GOPAKUMAR-VAFA INVARIANTS AND WALL-CROSSING

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Abstract

In this paper, we generalize a mathematical definition of Gopakumar-Vafa (GV) invariants on Calabi-Yau 3-folds introduced by Maulik and the author, using an analogue of BPS sheaves introduced by Davison-Meinhardt on the coarse moduli spaces of one dimensional twisted semistable sheaves with arbitrary holomorphic Euler characteristics. We show that our generalized GV invariants are independent of twisted stability conditions, and conjecture that they are also independent of holomorphic Euler characteristics, so that they define the same GV invariants. As an application, we will show the flop transformation formula of GV invariants.

1. Introduction

1.1. Background. Let $X$ be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$. For $g \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(X, \mathbb{Z})$, Gopakumar-Vafa (GV) conjectured the existence of integer valued invariants (called Gopakumar-Vafa (GV) invariants)

$$(1.1) \quad n_{g, \beta} \in \mathbb{Z}, \quad g \in \mathbb{Z}_{\geq 0}, \quad \beta \in H_2(X, \mathbb{Z})$$

which determine Gromov-Witten invariants [Beh97] and Pandharipande-Thomas invariants [PT09] by taking their generating series (see [MT18 Section 3.3] for precise conjectures). The original approach of Gopakumar-Vafa (GV) toward defining (1.1) was to use the $sl_2 \times sl_2$-action on the cohomology of the moduli space of D2-branes, which may be mathematically interpreted as the moduli space of one dimensional (semi)stable sheaves on $X$. In [MT18], Maulik and the author proposed a mathematical definition of the invariants (1.1) along with the idea of Gopakumar-Vafa (GV), based on earlier works by Hosono-Saito-Takahashi [HST01], Katz [Kat08] (the $g = 0$ case) and Kiem-Li [KL]. As we will review shortly, the key ingredients of the definition in [MT18] are the perverse sheaf of vanishing cycles on the moduli space of one dimensional stable sheaves, and the character formula of $sl_2 \times sl_2$-action on its cohomology.

The relevant moduli space in the above works [HST01, Kat08, KL, MT18] is the moduli space $M(\beta, 1)$ of one dimensional Gieseker-stable sheaves $E$ on $X$ satisfying $l(E) = \beta$ and $\chi(E) = 1$, where $l(E)$ is
the fundamental one cycle of $E$. In this paper, we address the question whether we can also define the invariants (1.1) using some variants of the moduli space $M(\beta,1)$, i.e. different stability conditions, holomorphic Euler characteristics. For $m \in \mathbb{Z}$ and an element of the complexified ample cone

\begin{equation}
\sigma = B + i\omega \in A(X)_{\mathbb{C}}
\end{equation}

let $M_{\sigma}(\beta,m)$ be the coarse moduli space of $B$-twisted $\omega$-semistable one dimensional sheaves $E$ on $X$ satisfying $[l(E)] = \beta$ and $\chi(E) = m$. The moduli space $M_{\sigma}(\beta,m)$ is related to the previous one by $M(\beta,1) = M_{\sigma=i\omega}(\beta,1)$. The purpose of this paper is to extend the construction of GV invariants in [MT18] using the moduli space $M_{\sigma}(\beta,m)$, and study the independence of the resulting invariants of additional data $(\sigma,m)$.

1.2. Generalized GV invariants. In [MT18], by modifying the earlier work of Kiem-Li [KL], we defined the invariant (1.1) using a perverse sheaf of vanishing cycles on $M(\beta,1)$, and the perverse cohomologies of its push-forward to the Chow variety $\text{Chow}_X(\beta)$. Here the latter moduli space parametrizes effective one cycles on $X$ with homology class $\beta$. The basic fact behind this idea is that $M(\beta,1)$ is the truncation of a derived scheme with a $(-1)$-shifted symplectic structure [PTVV13], so by the derived Darboux theorem [BBBJ15] it has a $d$-critical structure introduced by Joyce [Joy15]. In particular it is locally written as a critical locus of some functions. The locally defined vanishing cycle sheaves can be glued if we choose an orientation data, which is a square root line bundle of the virtual canonical line bundle of $M(\beta,1)$.

We will apply the similar idea for the moduli space $M_{\sigma}(v)$, where $\sigma$ is an element (1.2) and $v = (\beta,m)$. A difference from the moduli space $M(\beta,1)$ is that $M_{\sigma}(v)$ is not a fine moduli space in general, so it is not locally written as a critical locus. Nevertheless, we will define the perverse sheaf\footnote{The definition of the perverse sheaf (1.3) was suggested to the author by Ben Davison.}

\begin{equation}
\phi_{M_{\sigma}(v)} \in \text{Perv}(M_{\sigma}(v))
\end{equation}

as an analogue of BPS sheaves introduced by Davison-Meinhardt [DM] on the coarse moduli spaces of semistable quiver representations with super-potentials. The perverse sheaf (1.3) is, roughly speaking, constructed as follows. Let $M_{\sigma}(v)$ be the moduli stack of $\sigma$-semistable sheaves on $X$ with Chern character $v$. Then the stack $M_{\sigma}(v)$ admits a $d$-critical structure [BBBJ15], so given an orientation data we can construct a perverse sheaf of vanishing cycles on the stack $M_{\sigma}(v)$. Then we push-forward it to the coarse moduli space $M_{\sigma}(v)$, and its first perverse cohomology defines the perverse sheaf (1.3). It will turn out that (1.3) is an analogue of BPS sheaves (see Subsection 2.8).
Let $\pi_M$ be the Hilbert-Chow map

$$\pi_M : M_\sigma(v) \rightarrow \text{Chow}_X(\beta)$$

sending $E$ to its fundamental one cycle. Then as a generalization of the construction in [MT18], we introduce the following definition:

**Definition 1.1.** (Definition 2.11) For $\gamma \in \text{Chow}_X(\beta)$, we define the invariant

$$\Phi_\sigma(\gamma, m) := \sum_{i \in \mathbb{Z}} \chi_i^\pi \left( R^{\pi_M} \phi_M(\sigma) \right)_i \gamma^i y^i \in \mathbb{Z}[y^\pm 1].$$

We note that the GV invariant $n_{g,\beta}$ defined in [MT18] is recovered from (1.4) by the character formula of $sl_2 \times sl_2$-action

$$\int_{\gamma \in \text{Chow}_X(\beta)} \Phi_\sigma = \omega(\gamma, 1) \, de = \sum_{g \geq 0} n_{g,\beta}(y^{1/2} + y^{-1/2})^{2g}.$$ 

The following is the main conjecture we address in this paper.

**Conjecture 1.2.** The invariant $\Phi_\sigma(\gamma, m)$ is independent of $\sigma$ and $m$.

If the above conjecture is true, then one can define the GV invariants (1.1) from the moduli space $M_\sigma(v)$ for an arbitrary element $\sigma$ in (1.2) and $v = (\beta, m)$. Namely if we define $n_{g,\beta,m}(\sigma)$ by the identity

$$\int_{\gamma \in \text{Chow}_X(\beta)} \Phi_\sigma(\gamma, m) \, de = \sum_{g \geq 0} n_{g,\beta,m}(\sigma)(y^{1/2} + y^{-1/2})^{2g},$$

then $n_{g,\beta,m}(\sigma)$ is independent of $\sigma$ and $m$, so we have $n_{g,\beta,m}(\sigma) = n_{g,\beta}$. When $(\gamma, m)$ is coprime, then Conjecture 1.2 was also proposed in [MT18] Conjecture 3.21, and Conjecture 1.2 is its generalization to the non-coprime $(\gamma, m)$ case. A similar result and conjecture were discussed in [JS12, Tod14] for generalized Donaldson-Thomas invariants [JS12, KS] counting one dimensional semistable sheaves, which in turn implies Pandharipande-Thomas’s strong rationality conjecture [PT09] (see [Tod12] for details).

Here we mention about a technical subtlety in defining the invariant (1.4). By their constructions, the perverse sheaf (1.3) and the invariant (1.4) depend on a choice of an orientation data of $M_\sigma(v)$, so we have to specify its choice. Similarly to [MT18], we impose the condition on an orientation data of $M_\sigma(v)$ so that it is trivial locally on the Chow variety. More precisely, we assume that the virtual canonical line bundle of the stack $M_\sigma(v)$ is trivial on the preimage of some open neighborhood $\gamma \in U \subset \text{Chow}_X(\beta)$ under the Hilbert-Chow map

$$\pi_M : M_\sigma(v) \rightarrow \text{Chow}_X(\beta).$$

In this case, we say that $M_\sigma(v)$ is CY at $\gamma$, and conjecture that this is always the case. Then we can take an orientation data of $\pi_M^{-1}(U)$
which is trivial as a line bundle. Such an orientation data is called a CY orientation data. Then by a local argument in an open neighborhood of γ, we can define the invariant (1.4) using a CY orientation data. The resulting invariant (1.4) is shown to be independent of a CY orientation data (see Lemma 2.14).

1.3. Results. We will study Conjecture 1.2 under a CY condition on the bigger stack MX(β) of pure one dimensional sheaves E with [l(E)] = β. Our main result is the independence of stability conditions in Conjecture 1.2:

**Theorem 1.3.** (Theorem 5.7) Let X be a smooth projective CY 3-fold. For an effective one cycle γ on X with homology class β, suppose that the stack MX(β) is CY at γ. Then the invariant Φσ(γ,m) is independent of σ.

By the above theorem, if MX(β) is CY at γ we can write

\[ \Phi_X(\gamma, m) := \Phi_\sigma(\gamma, m). \]

The result of Theorem 1.3 implies a trivial wall-crossing of the invariants \( \Phi_\sigma(\gamma, m) \). Namely there is a locally finite number of codimension one submanifolds in \( A(X)_C \) called walls such that \( M_\sigma(v) \) is constant if \( \sigma \) lies on a connected component of complement of walls (called a chamber) but may change if \( \sigma \) crosses a wall. The result of Theorem 1.3 implies that, although the moduli space \( M_\sigma(v) \) may change by wall-crossing, the associated invariants \( \Phi_\sigma(\gamma, m) \) are not changed, i.e. the wall-crossing formula is trivial.

The main idea of the proof of Theorem 1.3 is to reduce to the case of representations of quivers with formal but convergent super-potentials. In this reduction step, we use the result of the companion paper [Tod], where we prove that the moduli stack of semistable sheaves \( M_\sigma(v) \) is described analytic locally on \( M_\sigma(v) \) as the moduli stack of representations of the Ext-quiver with a convergent super-potential. Here the super-potential is defined from the minimal \( A_\infty \)-structure of the derived category of coherent sheaves on X. After the above reduction, we use the results and arguments of Davison-Meinhardt [DM], where a similar wall-crossing phenomena was investigated for representations of quivers with super-potentials.

The first application of Theorem 1.3 is to show the independence of \( m \) of the invariant \( \Phi_\sigma(\gamma, m) \), when \( \gamma \) is a primitive one cycle, i.e. \( \gamma \) is written as \( \sum_{1 \leq i \leq k} a_i[C_i] \) for irreducible curves \( C_i, a_i \in \mathbb{Z}_{\geq 1} \) with g.c.d.(\( a_1, \ldots, a_k \)) = 1.

**Theorem 1.4.** (Theorem 5.8) Under the situation of Theorem 1.3, suppose that \( \gamma \) is a primitive one cycle. Then \( \Phi_X(\gamma, m) \) is independent of \( m \).
The next application of Theorem 1.3 is to show the flop invariance of the invariant $\Phi_X(\gamma, m)$. Let

$$
\phi: X \xrightarrow{f} Y \overset{f^\dagger}{\leftarrow} X^\dagger
$$

be a flop between smooth projective CY 3-folds. In this situation, we have the following result:

**Theorem 1.5.** (Theorem 6.8) Let $\gamma$ be an effective one cycle on $X$ with homology class $\beta$ such that $f_*\gamma \neq 0$. Suppose that the stacks $\mathcal{M}_X(\beta)$, $\mathcal{M}_X^\dagger(\phi_*\beta)$ are CY at $\gamma$, $\phi_*\gamma$ respectively. Then we have the identity

$$
\Phi_X(\gamma, m) = \Phi_X^\dagger(\phi_*\gamma, m).
$$

If the assumption of Theorem 1.5 holds for $m = 1$ and any $\gamma \in \text{Chow}_X(\beta)$, we in particular obtain

$$
n_{g,\beta} = n_{g,\phi_*\beta}
$$

for the curve class $\beta$ with $f_*\beta \neq 0$. The above flop invariance of GV invariants was proved in [MT18] when the curve class $\beta$ is irreducible.

The result of Theorem 1.5 gives a complete answer to the flop invariance of GV invariants for any curve class, assuming the CY properties of the relevant moduli stacks of one dimensional sheaves.

So far the results in Theorem 1.3, Theorem 1.4 and Theorem 1.5 are conditional to the conjectural CY property of the stack $\mathcal{M}_X(\beta)$. We will show that the above CY property holds for the non-compact CY 3-fold

$$
X = \text{Tot}_S(K_S)
$$

where $S$ is a smooth projective surface. Although $X$ is non-compact in this case, we can similarly define the invariant $\Phi_{\sigma}(\gamma, m)$ and can ask its independence of $(\sigma, m)$ as in Conjecture 1.2. Then the analogy of the results in Theorem 1.3, Theorem 1.4 and Theorem 1.5 hold without assuming the CY properties:

**Theorem 1.6.** (Theorem 7.1, Theorem 7.3, Theorem 7.4) Let $S$ be a smooth projective surface and $X = \text{Tot}_S(K_S)$ the non-compact CY 3-fold. Then we have the following:

(i) For any effective compactly supported one cycle $\gamma$ on $X$ with homology class $\beta$, the stack $\mathcal{M}_X(\beta)$ is CY at $\gamma$. In particular for any element $\sigma = B + i\omega \in A(S)_C$ and $m \in \mathbb{Z}$, the invariant $\Phi_{\sigma}(\gamma, m) \in \mathbb{Z}[y^{\frac{1}{2}}]$ is defined as in (1.4) with the CY orientation data.

(ii) The invariant $\Phi_{\sigma}(\gamma, m)$ is independent of $\sigma$. So we can write it as $\Phi_X(\gamma, m)$.

(iii) $\Phi_X(\gamma, m)$ is also independent of $m$ if $\gamma$ is a primitive one cycle.
Suppose that $\gamma$ is supported on the zero section $S \subset X$, and let $h: S^\dagger \to S$ be a blow-up at a point. Then for $X^\dagger = \text{Tot}_{S^\dagger}(K_{S^\dagger})$ and the one cycle $h^*\gamma$ on $X^\dagger$ supported on the zero section $S^\dagger \subset X^\dagger$, we have $\Phi_X(\gamma, m) = \Phi_{X^\dagger}(h^*\gamma, m)$.

1.4. Plan of the paper. The organization of this paper is as follows. In Section 2 we introduce the invariant $\Phi_\sigma(\gamma, m)$ and propose the conjecture that it is independent of $\sigma$ and $m$. In Section 3 we compute the invariant $\Phi_\sigma(\gamma, m)$ in some examples. In Section 4 we discuss a wall-crossing formula of perverse sheaves of vanishing cycles of representations of quivers with convergent super-potentials. In Section 5 we prove Theorem 1.3 and Theorem 1.4. In Section 6 we prove Theorem 1.5. In Section 7 we prove Theorem 1.6.

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1.6. Notation and convention. In this paper, all schemes and stacks are defined over $\mathbb{C}$. For a scheme or stack $M$, we will only consider constructible sheaves on it with $\mathbb{Q}$-coefficients. We denote by $\text{Perv}(M)$ the category of perverse sheaves on $M$, which is the heart of a $t$-structure on the derived category of constructible sheaves on $M$ (see [BBD82, LO09]). Let $\iota: M^{\text{red}} \to M$ be the reduced part of $M$. Since $\iota$ is a homeomorphism, we always identify $\text{Perv}(M)$ with $\text{Perv}(M^{\text{red}})$ in a natural way.

For a bounded complex $E$ of constructible sheaves on $M$, we denote by $^{p}H^i(E)$ the $i$-th cohomology with respect to the perverse $t$-structure, and $\chi(E)$ is the the Euler characteristic of $R\Gamma(M, E)$. For a constructible function $\nu$ on a scheme $M$, the weighted Euler characteristic is denoted by

$$\int_M \nu \cdot \text{de} := \sum_{m \in \mathbb{Z}} m \cdot e(\nu^{-1}(m)).$$

Here $e(-)$ is the topological Euler characteristic.

2. Generalized GV invariants

In this section, we recall some necessary background on moduli spaces of semistable sheaves [HL97], and Joyce’s $d$-critical schemes and stacks [Joy15]. We then introduce the invariant $\Phi_\sigma(\gamma, m) \in \mathbb{Z}[y^\pm 1]$, using an analogy of BPS sheaves [DM].
2.1. Twisted semistable sheaves. Let $X$ be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. $\dim X = 3$ and $K_X = 0$. We denote by

$$\text{Coh} \leq 1(X) \subset \text{Coh}(X)$$

the abelian subcategory of coherent sheaves $E$ on $X$ whose supports have dimensions less than or equal to one. Let $A(X)_\mathbb{C}$ be the complexified ample cone of $X$ defined by

$$A(X)_\mathbb{C} := \{ B + i\omega \in \text{NS}(X)_\mathbb{C} : \omega \text{ is ample} \}.$$ 

For an object $E \in \text{Coh} \leq 1(X)$ and an element

$$(B + i\omega) \in A(X)_\mathbb{C}$$

the $B$-twisted $\omega$-slope $\mu_{B,\omega}(E) \in \mathbb{R} \cup \{\infty\}$ is defined by

$$\mu_{B,\omega}(E) := \frac{\text{ch}^3_B(E) - \omega \cdot \text{ch}^2_B(E)}{\omega \cdot l(E)} = \frac{\chi(E) - B \cdot l(E)}{\omega \cdot l(E)}.$$

Here $\text{ch}^B(\cdot) := e^{-B} \text{ch}(\cdot)$ is the $B$-twisted Chern character and $\mu_{B,\omega}(E) = \infty$ if $\omega \cdot \text{ch}^B_2(E) = 0$. Also $l(E)$ is the fundamental one cycle of $E$, defined by

$$l(E) := \sum_{\eta \in X, \dim \{\eta\} = 1} \text{length}(E_\eta) : \{\eta\}.$$ 

**Definition 2.1.** An object $E \in \text{Coh} \leq 1(X)$ is called $(B, \omega)$-(semi)stable if for any subsheaf $0 \neq F \subset E$, we have $\mu_{B,\omega}(F) < (\leq) \mu_{B,\omega}(E)$.

The above stability condition can be interpreted in terms of Bridgeland stability conditions \cite{Bri07} as follows. Let

$$N_1(X) \subset H_2(X, \mathbb{Z})$$

be the group of numerical classes of algebraic one cycles on $X$ and set

$$\Gamma_X := N_1(X) \oplus \mathbb{Z}.$$ 

The Chern character of an object in $D^b(\text{Coh} \leq 1(X))$ takes its value in $\Gamma_X$, and given by

$$(2.2) \quad \text{ch}(E) = (\text{ch}_2(E), \text{ch}_3(E)) = ([l(E)], \chi(E)).$$

By definition, a Bridgeland stability condition on $D^b(\text{Coh} \leq 1(X))$ w.r.t. the Chern character map (2.2) consists of data

$$(2.3) \quad \sigma = (Z, A), \ Z: \Gamma_X \to \mathbb{C}, \ A \subset D^b(\text{Coh} \leq 1(X))$$

where $Z$ is a group homomorphism, $A$ is the heart of a bounded t-structure satisfying some axioms (see \cite{Bri07, KS} for details). It determines the set of $\sigma$-(semi)stable objects: $E \in D^b(\text{Coh} \leq 1(X))$ is $\sigma$-(semi)stable if $E[k] \in A$ for some $k \in \mathbb{Z}$, and for any non-zero subobject
0 \neq F \subset E[k] in \mathcal{A}, we have the inequality in (0, \pi):
\arg Z(\text{ch}(F)) < (\leq) \arg Z(\text{ch}(E[k))).

The set of Bridgeland stability conditions \((2.3)\) forms a complex manifold, which we denote by \(\text{Stab}_{\leq}(X)\). The forgetting map \((Z, A) \mapsto \gamma\)

gives a local homeomorphism

\[\text{Stab}_{\leq}(X) \rightarrow (\Gamma_X)_{\mathbb{C}}.\]

For a given element \((2.1)\), let \(Z_{B, \omega}\) be the group homomorphism \(\Gamma_X \rightarrow \mathbb{C}\) defined by

\[Z_{B, \omega}(\beta, m) := -m + (B + i\omega)\beta.\]

\[(2.4)\]

Then the pair

\[\sigma_{B, \omega} := (Z_{B, \omega}, \text{Coh}_{\leq}(X))\]

determines a point in \(\text{Stab}_{\leq}(X)\).

It is obvious that an object in \(\text{Coh}_{\leq}(X)\) is \((B, \omega)-(semi)stable iff it is Bridgeland \(\sigma_{B, \omega}-(semi)stable. We also call \((B, \omega)-(semi)stable sheaves as \(\sigma_{B, \omega}-(semi)stable objects. Moreover the map

\[A(X)_{\mathbb{C}} \rightarrow \text{Stab}_{\leq}(X), (B, \omega) \mapsto \sigma_{B, \omega}\]

is a continuous injective map, whose image is denoted by

\[U(X) \subset \text{Stab}_{\leq}(X).\]

We sometimes write \(\sigma = \sigma_{B, \omega} \in U(X)\) as \(\sigma = B + i\omega.\)

\subsection*{2.2. Moduli stacks of twisted semistable sheaves.}

For \(\beta \in \mathbb{N}_1(X)\), let \(\mathcal{M}_X(\beta)\) be the 2-functor

\[(2.6)\]

\[\mathcal{M}_X(\beta) : \text{Sch}/\mathbb{C} \rightarrow \text{Groupoid}\]

sending a \(\mathbb{C}\)-scheme \(S\) to the groupoid of \(S\)-flat sheaves \(\mathcal{E} \in \text{Coh}(X \times S)\) such that for each closed point \(s \in S\), the sheaf \(\mathcal{E}_s := \mathcal{E}|_{X \times \{s\}}\) is an object in \(\text{Coh}_{\leq}(X)\) satisfying \([l(\mathcal{E}_s)] = \beta\). It is well-known that the

2-functor \(\mathcal{M}_X(\beta)\) is an algebraic stack locally of finite type, though it is neither of finite type nor separated.

For \(m \in \mathbb{Z}\) and \(\sigma = \sigma_{B, \omega} \in U(X)\), let \(v = (\beta, m) \in \Gamma_X\) and

\[(2.7)\]

\[\mathcal{M}_\sigma(v) \subset \mathcal{M}_X(\beta)\]

be the substack of \(\sigma\)-semistable objects \(E \in \text{Coh}_{\leq}(X)\) satisfying

\[(2.8)\]

\[\text{ch}(E) = v = (\beta, m).\]

The stack \(\mathcal{M}_\sigma(v)\) is a finite type open substack of \(\mathcal{M}_X(\beta)\). Indeed, the stack \(\mathcal{M}_\sigma(v)\) is constructed as a GIT quotient stack (see [Tod, Lemma 7.4]), hence we have the projective coarse moduli space \(M_\sigma(v)\) together with the natural morphism

\[(2.9)\]

\[p_M : M_\sigma(v) \rightarrow M_\sigma(v).\]
For $\beta \in N_1(X)$, the Chow functor

$$\text{Chow}_X(\beta) : \text{Sch}^{\text{red}}/\mathbb{C} \to \text{Set}$$

is defined in [Ryd] by associating a reduced $\mathbb{C}$-scheme $S$ to the set of relative cycles on $X \times S$ over $S$, whose restriction to $X \times \{s\}$ for any closed point $s \in S$ is pure one dimensional with homology class $\beta$ (see [Ryd], Section 4). The functor (2.10) is represented by a reduced projective scheme

$$\text{Chow}_X(\beta)$$
called Chow variety, whose closed points correspond to effective one or zero cycles on $X$ with homology class $\beta$.

Let $S$ be a reduced $\mathbb{C}$-scheme and $E \in \text{Coh}(X \times S)$ be a $S$-valued point of $\mathcal{M}_X(\beta)$. Then by [Ryd], Theorem 7.14 there is a canonical relative cycle on $X \times S$ whose support is $\text{Supp}(E)$. It induces a morphism

$$S \to \text{Chow}_X(\beta)$$

which sends $s \in S$ to the fundamental cycle of $E_s$. The above morphism only depends on the isomorphism class of the sheaf $E$, thus induces the morphism of reduced stacks

$$\pi_M : \mathcal{M}^{\text{red}}_X(\beta) \to \text{Chow}_X(\beta).$$

The above morphism is called Hilbert-Chow (HC) map. By restricting the above morphism to the open substack $\mathcal{M}_{\sigma}(v) \subset \mathcal{M}_X(\beta)$ for $\sigma \in U(X)$ and $v = (\beta, m) \in \Gamma_X$, we obtain the morphism

$$\pi_M : \mathcal{M}^{\text{red}}_{\sigma}(v) \to \text{Chow}_X(\beta).$$

By the universality of the coarse moduli space (see [HL97], Definition 2.2.1, Theorem 4.3.4), the morphism $\pi_M$ uniquely factors through the (reduced part of the) morphism $p_M$, where $p_M$ is the natural morphism (2.9). So we have the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}^{\text{red}}_{\sigma}(v) & \xrightarrow{p_M} & \mathcal{M}^{\text{red}}_X(\beta) \\
\downarrow & & \downarrow \pi_M \\
\mathcal{M}^{\text{red}}_{\sigma}(v) & \xrightarrow{\pi_M} & \text{Chow}_X(\beta).
\end{array}$$

2.3. $d$-critical schemes. We recall the notion of $d$-critical schemes and $d$-critical stacks introduced in [Joy15]. For any complex scheme $T$, Joyce [Joy15] shows that there exists a canonical sheaf of $\mathbb{C}$-vector spaces $\mathcal{S}_T$ on $T$ satisfying the following property: for any Zariski open subset $R \subset T$ and a closed embedding $i : R \hookrightarrow V$ into a smooth scheme $V$, there is an exact sequence

$$0 \to \mathcal{S}_T|_R \to \mathcal{O}_V/I^2 \overset{dR}{\to} \Omega_V/I \cdot \Omega_V.$$
Here $I \subset \mathcal{O}_V$ is the ideal sheaf which defines $R$ and $d_{DR}$ is the de-Rham differential. Moreover there is a natural decomposition

$$\mathcal{S}_T = \mathcal{S}_T^0 \oplus \mathbb{C}_T$$

where $\mathbb{C}_T$ is the constant sheaf on $T$. The sheaf $\mathcal{S}_T^0$ restricted to $R$ is the kernel of the composition

$$\mathcal{S}_T|_R \hookrightarrow \mathcal{O}_V/I^2 \twoheadrightarrow \mathcal{O}_{\text{red}}.$$ 

For example, suppose that $f : V \to \mathbb{A}^1$ is a regular function such that

$$(2.15) \quad R = \{df = 0\}, \quad f|_{\text{red}} = 0.$$ 

Then we have $I = (df) := \text{Im}(T_V \xrightarrow{df} \mathcal{O}_V)$ and $f + (df)^2$ is an element of $\Gamma(R, \mathcal{S}_T^0|_R)$.

**Definition 2.2.** ([Joy15]) A pair $(T, s)$ for a complex scheme $T$ and $s \in \Gamma(T, \mathcal{S}_T^0)$ is called a $d$-critical scheme if for any $x \in T$, there is an open neighborhood $x \in R \subset T$, a closed embedding $i : R \hookrightarrow V$ into a smooth scheme $V$, a regular function $f : V \to \mathbb{A}^1$ satisfying (2.15) such that $s|_R = f + (df)^2$ holds. In this case, the data

$$(2.16) \quad (R, V, f, i)$$

is called a $d$-critical chart. The section $s$ is called a $d$-critical structure of $T$.

Given a $d$-critical scheme $(T, s)$, there exists a line bundle $K_{T,s}$ on $T^\text{red}$ called virtual canonical line bundle (see [Joy15] Section 2.4)\footnote{In [Joy15] Section 2.4, this was just called canonical bundle.}, such that for any $d$-critical chart (2.16) there is a natural isomorphism

$$(2.17) \quad K_{T,s}|_{T^\text{red}} \cong K_V^{\otimes 2}|_{T^\text{red}}.$$ 

**Definition 2.3.** ([Joy15]) An orientation of a $d$-critical scheme $(T, s)$ is a choice of a square root line bundle $K_{T,s}^{1/2}$ for $K_{T,s}$ on $T^\text{red}$ and an isomorphism

$$(2.18) \quad (K_{T,s}^{1/2})^{\otimes 2} \cong K_{T,s}.$$ 

A $d$-critical scheme with an orientation is called an oriented $d$-critical scheme.

**2.4. $d$-critical stacks.** Let $\mathcal{M}$ be an algebraic stack over $\mathbb{C}$. The category of sheaves of $\mathbb{C}$-vector spaces on $\mathcal{M}$ is defined in the lisse-étale site of $\mathcal{M}$. This is equivalent to the category $\text{Sh}(\mathcal{M})$ defined as follows (see [LMB00] for details): an object $\mathcal{F}$ of $\text{Sh}(\mathcal{M})$ consists of data

(i) For each $\mathbb{C}$-scheme $T$ and a smooth 1-morphism $t : T \to \mathcal{M}$, we are given a sheaf of $\mathbb{C}$-vector spaces $\mathcal{F}(T, t)$ on $T$ in étale topology.
(ii) For \( \mathbb{C} \)-schemes \( T, U \) and smooth 1-morphisms \( t: T \to \mathcal{M}, u: U \to \mathcal{M} \) with a 2-commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\phi} & U \\
t \downarrow & & \downarrow u \\
\mathcal{M} & & 
\end{array}
\]

we are given a morphism

\[
\phi^{-1} F(U, u) \to F(T, t)
\]

of sheaves of \( \mathbb{C} \)-vector spaces on \( T \) in \( \acute{e} \)tale topology.

The above data should satisfy several compatibility conditions. A global section \( s \in H^0(\mathcal{F}) \) of an object \( \mathcal{F} \in \text{Sh}(\mathcal{M}) \) consists of global sections \( s(T, t) \in H^0(F(T, t)) \) for each \( \mathbb{C} \)-scheme \( T \) and a smooth morphism \( t: T \to \mathcal{M} \), such that the morphism (2.19) sends \( s(U, u) \) to \( s(T, t) \).

By [Joy15, Corollary 2.52], there is a canonical sheaf of \( \mathbb{C} \)-vector spaces \( S^0_{\mathcal{M}} \) on an algebraic stack \( \mathcal{M} \), such that for any scheme \( T \) and a smooth 1-morphism \( t: T \to \mathcal{M} \) we have

\[
S^0_{\mathcal{M}}(T, t) = S^0_T.
\]

**Definition 2.4.** ([Joy15]) A pair \((\mathcal{M}, s)\) for an algebraic stack \( \mathcal{M} \) over \( \mathbb{C} \) and a global section \( s \in H^0(S^0_{\mathcal{M}}) \) is called a d-critical stack if for any \( \mathbb{C} \)-scheme \( T \) and a smooth 1-morphism \( t: T \to \mathcal{M} \), the pair \((T, s(T, t))\) is a d-critical scheme.

Given a d-critical stack \((\mathcal{M}, s)\), there exists a line bundle \( K_{\mathcal{M}, s} \) on \( \mathcal{M}^{\text{red}} \), called virtual canonical line bundle, such that for any \( \mathbb{C} \)-scheme \( T \) and a smooth 1-morphism \( t: T \to \mathcal{M} \), so that \( t^{\text{red}}: T^{\text{red}} \to \mathcal{M}^{\text{red}} \) is also smooth, there is a natural isomorphism

\[
K_{\mathcal{M}, s}(T^{\text{red}}, t^{\text{red}}) \cong K_{T, s(T, t)} \otimes (\Omega_{T/\mathcal{M}}|_{T^{\text{red}}})^{\otimes -2}.
\]

Here \((T, s(T, t))\) is the d-critical scheme in Definition 2.4.

**Definition 2.5.** ([Joy15]) An orientation of a d-critical stack \((\mathcal{M}, s)\) is a choice of a square root line bundle \( K^{1/2}_{\mathcal{M}, s} \) for \( K_{\mathcal{M}, s} \) on \( \mathcal{M}^{\text{red}} \) and an isomorphism

\[
(K^{1/2}_{\mathcal{M}, s})^{\otimes 2} \cong K_{\mathcal{M}, s}.
\]

A d-critical stack with an orientation is called an oriented d-critical stack.

For an oriented d-critical stack \((\mathcal{M}, s, K^{1/2}_{\mathcal{M}, s})\), let \((T, t)\) be as in Definition 2.4. Then we have the line bundle on \( T^{\text{red}} \)

\[
K^{1/2}_{T, s(T, t)} = K^{1/2}_{\mathcal{M}, s}(T^{\text{red}}, t^{\text{red}}) \otimes (\Omega_{T/\mathcal{M}}|_{T^{\text{red}}}).
\]
Then the isomorphism (2.20), induces the isomorphism
\[(K_{1/2}^{1/2}) \otimes 2 \cong K_{T,s(T,t)}\]
which gives an orientation of the \(d\)-critical scheme \((T, s(T, t))\).

As mentioned in [Joy15], the above notions of \(d\)-critical structures on schemes and stacks are naturally extended to those of analytic \(d\)-critical structures for complex analytic spaces and stacks respectively. Moreover, given an algebraic \(d\)-critical structure on an algebraic stack \(M\), it naturally gives an analytic \(d\)-critical structure on the analytification of \(M\).

**Example 2.6.** Suppose that an algebraic \(\mathbb{C}\)-group \(G\) acts on a complex analytic space \(R\), and set \(M = [R/G]\). Then as in [Joy15], Example 2.55, we have \(H^0(S^0_M) = H^0(S^0_R)^G\). Let \(V\) be a complex manifold with \(G\)-action, \(f : V \to \mathbb{C}\) be a \(G\)-invariant analytic function such that \(R = \{df = 0\}\). Then we have
\[s = f + (df)^2 \in H^0(S^0_R)^G.\]

The pair \((\mathcal{M}, s)\) is an example of an analytic \(d\)-critical stack.

**2.5. Perverse sheaves of vanishing cycles.** Let \(f : V \to \mathbb{C}\) be a holomorphic function on a complex manifold \(V\), and set \(R = \{df = 0\}\). Suppose that \(f|_{R_{red}} = 0\) and set \(V_0 = f^{-1}(0)\). We have the associated vanishing cycle functor (see [Dim04], Theorem 5.2.21)
\[\phi_f : \text{Perv}(V) \to \text{Perv}(V_0).\]

Let \(\text{IC}(V) \in \text{Perv}(V)\) be the intersection complex on \(V\), which coincides with \(\mathbb{Q}_V[\dim V]\) since \(V\) is smooth. We have the perverse sheaf of vanishing cycles supported on \(R_{red} \subset V_0\)
\[(2.23) \quad \phi_f(\text{IC}(V)) \in \text{Perv}(R) \subset \text{Perv}(V_0).\]

Let \((T, s)\) be a \(d\)-critical scheme or \(d\)-critical analytic space. For a \(d\)-critical chart \((R, V, f, i)\) as in (2.16), we have the perverse sheaf of vanishing cycles (2.23) on \(R\). In [BBD+15], it is proved that if \((T, s)\) is oriented, then the perverse sheaves of vanishing cycles (2.23) glue to give a global perverse sheaf on \(T\). Let
\[(2.24) \quad (K_{1/2}^{1/2}|_{R_{red}}) \otimes 2 \cong K_V^{\otimes 2}|_{R_{red}}\]
be the isomorphism given by the composition of (2.17) and (2.18). Then there is a \(\mathbb{Z}/2\mathbb{Z}\)-principal bundle \(\tau_R : R_{red} \to R_{red}\) which parametrizes local square roots of the isomorphism (2.24). We have the decomposition
\[\tau_R^*\mathbb{Q}_{R_{red}} = \mathbb{Q}_{R_{red}} \oplus \mathcal{L}\]
for a rank one local system \(\mathcal{L}\) on \(R_{red}\). The following result is proved in [BBD+15] (also see [KL] for the similar result in the framework of virtual critical structures):
Theorem 2.7. ([BBD$^+15$ Theorem 6.9], [BBBJ15 Theorem 4.12])

(i) For an oriented $d$-critical scheme or an oriented $d$-critical analytic space $(T, s, K_{T,s}^{1/2})$, there exists a natural perverse sheaf $\phi_T$ on $T$ such that for any $d$-critical chart (2.16) there is a natural isomorphism\(^3\)

$$\phi_T|_R \cong \phi_f(\text{IC}(V)) \otimes L.$$ 

(ii) For an oriented $d$-critical algebraic or analytic stack $(M, s, K_{M,s}^{1/2})$, there exists a natural perverse sheaf $\phi_M$ on $M$ such that for any $(T, t)$ as in Definition 2.4 we have

$$\phi_M(T, t) = \phi_T[-d_t].$$

Here $d_t$ is the relative dimension of $t: T \to M$ and $\phi_T$ is the perverse sheaf in (i) for $(T, s(T, t))$ with orientation given by (2.22).

2.6. CY properties of $d$-critical stacks. Let $X$ be a smooth projective CY 3-fold. For $\beta \in N_1(X)$, let $M_X(\beta)$ be the moduli stack defined as in (2.6). By the result of [BBBJ15], we have the following:

**Theorem 2.8.** ([BBBJ15]) There is a canonical $d$-critical stack structure $s \in H^0(S^0_M)$ on $M_X(\beta)$, whose virtual canonical line bundle is given by

$$K_{M_X(\beta),s} = K_{M_X(\beta)}^{\text{vir}} := \det R \text{Hom}_{pr_M}(\mathcal{E}, \mathcal{E}).$$

Here $\mathcal{E}$ is the universal family on $X \times M_X(\beta)$ and $pr_M: X \times M_X(\beta) \to M_X(\beta)$ is the projection.

We next consider CY conditions on our moduli stacks of one dimensional sheaves. In general, we introduce the following definition:

**Definition 2.9.** Let $(M, s)$ be a $d$-critical stack with a morphism

$$\pi_M: M^{\text{red}} \to \text{Chow}_X(\beta)$$

for $\beta \in N_1(X)$. We say $M$ is Calabi-Yau (CY) at $\gamma \in \text{Chow}_X(\beta)$ if there is an analytic open neighborhood $\gamma \in U \subset \text{Chow}_X(\beta)$ such that $K_{M,s}$ is trivial on $\pi_M^{-1}(U)$.

We propose the following conjecture:

**Conjecture 2.10.** The stack $M_X(\beta)$ is CY at any point $\gamma \in \text{Chow}_X(\beta)$ for the HC map (2.11).

\(^3\)In [BBD$^+15$ Theorem 6.9], the tensor product in the right hand side of (2.25) is denoted as $\otimes_{\mathbb{Z}/2\mathbb{Z}} B^{\text{red}}$. The latter tensor product is defined as the one given in the right hand side of (2.25) in [BBD$^+15$ Definition 2.9].
Note that the $d$-critical structure on $M_X(\beta)$ in Theorem 2.8 induces the one on $M_\sigma(v)$ by the open embedding (2.7). Moreover if Conjecture 2.10 is true, then the open substack $M_\sigma(v) \subset M_X(\beta)$ is also CY at $\gamma \in \text{Chow}_X(\beta)$ for the HC map (2.12).

Suppose that $M_\sigma(v)$ is CY at $\gamma \in \text{Chow}_X(\beta)$, and take an open subset $\gamma \in U \subset \text{Chow}_X(\beta)$ as in Definition 2.9. Below we denote the pull-back of the diagram (2.13) to $U$ by (omitting ‘red’ for simplicity)

\begin{equation}
(2.27) \quad \xymatrix{ M_\sigma(v)|_U \ar[r]^{p_M} & M_X(\beta)|_U \\
M_\sigma(v)|_U \ar[u]^{\pi_M} & U. \ar[u]_{\pi_M}
}
\end{equation}

By the CY condition of $M_\sigma(v)$, there is an orientation data of $M_\sigma(v)|_U$ satisfying

\begin{equation}
(2.28) \quad (K_{M_\sigma(v)|_U}^{\text{vir}})^{1/2} \cong \mathcal{O}_{M_\sigma(v)|_U}
\end{equation}

as line bundles on the stack $M_\sigma(v)|_U$. Such an orientation data is called a Calabi-Yau (CY) orientation data. Given a CY orientation data of $M_\sigma(v)|_U$, by Theorem 2.7 (ii) we have the associated perverse sheaf of vanishing cycles

\begin{equation}
(2.29) \quad \phi_{M_\sigma(v)|_U} \in \text{Perv}(M_\sigma(v)|_U).
\end{equation}

2.7. Definition of generalized GV invariants. We keep the situation and notation in the previous subsection. Using the perverse sheaf (2.29), we give the following definition:

**Definition 2.11.** For $v = (\beta, m) \in \Gamma_X$ and $\sigma \in U(X)$, suppose that the $d$-critical stack $M_\sigma(v)$ is CY at $\gamma \in \text{Chow}_X(\beta)$. Then we define the perverse sheaf $\phi_{M_\sigma(v)|_U}$ on $M_\sigma(v)|_U$ in the diagram (2.27) by

\begin{equation}
(2.30) \quad \phi_{M_\sigma(v)|_U} := p_H^!(R\pi_{M*}\phi_{M_\sigma(v)|_U}) \in \text{Perv}(M_\sigma(v)|_U).
\end{equation}

As we will see in the next subsection, the perverse sheaf (2.27) is an analogue of BPS sheaves introduced in [DM] for representations of quivers with super-potentials. The following lemma implies that (2.27) is a generalization of the perverse sheaf in Theorem 2.7 (i), and explains the reason of taking the first perverse cohomology (which is due to the shift convention of perverse sheaves by $\dim B\mathbb{C}^* = -1$):

**Lemma 2.12.** Suppose that $M_\sigma(v)|_U$ is a trivial $\mathbb{C}^*$-gerbe over $M_\sigma(v)|_U$, i.e. $M_\sigma(v)|_U = M_\sigma(v)|_U \times B\mathbb{C}^*$, e.g. the case of $\sigma = i\omega$ and $v = (\beta, 1)$. In this case, $M_\sigma(v)|_U$ itself is a $d$-critical scheme, so we have the associated perverse sheaf $\phi'_{M_\sigma(v)|_U}$ on $M_\sigma(v)|_U$ defined in Theorem 2.7 (i) using a CY orientation data. In this case, we have

\begin{equation}
(2.31) \quad \phi_{M_\sigma(v)|_U} = \phi'_{M_\sigma(v)|_U}.
\end{equation}
Proof. The perverse sheaf $\phi_{M_\sigma(v)|_U}$ regarded as a $C^*$-equivariant perverse sheaf on $M_\sigma(v)|_U$ coincides with $\phi'_{M_\sigma(v)|_U}[-1]$, because $\dim BC^* = -1$. Since $H^*(BC^*, \mathbb{Q}) = \mathbb{Q}[t]$ where $t$ is of degree two, we have

$$R^p M_\sigma M_\sigma(v)|_U \cong \phi'_{M_\sigma(v)|_U}[-1] \otimes \mathbb{Q}[t].$$

By taking the first perverse cohomologies, we obtain (2.31). q.e.d.

We then introduce our invariant $\Phi_\sigma(\gamma, m)$ as follows (see the diagram (2.27)):

**Definition 2.13.** In the situation of Definition 2.11, we define the following Laurent polynomial in $y$

$$(2.32) \quad \Phi_\sigma(\gamma, m) := \sum_{i \in \mathbb{Z}} \chi(p^H_i (R^p M_\sigma M_\sigma(v)|_U)|_\gamma) y^i \in \mathbb{Z}[y^{\pm 1}].$$

We have the following lemma:

**Lemma 2.14.** The Laurent polynomial $\Phi_\sigma(\gamma, m)$ is independent of a choice of a CY orientation data of $M_\sigma(v)|_U$.

Proof. The proof is similar to [MT18, Lemma 2.7]. By fixing an isomorphism (2.28), a CY orientation data of $M_\sigma(v)|_U$ is regarded as an isomorphism

$$(2.33) \quad \vartheta: \mathcal{O}_{M_\sigma(v)|_U} \otimes \mathcal{O}_{M_\sigma(v)|_U} \mathcal{O}_{M_\sigma(v)|_U} \cong \mathcal{O}_{M_\sigma(v)|_U}, \quad x \otimes y \mapsto xy.$$

Therefore an isomorphism (2.33) is regarded as an invertible element $\vartheta \in H^0(\mathcal{O}_{M_\sigma(v)|_U})$.

Let us take two invertible elements $\vartheta^{(1)}, \vartheta^{(2)} \in H^0(\mathcal{O}_{M_\sigma(v)|_U})$, and

$$\phi^{(i)}_{M_\sigma(v)|_U} \in \text{Perv}(M_\sigma(v)|_U), \quad \phi^{(i)}_{M_\sigma(v)|_U} \in \text{Perv}(M_\sigma(v)|_U)$$

the associated perverse sheaves (2.29), (2.30) w.r.t. the orientation data $\vartheta^{(i)}$ respectively. We consider the following commutative diagram

```
M_\sigma(v)|_U \xrightarrow{PM} M_\sigma(v)|_U
\downarrow \pi_M \quad \downarrow \pi_M
B \xrightarrow{\pi_B} U.
```

Here $\pi_M$ is the Stein factorization of $\pi_M$ and $\pi_M = \pi_M \circ p_M = \pi_B \circ \pi_M$. Since we have

$$p_M \circ \mathcal{O}_{M_\sigma(v)|_U} = \mathcal{O}_{M_\sigma(v)|_U}, \quad \pi_M \circ \mathcal{O}_{M_\sigma(v)|_U} = \mathcal{O}_B,$$
we have \( \pi_M^* \mathcal{O}_{M_\sigma(v)}(v) = \mathcal{O}_B \). Therefore the element

\[
\psi := \vartheta^{(1)} \circ (\vartheta^{(2)})^{-1} \in H^0(\mathcal{O}_{M_\sigma(v)}|_{\tilde{\mathcal{U}}})
\]

is written as \( \psi = \pi_M^* \overline{\psi} \) for some invertible element \( \overline{\psi} \in H^0(\mathcal{O}_B) \). Let \( \iota : \tilde{\mathcal{U}} \to B \)

be the \( \mu_2 \)-torsor given by local square roots of \( \overline{\psi} \), and \( \mathcal{L}_B \) a rank one local system on \( B \) given by \( \iota_* \mathcal{Q}_B = \mathcal{Q}_B \oplus \mathcal{L}_B \). Then by (2.25), we have

\[
\phi_M^{(2)}|_{\mathcal{U}} = \phi_M^{(1)} \otimes \pi_M^* \mathcal{L}_B, \ \phi_M^{(2)} = \phi_M^{(1)} \otimes \pi_M^* \mathcal{L}_B.
\]

Since \( \pi_B \) is a finite map, \( \pi_B^* \) preserves the perverse t-structures. It follows that

\[
\rho H^i(\mathcal{R}\pi_M^* \phi_M^{(2)}|_{\mathcal{U}}) = \rho H^i(\mathcal{R}\pi_B^*(\mathcal{R}\pi_M^* \phi_M^{(1)}|_{\mathcal{U}} \otimes \mathcal{L}_B)).
\]

Since \( \mathcal{L}_B \) is a rank one local system, we have

\[
\chi(\rho H^i(\mathcal{R}\pi_B^*(\mathcal{R}\pi_M^* \phi_M^{(1)}|_{\mathcal{U}} \otimes \mathcal{L}_B)))|_\gamma = \chi(\rho H^i(\mathcal{R}\pi_M^* \phi_M^{(1)}|_{\mathcal{U}} \otimes \mathcal{L}_B))|_\gamma.
\]

Therefore the lemma follows.

\[ \text{q.e.d.} \]

The following is the main conjecture we address in this paper.

**Conjecture 2.15.** For \( v = (\beta, m) \in \Gamma_X \) with \( \beta \neq 0 \) and \( \sigma \in U(X) \), suppose that \( M_\sigma(v) \) is CY at \( \gamma \in \text{Chow}_X(\beta) \). Then the Laurent polynomial \( \Phi_{\sigma}(\gamma, m) \) is independent of \( \sigma \) and \( m \).

By Lemma 2.12, the local GV invariants \( n_{g,\gamma} \) defined in [MT18] is characterized by the identity

\[
\Phi_{\sigma=\mathsf{i}\omega}(\gamma, 1) = \sum_{g \geq 0} n_{g,\gamma}(y^{1/2} + y^{-1/2})^{2g}.
\]

Suppose that Conjecture 2.15 holds. Then \( \Phi_{\sigma}(\gamma, m) = \Phi_{\sigma=\mathsf{i}\omega}(\gamma, 1) \), so if we define \( n_{g,\gamma,m}(\sigma) \in \mathbb{Z} \) by the identity

\[
\Phi_{\sigma}(\gamma, m) = \sum_{g \geq 0} n_{g,\gamma,m}(\sigma)(y^{1/2} + y^{-1/2})^{2g}
\]

then \( n_{g,\gamma,m}(\sigma) \) is independent of \( (\sigma, m) \).

**Remark 2.16.** In the case of \( \sigma = \mathsf{i}\omega \) and \( m = 1 \), the perverse sheaf \( \mathcal{O}_{M_\sigma(v)}|_{\mathcal{U}} \) in Lemma 2.12 is known to be self dual, so \( \Phi_{\sigma=\mathsf{i}\omega}(\gamma, 1) \) is invariant under \( y \mapsto 1/y \). However in other cases, it is not clear whether \( \Phi_{\sigma}(\gamma, m) \) is invariant under \( y \mapsto 1/y \). So a priori, we cannot define \( n_{g,\gamma,m}(\sigma) \) as in (2.35) without assuming Conjecture 2.15.

**Remark 2.17.** In Definition 2.13 it was enough to assume that the stack \( M_\sigma(v) \) is CY at \( \gamma \) to define the invariant \( \Phi_{\sigma}(\gamma, m) \). The CY condition of the bigger stack \( M_X(\beta) \) in Conjecture 2.10 will play a role in showing the independence of \( \sigma \) of the invariant (2.32) (see the proof of Theorem 5.7).
2.8. Relation to BPS sheaves. The definition of the perverse sheaf \(2.30\) was suggested to the author by Ben Davison, as it is an analogue of BPS sheaves introduced in [DM] in the case of representations of quivers with super-potentials. Here we explain its relationship. Let \(Q\) be a symmetric quiver (see Definition 4.1 below) with super-potential \(W\). Let \(\mathcal{M}_Q, \mathcal{M}_{(Q,\partial W)}\) be the moduli stack of finite dimensional \(Q\)-representations, \((Q,W)\)-representations respectively, and \(\mathcal{M}_Q, \mathcal{M}_{(Q,\partial W)}\) their coarse moduli spaces. We have the commutative diagram

\[
\begin{array}{c}
\mathcal{M}_{(Q,\partial W)} \xrightarrow{p_{(Q,\partial W)}} \mathcal{M}_Q \\
\mathcal{M}_{(Q,\partial W)} \xrightarrow{\text{tr } W} \mathcal{M}_Q \xrightarrow{\text{tr } W} C.
\end{array}
\]

Here the horizontal arrows are closed immersions, the vertical arrows are natural morphisms to the coarse moduli spaces. The function \(\text{tr } W\) is defined from the super-potential \(W\) as in Subsection 4.4 below, and \(\text{tr } W\) is the induced function by \(p_{Q}*\mathcal{O}_{\mathcal{M}_Q} = \mathcal{O}_{\mathcal{M}_Q}\). We have

\[
\mathcal{M}_{(Q,\partial W)} = \{d(\text{tr } W) = 0\}.
\]

Since \(\mathcal{M}_Q\) is a smooth stack, the stack \(\mathcal{M}_{(Q,\partial W)}\) has a \(d\)-critical structure whose virtual canonical line bundle is trivial by Proposition 4.8 below. Using a CY orientation data, we obtain the associated perverse sheaf on the stack \(\mathcal{M}_{(Q,\partial W)}\)

\[
\phi_{\mathcal{M}_{(Q,\partial W)}} \in \text{Perv}(\mathcal{M}_{(Q,\partial W)}).
\]

Then by [DM, Theorem 4.7], we have

\[
(2.36) \quad p\mathcal{H}^1(Rp_{(Q,\partial W)}^*\phi_{\mathcal{M}_{(Q,\partial W)}}) = \phi_{\mathcal{M}_{(Q,\partial W)}}(j^*_{\mathcal{M}_Q}) IC_{\mathcal{M}_Q^*}.
\]

Here \(j^*: \mathcal{M}_Q^* \subset \mathcal{M}_Q\) is the open immersion of the simple part. The RHS of (2.36) is supported on \(\mathcal{M}_{(Q,\partial W)}\), and defined to be the BPS sheaf in [DM]. The identity (2.36) explains that the perverse sheaf (2.30) is an analogue of the BPS sheaf introduced in [DM].

In Remark 5.5, we will return to this point of view and see that (under some matching of \(d\)-critical structures) the perverse sheaf (2.30) is analytic locally the BPS sheaf for the representations of a quiver with a (formal but convergent) super-potential.

3. Examples

In this section, we compute the invariants \(\Phi_\sigma(\gamma, m)\) in some cases.

3.1. The case of zero-cycles. In this subsection, we compute \(\Phi_\sigma(\gamma, m)\) when \(\gamma\) is a zero cycle, i.e. \(\gamma \in S^m(X)\). For a (not necessary projective) CY 3-fold \(X\), let \(\mathcal{M}_0(m)\) be the moduli stack of zero dimensional coherent sheaves on \(X\) with length \(m\). Since any zero dimensional sheaf has
a compact support, the stack $\mathcal{M}_0(m)$ has a $d$-critical structure\footnote{Although $\mathcal{M}_0(m)$ is non-compact, its derived enhancement admits a $(-1)$-shifted symplectic structure $\mathbb{P}\mathbb{R}$ and the associated $d$-critical structure by BBBBJ15 Theorem 1.2.} with the natural morphism to the coarse moduli space
\begin{equation}
 p: \mathcal{M}_0^{\text{red}}(m) \to S^n(X).
\end{equation}
If $X$ is furthermore projective, $\mathcal{M}_0(m) = \mathcal{M}_0(v)$ for $v = (0, m)$ and any $\sigma \in U(X)$.

We first show the CY condition of $\mathcal{M}_0(m)$ as in Definition 2.9. Indeed in this case, we have the following global CY property:

**Proposition 3.1.** For any CY 3-fold $X$, the virtual canonical line bundle $K^{\text{vir}}_{\mathcal{M}_0(m)}$ of the $d$-critical stack $\mathcal{M}_0(m)$ is trivial, i.e. $K^{\text{vir}}_{\mathcal{M}_0(m)} \cong \mathcal{O}_{\mathcal{M}_0^{\text{red}}(m)}$.

**Proof.** For the morphism (3.1), we have the natural morphism
\begin{equation}
 p^* p_* K^{\text{vir}}_{\mathcal{M}_0(m)} \to K^{\text{vir}}_{\mathcal{M}_0(m)}.
\end{equation}
We show that $p^* K^{\text{vir}}_{\mathcal{M}_0(m)}$ is a line bundle on $S^n(X)$ and the morphism (3.2) is an isomorphism of line bundles on $\mathcal{M}_0^{\text{red}}(m)$. It is enough to show these claims analytic locally\footnote{Since the virtual canonical line bundle is determined by a universal sheaf (see Theorem 2.8), it is enough to have analytic local isomorphism on $S^n(X)$, i.e. don’t have to compare these $d$-critical structures.} on $S^n(X)$, so we may assume that $X = \mathbb{C}^3$. In this case, the stack $\mathcal{M}_0(m)$ is the moduli stack of $m$-dimensional representations of a quiver with a super-potentail
\begin{equation}
 (Q(3), W(3))
\end{equation}
where the quiver $Q(3)$ has one vertex, three loops $A, B, C$, and the super-potentail $W(3)$ is given by $W(3) = A[B, C]$. Therefore by Proposition 4.8 below, we see that $K^{\text{vir}}_{\mathcal{M}_0(m)}$ is a trivial line bundle when $X = \mathbb{C}^3$. Since $S^n(X)$ is the coarse moduli space of $\mathcal{M}_0^{\text{red}}(m)$, we have $p_* \mathcal{O}_{\mathcal{M}_0^{\text{red}}(m)} \cong \mathcal{O}_{S^n(X)}$, therefore the isomorphism of (3.2) follows when $X = \mathbb{C}^3$. Hence (3.2) is an isomorphism for any CY 3-fold $X$.

Next we show that $K^{\text{vir}}_{\mathcal{M}_0(m)}$ is trivial for any CY 3-fold $X$. We consider the following morphisms
\begin{equation}
 [X^m / S_m] \xrightarrow{r} \mathcal{M}_0^{\text{red}}(m) \xrightarrow{p} S^n(X).
\end{equation}
Here $S_m$ acts on $X^m$ by permutation, $r$ sends $(x_1, \ldots, x_m)$ to $\oplus_{i=1}^m \mathcal{O}_{x_i}$ and $q$ is the natural morphism to the coarse moduli space. We will show
that \( r^*K_{\mathcal{M}_0(m)}^{\text{vir}} \) is a trivial line bundle on \([X^{\times m}/S_m]\). If this is true, then by pulling the isomorphism (3.2) back by \( r \) we have the isomorphism
\[
q^*p_*K_{\mathcal{M}_0(m)}^{\text{vir}} \cong \mathcal{O}_{[X^{\times m}/S_m]}.
\]
Since \( q_*\mathcal{O}_{[X^{\times m}/S_m]} \cong \mathcal{O}_{S^m(X)} \), by pushing forward the above isomorphism to \( S^m(X) \) we obtain \( p_*K_{\mathcal{M}_0(m)}^{\text{vir}} \cong \mathcal{O}_{S^m(X)} \). Therefore \( K_{\mathcal{M}_0(m)}^{\text{vir}} \) is trivial by the isomorphism (3.2).

Let \( p_i : X^{\times m} \to X \) be the projection to the \( i \)-th component, and set \( \Delta_i = (\text{id}_X \times p_i)^*\Delta \) for \( \text{id}_X \times p_i : X \times X^{\times m} \to X \times X \) and \( \Delta \) is the diagonal in \( X \times X \). The line bundle \( r^*K_{\mathcal{M}_0(m)}^{\text{vir}} \) is a \( S_m \)-equivariant line bundle on \( X^{\times m} \) given by
\[
r^*K_{\mathcal{M}_0(m)}^{\text{vir}} = \det R\mathcal{H}om_{pr} (\bigoplus_{i=1}^m \mathcal{O}_{\Delta_i}, \bigoplus_{i=1}^m \mathcal{O}_{\Delta_i})
\]
(3.4)
\[
= \bigotimes_{1 \leq i,j \leq m} L_{ij}.
\]
Here \( pr : X \times X^{\times m} \to X^{\times m} \) is the projection, and \( L_{ij} \) is the line bundle
\[
L_{ij} := \det R\mathcal{H}om_{pr} (\mathcal{O}_{\Delta_i}, \mathcal{O}_{\Delta_j})
\]
It is easy to see that the line bundle \( L_{ij} \) is a trivial line bundle on \( X^{\times m} \) (either by direct calculation using Hochschild resolution of the diagonal, or using a dimension filtration of K-theory as in \([\text{MT18} \text{ Proposition 3.13}]\)). Let \( e_{ij} \) be a nowhere vanishing global section of \( L_{ij} \). Then \( \sigma \in S_m \) acts on the trivial line bundle (3.4) by sending the basis \( \prod_{i,j} e_{ij} \) to \( \prod_{i,j} e_{\sigma(i)\sigma(j)} = \prod_{i,j} e_{ij} \). Thus the \( S_m \)-equivariant structure of the trivial line bundle (3.4) is also trivial, and the proposition holds. q.e.d.

By Proposition 3.1, we have a global orientation data of \( \mathcal{M}_0(m) \) satisfying
\[
(K_{\mathcal{M}_0(m)}^{\text{vir}})^{1/2} \cong \mathcal{O}_{\mathcal{M}_0^{\text{red}}(m)}.
\]

Using the above orientation data, by Theorem 2.7 we have the global perverse sheaf
\[
\phi_{\mathcal{M}_0(m)} \in \text{Perv}(\mathcal{M}_0(m)).
\]
Then the perverse sheaf \( \phi_{S^m(X)} \) on \( S^m(X) \) is defined by
\[
\phi_{S^m(X)} := [p^*] R\mathcal{H}^1 (Rq_* \phi_{\mathcal{M}_0(m)}) \in \text{Perv}(S^m(X)).
\]

**Lemma 3.2.** The perverse sheaf \( \phi_{S^m(X)} \) is isomorphic to \( \Delta_X \ast L[3] \) for a rank one local system \( L \) on \( X \), where \( \Delta_X : X \to S^m(X) \) is the diagonal embedding.

**Proof.** The question is local on \( X \), so we may assume that \( X = \mathbb{C}^3 \). In this case, the stack \( \mathcal{M}_0(m) \) is the moduli stack of \( m \)-dimensional
representations of \((Q^{(3)}, W^{(3)})\) defined in \((3.3)\), and in this case the BPS sheaf is computed in \([\text{Dav}]\) Section 5.1:

\[
pH^1(\mathbb{R}p_\ast \phi_{M_0(m)}) = \phi_{\text{tr}} W^{(3)}(\text{IC}_{M^{(3)}_Q(m)}) = \Delta_{C^3} \text{IC}_{C^3}.
\]

Here \(M^{(3)}_Q(m)\) is the coarse moduli space of \(Q^{(3)}\)-representations of dimension \(m\), the first identity is due to \((2.36)\) and the second one is computed in \([\text{Dav}]\) Section 5.1. The above local result shows the lemma.

For the case of zero cycles, the identity map \(S^m(X) \rightarrow S^m(X)\) is the HC map from the coarse moduli space of \(M^{\text{red}}_0(m)\) to the Chow variety of zero cycles. Therefore Lemma 3.2 immediately shows the following:

**Theorem 3.3.** For a smooth projective CY 3-fold and \(\gamma \in S^m(X)\), the invariant \(\Phi_\sigma(\gamma, m)\) is zero unless \(\gamma = \Delta_X(x)\) for \(x \in X\) and \(\Delta_X : X \hookrightarrow S^m(X)\) is the diagonal embedding. When \(\gamma = \Delta_X(x)\), we have

\[
\Phi_\sigma(\gamma, m) = \chi(\Delta_X \ast L[3]|_{\Delta_X(x)}) = -1.
\]

**3.2. The case of elliptic fibrations.** Here we give an example of the invariant \((2.32)\) in the case of a CY 3-fold with an elliptic fibration, and see that Conjecture 2.15 holds in this case.

Let \(S = \mathbb{P}^2\) and take general elements \(u \in H^0(S, \mathcal{O}_S(-2K_S)), v \in H^0(S, \mathcal{O}_S(-3K_S))\). Then as in \([\text{Tod12}]\) Section 6.4, we have a simply connected CY 3-fold \(X\) with a flat elliptic fibration

\[
(3.7) \quad \pi_X : X \rightarrow S
\]
defined by the equation \(zy^2 = uxz^2 + vz^3\) in the projective bundle

\[
\mathbb{P}_S(\mathcal{O}_S(-2K_S) \oplus \mathcal{O}_S(-3K_S) \oplus \mathcal{O}_S) \rightarrow S.
\]

Here \([x : y : z]\) is the homogeneous coordinate of the above projective bundle. Note that \(\pi_X\) admits a section

\[
\iota : S \rightarrow X
\]
whose image correspond to the fiber point \([0 : 1 : 0]\). By the construction, every scheme theoretic fiber \(X_s = \pi_X^{-1}(s)\) for \(s \in S\) is an integral curve, which is either a smooth elliptic curve, or nodal rational curve with one node, or a cuspidal rational curve. Let \([F] \in N_1(X)\) be a fiber class of \(\pi_X\), and set \(\beta = d[F]\) for \(d \in \mathbb{Z}_{\geq 1}\), and \(v = (\beta, m) \in \Gamma_X\). Let \(k \in \mathbb{Z}_{\geq 1}\) be the greatest common divisor of \((d, m)\), and set \(d' = d/k, m' = m/k\). For \(B + \iota \omega \in A(X)_{\mathbb{C}}\), let

\[
\pi_Y : Y \rightarrow S
\]
be the \(\pi_X\)-relative moduli space of \((B, \omega)\)-stable sheaves \(E\) on the fibers of \((3.7)\) such that \(v(E) = (d'[F], m')\).

**Lemma 3.4.** We have an isomorphism \(X \xrightarrow{\cong} Y\) over \(S\).
Proof. By the result of [BM02], the moduli space $Y$ is a smooth projective CY 3-fold. Let $J \to S$ be the $\pi_X$-relative moduli space of rank one torsion free sheaves $E$ on the fibers of $\pi_X$ satisfying $v(E) = ([F], m')$. We have birational maps over $S$

\[ \phi_1: Y \dashrightarrow J, \quad \phi_2: X \dashrightarrow J. \]

Here the birational map $\phi_1$ is given by sending a general point $y \in Y$ to $\det(E_y)$, where $E_y$ is the sheaf on $X_{\pi_Y(y)}$ corresponding to $y$, and the determinant is taken in the fiber $X_{\pi_Y(y)}$. The birational map $\phi_2$ is given by sending a general point $x \in X$ to $O_{X_{\pi_X(x)}}(x + (m - 1) \cdot \iota \circ \pi_X(x))$. It follows that we have a birational map

\[ (3.8) \quad \phi_1^{-1} \circ \phi_2: X \dashrightarrow Y \]

over $S$. Since both of $X, Y$ are smooth projective CY 3-folds, the birational map (3.8) has to be decomposed into flops over $S$. However as $\rho(X) = 2$ and the Mori cone $NE(X)$ of $X$ is spanned by the fiber $[F]$ and $[\iota(l)]$ for a line $l \subset S$, there is no extremal ray on $NE(X)$ corresponding to a flop. So the birational map (3.8) extends to the isomorphism $X \cong Y$ over $S$. q.e.d.

Note that if $\beta = d[F]$ we have the isomorphism

\[ \pi_X^*: S^d(S) \cong \text{Chow}_X(d[F]) \]

by sending a zero cycle $Z \subset S$ to $\pi_X^{-1}(Z)$. We identity $\text{Chow}_X(d[F])$ with $S^d(S)$ by the above isomorphism. Now we show the main result in this subsection:

**Theorem 3.5.** In the above situation, for $\sigma \in U(X), \gamma \in \text{Chow}_X(d[F]) = S^d(S)$ and $m \in \mathbb{Z}$, we have $\Phi_\sigma(\gamma, m) \neq 0$ only if $\gamma$ is in the image of the diagonal map $\Delta_S: S \hookrightarrow S^d(S)$. If $\gamma = \Delta_S(s)$ for $s \in S$, we have

\begin{equation}
(3.9) \quad \Phi_\sigma(\gamma, m) = y^{-1} + (2 - e(X_s)) + y.
\end{equation}

Here $e(-)$ is the topological euler number. In particular Conjecture 2.15 holds in this case.

**Proof.** By Lemma 3.4 and the result of [BM02], there is an auto-equivalence

\[ \Psi: D^b(\text{Coh}(X)) \xrightarrow{\sim} D^b(\text{Coh}(X)) \]

sending $\mathcal{O}_x$ for a closed point $x \in X$ to a stable sheaf $E_x$ with $v(E_x) = (d'[F], m')$. Let $\mathcal{M}_0(k)$ be the moduli stack of zero dimensional sheaves on $X$ with length $k$. Then as in [Tod12, Section 6.4], the equivalence $\Psi$ induces the isomorphism of stacks

\[ \Psi_*: \mathcal{M}_0(k) \xrightarrow{\cong} \mathcal{M}_\sigma(v) \]
for any \( \sigma \in U(X) \). Since the above isomorphism is induced by the derived equivalence \( \Psi \), it preserves the \( d \)-critical structures and the virtual canonical line bundles (see \[MT18\] Remark 8.6, where the announced work \[BD19\] Theorem 1.2] is now available in \[BD\] Theorem 5.5]. Moreover we have the commutative diagram

\[
\begin{align*}
\mathcal{M}_0 \mathcal{r}(k) & \xrightarrow{\Psi_*} \mathcal{M}_\sigma \mathcal{r}(v) \\
X & \xrightarrow{\Delta_X} S^k(X) & \xrightarrow{\Psi_*} \mathcal{M}_\sigma \mathcal{r}(v) \\
\pi_X & \xrightarrow{h} S^d(S) & \xrightarrow{\pi_X} \text{Chow}(d[F]).
\end{align*}
\]

Here the maps \( p, p_M \) are natural maps to the coarse moduli spaces, the middle horizontal arrow is the induced isomorphism on the coarse moduli spaces, and the map \( h \) is given by

\[(x_1, \ldots, x_k) \mapsto (d'\pi_X(x_1), \ldots, d'\pi_X(x_k)).\]

By Proposition 3.1 and the diagram (3.10), we have a global orientation data of \( \mathcal{M}_\sigma(v) \) satisfying

\[(K_{\mathcal{M}_\sigma(v)})^{1/2} \cong \mathcal{O}_{\mathcal{M}_\sigma(v)}.\]

Using the above orientation data, we have the global perverse sheaf \( \phi_{\mathcal{M}_\sigma(v)} \in \text{Perv}(\mathcal{M}_\sigma(v)) \).

Then we have the perverse sheaf on \( S^k(X) \) as in (3.5):

\[\phi_{S^k(X)} = \mathcal{O}^1 \text{H}^1(Rp_* \Psi_*^{-1}) \mathcal{O}_{\mathcal{M}_\sigma(v)} \in \text{Perv}(S^k(X)).\]

By Lemma 3.2 and the simply connectedness of \( X \), we have \( \phi_{S^k(X)} = \Delta_X \text{IC}_X \). By the commutative diagram (3.10) and the definition of \( \Phi_\sigma(\gamma, m) \), we have

\[\Phi_\sigma(\gamma, m) = \sum_{i \in \mathbb{Z}} \chi(p^i \mathcal{H}^i(Rh_* \Delta_X \text{IC}_X)|\gamma)y^i.\]

Therefore \( \Phi_\sigma(\gamma, m) = 0 \) if \( \gamma \) is not in the image of \( \Delta_S \). If \( \gamma = \Delta_S(s) \) for \( s \in S \), then by the left bottom diagram of (3.10), we have

\[\Phi_\sigma(\gamma, m) = \sum_{i \in \mathbb{Z}} \chi(p^i \mathcal{H}^i(R\pi_{X*} \text{IC}_X)|s)y^i.\]

Since by our assumption each \( X_s \) is either a smooth elliptic curve, or a rational nodal curve with one node, or a cuspidal rational curve, the perverse decomposition of \( R\pi_{X*} \text{IC}_X \) becomes

\[R\pi_{X*} \text{IC}_X = \text{IC}_S[1] \oplus V \oplus \text{IC}_S[-1].\]
Here $V$ is a constructible sheaf on $S$ such that for $s \in S$, we have $V|_s = Q^{2-e(X_s)}$. Therefore the identity (3.9) holds. q.e.d.

4. Wall-crossing formula for quivers with convergent super-potentials

The wall-crossing formula of cohomological DT invariants for representations of quivers with super-potentials was studied in [DM]. Our approach toward Conjecture 2.15 is to reduce the problem to the similar problem for representations of quivers with formal but convergent super-potentials, using the result of [Tod]. In this section, we review the work of [DM] and prove some necessary results in the case of quivers with convergent super-potentials.

4.1. Representations of quivers. Recall that a quiver $Q$ consists of data

\[ Q = (V(Q), E(Q), s, t) \]

where $V(Q), E(Q)$ are finite sets and $s, t$ are maps

\[ s, t : E(Q) \to V(Q). \]

The set $V(Q)$ is the set of vertices and $E(Q)$ is the set of edges. For $e \in E(Q)$, $s(e)$ is the source of $e$ and $t(e)$ is the target of $e$. For $i, j \in V(Q)$, we use the following notation

\[ E_{i,j} := \{ e \in E(Q) : s(e) = i, t(e) = j \} \]

(4.1)

i.e. $E_{i,j}$ is the set of edges from $i$ to $j$.

A $Q$-representation consists of data

\[ V = \{ (V_i, u_e) : i \in V(Q), e \in E(Q), u_e : V_{s(e)} \to V_{t(e)} \} \]

(4.2)

where $V_i$ is a finite dimensional $\mathbb{C}$-vector space and $u_e$ is a linear map. For a $Q$-representation (4.2), the vector

\[ \vec{m} = (m_i)_{i \in V(Q)}, \quad m_i = \dim V_i \]

(4.3)

is called the dimension vector.

Given a dimension vector (4.3), let $V_i$ be a $\mathbb{C}$-vector space with dimension $m_i$. Let us set

\[ G := \prod_{i \in Q(V)} \mathrm{GL}(V_i), \quad \mathrm{Rep}_Q(\vec{m}) := \prod_{e \in E(V)} \mathrm{Hom}(V_{s(e)}, V_{t(e)}). \]

The algebraic group $G$ acts on $\mathrm{Rep}_Q(\vec{m})$ by

\[ g \cdot u = \{ g_{t(e)}^{-1} \circ u_e \circ g_{s(e)} \}_{e \in E(Q)} \]

(4.4)

for $g = (g_i)_{i \in V(Q)} \in G$ and $u = (u_e)_{e \in E(Q)}$. A $Q$-representation with dimension vector $\vec{m}$ is determined by a point in $\mathrm{Rep}_Q(\vec{m})$ up to $G$-action. The moduli stack of $Q$-representations with dimension vector $\vec{m}$
is given by the quotient stack
\[ \mathcal{M}_Q(\bar{m}) := \left[ \text{Rep}_Q(\bar{m}) / G \right]. \]

It has the coarse moduli space, given by
\[ M_Q(\bar{m}) := \text{Rep}_Q(\bar{m}) \# G. \]

(4.5)

Here in general, if a reductive algebraic group \( G \) acts on an affine scheme \( Y = \text{Spec} R \), its GIT quotient is given by \( Y / / G := \text{Spec}(R^G) \). A closed point of \( M_Q(\bar{m}) \) corresponds to a semi-simple \( Q \)-representation, i.e. direct sum of simple \( Q \)-representations. We have the natural commutative diagram
\[ \begin{array}{ccc}
\text{Rep}_Q(\bar{m}) & \xrightarrow{\pi_Q} & \mathcal{M}_Q(\bar{m}) \\
| & & | \\
\downarrow{\rho_Q} & & \downarrow{\rho_Q} \\
M_Q(\bar{m}). & & M_Q(\bar{m}).
\end{array} \]

(4.6)

In what follows, we consider only symmetric quivers defined below.

**Definition 4.1.** A quiver \( Q \) is called symmetric if \( \sharp E_{i,j} = \sharp E_{j,i} \) for any \( i, j \in V(Q) \). Here \( E_{i,j} \) is defined as in (4.1).

Let \( \mathcal{M}_Q \) and \( M_Q \) be defined by
\[ \mathcal{M}_Q := \bigsqcup_{\bar{m} > 0} \mathcal{M}_Q(\bar{m}), \quad M_Q := \bigsqcup_{\bar{m} > 0} M_Q(\bar{m}). \]

Here \( \bar{m} > 0 \) means \( m_i \geq 0 \) for all \( i \) and \( \bar{m} \neq 0 \). For each \( n \geq 1 \), there is a natural map
\[ \oplus: \mathcal{M}_Q \times \cdots \times \mathcal{M}_Q \to M_Q \]
by taking the direct sum of the corresponding semi-simple \( Q \)-representations.

Then we have the map
\[ \boxtimes_{\oplus}: \text{Perv}(M_Q)^\times n \to \text{Perv}(M_Q) \]
by sending \((\mathcal{F}_1, \ldots, \mathcal{F}_n)\) to
\[ \boxtimes_{\oplus}(\mathcal{F}_1, \ldots, \mathcal{F}_n) := \oplus_{\oplus}(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n). \]

Here we note that, since the map (4.7) is a finite map, we have \( \oplus_{\oplus} = R \oplus_{\oplus} \) and it takes perverse sheaves to perverse sheaves. Then the map
\[ \operatorname{Sym}^*: \text{Perv}(M_Q) \to \text{Perv}(M_Q) \]
(4.9)

is defined by sending \( \mathcal{F} \) to
\[ \operatorname{Sym}^*(\mathcal{F}) := \bigoplus_{n \geq 1} (\boxtimes_{\oplus}(\mathcal{F}, \ldots, \mathcal{F}))^S_n. \]
It is easy to see that $\text{Sym}^\bullet(F)$ is a finite sum on each component $M_Q(\vec{m})$, so it is well-defined (see [DM] Section 3.2). The following result was proved in [DM].

**Theorem 4.2.** ([DM Theorem 4.7]) For a symmetric quiver $Q$, we have an isomorphism of perverse sheaves on $M_Q$

\[
\bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(Rp^*_{pQ*} \mathcal{I}C_{M_Q})[-i] \cong \text{Sym}^\bullet(j^*_s \mathcal{I}C_{M_Q} \otimes H^*(\mathbb{P}^\infty)_{\text{vir}}).
\]

Here $j^*: M_Q^s \subset M_Q$ is the open immersion of the simple part, and $H^*(\mathbb{P}^\infty)_{\text{vir}}$ is defined by

\[
H^*(\mathbb{P}^\infty)_{\text{vir}} := \bigoplus_{k \geq 0} \mathbb{Q}[-2k - 1].
\]

### 4.2. Semistable quiver representations.

For a quiver $Q$, let $K(Q)$ be the Grothendieck group of the abelian category of finite dimensional $Q$-representations. For each $i \in V(Q)$ let $S_i$ be the one dimensional $Q$-representation corresponding to the vertex $i$, whose dimension vector is denoted by $\text{dim}(S_i)$. By taking the dimension vectors of $Q$-representations, we have the group homomorphism

\[
\text{dim}: K(Q) \to \Gamma_Q := \bigoplus_{i \in V(Q)} \mathbb{Z} \cdot \text{dim}(S_i).
\]

Let $\mathcal{H} \subset \mathbb{C}$ be the upper half plane, and take

\[
\xi = (\xi_i)_{i \in V(Q)}, \xi_i \in \mathcal{H}.
\]

Let $Z_\xi$ be the group homomorphism

\[
Z_\xi: K(Q) \xrightarrow{\text{dim}} \Gamma_Q \to \mathbb{C}, \ [S_i] \mapsto \xi_i.
\]

Then $Z_\xi$ defines a Bridgeland stability condition on the category of finite dimensional $Q$-representations w.r.t. the group homomorphism (4.11). The associated (semi)stable representations are described in terms of the slope function $\mu_\xi$ defined by

\[
\mu_\xi(-) := -\frac{\text{Re} Z_\xi(-)}{\text{Im} Z_\xi(-)}.
\]

**Definition 4.3.** A $Q$-representation $V$ is called $\mu_\xi$-(semi)stable if for any non-zero sub $Q$-representation $V' \subsetneq V$, we have the inequality

\[
\mu_\xi(V') < (\leq) \mu_\xi(V).
\]

For a choice of $\xi$ as in (4.12), let

\[
\text{Rep}_Q^\xi(\vec{m}) \subset \text{Rep}_Q(\vec{m})
\]

be the open locus consisting of $\mu_\xi$-semistable objects. We take the associated GIT quotients:

\[
\mathcal{M}_Q^\xi(\vec{m}) := [\text{Rep}_Q^\xi(\vec{m})/G], \ M_Q^\xi(\vec{m}) := \text{Rep}_Q^\xi(\vec{m})/G.
\]
We have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{M}_\xi^Q(\vec{m}) & \xrightarrow{j_\xi^Q} & \mathcal{M}_Q(\vec{m}) \\
\downarrow p_\xi^Q & & \downarrow p_Q \\
M_\xi^Q(\vec{m}) & \xrightarrow{q_\xi^Q} & M_Q(\vec{m}).
\end{array}
\]

Here \(j_\xi^Q\) is an open immersion, \(p_Q, p_\xi^Q\) are natural morphisms to the coarse moduli spaces, and \(q_\xi^Q\) is the induced morphism by the universality of the coarse moduli spaces.

Recall that a morphism of algebraic varieties \(f: S \to T\) is semismall if \(f\) is proper, surjective, and there is a stratification \(\{S_\theta\}_\theta\) of \(T\) such that for any \(x \in T\) we have

\[(4.14) \quad \dim f^{-1}(x) \leq \frac{1}{2} \text{codim } S_\theta.\]

We call \(f\) a quasi semismall if there is a stratification \(\{S_\theta\}_\theta\) of \(T\) such that the condition \((4.14)\) holds (i.e. \(f\) is not assumed to be proper nor surjective). We will use the following lemma:

**Lemma 4.4.** For a symmetric quiver \(Q\), the morphism \(q_\xi^Q: M_\xi^Q(\vec{m}) \to M_Q(\vec{m})\) is a semismall map if \(M_\xi^Q(\vec{m}) \neq \emptyset\).

**Proof.** It is well-known that \(q_\xi^Q\) is projective (in particular proper) and surjective (see [HdlPn02, Theorem 4.1]). It is enough to show that \(q_\xi^Q\) is quasi semismall. For a dimension vector \(\vec{w}\) of \(Q\), let \(M_\vec{w}^\xi(\vec{m})\) be the moduli space of \(Q\)-representations \(V\) as in \((4.2)\) with dimension vector \(\vec{m}\) together with linear maps

\[(4.15) \quad W_i \to V_i, \quad i \in V(Q)
\]

where \(W_i\) is a \(w_i\)-dimensional vector space, such that the image of \((4.15)\) generates \(V\) as \(\mathbb{C}[Q]\)-module, where \(\mathbb{C}[Q]\) is the path algebra of \(Q\) (see Subsection 4.4 below). The moduli space \(M_\vec{w}^\xi(\vec{m})\) is a non-singular variety, and we have the forgetting morphism

\[\pi_\vec{w}^\xi: M_\vec{w}^\xi(\vec{m}) \to \mathcal{M}_Q(\vec{m}) \xrightarrow{p_Q} M_Q(\vec{m}).\]

Here the first arrow is a smooth morphism of relative dimension \(\vec{w} \cdot \vec{m}\), which is surjective when \(\vec{w} \gg 0\). Moreover by [MR19, Theorem 1.4], the morphism \(\pi_\vec{w}^\xi\) satisfies the following; there is a stratification \(\{S_\theta\}_\theta\) of \(M_Q(\vec{m})\) such that for any \(x \in S_\theta\) we have

\[\dim(\pi_\vec{w}^\xi)^{-1}(x) \leq \frac{1}{2} \text{codim } S_\theta + \vec{w} \cdot \vec{m} - 1.\]
Therefore for \( x \in S_\theta \), we have
\[
\dim(p_Q)^{-1}(x) \leq \frac{1}{2} \operatorname{codim} S_\theta - 1.
\]
Let \( M_Q^{\xi,s}((\vec{m})) \), \( M_Q^{\xi,s}((\bar{m})) \) be the stable locus of \( M_Q^{\xi}((\bar{m})) \), \( M_Q^{\xi}((\vec{m})) \) respectively. Then \( M_Q^{\xi,s}((\bar{m})) \) is a \( C^* \)-gerbe over \( M_Q^{\xi,s}((\vec{m})) \), so the diagram (4.13) and (4.16) imply that
\[
q^\xi|_{M_Q^{\xi,s}((\bar{m}))}: M_Q^{\xi,s}((\bar{m})) \to M_Q((\bar{m}))
\]
is a quasi semismall map. For dimension vectors \( \vec{m}_1, \ldots, \vec{m}_s \) whose sum equals to \( \vec{m} \) and \( \mu_\xi(\vec{m}_i) = \mu_\xi(\bar{m}) \), we have the commutative diagram
\[
\begin{array}{ccc}
M_Q^{\xi,s}((\vec{m}_1)) \times \cdots \times M_Q^{\xi,s}((\vec{m}_s)) & \oplus & M_Q^{\xi}((\vec{m})) \\
| & & | \\
(q_{1}^{\xi} \cdots q_{s}^{\xi}) & & q_{Q}^{\xi} \\
M_Q((\vec{m}_1)) \times \cdots \times M_Q((\vec{m}_s)) & \oplus & M_Q((\bar{m}))
\end{array}
\]
Since the images of the top horizontal arrows for various \( (\vec{m}_1, \ldots, \vec{m}_s) \) give a stratification of \( M_Q^{\xi}((\vec{m})) \), and the left vertical arrow is a quasi semismall map, we conclude that \( q_{Q}^{\xi} \) is quasi semismall. q.e.d.

4.3. Wall-crossing formula for IC sheaves. We keep the notation in the previous subsection. For each \( \mu \in (-\infty, \infty) \), let \( M_Q^{\xi}((\mu)) \), \( M_Q^{\xi}((\mu)) \) be defined by
\[
M_Q^{\xi}((\mu)) := \coprod_{\mu_\xi(\bar{m})=\mu} M_Q^{\xi}(\bar{m}), \ M_Q^{\xi}((\mu)) := \coprod_{\mu_\xi(\vec{m})=\mu} M_Q^{\xi}((\vec{m})).
\]
Similarly to \( M_Q \), a closed point of \( M_Q^{\xi}((\mu)) \) corresponds to a \( \mu_\xi \)-polystable \( Q \)-representation, i.e. direct sum of \( \mu_\xi \)-stable \( Q \)-representations with slope \( \mu \). Therefore we have the natural maps
\[
\boxtimes: \operatorname{Perv}(M_Q^{\xi}((\mu)))^\times \to \operatorname{Perv}(M_Q^{\xi}((\mu))),
\]
\[
\operatorname{Sym}^\ast: \operatorname{Perv}(M_Q^{\xi}((\mu))) \to \operatorname{Perv}(M_Q^{\xi}((\mu)))
\]
declared similar to (4.8), (4.9). We refer to the following results proved in [DM].

**Theorem 4.5.** ([DM, Theorem 4.11]) For a symmetric quiver \( Q \), we have the following:

(i) We have an isomorphism of perverse sheaves on \( M_Q^{\xi}((\mu)) \)
\[
\bigoplus_{i \in \mathbb{Z}} \rho_i^* \mathcal{H}(R\mathcal{I}C_{M_Q^{\xi}((\mu))})[-i] \cong \operatorname{Sym}^\ast(\mathcal{I}C_{M_Q^{\xi}((\mu))} \otimes \mathcal{H}(\mathbb{P}^\infty)_{\text{vir}}).
\]
Here $j_*^{\xi,s}: M^{\xi,s}_Q(\mu) \subset M^s_Q(\mu)$ is the open immersion of the stable part.

(ii) We have an isomorphism of perverse sheaves on $M^s$:

$$\bigoplus_{i \in \mathbb{Z}} \mathcal{P}^i(R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\mu)})[-i]$$

$$\cong \bigotimes_{\infty, \ldots, -\infty} R_q^{\xi} \text{Sym}^\bullet (j_*^{\xi,s} \mathcal{I}^s_{M^{\xi,s}_Q(\mu)}) \otimes H^*(\mathbb{P}^\infty_{\text{vir}}).$$

We will use the following lemma:

**Lemma 4.6.** In the diagram (4.13), for any $i \in \mathbb{Z}$ we have

$$R_q^{\xi} R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\mu)} \in \text{Perv}(M^s_Q(\bar{m})).$$

**Proof.** By Lemma 4.4 and a general fact that the derived push-forward of semismall maps take intersection complexes to perverse sheaves (for example see [MAM09]), we have

$$R_q^{\xi} j_*^{\xi,s} \mathcal{I}^s_{M^{\xi,s}_Q(\mu)} \in \text{Perv}(M^s_Q(\bar{m})).$$

Then the lemma follows from Theorem 4.5 (i), the condition (4.19) and the commutative diagram

$$M^{\xi}_Q(\mu) \times \cdots \times M^{\xi}_Q(\mu) \xrightarrow{\oplus} M^{\xi}_Q(\mu)$$

$$(q^\xi_s, \ldots, q^\xi_s) \downarrow \quad \downarrow q^\xi_s$$

$$M_Q \times \cdots \times M_Q \xrightarrow{\oplus} M_Q.$$

q.e.d.

In the diagram (4.13), we have the canonical morphism

$$\mathcal{I}^s_{M^s_Q(\bar{m})} \to R_{j_{Q*} j_{Q*}^\xi} \mathcal{I}^s_{M^s_Q(\bar{m})} = R_{j_{Q*}^\xi} \mathcal{I}^s_{M^s_Q(\bar{m})}$$

by adjunction. By pushing forward it to $M^s_Q(\bar{m})$, we have the morphism

$$R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})} \to R_{p_{Q*}} R_{j_{Q*}^\xi} \mathcal{I}^s_{M^s_Q(\bar{m})} = R_q^{\xi} R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})}.$$

By Lemma 4.6, taking the first perverse cohomology of the above morphism gives the morphism of perverse sheaves on $M^s_Q(\bar{m})$:

$$p^\xi H^1(R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})}) \to R_{p_{Q*}} p^\xi H^1(R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})}).$$

**Lemma 4.7.** The morphism (4.20) is an isomorphism.

**Proof.** By taking the first perverse cohomologies of the isomorphisms in Theorem 4.2 and Theorem 4.5 (i), we have

$$p^\xi H^1(R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})}) \cong j_*^{\xi,s} \mathcal{I}^s_{M^s_Q(\bar{m})},$$

$$p^\xi H^1(R_{p_{Q*}} \mathcal{I}^s_{M^s_Q(\bar{m})}) \cong j_*^{\xi,s} \mathcal{I}^s_{M^s_Q(\bar{m})}.$$
Moreover by Theorem 4.5 (ii) and (4.19), we have an isomorphism
\[ pH^1(Rp^*_Q IC_{M_Q(\vec{m})}) \cong Rq^*_Q j^* s IC_{M^s_Q(\vec{m})}. \]
Therefore if \( M^s_Q(\vec{m}) = \emptyset \), then both sides of (4.20) are zero. Otherwise
the morphism (4.20) is a non-zero endomorphism of the simple perverse sheaf \( j^s IC_{M^s_Q(\vec{m})} \), so it is an isomorphism. \( \text{q.e.d.} \)

4.4. Quivers with convergent super-potentials. Recall that a path of a quiver \( Q \) is a composition of edges in \( Q \)
\[ e_1 e_2 \ldots e_n, \quad e_i \in E(Q), \quad t(e_i) = s(e_{i+1}). \]
The number \( n \) above is called the length of the path. The path algebra of a quiver \( Q \) is a \( \mathbb{C} \)-vector space spanned by paths in \( Q \):
\[ \mathbb{C}[Q] := \bigoplus_{n \geq 0} \bigoplus_{\{e_1, \ldots, e_n\} \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n. \]
Here a path of length zero is a trivial path at each vertex of \( Q \), and the product on \( \mathbb{C}[Q] \) is defined by the composition of paths. By taking the completion of \( \mathbb{C}[Q] \) with respect to the length of the path, we obtain the formal path algebra:
\[ \mathbb{C}[[Q]] := \prod_{n \geq 0} \bigoplus_{\{e_1, \ldots, e_n\} \in E(Q), t(e_i) = s(e_{i+1})} \mathbb{C} \cdot e_1 e_2 \ldots e_n. \]
Note that an element \( f \in \mathbb{C}[[Q]] \) is written as
\[ (4.21) \quad f = \sum_{n \geq 0} \sum_{\{e_1, \ldots, e_n\} \in E(Q)} a_{\psi, e_1, \ldots, e_n} \cdot e_1 e_2 \ldots e_n. \]
Here \( a_{\psi, e_1, \ldots, e_n} \in \mathbb{C} \), \( e_i = (e_1, \ldots, e_n) \) and \( E_{\psi(i), \psi(i+1)} \) is defined as in (4.1). The above element \( f \) lies in \( \mathbb{C}[Q] \) iff \( a_{\psi, e_1, \ldots, e_n} = 0 \) for \( n \gg 0 \).

The subalgebra
\[ \mathbb{C}\{Q\} \subset \mathbb{C}[[Q]] \]
is defined to be elements (4.21) such that \( |a_{\psi, e_1, \ldots, e_n}| < C^n \) for some constant \( C > 0 \) which is independent of \( n \). Note that \( \mathbb{C}\{Q\} \) contains \( \mathbb{C}[Q] \) as a subalgebra. A convergent super-potential of a quiver \( Q \) is an element
\[ W \in \mathbb{C}\{Q\}/[\mathbb{C}\{Q\}, \mathbb{C}\{Q\}]. \]
A convergent super-potential \( W \) of \( Q \) is represented by a formal sum
\[ W = \sum_{n \geq 1} \sum_{\{e_1, \ldots, e_n\} \in E(Q)} a_{\psi, e_1, \ldots, e_n} \cdot e_1 e_2 \ldots e_n \]
with \( |a_{\psi, e_1, \ldots, e_n}| < C^n \) for a constant \( C > 0 \).
For a dimension vector \( \vec{m} \) of \( Q \), let \( \text{tr} W \) be the formal function of \( u = (u_e)_{e \in E(Q)} \in \text{Rep}_Q(\vec{m}) \) defined by
\[
\text{tr} W(u) := \sum_{n \geq 1} \sum_{\{1, \ldots, n+1\} \to V(Q)} \sum_{\psi \in E_{\psi(i), \psi(i+1)}} a_{\psi, e} \cdot \text{tr}(u_n \circ u_{n-1} \circ \cdots \circ u_1).
\]

The above formal function on \( \text{Rep}_Q(\vec{m}) \) is \( G \)-invariant. By the argument of [Tod, Lemma 2.10], there is an analytic open neighborhood
\[
0 \in V \subset M_Q(\vec{m})
\]
such that the formal function \( \text{tr} W \) absolutely converges on \( \pi_Q^{-1}(V) \) to give a \( G \)-invariant analytic function
\[
\text{tr} W: \pi_Q^{-1}(V) \to \mathbb{C}.
\]
Here \( \pi_Q \) is given in the diagram (4.3). Then we set
\[
\text{Rep}_{(Q, \partial W)}(\vec{m})|_V := \{ d(\text{tr} W) = 0 \},
\]
\[
M_{(Q, \partial W)}(\vec{m})|_V := \{ [d(\text{tr} W) = 0]/G \},
\]
\[
M_{(Q, \partial W)}(\vec{m})|_V := \{ d(\text{tr} W) = 0 \}/G.
\]

Here \(-//G\) above is an analytic Hilbert quotient (see [HMP98, Gre15, Tod]). We have the natural commutative diagram
\[
\begin{align*}
\text{Rep}_{(Q, \partial W)}(\vec{m})|_V & \xrightarrow{\pi_Q^{-1}(V)} \text{Rep}_Q(\vec{m}) \\
\pi_{(Q, \partial W)} & \xrightarrow{\pi_Q^{-1}(V)} \pi_Q \\
M_{(Q, \partial W)}(\vec{m})|_V & \xrightarrow{\pi_Q^{-1}(V)} M_Q(\vec{m})
\end{align*}
\]
Here the right horizontal arrows are open immersions and the left horizontal arrows are closed immersions.

Let \( \xi \) be as in (4.12) which defines the \( \mu_\xi \)-stability on the category of \( Q \)-representations, and
\[
\text{Rep}_{(Q, \partial W)}^\xi(\vec{m})|_V \subset \text{Rep}_{(Q, \partial W)}(\vec{m})|_V
\]
be the open locus consisting of \( \mu_\xi \)-semistable \( Q \)-representations. Similarly to (4.21), we define
\[
M_{(Q, \partial W)}^\xi(\vec{m})|_V := [\text{Rep}_{(Q, \partial W)}^\xi(\vec{m})|_V]/G,
\]
\[
M_{(Q, \partial W)}^\xi(\vec{m})|_V := \text{Rep}_{(Q, \partial W)}^\xi(\vec{m})|_V//G.
\]
Then we have the commutative diagram

\[ (4.27) \]

\[
\begin{array}{ccc}
\mathcal{M}_Q^\xi((\bar{\bar{m}}))|_V & \xrightarrow{j^\xi_{Q,\partial W}} & \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \\
\mathcal{M}_Q^\xi((\bar{\bar{m}}))|_V & \xrightarrow{p_{Q,\partial W}} & \mathcal{M}_Q((\bar{\bar{m}}))|_V \\
\mathcal{M}_Q^\xi((\bar{\bar{m}})) & \xrightarrow{\tilde{q}^\xi_{Q,\partial W}} & \mathcal{M}_Q((\bar{\bar{m}})) \\
\end{array}
\]

Here \( j^\xi_{Q,\partial W} \) is an open immersion, the morphisms \( p^\xi_{Q,\partial W}, p_{Q,\partial W} \) are the natural morphisms to the coarse moduli spaces, \( \tilde{q}^\xi_{Q,\partial W} \) is the induced morphism by the universality of analytic Hilbert quotients and the slanting arrows are locally closed embeddings.

### 4.5. Vanishing cycles for quivers with convergent super-potentials.

We have the following proposition on the analytic stack \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \) given in (4.24):

**Proposition 4.8.** The analytic stack \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \) has an analytic \( d \)-critical structure given by

\[
d(\text{tr } W) + (d(\text{tr } W))^2 \in H^0(S^0_{\mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V}).
\]

Here \( \text{tr } W \) is the function \( (4.25) \). Moreover if \( Q \) is symmetric, there is an orientation data of \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \) which is trivial as a line bundle.

**Proof.** The first statement follows from the definition of \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \) in (4.24) and Example 2.6. Below we prove the second statement. The virtual canonical line bundle of \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \), regarded as a \( G \)-equivariant line bundle on \( \text{Rep}_{Q,\partial W}((\bar{\bar{m}}))|_V \), is given by

\[
K_{\mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V}^{\text{vir}} = K_{\text{Rep}_{Q}((\bar{\bar{m}}))|_V}^{\text{red}}|_{\text{Rep}_{Q,\partial W}((\bar{\bar{m}}))|_V} \in \text{Pic}_G(\text{Rep}_{Q,\partial W}((\bar{\bar{m}}))|_V).
\]

Here \( \text{Pic}_G(\cdot) \) is the group of \( G \)-equivariant line bundles of \( (\cdot) \). Therefore we have an orientation data of \( \mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V \) given by

\[
(4.28) \quad K_{\mathcal{M}_{Q,\partial W}((\bar{\bar{m}}))|_V}^{\text{vir},1/2} = K_{\text{Rep}_{Q}((\bar{\bar{m}}))|_V}^{\text{red}}|_{\text{Rep}_{Q,\partial W}((\bar{\bar{m}}))|_V}.
\]

So the second statement follows if for a symmetric quiver \( Q \) we have

\[
(4.29) \quad K_{\text{Rep}_{Q}((\bar{\bar{m}}))|_V} \cong O_{\text{Rep}_{Q}((\bar{\bar{m}}))|_V}.
\]

as \( G \)-equivariant line bundles on \( \text{Rep}_{Q}((\bar{\bar{m}})) \). Since \( \text{Rep}_{Q}((\bar{\bar{m}})) \) is an affine space, we have an isomorphism \( (4.29) \) as line bundles. The \( G \)-equivariant
structure on $K_{\text{Rep}_Q}(\vec{m})$ is given by the following character $G \to \mathbb{C}^*$

$$(g_i)_{i \in V(Q)} \mapsto \prod_{e \in E(Q)} (\det g_{s(e)})^{m_t(e)} \cdot (\det g_{t(e)})^{-m_s(e)}.$$ 

The symmetric condition of $Q$, $\sharp E_{i,j} = \sharp E_{j,i}$ implies that the above character is trivial $G \to 1 \in \mathbb{C}^*$. Therefore we have an isomorphism (4.29) as $G$-equivariant line bundles. q.e.d.

Using the orientation data (4.28) of $\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V$, which is trivial as a line bundle, we have the perverse sheaf of vanishing cycles

(4.30) \[ \phi_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V} \in \text{Perv}(\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V). \]

The above perverse sheaf, regarded as a $G$-equivariant perverse sheaf on $\text{Rep}_{(Q,\partial W)}(\vec{m})$, is nothing but the perverse sheaf of vanishing cycles of the $G$-invariant function (4.23). The above oriented $d$-critical stack structure on $\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V$ induces the one on its open substack $\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V$. The associated perverse sheaf of vanishing cycles

$$\phi_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V} \in \text{Perv}(\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V)$$

is a pull-back of (4.30) by $j_{(Q,\partial W)}^\xi$ in the diagram (4.27).

**Lemma 4.9.** For any $i \in \mathbb{Z}$, in the diagram (4.27) we have

$$Rq_{(Q,\partial W)*}^\xi p^H (Rp_{(Q,\partial W)*}^\xi \phi_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V}) \in \text{Perv}(\mathcal{M}_{(Q,\partial W)}(\vec{m})|_V).$$

**Proof.** Since we have $p_{Q*}\mathcal{O}_{\mathcal{M}_Q}(\vec{m}) = \mathcal{O}_{\mathcal{M}_Q}($, which also holds after analytification (see [Tod] Lemma 2.7), the function (4.23) descends to the analytic function $\text{tr } W$ on $V$

$$\text{tr } W: \pi^{-1}_Q(V) \to V \triangleright W \subseteq \mathbb{C}.$$ 

We restrict the perverse sheaf (4.18) to $V$ and apply the vanishing cycle functor of $\text{tr } W$. Since the vanishing cycle functor preserves the perverse t-structures, for any $i \in \mathbb{Z}$ we have

$$\phi_{\text{tr } W}(Rq_{(Q,\partial W)*}^\xi p^H (Rp_{(Q,\partial W)*}^\xi IC_{\mathcal{M}_{(Q,\vec{m})}})|_V) \in \text{Perv}(V).$$
On the other hand, by pulling back the diagram (4.13) to \( V \), we obtain the diagram

\[
\begin{array}{c}
(r_Q^\xi)^{-1}(V) \\
\downarrow p_Q^\xi \\
(q_Q^\xi)^{-1}(V)
\end{array}
\begin{array}{c}
\rightarrow p_Q^{-1}(V) \\
\Rightarrow r_Q \Rightarrow q_Q \\
\Rightarrow \text{tr } W \\
\Rightarrow \text{tr } W \\
\Rightarrow C.
\end{array}
\]

We have isomorphisms

\[
\phi_{\text{tr } W}((Rq_Q^\xi)^{pH^i}(Rp_Q^\xi IC_{M_Q^\xi(m)})|_V) \cong Rq_Q^\xi(\phi_{\text{tr } W}^{pH^i}(Rp_Q^\xi IC_{(r_Q^\xi)^{-1}(V)}))
\]

\[
\cong Rq_Q^{\xi} pH^i(\phi_{\text{tr } W} Rp_Q^{\xi} IC_{(r_Q^\xi)^{-1}(V)})
\]

\[
\cong Rq_Q^\xi pH^i(Rp_Q^\xi \phi_{\text{tr } W}(IC_{(r_Q^\xi)^{-1}(V)})).
\]

Here the first isomorphism follows from the compatibility of vanishing cycle functors with proper push-forward (see [Dim04, Proposition 4.2.11]). The second isomorphism follows from that the vanishing cycle functor preserves the perverse t-structure. The third isomorphism is proved in [DM, Proposition 4.3] when \( W \) is a usual super-potential \( W \in \mathbb{C}[Q]/[\mathbb{C}[Q], \mathbb{C}[Q]] \), and the same argument applies for the convergent super-potential case. Therefore the lemma holds.

In the diagram (4.27), we have the canonical morphism

\[
\phi_{M_{(Q, \partial W)}(\bar{m})|_V} \rightarrow Rj_{(Q, \partial W)^*}^{\xi} \phi_{M_{(Q, \partial W)}(\bar{m})|_V}.
\]

By pushing forward it to \( M_{(Q, \partial W)}(\bar{m})|_V \), we obtain the morphism

\[
Rp_{(Q, \partial W)^*} \phi_{M_{(Q, \partial W)}(\bar{m})|_V} \rightarrow Rq_{(Q, \partial W)^*}^{\xi} Rp_{(Q, \partial W)^*} \phi_{M_{(Q, \partial W)}(\bar{m})|_V}.
\]

By Lemma 4.9 taking the first perverse cohomologies above gives the morphism

\[
(4.32) \quad pH^i(Rp_{(Q, \partial W)^*} \phi_{M_{(Q, \partial W)}(\bar{m})|_V})
\]

\[
\rightarrow Rq_{(Q, \partial W)^*}^{\xi} pH^1(Rp_{(Q, \partial W)^*} \phi_{M_{(Q, \partial W)}(\bar{m})|_V}).
\]

**Lemma 4.10.** The morphism (4.32) is an isomorphism.
Proof. We use the notation in the proof of Lemma 4.9. By the functoriality of vanishing cycle functors, we have the following natural commutative diagram

\[
\begin{array}{ccc}
\phi_{tr} W (R^p Q_* IC_{M_Q(m)} | V) & \rightarrow & R^p Q_* \phi_{tr} W (IC_{p^{-1}}(V)) \\
\downarrow & & \downarrow \\
\phi_{tr} W (R^p Q_* j_{Q*}^\xi IC_{M_{Q}(m)} | V) & \rightarrow & R^p Q_* j_{Q*}^\xi \phi_{tr} W (IC_{(r_Q - 1)}(V)).
\end{array}
\]

We take the first perverse cohomologies of the above diagram. Noting that the vanishing cycle functor commutes with perverse cohomology functors, the natural isomorphism

\[ R^p Q_* j_{Q*}^\xi \approx R^q Q_* R^p \xi Q_* \]

by the diagram (4.31), and using Lemma 4.9, the above diagram gives the following commutative diagram

\[
\begin{array}{ccc}
\phi_{tr} W (pH^1 (R^p Q_* IC_{M_Q(m)} | V)) & \rightarrow & pH^1 (R^p Q_* \phi_{tr} W (IC_{p^{-1}}(V))) \\
\downarrow & & \downarrow \\
\phi_{tr} W (pH^1 (R^p Q_* j_{Q*}^\xi IC_{M_{Q}(m)} | V)) & \rightarrow & pH^1 (R^p Q_* j_{Q*}^\xi \phi_{tr} W (IC_{(r_Q - 1)}(V))).
\end{array}
\]

The left vertical arrow is an isomorphism by Lemma 4.7. Moreover the horizontal arrows can be shown to be isomorphisms by the same argument of Lemma 4.9. Therefore the right vertical arrow is also an isomorphism, which implies that (4.32) is an isomorphism. q.e.d.

5. Wall-crossing formula for GV type invariants

In this section, using the results in the previous sections and the results in [Tod], we prove Theorem 1.3 and Theorem 1.4.

5.1. Ext-quiver. Let $X$ be a smooth projective CY 3-fold. For $\sigma = \sigma_{B, \omega} \in U(X)$ and $v = (\beta, m) \in \Gamma_X$, we consider the stack $M_\sigma(v)$ and its coarse moduli space $M_\sigma(v)$ given in Subsection 2.2. We have the natural morphism

\[ p_M : M_\sigma(v) \rightarrow M_\sigma(v). \]

For a closed point $p \in M_\sigma(v)$, it is represented by a $(B, \omega)$-polystable sheaf $E$ of the form

\[
E = \bigoplus_{i=1}^{k} V_i \otimes E_i
\]

where $E_i \in \text{Coh}_{\leq 1}(X)$ is $(B, \omega)$-stable with $\mu_{B, \omega}(E_i) = \mu_{B, \omega}(E)$, and $E_i \not\approx E_j$ for $i \neq j$.

For each $1 \leq i, j \leq k$, we fix a finite subset

\[ E_{i,j} \subset \text{Ext}^1(E_i, E_j) \]

\[ (5.2) \]
giving a basis of $\text{Ext}^1(E_i, E_j)^\vee$. The *Ext-quiver* $Q_{E\bullet}$ of $E\bullet$ is defined as follows. The set of vertices and edges are given by

$$V(Q_{E\bullet}) = \{1, 2, \ldots, k\}, \quad E(Q_{E\bullet}) = \coprod_{1 \leq i,j \leq k} E_{i,j}.$$  

The maps $s, t : E(Q_{E\bullet}) \to V(Q_{E\bullet})$ are given by

$$s|_{E_{i,j}} = i, \quad t|_{E_{i,j}} = j.$$  

**Lemma 5.1.** The quiver $Q_{E\bullet}$ is symmetric.

*Proof.* For $1 \leq i, j \leq k$ with $i \neq j$, we have $\text{Hom}(E_i, E_j) = 0$. Therefore we have

$$\dim \text{Ext}^1(E_j, E_i) - \dim \text{Ext}^1(E_i, E_j) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \text{Ext}^k(E_i, E_j) = 0.$$  

Here the first equality follows from the Serre duality and the second equality follows from the Riemann-Roch theorem. Therefore the lemma holds. q.e.d.

**5.2. Moduli spaces of semistable sheaves via Ext-quivers.** Let us take another stability condition

$$\sigma^+ = \sigma_{B^+} = (Z_{B^+, \omega^+}, \text{Coh}_{\leq 1}(X)) \in U(X).$$  

We take $\sigma^+$ sufficiently close to $\sigma$. Then by wall-chamber structure on the space of stability conditions, any $\sigma^+$-semistable object $E$ with $\text{ch}(E) = v$ is $\sigma$-semistable. Then we have the commutative diagram

$$\begin{array}{ccc}
M_{\sigma^+}(v) & \xrightarrow{r_M} & M_\sigma(v) \\
p_M^+ \downarrow & & \downarrow p_M \\
M_{\sigma^+}(v) & \xrightarrow{q_M} & M_\sigma(v).
\end{array}$$  

Here the top arrow is an open immersion, the vertical arrows are natural morphisms to the coarse moduli spaces and the bottom arrow is induced by the universality of the coarse moduli spaces.

Locally on $M_\sigma(v)$, we can compare the above diagram with a similar diagram for representations of the Ext-quiver with a convergent superpotential. For a closed point $p \in M_\sigma(v)$ corresponding to a polystable sheaf (5.1), let $Q_{E\bullet}$ be the associated Ext-quiver. We take data (4.12) for the Ext-quiver $Q_{E\bullet}$ by

$$\xi = (\xi_i)_{1 \leq i \leq k}, \quad \xi_i = Z_{B^+, \omega^+}(E_i), \quad 1 \leq i \leq k.$$  

Then we have the associated $\mu_\xi$-stability condition on the category of $Q_{E\bullet}$-representations. Let $\tilde{m}$ be the dimension vector of $Q_{E\bullet}$ given by

$$m_i = \dim V_i, \quad 1 \leq i \leq k.$$  

We have the following result:
Theorem 5.2. ([Tod, Theorem 7.7]) For a closed point \( p \in M_\sigma(v) \) corresponding to a polystable sheaf \( (\mathcal{F}, \mathcal{E}) \), let \( Q = Q_{E_\bullet} \) be the associated Ext-quiver. Then there exist a convergent super-potential \( W \) of \( Q \) and analytic open neighborhoods

\[ p \in T \subset M_\sigma(v), \quad 0 \in V \subset M_Q(\vec{m}) \]

where \( \vec{m} \) is the dimension vector \( (5.5) \), such that the commutative diagram \( (5.4) \) pulled back to \( T \)

\[
\begin{array}{ccc}
r_M^{-1}(T) & \longrightarrow & p_M^{-1}(T) \\
p_M^* & & p_M \\
q_M^{-1}(T) & \downarrow & T
\end{array}
\]

is isomorphic to the commutative diagram (see the diagram \( (4.27) \)):

\[
\begin{array}{ccc}
M_{(Q,\partial \mathcal{W})}(\vec{m})|_V & \overset{\kappa}{\longrightarrow} & M_{(Q,\partial \mathcal{W})}(\vec{m})|_V \\
p_{(Q,\partial \mathcal{W})}^* & \downarrow & p_{(Q,\partial \mathcal{W})} \\
M_{(Q,\partial \mathcal{W})}(\vec{m})|_V & \overset{q_{(Q,\partial \mathcal{W})}}{\longrightarrow} & M_{(Q,\partial \mathcal{W})}(\vec{m})|_V.
\end{array}
\]

By Theorem 2.8, the stack \( M_\sigma(v) \) has a canonical \( d \)-critical structure \( s \in H^0(S^0_{M_\sigma(v)}) \), which induces the one on its open substack \( p_M^{-1}(T) \). The stack \( M_{(Q,\partial \mathcal{W})}(\vec{m})|_V \) also has a \( d \)-critical structure by Proposition 4.8. As we will mention in Remark 5.4, these \( d \)-critical structures should be the same. However we give a weaker statement for the comparison of \( d \)-critical structures, which is enough for our purpose.

Proposition 5.3. Under the isomorphism

\[
(5.6) \quad \nu: M_{(Q,\partial \mathcal{W})}(\vec{m})|_V \overset{\cong}{\longrightarrow} p_M^{-1}(T)
\]

in Theorem 5.2, by shrinking \( V \) if necessary, there is a \( G \)-invariant analytic function \( \text{tr}'(W): \pi^{-1}_Q(V) \rightarrow \mathbb{C} \) and the identity in \( H^0(S^0_{M_{(Q,\partial \mathcal{W})}(\vec{m})|_V}) = H^0(S^0_{\text{Rep}(Q,\partial \mathcal{W})}(\vec{m})|_V)^G \)

\[
(5.7) \quad \iota^*(s|_{p_M^{-1}(T)}) = \text{tr}' W + (d(\text{tr}' W))^2.
\]

Here we have used the notation in \( (4.25) \) for \( Q = Q_{E_\bullet} \).

Proof. Note that \( \iota^*(s|_{p_M^{-1}(T)}) \) is a \( G \)-invariant global section of \( \mathcal{O}_{\pi^{-1}_Q(V)}/I^2 \) where \( I = (d \text{tr}' W) \subset \mathcal{O}_{\pi^{-1}_Q(V)} \). Since \( \pi^{-1}_Q(V) \) is Stein and \( G \) is reductive, the map \( H^0(\mathcal{O}_{\pi^{-1}_Q(V)}/I^2)^G \rightarrow H^0(\mathcal{O}_{\pi^{-1}_Q(V)}/I^2)^G \) is surjective. Therefore there is a \( G \)-invariant analytic function \( f: \pi^{-1}_Q(V) \rightarrow \mathbb{C} \) such that
\(\iota^*(s|_{p^M}(T)) = f + I^2\). On the other hand, since we can take a minimum Darboux chart of a derived enhancement of \(\mathcal{M}_\sigma(v)\) at \(E\) (see [BBBJ15 Theorem 2.10]), we can take a \(d\)-critical chart for \(\iota^*(s|_{p^M}(T))\) at \(0 \in \pi^{-1}Q(V)\) of the form \((\{d \tau W = 0\} \cap U, U, g, i)\) for an analytic open neighborhood \(0 \in U \subset \pi^{-1}Q(V)\) and an analytic function \(g: U \to \mathbb{C}\) which has vanishing order at least three. In particular on \(U\), we have \(I = (dg)\) and \(\iota^*(s|_{p^M}(T)) = g + I^2\). It follows that \(f = g + I^2\) holds on \(U\). Then noting that \(g\) has vanishing order at least three at 0, by shrinking \(U\) if necessary, the above identity implies that \((df) = (dg) = I\) on \(U\). Since both of \(f, \tau W\) are \(G\)-invariant, it follows that \((df) = I\) holds on \(\cup_{i \in G}^{-1}(U)\). We take an analytic open neighborhood \(0 \in V' \subset V\) such that \(\pi^{-1}Q(V') \subset \cup_{i \in G}^{-1}(U)\) holds, which is possible by [Tod Lemma 5.1]. By setting \(\tau' W = f|_{\pi^{-1}Q(V)}\) and replacing \(V\) with \(V'\), we obtain the proposition. \(\quad \text{q.e.d.}\)

**Remark 5.4.** It should be true that we can take \(\tau' W = \tau W\). Indeed \((-1)\)-shifted symplectic structure for \(\mathcal{M}_\sigma(v)\) is canonically determined by a left Calabi-Yau structure for a derived enhancement of \(D^b \text{Coh}(X)\), which is a choice of a non-zero element of \(H^0(X, K_X) = \mathbb{C}\) (see [BD19]). It gives a \(d\)-critical structure in the left hand side of (5.7). On the other hand, the left Calabi-Yau structure also determines cyclic \(L_\infty\)-structure on a minimal model of \(R\text{Hom}(E, E)[1]\), which should give a standard Darboux form for the formal completion of \(\mathcal{M}_\sigma(v)\) at \([E]\) giving a \(d\)-critical structure determined by \(\tau W\) (see [CS15 Appendix A]). By the canonicity of constructions of \((-1)\)-shifted symplectic structures, they should give the same \(d\)-critical structures. Some details may be pursued elsewhere.

**Remark 5.5.** By the proof of Lemma [4.10] under the isomorphism

\[\varphi: M_{(Q, \partial W)}(\overline{\mathfrak{m}})|_V \xrightarrow{\cong} T\]

in Theorem 5.3, we have

\[\varphi^!(\phi_{\sigma}(v)|_U)|_T = \phi_{\tau V}(j^*_V IC(V^s)).\]

Here \(j: V^s \subset V\) is the simple part, \(\phi_{\sigma(v)|_U}\) is given in Definition 2.17 and we have used the notation of the diagram (4.31). If \(\tau' W = \tau W\), then the right hand side is nothing but the BPS sheaf defined in [DM]. Therefore in this case, the perverse sheaf \(\phi_{\sigma(v)|_U}\) is interpreted as a gluing of BPS sheaves.

**Remark 5.6.** We can replace \(\tau W\) with \(\tau' W\) in Lemma 4.9, Lemma 4.10 so that we have the same results for vanishing cycle sheaves associated
with $\text{tr}' W$. Namely let
\[
\phi'_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|V} \in \text{Perv}(\mathcal{M}_{(Q,\partial W)}(\vec{m})|V), \\
\phi'_{\mathcal{M}_{\xi(\partial W)}(\vec{m})|V} \in \text{Perv}(\mathcal{M}_{\xi(\partial W)}(\vec{m})|V)
\]
be the perverse sheaves determined by the $d$-critical structure (5.7) and its restriction to $\mu_\xi$-semistable locus. Then the same conclusions in Lemma 4.9, Lemma 4.10 hold after replacing $\phi_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|V}$ with $\phi'_{\mathcal{M}_{(Q,\partial W)}(\vec{m})|V}$, $\phi'_{\mathcal{M}_{\xi(\partial W)}(\vec{m})|V}$ respectively. Indeed the only required property for $\text{tr}' W$ in the above lemmas is a $G$-invariance so that it descends to an analytic function on $V$. As $\text{tr}' W$ is also $G$-invariant, the same arguments apply.

### 5.3. Independence of stability conditions

We now prove the wall-crossing formula of the invariant $\Phi_\sigma(\gamma, m)$ defined in Definition 2.11. Namely we show that it is independent of $\sigma$.

**Theorem 5.7.** In the situation of Definition 2.11, the Laurent polynomial $\Phi_\sigma(\gamma, m)$ is independent of $\sigma \in U(X)$.

**Proof.** By the wall-chamber structure on $U(X)$, it is enough to show the following: for a fixed $\sigma \in U(X)$, if we take $\sigma^+ \in U(X)$ to be sufficiently close to $\sigma$, then we have the identity
\[
\Phi_\sigma(\gamma, m) = \Phi_{\sigma^+}(\gamma, m).
\]
Let $\gamma \in U \subset \text{Chow}_X(\beta)$ be an open subset as in Definition 2.9, and consider stacks in the diagram (2.27). By pulling the diagram (5.4) back to $U$, we obtain the commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}_\sigma(v)|U & \xrightarrow{j_M} & \mathcal{M}_\sigma(v)|U \\
p_M^* & & p_M^* \\
M_{\sigma^+}(v)|U & \xrightarrow{r_M} & M_\sigma(v)|U.
\end{array}
\]
We take a CY orientation data of $\mathcal{M}_X(\beta)|U$, which induces CY orientation data of its open substacks $\mathcal{M}_\sigma(v)|U, M_{\sigma^+}(v)|U$. Let $\phi_{\mathcal{M}_\sigma(v)|U}, \phi_{M_{\sigma^+}(v)|U}$ be the associated perverse sheaves (2.29) respectively. Then by our choice of orientation data, we have
\[
\phi_{\mathcal{M}_{\sigma^+}(v)|U} = j_M^* \phi_{\mathcal{M}_\sigma(v)|U}.
\]
Therefore we have the canonical morphism
\[
\phi_{\mathcal{M}_\sigma(v)|U} \to Rj_{M*}j_M^* \phi_{\mathcal{M}_\sigma(v)|U} \to Rj_{M*}\phi_{M_{\sigma^+}(v)|U}.
\]
By pushing forward it to $M_\sigma(v)|U$, we obtain the morphism
\[
Rp_{M*}\phi_{\mathcal{M}_\sigma(v)|U} \to R\text{tr}_M, Rj_{M*}\phi_{M_{\sigma^+}(v)|U} = Rq_{M*}R\text{tr}_M^* \phi_{M_{\sigma^+}(v)|U}.
\]
We claim that, for any \( i \in \mathbb{Z} \) we have
\[
R_\lambda M_\ast \mathcal{H}^i(\mathcal{R} \phi M_\ast (v)|U) \in \text{Perv}(M_\sigma(v)|U).
\]
Suppose that (5.10) holds. Then by taking the first perverse cohomologies of (5.9), we obtain the morphism
\[
\mathcal{H}^i(\mathcal{R} \phi M_\ast (v)|U) \rightarrow R_\lambda M_\ast \mathcal{H}^i(\mathcal{R} \phi M_\ast (v)|U).
\]
By Definition 2.11, the above morphism is
\[
\phi M_\sigma(v)|U \rightarrow R_\lambda M_\ast \phi M_\sigma(v)|U.
\]
We also claim that the morphism (5.11) is an isomorphism. In order to show the condition (5.10) and (5.11) is an isomorphism, it is enough to show these properties analytic locally on \( M_\sigma(v)|U \). By Theorem 5.2, Proposition 5.3 and Remark 5.6, we can essentially reduce these claims to claims for representations of symmetric quivers with convergent super-potentials. By Lemma 4.9 and Lemma 4.10, and also noting Remark 5.6, we obtain the desired claims, i.e. (5.10) holds and (5.11) is an isomorphism.

We have the commutative diagram
\[
\begin{array}{ccc}
M_\sigma(v)|U & \xrightarrow{q_\lambda} & M_\sigma(v)|U \\
\pi_M^+ \downarrow & & \downarrow \pi_M \\
U & &
\end{array}
\]
where \( \pi_M \) and \( \pi_M^+ \) are HC maps. By the isomorphism (5.11), we have the isomorphism
\[
R \pi_\lambda M_\ast \phi M_\sigma(v)|U \cong R \pi_M^+ M_\ast \phi M_\sigma(v)|U.
\]
Therefore the identity (5.8) holds by the definition of \( \Phi_\sigma(\gamma, m) \). q.e.d.

By Theorem 5.7, we can define \( \Phi_X(\gamma, m) \) by
\[
\Phi_X(\gamma, m) := \Phi_\sigma(\gamma, m).
\]
for \( \sigma \in U_X \).

5.4. Independence of Euler characteristics. As an application of Theorem 5.7 we show that \( \Phi_X(\gamma, m) \) is also independent of \( m \) when \( \gamma \) is primitive. Here a one cycle \( \gamma \) on \( X \) is called primitive if it is written as
\[
\gamma = \sum_{i=1}^{k} a_i [C_i]
\]
for irreducible curves \( C_i \) with \( C_i \neq C_j \) for \( i \neq j \), and \( a_i \in \mathbb{Z}_{>0} \) such that the greatest common divisor of \( (a_1, \ldots, a_k) \) is one.

**Theorem 5.8.** For a primitive one cycle \( \gamma \in \text{Chow}_X(\beta) \), suppose that \( M_X(\beta) \) is CY at \( \gamma \). Then \( \Phi_X(\gamma, m) \) is independent of \( m \).
Proof. We write $\gamma$ as (5.12) and take divisors $D_i$ for $1 \leq i \leq k$ defined in an analytic neighborhood of the support of $\gamma$ satisfying $D_i \cdot C_j = \delta_{ij}$. Since $\gamma$ is primitive, there is a divisor $D = \sum_{i=1}^{k} d_i D_i$, $d_i \in \mathbb{Z}$ such that $D \cdot \gamma = 1$. Then for a sufficiently small analytic open neighborhood $\gamma \in U \subset \text{Chow}_X(\beta)$, the map $F \mapsto F(D)$ gives an isomorphism $\otimes \mathcal{O}(D) : \mathcal{M}_{\sigma}(v)|_U \cong \mathcal{M}_{\sigma'}(v')|_U$ (5.13) which commutes with the HC maps to $U$. Here $\sigma = \sigma_{B,\omega}$, $\sigma' = \sigma_{B+D,\omega}$, $v = (\beta, m)$ and $v' = (\beta, m + 1)$. The isomorphism (5.13) is an isomorphism as $d$-critical stacks (see [MT18, Remark 8.6]). We take CY orientation data of $\mathcal{M}_{\sigma}(v)|_U$ which induces the one on $\mathcal{M}_{\sigma'}(v)|_{U'}$ by the isomorphism (5.13). Then the isomorphism (5.13) induces an isomorphism $\Phi_{\sigma}(\gamma, m) = \Phi_{\sigma'}(\gamma, m + 1)$. By Theorem 5.7 we have $\Phi_X(\gamma, m) = \Phi_X(\gamma, m + 1)$ and the theorem follows. q.e.d.

6. Flop invariance of GV invariants

As an application of Theorem 5.7 we show the flop invariance of generalized GV invariants. A similar result was already obtained for irreducible one cycles in [MT18], and the result here generalizes this result to arbitrary one cycles.

6.1. 3-fold flops. Let $X$, $X^\dagger$ be smooth projective CY 3-folds. A diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^\dagger \\
\downarrow f & & \downarrow f^\dagger \\
Y & &
\end{array}
\]

is called a flop if $f, f^\dagger$ are projective birational morphisms which are isomorphic in codimension one with relative Picard number one, $Y$ has only Gorenstein singularities, and $\phi$ is a non-isomorphic birational map. The exceptional loci of $f, f^\dagger$ are chains of smooth rational curves. In [Bri02], Bridgeland showed that there is an equivalence of derived categories (also see [Tod08, Proposition 5.2])

\[
\Phi : D^b(\text{Coh}_{\leq 1}(X)) \cong D^b(\text{Coh}_{\leq 1}(X^\dagger)).
\]
The above equivalence is given by the Fourier-Mukai transform whose kernel is $O_{X \times Y, X^1}$, and it fits into the commutative diagram

$$D^b(\text{Coh}_{\leq 1}(X)) \xrightarrow{\Phi} D^b(\text{Coh}_{\leq 1}(X^1))$$

$$\xrightarrow{\text{ch}} \Gamma_X \xrightarrow{\phi_c} \Gamma_{X^1}.$$

Here $\Phi$ takes $(\beta, n)$ to $(\phi_\ast \beta, n)$, where $\phi_\ast \beta$ is characterized as $\phi_\ast \beta \cdot D = \beta \cdot \phi^{-1}_\ast D$ for any divisor $D$ on $X^1$ and $\phi^{-1}_\ast D$ is the strict transform.

**Lemma 6.1.** For any one cycle $\gamma$ on $X$, there is a one cycle $\phi_\ast \gamma$ on $X^\dagger$ such that for any object $F \in D^b(\text{Coh}_{\leq 1}(X))$ with $l(F) = \gamma$, we have $l(\Phi(F)) = \phi_\ast \gamma$.

*Proof.* For $F, F' \in D^b(\text{Coh}_{\leq 1}(X))$, suppose that we have $l(F) = l(F')$. Then $[F] - [F']$ in $K(\text{Coh}_{\leq 1}(X))$ is represented by a linear combination of skyscraper sheaves. Therefore it is enough to show that $l(\Phi(O_x)) = 0$ for any $x \in X$. If $x \not\in \text{Ex}(f)$, then $\Phi(O_x) = O_{\phi(x)}$, so $l(\Phi(O_x)) = 0$ holds. If $x \in \text{Ex}(f)$, then $l(\Phi(O_x))$ is a one cycle supported on $\text{Ex}(f)$ whose homology class equals to zero in a neighborhood of $\text{Ex}(f)$. Therefore $l(\Phi(O_x)) = 0$ and the lemma holds. q.e.d.

**6.2. Perverse coherent sheaves.** The equivalence (6.2) restricts to the equivalence of perverse coherent sheaves, defined below.

**Definition 6.2.** ([Bri02 Section 3]) The subcategories $^p\text{Per}_{\leq 1}(X/Y) \subset D^b(\text{Coh}_{\leq 1}(X))$ are defined by

$$(6.3) \quad \left\{ E \in D^b(\text{Coh}_{\leq 1}(X)) : \text{Hom}^{\leq -p}(E, C_X) = \text{Hom}^{\leq p}(C_X, E) = 0 \right\}.$$  

Here $C_X := \{ F \in \text{Coh}(X) : Rf_\ast F = 0 \}$.

The subcategories (6.3) are known to be the hearts of bounded t-structures on $D^b(\text{Coh}_{\leq 1}(X))$. In particular they are abelian categories. Indeed when $p \in \{-1, 0\}$, they are obtained as a tilting of $\text{Coh}_{\leq 1}(X)$, so any object in $^p\text{Per}_{\leq 1}(X/Y)$ is concentrated on $[-1, 0]$. Below we always take $p \in \{-1, 0\}$. These t-structures fit into Bridgeland stability conditions on the boundary points of $U(X)$ as follows:

**Lemma 6.3.** Let us take data

$$(6.4) \quad \omega \in \text{NS}(Y)_\mathbb{R}, \quad H \in \text{NS}(X)_\mathbb{R}$$

such that $\omega$ is ample and $H$ is $f$-ample. Then for $p \in \{-1, 0\}$, there is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ we have

$$(6.5) \quad ^p\tau_{(-1)^p+1}\delta H, \omega := (Z_{(-1)^p+1}\delta H, f^\ast \omega, ^p\text{Per}_{\leq 1}(X/Y)) \in U(X).$$

Here $Z_{\ast, \ast}$ is the group homomorphism $\Gamma_X \to \mathbb{C}$ defined as in (2.4).
The equivalence \([6.2]\) induces the isomorphism on the space of stability conditions
\[
(6.6) \quad \Phi_*: \text{Stab} \leq 1(X) \cong \text{Stab} \leq 1(X^\dagger)
\]
sending \((Z, A)\) to \((Z \circ \Phi^{-1}, \Phi(A))\). On the other hand, the equivalence \([6.2]\) restricts to the equivalence (see \([Bri07]\))
\[
(6.7) \quad \Phi: \text{Per} \leq 1(X/Y) \cong -1\text{Per} \leq 1(X^\dagger/Y).
\]
Hence we have the following lemma:

**Lemma 6.4.** Under the isomorphism \([6.6]\), we have
\[
\Phi_* \tau^{-\delta H, \omega} = -1\tau^{\delta H^\dagger, \omega}.
\]
Here \(H^\dagger = -\phi_* H\), which is \(f^\dagger\)-ample by the definition of a flop.

**Proof.** See \([Tod08]\) Proposition 5.2 for details. q.e.d.

### 6.3. Moduli stacks of semistable perverse coherent sheaves

Let us write stability conditions \([6.5]\) as
\[
p_\tau = (pZ, p\text{Per} \leq 1(X/Y)), \quad p = -1, 0,
\]
for simplicity. For \(v = (\beta, m) \in \Gamma_X\), let \(\mathcal{M}_{p_\tau}(v)\) be the moduli stack of \(p_\tau\)-semistable objects \(E \in p\text{Per} \leq 1(X/Y)\) with \(\text{ch}(E) = v\).

**Lemma 6.5.** Suppose that \(f_* \beta \neq 0\). Then we have \(\mathcal{M}_{p_\tau}(v) \subset \mathcal{M}_X(\beta)\), i.e. any object \([E] \in \mathcal{M}_{p_\tau}(v)\) is a pure one dimensional sheaf. In particular if \(\mathcal{M}_{p_\tau}(v) \neq \emptyset\), then \(\beta\) is an effective class.

**Proof.** For \([E] \in \mathcal{M}_{p_\tau}(v)\), we have the exact sequence in \(p\text{Per} \leq 1(X/Y)\)
\[
0 \to \mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to 0.
\]
Suppose that \(\mathcal{H}^{-1}(E) \neq 0\). Since \(\mathcal{H}^{-1}(E)\) is supported on \(\text{Ex}(f)\) and \(Rf_* E\) is a one dimensional sheaf on \(Y\), we have
\[
\pi = \arg^p Z(\mathcal{H}^{-1}(E)[1]) > \arg^p Z(E).
\]
The above inequality contradicts to the \(p_\tau\)-semistability of \(E\). So we have \(\mathcal{H}^{-1}(E) = 0\), and \(E \in \text{Coh} \leq 1(X)\) holds. Similar argument shows that \(E\) is a pure sheaf. q.e.d.

For \(\sigma \in U(X)\) which is sufficiently close to \(p_\tau\), we have an open embedding
\[
(6.9) \quad \mathcal{M}_\sigma(v) \subset \mathcal{M}_{p_\tau}(v).\]
We show that the above embedding \([6.9]\) is an isomorphism for suitable choice of data \([6.4]\). We prepare using the following lemma:
Lemma 6.6. For an effective class $\beta \in N_1(X)$ with $f_*\beta \neq 0$, there exists $(H, \omega)$ such that, after replacing $\delta_0 > 0$ by a smaller one if necessary the following holds: for any decomposition $\beta = \beta_1 + \beta_2$ into effective classes $\beta_1, \beta_2$ which are not proportional in $N_1(X)_{\mathbb{R}}$, $(m_1, m_2) \in \mathbb{Z}^2$ and $0 < \pm \delta < \delta_0$, we have
\begin{equation}
\frac{m_1 + \delta(H \cdot \beta_1)}{f^*\omega \cdot \beta_1} \neq \frac{m_2 + \delta(H \cdot \beta_2)}{f^*\omega \cdot \beta_2}.
\end{equation}

Proof. We first note that a general $(H, \omega)$ satisfies the following: for any decomposition $\beta = \beta_1 + \beta_2$ into effective classes $\beta_1, \beta_2$ which are not proportional in $N_1(X)_{\mathbb{R}}$, we have
\begin{equation}
\alpha := \frac{H \cdot \beta_1}{f^*\omega \cdot \beta_1} - \frac{H \cdot \beta_2}{f^*\omega \cdot \beta_2} \neq 0.
\end{equation}
Indeed, suppose that (6.11) fails for any $(H, \omega)$. Then for any $(D_1, D_2) \in \text{NS}(X) \times \mathbb{R}$, we have
\begin{equation}
(D_1 \cdot \beta_1) \cdot (D_2 \cdot \beta_2) = (D_1 \cdot \beta_2) \cdot (D_2 \cdot \beta_1).
\end{equation}
An easy linear algebra argument shows that the above condition implies that $\beta_1$ and $\beta_2$ are proportional. Therefore for a fixed non-proportional $(\beta_1, \beta_2)$, a general choice of $(H, \omega)$ satisfies (6.11). Since the possible pairs $(\beta_1, \beta_2)$ are finite, a general $(H, \omega)$ satisfies (6.11) for any decomposition $\beta = \beta_1 + \beta_2$.

Let us take $(H, \omega)$ as above such that $\omega$ is rational. If for $\delta \neq 0$ the condition (6.10) fails, we have
\begin{equation}
\delta = \frac{1}{\alpha} \left( \frac{m_2}{f^*\omega \cdot \beta_2} - \frac{m_1}{f^*\omega \cdot \beta_1} \right).
\end{equation}
Since the RHS takes discrete values, we can find $\delta_0 > 0$ such that for $0 < \pm \delta < \delta_0$ the condition (6.10) holds. q.e.d.

Lemma 6.7. For an effective class $\beta \in N_1(X)$ with $f_*\beta \neq 0$, we take $(H, \omega)$ and $\delta_0 > 0$ as in Lemma 6.6. Then the open embedding (6.9) is an isomorphism of stacks.

Proof. Let $E \in \mathcal{P}\text{er}_{\leq 1}(X/Y)$ be a $\mathbb{P}\tau$-semistable object with $\text{ch}(E) = (\beta, m)$. By Lemma 6.5, we have $E \in \text{Coh}_{\leq 1}(X)$. Suppose that $E$ is not $\sigma$-semistable. Since $\text{Im}^\mathbb{P}Z(E) > 0$, there is an exact sequence
\begin{equation}
0 \to E_1 \to E \to E_2 \to 0
\end{equation}
in both of $\text{Coh}_{\leq 1}(X)$ and $\mathcal{P}\text{er}_{\leq 1}(X/Y)$ which destabilizes $E$ in $\sigma$-stability and
\begin{equation}
\text{arg}^\mathbb{P}Z(E_1) = \text{arg}^\mathbb{P}Z(E_2).
\end{equation}
By setting $\text{ch}(E_i) = (\beta_i, m_i)$, the above condition implies that $(\beta_1, m_1)$ and $(\beta_2, m_2)$ are proportional by Lemma 6.6. Then it contradicts to that (6.12) destabilizes $E$ in $\sigma$-stability. q.e.d.
6.4. Flop invariance formula. The following is the main result in this section.

**Theorem 6.8.** Suppose that the stack $\mathcal{M}_X(\beta)$ is CY at $\gamma \in \text{Chow}_X(\beta)$ and $\Phi_X(\gamma, m)$ is non-zero. Then $f_*\gamma$ is an effective one cycle on $X^\dagger$. If $\mathcal{M}_{X^\dagger}(\phi_*\beta)$ is also CY at $\phi_*\gamma \in \text{Chow}_{X^\dagger}(\phi_*\beta)$, then we have the identity

$$\Phi_X(\gamma, m) = \Phi_{X^\dagger}(\phi_*\gamma, m).$$

(6.13)

**Proof.** We take $(H, \omega)$ and $\delta_0 > 0$ as in Lemma 6.7. We write stability conditions (6.8) as $p\tau_X$, and let $\sigma(\cdot \tau_X) \in U(\cdot)$ be sufficiently close to $p\tau_X$. If $\Phi_X(\gamma, m)$ is non-zero, then by Theorem 5.7 there is an object $[E] \in \mathcal{M}_{\sigma^0}(\beta)$ such that $l(E) = \gamma$. By Lemma 6.4 and Lemma 6.7, we have the isomorphisms

$$\mathcal{M}_{\sigma^0}(\beta) \xrightarrow{\cong} \mathcal{M}_0(\beta) \xrightarrow{\Phi_*} \mathcal{M}_{-1}(\phi_* \beta) \xrightarrow{\cong} \mathcal{M}_{\sigma^1}(\phi_* \beta).$$

(6.14)

By the above isomorphisms, the object $\Phi(E)$ is also a sheaf so the one cycle $\phi_*\gamma = l(\Phi(E))$ on $X^\dagger$ is effective. Let $T^{(0)}_X(\beta) \subset \text{Chow}_X(\beta)$ be the image of the HC map $\mathcal{M}^\text{red}_{\sigma^0}(\beta) \rightarrow \text{Chow}_X(\beta)$. We have the commutative diagram

$$\mathcal{M}^\text{red}_{\sigma^0}(\beta) \xrightarrow{\Phi_*} \mathcal{M}^\text{red}_{\sigma^1}(\phi_* \beta)$$

where the vertical arrows are HC maps. The isomorphisms (6.14) preserve the $d$-critical structures and the virtual canonical line bundles. Therefore $\mathcal{M}_{\sigma^1}(\phi_* \beta)$ is also CY at $\phi_*\gamma$, and we have the identity

$$\Phi_{\sigma^0}(\gamma, m) = \Phi_{\sigma^1}(\phi_*\gamma, m).$$

If we furthermore assume that $\mathcal{M}_{X^\dagger}(\phi_* \beta)$ is CY at $\phi_*\gamma$, then by Theorem 5.7 we obtain the identity (6.13). q.e.d.

By Theorem 6.8 and the identity (2.34), we have the following corollary:

**Corollary 6.9.** For $\beta \in N_1(X)$ with $f_*\beta \neq 0$, suppose that Conjecture 2.10 holds for one cycles $\gamma$ and $\phi_*\gamma$. Then the local GV invariant $n_{g, \gamma}$ defined in [MT18] satisfy $n_{g, \gamma} = n_{g, \phi_*\gamma}$. In particular if Conjecture 2.17 holds for $X$ and $X^\dagger$, then we have $n_{g, \beta} = n_{g, \phi_* \beta}$.

7. The case of local surfaces

The results so far are conditional to Conjecture 2.10. In this section, we prove Conjecture 2.10 for local surfaces, which shows similar results.
as Theorem 5.7, Theorem 5.8 in this case without assuming Conjecture 2.10.

7.1. Moduli stacks of one dimensional semistable sheaves on local surfaces. Let $S$ be a smooth projective surface and consider the non-compact CY 3-fold $X$

$$X := \text{Tot}_S(K_S) \xrightarrow{p} S$$

where $p$ is the projection. The above CY 3-fold is compactified by adding the section at the infinity:

$$X \subset \overline{X} := \mathbb{P}_S(\mathcal{O}_S \oplus K_S).$$

Let $\text{Coh}_{c, \leq 1}(X)$ be the category of compactly supported coherent sheaves on $X$ whose supports have dimensions less than or equal to one. Note that $\text{Coh}_{c, \leq 1}(X)$ is the subcategory

$$\text{Coh}_{c, \leq 1}(X) \subset \text{Coh}_{\leq 1}(\overline{X})$$

consisting of sheaves whose supports do not intersect with the divisor $D_\infty := \overline{X} \setminus X$.

Let $s: S \to X$ be the zero section of $p$. For $\beta \in N_1(S)$, let $\mathcal{M}_{\overline{X}}(s, \beta)$ be the stack as in (2.6) for $\overline{X}$, and

$$\mathcal{M}_X(\beta) \subset \mathcal{M}_{\overline{X}}(s, \beta)$$

the open substack consisting of sheaves whose supports do not intersect with $D_\infty$. By its construction, the stack $\mathcal{M}_X(\beta)$ is nothing but the moduli stack of the following objects:

$$F \in \text{Coh}_{c, \leq 1}(X), \quad [l(p_*F)] = \beta.$$ 

By [PTVV13] (also see [Bus, Theorem 5.2] for the noncompact CY 3-fold case), the stack $\mathcal{M}_X(\beta)$ is a truncation of a smooth derived scheme with a $(-1)$-shifted symplectic structure. Therefore by [BBBJ15], there is a canonical $d$-critical structure on $\mathcal{M}_X(\beta)$, with virtual canonical line bundle given by the same formula (2.20) as in the compact CY 3-fold case.

Let $\text{Chow}_X(\beta)$ be the open subscheme of $\text{Chow}_{\overline{X}}(s, \beta)$ consisting of one cycles $\gamma$ on $\overline{X}$ which do not intersect with $D_\infty$. The HC map

$$(7.1) \quad \mathcal{M}_{\overline{X}}^{\text{red}}(s, \beta) \to \text{Chow}_{\overline{X}}(s, \beta)$$

for $\overline{X}$ restricts to the HC map

$$(7.2) \quad \pi_{\mathcal{M}}: \mathcal{M}_X^{\text{red}}(\beta) \to \text{Chow}_X(\beta)$$

by pulling back (7.1) to the open locus $\text{Chow}_X(\beta)$ in $\text{Chow}_{\overline{X}}(s, \beta)$. 
7.2. CY property for local surfaces. In this subsection, we show Conjecture 2.10 for the local surface case.

**Theorem 7.1.** For the local surface \( X = \text{Tot}_S(K_S) \), the stack \( \mathcal{M}_X(\beta) \) is CY at any \( \gamma \in \text{Chow}_X(\beta) \).

**Proof.** Let us fix a one cycle \( \gamma \in \text{Chow}_X(\beta) \). We first note that there exist smooth curves \( C_1, C_2 \) on \( S \) which are not contained in the support of \( p_*\gamma \), and admit an isomorphism

\[
\omega_S \cong \mathcal{O}_S(C_2 - C_1).
\]

Indeed it is enough to take a sufficiently ample divisor \( H \) on \( S \) and take general sections \( C_1 \in |H| \) and \( C_2 \in |H + K_S| \). Below we fix such \( C_1, C_2 \) and an isomorphism (7.3).

Let \( T \) be a complex analytic space and \( \mathcal{E} \in \text{Coh}(X \times T) \) a \( T \)-flat family of coherent sheaves on \( X \) giving a \( T \)-valued point of (analytification of) \( \mathcal{M}_X(\beta) \), i.e. a 1-morphism \( T \to \mathcal{M}_X(\beta) \). Suppose that its composition with the HC map (7.2) is contained in a sufficiently small open neighborhood of \( \gamma \in \text{Chow}_X(\beta) \). We use the following commutative diagrams

\[
(X_T := X \times T) \quad \pi_T : (C_i)_T := C_i \times T \quad \rho_T : S_T := S \times T
\]

where \( \pi_T, p_T, q_T, \rho_T, i_T \) are the projections, and \( j_{T,i} \) is the natural closed embedding. We have the canonical exact sequence of sheaves on \( X_T \)

\[0 \to p_T^*(\omega_S^{-1} \boxtimes p_T^*\mathcal{E}) \to p_T^*p_T^*\mathcal{E} \to \mathcal{E} \to 0.\]

We apply \( \mathbf{R}\text{Hom}_{p_T}(-, \mathcal{E}) \) to the above exact sequence. By setting \( \mathcal{F} = p_T^*\mathcal{E} \) and using the adjunction, we obtain the distinguished triangle in \( D^b(\text{Coh}(T)) \)

\[\mathbf{R}\text{Hom}_{\pi_T}(\mathcal{E}, \mathcal{E}) \to \mathbf{R}\text{Hom}_{q_T}(\mathcal{F}, \mathcal{F}) \to \mathbf{R}\text{Hom}_{q_T}(\omega_S^{-1} \boxtimes \mathcal{F}, \mathcal{F}).\]

Therefore we have the canonical isomorphism

\[K^\text{vir}_{\mathcal{M}_X(\beta)}|_T \xrightarrow{\cong} \det \mathbf{R}\text{Hom}_{q_T}(\mathcal{F}, \mathcal{F}) \otimes \det \mathbf{R}\text{Hom}_{q_T}(\omega_S^{-1} \boxtimes \mathcal{F}, \mathcal{F})^\vee.\]

Using the isomorphism (7.3), we have the distinguished triangle

\[\mathcal{O}_S(-C_1) \boxtimes \mathcal{F} \to \omega_S \boxtimes \mathcal{F} \to \omega_S|_{C_2} \boxtimes \mathcal{F}.
\]

By applying \( \mathbf{R}\text{Hom}_{q_T}(\mathcal{F}, -) \), we obtain the distinguished triangle

\[\mathbf{R}\text{Hom}_{q_T}(\mathcal{O}_S(C_1) \boxtimes \mathcal{F}, \mathcal{F}) \to \mathbf{R}\text{Hom}_{q_T}(\omega_S^{-1} \boxtimes \mathcal{F}, \mathcal{F}) \to \mathbf{R}\text{Hom}_{r_{T,2}}(Lj_{T,2}^*\mathcal{F}, \omega_S|_{C_2} \boxtimes Lj_{T,2}^*\mathcal{F})\]
which gives the isomorphism
\[ \det R\text{Hom}_{T,q}(\omega_{S}^{-1} \boxtimes F, F) \cong \det R\text{Hom}_{T,q}(O_{S}(C_{1}) \boxtimes F, F) \otimes \det R\text{Hom}_{rT,2}(L_{jT,2}^{*}F, \omega_{S}|C_{2} \boxtimes L_{jT,2}^{*}F). \]

Similarly from the distinguished triangle
\[ O_{S}(-C_{1}) \boxtimes F \to F \to O_{C_{1}} \boxtimes F \]
we have the distinguished triangle
\[ R\text{Hom}_{T,q}(O_{S}(C_{1}) \boxtimes F, F) \to R\text{Hom}_{T,q}(F, F) \]
\[ \to R\text{Hom}_{rT,1}(L_{jT,1}^{*}F, L_{jT,1}^{*}F) \]
which gives the isomorphism
\[ \det R\text{Hom}_{T,q}(O_{S}(C_{1}) \boxtimes F, F) \]
\[ \cong \det R\text{Hom}_{T,q}(F, F) \otimes \det R\text{Hom}_{rT,1}(L_{jT,1}^{*}F, L_{jT,1}^{*}F)^{\vee}. \]

By combining the above isomorphisms, we have the isomorphism
\[ K_{\text{vir}}^{\text{vir}}(M_{X}(\beta))_{T} \cong \det R\text{Hom}_{rT,1}(L_{jT,1}^{*}F, L_{jT,1}^{*}F) \]
\[ \otimes \det R\text{Hom}_{rT,2}(L_{jT,2}^{*}F, \omega_{S}|C_{2} \boxtimes L_{jT,2}^{*}F)^{\vee}. \]

By our assumption that \( C_{i} \) is not contained in the support of \( p_{*}\gamma \), the object \( L_{jT,1}^{*}F \) is a \( T \)-flat family of zero-dimensional sheaves on \( C_{i} \). Moreover the above isomorphism is compatible with complex analytic maps \( T' \to T \) for other complex analytic space \( T' \). Therefore the result follows from Lemma 7.2 below. q.e.d.

We have used the following lemma:

**Lemma 7.2.** Let \( C \) be a smooth projective curve and \( L \) a line bundle on it. Let \( M_{0} \) be the stack of zero dimensional sheaves on \( C \) and consider the HC map
\[ \pi_{M_{0}} : M_{0} \to \text{Sym}(C) \]
sending a zero-dimensional sheaf to its support. Let \( U \in \text{Coh}(C \times M_{0}) \) be the universal family, and \( r_{M_{0}} : C \times M_{0} \to M_{0} \) the projection. Let \( K_{M_{0},L}^{\text{vir}} \) be the line bundle on \( M_{0} \) defined by
\[ K_{M_{0},L}^{\text{vir}} := \det R\text{Hom}_{rM_{0}}(U, L \boxtimes U). \]

Then for each \( [Z] \in \text{Sym}(C) \), there is an analytic open neighborhood \( [Z] \in U \subset \text{Sym}(C) \) such that \( K_{M_{0},L}^{\text{vir}} \) is trivial on \( \pi_{M_{0}}^{-1}(U) \).
Proof. Let \( Z \in \text{Sym}(C) \) be given by \( Z = \sum_{i=1}^{k} a_i [p_i] \) for distinct points \( p_1, \ldots, p_k \in C \) and \( a_i \in \mathbb{Z}_{>0} \). We take analytic open neighborhoods \( p_i \in U_i \subset C \) such that \( U_i \cap U_j = \emptyset \) for \( i \neq j \) and \( L|_{U_i} \) is trivial. Let \( [Z] \in U \subset \text{Sym}(C) \) be an open neighborhood given by

\[
U = \prod_{i=1}^{k} \text{Sym}^{a_i}(U_i) \subset \text{Sym}(C)
\]

where the right inclusion is given by \( (Z_i)_{1 \leq i \leq k} \mapsto \sum Z_i \). Then we have the isomorphism

\[
\oplus: \prod_{i=1}^{k} \pi_{-1}^{-1}(\text{Sym}^{a_i}(U_i)) \cong \pi_{-1}^{-1}(U)
\]

given by taking the direct sum of zero dimensional sheaves. Under the above isomorphism, we have

\[
\oplus^* (K_{\mathcal{M}_0, \mathcal{O}_C}^{\text{vir}}|_{\pi_{-1}^{-1}(U)}) \cong \bigotimes_{i=1}^{k} K_{\mathcal{M}_0, \mathcal{O}_C}^{\text{vir}}|_{\pi_{-1}^{-1}(\text{Sym}^{a_i}(U_i))}.
\]

Therefore it is enough to show that \( K_{\mathcal{M}_0, \mathcal{O}_C}^{\text{vir}} \) is trivial when \( C = \mathbb{A}^1 \). In this case, the stack \( \mathcal{M}_0(k) \) of zero dimensional sheaves on \( \mathbb{A}^1 \) with length \( k \) is given by

\[
\mathcal{M}_0(k) = [\text{Hom}(V, V)/\text{GL}(V)]
\]

where \( V \) is a \( k \)-dimensional vector space and \( \text{GL}(V) \) acts on \( W := \text{Hom}(V, V) \) by conjugation. The universal sheaf \( \mathcal{U} \) is a \( \text{GL}(V) \)-equivariant sheaf on \( W \times \mathbb{A}^1 \), which admits a \( \text{GL}(V) \)-equivariant exact sequence

\[
0 \to V \otimes \mathcal{O}_{W \times \mathbb{A}^1} \xrightarrow{i} V \otimes \mathcal{O}_{W \times \mathbb{A}^1} \to \mathcal{U} \to 0.
\]

Here the map \( i \) corresponds to the \( \text{GL}(V) \)-invariant section of \( \text{Hom}(V, V) \otimes \mathcal{O}_{W \times \mathbb{A}^1} \) given by

\[
\delta \otimes 1 - \text{id}_V \otimes 1 \otimes t \in \text{Hom}(V, V) \otimes \text{Sym}^*(\text{Hom}(V, V)^{\vee}) \otimes \mathbb{C}[t]
\]

where \( \delta \in \text{Hom}(V, V) \otimes \text{Hom}(V, V)^{\vee} \) is the tautological element. Therefore \( \mathcal{U} \) is zero in the \( \text{GL}(V) \)-equivariant K-theory of \( \text{Hom}(V, V) \), thus \( K_{\mathcal{M}_0, \mathcal{O}_C}^{\text{vir}} \) is trivial when \( C = \mathbb{A}^1 \).

q.e.d.

7.3. GV type invariants for local surfaces. The GV type invariants for the local surface \( X = \text{Tot}_{S}(K_S) \) is defined similarly to the projective CY 3-fold case. For an element

\[
B + i\omega \in A(S)_C
\]

and \( F \in \text{Coh}_{c, \leq 1}(X) \), let \( \mu_{B, \omega}(F) \in \mathbb{R} \cup \{ \infty \} \) be defined by

\[
\mu_{B, \omega}(F) := \frac{\chi(F) - B \cdot l(p_* F)}{\omega \cdot l(p_* F)}.
\]
The above slope function defines the \((B, \omega)\)-stability on \(\text{Coh}_{c, \leq 1}(X)\). Let \(\Gamma_S := N_1(S) \oplus \mathbb{Z}\) and for \(F \in \text{Coh}_{c, \leq 1}(X)\) we set
\[
ch(F) := ([l(p_* F)], \chi(F)).
\]
Let \(Z_{B, \omega}\) be the group homomorphism \(\Gamma_S \to \mathbb{C}\) defined by
\[
Z_{B, \omega}(\beta, m) := -m + (B + i\omega)\beta.
\]
Then the pair
\[
\sigma_{B, \omega} := (Z_{B, \omega}, \text{Coh}_{c, \leq 1}(X))
\]
is a Bridgeland stability condition on \(D^b(\text{Coh}_{c, \leq 1}(X))\) w.r.t. the Chern character map \((7.5)\). Similarly to Subsection 2.1, we denote by \(\text{Stab}_{c, \leq 1}(X)\) the space of Bridgeland stability conditions on \(D^b(\text{Coh}_{c, \leq 1}(X))\) w.r.t. the Chern character map \((7.5)\), and
\[
U(X) \subset \text{Stab}_{c, \leq 1}(X)
\]
the subset consisting of stability conditions of the form \((7.6)\).

For \(v = (\beta, m) \in N_1(S) \oplus \mathbb{Z}\), we have the open substack
\[
\mathcal{M}_\sigma(v) \subset \mathcal{M}_X(\beta)
\]
consisting of \(\sigma\)-semistable objects in \(\text{Coh}_{c, \leq 1}(X)\). For \(\gamma \in \text{Chow}_X(\beta)\), let \(\gamma \in U \subset \text{Chow}_X(\beta)\) be a sufficiently small open neighborhood. Using the diagram \((2.27)\) and CY orientation data of \(\mathcal{M}_X(\beta)|_U\) which exists by Theorem \((7.4)\), the perverse sheaf \(\phi_{M_\sigma(v)|_U}\) on \(\text{Perv}(M_\sigma(v)|_U)\) and the invariant
\[
\Phi_\sigma(\gamma, m) \in \mathbb{Z}[y^{\pm 1}]
\]
are defined as in Definition \((7.7)\). Similarly to Lemma \((2.14)\), the invariant \((7.7)\) is independent of a CY orientation data of \(\mathcal{M}_X(\beta)|_U\). Then the arguments of Theorem \((5.7)\) and Theorem \((5.8)\) show the following (which is not conditional to Conjecture \((2.10)\) by Theorem \((7.1)\)):

**Theorem 7.3.** Let \(X = \text{Tot}_S(K_S)\) for a smooth projective surface \(S\). Then for \(\sigma \in U(X), \gamma \in \text{Chow}_X(\beta)\) and \(m \in \mathbb{Z}\), the invariant \(\Phi_\sigma(\gamma, m)\) is independent of \(\sigma\), so we can write it as \(\Phi_X(\gamma, m)\). If furthermore \(\gamma\) is primitive, then \(\Phi_X(\gamma, m)\) is independent of \(m\).

### 7.4. Blow-up formula.

Let \(S\) be a smooth projective surface and take a blow-up
\[
h : S^\dagger \to S
\]
at a point \(p \in S\). Then there exist smooth projective 3-folds \(X, X^\dagger\) connected by a flop
\[
\phi : X \to Y \leftarrow X^\dagger
\]
satisfying the following conditions (see [Tod15, Lemma 4.2])
Both of the exceptional locus $Z = \operatorname{Ex}(f)$, $Z^\dagger = \operatorname{Ex}(f^\dagger)$ are irreducible $(-1, -1)$-curves.

There are closed embeddings $i: S \hookrightarrow X$, $i^\dagger: S^\dagger \hookrightarrow X^\dagger$

(7.8)

such that $S \cap Z$ consists of one point, the strict transform of $S$ in $X^\dagger$ coincides with $S^\dagger$, and $Z^\dagger \subset S^\dagger$ coincides with the exceptional locus of $h: S^\dagger \to S$.

There are open neighborhoods $S \subset X$, $S^\dagger \subset X^\dagger$ and isomorphisms

(7.9)

such that the embeddings (7.8) are identified with the zero sections.

We regard one cycles on $S$, $S^\dagger$ as one cycles on $X$, $X^\dagger$ by isomorphisms (7.9) and zero sections. Applying the argument of Theorem 6.8, we obtain the following (which is not conditional to Conjecture 2.10):

**Theorem 7.4.** Let $S$ be a smooth projective surface and $h: S^\dagger \to S$ a blow-up at a point. Let $X = \operatorname{Tot}_S(K_S)$ and $X^\dagger = \operatorname{Tot}_{S^\dagger}(K_{S^\dagger})$. Then for any effective one cycle $\gamma$ on $S$ and $m \in \mathbb{Z}$, we have the identity

$$
\Phi_X(\gamma, m) = \Phi_{X^\dagger}(h^*\gamma, m).
$$

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