A comparative study of overlap and staggered fermions in the Schwinger model

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We investigate the validity of the square rooting procedure of the staggered determinant in the context of the Schwinger model. We find some evidence that at fixed physical quark mass the square root of the staggered determinant becomes proportional to the overlap determinant in the continuum limit. We also find that at fixed lattice spacing moderate smearing dramatically improves the chiral behavior of staggered fermions.

1. Introduction

Recently, unquenched lattice QCD calculations with improved staggered fermions have had remarkable success in reproducing a variety of phenomenologically interesting quantities \cite{1}. In these calculations, the determinant of a single fermion is obtained as the fourth root of the staggered determinant. It is a priori unclear whether there exists a local operator $D$ describing a single fermion flavor with

$$\text{det} \ (D) \propto \text{det} \ (D_{\text{stag}})^{1/4}$$

and it has been shown \cite{2} that a naive guess leads to a nonlocal operator. Furthermore, the lack of an (exact) index theorem for staggered fermions raises the question how far into the chiral regime one can push calculations at fixed lattice spacing.

Given this situation, we decided to study the behavior of staggered fermions in the simple and well known Schwinger model (QED\textsubscript{2}). We focus on observables that - like the determinant - are obtainable from the Dirac spectrum alone. As a point of reference, we compare the staggered data to results obtained with overlap fermions, which are known to be free of conceptual problems. We also investigate UV-filtered staggered and overlap fermions, which are technically realized by APE-smearing the gauge backgrounds. For the $U(1)$ gauge group smearing consists of taking a weighted average of the phases of the original link and the staple. Due to the twofold fermion doubling in 2D, the $N_f = 1$ case is obtained by weighting the gauge configurations with the square root of the staggered determinant. Technical details of our simulations can be found in \cite{3}, related studies in QCD are reported in \cite{4,5}.

2. Chiral condensate

Due to the Mermin-Wagner theorem, the chiral condensate of the continuum massless Schwinger model vanishes for $N_f \geq 2$ \cite{6}. For $N_f = 1$ the formation of a condensate does not imply spontaneous symmetry breaking. In the massless case, its infinite volume value is analytically known \cite{7}

$$\frac{\langle \bar{\psi} \psi \rangle}{e} = \frac{e^7}{2\pi^{3/2}} \simeq 0.1599...$$

Away from the chiral limit one expects an additional contribution to the condensate proportional to the quark mass and at most logarithmically divergent in the cutoff. Also, due to the exact/remnant chiral symmetry of overlap/staggered fermions, the chiral condensate renormalizes multiplicatively and due to the dimensionful coupling $[e] = [a]^{-1}$ all renormalization factors are $Z = 1 + O(a^2 e^2)$.

In fig. 1 we plot the $N_f = 1$ chiral condensate versus the quark mass, as obtained from staggered and overlap fermions. Our results for the unfiltered operators (dotted lines) indicate that
overlap fermions show a nice chiral behavior while staggered fermions behave qualitatively wrong in the chiral limit. By contrast, the UV-filtered staggered fermions (full line) exhibit a much better behavior down to comparatively small quark masses. They still show, ultimately, a qualitatively wrong behavior in the chiral limit, but the mass at which it sets in is drastically reduced.

3. Spectral mimicry

A finite condensate in the chiral limit is related to the existence of exact zero modes. The dramatically improved behavior of $\langle \bar{\psi} \psi \rangle/e$ for UV-filtered staggered fermions down to fairly small quark masses therefore suggests that they develop near-zero modes on topologically nontrivial gauge configurations and that very small quark masses are needed to expose these modes as not being true zero modes (see [3] for a more comprehensive discussion of this point). It is thus interesting to directly compare staggered and overlap spectra on individual configurations.

Fig. 2 shows a comparison of the IR part of massless staggered and overlap\(^1\) spectra on some selected configurations on a $20^2$ lattice at $\beta = 4$.

\(^1\)For the overlap operator, we plot the (purely imaginary) eigenvalues $\lambda = (1/\lambda - 1/2\rho)^{-1}$ of the chirally rotated operator (see [3] for more details).

The two plots on the left show typical configurations with topological charge $q = 0$ and $|q| = 1$, respectively. The unfiltered staggered spectra do not resemble the overlap spectra and it is difficult to see the qualitative difference between $q = 0$ and $|q| = 1$. After moderate UV-filtering however, the eigenvalues of the staggered operator form near-degenerate pairs which sit close to a single overlap eigenvalue. In particular, in the $|q| = 1$ case a pair of eigenmodes moves very close to the real axis, confirming the expectation about the existence of near-zero modes that we had from the chiral condensate analysis.

The third column of fig. 2 shows a typical configuration with higher topological charge ($|q| = 4$). Here, after 3 smearing steps only 3 pairs of eigenmodes have managed to come close to the real axis and the fourth one is still further out.

The last column of fig. 2 presents a selected “worst case” configuration on which the topological charge varies repeatedly under subsequent smearing steps. Qualitative resemblance between the two types of spectra is vague at best and there is no evidence for a pair of staggered eigenvalues moving close to the real axis. Such configurations are rare already at $\beta = 4.0$. 

Figure 1. $\langle \bar{\psi} \psi \rangle/e$ of staggered and overlap fermions with $N_f = 1$ on a $24^2$ lattice at $\beta = 7.2$. The star denotes the analytically known value of the continuum chiral condensate from [7].
4. Determinant ratio

The qualitative similarity between the IR part of staggered and overlap spectra on individual configurations leads to an interesting suggestion regarding the local operator \( D \) in (1). If eigenvalue pairs become truly degenerate in the continuum limit and topologically undecided configurations are are strictly \( O(\alpha^2) \) effects, it could be that \( D_{\text{overlap}} \) is a local operator with

\[
\det (D_{\text{stag}})^{1/2} \propto \det (D_{\text{overlap}}) + O(\alpha^2) \quad (3)
\]

Fig. 3 shows a scatter plot of \( \log(\det(D_{\text{stag}})^{1/2}) \) vs. \( \log(\det(D_{\text{overlap}})) \) for different lattice spacings and UV-filtering levels at one fixed physical quark mass. One can clearly see that there is a tendency for the determinants to be proportional in the continuum limit and that the proportionality is better developed for the UV-filtered operators.

With a careful scaling study one might be able to test (3). The real goal is, of course, to see whether such a relation holds in QCD. Preliminary studies in this direction can be found in [4].

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Figure 3. Log-Log scatter plot of the overlap determinant versus the square root of the staggered determinant at 4 lattice spacings and 3 UV-filtering levels and fixed fermion mass \( m/e = 0.035 \). Every plot shows 1000 decorrelated quenched configurations at fixed physical volume. The individual plots are offset for better visibility. In all cases the slope is consistent with 1.