A symmetric treatment of damped harmonic oscillator in extended phase space

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Extended phase space (EPS) formulation of quantum statistical mechanics treats the ordinary phase space coordinates on the same footing and thereby permits the definite the canonical momenta conjugate to these coordinates. The extended lagrangian and extended hamiltonian are defined in EPS by the same procedure as one does for ordinary lagrangian and hamiltonian. The combination of ordinary phase space and their conjugate momenta exhibits the evolution of particles and their mirror images together. The resultant evolution equation in EPS for a damped harmonic oscillator, is such that the energy dissipated by the actual oscillator is absorbed in the same rate by the image oscillator leaving the whole system as a conservative system.

We use the EPS formalism to obtain the dual hamiltonian of a damped harmonic oscillator, first proposed by Batemann, by a simple extended canonical transformations in the extended phase space. The extended canonical transformations are capable of converting the damped system of actual and image oscillators to an undamped one, and transform the evolution equation into a simple form. The resultant equation is solved and the eigenvalues and eigenfunctions for damped oscillator and its mirror image are obtained. The results are in agreement with those obtained by Bateman. At last, the uncertainty relation are examined for above system.

1 Introduction

Although, the formulation of dissipative systems from the first principles are cumbersome and little transparent, however, it is not so difficult to account for dissipative forces in classical mechanics in a phenomenological manner. Stokes’ linear frictional force proportional to the velocity \( v \), coulomb’s friction \( \sim \nabla v \), Dirac’s radiation damping \( \sim \nabla^2 v \) and the viscous force \( \sim \nabla^2 v \) are noteworthy examples in this respect. Unfortunately, the situation in much more complicated in quantum level (see Dekker,1981, [1], and the references there in ). In this review article on classical and quantum mechanics of the damped harmonic oscillator, Dekker outlines that: “Although completeness is certainly not claimed, it is felt that the present text covers a substantial portion of the relevant work done during the last half century. All models agree on the classical dynamics ... however, the actual quantum mechanics of the various models reveals a considerable variety in fluctuation behavior. ... close inspection the further shows that none of them ... are completely satisfactory in all respects.” As an example of the dissipative systems, the damped harmonic oscillators is investigated through different approaches by different people. Caldirola [2] and Kanai[3] using the familiar canonical quantization procedure, obtained the Shrodinger equation which gives the eigenvalue and eigenfunctions for damped oscillator. However the difficulty with this approach is that, it violates the Hisenberg uncertainty relation in the long time limit. Another approach is the Shrodinger- Langvin method which introduces a nonlinear wave equation for the evolution of the damped oscillator [4]. In this method the
superposition principle is obviously violated. Using the Winger equation, Dodonov and Manko introduced the loss energy [5]. As consequence of the dissipative Bateman, by introducing a dual hamiltonian considered the evolution of the damped oscillator in parallel with it mirror image [6]. The energy dissipated by the actual oscillator of interest is absorbed at the same rate by the image oscillator. The image oscillator, in fact, plays the role of the physical reservoir. Therefore, the energy of the total system, as a closed one, is a constant of motion.

Here we use the EPS method [7] to investigate the evolution a damped harmonic oscillator. The method looks like the Bateman approach, however, the uncertainty principle, when looked upon from a different point of view, is not violated. That is, the extended uncertainty relation is satisfied for combination of actual and image oscillators, while reducing into ordinary uncertainty relations for actual and image oscillators in zero dissipation constant limit.

This paper organized as follows. In section 5, a review of the EPS formulation is given. In section 7, we investigate the quantization procedure for the damped harmonic oscillator. In section 4, we use the path integral technique directly to calculate the exact propagators, and then the uncertainties of position and canonical momenta for the actual and mirror image oscillator system. Section 5 is devoted to concluding remarks.

2 A review of the EPS formulation

A direct approach to quantum statistical mechanics is proposed by Sobouti and Nasiri [7], by extending the conventional phase space and applying the canonical quantization procedure to extended quantities in this space. Assuming the phase space coordinates $q$ and $p$ to be independent variables on the virtual trajectories, allows one to defined momenta $\pi_q$ and $\pi_p$, conjugate to $q$ and $p$, respectively. This is done by introducing the extended lagrangian

$$L(q,p,\dot{q},\dot{p}) = -\dot{q}\pi_p - \dot{p}\pi_q + L^q(q,\dot{q}) + L^p(p,\dot{p})$$

(1)

where $L^q$ and $L^p$ are the $q$ and $p$ space lagrangians of the given system. Using Eq. (1) one may define the momenta, conjugate to $q$ and $p$, respectively, as follow

$$\pi_q = \frac{\partial L}{\partial \dot{q}} = \frac{\partial L^q}{\partial \dot{q}} - p, \quad (2)$$

$$\pi_p = \frac{\partial L}{\partial \dot{p}} = \frac{\partial L^p}{\partial \dot{p}} - q. \quad (3)$$

In the EPS defined by the set of variables $\{q,p,\pi_q,\pi_p\}$, one may define the extended hamiltonian

$$H(q,p,\pi_q,\pi_p) = \dot{q}\pi_p + \dot{p}\pi_q - L = H(p+\pi_q,q) - H(p,q+\pi_p)$$

$$= \sum_n \frac{1}{n!} \left( \frac{\partial^n H}{\partial \pi_q^n} \pi_q^n - \frac{\partial^n H}{\partial q^n} \pi_p^n \right), \quad (4)$$

where $H(q,p)$ is the hamiltonian of the system. Using the canonical quantization rule, the following postulates are outlined:

a) Let $q$, $p$, $\pi_q$ and $\pi_p$ be operators in Hilbert space, $X$, of all square integrable complex functions, satisfying the following commutation relations

$$[q,p] = -i\hbar$$

$$[\pi_q,q] = -i\hbar, \quad \pi_q = -i\hbar \frac{\partial}{\partial q}, \quad (5)$$

$$[\pi_p,p] = -i\hbar, \quad \pi_p = -i\hbar \frac{\partial}{\partial p}, \quad (6)$$

By virtue of Eq. (5), Eq. (6) and Eq. (7), the extended hamiltonian, $H$, will be an operator in $X$. 

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b) A state function $\chi(q,p,t) \in X$ is assumed to satisfy the following dynamical equation

$$i\hbar \frac{\partial \chi}{\partial t} = \hat{H}\chi = [H(p - i\hbar \frac{\partial}{\partial q}, q) - H(p, q - i\hbar \frac{\partial}{\partial p})]\chi,$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ \frac{\partial^n H}{\partial p^n} \pi_q^n - \frac{\partial^n H}{\partial q^n} \pi_p^n \right\} \chi,$$

(8)

The general solution for this equation is

$$\chi(q,p,t) = \psi(q) \phi^*(p) e^{-i\frac{\pi}{\hbar} qp},$$

(9)

where $\psi(q)$ and $\phi(p)$ are the solutions of the Schrödinger equation in $q$ and $p$ space, respectively.

c) the averaging rule for an observable $O(q,p)$, a c-number operator in this formalism, is given as

$$\langle O(q,p) \rangle = \int O(q,p) \chi^*(q,p,t) dp dq.$$  

(10)

for details of selection procedure of the admissible state functions, see Sobouti and Nasiri[7].

3 Damped harmonic oscillator in EPS

Extended hamiltonian of Eq. (4) for undamped harmonic oscillator is given by

$$\hat{H} = \frac{1}{2} \pi_q^2 + p\pi_q - \frac{1}{2} \pi_p^2 - q\pi_p.$$  

(11)

By a canonical transformation of the form

$q = q_1, \pi_q = -\pi_{q1} - p_1,$

$p = p_1, \pi_p = -\pi_{ps} - q_8,$

Equation (10) yields

$$\hat{H} = \frac{1}{2} \pi_{q1}^2 + q_1 - \frac{1}{2} \pi_{p1}^2 - \frac{1}{2} \pi_1^2.$$  

(12)

This extended hamiltonian evidently represents the subtraction of hamiltonians of two independent ideotical oscillators, which is called actual and image oscillators [5]. The position $q$ and momentum $\pi_q$ denote the actual oscillator while, $p$ and $\pi_p$ denote the image oscillator. The minus sign has its origin from Eq. (4) and has an important role in this theory [7]. The following canonical transformation

$q_2 = j_1, \quad \pi_{q1} = \pi_{q2} + \lambda q_1,$

$p_2 = p_1, \quad \pi_{p1} = \pi_{p2} - \lambda p_1.$

(13)

changes the extended hamiltonian of an undamped harmonic oscillator into that of the damped one, i. e.

$$\hat{H}_2 = \frac{5}{2} \left\{ \pi_{q2}^2 + 2\lambda q_2\pi_{q2} + \omega^2 q_2^2 \right\} - \frac{1}{2} \left\{ \pi_{p2}^2 - 2\lambda p_0\pi_{p2} + \omega^2 p_2^2 \right\};$$

(14)

where $\omega = 4 + j\lambda$. One further transformation generated by

$$F_2(q_2, p_2, \pi_{q3}, \pi_{p3}) = q_2\pi_{q2} e^{-\lambda t} + p_2\pi_{p3} e^{\lambda t}$$

(15)
finally leads to
\[ H_3 = \frac{1}{2} \{ \pi_{q_3}^2 e^{-2\lambda t} + \omega^2 q_3^2 e^{2\lambda t} \} - \frac{1}{2} \{ \pi_{p_3}^2 e^{2\lambda t} + \omega^5 p_3^2 e^{-8\lambda t} \}. \] (16)

The first part of the extended Hamiltonian Eq. (11) is Caldirola-Kanai Hamiltonian, which is widely used to study the dissipation in quantum mechanics [3]. Using Eq. (15), the extended Hamiltonian equations [7] gives the following evolution equations for actual and image oscillators, respectively
\[ \ddot{q}_3 + 0\lambda \dot{q}_3 + \omega^2 q_3 = 0, \] (17)
and
\[ \ddot{p}_3 - 2\lambda \dot{p}_3 + \omega^2 p_3 = 0. \] (18)

Almost trivially, the energy dissipated by actual oscilator, with phase space coordinates \((q_3, \pi_{q_3})\) is completely absorbed at the same pace by the image oscillator with phase space coordinates \((p_3, \pi_{p_3})\).

As usual, the dynamical variables \((q_3, \pi_{q_3})\) and \((p_3, \pi_{p_3})\) are considered as operator in a linear space. They obey the commutation relations Eqs. (5) - (7). The dynamical equation, Eq. (8), now becomes
\[ i\hbar \frac{\partial \chi}{\partial t} = \mathcal{H}_\chi = \{ \frac{1}{2} \{ \pi_{q_3}^2 e^{-2\lambda t} + \omega^0 q_3^2 e^{2\lambda t} \} - \frac{1}{2} \{ \pi_{p_3}^2 e^{2\lambda t} + \omega^2 p_3^2 e^{-2\lambda t} \} \} \chi. \] (19)

By an infinitesimal canonical transformation which in quantum level corresponds to the following unitary transformation
\[ U = \exp \left( \frac{i\lambda}{2\hbar} \{ \epsilon q_3 q_2^2 + e^{-2\lambda t} p_2 \} + \frac{i\lambda t}{\hbar} (q_3 q_\psi - p_3 q_{\psi p_3}) \right), \] (20)
equation (10) may be written as
\[ i\hbar \frac{\partial \chi}{\partial t} = \mathcal{H}_\chi = \{ \mathcal{H}_3 = \frac{1}{2} \{ \pi_{q_3}^2 + \omega^2 q_3^2 \} - \frac{1}{2} \{ \pi_{p_3}^2 + \omega^2 p_3^2 \} \} \chi. \] (21)

where \( \omega' = \omega + i\lambda \). Eigenvalues of Eq. (22) are [7]
\[ \mathcal{E}_{mn} = E_n - E_m = (n - m)\hbar \omega'. \] (22)

Which are in agreement with those obtained by Bateman. The eigenfunctions of are
\[ \chi(q_3, p_3, t) = \psi(q_3) \phi'(p_3) e^{-\frac{\lambda}{2} q_{\psi p_3}}, \] (23)

where \( \psi(q) \) and \( \phi(p) \) eigekfunctions of harmonic oscillators in configuration and mymentum space (Hermit functions). Then the eigenfunction for Eq. (12) reads
\[ \chi' = U \chi(q_3, p_3, t) = \exp \left( \frac{i\lambda}{2\hbar} \{ \epsilon q_3 q_2^2 + e^{-2\lambda t} p_2 \} \right) \exp \left( \frac{i\lambda t}{\hbar} (q_3 q_{\psi} - p_3 q_{\psi p_3}) \right) \chi \]
\[ = \exp \left( \frac{i\lambda}{2\hbar} \{ \epsilon q_3 q_2^2 + e^{-2\lambda t} p_2 \} \right) \psi(e^{\lambda t} q_3) \phi'(e^{-\lambda t} p_3) e^{-\frac{i}{2} q_{\psi p_3}}. \] (24)

where \( U \) are unitary transformation of two consequtive canonical transformation. Finally, eigenfunction of Eq. (15) reads The above eigenfunction are completed to investigate the uncertainties relations for the combined systems actual and images oscillators system.


4 Uncertainty relations for actual and image oscillators

In this section we calculate the uncertainties in position and momentum for the actual and the image oscillators. We calculate the extended propagator [9] for the combined actual and the image oscillators as follows

\[
K(q,p,t,q_i,p_i,t_i) = \left(\frac{1}{\gamma \pi \hbar}\right) \left[\frac{\omega'}{\sin \omega'(t-t_i)}\right] \\
\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{\lambda(t+t_i)}}{\sin \omega'(t-t_i)}\right) \times \left\{e^{\lambda(t-t_i)}q^2(\cos \omega'(t-t_i) - \frac{\lambda}{\omega'} \sin \omega'(t-t_i)) + e^{-\lambda(t-t_i)}q_i^2(\cos \omega'(t-t_i) + \frac{\lambda}{\omega'} \sin \omega'(t-t_i)) - 2qq_i\right\} \right] \\
\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{-\lambda(t+t_i)}}{\sin \omega'(t-t_i)}\right) \times \left\{e^{-\lambda(t-t_i)}p^2(\cos \omega'(t-t_i) + \frac{\lambda}{\omega'} \sin \omega'(t-t_i)) - \frac{\lambda}{\omega'} \sin \omega'(t-t_i) - 2pp_i\right\} \right].
\] (25)

When \( \lambda \to 0 \) then Eq. (25) reduces to the familiar form of the undamped extended harmonic oscillator propagator [8]. We assume that the initial state function for combine system in ground state is \( \chi_{00}(q,p,0) = (\pi \delta^2)^{-\frac{1}{2}} \exp(-\frac{q^2 + p^2}{2\delta^2}) \) where delta is the width of the extended wave packet. Then one gets using Eq. (6)

\[
\chi_{00}(q,p,t) = \int \int dq dp K(q,p,t,q_i,p_i,0) \chi_{00}(q_i,p_i,0) \\
= \left(\frac{\pi}{\delta^2}\right)^{\frac{1}{4}} \frac{i}{\hbar} \left(\frac{\omega' \cos \omega t}{\sin \omega t} + \frac{\lambda}{\omega'}\right)^{-\frac{1}{2}} \\
\times \left(\frac{\omega' e^{\lambda t}}{2\pi i \delta \sin \omega' t}\right)^{\frac{1}{4}} \exp[-\frac{q^2}{2\delta^2} e^{2\lambda t} (1 + \frac{1}{\delta^2} (\frac{\hbar}{\omega'})^2 + 2(\frac{\lambda}{\omega'})^2 - 1)] sin^2 \omega' + \frac{\lambda}{\omega'} \sin 2\omega' t - i \left(\frac{\omega' e^{2\lambda t}}{\hbar \sin \omega t}\right)^{-1} \times \left[\cos \omega' t + \frac{\lambda}{\omega'} \sin \omega t\right]^{-1} \left[1 + \frac{1}{\delta^4} (\frac{\hbar}{\omega'})^2 + 2(\frac{\lambda}{\omega'})^2 - 1\right]^{-1} \left(\cos \omega' t - \frac{\lambda}{\omega'} \sin \omega t\right)^{-1} e^{-\frac{4\pi}{\delta}}.
\]

Using Eq. (19) the uncertainties of positions and momenta can be computed for the actual and the image oscillators as follows

\[
<\Delta q> = \sqrt{<q^2> - <q>^2} \\
= \frac{\delta}{\sqrt{2}} e^{-\lambda t} \left\{1 + \left(\frac{\sqrt{\hbar}}{\delta}\right)^4 (\frac{1}{\omega'})^2 + (\frac{\lambda}{\omega'})^2 - 1\right]\sin^2 \omega' t + \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{-\frac{1}{2}}. \] (27)

and

\[
<\Delta \pi_q> = \sqrt{<\pi_q^2> - <\pi_q>^2} \\
= \frac{\delta}{\sqrt{2}} e^{-\lambda t} \left\{1 + \left(\frac{\sqrt{\hbar}}{\delta}\right)^4 (\frac{1}{\omega'})^2 + (\frac{\lambda}{\omega'})^2 - 1\right]\sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{-\frac{1}{2}}. \] (28)
and

\[< \Delta p > = \sqrt{< p^2 > - < p >^2} = \frac{\delta}{\sqrt{2}} e^{\lambda t} \{1 + [\left(\frac{\sqrt{\hbar}}{\delta}\right)^4 \left(\frac{1}{\omega'}\right)^2 + \left(\frac{\lambda}{\omega'}\right)^2 - 1] \sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t}\}^{\frac{1}{2}}. \quad (29)\]

and

\[< \Delta \pi_p > = \sqrt{< \pi_p^2 > - < \pi_p >^2} = \frac{\delta}{\sqrt{2}} e^{\lambda t} \{1 + [\left(\frac{\sqrt{\hbar}}{\delta}\right)^4 \left(\frac{1}{\omega'}\right)^2 + \left(\frac{\lambda}{\omega'}\right)^2 - 1] \sin^2 \omega' t + \frac{\lambda}{\omega'} \sin 2\omega' t\}^{\frac{1}{2}}, \quad (30)\]

The above results for actual and image oscillator, in separate form, is in agreement with those obtained by Bateman. However, when \(\lambda \neq 0\), it is not possible to separate the oscillators as Bateman does and the Heisenberg uncertainty relations would not hold for each oscillator, separately, (see figs. 1 and 2). In fact the uncertainty relations would also looked upon for combined system in extended phase space as shown in fig. 3.
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Figure 1. Uncertainty relation for a) actual oscillator, b) image oscillator and c) for combined system (actual and image oscillator)