Localization for the Ising model in a transverse field with generic aperiodic disorder

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Abstract

We study the ground state of the quantum Ising model with an aperiodic transverse field. We show for a dense $G_δ$ set of transverse fields there is a phase transition of an ordered phase for large coupling to a disordered phase for small coupling. In particular, we apply our results to the setting of quasiperiodic transverse fields and find conditions in that case for the phase transition.

1 Introduction

The ground state of the quantum Ising model on $\mathbb{Z}^d$ in a random transverse field,

$$H_\Lambda = - \sum_{\|x-y\|=1} \lambda \sigma_x^{(3)} \sigma_y^{(3)} - \sum_x \delta(x) \sigma_x^{(1)}$$

where, for any $x \in \mathbb{Z}^d$

$$\sigma_x^{(3)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the sums are over $x, y \in \Lambda \subset \mathbb{Z}^d$. At low interaction $\lambda \searrow 0$ between spins is known to exhibit both long range order and disordered phases. The phases depend on the behavior $g(s) = \mathbb{P}(\delta < s)$ as $s \to 0$, here $g(s) = 0$ for $s \leq 0$. In the long range order regime (ie corresponding to spontaneous magnetization), which occurs for sufficiently strong disorder of the transverse field, spin correlations are almost surely bounded below for any nonzero ferromagnetic interaction $\lambda$ \cite{6,7}. On the other hand, for weak disorder, and sufficiently small interaction $\lambda < \lambda_m$, the localized regime is obtained almost surely with exponential decay of correlations \cite{6,7}. In the weakly disordered system, for $\lambda$ large, the system again obtains a long range order phase (in this paper we will always assume $\delta$ is bounded above). See equations (1.10) and (1.11) for the definition of weak disorder and equations (1.12) and (1.13) for the definition of strong disorder. The existence of both phases in the weakly disordered case may be viewed as the persistence of phase transitions of the ground state in the clean system $\delta(x) \equiv \delta$, which is indeed sharp \cite{5,10}. In the clean system, $0 < \rho_c \lambda < \delta$ corresponds to a

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localized phase and $0 < \delta < \rho, \lambda$ corresponds to a long range order phase. In the weakly disordered case, whether the phase transition from the long range order phase to the exponential clustering phase is sharp is an open question.

As an alternative to characterizing the (almost sure) phase diagram of spin models with disordered defects one may consider the phase diagram for (a generic class of) ordered defects. In this paper, we show the ground state phase transition of the spontaneous magnetization persists for topologically generic, ie dense $G_\delta$, ordered defects of the transverse field. As with the disordered case we find there are atypical, yet in a sense dense, environments of ordered defects which have, for all $\lambda$, a long range ordered phase in the ground state. We discuss the topology of the defects below, the defects will be dynamically defined, see Section 1.1.1 for the metric defining the topology of sampling functions. As a principle example, we compare moment conditions of the disordered case admitting a phase transition to the case of quasiperiodic order. Thus we generalize the results of [20] which show the existence of a phase transition for quasiperiodic order for sufficiently nice sampling functions.

Absence of spontaneous magnetization is a relatively weak indicator of localization. One may ask for stronger indicators of localization such as exponential clustering of correlations or exponential decay of entanglement. Indeed we appeal to a multiscale argument to obtain exponential decay of correlations in a dense set of dynamically ordered environments. The multiscale argument is similar to the approach developed for the disordered model [6], [7]. We note that exponential decay of entanglement has been demonstrated for the ground state of (1.1) for small $\lambda$ in the weak disorder regime [10], the results of that paper rely on the multiscale method developed in [6], [7] and should carry into environments with dynamically ordered defects we consider in this paper.

The transverse field Ising model (1.1) is a boundary case of the anisotropic XY model in a transverse field, which, which for $\sigma^{(2)}_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, is defined as,

$$H_{XY} = -\sum_{x \sim y} \mu \left[ (1 + \gamma)\sigma_x^{(1)} \sigma_y^{(1)} + (1 - \gamma)\sigma_x^{(2)} \sigma_y^{(2)} \right] - \sum_x \delta_x \sigma_x^{(3)}. \quad (1.2)$$

Note that conjugating (1.2) by $U = \bigotimes_{x \in \Lambda} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ and sending $\gamma \to 1$ and $\mu = \lambda/4$ recovers (1.1). For $\gamma = 0$, (1.2) becomes the isotropic XY model in a transverse magnetic field. Under the Jordan Wigner transformation, the one dimensional isotropic model is equivalent to the Anderson model. The two point correlations properties of (1.2) are closely related to those of the effective one particle system, several papers have recently utilized this connection to study the Lieb Robinson velocity of (1.2), which in turn may be extended to properties of the ground state.

For random $\delta$ it was shown in [18] that a zero velocity Lieb Robinson bound holds, which can be used to prove exponential clustering of correlations which depend on the spectral gap in the finite system. The dependence on the spectral gap was removed in [24], where it is shown that exponential decay of correlations in one dimensional systems follow from exponential decay of fractional moment bounds of the Greens’ function of the one particle representation. (1.1) with random $\delta$ under a Jordan Wigner transformation is equivalent to a Jacobi matrix with randomness on alternate off diagonal entries, see [8] for discussion on this approach. Alternately the one particle model may be cast in the form of a Schrodinger operator with matrix valued potentials, the resulting model may be considered under the methods of [13] and [24] to conclude exponential clustering of correlations. Similarly, the properties of (1.2) with quasiperiodic $\delta$ have been studied through the Jordan Wigner transformation. An anomalous Lieb Robinson bound was found for the ground state of the XY spin model with a Fibonacci transverse field, this in turn implies that the ground state exhibits stretched exponential decay of correlations [11]. On the other hand, a lower bound for the Lieb Robinson velocity is
demonstrated [21], for a class of quasiperiodic \( \delta \) including those defined by an analytic sampling function and a Diophantine frequency and \( \mu \) sufficiently large. Prior to this, a lower bound of the Lieb Robinson velocity was found for [12] with periodic \( \delta \) [12], moreover \( \mu_x \neq 0 \) and \( \gamma_x \notin \{1, -1\} \) were permitted to be periodic as well, the result follows from ballistic transport in the effective one particle system.

We emphasize that the above results pertain to the one dimensional model whereas the results in this paper hold for general \( d \)-dimensional models.

1.1 The Ising model

We are interested in the ground state in the infinite volume limit. The relevant observable in finite volumes and temperature are defined as

\[
\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_{\Lambda, \beta} := \frac{\text{tr}(\sigma_x^{(3)} \sigma_y^{(3)} e^{-\beta H_\Lambda})}{\text{tr}(e^{-\beta H_\Lambda})}
\]

where we take free boundary conditions on finite sets \( \Lambda \subset \mathbb{Z}^d \). The existence of this observable, in the zero temperature limit \( \beta \to \infty \) and the limit \( \Lambda \to \infty \) is a standard fact [1], [7].

For a uniform transverse field \( \delta(x) \equiv \delta \) one may define an order parameter as the ratio \( \rho = \lambda / \delta \). In all dimensions there is a critical \( \rho_c \) (depending on the dimension) so that \( \rho < \rho_c \) implies exponential decay of correlations functions and \( \rho > \rho_c \) implies long range order [5], [10], in the sense of a lower bound of the correlation function. In particular for \( d = 1 \) the critical point is \( \rho_c = 2 \) [5]. Indeed there is scale invariance of parameters, so \( \lambda \to \lambda \) and \( \delta \to \delta \) correspond to a dilation of ‘imaginary time’ by a factor of \( 1/\tau \), this remains true in the inhomogeneous regime.

For \( \delta \) bounded over \( \mathbb{Z}^d \), for coupling \( \lambda > \rho_c \text{sup}_x \delta(x) \) the environment exhibits long range order. If \( \text{inf}_x \delta(x) > 0 \) then \( \lambda < \rho_c \text{inf}_x \delta(x) \) implies absence of long range order. These cases can be inferred by measure domination principles, there are no soft arguments obtaining a similar phase of absence of long range order in the case of \( \delta \) with \( \text{inf}_x \delta(x) = 0 \). On the other hand, if \( \sup \delta(x) \) is finite \( \lambda > \rho_c \text{sup}_x \delta(x) \) implies a long range order regime.

We will show that generically \( \delta \) which are defined by an ergodic process so that \( \text{inf}_x \delta(x) = 0 \) there is a phase characterized by absence of long range order. I.e in this case, a \( \lambda_c \) so that \( \lambda_c > 0 \) and \( 0 < \lambda < \lambda_c \) implies almost surely for all \( x \in \mathbb{Z}^d \) that \( \lim_{L \to \infty} \sup_y \{\langle \sigma_x^{(3)} ; \sigma_y^{(3)} \rangle : \|y - x\| > L\} = 0 \). On the other hand, the set of ergodically defined environments \( \delta \) with \( \text{inf}_x \delta(x) = 0 \), which maintain long range order for all \( \lambda > 0 \).

1.1.1 Description of the model and main results

We will define the transverse field \( \delta \) by a sampling function over a dynamic system on a compact metric space \((\Theta, r)\). Let \( T \) be a group action \( T : (\Theta, \mathbb{Z}^d) \to \Theta \) defined by a set of continuous commuting automorphisms, \( \{T_i\}_{i=1}^d \). We will write, for \( x \in \mathbb{Z}^d \) and \( \theta \in \Theta \), \( T : (\theta, x) \mapsto T_x \theta := T_1^{x_1} \ldots T_d^{x_d} \theta \).

We require the group generated by \( \{T_i\} \) to be aperiodic, that is, for all \( x \neq 0 \), the map \( T_x \) has no fixed points. Moreover, we require the set of automorphisms to share an ergodic measure \( \mu \) on \((\Theta, r)\). The sampling function \( h \) is a continuous function \( h : \Theta \to [0, \infty) \), so that \( h^{-1}(0) \neq \emptyset \). For initial condition \( \theta \), define the transverse field by \( \delta(x) = h(T^x \theta) \). Note the aperiodic assumption rules out, in particular, iid random fields from consideration.

The sampling functions we consider belong to the set \( C^+ := C^+(\Theta) \), the non-negative valued continuous functions on \( \Theta \). For \( h, h' \in C^+ \) define a distance function \( d_S(h, h') = \|h - h'\|_\infty \). For most
results we restrict the sampling functions to those with finite zero sets, thus define \( \mathcal{F} \) to be the set of nonempty finite sets \( F \subset \Theta \). For \( F \in \mathcal{F} \) a finite set, let \( \mathcal{S}_F \) be the set of sampling functions with zero set \( F \), i.e., the set of \( h \in \mathcal{C}^+ \) such that \( h^{-1}(\{0\}) = F \), finally, define the set \( \mathcal{S}_F \) of sampling functions so that \( h^{-1}(0) \) is nonempty and finite, that is, let \( \mathcal{S}_F = \bigcup_{F \in \mathcal{F}} \mathcal{S}_F \).

As stated above, the behavior of the two point function (1.3) is invariant under scaling of the sampling function and coupling parameter. Therefore, it is useful to identify a sampling function with its normalization, which we will define as \( \tau(h) := h/\|h\|_\infty \). For \( h, h' \in \mathcal{C}^+ \), let us define distance with respect to this normalization, \( d_\tau(h, h') = d_\mathcal{S}(\tau(h), \tau(h')) \). That is, sampling functions \( h \) and \( h' \) are close if their images under \( \tau \) are close.

Observe that the map \( \iota : (\mathcal{C}^+, d_\mathcal{S}) \to (\mathcal{C}^+, d_\tau) \) is an open continuous map. Continuity is clear, to see \( \iota \) is open, we will show the image of the set \( B_{\mathcal{S}}(h, \epsilon) = \{ h' \in \mathcal{C}^+ : d_\mathcal{S}(h', h) < \epsilon \} \) under \( \iota \) is open. If \( \epsilon > \|h\|_\infty \) the image of \( B_{\mathcal{S}}(h, \epsilon) \) is all normalized functions, i.e. \( \tau(B_{\mathcal{S}}(h, \epsilon)) = \mathcal{C}^+ \). Thus, let \( \epsilon \leq \|h\|_\infty \), then \( h' - h = g \) where \( \|g\|_\infty < \epsilon \). Then if \( \delta < \frac{\epsilon - \|g\|_\infty}{\|h\|_\infty + \|g\|_\infty} \) and \( d_\tau(h', h'') < \delta \), there is a \( h \) so that \( \tau(h'') = \tau(h) \) and \( d_\mathcal{S}(h, h'') < \epsilon \), so that \( \{ h'' \in \mathcal{C}^+ : d_\tau(h, h'') < \delta \} \subset \iota B_{\mathcal{S}}(h, \epsilon) \). Indeed, let \( k = \tau(h'') - \tau(h') \) and define \( h^* = \|h'||_\infty(k + \tau(h')) \) then we have

\[
\|h^* - h\|_\infty = \|g + \|h'||_\infty k\|_\infty \leq \|g\|_\infty + \|h'||_\infty \delta < \epsilon.
\]

Let us define the relevant observables and phases for (1.1) for transverse fields with \( \inf_x \delta(x) = 0 \). Given a phase \( \theta \), we define the long range order parameter for given interaction \( \lambda \), and field \( \delta(x) = h(T^x \theta) \),

\[
M_{h, \lambda}(\theta) := \lim_{L \to \infty} \sup_y \{ \langle \sigma_0^{(3)} \sigma_y^{(3)} \rangle : \|y\| > L \}.
\] (1.4)

If \( h, \lambda \) and \( \theta \) are such that \( M_{h, \lambda}(\theta) > 0 \) we say (1.1) is in the long range ordered phase and if \( M_{h, \lambda}(\theta) = 0 \) we say (1.1) is in the absence of long range order phase. Note that it follows from standard FKG inequalities that \( M_{h, \lambda}(\theta) > 0 \) implies \( M_{h, \lambda}(T^x \theta) > 0 \) for all \( x \in \mathbb{Z}^d \).

It follows from the Feynman Kac representation (a percolation model described in Section 2) of the two point function (1.3) that \( M_{h, \lambda}(T^x \theta) \geq 0 \). For constant \( \delta, \lambda \) the existence of at most one infinite component in the FK representation follows from standard techniques, the uniqueness of the infinite component may be extended to ergodically defined \( \delta \) as well, for background see [1].

We say a sampling function \( h \) admits a localized phase transition if there is some critical \( \lambda_h \) so that for \( 0 < \lambda < \lambda_h \), we have, for almost every \( \theta \in \Theta \), that \( M_{h, \lambda}(\theta) = 0 \), and for \( \lambda > \lambda_h \) we have for almost every \( \theta \in \Theta \) that \( M_{h, \lambda}(\theta) > 0 \). We say a sampling function is localization free if for all \( \lambda > 0 \) we have for almost every \( \theta \in \Theta \) that \( M_{h, \lambda}(\theta) > 0 \). Define \( \mathcal{S}_T \subset \mathcal{S}_F \) to be the set of sampling functions \( h \) which admit a localized phase transition. Define \( \mathcal{H}_T \subset \mathcal{S}_F \) to be the set of sampling functions \( h \) which are localization free. For any metric space \( (\Theta, r) \) with aperiodic action \( T \), \( \mathcal{S}_T \) is a dense \( G_\delta \) in \( \mathcal{S}_F \) and, the sets \( \mathcal{S}_T \) and \( \mathcal{H}_T \) partition \( \mathcal{S}_F \).

**Theorem 1.1** Let \( \Theta \) be a compact metric space and let the set of aperiodic commuting automorphisms \( \{T_i\}_{i=1}^d \) be mutually ergodic with respect to a measure \( \mu \). Then \( \mathcal{S}_T \) is a dense \( G_\delta \) in the topology of \( (\mathcal{S}_F, d_\tau) \). Moreover, \( \mathcal{S}_F \) partitions into \( \mathcal{S}_T \sqcup \mathcal{H}_T \) and \( \mathcal{H}_T \) is dense in \( (\mathcal{S}_F, d_\tau) \).

**Remark** In fact, the placement of a function into \( \mathcal{S}_T \) or \( \mathcal{H}_T \) depends only on the behavior of the function near the zero set. In principle it is possible to sharpen the density claims by inserting a penalty, for example by a weighting, for differences near the zero set. However, for the sake of simplicity we will not introduce more metrics on the space than necessary.
For the quasiperiodic case in particular and for uniquely ergodic transformations more generally we can further characterize the long range phase for sampling functions in $S_F$.

**Theorem 1.2** For $T$ is such that at least one $T_i$ is uniquely ergodic then for any $\lambda$ we have $M_{h,\lambda} \equiv 0$ or there is some $m > 0$ so that uniformly $M_{h,\lambda}(\theta) \geq m$.

Note that our result does not apply to the random case and indeed is not true in that case due to the presence of Griffiths singularities.

The primary example our set up applies to the quasiperiodic case, i.e. $\delta$ defined by irrational rotations on the torus. That is, let $A \in M_{n \times d}$ with rationally independent column vectors. Let $\Theta = \mathbb{T}^n$, the $n$ dimensional torus, and let the group action be defined as $T^x \theta = Ax + \theta$ for $\theta \in \mathbb{T}^n$. Our results also apply to the skew shift, for $d = 1$, which is the process on $\mathbb{T}^2$ defined as $(\theta_1, \theta_2) \mapsto (\theta_1 + \alpha, \theta_1 + \theta_2)$ with irrational $\alpha$. More generally, our results apply to any minimal ergodic system.

A multiscale analysis is used in [20] to demonstrate localization in the quasiperiodic case. It was shown that if $h$ satisfies transversal conditions at $h = 0$ and is sampled over irrational rotations, then correlations fall off exponentially in $Z^d$. The transversality conditions depend on the number theoretic properties of the irrational frequency. The transversal conditions for $h$ are similar to the moment conditions for localization demonstrated in [23] in the random case, we will comment on this relation in Section 1.1.3.

### 1.1.2 Multiscale proof of localization

The multiscale analysis takes place in the Fortuin Kasteleyn (FK) representation of the ground state of (1.1). The FK representation in this case is a percolation model which takes place on $\mathbb{Z}^d \times \mathbb{R}$. We write $Q_{\delta,\lambda} ((x, t) \leftrightarrow (y, s))$ for the probability that $(x, t)$ and $(y, s)$ are in the same component given the transverse field $\delta$ and coupling $\lambda$. A complete definition of the model is given in Section 2.

The multiscale method relies on a transversality condition on $h$ at $h = 0$. To evaluate the behavior at the zero set, we define the following functions bounding the recurrence

$$
\zeta_T(k) = \frac{1}{2} \inf_{\theta \in \Theta} \left( \min_{0 < |x| \leq k} r(\theta, T^x \theta) \right), \quad \hat{\zeta}_T(k) = \sup_{\theta' \in \Theta} \left( \min_{0 < |x| \leq k} r(\theta, T^x \theta') \right). \tag{1.5}
$$

Notice that, as $\Theta$ is compact, and $r(\theta, T^x \theta) > 0$ by the aperiodicity assumption, $\zeta_T(k) > 0$ for all $k$, moreover clearly, $\hat{\zeta}_T(k) \geq \zeta_T(k)$. We define the transversality behavior with the following parameter,

**Definition** A sampling function $h \in S_F$ is **admitted** by a strictly increasing continuous function $Z : [0, 1) \to [0, 1)$, if there is some $r' > 0$ so that for $\theta$ so that $r(\theta, h^{-1}(0)) < r'$ the lower bound $h(\theta) \geq Z(r(\theta, h^{-1}(0)))$ holds.

A sampling function $h \in S_F$ is **restricted** by a strictly increasing continuous function $\tilde{Z} : [0, 1) \to [0, 1)$, if there is some $r'' > 0$ so that for $\theta$ so that $r(\theta, h^{-1}(0)) < r''$ the upper bound $h(\theta) \leq \tilde{Z}(r(\theta, h^{-1}(0)))$ holds.

Given $\nu, m > 0$, we say a pair of a sampling function and coupling value $(h, \lambda)$ satisfy property $C_{\nu, m}$ if for almost all $\theta \in \Theta$ and transverse field $\delta(x) = h(T^x \theta)$, for all $x \in \mathbb{Z}^d$ there is some $C_x$ so that for any $y \in \mathbb{Z}^d$ and $t, s \in \mathbb{R}$ so that $|x - y| + |\ln(1 + |t - s|)|^{1/\nu} > C_x$ we have

$$
Q_{\delta,\lambda} ((x, t) \leftrightarrow (y, s)) < \exp \left\{ -m \left( |x - y| + |\ln(1 + |t - s|)|^{1/\nu} \right) \right\}. \tag{1.6}
$$


We will first discuss the results for the FK representation of the one dimensional transverse field Ising model, with quasiperiodic \( \delta \). To prepare for this we will introduce necessary number theoretic properties of the frequency.

In one dimension, the rotations are given by \( T \theta = T_\omega \theta = \theta + \omega \) for \( \theta \in \Theta = \mathbb{T} \). For \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) we can write \( \omega \) in the continued fraction expansion

\[
\omega = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{1.7}
\]

where \( a_i \) are positive integers, and for simplicity we will consider \( a_0 = 0 \). We encode this expansion as \( \omega = [a_1, a_2, \ldots] \), let \( p_n/q_n \) be the fraction given by the truncation to the \( n \)th term, i.e. \( p_n/q_n = [a_1, \ldots, a_n] \). \( \omega \) is said to be of finite type if \( a_i \) are uniformly bounded, in particular, the Fibonacci number \( \omega = (\sqrt{5} - 1)/2 \) is of finite type with \( a_i \equiv 1 \). On the other hand \( \omega \) is \( \gamma \)-Diophantine for \( \gamma > 0 \) if there is \( C_\omega < \infty \) so that for all \( n \) we have \( q_{n+1} < C_\omega q_n^{1+\gamma} \). For any \( \gamma > 0 \), almost all real numbers are \( \gamma \)-Diophantine, \( \omega \) of finite type are \( \gamma \)-Diophantine for all \( \gamma > 0 \). For frequencies of finite type, we can specify a phase transition in the transversality condition.

In [20] it was shown that given \( T \) defined by any irrational \( \omega \) and \( h \) admitted by \( Z(r) = e^{-|\ln r|^\nu} \) for \( \nu > 1/3 \), then for any \( m > 0 \) and sufficiently small \( \lambda \) the pair \( h, \lambda \) satisfy the \( C_{\nu, m} \) property with \( \nu = 1 \). Moreover, in [20], if \( \omega \) is \( \gamma \)-Diophantine a \( C_{\nu, m} \) property is shown to hold for \( Z(r) \) admitted \( h \) and small enough \( \lambda \) if \( (1 + \gamma)a < \nu < 1 \).

**Theorem 1.3** Let \( \Theta = \mathbb{T} \) be the one dimensional torus and \( d = 1 \) and let the group \( \{ T^x : x \in \mathbb{Z} \} \) be generated by \( T^0 \theta = \theta + \omega \) for some \( \omega \in \mathbb{R} \setminus \mathbb{Q} \).

1. If there is some point \( \theta_0 \in \mathbb{T} \) so that there exists \( a < 1 \) and sampling function \( h \) obeys

\[
\liminf_{\theta \to \theta_0} \frac{\log |\log h(\theta)|}{\log r(\theta, \theta_0)} \geq \frac{1}{a}
\]

then, for any irrational frequency \( \omega \), \( h \in H_T \).

2. On the other hand, if \( \omega \) is a rotation on \( \mathbb{T} \) of finite type, and and sampling function \( h \) obeys

\[
\limsup_{\theta \to \theta_0} \frac{\log |\log h(\theta)|}{\log r(\theta, \theta_0)} \leq a
\]

for \( a < 1 \) then \( h \in S_T \). Moreover, given \( m > 0 \) and \( \nu < 1 \), there is \( \lambda_m \) so that, for \( \lambda < \lambda_m \), the pair \( h, \lambda \) satisfies condition \( C_{\nu, m} \).

3. If \( \omega \) is a rotation on \( \mathbb{T} \) so that there exists \( b < \infty \) so that for large \( n \), \( q_{n+2} < e^{b^n} \), then, if \( a < b^{-1} \) and \( h \in S_T \) is admitted by \( Z(r) = e^{-|\ln r|^\nu} \) then \( h \in S_T \). Moreover, for any \( \frac{ab + 1}{2} < \nu < 1 \) and \( m > 0 \), there is \( \lambda_m \) so that for \( \lambda < \lambda_m \), the pair \( h, \lambda \) satisfy the \( C_{\nu, m} \) property.

The proof of (2.) is contained in [20], taken with (1.) it demonstrates a critical disorder at \( a = 1 \) for rotations of finite type. The final item (3.) extends faster than power law localization beyond Diophantine rotations demonstrated in [20].

These results hold as a corollary of the analysis in Section 4 and the following general theorems establishing conditions on localization and long range order.

Assume \( T \) is aperiodic and has recurrence bound \( \zeta_T \). The following theorem states a sufficient condition for a sampling function \( h \) to admit a localized phase, which, for an admitting function \( Z \), is specified by a condition on the function \( \Psi = Z \circ \zeta_T \).
Theorem 1.4 Suppose, for some $\xi < 1/d$, $\Psi$ satisfies
\[\limsup_{k \to \infty} \frac{\log |\log \Psi(k)|}{\log k} \leq \xi\] (1.8)
and let $\nu$ satisfy $\frac{1+\xi}{1+\nu} < \nu < 1$. Then, for any $\infty > m > 0$ there is $\nu_m$ so that for $0 < \lambda < \nu_m$, the pair $h, \lambda$ satisfy property $C_{\nu,m}$ defined in (1.6).

A complementary parameter provides a condition for long range order. In this case, we assume the sampling function is restricted by the function $\hat{Z}$. Thus we have long range order under a ‘large defect’ assumption specified by the function $\hat{\Psi}^\theta = \hat{Z} \circ \hat{\zeta}^\theta_T$.

Theorem 1.5 Suppose, $h(\theta) = 0$ and for some $\xi > 1$, $\hat{\Psi}^\theta$ satisfies
\[\limsup_{k \to \infty} \frac{\log |\log \hat{\Psi}(k)|}{\log k} \geq \xi.\] (1.9)
Then for every $\lambda > 0$ and every $\theta \in \Theta$, $M(\theta, \lambda) > 0$.

1.1.3 Phase transition in the random case

Localization in the disordered transverse field Ising model with i.i.d. parameters $\lambda(x_i,y_i)$ and $\delta_i$, was shown first shown by Campanino and Klein [6] by a multiscale analysis argument. The multiscale argument was optimised by Klein [23], where it is shown that for $\beta > \beta_d = 2d^2(1 + \sqrt{1 + \frac{1}{d} + \frac{1}{2d}})$ if the moment conditions
\[\langle \ln (1 + \frac{1}{\delta})^\beta \rangle < \infty; \quad \langle \ln(1 + \lambda)^\beta \rangle < \infty\] (1.10)
are satisfied then almost sure localization holds, with correlation functions decaying exponentially space wise provided a certain low density assumption
\[\left\langle \left( \ln \left( 1 + \frac{\lambda}{\delta} \right) \right)^\beta \right\rangle < \epsilon,\] (1.11)
for some sufficiently small $\epsilon > 0$. In the case that the distributions of the cut rate $\delta$ are in a strong disorder regime for any $d \geq 1$, then for any $\lambda > 0$ there is a unique infinite component almost surely Aizenman, Klein and Newman [1]. For $d = 1$ the strong disorder condition is
\[\lim_{u \to \infty} \frac{u}{|\ln(u)|} P \left( \left\{ \ln \left( 1 + \frac{1}{\delta} \right) > u \right\} \right) = \infty; \quad E_P \left( \delta + \frac{1}{\lambda} \right) < \infty.\] (1.12)
For $d \geq 2$ the strong disorder condition is
\[\lim_{u \to \infty} u^d P \left( \left\{ \ln \left( 1 + \frac{1}{\delta} \right) > u \right\} \right) = \infty.\] (1.13)
As indicated by Theorems [1,4] and [1,3] the disorder conditions are similar in aperiodic systems, there is a ‘low density’ phase for small $\lambda$ for generic aperiodic order. And persistent high density phase only for pathological behavior of the distribution near zero.
Let us compare these results to the dynamically defined system. In the quasiperiodic system, discussed in Theorem 1.3, the ergodic measure \( \mu \) is just the Lebesgue measure, conclusion (1.) states that \( h \in \mathcal{H}_T \) corresponds to existence of \( a < 1 \) so that \( \lim_{m \to \infty} m^a \mu(\{ \theta : \log[1 + h^{-1}(\theta)] > m \}) = \infty \). On the other hand, conclusion (2.) implies that \( h \) in which admits a localized phase corresponds to \( \mu(\{ \theta : \log[1 + h^{-1}] > m \}) < m^{−1/a} \) for some \( a < 1 \). Thus if \( a^{-1} > \beta > 1 \) then \( \mu(\{ \theta : \log[1 + h^{-1}] > m \}) < m^{−[\beta]} \) then \( \mathbb{E} [\log \beta(1+h^{-1})] < \infty \) which corresponds to (1.10). Notice the requirement \( \beta > 1 \) is less stringent than the requirement \( \beta > \beta_d \).

### 1.1.4 Organization of the paper

The rest of the paper is organized as follows. In Section 2 we introduce the continuum random cluster model and relate it to the quantum Ising model. We begin Section 3 by proving Theorems 1.1 which follow from Theorems 3.1 and 3.2. Theorem 3.1 follows from Propositions 3.3 and Proposition 3.4. In Section 3.1 we show regularity properties of the continuum percolation model, in Section 3.2 we prove Proposition 3.4. Theorem 1.2 follows from Proposition 3.11 in Section 3.2. We prove Proposition 3.3 in Section 3.3.

We carry out the multiscale analysis in Section 4.1 and relate it to the dynamics in Section 4.2. We conclude the paper applying the results to the phase transition in the quasiperiodic case and prove Theorem 1.3 in Section 4.4.

### 2 FK representation

Quantum spin models may be related by a Fortuin Kasteleyn representation to a percolation processes called the continuous time random cluster model. Measurements of observables in the Ising model, such as the correlation function \( \langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle \) are equal to communication probabilities in the continuous time random cluster model.

The continuous time random cluster model can be defined in terms of a product measure percolation \( \mathcal{Z} \), which we will introduce first. Despite the name, the ‘time’ dimension in the model is non-oriented. The oriented version of continuous time percolation is the well known contact process, essentially, allowing travel in the negative time direction recovers continuous-time percolation. Moreover, there are stochastic dominations between the continuum random cluster model and continuous-time percolation, which become useful for demonstrating localization and percolation in various regimes.

We will take all boundary conditions, for the random cluster measure and the quantum spin model to be free.

**Continuous-time percolation** We begin with the graph \( \mathbb{L} = (\mathbb{Z}^d, \mathbb{E}^d) \), where \( \mathbb{E}^d \) is the set of nearest neighbor pairs \( \{x, y\} \) in \( \mathbb{Z}^d \) so that \( |x - y|_1 = 1 \). The inhomogeneous continuous time percolation process on \( \mathbb{L} \) is defined via the parameters \( \delta : \mathbb{Z}^d \to [0, \infty) \) and \( \lambda > 0 \) which we will call the environment. For every element \( x \in \mathbb{Z}^d \) there is a Poisson process of deaths on \( \{x\} \times \mathbb{R} \) at rate \( \delta_x \), and for every edge \( \{x, y\} \in \mathbb{E}^d \), there is a Poisson process of bonds at a rate \( \lambda \) on \( \{x, y\} \times \mathbb{R} \). The measure of the Poisson process of deaths on \( \{x\} \times \mathbb{R} \) will be denoted by \( \bar{Q}^x_\delta, \lambda \), similarly the Poisson process of bonds for any \( u \in \mathbb{E}^d \) will be labeled \( \bar{Q}^u_\delta, \lambda \). The space of realizations for the Poisson measures \( \bar{Q}^u_\delta, \lambda \) for each \( \ast \in \mathbb{Z}^d \), respectively \( \ast \in \mathbb{E}^d \), is the set of all locally finite sets of points in \( \{x\} \times \mathbb{R} \) denoted \( \Omega_x \), respectively locally finite sets in \( u \times \mathbb{R} \) denoted \( \Omega_u \). Any locally finite set \( \omega \)
in \((\mathbb{Z}^d \cup \mathbb{E}^d) \times \mathbb{R}\) is called a configuration, and the space of all configurations is denoted \(\Omega\), it is the product of all sets \(\Omega_v\), that is \(\Omega = (\times_{x \in \mathbb{Z}^d} \Omega_x) \times (\times_{u \in \mathbb{E}^d} \Omega_u)\). The percolation measure on \(\Omega\) is the product measure of these Poisson processes,

\[
Q_{\delta,\lambda} = \left( \prod_{x \in \mathbb{Z}^d} \overline{Q}_{\delta,\lambda}^x \right) \left( \prod_{u \in \mathbb{E}^d} \overline{Q}_{\delta,\lambda}^u \right).
\]

(2.1)

For any configuration \(\omega \in \Omega\), two points \(X, Y \in \mathbb{Z}^d \times \mathbb{R}\) communicate in \(\omega\) if there is a path

\[ X = W_0, W_1, W_2, \ldots, W_m = Y \]

so that, for \(1 \leq i \leq m\), \(W_i = (x_i, t_i) \in \mathbb{Z}^d \times \mathbb{R}\); either \(t_i = t_{i-1}\) and there is a bond \(\{x_i, x_{i-1}\}, t_i \in \omega\), or \(t_i \neq t_{i-1}\) but \(x_i = x_{i-1}\) and \(\omega\) has no cut in the interval \(\{x\} \times \mathbb{R}\) between \(t_i\) and \(t_{i-1}\). Note we allow both \(t_i < t_{i-1}\) and \(t_{i-1} < t_i\). Equivalently, if we consider \(\mathbb{Z}^d \times \mathbb{R} \subset \mathbb{R}^{d+1}\). All configurations \(\omega \in \Omega\) have a graphical representation \(\Gamma(\omega)\), similar to the graphical representation of the contact process, formed by removing all cuts \((x, t) \in \omega\) from \(\mathbb{Z}^d \times \mathbb{R}\), and for every bond \((\{x, y\}, t) \in \omega\), adding a line segment between \((x, t)\) and \((y, t)\). Thus, two points \(X, Y \in \mathbb{Z}^d \times \mathbb{R}\) communicate in \(\omega\) if they are in the same path connected subset of \(\Gamma(\omega)\). For \(X \in \mathbb{Z}^d \times \mathbb{R}\) the cluster \(C_\omega(X)\) for \(\omega \in \Omega\) is the set of points \(Y \in \mathbb{Z}^d \times \mathbb{R}\) in the path connected component containing \(X\). When it is not necessary to emphasize the configuration \(\omega\) we write \(C(X)\) for \(C_\omega(X)\). For \(X, Y \in \mathbb{Z} \times \mathbb{R}\), we write \(X \leftrightarrow Y\) if \(Y \in C(X)\); similarly for subsets \(W_1, W_2 \subset \mathbb{Z} \times \mathbb{R}\) we write \(W_1 \leftrightarrow W_2\) if there are \(X \in W_1\) and \(Y \in W_2\) so that \(Y \in C(X)\).

The proper topology of \(\Theta\) is the Skorohod topology, roughly speaking a configuration \(\omega\) is close to \(\omega'\) if bonds and cuts in the configurations are close on bounded sets. For a detailed construction of the Skorohod topology in general see [14]. For further background in the context of the FK representation see [15].

The continuous time random cluster model The continuous time random cluster measure is first constructed on bounded subsets in \(\mathbb{Z}^d \times \mathbb{R}\). For a subset \(W \subset \mathbb{Z}^d\), set the boundary to be

\[ \partial W = \{ y \in W | \exists y' \in W^c \text{ so that } (y, y') \in \mathbb{E} \} \]

A cylinder of \(W\) is defined as \(B = W \times I \subset \mathbb{Z}^d \times \mathbb{R}\) for some interval \(I = [a, b] \subset \mathbb{R}\). Let the horizontal boundary be \(\partial_H B = W \times \{a, b\}\) and let the vertical boundary be \(\partial_V B = \partial W \times I\) the boundary of \(B\) is then \(\partial B = \partial_H B \cup \partial_V B\).

The number of deaths in a bounded box is almost surely finite so we may define \(k_B : \Omega \to \mathbb{Z}^+\) the function counting the number of clusters in \(B = \Lambda \times [-T, T]\) for almost every \(\omega \in \Omega\). To see that \(k_B\) is a.s. finite notice that the number of clusters is bounded by \(|\Lambda| + K\) where \(K\) is the total number of deaths in \(B\). Let \(K = \max_{x \in \Lambda} \delta(x)\), then \(K\) is bounded by a Poisson random variable with rate \(2TK|\Lambda|\). Thus, for \(q > 0\), \(q^{k_B(\cdot)} \in L_1(\Omega|B, Q_{\delta,\lambda}|B)\). In fact, it is an easy calculation, by the above observation, to show

\[
Q_{\delta,\lambda}|_B(q^{k_B}) \leq Q_{\delta,\lambda}|_B(q^{K+|\Lambda|}) \leq q^{|\Lambda|} \exp\{(q - 1) \times 2TK|\Lambda|\}
\]

(2.2)

So we can define the continuous time random cluster measure, on a bounded \(B\), by

\[
Q_{\delta,\lambda}\ {}^{(q)}|_B(A) = \frac{\int_{\Omega} 1_A(\omega) q^{k_B(\omega)} dQ_{\delta,\lambda}(\omega)}{\int_{\Omega} q^{k_B(\omega)} dQ_{\delta,\lambda}(\omega)}.
\]

Note that if we set \(q = 1\) we recover the independent percolation model.
Relations of the measures  The following is a standard relation of the random cluster model and the quantum Ising model and the motivation for introducing the random cluster model, (see, for example, [19])

\[ \langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle = \lim_{B \to \mathbb{Z}^d \times \mathbb{R}} Q^{(2)}_{\delta,\lambda | B} \{ (x, 0) \leftrightarrow (y, 0) \}. \] (2.3)

where the limit is over a sequence \( B \) of increasing subsets of the infinite lattice \( \mathbb{Z}^d \times \mathbb{R} \).

The set of configurations \( \Omega \) has a partial ordering property. Let \( \omega, \omega' \in \Omega \) if every bond in \( \omega \) is in \( \omega' \) and every death in \( \omega' \) is in \( \omega \) then \( \omega \) and \( \omega' \) are ordered and we write \( \omega \leq \omega' \). A set \( U \subset \Omega \) is positive if \( \omega \in U \) and \( \omega \leq \omega' \) together imply \( \omega' \in U \). Notice that \( C_{\omega}(X) = C_{\omega}(Y) \) and \( \omega \leq \omega' \) imply \( C_{\omega'}(X) = C_{\omega'}(Y) \), so that communication events are positive. The measures \( Q \) on \( \Omega \) enjoy a partial ordering property as well called stochastic ordering. Two measures \( Q \) and \( Q' \) have an ordering \( Q \leq Q' \) if, for all positive sets \( U \), \( Q(U) \leq Q'(U) \).

The random cluster models have the following orderings.

**Proposition 2.1** Let \( q \geq q' \geq 1 \), \( \lambda, \lambda' \in \mathbb{R}^+ \) and \( \delta, \delta' : \mathbb{Z}^d \to \mathbb{R}^+ \). If \( \lambda' \geq \lambda \) and \( \delta' \leq \delta \) we have the ordering,

\[ Q^{(q)}_{\delta,\lambda} \leq Q^{(q')}_{\delta',\lambda'} \] (2.4)

And if \( \lambda' \leq \lambda q' / q \) and \( \delta' \geq \delta q / q' \)

\[ Q^{(q')}_{\delta',\lambda'} \leq Q^{(q)}_{\delta,\lambda} \] (2.5)

This ordering is similar to the ordering for the measure of the discrete random cluster model [17], a proof of Proposition 2.1 in the continuous context can be found in [4]. For our purposes, communication events \( A = \{ \omega \mid C_{\omega}(X) = C_{\omega}(Y) \} \) are the relevant positive events.

We also find it useful to recall the FKG inequality describing intersections of positive events, given two positive events \( U, V \), we have for any \( \delta : \mathbb{Z}^d \to \mathbb{R}^+ \) and \( \lambda > 0 \) for any \( q \geq 1 \)

\[ Q^{(q)}_{\delta,\lambda} (U \cap V) \geq Q^{(q)}_{\delta,\lambda}(U) Q^{(q)}_{\delta,\lambda}(V). \] (2.6)

Again, this result is similar to the inequality for the discrete model [17].

### 3 Regularity of random cluster measures

The main goal of this section is to show the Borel regularity of \( \mathcal{H}_T \) and \( \mathcal{S}_T \). Essentially the strategy is to show regularity of the percolation measures with respect to \( \delta \) and \( \lambda \) in finite subsets and extend these properties to the infinite lattice.

**Theorem 3.1** \( \mathcal{S}_T \) is a \( G_\delta \) in the topology of \( (\mathcal{S}_F, d_T) \) and \( \mathcal{S}_F \) is partitioned into \( \mathcal{S}_T \sqcup \mathcal{H}_T \).

We state conditions for an \( \mathcal{S}_F \) sampling function to satisfy condition \( C_{\nu,m} \).

**Theorem 3.2** Let \( T \) be any aperiodic environment process and \( F \in \mathcal{F} \) a finite set in \( \Theta \). Then, for any \( \nu \) such that \( \frac{1}{1+1/d} < \nu < 1 \), there exists a function \( \psi_F \in \mathcal{S}_F(\Theta) \) so that for sufficiently small \( \lambda \), the pair \( \psi_F, \lambda \) have the property \( C_{\nu,m} \).
Let \( B_n = \{ x \in \mathbb{Z}^d : |x| \leq L_n \} \times [-T_n, T_n] \) for some non decreasing sequences \( L_n \) and \( T_n \). Let
\[
\widehat{M}^{(n)}_{q; h, \lambda}(\theta) = Q_{h, \theta, \lambda}^{(q)}(\{0, 0\} \leftrightarrow \partial B_n)).
\]
We define the limit, \( \widehat{M}_{q; h, \lambda}(\theta) = \lim_{n \to \infty} \widehat{M}^{(n)}_{q; h, \lambda}(\theta) \). For fixed \((h, \theta, \lambda)\) we define the set of phases defining absence of long range order to be
\[
\Theta_0 = \Theta_0(h, \lambda) = \{ \theta \in \Theta : \widehat{M}_{q; h, \lambda}(\theta) = 0 \}.
\]
As the analysis is based on a random cluster expansion, our conclusions hold for general \( q \)-Potts type models. We state our results in this context.

**Proposition 3.3** For any \( q \geq 1 \), the set of \( h \in S_T \) so that there exists a \( \lambda \) so that \( \mu(\Theta_0(h, \lambda)) = 1 \) is a \( G_\delta \) in the \( (S_T, d_r) \) topology.

**Remark** A similar result holds for non-normalized functions, however, the topology would include the coupling parameter, thus define a metric \( \rho\{(h, \lambda), (h', \lambda')\} = m_T(h, h') + |\lambda - \lambda'| \). In this case, a similar proof would show that the set of \((h, \lambda) \in S_T \times \mathbb{R}^+ \) so that \( \mu(\Theta_0(h, \lambda)) = 1 \) is a \( G_\delta \) in the \((S_T \times \mathbb{R}^+, \rho)\) topology.

Let us define \( \widehat{\rho}\{(h, \theta, \lambda), (h', \theta', \lambda')\} = d_S(h, h') + |\lambda - \lambda'| + r(\theta, \theta') \).

**Proposition 3.4** For any \( q \geq 1 \), \( \widehat{M}_{q; h, \lambda}(\theta) \) is upper semicontinuous in \( \widehat{\rho} \). Moreover, for any \( q \geq 1 \), \( \lambda > 0 \), \( h \in C^+(\Theta) \), we have \( \mu(\Theta_0) = 0 \) or \( \mu(\Theta_0) = 1 \).

From these propositions we can prove Theorem 3.1.

**Proof** From (2.3) we have \( \widehat{M}_{2, h, \lambda}(\theta) \geq M_{h, \lambda}(\theta) \). On the other hand, with \( Q_{h, \theta, \lambda}^{(q)} \) probability one there exists at most 1 infinite cluster, this is a standard fact in many percolation models, for the result in the context of iid \( \delta \) see [1], a proof for ergodic \( \delta \) is similar. Thus, by the FKG inequality,
\[
Q_{h, \theta, \lambda}^{(q)}\{(x, 0) \leftrightarrow (y, 0)\} \geq \widehat{M}_{q, h, \lambda}(\mathbb{T}^x \theta) \widehat{M}_{q, h, \lambda}(\mathbb{T}^y \theta).
\]
But again by (2.3) the left hand side is \( \langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle \) thus, by choosing a sequence of \( y_i \) moving to infinity so that \( \widehat{M}_{2, h, \lambda}(\mathbb{T}^y \theta) \to \sup_{\theta} \widehat{M}_{2, h, \lambda}(\theta) \) we have
\[
\widehat{M}_{2, h, \lambda}(\theta) \geq M_{h, \lambda}(\theta) \geq \widehat{M}_{2, h, \lambda}(\theta) \|M_{h, \lambda}\|.
\]
On the other hand, from Proposition 3.4 we have \( \mu(\cap_{\lambda > 0} \Theta_0(h, \lambda)) \) is 0 or 1. It follows that every \( h \in S_T \) belongs to \( S_T \) or \( H_T \). The statement that \( S_T \) is a \( G_\delta \) is exactly the second statement of Proposition 3.3.

### 3.1 Continuity of measures

This section is devoted to showing communication events for fixed \( q \geq 1 \) are continuous in the space of continuous sample functions, \( C^+(\Theta) \). The culmination of the results of this section is contained in Proposition 3.5 which follows from Lemma 3.9 and Proposition 3.10.

**Proposition 3.5** Let \( A = \{ \omega | C_\omega(X) = C_\omega(Y) \} \) or \( A = \{ W_1 \leftrightarrow W_2 \} \) for bounded \( W_1 \) and \( W_2 \) be a communication event. Given \((h, \theta, \lambda)\) and \( \epsilon \), there is some \( \eta > 0 \) so that \( d_S(h, h') + |\lambda - \lambda'| + r(\theta, \theta') < \eta \) implies that
\[
|Q_{(h', \theta'), \lambda'}^{(q)}(A) - Q_{(h, \theta), \lambda}^{(q)}(A)| < \epsilon
\]
3.1.1 regularity with respect to parameters

Consider bounded sets $\Lambda \subset \mathbb{Z}^d$. Let us introduce a topology on the environments $(\delta, \lambda)$ where $\delta : \Lambda \to (0, \infty)$ $\lambda > 0$,

$$
\| (\delta, \lambda) - (\delta', \lambda') \|_\ell = |\lambda - \lambda'| + \sup_{x \in \mathbb{Z}^d} |\delta'(x) - \delta(x)| 2^{-|x|}
$$

the local convergence of environments.

We will show weak convergence under this metric, that is, the measures of sets of continuity converge as $(\delta_k, \lambda_k) \to (\delta, \lambda)$. By sets of continuity we mean $U \subset \Omega$ so that $\partial U$ has zero measure for all measures in some tight family.

The following propositions and their proofs are similar in spirit to [17] (also see [4] for development in the continuum case). We need some modifications since we are considering continuity in environments. In the following we may let $B(n)$ be any sequence of increasing sets converging to $\mathbb{Z}^d \times \mathbb{R}$, for convenience one may simply take $B(n) = \{(x, t) : \|x\| \leq n; |t| \leq n\}$.

**Proposition 3.6** Let $0 < K < \infty$, and $\delta_k : \mathbb{Z}^d \to (0, K)$, $0 < \lambda_k < K$ then $Q_{\delta_k;\lambda_k}^{(q)}|_{B(n)}$ is a tight family.

**Proof** Recall that a family of measures $\mathcal{M}$ on $\Omega$ is a tight family if, for every $\epsilon > 0$, there exists a set $R_\epsilon \subset \Omega$, compact in the Skorohod topology, so that

$$
\inf_{P \in \mathcal{M}} P(R_\epsilon) > 1 - \epsilon.
$$

We have $\delta_k(x) < M$ for all $k$ and $x$ so by Proposition 2.1 for all $k$, $Q_{\delta_k;\lambda_k}^{(q)} \geq Q_{\delta_k;\lambda_k}^{(q)} \geq Q_{\delta_k;\lambda_k}^{(q)}$. We introduce a function $\xi_u : \mathbb{Z} \to \mathbb{R}^+$, for all $u \in \mathbb{Z}^d \cup \mathbb{E}^d$, to be specified. Let $V'_x \subset \Omega|_x \times \mathbb{R}$ be the event that for all $j$ cuts in $\omega \cap (\{x\} \times [j, j+1))$ are separated by at least $\xi_u(j)$. Define similarly $V'_{x,y}$ the set with bonds spaced out by $\xi_{x,y}(j)$. The closure $V_u$ of $V'_u$ is compact [14] (Theorem 3.6.3). For $\xi_u(j)$ decreasing quickly enough $V_x$ has probability greater than $1 - \frac{1}{2d(|x|+1)^{d+1}}; similarly let $\xi_{x,y}(j)$ decrease quickly so that (let $|x| \geq |y|$) $V(x,y)$ has probability greater than $1 - \frac{1}{2d(|x|+1)^{d+1}}$.

By the ordering of measures and the observation that $V_{x;\mathbb{Z}^d} = \cap_{x \in \mathbb{Z}^d} V_x$ is an increasing set, we have for all $n$,

$$
Q_{\delta_k;\lambda_k}(V_{x;\mathbb{Z}^d}) \geq Q_{\delta_k;\lambda_k}(V_{x;\mathbb{Z}^d}) \geq 1 - \epsilon/2.
$$

On the other hand $V_{x;\mathbb{Z}^d} = \cap_{x \in \mathbb{Z}^d} V_x$ is decreasing, so

$$
Q_{\delta_k;\lambda_k}^{(q)}(\Omega \setminus V_{x;\mathbb{Z}^d}) \leq Q_{\delta_k;\lambda_k}(\Omega \setminus V_{x;\mathbb{Z}^d}) \leq \epsilon/2.
$$

Finally the set $V_x = V_{x;\mathbb{Z}^d} \cap V_{x;\mathbb{Z}^d}$ is compact and $Q_{\delta_k;\lambda_k}^{(q)}(V_x) \geq 1 - \epsilon$ for all $k$ and $n$. \qed

We refer to [4] for the following fact.

**Proposition 3.7** Let $0 < K < \infty$, and $\delta : \mathbb{Z}^d \to (0, K)$, there is some limit $Q_{\delta;\lambda}$ so that $Q_{\delta_k;\lambda_k}^{(q)}|_{B(n)} \to Q_{\delta;\lambda}^{(q)}$ weakly.

We consider sequences $\delta_k : \mathbb{Z}^d \to (0, K)$ and $0 < \lambda_k < K$. We will show weak convergence of the measures as $(\delta_k, \lambda_k)$ to $(\delta, \lambda)$ in the $\| \cdot \|_\ell$. First we will show weak convergence for bounded subsets.

**Proposition 3.8** Let $0 < K < \infty$, and $\delta : \mathbb{Z}^d \to (0, K)$. $Q_{\delta_k;\lambda_k}^{(q)}|_B \to Q_{\delta;\lambda}^{(q)}|_B$ weakly.
Proof It is enough to show the finite dimensional distributions converge \(3\) (Theorem 12.6). The finite dimensional distributions in this case are events counting the number of bonds and cuts in bounded intervals, that is events of the type,
\[
U^r_{\xi_1,\ldots,\xi_n} = \{ \omega : |\omega \cap \{z_i \times (-t_i, s_i)\}| = r_i, i = 1, \ldots, n \},
\]
where \(z_i, s_i \in \mathbb{Z}^d \cup \mathbb{E}, t_i, s_i \in \mathbb{R}^+\) and \(r_i \in \mathbb{Z}^+\). The difficulty is in the case \(q > 1\) as the case \(q = 1\) is simply a product of Poisson distributions. Recall, for any \(\epsilon > 0\) we define \(V_\xi\) to be the compact event from Proposition 3.6 so that \(|V_\xi| > 1 - \epsilon\); the spacing of cuts implies for any bounded \(B \subset \mathbb{Z}^d \times \mathbb{R}\), there is some \(K_\xi < \infty\) so that \(k_B(\omega) < K_\xi\) for any \(\omega \in V_\xi\).

For \(\eta > 0\) and \(\xi\) as above, let \(f_{\xi, \eta}\) be a continuous function on \(V_\xi\) approximating \(k_B\). \(k_B\) takes on positive integer values and only may change value where a cut ‘moves past’ a bond, that is, \(k_B\) is discontinuous at \(\omega\) only if there are \(x, y, t\) so that \((x, t) \in \omega\) and \((\{x, y\}, t) \in \omega\). Therefore for \(1/2 > \eta > 0\) we can require \(f_{\xi, \eta}\) to be bounded by \(k_B\) and equal to \(k_B\) for \(\omega\) such that, for any \(x, y, t, s\) so that \((x, t) \in \omega\) and \((\{x, y\}, s) \in \omega\) then \(|s - t| > \eta \xi(t)|\).

As \(\epsilon, \eta \to 0\), by dominated convergence and (2.2), \(I_{\xi, \eta} = \text{Q}_{\delta_k, \lambda} \equiv |B| (|q^{f_{\xi, \eta}} - q^{k_B}|) \to 0\). In fact, convergence of \(I_{\xi, \eta}\) to 0 is uniform in \(\delta_k\). Indeed, for any \(\epsilon\) let \(\xi\) be chosen so that \(Q_{K, \lambda} (q^{K\xi}; V_\xi) < \epsilon/2\). Similarly, let \(f_{\xi, \eta}\) be chosen so that \(O = \{ \omega \in V_\xi : f_{\xi, \eta} \neq k_B \}\), an open set in \(V_\xi\) so that \(Q_{K, \lambda} (q^{K\xi}; O) < \epsilon/2\). Then, since \(V_\xi\) is compact, the set \(V_\xi = V_\xi \cap O^c\) is compact, and \(Q_{K, \lambda} (q^{K\xi}; V_\xi^c) < \epsilon\). Thus, for \(f_\xi = k_B\) on \(V_\xi\) and bounded by \(k_B\) on \(V_\xi^c\), then we have for all \(k\)
\[
\text{Q}_{\delta_k, \lambda} \equiv (|q^{k_B} - q^{f_\xi}|) \leq \text{Q}_{K, \lambda} (|q^{k_B} - q^{f_\xi}|) < 2\epsilon.
\]

Therefore, using weak convergence for such continuous functions, \(\text{Q}_{\delta_k, \lambda} \equiv |B| (g) \to \text{Q}_{\delta, \lambda} \equiv |B| (g)\) for \(g = q^{f_\xi}\) or \(g = q^{k_B} 1_U\), for a set \(U\) as in (3.1). Thus,
\[
\left| \text{Q}_{\delta_k, \lambda} \equiv |B| (U) - \text{Q}_{\delta, \lambda} \equiv |B| (U) \right| = \left| \frac{\text{Q}_{\delta_k, \lambda} \equiv |B| (1_U q^{k_B})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{k_B})} - \frac{\text{Q}_{\delta, \lambda} \equiv |B| (1_U q^{k_B})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{k_B})} \right|
\]
\[
\leq \left| \frac{\text{Q}_{\delta_k, \lambda} \equiv |B| (1_U q^{f_\xi})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{f_\xi})} - \frac{\text{Q}_{\delta, \lambda} \equiv |B| (1_U q^{f_\xi})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{f_\xi})} \right| + \left| \frac{\text{Q}_{\delta_k, \lambda} \equiv |B| (1_U q^{k_B})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{k_B})} - \frac{\text{Q}_{\delta, \lambda} \equiv |B| (1_U q^{k_B})}{\text{Q}_{\delta, \lambda} \equiv |B| (q^{k_B})} \right|
\]

The second term vanishes by sending \(k \to \infty\), by weak convergence of the independent percolation model. The first and third term vanish by sending \(\epsilon, \eta \to 0\) by the dominated convergence theorem. □

Lemma 3.9 Let \(q \geq 1\) and let \(0 < K < \infty\), and \(\delta, \delta_k : \mathbb{Z}^d \to (0, K)\) so that \(\| (\delta_k, \lambda_k) - (\delta, \lambda) \|_\ell \to 0\). Then for any \(U\) which is an increasing set of continuity, and \(q \geq 1\),
\[
\text{Q}_{\delta_k, \lambda_k} (U) \to \text{Q}_{\delta, \lambda} (U)
\]

Here these infinite volume measures are limits guaranteed to exist by Proposition 3.7. This extends the conclusion of Proposition 3.8 to the infinite volume \(\mathbb{Z}^d \times \mathbb{R}\).

Proof By Proposition 3.8 we have for all \(n\)
\[
\text{Q}_{\delta_k, \lambda_k} \equiv |B(n)| (U) \to \text{Q}_{\delta, \lambda} \equiv |B(n)| (U).
\]
We use a diagonalization argument. Let \((k^1_i)\) be a sequence so that
\[
\left| Q_{\delta,i:1;\lambda}^{(q)}|_{B(1)}(U) - Q_{\delta,i;\lambda}^{(q)}|_{B(1)}(U) \right| < i^{-1}.
\]
And let \((k^{n+1}_i)\) be a subsequence of \((k^n_i)\) so that
\[
\left| Q_{\delta,i+1:1;\lambda}^{(q)}|_{B(n+1)}(U) - Q_{\delta,i;\lambda}^{(q)}|_{B(n+1)}(U) \right| < i^{-1}.
\]
Let \(\ell(i) = k^1_i\) then
\[
\left| Q_{\delta,\ell(i):\lambda(i)}^{(q)}(U) - Q_{\delta;\lambda}^{(q)}(U) \right| \leq \left| Q_{\delta,\ell(i):\lambda(i)}^{(q)}(U) - Q_{\delta,\ell(i):\lambda(i)}^{(q)}|_{B(n)}(U) \right| + \left| Q_{\delta,\ell(i):\lambda(i)}^{(q)}|_{B(n)}(U) - Q_{\delta,\lambda}^{(q)}|_{B(n)}(U) \right| + \left| Q_{\delta;\lambda}^{(q)}|_{B(n)}(U) - Q_{\delta;\lambda}^{(q)}(U) \right|.
\]
The last term is small for large \(n\) by Proposition 3.7. Then the first term is less than \(1/n\) for all \(i \geq n\) by (3.2) and the definition of \(\ell(i)\). Finally, the second term can be made small for large \(i\).

The boundary of communication events has measure zero:

**Proposition 3.10** Let \(A = \{\omega | C_\omega(X) = C_\omega(Y)\}\) or \(A = \{W_1 \leftrightarrow W_2\}\) for bounded \(W_1\) and \(W_2\) be a communication event. For any \(q \geq 1\) and cut parameter \(\delta\) and \(\lambda > 0\), the communication event \(A_B\) for \(B\) bounded in \(\mathbb{Z}^d \times \mathbb{R}\) is a set of continuity.

**Proof** First let \(q = 1\). We define two events \(D_i\), \(i = 1, 2\). The first event \(D_1\) is the case that along some line \(\{s\} \times \mathbb{R}\) for \(s \in \mathbb{Z}^d \cup \mathbb{R}^d\) \(\omega\) has an accumulation point of cuts or bonds. And the second, \(D_2\) is that \(\omega\) has a bond and cut which coincide, that is there is some time \(t \in \mathbb{R}\) and pair \((x, y)\) so that \((x, t)\) is a death and \((\{x, y\}, t)\) is a bond in \(\omega\). Observe that, the boundary of \(A\) is contained in the union, \(D_1 \cup D_2\).

To see that \(Q_{\delta,\lambda}(D_1) = 0\) notice this follows from the construction of Proposition 3.6 as for any \(\epsilon\) the associated \(\xi\) has \(D_1 \subset V_\epsilon\) and \(V_\epsilon\) has measure \(1 - \epsilon\) so the union (over \(\xi\) chosen for \(\epsilon > 0\)) \(\cup_\epsilon V_\xi\) has measure 1. A similar construction spacing bonds and deaths obtains \(Q_{\delta,\lambda}(D_1) = 0\), for further discussion, see [2].

To obtain the statement for \(q > 1\), note \(Q_{\delta,\lambda}|_B\) has Radon Nikodym derivative
\[
g^{k,B}(\omega)\frac{dQ_{\delta,\lambda}(\omega)}{dQ_{\delta,\lambda}(\omega')},
\]
so it is absolutely continuous with respect to \(Q_{\delta,\lambda}|_B\); it follows that \(Q_{\delta,\lambda}|_B(D_1) = 0\).

Proposition 3.5 now follows directly from Proposition 3.10 and Lemma 3.9.

### 3.2 Regularity

**Proposition 3.11** For any \(h\), \(\lambda\) the set \(\Theta_0\) has full measure or zero measure. If one of \(T_1, \ldots, T_d\) is uniquely ergodic, then \(\Theta_0\) is full measure or there is some \(\epsilon > 0\) so that \(M_{q,h,\lambda}(\theta) > \epsilon\) for every \(\theta \in \Theta\).

**Proof** Let \(\Theta_\epsilon = \Theta_\epsilon(h, \lambda) = \{\theta \in \Theta : \widetilde{M}_{q,h,\lambda}(\theta) > \epsilon\}\) suppose, \(\Theta_\epsilon\) has positive measure in \(\Theta\). As \(T_1\) is ergodic, for almost every \(\theta \in \Theta\) then there is some \(N_0\) so that \(T_1^{N_0}\theta \in \Theta\).

Let \(L_{1,N}\) be the event defined by no cuts along \(k_{c_1} \times [-1, 1]\) for \(k = 0, \ldots, N\) and at least one bond between \(k_{c_1} \times [-1, 1]\) and \((k+1)c_1 \times [-1, 1]\). The event \(L_{1,N}\) has positive measure, writing \(\|h\|_\infty = K\), we have \(Q_{h,\theta,\lambda}(L_{1,N}) > e^{-2K N_0 (1 - e^{-2\lambda})N_0 + 1}\).
Thus for almost every $\theta \in \Theta$ we have, for any $n$, and $x$ so that $x_1 = N_\theta$ and $x_i = 0$ for $i = 2, \ldots, d$, 
\[
\widehat{M}_{q,h,\lambda}^{(n)}(\theta) \geq Q_{h,\theta,\lambda}^{(q)}(L_1,N_\theta \cap ((x,0) \leftrightarrow \partial B_n)) \geq Q_{h,\theta,\lambda}^{(q)}(L_1,N_\theta) \geq \epsilon
\]
where we applied the FKG inequality on the last step. Here $e_i \in \mathbb{Z}^d$ is the vector with is 1 at index $i$ and 0 at other indices. But $T_1^{N_\theta} \theta \in \Theta_\epsilon$ so 
\[
Q_{h,\theta,\lambda}^{(q)}((x,0) \leftrightarrow \partial B_n) = M_{q,h,\lambda}^{(n)}(T_1^{N_\theta} \theta) \geq \epsilon
\]
for all $n$.

Let $p_\theta = Q_{h,\theta,\lambda}^{(q)}(L_1,N_\theta)$, we then have for almost any $\theta$, $p_\theta > 0$ so $\widehat{M}_{q,h,\lambda}(\theta) \geq \epsilon p_\theta > 0$.

Finally, if $T_i$ is uniquely ergodic then there is some $N$ so that for any $\theta$ there is some $n$ so that $0 < n \leq N$ and $T_1^n \theta \in \Theta_\epsilon$.

We therefore have $p_\theta > p > 0$ uniformly so that for all $\theta$, $\widehat{M}_{q,h,\lambda}(\theta) \geq \epsilon p > 0$. 

We now prove Proposition 3.4

**Proof** For $(h_k, \lambda_k, \theta_k) \to (h, \lambda, \theta)$ in $(C^+(\Theta) \times \mathbb{R}^+ \times \Theta, \rho)$ we have, for each $x \in \mathbb{Z}^d$ that $h_k(T^x \theta_k) \to h(T^x \theta)$ so that $\widehat{M}_{q,h_k,\lambda_k}^{(n)}(\theta_k) \to \widehat{M}_{q,h,\lambda}^{(n)}(\theta)$. Thus, $\widehat{M}^{(n)}$ is continuous in $(h, \theta, \lambda)$, in the $\rho$ metric by Proposition 3.5. Moreover, $\widehat{M}^{(n)}$ is decreasing in $n$. It follows that $\widehat{M}_{q,h,\lambda}(\theta)$ is upper semi-continuous in $\rho$. The statement $\mu(\Theta_0) \in \{0, 1\}$ follows from Proposition 3.11

3.3 The $G_\delta$ construction

In this section we will prove Proposition 3.3 fix $q \geq 1$ for this section.

Let $X$ be a space with a Borel topology and probability measure $\nu$. Let $Y$ be a space with a Borel topology. Let $V$ be a map from $X$ to the Borel topology of $Y$, ie for $x \in X$ let $V_x \subset Y$ and let $W$ be an ‘inverse’ i.e. $W_y = \{x \in X : y \in V_x\}$. Let $J_\eta = \{y : \nu(W_y) > \eta\}$.

**Lemma 3.12** If $V_x$ is open for all $x \in X$ then $J_\eta$ is open for all $\eta > 0$.

**Proof** Suppose $y_i \in J_\eta$ and $y_i \to y$. Suppose there is $x$ so that $y \in V_x$, then $y_i \in V_x$ for all large $i$. Therefore $\liminf W_{y_i} \supset W_y$. Then $\nu(W_y) \leq \nu(\liminf W_{y_i}) \leq \eta$ so $y \in J_\eta$.

Now we apply this result to obtain the proof of Proposition 3.3. We relate Proposition 3.12 $\Theta$ takes the role of $X$ with measure $\nu$ replaced with $\mu$, the space $Y$ is replaced with $S_F$ or $S_F \times \mathbb{R}^+$. Let $\overline{\rho}\{(h, \theta), (h', \theta')\} = d_{S}(h, h') + r(\theta, \theta')$, it follows from Proposition 3.4 that $\widehat{M}$ is upper semi-continuous in $\overline{\rho}$.

**Proof** To prove the first statement, let us fix $\lambda = 1$. Let $V^{(c)}_\theta = \{h \in S_F : \widehat{M}_{q,h,\lambda}(\theta) < \epsilon\}$, as $\widehat{M}$ is upper semi continuous in $\overline{\rho}$, we have $V^{(c)}_\theta$ is open. Let $W^{(c)}_h = \{\theta \in \Theta : h \in V^{(c)}_\theta\}$ then, $J^{(c)}_\eta = \{h \in S_F : W^{(c)}_h > \eta\}$ is open by Proposition 3.12. As discussed above, the inclusion $(S_F, d_S) \to (S_F, d_\tau)$ maps open sets to open sets so $J^{(c)}_\eta$ is open in $(S_F, d_\tau)$, moreover, $\widehat{M}^{(c)}_\eta = \tau^{-1} \tau J^{(c)}_\eta$ is open in $(S_F, d_\tau)$. Let $\epsilon_i \downarrow 0$, $\eta_i \uparrow 1$ then $\cap_i J^{(c)}_{\eta_i}$ is a $G_\delta$ in $(S_F, d_\tau)$. Finally $\cap_i \widehat{M}^{(c)}_{q,h,\lambda} = \{h \in S_F : \widehat{M}_{q,h,\lambda}(\theta) = 0\}$ for $\mu$ almost every $\theta$. To check that $\cap_i \widehat{M}^{(c)}_{q,h,\lambda} = S_F$, consider $g \in S_F$, let $\lambda > 0$ be small enough that $\widehat{M}_{q,g,\lambda}(\theta) = 0$ for $\mu$ almost every $\theta$. As $\widehat{M}$ is invariant under scaling, $\widehat{M}_{q,g,\lambda}(\theta) = 0$ for $\mu$ almost every $\theta$, but $\tau(g/\lambda) = \tau(g)$ so $g \in \cap_i \widehat{M}^{(c)}_{q,h,\lambda}$.

To prove the remark, let $\widehat{W}^{(c)}_\theta = \{h \in S_F : \widehat{M}_{q,h,\lambda}(\theta) < \epsilon\}$, again this set is open. The rest of the proof is similar to the first part by setting $\widehat{W}^{(c)}_h = \{\theta \in \Theta : h \in \widehat{W}^{(c)}_\theta\}$ and $\widehat{J}^{(c)}_\eta = \{h \in S_F : \widehat{W}^{(c)}_h > \eta\}$. 

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3.4 Theorem [1.1]

We complete this section by finishing the proof of Theorem [1.1] as the $G_d$ statement follows from Proposition [3.3] we only need to show the density of the $S_T$ and $H_T$ sets. We first prove Theorem [3.2] which allows us to construct sampling functions in $S_T$.

Proof Let $\zeta_T$ be defined on $\mathbb{N}$ as above, let $\xi > 0$ satisfy $1 + \xi < \nu(1 + 1/d)$. Then extend $\zeta_T$ to a continuous decreasing function $\zeta$ on $[1, \infty)$ by linearly interpolating the values of $\zeta_T$ on $\mathbb{N}$. Let $\tilde{Z}$ be defined as

$$\tilde{Z}(r) = \begin{cases} \exp \left\{ 1 - (\zeta^{-1}(r))^\xi \right\}, & r < \zeta_T(1) \\ 1, & \zeta_T(1) \leq r \end{cases}.$$ 

Now let, $\psi_F(\theta) = \tilde{Z}(r(\theta, F))$, then $\psi_F^{-1}(0) = F$ and $\|\psi_F\|_\infty = 1$. Moreover, $\psi_F$ satisfies the conditions of Theorem [1.4] indeed, $\psi_F$ is admitted by $\tilde{Z}(r)$ and

$$\lim_{k \to \infty} \frac{\log |\log \tilde{Z} \circ \zeta_T(k)|}{\log k} = \xi.$$ 

This completes the proof of Theorem [3.2].

We now complete the proof of Theorem [1.1] by finding functions in $H_T$ and $S_T$ close to a given function $h \in S_F$ in the $d_\tau$ topology.

Proof Let $h \in S_F$ let us write $F = h^{-1}(0)$, the nonempty and finite zero set of $h$, moreover, without loss of generality we may assume $\|h\|_\infty = 1$. For any $\epsilon > 0$, we will construct an $h_\epsilon \in H_T$ so that $\|h - h_\epsilon\| < \epsilon$ and $\|h_\epsilon\| = 1$ from which we will have $d_\tau(h, h_\epsilon) < \epsilon$.

In order to find a function in $S_F$, uniformly close to $h$ with percolation for all $\lambda > 0$ we only need to consider $h$ in a neighborhood of a single point $\theta_0 \in F$. In particular, let $\eta > 0$ be so that $r(\theta_0, \theta) < \eta$ implies $h(\theta) < \epsilon$. Now let $\eta > \eta_0 > \eta_1 > \eta_2 > \cdots$, so that $\eta_i \searrow 0$ and for all $i$, the $\eta_i$ ball centered at $\theta_0$ has $\mu$ measure 0 boundary.

Now, as $\Theta$ is compact for every $i$ there is some $L_i$ so that

$$\Theta \subset \bigcup_{x \in [0, L_i - 1]^d} \mathbb{T}^d B(\eta_i; \theta_0),$$

note the statement holds if $\Theta$ is not compact but some $T_i$ is uniquely ergodic with respect to $\mu$. That is, in a box of length $L_i$ we arrive $\eta_i$ close to $\theta'$. For $a > 1$ we construct the following function. Let $\nu(x) = 1$ on $1 \geq x > \eta$ and

$$\nu(x) = e^{\left(\frac{x - \eta_0}{\eta - \eta_0} + \frac{\eta - x}{\eta - \eta_0}L_0\right)^a}$$
on $\eta \geq x > \eta_0$ and

$$\nu(x) = e^{\left(\frac{x - \eta_i}{\eta_i - \eta_i}L_i + \frac{\eta_i - x}{\eta_i - \eta_i + 1}L_{i+1}\right)^a}$$
on $\eta_i \geq x > \eta_{i+1}$, for $i \geq 0$. And let

$$h_\epsilon(\theta) = \frac{h(\theta)}{\nu(r(\theta, \theta_0)).}$$

By construction, $h_\epsilon$, is already scaled so that $\|h_\epsilon\| = 1$, and $\|h_\epsilon - h\|_\infty < \epsilon$.

Now we prove the density of $S_T$, again let $h \in S_F$ so that $\|h\|_\infty = 1$ and $F = h^{-1}(0)$ is nonempty and finite. Let $\nu$ and $\xi$ satisfy $1 + \xi < \nu(1 + 1/d)$ and let $m < \infty$ then let $\psi_F$ be the sampling
function introduced in Proposition 3.2 so that the pair $\psi_F, \lambda$ satisfies a $C_{m,\nu}$ property. By the scaling property, $(t\psi_F, t\lambda)$ also satisfies a $C_{m,\mu}$ property, and by monotonicity, $(h + t\psi_F, t\lambda)$ satisfies the $C_{m,\nu}$ property. Let $h_t = h + t\psi_F$, then $\|h - h_t\|_\infty \to 0$ as $t \to 0$. It follows that $d_\tau(h, h_t) \to 0$ as $t \to 0$ which establishes density of $S_T$ in $(S_F, d_\tau)$.

Finally, the statement that $S_F$ partitions into $S_F = S_T \sqcup H_T$ follows immediately from Proposition 3.3.

4 Analysis

For small $\lambda$ relative to $\delta$ in the bulk we obtain by comparison to the constant system uniform decay of correlation in the bulk of the spatial environment. In regions where $\delta$ is small however one will have locally extended correlations in space. Thus, we only need to show the portions of the environment where $\delta$ is small do not ruin the decay of correlations. The goal of the multiscale method then is to show these bad regions are well enough separated that these locally ordered regions cannot percolate over infinitely large regions in space.

We define geometric objects necessary to carry out the multiscale analysis. Let a box of length $2L + 1$ and centered at $x \in \mathbb{Z}^d$ be denoted as

$$\Lambda_L(x) = \{y \in \mathbb{Z}^d : |y - x| < L\};$$

this set’s cylinder in the time dimension of length $2T(L)$ is written as

$$B_L(x, t) = \Lambda_L(x) \times [t - T(L), t + T(L)]$$

where the increasing function $T$ will be specified. The environment is invariant in the time dimension so the choice of $t$ above will often not affect the discussion, thus we fix the notation $B_L(x) = B_L(x, 0)$.

4.1 Multiscale Analysis

By the ordering Lemma 2.1 we need only demonstrate localization for the product measure $q = 1$ to infer similar results for $q > 1$. Thus for this section we fix $q = 1$.

Definition Let $m > 0$ and $L \in \mathbb{Z}^+$. For a fixed environment $\delta, \lambda$; a site $x \in \mathbb{Z}^d$ is $(m, L)$-regular if

$$Q_{\delta, \lambda}(x \leftrightarrow \partial B_L(x)) \leq \exp\{-mL\}$$

otherwise it is $(m, L)$-singular. A set $A \subset \mathbb{Z}^d$ is $(m, L)$-regular if every $y \in A$ is $(m, L)$-regular. Otherwise it is $(m, L)$-singular.

Definition A site $x \in \mathbb{Z}^d$ is $\epsilon$-resonant if $\delta(x) < \epsilon$. A set $A \subset \mathbb{Z}^d$ is $\epsilon$-resonant if there exists $x \in A$ which is $\epsilon$-resonant.

Definition The pair $(\epsilon, L)$ is $m$-simple if $x \in \mathbb{Z}^d$ is $(m, L)$-singular implies $\Lambda_L(x)$ is $\epsilon$-resonant.

From the assumptions in Proposition 1.4 we have $\nu(d + 1) - (1 + \xi)d > 0$, let us introduce $\alpha > d$ which satisfies,

$$\alpha > \frac{(\xi + \nu)d}{\nu(d + 1) - (1 + \xi)d}.$$
It follows that
\[ 0 < \xi(\alpha + 1)d < \alpha \nu - (\nu + \alpha - \alpha \nu)d \]

Let \( \gamma \) and \( \kappa \) be parameters so that \( \alpha \nu / d - \xi(\alpha + 1) > \kappa > \nu + \alpha - \alpha \nu \) and
\[ \xi(\alpha + 1)d < \gamma < \alpha \nu - \kappa d \]

Finally, let \( \tau \) satisfy \( \nu < \tau < \kappa - \alpha(1 - \nu) \), and define \( T(L) = \exp\{L^\nu\} \).

We utilize induction steps for the multiscale analysis which were demonstrated in [23], the following Proposition is essentially the statement of Sublemmas 4.2 and 4.3 of that paper.

Let \( R \) be a fixed positive integer and let \( L \) be sufficiently large. Let \( y_1, \ldots, y_R \in \mathbb{Z}^d \) be centers of the resonant regions in space and let \( \Lambda = \Lambda_{L^\alpha}(x) \cap (\cup_{i=1}^R \Lambda_{L^{\nu}}(y_i)) \). We use regularity in the bulk of \( \Lambda_{L^\alpha}(x) \) and limited resonance in \( \Lambda \) to obtain regularity at \( x \) on the scale \( L^\alpha \).

**Proposition 4.1** Suppose \( \Lambda \) is \( e^{-L^\gamma} \) nonresonant and every \( y \in \Lambda_{L^\alpha}(x) \setminus \cup_{i=1}^R \Lambda_{2L^{\nu+1}}(y_i) \) is \((m, L)\)-regular. Then \( x \) is \((m - L^{-\tau}, L^\alpha)\) regular.

Let us note that we can initialize the multiscale analysis with any \( m_1 \) and large \( L_1 \) by selecting \( \lambda > 0 \) sufficiently small. Let \( \delta' = \min_{x \in \Lambda_{L^\alpha}(0)} h(T^x \theta) \) then, for \( \lambda / \delta' \) sufficiently small,
\[ Q_{h, \theta, \lambda}(0 \leftrightarrow \partial B_L(0)) \leq Q_{\delta', \lambda}(0 \leftrightarrow \partial B_L(0)) < e^{-m_1 L_1}. \] (4.1)

This inequality follows from Corollary 2.2 in [23].

The following theorem is the standard use of the Borel-Cantelli lemma in multiscale arguments [6, 23]. It is also stated as Theorem 2.1 in [20] in a more general form. The sequences \((m_i)\) and \((L_i)\) correspond to the sequences defined in Proposition 4.1 and are defined for initial \( m_1 \) and \( L_1 \) and induction \( m_{i+1} = m_i - L_i^{-\kappa} \) and \( L_{i+1} = L_i^\alpha \).

Now we state the analogy of the probabilistic theorem for decay of correlations from every point in \( \mathbb{Z}^d \). In the dynamical setting it is indeed still a Borel Cantelli argument with respect to the Haar measure \( \mu \).

We say \( \theta \in \Theta \) is \((m, L)\)-singular for coupling \( \lambda \) and sampling function \( h \) if the site \( 0 \in \mathbb{Z}^d \) is \((m, L)\)-singular in the environment defined by coupling \( \lambda \) and transverse field \( \delta(x) = h(T^x \theta) \). We call the environment \( \delta(x) = h(T^x \theta) \) the environment initialized at \( \theta \). For given sampling function \( h \) and coupling \( \lambda \) let us write
\[ a_k = \mu(\{ \theta : \text{Environment initialized at } \theta \text{ is } (m_k, L_k)\text{-singular}\}) \]

In this case that \( a_k \) decays sufficiently fast, we can apply the following theorem, which follows from Theorem 3.3 in [23].

**Theorem 4.2** Fix coupling \( \lambda > 0 \) and sampling function \( h \). Let \( p > \alpha d \), then if \( \limsup a_k L_k^p \to \infty \) we have that for any \( 0 < m < m_\infty = \inf_k m_k \) with probability one every \( x \in \mathbb{Z}^d \) has some constant \( C_x < \infty \) so that for any \( y \in \mathbb{Z}^d \) and \( t, s \in \mathbb{R} \) so that \( |x - y| + [\ln(1 + |t - s|)]^{1/\nu} > C_x \) we have
\[ Q_{\delta, \lambda}((x, t) \leftrightarrow (y, s)) < \exp\{-m\left(|x - y| + [\ln(1 + |t - s|)]^{1/\nu}\right)\}. \]
4.2 Recurrence

We carry out the generalization of the arguments in [20] for the specified controlled recurrence models. First we state bound for the probability $\Lambda_L$ is $\varepsilon$-resonant.

**Proposition 4.3** For any $F \in \mathcal{F}$ and $h \in \mathcal{S}_F$, admitted by $Z$ and $\Psi = Z \circ \zeta_T$, the set of phases $\theta \in \Theta$ so that $\Lambda_L(0)$ is $\varepsilon$-resonant has $\mu$-measure bounded above by $C_dL^d|F|\left(\Psi^{-1}(\varepsilon)\right)^{-1}$ for some constant $C_d$.

**Proof** We assume that $h$ is admitted by $Z$, thus $h(\theta) < \varepsilon$ implies $Z(r(\theta, F)) < \varepsilon$. Let $F = \{\theta_1, \ldots, \theta_{|F|}\}$, we can write, $A_i = h^{-1}(0, \varepsilon) \cap B_{Z^{-1}}(\varepsilon)(\theta_i)$ thus we have

$$h^{-1}(0, \varepsilon) = \bigcup_{1 \leq i \leq |F|} A_i. \quad (4.2)$$

We consider each $A_i$ separately. Suppose for some $\theta$, there are $x, y \in \mathbb{Z}^d$ so that $T^x\theta, T^y\theta \in B_{Z^{-1}}(\varepsilon)(\theta_i)$ then $r(T^x\theta, T^y\theta) < 2Z^{-1}(\varepsilon)$. By definition of $\zeta_T$, this implies that $\zeta_T(|x - y|) < Z^{-1}(\varepsilon)$ or $|x - y| \geq \Psi^{-1}(\varepsilon)$, which is a lower bound on return times. On the other hand, the return times are related to the size of the set in the ergodic measure by Kac’s lemma, (see e.g. [9]) thus,

$$\Psi^{-1}(\varepsilon) \leq \mathbb{E}\{\text{Return time to } A_i\} = \frac{1}{\mu(A_i)}. \quad (4.3)$$

But using (4.2) and (4.3)

$$\mu(h^{-1}(0, \varepsilon)) \leq \sum_i \mu(A_i) \leq \frac{|F|}{\Psi^{-1}(\varepsilon)}.$$

Finally, the conclusion follows from the fact that the probability of a set $\Lambda \subset \mathbb{Z}^d$ being $\varepsilon$-resonant is bounded by $|\Lambda| \cdot \mu(h^{-1}(0, \varepsilon))$. \qed

We show that simplicity at scale $L$ implies regularity in the bulk at scale $L^\alpha$. Essentially, we show that at the chosen sequence of scales, at most one resonance occurs per $L_i$ box per zero of $h$. Let $h \in \mathcal{S}_F$ for some $F \in \mathcal{F}$, be admitted by a function $Z$. Then for $\Psi = Z \circ \zeta_T$ suppose (1.8) holds.

**Proposition 4.4** Suppose $(\exp\{-L^\gamma\}, L)$ is $m$-simple and $L$ is large. Then for any $x \in \mathbb{Z}^d$ there exists $y_i \in \Lambda_{L^\alpha}$, for $i = 1, \ldots, |F|$, so that $\Lambda_{L^\alpha}(x) \cup_{i=1}^{|F|} \Lambda_L(y_i)$ is $(m, L)$-regular.

**Proof** Fix a phase $\theta$. Let $F = \{\theta_1, \ldots, \theta_{|F|}\}$. If $x \in \mathbb{Z}^d$ is $\varepsilon$-resonant then $r(T^x\theta, \theta_i) < F^{-1}(\varepsilon)$ for some $i = 1, \ldots, |F|$. If $T^x\theta, T^y\theta$ are in $B_{F_{\varepsilon}^\alpha}(\theta_1)$ then $|x - y| \geq \Psi^{-1}(\varepsilon)$, as in the proof of Proposition 4.3. Let $\varepsilon = \exp\{-L^\gamma\}$, then from the assumption on $\Psi$ we get, for $L$ sufficiently large, $|x - y| > L^{\gamma/\xi} > 2L^\alpha$. Thus for each $\theta_i \in h^{-1}(0)$ there exists at most one $\exp\{-L^\gamma\}$-resonant $y_i \in \Lambda_{L^\alpha}(x)$. By definition of $m$-simple, the result follows. \qed

**Proposition 4.5** Suppose $(\exp\{-L^\gamma\}, L)$ is $m$-simple, then $(\exp\{-L^\alpha^\gamma\}, L^\alpha)$ is $m' = m - \tau_L$ simple.

**Proof** By Proposition 4.4 we have that there exists $y_1, \ldots, y_{|F|} \in \Lambda_{L^\alpha}$ so that $\Lambda_{L^\alpha}(x) \cup_{i=1}^{|F|} \Lambda_L(y_i)$ is $(m, L)$-regular. Now if $x \in \mathbb{Z}^d$ is so that $\Lambda_{L^\alpha^\gamma}$ is $\exp\{-L^\alpha^\gamma\}$ non-resonant, Theorem 4.1 implies $x$ is $(m', L^\alpha)$ regular. \qed
Now we can prove Theorem 4.4. Take a closed neighborhood around the zero set of $h$, that is for some small $r_1 > 0$ let $B_0 = B(F; r_1)$ and let $h = \min h|_{B_0}$ be the minimum value outside $B_0$.

**Proof** To prove the theorem we need to show the hypothesis of theorem 1.2 holds. Let $\gamma, \alpha, \tau, \kappa$ satisfy the assumptions of Proposition 1.1 so that $\xi(\alpha + 1)\delta < \gamma$. Let parameters $\xi' > \xi$ and $p > \alpha d$ be such that $\xi'(p + d) < \gamma$.

Let $m$ be as defined in Theorem 1.4, let $m_0 = m + 1$. For chosen $L_0$ we have a sequence of scales $L_{k+1} = L_k^p$, $\epsilon_k = \exp \{-L_k\}$ and $m_{k+1} = m_k - L_k^{-\tau}$. Let $L_0$ be large enough that $m < m_{\infty} = m_0 - \sum_{i=0}^{\infty} L_i^{-\tau}$. For any $\xi' > \xi$ for large enough $k$ it follows from assumption (1.3) that $\Psi(k) \geq e^{-k\xi'}$. As $\Psi$ is decreasing, this implies for large enough $k$, $\Psi^{-1}(e^{-k\xi'}) \geq k$. It follows that $\Psi^{-1}(\epsilon_k) \geq L_k^{\gamma/\xi'}$ for all $k$ provided $L_0$ is chosen sufficiently large.

The probability that $\Lambda_{L_0}$ is $\epsilon_0$ resonant is of order $L_0^{d-\gamma/\xi'}$ for large $L_0$. That is, for large $L_0$ the probability that $\Lambda_{L_0}$ is $\epsilon_0$ resonant approaches 0. On the other hand, in the uniform case, as density $\rho = \lambda/\delta \to 0$ the probability of escaping a box of size $L_0$ approaches 0, in fact, the radius of exponential localization approaches 0 [2]. Thus, take bond rate $\lambda > 0$ so small that the uniform environment $\lambda$ and $\delta = \epsilon_0$ has probability of escape $Q_{\epsilon_0,\lambda}(0 \longleftrightarrow \partial B_{L_0}(0)) < e^{-m_{\infty}L_0}$.

Thus in the disordered environment, by comparison to the homogeneous environment using (2.4), $(\exp(-L_0^p), L_0)$ is $m_0$-simple.

Using proposition 4.3 notice for large enough $L_0$, we have for all $k$, $(\epsilon_k, L_k)$ is $m_k$-simple. We only need to check that the sequence $a_k$ decays sufficiently fast. Since $(\epsilon_k, L_k)$ is $m_k$-simple, the origin is $(m_k, L_k)$ regular unless $\Lambda_{L_k}(0)$ contains a site $x$ which is $\epsilon_k$ resonant. By Proposition 4.3 and the calculation above, the $\mu$-probability $\Lambda_{L_k}$ is $\epsilon_k$ resonant is of order $L_k^{d-\gamma/\xi'}$.

Finally, to apply Theorem 4.2 check that $\limsup \gamma L_k^{p+d-\gamma/\xi'} < \infty$, which is indeed true. This completes the proof. $\blacksquare$

### 4.3 Infinite components

In this section we summarize results from [1] as they apply to our model. First we state the uniqueness of the infinite cluster, and an immediate corollary, a lower bound on the probability the two sites communicate.

**Theorem 4.6** Let $\delta(x) = h(T^x\theta)$. There exists with probability one either zero or one unbounded components. Furthermore, for $x, y \in \mathbb{Z}^d$ and any $t, s \in \mathbb{R}$,

$$Q_{h,\theta,\lambda}(\{(x,t) \leftrightarrow (y,s)\}) \geq M_\lambda(T^x\theta)M_\lambda(T^y\theta).$$

The uniqueness of the infinite cluster is shown in [1] for random independent environments. In [1] the bond rates as well as the death rates are chosen at random which makes establishing uniqueness considerably more difficult, however allowing the environment to be chosen ergodically by a sampling function adds no difficulty to the proof.

The recurrence condition in Theorem 1.5 suffices to ensure that there are a sequence of scales with high resonances.

The inequality follows from the fact that $Q_{\delta,\lambda}$ obeys the FKG inequality (2.6) and $\{X \leftrightarrow \infty\}$ is an increasing event and the unbounded component is unique. The strategy in [1] for proving the existence of an infinite component in $\mathbb{Z}^d \times \mathbb{R}$ is by coupling the model to a bond site percolation model on $\mathbb{Z}^{d+1}$.
Proof of 1.5 From the assumption, for any $1 < \xi' < \xi$ there is a sequence $(L_i)$ in $\mathbb{N}$ so that $\hat{\Psi}(L_i) < e^{-L_i^{\xi'}}$ for all $i$. Let $F = h^{-1}(0)$, and let $\theta' \in F$. Let $|x| < L_i$ minimize $r(\theta', \theta^x \theta)$ over $|x| \leq L_i$, then $h(\theta^x \theta) < \hat{Z} \circ \zeta_T(h) < e^{-L_i^{\xi'}}$.

As discussed above we use a strategy similar to [11], choose as a parameter for scaling time $\tau$ such that $\xi' > \tau > 1$. For each $L_i$ in the sequence we construct a bond-site percolation on $\hat{Z}^{d+1}$ with a measure $\hat{P}$ coupled to the original percolation measure $Q_{\delta, \lambda}$ on $\mathbb{Z}^d \times \mathbb{R}$. For $\tilde{x} \in \mathbb{Z}$ we associate a box in $\mathbb{Z}^d \times \mathbb{R}$ of length $L$ in the space dimension and height $T = \exp\{L^x\}$ in the time dimension. Roughly, the strategy of the proof is to declare a site occupied if there is a line through this box with out deaths, a bond between neighbors is occupied if their occupied lines communicate. For $\tilde{R}$ roughly the strategy of the proof is to declare a site occupied if there is a line through this box with both $\tilde{\eta}$ and $\tilde{\xi}$ that $\xi$ that $\xi > \tau > 1$. Observe, for neighbors $\tilde{x}$ and $\tilde{y}$ occluded only if both $\tilde{x}$ and $\tilde{y}$ are occupied. If $\tilde{x}$ and $\tilde{y}$ are consecutive in time, i.e. $|\tilde{x} - \tilde{y}| = 1$ and $|\tilde{x} + 1 - \tilde{y} + 1| = 1$, then the bond between them is occupied if both $\tilde{x}$ and $\tilde{y}$ are occupied. If $\tilde{x}$ and $\tilde{y}$ are neighbors with the same time value, i.e. $|\tilde{x} - \tilde{y}| = 1$ and $\tilde{x} + 1 = \tilde{y} + 1$, then the bond between them if both are occupied and the event $D(\tilde{x}, \tilde{y})$ that $\tilde{u}$ communicates with $\tilde{u}$ within the box $B(L; \tilde{x}, \tilde{y})$.

For $\tilde{x}$, occupation probabilities to the 2d spatial neighbors are not independent, however, they are positively correlated; that is, the occupation events in $\hat{Z}^{d+1}$ are defined by increasing events in $\mathbb{Z}^d \times \mathbb{R}$. We will calculate occupation probabilities for bonds in $\hat{Z}^{d+1}$. By the FKG inequality, it is enough to show that percolation occurs for occupation probabilities taken independently to obtain that percolation occurs in the non-independent model.

We set ourselves now to estimate the occupation probabilities. For $\tilde{x}, \tilde{y} \in \hat{Z}^{d+1}$ so that $\tilde{x} + 1 = \tilde{y} + 1$, split $B(L; \tilde{x}, \tilde{y})$ into $T$ similar time ‘slices’, for $i \in \{1, \ldots, T\}$ let

$$B_i(\tilde{x}, \tilde{y}) = B(L; \tilde{x}, \tilde{y}) \cap (\mathbb{Z}^d \times ([\tilde{x} + iT - 1, \tilde{x} + iT + i]))$$

then consider $D_i(\tilde{x}, \tilde{y})$ the event that $\tilde{u}$ communicates to $\tilde{u}$ within $B_i(\tilde{x}, \tilde{y})$. Observe, for neighbors $\tilde{x}$, $\tilde{y}$, within $\Lambda_\tilde{x}(\tilde{x}) \cup \Lambda_\tilde{y}(\tilde{y})$ there exists a path of length less than $5dL$. Consider the event that there are no deaths in $B_i(\tilde{x}, \tilde{y})$ on each point in the path and and there is a bond in $B_i(\tilde{x}, \tilde{y})$ for every step in the path. Therefore, it is easy to see that, for all $\theta \in \Theta$,

$$Q_{h, \theta, \lambda}(D_i(\tilde{x}, \tilde{y})) \geq (1 - e^{-\lambda})^{5dL} (e^{-|h|_{\infty}})^{5dL} \geq \exp\{-cL\},$$

for some $c < \infty$ depending only on $|h|_{\infty}$ and $\lambda$. Observe, the bond between $\tilde{x}$ and $\tilde{y}$ is occupied if $D_i(\tilde{x}, \tilde{y})$ holds for some $i$ therefore $D^x(\tilde{x}, \tilde{y}) \subset \cap_{i=1}^T D_i(\tilde{x}, \tilde{y})$. The events $D_i(\tilde{x}, \tilde{y})$ are independent by construction, so we can estimate, for all $\theta \in \Theta$,

$$Q_{h, \theta, \lambda}(D(\tilde{x}, \tilde{y})) \geq 1 - (1 - e^{-cL})^T \geq 1 - e^{-T e^{-cL}} = 1 - C e^{-e^{L^x - cL}},$$

for some $C < \infty$. By the scaling $\tau > 1$ the above quantity goes to one as $L \to \infty$. Now consider the probability that $\tilde{x}$ is occupied, by assumption $\delta_{u_\tilde{x}} < \exp\{-L^\xi\}$ so we have

$$\hat{P}(\tilde{x} \text{ is occupied}) > \exp \left\{ -T \exp\{-L^\xi\} \right\} = \exp \left\{ -\exp\{L^\tau - L^\xi\} \right\}$$

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Which by the scaling $\xi' > \tau$ goes to 1 as $L \to \infty$.

Therefore, we have a bond site percolation model where the occupation probabilities are close to one as $L \to \infty$. This model is in fact ‘more connected’ than the analogous model with probability of occupations defined independently. Indeed, the events $D(\tilde{x}, \tilde{y})$ defined on the original model $\mathbb{Z}^d \times \mathbb{R}$ causing edges to be occupied in the coupled model on $\tilde{\mathbb{Z}}^{d+1}$ are positive, therefore, by the FKG inequality, the probability that edges incident to the same site in $\tilde{\mathbb{Z}}^{d+1}$ are occupied is greater than the probability the product of each of the bonds is occupied. Consider an independent bond site percolation model with uniform parameter $p \to 1$, where $p$ depends on $L$, by a Peierls argument, there is some $p < 1$ so that there exists an infinite component. Therefore, there is an infinite component in $\mathbb{Z}^d \times \mathbb{R}$.

4.4 Rotations on $\mathbb{T}$

In this section we prove Theorem 1.3.

First we apply Theorem 1.5 to rotations in $d = 1$ to obtain statement (1.). Let us review facts from the theory of continued fractions [22]. Let $p_n/q_n$ be the sequence of approximants defined in (1.7) the sequence of denominators is defined as $q_{n-2} = 0, q_{n-1} = 1$ and $q_n = a_n q_{n-1} + q_{n-2}$. Moreover, for any interval $I \subset \mathbb{T}$ so that $|I| > 1/q_n$, and any $\theta \in \mathbb{T}$, there is some $k$ so that $1 \leq k \leq q_n + q_{n-1}$ and $\theta + k \omega \in I$. It follows, for any $\theta$, that $\hat{\zeta}(q_n) \leq 1/q_n$. From the assumption on $h$, for any $1 < \xi' < 1/a$, we have that $h$ is restricted above by a function $\hat{Z}(r) = e^{-r - \xi'}$. Thus we have, for large $n$, $\log |\log \hat{\Psi}(q_n)| \geq \log |\log \hat{Z} \circ \hat{\zeta}| \geq \log |\log \hat{Z}(1/q_n)| = \xi' \log q_n$. Thus $\hat{\Psi}$ satisfies (1.8), this completes the proof of statement (1.).

We apply Theorem 1.4 to obtain statement (3.). Again we recall a standard result in continued fraction theory, for all $0 < q < q_n, \sin(q \pi \omega) > |\sin(q_n \pi \omega)| > \frac{1}{4q_{n+1}}$ which implies for all $q_{n-1} < q \leq q_n$, that $\zeta(q) > C q_{n+1}^{-1} > e^{-q_{n-1}}$. So for $\Psi = \hat{Z} \circ \hat{\zeta}$ we have $\log |\log \Psi(q)| \leq \log |\log \hat{Z}(Ce^{-q_{n-1}})|$. Thus for $ab < \xi' < 2\nu - 1$ we have, for large enough $n$, that $\log |\log \Psi(q)| \leq \xi' \log q_{n-1} \leq \xi' \log q$. Thus $\Psi$ satisfies (1.8) so for $\frac{1+ab}{2} < \nu < 1$ and $m > 0$ so that $h$ has the $C_{m, \nu}$ property.

References

[1] Michael Aizenman, Abel Klein, and Charles Newman. Percolation methods for disordered quantum Ising models. In R. Kotecky, editor, Phase Transitions: Mathematics, Physics, Biology. World Scientific, 1993.

[2] Carol Bezuidenhout and Geoffrey Grimmett. Exponential decay for subcritical contact and percolation processes. The Annals of Probability, 19(3):984–1009, 1991.

[3] Patrick Billingsley. Convergence of probability measures. Wiley, 1999.

[4] Jakob Bjornberg. Graphical representations of Ising and Potts models. PhD thesis, KTH Matematik, 2009.

[5] JE Björnberg and GR Grimmett. The phase transition of the quantum Ising model is sharp. Journal of Statistical Physics, 136(2):231–273, 2009.
[6] Massimo Campanino and Abel Klein. Decay of two-point functions for \((d + 1)\)-dimensional percolation, ising and potts models with \(d\)-dimensional disorder. *Communications in Mathematical Physics*, 135:438 – 497, 1991.

[7] Massimo Campanino, Abel Klein, and J. Fernando Perez. Localization in the ground state of the ising model with a random transverse field. *Communications in Mathematical Physics*, 135:499–515, 1991.

[8] Jacob Chapman and Günter Stolz. Localization for random block operators related to the xy spin chain. In *Annales Henri Poincaré*, volume 16, pages 405–435. Springer, 2015.

[9] I.P. Cornfeld, S.V. Fomin, and Ya.G. Sinai. *Ergodic theory*. Springer-Verlag, 1982.

[10] Nicholas Crawford and Dmitry Ioffe. Random current representation for transverse field ising model. *Communications in Mathematical Physics*, 296(2):447–474, 2010.

[11] David Damanik, Marius Lemm, Milivoje Lukic, and William Yessen. New anomalous lieb-robinson bounds in quasiperiodic x y chains. *Physical review letters*, 113(12):127202, 2014.

[12] David Damanik, Milivoje Lukic, and William Yessen. Quantum dynamics of periodic and limit-periodic jacobi and block jacobi matrices with applications to some quantum many body problems. *Communications in Mathematical Physics*, 337(3):1535–1561, 2015.

[13] Alexander Elgart, Mira Shamis, and Sasha Sodin. Localization for non-monotone schrödinger operators. *arXiv preprint arXiv:1201.2211*, 2012.

[14] Stewart Ethier and Thomas Kurtz. *Markov processes: characterization and convergence*. Wiley, 1986.

[15] Geoffrey Grimmett. *Probability on graphs: random processes on graphs and lattices*, volume 1. Cambridge University Press, 2010.

[16] Geoffrey R Grimmett, Tobias J Osborne, and Petra F Scudo. Entanglement in the quantum ising model. *Journal of Statistical Physics*, 131(2):305–339, 2008.

[17] G.R. Grimmett. *The Random-Cluster Model*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2010.

[18] Eman Hamza, Robert Sims, and Günter Stolz. Dynamical localization in disordered quantum spin systems. *Communications in Mathematical Physics*, 315(1):215–239, 2012.

[19] Dmitry Ioffe. Stochastic geometry of classical and quantum ising models. In *Methods of contemporary mathematical statistical physics*, pages 1–41. Springer, 2009.

[20] Svetlana Jitomirskaya and Abel Klein. Ising model in a quasiperiodic transverse field, percolation, and contact processes in quasiperiodic environments. *Journal of Statistical Physics*, 73(1):319 – 344, 1993.

[21] Ilya Kachkovskiy. On transport properties of isotropic quasiperiodic xy spin chains. *Communications in Mathematical Physics*, pages 1–15, 2015.
[22] A.Ya. Khinchin and H. Eagle. *Continued Fractions*. Dover books on mathematics. Dover Publications, 1964.

[23] Abel Klein. Extinction of contact and percolation processes in a random environment. *The Annals of Probability*, 22(2):1227–1251, 1994.

[24] Robert Sims and Simone Warzel. Decay of determinantal and pfaffian correlation functionals in one-dimensional lattices. *Communications in Mathematical Physics*, pages 1–29, 2015.