S^*_i} w_i c_r(w_r(S_{<i}) + w_i).
\]
Using this inequality \( n \) times we obtain
\[
C(S) = \sum_{i \in N} C(S_{\leq i}) - C(S_{\leq i-1}) \leq \sum_{i \in N} \sum_{r \in S_i} \beta w_r c_r(w_r(S_{\leq i}) + w_i)
\]
\[
\leq \sum_{i \in N} \sum_{r \in S_i} \beta w_r c_r(w_r(S_r) + w_r(S^*)) = \beta \sum_{r \in R} w_r(S^*) c_r(w_r(S_r) + w_r(S^*))
\]
\[
\leq \beta \sum_{r \in R} \lambda w_r(S^*) c_r(w_r(S^*)) + \mu w_r(S_r) c_r(w_r(S)) = \beta \lambda C(S^*) + \beta \mu C(S).
\]

Rearranging terms gives the claimed result. \( \square \)

We proceed to use Lemma 1 in order to give an upper bound on the competitive ratio of PersonalCostWalk. We will express the competitive ratio in terms of the unique solution to the equation \((d+1)(x+1)^d = x^{d+1}\) which we will denote by \(\Psi_d\).

**Theorem 2.** For polynomial cost functions with non-negative coefficients and maximal degree \(d\), the competitive ratio of the PersonalCostWalk is at most \(\Psi_d\) where \(\Psi_d\) is the unique solution to the equation \((d+1)(x+1)^d = x^{d+1}\).

**Proof.** By splitting up resources with cost functions \(c_i\) into several resources, we may assume that each resource has a cost function of the form \(c_r(x) = a_r x^{d_r}\) with \(d_r \in [0, d]\) and \(a_r \in \mathbb{R}_{\geq 0}\). We then obtain
\[
\beta = \sup_{c \in C} \sup_{x \in \mathbb{R}_{\geq 0}} \frac{c(x) + xc'(x)}{c(x)} = \sup_{a \in \mathbb{R}_{\geq 0}, k \in [0, d]} \sup_{x \in \mathbb{R}_{\geq 0}} \frac{ax^k + ak x^k}{ax^k} = d + 1.
\]

In light of Lemma 1 we can bound the competitive ratio \(\rho\) of PersonalCostWalk by solving the following minimization problem
\[
\rho \leq \inf_{\lambda \in \mathbb{R}_{\geq 0}, \mu \in [0, \frac{1}{d+1})} \left\{ \frac{(d+1)\lambda}{1 - (d+1)\mu} : x(x+y)^k \leq \lambda x^{k+1} + \mu y^{k+1} \text{ for all } x, y \in \mathbb{R}_{\geq 0}, k \in [0, d] \right\}.
\]

Dividing the inequality by \(x^{k+1}\) and substituting \(z = y/x\) gives the equivalent formulation
\[
\rho \leq \inf_{\lambda \in \mathbb{R}_{\geq 0}, \mu \in [0, \frac{1}{d+1})} \left\{ \frac{(d+1)\lambda}{1 - (d+1)\mu} : \lambda \geq (z+1)^k - \mu z^{k+1} \text{ for all } z \in \mathbb{R}_{\geq 0}, k \in [0, d] \right\}
\]
\[
= \inf_{\mu \in [0, \frac{1}{d+1})} \sup_{z \in \mathbb{R}_{\geq 0}} \frac{(z+1)^k - \mu z^{k+1}}{z}.
\]

Aland et al. [12] Lemma 5.1] show that \((z+1)^d - \mu z^{d+1} \geq (z+1)^k - \mu z^{k+1}\) for all \(k \in [0, d]\) for which the latter term is non-negative. They further show [12] Lemma 5.2] that the function \((z+1)^d - \mu z^{d+1}\) has a unique maximum on \(\mathbb{R}_{\geq 0}\). We thus obtain
\[
\rho \leq \inf_{\mu \in [0, \frac{1}{d+1})} \sup_{z \in \mathbb{R}_{\geq 0}} \left\{ (z+1)^d - \mu z^{d+1} \right\}.
\]

Let \(\Psi_d\) be the solution to the equation \((d+1)(x+1)^d = x^{d+1}\). We claim that the term above can be upper bounded by \(\Psi_d\). To this end, consider the functions \(g : \mathbb{R}_{\geq 0} \times [0, \frac{1}{d+1}) \rightarrow \mathbb{R}\) and \(h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) defined as
\[
g(x, \mu) = \frac{(x+1)^d - \mu x^{d+1}}{d+1 - \mu}, \quad h(x) = \frac{d(x+1)^{d-1}}{(d+1)x^d}.
\]

We first show that \(h(\Psi_d) \in [0, \frac{1}{d+1})\). To see this, note that
\[
h(\Psi_d) = \frac{d(\Psi_d + 1)^{d-1}}{(d+1)\Psi_d^{d-1}} = \frac{d(\Psi_d + 1)^d}{\Psi_d^d(\Psi_d + 1)} = \frac{d(\Psi_d + 1)^d}{\Psi_d^{d+1}} = \frac{1}{d+1} \cdot \frac{\Psi_d}{\Psi_d + 1} \in \left(0, \frac{1}{d+1}\right).
\]
Second, we claim that if there is a pair \((\hat{x}, \hat{\mu}) \in \mathbb{R}_{\geq 0} \times (0, \frac{1}{\mu+1})\) with \(\hat{\mu} = h(\hat{x})\), then 
\[g(\hat{x}, \hat{\mu}) = \max_{\theta \in \mathbb{R}_{\geq 0}} g(x, \hat{\mu}).\]
To see this claim, note that by construction of \(h\), \(\hat{x}\) satisfies the first order maximality conditions of \(g(\cdot, \hat{\mu})\). As shown by Aland et al. [11] Lemma 5.2, the function \((x + 1)^d - \mu x^{d+1}\) has a unique maximum and is increasing for values smaller than the maximum and decreasing for all values larger than the maximum. This implies the claim.

For \(\hat{\mu} = h(\Psi_d)\), we obtain
\[\rho \leq \max_{x \in \mathbb{R}_{\geq 0}} g(x, \hat{\mu}) = g(\Psi_d, \hat{\mu}) = \frac{(\Psi_d + 1)^d - \hat{\mu} \Psi_{d+1}^d}{\frac{\Psi_d}{d} - \hat{\mu}} = \Psi_{d+1},\]
which completes the proof. Furthermore, we have for all \(\mu \in (0, \frac{1}{\mu+1})\) that \(\rho \geq \Psi_{d+1}^d\).

### 3.2 Social Cost Upper Bound

We will prove an upper bound on the competitive ratio of \(\text{SocialCostWalk}\) by using Inequality 2. We will derive the following \((\lambda, \mu)\)-optimization problem that has also been considered in [7]. Subsequently we can use the results of [7] to derive an upper bound on the competitive ratio of \((\sqrt{d+1} - 1)^{-(d+1)} \in \mathcal{O}(\frac{d}{\ln 2})^{d+1}\).

**Lemma 3.** Let \(C\) be a set of semi-convex and non-decreasing cost functions, \(\lambda > 0\) and \(\mu \in [0, 1]\) such that
\[(x + y)c(x + y) - yc(y) \leq \lambda xc(x) + \mu yc(y) \text{ for all } x, y \in \mathbb{R} \text{ and } c \in C.\] (4)

**Then \(\text{SocialCostWalk}\) is \(\frac{1}{\mu\lambda}\)-competitive.**

**Proof.** Let \(S_{<i}\) denote the allocation vector before the \(i\)-th request is revealed, and let 
\[C_r(w_r(S_{<i})) = w_r(S_{<i})c_r(w_r(S_{<i})),\]
as above. Upon arrival of request \(i\), we allocate a set of resources to request \(i\), such that the marginal increase in the total social cost is minimized.

Inequality 2 allows us to bound the cost incurred by the algorithm with
\[\text{ALG}(\mathcal{R}) = \sum_{i \in [n]} \sum_{r \in S_i} C_r(w_r(S_{<i}) + w_i) - C_r(w_r(S_{<i}))\]
\[\leq \sum_{i \in [n]} \sum_{r \in S_i^*} C_r(w_r(S_{<i}) + w_i) - C_r(w_r(S_{<i}))\]
\[\leq \sum_{i \in [n]} \sum_{r \in S_i^*} C_r(w_r(S) + w_i) - C_r(w_r(S))\]
\[\leq \sum_{r \in \mathcal{R}} C_r(w_r(S) + w_r(S^*)) - C_r(w_r(S)).\]

The first inequality uses 2 and the second and third inequality use the convexity of \(C_r(x)\). Using 1, we have
\[C_r(w_r(S) + w_r(S^*)) - C_r(w_r(S)) \leq \mu C_r(w_r(S)) + \lambda C_r(w_r(S^*))\]
so that we obtain
\[\text{ALG}(\mathcal{R}) \leq \mu\text{ALG}(\mathcal{R}) + \lambda\text{OPT}(\mathcal{R}),\]
which implies the claimed result. \(\square\)

Using this lemma, we establish the competitiveness of \(\text{SocialCostWalk}\) for polynomial cost functions with non-negative coefficients and maximal degree \(d\).

**Theorem 4.** For polynomial cost functions with non-negative coefficients and maximal degree \(d\), \(\text{SocialCostWalk}\) is \((d+\sqrt{2} - 1)^{-(d+1)}\)-competitive where \((d+\sqrt{2} - 1)^{-(d+1)} \in \mathcal{O}(\frac{d}{\ln 2})^{d+1}\).
Proof. With the same arguments as in the proof of Theorem 2, it is without loss of generality to assume that for each resource the cost function is of the form \( c_r(x) = a_r x^d \) with \( a_r \in \mathbb{R}_{>0} \) and \( d_r \in [0, d] \). In light of Lemma 3, we are interested in solving the following minimization problem

\[
\min_{\lambda > 0} \left\{ \frac{\lambda}{1 - \mu} \right\} \quad \text{such that} \quad (x + y)^{k+1} - y^{k+1} \leq \mu x^{k+1} + \lambda y^{k+1} \quad \text{for all} \quad x, y \in \mathbb{R}_{\geq 0}, \ k \in [0, d].
\]

In Bjelde et al. [7], it is shown that the optimal solution of this optimization problem is \( (\sqrt{2} - 1)^{-d+1} \) which is attained for \( \lambda = 2 \pi + (2 \pi - 1)^{-d} \) and \( \mu = 2 \pi - 1 \).

3.3 Personal Cost Lower Bound

Our lower bound construction on the competitive ratio of the PERSONALCOSTWALK algorithm is a weighted singleton resource allocation problem with identical cost functions. This bound also applies to SOCIALCOSTWALK. We also show that the lower bound construction for the locality gap provided by [7] also works in the online setting.

The lower bound construction for PERSONALCOSTWALK is as follows. In every step, requests can be allocated to exactly two different single resources and both the cost functions and the current load for both options is identical. Therefore requests are indifferent between these options for both algorithms. We assume all ties are broken arbitrarily and chose the player weights in a way such as to maximize the ratio up to a constant factor between the incured by the algorithm and the cost of the optimal solution.

**Theorem 5.** There is a weighted resource allocation problem with cost functions in \( C_d \), where both PERSONALCOSTWALK and SOCIALCOSTWALK calculate a solution with cost

\[
\frac{d}{2^{(\pi^2 - 1)^{d+1}}} \in \Omega\left(\frac{d}{\log d}\right)^{d+1}
\]

times the optimal cost.

**Proof Outline.** The structure of the instance is the same as for Theorem 3. There are \( n \) requests \( N \) with weights \( w_i = c^{(\log(n) - \log(i))} \), where \( c \) is a non-negative constant that will be chosen later. In addition, there are \( n + 1 \) resources \( r_j \), each resource with cost \( c_{r_j}(x) = x^d \).

Request \( i \in [n] \) can be either allocated to \( r_{i - 2^{\log(n) + 1}} \) or \( r_{i + 1} \).

In the optimal solution, every request \( i \) gets assigned to resources \( r_{i+1} \), so no resource has more than one player. Therefore, the cost of the optimal solution is

\[
C(S^*) = \sum_{i=0}^{[\log(n)]} 2^i \left(c^{[\log(n) - i]}\right)^{d+1}.
\]

The order of arrival of the request is in decreasing order from \( n \) to 1. We assume that the tie-breaking is towards the resource with the lower index. In this way, whenever a request
arrives, the load on both resources is identical. Therefore, in both algorithms PERSONAL\-COSTWalk and SOCIAL\-COSTWalk, all requests are assigned to resources \( r_{i-2^{\lceil \log(i) \rceil}+1} \).

In the end, the load on resource \( r_j \) is \( \sum_{l=0}^{\lceil \log(j) \rceil} c^j \).

In Appendix 5.2 we show that for \( c = 2^{1/(d+1)} \), we get

\[
\frac{C(S)}{C(S^*)} \geq \sum_{i=0}^{\lceil \log(n) \rceil - 1} \left( 1 - \frac{2^{i+1}}{(1 + |\log(n)|)} \right)^{d+1}
\]

We show that \( \left( 2^{d+1} - 1 \right)^{d+1} \geq 2^i \) for all \( i \geq \log(e)(d + 1)^2 \). In the limit, we have

\[
\lim_{n \to \infty} \frac{C(S)}{C(S^*)} = \lim_{n \to \infty} \frac{\left( \left\lceil \log(n) \right\rceil - \log(e)(d + 1)^2 - 1 \right)}{2(1 + |\log(n)|)(2\sqrt{n+1} - 1)}^{d+1} = \frac{1}{2(2\sqrt{2} - 1)^{d+1}}.
\]

\[\square\]

### 3.4 Social Cost Lower Bound

To obtain a lower bound on the competitive ratio of the social cost greedy algorithm, we revisit a construction due to Bjelde et al. [7]. They use this construction to show that the locality gap is \( (\sqrt{n} - 1)^{-(d+1)} \), but it is not hard to verify that the same construction also provides a lower bound for the competitiveness of the social cost greedy allocation procedure. This proves that the upper bound given in Section 5.2 is tight. The reverse is not true, the bound described here does not translate to an identical lower bound for PERSONAL\-COSTWalk.

**Theorem 6.** For every \( d \in \mathbb{R}_{\geq 0} \), there is a weighted resource allocation problem with polynomial cost functions in \( C_d \), so that the cost of the solution computed by SOCIAL\-COSTWalk is \( (\sqrt{n} - 1)^{-(d+1)} \) times the optimal cost.

**Proof.** We start the proof by restating the construction of the example provided by Bjelde et al. [7]. For any given \( d \) and \( k \in \mathbb{Z}_{\geq 0} \), we define a resource allocation problem with \( k + 1 \) requests and \( k + 2 \) resources. Analogously to the description in [7], let \( \beta = \sqrt{2} - 1 \). We are given resources \( R = \{r_1, \ldots, r_{k+2}\} \), where \( c_{r_i}(x) = \beta^{i-1}(d+1)x^d \) for \( i \in \{1, \ldots, k + 1\} \) and \( c_{r_{k+1}}(x) = c_{r_{k+2}}(x) = \beta^{k+1}x^d \). Every request \( i \) can only be assigned to either \( r_i \) or \( r_{i+1} \).

Bjelde et al. have shown that the allocation vector \( S = (r_1, \ldots, r_{k+1}) \) is a local optimal solution. They have argued that assigning request \( k + 1 \) to \( r_{k+2} \) instead of \( r_k \) does not improve the total cost, since the cost functions are the same. They have also shown that assigning request \( i \in \{1, \ldots, k\} \) to \( r_{i+1} \) instead of \( r_i \), does not change the total cost, since there is already a request on resource \( r_{i+1} \). If we now let the requests arrive in decreasing order and always assign the resource with smaller index, it is clear that our algorithm would construct the local optimal solution \( S \). Since the cost functions of [7] and our work are the same, we get

\[
\lim_{k \to \infty} \frac{C(S)}{C(S^*)} = \frac{1}{(\sqrt{n} - 1)^{d+1}},
\]

which implies the result. \[\square\]

### 4 Unweighted Resource Allocation Problems

In this section, we consider unweighted resource allocation problems and derive upper and lower bounds on the competitive ratio for the unweighted case. Interestingly, we show a significant difference to the weighted setting, where PERSONAL\-COSTWalk and the local search procedure achieve exactly the same approximation ratio. We show in this section that this is no longer true in the unweighted case by providing a lower bound example, that both holds for PERSONAL\-COSTWalk and SOCIAL\-COSTWalk. Here, we can show that the competitive ratio is at least in \( \Omega \left( \left( \frac{L}{\ln 2} \right)^{d+1} \right) \), and thus significantly larger than the upper
bound on the local search procedure of $O\left((\frac{2d}{ln d})^{d+1}\right)$. We complete the section with a refined upper bound for PersonalCostWalk for unweighted instances. An improved analysis of the upper bound on the competitive ratio of SocialCostWalk for unweighted instances remains open.

Recall that the sequence of requests is given by $R = (1,S_1),\ldots,(1,S_n)$ for unweighted resource allocation problems, i.e. $w_i = 1$ for all $i \in N$. This implies that the cost functions $c_r$ directly define the per request cost on this resource, since $w_c c_r = c_r$ in this case. The cost functions $c_r$ now do only depend on the number of requests that have chosen some resource $r$.

4.1 Personal Cost Upper Bound

In the following, we show an upper bound of $O((\Xi d)^{d+1})$ on the competitive ratio of PersonalCostWalk for unweighted resource allocation problems with cost functions in $C_d$. Here, $\Xi$ is the unique solution to $2xe^{1/x} + x^2 = e^{2/x} + x^2 e^{1/x}$ and $\Xi \approx 1.523$. Analogously to Section 5.3 we use Inequality 1 to derive an inequality that fits the $(\lambda,\mu)$-smoothness framework. In contrast to the previous section, this inequality only has to hold for integral $x = w_r (S^*)$ and $y = w_r (S)$.

The smoothness framework allows us to bound the loss of the algorithm compared to the optimal solution on every resource. To retain generality, let $d_r \leq d$ denote the maximal degree of the cost function $c_r$. We show that for every resource $r$, the inequality is fulfilled for $\lambda_d \in O((\Xi d)^{d+1})$ and a constant $0 < \mu < 1$. Towards this end, we make a case distinction between $y \leq d_r$, $y > d_r$ and $x = 1$, and $y > d_r$, and $x > 1$. The bound on the second case dominates the other two and gives the result.

**Theorem 7.** The algorithm PersonalCostWalk is asymptotically $O\left((\Xi d)^{d+1}\right)$ competitive for unweighted instances cost functions in $C_d$. Here, $\Xi$ is the unique solution to $2xe^{1/x} + x^2 = e^{2/x} + x^2 e^{1/x}$ and $\Xi \approx 1.523$.

In this section, we only give a short overview over the proof of Theorem 7. Please see Appendix 5.3 for the full proof.

**Proof Outline.** We sum the marginal cost over all players and get

$$C(S) = \sum_{i \in N} \sum_{r \in S_i} \left((w_r(S_{<i} + 1))^{d_r+1} - w_r(S_{<i})^{d_r+1}\right)$$

$$= \sum_{i \in N} \sum_{r \in S_i} \left((d_r + 1)(w_r(S_{<i} + 1))^{d_r} + \sum_{k=0}^{d_r} w_r(S_{<i})^k \frac{(d_r + 1)!}{k!(d_r + 1 - k)!}\right)$$

$$- \sum_{i \in N} \sum_{r \in S_i} \sum_{k=0}^{d_r} w_r(S_{<i})^k \frac{d_r!}{k!(d_r - k)!} \cdot (d_r + 1).$$

Then we apply Inequality 1 our condition derived from the properties of PersonalCostWalk, and obtain

$$C(S) \leq \sum_{r \in R} (d_r + 1)w_r(S^*) (w_r(S) + 1)^{d_r} - \sum_{r \in R} \frac{w_r(S)^{d_r+2} - (w_r(S) - 1)^{d_r+2}}{d_r + 2}.$$

Then we show, that we can find suitable $\lambda_d$ and $\mu_d$ such that

$$(d_r + 1)(y + 1)^{d_r} x - y^{d_r - 1} + \frac{y^{d_r+2} - (y - 1)^{d_r+2}}{d_r + 2} \leq \lambda_d x^{d_r + 1} + \mu_d y^{d_r + 1}$$

for all $x, y \in \mathbb{N}_{\geq 0}$ and $d_r \leq d$. Applying this to all $r \in R$ yields a competitive ratio of $\frac{\lambda_d}{1 - \mu_d}$. So we seek to minimize $\frac{\lambda_d}{1 - \mu_d}$ such that the inequality is fulfilled. We will show in Lemma 3 that there is a solution to the minimization problem such that $\frac{\lambda_d}{1 - \mu_d}$ is upper bounded by $(\Xi d)^{d+1}$.  \[\square\]
4.2 Unweighted Social Cost Lower Bound

We give a lower bound instance that uses the graph structure sketched in Figure 3 and Figure 4. It is based on a singleton congestion game with identical cost functions on all resources. On arrival, all requests have two options that, at this point in time, have identical load. Therefore, SocialCostWalk is indifferent between both options. We assume that ties are broken adversarily.

**Theorem 8.** There is a unweighted resource allocation problem with cost functions in $C_d$ where SocialCostWalk calculates a solution with cost $A_{d+1} \in \Omega\left(\left(\frac{d+1}{\ln(d+1)}\right)^{d+1}\right)$ times the optimal cost. Here $A_{d+1}$ is the $(d+1)$-th ordered Bell number.

**Proof.** In the lower bound instance, there are $n$ requests $N$ and $n+1$ resources $R$ where $n$ is a multiple of 2. The two possible strategies for request $i$ is either the singleton strategy $r_{i-2\lfloor \log(i) \rfloor + 1}$ or $r_{i+1}$. Requests arrive in increasing order of indices.

In the optimal solution OPT, all requests $i$ are matched to resource $r_{i+1}$. So every resource serves one request and therefore $C(OPT) = n$.

Whereas in the algorithm, when request $i$ arrives, the load on both available resources is identical, so the algorithm is indifferent between both choices. We assume that all ties are broken adversarily and $i$ is assigned to resource $r_{i-2\lfloor \log(i) \rfloor + 1}$. In this way, the load on resource $r_j$ is $\lfloor \log(n) \rfloor - \lfloor \log(j) \rfloor$.

With $c_{r_j}(x) = x^{d+1}$, the total cost of the algorithm is

$$C(ALG) = \sum_{t=1}^{\log(n+1)-1} \left( \frac{t^{d+1}}{2^{t+1}}(n+1) \right) + \log(n+1)^{d+1}.$$ 

In the limit, this gives the following bound on the competitive ratio

$$\frac{C(ALG)}{C(OPT)} \geq \lim_{n \to \infty} \frac{1}{2} \sum_{t=1}^{\log(n+1)-1} \frac{t^{d+1}}{2^t} = \frac{1}{2} \sum_{t=1}^{\infty} \frac{t^{d+1}}{2^t} = \frac{1}{2} A_{d+1},$$ 

where $A_{d+1} = \frac{(d+1)!}{2^{m(d+1)}} + o(d!)$ is the $(d+1)$th ordered Bell number. With $(d+1)! \geq \left(\frac{d+1}{e}\right)^{d+1}$, the competitive ratio is in $\Omega\left(\left(\frac{d+1}{\ln(d+1)}\right)^{d+1}\right)$.

Please note that this lower bound also holds for PersonalCostWalk. For this algorithm, the construction by Harks et al. [19] gives a lower bound of $(d+1)^{d+1}$.

---

1 The $n$-th ordered Bell number counts the number of partitions of an $n$-element set into $k$ nonempty subsets. 
$$A_n = \frac{1}{2} \sum_{m=0}^{n} \frac{n^m}{2^m} = \frac{n!}{2^{n+1}} + o((n-1)!)$$ [5].
References

[1] Sebastian Aland, Dominic Dumrauf, Martin Gairing, Burkhard Monien, and Florian Schoppmann. Exact price of anarchy for polynomial congestion games. *SIAM Journal on Computing*, 40:1211–1233, 2011.

[2] Baruch Awerbuch, Yossi Azar, and Amir Epstein. The price of routing unsplittable flow. *SIAM Journal on Computing*, 42(1):160–177, 2013.

[3] Baruch Awerbuch, Yossi Azar, Amir Epstein, Vahab S. Mirrokni, and Alexander Skopaklik. Fast convergence to nearly optimal solutions in potential games. In *Proceedings of the 9th ACM Conference on Electronic Commerce (EC)*, pages 264–273, 2008.

[4] Baruch Awerbuch, Yossi Azar, Edward F. Grove, Ming-Yang Kao, P. Krishnan, and Jeffrey Scott Vitter. Load balancing in the lp norm. In *Proceedings of the 36th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 383–391, 1995.

[5] Jean-Pierre Barthélémy. An asymptotic equivalent for the number of total preorders on a finite set. *Discrete Mathematics*, 29(3):311–313, 1980.

[6] Vittorio Bilò, Angelo Fanelli, Michele Flammini, and Luca Moscardelli. Performance of one-round walks in linear congestion games. *Theory of Computing Systems*, 49(1):24–45, 2011.

[7] Antje Bjelde, Max Klimm, and Daniel Schmand. Brief announcement: Approximation algorithms for unsplittable resource allocation problems with diseconomies of scale. In *Proceedings of the 29th ACM Symposium on Parallelism in Algorithms and Architectures (SPAA)*, pages 227–229, 2017.

[8] Ioannis Caragiannis, Michele Flammini, Christos Kaklamanis, Panagiotis Kanellopoulos, and Luca Moscardelli. Tight bounds for selfish and greedy load balancing. *Algorithmica*, 61(3):606–637, 2011.

[9] Po-An Chen, Bart de Keijzer, David Kempe, and Guido Schäfer. The robust price of anarchy of altruistic games. In *Proceedings of the 7th International Workshop on Internet and Network Economics (WINE)*, pages 383–390, 2011.

[10] George Christodoulou and Martin Gairing. Price of stability in polynomial congestion games. *ACM Transactions on Economics and Computation*, 4(2):10:1–10:17, 2016.

[11] George Christodoulou, Martin Gairing, Yiannis Giannakopoulos, and Paul G. Spirakis. The price of stability of weighted congestion games. *CoRR*, abs/1802.09952, 2018.

[12] George Christodoulou and Elias Koutsoupias. The price of anarchy of finite congestion games. In *Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC)*, pages 67–73, 2005.

[13] George Christodoulou, Vahab S. Mirrokni, and Anastasios Sidiropoulos. Convergence and approximation in potential games. *Theoretical Computer Science*, 438:13–27, 2012.

[14] Alex Fabrikant, Christos H. Papadimitriou, and Kunal Talwar. The complexity of pure nash equilibria. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC)*, pages 604–612, 2004.

[15] Angelo Fanelli, Michele Flammini, and Luca Moscardelli. The speed of convergence in congestion games under best-response dynamics. *ACM Transactions on Algorithms*, 8(3):25:1–25:15, 2012.

[16] Babak Farzad, Neil Olver, and Adrian Vetta. A priority-based model of routing. *Chicago Journal of Theoretical Computer Science*, 1, 2008.

[17] Michel X. Goemans, Vahab S. Mirrokni, and Adrian Vetta. Sink equilibria and convergence. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 142–154, 2005.

[18] Tobias Harks, Stefan Heinz, and Marc E. Pfetsch. Competitive online multicommodity routing. *Theory of Computing Systems*, 45(3):533–554, 2009.

[19] Tobias Harks, Stefan Heinz, Marc E. Pfetsch, and Tjark Vredeveld. Online multicommodity routing with time windows. In *ZIB Report 07-22, Zuse Institute Berlin, 2007*. 2007.
5 Appendix

5.1 Missing Details in Section 3.1

Lemma 9. The equation $\frac{d+1}{d}x^d + 1 = x^{d+1}$ has a unique solution $\Psi_d$ in $\mathbb{R}_{\geq 0}$ for all $d \in \mathbb{R}_{\geq 0}$. Moreover, $\Psi_d \in \left[\frac{d}{W\left(\frac{d}{d+1}\right)} - 1, d/W\left(\frac{d}{d+1}\right)\right]$, where $W : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is the product logarithm function on $\mathbb{R}_{\geq 0}$.

Proof. We first show that the equation $(d + 1)(x + 1)^d = x^{d+1}$ has a unique solution. Since this solution is not $x = 0$ we may assume $x \neq 0$, take logarithms and obtain the equivalent equation

$$\log(d + 1) + d \log(x + 1) = \log x + (d + 1) \log x.$$  

Rearranging terms yields

$$\log(x + 1) - \log(x) = \frac{\log x - \log(d + 1)}{d}. \quad (5)$$

The left hand side of this equation is decreasing in $x$ and takes values in $(0, \infty)$. The right hand side is increasing in $x$ and for $x \in [d + 1, \infty)$, it takes values in $(0, \infty)$. Thus, the equation has a unique solution which we denote by $\Psi_d$.

To get an approximate closed form expression for $\Psi_d$, we use (5) and the mean value theorem to obtain

$$\frac{1}{\xi} = \frac{\log \Psi_d - \log(d + 1)}{d} \text{ for some } \xi \in (\Psi_d, \Psi_d + 1).$$

We obtain

$$\Psi_d \in \left\{ x \in [d + 1, \infty) : \frac{1}{x + 1} \leq \frac{\log x - \log(d + 1)}{d} \leq \frac{1}{x} \right\}.$$

As $\log x$ is strictly increasing in $x$ and both $\frac{1}{x + 1}$ and $\frac{1}{x}$ are decreasing, we obtain $\Psi_d \in [a, b]$ where $a$ is the unique solution to the equation $\frac{d}{x + 1} = \log x - \log(d + 1)$ and $b$ is the unique solution to the equation $\frac{d}{x} = \log x - \log(d + 1)$. The latter equation gives

$$e^{d/b} = \frac{b}{d + 1} \iff e^{d/d} = \frac{b}{d + 1}.$$  

Using that $W$ is bijective on $\mathbb{R}_{\geq 0}$ and that $W(xe^{x}) = x$ for all $x \in \mathbb{R}_{\geq 0}$, this implies $\Psi = W\left(\frac{d}{d+1}\right)$ and, hence, $b = d/W\left(\frac{d}{d+1}\right)$. To get a bound on $a$, note that $a \geq a'$ where $a'$ solves $\frac{d}{x^{a'}} = \log(a' + 1) - \log(d + 1)$. Substituting $b = a' + 1$, we obtain $a' = d/W\left(\frac{d}{d+1}\right) - 1$ as before.
5.2 Missing Details in Section 3.3

Proof of Theorem 3. The structure of the instance is the same as for Theorem 8. There are \(n\) requests with weights \(w_i = c(|\log(n)| - |\log(i)|)\), where \(c\) is a non-negative constant that will be chosen later. In addition, there are \(n+1\) resources \(r_j\), each resource with cost \(c_{ij}(x) = x^d\). Request \(i \in [n]\) can be either allocated to \(r_{i-2(\log(i)+1)+1}\) or \(r_{i+1}\).

In the optimal solution, every request \(i\) gets assigned to resources \(r_{i+1}\), so no resource has more than one player. Therefore, the cost of the optimal solution is

\[
C(S^*) = \sum_{i=0}^{\lfloor \log(n) \rfloor} 2^i \left( c^{\lfloor \log(n) \rfloor - i} \right)^{d+1}.
\]

The order of arrival of the request is in decreasing order from \(n\) to 1. We assume that the tie-breaking is towards the resource with the lower index. In this way, whenever a request arrives, the load on both resources is identical. Therefore, in both algorithms Personal-CostWalk and SocialCostWalk, all requests \(i\) get assigned to resources \(r_{i-2(\log(i)+1)+1}\).

In the end, the load on resource \(r_j\) is \(\sum_{i=0}^{\lfloor \log(n) \rfloor - \lfloor \log(j) \rfloor} c_i\). So the total cost of the algorithm is

\[
C(S) = \sum_{j=1}^{n+1} \left( \sum_{i=0}^{\lfloor \log(n) \rfloor - \lfloor \log(j) \rfloor} c_i \right)^{d+1}
= \sum_{j=0}^{\lfloor \log(n) \rfloor - 1} \left( 2^{\lfloor \log(n) \rfloor - j} \left( \sum_{i=0}^{j} c_i \right)^{d+1} \right) + \left( \sum_{i=0}^{\lfloor \log(n) \rfloor} c_i \right)^{d+1}.
\]

This gives the ratio

\[
\frac{C(S)}{C(S^*)} \geq \frac{\sum_{j=0}^{\lfloor \log(n) \rfloor - 1} 2^{\lfloor \log(n) \rfloor - j - 1} \left( \sum_{i=0}^{j} c_i \right)^{d+1}}{\sum_{i=0}^{\lfloor \log(n) \rfloor} 2^i \left( c^{\lfloor \log(n) \rfloor - i} \right)^{d+1}}
= \frac{\sum_{j=0}^{\lfloor \log(n) \rfloor - 1} 2^{\lfloor \log(n) \rfloor - j - 1} \left( \frac{j+1}{c \cdot j} \right)^{d+1}}{\sum_{i=0}^{\lfloor \log(n) \rfloor} 2^{i+1} \left( c^{d+1} \right)^{d+1}}
= \frac{\sum_{j=0}^{\lfloor \log(n) \rfloor} 2^{-j} c^{d+1}}{\sum_{j=0}^{\lfloor \log(n) \rfloor} 2^{-j} c^{d+1} (c-1)^{d+1}}.
\]

Here we used the geometric sum and reordered the terms. We choose \(c = 2^{1/(d+1)}\) and get

\[
\frac{C(S)}{C(S^*)} \geq \frac{\sum_{j=0}^{\lfloor \log(n) \rfloor} \left( 2^{-i-1} \left( \frac{2^{i+1}}{2^{d+1}} - 1 \right)^{d+1} \right)}{\sum_{j=0}^{\lfloor \log(n) \rfloor} \left( \frac{2^{i+1}}{2^{d+1}} - 1 \right)^{d+1}}
= \frac{\sum_{j=0}^{\lfloor \log(n) \rfloor} \left( 2^{i+1} - 1 \right)^{d+1}}{\sum_{j=0}^{\lfloor \log(n) \rfloor} \left( \frac{2^{i+1}}{2^{d+1}} - 1 \right)^{d+1}}.
\]

At this point, we rewrite \(\left( 1 - 2^{-2d+1} \right)^{d+1} = \left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1} = \left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1} \). Applying Lemma 10 gives us \(\left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1} \geq 2^i\) for all \(i \geq \log_2(e)(d+1)^2\), this yields

\[
\frac{C(S)}{C(S^*)} \geq \frac{\sum_{i=\log_2(e)(d+1)^2}^{\lfloor \log(n) \rfloor} \left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1}}{\sum_{i=\log_2(e)(d+1)^2}^{\lfloor \log(n) \rfloor} \left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1}}
= \frac{\left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1}}{\left( \frac{2^{2d+1} - 1}{2^{2d+1}} \right)^{d+1}}.
\]

In the limit for \(n \to \infty\) this is \(\frac{1}{2^{(d+1)(\sqrt{2}-1)^{d+1}}}\).

Lemma 10. The inequality \(e^x - 1^x \geq e^{x^2} - 1^x\) holds for all \(x \geq 2\).
Proof. The inequality trivially holds for \( x = 2 \) with \((e^2 - 1)^2 = e^4 - 2e^2 + 1 \geq e^3\). Next we consider the derivatives of both sides of the inequality

\[
\frac{\partial}{\partial x} (e^x - 1)^x = (e^x - 1)^x \left( \frac{e^x}{e^x - 1} + \log (e^x - 1) \right),
\]

and

\[
\frac{\partial}{\partial x} e^{x^2-1} = 2e^{x^2-1} x.
\]

Obviously both derivatives are positive for \( x > 2 \). Furthermore, the ratio of the derivatives is greater than 1 for all \( x > 2 \) and therefore the left side grows faster than the right one

\[
\frac{(e^x - 1)^x \left( \frac{e^x}{e^x - 1} + \log (e^x - 1) \right)}{2e^{x^2-1} x} = \frac{(e^x - 1)^{x-1}}{2e^{x-1}} + \frac{\log (e^x - 1)}{2e^{x^2-1} x} \geq \frac{(e^x - 1)^{x-1}}{x^{\sqrt{2}}} \geq 1,
\]

for \( x > 2 \).

5.3 Missing Details in Section 4.1

Proof of Theorem 4.1. We will prove that there is a constant \( d_0 \) such that we can upper bound the competitive ratio for all online resource allocation problems with polynomial cost functions with maximal degree \( d \) by \((\Xi d)^{d+1}\) for all \( d \geq d_0 \).

Fix \( d \geq d_0 \) and assume that resource \( r \) has cost equal to \( x^{d_r} \) with \( d_r < d \). The algorithm minimizes the current user’s cost in each step, that is, we have

\[
\sum_{r \in S_i} (w_r(S_{<i} + 1))^{d_r} \leq \sum_{r \in S_i} (w_r(S_{<i} + 1))^{d_r}.
\]

The total cost can be written as the sum of the marginal increases to the total cost functions, i.e. we can also write

\[
C(S) = \sum_{i \in N} \sum_{r \in S_i} \left( (w_r(S_{<i} + 1))^{d_r+1} - w_r(S_{<i})^{d_r+1} \right)
\]

\[
= \sum_{i \in N} \sum_{r \in S_i} \left( \sum_{k=0}^{d_r} w_r(S_{<i})^k \binom{d_r + 1}{k} \right) + \sum_{i \in N} \sum_{r \in S_i} \left( (w_r(S_{<i}) + 1)^{d_r} \cdot (d_r + 1) \right)
\]

\[
- \sum_{i \in N} \sum_{r \in S_i} \left( (w_r(S_{<i}) + 1)^{d_r} \cdot (d_r + 1) \right)
\]

\[
= \sum_{i \in N} \sum_{r \in S_i} \left( (d_r + 1) (w_r(S_{<i}) + 1)^{d_r} + \sum_{k=0}^{d_r} w_r(S_{<i})^k \frac{(d_r + 1)!}{k!(d_r + 1 - k)!} \right)
\]

\[
- \sum_{i \in N} \sum_{r \in S_i} \sum_{k=0}^{d_r} \left( w_r(S_{<i})^k \frac{d_r!}{k!(d_r - k)!} \cdot (d_r + 1) \right),
\]

where we used that \( \sum_{k=0}^{d_r+1} a^k \binom{d_r}{k} = (a + 1)^{d_r+1} \). We get

\[
C(S) \leq \sum_{i \in N} \sum_{r \in S_i} \left( (d_r + 1) (w_r(S_{<i}) + 1)^{d_r} + \sum_{i \in N} \sum_{r \in S_i} \sum_{k=0}^{d_r-1} w_r(S_{<i})^k \binom{d_r + 1}{k} (d_r - k) \right),
\]

by using the definition of the algorithm. In the following, we use \( w_r(S_{<i}) \leq w_r(S) \) and that \( w_r(S_{<i})^k \) can be written as \( j - 1 \) in the second sum, if \( i \) is the \( j - 1 \)th request that has been allocated to \( r \) in \( S \). We get

\[
C(S) \leq \sum_{r \in R} \left( (d_r + 1) w_r(S^*) (w_r(S) + 1)^{d_r} - \sum_{r \in R} \sum_{j=1}^{w_r(S)^{d_r-1}} \sum_{k=0}^{j-1} (j - 1)^k \binom{d_r + 1}{k} (d_r - k) \right).
\]

16
Now we bound the triple sum as follows

\[
\sum_{r \in R} \sum_{j=1}^{w_r(S)d_r-1} \sum_{k=0}^{d_r-1} (j-1)^k \left( \frac{d_r + 1}{k} \right) (d_r - k) \geq \sum_{r \in R} \sum_{k=0}^{d_r-1} \left( \frac{d_r + 1}{k} \right) (d_r - k) \int_1^{w_r(S)} (j-1)^k dj
\]

\[
\geq \sum_{r \in R} \sum_{k=0}^{d_r-1} \left( \frac{d_r + 1}{k} \right) (d_r - k) \left( \frac{1}{k+1} (w_r(S) - 1)^{k+1} - \frac{1}{k+1} \right)
\]

\[
\geq \sum_{r \in R} \sum_{k=0}^{d_r-1} \left( \frac{d_r + 1}{k+1} \right) \frac{d_r - k + 1}{d_r - k + 2} \left( (w_r(S) - 1)^k - 1 \right)
\]

\[
= \sum_{r \in R} \frac{w_r(S)d_r+2 - (w_r(S) - 1)d_r+2}{d_r + 2}.
\]

In Lemma 11, we will show, that we can find suitable \( \lambda_d \) and \( \mu_d \) such that

\[
(d_r + 1)(y + 1)^{d_r} x - y^{d_r} - y^{d_r+2} - (y - 1)^{d_r+2} \leq \lambda_d x^{d_r+1} + \mu_d y^{d_r+1}
\]

for all \( x, y \in \mathbb{N}_{\geq 0} \) and \( d_r \leq d \). Applying this to all \( r \in R \) yields a competitive ratio of \( \frac{\lambda_d}{1-\mu_d} \).

So we seek to minimize \( \frac{\lambda_d}{1-\mu_d} \) such that the inequality is fulfilled. We will show in Lemma 11 that there is a solution to the minimization problem such that \( \frac{\lambda_d}{1-\mu_d} \) is upper bounded by \( (\Xi d)^{d+1} \).

**Lemma 11.** For sufficiently large \( d \), we have

\[
\min_{\lambda_d \geq 0, \mu_d \in (0,1)} \frac{\lambda_d}{1-\mu_d} \leq a \cdot (\Xi d)^{d+1},
\]

and \((d_r + 1)(y + 1)^{d_r} x - y^{d_r} - y^{d_r+2} - (y - 1)^{d_r+2} \leq \lambda_d x^{d_r+1} + \mu_d y^{d_r+1}\) \( \forall x, y \in \mathbb{N}_{\geq 1} \) and \( d_r \leq d \), where \( \Xi \) is the solution to \( 0 = 2xe^x + x^2 - e^x - x^2e^x \), i.e. \( \Xi \approx 1.523 \).

**Proof.** Let \( d_r \leq d \) and \( d \) sufficiently large. In this proof, we will distinguish 3 cases. First, we show the Lemma for \( x, y \in \mathbb{N}_{\geq 1} \) with \( y \leq d_r \). Second, we will consider the case \( y > d_r, x \geq 2 \) and we will finish the proof with the case \( y > d_r, x = 1 \). For all 3 cases, we will choose \( \mu_d = 1 - \epsilon \) with a sufficiently small, but constant \( \epsilon \) (i.e. \( \mu_d \) is independent of \( d \)) and \( \lambda_d = (\Xi d)^{d+1} \). We will show that

\[
\max_{x,y} \left\{ \frac{(d_r + 1)(y + 1)^{d_r} x - y^{d_r+1} + \frac{y^{d_r+2} - (y - 1)^{d_r+2}}{d_r+2} - (1 - \epsilon)y^{d_r+1}}{x^{d_r+1}} \right\} \leq (\Xi d)^{d+1}.
\]

for sufficiently small \( \epsilon \).

**Case 1:** \( y \leq d_r \). First, we use that \( \frac{y^{d_r+2} - (y - 1)^{d_r+2}}{d_r+2} \leq y^{d_r+1} \). We get

\[
\max_{x,y} \left\{ \frac{(d_r + 1)(y + 1)^{d_r} x - y^{d_r+1} + \frac{y^{d_r+2} - (y - 1)^{d_r+2}}{d_r+2} - (1 - \epsilon)y^{d_r+1}}{x^{d_r+1}} \right\}
\]

\[
\leq \max_{x,y} \left\{ \frac{(d_r + 1)(y + 1)^{d_r} x - (1 - \epsilon)y^{d_r+1}}{x^{d_r+1}} \right\}
\]

\[
\leq \max_{x,y} \left\{ \frac{(d_r + 1)(y + 1)^{d_r} x}{x^{d_r+1}} \right\} \leq \max_{y} \left\{ (d_r + 1)(y + 1)^{d_r} \right\}
\]

\[
\leq (d_r + 1)^{d_r+1} \leq (d + 1)^{d+1} \leq (\Xi d)^{d+1},
\]

where the last inequality holds for all \( d \geq 8 \).
Case 2: $y > d_r, x = 1\quad$ For the case $y > d_r, x = 1$, we have

$$\max_y \left\{ (d_r + 1)(y + 1)^{d_r} - (2 - \epsilon) y^{d_r + 1} + \frac{y^{d_r + 2} - (y - 1)^{d_r + 2}}{d_r + 2} \right\}$$

$$\leq \max_y \left\{ y^{d_r} \left( (d_r + 1)\left(1 + \frac{1}{y}\right)^{d_r} - (2 - \epsilon) y + y^2 \frac{1 - \frac{1}{y}^{d_r + 2}}{d_r + 2} \right) \right\}.$$  

We use that $(1 + \frac{1}{y})^y \leq e \leq (1 + \frac{1}{y})^{y+1}$ and get

$$\max_y \left\{ y^{d_r} \left( (d_r + 1)\left(1 + \frac{1}{y}\right)^{d_r} - (2 - \epsilon) y + y^2 \frac{1 - e^{-\frac{d_r+2}{d_r+2}}}{d_r + 2} \right) \right\}.$$  

As in the other cases, we replace $y$ with $cd_r$, this gives us

$$\max_c \left\{ (cd_r)^{d_r} \left( (d_r + 1)(e)^{d_r} - (2 - \epsilon) cd_r + (cd_r)^2 \frac{1 - e^{-\frac{d_r+2}{d_r+2}}}{d_r + 2} \right) \right\}$$

$$\leq \max_c \left\{ (cd_r)^{d_r} \left( (d_r + 1)(e)^{d_r} - (2 - \epsilon) cd_r + e^2 d_r \left( 1 - e^{-\frac{d_r+2}{d_r+2}} \right) \right) \right\}.$$  

Here we use $\frac{1 - e^{-\frac{d_r+2}{d_r+2}}}{d_r + 2} \geq 2 \geq (d_r + 2)e^{-\frac{d_r+2}{d_r+2}} - e^{-\frac{d_r+2}{d_r+2}}$, which holds true for all $c$ and $d_r$ such that $cd_r > 1$.

We observe that the first part of the product is increasing in $c$ and the second part decreasing in $c$. In the following, we derive an upper bound $\tilde{c}(d_r)$ on the maximum $c$ that depends on $d_r$. Note that the expression is negative, if the inner bracket is negative.

$$(d_r + 1)(e)^{\frac{d_r}{2}} - (2 - \epsilon) cd_r + e^2 d_r \left( 1 - e^{-\frac{d_r+2}{d_r+2}} \right) < 0$$

$$\Leftrightarrow d_r < \frac{e^{\frac{d_r}{2}}}{2e^{\frac{d_r}{2}} - c^2 - e^{\frac{d_r}{2}}} = \frac{e^{\frac{d_r}{2}}}{2ce^{\frac{d_r}{2}} - e^{\frac{2d_r}{2}} - c^2e^{\frac{d_r}{2}}}.$$  

Thus for these pairs $(c, d_r)$, the lemma is fulfilled trivially. Let

$$d(c) = \frac{e^{\frac{d_r}{2}}}{2ce^{\frac{d_r}{2}} - e^{\frac{d_r}{2}}} - c^2e^{\frac{d_r}{2}}.$$  

and note that the right hand side is decreasing in $c$. Thus, there is a bijection between $c$ and $d(c)$, so we can also consider the inverse function $\tilde{c}(d_r)$ that is monotonically decreasing in $d_r$. So, we get an upper bound on $c$ dependent on $d_r$ and have

$$\max_c \left\{ (cd_r)^{d_r} \left( (d_r + 1)(e)^{d_r} - (2 - \epsilon) cd_r + e^2 d_r \left( 1 - e^{-\frac{d_r+2}{d_r+2}} \right) \right) \right\}$$

$$\leq \max_c \left\{ (\tilde{c}(d_r))^{d_r} \left( (d_r + 1)(e)^{d_r} - (2 - \epsilon) cd_r + e^2 d_r \left( 1 - e^{-\frac{d_r+2}{d_r+2}} \right) \right) \right\}.$$  

In the following, we argue that $\tilde{c}(d_r)d_r$ is monotonically increasing in $d_r$ and thus, can be upper bounded by $\tilde{c}(d)d$. In order to do so, note that $cd(c)$ is monotonically decreasing in $c$. Now, $\tilde{c}(d_r) \cdot d(\tilde{c}(d_r)) = \tilde{c}(d_r) \cdot d_r$ is monotonically increasing in $d_r$, since $\tilde{c}(d_r)$ is decreasing in $d_r$. We use that the second part of the product is decreasing in $c$ to get

$$\tilde{c}(d)\max_c \left\{ \left( (d_r + 1)(e)^{d_r} - (2 - \epsilon) cd_r + e^2 d_r \left( 1 - e^{-\frac{d_r+2}{d_r+2}} \right) \right) \right\}$$

$$\leq \tilde{c}(d) \max_d \left\{ (d_r + 1)(e)^{1 - \frac{d_r+2}{d_r+2}} - (2 - \epsilon) d_r + (1 - e^{-1}) \right\}$$

$$\leq \tilde{c}(d) \max \left\{ (e - 2 + \epsilon)d + (e + 1 - e^{-1}) \right\}$$

$$\tilde{c}(d) \max \left\{ (e - 2 + \epsilon)d + (e + 1 - e^{-1}) \right\} \leq (\Xi d)^{d+1},$$  

for sufficiently large $d$. 

18
Case 3: $y > d_r, x \geq 2$ We start again by using \(\frac{y^{d_r+2} - (y-1)^{d_r+2}}{d_r+2} \leq y^{d_r+1}\) and derive

\[
\max_{x,y} \left\{ (d_r + 1)(y + 1)^{d_r} x - y^{d_r+1} \right\} \leq \frac{(d_r + 1)(y + 1)^{d_r} x - (1 - \epsilon) y^{d_r+1}}{y^{d_r+1}}
\]

Now, let $x^*, y^*$ be the maximizer of the term above. For fixed $d_r$ and $y^*$, we write $y^* = c \cdot d_r$ with some $c > 1$. We can rewrite the term as

\[
\max_{x,c} \left\{ \left( d_r + 1 \right) \left( x + 1 \right)^{d_r} - \left( 1 - \epsilon \right) \left( x + 1 \right)^{d_r} \right\}
\]

In the following, we again derive an upper bound $\bar{c}(d_r) \cdot x^*$ on the maximal possible factor $c$ dependent on $d_r$ and $x^*$. This upper bound fulfills

\[
0 \leq (d_r + 1) e^{\frac{1}{2} \bar{c}(d_r)} - \left( 1 - \epsilon \right) \bar{c}(d_r) \cdot d_r \leq (d_r + 1) e^{\frac{1}{2} \bar{c}(d_r)} - \left( 1 - \epsilon \right) \bar{c}(d_r) \cdot d_r
\]

We can show analogously to Case 2 that the right hand side is decreasing in $\bar{c}(d_r)$ and $\frac{x^* e^{\bar{c}(d_r)}}{d_r}$ is increasing in $d_r$. We derive

\[
\max_{x} \left\{ \left( \bar{c}(d_r) x \right) d_r \left( d_r + 1 \right) - \left( 1 - \epsilon \right) d_r \frac{1}{x} \right\}
\]

In the following, we can upper bound $\bar{c}(d)$ that solves $0 = (d + 1) e^{\frac{1}{2} \bar{c}(d)} - (1 - \epsilon) \bar{c}(d) \cdot d$ by 1.389 for sufficiently large $d$. This yields

\[
\left( \bar{c}(d) d \right) \left( (d + 1) e \right) \leq \left( \Xi d \right)^{d+1}
\]

for $d$ large enough, which completes the proof.