Finite groups of bimeromorphic selfmaps of uniruled Kähler threefolds

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Abstract. We classify uniruled compact Kähler threefolds whose groups of bimeromorphic selfmaps do not have the Jordan property.

Keywords: Jordan group, Kähler manifold, bimeromorphic map, rationally connected fibration.

§ 1. Introduction

Groups of automorphisms and bimeromorphic selfmaps of complex manifolds can have a very complicated structure. In many cases it is relatively easy to study them on the level of finite subgroups. Although even in the simplest situations such groups can contain infinitely many non-isomorphic finite subgroups, it often happens that certain important parameters of these subgroups are bounded. An example of such behavior is provided by the Jordan property.

Definition 1.1 (see [1], Definition 2.1). A group $\Gamma$ is said to be Jordan (or has the Jordan property) if there exists a constant $J = J(\Gamma)$ such that any finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \subset G$ of index at most $J$.

An old theorem due to C. Jordan states that the groups $\text{GL}_n(\mathbb{C})$ enjoy this property (see, for example, [2], Theorem 36.13). J.-P. Serre pointed out that this is also the case for certain groups of geometric origin; namely, he proved that the group of birational selfmaps of the projective plane over a field of zero characteristic is Jordan (see [3], Theorem 5.3, and [4], Theorem 3.1). Yu.G. Zarhin found an example of an algebraic surface whose group of birational selfmaps is not Jordan [5] and V.L. Popov classified all such surfaces.

Theorem 1.2 (see [1], Theorem 2.32). Let $X$ be an algebraic surface over the field $\mathbb{C}$ of complex numbers. Then the group of birational selfmaps of $X$ is not Jordan if and only if $X$ is birational to the direct product $E \times \mathbb{P}^1$, where $E$ is an elliptic curve.

There are certain results concerning the Jordan property for birational automorphism groups of higher dimensional algebraic varieties (see [6], Theorem 1.8, [7], Theorem 1.8, and [8], Theorem 1.1). Furthermore, for birational automorphism groups of the projective plane and the three-dimensional projective space the bounds for the corresponding constants are known (see [9] and [10]). Some of

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these bounds can be made more precise if instead of arbitrary finite groups one considers a more restricted class of groups, for instance, finite $p$-groups; see [11] and [12], §3.3. Also, there are numerous results on the Jordan property for diffeomorphism groups of smooth manifolds and other groups of this kind; see [13]–[20].

For algebraic threefolds one has the following analogue of Theorem 1.2.

**Theorem 1.3** (see [21], Theorem 1.8). Let $X$ be a three-dimensional algebraic variety over $\mathbb{C}$. Then its group of birational selfmaps is not Jordan if and only if $X$ is birational either to $E \times \mathbb{P}^2$, where $E$ is an elliptic curve, or to $S \times \mathbb{P}^1$, where $S$ is a surface of one of the following types:

1) an abelian surface,
2) a bielliptic surface,
3) a surface of Kodaira dimension 1 such that the Jacobian fibration of its pluricanonical fibration is locally trivial (in the Zariski topology).

Attempts have recently been made to study the groups of automorphisms and bimeromorphic selfmaps of complex manifolds from the point of view of the Jordan property. In particular, in [22] (see also [23]) the authors obtained a generalization of Theorem 1.2 for the case of compact complex surfaces. The Jordan property for automorphism groups of three-dimensional Moishezon compact complex manifolds was proved in [24]. However, for arbitrary compact complex manifolds of higher dimension the situation is still unclear because of the lack of appropriate techniques that would enable one to work with their automorphism groups. On the other hand, it is known that compact Kähler manifolds exhibit many similarities with algebraic varieties, in particular, on the level of automorphism groups (cf. [25]).

In this paper we prove the following companion result of Theorem 1.3.

**Theorem 1.4.** Let $X$ be a non-algebraic three-dimensional uniruled compact Kähler manifold. Suppose that the group of its bimeromorphic selfmaps is not Jordan. Then $X$ is bimeromorphic to the projectivization of a holomorphic vector bundle of rank 2 on a two-dimensional complex torus $S$ of algebraic dimension 1. Moreover, if the algebraic dimension of $X$ is equal to 2, then $X$ is bimeromorphic to the direct product $\mathbb{P}^1 \times S$.

**Remark 1.5.** Let $S$ be a complex torus of positive algebraic dimension. Then the group of bimeromorphic selfmaps of $\mathbb{P}^1 \times S$ is not Jordan; see [26], Theorem 1.9. Furthermore, there are examples of (non-trivial) decomposable holomorphic vector bundles $\mathcal{E}$ of rank 2 on $S$ such that the group of bimeromorphic selfmaps of the projectivization of $\mathcal{E}$ is not Jordan; see [26], Theorems 1.10 and 1.12. We do not know whether one can choose such a projectivization in such a way that it has algebraic dimension 1 in the case when $\dim S = 2$ nor whether one can construct an example like this with an indecomposable vector bundle $\mathcal{E}$.

In §2 we recall some necessary auxiliary assertions. In §3 we recall the definitions and basic properties of uniruled and rationally connected manifolds and also the basic properties of maximal rationally connected fibrations. In §4 we study the interaction between the maximal rationally connected fibration and the group of bimeromorphic selfmaps of a compact complex manifold. In §5 we study properties of conic bundles over non-algebraic compact complex surfaces. Many of the results in this section are stated and proved in a more general form than we actually need here; we hope that this will fill an existing gap in the literature. In §6
we study the groups of bimeromorphic selfmaps of three-dimensional compact complex manifolds fibred into rational curves over a surface of algebraic dimension 0. In § 7 we study the groups of bimeromorphic selfmaps of three-dimensional compact complex manifolds fibred into rational curves over a surface of algebraic dimension 1 and complete the proof of Theorem 1.4.

We use the following notation and conventions. A complex manifold is an irreducible smooth reduced complex space. A morphism is a holomorphic map of complex manifolds. We denote by \(a(X)\) the algebraic dimension of a compact complex manifold \(X\) and by \(\kappa(X)\) its Kodaira dimension. By a typical point of a complex manifold \(X\) we mean a point in a non-empty subset of the form \(Z \setminus \Delta\), where \(\Delta\) is a closed analytic subset in \(Z\). A typical fibre of a morphism of complex manifolds is defined in a similar way.

Let \(\tau: X \to Y\) be a meromorphic map of complex manifolds and \(\gamma: X \to X\) a bimeromorphic map. Then we will say that the action of \(\gamma\) is fibrewise with respect to \(\tau\) if for every fibre \(F\) of \(\tau\) such that \(\gamma\) is defined at at least one point of \(F\), the image of every such point under \(\gamma\) is again contained in \(F\). If \(X\) is a compact complex manifold, then \(K_X\) will denote the canonical line bundle on \(X\). We denote by \(\Omega^p_X\) the vector bundle of holomorphic \(p\)-forms on \(X\). The Hodge numbers \(h^{p,q}(X)\) are defined as the dimensions of the cohomology groups \(H^q(X, \Omega^p_X)\). In particular, if \(\dim X = n\), then the geometric genus of \(X\) is defined as \(p_g(X) = h^{n,0}(X)\).

We denote the arithmetic genus of a projective scheme \(D\) by \(p_a(D)\).

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§ 2. Preliminaries

In this section we recall some necessary facts about complex manifolds and meromorphic maps.

Let us say that a group \(\Gamma\) has bounded finite subgroups if there exists a constant \(B = B(\Gamma)\) such that the order of any finite subgroup of \(\Gamma\) does not exceed \(B\).

**Lemma 2.1.** Let

\[1 \to \Gamma' \to \Gamma \to \Gamma''\]

be an exact sequence of groups. Suppose that \(\Gamma''\) has bounded finite subgroups. Then \(\Gamma\) is Jordan (respectively, has bounded finite subgroups) if and only if \(\Gamma'\) is Jordan (respectively, has bounded finite subgroups).

**Proof.** Obvious. \(\square\)

We will need some basic facts about the bimeromorphic geometry of complex manifolds. The reader can find more details in [27] and [28].

**Definition 2.2.** A proper morphism \(f: X \to Y\) of (not necessary compact) complex manifolds is called a (proper) modification if there exist closed analytic subsets \(V \subset X\) and \(W \subset Y\) such that \(f\) induces an isomorphism \(X \setminus V \cong Y \setminus W\).

**Definition 2.3.** Let \(X\) and \(Y\) be complex manifolds. A meromorphic map \(f: X \to Y\) is a map \(X \to 2^Y\) from \(X\) to the set \(2^Y\) of subsets of \(Y\) such that its graph

\[\text{Graph}_f = \{(x, y) \mid y \in f(x)\} \subset X \times Y\]
is an irreducible closed analytic subset of the complex manifold $X \times Y$ and the first projection
\[ \text{pr}_X : \text{Graph}_f \to X \]
is a modification. For every meromorphic map $f : X \to Y$ there exists a minimal closed analytic subset $V \subset X$ such that the restriction $f|_{X \setminus V}$ is holomorphic. This subset is called the indeterminacy locus of $f$. A typical fibre of $f$ is a typical fibre of the projection
\[ \text{pr}_Y : \text{Graph}_f \to Y. \]

A meromorphic map is said to be bimeromorphic if the projection $\text{pr}_Y$ is also a modification. The set of bimeromorphic maps $X \to Y$ is a group, which we denote by $\text{Bim}(X)$.

Note that in the case when $X$ and $Y$ are smooth complex projective algebraic varieties, according to the GAGA principle, the graph $\text{Graph}_f$ of a meromorphic map $f : X \to Y$ is an algebraic subvariety of $X \times Y$ and thus $f$ is also a rational map in this case. In particular, for a smooth complex projective algebraic variety $X$ the group $\text{Bim}(X)$ defined above coincides with the group of birational automorphisms of $X$.

It is clear that a bimeromorphic map $f : X \to Y$ of compact complex manifolds induces an isomorphism
\[ f^* : \mathcal{M}(Y) \cong \mathcal{M}(X) \]
of the fields of meromorphic functions. However, in contrast to the algebraic case, such an isomorphism does not usually define the map $f$.

Recall that a complex manifold is said to be Kähler if it has a Hermitian metric such that the corresponding $(1,1)$-form $\omega$ is closed, and in this case $\omega$ is called the Kähler form. All complex tori are Kähler manifolds. Examples of Kähler manifolds covered by rational curves can be obtained from the following statement.

**Theorem 2.4** (see [29], Proposition 3.18). Let $X$ be a compact Kähler manifold and $E$ a holomorphic vector bundle on $X$. Then the projectivization of $E$ is a Kähler manifold.

We need the following sufficient condition for the algebraicity of Kähler manifolds.

**Proposition 2.5.** Let $X$ be a compact Kähler manifold such that $H^0(X, \Omega^2_X) = 0$. Then $X$ is a projective algebraic variety.

*Proof.* In this case $H^2(X, \mathbb{C}) = H^{1,1}(X)$ and so the $(1,1)$-form associated with a Kähler metric is integral (that is, $X$ is a Hodge manifold). Therefore, it is projective by the Kodaira criterion [30]. □

We will also use another general sufficient condition for algebraicity. Recall that an $n$-dimensional complex manifold is said to be Moishezon if the transcendence degree of its field of meromorphic functions is equal to $n$. It is well known that a compact Kähler Moishezon manifold is a projective algebraic variety; see [31], Theorem 11.

**Lemma 2.6** (see [27], Proposition 12.2). Let $X$ be a compact complex manifold, $Y$ a Moishezon manifold and $h : X \to Y$ a morphism. Assume that a typical fibre
of $h$ is a rational curve. Then $X$ is Moishezon. Moreover, if $X$ is Kähler, then it is a projective algebraic variety.

The automorphism groups of Kähler manifolds have some nice properties.

**Theorem 2.7** (see [25]). Let $X$ be a compact Kähler manifold. Then the group $\text{Aut}(X)$ is Jordan.

The Jordan property is also known to hold for automorphism groups of certain special compact complex manifolds.

**Lemma 2.8** (see [32], Corollary 5.9). Let $S$ be a complex torus, $X$ a compact complex manifold and $\tau: X \to S$ a flat surjective morphism a typical fibre of which is isomorphic to $\mathbb{P}^1$. Suppose that $X$ is not bimeromorphic to the projectivization of a holomorphic vector bundle of rank 2 on $S$. Then the group $\text{Bim}(X)$ is Jordan.

Throughout the paper we will frequently use the notion of the algebraic reduction of a compact complex manifold. For such a manifold $X$, its algebraic reduction is a meromorphic map $\theta: X \to Y$ with connected fibres to a projective variety $Y$ of dimension $\dim Y = a(X)$ such that the fields of meromorphic functions of $X$ and $Y$ are isomorphic to each other; we refer the reader to [27], §I.3 for details. Note that in the case when $\dim X = 2$ the algebraic reduction is a holomorphic map and a typical fibre is an elliptic curve provided that $a(X) = 1$; see [33], Proposition VI.5.1.

We will need some auxiliary assertions about compact complex surfaces. The following fact is well known.

**Lemma 2.9** (see, for example, [34], Lemma 2.1). Let $S$ be a compact complex surface. Suppose that $S$ contains two divisors $C_1$ and $C_2$ such that $C_2^2 \geq 0$ and $C_1 \cdot C_2 > 0$. Then $S$ is algebraic.

**Corollary 2.10.** Let $S$ be a compact complex surface of algebraic dimension 1. Then all the curves on $S$ are contained in the fibres of its algebraic reduction.

**Proof.** Suppose that there exists an irreducible curve $Z$ on $S$ that is not contained in a fibre of the algebraic reduction $\theta$ of $S$. Let $F$ be a typical fibre of $\theta$. Then $F^2 = 0$ and $F \cdot Z > 0$. Thus when $n \gg 0$ one has $(Z + nF)^2 > 0$ and $(Z + nF) \cdot F > 0$. The assertion now follows from Lemma 2.9. □

Recall that a compact complex surface $S$ is said to be minimal if it does not contain smooth rational curves with self-intersection index $-1$; see the Kodaira–Enriques classification of minimal compact complex surfaces in [33], Chapter VI. Recall, in particular, that a compact complex surface of non-negative Kodaira dimension is non-ruled (that is, it is not covered by rational curves). Any non-algebraic Kähler compact complex surface has non-negative Kodaira dimension.

**Lemma 2.11** (see, for example, [22], Proposition 3.5). Let $S$ be a non-ruled minimal compact complex surface. Then $\text{Bim}(S) = \text{Aut}(S)$.

**Lemma 2.12** (see [34], Proposition 1.2 and Lemma 2.4). Let $S$ be either a Kodaira surface or a minimal compact complex surface with $\kappa(S) = 1$. In the former case we define the elliptic fibration $\phi: S \to C$ as the algebraic reduction of $S$ and in the latter as the pluricanonical fibration. Then the image of the group $\text{Aut}(S)$ in $\text{Aut}(C)$ is finite.
**Proposition 2.13** (see [34], Theorem 1.1). *Let $S$ be a compact complex surface with $\kappa(S) \geq 0$. When the group $\text{Bim}(S)$ of bimeromorphic selfmaps of $S$ has unbounded finite subgroups, $S$ is bimeromorphic to a surface of one of the following types:*

1) a complex torus,
2) a bielliptic surface,
3) a Kodaira surface,
4) a surface of Kodaira dimension 1.

Moreover, for the first three surfaces the group $\text{Bim}(S)$ always has unbounded finite subgroups.

The following assertion is well known but we provide a proof for the reader’s convenience.

**Lemma 2.14.** *Let $S$ be a non-algebraic compact complex surface and $D$ a non-zero effective divisor on $S$. Then $p_a(D) \leq 1$.***

**Proof.** If $a(S) = 1$, then by Corollary 2.10 every effective divisor $D$ on $S$ is contained in the fibres of the algebraic reduction of $S$. The latter is an elliptic fibration and thus in this case it is obvious that $p_a(D) \leq 1$. Therefore, we can assume that $a(S) = 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

of sheaves. It gives the exact sequence

$$\cdots \to H^1(S, \mathcal{O}_S) \to H^1(D, \mathcal{O}_D) \to H^2(S, \mathcal{O}_S(-D)) \to \cdots$$

of cohomology groups. Thus we obtain that

$$h^1(D, \mathcal{O}_D) \leq h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S(-D)).$$

Serre duality implies that

$$h^1(D, \mathcal{O}_D) \leq h^1(S, \mathcal{O}_S) + h^0(S, \mathcal{K}_S \otimes \mathcal{O}_S(D)).$$

Since $a(S) = 0$, the space of global sections of any line bundle on $S$ has dimension at most 1. Hence,

$$h^1(D, \mathcal{O}_D) \leq h^1(S, \mathcal{O}_S) + 1.$$

Recall that

$$h^1(S, \mathcal{O}_S) = h^{0.1}(S) \leq h^{1.0}(S) + 1;$$

see, for example, [33], Theorem IV.2.7. Furthermore, since $a(S) = 0$, one has $h^{1.0}(S) \leq 2$; see [33], Proposition IV.8.1(ii). Therefore, we obtain that

$$h^1(D, \mathcal{O}_D) \leq h^{0.1}(S) + 1 \leq h^{1.0}(S) + 2 \leq 4. \quad (2.1)$$

This gives

$$p_a(D) = h^1(D, \mathcal{O}_D) - h^0(D, \mathcal{O}_D) + 1 \leq h^1(D, \mathcal{O}_D) \leq 4. \quad (2.2)$$
Suppose that \( p_a(D) > 1 \). Then (2.2) implies that
\[
h^1(D, \mathcal{O}_D) > 1
\]
and thus \( h^{0,1}(S) > 0 \) by (2.1). In particular, we have \( \text{rk} H^1(S, \mathbb{Z}) > 0 \). Hence for any positive integer \( n \) there exists an unramified \( n \)-fold covering \( \pi: S' \to S \) (see, for example, [33], Proposition I.18.1(i)). Put \( D' = \pi^*(D) \). Then \( D' \) is a non-zero effective divisor on a compact complex surface \( S' \). Since \( S' \) has zero algebraic dimension, the above argument shows that \( p_a(D') \leq 4 \). On the other hand, we have
\[
6 \geq 2p_a(D') - 2 = \deg(\mathcal{K}_{S'} \otimes \mathcal{O}_{S'}(D'))|_{D'} = \deg(\pi^*\mathcal{K}_S \otimes \pi^*\mathcal{O}_S(D))|_{D'} = n \cdot \deg(\mathcal{K}_S \otimes \mathcal{O}_S(D))|_{D} = n \cdot (2p_a(D) - 2) \geq 2n.
\]
This contradiction shows that the inequality \( p_a(D) > 1 \) is impossible. □

Lemma 2.15. Let \( S \) be a compact Kähler surface of algebraic dimension 0 and \( D \) a connected reduced effective divisor on \( S \). Then \( p_a(D) = 0 \).

Proof. In this case the minimal model \( S_{\text{min}} \) of \( S \) is either a complex torus or a K3 surface. If \( p_a(D) > 0 \), then \( D \) varies in a positive-dimensional algebraic family. On the other hand, since \( a(S_{\text{min}}) = 0 \), the surface \( S_{\text{min}} \) contains at most a finite number of curves; see [33], Theorem IV.8.2. Hence the support of any connected divisor on \( S_{\text{min}} \) is a tree of smooth rational curves. Therefore, the same holds for \( S \). □

Remark 2.16. Let \( S \) be a non-algebraic compact complex surface. Then \( S \) contains at most a finite number of rational curves. Indeed, if \( a(S) = 0 \), then \( S \) contains only a finite number of curves altogether [33], Theorem IV.8.2. If \( a(S) = 1 \), then it follows from Corollary 2.10 that all the curves on \( S \) are contained in the fibres of its algebraic reduction and it remains to notice that a typical fibre of the latter is an elliptic curve. A similar argument shows that a non-algebraic compact complex surface contains at most a finite number of singular curves.

§ 3. Uniruled manifolds

In this section we recall the definitions and main properties of uniruled and rationally connected manifolds as well as the main properties of rationally connected fibrations.

A compact complex manifold is said to be uniruled if it can be covered by rational curves. More precisely, a compact complex manifold \( X \) is uniruled if there exist compact complex manifolds \( \mathcal{U} \) and \( \mathcal{Z} \) and morphisms
\[
\pi: \mathcal{Z} \to \mathcal{U} \quad \varphi: \mathcal{U} \to X
\]
such that a typical fibre of \( \pi \) is a smooth rational curve and \( \varphi \) is surjective and does not contract a typical fibre of \( \pi \). This is equivalent to the existence of a compact complex manifold \( Y \), a holomorphic vector bundle \( \mathcal{E} \) of rank 2 on \( Y \) and a dominant meromorphic map
\[
f: \mathbb{P}_Y(\mathcal{E}) \to X
\]
that does not factor through the projection \( \mathbb{P}_Y(\mathcal{E}) \to Y \) (see [35], Lemma 2.2). It is clear that uniruledness is a bimeromorphic invariant.
Proposition 3.1 ([35], p. 691, Remark, and [36], Corollary IV.1.11). Let $X$ be a uniruled compact complex manifold. Then

$$H^0(X, (\mathcal{K}_X)^\otimes p) = 0$$

for all $p > 0$, that is, $\kappa(X) = -\infty$.

Theorem 3.2 (see [37] and [38]). Let $X$ be a compact Kähler manifold of dimension at most 3. Then $X$ is uniruled if and only if $\kappa(X) = -\infty$.

The following assertion is an analytic version of a well-known result of Abhyankar (see [36], VI.1.2).

Lemma 3.3. Let $f: X \to Y$ be a bimeromorphic map of compact complex manifolds and $D \subset X$ an irreducible divisor contracted by $f$. Then $D$ is bimeromorphic to a uniruled compact complex manifold.

Proof. Replacing $X$ by a resolution of singularities of the graph of $f$ (see [39]), we can assume that $f$ is holomorphic. According to the relative complex analytic version of Chow’s lemma [40], Corollary 2, $f$ is dominated by a projective morphism and we can replace it by this projective morphism. In other words, we can assume that $f$ is a blow-up of some coherent ideal sheaf $\mathcal{F}$ on $Y$. Then it follows from [39], Theorem 1.10, that the morphism $f$ is dominated by a composite $f': X' \to Y$ of blow-ups with smooth centres. All the exceptional divisors of the latter morphism are uniruled compact complex manifolds. □

A compact complex manifold $X$ is said to be rationally connected if two typical points of $X$ can be connected by a rational curve. More precisely, $X$ is rationally connected if there exist compact complex manifolds $U$ and $Z$ and morphisms

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi} & U \\
\downarrow & & \downarrow \varphi \\
& X & 
\end{array}
$$

such that a typical fibre of $\pi$ is a rational curve and the induced map

$$\varphi^2: U \times_Z U \to X \times X$$

is surjective. This definition is taken from [36], Definition IV.3.2, and is rather different from Definition 3.1 in [41]. It is easy to see that they are equivalent. It is also clear that rational connectedness is a bimeromorphic invariant.

Proposition 3.4 (see [36], Corollary IV.3.8). Let $X$ be a rationally connected compact complex manifold. Then $X$ carries no non-trivial global holomorphic (pluri)forms:

$$H^0(X, (\Omega_X^1)^\otimes p) = 0$$

for all $p > 0$.

Proof. This fact was proved in [36], Corollary IV.3.8, for projective algebraic manifolds. The proof works in the general case as well. □
Let $X$ and $S$ be compact complex manifolds. A dominant meromorphic map $f: X \rightarrow S$ is called a rationally connected fibration if a typical fibre is irreducible and rationally connected. A rationally connected fibration is said to be maximal if for a sufficiently general fibre $X_s$ and a sufficiently general point $x \in X_s$ there is no rational curve $C \subset X$ that passes through $x$ and is not contained in $X_s$. Here a sufficiently general point means a point in the complement of the union of a countable number of proper closed analytic subsets, and a sufficiently general fibre means a fibre over a sufficiently general point. If such a map exists, then it is unique; in particular, it is equivariant under the action of the group of bimeromorphic automorphisms of the complex manifold. Maximal rationally connected fibrations exist for arbitrary compact Kähler manifolds [42], Theorem 2.3 and Remark 2.8.

Remark 3.5. A compact complex manifold is rationally connected if and only if the base of its maximal rationally connected fibration is a point.

Using Proposition 2.5, it is easy to deduce the following result. Recall that in the category of compact complex spaces there exists a resolution of singularities (see, for example, [39]).

Corollary 3.6. Let $X$ be a compact Kähler manifold and $\tau: X \rightarrow S$ a rationally connected fibration. Assume that $X$ is not algebraic. Then $h^{2,0}(S) \neq 0$.

Proof. Replacing $X$ by another bimeromorphic model, we can assume that the map $\tau$ is holomorphic. If $h^{2,0}(S) = 0$, then $h^{2,0}(X) = 0$ (because the fibres of $\tau$ are covered by rational curves). Therefore, $X$ is algebraic by Proposition 2.5, which contradicts our assumptions. □

Corollary 3.7. Let $X$ be a non-algebraic compact Kähler manifold and $\tau: X \rightarrow S$ a rationally connected fibration. Then $\dim S \geq 2$. In particular, $X$ is not rationally connected. If $\dim S = 2$, then $S$ is a Kähler surface with $p_g(S) > 0$. In particular, the Kodaira dimension of $S$ is non-negative.

Proof. According to Corollary 3.6, we have $h^{2,0}(S) \neq 0$ and therefore the dimension of $S$ cannot be less than 2. In particular, $S$ is not a point, that is, by Remark 3.5, $X$ is not rationally connected. If $S$ is a surface, then according to the inequality above, $p_g(S) > 0$ holds and so $\kappa(S) \geq 0$. Moreover, the surface $S$ is Kähler by [43], Theorem 5, because Kählerness is preserved under bimeromorphic maps of compact complex surfaces. □

The following assertion is a partial generalization of Corollary 3.7. Note that in the proofs of the main results of this paper such generality is not needed.

Proposition 3.8 (see [44] and [45], §3). Let $X$ be a compact Kähler manifold and $\tau: X \rightarrow Y$ the maximal rationally connected fibration. Then $Y$ is not uniruled.

Proof. Let $\phi: Y \rightarrow Z$ be the maximal rationally connected fibration of $Y$. Suppose that $\dim Z < \dim Y$. We can assume that $\tau$ and $\phi$ are holomorphic maps. Also, we may assume that the manifold $Y$ is Kähler; see [43], Theorem 5. Let $F$ be a typical fibre of the composite $\phi \circ \tau$. Then $F$ is a Kähler manifold. Hence there exists a rationally connected fibration $F \rightarrow \tau(F)$ over the rationally connected base $\tau(F)$. As in the proof of Corollary 3.6, we obtain $h^{2,0}(F) = h^{2,0}(\tau(F)) = 0$. Thus $F$ is a projective algebraic variety. According to [44], Corollary 1.3, the
variety $F$ is rationally connected. Hence, we have the equality $\dim Z = \dim Y$, a contradiction \(\square\)

**Theorem 3.9** (cf. Corollary 3.7). Let $X$ be a compact Kähler rationally connected manifold. Then $X$ is a projective algebraic variety.

**Proof.** Note that the vector bundle $\Omega_X^2$ is a direct summand of the vector bundle $(\Omega_X^1)^{\otimes 2}$. Hence it follows from Proposition 3.4 that $H^0(X, \Omega_X^2) = 0$. It remains to apply Proposition 2.5. \(\square\)

There is a classification of non-algebraic uniruled compact Kähler threefolds which describes them in terms of their maximal rationally connected fibrations. We do not use this classification but provide it here for completeness.

**Theorem 3.10** (see [46], [47], Theorem 9.1, and [48], Theorem 1.2). Let $X$ be a compact Kähler threefold. Assume that $X$ is not algebraic and $\kappa(X) = -\infty$. Let $\eta: X \dashrightarrow B$ be the algebraic reduction and $\tau: X \dashrightarrow S$ the maximal rationally connected fibration. Then $\dim S = 2$ and one of the following cases occurs.

(i) $a(X) = 0$ and $a(S) = 0$: in this case $S$ is either a complex torus or a $K3$ surface.

(ii) $a(X) = 1$ and $a(S) = 0$: in this case $X$ is bimeromorphic to $\mathbb{P}^1 \times S$, where $S$ is either a complex torus or a $K3$ surface.

(iii) $a(X) = 1$ and $a(S) = 1$: in this case the algebraic reduction $\eta: X \dashrightarrow B$ can be decomposed as follows:

$$
\eta: X \dashrightarrow S \dashrightarrow B,
$$

where $\tau$ coincides with the relative Albanese map and $\beta$ is the algebraic reduction of $S$. A typical fibre $\eta$ has the form $\mathbb{P}(O \oplus \mathcal{L})$, where $\mathcal{L}$ is a non-torsion line bundle of degree 0 over an elliptic curve.

(iv) $a(X) = 2$ and $a(S) = 1$. There is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & B \\
\downarrow{\tau} & & \downarrow{C} \\
S & \xrightarrow{\beta} & B,
\end{array}
\]

where $C$ is a curve, $B \to C$ is a fibration with typical fibre $\mathbb{P}^1$ and $S \to C$ is an algebraic reduction. The induced map

$$X \dashrightarrow S \times_C B$$

is dominant.

§ 4. Maximal rationally connected fibrations

In this section we study a relation between maximal rationally connected fibrations and the structure of groups of bimeromorphic automorphisms of complex manifolds.

For a meromorphic map $\gamma: X \dashrightarrow Y$ of compact complex manifolds, we denote by $\text{ind}(\gamma)$ its indeterminacy locus. This is the minimal proper closed analytic subset
of $X$ such that the restriction $\gamma|_{X\setminus \text{ind}(\gamma)}$ is holomorphic. Recall that the codimension of $\text{ind}(\gamma)$ is at least 2.

Given a surjective morphism of compact complex manifolds $h: X \to Z$, we consider the subgroup $\text{Bim}(X)_h$ of $\text{Bim}(X)$ that consists of bimeromorphic self-maps whose action is fibrewise with respect to $h$. We also consider the subgroup $\text{Bim}(X)_{h}^{\text{hol}}$ of $\text{Bim}(X)_h$ consisting of bimeromorphic selfmaps $\gamma$ that are holomorphic on a typical fibre of $h$ (the set of such fibres may depend on $\gamma$).

**Lemma 4.1.** Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Let $G_i$, $i \in \mathbb{N}$, be a countable family of finite subgroups in $\text{Bim}(X)_h$ (respectively, in $\text{Bim}(X)_h^{\text{hol}}$). Then there exists a smooth, irreducible and reduced fibre $F$ of $h$ of dimension $\dim X - \dim Z$ such that all the groups $G_i$ are embedded in $\text{Bim}(F)$ (respectively, in $\text{Aut}(F)$). Moreover, if $\dim Z > 0$ and $\Xi$ is a countable union of proper closed analytic subsets of $Z$, then $F$ can be chosen in such a way that the point $h(F)$ does not lie in $\Xi$.

**Proof.** Let $\Delta \subset Z$ be the set of those points over which $h$ is not smooth.

Choose a non-trivial bimeromorphic map $\gamma$ contained in $\text{Bim}(X)_h$. Consider the set $\nabla_\gamma \subset Z$ consisting of all the points $P$ for which $\text{ind}(\gamma) \supset h^{-1}(P)$. Thus the map $\gamma$ is defined at a typical point of the fibre $h^{-1}(P)$ over $P \in Z \setminus \nabla_\gamma$.

Also consider the subset $\Delta_\gamma \subset Z \setminus \nabla_\gamma$, consisting of all the points $P$ such that for a typical point $Q \in h^{-1}(P)$ one has $\gamma(Q) = Q$. Then for any point

$$P \notin \Delta \cup \overline{\Delta_\gamma} \cup \nabla_\gamma$$

the map $\gamma$ defines an element $\gamma|_F$ of $\text{Bim}(F)$, where $F = h^{-1}(P)$. Since $\gamma$ is not trivial, neither is $\gamma|_F$. Moreover, if $\gamma \in \text{Bim}(X)_h^{\text{hol}}$, then

$$\gamma|_F \in \text{Aut}(F) \subset \text{Bim}(F).$$

It is obvious that $\Delta$, $\overline{\Delta_\gamma}$ and $\nabla_\gamma$ are proper closed analytic subsets of $Z$. In the case when $\dim Z > 0$, also fix a countable union $\Xi$ of proper closed analytic subsets of $Z$. Since $\mathbb{C}$ is uncountable, $Z$ cannot be represented as a union of a countable number of proper closed analytic subsets. Hence the complement

$$U = Z \setminus \left( \Delta \cup \bigcup_{\gamma \in \bigcup_i G_i, \gamma \neq \text{id}} (\overline{\Delta_\gamma} \cup \nabla_\gamma) \cup \Xi \right)$$

is not empty. It remains to note that the fibre of $h$ over any point of $U$ has the desired properties. □

**Corollary 4.2.** Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Assume that $\text{Bim}(X)_h$ (resp. $\text{Bim}(X)_h^{\text{hol}}$) is not Jordan. Then for a typical fibre $F$ of $h$, $\text{Bim}(F)$ (resp. the group $\text{Aut}(F)$) is not Jordan.

**Proof.** One can find countably many subgroups $G_i$, $i \in \mathbb{N}$, in $\text{Bim}(X)_h$ (resp. $\text{Bim}(X)_h^{\text{hol}}$) such that all the $G_i$ cannot simultaneously appear as subgroups of any Jordan group. On the other hand, by Lemma 4.1, all the groups $G_i$ are embedded in $\text{Bim}(F)$ (resp. $\text{Aut}(F)$) for a typical fibre $F$ of $h$. □
Corollary 4.3. Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Assume that $X$ is Kähler and a typical fibre of $h$ is rationally connected. Then $\text{Bim}(X)_h$ is Jordan.

Proof. Assume that $\text{Bim}(X)_h$ is not Jordan. We obtain from Corollary 4.2 that, for a typical fibre $F$ of $h$, $\text{Bim}(F)$ is not Jordan. On the other hand, the complex manifold $F$ is Kähler and rationally connected and is thus a projective algebraic variety by Theorem 3.9. Thus, $\text{Bim}(F)$ is Jordan by [6], Theorem 1.8, and [8], Theorem 1.1, a contradiction. \qedsymbol

Corollary 4.4. Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Assume that a typical fibre of $h$ has dimension $1$. Then $\text{Bim}(X)_h$ is Jordan.

Proof. Assume that $\text{Bim}(X)_h$ is not Jordan. We obtain from Corollary 4.2 that, for some smooth irreducible one-dimensional fibre $F$ of $h$, the group $\text{Bim}(F) = \text{Aut}(F)$ is not Jordan. This gives an obvious contradiction. \qedsymbol

Corollary 4.5. Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Assume that a typical fibre of $h$ has dimension $2$. Then $\text{Bim}(X)_{\text{hol}}^h$ is Jordan.

Proof. Assume that $\text{Bim}(X)_{\text{hol}}^h$ is not Jordan. We obtain from Corollary 4.2 that, for some smooth irreducible two-dimensional fibre $F$ of $h$, $\text{Aut}(F)$ is not Jordan. However, this is impossible by [22], Theorem 1.6. \qedsymbol

Corollary 4.6. Let $X$ and $Z$ be compact complex manifolds and $h: X \to Z$ a surjective morphism with connected fibres. Assume that $X$ is Kähler. Then $\text{Bim}(X)_{\text{hol}}^h$ is Jordan.

Proof. Assume that $\text{Bim}(X)_{\text{hol}}^h$ is not Jordan. We obtain from Corollary 4.2 that, for some smooth irreducible fibre $F$ of $h$, $\text{Aut}(F)$ is not Jordan. However this is impossible by Theorem 2.7 because $F$ is a compact Kähler manifold. \qedsymbol

We now prove the main result of this section.

Proposition 4.7. Let $X$ be a compact complex threefold and $S$ a compact complex surface. Let $\tau: X \dasharrow S$ be a rationally connected fibration. Assume that $S$ has non-negative Kodaira dimension and is not an algebraic surface. Finally, assume that $\text{Bim}(X)$ is not Jordan. Then $S$ is bimeromorphic to a complex torus.

Proof. Since $\kappa(S) \geq 0$, $S$ is not ruled. Therefore, $\tau$ is equivariant with respect to the whole of $\text{Bim}(X)$. Hence there is an exact sequence

$$1 \to \text{Bim}(X)_\tau \to \text{Bim}(X) \to \text{Bim}(S)$$

of groups.

We may assume that $\tau$ is holomorphic. Since $\text{Bim}(X)$ is not Jordan, $\text{Bim}(S)$ has unbounded finite subgroups. This follows from Corollary 4.3 (or Corollary 4.4) and Lemma 2.1.

Since the Kodaira dimension of $S$ is non-negative and $\text{Bim}(S)$ has unbounded finite subgroups, $S$ is bimeromorphic to a complex torus, a bielliptic surface, a Kodaira surface or a surface of Kodaira dimension 1; see Proposition 2.13. Since
S is not algebraic, it cannot be bimeromorphic to a bielliptic surface. We can assume that $S$ is minimal.

Suppose that either $S$ is a Kodaira surface or $\kappa(S) = 1$. In the former case consider its algebraic reduction $\phi: S \to C$. In the latter case consider the pluricanonical fibration $\phi: S \to C$ (which also coincides with the algebraic reduction under our assumptions). In both cases put $\psi = \phi \circ \tau$. These maps form a Bim($X$)-equivariant commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & C \\
\downarrow{\tau} & & \\
S & \xrightarrow{\phi} & C \\
\end{array}
$$

(4.1)

Consider the subgroups Bim($X$)$_\psi$ and Bim($X$)$_{\psi}^{\text{hol}}$ in Bim($X$). Let $\gamma$ be an arbitrary element of Bim($X$)$_\psi$. Since $a(S) = 1$, it follows from Corollary 2.10 that $S$ contains no curves that are projected surjectively to $C$. Therefore, $\tau(\text{ind}(\gamma))$ is contained in the union of finitely many fibres of $\phi$, and $\text{ind}(\gamma)$ is contained in the union of finitely many fibres of $\psi$. In other words, $\gamma \in \text{Bim}(X)_{\psi}^{\text{hol}}$. This implies that Bim($X$)$_\psi$ coincides with its subgroup Bim($X$)$_{\psi}^{\text{hol}}$. In particular, Bim($X$)$_\psi$ is Jordan by Corollary 4.5.

Since $S$ is minimal, Bim($S$) = Aut($S$) by Lemma 2.11. Consider the subgroup Aut($S$)$_{\phi}$ of Aut($S$) consisting of those automorphisms of $S$ whose action is fibrewise with respect to $\phi$. There are exact sequences

$$
1 \to \text{Bim}(X)_{\psi} \to \text{Bim}(X) \to \Gamma_{\psi} \quad \text{and} \quad 1 \to \text{Aut}(S)_{\phi} \to \text{Aut}(S) \to \Gamma_{\phi}
$$

of groups, where $\Gamma_{\psi}$ and $\Gamma_{\phi}$ are subgroups of Aut($C$). Since the diagram (4.1) is commutative, we see that $\Gamma_{\psi} \subset \Gamma_{\phi}$. On the other hand, $\Gamma_{\phi}$ is finite by Lemma 2.12. We now conclude from Lemma 2.1 that Bim($X$) is Jordan. The contradiction shows that $S$ is not a Kodaira surface and $\kappa(S) \neq 1$, that is, it is a complex torus. \(\square\)

Remark 4.8. If, in the notation of Proposition 4.7, the manifold $X$ is Kähler and $\tau$ is the maximal rationally connected fibration, then certain other hypotheses are satisfied automatically. Namely, $S$ has non-negative Kodaira dimension by Corollary 3.7 (and also $S$ is non-algebraic by Lemma 2.6 under the additional assumption that the dimension of $X$ is equal to 3). Moreover, in this case $S$ is Kähler by Corollary 3.7 and so we do not need to consider the case when $S$ is a Kodaira surface in the proof. Also, if $X$ is Kähler, we can use Corollary 4.6 instead of Corollary 4.5 in the proof.

Remark 4.9. If, in the notation of Proposition 4.7, $S$ is a complex torus, then $\tau: X \to S$ is nothing but the Albanese map. In particular, in this case $\tau$ is a holomorphic map.

§ 5. Conic bundles over non-algebraic surfaces

In this section we study the properties of conic bundles over non-algebraic compact complex surfaces. Many of the results in this section are stated and proved in greater generality than we need here.
Definition 5.1. A proper surjective morphism $f: X \to S$ of complex manifolds is called a conic bundle if any fibre of $f$ is isomorphic to a conic in $\mathbb{P}^2$. A conic bundle $f: X \to S$ is said to be standard if for any prime divisor $D \subset S$ its inverse image $f^{-1}(D)$ is irreducible.

Remark 5.2. According to Definition 5.1, for a conic bundle $f: X \to S$ the anti-canonical line bundle $K_X^{-1}$ is $f$-ample. In particular, $f$ is a projective morphism.

For a conic bundle $f: X \to S$ the set of points at which $f$ is not smooth forms a divisor $\Delta$ on $S$ called the discriminant divisor or the degeneracy divisor. The following assertion is well known; see, for example, [49], Corollary 3.3.3, (3.3.2), Corollary 3.9.1 and (3.8.2).

Lemma 5.3. Let $f: X \to S$ be a standard conic bundle over a compact complex surface $S$ and $\Delta$ its discriminant curve. Suppose that $\Delta \neq \emptyset$. Then $\Delta$ has only ordinary double singularities. Moreover, the fibre of $f$ over a point $o \in \Delta$ is reduced (and has two irreducible components) if and only if $o$ is a smooth point of $\Delta$.

Assume further that there exists an irreducible component $\Delta_1 \subset \Delta$ that is a smooth rational curve. Then the intersection of $\Delta_1$ and $\Delta - \Delta_1$ is non-empty and consists of an even number of points.

Remark-definition 5.4. Let $S$ be a non-algebraic compact complex surface and $S_{\text{min}}$ its minimal model. Then there is at most a finite number of singular curves on $S_{\text{min}}$ by Remark 2.16. Hence there exists a sequence of blow-ups $S' \to S_{\text{min}}$ such that any curve on $S'$ has only ordinary double singularities and $S'$ satisfies the following universal property: if $S'$ is a compact complex surface bimeromorphic to $S$ and such that any curve on $S'$ has only ordinary double singularities, then there is a modification $S' \to S$. In particular, such a surface $S'$ is unique. We call it the almost minimal surface or the almost minimal model of $S$.

Remark 5.5. For applications in the framework of this paper we need only conic bundles over surfaces bimeromorphic to complex tori. It is easy to see that the almost minimal model of any such surface is just a complex torus.

Remark 5.6. If a compact complex surface $S$ is non-algebraic, then by Lemma 2.14, any curve on $S_{\text{min}}$ (and also on $\overline{S}$ and $S$) has arithmetic genus at most 1. If in addition $S$ is Kähler and $a(S) = 0$, then any connected curve on $S_{\text{min}}$, $\overline{S}$ or $S$ has arithmetic genus 0 by Lemma 2.15.

The projective case of the following assertion was proved in [50]. An alternative proof was given in [51]. It is based on a result concerning three-dimensional extremal contraction; see, for example, [52].

Proposition 5.7 (see [53], Proposition 3.8). Let $X$ be a compact complex threefold and $S$ a non-algebraic compact complex surface. Let $f: X \to S$ be a rationally connected fibration. Let $\overline{S}$ be the almost minimal model of $S$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
X & \to & \overline{X} \\
\downarrow f & & \downarrow \overline{f} \\
S & \to & \overline{S},
\end{array}
$$

Finite groups of bimeromorphic selfmaps
where the horizontal arrows are bimeromorphic maps and \( \overline{f}: \overline{X} \to \overline{S} \) is a standard conic bundle. Here the discriminant curve of the conic bundle \( \overline{f} \) is either empty or a disjoint union of smooth elliptic curves, rational curves with one ordinary double point, and combinatorial cycles of smooth rational curves.

Proof. According to \([53]\), Proposition 3.8, there exists a standard conic bundle \( f': X' \to S' \) fibrewise bimeromorphic to \( f \), where \( S' \) is some bimeromorphic model of \( S \). Let \( \Delta' \) be the discriminant curve of \( f' \). By Lemma 5.3 and Remark 5.6, each connected component \( \Delta'_{(i)} \subset \Delta' \) is a smooth elliptic curve, a rational curve with one ordinary double point or a combinatorial cycle of smooth rational curves.

According to Remark-Definition 5.4, we have a modification \( \sigma: S' \to \overline{S} \). Assume that \( S' \neq \overline{S} \). Then the \( \sigma \)-exceptional locus contains a \((-1)\)-curve \( C' \). Again according to Lemma 5.3 and Remark 5.6, \( C' \) is disjoint from \( \Delta' \) or contained in it, and then

\[
C' \cdot (\Delta' - C') = 2.
\]

In this case we can apply to \( X' \) a sequence of bimeromorphic transformations as described in \([54]\), Lemma 4, or \([49]\), Proposition 8.5, contract all \((-1)\)-curves on \( S' \) and obtain a standard conic bundle over \( \overline{S} \). \( \square \)

Remark 5.8. Proposition 5.7 cannot be directly generalized to the case of algebraic surfaces. For example, over any projective algebraic surface \( S \) there exist non-standard conic bundles that are not bimeromorphic to standard conic bundles over a minimal model of \( S \).

Proposition 5.7 implies the following result.

Corollary 5.9. Let \( X \) be a compact complex threefold and \( S \) a non-algebraic compact complex surface bimeromorphic to a complex torus \( S_0 \). Let \( \tau: X \dashrightarrow S \) be a rationally connected fibration. Assume that \( X \) is not bimeromorphic to the projectivization of a vector bundle of holomorphic rank 2 on \( S_0 \). Then the group \( \text{Bim}(X) \) is Jordan.

Proof. By Proposition 5.7 and Remark 5.5, we can assume that the surface \( S = S_0 \) is a complex torus and the map \( \tau \) is holomorphic and is a conic bundle. In this case the desired assertion follows from Lemma 2.8. \( \square \)

The following proposition refines some of the results obtained in \([53]\). We do not use it in the proofs of main results of this paper.

Proposition 5.10. Let \( f: X \to S \) be a standard conic bundle, where \( X \) is a compact complex threefold and \( S \) is an almost-minimal non-algebraic compact complex surface. Let \( \Delta \subset S \) be the discriminant curve of \( f \). Then any bimeromorphic map \( \varphi: X \dashrightarrow X \) fits into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & X \\
\downarrow f & & \downarrow f \\
S & \xrightarrow{\delta} & S,
\end{array}
\]
where $\delta$ is an isomorphism. Moreover, $\varphi$ does not contract components of the divisor $f^{-1}(\Delta)$ and so $\delta(\Delta) = \Delta$.

**Proof.** Since $S$ is non-algebraic, the number of rational curves on $S$ is finite by Remark 2.16. Therefore, $f : X \to S$ is the maximal rationally connected fibration. Hence the map $\varphi$ is fibrewise, that is, we have the diagram (5.1) with a bimeromorphic map $\delta$. According to the universal property (see Remark-Definition 5.4), this map is an isomorphism.

For convenience, we write our map as $\varphi : X' \dashrightarrow X''$, where $X' = X'' = X$. Put

$$f' = \delta \circ f, \quad \Delta' = \delta(\Delta), \quad f'' = f \quad \text{and} \quad \Delta'' = \Delta.$$

Thus, $\Delta'$ (resp. $\Delta''$) is the discriminant curve of the conic bundle $f' : X' \to S$ (resp. $f'' : X'' \to S$) and the diagram (5.1) can be rewritten as the commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\varphi} & X'' \\
\downarrow f' & & \downarrow f'' \\
S & & \\
\end{array}
$$

Here the action of $\varphi$ is fibrewise. Assume that $\varphi$ contracts the irreducible surface $D' = f'^{-1}(\Lambda)$, where $\Lambda$ is an irreducible component of the discriminant curve $\Delta'$. Then the map $\varphi^{-1}$ must contract the irreducible surface $D'' = f''^{-1}(\Lambda)$. It follows from the commutativity of the diagram that

$$\Theta' = \varphi^{-1}(D'') \subset X'$$

is an irreducible curve and $f'(\Theta') = \Lambda$.

Let $o$ be a typical point of the curve $\Lambda$ and let $C' = f'^{-1}(o)$ and $C'' = f''^{-1}(o)$ be the fibres over the point $o$ of the maps $f'$ and $f''$, respectively. Since $o$ is a smooth point of $\Delta'$, Lemma 5.3 implies that the conic $C'$ has two irreducible components $C'_1$ and $C'_2$ meeting transversally at a point. If $\Theta'$ meets only one of the components of the fibre $C'$, then the same holds for nearby fibres over points of $\Lambda$. Consider the double cover parameterizing irreducible components of the fibres over the points of $\Lambda$, that is, the cover by the corresponding component of the Douady space, which is compact by [55] because the (singular) surface $D' = f'^{-1}(\Lambda)$ is bimeromorphic to a ruled surface (and, in particular, is Moishezon). According to the above, this double cover splits. We conclude that $D'$ is reducible, which gives a contradiction.

Therefore, both $C'_1$ and $C'_2$ must intersect $\Theta'$. On the other hand, we claim that $\text{ind}(\varphi) \supset \Theta'$. Indeed, otherwise $\varphi$ is defined at a typical point of $\Theta'$. By considering the graph

$$
\begin{array}{ccc}
\hat{X} & \xleftarrow{p'} & X' \\
\downarrow & & \downarrow \varphi \\
& X'' & \\
\end{array}
$$

of $\varphi$, we see that the map $p'^{-1}$ is defined at a typical point of $\Theta'$ and its inverse morphism $p'$ contracts on $\Theta'$ the irreducible divisor $\hat{D} \subset \hat{X}$ that is the proper transform of $D''$ under $p''$. This is clearly impossible. Therefore, one has $\text{ind}(\varphi) \supset \Theta'$ and thus both $C'_1$ and $C'_2$ intersect $\text{ind}(\varphi)$. 


Consider the germ of an analytic curve $\Upsilon \subset S$ meeting $\Lambda$ transversally at $o$ and consider the surfaces

$$V' = f'^{-1}(\Upsilon) \subset X' \quad \text{and} \quad V'' = f''^{-1}(\Upsilon) \subset X''.$$

Thus, $V'$ is the inverse image of $V''$ with respect to $\varphi$ and we have a bimeromorphic map $\varphi_V : V' \dashrightarrow V''$. The natural projections

$$f'_V : V' \to \Upsilon \quad \text{and} \quad f''_V : V'' \to \Upsilon$$

are fibrations whose typical fibre is a smooth rational curve and the curves $C' = V' \cap D'$ and $C'' = V'' \cap D''$ are fibres over the point $o$ of the maps $f'_V$ and $f''_V$, respectively. Note that $C''$ is contracted by $\varphi_V^{-1}$ so that $\varphi_V$ is not holomorphic. On the other hand, its indeterminacy locus $\text{ind}(\varphi_V)$ is contained in $C'$. Moreover, the set $\text{ind}(\varphi_V)$ coincides with $\text{ind}(\varphi)$ in a neighborhood of $C'$. Consider the resolution of indeterminacies

$$\begin{array}{ccc}
V' & \xrightarrow{\varphi_V} & V'' \\
\xrightarrow{f'_V} & & \xrightarrow{f''_V} \\
\xrightarrow{p} & & \xrightarrow{q} \\
\xrightarrow{\tilde{V}} & & \\
V' & \xrightarrow{\varphi} & V''.
\end{array} \quad (5.2)
$$

We may assume that this resolution is minimal, that is, it has the minimal possible Picard number. Then none of the $(-1)$-curves on $\tilde{V}$ can be simultaneously contracted by both $p$ and $q$. Hence there exists a $(-1)$-curve $\tilde{E}$ that is $q$-exceptional but not $p$-exceptional. Since $\varphi_V^{-1}$ is an isomorphism on the complement of $C''$, we conclude from the commutativity of the diagram (5.2) that $p(\tilde{E})$ is contained in $C'$. Then $p(\tilde{E})$ coincides with one of the irreducible components of $C'$, say, $p(\tilde{E}) = C'_1$. Since

$$C'_{1}^2 = -1 = \tilde{E}^2,$$

$p$ must be an isomorphism near $C'_1$. Thus, $\text{ind}(\varphi_V) = \varphi_V^{-1}(C'')$ does not intersect $C'_1$. Therefore, $\text{ind}(\varphi)$, which coincides with $\varphi^{-1}(D'')$ in a neighborhood of $C''$, also does not intersect $C'_1$. But this contradicts the observation made above. Hence $\varphi$ cannot contract the divisor $f'^{-1}(\Lambda)$. This proves the proposition. \(\square\)

§ 6. The case when $a(S) = 0$

In this section we consider compact complex threefolds having the structure of a rational curve fibration over a surface of algebraic dimension 0.

Remark 6.1. Let $f : X \to S$ be a standard conic bundle over a compact Kähler surface of algebraic dimension 0. Then $f$ is a smooth morphism such that all the fibres of $f$ are isomorphic to $\mathbb{P}^1$; see [53], Proposition 3.10. For proofs of our results we do not need this assertion; we will use the weaker (but more general) Proposition 5.7.

The following assertion is a particular case of [56], Corollary 3.1. We provide a proof for the reader’s convenience.
**Lemma 6.2.** Let $S$ be a compact complex surface that does not contain any curves (and, in particular, has algebraic dimension 0). Let $f : X \to S$ be a standard conic bundle. Then the group $\text{Bim}(X)$ acts on $X$ biholomorphically.

**Proof.** Since $S$ does not contain any (rational) curves, $f$ is the maximal rationally connected fibration. In particular, $f$ is equivariant under $\text{Bim}(X)$. Consider an arbitrary element $\varphi \in \text{Bim}(X)$. It induces a biholomorphic map $\delta : S \to S$ by Lemma 2.11 (because $S$ is minimal). As in the proof of Proposition 5.10, we write the map as $\varphi : X' \to X''$, where $X' = X'' = X$. Put

$$f' = \delta \circ f \quad \text{and} \quad f'' = f.$$

We have the commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\varphi} & X'' \\
\downarrow{f'} & & \downarrow{f''} \\
S & \xleftarrow{f} & S
\end{array}
$$

where the map $\varphi$ is fibrewise.

Assume that $\varphi$ contracts an irreducible divisor $D' \subset X'$. Then $D'$ is bimeromorphic to a ruled surface (see Lemma 3.3). This implies that $f'(D') \neq S$. Then since $S$ does not contain any curves, the image $f'(D')$ must be a point. But this is impossible because all the fibres of $f'$ are one-dimensional. Thus, $\varphi$ does not contract any divisors. The same holds for the inverse map $\varphi^{-1}$. Therefore, $\varphi$ is an isomorphism in codimension 1, that is, there exist closed analytic subsets $Z' \subset X'$ and $Z'' \subset X''$ of codimension 2 such that the restriction

$$\varphi_{|U'} = \varphi_{|U'} : U' \to U''$$

to the open subset $U' = X \backslash Z'$ is an isomorphism with the open subset $U'' = X \backslash Z''$. From the commutativity of the diagram we obtain that $f'(Z') = f''(Z'')$. Put $\Xi = f'(Z')$. Since $S$ does not contain any curves, $\Xi$ is a finite subset of $S$. Put $S_0 = S \backslash \Xi$. The sheaves $\mathcal{E}' = f_* \mathcal{K}_X^{-1}$ and $\mathcal{E}'' = f''_* \mathcal{K}_X^{-1}$ are locally free of rank 3. Moreover, there are natural isomorphisms

$$\mathcal{E}'|_{S_0} = f'_* \mathcal{K}_X^{-1}|_{U'} = f'_* \circ \varphi^* \mathcal{K}_X^{-1}|_{U'} \cong \delta^* \circ f''_* \mathcal{K}_X^{-1}|_{U''} = \delta^* \mathcal{E}''|_{S_0}.$$

Thus, the vector bundles $\mathcal{E}'$ and $\delta^* \mathcal{E}''$ coincide on the open subset $S_0 \subset S$, whose complement is zero-dimensional. We claim that they coincide everywhere, that is, $\mathcal{E}' = \delta^* \mathcal{E}''$. Indeed, the problem is local along the base and so we can assume that $S \ni s$ is a small analytic neighborhood of a point $s$ and our vector bundles are trivial: $\mathcal{E}' = S \times \mathbb{C}^3$ and $\mathcal{E}'' = S \times \mathbb{C}^3$. The isomorphism $\mathcal{E} \cong \mathcal{E}''$ on $S_0 = S \backslash \{s\}$ is given by a matrix $\|g_{i,j}\|_{1 \leq i, j \leq 3}$ whose entries are functions holomorphic on $S_0$. By Hartogs’ extension theorem, they can be uniquely extended to holomorphic functions on $S$. Moreover, the matrix $\|g_{i,j}\|$ is invertible on $S_0$ and hence on the whole of $S$. This shows that there is an isomorphism $\mathcal{E}' \cong \mathcal{E}''$ of vector bundles.

Since the anticanonical bundle $\mathcal{K}_X^{-1}$ is very ample relative to $f'$ and $f''$, it induces embeddings $i' : X \hookrightarrow \mathbb{P}_S(\mathcal{E}')$ and $i'' : X \hookrightarrow \mathbb{P}_S(\mathcal{E}'')$. Note that there are isomorphisms

$$\varphi^*_{|U'} \mathcal{K}_X^{-1}|_{U''} \cong \varphi^*_{|U'} \mathcal{K}_U^{-1} \cong \mathcal{K}_U^{-1} \cong \mathcal{K}_U^{-1}|_{U''}.$$
This implies that the restrictions
\[ i'_U : U' \hookrightarrow \mathbb{P}_S(E') \quad \text{and} \quad i''_U : U'' \hookrightarrow \mathbb{P}_S(E'') \]
of \( i' \) and \( i'' \) commute with the isomorphisms \( \varphi_U: U' \sim U'' \) and \( \mathbb{P}_S(E') \cong \mathbb{P}_S(E'') \). Thus, we have a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}_S(E') & \xrightarrow{\cong} & \mathbb{P}_S(E'') \\
\downarrow \cong & & \downarrow \cong \\
\mathbb{P}_S(E'') & \xrightarrow{\cong} & \mathbb{P}_S(E''') \\
\end{array}
\]

which shows that \( \varphi \) is a holomorphic map. Applying a similar argument to \( \varphi^{-1} \), we obtain that \( \varphi \) is an isomorphism. \( \square \)

**Remark 6.3.** In Lemma 6.2 it is not sufficient to assume that the surface \( S \) has algebraic dimension 0. Indeed, if \( S \) contains a curve \( C \), then passing to a suitable bimeromorphic model of \( S \) we can assume that \( C \) is non-singular. In this case the conic bundle \( X = S \times \mathbb{P}^1 \) admits elementary transformations over \( C \).

We can now prove the main result of this section. Note that it can be obtained as a particular case of [56], Theorem 1.2.

**Theorem 6.4.** Let \( X \) be a compact complex threefold and \( S \) a compact complex surface with \( \kappa(S) \geq 0 \) and \( a(S) = 0 \). Let \( \tau: X \rightarrow S \) be a rationally connected fibration. Then the group \( \text{Bim}(X) \) is Jordan.

**Proof.** By Proposition 4.7, it is sufficient to consider the case when the surface \( S \) is bimeromorphic to a complex torus. Moreover, by Proposition 5.7 and Remark 5.5, we can assume that \( S \) is a complex torus and \( \tau \) is holomorphic. Furthermore, by Lemma 2.8, it is sufficient to consider the case when \( X \) is the projectivization a vector bundle of holomorphic rank 2 on \( S \). In this case \( X \) is Kähler by Theorem 2.4. Recall that a complex torus of algebraic dimension zero does not contain any curves. Thus, according to Lemma 6.2, the group \( \text{Bim}(X) \) acts biholomorphically on \( X \). The assertion now follows from Theorem 2.7. \( \square \)

**§ 7. The case when \( a(S) = 1 \)**

In this section we prove Theorem 1.4. Our main purpose is to study compact complex threefolds of algebraic dimension 2 for which the base of the maximal rationally connected fibration has algebraic dimension 1. The proof of the following fact was explained to us by F. Campana.

**Lemma 7.1.** Let \( X \) be a compact complex threefold, \( B \) and \( S \) compact complex surfaces, and \( C \) a smooth curve. Let \( \tau: X \rightarrow S \) and \( \eta: X \rightarrow B \) be dominant meromorphic maps, where \( \eta \) has connected fibres. Let \( \theta: S \rightarrow C \) and \( \sigma: B \rightarrow C \) be surjective morphisms with connected fibres such that \( \theta \circ \tau = \sigma \circ \eta \). Assume that the \( B \) is algebraic and the algebraic dimension of \( S \) is equal to 1. Then \( X \) is bimeromorphic to the fibre product \( Y = S \times_C B \).
Proof. We can assume that the maps $\tau$ and $\eta$ are holomorphic. There is a natural morphism $\zeta: X \to Y$. It is easy to see that $Y$ is a three-dimensional reduced irreducible compact complex space (with at worst hypersurface singularities) and $\zeta$ is surjective. In particular, a typical fibre of $\zeta$ is finite. Denote by $\tau_Y$ and $\eta_Y$ the natural projections of $Y$ to $S$ and $B$, respectively:

\[
\begin{array}{c}
\text{X} \\
\downarrow \tau \\
\downarrow \zeta \\
\downarrow \eta_Y \\
\text{Y} \\
\downarrow \tau_Y \\
\downarrow \eta \\
\downarrow \theta \\
\text{S} \\
\downarrow \sigma \\
\text{B}, \\
\end{array}
\]

(7.1)

Assume that the map $\zeta$ is not bimeromorphic. Let $R \subset Y$ be the ramification divisor of $\zeta$. If $\eta_Y$ maps $R$ to $B$ non-surjectively (in particular, if $R = \emptyset$), then $\eta = \eta_Y \circ \zeta$ has non-connected fibres over the points of the open set $B \setminus \eta_Y(R)$, which is impossible by our assumption. Let $R'$ be an irreducible component of $R$ that is mapped surjectively to $B$. Then the restriction of $\eta_Y$ to $R'$ is finite over a typical point of $B$, which implies that $R'$ is an algebraic surface. Since $S$ is not algebraic, the restriction of $\tau_Y$ to $R'$ cannot be finite over a typical point of $S$. Therefore, the image $\tau_Y(R')$ is contained in a curve on $S$.

Note that $a(S) = 1$ and $\theta$ is the algebraic reduction for $S$. Hence none of the curves on $S$ is mapped to $C$ surjectively by Corollary 2.10. From this we see that $\theta \circ \tau_Y(R')$ is a point. On the other hand, the morphism $\sigma \circ \eta_Y$ maps $R'$ to $C$ surjectively, which gives a contradiction to the commutativity of the diagram (7.1).

Corollary 7.2. Let $X$ be a compact complex threefold of algebraic dimension 2 and $S$ a compact complex surface bimeromorphic to a complex torus. Assume that $a(S) = 1$. Let $\tau: X \dashrightarrow S$ be a rationally connected fibration. Then $X$ is bimeromorphic to $S \times \mathbb{P}^1$.

Proof. Consider the algebraic reduction $\eta: X \dashrightarrow B$, where $\dim B = 2$ by our assumption. We can assume that the maps $\tau$ and $\eta$ are holomorphic and $B$ is a non-singular surface. Moreover, by Proposition 5.7 and Remark 5.5, we can assume that $S$ is a complex torus. Then its algebraic reduction $\theta: S \to C$ is an elliptic fibration over some elliptic curve $C$.

Since $\eta$ does not contract a typical fibre of $\tau$, the surface $B$ is ruled. The embedding

\[
\mathcal{M}(C) \subset \mathcal{M}(X) = \mathcal{M}(B)
\]

of the fields of meromorphic functions induces a map $\sigma: B \dashrightarrow C$ that must be a morphism with rational fibres (in fact, $\tau: X \to S$ and $\sigma: B \to C$ are Albanese maps). By Lemma 7.1, $X$ is bimeromorphic to the fibre product $S \times_C B$. Since the ruled surface $B$ is bimeromorphic to $C \times \mathbb{P}^1$, $X$ is bimeromorphic to $S \times \mathbb{P}^1$. \qed
Corollary 7.3. Let $X$ be a compact complex threefold and $S$ a non-algebraic compact complex surface of non-negative Kodaira dimension. Let $\tau: X \to S$ be a rationally connected fibration. Assume that the group $\text{Bim}(X)$ is not Jordan. Then $S$ is bimeromorphic to a complex torus and $X$ is bimeromorphic to the projectivization of a vector bundle of holomorphic rank 2 on this complex torus. Moreover, if $a(X) = 2$, then $X$ is bimeromorphic to the product $S \times \mathbb{P}^1$.

Proof. By Proposition 4.7, $S$ is bimeromorphic to some complex torus $S_0$, and $a(S) = 1$ by Theorem 6.4. We conclude from Corollary 5.9 that $X$ is bimeromorphic to the projectivization of a vector bundle of holomorphic rank 2 on $S_0$. Moreover, if $a(X) = 2$, then by Corollary 7.2, the threefold $X$ is bimeromorphic to $S \times \mathbb{P}^1$. □

We now are ready to prove the main result of this paper.

Proof of Theorem 1.4. Let $\tau: X \to S$ be the maximal rationally connected fibration. Since $X$ is not algebraic, we have $\dim S = 2$ and $\kappa(S) \geq 0$ by Corollary 3.7. Also we know that $S$ is not algebraic by Lemma 2.6. It remains to apply Corollary 7.3. □

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