CONGLOMERABILITY AND THE REPRESENTATION OF LINEAR FUNCTIONALS

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Abstract. We prove results concerning the representation of linear functionals as integrals of a given random quantity $X$. The existence of such representation is related to the notion of conglomerability, originally introduced by de Finetti and Dubins. We show that this property has interesting applications in probability and in analysis. These include a version of the extremal representation theorem of Choquet, a proof of Skorohod theorem and of the statement that Brownian motion assumes whatever family of finite dimensional distributions upon a change of the probability measure.

1. Introduction.

In this paper we study the classical problem of the integral representation of linear functionals with a degree of generality which does not permit the direct application of classical techniques. Conglomerability is then necessary and sufficient to conveniently transform the original problem into one in which integral representation is indirectly possible. Although the fields in which our results may be fruitfully applied are disparate, we were motivated by the problem of existence of companions that arises in several places in probability and statistics.

Let $S$ and $\Omega$ be given, non empty sets, $\mathcal{H}$ a family of real valued functions on $S$ and $(X, m)$ a pair, with $X$ a mapping of $\Omega$ into $S$ and $m$ a positive, finitely additive set function $m$ on $\Omega$. Following Dubins and Savage [16], we say that a pair $(X', \mu)$ on a set $\Omega'$ is a companion to $(X, m)$, relatively to $\mathcal{H}$, if it solves the equation

\[ h(X') \in L^1(\mu) \quad \text{and} \quad \int h(X)dm = \int h(X')d\mu \quad h \in \mathcal{H}, h(X) \in L^1(m). \]

The collection $\mathcal{H}$ is interpreted as a model of the information available.

Finding a correct statistical model $X'$ for a given data sample is a problem fitting into (1): set $S = \Omega = \mathbb{R}$, let $X$ be the identity, $m$ the sample distribution and each $h \in \mathcal{H}$ a statistic. Given a predictive marginal $m$ on an algebra $\mathcal{A}$, a similar problem in Bayesian statistics is that of finding a parametric family $Q = \{Q_\theta : \theta \in \Theta\}$ of probabilities and a prior $\lambda$ on the parameter space $\Theta$ such that

\[ m(A) = \int_\Theta Q_\theta(A)d\lambda \quad A \in \mathcal{A}. \]

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Dubins [17] proved long ago that the existence of a disintegration formula similar to (2) is equivalent to conglomerability, a notion originally due to de Finetti [13] that has remained undeservedly neglected outside a limited number of distinguished authors (which include Schervish et al. [30], Hill and Lane [22] and Zame [31]). The conglomerability property, we believe, may be formulated in more general terms than those in which it was originally stated and it may be applied to more ambitious problems in probability and in analysis than those for which it had been originally devised.

We found it useful to write problem (1) in more abstract terms, replacing the integral on left hand side with a linear functional and modelling the action of $X'$ on $\mathcal{H}$ as a linear transformation. We solve this version of our problem in Theorem 1, obtaining a special integral representation for a conglomerative linear functional on an arbitrary vector space. The absence of any structure on the underlying space, save linearity, makes the claim significantly more general than classical Riesz representation theorems. In this functional analytic formulation, conglomerability may be nicely restated as a geometric property. In Corollary 2 we show that if $\Phi$ and $\Psi$ are two sets of positive, linear functionals on a vector lattice then $\Phi$ is $\Psi$-conglomerative if and only if each $\phi \in \Phi$ is the barycentre of a measure supported by $\Psi$. In Corollary 3 we obtain a generalization of the original theorem of Choquet [9].

Theorem 1 admits a large number of implications, the most immediate of which is the existence of companions with or without additional conditions on the representing measure $\mu$, such as countable additivity or absolute continuity with respect to some given, reference set function. An immediate corollary is that, relatively to continuous functions, a normally distributed random quantity is companion to any $X$ and that Brownian motion can assume whatever family of finite dimensional distributions on $\mathbb{R}$ upon an appropriate choice of the underlying probability. We also provide applications to the classical Skhorohod representation theorem in the case in which $S$ is separable.

In the closing section we prove some results concerning the representation of convex functions as integrals. We show that any convex function on $\mathbb{R}$ decomposes into the sum of a piece wise linear component and an integral part, a representation curiously near to the one popular in mathematical finance as a model for option prices.

All proofs are quite simple and, despite a natural interest for countable additivity, they are obtained by exploiting the theory of the finitely additive integral in which the measurability constraint is much less burdensome. We hope to disprove thus, at least partially, the harsh judgement of Bourgin [6, p. 173] that “an integral representation theory based on finitely additive measures is virtually useless”.

2. Notation and Preliminaries.

Throughout the paper the symbol $\mathcal{F}(\Omega, S)$ (resp. $\mathcal{F}(\Omega)$) denotes the family of functions mapping $\Omega$ into $S$ (resp. into $\mathbb{R}$) and $\mathcal{F}$ is replaced with $\mathcal{L}$, $\mathcal{C}$ or $\mathcal{C}_K$ when restricting to linear, continuous or continuous functions with compact support, respectively. A collection $\{f_y : y \in Y\} \subset \mathcal{F}(X, S)$ is
also written as a function \( f \in \mathcal{F}(X \times Y, S) \) with \( f(x, y) = f_y(x) \) and vice versa. If \( f \in \mathcal{F}(\Omega, S) \) and \( A \subset \Omega \) the symbols \( f|A \) and \( f[A] \) designate the restriction of \( f \) to \( A \) and the image of \( A \) under \( f \). A subset \( \mathcal{H} \) of \( \mathcal{F}(S) \) is Stonean if \( h \in \mathcal{H} \) implies \( h \wedge 1 \in \mathcal{H} \), where \( 1 \in \mathcal{F}(S) \) indicates the function constantly equal to 1.

If \( \mathcal{A} \) is a ring of subsets of \( \Omega \), then \( \mathcal{F}(\mathcal{A}) \) and \( \mathfrak{B}(\mathcal{A}) \) denote the families of \( \mathcal{A} \) simple functions and its closure in the topology of uniform convergence, respectively. \( fa(\mathcal{A}) \) (resp. \( fa(\Omega) \)) is the space of real valued, finitely additive set functions on \( \mathcal{A} \) (resp. \( 2^\Omega \)) and \( ba(\mathcal{A}) \) the subspace of set functions of bounded variation. To indicate that \( \mathcal{A} \) is a ring of subsets of \( \Omega \) and that \( \lambda \in fa(\mathcal{A})_+ \), i.e. that \((\mathcal{A}, \lambda)\) is a measure structure on \( \Omega \) we write more compactly \((\mathcal{A}, \lambda) \in \mathcal{M}(\Omega)\).

We recall a few definitions and facts relative to the finitely additive integral (see [4] and [19]) given \((\mathcal{A}, \lambda) \in \mathcal{M}(\Omega)\). \( X \in \mathcal{F}(\Omega) \) is \( \lambda \)-measurable if there exists a sequence \( \langle X_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{F}(\mathcal{A}) \) that \( \lambda \)-converges to \( X \), i.e. such that

\[
\text{lim}_{n} \lambda^* (|X_n - X| > c) = 0 \quad \text{for every} \quad c > 0
\]

where the set function \( \lambda^* \) and its conjugate \( \lambda_* \) are defined (with the convention \( \inf \emptyset = \infty \)) as

\[
\lambda^*(E) = \inf_{\{A \in \mathcal{A} : E \subset A\}} \lambda(A) \quad \text{and} \quad \lambda_*(E) = \sup_{\{B \in \mathcal{A} : B \subset E\}} \lambda(B) \quad E \subset \Omega.
\]

If \( S \) is a topological space, \( X \) is \( \lambda \)-tight if for all \( \varepsilon > 0 \) there exists \( K \subset S \) compact such that \( \lambda^*(X \notin K) < \varepsilon \). \( X \) is \( \lambda \)-integrable, \( X \in L^1(\lambda) \), if there is a sequence \( \langle X_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{F}(\mathcal{A}) \) that \( \lambda \)-converges to \( X \) and is Cauchy in \( L^1(\lambda) \); we then write \( \int X d\lambda \) or \( \int_{\Omega} X(\omega) d\lambda(\omega) \). We notice that if \( A, B \subset \Omega \) and \( f \in L^1(\lambda) \), then

\[
\mathbbm{1}_A \leq f \leq \mathbbm{1}_B \quad \text{implies} \quad \lambda^*(A) \leq \int f d\lambda \leq \lambda_*(B).
\]

The following collections are important:

\[
\text{(6a)} \quad \mathcal{D}(X, \lambda) = \left\{ t > 0 : \lim_{n} \lambda_* (X > t - 2^{-n}) = \lim_{n} \lambda_* (X > t + 2^{-n}) \right\};
\]

\[
\text{(6b)} \quad \mathcal{R}_0(X, \lambda) = \left\{ \{X > t\} : t \in \mathcal{D}(X, \lambda) \right\} \cup \left\{ \{-X > u\} : u \in \mathcal{D}(-X, \lambda) \right\};
\]

\[
\text{(6c)} \quad \mathcal{A}(\lambda) = \left\{ E \subset \Omega : \lambda^*(E) = \lambda_*(E) < \infty \right\}.
\]

There is clearly just one extension of \( \lambda \) to \( \mathcal{A}(\lambda) \) and \( X \) is \( \lambda \)-measurable if and only if it is measurable with respect to such extension, which we shall denote, accordingly, again by \( \lambda \). A sequence \( \langle X_n \rangle_{n \in \mathbb{N}} \) in \( L^1(\lambda) \) converges to \( X \) in norm if and only if it \( \lambda \)-converges to \( X \) and is Cauchy in the norm of \( L^1(\lambda) \). \[19 \text{ III.3.6].} \]

\[1\] To be formal, we depart from the classical theory of Dunford and Schwartz which has an extended real valued set function on an algebra of sets as its starting point. Our notion of a simple function is obtained from theirs after restricting to the family of sets of finite measure, a ring, and coincides therefore with the notion of integrable simple functions of Dunford and Schwartz. Thus, our notion of measurability is more restrictive than that of total measurability given in \[19 \text{ III.2.10]} \) although integrable functions are defined by Dunford and Schwartz as being measurable in our restrictive sense.
We shall use the following results on measurability and integrability of a positive function.

**Lemma 1.** Let $X \geq 0$. $X$ is $\lambda$-measurable if and only if it is $\lambda$-tight and either (i) $\infty > \lambda_s(X > s)$ $\geq \lambda^*(X \geq t)$ for all $0 < s < t$, (ii) $\mathcal{R}_0(X, \lambda) \subset \mathcal{A}(\lambda)$, or (iii) the set $\{t > 0 : \{X > t\} \in \mathcal{A}(\lambda)\}$ is dense in $\mathbb{R}_+$.

**Proof.** If $X$ is $\lambda$-measurable it is $\lambda$-tight, [44 p. 190]. Choose $\langle X_k \rangle_{k \in \mathbb{N}}$ in $\mathcal{R}(\mathcal{A})$ $\lambda$-convergent to $X$, fix $s, \eta > 0$ and $A^\eta_k \subset \mathcal{A}$ such that $\{|X - X_k| \geq \eta\} \subset A^\eta_k$ and $\lambda(A^\eta_k) \leq \lambda^*(|X - X_k| \geq \eta) + 2^{-k}$.

$$\{X \geq s + 2\eta\} \subset \{X_k \geq s + \eta\} \cup A^\eta_k \subset \{X > s\} \cup A^\eta_k$$

so that $\lambda^*(X \geq s + 2\eta) \leq \lambda(\{X_k \geq s + \eta\} \cup A^\eta_k) \leq \lambda_s(X > s) + \lambda(A^\eta_k)$ and $\lambda^*(X \geq s + 2\eta) < \infty$. Assume (i). If $t \in D(X, \lambda)$ then $\infty > \lambda_s(X > t) = \lim_n \lambda_s(X > t - 2^{-n}) \geq \lambda^*(X \geq t) \geq \lambda^*(X > t)$.

(ii) $\Rightarrow$ (iii) is obvious. Assuming (iii), choose $\{0 = t^0_1 \leq t^1_1 \leq \ldots \leq t^n_1 \leq t^n_{i+1} = 2^n\}$ such that $\{X > t^n_i\} \in \mathcal{I}(\lambda)$ for $i = 1, \ldots, n$ and $\sup_{0 \leq i \leq n} |t^n_i - t^n_{i+1}| < 2^{-n}$. Define

$$X_n = \sum_{i=1}^{2^{n-1}} t^n_i 1_{\{t^n_i < X \leq t^n_{i+1}\}} \in \mathcal{I}(\mathcal{A}(\lambda)).$$

Then $\{|X - X_n| \geq 2^{-n}\} \subset \{X > 2^{-n}\}$ so that $X_n \lambda$-converges to $X$ whenever $X$ is $\lambda$-tight. □

**Lemma 2.** Let $X \geq 0$. $X \in L^1(\lambda)$ if and only if $\int_0^\infty \lambda_s(X > t)dt = \int_0^\infty \lambda^*(X > t)dt < \infty$. Then,$$\int Xd\lambda = \int_0^\infty \lambda_s(X > t)dt.$$  

**Proof.** Assume $\int \lambda_s(X > t)dt = \int \lambda^*(X > t)dt < \infty$. Then $X$ is $\lambda$-tight and $\{t \in \mathbb{R} : \{X > t\} \in \mathcal{A}(\lambda)\}$ is dense in $\mathbb{R}_+$ so that $X$ is $\lambda$-measurable. As in (7) we can construct an increasing sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{I}(\mathcal{A}(\lambda))$ such that $0 \leq X_n \leq X$ and $\lambda$-converges to $X$. But then,

$$\infty > \int_0^\infty \lambda_s(X > t)dt \geq \lim_n \int_0^\infty \lambda(X_n > t)dt = \lim_n \int X_n d\lambda = \int X d\lambda$$

as $\langle X_n \rangle_{n \in \mathbb{N}}$ is Cauchy in $L^1(\lambda)$. Assume conversely that $X \in L^1(\lambda)$ and take $b > a > \varepsilon > 0$. If $\langle X_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{I}(\mathcal{A}(\lambda))$ converges to $X$ in $L^1(\lambda)$, then

$$\int_{a + \varepsilon}^{b + \varepsilon} \lambda^*(X > t)dt \leq \int_{a + \varepsilon}^b \lambda(X_n > t)dt + (b - a)\lambda^*(|X - X_n| > \varepsilon) \leq \int_{a - \varepsilon}^{b - \varepsilon} \lambda_s(X > t)dt + 2(b - a)\lambda^*(|X - X_n| > \varepsilon)$$

by [4] 3.2.8.(iii). Thus, $\int_{a}^{b} \lambda_s(X > t)dt = \int_{a}^{b} \lambda^*(X > t)dt$ and

$$\int_{a}^{b} \lambda^*(X > t)dt = \lim_n \int_{a}^{b} \lambda(X_n > t)dt = \lim_n \int (b \land X_n - a)^+ d\lambda = \int (b \land X - a)^+ d\lambda.$$  

Thus $\int_0^\infty \lambda_s(X > t)dt = \int_0^\infty \lambda^*(X > t)dt = \int X d\lambda < \infty$ and (5) holds. □
Proving uniqueness of the set function generating a given class of integrals requires to identify a minimal element in $\mathcal{M}(\Omega)$ as $\mathcal{H}$ is closed with respect to set difference and, from (10) with respect to intersection and this fact together the linear structure of $\lambda$ and $t > s > 0$ and $\lambda_{\phi}$ coincide on the ring $\mathcal{R}$ and $\lambda_{\phi}$ coincides on $\mathcal{R}_{\phi}$ with $\lambda_{\phi}$ for all $\mathcal{R}_{\phi}$-measurable and therefore $\int h d\lambda_{\phi} = \int h d\lambda_{\phi}$, by [19] II.8.1(e).
Although the minimal structure \((\mathcal{A}_\phi, \lambda_\phi)\) will generally depend on \(\phi\), the generated \(\sigma\) ring corresponds to the usual notion, as \(D(h, \lambda)\) is dense.

The next result, and its use in Theorem \([\text{II}]\) provides the best illustration of our interest for set functions defined on rings.

**Lemma 4.** Let \(g \in \mathcal{F}(\Omega)_+\) be \(\lambda\)-measurable and define the ring \(\mathcal{B}_g = \{A \in \mathcal{A}(\lambda) : g1_A \in L^1(\lambda)\}\). There exists a unique \(\lambda_g \in \mathcal{F}(\Omega)_+\) such that

\[
\int f\lambda_g = \int fgd\lambda \quad f \in \mathcal{B}(\lambda), \; fg \in L^1(\lambda).
\]

**Proof.** \([\text{14}]\) implies \(\lambda_g(A) = \int 1_Agd\lambda\) for every \(A \in \mathcal{B}_g\) and thus uniqueness. In proving \([\text{14}]\) we may assume \(f \in \mathcal{B}(\lambda)_+\). Let \((f_n)_{n \in \mathbb{N}}\) be an increasing sequence in \(\mathcal{F}(\mathcal{A}(\lambda))\) such that \(0 \leq f_n \leq f\) and \(f_n\) converges to \(f\) uniformly, obtained as in \([\text{7}]\). Then \(f_n\) is \(\lambda\)- and \(\lambda_g\)-convergent to \(f\). Moreover, \(f_n\) and \(f_ng\) are Cauchy sequence in \(L^1(\lambda_g)\) and \(L^1(\lambda)\). \(\square\)

3. INTEGRAL REPRESENTATION OF LINEAR FUNCTIONALS.

First we make the notion of conglomerability precise.

**Definition 1.** Let \(\mathcal{H}\) be a vector space. Then \(\phi \in \mathcal{L}(\mathcal{H})\) is said to be conglomerable with respect to \(T \in \mathcal{F}(\mathcal{H}, \mathcal{F}(\Omega))\) (or \(T\)-conglomerative) if \(\phi(h) < 0\) implies \(\inf_\omega (Th)(\omega) < 0\) for all \(h \in \mathcal{H}\).

\(T\)-conglomerative linear functionals form a convex cone in \(\mathcal{L}(\mathcal{H})\) which is \(\mathcal{H}\)-closed, i.e. closed in the topology induced by \(\mathcal{H}\) on \(\mathcal{L}(\mathcal{H})\). Another key property is the following:

**Definition 2.** Let \(\mathcal{H}\) be a vector space. A map \(T \in \mathcal{L}(\mathcal{H}, \mathcal{F}(\Omega))\) is said to be directed if:

\[
\forall h \in \mathcal{H}, \; \exists h' \in \mathcal{H} \quad \text{such that} \quad |Th| \leq Th'.
\]

Proving property \([\text{15}]\) will be a delicate step in most of the applications that follow. Two easy special cases are: (a) when \(\mathcal{H}\) is a vector lattice and \(T\) is positive and (b) when \(T[\mathcal{H}] \subset \mathcal{B}(\Omega)\) and \(\sup_h \inf_\omega (Th)(\omega) > 0\) – e.g. if \(T[\mathcal{H}]\) contains the constants. In general, there are several important situations in which \(\mathcal{H}\) is an ordered vector space but not a lattice. In such general situations a possibility is to restrict to the vector space

\[
\mathcal{H}(T) = \{h \in \mathcal{H} : |Th| \leq Th' \text{ for some } h' \in \mathcal{H}\}
\]
on which \(T\) is directed, by construction.

Most results in this paper follow from the next claim.

**Theorem 1.** Let \(\mathcal{H}\) be a vector space and let \(T \in \mathcal{L}(\mathcal{H}, \mathcal{F}(\Omega))\) be directed. Write \(L = \{f \in \mathcal{F}(\Omega) : |f| \leq Th \text{ for some } h \in \mathcal{H}\}\). Then \(\phi \in \mathcal{L}(\mathcal{H})\) is \(T\)-conglomerative if and only if there exist (i) \(F^\perp \in \mathcal{L}(L)_+\) with \(F^\perp[L \cap \mathcal{B}(\Omega)] = \{0\}\) and (ii) \((\mathcal{A}, \mu) \in \mathcal{H}(\Omega)\) such that

\[
L \subset L^1(\mu) \quad \text{and} \quad \phi(h) = F^\perp(Th) + \int Thd\mu \quad h \in \mathcal{H}.
\]

Moreover,
(a) \(\|\mu\| = 1\) if and only if \(\inf_\omega (Th)(\omega) \leq \phi(h)\) for all \(h \in \mathcal{H}\),
(b) \(\mu\) may be chosen to be countably additive if \(\limsup_n \phi(h_n) \leq 0\) for all sequences \((h_n, f_n)_{n \in \mathbb{N}}\)
in \(\mathcal{H} \times L\) satisfying
\[
1 \geq f_n \downarrow 0 \quad \text{and} \quad \limsup_n \sup \{\phi(g) : g \in \mathcal{H}, Tg \leq |f_n - Th_n|\} \leq 0,
\]
(c) for each \(L_0 \subset L \cap \mathfrak{B}(\Omega)\), \(\mu\) may be chosen to be \(L_0\)-maximal, i.e. maximal as a map on \(L_0\).

Proof. \(T[\mathcal{H}]\) is a majorizing subspace of the vector lattice \(L\), by \((15)\). If \(\phi\) is \(T\)-conglomerative
\[
F(Th) = \phi(h) \quad h \in \mathcal{H}
\]
implicitly defines a positive linear functional \(F\) on \(T[\mathcal{H}]\). By \([1, \text{Theorem 1.32}]\), \(F\) extends as a positive linear functional (still denoted by \(F\)) to the whole of \(L\). For each \(\alpha \subset \mathcal{H}\) finite, let \(h_\alpha \in \mathcal{H}\) be such that \(Th_\alpha \geq \bigvee_{h \in \alpha} |Th|\), \(\Omega_\alpha = \{Th_\alpha \neq 0\}\) and define \(I_\alpha \in \mathfrak{F}(L, \mathfrak{F}(\Omega_\alpha))\) by letting
\[
I_\alpha(f)(\omega) = \frac{f(\omega)}{Th_\alpha(\omega)} \quad f \in L, \, \omega \in \Omega_\alpha.
\]
Let also
\[
L_\alpha = \{f \in L : |f| \leq cTh_\alpha \text{ for some } c > 0\} \quad \text{and} \quad H_\alpha = I_\alpha[L_\alpha].
\]

\(H_\alpha\) is a sublattice of \(\mathfrak{B}(\Omega_\alpha)\) containing the constants; \(f, g \in L_\alpha\) and \(I_\alpha(f) \geq I_\alpha(g)\) imply \(f \geq g\).

Thus, upon writing
\[
U_\alpha(I_\alpha(f)) = F(f) \quad f \in L_\alpha
\]
we obtain yet another positive, linear functional \(U_\alpha\) on \(H_\alpha\). \([8, \text{Theorem 1}]\) implies
\[
U_\alpha(I_\alpha(f)) = \int I_\alpha(f) d\bar{m}_\alpha \quad f \in L_\alpha
\]
for some \(\bar{m}_\alpha \in ba(\Omega_\alpha)_+\). Let \(m_\alpha(A) = \bar{m}_\alpha(A \cap \Omega_\alpha)\) for each \(A \subset \Omega\). By Lemma \([4]\) we can write
\[
F(f) = \int \frac{f}{Th_\alpha} I_{\Omega_\alpha} dm_\alpha = \int f d\bar{m}_\alpha \quad f \in L_\alpha \cap \mathfrak{B}(\Omega)
\]
with \(\bar{m}_\alpha = m_\alpha, g\) defined as in \([14]\) with \(g = I_{\Omega_\alpha}/Th_\alpha\). Since \(L_\alpha \cap \mathfrak{B}(\Omega)\) is a Stonean lattice, we deduce from Lemma \([3]\) the existence of a minimal \((\mathcal{R}_\alpha, \mu_\alpha) \in \mathcal{M}(\Omega)\) supporting the representation \((23)\). Define \(\mathcal{R} = \bigcup_\alpha \mathcal{R}_\alpha\) and \(\mu(A) = \lim_\alpha \mu_\alpha(A)\) for all \(A \in \mathcal{R}\). \(\alpha \subset \alpha'\) implies \(L_\alpha \subset L_{\alpha'}\), \((\mathcal{R}_\alpha, \mu_\alpha) \leq (\mathcal{R}_{\alpha'}, \mu_{\alpha'})\) as well as the martingale restriction
\[
\mu_\alpha = \mu_{\alpha'}|\mathcal{R}_\alpha = \mu|\mathcal{R}_\alpha \quad \alpha \subset \alpha'.
\]
But then for each \(f \in L_\alpha\) with \(f \geq 0\),
\[
F(f) = \lim_k F(f \land k) + \lim_k F((f - k)^+) \\
= \lim_k \int (f \land k) d\mu + F^+(f) \\
= \int f d\mu + F^+(f)
\]
where we have set $F^\perp(f) = \lim_k F((f - k)^+)$ and the inequality $\mu^+(f > k) \leq k^{-1} \int f \wedge k d\mu \leq k^{-1} F(f)$ induces the conclusion that $f \wedge k$ is $\mu$-convergent to $f$ and is Cauchy in $L^1(\mu)$. $\int |f| d\mu \leq F(|f|)$ follows from (25) and implies $L \subset L^1(\mu)$. (17) is a consequence of (19) and (25).

Necessity is obvious as the right hand side of (17) defines a positive linear functional on $L$.

(a). Suppose that $\phi(h) < a < \inf_\omega T h(\omega)$ for some $a \in \mathbb{R}$ and $h \in \mathcal{H}$. Then, by (17) and properties of $F^\perp$, $a > \int Thd\mu \geq a\|\mu\|$ which is contradictory if $\|\mu\| = 1$. Conversely, define $\hat{\phi} \in \mathcal{L}(\mathbb{R} \times \mathcal{H})$ and $\hat{T} \in \mathcal{L}(\mathbb{R} \times \mathcal{H}, \mathfrak{F}(\Omega))$ implicitly by letting

$$
(26) \quad \hat{\phi}(r, h) = r + \phi(h) \quad \text{and} \quad \hat{T}(r, h) = r + Th \quad (r, h) \in \mathbb{R} \times \mathcal{H}.
$$

By assumption $\hat{\phi}$ is $\hat{T}$-conglomerative and thus admits a pair $\hat{F}^\perp$ and $\hat{\mu}$ as above. Therefore

$$
(27) \quad r + \phi(h) = \hat{F}^\perp(r + Th) + \int (r + Th) d\hat{\mu} = \hat{F}^\perp(Th) + \int (r + Th) d\hat{\mu} \quad (r, h) \in \mathbb{R} \times \mathcal{H}.
$$

Letting $h = 0$ we deduce $\|\hat{\mu}\| = 1$ and, from this, $\phi(h) = \hat{F}^\perp(Th) + \int Thd\hat{\mu}$ for every $h \in \mathcal{H}$.

(b). Fix a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ as in (18). By [1, theorem 1.33] the extension of $F$ from $T[\mathcal{H}]$ to $L$ constructed above may be chosen such that $\inf_{h \in \mathcal{H}} F(|f - Th|) = 0$ for every $f \in L$. Let $F^\perp$ and $\mu$ be the corresponding components of $F$ according to (17). Thus, for each $n \in \mathbb{N}$, let $h_n \in \mathcal{H}$ be such that $F(|f_n - Th_n|) \leq 2^{-n}$. If $g \in \mathcal{H}$ and $Tg \leq |f_n - Th_n|$, then

$$
\phi(g) = F(Tg) \leq F(|f_n - Th_n|) \leq 2^{-n}.
$$

Thus $\langle(h_n, f_n)\rangle_{n \in \mathbb{N}}$ satisfies (18) and, by assumption, $\limsup_n \phi(h_n) \leq 0$. The inequality $\int f_n d\mu = F(f_n) \leq \phi(h_n) + F(|f_n - Th_n|)$ then proves that the functional $f \to \int f d\mu$ is a Daniel integral on the Stonean lattice $L \cap \mathfrak{B}(\Omega)$ and it may thus be represented by some countably additive $(\mathcal{R}, \hat{\mu}) \in \mathcal{M}(\Omega)$. To prove that $\hat{\mu}$ agrees with $\mu$ over the whole of $L$ it is enough to remark that when $f \in L$ and $f \geq 0$, then $\hat{\mu}^+(f > k) \leq k^{-1} \int f \wedge k d\hat{\mu} = k^{-1} \int f \wedge k d\mu \leq k^{-1} F(f)$ and therefore $f \wedge k$ converges to $f$ in $L^1(\hat{\mu})$.

(c). For each $\alpha$ in a directed set $\mathfrak{A}$, let $F_\alpha \in \mathcal{L}(L_+)$ be such that $F_\alpha(Th) = \phi(h)$ for each $h \in \mathcal{H}$. Given that $F_\alpha$ is conglomerative with respect to the identity on $L$, it is of the form

$$
(28) \quad F_\alpha(f) = F^\perp_\alpha(f) + \int f d\mu_\alpha \quad f \in L
$$

with $F^\perp_\alpha[L \cap \mathfrak{B}(\Omega)] = \{0\}$ and $(\mathcal{R}_\alpha, \mu_\alpha) \in \mathcal{M}(\Omega)$ such that $L \subset L^1(\mu_\alpha)$. Observe that if $f \in L$ then there exists $h \in \mathcal{H}$ such that $|f| \leq Th$ and thus such that $F_\alpha(|f|) \leq \phi(h)$. The net $\langle F_\alpha \rangle_{\alpha \in \mathfrak{A}}$ admits then a subnet (still indexed by $\alpha$ for convenience) such that

$$
F(f) = \lim_\alpha F_\alpha(f) \quad f \in L.
$$

Since $F$ is positive we write it as $F(f) = F^\perp(f) + \int f d\mu$. If the net $\langle \mu_\alpha \rangle_{\alpha \in \mathfrak{A}}$ is increasing on $L_0 \subset L \cap \mathfrak{B}(\Omega)$ then

$$
\lim_\alpha \int f d\mu_\alpha = \lim_\alpha F_\alpha(f) = F(f) = \int f d\mu \quad f \in L_0.
$$
It is clear that \( \int fd\mu \geq \int fd\mu_\alpha \) for each \( \alpha \in \mathfrak{A} \) and \( f \in L_0 \). By Zorn lemma this proves the existence of a representing measure \( \mu \) which is \( L_0 \)-maximal. \( \square \)

Before moving to applications we can generalize Theorem 1 by dropping the assumption of linearity.

**Corollary 1.** Let \( \mathcal{H} \) be a non empty set, let \( T \in \mathfrak{F} (\mathcal{H}, \mathfrak{F} (\Omega)) \) be directed and denote by \( L \) the ideal generated by \( T[\mathcal{H}] \). Then \( \phi \in \mathfrak{F} (\mathcal{H}) \) is \( T \)-conglomerative in the sense that

\[
(29) \quad \sum_{n=1}^{N} a_n \phi (h_n) < 0 \quad \text{implies} \quad \inf_\omega \sum_{n=1}^{N} a_n (Th_n) (\omega) < 0 \quad h_1, \ldots, h_N \in \mathcal{H}, a_1, \ldots, a_N \in \mathbb{R}
\]

if and only if there exist (i) \( (\mathcal{H}, \mu) \in \mathcal{M} (\Omega) \) and (ii) \( F^\perp \in \mathcal{L} (L)_+ \) such that \( F^\perp [L \cap \mathfrak{B} (\Omega)] = 0 \).

(30) \( L \subset L^1 (\mu) \) and \( \phi (h) = F^\perp (Th) + \int Thd\mu \quad h \in \mathcal{H} \).

Moreover, \( \mu \) is a probability if and only if

\[
(31) \quad \sum_{n=1}^{N} a_n \phi (h_n) \geq \inf_\omega \sum_{n=1}^{N} a_n (Th_n) (\omega) \quad h_1, \ldots, h_N \in \mathcal{H}, a_1, \ldots, a_N \in \mathbb{R}.
\]

**Proof.** Let \( e_h \) be the evaluation on \( \mathfrak{F} (\mathcal{H}) \) corresponding to \( h \in \mathcal{H} \), that is \( e_h (G) = G (h) \). If \( V \) is a linear space, then each \( G \in \mathfrak{F} (\mathcal{H}, V) \) may be associated with a map \( \hat{G} \) from the span of \( \{e_h : h \in \mathcal{H}\} \) into \( V \) by letting

\[
(32) \quad \hat{G} (a_1 e_{h_1} + \ldots + a_N e_{h_N}) = \sum_{n=1}^{N} a_n G (h_n) \quad a_1, \ldots, a_N \in \mathbb{R}.
\]

It is immediate that \( \hat{G} \) is well defined and linear. Letting \( \hat{\phi} \) and \( \hat{T} \) be defined via (32), then (29) is equivalent to the statement that \( \hat{\phi} \) is \( \hat{T} \)-conglomerative while \( \hat{T} \) is directed if and only if so is \( T \). The claim follows from Theorem 1. \( \square \)

A special case of Corollary 1 applies to the case in which \( \langle \mathcal{H}_\alpha \rangle_{\alpha \in \mathfrak{A}} \) is a family of sets and, for each \( \alpha \in \mathfrak{A} \), \( \phi_\alpha \in \mathfrak{F} (\mathcal{H}_\alpha) \) and \( T_\alpha \in \mathfrak{F} (\mathcal{H}_\alpha, \mathfrak{F} (\Omega)) \). Just let \( \mathcal{H} = \{(h, \alpha) : u \in \mathcal{H}_\alpha, \alpha \in \mathfrak{A}\} \), \( \phi (h, \alpha) = \phi_\alpha (h) \) and \( T (h, \alpha) = T_\alpha (h) \).

As pointed out by Choquet [10] p. 325], not all linear functionals admit an integral representation, not even finitely additive. This occurs, e.g., when \( \mathcal{H} \) consists of polynomials and \( \phi \) associates to each \( h \in \mathcal{H} \) the coefficient of its term of degree \( n \), for some fixed \( n \geq 1 \). With the aim of extending the classical Riesz-Markoff theorem, Choquet assumes that \( \Omega \) is a compact topological space, \( \mathcal{H} \) a positively generated linear space of extended real-valued, continuous functions on \( \Omega \) and takes \( T \) to be a quotient \( T (h) = h / g \). This construction permits to characterize positive linear functionals on \( \mathcal{H} \) as a summable family of submeasures [11] theorem 42].

Theorem 1 bears a closer relation to another result of Choquet, the extremal representation theorem, that was originally proved in [9] and later variously extended and reformulated (see [12], [25] or [26] for an overview of this literature). To see this connection clearly, fix \( \Omega = \Psi \subset \mathcal{L} (\mathcal{H}) \) and define \( T \in \mathcal{L} (\mathcal{H}, \mathfrak{F} (\Psi)) \) by letting \( Th (\psi) = \psi (h) \) i.e. as the map that associates each \( h \in \mathcal{H} \).
with the (restriction to $\Psi$ of the) corresponding evaluation $e_h$ on $\mathfrak{F}(\mathcal{H})$. It is then easily seen that conglomerability may be nicely restated in geometric terms as the condition

$$
\phi \in \overline{\text{con}}_{\mathcal{H}}(\Psi),
$$

i.e. as $\phi$ being an element of the closed, conical hull of $\Psi$, the closure being in the $\mathcal{H}$ topology. Likewise, the inequality $\inf_{\omega} T\omega(h) \leq \phi(h)$ for all $h \in \mathcal{H}$ is equivalent to the condition $\phi \in \overline{\text{co}}_{\mathcal{H}}(\Psi)$.

In the light of these remarks the following result becomes obvious.

**Corollary 2.** Let $\mathcal{H}$ be a vector lattice and $\Psi \subset \mathcal{L}(\mathcal{H})_+$. Then, $\phi \in \overline{\text{con}}_{\mathcal{H}}(\Psi)$ if and only if there exist (i) $\phi^+ \in \mathcal{L}(\mathcal{H})$ with $\phi^+(h) \geq 0$ when $\inf_{\psi} \psi(h) > -\infty$ and (ii) $(\mathcal{R}, \mu) \in \mathcal{M}(\Psi)$ such that

$$
e_h |_{\Psi} \in L^1(\mu) \quad \text{and} \quad \phi(h) = \phi^+(h) + \int_{\Psi} \psi(h) d\mu \quad h \in \mathcal{H}.
$$

Moreover, $\mu$ is a probability if and only if $\phi \in \overline{\text{co}}_{\mathcal{H}}(\Psi)$.

The lattice structure of $\mathcal{H}$ guarantees that the map $T$ defined above is directed, as in $(\alpha)$.

To compare this result with the classical extremal or barycentric representation, we remark that the conical structure and the choice of the $\mathcal{H}$ topology make the conglomerability condition (33) a very weak restriction not requiring compactness nor boundedness and not relying as a consequence on the existence of extreme points. The first to obtain a proof of Choquet theorem without assuming compactness was Edgar [20, theorem p. 355] who considered a bounded, closed, convex, separable subset of a Banach space possessing the Radon Nikodym property and constructed his proof exploiting norm convergence of vector valued martingales.

Another version of Choquet theorem is obtained starting from condition $(\beta)$ for directedness of $T$ and requires boundedness.

**Corollary 3.** Let $\mathcal{H} \subset \mathfrak{F}(S)$ be a vector subspace, $\phi \in \mathcal{L}(\mathcal{H})$ and let $V \subset S$ be $\mathcal{H}$-bounded, i.e. such that $\sup_{v \in V} |h(v)| < \infty$ for all $h \in \mathcal{H}$. Then,

$$
\phi(h) \geq \inf_{v \in V} h(v) \quad h \in \mathcal{H}
$$

if and only if there exists a probability structure $(\mathcal{R}, \mu)$ on $V$ such that

$$
h \in L^1(\mu) \quad \text{and} \quad \phi(h) = \int_V h(v) d\mu \quad h \in \mathcal{H}.
$$

**Proof.** Consider the vector space $\mathcal{H} \times \mathbb{R}$ as acting on $S$ via $(h, r)(s) = h(s) + r$. In the notation of Theorem [1] let $\Omega = V$, $T(h, r) = (h, r)|V$ and $\hat{\phi}(h, r) = \phi(h) + r$. By $(\beta)$, $T$ is directed as $T[\mathcal{H} \times \mathbb{R}]$ is a subset of $\mathfrak{B}(\Omega)$ containing the constants. (35) is equivalent to $\hat{\phi}(h, r) \geq \inf_{v} (Th)(v)$, i.e. to the representation of $\hat{\phi}$ in the form (17) form some probability structure $(\mathcal{R}, \mu)$ and with $F^1 = 0$ as $\hat{T}[\mathcal{H} \times \mathbb{R}] \subset \mathfrak{B}(V)$. (36) follows upon restricting to elements of the form $(h, 0)$. The converse implication is obvious.

A clear example in which (35) holds is the one in which $\mathcal{H}$ consists of affine functions and $\phi(h) = h(u)$ for some $u \in \overline{\text{co}}_{\mathcal{H}}(V)$. We highlight that Corollary 3 does not require topological
assumptions; as a drawback, the characterization of the mapping \( u \to \mu_u \) is rather difficult. In the case in which \( \mathcal{H} \) is a Stonean sublattice, however, the minimality property is enough to imply that \( u \in V \) if and only if \( \mu_u \) is the point mass measure at \( u \).

4. Finitely Additive Companions.

In this section we return to the problem of the existence of companions.

**Theorem 2.** Let \((\mathcal{A}, m) \in \mathcal{M}(\Omega), X \in \mathfrak{F}(\Omega, S)\) and \( \mathcal{H} \) a Stonean vector sublattice of \( \mathfrak{F}(S) \). Let \( X' \in \mathfrak{F}(\Omega', S) \). There is equivalence between the condition

\[
\int h(X)dm < 0 \implies \inf_{\omega' \in \Omega'} h(X'(\omega')) < 0 \quad h \in \mathcal{H}, \ h(X) \in L^1(m)
\]

and the existence of a minimal \((\mathcal{R}, \mu) \in \mathcal{M}(\Omega')\) satisfying

\[
h(X') \in L^1(\mu) \quad \text{and} \quad \int h(X)dm = \int h(X')d\mu \quad h \in \mathcal{H}, \ h(X) \in L^1(m).
\]

In addition,

(a) \( \mu \) is a probability if and only if

\[
\int h(X)dm \geq \inf_{\omega' \in \Omega'} h(X'(\omega')) \quad h \in \mathcal{H}, \ h(X) \in L^1(m);
\]

(b) if \( X'[\Omega'] \) is closed in the topological space \( S \) and \( \mathcal{H} \subset \mathcal{C}(S) \) then \( \mu \) is countably additive if either (i) \( \mathcal{H} \subset \mathcal{C}_K(S) \), (ii) \( X' \) is \( \mu \)-tight or (iii) \( X \) is \( m \)-tight and \( m_s(X \notin X'[\Omega']) = 0 \).

**Proof.** \((37)\) is equivalent to \( \phi \) being \( T \)-conglomerative with \( \phi(h) = \int h(X)dm + Th = h(X') \) for every \( h \in \mathcal{H} \). Thus, \((38)\) follows from \((17)\) after noting that, in the present setting, \( \phi(h) = \lim_k \phi(h \wedge k) \) for every \( h \in \mathcal{H}_+ \). That \((39)\) is necessary and sufficient for \( \mu \) to be a probability follows directly from Theorem 1(a).

Let \( X'[\Omega'] \) be closed and \( \langle h_n \rangle_{n \in \mathbb{N}} \) a sequence in \( \mathcal{H} \subset \mathcal{C}(S) \) with \( h_n(X') \) decreasing to 0, i.e. \( h_n \) decreasing to 0 on \( X'[\Omega'] \). We claim that (i), (ii) or (iii) imply \( \lim_n \int h_n(X')d\mu = 0 \). If \( \mathcal{H} \subset \mathcal{C}_K(S) \), then in computing such limit one may replace \( S \) with some compact subset so that (i) follows from (ii). Fix \( \varepsilon > 0 \). Under (ii) there exists \( K' \subset S \) compact and \( B^c \in \mathcal{A} \) such that \( B' \subset \{X' \in K'\} \) and

\[
\int h_n(X')d\mu \leq \int h_n(X')1_Bd\mu + \varepsilon \quad n \in \mathbb{N}.
\]

But then \( \lim_n \sup_{\omega' \in B'} h_n(X') \leq \lim_n \sup_{s \in X'[\Omega'] \cap K'} h_n(s) = 0 \), by Dini’s theorem. Under (iii), we can find an extension \( \tilde{m} \) of \( m \) to the minimal ring containing the set \( F = \{X \notin X'[\Omega']\} \) such that \( \tilde{m}(F) = 0 \). We can also find \( K \subset S \) compact and \( B^c \in \mathcal{A} \) such that \( B \subset \{X \in K\} \) and that

\[
\int h_n(X')d\mu = \int h_n(X)dm = \int h_n(X)d\tilde{m} \leq \int h_n(X)1_{B \setminus F}d\tilde{m} + \varepsilon \quad n \in \mathbb{N}
\]

so that again \( \lim_n \sup_{\omega \in B \setminus F} h_n(X) \leq \lim_n \sup_{s \in X'[\Omega'] \cap K} h_n(s) = 0 \). In either case the positive linear functional \( \int h(X')d\mu \) on the Stonean lattice \( \mathcal{H}[X'] \) is a Daniell integral and it may be represented via a countably additive set function. Since \( \mu \) is minimal, it must then be countably additive. \( \square \)
To clarify the connection with Doob’s work, consider a π-strategy, i.e. a function \( \sigma(h|B) \) where \( h \) runs across the family \( \mathcal{B}(\Omega) \) of bounded functions on \( \Omega \) and \( B \) is an element of the partition \( \pi \) of \( \Omega \).

As in other papers on finitely additive probability (see e.g. Regazzini [27]) conditional expectation is defined setwise rather than as a measurable function, as in Kolmogorov classical construction. One notices that \( m \) is defined setwise rather than as a measurable function, as in Kolmogorov classical construction.

One notices that \( m \) is \( \sigma \)-conglomerative in the sense of [17] p. 90 if and only if \( 37 \) holds with \( \mathcal{H} = \mathcal{B}(\Omega) \), \( S = \Omega \), \( X \) the identity map and \( h(X') = \sum_{B \in \pi} \sigma(h|B)1_B \).

In the absence of restrictions on \( \mu \), the existence of companions is guaranteed under a weak condition such as \( 37 \), namely if \( X \) is \( X' \)-conglomerative. An obvious companion to any \( X \) is the identity map on \( \Omega' = S \). Given that being companion (relatively to the one given family \( \mathcal{H} \)) is a transitive property, the problem in Theorem 2 may be simplified with no loss of generality by assuming that \( X \) is the identity map on \( \Omega = S \). In this case, if \( m \) consists of sample frequencies, then the condition \( m^*(X'|\Omega') = 0 \) sufficient for \( m \) to be \( X' \)-conglomerative means that all the observations in the given sample must belong to the range of \( X' \).

The existence of a countably additive companion was proved under (ii) by Dubins and Savage [16], p. 190, for the case \( \Omega = \Omega' = S = \mathbb{R} \), and has then been revived and extended to the case \( S = \mathbb{R}^n \) by Karandikar, [23] and [24], who used it in the proof of finitely additive limit theorems. The conditions for the existence of a countably additive companion obtained in Theorem 2 may be employed to refine the results of the preceding section. In particular if the set \( \Psi \) in Corollary 2 is \( \mathcal{H} \)-compact then in \( 34 \) one has \( \phi^\perp = 0 \) and \( \mu \) can be chosen to be countably additive.

An interesting issue concerns the construction of an auxiliary state space on which every function \( X \) admits a countably additive companion.

**Lemma 5.** Let \( (\mathcal{A}, m) \in \mathcal{M}(\Omega), S \) be a metric space, \( s_0 \in S, X \in \mathcal{F}(\Omega, S) \) and \( \tilde{\Omega} = \mathcal{F}(\mathbb{N}, \Omega) \). Define \( \tilde{X} \in \mathcal{F}(\tilde{\Omega}) \) as

\[
\tilde{X}(\tilde{\omega}) = \lim_k X(\omega_k) \quad \text{if the limit exists or else} \quad \tilde{X}(\tilde{\omega}) = s_0, \quad \tilde{\omega} = (\omega_k)_{k \in \mathbb{N}} \in \tilde{\Omega}.
\]

There exists \( (\mathcal{R}, \mu) \in \mathcal{M}(\tilde{\Omega}) \) countably additive and such that \( (\tilde{X}, \mu) \) is companion to \( (X, m) \) relatively to \( \mathcal{C}_K(S) \). Moreover, if \( S = \mathcal{F}(\mathbb{N}) \) and \( X_n \) is \( m \)-convergent (resp. converges in \( L^1(m) \)) to 0 then \( \tilde{X}_n \) is \( \mu \)-convergent (resp. converges in \( L^1(\mu) \)) to 0.

**Proof.** \( X \) is \( \tilde{X} \)-conglomerative relatively to any \( \mathcal{H} \subset \mathcal{F}(S) \) since \( X[\Omega] \subset \tilde{X}[\tilde{\Omega}] \); moreover, \( \tilde{X}[\tilde{\Omega}] \) is closed. The first claim follows from Theorem 2(b).

Let \( S = \mathcal{F}(\mathbb{N}) \) and replace \( m \) with some positive extension \( \tilde{m} \) to the ring \( \{ A \subset \Omega : m^*(A) < \infty \} \).

By the first claim there exists \( (\mathcal{R}, \mu) \in \mathcal{M}(\tilde{\Omega}) \) countably additive such that \( (X, \tilde{m}) \) and \( (\tilde{X}, \mu) \) are companions relatively to \( \mathcal{C}_K(S) \) – and a fortiori so are \( (X, m) \) and \( (\tilde{X}, \mu) \). Fix \( b > a > 0 \) and \( k > 0 \) and let \( g, f_k \in \mathcal{C}(\mathbb{R}) \) be such that \( \mathbb{1}_{\{x > b\}} < g(x) \leq \mathbb{1}_{\{x > a\}} \) and \( \mathbb{1}_{\{x < k-1\}} < f_k(x) < \mathbb{1}_{\{x < k\}} \) so that \( f_k \uparrow 1 \). Writing \( h_n(X) = g(|X_n|) \) and \( h_n^k(X) = h_n(X)f_k(|X_n|), h_n \in \mathcal{C}(S) \) and \( h_n^k \in \mathcal{C}_K(S) \). But then,

\[
m^*(|X_n| > a) \geq \int h_n(X)d\tilde{m} \geq \lim_k \int h_n^k(X)d\tilde{m} = \lim_k \int h_n^k(\tilde{X})d\mu = \int h_n(\tilde{X})d\mu \geq m^*(|\tilde{X}_n| > b)
\]
and, consequently,
\[
\int_{-\infty}^{\infty} m^*(|X_n| > t)\,dt \geq \int_{0}^{\infty} \mu^*(|\tilde{X}_n| > t)\,dt \geq \int_{0}^{\infty} \mu^*(|\tilde{X}_n| > t)\,dt
\]
so that \( \int |X_n|d\mu \geq \int |\tilde{X}_n|d\mu \) whenever \( X_n \in L^1(m) \), by Lemma 2.

Lemma 5 may help understanding the connection between convergence pointwise and in measure under finite additivity, i.e. when Egoroff theorem fails. We establish that a condition weaker than uniform convergence may be assumed.

**Corollary 4.** Let \((\Omega, \mathcal{A}, m)\) be a probability space and \(\langle X_n \rangle_{n \in \mathbb{N}}\) a \(m\)-measurable sequence in \(\mathfrak{F}(\Omega)\). Assume that
\[
(41) \quad \lim \lim_n X_n(\omega_k) = 0,
\]
whenever \( \lim_k X_n(\omega_k) \) exists for all \( n \in \mathbb{N} \). Then, \( X_n \) \(m\)-converges to 0.

**Proof.** Write \( Y = (|X_n| \land 1)_{n \in \mathbb{N}}\) and define \( \tilde{\Omega} \) and \( \tilde{Y} \) as in (40), with \( s_0 = 0 \). By Lemma 5 there exists a countably additive \((\mathcal{R}, \mu) \in \mathcal{M}(\tilde{\Omega})\) such that \((Y, m)\) and \((\tilde{Y}, \mu)\) are companions relatively to \(\mathcal{E}_K(\mathfrak{F}(\mathbb{N}))\). Fix \( \tilde{\omega} = (\omega_k)_{k \in \mathbb{N}}\) in \( \tilde{\Omega} \). If \( Y \) does not converge along \( \tilde{\omega} \) then \( Y_n(\tilde{\omega}) = 0 \), otherwise \( \lim_n \tilde{Y}_n(\tilde{\omega}) = \lim_n \lim_k Y_n(\omega_k) = 0 \), by (41). But then countable additivity implies
\[
0 = \lim_n \int \tilde{Y}_n d\mu = \lim_n \int Y_n dm \quad \text{so that } X_n \text{ \(m\)-converges to 0}.
\]

In Theorem 2 the set function \( \mu \) is completely unrestricted. A possible mitigation is to require that \( \mu \) vanishes on some suitable, given collection \( \mathcal{N} \) of subsets of \( \Omega \).

**Theorem 3.** In the same setting as Theorem 2 let \( \mathcal{N} \) an ideal of subsets of \( \Omega' \). The condition
\[
(42) \quad \int h(X)dm < 0 \quad \text{implies} \quad \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} h(X'(\omega')) < 0 \quad h \in \mathcal{H}
\]
is equivalent to the existence of a minimal \((\mathcal{R}, \mu) \in \mathcal{M}(\Omega')\) which satisfies \( \mathcal{N} \subset \mathcal{R} \),
\[
(43) \quad \mu[\mathcal{N}] = \{0\}, \quad h(X') \in L^1(\mu) \quad \text{and} \quad \int h(X)dm = \int h(X')d\mu \quad h \in \mathcal{H}.
\]
Moreover, (a) \( \mu \) is a probability if and only if
\[
(44) \quad \int h(X)dm = \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} h(X'(\omega')) \quad h \in \mathcal{H},
\]
(b) if \( \mathcal{A} \) is a \( \sigma \) ring, \( m \) is countably additive and \( \mathcal{N} \) a \( \sigma \) ideal then \( \mu \) is countably additive provided \( m_*(X \notin X'[N^c]) = 0 \) for all \( N \in \mathcal{N} \).

**Proof.** Since \( \mathcal{N} \) is an ideal, the binary relation \( \geq \) on \( \mathfrak{F}(\Omega') \) defined by letting
\[
(45) \quad f \geq g \quad \text{if and only if} \quad \sup_{N \in \mathcal{N}} \inf_{\omega \in N^c} (f-g)(\omega') \geq 0 \quad f, g \in \mathfrak{F}(\Omega')
\]
is a partial order and \( f \geq g \) implies \( f \geq g \). Moreover, \( f_i \geq g_i \) for \( i = 1, 2 \) implies \( f_1 \lor f_2 \geq g_1 \lor g_2 \). In fact, \( f_1 \lor f_2 \geq f_i \geq g_i \) i.e. \( f_1 \lor f_2 \geq g_i - \varepsilon \) outside of some \( N_i \in \mathcal{N} \). Thus, \( f_1 \lor f_2 \geq g_1 \lor g_2 - \varepsilon \).
outside of \( N_1 \cup N_2 \in \mathcal{N} \) which, by (15), is equivalent to \( f_1 \vee f_2 \preceq g_1 \vee g_2 \). It is easy to see that, relatively to pointwise ordering, the set

\[
\mathcal{F} = \{ f \in \mathfrak{F}(\Omega') : f \sim h(X') \text{ for some } h \in \mathcal{H} \}
\]

is a Stonean vector sublattice of \( \mathfrak{F}(\Omega') \). Writing

\[
(47) \quad \phi(f) = \int h(X)dm \quad f \sim h(X'), \; h \in \mathcal{H}
\]

implicitly defines, via (12), a positive linear functional on \( \mathcal{F} \) so that, by Corollary 2, we conclude that there exists a minimal measurable structure (\( \mathcal{A}, \mu \)) on \( \Omega' \) satisfying

\[
(48) \quad f \in L^1(\mu) \quad \text{and} \quad \phi(f) = \int fd\mu \quad f \in \mathcal{F}.
\]

Observe that if \( N \in \mathcal{N} \) then \( 1_N \sim 0 \); thus, \( 1_N \in \mathcal{F}, \; N \in \mathcal{A} \) and \( \mu(N) = 0 \). This proves (43) while the converse implication, is obvious. The proof of claim (a) is easily obtained from the one of the corresponding claim in Theorem 1. Eventually we prove (b), once again, by showing that under the stated conditions the functional \( \phi \) defined in (17) is a Daniell integral over \( \mathcal{F} \). In fact, let \( \{ f_n \}_{n \in \mathbb{N}} \) be sequence in \( \mathcal{F} \) decreasing pointwise to 0 with \( f_n \sim h_n(X') \) and \( h_n \in \mathcal{H}, \; n = 1, 2, \ldots \). Define \( g_n = \bigwedge_{1 \leq j \leq n} h_j \) and \( g = \lim_n g_n \). As shown above, \( f_n \sim g_n(X') \preceq g(X') \) so that, by the assumption that \( \mathcal{N} \) is a \( \sigma \) ideal, \( \{ g(X') > \varepsilon \} \subset \bigcup_n \{ g(X') \geq f_n + \varepsilon \} \in \mathcal{N} \) and \( \{ g > \varepsilon \} \subset X'[\{ g(X') \leq \varepsilon \}]^c \). Given that \( \mathcal{A} \) is a \( \sigma \) ring, we conclude that \( m(g(X) > \varepsilon) = 0 \) and so \( \lim_n \phi(f_n) = \lim_n \int g_n(X')d\mu = \lim_n \int g_n(X)d\mu = \int g(X)d\mu = 0 \).

Example 1. Let \( (\Omega', \mathcal{A}, P) \) be a classical probability space, \( S = \mathbb{R} \) and let \( X' \) be a normally distributed random quantity on \( \Omega' \). Fix \( m \in \text{fa}(\mathcal{B}(\mathbb{R}))_+ \) arbitrarily and let \( \mathcal{H} = \mathcal{C}(\mathbb{R}) \cap L^1(m) \). Given that \( P(X' \in B) > 0 \) for every \( B \) open, we conclude that \( m \) is \( X' \)-conglomerative relatively to \( \mathcal{H} \). In other words a normally distributed random quantity can assume any arbitrary distribution (relatively to the continuous functions) upon an accurate choice of the reference measure.

In addition, let \( \mathcal{N} \) consist of all \( P \) null sets and observe that \( X'[\mathcal{N}^c]^c \) has 0 Lebesgue measure – as \( P(X' \in X'[\mathcal{N}^c]^c) = P(N) = 0 \) and the \( P \) distribution of \( X' \) is mutually absolutely continuous with respect to Lebesgue measure – and has therefore empty interior – so that \( X'[\mathcal{N}] = \mathbb{R} \). Therefore,

\[
\sup_{N \in \mathcal{N}} \inf_{\omega \in \mathcal{N}^c} h(X'(\omega)) = \sup_{N \in \mathcal{N}} \inf_{s \in X'[\mathcal{N}^c]} h(s) = \sup_{N \in \mathcal{N}} \inf_{s \in \mathcal{R}} h(s) = \inf_{s \in \mathcal{R}} h(s) \quad h \in \mathcal{C}(\mathbb{R}).
\]

Property (12) then holds for every \( m \in \text{fa}(\mathcal{B}(\mathbb{R}))_+ \) with \( \mathcal{H} = \mathcal{C}(\mathbb{R}) \). One may then find \( \mu \) vanishing on \( \mathcal{N} \) and such that \( (X', \mu) \) is companion to \( m \).

Even if \( m \) were countably additive, \( \mu \) need not be so. The Dirac measure is a good case in point of a regular, countably additive measure that cannot be represented as the distribution of \( X' \) with respect to some countably additive representing measure \( \mu \) which vanishes on \( P \) null sets. To this end we may assume in addition that \( m \) does not charge sets with empty interior. Under this further assumption, \( m_*(X'[\mathcal{N}^c]^c) = 0 \) so that \( \mu \) is countably additive by virtue of Theorem 5(b) and vanishes on \( N \in \mathcal{N} \). Of course the same conclusion holds upon replacing \( X' \) with any variable.
Theorem 4. Let \( \langle \phi \rangle \) in its turn be a directed map. The representation of an increasing sequence \( \langle f_n \rangle_{n \in \mathbb{N}} \) of continuous functions, and conclude

\[
\mu(X' \in B) = \lim_n \int f_n(X')d\mu = \lim_n \int f_n dm = m(B).
\]

The preceding example may be generalized into the following:

**Theorem 4.** Let \( \mathcal{H} \subset \mathcal{C}(\mathbb{R}) \) be a Stonean sublattice, \( \phi \in \mathcal{L}(\mathcal{H})_+ \), \( X' \) a normally distributed random quantity on a standard probability space \( (\Omega, \mathcal{A}, P) \) and \( \mathcal{N} \) the collection of all \( P \) null sets. There exists a minimal \( (\mathcal{R}, \mu) \in \mathcal{M}(\Omega') \) such that \( \mathcal{N} \subset \mathcal{R} \), \( \mu \) vanishes on \( \mathcal{N} \) and

\[
\phi(h) = \int h(X')d\mu \quad h \in \mathcal{H}.
\]

Moreover, if \( \mathcal{H} \) is an ideal in \( \mathcal{C}(\mathbb{R}) \) then

(i) \( \mu \) is countably additive if and only if \( \lim_n \phi(h_n) = 0 \) for any decreasing sequence \( \langle h_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{H}_+ \) which converges to 0 in Lebesgue measure,

(ii) \( \mu \) is countably additive and \( \mu^*(X' \in C) = 0 \) when \( C \) has empty interior if and only if \( \lim_n \phi(h_n) = 0 \) for any decreasing sequence \( \langle h_n \rangle_{n \in \mathbb{N}} \) in \( \mathcal{H}_+ \) admitting 0 as the largest continuous function dominated by \( \inf_n h_n \).

**Proof.** A positive linear functional on a vector lattice is conglomerative with respect to the identity, in its turn a directed map. The representation of \( \phi \) as \( \int h dm \), with \( m \) minimal, follows from Theorem 3. If \( \mathcal{H} \) is an ideal and \( \phi \) meets either property, (i) or (ii), then it is a Daniell integral and \( m \) is a countably additive, regular measure on the generated \( \sigma \) ring, still denoted by \( \mathcal{R} \). We also notice that the indicator of a closed set \( F \in \mathcal{R} \) may be expressed as the pointwise limit of a decreasing sequence \( \langle h_n \rangle_{n \in \mathbb{N}} \) of positive, continuous functions with \( 0 \leq h_n \leq 1 \). Fix \( h \in \mathcal{H}_+ \). Since \( \mathcal{H} \) is an ideal, \( hh_n \in \mathcal{H} \) for each \( h \in \mathcal{H}_+ \) and thus \( \int hh_n dm = \lim_n \int hh_n dm = \lim_n \phi(hh_n) \). Then \( \int h F dm = 0 \) in two different situations: when \( F \) has 0 Lebesgue measure and \( \phi \) satisfies (i) (as \( hh_n \) converges then to 0 in Lebesgue measure) or if \( F \) is nowhere dense and \( \phi \) satisfies (ii) (as 0 is then the largest, continuous function dominated by \( h \)).

In either case the restriction of \( m \) to \( F^c \) is another representing measure for \( \phi \) so that, by minimality, \( m(F) = 0 \). Given that \( X'[N^c]^c \) has 0 Lebesgue measure and empty interior when \( N \in \mathcal{N} \) and that \( m \) is regular, then (i) and (ii) imply \( m^*(X'[N^c]^c) = 0 \) and, by Theorem 3, that \( \mu \) is countably additive. Assume, conversely, that \( \mu \) is countably additive and let \( \langle h_n \rangle_{n \in \mathbb{N}} \) be a decreasing sequence in \( \mathcal{H}_+ \) with pointwise limit \( h \). For each fixed \( \varepsilon > 0 \) we obtain that \( \mu^*(h(X') > \varepsilon) = 0 \) in the following two cases: when \( h_n \) decreases to 0 in Lebesgue measure and \( \mu \) meets (i) (as the set \( \{ h > \varepsilon \} \) has 0 Lebesgue measure and thus \( \{ h(X') > \varepsilon \} \in \mathcal{N} \)) or when 0 is the largest, continuous function dominated by \( h \) and \( \mu \) meets (ii) (as \( \{ h > \varepsilon \} \) has then empty interior). In either case \( \lim_n \phi(h_n) = \lim_n \int h_n(X')d\mu = \int h(X')d\mu = 0 \). \( \square \)
It is implicit in Theorem 4 that a normally distributed random quantity may assume whatever distribution upon a change of the reference measure and whatever distribution absolutely continuous with respect to Lebesgue measure upon an absolutely continuous change of the original probability measure upon a change of the reference measure and whatever distribution absolutely continuous with respect to Lebesgue measure.

We now show that the existence of companions may be obtained even outside of the linear case. Eventually, we turn attention to convex functions. For \( f \in \mathcal{F}(\mathbb{R}) \) we denote by \( D^+ f \) and \( D^- f \) the right and left derivatives and by \( f(x+) \) and \( f(x-) \) the right and left limits at \( x \), provided such quantities exist. We also set conventionally

\[
D^+ f(\infty) = D^- f(\infty) = \lim_{x \to \infty} D^+ f(x) \quad \text{and} \quad D^+ f(-\infty) = D^- f(-\infty) = \lim_{x \to -\infty} D^+ f(x).
\]

Observe that if \( x_0 \in \arg\inf_{x \in \mathbb{R}} f(x) \), then \( D^+ f(x_0), D^- f(x_0) \in \mathbb{R} \) and that for this reason, upon replacing \( f \) with the function \( \hat{f}(x) = f(x) - [D^+ f(x_0)\mathbb{1}_{\{x > x_0\}} + D^- f(x_0)\mathbb{1}_{\{x \leq x_0\}}] \) we may assume \( D^+ f(x_0) = D^- f(x_0) = 0 \).

**Theorem 5.** Let \( \varphi \in \mathcal{F}(\mathbb{R}) \), \( x_0 \in \arg\inf_{x \in \mathbb{R}} \varphi(x) \) and assume \( D^+ \varphi(x_0) = D^- \varphi(x_0) = 0 \). Define

\[
h_u^v(x) = (v - x \vee u)^+ \mathbb{1}_{\{x > x_0\}} - (v \wedge x - u)^+ \mathbb{1}_{\{x \leq x_0\}} \quad x, u, v \in \mathbb{R}.
\]

Let \( \mathcal{N} \) be an ideal of subsets of \( \Omega \) and \( X \in \mathcal{F}(\Omega) \). The following properties are mutually equivalent:

(i) \( \varphi \) is convex and \( \{u < X < v\} \in \mathcal{N} \) implies \( D^- \varphi(v) \leq D^+ \varphi(u) \);

(ii) there exists a \( (\mathcal{R}, \lambda) \in \mathcal{M}(\Omega) \) such that (a) \( \mathcal{N} \subset \mathcal{R} \) and \( \lambda[\mathcal{N}] = \{0\} \), (b) \( \lim_{n \to \infty} \lambda^n(|X - x_0| < 2^{-n}) = 0 \), (c) \( \{h_u^v(X) : v, u \in \mathbb{R}\} \subset L^1(\lambda) \) and

\[
\varphi(v) = \varphi(u) + \int h_u^v(X)d\lambda \quad v \geq u;
\]

(iii) there exists \( \nu \in f\lambda(\mathcal{R}(\mathbb{R}))_+ \), countably additive such that (a) \( \nu(A) = 0 \) for \( A \) open and \( X^{-1}(A) \in \mathcal{N} \), (b) \( \nu^{\ast}(\{x_0\}) = 0 \), (c) \( \{h_u^v : v, u \in \mathbb{R}\} \subset L^1(\nu) \) and

\[
\varphi(v) = \varphi(u) + \int h_u^v d\nu \quad v \geq u.
\]

**Proof.** \( \ref{item:1} \Rightarrow \ref{item:2} \). Write \( \mathcal{D} = \{t : D^- \varphi(t) = D^+ \varphi(t)\} \cup \{x_0\} \) and define \( A_u = \{u < X \leq x_0\} \), \( A^v = \{x_0 < X \leq v\} \) and

\[
\mathcal{R}_0 = \left\{ (A_u \cap N_u^c) \cup (A^v \cap N_v^c) \cup N : u, v \in \mathcal{D}, \ N_u, N_v, N \in \mathcal{N} \right\}.
\]

It is clear that \( \mathcal{R}_0 \) contains \( \mathcal{N} \) (upon taking \( u = v = x_0 \)) as well as \( A_u, A^v : u, v \in \mathcal{D} \). Moreover, it is routine to verify that \( \mathcal{R}_0 \) is closed with respect to union and intersection with

\[
H_1 \cup H_2 = (A_{u_1} \cap u_2 \cap N_u^c) \cup (A^{u_1 \vee u_2} \cap N_v^c) \cup N
\]

and

\[
H_1 \cap H_2 = (A_{u_1 \vee u_2} \cap N_u^c) \cup (A^{u_1 \wedge u_2} \cap N_v^c) \cup N
\]
whenever $H_i = (A_{u_i} \cap N_{u_i}^c) \cup (A_v \cap N_v^c) \cup N_i \in \mathcal{R}_0$ for $i = 1, 2$. Write $F(x) = D^+ \varphi(x \lor x_0) + D^- \varphi(x \land x_0)$ and

$$\lambda_0(H) = F(v \lor x_0) - F(u \land x_0) \quad \text{when} \quad H = (A_u \cap N_u^c) \cup (A_v \cap N_v^c) \cup N \in \mathcal{R}_0. \quad (56)$$

To see that $\lambda_0$ is well defined observe that if $u_1 \land x_0 < u_2 \land x_0$ and

$$(A_{u_1} \cap N_{u_1}^c) \cup (A_{v_1} \cap N_{v_1}^c) \cup N_1 = (A_{u_2} \cap N_{u_2}^c) \cup (A_{v_2} \cap N_{v_2}^c) \cup N_2 \in \mathcal{R}_0$$

then $\{u_1 \land x_0 < X \leq u_2 \land x_0\} \in \mathcal{N}$. Thus by (i) and the fact that $u_1, u_2 \in D$ and that $u_1 < x_0$,

$$D^- \varphi(u_1 \land x_0) = D^- \varphi(u_2 \land x_0) \quad \text{i.e.} \quad F(u_1 \land x_0) = F(u_2 \land x_0)$$

and likewise $F(v_1 \lor x_0) = F(v_2 \lor x_0)$. In other words $\lambda_0 \in fa(\mathcal{R}_0)_+$ with $\lambda[\mathcal{N}] = \{0\}$. Moreover, if $H_1, H_2 \in \mathcal{R}_0$ then by (55)

$$\lambda_0(H_1) + \lambda_0(H_2) = F(v_1 \lor x_0) + F(v_2 \lor x_0) - F(u_1 \land x_0) - F(u_2 \land x_0)$$

$$= F(v_1 \lor v_2 \lor x_0) + F((v_1 \lor v_2) \lor x_0) - F((u_1 \lor u_2) \lor x_0) - F(u_1 \land u_2 \land x_0)$$

$$= \lambda_0(H_1 \cup H_2) + \lambda_0(H_1 \cap H_2)$$

i.e. $\lambda_0$ is strongly additive on $\mathcal{R}_0$. It follows from [4, 3.1.6 and 3.2.4] that $\lambda_0$ admits a unique extension $\lambda_1 \in fa(\mathcal{R}_1)_+$ to the generated ring $\mathcal{R}_1$. Let $I$ be an interval with endpoints in $\mathbb{R} \cup \{x_0\}$. Given that $D$ is dense in $\mathbb{R} \cup \{x_0\}$, $\lambda^*(X \in I) < \infty$. By [4, 3.4.1 and 3.4.4] we obtain a further extension $\lambda \in fa(\mathcal{R})_+$ to the ring $\mathcal{R} = \{A \subset \Omega : \lambda^*(A) < \infty\}$. Then $\{X \in I\} \in \mathcal{R}$ and $X I_\Lambda$ is $\lambda$-measurable whenever $I$ is as above, by Lemma 1. Therefore,

$$\int_{u \lor x_0}^{v \lor x_0} D^+ \varphi(t)dt = \int_{u \lor x_0}^{v \lor x_0} \mathbb{1}_D[D^+ \varphi(t) - y_0^+]dt$$

$$= \int_{u}^{v} \mathbb{1}_D \lambda_1(x_0 < X \leq t)dt$$

$$= \int_{u}^{v} \lambda(x_0 < X \leq t)dt$$

$$= \int_{x_0}^{\infty} (v - u \lor X)^+d\lambda \quad \text{(by Lemma 2)}$$

and similarly $\int_{u \land x_0}^{v \land x_0} D^+ \varphi(t)dt = - \int_{-\infty}^{x_0}(v \land X - u)^+d\lambda$. We conclude

$$\varphi(v) - \varphi(u) = \int_{u \lor x_0}^{v \lor x_0} D^+ \varphi(t)dt + \int_{u \land x_0}^{v \land x_0} D^- \varphi(t)dt = \int h^w_u(X)d\lambda.$$

Fix an increasing $(u_n)_{n \in \mathbb{N}}$ and a decreasing $(v_n)_{n \in \mathbb{N}}$ sequence in $D$ converging to $x_0$, with $u_n < u_{n+1} < x_0$ if $x_0 > -\infty$ and $v_n > v_{n+1} > x_0$ if $x_0 < \infty$. Then,

$$\lim_n \lambda^*(u_n < X < v_n) \leq \lim_n D^+ \varphi(v_n) - D^- \varphi(u_n) = 0$$

so that $\lim_n \lambda^*(|X - x_0| < 2^{-n}) = 0$. 

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(ii)$\Rightarrow$ (iii). With $u_n$ and $v_n$ defined as above, define the function

$$h^u_n(x; n) = \begin{cases} h^u_0(x), & \text{if } x \not\in (u_n, v_n) \\ h^u_n(u_n) \frac{u_n+1-x}{u_n+1-u_n}, & \text{if } x \in (u_n, u_{n+1}] \\ h^u_n(v_n) \frac{x-v_n+1}{v_n-v_{n+1}}, & \text{if } x \in (v_{n+1}, v_n]. \end{cases}$$

Then, $h^u_n(\cdot; n)$ is a continuous function vanishing outside of the interval $[u \wedge u_{n+1}, v \vee v_{n+1}]$. Moreover:

(a) $\{ |h^u_n(x; n) - h^u_n(x)| > c \} \subset (u_n, v_n)$ so that $h^u_n(X; n) = \lambda$-convergent to $h^u_n(X)$, (b) $|h^u_n(x; n)| \leq |h^u_n(x; n + 1)| \leq |h^u_n(x)|$, (c) $\lim_n h^u_n(x; n) = h^u_n(x)$ for all $x \neq x_0$ and (d) $h^u_n(X; n)$ is $\lambda$-measurable and therefore an element of $L^1(\lambda)$. Let $(\mathcal{X}', \nu)$, with $\mathcal{X}' = \mathbb{R}$ and $X'$ the identity, be the countably additive companion of $(X, \lambda)$ relatively to the family $\{ \mathcal{h}(X) : h \in \mathcal{C}_K(\mathbb{R}) \}$. It follows that

$$\int h^u_n(X) d\lambda = \lim_n \int h^u_n(X; n) d\lambda = \lim_n \int h^u_n(x; n) d\nu = \int h^u_n d\nu.$$

Observe that if $x_0 \in \mathbb{R}$ and $g_n \in \mathcal{C}_K(\mathbb{R})$ is such that $1_{[u_n, v_n]} \geq g_n \geq 1_{(u_{n+1}, v_{n+1}]}$, then

$$\nu^*\{(x_0)\} \leq \lim_n \int g_n(X) d\lambda \leq \lim_n \lambda(u_n < X \leq v_n) = 0.$$

Let $I \subset \mathbb{R}$ be an open interval with $X^{-1}(I) \in \mathcal{N}$ and $\langle g_n \rangle_{n \in \mathbb{N}}$ a sequence of non-negative, continuous functions which increases to $1_I$. It is then obvious that

$$0 = \lim_n \int g_n(X) d\lambda = \lim_n \int g_n d\nu = \nu(I).$$

The conclusion extends to open sets.

(2(ii)$\Rightarrow$(2(i)). If $\varphi$ satisfies (53), it is clearly convex since the function $v \rightarrow h^u_n(x)$ is convex for every $u \leq v$. Assume that $u < v$ and $\{u < X < v\} \in \mathcal{N}$. Then, $\nu((u, v)) = 0$ so that, for arbitrary $u < t < v$

$$(57) \frac{\varphi(v) - \varphi(u)}{v-u} = \begin{cases} \nu([x_0, t)), & \text{if } v > u \geq x_0 \\ \nu([t, x_0]), & \text{if } x_0 \geq v > u \\ 0, & \text{if } v > x_0 > u \end{cases}$$

and (i) follows. $\square$

If, e.g., $\varphi$ is differentiable at $x_0$, then (52) simplifies into:

$$(58) \varphi(v) = \varphi(x_0) + \int_{\{v < X \leq x_0\}} (X - v) d\lambda + \int_{\{x_0 < X \leq v\}} (v - X) d\lambda.$$

The above result can be stated in a slightly different way:

**Corollary 5.** Let $X \in \mathfrak{F}(\Omega)$ with $\overline{X[\Omega]} = \mathbb{R}$, $\varphi \in \mathfrak{F}(\mathbb{R})$. Define $x_0$ and $h^u_n$ as in Theorem 2 and assume $D^+\varphi(x_0) = D^-\varphi(x_0) = 0$. $\varphi$ is convex if and only if there exists a measure structure $(\mathcal{R}, \lambda)$ on $\Omega$ such that (a) $\lambda(u < X < v) = 0$ when $D^+\varphi(v) \leq D^-\varphi(u)$, (b) $\{h^u_n(X) : v \geq u\} \subset L^1(\lambda)$ and

$$\varphi(v) = \varphi(u) + \int h^u_n(X) d\lambda \quad v \geq u.$$
Proof. Define $\mathcal{N} = \{ \{ u < X < v \} : u, v \in \mathbb{R}, D^+\varphi(v) \leq D^-\varphi(u) \}$. From $X[\Omega] = \mathbb{R}$ follows that $\{ u < X < v \} \in \mathcal{N}$ if and only if $D^+\varphi(v) \leq D^-\varphi(u)$ and that $\mathcal{N}$ is an ideal of sets. Then (59) follows from Theorem $5(iii)$. □

5. APPLICATIONS TO STATISTICS AND PROBABILITY.

Returning to the Bayesian problem described in the Introduction, fix a $(\mathcal{A}, m) \in \mathcal{M}(\Omega)$.

**Theorem 6.** Let $X \in \mathfrak{F}(\Omega, \mathcal{S})$. The following properties are equivalent: (i) there exist a family $\{ Q_\theta : \theta \in \Theta \}$ of probabilities on $\mathcal{A}$ and an injective map $G \in \mathfrak{F}(\Theta, \mathcal{S})$ satisfying

\begin{equation}
(60a) \int hdm < 0 \quad \text{implies} \quad \inf_{\theta \in \Theta} \int hdQ_\theta < 0 \quad h \in \mathcal{S}(\mathcal{A}),
\end{equation}

\begin{equation}
(60b) \quad Q_\theta^*(A \cap \{ X \neq G(\theta) \}) = 0 \quad A \in \mathcal{A}, \theta \in \Theta;
\end{equation}

(ii) there exist $K \in \mathfrak{F}(\mathcal{A} \times \mathcal{S})$ and $(\mathcal{R}, \mu) \in \mathcal{M}(\Omega)$ such that $\{ K_s : s \in \mathcal{S} \} \subset ba(\mathcal{A})_+$ and, for each $A \in \mathcal{A}$, $E \subset \mathcal{S}$ and $s \in \mathcal{S}$,

\begin{equation}
(61a) \quad K(A, X) \in L^1(\mu) \quad \text{and} \quad m(A) = \int K(A, X)d\mu,
\end{equation}

\begin{equation}
(61b) \quad A \cap \{ X \in E \} \in \mathcal{A}(K_s) \quad \text{and} \quad K(A \cap \{ X \in E \}; s) = K(A; s)1_E(s).
\end{equation}

Proof. (i)$\Rightarrow$(ii). Since $G$ is injective we may define $K \in \mathfrak{F}(\mathcal{A} \times \mathcal{S})$ by letting

\begin{equation}
(62) \quad K(A, s) = Q_{G^{-1}(s)}(A) \quad A \in \mathcal{A}, s \in G[\Theta]
\end{equation}

or $K(A, s) = 0$ if $s \notin G[\Theta]$. By (60b), $\inf_{h} K(h; X(\omega)) \leq \inf_{h} Q_\theta(h)$ for every $h \in \mathcal{S}(\mathcal{A})$ so that, letting $(Th)(\omega) = K(h; X(\omega))$ in Theorem $1$, we conclude that $T$ is directed and $m$ is $T$-conglomerative. There exists then $(\mathcal{R}, \mu) \in \mathcal{M}(\Omega)$ such that

\begin{equation}
K(h, X) \in L^1(\mu) \quad \text{and} \quad \int hdm = \int K(h, X)d\mu \quad h \in \mathcal{S}(\mathcal{A}).
\end{equation}

If $A \in \mathcal{A}$ and $E \subset \mathcal{S}$, then either $Q_{G^{-1}(s)}^*(A \cap \{ X \in E \}) = 0$ (if $G(\theta) \notin E$) or $Q_{G^{-1}(s)}^*(A \cap \{ X \in E^c \}) = 0$. In either case $A \cap \{ X \in E \} \in \mathcal{A}(K_s) \cap \mathcal{A}(Q_\theta)$ and

\begin{equation}
K(A \cap \{ X \in E \}; s) = Q_{G^{-1}(s)}(A \cap \{ X \in E \}) = Q_{G^{-1}(s)}(A \cap \{ X \in E \})1_E(s) = K(A, s)1_E(s).
\end{equation}

(ii)$\Rightarrow$(i). Take $S_0 = \{ s \in \mathcal{S} : K_s \neq 0 \}$, $\Theta = S_0$, $Q_\theta = K_s$ and $G$ the identity. Then, (60b) follows from (61b). To deduce (60a) from (61a) it is enough to remark, via Theorem $2$, that the identity on $S$ is trivially a companion to $X$ (relatively to the whole of $L^1(\mu)$). □

The kernel $K(A, s)$ in Theorem $6$ plays a prominent role in statistics in which it is interpreted as the prevision of $A$ conditional on the occurrence of $X = s$. Its existence is generally deduced from that of regular conditional expectation and requires some classical properties such as $S$ being a Blackwell space. In Theorem $6$ instead, the existence of $K$ follows from $X$ strictly separating priors, so that each $\theta \in \Theta$ may be interpreted as a corresponding hypothesis concerning $X$.

The following is an example of (60b) in the classical setting.
Example 2. Let $X_1, X_2, \ldots$ be $m$-measurable random quantities on $\Omega$. Define implicitly the map

$$F(\omega, t) = \lim_k \lim_n \inf \frac{1}{n} \sum_{j=1}^n \mathbb{1}_{\{X_i \leq t + 2^{-k}\}}(\omega)$$

of $\Omega$ into the set $\mathcal{X}$ of increasing, right continuous, $[0,1]$-valued functions on $\mathbb{R}$, the limiting empirical distribution. For each $\theta \in \Theta$, let $G(\theta)$ be a candidate distribution. In the classical case, with each $Q_\theta$ countably additive and the sequence $X_1, X_2, \ldots$ independently and identically distributed under each $Q_\theta$, condition (60), with $X = F$, is a simple consequence of the strong law of large numbers, examined by Doob [15]. In order to guarantee that the inverse of $G$ is Borel measurable Doob assumes that $G$ is Borel measurable and that $\Theta$ is a subset of a complete and separable metric space, see also [4].

We pass now to the classical problem of Skhorohod which has been studied by a number of authors too large to give exact references. We have been influenced by the work of Berti, Pratelli and Rigo [3]. The starting point is the construction of a universal representation for the case of a separable space.

Corollary 6. Let $U \in \mathcal{F}(\Omega)$ with $\overline{U[\Omega]}$ having non empty interior and let $S$ be a separable, topological space. There exists a Borel function $H \in \mathcal{F}(\mathbb{R}, S)$ with countable range and such that $X' = H(U)$ is companion to any pair $(X, m)$ relatively to $\mathcal{E}(S)$.

Proof. By the remarks following Theorem 2, we can assume with no loss of generality that $X$ is the identity. Given that $[a, b] \subset \overline{U[\Omega]}$ for some $a, b \in \mathbb{R}$ then, upon replacing $U$ with a suitable continuous transformation, we can assume that $\overline{U[\Omega]} = [0,1]$. Let $S_0$ be a countable, dense subset of $S$ and $\iota \in \mathcal{F}(\mathbb{N}, S_0)$ an enumeration of $S_0$. Define,

$$G(x) = \inf \{n \in \mathbb{N} : 1 - 2^{-n} \geq x\} \quad x \in (0,1) \quad \text{and} \quad H = \iota \circ G.$$  

$H$ is a Borel function mapping $(0,1)$ onto $S_0$ — since $G^{-1}(n) = (1 - 2^{-\lfloor n \rfloor}, 1 - 2^{-n}]$. If $h \in \mathcal{H}$ and $\int h \, dm < 0$ then $\{h < 0\}$ is an open, non empty subset of $S$ and as such it contains some element $\iota(n_h)$ of $S_0$. The set $B_h = \{U \in G^{-1}(n_h)\}$ is non empty (as $\overline{U[\Omega]} = [0,1]$) and coincides with $\{X' = \iota(n_h)\}$. Thus, $B_h \subset \{h(X') < 0\}$ so that $m$ is $X'$-conglomerative relatively to $\mathcal{H}$. □

Corollary 6 extends to the case of finite additivity and of a separable state space the classical idea of generating a random quantity with given distribution by applying to a uniformly distributed random quantity the inverse of the corresponding cumulative density function. Interestingly, we obtain that the same function $X$ represents all possible distributions relatively to the class of continuous functions and for some suitable set function $\mu$. Let us also mention the possibility of dropping the condition that $S$ is separable by assuming that $m$ is supported by a measurable, separable subset of $S$.

We highlight the advantage of doing without measurability. Constructing a function such as $U$ in Corollary 6 is a rather trivial exercise as long as $\Omega$ has the right cardinality. Requiring that $U$ is uniformly distributed on the unit interval under some classical probability measure $P$, as in the
following Theorem\textsuperscript{7} requires, in contrast, additional assumptions. The following result is inspired by \cite[theorem 3.1]{3}.

**Theorem 7.** Let $S$ be a normal, separable topological space, $\Sigma$ a ring of subsets of $S$ and $(\Omega, \mathcal{A}, P)$ a classical probability space supporting a random quantity $U$ uniformly distributed on $(0, 1)$. Let either $m \in \text{fa}(\Sigma)_+$ be countably additive or $S$ be compact and write $\mathcal{H} = \mathcal{C}(S) \cap L^1(m)$. There exists a Borel function $g \in \mathfrak{F}((0, 1), S)$ such that $X = g(U)$ is supported by $(\Omega, \mathcal{A}, P)$ and

$$
\int h \, dm = \int h(X) \, dP \quad h \in \mathcal{H}.
$$

**Proof.** If $S$ is compact then the restriction of $m$ to the minimal ring $\mathcal{R}_\mathcal{H}$ is countably additive. Let $H$ be the map defined in (64). Then, as was shown in the proof of Corollary 6, $m$ is $H$-conglomerative relatively to $\mathcal{C}(S)$ so that, by Theorem 2,

$$
\int h \, dm = \int h(H) \, d\mu \quad h \in \mathcal{H}
$$

for some $(\mathcal{R}, \mu) \in \mathcal{M}((0, 1))$. We claim that $\sigma \mathcal{R} = \mathcal{B}((0, 1))$. Recall that $\sigma \mathcal{R}$ is generated by sets of the form $\{ h(H) > t \}$ which are Borel since $h$ is continuous and $H$ is Borel. Conversely, if $0 \leq a \leq b \leq 1$ then the set $H[(a, b)]$ is a finite subset of $S$ and therefore closed. Since $S$ is normal, for any other finite subset $F$ of $H[(a, b)^c]$ we can find a function $f \in \mathfrak{F}(S, [0, 1])$ such that $f = 1$ on $H[(a, b)]$ and $f = 0$ on $F$. Thus $(a, b) \subset \{ f(H) \geq 1 \} \in \sigma \mathcal{R}$. Since $H[(0, 1)]$ is countable we find a sequence $(f_n)_{n \in \mathbb{N}}$ of such functions each vanishing on a finite subset of $H[(a, b)^c]$ so that the intersection $\bigcap_n \{ f_n(H) \geq 1 \}$ is again an element of $\sigma \mathcal{R}$ and coincides with $(a, b)$. In other words, we can assume that $\mu$ is defined on the Borel subsets of $(0, 1)$. From the classical Skhorohod theorem, we deduce the existence of an $S$ valued random quantity $Z$ supported by $((0, 1), \mathcal{R}((0, 1)), \Lambda)$ (with $\Lambda$ the Lebesgue measure on $(0, 1)$) and admitting $\mu$ as its distribution. On its turn, $\Lambda$ is the distribution of $U$ under $P$. A repeated application of Theorem 2 with $g = H \circ Z$ and $X = g(U)$ gives

$$
\int h \, dm = \int h(H) \, d\mu = \int h(g) \, d\Lambda = \int h(X) \, dP \quad h \in \mathcal{H}.
$$

Thus the random quantity $X$ is supported by $(\Omega, \mathcal{A}, P)$ and represents $m$ relatively to $\mathcal{C}(S)$. \hfill \Box

6. Applications to stochastic processes.

We start this section with a result closely related to Theorem\textsuperscript{4}.

**Theorem 8.** Let $\mathcal{H} \subset \mathcal{C}(\mathbb{R})$ be a Stonean sublattice, $X' = (X'_t : t \in \mathbb{R}_+)$ Brownian motion on some, filtered, standard probability space $(\Omega', \mathcal{A}', P)$. Write $\mathcal{N}$ to denote the family of sets $A \subset \Omega \times \mathbb{R}_+$ such that $P^\star(\pi_\Omega A) = 0 \\Leftrightarrow \phi \in \mathcal{L}(\mathcal{H})_+$ if and only if there exists a minimal $(\mathcal{R}', \mu) \in \mathcal{M}(\Omega' \times \mathbb{R}_+)$ with $\mathcal{N} \subset \mathcal{R}$, $\mu(N) = 0$ for all $N \in \mathcal{N}$,

$$
(67) \quad h(X') \in L^1(\mu) \quad \text{and} \quad \phi(h) = \int h(X') \, d\mu \quad h \in \mathcal{H}.
$$
Moreover, $\mu$ is countably additive if and only if $\lim_n \phi(h_n) = 0$ for every decreasing sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{H}$ which converges to 0 in Lebesgue measure.

Proof. By Theorem 4 if $\phi(h) < 0$ and $N \in \mathcal{N}$ then, since $N_t = \{\omega : (\omega, t) \in N\}$ is $P$ null

$$0 > \inf_{\omega \in N_t} h(X'_t)(\omega) \geq \inf_{(\omega, s) \in N} h(X'_s)(\omega).$$

The main claim follows immediately. The last claim may be proved as in Theorem 4 upon noting that $X'[N^c]$ has 0 Lebesgue measure when $N \in \mathcal{N}$. But this is again clear since $\{X'_t \in X'[N^c]\}$ is $P$ null. The rest of that proof remains unchanged.

Let $\mathcal{I}$ be the family of finite subsets of $\mathbb{R}$. For each $\alpha = \{t_1, \ldots, t_n\} \in \mathcal{I}$, let $\pi_\alpha$ be the projection

$$(68) \quad \pi_\alpha(s) = (s_{t_1}, s_{t_2}, \ldots, s_{t_n}) \quad s \in \mathfrak{F}(\mathbb{R}).$$

If $X = (X_t : t \in \mathbb{R})$, write $X_\alpha = (X_t : t \in \alpha)$.

**Corollary 7.** Let $X' = (X'_t : t \in \mathbb{R})$ be Brownian motion on some classical probability space $(\Omega', \mathcal{A}, P)$ and $(m_\alpha : \alpha \in \mathcal{I})$ a projective family of probabilities (namely $m_\alpha \in \mathcal{P}(\mathcal{B}(\mathbb{R}))$ is the marginal of $m_\beta$ whenever $\alpha \subset \beta$). There exists a probability structure $(\mathcal{A}, \mu)$ on $\Omega$ such that

$$(69) \quad h(X'_\alpha) \in L^1(\mu) \quad \text{and} \quad \int hdm_\alpha = \int h(X'_\alpha) d\mu \quad \alpha \in \mathcal{I}, \ h \in C(\mathbb{R}) \cap L^1(m_\alpha).$$

If $m_\alpha$ is countably additive, then

$$(70) \quad m_\alpha(B) = \mu(X'_\alpha \in B) \quad B \in \mathcal{B}(\mathbb{R}).$$

Proof. As usual, a projective family of probabilities induces a unique probability on the algebra $\Sigma = \{\pi_\alpha^{-1}(B) : \alpha \in \mathcal{I}, B \in \mathcal{B}(\mathbb{R})\}$ of finite dimensional cylinders obtained by letting

$$(71) \quad m(\pi_\alpha^{-1} A) = m_\alpha(A) \quad A \in \mathcal{B}(\mathbb{R}), \ \alpha \in \mathcal{I}.$$ 

If $g \in \mathfrak{F}(\mathbb{R})$ and $h = g \circ \pi_\alpha$ then $\{h > t\} = \pi_\alpha^{-1}(\{g > t\})$ so that from Lemma 2 we conclude

$$\int hdm = \int gdm_\alpha$$

whenever either side is well defined. Let $\mathcal{H} = \{g \circ \pi_\alpha : g \in C(\mathbb{R}), \ \alpha \in \mathcal{I}\}$ and $L^1(m)$.

If $h \in \mathcal{H}$ and $\int hdm < 0$, then $\int hdm_\alpha < 0$ for some $\alpha = \{t_1 < \ldots < t_n\} \in \mathcal{I}$. Since $\{h_\alpha < 0\}$ is open and nonempty, there exist open, nonempty sets $B_1, \ldots, B_n \subset \mathbb{R}$ such that $x_i - x_{i-1} \in B_i$ for $i = 1, \ldots, n$ (and $x_0 = 0$) implies $h_\alpha(x_1, \ldots, x_n) < 0$. Therefore, $P(X_{t_1}, \ldots, X_{t_n} \in \{h_\alpha < 0\}) \geq \prod_{i=1}^n P(X_t - X_{t-1} \in B_i) > 0$ so that $\inf \omega h_\alpha(X'_\alpha) < 0$ and $m$ is $X'$-conglomerative. The second claim, as in Example 1, follows from metric spaces being normal.

**Corollary 7** is related to [12, Theorem 1] and, in Dubins’ peculiar terminology, it asserts that Brownian motion is cousin to any stochastic process. Dubins main finding is a necessary and sufficient condition for the existence of cousins with almost all paths in a given class. His claim is an easy corollary of our previous results. We give a simple proof for completeness.
Corollary 8 (Dubins). Let $X$ be a stochastic process on a probability space $(\Omega, \mathcal{A}, m)$ and let $\mathbb{F} \subset \mathfrak{F}(\mathbb{R}_+)$ satisfy:

\[ \forall (\omega, \alpha) \in \Omega \times \mathcal{I}, \quad \exists Y \in \mathbb{F} \text{ such that } Y(t) = X(\omega, t) \quad t \in \alpha. \tag{72} \]

There is a process $X'$ on a probability space $(\Omega', \Sigma, \mu)$ with $\mu$-a.a. paths in $\mathbb{F}$ and such that

\[ g(X'_\alpha) \in L^1(\mu) \quad \text{and} \quad \int g(X'_\alpha) d\mu = \int g(X_\alpha) d\mu \quad \alpha \in \mathcal{I}, \; g \in \mathfrak{F}(\mathbb{R}_\alpha), \; g(X_\alpha) \in L^1(m). \tag{73} \]

Proof. Write

\[ \mathcal{H} = \{ g \circ \pi_\alpha : \alpha \in \mathcal{I}, \; g \in \mathfrak{F}(\mathbb{R}_\alpha), \; g(X_\alpha) \in L^1(m) \}, \]

$\Omega' = \mathfrak{F}(\mathbb{R}_+)$ and define $T \in \mathcal{L}(\mathcal{H}, \mathfrak{F}(\mathbb{F}))$ by letting $T(h)(Y) = h(Y)$ for each $h \in \mathcal{H}$ and $Y \in \mathbb{F}$. Then, $T$ is directed and, by (72), the linear functional $\phi(h) = \int h(X) dm$ is $T$-conglomerative. By Theorem 1 there exists a minimal $(\mathcal{R}_0, \mu_0) \in \mathcal{M}(\mathbb{F})$ such that

\[ \int h(X) dm = \int h d\mu_0 \quad \text{for all } h \in \mathcal{H}. \]

Since $m$ is a probability, then $\mathcal{R}_0$ is an algebra and $\mu_0$ a probability. Let

\[ \Sigma = \{ A \subset \Omega' : A \cap \mathbb{F} \in \mathcal{R}_0 \} \quad \text{and} \quad \mu(A) = \mu_0(A \cap \mathbb{F}) \quad A \in \Sigma. \]

Then, $\Sigma$ is an algebra of subsets of $\Omega'$, $\mu$ a probability on $\Sigma$ with $\mu(\mathbb{F}^c) = 0$ and $X'(w, t) = w(t)$ a stochastic process on $(\Omega', \Sigma, Q)$ with $X'_w = w$. Moreover, $\int h d\mu_0 = \int h(X') d\mu$ for all $h \in \mathcal{H}$. \(\square\)

Dubins deduces from this result that any stochastic process admits cousins having continuous or polynomial or stepwise linear paths.

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