Observers of Disturbances and Measurement Noises for Sector-bound Nonlinear System

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Abstract. Observers of disturbances and measurement noises are developed for dynamical systems with sector-bound nonlinearity and parametric uncertainty. Conditions for the existence of these observers are proposed in the form of solvability of linear matrix inequalities (LMIs). Using the observers, a control law is designed to ensure system stabilization. The proposed algorithm is experimentally studied on a stand for multi-machine power systems control.

1. Introduction
The control problems under disturbances and measurement noises are widespread in practice, for example, in network control [1], aircraft control [2], multi-machine power system control [3], control in navigation systems [4], control in chemical industry [5], etc. This problem becomes more complicated if only the plant output is available for measurement, but not the full state vector. Currently, many methods have been proposed to solve this problem. In [6, 7] linear and nonlinear filters are used for removing some frequencies or frequency bands in the measurements. In [8] the algorithm synthesis is based on Kalman filter for estimation of the plant state vector under random disturbances and noises. However, in [6–8] the plant parameters should be known.

In presence of parametric uncertainties in [9–13] the high-gain observer-based algorithms is designed under bounded high-frequency noises. In [14, 15] the optimal algorithms for linear plants with bounded disturbances and noises is proposed.

The present paper basically focuses on the solution of the following two main problems:

(i) design of the control algorithm for sector-bound nonlinear systems with disturbances and measurement noises;
(ii) obtaining the conditions in terms of LMIs providing the input-to-state stability of the closed-loop system.

The following notations are used in the paper: \( \mathbb{R} \) is the set of real numbers; \( I \) is the identity matrix; \( A^+ \) is the left pseudo-inverse matrix to \( A \); \( E_j = [0, \ldots, 0, 1, 0, \ldots, 0]^T \) is the vector of the
corresponding dimension where the \( j \)th component is equal to 1 and other components are equal to zero: \( \tilde{E} = [E_1, ..., E_{r-1}, E_r, ..., E_m] \); \( | \cdot | \) and \( \left\| \cdot \right\| \) denote Euclidean norm of a vector and matrix, respectively; \( p = \frac{d}{dt}, \quad \dot{w} = \frac{dw}{dt}, \quad w'_t = \frac{\partial w}{\partial t} \) and \( w'_x = \frac{\partial w}{\partial x} \).

2. Problem statement
Consider a plant model in the form

\[
\dot{x} = Ax + Bu + D(\psi(x,t) + \varphi(t) + c_0u(t)), \quad y = Lx,
\]

\[
z = y + \xi,
\]

where \( x = x(t) \in \mathbb{R}^n \) is the state, \( u = u(t) \in \mathbb{R}^l \) is the control signal, \( y = y(t) \in \mathbb{R}^m \) is the unmeasured output signal (\( m \geq 2 \)), the signal \( z = z(t) \in \mathbb{R}^m \) is available for measurement, \( \xi = \xi(t) = [\xi_1(t), ..., \xi_m(t)]^T \) is the bounded noise with bounded component \( \tilde{\xi}_r, r \in \{1, 2, ..., m\} \). Denote \( \chi_1 = \lim_{t \to \infty} \sup_{t \geq 0} |\xi_1(t)| \) and \( \chi_2 = \lim_{t \to \infty} \sup_{t \geq 0} |\xi_2(t)| \). Additionally, \( |\psi| \leq \alpha_1|x|, \quad |\psi'_x| \leq \alpha_2, \quad |\psi'_s| \leq \alpha_3|x|, \quad \chi_3 = \lim_{t \to \infty} \sup_{t \geq 0} |\varphi|, \quad \chi_4 = \lim_{t \to \infty} \sup_{t \geq 0} |\dot{\varphi}| \). The coefficient \( c_0 \) is unknown, but \( c_0 \in [c_{\min}, c_{\max}] \) where \( c_{\min} \) and \( c_{\max} \) are known. The matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times l}, D \in \mathbb{R}^{n \times l}, L \in \mathbb{R}^{m \times n} \) are known and \( A \) is Hurwitz.

It is necessary to design the algorithm that provides the input-to-state stability of (1) leading to ultimate bound

\[
\lim_{t \to \infty} \sup_{t \geq 0} |y(t)| \leq \delta,
\]

where \( \delta = \Theta(\sum_{i=1}^{4} \chi_i) \) is obtained in Theorem 4.1 below, \( \Theta(\chi) \) for \( \chi \in \mathbb{R} \) means that \( \lim_{\chi \to 0} \frac{\Theta(\chi)}{\chi} = C, \ C \) is a constant.

3. Measurement Noise Observer
Eliminate the \( r \)th equation in (2) and rewrite the result w.r.t. \( \tilde{\xi} \). To this end, pre-multiplying (2) by \( \tilde{E}^T \), we have

\[
\tilde{\xi} = \tilde{E}^T z - \tilde{E}^T Lx.
\]

It follows from (2) that the variables \( x \) and \( \xi \) are related, and \( x \) is contained in (4). However, the variable \( x \) cannot be expressed through \( \xi \) from (2). Therefore, according to the structures of (1), (2) and taking into account \( \xi = \tilde{E}\xi + E_r\xi_r \), we introduce the new variable \( \tilde{x} = L^+[z - \tilde{E}\xi - E_r\xi_r] \). Considering (2) and (4), we have \( x - \tilde{x} = \tilde{E}^T LA(I - L^+L)x \). Integrating (1) w.r.t. \( t \) and substituting the results in (4), we get

\[
\tilde{\xi} = \int_0^t \left[ \tilde{A}\tilde{\xi}(s) - \tilde{A}_1 z(s) \right] ds + \tilde{E}^T z - \tilde{E}^T Lx(0)
\]

\[
- \int_0^t \left[ Bu(s) + Df(s) - \tilde{A}_2 \xi_r(s) + \tilde{A}_3 x(s) \right] ds,
\]

where \( \tilde{A} = \tilde{E}^T LAL^+\tilde{E}, \tilde{A}_1 = \tilde{E}^T LAL^+, \tilde{A}_2 = \tilde{E}^T LAL^+ E_r, \tilde{A}_3 = \tilde{E}^T LA(I - L^+L), \tilde{B} = \tilde{E}^T LB, \tilde{D} = \tilde{E}^T LD, \ f = \psi(x,t) + \varphi(t) + c_0u(t) \).

Following the structure of (5), we introduce the algorithm for estimation of the vector \( \tilde{\xi} \) in the form

\[
\tilde{\xi} = \int_0^t \left[ \tilde{A}\tilde{\xi}(s) - \tilde{A}_1 z(s) \right] ds + \tilde{E}^T z,
\]
where $\hat{\xi}$ is the estimate of $\tilde{\xi}$. To evaluate the performance of noise estimator (6), consider the following error

$$e = \hat{\xi} - \tilde{\xi}. \quad (7)$$

Differentiate (7) along the trajectories of (5), (6) and rewrite the result in the form

$$\dot{e} = \hat{A}e - \hat{B}u - \hat{D}f + \hat{A}_2\xi_r - \hat{A}_3x. \quad (8)$$

### 4. Disturbance Observer and Control Law

First, let us clarify the information about the output signal $y$ by using the signal $\hat{\xi}$ given by (6). Let $\hat{y}$ be the estimate of $y$ which is introduced as

$$\hat{y} = z - \hat{E}\hat{\xi}. \quad (9)$$

The following assumption is needed to derive the control law.

**Assumption 1.** The matrices $B$, $D$ and $L$ in (1) satisfy the following conditions

$$D = FR, \quad (LF)^+(LF) = I, \quad (LF)^+(LB) = kI,$$

where $F \in \mathbb{R}^{n \times l}$, $R \in \mathbb{R}^{l \times g}$, $k \in \mathbb{R}$ and $c_{\min} + k > 0$.

According to (2), (7) and with $\xi = \hat{E}\hat{\xi} + \hat{E}r\xi_r$, we rewrite (9) in the form

$$\hat{y} = Lx + \hat{E}\dot{e} + \hat{E}r\xi_r.$$

Taking into account (1) and Assumption 1, we differentiate $\hat{y}$ w.r.t. $t$ and rewrite the result as follows:

$$LFRf = \dot{\hat{y}} - LAx - LBu - \hat{E}\dot{e} - \hat{E}r\xi_r. \quad (10)$$

Consider the method [16] for designing the control law. It follows from (1) that unknown function $f$ can be compensated if the control law is chosen such that $Bu + Df = 0$. Pre-multiplying $Bu + Df = 0$ by $(LF)^+L$ and taking into account Assumption 1, we have $u = -\frac{1}{k}RFf$. However, the signal $f$ cannot be used from (10), because it depends on unmeasured signals $x$, $\dot{e}$ and $\xi_r$. Therefore, the control law $u$ is defined as follows:

$$u = -\frac{1}{k}RF\hat{f}, \quad (11)$$

where $\hat{f}$ is the estimate of $f$. According to the structure of (10) and the second equation of (1), we define the signal $\hat{f}$ in the form

$$R\hat{f} = (LF)^+\left(\dot{\hat{y}} - LAL+\hat{y} - \alpha(p)LBu\right). \quad (12)$$

Here, $\alpha(p)$ is a scalar differential operator and $p = d/dt$. Let us explain the choice of the operator $\alpha(p)$. Substituting (12) into (11), we have

$$[1 - \alpha(p)]u = -\frac{1}{k}(LF)^+\left[\dot{\hat{y}} - LAL+\hat{y}\right]. \quad (13)$$

Considering $\alpha(p) = 1 - \mu p$ and taking into account (13), introduce the control law in the form

$$u = -\frac{1}{\mu k}(LF)^+\left[\dot{\hat{y}} - LAL+\int_0^t \dot{\hat{y}}(s)ds\right]. \quad (14)$$

**Assumption 2.** The relation $RG = I$ holds.
Let us introduce the following notations

\[ w = c_0 + k, \quad W = (B + c_0 F)(LF)^+, \]
\[ A_{21} = \frac{1}{k\mu}(wA - WLAL(I - L^+L)), \]
\[ A_{22} = \frac{1}{k\mu}[-wI + k\mu A], \quad A_{23} = \frac{1}{k\mu}WLAL^+\tilde{E}, \]
\[ A_{24} = \frac{1}{k\mu}W\tilde{E}, \quad B_{21} = \frac{1}{k\mu}(kF - B)R, \]
\[ B_{23} = \frac{1}{k\mu}WLAL^+Er, \quad B_{24} = -\frac{1}{k\mu}WE_r, \]
\[ \tilde{W} = (\tilde{B} + c_0\tilde{E}^T(LF)(LF)^+, \]
\[ A_{41} = \frac{1}{k\mu}(\tilde{W} - w\tilde{E})LA(I - L^+L), \]
\[ A_{43} = \frac{1}{k\mu}[w\tilde{A} - W\tilde{LAL}^+\tilde{E}], \]
\[ A_{44} = \frac{1}{k\mu}[-wI + k\mu\tilde{A} + W\tilde{E}], \]
\[ B_{41} = \frac{1}{k\mu}(\tilde{B} - k\mu\tilde{E}^T(LF))R, \]
\[ B_{43} = \frac{1}{k\mu}[\tilde{W}\tilde{A}_2 - \tilde{W}\tilde{LAL}^+Er], \]
\[ B_{44} = \frac{1}{k\mu}[\tilde{W}E_r + k\mu\tilde{A}_2], \]
\[ A_e = \begin{bmatrix} 0 & I & 0 & 0 \\
A_{21} & A_{22} & A_{23} & A_{24} \\
0 & 0 & 0 & I \\
A_{41} & -\tilde{A}_3 & A_{43} & A_{44} \end{bmatrix}, \]
\[ B_e = \begin{bmatrix} 0 & 0 & 0 & 0 \\
B_{21} & D & B_{23} & B_{24} \\
0 & 0 & 0 & 0 \\
B_{41} & -D & B_{43} & B_{44} \end{bmatrix}, \]
\[ G_e = \begin{bmatrix} 0 \\
B_{21} \\
0 \\
B_{41} \end{bmatrix}, \quad F_e = \begin{bmatrix} 0 \\
D \\
0 \\
-\tilde{D} \end{bmatrix}, \]
\[ C_1 = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix}, \]
\[ \Psi_1 = \alpha_1^2C_2^TC_1, \quad \Psi_2 = 2\alpha_2^2C_1^TC_2 + 2\alpha_3^2C_1^TC_2, \]
\[ \Psi_{11} = A_e^TP + PA_e + 2\beta P + \tau_1\Psi_1 + \tau_2\Psi_2. \]

Setting \( x_e = \text{col}\{x, \dot{x}, e, \dot{e}\} \) and \( d = \text{col}\{\varphi, \dot{\varphi}, \xi_r, \dot{\xi}_r\} \), we rewrite the closed-loop system in the form

\[ \dot{x}_e = A_ex_e + G_e\psi + F_e\dot{\psi} + B_ed. \]  \hspace{1cm} (16)

**Theorem 4.1** Let Assumption 1 and 2 hold. Consider the control system consisting of plant (1), (2), noise estimator (6) and control law (14), (9). Given the coefficients \( \beta > 0 \) and \( \mu > 0 \), let there exist \( \tau_1 > 0, \tau_2 > 0, \rho > 0 \), and the matrix \( P > 0 \) such that the following LMI holds:

\[ \Psi = \begin{bmatrix} \Psi_{11} & PG_e & PF_e & PB_e \\
* & -\tau_1I & 0 & 0 \\
* & * & -\tau_2I & 0 \\
* & * & * & -\rho I \end{bmatrix} < 0. \]  \hspace{1cm} (17)

Here "**" denotes a symmetrical block of a symmetric matrix. Then algorithm (6), (9), (14) ensures the goal (3) with the accuracy

\[ \delta = \|L\|\sqrt{\frac{\rho\sum_{i=1}^{4}\lambda_i^2}{2\beta^2\lambda_{\min}(P)}}. \]  \hspace{1cm} (18)
where \( \lambda_{\text{min}}(P) \) is the smallest eigenvalue of the matrix \( P \).

The proof of Theorem 4.1 is similar to the proof in [16]. Note that the proposed method is efficient with noises in data transmission and reception channels with a sufficiently small unknown time-varying delays in input/output signals.

5. Control of Multi-machine Power Systems

**Numerical study.** Consider an electrical generator model in the form proposed in [3, 17]:

- mechanical dynamics equations:
  \[
  \dot{\theta}(t) = \omega(t),
  \]
  \[
  \dot{\omega}(t) = -\frac{D}{2\pi} \omega(t) - \frac{\omega_0}{2\pi} \Delta P_c(t).
  \]

- electrical dynamics equation:
  \[
  E'_q(t) = \frac{1}{T_{d0}} (E_f(t) - E_q(t)).
  \]

- electrical equations:
  \[
  E_q(t) = \frac{x_{ds}}{x_{qds}} E'_q(t) - \frac{x_d-x'_d}{x_{qds}} V_s \cos \theta(t),
  \]
  \[
  E_f(t) = k_c u_f(t - h),
  \]
  \[
  P_c(t) = \frac{V_s E_q(t)}{x_{ds}} \sin \theta(t),
  \]
  \[
  I_q(t) = \frac{V_s}{x_{qds}} \sin \theta(t) = \frac{P_c(t)}{x_{ad} I_f(t)},
  \]
  \[
  Q_c(t) = \frac{V_s}{x_{qds}} E_q(t) \cos \theta(t) - \frac{V_s}{x_{qds}},
  \]
  \[
  E_q(t) = x_{ad} I_f(t),
  \]
  \[
  V_I(t) = \frac{1}{x_{qds}} \sqrt{x_{qds}^2 E_q^2(t) + V_s^2 x_{qds}^2 + 2 x_{qds} x_{qad} x_{qds} P_c(t) \cot \theta(t)}.
  \]

Here, \( \theta \) is the power angle of the generator (rad), \( \omega \) is the relative speed (rad/s), \( \Delta P_c = P_c - P_m \) (p.u.), \( P_c \) is the electrical power (p.u.), \( P_m \) is the mechanical power input of the generator (p.u.), \( D \) is a damping constant (p.u.), \( H \) is the inertia coefficient (s), \( \omega_0 \) is the rotor speed in synchronous mode (rad/s), \( T_{ad0} = T_{d0} x_{ad}' x_{ds} / x_{ds} \) is the direct axis transient short-circuit time constant (s), \( T_{d0} \) is the direct axis transient open-circuit time constant (s), \( x_{ds}' = x_T + 0.5 x_L + x'_d \) (p.u.), \( x_T \) is the reactance of the transformer (p.u.), \( x_L \) is the reactance of the power line (p.u.), \( x_q \) is a direct axis transient reactance (p.u.), \( x_{ds} = x_T + 0.5 x_L + x_d \), \( x_{ad} \) is the mutual reactance between the excitation coil and the stator (p.u.), \( V_s \) is the voltage on the infinite bus (p.u.), \( k_c \) is the inertia constant (p.u.), \( u_f \) is the input of the SCR amplifier of the generator (control signal) (p.u.), \( E_q \) is the EMF of the generator in the quadratic axis (p.u.), \( E_f \) is the equivalent EMF in the excitation coil (p.u.), \( E'_q \) is the transient EMF in quadratic axis (p.u.), \( I_q \) is a quadratic axis current (p.u.), \( I_f \) is the excitation current (p.u.), \( Q_c \) is the reactive power (p.u.), \( V_I \) is a generator terminal voltage (p.u.), \( h \) is the delay in the control signal. The model of the electrical generator is shown in Fig. 1.

In [3] it is noted that the measurements of angle and relative speed have a large value of noises in faults. The value of \( E'_q \) is measured with small values of noises in faults. Additionally, in the normal mode of generator operation, these signals are measured with small value of noises.

Apply the control law from the previous example. Consider the following fault. Assume that before \( t = 10 \) (s), \( f = 1 + 0.2 \sin(t) + 0.1 \sin(0.3t) + 0.1 \sin(0.8t) + 0.1 \sin(1.1t) \). The fault occurs at \( t = 10 \) (s), where \( f = 10(1 + 0.2 \sin(t) + 0.1 \sin(0.3t) + 0.1 \sin(0.8t) + 0.1 \sin(1.1t)) \). The fault is removed by opening the breaker of the fault line at time \( t = 20 \) (s) with \( f = 1 + 0.2 \sin(t) \).
According to [3] the large changes in $f$ in faults is caused by the short circuit faults occurring in transmission lines.

Compare the proposed control system with algorithm from [3]. According to [3], the control law can be defined as $u = -[1 1.5 2.1]x$. In Figs. 2 and 3 the simulations results are given for the proposed control system and the control system from [3]. It follows from Figs. 2 and 3 that the proposed control system compensates disturbances after 6 (s) with the accuracy of 0.05.

**Figure 1.** Single machine-infinite bus model.

**Figure 2.** The transients of the signals $\theta - \theta^*$ (black curve), $\omega$ (green curve) and $E_q' - E_{q'}^*$ (red curve) for the proposed control law.

**Figure 3.** The transients of the signals $\theta - \theta^*$ (black curve), $\omega$ (blue curve) and $E_q' - E_{q'}^*$ (red curve) for the control law from [3], $\theta^*$ and $E_{q'}^*$ are the equilibrium points.
Experimental study. Now let us study the proposed algorithm on experimental stand to study the power system (see Fig. 4). The stand consists of three electrical generators. Each generator has direct mechanical connection to the electric motor to drive the generator, a rotor field winding for controlling the energy generated by the generator, and 3 phases of output current for load removal.

![Figure 4. The experimental stand to study the power system at IPME RAS.](image)

The role of the source of mechanical impact on the generators is played by electric AC motors powered by 220 V. The shaft of the engine and generator is connected by a flexible coupling on which a disk is mounted for the operation of the optical encoder.

The encoder is used to obtain data on the rotation speed of the generator shaft. The received data is transmitted directly to the control board.

To remove the load of the generators, ballast rheostats and inductors from welding machines are used. In this case, the same phases with 3 generators are connected to 3 different load devices.

Data on the current flowing through the load is provided by three AC sensors connected each to the corresponding load; the signal from the sensors goes to the control board.

The control board consists of two devices interconnected via UART. The role of the machine on which the control signals are calculated is performed by a personal computer with the MATLAB Simulink software package installed. The second STM32 board is designed to collect information from sensors, exchange the information with the computer and generate a PWM signal for controlling transistors.

MOSFET transistors provide the passage of required current through control winding of the rotor of the generator. They have an external power supply of 12 V.

To configure the operation of transistors, current sensors are used in the field winding circuit, the signal from which goes to the control board.

Thus, we obtain a general control scheme that allows you to bring the generators into motion, receive feedback and control each of the generators individually. The control laws are implemented in MATLAB Simulink.

Figure 5 shows the transients of rotor relative degrees for each generator. At $t = 10$ s, the load sufficiently changed. However, with the proposed algorithm, it is possible to synchronize each generator.
6. Conclusions

Observers of disturbances and measurement noises are developed for dynamical systems with sector-bound nonlinearity and parametric uncertainty. Conditions for the existence of these observers are proposed in the form of solvability of linear matrix inequalities (LMIs). Using the observers of disturbances and measurement noises, a control law is developed to ensure system stabilization. The proposed algorithm is experimentally studied on a stand for multi-machine power systems control.

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