Casimir effect in $E^3$ closed spaces

Mariana P Lima and Daniel Müller

Instituto de Física, UnB, Campus Universitário Darcy Ribeiro, Cxp 04455, 70919-970, Brasília DF, Brazil

E-mail: mariana@unb.br and muller@fis.unb.br

Received 1 September 2006, in final form 29 December 2006
Published 24 January 2007
Online at stacks.iop.org/CQG/24/897

Abstract
As it is well known the topology of space is not totally determined by Einstein’s equations. It is considered a massless scalar quantum field in a static Euclidean space of dimension 3. The expectation value for the energy density in all compact orientable Euclidean 3-spaces are obtained in this work as a finite summation of Epstein type zeta functions. The Casimir energy density for these particular manifolds is independent of the type of coupling with curvature. A numerical plot of the result inside each Dirichlet region is obtained.

PACS numbers: 04.62.+v, 03.70.+k

1. Introduction

The universe today is very close to homogeneous and isotropic. Then the most important three manifolds in connection with cosmology are $S^3$, $H^3$ and $E^3$. It is well known that $R \times S^3$, $R \times H^3$ and $R \times E^3$ are solutions of Einstein’s equations (EQ). It is irrelevant for the EQ whether the spatial sections have a nontrivial topology modelled on them or not.

There are many effects that arise if space is compact. The first astrophysical limits on the topology of the universe were obtained for a 3-torus $T^3$. Accordance with the homogeneity of the cosmic microwave background radiation CMBR puts a lower limit on the size of the fundamental cell, about 3000 Mpc, which is a cube in the cases of [1] and [2]. Later on, it was shown that this result is very sensitive to the type of the compactifications of the spatial sections. For a universe with spatial sections $T^2 \times R$, the fundamental cell’s size is about 1/10 of the horizon, and is compatible with the homogeneity of the CMBR [3]. In compact universes, the pair separation histogram would present spikes for characteristic distances. At first it was thought that this technique, known as the crystallographic method, was able to totally determine the topology of the universe [4]. It turned out that the crystallographic method only applied when the group that defines the manifold, contained at least a Clifford translation, i.e. a translation which moves all the points by the same distance [5] and [6]. Generalizations of the crystallographic method were proposed, for example in [7].
Also in compact universes the light front of the CMBR interacts with itself producing circles in its sky pattern [8].

Recent results have called our attention to the possibility that methods based on multiple images will not prove to be efficient [9]. The reasoning is that, according to observations, the curvature is very small, so the fundamental regions are so big that there has not been enough time for the formation of ghost images. The result is that for low curvature universes such as ours, only compact universes with the smallest volumes could be detected by pattern repetitions.

The knowledge of the Laplacian spectrum for compact spaces has some motivations. One of them is the decomposition of the CMBR in right set of modes, if space is nontrivial. It has been studied numerically for compact hyperbolic space by [10, 11]. And more recently [12] for compact spherical space.

In 1948, Casimir calculated an attraction force between parallel noncharged plates in vacuum [13]. This force is universal, in the sense that it does not depend on the physical properties of the plates, or the electric charge. Its origin is of topological nature, connected to the boundary conditions. There are interesting articles, in which the imposition of boundary conditions is addressed in a more rigorous and realistic approach, for instance [14].

The Casimir effect also occurs in compact spaces. For the present era where the size of the universe is $\sim 3000$ Mpc the Casimir effect is irrelevant. However, soon after the cosmological singularity the co-moving distances were on the order of the Planck scale. It is generally believed that the very primordial universe was dominated by strict quantum effects including Casimir effect. Although today quantum effects are commonly disregarded, they must have been fundamentally important in the primordial universe. We shall use the massless scalar field as a toy model.

The Casimir effect can be obtained by analytic continuation of the zeta function. This procedure is known as the zeta function regularization, see for example [15, 16]. However, in a more realistic context, the ability of the zeta function method has been questioned [17]. For a calculation of the Casimir energy in the spherical space, see [18].

In this work we expose a different technique which was previously used by us for the hyperbolic space [19–21].

Previous calculations for the Casimir effect for the Euclidean space were done, for instance by, DeWitt, Hart, Isham and others [22]. Particularly interesting is Dowker’s article in which the result is also written in terms of Epstein’s zeta functions. A topological classification of possible ‘gauge fields’ over compact base manifolds is also given in [22]. These and other very interesting topological questions are not going to be addressed in this present work.

The crystallographic groups are discrete subgroups of the full isometry group of the Euclidean space. Among all crystallographic groups, only a few of them are of interest in connection to manifolds. These groups must be torsion-free, for the Euclidean and hyperbolic spaces. In the celebrated list of problems proposed by Hilbert, the 18th was answered affirmatively by Bieberbach in 1910–1912 [23, 24], see also [25]. Bieberbach showed that there are only a finite number of crystallographic groups for a Euclidean space of fixed dimension. For example, when the dimension of the Euclidean space is 3 there are only ten compact manifolds, of which six are orientable and four nonorientable.

In this present work we shall investigate the six compact orientable manifolds and obtain the Casimir energy for each of them. Due to the more simple structure connected to these topological spaces compared to the hyperbolic or spherical case, an analytical result is obtained in terms of Epstein zeta functions. It is explicitly obtained that our result is independent of the type of coupling with curvature $\xi$ for these particular manifolds, which means that the calculated Casimir energy density is the same even in an inflationary de Sitter regime.
The paper is organized as follows. Section 2 is included for completeness. In this section, the expectation value for the energy–momentum tensor, for a massless scalar field in closed Minkowski spaces is obtained. Section 3 presents a brief description of the compact flat manifolds. The Casimir energy is obtained in each subsection, 3.1–3.6 and a numerical plot is given. Natural units are used, $c = G = \hbar = 1$, and the metric $\eta_{\mu\nu} = [-1, 1, 1, 1]$

2. Casimir effect in closed spaces

The point splitting method was constructed to obtain the renormalized (finite) expectation values for the quantum mechanical operators. It is based on the Schwinger formalism [26], and was developed in the context of curved space by DeWitt [27]. Further details are contained in the articles by Christensen [28, 29]. For a review, see [31].

Metric variations in the scalar action

$$S = -\frac{1}{2} \int \sqrt{-g} (\phi,\phi^* + \xi R \phi^2 + m^2 \phi^2) \, d^4 x,$$

give the classical energy–momentum tensor

$$T_{\mu\nu} = (1 - 2\xi) \phi,\phi,_{\mu}\phi,\phi,_{\nu} - \left( \frac{1}{2} - 2\xi \right) \phi,\phi,^\rho g_{\mu\nu} - 2\xi \phi,\phi,_{\mu\nu} + 2\xi g_{\mu\nu} \phi,\phi, + \xi G_{\mu\nu} \phi^2 - \frac{1}{2} m^2 g_{\mu\nu} \phi^2,$$

(1)

where $G_{\mu\nu}$ is the Einstein tensor. As expected, for conformal coupling, $\xi = 1/6$ with massless fields, it can be verified that the trace of the above tensor is identically zero if $m = 0$. In the following we shall restrict to spatial Euclidean sections and massless fields.

The renormalized energy–momentum tensor involves field products at the same spacetime point. Thus the idea is to calculate the averaged products at separate points, $x$ and $x'$, taking the limit $x' \to x$ in the end.

$$\langle 0 | T_{\mu\nu} (x) | 0 \rangle = \frac{1}{2} \lim_{x' \to x} T(x, x')_{\mu\nu}$$

(2)

where for the particular manifolds addressed in this paper,

$$T(x, x')_{\mu\nu} = (1 - 2\xi) \{ (\partial_\mu \partial_\nu' + \partial_\mu' \partial_\nu) - 2\xi (\partial_\beta \partial_\alpha' + \partial_\alpha' \partial_\beta) \} G_{M}^{(1)}(x, x'),$$

(3)

where $G_{M}^{(1)}$ is the Hadamard function, which is the expectation value of the anti-commutator of $\phi(x)$ and $\phi(x')$ on the manifold $M$, see below.

The causal Green function or Feynman propagator, for the infinite covering space $R^3$, is obtained as

$$G(x, x') = i \langle 0 | T \phi(x) \phi(x') | 0 \rangle,$$

where $T$ is the time-ordering operator. Taking its real and imaginary parts,

$$G(x, x') = G_{r}(x, x') + \frac{i}{2} G^{(1)}(x, x'),$$

(4)

we get for the Hadamard function

$$G^{(1)}(x, x') = \langle 0 | [\phi(x), \phi(x')] | 0 \rangle = 2 \text{Im} G(x, x').$$

For a massless scalar field

$$G^{(1)}(x, x') = \frac{1}{2\pi^2} \frac{1}{-(t - t')^2 + r^2} = \frac{1}{2\pi^2} \frac{1}{(x - x')^2}.$$
Green functions, as any other function defined in the spatially compact spacetime \( R \times M \), must have the same periodicities of the manifold \( M \) itself. One way of imposing this periodicity is by the direct summation of all the elements of the group \( \Gamma \) which defines the manifold

\[
G^{(1)}_{M}(x, x') = \frac{1}{2\pi^2} \sum_{\gamma_i \in \Gamma} \frac{1}{-\left(t - t'\right)^2 + \left(\vec{r} - \vec{r}'\right)^2}.
\] (5)

The above expression is named the Poincaré series, and when it converges, it defines functions on the manifold \( M \), in this particular case, the Hadamard function.

There are recent articles where the physical context connected to particular boundary conditions, Dirichlet, Neumann and periodic, is addressed [32]. Edery developed a very interesting cut-off technique and obtained analytical formulae as sums over gamma and Riemann zeta functions plus an exact multidimensional remainder expressed as sums over modified Bessel functions.

Anyway if the spatial sections are compact, the point \( x \) and the point \( \gamma x \) where \( \gamma \in \Gamma \), \( x \equiv \gamma x \) for all the elements of the fundamental group \( \Gamma \). That is, the points \( x \) and \( \gamma x \) are the same and identical points. This means that the only possibility is that the function (5) must be periodic in some sense. Mathematically, the functions defined on a closed manifold \( M \) are called automorphic, see for example [33].

In [32], for instance, the summations are over the spectrum of the Laplace operator. It is well known that summations over the spectrum are equivalent to summation over the closed geodesics, also known as the method of images [33]. This result lies at the heart of the Selberg formalism for the calculations of functional traces and is valid at least for manifolds, Lie groups included [34], see also [16].

Wald’s axioms for the expectation value of the energy–momentum tensor are satisfied [35]. Albeit this does not prove that the chosen state is the correct one, it does indeed indicate a criterion for a candidate physical state.

The Casimir energy density is independent of \( \xi \) and is given by the 00 component of (3) which is the projection to static observers \( \langle 0 | T_{\mu\nu}(x) | 0 \rangle u^\mu u^\nu \) with 4-velocity \( u = (1, 0, 0, 0) \)

\[
\rho_{M}^{c} = -\frac{1}{\pi^2} \lim_{\gamma \to 1} \sum_{\gamma \in \Gamma} \frac{1}{(\vec{r} - \vec{r}')^4},
\] (6)

where \( \gamma \) means that the direct path with \( \gamma = 1 \) is to be excluded. It can be shown that the exclusion of the direct path corresponds to a renormalization of the cosmological constant [20].

3. The closed oriented Euclidean spaces

Different from the hyperbolic or spherical spaces, only a finite number of topologies can be modelled in the Euclidean space. A representation for an arbitrary element for the rigid motion subgroup of the full isometry group can be given

\[
\gamma = \begin{pmatrix}
R & a_1 \\
\vec{a}_2 & a_2 \\
0 & a_3 \\
0 & 0 & 0 & 1
\end{pmatrix},
\] (7)

where \( R \in SO(3) \) is an orthogonal rotation matrix and \( \vec{a} \) is the translation. The action of the group on any point \( x = \{x_1, x_2, x_3\} \) of \( E^3 \) is linear and multiplicative.
The discrete subgroups of the isometry group define the manifolds modelled in the space of interest. A group \( \Gamma \) is said to be torsion-free if for each element \( \gamma \in \Gamma \), \( \gamma^n \neq 1 \) for any integer \( n \). In the Euclidean case, the torsion-free elements must be ‘screw motions’, i.e. a combined rotation and translation along the rotation axis. The holonomy group is the rotational part of \( \Gamma \), \( \Psi = \Gamma / T \), where \( T \) is the translational part of the group. Two oriented manifolds are diffeomorphic if and only if the discrete torsion-free groups have isomorphic holonomy groups, \( \Psi \). Of course, \( \Psi \subset SO(3) \) is a finite group. According to lemma 3.5.4 of Wolf’s book [23], for the \( E^3 \) case, the only possibilities for \( \Psi \) are \( \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6, \mathbb{Z}_2 \otimes \mathbb{Z}_2 \), which give rise to the six closed oriented manifolds [23]. These manifolds and the Casimir energy are given in the subsections 3.1–3.6.

The very important theorem 8.2.5 of Ratcliffe’s book [24] is reproduced in the following:

Every compact, \( n \)-dimensional, Euclidean space form is finitely covered by a Euclidean \( n \)-torus.

This theorem is in fact Bieberbach’s first theorem, which states that \( T \) is a normal subgroup of finite index in \( \Gamma \). This finite index is given by the order of \( \Psi \) and corresponds to the number of coverings of the space in question by the Euclidean \( n \)-torus. The above theorem is used to rewrite the summations in the expectation values over all the elements of the group as a finite sum of Epstein type zeta functions. The Epstein zeta function is already well known to give the appropriate result for the Casimir energy for the \( n \)-torus, see for example [30] [16]. In the original Epstein’s notation let \( p \) be a positive integer and

\[
\tilde{g} = (g_1, \ldots, g_p), \quad g_i \in \mathbb{R}, \quad \tilde{g} \in \mathbb{R}^p \\
\tilde{h} = (h_1, \ldots, h_p), \quad h_i \in \mathbb{R}, \quad \tilde{h} \in \mathbb{R}^p \\
\tilde{m} = (m_1, \ldots, m_p), \quad m_i \in \mathbb{Z}, \quad \tilde{m} \in \mathbb{Z}^p.
\]

The scalar product is defined as

\[
(g, h) = \sum_{i=1}^{p} g_i h_i,
\]

and \( c_{\mu\nu} \) an invertible \( p \times p \) matrix associated with a quadratic form as follows:

\[
\varphi(x) = \sum_{\mu, \nu=1}^{p} c_{\mu\nu} x_\mu x_\nu.
\]

With the above definitions Epstein’s zeta function is written as

\[
Z \left| \begin{array}{cc}
g_1 & \cdots & g_p \\
h_1 & \cdots & h_p \\
\end{array} \right| \left( s \right) \varphi = \sum_{m_1, \ldots, m_p = -\infty}^{\infty} \left[ \varphi(\tilde{m} + \tilde{g}) \right]^{-\frac{s}{2}} e^{2\pi i (\tilde{m}, \tilde{h})},
\]

where \( \cdot \) means that \( m_1 = 0, m_2 = 0, \ldots, m_p = 0 \) is to be excluded. Generalizations for nonhomogeneous Epstein zeta functions and reflection formulae, can be found in [36, 16].
3.1. $\Psi = \mathbb{Z}_1$

The volume of this manifold is $V = ab_2c_3$. The Dirichlet region is the parallelepiped given in figure 1. According to the representation (7), the matrices

$$
\gamma_1 = \begin{pmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_2 = \begin{pmatrix}
1 & 0 & 0 & b_1 \\
0 & 1 & 0 & b_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_3 = \begin{pmatrix}
1 & 0 & 0 & c_1 \\
0 & 1 & 0 & c_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(9)

with the respective inverses are also the generators of the translational part of the group $\Gamma = T = \{\gamma_1, \gamma_2, \gamma_3\}$. The identifications are as follows:

\[ \vec{r} = (x, y, z) \rightarrow (x + ma + nb_1 + kc_1, y + nb_2 + kc_2, z + kc_3) \]

and according to (6)

$$
\rho_{\mathbb{Z}_1} = -\frac{1}{\pi^2} \sum_{m,n,k=-\infty}^{+\infty} \left\{ \left[ (ma + nb_1 + kc_1)^2 + (nb_2 + kc_2)^2 + (kc_3)^2 \right]^{-2} \right\}
$$

(10)

where according to (8) the matrix

$$
c_{\mu\nu} = \begin{pmatrix}
a^2 & ab_1 & ac_1 \\
ab_1 & b_1^2 + b_2^2 & b_1c_1 + b_2c_2 \\
ac_1 & b_1c_1 + b_2c_2 & c_1^2 + c_2^2 + c_3^2
\end{pmatrix}
$$

defines the positive quadratic form $\varphi(m) = c_{11}m^2 + c_{22}n^2 + c_{33}k^2 + 2c_{12}mn + 2c_{13}mk + 2c_{23}nk$.

3.2. $\Psi = \mathbb{Z}_2$

The volume of this manifold is $V = ab_2c$ (figure 2). Now the matrices and their inverses

$$
\gamma_1 = \begin{pmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_2 = \begin{pmatrix}
1 & 0 & 0 & b_1 \\
0 & 1 & 0 & b_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_3 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(11)

generate the group $\Gamma$. 
The generators of the translational part \( T \), now, are given by \( T = \Gamma/\mathbb{Z}_2 = \{ \gamma_1, \gamma_2, \gamma_3^2 \} \), together with the inverses. The fundamental domain, \( FD \), for \( \Gamma/\mathbb{Z}_2 \) has volume \( V = 2ab_2c \) and is given by two copies of the Dirichlet region of figure 2; \( FD = \{ DR, \gamma_3 DR \} \).

The identifications in this manifold are provided by

\[
\vec{r} = (x, y, z) \rightarrow (x + ma + nb_1, y + nb_2, z + 2k) \rightarrow (-x + ma + nb_1, -y + nb_2, z + (2k + 1)c).
\]

Using (6), the Casimir energy is given by

\[
\rho^\Sigma_{\mathbb{Z}_2} = -\frac{1}{\pi^2} \sum_{m,n,k=\infty}^{\infty} \left[ (ma + nb_1)^2 + (nb_2)^2 + (2k)^2 \right]^{-2}
+ \left[ (-2x + ma + nb_1)^2 + (-2y + nb_2)^2 + ((2k + 1)c)^2 \right]^{-2}
\]

\[
\rho^\Sigma_{\mathbb{Z}_2} = -\frac{1}{\pi^2} \left( \begin{array}{ccc}
Z & 0 & 0 \\
g_x & g_y & g_z \\
(4\phi) + Z & 0 & 0 \\
0 & 0 & (4\phi)
\end{array} \right),
\]

where

\[
\vec{g} = \left( \frac{2x}{a}, \frac{2yb_1}{ab_2}, \frac{2y}{b_2}, 1/2 \right)
\]

\[
c_{\mu\nu} = \begin{pmatrix}
a^2 & ab_1 & 0 \\
ab_1 & b_1^2 + b_2^2 & 0 \\
0 & 0 & 4c^2
\end{pmatrix}
\]

The values of \( a = 1.2, b_1 = 0.7, b_2 = 1.4 \) and \( c = 1.3 \) are chosen. The truncation in (8) is done such that the summation is over a geodesic sphere of radius \( R \sim 100 \) of integer numbers. The corresponding truncation error on the energy density should be less than 1%. The result is shown in figure 3, and is independent of the coordinate \( z \).
3.3. $\Psi = \mathbb{Z}_4$

The volume of this manifold is $V = a^2 b$ (figure 4). The matrices

$$
\gamma_1 = \begin{pmatrix}
1 & 0 & 0 & a \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & a \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\gamma_3 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & b \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(15)

together with their inverses are the generators of $\Gamma$. The translational part is generated by $T = \Gamma/\mathbb{Z}_4 = \{\gamma_1, \gamma_2, \gamma_3\}$, together with the inverses. The fundamental region, $FR$ for $\Gamma/\mathbb{Z}_4$ has volume $V = 4a^2 b$ and is given by four copies of the DR in figure 4: $FR = \{DR, \gamma_1 DR, \gamma_2^2 DR, \gamma_3 DR\}$. The identifications and the direct summation are given in the A.1. The result for the Casimir energy according to (6) and (8) is the following:

$$
\rho_{\mathbb{Z}_4} = -\frac{1}{\pi^2} \left( Z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{(4)_{\psi}} + Z \begin{pmatrix} 0 & 0 & 0 & 0 \\ s_z^4 & s_y^4 & s_z^4 \end{pmatrix}_{(4)_{\psi}} + Z \begin{pmatrix} 0 & 0 & 0 & 0 \\ s_x^4 & s_y^4 & s_z^4 \end{pmatrix}_{(4)_{\psi}} + Z \begin{pmatrix} 0 & 0 & 0 & 0 \\ s_x^4 & s_y^4 & s_z^4 \end{pmatrix}_{(4)_{\psi}} \right),
$$

(16)
Figure 5. The Casimir energy density in natural units, given in (16) inside the DR in figure 4, with the coordinate $z = 0.6$.

where

\[
\begin{align*}
\vec{g}_1 &= \left(\frac{x + y}{a}, \frac{x - y}{a}, \frac{1}{4}\right) \\
\vec{g}_2 &= \left(-\frac{2x}{a}, -\frac{2y}{a}, \frac{2}{4}\right) \\
\vec{g}_3 &= \left(\frac{y - x}{a}, -\frac{x + y}{a}, \frac{3}{4}\right) \\
c_{\mu\nu} &= \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 16b^2 \end{pmatrix}.
\end{align*}
\]

The values of $a = 1.2, b = 1.4$ are chosen. The truncation in (8) is done such that the summation is over a geodesic sphere of radius $R \sim 100$ of integer numbers. The corresponding truncation error on the energy density should be less than 1%. The result is shown in figure 5, and is independent of the coordinate $z$.

3.4. $\Psi = \mathbb{Z}_3$

The volume of this manifold is $V = \sqrt{3}a^2b$. The Dirichlet domain is a hexagonal prism of side $a/\sqrt{3}$ (figure 6). The face pairing matrices

\[
\begin{align*}
\gamma_1 &= \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} 1 & 0 & 0 & -a/2 \\ 0 & 1 & 0 & \sqrt{3}a/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 1 & 0 & 0 & a/2 \\ 0 & 1 & 0 & \sqrt{3}a/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \gamma_4 &= \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{align*}
\]

(18)

together with the inverses generate the group $\Gamma$.

The generators of the translational part are given by $T = \Gamma/\mathbb{Z}_3 = \{\gamma_1, \gamma_2, \gamma_3\}$, with the inverses. The fundamental region, $FR$ for $\Gamma/\mathbb{Z}_3$ has volume $V = 3\sqrt{3}a^2b$ and is given by three copies of the DR in figure 6; $FR = \{DR, \gamma_4.DR, \gamma_3.DR\}$. The identifications are
Figure 6. Dirichlet region. The top and bottom faces in the \( z \) direction are identified with a rotation of \( 2\pi/3 \).

Figure 7. The Casimir energy density in natural units, given in (19) inside the DR in figure 6 with the coordinate \( z = 0.6 \). The region is a disc of radius \( r = 0.6 \), \( \theta \) is the angle between the \( x \) and \( y \) coordinates: \( \theta = 0 \) corresponds to the \( x \) axis of figure 6.

given in A.2. The Casimir energy, according to (6) and (8) is

\[
\rho_{\Sigma z}^c = -\frac{1}{\pi^2} \left( Z | 0 0 0 \rangle \langle 0 0 0 | (4)_\rho + Z | g^1_z g^1_y g^1_c \rangle \langle 4)_\rho + Z | g^2_z g^2_y g^2_c \rangle \langle 4)_\rho \right),
\]

(19)

where

\[
\bar{g}^1 = \left( \frac{-2x}{a}, \frac{x - y\sqrt{3}}{a}, 1/3 \right)
\]

\[
\bar{g}^2 = \left( \frac{x - y\sqrt{3}}{a}, \frac{x + y\sqrt{3}}{a}, 2/3 \right)
\]

(20)

The values of \( a = 1.2, b = 1.4 \) are chosen. The truncation in (8) is done such that the summation is over a geodesic sphere of radius \( R \sim 100 \) of integer numbers. The corresponding truncation error on the energy density should be less than 1%. The result is shown in figure 7, and is independent of the \( z \) coordinate.
3.5. \( \Psi = \mathbb{Z}_6 \)

The Dirichlet region is a prism with hexagonal base of side \( a/\sqrt{3} \), figure 8. The volume of this manifold is \( V = \sqrt{3}a^2b \). The face pairing matrices are

\[
\begin{align*}
\gamma_1 &= \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\gamma_2 &= \begin{pmatrix} 1 & 0 & 0 & -a/2 \\ 0 & 1 & 0 & \sqrt{3}a/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\gamma_3 &= \begin{pmatrix} 1 & 0 & 0 & a/2 \\ 0 & 1 & 0 & \sqrt{3}a/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\gamma_4 &= \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{align*}
\]

which together with their inverses generate \( \Gamma \). The translational part is generated by \( T = \Gamma/\mathbb{Z}_6 = \{ \gamma_1, \gamma_1^6 \} \), together with the inverses. The fundamental region, \( FR \) for \( \Gamma/\mathbb{Z}_6 \) has volume \( V = 6\sqrt{3}a^2b \) and is given by six copies of the DR in figure 8: \( FR = \{ DR, \gamma_4. DR, \gamma_4^2. DR, \gamma_4^3. DR, \gamma_4^4. DR, \gamma_4^5. DR \} \). The identifications are given in A.3.

With a reasoning analogous to the previous sections the following result is obtained using (6) and (8),

\[
\rho_{\varepsilon e}^C = -\frac{1}{\pi^2} \left( Z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (4)_{\phi} + Z \begin{pmatrix} g_x^1 & g_y^1 & g_z^1 \end{pmatrix} (4)_{\phi} \\ + Z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (4)_{\phi} + Z \begin{pmatrix} g_x^2 & g_y^2 & g_z^2 \end{pmatrix} (4)_{\phi} \\ + Z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (4)_{\phi} + Z \begin{pmatrix} g_x^3 & g_y^3 & g_z^3 \end{pmatrix} (4)_{\phi} \right),
\]

where

\[
\begin{align*}
\bar{g}^1 &= \begin{pmatrix} x + y/\sqrt{3} \\ a \\ x - y/\sqrt{3} \\ a \end{pmatrix}, \\
\bar{g}^2 &= \begin{pmatrix} -2x/\sqrt{3} \\ a \\ x - y/\sqrt{3} \\ a \end{pmatrix}, \\
\end{align*}
\]
The Casimir energy density in natural units, given in (22) inside the DR in figure 8, with the coordinate $z = 0.6$. The region is a disc of radius $r = 0.6$, $\theta$ is the angle between the $x$ and $y$ coordinates: $\theta = 0$ corresponds to the $x$ axis of figure 8.

\[
\vec{g}_3 = \left( \frac{-2x + 2y/\sqrt{3}}{a}, \frac{-4y/\sqrt{3}}{a}, \frac{3}{6} \right)
\]
\[
\vec{g}_4 = \left( \frac{-x + y\sqrt{3}}{a}, \frac{-x + y\sqrt{3}}{a}, \frac{4}{6} \right)
\]
\[
\vec{g}_5 = \left( \frac{2y/\sqrt{3}}{a}, \frac{-x + y/\sqrt{3}}{a}, \frac{5}{6} \right)
\]
\[
c_{\mu\nu} = \begin{pmatrix}
a^2 & a^2/2 & 0 \\
a^2/2 & a^2 & 0 \\
0 & 0 & 36b^2
\end{pmatrix}.
\]

The values of $a = 1.2, b = 1.4$ are chosen. The truncation in (8) is done such that the summation is over a geodesic sphere of radius $R \sim 100$ of integer numbers. The corresponding truncation error on the energy density should be less than 1%. The result is shown in figure 9, and is independent of the $z$ coordinate.

### 3.6. $\Psi = \mathbb{Z}_2 \otimes \mathbb{Z}_2$

The face pairing matrices are now
\[
\gamma_1 = \begin{pmatrix}
1 & 0 & 0 & a \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \gamma_2 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & b \\
0 & 0 & -1 & c \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \gamma_3 = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & -b \\
0 & 0 & -1 & -c \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which together with their inverses generate $\Gamma$, and the volume of the manifold is $V = 2abc$, figure 10. The translational part is generated by $T = \Gamma/(\mathbb{Z}_2 \otimes \mathbb{Z}_2) = \{\gamma_1^0, \gamma_2^0, \gamma_3^0\}$, with the inverses. The fundamental region, $FR$ for $\Gamma/(\mathbb{Z}_2 \otimes \mathbb{Z}_2)$ has volume $V = 8abc$ and is given by four copies of the DR in figure 10; $FR = \{DR, \gamma_1DR, \gamma_2DR, \gamma_3\gamma_3DR\}$. The identifications are given in A.4. With (6) and (8) the Casimir energy results in

\[
\rho_{\mathbb{Z}_2 \otimes \mathbb{Z}_2} = -\frac{1}{\pi^2} \left( Z \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} + Z \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} (4)_{\Psi} + Z \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} (4)_{\Psi} + Z \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} (4)_{\Psi} \right),
\]

where $Z = \frac{1}{16\pi^2}$.
Figure 10. Dirichlet region. The opposite faces in the $x$ direction are identified with a rotation of $\pi$. The adjacent faces are identified with a rotation of $\pi$, also.

Figure 11. The Casimir energy density in natural units, given in (25) inside the DR in figure 10, a cylindrical region surrounding the $x$ axis, of radius 0.762. $\phi$ is the angle.

where

\[
\vec{g}^1 = \left( \frac{1}{2}, -\frac{y}{b}, -\frac{z}{c} \right)
\]

\[
\vec{g}^2 = \left( -\frac{x}{a}, 1/2, \frac{c - 2z}{2c} \right)
\]

\[
\vec{g}^3 = \left( \frac{a - 2x}{2a}, \frac{b - 2y}{2b}, 1/2 \right)
\]

\[
c_{\mu\nu} = \begin{pmatrix} 4a^2 & 0 & 0 \\ 0 & 4b^2 & 0 \\ 0 & 0 & 4c^2 \end{pmatrix}.
\]

The values of $a = 1.2$, $b = 1.4$ and $c = 1.3$ are chosen. The truncation in (8) is done such that the summation is over a geodesic sphere of radius $R \sim 100$ of integer numbers. The corresponding truncation error on the energy density should be less than 1%. The result is
shown in figure 11. This is the only manifold for which there is not an axis such that the energy density expectation value is constant.

4. Conclusions

The Casimir energy for the oriented closed Euclidean spacetimes are obtained in this work. Closed manifolds are described by the discrete subgroups of the isometry group. The crystallographic groups are discrete subgroups of the full isometry group of Euclidean space. Among all crystallographic groups, only a few of them are of interest in connection to manifolds. These groups must be torsion-free, for the Euclidean and hyperbolic spaces. In the celebrated list of problems proposed by Hilbert, the 18th was answered affirmatively by Bieberbach in 1910–1912 [23, 24], see also [25]. Bieberbach showed that there are only a finite number of crystallographic groups for a Euclidean space of fixed dimension.

When the dimension of the Euclidean space is 3 there are only ten compact manifolds, of which six are orientable and four nonorientable.

In this present work we have investigated the six compact orientable manifolds and obtain the Casimir energy for each of them. Due to the more simple structure connected to these topological spaces compared to the hyperbolic or spherical case, an analytical result is obtained in terms of Epstein zeta functions. It is explicitly obtained that our result is independent of the type of coupling with curvature $\xi$ for these particular manifolds, which means that the calculated Casimir energy density is the same even in an inflationary de Sitter regime.

According to the theorem in section 2, it is always possible to find the normal subgroup of finite index, $T$. With this, the $n$-torus that covers the manifold a finite number of times is obtained. The Casimir energy is given by a finite sum of Epstein type zeta functions. The number of distinct zeta functions is given by the index of $T$ in $\Gamma$, which is the number of times the manifold is covered by this $n$-torus.

A numerical plot inside the Dirichlet region for each manifold is given. The manifold in section 3.1 is the only one for which the Casimir energy density is homogeneous. For the manifolds in sections 3.2–3.5 the Casimir energy density depends on the $x$ and $y$ coordinates inside the fundamental domain. For the last manifold in section 3.6, the Casimir energy density depends in all the coordinates.

The sign of Casimir energy density is negative for all manifolds. Recall that there is a subtraction of the divergence of the infinite $R^3$ covering space, which means that the energy of the manifold is less than the energy of the infinite $R^3$. The effect of choosing different representations for the fundamental group $\Gamma$ acting on the field, like twisting, can increase the energy of the vacuum [22]. De Witt finds that the sign of the energy depends on the representation of the fundamental group, the orientability of the manifold and the spin of the field. This must have a more fundamental reason connected to the Atiyah–Singer theorem, and is beyond our knowledge.

It is explicitly checked that the truncation error should be on the order of $1/R$ where $R$ is the geodesic radius of the sphere of integer numbers where the truncation of (6) was performed. $R \sim 100$ was used such that the truncation error should be in the order of 1%. The respective values of the parameters of the manifold are indicated in each case.

Acknowledgments

MPL wishes to thank CNPq for financial support. DM wishes to thank the Brazilian project Nova Física no Espaço of FAPESP.
Appendix

A.1. $\Psi = \mathbb{Z}_4$

$$\vec{r} = (x, y, z) \rightarrow (x + m\alpha, y + n\alpha, z + 4kb)$$

$$\rightarrow (-y + ma, x + na, z + (4k + 1)b)$$

$$\rightarrow (-x + ma, -y + na, z + (4k + 2)b)$$

$$\rightarrow (y + ma, -x + na, z + (4k + 3)b).$$

According to the above the Casimir energy is given by

$$\rho_{\mathbb{Z}_4} = -\frac{1}{\pi^2} \sum_{m,n,k=-\infty}^{\infty} ((ma)^2 + (na)^2 + (4kb)^2)^{-2}$$

$$+ [(-x - y + ma)^2 + (x - y + na)^2 + (4k + 1)b]^2$$

$$+ [(-2x + ma)^2 + (-2y + na)^2 + (4k + 2)b]^2$$

$$+ [(y - x + ma)^2 + (-x - y + na)^2 + (4k + 3)b]^2].$$

A.2. $\Psi = \mathbb{Z}_3$

$$\vec{r} = (x, y, z) \rightarrow [x + ma + na/2, y + na\sqrt{3}/2, z + 3kb]$$

$$\rightarrow [-x + y\sqrt{3}/2 + ma + na/2, (x\sqrt{3} - y)/2 + na\sqrt{3}/2, z + (3k + 1)b]$$

$$\rightarrow [-x - y\sqrt{3}/2 + ma + na/2, -(x\sqrt{3} + y)/2 + na\sqrt{3}/2, z + (3k + 2)b].$$

According to the above the Casimir energy is given by

$$\rho_{\mathbb{Z}_3} = -\frac{1}{\pi^2} \sum_{m,n,k=-\infty}^{\infty} ((ma + na/2)^2 + (na\sqrt{3}/2)^2 + (3kb)^2)^{-2}$$

$$+ [(-3x + y\sqrt{3}/2 + ma + na/2)^2$$

$$+ ((\sqrt{3}x - 3y)/2 + na\sqrt{3}/2)^2 + (3k + 1)b]^2$$

$$+ [(-3x - y\sqrt{3}/2 + ma + na/2)^2$$

$$+ (-\sqrt{3}x + 3y)/2 + na\sqrt{3}/2)^2 + (3k + 2)b]^2].$$

A.3. $\Psi = \mathbb{Z}_6$

$$\vec{r} = (x, y, z) \rightarrow [x + ma + na/2, y + na\sqrt{3}/2, z + 6kb]$$

$$\rightarrow [(x - y\sqrt{3}/2 + ma + na/2, (\sqrt{3}x + y)/2 + na\sqrt{3}/2, z + (6k + 1)b]$$

$$\rightarrow [-x + y\sqrt{3}/2 + ma + na/2, (x\sqrt{3} - y)/2 + na\sqrt{3}/2, z + (6k + 2)b]$$

$$\times [-x + ma + na/2, -y + na\sqrt{3}/2, z + (6k + 3)b]$$

$$\rightarrow [-x - y\sqrt{3}/2 + ma + na/2, -(x\sqrt{3} + y)/2 + na\sqrt{3}/2, z + (6k + 4)b]$$

$$\rightarrow [(x + y\sqrt{3}/2 + ma + na/2, -\sqrt{3}x - y)/2 + na\sqrt{3}/2, z + (6k + 5)b].$$

According to the above the Casimir energy is given by

$$\rho_{\mathbb{Z}_6} = -\frac{1}{\pi^2} \sum_{m,n,k=-\infty}^{\infty} ((ma + na/2)^2 + (na\sqrt{3}/2)^2 + (6kb)^2)^{-2}$$

$$+ [(-x/2 + y\sqrt{3}/2 + ma + na/2)^2$$

$$+ (-x - y/2 + na\sqrt{3}/2)^2 + (6k + 1)b]^2.$$
According to the above the Casimir energy is given by

\[ +((\sqrt{3}x - y)/2 + na\sqrt{3}/2)^2 + ((6k + 1)b)^2 \]^{-2}
\[ +[(-(3x + y\sqrt{3})/2 + ma + na/2)^2
\[ +((\sqrt{3}x - 3y)/2 + na\sqrt{3}/2)^2 + ((6k + 2)b)^2 \]^{-2}
\[ +[(-2x + ma + na/2)^2 + (-2y + na\sqrt{3}/2)^2
\[ +((6k + 3)b)^2 \]^{-2} + [(-(3x - y\sqrt{3})/2 + ma + na/2)^2
\[ +((\sqrt{3}x + 3y)/2 + na\sqrt{3}/2)^2 + ((6k + 4)b)^2 \]^{-2}
\[ +[(-x - y\sqrt{3})/2 + ma + na/2)^2
\[ +(-\sqrt{3}x + y)/2 + na\sqrt{3}/2)^2 + ((6k + 5)b)^2 \]^{-2}. \]

A.4. \( \Psi = \mathbb{Z}_2 \otimes \mathbb{Z}_2 \)

\[ \tilde{r} = (x, y, z) \rightarrow (x + 2ma, y + 2nb, z + 2kc)
\[ \rightarrow (x + a + 2ma, -y + 2nb, -z + 2kc)
\[ \rightarrow (-x + 2ma, y + b + 2nb, -z + c + 2kc)
\[ \rightarrow (-x + a + 2ma, -y + b + 2nb, z + c + 2kc). \]

According to the above the Casimir energy is given by

\[ \rho_{\mathbb{Z}_2 \otimes \mathbb{Z}_2} = -\frac{1}{\pi^2} \sum_{m,n,k=-\infty}^{\infty} \left[(2ma)^2 + (2nb)^2 + (2kc)^2\right]^{-2}
\[ +[(-a + ma)^2 + (-2y + 2nb)^2 + (-2z - 2kc)^2 \]^{-2}
\[ +[(-2x + ma)^2 + (b + 2nb)^2 + (-2z + c + 2kc)^2 \]^{-2}
\[ +[(-2x + a + 2ma)^2 + (-2y + b + 2nb)^2 + (c + 2kc)^2 \]^{-2}. \]

References

[1] Sokolov I Y 1993 Sov. Phys. JETP Lett. 57 617
[2] de Oliveira-Costa A and Smoot G F 1995 Astrophy. J. 448 577
[3] Roukema B F 2000 Class. Quantum Grav. 17 3951
[4] Lehoucq R, Lachièze-Rey M and Luminet J-P 1996 Astron. Astrophys. 313 330
[5] Lehoucq R, Luminet J-P and Uzan J-P 1999 Astron. Astrophys. 344 735
[6] Gomero G I, Teixeira A F F, Rebouças M J and Bernui A 1998 Preprint gr-qc/9811038
[7] Fagundes H V and Gausmann E 1999 Phys. Lett. A 261 235
[8] Cornish N J, Spergel D and Starkman G 1998 Class. Quantum Grav. 15 2657
[9] Gomero G I, Rebouças M J and Tavakol R 2001 Class. Quantum Grav. 18 4461
[10] Cornish N J and Spergel D N 1998 Preprint math.DG/9906017
[11] Inoue K T 1999 Class. Quantum Grav. 16 3071
[12] Lehoucq R et al 2002 Class. Quantum Grav. 19 4683
[13] Casimir H G B 1948 Proc. K. Ned. Akad. Wet. B 51 793
[14] Graham N et al 2003 Phys. Lett. B 572 196
[15] Bordag M, Mohideen U and Mostepanenko V M 2001 Phys. Rep. 353 1
[16] Elizalde E et al 1994 Zeta Regularization Techniques with Applications (Singapore: World Scientific)
[17] Graham N et al 2002 Nucl. Phys. B 645 49
[18] Elizalde E and Tort A C 2004 Mod. Phys. Lett. A 19 111
[19] Müller D, Fagundes H V and Opher R 2001 Phys. Rev. D 63 123508
[20] Müller D, Fagundes H V and Opher R 2002 Phys. Rev. D 66 083507
[21] Müller D and Fagundes H V 2002 Int. J. Mod. Phys. A 17 4385
[22] Dowker J S and Banach R 1978 J. Phys. A. Math. Gen. 11 2255
[23] Wolf J A 1984 Spaces of Constant Curvature 5th edn (Wilmington, DE: Publish or Perish)
Casimir effect in $E^3$ closed spaces

[24] Ratcliffe J G 1994 *Foundations of Hyperbolic Manifolds* (New York: Springer)

[25] Gomero G I and Rebouças M J 2003 *Phys. Lett. A* 311 319
    Mota B, Gomero G I, Rebouças M J and Tavakol R 2004 *Class. Quantum Grav.* 21 3361

[26] Schwinger J 1951 *Phys. Rev.* 82 664

[27] De Witt B S 1975 *Phys. Rep.* 19 296

[28] Christensen S M 1978 *Phys. Rev. D* 17 946

[29] Christensen S M 1976 *Phys. Rev. D* 14 2490

[30] Blau S K, Visser M and Wipf A 1991 *Int. J. Mod. Phys. A* 6 5409

[31] Grib A A, Mamayev S G and Mostepanenko V M 1994 *Vacuum Quantum Effects in Strong Fields* (St. Petersburg: Friedmann Laboratory Publishing)

[32] Edery A 2006 *J. Phys. A: Math. Gen.* 39 685
    Nesterenko V V, Lambiase G and Scarpetta G 2004 *Riv. Nuovo Cim.* 27 1

[33] Hejhal D A 1976 *Duke Math. J.* 43 441
    Venkov A B 1979 *Russ. Math. Surv.* 34 79
    Banach R and Dowker J S 1979 *J. Phys. A: Math. Gen.* 12 2545
    Banach R and Dowker J S 1979 *J. Phys. A: Math. Gen.* 12 2527
    Balazs N L and Voros A 1986 *Phys. Rep.* 143 109

[34] Camporesi R 1990 *Phys. Rep.* 196 1
    Dowker J S 1971 *Ann. Phys.* 62 361

[35] Wald R M 1977 *Commun. Math. Phys.* 54
    Wald R M 1978 *Phys. Rev. D* 17 1477

[36] Elizalde E 1998 *Commun. Math Phys.* 198 83