PATH INTEGRAL SOLUBILITY OF A GENERAL
TWO-DIMENSIONAL MODEL

Ashok Das
and
Marcelo Hott†
Department of Physics and Astronomy
University of Rochester
Rochester, NY 14627

Abstract

The solubility of a general two dimensional model, which reduces to various models in different limits, is studied within the path integral formalism. Various subtleties and interesting features are pointed out.

†On leave of absence from UNESP - Campus de Guaratinguetá, P.O. Box 205, CEP : 12.500, Guaratinguetá, S.P., Brazil
There are a number of 1 + 1 dimensional field theoretic models that can be solved exactly. The solubility of these models have been studied from various points of view [1-11]. Normally, these models are formulated in terms of a fermion field with a vector or axial-vector or a chiral coupling. More recently, however, there has been an interest in a model [12,13] where the fermion has both vector and axial vector couplings of arbitrary strength. This model reduces to all other known models in various limits. In this brief report, we show how this model can be solved in its generality within the path integral formalism. We compare our results with those obtained through a point-splitting regularization [13] and point out various characteristics of the model.

To begin with, let us consider a fermion in 1 + 1 dimension interacting with an external spin 1 field, described by

\[ L = i \bar{\psi} \gamma^\mu \left( \partial_\mu - i (1 + r \gamma_5) A_\mu \right) \psi \]

(1)

Here ‘r’ is an arbitrary real parameter and we have used the familiar identities of (1 + 1) dimensions in the last line of Eq. (1). (See ref. 10 for notations, identities and details.) Let us define

\[ \tilde{A}_\mu = (\eta_{\mu\nu} + r \epsilon_{\mu\nu}) A^\nu \]

(2)

and note that in 1 + 1 dimensions, we can write

\[ \tilde{A}_\mu = \partial_\mu \sigma + \epsilon_{\mu\nu} \partial^\nu \rho \]

(3)

so that the Lagrangian in Eq. (1) takes the form

\[ L = i \bar{\psi} \gamma^\mu \left( \partial_\mu - i \tilde{A}_\mu \right) \psi = i \bar{\psi} \gamma^\mu \left( \partial_\mu - i \partial_\mu \sigma - i \gamma_5 \partial_\mu \rho \right) \psi \]

(4)

It is clear now that if we define

\[ \psi = e^{i(\sigma + \gamma_5 \rho)} \psi' \]

\[ \bar{\psi} = \bar{\psi}' e^{-i(\sigma - \gamma_5 \rho)} \]

(5)

then the Lagrangian in Eq. (4) reduces to a free theory, namely,

\[ L = i \bar{\psi} \gamma^\mu \left( \partial_\mu - i \partial_\mu \sigma - i \gamma_5 \partial_\mu \rho \right) \psi = i \bar{\psi}' \gamma^\mu \partial_\mu \psi' \]

(6)
In the path integral formalism, the Jacobian under the field redefinition in Eq. (5) is nontrivial [14] and we obtain the effective action by evaluating this Jacobian [9-11].

The evaluation of the Jacobian is straightforward and can be read off from ref. 10. However, we would like to emphasize that for the present case, we can define

\[ \tilde{A}_\mu^D = \epsilon_{\mu\nu} \tilde{A}^\nu \]  

which leads to the identity

\[ \gamma^\mu \tilde{A}_\mu = \gamma^\mu \left( \eta \tilde{A}_\mu + \xi \gamma_5 \tilde{A}_\mu^D \right) \]

with

\[ \eta + \xi = 1 \]  

and one can use the Euclidean Dirac operator

\[ \tilde{D}_E = \gamma_\mu \left( \partial_\mu - i \eta \gamma_5 \tilde{A}_\mu - i \xi \gamma_5 \gamma_\mu \tilde{A}_\mu^D \right) \]

(9)

to evaluate the Jacobian for the change of variables. We note here that it is this operator which provides the most general regularization which is consistent.

For an infinitesimal field redefinition

\[ \psi = e^{i(\epsilon(x) + \gamma_5 \bar{\epsilon}(x))} \psi' \]

\[ \bar{\psi} = \bar{\psi}' e^{-i(\epsilon(x) - \gamma_5 \bar{\epsilon}(x))} \]

(10)

the Jacobian with the regularization in Eq. (9) can be read off from ref. 10 (simply replace \( A_\mu \rightarrow \tilde{A}_\mu \) and \( A_{5\mu} \rightarrow \tilde{A}_\mu^D \)) and has the form

\[ J = \exp \left[ -\frac{i}{2\pi} \int d^2 x_E \left( \eta \epsilon(x) \epsilon_{\mu\nu} \tilde{F}_{\mu\nu}^D + \xi \bar{\epsilon}(x) \epsilon_{\mu\nu} \tilde{F}_{\mu\nu} \right) \right] \]

(11)

which when rotated to Minkowski space has the form

\[ J = \exp \left[ \frac{i}{\pi} \int d^2 x \left( \eta \epsilon(x) \partial_\mu \tilde{A}_\mu + \xi \bar{\epsilon}(x) \partial_\mu \tilde{A}_\mu^D \right) \right] \]

(12)

The anomaly equations for the vector and the axial-vector currents, then, follow to be \((j_V^\mu = \bar{\psi} \gamma^\mu \psi, j_A^\mu = \bar{\psi} \gamma_5 \gamma^\mu \psi)\)

\[ \partial_\mu j_V^\mu = \frac{\eta}{\pi} \partial_\mu \tilde{A}_\mu = \frac{\eta}{\pi} \left( \partial_\mu A_\mu + r \epsilon^{\mu\nu} \partial_\nu A_\nu \right) \]

\[ \partial_\mu j_A^\mu = -\frac{\xi}{\pi} \partial_\mu \tilde{A}_\mu^D = -\frac{\xi}{\pi} \left( r \partial_\mu A_\mu^D + \epsilon^{\mu\nu} \partial_\nu A_\nu \right) \]

(13)
These, of course, reduce to the well known results [10] when \( r = 0 \) and we note that for \( j^\mu = j^\mu_\psi - r j^\mu_A = \overline{\psi} \gamma^\mu (1 + r \gamma_5) \psi, \)

\[
\partial_\mu j^\mu = \frac{1}{\pi} (\eta + \xi r^2) \partial_\mu A^\mu + \frac{r}{\pi} \epsilon^{\mu\nu} \partial_\mu A_\nu
\]  
(14)

Here \( j^\mu \) is the current of our theory in Eq. (1) and we note that it is anomalous for \( r \neq 0 \) for any choice of regularization.

The Jacobian for the finite field redefinition in Eq. (5) is again straightforward following ref. 9 and we obtain

\[
J = \exp \left[ -\frac{i}{2\pi} \int d^2x (\eta_{\mu\sigma} + r \epsilon_{\mu\sigma})(\eta_{\nu\tau} + r \epsilon_{\nu\tau}) A^\sigma \left( \eta \frac{\partial^\mu \partial^\nu}{\Box} + \xi \epsilon^\lambda \epsilon^{\rho\nu} \frac{\partial_\lambda \partial_\rho}{\Box} \right) A^\tau \right] \]  
(15)

which leads to the effective action

\[
Z[A_\mu] = N \int \mathcal{D}\overline{\psi} \mathcal{D}\epsilon e^{i \int d^2x \mathcal{L}}
\]

\[
= N' \exp \left[ -\frac{i}{2\pi} \int d^2x (\eta_{\mu\sigma} + r \epsilon_{\mu\sigma})(\eta_{\nu\tau} + r \epsilon_{\nu\tau}) A^\sigma \left( \eta \frac{\partial^\mu \partial^\nu}{\Box} + \xi \epsilon^\lambda \epsilon^{\rho\nu} \frac{\partial_\lambda \partial_\rho}{\Box} \right) A^\tau \right] \]  
(16)

It is straightforward to check that this generating functional yields the anomaly equation (14). This can also be checked to coincide with the result obtained through the point-splitting regularization [13].

Next, let us consider the general model described by [12,13]

\[
\mathcal{L}_{\text{TOT}} = -\frac{1}{4} \left( \partial_\mu B_\nu - \partial_\nu B_\mu \right) \left( \partial^\mu B^\nu - \partial^\nu B^\mu \right) + \frac{\mu_0^2}{2} B_\mu B^\mu + i \overline{\psi} \gamma^\mu \left( \partial_\mu - i (1 + r \gamma_5) (A_\mu + e B_\mu) \right) \psi + J_\mu B^\mu
\]  
(17)

In the generating functional, the fermionic fields can be integrated out to give the effective action in Eq. (16) with the substitution

\[
A_\mu \rightarrow A_\mu + e B_\mu
\]  
(18)

As a result, we can write

\[
Z_{\text{TOT}} (A_\mu, J_\mu) = N \int \mathcal{D}B_\mu e^{i \mathcal{S}_{\text{eff}}(B_\mu, A_\mu, J_\mu)} \]  
(19)
where
\[ S_{\text{eff}} = \int d^2 x \left( \frac{1}{2} B_\mu P^{\mu \nu} B_\nu + B_\mu Q^\mu + \frac{1}{2} A_\mu R^{\mu \nu} A_\nu \right) \] (20)

with
\[ P^{\mu \nu} = \eta^{\mu \nu} \left( \square + \mu_0^2 + \frac{\epsilon^2}{\pi} \left( \xi + \eta r^2 \right) \right) \]
\[ - \left( 1 + \frac{\epsilon^2}{\pi} (1 + r^2) \square^{-1} \right) \partial^\mu \partial^\nu - \frac{\epsilon^2 r}{\pi} \left( \epsilon^\sigma \partial^\nu + \epsilon^\sigma \partial^\mu \right) \partial_\sigma \square^{-1} \]
\[ Q^\mu = J^\mu - \frac{\epsilon}{\pi} \left( - (\xi + \eta r^2) A^\mu \right) \]
\[ + \left( 1 + r^2 \right) \frac{\partial^\mu \partial \cdot A}{\square} + r \left( \epsilon^\sigma \partial^\nu + \epsilon^\sigma \partial^\mu \right) \partial_\sigma \square^{-1} A_\nu \] (21)
\[ R^{\mu \nu} = - \frac{1}{\pi} \left( \epsilon \eta^\mu \nu + (1 + r^2) \frac{\partial^\mu \partial^\nu}{\square} + r \left( \epsilon^\sigma \partial^\nu + \epsilon^\sigma \partial^\mu \right) \partial_\sigma \square^{-1} \right) \]

The action in Eq. (20) is quadratic in \( B_\mu \) and hence the generating functional is easily obtained to be
\[ Z_{\text{TOT}}(A_\mu, J_\mu) = N' \exp \left[ - \frac{i}{2} \int d^2 x \left( Q^\mu P^{-1}_{\mu \nu} Q_\nu + A_\mu R^{\mu \nu} A_\nu \right) \right] \] (22)

Note that if we define
\[ P^{-1}_{\mu \nu} = a \eta_{\mu \nu} + b \partial_\mu \partial_\nu + c \left( \epsilon_\sigma \partial_\nu + \epsilon_\sigma \partial_\mu \right) \partial_\sigma \]
(23)

then, from
\[ P^{\mu \nu} P^{-1}_{\nu \lambda} = \delta^\mu_\lambda \]
(24)

we can determine
\[ a = \frac{1}{\square + \mu_0^2 + \frac{\epsilon^2}{\pi} \left( \xi + \eta r^2 \right) + \frac{\epsilon^2 r}{\pi} \left( \mu_0^2 + \frac{\epsilon^2}{\pi} \left( \xi + \eta r^2 \right) - \frac{\epsilon^2 r}{\pi} \right)} = \frac{1}{\square + m_{\text{phys}}^2} \]

\[ b = \frac{1}{\mu_0^2 - \frac{\epsilon^2}{\pi} \left( \eta + \xi r^2 \right)} \frac{\left( \square + \frac{\epsilon^2 r}{\pi} \right)}{\left( \square + m_{\text{phys}}^2 \right)} \frac{1}{\square} \]

\[ c = \frac{\epsilon^2 r}{\pi \left( \mu_0^2 - \frac{\epsilon^2}{\pi} \left( \eta + \xi r^2 \right) \right)} \frac{1}{\left( \square + m_{\text{phys}}^2 \right)} \frac{1}{\square} \]

5
We can also rewrite

\[ m_{\text{phys}}^2 = \mu_0^2 \left( 1 - \frac{\eta e^2}{\pi \mu_0^2} (1 - r^2) \right) \left( 1 + \frac{\xi e^2}{\pi \mu_0^2} (1 - r^2) \right) \frac{1}{\left( 1 - \frac{e^2}{\pi \mu_0^2} (\eta + \xi r^2) \right)} \]  

which coincides with the result obtained through the point-splitting regularization [13].

The propagator for the \( B_\mu \)-field is now seen to be

\[
D_{\mu\nu} = \frac{1}{\left( \Box + m_{\text{phys}}^2 \right)} \left[ \eta_{\mu\nu} + \frac{\left( \Box + \frac{e^2}{\pi} (1 + r^2) \right)}{\left( \mu_0^2 - \frac{e^2}{\pi} (\eta + \xi r^2) \right)} \partial^\mu \partial^\nu - \frac{e^2 r}{\pi} \left( \epsilon_{\sigma\mu} \partial_{\nu} + \epsilon_{\sigma\nu} \partial_{\mu} \right) \partial^\sigma \partial^\nu - 1 \right]
\]

We end our discussion by noting that the term quadratic in \( Q_\mu \) in Eq. (22) gives rise to a term which is quadratic in \( A_\mu \). (See definition in Eq. (21).) Consequently, the term quadratic in \( A_\mu \) will have a structure of the form

\[
\frac{1}{2} \int d^2 x \ A_\mu R^{\mu\nu} A_\nu
\]

in the exponent of the generating functional. Consequently, the anomaly equation derived from this generating functional will differ from that in Eq. (14). The reason for this is not hard to understand. Since both \( A_\mu \) and \( B_\mu \) couple to the same current, the one-loop diagrams contributing to the anomaly will have two parts.

While Eq. (14) contains the contribution from the first diagram alone, it is the second diagram which is responsible for the modification in the anomaly (also in Eq. (28)).

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