String model with dynamical geometry
and torsion

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Abstract

A string model with dynamical metric and torsion is proposed. The geometry of the string is described by an effective Lagrangian for the scalar and vector fields. The path integral quantization of the string is considered.

1. Superstring theory now appears as a candidate for a unified theory of elementary particles [1–5]. The string concept still plays an important role in strong interaction and statistical physics [6–10]. The fundamental problem of the bosonic string model is that it may be quantized only in 26 spacetime dimensions. Furthermore it contains a tachyon state. In this note we consider a new bosonic string model which has some appealing properties and perhaps circumvents the difficulties mentioned.

The classical string model is defined by the Lagrangian

$$L_0 = \frac{1}{2} \sqrt{g} g^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \quad g = \det g_{\alpha \beta},$$

(1)

which contains the string position variable $X^\mu (\zeta), \mu = 1, \ldots, D$, and the metric tensor $g_{\alpha \beta} (\zeta)$ on the two-dimensional space of the parameters $\zeta^\alpha, \alpha = 0, 1$, which describe the string. The metric $g_{\alpha \beta}$ acts like a Lagrange multiplier in (1) and may be eliminated algebraically via its equation of motion.

The bosonic part of the action for known string models is essentially the same and is proportional to the area of the string surface. This is true for (1) after elimination of the metric via its equation of motion. The action for (1) is invariant under (i) reparametrization of the string surface or general coordinate transformation of the parameters $\zeta^\alpha$, (ii) Poincaré transformation of the string position variable $X^\mu$, (iii) and Weyl transformation of the metric $g_{\alpha \beta}$.

We try to extend the list of possible string models from the geometric point of view. It seems that there is no way to generalize the Lagrangian (1) without introducing the internal degrees of freedom for a string. One commonly acknowledged generalization is fermionic or superstring theory which attributes the string with fermionic degrees of freedom in a supersymmetric fashion. The guideline for such a generalization is supersymmetry. Below we follow another way and generalize the Lagrangian (1) from a purely

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geometric point of view. Namely the introduction of metric and torsion for a string surface attributes new bosonic degrees of freedom describing the internal structure of the string. The torsion degree of freedom is dynamical. The choice of the geometric Lagrangian seems to be unique in the framework of Riemann–Cartan geometry. There we have looked over all invariants which yield the second order equations of motion. Note that there is no geometric Lagrangian for a metric alone without torsion which gives rise to the second order equations of motion. The resulting theory has invariance (i) and (ii) but Weyl invariance is broken already at the classical level.

It was shown [7] that the Weyl invariance of the model (1) is broken in the quantum domain. As a result, the conformal piece of the metric participates in a quantum dynamics and the theory is reduced to the quantum theory of the Liouville equation. The absence of a simple vacuum solution of the Liouville equation presents difficulties in the quantization of the theory. Attempts to construct a satisfactory bosonic string theory have led us to the investigation of the following possibility. The Lagrangian (1) may be considered as describing the two-dimensional gravity field $g_{\alpha\beta}$ interacting with the scalar fields $X^\mu$. From this point of view the dynamical equation of motion for the metric $g_{\alpha\beta}$ is needed.

Adding to the Lagrangian $L_0$ the Hilbert–Einstein Lagrangian $\sqrt{g}R$ does not give rise to the dynamical equation for the metric because the latter Lagrangian in two dimensions equals to a total divergence. But if one admits a torsion to be nontrivial and dynamical then there exists the possibility of obtaining a dynamical equation of motion for the metric or for the corresponding zweibein. In conformal gauge this theory is reduced to the theory of interacting scalar and vector fields and has a simple vacuum solution.

2. The geometry of the string is described by the zweibein $e^a_\alpha \ (a = 0, 1), \ g_{\alpha\beta} = e^a_\alpha e^b_\beta$ and the Lorentz connection $\omega^a_\alpha \ (a = 0, 1)$. Curvature and torsion have the following form

$$
R_{\alpha\beta}^{\ a\ b} = \partial_\alpha \omega^a_\beta - \omega^a_\gamma \omega^b_\beta - (\alpha \leftrightarrow \beta),
$$

$$
T_{\alpha\beta}^a = \partial_\alpha e^a_\beta - \omega^a_\gamma e^b_\beta - (\alpha \leftrightarrow \beta),
$$

$$
R_{abcd} = R_{\alpha\beta\gamma\delta} e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta, \quad T_{abc} = T_{\alpha\beta\gamma} e^a_\alpha e^b_\beta e^c_\gamma.
$$

Let us consider the most general reparametrization-invariant and parity-conserving Lagrangian quadratic in curvature and torsion

$$
L_1 = \sqrt{g} \left( \frac{1}{4} \mu^2 R^2_{abcd} + \frac{1}{4} \gamma T^2_{abc} + \lambda \right),
$$

which yields the second-order differential equations. There are no parity-violating terms in two dimensions giving rise to equations of motion of no more then second order. Thus the Lagrangian (2) represents the most general reparametrization-invariant Lagrangian which yields the second-order equations of motion for the zweibein and the Lorentz connection. The curvature squared term includes the kinetic one for the Lorentz connection and the torsion squared term includes the kinetic term for the zweibein and the mass for the Lorentz connection. Note that only three parameters $\mu, \gamma$ and $\lambda$ survive in two dimensions, in contrast to the ten parameters [11] of a four-dimensional Lagrangian.

The string action

$$
S = \int d^2 \zeta L, \quad L = L_0 + L_1
$$

(3)
yields the following equations of motion:

\[ g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} X^\mu = 0, \quad (4a) \]

\[ -\gamma^2 \nabla_\beta T^{\beta\alpha} + \gamma^2 \left( \frac{1}{4} T_{bcd} e^\alpha_a - T^{abc} T_{abc} \right) + 2\mu^2 \left( \frac{1}{4} F^2_{bc} e^\alpha_a - F^{abc} F_{abc} \right) + \lambda e^\alpha_a \]

\[ + \frac{1}{2} g^{\beta\gamma} p_{\beta} X^\mu \partial_\gamma X_\mu e^\alpha_a - \partial^\alpha X^\mu \partial_\beta X_\mu e^\beta_a = 0, \quad (4b) \]

\[ 2\mu^2 \nabla_\beta F^{\beta\alpha} - \gamma^2 T^{\alpha\beta} \varepsilon_{\beta} = 0. \quad (4c) \]

Here \( \nabla_\alpha \) means a covariant derivative with Lorentz connection \( \omega^{\alpha}_{\beta\gamma} \) when it acts on the tensors with Latin indices, and with the metrical connection with out torsion \( \Gamma^{\alpha\beta\gamma} \) (Christoffel’s symbols) when it acts on the tensors with Greek indices \( \alpha, \beta \). We have parametrized the Lorentz connection by a vector field \( \omega^{\alpha}_{\beta\gamma} = A_\alpha \varepsilon_{\beta\gamma} \) where \( \varepsilon_{\beta\gamma} = -\varepsilon_{\gamma\beta} \), and \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). We consider the string with Euclidean metric. Note that we have the equation of motion (4b) instead of the constraints in the usual approach. It leads to a different quantization scheme. The corresponding Virasoro algebra will be considered in a separate publication.

In conformally flat gauge \( e^{a}_\alpha = e^\varphi \delta^a_\alpha \) the Lagrangian (2) is reduced to the following expression up to a total divergence:

\[ L_1 = \frac{1}{2} \mu^2 e^{-2\varphi} F^2_{\alpha\beta} + \frac{1}{2} \gamma^2 \left[ (\partial_\alpha \varphi)^2 + \varphi F_{\alpha\beta} \varepsilon^{\alpha\beta} + A^2_\alpha \right] + \lambda e^{2\varphi}. \quad (5) \]

It describes in flat Euclidean space the interaction of the scalar field \( \varphi \) appearing from the zweibein and the vector field \( A_\alpha \) appearing from the Lorentz connection.

A special case of the Lagrangian (5) is of particular interest. Rescaling the fields \( \varphi \rightarrow \gamma \varphi, A_\alpha \rightarrow \mu A_\alpha \) and taking the limit \( \gamma/\mu \rightarrow 0 \) one has the following Lagrangian:

\[ L_1 = \frac{1}{2} e^{-2\varphi/\gamma} F^2_{\alpha\beta} + \frac{1}{2} (\partial_\alpha \varphi)^2 + \lambda e^{2\varphi/\gamma}. \quad (6) \]

The remarkable property of this theory is its integrability. The Lagrangian (6) yields the following equations of motion:

\[ \partial_\alpha (e^{-2\varphi/\gamma} F_{\alpha\beta}) = 0, \quad (7a) \]

\[ \Delta \varphi = \partial_\alpha \partial_\alpha \varphi = \frac{1}{\gamma} \left( 2\lambda e^{2\varphi/\gamma} - F^2_{\alpha\beta} e^{-2\varphi/\gamma} \right). \quad (7b) \]

Eq. (7a) has the general solution

\[ F_{\alpha\beta} = c \varepsilon_{\alpha\beta} e^{2\varphi/\gamma}, \quad (8) \]

with arbitrary constant \( c \). Substitution of (8) into (7b) yields the integrable Liouville equation

\[ \Delta \varphi = \frac{2}{\gamma} (\lambda - c^2) e^{2\varphi/\gamma}. \]

Thus the Lagrangian (6) gives integrable equations of motion. It is important that eqs. (7) have a simple vacuum solution

\[ \varphi = \varphi_0 = \text{const}, \quad F_{\alpha\beta} = \sqrt{\lambda} \varepsilon_{\alpha\beta} e^{2\varphi_0/\gamma}, \quad (9) \]

in contrast to the Liouville equation.
It is known that the Liouville equation closely relates to the theory of surfaces with constant curvature. The Lagrangians (5) and (6) seem to be related by analogy to the geometry of surfaces with torsion. The Lagrangian (2) is of interest also as a two-dimensional theory of gravity (about lower dimensional theories of gravity, see ref. [12]).

3. Let us proceed to the quantization of the string model described by the action (3). The partition function has the following form

\[ Z = \int \mathcal{D}(X^\mu)\mathcal{D}(e_\alpha^a)\mathcal{D}(\omega_{\alpha}^{ab})e^{-S}. \]

We consider here only closed strings with the topology of a sphere. The measure \( \mathcal{D}(e_\alpha^a) \) takes the following form [7]:

\[ \mathcal{D}(e_\alpha^a) = \mathcal{D}\varphi\mathcal{D}e^\alpha\mathcal{D}l\exp \left( -\frac{26}{12\pi} \int d^2\zeta \left[ \frac{1}{2}(\partial_\alpha\varphi)^2 + \lambda_1 e^{2\varphi} \right] \right). \]

Integration over \( X^\mu \) for a sphere gives [7]

\[ \int \mathcal{D}X^\mu \exp \left( -\frac{1}{2} \int d^2\zeta \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu \right) = \exp(-\mathcal{F}), \]

\[ \mathcal{F} = -\frac{D}{12\pi} \int d^2\zeta \left[ \frac{1}{2}(\partial_\alpha\varphi)^2 + \lambda_2 e^{2\varphi} \right]. \]

The above \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants. The measure \( \mathcal{D}(\omega_{\alpha}^{ab}) = \mathcal{D}(A_\alpha) \). Thus the partition function takes the form

\[ Z = \int \mathcal{D}\varphi\mathcal{D}(A_\alpha) \exp \left( -\int d^2\zeta L_{\text{eff}} \right), \]

where

\[ L_{\text{eff}} = \frac{1}{2} \mu^2 e^{-2\varphi} F_{\alpha\beta}^2 + \frac{1}{2} \kappa^2 (\partial_\alpha \varphi)^2 + \frac{1}{2} \gamma^2 \varphi F_{\alpha\beta} \varepsilon^{\alpha\beta} + \frac{1}{2} \gamma^2 A_\alpha^2 + \tilde{\lambda} e^{2\varphi}, \]

\[ \kappa^2 = \gamma^2 + \frac{1}{12\pi} (26 - D), \quad \tilde{\lambda} = \lambda + \lambda_1 - \lambda_2. \]

The theory takes a particular simple form after rescaling the fields \( \varphi \rightarrow \kappa \varphi, A_\alpha \rightarrow \mu A_\alpha \) and taking the limit \( \gamma/\mu \rightarrow 0 \). Then the theory results in an integrable one with an effective Lagrangian

\[ L_{\text{eff}} = \frac{1}{2} e^{-2\varphi/\kappa} F_{\alpha\beta}^2 + \frac{1}{2} (\partial_\alpha \varphi)^2 + \tilde{\lambda} e^{2\varphi/\kappa}, \]  

as it was discussed earlier. This Lagrangian may be quantized in powers of \( 1/\kappa \) around vacuum [9] \( \varphi = \varphi_0 \), \( F_{\alpha\beta} = \sqrt{\tilde{\lambda}} \varepsilon_{\alpha\beta} e^{2\varphi_0/\kappa} \) in any dimension \( D \) because the inequality

\[ \kappa^2 = \gamma^2 + \frac{1}{12\pi} (26 - D) > 0 \]

may be satisfied by an appropriate choice of \( \gamma \).

So there exists an interesting string model which is described by the action (9) and seems to be very promising. Further details of the present model, in particular the spectrum of the theory, will be discussed in a separate publication.

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