Hierarchy of one and many-parameter families of elliptic chaotic maps of cn and sn types

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Abstract

We present hierarchy of one and many-parameter families of elliptic chaotic maps of $\text{cn}$ and $\text{sn}$ types at the interval $[0, 1]$. It is proved that for small values of $k$ the parameter of the elliptic function, these maps are topologically conjugate to the maps of references [1, 2], where using this we have been able to obtain the invariant measure of these maps for small $k$ and thereof it is shown that these maps have the same Kolmogorov-Sinai entropy or equivalently Lyapunov characteristic exponent of the maps [1, 2]. As this parameter vanishes, the maps are reduced to the maps presented in above-mentioned reference. Also in contrary to the usual family of one-parameter maps, such as the logistic and tent maps, these maps do not display period doubling or period-n-tupling cascade transition to chaos, but they have single fixed point attractor at certain parameter values where they bifurcate directly to chaos without having period-n-tupling scenario exactly at these values of parameters whose Lyapunov characteristic exponent begin to be positive.

**Keywords:** Chaos, Jacobian elliptic function, Invariant measure, Entropy, Lyapunov characteristic exponent, Ergodic dynamical systems.

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1 Introduction

In the past twenty years dynamical systems, particularly one dimensional iterative maps have attracted much attention and have become an important area of research activity \[3\] specially elliptic maps \[4, 5\]. Here in this paper we propose a new hierarchy of families of one and many-parameter elliptic chaotic maps of the interval \([0, 1]\). By replacing trigonometric functions with Jacobian elliptic functions of \(cn\) and \(sn\) types, we have generalized the maps presented in references \[1, 2\] such that for small values of \(k\) the parameter of the elliptic function these maps are topologically conjugate to the maps of references \[1, 2\], where using this we have been able to obtain the invariant measure of these maps for small \(k\) and thereof it is shown that these maps have the same Kolmogorov-Sinai (KS) entropy \[6\] or equivalently Lyapunov characteristic exponent of the trigonometric chaotic maps of references \[1, 2\]. As this parameter vanishes, these maps are reduced to trigonometric chaotic maps. Also it is shown that just like the maps of references \[1, 2\], the new hierarchy of elliptic chaotic maps displays a very peculiar property, that is, contrary to the usual maps, these maps do not display period doubling or period-n-tupling, cascade transition to chaos \[9\] as their parameter \(\alpha\) (parameters) varies, but instead they have single fixed point attractor at certain region of parameters values, where they bifurcate directly to chaos without having period-n-tupling scenario exactly at the values of parameter whose Lyapunov characteristic exponent begins to be positive.

The paper is organized as follows: In section 2 we introduce new hierarchy of one-parameter families of elliptic chaotic maps of \(cn\) and \(sn\) types, then in order to make more general class of these families, by composing these maps, we generate hierarchy of families of many-parameters elliptic chaotic maps. For small values of the parameter of the elliptic functions, we have presented, in section 3, the equivalence of the elliptic chaotic maps with the trigonometric chaotic maps of \[1, 2\]. In section 4 we obtain Sinai-Ruelle-Bowen measure for hierarchy of one and many-parameter of elliptic chaotic maps for small \(k\). Section 5 is devoted to explain KS-entropy of elliptic chaotic maps. Paper ends with a brief conclusion.
2 One-parameter and many-parameter families of elliptic chaotic maps of cn and sn types

The families of one-parameter elliptic chaotic maps of \( \text{cn} \) and \( \text{sn} \) at the interval \([0, 1]\) are defined as the ratio of Jacobian elliptic functions of \( \text{cn} \) and \( \text{sn} \) types \([10]\) through the following equation:

\[
\Phi_N(x, \alpha) = \frac{\alpha^2 (Ncn^{-1}(\sqrt{x}))^2}{1 + (\alpha^2 - 1)(Ncn^{-1}(\sqrt{x}))^2},
\]

\[
\Phi_N(x, \alpha) = \frac{\alpha^2 (Nsn^{-1}(\sqrt{x}))^2}{1 + (\alpha^2 - 1)(Nsn^{-1}(\sqrt{x}))^2}.
\]

(2.1)

Obviously, these equations map the unit interval \([0, 1]\) into itself. Defining schwartzian derivative \([11]\)

\[
S(\Phi_N^\omega(x)) = \frac{\Phi_N^{\omega''}(x)}{\Phi_N^{\omega'}(x)} - \frac{1}{2} \left( \frac{\Phi_N^{\omega'}(x)}{\Phi_N^{\omega''}(x)} \right)^2,
\]

with a prime denoting a single differential, one can show that:

\[
S(\Phi_N^\omega(x)) = S\left(sn(Nsn^{-1}(\sqrt{x}))^2\right) \leq 0,
\]

since \( \frac{d}{dx}(sn(Nsn^{-1}(\sqrt{x})^2)) \) can be written as:

\[
\frac{d}{dx}(sn(Nsn^{-1}(\sqrt{x})^2)) = A \prod_{i=1}^{N-1} (x - x_i),
\]

with \(0 \leq x_1 < x_2 < x_3 < ... < x_{N-1} \leq 1\), then we have:

\[
S\left(sn(Nsn^{-1}(\sqrt{x})^2)\right) = -\frac{1}{2} \sum_{j=1}^{N-1} \frac{1}{(x - x_j)^2} - \left( \sum_{j=1}^{N-1} \frac{1}{(x - x_j)} \right)^2 < 0.
\]

Therefore, the maps \( \Phi_N^\omega(\alpha, x), \omega = 1, 2 \) are \((N-1)\)-nodal maps, that is, they have \((N-1)\) critical points in unit interval \([0, 1]\) \([11]\) and they have only a single period one stable fixed point or they are ergodic (See Figure 1).

As an example, we give below some of these maps \( \Phi_2^{(1)}(x, \alpha) \) and \( \Phi_2^{(2)}(x, \alpha) \):

\[
\Phi_2^{(1)}(x, \alpha) = \frac{\alpha^2 ((1 - k^2)(2x - 1) + k^2 x^2)^2}{(1 - k^2 + 2k^2 x - k^2 x^2)^2 + (\alpha^2 - 1)(1 - k^2)(2x - 1) + k^2 x^2)^2},
\]

\[
\Phi_2^{(2)}(x, \alpha) = \frac{\alpha^2 ((1 - k^2)(2x - 1) + k^2 x^2)^2}{(1 - k^2 + 2k^2 x - k^2 x^2)^2 + (\alpha^2 - 1)(1 - k^2)(2x - 1) + k^2 x^2)^2}.
\]
Elliptic Chaotic Maps

\[ \Phi^{(2)}_2(x, \alpha) = \frac{4\alpha^2 x(1 - k^2 x)(1 - x)}{(1 - k^2 x^2)^2 + 4x(1 - x)(\alpha^2 - 1)(1 - k^2 x)}. \]

Below we also introduce their conjugate or isomorphic maps which can be very useful in derivation of their invariant measure and calculation of their KS-entropy for small values of parameter \( k \). Conjugacy means that the invertible map \( h(x) = \frac{1-x}{x} \) (which maps \( I = [0, 1] \) into \([0, \infty)\)) transform maps \( \Phi^{(\omega)}_{N_1 N_2 \ldots N_n} (x, \alpha) \) into \( \tilde{\Phi}^{(\omega)}_{N_1 N_2 \ldots N_n} (x, \alpha), \omega = 1, 2 \) defined as:

\[
\begin{align*}
\tilde{\Phi}^{(1)}_{N_1 N_2 \ldots N_n} (x, \alpha) &= \left( h \circ \Phi^{(1)}_{N_1} \circ h^{-1} \right)(x, \alpha) = \frac{1}{\alpha^2} \text{cs}^2(N\text{sc}^{-1}(\sqrt{x})), \\
\tilde{\Phi}^{(2)}_{N_1 N_2 \ldots N_n} (x, \alpha) &= \left( h \circ \Phi^{(2)}_{N_1} \circ h^{-1} \right)(x, \alpha) = \frac{2}{\alpha} \text{cs}^2(N\text{cs}^{-1}(\sqrt{x})).
\end{align*}
\] (2.3)

Finally, by composing the maps introduced in (2.1) we can define many-parameter families of elliptic chaotic maps, where these many-parameter maps belong to different universal class than the single parameters ones (as it is shown at the end of section 5). Therefore, by denoting their composition by \( \Phi^{(\omega_1, \alpha_1), (\omega_2, \alpha_2), \ldots, (\omega_n, \alpha_n)}_{N_1 N_2 \ldots N_n} (x) \) we can write:

\[
\begin{align*}
\Phi^{(\omega_1, \alpha_1), (\omega_2, \alpha_2), \ldots, (\omega_n, \alpha_n)}_{N_1 N_2 \ldots N_n} (x) &= \left( \Phi^{(\omega_1)}_{N_1} \circ \Phi^{(\omega_2)}_{N_2} \circ \cdots \circ \Phi^{(\omega_n)}_{N_n} \right)(x) = \\
&= \Phi^{(\omega_1)}_{N_1} \left( \Phi^{(\omega_2)}_{N_2} \left( \cdots \left( \Phi^{(\omega_n)}_{N_n} (x, \alpha_n, \alpha_{(n-1)}, \ldots, \alpha_2, \alpha_1) \right) \right) \right) \\
&= \Phi^{(\omega_1, \alpha_1), (\omega_2, \alpha_2), \ldots, (\omega_n, \alpha_n)}_{N_1 N_2 \ldots N_n} (x).
\end{align*}
\] (2.4)

Thus obtained maps are many-parameter generalization of Ulam and von Neumann maps [7]. Since these maps consist of the composition of the \((N_k - 1)\)-nodals \((N_k = 1, 2, \ldots, n)\) with negative shwarzian derivative, they are \( N_1 N_2 \ldots N_n - 1 \)-nodals map and their shwarzian derivative is negative, too [11]. Therefore, these maps have at most \( N_1 N_2 \ldots N_N + 1 \) attracting periodic orbits [11]. Once again the composition maps have also a single period one fixed point or they are ergodic (see Figure 2).

As an example, we give below some of them: \( \Phi^{(1, \alpha_1), (1, \alpha_2)}_{2,2} (x) \) and \( \Phi^{(1, \alpha_1), (2, \alpha_2)}_{2,2} (x) \):

\[
\Phi^{(1, \alpha_1), (1, \alpha_2)}_{2,2} (x) = \frac{\alpha_1^2 (-Y^2 + 2XY + k_1^2 (Y - X)^2)^2}{(Y^2 - k_1^2 (Y - X)^2)^2 + (\alpha_1^2 - 1) (-Y^2 + 2XY + k_1^2 (Y - X)^2)^2},
\]

where:

\[
X = \alpha_2^2 (-1 + 2x + k_1^2 (1 - x)^2)^2 \quad \text{and} \quad Y = \left(1 - k_1^2 (1 - x)^2\right)^2 + \left(\alpha_2^2 - 1\right)(-1 + 2x + k_1^2 (1 - x)^2)^2
\]
Elliptic chaotic Maps

\[
\Phi^{(1,\alpha_1),(2,\alpha_2)}_{2,2}(x) = \frac{\alpha_1^2 (-Y^2 + 2XY + k_1^2(Y - X)^2)^2}{(Y^2 - k_1^2(Y - X)^2)^2 + (\alpha_1^2 - 1)(-Y^2 + 2XY + k_1^2(Y - X)^2)^2},
\]

where:

\[
X = 4\alpha_2^2x(1 - x)(1 - k_2^2x) \quad \text{and} \quad Y = (1 - k_2^2x^2)^2 + 4(\alpha_2^2 - 1)x(1 - x)(1 - k_2^2x).
\]

3 Topological conjugacy of elliptic chaotic maps with trigonometric ones for small values of elliptic parameter K

In order to obtain the SRB-measure of one and many-parameter families of elliptic chaotic maps for small value of elliptic parameter \( k \), we prove that elliptic chaotic maps are topologically conjugated with trigonometric chaotic maps of references [1, 2] for small value of elliptic parameter.

To do so, we consider the first order differential equation of elliptic chaotic maps given in (2.1). This differential equation can be obtained simply by taking derivation with respect to \( x \) from both sides of the relations (2.1), so we have:

\[
\frac{d\tilde{\Phi}^{(\omega)}_N}{dx} = \frac{N}{\alpha} \times \frac{\sqrt{\tilde{\Phi}^{(\omega)}_N(x, \alpha)} (1 + \frac{(1-k_2^2(1-(-1)^\omega))\alpha^2\tilde{\Phi}^{(\omega)}_N(x, \alpha)}{1-4(1+(-1)^\omega)})}{\sqrt{x} \left(1 + \frac{(1-k_2^2(1-(-1)^\omega))\alpha^2}{1-4(1+(-1)^\omega)}\right)}. \tag{3.1}
\]

For small values of \( k \), the above differential equation is reduced to:

\[
\frac{d\tilde{\Phi}^{(\omega)}_N(x, \alpha)}{dx} = \frac{N}{\alpha} \times \frac{\sqrt{\tilde{\Phi}^{(\omega)}_N(x, \alpha)} (1 + \frac{(1-k_2^2(1-(-1)^\omega))\alpha^2\tilde{\Phi}^{(\omega)}_N(x, \alpha)}{1-4(1+(-1)^\omega)})}{\sqrt{x} \left(1 + \frac{(1-k_2^2(1-(-1)^\omega))\alpha^2}{1-4(1+(-1)^\omega)}\right)}. \tag{3.2}
\]
Now, the dilatation map:

\[ x' = \frac{(1 - \frac{k^2}{4}(1 - (-1)^{\omega}))\alpha^2 x}{1 - \frac{k^2}{4}(1 + (-1)^{\omega})}, \quad \Phi'_N(x', \alpha) = \frac{(1 - \frac{k^2}{4}(1 - (-1)^{\omega}))\alpha^2 \Phi_N(x, \alpha)}{1 - \frac{k^2}{4}(1 + (-1)^{\omega})}, \]

reduces the differential equation (3.1) to:

\[ \frac{d\Phi'_N(x', \alpha)}{dx'} = \frac{N}{\alpha} \frac{\sqrt{\Phi'_N(x', \alpha)} (1 + \alpha^2 \Phi'_N(x', \alpha))}{\sqrt{x'}(1 + x')} \]

Integrating it, we get:

\[ \Phi'_N^{(1)}(x', \alpha) = \frac{1}{(1 - \frac{k^2}{2})} \Phi_N^{(1)}((1 - \frac{k^2}{2})x, \alpha) = \frac{1}{\alpha^2} \tan^2 \left( N \arctan(\sqrt{x'}) \right), \quad (3.3) \]

\[ \Phi'_N^{(2)}(x', \alpha) = (1 - \frac{k^2}{2})\Phi_N^{(2)}(\frac{x}{1 + \frac{k^2}{2}}, \alpha) = \frac{1}{\alpha^2} \cot^2 \left( N \arctan(\sqrt{\frac{1}{x'}}) \right). \quad (3.4) \]

Therefore, for small values of \( k \) (the parameter of the elliptic functions) elliptic chaotic maps are topologically conjugate with trigonometric chaotic maps. Hence, for small \( k \) their KS-entropy or equivalently Lyapunov characteristic exponent is the same with the KS-entropy and Lyapunov exponent of chaotic maps of reference [1, 2], where the numerical simulations of section 5 approve the above assertion. Actually, the simulations of section 5 indicate that, except for the values of \( k \) near one, the elliptic and trigonometric maps are topologically conjugate. With a reasoning similar to one given above, we can prove that the combination of elliptic maps given in section 2 is almost topologically conjugate with the combination of trigonometric maps of reference [2].

### 4 Invariant measure

Characterizing invariant measure for explicit nonlinear dynamical systems is a fundamental problem which connects dynamical theory to statistics and statistical mechanics. A well-known example is Ulam and von Neumann map which has an ergodic measure \( \mu = \frac{1}{\sqrt{x(1-x)}} \) [7]. The probability measure \( \mu \) on \([0, 1]\) is called an SRB or invariant measure [6]. For deterministic system such as \( \Phi_N^{(\omega)}(x, \alpha) \)-map, the \( \Phi_N^{(\omega)}(x, \alpha) \)-invariance means that, its invariant
measure $\mu(x)$ fulfills the following formal Frobenius-Perron (FP) integral equation:

$$\mu(y) = \int_0^1 \delta(y - \Phi_N^{(1,2)}(x, \alpha)) \mu(x) dx.$$ 

This is equivalent to:

$$\mu(y) = \sum_{x \in \Phi_N^{-1}(y, \alpha)} \mu(x) \frac{dx}{dy}, \quad (4.1)$$

defining the action of standard FP operator for the map $\Phi_N(x)$ over a function as:

$$P_{\Phi_N^{(\omega)}} f(y) = \sum_{x \in \Phi_N^{-1}(y, \alpha)} f(x) \frac{dx}{dy}. \quad (4.2)$$

We see that, the invariant measure $\mu(x)$ is actually the eigenstate of the FP operator $P_{\Phi_N^{(\omega)}}$ corresponding to the largest eigenvalue 1.

One can show that $\tilde{\mu}(x)$, the invariant measure of conjugate map, $\tilde{\Phi} = h \circ \Phi \circ h^{-1}$ can be written in terms of $\mu(x)$, the invariant measure of chaotic map $\Phi$, as:

$$(\tilde{\mu} \circ h')(x) = \mu(x) \quad (4.3)$$

Therefore, considering the conjugacy relation (3.3) between the maps $\Phi_N^{(1)}(x, \alpha)$ and $\tilde{\Phi}_N(x, \alpha) = \frac{1}{\alpha^2} \tan^2 \left(N \arctan(\sqrt{x'})\right)$ and using the relation (4.3) with invertible map $h(x) = \frac{1}{(1 - k^2)}$ together with taking into account that the former one, $\Phi(x, \alpha)$, one has the following invariant measure [1]

$$\mu(x, \beta) = \frac{1}{\pi} \frac{\sqrt{\beta}}{\sqrt{x(1 - x)(\beta + (1 - \beta)x)}} \quad \beta > 0, \quad (4.4)$$

we obtain the following expression for the invariant measure of chaotic maps $\Phi_N^{(1)}(x, \alpha)$:

$$\mu(x) = \frac{2\sqrt{2\beta}}{(2 + (2 + k^2)\beta x)\sqrt{(2 - k^2)x}}, \quad (4.5)$$

for small values of $k$, where:

$$\alpha = \sqrt{\beta} \tan \left(N \arctan\left(\sqrt{\frac{1}{\beta}}\right)\right). \quad (4.6)$$

With the same prescription as mentioned above, we can, for small values of elliptic parameter $k$, obtain the invariant measure of all other types of elliptic chaotic maps which are
generalizations of trigonometric chaotic maps. It should be mentioned that for trigonometric chaotic maps \[1\], their composition \[2\] and their coupling \[8\] the invariant measure has already been obtained and presented in our previous papers.

## 5 KS-Entropy and Lyapunov exponent

KS-entropy or metric entropy measures how chaotic a dynamical system is and it is proportional to the rate at which information about the state of system is lost in the course of time or iteration \[12\]. As it is proved in Appendix A, for small values of \(k\), the KS-entropy of elliptic chaotic maps \(h(\mu, \Phi_N^{(\omega)}(x, \alpha))\) is equal to KS-entropy of trigonometric chaotic maps \[1\], where for one-parameter elliptic chaotic maps we have:

\[
h(\mu, \Phi_N^{(\omega)}(x, \alpha)) = \ln \left( \frac{N(1 + \beta + 2\sqrt{\beta})^{N-1}}{(\sum_{k=0}^{[N/2]} C_{2k}^N \beta^k)(\sum_{k=0}^{[N/2]} C_{2k+1}^N \beta^k)} \right).
\]

(5.1)

Also, in order to study discrete dynamical system, we could refer to Lyapunov exponent which is, in fact, the characteristic exponent of the rate of average magnificent of the neighborhood of an arbitrary point \(x_0\) and it is shown by \(\Lambda(x_0)\) which is written as:

\[
\Lambda(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{d}{dx} \Phi_N^{(\omega)}(x_k, \alpha) \right|,
\]

(5.2)

where \(x_k = \Phi_N \circ \Phi_N \circ ... \circ \Phi_N^{k}(x_0)\). It is obvious that \(\Lambda(x_0) < 0\) for an attractor, \(\Lambda(x_0) > 0\) for a repeller and \(\Lambda(x_0) = 0\) for marginal situation \[12\]. Also, the Lyapunov number is independent of initial point \(x_0\), provided that the motion inside the invariant manifold is ergodic, thus \(\Lambda(x_0)\) characterizes the invariant manifold of \(\Phi_N^{(\omega)}\) as a whole.

For small values of the elliptic parameter, the map \(\Phi_N^{(\omega)}\) and its combination are measurable. Birkhoff ergodic theorem implies the equality of KS-entropy and Lyapunov number, that is \[12\]:

\[
h(\mu, \Phi_N^{(\omega)}(x, \alpha)) = \Lambda(x_0, \Phi_N^{(\omega)}(x, \alpha)).
\]

(5.3)
A comparison of analytically calculated KS-entropy of maps $\Phi^{(\omega)}_{N}(x, \alpha)$ (5.1) and their combinations (first kind) for small values of the elliptic parameter, with the corresponding Lyapunov characteristic exponent obtained by simulation (see Figures 1–3), indicates that in chaotic region these maps are ergodic as Birkhoff ergodic theorem predicts. In non-chaotic regions of the parameters, Lyapunov characteristic exponent is negative, since in this region we have only a single period fixed point without transition to chaos.

Also, numerical calculation shows that this class of maps have different asymptotic behavior. Actually one can show that the KS-entropy of one-parameter family of elliptic chaotic maps of $\text{sn}$ and $\text{cn}$ types (5.1) have the following asymptotic behavior:

\[
\begin{align*}
\frac{h(\mu, \Phi_{N}^{(\omega)}(x, \alpha = N + 0^{-}))}{(N - \alpha)^{\frac{1}{2}}} & \sim (N - \alpha)^{\frac{1}{2}}, \\
\frac{h(\mu, \Phi_{N}^{(\omega)}(x, \alpha = 1 + 0^{+}))}{(\alpha - \frac{1}{N})^{\frac{1}{2}}} & \sim (\alpha - \frac{1}{N})^{\frac{1}{2}},
\end{align*}
\]

Therefore, the above relation (5.4) implies that: all one-parameter elliptic chaotic maps belong to the same universal class, which is different from the universality class of pitch fork bifurcation maps, actually the asymptotic behavior elliptic ones is similar to the class of intermittent maps [13]. But intermittency can not occur in this family of maps for any values of parameter $\alpha$ and for small values of parameter $k$ since elliptic chaotic maps and their $n$-composition do not have minimum values other than zero and maximum values other than one in the interval $[0, 1]$.

It is interesting that the numerical and theoretical calculations predict different asymptotic behavior for many-parameter elliptic chaotic maps. As an example of asymptotic of the composed maps, the KS-entropy of $\Phi_{2,2}^{(1,\alpha_1)(1,\alpha_2)}(x)$ is presented below [4]:

\[
h(\mu, \Phi_{2,2}^{(1,\alpha_1)(1,\alpha_2)}(x)) = \ln \frac{(1 + \sqrt{\beta}^2(2\sqrt{\beta} + \alpha_2(1 + \beta))^2}{(1 + \beta)(4\beta + \alpha_2^2(1 + \beta)^2)},
\]

With choosing $\beta = \alpha_2^\nu$, $0 < \nu < 2$, entropy given by (5.5) reads:

\[
h(\mu, \Phi_{2,2}^{(1,\alpha_1)(1,\alpha_2)}(x)) = \ln \frac{(1 + \alpha_2^\nu)^2(2\alpha_2^\nu + \alpha_2(1 + \alpha_2^\nu))^2}{(1 + \alpha_2^\nu)(4\alpha_2^\nu + \alpha_2^2(1 + \alpha_2^\nu)^2)},
\]

which has the following asymptotic behavior near $\alpha_2 \to 0$ and $\alpha_2 \to \infty$:

\[
\begin{align*}
h(\mu, \Phi_{2,2}^{(1,\alpha_1)(1,\alpha_2)}(x)) & \sim \alpha_2^\nu \quad \text{as} \quad \alpha_2 \to 0, \\
h(\mu, \Phi_{2,2}^{(1,\alpha_1)(1,\alpha_2)}(x)) & \sim (\frac{1}{\alpha_2})^\nu \quad \text{as} \quad \alpha_2 \to \infty.
\end{align*}
\]
The above asymptotic behaviours indicate that, for an arbitrary value of $0 < \nu < 2$, the maps $\Phi^{(1,\alpha_1)/(1,\alpha_2)}_{2,2}(x)$ belong to the universal class which is different from that of one-parameter elliptic chaotic maps (5.4) or that of pitch fork bifurcating maps.

In summary, combining the analytic discussion of section 2 with the numerical simulation, we deduce that these maps are ergodic in certain values of their parameters as explained above and in the complementary interval of parameters they have only a single period one attractive fixed point in a way that, in contrary to the most of usual one-dimensional one-parameter families of maps, they have only a transition to chaos from a period one attractive fixed points to chaotic state or vise versa.

6 Conclusion

We have given hierarchy of one and many-parameter families of one-dimensional elliptic chaotic maps having the interesting property of being either chaotic (proper to say ergodic) or having stable period one fixed point and they go to chaotic state from a stable single periodic state without having usual period doubling or period-n-tupling scenario. Perhaps this interesting property is again due to existence of invariant measure for small values of the elliptic parameter.

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Appendix A: KS-entropy of elliptic chaotic maps:

In order to prove that KS-entropy for one and many-parameter elliptic chaotic maps for small values of elliptic parameter would be equal to KS-entropy of trigonometric chaotic maps [1, 2], the following statement should be considered taking into account that $y = \Phi_N^{(\omega)}(x, \alpha)$:

\[
\mu(x) dy = \sum_{x_i \in f^{-1}(y)} \mu(x_i) dx_i,
\]

\[
\tilde{y} = h(y) \quad \tilde{x} = h(x) \quad \tilde{y} = \tilde{f}(\tilde{x}),
\]

with

\[
\tilde{f} = h \circ f \circ h^{-1},
\]

\[
(\tilde{\mu} \circ h')(x) = \mu(x),
\]

\[
\tilde{h} \left( \mu, \Phi_N^{(\omega)}(x, \alpha) \right) = \int d\tilde{x} \tilde{\mu}(x) \ln \left| \frac{d\tilde{y}}{dx} \right|
\]

\[
= \int dx (\tilde{\mu} \circ h)(x) h'(x) \ln \left( \frac{dy'(y)}{dx h'(x)} \right)
\]

\[
= \int dx \mu(x) \ln \left( \frac{dy}{dx} \right) + \int dx \mu(x) \ln \left( \frac{dy}{dx h'(x)} \right)
\]

\[
= \int dx \mu(x) \ln \left( \frac{dy}{dx} \right).
\]

since

\[
\int dx \mu(x) \ln h'(y) = \int \ln \left( \sum_{x_i \in f^{-1}(y)} \mu(x_i) dx_i \right) = \int dx \mu(x) \ln (h'(y)) = \int dx \mu(x) \ln (h'(x)).
\]

In the same way, one can show that for small values of elliptic parameters, the KS-entropy of many-parameter families of elliptic chaotic maps would be equal to KS-entropy of many-parameter families of trigonometric chaotic maps of Reference [3].
Figures Captions

Fig.1. The plot of Lyapunov exponent of $\Phi_2^{(1)}(x, \alpha)$, versus the parameters $\alpha$.

Fig.2. The plot of Lyapunov exponent of $\Phi_2^{(2)}(x, \alpha)$, versus the parameters $\alpha$.

Fig.3. The plot of Lyapunov exponent of $\Phi_2^{(1,\alpha_1),(1,\alpha_2)}(x)$, versus the parameters $\alpha_1$ and $\alpha_2$. 
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