Location-Aided Fast Distributed Consensus in Wireless Networks

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Abstract

Existing works on distributed consensus explore linear iterations based on reversible Markov chains, which contribute to the slow convergence of the algorithms. It has been observed that by overcoming the diffusive behavior of reversible chains, certain nonreversible chains lifted from reversible ones mix substantially faster than the original chains. In this paper, we investigate the idea of accelerating distributed consensus via lifting Markov chains, and propose a class of Location-Aided Distributed Averaging (LADA) algorithms for wireless networks, where nodes’ coarse location information is used to construct nonreversible chains that facilitate distributed computing and cooperative processing. First, two general pseudo-algorithms are presented to illustrate the notion of distributed averaging through chain-lifting. These pseudo-algorithms are then respectively instantiated through one LADA algorithm on grid networks, and one on general wireless networks. For a $k \times k$ grid network, the proposed LADA algorithm achieves an $\epsilon$-averaging time of $O(k \log(\epsilon^{-1}))$. Based on this algorithm, in a wireless network with transmission range $r$, an $\epsilon$-averaging time of $O(r^{-1} \log(\epsilon^{-1}))$ can be attained through a centralized algorithm. Subsequently, we present a fully-distributed LADA algorithm for wireless networks, which utilizes only the direction information of neighbors to construct nonreversible chains. It is shown that this distributed LADA algorithm achieves the same scaling law in averaging time as the centralized scheme in wireless networks for all $r$ satisfying the connectivity requirement. The constructed chain attains the optimal scaling law in terms of an important mixing metric, the fill time, among all chains lifted from one with an approximately uniform stationary distribution on geometric random graphs. Finally, we

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propose a cluster-based LADA (C-LADA) algorithm, which, requiring no central coordination, provides the additional benefit of reduced message complexity compared with the distributed LADA algorithm.

**Index Terms**

Clustering, Distributed Computation, Distributed Consensus, Message Complexity, Mixing Time, Nonreversible Markov Chains, Time Complexity

**I. INTRODUCTION**

As a basic building block for networked information processing, distributed consensus admits many important applications in various areas, such as distributed estimation and data fusion, coordination and cooperation of autonomous agents, as well as network optimization. The distributed averaging problem where nodes try to reach consensus on the average value\(^1\) through iterative local information exchange has been vigorously investigated recently [1]–[6]. Compared with centralized counterparts, such distributed algorithms scale well as the network grows, and exhibit robustness to node and link failures. Distributed consensus can be realized through linear iteration in the form \(x(t + 1) = W(t)x(t)\) where \(W(t)\) is a graph conformant matrix\(^2\). Distributed averaging through linear iteration with a deterministic \(W\) is studied in [1]. For time-varying \(W(t)\), convergence is guaranteed under mild conditions [2], [3]. The class of randomized gossip algorithms recently studied by Boyd et al [4], [5] realizes consensus through iterative pairwise averaging, and allows for asynchronous operation. In their study, independent and identically distributed random matrices \(W(t)\) are considered, and performance of the proposed algorithms is governed by the second largest eigenvalue of \(E[W(t)]\).

Typically, governing matrices in distributed consensus algorithms are chosen to be stochastic, which connects them closely to Markov chain theory. It is also convenient to view the evolution of a Markov chain \(P\) as a random walk on a graph (with vertex set \(V\) being the state space of the chain, and edge set \(E = \{(u, v) : P_{uv} > 0\}\)). In both fixed and random algorithms studied in [1], [4], [5], mainly a symmetric, doubly stochastic weight matrix is used, hence the convergence time of such algorithms is closely related to the mixing time of a reversible random walk, which is usually slow due to its diffusive behavior. It has been shown in [5] that in a wireless network of size \(n\) with a common transmission range \(r\), the

\(^1\)With appropriate modification, such algorithms can also be extended to computation of weighted sums, linear synopses, histograms and types, and can address a large class of distributed computing and statistical inferencing problems.

\(^2\)For a graph \(G = (V, E)\) with the vertex set \(V\) and edge set \(E\), a matrix \(W\) of size \(|V| \times |V|\) is \(G\)-conformant, if \(W_{ij} \neq 0\) only if \((i, j) \in E\).
optimal gossip algorithm requires $\Theta(r^{-2}\log(\epsilon^{-1}))^3$ time for the relative error to be bounded by $\epsilon$. This means that for a small radius of transmission, even the fastest gossip algorithm converges slowly.

Reversible Markov chains are dominant in research literature, as they are mathematically more tractable – see [7] and references therein. However, it is observed by Diaconis et al. [8] and later by Chen et al. [9] that certain nonreversible chains mix substantially faster than corresponding reversible chains, by overcoming the diffusive behavior of reversible random walks. Our work is directly motivated by this finding, as well as the close relationship between distributed consensus algorithms and Markov chains. We first show that by allowing each node in a network to maintain multiple values, mimicking the multiple lifted states from a single state, a nonreversible chain on a lifted state space can be simulated, and we present two general pseudo-algorithms for this purpose. The next and more challenging step is to explicitly construct fast-mixing non-reversible chains given the network graphs. In this work, we propose a class of Location-Aided Distributed Averaging (LADA) algorithms that result in significantly improved averaging times compared with existing algorithms. As the name implies, the algorithms utilize (coarse) location information to construct nonreversible chains that prevent the same information being “bounced” forth and back, thus accelerating information dissemination.

Two important types of networks, grid networks and general wireless networks modeled by geometric random graphs, are considered in this work. For a $k \times k$ grid, we propose a LADA algorithm as an application of our Pseudo-Algorithm 1, and show that it takes $O(k\log(\epsilon^{-1}))$ time to reach a relative error within $\epsilon$. Then, for the celebrated geometric random graph $G(n,r)$ with a common transmission range $r$, we present a centralized grid-based algorithm which exploits the LADA algorithm on the grid to achieve an $\epsilon$-averaging time of $O(r^{-1}\log(\epsilon^{-1}))$.

In practice, purely distributed algorithms requiring no central coordination are typically preferred. Consequently, we propose a fully-distributed LADA algorithm, as an instantiation of Pseudo-Algorithm 2. On a wireless network with randomly distributed nodes, the constructed chain does not possess a uniform stationary distribution desirable for distributed averaging, due to the difference in the number of neighbors a node has in different directions. Nevertheless, we show that the non-uniformity for the stationary distribution can be compensated by weight variables which estimate the stationary probabilities, and that the algorithm achieves an $\epsilon$-averaging time of $O(r^{-1}\log(\epsilon^{-1}))$ with any transmission range $r$.

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3We use the following order notations in this paper: Let $f(n)$ and $g(n)$ be nonnegative functions for $n \geq 0$. We say $f(n) = O(g(n))$ and $g(n) = \Omega(f(n))$ if there exists some $k$ and $c > 0$, such that $f(n) \leq cg(n)$ for $n \geq k$; $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ as well as $f(n) = \Omega(g(n))$. We also say $f(n) = o(g(n))$ and $g(n) = \omega(f(n))$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$.  

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guaranteeing network connectivity. Although it is not known whether the achieved averaging time is optimal for all $\epsilon$, we demonstrate that the constructed chain does attain the optimal scaling law in terms of another mixing metric $T_{\text{fill}}(P, c)$ (c.f. (3)), among all chains lifted from one with an approximately (on the order sense) uniform stationary distribution on $G(n, r)$. In Appendix C, we provide another algorithm, the LADA-U algorithm, where the nonreversible chain is carefully designed to ensure an exact uniform stationary distribution (which accounts for the suffix “U”), by allowing some controlled diffusive behavior. It is shown that LADA-U can achieve the same scaling law in averaging time as the centralized and distributed LADA algorithm, but needs a larger transmission range than minimum connectivity requirement, mainly due to the induced diffusive behavior.

Finally, we propose a cluster-based LADA (C-LADA) variant to further improve on the message complexity. This is motivated by the common assumption that nodes in some networks, such as wireless sensor networks, are densely deployed, where it is often more efficient to have co-located nodes clustered, effectively behaving as a single entity. In this scenario, after initiation, only inter-cluster communication and intra-cluster broadcast are needed to update the values of all nodes. Different from the centralized algorithm, clustering is performed through a distributed clustering algorithm; the induced graph is usually not a grid, so the distributed LADA algorithm, rather than the grid-based one, is suitably modified and applied. The same time complexity as LADA is achieved, but the number of messages per iteration is reduced from $\Theta(n)$ to $\Theta(r^{-2})$.

In this paper, for ease of exposition we focus on synchronous algorithms without gossip constraints, i.e., in each time slot, every node updates its values based on its neighbors’ values in the previous iteration. Nonetheless, these algorithms can also be realized in a deterministic gossip fashion, by simulating at most $d_{\text{max}}$ matchings for each iteration, where $d_{\text{max}}$ is the maximum node degree. Also note that while most of our analysis is conducted on the geometric random graph, the algorithms themselves can generally be applied on any network topology.

Our paper is organized as follows. In Section II, we formulate the problem and review some important results in Markov chain theory. In Section III, we introduce the notion of lifting Markov chains and present two pseudo-algorithms for distributed consensus based on chain-lifting. In Section IV, the LADA algorithm for grid networks is proposed, which is then extended to a centralized algorithm for geometric random graphs. In Section V, we present the distributed LADA algorithm for wireless networks and analyze its performance. The C-LADA algorithm is treated in Section VI. Several important related works are discussed in Section VII. Finally, conclusions are given in Section VIII.
II. PROBLEM FORMULATION AND PRELIMINARIES

A. Problem Formulation

Consider a network represented by a connected graph $G = (V, E)$, where the vertex set $V$ contains $n$ nodes and $E$ is the edge set. Let vector $\mathbf{x}(0) = [x_1(0), \cdots, x_n(0)]^T$ contain the initial values observed by the nodes, and $x_{\text{ave}} = \frac{1}{n} \sum_{i=1}^{n} x_i$ denote the average. The goal is to compute $x_{\text{ave}}$ in a distributed and robust fashion. As we mentioned, such designs are basic building blocks for distributed and cooperative information processing in wireless networks. Let $\mathbf{x}(t)$ be the vector containing node values at the $t$th iteration. Without loss of generality, we consider the set of initial values $\mathbf{x}(0) \in \mathbb{R}^n_+$, and define the $\epsilon$-averaging time as

$$T_{\text{ave}}(\epsilon) = \sup_{\mathbf{x}(0) \in \mathbb{R}^n_+} \inf \left\{ t : \|\mathbf{x}(t) - x_{\text{ave}}\|_1 \leq \epsilon \|\mathbf{x}(0)\|_1 \right\} \tag{1}$$

where $\|\mathbf{x}\|_1 = \sum_i |x_i|$ is the $l_1$ norm\(^5\).

We will mainly use the geometric random graph [10], [11] to model a wireless network in our analysis. In the geometric random graph $G(n, r(n))$, $n$ nodes are uniformly and independently distributed on a unit square $[0, 1]^2$, and $r(n)$ is the common transmission range of all nodes. It is known that the choice of $r(n) \geq \sqrt{\frac{2 \log n}{n}}$ is required to ensure the graph is connected with high probability (w.h.p.)\(^6\) [10], [11].

B. Markov Chain Preliminaries

The averaging time of consensus algorithms evolving according to a stationary Markov chain is closely related to the chain’s convergence time. In this section, we briefly review two metrics that characterize the convergence time of a Markov chain, i.e., the mixing time and the fill time. For $\epsilon > 0$, the $\epsilon$-mixing time of an irreducible and aperiodic Markov chain $\mathbf{P}$ with stationary distribution $\pi$ is defined in terms of the total variation distance as [7]

$$T_{\text{mix}}(\mathbf{P}, \epsilon) \triangleq \sup_i \inf \left\{ t : \|\mathbf{P}^t(i, \cdot) - \pi\|_{TV} \triangleq \frac{1}{2} \|\mathbf{P}^t(i, \cdot) - \pi\|_1 \leq \epsilon \right\} = \sup_{\mathbf{p}(0)} \inf \left\{ t : \|\mathbf{p}(t) - \pi\|_1 \leq 2\epsilon \right\},$$

\(^4\)For the more general case $\mathbf{x}(0) \in \mathbb{R}^n$, the corresponding expression in (1) is $\|\mathbf{x}(t) - x_{\text{ave}}\|_1 \leq \epsilon \|\mathbf{x}(0) - \min_i x_i(0)\|_1$.
\(^5\)In the literature of distributed consensus, the $l_2$ norm $\|\mathbf{x}\|_2 = \sqrt{\sum_i |x_i|^2}$ has also been used in measuring the averaging time [1], [5]. The two metrics are closely related. Define $T_{\text{ave}, 2}(\epsilon) = \sup_{\mathbf{x}(0) \in \mathbb{R}^n} \inf \{ t : \|\mathbf{x}(t) - x_{\text{ave}}\|_2 \leq \epsilon \|\mathbf{x}(0)\|_2 \}$. It is not difficult to show that when $\epsilon = O \left( \frac{1}{n} \right)$, then $T_{\text{ave}, 2}(\epsilon) = O(T_{\text{ave}}(\epsilon))$.
\(^6\)with probability approaching 1 as $n \to \infty$
where \( p(t) \) is the probability distribution of the chain at time \( t \), and \( P^t(i, \cdot) \) is the \( i \)th row of the \( t \)-step transition matrix (i.e., \( p(t) \) given \( p(0) = e^T_i \)). The second equality is due to the convexity of the \( l_1 \) norm.

Another related metric, known as the fill time [12] (or the separate time [13]), is defined for \( 0 < c < 1 \) as

\[
T_{\text{fill}}(P, c) \triangleq \sup_i \inf \{ t : P^t(i, \cdot) > (1 - c)\pi \}. \tag{3}
\]

For certain Markov chains, it is (relatively) easier to obtain an estimate for \( T_{\text{fill}} \) than for \( T_{\text{mix}} \). The following lemma comes handy in establishing an upper bound for the mixing time in terms of \( T_{\text{fill}} \), and will be used in our analysis.

**Lemma 2.1:** For any irreducible and aperiodic Markov chain \( P \),

\[
T_{\text{mix}}(P, \epsilon) \leq \left\lfloor \log(\epsilon^{-1})/\log(c^{-1}) + 1 \right\rfloor T_{\text{fill}}(P, c). \tag{4}
\]

**Proof:** The lemma follows directly from a well-known result in Markov chain theory (see the fundamental theorem in Section 3.3 of [14]). It states that for a stationary Markov chain \( P \) on a finite state space with a stationary distribution \( \pi \), if there exists a constant \( 0 < c < 1 \) such that \( P(i, j) > (1 - c)\pi_j \) for all \( i, j \), then the distribution of the chain at time \( t \) can be expressed as a mixture of the stationary distribution and another arbitrary distribution \( r(t) \) as

\[
p(t) = (1 - c^t)\pi + c^t r(t). \tag{5}
\]

Thus

\[
\|p(t) - \pi\|_1 = c^t \|\pi - r(t)\|_1 \leq 2c^t. \tag{6}
\]

Now, for any irreducible and aperiodic chain, by (3), we have \( P^\tau(i, j) > (1 - c)\pi_j \) for any \( i, j \) when \( \tau > T_{\text{fill}}(P, c) \). It follows from the above that for any starting distribution,

\[
\frac{1}{2}\|p(t) - \pi\|_1 \leq e^{t/T_{\text{fill}}(P, c)}, \tag{7}
\]

and the desired result follows immediately by equating the right hand side of (7) with \( \epsilon \).

\( e_i \) is the vector with 1 at the \( i \)th position and 0 elsewhere.
III. FAST DISTRIBUTED CONSENSUS VIA LIFTING MARKOV CHAINS

The idea of the Markov chain lifting was first investigated in [8], [9] to accelerate convergence. A lifted chain is constructed by creating multiple replica states corresponding to each state in the original chain, such that the transition probabilities and stationary probabilities of the new chain conform to those of the original chain. Formally, for a given Markov chain $\mathbf{P}$ defined on state space $V$ with stationary probabilities $\pi$, a chain $\tilde{\mathbf{P}}$ defined on state space $\tilde{V}$ with stationary probability $\tilde{\pi}$ is a lifted chain of $\mathbf{P}$ if there is a mapping $f : \tilde{V} \rightarrow V$ such that

$$
\pi_v = \sum_{\tilde{v} \in f^{-1}(v)} \tilde{\pi}_{\tilde{v}}, \quad \forall v \in V \quad (8)
$$

and

$$
P_{uv} = \sum_{\tilde{u} \in f^{-1}(u), \tilde{v} \in f^{-1}(v)} \frac{\tilde{\pi}_{\tilde{u}}}{\pi_u} \tilde{P}_{\tilde{u}\tilde{v}}, \quad \forall u, v \in V. \quad (9)
$$

Moreover, $\mathbf{P}$ is called a collapsed chain of $\tilde{\mathbf{P}}$.

Given the close relationship between Markov chains and distributed consensus algorithms, it is natural to ask whether the nonreversible chain-lifting technique could be used to speed up distributed consensus in wireless networks. We answer the above question in two steps. First, we show that by allowing each node to maintain multiple values, mimicking the multiple lifted states from a single state, a nonreversible chain on a lifted state space can be simulated. In this section, we provide two pseudo-algorithms to illustrate this idea. With such pseudo-algorithms in place, the second step is to explicitly construct fast-mixing non-reversible chains that result in improved averaging times compared with existing algorithms. The latter part will be treated in Section IV and V, where we provide detailed algorithms for both grid networks as well as general wireless networks modeled by geometric random graphs.

Consider a wireless network modeled as $G(V, E)$ with $|V| = n$. A procedure that realizes averaging through chain-lifting is given in Pseudo-algorithm 1, where $\mathbf{P}$ is some $G$-conformant ergodic chain on $V$ with a uniform stationary distribution.

Lemma 3.1: Using Pseudo-algorithm 1, $x(t) \rightarrow x_{ave} \mathbf{1}$ and the averaging time $T_{ave}(\epsilon) \leq T_{mix}(\tilde{\mathbf{P}}, \epsilon/2)$.

Proof: Let $\tilde{\mathbf{p}}(t)$ be the distribution of $\tilde{\mathbf{P}}$ at time $t$, and $\tilde{\pi}$ the stationary distribution of $\tilde{\mathbf{P}}$. As $\tilde{\mathbf{P}}$ is ergodic and the linear iteration in Pseudo-algorithm 1 is sum-preserving, it can be shown that

8Although sometimes used interchangeably in related works, in this study it is better to differentiate between nodes (in a network) and states (in a Markov chain), since several states in the lifted chain correspond to a single node in a network.
Algorithm 1 Pseudo-Algorithm 1.

1) Each node \(v \in V\) maintains \(b_v\) copies of values \(y^1_v, \cdots, y^{b_v}_v\), the sum of which is initially set equal to \(x_v(0)\). Correspondingly, we obtain a new state space \(\tilde{V}\) and a mapping \(f : \tilde{V} \rightarrow V\) with the understanding that \(\{y^i_v\}_{i=1}^{b_v}\) can be alternatively represented as \(\{y_v\}_{\tilde{v} \in f^{-1}(v)}\).

2) At each time instant \(t\), each node updates its values based on the values of its neighbors. Let the vector \(y\) contain the copies of values of all nodes, i.e., \(y = [y^1_T, \cdots, y^{|V|}_T]^T\) with \(y_v = [y^1_v, \cdots, y^{b_v}_v]^T\). The values are updated according to the linear iteration \(y(t+1) = \tilde{P}^T y(t)\), where \(\tilde{P}\) is some ergodic chain on \(\tilde{V}\) lifted from \(P\).

3) At each time instant \(t\), each node estimates the average value by summing up all its copies of values: \(x_v(t) = \sum_{l=1}^{b_v} y^l_v(t)\).

\[
y(t) \rightarrow nx_{\text{ave}} \pi, \text{ and } x(t) \rightarrow x_{\text{ave}} 1 \text{ due to the lifting property (8) and the uniform stationary distribution of } P. \text{ Furthermore, we have } y(t) = nx_{\text{ave}} \tilde{p}(t), \text{ and for } t \geq T_{\text{mix}}(\tilde{P}, \epsilon/2),
\]

\[
\|x(t) - x_{\text{ave}} 1\|_1 = \sum_{v \in V} |x_v(t) - x_{\text{ave}}| = \sum_{v \in V} \sum_{l=1}^{b_v} |y^l_v(t) - x_{\text{ave}}| = \sum_{v \in V} \sum_{\tilde{v} \in f^{-1}(v)} (y_{\tilde{v}}(t) - \pi_{\tilde{v}} nx_{\text{ave}}) \leq \sum_{v \in V, \tilde{v} \in f^{-1}(v)} |y_{\tilde{v}}(t) - \pi_{\tilde{v}} nx_{\text{ave}}| = nx_{\text{ave}} \sum_{\tilde{v} \in V} |\tilde{p}_{\tilde{v}}(t) - \pi_{\tilde{v}}| \leq nx_{\text{ave}} \epsilon = \epsilon \|x(0)\|_1,
\]

where the third equality is by \(\pi_{\tilde{v}} = \sum_{\tilde{v} \in f^{-1}(v)} \pi_{\tilde{v}} = \frac{1}{n}, \forall v \in V\), the first inequality is by the triangular inequality, and the last inequality is by the definition of mixing time in (2).

From the above discussion, we see that for a wireless network modeled as \(G = (V, E)\), as long as we can find a fast-mixing chain whose collapsed chain is \(G\) conformant and has a uniform stationary distribution on \(V\), we automatically obtain a fast distributed averaging algorithm on \(G\). The crux is then to design such lifted chains which are typically nonreversible to ensure fast-mixing. While the fact that the collapsed Markov chain possesses a uniform stationary distribution facilitates distributed consensus, this does not preclude the possibility of achieving consensus by lifting chains with non-uniform stationary distributions. In fact, the non-uniformity of stationary distribution can be “smoothen out” by incorporating some auxiliary variables that asymptotically estimate the stationary distribution. Such a procedure allows us more flexibilities in finding a fast-mixing chain on a given graph. This idea is presented in Pseudo-algorithm 2, where \(P\) is some \(G\)-conformant ergodic chain on \(V\).

**Lemma 3.2:** a) Using Pseudo-algorithm 2, \(x(t) \rightarrow x_{\text{ave}} 1\).

b) Suppose for the collapsed chain \(P\), there exists some constant \(c' > 0\) such that the stationary distri-
Algorithm 2 Pseudo-Algorithm 2.

1) Each node \( v \in V \) maintains \( b_v \) pairs of values \((y^l_v, w^l_v)\), \( l = 1, \cdots, b_v \), whose initial values satisfy \( \sum_l y^l_v(0) = x_v(0) \) and \( \sum_l w^l_v(0) = 1 \). Correspondingly, we obtain a new state space \( \tilde{V} \) and a mapping \( f : \tilde{V} \to V \).

2) Let the vector \( y \) contain the copies \( y^l_v \) for all \( v \in V \) and \( l_v = 1, \cdots, b_v \), and similarly denote \( w \).

At each time instant, the values are updated with

\[
y(t + 1) = \tilde{P}^T y(t),
\]

\[
w(t + 1) = \tilde{P}^T w(t),
\]

where \( \tilde{P} \) is some ergodic chain on \( \tilde{V} \) lifted from \( P \).

3) At each time instant, each node estimates the average value by

\[
x_v(t) = \frac{\sum_{l=1}^{b_v} y^l_v(t)}{\sum_{l=1}^{b_v} w^l_v(t)}.
\]

Proof: a) Denote the stationary distribution of \( \tilde{P} \) by \( \tilde{\pi} \). By a similar argument as that of Lemma 3.1, \( \lim_{t \to \infty} y(t) = nx_{\text{ave}} \tilde{\pi} \) and \( \lim_{t \to \infty} w(t) = n \tilde{\pi} \). It follows that \( \lim_{t \to \infty} x(t) = x_{\text{ave}} \).

b) Let \( \tilde{p}(t) \) be the distribution of \( \tilde{P} \) at time \( t \). For any \( \epsilon > 0 \) and any constant \( 0 < c < 1 \), Lemma 2.1 says that there exists some time \( \tau = O \left( \log \epsilon^{-1} T_{\text{fill}}(\tilde{P}, c) \right) \), such that for any \( t \geq \tau \) and any initial distribution \( \tilde{p}(0) \),

\[
\| \tilde{p}(t) - \pi \|_1 \leq \frac{\epsilon(1-c)e'}{2}.
\]

Moreover, for \( t \geq T_{\text{fill}}(\tilde{P}, c) \), we have for \( \forall v \in V \),

\[
\sum_{\tilde{v} \in f^{-1}(v)} w_{\tilde{v}}(t) \geq (1-c) \sum_{\tilde{v} \in f^{-1}(v)} \tilde{\pi}_{\tilde{v}}(t)n = (1-c)\pi_v n \geq (1-c)e'.
\]
Thus, for $\forall t \geq \tau$,
\[
\|x(t) - x_{\text{ave}}\|_1 = \sum_{v \in V} |x_v(t) - x_{\text{ave}}| \\
= \sum_{v \in V} \left| \sum_{\tilde{v} \in f^{-1}(v)} y_{\tilde{v}}(t) - \sum_{\tilde{v} \in f^{-1}(v)} w_{\tilde{v}}(t) \right| x_{\text{ave}} \\
\leq \frac{1}{(1 - c)c'} \sum_{v \in V} \left( \sum_{\tilde{v} \in f^{-1}(v)} |y_{\tilde{v}}(t) - w_{\tilde{v}}(t)x_{\text{ave}}| \right) \\
\leq \frac{1}{(1 - c)c'} \sum_{\tilde{v} \in V} |y_{\tilde{v}}(t) - w_{\tilde{v}}(t)x_{\text{ave}}| \\
\leq \frac{1}{(1 - c)c'} \left[ \sum_{\tilde{v} \in V} |y_{\tilde{v}}(t) - n\tilde{\pi}_{\tilde{v}}x_{\text{ave}}| + \sum_{\tilde{v} \in V} |w_{\tilde{v}}(t) - n\tilde{\pi}_{\tilde{v}}x_{\text{ave}}| \right] \\
\leq \frac{1}{(1 - c)c'} \left[ \frac{\epsilon(1 - c)c'}{2} n x_{\text{ave}} + \frac{(1 - c)c'}{2} n x_{\text{ave}} \right] = \epsilon \|x(0)\|_1.
\]

Remark: It is clear that $w_{\tilde{v}}$ serves to estimate the scaling factor $n\tilde{\pi}_{\tilde{v}}$ at each iteration. Alternatively, a pre-computation phase can be employed where each node $v$ computes $\sum_{\tilde{v} \in f^{-1}(v)} \tilde{\pi}_{\tilde{v}}$. Then only the $y$ values need to be communicated.

In the above, we have proposed two pseudo-algorithms to illustrate the idea of distributed consensus through lifting Markov chains, leaving out the details of constructing fast-mixing Markov chains. In the following two sections, we present one efficient realization for each of these two pseudo-algorithms, on regular networks and geometric random networks, respectively.

IV. LADA ALGORITHM ON GRID

In this section, we present a LADA algorithm on a $k \times k$ grid. This algorithm utilizes the direction information (not the absolute geographic location) of neighbors to construct a fast-mixing Markov chain, and is a specific example of Pseudo-Algorithm 1 described in Section III. While existing works typically assumes a torus structure to avoid edge effects and simplify analysis, we consider the grid structure which is a more realistic model for planar networks, and explicitly deal with the edge effects. This algorithm is then extended to a centralized algorithm for general wireless network as modeled by a geometric random graph. Our analysis directly addresses the standard definition of mixing time in (2). Besides interest in its own right, results in this section will also facilitate our analysis in the following sections.
A. Algorithm

Consider a $k \times k$ grid. For each node $i$, denote its east, north, west and south neighbor (if exists) respectively by $N_i^0, N_i^1, N_i^2$ and $N_i^3$, as shown in Fig. 1. Each node $i$ maintains four values indexed according to the four directions counter-clockwise (see Fig. 1). The east, north, west and south value of node $i$, denoted respectively by $y_i^0, y_i^1, y_i^2$ and $y_i^3$, are initialized to

$$y_i^l(0) = \frac{x_i(0)}{4}, \quad l = 0, \ldots, 3. \quad (12)$$

At each time instant $t$, the east value of node $i$ is updated with

$$y_i^0(t + 1) = \left(1 - \frac{1}{k}\right) y_i^0(t) + \frac{1}{2k} \left(y_i^1(t) + y_i^3(t)\right). \quad (13)$$

That is, the east value of $i$ is updated by a weighted sum of the previous values of its west neighbor, with the majority $\left(1 - \frac{1}{k}\right)$ coming from the east value, and a fraction of $\frac{1}{2k}$ coming from the north value as well as the south value. If $i$ is a west border node (i.e., one without a west neighbor), then the west, north and south value of itself are used as substitutes:

$$y_i^0(t + 1) = \left(1 - \frac{1}{k}\right) y_i^0(t) + \frac{1}{2k} \left(y_i^1(t) + y_i^3(t)\right). \quad (14)$$

The above discussion is illustrated in Fig. 2. Intuitively the west value is “bounced back” when it reaches the west boundary and becomes the east value. As we will see, this is a natural procedure on the grid.
structure to ensure that the iteration evolves according to a doubly stochastic matrix which is desirable for averaging. Moreover, the fact that the information continues to propagate when it reaches the boundary is essential for the associated chain to mix rapidly. Similarly, the north value of \( i \) is updated by a weighted sum of the previous values of its south neighbor, with the majority coming from the north value, and so on. Each node then calculates the average of its four values as an estimate for the global average:

\[
x_i(t + 1) = \frac{1}{2} \sum_{l=0}^{3} y_l^i(t + 1).
\]

\[
B. \text{Analysis}
\]

Assume nodes in the \( k \times k \) grid are indexed by \( (x, y) \in [0, k - 1] \times [0, k - 1] \), starting from the south-west corner. The nonreversible Markov chain \( \tilde{P} \) underlying the above algorithm is illustrated in Fig. 3. Each state \( s \in S \) is represented by a triplet \( s = (x, y, l) \), with \( l \in \{E, W, N, S\} \) denoting the specific state within a node in terms of its direction. The transition probabilities of \( \tilde{P} \) for an east node are as follows (similarly for \( l \in \{N, W, S\} \)):

\[
\tilde{P}((x, y, E), (x + 1, y, E)) = 1 - \frac{1}{k}, \quad x < k - 1
\]

\[
\tilde{P}((x, y, E), (x, y, W)) = 1 - \frac{1}{k}, \quad x = k - 1
\]

\[
\tilde{P}((x, y, E), (x, y + 1, N)) = \tilde{P}((x, y, E), (x, y - 1, S)) = \frac{1}{2k}, \quad 0 < y < k - 1
\]

\[
\tilde{P}((x, y, E), (x, y, S)) = \tilde{P}((x, y, E), (x, y - 1, S)) = \frac{1}{2k}, \quad y = k - 1
\]

\[
\tilde{P}((x, y, E), (x, y + 1, N)) = \tilde{P}((x, y, E), (x, y, N)) = \frac{1}{2k}, \quad y = 0.
\]

It can be verified that \( \tilde{P} \) is doubly stochastic, irreducible and aperiodic. Therefore, \( \tilde{P} \) has a uniform stationary distribution on its state space, and so does its collapsed chain. Consequently each \( x_i(t) \rightarrow \cdots \)
Fig. 3. Nonreversible chain used in the LADA algorithm on a grid: outgoing probabilities for the states of node $i$ are depicted.

Moreover, since the nonreversible random walk $\tilde{P}$ most likely keeps its direction, occasionally makes a turn, and never turns back, it mixes substantially faster than a simple random walk (where the next node is chosen uniformly from the neighbors of the current node). Our main results on the mixing time of this chain, and the averaging time of the corresponding LADA algorithm are given below.

**Lemma 4.1:** The $\epsilon$-mixing time of $\tilde{P}$ is

a) $T_{\text{mix}}(\tilde{P}, \epsilon) = O(k \log(\epsilon^{-1}))$, for any $\epsilon > 0$;

b) $T_{\text{mix}}(\tilde{P}, \epsilon) = \Theta(k)$, for a sufficiently small constant $\epsilon$.

**Proof:**

a) See Appendix A. The key is to show that $T_{\text{fill}} = O(k)$. The desired result then follows from Lemma 2.1.

b) We are left to show that $T_{\text{mix}}(\tilde{P}, \epsilon) = \Omega(k)$ for a constant $\epsilon$ which is sufficiently small (less than $2/32$ in this case). For the random walk starting from $s_0 \in S$, denote by $\hat{s}_t$ the state it visits at time $t$ if it never makes a turn. Note that $\left(1 - \frac{1}{k}\right)^k$ is an increasing function in $k$, hence $\left(1 - \frac{1}{k}\right)^k \geq \frac{1}{4}$ for $k \geq 2$. 

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Thus we have for \( t \leq k \),
\[
\| \tilde{P}^t(s_0, \cdot) - \frac{1}{4k^2} \cdot 1 \|_1 \geq \| \tilde{P}^t(s_0, \hat{s}_t) - \frac{1}{4k^2} \|_1 = \left| \left( 1 - \frac{1}{k} \right)^t - \frac{1}{4k^2} \right | \ 
\]
(21)
\[
\geq \left( 1 - \frac{1}{k} \right)^k - \frac{1}{4k^2} \geq \frac{1}{4} - \frac{1}{16} = \frac{3}{16} > 2\epsilon, \ 
\]
(22)
for \( 0 < \epsilon < \frac{3}{32} \), where the second inequality follows from \( (1 - \frac{1}{k})^t \geq (1 - \frac{1}{k})^k \geq \frac{1}{4} \geq \frac{1}{4k^2} \). The result follows from the definition of mixing time in (2).

**Theorem 4.1:** For the LADA algorithm on a \( k \times k \) grid, a) \( T_{\text{ave}}(\epsilon) = O(k \log(\epsilon^{-1})) \) for any \( \epsilon > 0 \); b) \( T_{\text{ave}}(\epsilon) = \Theta(k) \) for a sufficiently small constant \( \epsilon \).

**Proof:** a) Follows from Lemma 3.1 and Lemma 4.1 a).

b) Note that the proof of Lemma 4.1 b) also implies that for \( k \geq 3 \), for any initial state \( s_0 \in S \), when \( t \leq k \), there is at least one state \( \hat{s} \in S \) with which \( \tilde{P}^t(s_0, \hat{s}) \geq \left( 1 - \frac{1}{k} \right)^k \geq \frac{8}{27} \). Suppose state \( \hat{s} \) is some state belonging to some node \( v \). Thus for \( t \leq k \ (k \geq 3) \)
\[
| x_v(t) - x_{\text{ave}} | = | \sum_{s \in f^{-1}(v)} \tilde{P}^t(s_0, s) - \frac{1}{k^2} | \cdot \| x(0) \|_1 \geq | \tilde{P}^t(s_0, \hat{s}) - \frac{1}{k^2} | \cdot \| x(0) \|_1 \geq \frac{5}{27} \| x(0) \|_1, \ 
\]
(23)
i.e., node \( v \) has not reached an average estimate in this scenario (when \( 0 < \epsilon < \frac{5}{27} \)).

C. A Centralized Grid-based Algorithm for Wireless Networks

The regular grid structure considered above does appear in some applications, and often serves as a first step towards modeling a realistic network. In this section, we explore a celebrated model for wireless networks, geometric random graphs, and present a centralized algorithm which achieves an \( \epsilon \)-averaging time of \( O(r^{-1} \log(\epsilon^{-1})) \) on \( G(n, r) \). The algorithm relies on a central controller to perform tessellation and clustering, and simulates the LADA algorithm on the grid proposed above on the resultant 2-d grid.

This is a common approach in literature (e.g., [10]), where the main purpose is to explore the best achievable performance in wireless networks, with implementation details ignored.

Assume that the unit area is tessellated into \( k^2 \equiv \frac{r\sqrt{2} \gamma^2}{2} \) squares (clusters). By this tessellation, a node in a given cluster is adjacent to all nodes in the four edge-neighboring clusters. Denote the number of nodes in a given cluster \( m \) by \( n_m \). Then for a geometric random graph \( n_m \geq 1 \) for all \( m \) w.h.p. [10]. One node in each cluster is selected as a cluster-head. Denote the index of the cluster where node \( i \) lies by \( C_i \).

For each cluster \( m \), denote its east, north, west and south neighboring cluster (if exists) respectively by \( N^0_m, N^1_m, N^2_m \) and \( N^3_m \). Every cluster-head maintains four values corresponding to the four directions from east to south clockwise, denoted respectively by \( y^0_m, y^1_m, y^2_m \) and \( y^3_m \) for cluster \( m \). In the initialization
stage, every node transmits its value to the cluster-head. The cluster-head of cluster $m$ computes the sum of the values within the cluster and initializes all its four values to

$$y_m^l(0) = \frac{1}{4} \sum_{C_i = m} x_i(0), \quad l = 0, \cdots, 3.$$  \hspace{1cm} (24)

At each time instant $t$, the cluster-heads of neighboring clusters communicate and update their values following exactly the same rules as the LADA algorithm on the grid. Each cluster-head then calculates the average of its four values as an estimate for the global average, and broadcasts this estimate to its members, so that every node $i$ obtains

$$x_i(t + 1) = \frac{k^2}{n} \sum_{l=0}^{3} y_{C_i}^l(t + 1).$$  \hspace{1cm} (25)

**Theorem 4.2:** The centralized algorithm has an $\epsilon$-averaging time $T_{\text{ave}}(\epsilon) = O(r^{-1} \log(\epsilon^{-1}))$ on the geometric random graph $G(n, r)$ with common transmission radius $r > \sqrt{\frac{20 \log n}{n}}$ w.h.p. Moreover, for a sufficiently small constant $\epsilon$, $T_{\text{ave}}(\epsilon) = \Theta(r^{-1})$.

**Proof:** We can appeal to uniform convergence in the law of large numbers using Vapnik-Chervonenkis theory as in [10] to bound the number of nodes in each cluster as follows:

$$\Pr \left( \max_{1 \leq m \leq k^2} \left| \frac{n_m}{n} - \frac{1}{k^2} \right| \leq \epsilon(n) \right) > 1 - \delta(n)$$  \hspace{1cm} (26)

when $n \geq \max\{3 \epsilon(n) \log 16 \epsilon(n), \frac{2}{\epsilon(n)} \log \frac{2}{\delta(n)}\}$. This is satisfied if we choose $\epsilon(n) = \delta(n) = \frac{4 \log n}{n}$. Thus we have for all $m$, $n_m \geq \frac{n}{k^2} - 4 \log n = \frac{n^2}{k^2} - 4 \log n$, which is at least 1 for sufficiently large $n$ if $r > \sqrt{\frac{20 \log n}{n}}$. In this case, we have that $c_2 \frac{n_m}{k^2} \leq n_m \leq c_1 \frac{n}{k^2}$ for all $m$ for some constants $c_1, c_2 > 0$ w.h.p. By Lemma 4.1 a), for any $\epsilon > 0$, there exists some $\tau = T_{\text{mix}}(\tilde{\mathbf{P}}, \frac{\epsilon}{2c_1}) = O(r^{-1} \log(\epsilon^{-1}))$ such that for all $t \geq \tau$,

$$\|x(t) - x_{\text{ave}}\|_1 \leq \frac{k^2}{n} \sum_{m=1}^{k^2} \frac{3}{n} \sum_{l=0}^{3} y_{m}^l(t) - x_{\text{ave}} \leq \frac{k^2}{n} \sum_{m=1}^{k^2} \frac{3}{n} \sum_{l=0}^{3} |y_{m}^l(t) - \frac{n x_{\text{ave}}}{4k^2}| \leq \epsilon \|x(0)\|_1,$$

where the last inequality follows a similar argument as in the proof of Lemma 3.1.

To prove the latter part of the theorem, note that $\|x(t) - x_{\text{ave}}\|_1 \geq c_2 \sum_{m=1}^{k^2} | \sum_{l=0}^{3} y_{m}^l(t) - \frac{n x_{\text{ave}}}{k^2} |$. The rest follows a similar argument as in the proof of Theorem 4.1 b).

In large dynamic wireless networks, it is often impossible to have a central controller that maintains a global coordinate system and clusters the nodes accordingly. In the following sections, we investigate some more practical algorithms, which can be applied to wireless networks with no central controller or global knowledge available to nodes.
V. DISTRIBUTED LADA ALGORITHM FOR WIRELESS NETWORKS

In practice, purely distributed algorithms requiring no central coordination are typically preferred. In this section, we propose a fully distributed LADA algorithm for wireless networks, which is an instantiation of Pseudo-Algorithm 2 in Section III. As we mentioned, while our analysis is conducted on $G(n, r(n))$, our design can generally be applied to any network topology.

A. Neighbor Classification

As the LADA algorithm on a grid, LADA for general wireless networks utilizes coarse location information of neighbors to construct fast-mixing nonreversible chains. Due to irregularity of node locations, a neighbor classification procedure is needed. Specifically, a neighbor $j$ of node $i$ is said to be a Type-$l$ neighbor of $i$, denoted as $j \in N_i^l$, if

$$\angle(X_j - X_i) \in \left(\frac{l\pi}{2} - \frac{\pi}{4}, \frac{l\pi}{2} + \frac{\pi}{4}\right) \quad l = 0, \ldots, 3,$$

(27)

where $X_i$ denotes the geometric location of node $i$ (whose accurate information is not required). That is, each neighbor $j$ of $i$ belongs to one of the four regions each spanning 90 degrees, corresponding to east (0), north (1), west (2) and south (3). Note that if $i \in N_i^l$, then $j \in N_i^{l+2 \mod 4}$. We denote the number of type $l$ neighbors for node $i$ by $d_i^l \triangleq |N_i^l|$ (except for boundary cases discussed below).

In literature, wireless networks are often modeled on a unit torus or sphere to avoid the edge effects in performance analysis [5], [10]. In our study, we explicitly deal with the edge effects by considering the following modification, as illustrated in Fig. 4. A boundary node is a node within distance $r$ from one of the boundaries, e.g., node $i$ in Fig. 4. For a boundary node $i$, we create mirror images of its neighbors with respect to the boundary. If a neighbor $j$ has an image located within the transmission range of $i$, node $j$ (besides its original role) is considered as a virtual neighbor of $i$, whose direction is determined by the image’s location with respect to the location of $i$. For example, in Fig. 4, node $j$ is both a north and a virtual east neighbor of $i$, and node $i$ is a virtual east neighbor of itself. Specifically, we use $\tilde{N}_i^0$ to denote the set of virtual east neighbors of an east boundary node $i$, and use $\hat{N}_i^0$ to denote the set of virtual east neighbors of a north or south boundary node $i$. Similarly, $\tilde{N}_i^1$ denotes the set of virtual north neighbors of a north boundary node $i$, and $\hat{N}_i^1$ denotes that of an east or west boundary node, and so on for virtual west and south neighbors. Informally, $\sim$ is used for the case the direction of the virtual neighbors and the boundary “match”, while $\hat{\ }$ is used for the “mismatch” scenarios. As we will see, they play different roles in the LADA algorithm. For example, in Fig. 4, we have $i, j, k \in \tilde{N}_i^0$, and $l \in \tilde{N}_i^3$. It can be shown that if $i \in \tilde{N}_j^l$, then $j \in \tilde{N}_i^l$, while if $i \in \hat{N}_j^l$, then $j \in \hat{N}_i^{l+2 \mod 4}$. For a
boundary node $i$, $d_i^l$ is instead defined as the total number of physical and virtual neighbors in direction $l$, i.e., $d_i^l \triangleq |\mathcal{N}_i^l| + |\hat{\mathcal{N}}_i^l| + |\tilde{\mathcal{N}}_i^l|$. With this modification, every type-$l$ neighborhood has an effective area $\frac{\pi r^2}{4}$, hence $d_i^l$ is roughly the same for all $i$ and $l$. We also expect that as $n$ increases, the fluctuation in $d_i^l$ diminishes. This is summarized in the following lemma, which will be used in our subsequent analysis.

Lemma 5.1: With high probability, the number of type $l$ neighbors of $i$ satisfies

$$d_i^l = \begin{cases} 
\Theta(nr^2) & \text{if } r > \sqrt{\frac{16 \log n}{\pi n}} \\
\frac{n \pi r^2}{4} (1 \pm O(r)) & \text{if } r = \Omega\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right) 
\end{cases}$$

(28)

Proof: We can appeal to the Vapnik-Chervonenkis theory as in [10] to bound the number of nodes in each cluster as follows:

$$\Pr\{\sup_{i,l} \left| \frac{d_i^l}{n} - \frac{\pi r^2}{4} \right| \leq \frac{4 \log n}{n} \} > 1 - \frac{4 \log n}{n}.$$  

(29)

Hence, we have $|d_i^l - \frac{n \pi r^2}{4}| \leq 4 \log n$ with probability at least $1 - \frac{4 \log n}{n}$ for all node $i$ and direction $l$. Therefore, if $r > \sqrt{\frac{16 \log n}{\pi n}}$, we have $d_i^l = \frac{n \pi r^2}{4} \left(1 \pm O\left(\frac{\log n}{nr^2}\right)\right) = \Theta(nr^2).$ If $r = \Omega\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right)$, we have $d_i^l = \frac{n \pi r^2}{4} \left(1 \pm O\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right)\right) = \frac{n \pi r^2}{4} (1 \pm O(r)).$ 

---

The stronger result regarding $r = \Omega\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right)$ is required for the LADA-U algorithm presented in Appendix C.

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B. Algorithm

The LADA algorithm for general wireless networks works as follows. Each node $i$ holds four pairs of values $\(y_l^i, w_l^i\)$, $l = 0, \cdots, 3$ corresponding to the four directions counter-clockwise: east, north, west and south. The values are initialized with

\[
y_l^i(0) = \frac{x_i(0)}{4}, \quad w_l^i(0) = \frac{1}{4}, \quad l = 0, \cdots, 3.
\]

At time $t$, each node $i$ broadcasts its four values. In turn, it updates its east value $y_0^i$ with

\[
y_0^i(t+1) = \sum_{j \in N_2^i} \frac{1}{d_{j_2}^0} \left[ (1-p)y_0^j(t) + \frac{p^2}{2} (y_1^j(t) + y_3^j(t)) \right],
\]

where $p = \Theta(r)$ is assumed. This is illustrated in Fig. 5. That is, the east value of node $i$ is updated by a sum contributed by all its west neighbors $j \in N_2^i$; each contribution is a weighted sum of the values of node $j$ in the last slot, with the major portion $\frac{1-p}{d_{j_2}^0}$ coming from the east value, and a fraction of $\frac{p^2}{2d_{j_2}^0}$ coming from the north as well as the south value.

As in the grid case, boundary nodes must be treated specially. Let us consider two specific cases:

1) If $i$ is a west boundary node (as shown in Fig. 6), then we must include an additional term

\[
\sum_{j \in N_2^w} \frac{1}{d_{j}^2} \left[ (1-p)y_2^j(t) + \frac{p^2}{2} (y_1^j(t) + y_3^j(t)) \right]
\]
in (31), i.e. values from both physical and virtual west neighbors (of the first category) are used.
Moreover, for the virtual west neighbors, the west rather than east values are used. This is similar
to the grid case, where the west values are bounced back and become east values when they reach
the west boundary, so that the information continues to propagate. The factor \( \frac{1}{d_j} \) rather than \( \frac{1}{d_j^0} \) is
adopted here to ensure the outgoing probabilities of each state of each node \( j \in \tilde{N}_i^2 \) sum to 1.

2) If \( i \) is a north or south boundary node (as shown in Fig. 7), however, the sum in (31) is replaced
with
\[
\sum_{j \in N_i^2 \cup \tilde{N}_i^2} \frac{1}{d_j} \left[ (1 - p)y_j^0(t) + \frac{p}{2} (y_j^1(t) + y_j^3(t)) \right],
\]
i.e., the east, north and south values of both physical and virtual west neighbors (of the second
category) are used. Note that \( \tilde{N}_i^2 \) are meant only for compensating the loss of neighbors for north or
south boundary nodes, so unlike the previous case, their east or west values continue to propagate
in the usual direction.

If \( i \) is both a west and north (or south) boundary node, the above two cases should be combined. The
purpose of introducing virtual neighbors described above is to ensure the approximate regularity of the
underlying graph of the associated chain, so that the randomized effect is evenly spread out over the
network. The north, west and south values, as well as the corresponding \( w \) values are updated in the
same fashion. Node \( i \) computes its estimate of \( x_{\text{ave}} \) with
\[
x_i(t+1) = \frac{\sum_{l=0}^{3} y_i^l(t+1)}{\sum_{l=0}^{3} w_i^l(t+1)}.
\]

The detailed algorithm is given in Algorithm 3\(^{10}\).

We remark that even the exact knowledge of directions is not critical for the LADA algorithm. For
example, if a neighbor \( j \) of node \( i \) is roughly on the border of two regions, it is fine to categorize \( j \)
to either region, as long as \( j \) categorizes \( i \) correspondingly (i.e., \( i \in N_j^{l+2} \) if \( j \in N_i^l \)).

C. Analysis

Denote \( y = [y_0^T, y_1^T, y_2^T, y_3^T]^T \), with \( y_l = [y_1^l, y_2^l, \cdots, y_n^l]^T \), and similarly denote \( w \). The above iteration
can be written as \( y(t+1) = \tilde{P}_1^T y(t) \) and \( w(t+1) = \tilde{P}_1^T w(t) \). Using the fact that if \( i \in N_j^{l+2} \) \( (\mod 4) \) if \( j \in N_i^l \),
then \( j \in N_i^{l+2} \) \( (\mod 4) \) \( \cup \tilde{N}_i^{l+2} \) \( (\mod 4) \), and if \( i \in \tilde{N}_j^l \), then \( j \in \tilde{N}_i^l \), it can be shown that each row in

\(^{10}\) We do not explicitly differentiate between the non-boundary and boundary cases, since the corresponding terms are
automatically zero for non-boundary nodes.
\(\bar{P}_1\) (i.e., each column in \(\bar{P}_1^T\)) sums to 1, hence \(\bar{P}_1\) is a stochastic matrix (see Fig. 8 for an illustration). On a finite connected 2-d network, the formed chain \(\bar{P}_1\) is irreducible and aperiodic by construction. Since the incoming probabilities of a state do not sum to 1 (see Eq. (31) and Fig. 5)\(^{11}\), \(\bar{P}_1\) is not doubly stochastic and does not have a uniform stationary distribution. The LADA algorithm for general wireless networks is a special case of the Pseudo-Algorithm 2 in Section III, and it converges to the average of node values by Lemma 3.2 a). In the rest of this section, we analyze the performance of LADA algorithm on geometric random graphs.

**Lemma 5.2:** On the geometric random graph \(G(n, r)\) with \(r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)\), with high probability, the Markov chain \(\bar{P}_1\) constructed in the LADA algorithm has an approximately uniform stationary distribution, i.e., for any \(s \in S\), \(\pi(s) = \Theta\left(\frac{1}{\sqrt{n}}\right)\), and \(T_{\text{fill}}(\bar{P}_1, c) = O(r^{-1})\) for some constant \(0 < c < 1\).

The proof is given in Appendix B. Essentially, we first consider the expected location of the random walk \(\bar{P}_1\) (with respect to the node distribution), which is shown to evolve according to the random walk \(\bar{P}\) on a \(k \times k\) grid with \(k = \Theta(r^{-1})\) when \(p = \Theta(r)\). Thus the expected location of \(\bar{P}_1\) can be anywhere.

\(^{11}\)Due to irregularity of the network, all west neighbors of a node don’t have exactly the same number of east neighbors.
Algorithm 3 LADA Algorithm

for $i = 1$ to $n$ do
  $y_l^i(0) \leftarrow x_i(0), w_l^i(0) \leftarrow 1, l = 0, 1, 2, 3$
end for

$p \leftarrow \frac{p}{2}, t \leftarrow 0$

while $\|x(t) - x_{ave}\|_1 > \epsilon$ do
  for $i = 1$ to $n$ do
    for $l = 0$ to $3$ do
      $y_l^i(t + 1) \leftarrow \sum_{j \in \mathcal{N}_i} \frac{1}{d_j^l} \left[ (1 - p)y_j^i(t) + \frac{p}{2} \left( x_j^i(t) + x_{j+3}^i(t) \right) \right] + \frac{1}{d_j^l} \left[ (1 - p)y_{j+1}^i(t) + \frac{p}{2} \left( x_{j+1}^i(t) + x_{j+3}^i(t) \right) \right]$
      $w_l^i(t + 1) \leftarrow \sum_{j \in \mathcal{N}_i} \frac{1}{d_j^l} \left[ (1 - p)w_j^i(t) + \frac{p}{2} \left( w_{j+1}^i(t) + w_{j+3}^i(t) \right) \right] + \frac{1}{d_j^l} \left[ (1 - p)w_{j+1}^i(t) + \frac{p}{2} \left( w_{j+1}^i(t) + w_{j+3}^i(t) \right) \right]$
    end for
    $x_i(t + 1) \leftarrow \frac{\sum_{l=0}^{3} y_l^i(t+1)}{\sum_{l=0}^{3} w_l^i(t+1)}$
  end for
  $t \leftarrow t + 1$
end while

Fig. 7. Update of east value of a north boundary node $i$: east value of virtual west neighbor $j \in \mathcal{N}_i^2$ is used.
Theorem 5.1: On the geometric random graph $G(n, r)$ with $r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$, the LADA algorithm has an $\epsilon$-averaging time $T_{ave}(\epsilon) = O(\frac{1}{\epsilon^2} \log(\frac{1}{\epsilon}))$ with high probability.

Proof: Since when $r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$, the Markov chain $\tilde{P}_1$ constructed in the LADA algorithm has an approximately uniform stationary distribution from Lemma 5.2, so does its collapsed chain. Thus Lemma 3.2 b) can be invoked to show that $T_{ave}(\epsilon) = O\left(T_{fill}(\tilde{P}_1, c) \log(\epsilon^{-1})\right) = O(\frac{1}{\epsilon^2} \log(\epsilon^{-1}))$.

We have also explored a variant of the LADA algorithm, called LADA-U, which is a realization of Pseudo-Algorithm 1. The nonreversible chain is carefully designed to ensure a uniform stationary distribution (accounting for the suffix “U”), by allowing transitions between the east and the west, as well as between the north and south state for each node. It can be shown that LADA-U can achieve the same scaling law in averaging time as LADA, but requiring a transmission range larger than the minimum connectivity requirement, mainly due to the induced diffusive behavior. In particular, a sufficient condition for the same scaling law as LADA to hold is $r = \Omega\left(\frac{\log n}{n} \frac{1}{3}\right)$. The LADA-U algorithm and its performance analysis are summarized in Appendix C for possible interest of the reader.
D. $T_{fill}$ Optimality of LADA Algorithm

To conclude this section, we would like to discuss the following question: what is the optimal performance of distributed consensus through lifting Markov chains on a geometric random graph, and how close LADA performs to the optimum? A straightforward lower bound of the averaging time of this class of algorithms would be given by the diameter of the graph, hence $T_{\text{ave}}(\epsilon) = \Omega(r^{-1})$. Therefore, for a constant $\epsilon$, LADA algorithm is optimal in the $\epsilon$-averaging time. For $\epsilon = O(1/n)$, it is not known whether the lower bound $\Omega(r^{-1})$ can be further tightened, and whether LADA achieves the optimal $\epsilon$-averaging time in scaling law. Nevertheless, we provide a partial answer to the question by showing that the constructed chain attains the optimal scaling law of $T_{\text{fill}}(\tilde{\mathbf{P}}, c)$ for a constant $c \in (0, 1)$, among all chains lifted from one with an approximately uniform stationary distribution on $G(n, r)$. For our analysis, we first introduce two invariants of a Markov chain, the conductance and the resistance. The conductance measures the chance of a random walk leaving a set after a single step, and is defined for the corresponding chain $\mathbf{P}$ as

$$
\Phi(\mathbf{P}) = \min_{S \subset V, 0 < \pi(S) < 1} \frac{Q(S, \bar{S})}{\pi(S)\pi(\bar{S})},
$$

where $\bar{S}$ is the complement of $S$ in $V$, $Q(A, B) = \sum_{i \in A} \sum_{j \in B} Q_{ij}$, and for edge $e = ij$, $Q(e) = Q_{ij} = \pi_i P_{ij}$ is often interpreted as the capacity of the edge in combinatorial research. The resistance is defined in terms of multi-commodity flows. A flow\(^ {12} \) in the underlying graph $G(\mathbf{P})$ of $\mathbf{P}$ is a function $f : \Gamma \rightarrow \mathbb{R}^+$ which satisfies

$$
\sum_{\gamma \in \Gamma_{uv}} f(\gamma) = \pi(u)\pi(v), \quad \forall u, v \in V, u \neq v
$$

where $\Gamma_{uv}$ is the set of all simple directed paths from $u$ to $v$ in $G(\mathbf{P})$ and $\Gamma = \bigcup_{u \neq v} \Gamma_{uv}$. The congestion parameter $R(f)$ of a flow $f$ is defined as

$$
R(f) \triangleq \max_e \frac{1}{Q(e)} \sum_{\gamma \in \Gamma; \gamma \ni e} f(\gamma).
$$

The resistance of the chain $\mathbf{P}$ is defined as the minimum value of $R(f)$ over all flows,

$$
R(\mathbf{P}) = \inf_f R(f).
$$

It has been shown that the resistance of an ergodic reversible Markov chain $\mathbf{P}$ satisfies $R(\mathbf{P}) \leq 16T_{\text{mix}}(\mathbf{P}, 1/8)$ [15]. This result does not readily apply to nonreversible chains. Instead, a similar result exists for $T_{\text{fill}}$, as given below.

\(^{12}\) An alternative and equivalent definition of a flow as a function on the edges of graphs can be found in [16].
Lemma 5.3: For any irreducible and aperiodic Markov chain $\mathbf{P}$, the resistance satisfies

$$T_{\text{fill}}(\mathbf{P}, c) \geq \frac{R(\mathbf{P})}{1 - c}. \quad (39)$$

Proof: Let $t = T_{\text{fill}}(\mathbf{P}, c)$. Let $\Gamma_{uv}^{(t)}$ denote the set of all (not necessarily simple) paths of length exactly $t$ from $u$ to $v$ in the underlying graph $G(\mathbf{P})$. $\Gamma_{uv}^{(t)}$ is nonempty by the definition of $T_{\text{fill}}$. For each $\gamma \in \Gamma_{uv}^{(t)}$, let $p(\gamma)$ denote the probability that the Markov chain, starting in state $u$, makes the sequence of transitions defined in $\gamma$, thus $\sum_{\gamma \in \Gamma_{uv}^{(t)}} p(\gamma) = P^t(u, v)$. For each $u, v$ and $\gamma \in \Gamma_{uv}^{(t)}$, set

$$f(\gamma) = \frac{\pi(u)\pi(v)p(\gamma)}{P^t(u, v)} \quad (40)$$

and set $f(\gamma) = 0$ for all other paths. Thus, $\sum_{\gamma \in \Gamma_{uv}^{(t)}} f(\gamma) = \pi(u)\pi(v)$. Now, by removing cycles on all paths, we can obtain a flow $f'$ (consisting of simple paths) from $f$ without increasing the throughput on any edge. The flow routed by $f'$ through $e$ is

$$f'(e) = \sum_{\gamma \in \Gamma; \gamma \ni e} f'(\gamma) \leq \sum_{u,v} \sum_{\gamma \in \Gamma_{uv}^{(t)}} \frac{\pi(u)\pi(v)p(\gamma)}{P^t(u, v)} \leq \frac{1}{1 - c} \sum_{u,v} \sum_{\gamma \in \Gamma_{uv}^{(t)}} \pi(u)p(\gamma), \quad (41)$$

where the second inequality follows from the definition of $T_{\text{fill}}$. The final double sum in (41) is precisely the probability that the stationary process traverses the oriented edge $e$ within $t$ steps, which is at most $tQ(e)$. It then follows

$$R(f') = \max_e \frac{f'(e)}{Q(e)} \leq \frac{t}{1 - c}. \quad (42)$$

Lemma 5.4: For the geometric random graph $G(n, r)$ with $r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$, the resistance of any $G$-conformant Markov chain with $\pi(v) = \Theta\left(\frac{1}{n}\right)$, $\forall v \in V$ satisfies the following with high probability: a) the conductance $\Phi(\mathbf{P}) = O(r)$, and b) the resistance $R(\mathbf{P}) = \Omega(r^{-1})$.

Proof: Consider dividing the square with a line parallel to one of its sides into two halves $S$ and $\bar{S}$ such that $\pi(S) > 1/4$ and $\pi(\bar{S}) > 1/4$, as illustrated in Fig. 9. Note that such a line always exists and needs not to be at the center of the square. A node in $S$ must lie in the shadowed region to have a neighbor in $\bar{S}$. For any such node $i$, $\sum_{j \in \bar{S}} P_{ij} \leq 1$. Applying the Chernoff bound [17], it can be shown that when $r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$, the number of nodes in the shadowed area is upper bounded by $2rn$ w.h.p. Therefore, we have

$$\Phi(\mathbf{P}) < \frac{Q(S, \bar{S})}{\pi(S)\pi(\bar{S})} \leq \frac{2rn \cdot \Theta\left(\frac{1}{n}\right) \cdot 1}{0.25 \cdot 0.25} = \Theta(r), \quad (43)$$

i.e., $\Phi(\mathbf{P}) = O(r)$ w.h.p. By the the max-flow min-cut theorem [15], [18], the resistance $R$ is related to the conductance $\Phi$ as $R \geq \frac{1}{\Phi}$, thus we have $R(\mathbf{P}) = \Omega(r^{-1})$ w.h.p.
Note that the resistance cannot be reduced by lifting [9]. Combining this fact with Lemma 5.3 and Lemma 5.4 yields the following.

**Theorem 5.2:** Consider a chain $\mathbf{P}$ on the geometric random graph $G(n, r) = (V, E)$ with $r = \Omega\left(\sqrt{\frac{\log n}{n}}\right)$ and $\pi(v) = \Theta\left(\frac{1}{n}\right)$, $\forall v \in V$. For any chain $\tilde{\mathbf{P}}$ lifted from $\mathbf{P}$ and any constant $0 < c < 1$, $T_{\text{fill}}(\tilde{\mathbf{P}}, c) = \Omega(r^{-1})$ with high probability.

The above shows that the constructed chain in LADA is optimal in the scaling law for the mixing parameter $T_{\text{fill}}$ for any chains lifted from one with an approximately uniform stationary distribution on $G(n, r)$.

**VI. CLUSTER-BASED LADA ALGORITHM FOR WIRELESS NETWORKS**

In Section IV-C, we have presented a centralized algorithm, where the linear iteration is performed on the 2-d grid obtained by tessellating the geometric random graph. Only the cluster-heads are involved in the message exchange. Therefore, compared to the purely distributed LADA algorithm, the centralized algorithm offers an additional gain in terms of the message complexity, which translates directly into power savings for sensor nodes. However, as we have mentioned previously, the assumption of a central controller with knowledge of global coordinates might be unrealistic. This motivates us to study a more general cluster-based LADA (C-LADA) algorithm which alleviates such requirements, and still reaps the benefit of reduced message complexity.

**A. C-LADA Algorithm**

The idea of C-LADA can be described as follows. The nodes are first clustered using a distributed clustering algorithm given in Appendix D, where no global coordinate information is required. Two
clusters are considered adjacent (or neighbors) if there is a direct link joining them. Assume that through some local information exchange, a cluster-head knows all its neighboring clusters. In the case that two clusters are joined by more than one links, we assume that the cluster-heads of both clusters agree on one single such link being activated. The end nodes of active links are called gateway nodes. The induced graph \( \tilde{G} \) from clustering is a graph with the vertex set consisting of all cluster-heads and the edge set obtained by joining the cluster-heads of neighboring clusters. In Fig. 10, we illustrate the induced graph as a result of applying our distributed clustering algorithm to a realization of \( G(300, r(300)) \), where \( r(n) = \sqrt{2 \log n} \).

As can be seen, the induced graph typically has an arbitrary topology. Neighbor classification on the induced graph is based on the relative location of the cluster-heads, according to a similar rule as described in Section V-A. Let \( \mathcal{N}_m^l \) denote the set of type-\( l \) neighboring clusters (including virtual neighbors) for cluster \( m \), and \( d_m^l = |\mathcal{N}_m^l| \). It can be shown that \( d_m^l \geq 1 \) for any \( m \) and \( l \) w.h.p.. Let \( C_i \) be the index of the cluster node \( i \) belongs to, and \( n_m \) be the number of nodes in cluster \( m \). It is convenient to consider another relevant graph \( \hat{G} = (\hat{V}, \hat{E}) \) constructed from the original network graph \( G = (V, E) \) as follows: for any \( i, j \in V \), \( (i, j) \in \hat{E} \) if and only if \( C_i \) and \( C_j \) are neighbors. Moreover, \( j \) is considered as a type-\( l \) neighbor of \( i \) if and only if \( C_j \) is a type-\( l \) neighboring cluster of \( C_i \). It is easy to see that nodes in the...
same cluster have the same set of type-\( l \) neighbors, and hence they would follow the same updating rule if the LADA algorithm is applied. Furthermore, nodes in the same cluster would have the same values at any time, if their initial values are the same. Note that the initial values in a given cluster can be made equal through a simple averaging at the cluster-head. The above allows updating a cluster as a whole at the cluster-head, saving the transmissions of individual nodes. For any cluster \( m \), let \( \hat{d}_m = \sum_{m' \in N_m} n_{m'} \) be the total number of nodes in the type-\( l \) neighboring clusters of \( m \), which is equal to the number of type-\( l \) neighbors of any node in cluster \( m \) in \( \hat{G} \).

Every cluster-head maintains four pairs of values \( (y_{m}^{l}, w_{m}^{l}) \), \( l = 0, \cdots, 3 \), initialized with \( y_{m}^{0}(0) = \sum_{C_{i} = m} x_i(0)/(4n_{m}) \), and \( w_{m}^{l}(0) = 1/4 \), \( l = 0, \cdots, 3 \). At time \( t \), the gateways nodes of neighboring clusters exchange values and forward the received values to the cluster-heads. The cluster-head of cluster \( m \) updates its east \( y \) value according to

\[
y_{m}^{0}(t+1) = \sum_{m' \in N_{m}^{2}} \frac{n_{m'}}{\hat{d}_{m}^{0}} \left[ (1-p)y_{m'}^{0}(t) + \frac{p}{2} \left( y_{m'}^{1}(t) + y_{m'}^{3}(t) \right) \right],
\]

and similarly for other \( y \) values and \( w \) values, and broadcasts them to its members. Every node computes the estimate of the average with \( x_{i}(t) = \frac{\sum_{l=0}^{3} y_{C_{i}}^{l}(t)}{\sum_{l=0}^{3} w_{C_{i}}^{l}(t)} \).

It can be verified that, the above C-LADA algorithm essentially realizes the LADA algorithm on graph \( \hat{G} \) with the above neighbor classification rule; for any node in cluster \( m \), the update rule in (44) is equivalent to the update rule in (31). It follows that \( x(t) \) converges to \( x_{\text{ave}} \) as \( t \to \infty \), and C-LADA also achieves an \( \epsilon \)-averaging time of \( O(r^{-1}\log(\epsilon^{-1})) \) on geometric random graphs.

**B. Message Complexity**

Finally, we demonstrate that C-LADA considerably reduces the message complexity, and hence the energy consumption. For LADA, each node must broadcast its values during each iteration, hence the number of messages transmitted in each iteration is \( \Theta(n) \). For C-LADA, there are three types of messages: transmissions between gateway nodes, transmissions from gateway nodes to cluster-heads and broadcasts by cluster-heads. Thus, the number of messages transmitted in each iteration is on the same order as the number of gateway nodes, which is between \( Kd_{\text{min}} \) and \( Kd_{\text{max}} \), where \( K \) is the number of clusters, and \( d_{\text{min}} \) and \( d_{\text{max}} \) are respectively the maximum and the maximum number of neighboring clusters in the network.

**Lemma 6.1:** Using the Distributed Clustering Algorithm in Appendix D, the number of neighboring clusters for any cluster \( m \) satisfies \( 4 \leq d_{m} \leq 48 \), and the number of clusters satisfies \( \pi^{-1}r^{-2} \leq K \leq 2r^{-2} \).
Proof: The lower bound \( d_m \geq 4 \) follows from \( d_m^l \geq 1 \) for any \( m \) and \( l \). Note that the cluster-heads are at least at a distance \( r \) from each other (see Appendix D). Hence, the circles with the cluster-heads as the centers and radius 0.5\( r \) are non-overlapping. Note also that, for a cluster \( m \), the cluster-heads of all its neighboring clusters must lie within distance 3\( r \) from the cluster-head of \( m \). Within the neighborhood of radius 3.5\( r \) of a cluster-head, there are no more than \( \left( \frac{3.5}{0.5} \right)^2 \) non-overlapping circles of radius 0.5\( r \). This means that the number of neighboring clusters is upper bounded by 48.

Consider the tessellation of the unit square into squares of side \( \frac{\sqrt{2}}{2} \). Thus, every such square contains at most one cluster-head, so there are at most 2\( r^{-2} \) clusters. On the other hand, in order to cover the whole unit square, there must be at least \( \pi^{-1} r^{-2} \) clusters.

The theorem below on the message complexity follows immediately.

**Theorem 6.1:** The \( \epsilon \)-message complexity, defined as the total number of messages transmitted in the network to achieve \( \epsilon \)-accuracy, is \( O(n r^{-1} \log(\epsilon^{-1})) \) for the LADA algorithm, and \( O(r^{-3} \log(\epsilon^{-1})) \) for the C-LADA algorithm with high probability in the geometric random graph \( G(n, r) \) with \( r = \Theta(\sqrt{\log n/n}) \).

As a side note, cluster-based algorithms haven also been designed based on reversible chains [19] to reduce the message complexity.

**VII. RELATED WORKS**

In this section, we review several relevant works reflecting recent development on distributed consensus. The reader is referred to [2] for a systematic treatment of distributed computation. Xiao and Boyd [1] derived necessary and sufficient conditions for the deterministic weight matrix \( W \) such that the linear iteration \( x(t+1) = Wx(t) \) asymptotically computes \( x_{\text{ave}}1 \) as \( t \to \infty \). They formulated the fastest linear averaging problem as a semi-definite program, which is convex when \( W \) is restricted to be symmetric. Finding the optimal symmetric \( W \) with non-negative weights is closely tied to the problem of finding the fastest mixing reversible Markov chain on the graph. Recently, another class of distributed consensus algorithms, the gossip algorithms have received much interest [20], [21], [5]. Under the gossip constraint, a node can communicate with at most one node at a time. In particular, the randomized gossip algorithm studied by Boyd et al. [5] realizes distributed averaging through asynchronous pairwise relaxation. On a geometric random graph with transmission radius \( \Theta\left(\sqrt{\log n/n}\right) \), the time complexity and message complexity to reach \( \epsilon \)-accuracy are respectively \( \Theta\left(n \log \epsilon^{-1}/\log n\right) \) and \( \Theta\left(n^2 \log \epsilon^{-1}/\log n\right) \). A recent work by Moalleimi and Roy [6] proposed consensus propagation, a special form of Gaussian belief propagation, as an alternative for distributed averaging. By avoiding passing information back to where it is received, consensus propagation suppresses to some extent the diffusive nature of a reversible
random walk. However, the gain of consensus propagation in time complexity over gossip algorithms quickly diminishes as the average node degrees grow, in which case the diffusive behavior is not effectively reduced. In comparison, our LADA algorithms realize distributed consensus with time complexity $O\left(n^{0.5}\log \epsilon^{-1}/\sqrt{\log n}\right)$ and message complexity as low as $O\left(n^{1.5}\log \epsilon^{-1}/(\log n)^{1.5}\right)$ on a connected geometric random graph.

While the above works studied either synchronous or asynchronous parallel algorithms, the work by Savas et al. [22] explored distributed computation of decomposable functions through sequential algorithms, where a node does not transmit messages until it is activated by another node. They proposed two algorithms, SIMPLE-WALK and COALESCENT, with which the transmission tokens follow a simple and a coalescing random walk respectively. Both algorithms provide gain in message complexity at a cost of time complexity compared with gossip algorithms. The geographic gossip algorithm proposed by Dimakis et al. [23] is another work along this line. Motivated by the observation that standard gossip algorithms can lead to a significant energy waste by repeatedly circulating redundant information, the geographic gossip algorithm reduces the message complexity by greedy geographic routing, for which an overlay network is built so that every pair of nodes can communicate. Note that such a modification entails the absolute location (coordinates) knowledge of the node itself and its neighbors. A notable recent work by Bénézit et al. [24] further improves the geographic gossip algorithm by allowing averaging along routing paths. Under the box-greedy routing scheme they propose, further reduction in time and message complexity is achieved. Both time and message complexity of the algorithms in [24] are essentially $\Omega(n \log \epsilon^{-1})$ on geometric random graphs. In comparison, the class of LADA algorithms we propose reduce time complexity by a factor of $O\left(\sqrt{n \log n}\right)$ and increase message complexity by a factor of $O\left(\sqrt{n}/(\log n)^{1.5}\right)$ to $O\left(\sqrt{n}/(\log n)\right)$, and does not require global coordination. The optimal tradeoff between time and message complexity of distributed consensus warrants further study.

The independent work by Jung and Shah [25] also explored nonreversible chains for fast distributed consensus. However, our scheme is considerably different from theirs. Their algorithm adopts the non-reversible lifting of an existing Markov chain as proposed in [9], which is constructed from a multi-commodity flow of the chain with minimum congestion. For each path in the multi-commodity flow (at least one path between each ordered pair of nodes), a new replica node (state) is created for each internal node of the path. Therefore, the state space of the new chain is of a size up to $n^3$. Moreover, to construct the chain each node in the network must have global knowledge of the network – in particular, the paths

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13On the contrary, our algorithm only requires direction knowledge of neighbors.
in the optimal multi-commodity flow that pass through itself. On the other hand, the chain used in our algorithm is formed in a distributed fashion exploiting only local information and simple computation, and the size of the state space is linear in \( n \). As a result, our algorithm is more robust to topology changes: when a node joins or leaves the network, only its neighbors need to update their local processing rules. Therefore, the class of LADA algorithms we propose is more suited for distributed implementation in dynamic large-scale networks.

VIII. CONCLUSION

We propose a class of Location-Aided Distributed Averaging (LADA) algorithms for grid networks and wireless networks, which achieve fast convergence via constructing nonreversible lifting of Markov chains. Our algorithms can realize an \( \varepsilon \)-averaging time of \( O(r^{-1} \log(\varepsilon^{-1})) \) for all transmission range \( r \) that guarantees network connectivity, a significant improvement over existing algorithms based on reversible chains. The cluster-based LADA (C-LADA) variant requires no central controller to perform clustering, while reaps the benefit of reduced message complexity. Our constructed chain attains the optimal scaling law in terms of an important mixing metric, the fill time [12], among all chains lifted from one with an approximately uniform stationary distribution on geometric random graphs.

APPENDIX

A. Proof of Lemma 4.1

We will show that by time \( t = 6k \), the random walk starting from any state visits every state with probability at least \( \frac{C}{4k^2} \) for some constant \( C > 0 \). The desired result then follows from Lemma 2.1. Recall that in Section IV. B., we denote each state \( s \in S \) by a triplet \( s = (x, y, l) \). To facilitate the analysis, we define an auxiliary parameter \( z \) for a state \( s \) as follows:

\[
z \triangleq \begin{cases} 
  x & l = E \\
  2k - x - 1 & l = W \\
  y & l = N \\
  2k - y - 1 & l = S.
\end{cases}
\]  

(45)

For example, the numbering for east and west states in a given row is illustrated in Fig. 11. Due to the circular numbering, a horizontal movement of the random walk that keeps the direction (and bounces back at the boundary) can be written as \( (y, z) \rightarrow (y, z + 1 \mod 2k) \), and similarly for a vertical movement. Note that by defining the function

\[
g(z) = \min(z, 2k - z - 1),
\]

(46)
we have $g(z) = x$ when $l \in \{E, W\}$, and $g(z) = y$ when $l \in \{N, S\}$.

Without loss of generality, we assume that the chain starts from some horizontal state $s_0 = (x_0, y_0, l_0)$ with $l_0 \in \{E, W\}$. Let $T_1, T_2, \cdots$, $(1 \leq T_1 \leq T_2 \leq \cdots)$ be the times that the random walk makes a turn. Let $s_t$ be the state the random walk visits at the $t$th step\footnote{In our notation, in the $t$th step, the random walk goes from state $s_{t-1}$ to $s_t$.}, and $A_t$ be the number of turns made by the random walk up to time $t$. In the following, we consider two cases: (1) a target state $s = (x, y, l)$ with $l \in \{E, W\}$, i.e., a horizontal state, and (2) a target state with $l \in \{N, S\}$, i.e., a vertical state, and show that at $t = 6k$, for both cases

$$\Pr\{s_t = s\} \geq \frac{C}{4k^2}.$$  

1) $s$ is a horizontal state. In this case, we focus on $A_t = 2$ (so $s_t$ is also a horizontal state), and show that

$$\Pr\{s_t = s\} \geq \Pr\{s_t = s, A_t = 2\} \geq \frac{C}{4k^2}. \quad (47)$$

Note that a horizontal state $s$ is fully characterized by $y$ and $z$ (since $x = g(z)$). Thus, the state at time 0 can be represented as $(y_0, z_0)$, as illustrated in Fig. 12. Now, consider the state at time $t$. First, observe that $y_t$ is determined only by the direction of the first turn at $T_1$, which may be towards north or south, as illustrated by the two states labeled with $T_1$ in Fig. 12. If the turn is towards north, we have

$$y_t = g(y_0 + T_2 - T_1 \pmod{2k}); \quad (48)$$

if it is towards south, we have

$$y_t = g(2k - 1 - y_0 + T_2 - T_1 \pmod{2k}) = g(-y_0 + T_2 - T_1 - 1 \pmod{2k}). \quad (49)$$
Second, observe that $z_t$ is determined only by the direction of the second turn at $T_2$, which may be the same as the one in which the random walk is moving at time $T_1 - 1$, or the opposite. In the former case (the two east states at time $T_2$ shown in Fig. 12), it can be shown (by observing the two periods $[1, T_1 - 1]$ and $[T_2, t]$ within which the random walk is traveling horizontally) that

$$z_t = z_0 + T_1 - 1 + (t - T_2 + 1) \pmod{2k} = z_0 + T_1 - T_2 + t \pmod{2k}; \quad (50)$$

in the latter case (the two west states at time $T_2$ shown in Fig. 12), we have

$$z_t = 2k - 1 - (z_0 + T_1 - 1) + (t - T_2 + 1) \pmod{2k} = -z_0 - T_1 - T_2 + t + 1 \pmod{2k}. \quad (51)$$

Therefore, we have at $t = 6k$,

$$\Pr\{s_t = s\} \geq \Pr\{s_t = s, A_t = 2\}$$

$$\geq \Pr\{g(y_0 + T_2 - T_1) = y \pmod{2k}, -z_0 - T_1 - T_2 + t + 1 = z \pmod{2k}, A_t = 2\}$$

$$+ \Pr\{g(-y_0 + T_2 - T_1 - 1) = y \pmod{2k}, -z_0 - T_1 - T_2 + t + 1 = z \pmod{2k}, A_t = 2\},$$
where the second inequality comes from picking two combinations of \( y_t \) and \( z_t \) out of the four possible combinations formed from (48) - (51). Assuming that \( g(i) = i \) (the case for \( g(i) = 2k-1-i \) can be similarly argued), and letting \( a = y - y_0, b = t - z_0 - z + 1 \) and \( c = y + y_0 + 1 \), we get

\[
\Pr\{s_t = s\} \geq \Pr\{T_2 - T_1 = a \pmod{2k}, T_1 + T_2 = b \pmod{2k}, A_t = 2\} + \Pr\{T_2 - T_1 = c \pmod{2k}, T_1 + T_2 = b \pmod{2k}, A_t = 2\}.
\]

Note that \( T_2 - T_1 \) and \( T_1 + T_2 \) must have the same parity, so we need to consider two cases: if \( a \) and \( b \) have the same parity, then there exists at least a pair of \( (T_1, T_2) \) with \( 1 \leq T_1 < T_2 \leq t \) (e.g., \( T_1 = \frac{b-a}{2} - 1 \pmod{2k} + 1 \) and \( T_2 = \frac{a+b}{2} - 1 \pmod{2k} + 2k + 1 \)) such that \( T_2 - T_1 = a \pmod{2k} \) and \( T_1 + T_2 = b \pmod{2k} \) are satisfied; if \( a \) and \( b \) have different parities, then \( c \) and \( b \) must have the same parity, and there exists at least a pair of \( (T_1, T_2) \) with \( 1 \leq T_1 < T_2 \leq t \) such that the second set of equations above is satisfied. Either of the two cases occurs with a probability \( \frac{1}{4k^2} \left(1 - \frac{1}{k}\right)^{t-2} \). Using the fact that \( (1 - \frac{1}{k})^k \geq 1/4 \) for \( k > 2 \), at \( t = 6k \) we get

\[
\Pr\{s_t = s\} \geq \frac{1}{4k^2} \left(1 - \frac{1}{k}\right)^{t-2} > \frac{2^{-12}}{4k^2}. \tag{52}
\]

2) \textbf{s is a vertical state.} We show that in this case it is sufficient to consider the case of \( A_t = 3 \). Similarly as above, a vertical state \( s \) is fully characterized by \( x \) and \( z \). Note that \( x_t \) is only determined by the direction of the second turn. Similar to (50) and (51) two possible values for \( x_t \) are given by

\[
x_t = \begin{cases} 
 g(z_0 + T_1 - T_2 + T_3 - 1 \pmod{2k}) \\
 g(-z_0 - T_1 - T_2 + T_3 \pmod{2k}).
\end{cases} \tag{53}
\]

Also \( z_t \) is only determined by the direction of the first turn and third turn. It can be shown that the four possible values of \( z_t \) are given by

\[
z_t = \begin{cases} 
 y_0 + t - T_1 + T_2 - T_3 + 1 \pmod{2k} \\
 -y_0 + t + T_1 - T_2 - T_3 \pmod{2k} \\
 -y_0 + t - T_1 + T_2 - T_3 \pmod{2k} \\
 y_0 + t + T_1 - T_2 - T_3 + 1 \pmod{2k}.
\end{cases} \tag{54}
\]
Therefore,
\[
\Pr\{s_t = s\} \geq \Pr\{s_t = s, A_t = 3\}
\]
\[
\geq \Pr\{z_0 + T_1 - T_2 + T_3 - 1 = x \pmod{2k}, \ y_0 + t + T_1 - T_2 - T_3 + 1 = z \pmod{2k}, \ A_t = 3\}
+ \Pr\{z_0 + T_1 - T_2 + T_3 - 1 = x \pmod{2k}, \ -y_0 + t + T_1 - T_2 - T_3 = z \pmod{2k}, \ A_t = 3\}
= \Pr\{T_3 - (T_2 - T_1) = a \pmod{2k}, \ T_3 + (T_2 - T_1) = b \pmod{2k}, \ A_t = 3\}
+ \Pr\{T_3 - (T_2 - T_1) = a \pmod{2k}, \ T_3 + (T_2 - T_1) = c \pmod{2k}, \ A_t = 3\},
\]
where the second inequality comes from picking two combinations out of eight possible combinations formed from (53) and (54), and in the last inequality, we have substituted \(a = x - z_0 + 1,\) \(b = y_0 + t - z + 1\) and \(c = -y_0 + t - z.\) Same as 1), we must consider two cases on parity. For \(a\) and \(b\) with the same parity, consider the \(2k\) triplets of \((T_1, T_2, T_3)\) given by
\[
(T_1, \frac{b-a}{2} - 1 \pmod{2k} + 1 + T_1, \frac{b+a}{2} - 1 \pmod{2k} + 1 + 4k), \quad T_1 = 1, 2, \ldots, 2k.
\]
It is obvious that any such triplet satisfies \(1 \leq T_1 < T_2 < T_3 \leq 6k,\) as well as the conditions in (55). For \(a\) and \(b\) with different parity, \(a\) and \(c\) must have the same parity, and similarly there exists at least \(2k\) valid triplets of \((T_1, T_2, T_3)\) satisfying the conditions in (56). Thus, for any target vertical state \(s,\) we can always find \(2k\) turning times \((T_1, T_2, T_3)\) with proper turning directions to reach \(s\) at \(t = 6k\) with probability
\[
\Pr\{s_t = s\} \geq 2k \cdot \frac{1}{8k^3} \left(1 - \frac{1}{k}\right)^{t-3} \geq \frac{2^{-12}}{4k^2}.
\]
This completes the proof.

B. Proof of Lemma 5.2

Assume the unit square is coordinated by \((x, y)\) with \(x, y \in [0, 1],\) starting from the south-west corner. Denote the state space of the chain \(\tilde{P}_1\) by \(\mathcal{S}.\) A state \(s \in \mathcal{S}\) is represented with a triplet \(s = (x, y, l)\) following the grid case in Appendix A. Define an auxiliary parameter \(z\) for a state \(s\) as follows:
\[
z \triangleq \begin{cases} 
  x & l = E \\
  2 - x & l = W \\
  y & l = N \\
  2 - y & l = S.
\end{cases}
\]
We will show that by the time \(t = 6k + 1,\) for any state \(s \in \mathcal{S},\) \(\Pr\{s_t = s\} \geq c_1 \pi(s)\) for some positive constant \(c_1.\)
Consider a movement of the random walk. Denote the distance traveled in the direction of movement, and that orthogonal to the direction of movement at time $t$ respectively by $\alpha_t$ and $\beta_t$, as shown in Fig. 13. Since nodes are randomly and uniformly distributed and the transition probability is uniform for all neighbors in the same direction, we can calculate the expected value of $\alpha_t$ and $\beta_t$ (with respect to the node distribution) as follows:

$$E(\alpha_t) = \frac{4}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} x^2 \cos \theta \, dx \, d\theta = \frac{4\sqrt{2}}{3\pi} r \triangleq \mu_\alpha,$$

$$E(\beta_t) = \frac{4}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} x^2 \sin \theta \, dx \, d\theta = 0.$$  

Similarly, their second-order moments can be readily computed as

$$E(\alpha_t^2) = \frac{4}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} x^3 \cos^2 \theta \, dx \, d\theta = \frac{\pi + \sqrt{2}}{4\pi} r^2,$$

$$E(\beta_t^2) = \frac{4}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} x^3 \sin^2 \theta \, dx \, d\theta = \frac{\pi - \sqrt{2}}{4\pi} r^2,$$

and the variances of $\alpha_t$ and $\beta_t$ are given by

$$\left(\frac{\pi + \sqrt{2}}{4\pi} - \frac{32}{9\pi^2}\right) r^2 \triangleq \sigma^2_\alpha,$$

$$\left(\frac{\pi - \sqrt{2}}{4\pi}\right) r^2 \triangleq \sigma^2_\beta.$$  

Note that $\alpha_t$ and $\beta_t$ are uncorrelated, i.e.,

$$E((\alpha_t - \mu_\alpha)(\beta_t)) = E(\alpha_t \beta_t) = \frac{4}{\pi r^2} \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} x^3 \cos \theta \sin \theta \, dx \, d\theta = 0.$$  

In the following, we assume $k = \lceil \frac{1}{\mu_\alpha} \rceil$ and the turning probability $p = \frac{1}{k} = \Theta(r)$.

Without loss of generality, we assume that the random walk starts from some arbitrary horizontal state $s_0 = (x_0, y_0, l_0)$ with $l_0 \in \{E, W\}$, $y_0 = a_0 \mu_\alpha$ for some $a_0 \in \{0, 1, \cdots, k-1\}$ and the corresponding $z_0 = b_0 \mu_\alpha$ for some $b_0 \in \{0, 1 \cdots, 2k-1\}$.$^{15}$ Similar to Appendix A, we need to consider two cases: the target state $s$ being a horizontal state and the target state $s$ being a vertical state. In the following, we will focus on the the former case, and the proof for the latter case is similar.

First consider the expected location $E(s_t)$ of the random walk at $t$. It depends only on the turning times and turning directions, and evolves according to the random walk $\tilde{P}$ on the $k \times k$ grid (see Section IV).  

$^{15}$Recall that a horizontal node is completely characterized by $y$ and $z$. The proof is essentially the same for non-integer $a_0$ and $b_0$, with a little more complicated notation.
Thus, according to Appendix A, at $t = 6k$, for any $a' \in \{0, 1, \ldots, k - 1\}$ and $b' \in \{0, 1, \ldots, 2k - 1\}$, we have
\[
\Pr\{E(y_t) = a'\mu_\alpha, E(z_t) = b'\mu_\alpha\} \geq \Pr\{E(y_t) = a'\mu_\alpha, E(z_t) = b'\mu_\alpha, A_t = 2\} \geq \frac{c_2}{4k^2}
\] (65)
for some $c_2 > 0$.

In order to obtain a lower bound for the probability of reaching a target horizontal state $s$ at $t = 6k + 1$, we first obtain a lower bound for the probability of reaching any ancestor of $s$ in the underlying graph of the chain at $t = 6k$. For example, consider an east state $s$ of node $i$ as in Fig. 13. Note that the effective west neighboring region of node $i$ covers a circular sector of 90 degrees (for boundary nodes virtual neighbors are considered). It can be shown that such a circular sector contains a square of side $\mu_\alpha$ as depicted in Fig. 13 (for boundary nodes the corresponding square is folded along the boundary).

Denote the set of east states in $\hat{N}_i^2 \cup \tilde{N}_i^2$ and west states in $\tilde{N}_i^2$ in this square by $\hat{S} = \{\hat{s} : \hat{y} \in \hat{Y}, \hat{z} \in \hat{Z}, \hat{l} \in \{E, W\}\}$, where generally for a non-boundary node, we have $\hat{Y} = [a\mu_\alpha, (a + 1)\mu_\alpha)$ and $\hat{Z} = [b\mu_\alpha, (b + 1)\mu_\alpha)$ for some $a \in [0, k - 2]$ and $b \in [0, 2k - 2]$, and $\hat{l} = l$ (the direction of the target state). In the following, we assume $i$ is not a boundary node for simplicity, but the proof extends easily

$\text{Fig. 13. Illustration of moving distances and target set}$

---

\[16\] If $p = \frac{c}{k}$ for some positive $c \neq 1$, then the expected location would evolve according to another chain which differs from $\tilde{P}$ only in the turning probability, and has the same scaling law in the mixing time as $\tilde{P}$.

\[17\] In the above example, if $i$ is a west boundary node, then the square under consideration is folded along the west boundary, such that $\tilde{Z} = [0, (1 - b)\mu_\alpha] \cup [2 - b\mu_\alpha, 2)$ for some $b \in (0, 1)$, with the latter corresponding to west states of nodes in $\tilde{N}_i^2$. Note that in all cases, both $\hat{Y}$ and $\hat{Z}$ consist of intervals with a total length $\mu_\alpha$. 

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to the boundary nodes.

We claim that at \( t = 6k \),

\[
\sum_{a'=0}^{k-1} \sum_{b'=0}^{2k-1} \Pr \left\{ s_t \in \mathcal{S} \mid E(y_t) = a' \mu_\alpha, E(z_t) = b' \mu_\alpha, A_t = 2 \right\} \geq c'
\]

for some constant \( c' \) w.h.p. Based on this result and (65), we have at \( t = 6k \),

\[
\Pr \{ s_t \in \mathcal{S} \} \geq \sum_{a'=0}^{k-1} \sum_{b'=0}^{2k-1} \Pr \left\{ s_t \in \mathcal{S} \mid E(y_t) = a' \mu_\alpha, E(z_t) = b' \mu_\alpha, A_t = 2 \right\} \cdot \Pr \{ A_t = 2, E(y_t) = a' \mu_\alpha, E(z_t) = b' \mu_\alpha \} \geq c' c_2 \frac{4n}{4k^2}.
\]

By Lemma 5.1, when \( r > \sqrt{\frac{16 \log n}{\pi n}} \), \( d_{max} \triangleq \max_{i,l} d_l^i \leq c_3 n r^2 \) for some constant \( c_3 > 0 \) w.h.p., thus we have

\[
\Pr \{ s_{6k+1} = s \} \geq \frac{1}{2} \sum_{\hat{s} \in \mathcal{S}} \frac{\Pr \{ s_{6k} = \hat{s} \}}{d_{max}} \geq \frac{1}{2} \frac{c_2}{c_3 n r^2} \frac{4n}{4k^2} \triangleq \frac{c_4}{4n}.
\]

(68)

Note that, the random walk \( \tilde{P} \) has a uniform stationary distribution on the \( k \times k \) grid. Using the argument as above, it can be shown that for any set \( \mathcal{S} \) containing states of the same type in a square of side \( \mu_\alpha \), the stationary probability of \( \tilde{P}_1 \) satisfies \( \pi(\mathcal{S}) = \frac{1}{4k^2} \), and consequently the stationary probability of any state of \( \tilde{P}_1 \) is lower bounded by \( \frac{c_5}{4k} \) for some \( c_5 > 0 \) (c.f.(68)). For an upper bound, note that in Fig. 13 the effective west neighboring region of \( i \) is also contained in an area \( A \) consisting of \( 2 \times 3 \) squares of side \( \mu_\alpha \). Let \( \mathcal{S}^0, \mathcal{S}^1 \) and \( \mathcal{S}^3 \) respectively denote the set of east states\(^{18} \), the set of north states and the set of south states of (physical and virtual) west neighbors of \( i \) that lie in \( A \). By Lemma 5.1, when \( r > \sqrt{\frac{16 \log n}{\pi n}} \), \( d_{min} \triangleq \min_{i,l} d_l^i \geq c_6 n r^2 \) w.h.p. Hence for any state \( s \),

\[
\pi(s) \leq (1 - p) \sum_{s \in \mathcal{S}^0} \frac{\pi(s)}{d_{min}} + \frac{p}{2} \sum_{s \in \mathcal{S}^1} \frac{\pi(s)}{d_{min}} + \sum_{s \in \mathcal{S}^3} \frac{\pi(s)}{d_{min}} \leq \left[ (1 - p) + \frac{p}{2} \cdot 2 \right] \frac{1}{c_6 n r^2} \cdot \frac{6}{4k^2} \triangleq \frac{c_7}{4n}.
\]

(69)

We conclude that the stationary distribution of \( \tilde{P}_1 \) is approximately uniform, i.e., for any \( s \in \mathcal{S} \),

\[
\frac{c_5}{4k} \leq \pi(s) \leq \frac{c_7}{4n}
\]

for some \( c_5, c_7 > 0 \). It follows from (68) that \( \Pr \{ s_{6k+1} = s \} \geq \frac{c_4}{4n} = c_1 \pi(s) \) w.h.p., which implies that the fill time of \( \tilde{P}_1 \) is \( T_{fill}(\tilde{P}_1, \epsilon) = O(r^{-1}) \) w.h.p.

We are left to verify the claim (66). It is sufficient to consider the case that the random walk makes two turns in first \( 6k \) steps, with the turning times \( T_1 \) and \( T_2 \). Denote the distance vector traveled at the \( t \)th step by

\[
\Lambda_t \triangleq \begin{cases} 
[\alpha_t \beta_t]^T & t \in [1, T_1) \cup [T_2, 6k] \\
[\beta_t \alpha_t]^T & t \in [T_1, T_2),
\end{cases}
\]

(70)

\(^{18}\) For nodes in \( \mathcal{N}_r^2 \), their west states are considered instead.
with mean
\[
E(\Lambda_t) \triangleq \mu_\Lambda = \begin{cases}
[\mu_\alpha, 0]^T & t \in [1, T_1) \cup [T_2, 6k] \\
[0, \mu_\alpha]^T & t \in [T_1, T_2),
\end{cases}
\] (71)
and covariance matrix (note \(\alpha_t\) and \(\beta_t\) are uncorrelated)
\[
\Sigma_\Lambda = \begin{cases}
\begin{bmatrix}
\sigma_\alpha^2 & 0 \\
0 & \sigma_\beta^2
\end{bmatrix} & t \in [1, T_1) \cup [T_2, 6k] \\
\begin{bmatrix}
\sigma_\beta^2 & 0 \\
0 & \sigma_\alpha^2
\end{bmatrix} & t \in [T_1, T_2).
\end{cases}
\] (72)
As the distance vectors in different steps are independent, the covariance matrix of the total distance vector \(\Lambda = \sum_{t=1}^{6k} \Lambda_t\) is given by
\[
\Sigma_{\Lambda|T_1,T_2} = \begin{pmatrix}
\sigma_{\alpha|T_1,T_2}^2 & 0 \\
0 & \sigma_{\beta|T_1,T_2}^2
\end{pmatrix},
\] (73)
where
\[
\sigma_{\alpha|T_1,T_2}^2 = [T_1 + (6k - T_2)]\sigma_\alpha^2 + (T_2 - T_1)\sigma_\beta^2 = (\sigma_\beta^2 - \sigma_\alpha^2)(T_2 - T_1) + 6k\sigma_\alpha^2
\] (74)
and
\[
\sigma_{\beta|T_1,T_2}^2 = [T_1 + (6k - T_2)]\sigma_\beta^2 + (T_2 - T_1)\sigma_\alpha^2 = (\sigma_\alpha^2 - \sigma_\beta^2)(T_2 - T_1) + 6k\sigma_\beta^2
\] (75)
are the respective variance of the total distance traveled horizontally and vertically in 6k steps. As \(\sigma_\beta^2 > \sigma_\alpha^2\), it is easy to verify that the maximum of \(\sigma_{\alpha|T_1,T_2}^2\) and \(\sigma_{\beta|T_1,T_2}^2\) (with respect to \(T_1\) and \(T_2\)) are the same:
\[
\sigma_{\alpha,max}^2 = \sigma_{\beta,max}^2 = \sigma_\alpha^2 + (6k - 1)\sigma_\beta^2.
\] (76)
Let
\[
\Lambda_{k,t} \triangleq \sum_{\Lambda|T_1,T_2}^{-1/2}(\Lambda_t - \mu_\Lambda) = \begin{cases}
\frac{(\alpha_t - \mu_\alpha)}{\sigma_{\alpha|T_1,T_2}} & t \in [1, T_1) \cup [T_2, 6k] \\
\frac{\beta_t}{\sigma_{\beta|T_1,T_2}} & t \in [T_1, T_2),
\end{cases}
\] (77)
we have \(E(\Lambda_{k,t}) = 0\) and \(\lim_{n \to \infty} \sum_{t=1}^{6k} E(\Lambda_{k,t}^T \Lambda_{k,t}) = I\), where \(I\) is the 2 \(\times\) 2 identity matrix. In addition, by defining \(E(Y; C) = E(Y_1C)\) with \(1_C\) being the indicator function of \(C\), for any \(\epsilon > 0\)
\[
\lim_{n \to \infty} \sum_{t=1}^{6k} E(\left|\Lambda_{k,t}\right|^2; \left|\Lambda_{k,t}\right| > \epsilon) = 0,
\] (78)
since $|\Lambda_{k,t}|$ is always less than $\epsilon$ when $n$ is sufficiently large such that $\max\{\sigma_{\alpha|r_1,r_2}, \sigma_{\beta|r_1,r_2}\} < \epsilon/2$. Then according to the multivariate Lindeberg-Feller Theorem ([26] Proposition 2.27), the conditional probability density function (PDF) of

$$\sum_{t=1}^{6k} \Lambda_{k,t} = \Sigma^{-1/2}_{\Lambda|T_1,T_2} \sum_{t=1}^{6k} (\Lambda_t - \mu_\Lambda) = \left[ \begin{array}{c} (z_{6k} - \mathbb{E}(z_{6k}))/\sigma_{\alpha|T_1,T_2} \\ (y_{6k} - \mathbb{E}(y_{6k}))/\sigma_{\beta|T_1,T_2} \end{array} \right] ,$$

(79)

given $T_1$ and $T_2$ converges in distribution to the standard multivariate normal distribution $\mathcal{N}(0, I)$.

Suppose $T_{(a',b')}$ is the set of turning times combination that result in $\mathbb{E}(z_t) = b'\mu_\alpha$, $\mathbb{E}(y_t) = a'\mu_\alpha$, and

$$\{T_{1,(a',b')}, T_{2,(a',b')}\} = \arg\min_{\{T_1, T_2\}} \Pr \{ z_t \in [b\mu_\alpha, (b+1)\mu_\alpha), y_t \in [a\mu_\alpha, (a+1)\mu_\alpha] | T_1, T_2 \}$$

for any $a \in [0, k-2]$ and $b \in [0, 2k-2]$. Define

$$\Pi(X; \Lambda, \Sigma) = \frac{1}{2\pi \sqrt{|\Sigma|}} \exp\left\{ -\frac{1}{2} (X - \Lambda)^T \Sigma^{-1} (X - \Lambda) \right\}$$

as the PDF value of the multivariate normal distribution $\mathcal{N}(\Lambda, \Sigma)$ at $X$, and (c.f. (73))

$$\Pi'_{(a',b')}(X) = \Pi(X; [b'\mu_\alpha a'\mu_\alpha]^T, \Sigma_{\Lambda|T_1,(a',b')}, T_{2,(a',b')}).$$

Then for any $a \in [0, k-2]$ and $b \in [0, 2k-2]$, we can always find a matrix (c.f. (76))

$$\Sigma_0 = \left( \begin{array}{cc} \sigma^2_{\alpha_0} & 0 \\ 0 & \sigma^2_{\beta_0} \end{array} \right)$$

satisfying

$$\frac{1}{2\pi \sqrt{|\Sigma_0|}} \leq \min_{a'=0,...,k-1,b'=0,1,...,2k-1} \left\{ \begin{array}{l} \Pi'_{(a',b')}( [b\mu_\alpha a\mu_\alpha]^T), \\ \Pi'_{(a',b')}( [(b+1)\mu_\alpha a\mu_\alpha]^T), \\ \Pi'_{(a',b')}( [b\mu_\alpha (a+1)\mu_\alpha]^T), \\ \Pi'_{(a',b')}( [(b+1)\mu_\alpha (a+1)\mu_\alpha]^T) \end{array} \right\} .$$

(80)

This allows us to define an auxiliary normal distribution with an arbitrary mean and covariance matrix $\Sigma_0$ whose maximal PDF value is less than the minimum PDF values of all $\Pr\{z_{6k}, y_{6k} | \mathbb{E}(z_{6k}) =$

which determine $\mathbb{E}(z_t)$ and $\mathbb{E}(y_t)$ (for fixed turning directions), but not vice versa. There may exist multiple combinations of $\{T_1, T_2\}$ which can result in the same $\{\mathbb{E}(z_t), \mathbb{E}(y_t)\}$.
directions, all initialized to random graph. In LADA-U, each node \( n \rightarrow \infty \)
We then give some performance analysis for LADA-U. Denote $y_i$ as in LADA, the iteration can be written as

$$y(t+1) = \tilde{P}_{2}y(t),$$

where $\tilde{P}_{2}$ is a doubly stochastic matrix through our design. The exchange weights for an east value of some node $i$ are illustrated in Fig. 14: a fraction $p_{2,i}$ of the east value goes to the north and south value of the same node respectively, a total fraction of $d_{0,i}d_{\max}(1-p)$ goes uniformly to the east values of $d_{0,i}$ east neighbors, and the remaining $(1-d_{0,i}d_{\max})(1-p)$ goes to the west value of node $i$. The transitions between the east and west state make up for the difference in $d_{0,i}$ and $d_{2,i}$, and ensures that the incoming probabilities for each state also sum to 1. While such a design guarantees that the associated chain has a uniform stationary distribution, it also introduces some diffusive behavior, hence the centralized performance can only be achieved with a larger $r$. In the following, we show that for LADA-U, $T_{\text{ave}}(\epsilon) = O(r^{-1}\log(\epsilon^{-1}))$ when the transmission radius $r = \Omega\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right)$ with high probability.

It can be shown that the expected location of the random walk $\tilde{P}_{2}$ evolves according to a random walk $\tilde{P}'$ on the $k \times k$ grid, where $k = \frac{1}{\mu_{n}} + 1$ as defined in Appendix B. $\tilde{P}'$ differs from $\tilde{P}$ used in Section IV in two aspects: 1) there are additional probabilities of moving between states of opposite directions corresponding to the same node; 2) a 90 degree turn is towards a state corresponding to the same node instead of the next node in the turning direction. Recall that from Lemma 5.1, when $r = \Omega\left(\left(\frac{\log n}{n}\right)^{\frac{1}{3}}\right)$, we have $d_{l,i}^{l} = \frac{\pi n r}{4}(1 \pm O(1))$ for all $i$ and $l$ w.h.p. Thus for each move, the probability that the random walk $\tilde{P}'$ keeps the direction is at least $(1-p)\frac{d_{\min}}{d_{\max}} = (1-1/k)(1-\Omega(r)) > 1 - \frac{c}{k}$ for some constant $c$. 

Fig. 14. The Markov chain used in LADA-U: outgoing probabilities (solid lines) and incoming probabilities (dotted lines) for the east state are depicted.
$c_1 > 1$ w.h.p. During the first $6k$ moves, the probability that the random walk $\tilde{P}'$ makes exactly two 90 degree turns towards given directions at given times $T_1$ and $T_2$, and keeps direction for the remaining moves is at least $\frac{1}{4k^2} (1 - \frac{c_1}{k})^{6k-2} \geq \frac{2^{-12c_1}}{4k^2}$. Then, following the argument in Appendix A, if the random walk $\tilde{P}'$ starts from an east or west state, any east or west state can be reached with probability at least $\frac{2^{-12c_1}}{4k^2}$ in $6k$ steps (note that the modification in the 90 degree turns only causes constant shifts in the expressions of $s_i$, and does not affect the result). The case for north and south states can be similarly argued, and we conclude that the state distribution of the random walk $\tilde{P}'$ is approximately uniform at $t = 6k$ w.h.p. Then, following the analysis in Appendix B, it can be shown that the exact location of random walk $\tilde{P}_2$ is also approximately uniform at $t = 6k$, which by the uniformity of the stationary distribution of $\tilde{P}_2$ implies that the $\epsilon$-mixing time of $\tilde{P}_2$, as well as the $\epsilon$-averaging time of LADA-U is $O(r^{-1} \log(\epsilon^{-1}))$ w.h.p.

D. Distributed Clustering

We assume each node $i$ has an initial seed $s_i$ which is unique within its neighborhood. This can be realized through, e.g., drawing a random number from a large common pool, or simply using nodes’ IDs. From time 0, each node $i$ starts a timer with length $t_i = s_i$, which is decremented by 1 at each time instant as long as it is greater than 0. If node $i$’s timer expires (reaches 0), it becomes a cluster-head, and broadcasts a “cluster_initialize” message to all its neighbors. Each of its neighbors with a timer greater than 0 signals its intention to join the cluster by replying with a “cluster_join” message, and also sets the timer to 0. If a node receives more than one “cluster_initialize” messages at the same time, it randomly chooses one cluster-head and replies with the “cluster_join” message. At the end, clusters are formed such that every node belongs to one and only one cluster. The uniqueness of seeds within the neighborhood ensures that cluster-heads are at least of distance $r$ from each other. We assume that clusters are formed in advance and the overhead is amortized over the multiple computations. The detailed algorithm is given in Algorithm 4.
Algorithm 4 Distributed Clustering

\[ K \leftarrow 0 \{K: \text{number of clusters}\} \]

for all \( i \in V \) do

\[ t_i \leftarrow s_i \]

end for

repeat

for all \( i \) with \( t_i > 0 \) do

\[ t_i \leftarrow t_i - 1 \]

if \( t_i = 0 \) then

\[ K \leftarrow K + 1, \ C_K \leftarrow \{i\} \{C_k: \text{nodes in cluster } k\} \]

for all \( j \in N_i \) and with \( t_j > 0 \) do

\[ t_j \leftarrow 0, \ C_K \leftarrow C_K \cup \{j\} \]

end for

end if

end for

until \( \bigcup_k C_k = V \)

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