WHEN IS THE AUTOMORPHISM GROUP OF AN AFFINE VARIETY NESTED?

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Abstract. For an affine algebraic variety $X$, we study the subgroup $\text{Aut}_{\text{alg}}(X)$ of the group of regular automorphisms $\text{Aut}(X)$ of $X$ generated by all the connected algebraic subgroups. We prove that $\text{Aut}_{\text{alg}}(X)$ is nested, i.e., is a direct limit of algebraic subgroups of $\text{Aut}(X)$, if and only if all the $\mathbb{G}_a$-actions on $X$ commute. Moreover, we describe the structure of such a group $\text{Aut}_{\text{alg}}(X)$.

1. Introduction

It was proved in 1958 by Matsusaka \cite{13} that the neutral component $\text{Aut}^0(Y)$ of the automorphism group of a projective irreducible algebraic variety $Y$ is an algebraic group. For affine algebraic varieties the situation is quite different. For example the automorphism group $\text{Aut}(\mathbb{A}^n)$ of an affine $n$-space contains a copy of a polynomial ring in $n - 1$ variables. Hence, there is no way to put a structure of an algebraic group on $\text{Aut}(\mathbb{A}^n)$ for $n \geq 2$. In \cite{16} Shafarevich introduced the notion of ind-group. It is known that for an affine variety $X$ its automorphism group $\text{Aut}(X)$ has a natural structure of an ind-group (see \cite[Section 5]{5} and \cite[Section 2]{9} for details).

The base field $\mathbb{K}$ is algebraically closed of zero characteristic, and the additive group of $\mathbb{K}$ is denoted by $\mathbb{G}_a$. Through the whole paper $X$ denotes an irreducible affine algebraic variety. We call an element $g \in \text{Aut}(X)$ algebraic if there is an algebraic subgroup $G$ of the ind-group $\text{Aut}(X)$. We also denote by $U(X) \subset \text{Aut}(X)$ the (possibly trivial) subgroup generated by all the $\mathbb{G}_a$-actions. It is also called the special automorphism group and denoted by $S\text{Aut}(X)$.

In \cite{9} and \cite{15} the neutral component $\text{Aut}^0(X)$ of the ind-group of automorphisms $\text{Aut}(X)$ of an affine surface $X$ has been studied. Note that $\text{Aut}^0(X)$ is a closed subgroup of $\text{Aut}(X)$. The equivalence of the following conditions is claimed:

- all elements of $\text{Aut}^0(X)$ are algebraic;
- the subgroup $\text{Aut}^0(X) \subset \text{Aut}(X)$ is nested, i.e., is a direct limit of algebraic subgroups;
- $\text{Aut}^0(X) = \mathbb{T} \ltimes U(X)$, where $\mathbb{T}$ is a maximal subtorus of $\text{Aut}(X)$.

Our intention is to prove this result independently in arbitrary dimension. Originally, we were motivated by Conjecture 1.1 and Question 1.2.

Conjecture 1.1 (P. Zaidenberg, Feb.'13). An affine variety does not admit additive group actions if and only if the neutral component of the automorphism group is an algebraic torus.

The statement that the neutral component is a torus was proved in \cite[Theorem 1.3]{10} under the assumption that $\text{Aut}^0(X)$ is finite-dimensional and in \cite[Proposition 3.2]{11} for $\mathbb{T}$-varieties satisfying certain conditions.

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**Question 1.2 (Kraft).** Which affine varieties have automorphism groups comprised of algebraic elements?

We provide a partial answer to Question 1.2 in Theorem 5.1. In the direction of the intended generalization we prove in the present paper the following statement.

**Theorem 1.3.** Given an affine variety $X$, let $\text{Aut}_{\text{alg}}(X)$ be the subgroup of $\text{Aut}(X)$ generated by all connected algebraic subgroups. The following conditions are equivalent:

1. $\mathcal{U}(X)$ is abelian;
2. all elements of $\text{Aut}_{\text{alg}}(X)$ are algebraic;
3. the subgroup $\text{Aut}_{\text{alg}}(X) \subset \text{Aut}(X)$ is a closed nested ind-subgroup;
4. $\text{Aut}_{\text{alg}}(X) = \mathbb{T} \ltimes \mathcal{U}(X)$, where $\mathbb{T}$ is a maximal subtorus of $\text{Aut}(X)$, and $\mathcal{U}(X)$ is closed in $\text{Aut}(X)$.

**Remark 1.4.** If $\dim X \geq 2$, then $\mathcal{U}(X)$ is either trivial or infinite-dimensional. Indeed, if there exists a $\mathbb{G}_a$-action corresponding to a locally nilpotent derivation $\partial$, then we have an infinite-dimensional unipotent subgroup $\exp((\ker \partial) \cdot \partial) \subset \mathcal{U}(X)$, e.g. see [6, Theorem 6.3].

We expect that Theorem 1.3 holds if we replace $\text{Aut}_{\text{alg}}(X)$ by $\text{Aut}^o(X)$. In particular, we formulate the following extension of Conjecture 1.1.

**Conjecture 1.5.** If $X$ is an affine variety, then $\mathcal{U}(X)$ is abelian if and only if $\text{Aut}^o(X)$ is nested.

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## 2. Preliminaries

### 2.1. Derivations and group actions.

Recall that $X$ is an irreducible affine algebraic variety. A derivation $\delta$ is called *locally finite* if it acts locally finitely on $\mathcal{O}(X)$, i.e., for any $f \in \mathcal{O}(X)$ there is a finite-dimensional vector subspace $V \subset \mathcal{O}(X)$ such that $f \in V$ and $V$ is stable under action of $\delta$. A derivation $\delta \in \text{Der}(\mathcal{O}(X))$ is called *semisimple* if there exists a basis $\{f_i \mid i \in \mathbb{N}\}$ of the vector space $\mathcal{O}(X)$ such that $\delta(f_i) \in \mathbb{K}f_i$. Finally, a derivation $\delta \in \text{Der}(\mathcal{O}(X))$ is called *locally nilpotent* if for any $f \in \mathcal{O}(X)$ there exists $n \in \mathbb{N}$ (which depends on $f$) such that $\delta^n(f) = 0$. There is a one-to-one correspondence between locally nilpotent derivations on $\mathcal{O}(X)$ and $\mathbb{G}_a$-actions on $X$ given by the map $\delta \mapsto \{t \mapsto \exp(t\delta)\}$. We denote the set of locally nilpotent derivations (LNDs) on $\mathcal{O}(X)$ by $\text{LND}(X)$. We call two LNDs $\partial_1, \partial_2$ *equivalent* if their kernels coincide. By [9, Principle 12], equivalence of $\partial_1$ and $\partial_2$ implies that $\partial_1 = c\partial_2$ for some $c \in \text{Frac ker} \partial_1$. If $\partial_1$ and $\partial_2$ are equivalent, we call the corresponding $\mathbb{G}_a$-actions $\exp(t\partial_1)$ and $\exp(t\partial_2)$ equivalent as well. Note that these $\mathbb{G}_a$-actions have the same general orbits and hence commute. Indeed, a general orbit is an affine line, and any two unipotent elements of $\text{Aut}(\mathbb{A}^1) \cong \mathbb{G}_a \ltimes \mathbb{G}_m$ commute.

An element $u \in \text{Aut}(X)$ is called *unipotent* if $u = \exp(\partial)$ for some $\partial \in \text{LND}(X)$. An (ind-)subgroup $U \subset \text{Aut}(X)$ is called *unipotent* if each $u \in U$ is unipotent.

**Definition 2.1.** We denote the Lie subalgebra of the Lie algebra of derivations $\text{Der}(\mathcal{O}(X))$ on $X$ generated by all LNDs by

$$\mathfrak{u}(X) = \langle \partial \mid \partial \in \text{LND}(X) \rangle,$$
and the automorphism subgroup generated by the unipotent elements by
\[ U(X) = \langle \exp(\partial) \mid \partial \in \text{LND}(X) \rangle \subset \text{Aut}(X). \]

**Lemma 2.2.** The unipotent-generated subgroup \( U(X) \) is abelian if and only if all LNDs on \( X \) are equivalent.

**Proof.** If all LNDs are equivalent, then they share the same kernel and coincide up to a multiplication by kernel elements. Thus, their exponents commute and comprise \( \text{U}(X) \). Hence, we can assume that \( \partial_1 \) and \( \partial_2 \) commute. Since \( \partial_1 \) and \( \partial_2 \) are not equivalent, there exists \( f \in \ker \partial_1 \) that does not belong to \( \ker \partial_2 \). Hence, \([\partial_2, f\partial_1] = \partial_2(f)\partial_1 \neq 0 \). Non-commutativity of the LNDs \( \partial_2 \) and \( f\partial_1 \) implies non-commutativity of the \( \mathbb{G}_a \)-actions \( \{\exp(t\partial_2) \mid t \in \mathbb{K}\} \) and \( \{\exp(tf\partial_1) \mid t \in \mathbb{K}\} \). The proof follows. \( \square \)

### 2.2. Ind-groups

The notion of an ind-group goes back to Shafarevich who called these objects infinite dimensional groups (see [16]). We refer to [5] and [9, Section 2] for basic notions in this context.

**Definition 2.3.** By an affine ind-variety we mean an injective limit \( V = \lim_{\longrightarrow} V_i \) of an ascending sequence \( V_0 \hookrightarrow V_1 \hookrightarrow V_2 \hookrightarrow \ldots \) such that the following holds:

1. \( V = \bigcup_{k \in \mathbb{N}} V_k \);
2. each \( V_k \) is an affine algebraic variety;
3. for all \( k \in \mathbb{N} \) the embedding \( V_k \hookrightarrow V_{k+1} \) is closed in the Zariski topology.

For simplicity we will call an affine ind-variety simply an ind-variety.

An ind-variety \( V \) has a natural topology: a subset \( S \subset V \) is called open, resp. closed, if \( S_k := S \cap V_k \subset V_k \) is open, resp. closed, for all \( k \in \mathbb{N} \). A closed subset \( S \subset V \) has a natural structure of an ind-variety and is called an ind-subvariety.

The product of ind-varieties is defined in the obvious way. A morphism between ind-varieties \( V = \bigcup_k V_k \) and \( W = \bigcup_m W_m \) is a map \( \phi : V \to W \) such that for every \( k \in \mathbb{N} \) there is an \( m \in \mathbb{N} \) such that \( \phi(V_k) \subset W_m \) and that the induced map \( V_k \to W_m \) is a morphism of algebraic varieties. This allows us to give the following definition.

**Definition 2.4.** An ind-variety \( G \) is said to be an ind-group if the underlying set \( G \) is a group such that the map \( G \times G \to G, \ (g, h) \mapsto gh^{-1} \), is a morphism.

A closed subgroup \( H \) of \( G \) is a subgroup that is also a closed subset. Then \( H \) is again an ind-group with respect to the induced ind-variety structure. A closed subgroup \( H \) of an ind-group \( G = \lim_i G_i \) is called an algebraic subgroup if \( H \) is contained in \( G_i \) for some \( i \).

From [5, Proposition 5.6.5(1)] it follows that algebraic subgroups of \( \text{Aut}(X) \) are exactly algebraic groups that act regularly on \( X \).

The next result can be found in [5, Section 5] and [9, Section 2].

**Proposition 2.5.** Let \( X \) be an affine variety. Then \( \text{Aut}(X) \) has the structure of an ind-group such that a regular action of an algebraic group \( G \) on \( X \) induces an ind-group homomorphism \( G \to \text{Aut}(X) \).

**Definition 2.6.** An element \( g \in \text{Aut}(X) \) is called algebraic if there is an algebraic subgroup \( G \subset \text{Aut}(X) \) such that \( g \in G \).

**Definition 2.7.** An ind-group \( G \) is called nested if \( G = \lim_i G_i \), where \( G_i \) is an algebraic group and \( G_i \subset G_{i+1} \) is a closed subgroup for \( i = 1, 2, \ldots \).
If \( G \subset \text{Aut}(X) \) is nested, then there exists a Levi decomposition of the neutral component \( G^0 = L \times U \), where \( L \) is a reductive algebraic group and \( U \) is a normal ind-subgroup which consists of unipotent elements, see [9, Theorem 2.11]. In addition, there exists an algebraic subgroup \( H \subset G \) with same orbits on \( X \), i.e., \( G \cdot x = H \cdot x \) for any \( x \in X \), see [9, Proposition 2.17].

2.3. Lie algebras of ind-groups. For any ind-variety \( V = \bigcup_{k \in \mathbb{N}} V_k \) we can define the tangent space in \( x \in V \) in the obvious way: we have \( x \in V_k \) for \( k \geq k_0 \), and \( T_x V_k \subset T_x V_{k+1} \) for \( k \geq k_0 \), and then define
\[
T_x V := \bigcup_{k \geq k_0} T_x V_k,
\]
which is a vector space of countable dimension.

For an ind-group \( G \), the tangent space \( T_G \) has a natural structure of a Lie algebra which is denoted by \( \text{Lie} G \) (see [11, Section 4] and [5, Section 2] for details). By \( \text{Aut}_{\text{alg}}(X) \subset \text{Aut}(X) \) we denote the closure of the subgroup \( \text{Aut}_{\text{alg}}(X) \) in \( \text{Aut}(X) \) generated by all connected algebraic subgroups. By [5, Theorem 0.3.2] there is an injective antihomomorphism from the Lie algebra \( \text{Lie} \text{Aut}(X) \) into the Lie algebra \( \text{Der}(\mathcal{O}(X)) \) of derivations on \( X \). From now on, we will always identify \( \text{Lie} \text{Aut}(X) \) and \( \text{Lie} \text{Aut}_{\text{alg}}(X) \) with their images in \( \text{Der}(\mathcal{O}(X)) \). Note that \( \text{Lie} \text{Aut}_{\text{alg}}(X) \) contains all locally finite derivations because each such derivation \( \delta \) is contained in \( \text{Lie} G \) for some connected algebraic subgroup \( G \subset \text{Aut}(X) \).

3. The case: \( \mathcal{U}(X) \) is not abelian

Provided that the unipotent-generated subgroup \( \mathcal{U}(X) \) is not abelian, by Lemma 2.2 there exist non-equivalent \( \mathbb{G}_a \)-actions on \( X \). The aim of this section is to prove the following result.

**Proposition 3.1.** Assume that an affine variety \( X \) admits two non-equivalent \( \mathbb{G}_a \)-actions. Then

1. there exists a derivation \( \partial \) in the linear span of \( \text{LND}(X) \) which is not locally finite.
2. there exists a non-algebraic element in \( \mathcal{U}(X) \).

**Remark 3.2.** A variety \( X \) as in this Proposition cannot be of dimension \( \leq 1 \), otherwise all LNDs are equivalent. Thus, \( \dim X \geq 2 \).

Let \( \partial_1, \partial_2 \) be two locally nilpotent derivations corresponding to two non-equivalent \( \mathbb{G}_a \)-actions \( H_1, H_2 \) on \( X \), respectively, \( \mathfrak{p}_i = (\ker \partial_i) \cap (\text{Im} \partial_i) \), \( i = 1, 2 \) their plinth ideals, and \( v_1, v_2 \) their corresponding vector fields.

Let us consider a fibration \( X \to \mathbb{A}^1 \), \( p \mapsto f(p) \) for some \( f \in \ker \partial_1 \) such that \( f \notin \ker \partial_2 \). Then \( H_1 \)-orbits lie in its fibers, but general \( H_2 \)-orbits do not. Hence \( v_1(p) \) and \( v_2(p) \) are linearly independent at a general point. Since \( V(\mathfrak{p}_1), V(\mathfrak{p}_2) \) are proper closed subsets of \( X \), we can take a smooth point \( p \in X_{\text{reg}} \setminus (V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2)) \) such that \( v_1(p) \) and \( v_2(p) \) are linearly independent.

Let \( m_p \) be the maximal of \( \mathcal{O}(X) \) that corresponds to \( p \in X \). We operate in the \( m_p \)-adic completion of the local ring at \( p \)

\[
\hat{\mathcal{O}}_p(X) = \lim_{\longrightarrow} \mathcal{O}_p(X)/m_p^k \mathcal{O}_p(X).
\]
Lemma 3.4. For any homogeneous parts:

\[ \partial \]

Thus, we may take

\[ \partial \]

Proof. By Lemma 3.3, we may take \( \hat{\partial} \in \text{Der} \mathcal{O}(X) \) extended to \( \mathcal{O}_p(X) \). Moreover, each derivation of \( \mathcal{O}(X) \) is uniquely extended to \( \mathcal{O}_p(X) \) and each derivation of \( \mathcal{O}_p(X) \) is uniquely extended to a derivation of \( \hat{\mathcal{O}}_p(X) \) (see e.g., \([18\, \text{Tag 07PE}]\)) for each \( \delta \in \text{Der} \mathcal{O}(X) \) we denote its extension by \( \hat{\delta} \in \text{Der} \hat{\mathcal{O}}_p(X) \).

Since \( p \) is smooth, \( \hat{\mathcal{O}}_p(X) = \mathbb{Z}[x_1, x_2, \ldots, x_n] \) is a formal power series ring (by the Cohen structure theorem, e.g., see \([13]\)). Thus, we have a natural \( \mathbb{Z}_{\geq 0} \)-grading on \( \hat{\mathcal{O}}_p(X) \) by the minimum degree, which in turn induces the \( \mathbb{Z}_{\geq -1} \)-grading on \( \text{Der} \hat{\mathcal{O}}_p(X) \) via the formula

\[
\text{deg} \, \partial = \text{deg} \, \partial h - \text{deg} \, h
\]

for a homogeneous derivation \( \partial \) and any homogeneous element \( h \in \hat{\mathcal{O}}_p(X) \). Let \( f \) be an element of either \( \hat{\mathcal{O}}_p(X) \) or \( \text{Der} \hat{\mathcal{O}}_p(X) \). We denote by \( \text{LHC}(f) \) the homogeneous component of lowest degree and by \( f^{(d)} \) the \( d \)-th homogeneous component.

By our convention \( \hat{\partial}_i \in \text{Der} \hat{\mathcal{O}}_p(X) \) is the derivation induced by \( \partial_i, i = 1, 2 \). Since \( v_1(p) = (1, 0, 0, \ldots, 0) \) and \( v_2(p) = (0, 1, 0, \ldots, 0) \) indicate the lowest (linear) homogeneous components of \( \hat{\partial}_1, \hat{\partial}_2 \) respectively, we have \( \text{LHC}(\hat{\partial}_i) = \frac{\partial}{\partial x_i}, i = 1, 2 \).

Lemma 3.3. (1) \( \text{LHC}(g) \in \mathbb{K}[x_2, \ldots, x_n] \) for any \( g \in \ker \hat{\partial}_1 \).

(2) The map \( \ker \hat{\partial}_1 \to \mathbb{K}[x_2, \ldots, x_n] \) which maps \( g(x_1, \ldots, x_n) \in \ker \hat{\partial}_1 \) to \( g(0, x_2, \ldots, x_n) \) is an isomorphism of algebras.

The same holds if we switch \( x_1 \) with \( x_2 \) and \( \hat{\partial}_1 \) with \( \hat{\partial}_2 \) respectively.

Proof. The first assertion is straightforward:

\[
\hat{\partial}_1 g = 0 \implies \frac{\partial \text{LHC}(g)}{\partial x_1} = 0 \implies \text{LHC}(g) \in \mathbb{K}[x_2, \ldots, x_n].
\]

The second assertion is that for any \( g_0 \in \mathbb{K}[x_2, \ldots, x_n] \) there exists a unique element \( g \in \ker \hat{\partial}_1 \) such that \( g_0 = g(0, x_2, \ldots, x_n) \). Let us split the equation \( \hat{\partial}_1 g = 0 \) into homogeneous parts:

\[
0 = (\hat{\partial}_1 g)(k) = (\hat{\partial}_1 g(0)) + \cdots + (\hat{\partial}_1 g(k)) + \frac{\partial}{\partial x_1} g(k+1), \quad k = 0, 1, \ldots
\]

Thus, \( \frac{\partial}{\partial x_1} g(k+1) = -\sum_{i=0}^{k} (\hat{\partial}_i g(k)) \), and \( g(k+1) \) is uniquely determined by lower homogeneous components up to \( x_1 \)-free monomials. But the \( x_1 \)-free monomials of \( g \) comprise exactly \( g(0, x_2, \ldots, x_n) \). Thus, all homogeneous components of \( g \) are uniquely constructed by induction on the degree from the \( x_1 \)-free part \( g(0, x_2, \ldots, x_n) = g_0 \).

The statement for \( \hat{\partial}_2 \) is analogous. \( \square \)

Lemma 3.4. For any \( d > 1 \) there are elements \( f_i \in \ker \partial_i, i = 1, 2 \) such that \( \partial = f_1 \partial_1 + f_2 \partial_2 \in \mathcal{U}(X) \) satisfies

\[
\text{LHC}(\hat{\partial}) = x_2^d \frac{\partial}{\partial x_1} + x_1^d \frac{\partial}{\partial x_2}.
\]

Proof. By Lemma 3.3 we may take \( g_1 \in \ker \hat{\partial}_1 \) such that \( \text{LHC}(g_1) = x_2^d \). Since \( \mathfrak{p}_1 \notin \mathfrak{m}_p \), by \([13\, \text{Lem. 3.2}]\), \( \ker \hat{\partial}_1 \) equals the \( (\mathfrak{m}_p \cap \ker \partial_1) \)-adic completion of \( \ker \partial_1 \). Thus, the images of \( \ker \partial_1 \) and \( \ker \hat{\partial}_1 \) in \( \hat{\mathcal{O}}_p(X)/\mathfrak{m}_p^{d+1} = \mathcal{O}_p(X)/\mathfrak{m}_p^{d+1} \mathcal{O}_p(X) \) coincide, where \( \mathfrak{m}_p = \mathfrak{m}_p \hat{\mathcal{O}}_p(X) \).

Therefore, there exists \( f_1 \in \ker \partial_1 \) such that \( \text{LHC}(f_1) = \text{LHC}(g_1) = x_2^d \). Analogously, there exists \( f_2 \in \ker \partial_2 \) such that \( \text{LHC}(f_2) = x_1^d \). The statement follows. \( \square \)

Proof of Proposition 3.1. (1) Let us take a derivation \( \partial \) as in Lemma 3.4 for \( d = 2 \), i.e.,

\[
\text{LHC}(\hat{\partial}) = x_2^2 \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial x_2}.
\]

It is enough to prove that \( \partial \) is not locally finite. Let \( f \in \mathcal{O}(X) \)
be such that $\text{LHC}(f) = x_1 + x_2$. Then for each $k \geq 1$

$$\text{LHC}(\partial^{k-1}f) = \sum_{i=0}^{k} c_{k,i} x_1^i x_2^{k-i},$$

where $c_{k,i} \in \mathbb{Z}_{\geq 0}$ and $\sum_{i=0}^{k} c_{k,i} > 0$. Thus, ord $\partial^{k-1}f = k$, hence a sequence $\{\partial^k f \mid k = 0, 1, \ldots\}$ spans an infinite-dimensional subspace of $\mathcal{O}(X)$.

(2) In terms of Lemma 3.3, let

$$g = \exp(f_1 \partial_1) \circ \exp(f_2 \partial_2).$$

Then $g$ belongs to $\mathcal{U}(X)$, fixes $p$ and induces an automorphism $g^*$ of $\hat{\mathcal{O}}_p(X)$ that preserves the subalgebra $\mathcal{O}(X)$. A direct calculation shows that the linear operator $h = g^* - \text{id} \in \text{End} \hat{\mathcal{O}}_p(X)$ satisfies the following equality:

$$\text{LHC}(h(x_1^{q_1} x_2^{q_2})) = a_1 x_1^{a_1-1} x_2^{d+a_2} + a_2 x_1^{d+a_1} x_2^{-1},$$

where $x_i^{-1}$ is zero by definition, $i = 1, 2$. Moreover, $h(x_i)$ for $i > 2$ is of degree at least $d + 1$, if nonzero. Hence, for a given $f \in \hat{\mathcal{O}}_p(X)$ such that $\text{LHC}(f) = P(x_1, x_2)$ is a polynomial of degree $s > 0$ with positive integer coefficients, $\text{LHC}(h(f))$ is again a polynomial in $x_1, x_2$ of degree $s + d - 1$ with positive integer coefficients.

Let us take $f \in \mathcal{O}(X)$ such that $\text{LHC}(f) = x_1$ and let $F \subset \mathcal{O}(X)$ be a minimal subspace that contains $f$ and is $h$-stable. Since $h^i(f) \in F$ and $\deg(\text{LHC}(h^i(f))) = 1 + i(d - 1)$ for any $i \in \mathbb{Z}_{\geq 0}$, $F$ is infinite-dimensional. We claim that $g$ is not algebraic. Indeed, if $g$ were algebraic, then $g^*$ would act locally finitely on $\mathcal{O}(X)$, and so would $h$. The claim follows.

Example 3.5. By Jung–Van der Kulk’s theorem, the automorphism group of the affine plane $X = \mathbb{A}^2$ equals the amalgamated product

$$\text{Aut}(\mathbb{A}^2) = \text{Aff}(\mathbb{A}^2) \ast_C \text{Aut}_{\pi_1}(\mathbb{A}^2),$$

where $\text{Aff}(\mathbb{A}^2)$ is the subgroup of affine transformations,

$$\text{Aut}_{\pi_1}(\mathbb{A}^2) = \{(x, y) \mapsto (ax + P(y), by + c) \mid a, b \in \mathbb{K}, c \in \mathbb{K}, P \in \mathbb{K}[y]\}$$

is the subgroup preserving the projection $\pi_1: \mathbb{A}^2 \to \mathbb{A}^1$, $(x, y) \mapsto x$, and $C$ is their intersection. Thus, if $u \in \text{Aut}_{\pi_1}(\mathbb{A}^2) \setminus C$ and $g \in \text{Aff}(\mathbb{A}^2) \setminus C$, then $u \cdot g u^{-1}$ is a product of two unipotent elements which is not algebraic.

Remark 3.6. There exists an affine surface $X$ (see [2]) with the huge automorphism group, i.e., such that $\text{Aut}(X)/\text{Aut}_{\text{alg}}(X)$ is not countably generated. We believe that $\text{Aut}^o(X)/\text{Aut}_{\text{alg}}(X)$ is uncountably generated as well.

4. **The Case: $\mathcal{U}(X)$ is Abelian**

We denote $\mathfrak{g} = \text{Lie Aut}(X) \subset \text{Der}(\mathcal{O}(X))$. The following lemma is well known and appeared in similar form in [4] Lemma 3.1 and [1] Theorem 2.1.

**Lemma 4.1.** Assume that $\mathfrak{g}$ is $\mathbb{Z}^r$-graded for $r > 0$ and consider a locally finite element $z \in \mathfrak{g}$ that does not belong to the zero component $\mathfrak{g}_0$. Then there exists a locally nilpotent homogeneous component of $z$ of non-zero weight.

**Proof.** Let us take the convex hull $P(z) \subset \mathbb{Z}^r \otimes \mathbb{Q}$ of component weights of $z$. Then for any non-zero vertex $v \in P(z)$ the corresponding homogeneous component is locally nilpotent. The details are left to the reader. \qed
In this section we assume that \( \mathcal{U}(X) \) is abelian. The next lemma is an adaptation of [4, Lemma 3.6] for locally finite elements.

**Lemma 4.2.** Let \( \delta \) be a locally finite derivation, and \( \partial \) be a locally nilpotent derivation. If \( \mathcal{U}(X) \) is abelian, then \( \delta - \partial \) is locally finite.

**Proof.** Since \( \delta \in \text{Der}(\mathcal{O}(X)) \) is a locally finite element, there is the Jordan decomposition into a sum of a locally nilpotent element \( \delta_s \) and a semisimple element \( \delta_c \) that belongs to the Lie algebra of some torus \( T \), e.g., see [4, Section 2] or [5, Prop. 7.6.1]. The character lattice \( M \cong \mathbb{Z}^r \) of \( T \) induces an \( M \)-grading \( \mathcal{O}(X) = \bigoplus_{\chi \in M} \mathcal{O}(X)_\chi \). The map \( \chi : T \to \mathbb{K}^\times \) induces the tangent map \( \text{Lie}T \to \mathbb{K} \), which we denote by the same letter. So, \( \delta_s a = \chi(\delta_s) a \) for \( a \in \mathcal{O}(X)_\chi \). Consider the homogeneous decomposition of \( \delta \) with respect to this grading, i.e., \( \delta = \sum_{\chi \in M} \partial_\chi \), where \( [\delta_s, \partial_\chi] = \chi(\delta_s) \partial_\chi; \chi \) is called the degree of \( \partial_\chi \). Note that \([\delta, \partial'] = [\delta_s, \partial'] \) for any LND \( \partial' \), since \([\delta_s, \partial'] = 0 \).

If \( \partial = \partial_0 \), then \( [\delta, \partial] = 0 \) and the difference of two commuting locally finite derivations \( \delta - \partial \) is again locally finite. If \( \partial \neq \partial_0 \), then by Lemma 11 there exists a locally nilpotent homogeneous component of \( \partial \), \( v \neq 0 \). By Lemma 2.2 for each \( \chi \in M \) we have \( \partial_\chi = c_\chi \partial_v \) for some \( c_\chi \) from the field of fractions of \( \ker \partial \); thus, \( c_\chi \) is a homogeneous rational function of degree \( \chi - v \).

So,
\[
[\delta, \partial] = [\delta_s, \sum_{\chi \in M} \partial_\chi] = \sum_{\chi \in M} \chi(\delta_s) \partial_\chi = \left( \sum_{\chi \in M} \chi(\delta_s) c_\chi \right) \partial_v.
\]

Taking \( \partial' = \sum_{\chi \neq 0} \frac{\partial_\chi}{\chi(\partial')} \partial_v \), we have \([\delta, \partial'] = \sum_{\chi \neq 0} c_\chi \partial_v = \partial_0 - \partial_0 \), where the zero component \( \partial_0 = c_0 \partial_v \) might be trivial.

Derivations \([\delta, \partial'] \) and \( \partial' \) are locally nilpotent, hence commute. Thus, applying [4, Lemma 2.4] to \( \delta \) and \(-\partial' \), we conclude that \( \delta - \partial + \partial_0 = \exp(\partial') \delta \exp(-\partial') \) is locally finite. Since \( \partial_0 \) commutes with both \( \delta \) and \( \partial - \partial_0 \), the difference of locally finite elements \( \partial_0 \) and \( \delta - \partial + \partial_0 \) is again locally finite. The claim follows.

Recall that \( u = \langle \partial \mid \partial \in \text{LND}(X) \rangle \) is the Lie subalgebra of \( \text{Der}(\mathcal{O}(X)) \) generated by LNDs. By \( t \) we denote the Lie algebra of a maximal subtorus \( T \subset \text{Aut}(X) \).

**Proposition 4.3.** If \( \mathcal{U}(X) \) is abelian, then every locally finite derivation on \( X \) belongs to the semidirect product of \( t \) and \( u \).

**Proof.** First note that any locally finite derivation on \( X \) belongs to \( \mathfrak{g} = \text{Lie Aut}(X) \). Now, the adjoint action of \( t \) on \( \mathfrak{g} \) induces a grading on \( \mathfrak{g} \) by the character lattice \( M \cong \mathbb{Z}^r \), which we fix. We proceed by induction on the number of homogeneous components of \( z \). If \( z \in \mathfrak{g}_0 \), then \( z \) commutes with \( t \). Thus, the semisimple part \( z_s \) commutes with \( t \) and due to the maximality of \( T \), \( z_s \) belongs to \( t \). Therefore, \( z = z_s + z_n \) belongs to the semidirect product of \( t \) and \( u \). If \( z \notin \mathfrak{g}_0 \), then there exists a locally nilpotent homogeneous component \( z_v \) of \( z \) (see Lemma 11). Hence, \( z - z_v \) is locally finite by Lemma 1.2 which belongs to the semidirect product of \( t \) and \( u \) by the induction hypothesis. Therefore, \( z = (z - z_v) + z_v \) also belongs to the semidirect product of \( t \) and \( u \). \( \square \)

**Proposition 4.4.** If \( \mathcal{U}(X) \) is abelian, then the group \( \text{Aut}_{\text{alg}}(X) \) coincides with \( T \ltimes \mathcal{U}(X) \) and \( \text{Aut}_{\text{alg}}(X) \) is a closed normal subgroup of \( \text{Aut}(X) \).

**Proof.** Let \( G \subset \text{Aut}(X) \) be a connected algebraic subgroup. Then the Lie algebra \( \text{Lie}G \) consists of locally finite derivations and, by Proposition 4.3, \( \text{Lie}G \subset t \oplus u \) as a vector space.
Since \( u \) is \( t \)-stable, there exists a decomposition \( u = \bigoplus_{i=1}^{\infty} \mathbb{K}\partial_i \) such that \([t, \partial_i] \subset \mathbb{K}\partial_i\) for each \( i = 1, 2, \ldots \). Thus, \( \text{Lie} \, G \subset t \oplus \bigoplus_{k=1}^{\infty} \mathbb{K}\partial_k \) for some \( k \). Therefore, \( G \subset \mathbb{T} \ltimes U_k \), where \( U_k = \exp(\bigoplus_{k=1}^{\infty} \mathbb{K}\partial_k) \) is a finite-dimensional \( \mathbb{T} \)-stable unipotent group, see also [5 Remark 17.3.3]. This means that \( \text{Aut}_{\text{alg}}(X) \) coincides with \( \mathbb{T} \ltimes U(X) \).

The group \( \text{Aut}_{\text{alg}}(X) \) is normal in \( \text{Aut}^\circ(X) \), because the set of connected algebraic subgroups of \( \text{Aut}(X) \) is stable under conjugation.

Let us prove that \( \text{Aut}_{\text{alg}}(X) \) is a closed ind-subgroup in \( \text{Aut}(X) \). Assume that \( U(X) \) is non-trivial, otherwise \( \text{Aut}_{\text{alg}}(X) = \mathbb{T} \) and the statement follows.

Fix some \( \partial_0 \in \text{LND}(X) \) and let \( x \in \mathcal{O}(X) \) be such that \( \partial_0(x) \neq 0 \). Fix also a nonzero element of the plinth ideal \( b \in \text{Im}(\partial_0) \cap \ker(\partial_0) \). Then any \( \text{LND} \, \partial \in \text{LND}(X) \) is equal to \( a\partial_0 \) for some \( a \in \ker(\partial_0) \). Indeed, there exists \( f \) such that \( \partial_0(f) = b \), hence \( \partial = \frac{a}{b}\partial_0 \), see also [3 Principle 12].

Let \( H \subset \text{Aut}(X) \) be some closed algebraic subset, then \( V = \langle H \circ x \rangle_{\mathbb{K}} \) and \( V_{\mathbb{T}} = \mathbb{T} \circ V \) are finite-dimensional subspaces in \( \mathcal{O}(X) \). Take \( g \in H \cap \text{Aut}_{\text{alg}}(X) \), then \( g \circ x \in V \). Consider the decomposition \( g = t \cdot u \), where \( t \in \mathbb{T}, u \in U(X) \). Then \( u \circ x \in V_{\mathbb{T}} \).

There exists \( \partial \in \text{LND}(X) = u \) such that \( u = \exp(\partial) \). Let \( \partial = \frac{a}{b}\partial_0 \), then

\[
\exp(\partial_0) \circ x = \sum_{i=0}^{s} \frac{\partial_0^i(x)}{i!} \quad \text{and} \quad u \circ x = \sum_{i=0}^{s} \frac{a^i \partial_0^i(x)}{b^i i!},
\]

where \( s \) is such that \( \partial_0^s(x) \neq 0 \) and \( \partial_0^{s+1}(x) = 0 \).

Choose an embedding \( X \hookrightarrow \mathbb{A}^n \) for some \( n \) to define the degree on \( \mathcal{O}(X) \) as usual:

\[
\deg(f) = \min\{\deg(F) \mid F \in \mathcal{O}(\mathbb{A}^n), F|_X = f \} \quad \text{for} \quad f \in \mathcal{O}(X).
\]

Then \( \deg(u \circ x) \leq \max\{\deg(v) \mid v \in V_{\mathbb{T}}\} \). Therefore, if the degree of \( u \circ x \) is not less than the degree of any summand of its decomposition \( \frac{a^i \partial_0^i(x)}{b^i i!} \), then for any \( i \)

\[
\deg(u \circ x) = i(\deg(a) - \deg(b)) + \deg(\partial_0^i(x)) \leq \max_{v \in V_{\mathbb{T}}} \{\deg(v)\}.
\]

Otherwise, there is a cancellation in the decomposition of \( u \circ x \), hence two summands have the same degree, and for some different \( i, j \)

\[
i(\deg(a) - \deg(b)) + \deg(\partial_0^i(x)) = j(\deg(a) - \deg(b)) + \deg(\partial_0^j(x)).
\]

In both cases there holds

\[
\deg(a) - \deg(b) \leq \max_{v \in V_{\mathbb{T}}} \{\deg(v)\} + \max_{i \leq s} \deg(\partial_0^i(x)),
\]

so \( \deg(a) \) is bounded by some number, say, \( N \) for any \( u = \exp(\frac{a}{b}\partial_0) \in U(X) \) such that \( u \circ x \in V_{\mathbb{T}} \).

Then

\[
U_N = \{\exp\left(\frac{a}{b}\partial_0\right) \mid a \in \ker(\partial_0), \deg(a) \leq N, \frac{a}{b}\partial_0 \in \text{LND}(X)\}
\]

is a finite-dimensional subgroup in \( U(X) \) such that \( H \cap \text{Aut}_{\text{alg}}(X) = H \cap (\mathbb{T} \ltimes U_N) \), which is closed. Since \( H \) is an arbitrary closed algebraic subset, \( \text{Aut}_{\text{alg}}(X) \) is a closed ind-subgroup.

\[\square\]

**Example 4.5.** Given an algebraic curve \( C \), the configuration space \( X = \mathcal{C}^n(C) \) is the algebraic variety consisting of all \( n \)-point subsets of \( C \). By [12 Theorem 1.2], for \( n > 2 \) the neutral component \( \text{Aut}^\circ(X) \) is nested and equals the semidirect product \( \mathbb{T} \ltimes \mathcal{U}(X) \), where \( \mathbb{T} \) is a two-dimensional algebraic torus and \( \mathcal{U}(X) \) is abelian.
5. Conclusion

In the following theorem we reformulate our result geometrically in terms of fibrations. We define an $A^1$-fibration on $X$ to be a dominant morphism $f: X \to Y$ whose general fibers are isomorphic to the affine line $A^1$ (see for example [17]). A $G_a$-action $H$ on an affine variety $X$ induces the quasi-affine variety $Y = \text{Spec} \mathcal{O}(X)^H$, e.g. see [17 Theorem 1], and the $A^1$-fibration $\mu: X \to Y$, on the fibers of which $H$ acts. Moreover, equivalent $G_a$-actions induce the same fibration.

**Theorem 5.1.** If $\text{Aut}_{\text{alg}}(X)$ consists of algebraic elements, then one of the following holds:

(i) there exists a unique $A^1$-fibration with a quasi-affine base $$\mu: X \to Z$$ and $U(X)$ consists of equivalent $G_a$-actions that act by translations on fibers of $\mu$, see [9 Section 6.1]. Moreover, $\text{Aut}_{\text{alg}}(X) = T \ltimes U(X)$, where $T$ is an algebraic torus of dimension $\leq \dim X$ and $U(X) \subset \text{Aut}(X)$ is an abelian ind-subgroup which is of infinite dimension if $\dim X \geq 2$. In particular, $\text{Aut}_{\text{alg}}(X)$ is a nested ind-group.

(ii) $U(X)$ is trivial. Then $\text{Aut}_{\text{alg}}(X)$ is a torus, and there are no $A^1$-fibrations with quasi-affine base.

**Proof.** First, assume that $U(X)$ is non-trivial. Let us prove that the case (i) holds.

Since all elements of $\text{Aut}_{\text{alg}}(X)$ are algebraic, Proposition 4.1 implies that all $G_a$-actions on $X$ are equivalent. It is well known that any non-trivial $G_a$-action $H = \{\exp(t\partial)\}$, where $\partial \in \text{LND}(X)$, induces an $A^1$-fibration over an quasi-affine base $X//H$. Indeed, the invariant ring $\mathcal{O}(X)^H = \ker \partial$ is of codimension one in $\mathcal{O}(X)$, so $X \to \text{Spec} \mathcal{O}(X)^H$ is a dominant morphism, whose general fibers are one-dimensional, irreducible and coincide with $A^1$ by [6 Cor. 1.29]. Conversely, assume that there are two distinct $A^1$-fibrations $\pi_1: X \to B_1$ and $\pi_2: X \to B_2$ with quasi-affine bases $B_1$ and $B_2$. For each fibration $\pi_i$, there exists an affine trivialization chart $U_i \subset B_i$, $\pi_i^{-1}(U_i) \cong U_i \times A^1$. Thus, in terms of $[8]$, $X$ is cylindrical. Following [8, Proposition 3.5] for both fibrations, we obtain two non-equivalent $G_a$-actions. This proves the first part of (i).

To prove the second part of (i) we note that $\mathfrak{g} = t \oplus u$ as a vector space by Proposition 4.3. If $\dim X = 1$, then $X \simeq A^1$ by [6 Cor. 1.29]. Otherwise, by [6 Principle 7], $u$ contains an infinite-dimensional subspace $\{f\partial \mid f \in \ker \partial\}$ for any LND $\partial$. Moreover, $u$ is graded by the character lattice of $T$, and one can construct an increasing sequence $u_1 \subset u_2 \subset \ldots \subset u$ of finite-dimensional $t$-stable subalgebras that exhaust $u$. So, we obtain a filtration by finite-dimensional Lie subalgebras

$\mathfrak{g} = \bigcup_{i=1}^{\infty} t \oplus u_i$.

There exists a commutative unipotent subgroup $U_i \subset \text{Aut}(X)$ such that $\text{Lie} U_i = u_i$ and $G_i = T \ltimes U_i \subset \text{Aut} X$ is an algebraic subgroup with the tangent Lie algebra $t \oplus u_i$. We claim that $\text{Aut}_{\text{alg}}(X) = \lim G_i$. Indeed, for any connected algebraic subgroup $G \subset \text{Aut}(X)$ we have $\text{Lie} G \subset t \oplus u_i$, hence $G \subset G_i$ and the claim follows.

Now assume that $U(X)$ is trivial, i.e., $X$ does not admit a $G_a$-action. By Proposition 4.3, $\mathfrak{g} = t$, where $t = \text{Lie} T$ for a maximal subtorus $T \subset \text{Aut}(X)$. Hence, $\text{Aut}_{\text{alg}}(X) = T$. □
Proof of Theorem 1.3. Proposition 3.1(2) and Proposition 4.4 provide the implications (2) ⇒ (1) and (1) ⇒ (4) respectively. The implications (4) ⇒ (3) and (3) ⇒ (2) are clear. The proof follows. □

REFERENCES

[1] I. Arzhantsev, S. Gaifullin, The automorphism group of a rigid affine variety, Math.Nachrichten 290 (2017), no. 5–6, 662–671.
[2] J. Blanc, A. Dubouloz, Affine surfaces with a huge group of automorphisms, Int. Math. Res. Notices 2015 (2015), no. 2, 422–459.
[3] I. S. Cohen, On the structure and ideal theory of complete local rings, Trans. Amer. Math. Soc. 59 (1946), no. 1, 54–106.
[4] H. Flenner, M. Zaidenberg, On the uniqueness of C∗-actions on affine surfaces, in: Affine Algebraic Geometry, Contemporary Mathematics, Vol. 369, Amer. Math. Soc. Providence, R.I., 2005.
[5] J.-P. Furter, H. Kraft, On the geometry of the automorphism groups of affine varieties, arXiv:1809.04175.
[6] G. Freudenburg, Algebraic Theory of Locally Nilpotent Derivations, Encyclopaedia of Mathematical Sciences, Vol. 136, Invariant Theory and Algebraic Transformation Groups, VII, Springer-Verlag, Berlin, 2006.
[7] R.V.Gurjar, K. Masuda, M. Miyanishi, $A^1$-fibrations on affine threefolds, Journal of Pure and Applied Algebra 216 (2012), 296 – 313.
[8] T. Kishimoto, Y. Prokhorov, M. Zaidenberg, Group actions on affine cones, Transformation Groups 18 (2013), 1137–1153.
[9] S. Kovalenko, A. Perepechko, M. Zaidenberg, On automorphism groups of affine surfaces, in: Algebraic Varieties and Automorphism Groups, Advanced Studies in Pure Mathematics 75 (2017), 207–286.
[10] H Kraft, Automorphism Groups of Affine Varieties and a Characterization of Affine n-Space, Trans. Moscow Math. Soc. 78 (2017), 171–186.
[11] S. Kumar, Kac-Moody Groups, Their Flag Varieties and Representation Theory, Progress in Mathematics, Vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
[12] V. Lin, M. Zaidenberg, Configuration Spaces of the Affine Line and their Automorphism Groups. In: Automorphisms in Birational and Affine Geometry. Levico Terme, Italy, October 2012. Springer Proceedings in Mathematics and Statistics, Vol. 79, Springer, 2014.
[13] T. Matsusaka, Polarized varieties, fields of moduli and generalized Kummer varieties of polarized varieties, Amer. J. Math. 80 (1958), 45–82.
[14] M. Miyanishi, $G_a$-actions and completions, Journal of Algebra 319 (2008), 2845–2854.
[15] A. Perepechko, M. Zaidenberg, Automorphism groups of affine $\text{ML}_2$-surfaces: dual graphs and Thompson groups, preprint: in preparation.
[16] I. R. Shafarevich, On some infinite-dimensional groups, Rend. Mat. e Appl. (5) 25 (1966), no. 1-2, 208–212.
[17] J. Winkelmann, Invariant rings and quasiaffine quotients, Math. Z. 244, (2003), 163–174.
[18] The Stacks Project Authors, Stacks Project, http://stacks.math.columbia.edu, 2016.

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