ON THE GROUP OF AUTOMORPHISMS OF THE BRANDT $\lambda^0$-EXTENSION OF A MONOID WITH ZERO

OLEG GUTIK

ABSTRACT. The group of automorphisms of the Brandt $\lambda^0$-extension $B^0_\lambda(S)$ of an arbitrary monoid $S$ with zero is described. In particular we show that the group of automorphisms $\text{Aut}(B^0_\lambda(S))$ of $B^0_\lambda(S)$ is isomorphic to a homomorphic image of the group defines on the Cartesian product $\mathcal{S}_\lambda \times \text{Aut}(S) \times H^1_1$ with the following binary operation:

$$[\varphi, h, u] : [\varphi', h', u'] = [\varphi\varphi', hh', \varphi u : uh'],$$

where $\mathcal{S}_\lambda$ is the group of all bijections of the cardinal $\lambda$, $\text{Aut}(S)$ is the group of all automorphisms of the semigroup $S$ and $H^1_1$ is the direct $\lambda$-power of the group of units $H_1$ of the monoid $S$.

1. INTRODUCTION AND PRELIMINARIES

Further we shall follow the terminology of [2] [21].

Given a semigroup $S$, we shall denote the set of idempotents of $S$ by $E(S)$. A semigroup $S$ with the adjoined unit (identity) will be denoted by $S^1$ [S'] (cf. [2]). Next, we shall denote the unit (identity) and the zero of a semigroup $S$ by $1_S$ and $0_S$, respectively. Given a subset $A$ of a semigroup $S$, we shall denote by $A^* = A \setminus \{0_S\}$.

If $S$ is a semigroup, then we shall denote the subset of idempotents in $S$ by $E(S)$. If $E(S)$ is closed under multiplication in $S$ and we shall refer to $E(S)$ a band (or the band of $S$). If the band $E(S)$ is a non-empty subset of $S$, then the semigroup operation on $S$ determines the following partial order $\leq$ on $E(S)$: $e \leq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$.

If $h : S \to T$ is a homomorphism (or a map) from a semigroup $S$ into a semigroup $T$ and if $s \in S$, then we denote the image of $s$ under $h$ by $(s)h$.

Let $S$ be a semigroup with zero and $\lambda$ a cardinal $\geq 1$. We define the semigroup operation on the set $B^0_\lambda(S) = (\lambda \times S \times \lambda) \cup \{0\}$ as follows:

$$(\alpha, a, \beta) : (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B^1_\lambda(S)$ is called the Brandt $\lambda$-extension of the semigroup $S$ [4]. Obviously, if $S$ has zero then $J = \{0\} \cup \{(0_S, \beta) : 0_S$ is the zero of $S\}$ is an ideal of $B^1_\lambda(S)$. We put $B^0_\lambda(S) = B^1_\lambda(S) / J$ and the semigroup $B^0_\lambda(S)$ is called the Brandt $\lambda^0$-extension of the semigroup $S$ with zero [8].

If $I$ is a trivial semigroup (i.e. $I$ contains only one element), then we denote the semigroup $I$ with the adjoined zero by $I^0$. Obviously, for any $\lambda \geq 2$, the Brandt $\lambda^0$-extension of the semigroup $I^0$ is isomorphic to the semigroup of $\lambda \times \lambda$-matrix units and any Brandt $\lambda^0$-extension of a semigroup with zero which also contains a non-zero idempotent contains the semigroup of $\lambda \times \lambda$-matrix units. We shall denote the semigroup of $\lambda \times \lambda$-matrix units by $B^1_\lambda$. The $2 \times 2$-matrix semigroup with adjoined identity $B^1_2$ plays an impotent role in Graph Theory and its called the Perkins semigroup. In the paper [20] Perkins showed that the semigroup $B^1_2$ is not finitely based. More details on the word problem of the Perkins semigroup via different graphs may be found in the works of Kitaev and his coauthors (see [17] [18]).
We always consider the Brandt $\lambda^0$-extension only of a monoid with zero. Obviously, for any monoid $S$ with zero we have $B^0_\lambda(S) = S$. Note that every Brandt $\lambda$-extension of a group $G$ is isomorphic to the Brandt $\lambda^0$-extension of the group $G^0$ with adjoined zero. The Brandt $\lambda^0$-extension of the group with adjoined zero is called a Brandt semigroup \cite{2, 21}. A semigroup $S$ is a Brandt semigroup if and only if $S$ is a completely 0-simple inverse semigroup \cite{1, 19} (cf. also \cite{21}, Theorem II.3.5). We shall say that the Brandt $\lambda^0$-extension $B^0_\lambda(S)$ of a semigroup $S$ is finite if the cardinal $\lambda$ is finite.

In the paper \cite{14} Gutik and Repovš established homomorphisms of the Brandt $\lambda^0$-extensions of monoids with zeros. They also described a category whose objects are ingredients in the constructions of the Brandt $\lambda^0$-extensions of monoids with zeros. Here they introduced finite, compact topological Brandt $\lambda^0$-extensions of topological semigroups and countably compact topological Brandt $\lambda^0$-extensions of topological inverse semigroups in the class of topological inverse semigroups, and established the structure of such extensions and non-trivial continuous homomorphisms between such topological Brandt $\lambda^0$-extensions of topological monoids with zero. There they also described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt $\lambda^0$-extensions of topological monoids with zeros. These investigations were continued in \cite{10} and \cite{9}, where established countably compact topological Brandt $\lambda^0$-extensions of topological monoids with zeros and pseudocompact topological Brandt $\lambda^0$-extensions of semitopological monoids with zeros their corresponding categories. Some other topological aspects of topologizations, embeddings and completions of the semigroup of $\lambda \times \lambda$-matrix units and Brandt $\lambda^0$-extensions as semitopological and topological semigroups were studied in \cite{3, 5, 7, 11, 12, 13, 15, 16}.

In this paper we describe the group of automorphisms of the Brandt $\lambda^0$-extension $B^0_\lambda(S)$ of an arbitrary monoid $S$ with zero.

2. Automorphisms of the Brandt $\lambda^0$-extension of a monoid with zero

We observe that if $f: S \to S$ is an automorphism of the semigroup $S$ without zero then it is obvious that the map $\hat{f}: S^0 \to S^0$ defined by the formula

$$(s)\hat{f} = \begin{cases} (s)f, & \text{if } s \neq 0_S; \\ 0_S, & \text{if } s = 0_S, \end{cases}$$

is an automorphism of the semigroup $S^0$ with adjoined zero $0_S$. Also the automorphism $f: S \to S$ of the semigroup $S$ can be extended to an automorphism $f_B: B^0_\lambda(S) \to B^0_\lambda(S)$ of the Brandt $\lambda^0$-extension $B^0_\lambda(S)$ of the semigroup $S$ by the formulæ:

$$(\alpha, s, \beta) f_B = (\alpha, (s)f, \beta), \quad \text{for all } \alpha, \beta \in \lambda$$
and $(0)f_B = 0$. We remark that so determined extended automorphism is not unique.

The following theorem describes all automorphisms of the Brandt $\lambda^0$-extension $B^0_\lambda(S)$ of a monoid $S$.

**Theorem 1.** Let $\lambda \geq 1$ be cardinal and let $B^0_\lambda(S)$ be the Brandt $\lambda^0$-extension of monoid $S$ with zero. Let $h: S \to S$ be an automorphism and suppose that $\varphi: \lambda \to \lambda$ is a bijective map. Let $H_1$ be the group of units of $S$ and $u: \lambda \to H_1$ a map. Then the map $\sigma: B^0_\lambda(S) \to B^0_\lambda(S)$ defined by the formulæ

\begin{equation}
((\alpha, s, \beta)) \sigma = ((\alpha)\varphi, (\alpha) u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi) \quad \text{and} \quad (0)\sigma = 0,
\end{equation}

is an automorphism of the semigroup $B^0_\lambda(S)$. Moreover, every automorphism of $B^0_\lambda(S)$ can be constructed in this manner.

**Proof.** A simple verification shows that $\sigma$ is an automorphism of the semigroup $B^0_\lambda(S)$.

Let $\sigma: B^0_\lambda(S) \to B^0_\lambda(S)$ be an isomorphism. We fix an arbitrary $\alpha \in \lambda$.

Since $\sigma: B^0_\lambda(S) \to B^0_\lambda(S)$ is the automorphism and the idempotent $(\alpha, 1_S, \alpha)$ is maximal with respect to the natural partial order on $E(B^0_\lambda(S))$, Proposition 3.2 of \cite{14} implies that $((\alpha, 1_S, \alpha)) \sigma = (\alpha', 1_S, \alpha')$ for some $\alpha' \in \lambda$.

Since $(\beta, 1_S, \alpha)(\alpha, 1_S, \alpha) = (\beta, 1_S, \alpha)$ for any $\beta \in \lambda$, we have that

$$((\beta, 1_S, \alpha)) \sigma = ((\beta, 1_S, \alpha)) \sigma \cdot (\alpha', 1_S, \alpha').$$
and hence
\[(\beta, 1_s, \alpha)\sigma = ((\beta)\varphi, (\beta)u, \alpha'),\]
for some \((\beta)\varphi \in \lambda\) and \((\beta)u \in S\). Similarly, we get that
\[(\alpha, 1_s, \beta)\sigma = (\alpha', (\beta)v, (\beta)\psi),\]
for some \((\beta)\psi \in \lambda\) and \((\beta)v \in S\). Since \((\alpha, 1_s, \beta)(\beta, 1_s, \alpha) = (\alpha, 1_s, \alpha)\), we have that
\[(\alpha', 1_s, \alpha') = ((\alpha, 1_s, \alpha))\sigma = (\alpha', (\beta)v, (\beta)\psi) \cdot ((\beta)\varphi, (\beta)u, \alpha') = (\alpha', (\beta)v \cdot (\beta)u, \alpha'),\]
and hence \((\beta)\varphi = (\beta)\psi = \beta' \in \lambda\) and \((\beta)v \cdot (\beta)u = 1_s\). Similarly, since \((\beta, 1_s, \alpha) \cdot (\alpha, 1_s, \beta) = (\beta, 1_s, \beta)\), we see that the element
\[((\beta, 1_s, \beta))\sigma = ((\beta, 1_s, \alpha)(\alpha, 1_s, \beta))\sigma = (\beta', (\beta)v \cdot (\beta)u, \beta')\]
is a maximal idempotent of the subsemigroup \(S_{\beta', \beta'}\) of \(B^0_\lambda(S)\), and hence we have that \((\beta)v \cdot (\beta)u = 1_s\). This implies that the elements \((\beta)v\) and \((\beta)u\) are mutually invertible in \(H_1\), and hence \((\beta)v = ((\beta)u)^{-1}\).

If \((\gamma)\varphi = (\delta)\varphi\) for \(\gamma, \delta \in \lambda\) then
\[0 \neq (\alpha', 1_s, (\gamma)\varphi) \cdot ((\delta)\varphi, 1_s, \alpha') = ((\alpha, 1_s, \gamma))\sigma \cdot ((\delta, 1_s, \alpha))\sigma,\]
and since \(\sigma\) is an automorphism, we have that
\[(\alpha, 1_s, \gamma) \cdot (\delta, 1_s, \alpha) \neq 0\]
and hence \(\gamma = \delta\). Thus \(\varphi: \lambda \to \lambda\) is a bijective map.

Therefore for \(s \in S \setminus \{0_s\}\) we have
\[((\gamma, s, \delta))\sigma = ((\gamma, 1_s, \alpha) \cdot (\alpha, s, \alpha) \cdot (\alpha, 1_s, \delta))\sigma =
= ((\gamma, 1_s, \alpha))\sigma \cdot ((\alpha, s, \alpha))\sigma \cdot ((\alpha, 1_s, \delta))\sigma =
= ((\gamma)\varphi, (\gamma)u, \alpha') \cdot (\alpha', (s)h, \alpha') \cdot (\alpha', ((\delta)u)^{-1}, (\delta)\varphi) =
= ((\gamma)\varphi, (\gamma)u \cdot (s)h \cdot ((\delta)u)^{-1}, (\delta)\varphi).\]

Also, since 0 is zero of the semigroup \(B^0_\lambda(S)\) we conclude that \((0)\sigma = 0\). \(\square\)

Theorem \ref{thm:main} implies the following corollary:

**Corollary 1.** Let \(\lambda \geq 1\) be cardinal and let \(B_\lambda(G)\) be the Brandt semigroup. Let \(h: G \to G\) be an automorphism and suppose that \(\varphi: \lambda \to \lambda\) is a bijective map. Let \(u: \lambda \to G\) be a map. Then the map \(\sigma: B_\lambda(G) \to B_\lambda(G)\) defined by the formulae
\[((\alpha, s, \beta))\sigma = ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi) \quad \text{and} \quad (0)\sigma = 0,\]
is an automorphism of the Brandt semigroup \(B_\lambda(G)\). Moreover, every automorphism of \(B_\lambda(G)\) can be constructed in this manner.

Also, we observe that Corollary \ref{cor:main} implies the following well known statement:

**Corollary 2.** Let \(\lambda \geq 1\) be cardinal and \(\varphi: \lambda \to \lambda\) a bijective map. Then the map \(\sigma: B_\lambda \to B_\lambda\) defined by the formulae
\[((\alpha, \beta))\sigma = ((\alpha)\varphi, (\beta)\varphi) \quad \text{and} \quad (0)\sigma = 0,\]
is an automorphism of the semigroup of \(\lambda \times \lambda\)-matrix units \(B_\lambda\). Moreover, every automorphism of \(B_\lambda\) can be constructed in this manner.

The following example implies that the condition that semigroup \(S\) contains the identity is essential.

**Example 1.** Let \(\lambda\) be any cardinal \(\geq 2\). Let \(S\) be the zero-semigroup of cardinality \(\geq 3\) and \(0_s\) is zero of \(S\). It is easily to see that every bijective map \(\sigma: B^0_\lambda(S) \to B^0_\lambda(S)\) such that \((0)\sigma = 0\) is an automorphism of the Brandt \(\lambda^0\)-extension of \(S\).
Remark. By Theorem 1 we have that every automorphism $\sigma: B_0^\lambda(S) \rightarrow B_0^\lambda(S)$ of the Brandt $\lambda^0$-extension of an arbitrary monoid $S$ with zero identifies with the ordered triple $[\varphi, h, u]$, where $h: S \rightarrow S$ is an automorphism of $S$, $\varphi: \lambda \rightarrow \lambda$ is a bijective map and $u: \lambda \rightarrow H_1$ is a map, where $H_1$ is the group of units of $S$.

**Lemma 1.** Let $\lambda \geq 1$ be cardinal, $S$ be a monoid with zero and let $B_0^\lambda(S)$ be the Brandt $\lambda^0$-extension of $S$. Then the composition of arbitrary automorphisms $\sigma = [\varphi, h, u]$ and $\sigma' = [\varphi', h', u']$ of the Brandt $\lambda^0$-extension of $S$ defines in the following way:

$$[\varphi, h, u] \cdot [\varphi', h', u'] = [\varphi\varphi', hh', \varphi'\cdot u'h']$$

**Proof.** By Theorem 1 for every $(\alpha, s, \beta) \in B_0^\lambda(S)$ we have that

$$(\alpha, s, \beta)(\sigma\sigma') = ((\alpha)\varphi, (\alpha)u(s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi)\sigma' = (((\alpha)\varphi)\varphi', ((\alpha)\varphi)u' \cdot ((\alpha)u(s)h \cdot ((\beta)u)^{-1}) h' \cdot (((\beta)\varphi)u')^{-1}, ((\beta)\varphi)\varphi') =$$

and since $h'$ is an automorphism of the monoid $S$ we get that this is equal to

$$=((\alpha)(\varphi\varphi'), ((\alpha)\varphi)u' \cdot ((\alpha)u(s)h \cdot ((\beta)u)^{-1}) h' \cdot (((\beta)\varphi)u')^{-1}, ((\beta)\varphi)\varphi') =$$

This completes the proof of the requested equality. \( \square \)

**Theorem 2.** Let $\lambda \geq 1$ be cardinal, $S$ be a monoid with zero and let $B_0^\lambda(S)$ be the Brandt $\lambda^0$-extension of $S$. Then the group of automorphisms $\text{Aut}(B_0^\lambda(S))$ of $B_0^\lambda(S)$ is isomorphic to a homomorphic image of the group defines on the Cartesian product $\mathcal{S}_\lambda \times \text{Aut}(S) \times H_1^\lambda$ with the following binary operation:

$$(2) \quad [\varphi, h, u] \cdot [\varphi', h', u'] = [\varphi\varphi', hh', \varphi'\cdot u'h']$$

where $\mathcal{S}_\lambda$ is the group of all bijections of the cardinal $\lambda$, $\text{Aut}(S)$ is the group of all automorphisms of the semigroup $S$ and $H_1^\lambda$ is the direct $\lambda$-power of the group of units $H_1$ of the monoid $S$. Moreover, the inverse element of $[\varphi, h, u]$ in the group $\text{Aut}(B_0^\lambda(S))$ is defined by the formula:

$$[\varphi, h, u]^{-1} = [\varphi^{-1}, h^{-1}, \varphi^{-1}u^{-1}h^{-1}]$$

**Proof.** First, we show that the binary operation defined by formula (2) is associative. Let $[\varphi, h, u]$, $[\varphi', h', u']$ and $[\varphi'', h'', u'']$ be arbitrary elements of the Cartesian product $\mathcal{S}_\lambda \times \text{Aut}(S) \times H_1^\lambda$. Then we have that

$$([\varphi, h, u] \cdot [\varphi', h', u']) \cdot [\varphi'', h'', u''] = [\varphi\varphi', hh', \varphi'\cdot u'h'] \cdot [\varphi'', h'', u''] =$$

$$=[\varphi\varphi'\varphi'', hh'h'', \varphi'\varphi''u'' \cdot (\varphi'\cdot u'h')h''] =$$

and

$$[\varphi, h, u] \cdot ([\varphi', h', u'] \cdot [\varphi'', h'', u'']) = [\varphi, h, u] \cdot [\varphi'\varphi'', hh'h'', \varphi'\varphi''u'' \cdot u'h'h''] =$$

and hence so defined operation is associative.

Theorem 1 implies that formula (1) determines a map $\mathcal{S}_\lambda \times \text{Aut}(S) \times H_1^\lambda$ onto the group of automorphisms $\text{Aut}(B_0^\lambda(S))$ of the Brandt $\lambda^0$-extension $B_0^\lambda(S)$ of the monoid $S$, and hence the associativity of binary operation (2) implies that the map $\mathcal{S}_\lambda \times \text{Aut}(S) \times H_1^\lambda$ onto the group $\text{Aut}(B_0^\lambda(S))$.

Next we show that $[1_{\mathcal{S}_\lambda}, 1_{\text{Aut}(S)}, 1_{H_1^\lambda}]$ is a unit element with the respect to the binary operation (2), where $1_{\mathcal{S}_\lambda}$, $1_{\text{Aut}(S)}$ and $1_{H_1^\lambda}$ are units of the groups $\mathcal{S}_\lambda$, $\text{Aut}(S)$ and $H_1^\lambda$, respectively. Then we have
that
\[ [\varphi, h, u] \cdot [1_{\mathcal{X}}, 1_{\text{Aut}(S)} \cdot 1_{H_1}] = [\varphi 1_{\mathcal{X}}, h 1_{\text{Aut}(S)}, \varphi 1_{H_1}] \cdot u 1_{\text{Aut}(S)} = \\
[\varphi h, \varphi 1_{H_1} \cdot u] = \\
[\varphi, h, 1_{H_1} \cdot u] = \\
[\varphi, h, u] \]

and
\[ [1_{\mathcal{X}}, 1_{\text{Aut}(S)} \cdot 1_{H_1}] \cdot [\varphi, h, u] = [1_{\mathcal{X}}, \varphi, 1_{\text{Aut}(S)} h, 1_{\mathcal{X}} u \cdot 1_{H_1} \cdot h] = [\varphi, h, u], \]
because every automorphism \( h \in \text{Aut}(S) \) acts on the group \( H_1 \) by the natural way as a restriction of global automorphism of the semigroup \( S \) on every factor, and hence we get that \( 1_{H_1} h = 1_{H_1} \).

Also, similar arguments imply that
\[ [\varphi, h, u] : [\varphi, h, u]^{-1} = [\varphi, h, u] : [\varphi^{-1}, h^{-1}, \varphi^{-1} u^{-1} h^{-1}] = \\
[\varphi, h^{-1}, hh^{-1}, (\varphi^{-1})^{-1} u^{-1} h^{-1} \cdot uh^{-1}] = \\
[\varphi, h^{-1}, hh^{-1}, (1_{\mathcal{X}}) u^{-1} h^{-1} \cdot uh^{-1}] = \\
[\varphi, h^{-1}, hh^{-1}, u^{-1} h^{-1} \cdot uh^{-1}] = \\
[1_{\mathcal{X}}, 1_{\text{Aut}(S)} \cdot 1_{H_1}] \]

This implies that the elements \([\varphi^{-1}, h^{-1}, \varphi^{-1} u^{-1} h^{-1}]\) and \([\varphi, h, u]\) are invertible in \( \mathcal{X} \times \text{Aut}(S) \times H_1 \), and hence the set \( \mathcal{X} \times \text{Aut}(S) \times H_1 \) with the binary operation \((2)\) is a group.

Let \( \text{Id}: B^0_\lambda(S) \to B^0_\lambda(S) \) be the identity automorphism of the semigroup \( B^0_\lambda(S) \). Then by Theorem 1 there exist some automorphism \( h: S \to S \), a bijective map \( \varphi: \lambda \to \lambda \) and a map \( u: \lambda \to H_1 \) into the group \( H_1 \) of units of \( S \) such that
\[ (\alpha, s, \beta) = (\alpha, s, \beta) \text{Id} = ((\alpha)\varphi, (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1}, (\beta)\varphi), \]
for all \( \alpha, \beta \in \lambda \) and \( s \in S^* \). Since \( \text{Id}: B^0_\lambda(S) \to B^0_\lambda(S) \) is the identity automorphism we conclude that
\( (\alpha)\varphi = \alpha \) for every \( \alpha \in \lambda \). Also, for every \( s \in S^* \) we get that \( s = (\alpha)u \cdot (s)h \cdot ((\beta)u)^{-1} \) for all \( \alpha, \beta \in \lambda \), and hence we obtain that
\[ 1_S = (\alpha)u \cdot (1_S)h \cdot ((\beta)u)^{-1} = (\alpha)u \cdot ((\beta)u)^{-1} \]
for all \( \alpha, \beta \in \lambda \). This implies that \((\alpha)u = (\beta)u = \tilde{u}\) is a fixed element of the group \( H_1 \) for all \( \alpha, \beta \in \lambda \).

We define
\[ \ker N = \{ [\varphi, h, u] \in \mathcal{X} \times \text{Aut}(S) \times H_1 : \varphi: \lambda \to \lambda \ is \ an \ identity \ map, \ \tilde{u}(s)h\tilde{u}^{-1} = s \ for \ any \ s \in S \}. \]

It is obvious that the equality \( \tilde{u}(s)h\tilde{u}^{-1} = s \) implies that \( (s)h = \tilde{u}^{-1}s\tilde{u} \) for all \( s \in S \). The previous arguments implies that \([\varphi, h, u] \in \ker N\) if and only if \([\varphi, h, u][3] \) is the unit of the group \( \text{Aut}(B^0_\lambda(S)) \), and hence \( \ker N \) is a normal subgroup of \( \mathcal{X} \times \text{Aut}(S) \times H_1 \). This implies that the quotient group \((\mathcal{X} \times \text{Aut}(S) \times H_1) / \ker N \) is isomorphic to the group \( \text{Aut}(B^0_\lambda(S)) \). \( \square \)
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Faculty of Mathematics, National University of Lviv, Universytetska 1, Lviv, 79000, Ukraine
E-mail address: o_gutik@franko.lviv.ua, ovgutik@yahoo.com