Non-Hermitian nodal-line semimetals

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We investigate non-Hermitian nodal-line semimetals in the presence of a particle gain-and-loss perturbation. It is found that this perturbation will split the original nodal ring into two exceptional rings (ERs). Some of the ERs may annihilate eventually when further increasing the strength of the perturbation beyond a critical value. The topology of the bulk-band structure is characterized by two different topological invariants defined for a one-dimensional loop in the momentum space, namely, the vorticity and the winding number, both of which are shown to take half-integer [integer] values when an odd [even] number of ERs thread through the loop. The conventional bulk-surface correspondence in nodal-line semimetals is found to break down, where the surface zero-energy flat bands are no longer bounded by projections of bulk ERs. Interestingly, a macroscopic fraction of the bulk eigenstates can be localized near the surface, thus leading to significantly enhanced surface density of states, which creates an opportunity to study interaction effects in non-Hermitian systems.

I. INTRODUCTION

Recently, the studies on topological materials, including topological insulators1–3, topological superconductors4–6, and topological semimetals7–25, have profoundly deepened our understandings of symmetries and topology. Topological materials can be characterized by corresponding topological invariants calculated from bulk band structures, which ensure the existence of gapless boundary states through the celebrated bulk-boundary correspondence. Among various topological materials, nodal-line semimetals have attracted much interest, and have been intensively studied both theoretically26–35 and experimentally36–38. They have band degeneracies along lines in the momentum space, and possess exotic drumhead-like surface states or flat bands, which hold great promises for the realizations of surface superconductivity and surface magnetism when considering electron-electron correlation39–46.

Very recently, there has been a growing interest in non-Hermitian extensions of topological phases. Non-Hermiticity is ubiquitous in a diverse range of situations including open quantum systems47–51, optical systems with gain and loss52–55, and interacting or disordered systems56–60. The interplay between non-Hermiticity and topology may lead to quite distinct properties in non-Hermitian topological systems, such as the breakdown of the conventional bulk-boundary correspondence61–63, the emergence of anomalous edge states64–66, and anomalous localization of bulk eigenstates (“non-Hermitian skin effect”)67–69. It has also been shown that non-Hermitian topology could manifest itself in some interesting transport phenomena70,71, say, the deviation of the Hall conductance of the edge state from the quantized Chern number72,73, one-way transport in low-dimensional lattices by an imaginary gauge field74,75, and the recently proposed topological insulator laser76.

Though topological phenomena in one-dimensional (1D) and two-dimensional (2D) non-Hermitian systems have been extensively studied66–75, much less effort has been devoted to three-dimensional (3D) systems76–89. In this paper, we investigate both continuum and lattice models of non-Hermitian nodal-line semimetals in the presence of a particle gain-and-loss term. It is found that such a non-Hermitian perturbation will split each nodal ring into two exceptional rings (ERs). With increasing strength of this perturbation, some of the ERs may shrink and eventually vanish. To characterize the topological property of the bulk band structure, two different topological invariants are used: (1) One is the vorticity80 of a loop around the exceptional points (EPs) generated by cutting the ERs with a 2D slice in the cylinder coordinate. (2) The other is the winding number for a loop in the 3D momentum space, which stems from the chiral symmetry and can be calculated through the definition of a complex angle72,90. Both invariants take fractional [integer] values when the loop is threaded by an odd [even] number of ERs. Moreover, under open boundary conditions (OBC), the zero-energy surface flat bands are no longer bounded by the projections of bulk ERs, thus suggesting the breakdown of conventional bulk-surface correspondence in nodal-line semimetals. Intriguingly, not only the zero-energy flat bands but also a macroscopic fraction of bulk eigenstates are found to be localized near the surface, which could be well explained by dimensional-reduction to 1D non-Hermitian lattice models. Such a non-Hermitian skin effect leads to significantly enhanced surface density-of-states and offers a good opportunity to study surface physics related to interaction effects in non-Hermitian systems.

This paper is organized as follows. In Sec. II, we first study the bulk-band structure of non-Hermitian nodal-line semimetals through a simple continuum model in Sec. II A, and then introduce the two topological invariants, namely, the vorticity and the winding number, in Sec. II B and II C, respectively, to characterize the
bulk topology. In Sec. III, we address the issue of non-Hermitian bulk-boundary correspondence, where a lattice model is used to illustrate the band structures under periodic boundary conditions (PBC) and OBC, in Sec. III A and III B, respectively. The skin effect of non-Hermitian nodal-line semimetals is discussed in Sec. III C. Section IV concludes this paper.

II. BULK BAND FROM CONTINUUM MODEL

A. Model description

A typical two-band spinless nodal-line semimetal can be described by the simple continuum model Hamiltonian \[ H = (m - Bk^2)\tau_x + vk_z\tau_z, \] where \( k^2 = k_x^2 + k_y^2 + k_z^2 \), \( \tau_i \) \((i = x, y, z)\) are Pauli matrices acting in the two-orbital subspace, \( v \) denotes the Fermi velocity along the \( k_z \)-direction, and \( m\) and \( B\) are parameters with the dimension of energy and inverse energy, respectively. When \( mB > 0 \), the conduction and valence bands touch along the nodal ring located in the \( k_z = 0 \) plane at \( k_x^2 + k_y^2 = m/B \) (see Fig. 1(a)), while for \( mB < 0 \), the system lies in the trivial insulator phase with a full gap. Without loss of generality and for simplicity, henceforth, unless stated explicitly, \( m\), \( B\), and \( v \) are assumed to be positive. In the presence of a non-Hermitian term \( iv\tau_z\) \((\gamma_z > 0)\) associated with particle gain and loss for the two orbitals, respectively, the Hamiltonian becomes:

\[ H = (m - Bk^2)\tau_x + (vk_z + iv\gamma_z)\tau_z. \]

The energy is now obtained as

\[ E_{\pm} = \pm \sqrt{(m - Bk_{\pm}^2)^2 + v^2k_{\pm}^2 - \gamma_z^2 + 2vk_z\sqrt{B^2 - k_{\pm}^2}}, \]

which is generally complex for nonzero \( \gamma_z \). To see the fate of the original nodal ring, we focus on the \( k_z = 0 \) plane, where the energy becomes \( E_{\pm} = \pm \sqrt{(m - Bk_{\pm}^2)^2 - \gamma_z^2} \), with \( k_{\pm} = \sqrt{k_x^2 + k_y^2}. \) When \( \gamma_z < m \), the original nodal ring splits into two ERs characterized by \( Bk_{\pm}^2 = m \pm \gamma_z \), as shown in Fig. 1(b). In the \( k_z = 0 \) plane, the energy is purely real both inside the inner ER and outside the outer ER, while it is purely imaginary between the two ERs, as demonstrated in Figs. 1(c) and 1(d), respectively. With increasing \( \gamma_z \), the inner ER shrinks and vanishes beyond the critical value of \( \gamma_z = m \), where it becomes a point. Quite intriguingly, an ER appears even for the original gapped phase with negative \( m \), as long as \( \gamma_z > |m| \) is satisfied.

B. Vorticity

In contrast to Hermitian band degeneracies consisting of distinct eigenvectors, ERs are ubiquitous in non-Hermitian band structures, where not only the eigenvalues but also the eigenvectors coalesce with each other, thus rendering the corresponding Hamiltonian defective and nondiagonizable \[ [32, 33] \]. When encircling an ER, the constitutive bands get exchanged due to the square root taken in Eq. \[ (3) \], and two loops are required to return to the initial state \[ [61, 72, 80, 91, 93] \]. In order to characterize the ERs, we adopt the cylinder coordinate and divide each ER into a collection of EPs residing in the two-dimensional \( (2D) \) \( k_{\rho} - k_z \) slice. After this decomposition, we can then resort to the concept of vorticity introduced in Ref. \[ 80 \] to characterize each EP.

First, we consider the case with both the inner and outer ERs \((\gamma_z < m)\), which are located in the \( k_z = 0 \) plane at \( k_{\pm} = \sqrt{(m - \gamma_z)/B} \) and \( \sqrt{(m + \gamma_z)/B} \), respectively. For each \( k_{\rho} - k_z \) slice, altogether four EPs appear at \( (k_{\rho}, k_z) = (k_{\rho}^s, 0) \), as shown in Fig. 2(a), with

\[ k_{\rho}^s = \pm \sqrt{(m - s\gamma_z)/B}, \]

where \( s = +1 [-1] \) for the ERs from the inner [outer] ER. In fact, these EPs can be understood from the non-Hermitian-term-induced splitings of the original Dirac points at \((\pm \sqrt{m/B}, 0)\) in the 2D \( k_{\rho} - k_z \) slice. By expanding the low-energy effective Hamiltonian to linear-order around each EP, we obtain

\[ H_{\pm}^s(q) = (\pm \gamma_z - 2Bk_{\pm}^sq_{\rho})\sigma_x + (vq_z + i\gamma_z)\sigma_z. \]

The dispersion to the leading order of \( q \) is then derived.

![FIG. 1. Illustration of the nodal rings in the \( k_z = 0 \) plane in the (a) absence and (b) presence of the non-Hermitian term \( iv\gamma_z\). The parameters are chosen as \( m = 0.5, B = v = 1 \), and \( \gamma_z = 0.3 \). (c) The real part and (d) the imaginary part of the energy in the \( k_z = 0 \) plane with the same parameters as in (b).](image-url)
In the numerical calculation.

The evolution of the complex eigenvalues of the two bands along the loops (a) $\Gamma_1$, (c) $\Gamma_2$, and (e) $\Gamma_3$ in (a), which are parameterized by $\theta \in [0, 2\pi]$. The parameters are chosen as $B = v_x = 1$, $m = 0.5$, and $\gamma_z = 0.3$ in the numerical calculation.

as

$$E_{x,\lambda}^\pm(q) = \lambda \sqrt{2\gamma_z(s v_x^\pm q_\rho + i v_z q_z)},$$

where $v_x^\pm = 2Bk_0^\pm$ and $\lambda = \pm 1$ for the two branches of bands. Following Ref. [80], the vorticity of each EP can be calculated as

$$\nu_{\pm}^x = -\frac{1}{2\pi} \oint_{\Gamma} \nabla_{\theta} \arg[E_{x,\pm}^+ - E_{x,\pm}^-] \cdot dq = -\frac{1}{2\pi} \oint_{\Gamma} \frac{\partial \theta}{q} q \arg[q \rho^\pm(i \pi)] \cdot dq \theta$$

where $\Gamma$ is a loop enclosing the EP, $\tan \theta = \frac{v_zq_x - q_\rho}{v_zq_\rho + q_x}$, and $q = \sqrt{(v_zq_z)^2 + (v_x^\pm q_\rho)^2}$. As an illustration, in Figs. 2(b) and 2(c), we numerically plot the evolution paths of the two bands along the loops $\Gamma_1$ and $\Gamma_2$ around the inner and outer EPs $k^\pm$ and $k^\pm$, respectively, both of which are parameterized by $\theta \in [0, 2\pi]$. It can be seen that around both $k^\pm$ and $k^\pm$, the two bands get switched at $\theta = 2\pi$. However, they wind around each other in opposite directions, namely, clockwise [counterclockwise] for $k^\pm$ [k^\pm] with $v_{x}^\pm = -1/2$ [$v_{x}^\pm = 1/2$]. More generally, when the loop encloses an odd number of EPs, the two bands swap with each other and the vorticity takes a half-integer value, while when an even number of EPs is enclosed, the vorticity becomes an integer, and the two bands return to their original states, as exemplified by the loop $\Gamma_3$ enclosing both $k^+_3$ and $k^-_3$ in Fig. 2(a), with the bands’ evolution shown in Fig. 2(d).

In addition, with increasing $\gamma_z$, the two EPs from the inner ER approach each other until $\gamma_z = m$, where they meet and annihilate as a result of their opposite vorticities [80], which accounts for the disappearance of the inner ER when $\gamma_z > m$.

C. Winding number

To fully capture the topological property of the bulk band, the above calculation of vorticity is insufficient. Owing to the chiral symmetry of the Hamiltonian in Eq. (7), $\sigma_y H \sigma_y = -H$, by treating $k_x$ and $k_\rho$ as parameters, a 1D winding number for every 1D chain in the $k_z$ direction can be defined as

$$w = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_z \partial_{k_z} \phi,$$

where $\phi \equiv \arctan[h_z/(h_z)] = \arctan[(m - Bk^2)/(v_zk_x + i\gamma_z)]$, with $h_z$ and $h_z$ representing the components of the $\sigma_x$ and $\sigma_z$ terms, respectively, in $H$. (If the alternative definition $\phi \equiv \arctan[h_z/h_z]$ is used, the final result of the winding number will only differ by a sign reversal.) Note that the presence of the non-Hermitian term indicates that $\phi$ is generically complex.

When $m > \gamma_z$, two ERs appear in the $k_z = 0$ plane, as shown before. Considering the rotational symmetry of the system, we only numerically present the real part of $\phi$ for a 2D $k_x - k_z$ slice in Fig. 3(b), with the parameters $m = 0.5, B = v_z = 1$, and $\gamma_z = 0.3$. In Figs. 3(c)-(d), we plot $\text{Re}(\phi)$ as well as its derivative to $k_z$ for three representative lines $A$ (inside the inner ER), $B$ (between the two ERs), and $C$ (outside the outer ER), as schematically shown in Figs. 3(a) and 3(b). $\text{Re}(\phi)$ is an odd function of $k_z$, and at $k_z = 0$, it is continuous when $k_\rho$ lies between the two ERs, while for $k_\rho$ outside this range, it is discontinuous with a $\pi$-jump, as demonstrated in Fig. 3(c). However, this $\pi$-jump makes no contribution to the integral, which involves only smooth gradients [73]. In contrast, the real part of $\partial_{k_z} \phi$ is always continuous with no such jumps, thus validating Eq. (8). In addition, the imaginary part of $\phi$ is found to be an even function of $k_z$, suggesting its derivative $\text{Im}(\partial_{k_z} \phi)$ to be an odd function, which consequently does not contribute to the integral in Eq. (8). Finally, the winding number can be explicitly obtained as

$$w = \begin{cases} 
-1, & \text{for } |k_\rho| < k_{\text{in}}, \\
-\frac{1}{2}, & \text{for } k_{\text{in}} < |k_\rho| < k_{\text{out}}, \\
0, & \text{for } |k_\rho| > k_{\text{out}},
\end{cases}$$

where $k_{\text{in}} = \sqrt{(m - \gamma_z)/B}$, and $k_{\text{out}} = \sqrt{(m + \gamma_z)/B}$. This result is supported by the numerical calculation of $w$ as a function of $k_\rho$ in Fig. 3(e). The phase diagram of $w$ as a function of both $k_x$ and $k_\rho$ is also presented in Fig.
FIG. 3. (a) Illustration of representative 1D lines or loops in the momentum space for the calculation of the winding number. (b) The distribution of Re(φ) in the 2D k_y − k_x slice, where the red dots represent EPs. (c) Re(φ) and (d) Re(∂_k_y φ) as a function of k_x for the lines A, B, and C, in (b). The winding number as a function of (e) k_y and (f) k_x and k_y.

3(f), where the boundaries between regions of different w values exactly correspond to the bulk ER.

The emergence of the fractional value w = −1/2 and integer value w = −1 can be understood as follows. Though the values of φ differ by π for the two opposite limits k_z → ∞ and k_z → −∞, their derivatives ∂_k_z φ are found to be the same, thus enabling us to reasonably compactify the integral line into a loop by connecting k_z = ∞ to k_z = −∞ (The compactness will be quite natural for a Bloch Hamiltonian in a lattice model with PBC.

As a result, Lines A, B, and C, turn out to be topologically equivalent to the loops D, E, and F, respectively, which are threaded by two, one, and zero ERs. Since the D loop encloses two EPs, the winding number can be proved as ±1 [61]. For the E loop encircling only one EP, the winding number is found to take fractional values ±1/2 [61], which is related to the fact that the non-Hermitian-generalized Berry phase equals ±π after a path circles twice around an EP [61, 93, 94]. For the F loop enclosing no EPs, the winding number should obviously take the trivial value zero.

When m < γ_z, only the outer ER remains, and it is evident from the above analysis to predict that w = −1/2 [w = 0] inside [outside] this ER.

III. BREAKDOWN OF BULK-SURFACE CORRESPONDENCE

In Hermitian systems, by virtue of bulk-boundary correspondence, the emergence of topological surface (edge) states are ensured by calculating relevant topological invariants of bulk bands under PBC. This rule holds true for Hermitian nodal-line semimetals, where drumhead surface states (flat bands) are expected to be bounded by the projections of bulk nodal rings onto the surface Brillouin zone (BZ) [24–28]. However, the generalization of such correspondence to non-Hermitian systems is problematic, which has been shown to break down in certain systems [60–67], such as the non-Hermitian Su-Schrieffer-Heeger (SSH) model [62, 63] and the non-Hermitian Chern insulator [63, 64, 66]. Intriguingly, under OBC, even a macroscopic number of bulk eigenstates become localized near the boundary, producing so-called “non-Hermitian skin effect” [61, 62, 63, 66, 67]. In this section, we will inspect such effects for a lattice model of non-Hermitian nodal-line semimetal under PBC and OBC, respectively.

A. Bloch band from lattice model

By taking k_x → sin k_i and k_x^2 → 2(1 − cos k_i) in Eq. [20], the lattice model Hamiltonian can be obtained as

\[ H = \left[ m - 2B(3 - \cos k_x - \cos k_y) \right] \sigma_x + (v_z \sin k_z + i\gamma_z) \sigma_z. \]  (10)

Band degeneracies are found to occur in the k_z = 0 plane at

\[ \cos k_x + \cos k_y = 2 - \frac{m \pm \gamma_z}{2B}, \]  (11)

and in the k_z = π plane at

\[ \cos k_x + \cos k_y = 4 - \frac{m \pm \gamma_z}{2B}. \]  (12)

In the absence of the non-Hermitian iγ_z σ_z term, a nodal loop appears in the k_z = 0 plane when 0 < \( \frac{m}{2B} < 4 \), and in the k_z = π plane when 2 < \( \frac{m}{2B} < 6 \), as illustrated by the red and blue dashed lines, respectively, in Fig. 4(a) with m = 3, B = 0.5, v_z = 1. In the presence of a small iγ_z σ_z term, analogous to the continuum model, each nodal loop will split into two ERs (see the solid lines in Fig. 4(a) with γ_z = 0.6). The energy is also purely imaginary between the two ERs and purely real outside. With increasing γ_z, each inner [outer] ER shrinks towards \( \left( k_x, k_y \right) = \left( 0, 0 \right) \left[ \left( \pi, \pi \right) \right] \), and vanishes there beyond a critical value of γ_z determined by Eqs. (11) and (12).
example, if $\gamma_z$ is increased to 1.2 in Fig. 4(a), only one ER persists in both the $k_z = 0$ and $k_z = \pi$ planes, as shown in Fig. 4(b).

Similar to the continuum model, by treating $k_x$ and $k_y$ as parameters, the bulk band can also be characterized by the winding number in Eq. (8), where the integral interval of $k_z$ should now be replaced by $[−\pi, \pi]$. Depending on the model parameters, the winding number $w$ may take the value of $−1, −1/2$ or 0. Regions of distinct $w$ are bounded by the ERs, as can be seen from Figs. 4(c) and 4(d). The emergence of such values of $w$ also originate from encircling the EPs, as has already been clarified in the continuum model.

### B. Spectra under OBC

To examine the bulk-surface correspondence in the non-Hermitian nodal-line semimetals, as a first step, we choose OBC in the $z$-direction of $N = 20$ slabs with the same parameters as those in Fig. 4(a) to numerically calculate the spectrum as a function of both $k_x$ and $k_y$. Figure 5(a) presents the lowest band of the absolute complex energy spectra $|\epsilon|$, where the projections of the bulk ERs under PBC are provided as a comparison (blue dashed lines). It is obvious to see the discrepancy between the boundaries of the zero-energy flat bands (blue solid lines) and the projections of bulk ERs, which indeed reflects the breakdown of the usual bulk-surface correspondence of Hermitian nodal-line semimetals. This discrepancy can be well explained as follows.

By treating $k_x$ and $k_y$ as parameters, the Hamiltonian in Eq. (10) will be effectively reduced to a 1D one:

$$H_{xy} = (m_{xy} + 2B \cos k_x)\sigma_x + (v_z \sin k_z + i\gamma_z)\sigma_z,$$

where $m_{xy} = m - 2B(3 - \cos k_x - \cos k_y)$. This Hamiltonian takes the same form as the 1D non-Hermitian lattice model in Refs. [61, 67]. It also bears a very close resemblance to the well-studied non-Hermitian SSH model after taking a basis change $\sigma_y \rightarrow \sigma_z$ [62, 66, 81]. For simplicity and without loss of generality, we set the parameters $B = 0.5$ and $V_z = 1$. Following Refs. [62] and [66], under OBC in the $k_z$-direction, it can be shown that topological phase transitions accompanying the (dis)appearance of boundary zero modes take place at $m_{xy} = \pm \sqrt{\gamma_z^2 + 1}$ or $\pm \sqrt{\gamma_z^2 - 1}$. This is in striking contrast to the periodic case, where the bulk ERs are projected to $m_{xy} = 1 \pm \gamma_z$ and $-1 \pm \gamma_z$. Under OBC, the topologically nontrivial region with boundary zero modes corresponds to

$$\begin{cases}
|m_{xy}| < \sqrt{\gamma_z^2 + 1}, \\
\sqrt{\gamma_z^2 - 1} < |m_{xy}| < \sqrt{\gamma_z^2 + 1},
\end{cases}
\quad \text{for } \gamma_z < 1;
\begin{cases}
|m_{xy}| < \sqrt{\gamma_z^2 + 1}, \\
\sqrt{\gamma_z^2 - 1} < |m_{xy}| < \sqrt{\gamma_z^2 + 1},
\end{cases}
\quad \text{for } \gamma_z > 1.
(14)

This is numerically verified in Fig. 5(a), where surface flat bands are bounded by blue solid lines characterized by $m_{xy} = \pm \sqrt{\gamma_z^2 + 1}$ instead of the dashed lines representing bulk ERs. As a further illustration, we plot the absolute (Fig. 5(b)), real (Fig. 5(c)), and imaginary (Fig. 5(d)) values of the full complex energy spectra as a function of $k_x$ with fixed $k_y = 0$. Although the model
is non-Hermitian with gain and loss, during some parameter regions, the spectra become purely real, which results from the PT symmetry [67]. Moreover, since both the real and imaginary parts of the flat bands equal zero \(|\epsilon| = 0\), they should be dynamically stable.

![Image](image)

**FIG. 6.** When \(m = 3\), \(B = 0.5\), \(v_z = 1\), and \(\gamma_z = 0.6\), under OBC, the wavefunction distribution as a function of the position of the slab in the \(z\)-direction for (a) the zero mode and (b) bulk eigenstates at \((k_x, k_y) = (\pi/2, 0)\) and (c) for all the eigenstates at \((k_x, k_y) = (\pi/2, \arccos(0.6))\). (d) The inverse participation ratio (IPR) of a typical bulk eigenstate as a function of both \(k_x\) and \(k_y\). Extended states exist only in the black regions around \(m_{xy} = 0\) with IPR \(\approx 1/N = 0.05\) and IPR reaches its maximum value in the white region around \(m_{xy} = \pm \gamma_z\). The wavefunction distribution on the top and bottom slabs for the state nearest to zero energy when \(m = 3\), \(\gamma_z = 0.6\) and (f) \(m = 0.9\), \(\gamma_z = 1.1\).

### C. Non-Hermitian skin effect

We continue to investigate the exotic “non-Hermitian skin effect” under OBC in our system. For the 1D Hamiltonian in Eq. (13), it can be shown that when \(m_{xy} < 0\) \([m_{xy} > 0]\), not only the zero modes but also a macroscopic fraction of the bulk eigenstates will be localized near the top [bottom] boundary for a large parameter region \([67]\). For example, we use the same set of parameters as in Fig. 5(a) and choose the point \((k_x, k_y) = (\pi/2, 0)\) with \(m_{xy} = 1\) to plot the wavefunction distribution \(|\phi(z)|^2\) of both the zero mode (Fig. 6(a)) and bulk eigenstates (Fig. 6(b)) as a function of the slab position in the \(z\)-direction, where they indeed become localized near the bottom slab. It is also found that when \(m_{xy} = \pm \gamma_z\), all the eigenstates are totally localized at the boundary slab (see Fig. 6(c) with \(m_{xy} = \gamma_z\)), which could be related to the occurrence of higher-order EPs, as marked by red points in Fig. 5(c) \([67]\). As a further evidence, we calculate the inverse participation ratio (IPR) to measure the localization of a state \(\phi_i\), which is defined as 

\[\text{IPR} = \frac{\sum_z |\phi_i(z)|^4}{(\sum_z |\phi_i(z)|^2)^2}.\]

For extended states, it should be proportional to \(1/N\), where \(N\) is the total lattice number in the open-boundary direction. Figure 6(d) numerically shows the IPR of a typical bulk eigenstate with \(m = 3\) and \(\gamma_z = 0.6\), where the extended states exist only in the vicinity of the lines characterized by \(m_{xy} = 0\) (black regions), while the maximum IPR appears around the lines with \(m_{xy} = \pm \gamma_z = \pm 0.6\) (white regions).

Intriguingly, depending on the parameter \(m\), the surface flatbands and a macroscopic fraction of bulk eigenstates may be localized at: (i) bottom surface when \(m > 5\) \((m_{xy} > 0\) is always satisfied), (ii) top surface when \(m < 1\) \((m_{xy} < 0\) is always satisfied), (iii) both the top and bottom surfaces but at different surface BZ regions when \(1 < m < 5\). As an illustration, we plot the wavefunction distribution of the state nearest to zero energy on the top and bottom slabs, respectively, for \(m = 3\), \(\gamma_z = 0.6\) (Figs. 6(e)) and \(m = 0.9\), \(\gamma_z = 1.1\) (Fig. 6(f)), where distinct localization behaviors between them can be clearly observed. It should be emphasized that, compared to the surface flat bands in the Hermitian nodal-line semimetals, the bulk condensation resulting from non-Hermitian skin effect endows the nodal-line semimetal with further enhanced surface density-of-states.

### IV. CONCLUSION

In summary, we have theoretically investigated non-Hermitian nodal-line semimetals, where the non-Hermiticity originates from the introduced particle gain-and-loss perturbation. Through dimensional reduction, two different topological numbers have been used to describe the topology of the bulk bands. By comparing the band structures under PBC and OBC, the conventional bulk-surface correspondence in nodal-line semimetals is found to fail in the non-Hermitian case. Furthermore, the non-Hermitian skin effect in our system has also been discussed based on the knowledge from 1D non-Hermitian models. Such an effect could lead to significantly enhanced surface density-of-states in nodal-line semimetals, which provides a versatile platform to study various interaction effects in non-Hermitian systems.

In addition, since previous research \([92]\) has shown that nodal-line semimetals can be driven into Weyl semimetals by applying external perturbations, it would be quite
interesting to investigate the evolution of the band structure during this process in the presence of non-Hermitian perturbations. It is also worthy of attention to study the non-Hermitian skin effect in other topological semimetals such as Weyl and Dirac semimetals, which may be left for future work.

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