Symmetries and integrals of motion of the deformed oscillator

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Abstract

The symmetry structure of twodimensional nonlinear isotropic oscillator, introduced in Physica D237 (2008) 505, is discussed. It is shown that it possesses three independent integrals of motion which can be chosen in such a way that they span SU(2), e(2) or SU(1,1) algebras, depending on the value of total energy. They generate the infinitesimal canonical symmetry transformations; integrability of the latter is analyzed.

I Introduction

An interesting example of superintegrable system is provided by the radially symmetric nonlinear oscillator [1]. It can be viewed as describing either a particle moving on a space of nonconstant curvature or an oscillator with space-dependent mass. It has been further studied, both on classical and quantum levels, in Refs. [2] - [6]. In particular, in the recent paper [7] Anco et. al. analyzed in more detail the integrals of motion and symmetry transformations for this system.

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Our aim here is to reveal further interesting properties of the integrals of motion and symmetries of the nonlinear oscillator. Let us start recalling some well known facts [8], [9], [10]. An isolated classical dynamical system with $f$ degrees of freedom possesses $2f - 1$ independent integrals of motion which do not depend explicitly on time. However, in general these integrals are defined only locally; the simplest example is provided by two uncoupled harmonic oscillators ($f = 2$). Their individual energies are integrals of motion defined globally while the third global integral exists only provided the oscillator frequencies are commensurate. Such a behaviour is typical for integrable systems. Consider a twodimensional integrable system. For bounded motions all trajectories lie on the twodimensional tori parametrized by the values of two independent Poisson-commuting integrals of motion [9]. One can introduce the action variables $I_1, I_2$ and canonically conjugated angles $\Theta_1, \Theta_2$; the Hamiltonian depends on action variables only, $H = H(I_1, I_2)$. It is straightforward to construct the third integral

$$C = \omega_2 \Theta_1 - \omega_1 \Theta_2$$

where

$$\omega_i \equiv \frac{\partial H}{\partial I_i}, \quad i = 1, 2,$$  (2)

are the relevant frequencies. $C$ is defined only locally because the angles $\Theta_1, \Theta_2$ are defined up to the multiplicies of $2\pi$. However, if $\omega_1$ and $\omega_2$ are commensurate, $\omega_1/\omega_2 = n_1/n_2$ or $\omega_k = n_k \omega(I), \ k = 1, 2$, then

$$\frac{C}{\omega(I)} = n_2 \Theta_1 - n_1 \Theta_2$$  (3)

and any periodic function of $C/\omega(I)$ is a globally defined integral of motion. If the frequencies are not commensurate no additional independent globally defined integral of motion exists. In fact, generic trajectories cover then densely the relevant invariant torus so they cannot be viewed as the intersections of the latter with level hypersurfaces of some regular function on phase space. We conclude that the third integral exists iff

$$H = H(n_1 I_1 + n_2 I_2)$$  (4)

for some integers $n_1, n_2$. On the other hand, a local integral exists for any integrable Hamiltonian and is given by eqs. [11], [2]. It is also worth to
note that in the region of unbounded trajectories some of the variables are no longer angles and the relevant periodicity conditions are relaxed. This makes the existence of additional globally defined integrals of motion more likely.

The main source of global integrals is provided by the Noether theorem. It should be stressed that it applies not only to the point transformations but also to general canonical ones. In short, if \( G(q, p, t) \) is a generator of canonical symmetry transformations then

\[
\{G, H\} + \frac{\partial G}{\partial t} = 0 \tag{5}
\]

i.e. \( G \) is an integral of motion. The reverse is also true: if \( G \) obeys (5) then it generates symmetry transformations by

\[
\delta(\cdot) = \delta\varepsilon\{\cdot, G\} \tag{6}
\]

where ” \( \cdot \) ” stands for any canonical variable while \( \delta\varepsilon \) is an infinitesimal parameter. Eq. (6), when integrated, yields finite symmetry transformations.

II Deformed oscillator

We consider the deformed two-dimensional oscillator defined by the Hamiltonian

\[
H = \frac{\vec{p}^2 + \omega^2 \vec{q}^2}{2(1 + \lambda \vec{q}^2)} , \quad \lambda \geq 0 , \quad \vec{q} = (q_1, q_2)
\]

\[
\vec{p} = (p_1, p_2) \tag{7}
\]

Most results obtained below can be generalized to higher dimensions.

Our system is integrable, the Poisson-commuting independent integrals being \( H \) and \( J \), the angular momentum,

\[
J = q_1 p_2 - q_2 p_1 \tag{8}
\]

The level surfaces of constant \( H = E \) and \( J \) are tori for \( 2\lambda E < \omega^2 \) and planes for \( 2\lambda E \geq \omega^2 \).

The Hamiltonian (8) is an example of the so-called Liouville system. Therefore, the Hamilton-Jacobi equation

\[
\frac{(\frac{\partial S}{\partial \vec{q}})^2 + \omega^2 \vec{q}^2}{2(1 + \lambda \vec{q}^2)} + \frac{\partial S}{\partial t} = 0 \tag{9}
\]
is completely separable. Its solution may be described as follows. Let
\[ \tilde{S}(q, t; \omega^2, E) = \tilde{S}_0(q; \omega^2, E) - Et \]  
be the solution to Hamilton-Jacobi equation for harmonic oscillator with frequency \( \omega \); it can be analytically continued to the whole range \(-\infty < \omega^2 < \infty\). The solution to the eq. (10) now reads
\[ S(q, t; \omega^2, \varepsilon_1, \varepsilon_2) = \tilde{S}_0(q_1; \tilde{\omega}^2, \varepsilon_1) + \tilde{S}_0(q_2; \tilde{\omega}^2, \varepsilon_2) - Et \]
where
\[ \tilde{\omega}^2 \equiv \omega^2 - 2\lambda E \]  
\[ E \equiv \varepsilon_1 + \varepsilon_2 \]
The phase-space trajectories are given by
\[ p_i = \frac{\partial \tilde{S}_0(q_i; \tilde{\omega}^2, \varepsilon_i)}{\partial q_i}, \quad i = 1, 2 \]  
\[ \alpha_i = \frac{\partial \tilde{S}_0(q_i; \tilde{\omega}^2, \varepsilon_i)}{\partial \varepsilon_i} - 2\lambda \left( \frac{\partial \tilde{S}_0(q_1; \tilde{\omega}^2, \varepsilon_1)}{\partial \tilde{\omega}^2} + \frac{\partial \tilde{S}_0(q_2; \tilde{\omega}^2, \varepsilon_2)}{\partial \tilde{\omega}^2} \right) - t, \quad i = 1, 2 \]
with \( \alpha_1, \alpha_2 \) being arbitrary constants. One easily concludes from eqs. (14), (15) that the shapes of trajectories coincide with those for twodimensional isotropic oscillator with frequency \( \tilde{\omega} \); only the time dependence is modified.
Now, the latter is superintegrable and the shape of its trajectories is determined by the values of three independent integrals of motion with no explicit time dependence. Let us call these integrals \( C_i(\tilde{q}, \tilde{p}; \tilde{\omega}^2), \quad i = 1, 2, 3 \). Then
\( C_i(\tilde{q}, \tilde{p}; \tilde{\omega}^2) \) are the integrals of motion for deformed oscillator. Therefore,
\[ \{ C_i(\tilde{q}, \tilde{p}; \tilde{\omega}^2), H \} \bigg|_{H=E} = 0 \]
The latter formula can be rewritten as
\[ \{ C_i(\tilde{q}, \tilde{p}; \tilde{\omega}^2(H)), H \} = 0 \]
with (cf. eq. (12))
\[
\tilde{\omega}^2(H) \equiv \omega^2 - 2\lambda H
\]  
(18)

A convenient choice of the integrals of motion for isotropic oscillator reads:

\[ C_1 = \frac{1}{2}(p_1p_2 + \omega^2 q_1q_2) \]  
(19)

\[ C_2 = \frac{1}{2}(q_1p_2 - q_2p_1) \equiv \frac{1}{2}J \]  
(20)

\[ C_3 = \frac{1}{4}(p_1^2 - p_2^2 + \omega^2(q_1^2 - q_2^2)) \]  
(21)

Then

\[ \{C_1, C_2\} = C_3 \]  
(22)

\[ \{C_2, C_3\} = C_1 \]  
(23)

\[ \{C_3, C_1\} = \omega^2 C_2; \]  
(24)

therefore, the symmetry algebra is \( SU(2), e(2) \) or \( SU(1,1) \), depending on whether \( \omega^2 > 0, \omega^2 = 0 \) or \( \omega^2 < 0 \), respectively.

It is straightforward to check that

\[ \tilde{C}_i \equiv C_i(\tilde{q}, \tilde{p}; \tilde{\omega}^2(H)), \quad i = 1, 2, 3 \]  
(25)

are integrals of motion and obey the Poisson relations (24) provided the replacement \( \omega^2 \rightarrow \tilde{\omega}^2(H) \) has been made there.

We conclude that our dynamics exhibits deformed symmetry. On the hypersurfaces of constant energy it reduces to the symmetries described by \( SU(2), e(2) \) and \( SU(1,1) \) Lie algebras. In this respect the symmetry structure resembles that of the Kepler problem.

The integrals \( \tilde{C}_i \) can be used as generators of canonical symmetry transformations. Let us remind that in the undeformed case, \( \lambda = 0 \), these transformations are given by linear representations of the relevant Lie groups. It
is interesting to analyze their counterparts in the deformed case: in particular we would like to know if the infinitesimal transformations integrate to global ones which provide the realizations of relevant groups. $\tilde{C}_2$ continues to be the (one half of) angular momentum; so it generates rotations. Let us consider $\tilde{C}_3$. Infinitesimal transformations generated by $\tilde{C}_3$ read

$$\delta(\cdot) = \delta \epsilon \{\cdot, \tilde{C}_3\}$$  \hspace{1cm} (26)

Global transformations are obtained by solving ”dynamical” equations

$$q'_i = \{q_i, \tilde{C}_3\}$$  \hspace{1cm} (27)

$$p'_i = \{p_i, \tilde{C}_3\}$$  \hspace{1cm} (28)

where prime denotes differentiation with respect to the transformation parameter $\epsilon$. The dynamics described by eqs. (27), (28) admits two independent integrals of motion, $H, \tilde{C}_3$ (the actual value of the integral $\tilde{C}_3$ will be denoted by the same letter), and is therefore integrable. As an example consider the region $2\lambda E < \omega^2$. The invariant tori are given by the equations

$$p_1^2 + \tilde{\omega}^2(E)q_1^2 = E + 2\tilde{C}_3$$  \hspace{1cm} (29)

$$p_2^2 + \tilde{\omega}^2(E)q_2^2 = E - 2\tilde{C}_3$$  \hspace{1cm} (30)

The relevant action variables take the form

$$I_1 = \frac{1}{\pi} \int_{q_{\min}}^{q_{\max}} \sqrt{(E + 2\tilde{C}_3) - \tilde{\omega}^2(E)q^2} \, dq$$  \hspace{1cm} (31)

$$I_2 = \frac{1}{\pi} \int_{q_{\min}}^{q_{\max}} \sqrt{(E - 2\tilde{C}_3) - \tilde{\omega}^2(E)q^2} \, dq$$  \hspace{1cm} (32)

which yields

$$E = \sqrt{\omega^2 + \lambda^2(I_1 + I_2)^2} (I_1 + I_2) - \lambda(I_1 + I_2)^2$$  \hspace{1cm} (33)
Let us note that the "Hamiltonian" $\tilde{C}_3$ is not superintegrable! Eqs. (27), (28) are integrable by quadratures but the trajectories are generically not closed and cover densely the invariant tori. On the other hand the path in $SU(2)$ manifold generated by the counterpart of $\tilde{C}_3$ should be closed.

In order to understand what is happening let us remind some properties of canonical transformations. Denote collectively by $\zeta^\alpha$ the canonical variables $q_i, p_i$. Let $G(\zeta)$ be a generator of canonical transformations,

$$\delta\zeta^\alpha = \{\zeta^\alpha, G\}$$

(35)

The corresponding vector field on phase space reads

$$X_G \equiv \delta\zeta^\alpha \frac{\partial}{\partial \zeta^\alpha} = \{\zeta^\alpha, G\} \frac{\partial}{\partial \zeta^\alpha}$$

(36)

and it is straightforward to derive the following basic relation

$$[X_G, X_{G'}] = -X_{\{G, G'\}}$$

(37)

In particular, the counterpart of equation (24),

$$\{\tilde{C}_3, \tilde{C}_1\} = \omega^2(H)\tilde{C}_2$$

implies

$$[X_{\tilde{C}_3}, X_{\tilde{C}_1}] = -X_{\tilde{C}_2}(H)\tilde{C}_2 = -\tilde{\omega}^2(H)X_{\tilde{C}_2} + 2\lambda\tilde{C}_2X_H$$

(39)

so the infinitesimal action of $\tilde{C}_i$’s on the phase space is not that of $SU(2)$, even on the submanifold $H = E$. This can be cured by defining new generators (assuming $\omega^2 - 2\lambda E > 0$).

$$D_1 = \frac{\tilde{C}_1}{\tilde{\omega}(H)} \quad , \quad D_2 = \tilde{C}_2 \quad , \quad D_3 = \frac{\tilde{C}_3}{\tilde{\omega}(H)}$$

(40)

Then

$$\{D_\alpha, D_\beta\} = \varepsilon_{\alpha,\beta,\gamma}D_\gamma$$

(41)
and

\[ [X_{D_\alpha}, X_{D_\beta}] = -\varepsilon_{\alpha,\beta,\gamma} X_{D_\gamma} \]  \hspace{1cm} (42)

Therefore, the modified integrals of motion generate the action of \( SU(2) \) algebra. The infinitesimal action can be integrated to the global one, in accordance with Lie-Palais integrability theorem. In fact, it follows from eq. (34) that the new generator \( D_3 \) takes the form

\[ D_3 = \frac{1}{2}(I_1 - I_2) \]  \hspace{1cm} (43)

and generates superintegrable dynamics. The relevant trajectories are closed as it should be since \( SU(2) \) is simply connected. This conclusion holds true also for \( D_1 \); to see this it is sufficient to make the rotation by \( \pi/4 \) in the plane of motion. Finally, \( D_2 \) generates ordinary rotations.

The noncompact case (no periodicity condition in the noncompact directions) will be considered elsewhere.

### III \hspace{1cm} Polar coordinates

It is instructive to reconsider our dynamical system in polar coordinates,

\[ q_1 = r \cos \varphi \] \hspace{1cm} (44)
\[ q_2 = r \sin \varphi; \] \hspace{1cm} (45)

the Hamiltonian reads

\[ H = \frac{p_r^2 + \frac{p_\varphi^2}{r^2} + \omega^2 r^2}{2(1 + \lambda r^2)} \]  \hspace{1cm} (46)

Then \( p_\varphi = J \) and \( H=E \) obey

\[ E = \frac{p_r^2 + \frac{p_\varphi^2}{r^2} + \omega^2 r^2}{2(1 + \lambda r^2)} \geq \frac{p_\varphi^2}{2(1 + \lambda r^2)} + \omega^2 r^2 \]  \hspace{1cm} (47)

Assume \( \lambda > 0 \); for \( p_\varphi \neq 0 \) the right hand side tends to \( \infty \) for \( r \to 0^+ \) and to \( \frac{\omega^2}{2\lambda} \) for \( r \to \infty \). It has the unique minimum equal to

\[ \frac{\omega^2 |p_\varphi| \sqrt{\omega^2 + \lambda^2 p_\varphi^2}}{\omega^2 + \lambda^2 p_\varphi^2 + \lambda |p_\varphi| \sqrt{\omega^2 + \lambda^2 p_\varphi^2}} < \omega^2 \] \hspace{1cm} (48)
Therefore, for $E \geq \frac{\omega^2}{2\lambda}$ the angular momentum $p_\phi$ takes arbitrary values while in the confining region, $\frac{\omega^2}{2\lambda} > E$, one finds from (47) and (48)

$$|p_\phi| \leq \frac{E}{\sqrt{\omega^2 - 2\lambda E}} \quad (49)$$

In the confining region one can construct action-angle variables. First relation (47) yields

$$p_r = \pm \sqrt{2(1 + \lambda r^2)E - \frac{p_\phi^2}{r^2} - \omega^2 r^2} \quad (50)$$

Assume for definiteness $p_\phi \geq 0$. Then the action variables read:

$$I_\phi = \frac{1}{2\pi} \oint p_\phi d\varphi = p_\phi \quad (51)$$

$$I_r = \frac{1}{2\pi} \oint p_r dr = \frac{1}{\pi} \int_{r_{\text{min}}}^{r_{\text{max}}} \sqrt{2(1 + \lambda r^2)E - \frac{p_\phi^2}{r^2} - \omega^2 r^2} \, dr \quad (52)$$

By virtue of the inequality (49) one obtains

$$I_r = \frac{1}{2} \left( \frac{E}{\sqrt{\omega^2 - 2\lambda E}} - I_\phi \right) > 0 \quad (53)$$

and

$$H = (2I_r + I_\phi) \sqrt{\omega^2 + \lambda^2(2I_r + I_\phi)} - \lambda(2I_r + I_\phi)^2 \quad (54)$$

The Hamiltonian depends on specific combination of action variables, $2I_r + I_\phi$. Consequently, any periodic function of $\Theta_r - 2\Theta_\varphi$ of angle variables is an additional global integral of motion. Now,

$$\Theta_r - 2\Theta_\varphi = \frac{\partial S}{\partial I_r} - 2\frac{\partial S}{\partial I_\phi} \quad (55)$$

where the generating function $S$ reads

$$S(r, \varphi; I_r, I_\phi) = \int_r^r p_r dr + \int_\varphi^\varphi I_\phi d\varphi \quad (56)$$
\( I_r \) enters \( S \) only through \( E \) while \( \frac{\partial}{\partial I_r} - 2\frac{\partial}{\partial \phi} \) annihilates \( E \). Therefore,

\[
\Theta_r - 2\Theta_\phi = -2 \frac{\partial S}{\partial I_\phi} \bigg|_E
\]

(Eq. (57))

Eqs. (56), (57) lead to the following result

\[
\Theta_r - 2\Theta_\phi = -2\phi + \arcsin \left( \frac{Er^2 - I_\phi^2}{r^2 \sqrt{E^2 - I_\phi^2(\omega^2 - 2\lambda E)}} \right) + \text{const.}
\]

(Eq. (58))

which, in turn, implies that

\[
\tilde{C} = \frac{1}{2} \left( H - \frac{p_\phi^2}{r^2} \right) \cos 2\phi - \frac{p_rp_\phi}{2r} \sin 2\phi
\]

(Eq. (59))

is an integral of motion. It is easy to see that

\[
\tilde{C} = C_3(q, p; \tilde{\omega}^2(H))
\]

(Eq. (60))

Also

\[
\frac{1}{2} p_\phi = \frac{1}{2} I_\phi = C_2(q, p; \tilde{\omega}^2(H))
\]

(Eq. (61))

and, finally,

\[
C_1(q, p; \tilde{\omega}^2(H)) = \frac{1}{2} \{p_\phi, \tilde{C}\} = \frac{1}{2} \left( H - \frac{p_\phi^2}{r^2} \right) \sin 2\phi + \frac{p_rp_\phi}{2r} \cos^2 \phi
\]

(Eq. (62))

Let us note that the trajectories \( r = r(\phi) \) can be easily computed from (59), (61) and (62). As expected they are ellipses centered at the origin. It is also straightforward to derive the relation \( p_r = p_r(\phi) \) which, together with \( p_\phi = \text{const.} \), completes the full description of trajectory in phase space.

Similar analysis can be performed for \( 2\lambda E \geq \omega^2 \).

\section*{IV Summary}

We have shown that the nonlinear isotropic oscillator is superintegrable. Three independent integrals of motion can be chosen in analogy with the
harmonic oscillator case. They span (with respect to the Poisson bracket) a Lie algebra on any submanifold of constant energy; depending on the value of energy it is $SU(2)$, $e(2)$ or $SU(1, 1)$ algebra. In this respect the symmetry structure resembles that of the Kepler problem. The integrals of motion generate infinitesimal symmetry transformations; however, in general the latter are rather general canonical transformations and not the point ones. The infinitesimal transformations can be integrated to the global $SU(2)$ ones once the generators are suitably redefined to get rid of energy dependent structure constants. We have also presented the concise discussion of the integrals of motion in polar coordinates.

Two more general conclusions seem to be worth of stressing. First, in order to provide the general Noether theorem one should address to the Hamiltonian formalism. This allows us to relate the integrals of motion to canonical transformations which do not necessarily reduce to the point ones. Second, for the dynamical systems integrable in the Liouville sense it is easy to construct the additional local integrals of motion and find the necessary and sufficient conditions for the existence of global integrals as well as the algebra they obey. For example, consider a superintegrable twodimensional system in the confining region of phase-space. One can introduce action-angle variables $(I_i, \varphi_i)$, $i = 1, 2$; superintegrability implies the general form of the Hamiltonian as given by eq. (4). Obviously, one may assume that $n_1, n_2$ are coprime. Then there exist integers $m_1, m_2$ such that

$$n_1m_2 - n_2m_1 = 1$$

(63)

Therefore, one has

$$\left( \begin{array}{ll} n_1 & n_2 \\ m_1 & m_2 \end{array} \right) \in SL(2, \mathbb{Z})$$

(64)

and

$$\tilde{\varphi}_1 = m_2\varphi_1 - m_1\varphi_2$$
$$\tilde{\varphi}_2 = -n_2\varphi_1 + n_1\varphi_2$$
$$\tilde{I}_1 = n_1I_1 + n_2I_2$$
$$\tilde{I}_2 = m_1I_1 + m_2I_2$$

(65)

is a well-defined canonical transformation, and $\tilde{I}_1, \tilde{I}_2$ and $\tilde{\varphi}_2$ are integrals of motion.
motion. It is now straightforward to check that $C_\alpha, \alpha = 1, 2, 3$, defined as

$$
C_1 = \sqrt{-\tilde{I}_2^2 + A(\tilde{I}_1)\tilde{I}_2 + B(\tilde{I}_1) \cos \tilde{\varphi}_2}
$$

$$
C_2 = \sqrt{-\tilde{I}_2^2 + A(\tilde{I}_1)\tilde{I}_2 + B(\tilde{I}_1) \sin \tilde{\varphi}_2}
$$

$$
C_3 = \tilde{I}_2 - \frac{1}{2} A(\tilde{I}_1)
$$

(66)

with $A(\cdot), B(\cdot)$ being arbitrary, are integrals of motion obeying $SU(2)$ algebra

$$
\{C_\alpha, C_\beta\} = \varepsilon_{\alpha\beta\gamma} C_\gamma
$$

(67)

We would like $C_\alpha$ to be real; this imposes additional conditions on $A(\cdot)$ and $B(\cdot)$ Note that the original action variables $I_1, I_2$ (cf. eq. (65)) are (by definition) nonnegative. However, even with such a restriction it is in general not possible to arrange things in such a way that $C_{1,2}$ are real. This is, for example, possible if $n_{1,2} > 0$ (which includes the case of deformed harmonic oscillator). In the general case one has to consider the trajectories in $(I_1, I_2)$ space characterized by $\tilde{I}_1 = \text{const.}, I_1 \geq 0, I_2 \geq 0$.

For more degrees of freedom cf. [11]

The nonlinear superintegrable oscillators viewed as the bosonic parts of $N = 8$ supersymmetric mechanical systems have been discussed in the interesting paper by Krivonos et al [12].

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