A GENERIC ALGEBRA ASSOCIATED TO CERTAIN HECKE ALGEBRAS

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Abstract. We initiate the systematic study of endomorphism algebras of permutation modules and show they are obtainable by a descent from a certain generic Hecke algebra, infinite-dimensional in general, coming from the universal enveloping algebra of \(\mathfrak{gl}_n\) (or \(\mathfrak{sl}_n\)). The endomorphism algebras and the generic algebras are cellular (in the latter case, of profinite type in the sense of R.M. Green). We give several equivalent descriptions of these algebras, find a number of explicit bases, and describe indexing sets for their irreducible representations.

Introduction

We study the intertwining spaces \(\lambda S(n, r)_{\mu} := \text{Hom}_{\Sigma_r}(M^\mu, M^\lambda)\) between permutation modules \(M^\mu, M^\lambda\) for a symmetric group \(\Sigma_r\). Here \(\lambda, \mu\) are given \(n\)-part compositions of \(r\); that is, unordered \(n\)-part partitions with 0 allowed. We are particularly interested in the endomorphism algebras \(S(\lambda) := \lambda S(n, r)_\lambda\) of the modules \(M^\lambda\). These permutation modules are of central interest for the representation theory of symmetric groups, and moreover provide a natural link with the representation theory of general linear groups, via Schur algebras. Let \(E\) be a fixed \(n\)-dimensional vector space over a field \(K\). The symmetric group \(\Sigma_r\) acts on the right on \(E^{\otimes r}\) by place permutations. The endomorphism algebra

\[S(n, r) = \text{End}_{\Sigma_r}(E^{\otimes r})\]

is the Schur algebra. When \(K\) is infinite its module category is equivalent to the category of \(r\)-homogeneous polynomial representations of \(\text{GL}_n(K)\). As a module over \(K\Sigma_r\), the tensor space \(E^{\otimes r}\) is isomorphic to the direct sum \(\oplus M^\lambda\) where \(M^\lambda\) is the transitive permutation module corresponding to an \(n\)-part composition \(\lambda\), that is the \(\Sigma_r\)-orbit of weight \(\lambda\) on the standard basis of \(E^{\otimes r}\) (see 3.1). Hence \(S = S(n, r)\) decomposes into a direct sum of spaces \(\lambda S_{\mu} = \lambda S(n, r)_{\mu}\), and any such space is an \(S(\lambda)\)-\(S(\mu)\) bimodule. We expect that a systematic study of these algebras and bimodules will ultimately lead to a better understanding of Schur algebras; however, the finite-dimensional algebras \(S(\lambda)\) are interesting in their own right. Moreover, the group algebra of \(\Sigma_r\) appears in this theory as the special case \(S(\omega) = \omega S(r, r)_\omega\) for the particular partition \(\omega = (1^r)\), so our study may be regarded as an extension of the study of symmetric groups to a broader context.

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We also study, in the second part of the paper, certain infinite-dimensional analogues of the $S(\lambda)$, $\lambda S_\mu$. These analogous objects occur naturally in Lusztig’s construction of the modified form $\hat{U}$ of the universal enveloping algebra $U$ corresponding to the Lie algebra $\mathfrak{gl}_n$ (over base field $\mathbb{Q}$). By definition $U$ is the direct sum of spaces $\lambda U_\mu$ as $\lambda, \mu$ vary over the set $\mathbb{Z}^n$. For given $\lambda \in \mathbb{Z}^n$, the space $\hat{U}(\lambda) := \lambda U_\lambda$ is an infinite-dimensional algebra, and if $\lambda$ is an $n$-part composition then the finite-dimensional algebra $S(\lambda)$ is a quotient of $\hat{U}(\lambda)$. The space $\lambda U_\mu$ is a bimodule for $\hat{U}(\lambda)\hat{U}(\mu)$, and $\lambda S_\mu$ is a homomorphic image of $\lambda U_\mu$, by a linear map which is compatible with the bimodule structures. Similar statements apply if one replaces $\hat{U}(\mathfrak{gl}_n)$ by $\hat{U}(\mathfrak{sl}_n)$.

Knowing the dimensions of the simple $S(\lambda)$-modules for every $n$-part composition $\lambda$ is equivalent to knowing the formal characters of the simple $S(n, r)$-modules. Similarly, knowing the dimensions of the simple $\hat{U}(\lambda)$-modules for every $\lambda$ in $\mathbb{Z}^n$ is equivalent to knowing the formal characters of the simple unital $\hat{U}$-modules. See [1] [10] for details.

Part 1. Endomorphism algebras of permutation modules

We begin (in §2) by summarizing basic facts about weight spaces and Schur algebras, following ideas of J.A. Green and S. Donkin. Then (in §3) we show that the intertwining space $\lambda S(n, r)_\mu = \text{Hom}_{\mathbb{C}}(M^\mu, M^\lambda) \simeq 1_\lambda S(n, r)1_\mu$ is isomorphic with the space

$$\text{Hom}_{\text{GL}_n}(S^\mu \mathbb{E}, S^\lambda \mathbb{E})$$

for $\lambda, \mu$ arbitrary $n$-part compositions of $r$, where $S^\mu \mathbb{E}$ is a generalized symmetric power. Here $1_\lambda \in S(n, r)$ corresponds to the projection onto $M^\lambda$ with kernel $\oplus_{\mu \neq \lambda} M^\lambda$. For $n \geq r$ this was proved in [Do2] but it holds in general. In particular, it follows that $S(\lambda) \simeq \text{End}_{\text{GL}_n}(S^\lambda \mathbb{E})$. We describe a connection between Kostka duality and an indexing set for the simple $S(\lambda)$-modules (see §4).

One can study Schur algebras (over $\mathbb{Q}$) as quotients of $\hat{U}$ and can locate within $S_Q(n, r)$ a certain integral form $S_Z(n, r)$, a quotient of a certain $\mathbb{Z}$-form $U_Z$ in $\hat{U}$ (see §1). Then $S_K(n, r) \simeq K \otimes \mathbb{Z} S_Z(n, r)$ for any field $K$. There are similar integral forms for $\lambda S_\mu$ and $S(\lambda)$, and we give several natural $\mathbb{Z}$-bases for these spaces.

1. Weight spaces

1.1. Fix an arbitrary infinite field $K$. We consider the affine algebraic group $\text{GL}_n = \text{GL}_n(K)$ or the affine algebraic monoid $M_n = M_n(K)$. Let $T_n \subset \text{GL}_n$ (resp., $D_n \subset M_n$) denote the subgroup (resp., submonoid) of diagonal matrices. Any rational left $\text{GL}_n$-module (or $M_n$-module) $V$ has an eigenspace decomposition relative to $T_n$ (or $D_n$): $V = \bigoplus_{\lambda \in \mathbb{Z}^n} \lambda V$ (resp., $V = \bigoplus_{\lambda \in \mathbb{N}^n} \lambda V$), where

$$(1.1.1) \quad \lambda V = \{v \in V \mid t \cdot v = t^\lambda v, \text{ all } t \in T_n \text{ (or } D_n)\}$$

Similarly, if $V$ is a rational right $\text{GL}_n$-module (resp., $M_n$-module) then $V$ has the decomposition: $V = \bigoplus_{\lambda \in \mathbb{Z}^n} V_\lambda$ (resp., $V = \bigoplus_{\lambda \in \mathbb{N}^n} V_\lambda$), where

$$(1.1.2) \quad V_\lambda = \{v \in V \mid v \cdot t = t^\lambda v, \text{ all } t \in T_n \text{ (resp., } D_n)\}.$$
Here $t^\lambda$ means $t_1^{\lambda_1} \cdots t_n^{\lambda_n}$, for $t = \text{diag}(t_1, \ldots, t_n)$.

Taking $V = K[\mathcal{G}L]_n$ (or $K[M]_n$), regarded as bimodules for $\mathcal{G}L_n$ (or $M_n$) via left and right translation of functions $(g \cdot f)(x) = f(xg)$ and $(f \cdot g)(x) = f(gx)$, respectively, we obtain direct sum decompositions of $K[\mathcal{G}L]_n$ (or $K[M]_n$), both as left and right modules. Let $\lambda \mathcal{G}L$ (or $\lambda K$) denote the 1-dimensional left (or right) $T_n^-$ or $D_n^-$-module afforded by the character $\lambda$. The following (see [D1], 3.5) is an immediate consequence of the definitions.

**Lemma 1.2.** (a) For any $\lambda \in \mathbb{Z}^n$, the left (resp., right) weight space $\lambda K[\mathcal{G}L]_n$ (resp., $K[\mathcal{G}L]_n \lambda$) is isomorphic with the module $\text{ind}_{T_n}^{T_n^\text{GL}_n} \lambda K$ (or $\text{ind}_{T_n}^{T_n^\text{GL}_n \lambda}$), as right (or left) rational $\mathcal{G}L_n$-modules.

(b) For any $\lambda \in \mathbb{N}^n$, the left (or right) weight space $\lambda K[M]_n$ (resp., $K[M]_n \lambda$) is isomorphic with $\text{ind}_{D_n}^{D_n^\text{M}_n} \lambda K$ (or $\text{ind}_{D_n}^{D_n^\text{M}_n \lambda}$) as right (or left) rational $M_n$-modules.

1.3. Although the decompositions in the preceding lemma are similar, it should be noted that for $\lambda \in \mathbb{N}^n$ the modules $K[\mathcal{G}L]_n \lambda$, $\lambda K[\mathcal{G}L]_n$ are infinite-dimensional while $K[M]_n \lambda$, $\lambda K[M]_n$ are finite-dimensional. Let $c_{ij}$ be the element of $K[M]_n$ given by evaluation of a matrix at its $(i, j)$th entry. Then $K[M]_n$ may be identified with the polynomial algebra $K[c_{ij}]$. The algebra $K[\mathcal{G}L]_n$ may be identified with the localization of $K[c_{ij}]$ at the element $\det(c_{ij})$, and one may regard $K[M]_n$ as a subalgebra (in fact it is a sub-bialgebra) of $K[\mathcal{G}L]_n$ via the map $K[M]_n \to K[\mathcal{G}L]_n$ given by restricting functions from $M_n$ to $\mathcal{G}L_n$. This gives a categorical isomorphism between polynomial $\mathcal{G}L_n$-modules and rational $M_n$-modules; this justifies our interest in $M_n$-modules. The bimodule structure on $K[M]_n$ is given explicitly by

\begin{equation}
1.3.1 \quad g \cdot c_{ij} = \sum_k c_{ik} g_{kj}; \quad c_{ij} \cdot g = \sum_k g_{ik} c_{kj} \quad (g \in M_n).
\end{equation}

These formulas also give the bimodule structure on $K[\mathcal{G}L]_n$, simply by restricting $g$ to $\mathcal{G}L_n$. Moreover, by restricting to $D_n$ we obtain the equalities

\begin{equation}
1.3.2 \quad t \cdot c_{ij} = c_{ij} t_{jj} = t^{e_j} c_{ij}; \quad c_{ij} \cdot t = t_{ii} c_{ij} = t^{e_i} c_{ij} \quad (t \in D_n).
\end{equation}

This shows that $c_{ij}$ belongs to the weight space $\varepsilon_j K[M]_{\varepsilon_i}$, i.e., $c_{ij}$ has left (resp., right) weight $\varepsilon_j$ (resp., $\varepsilon_i$), where $\varepsilon_1, \ldots, \varepsilon_n$ is the standard basis of $\mathbb{Z}^n$.

We have the following version of [DW] (2.7); (2.12) or [Do3] proof of (3.4)(i), which provides an alternative description of the weight spaces in $K[M]_n$, in terms of symmetric powers of the natural representation $E$.

**Lemma 1.4.** Let $E$ be the $n$-dimensional vector space $K^n$, regarded as left (or right) $M_n$-module by left (or right) matrix multiplication. For any $\lambda \in \mathbb{N}^n$ there is an isomorphism between the induced module $\text{ind}_{D_n}^{M_n} \lambda K$ (resp., $\text{ind}_{D_n}^{M_n} \lambda K$) and

\begin{equation}
S^\lambda E := (S^{\lambda_1} E) \otimes \cdots \otimes (S^{\lambda_n} E).
\end{equation}

as right (or left) rational $M_n$-modules.

**Proof.** Let $(e_i)$ be the canonical basis of $E$. We may identify $S^a E$ with homogeneous polynomials in the $e_i$ of degree $a$. The left and right actions of $M_n$ on $S^a E$ are by linear substitutions, and the natural action on $E$. 

The distinct idempotents in this family are parametrized by the set
\[(2.1.3) \lambda \in \mathbb{N}^n \mid \sum_i \lambda_i = r \]
We shall write
\[(2.1.1) c_{ij} = c_{i_1 j_1} c_{i_2 j_2} \cdots c_{i_r j_r} \]
with \(M_n\) acting on the right as linear substitutions by the second formula in \[(2.1.2)\] above. Similarly, the right weight space \(K[M_n]_{\lambda}\) is spanned by all monomials of the form
\[(2.1.2) \prod_{i,j} c_{ij} \quad (\sum_j a_{ij} = \lambda_j, \text{ for all } j)\]
with \(M_n\) acting on the left as linear substitutions by the first formula in \[(2.1.1)\] above.

The map taking an element of form \[(1.4.1)\] onto the element
\[(1.4.2) \prod_i e_{i_1}^j \otimes (\prod_i e_{i_2}^j) \otimes \cdots \otimes (\prod_i e_{i_n}^j)\]
defines the desired isomorphism of right \(M_n\)-modules. Similarly, the map taking an element of form \[(1.4.2)\] onto the element
\[(1.4.3) \prod_j e_{j_1}^i \otimes (\prod_j e_{j_2}^i) \otimes \cdots \otimes (\prod_j e_{j_n}^i)\]
defines the desired isomorphism of left \(M_n\)-modules.

**Remark 1.5.** The proof shows that \(\lambda K[M_n] \) (or \(K[M_n]_{\lambda}\)) has a basis in one-one correspondence with the set of \(n \times n\) matrices over \(\mathbb{N}\) with column (resp., row) sums equal to \(\lambda\).

## 2. Schur algebras

### 2.1. The Schur algebra

The Schur algebra can alternatively be constructed as the linear dual of the coalgebra \(A_K(n,r)\), the \(K\)-linear span of the monomials in the \(c_{ij}\) in \(K[M_n]\) of total degree \(r\) (see \([G1\ (2.4b)]\)). This provides a basis for \(S_K(n,r)\), as follows. Given a pair of multi-indices \(i,j\) in \(I(n,r)\), define
\[(2.1.1) c_{ij} = c_{i_1 j_1} c_{i_2 j_2} \cdots c_{i_r j_r} .\]
Here \(I(n,r)\) is the set of \(i = (i_1, \ldots, i_r)\) where each \(i_k\) belongs to \(\{1, \ldots, n\}\). The commutativity of the variables \(c_{ij}\) implies that we have to take into account the equality rule
\[(2.1.2) c_{ij} = c_{i' j'} \iff i' = i\pi, j' = j\pi, \text{ some } \pi \in \Sigma_r ,\]
with respect to the obvious right action of \(\Sigma_r\) on \(I(n,r)\). As a \(K\)-space, \(A_K(n,r)^*\) has basis \(\{\xi_{ij}\}\) dual to the basis \(\{c_{ij}\}\). As with the \(c_{ij}\) there is a similar equality rule for the \(\xi_{ij}\). The distinct \(\xi_{ij}\) provide the desired basis for the Schur algebra \(S_K(n,r)\).

In particular, by \([G1\ 3.2]\) the distinct elements of the form \(\xi_{ii}\) for \(i \in I(n,r)\) provide a set of orthogonal idempotents which add up to the identity in \(S_K(n,r)\). Given \(i\) we set \(\text{wt}(i) = (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^n\) where \(\lambda_j\) is defined to be the number of \(i_k\) which equal \(j\). We shall write
\[(2.1.3) 1_\lambda := \xi_{ii} \quad (\lambda = \text{wt}(i) \text{ as above}) .\]
The distinct idempotents in this family are parametrized by the set
\[(2.1.4) \Lambda(n,r) := \{ \lambda \in \mathbb{N}^n \mid \sum_i \lambda_i = r \}\]
of $n$-part compositions of $r$. For any $S_K(n,r)$-module $V$, one clearly has decompositions $V = \oplus \lambda 1_\lambda V$ and $V = \oplus \lambda V 1_\lambda$. Moreover, one has by [11.3.2] identifications

\[(2.1.5) \quad 1_\lambda V = \lambda V, \quad V 1_\lambda = V_\lambda \]

with the weight spaces defined in (1.1.1), (1.1.2), for any $\lambda \in \Lambda(n,r)$.

2.2. It follows from [1.3.1] that $A_K(n,r)$, regarded as $M_n$-$M_n$ bimodule, is $r$-homogeneous for either action. Moreover, $A_K(n,r)$ is an $S_K(n,r)$-$S_K(n,r)$ bimodule (see [11.2.8]). In particular, this means that the right (resp., left) weight space $A_K(n,r)_\lambda$ (resp., $\lambda A_K(n,r)$) is a left (resp., right) $S_K(n,r)$-module. For an $S_K(n,r)$-module $V$, we denote by $V^o$ the contravariant dual of $V$ (see [11.1]).

Lemma 2.3. Let $\lambda \in \Lambda(n,r)$.

(a) $\lambda A_K(n,r) = \lambda K[M_n]$ and $A_K(n,r)_{\lambda} = K[M_n]_\lambda$.

(b) There are isomorphisms $(\lambda A_K(n,r))^o \simeq 1_\lambda S_K(n,r)$ (as right $S_K(n,r)$-modules) and $(A_K(n,r)_{\lambda})^o \simeq S_K(n,r) 1_{\lambda}$ (as left $S_K(n,r)$-modules).

Proof. (a) The space $\lambda K[M_n]$ is spanned by $\{ c_{i,u} \mid i \in I(n,r) \}$, where $u$ is a fixed element of $I(n,r)$ of weight $\lambda$, so there is an inclusion $\lambda K[M_n] \subset \lambda A_K(n,r)$. The opposite inclusion is obvious. This proves the first equality in (a), and the second is similar.

(b) It is enough to prove the first statement, since the other case is similar. One can check that the contravariant form $(\ , \ ) : S_K(n,r) \times A_K(n,r) \rightarrow K$ defined in [11.2.8], upon restriction to $1_\lambda S_K(n,r) \times \lambda A_K(n,r)$, remains nonsingular. In fact, $\{ c_{i,u} \mid i \in I(n,r) \}$ gives a basis for $\lambda A_K(n,r)$ and $\{ \xi_{uj} \mid j \in I(n,r) \}$ gives a basis for $1_\lambda S_K(n,r)$, and these two bases are dual relative to the contravariant form. The result now follows from [11.2.7e].

3. Hecke algebras associated to permutation modules

We give a number of descriptions of the Hecke algebra $S_K(\lambda) = \text{End}_{\Sigma_r}(M^\lambda)$ and the bimodule $\lambda S_K(n,r)_\mu = \text{Hom}_{\Sigma_r}(M^\mu, M^\lambda)$, for $\lambda, \mu \in \Lambda(n,r)$.

3.1. As right $K\Sigma_r$-modules, one has a decomposition

\[(3.1.1) \quad E^{\otimes r} = \oplus \lambda M^\lambda. \]

Here $M^\lambda$ is the transitive permutation module with basis all $e_i = e_i \otimes \cdots \otimes e_i$ such that $\text{wt}(i) = \lambda$. The idempotents $1_\lambda$ of $S(n,r)$ as in (2.1.3) are precisely the projections corresponding to the direct sum decomposition (3.1.1), see [11.1].

By standard arguments we have an isomorphism of $S_K(\lambda)$-$S_K(\mu)$ bimodules

\[(3.1.2) \quad 1_\lambda S_K(n,r) 1_\mu \simeq \text{Hom}_{\Sigma_r}(M^\mu, M^\lambda) \]

and in particular an algebra isomorphism

\[(3.1.3) \quad 1_\lambda S_K(n,r) 1_\lambda \simeq \text{End}_{\Sigma_r}(M^\lambda). \]
So we may identify $S_K(\lambda)$ and $\lambda(S_K)$ with $1_\lambda S_K(n,r)1_\lambda$ and $1_\lambda S_K(n,r)1_\mu$. This shows in particular that $S_K(\lambda)$ is a type of Iwahori-Hecke algebra; see [G1 6.1, Remark]. It also shows that, whenever $n \geq r$, we have an isomorphism

$$1_\omega S_K(n,r)1_\omega \simeq K \Sigma_r,$$

where $\omega \in \Lambda(n,r)$ is the special weight

$$\omega = (1, \ldots, 1, 0, \ldots, 0) \quad (r \, 1's),$$

since $M^\omega$ is the regular representation. (See [G1 (6.1d)] for an explicit isomorphism.)

One needs to consider $S_K(\lambda)$ and $\lambda(S_K)$ only in case $\lambda, \mu$ are dominant, that is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, since these label the orbits of the Weyl group $\Sigma_n$ acting on $\Lambda(n,r)$.  

**Lemma 3.2.** Let $\lambda, \mu \in \Lambda(n,r)$.

(a) For any $w \in \Sigma_n$, there is an algebra isomorphism $S_K(\lambda) \simeq S_K(w\lambda)$.

(b) For any $w, w' \in \Sigma_n$, there is an isomorphism $\lambda(S_K) \simeq \omega \lambda(S_K) w'\mu$ of $S_K(\lambda)$-$S_K(\mu)$ bimodules.

**Proof.** It is enough to show that $M^\lambda \simeq M^w$ for any $w \in \Sigma_r$. But this is clear from the definition of $M^\lambda$ (see 3.1). \hfill \Box

From the isomorphism (3.2) and the multiplication rule [G1 (2.3c)] we immediately obtain the following result, which provides a basis for $\lambda(S_K)$.

**Proposition 3.3.** Let $\lambda, \mu \in \Lambda(n,r)$. A basis for $1_\lambda S_K(n,r)1_\mu$ is given by the set of all $\xi_{i,j}$ such that $\text{wt}(i) = \lambda$, $\text{wt}(j) = \mu$. In particular, the set of all $\xi_{i,j} (i,j \in I(n,r))$ satisfying $\text{wt}(i) = \lambda = \text{wt}(j)$ is a basis for $1_\lambda S_K(n,r)1_\lambda \simeq S_K(\lambda)$.

In [G2], J.A. Green introduced the codeterminant basis of $S_K(n,r)$. There is a similar basis for $S_K(\lambda)$. Let $\nu \in \Lambda(n,r)$ be a composition. Given a word $i \in I(n,r)$ let $T_i^\nu$ denote the $\nu$-tableau obtained by writing the components of $i$ in order in the Young diagram of shape $\nu$. The weight of a tableau $T_i^\nu$ is simply $\text{wt}(i)$. The multi-index $\ell(\nu)$ is the word consisting of $\nu_1$ 1’s, followed by $\nu_2$ 2’s, and so forth. The following is immediate from the main result of [G2].

**Proposition 3.4.** For $\lambda, \mu \in \Lambda(n,r)$, there is a basis for $1_\lambda S_K(n,r)1_\mu$ consisting of the codeterminants of the form $Y_{i,j}^\nu = \xi_{i,\ell(\nu)} \xi_{\ell(\nu),j}$, such that $\nu \in \Lambda(n,r)$ is dominant and $T_i^\nu$, $T_j^\nu$ are semistandard tableaux of weight $\lambda, \mu$, respectively.

Thus the dimension of $S_K(\lambda)$ is the number of pairs of semistandard tableaux (of some dominant shape $\nu$) of weight $\lambda$.

3.5. We can view $M^\lambda$ as a left $K\Sigma_r$-module. (Any right $K\Sigma_r$-module can be viewed as a left $K\Sigma_r^{op}$-module, but $K\Sigma_r^{op}$ is naturally isomorphic to $K\Sigma_r$, via the involution induced by $\pi \to \pi^{-1}$ for $\pi \in \Sigma_r$.)

Assume that $n \geq r$. The Schur functor $\mathcal{F}$ (see [G1 6.1]) is the functor from left $S_K(n,r)$-modules to left $K\Sigma_r$-modules defined by $V \to 1_\omega V$; it takes $S^\lambda E$ to $M^\lambda$. Indeed, by [L2 and L3] we have isomorphisms $\mathcal{F} S^\lambda E \simeq \omega K[M_{n}]_{\lambda} \simeq (E^{\otimes r})_{\lambda} = M^\lambda$. 

Since $S^\lambda E$ has a good filtration, the natural map
\[(3.5.1) \quad \text{Hom}_{S_K(n,r)}(S^\mu E, S^\lambda E) \to \text{Hom}_{\Sigma_1}(M^\mu, M^\lambda)\]
is injective, for any $\lambda, \mu \in \Lambda(n, r)$. (See [1.8] for details.) Thus, by comparing dimensions we see that the map (3.5.1) is an isomorphism. (See [D62 2.4].) In particular, this gives an algebra isomorphism
\[(3.5.2) \quad \text{End}_{S_K(n,r)}(S^\lambda E) \simeq \text{End}_{\Sigma_1}(M^\lambda)\]
for any $\lambda \in \Lambda(n, r)$. Thus $S_K(\lambda) \simeq \text{End}_{S_K(n,r)}(S^\lambda E)$ (provided $n \geq r$).

We would like to remove the restriction $n \geq r$ in the above. So suppose that $n < r$ and let $E' = K^n$. There is another functor taking $S_K(r, r)$-modules to $S_K(n, r)$-modules given by $M \to eM$ where $e$ is a certain idempotent in $S_K(r, r)$ (see [G1 6.5]). The idempotent $e$ is defined as follows. We regard $\Lambda(n, r)$ as a subset of $\Lambda(r, r)$ via the embedding $(\lambda_1, \ldots, \lambda_n) \to (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$. Let $\Gamma$ be the image of this map; then $e = \sum_{\beta \in \Gamma} 1_{\beta}$. The functor takes $E'$ to $E$ and $S^\lambda E'$ to $S^\lambda E$, for any $\lambda \in \Lambda(n, r)$. By Lemma 3.8, the functor induces a surjection
\[(3.5.3) \quad \text{Hom}_{S_K(r,r)}(S^\mu E', S^\lambda E') \to \text{Hom}_{S_K(n,r)}(S^\mu E, S^\lambda E)\]
for any $\lambda, \mu \in \Lambda(n, r)$. We claim that this is an isomorphism. To see this, observe that from [1.2, 1.4] and Frobenius reciprocity, the dimension of the left-hand-side (right-hand-side) of (3.5.3) is $\dim_K \lambda K[M_\mu]_\mu$ ($\dim_K \lambda K[M_\mu]_\mu$). Since $\lambda, \mu$ are both in $\Lambda(n, r)$, embedded in $\Lambda(r, r)$ as above, we see that these dimensions coincide. In fact, these dimensions are given by the number of $r \times r$ matrices ($n \times n$ matrices) over $\mathbb{N}$ with row sums $\mu$ and column sums $\lambda$; see Remark 1.5. Hence (3.5.3) is an isomorphism, as claimed.

In particular, this gives an algebra isomorphism
\[(3.5.4) \quad \text{End}_{S_K(r,r)}(S^\lambda E') \simeq \text{End}_{S_K(n,r)}(S^\lambda E)\]
for any $\lambda \in \Lambda(n, r)$. We have removed the restriction $n \geq r$ in (3.5.1) and (3.5.2) above. In summary, we have proved the following general result, with no restriction on $n, r$.

**Proposition 3.6.** Let $n$ and $r$ be arbitrary positive integers, $E = K^n$.

(a) For $\lambda \in \Lambda(n, r)$, the algebra $S_K(\lambda)$ is isomorphic with
\[\text{End}_{S_K(n,r)}(S^\lambda E) \simeq \text{End}_{\text{Gl}_n}(S^\lambda E)\]
(b) For $\lambda, \mu \in \Lambda(n, r)$, the $S_K(\lambda)$-$S_K(\mu)$ bimodule $\lambda(S_K)_\mu$ is isomorphic with
\[\text{Hom}_{S_K(n,r)}(S^\mu E, S^\lambda E) \simeq \text{Hom}_{\text{Gl}_n}(S^\mu E, S^\lambda E)\]

**Remark 3.7.** (a) The above result, along with (3.1.4), proves that $\text{End}_{\text{Gl}_n}(E^{\otimes r}) \simeq S_K(\omega) \simeq K\Sigma_r$, provided $n \geq r$.

(b) Since $\oplus_\mu S^\mu E \simeq \oplus_\mu K[M_\mu]_\mu = A_K(n, r)$ (sums taken over all $\mu \in \Lambda(n, r)$), we see that there is an algebra isomorphism $S_K(n, r) \simeq \text{End}_{\text{Gl}_n}(A_K(n, r))$. (This can be proved by other means.)

It remains to formulate the lemma referred to in the proof of the proposition. In fact, we have a more general statement than what is needed above.
Lemma 3.8. Let $S$ be a finite-dimensional algebra, $e \in S$ a nonzero idempotent. Suppose that $N$ is an injective $S$-module, and $eN$ is an injective $eSe$-module. Then for any finite-dimensional $S$-module $M$ the restriction map
\[ \text{Hom}_S(M, N) \to \text{Hom}_{eSe}(eM, eN) \]
is surjective.

Proof. The argument is similar to [E, 1.7]. We proceed by induction on the composition length of $M$. The inductive step is clear since the functors $\text{Hom}_S(\_ , N)$ and $\text{Hom}_{eSe}(\_ , eN)$ are exact. So we only have to prove surjectivity when $M$ is simple. Then either $eM = 0$, or $eM$ is simple and if $eM = 0$ the statement follows trivially. So assume now that $eM$ is simple. Clearly we may also assume that $N$ is indecomposable injective.

We have a surjection $Se \otimes_{eSe} eM \to SeM$ and $SeM = M$ in this case. So we get an inclusion
\[ (3.8.1) \quad 0 \to \text{Hom}_S(M, N) \to \text{Hom}_S(Se \otimes_{eSe} eM, N) \cong \text{Hom}_{eSe}(eM, eN) \]
where the isomorphism comes from adjointness. The general properties of adjointness give that the composition of the inclusion and the isomorphism is precisely the map induced by the functor $X \to eX$.

If $N$ is not the injective hull of $M$ then all the Hom spaces in (3.8.1) are zero. Otherwise they are 1-dimensional and then the map of interest is an isomorphism (and hence is surjective). \hfill \Box

4. Simple $S_K(\lambda)$-modules

There is a connection between Kostka duality and the problem of parametrizing the simple $S_K(\lambda)$-modules.

4.1. Let $\lambda \in \Lambda(n,r)$ be given. There is a functor $F_{\lambda}$ from left $S_K(n,r)$-modules to left $S_K(\lambda)$-modules defined by $V \to 1_{\lambda}V$ (see [Gl, 6.2]). The functor $F$ considered earlier is $F_\omega$. For general $\lambda$, $F_{\lambda}$ is an exact covariant functor mapping simple modules to simple modules or 0. Let $L(\mu)$ be the simple $S_K(n,r)$-module corresponding to a dominant weight $\mu \in \Lambda(n,r)$. By [Gl, (6.2g)], the collection of all nonzero $F_{\lambda}L(\mu)$ forms a complete set of simple $S_K(\lambda)$-modules.

4.2. For a dominant $\mu \in \Lambda(n,r)$, let $J(\mu)$ denote the injective envelope of $L(\mu)$ in the category of rational $M_n$-modules. This is the contravariant dual of the projective cover of $L(\mu)$, in the category of rational $M_n$-modules (or the category of $S_K(n,r)$-modules). Since algebraic monoid induction takes injectives to injectives, the generalized symmetric power $S^{\lambda}E \simeq \text{ind}_D M_n K_\lambda$ is injective as $S(n,r)$-module, for any $\lambda \in \Lambda(n,r)$. Write $(S^{\lambda}E : J(\mu))$ for the multiplicity of $J(\mu)$ in a Krull-Schmidt decomposition of $S^{\lambda}E$. By Frobenius reciprocity one has an equality
\[ (S^{\lambda}E : J(\mu)) = \dim_K 1_{\lambda}L(\mu). \]
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(For more details see [DW] or [Do3] (3.4).) The equality is known as Kostka duality; the nonnegative integer in the equality is the Kostka number, denoted by \( K_{\mu \lambda} \). Note that \( K_{\mu \lambda} \) may be equivalently defined to be the multiplicity of a Young module \( Y^\mu \) in a Krull-Schmidt decomposition of \( M^\lambda \); see [Do3] (3.5), (3.6).

**Proposition 4.3.** Let \( \lambda \in \Lambda(n, r) \) be fixed. The isomorphism classes of simple \( S(\lambda) \)-modules are the \( \lambda L(\mu) = 1_{\lambda} L(\mu) \) for which \( K_{\mu \lambda} \neq 0 \).

**Proof.** Combine [G1] (6.2g) with Kostka duality. \( \square \)

**Remark 4.4.** Donkin [Do3, p. 55, Remark] gives a more precise necessary and sufficient condition on \( \lambda, \mu \) for \( K_{\mu \lambda} \neq 0 \), obtained by combining Steinberg’s tensor product theorem with Suprunenko’s theorem [S].

5. **PBW basis**

In this section, we work with the Lie algebra \( \mathfrak{gl}_n \) and its enveloping algebra \( \mathcal{U} \) over the rational field \( \mathbb{Q} \). We also consider the algebra of distributions (the hyperalgebra) \( \mathcal{U}_\mathbb{Z} \) of the algebraic \( \mathbb{Z} \)-group scheme \( \text{GL}_n \) defined by \( \mathbb{Z}[c_{ij}; (\det(c_{ij}))^{-1}] \).

5.1. Let \( e_{ij} \) be the \( n \times n \) matrix whose unique nonzero entry is a 1 in the \((i, j)\)th position. Set \( f_{ij} = e_{ji} \) and \( H_i = e_{ii} \). The set

\[
\{f_{ij} \mid i < j\} \cup \{H_i\} \cup \{e_{ij} \mid i < j\}
\]

is a Chevalley basis for \( \mathfrak{gl}_n \). We regard these as elements of the universal enveloping algebra \( \mathcal{U} = \mathcal{U}(\mathfrak{gl}_n) \). Note that \( \mathcal{U} \) is generated by the elements

\[
e_i = e_{i,i+1}, \quad f_i = e_{i+1,i} \quad (1 \leq i \leq n-1), \quad H_i \quad (1 \leq i \leq n).
\]

For an element \( X \) and an integer \( a \geq 0 \), set \( X^{(a)} := X^a/(a!) \), \( (X)_a := X(X-1)\cdots(X-a+1)/(a!) \). The hyperalgebra \( \mathcal{U}_\mathbb{Z} \) of the \( \mathbb{Z} \)-group \( \text{GL}_n \) is the \( \mathbb{Z} \)-subalgebra of \( \mathcal{U} \) generated by all \( f_i^{(a)}, (H_i)_b, e_i^{(c)} (a, b, c \geq 0) \); see [J] II, Chapter 1]. The set of all products of the form

\[
\prod_{i<j} f_{ij}^{(a_{ij})} \prod_i \left( \frac{H_i}{b_i} \right) \prod_{i<j} e_{ij}^{(c_{ij})}
\]

for nonnegative integers \( a_{ij}, b_i, c_{ij}, \) forms a \( \mathbb{Z} \)-basis of \( \mathcal{U}_\mathbb{Z} \), where the products among the \( f \)'s (resp., \( e \)'s) are taken in some arbitrary, but fixed, order. With similar conventions on the order of products, the set

\[
\prod_{i<j} e_{ij}^{(c_{ij})} \prod_i \left( \frac{H_i}{b_i} \right) \prod_{i<j} f_{ij}^{(a_{ij})}
\]

is another \( \mathbb{Z} \)-basis of \( \mathcal{U}_\mathbb{Z} \).
5.2. By differentiating the homomorphism \( \rho : \mathfrak{gl}_n \to \text{End}_\mathbb{C}(E^{\otimes r}) \) one obtains a representation \( \mathfrak{gl}_n \to \text{End}(E^{\otimes r}) \); this extends uniquely to a representation

\[
\mathfrak{U} \to \text{End}(E^{\otimes r})
\]

and \( S(n, r) \) (over \( \mathbb{Q} \)) is the image of this representation. Moreover, \( S_{\mathbb{Z}}(n, r) \) is the image of \( \mathfrak{U}_{\mathbb{Z}} \) under the same map. In fact, one can consider the \( \mathfrak{U}_{\mathbb{Z}} \)-invariant lattice \( E_{\mathbb{Z}} \) (the \( \mathbb{Z} \)-span of the standard basis \( (e_i) \) on \( E \)) and check that the map \( d \rho \) takes \( \mathfrak{U}_{\mathbb{Z}} \) into \( \text{End}(E_{\mathbb{Z}}^{\otimes r}) \); then \( S_{\mathbb{Z}}(n, r) \) is the image \( d \rho(\mathfrak{U}_{\mathbb{Z}}) \). One has isomorphisms

\[
S_K(n, d) \simeq K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n, r), \quad \mathfrak{U}_K \simeq K \otimes_{\mathbb{Z}} \mathfrak{U}_{\mathbb{Z}}
\]

for a field \( K \), where \( \mathfrak{U}_K \) is the hyperalgebra of the \( K \)-group \( \mathfrak{gl}_n \). Thus \( S_K(n, r) \) is a homomorphic image of \( \mathfrak{U}_K \). Denote the images in \( S(n, r) \) of the elements \( f_{ij}^{(a)}, (h_i^k), e_{ij}^{(c)} \) by the same symbols.

**Lemma 5.3.** The element \( 1_\lambda \) in \( S(n, r) \) coincides with the element given by the product \( \prod (h_i^k) \), for any \( \lambda \in \Lambda(n, r) \).

**Proof.** The element \( 1_\lambda \) was defined in (2.4.3). It is known (see [GL 3.2]) that, if \( M \) is any \( S(n, r) \)-module, then \( 1_\lambda M \) coincides with the \( \lambda \)-weight space of \( M \) (for \( \lambda \in \Lambda(n, r) \)), considered as left \( \mathfrak{gl}_n \)-module. Similarly, one can show that the element \( 1_\lambda := \prod (h_i^k) \) acts as zero on all weight space except the \( \lambda \) one, and acts as 1 there. This proves that \( 1_\lambda \) and \( 1'_\lambda \) act the same on all modules \( M \). To finish, we need the existence of a faithful module, since on such a module the difference \( 1_\lambda - 1'_\lambda \) acts as zero, hence must equal zero in the algebra. Since \( E^{\otimes r} \) is faithful, as \( S(n, r) \)-module, the proof is complete. \( \Box \)

4. [DG Theorem 2.3] can be reformulated as follows. Given an \( n \times n \) matrix \( A = (a_{ij}) \), let \( \lambda^+(A) := (\lambda^+_1, \ldots, \lambda^+_n) \), where \( \lambda^+_j = a_{jj} + \sum_{i<j} (a_{ii} + a_{ji}) \) for each \( j \). Similarly, \( \lambda^-(A) := (\lambda^-_1, \ldots, \lambda^-_n) \), where \( \lambda^-_j = a_{jj} + \sum_{i>j} (a_{ii} + a_{ji}) \) for each \( j \). Let \( \Theta(n, r) \) be the set of \( n \times n \) matrices over \( \mathbb{N} \) whose entries sum to \( r \). Then by DG Theorem 2.3] the set of products of the form

\[
(\prod_{i<j} f_{ij}^{(a_{ij})}) 1_{\lambda^-}(A) \left( \prod_{i<j} e_{ij}^{(a_{ij})} \right) \quad (A \in \Theta(n, r))
\]

is a \( \mathbb{Z} \)-basis for \( S_{\mathbb{Z}}(n, r) \). Similarly, the set

\[
(\prod_{i<j} e_{ij}^{(a_{ij})}) 1_{\lambda^+}(A) \left( \prod_{i<j} f_{ij}^{(a_{ij})} \right) \quad (A \in \Theta(n, r))
\]

is another basis for \( S_{\mathbb{Z}}(n, r) \). As usual, one has to take products of \( f \)'s (resp., \( e \)'s) with respect to an arbitrary, but fixed, order. One can easily see that any element of the form \( \prod f_{ij}^{(a_{ij})} \) has content (in the sense defined in [DG p. 1911]) not exceeding \( \lambda^+(A) \); conversely, given a monomial of the form \( e_{A1} f_C \) with content not exceeding \( \mu \), one can find a matrix \( A \in \Theta(n, r) \) which defines that monomial. This proves that the first basis given in [DG (2.7)] coincides with the basis described in (4.4.2) above. One can argue similarly that the second basis in [DG (2.7)] coincides with the basis (5.3.1).\(^1\)

\(^1\)There is an error in the definition of the second basis in [DG (2.7)] (and similarly in [DG (3.9)]): one needs a notion of content opposite to the one used to define the first basis there.
By commutation relations given in \cite[Proposition 4.5]{DG} one can rewrite a given basis element of the form (5.4.1) in the form
\begin{equation}
1_{\text{row}(A)} \left( \prod_{i<j} f_{ij}^{(a_{ij})} \prod_{i<j} e_{ij}^{(a_{ij})} \right) 1_{\text{col}(A)} \quad (A \in \Theta(n, r));
\end{equation}
similarly one can rewrite a given basis element of the form (5.4.2) in the form
\begin{equation}
1_{\text{row}(A)} \left( \prod_{i<j} e_{ij}^{(a_{ij})} \prod_{i<j} f_{ij}^{(a_{ij})} \right) 1_{\text{col}(A)} \quad (A \in \Theta(n, r))
\end{equation}
where \text{row}(A) (resp., \text{col}(A)) is the vector of row (column) sums in the matrix \( A \).

Identify \( S(\lambda) \) with \( 1_\lambda S(n, r) 1_\lambda \) and write \( S_Z(\lambda) = 1_\lambda S_Z(n, r) 1_\lambda \) and \( \lambda(S_Z)_{\mu} = 1_\lambda S_Z(n, r) 1_\mu \). Then \( \lambda(S_K)_{\mu} \cong K \otimes_Z \lambda(S_Z)_{\mu} \). The following result provides a basis for \( S(\lambda) \) specialized to any field \( K \).

**Theorem 5.5.** For \( \lambda, \mu \in \Lambda(n, r) \) the bimodule \( \lambda(S_Z)_{\mu} \) has a \( Z \)-basis consisting of all elements
\begin{equation}
1_\lambda \left( \prod_{i<j} f_{ij}^{(a_{ij})} \prod_{i<j} e_{ij}^{(a_{ij})} \right) 1_\mu \quad (A \in \lambda \Theta_{\mu})
\end{equation}
and another such basis consisting of all
\begin{equation}
1_\lambda \left( \prod_{i<j} e_{ij}^{(a_{ij})} \prod_{i<j} f_{ij}^{(a_{ij})} \right) 1_\mu \quad (A \in \lambda \Theta_{\mu})
\end{equation}
where \( \lambda \Theta_{\mu} \) is the set of all \( n \times n \) matrices over \( \mathbb{N} \) with row and column sums equal to \( \lambda \), \( \mu \), resp., and the products of \( f \)'s (resp., \( e \)'s) is taken with respect to some fixed, but arbitrary, order.

**Proof.** This follows immediately from (5.5.1), (5.5.2) above. \( \square \)

### Part 2. Generic Hecke algebras

This part is concerned mainly with certain infinite-dimensional quotients of a subalgebra \( \mathfrak{U}[0] \) of the enveloping algebra \( \mathfrak{U} = \mathfrak{U}(\mathfrak{gl}_n) \), closely related to a modified form \( \hat{\mathfrak{U}} \) of \( \mathfrak{U} \). The construction of \( \hat{\mathfrak{U}} \) was given by Lusztig in the quantum case; its definition here, in §6, is virtually the same. In particular, \( \hat{\mathfrak{U}} \) is an algebra over \( \mathbb{Q} \) (without 1) which is by definition the direct sum of various quotient spaces \( \lambda \mathfrak{U}_{\mu} \) of \( \mathfrak{U} \). There is an infinite family \( \{ 1_\lambda \}_{\lambda \in \mathbb{Z}^n} \) of pairwise orthogonal idempotents in \( \hat{\mathfrak{U}} \) which serves as a replacement for the identity, and \( \lambda \mathfrak{U}_{\mu} = 1_\lambda \mathfrak{U} 1_\mu \) for each \( \lambda, \mu \). We call \( \hat{\mathfrak{U}}(\lambda) := \lambda \mathfrak{U}_{\lambda} \) generic Hecke algebras. Each \( \lambda \mathfrak{U}_{\mu} \) is a bimodule for \( \hat{\mathfrak{U}}(\lambda) \)-\( \hat{\mathfrak{U}}(\mu) \).

In \[\text{7}\] we show that \( \lambda S_{\mu} \) and \( S(\lambda) \) are homomorphic images of \( \lambda \mathfrak{U}_{\mu} \) and \( \hat{\mathfrak{U}}(\lambda) = \lambda \mathfrak{U}_{\lambda} \), and we obtain explicit integral PBW bases for \( \lambda \mathfrak{U}_{\mu} \). As an application, we find (see \[\text{7.5}\]) that the group algebra \( \mathbb{Z} \Sigma_r \) is a homomorphic image of \( \mathfrak{U}[0] \), a certain subalgebra of \( \mathfrak{U}_Z \). In \[\text{8}\] we consider the modified form of \( \mathfrak{U}(\mathfrak{sl}_n) \), and show that the generic algebras and bimodules are the same as the ones constructed in terms of \( \hat{\mathfrak{U}}(\mathfrak{gl}_n) \). It makes little difference whether one works with \( \hat{\mathfrak{U}}(\mathfrak{gl}_n) \) or \( \hat{\mathfrak{U}}(\mathfrak{sl}_n) \). We show (see \[\text{8}\]) that \( \mathfrak{U}_Z(\lambda) \), \( S_Z(\lambda) \)
are cellular algebras in the sense of Graham and Lehrer \([Gl]\). Finally, we describe an indexing set for the simple \(U(\lambda)\)-modules (see \([10]\)).

6. Lusztig’s modified form of \(U(\mathfrak{g}_n)\)

We work over \(\mathbb{Q}\) in this section. We consider the modified form of \(U = U(\mathfrak{g}_n)\).

6.1. Recall that \(U\) is the associative algebra with 1 given by generators \(e_i, f_i\) \((1 \leq i \leq n - 1)\), and \(H_i\) \((1 \leq i \leq n)\) subject to the usual relations given by the Lie algebra structure (see e.g. \([DG]\) relations (R1)–(R5))

For \(\lambda, \mu \in \mathbb{Z}^n\) we define \(\lambda\mathcal{U}_\mu (\text{as a vector space})\) to be the following quotient:

\[
\lambda\mathcal{U}_\mu = U/(\sum_i (H_i - \lambda_i)U + \sum_i U(H_i - \mu_i)).
\]

Let \(\pi_{\lambda,\mu} : U \to \lambda\mathcal{U}_\mu\) be the canonical projection, and set \(\check{U} = \bigoplus_{\lambda,\mu} \lambda\mathcal{U}_\mu\).

Set \(\alpha_i = \varepsilon_i - \varepsilon_{i+1}\) for \(1 \leq i \leq n - 1\), where \(\{\varepsilon_1, \ldots, \varepsilon_n\}\) is the canonical basis of \(\mathbb{Z}^n\).

Consider the grading on \(U\) defined by putting \(H_i \in U[0], e_i \in U[\alpha_i], f_i \in U[-\alpha_i]\) subject to the requirement \(U[\nu']U[\nu'] \subset U[\nu' + \nu']\). Then \(U = \bigoplus_{\nu} U[\nu]\), where \(\nu\) runs over the set \(\sum \mathbb{Z}\alpha_i\) (the root lattice). We note that \(\lambda U[\nu]_\mu := \pi_{\lambda,\mu} U[\nu]\) is zero unless \(\lambda - \mu = \nu\).

The above grading allows one to define a natural associative \(\mathbb{Q}\)-algebra structure on \(\check{U}\) inherited from that of \(U\). For any \(\lambda, \mu, \lambda', \mu' \in \mathbb{Z}^n\) and any \(t \in U[\lambda - \mu], s \in U[\lambda' - \mu']\), the product \(\pi_{\lambda,\mu}(t)\pi_{\lambda',\mu'}(s)\) is defined to be equal to \(\pi_{\lambda,\mu'}(ts)\) if \(\mu = \lambda'\) and is zero otherwise.

The elements \(1_\lambda = \pi_{\lambda,\lambda}(1)\) satisfy the relation \(1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda\), i.e., \(\{1_\lambda\}\) is a family of orthogonal idempotents in \(\check{U}\). Moreover,

\[
\lambda\mathcal{U}_\mu = 1_\lambda \check{U} 1_\mu \quad (\lambda, \mu \in \mathbb{Z}^n).
\]

The algebra \(\check{U}\) does not have 1, since the infinite sum \(\sum 1_\lambda\) does not belong to \(\check{U}\).

6.2. There is a natural \(U\)-bimodule structure on \(\check{U}\), defined by the requirement

\[
t' \pi_{\lambda,\mu}(s) = \pi_{\lambda+\nu,\mu-\nu'}(ts')
\]

for all \(t \in U[\nu], t' \in U[\nu'], \lambda, \mu \in \mathbb{Z}^n\). Moreover, the following identities hold in the algebra \(\check{U}\):

\[
e_i 1_\lambda = 1_{\lambda + \alpha_i} e_i; \quad f_i 1_\lambda = 1_{\lambda - \alpha_i} f_i;
\]

\[
(e_i f_j - f_j e_i) 1_\lambda = \delta_{ij} \lambda 1_\lambda
\]

for all \(\lambda \in \mathbb{Z}^n\), all \(i, j\). One can show that the algebra \(\check{U}\) is generated by all elements of the form

\[
e_i 1_\lambda, \quad f_i 1_\lambda \quad (1 \leq i \leq n - 1, \lambda \in \mathbb{Z}^n).
\]

The algebra \(\check{U}\) inherits a “comultiplication” from the comultiplication on \(U\) (follow \([22, 23.1.5]\)) but we shall not need it here.
6.3. Following Lusztig \[2\] 23.1.4 we say that a $\hat{U}$-module $M$ is unital if: (a) for all $m \in M$ one has $1_\lambda m = 0$ for all but finitely many $\lambda \in \mathbb{Z}^n$; (b) for any $m \in M$ one has $\sum_\lambda 1_\lambda m = m$ (sum over all $\lambda \in \mathbb{Z}^n$).

If $M$ is a unital $\hat{U}$-module, then one may regard it as a $\mathcal{U}$-module with weight space decomposition, as follows. The weight space decomposition setting $\lambda$ algebra.

\[(6.4.4) (\text{basis for } \hat{U}) \]

and another such basis consisting of the elements of the form

\[(6.4.2) \]

Let $b^+1_\lambda b^-$ (for any $b^+, b^- \in B^-$, $\lambda \in \mathbb{Z}^n$)

and another such basis consisting of the elements of the form

\[(6.4.3) \]

It follows that there is a $\mathbb{Z}$-basis for $\hat{U}_\lambda$ consisting of all elements of the form

\[(6.4.4) \]

and another $\mathbb{Z}$-basis consisting of all elements of the form

\[(6.4.5) \]

where $\tilde{\Theta}(n)$ is the set of $n \times n$ integral matrices with off-diagonal entries $\geq 0$, and where as usual products of $f$’s (resp., $e$’s) are taken with respect to some fixed order. The definition of $\lambda^+(A)$, $\lambda^-(A)$ here is the same as in 5.3.

7. THE GENERIC ALGEBRA $\hat{U}(\lambda) = \lambda \mathcal{U}_\lambda$

7.1. From the definition of the multiplication in $\hat{U}$ it is clear that $\lambda \mathcal{U}_\lambda = 1_\lambda \mathcal{U} \lambda_\lambda$ is an algebra and $\lambda \mathcal{U}_\mu = 1_\lambda \mathcal{U} \mu_\mu$ is a bimodule (for $\lambda \mathcal{U} \mu_\mu$). By definition, $\hat{U}$ is the direct sum of these bimodules, as $\lambda, \mu$ range over $\mathbb{Z}^n$. We call the algebra $\hat{U}(\lambda) = \lambda \mathcal{U}_\lambda$ a generic algebra.

We have the following analogue of Lemma 3.2 which shows that one needs to consider $\hat{U}(\lambda)$ and $\lambda \mathcal{U}_\mu$ only in case $\lambda, \mu \in \mathbb{Z}^n$ are dominant.

Lemma 7.2. Let $\lambda, \mu \in \mathbb{Z}^n$.

(a) For any $w \in \Sigma_n$, one has an algebra isomorphism $\hat{U}(\lambda) \simeq \hat{U}(w \lambda)$.

(b) For any $w, w' \in \Sigma_n$, one has an isomorphism $\lambda \mathcal{U}_\mu \simeq w_\lambda \mathcal{U} w'_\mu$ of $\hat{U}(\lambda)$-$\hat{U}(\mu)$ bimodules.
Proof. The Weyl group $W \simeq \Sigma_n$ acts as permutations on the Lie algebra $\mathfrak{gl}_n$ via the rule: $w e_{ij} = e_{w^{-1}(i),w^{-1}(j)}$. (Here the $e_{ij}$ are the matrix units defined in 5.1.) This induces a corresponding action of $\Sigma_n$ on $\mathfrak{U}$. Using this action, the statements are easily checked.

Proposition 7.3. For any $\lambda, \mu \in \Lambda(n, r)$ the linear map $\psi_{\lambda, \mu} : \mathfrak{U} \to 1\lambda S(n, r)1_\mu$ given by $u \to 1\lambda \pi_{n,r}(u)1_\mu$ is surjective, where $\pi_{n,r} : \mathfrak{U} \to S(n, r)$ is the map defined in (5.2.1). Moreover, the above linear map induces a linear surjection $\lambda \mathfrak{U}_\mu \to 1\lambda S(n, r)1_\mu$.

Proof. By [DG Proposition 4.3(a)], in $S(n, r)$ we have the relation $H_i 1_\mu = 1_\mu H_i = \mu_i 1_\mu$, for any $i$ and any $\mu \in \mathbb{Z}^n$. It follows that $H_i - \lambda_i$ acts on the left on $1\lambda S(n, r)1_\mu$ as zero; similarly, $H_i - \mu_i$ acts as zero on the right. Thus the map $\psi_{\lambda, \mu}$ sends any element of $\sum_i(H_i - \lambda_i)\mathfrak{U} + \sum_i \mathfrak{U}(H_i - \mu_i)$ to zero, so its kernel contains the kernel of $\lambda \mathfrak{U}_\mu$, and so $\psi_{\lambda, \mu}$ factors through $\lambda \mathfrak{U}_\mu$. Moreover, $\psi_{\lambda, \mu}$ is surjective since the map

$$u \to \sum_{\lambda, \mu \in \Lambda(n, r)} 1\lambda \pi_{n,r}(u)1_\mu$$

is the surjection $\mathfrak{U} \to S(n, r)$ of (5.2.1). All the assertions of the proposition are now clear.

Corollary 7.4. (a) The restriction to $\mathfrak{U}_\mathbb{Z}$ of $\psi_{\lambda, \mu}$ surjects onto $1\lambda S_{\mathbb{Z}}(n, r)1_\mu$.

(b) For $\lambda \in \Lambda(n, r)$, $S(\lambda)$ is a homomorphic image of $\hat{\mathfrak{U}}(\lambda)$. Similarly, $S_{\mathbb{Z}}(\lambda)$ is a homomorphic image of $\mathfrak{U}_\mathbb{Z}(\lambda)$.

Remarks 7.5. (a) The map $\psi_{\lambda, \lambda} : \mathfrak{U} \to S(\lambda)$ is not in general an algebra map. (For instance, the product $f_i e_i$ will map to something nonzero but $e_i$, $f_i$ themselves map to zero.) However, the restriction of $\psi_{\lambda, \lambda}$ to the subalgebra $\mathfrak{U}[0]$ (see 6.1) of $\mathfrak{U}$ will be an algebra map. So, for $\lambda \in \Lambda(n, r)$, $S(\lambda)$ is a homomorphic image of $\mathfrak{U}[0]$ and $S_{\mathbb{Z}}(\lambda)$ is a homomorphic image of $\mathfrak{U}_\mathbb{Z}[0]$.

(b) In particular, this shows that $\mathbb{Z}\Sigma_r$ is a quotient of $\mathfrak{U}_\mathbb{Z}[0]$; in other words, $\mathbb{Z}\Sigma_r$ is a subquotient of $\mathfrak{U}_\mathbb{Z}$. Similarly, $\mathbb{Z}\Sigma_r$ is a subquotient of $\hat{\mathfrak{U}}_\mathbb{Z}$, since it is a homomorphic image of $\lambda(\mathfrak{U}_\mathbb{Z}) = \hat{\mathfrak{U}}_\mathbb{Z}(\lambda)$. Thus for any field $K$ the group algebra $K\Sigma_r$ is a subquotient of the hyperalgebra $\mathfrak{U}_K = K \otimes_{\mathbb{Z}} \mathfrak{U}_\mathbb{Z}$ (and of $\hat{\mathfrak{U}}_K = K \otimes_{\mathbb{Z}} \hat{\mathfrak{U}}_\mathbb{Z}$).

The following result provides a basis for $\hat{\mathfrak{U}}_K(\lambda) := \hat{\mathfrak{U}}_\mathbb{Z}(\lambda) \otimes_{\mathbb{Z}} K$ under specialization to any field $K$.

Proposition 7.6. For any $\lambda, \mu \in \mathbb{Z}^n$, a $\mathbb{Z}$-basis for the bimodule $\lambda(\hat{\mathfrak{U}}_\mathbb{Z})_\mu = 1\lambda \hat{\mathfrak{U}}_\mathbb{Z}1_\mu$ is given by the set of all elements of the form

$$(7.6.1) \quad 1\lambda \left( \prod_{i<j} f_{ij}^{(a_{ij})} \prod_{i<j} e_{ij}^{(a_{ij})} \right) 1_\mu \quad (A \in \lambda \Theta_{\mu})$$

and another such basis is given by all elements of the form

$$(7.6.2) \quad 1\lambda \left( \prod_{i<j} e_{ij}^{(a_{ij})} \prod_{i<j} f_{ij}^{(a_{ij})} \right) 1_\mu \quad (A \in \lambda \widetilde{\Theta}_{\mu})$$
where \( \tilde{\Theta}_\mu \) is the set of all matrices \( A \in \tilde{\Theta}(n) \) with row and column sums equal to \( \lambda \), \( \mu \), resp. As usual, products of \( f \)'s (resp., \( e \)'s) is taken with respect to some fixed, but arbitrary, order.

**Proof.** This follows from (6.4.4), (6.4.5) and commutation formulas (6.2.2). The argument is similar to the argument for 5.5. \( \square \)

**Example 7.7.** We consider the case \( n = 2 \). Let \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \). Then the basis of \( \dot{\mathfrak{U}}_Z(\lambda) \) described in (7.6.1) consists of all elements of the form

\[
1^\lambda f^{(a)} e^{(a)} 1^\lambda \quad (a \geq 0)
\]

and the basis described in (7.6.2) consists of all elements of the form

\[
1^\lambda e^{(a)} f^{(a)} 1^\lambda \quad (a \geq 0).
\]

If \( \lambda \in \Lambda(2, r) \), the nonzero images in \( S_Z(\lambda) \) of the elements (7.7.1) give the following set of elements

\[
1^\lambda f^{(a)} e^{(a)} 1^\lambda \quad (0 \leq a \leq \min(\lambda_1, \lambda_2))
\]

which was described in (5.5.1); the nonzero images in \( S_Z(\lambda) \) of the elements (7.7.2) give the following set of elements

\[
1^\lambda e^{(a)} f^{(a)} 1^\lambda \quad (0 \leq a \leq \min(\lambda_1, \lambda_2))
\]

which was described in (5.5.2). The sets in (7.7.3), (7.7.4) are bases of \( S_Z(\lambda) \). Thus \( \dim S(\lambda) = 1 + \min(\lambda_1, \lambda_2) \).

Note that from this description it follows that \( \dot{\mathfrak{U}}(\lambda) \) (for \( \lambda \in \mathbb{Z}^2 \)) is generated by the element \( 1^\lambda f e 1^\lambda \). It is also generated by the element \( 1^\lambda e f 1^\lambda \). It follows that the algebra \( \mathfrak{U}(\lambda) \) is commutative. The same statements apply to \( S(\lambda) \) (for \( \lambda \in \Lambda(2, r) \)). This does not hold for partitions with more than 2 parts, in general.

8. The modified form of \( \mathfrak{U}(\mathfrak{sl}_n) \)

8.1. Note that \( \mathfrak{U}(\mathfrak{sl}_n) \) is the subalgebra of \( \mathfrak{U} = \mathfrak{U}(\mathfrak{gl}_n) \) generated by all \( e_i, f_i, h_i := H_i - H_{i+1} \) (\( 1 \leq i \leq n - 1 \)); see e.g. [DG, p. 1909], sentence following Theorem 2.1. The definition of \( \dot{\mathfrak{U}}(\mathfrak{sl}_n) \) is nearly the same as the definition of \( \dot{\mathfrak{U}}(\mathfrak{gl}_n) \). For \( \lambda, \mu \in \mathbb{Z}^{n-1} \) one defines \( \lambda \mathfrak{U}(\mathfrak{sl}_n)_\mu \) (as a vector space) to be the following quotient space:

\[
\mathfrak{U}(\mathfrak{sl}_n)/\left( \sum_i (h_i - \langle \lambda, \alpha_i^\vee \rangle) \mathfrak{U}(\mathfrak{sl}_n) + \sum_i \mathfrak{U}(\mathfrak{sl}_n) (h_i - \langle \mu, \alpha_i^\vee \rangle) \right).
\]

Set \( \dot{\mathfrak{U}}(\mathfrak{sl}_n) = \bigoplus_{\lambda, \mu}(\lambda \mathfrak{U}(\mathfrak{sl}_n)_\mu) \). The rest of the construction is exactly the same as for \( \dot{\mathfrak{U}}(\mathfrak{gl}_n) \). The only difference between \( \mathfrak{U} \) and \( \dot{\mathfrak{U}}(\mathfrak{sl}_n) \) is that the former has more idempotents than the latter.
8.2. Given a weight \( \lambda \in \mathbb{Z}^n \) (for \( \mathfrak{g}l_n \)) we obtain a corresponding weight \( \tilde{\lambda} \in \mathbb{Z}^{n-1} \) (for \( \mathfrak{sl}_n \)) as follows:

\[
\lambda \rightarrow \tilde{\lambda} := (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n).
\]

The restriction to \( \mathfrak{U}(\mathfrak{sl}_n) \) of the map \( \pi_{n,r} \) is still a surjection onto \( S^{Q}(n,r) \). This follows from the decomposition \( \mathfrak{g}l_n = \mathfrak{sl}_n \oplus Q I \); the image of \( I \) in \( \mathfrak{U}(\mathfrak{g}l_n) \) acts as scalars.

Hence the Schur algebra \( S^{Q}(n,r) \) is a homomorphic image of \( \mathfrak{U}(\mathfrak{sl}_n) \). This restriction is compatible with integral forms; i.e., the restriction of \( \pi_{n,r} \) to \( \mathfrak{U}_Z(\mathfrak{sl}_n) \) surjects onto \( S^{Z}(n,r) \).

**Proposition 8.3.** For \( \lambda, \mu \in \mathbb{Z}^n \), the natural map \( \tilde{\lambda} \mathfrak{U}_Z(\mathfrak{sl}_n) \tilde{\mu} \rightarrow \lambda \mathfrak{U}_Z(\mathfrak{g}l_n) \mu \) is an isomorphism of bimodules. In case \( \mu = \lambda \) it is an isomorphism of algebras.

**Proof.** One can adapt the argument for [23.2.5] to prove the map is a \( \mathbb{Z} \)-linear isomorphism. In fact, one can easily check that there are \( \mathbb{Z} \)-bases of \( \tilde{\lambda} \mathfrak{U}_Z(\mathfrak{sl}_n) \tilde{\mu} \) of the form (7.6.1) and (7.6.2), with \( 1_{\lambda}, 1_{\mu} \) there replaced by \( 1_{\tilde{\lambda}}, 1_{\tilde{\mu}} \). The bijection is now clear.

Thus we see that, for the study of the Hecke algebras \( S(\lambda) \), one can use a descent from generic subalgebras of either algebra \( \mathfrak{U}(\mathfrak{sl}_n) \) or \( \mathfrak{U}(\mathfrak{g}l_n) \). The difference between these viewpoints is merely notational.

9. **Cellularity of \( \mathfrak{U}(\lambda), S(\lambda) \)**

Lusztig [1] showed that the positive part \( U^+ \) of a quantized enveloping algebra \( U \) has a canonical basis. In [2, Part IV] the canonical basis is extended to \( \mathfrak{U} \). We show that the canonical basis on \( \mathfrak{U} \) induces compatible canonical bases on \( \mathfrak{U}(\lambda), S(\lambda) \). We apply this information to show that \( \mathfrak{U}(\lambda) \) and \( S(\lambda) \) inherit canonical bases from the canonical basis on \( \mathfrak{U} \), and that these bases are cellular bases. In particular, this shows that \( \mathfrak{U}(\lambda) \) and \( S(\lambda) \) are cellular algebras. (One can see the cellularity in other ways; see e.g. [DJM] Theorem 6.6, [M] Chapter 4, Exer. 13.)

9.1. We recall from [GL] the definition of cellularity. Let \( A \) be an associative algebra over a ring \( R \) (commutative with 1). We do not insist that \( A \) has 1, nor do we insist that \( A \) be finite-dimensional. A cell datum for \( A \) is a quadruple \( (\Lambda, M, C, \iota) \) where:

(a) \( \Lambda \) (the set of weights) is partially ordered by \( \geq \), \( M \) is a function from \( \Lambda \) to the class of finite sets, and \( C \) is an injective function

\[
C : \coprod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \rightarrow A
\]

with image an \( R \)-basis of \( A \). If \( \lambda \in \Lambda \) and \( S, T \in M(\lambda) \) then one writes \( C_{S,T}^\lambda \) for \( C(S, T) \).

(b) The map \( \iota \) is an \( R \)-linear involutory antiautomorphism of \( A \) such that \( \iota(C_{S,T}^\lambda) = C_{T,S}^\lambda \).

\[
\mathfrak{g}l_n = \mathfrak{sl}_n \oplus \mathbb{Q} I
\]
(c) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$ then for all $a \in A$ one has
\[
a C_{S,T}^\lambda \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C_{S',T}^\lambda \mod A(> \lambda),
\]
where $r_a(S', S)$ is independent of $T$ and $A(> \lambda)$ is the $R$-submodule of $A$ generated by
the set of all $C_{S',T}^\mu$ such that $\mu > \lambda$, $S'', T'' \in M(\mu)$.

Any algebra $A$ that possesses a cell datum is said to be cellular and the basis $\{C_{S,T}^\lambda\}$
produced by its cell datum is its cellular basis.

Remark 9.2. For our applications, it is convenient to use the order on $\Lambda$ opposite to
the order in the usual definition, so we have reversed the usual ordering of weights in
the definition above.

9.3. Following R.M. Green [3] we say that a cell datum $(\Lambda, M, C, \iota)$ is of profinite
type if $\Lambda$ is infinite and if for each $\lambda \in \Lambda$, the set $\{\mu \in \Lambda : \mu \leq \lambda\}$ is finite.

The following useful lemma follows immediately from the definitions.

Lemma 9.4. (a) Let $(\Lambda, M, C, \iota)$ be a cell datum for $A$. Let $e \in A$ be an idempotent
fixed by the involution $\iota$. Then $eAe$ is cellular, with cell datum $((\overline{\Lambda}, \overline{M}, \overline{C}, \overline{\iota})$, where
$\overline{\Lambda} = \{\lambda \in \Lambda : e C_{S,T}^\lambda e \neq 0, \text{ some } S, T \in M(\lambda)\}$, $\overline{M}(\lambda) = \{S, T \in M(\lambda) : e C_{S,T}^\lambda e \neq 0\}$ for
any $\lambda \in \overline{\Lambda}$; $\overline{C}$ is defined by $\overline{C}_{S,T}^\lambda = e C_{S,T}^\lambda e$ whenever $\lambda \in \overline{\Lambda}$, $S, T \in \overline{M}(\lambda)$; and $\overline{\iota}$ is the
restriction of $\iota$ to $eAe$.

(b) If the original cell datum $(\Lambda, M, C, \iota)$ is of profinite type, then $(\overline{\Lambda}, \overline{M}, \overline{C}, \overline{\iota})$ is also
of profinite type, provided $\overline{\Lambda}$ is infinite.

9.5. The remarks in this subsection apply to a quantized enveloping algebra $U = U(g)$
corresponding to an arbitrary reductive Lie algebra $g$. We follow the setup and notation
of [2]. Let $\mathfrak{U} = \mathfrak{U}(g)$. Lusztig [2] shows that the canonical basis on $U^+$ can be
extended to a canonical basis of the modified form $\hat{U}$. Moreover, in [2] Chapter 29
Lusztig shows that the canonical basis of $\hat{U}$ is a cellular basis (of profinite type). This
is spelled out in greater detail in [2] 2.5.

In [1] it was pointed out that the canonical basis on $U^+$ (which is an $A$-basis for
$U^+_A$) corresponds under specialization to a $\mathbb{Z}$-basis of the plus part $U^+_1 = U^+_\mathbb{Z}$ of $U_\mathbb{Z}$. By
[2] 23.2, one has an $A$-basis of $\hat{U}$ consisting of elements of the form
\[
b^+ 1_\lambda b^-.
\]
Similarly one has another $A$-basis consisting of elements of the form
\[
b^- 1_\lambda b^+.
\]
In both sets of elements above, $b^+$, (resp., $b^-$) vary independently over any $A$-basis of
$U^+_A$, (resp., $U^-_A$). It follows that specializing $v$ to 1 takes $\hat{U}_A$ to $\hat{U}_\mathbb{Z}$. Moreover, any
$A$-basis for $\hat{U}_A$ of the form (9.5.1) or (9.5.2) will correspond to a $\mathbb{Z}$-basis of $U_\mathbb{Z}$. In
particular, the canonical basis on $\hat{U}_A$ corresponds under specialization to a $\mathbb{Z}$-basis of
$U_\mathbb{Z}$; we call this the canonical basis of $\mathfrak{U}$. It is a cellular basis of $U_\mathbb{Z}$ (of profinite type).
In the rest of this section, \( \mathfrak{U} = \mathfrak{U}(\mathfrak{gl}_n) \) or \( \mathfrak{U}(\mathfrak{sl}_n) \), and we set \( X = \mathbb{Z}^n \) or \( \mathbb{Z}^{n-1} \).

**Lemma 9.6.** The image of the canonical basis under the quotient map \( \hat{\mathfrak{U}}_Z \to S_Z(n, r) \) is a \( \mathbb{Z} \)-basis of \( S_Z(n, r) \). (We call it the canonical basis of \( S_Z(n, r) \)).

**Proof.** By Donkin [Do1, Do2], \( S_Z(n, r) \) is a generalized Schur algebra, defined by the saturated set \( \pi = \Pi^+(E_Z^{\otimes r}) \) (the set of dominant weights occurring in the tensor space \( E_Z^{\otimes r} \)). Since \( S_Z(n, r) \simeq \hat{\mathfrak{U}} / \mathfrak{U}[P] \), where \( P \) is the complement of \( \pi \) in the set of dominant weights, the claim follows by the analogue of [L2] 29.2. \( \square \)

**Remark 9.7.** We have chosen, for simplicity, to obtain the canonical basis on \( S(n, r) \) by descent from the canonical basis of \( \hat{\mathfrak{U}} \). In [Du] this is approached (for the \( q \)-Schur algebra) the other way around, by building up from the Kazhdan-Lusztig basis for the Hecke algebra of type \( A \). See [BLM] for yet another approach.

**Theorem 9.8.** (a) For \( \lambda \in X \), the generic algebra \( \hat{\mathfrak{U}}_Z(\lambda) \) is a cellular subalgebra of \( \hat{\mathfrak{U}} \), with cell datum of procellular type. The cellular basis determined by the cell datum is inherited from the canonical basis of \( \hat{\mathfrak{U}}_Z \); we call this cellular basis the canonical basis of \( \hat{\mathfrak{U}}_Z(\lambda) \).

(b) For \( \lambda \in \Lambda(n, r) \), the Hecke algebra \( S_Z(\lambda) \) is a cellular subalgebra of the Schur algebra \( S_Z(n, r) \). The cellular basis determined by its cell datum is inherited from the canonical basis of \( S_Z(n, r) \); we call this cellular basis the canonical basis of \( S_Z(\lambda) \).

(c) For \( \lambda \in \Lambda(n, r) \), the quotient map \( \mathfrak{U} \to S(n, r) \) maps the canonical basis of \( \hat{\mathfrak{U}}_Z(\lambda) \) (or of \( \hat{\mathfrak{U}}_Z(\lambda) \)) onto the canonical basis of \( S_Z(\lambda) \).

**Proof.** This follows immediately from Lemmas 9.4 and 9.6 since the idempotent \( 1_\lambda \) is fixed by the involution. \( \square \)

**10. Simple \( \hat{\mathfrak{U}}_K(\lambda) \)-modules**

We work over an arbitrary infinite field \( K \) in this section. Set \( \hat{\mathfrak{U}}_K = K \otimes \hat{\mathfrak{U}}_Z \), and similarly for \( \hat{\mathfrak{U}}_K(\lambda) \), \( \lambda(\hat{\mathfrak{U}}_K)_\mu \). The simple unital \( \hat{\mathfrak{U}}_K \)-modules are the same as the simple rational \( \mathfrak{GL}_n(K) \)-modules. This follows, for instance, from the \( v = 1 \) analogue of [L2] Chapter 31. (Alternately, see the discussion in [J] II, 1.20.)

10.1. Let \( \lambda \in \mathbb{Z}^n \) be given. Let \( \mathcal{F}_\lambda \) be the functor \( V \to 1_\lambda V \) from unital \( \hat{\mathfrak{U}}_K \)-modules to left \( \hat{\mathfrak{U}}_K(\lambda) \)-modules. This is an exact covariant functor mapping simple modules to simple modules or 0. Let \( L(\mu) \) be the simple \( \hat{\mathfrak{U}}_K \)-module corresponding to a dominant weight \( \mu \in \mathbb{Z}^n \). By [GL] (6.2g)], the collection of all nonzero \( \mathcal{F}_\lambda L(\mu) \) forms a complete set of simple \( \hat{\mathfrak{U}}_K(\lambda) \)-modules.

10.2. We write \( \mathfrak{GL}_n = \mathfrak{GL}_n(K) \). For a dominant \( \mu \in \mathbb{Z}^n \), let \( I(\mu) \) denote the injective envelope of \( L(\mu) \) in the category of rational \( \mathfrak{GL}_n \)-modules. Since induction takes injectives to injectives, the module \( \text{ind}_{\hat{\mathfrak{U}}_n}^{\mathfrak{GL}_n} K(\lambda) \) is injective as rational \( \mathfrak{GL}_n \)-module, for
any $\lambda \in \mathbb{Z}^n$. Write $(\text{ind}_{T_n}^{GL_n} K_\lambda : I(\mu))$ for the multiplicity of $I(\mu)$ in a Krull-Schmidt decomposition of $\text{ind}_{T_n}^{GL_n} K_\lambda$. By Frobenius reciprocity one has an equality

(10.2.1) \[ (\text{ind}_{T_n}^{GL_n} K_\lambda : I(\mu)) = \dim_K 1_\lambda L(\mu) \]

and this shows, in particular, that the multiplicities on the left-hand-side are finite. Let us denote the integer in the equality by $K_{\mu, \lambda}'$. The following analogue of 4.3 is now clear.

**Proposition 10.3.** Let $\lambda \in \mathbb{Z}^n$ be fixed. The isomorphism classes of simple $\hat{\mathfrak{U}}_K(\lambda)$-modules are the $\lambda L(\mu) = 1_\lambda L(\mu)$ for which $K_{\mu, \lambda}' \neq 0$.

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