Multiplicativity of maximal output purities of Gaussian channels under Gaussian inputs

A. Serafini1,2, J. Eisert2,3 and M.M. Wolf4

1 Dipartimento di Fisica “E.R. Caianiello”, Università di Salerno, INFN Sezione di Napoli, Gruppo Collegato di Salerno, Via S. Allende, 84081, Baronissi (SA), Italy
2 Institut für Physik, Universität Potsdam, Am Neuen Palais 10, D-14469 Potsdam, Germany
3 Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, UK and
4 Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Straße 1, 85748 Garching, Germany

(Dated: October 24, 2018)

We address the question of the multiplicativity of the maximal $p$-norm output purities of bosonic Gaussian channels under Gaussian inputs. We focus on general Gaussian channels resulting from the reduction of unitary dynamics in larger Hilbert spaces. It is shown that the maximal output purity of tensor products of single-mode channels under Gaussian inputs is multiplicative for any $p \in (1, \infty)$ for products of arbitrary identical channels as well as for a large class of products of different channels. In the case of $p = 2$ multiplicativity is shown to be true for arbitrary products of generic channels acting on any number of modes.

PACS numbers: 03.67.-a, 42.50.-p, 03.65.Ud

I. INTRODUCTION

Additivity and multiplicativity questions play a central role in the field of quantum information theory: does it help to make joint use of a quantum channel when transmitting quantum or classical information in form of entangled inputs, or is one better off by merely invoking the channel many times with uncorrelated inputs? Or, what is the entanglement cost, the rate at which maximally entangled pairs need to be invested in the asymptotic preparation of a mixed bi-partite state using local operations and classical communication only? If the so-called entanglement of formation turned out to be additive, the evaluation of this quantity, which amounts to a much simpler optimization problem, would be sufficient to provide the complete answer to this question. Most instances of such additivity problems in quantum information share the common feature of being notoriously difficult to solve. Recently, yet, a picture emerged that made clear that several of these problems share more than a formal similarity. It has actually been shown that at least four instances of such additivity problems are logically equivalent, being either all wrong or all true. Besides the fundamental insight that this equivalence provides, such an observation is practically helpful, since it links isolated additivity results to other instances of such problems.

This paper is concerned with a specific multiplicativity question for quantum channels of bosonic systems: it deals with the maximal output purity of Gaussian channels under Gaussian inputs. This quantity specifies how well the purity of an input state, measured in terms of $p$-norm purities (or equivalently Rényi entropies), can be preserved under the application of the (generally decohering) channel and provides a way to characterize the decoherence rate of the channel. If this output purity turns out to be multiplicative for a tensor product of channels, then input entanglement cannot help to better preserve the coherence of the output states. The multiplicativity of the maximal output purity for $p \to 1+$ corresponds to the additivity of the minimal von Neumann entropy. Furthermore, if general inputs are allowed for, the multiplicativity for such a limiting instance is strictly related to the additivity of the (appropriately constrained) Holevo capacity and of the entanglement of formation (EoF). In turn, the quantum information properties of bosonic Gaussian channels have attracted strong theoretical attention in recent years. This class of quantum channels is practically very important: indeed, the transmission of light through a fiber is described to a very good approximation by a Gaussian bosonic channel. The unavoidable coupling to external field modes yield losses, whereas excess noise can be incorporated as random classical Gaussian noise, reflecting random displacements in phase space. The estimation of various information capacities, both quantum and classical, has been thoroughly addressed for these Gaussian channels. As for the multiplicativity of the maximal output purity, the quest has been challenged in a series of recent works, and strong arguments have been provided to support it, addressing a subset of channels investigated in the present paper. In fact, for a specific channel model describing a beam splitter interaction of a bosonic mode with a thermal noise source and for integer $p \geq 2$ multiplicativity of the maximal $p$-norm output purity was recently proven. However, a definitive proof of the multiplicativity conjecture for tensor products of general Gaussian channels and non-integer $p$ is still missing.

In the present paper we aim to make a step towards a theory of channel capacities of Gaussian channels under Gaussian inputs, dealing with more general instances of channels. Such a setting, besides being interesting in its own right, yields obvious bounds for the unconstrained maximal output purity for Gaussian bosonic channels, which in turn may be conjectured to be tight as it is the case for particular single-mode channels and integer $p \geq 2$. Moreover, the present paper is meant to be a further step towards a clear picture of a general quantum information theory of Gaussian states, linking channel capacities with entanglement properties. This picture could provide a powerful laboratory, when a complete solution to the specified additivity problems is lacking.

The paper is structured as follows. In section we introduce the notation, basic facts about Gaussian states and we...
define the class of Gaussian channels we will deal with. In
section III the $p$-norms as measures of purity are presented
and determined for Gaussian states. In section IV the Gauss-
ian multiplicity of the maximal output purity is defined,
while section V contains all the analytical results about mul-
tiplicativity. Finally, in section VI we review our results and
provide some comments and perspectives.

II. GAUSSIAN STATES AND CHANNELS

A. Gaussian states

We consider quantum systems with $n$ canonical degrees
of freedom, i.e., a system consisting of $n$ modes. The canonical
coordinates corresponding to position and momentum will be
denoted as $\hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$, where in terms of the
usual creation and annihilation operators we have that $\hat{x}_i =
(\hat{a}_i + \hat{\bar{a}}_i)/\sqrt{2}$ and $\hat{p}_i = -i(\hat{a}_i - \hat{\bar{a}}_i)/\sqrt{2}$. In terms of the
Weyl operators

$$W_\xi = e^{i\xi^T \sigma \hat{R}}, \quad \xi \in \mathbb{R}^{2n}$$

(1)

the canonical commutation relations (CCR) can be written as

$$W_{\xi}^T W_{\xi'} = W_{\xi}^T e^{i\xi^T \sigma \xi'},$$

(2)

where

$$\sigma = \bigoplus_{i=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(3)

The latter matrix $\sigma$ is the symplectic matrix. States can be
fully characterized by functions in phase space $(\mathbb{R}^{2n}, \sigma)$. The
characteristic function is defined as

$$\chi_{\rho}(\xi) = \text{tr}[\rho W_{\xi}],$$

(4)

where the state $\rho$ can in turn be expressed as

$$\rho = \frac{1}{(2\pi)^n} \int d^{2n} \chi_{\rho}(-\xi) W_\xi;$$

(5)

The characteristic function is the ordinary Fourier transform of
the Wigner function commonly employed in the phase
space description of quantum optics [12][13].

Gaussian states are, by definition, the states with Gaussian
characteristic function (and therefore Gaussian Wigner function)

$$\chi(\xi) = \chi(0)e^{-\frac{1}{2} \xi^T \Gamma^T \xi + \xi D}.$$ 

(6)

Gaussian states are fully determined by first and second
moments of the quadrature operators, respectively embodied by
the vector $d = \sigma D$ and by the real symmetric $2n \times 2n$ matrix
$\Gamma = \sigma^T \gamma \sigma$, with

$$\gamma_{ij} = \frac{1}{2} \langle \hat{R}_i \hat{R}_j + \hat{\bar{R}}_j \hat{\bar{R}}_i \rangle_\rho - \langle \hat{R}_i \rangle_\rho \langle \hat{\bar{R}}_j \rangle_\rho,$$

(7)

where $\langle \hat{O} \rangle_\rho = \text{tr}[\rho \hat{O}]$ for the operator $\hat{O}$. First moments can be
set to zero by local unitaries, so that they play no direct
role in properties related to entanglement and mixedness of
Gaussian states. To our aims a Gaussian state will be charac-
terized by its covariance matrix $\gamma$. The covariance matrix $\gamma$ of
a Gaussian state $\rho$ (and, indeed, any covariance matrix related
to a physical state), has to satisfy the uncertainty principle

$$\gamma + i \sigma \geq 0,$$  

(8)

reflecting the positivity of $\rho$. Subsequently, $\gamma$ will stand for
the set of Gaussian states with vanishing first moments (for simplicity, the underlying number of canonical degrees of freedom will not be made explicit). Pure Gaussian states are
those for which det $\gamma = 1$. These are the minimal uncertainty
states, saturating Ineq. (8). The subset of pure Gaussian states with
vanishing first moments will be denoted as $\mathcal{G}$.

Any unitary $U$ generated by polynomials of degree two in
the canonical coordinates is, by virtue of the Stone–von Neu-
mann theorem, the metaplectical representation of a real sym-
plectic transformation $S \in Sp(2n, \mathbb{R})$. We recall that the real
symplectic group $Sp(2n, \mathbb{R})$ consist of those real $2n \times 2n$
mats for which $S^T \sigma S = \sigma$. Such symplectic operations
preserve the Gaussian character of the states and act by congruence on covariance matrices

$$\gamma \longmapsto S^T \gamma S.$$ 

(9)

On Weyl operators such an operation is reflected by

$$W_\xi \longmapsto W_{S^{-1} \xi}.$$  

(10)

We mention that ideal beam splitters and squeezers are de-
scribed by symplectic transformations. The expression of the
generators of $Sp(2n, \mathbb{R})$ will be useful in the following and is
detailed in App. [12]. Moreover, we recall that a useful way to
express a generic symplectic transformation $S$ is provided by the Euler decomposition [13]

$$S = O'ZO'',$$ 

(11)

where $O', O'' \in K(n) = Sp(2n, \mathbb{R}) \cap SO(2n)$ are ortho-
gonal symplectic transformations, whose set forms the maxi-
mal compact subgroup of $Sp(2n, \mathbb{R})$. They are those op-
tations typically referred to as being passive, again, in opti-
cal systems corresponding to beam splitters and phase shifts.
The group of all $Z = \text{diag}(z_1, 1/z_1, \ldots, z_n, 1/z_n)$ with
$z_1, \ldots, z_n \in \mathbb{R} \setminus \{0\}$ is the non-compact group of all such
$Z$, reflecting local squeezings; this group will be denoted by
$Z(n)$ in the following.

A frequently used tool will be the fact that any CM $\gamma$ can be
brought to the Williamson normal form [13]

$$\gamma \longmapsto \begin{pmatrix} \nu_i^+ & 0 \\ 0 & \nu_i^- \end{pmatrix},$$ 

(12)

with $\nu_i^+ \in [1, \infty)$ and $S \in Sp(2n, \mathbb{R})$. The vector $(\nu_1^+, \ldots, \nu_n^+)$
is the vector of decreasingly ordered symplectic eigenvalues,
which can be computed as the spectrum of the matrix $|i \sigma \gamma|$. 

The previous decomposition is nothing but the normal mode decomposition. Choosing the standard number basis \( \{ |n\rangle : n \in \mathbb{N} \} \) of the Hilbert space associated with each mode, the Gaussian state with vanishing first moments and the second moments as in the right hand side of Eq. (12) is given by

\[
\rho = \bigotimes_{i=1}^{n} \frac{2}{\nu_i^2} \sum_{k=0}^{\infty} \left( \frac{\nu_i^k - 1}{\nu_i^k + 1} \right) |k\rangle \langle k|.
\]

Recalling the Euler decomposition and Williamson’s theorem, expressed by Eq. (11) and Eq. (12), one then finds that the CM \( \gamma \) of an arbitrary pure Gaussian state reads

\[
\gamma = O^TZO, \quad \text{with} \quad O \in K(n), \quad Z \in Z(n).
\]

We finally mention that, as it is evident from the definition of the characteristic function, tensor products of Hilbert spaces correspond to direct sums of phase spaces. Therefore an uncorrelated tensor product of Gaussian states with CMs \( \gamma_i \), \( i = 1, \ldots, n \), has the CM \( \gamma = \bigotimes_{i=1}^{n} \gamma_i \). Likewise, for a ‘local’ tensor product of symplectic transformations \( S_t \) one has \( S = \bigotimes_{i=1}^{n} S_t \).

### B. Gaussian channels

In general, a Gaussian channel is a trace-preserving completely positive map that maps Gaussian trace-class operators onto Gaussian trace-class operators. A Gaussian channel is defined by its action on the Weyl operators, according to

\[
W_\xi \mapsto W_X \xi e^{-y(\xi)},
\]

where \( X \) is a real \( 2n \times 2n \)-matrix, and \( y \) is a quadratic form. We do not consider linear terms in this quadratic form, which would merely correspond to a displacement, i.e., a change in first moments. We can hence write \( y(\xi) = \xi^T \gamma \xi / 2 \). Complete positivity of the channel requires that

\[
Y + i\sigma - iX^T \sigma X \geq 0.
\]

For single-mode channels, this requirement is equivalent to

\[
Y \geq 0, \quad \det[Y] \geq (\det[X] - 1)^2.
\]

The second moments are transformed under the application of such a channel according to

\[
\gamma \mapsto X^T \gamma X + Y.
\]

Any channel of the form of Eq. (18) corresponds to the reduction of a symplectic (unitary) evolution acting on a larger Hilbert space and, vice versa, any evolution of this kind is described by Eq. (18) for some \( X \) and \( Y \).

In the Schrödinger picture, we will denote such channels (characterized by the matrices \( X \) and \( Y \)) by \( \Phi_{X,Y} \), acting as

\[
\rho \mapsto \Phi_{X,Y}(\rho).
\]

This class of channels includes the classical case of random displacements with a Gaussian weight

\[
\rho \mapsto \int d^{2n} \xi P(\xi) W_\xi^\dagger \rho W_\xi,
\]

where \( P(\xi) = P(0) e^{-\frac{i}{2} \xi^T \sigma^{-1} \sigma \xi} \) is a multivariate Gaussian with positive covariance matrix \( Y \). In our notation, such a channel corresponds to \( X = 1 \), \( Y \geq 0 \). The amplification and attenuation channels can be described by Eq. (18) too, with

\[
X = \varepsilon \mathbb{1}_2, \quad Y = |1 - \varepsilon^2| \mathbb{1}_2,
\]

with \( \varepsilon < 1 \) (attenuation) or \( \varepsilon > 1 \) (amplification). Note that this instance encompasses the case of white noise as well.

Also the dissipation in Gaussian reservoirs after a time \( t \) is included in such a class of channels, with the choices

\[
X = e^{-\Gamma t / 2} \mathbb{1}, \quad Y = (1 - e^{-\Gamma t}) \mathbb{1}_B,
\]

where \( \Gamma \) is the coupling to the bath and \( \mathbb{1}_B \) is the covariance matrix describing the reservoir.

The channel model on which the papers of Ref. [9] are focused on is characterized by \( X = c \mathbb{1}, c \leq 1 \) and \( Y \) diagonal (mainly \( Y \propto \mathbb{1} \)).

The additional noise term \( Y \) of a general channel incorporates both the noise that is due to the Heisenberg uncertainty in a dilation, and the additional classical noise. In the same manner as minimal uncertainty Gaussian states can be introduced, pure channels can be considered, satisfying

\[
Y = -(X^T \sigma X - \sigma)Y^{-1}(X^T \sigma X - \sigma),
\]

where the inverse has to be understood as the Moore-Penrose inverse.

### III. MEASURES OF PURITY

Generally, the degree of purity of a quantum state \( \rho \) can be characterized by its Schatten \( p \)-norm [14]

\[
\|\rho\|_p = (\text{tr} |\rho|^p)^{\frac{1}{p}} = (\text{tr} \rho^p)^{\frac{1}{p}}, \quad p \in (1, \infty).
\]

We mention that the case \( p = 2 \) is directly related to the quantity often referred to as linear entropy or purity in the closer sense, \( \mu = \text{tr} \rho^2 = \|\rho\|_2^2 \). The \( p \)-norms are multiplicative on tensor product states and determine the family of Rényi entropies \( S_p \) [15], given by

\[
S_p = \frac{\ln \text{tr} \rho^p}{1 - p},
\]

quantifying the degree of mixedness of the state \( \rho \). It can be easily shown that

\[
\lim_{p \to 1^+} S_p = -\text{tr} [\rho \ln \rho] = S_V(\rho).
\]
Thus the von-Neumann entropy $S_V$ is determined by $p$-norms, as it is given by the first derivative of $\|\rho\|_p$ at $p \to 1^+$. The quantities $S_p$ are additive on tensor product states. It is easily seen that $S_p(\rho) \in (0, \infty)$, taking the value 0 exactly on pure states.

Because of the unitary invariance of the $p$-norms, the quantities $\text{tr} \rho^p$ of a $n$-mode Gaussian state $\rho$ can be simply computed in terms of its symplectic eigenvalues. In fact, due to Eq. (12), $\text{tr} \rho^p$ can be computed exploiting the diagonal state of Eq. (13). One obtains
\[
\text{tr} \rho^p = \prod_{i=1}^{n} \frac{2^p}{f_p(\nu_i^{\downarrow})} = \frac{2^{pn}}{F_p(\gamma)} ,
\] where
\[
f_p(x) = (x + 1)^p - (x - 1)^p
\] and we have defined $F_p(\gamma) = \prod_{i=1}^{n} f_p(\nu_i)$, in terms of the symplectic eigenvalues of the covariance matrix $\gamma$ of $\rho$. A first consequence of Eq. (27) is that the purity $\mu$ of a Gaussian state is fully determined by the symplectic invariant $\text{det} \gamma$ alone:
\[
\mu(\rho) = \frac{1}{\prod_{i=1}^{n} \nu_i^{\downarrow}} \sqrt{\text{det} \gamma} .
\] Every Rényi entropy, or $p$-norm respectively, yields an order within the set of density operators with respect to the purity of the states. Yet a stronger condition for one state being more ordered than another one is given by the majorization relation which gives rise to a half-ordering in state space. A density operator $\rho$ is said to majorize $\hat{\rho}$, i.e., $\rho \succ \hat{\rho}$ if
\[
\sum_{i=1}^{r} \lambda_i^{\downarrow} \geq \sum_{i=1}^{r} \hat{\lambda}_i^{\downarrow},
\] for all $r \geq 1$, where $\lambda_i^{\downarrow}$ is the decreasingly ordered spectrum of $\rho$. Majorization is the strongest ordering relation in the sense that if $\rho \succ \hat{\rho}$, then $\text{tr} f(\rho) \leq \text{tr} f(\hat{\rho})$ holds for any concave function $f$ and in particular for every Rényi entropy [14]. It has recently been conjectured [8] for a special class of Gaussian single-mode channels that the maximal output purity is not only achieved for a Gaussian input state but that the optimal Gaussian output even majorizes any other possible output state.

IV. MULTIPlicativeITY OF THE MAXIMAL OUTPUT PURITIES

We define now for $p \in (1, \infty)$ the Gaussian maximal output $p$-purity of a Gaussian channel $\Phi_{X,Y} : \mathcal{G} \to \mathcal{G}$, as
\[
\xi_p(\Phi_{X,Y}) = \sup_{\rho \in \mathcal{G}} \|\Phi_{X,Y}(\rho)\|_p ,
\] where the supremum is taken over the set of Gaussian states. In terms of covariance matrices and of the function $F_p$, one has
\[
\left(\frac{2^n}{\xi_p(\Phi_{X,Y})}\right)^p = \inf_{\rho} F_p(\phi_{X,Y}(\gamma)) .
\] Thus, on the level of second moments, the multiplicativity of the Gaussian maximal output $p$-purity under Gaussian inputs corresponds to the multiplicativity of the infimum of $F_p$
\[
\inf_{\gamma} F_p(\phi_{X,Y}(\gamma)) = \prod_{i=1}^{n} \inf_{\gamma} F_p(\phi_{X_i,Y_i}(\gamma)) ,
\] where the infimum is taken over all covariance matrices.

For finite dimensional systems and the usual definition of maximal output purity (allowing for input on the whole convex set of trace class operators), the convexity of the $p$-norms guarantees that the supremum of Eq. (31) is that the infimum of $F_p$ can be approached restricting to pure states. The set $\mathcal{G}$ of Gaussian states is not convex. However, every Gaussian state still has a convex decomposition into pure Gaussian states such that it is again sufficient to consider pure input states only:

Lemma 1. For any Gaussian channel $\Phi_{X,Y} : \mathcal{G} \to \mathcal{G}$ and any $p \in (1, \infty)$, one has
\[
\sup_{\rho \in \mathcal{G}} \|\Phi_{X,Y}(\rho)\|_p = \sup_{\rho \in \mathcal{G}} \|\Phi_{X,Y}(\rho)\|_p .
\] 

Proof. Consider the Williamson standard form of the covariance matrix $\gamma = S^T \nu S$. By rewriting this as
\[
\gamma = S^T S + S^T (\nu - 1)S =: \gamma_p + V
\] one infers from Eq. (31) that the state corresponding to $\gamma$ can be generated by randomly displacing a pure state with covariance matrix $\gamma_p$ in phase space according to a classical Gaussian probability distribution with covariance $V$. Hence, the state has a convex decomposition into pure Gaussian states and the Lemma follows from the convexity of the $p$-norms. □

Let us now consider a channel $\Phi_{X,Y}$ resulting from the tensor product of the channels $\Phi_{X_i,Y_i}, i = 1, \ldots, n$,
\[
\Phi_{X,Y} = \bigotimes_{i=1}^{n} \Phi_{X_i,Y_i},
\] acting on Gaussian states associated with the tensor product Hilbert space. Since tensor products in Hilbert spaces correspond to direct sums in phase space, we have that $\Phi_{X,Y} = \Phi_{\otimes X_i \otimes \otimes Y_i}$. We will say that the Gaussian maximal output $p$-purity of the channel $\Phi_{X,Y}$ is multiplicative if
\[
\xi_p(\Phi_{X,Y}) = \prod_{i=1}^{n} \xi_p(\Phi_{X_i,Y_i}) .
\] Let us remark that this is equivalent to stating that the maximal output purity can be attained by means of uncorrelated input states. More precisely, for tensor products of channels one has that, denoting by $S$ the set of product Gaussian states with respect to each of the modes, the multiplicativity of the maximal output $p$-purity of the channel $\Phi_{X,Y}$ is equivalent to
\[
\xi_p(\Phi_{X,Y}) = \inf_{\rho \in \mathcal{S}} \|\Phi_{X,Y}(\rho)\|_p .
\]
V. MULTIPLICIVITY STATEMENTS

From now on, we will mainly restrict to tensor products of single-mode channels, for which the matrices describing the channels are direct sums of $2 \times 2$ matrices $X_i$ and $Y_i$, $i = 1, \ldots, n$: $X = \bigoplus_{i=1}^n X_i$ and $Y = \bigoplus_{i=1}^n Y_i$. Moreover, we will assume that the determinants of the $X_i$ have equal sign. In this case the invariance of the $p$-norms and of the correlations of quantum states under local unitary operations can be exploited to simplify the problem, according to the following.

**Lemma 2.** Let the determinants of the matrices $\{X_i : 1, \ldots, n\}$ have equal signs. Then the Gaussian maximal output $p$-purity of the tensor product of single-mode channels $\Phi_{\otimes X_i, \oplus Y_i}$ is multiplicative if and only if the Gaussian maximal output $p$-purity of the channel $\Phi_{X_i, Y_i}$ is multiplicative, with

$$\hat{X} = \bigoplus_{i=1}^n \hat{X}_i, \quad \hat{Y} = \bigoplus_{i=1}^n \hat{Y}_i, \quad (39)$$

$$\hat{X}_i = \sqrt{\det|X_i|} \|2, \quad \hat{Y}_i = \sqrt{\det|Y_i|} \|2. \quad (40)$$

**Proof.** Let us first reduce the case of negative determinants to that of $\det|X_i| > 0$. To this end we write $X_i = \sigma Z_i - \sigma^T Z_i$, so that $\det|X_i^\dagger| = -\det|X_i|$. Since $\theta \gamma \theta$ with $\theta = \bigoplus_{i=1}^n \sigma$ is again an admissible covariance matrix (corresponding to the time reversed state) and in addition $\theta^2 = 1$, we have indeed that

$$\inf \gamma F_p(\gamma X + Y) = \inf \gamma F_p(\gamma X^T + Y). \quad (41)$$

Now let us see how the case $\det|X_i| > 0$ can be reduced to the standard form given in the Lemma. Due to the unitary invariance of the $p$-norm we can replace $X, Y$ by $\tilde{X} = S' X S$, $\tilde{Y} = S^T Y S$, with $S, S'$ being any symplectic transformations and

$$\inf \gamma F_p(\gamma X + Y) = \inf \gamma F_p(\gamma \tilde{X} + \tilde{Y}). \quad (42)$$

In particular we may choose $S = \bigoplus_{i=1}^n S_i O_i$, such that $O_i \in SO(2)$ and $S_i \in Sp(2, \mathbb{R})$ bring $Y$ in standard form: $S Y S^T Y = \sqrt{\det Y} \mathbb{1} = \hat{Y}$. Furthermore, we choose $S'$ to consist of blocks $S'_i = Z_i O'_i$, $O'_i \in SO(2)$, $Z_i \in Z(2)$ such that the orthogonal matrices $O_i, O'_i$ diagonalize $X_i, S_i$ in

$$\tilde{X}_i = Z_i [O'_i (X_i S_i) O_i] \quad (43)$$

and the squeezing transformation $Z_i$ gives rise to equal diagonal entries yielding $\check{X}_i = \sqrt{\det|X_i| \|1}$. According to the block structure of the involved transformations (corresponding to the direct sum of ‘local’ single mode operations) one has

$$\prod_{i=1}^n \inf \gamma F_p(\gamma X_i + Y_i) = \prod_{i=1}^n \inf \gamma F_p(\gamma \tilde{X}_i + \tilde{Y}_i). \quad (44)$$

Eqs. (42) and (44) straightforwardly imply that $\inf \gamma F_p(\gamma X + Y) = \prod_{i=1}^n \inf \gamma F_p(\gamma X_i + Y_i)$ if and only if $\inf \gamma F_p(\gamma X + Y) = \prod_{i=1}^n \inf \gamma F_p(\gamma X_i + Y_i)$. This proves the claimed equivalence of the multiplicativity statements. □

One remark is in order concerning the case $\det X_i = 0$ in the above Lemma. In fact, in this case multiplicativity is trivial and the maximal $p$-norm output purity does not at all depend on the $X_i$. To see this note that for two positive matrices $A \geq B \geq 0$ we have $\nu^p_i (A) \geq \nu^p_i (B)$, which implies that $\inf \gamma F_p(J X_i Y_i + Y_i) \geq F_p(Y)$, which becomes, however, an equality in the case $\det X_i = 0$ since we can always choose the input state to be a product of squeezed states such that in the limit of infinite squeezing $X_i Y_i \to 0$.

A first relevant consequence of Lemma 2 follows.

**Proposition 1.** The Gaussian maximal output $p$-purity of a tensor product of $n$ identical single-mode Gaussian channels $\Phi_{\otimes X_i, \oplus Y_i}$ is multiplicative for any $p \in (1, \infty)$. Moreover, the output corresponding to the optimal product input majorizes any other Gaussian output state of the channel.

**Proof.** We recall that, because of Euler decomposition, the covariance matrix of any pure Gaussian state $\rho$ can be written as $\gamma = O^T Z O$, where $O \in K(n) = Sp(2n, \mathbb{R}) \cap SO(2n)$ is an orthogonal symplectic transformation and $Z \in Z(n)$ corresponds to a tensor product of local squeezings, $Z = \diag(z_1, 1/ z_1, \ldots, z_n, 1/ z_n)$. Clearly, if $O = 1$ then the state is uncorrelated. For a tensor product of identical channels, Eq. (40) holds globally, $\tilde{X} = x_1 Z_n$, $\tilde{Y} = y_1 Z_n$. Therefore, exploiting Lemma 2 and the invariance of $F_p$ under symplectic transformations, one has

$$\inf \gamma F_p(\gamma \tilde{X} X + Y) = \inf \gamma F_p(x^T \gamma Z O + y) \quad (45)$$

$$= \inf \gamma Z \in Z(n) F_p(x^T Z + y). \quad (46)$$

Due to the block structure of elements in $Z(n)$ this proves the first part of the proposition.

For the majorization part we exploit the fact that a componentwise inequality for the symplectic eigenvalues $\nu_i^p \leq \nu_i^p$ for all $i$ implies majorization on the level of density operators, i.e. $\rho \succ \tilde{\rho}$ minimizes all $\inf \gamma F_p(x^T \gamma Z O + y)$ for all $x$.

The symplectic eigenvalue $\nu_i^p$ of the output covariance matrix $\tilde{\gamma}^p = X_i Y_i + Y_i$ is given by the square root of the ordinary eigenvalue $\lambda_i^p$ of the matrix $\sigma^T \sigma^T \gamma$ (when appropriately taking degeneracies into account). Continuing with the expression in Eq. (46) we have thus to consider the dependence of

$$\lambda_i^p (\sigma \gamma^T \gamma) = x^i + y^i + x^i y \lambda_i^p (Z + Z^{-1}) \quad (47)$$

on $Z$. However, choosing $Z = 1$ in Eq. (47) minimizes all the eigenvalues simultaneously and thus proves the desired inequalities between the optimal and any other Gaussian output state. □

In the following we will investigate the multiplicativity issue in the case of tensor products of different Gaussian channels. To proceed in this direction, we aim to turn our optimization problem over the non-convex set of Gaussian states into an analytical one. To do so two simple remarks, giving
rise to alternative parametrizations of pure covariance matrices, will be exploited. Firstly, because of the Euler decomposition given by Eq. (A.14), the set of the pure $n$-mode covariance matrices can be parametrized by means of the functions

$$\hat{\gamma} : (\mathbb{R}^+)^n \times \mathbb{R}^{n^2} \rightarrow \hat{G}$$

defined as

$$\hat{\gamma}(l, z) = e^{\sum_{i=1}^d l_i K_i^T \Gamma(z) e^{-\sum_{i=1}^d l_i L_i}},$$

where the $L_i$, $i = 1, \ldots, n^2$, are the generators of the compact subgroup $K(n)$ (see App. A) and $D(z) = \text{diag}(z_1, 1/z_1, \ldots, z_n, 1/z_n) \in Z(n)$, with $z_i > 0$ for all $i$. Here, $l = (l_1, \ldots, l_{n^2})$ is a vector of $n^2$ real parameters while $z = (z_1, \ldots, z_n)$ is a vector of $n$ real strictly positive parameters.

Otherwise, the set of pure covariance matrices admits the following parametrization

$$\hat{\gamma}(k) = e^{\sum_{i=1}^d k_i K_i^T \Gamma} e^{\sum_{i=1}^d k_i K_i},$$

where $\gamma$ is an arbitrary pure covariance matrix, the $K_i$ are the $d = 2n^2 + n$ generators of the symplectic group (detailed in App. A) and $k = (k_1, \ldots, k_d)$ is a real vector of dimension $d$.

We are now in a position to prove our main result.

**Proposition 2.** The Gaussian maximal output $p$-purity of a tensor product of single-mode Gaussian channels $\Phi_{X_1, \ldots, X_n}$, with $Y_i^\dagger > 0$ and identical $\text{det}[X_i]$ for all $i$ is multiplicative for any $p \in (1, \infty)$.

**Proof:** Because of Lemma 2, this multiplicity issue is equivalent to the one for the ‘simplified channel’ $\Phi_{\tilde{X}, \tilde{Y}}$, with $\tilde{X} = x_{12n}$, $x = \sqrt{\text{det}[X]}$ and $\tilde{Y} = \oplus_i \sqrt{\text{det}[Y_i]} I_2$ according to Eqs. (A.19) and (A.10). For ease of notation, and since the subsequent argumentation does not depend on the value of $x$, we will state the proof for $x = 1$.

In a first step we aim to show that the infimum of Eq. (A.33) is indeed a minimum, that is, the infimum of $F_p(\Phi_{\tilde{X}, \tilde{Y}}(\gamma))$ is achieved for a defined input and not asymptotically approached in the non-compact set of pure covariance matrices.

Therefore, we will analyse the asymptotic behaviour of the output purity of the channel $\Phi_{\tilde{X}, \tilde{Y}}$ in the limiting case of infinite squeezing. For a given channel of this kind, let us define the function $G_{p, \tilde{Y}} : (\mathbb{R}^+) \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}$ as

$$G_{p, \tilde{Y}}(l, z) = (F_p \circ \Phi_{\tilde{X}, \tilde{Y}})(\gamma)(l, z)), \quad p > 1,$$

where the function $\gamma(l, z)$ has been defined in Eq. (A.49).

To show that the function is indeed attained by a (possibly not unique) given covariance matrix, we address the asymptotic behaviour of $G_{p, \tilde{Y}}$, showing that its infimum cannot be asymptotically approached. To see this, let us investigate the product of the sympletic eigenvalues

$$\prod_{i=1}^n (\nu_i^\dagger)^2 = \text{det}[\gamma(l, z) + \tilde{Y}]_\dagger.$$
For a given channel and covariance matrix $\tilde{\gamma}$, let us define the function $H_p = F_p \circ \Phi_{1:T}^{1}(\tilde{\gamma};(k)) : \mathbb{R}^d \to \mathbb{R}$ (with $d = 2n^2 + n$)

$$H_p(k) = F_p(\sum_{i=1}^{d} K_i k_i V^T \tilde{\gamma} V \sum_{i=1}^{d} K_i k_i + V^T \tilde{Y} V),$$  \hspace{1cm} (59)

with $p \in (1, \infty)$. Such a function is well defined for any channel and input covariance matrix. The form of the $K_i$ is the one given in App. A. By definition, $M$ is the minimum of the function $H_p(k)$ and, because of Eq. (57), $H_p(0) = M$. Furthermore, such a function is differentiable in $k = 0$. Therefore, if the covariance matrix $\tilde{\gamma}$ is indeed optimal the function $H_p(k)$ has to be stationary (critical) in $k = 0$. This constraint is explicitly expressed by

$$\frac{\partial}{\partial k_i} \bigg|_{k=0} F_p(\sum_{i=1}^{d} K_i k_i \gamma' e^\sum_{i=1}^{d} k_i K_i) + Y') = 0,$$  \hspace{1cm} (60)

for all $i = 1, \ldots, d$. We have that

$$\frac{\partial}{\partial k_i} \bigg|_{k=0} F_p(\sum_{i=1}^{d} K_i k_i \gamma' e^\sum_{i=1}^{d} k_i K_i) + Y') = (K_i^T \gamma' + \gamma' K_i).$$  \hspace{1cm} (61)

Since $\gamma' + Y'$ is in Williamson form, we can apply the results of App. B. On using Eq. (61) and Eq. (58) we obtain for the first derivative of the symplectic eigenvalues of the output

$$\frac{\partial}{\partial k_i} \bigg|_{k=0} \nu_j = \text{tr}_j[K_i^T \gamma' + \gamma' K_i],$$

where tr$_j$ denotes the trace of the leading principle submatrix corresponding to the mode $j$ (see App. B). For the derivative of the output $p$-purity we hence obtain

$$\frac{\partial}{\partial k_i} \bigg|_{k=0} F_p(\sum_{i=1}^{d} K_i k_i \gamma' e^\sum_{i=1}^{d} k_i K_i) + Y')) = \frac{f'(\nu_j)}{f'(\nu_j)} \text{tr}_j[K_i^T \gamma' + \gamma' K_i].$$  \hspace{1cm} (63)

where $f' = df/dx$. In order for $\gamma'$ to be the an input corresponding to a critical point, this derivative has to be zero for all generators $K_i$ of $Sp(2n, \mathbb{R})$. As can be promptly verified exploiting the form of the generators and Eq. (63), the condition of Eq. (63) results, for the symmetric $K_i$, in the following constraints on the submatrices of $\gamma' \alpha_i$, and $\beta_{kl}$

$$\alpha_i = a_i I_2, \quad \beta_{kl} = \begin{pmatrix} b_{kl}^l & b_{kl}^r \\ -b_{kl}^r & b_{kl}^l \end{pmatrix},$$  \hspace{1cm} (64)

for some real $a_i \geq 1$ and $b_{kl}^l, b_{kl}^r \in \mathbb{R}$. Since $\gamma'$ is the covariance matrix of a pure state, all its symplectic eigenvalues are equal to 1, implying that $|i \sigma \gamma'| = I_{2n}$. Therefore

$$-\sigma \gamma' \sigma \gamma' = I_{2n}.$$  \hspace{1cm} (65)

Applying the previous condition to the submatrices of Eq. (64) yields

$$a_i^2 + \sum_{l \neq i} (b_{kl}^l + b_{kl}^r)^2 = 1$$  \hspace{1cm} (66)

which is equivalent to

$$a_i = 1, \quad b_{kl}^l = b_{kl}^r = 0.$$  \hspace{1cm} (67)

Eq. (67) shows that the unique $\gamma'$ that is consistent with a critical point is the identity $I_{2n}$, corresponding to the $n$-fold tensor product of a coherent state. It is easy to verify that the identity also satisfy Eq. (65) for the antisymmetric $K_i$.

Summarizing, we have shown that the unique optimal input $\tilde{\gamma}$ corresponds to $\gamma' = V^T \tilde{Y} V = I$, where $V$ is a symplectic transformation for which

$$V^T \tilde{Y} V + V^T \tilde{Y} V$$  \hspace{1cm} (68)

is in Williamson standard form. However, as the identity and $\tilde{Y}$ themselves are in Williamson form, it is immediate to see that $V' = I$, yielding $\tilde{\gamma} = I$, which completes the proof. □

For $p = 2$ the above multiplicativity result can easily be extended to products of arbitrary channels acting on any number of modes without imposing additional constraints on the determinants of the $X_i$ (apart from being non-zero):

**Proposition 3.** The maximal Gaussian output 2-purity of a tensor product of arbitrary multi-mode Gaussian channels $\Phi_{\otimes X_i \otimes Y_i}$, with $\det[X_i] \neq 0$ for all $i$ is multiplicative.

**Proof.** Because of Eq. (29), the Gaussian multiplicativity issue reduces for $p = 2$ to the multiplicativity of the infimum of $\det[\Phi_{\otimes X_i \otimes Y_i}]$ over all covariance matrices corresponding to pure Gaussian states. For a given Gaussian channel $\Phi_{X \otimes Y}$, making use of the Binet theorem ($\det AB = \det A \det B$) and defining

$$Y' = X^{-1} Y X^{-1},$$  \hspace{1cm} (69)

one gets

$$\det[X^T \tilde{\gamma}(l, z) X + Y] = \det[X]^2 \det[\tilde{\gamma}(l, z) + Y'].$$

However, $Y'$ can be diagonalized to $\tilde{Y}'$ by a symplectic block matrix, which in turn does not change the determinant:

$$\tilde{Y}' = SYS^T, \quad S = \bigoplus_i S_i.$$  \hspace{1cm} (70)

Therefore, the problem is equivalent to verifying the multiplicativity of the infimum of

$$\det[\tilde{\gamma}(l, z) + \tilde{Y}'] = F_2(\Phi_{1, Y})/4^n,$$  \hspace{1cm} (71)

which we know to hold true because of Proposition 2. □

Note that Proposition 2 also implies multiplicativity for other multi-mode Gaussian channels. In particular if $X_i = x_i S_i$ are proportional to symplectic transformations with $x_i > 0$, then multiplicativity holds for any $\Phi_{\otimes X_i \otimes Y_i}$ within the entire range $p \in (1, \infty)$. 


VI. COMMENTS AND OUTLOOK

We have addressed the multiplicativity of the maximal output $p$-purities of tensor products of Gaussian channels described by Eq. (13). We have proved that, restricting to Gaussian inputs, the maximal output $p$-purities are multiplicative for any $p \in (1, \infty)$ for single-mode channels with $X = \oplus_{i=1}^{n} X_i$, $Y = \oplus_{i=1}^{n} Y_i$ if $\det X_i$ is the same for $i = 1, \ldots, n$, and that the ordinary ‘purity’, corresponding to $p = 2$, is multiplicative for any generic choice of multi-mode Gaussian channels. In particular, the maximal output purity is multiplicative for identical $n$-fold single-mode Gaussian channels for all values of $p \in (1, \infty)$ and in this case the optimal product output (which is independent of $p$) majorizes any other Gaussian output state.

The restriction to Gaussian states, formally expressed by the definition of Gaussian maximal output purity of Eq. (31), is here motivated by essentially three arguments. Firstly, the question is interesting in its own right: Gaussian states have a prominent role in quantum information and communication with continuous variables, where many protocols completely rely on such states. Colloquially, one may say that the results indicate that entangled Gaussian input states suffer more decoherence than uncorrelated ones. The results and arguments presented in this paper constitute a strong hint towards the multiplicativity of the maximal output purities of general products of Gaussian channels under Gaussian inputs. In other words, it seems plausible that, in a fully Gaussian setting, input entanglement does not help to better preserve output purity of quantum channels. Notice also that our proofs of multiplicativity encompass several instances of interest. In particular, Proposition 1 represents the case of the subsequent uses of a single channel, where input correlations could be distributed in time over the global input. In such an instance our result proves that Gaussian entangled input states suffer more decoherence than uncorrelated states. We mention as well that Proposition 3 includes the relevant case of dissipation of multi-mode systems in Gaussian reservoirs, provided that the coupling to the reservoir is the same for any mode, but allowing for generally different reservoir states in different modes [see Eq. (22)].

Secondly, the maximal output purities under Gaussian inputs readily deliver bounds for the maximal output purities of Gaussian channels, not restricting to Gaussian inputs. Following the results presented in Refs. [2] one might conjecture that these bounds are tight and that Gaussian input states are already optimal.

Thirdly, the issues considered here could be the first steps towards a general theory of quantum information of Gaussian states, linking output purities to the Gaussian instance of the entanglement of formation [13] and Gaussian versions of channel capacities. As such, the Gaussian picture would deliver a convenient and powerful testbed in entanglement theory, in an instance for which a complete solution for the seemingly unrelated additivity and multiplicativity problems may be anticipated. It is the aim for future work to establish this connection in generality.

Acknowledgements

This work has been supported by the DFG (Schwerpunktprogramm QIV, SPP 1078) and the European Commission (IST-2001-38877). We thank K.M.R. Audenaert for motivating us to make these notes public, and M.B. Plenio for discussions.

APPENDIX A: GENERATORS OF THE SYMPLECTIC GROUP

As can be easily verified from the expression of the condition $S^T \sigma S = \sigma$, the symplectic group $Sp(2n, \mathbb{R})$ is generated by those matrices which can be written as $K = \sigma J$, where $J$ is a symmetric $2n \times 2n$ matrix [12]. The antisymmetric generators result in orthogonal symplectic transformations, giving rise to the compact subgroup $K(n) = Sp(2n, \mathbb{R}) \cap SO(2n)$. Such a subset of transformations is constituted by ‘energy preserving’ or passive operations. In contrast, the symmetric generators generate the non compact subset of the group (made up of active transformations, like squeezings). A basis of such generators can be built by means of transformations affecting only 1 or 2 modes at a time. We define

$$ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} $$

and recall the definition of $\sigma$ in Eq. (5) (to be understood in the single mode, $2 \times 2$ instance). Single mode transformations are generated by

$$ \sigma, \quad -\sigma \beta = \delta, \quad \sigma \delta = \beta, $$

where $\sigma$ generates the compact single mode rotations while $\beta$ and $\delta$ generate single mode squeezings.

Two-mode transformations (corresponding to the compact set) are generated by

$$ \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. $$

Whereas two-mode transformations (corresponding to the non-compact set) are generated by

$$ \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}. $$

The complete set of generators

$$ \{ K_i : i = 1, \ldots, 2n^2 + n \} $$

is described by Eq. (A2) for any mode and by Eqs. (A3) and (A4) for any couple of modes. The total number of independent generators is

$$ 3n + 4n(n - 1)/2 = 2n^2 + n. $$

The number of generators of the compact subgroup, which we refer to as $\{ L_i : i = 1, \ldots, n^2 \}$ in this paper, is $n + 2n(n - 1)/2 = n^2$. 

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APPENDIX B: SYMPLECTIC PERTURBATIONS

We consider a covariance matrix of $n$ modes in Williamson form

$$\gamma = \bigoplus_{j=1}^{n} \begin{pmatrix} \nu_j^+ & 0 \\ 0 & \nu_j^- \end{pmatrix},$$

and investigate the variations of the symplectic eigenvalues $\nu_1^+, \ldots, \nu_n^+$ under an additive perturbation. Let us consider $\gamma + kP$, with $k \in \mathbb{R}^+$, and $P$ being a symmetric $2n \times 2n$ matrix, partitioned in terms of $2 \times 2$ submatrices $P_{ij}$ as

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{12}^T & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & P_{n-1,n} \\ P_{n1}^T & \cdots & P_{n1}^T & P_{nn} \end{pmatrix}.$$ (B2)

The eigenvalues of the matrix $i\sigma \gamma$ are given by

$$\{+\nu_1^+, -\nu_1^+, \ldots, +\nu_n^+, -\nu_n^+\},$$

with eigenvectors

$$v_{j+} = (0, \ldots, 0, i, 0, \ldots, 0)^T, \quad \text{mode } j$$

$$v_{j-} = (0, \ldots, 0, 1, 0, \ldots, 0)^T, \quad \text{mode } j$$

for $j = 1, \ldots, n$, so that $i\sigma \gamma v_{j+} = \mp \nu_j^+ v_{j+}$. Now, one has

$$\left. \frac{d}{dk} \right|_{k=0} \nu_j^+ = v_{j+} (i\sigma P) v_{j+}^T = v_{j+} P v_{j+}^T = \text{tr} P_{jj} = \text{tr}_j P,$$ (B6)

where we have defined $\text{tr}_j$ as the trace of the leading submatrix associated to mode $j$. The first order derivative of the symplectic eigenvalue $\nu_j^+$ is just given by the trace of the $2 \times 2$ principal submatrix related to mode $j$ of the matrix embodying the perturbation.

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