A note on sublinear separators and expansion

Zdeněk Dvořák*

Abstract

For a hereditary class $G$ of graphs, let $s_G(n)$ be the minimum function such that each $n$-vertex graph in $G$ has a balanced separator of order at most $s_G(n)$, and let $\nabla_G(r)$ be the minimum function bounding the expansion of $G$, in the sense of bounded expansion theory of Nešetřil and Ossona de Mendez. The results of Plotkin, Rao, and Smith (1994) and Esperet and Raymond (2018) imply that if $s_G(n) = \Theta(n^{1-\varepsilon})$ for some $\varepsilon > 0$, then $\nabla_G(r) = \Omega(r^{\frac{1}{2\varepsilon}-1}/\text{polylog } r)$ and $\nabla_G(r) = O(r^{\frac{1}{2\varepsilon}-1}\text{polylog } r)$. Answering a question of Esperet and Raymond, we show that neither of the exponents can be substantially improved.

For an $n$-vertex graph $G$, a set $X \subseteq V(G)$ is a balanced separator if each component of $G - X$ has at most $2n/3$ vertices. Let $s_G$ denote the minimum size of a balanced separator in $G$, and for a class $G$ of graphs, let $s_G : N \rightarrow N$ be defined by

$$s_G(n) = \max\{s(G) : G \in G, |V(G)| \leq n\}.$$ 

Classes with sublinear separators (i.e., classes $G$ with $s_G(n) = o(n)$) are of interest from the computational perspective, as they naturally admit divide-and-conquer style algorithms. They also turn out to have a number of intriguing structural properties; of interest for this note is the connection to the density of shallow minors.

For a graph $G$ and an integer $r \geq 0$, an $r$-shallow minor of $G$ is any graph obtained from a subgraph of $G$ by contracting pairwise vertex-disjoint subgraphs, each of radius at most $r$. The density of a graph $H$ is $|E(H)|/|V(H)|$. We let $\nabla_r(G)$ denote the maximum density of an $r$-shallow minor of $G$. For a class $G$ of graphs, let $\nabla_G : N \rightarrow \mathbb{R} \cup \{\infty\}$ be defined by

$$\nabla_G(r) = \sup\{\nabla_r(G) : G \in G\}.$$ 



*Computer Science Institute, Charles University, Prague, Czech Republic. E-mail: raka@iuuk.mff.cuni.cz. Supported in part by ERC Synergy grant DYNASNET no. 810115.
If $\nabla_G(r)$ is finite for every $r$, we say that the class $G$ has bounded expansion. The classes with bounded expansion have a number of common properties and computational applications; we refer the reader to [8] for more details. The first connection between sublinear separators and bounded expansion comes from the work of Plotkin, Rao, and Smith [9].

**Theorem 1** (Plotkin, Rao, and Smith [9]). For each $n$-vertex graph $G$ ($n \geq 2$) and all integers $l, h \geq 1$, either $G$ has a balanced separator of order at most $n/l + 2h^2l \log_2 n$, or $G$ contains a $(2l \log_2 n)$-shallow minor of $K_h$.

As observed in [5] (and qualitatively in [7, 4]), this has the following consequence.

**Corollary 2.** Suppose $G$ is a class of graphs such that $s_G(n) = \Omega(n^{1-\varepsilon})$ for some $\varepsilon > 0$. Then $\nabla_G(r) = \Omega(r^{1-\varepsilon\over \log_2 r})$.

**Proof.** Consider a sufficiently large integer $r$, and let $n = \lfloor r^{1-\varepsilon\over \log_2 r} \rfloor$, $l = \lfloor r^{1-\varepsilon\over \log_2 r} \rfloor$, and $h = \lfloor n^{1/2l-1} \log_2 n \rfloor$. Note that

$$2h^2l \log_2 n < n/l \leq n^\varepsilon \leq r \log_2^{-2}r \leq r \leq \log_2 n \leq \frac{1}{\varepsilon} \log_2 r \leq 2l \log_2 n \leq r \leq 3\log_2 n.$$

Consequently, we have

$$\frac{n}{l} + 2h^2l \log_2 n \leq \frac{2n}{l} \leq \frac{6n \log_2 n}{r} \leq 6n^{1-\varepsilon\log_2 n \over \log_2 r} \leq \frac{6}{\varepsilon \log_2 r} n^{1-\varepsilon} < s_G(n).$$

Let $G \in G$ be an $n$-vertex graph with no balanced separator of order less than $s_G(n)$; then Theorem 1 implies $G$ contains an $r$-shallow minor of $K_h$, implying that

$$\nabla_G(r) \geq \nabla_r(G) \geq \frac{|E(K_h)|}{|V(K_h)|} = \Omega(h) = \Omega(r^{1-\varepsilon\over \log_2 r}).$$

Dvořák and Norin [4] proved that surprisingly, a converse to Corollary 2 holds as well. Subsequently, Esperet and Raymond [5] gave a simpler argument with a better exponent: they state their result with $O(r^{1-\varepsilon\log_2 r})$ bound, but an analysis of their argument shows that the exponent can be improved by 1. We include the short proof for completeness; the proof uses the following result establishing the connection between separators and treewidth.
**Theorem 3** (Dvořák and Norin [3]). Let $G$ be a graph and $k$ an integer. If $s(H) \leq k$ for every induced subgraph $H$ of $G$, then $\text{tw}(G) \leq 15k$.

For $\alpha > 0$, a graph $G$ is an $\alpha$-expander if $|N(S)| \geq \alpha |S|$ holds for every set $S \subseteq V(G)$ of size at most $|V(G)|/2$.

**Theorem 4** (Esperet and Raymond [5]). Suppose $\mathcal{G}$ is a hereditary class of graphs such that $s_G(n) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$. Then $\nabla_G(r) = O(r^{\frac{1}{\varepsilon}} \cdot \text{polylog } r)$.

**Proof.** Let $H$ be an $r$-shallow minor of a graph $G \in \mathcal{G}$ and let $d$ be the density of $H$. By the result of Shapira and Sudakov [11], there exists a subgraph $H_1 \subseteq H$ of average degree $\Omega(d)$ such that, letting $n = |V(H_1)|$, the graph $H_1$ is a $(1/polylog n)$-expander. Consequently, $H_1$ has treewidth $\Omega(n/polylog n)$. As Chekuri and Chuzhoy [2] proved, $H_1$ has a subcubic subgraph $H_2$ of treewidth $\Omega(n/polylog n)$. Since $H_2$ is a subcubic $r$-shallow minor of $G$, we conclude that $G$ has a subgraph $G_2$ obtained from $H_2$ by subdividing each edge at most $4r$ times, and thus $|V(G_2)| = O(r|V(H_2)|) = O(rn)$. Furthermore, since $H_2$ is a minor of $G_2$, we have

$$\text{tw}(G_2) \geq \text{tw}(H_2) = \Omega(n/polylog n).$$

On the other hand, since $\mathcal{G}$ is hereditary, $G_2$ is a spanning subgraph of a graph from $\mathcal{G}$, and thus every subgraph of $G_2$ has a balanced separator of size $O(|V(G_2)|^{1-\varepsilon}) = O((rn)^{1-\varepsilon})$. By Theorem 3, this implies $\text{tw}(G_2) = O((rn)^{1-\varepsilon})$. Combining this inequality with (1), this gives $n = O(r^{\frac{1}{\varepsilon}} \cdot \text{polylog } n) = O(r^{\frac{1}{\varepsilon}} \cdot \text{polylog } r)$. Since $H_1$ has $n$ vertices and average degree $\Omega(d)$, we have $d = O(n) = O(r^{\frac{1}{\varepsilon}} \cdot \text{polylog } r)$. This holds for every $r$-shallow minor of a graph from $\mathcal{G}$, and thus $\nabla_G(r) = O(r^{\frac{1}{\varepsilon}} \cdot \text{polylog } r)$.

For $0 < \varepsilon \leq 1$,

- let $b_\varepsilon$ denote the supremum of real numbers $b$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_G(n) = \Theta(n^{1-\varepsilon})$ satisfies $\nabla_G(r) = \Omega(r^b)$, and

- let $B_\varepsilon$ denote the infimum of real numbers $B$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_G(n) = \Theta(n^{1-\varepsilon})$ satisfies $\nabla_G(r) = O(r^B)$.

Corollary 2 and Theorem 4 give the following bounds.

**Corollary 5.** For $0 < \varepsilon \leq 1$,

$$\max\left(\frac{1}{2^{1-\varepsilon}}, 1, 0\right) \leq b_\varepsilon \leq B_\varepsilon \leq \frac{1}{\varepsilon} - 1.$$
Esperet and Raymond [5] asked whether either of these bounds (in particular, in terms of multiplicative constants) can be improved. They suggest some insight into this question could be obtained by investigating the $d$-dimensional grids. While the grids ultimately do not give the best bounds we obtain, their analysis is instructive and we give it (for even $d$) in the following lemma.

Note that $b_{1/2} = 0$, matching the lower bound from Corollary 5. Indeed, as proved by Lipton and Tarjan [6], the class $\mathcal{P}$ of planar graphs satisfies $s_{\mathcal{P}}(n) = \Theta(n^{1/2})$, and on the other hand, every minor of a planar graph is planar, implying $\nabla_{\mathcal{P}}(r) \leq 3 = O(r^0)$. However, 2-dimensional grids with diagonals give $B_{1/2} = 1$, as we will show in greater generality in the next lemma. Hence, $b_c$ is not always equal to $B_c$.

**Lemma 6.** For every even integer $d$,

$$b_{1/d} \leq \frac{d}{2} \leq B_{1/d}.$$  

**Proof.** Let $Q_n^d$ denote the graph whose vertices are elements of $\{1, \ldots, n\}^d$ and two distinct vertices are adjacent if they differ by at most 2 in each coordinate. Let $\mathcal{G}_d$ denote the class consisting of graphs $Q_n^d$ for all $n \in \mathbb{N}$ and their induced subgraphs. Note that $s_{\mathcal{G}}(n) = \Theta(n^{1-1/d})$: Each induced subgraph $H$ of $Q_n^d$ can be represented as an intersection graph of axis-aligned unit cubes in $\mathbb{R}^d$ where each point is contained in at most 3 $d$ cubes, and such graphs have balanced separators of order $O(|V(H)|^{1-1/d})$, see e.g. [12]. Conversely, standard isoperimetric inequalities show that $Q_n^d$ does not have a balanced separator smaller than $\Omega(n^{d-1}) = \Omega(|V(Q_n^d)|^{1-1/d})$. We claim that $\nabla_{\mathcal{G}}(r) = \Theta(r^{d/2})$.

Consider any $r$-shallow minor $H$ of $Q_n^d$, and for $v \in V(H)$, let $B_v$ denote the subgraph of $Q_n^d$ of radius at most $r$ contracted to form $v$. We have $\Delta(Q_n^d) < 5^d$, and thus $\deg_H(v) < 5^d|V(B_v)|$ holds for every $v \in V(H)$. Let $v$ be the vertex of $H$ with $|V(B_v)|$ minimum, and let $c$ be a vertex of $Q_n^d$ such that each vertex of $B_v$ is at distance at most $r$ from $c$. Note that if $uv \in V(H)$, then every vertex of $B_u$ is at distance at most $3r + 1$ from $c$. Consequently, the pairwise vertex-disjoint subgraphs $B_u$ for $u \in N(v)$ are all contained in a cube with side of length $12r + 4$ centered at $c$, implying

$$(\deg_H(v) + 1)|V(B_v)| \leq |V(B_v)| + \sum_{u \in N(v)} |V(B_u)| \leq (12r + 5)^d,$$

and thus $\deg_H(v) < (12r + 5)^d/|V(B_v)|$. Therefore,$$

\deg_H(v) < \min(5^d|V(B_v)|, (12r + 5)^d/|V(B_v)|) \leq (60r + 25)^{d/2}.$$
Since each $r$-shallow minor of $Q^d_n$ has minimum degree $O(r^{d/2})$, we have $\nabla_b(r) = O(r^{d/2})$.

On the other hand, consider the graph $Q^d_{2r}$. For $x \in \{1, \ldots, 2r\}^{d/2}$, let $A_x$ be the subgraph of $Q^d_{2r}$ induced by vertices $(i_1, \ldots, i_d)$ such that $i_j = x_j$ for $j = 1, \ldots, d/2$ and $i_j \in \{1, \ldots, 2r\}$ for $j = d/2 + 1, \ldots, d$, and let $B_x$ be the subgraph induced by vertices $(i_1, \ldots, i_d)$ such that $i_1 \in \{2, 4, \ldots, 2r\}$, $i_j \in \{1, \ldots, 2r\}$ for $j = 2, \ldots, d/2$, and $i_j = x_{j-d/2}$ for $j = d/2 + 1, \ldots, d$. Each of these subgraphs has radius at most $r$, for all distinct $x, x' \in \{1, \ldots, 2r\}^{d/2}$ we have $V(A_x) \cap V(A_{x'}) = \emptyset$ and $V(B_x) \cap V(B_{x'}) = \emptyset$, and for all $x \in \{1, \ldots, 2r\}^{d/2}$ such that $x_1$ is odd and $y \in \{1, \ldots, 2r\}^{d/2}$, the graphs $A_x$ and $B_y$ are vertex-disjoint and $Q^d_{2r}$ contains an edge with one end $(x, y) \in V(A_x)$ and the other end in $(x + e_1, y) \in V(B_y)$. Consequently, $K_r^{(2r)^{d/2}/2,(2r)^{d/2}}$ is an $r$-shallow minor of $Q^d_{2r}$, implying $\nabla_b(r) = \Omega(r^{d/2})$.

Lemma 6 implies that $b_5 \leq \frac{1}{r^2}$ when $\frac{1}{r}$ is an even integer, and thus at these points the lower bound from Corollary 5 cannot be improved by more than 1. Actually, we can prove even better bound for all values of $\varepsilon > 0$. To this end, let us first establish bounds on the size of balanced separators in certain graph classes.

**Lemma 7.** Let $f, t : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing functions, and let $p : \mathbb{R}^+ \to \mathbb{R}^+$ be the inverse to the function $x \mapsto xt(x)$. Let $G$ be a graph such that every induced subgraph $H$ of $G$ satisfies $s(H) \leq f(|V(H)|)$. Let $G'$ be a graph obtained from $G$ by subdividing each edge at least $t(|V(G)|)$ times. Then $s(H') \leq 15f(p(2|V(H')|)) + 1$ for every induced subgraph $H'$ of $G'$.

**Proof.** Without loss of generality, we can assume $H$ is connected, as otherwise it suffices to consider the size of a balanced separator in the largest component of $H$. Let $B$ be the set of vertices of $G'$ created by subdividing the edges, and let $A = V(H') \setminus B$ and $a = |A|$. If $a \leq 1$, then $H'$ is a tree, and thus it has balanced separator of size at most 1. Hence, assume that $a \geq 2$. Since $H'$ is connected, we have $|V(H')| \geq (a-1)t(|V(G)|) \geq at(a)/2$, and thus $a \leq p(2|V(H')|)$. Note that $H'$ is obtained from a subgraph $H$ of $G$ with $a$ vertices by subdividing edges and repeatedly adding pendant vertices. By Theorem 3 we have $\text{tw}(H) \leq 15f(a)$, and thus $\text{tw}(H') \leq \text{tw}(H) \leq 15f(a) \leq 15f(p(2|V(H')|))$. As proved in Theorem 10, every graph of treewidth at most $c$ has a balanced separator of order at most $c + 1$. Consequently, $H'$ has a balanced separator of order at most $15f(p(2|V(H')|)) + 1$. 

For a graph $G$ with $m$ vertices and $0 < \varepsilon < 1$, let $G^\varepsilon$ denote the graph obtained from $G$ by subdividing each edge $[m^{\varepsilon/(1-\varepsilon)}]$ times. For a class of
graphs $\mathcal{G}$, let $\mathcal{G}^\varepsilon$ denote the class consisting of all induced subgraphs of the graphs $G^\varepsilon$ for $G \in \mathcal{G}$.

**Lemma 8.** For every class of graphs $\mathcal{G}$ and every $0 < \varepsilon < 1$, we have $s_{\mathcal{G}^\varepsilon}(n) = O(n^{1-\varepsilon})$. If $\mathcal{G}$ contains all 3-regular graphs, then $s_{\mathcal{G}^\varepsilon}(n) = \Omega(n^{1-\varepsilon})$.

**Proof.** Applying Lemma 7 with $f(n) = n$ and $t(m) = \lceil m^{\varepsilon/(1-\varepsilon)} \rceil$ (so that $p(n) = \Theta(n^{1-\varepsilon})$), we have $s_{\mathcal{G}^\varepsilon}(n) = O(n^{1-\varepsilon})$.

Conversely, let $G \in \mathcal{G}$ be a 3-regular $\frac{6}{20}$-expander with $m = \Theta(n^{1-\varepsilon})$ vertices (such a graph exists for every sufficiently large even number of vertices $n$). Note that $|V(G^\varepsilon)| = \Theta(m \cdot n^{\varepsilon/(1-\varepsilon)}) = \Theta(m^{1/(1-\varepsilon)}) = \Theta(n)$. We now argue that $s(G^\varepsilon) = \Omega(m)$, which implies $s_\varepsilon(n) = \Omega(m) = \Omega(n^{1-\varepsilon})$.

Let $M$ be the set of vertices of $G^\varepsilon$ of degree three. Suppose for a contradiction $X$ is a balanced separator in $G^\varepsilon$ of size $o(m)$. For sufficiently large $n$, this implies $V(G^\varepsilon)$ can be expressed as disjoint union of $X$, $C_1$, and $C_2$, where $C_1$ and $C_2$ are unions of components of $G^\varepsilon - X$ and $|C_1|, |C_2| \geq |V(G^\varepsilon)|/4 = \Omega(n)$. Each component of $G^\varepsilon - X$ disjoint from $M$ has two neighbors in $X$, implying the total number of vertices in such components is at most $\frac{3}{4}|X| \lceil m^{\varepsilon/(1-\varepsilon)} \rceil = o(n)$. Furthermore, a component of $G - X$ containing $k \geq 1$ vertices of $M$ has $O(km^{\varepsilon/(1-\varepsilon)})$ vertices. Consequently, $|C_1 \cap M|, |C_2 \cap M| = \Omega(n/m^{\varepsilon/(1-\varepsilon)}) = \Omega(m)$. By symmetry, we can assume $|C_1 \cap M| \leq m/2$, and since $G$ is a $\frac{3}{20}$-expander, we have $N_G(C_1 \cap M) = \Omega(m)$. However, this implies $|X| = \Omega(m)$, which is a contradiction.

Applying this lemma with $\mathcal{G}$ consisting of all 3-regular graphs, we obtain the following bound.

**Lemma 9.** For $0 < \varepsilon \leq 1$, $b_\varepsilon \leq \frac{1}{2\varepsilon} - \frac{1}{2}$.

**Proof.** We have $b_1 = 0$ by Corollary 5 and thus we can assume $\varepsilon < 1$. Let $\mathcal{G}_3$ be the class of all 3-regular graphs. By Lemma 8, we have $s_{\mathcal{G}_3^\varepsilon}(n) = \Theta(n^{1-\varepsilon})$.

Let $G$ be a 3-regular graph with $m$ vertices, and consider any $r$-shallow minor $F$ of $G^\varepsilon$. If $4r < \lceil m^{\varepsilon/(1-\varepsilon)} \rceil$, then $F$ is 2-degenerate, and thus it has density at most 2. Hence, we can assume $r = \Omega(m^{\varepsilon/(1-\varepsilon)})$. Let $M$ be the set of vertices of $G^\varepsilon$ of degree three, for each vertex $v \in V(F)$ let $B_v$ be the vertex set of the subgraph of $G^\varepsilon$ contracted to $v$, and let $v$ be the vertex of $F$ with $|B_v \cap M|$ minimum. Note that $\deg_F(v) \leq 2 + |B_v \cap M|$. Furthermore, since the sets $B_u$ for $u \in V(F)$ are pairwise disjoint, we have $(\deg_F(v) + 1)|B_v \cap M| \leq |B_v \cap M| + \sum_{u \in N(v)} |B_u \cap M| \leq |M| = m$. Consequently, $\deg_F(v) \leq 2 + \min(|B_v \cap M|, m/|B_v \cap M|) \leq 2 + \sqrt{m} = O(r^{1/2\varepsilon} - \frac{1}{2})$. Therefore, $\nabla_{\mathcal{G}_3^\varepsilon}(r) = O(r^{1/2\varepsilon} - \frac{1}{2})$. \qed
Furthermore, note that if $G$ is an expander, then $G$ contains as an $O(\log m)$-shallow minor a clique with $\Omega(\sqrt{m/\log m})$ vertices by Theorem 11 and thus if $m = \Theta((r^{1-\varepsilon}))$, we conclude $G^\varepsilon$ contains as an $O(r \log r)$-shallow minor a clique with $\Omega(r^{1-\frac{1}{2}}/\log r)$ vertices. Consequently, $\nabla G^\varepsilon(r) = \Omega(r^{1-\frac{1}{2}}/\log r)$; hence, the analysis of this example cannot be substantially improved.

This construction also gives a lower bound for $B_\varepsilon$ that matches the upper bound from Corollary 5.

**Lemma 10.** For $0 < \varepsilon \leq 1$, $B_\varepsilon \geq \frac{1}{2} - 1$.

*Proof.* Since $B_\varepsilon = 0$ by Corollary 5, we can assume $\varepsilon < 1$. Let $G_a$ be the class of all graphs. By Lemma 8 we have $s_{G_a}(n) = \Theta(n^{1-\varepsilon})$. For sufficiently large integer $r$, let $m = \lfloor r^{1-\varepsilon} \rfloor$. The graph $K_m^\varepsilon$ contains the clique $K_m$ as an $r$-shallow minor, implying $\nabla G_a^\varepsilon(r) = \Omega(r^{1-\frac{1}{2}})$. Finally, a similar idea enables us to obtain a better bound for $b_\varepsilon$ in the range $\frac{1}{2} \leq \varepsilon \leq 1$.

**Lemma 11.** For $\frac{1}{2} \leq \varepsilon \leq 1$, $b_\varepsilon = 0$.

*Proof.* Since $b_\varepsilon = 0$ by Corollary 5, we can assume $\varepsilon < 1$. For a graph $G$ with $m$ vertices, let $G'$ denote the graph obtained from $G$ by subdividing each edge $\lceil m^{\frac{1}{2\varepsilon-1}} \rceil$ times. Let $G$ consist of all induced subgraphs of the graph $G'$ for $G$ planar. All graphs in $G$ are planar, and thus $\nabla G(r) \leq 3 = O(r^0)$ holds for every $r \geq 0$. Standard isoperimetric inequalities applied with $G$ being a $(t \times t)$-grid for $t = \Theta(n^{1-\varepsilon})$ (so that $|V(G')| = \Theta(n)$) show that every balanced separator in $G'$ has size $\Omega(t) = \Omega(n^{1-\varepsilon})$, implying $s_{G'}(n) = \Omega(n^{1-\varepsilon})$. Conversely, Lemma 7 applied with $f(n) = O(\sqrt{n})$ and $t(m) = \lceil m^{\frac{2}{2\varepsilon-1}} \rceil$ (so that $p(n) = \Theta(n^{2-2\varepsilon})$) implies $s_{G'}(n) = O(n^{1-\varepsilon})$.

Let us summarize our findings: We have $b_\varepsilon = 0$ when $\frac{1}{2} \leq \varepsilon \leq 1$,

$$\frac{1}{2\varepsilon} - 1 \leq b_\varepsilon \leq \frac{1}{2\varepsilon} - \frac{1}{2}$$

when $0 < \varepsilon < \frac{1}{2}$, and

$$B_\varepsilon = \frac{1}{\varepsilon} - 1$$

when $0 < \varepsilon \leq 1$. In particular, if $\varepsilon < 1$, then $b_\varepsilon \neq B_\varepsilon$.

The bounds for $b_\varepsilon$ differ by at most $1/2$. It is unclear whether the upper or the lower bound can be improved. While fact that $b_{1/2} = 0$ matches the
lower bound suggests that a better construction improving the upper bound in general could exist, it is also plausible that this is just a “dimension 2” artifact and in fact the lower bound might be possible to improve for $\varepsilon < 1/2$ (possibly leading to discontinuity of $b_\varepsilon$ at $\varepsilon = 1/2$).

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