RICCI FLOW OF WARPED BERGER METRICS ON $\mathbb{R}^4$

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Abstract. We study asymptotically flat SU(2)-cohomogeneity 1 solutions of Ricci flow on $\mathbb{R}^4$ whose restrictions to any Euclidean hypersphere are left-invariant Berger spheres. We show that these solutions are never Type-I and that any finite-time singularity is modelled by the Bryant soliton once suitably dilated. Moreover we identify a large family of warped Berger Ricci flow solutions that exist for any positive time; the emphasis is on the existence of immortal solutions with no maximal volume growth, no maximal symmetry and mixed sign for the scalar curvature.

1. Introduction

Given a smooth Riemannian metric $(M, g_0)$, Hamilton’s Ricci flow starting at $g_0$ is the geometric heat-type evolution equation

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g(t)), \quad g(0) = g_0.$$ 

If $(M, g_0)$ is a complete noncompact manifold with bounded curvature Shi proved that the Ricci flow problem starting at $g_0$ satisfies the short-time existence property [34]; twenty years later, Chen and Zhu showed that Shi’s solution is unique in the class of complete solutions with curvature bounded on compact time intervals [10].

According to Hamilton’s programme, a complete Riemannian metric $g_0$ with bounded curvature should evolve along the Ricci flow to some more regular metric from which topological information about the underlying manifold can often be inferred.

A crucial step in this approach consists in understanding which assumptions for $g_0$ might prevent the Ricci flow solution from developing finite-time singularities. General long-time existence results on complete noncompact manifolds usually rely on a strong control on both the sign of the curvature and the volume growth [7], [33]. Alternatively, similar conclusions may be reached when the analysis of the Ricci flow is restricted to families of Riemannian metrics with many symmetries. Lauret proved that the Ricci flow on $\mathbb{R}^n$ starting at some metric that admits a transitive action of a nilpotent Lie group is immortal [27]; this result was then generalized to any homogeneous Ricci flow on $\mathbb{R}^n$ by Lafuente [26]. Instead of assuming a transitive action of some Lie group on $\mathbb{R}^n$, one may examine the Ricci flow of cohomogeneity 1 Riemannian metrics on $\mathbb{R}^n$; in this direction, Oliynyk and Woolgar proved that the Ricci flow on $\mathbb{R}^n$, with $n \geq 3$, starting at an asymptotically flat spherically symmetric metric which does not have minimal embedded hyperspheres exists for any positive time [31]. While the requirement of no minimal embedded hyperspheres naturally arises to prevent the formation of neckpinches [2], it is worth asking if a similar result still holds for a class of cohomogeneity 1 metrics without maximal symmetry.

In this work we show that we can answer the previous question affirmatively once we restrict to the 4-dimensional setting. We obtain a long-time existence result for a family of SU(2)-cohomogeneity 1 metrics on $\mathbb{R}^4$; as a consequence of the lack of maximal symmetry for the principal orbits, this family includes metrics that do not have Euclidean volume growth. The Ricci flow on 4-dimensional cohomogeneity 1 manifolds has been recently studied on various (compact) topologies [5], [23], [24]. In [23] Isenberg, Knopf and Sáez analysed the Ricci flow problem in a subclass of generalized warped Berger metrics on $S^4 \times S^3$ supporting an isometric cohomogeneity 1 action of SU(2); they proved that the solution to the Ricci flow is Type-I and becomes locally rotationally symmetric around any singularity.

In line with their study, we investigate the behaviour of generalized warped solutions to the Ricci
The last property along with the bounds in Section 2 force the singularity model to be rotationally one of the points in the paper where the fact that we work in real dimension 4 plays a crucial role). Su defines a copy of first show that a fixed SU(2) Milnor frame of left-invariant vector fields passes to the limit and finally, we devote Section 5 to prove Theorem 1.1 and Theorem 1.2 by a blow-up analysis. We no creation of necks along the flow via a Sturmian-type of argument [1].

In Section 4 we consider the class of solutions satisfying Assumption 1 and we prove that there is same phenomenon of enhancement of the symmetries happens in this setting; we also derive symmetric around any singularity at some rate that breaks scale-invariance, thus proving that the derived in [23] and [24] to the Ricci flow equations; we mainly extend the estimates no the class of generalized warped Berger metrics; a stronger requirement is then needed, which also per spheres are created along the Ricci flow [2], [31], we are not able to extend this property to although in the spherically symmetric case it has been shown that no minimal embedded hyperspheres can be immortal. In the setting of Theorem 1.1, the existence of minimal hyperspheres for the initial metric $g_0$ may lead to the formation of degenerate neckpinches for the Ricci flow.

According to the examples of neckpinches studied in [2], as observed in [31] one may expect that Ricci flow solutions starting at initial metrics with sufficiently pinched minimal embedded hyperspheres cannot be immortal. In the setting of Theorem 1.1 the existence of minimal hyperspheres for the initial metric $g_0$ may lead to the formation of degenerate neckpinches for the Ricci flow. Although in the spherically symmetric case it has been shown that no minimal embedded hyperspheres are created along the Ricci flow [2], [31], we are not able to extend this property to the class of generalized warped Berger metrics; a stronger requirement is then needed, which also guarantees a control on the growth of the metric along those SU(2)-directions that are orthogonal to the $S^1$-fibers of the Hopf fibration (see Section 2 for the formulation of Assumption 1).

Once we restrict the analysis to this subclass, we are able to show that the long-time existence property holds.

**Theorem 1.1.** Let $g : [0, T) \to \mathcal{M}(\mathbb{R}^4)$ be the maximal solution to the Ricci flow starting at some asymptotically flat $g_0 \in \mathcal{WB}(\mathbb{R}^4)$. The following conditions hold:

(i) The solution is not Type-I.

(ii) If $T < \infty$, then for any sequence $T_j \nearrow T$ there exist sequences $\lambda_j \nearrow \infty$, $t_j \nearrow T$ and $\{p_j\} \subset \mathbb{R}^4$ such that the rescaled Ricci flows $(\mathbb{R}^4, g_j(t_j), p_j)$ defined on $[-\lambda_j t_j, (T_j - t_j) \lambda_j]$ by $g_j(t) \equiv \lambda_j g(t_j + \frac{t}{\lambda_j})$ subconverge to the Bryant Soliton (up to a fixed homothety).

We point out that the family of warped Berger initial data given by Theorem 1.2 also includes Riemannian metrics with cubic volume growth and mixed sign for the scalar curvature. It is worth mentioning that any $g_0$ sufficiently close to the Taub-NUT metric satisfies Assumption 1 in Theorem 1.2, however, in a given topology, the statement still holds for initial data that are far away from the Taub-NUT metric.

We also note that when $g_0$ is spherically symmetric Theorem 1.2 fully recovers the result of Oliynyk and Woolgar in the 4-dimensional setting.

In the following we provide an overview of the paper.

In Section 2 we describe the class of asymptotically flat warped Berger metrics on $\mathbb{R}^4$ and we prove that the Ricci flow is well defined within this class.

Section 3 is dedicated to the analysis of the Ricci flow equations; we mainly extend the estimates derived in [23] and [24] to the $\mathbb{R}^4$-topology. Namely, we show that the solution becomes rotationally symmetric around any singularity at some rate that breaks scale-invariance, thus proving that the same phenomenon of enhancement of the symmetries happens in this setting; we also derive appropriate lower bounds for the sectional curvatures.

In Section 4 we consider the class of solutions satisfying Assumption 1 and we prove that there is no creation of necks along the flow via a Sturmian-type of argument [1].

Finally, we devote Section 5 to prove Theorem 1.1 and Theorem 1.2 by a blow-up analysis. We first show that a fixed SU(2) Milnor frame of left-invariant vector fields passes to the limit and defines a copy of $su(2)$ in the Lie algebra of Killing vector fields of the singularity model (this is one of the points in the paper where the fact that we work in real dimension 4 plays a crucial role); the last property along with the bounds in Section 2 force the singularity model to be rotationally
symmetric with nonnegative curvature operator. From a topological argument we derive that if the flow is Type-I then we can always dilate it around a singularity so that the singularity model must be $\mathbb{R}^4$. The first part of Theorem 1.1 follows then from the rigidity result in [25]; once we know that the singularity model is an eternal flow with strictly positive curvature operator, we show the convergence to the Bryant soliton by a point-picking argument. Finally we prove that if the solution has no necks then the singularity model has positive asymptotic volume ratio, which contradicts Perelman’s result for non-collapsed ancient solutions to the Ricci flow with nonnegative curvature operator [32].

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2. Setting

2.1. Asymptotically flat warped Berger metrics on $\mathbb{R}^4$. A compact Lie Group $G$ acting on a Riemannian manifold $(M, g)$ via isometries is said to act with cohomogeneity one if the orbit space $M/G$ is 1-dimensional. If $M$ is a noncompact manifold, the quotient space is either homeomorphic to $[0, 1)$ or to $\mathbb{R}$, depending on whether there exists a singular orbit of codimension greater than one (see, e.g., [17]): we analyse the first case, with $G$ and $M$ being $SU(2)$ and $\mathbb{R}^4$ respectively.

Let $\{X_1, X_2, X_3\}$ be a Milnor frame on $SU(2)$ and let $\{\sigma_1, \sigma_2, \sigma_3\}$ be its associated Milnor coframe [13, Chapter 1]. Once we identify $S^3$ with the Lie group $SU(2)$, any left-invariant metric on $S^3$ can be written as

$$\hat{g} = \lambda_1 \sigma_1 \otimes \sigma_1 + \lambda_2 \sigma_2 \otimes \sigma_2 + \lambda_3 \sigma_3 \otimes \sigma_3,$$

for some positive constants $\lambda_1, \lambda_2$, and $\lambda_3$.

We note that the choice $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \varepsilon < 1$ parametrizes the classic family of Berger spheres collapsing along the $S^1$-fibers of the Hopf fibration as $\varepsilon \to 0$.

In the following we describe generalized warped Berger metrics on $\mathbb{R}^4$.

We consider $SU(2)$-cohomogeneity 1 Riemannian manifolds $(I \times S^3, g)$, with $I = (0, +\infty)$ and $g$ of the form

$$g = dx \otimes dx + \hat{g}_x$$

$$= dx \otimes dx + a^2(x) \sigma_1 \otimes \sigma_1 + b^2(x) \sigma_2 \otimes \sigma_2 + c^2(x) \sigma_3 \otimes \sigma_3,$$

where we have normalized the metric along the radial direction. For the smooth metric (1) to be defined on some topology boundary conditions need to be prescribed at $x = 0$.

Lemma 2.1. The metric $g$ in (1) extends to a smooth and complete metric on $\mathbb{R}^4$ if and only if

$$\lim_{x \to 0} a(x) = \lim_{x \to 0} b(x) = \lim_{x \to 0} c(x) = 0, \quad \lim_{x \to 0} a_x(x) = \lim_{x \to 0} b_x(x) = \lim_{x \to 0} c_x(x) = 1,$$

$$\lim_{x \to 0} \frac{d^{2k} a}{dx^{2k}}(x) = \lim_{x \to 0} \frac{d^{2k} b}{dx^{2k}}(x) = \lim_{x \to 0} \frac{d^{2k} c}{dx^{2k}}(x) = 0, \quad \forall k \in \mathbb{N}.$$

Proof. See, e.g., [13, Lemma 2.10]. □

From now on we assume the constraint (2) to hold, so that the metric $g$ in (1) extends to $\mathbb{R}^4$, with the origin representing the singular orbit for the $SU(2)$-action where the vector fields $\{X_1, X_2, X_3\}$ vanish. We also restrict the analysis of the Ricci flow to the subclass of biaxial Bianchi IX metrics, which consists of those metrics of the form (1) that satisfy the further condition $a = b$.

Since we are interested in generalized warped Berger metrics on $\mathbb{R}^4$, we require the ordering $c \leq b$: for any $x > 0$ the metric

$$g_x = b^2(x) (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(x) \sigma_3 \otimes \sigma_3$$

is then a left-invariant metric on the Euclidean hypersphere $S(0, x)$ with the $S^1$-fiber squashed by a factor $c(x)/b(x) \in (0, 1]$. For notational reasons, we make the following

Definition 2.2. We define the class of generalized warped Berger metrics $\mathcal{WB}(\mathbb{R}^4) \subset \mathcal{M}(\mathbb{R}^4)$ as the set of (smooth) complete Riemannian metrics on $\mathbb{R}^4$ of the form (1) and satisfying the constraints $a = b$ and $c \leq b$.

In order to study the Ricci flow starting at $g \in \mathcal{WB}(\mathbb{R}^4)$ some control on the geometry at spatial infinity is required; that accounts for the need to focus our analysis on the family of asymptotically flat metrics in $\mathcal{WB}(\mathbb{R}^4)$. 
**Definition 2.3.** We say that \( g \in \mathcal{W}((R^4) \) is asymptotically flat if there exists \( \epsilon > 0 \) such that for any \( x \geq 1 \)

\[
|\text{Rm}_g(x)| \leq \frac{\alpha}{x^{2+\epsilon}},
\]

for some \( \alpha > 0 \).

The stronger than quadratic decay of the curvature determines the asymptotic behaviour of any warped Berger metric at infinity. For any asymptotically flat metric \( g \in \mathcal{W}(R^4) \) the quantity \( m(g) = (\lim_{x \to \infty} c(x))^{-1} \) is well defined and finite. We note that \( m \) is the inverse of the length at infinity of the \( S^1 \) fiber given by the Hopf fibration; in line with the physical monopole interpretation, we refer to \( m \) as to the mass of \( g \). We prove the next Lemma in the Appendix A.

**Lemma 2.4.** Let \( g \in \mathcal{W}(R^4) \) be asymptotically flat. Then \( g \) satisfies one of the two conditions below:

(i) There exist the limits

\[
\lim_{x \to \infty} b_x(x) = 2, \quad \lim_{x \to \infty} c_x(x) = 0,
\]

and \( m(g) \in (0, \infty) \).

(ii) There exist the limits

\[
\lim_{x \to \infty} b_x(x) = 1, \quad \lim_{x \to \infty} c_x(x) = 1,
\]

and \( m(g) = 0 \).

In both cases the injectivity radius of \( g \) is bounded away from zero.

**Remark 2.5.** Given \( g \in \mathcal{W}(R^4) \) asymptotically flat, from the SU(2)-symmetry it immediately follows that \( m \) vanishes if and only if \( g \) has Euclidean volume growth; if instead the mass of \( g \) is positive, then \( g \) has cubic volume growth.

As discussed in the Introduction, when addressing the long-time existence problem it seems natural to exclude initial geometries that may give rise to Ricci flow solutions with minimal embedded hyperspheres. The next Assumption is meant to prevent this phenomenon.

**Assumption 1.** We say that \( g \in \mathcal{W}(R^4) \) of the form (1) satisfies Assumption 1 if

(i) There are no minimal embedded hyperspheres in \((R^4, g)\).

(ii) \( b' > 0 \) in \( R^4 \).

**Remark 2.6.** The condition (i) is equivalent to requiring

\[
\left( 2 \frac{b'}{b} + \frac{c'}{c} \right) > 0
\]

in \( R^4 \). Assumption 1 is thus weaker than asking for the monotonicity of both \( b \) and \( c \). We point out that according to Lemma 2.4 Assumption 1 can be rephrased by replacing the condition \( b' > 0 \) with

\[
\left( \frac{b'}{b} \right) \geq \delta \left( \frac{c'}{c} \right),
\]

for some \( \delta > 0 \); therefore \( g \) satisfies Assumption 1 if it does not admit minimal embedded hyperspheres and if its growth along the \( S^1 \) fibers of the Hopf fibration is controlled in terms of the growth along the other two directions in a global Milnor frame on \( R^4 \). By choosing \( b = c \) we fully recover the case analysed in [31] for the four dimensional setting.

**Remark 2.7.** From Lemma 2.4 it follows that there exist generalized warped Berger metrics with cubic volume growth that satisfy Assumption 1.
2.2. The Ricci Flow PDEs. In this subsection we explicitly write down the Ricci flow equations on \( \mathbb{R}^4 \) with initial data \( g_0 \in \mathcal{WBF}(\mathbb{R}^4) \). A simple application of the O’Neill formula to the Riemannian submersion \( \pi : (\mathbb{R}_+ \times S^3, g_0) \to (\mathbb{R}_+, (dx)^2) \) allows to compute the sectional curvatures of the vertical planes \([23]\ Appendix A]\)

\[
\begin{align*}
   k_{12} &= \frac{4b^2 - 3c^2}{b^4} - \frac{b^2}{b^2}, \\
   k_{13} &= k_{23} = \frac{c^2}{b^4} - \frac{c b^2}{c b},
\end{align*}
\]

and of the mixed vertical-horizontal ones

\[
\begin{align*}
   k_{01} &= k_{02} = -\frac{b x}{b}, \\
   k_{03} &= -\frac{c x}{c}.
\end{align*}
\]

By the SU(2)-symmetry we can write the scalar curvature as

\[
R = 2(k_{01} + k_{02} + k_{03} + k_{12} + k_{13} + k_{23}).
\]

Therefore, it follows that there exist asymptotically flat metrics in \( \mathcal{WBF}(\mathbb{R}^4) \) that satisfy Assumption 1 and whose scalar curvature has mixed sign.

We note that due to the symmetries the final term one has to take into account for describing the full curvature is

\[
R_{m\times 123} = \frac{1}{b} \left( \frac{c}{b} \right)_{x}.
\]

The Ricci flow diffeomorphism invariance ensures that the symmetries persist; moreover, since the Ricci tensor is diagonal along the global frame \( \{ \partial_x, X_1, X_2, X_3 \} \), any complete and bounded curvature solution to the Ricci flow starting at \( g_0 \in \mathcal{WBF}(\mathbb{R}^4) \) must be of the form

\[
g(t) = \xi^2(x, t) dx \otimes dx + b^2(x, t) (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(x, t) \sigma_3 \otimes \sigma_3
\]

\[
= ds \otimes ds + b^2(s, t) (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2(s, t) \sigma_3 \otimes \sigma_3,
\]

where similarly to \([23]\) we have introduced the geometric time-dependent coordinate \( s(x, t) \) representing the \( g(t) \)-distance of the Euclidean hypersphere of radius \( x \) from the origin. In terms of the variables \( s \) and \( t \) the Ricci flow equations become

\[
\begin{align*}
   b_t &= b_{ss} + \left( \frac{c_s}{c} + \frac{b_s}{b} \right) b_s + \frac{2(c^2 - 2b^2)}{b^4} \\
   c_t &= c_{ss} + \frac{2b_s}{b} c_s - \frac{2c^2}{b^4}.
\end{align*}
\]

The choice of a meaningful geometric coordinate \( s \) provides us with a parabolic form of the Ricci flow equations; however, we get a non vanishing commutator between \( \partial_t \) and \( \partial_s \) given by

\[
\left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -(\ln(\xi)) \frac{\partial}{\partial s} = -\left( \frac{2 b_s}{b} + \frac{c_s}{c} \right) \frac{\partial}{\partial s}.
\]

We also report the formula for the (time-dependent) laplacian along the Ricci flow: for any smooth function \( f \in C^\infty(\mathbb{R}^4) \) we have

\[
\Delta f \equiv f_{ss} + \left( \frac{b_s}{b} + \frac{c_s}{c} \right) f_s.
\]

We dedicate the end of this subsection to prove that the set of asymptotically flat metrics in \( \mathcal{WBF}(\mathbb{R}^4) \) is consistent with the Ricci flow.

**Lemma 2.8.** Let \( g_0 \in \mathcal{WBF}(\mathbb{R}^4) \) be asymptotically flat. There exists a unique complete maximal solution \( g : [0, T) \to \mathcal{WBF}(\mathbb{R}^4) \) to the Ricci flow starting at \( g_0 \) with curvature bounded on any compact time interval. The maximal time \( T \) is either \( \infty \) or is characterised by the following:

\[
\limsup_{t \to T} \left( \sup_{\mathbb{R}^4} |Rm_{g(t)}|_{g(t)} \right) = \infty.
\]

\[\text{In order to avoid a notational quagmire, we generally omit to explicitly write the time dependence of } s. \text{ In the following we interchangeably use both the coordinates } x \text{ and } s.\]
Moreover, \( g(t) \in \mathcal{W}^m(\mathbb{R}^4) \) and satisfies (8) with the same \( \epsilon \) for any \( 0 \leq t < T \).

Proof. The first part of the statement is a standard fact that follows from \([33]\) and \([10]\). By Lemma [A.1] (see Appendix A) we can apply Proposition B.10 in \([28]\) to our setting; therefore, the rate of decay of the curvature persists along the flow.

Since we have already seen that the solution \( g(t) \) must be of the form (12), to conclude the proof it remains to show that the ordering \( c \leq b \) is preserved.

As long as a (smooth) solution exists the boundary conditions (2) are satisfied, which then imply that \( c/b \) is identically 1 at \( s = 0 \). Moreover, according to Lemma [2.4] \( c/b \) converges to either 0 (cubic volume growth) or 1 (Euclidean volume growth) at infinity. From the uniform control along the parabolic boundary we derive that if the ordering fails to be preserved, then there exist a sufficiently small \( \epsilon > 0 \), \( t_\epsilon \in (0,T) \), and an interior maximum point \( x_\epsilon \in (0,\infty) \) where \( (c/b)(x_\epsilon,t_\epsilon) = 1 + \epsilon \) for the first time. A direct calculation gives:

\[
\left( \frac{c}{b} \right)_t = \left( \frac{c}{b} \right)_{ss} + 3 \frac{b}{c} \left( \frac{c}{b} \right)_s + 4 \frac{c}{b^2} \left( 1 - \frac{c^2}{b^2} \right).
\]

Evaluating (18) at \( (x_\epsilon,t_\epsilon) \) we get \( (c/b)(x_\epsilon,t_\epsilon) < 0 \), which is a contradiction.

In fact, once we know that the ordering \( c \leq b \) is preserved, a standard application of the strong Maximum Principle shows that if \( c = b \) at some \( (x_0,\tau_0) \in (0,\infty) \times (0,T) \), then \( c = b \) in a space-time neighbourhood of the point and thus \( c = b \) everywhere for all earlier times by real analyticity of solutions to the Ricci flow equations [31]. \( \square \)

2.3. Taub-NUT metrics. An important curve lying in \( \mathcal{W}_m^4(\mathbb{R}^4) \) is given by the one-parameter family of gravitational instantons called Taub-NUT metrics \( \{g_m\}_{m > 0} \) [22]; equivalently, \( g_m \) is an asymptotically flat hyperkähler metric for any \( m > 0 \). A way to define Taub-NUT metrics is via self-dual 2 forms (see, e.g., [16], Section 6). Given a Riemannian manifold \((M^4,g)\) the Hodge star operator is a bundle isomorphism sending the bundle of 2-forms to itself, thus leading to a splitting \( \Lambda^2_t(M) = \Lambda^+_4(M) \oplus \Lambda^-_4(M) \). In our setting, given \( g \in \mathcal{W}^m_4(\mathbb{R}^4) \), if we let \( \{\sigma_1,\sigma_2,\sigma_3\} \) denote again an SU(2)-left invariant coframe, then we can define a basis of self-dual 2 forms in \( \Lambda^+_4(\mathbb{R}^4) \) by

\[
\omega_1 = bdx \wedge \sigma_1 + bcs_2 \wedge \sigma_3, \quad \omega_2 = bdx \wedge \sigma_2 + bcs_3 \wedge \sigma_1, \quad \omega_3 = cdx \wedge \sigma_3 + bcs_1 \wedge \sigma_2.
\]

According to [16], one can either look for hyperkähler metrics where the self dual 2-forms \( \{\omega_1,\omega_2,\omega_3\} \) are closed or consider metrics where the SU(2)-action rotates the triple. In the second case, one obtains the two ODEs:

\[
\frac{db}{dx} = \frac{c^2 - 2bc}{bc}, \quad \frac{dc}{dx} = -\frac{c^2}{b^2}.
\]

A Taub-NUT metric \( g_m \) on \( \mathbb{R}^4 \) is then defined to be an element in \( \mathcal{W}^m_4(\mathbb{R}^4) \) of the form

\[
g_m = dx \otimes dx + b^2 (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + c^2 \sigma_3 \otimes \sigma_3,
\]

where \( b \) and \( c \) solve the differential equations (19) (note that the sign in (19) has no geometric meaning). From (19) we also derive that \( g_m \) satisfies Assumption 1.

By direct computation one may check that the curvature of \( g_m \) decays as

\[
\lvert R_{g_m} \rvert g_m (x) = O(x^{-3}),
\]

as \( x \to \infty \) and that any Taub-NUT metric has cubic volume growth. Therefore, from Lemma [2.4] we derive that \( m \doteq \lim_{x \to \infty} c(x) \) is a positive quantity which we may use to parametrize this family.

3. Analysis of the Ricci flow equations

In the following we assume that \( g_0 \in \mathcal{W}^m_4(\mathbb{R}^4) \) is asymptotically flat and that the maximal solution \( g(t) \) to the Ricci flow starting at \( g_0 \) given by Lemma [2.3] develops a finite-time singularity at some \( T < \infty \); we emphasize that any estimate derived in this section does not require the initial metric to satisfy Assumption 1.

Similarly to the analyses in \([23]\) and \([24]\) (which are performed on the compact topologies of \( S^1 \times S^3 \) and \( S^4 \times S^2 \) respectively) we prove that away from the origin the Ricci flow is controlled by \( b \) (and not its derivatives): the formation of a singularity at some positive \( x \) (i.e. along the Euclidean
hypersphere of radius $x$) is equivalent to $b(x, t)$ converging to zero as $t \to T$. In fact, one can say more about the behaviour of the flow: around any singularity the solution becomes rotationally symmetric at some rate that breaks scale-invariance.

By the boundary conditions (2) and the asymptotics in Lemma 2.4 we can also extend some of the lower bounds for the sectional curvatures obtained in [23] to the $\mathbb{R}^4$-topology.

We first show that the flow is uniformly controlled outside some compact region.

For notational reasons we always let $\alpha$ denote a positive constant only depending on $g_0$ that may change from line to line.

**Lemma 3.1.** There exist $\alpha > 0$ and $\rho > 0$ such that
\[
\sup_{(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)} |\text{Rm}_{g(t)}|_{g(t)} \leq \alpha,
\]
with $B(0, \rho)$ the Euclidean ball of radius $\rho$.

**Proof.** According to Lemma 2.4 the injectivity radius of any asymptotically flat $g_0 \in \mathbb{W}(\mathbb{R}^4)$ is bounded away from zero; the result then follows from [8, Theorem 1.1]. □

**Remark 3.2.** We note that as a consequence to the previous Lemma the asymptotic behaviour of the solution to the Ricci flow persists in time; namely, conditions (11) and (15) are preserved along the flow once we replace the fixed radial coordinate $x$ with the geometric time dependent coordinate $s$.

### 3.1. First order estimates

We start by proving that $b_s$ and $c_s$ are uniformly bounded in the space-time. From the Ricci flow equations (13) and (14) and the formulas (15) and (10), we get (see also [23]):

\[
(b_s)_t = \Delta(b_s) - 2\frac{b_s}{b}(b_s)_s + \left(\frac{4}{b^2} - \frac{b^2}{b^2} - \frac{c^2}{c^2} - \frac{6c^2}{b^4}\right)b_s + 4c \frac{c}{b^3}c_s
\]

and

\[
(c_s)_t = \Delta(c_s) - 2\frac{c_s}{c}(c_s)_s - \left(\frac{6c^2}{b^4} + \frac{2b^2}{b^2}\right)c_s + 8\frac{c^3}{b^5}b_s.
\]

**Lemma 3.3.** There exists $\alpha > 0$ such that $|b_s| \leq \alpha$ and $|c_s| \leq \alpha$.

**Proof.** Consider the upper bound for $b_s$. By the boundary conditions (2) and Lemma 3.1 $b_s$ is uniformly controlled at the origin and at spatial infinity. Suppose that $(p_0, t_0) \in \mathbb{R}^4 \times (0, T)$ is a critical point for $b_s(\cdot, t_0)$ where $b_s = \alpha$ for the first time, for some $\alpha$ to be chosen below. Evaluating (20) at $(p_0, t_0)$ we get

\[
(b_s)(p_0, t_0) \leq \frac{1}{b^2}\left(4\alpha - \alpha^3 - \alpha c_s^2 - 6\alpha \left(\frac{c^2}{b^2}\right) + 4\frac{c}{b}c_s\right).
\]

By choosing $\alpha > \max\{\max|b_s|(\cdot, 0), 2\}$ works one easily verifies that $c_s$ - quadratic polynomial in the brackets does not admit roots, thus proving $b_s(p_0, t_0) < 0$.

The exact same argument works for the lower bound of $b_s$; since by Lemma 2.8 the ordering $c \leq b$ is preserved, one deals with the case of $c_s$ analogously. □

From the previous Lemma and the condition $c \leq b$ we immediately derive the following bounds for the vertical sectional curvatures.

**Corollary 3.4.** There exists $\alpha > 0$ such that
\[
|k_{12}| + |k_{13}| \leq \frac{\alpha}{c^2}
\]
on $(\mathbb{R}^4 \setminus \{0\}) \times [0, T)$.

The following estimate is a necessary step to prove that the solution to the Ricci flow becomes spherically symmetric at any singularity forming away from the origin.

**Lemma 3.5.** There exists $\alpha > 0$ such that
\[
\varphi \geq \frac{1}{b}(b_s^2 - 4) \leq \alpha.
\]
3.2. Rotational symmetry at any singular point.

Given $\rho > 0$ only depending on $g_0$ as in Lemma 3.1 there exists $\gamma > 0$ satisfying

$$\gamma = \frac{1}{\gamma} \leq \frac{c}{b} \leq 1$$

in $B(0, \rho) \times [0, T]$; for the function $c/b$ is uniformly bounded from below at $0$ and at $\rho$ and the evolution equation (13) prevents $c/b$ from attaining interior minima approaching zero. In the next Lemma we actually prove that if $c(x, t)$ (and hence $b(x, t)$) as we read from (23) is converging to zero as $t \to T$ (along some sequence of times) for some $x > 0$, then the metric must become rotationally symmetric at $x$.

Lemma 3.6. There exists a uniform constant $\alpha > 0$ such that

$$\frac{1}{\gamma} \leq \frac{b}{c} - 1 \leq \alpha.$$ 

Proof. Set $\phi \equiv (b/c - 1)/b$. We first observe that (2) ensures that $\phi$ is $C^2$-defined in $\mathbb{R}^4 \times [0, T)$ with $\phi(0, t) = 0$ as long as the flow exists$^2$. From Lemma 3.1 we also get that $\phi$ is bounded from above in $[0, T) \setminus B(0, \rho) \times [0, T]$. Let us introduce the quantity $\bar{\alpha} \equiv L_\gamma (\gamma - 1) + \max \phi(., 0)$, where $L$ and $\gamma$ ($> 1$) are chosen to satisfy the bounds in (22) and (23) respectively. Suppose that $(p_0, t_0) \in B(0, \rho) \times (0, T)$ is the space-time maximum point where $\phi$ attains the value $\bar{\alpha}$ for the first time. By direct computation we get

$$\phi_t = \Delta \phi + \frac{b^2}{c^3} - \frac{c^2}{b^2} + \frac{1}{b^3} (-4 - \frac{c}{b} + 2 \frac{c^2}{b^2}) \cdot$$

Evaluating the evolution equation at $(p_0, t_0)$ we have

$$\phi_t(p_0, t_0) \leq \frac{1}{b^3} \left( (L b + 4) \left( 1 - \frac{c}{b} \right) - 4 + \frac{2 c}{b} + 2 \frac{c^2}{b^2} \right).$$

Using (22), we can estimate the time derivative from above as

$$\phi_t(p_0, t_0) \leq \frac{1}{b^3} \left( (L b + 4) \left( 1 - \frac{c}{b} \right) - 4 + \frac{2 c}{b} + 2 \frac{c^2}{b^2} \right)$$

$$= \frac{1}{b^3} \left( -2 \frac{c}{b} + 2 \frac{c^2}{b^2} + L(b - c) \right)$$

$$= \frac{c}{b^3} \phi \left( -2 c + L b \right) \leq \frac{c}{b^3} \phi \left( -2 \gamma + L b \right) \equiv \frac{c}{b^3} \bar{\alpha} \left( -2 \frac{\gamma}{b} + L b \right).$$

\footnote{Namely, according to (2) we have $b/c - 1 = O(x^2)$ near the origin, for any $t \in [0, T)$.}
From the definition of $\phi$ we derive
\[
b \leq \frac{1}{\phi} (\gamma - 1) \equiv \frac{1}{\alpha} (\gamma - 1),
\]
which then yields
\[
\phi_t(p_0, t_0) \leq \frac{c}{b^4} \phi \left( -\frac{2}{\gamma} + \frac{L}{\alpha} (\gamma - 1) \right) \leq \frac{c}{b^4} \phi \left( -\frac{2}{\gamma} + \frac{1}{\gamma} \right) < 0.
\]
□

An analogous bound also holds for the first order spatial derivatives; namely, we have the following

**Lemma 3.7.** There exists $\alpha > 0$ such that
\[
\left| \frac{c_s}{c} \right| \frac{b_s}{b} \leq \alpha.
\]

**Proof.** Let us define $\psi \doteq c_s/c - b_s/b$; note that $\psi$ extends continuously to zero at the origin due to (2). By Lemma 3.1 and Lemma 3.3 $|\psi|$ is uniformly bounded in $\left(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)\right)$. Consider the upper bound; suppose that there exists a large value $\bar{\alpha}$ which $\psi$ attains for the first time at some maximum space-time point $(p_0, t_0) \in B(0, \rho) \times (0, T)$. The evolution equation of $\psi$ is
\[
\psi_t = \Delta \psi - \psi \left( \frac{c^2}{c^2} + 8 \frac{c^2}{b^2} + 2 \frac{b^2}{b^2} \right) - \frac{b_s}{b^2} \left( 1 - \frac{c^2}{b^2} \right).
\]

We can then evaluate both sides at $(p_0, t_0)$ and use Lemma 3.6 to get
\[
\psi_t(p_0, t_0) \leq \frac{1}{b^2} \left( -\bar{\alpha} \left( \frac{b^2 c^2}{c^2} + 8 \frac{c^2}{b^2} + 2 \frac{b^2}{b^2} \right) + 8 \frac{c b_s}{b} \left( 1 - \frac{b}{c} - 1 \right) \right)
\leq \frac{8}{b^2} \left( -\bar{\alpha} \frac{c^2}{b^2} + 2 \alpha |b_s| \right) \leq \frac{8}{b^2} \left( -\bar{\alpha} + 2 \alpha \right) < 0,
\]
for $\bar{\alpha}$ sufficiently large, with $\gamma$ defined as in (2). The same argument shows the existence of a uniform lower bound. □

The control at any point in the space-time where $c$ (and thus $b$) is sufficiently small provided by Lemma 3.6 and Lemma 3.7 enables us to improve the estimate (22).

**Lemma 3.8.** There exists a uniform constant $\alpha > 0$ such that
\[
\frac{1}{b} \left( b_s^2 - 1 \right) \leq \alpha.
\]

**Proof.** Let us denote $(b_s^2 - 1)/b$ by $\varphi$. A direct computation gives
\[
\varphi_t = \Delta \varphi - \frac{2 b_s^2}{b} - 3 \frac{b_s}{b^2} + \frac{b^2}{b^3} \left( 13 - 14 \frac{c^2}{b^2} \right) - 2 \frac{b_s c^2}{b c^2} + 8 \frac{c b_s c_s}{b^2} - \frac{4}{b^2} + \frac{2 c^2}{b^2}.
\]

Since $\varphi$ is uniformly bounded at the origin and in $\left(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)\right)$, we let $(p_0, t_0) \in B(0, \rho) \times (0, T)$ be the maximum space-time point where $\varphi = \bar{\alpha}$ for the first time, for some positive $\bar{\alpha}$ to be chosen below. We have
\[
\varphi_t(p_0, t_0) \leq \frac{1}{b^2} \left( -\frac{2 b_s c^2}{c^2} + 8 \frac{c b_s c_s}{b^2} - \frac{7 b_s^2}{2 b} + \frac{14 b_s}{b^2} \left( 1 - \frac{c_s}{b^2} \right) - \frac{9}{2 b} + \frac{2 c^2}{b^2} \right).
\]

Without loss of generality we assume that $b_s$ and $c_s$ are positive. According to Lemma 3.7 we can bound $c_s$ in terms of $b_s$; it follows that there exists some positive constant $\alpha > 0$ such that
\[
\varphi_t(p_0, t_0) \leq \frac{1}{b^2} \left( -\frac{11 b_s^2}{2 b} + \frac{b_s}{b} \left( 14 - 6 \frac{c_s}{b^2} \right) - \frac{9}{2 b} + \frac{2 c^2}{b^2} \right)
\leq \frac{1}{2 b^2} \left( -\frac{11 b_s^2}{b} (b_s^2 - 1) + \frac{5}{b} (b_s^2 - 1) + \frac{12 b_s^2}{b} (1 - \frac{c_s}{b^2}) - \frac{4}{b} (1 - \frac{c^2}{b^2}) \right).
\]

We finally use Lemma 3.3, Lemma 3.6 and the fact that $\varphi > 0$ implies $b_s^2 > 1$ to derive
\[
\varphi_t(p_0, t_0) \leq \frac{1}{2 b^2} \left( \alpha - 6 \bar{\alpha} \right) < 0,
\]
for $\bar{\alpha}$ large enough.

We can reformulate the previous estimates in a more geometric way: there exist lower bounds for the vertical sectional curvatures that break scale-invariance.

**Corollary 3.9.** There exists a uniform constant $\alpha > 0$ such that

$$k_{12} \geq -\frac{\alpha}{b}, \quad k_{13} \geq -\frac{\alpha}{b^2}.$$  

3.3. **Second order estimates.** In this subsection we extend the previous arguments to the second spatial derivatives. We show that away from the origin the mixed sectional curvatures are controlled by $b$ as in Corollary 3.3 any singular radius $x$ for the flow is characterised by the property that $c(x,t)$ and $b(x,t)$ converge to zero as $t$ approaches $T$.

We then prove that the metric becomes rotationally symmetric at the second order at any singular point. Finally, we obtain lower bounds for the mixed sectional curvatures that break scale-invariance.

Throughout this subsection we use the boundary conditions, the uniform control at infinity given by Perelman’s Pseudolocality Theorem, and the bounds already derived to adapt the proofs in [23] and [24] to our setting.

We preliminary prove that the full curvature is indeed controlled by $b$ outside $B(0, \rho)$, with $\rho$ given by Lemma 3.11.

**Lemma 3.10.** Let $g_0 \in \mathfrak{W}(\mathbb{R}^4)$ satisfy the asymptotic flatness condition for some $\epsilon > 0$. There exists $M > 0$ such that

$$\sup_{(\mathbb{R}^4 \setminus B(0, \rho)) \times [0,T)} b^{2+\epsilon} |Rm_{g(t)}|_{g(t)} \leq M.$$  

**Proof.** Define $f \doteq b^\nu |Rm|^2$, with $\nu \doteq 2(2 + \epsilon/2)$; since by Lemma 2.8 the rate of decay of the curvature persists in time, $f$ converges to zero at infinity for any $t \in [0,T)$. By Lemma 3.1 we also derive that there exists $A > 0$ such that $f(\rho, t) \leq A < \infty$ in $[0,T)$.

The evolution equation of $f$ in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0,T)$ is given by

$$f_t \leq \Delta f - 4(\nu b^{\nu-1} b_\nu |Rm|)(|Rm|)_\nu - \nu^2 b^{\nu-2} b_\nu^2 |Rm|^2 + 2\nu b^{\nu-2} |Rm|^2 \left(\frac{c^2}{b^2} - 2\right) + 16b^\nu |Rm|^3,$$

where we have used a standard estimate for the evolution of the curvature along the Ricci flow (see, e.g., [13]). Suppose that $f$ attains a maximum at a space-time point $(p_0, t_0)$ belonging to the interior of $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0,T)$; then there exists a positive constant $\alpha$ satisfying

$$f_t(p_0, t_0) \leq f \left(16|Rm| + \frac{1}{b^2} (\nu^2 - 2\nu)\right) \leq \alpha f,$$

where the last inequality follows from Lemma 3.3 and the uniform control of the curvature in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0,T)$. Since $f$ is bounded from above along the parabolic boundary, a standard application of the Maximum Principle allows us to conclude that $f$ must stay bounded on $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0,T)$. \hfill \Box

The evolution equations of the mixed sectional curvatures (8) and (9) are given by

$$\begin{align*}
(k_{01})_t &= \Delta(k_{01}) + 2k_{01}^2 + k_{01} \left(\frac{8c^2}{b^2} - \frac{8c^2}{b^4} - \frac{2c^2}{c^2} - \frac{4b^2}{b^2}\right) + k_{03} \left(\frac{4c^2}{b^4} - \frac{2b_c c_s}{bc}\right) \\
&= -\frac{4c^2}{b^4} + \frac{24b_c c_s}{b^5} - \frac{2b_c c_s}{bc^3} + 24c^2 b_s^2 + \frac{8b_c c_s}{b^4} + \frac{2b_c c_s}{b^4}
\end{align*}$$

(24)

and

$$\begin{align*}
(k_{03})_t &= \Delta(k_{03}) + 2k_{03}^2 - 4k_{03} \left(\frac{b_c c_s}{b^2} + \frac{c^2}{b^3}\right) + 4k_{01} \left(\frac{2c^2}{b^2} - \frac{b_c c_s}{bc}\right) \\
&= \frac{12c^2}{b^4} + \frac{40c^2 b_c}{b^6} - \frac{48b_c c_s}{b^5} - \frac{4b_c c_s}{b^3 c}
\end{align*}$$

(25)

**Lemma 3.11.** There exists $\alpha > 0$ such that

$$|k_{01}| + |k_{03}| \leq \frac{\alpha}{b^2}.$$
Proof. We show that $bb_{ss} \equiv -b^2 k_{01}$ admits a uniform lower bound. In the following we prove that Lemma 3.10 and the boundary conditions [2] enable us to make the exact same proof in [21] work in our setting too. Define the map $\psi \equiv b_{ss} - \mu b^2 - \nu c^2$, for some $\mu$ and $\nu$ positive constants we will determine below. Then at any stationary point of $\psi(t, t)$ we have (see also [21])

$$
\psi_t = \Delta \psi - bb_{ss} \left( \frac{4c^2}{b^4} + \frac{2c^2}{c^2} + \frac{4b^2}{b^2} \right) - \frac{16\nu c^3 b c_s}{b^2} - (24 + 8\mu) \frac{cb c_s}{b^3} + \frac{2b b_{ss} c^3}{c^3} \\
- \frac{8b^2(\mu + 1)}{b^2} + \frac{4cc_{ss}}{b^2} + \frac{4\nu c^2 c_{ss}}{c} - \frac{8b c c_{ss}}{b^2} - \frac{2bb_{ss} c c_{ss}}{b^2} + \frac{12\nu c^2 c}{b^4} + \frac{4c^2}{b^2} \\
+ \frac{24c^2 b^2}{b^4} + \frac{12\nu c^2 b^2}{b^4} + \frac{2b b^2 c^2}{b^2} + \frac{4b b c^2}{b^2} + \frac{2b^4}{b^2} + \frac{2\nu c^2}{b^2} + 2(\mu - 1)b^2.
$$

Suppose $\psi$ attains some negative value $-\bar{\alpha}$ for the first time at $t_0 \in (0, T)$; since $\psi(0, t) = -\mu - \nu$ for $0 \leq t < T$ and according to Lemma 3.10 and Lemma 3.11 $\psi$ is uniformly bounded in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$, we derive that $\psi(p_0, t_0) = -\bar{\alpha}$ at some minimum point $p_0 \in B(0, \rho)$. By Lemma 3.8 we can choose $\bar{\alpha}$ sufficiently large such that $bb_{ss} \leq -\bar{\alpha}/4$; therefore we get

$$
-bb_{ss} \left( \frac{4c^2}{b^4} + \frac{2c^2}{c^2} + \frac{4b^2}{b^2} \right) \geq \bar{\alpha} \left( \frac{c^2}{2b^4} + \frac{c^2}{2c^2} + \frac{\mu b^2}{b^4} \right).
$$

Evaluating the evolution equation of $\psi$ at $(p_0, t_0)$, we have (provided we set $\mu \geq 1$)

$$
\psi_t(p_0, t_0) \geq 2\nu c^2 + \bar{\alpha} \left( \frac{c^2}{2b^4} + \frac{c^2}{2c^2} + \frac{\mu b^2}{b^4} \right) - \frac{16\nu c^3 b c_s}{b^2} - (24 + 8\mu) \frac{cb c_s}{b^3} \\
+ \frac{2b b_{ss} c^3}{c^3} - \frac{8b^2(\mu + 1)}{b^2} + \frac{4cc_{ss}}{b^2} + \frac{4\nu c^2 c_{ss}}{c} - \frac{8b c c_{ss}}{b^2} - \frac{2bb_{ss} c c_{ss}}{b^2}.
$$

One can then estimate the remaining terms exactly as in [21] by using the uniform boundedness of the first spatial derivatives and (23). Namely, we can use the weighted Cauchy-Schwarz inequality to get

$$
\left| \frac{2b b_{ss} c c_{ss}}{c^2} \right| \leq \frac{c^2}{2} + 2 \frac{b^2 c^2 c_{ss}}{c^4} \leq \frac{c^2}{2} + 2 \frac{\alpha^2 b^2 c^2}{c^4} \leq \frac{c^2}{2} + 2 \frac{\alpha^2 - \gamma c^2}{c^2},
$$

and similarly for the others. Therefore there exists a uniform constant $\alpha$ such that the time derivative at $(p_0, t_0)$ is bounded from below as

$$
\psi_t(p_0, t_0) \geq (\nu - 1)c^2 + \bar{\alpha} \left( \frac{c^2}{2b^4} + \frac{c^2}{2c^2} + \frac{\mu b^2}{b^4} \right) - \alpha(\mu + \nu + 1) \left( \frac{c^2}{2b^4} + \frac{c^2}{2c^2} + \frac{\mu b^2}{b^4} \right) > 0,
$$

once we choose $\mu = 1, \nu = 2$ and $\bar{\alpha}$ sufficiently large. The existence of a uniform upper bound follows from a similar argument by considering $\psi \equiv bb_{ss} + \mu b^2 + \nu c^2$. The very same proof applies for $k_{03}$.

We can prove that both $c$ and $b$ admit limits as the flow approaches its maximal time of existence.

**Corollary 3.12.** For any $x \geq 0$ the limits $\lim_{t \to T} c(x, t)$ and $\lim_{t \to T} b(x, t)$ exist finite.

**Proof.** Using Lemmas 3.8 and 3.11 we get

$$
|c^2(t)| \leq |cc_{ss}| + 4 \left| \frac{c}{b} b c_s - 4 \frac{c^2}{b^2} \right| \leq \alpha.
$$

The same argument works for $b$ as well. \(\square\)

The curvature is thus uniformly controlled along any Euclidean hypersphere where the components $b$ and $c$ do not converge to zero as $t \nearrow T$; namely, from Corollary 3.3 Lemma 3.11 and Lemma 3.7 applied to (11) we derive that there exists a positive constant $\alpha$ such that

\[
(26) \quad \sup_{\mathbb{R}^4 \times [0, T)} b^2 |\text{Rm}| \leq \alpha.
\]

The following bound shows that around a singularity the solution becomes spherically symmetric at the order 2 at a rate that breaks scale invariance.
Lemma 3.13. There exists $\alpha > 0$ such that
\[ |k_{01} - k_{03}| \leq \frac{\alpha}{b}. \]

Proof. We adapt the proof in [23], whose argument works in the Type-I Ricci flow setting. Once we define $\psi \equiv c_s/c - b_s/b$, we consider the map
\[ \phi \equiv (\psi, b)^2 \equiv \left( \frac{c_{ss}}{c} - \frac{c_s^2}{c^2} - \frac{b_{ss}}{b} + \frac{b_s^2}{b^2} \right)^2 b^2. \]

The boundary conditions (2) ensure that $\phi(0, t) = 0$ for any $t$. From Lemma 3.12 we derive that any sufficiently large value $\alpha$ is attained by $\phi$ for some first time $t_0$ at an interior point $p_0 \in B(0, \rho)$. The evolution equation for $\phi$ is
\[ \phi_t = \Delta \phi - 2(b^2)_{ss}(\psi^2)_{ss} - 2b^2 \psi_{ss} - 2b^2 \psi \psi_{ss} + 2Hb^2 \psi_s - 4b^2 \psi_{ss} - 8\psi^2 + 4\psi^2 \frac{c^2}{b^2}, \]
where
\[ F \equiv 4\frac{b^2}{b^2} + 2\frac{c^2}{c^2} + 8\frac{c^2}{b^2}, \quad G \equiv \left( \frac{2b^2}{b^2} + \frac{c^2}{c^2} + 8\frac{c^2}{b^2} \right), \quad H \equiv - \left( 8\frac{b^2}{b^2} \left( 1 - \frac{c^2}{b^2} \right) \right). \]

Evaluating $\phi$ at the maximum point $(p_0, t_0)$ we get
\[ \phi_t(p_0, t_0) \leq 2b^2 \psi^2 - \psi^2 \left( 8 - 4\frac{c^2}{b^2} \right) - 2b^2 \phi - 2Gb^2 \psi_s + 2Hb^2 \psi_s \]
\leq \psi_{ss}^2 \left( 8 + 12\frac{c^2}{b^2} \right) - 2Gb^2 \psi_s + 2Hb^2 \psi_s.

From (26) it easily follows that there exists some uniform constant $\alpha > 0$ such that $|G| \leq \alpha/b^2$; being $\psi$ uniformly bounded in the space-time (Lemma 3.7), we have

\[ -2Gb^2 \psi_s \leq \alpha |\psi_s|^2. \]

According to Lemma 3.10 and Lemma 3.7 an analogous estimate can be found for $|Hb^2 \psi_s|$. Then
\[ \phi_t(p_0, t_0) \leq \psi_{ss}^2 \left( 8 + 12\frac{c^2}{b^2} \right) + \alpha \frac{|\psi_s|}{b} \left( \left( -8 - 12\frac{c^2}{b^2} \right) \sqrt{\alpha} + \alpha \right) < 0, \]
for $\alpha$ sufficiently large. Therefore $\phi$ is uniformly bounded in the space-time; in $B(0, \rho) \times [0, T)$ we then have
\[ b|k_{01} - k_{03}| \leq \alpha + \left| \frac{b^2}{b} - \frac{b_s^2}{b^2} \right| \leq \alpha + \left| b_s \right| \left( \frac{b_s}{b} - \frac{c_s}{c} \right) + |c_s| \frac{b_s}{b} \left( \frac{b_s}{b} - \frac{c_s}{c} \right) \leq \alpha \left( 1 + \gamma \right), \]
where we have used Lemma 3.7 and the bound (23). Since from Lemma 3.10 it follows that $b|k_{01} - k_{03}|$ is uniformly bounded in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$, the proof is complete.

We finally discuss the existence of lower bounds for the mixed sectional curvatures; the next estimate follows by adapting the analogous argument in [23].

Lemma 3.14. There exists $\alpha > 0$ such that
\[ c_{ss}c \log c \geq -\alpha, \quad b_{ss}b \log b \geq -\alpha. \]

Proof. Consider the map $f \equiv c_{ss}c \log c$, which is smooth in $\mathbb{R}^4 \setminus \{0\} \times [0, T)$; moreover $f$ extends continuously to zero and $f(0, t) = 0$ as long as a solution exists. By Lemma 3.10 we get that $f$ is controlled in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$. Let $(p_0, t_0) \in B(0, \rho) \times (0, T)$ be the first minimum point where $f = -\alpha$ for some large $\bar{\alpha} > 0$. From (9) and (26) it follows
\[ \bar{\alpha} = \left| f(p_0, t_0) \right| \leq \alpha |\log c(p_0, t_0)|. \]

Since by Corollary 3.12 $c$ is uniformly bounded from above in $B(0, \rho) \times [0, T)$, the last inequality implies
\[ c_{ss}(p_0, t_0) = \frac{-\bar{\alpha}}{c|\log c|(p_0, t_0)} \geq \bar{\alpha}, \]

(27)
Evaluating the evolution equation of $f_t$ in the following the signs of $b$ and $c$ are not relevant; we assume without loss of generality that $b > 0$ and $c > 0$. By Lemma 3.6 we have

$$-8 \frac{c^2}{b^4} \log c \left( \frac{c_{ss}}{c} - \frac{b_{ss}}{b} \right) \geq -\alpha \frac{\log c}{c},$$

where we have used Lemma 3.6 and Lemma 3.8 to derive the last inequality. According to Lemma 3.6, Lemma 3.7, and Lemma 3.13 yield the bounds

$$12 \frac{c^2}{b^4} - 48 \frac{c^2}{b^6} b_{ss} + 40 \frac{c^3}{b^4} b_{ss}^2 - 4 \frac{b^3}{b^4} c_{ss} \geq \frac{4}{b^4} \left( -\alpha - b^2 \frac{c}{b^2} \left( b^2 - \frac{c^2}{b^2} \right) \right) \geq -\frac{\alpha}{c^2},$$

for sufficiently large (i.e. $c$ small enough). The case of $b_{ss} \log b$ does not require modifications.

We can rephrase the previous quantitative estimates in a more geometric way.

**Corollary 3.15.** In the region where $b < 1$

$$b^2 k_0 \geq -\frac{\alpha}{\log b}, \quad c^2 k_0 \geq -\frac{\alpha}{\log c}.$$

4. **Ricci flow without necks**

In this section we focus the analysis on the family of asymptotically flat warped Berger metrics in $\mathcal{W}(\mathbb{R}^4)$ satisfying Assumption 1: we demonstrate that in this case the spatial derivative $b_s(\cdot, t)$ admits a uniform positive lower bound in $\mathbb{R}^4 \times [0, T]$.

We first prove that this class is preserved along the flow. To this aim, we introduce the quantity $H \doteq 2b_s + c_s : \mathbb{R}^4 \setminus \{0\} \times [0, T) \to \mathbb{R}$, representing the mean curvature of the embedded hyperspheres in $(\mathbb{R}^4, g(t))$; the evolution equation of $H$ is given by

$$H_t = \Delta H + \frac{1}{b^2} \left( H \left( 4 \frac{c^2}{b^2} - 2b^2 - 2 \frac{b^2}{c^2} \right) + 10 \frac{b_s}{b} \left( 1 - \frac{c^2}{b^2} \right) \right).$$

**Lemma 4.1.** Let $g_0 \in \mathcal{W}(\mathbb{R}^4)$ be asymptotically flat and satisfy Assumption 1. Then the maximal solution $g(t)$ to the Ricci flow starting at $g_0$ satisfies Assumption 1 for any $t \in [0, T)$. Namely, it holds:

$$H(\cdot, t) > 0, \quad b_s(\cdot, t) > 0, \quad \forall t \in [0, T).$$
Proof. From Lemma 3.10 it follows that there exists $M > 0$ such that

$$b^{-2} M \geq |1 - b^2|$$

in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$. According to the asymptotics (4) and (5) we can then choose $\rho > 0$ sufficiently large such that $b(x, t) \geq \alpha > 0$ for any $x \geq \rho$ and for any $t \in [0, T)$, with $\alpha$ some positive constant. Similarly, in the Euclidean volume growth case one can take $\rho > 0$ large enough such that an analogous uniform lower bound for $c_s$ can be found; in the cubic volume growth case the estimate

$$b^{-2} M \geq \left| \frac{c^2}{b^2} - \frac{c_s b_0}{c} \right|$$

in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$, implies that for any small $\delta > 0$ there exists $\rho > 0$ such that $|c_s b/c|(x, t) \leq \delta$ for $x \geq \rho$ and $t \in [0, T)$. We can thus choose $\rho > 0$ in Lemma 3.1 only depending on the initial metric $g_0$ such that $bH \geq \alpha > 0$ and $b_s \geq \alpha > 0$ in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$, with $\alpha = \alpha(g_0) > 0$ a constant. Therefore, since both $H$ and $b_s$ are also uniformly bounded from below at the origin, if Assumption 1 fails to be preserved along the Ricci flow, then there exist two space-time minima $(p_0, t_0), (p_1, t_1) \in B(0, \rho) \times (0, T)$ for $H$ and $b_s$ respectively, such that $H(p_0, t_0) = 0$ and $b_s(p_1, t_1) = 0$ for the first time. From (20) and (25) we derive

$$(b_s)(p_1, t_1) \geq 4 \frac{c}{b^2} c_s(p_1, t_1), \quad H_t(p_0, t_0) \geq 16 \frac{b_s}{b} \left( 1 - \frac{c^2}{b^2} \right) (p_0, t_0),$$

which then imply $H(p_1, t_1) = (c_s/c)(p_1, t_1) = 0$ and $b_s(p_0, t_0) = 0$; thus we only need to consider the case $(p_0, t_0) = (p_1, t_1)$.

We note that $(b_s)^{(j)}(s(p_0, t_0), t_0) = (c_s)^{(j)}(s(p_0, t_0), t_0) = 0$, for $j = 0, 1, 2$. Moreover, from the real analyticity of solutions to the Ricci flow equations (4) (and the boundary conditions (2)) we derive that there exist two even integers $n > 2$ and $\bar{n} > 2$ such that

$$\begin{align*}
(b_s)^{(j)}(p_0, t_0), t_0) &= 0, \quad j < n, \\
(b_s)^{(n)}(p_0, t_0), t_0) &> 0 \\
(c_s)^{(j)}(s(p_0, t_0), t_0) &= 0, \quad j < \bar{n}, \\
(c_s)^{(\bar{n})}(s(p_0, t_0), t_0) &> 0.
\end{align*}$$

Suppose that $n \leq \bar{n}$. Then

$$\partial_s^j((b_s)(s(p_0, t_0), t_0)), t_0) = 0, \quad j < n - 2, \quad \partial_s^{n-2}(b_s)(s(p_0, t_0), t_0) = b_s^{(n)}(s(p_0, t_0), t_0) > 0.$$ 

It follows that there exists a space-time neighbourhood $U$ of $(p_0, t_0)$ where $b_s > 0$ in $U \setminus \{(p_0, t_0)\}$; by applying l'Hôpital rule (with respect to both time and space variables) we get that $c_s/bs$ is a continuous function in $U$. We can then rewrite the evolution equation of $b_s$ in $U$ as

$$(b_s)_t = (b_s)_s + F(b_s)_s + b_s \left( G + 4 \frac{c}{b^2} c_s \right),$$

with $F$ and $G$ smooth in $U$ (we can always choose a space-time neighbourhood $U$ that stays away from the origin for some short time); Theorem D in [11] guarantees that $b_s$ cannot vanish in the interior of $U$, which is a contradiction. The case $n > \bar{n}$ only requires to adapt the argument above to the function $b_s/H$. 

Before we address the existence of a uniform positive lower bound for $b_s$ we show that at any singular hypersphere for the flow the solution becomes rotationally symmetric at a rate faster than the one derived in Lemma 3.6 and Lemma 3.7. We note that the following sharper estimates are in fact valid without requiring the Assumption 1 to hold at time 0.

**Lemma 4.2.** For any $k \in [0, 1)$ there exists $\alpha_k > 0$ such that:

$$\frac{1}{b^{1+k}} \left( \frac{b}{c} - 1 \right) \leq \alpha_k.$$

**Proof.** Let $f \equiv 1/b^{1+k}(b/c - 1)$ for some $0 < k < 1$. The boundary conditions (2), the control at infinity (4), (5) and Lemma 3.1 ensure that $f$ is uniformly bounded in $(\mathbb{R}^4 \setminus B(0, \rho)) \times [0, T)$. A
simple computation yields
\[ f_t = \Delta f + \frac{1}{b^2 + \epsilon C} \left\{ b_s^2 \left(-k^2 + \frac{c}{b} (k + 1)^2 - \frac{c^2 b^2}{c^2} - 2bksc_b \right) + \frac{1}{b^2 + \epsilon C} \left\{ 4k + \frac{c}{b} \left(-4(k + 1) - \frac{c^2 b^2}{b^2} + 2(k + 1)\frac{c^2}{b^2}(k + 1)\right) \right\} \]
Evaluating the previous evolution equation at any interior maximum point \((p_0, t_0) \in B(0, \rho) \times (0, T)\) we get
\[ f_t(p_0, t_0) \leq \frac{1}{b^2 + 2\epsilon C} \left\{ \left( \frac{b}{c} - 1 \right) \left(b^2_s(k + 1)^2\frac{c^2}{b^2} + \frac{c}{b} \left(4k - 4\frac{c}{b} - \frac{2c^2}{b^2}(k + 1)\right) \right) \right\} \]
According to Lemma 3.6 and Lemma 3.7 we can bound the right hand side from above as
\[ f_t(p_0, t_0) \leq \frac{1}{b^2 + \epsilon C} \left\{ \left( \frac{b}{c} - 1 \right) \left(k^2 + 4k - 5 + ab\right) \right\} , \]
for some uniform constant \(\alpha > 0\). For any fixed \(k \in (0, 1)\) we can then take \(\alpha_k\) sufficiently large (thus \(b\) sufficiently small due to (23)) so that \(f_t(p_0, t_0) < 0\). □

Arguing as in the proofs of Lemma 3.7 and Lemma 3.8 respectively, we obtain analogous first order improved estimates.

**Corollary 4.3.** For any \(k \in [0, 1]\) there exists \(\alpha_k > 0\) such that:
\[ \frac{1}{b^2} \left[ \frac{c_s}{c} - \frac{b_s}{b} \right] \leq \alpha_k, \quad \frac{1}{b^2 + \epsilon C} \left( b_s^2 - 1 \right) \leq \alpha_k. \]
We can now prove that the mean curvature of any embedded hypersphere is bounded away from zero as long as the radius stays finite.

**Lemma 4.4.** Let \(g_0 \in \mathcal{M}(\mathbb{R}^4)\) be asymptotically flat and satisfy Assumption 1. Then there exists \(\xi > 0\) such that \(H \geq \xi\) in \(B(0, \rho) \setminus \{0\} \times [0, T)\).

**Proof.** We have already seen in the proof of Lemma 4.1 that \(H\) is bounded from below by a positive constant \(\delta = \delta(g_0)\) along the parabolic boundary of \((B(0, \rho) \setminus \{0\}) \times [0, T)\). Let us choose \(n\) positive integer such that it holds
\[ \xi \geq \frac{1}{n} \min \{ \min_{B(0, \rho)} H(\cdot, 0, \delta) \} \geq \left( \frac{\max_{B(0, \rho) \times (0, T)} b}{\min_{B(0, \rho)} b} \right) - 1 \sqrt{\frac{2}{\gamma}}, \]
with \(\gamma\) defined as in (23). Suppose \(H\) attains a minimum value \(\xi \leq \xi\) at some positive time \(t_0\); then \(\xi = H_{\min}(t_0) = H(x_0, t_0)\) for some \(x_0 \in (0, \rho)\). From (23) we get
\[ H_t(x_0, t_0) \geq \frac{H}{b^2} \left( 4\frac{c^2}{b^2} - \frac{2c^2}{b^2}(\bar{b}^2 - c^2) \right) , \]
where we have also used that Lemma 4.1 implies that \(b > 0\) as long as the solution exists. From Lemma 3.7 and the definition of \(\xi\) it follows that
\[ H_t(x_0, t_0) \geq \frac{H}{b^2} \left( 4\frac{c^2}{b^2} - (b\bar{H})^2 + 6b^2 - ab_b \right) \geq \frac{H}{b^2} \left( \frac{2}{\gamma} - ab_b \right) . \]
Being \(b_s\) uniformly controlled along the flow, for the right hand side to be negative \(b\) must be bounded away from zero by some positive quantity \(\nu > 0\) only depending on \(g_0\); we conclude that as long as \(H \leq \xi\) the minimum of \(H\) in space is a Lipschitz function of time satisfying the differential inequality
\[ H_{\min}(t) \geq -\nu H_{\min}(t), \]
for some \(\nu > 0\). The Maximum Principle then implies that \(H\) cannot approach zero in the interior of \(B(0, \rho)\) as \(t \to T\). □

In the next result we demonstrate that at any radius where the flow develops a singularity the spatial derivative \(b_s\) cannot converge to zero faster than \(O(|\log(b)|^{-1})\).
Claim 4.6. There exists $x \exists \frac{b}{\nu} \leq 2 \frac{\nu}{\nu} \geq b \geq \delta > 0$ such that

$\nu > 0$ and $\delta > 0$ such that

uniformly in $(B(0, \rho) \setminus \{0\}) \times [0, T)$.

Proof. We first show that the rate of convergence of $b_\nu$ to zero is controlled by $b$.

Claim 4.6. There exists $\alpha > 0$ such that

in $(B(0, \rho) \setminus \{0\}) \times [0, T)$.}

Proof of Claim 4.6 Define $f \doteq b_\nu/b$ in $(B(0, \rho) \setminus \{0\}) \times [0, T)$. Once again we only need to show the existence of a lower bound on the interior of the space-time region. At any minimum point $(x_0, t_0)$ we have

$\nu > 0$ as in the statement of Lemma 4.5 follows from Corollary 3.12. Set

Lemma 3.3 and equation (23) finally yield

consider the value

by

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for some uniform constant $\alpha > 0$; therefore for $f$ sufficiently small the right hand side is strictly positive.

The existence of $\nu > 0$ as in the statement of Lemma 4.5 follows from Corollary 3.12. Set

$\phi \doteq \frac{|\log \left( \frac{b}{\nu} \right) |}{b \nu}$; the boundary conditions 2 and Lemma 3.10 imply that $\phi$ is uniformly bounded from below along the parabolic boundary of $(B(0, \rho) \setminus \{0\}) \times [0, T)$. The evolution of $\phi$ is given by

Consider the value

$\bar{\varepsilon} \doteq \frac{1}{n} \min \left\{ \min_{B(0, \rho) \setminus \{0\}} \phi(\cdot, 0), \inf_{t \in [0, T]} \phi(\rho, t) \right\}$,

with $n$ some positive integer to be chosen below. If $\phi < \bar{\varepsilon}$ at some positive time $t_0$ then there exists $x_0 \in (0, \rho)$ such that $\phi_{\min}(t_0) = \phi(x_0, t_0)$ and

$\phi_t(x_0, t_0) \geq \phi \left( 4 - b_\nu^2 - \frac{b_\nu^2}{\nu^2} - 6 \frac{c_\nu^2}{b_\nu^2} \right) + \frac{4c_\nu |\log \left( \frac{b}{\nu} \right) | c_\nu}{b_\nu^2} + \frac{2b_\nu}{b_\nu^2} \left( 2 - \frac{c_\nu^2}{b_\nu^2} \right)$.

By choosing $n$ sufficiently large we can assume that $b_\nu^2 \leq 1/4$ at $(x_0, t_0)$. Moreover, from the sharper estimates in Lemma 4.2 and Corollary 4.3 we derive

$\phi_t(x_0, t_0) \geq \frac{|\log \left( \frac{b}{\nu} \right) |}{b_\nu^2} \left( 2b_\nu(1 - b_\nu^2) - \alpha_\nu b^{k+1} \right) \geq \frac{|\log \left( \frac{b}{\nu} \right) |}{b_\nu^2} \left( \frac{3}{2} b_\nu - \alpha_\nu b^{k+1} \right)$,

for some uniform constant $\alpha_\nu > 0$, with $k \in (0, 1)$. According to Claim 4.6 we can estimate the right hand side by

$\phi_t(x_0, t_0) \geq \frac{|\log \left( \frac{b}{\nu} \right) |}{b_\nu^2} \left( \frac{3}{2} b_\nu - \alpha_\nu b^{k+1} \right) \phi \left( \frac{3}{2} b_\nu - \alpha_\nu b^{k+1} \right)$.

For the right hand side to be negative we must have $2\alpha_\nu b^{k+1} \geq 3$, which then implies

$\phi_t(x_0, t_0) \geq -\alpha_\nu \frac{\phi}{b_\nu^{k+1}} \geq -\alpha_\nu \phi$.

Arguing as in the proof of Lemma 4.3 we conclude that $\phi$ is bounded from below by a uniform positive quantity in the interior of $(B(0, \rho) \setminus \{0\}) \times [0, T)$. \qed

\begin{align*}
\phi_t(x_0, t_0) & \geq \frac{|\log \left( \frac{b}{\nu} \right) |}{b_\nu^2} \left( 2b_\nu(1 - b_\nu^2) - \alpha_\nu b^{k+1} \right) \\
& \geq \frac{|\log \left( \frac{b}{\nu} \right) |}{b_\nu^2} \left( \frac{3}{2} b_\nu - \alpha_\nu b^{k+1} \right) \\
& \geq -\alpha_\nu \frac{\phi}{b_\nu^{k+1}} \geq -\alpha_\nu \phi.
\end{align*}
We can finally show that along any singular hypersphere for the flow the spatial derivative $b_s$ stays away from zero at a rate that breaks scale invariance.

**Lemma 4.7.** Let $g_0 \in \mathcal{M}^2(\mathbb{R}^4)$ be asymptotically flat and satisfy Assumption 1. There exist $\nu > 0$, $\alpha > 0$ and $\beta > 0$ such that

$$\frac{b}{\nu} \leq \frac{2}{3}, \quad \left| \log \left( \frac{b}{\nu} \right) \right| (b_s - \beta) \geq -\alpha > -\infty,$$

uniformly in $(B(0, \rho) \setminus \{0\}) \times [0, T]$.

**Proof.** We define $\psi = \log \left( \frac{b}{\nu} \right) (b_s - \beta)$ on $(B(0, \rho) \setminus \{0\}) \times [0, T)$; we also choose

$$\psi < \min \{ \inf_{t \in [0, T)} b_s(\rho, t), \min_{B(0, \rho)} b_s(\cdot, 0) \}.$$

Then $\psi$ is bounded from below uniformly along the parabolic boundary of its space-time domain. Suppose that $\psi$ attains a negative value $-\alpha$ at some interior minimum point $(x_0, t_0)$ for the first time; a straightforward computation gives

$$\psi_t(x_0, t_0) \geq \frac{\log \left( \frac{b}{\nu} \right)}{b^2} \left( b_s \left( 4 - b_s^2 - c_s^2 \frac{b^2}{c^2} - 6 \frac{c^2}{b^2} \right) + 4 \frac{c^2}{b} \right) + \frac{(b_s - \beta)}{b^2} \left( 4 - 2 \frac{c^2}{b^2} - 2 b_s^2 + \frac{2b_s^2}{\log \left( \frac{b}{\nu} \right)} \right).$$

**Lemma 5.1.** Corollary 4.8 and Lemma 5.5 then yield the following bound (we also take $\beta < 1$):

$$\psi_t(x_0, t_0) \geq \frac{\log \left( \frac{b}{\nu} \right)}{b^2} \left( 2b_s(1 - b_s^2) - \alpha k b^{k+1} + 2 \frac{(b_s - \beta)}{b^2} \right) \geq \frac{1}{b^2} \left( 2\delta(1 - b_s^2) - \log \left( \frac{b}{\nu} \right) \alpha \delta b^{k+1} - 2\beta \right)$$

with $k \in (0, 1)$. We note that $|\psi(x_0, t_0)| = \alpha \leq \alpha \log \left( \frac{b}{\nu} \right)$; we can then take $(b^{k+1} \log \left( \frac{b}{\nu} \right)) \leq \delta/2$ at $(x_0, t_0)$, where we assume without loss of generality that $\delta < 1$. Choose $\beta < \delta/10$. Then $\psi_t(x_0, t_0) > 0$ which gives a contradiction. \( \square \)

## 5. Long-Time Existence

In this section we address the proofs of Theorem 1.1 and Theorem 1.2. Let $g(t)$ be the maximal solution to the Ricci flow equation starting at some asymptotically flat metric $g_0 \in \mathcal{M}^2(\mathbb{R}^4)$, for $t \in [0, T)$. We say that $g(t)$ is Type-I if $T < \infty$ and if there exists $M > 0$ such that for all $t \in [0, T)$

$$\sup_{\mathbb{R}^4} |\mathcal{R}_{g(t)}|_{g(t)} \leq \frac{M}{T - t}. \quad (29)$$

Conversely, if the flow develops a finite-time singularity where the curvature blows up at a faster rate than the one given in (29), we refer to $g(t)$ as to a Type-II Ricci flow.

We assume that the maximal time of existence $T$ is finite and we characterize the formation of singularities via a blow-up analysis.

We first define the singular set $\Sigma \subset \mathbb{R}^4$ by the following property: given a point $p \in \mathbb{R}^4$ then $|\mathcal{R}_{g(t)}|_{g(t)}$ stays bounded in some neighbourhood of $p$ as $t \uparrow T$ if and only if $p \in \mathbb{R}^4 \setminus \Sigma$.

**Lemma 5.1.** Let $g_0 \in \mathcal{M}^2(\mathbb{R}^4)$ be asymptotically flat and let $g(t)$ be the maximal solution starting at $g_0$. Assume that the flow develops a finite-time singularity at some $T < \infty$. Then either one of the conditions below holds:

(i) The flow is Type-I. Let $t_j \uparrow T$ be an arbitrary sequence and set $\lambda_j \downarrow (T - t_j)^{-1}$. Then for any singular point $p \in \Sigma$ the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), p)$ defined on $[-\lambda_j t_j, 1]$ by $g_j(t) \equiv \lambda_j g(t_j + \frac{1}{\lambda_j})$ subconverge to a nontrivial gradient shrinking soliton $(M_\infty, g_\infty(t), p_\infty) \in (\mathbb{R}^4, \mathcal{M}^2(\mathbb{R}^4, \mathcal{M}_\infty(\mathbb{R}^4), \mathcal{M}_\infty(\mathbb{R}^4))$. 

(ii) The flow is Type-II. For any space-time sequence $T_j \uparrow T$ there exists a space-time sequence $(p_j, t_j)$ such that if we set $\lambda_j \equiv |\mathcal{R}_{g(t_j)}|_{g(t_j)}(p_j, t_j)$ then the rescaled Ricci flows $(\mathbb{R}^4, g_j(t), p_j)$ defined on $[-\lambda_j t_j, (T_j - t_j) \lambda_j]$ by $g_j(t) \equiv g(t_j + \frac{1}{\lambda_j})$ subconverge to a complete eternal solution to the Ricci flow $(M_\infty, g_\infty, p_\infty) \in (\mathbb{R}^4, \mathcal{M}_\infty(\mathbb{R}^4), \mathcal{M}_\infty(\mathbb{R}^4))$ satisfying

$$\sup_{M_\infty \times (-\infty, +\infty)} |\mathcal{R}_{g_\infty}|_{g_\infty} \leq 1, \quad |\mathcal{R}_{g_\infty}|_{g_\infty} |p_\infty, 0| = 1.$$
Proof. (i) We apply [29, Theorem 1.4] and [15, Theorem 1.1, Theorem 1.2].
(ii) The following is a standard point-picking argument (see, e.g., [13, Chapter 8]). Let \( T_j \not\sim T \). There exists a space-time sequence \((p_j, t_j)\) such that if we set \( \lambda_j = |\text{Rm}_{g(t_j)}|g(t_j)(p_j, t_j)\), then
\[
|\text{Rm}_{g(t_j)}|g(t_j)(p, t) \leq \left(1 - \frac{1}{\lambda_j} \right) \left(1 - \frac{1}{(t_j - t_j) \lambda_j} \right),
\]
for any \( p \in \mathbb{R}^4 \) and \( t \in [-\lambda_j t_j, (T_j - t_j) \lambda_j) \).

According to Lemma [24] there exists \( \alpha = \alpha(g_0) > 0 \) such that \( \text{Vol}_{g_0}(B_{g_0}(p, 1)) \geq \alpha \) for any \( p \in \mathbb{R}^4 \); we can therefore apply the adaptation of Perelman’s No local collapsing theorem [32] to the complete bounded curvature Ricci flow \([11, \text{Theorem 8.26}]\). We obtain that the solution \( k > 0 \) namely, we can find some uniform constant \( p \).

Claim 5.3. There exists a positive radius \( \bar{r} \) independent of \( j \) such that
\[
B_{\bar{r}}(p_j, \nu) \subset V(j, \nu) \subset B_{\bar{r}}(p_j, 2\nu),
\]
for any \( \nu > 0 \). The SU(2)-symmetry of the Ricci flow ensures that the regions \( V(j, \nu) \) are annular regions in \( \mathbb{R}^4 \), hence simply connected.

Lemma 5.2. \( \pi_1(M_\infty) = 0 \).
Proof. We first prove a preliminary property.

Claim 5.3. There exists a positive radius \( \bar{r} \) independent of \( j \) such that
\[
B_{\bar{r}}(p_j, \nu) \subset V(j, \nu) \subset B_{\bar{r}}(p_j, 2\nu),
\]
for any \( \nu \geq \bar{r} \).

Proof of Claim 5.3. By the triangle inequality, it suffices to show that for any \( \eta, \eta' \) in \( S^3 \) the distance \( d_{g_j(0)}((x_j, \eta), (x_j, \eta')) \leq \alpha < \infty \), uniformly in \( j \). Such a bound easily follows from [26]; namely, we can find some uniform constant \( k > 0 \) such that\footnote{For the Type-I case we note that at any singular point the curvature is blowing up at a Type-I rate [15, Theorem 1.2]. By [26] we conclude that \( \frac{d}{dt} \) is blowing up at a Type-I rate as well along the singular set \( \Sigma \).}
\[
d_{g_j(0)}((x_j, \eta), (x_j, \eta')) = \sqrt{\lambda_j}d_{g(t_j)}((x_j, \eta), (x_j, \eta')) \leq k\sqrt{\lambda_j} b(x_j, t_j) \leq \alpha < \infty.
\]
We note that by Claim 5.3 the maps $\Phi_j^{-1}$ given by Hamilton’s Compactness theorem are well defined on $V(j, 2^{-1})$ for $j > j_0$, for some $j_0$. We can pick $j_0 > j_0$ sufficiently large such that for any $q \in U_{j_0}$ we have
\[ d_{g_{j_0 + 2}}(p_\infty, q) \leq 1 + d_{g_{j_0}}(p_\infty, q) \leq 1 + d_{g_{j_0}}(q, \Phi_j(q)) \leq 1 + 2^\lambda < 2^\lambda + 1. \]

We thus obtain the following inclusions
\[ U_{j_0} \subset \Phi_j^{-1}(B_{g_{j_0 + 2}}(p_{j_0 + 2}, 2^\lambda + 1)) \subset \Phi_j^{-1}(V(j_0 + 2, 2^\lambda + 1)) = \tilde{V}_{j_0} \subset M_\infty. \]

Since we can iterate the method by replacing $U_{j_0}$ with $U_{j_0 + 2}$, we may conclude that the $M_\infty$ admits an exhaustion $\{\tilde{V}_j\}$ of simply connected open sets; that completes the proof.

Next, we prove that the symmetries of the flow are enhanced when dilating. To this aim, we first show that the Milnor frame passes to the singularity model; since the proof of that relies on an Ascoli-Arzelà argument, we need $C^3$-bounds with respect to the rescaled solutions.

**Lemma 5.4.** There exists a continuous positive function $f : (-\infty, \omega) \times (0, +\infty) \to \mathbb{R}$ such that for any $t \in (-\infty, \omega)$ and $\nu > 0$ the following holds uniformly with respect to $j$:
\[ \sup_{B_{g_j}(0,\nu)} \sum_{k=0}^3 |\nabla^k g_{j}X_i|_{g_j(t)} \leq f(t, \nu), \]
with $i = 1, 2, 3$. Furthermore, there exists $\alpha > 0$ such that for $i = 1, 2, 3$
\[ |\nabla g_{j}X_i|_{g_j}(p_j, 0) \geq \alpha, \]
up to passing to a subsequence.

**Proof.** We first fix $t = 0$ and we let $\nu > 0$ and $q \in B_{g_j}(0, \nu)$. In the following we only analyse the case of $X_1$ since the others do not require modifications. We deal with the bounds for $|\nabla g_{j}X_i|_{g_j}(0)$ with $k = 0, 1, 2, 3$ separately.

**Case $k = 0$.** We consider a unit speed geodesic from $p_j$ to $q$; from Lemma 3.3 we get
\[ \sqrt{\lambda_j} (b(q, t_j) - b(p_j, t_j)) \leq \sqrt{\lambda_j} \left( \sup_{B_{g_j}(p_j, \nu)} |b| \right) d_{g_j}(p_j, q) \leq \alpha \nu. \]

The desired estimate then follows from (26) which gives $\lambda_j b^2(p_j, t_j) \leq \alpha$.

**Case $k = 1$.** By direct computation we get
\[ |\nabla g_{j}X_i(t, t)|^2 = 2 \left( b_j^2 \frac{b_{ss}}{c_s} + c_j^2 \right), \]
for any $t \in [0, T]$.

**Lemma 5.1, Lemma 3.3 and (23)** imply that $|\nabla g_{j}X_i(t, t)|$ is uniformly bounded in any compact region.

By (29) we have $b(p_j, t_j) \to 0$; the estimate (30) then follows from Lemma 3.6.

**Case $k = 2$.** We analyse in detail only one exemplificative instance. One of the terms appearing in the computation of the norm $(\lambda_j)^{-\frac{1}{2}}|\nabla^2 g_{j(t)}X_1|$ is
\[ (\lambda_j)^{-\frac{1}{2}} |\nabla^2 g_{j(t)}X_1(\partial s, \partial s, \sigma_1)| b(q, t_j) \equiv (\lambda_j)^{-\frac{1}{2}} |b_{ss}|(q, t_j) \leq \alpha(\lambda_j) \frac{b}{b_s}(q, t_j), \]
where we have used (8) and the definition of $\lambda_j$ in Lemma 5.1; the last term is then bounded because it coincides with the case $k = 0$ we have already discussed.

**Case $k = 3$.** One of the terms appearing in the computation of the norm $(\lambda_j)^{-\frac{1}{2}}|\nabla^3 g_{j(t)}X_1|$ is
\[ (\lambda_j)^{-\frac{1}{2}} b |\nabla^3 g_{j(t)}X_1(\partial s, \partial s, \partial s, \sigma_1)|(q, t_j) = (\lambda_j)^{-\frac{1}{2}} b \left( \frac{b_{ss}}{b_s} \right)|(q, t_j). \]

According to Shi’s first derivative estimate and Lemma 5.1, there exists a uniform constant $\alpha$ such that
\[ |\nabla g_{j(t)} Rm_g(t)|_{g(t)} \leq 2(\lambda_j)^{\frac{1}{2}}. \]
We thus have
\[
\left| \frac{b_{s+1}}{b}(q, t_j) \right| \leq \left( \frac{b_{s+1}}{b^2} + |(k_{01})_{s+1}| \right)(q, t_j) \leq \alpha \frac{\lambda_j}{b} + \alpha(\lambda_j)^{\frac{s}{2}}(q, t_j).
\]
We can then bound the right hand side of (31) as
\[
(\lambda_j)^{-1}b \left| \frac{b_{s+1}}{b}(q, t_j) \right| \leq \alpha(1 + \sqrt{\lambda_j}b)(q, t_j) \leq f(\nu),
\]
where the last inequality follows again from the case \( k = 0 \). The other terms are dealt with similarly.

Let now \( t \in (\infty, \omega) \). By Corollary 3.12 we get
\[
\lambda_j \left| b^2(p_j, t_j) - b^2(p_j, t_j + \frac{t}{\lambda_j}) \right| \leq \alpha \lambda_j \left| \frac{t}{\lambda_j} \right| \leq \alpha|t|.
\]
We may then extend the proof of the bound for the case \( k = 0 \) for any \( t \in (\infty, \omega) \). According to the definition of \( \lambda_j \) given in Lemma 5.1 we easily generalize the cases \( k = 1, 2, 3 \) as well.

Since the rescaled Ricci flows (sub)converge to the limit ancient flow in the pointed Cheeger-Gromov sense, from Lemma 5.4 it follows that the sequence \( (\Phi_j)^{-1}X_1 \) is uniformly \( C^4 \)-bounded in \( B_{g_{\infty}(0)}(p_{\infty}, 1) \) with respect to \( g_{\infty}(0) \). We can then apply the Ascoli-Arzelà theorem and get the following

**Corollary 5.5.** There exists a subsequence \( (\Phi_j)^{-1}_1X_1 \) that converges in \( C^2 \) to a vector field \( Y_1 \) on \( B_{g_{\infty}(0)}(p_{\infty}, 1) \) with respect to \( g_{\infty}(0) \).

From now on we re-index the subsequence given by the previous Corollary.

In order to prove that \( Y_1 \) is actually a killing vector field for \( g_{\infty}(0) \) we need a preliminary result; the following shows that the singularity model cannot be Ricci flat.

**Lemma 5.6.** For any \( q \in M_\infty \) and for any \( t \in (\infty, \omega) \) it holds
\[
b\left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right) \to 0, \quad j \to \infty.
\]

**Proof.** Suppose for a contradiction that there exist \( q \in M_\infty \), \( t \in (\infty, \omega) \) a subsequence (which we still denote by \( j \)) and a positive quantity \( \mu \) (which without loss of generality we may take to be finite) such that \( b(\Phi_j(q), t_j + \frac{t}{\lambda_j}) \to \mu \).

By (20) we immediately derive that \( |\text{Rm}_{g_{\infty}(\cdot)}g_{\infty}(\cdot)(q, t)| = 0 \). Since any complete ancient solution to the Ricci flow has nonnegative scalar curvature [9], a standard application of the Maximum Principle and the uniqueness of the flow among complete and bounded curvature solutions yield \( \text{Ric}_{\infty} \equiv 0 \) everywhere in the space-time.

Let us then fix the time to be 0 and assume that \( g_{\infty}(0) \) is not flat.

By the uniform \( C^1 \)-lower bound in (30) there exists an open subset \( U \subset B_{g_{\infty}(0)}(p_{\infty}, 1) \) where \( |Y_1|_{g_{\infty}(0)}|U| > 0 \), with \( Y_1 \) given by Corollary 5.3. From the real analyticity of the ancient limit flow [4] it follows that there exists \( \bar{q} \in U \) such that \( |\text{Rm}_{g_{\infty}(0)}|_{g_{\infty}(0)}(\bar{q}) > 0 \), otherwise the limit would be flat.

Moreover, \( b(\Phi_j(\bar{q}), t_j) \to 0 \) as \( j \to \infty \); for if such condition did not hold, then by (20) the Riemann tensor would vanish at \( \bar{q} \).

Since \( g_{\infty}(0) \) is Ricci flat, by the SU(2)-symmetry we get
\[
0 = |\text{Ric}_{g_{\infty}(0)}|_{g_{\infty}(0)}(\bar{q}, 0)
\]
\[
= \lim_{j \to \infty} \frac{1}{\lambda_j} \left( (k_{01} + k_{02} + k_{03})^2 + 2(k_{01} + k_{12} + k_{13})^2 + (k_{03} + k_{13} + k_{23})^2 \right)^{\frac{1}{2}} (\Phi_j(\bar{q}), t_j)
\]
\[
= \lim_{j \to \infty} \frac{1}{b^2} \left( (b^2(2k_{01} + k_{03}))^2 + 2(b^2(k_{01} + k_{12} + k_{13}))^2 + (b^2(k_{03} + k_{23}))^2 \right)^{\frac{1}{2}} (\Phi_j(\bar{q}), t_j).
\]

By the Cheeger-Gromov convergence we get \( \Phi_j(\bar{q}) \in B_{g_{\infty}(0)}(p_j, 2) \) for \( j \) large enough. Therefore, from Lemma 5.3 (the case of \( k = 0 \)) it follows that \( \lambda_j b^2(\Phi_j(\bar{q}), t_j) \leq \alpha \) for \( j \) large and for some positive \( \alpha \). From the estimate in Lemma 3.13 we finally derive that (32) is equivalent to
\[
\lim_{j \to \infty} (b^2|\text{sec}_{g(t_j)}|) (\Phi_j(\bar{q}), t_j) = 0,
\]
with $\sec g(t_j)$ the sectional curvatures of $g(t_j)$. We note that due to the SU(2)-symmetry the only term in the Riemann tensor which is not the sectional curvature of a 2-plane is given in (11); since $b(\Phi_j(q), t_j) \to 0$ we get

$$\lim_{j \to \infty} (b^2|\text{Rm}|_{g(t_j)}) (\Phi_j(q), t_j) = \lim_{j \to \infty} \left( c \frac{c_s - b}{b} \right) (\Phi_j(q), t_j) = 0,$$

where we have used Lemma 5.7. Therefore by Corollary 5.3 and the choice of $q$ we conclude that

$$0 < |Y_1|^2_{g(0)}(0) |\text{Rm}|_{g(0)}(0) (q, 0) = \lim_{j \to \infty} |X_1|^2_{g(0)}(0) |\text{Rm}|_{g(0)}(0) (\Phi_j(q), 0)$$

$$= \lim_{j \to \infty} (b^2|\text{Rm}|_{g(t_j)}(0)) (\Phi_j(q), t_j),$$

which is a contradiction because we have just proved that the right hand side must vanish. □

We can now show that $Y_1$ is a killing vector field on the limit manifold for any time.

**Lemma 5.7.** There exists a unique smooth extension of $Y_1$ to the entire limit manifold $M_\infty$ such that $(\Phi_j^{-1})_* X_1$ converges in $C^2$ to $Y_1$ on compact sets with respect to $g_\infty(t)$, for any $t \in (-\infty, \omega)$. Moreover $Y_1$ is a $g_\infty(t)$-killing vector field for any $t \in (-\infty, \omega)$.

**Proof.** First we show that $Y_1$ is a killing vector field in $B_{g(0)}(p, 1)$ with respect to $g_\infty(0)$. Suppose for a contradiction that there exist $q \in B_{g(0)}(p, 1), \delta > 0$ and $Z, W \in C^\infty(TM_\infty)$ such that

$$g(0) \left( \nabla_{g(0)} Z, Y_1 \right) + g(0) \left( Z, \nabla_{g(0)} Y_1 \right) \geq \delta > 0$$

in some compact neighbourhood $\Omega$ of $q$. By Corollary 5.3 and the Cheeger-Gromov convergence we get

$$\left( g_j(0) \left( \nabla_{g_j(0)} Z, Y_1 \right), (\Phi_j)_* W \right) + g_j(0) \left( (\Phi_j)_* Z, \nabla_{g_j(0)} Y_1 \right) \geq \delta_j,$$

for some $j$ large enough. If at $\Phi_j(q)$ we write $(\Phi_j)_* Z = z_j^i \partial_{\Phi_j(t_j)} + \sum_{k=1}^3 z_j^k X_k$ and similarly for $(\Phi_j)_* W$, then by the Koszul formula we get

$$\delta_j \leq \left( 2\lambda_j b^2 \left( 1 - \frac{c^2}{b^2} \right) \left( z_j^3 w_j^3 + z_j^2 w_j^2 \right) \right) \geq \frac{\delta}{3}.$$

Since $|Z|_{g(0)}(q, 0) \geq \lim_{j \to \infty} \sqrt{\lambda_j} z_j^i b(\Phi_j(q), t_j)$ and similarly for $W$, there exists a positive constant $\beta$ depending on $q$ such that:

$$\frac{\delta}{3} \leq \beta \left( 1 - \frac{c^2}{b^2} \right) (\Phi_j(q), t_j) \leq \beta \left( 1 - \frac{c^2}{b^2} \right) \leq ab (\Phi_j(q), t_j),$$

where we have used Lemma 5.4. According to Lemma 5.6 we can choose $j$ sufficiently large such that the right hand side is as small as we need, thus obtaining the contradiction.

Since the limit ancient flow is real analytic and by Lemma 5.4 $M_\infty$ is simply connected, it is a classic result that $Y_1$ extends uniquely to a global killing vector field on $(M_\infty, g_\infty(0))$; being $g_\infty(0)$ complete, we also get that $Y_1$ is smooth.

Given $\nu > 1$, Lemma 5.4 implies that for any subsequence of $(\Phi_j^{-1})_* X_1$ there exists a subsubsequence that converges in $C^2$ to some vector field on $B_{g(0)}(p, \nu)$ with respect to $g_\infty(0)$. The argument above shows that the limit vector field must be a killing field for $g_\infty(0)$; by the uniqueness result in 3.2 we conclude that such limit vector field is indeed $Y_1$. The statement is then proved when $t = 0$.

Consider an arbitrary time $t \in (-\infty, \omega)$. Since $g_\infty(t)$ and $g_\infty(0)$ are equivalent, we get that the sequence $(\Phi_j^{-1})_* X_1$ converges in $C^0$ on compact sets to $Y_1$. The very same proof for the case $t = 0$ shows that we can improve the convergence up to $C^2$ and that $Y_1$ is a killing vector field for $g_\infty(t)$.

The lower bound (5.2) and the previous Lemma extend to the sequences $(\Phi_j^{-1})_* X_2$ and $(\Phi_j^{-1})_* X_3$ which then define analogous killing vector fields $Y_2$ and $Y_3$ for the limit flow. Moreover, from the Cheeger-Gromov-Hamilton convergence we derive that the system $(Y_i)_{i=1}^3$ is an orthogonal frame with respect to $g_\infty(t)$ for any $t \in (-\infty, \omega)$. We can now prove that this frame of killing fields forces the limit to be spherically symmetric.
Lemma 5.8. The metric $g_\infty(t)$ is rotationally symmetric for any $t \in (-\infty, \omega)$. Moreover $M_\infty = \mathbb{R}^4$ or $M_\infty = \mathbb{R} \times S^3$.

Proof. According to (30) and the orthogonality of the $Y_i$’s there exists at least a point in $M_\infty$ where this frame spans a three-dimensional subspace. Therefore, since the Lie brackets are preserved in the limit, Lemma 5.7 implies that there exists a (non-trivial) copy of $\mathfrak{su}(2)$ in the Lie algebra of killing fields $\text{iso}(M_\infty, g_\infty(t))$; by integrating the killing fields we derive that SU(2) acts isometrically with cohomogeneity 1 on $(M_\infty, g_\infty(t))$ for any $t \in (-\infty, \omega)$.

By the standard classification of connected cohomogeneity 1 manifolds there exists at most one singular orbit $O_{\text{sing}}$ for the SU(2) action on $M_\infty$ \cite{17}. Moreover, we can write $M_\infty = O_{\text{sing}} \cup M_{\text{prin}}$, where $M_{\text{prin}}$ is an open dense submanifold (still connected) foliated by maximal orbits of the form

\begin{equation}
M_{\text{prin}} = \mathbb{R} \times SU(2)/H,
\end{equation}

with $H$ the isotropy group of the action along principal orbits \cite{17}; we note that when $O_{\text{sing}} = \emptyset$ Lemma 5.9 implies that $M_\infty = \mathbb{R} \times S^3$.

All the information about $g_\infty(t)$ can be obtained by restricting it to a geodesic starting at the singular orbit and meeting the principal orbits orthogonally; namely, once we denote the dual coframe associated with $\{Y_i\}_{i=1}^3$ by $\{\sigma_i\}_{i=1}^3$, we have

\[ g_\infty(t)|_{M_{\text{prin}}} = (dy)^2 + g(q(t)) = (dy)^2 + \phi_1^2(y, t) \sigma_1 \otimes \sigma_1 + \phi_2^2(y, t) \sigma_2 \otimes \sigma_2 + \phi_3^2(y, t) \sigma_3 \otimes \sigma_3, \]

with $\phi_i(y, t) = |Y_i|_{g_\infty(t)}(y)$ for any $y > 0$, $t \in (-\infty, \omega)$ and $i = 1, 2, 3$.

Let $q \in M_{\text{prin}}$, according to the convergence of $(\Phi_j^{-1})_* X_i$ in $C^2$ to $Y_i$ with respect to $g_\infty(t)$, we get

\[ \frac{1}{\phi_1} \left( \phi_1 - 1 \right) (q, t) = \frac{1}{\sqrt{1 + \frac{b}{c}} - 1} \left( \phi_j(q), t_j + \frac{t}{\lambda_j} \right) \leq \lim_{j \to \infty} \frac{\alpha}{\sqrt{\lambda_j}} = 0, \]

where we have used the bound in Lemma 5.6. Since the ratio $b/c \geq 1$ is scale invariant we get $\phi_1 = \phi_3$. The final identity $\phi_1 = \phi_2$ is a consequence of the extra $U(1)$-symmetry; therefore $g_\infty(t)$ is rotationally symmetric. From the classification in \cite{17} it also follows that if there exists a singular orbit then $\phi \equiv \phi$ is an odd function with $\phi_0(0) = 1$; Lemma 2.4 thus implies that $M_\infty = \mathbb{R}^4$. We may finally conclude that $M_\infty = \mathbb{R}^4$ or $M_\infty = \mathbb{R} \times S^3$ with

\begin{equation}
q_\infty(t) = (dy)^2 + \phi^2(y, t) g_{S^3},
\end{equation}

where $g_{S^3}$ is the standard constant curvature 1 metric on $S^3$ and

\[ \phi(y, t) = \lim_{j \to \infty} \sqrt{\lambda_j} \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right), \]

for any $(q, t) \in M_\infty \times (-\infty, \omega)$. \hfill \Box

The existence of lower bounds for the sectional curvatures that break scale invariance provides a control on the sign of the curvature of the limit.

Lemma 5.9. The limit ancient flow $(M_\infty, g_\infty(t), p_\infty)$ satisfies either one of the two conditions below:

(i) $(M_\infty, g_\infty(t))$ is the round shrinking cylinder.

(ii) $\text{Rm}_{g_\infty(t)} > 0$ everywhere in $M_\infty \times (-\infty, \omega)$, with $M_\infty = \mathbb{R}^4$. 

Proof. Let \((q, t) \in M_{\infty} \times (-\infty, \omega).\) According to Corollary 5.15, Lemma 5.6 and Lemma 5.8 we have
\[
(Y_{\infty}^j k^\infty_{\alpha \alpha}) (q, t) = \lim_{j \to \infty} \left( |X_1|^2 k_{12} \right) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right)
\]
\[
= \lim_{j \to \infty} (-b_{a \alpha} b^a) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right) \geq \lim_{j \to \infty} \left( \frac{\alpha}{|\log b|} \right) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right) = 0.
\]
Similarly, from Lemma 3.8 it follows
\[
(Y_{\infty}^j k_{12}^\infty) (q, t) = \lim_{j \to \infty} \left( |X_1|^2 k_{12} \right) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right)
\]
\[
\geq \lim_{j \to \infty} (1 - b_2^2) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right) \geq \lim_{j \to \infty} (-\alpha b) \left( \Phi_j(q), t_j + \frac{t}{\lambda_j} \right) = 0.
\]
The condition \(\text{sec}_{\infty}(\cdot, t) \geq 0\) is then equivalent to \(R_{m\infty}(\cdot, t) \geq 0\) because \(g_{\infty}(t)\) is rotationally symmetric.

Since \(M_{\infty}\) is simply connected and noncompact and \(g_{\infty}(t)\) is nonflat, if the curvature operator is never strictly positive by Hamilton’s strong maximum principle for systems the holonomy is reduced to a line and therefore by de Rham’s decomposition theorem \((M_{\infty}, g_{\infty}(t))\) splits a line. Lemma 5.8 finally shows that \((M_{\infty}, g_{\infty}(t))\) must be the round shrinking cylinder.

If the curvature operator is strictly positive at some \((q, t) \in M_{\infty} \times (-\infty, \omega)\) then the strong maximum principle implies that it must be positive everywhere in the space-time. By the Soul theorem we get \(M_{\infty} = \mathbb{R}^4\).

We now address the proof of Theorem 1.1.

Proof of Theorem 1.1 (i) Assume the flow to be Type-I. According to Lemma 5.4 we set \(\lambda_j = (T - t_j)^{-1}\) for any \(t_j \to T\) and we dilate the flow at a fixed singular point \(p = (x, \bar{\eta}) \in [0, \rho] \times S^3,\) where \(\rho\) is given by Lemma 5.4.

Since any rotationally symmetric shrinking gradient soliton on \(\mathbb{R}^4\) must be flat [25], by Lemma 5.9 we may conclude that \((M_{\infty}, g_{\infty}(t))\) is the round shrinking cylinder. We show that we can always choose a sequence of times approaching \(T\) for which the cylinder cannot arise as a singularity model.

Claim 5.10. If the singularity is Type-I, then there exists a sequence \(t_j \nearrow T\) such that
\[
V(j, 2^{j-1}) = \bigcup_{\eta \in S^3} B_{g_j(0)}((x, \eta), 2^{j-1}) \ni 0
\]

for all \(j\) large.

Proof of Claim 5.10. For any positive \(x\) the \(g(t)\)-distance of the Euclidean hypersphere of radius \(x\) from the origin is given by \(\int_0^x \xi(y, t) dy\), with \(\xi\) the (unnormalized) radial component of the solution to the Ricci flow defined as in (12). The Type-I condition and a comparison argument applied to the evolution equation of \(\xi\) (13) yield the bound
\[
d_{g_j(0)}(0, x) = \sqrt{\lambda_j} d_{g(t_j)}(0, x) \equiv \sqrt{\lambda_j} \int_0^x \xi(y, t_j) dy \leq \rho(T) \alpha(T - t_j)^{-(\alpha + \frac{1}{2})}
\]

for any \(j\), with \(\alpha\) some uniform positive constant. Therefore the Claim follows provided that we can choose a sequence \(t_j \nearrow T\) such that
\[
\rho(T) < 2^{1/2} (T - t_j)^{\alpha + \frac{1}{2}};
\]

for \(j\) larger than some fixed \(j_0\).

---

4An argument similar to the one used for proving Lemma 5.7 shows that the radial direction \(\partial_y\) orthogonal to the \(S^3\) orbits is pushed forward along the diffeomorphisms \(\Phi_j\) to the radial direction \(\partial_y\) on \(\mathbb{R}^4\).

5If \(M_{\infty}\) was compact it would then follow from the Cheeger-Gromov convergence that for \(j\) large enough \(\Phi_j\) is a diffeomorphism from \(M_{\infty}\) to \(\mathbb{R}^4\).

6We note that if \(0\) is a singular point for the flow, then the conclusion is trivially true.
Since \( V(j, 2^{j-1}) \) are annular regions, it follows that for the sequence given by Claim 5.10 \( V(j, 2^{j-1}) \cong \mathbb{R}^4 \) for \( j \) larger than some fixed \( j_0 \); we can then extend the proof of Lemma 5.2 to conclude that \( M_\infty \) admits an exhaustion of open sets diffeomorphic to \( \mathbb{R}^4 \), thus ruling out the cylinder topology. 

(ii) If the flow develops a finite-time singularity, then part (i) shows that the condition \( (29) \) fails; therefore by Lemma 5.1 we consider rescaled Ricci flows that (sub)converge to an eternal solution with curvature attaining its supremum in the space-time. 

We may now prove that any Ricci flow solution starting at some asymptotically flat metric \( g \in \mathfrak{M}^2(\mathbb{R}^4) \) is immortal.

7From the non triviality of the rank 1 compactly supported de Rham cohomology group we derive that \( \mathbb{R} \times S^3 \) cannot be exhausted by open sets diffeomorphic to \( \mathbb{R}^4 \).
We may then suppose that $g$. In the following we always take $R$ on $b$. For the case (ii) there is nothing to prove because the asymptotic behaviour of the derivatives is a known fact \[36\]. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. There exists $g$ in (3) and a uniform constant that may change from line to line respectively. We first note that $\gamma > 1$ is easier to deal with. Therefore, we get the following bound: 

$$ x^{2+\epsilon} c_x \leq \alpha x^{-2\epsilon/3} c_x \leq \alpha x^{-2\epsilon/3} \leq \alpha x^{-2\epsilon+2-4\epsilon} = \alpha x^{2\epsilon}, $$

where we have used that $b$ grows linearly at infinity. The previous estimate, the condition $b_x \to 2$ and formula (7) guarantee that

$$ |x^{1+\epsilon} c_x| \leq \alpha. $$

Integrating we get

$$ |\ln(c(x)) - \ln(c(1))| \leq \alpha - \frac{\alpha}{x}, $$

for any $x \geq 1$. Thus there exist $0 < \delta < M < \infty$ such that

$$ \delta \leq c \leq M $$

for any $x \geq 1$. The uniform upper bound for $c$ and (3) give $|c_{xx}| \leq \alpha x^{-2-\epsilon}$; integrating and using that $c_x \to 0$ at infinity we obtain $|c_x| \leq \alpha x^{-1-\epsilon}$. Therefore $c$ admits a finite limit at infinity; the uniform lower bound found above shows that such limit needs to be positive.

In order to prove that for such $g$ there are no collapsed regions at infinity, we first consider the metric

$$ \tilde{g} = (dx)^2 + (2x)^2 (\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + \sigma_3 \otimes \sigma_3, $$

in $\mathbb{R}^4 \setminus B(0,1)$. We can then extend the Hopf fibration to a Riemannian submersion $(\mathbb{R}^4 \setminus B(0,1), \tilde{g}) \to (\mathbb{R}^3 \setminus B(0,1), \tilde{g})$, with $\tilde{g}$ the standard Euclidean metric on $\mathbb{R}^3$. Since the $\tilde{g}$-length of the $S^1$-fiber stays constant (and positive) at infinity there exists $\nu > 0$ such that

$$ Vol_{\tilde{g}}(B_{\tilde{g}}(p,1)) \geq \nu > 0 $$
for any \( p \in \mathbb{R}^4 \setminus B(0, 1) \); the bound above and the control on the curvature guarantee that \( \text{inj}_g \geq \delta \) for some uniform \( \delta > 0 \). The same conclusion holds for any asymptotically flat positive mass metric \( g \in \mathcal{Mfd}(\mathbb{R}^4) \), because any such \( g \) is equivalent to \( \tilde{g} \) in \( \mathbb{R}^4 \setminus B(0, 1) \).

We recall that given a complete Riemannian manifold \((M, g)\) with bounded curvature there exist a smooth function \( \psi \) and a constant \( \lambda > 0 \) such that

\[
\lambda^{-1}(d_g(0, x) + 1) \leq \psi(x) \leq \lambda(d_g(0, x) + 1),
\]

\[
|\nabla g \psi| \leq \lambda,
\]

\[
\text{Hess}_g(\psi) \leq \lambda g.
\]

We call \( \psi \) a distance-like function (with respect to the given metric).

The following property allows us to apply Proposition B.10 in [28] to our setting, thus proving that the power law decay of the curvature in [3] persists along the flow.

**Lemma A.1.** Any asymptotically flat \( g \in \mathcal{Mfd}(\mathbb{R}^4) \) admits a distance-like function \( \tilde{\psi} \) satisfying

\[
\phi^{-1} \text{Hess}_g(\phi) \geq \gamma g,
\]

for some \( \gamma \in \mathbb{R} \).

**Proof.** We first note that one can always choose the \( g \)-distance of hyperspheres from the origin as radial coordinate \( x \). Let \( \psi : [0, \infty) \to \mathbb{R} \) be a smooth non decreasing function which is sufficiently flat at 0 (say that \( \lim_{x \to 0} (x^2 \psi^{(k)}(x)) = 0 \) for any \( k \in \mathbb{N} \)) and is identically 1 for \( x \geq x_0 \) for some sufficiently small \( x_0 \) only depending on \( g \). From the boundary conditions \([2], \) the asymptotics \([4], \) and \([5], \) and the finiteness of \( m(g) \) it then follows that the map \( \phi \doteq x \psi + 1 \) is a distance-like function satisfying the required lower bound. \( \square \)

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