Energy and momentum entanglement in parametric downconversion

Pablo L. Saldanha
Departamento de Física, Universidade Federal de Pernambuco, 50670-901, Recife, PE, Brazil

C. H. Monken
Departamento de Física, Universidade Federal de Minas Gerais, Caixa Postal 702, 30161-970, Belo Horizonte, MG, Brazil

We present a simple treatment of the phenomenon of spontaneous parametric downconversion consisting of the coherent scattering of a single pump photon into an entangled photon pair inside a nonlinear crystal. The energy and momentum entanglement of the quantum state of the generated twin photons are seen as a consequence of the fundamental indistinguishability of the time and the position in which the photon pair is created inside the crystal. We also discuss some consequences of the photon entanglement.

PACS numbers: 42.50.Ar, 03.65.Ud

I. INTRODUCTION

The generation of twin photons inside a nonlinear crystal by the phenomenon of spontaneous parametric downconversion (SPDC) has been extremely useful for studying fundamental aspects of quantum mechanics and to physically implement quantum information protocols in recent decades. This usefulness is due to the photon pair being entangled in many degrees of freedom, including energy, momentum, angular momentum and polarization. For a pair of particles in an entangled state Bell’s theorem states that we cannot associate an objective reality to each particle, and this has profound consequences for the way we view nature. Entanglement is also useful in quantum information science, permitting the execution of quantum algorithms that are more efficient and secure than classical algorithms.

In this paper we derive an expression for the quantum state of the twin photons generated in the process of SPDC using a simplified version of the example presented in our recent work in which the interaction between light and matter is treated using the Bialinicki-Birula–Sipe photon wave function formalism. (For didactical discussions of the photon wave function formalism, see Refs. 11 and 12.) The reader interested in a more formal treatment should consult Ref. 8. The main idea is that when a single photon is converted into two photons inside a nonlinear crystal, there is a fundamental uncertainty in the location and the time when the photon pair is generated. This fact requires that we coherently sum all possible probability amplitudes for this event. The interference of amplitudes for generation at different positions leads to energy entanglement while the interference of amplitudes for generation at different times leads to momentum entanglement.

The remainder of this paper is organized as follows. In Sec. II we discuss the physical arguments of our approach and arrive at a general expression for calculating the twin-photon wave function. In Sec. III we compute the twin-photon wave function using reasonable approximations, and in Sec. IV we discuss some consequences of the system entanglement. Finally, in Sec. V we present some concluding remarks.

II. PARAMETRIC DOWNCONVERSION AS A SCATTERING PHENOMENON

If an electromagnetic wave interacts with a small transparent object that responds linearly to the electric field, such that the induced electric dipole moment of the object is proportional to the incident electric field, the resultant scattered electromagnetic field is a superposition of the incident field with the field generated by the oscillating dipole moment of the object. The scattered field will in general have a diffraction pattern from which we can deduce some properties of the scatterer. If the incident electromagnetic field is composed by only one photon and we use a quantum language, we can say that there is a probability amplitude for the incident photon to be instantaneously absorbed and re-emitted by the object that must be coherently summed with the probability amplitude for the photon to pass without any interaction with the object in order to compute the wave function of the scattered photon. The state of the transparent object before and after the interaction with the photon is the same, such that no correlation between the state of the object and the state of the photon appears.

This instantaneous absorption and re-emission of a photon can be interpreted as the absorption of the incident photon inducing the object to have an oscillating dipole moment and this oscillating dipole moment creating the scattered photon. For instance, the propagation of an electromagnetic wave through a transparent linear medium can be seen as a superposition of the incident wave being transmitted directly through the medium with the waves generated by the oscillating charges in the medium. Each component of the resultant wave propagates at the speed of light in vacuum, but the coherent superposition of these components causes the resultant wave to propagate with a reduced velocity.
In the phenomenon of parametric downconversion, light interacts with a nonlinear medium whose nonlinear polarization is proportional to the square of the incident electric field (or to the product of different components of the electric field). A nonlinear scatterer like this can absorb two photons and emit one. We can interpret the phenomenon as the induced nonlinear dipole oscillations creating one photon, but these oscillations occurring at the expense of the absorption of two photons from the incident field. This situation corresponds to the phenomenon of second-harmonic generation. By symmetry it must also be possible that the nonlinear scatterer does the opposite—emits two photons while absorbing one. Such a process corresponds to parametric downconversion.

If we have a medium with a small nonlinear scatterer placed at a fixed position $\mathbf{r}'$, the interaction between an incident electromagnetic field and this nonlinear scattering element leads to a probability amplitude for one incident pump photon to be converted into two photons during its passage through the medium. This situation is illustrated in Fig. 1, where photon 1 is observed at position $\mathbf{r}_1$ and photon 2 at position $\mathbf{r}_2$, both at time $t$. Of course, there is also a probability amplitude that no downconversion occurs, which is what happens in most cases, but from now on we will assume we are dealing with the case when a photon pair is created.

We assume that the incident photon has a wave function $\psi_p(\mathbf{r}',t')$ in the nonlinear scatterer element and that the downconversion occurs at time $t'$, such that the probability amplitude of generating the twin photons at time $t'$ is proportional to $\psi_p(\mathbf{r}',t')$. If we assume that both photons are created at a precise position $\mathbf{r}'$ and time $t'$, then according to the uncertainty relations there must be a large uncertainty in their momenta $\hbar \mathbf{k}_i$ and energies $\hbar \omega_i = \hbar c k_i/ n_i$. Here, $\mathbf{k}_i$ is the wavevector, $\omega_i$ the angular frequency, and $n_i$ the refractive index for photon $i (= \{1,2\})$ in the propagation medium, $k_i = |\mathbf{k}_i|$, $\hbar$ is Planck's constant divided by $2\pi$, and $c$ is the speed of light in vacuum. Thus, to compute the total probability amplitude for finding photons 1 and 2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at time $t$, we must coherently add the probability amplitudes for the propagation of photon 1 from $(\mathbf{r}',t')$ to $(\mathbf{r}_1,t)$ with all possible wavevectors $\mathbf{k}_1$ and angular frequencies $\omega_1 = k_1 c / n_1$, and similarly for photon 2.

Now, when propagating from $(\mathbf{r}',t')$ to $(\mathbf{r}_1,t)$, a photon in a plane-wave mode with wavevector $\mathbf{k}_1$ and angular frequency $\omega_1$ accumulates a phase $\phi = \mathbf{k}_1 \cdot (\mathbf{r}_1 - \mathbf{r}') - \omega_1 (t - t')$ in the probability amplitude. So we can write the total probability amplitude of finding the generated photons in the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at time $t$ as:

$$A_c(\mathbf{r}_1,\mathbf{r}_2,t;\mathbf{r}',t') \propto \psi_p(\mathbf{r}',t') \int d^3k_1 \int d^3k_2 \exp\left(i \mathbf{k}_1 \cdot (\mathbf{r}_1 - \mathbf{r}') - i \omega_1 (t - t')\right) \exp\left(i \mathbf{k}_2 \cdot (\mathbf{r}_2 - \mathbf{r}') - i \omega_2 (t - t')\right).$$

(1)

Here and elsewhere in this paper, unless explicitly stated the integration is carried out over the entire domain of the integration variables.

In the above probability amplitude, we are coherently superimposing possibilities where the energy and momentum of photons 1 and 2 have a sum that is different from the energy and momentum of the incident photon, thus being possibilities that do not conserve energy or momentum. This treatment is closely related to the Feynman path-integral formalism, and the conservation of energy and momentum in the process appears somewhat surprisingly as a consequence of the coherent sum of all possibilities. The point is that possibilities that do not conserve say, energy, will interfere destructively among themselves, as will become clearer in the remainder of the paper.

Let us now consider that instead of a single nonlinear scatterer we have a continuous set of nonlinear scatterers forming a nonlinear crystal, as depicted in Fig. 2. For simplicity, we will assume that the linear response of the medium outside the crystal is the same as inside, but the nonlinear response is absent. In this situation, there is a fundamental quantum indistinguishability of the position and the time in which the photon pair is created in the process of parametric downconversion. This condition forces us to coherently sum the probability amplitudes of generating the photons in any part of the crystal and at any time to find the total probability amplitude of finding photons 1 and 2 at the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ at time $t$. Figure 2 illustrates the probability amplitudes for generating the photon pair at the spacetime points $(\mathbf{r}_1, t_1)$ and $(\mathbf{r}_2, t_2)$. The twin-photon wave function $\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2,t)$ is proportional to this total probability amplitude, and we can write

$$\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2,t) \propto \int_V d^3\mathbf{r}' \int_{-\infty}^t dt' A_c(\mathbf{r}_1,\mathbf{r}_2,t;\mathbf{r}',t'),$$

(2)

where the spatial integral is taken over the crystal volume $V$, and $A_c(\mathbf{r}_1,\mathbf{r}_2,t;\mathbf{r}',t')$ is given by Eq. (1).

Equations (1) and (2) are the essential equations of our treatment so we discuss their physical meaning once again. The twin-photon wave function is obtained from the coherent scattering of a single pump photon into a photon pair inside the nonlinear crystal. Because there is a fundamental indistinguishability of the position and time in which the photon pair is created, the probability amplitudes for generation at different times and positions must be coherently added, leading to the volume and
time integrals in Eq. (2). The probability amplitude for a precise creation time and position is given by Eq. (1), which takes into account the fact that these photons must have large uncertainties in their energies and momenta. Thus, all possibilities for different energies and momenta are coherently added, taking into account the corresponding phases accumulated during the propagation from the creation (spacetime) point to the observation points.

In the following section we will calculate the twin-photon wavefunction by evaluating the integrals in Eq. (2) using reasonable approximations.

III. CALCULATING THE WAVE FUNCTION OF THE TWIN PHOTONS

To calculate the wave function of the twin photons using Eq. (2), we first make some assumptions. If the incident pump photon is almost monochromatic, we can approximately write its position-space wave function from the momentum-space wave function as in the traditional quantum mechanics of massive particles. The result is

\[
\psi_p(r', t') \propto \int d^3k_p \phi_p(k_p)e^{i\mathbf{k}_p \cdot \mathbf{r}' - i\omega_p t'},
\]

where we write the decomposition in terms of the wavevectors \( \mathbf{k}_p \) and angular frequencies \( \omega_p = \hbar \mathbf{k}_p / c / n_p \) (instead of the momenta \( \hbar \mathbf{k}_p \) and energies \( \hbar \omega_p \)) to simplify the notation. Decomposing \( \mathbf{k}_p \equiv q_p + \sqrt{k_p^2 - q_p^2} \hat{\mathbf{z}} \), with \( q_p \) being the component of \( \mathbf{k}_p \) in the \( xy \)-plane, we note that \( q_p \) and \( \omega_p \) completely determine \( \mathbf{k}_p \). Thus, we can write the momentum-space wave function in terms of \( q_p \) and \( \omega_p \), changing the integrals in Eq. (3) from \( d^3k_p \) to integrals in \( dq_p \) and \( d\omega_p \), giving

\[
\psi_p(r', t') \propto \int dq_p \int d\omega_p \phi_p(q_p, \omega_p)e^{i\mathbf{k}_p \cdot \mathbf{r}' - i\omega_p t'}.
\]

This decomposition in terms of \( q_p \) and \( \omega_p \) is useful when the incident pump photon is in a beam mode that propagates nearly parallel to the \( z \)-direction, in the paraxial regime. We will assume that we are operating in the paraxial regime so that \( \phi_p(q_p, \omega_p) \) has non-negligible values only for \( q_p \ll k_p \). Writing \( r' \equiv \mathbf{r}' + z' \hat{\mathbf{z}} \), where \( z' \) is the component of \( r' \) in the \( xy \)-plane, we have

\[
k_p \cdot r' = q_p \cdot \mathbf{r}' + \sqrt{k_p^2 - q_p^2} z'.
\]

We will use a similar notation for the wavevectors of photons 1 and 2. We will also consider that the nonlinear crystal has \( x \)- and \( y \)-dimensions much larger than the width of the beam mode so that the \( x' \) and \( y' \) integrals can be extended up to infinity in Eq. (2), and a small dimension in the \( z \) direction. Under these approximations, the wave function of the twin photons from Eq. (2) can be written as

\[
\psi(2)(r_1, r_2, t) \propto \int dq_1 \int d\omega_1 \int dq_2 \int d\omega_2 \int dq_p \int d\omega_p \int dq'_p \int d\omega'_p \phi_p(q_p, \omega_p) \phi'_p(q'_p, \omega'_p) e^{i\mathbf{k}_p \cdot \mathbf{r}_1 - i\omega_p t_1} e^{i\mathbf{k}_p \cdot \mathbf{r}_2 - i\omega_p t_2} e^{i\mathbf{k}_p \cdot \mathbf{r}'_1 - i\omega_p t'_1} e^{i\mathbf{k}_p \cdot \mathbf{r}'_2 - i\omega_p t'_2}.
\]

where the integral in \( z' \) gives a constant if we also have \( k_{1z} \gg q_{1z} \) and \( k_{2z} \gg q_{2z} \). In the appendix we discuss the relation between the omitted integral in \( z' \) and the phase matching condition for the efficient generation of photon pairs. The integral in \( \mathbf{r}' \) results in a term proportional to the 2-dimensional delta function \( \delta(2)(q_p - q_1 - q_2) \). If the time light takes to propagate from the crystal to the observation points \( r_1 \) and \( r_2 \) is greater than the duration of the incident photon pulse, the \( t' \) integral can be extended to \( t' = \infty \), resulting in a term proportional to \( \delta(\omega_p - \omega_1 - \omega_2) \). So the wave function of the twin photons can be written as

\[
\psi(2)(r_1, r_2, t) \propto \int dq_1 \int d\omega_1 \int dq_2 \int d\omega_2 \phi_p(q_1 + q_2, \omega_1 + \omega_2) e^{i\mathbf{k}_p \cdot r_1 - i\omega_1 t_1} e^{i\mathbf{k}_p \cdot r_2 - i\omega_2 t_2}.
\]
way of the delta function $\delta(\omega_p - \omega_1 - \omega_2)$ being a consequence of the $t'$ integral in Eq. (9). In the same way, conservation of momentum is a consequence of the coherent superposition of the probability amplitudes of the pair creation at any position inside the crystal, by way of the 2-dimensional delta function $\delta(z)$, being a consequence of the $\rho'$ integral in Eq. (3). These conservation laws can be obtained simply from the interference of the probability amplitudes associated with indistinguishable situations.

IV. ENERGY AND MOMENTUM ENTANGLEMENT OF THE SYSTEM

The twin-photon wave function of Eq. (7) cannot be written as a product of a wave function for photon 1 and a wave function for photon 2. This condition characterizes the entanglement of the system. Thus, even if the incident photon has narrow energy ($\hbar\omega_p$) and momentum ($\hbar k_p$) distributions, we see that the sum of the energies of the generated photons is equal to the energy of the incident photon, even though the energy of each photon can assume a large range of values. Similarly, the sum of the momenta of the generated photons is equal to the momentum of the incident photon, even though the momentum of each photon can also assume a large range of values. This situation characterizes a state where the twin photons are highly entangled in both energy and momentum.

An entangled state of the form of Eq. (7) was used by Einstein, Podolsky, and Rosen (EPR) in their famous paper on the completeness of quantum mechanics. Let us consider the wave function for the $x$-component of the photon momenta when the incident photon is a plane wave propagating in the $z$-direction. In this case, we can write $\phi(\omega_p, p_z) \propto \delta(p_z - p_2)$, with $p_z = hq_{ix}$. Of course, in a realistic situation the delta function must be seen as an approximation for a very narrow distribution. On the other hand, if we consider the position-space wave function at the exit face of the crystal (assumed to be extremely thin), we have $\psi(x_1, x_2) \propto \delta(x_1 - x_2)$, because the twin photons are born at the same position. Again, the delta function must be seen as an approximation.

Consider now that the generated photons reach two observers Alice and Bob who are placed in arbitrarily separated regions of space. If Bob measures the $x$-momentum component of his photon and obtains the value $P$, we can state with certainty that if Alice measures the $x$-momentum component of her photon she will obtain the value $-P$. So EPR says there is an element of physical reality associated with the momentum of Alice’s photon. On the other hand, if Bob measures the $x$-position of his photon using lenses to project an image of the crystal in the observation plane and obtains the value $X$, we can state with certainty that if Alice measures the $x$-position of her photon in a similar way she will also obtain the value $X$, and EPR says that there is an element of physical reality associated with the position of Alice’s photon. However, if Bob is arbitrarily far away from Alice, his measurement cannot affect her photon in any way, so EPR argues that Alice’s photon should have elements of physical reality associated with both position and momentum. However, in quantum mechanics two observables represented by noncommuting operators cannot have definite and (in principle) predictable values simultaneously. Therefore, EPR argues that quantum mechanics is not a complete theory because “hidden variables” not considered by the theory would be essential to guarantee the simultaneous reality for the position and momentum of Alice’s photon in the gedanken experiment discussed above.

Bohr’s reply to the EPR argument was that the measurements of position or momentum made by Bob are mutually incompatible experiments. Bob’s choice of which experiment he will perform on his photon determines different types of predictions he can make for experiments made by Alice. And there is no experiment that Alice can perform on her photon which would reveal the experiment performed by Bob. Bohr argued that quantum mechanics is indeed a complete theory because it can predict the probability of the experimental results of any combination of compatible experiments, which is the actual objective of the theory (such that there are no paradoxes).

This discussion continued on philosophical grounds until the seminal work of John Bell. Bell showed that there are quantum entangled states with correlations between two parties that are stronger than what is allowed by “hidden variables” theories. The fact that experiments have confirmed the quantum mechanical predictions (although there are still some loopholes) has changed the way we view the world. We cannot have a consistent local and realistic description of nature because in an entangled state like Eq. (7), we cannot attribute reality to each of two separated photons. We must consider either that Bob’s measurement instantaneously changes the state of Alice’s photon, violating locality (but not permitting any instantaneous transmission of information between Bob and Alice), or that the experimental results of the measurements performed by Alice do not depend only on the properties of her photon (independently of Bob’s photon) and on her measuring apparatus (independently of Bob’s apparatus), violating any reasonable definition of reality for her photon properties. Accessible discussions of these fundamental aspects of quantum mechanics can be found in chapter 6 of Ref. 5 and in Ref. 6.

A physical implementation of the EPR state was performed by Howell and co-workers using the twin photons produced by parametric downconversion as described above. The high degree of entanglement in Eq. (7) renders the twin photons generated by parametric downconversion an extremely valuable resource for testing such fundamental aspects of quantum mechanics. Moreover,
entangled states have correlations among parties of the system that are stronger than what is allowed by classical physics, and some quantum information protocols take advantage of these quantum correlations to be more secure or efficient than their classical counterparts. The twin photons generated by parametric downconversion have also been extensively used in the implementation of these quantum information protocols.

V. CONCLUSIONS

We derived an expression for the quantum state of the twin photons generated in the process of parametric downconversion by a scattering procedure that coherently sums the probability amplitudes for the photon pair to be generated at any position inside the crystal and at any time. The quantum state we find is equivalent to the one obtained using the traditional perturbative approach to calculate the Hamiltonian evolution of an electromagnetic field interacting with a nonlinear medium. Our treatment, however, is more intuitive, and provides useful physical insight to the problem. The twin-photon state is highly entangled in both energy and momentum, making this system very useful for experimental tests of fundamental aspects of quantum mechanics and for the implementation of quantum information protocols. Many of the exciting properties of the system entanglement can be verified in undergraduate laboratories. A more formal treatment of the work presented here can be found in Ref. 8.

Acknowledgments

The authors acknowledge Júlia E. Parreira for very useful comments on the manuscript. This work was supported by the Brazilian agencies CNPq, FAPEMIG and FACEPE.

Appendix: Phase matching in parametric downconversion

In this appendix we discuss the necessary phase matching for the efficient generation of photon pairs in parametric downconversion.

The phase matching condition can be obtained from Eq. (5). Approximating $\sqrt{k_i^2 - q_i^2} \approx k_i - q_i^2 / (2k_i)$, we have

$$I \propto \int_{-L/2}^{L/2} dz' e^{i(k_p - k_1 - k_2)z'} e^{-i(q_p^2/k_p - q_1^2/k_1 - q_2^2/k_2)z'/2},$$

(A.1)

where the width of the crystal in the z direction is L. The efficiency of photon-pair generation is proportional to $I$, so we must have $k_p = k_1 + k_2$ to obtain collinear generation, in which the generated photons propagate close to the z-axis; otherwise the oscillations of the first exponential above with $z'$ will decrease the value of $I$. The wavenumber of each photon can be written as $k_i(\omega_i) = \omega_i n_i(\omega_i)/c$, where $n_i(\omega_i)$ is the refractive index for photon $i$ with angular frequency $\omega_i$. Let us assume that each of the photons is post-selected in a relatively narrow range of frequencies such that the refractive index for each photon can be considered a constant $n_i$; this can be accomplished with the use of interference filters in the photon detectors that are used to detect the twin photons. Because Eq. (7) states that the sum of the frequencies of the generated photons is equal to the frequency of the incident photon, the phase matching condition for collinear generation can be written as $n_p = (n_1 + n_2)/2$. However, the inherent dispersion of the material makes the refractive index increase with frequency, so that in an isotropic medium we have $n_p > (n_1 + n_2)/2$, thus making it impossible to efficiently generate photon pairs. This problem is solved with the use of a nonlinear and birefringent material such that the refractive index also depends on the polarization of the photons, giving $n_p = (n_1 + n_2)/2$ for photons with different polarization states. Type-I parametric downconversion corresponds to the case in which both generated photons have polarization orthogonal to the polarization of the incident photon, and type-II parametric downconversion has one of the generated photons with polarization orthogonal to, and the other one parallel to, the incident photon.

In cases where $k_p - k_1 - k_2 = -\alpha < 0$, we can have an efficient generation of photon pairs that propagate in directions making angles with the z-axis such that $q_i^2/k_i + q_1^2/k_1 = 2\alpha$ and the second exponential in Eq. (A.1) cancels the first ($q_p^2/k_p$ is assumed to be negligible). For this reason the generated photons are usually emitted around cones. In any case we have $k_{1z} + k_{2z} \approx k_{pz}$, where $k_{iz}$ corresponds to the z-component of the wavenumber of photon $i$.

Another important approximation that we make in our treatment is to consider that the integrals in $dq_1$ and $dq_2$ in Eq. (5) can be computed without considering the dependence of $I$ on $q_1$ and $q_2$ in Eq. (A.1). This is a good approximation for crystals with small width $L$, because the accumulated phase in the second exponential in Eq. (A.1) becomes negligible once the phase matching condition is established.
J. E. Sipe, “Photon wave functions,” Phys. Rev. A 11, 210403 (2004).

M. B. James and D. J. Griffiths, “Why the speed of light is reduced in a transparent medium,” Am. J. Phys. 60, 309–313 (1992).

R. N. C. Pfeifer, T. A. Nieminen, N. R. Heckenberg, H. Rubinsztain-Dunlop, “Colloquium: Momentum of an electromagnetic wave in dielectric media,” Rev. Mod. Phys. 79, 1197–1216 (2007).

P. L. Saldanha, “Division of the momentum of electromagnetic waves in linear media into electromagnetic and material parts,” Opt. Express 18, 2258–2268 (2010).

A. Einstein, B. Podolsky, and N. Rosen, “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?,” Phys. Rev. 47, 777–780 (1935).

N. Bohr, “Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?,” Phys. Rev. 48, 696–702 (1935).

J. S. Bell, “On the Einstein Podolsky Rosen paradox,” Physics 1, 195–200 (1964).

A. Aspect, J. Dalibard, and G. Roger “Experimental Test of Bell’s Inequalities Using Time-Varying Analyzers,” Phys. Rev. Lett. 49, 1804–1807 (1982).

L. Mandel and E. Wolf, Optical Coherence and Quantum Optics (Cambridge, New York, 1995), chapter 22.

C. K. Hong and L. Mandel, “Theory of parametric frequency down conversion of light,” Phys. Rev. A 31, 2409–2418 (1985).

C. H. Monken, P. H. S. Ribeiro, and S. Pádua, “Transfer of angular spectrum and image formation in spontaneous parametric down-conversion,” Phys. Rev. A 57, 3123–3126 (1998).

D. Dehlinger and M. W. Mitchell, “Entangled photons, nonlocality, and Bell inequalities in the undergraduate laboratory,” Am. J. Phys. 70, 903–910 (2002).

E. J. Galvez, C. H. Halbrow, M. J. Pysher, J. W. Martin, N. Courtemanche, L. Heilig, and J. Spencer, “Interference with correlated photons: Five quantum mechanics experiments for undergraduates,” Am. J. Phys. 73, 127–140 (2005).

J. A. Carlson, M. D. Olmstead, and M. Beck, “Quantum mysteries tested: An experiment implementing Hardy’s test of local realism,” Am. J. Phys. 74, 180–186 (2006).

B. J. Pearson and D. P. Jackson, “A hands-on introduction to single photons and quantum mechanics for undergraduates,” Am. J. Phys. 78, 471–484 (2010).

E. J. Galvez, “Qubit quantum mechanics with correlated-photon experiments,” Am. J. Phys. 78, 510–519 (2010).

It is important to stress that the form $\hbar k$ is not the only possibility for the momentum of a photon with wavevector $k$ in a medium, as discussed in Refs. 3 and 10. There are many different ways of dividing the total momentum of an electromagnetic wave in a linear medium into electromagnetic and material parts, all of which are compatible with momentum conservation, and for each chosen division the photon momentum will have a different expression. The form $\hbar k$ is the simplest one for the present case, which is why we have used it.