Generalization of multivariable Laplace transform based on Tsallis $q$-exponential and its inverse using Post-Widder’s method

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A generalization of the multivariate Laplace transform based on the Tsallis $q$-exponential is given in the present work for a new type of kernel. We also define the inverse transform for this generalized multivariate transform based on the complex integration method. We prove identities corresponding to the Laplace transform and inverse transform like the $q$-convolution theorem, the action of generalized derivative and generalized integration on the Laplace transform. We then derive a $q$-generalization of the inverse Laplace transform based on the Post-Widder’s method which bypasses the necessity for a complex contour integration. This extension of the Post-Widder’s method has been carried out for the multivariate case as well. We demonstrate the usefulness of this in computing the Laplace and inverse Laplace transform of some elementary functions. The inverse Laplace transform has been calculated using both the complex integration method as well as the Post-Widder’s methods and the results obtained through these two methods agree with each other.
I. INTRODUCTION

A mathematical function ‘\( f \)’ in the ‘\( x \)’ domain can be transformed to a function ‘\( F \)’ in the ‘\( u \)’ domain using the integral

\[
F(u) = \int_{a}^{b} f(x) K(u, x) dx.
\]

This process is known as an integral transform and the function \( K(u, x) \) is the kernel of the transformation. An extension of Eq. (1) to the multivariate case reads:

\[
F(u_1, \ldots, u_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n) K(u_1, \ldots, u_n, x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]

Here the integral transform maps the problem in a given domain to another domain in which it is simpler to solve. Laplace transform is one of the widely used integral transforms in physics. It is very useful in solving convolution integral equations and differential equations. The inverse Laplace transform is an equally important transform with several applications. In general an inverse Laplace transform is carried out using a Bromwich contour integral in the complex plane. But this practice of using a complex variable technique is done only for the sake of convenience. In fact an inverse Laplace transform based only on real variables was introduced by Post and later it was refined by Widder. This method has been investigated in several works considering different applications.

The kernel of a Laplace transform is an exponential function of the form \( \exp(-st) \). Tsallis introduced generalizations of the logarithm and the exponential functions as follows:

\[
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q}; \quad \exp_q(x) \equiv [1 + (1 - q)x]^{\frac{1}{1-q}},
\]

where \( q \in \mathbb{R}_+ \) is the generalization parameter. These functions are generally referred to as Tsallis \( q \)-logarithm and Tsallis \( q \)-exponential and are inversely related. The \( q \)-logarithm and the \( q \)-exponential functions reduce to the usual logarithm and exponential functions in the \( q \to 1 \) limit. These generalized functions have been investigated in a wide variety of fields like astrophysics, high energy physics, neutrino physics, mathematical physics and nonequilibrium statistical physics.

A generalization of the Laplace transform has been done in Ref.\[26-28\] using the Tsallis \( q \)-exponential. So far only the single variable Laplace transform has been considered in these works. In our present work we study the multivariable Laplace transform based on Tsallis \( q \)-exponential. Further the inverse transforms has not been defined for even the single variable
Laplace transform introduced in 19. First we define the inverse using a contour integral in the complex plane for the Laplace transform defined in 19. Then we use the Post-Widder’s method to compute the inverse transform using real variable. In our work we introduce the inverse for the Laplace transform based on the generalized exponential function. We also extend this generalized Laplace transform and its inverse to the case of multivariable functions.

The work is organized as follows: In Section 2, we give a brief introduction to the Post-Widder’s technique and the Tsallis $q$-exponential. The single and multivariable Laplace transform based on type - I kernel is defined in the third section. The inverse transform based on complex integration method is also defined in the same section. The properties of Laplace transform are derived in Section 4. In Section 5, we give the $q$-generalization of the Post-Widder’s method for computing the inverse Laplace transform for both the single variable and the multivariable case. The Laplace transform and the inverse Laplace transform based on the Widder’s method for a simple function is computed in Section 6. We also compile a table consisting of the $q$-generalized Laplace transform and its inverse for some elementary functions. For the inverse transform the calculations were done using both the complex integration method and the Post-Widder’s method and the results obtained through these two different methods agree with each other.

II. A PRIMER ON LAPLACE TRANSFORM, POST-WIDDER’S TECHNIQUE AND TSALLIS $q$-EXPONENTIAL

In this section we briefly review the concept of Laplace transform and its inverse. Then we describe the Post-Widder’s method to calculate the inverse Laplace transform. The Laplace transform of a function $f(t)$ denoted by $\mathcal{L}[f(t)] = F(s)$ is defined as

$$F(s) = \int_0^\infty \exp(-st)f(t)dt. \quad (4)$$

Here $f \in S(\mathbb{R})$ where $S(\mathbb{R})$ represents the Schwartz space of functions $f : \mathbb{R} \to \mathbb{C}$ with $f \in C^\infty(\mathbb{R})$, i.e., $f$ is infinitely differentiable on $\mathbb{R}$. In general $s = \sigma + i\tau$ with $\sigma$ and $\tau$ being real numbers. The integral converges when $\text{Re}[s] = \sigma > 0$ and for $\sigma < 0$, $F(s) = 0$. For the Laplace transform of $F(s)$, the inverse transform is defined as

$$f(t) = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{\sigma-i\tau}^{\sigma+i\tau} \exp(st)F(s)ds. \quad (5)$$
The direct Laplace transform Eq. (4) and the inverse Laplace transform Eq. (5) are inverses of each other for the functions $f \in S(\mathbb{R})$. In order to calculate the inverse Laplace transform we need to perform a Bromwich contour integration over the complex plane. Below we explain an alternative method introduced by Post and Widder which does a Laplace inverse using only real variables.

Let us consider the $n^{th}$ derivative of the Laplace transform $F(s)$ with respect to the variable ‘s’,

$$\frac{d^n}{ds^n}F(s) \equiv F^{(n)}(s) = (-1)^n \int_0^{\infty} t^n \exp(-st)f(t)dt. \quad (6)$$

We use the following three steps, (i) First we use the variable transformation $s = n/x$, (ii) We follow it by multiplying the numerator and the denominator by $x^n$ and (iii) Finally we use the variable change $y = t/x$, and recast the integral in Eq. (6) to

$$F^{(n)} \left( \frac{n}{x} \right) = (-1)^n x^{n+1} \int_0^{\infty} (ye^{-y})^n f(xy)dy. \quad (7)$$

Here we would like to point out that the function $ye^{-y}$ has a single maximum at $y = 1$ and for the function $(ye^{-y})^n$ this maximum is sharply peaked at $y = 1$ and hence Eq. (7) can be rewritten as

$$F^{(n)} \left( \frac{n}{x} \right) \approx (-1)^n x^{n+1} f(x) \int_0^{\infty} (ye^{-y})^n dy. \quad (8)$$

Evaluating the integral we get

$$F^{(n)} \left( \frac{n}{x} \right) \approx (-1)^n n! \left( \frac{x}{n} \right)^{n+1} f(x). \quad (9)$$

In the limit $n \to \infty$, we can use the transformation $y = \frac{t}{x}|_{y=1} \Rightarrow x = t$ and so $s = \frac{n}{x} \Rightarrow s = \frac{n}{t}$ we can rewrite Eq. (9) as

$$f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} s^{n+1} F^{(n)}(s)|_{s=n/t}. \quad (10)$$

Thus we observe that the Laplace transform and the inverse Laplace transform can be expressed as functions of real variables alone. This sequence converges very fast since the rate of convergence is at least $1/n$.

The Laplace transform exists for a piecewise continuous function of exponential order. Similarly, the $q$-Laplace transform has been defined for a piecewise continuous function of $q$-exponential order. The $q$-exponential based Laplace transform can be defined using three
different types of kernels as noted in Ref.\textsuperscript{16} and these kernels are
\begin{align}
K_I(q; s, t) &= \exp_q(-st), \\
K_{II}(q; s, t) &= \exp_q(-t)^{s}, \\
K_{III}(q; s, t) &= \exp_q(t)^{-s}.
\end{align}

Of these three kernels, Laplace transform has been defined using the first and the second kernels in Ref.\textsuperscript{16-19}. In the extensive limit ($q \to 1$), both these generalized Laplace transforms reduce to the ordinary Laplace transform. For the generalized Laplace transform based on the first kernel\textsuperscript{19}, the inverse Laplace transform has not been defined. But for the case where the Laplace transform was defined using the second kernel, the inverse transform was defined using the complex integration method.

III. LAPLACE AND INVERSE TRANSFORM BASED ON TYPE - I KERNEL

The Laplace transform due to type-I kernel was introduced in Ref.\textsuperscript{12} for a single variable. But an introduction of the inverse transform has not been done so far. In the present section we have two subsections, where the first one discusses the Laplace and inverse Laplace transform of single variable function. The second subsection introduces the multivariable Laplace transform and its inverse. Throughout this section we restrict and present the results only for the $q < 1$ case.

A. Single variable Laplace transform:

The single variable Laplace transform introduced in Ref.\textsuperscript{19} is revisited here. However an inverse Laplace transform has not been defined so far. Here in our work we define the inverse Laplace transform and also prove its inverse property. A function ‘$f$’ is said to be of $q$-exponential order ‘$c$’, if there exists $c$, $M > 0$, $T > 0$, such that $|f(t)| \leq M \exp_q(ct) \forall t > T$. If a function is piecewise continuous and is of $q$-exponential order $c$, then $F_q(s) = L_q[f(t)]$ exists for $s > c$ and $\lim_{s \to \infty} F(s) = 0$. Under these conditions, the $q$-Laplace transform is defined as
\[ L_q[f(t)](s) = F_q(s) = \int_0^\infty f(t) \exp_q(-st)dt. \]
The corresponding inverse Laplace transform is defined as

\[ L_q^{-1}[F_q(s)](t) = f(t) = \frac{2 - q}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_q(s) [\exp_q(-st)]^{2q-3} ds, \quad (15) \]

where in the limit \( q \to 1 \), we have \([\exp_q(-st)]^{2q-3} \to \exp(st)\). Here \( c \) is a real constant that exceeds real part of all the singularities of \( F_q(s) \). To prove the inverse relationship between \( L_q \) and \( L_q^{-1} \), we verify the following two identities:

\[ f(t) = L_q^{-1}[L_q[f(t)]], \quad (16) \]

\[ F_q(s) = L_q[L_q^{-1}[F_q(s)]]. \quad (17) \]

Using the inverse Laplace transform we can write (17),

\[ L_q[L_q^{-1}[F_q(s)]] = \int_0^\infty \exp_q(-st) L_q^{-1}[F_q(s)] dt \]

\[ = \int_0^\infty dt \exp_q(-st) \frac{(2 - q)}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds' F_q(s') [\exp_q(-s't)]^{2q-3}. \quad (18) \]

We can rewrite the above expression as

\[ L_q[L_q^{-1}[F_q(s)]] = \frac{(2 - q)}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds' F_q(s') \int_0^\infty dt \exp_q(-st) [\exp_q(-s't)]^{2q-3}. \quad (19) \]

The second integral converges when \( \text{Re}[s'] = c < \text{Re}[s] \) and the resulting solution is

\[ I_q(s, s') = \int_0^\infty dt \exp_q(-st) [\exp_q(-s't)]^{2q-3} = \frac{1}{(2 - q)} \frac{1}{(s - s')}. \quad (20) \]

Substituting this in Eq. (19) we get

\[ L_q[L_q^{-1}[F_q(s)]] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds' \frac{F_q(s')}{(s - s')} . \quad (21) \]

The integrand \( F_q(s')/(s - s') \) has a simple pole at \( s = s' \). To evaluate the integral we draw a straight line at \( s' \) and an arc enclosing the pole of the integrand. The solution of the integral is

\[ \int_{c-i\infty}^{c+i\infty} ds' \frac{F_q(s')}{(s - s')} = 2\pi i F_q(s). \quad (22) \]

Substituting this in Eq. (21) we can observe that \( L_q[L_q^{-1}[F_q(s)]] = F_q(s) \). Next we verify the first identity Eq. (16) as follows:

\[ L_q^{-1}[L_q[f(t)]] \equiv L_q^{-1}[F_q(s)] = L_q^{-1} \left[ \int_0^\infty dt \exp_q(-st) f(t) \right] . \quad (23) \]
Using the definition of the inverse Laplace transform

\[ L_q^{-1}[L_q[f(t)]] = \frac{2 - q}{2\pi i} \int_{c-i\infty}^{c+i\infty} [\exp_q(-st)]^{2q-3} \int_0^\infty dt' f(t') \exp_q(-st') \, ds. \]  

(24)

Rewriting the above equation (24) we arrive at

\[ L_q^{-1}[L_q[f(t)]] = \frac{2 - q}{2\pi i} \int_0^\infty dt' f(t') \int_{c-i\infty}^{c+i\infty} [\exp_q(-st)]^{2q-3} \exp_q(-st') \, ds. \]  

(25)

Using the definition of the Dirac delta function based on the \( q \)-exponential described in Ref.\textsuperscript{24,25} we get

\[ L_q^{-1}[L_q[f(t)]] = \int_0^\infty dt' f(t') \delta(t - t') \equiv f(t). \]  

(26)

Thus in the present section, we have introduced the inverse for the Laplace transform defined using the type-I kernel using the complex integration technique.

B. Multivariable Laplace transform:

A generalization of the \( q \)-Laplace transform to the case of a multivariable function is carried out in this section. Let us consider a multivariable function \( f(t_1, \ldots, t_n) \) for which the \( q \)-generalization of the multivariable Laplace transform is defined as follows:

\[ L_q[f(t_1, \ldots, t_n)] = \int_0^\infty dt_1 \ldots \int_0^\infty dt_n \exp_q \left( - \sum_{l=1}^n s_l t_l \right) f(t_1, \ldots, t_n) \]

\[ = F_q(s_1, \ldots, s_n) \]  

(27)

In the limit \( q \to 1 \), the generalized Laplace transform defined in Eq. (27) reduces to the ordinary Laplace transform. The inverse transform corresponding to the multivariable Laplace transform is

\[ f(t_1, \ldots, t_n) = \frac{Q_n(2 - q)}{(2\pi i)^n} \int_{c_1-i\infty}^{c_1+i\infty} ds_1 \ldots \int_{c_n-i\infty}^{c_n+i\infty} ds_n \left[ \exp_q \left( - \sum_{l=1}^n s_l t_l \right) \right]^{(n+1)q-(n+2)} 1_q^{-m+1} F_q(s_1, \ldots, s_n) \]  

(28)

where \( c_1, \ldots, c_n \) are real constants that exceeds real parts of the singularities and the factor

\[ Q_{m+1}(2 - q) = (1 - q)^{m+1} \frac{\Gamma \left( \frac{1}{2} + m + 2 \right)}{\Gamma \left( \frac{1}{1-q} + 1 \right)}. \]
IV. PROPERTIES OF $q$-LAPLACE TRANSFORM

In this section, we list some of the properties of the $q$-Laplace transform:

1. 1st Identity on Limits:
   \[
   \lim_{s \to \infty} s L_q[f(t)] = \lim_{t \to 0} \frac{f(t)}{1 + (1 - q)}. \tag{29}
   \]

   **Proof:** Let us consider a general convergent function which can be expressed in terms of a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$. The $q$-Laplace transform of this general function is
   \[
   L_q[f(t)] = \int_0^\infty \left( \sum_{n=0}^{\infty} a_n t^n \right) \left[ 1 - (1 - q)st \right] \frac{1}{1-q} \, dt. \tag{30}
   \]
   Since the function $f(t)$ is a convergent function we can rewrite the above Equation (30) as
   \[
   L_q[f(t)] = \sum_{n=0}^{\infty} a_n \int_0^\infty t^n \left[ 1 - (1 - q)st \right] \frac{1}{1-q} \, dt. \tag{31}
   \]
   To solve Eq. (31) we use the integration by parts method choosing $u = t^n$ and $dV = \left[ 1 - (1 - q)st \right] \frac{1}{1-q}$. The resulting solution reads:
   \[
   s L_q[f(t)] = -\frac{1}{1 + (1-q)} \left( f(t) \left[ 1 - (1 - q)st \right] \frac{1}{1-q} \right)_{0}^{\infty} 
   - \int_0^\infty \left[ 1 - (1 - q)st \right] \left[ \frac{d}{dt} f(t) \right] \exp_q(-st) \, dt. \tag{32}
   \]
   On applying the limits corresponding to the integration in Eq. (32) we get
   \[
   s L_q[f(t)] = \frac{f(0)}{1 + (1-q)} + \frac{1}{1 + (1-q)} \int_0^\infty \left[ 1 - (1 - q)st \right] \frac{df(t)}{dt} \left[ 1 - (1 - q)st \right] \frac{1}{1-q} \, dt. \tag{33}
   \]
   Under the limiting condition $s \to \infty$, Eq. (33) gives
   \[
   \lim_{s \to \infty} s L_q[f(t)] = \lim_{s \to \infty} \frac{f(0)}{1 + (1-q)}. \tag{34}
   \]
   We can observe that the RHS is independent of $s$ and can be expressed as a limiting value of the parameter $t$ and this gives us
   \[
   \lim_{s \to \infty} s L_q[f(t)] = \lim_{t \to 0} \frac{f(t)}{1 + (1-q)}. \tag{35}
   \]
   Thus we prove the first identity on the limits of a Laplace transform.
2. II\textsuperscript{nd} Identity on Limits:

\[
\lim_{s \to 0} s L_q[f(t)] = \lim_{t \to \infty} \frac{f(t)}{1 + (1 - q) t}.
\] (36)

\textit{Proof:} To prove this limit let us consider the Eq. (32) and evaluate the limit, \(s \to 0\) and the resulting expression is

\[
\lim_{s \to 0} s L_q[f(t)] = \lim_{s \to 0} \frac{f(0)}{1 + (1 - q) t} + \frac{1}{1 + (1 - q)} \int_0^\infty \frac{df}{dt} dt
\]

\[
= \frac{f(0)}{1 + (1 - q)} + \frac{f(t)}{1 + (1 - q)} \bigg|_0^\infty.
\] (37)

Applying the limits we get

\[
\lim_{s \to 0} s L_q[f(t)] = \frac{f(\infty)}{1 + (1 - q)}.
\] (38)

The RHS of the equation can be rewritten as

\[
\lim_{s \to 0} s L_q[f(t)] = \lim_{t \to \infty} \frac{f(t)}{1 + (1 - q)}.
\] (39)

Thus we prove the second identity on the Laplace transform.

3. Scaling:

\[ L_q[f(at)] = \frac{1}{a} F_q(s/a). \] (40)

\textit{Proof:} Let us consider the Laplace transform of a function \(f(at)\)

\[ L_q[f(at)] = \int_0^\infty dt [1 - (1 - q) st]^{\frac{1}{t} - q} f(at). \] (41)

Substituting \(at = x\), we get

\[ L_q[f(at)] = \frac{1}{a} \int_0^\infty dx \left[1 - (1 - q) \frac{sx}{a} \right]^{\frac{1}{t} - q} f(x) = \frac{1}{a} F_q(s/a). \] (42)

Hence the scaling relation is proved.

4. Shifting:

\[ F_q(s - s_0) = L_q \left[ f(t) \exp_q \left( \frac{s_0 t}{1 - (1 - q) st} \right) \right]. \] (43)

\textit{Proof:} Let us consider the definition of the Laplace transform

\[ F_q(s) = \int_0^\infty \exp_q(-st) f(t) dt, \] (44)
and introduce a shift in ‘s’ as ‘s → s − s₀’ and this yields

\[ F_q(s - s_0) = \int_0^\infty \exp_q(-(s - s_0)t)f(t)dt. \]  (45)

This can be rewritten as

\[ F_q(s - s_0) = \int_0^\infty \exp_q(-st)\exp_q\left(\frac{s_0t}{1 - (1 - q)st}\right)f(t)dt. \]  (46)

Hence we have proved

\[ F_q(s - s_0) = L_q\left[f(t)\exp_q\left(\frac{s_0t}{1 - (1 - q)st}\right)\right]. \]  (47)

5. q-translation:

\[ L_q\left[f(t)[\exp_q(st_0)]^{1 + (1 - q)}\right] = L_q\left[f\left(\frac{t - t_0}{1 - (1 - q)st_0}\right)\Theta\left(\frac{t - t_0}{1 - (1 - q)st_0}\right)\right]. \]  (48)

**Proof:** The RHS of the above identity gives

\[ I = L_q\left[f\left(\frac{t' - t_0}{1 - (1 - q)st_0}\right)\Theta\left(\frac{t' - t_0}{1 - (1 - q)st_0}\right)\right]\]
\[ = \int dt'f\left(\frac{t' - t_0}{1 - (1 - q)st_0}\right)\Theta\left(\frac{t' - t_0}{1 - (1 - q)st_0}\right)\exp_q(-st'), \]

where \(\Theta(x)\) is the Heaviside step function such that \(\Theta(x) = 0\) when \(x < 0\) and \(\Theta(x) = 1\) for \(x \geq 0\). Using the scaling \(t = \frac{t' - t_0}{1 - (1 - q)st_0}\), we can rewrite the integral as

\[ I = \int_{-\alpha}^\infty dt \exp_q(-st)f(t)[\exp_q(-st_0)]^{2-q} \Theta(t). \]  (49)

where \(\alpha = \frac{t_0}{1 - (1 - q)st_0}\). On applying the Heavisides step function we get

\[ I = L_q[f(t)[\exp_q(-st_0)]^{2-q}]. \]  (50)

Hence the q-translation identity has been proved.

The properties of linearity and q-convolution have been established in Ref. and here we are stating them just for the sake of completeness. The two properties of linearity and q-convolution are

1. Linearity:

\[ L_q[a_1f_1(t) + a_2f_2(t)] = a_1L_q[f_1(t)] + a_2L_q[f_2(t)]. \]  (51)
2. \( q \)-convolution:

Let \( f(t) \) and \( g(t) \) be two positive scalar functions of \( 't' \), and \( F_q(s) \) and \( G_q(s) \) be their \( q \)-Laplace transforms, then

\[
L_q[f(t) \ast g(t)] = F_q(s) \ast G_q(s),
\]

where \( f(t) \ast g(t) = \int_0^t f(\tau) \ast g(t-\tau)d\tau \).

\textit{Laplace transform of} \( q \)-\textit{derivatives and} \( q \)-\textit{integrals:} The properties of the \( q \)-Laplace transform based on derivatives and integrals have been derived in Ref. \ref{19}. For the sake of completeness, we give below the expression for the generalized Laplace transform of a derivative function

\[
L_q \left[ \frac{d^n}{dt^n} f(t) \right] = - \left( f^{(n-1)}(t) + (1 - \delta_{1n}) \sum_{\ell=1}^{n-1} Q_{\ell-1}(q)s^{\ell} f^{(n-\ell-1)}(t) \right) \bigg|_{t=0} + \sum_{\ell=1}^{n-1} Q_{n-\ell}(q)s^{\ell} \int L_{\frac{n+1}{n}} \left( f(t) \right)(a_n s),
\]

where, \( Q_n(q) = \prod_{j=0}^{n} a_j \) with \( a_j = jq - (j - 1) \) and \( \delta_{1j} \) is the Kronecker delta function. The Laplace transform of the integral reads:

\[
L_q \left[ \int_0^t f(x)dx \right] = \frac{2-q}{s} L_{\frac{1}{2-q}} \left[ f(t) \right] (s(2 - q)).
\]

In the present work, we derive the \( q \)-Laplace transform of \( q \)-calculus of functions. The concept of \( q \)-calculus was first introduced in \ref{26}, where the authors defined the derivative and integral based on \( q \)-deformation. Many such derivatives and integrals were investigated in subsequent works \ref{13,16,19}. For the present work we use the \( q \)-derivative defined in \ref{27} and its corresponding \( q \)-deformed integral. The relevant \( q \)-derivative and \( q \)-integral operators are

\[
D_q(s) = \frac{1}{1 - (1 - q) \left( \frac{d}{ds} \right)}; \quad \int d_q x = \int dx \left[ 1 - (1 - q)x \frac{d}{dx} \right].
\]

1. Derivative of Laplace transform:

\[
D_q^{(n)}(s) \{ L_q[f(t)] \}(s) = L_q[(-t)^n f(t)].
\]

\textbf{Proof:} Let us consider the first derivative of the \( q \)-Laplace transform

\[
D_q \{ L_q[f(t)] \}(s) = D_q(s) \left\{ \int_0^\infty dt \exp_q(-st)f(t) \right\} = \int_0^\infty dt D_q(s) \exp_q(-st)f(t) = L_q[(-t)f(t)].
\]
The second derivative of the $q$-Laplace transform is
\[ D_q^{(2)} \{ L_q[f(t)] \}(s) = L_q[(-t)^2 f(t)]. \] (58)
For the $n^{th}$ $q$-derivative we get
\[ D_q^{(n)} \{ L_q[f(t)] \}(s) = L_q[(-t)^n f(t)]. \] (59)

2. Integral of a Laplace transform:
\[ \int_s^\infty d_q s' F_q(s') = L_q \left\{ \frac{f(t)}{t} \right\}. \] (60)

Let us consider the definition of the generalized Laplace transform apply the $q$-integral operator
\[ \int_s^\infty d_q s' F_q(s') = \int_s^\infty d_q s' \int_0^\infty \exp_q(-s't)f(t)dt. \] (61)
Rearranging the order of the integrals and evaluating the $q$-integral with respect to ‘s’ gives
\[ \int_s^\infty d_q s' F_q(s') = \int_0^\infty \frac{1}{t} \exp_q(-st)f(t)dt = L_q \left\{ \frac{f(t)}{t} \right\}. \] (62)
Hence proved.

V. POST-WIDDER’S METHOD OF INVERSE LAPLACE TRANSFORM

The inverse Laplace transform is usually calculated using a Bromwich contour integral over the complex plane. An alternative method is to introduce a method based on real variables, which is the Post-Widder’s method. In this section we introduce the generalization of Post-Widder’s method for the inverse of $q$-Laplace transfrom for both the single variable and the multivariable case.

A. Single variable inverse Laplace transform:

For the single variable inverse $q$-Laplace transform to derive the $q$-Widder’s formula we will have to rewrite the type I kernel as $K_q(s, t) = \exp \left( \frac{1}{1-q} \ln(1 - (1-q)st) \right)$. We then have to take the $k^{th}$ derivative of the Laplace transform and scale the function $f(t)$ to $f(\xi_m t)$ and the resulting expression is
\[ F_q^{(k)}(s) = \int_0^\infty dt[-(1-q)t]^k \frac{\Gamma \left( \frac{2-q}{1-q} \right)}{\Gamma \left( \frac{2-q}{1-q} - k \right)} (1 - (1-q)st)^{-\frac{1}{1-q}-k} f(\xi_m). \] (63)
Through a change of variables $t = xy$ and $s = k/x$ we get:

$$F_q^{(k)} \left( \frac{k}{x} \right) = \left[ -(1 - q) \right]^k x^{k+1} \frac{\Gamma \left( \frac{2q}{1-q} \right)}{\Gamma \left( \frac{2q}{1-q} - k \right)} \int_0^\infty (1 - (1 - q)ky)^{\frac{1}{1-q} - k} y^k dy f(xy \xi_m). \quad (64)$$

The function $y^k(1 - (1 - q)ky)^{\frac{1}{1-q} - k}$ has a single maximum which is sharply peaked at $y = 1$. Hence we can replace the function $f(\xi_mxy)$ by $f(\xi_mx)$ and get

$$F_q^{(k)} \left( \frac{k}{x} \right) = f(\xi_mx)[-(1 - q)]^k x^{k+1} \frac{\Gamma \left( \frac{2q}{1-q} \right)}{\Gamma \left( \frac{2q}{1-q} - k \right)} \int_0^\infty (1 - (1 - q)ky)^{\frac{1}{1-q} - k} y^k dy. \quad (65)$$

The solution of the integral in the above equation is

$$\int_0^\infty (1 - (1 - q)ky)^{\frac{1}{1-q} - k} y^k dy = \frac{\Gamma(k + 1) \Gamma \left( \frac{2q}{1-q} - k \right)}{\Gamma \left( \frac{3-2q}{1-q} \right) (1 - q)^{k+1} k^{k+1}}. \quad (66)$$

Replacing the value of the integral in Eq. (65) and simplifying we have

$$F_q^{(k)} \left( \frac{k}{x} \right) = (-1)^k f(x \xi_m)x^{k+1} \frac{\Gamma(k + 1)}{k^{k+1}(2 - q)}. \quad (67)$$

where $\xi_m = \left[ \frac{1+(1-q)}{Q_m(2-q)} \right]^{m-1}$ valid for $m \geq 2$ and ill defined for $m = 1$ and the polynomial $Q_m(q) = \prod_{j=1}^m (1 - (1 - q)j)$ exists for only for $j \geq 1$. Substituting $t = \xi_mx$, the $q$-deformed Widder’s formula for the inverse Laplace transform is

$$f(t) = \lim_{k \to \infty} \frac{(-1)^k}{\Gamma(k + 1)} F_q^{(k)}(s)(2 - q)s^{k+1} |_{s = \frac{\xi_m}{x}}. \quad (68)$$

**B. Multivariable inverse Laplace transform:**

The multivariable inverse $q$-Laplace transform based on real variables i.e., the Post-Widder’s method for the type-I kernel is

$$f(t_1, t_2, \ldots, t_n) = Q_1(2 - q) \left( \prod_{i=1}^n \lim_{k_i \to \infty} \frac{(-1)^{k_i}}{\Gamma(k_i + 1)} t_i^{k_i+1} \right) \frac{d^{k_1}}{ds_1^{k_1}} \left( \frac{d^{k_2}}{ds_2^{k_2}} \left( \ldots \frac{d^{k_n}}{ds_n^{k_n}} F_q(s_1, s_2, \ldots, s_n) \bigg|_{s_1} \ldots \bigg|_{s_n} \right) \right) |_{s_1}. \quad (69)$$

Here $s_1 = (k_1/t_1)\xi_r$, where $r = \sum_{\ell=1}^n m_\ell + n$, $s_2 = (k_2/t_2)(t_1s_1/k_1)^{m_2+1/m_1}$ and $s_n = (k_n/t_n)(t_1s_1/k_1)^{m_n+1/m_1}$. The factor $\xi$ is as defined in the previous section on single variable inverse $q$ Laplace transform based on Widder’s method.
VI. INVERSE LAPLACE TRANSFORM FOR SOME ELEMENTARY FUNCTIONS

In this section we evaluate the Laplace transform of a simple algebraic function and calculate its inverse.

A. Laplace transform of single variable

Let us consider an algebraic function \( f(t) = t^{m-1} \), the \( q \)-Laplace transform of this function is

\[
L_q[t^{m-1}] = \int_0^\infty dt \exp_q(-st)t^{m-1}. 
\]

Evaluating the Laplace transform we get

\[
F_q(s) = \frac{\Gamma(m)}{Q_m(2-q) s^m}, \quad \text{for } m \geq 2. 
\]

The inverse of the \( q \)-Laplace transform can be computed using a \( q \)-version of the Widder’s formula. For this we calculate the \( k^{th} \)-derivative of \( F_q(s) \)

\[
F_q^{(k)}(s) = \frac{1}{Q_m(2-q)}(-1)^k \Gamma(m + k) s^{m+k}. 
\]

Substituting Eq. (72) in the \( q \)-Widder’s formula we get

\[
f(t) = \lim_{k \to \infty} \frac{(-1)^k}{\Gamma(k+1)} \left( \frac{1}{Q_m(2-q)}(-1)^k \Gamma(m + k) s^{m+k} \right) Q_1(2-q)|_{s = \frac{k \xi_m}{m}} \approx \frac{1}{Q_m(2-q) \xi^{m-1}} \left( \lim_{k \to \infty} \frac{k^{1-m} \Gamma(m + k)}{\Gamma(k + 1)} \right). 
\]

On substitution of the limits and simplifying Eq. (74) we get the algebraic limit \( f(t) = t^{m-1} \). This validates the Post-Widder’s method of computing inverse Laplace transform for any Algebraic function. The Laplace transform and inverse Laplace transform of some common functions is given in Table I below. Here we have calculated the inverse transform using the contour integration method as well as the Post-Widder’s technique.
To illustrate the multivariate Laplace transform let us choose the function \( f(t_1, t_2, \ldots, t_n) = \alpha \prod_{l=1}^{n} t_i^{m_l} \) where \( \alpha \) is a constant. The multivariate \( q \)-Laplace transform of this function is

\[
F_q(s_1, s_2, \ldots, s_n) = \alpha \frac{\Gamma \left( \frac{1}{1-q} + n \right)}{\Gamma \left( \frac{1}{1-q} + \sum_{l=1}^{n} m_l + n + 1 \right)} \times \prod_{l=1}^{n} \frac{\Gamma (m_l + 1)}{(1-q)^{m_l + 1} s_l^{m_l + 1}} \tag{75}
\]
To calculate the inverse we can substitute $F_q(s_1, s_2, \ldots, s_n)$ in the expression for general inverse

$$f(t_1, t_2, \ldots, t_n) = \alpha \frac{Q_n(2 - q)}{(2\pi i)^n} \frac{\Gamma\left(\frac{1}{1-q} + 1\right)}{\Gamma\left(\frac{1}{1-q} + \sum_{l=1}^{n} m_l + n + 1\right)} \prod_{l=1}^{n} \frac{\Gamma(m_l + 1)}{(1 - q)^{m_l + 1}} \times$$

$$\int_{\Gamma_1} ds_1 \int_{\Gamma_2} ds_2 \ldots \int_{\Gamma_n} ds_n \left[ 1 - (1 - q) \left( \sum_{l=1}^{n} s_l t_l \right) \right] \left( \frac{1}{s_l^{m_l + 1}} \right)$$

$$= \alpha \frac{Q_n(2 - q)}{(2\pi i)^n} \left( \sum_{j=1}^{n} m_j + n \right) \prod_{l=1}^{n} \frac{\Gamma(m_l + 1)}{(1 - q)^{m_l + 1}}$$

Using complex contour integration we get $f(t_1, t_2, \ldots, t_n) = \alpha \prod_{l=1}^{n} t_l^{m_l}$, where $\alpha$ is a constant.

Alternatively we can use the Post-Widder’s method to get

$$f(t_1, t_2, \ldots, t_n) = Q_1(2 - q) \left( \prod_{l=1}^{n} \lim_{k_l \to \infty} \frac{(-1)^{k_l}}{\Gamma(k_l + 1)} s_l^{k_l + 1} \right)$$

$$\frac{d^{k_1}}{d s_1^{k_1}} \left( \frac{d^{k_2}}{d s_2^{k_2}} \left( \ldots \frac{d^{k_n}}{d s_n^{k_n}} F_q(s_1, s_2, \ldots, s_n) \bigg|_{s_n} \ldots \bigg|_{s_2} \bigg|_{s_1} \right) \right)$$

Substituting $F_q(s_1, s_2, \ldots, s_n)$ in Eq. (77), we get $f(t_1, t_2, \ldots, t_n) = \alpha \prod_{l=1}^{n} t_l^{m_l}$. Thus we evaluate the inverse Laplace transform of the multivariable algebraic function using complex contour integration method as well real variable based Post-Widder’s method. We find identical results which leads to conclude that both these methods are equivalent.

VII. CONCLUDING REMARKS

A generalization of the Laplace transform based on Tsallis $q$-exponential is investigated in the present work. We use $K_l(q; s, t) = \exp_q(-st)$ as the kernel of the transform. A single variable Laplace transform has been defined in [14] based on this Kernel. But the inverse transformation has not been defined so far. In the present work we define the inverse Laplace transform of a single variable function using the complex integration method as well as the
Post-Widder’s method which uses real variables. Then we introduce a multivariable Laplace transform for the same kernel and also define the inverse transform through the complex integration method as the real variable based method. We verify the properties of the Laplace transform as well as the inverse transform and compute them for some elementary functions. The results are given in Table: II where the inverse transforms were calculated using the complex integration method and the real variable based Post-Widder’s method.

An application of the Laplace transform and its inverse in statistical mechanics is to interrelate the partition function and the density of states. To describe a physical system in thermodynamic equilibrium with its surroundings we need to describe its thermal, mechanical and chemical properties. Each of this property is characterized using a pair of quantities of which one is an extensive variable and the other is an intensive variable. So, in total we have 8 \(2^3\) different ensembles. Of the eight ensembles there are four of them where the temperature is fixed and these are known as the isothermal ensembles and the remaining four where the heat is fixed are the adiabatic ensembles. The Laplace transform and the inverse transform can be used to interrelate the partition functions and density of states of the different ensembles. For example let us consider a classical ideal gas in \(D\)-dimensions with the Hamiltonian 
\[
H = \sum_{i=1}^{D_N} \frac{p_i^2}{2m} + \frac{1}{2m} \omega^2 x_i^2.
\]
The partition function of this system is
\[
Z_q(\beta) = \frac{V^N (2\pi m)^{D_N}}{h^{D_N} N!} \left( \frac{\Gamma \left( \frac{1}{1-q} + 1 \right)}{(1-q)^{D_N/2} \Gamma \left( \frac{1}{1-q} + \frac{D_N}{2} + 1 \right)} \right) \frac{1}{\beta^{D_N/2}}.
\] (78)

From the generalized Post-Widder’s method we get:
\[
g(E) = \lim_{k \to \infty} \frac{(-1)^k}{\Gamma(k+1)} Z_q^{(k)}(\beta)^{\beta/k+1} Q_1(2-q) \bigg|_{\beta = \frac{k\xi}{m}} = \frac{V^N (2\pi m)^{D_N}}{h^{D_N} N! (\frac{D_N}{2} - 1)!} E^{D_N - 1}.
\] (79)

which is the density of states of the classical ideal gas. Similarly we can consider a collection of \(N\) harmonic oscillators in \(D\)-dimensions with the Hamiltonian 
\[
H = \sum_{i=1}^{D_N} \left( \frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 \right).
\]
The partition function of this system is
\[
Z_q(\beta) = \frac{1}{(\hbar \omega)^{D_N}} \left( \frac{\Gamma \left( \frac{1}{1-q} + 1 \right)}{(1-q)^{D_N} \Gamma \left( \frac{1}{1-q} + D_N + 1 \right)} \right) \frac{1}{\beta^{D_N}}.
\] (80)

Using the Post-Widder’s method of inverse transform we get
\[
g(E) = \lim_{k \to \infty} \frac{(-1)^k}{\Gamma(k+1)} Z_q^{(k)}(\beta)^{\beta/k+1} Q_1(2-q) \bigg|_{\beta = \frac{k\xi}{\hbar \omega}} = \frac{1}{(\hbar \omega)^{D_N} (D_N - 1)!} E^{D_N - 1}.
\] (81)
which is the expression for the density of states of the system in the microcanonical ensemble. While these applications demonstrate the generalized Laplace transform and its inverse, we can also use them in the fields of atomic physics, open quantum systems and also in finding the solutions of differential equations.

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AUTHOR DECLARATIONS

Conflict of interest

The authors have no conflicts to disclose

Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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