RULED QUARTIC SURFACES, MODELS AND CLASSIFICATION

IRENE POLO-BLANCO, MARIUS VAN DER PUT AND JAAP TOP

Abstract. New historical aspects of the classification, by Cayley and Cremona, of ruled quartic surfaces and the relation to string models and plaster models are presented. In a ‘modern’ treatment of the classification of ruled quartic surfaces the classical one is corrected and completed. A conceptual proof is presented of a result of Rohn concerning curves in \( \mathbb{P}^1 \times \mathbb{P}^1 \) of bi-degree \((2,2)\). The string models of Series XIII (of some ruled quartic surfaces) are based on Rohn’s result.

Motivation and History

The collection of string models of ruled quartic surfaces, present at some mathematical institutes (for instance at the department of mathematics in Groningen) is the direct motivation for this paper. This Series XIII, produced by Martin Schilling in 1886, is based upon a paper of K. Rohn \[13\] containing a classification of ruled quartic surfaces over \( \mathbb{C} \) and \( \mathbb{R} \). Some authors before Rohn (e.g., M. Chasles \[4\], A. Cayley \[3\], L. Cremona \[5\], R. Sturm \[17\], G. Salmon \[14\]) and many after his time (e.g., B.C. Wong \[23\], H. Mohrmann \[10\], W.Fr. Meyer \[9\], W.L. Edge \[7\], O. Bottema \[2\], T. Urabe \[20\]) have contributed to this beautiful topic of 19th century geometry.

Cremona classified the ruled quartic surfaces in 12 types. He states in \[5\] that Cayley produced 8 of these without revealing his method. However, Cayley’s third memoir on this subject \[3\] was written earlier the same year 1868 and contains 10 types. In an addition to this memoir (May 18, 1869), Cayley gives the comparison between his own classification and the one by Cremona and makes it clear what the two types he missed are. The method of Cayley consists of taking three curves in \( \mathbb{P}^3 \) and to consider the ruled surface \( S \) which is the union of the lines meeting all three curves. Using a formula for the degree of \( S \), he now computes possibilities of ruled quartic surfaces. The expression ‘the six coordinates of a line’ in Cayley’s work indicates that the Grassmann variety \( Gr(2,4) \) of the lines in \( \mathbb{P}^3 \) plays a role. The work of Cayley contains also explicit calculations for reciprocal surfaces (see

Date: April 15, 2009.
The results of Cremona can be explained as follows. Let $S \subset \mathbb{P}^3$ be a ruled quartic surface (reduced, irreducible and defined over $\mathbb{C}$). The fact that through a general point of $S$ there is only one line of $S$ is tacitly assumed (compare Lemma 1.2). The locus $D$ of the points on $S$ through which there are at least two lines of $S$ (in the 1-parameter family) is called the ‘double curve’. Cremona states that $D$ is indeed a curve (hereby excluding cones) and has ‘in general’ degree 3.

(We note that $D$ need not coincide with the singular locus of $S$, that $D$ can also have degree 2 (see Number 15 in Subsection 1.6) and that $D = \emptyset$ is possible; compare, for example, Corollary 1.7(2).)

Two intersecting lines on $S$ determine a plane. The collection $\mathcal{D}$ of all these planes is called the ‘bitangent developable’. This 1-dimensional family (assuming $D \neq \emptyset$) can be seen as a curve in the dual projective space. The genus of $S$ is defined as the genus of the (irreducible, singular) curve $H \cap S$ of degree 4, where $H$ is a general plane. Cremona states that the genus can only be 0 or 1. Missing is the nontrivial argument showing that genus 2 is impossible (see Observation 1.5 and [20, Proposition 2.6]). Cremona classifies $S$ according to the nature (degrees and multiplicities of the irreducible components) of the curves $D$ and $\mathcal{D}$ (and in one case a relation between $D$ and $\mathcal{D}$). He obtains his list of possibilities via the following construction:

Consider a tuple $(C_1, C_2, f)$ consisting of two conics $C_1, C_2 \subset \mathbb{P}^3$, not in the same plane, and an isomorphism $f : C_1 \to C_2$. This defines a ruled surface $S$ which is the union of the lines through the pairs of points $\{c_1, f(c_1)\}$ with $c_1 \in C_1$. In the general case, the line $H_i \cap H_2$, where $C_i$ lies in the plane $H_i$ for $i = 1, 2$, intersects $C_1$ in two points $p_1 \neq q_1$ and intersects $C_2$ in two points $p_2 \neq q_2$. Now $H_2 \cap S$ is the union of the conic $C_2$ and the two lines through the pairs of points $(p_1, f(p_1))$ and $(q_1, f(q_1))$. Thus $S$ is an irreducible ruled surface of degree 4. Moreover, the two lines intersect in a point of the ‘double curve’ and $H_2$ is a ‘bitangent plane’, i.e., a point on the ‘bitangent developable.’ The same holds of course for $H_1$. Cremona’s examples are obtained by varying and degenerating $C_1, C_2, f$. His assertion to have found all types in this way is not correct since some ruled quartic surfaces are only obtained from a line and a curve of degree 3. However by including ‘reciprocal surfaces’ and maybe stretching the meaning of ‘degeneration’ some of the latter surfaces can be obtained.

The approach of Cayley (and of Rohn) has the classical name “analytic geometry”, indicating the use of coordinates and algebraic operations with formulas. In contrast, Cremona’s (and Sturm’s) approach
is purely “synthetic”. As a consequence, Cremona’s paper is difficult to read and it is hard to verify the results.

Bottema [2, p. 349] remarks that Rohn claims to have discovered a type overlooked by his predecessors. Indeed, on p. 147 of Rohn’s paper [12], there is an explicit equation and the remark in a footnote: “this ruled surface is not mentioned by Cremona in his treatise”. However, it is easy to verify that Rohn’s equation (in the homogeneous coordinates $x, y, z, w$)

$$wx^2(x + 3Nz) + F_4(x, y) = 0, \text{ } F_4 \text{ a binary quartic, } N \text{ a constant},$$

does not define a ruled surface, since a general point $(a : b : c : 1)$ on it is not contained in any line of the surface. Actually, Rohn’s geometric construction is valid but his formula happens to be mistaken. The construction gives indeed a case which is not explicitly mentioned by Cremona. However it can be interpreted as hiding in Cremona’s species 10. Pascal’s well written Repertorium reviews the classification of Cayley and Cremona, [11], XII, §10. Here Rohn’s extra case reappears on p. 338-339 with the same correct geometric construction and another mistake in the formula. In the classification of the present paper Rohn’s example is Number 5.

There are also critical comments by R. Sturm to the list of Cremona. Moreover, some of the 12 species of Cremona contain surfaces of a rather different nature, as we will see in Section 3.

The classification of ruled quartic surfaces in the book of W.L. Edge [7] is identical with the one of Cremona. Two methods are developed there. The first one classifies the curves (irreducible and of degree 4), corresponding to ruled quartic surfaces, in the Grassmann variety $Gr(2, 4)$ (parametrizing the 2-dimensional subspaces of a 4-dimensional vector, or, equivalently, the lines in $\mathbb{P}^3$). The second method obtains the ruled quartic surfaces in $\mathbb{P}^3$ as projections of certain ruled quartic surfaces in $\mathbb{P}^5$ or $\mathbb{P}^4$. This is related to a paper of C. Segre [10] and to a paper by Swinnerton–Dyer [18].

In the thesis of Wong [23], a rational morphism $\mathbb{P}^3 \cdots \rightarrow Gr(2, 4)$, associated to the classical ‘tetrahedral complex’, is considered. Certain plane curves in $\mathbb{P}^3$ of degree 2 and 3 have as images in $Gr(2, 4)$ curves of degree 4 and correspond therefore to ruled quartic surfaces. It is claimed in this thesis that every ‘species’ in Cremona’s list can be obtained in this way.

For other details on the early history of the subject we refer to the contribution of W.Fr. Meyer in [9].
A ruled surface in modern terminology (see [8, Section V.2.]), is a morphism $Z \to C$ of a smooth projective surface $Z$ to a smooth curve $C$ such that all fibres are isomorphic to $\mathbb{P}^1$. A classical ruled surface $S \subset \mathbb{P}^3$ is obtained as the image of a suitable morphism $Z \to \mathbb{P}^3$. This method and the papers of T. Urabe [19], [20], [21], [22] may lead to a modern classification of ruled quartic surfaces (including moduli). We note, in passing, that Urabe’s important work concerns the discovery of new aspects in the classification of the singularities of quartic surfaces in $\mathbb{P}^3$ (which, generally, are not ruled) and their relation to Dynkin diagrams.

One aim of the present paper is to give a modern treatment of Rohn’s paper [13], namely the ‘symmetrization’ of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(2, 2)$ and the classification of some ruled quartic surfaces over $\mathbb{R}$. The latter is used to obtain explicit equations explaining the visual features of the models of Series XIII.

The possibilities for the 1-dimensional part of the singular locus of a ruled surface $S$ can be read off from the intersection of $S$ with a general plane. This leads to the elegant elementary treatment of ruled cubic surfaces in Dolgachev’s book [6]. As a didactical step towards ruled quartic surfaces, we present here another method valid over any base field and obtain the three types of ruled cubic surfaces over $\mathbb{R}$.

The other aim of this paper is to present a classification of the quartic ruled surfaces, such that each class is determined by discrete data and the surfaces belonging to a given class give rise to a connected moduli space. This leads to 29 cases. A combination of the following methods leads to this classification.

1. Deriving some properties of the curves $C$ of degree 4 (corresponding to ruled quartic surfaces) lying on the Grassmann variety $Gr(2, 4) \subset \mathbb{P}^5$.
2. Determining the possibilities for the singular locus of a quartic ruled surface.
3. The normalization $C^{\text{norm}}$ of $C$ carries a vector bundle of rank two. In case the genus of $C^{\text{norm}}$ is 0, there are two possibilities for this vector bundle. Two ‘generating’ meromorphic sections of this vector bundle are brought in some standard form, by some linear base changes. This leads to explicit equations for the corresponding quartic ruled surfaces.
4. Classification of the position of $C$ w.r.t. the tangent spaces of $Gr(2, 4)$.

Part (4) is in fact one of the two methods of [7] in deriving Cremona’s list. Although we could not verify this in detail because of a certain
vagueness in Edge’s arguments, the results agree with our computations.

Before giving Cremona’s list we need to explain to notion of ‘reciprocal surface’ or ‘dual surface’ in more modern terms, of a surface \( S \subset \mathbb{P}^3 \). It is obtained by considering all tangent planes at the non-singular points of \( S \). Each tangent plane is a point in the dual projective space \( \tilde{\mathbb{P}}^3 \). The Zariski closure of all these points is the dual surface \( \tilde{S} \subset \tilde{\mathbb{P}}^3 \). In the case that \( S \) is ruled, also \( \tilde{S} \) is ruled and has the same degree as \( S \). Moreover, the ‘double line’ of \( \tilde{S} \) can be seen to be the ‘bitangent developable’. Cremona’s list shows that the species 3 and 4 are dual, as well as the species 7 and 8. The other species are ‘selfdual’. In the table one has to give ‘double curve’ the interpretation ‘singular locus’. The genus \( g \) of the surface is 0, except for Cremona 11, 12 where it is 1.

We adopt a notation of Cayley, namely the expression \( d^m \) stands for an irreducible component of the singular locus of degree \( d \) and with multiplicity \( m \). The difference between Cremona 6 and Cremona 11 is that the two lines intersect in the first case and are skew in the second one. The difference between Cremona 9 and Cremona 10 is somewhat subtle. In case 10 the bitangent planes are the planes containing the singular line, denoted by 1. In case 9, the bitangent planes are the planes containing another line, denoted by 1’.

| Double curve | Double curve recip. surface | Cremona type | Double curve | Double curve recip. surface | Cremona type |
|--------------|----------------------------|--------------|--------------|----------------------------|--------------|
| 3^2          | 3^2                        | 1            | 3^2          | 1^3                        | 7            |
| 2^2, 1^3     | 2^2, 1^3                   | 2            | 1^3          | 3^2                        | 8            |
| 1^3          | 2^2, 1^2                   | 3            | 1^3          | 1^9                        | 9            |
| 2^2, 1^3     | 1^3                        | 4            | 1^3          | 1^3                        | 10           |
| 1^2, 1^2, 1^2| 1^2, 1^2, 1^2              | 5            | 1^2, 1^2     | 1^2, 1^2                   | 11, \( g = 1 \) |
| 1^2, 1^2 int | 1^2, 1^2 int               | 6            | 1^2          | 1^2                        | 12, \( g = 1 \) |

Some models from Series XIII: nr. 1, 4, and 5.
1. Curves on the Grassmann variety $Gr(2,4)$

1.1. Properties of the Grassmann variety. Let $V$ be a vector space of dimension 4 over the (algebraically closed) field $K$. The lines in the projective space $\mathbb{P}(V) \cong \mathbb{P}^3$ are points of the Grassmann variety $Gr := Gr(2, V) = Gr(2, 4)$ and the natural way to study a ruled surface $S \subset \mathbb{P}(V)$ is to consider the set of the lines on $S$ as subset of $Gr$. We briefly define $Gr$ and summarize its main properties.

For notational convenience we fix a basis $e_1, e_2, e_3, e_4$ of $V$ and we identify the exterior power $\Lambda^2 V$ with $K$ by $e_1 \wedge \ldots \wedge e_4 \mapsto 1$. The obvious symmetric bilinear map $\Lambda^2 V \times \Lambda^2 V \to \Lambda^2 V = K$ is nondegenerate. A line in $\mathbb{P}(V)$ correspond to a plane $W \subset V$, a line $\Lambda^2 W \subset \Lambda^2 V$ and to a point in $\mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^5$. If $W$ has basis $v_1, v_2$, then $w = v_1 \wedge v_2$ is a basis vector for $\Lambda^2 W$ and $\overline{w} := Kw$ is this point of $\mathbb{P}(\Lambda^2 V)$. By definition $Gr = Gr(2, V) \subset \mathbb{P}(\Lambda^2 V)$ consists of all these points. Now $\overline{w}$ (with $w \in \Lambda^2 V$, $w \neq 0$) belongs to $Gr$ if and only if $w$ is decomposable, i.e., has the form $v_1 \wedge v_2$. The latter is equivalent to $w \wedge w = 0$. We use the six elements $e_{ij} := e_i \wedge e_j$, $i < j$ as basis for $\Lambda^2 V$ and write an element of this vector spaces $\sum_{i<j} p_{ij} e_{ij}$. The $p_{ij}$ are called the Plücker coordinates. They also serve as homogeneous coordinates for $\mathbb{P}(\Lambda^2 V)$. One finds that $Gr$ is the nondegenerate quadric given by the equation $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$. For notational purposes and for convenience of the reader we recall the following.

**List of properties of $Gr$** (of importance for our purposes).

(i) $p_0, \ell_0, h_0$ are a point, a line and a plane of $\mathbb{P}(V) = \mathbb{P}^3$. One identifies $p_0$ with a $\overline{v}_0$, $v_0 \in V$, $v_0 \neq 0$ and $\ell_0$ with a $\overline{w}_0$, $w_0 \in \Lambda^2 V$, $w_0 \wedge w_0 = 0$. We note that $\overline{w} \in \mathbb{P}(\Lambda^2 V)$ with $w \wedge w = 0$ is both seen as a point of $Gr$ and as a line in $\mathbb{P}(V)$.

(ii) Two lines $\overline{w}_1, \overline{w}_2$ of $\mathbb{P}(V)$ intersect if and only if $w_1 \wedge w_2 = 0$.

(iii) Every hyperplane of $\mathbb{P}(\Lambda^2 V)$ has the form $\{ z \mid w \wedge z = 0 \}$ with $w \in \Lambda^2 V$, $w \neq 0$ and unique $\overline{w}$. If $w$ is indecomposable, then the intersection of the hyperplane with $Gr$ is a nondegenerate quadric. If $w$ is decomposable, i.e., $\overline{w} = \overline{w}_0 \in Gr$ and correspond to the line $\ell_0$, then the hyperplane is the tangent plane $T_{Gr, \overline{w}_0}$ of $Gr$ at $\overline{w}_0$. The intersection $T_{Gr, \overline{w}_0} \cap Gr$ is singular and can be identified with the cone in $\mathbb{P}^3$ over a nonsingular quadric in $\mathbb{P}^4$. This intersection identifies with $\sigma_1(\ell_0) := \{ \text{the collection of all lines } \ell \text{ with } \ell \cap \ell_0 \neq \emptyset \}$. Consider for example $w_0 = e_1 \wedge e_2$. This intersection is now $\{ \{ p_{ij} \mid p_{34} = 0, -p_{13}p_{24} + p_{14}p_{23} = 0 \}$.

The vertex $\overline{w}_0$ of this cone is its only singular point.
(iv) $\sigma_2(p_0) := \text{the collection of all lines through } p_0; \text{ this is a 2-plane in } Gr. \text{ Indeed, take } p_0 = \bar{e}_1. \text{ Then}
\[\sigma_2(p_0) = \left\{ \sum_{1 < j \leq 4} p_{ij}e_1 \wedge e_j \right\} \text{ no relations}\]
(called a $\omega$-plane in [7]).
(v) $\sigma_{1,1}(h_0) := \text{the collection of all lines in the plane } h_0. \text{ This is a 2-plane in } Gr. \text{ Indeed, take } h_0 = \langle e_1, e_2, e_3 \rangle. \text{ Then this collection identifies with } \{ \sum_{1 \leq i < j \leq 3} p_{ij}e_i \wedge e_j \text{ no relations} \} \text{ (called a } \rho \text{-plane in [7]).}
(vi) $\sigma_{2,1}(p_0, h_0) := \text{the collection of all lines in } h_0 \text{ through } p_0. \text{ This is a line on } Gr. \text{ Indeed, take } h_0 = \langle e_1, e_2, e_3 \rangle; \text{ } p_0 = \bar{e}_1. \text{ Then this collection identifies with } \{ \sum_{j=2,3} p_{ij}e_i \wedge e_j \text{ no relations} \}.
(vii) Every plane in $Gr$ has the form $\sigma_2(p_0)$ or $\sigma_{1,1}(h_0)$. Every line in $Gr$ has the form $\sigma_{2,1}(p_0, h_0)$ and is thus the intersection of a (uniquely determined) pair of 2-planes in $Gr$ of different type.
(viii) The are three types of projective subspaces $P \subset \mathbb{P}(\Lambda^2V)$ of dimension 3 with respect to their relation with $Gr$, namely:
(a) $Gr \cap P$ is a nondegenerate quartic surface. The equations of $P$ are $p_{12} = p_{34} = 0$ for a suitable basis of $V$. Moreover $P$ lies in precisely two tangent space, namely $T_{Gr, \bar{e}_1}$ and $T_{Gr, \bar{e}_2}$.
(b) $Gr \cap P$ is an irreducible degenerate quartic surface. The equations of $P$ are $p_{34} = p_{13} + p_{24} = 0$ for a suitable basis of $V$. Now $P$ lies on only one tangent space, namely $T_{Gr, \bar{e}_2}$ and $Gr \cap P$ is the cone $p_{13}^2 + p_{14}p_{23} = p_{34} = p_{13} + p_{24} = 0$ over the quadric curve $p_{13}^2 + p_{14}p_{23} = 0$.
(c) $Gr \cap P$ is reducible. The equations for $P$ are $p_{12} = p_{13} = 0$ for a suitable basis of $V$. Further, $Gr \cap P$ is the union of the planes $p_{14} = 0$ and $p_{23} = 0$. \hfill \Box

1.2. Ruled surfaces and curves on $Gr$. 

Lemma 1.1. (1). Let $C \subset Gr$ be an irreducible curve of degree $d \geq 2, \text{ not lying in some 2-plane } \sigma_2(p_0). \text{ Then } \tilde{S} := \{(\bar{w}, \bar{v}) \in C \times \mathbb{P}(V) | w \wedge v = 0\}$ is an irreducible variety of dimension 2. Its image $S$ under the projection map $pr_2 : \tilde{S} \to \mathbb{P}(V)$ is an irreducible surface of degree $e$. Suppose that through a general point of $S$ there are $f$ lines $\bar{w} \in C$. Then $d = e \cdot f$.
(2). Let $P(C)$ denote the smallest projective subspace of $\mathbb{P}(\Lambda^2V)$, containing $C$. If $d \geq 3$ and $S$ is not a cone, a plane or a quadric, then $\dim P(C) \geq 3$.

Proof. (1). We note that $C \subset \sigma_2(p_0)$ is not interesting since then $S$ is a cone. The fibers of $pr_1 : \tilde{S} \to C$ are lines in $\mathbb{P}(V)$ and the fibers of $pr_2 : \tilde{S} \to S$ are finite. Thus $S$ is an irreducible ruled surface of some degree $e$. A general line $\bar{w}_0$ in $\mathbb{P}(V)$ intersects $S$ in $e$ points. Through

\[
(2)
\]

Then (1). We note that $C \subset \sigma_2(p_0)$ is not interesting since then $S$ is a cone. The fibers of $pr_1 : \tilde{S} \to C$ are lines in $\mathbb{P}(V)$ and the fibers of $pr_2 : \tilde{S} \to S$ are finite. Thus $S$ is an irreducible ruled surface of some degree $e$. A general line $\bar{w}_0$ in $\mathbb{P}(V)$ intersects $S$ in $e$ points. Through
each of these $e$ points there are $f$ lines $\bar{w} \in C$. Thus the intersection of $C$ with the general hyperplane \{\(\bar{w} \in \mathbb{P}(\Lambda^2 V)\mid w \wedge w_0 = 0\}\) consists of $e \cdot f$ points and therefore $d = e \cdot f$.

(2). Since $d > 1$, one has $\dim P(C) > 1$. Suppose that $\dim P(C) = 2$. If $P(C) \subset Gr$, then either $P(C)$ is a $\sigma_2(p_0)$ and $S$ is a cone, or $P(C)$ is a $\sigma_{1,1}(h_0)$ and $S$ is the plane $h_0$. If $P(C) \not\subset Gr$, then $C \subset P(C) \cap Gr$ is a curve of degree at most 2 and $S$ is a plane or a quadric. Hence $\dim P(C) \geq 3$.

In the sequel we consider ruled surfaces (reduced, irreducible) $S \subset \mathbb{P}(V)$ of some degree $d \geq 3$ which are not cones. One associates to $S$ the subset $\hat{C}$ of $Gr$ corresponding to the lines on $S$.

**Lemma 1.2.** $\hat{C}$ is the union of an irreducible curve $C$ (not lying in some 2-plane $\sigma_2(p_0)$) of degree $d$ and a finite, possibly empty, set. Moreover, through a general point of $S$ there is one line of the surface.

**Proof.** Consider the affine open part of $Gr$ given by $p_{12} \neq 0$. The points of this affine part, actually $\cong \mathbb{A}^4$, can uniquely be written as planes in $V$ with basis $e_1 + ae_3 + be_4$, $e_2 + ce_3 + de_4$ and correspond to the vectors $e_{12} + ce_{13} + de_{14} + ae_{23} + be_{24} + (ad - bc)e_{34}$.

Let $F(t_1, \ldots, t_4) = 0$ be the homogeneous equation of $S$. The intersection of $\hat{C}$ with this affine part consists of the tuples $(a, b, c, d)$ such that $F(s, t, as + ct, bs + dt) = 0$ for all $(s, t) \neq (0, 0)$. Write this expression as a homogeneous form in $s, t$ and coefficients polynomials in $a, b, c, d$. Then the ideal generated by these polynomials in $a, b, c, d$ defines the intersection of $\hat{C}$ with this affine part of $Gr$. Thus $\hat{C}$ is Zariski closed.

Clearly $\hat{C}$ has dimension 1 and can be written as the union of irreducible curves $C_i$, $i = 1, \ldots, r$ and a finite set. The image of the projection $\{(\bar{w}, \bar{v}) \in C_1 \times \mathbb{P}(V)\mid w \wedge v = 0\} \to \mathbb{P}(V)$ is a ruled surface contained in $S$. Since $S$ is irreducible, the image is $S$. If $r \geq 2$, then, for through a point $\bar{v}$ of a line $\bar{w}_2 \in C_2$, $\bar{w}_2 \not\in C_1$ passes a line $\bar{w}_1 \in C_1$. Hence $w_1 \wedge w_2 = 0$ for all $\bar{w}_1 \in C_1$ and thus $w \wedge w_2 = 0$ for all $\bar{w} \in P(C_1)$. By symmetry $w_1 \wedge w_2 = 0$ for all $\bar{w}_1 \in P(C_1)$, $\bar{w}_2 \in P(C_2)$. Since the symmetric bilinear form $(w_1, w_2) \mapsto w_1 \wedge w_2$ on $\Lambda^2 V$ is not degenerate, one obtains a contradiction by comparing dimensions: $\dim P(C_1) \geq 3$, $\dim P(C_2) \geq 3$, $\dim \mathbb{P}(\Lambda^2 V) = 5$. We conclude that the $f$ of Lemma 1.1 is 1 and that the degree of $C$ is $d$. \hfill \Box

**Lemma 1.3.** Let $\bar{w}_0 \in \hat{C} \setminus C$, then $C$ lies in the tangent space of $Gr$ at $\bar{w}_0$. In other words, the line $\bar{w}_0$ intersects every line on $S$, belonging to $C$.

**Proof.** If the tangent space at $\bar{w}_0$ does not contain $C$, then the intersection $C \cap T_{\bar{w}_0}$ consists of $d$ points, counted with multiplicity. Thus the line $\bar{w}_0$ on $S$ intersects $d$ lines of $S$, corresponding to points of $C$. 

Let $H \subset \mathbb{P}(V)$ be a plane through $\varpi_0$. The intersection $H \cap S$ consists of $\varpi_0$ and a curve $\Gamma$ of degree $d - 1$. Therefore $\Gamma \cap \varpi_0$ consists of $d - 1$ points (counted with multiplicity), instead of the $d$ points that we expect. This contradiction proves the lemma.

**Definition 1.4.** The lines on $S$ corresponding to the points of $\mathring{C} \setminus C$ will be called here isolated lines. A line $\varpi_1$ on $S$ is, classically, called a directrix if $\varpi_1$ meets every line $\varpi_2$ with $\varpi_2 \in C$. Thus an isolated line is a directrix. It is also possible that a line $\varpi_1 \in C$ is a directrix. The classical concept of ‘double curve’ on $S$ is, according to [1], p. 8, (the Zariski closure of) the set of points on $S$ lying on at least two, not isolated, lines of $S$.

**Observation 1.5.** Let $C \subset P = \mathbb{P}^r, r \geq 3$ be an irreducible curve of degree 4 and such that $C$ does not lie in a proper subspace of $P$. Let $g \leq 2$ be the genus of the normalization $n : C^{\text{norm}} \rightarrow C$. Then one of the following holds:

1. $g = 0$, $r = 4$, $C$ is the, nonsingular, rational normal quartic.
2. $g = 0$, $r = 3$, $C$ is nonsingular or has one singular point which is a node or ordinary cusp.
3. $g = 1$, $r = 3$ and $C$ is nonsingular.

Moreover, if $C$ lies on a quadratic cone in $\mathbb{P}^3$, then $g = 1$ or $g = 0$ and $C$ has a singular point.

**Proof.** Let $I$ be the sheaf of ideals of $C$. The exact sequence $0 \rightarrow I \otimes \mathcal{O}_P(1) \rightarrow \mathcal{O}_P(1) \rightarrow \mathcal{L} \rightarrow 0$ with $\mathcal{L} = \mathcal{O}_P(1) \otimes \mathcal{O}_C$ and the minimality of $r$ implies that $H^0(\mathcal{O}_P(1)) \rightarrow H^0(\mathcal{L})$ is injective and thus $1 + r \leq \dim H^0(\mathcal{L})$. Define the skyscraper sheaf $\mathcal{Q}$ on $C$ by the exact sequence of sheaves on $C$, $0 \rightarrow \mathcal{L} \rightarrow n_*n^*\mathcal{L} \rightarrow \mathcal{Q} \rightarrow 0$. Denoting $\dim H^i$ by $h^i$, one finds

$$4 \leq 1 + r \leq h^0(C, \mathcal{L}) \leq h^0(C^{\text{norm}}, n^*\mathcal{L}) = 1 - g + 4 + \dim H^1(C^{\text{norm}}, n^*\mathcal{L}).$$

Now $H^1(C^{\text{norm}}, n^*\mathcal{L}) = 0$, since the degree of $n^*\mathcal{L}$ is 4 and $g \leq 2$. Thus $g = 2$ is not possible. For $g = 1$, one has $H^0(C, \mathcal{L}) = H^0(C^{\text{norm}}, n^*\mathcal{L})$ and $C = C^{\text{norm}}$ since $n^*\mathcal{L}$ is very ample on $C^{\text{norm}}$. Let $E$ be an elliptic curve with neutral element $e$, then $H^0(E, 4[e])$ has basis $t_1 = 1, t_2 = x, t_3 = y, t_4 = x^2$ (in the standard notation) and $E$ lies on the quadratic cone $t_2^2 - t_1t_4 = 0$.

For $g = 0$, the curves $C \subset P$ are parametrized by polynomials of degree $\leq 4$ in a variable $t$. Hence $r \leq 4$. For $r = 4$, the only possibility is $t \mapsto (1, t, t^2, t^3, t^4)$. For $r = 3$, one has the examples:

- $t \mapsto (1, t, t^3, t^4)$ and $C$ is nonsingular,
- $t \mapsto (1, t^2, t^3, t^4)$ and $C$ has an ordinary cusp,
- $t \mapsto (t, t^2, t^3, t^4 - 1)$ and $C$ has a node.

In general, by intersecting $C$ with planes $H \subset \mathbb{P}^3$, through one singular point (or more), one can verify that $C$ has at most one singular
point and that such a point can only be a node or an ordinary cusp. Finally, if $g = 0$ and $C$ is contained in a quadratic cone in $\mathbb{P}^3$, then $C$ is singular (see \cite{schoenemann}, exercise IV, 6.1). According to the examples, this singular point can be either a node or a cusp. We note that the exercises IV, 3.4, 3.6 and II, Example 7.8.6 of \cite{schoenemann} are closely related to the above reasoning.

\begin{corollary}
A ruled surface of degree $d \geq 3$ can have at most two isolated lines. If $S$ has two isolated lines $\overline{w}_1$, $\overline{w}_2$, then $\overline{w}_1 \cap \overline{w}_2 = \emptyset$.
\end{corollary}

\textbf{Proof.} The first statement follows from $\dim P(C) \geq 3$. If $\overline{w}_1 \cap \overline{w}_2 \neq \emptyset$, then $C$ lies in $\text{Gr} \cap T_{\text{Gr}, \overline{w}_1} \cap T_{\text{Gr}, \overline{w}_2}$. According to the list of properties of $\text{Gr}$, (viii) part (c), the latter is the union of two planes. One of them contains $C$ and this contradicts $\dim P(C) \geq 3$.

\begin{corollary}
(1) A general line of a ‘general’ ruled surface $S$ of degree $d \geq 3$ meets $d - 2$ other lines of $S$, corresponding to points of $C$. In particular, the ‘double curve’ is not empty. However:

(2) Let $TC \subset \mathbb{P}^3$ be the twisted cubic curve. The equation of the surface $S$ consisting of the tangents of $TC$ is $(t_1 t_4 - t_2 t_3)^2 - 4(t_1 t_3 - t_2^2)(t_2 t_4 - t_1 t_3) = 0$. The singular locus of $S$ is $TC$ and no two distinct lines of the surface intersect.
\end{corollary}

\textbf{Proof.} (1) For a general point $\overline{w}_0 \in C$, the intersection $C \cap T_{\text{Gr}, \overline{w}_0}$ is a positive divisor on $C$ of degree $d$, with support in the nonsingular locus of $C$ and $\geq 2|\overline{w}_0|$. For a ‘general’ $S$ the divisor will be $2|\overline{w}_0| + \sum_{i=1}^{d-2}|\overline{w}_i|$ with distinct points $\overline{w}_i \in C$, $i = 0, \ldots, d - 2$. Thus $\overline{w}_0$ meets precisely $d - 2$ other lines corresponding to points of $C$.

(2) Let $t \mapsto (1, t, t^2, t^3) \in \mathbb{P}^3$ be $TC$ in parametrized form. The tangent line $\overline{w}_t$ contains the point $(0, 1, 2t, 3t^2)$ and has Plücker coordinates

$$p_{12} = 1, \quad p_{13} = 2t, \quad p_{14} = 3t^2, \quad p_{23} = t^2, \quad p_{24} = 2t^3, \quad p_{34} = t^4.$$ 

This defines the nonsingular curve $C \subset \text{Gr}$ corresponding to $S$. From $\overline{w}_t \wedge \overline{w}_s = (t - s)^4$ it follows that the tangent lines do not intersect for $t \neq s$. In other terms $T_{\text{Gr}, \overline{w}_0} \cap C = 4|\overline{w}_0|$ for every $\overline{w}_0 \in C$.

\begin{proposition}
(1) $pr_2 : \tilde{S} := \{(\overline{w}, \overline{v}) \in C \times \mathbb{P}(V) | w \wedge v = 0\} \to S$ is a birational morphism. Let $C^{\text{norm}} \to C$ denote the normalization of $C$ and let $\tilde{S} = \tilde{S} \times_C C^{\text{norm}}$ be the pullback of $\tilde{S} \to C$. Then $\tilde{S} \to C^{\text{norm}}$ is a ruled surface (in the modern sense) and $\tilde{S} \to S$ is the normalization of $S$.

(2) The singular locus of $S$ is purely 1-dimensional or empty.

(3) Suppose that the line $\overline{w}$ belongs to the singular locus of $S$ and does not correspond to a singular point of $C$. Then $C$ lies in the tangent space of $\text{Gr}$ at the point $\overline{w}$.
\end{proposition}

\textbf{Proof.} (1) The morphism is finite since $pr_2^{-1}(\overline{v})$ is the finite set of lines of $S$ through $\overline{v} \in S$. For a general $\overline{v} \in \tilde{S}$, this set has one element and
therefore the degree of $pr_2$ is 1 and so $pr_2$ is birational. The fibres of $pr_1 : ˜S → C$ are isomorphic to $\mathbb{P}^1$ and the same holds for the fibres of $S → C^{\text{norm}}$. Therefore the latter is a ruled surface in the modern terminology. Moreover the morphism $S → S$ is normal and will remain a singular point of the normalization of $S$. Since $S$ is smooth, $S$ has no isolated singularities.

(2) The local ring of an isolated singular point of $S$ is the normalization.

(3) The assumption that $w$ does not correspond to a singular point of $C$ implies that through any point of $w$ there are at least two lines corresponding to points of $C$ (one of them could be $w$ itself). Hence $w$ meets every line corresponding to a point of $C$ and thus $C \subset T_{Gr,w}$. □

Remarks 1.9. The ‘double curve’, as defined above, is seen, by Proposition 1.8, to be part of the singular locus of $S$. The genus of $S$ is defined as the genus of $S$ and thus equal to the genus of $C^{\text{norm}}$. □

Lemma 1.10. Suppose that $\dim P(C) = 3$ and that $P(C)$ is the intersection of two tangent spaces of $Gr$ at points $w_1 \neq w_2$. Then the lines $w_1$, $w_2$ do not intersect. For a suitable choice of the homogeneous coordinates $t_1,t_2,t_3,t_4$ of $\mathbb{P}(V)$, the equation $F$ of $S$ is bi-homogeneous of degree $(a_1,a_2)$, with $a_1 + a_2 = d$, in the pairs $t_1,t_2$ and $t_3,t_4$. Further $C \setminus C = \{w_1,w_2\}$.

The lines $w_1,w_2$ are ‘directrices’. The singular locus of $S$ consists of the lines $w_i$ with $a_i > 1$ and for each singular point $w \in C$, the line $w \subset S$.

Proof. The assumption that the lines $w_1$, $w_2$ intersect, yields, according to (viii) part (c), the contradiction that $C$ lies in a plane. Take $w_1 = e_{12}$ and $w_2 = e_{34}$, then $P(C) = T_{Gr,w} \cap T_{Gr,w'}$ is the projective space with coordinates $p_{13},p_{14},p_{23},p_{24}$ and $C$ lies on the quadric surface $Gr \cap P(C)$ given by $-p_{13}p_{24} + p_{14}p_{23} = 0$. Identifying $Gr \cap P(C)$ with $\mathbb{P}^1 \times \mathbb{P}^1$ leads to $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(a_1,a_2)$ with $a_1 + a_2 = d$.

Consider the rational map $f : \mathbb{P}(V) \cdots \to \mathbb{P}^1 \times \mathbb{P}^1$, given by

$$(t_1,t_2,t_3,t_4) \mapsto ((t_1,t_2),(t_3,t_4)),$$

which is defined outside the two lines $w_1,w_2$. The surface $S$ is the Zariski closure of $f^{-1}(C)$ and so the equation $F$ of $S$ is bi-homogeneous and coincides with the equation for $C \subset \mathbb{P}^1 \times \mathbb{P}^1$. The other statements of the lemma are easily verified. □

Observation 1.11. $\dim P(C) = 3$ and $P(C)$ in a single tangent space of $Gr$.

For a suitable basis of $V$ the projective subspace $P(C) \subset T_{Gr,w}$ is given by the equations $p_{34} = 0, p_{13} + p_{24} = 0$ and $p_{12}, p_{13}, p_{14}, p_{23}$ are the homogeneous coordinates of $P(C)$. Further $Gr \cap P(C)$ is the cone with equation $p_{13}^2 + p_{14}p_{23} = 0$ with vertex $e_{12}$. Since $C$ lies on this
cone we have a rational map $f : C \cdots \to E := \{p_{13}^2 + p_{14}p_{23} = 0\}$. This map can be identified with the rational map $C \cdots \to \overline{e_{12}}$, given by $w \mapsto w \cap \overline{e_{12}}$. The rational map $f$ is a morphism if $\overline{e_{12}} \notin C$ or if $\overline{e_{12}} \in C$ and this is a regular point of $C$.

In case $\overline{e_{12}} \notin C$ the morphism $f$ has degree $e$. Take two unramified points $e_1, e_2 \in E$ and the plane through the corresponding two lines through $\overline{e_{12}}$. This plane meets $C$ in $2e$ points. Hence $d = 2e$. In case $\overline{e_{12}} \in C$ and is not a singular point, the same reasoning yields $d − 1 = 2e$.

It seems difficult to investigate the possibilities for general $d$. The cases $d = 3$ and $d = 4$ will be presented later on.

1.3. **The vector bundle $B$ on $C^{\text{norm}}$.** Let again $d \geq 3$ denote the degree of the ruled surface $S$ and let $C \subset Gr$ be the corresponding curve. Put

$$B := \{(p, v) \in C^{\text{norm}} \times V \mid p \mapsto \overline{w} \in C, w \wedge v = 0\} \subset C^{\text{norm}} \times V.$$  

This is a (geometric) vector bundle of rank two on $C^{\text{norm}}$. We will identify $B$ with its sheaf of sections. We note that $\text{Proj}(B) = \tilde{\mathbb{S}}$. The line bundle $\Lambda^2 B$ on $C^{\text{norm}}$ is the pullback of the restriction of $O_{\text{proj}(\Lambda^2 V)}(-1)$ to $C$ and has therefore degree $−d$. The vector space $H^0(C^{\text{norm}}, B) = 0$, otherwise all the lines of $C$ pass through one point and $S$ is a cone.

The vector bundle $B$ is an important tool in case $C^{\text{norm}}$ has genus 0. For the case $d = 3$ it is easily seen that $C^{\text{norm}}$ has genus 0. Let $t$ parameterize $C^{\text{norm}}$. Then $B$ is isomorphic to $O_{C^{\text{norm}}}(-1) \oplus O_{C^{\text{norm}}}(-2)$. In particular, $\tilde{\mathbb{S}}$ is isomorphic to $\mathbb{P}^2$ with one point blown up (see [S, V, Example 2.11.5]). The sections of $B$ with a pole of order 1 at $t = \infty$ are $Ca$ and those with a pole of order $\leq 2$ at $t = \infty$ are $Ca + \mathbb{C}b$. By choosing a suitable basis of $V$ one can normalize to the following two cases:

$a = (1, t, 0, 0)$, $b = (0, 0, 1, t^2)$ and $S$ has the equation $t_1^2t_4 - t_2^2t_3 = 0$;

$a = (1, t, 0, 0)$, $b = (0, 1, t, t^2)$ and $S$ has the equation $t_3^3 + t_4(t_1t_4 - t_2t_3) = 0$.

This gives the classification of the ruled cubic surfaces over, say, $\mathbb{C}$. In Section 2 we will follow another method to obtain the classification of ruled cubic surfaces over any field and compare this with Dolgachev’s method.

For $d = 4$ and assuming that $C^{\text{norm}}$ has genus 0, there are two possibilities for the vector bundle $B$, namely:

$B \cong O_{C^{\text{norm}}}(-1) \oplus O_{C^{\text{norm}}}(-3)$ and $\tilde{\mathbb{S}}$ is the Hirzebruch surface $\Sigma_2$,

$B \cong O_{C^{\text{norm}}}(-2) \oplus O_{C^{\text{norm}}}(-2)$ and $\tilde{\mathbb{S}}$ is $\mathbb{P}^1 \times \mathbb{P}^1$.

We note in passing that the first possibility was overlooked by Cremona. The method of Cayley can be interpreted as taking three sections of the vector bundle $B(d)$ for a certain values of $d \geq 1$. Normalizing
sections of $B$ with poles of order 1,2,3 at $t = \infty$, by a choice of the basis of $V$ and possibly changing $t$, we will arrive in Subsection 3.2 at a classification of the corresponding ruled quartic surfaces.

If the genus of $C^{\text{norm}}$ is 1, the vector bundle $B$ is not helpful for the computation. However, $B$ and also $\tilde{S} = \text{Proj}(B)$ will be identified.

1.4. The possibilities for the singular locus. It is helpful for the classification of the ruled surfaces to consider $Q := S \cap H$ with $H \subset \mathbb{P}(V)$ a general plane. By Bertini’s theorem, $Q$ is an irreducible reduced curve of degree $d$. The morphism $C \to Q$, given by $[w] \in C \mapsto [w] \cap H \in Q$, is birational. Thus $C^{\text{norm}}$ is the normalization of $Q$. The singular locus of $S$ is written as a union of its irreducible components $C_i$, $i = 1, \ldots, s$ of degree $d_i$ and generic multiplicity $m_i \geq 2$. The curve $Q$ meets every $C_i$ with multiplicity in $d_i$ points with multiplicity $m_i$. For every singular point $q \in Q$ one defines a number $\delta_q$ which is the sum of the integers $\frac{k(k-1)}{2}$ taken over the multiplicities $k$ of $q$ itself and of all the singular points that occur in the successive blow ups of $q$. The Plücker formula states that the genus of the normalization $C^{\text{norm}}$ of $Q$ is

$$\frac{(d-1)(d-2)}{2} - \sum \delta_q.$$ 

For $d = 3$, there is a single singular point $q$ and $\delta_q = 1$ (and $q$ is a node or cusp). The singular locus is described by $s = 1, d_1 = 1, m_1 = 2$.

For $d = 4$, there are more possibilities. The singularities of a simple plane curve (i.e., reduced, multiplicity $\leq 3$ and in the blow ups there are only singularities of multiplicity $\leq 3$) are classified, see [1], p. 62, by formal standard equations $F \in K[[x,y]]$. The condition that $Q$ is irreducible, has degree 4 and the genus of its normalization $C^{\text{norm}}$ is 0 or 1, leads to the list of possibilities (with their symbols or names): for $m = 2$:

A2 : $x^2 - y^2, \delta = 1$; A3 : $x^2 - y^3, \delta = 1$; A4 : $x^2 - y^4, \delta = 2$;

for $m = 3$:

D4 : $y(x^2 - y^2), \delta = 3$; D5 : $y(x^2 - y^3), \delta = 3$; E6 : $x^3 - y^4, \delta = 3$

and the last case $E_7 : x(x^2 - y^3)$, which is ruled out by $\delta = 4$.

The inequalities

$$\sum_{i=1}^{s} d_i \frac{m_i(m_i-1)}{2} \leq \sum \delta_q \leq \frac{(d-1)(d-2)}{2}$$

lead to a list of possibilities for the singular locus, again with Cayley’s convention that $d^m$ stands for an irreducible curve of degree $d$ and with multiplicity $m$ and ‘int’ meaning intersecting lines: $1^2; 1^3; 2^2; 2^3; 3^2; 1^2 2^2; 1^2 2^2; 1^2 1^2; 1^2 1^2$. 

1.5. The reciprocal of a ruled surface. As before, $V$ is a vector space of dimension 4 over a field $K$. One identifies $\Lambda^4V$ with $K$. The nondegenerate symmetric bilinear form on $\Lambda^2V$, given by $(w_1, w_2) = w_1 \wedge w_2 \in \Lambda^4V = K$, yields an isomorphism $f : \Lambda^2V \to \Lambda^2V^* = (\Lambda^2V)^*$ by $f(w_1)(w_2) = w_1 \wedge w_2 \in K$. This isomorphism maps decomposable vectors of $\Lambda^2V$ to decomposable vectors of $\Lambda^2V^*$. 
Indeed, consider \( f(v_1 \wedge v_2) \). Let \( v_1, v_2, v_3, v_4 \) be a basis of \( V \). The kernel of \( f(v_1 \wedge v_2) \) has basis \( v_1 \wedge v_2, v_1 \wedge v_3, v_1 \wedge v_4, v_2 \wedge v_3, v_2 \wedge v_4 \). Let \( \ell_1, \ell_2 \) be a basis of \( (V/Kv_1 + Kv_2)^* \subset V^* \). Then \( \ell_1 \wedge \ell_2 \) has the same basis vectors in the kernel. Hence \( f(v_1 \wedge v_2) \) is a multiple of \( \ell_1 \wedge \ell_2 \).

Thus \( f \) induces an isomorphism \( \tilde{f} : Gr(2, V) \to Gr(2, V^*) \). Let \( P \subset V \) be a 2-dimensional subspace. Then \( \tilde{f}(P) \) is the 2-dimensional subspace \((V/P)^*\) of \( V^* \). For 2-dimensional subspaces \( P_1, P_2 \subset V \) with \( P_1 \cap P_2 \neq 0 \) one has \((V/P_1)^* \cap (V/P_2)^* \neq 0 \). This also follows from the formula \( f(w_1) \wedge f(w_2) = w_1 \wedge w_2 \) for any \( w_1, w_2 \in \Lambda^2 V \) (for a suitable identification of \( \Lambda^4 V^* \) with \( K \)).

Any 1-dimensional subspace \( L \subset V \) determines the plane in \( Gr(2, V^*) \) consisting of all 2-dimensional \( P \subset V \), \( P \supset L \) (an \( \omega \)-plane in \( \mathbb{A} \)). The image of this plane under \( \tilde{f} \) is the plane in \( Gr(2, V^*) \) consisting of all 2-dimensional \( Q \subset (V/L)^* \subset V^* \). Since the latter is a plane of the ‘opposite type’ (a \( \rho \)-plane in \( \mathbb{A} \)), there is no isomorphism \( V \to V^* \) inducing \( f \).

Let \( e_1, \ldots, e_4 \) denote a basis of \( V \) and \( e_1^*, \ldots, e_4^* \) the dual basis of \( V^* \). Then \( e_{ij} := e_i \wedge e_j, \ i < j \) is a basis of \( \Lambda^2 V \) and \( e_{ij}^* = e_i^* \wedge e_j^*, \ i < j \) is a basis of \( \Lambda^2 V^* \). From the Plücker coordinates \( \{p_{ij}\} \) of a line \( \overline{w} \in Gr(2, V) \) one easily finds the Plücker coordinates of \( f(\overline{w}) \in Gr(2, V^*) \) by the identities

\[
\begin{align*}
f(e_{12}) &= e_{34}^*; \ f(e_{13}) = -e_{24}^*; \ f(e_{14}) = e_{23}^*; \ f(e_{23}) = e_{14}^*; \\
f(e_{24}) &= -e_{13}^*; \ f(e_{34}) = e_{12}^*.
\end{align*}
\]

Let an irreducible ruled surface \( S \subset \mathbb{P}(V) \) (of degree \( d \)) be given by an irreducible curve \( C \subset Gr(2, V) \) of degree \( d \). Consider a nonsingular point \( s \in S \) lying on a single line \( \ell \) of the surface. The tangent plane \( T_{S,s} \) contains the line \( \ell \). The same holds for the tangent planes \( T_{S,s'} \) for almost all points \( s' \in \ell \). Since \( T_{S,s'} \) cannot be all equal, the reciprocal (or dual) surface contains all planes \( H \supset \ell \). It now follows that the reciprocal surface \( \tilde{S} \subset \mathbb{P}(V^*) \) is ruled. The corresponding curve in \( Gr(2, V^*) \) is \( f(C) \). It has also degree \( d \) since the degree of the curve \( f(C) \) is \( d \).

Using Plücker coordinates one easily finds \( \tilde{S} \). Another useful computation of the reciprocal surface is the following. Consider \( \tilde{S} = \{ (\overline{w}, \overline{v}) \in C \times \mathbb{P}(V) | v \wedge v = 0 \} \to S \subset \mathbb{P}(V) \) and a nonsingular point \( \overline{v_0} \in S \) and the nonsingular point \( \overline{w_0} \in C \) with \( w_0 \wedge v_0 = 0 \). (We note that the tangent plane of \( \tilde{S} \) at the point \( (\overline{w_0}, \overline{v_0}) \) is mapped isomorphically to the tangent plane of \( S \) at the point \( \overline{v_0} \). The first tangent plane is the product of the tangent line of \( C \) at the point \( \overline{w_0} \) and the line \( \overline{v_0} \)).

Let a (local) parametrization \( t \mapsto w(t) \) for \( C \) be given, such that \( w_0 = w(t_0) \). Choose a decomposition \( w(t) = a(t) \wedge b(t), \) locally at \( t_0 \). Then \( v_0 = s_0 a(t_0) + (1 - s_0) b(t_0) \) and \( S \) has the local parametrization
(t, s) \mapsto sa(t) + (1 − s)b(t). The linearization of this, i.e.,

\[ v_0 + (s − s_0)(a(t_0) − b(t_0)) + (t − t_0)(s_0a'(t_0) + (1 − s_0)b'(t_0)), \]

is a parametrization of the tangent plane \( T_{S, v_0} \). This corresponds with the 3-dimensional subspace of \( V \) with basis \( v_0, v_0 + a(t_0) − b(t_0), v_0 + s_0a'(t_0) + (1 − s_0)b'(t_0). \)

The exterior product \( a(t_0) \land b(t_0) \land (s_0a'(t_0) + (1 − s_0)b'(t_0)) \) of these vectors is an element of \( \Lambda^3 V = V^* \). This defines a point in \( \mathbb{P}(V^*) \) corresponding to the tangent plane \( T_{S, v_0} \). The reciprocal surface \( \tilde{S} \) consists of all these points. In varying \( s_0 \) one finds a line on \( \tilde{S} \), through the points \( a(t_0) \land b(t_0) \lor a'(t_0) \) and \( a(t_0) \land b(t_0) \lor b'(t_0) \). By varying \( t_0 \) one obtains \( \tilde{S} \).

For the two cases of ruled quartic surfaces \( S \) with genus 1, it is easily seen that \( \tilde{S} \cong S \). For the ruled quartic surfaces cases of genus 0 there are explicit global expressions \( w(t) = a(t) \land b(t) \) (with \( t \in \mathbb{P}^1 \)) and the above can be used for the computation of \( \tilde{S} \).

### 1.6. The classification of quartic ruled surfaces.

The Number appearing in the tables are introduced for notational reasons in the computation of Subsection [3.1]. It has no intrinsic meaning. The cases with \( C^\text{norm} \) of genus 0 and \( B \) of type \(-1, -3\) are:

| Number | singular locus | dim \( P(C) \) | singularities of \( C \) | tangent spaces | Cremona | XIII |
|--------|---------------|----------------|--------------------------|----------------|---------|------|
| 1 \( a, b, c \) | 1^3 | 3 | - | 2 | 9 |
| 2 \( a, b, c \) | 3^2 | 4 | - | 1 | 7 |
| 3 \( a, b \) | 1^2, 2^2 | 4 | - | 1 | 4 |
| 4 | 1^3 | 3 | node | 1 | 10 | 7 |
| 5 | 1^3 | 3 | cusp | 1 | 10 |

The cases with \( C^\text{norm} \) of genus 0 and \( B \) of type \(-2, -2\) are:

| Number | singular locus | dim \( P(C) \) | singularities of \( C \) | tangent spaces | Cremona | XIII |
|--------|---------------|----------------|--------------------------|----------------|---------|------|
| 6 | 1^2, 1^2, 1^2 | 3 | node | 2 | 5 |
| 7 | 1^2, 1^2, 1^2 | 3 | cusp | 2 | 5 |
| 8 | 1^3 | 4 | - | 1 | 3 |
| 9 | 1^3 | 4 | - | 1 | 3 |
| 10 \( a \) | 1^2, 1^2, int | 3 | node | 1 | 6 |
| 10 \( b \) | 1^2, 1^2, int | 3 | cusp | 1 | 6 |
| 11 | 1^2, 2^2 | 4 | - | 1 | 2 |
| 12 | 1^2, 2^2 | 4 | - | 1 | 2 | 8 |
| 13 \( a, b, c \) | 1^3 | 4 | - | 1 | 8 | 6 |
| 14 ... | 3^2 | 4 | - | - | 1 | 9, 10 |
In §3.2.4 it is shown that Number 14 consists of six distinct cases. In the text it is explained that the cases 11 and 12 are different. The reciprocals of $a, b, c$ are $13_a, b, c$ and the reciprocals of $3 a, b$ are $8, 9$.

The other examples are ‘selfdual’.

The cases with $C^{\text{norm}}$ of genus 1 are:

| Number | singular locus | dim $P(C)$ | tangent spaces | Cremona | XIII |
|--------|---------------|------------|--------------|---------|------|
| 15     | $t^2$         | 3          | 1            | 12      | 5    |
| 16     | $t^2, t^2$    | 3          | 2            | 11      | 1, 2, 3, 4 |

2. Ruled surfaces of degree 3

Here we give the classification over an arbitrary field $K$. The singular locus of $S$ is a line, $C^{\text{norm}}$ has genus 0 and dim $P(C) = 3$. This implies that $C = C^{\text{norm}}$ is the twisted cubic curve in $P(C)$.

In the first case $P(C)$ lies in two tangent spaces at the points $\bar{w}_1, \bar{w}_2 \in Gr$. From Lemma 1.10 we conclude that $S$ is given by a bi-homogeneous equation $F$ in the pairs of variables $t_1, t_2$ and $t_3, t_4$ of bi-degree $(2, 1)$, corresponding to a morphism $f : \bar{w}_1 \rightarrow \bar{w}_2$ of degree 2. The line $\bar{w}_1$ is nonsingular and a ‘directrix’. The line $\bar{w}_2$ is the singular locus. Further $\bar{C} \setminus C = \{\bar{w}_1, \bar{w}_2\}$.

If the field $K$ has characteristic $\neq 2$, then $C, P(C), \bar{w}_1, \bar{w}_2$ are all defined over $K$ and can be put in a standard form. The morphism $f$ is defined over $K$. The ramification points of $f$ are either both in $K$ or belong to a quadratic extension of $K$ and are conjugated.

If the field $K$ has characteristic 2, then $f$ is either separable and has one point of ramification, or $f$ is inseparable. In both cases $f$ can be put into a standard form.

In the second case $P(C)$ lies in only one tangent space, namely at the point $\bar{w}_0$ which is the singular line of $S$. Then $C$ lies on the quadratic cone in $P(C)$ and $\bar{w}_0 \in C$. In this case $\bar{C} = C$. Now $C$ and $S$ can be put into a standard form. We arrive at the following result.

**Proposition 2.1.** The standard equations for ruled cubic surfaces $S/K$, which are not cones, are the following:

1. $t_3 t_1^2 + t_4 t_2^2 = 0$. If char$K = 2$, then there are no twist. For char$K \neq 2$ the twists are $t_3(t_1 t_2) + t_4(at_1^2 + t_2^2) = 0$ with $a \in K^*$ not a square.

2. $t_3 t_1 t_2 + t_4 t_1^2 + t_3^2 = 0$ (there are no twists).

3. $t_3 t_1 t_2 + t_4(t_1^2 + t_1 t_2) = 0$ if char$K = 2$ (there are no twists).

The curves $C$ for (1) and (2) are in parameter form

$p_{12} = 0$, $p_{13} = -t^2$, $p_{14} = 1$, $p_{23} = -t^3$, $p_{24} = t$, $p_{34} = 0$ and

$p_{12} = 0$, $p_{13} = t^3$, $p_{14} = t^2$, $p_{23} = -t^2$, $p_{24} = -t$, $p_{34} = -1$. 

The above equations for $S$ are derived in an elegant way by I. Dolgachev \cite{6}, using only the information that the singular locus of $S$ is a line with multiplicity 2.

For $K = \mathbb{R}$, there are three types of cubic ruled surfaces (omitting cones). Case (1) of Proposition 2.1, without twist, is represented by the plaster model VII, nr. 21 and by the string models XVIII, nr. 2 and 3. The twisted case ($a = -1$) is represented by VII, nr. 20 and XVIII, nr. 1. Finally, case (2) carries the name Cayley’s ruled cubic surface. It is represented by VII, nr. 22 and 23 and XVIII, nr. 4.

3. Ruled surfaces of degree 4

The base field $K$ is supposed to be algebraically closed. The only role that the characteristic of $K$ plays is in the classification of the morphisms $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 and 3. For convenience we suppose that $K$ has characteristic 0 or $> 3$. We need both the vector bundle $B$ and the possibilities for the singular locus in order to find all cases and to verify the computations by comparison.

3.1. Classification of $S$, using the vector bundle $B$. 

3.1.1. $C^{\text{norm}}$ of genus 0 and $B$ of type $-1$, $-3$. Choose a parameter $t$ for $C^{\text{norm}} \cong \mathbb{P}^1$ and let $p$ be the pole of $t$. Then $H^0(C^{\text{norm}}, B([p]))$ has basis $a$ and $H^0(C^{\text{norm}}, B(3 \cdot [p]))$ has basis $a, ta, t^2a, b$. Now $a, b \in K[t] \otimes_K V$ have degrees 1 and 3. We note that $b$ is not unique and can be replaced by $\mu b + \lambda_0 a + \lambda_1 ta + \lambda_2 t^2a$ with $\mu \neq 0$. We will derive normal forms for the possibilities of the pair $a, b$. These will depend on the choice of $t$. There is a unique subspace $W \subset V$ of dimension 2 with $a \in K[t] \otimes_W V$ and the image $b'$ of $b$ in $K[t] \otimes_V W$ is unique up to multiplication by some $\mu \in K^*$ and has degree $\geq 1$.

The above normal form is obtained by choosing $t$ and a basis $e_1, e_2, e_3, e_4$ of $V$ such that $\{e_1, e_2\}$ is a basis of $W$ and such that $a = (a_1, a_2, 0, 0)$ and a suitable $b = (b_1, b_2, b_3, b_4)$ w.r.t. this basis are as simple as possible.

The Plücker coordinates of the line through $a(t)$ and $b(t)$ are easily computed and this yields $C \subset Gr$ in parametrized form. From this one deduces $\dim P(C)$, possible singularities of $C$ and the relation of $C$ w.r.t. the tangent spaces of $Gr$. The reciprocal surface (needed for the comparison with Cremona’s list) is computed by the method of Subsection 1.5, again using $a(t), b(t)$. For some cases (especially when the singular locus is $3^2$), the equation of the surface $S$ is rather long and requires a MAPLE computation. We avoid this and compute the singular locus by other means. We start by classifying the pairs $(b_3, b_4)$ which are uniquely determined by $S$, up to taking linear combinations. 

$max(\deg b_3, \deg b_4) = 3$ and $\gcd(b_3, b_4) = 1$.

The morphism $C^{\text{norm}} \to \mathbb{P}^1, t \mapsto (b_3 : b_4)$ has degree 3. The possibilities
for the ramification indices are: (a) 3, 3, (b) 3, 2, 2 or (c) 2, 2, 2. A change of $t$ and a linear change of $e_3, e_4$ brings the pair $(b_3, b_4)$ into a standard form

$$(1, t^3), \quad (1, t^2(t+1)), \quad \text{or} \quad (t - \mu, (2\mu - 1)t^3 + (2 - 3\mu)t^2) \quad \text{with} \quad \mu \neq 0, 1, 1/2.$$  

(In the last case the 4 ramification points are 0, 1, $\pm\sqrt{\frac{4}{3\mu-1}}$.) One normalizes $a = (1, t, 0, 0), \quad b = (b_1, b_2, b_3, b_4)$ and max(deg $b_1$, deg $b_2$) $\leq 2$. Then $b - b_1 \cdot a$ produces a new $b = (0, b_2, b_3, b_4)$. There are now again two cases:

1. $(a, b, c)$. $b_2 = 0$ and the data are: $C$ is nonsingular, $P(C) = T_{Gr, \overline{e_{34}}} \cap T_{Gr, \overline{e_{34}}}$ and $\overline{e_{34}}, \overline{e_{34}} \not\subset C$; equation $t_1^3b_3(t_2(t_2 - 1))t_4 - t_1^3b_4(t_2(t_2 - 1))t_3 = 0$; the singular locus of $S$ is the line $\overline{e_{34}}$ with multiplicity 3. Then $I^3$ and Cremona 9.

2. $(a, b, c)$. $b_2 \neq 0$ and the data are: $C$ nonsingular, $P(C) = T_{Gr, \overline{e_{34}}}$ and $\overline{e_{12}}, \overline{e_{34}} \not\subset C$. A direct computation of the equation seems difficult. The points of the line through $a(t)$ and $b(t)$ can be written as $(1, t + \lambda \frac{b_2}{b_3}(t), \lambda, \lambda \frac{b_4}{b_3}(t))$. Computing with the equality

$$(1, t + \lambda \frac{b_2}{b_3}(t), \lambda, \lambda \frac{b_4}{b_3}(t)) = (1, s + \lambda \frac{b_2}{b_3}(s), \lambda, \lambda \frac{b_4}{b_3}(s)), \quad \text{with} \quad s \neq t$$

leads to the result that the ‘double curve’ and thus also the singular locus is a twisted cubic curve. Then $I^3$ and Cremona 7.

3. $(a, b)$ max(deg $b_3$, deg $b_4$) = 2 and gcd($b_3$, $b_4$) = 1. A normalization is $a = (t+\beta, 1, 0, 0)$ and $b = (0, t^3 + \alpha t, t^2, 1)$. Equation

$$t_3t_4(t_2 - \beta(t_3 + \alpha t_4))^2 - (t_3(t_3 + \alpha t_4) - \beta t_2t_4 + t_1t_4)^2 = 0.$$  

One has $P(C) = T_{Gr, \overline{e_{12}}} \cap \overline{e_{12}} \subset C$. The singular locus is the union of the line $\overline{e_{12}}$ and the conic $t_2 - \beta(t_3 + \alpha t_4) = 0$, $t_3(t_3 + \alpha t_4) - \beta t_2t_4 + t_1t_4 = 0$. Then $I^2, 2^2$ and Cremona 4. The degree morphism $C \to D$ has two points of ramification. The point $L \cap D$ is a ramification point on $D$ if and only if $\beta = 0$. We consider this as two cases.

4. max(deg $b_3$, deg $b_4$) = 2 and gcd($b_3$, $b_4$) has degree 1. A normalization is $a = (t, 1, 0, 0), \quad b = (0, t^3 + \alpha t, t(t + \beta), t + \beta)$ with gcd($t^3 + \alpha t, t + \beta$) = 1. The equation is

$$t_1^2(t_3 + \beta t_4) - t_2t_3t_4(t_3 + \beta t_4) + \alpha t_3t_4^3 + t_4^4 = 0.$$  

Further $\overline{e_{12}} \subset C$ is a node (for $t = \infty, t = -\beta$), dim $P(C)$ = 3 and $P(C)$ lies in only one tangent space $T_{Gr, \overline{e_{12}}}$. Then $I^3$ and Cremona 10.

5. max(deg $b_3$, deg $b_4$) = 1.

A normalization is $a = (t, 1, 0, 0), \quad b = (0, t^3 + \alpha t^2, t, 1)$. The equation is

$$t_1t_4^3 - t_2t_3t_4^2 + \alpha t_3^3t_4 + t_3^4 = 0.$$  

Further $\overline{e_{12}} \subset C$ is a cusp (for $t = \infty$), dim $P(C)$ = 3 and $P(C)$ lies in only one tangent space, namely $T_{Gr, \overline{e_{12}}}$. Then $I^3$ and Cremona 10.

Finally we show that the omitted cases can be reduced to the above.
max(deg \(b_3\), deg \(b_4\)) = 3 and gcd(\(b_3, b_4\)) has degree 1. A normalization is \(a = (1, t, 0, 0)\), \(b = (b_1, b_2, t, (t + \mu)^2)\). Replacing \(t\) by \(s^{-1}\) and multiplying by a suitable power of \(s\) yields \(a = (s, 1, 0, 0)\), \(b = (s^3b_1(s^{-1}), s^2b_2(s^{-1}), s^2, (1 + \mu s)^2)\). Thus reduction to max(deg \(b_3\), deg \(b_4\)) = 2.

max(deg \(b_3\), deg \(b_4\)) = 3 and gcd(\(b_3, b_4\)) has degree 2. A normalization is \(a = (1, t, 0, 0)\), \(b = (b_1, b_2, t(t + \mu), t(t + \mu)(t + \lambda))\). Replacing \(t\) by \(s^{-1}\) and multiplying by a suitable power of \(s\) gives a reduction to max(deg \(b_3\), deg \(b_4\)) = 2.

3.1.2. \(C^{\text{norm}}\) of genus 0 and \(B\) of type \(-2, -2\). \(V, t, p\) have the same meaning as in § 3.1.1. The vector space \(H^0(C^{\text{norm}}, B(2[p]))\) has dimension 2 and consists of elements in \(K[t] \otimes V\) of degree \(\leq 2\) and the only element of degree \(\leq 1\) is 0. We are interested in lines \(Ka \subset H^0(C^{\text{norm}}, B(2[p]))\) such that \(a \in K[t] \otimes W\) with \(\text{dim} W = 2\).

Suppose that there are two such lines \(Ka\) and \(KB\).

One can normalize by \(a = (a_1, a_2, 0, 0)\), \(b = (0, 0, b_3, b_4)\). The two morphisms \(\bar{a}, \bar{b}: C^{\text{norm}} \rightarrow \mathbb{P}^1, t \mapsto (a_1 : a_2)\) and \(t \mapsto (b_3, b_4)\) of degree 2 are distinct and their sets of ramification points can be either disjoint or have one point of intersection. Choosing \(t\) leads to the following normalizations.

6. \(a = (1, t^2, 0, 0)\), \(b = (0, 0, (t - 1)^2, (t - \lambda)^2)\). The singular locus consists of the lines \(\overline{e_{12}}, \overline{e_{34}}\) and a third line corresponding to \(t = \pm \lambda\). The morphism \(C^{\text{norm}}(\overline{e_{12}}, \overline{e_{34}}) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) maps \(t = \pm \lambda\) to the same point of \(C\). Thus \(C\) has a node, \(P(C) = T_{Gr, \overline{e_{12}}} \cap T_{Gr, \overline{e_{34}}}, 1^2, 1^2, 1^2\) and Cremona 5.

7. \(a = (1, t^2, 0, 0)\), \(b = (0, 0, 1, (t - 1)^2)\). The image \(C\) of \(C^{\text{norm}}(\overline{e_{12}}) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1\) has a cusp corresponding to \(t = \infty\). The singular locus consists of three lines \(\overline{e_{12}}, \overline{e_{34}}\) and the line corresponding to \(t = \infty\). Thus \(P(C) = T_{Gr, \overline{e_{12}}} \cap T_{Gr, \overline{e_{34}}}, 1^2, 1^2, 1^2\) and Cremona 5.

Suppose that there exists only one such line \(Ka\).

Normalize by \(a = (1, t^2, 0, 0)\), \(b = (b_1, b_2, b_3, b_4)\) with deg \(b_2 < 2\). The pair \((b_3, b_4)\) is, up to taking linear combinations, uniquely determined by the surface. The morphism \(m : C^{\text{norm}} \rightarrow \mathbb{P}^1, t \mapsto (b_3 : b_4)\) cannot be constant and has degree 1 or 2. There are the following cases.

8. \((b_3, b_4) = (1, t)\). Then \(P(C) = T_{Gr, \overline{e_{12}}}\) and \(\overline{e_{12}} \notin C\). The equation is

\[
t_1t_3t_4^2 - t_2t_3^3 - t_3^2t_4^3b_1(t^4_4) + t_3^3b_2(t^4_4) = 0.
\]

Thus \(1^2\) and Cremona 3.
9. \((b_3, b_4) = (t - \alpha, t(t - \alpha))\). Then \(P(C) = T_{\text{Gr}, \overline{\mathbb{E}}^{12}}\) and \(\overline{\mathbb{E}}^{12} \subset C\). The equation is
\[
t_1(t_4 - \alpha t_3)t_4^2 - t_2t_3^2(t_4 - \alpha t_3) - t_3^2t_4b_1(t_4) + t_3b_2(t_4) = 0.
\]
Thus \(1^3\) and Cremona 3.

10. \((a, b)\) Now the morphism \(m\) has degree two. If \(t = 0, \infty\) are the ramification points of \(m\), then one normalizes to \((b_3, b_4) = (1, t^2)\). Then \(\dim P(C) = 3\) and \(P(C)\) lies in only one tangent space, namely \(T_{\text{Gr}, \overline{\mathbb{E}}^{12}}\), and \(\overline{\mathbb{E}}^{12} \notin C\). Write \(b_1 = b_{12}t^2 + b_{11}t + b_{10}\) and \(b_2 = b_{21}t + b_{20}\). One can normalize further to \(b_1 = b_{11}t, b_2 = b_{21}t\). Then \(C\) has a node (case (a)) if \(b_{21}b_{11} \neq 0\) and has a cusp otherwise (case (b)). Then \(1^2, 1^2, \text{ int}\) and Cremona 6.

11. If \(m : C^{\text{norm}} \to \mathbb{P}^1\) is ramified for, say, \(t = 1, \infty\), then one can normalize \(a = (1, t^2, 0, 0), b = (b_1(t - 1), b_2(t - 1), 1, (t - 1)^2)\) with \(b_1, b_2 \in K\). The equation is
\[
t_3t_4(2t_1 + (b_2 - b_1)t_3 - b_1t_4)^2 - (t_2t_3 - t_1t_4 - t_4 + 2b_1t_3t_4)^2 = 0.
\]
The singular locus is the union of the line \(L = \overline{\mathbb{E}}^{12}\) and the conic \(D\) given by \(2t_1 + (b_2 - b_1)t_3 - b_1t_4 = 0, t_2t_3 - t_1t_4 - t_4 + 2b_1t_3t_4 = 0\). Now \(P(C) = T_{\text{Gr}, \overline{\mathbb{E}}^{12}}, \overline{\mathbb{E}}^{12} \notin C\), the image of \(C \to L \times D, \overline{w} \mapsto (\overline{w} \cap L, \overline{w} \cap D)\) is a rational curve having a cusp. Then \(1^2, 2^2\) and Cremona 2.

12. If \(m\) is ramified for, say, \(t = 1, \mu\) with \(\mu \neq 0, 1, \infty\), then one can normalize to \(a = (1, t^2, 0, 0) b = (b_1, b_2, (t - 1)^2, (t - \mu)^2)\) with \(b_1, b_2 \in K\). One can replace \(b\) by \(b - b_1 a\) and normalize further to \(a = (1, t^2, 0, 0), b = (0, 1, (t - 1)^2, (t - \mu)^2)\). A somewhat long computation yields the equation
\[
4t_3t_4((\mu - 1)^2(t_2 - \mu t_1) - 2t_3 - 2t_4)^2 - ((\mu - 1)^2(-\mu^2t_3 - t_1t_4 + t_2t_3 + t_4) - t_3^2 - 6t_3t_4 - t_4^2)^2 = 0.
\]
The singular locus is the union of the line \(L = \overline{\mathbb{E}}^{12}\) and the conic \(D\) given by the equations
\[
(\mu - 1)^2(t_2 - \mu t_1) - 2(t_3 + t_4) = 0, (\mu - 1)^3t_1(t_3 - \mu t_4) - (t_3 - t_4)^2 = 0.
\]
Further \(P(C) = T_{\text{Gr}, \overline{\mathbb{E}}^{12}}, \overline{\mathbb{E}}^{12} \notin C\), the image of the morphism \(C \to L \times D\) is a rational curve having a node. Then \(1^2, 2^2\) and Cremona 2.

Suppose that there is no such line and that \(P(C)\) lies in a tangent space.

The inclusion \(P(C) \subset T_{\text{Gr}, \overline{\mathbb{E}}^{12}}\) yields a morphism \(f : C^{\text{norm}} \to \overline{\mathbb{E}}^{12}\) induced by \(\overline{w} \notin C \mapsto \overline{w} \cap \overline{\mathbb{E}}^{12}\). If the degree of \(f\) is 1, then we may assume that \((1, t, 0, 0)\) lies on \(S\). Combining with a nonzero element \(a \in H^0(C^{\text{norm}}, B(2[p]))\), one finds a surface of degree 3 instead of 4.

The possibility that the degree of \(f\) is 2 is excluded by the following reasoning. Let \(t\) be a parameter for \(C^{\text{norm}}\) and write \(f = ((\alpha t + \beta)^2, (\gamma t + \delta)^2, 0, 0)\). Let \(a, b\) be a basis of \(H^0(C^{\text{norm}}, B(2[p]))\). Then
\( \lambda_0(t)f = \lambda_1(t)a + \lambda_2(t)b \) holds for some \( \lambda_0(t), \lambda_1(t), \lambda_2(t) \in K[t] \) with \( \gcd(\lambda_1(t), \lambda_2(t)) = 1 \). The Plücker coordinates of \( a \wedge b \) are polynomials in \( t \) with greatest common divisor 1 and maximal degree 4, since these parametrize \( C \). The same holds for the Plücker coordinates of \( f \wedge a \) and \( f \wedge b \). The equality \( \lambda_0(t)f \wedge a = -\lambda_0(t) \cdot a \wedge b \) implies that \( \lambda_0(t) \) is a constant multiple of \( \lambda_2(t) \). Similarly, \( \lambda_0(t) \) is a constant multiple of \( \lambda_1(t) \). We conclude that the \( \lambda_i(t) \) are constant. Then \( f \in H^0(C^\text{norm}, B(2[p])) \) and this contradicts the assumption.

13. (a,b,c). If the degree of \( f \) is 3, then \( \overline{c_{12}} \) has multiplicity 3 and thus \( 1^3 \). As in case 2, there are three different possibilities for the ramification of \( f \). One writes \( f(t) = (c_1, c_2, 0, 0) \) where \( c_1, c_2 \) are relatively prime polynomials in \( t \) and, say, \( \deg c_1 < \deg c_2 = 3 \). Let \( a(t) = (a_1, a_2, a_3, a_4) \) be a nonzero section of \( B(2[p]) \). An inspection of the Plücker coordinates of \( f \wedge a \) implies that \( \max(\deg a_3, \deg a_4) \leq 1 \). Moreover \( a_3, a_4 \) are linearly independent. Thus we may normalize to \( (a_3, a_4) = (1, t) \). Because \( \overline{c_{12}} \) has multiplicity 3, the equation for \( S \) has the form \( t_1A_1 + t_2A_2 + A_3 = 0 \), where \( A_1, A_2, A_3 \) are homogeneous polynomials in \( t_3, t_4 \) of degrees 3, 3, 4. Substitution of \( (\lambda c_1 + a_1, \lambda c_2 + a_2, 1, t) \) in this equation yields \( c_1(1), A_1(1, t) + c_2(t)A_3(1, t) = 0 \) and we can normalize to \( A_1(1, t) = c_2(t), A_4(1, t) = -c_1(t) \). In particular, \( \gcd(A_1, A_2) = 1 \). Further \( A_3(1, t) = -a_1(t)c_2(t) + a_2(t)c_1(t) \). The term \( A_3 \) cannot be made 0 by a transformation of the form \( t_1 \mapsto t_1 + *t_3 + *t_4, t_2 \mapsto t_2 + *t_3 + *t_4, \) since \( P(C) \) does not lie in another tangent space. Therefore, \( \max(\deg a_1, \deg a_2) = 2 \). Further \( \dim P(C) = 4 \) and \( \overline{c_{12}} \notin C \). One verifies that the equations belong to the case that \( B \) has type \(-2,-2\) by comparing with the cases \( 1^3 \) where \( B \) has type \(-1,-3\). Further Cremona 8.

Suppose that there is no such line and \( P(C) \) does not lie in a tangent space.

14. We claim that the singular locus is \( 3^2 \) and is of species Cremona 1. The conditions imply that \( \dim P(C) = 4 \) and \( C \) is nonsingular. Suppose that the singular locus of \( S \) contains a line. This line belongs to \( C \) (because of Lemma 1.3) and is, say, \( \overline{a}(0) \in C \). Take a plane \( H \) containing \( \overline{a}(0) \). The intersection \( H \cap S \) consists of \( \overline{a}(0) \) with multiplicity \( \geq 2 \) and a remaining curve \( R \) which is a conic or two lines or one line. For \( t \neq 0 \) the intersection \( \overline{a}(t) \cap H \) lies on \( R \). The possibility that \( R \) is one or two lines contradicts that \( P(C) \) does not lie in a tangent space. Thus \( R \) can only be a conic. For \( t \neq 0 \), the positive divisor \( \overline{a}(t) \cap R \) has degree 1 and has degree 2 for \( t = 0 \). This is a contradiction.

We conclude that the singular locus of \( S \) does not contain a line. Then, because of Lemma 3.1 and Subsection 1.4, the singular locus of \( S \) is the twisted cubic curve. In § 3.2.4 it is shown that this Number 14 consists of six subclasses.
3.1.3. The vector bundle $B$ for a genus 1 curve $C^\text{norm}$. Here we use the information from §3.2.5 and §3.2.7 below, and deduce the structure of the vector bundle $\tilde{B}$ on the genus 1 curve $C = C^\text{norm}$.

15. Case $1^2$. The equation is $(t_1t_4 - t_2t_3)^2 + H(t_3, t_4)$, where $H$ is homogeneous of degree 4 and defines 4 distinct points on $\mathbb{P}(Ke_3 + Ke_4) = \mathbb{P}^1$. We may suppose that these points are $0, 1, \lambda, \infty$. The lines $\overline{w}(t) \in C$ on $S$ are computed to be the lines passing through the points $(1, t, 0, 0)$ and $(0, y, 1, t)$, with $y^2 = H(1, t)$. The genus one curve $C$ is made into an elliptic curve by the choice of the neutral element $e$ to correspond to $t = y = \infty$. We note that $\overline{w}_{12} \notin C$. $(1, t, 0, 0)$ is a section of $B(2[e])$ and $(0, y, 1, t)$ is a section of $B(3[e])$. Further $w(t) = (1, t, 0, 0) \wedge (0, y, t, 1) = ye_{12} + e_{13} + te_{14} + te_{23} + t^2e_{24} + 0e_{34}$ is a section of $\Lambda^2B([4]e)$. Consider the exact sequence

$$0 \to O_C(1, t, 0, 0) \to B(2[e]) \to O_C(0, 0, 1, t) \to 0.$$

From $O_C(1, t, 0, 0) \cong O_C(0, 0, 1, t) \cong O_C$ and $H^0(C, B(2[e])) = K(1, t, 0, 0)$ one concludes that the sequence does not split. Therefore the ruled surface (in the modern sense) $\tilde{S} \to C$ corresponds to the unique indecomposable vector bundle on $C$ which is an extension of $O_C$ by $O_C$. (see [N]).

16. Case $1^2, 1^2$. The equation $F$ for $S$ is bi-homogeneous of degree $(2, 2)$ in the pairs of variables $t_1, t_2$ and $t_3, t_4$. The equation $F$ also defines a genus one curve $E \subset \mathbb{P}(Ke_1 + Ke_2) \times \mathbb{P}(Ke_3 + Ke_4)$. Further $E \to C \subset Gr$ is the isomorphism which sends $p \in E$ to the line through the points $(pr_1(p), 0, 0)$ and $(0, 0, pr_2(p))$. The vector bundle $B$ is the direct sum of the line bundles $\mathcal{L}_1 := \{[\overline{w}, v] | \overline{w} \in C, v \in Ke_1 + Ke_2, w \wedge v = 0\}$ and $\mathcal{L}_2 := \{[\overline{w}, v] | \overline{w} \in C, v \in Ke_3 + Ke_4, w \wedge v = 0\}$ of degree $-2$.

A line bundle $\mathcal{L}$ on $E$ of degree $-2$ induces a degree 2 morphism $E \to \mathbb{P}(H^0(E, \mathcal{L}^*))$. This yields a bijection between the isomorphy classes of line bundles of degree $-2$ and the equivalence classes of nonconstant morphisms $E \to \mathbb{P}^1$ of degree 2. Then $\mathcal{L}_1$ is not isomorphic to $\mathcal{L}_2$, since the two morphisms are not equivalent. The ruled surface $\tilde{S} \to E$ is equal to $Proj(O_E \oplus \mathcal{L})$, where $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ is any line bundle of degree 0, not isomorphic to $O_E$. In particular, $\tilde{S} \not\cong \mathbb{P}^1 \times E$.

3.2. The classification, using the singular locus.

3.2.1. $2^2$ does not occur as singular locus.

Lemma 3.1. The singular locus of a quartic ruled surface cannot be a conic.

Proof. Suppose that the conic $D$, lying in a plane $H \subset \mathbb{P}(V)$, is the singular locus of some ruled quartic surface $S$, corresponding to a curve $C \subset \mathbb{P}(A^2V)$. If $C$ has genus 1, then $P(C)$ is contained in a tangent space of $Gr$ at some point $\overline{w}_0$. The morphism $\overline{w} \in C \mapsto \overline{w} \cap \overline{w}_0 \in \overline{w}_0$ has degree at least 2 and thus $\overline{w}_0$ belongs to the singular locus. Hence
$C^{\text{norm}}$ has genus 0. The morphism $f : C^{\text{norm}} \rightarrow D$, given by $\varpi \in C^{\text{norm}} \mapsto \varpi \cap H \in D$, has degree at most 2, since the multiplicity of $D$ is 2.

Suppose that the degree of $f$ is 1. One can parametrize $C^{\text{norm}}$ with a parameter $t$ and choose coordinates for $\mathbb{P}(V)$ such that the line $\varpi(t) \in C^{\text{norm}}$ intersects the conic $D$ in the point $(0,1,t,t^2)$. Let $(1,0,a,b)$ with $a,b \in K(t)$ be another point of this line $\varpi(t)$. The Plücker coordinates of $\varpi(t)$ are

$$p_{12} = 1, \ p_{13} = t, \ p_{14} = t^2, \ p_{23} = -a, \ p_{24} = -b, \ p_{34} = t^2a - tb.$$  

Let $d$ be the common denominator of $a$ and $b$. Then $\{dp_{ij}\}$ are polynomials of degree at most 4 and with $\gcd = 1$. If $\alpha$ is a zero of $d$, then the line $\varpi(\alpha)$ lies in the plane $H$. Since this is not possible, $d = 1$ and $a,b \in K[t]$. One obtains the contradiction that the line $\varpi(\infty)$ lies in the plane $H$.

Suppose that the degree of $f$ is 2. One can parametrize $C^{\text{norm}}$ with parameter $t$, and choose coordinates for $\mathbb{P}(V)$ such that $\varpi(t) : (0,1,t,t^2,t^4) \in D$. The line $\varpi(t)$ goes through a point $(1,0,a,b)$ where $a,b \in K(t)$. The Plücker coordinates of $\varpi(t)$ are

$$p_{12} = 1, \ p_{13} = t^2, \ p_{14} = t^4, \ p_{23} = -a, \ p_{24} = -b, \ p_{34} = t^4a - t^2b.$$  

Let $d$ be the common denominator of $a$ and $b$. After multiplying the Plücker coordinates with $d$, the degrees are bounded by 4. Hence $d = 1$ and $a,b \in K[t]$. Further the degree of $a$ is at most 2 and the degree of $c := b - t^2a$ is at most 2. The symmetric polynomial $w(s) \cap w(t)$ in $s,t$ can only have the factors $s+t$ and $s-t$. Indeed, $t \neq s$ and $w(s) \cap w(t) = 0$ implies that $w(s) \cap w(t) \in D$ and thus $s = -t$. It follows that $a = a_0 + a_2t^2$, $c = c_0 + c_2t^2$ and this contradicts that $C^{\text{norm}}$ is parametrized by $t$. $\square$

3.2.2. $1^2, 2^2$. The curve $C^{\text{norm}}$, corresponding to a ruled quartic surface $S$ with this type of singular locus, has genus 0 by Observation [1,1]. The singular locus is the union of a conic $D$ and a line $L$. The plane $H \supset D$ satisfies $S \cap H = C$. Thus $L$ does not lie in $H$ and the intersection $L \cap H$ is a point of $D$. As in the proof of Lemma [3,1], one shows that the morphism $C^{\text{norm}} \rightarrow D$, given by $\varpi \in C \mapsto \varpi \cap H \in D$, has degree 2. Let $D = \{(0,1,\mu^2,\mu) \mid \mu \in \mathbb{P}^1\}$ and $L = \{(1,\lambda,0,0) \mid \lambda \in \mathbb{P}^1\}$. The equations for $D$ and $L$ are $t_1 = t_2t_3 - t_4^2 = 0$ and $t_3 = t_4 = 0$. The equation $F$ for $S$ lies in the ideal $(t_1,t_2t_3 - t_4^2) \cap (t_3,t_4)$, hence $F = t_2^2A_2 + t_1(t_2t_3 - t_4^2)A_1 + (t_3t_2 - t_4^2)A_1$ where $A_2$ and $A_1$ are homogeneous of degrees 2 and 1. One may suppose that $A_1$ does not contain $t_1$. If $A_1$ contains $t_2$, then $F$ contains the monomial $t_4t_2^2t_3$ which is not possible. Hence $A_1$ is linear in $t_3,t_4$ and it follows that $A_2$ is homogeneous of
degree 2 in $t_3, t_4$. Thus

$$F = t_1^2(t_1^2 + c_2t_3t_4 + c_3t_4^2) + t_1(t_2t_3 - t_4^2)(c_4t_3 + c_5t_4) + (t_2t_3 - t_4^2)^2.$$  

We will show that an irreducible equation $F$ as above, defines a ruled surface. Consider a point $(0, 1, \mu^2, \mu) \in D$, $\mu \neq 0, \infty$, then there is a $(1, \lambda, 0, 0) \in L$ such that the line $\{(s, s\lambda + 1, \mu^2, \mu) \mid s \in \mathbb{P}^1\}$ lies on the surface $F = 0$. Indeed, substitution in $F$ yields the equation

$$s^2(c_1\mu^4 + c_2\mu^3 + c_3\mu^2) + s^2\lambda\mu^2(c_4\mu^2 + c_5\mu) + s^2\lambda^2\mu^4 = 0.$$  

For general constants $c_i$ and general $\mu \neq 0, \infty$, this equations has two solutions for $\lambda$. If the equation has for every $\mu$ only one solution for $\lambda$, then one easily verifies that $F$ is reducible (in fact a square).

Suppose now that $(1, \lambda, 0, 0) \in L$ is given. The $\mu \neq 0, \infty$ such that the line $\{(s, s\lambda + 1, \mu^2, \mu) \mid s \in \mathbb{P}^1\}$ lies on $F = 0$ are solutions of the equation

$$\mu^2(\lambda^2 + c_4\lambda + c_1) + \mu(c_2 + \lambda c_5) + c_3 = 0.$$  

(a) Suppose $c_3 \neq 0$. If the equation has only one solution for $\mu \neq 0, \infty$, then $F$ is easily verified to be a square. The assumption that $F$ is irreducible, implies that there are for general $\lambda$ two solutions $\mu$. We conclude that the maps $C \to D$ and $C \to L$, given by $\overline{w} \in C \mapsto \overline{w} \cap H \in D$ and $\overline{w} \in C \mapsto \overline{w} \cap L \in L = \overline{e_{12}}$ have both degree 2. A further calculation shows that $P(C) = T_{Gr, \overline{e_{12}}}$, $\overline{e_{12}} \notin C$ and the vector bundle $B$ has type $-2, -2$. There are still two cases, Number 11 and 12.

(b) If $c_3 = 0$, then $c_2 = c_5 = 0$ is excluded by $F$ is irreducible. Thus there is only one solution $\mu$. The maps $C \to D$ and $C \to L = \overline{e_{12}}$ have degrees 2 and 1. Further calculation shows that $P(C) = T_{Gr, \overline{e_{12}}}$, $\overline{e_{12}} \in C$ and the vector bundle $B$ has type $-1, -3$. This is Number 3.

In Rohn’s paper only case (a) is considered and this is treated as follows. The image $E$ of the morphism $C^{\text{norm}} \to D \times L$ is given by an irreducible bi-homogeneous form of bi-degree $(2, 2)$. Since $C^{\text{norm}}$ has genus 0, the curve $E$ has a singular point which is a node or a cusp. The embedding $E \subset D \times L \cong \mathbb{P}^1 \times \mathbb{P}^1$ can be chosen to be symmetric if the field $K$ is algebraically closed. For $K = \mathbb{R}$ one can have a symmetric or an anti-symmetric embedding.

If $E$ has a node, then the equation $A$, symmetric in $\lambda, \mu$, for the embedding is written as $a_1\lambda^2\mu^2 + a_2(\mu^2 \pm \lambda^2) + 2a_3\lambda\mu$, where $\lambda$ and $\mu$ are inhomogeneous coordinates for the rational curves $L$ and $D$. The $\pm$ sign takes care of the real case where one also has to consider an anti-symmetric embedding. The singular point of $E$ corresponds to $\lambda = \mu = 0$, which is the point $(0, 1, 0, 0)$. The surface $S$ containing the family of the lines through the the pairs of points $\{(\lambda, 1, 0, 0), (0, 1, \mu^2, \mu)\}$
satisfying \( A(\lambda, \mu) = 0 \), \( \lambda, \mu \neq 0 \) has the equation

\[
a_1 t_1^3 t_2^3 + a_2 (t_2 t_3 - t_4^2)^2 + t_1^2 t_2^2 + 2a_3 t_1 t_4 (t_2 t_3 - t_4^2) = 0.
\]

There are various possibilities over \( \mathbb{R} \) of the ‘pinch points’ on \( L \) and \( D \), i.e., the ramification points for the two projections \( pr_1 : E \rightarrow D \), \( pr_2 : E \rightarrow L \).

(i) All four are real if \( \pm = + \) and \( \frac{a_1^2 - a_2^2}{a_1 a_2} > 0 \). Series XIII, no 8, corresponds to this case with additionally \( a_1, a_2 > 0 \).

(ii) No real ones, if \( \pm = + \) and \( \frac{a_1^2 - a_2^2}{a_1 a_2} < 0 \).

(iii) Real on \( L \) and not real on \( D \) if \( \pm = - \) and \( a_1 > 0 \), \( a_2 < 0 \).

(iv) Not real on \( D \) and real on \( L \) if \( \pm = - \) and \( a_1 > 0 \), \( a_2 > 0 \).

If \( E \) has a cusp, then the equation \( A \), symmetric in \( \lambda, \mu \), for the embedding \( E \subset \mathbb{P}^1 \times \mathbb{P}^1 \) can be normalized (following Rohn) to

\[
(\lambda - \mu)^2 - 2\lambda \mu (\lambda + \mu) + \lambda^2 \mu^2 = 0.
\]

This leads to the equation

\[
t_1^2 t_3^2 - 2t_1 t_3 (t_1 t_4 + t_2 t_3 - t_4^2) + (t_4 - t_2 t_3 + t_3^2)^2 = 0
\]

for \( S \).

3.2.3. 3. The line with multiplicity 3 is chosen to be \( t_3 = t_4 = 0 \). Then the equations have the form \( t_1 A_1 + t_2 A_2 + A_3 = 0 \) with \( A_1, A_2, A_3 \) homogeneous in \( t_3, t_4 \) of degree 3, 4; \( \gcd(A_1, A_2, A_3) = 1 \) and \( A_1, A_2, A_3 \) are linearly independent. Conversely, one easily verifies that the above equation defines a ruled surface of degree 4.

The pair \( (A_1, A_2) \) is unique up to taking linear combinations (and linear changes of \( t_3, t_4 \)). In other words the morphism \( f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), given by \( (t_3 : t_4) \mapsto (A_1 : A_2) \), is unique and can have degree 3, 2 or 1. In the first case there are many possibilities for \( f \). In the second case one can normalize \( (A_1, A_2) = d(t_3, t_4) \cdot (t_3^2, t_4^2) \) and in the third case \( (A_1, A_2) = d(t_3, t_4) \cdot (t_3, t_4) \). The term \( A_3 \) can be changed into \( A_3 + \ell_1 A_1 + \ell_2 A_2 \) with \( \ell_1, \ell_2 \) homogeneous in \( t_3, t_4 \) of degrees 1, by replacing \( t_1, t_2 \) by \( t_1 + \ell_1, t_2 + \ell_2 \).

1. \( \gcd(A_1, A_2) = 1 \) and \( A_3 = 0 \). Number 1 a, b, c.

2. \( \gcd(A_1, A_2) = 1 \) and \( A_3 \notin \{\ell_1 A_1 + \ell_2 A_2\} \). Number 13 a, b, c, XIII 6.

3. \( \gcd(A_1, A_2) \) has degree 1. Number 8, 9.

4. \( \gcd(A_1, A_2) \) has degree 2, not a square. \( \overline{\mathbb{P}^1} \) is a node. Number 4, XIII 7.

5. \( \gcd(A_1, A_2) \) has degree 2 and is a square. \( \overline{\mathbb{P}^1} \) is a cusp. Number 5.

3.2.4. 3. The twisted cubic curve \( TC := \{(1, \lambda, \lambda^2, \lambda^3) | \lambda \in \mathbb{P}^1\} \) is the singular locus of \( S \). The homogeneous ideal of \( TC \) is generated by the three homogeneous forms \( X = t_1 t_3 - t_2^2 \), \( Y = t_2 t_3 - t_1 t_4 \), \( Z = t_2 t_4 - t_3^2 \). There are two relations \( t_3 X + t_2 Y + t_1 Z = t_4 X + t_3 Y + t_2 Z = 0 \). The equation \( F \) of \( S \) is homogeneous of degree 4 and lies in the ideal \( (X, Y, Z)^2 \subset K[t_1, t_2, t_3, t_4] \). A computation in the ring \( R := \)
shows that the element \( G := F(1, t_2, t_3, t_4) \) of total degree \( \leq 4 \), lying in the ideal \((R_X^Y + R_Y^Z)^2 \subset R\), is a homogeneous polynomial in the terms \( \frac{X}{t_1}, \frac{Y}{t_1}, \frac{Z}{t_1} \) of degree 2. It follows that \( F(t_1, t_2, t_3, t_4) = H(X, Y, Z) \), where \( H \) is a homogeneous form of degree 2.

Consider the morphism \( f : \mathbb{P}(V) \setminus TC \to \mathbb{P}^2 \), given by \((t_1, t_2, t_3, t_4) \mapsto (X, Y, Z)\). The fibres of \( f \) are the lines of \( \mathbb{P}(V) \) intersecting \( TC \) with multiplicity 2. Thus a fibre is a conic of \( TC \) or a tangent line of \( TC \).

Let \( H(X, Y, Z) \) be homogeneous of degree 2. Then the closure of the preimage under \( f \) of the subscheme \( H = 0 \) of \( \mathbb{P}^2 \) is the ruled surface \( S \) given by the equation \( F(t_1, t_2, t_3, t_4) = H(X, Y, Z) \). Further \( F \) is irreducible and reduced if and only if \( H = 0 \) is a conic. In the sequel we suppose that \( \{H = 0\} \) is a conic and we classify the possibilities. The surface with \( H = T := Y^2 - 4XZ \) is rather special. It consists of all tangent lines of \( TC \) (see Corollary \[1.6\]). For any other conic \( H = 0 \), the intersection with \( T = 0 \) has multiplicity 4. In the general case, the intersection of the two conics consists of 4 points. XIII 9, 10.

Suppose that the intersection has at least one point with multiplicity > 1. The projective space \( \mathbb{P}^3 \) admits an automorphism which preserves the curve \( T = Y^2 - 4XZ = 0 \) and brings this point to \((0, 0, 1)\). Then \( H \) has the form \( XZ + aX^2 + bXY + cY^2 \). One has the following cases for the intersection.

(i) \( aX^2 + bXY + (c + 1/4)Y^2 = 0 \) has two distinct solutions (i.e., \( b^2 - a(4c + 1) \neq 0 \)) and \( (c + 1/4) \neq 0 \). Then the intersection consists of one point with multiplicity 2 and two points with multiplicity 1.

(ii) \( aX^2 + bXY + (c + 1/4)Y^2 = 0 \) has two distinct solutions and \( (c + 1/4) = 0 \). Then the intersection consists of one point with multiplicity 3 and one point with multiplicity 1.

(iii) \( aX^2 + bXY + (c + 1/4)Y^2 = 0 \) has one solution (i.e. \( b^2 - a(4c + 1) = 0 \)) and \( (c + 1/4) \neq 0 \). Then the intersection consists of two points with multiplicity 2.

(iv) \( aX^2 + bXY + (c + 1/4)Y^2 = 0 \) has one solution (i.e. \( b^2 - a(4c + 1) = 0 \)), \( (c + 1/4) = 0 \) and \( a \neq 0 \). Then the intersection consists of one point with multiplicity 4.

Thus we found in total six distinct cases for \( 3^2 \) (compare \[2\]). As we will show below, there is a further natural subdivision of these classes.
Moreover, the ruled surface defined by this equation has of this form, namely \( H \) locus if and only if \( P \) is irreducible. We may suppose that \( P \) denotes the plane through these two lines, has degree 1. We may obtain for, say, \( t = 0 \). The possibilities for intersection of \( H = 0 \) with \( t = 0 \) reflects the possibilities for the ramification of the degree 3 morphism in 2 a, b, c. Case 2 a corresponds to (iii) above; case 2 b to (i) above; case 2 c to the case where the intersection consists of 4 points.

3.2.5. 1². From the Observations 1.5 and Subsection 1.4 one obtains that the genus of \( C \) is 1. Further \( P(C) \) lies in only one tangent space of \( Gr \), say at the point \( \overline{e_{12}} \), since otherwise the surface \( S \) has two skew singular lines. The morphism \( C \to \overline{e_{12}} \), given by \( \overline{w} \in C \leftrightarrow \overline{w} \cap \overline{e_{12}} \), has degree 2 since \( m = 2 \). This map has 4 ramification points and we obtain for, say, \( t \neq 0,1,\lambda, \infty \) two lines of \( C \) through the point \((1,t,0,0) \in \overline{e_{12}} \). The map \( t \neq 0,1,\lambda, \infty \mapsto P(t) \), where \( P(t) \supset \overline{e_{12}} \) denotes the plane through these two lines, has degree 1. We may suppose that \( P(t) \cap \{(0,0,*,*)\} = (0,0,1,t) \). The equation for \( S \) is

\[
t_1^2A + t_2^2b + t_1t_2c + t_1d + t_2e + f; \quad A, B, C, D, E, F \text{ homogeneous in } t_3, t_4.
\]

For any point \((a_1,a_2,0,0) \in \overline{e_{12}} \), the plane \( a_2t_3 - a_1t_4 = 0 \) meets \( S \) in \( \overline{e_{12}} \) and two lines (or one with multiplicity 2) through \((a_1,a_2,0,0) \). This implies that \( t_1^2A(t_3,t_4) + t_2^2B(t_3,t_4) + t_1t_2C(t_3,t_4) \) is a multiple of \((t_3t_1 - t_3t_2)^2 \) and that \( t_1D(t_3,t_4) + t_2E(t_3,t_4) \) is divisible by \((t_4t_1 - t_3t_2) \). After changing the variables \( t_1, t_2 \) we are reduced to two possible equations for \( S \):

\[
(t_3t_1 - t_3t_2)^2 + H(t_3,t_4) = 0 \quad \text{and} \quad G(t_3,t_4)(t_3t_1 - t_3t_2) + H(t_3,t_4) = 0.
\]

The line \( \overline{e_{12}} \) has multiplicity 3 for the second equation. Thus only the first equation is possible with \( H \) not a square since \( S \) is irreducible. Moreover, the ruled surface defined by this equation has \( \overline{e_{12}} \) as singular locus if and only if \( H \) has no multiple factor. Rohn found an equation of this form, namely

\[
a(t_3^2 \pm t_4^2) + 2bt_3t_4 + c(t_4t_2 - t_4t_1)^2 = 0.
\]

The sign \pm distinguishes two classes of real cases. For \( \pm = + \) and \( \frac{b}{a} < -1 \), the four ramification points of \( C \to \overline{e_{12}} \) are real. This is Number 15 and Series XIII, no 5.

Remark. The equation \((t_3t_1 - t_3t_2)^2 + H(t_3,t_4) = 0 \) where \( H \) has no multiple factors, is valid for any field \( K \). If \( K \) is algebraically closed, then \( H \) is determined by the \( j \)-invariant of the four zeros of \( H \) in \( \mathbb{P}^1 \).
For a general field $K$ there are forms for $H$.

3.2.6. 1², 1², int, intersecting lines. The two intersecting lines $L_1, L_2$, making up the singular locus of the ruled quartic surface $S$, lie in a plane $H$. For $\varpi \in C$ and $\varpi \neq L_1, L_2$ the intersection $\varpi \cap H$ is a point of $L_1 \cup L_2$. The induced morphism $C^{\text{norm}} \to L_1 \cup L_2$ has, say, the line $L_1$ as image. Thus we find a nonconstant morphism $f : C^{\text{norm}} \to L_1$ and $P(C)$ lies in the tangent space of $Gr$ at the point $L_1$. For $q \in L_2$ and $q \notin L_1$, there is no $\varpi \in C, \varpi \neq L_1, L_2$ with $q \in \varpi$. One concludes that $L_2 \in C$. Moreover $L_2$ is a singular point $s$ of $C$ since $L_2$ belongs to the singular locus. In particular, $C$ is a rational curve and $\dim P(C) = 3$.

If $P(C)$ lies in the tangent space of $Gr$ at another point $M \in Gr$, then one obtains a morphism $C \to M$ by $\varpi \mapsto \varpi \cap M$. Since $C$ has a singular point, this morphism has degree $> 1$ and one finds the contradiction that $M$ belongs to the singular locus. Thus $P(C)$ lies in a single tangent space. The rational map $C \to L_1$, given by $\varpi \mapsto \varpi \cap L_1$, is well defined at the singular point $s \in C$. Then $f$ has degree $> 1$ and its degree is 2, since $L_1$ has multiplicity 2. Further $L_1 \notin C$, otherwise the multiplicity of $L_1$ would be 3.

For a suitable basis of $V$ and parametrization of $C^{\text{norm}}$, the morphism $C^{\text{norm}} \to L_1$ has the form $\varpi(t) \mapsto (1, t^2, 0, 0)$. Let $b := (b_1, b_2, b_3, b_4)$, with all $b_i \in K[t]$ and $\gcd(b_1, \ldots, b_4) = 1$, be another point of the line $\varpi(t)$. By subtracting a multiple of $(1, t^2, 0, 0)$ one arrives at $\deg b_2 \leq 1$. The Plücker coordinates of $\varpi(t)$ are $(b_2 - t^2 b_1, b_3, b_4, t^2 b_3, t^2 b_4, 0)$ and thus $\deg b_1, \deg b_3, \deg b_4 \leq 2$. The morphism $C \to \mathbb{P}^1$, by $\varpi(t) \mapsto (b_3(t) : b_4(t))$, is well defined and not constant. Since $C$ is singular, this morphism has degree 2. The corresponding degree 2 morphism $g : C^{\text{norm}} \to \mathbb{P}^1$ factors over $C$. If the singular point of $C$ is a cusp for $t = \infty$, then $t = \infty$ is a ramification point and $g$ has the form $g(t) = (1 : (t + \alpha)^2)$. If the singular point of $C$ is a node, corresponding to $t = \pm 1$, then $g(t) = (1 : (\frac{\alpha + \beta}{t})^2)$ also satisfies $g(1) = g(-1)$. Hence $g(t) = (1 : (\frac{1 + \beta}{t + 1})^2)$ with $\beta^2 \neq 1$.

Suppose that $C$ has a cusp, then $(b_3(t), b_4(t)) = (1, (t + \alpha)^2)$ and $b_1, b_2$ can be normalized to constant multiples of $t$. The condition that $t = \infty$ is a cusp for $C$ implies $b_1 = 0$ and so we arrive at $b = (0, t, 1, (t + \alpha)^2)$. The equation reads

$$(t_2 t_3 - t_1 t_4 - \alpha^2 t_1 t_3 + \alpha^2 t_3^2)^2 - t_1 t_3 t_4 (t_3 - 2 \alpha t_1)^2 = 0. \text{ Number 10 b.}$$

Suppose that $C$ has a node, then $(b_3, b_4) = ((\beta t + 1)^2, (t + \beta)^2)$ with $\beta^2 \neq 1$. For $\beta = 0$, one can normalize $b_1, b_2$ to constant multiples of $t$. The condition $\varpi(1) = \varpi(-1)$ implies that $b_1 = b_2 = ct \neq 0$. The equation reads

$$c^2 t_3 t_4 (t_3 - t_4)^2 - (t_1 t_4 - t_2 t_3)^2 = 0. \text{ Number 10 a.}$$
For $\beta^2 \neq 0, 1$, one can normalize $b_1, b_2$ to constants and the condition $\bar{w}(1) = \bar{w}(-1)$ implies $b_1 = b_2 = c \neq 0$. The equation reads
\[ t_3 t_4 \{ 2 \beta (t_1 - t_2) + e (1 - \beta^2) (t_3 - t_4) \}^2 - \{ - t_1 (\beta^2 t_3 + t_4) + t_2 (t_3 + \beta t_4) \}^2 = 0. \]
Again Number 10 a. Rohn found the two similar equations
\[ a t_3 t_4^2 + (t_1 t_4 - t_2 t_3)^2 = 0, \quad \text{and} \quad a t_4^3 + 2 b t_3^2 t_4 + (t_1 t_4 - t_2 t_3)^2 = 0. \]

3.2.7. $1^2, 1^2$, skew lines. The skew lines can be supposed to be $\bar{e}_{12}, \bar{e}_{34}$. Every monomial in the equation $F$ of $S$ is divisible by one of the terms $t_1^2, t_1 t_2, t_2^2$ and by one of the terms $t_3^2, t_3 t_4, t_4^2$. Therefore $F$ is bi–homogeneous of degree $(2, 2)$ and $F$ defines a Zariski closed subset of $\bar{e}_{12} \times \bar{e}_{34} \cong \mathbb{P}^1 \times \mathbb{P}^1$, which is an irreducible curve $E$. One considers the morphism $f : \mathbb{P}(V) \setminus \bar{e}_{12} \cup \bar{e}_{34} \to \mathbb{P}^1 \times \mathbb{P}^1$, given by $(a_1, a_2, a_3, a_4) \mapsto ((a_3, a_2), (a_3, a_4))$. Then $S$ is the Zariski closure of $f^{-1}(E)$. The curve $E$ has no singularities since otherwise the singular locus of $S$ would contain another line. Thus $E$ is a curve of genus 1. One easily sees that $C$ identifies with $E$ and that $P(C)$ lies in the two tangent space of $Gr$ at the points $\bar{e}_{12}$ and $\bar{e}_{34}$. Number 16.

In the above the bases of the two vector spaces $Ke_1 + Ke_2$ and $Ke_3 + Ke_4$ (or equivalently the parametrization of $\bar{e}_{12}$ and $\bar{e}_{34}$) can be chosen in a suitable way. Rohn (see Section 4) shows that for $K = \mathbb{C}$ these bases can be chosen such that the equation $F$ becomes symmetric, i.e., $F(t_1, t_2, t_3, t_4) = F(t_3, t_4, t_1, t_2)$. For $K = \mathbb{R}$ the results of Rohn are more complicated. These results are essential for the understanding of the models in Series XIII, 1,2,3,4 of quartic ruled surfaces with two skew lines of singularities.

3.2.8. $1^2, 1^2, 1^2$. Let $L_1, L_2, L_3$ denote the singular lines with multiplicity 2.

Suppose that the lines $L_1, L_2$ are skew. Then we may suppose $L_1 = \bar{e}_{12}, \quad L_2 = \bar{e}_{34}$. From Lemma 1.10 it follows that the equation $F$ of the surface $S$ is bihomogeneous of degree $(2, 2)$ in the pairs of variables $t_1, t_2$ and $t_3, t_4$. The curve $E \subset \bar{e}_{12} \times \bar{e}_{34} \cong \mathbb{P}^1 \times \mathbb{P}^1$, defined by $F$, has one singular point corresponding to the line $L_3$. This point is a node or a cusp. Number 6, 7. Not in Series XIII. The parametrization of $\bar{e}_{12}$ and $\bar{e}_{34}$ can be chosen (see Section 4) in order to obtain the standard equations of Rohn
\[ a_1 \lambda^2 \mu^2 + a_2 (\lambda^2 \pm \mu^2) + 2 a_3 \lambda \mu = 0 \quad \text{and} \quad \lambda^2 \mu^2 + (\lambda - \mu)^2 - 2 \lambda \mu (\lambda + \mu) = 0, \]
where $\lambda = \frac{t_2}{t_1}, \quad \mu = \frac{t_4}{t_3}$.

The next case to consider is $L_1 \cap L_2, \quad L_1 \cap L_3, \quad L_2 \cap L_3 \neq \emptyset$. The three lines cannot lie in a plane $H$ since otherwise the curve $H \cap S$ has degree 6. It follows that $L_1 \cap L_2 \cap L_3$ is one point. We may suppose that $L_1$ is given by $t_1 = t_2 = 0, L_2$ by $t_1 = t_3 = 0$ and $L_3$ by $t_2 = t_3 = 0$. Every monomial of the equation $F$ is divisible by $t_1^a t_2^b$ with $a_0 + a_1 = 2$, by
with \(b_0 + b_1 = 2\) and by \(t_2^0 t_3^1\) with \(c_0 + c_1 = 2\). The \(t_4\)-part of \(F\) can only be \(c \cdot t_1 t_2 t_3 t_4\). If \(c = 0\), then \(F\) defines a cone. Otherwise one can reduce to the equation \((t_2^2 t_3^0 + t_1^1 t_3^0 + t_1^1 t_2^0) + t_4 t_1 t_3 t_4 = 0\) (or equivalently \((t_2 t_3 + t_1 t_3 + t_1 t_2)^2 + t_4 t_1 t_3 t_4 = 0\)). This equation defines the Steiner’s Roman surface and the three singular lines are in fact the only lines on this surface.

4. Rohn’s symmetric form for bi–degree \((2, 2)\)

K. Rohn proves that over the field \(K = \mathbb{C}\), there is an identification of \(\mathbb{C}^{12} \times \mathbb{C}^{34}\) with \(\mathbb{P}^1 \times \mathbb{P}^1\) such that the equation \(F\) of bi–degree \((2, 2)\) is symmetric in the pairs of variables \(t_1, t_2\) and \(t_3, t_4\), i.e., \(F(t_3, t_4, t_1, t_2) = F(t_1, t_2, t_3, t_4)\). This leads to only a few standard forms for \(F\). Over the field \(\mathbb{R}\), there are more possibilities. First of all, \(\mathbb{C}^{12}, \mathbb{C}^{34}\) can be a pair of conjugated lines over \(\mathbb{C}\). Secondly, even if \(\mathbb{C}^{12}, \mathbb{C}^{34}\) are real lines, then the above identification need not be defined over \(\mathbb{R}\). Thirdly, there are various possibilities over \(\mathbb{R}\) for the ramification points of the two projections \(C \to \mathbb{P}^1\). The models Series XIII, nr. 1,2,3,4 represent some of these cases. A ‘modern version’ of this work of Rohn is as follows.

Consider the closed subset \(E\) of \(\mathbb{P}^1 \times \mathbb{P}^1\), defined by a bi–homogeneous form \(F\) of bi–degree \((2, 2)\). To start we consider the case that \(F\) is irreducible and \(E\) is nonsingular and thus \(E\) has genus 1. We call the embedding \(E \subset \mathbb{P}^1 \times \mathbb{P}^1\) symmetric if \((p, q) \in E \Rightarrow (q, p) \in E\).

Theorem 4.1 (K. Rohn). For a given embedding \(E \subset \mathbb{P}^1 \times \mathbb{P}^1\) as above, there exists an automorphism \(f\) of the first factor, such that the new embedding \(E \subset \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{f^{-1}} \mathbb{P}^1 \times \mathbb{P}^1\) is symmetric.

Proof. The required automorphism \(f\) of \(\mathbb{P}^1\) has the property \((p, q) \in E \Rightarrow (f^{-1} q, f p) \in E\). In particular, the morphism \(C : (p, q) \mapsto (f^{-1} q, f p)\) is an automorphism of \(E\) of order 2. We assume that \(f\) exists, try to find its explicit form and then use this form to produce an \(f\) with the required property. Some explicit information concerning the automorphisms of order 2 of \(E\) is needed. For this purpose, we choose a point \(e_0 \in E\). This makes \(E\) into an elliptic curve (and the addition of two points \(a, b\) is written as \(a + b\)). Consider the automorphisms \(\sigma\) and \(\tau_a\) (any \(a \in E\)), given by \(\sigma(p) = -p\) and \(\tau_a(p) = p + a\). One verifies that the automorphisms of order 2 of \(E\) are:

(a) \(\sigma \tau_a\) for any point \(a\) on \(E\),
(b) \(\tau_a\) where \(a \neq 0\) is a point of order two on \(E\).

Division of \(E\) by the action of an element in the first class yields \(\mathbb{P}^1\) and division by the action of an element in the second class yields an
elliptic curve. Thus the two projections \( pr_1 : E \to \mathbb{P}^1 \) correspond to distinct elements \( \sigma \tau_{a_1} \) and \( \sigma \tau_{a_2} \) of order 2 with the property \( pr_i \circ \sigma \tau_{a_i} = pr_i \) for \( i = 1, 2 \).

The assumption on \( f \) and the definition of \( C \) are equivalent to \( pr_2(Ce) = f(pr_1(e)) \) for any \( e \in E \). Replacing \( e \) by \( \sigma \tau_{a_1}e \) does not change the right hand side. Thus \( C \sigma \tau_{a_1}e \) is either \( Ce \) or \( \sigma \tau_{a_2}Ce \). The first equality can only hold for four elements \( e \in E \). Hence the second equality holds for almost all \( e \) and thus holds for all \( e \). We conclude that \( C \sigma \tau_{a_1} = \sigma \tau_{a_2} C \).

Suppose that \( C = \sigma \tau_c \). The equality \( \sigma \tau_c \sigma \tau_{a_1} = \sigma \tau_{a_2} \sigma \tau_c \) is equivalent to \( 2c = a_1 + a_2 \). There are 4 solutions \( c \) of this equation.

Suppose that \( C = \tau_c \) with \( c \) an element of order 2. Then one finds the contradiction \( a_1 = a_2 \).

Take \( C = \sigma \tau_c \) for some \( c \) with \( 2c = a_1 + a_2 \). Define \( f \) by the formula \( f(pr_1(e)) := pr_2(Ce) \). This is well defined because of \( C \sigma \tau_{a_1} = \sigma \tau_{a_2} C \).

It is easily verified that \( f \) is an isomorphism and has the required property. \( \square \)

Let \( E \subset \mathbb{P}^1 \times \mathbb{P}^1 \) be a symmetric embedding and the homogeneous coordinates of the two projective lines are denoted by \( x_0, x_1 \) and \( y_0, y_1 \). Let \( \{p_1, p_2, p_3, p_4\} \subset \mathbb{P}^1 \) denote the four ramification points of the project \( pr_1 : E \to \mathbb{P}^1 \). There is an automorphism \( s \) of order two which permutes each of the pairs \( \{p_1, p_2\} \) and \( \{p_3, p_4\} \). The two fixed points of \( s \) can be supposed to be 0, \( \infty \) and thus \( s \) has the form \( s(x, y) = (x, -x) \). The four ramification points are then \( \{(1, \pm d)\} \) and \( \{(1, \pm e)\} \). By scaling \( (x_0, x_1) \leftrightarrow (x_0, \lambda x_1) \) with \( \lambda^2 ed = \pm 1 \) we arrive at four ramification points \( \{(1, \pm d \pm e)\} \) (with of course \( d \neq 1 \)). Write \( F = Ay_0^2 + By_0y_1 + Cy_1^2 \). Then the four ramification points of \( pr_1 \) are the zeros of the discriminant \( B^2 - 4AC \) and thus \( B^2 - 4AC = x_1^2 + bx_0^2 x_1^2 + x_0^4 \) with \( b = -(d^2 + d^{-2}) \). Then we obtain the normal form of \( K \). Rohn for \( F \):

\[ a_1(x_0^2 y_0^2 + x_1^2 y_1^2) + a_2(x_0^2 y_1^2 + x_1^2 y_0^2) + 2a_3 x_0 x_1 y_0 y_1 \quad \text{with} \quad a_1 a_2 \neq 0 \]

or in Rohn’s notation \( a_1(\lambda^2 \mu^2 + 1) + a_2(\lambda^2 + \mu^2) + 2a_3 \lambda \mu \) with \( \lambda = \frac{x_1}{x_0}, \mu = \frac{y_1}{y_0}, \)

and with discriminant \( x_1^4 + bx_0^2 x_1^2 + x_0^4 \) and \( b = \frac{a_1^2 + a_2^2 - a_3^2}{a_1 a_2} \neq \pm 2 \).

The above calculations are valid over any algebraically closed field of characteristic \( \neq 2 \). Now we analyze the more complicated situation over the field \( \mathbb{R} \). Assume that the two lines \( \overline{e_{12}}, \overline{e_{34}} \) and \( E \) are defined over \( \mathbb{R} \). Assume moreover that \( E(\mathbb{R}) \) is not empty (indeed otherwise the real model for the corresponding surface has no points). Fix a real point \( e_0 \) as the neutral element of \( E \). The group \( E(\mathbb{R}) \) is either isomorphic to the circle \( \mathbb{R}/\mathbb{Z} \) (the connected case) or to \( \mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) (the disconnected case). In the connected case \( E(\mathbb{R}) \) has two elements of order dividing 2 and in the disconnected case there are 4 such elements.
The collection of the real automorphisms of order two of $E$ consists of the $\sigma \tau_a$ with $a \in E(\mathbb{R})$ and $\tau_v$ where $v$ is a real point of order 2.

Now we revisit the proof of the theorem for the case $K = \mathbb{R}$.

The connected case. The fixed points $b$ of $\sigma \tau_a$ (note that $a_1 \in E(\mathbb{R})$) are the solutions of $2b = -a_1$. Two of the $b$’s are real. The other two are complex conjugated. Hence two of the ramification points for $pr_1 : E \to \mathbb{P}^1$ are real, the other two are complex conjugated. The same holds for the ramification points of $pr_2 : E \to \mathbb{P}^1$. For the element $c$ with $2c = a_1 + a_2$ there are two real choices. Thus the real version of the theorem remains valid in this case. Two of the four ramification points are real and the other two are complex conjugated. One can normalize such that the ramification points are $\pm d$, $\pm id$. This leads to Rohn’s normal equation

$$
\begin{align*}
& a_1 (-x_0^2y_0^2 + x_1^2y_1^2) + a_2 (x_0^2y_1^2 + x_1^2y_0^2) + 2a_3x_0x_1y_0y_1 \quad \text{with real } a_1, a_2, a_3.
\end{align*}
$$

The disconnected case. There are 4 real fixed points of $\sigma \tau_a$ if $a_1$ lies in the component of the identity of $E(\mathbb{R})$. In the opposite case there are no real solutions of $2b = a_1$. The same holds for $\sigma \tau_a$ and for the solutions of the equation $2c = a_1 + a_2$. Hence there are cases where no real automorphism $f$ exist. All cases can be listed by:

(a) 4 real ramification points for $pr_1$ and for $pr_2$ and 4 real solutions for $c$,
(b) no real ramification points for $pr_1$ and $pr_2$ and 4 real solutions for $c$,
(c) 4 real ramification points for $pr_1$, none for $pr_2$ and no real solution for $c$,
(d) 4 real ramification points for $pr_2$, none for $pr_1$ and no real solution for $c$.

Suppose that $c$ can be chosen to be real. For Rohn’s normal form one needs an automorphism $s$ permuting each pair $\{p_1, p_2\}$ and $\{p_3, p_4\}$. One may suppose that each pair is invariant under complex conjugation. Then the resulting $s$ is also real. For the cases (a) and (b) the standard equation is indeed

$$
\begin{align*}
& a_1 (x_0^2y_0^2 + x_1^2y_1^2) + a_2 (x_0^2y_1^2 + x_1^2y_0^2) + 2a_3x_0x_1y_0y_1
\end{align*}
$$

and discriminant $x_1^4 + \frac{a_1^2 + a_2^2 - a_3^2}{a_1a_2}x_1^2x_0^2 + x_0^4$,

with $a_1, a_2, a_3 \in \mathbb{R}$. One easily calculates that $\frac{(a_1 + a_2)^2 - a_3^2}{a_1a_2} < 0$ corresponds to (a) and $\frac{(a_1 + a_2)^2 - a_3^2}{a_1a_2} > 0$ corresponds to (b).

For the cases (c) and (d) there is no real symmetric normal form for $F$. In case (c) (case (d) is similar), Rohn’s real normal form could be
called \emph{half-symmetric}, because of its form
\begin{equation*}
a_1(x_0^2y_0^2 - x_1^2y_1^2) - a_2(x_0^2y_1^2 - x_1^2y_0^2) + 2a_3x_0x_1y_0y_1.\end{equation*}

The models 1, 2 and 3 of Series XIII deal with a pair of real skew double lines. In the terminology of Rohn, a \textit{pinch point} is a ramification point for one of the two projections \(pr_1, pr_2\) and situated on \(L_1\) and \(L_2\) with the obvious identification of these lines with the two \(\mathbb{P}^1\)'s. Series XIII nr. 1 corresponds to (a), Series XIII nr 2 corresponds to (b) and Series XIII nr 3 to (c). Rohn also considers the situation where the ruled surface has a pair of complex conjugated lines as double lines and produces a standard form and an example, namely model 4 of series XIII.

\emph{Rohn’s normal form for other curves} \(E \subset \mathbb{P}^1 \times \mathbb{P}^1\) of type \((2, 2)\).

These normal forms are useful for §§ 3.2.2, 3.2.5, 3.2.6 and 3.2.8. There are three cases:

(a) \(E\) is irreducible and has a node,

(b) \(E\) is irreducible and has a cusp and

(c) \(E\) is reducible or is not reduced.

In the following we use the notation and the ideas of the proof of the theorem.

(a). The nonsingular locus of \(E\) is, after a choice of a point \(e_0\), the group \(\mathbb{G}_m\). Let \(\sigma\) denote the automorphism \(x \mapsto -x\) and define \(\tau_a(x) = ax\). The automorphisms of order 2 of \(E\) are \(\sigma \tau_a\) (any \(a \in \mathbb{G}_m\)) and \(\tau_{-1}\). Dividing \(E\) by the action of \(\sigma \tau_a\) yields the quotient \(\mathbb{P}^1\) and dividing by the action of \(\tau_{-1}\) yields a rational curve with a double point. Thus the two projections \(pr_i : E \to \mathbb{P}^1\) correspond to order two elements \(\sigma \tau_{a_i}\) for \(i = 1, 2\) with \(a_1 \neq a_2\). The required automorphism \(C\) of order two should satisfy \(C \sigma \tau_{a_1} = \sigma \tau_{a_2} C\). There are two possibilities for \(C\), namely \(C = \sigma \tau_c\) with \(c^2 = a_1a_2\). Thus we find a symmetric embedding \(E \subset \mathbb{P}^1 \times \mathbb{P}^1\) for any algebraically closed field of characteristic \(\neq 2\).

For \(\mathbb{R}\) as base field, the situation is more complicated. Suppose that both lines, i.e., the two factors \(\mathbb{P}^1\), and \(E\) are defined over \(\mathbb{R}\). We assume that the nonsingular locus \(E^*\) has a real point \(e_0\). There are two possibilities for \(E^*(\mathbb{R})\), namely: (i) \(\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^*\) and (ii) \(\mathbb{R}/\mathbb{Z}\).

In case (i), one has to solve the equation \(c^2 = a_1a_2\) with \(c \in \mathbb{R}^*\). If there is a solution, then one has a symmetric embedding \(E \to \mathbb{P}^1 \times \mathbb{P}^1\), defined over \(\mathbb{R}\). In the opposite case, one makes an anti-symmetric embedding (by adding some minus signs). The \textit{two standard equations} are
\begin{equation*}
a_1\lambda^2\mu^2 + a_2(\lambda^2 \pm \mu^2) + 2a_3\lambda\mu = 0, \quad \text{with} \quad \lambda = \frac{x_1}{x_0}, \quad \mu = \frac{y_1}{y_0}.\end{equation*}
In case (ii), the automorphisms of order two are the maps $f_a : x \mapsto -x + a$ (any $a \in \mathbb{R}/\mathbb{Z}$) and $x \mapsto x + 1/2$. The last automorphism is ruled out because it does not give a $\mathbb{P}^1$ as quotient. Now we have to solve $Cf_{a_1} = f_{a_2}C$ for some order two automorphism $C$. The two solutions for $C$ are $f_c$ with $2c = a_1 + a_2$. There are two solutions for $c \in \mathbb{R}/\mathbb{Z}$ and therefore there is a symmetric embedding. The standard equation is

$$a_1\lambda^2\mu^2 + a_2(\lambda^2 + \mu^2) + 2a_3\lambda\mu = 0,$$
with $\lambda = \frac{x_1}{x_0}$, $\mu = \frac{y_1}{y_0}$.

Finally, there is the possibility that the two lines form a conjugate pair over $\mathbb{R}$. [We do not work out the details here.]

(b). The nonsingular locus $E^*$ of $E$ is isomorphic to the additive group $\mathbb{G}_a$. The automorphism of order two are $f_a : x \mapsto -x + a$ (any $a \in \mathbb{G}_a$). The equation $Cf_{a_1} = f_{a_2}C$ (with $a_1 \neq a_2$) has a unique solution $C = f_c$ with $2c = a_1 + a_2$. Thus there exists a symmetric embedding $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ and this embedding is unique. The above is valid for any field of characteristic $\neq 2$, because the group $\mathbb{G}_a$ has no forms. The standard equation is

$$\lambda^2\mu^2 + (\lambda - \mu)^2 - 2\lambda\mu(\lambda + \mu) = 0.$$

(c) For a reducible or nonreduced $E$, Rohn obtains the following standard equations

$$(\lambda + \mu)^2 + 2a\lambda\mu = 0,$$ 

$$(\lambda - \mu)^2 \pm (\lambda - \mu)^2 = 0,$$ 

$$(\lambda - \mu)^2 = 0.$$

References

[1] Barth, W. Peters, C., van der Ven, A., Compact complex surfaces, Springer-Verlag, Berlin and New York, 1984.
[2] Bottema, O., A Classification of Rational Quartic Ruled Surfaces, Geometriae Dedicata 1, no. 3, (1973), 349-355.
[3] Cayley, A., A Third Memoir on Skew Surfaces, Otherwise Scrolls, Philosophical Transactions of the Royal Society of London 159 (1869), 111-126.
[4] Chasles, M., Sur les six droites qui peuvent ê être les directions de six forces en équilibre. Propriétés de l’hyperbolôide à une nappe et d’une certaine surface du quatrième ordre. Comptes Rendus des Séances de l’Académie des Sciences. Paris. bf 52 (1861), 1094-1104.
[5] Cremona, L., Sulle Superficie Gobbe di Quarto Grado. Memorie dell’ Academia delle Science dell’ Istituto di Bologna, serie II, tomo VIII (1868), 235-250. Opere, II, 420.
[6] Dolgachev, I. V., Topics in Classical Algebraic Geometry, www.math.lsa.umich.edu/~idolga/topics1.pdf
[7] Edge, W. L., The theory of ruled surfaces, Cambridge, 1931.
[8] Hartshorne, R., Algebraic Geometry, Springer-Verlag, Berlin etc., 1977.
[9] Meyer, W. Fr., Flächen vierter und höherer Ordnung. Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, IIIc, 10b (1930), 1744-1759.
[10] Mohrmann, H., Die Flächen vierter Ordnung mit gewundener Doppelkurve, Mathematische Annalen 89 (1923), 1-31.
[11] Pascal, E., Repertorium der Höheren Mathematik (Definitionen, Formeln, Theoreme, Literatur). II. Theil: die Geometrie. Teubner, 1902.
[12] Rohn, K., Ueber die Flächen vierter Ordnung mit dreifachem Punkte, Mathematische Annalen 24, No. 1 (1884), 55-151.
[13] Rohn, K., Die verschiedenen Arten der Regelflächen vierter Ordnung, (1886). Mathematische Abhandlungen aus dem Verlage Mathematischer Modelle von Martin Schilling. Halle a. S., 1904. Mathematische Annalen 28, No. 2 (1886), 284-308.
[14] Salmon, G., A Treatise on the Analytic Geometry of Three Dimensions, Dublin, 1882.
[15] Schilling, M., Catalog mathematischer Modelle für den Höheren mathematischen Unterricht, Leipzig, 1911.
[16] Segre, C., Etude des différentes surfaces du 4e ordre à conique double ou cuspidale (générale ou décomposée) considérées comme des projections de l’intersection de deux variétés quadratiques del’espace à quatre dimensions. Mathematische Annalen 24, 3 (1884), 313-444.
[17] Sturm, R., Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung, I, B.G. Teubner (1892).
[18] Swinnerton-Dyer, H. P. F., An Enumeration of All Varieties of Degree 4. American Journal of Mathematics 95, No. 2 (1973), 403-418.
[19] Urabe, T., Dynkin Graphs and Combinations of Singularities on Quartic Surfaces, Proc. Japan. Acad., Ser. A 61 (1985), 266-269.
[20] Urabe, T., Classification of Non-normal Quartic Surfaces. Tokyo Journal of Mathematics 9, No. 2 (1986), 265-295.
[21] Urabe, T., Elementary transformations of Dynkin graphs and singularities on quartic surfaces, Inventiones mathematicae 87 (1987) 549-572.
[22] Urabe, T., The transformations of Dynkin graphs and singularities on quartic surfaces. Inventiones mathematicae 100 (1990) 207-230.
[23] Wong, B. C., A study and Classifications of Ruled Quartic Surfaces by Means of a Point-to-Line Transformation. University of California Publications of Mathematics 1, No. 17 (1923), 371-387.

Department Matemáticas, University of Cantabria, Avda. Castros s/n, 39005 Santander, Spain.

IWI-RUG, University of Groningen, Nijenborgh 9, 9747 AG Groningen, the Netherlands.

E-mail address: irene.polo@unican.es, mvdput@math.rug.nl, j.top@rug.nl