THE TOPOLOGY ON BERKOVICH AFFINE LINES OVER COMPLETE VALUATION RINGS

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Abstract. In this article, we give a full description of the topology of the one dimensional affine analytic space \( A_1^R \) over a complete valuation ring \( R \) (i.e., a valuation ring with “real valued valuation” which is complete under the induced metric), when its field of fractions \( K \) is algebraically closed. In particular, we show that \( A_1^R \) is both connected and locally path connected. Furthermore, \( A_1^R \) is the completion of \( K \times (1, \infty) \) under a canonical uniform structure. As an application, we describe the Berkovich spectrum \( \mathcal{M}(\mathbb{Z}_p[G]) \) of the Banach group ring \( \mathbb{Z}_p[G] \) of a cyclic \( p \)-group \( G \) over the ring \( \mathbb{Z}_p \) of \( p \)-adic integers.

1. INTRODUCTION AND NOTATION

Let \( S \) be a commutative unital Banach ring with its norm being denoted by \( \| \cdot \| \). The Berkovich spectrum \( \mathcal{M}(S) \) as well as the \( n \)-dimensional affine analytic space \( A^n_S \) over \( S \) was introduced by Vladimir Berkovich (see, e.g., [3]) and was further studied by Jérôme Poineau in [8]. More precisely, \( \mathcal{M}(S) \) is the set of all non-zero multiplicative seminorms on \( S \), while \( A^n_S \) is the set of all non-zero multiplicative seminorms on the \( n \)-variables polynomial ring \( S[t_1, \ldots, t_n] \) whose restrictions on \( S \) are contractive. The topologies on both \( \mathcal{M}(S) \) and \( A^n_S \) are the ones given by pointwise convergence. The topology on \( A^n_S \) is also induced by the Berkovich uniform structure which is given by a fundamental system of entourages consisting of sets of the form

\[ E^X_\epsilon := \{(\mu, \nu) \in A^n_S \times A^n_S : ||p|\mu - |p|\nu| < \epsilon, \text{ for any } p \in X\}, \]

where \( \epsilon \) runs through all strictly positive real numbers and \( X \) runs through all non-empty finite subsets of \( S[t_1, \ldots, t_n] \). It is not hard to see that \( A^n_S \) is complete under this uniform structure.

In the case of a non-Archimedean field \( L \), the properties of \( A^n_L \) play an important role in the study of non-Archimedean geometry. For example, the Bruhat-Tits tree of \( SL_2(\mathbb{Q}_p) \) can be realized as a subspace of the Berkovich projective line, which is a glue of two copies of \( A^n_L \) (see [10]).

When \( L \) is an algebraically closed non-Archimedean complete valued field, Berkovich gave in [3] a full description of the space \( A_1^L \). This may then be used to describe the one-dimensional affine analytic spaces over not necessarily algebraically closed fields.

The aim of this article is to give a full description of the topology of the space \( A_1^R \) of a “complete valuation ring” \( R \). Recall that an integral domain \( R \) is a valuation ring if for every element \( x \) in its field of fractions \( K \), either \( x \in R \) or \( x^{-1} \in R \) (see e.g., [11], p.65) or Definition 2 of [4, §VI.1]). It is easy to see that if \( R^\times \) and \( K^\times \) are the sets of invertible elements in \( R \) and \( K \) respectively, then \( K^\times / R^\times \) is a totally ordered abelian group and the canonical map \( \nu_R : K^\times \to K^\times / R^\times \) is a “valuation” such that \( R = \{ x \in K^\times : \nu_R(x) \geq 0 \} \cup \{0_R\} \).

Definition 1.1. A valuation ring \( R \) is called a complete valuation ring if \( K^\times / R^\times \) is isomorphic to an ordered subgroup of \( \mathbb{R} \) and \( R \) is complete under the induced norm.

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If \( R \) is a complete valuation ring, then \( K \) is a non-Archimedean complete valued field and \( R \) coincides with the ring of integers, \( \{ a \in K : |a| \leq 1 \} \), of \( K \). Conversely, the ring of integers of a non-Archimedean complete valued field is always a complete valuation ring.

Throughout this article, for a commutative unital Banach ring \( S \), we denote
\[
S^* := S \setminus \{ 0_S \},
\]
where \( 0_S \) is the zero element of \( S \). The identity of \( S \) will be denoted by \( 1_S \). For any \( n \in \mathbb{N} \), we define
\[
S\{ n^{-1}t \} := \left\{ \sum_{k=0}^{\infty} a_k t^k : \lim_k \|a_k\| n^k = 0 \right\}
\]
and equip it with the norm \( \| \sum_{k=0}^{\infty} a_k t^k \| := \max_k \|a_k\| n^k \). It is well-known that \( S\{ n^{-1}t \} \) is a commutative unital Banach ring. For simplicity, we will use the notation \( S(t) \) for \( S\{ 1^{-1}t \} \).

It was shown in \([3]\) that for each \( n \in \mathbb{N} \), the compact space \( M(S\{ n^{-1}t \}) \) can be identified with the subspace \( \{ \mu \in A^1_S : |t|_{\mu} \leq n \} \) of \( A^1_S \). If we set
\[
U^S_n := \{ \mu \in A^1_S : |t|_{\mu} < n \},
\]
then \( A^1_S = \bigcup_{n \in \mathbb{N}} U^S_n = \bigcup_{n \in \mathbb{N}} M(S\{ n^{-1}t \}) \) and hence \( A^1_S \) is both locally compact and \( \sigma \)-compact.

From now on, \( \bar{R} \) is a complete valuation ring and \( K \) is its field of fractions. The absolute value on \( K \) induced by \( | \cdot | \) will be denoted by \( | \cdot | \). The residue field of \( R \) is denoted by \( F \) and \( \bar{Q} : \bar{R} \rightarrow F \) is the quotient map. We set
\[
\bar{Q} : R[t] \rightarrow F[t]
\]
to be the map induced by \( Q \).

Suppose that \( s \in K \) and \( p \in K[t] \). If \( r_0, \ldots, r_s \in K \) are the unique elements with \( p = \sum_{k=0}^{s} r_k t^k \), then we put \( p_s := \sum_{k=0}^{s} r_k (t + s)^k \). For any \( \lambda \in A^1_K \), we define
\[
|p|_{\lambda+s} := \max \{ |p|_\lambda \cdot |p|_{t|t} \} \quad (p \in K[t]) \quad \text{and} \quad \lambda - s := \lambda + (-s).
\]
It is easy to see that \( \lambda + s \in A^1_K \), and that \( \lambda \mapsto \lambda + s \) is a bicontinuous bijection from \( A^1_K \) to itself.

For every \( s \in K \) and \( \tau \in \mathbb{R}_+ \), we denote the closed ball with center \( s \) and radius \( \tau \) by \( D(s, \tau) \), i.e.
\[
D(s, \tau) := \{ t \in K : |t - s| \leq \tau \},
\]
and define (as in \([2]\)) \( \zeta_{s, \tau} \in A^1_K \) by \( |p|_{\zeta_{s, \tau}} := \sup_{t \in D(s, \tau)} |p(t)| \) \( (p \in K[t]) \). Because of the maximum modulus principle, one has
\[
\sum_{k=0}^{n} a_k (t - s)^k \bigg|_{\zeta_{s, \tau}} = \max_{k=0, \ldots, n} |a_k| \tau^k \quad (n \in \mathbb{N}; a_0, \ldots, a_n \in K);
\]
here, we use the convention that \( 0^0 := 1 \).

Let us recall the following result from \([3]\).

**Theorem 1.2.** (Berkovich) Let \( L \) be an algebraically closed non-Archimedean complete valued field with a non-trivial norm \( | \cdot | \), and \( | \cdot | : L[t] \rightarrow \mathbb{R}_+ \) be a function. Then \( \lambda \in A^1_L \) if and only if there is a decreasing sequence \( \{ D(s_n, \tau_n) \}_{n \in \mathbb{N}} \) of closed balls in \( L \) such that \( |p|_{\lambda} = \inf_{n \in \mathbb{N}} |p|_{\zeta_{s_n, \tau_n}} \).

The topology on \( A^1_L \) was also described in \([3]\). Moreover, as noted in \([3]\), by using the fact that \( D(t, \tau) = D(s, \tau) \) whenever \( D(s, \tau) \subseteq D(t, \tau) \), one can show easily that \( A^1_L \) is path connected.

It is also well-known that \( A^1_L \) is locally path connected, in the sense that for every point in this topological space, there is a local neighborhood basis at that point consisting of open sets that are path connected under the induced topologies. Indeed, as noted in \([2]\), any two points in \( A^1_L \) are joined by a unique path, and hence \( A^1_L \) is a \( \mathbb{R} \)-tree. The “weak topology” induced by this \( \mathbb{R} \)-tree structure coincides with the pointwise convergence topology on \( M(L[t]) \) (see e.g. \([2]\), Proposition 1.13]). Since the canonical basic neighborhoods of the “weak topology” are path connected, we know
that $U_t^1 \subseteq M(L\{t\})$ (see (3)) is locally path connected. Furthermore, as $\zeta_{s,0} = \zeta_{0,0} + s \in U_t^1 + s$ (see (4)), the density of the image of $L$ in $A^1_L$ implies $A^1_L = \bigcup_{s \in L} U_t^1 + s$, and this gives the local path connectedness of $A^1_L$.

Observe that if the complete valuation ring $R$ is actually a field, then the absolute value $| \cdot |$ on $K$ is trivial, and the structure of $A^1_R$ is already given in [3, 1.4.4]. However, because we need a concrete presentation of this space for the general case, we will first have a closer look at this case in Proposition 2.1. As a sidetrack, we verify the fact that if two fields $k_1$ and $k_2$ are endowed with the trivial norm, then $A^1_{k_1} \cong A^1_{k_2}$ if and only if the cardinalities of the sets of monic irreducible polynomials over them are the same.

Suppose that $R$ is not a field, or equivalently, the absolute value $| \cdot |$ on $K$ is non-trivial. Let us pick an arbitrary number $\omega \in [1, \infty)$, and set $K^\omega$ to be the field $K$ equipped with the equivalent norm $| \cdot |^\omega$.

As $|a|^\omega \leq |a|$ ($a \in R$), we know that every semi-norm $\lambda \in A^1_{K^\omega}$ restricts to an element in $A^1_R$ and this gives a map

$$J_{K^\omega}^R : A^1_{K^\omega} \rightarrow A^1_R.$$

The map $J_{K^\omega}^R$ is injective, because for any $p \in K[t]$, there exists $a \in R$ with $ap \in R[t]$. On the other hand, the surjection $\bar{Q}$ as in (3) produces an injection

$$Q^\lambda : A^1_F \rightarrow A^1_R.$$

It is not hard to see that one actually has $A^1_R = Q^\lambda(A^1_F) \cup \bigcup_{\omega \in [1, \infty]} J_{K^\omega}^R(A^1_{K^\omega})$ (Proposition 2.4).

Note, however, that in the case of a general Banach integral domain $S$, elements in $A^1_S$ cannot be described in such an easy way; for example, if $S$ is the ring $\mathbb{Z}$ equipped with the trivial norm, the description of elements in $A^1_S$ requires the knowledge of all multiplicative ultrametric norms on $\mathbb{Q}$, instead of just the trivial norm on $\mathbb{Q}$ (which is the one induced from $S$).

The topology on $A^1_R$ is more difficult to describe, and we will give a full presentation of it in Theorem 2.8 in the case when $K$ is algebraically closed. Using this description, we obtain in Theorem 2.8(c) that $A^1_R$ is first countable if and only if $F$ is countable and $A^1_K$ is first countable. We will also verify, in Proposition 2.10, that $A^1_R$ is second countable if and only if $R$ is separable as a metric space (or equivalently, $K$ is a separable metric space). Moreover, the Berkovich uniform space $A^1_R$ is the completion of $K \times (1, \infty)$ under the induced uniform structure (see Remark 2.7(c)).

We also show that $A^1_R$ is both connected and locally path connected (parts (a) and (b) of Theorem 2.8). Notice that, unlike the case of $A^1_K$, any two points in a connected open subset of $A^1_R$ are joined by infinitely many paths inside that subset (see Remark 2.9). Consequently, the topology on $A^1_R$ cannot be described using the “weak topology” of a $\mathbb{R}$-tree structure.

Finally, we will apply our main result to give a description of the Berkovich spectrum of the Banach group ring $R[G]$ of a cyclic group $G$ over $R$ (Corollary 2.13). In the case when $K$ is not necessarily algebraically closed, one may obtain information about $\mathcal{M}(R[G])$ by looking at the corresponding spectrum over the completion of the algebraic closure of $K$. In particular, we will take a closer look at the case when $R = \mathbb{Z}_p$ and $G$ is a cyclic $p$-group, for a fixed prime number $p$ (Example 2.13).

2. The main results

Let us begin with a careful presentation of the content of the second line of [3, 1.4.4]. More precisely, we will give a concrete description of $A^1_F$ when $t$ is a field (not necessarily algebraically closed) equipped with the trivial norm.

In the following, $t[t]_{irr}$ is the set of all monic irreducible polynomials in $t[t]$. Consider $q, q' \in t[t]_{irr}$ as well as $\kappa, \kappa' \in \mathbb{R}_+$. We define a semi-norm $\gamma_{q, \kappa}$ on $t[t]$ by

$$\sum_{i=0}^n r_iq^i \gamma_{q, \kappa} := \max_{r_i \neq 0} \kappa'$$

(6)
(again, \(0^0 := 1\)), where \(r_0, \ldots, r_{n-1} \in \mathfrak{t}[t]\) and \(r_n \in \mathfrak{t}[t]^*\) are elements with degrees strictly less than \(\deg q\). Note that, because the absolute value on \(\mathfrak{t}\) is trivial, one has (see (9))

\[
\gamma_{t-x,\kappa} = \zeta_{x,\kappa} \quad (x \in \mathfrak{t}; \kappa \in \mathbb{R}_+). \tag{7}
\]

The semi-norm \(\gamma_{q,1}\) is independent of \(q\) and equals the trivial norm on \(\mathfrak{t}[t]\). Furthermore, when \(\gamma_{q,k} = \gamma_{q',k'}\), we have \(k = k'\), and we will also have \(q = q'\) if, in addition, \(k = k' < 1\). For \(k \in (1, \infty)\) and \(x \in \mathfrak{t}\), one has \(\gamma_{t-x,\kappa}(p) = \gamma_{t,\kappa}(p) = k^{\deg p} (p \in \mathfrak{t}[t])\).

Notice that \(\gamma_{t,\kappa} \in A^1_{t}\) for any \(\kappa \in \mathbb{R}_+\), but \(\gamma_{q,\kappa}\) is not submultiplicative when \(\deg q > 1\) and \(\kappa > 1\). Nevertheless, if \(\kappa \in [0, 1]\), then \(\gamma_{q,\kappa} \in A^1_{q}\) (regardless of the degree of \(q\)).

On the other hand, for any \(\tau \in [-1, \infty)\) and \(q \in \mathfrak{t}[t]_{\text{irr}}\), we consider \(\delta_{q,\tau}\) to be the function from \(\mathfrak{t}[t]_{\text{irr}}\) to \([-1, \infty)\) that vanishes outside the point \(q\) and sends \(q\) to \(\tau\). Observe that \(\delta_{q,0}\) is the constant zero function for all \(q \in \mathfrak{t}[t]_{\text{irr}}\).

**Proposition 2.1.** Let \(\mathfrak{t}\) be a field endowed with the trivial norm. Then \(A^1_{t}\) is canonically homeomorphic to the subspace \(X := \{\delta_{q,\tau} : q \in \mathfrak{t}[t]_{\text{irr}} \setminus \{1\}; \tau \in [-1, 0) \cup \{\delta_{t,\tau} : \tau \in [-1, \infty)\}\} of the product space \(\coprod_{t} \mathfrak{t}[t], \tau \in [-1, \infty)\). Consequently, \(A^1_{t}\) is connected and locally path connected. Moreover, \(A^1_{t}\) is first countable if and only if \(\mathfrak{t}\) is at most countable.

**Proof.** Since the second and the third statements follow easily from the first one, we will only establish the first statement (observe that \(\mathfrak{t}[t]_{\text{irr}}\) is countable if and only if \(\mathfrak{t}\) is at most countable). Let us show that

\[A^1_{t} = \{\gamma_{q,\kappa} : q \in \mathfrak{t}[t]_{\text{irr}} \setminus \{1\}; \kappa \in [0, 1]\} \cup \{\gamma_{t,\kappa} : \tau \in \mathbb{R}_+\}. \tag{8}
\]

In fact, consider any \(\lambda \in A^1_{t}\) and set \(\tau := |t|\). If \(\tau \in \mathbb{R}_+ \setminus \{1\}\), one may deduce from the Isosceles Triangle Principle that \(\lambda = \gamma_{t,\tau}\).

Suppose that \(\tau = 1\). Clearly, \(|p|_{\lambda} \leq |p|_{1,\lambda}\) (\(p \in \mathfrak{t}[t]\)) and we consider the case when \(\lambda \neq \gamma_{t,1}\). Let \(P^\lambda := \{p \in \mathfrak{t}[t] : |p|_{\lambda} < 1\}\). As \(P^\lambda\) is a non-zero prime ideal of \(\mathfrak{t}[t]\), we know that \(P^\lambda = q \cdot \mathfrak{t}[t]\) for a unique element \(q \in \mathfrak{t}[t]_{\text{irr}} \setminus \{1\}\) (note that \(|t|_{\lambda} = |t| = 1\)), and we put \(\kappa := |q|_{\lambda} \in [0, 1]\). For any \(n \in \mathbb{Z}_+\) and \(r_0, \ldots, r_n \in \mathfrak{t}[t]\) with \(r_n \neq 0\) and \(\deg r_k < \deg q\ (k = 0, \ldots, n)\), we know from the Isosceles Triangle Principle that \(\sum_{k=0}^n r_k q^k |_{\gamma_{q,\kappa}} = \sum_{k=0}^n r_k q^k |_{\gamma_{q,\kappa}}\).

Next, we define a map \(\Phi : A^1_{t} \rightarrow \mathcal{X}\) by \(\Phi(\gamma_{q,\kappa}) := \delta_{q,\kappa-1}\) (\(\gamma_{q,\kappa} \in A^1_{q}\)). Clearly, \(\Phi\) is bijective, and is continuous on the two subsets \(\{\gamma_{q,\kappa} : q \in \mathfrak{t}[t]_{\text{irr}}; \kappa \in [0, 1]\}\) and \(\{\gamma_{t,\kappa} : \tau \in [1, \infty)\}\). As \(\Phi\) restricts to a homeomorphism on the compact subset \(\mathcal{X}(\mathfrak{t}[t])\) of \(A^1_{t}\), it is not hard to verify that \(\Phi\) is actually bicontinuous.

If \(\mathfrak{t}\) is algebraically closed, then we have, by Equalities (7) and (8),

\[A^1_{t} = \{\gamma_{s,\kappa} : s \in \mathfrak{t}^*; \kappa \in [0, 1]\} \cup \{\delta_{t,\tau} : \tau \in \mathbb{R}_+\}. \tag{9}
\]

Now, consider \(t_1\) and \(t_2\) to be two (not necessarily algebraically closed) fields equipped with the trivial norms. If the cardinalities of \(\mathfrak{t}_1[t]_{\text{irr}}\) and \(\mathfrak{t}_2[t]_{\text{irr}}\) are the same, then Proposition 2.1 implies that \(A^1_{t_1}\) is homeomorphic to \(A^1_{t_2}\). Conversely, suppose that we have a homeomorphism \(\Psi : A^1_{t_1} \rightarrow A^1_{t_2}\). For any \(q_1 \in \mathfrak{t}_1[t]_{\text{irr}}\), it is easy to see that \(\Psi\) will send \(\gamma_{q_1,0}\) to \(\gamma_{q_2,0}\) for some \(q_2 \in \mathfrak{t}_2[t]_{\text{irr}}\) (and vice-versa), because elements of the form \(\gamma_{q,0}\) are all the free end points of maximal line-segments of \(A^1_{t_i}\ (i = 1, 2)\). Hence, we have a bijection between \(t_1[t]_{\text{irr}}\) and \(t_2[t]_{\text{irr}}\). These produce part (a) of the following corollary. The other corollaries follow from part (a) and some well-known facts.

**Corollary 2.2.** Suppose that \(t_1\) and \(t_2\) are two fields equipped with the trivial norm.

(a) \(A^1_{t_1}\) and \(A^1_{t_2}\) are homeomorphic if and only if the cardinality of \(t_1[t]_{\text{irr}}\) equals that of \(t_2[t]_{\text{irr}}\).
Proof. We have already seen that $\Lambda$ is actually a homeomorphism. We will also describe how the subspaces $Q^0(\mathbb{A}^1_K)$ and $\bigcup_{\omega \in [1,\infty)} J^0_\omega(\mathbb{A}^1_K)$ sit together in $\mathbb{A}^1_K$.

Later on, we will verify that $\Lambda$ is actually a homeomorphism. We will also describe how the subspaces $Q^0(\mathbb{A}^1_K)$ and $\bigcup_{\omega \in [1,\infty)} J^0_\omega(\mathbb{A}^1_K)$ sit together in $\mathbb{A}^1_K$.
From now on, we will identify \( \mathbb{A}^1_K \) as subspaces of \( \mathbb{A}^1_R \), and may sometimes ignore the map \( Q^k \) if no confusion arises.

**Lemma 2.5.** Suppose that \( F \) is algebraically closed. Consider a net \( \{ (\lambda_i, \omega_i) \}_{i \in I} \) in \( \mathbb{A}^1_K \times [1, \infty) \).

(a) If \( \{ \lambda_i^{\omega_i} \}_{i \in I} \) converges to an element in \( \mathbb{A}^1_F \), then \( \lim_i \omega_i = \infty \) and \( \limsup_{i \in I} |t|_{\lambda_i} \leq 1 \).

(b) If \( \lambda_i \in U_F^{\tau} \) for all \( i \in I \) (see (2)), \( \lim_i \omega_i = \infty \) and \( \lim_i |t|_{\lambda_i}^{\omega_i} = \tau \in [0, 1] \), then \( \lambda_i^{\omega_i} \to \zeta_{0_F, \tau} \).

**Proof.** (a) By (9) and the assumption, one can find \( x, \tau \) in \( F^* \times [0, 1) \cup \{ 0_F \} \times R_+ \) such that \( \lambda_i^{\omega_i} \to \zeta_{x, \tau} \). Assume on the contrary that \( \{ \omega_i \}_{i \in I} \) has a bounded subnet. Then there is a subnet \( \{ \omega_{i_j} \}_{j \in J} \) of \( \{ \omega_i \}_{i \in I} \) converging to a number \( \omega_0 \in [1, \infty) \). For any \( a \in F \) with \( |a| < 1 \), one gets from

\[
|a|^{\omega_{i_j}} = |a|^{\omega_{i_j}} \to |Q(a)|_{\zeta_{x, \tau}} = 0,
\]

that \( |a|^{\omega_0} = 0 \), and this contradicts the non-trivial assumption of \( | \cdot | \). Hence, \( \omega_i \to \infty \). On the other hand, since \( \{ |t|_{\lambda_i}^{\omega_i} \}_{i \in I} \) converges to \( |Q(t)|_{\zeta_{x, \tau}} \), one concludes that \( \limsup_{i \in I} |t|_{\lambda_i} \leq 1 \), because otherwise, there exist \( r > 1 \) and a subnet \( \{ \lambda_{i_j} \}_{j \in J} \) with \( |t|_{\lambda_{i_j}} \geq r \), which produces the contradiction that \( |t|_{\lambda_{i_j}} \to r \to \infty \).

(b) If \( b \in R \) with \( |b| = 1 \), then \( |t - b|_{\lambda_i}^{\omega_i} = 1 \) (because \( |t|_{\lambda_i} \leq 1 \)) for all \( i \in I \). On the other hand, consider a polynomial \( q \in \ker \hat{Q} \). There exist \( a_0, \ldots, a_n \in \ker Q \) with \( q = \sum a_k t^k \). Hence,

\[
|q|_{\lambda_i} = \max_{k=0, \ldots, n} |a_k||t|_{\lambda_i}^k \leq \max_{k=0, \ldots, n} |a_k| < 1,
\]

and one has \( |q|_{\lambda_i} \leq \max_{k=0, \ldots, n} |a_k||t|_{\lambda_i}^k \to 0 \) (along \( i \)).

Now, let \( p \in R[t]^{(n)} \) and \( \{ x_1, \ldots, x_n \} \) be all the non-zero roots of \( \hat{Q}(p) \) in \( F \) (counting multiplicity). Pick \( b_1, \ldots, b_m \in R \) with \( Q(b_l) = x_l \) \( (l = 1, \ldots, m) \). Then \( |b_l| = 1 \) for all \( l \in \{ 1, \ldots, m \} \), and one can find \( k \in \mathbb{Z}_+ \) as well as \( q_0 \in \ker Q \) satisfying

\[
p = t^k \cdot (t - b_1) \cdots (t - b_m) + q_0.
\]

The above and the hypothesis will then tell us that \( |t^k \cdot (t - b_1) \cdots (t - b_m)|_{\lambda_i}^{\omega_i} \to \tau^k \) and \( |q_0|_{\lambda_i}^{\omega_i} \to 0 \).

Consequently, \( |p|_{\lambda_i}^{\omega_i} \to \tau^k = |\hat{Q}(p)|_{\zeta_{0_F, \tau}} \), as is required (observe that \( 0 \leq \tau \leq 1 \)).

The following is our first main theorem that gives a full description of the topological space \( \mathbb{A}^1_R \).

**Theorem 2.6.** Let \( R \) be a complete valuation ring which is not a field, \( F \) be the residue field of \( R \), and \( K \) be the field of fractions of \( R \) equipped with the induced absolute value \( | \cdot | \). Suppose that \( K \) is algebraically closed. We fix a cross section \( \hat{F} \subseteq R \) of \( Q : R \to F \) that contains \( 0_R \).

(a) \( \mathbb{A}^1_F \) is closed in \( \mathbb{A}^1_R \).

(b) The map \( \Lambda : \mathbb{A}^1_K \times [1, \infty) \to \bigcup_{\omega \in [1, \infty)} J^\omega_K(\mathbb{A}^1_K) \) in (10) is a homeomorphism.

(c) Suppose that \( \{ (\lambda_i, \omega_i) \}_{i \in I} \) is a net in \( \mathbb{A}^1_K \times [1, \infty) \). Then \( \{ \lambda_i^{\omega_i} \}_{i \in I} \) converges to an element \( \lambda_0 \in \mathbb{A}^1_F \) if and only if \( \omega_i \to \infty \) and either one of the following holds:

- \( C1) \), there exist \( \tau_1 \in [0, 1) \) and \( b \in R \) such that \( |t - b|_{\lambda_i}^{\omega_i} \to \tau_1 \) (in this case, \( \lambda_0 = \zeta_{Q(b), \tau_1} \));
- \( C2) \), one can find \( \tau_2 \in (1, \infty) \) such that \( |t|_{\lambda_i}^{\omega_i} \to \tau_2 \) (in this case, \( \lambda_0 = \zeta_{0_F, \tau_2} \));
- \( C3) \), \( |t - c|_{\lambda_i}^{\omega_i} \to 1 \) for any \( c \in \hat{F} \) (in this case, \( \lambda_0 = \zeta_{0_F, 1} \)).

(d) Under the homeomorphism in part (b), one may regard \( \mathbb{A}^1_K \times [1, \infty) \) as an open dense subset of \( \mathbb{A}^1_R \). Consequently, the image of \( K \times (1, \infty) \) is dense in \( \mathbb{A}^1_R \).

**Proof.** The non-triviality of \( | \cdot | \) produces \( a_0 \in R \) with \( |a_0| \in (0, 1) \). Moreover, since \( K \) is algebraically closed, so is \( F \). As in (9), one has

\[
\mathbb{A}^1_F = \{ \zeta_{x, \tau} : (x, \tau) \in F \times [0, 1) \cup \{ 0_F \} \times [1, \infty) \}.
\]
(a) Suppose on the contrary that there is a net \( \{ (x_i, \tau_i) \}_{i \in \mathbb{N}} \) in \( F \times [0, 1) \cup \{ \infty \} \times [1, \infty) \) with \( \zeta_{x_i, \tau_i} \rightarrow \lambda^\omega \) for some \( (\lambda, \omega) \in K^1 \times [1, \infty) \). Then, \( 0 = |Q(a_0)|_{\zeta_{x_i, \tau_i}} \rightarrow |a_0|\omega \), which is absurd.

(b) As said in the paragraph preceding Lemma 2.5, the map \( \Lambda \) is a continuous bijection. If \( \{ (\lambda_i, \omega_i) \}_{i \in \mathbb{N}} \) is a net in \( K^1 \times [1, \infty) \) satisfying \( \lambda_i^\omega \rightarrow \lambda_0^\omega \) for some \( (\lambda_0, \omega_0) \in K^1 \times [1, \infty) \), then

\[
|a_0|^{\omega_i/\omega_0} = |a_0|^{1/\omega_i \rightarrow 1/\omega_0}
\]

and so \( \omega_i \rightarrow \omega_0 \). From this, one can also deduce that \( |p|_{\lambda_i} \rightarrow |p|_{\lambda_0} \) for every \( p \in R[t] \). Consequently, the inverse of \( \Lambda \) is continuous.

(c) \( \Rightarrow \). Suppose that \( \{ \lambda_i^\omega \}_{i \in \mathbb{N}} \) converges to \( \zeta_{Q(b), \tau} \) for some \( b \in R \) and \( \tau_1 \in [0, 1) \). Then we have \( \omega_i \rightarrow \infty \) (by Lemma 2.5(a)) and \( |t - b|_{\lambda_i} \rightarrow |t - Q(b)|_{\zeta_{Q(b), \tau} = \tau_1} \). This verifies Condition (C1).

On the other hand, suppose that \( \{ \lambda_i^\omega \}_{i \in \mathbb{N}} \) converges to \( \zeta_{0, \kappa} \) for some \( \kappa \in [1, \infty) \). Again, one has \( \omega_i \rightarrow \infty \) because of Lemma 2.5(a). Moreover, as \( \kappa \geq 1 \), one knows that

\[
|t - c|_{\lambda_i} \rightarrow |t - Q(0)|_{\zeta_{0, \kappa}} \Rightarrow \kappa \quad (c \in R).
\]

Hence, either Conditions (C2) or (C3) holds (depending on whether \( \kappa = 1 \)).

\( \Leftarrow \). Suppose that \( \omega_i \rightarrow \infty \) and Condition (C1) holds. Then \( |t|_{\lambda_i} \rightarrow \infty \) and \( |t - b|_{\lambda_i} \rightarrow \tau_1 < 1 \) (see \( (2) \)), which implies that \( |t|_{\lambda_i - b} \rightarrow \tau_1 \). By Lemma 2.5(b), we know that \( \lambda_i - b \rightarrow \zeta_{0, \kappa} \) and hence \( \lambda_i^\omega \rightarrow \zeta_{Q(b), \tau} \).

Secondly, suppose that \( \omega_i \rightarrow \infty \) and Condition (C2) holds. We first note that

\[
\lim \sup_{i \in \mathbb{N}} |t|_{\lambda_i} \leq 1,
\]

because otherwise, one can find a subnet such that \( |t|_{\lambda_i} \rightarrow \infty \). Let us also show that

\[
|t - c|_{\lambda_i} \rightarrow \tau_2 \quad (c' \in R).
\]

In fact, as \( \tau_2 > 1 \), when \( i \) is large, one has \( |t|_{\lambda_i} > 1 \), and hence \( |t - c'|_{\lambda_i} = |t|_{\lambda_i} (c' \in R) \) and Relation (12) follows.

Now, for any \( q \in \ker \tilde{Q} \), we let \( a_0, \ldots, a_n \) \( \in \ker Q \) be the elements with \( q = \sum_{k=0}^n a_k t^k \). As \( k := \max\{|a_0|, \ldots, |a_n|\} < 1 \), we obtain from (11) an element \( i_0 \in J \) satisfying \( \sup_{i \geq i_0} |t|_{\lambda_i} < 1/\kappa^{1/\tau_2} \)
and hence for \( i \geq i_0 \),

\[
|q|_{\lambda_i} \leq \max_{k=0, \ldots, n} |a_k||t|_{\lambda_i}^k < \kappa^{-1/\tau_2}.
\]

This means that \( |q|_{\lambda_i} \rightarrow 0 \). Consider now a polynomial \( p \in R[t] \) of degree \( m \). Let \( \{ y_1, \ldots, y_m \} \) be all the roots of \( \tilde{Q}(p) \) in \( F \) (counting multiplicity) and let \( c_1, \ldots, c_m \) be elements in \( R \) satisfying \( Q(c_l) = y_l \) \((l = 1, \ldots, m)\). Then one can find \( q_0 \in \ker \tilde{Q} \) with

\[
p = (t - c_1) \cdots (t - c_m) + q_0.
\]

Thus, (12) and the above imply that \( |p|_{\lambda_i} \rightarrow \tau_2^{n} = \zeta_{0, \tau_2}(\tilde{Q}(p)) \) (notice that \( \tau_2 \geq 1 \)).

Finally, consider the case when \( \omega_i \rightarrow \infty \) and Condition (C3) holds. As \( 0 \in \bar{F} \), we know that \( |t|_{\lambda_i} \rightarrow 1 \) and Relation (11) is satisfied. Moreover, we have \( |t - c'|_{\lambda_i} \rightarrow 1 \) \((c' \in R) \). Indeed, if \( c' \in R \), one can find \( c \in \bar{F} \) satisfying \( |c' - c| < 1 \). As \( \omega_i \rightarrow \infty \), there exists \( i_0 \in J \) such that for any \( i \geq i_0 \), one has both \( |c' - c|^\omega_i < 1/2 \) as well as \( |t - c|_{\lambda_i}^\omega_i < 1/2 \), and hence

\[
|t - c'|_{\lambda_i} = |t - c|_{\lambda_i} \quad (i \geq i_0).
\]

This implies that \( |t - c'|_{\lambda_i} \rightarrow 1 \) as required. Now, it follows from the same argument as that for Condition (C2) that \( |p|_{\lambda_i} \rightarrow 1 = \zeta_{0, 1}(\tilde{Q}(p)) \), for any \( p \in R[t] \).

(d) If \( \tau_1 < 1 \), then it follows easily from part (c) that \( \zeta_{0, \tau_i/n} \rightarrow \zeta_{Q(b), \tau_i} \) for every \( b \in R \). If \( \tau_2 > 1 \), it is not hard to see from part (c) that \( \zeta_{0, \tau_i}^{n} \rightarrow \zeta_{0, \tau_2} \). Furthermore, since \( \zeta_{0, 1}(t - c) = 1 \) for all
c ∈ R, we know from part (c) that ζ_{0,1} → ζ_{0,p,1}. These establish the first statement. The second statement follows from the first one, part (b) as well as the fact that K is dense in Λ_K. □

From now on, we will also identify Λ_K × [1, ∞) with \( \bigcup_{\omega \in [1,\infty)} J^K_\omega(\Lambda_K) \) (i.e. \( (\lambda, \omega) \)).

Remark 2.7. (a) In the proof of Theorem 2.5(c), we see that \( \lambda'^{\omega_i} \rightarrow \zeta_{0,p,\kappa} \) for some \( \kappa \in [1, \infty) \) if and only if \( \omega_i \rightarrow \infty \) and \( |t - c'|^{\omega_i} \rightarrow \kappa \), for all \( c' \in R \).

(b) Unlike the case of \( \tau_2 \in (1, \infty) \), the requirement \( |t|^{\omega_1} \rightarrow 1 \) does not imply \( \lambda'^{\omega_i} \rightarrow \zeta_{0,p,1} \). For example, if we set \( \lambda_i := \zeta_{1,0} \) (i ∈ I, then \( |t|^{\omega_i} = 1 \) but \( |t - 1|^{\omega_i} = 0 \) for all \( i \in J \).

(c) Consider the uniform structure on \( K \times (1, \infty) \) that is induced by the fundamental system of entourages of the form:

\[
\{(s_1, \omega_1), (s_2, \omega_2) : |p(s_1)|^{\omega_1} - |p(s_2)|^{\omega_2} < \epsilon, \text{ for each } p \in X\},
\]

where \( \epsilon \) runs through all positive numbers and \( X \) runs through all non-empty finite subsets of \( R[t] \). As a uniform space, \( K \) is the completion of \( K \times (1, \infty) \) under this structure (because of Theorem 2.6(d)).

Before continuing our discussion, let us set some more notations. For any \( p \in R[t] \) as well as \( \tau, \epsilon \in R_+ \), we denote

\[
U_{t,\epsilon} := \{ \mu \in \Lambda_K : \tau - \epsilon < \mu(p) < \tau + \epsilon \}.
\]

It follows from the definition that \( U_{t,\epsilon} \) an open subset of \( \Lambda_K \).

Proposition 2.4 as well as parts (a) and (b) of Theorem 2.5 tell us that \( \Lambda_K \) contains the product space \( \Lambda_K \times [1, \infty) \) as an open subset with its complement being \( \Lambda_K \). Moreover, by Lemma 2.5(a), we know that for every \( \kappa \geq 1 \), the subset \( \Lambda_K \times [1, \kappa] \) is closed in \( \Lambda_K \).

Since the topology on the product space \( \Lambda_K \times [1, \infty) \) is well-known and Proposition 2.4 describes the topological space \( \Lambda_K \), the topology of \( \Lambda_K \) can be determined if one knows the description of neighborhood bases over elements in \( \Lambda_K \). Through Theorem 2.6, these bases will be described as follows:

N1) Consider \( t \in [0, 1) \) and \( b \in F \) (see Theorem 2.6). If \( \kappa \geq 1 \) and \( 0 < \epsilon < 1 - \tau \), one can easily verify Relation (7), that

\[
U_{t,\epsilon} \setminus \Lambda_K = \{ (\lambda, \omega) \in \Lambda_K : \omega > \kappa ; |t - b|^{\omega} < \epsilon \} \cup \{ \zeta_{Q(b),v} : v \in [0, 1]; |v| < \epsilon \}.
\]

Thus, by Theorem 2.6(c) and Proposition 2.4, the countable collection:

\[
\{ U_{t,\epsilon} \setminus \Lambda_K : \kappa \in [1, m] : m, n \in N \text{ such that } 1/n \in (0, 1 - \tau) \}
\]

constitutes an open neighborhood basis of \( \zeta_{Q(b),\tau} \).

N2) Consider \( \tau > 1 \). If \( \kappa \geq 1 \) and \( 0 < \epsilon < \tau - 1 \), one may verify that

\[
U_{t,\epsilon} \setminus \Lambda_K = \{ (\lambda, \omega) \in \Lambda_K : \omega > \kappa ; |t|^{\omega} < \epsilon \} \cup \{ \zeta_{0,p,v} : \tau - \epsilon < v < \tau + \epsilon \}.
\]

Again, Theorem 2.6(c) and Proposition 2.4 imply that the countable collection:

\[
\{ U_{t,\epsilon} \setminus \Lambda_K : \kappa \in [1, m] : m, n \in N \text{ such that } 1/n \in (0, \tau - 1) \}
\]

constitutes an open neighborhood basis of \( \zeta_{0,p,\tau} \).
Let $F$ be the collection of all finite subsets of $\tilde{F}$ such that each of them contains $0_R$. If $X \in \mathcal{F}$, $\kappa \geq 1$ and $c \in (0, 1)$, it is not hard to see that
\[
\bigcup_{c \in X} U_{1,c}^t \setminus A^K_1 \times [1, \kappa] = \{ (\lambda, \omega) \in A^K_1 \times [1, \infty) : \omega > \kappa; |t - c|^{\omega}_\lambda < 1 \text{ for any } c \in X \} \bigcup \{ \zeta_{Q(v)} : v \in X; v \in (1 - \epsilon, 1) \} \bigcup \{ \zeta_{Q(v)} : v \in [1, 1 + \epsilon) \} \bigcup \{ \zeta_{Q(v)} : b \in \tilde{F} \setminus X; v \in [0, 1) \}.
\]

It now follows from Theorem (2.6) and Proposition (2.7) that the collection:
\[
\left\{ \bigcap_{c \in X} U_{1,c}^t \setminus A^K_1 \times [1, m] : m, n \in \mathbb{N}; X \in \mathcal{F} \right\}
\]

constitutes an open neighborhood basis of $\zeta_{Q,F,1}$.

One may use the above information to obtain certain topological properties of $A^K_1$. The following is an illustration.

**Theorem 2.8.** Let $R$ be a complete valuation ring which is not a field, with its field of fractions being algebraically closed. The following properties hold.

(a) $A^K_1$ is path connected.

(b) $A^K_1$ is locally path connected.

(c) $A^K_1$ is first countable if and only if $F$ is countable and $A^K_1$ is first countable.

**Proof.** (a) By Proposition (2.1), $A^K_1$ is path connected. Moreover, as said in the Introduction, $A^K_1$ is path connected, and this gives the path connectedness of $A^K_1 \times [1, \infty)$. Consider $(x_0, \tau, 0) \in F \times [0, 1) \cup \{0_F\} \times [1, \infty)$ and $(x_0, \omega, 0) \in A^K_1 \times [1, \infty)$. As in the proof of Theorem 2.6(d), one has $\zeta_{\omega,0} \to \zeta_{0,F,0} (\text{when } \omega \to \infty)$, and this produces a path joining $\zeta_{0,F,0}$ to $\zeta_{0,F,0}$. By considering a path in $A^K_1$ (respectively, $A^K_1 \times [1, \infty)$) joining $\zeta_{x_0,\tau,0}$ to $\zeta_{0,\tau,0}$ (respectively, joining $\lambda_{0,0}^0$ to $\zeta_{0,F,0}$), one obtains a path that joins $\zeta_{x_0,\tau,0}$ to $\lambda_{0,0}^0$.

(b) As recalled in the Introduction, $A^K_1$ is locally path connected, and hence so is the open subset $A^K_1 \times [1, \infty)$ of $A^K_1$.

Let us now verify the path connectedness of open sets of the form as in (N1). Let $\tau \in [0, 1)$ and $b \in F$. Fix $\kappa \geq 1$ as well as $\epsilon \in (0, 1 - \tau)$. For simplicity, we denote
\[
V^K := A^K_1 \times (\kappa, \infty).
\]

It is easy to see that $\{ \zeta_{Q(v)} : v \in [0, 1); |v - \tau| < \epsilon \}$ is path connected (see Proposition 2.1). Thus, in order to verify the path connectedness of $U_{t,c}^t \setminus A^K_1 \times [1, \kappa]$, it suffices to show that an arbitrary fixed element $\lambda_{0}^\omega \in U_{t,c}^t \cap V^K$ can be joined to $\zeta_{Q(v),\tau}$ through a path inside $U_{t,c}^t \setminus A^K_1 \times [1, \kappa]$.

Notice that the subset
\[
\{ \lambda \in A^K_1 : |t - b|^{\omega}_\lambda < \epsilon \}
\]

is open in $A^K_1$ and contains $\lambda_{0}^\omega$. Hence, by the locally path connectedness of $A^K_1$ and the density of the image of $K$ in $A^K_1$, one may assume that $\lambda_0 = \zeta_{s_0,0}$ for an element $s_0 \in K$. The requirement of $\zeta_{s_0,0} \in U_{t,c}^t \setminus A^K_1 \times [1, \kappa]$ implies that
\[
\tau - \epsilon < |s_0 - b|^\omega < \tau + \epsilon.
\]

For every $v \in [0, |s_0 - b|]$, the relation
\[
|t - b|^{\omega}_{\zeta_{s,v}} = \max\{v, |s_0 - b|\}^{\omega}_v = |s_0 - b|^{\omega}_v \in (\tau - \epsilon, \tau + \epsilon)
\]
tells us that $\zeta_{s,v}^{\omega} \in U_{t,c}^t \setminus A^K_1 \times [1, \kappa]$. Thus, $\{ \zeta_{s,v}^{\omega} : v \in [0, |s_0 - b|] \}$ is a path in $U_{t,c}^t$ joining $\zeta_{s_0,0}$ to $\zeta_{s_0,|s_0 - b|}^{s_0} = \zeta_{b,|s_0 - b|}$.
Similarly, the path \( \{ \zeta_{0,v}^{\omega_0} : v \text{ is in between } |s_0 - b| \text{ and } \tau^{1/\omega_0} \} \) that joins \( \zeta_{0,|s_0-b|/\omega_0}^{\omega_0} \) to \( \zeta_{0,\tau^{1/\omega_0}}^{\omega_0} \) also lies inside \( U_{1,1}^{-c} \). Moreover, it follows from
\[
|t-b|_{\zeta_{0,\tau^{1/\omega_0}}}^{\omega_0} = \tau
\]
and Theorem 2.9(c) that the net \( \{ \zeta_{0,\tau^{1/\omega_0}}^{\omega_0} : \omega \geq \omega_0 \} \) converges to \( \zeta_{Q(b),\tau} \) when \( \omega \to \infty \). Therefore,
\[
\{ \zeta_{0,\tau^{1/\omega_0}}^{\omega_0} : \omega \geq \omega_0 \} \cup \{ \zeta_{Q(b),\tau} \}
\]
is a path in \( A^1_K \) joining \( \zeta_{0,\tau^{1/\omega_0}}^{\omega_0} \) to \( \zeta_{Q(b),\tau} \). Furthermore, (13) also ensures that this path lies inside \( U_{1,1}^{-c} \). Consequently, \( U_{1,1}^{-c} \backslash A^1_K \times [1, \kappa] \) is path connected (note that \( \omega_0 \) as well as all the \( \omega \) in the above are strictly bigger than \( \kappa \)).

In the same way, for any fixed \( \tau \in (1, \infty) \), one can establish the path connectedness of open sets as in (N2), i.e., \( U_{1,1}^{-c} \backslash A^1_K \times [1, \kappa] \). It remains to show that open sets of the form as in (N3), namely, the set \( \bigcap_{e \in X} U_{1,1}^{-c} \backslash A^1_K \times [1, \kappa] \) is path connected. As in the above, we need to find a path in \( \bigcap_{e \in X} U_{1,1}^{-c} \backslash A^1_K \times [1, \kappa] \) that joins an arbitrary chosen element \( \lambda_0^{\omega_0} \in \bigcap_{e \in X} U_{1,1}^{-c} \cap V^\kappa \) to \( \zeta_{0,1}^{a_0} \). Again, we may assume that \( \lambda_0 = \zeta_{s_0,0} \) for an element \( s_0 \in K \). The condition \( \zeta_{s_0,0}^{\omega_0} \in \bigcap_{e \in X} U_{1,1}^{-c} \) implies that
\[
(1 - \epsilon)^{1/\omega_0} < |s_0 - c| < (1 + \epsilon)^{1/\omega_0} \quad (c \in X),
\]
and, in particular, \( |s_0|^{\omega_0} \in (1 - \epsilon, 1 + \epsilon) \) (because \( 0_R \in X \) by the assumption of \( \mathcal{F} \)). For every \( v \in [0, |s_0|] \), the equality
\[
|t - c|_{\zeta_{s_0,v}^{\omega_0}}^{\omega_0} = \max\{v, |s_0 - c|\}^{\omega_0} \quad (c \in X)
\]
will then ensure that \( \zeta_{s_0,v}^{\omega_0} \in \bigcap_{e \in X} U_{1,1}^{-c} \). In addition, a similar relation as (14) tells us that the path \( \{ \zeta_{s_0,v}^{\omega_0} : v \text{ is in between } |s_0| \text{ and } 1 \} \) lies inside \( \bigcap_{e \in X} U_{1,1}^{-c} \). On the other hand, using Theorem 2.9(c), we see that \( \bigcap_{e \in X} \{ \zeta_{s_0,v}^{\omega_0} : v \text{ is in between } |s_0| \text{ and } 1 \} \) converges to \( \zeta_{Q_{1,1}} \) when \( \omega \to \infty \), and this produces a path joining \( \zeta_{Q_{1,1}}^{a_0} \) to \( \zeta_{Q_{1,1}} \). Moreover, the equalities \( \zeta_{Q_{1,1}}(t - c) = 1 \) \( (c \in X) \) gives \( \zeta_{Q_{1,1}}^{a_0} \in \bigcap_{e \in X} U_{1,1}^{-c} \) \( (\omega \geq \omega_0) \). Consequently, we obtain a path in \( \bigcap_{e \in X} U_{1,1}^{-c} \backslash A^1_K \times [1, \kappa] \) joining \( \lambda_0^{\omega_0} \) to \( \zeta_{Q_{1,1}} \).

(c) Clearly, the countability of \( F \) and the first countability of \( A^1_K \) will imply the first countability of \( A^1_K \). Conversely, suppose that \( A^1_K \) is first countable. Then \( A^1_K \times [1, \infty) \) is first countable and so is \( A^1_K \). Moreover, there is a countable subcollection
\[
\left\{ \bigcap_{e \in X} U_{1,1}^{-c} : \kappa_{e,\lambda} \times [1, m_k] : k \in \mathbb{N} \right\}
\]
that form a neighborhood basis for \( \zeta_{Q_{1,1}} \) in \( A^1_K \). Let \( C := \bigcup_{k \in \mathbb{N}} X_k \). We know that whenever a net \( \{ \lambda_i, \omega_i \}_{i \in \mathbb{N}} \) in \( A^1_K \) satisfying \( \omega_i \to \infty \) as well as \( |t - b|_{\lambda_i} \to 1 \) \( (b \in C) \), we have \( \lambda_i \to \zeta_{Q_{1,1}} \). Assume on the contrary that \( \tilde{F} \) is uncountable, then there exists \( c_0 \in \tilde{F} \backslash C \) (which implies \( |c_0 - b| = 1 \) for all \( b \in C \)). If we set \( \omega_n := n \) and \( \lambda_n := \zeta_{c_0,0} \) \( (n \in \mathbb{N}) \), then \( |t - b|_{\lambda_n} = 1 \) \( (n \in \mathbb{N}) \), for each \( b \in C \), but \( |t - c_0|_{\lambda_n} = 0 \) \( (n \in \mathbb{N}) \), which contradicts \( \zeta_{Q_{1,1}} \to \zeta_{Q_{1,1}} \).

\[ \Box \]

**Remark 2.9.** Unlike \( A^1_K \), the topological space \( A^1_K \) is far from having a \( \mathbb{R} \)-tree structure. Actually, for any connected open subset \( V \subset A^1_K \) and any \( \lambda_1, \lambda_2 \in V \), there exist infinitely many paths in \( V \) joining \( \lambda_1 \) and \( \lambda_2 \).

In fact, consider \( i \in \{ 1, 2 \} \). Let \( U_i \subset V \) be a path connected open neighborhood of \( \lambda_i \) (see Theorem 2.9(b)) and fix any \( (\mu_i, \kappa_i) \in A^1_K \times (1, \infty) \cap U_i \) (see Theorem 2.9). Thus, \( \lambda_i \) is joined to \( (\mu_i, \kappa_i) \) through a path inside \( U_i \). Choose a connected open neighborhood \( W_i \) of \( \mu_i \) in \( A^1_K \) as well as a number \( \epsilon \in (0, 1 - \kappa_i) \) such that \( W_i \times (\kappa_i - \epsilon, \kappa_i + \epsilon) \subset U_i \). Pick any \( \nu_i \in W_i \). There exist infinitely many
paths in $W_i \times (\kappa_i - \epsilon, \kappa_i + \epsilon)$ joining $(\mu_i, \kappa_i)$ to $(\nu_i, \kappa_i)$. As $V$ is path connected, there exists a path in $V$ joining $(\nu_1, \kappa_1)$ to $(\nu_2, \kappa_2)$. In this way, we obtain infinitely many paths joining $\lambda_1$ and $\lambda_2$.

Similar to Theorem 2.8(c), there is also a description of the second countability of $\mathbb{A}_k^1$. In fact, one has the following more general result. This result could be a known fact, but since we do not find an explicit reference for it, we present its simple argument here.

**Proposition 2.10.** If $(S, \| \cdot \|)$ is a commutative unital Banach ring with multiplicative norm, the following statements are equivalent.

1. $S$ is separable as a metric space.
2. $S$ is separable as a metric space.
3. $M(S\{n^{-1}t\})$ is metrizable for every $n \in \mathbb{N}$.

**Proof.** $(S1) \Rightarrow (S2)$. For any $a \in S$, we defined $\zeta_a \in \mathbb{A}_k^1$ by $\zeta_a(p) := \|p(a)\| (p \in S[t])$. It is easy to see that $a \mapsto \zeta_a$ is a homeomorphism from $S$ onto its image in $\mathbb{A}_k^1$. Thus, $S$ is also second countable and hence is separable.

$(S2) \Rightarrow (S3)$. Suppose that $S$ contains a countable dense subset $S_0$. We denote by $S_0$ the collection of non-empty finite subsets of $S[t]$ consisting of polynomials with coefficients in $S_0$. For a fixed $n \in \mathbb{N}$, since $\| \cdot \|_n$ is a metric on $S$, it is not hard to see that the countable family

$$\{E_{1/k}^{S[n^{-1}t]} \cap M(S\{n^{-1}t\}) : k \in \mathbb{N}; X \in S_0\}$$

(see Relation (1) for the meaning of $E_{1/k}^{S[n^{-1}t]}$) forms a fundamental system of entourages for the Berkovich uniform structure on $M(S\{n^{-1}t\})$. Consequently, the Berkovich uniform structure (and hence the topology defined by it) is pseudo-metrizable (see, e.g., Theorem 13 in chapter 6 of [6]). However, since the topology on $M(S\{n^{-1}t\})$ is Hausdorff, we conclude that this topology is indeed metrizable.

$(S3) \Rightarrow (S1)$. Since $M(S\{n^{-1}t\})$ is a compact metric space, it is second countable, and so, the open subset $U_n^S \subseteq M(S\{n^{-1}t\})$ (see (2)) is also second countable. Consequently, $\mathbb{A}_k^1$ is second countable.

Clearly, the Banach ring $\mathbb{A}_k^1$ is separable (equivalently, second countable) if and only if $K$ is separable as a metric space. Note also that if a complete valued field is algebraically closed and spherically complete, then it is not separable (see e.g. [3] Remark 1.4). However, the completion of the algebraic closure of a separable complete valued field is again separable (see e.g. [3] Remark 1.3).

**Example 2.11.** Let $\mathfrak{t}$ be a field equipped with the trivial norm. By Proposition 2.7, one can define a metric on $\mathbb{A}_k^1$ through the “geodesic distance” $d_G$ of two given points. If $\mathfrak{t}$ is uncountable, then Proposition 2.7 tells us that $\mathbb{A}_k^1$ is not metrizable, and hence the topology induced by $d_G$ is different from the pointwise convergence topology on $\mathbb{A}_k^1$.

On the other hand, suppose that $\mathfrak{t}$ is at most countable and $p_1, p_2, \ldots$ are all the elements in $\mathfrak{t}[t]_{\mathfrak{t}^\ast}$. If we rescale the “geodesic distance” so that $d_G(\gamma_{p_0, 0}, \gamma_{t, 1}) \rightarrow 0$ when $k \rightarrow \infty$, then it is not hard to see that this metric defines the topology of pointwise convergence on $\mathbb{A}_k^1$. In this case, $\mathbb{A}_k^1$ is the following closed subspace of $\mathbb{R}^2$:

$$
\begin{array}{c}
\cdots \\
\gamma_{t, p_0} \\
\gamma_{t, 1} \\
\gamma_{p_2, 0} \\
\gamma_{p_1, 0} \\
\gamma_{p_2, \infty} \\
\gamma_{p_1, \infty} \\
\cdots
\end{array}
$$

In the following, we will have a look at the set of "type I points" of $\mathbb{A}_k^1$, i.e. those multiplicative semi-norms $\lambda \in \mathbb{A}_k^1$ with $\ker| \cdot |_\lambda \neq \{0_K\}$. 


Corollary 2.12. Let $R$, $(K, |·|)$, $F$ and $\tilde{F}$ be as in Theorem 2.6. Denote $$\mathbb{A}^1_{R,Z} := \{ \lambda \in \mathbb{A}_R^1 : \ker |·| \neq \{0_R\} \}.$$  

(a) $\mathbb{A}^1_{R,Z} = \mathbb{A}_{F}^1 \cup K \times [1, \infty)$ (and hence, $\mathbb{A}^1_{R,Z}$ is dense in $\mathbb{A}^1_R$).

(b) Suppose that $\{ (s_i, \omega_i) \}_{i \in I}$ is a net in $K \times [1, \infty)$.
   
   • $\zeta_{s_i,0}^\omega \rightarrow \zeta_{Q(b),\tau_1}$ for some $b \in R$ and $\tau_1 \in [0, 1)$ if and only if $\omega_i \rightarrow \infty$ and $|s_i - b|^\omega_i \rightarrow \tau_1$;
   
   • $\zeta_{s_i,0}^\omega \rightarrow \zeta_{0,\tau_2}$ for a number $\tau_2 \in (1, \infty)$ if and only if $\omega_i \rightarrow \infty$ and $|s_i|^\omega_i \rightarrow \tau_2$;
   
   • $\zeta_{s_i,0}^\omega \rightarrow \zeta_{0,1}$ if and only if $\omega_i \rightarrow \infty$ and $|s_i - c|^\omega_i \rightarrow 1$, for any $c \in F$.

(c) If $\{ (s_i, \omega_i) \}_{i \in I}$ is a net in $K \times [1, \infty)$ such that $\zeta_{s_i,0}^\omega \rightarrow \zeta_{Q(b),\tau_1}$ for some $b \in R$ and $\tau_1 \in [0, 1)$, then $s_i$ eventually belongs to the “open ball” of $K$ of radius $1$ and center $b$.

Proof. Since part (b) follows from Theorem 2.6 and part (c) follows directly from part (b), we will only establish part (a). In fact, it is clear that $\mathbb{A}^1_{F} \cup K \times [1, \infty) \subseteq \mathbb{A}^1_{R,Z}$. Consider any element $\lambda \in \mathbb{A}^1_{R,Z}$. If $\ker |·| \cap R \neq \{0_R\}$, the argument of Proposition 2.4 tells us that $\lambda \in \mathbb{A}^1_F$. Suppose that $\ker |·| \cap R = \{0_R\}$. As in the proof of Proposition 2.4, there is a unique positive number $\omega \in [1, \infty)$ such that $\lambda$ extends to an element $\bar{\lambda} \in \mathbb{A}^1_{R,Z}$. Since $\ker |·| \bar{\lambda}$ (which contains $\ker |·| \lambda$) is a non-zero prime ideal of $K[t]$ and $K$ is algebraically closed, one can find a (unique) element $s \in K$ with $\ker |·| \bar{\lambda} = (t - s) \cdot K[t]$ and it is not hard to check that $\lambda = \zeta_{s,0}^\omega$.

In the following, we will use Theorem 2.6 to obtain the Berkovich spectra of Banach group rings of finite cyclic groups over $R$. Let $G$ be a finite abelian group and $(S, ||·||)$ be a commutative unital Banach ring. We denote by $S[G]$ the group ring of $G$ over $S$, and endowed it with the norm $\| \sum_{g \in G} a_g g \| := \max_{g \in G} \| a_g \|$. Clearly, $S[G]$ is a commutative unital Banach ring.

Corollary 2.13. Let $R$, $F$, $Q$ and $(K, |·|)$ be as in Theorem 2.6. Denote by $|·|_0$ the trivial norm on $F$. Let $G$ be a cyclic group of order $M$ with $u$ being a generator of $G$. Suppose $b_1, \ldots, b_n$ are all the distinct $M$-th roots of unity in $K$.

(a) If we set $\alpha_k^\omega := \left| \sum_{i=0}^{M-1} a_i b_i^k \right|^\omega$ and $\beta_k^\omega := \left| \sum_{i=0}^{M-1} Q(a_i b_i^k) \right|^\omega$, then $\mathcal{M}(R[G]) = \{ \alpha_k^\omega : \omega \in [1, \infty); k = 1, \ldots, n \} \cup \{ \beta_k^\omega : k = 1, \ldots, n \}$.

(b) As a topological space, $\mathcal{M}(R[G])$ consists of $n$ intervals of the form $[1, \infty)$ corresponding to the elements $b_1, \ldots, b_n$ such that the “1-ends” of all these intervals are free, while the “∞-ends” of the two intervals corresponding to $b_k$ and $b_l$ are identified with each other if $Q(b_k) = Q(b_l)$.

Proof. (a) There is a contractive and surjective ring homomorphism $q_G : R\{t\} \rightarrow R[G]$ sending $t$ to $u$, and it is not hard to check that $\ker q_G = (t^M - 1) \cdot R\{t\}$. Hence, one may regard $\mathcal{M}(R[G])$ as a topological subspace of $\mathcal{M}(R\{t\})$ through $q_G$ in the following way:

$$\mathcal{M}(R[G]) = \{ \lambda \in \mathbb{A}^1_R : |t|\lambda \leq 1; \ |t^M - 1|\lambda = 0 \} \subseteq \mathbb{A}^1_{R,Z}.$$  

It is obvious that $\alpha_k^\omega$ and $\beta_k^\omega$ are well-defined elements in $\mathcal{M}(R[G])$, and they can be identified, respectively, with the elements $\zeta_{0,0}^\omega$ and $\zeta_{Q(b),0}^\omega$ in $\mathbb{A}^1_{R,Z}$.

On the other hand, let us pick an arbitrary element $\lambda \in \mathcal{M}(R[G])$. By Corollary 2.12(a), either $\lambda = \zeta_{s,\omega}$ for a unique $(s, \omega) \in K \times [1, \infty)$ or $\lambda \in \mathbb{A}^1_F$. In the first case, the condition $|t^M - 1|\zeta_{s,\omega} = 0$ will force $s = b_k$ for some $k \in \{1, \ldots, n\}$, which means that $\lambda = \alpha_k^\omega$. In the second case, there exist $x \in F$ and $\tau \in \mathbb{R}_+$ satisfying $\lambda = \zeta_{x,\tau}$, and the condition $|t^M - 1|\zeta_{x,\tau} = 0$ tells us that $\tau = 0$ and $x^M = 1$. Since $Q(b_1), \ldots, Q(b_n)$ are all the roots of $t^M - 1$ in $F$, we conclude that $\lambda = \beta_k^\omega$ for some $k = 1, \ldots, n$. 


(b) By Corollary 2.12(a), it is not hard to see that the subset \( \{ \zeta_{q,0}^i : \omega \in [1, \infty); k = 1, \ldots, n \} \) of \( \mathbb{A}^1_{R,p} \) are \( n \) disjoint intervals of the form \([1, \infty)\). Assume that \((s_1, \omega_i) \in \{b_1, \ldots, b_n\} \times [1, \infty) \) \((i \in 3)\) such that \( \{ \zeta_{q,0}^i \}_{i \in 3} \) converges to \( \zeta_{Q(b_1),0} \) for some \( k \in \{1, \ldots, n\} \). Then Corollary 2.12(c) tells us that \(|s_1 - b_k| < 1\) eventually. In other words, \( Q(s_1) = Q(b_k) \) eventually. Conversely, it follows from Corollary 2.12(b) that the conditions \( Q(s_1) = Q(b_k) \) for large \( i \) and \( \omega_i \to \infty \) will imply \( \zeta_{s_i,0}^i \to \zeta_{Q(b_k),0} \). This completes the proof. \( \square \)

In the case when the field \( K \) is not algebraically closed, one may use the above as well as [3, Corollary 1.3.6] to describe \( M(R[G]) \). In the following, we will consider the case when \( R \) is the ring \( \mathbb{Z}_p \) of \( p \)-adic integers and \( G \) is a cyclic \( p \)-group (for a fixed prime number \( p \)). Let us start with the following possibly known lemma.

**Lemma 2.14.** For any \( l \in \mathbb{Z}_+ \), the polynomial \( q_{l+1} := t^{p(l-1)} + t^{p(l-2)} + \cdots + t^p + 1 \) is irreducible in \( \mathbb{Z}_p[t] \).

**Proof.** We first consider the case when \( l = 0 \). The equalities

\[
((t + 1) - 1) = \sum_{i=1}^{p} (t + 1)^{p-1} = (t + 1)^p - 1 = t \cdot \sum_{i=1}^{p} \left( p \cdot \frac{1}{i - 1} \right) t^{p-1},
\]

gives \( \sum_{i=1}^{p} (t + 1)^{p-1} = \sum_{i=1}^{p} \left( \frac{p}{i - 1} \right) t^{p-1} \). Thus, it follows from the Eisenstein’s criterion that the polynomial \( \sum_{i=1}^{p} (t + 1)^{p-1} \) is irreducible in \( \mathbb{Z}_p[t] \), and hence so is \( \sum_{i=1}^{p} t^{p-1} \).

In the case of \( l \geq 1 \), we set \( s := (t + 1)^l - 1 \). Since \( \sum_{i=1}^{p} (s + 1)^{p-1} = \sum_{i=1}^{p} \left( \frac{p}{i - 1} \right) s^{p-1} \), we have

\[
\sum_{i=1}^{p} (t + 1)^{p(p-1)} = \sum_{i=1}^{p} \left( \frac{p}{i - 1} \right) ((t + 1)^l - 1)^{p-1}.
\]

From the left hand side, we see that the coefficient of \( t^{p(l-1)} \) is 1 and the constant coefficient is \( p \). From the right hand side, we know that all the other coefficients of \( t^k \) are divisible by \( p \). Thus, the Eisenstein’s criterion tells us that \( \sum_{i=1}^{p} (t + 1)^{p(p-1)} \) is an irreducible polynomial in \( \mathbb{Z}_p[t] \) and hence so is \( \sum_{i=1}^{p} t^{p(p-1)} \). \( \square \)

**Example 2.15.** Let \( G \) be the cyclic group of order \( p^N \) for a positive integer \( N \). We set \( \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z} \) and consider \( q_G : \mathbb{Z}_p[t] \to \mathbb{Z}_p[G] \) to be the canonical quotient map. As in the argument of Corollary 2.12, we may identify, through \( q_G \):

\[
M(\mathbb{Z}_p[G]) = \{ \lambda \in Q^b(\mathbb{A}^1_{\mathbb{Z}_p}) : \bigcup_{\omega \in [1, \infty)} J^b(\mathbb{A}^1_{\mathbb{Z}_p}) : |t|_\lambda \leq 1; |t^{p^N} - 1|_\lambda = 0 \}.
\]

(see Proposition 2.4). By Lemma 2.14, the prime factorization of \( t^{p^N} - 1 \) in \( \mathbb{Z}_p[t] \) is

\[
t^{p^N} - 1 = q_0 \cdot q_1 \cdot q_2 \cdots q_{N},
\]

where \( q_0 := t - 1 \). On the other hand, the following is the prime factorization of \( t^{p^N} - 1 \) in \( \mathbb{F}_p[t] \):

\[
t^{p^N} - 1 = (t - 1)^{p^N}.
\]

Let \( \mathbb{C}_p \) be the completion of the algebraic closure of \( \mathbb{Q}_p \), let \( R_p \) be the ring of integers of \( \mathbb{C}_p \) and let \( F_p \) be the residue field of \( R_p \). For \( 1 \leq k \leq N \), we consider \( \{r_{k,1}, \ldots, r_{k,n_k}\} \) to be the set of all distinct roots of \( q_k \) in \( \mathbb{C}_p \). It follows from (13) that \( t^{p^N} - 1 = (t - 1)^{p^N} \) in \( F_p[t] \). Hence, \( Q(r_{k,i}) = 1 \) for all possible \( k \) and \( i \). Therefore, Corollary 2.11(a) tells us that

\[
M(R_p[G]) = \{ a_r^{\omega} : \omega \in [1, \infty); 1 \leq k \leq N; 1 \leq i \leq n_k \} \cup \{ a_r^{\beta} : \omega \in [1, \infty) \} \cup \{ \beta_1 \}.
\]
As in Corollary 2.13(b), the topological space $M(R_p[G])$ consists of $1 + \sum_{k=1}^{N} n_k$ intervals of the form $[1, \infty]$ with all the “1-ends” being free but with all the “$\infty$-ends” being identified with one point, namely, $\beta_1$.

Again, the prime factorization as in [10] ensures that $M(Z_p[G]) \cap Q^\Lambda(A_{Q_p}^1) = \{ Q^\Lambda(\zeta_{1p}, 0) \}$, and the semi-norm $\bar{\beta}_1 := Q^\Lambda(\zeta_{1p}, 0)$ coincides with the one induced by $\bar{\beta}_1 \in M(R_p[G])$ through restriction. On the other hand, as in [3] Corollary 1.3.6, any element $\lambda \in M(Z_p[G]) \cap J_0^\Lambda(A_{Q_p}^1)$ can be extended to an element $\tilde{\lambda} \in J_0^\Lambda(A_{Q_p}^1)$. It follows from $|t| - 1 \leq 1$ and $|t^N - 1| = 0$ that $\tilde{\lambda}$ is either $\alpha_1^0$ or $\alpha_1^\omega$ for suitable $k$ and $i$.

Let us set $\tilde{\alpha}_0^k$ to be the element in $M(Z_p[G])$ induced by $\alpha_1^0$. On the other hand, for a fixed $k \in \{1, 2, \ldots, N\}$, by considering an automorphism in the Galois group of the splitting field of the irreducible polynomial $q_k$ over $\mathbb{Q}_p$, we know that $\alpha_{r, i, j}^0$ and $\alpha_{r, i, j}^\omega$ restrict to the same element in $M(Z_p[G])$ for any $i, j \in \{1, \ldots, n_k\}$. We denote the resulting element by $\tilde{\alpha}_k^0$. As $\tilde{\alpha}_k^0$ comes from different irreducible factors of $t^N - 1$ for different $k$, they are all distinct.

Consequently, as a quotient space of $M(R_p[G])$, the topological space $M(Z_p[G])$ is the following subspace of $\mathbb{R}^2$:

\[
\begin{array}{c}
\tilde{\alpha}_0^1 & \tilde{\alpha}_0^2 & \tilde{\alpha}_1^1 & \tilde{\alpha}_2^1 & \tilde{\alpha}_2^0 \\
\beta_1 & \tilde{\alpha}_0^N & \tilde{\alpha}_N^N & \tilde{\alpha}_N^N & \tilde{\alpha}_N^N
\end{array}
\]

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