THE BOUNDEDNESS OF GENERAL ALTERNATIVE GAUSSIAN
SINGULAR INTEGRALS ON VARIABLE LEBESGUE SPACES WITH
GAUSSIAN MEASURE

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ABSTRACT. In a previous paper [13], we introduced a new class of Gaussian sin-
gular integrals, that we called the general alternative Gaussian singular integrals
and study the boundedness of them on \(L^p(\gamma_d), 1 < p < \infty\). In this paper, we
study the boundedness of those operators on Gaussian variable Lebesgue spaces
under a certain additional condition of regularity on \(p(\cdot)\) following [6].

1. INTRODUCTION AND PRELIMINARIES

The general Gaussian singular integrals, generalizing the Gaussian higher order
Riesz transforms were initially introduced by W. Urbina in [18] and later, S. Pérez
[14] extend to a much larger class.

**Definition 1.1.** Given a \(C^1\)-function \(F\), satisfying the orthogonality condition

\[
\int_{\mathbb{R}^d} F(x) \gamma_d(dx) = 0,
\]

and such that for every \(\varepsilon > 0\), there exist constants, \(C_\varepsilon\) and \(C'_\varepsilon\) such that

\[
|F(x)| \leq C_\varepsilon e^{\varepsilon|x|} \quad \text{and} \quad |\nabla F(x)| \leq C'_\varepsilon e^{\varepsilon|x|}.
\]

Then, for each \(m \in \mathbb{N}\) the generalized Gaussian singular integral is defined as

\[
T_{F,m}f(x) = \int_{\mathbb{R}^d} \int_0^1 \left( -\log \frac{r}{1-r^2} \right)^{m-2} r^m F\left( \frac{y-rx}{\sqrt{1-r^2}} \right) e^{-\frac{|y|^2}{1-r^2}} \frac{1}{r} f(y)dy.
\]

\(T_{F,m}\) can be written as

\[
T_{F,m}f(x) = \int_{\mathbb{R}^d} \mathcal{K}_{F,m}(x,y) f(y)dy,
\]

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Theorem 1.1. The operators $T_{F,m}$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is to say there exists $C > 0$, depending only in $p$ and dimension such that

\begin{equation}
\|T_{F,m,f}\|_{p,\gamma} \leq C\|f\|_{p,\gamma},
\end{equation}

for any $f \in L^p(\gamma_d)$.

Regarding the weak $(1,1)$ boundness with respect to the Gaussian variable, she proved a negative result,

**Theorem 1.2.** Let $\Omega_t = \{z \in \mathbb{R}^d : \min_{1 \leq i \leq d} |z_i| \geq t\}$ and $\Theta(t) = \inf_{\Omega_t \rightarrow \infty} F(\cdot) / t^2$, if $\limsup_{t \rightarrow \infty} \Theta(t) = \infty$, then the operator $T_{F,m}$ is not of weak type $(1,1)$ with respect to the Gaussian measure.

Also, she obtained a positive result that is contained in the following theorem. In order to get sufficient conditions on $F$ for the weak type $(1,1)$ of $T_{F,m}$, since it is known that the Gaussian Riesz transform $R_{\beta}$ for $|\beta| \geq 3$ are not weak $(1,1)$ with respect to the Gaussian measure. Thus, since the weak type is not true, the natural question is what weights can be put in order to get a weak type inequality. She got the the weight should be of the form $w(y) = 1 + |y|^{\beta-2}$. Moreover, for every $0 < \varepsilon < |\beta| - 2$, there exists a function $F \in L^1((1 + |\cdot|^\varepsilon)\gamma_d)$ such that $T_{F,m}f \notin L^{1,\infty}(\gamma_d)$, see [10]. The weights $w$ that will be considered, in order to ensure that $T_{F,m}$ is bounded from $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$, depend on the function $\Phi$.

**Theorem 1.3.** The operator $T_{F,m}$ maps continuously $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$ with $w(y) = 1 \vee \max_{1 \leq i \leq d} \eta(i)$ and

\[
\eta(i) = \begin{cases} 
\Phi(i)/t & \text{if } 1 \leq m < 2, \\
\Phi(i)/t^2 & \text{if } m \geq 2,
\end{cases}
\]

In a previous paper [13], we introduced a new class of Gaussian singular integrals, the general alternative Gaussian singular integrals as follows:
Definition 1.2. Given a $C^1$-function $F$, satisfying the orthogonality condition
\begin{equation}
\int_{\mathbb{R}^d} F(x)\gamma_d(dx) = 0,
\end{equation}
and such that for every $\varepsilon > 0$, there exist constants, $C_\varepsilon$ and $C'_\varepsilon$ such that
\begin{equation}
|F(x)| \leq C_\varepsilon e^{\varepsilon|x|^2} \quad \text{and} \quad |\nabla F(x)| \leq C'_\varepsilon e^{\varepsilon|x|^2}.
\end{equation}
Then, for each $m \in \mathbb{N}$ the general alternative Gaussian singular integral is defined as
\begin{equation}
\overline{T}_{F,m} f(x) = \int_{\mathbb{R}^d} \int_0^1 \left( -\log \frac{r}{1-r^2} \right) e^{-\frac{|x-ry|^2}{1-r^2}} F\left( \frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{-\frac{\beta - xy^2}{1-r^2}}}{(1-r^2)^{y^2/2+1}} dr df(y)dy.
\end{equation}
Thus, $\overline{T}_{F,m}$ can be written as
\begin{align}
\overline{T}_{F,m} f(x) &= \int_{\mathbb{R}^d} \mathcal{K}_{F,m}(x,y) f(y) dy,
\end{align}
where,
\begin{align}
\mathcal{K}_{F,m}(x,y) &= \int_0^1 \left( -\log \frac{r}{1-r^2} \right) e^{-\frac{|x-ry|^2}{1-r^2}} F\left( \frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{-\frac{\beta - xy^2}{1-r^2}}}{(1-r^2)^{y^2/2+1}} dr \\
&= \int_0^1 \varphi_m(r) F\left( \frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{-\frac{\beta - xy^2}{1-r^2}}}{(1-r^2)^{y^2/2+1}} dr \\
&= \frac{1}{2} \int_0^1 \psi_m(t) F\left( \frac{x-\sqrt{1-t} y}{\sqrt{t}} \right) e^{-u(t)} p/t^{y^2/2+1} dt,
\end{align}
with $\varphi_m(r) = \left( -\log \frac{r}{1-r^2} \right) r^{d-1}$; and after taking the change of variables $t = 1 - r^2$, $\psi_m(t) = \varphi_m(\sqrt{1-t})/\sqrt{1-t}$, and $u(t) = \frac{\beta - xy^2}{1-t}$. Additionally, in [13] it was proved the boundedness of them in $L^p(\gamma_d)$, for $d > 1$ and $1 < p < \infty$,

**Theorem 1.4.** For $d > 1$, the operators $\overline{T}_{F,m}$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is to say there exists $C > 0$, depending only in $p$ and dimension such that
\begin{equation}
\|\overline{T}_{F,m} f\|_{p,\gamma} \leq C\|f\|_{p,\gamma},
\end{equation}
for any $f \in L^p(\gamma_d)$.

In [2], H. Aimar, L. Forzani and R. Scotto obtained a surprising result: the alternative Riesz transforms $\overline{R}_\beta$ are weak type $(1,1)$ for all multi-index $\beta$, i.e. independently of their orders which is a contrasting fact with respect to the anomalous behavior of the higher order Riesz transforms $R_\beta$. For the general alternative Gaussian singular integrals $\overline{T}_{F,m}$ we also proved...
Theorem 1.5. For \( d > 1 \), there exists a constant \( C \) depending only on \( d \) and \( m \) such that for all \( \lambda > 0 \) and \( f \in L^1(\gamma_d) \), we have

\[
\gamma_d\left( \{ x \in \mathbb{R}^d : T_{F,m}(x) > \lambda \} \right) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(y)| \gamma_d(dy).
\]

On the other hand, in [6] E. Dalmaso and R. Scotto proved the boundedness of the general Gaussian singular integrals \( T_{F,m} \), on Gaussian variable Lebesgue spaces under certain condition of regularity on \( p(\cdot) \). In order to understand their result we need to get more background on variable Lebesgue spaces with respect to a Borel measure in general and the Gaussian measure in particular.

As usual in what follows \( C \) represents a constant that is not necessarily the same in each occurrence; also we will used the notation: given two functions \( f, g \), the symbols \( \leq \) and \( \geq \) denote, that there is a constant \( c \) such that \( f \leq cg \) and \( cf \geq g \), respectively. When both inequalities are satisfied, that is, \( f \leq g \leq f \), we will denote \( f \approx g \).

Any \( \mu \)-measurable function \( p(\cdot) : \mathbb{R}^d \to [1, \infty] \) is an exponent function; the set of all the exponent functions will be denoted by \( \mathcal{P}(\mathbb{R}^d, \mu) \). For \( E \subset \mathbb{R}^d \) we set

\[
p_-(E) = \text{ess inf}_{x \in E} p(x) \quad \text{and} \quad p_+(E) = \text{ess sup}_{x \in E} p(x).
\]

We use the abbreviations \( p_+ = p_+(\mathbb{R}^d) \) and \( p_- = p_- (\mathbb{R}^d) \).

Definition 1.3. Let \( E \subset \mathbb{R}^d \). We say that \( \alpha(\cdot) : E \to \mathbb{R} \) is locally log-Hölder continuous, and denote this by \( \alpha(\cdot) \in LH_0(E) \), if there exists a constant \( C_1 > 0 \) such that

\[
|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log(e + \frac{1}{|x-y|})}
\]

for all \( x, y \in E \). We say that \( \alpha(\cdot) \) is log-Hölder continuous at infinity with base point at \( x_0 \in \mathbb{R}^d \), and denote this by \( \alpha(\cdot) \in LH_\infty(E) \), if there exist constants \( \alpha_\infty \in \mathbb{R} \) and \( C_2 > 0 \) such that

\[
|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{\log(e + |x-x_0|)}
\]

for all \( x \in E \). We say that \( \alpha(\cdot) \) is log-Hölder continuous, and denote this by \( \alpha(\cdot) \in LH(E) \) if both conditions are satisfied. The maximum, \( \max\{C_1, C_2\} \) is called the log-Hölder constant of \( \alpha(\cdot) \).

Definition 1.4. We denote \( p(\cdot) \in \mathcal{P}_d^{log}(\mathbb{R}^d) \), if \( \frac{1}{p(\cdot)} \) is log-Hölder continuous and denote by \( C_{log}(p) \) or \( C_{log} \) the log-Hölder constant of \( \frac{1}{p(\cdot)} \).

We will need the following technical result, for its proof see Lemma 3.26 in [5].

Lemma 1.1. Let \( p(\cdot) : \mathbb{R}^d \to [0, \infty) \) be such that \( p(\cdot) \in LH_\infty(\mathbb{R}^d) \), \( 0 < p_\infty < \infty \), and let \( R(x) = (e + |x|)^N \), \( N > d/p_- \). Then there exists a constant \( C \) depending on \( d, N \) and the \( LH_\infty \) constant of \( r(\cdot) \) such that given any set \( E \) and any function \( F \)
with $0 \leq F(y) \leq 1$ for $y \in E$,

\begin{align}
(1.11) \quad \int_E F^p(y)dy & \leq C \int_E F(y)^{p^*}dy + \int_E R^{p^*}(y)dy, \\
(1.12) \quad \int_E F^{p^*}(y)dy & \leq C \int_E F(y)p(y)dy + \int_E R^p(y)dy.
\end{align}

**Definition 1.5.** For a $\mu$-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we define the modular

\begin{equation}
(1.13) \quad \rho_{p(\cdot),\mu}(f) = \int_{\mathbb{R}^d \Omega} |f(x)|^{p(x)} \mu(dx) + \|f\|_{L^{p^*(\cdot),\mu}},
\end{equation}

and the norm

\begin{equation}
(1.14) \quad \|f\|_{L^{p^*(\cdot),\mu}} = \inf \left\{ \lambda > 0 : \rho_{p(\cdot),\mu}(f/\lambda) \leq 1 \right\}.
\end{equation}

**Definition 1.6.** The variable exponent Lebesgue space on $\mathbb{R}^d$, $L^{p(\cdot)}(\mathbb{R}^d, \mu)$ consists on those $\mu$-measurable functions $f$ for which there exists $\lambda > 0$ such that $\rho_{p(\cdot),\mu}(f/\lambda) < \infty$, i.e.

\[ L^{p(\cdot)}(\mathbb{R}^d, \mu) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ measurable } \rho_{p(\cdot),\mu}(f/\lambda) < \infty, \text{ for some } \lambda > 0 \right\}. \]

It is well known that, if $p(\cdot) \in LH\left(\mathbb{R}^d\right)$ with $1 < p_- \leq p^+ < \infty$ the classical Hardy-Littlewood maximal function $M$ is bounded on the variable Lebesgue space $L^{p(\cdot)}$, see [4]. However, it is known that even though these are the sharpest possible point-wise conditions, they are not necessary. In [7] a necessary and sufficient condition is given for the $L^{p(\cdot)}$-boundedness of $M$, but it is not an easy to work condition. The class $LH(\mathbb{R}^d)$ is also sufficient for the boundedness on $L^{p(\cdot)}$-spaces of classical singular integrals of Calderón-Zygmund type, see [5, Theorem 5.39].

If $\mathcal{B}$ is a family of balls (or cubes) in $\mathbb{R}^d$, we say that $\mathcal{B}$ is $N$-finite if it has bounded overlappings for $N$, this is \( \sum_{B \in \mathcal{B}} \chi_B(x) \leq N \) for all $x \in \mathbb{R}^d$; in other words, there is only $N$ balls (resp cubes) that intersect at the same time.

The following definition was introduced for the first time by Bereznoi in [3], defined for family of disjoint balls or cubes. In the context of variable spaces, it has been considered in [7], allowing the family to have bounded overlappings.

**Definition 1.7.** Given an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, we will say that $p(\cdot) \in \mathcal{G}$, if for every family of balls (or cubes) $\mathcal{B}$ which is $N$-finite,

\[ \sum_{B \in \mathcal{B}} \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \]

for all functions $f \in L^{p(\cdot)}(\mathbb{R}^d)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^d)$. The constant only depends on $N$.

**Lemma 1.2** (Teorema 7.3.22 in [7]). If $p(\cdot) \in LH(\mathbb{R}^d)$, then $p(\cdot) \in \mathcal{G}$.
We will consider only variable Lebesgue spaces with respect to the Gaussian measure \( \gamma_d \), \( L^{p}(\mathbb{R}^d, \gamma_d) \). The next condition was introduced by E. Dalmasso and R. Scotto in [6].

**Definition 1.8.** Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d) \), we say that \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \) if there exist constants \( C_{\gamma_d} > 0 \) and \( p_{\infty} \geq 1 \) such that

\[
|p(x) - p_{\infty}| \leq \frac{C_{\gamma_d}}{|x|^2},
\]

for \( x \in \mathbb{R}^d \setminus \{(0, 0, \ldots, 0)\} \).

**Observation 1.1.** If \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \), then \( p(\cdot) \in LH_{\infty}(\mathbb{R}^d) \)

**Lemma 1.3.** If \( 1 < p_- \leq p_+ < \infty \), the following statements are equivalent

(i) \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \)

(ii) There exists \( p_{\infty} > 1 \) such that

\[
C^{-1}_1 \leq e^{-|x|^2(p(x)/p_{\infty} - 1)} \leq C_1 \quad \text{and} \quad C^{-1}_2 \leq e^{-|x|^2(p'(x)/p'_{\infty} - 1)} \leq C_2,
\]

for all \( x \in \mathbb{R}^d \), where \( C_1 = e^{C_{\gamma_d}/p_{\infty}} \) and \( C_2 = e^{C_{\gamma_d}(p'_{\infty})/p_{\infty}} \).

Definition 1.8 with Observation 1.1 and Lemma 1.3 end up strengthening the regularity conditions on the exponent functions \( p(\cdot) \) to obtain the boundedness of the Ornstein-Uhlenbeck semigroup \( \{T_t\} \), see [12]. As a consequence of Lemma 1.2, we have

**Corollary 1.1.** If \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \), then \( p(\cdot) \in \mathcal{G} \)

As it has been mentioned already, in [6] E. Dalmasso and R. Scotto proved the boundedness of \( T_{F,m} \) on Gaussian variable Lebesgue spaces under the additional condition of regularity \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \).

**Theorem 1.6.** Let \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \). Then there exists a constant \( C > 0 \), depending only in \( p \) and dimension such that

\[
\|T_{F,m}f\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma},
\]

for any \( f \in L^{p(\cdot)}(\gamma_d) \).

The main result in this paper is the proof, following the arguments of Dalmasso and Scotto [6], that the general alternative Gaussian singular integrals \( \overrightarrow{T}_{F,m} \) are also bounded on Gaussian variable Lebesgue spaces under the same condition of regularity on \( p(\cdot) \) considered by Dalmasso and Scotto.

**Theorem 1.7.** Let \( d > 1 \) and \( p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d) \) with \( 1 < p_- \leq p_+ < \infty \). Then there exists a constant \( C > 0 \), depending only in \( p \) and dimension such that

\[
\|\overrightarrow{T}_{F,m}f\|_{p(\cdot),\gamma} \leq C\|f\|_{p(\cdot),\gamma},
\]

for any \( f \in L^{p(\cdot)}(\gamma_d) \).
2. Proof of the main result.

We are ready for the proof of our main result, Theorem 1.7. As usual we split operator $\overline{T}_{F,m}$ into a local and a global part,

$$\overline{T}_{F,m}f(x) = C_d \int_{|x-y|<Cd(x)} \mathcal{K}_{F,m}(x,y)f(y)dy + C_d \int_{|x-y|\geq Cd(x)} \mathcal{K}_{F,m}(x,y)f(y)dy$$

$$= \overline{T}_{F,m,L}f(x) + \overline{T}_{F,m,G}f(x)$$

where

$$\overline{T}_{F,m,L}f(x) = \overline{T}_{F,m}(f\chi_{B_h(x)}) (x)$$

is the local part and

$$\overline{T}_{F,m,G}f(x) = \overline{T}_{F,m}(f\chi_{\tilde{B}_h(x)}) (x)$$

is the global part of $\overline{T}_{F,m}$, and

$$B_h(x) = B(x, Cd(x)) = \{y \in \mathbb{R}^d : |y - x| < Cd(x)\},$$

with $m(x) = 1 \wedge \frac{1}{|x|}$, is an admissible (hyperbolic) ball for the Gaussian measure, see [19, Chapter 1].

Proof. We need to bound the local and the global part.

1. The study of the local part $\overline{T}_{F,m,L}$ is similar to the one done in the proof of [13, Theorem 1.2], see also [6, Lemma 3.2], obtaining the inequality

$$\left| \overline{T}_{F,m,L}f(x) \right| = \left| \overline{T}_{F,m} (f\chi_{B_h(x)}) (x) \right| = \left| \int_{B_h(x)} \mathcal{K}_{F,m}(x-y)f(y)dy \right|$$

$$\leq \sum_{B \in \mathcal{F}} \left| T (f\chi_{\tilde{B}_h(x)}) (x) \right| + M(f\chi_{\tilde{B}_h(x)}) (x).$$

(2.1)

where $M(g)$ is the classical Hardy-Littlewood maximal function of the function $g$, and

$$T f(x) = \text{p.v.} \int_{\mathbb{R}^d} \mathcal{K}(x-y)f(y)dy$$

is a (convolution type) Calderón-Zygmund operator with kernel

$$\mathcal{K}(x) = \int_0^\infty F \left( -\frac{x}{t^{1/2}} \right) e^{-|x|^2/2t} \frac{dt}{t^{d/2+1}}$$

and $\mathcal{F}$ is a countable family of admissible balls such that satisfies the conditions of (covering) Lemma 4.3 of [19] in particular, $\mathcal{F}$ verify

i) For each $B \in \mathcal{F}$ let $\tilde{B} = 2B$, then, the family of those balls $\tilde{\mathcal{F}} = \{B(0,1), \{\tilde{B}\}_{B \in \mathcal{F}}\}$ is a covering of $\mathbb{R}^d$;

ii) $\mathcal{F}$ has a bounded overlaps property;

iii) Every ball $B \in \mathcal{F}$ is contained in an admissible ball, and therefore for any pair $x, y \in B, e^{-|x|^2} \sim e^{-|y|^2}$ with constants independent of $B$

iv) There exists a uniform positive constant $C_d$ such that, if $x \in B \in \mathcal{F}$ then $B_h(x) \subset C_d B := \hat{B}$. Moreover, the collection $\hat{\mathcal{F}} = \{\hat{B}\}_{B \in \mathcal{F}}$ also satisfies the properties ii) and iii).
Then, for \( f \in L^{p_v}(\mathbb{R}^n, \gamma_d) \) we will use the norm on the dual space \( L^{p'_v}(\mathbb{R}^n, \gamma_d) \)

\[
\|T_{F_{m}}(f)(x)\|_{p'_{v}, \gamma_d} \leq 2 \sup_{\|g\|_{p'_{v}}, \gamma_d} \int_{\mathbb{R}^n} |T_{F_{m}}(f)(x) - g(x)| \, \gamma_d(dx)
\]

Using the pointwise inequality (2.1) we split the integral as

\[
\int_{\mathbb{R}^d} |T_{F_{m}}(f)(x) - g(x)| \, \gamma_d(dx) \lesssim \sum_{B \in F} \int_{\mathbb{R}^n} |T_{F_{m}}(f)(x) - g(x)| \, \gamma_d(dx)
\]

where \( c_B \) is the center of \( B \) and \( \hat{B} \) and we have used property iii) above, i.e. that over each ball of the family \( F \), the values of \( \gamma_d \) are all equivalent. Applying Hölder’s inequality with \( p'_{v} \) and \( p'_{v'} \) with respect of the Lebesgue measure in each integral and the boundedness of \( T \) and \( M \) on \( L^{p_{v}}(\mathbb{R}^n) \), we get

\[
\int_{\mathbb{R}^d} |T_{F_{m}}(f)(x) - g(x)| \, \gamma_d(dx) \lesssim \sum_{B \in F} e^{-|x|^2} |T_{F_{m}}(f)(x) - g(x)| \, \gamma_d(dx)
\]

since \( p \in P^\gamma_{\gamma'}(\mathbb{R}^n) \) and \( p'_{v} \). Thus, from Lemma 1.4, for every \( x \in \mathbb{R}^d \),

(2.4) \[
e^{-|x|^2} \lesssim C_1 \text{ and } e^{-|x|^2} \lesssim C_2.
\]

Moreover, since the values of the Gaussian measure \( \gamma_d \) are all equivalent on each ball \( \hat{B} \), we have

\[
\int_{\mathbb{R}^d} \left( \frac{|f(y)|}{e^{k|y|^2/p_{v}} \left\| f_{\hat{B}} \right\|_{p_{v}, \gamma_d}} \right)^{p_{v}} \, d\gamma_d(dy) \lesssim 1
\]
which yields
\[ e^{-|y|^2/p_y} \| f_x \|_{p(\cdot)} \leq \| f_x \|_{p(\cdot),Y}. \]

Similarly, by applying the second inequality of (2.4) we get
\[ e^{-|y|^2/p_y'} \| g_y \|_{p'(\cdot)} \leq \| g_y \|_{p'(\cdot),Y}. \]

Replacing both estimates in (2.3) we obtain
\[
\int_{\mathbb{R}^d} |T_{F,m} f_x \hat{g} dy d(x) \leq \sum_{B \in F} \| f_x \|_{p(\cdot),Y} \| g_y \|_{p'(\cdot),Y} \\
= \sum_{B \in F} \| f_x \|_{p(\cdot)} \| g_y \|_{p'(\cdot)} \|
\]

since the family of balls \( \hat{F} \) has bounded overlaps, from Corollary 1.1 applied to \( e^{-|y|^2/p_y} \in L^{p(\cdot)}(\mathbb{R}^d) \) and \( e^{-|x|^2/p_x} \in L^{p'(\cdot)}(\mathbb{R}^d) \), it follows that
\[
\int_{\mathbb{R}^d} |T_{F,m} f_x \hat{g} dy d(x) \leq \| f \|_{p(\cdot),Y} \| g \|_{p'(\cdot),Y}
\]

Taking the supremum over all functions \( g \) with \( \| g \|_{p'(\cdot),Y} \leq 1 \), from (2.2) we get finally
\[
\| T_{F,m,L} f \|_{p(\cdot),Y} = \| T_{F,m} f_x \|_{p(\cdot),Y} \leq C \| f \|_{p(\cdot),Y}.
\]

ii) For the global part, to handle the kernel \( \overline{K} \), we need the following result

**Lemma 2.1.** Let us consider the kernel \( \overline{K}_{F,m}(x,y) \) in the global region, i.e. \( y \notin B_x(y) = \{ y \in \mathbb{R}^d : |y - x| < C_d m(x) \} \). If \( a = |x|^2 + |y|^2 \) and \( b = 2\langle x, y \rangle \), we have the following inequalities:

i) If \( b \leq 0 \), for each \( 0 < \epsilon < 1 \), there exists \( C_\epsilon > 0 \) such that
\[
|K_{F,m,x,y}| \leq C_\epsilon e^{a\epsilon} + \epsilon |x|^2
\]

ii) If \( b > 0 \), for each \( 0 < \epsilon < \frac{1}{b} \), there exists \( C_\epsilon > 0 \) such that
\[
|K_{F,m,x,y}| \leq C_\epsilon e^{\frac{1}{(1-\epsilon)^2} u_0(t_0)} \epsilon^{\frac{1}{b}}
\]

where \( t_0 = 2 \sqrt{\frac{|y|^2 - |x|^2}{a + \sqrt{a^2 - b^2}}} \) and \( u_0 = \frac{1}{2} \left( |y|^2 - |x|^2 + |x + y||x - y| \right) \).

Now, for the study of the global part \( \overline{T}_{F,m,G} \) let us consider the set
\[
E_x = \{ y : \langle x, y \rangle > 0 \}
\]
and consider two cases:
Case \( b = 2(x, y) \leq 0 \): let \( 0 < \epsilon < \frac{1}{p_d} \), then for \( f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d) \) with \( \|f\|_{p(\cdot), \gamma_d} = 1 \) we have, by Lemma 2.1

\[
\int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} |\mathcal{F}_{F_m}(x, y)||f(y)|dy \right)^{p(x)} \gamma_d(dx) \leq \int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} e^{-|b|^2 + \epsilon |y|^2} |f(y)|dy \right)^{p(x)} \gamma_d(dx)
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(y)\gamma_d(dy)\right)^{p(\cdot)} e^{p(\cdot)|x|^2} |dy|\gamma_d(dx)
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(y)\gamma_d(dy)\right)^{p(\cdot)} e^{(p(\cdot)-1)|x|^2} dx.
\]

Since, by hypothesis, \( \int_{\mathbb{R}^d} |f(x)|^{p(x)} \gamma_d(dx) \leq 1 \) and therefore,

\[
\int_{\mathbb{R}^d} |f(x)|^{p(\cdot)} \gamma_d(dx) \leq \int_{|f| > 1} |f(x)|^{p(x)} \gamma_d(dx) + \int_{|f| \leq 1} \gamma_d(dx)
\]

\[
\leq 1 + C_d.
\]

Then,

\[
\int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} |\mathcal{F}_{F_m}(x, y)||f(y)|dy \right)^{p(x)} \gamma_d(dx) \leq \int_{\mathbb{R}^d} (1 + C_d)^{p(\cdot)} e^{(p(\cdot)-1)|x|^2} dx
\]

\[
\leq (1 + C_d)^{p(\cdot)} \int_{\mathbb{R}^d} e^{(p(\cdot)-1)|x|^2} dx = C_{p,d}.
\]

In other words,

\[
\|\mathcal{T}_{F_m}(f \chi_{B^d(\cdot) \cap E^d})\|_{p(\cdot), \gamma_d} \leq C_{p,d},
\]

for all \( f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d) \) with \( \|f\|_{p(\cdot), \gamma_d} = 1 \).

Case \( b = 2(x, y) > 0 \): for \( f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d) \) with \( \|f\|_{p(\cdot), \gamma_d} = 1 \) we have

\[
(I) = \int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} |\mathcal{F}_{F_m}(x, y)||f(y)|dy \right)^{p(x)} \gamma_d(dx)
\]

\[
\leq C \int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} e^{\epsilon |x|^2 - |y|^2} \frac{e^{-(1-\epsilon)\sigma(t_0)}}{t_0^{d/2}} |f(y)|dy \right)^{p(x)} e^{-\epsilon |x|^2} (dx)
\]

\[
= C \int_{\mathbb{R}^d} \left( \int_{B^d(\cdot) \cap E^d} e^{-\epsilon(1-\epsilon)\sigma(t_0)} \frac{|x|^2}{p_0^{d/2}} e^{-\frac{|x|^2}{p_0 \sigma(t_0)}} e^{-\epsilon |y|^2} |f(y)|e^{-\frac{|y|^2}{p_0 \sigma(t_0)}} dy \right)^{p(x)} (dx).
\]
Now, using the inequality \(|y|^2 - |x|^2| \leq |x + y||x - y|\), and that on the global region, \(|x + y||x - y| > d\), as \(b > 0\), we have

\[
\int_{C} e^{-(1-\epsilon)u(t_0)} e^{\frac{|y|^2}{p_0}} e^{\frac{|x|^2}{p_0^\epsilon}} e^{-\epsilon(|y|^2 - |x|^2)} \leq C e^{-(1-\epsilon)u(t_0)} e^{\frac{|y|^2}{p_0}} e^{\frac{|x|^2}{p_0^\epsilon}} \int_{C} e^{-\epsilon(|y|^2 - |x|^2)}
\]

where,

\[
\alpha_{\infty} = \frac{(1-\epsilon)}{2} - \left| \frac{1}{p_0^\epsilon} - \frac{1}{1-\epsilon} \right| > 0, \quad \text{if} \quad \epsilon < \frac{1}{p_0^\epsilon}.
\]

Thus, we take \(0 < \epsilon < \min\left(\frac{1}{p_0^\epsilon}, \frac{1}{p_x}\right)\). Now, let us consider the kernel

\[
P(x,y) := |x + y|^d e^{-\alpha_{\infty} |x + y||x - y|},
\]

which is integrable in each variable (since it is symmetric), with constant independent of \(x\) and \(y\). Then,

\[
\int_{B'_{\epsilon}(x) \cap E_x} e^{-(1-\epsilon)u(t_0)} e^{\frac{|y|^2}{p_0}} e^{\frac{|x|^2}{p_0^\epsilon}} e^{-\epsilon(|y|^2 - |x|^2)} |f(y)| e^{-\frac{|y|^2}{p_0^\epsilon}} dy \leq C \int_{B'_{\epsilon}(x)} P(x,y) |f(y)| e^{-\frac{|y|^2}{p_0^\epsilon}} dy
\]

Set \(A_{\epsilon} = \left\{ y : |y - x| < \frac{\epsilon}{4} \right\}\) and \(C_{\epsilon} = B_{\epsilon}(x, 1/2) = \left\{ y : |y - x| > \frac{\epsilon}{2} \right\}\).

Therefore, \(B'_{\epsilon}(x) \subset A_{\epsilon} \cup C_{\epsilon}\). Define

\[
J_1 = \int_{A_{\epsilon} \cap E_x} P(x,y) f(y) e^{-|y|^2/p(x)} dy,
\]

and

\[
J_2 = \int_{C_{\epsilon} \cap E_x} P(x,y) f(y) e^{-|y|^2/p(x)} dy.
\]

Let us estimate \(J_1\) first. Observe that, if \(y \in A_x, \frac{3}{4}|x| \leq |y| \leq \frac{5}{4}|x|\) and then \(|x| \approx |y|\) hence \(|x| \approx |x + y|\), and thus

\[
J_1 \leq \int_{\frac{3}{4}|x| \leq |y|} |x| e^{-\alpha_{\infty} |x - y|} f(y) e^{-|y|^2/p(x)} dy
\]

\[
\leq M(f e^{-\epsilon |y|^2/p(x)})(x).
\]

(2.5)

It is known that the Hardy-Littlewood maximal function is weak \((1, 1)\) in in Lebesgue variable spaces, see [4], see also [12], then from the hypothesis on \(p(x)\) we get

\[
\|M(f e^{-\epsilon |y|^2/p(x)})\|_{p(x)} \leq \|f e^{-\epsilon |y|^2/p(x)}\|_{p(x)} = \|f\|_{p(x), y_d} = 1,
\]
and then
\[ \rho_{p(\cdot)} \left( M(f e^{-|\cdot|^2/p(\cdot)}) \right) \lesssim 1. \]

Now, in order to estimate \( J_2 \), we have
\[ J_2 \leq \| P(x, \cdot) \chi_{C_i} \|_{p(\cdot)} \leq C, \]
for details see [6]. This implies that there exists a constant independent on \( x \) such that,
\[ J_2 = \int_{C_i \cap E_i} P(x, y) f(y) e^{-|y|^2/p(y)} dy \leq C \]
thus
\[ \frac{1}{C} \int_{C_i \cap E_i} P(x, y) f(y) e^{-|y|^2/p(y)} dy \leq 1. \]

We set \( g(y) = f(y) e^{-|y|^2/p(y)} = g_1(y) + g_2(y) \), where \( g_1 = g \chi_{\{g \leq 1\}} \) and \( g_2 = g \chi_{\{g > 1\}} \); applying (2.5), we have

\[
(I) \quad \lesssim \int_{\mathbb{R}^d} \left( \int_{B(y) \cap E_i} P(x, y) f(y) e^{-|y|^2/p(y)} dy \right)^{p(x)} dx
\]
\[ \lesssim \int_{\mathbb{R}^d} (J_1)^{p(x)} dx + \int_{\mathbb{R}^d} (J_2)^{p(x)} dx \]
\[ \lesssim \rho_{p(\cdot)} \left( M_{H-L}(f e^{-|\cdot|^2/p(\cdot)}) \right) + \int_{\mathbb{R}^d} \left( \frac{1}{C} \int_{C_i \cap E_i} P(x, y) g_1(y) dy \right)^{p(x)} dx \]
\[ + \int_{\mathbb{R}^d} \left( \frac{1}{C} \int_{C_i \cap E_i} P(x, y) g_2(y) dy \right)^{p(x)} dx \]
\[ \leq 1 + (II) + (III) \]

Now, we study the terms (II) and (III).

\[
(II) = \int_{\mathbb{R}^d} \left( \frac{1}{C} \int_{C_i \cap E_i} P(x, y) g_1(y) dy \right)^{p(x)} dx \leq \int_{\mathbb{R}^d} \left( \int_{C_i \cap E_i} P(x, y) g_1(y) dy \right)^{p(x)} dx
\]

On the other hand, using Lemma 1.1 with \( G(x) = \frac{1}{C} \int_{C_i \cap E_i} P(x, y) g_2(y) dy \leq 1 \) and applying the inequality (1.11), we obtain

\[
(III) = \int_{\mathbb{R}^d} \left( \frac{1}{C} \int_{C_i \cap E_i} P(x, y) g_2(y) dy \right)^{p(x)} dx = \int_{\mathbb{R}^d} (G(x))^{p(x)} dx
\]
\[ \lesssim \int_{\mathbb{R}^d} (G(x))^{p_\infty} dx + \int_{\mathbb{R}^d} \frac{dx}{(e + |x|)^d p_\infty} \]
\[ = \int_{\mathbb{R}^d} \left( \int_{C_i \cap E_i} P(x, y) g_2(y) dy \right)^{p_\infty} + C_{d,p}. \]

Therefore
\[ (I) \lesssim \int_{\mathbb{R}^d} \left( \int_{C_i \cap E_i} P(x, y) g_1(y) dy \right)^{p(x)} dx + \int_{\mathbb{R}^d} \left( \int_{C_i \cap E_i} P(x, y) g_2(y) dy \right)^{p(x)} dx + C_{d,p} \]
Now, in order to estimate the last two integrals, we apply Hölder’s inequality.

\[
\int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_1(y) dy \right)^{p_{-}} dx \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} P(x, y)^{\frac{1}{p_{-}}} \int_{\mathbb{R}^d} P(x, y)^{\frac{1}{p_{-}}} g_1(y) dy \right)^{p_{-}} dx
\]

\[
\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} (P(x, y))^{\frac{1}{p_{-}/p_{-}} - \frac{1}{p_{-}}} dy \right)^{p_{-}/p_{-}} \left( \int_{\mathbb{R}^d} (P(x, y))^{\frac{1}{p_{-}/p_{-}} - \frac{1}{p_{-}}} g_1(y) dy \right)^{p_{-}/p_{-}} dx
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} P(x, y) dy \right)^{p_{-}/p_{-}} \left( \int_{\mathbb{R}^d} P(x, y)g_1(y) dy \right)^{p_{-}/p_{-}} dx
\]

\[
\leq \int_{\mathbb{R}^d} P(x, y)g_1(y) dy dx.
\]

Then, by Fubini’s theorem we get,

\[
\int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_1(y) dy \right)^{p_{-}} dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P(x, y)g_1(y) dy dx
\]

\[
= \int_{\mathbb{R}^d} S_1^{p_{-}}(y) \left( \int_{\mathbb{R}^d} P(x, y) dx \right) dy
\]

\[
\leq \int_{\mathbb{R}^d} (g_1(y))^{p_{-}} dy
\]

\[
\leq \int_{\mathbb{R}^d} f(y)^{p_{-}} e^{-b^2} dy \leq \rho_{p(-),y_d}(f)
\]

To estimate the integral \( \int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_2(y) dy \right)^{p_{-}} dx \), we proceed in analogous way, but applying the Hölder’s inequality to the exponent \( p_{-}\omega \), and applying the inequality (1.12) in Lemma 1.1. In consequence we obtain

\[
\int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_2(y) dy \right)^{p_{-}\omega} dx \leq \int_{\mathbb{R}^d} S_2^{p_{-}\omega}(y) dy + C_{d,p}
\]

\[
\leq \rho_{p(-),y_d}(f) + C.
\]

Therefore,

\[
(I) \leq \int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_1(y) dy \right)^{p_{-}} dx + \int_{\mathbb{R}^d} \left( \int_{C_{i} \cap E_{i}} P(x, y)g_2(y) dy \right)^{p_{-}\omega} dx + C_{d,p}
\]

\[
\leq 2 \rho_{p(-),y_d}(f) + C_{d,p}
\]

Hence, we obtain that

\[
\|T_{F,m} f(x_{B_{h}^{(c)} \cap E_{i}})\|_{p(-),y} \leq C,
\]

for \( \|f\|_{p(-),y_d} = 1 \).

Putting together both cases we get

\[
\|T_{F,m} f(x_{B_{h}^{(c)}})\|_{p(-),y} \leq C,
\]
for $\|f\|_{p(.),\gamma_d} = 1$. Then by homogeneity of the norm the result holds for all function $f \in L^{p(.)}(\mathbb{R}^d, \gamma_d)$. Now the proof of the theorem is complete.

□

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