\textbf{ABSTRACT}

Using the zero curvature formulation, it is shown that $\mathcal{W}$-algebra transformations are symmetries of corresponding generalised Drinfel’d-Sokolov hierarchies. This result is illustrated with the examples of the KdV and Boussinesque hierarchies, and the hierarchy associated to the Polyakov-Bershadsky $\mathcal{W}$-algebra.
1. Introduction

Integrable hierarchies of differential equations associated to Lie algebras have been described by Drin’feld and Sokolov [1]. These hierarchies have been increasingly studied in recent work in conformal field theory. Much of this research has been spurred by the formulation of the continuum limit of the one-matrix model in terms of the formalism associated to the KdV hierarchy [2, 3]. More general matrix models, describing two-dimensional conformal field theories coupled to two-dimensional gravity, have also been described in terms of Drinfel’d-Sokolov hierarchies [2-4]. Using this approach, a stable non-perturbative definition of these theories has been proposed recently in ref. [5].

In the further study of integrable hierarchies, an important question is that of symmetries. In this letter, it will be argued that \(W\)-algebra transformations are symmetries of corresponding integrable hierarchies of differential equations. (Note that these \(W\)-algebra transformations are different to the \(W\)-algebra constraints associated to matrix models, which have been discussed in ref. [6] and follow from the \(L_{-1}\) constraint, which is related to the Galilean invariances of hierarchies.) The argument given below that a given \(W\)-algebra is a symmetry algebra of the corresponding (generalised) Drinfel’d-Sokolov hierarchy can be simply summarised in words - each equation in an integrable hierarchy of differential equations can be written as a zero-curvature condition on a two-dimensional ‘Lax connection’. Symmetries of the hierarchy are then the set of two-dimensional gauge transformations which preserve the form of these Lax connections - this symmetry turns out to be the \(W\)-algebra transformations, with additional conditions fixing the dependence of the transformation parameters upon the time variables. In the final section, this result will be illustrated using the hierarchies of Korteweg-deVries and Boussinesque, and the hierarchy associated to the Polyakov-Bershadsky \(W\)-algebra.

The Drinfel’d-Sokolov classification of integrable hierarchies has recently been generalised and includes hierarchies corresponding to all embeddings of \(sl(2)\) in Lie algebras [7] (the Drinfel’d-Sokolov hierarchies are those associated to the ‘principal’
embeddings). There is a similar classification of $W$-algebras [8]. The results of this paper show how these classifications are related - a given generalised $W$-algebra is a symmetry algebra of the corresponding generalised Drinfel’d-Sokolov hierarchy.

2. Hierarchy Symmetries

Integrable hierarchies of differential equations of the Drinfel’d-Sokolov type are neatly formulated as matrix equations [1, 7]. This is the ‘Lax pair’ formulation, the $k$-th equation of the hierarchy being written as

$$\frac{\partial L}{\partial t_k} = [P_k, L],$$

(1)

where the ‘Lax operator’ $L$, and the $P_k$, are matrices which depend on the basic hierarchy fields $\{u^I\}$, for some index set labelled by $I$. $k$ is a non-negative integer. The fields $u^I$ are functions of the ‘times’ $t_k$; as the first equations of the hierarchy are $(\frac{\partial}{\partial x} - \frac{\partial}{\partial t_0})u^I = 0$, the time $t_0$ is usually identified with $x$. The pair of matrices $(L, P_k)$ are often called the ‘Lax pair’. The Lax operator can in fact be written as $L = \partial + A$, where $\partial = \frac{\partial}{\partial x}$, and $A$ is a matrix not containing freely acting differential operators. The Lax pair equation (1) can then be written as the condition

$$\left[ - \frac{\partial}{\partial t_k} + P_k, \partial + A \right] = 0.$$

(2)

If, for each $k > 0$, the coordinates $(t_k, x)$ are identified as the coordinates of a two-dimensional space, and the pair $(-P_k, A)$ as the components of a two-dimensional connection, then the conditions (2) are simply the requirement that the curvature of this connection vanishes. This concise description of the equations of integrable hierarchies will be crucial in the following. The pair $(-P_k, A)$ will henceforth be called the ‘Lax connection’, and the corresponding curvature the ‘Lax curvature’.

The study of the symmetries of an integrable hierarchy then reduces to the study of the transformations which preserve the zero-curvature conditions (2).
Modulo global questions not addressed here, it is immediately seen that such symmetries will be realised by those two-dimensional gauge transformations of the Lax connection which preserve the form of the Lax pair.

Via the Miura transform, the Lax pair of Drinfel’d-Sokolov corresponds to the Lax pair of Toda field theory [1, 7]. In ref. [9] it was shown that the $\mathcal{W}A_n$-algebra symmetry of $A_n$ Toda field theory arises as the algebra of gauge transformations preserving the form of the $A_n$ Toda Lax pair. Recently [10] the relationship of this approach to Toda theory and the Drinfel’d-Sokolov formalism has been clarified. The Lax connection component $A$ of eqn. (1) is precisely the constrained WZW current $J$ in the ‘DS gauge’ of refs. [10-12]. The WZW field equations $\bar{\partial}J = 0$ can be written as the zero curvature equation $[\bar{\partial}, \partial + J] = 0$. The $\mathcal{W}$-algebra is the algebra of residual Kač-Moody gauge transformations of the Hamiltonian reduction of the WZW model which preserves the constraints and gauge fixing [10,11]. Hence it follows immediately that this is the algebra of gauge transformations which preserves the form of the Lax connection component $A$ of eqns. (1) and (2). For those gauge transformations to represent symmetries of the (generalised) Drinfel’d-Sokolov hierarchy, expressed as the zero curvature equations (2), they must also preserve the form of the other component $P_k$ of the Lax connection. This fixes the dependence of the $\mathcal{W}$-algebra transformation parameters upon the time variables $t_k$.

The transformations of the fields $u^I$ are those determined by the $\mathcal{W}$-algebra, and are the same for each equation in a given hierarchy. With the dependence of the $\mathcal{W}$-algebra transformation parameters on the time variables $t_k$ fixed by the form-invariance of the $P_k$, it then follows that these $\mathcal{W}$-algebra transformations are symmetries of the entire set of equations of the generalised Drinfel’d-Sokolov hierarchy. Thus a given (generalised) Drinfel’d-Sokolov hierarchy is invariant under the transformations generated by the corresponding (generalised) $\mathcal{W}$-algebra.
3. Examples

The above argument will now be illustrated with some examples.

Virasoro Symmetries of the KdV Hierarchy

The \(k\)-th equation of the KdV hierarchy is

\[
\frac{\partial u}{\partial t_k} = D_2 R_k,
\]

where \(u\) is a function of \(x\) and the \(t_k\). \(R_k\) is the \(k\)-th Gel’fand-Dikii polynomial [13], and \(D_2\) is the second Hamiltonian structure of the KdV system

\[
D_2 = \frac{1}{4} \partial^3 - u \partial - \frac{1}{2} u'.
\]

\(\partial = \partial/\partial x\) and \(u' = \partial u/\partial x\) in eqn. (4). Using \(D_2\), the Poisson bracket of two \(u\)’s is the Virasoro algebra, with a central term.

In the Drinfel’d-Sokolov approach to the KdV hierarchy, the Lax operator is

\[
L = \partial + \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \equiv \partial + A.
\]

The above matrix \(A\) is the \(x\) component of the Lax connection \((-P_k, A)\). The matrix \(P_k\) will be defined in a moment. The Virasoro algebra is in fact the algebra of two-dimensional gauge transformations which preserves the form of the Lax connection. In the case of \(A\), form invariance means that the gauge parameter matrix \(\Lambda\) must satisfy

\[
\begin{pmatrix} 0 & \delta u \\ 0 & 0 \end{pmatrix} = \Lambda' + [A, \Lambda]
\]

for some variation \(\delta u\). It can be checked that this equation fixes \(\Lambda\) to be

\[
\Lambda = -\frac{1}{2} \begin{pmatrix} -\frac{1}{2} \epsilon' & -\frac{1}{2} \epsilon'' - u \epsilon \\ \epsilon & \frac{1}{2} \epsilon' \end{pmatrix}
\]

(7)
and the variation $\delta u$ to be

$$\delta u = D_2 \epsilon, \quad (8)$$

where $\epsilon$ is some function of $x$ and the $t_k$. The algebra of the transformations (8), at fixed times $t_k$, is the Virasoro algebra with central charge $c = 1/6$.

The same manipulations may be used to find the $t_k$ connection component $P_k$. If $P_k$ is required to satisfy the Lax pair condition

$$\frac{\partial L}{\partial t_k} = [P_k, L], \quad (9)$$

then one finds the solution, using the $k$-th KdV equation (3),

$$P_k = \frac{1}{2} \begin{pmatrix} -\frac{1}{2} R'_k & -\frac{1}{2} R''_k + uR_k \\ R_k & \frac{1}{2} R'_k \end{pmatrix}. \quad (10)$$

The $k$-th KdV equation (3) may then be written as the zero-curvature condition ($\partial_{t_k} \equiv \partial/\partial t_k$)

$$[-\partial_{t_k} + P_k, \partial + A] = 0. \quad (11)$$

Gauge transformations preserving the form of the Lax connection component $A$ were just seen to be the Virasoro transformations (8). As discussed in the preceding section, the additional requirement that these gauge transformations preserve the form of the other Lax connection component $P_k$, for all $k > 0$, fixes the dependence of the gauge parameter variables - here just $\epsilon$ - on the variables $t_k$. A calculation shows that the form of $P_k$ is preserved by the gauge transformation with parameter matrix (7) if $\epsilon$ satisfies

$$\frac{\partial \epsilon}{\partial t_k} = \delta R_k + \frac{1}{2} \epsilon R'_k - \frac{1}{2} \epsilon' R_k, \quad (12)$$

where $\delta R_k$ is the variation of the $k$-th Gel’fand-Dikii polynomial $R_k[u]$ induced by the variation (8) of $u$. Note that the consistency of eqns. (12) is guaranteed by the realisation of the Virasoro transformations as gauge transformations.
Since the $k$-th KdV equation (3) can be written as the condition (11) for vanishing curvature of the Lax connection, it follows that gauge transformations preserving this connection are symmetries of the KdV equation. With the dependence of the parameter $\epsilon$ upon the variables $t_k$ determined from eqn. (12), it then follows that the entire KdV hierarchy of equations is invariant under the transformations (8), which generate the Virasoro algebra. This is the Virasoro symmetry of the KdV hierarchy. Note that, by construction, this is the full set of KdV hierarchy symmetries which act only on the functions $u$ (the well-known Galilean and scaling symmetries act also on the variables $(x,t_k)$, the former additionally mixing the different equations of the hierarchy). The Virasoro symmetry of the KdV hierarchy can be realised as a (time dependent) canonical transformation with generating function $\int \epsilon u$. (These comments generalise to the other hierarchies, with the generating function being $\int \epsilon^I u^I$.)

As examples, the first equation of the KdV hierarchy is just $\partial u/\partial t_0 = -\frac{1}{4} u'$, which is invariant under the transformation (8) if $\epsilon$ satisfies $\partial \epsilon/\partial t_0 = -\frac{1}{4} \epsilon'$. The second equation of the KdV hierarchy is the KdV equation itself $\partial u/\partial t_1 = -\frac{1}{16} u'' + \frac{3}{8} uu'$, and is invariant under the transformations (8) if $\partial \epsilon/\partial t_1 = -\frac{1}{16} \epsilon'' + \frac{3}{8} \epsilon' u$. Further explicit examples may be worked out from the formula (12), using eqn. (8) and the explicit forms of the polynomials $R_k$ in ref.[13].

$\mathcal{W}_3$ Symmetries of the Boussinesque Hierarchy

The Lax operator in this case is

$$L = \partial + A = \partial + \begin{pmatrix} 0 & 0 & -u \\ 1 & 0 & -v \\ 0 & 1 & 0 \end{pmatrix},$$

(13)

with the Boussinesque hierarchy variables $u, v$. The $\mathcal{W}_3$ algebra is the set of gauge transformations preserving $A$ [11] - a similar calculation to that above proves that a gauge transformation, with parameter matrix $\Lambda$, preserves the form of $A$,

$$\delta A = \partial \Lambda + [A, \Lambda],$$

(14)
if $\Lambda$ is given by

$$
\Lambda = \begin{pmatrix}
-\frac{d'}{3}g'' + \frac{2}{3}gv & -\frac{d''}{3}g'' + \frac{2}{3}(vg)' - ug & -d'' - \frac{1}{3}g''' + \frac{2}{3}(vg)'' \\
- \frac{1}{3}g'' - \frac{1}{3}vg & -d'' - \frac{2}{3}g''' - \frac{2}{3}vg' & \frac{1}{3}v' - ug - vd \\
g & g' + d & \frac{2}{3}g'' + d' - \frac{1}{3}vg
\end{pmatrix},
$$

and if the variations of $u$ and $v$ are given by

$$
\left( \begin{array}{c}
\delta u \\
\delta v
\end{array} \right) = D_2 \left( \begin{array}{c}
d \\
g
\end{array} \right).
$$

$d$ and $g$ in eqns. (15) and (16) are arbitrary functions of $(t_k, x)$. $D_2$ in eqn. (16) is the second Hamiltonian structure matrix, given by

$$
D_2 = \begin{pmatrix}
\partial^4 + v\partial^2 + 3u\partial + u' & \frac{1}{3}\partial^5 - \frac{1}{3}v\partial^3 + (3u - 2v')\partial^2 \\
2\partial^3 + 2v\partial + v' & \partial^4 + v\partial^2 + (3u - v')\partial + (2u' - v'')
\end{pmatrix}.
$$

Identifying $T = v$ and $W = u - \frac{1}{3}v'$, it can be checked that this is the matrix operator defining the usual $W_3$ algebra, with generators $T$ the stress-tensor and $W$ the primary spin three generator.

The $k$-th equation of the Boussinesque hierarchy can be written

$$
\frac{\partial}{\partial t_k} \begin{pmatrix} u \\ v \end{pmatrix} = D_2 \begin{pmatrix} R_k \\ S_k \end{pmatrix},
$$

with $R_k, S_k$ generalised Gel'fand-Dikii polynomials [14]. If a matrix $P_k(R_k, S_k)$ is defined by

$$
P_k(R_k, S_k) = \Lambda(R_k, S_k),
$$

where $\Lambda(d, g)$ is given by eqn. (15), then an exactly analogous calculation to that yielding eqn. (17) shows that the Boussinesque hierarchy equation (18) can be
expressed as the vanishing curvature equation

$$\left[-\partial_{t_k} + P_k, \partial + A\right] = 0.$$  \tag{20}$$

The condition that the gauge transformation represented by the parameter matrix \( \Lambda \) preserves the form of \( P \) then fixes the time dependence of the parameters \( d, g \) by the requirements

\[
\frac{\partial g}{\partial t_k} = -\delta S_k + \left[ S'_k d - S_k g'' - 2S_k d' - (R_k, S_k) \leftrightarrow (d, g) \right],
\]

\[
\frac{\partial d}{\partial t_k} = -\delta R_k + \left[ R'_k d + S_k d'' + \frac{2}{3} (S_k g''' + S_k v g') - (R_k, S_k) \leftrightarrow (d, g) \right].
\] \tag{21}

\( \delta S_k \) and \( \delta R_k \) in eqn. (21) are the variations in the generalised Gel’fand-Dikii polynomials induced by the variations (16) in \( u \) and \( v \). Thus it has been shown explicitly that the \( \mathcal{W}_3 \)-algebra transformations (16), with the \( t_k \) dependence of the \( \mathcal{W}_3 \)-algebra parameters \( g, d \) determined by eqn. (21), are symmetries of the Boussinesque hierarchy (18).

\underline{Polyakov-Bershadsky Symmetries of the Associated Hierarchy}

The simplest generalised Drinfel’d-Sokolov hierarchy which arises from a non-principal \( sl(2) \) embedding in a simple Lie algebra is the hierarchy associated to the \( \mathcal{W} \)-algebra of Polyakov and Bershadsky [15]. This hierarchy was first investigated by Bakas and Depireux [16], who also studied the zero-curvature formulation from the point of view of four-dimensional self-dual Yang-Mills equations [17]. For simplicity, here only the first non-trivial set of equations in this hierarchy will be considered. There are four fields \( U, G^+, G^-, \) and \( T \), functions of \( x \) and \( t \), and the equations are

\[
\dot{U} = G^+ - G^-,
\]

\[
\dot{G}^+ = 3U^2 - T + \frac{3}{2} U',
\]

\[
\dot{T} = \frac{1}{2}(G^+ + G^-)',
\]

\[
\dot{G}^- = -3U^2 + T + \frac{3}{2} U'.
\] \tag{22}
Following ref. [16], these can be written as the zero curvature equation

\[ [\partial_x + A_x, \partial_t + A_t] = 0, \]  

(23)

with

\[
A_x = \begin{pmatrix} \frac{-1}{2} U & 0 & 1 \\ G^+ & U & 0 \\ T - \frac{3}{4} U^2 & G^- & -\frac{1}{2} U \end{pmatrix}, \quad A_t = \begin{pmatrix} 0 & 1 & 0 \\ \frac{3}{2} U & 0 & 1 \\ \frac{1}{2} (G^+ + G^-) & \frac{3}{2} U & 0 \end{pmatrix}. \]  

(24)

A general gauge transformation \( \delta A = \partial \Lambda + [A, \Lambda] \) preserves the form of the connection \( A_x \) above if \( \Lambda \) takes the form

\[
\Lambda = \begin{pmatrix} a \\ f' + cG^+ \frac{3}{2} f U \\ -\frac{1}{2} c'' + \frac{1}{2} (bG^+ + fG^-) + c (T - \frac{3}{4} U^2) - b' + \frac{3}{2} b U + cG^- - a - c' \end{pmatrix},
\]  

(25)

and if the variations are given by

\[
\delta U = c'' - 2a' + bG^+ - fG^-,
\]

\[
\delta G^+ = f'' + 3aG^+ + cG^+ + 3f'U + \frac{3}{2} f U' + \frac{3}{2} cG^+ U + f(3U^2 - T),
\]

\[
\delta G^- = -b'' + 3(c' - a)G^- + cG^- + 3b'U + \frac{3}{2} b U' - \frac{3}{2} cG^- U + b(T - 3U^2),
\]

\[
\delta T = -\frac{1}{2} c'' + 2c'T + cT' - \frac{3}{2} cU^2 - \frac{3}{2} c U U' + \left( \frac{3}{2} c'' - 3a' \right) U + \frac{1}{2} bG^+ + \frac{1}{2} fG^- + \frac{3}{2} \left( b'G^+ + f'G^- \right),
\]

(26)

with \( a, b, c \) and \( f \) arbitrary functions of \((t_k, x)\). The variations (26) realise the \( \mathcal{W} \)-algebra of Polyakov and Bershadsky. (The functions \( b, f \) and \( c \) parameterise the variations induced by the generators \( G^+, G^- \) and \( T \) respectively. To obtain the canonical form of the action of the generator \( U \) as given in ref. [15] one needs to shift its parameter \( a \) by a term \(-\frac{3}{4} cU\).)
Finally, the connection coefficient $A_t$ in eqn. (24) is invariant under the gauge transformations represented by eqn. (25) if the parameters $a, b, c$ and $f$ have time dependence determined by

$$\dot{a} = \frac{3}{2}(b - f)U + \frac{1}{2}c(G^+ - G^+) - f', \quad \dot{b} = 3a - c' + \frac{3}{2}cU, \quad \dot{c} = b - f, \quad \dot{f} = 2c' - 3a - \frac{3}{2}cU. \tag{27}$$

The transformations (26) are thus symmetries of the equations (22) of this hierarchy if the transformation parameters satisfy (27). As before, this is a consequence of the fact that these transformations can be realised as gauge transformations, and that the original equations can be expressed as a zero-curvature condition. Again, from the general argument given earlier, the transformations (26) are symmetries of the entire hierarchy of differential equations, with the parameters $a, b, c, f$ having $t \equiv t_1$ dependence determined by eqn. (27), and $t_k, (k > 1)$ dependence determined by generalisations of this equation. Details can be worked out straightforwardly from the formulation of this hierarchy in ref. [16].

4. FURTHER REMARKS

It has been shown above that a given generalised $\mathcal{W}$-algebra generates symmetries of the corresponding generalised Drinfel’d-Sokolov hierarchy of differential equations, and this was shown in detail in some examples. This connection between $\mathcal{W}$-algebras and integrable hierarchies is another intriguing interconnection in the whole area of integrable models. Elucidation of a common underlying structure is an obvious task. The bi-Hamiltonian structure fundamental to integrable hierarchies has a geometric meaning (see, for example, ref. [18]), which may be an important clue to the direction of further progress.

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