Thermodynamics in Closed Universe with Entropy Corrections

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Abstract

We discuss the generalized second law of thermodynamics in three different systems by taking quantum corrections (logarithmic and power law) to cosmological horizon entropy as well as black hole entropy. Firstly, we consider phantom energy accretion onto the Schwarzschild black hole in the closed FRW universe and investigate validity of the generalized second law of thermodynamics on the apparent and event horizons. In another scenario, we evaluate the critical mass of the Schwarzschild black hole with upper and lower bounds under accretion process due to phantom-like modified generalized chaplygin gas. It is found that the generalized second law of thermodynamics is respected within these bounds and black hole cannot accrete outside them. Finally, we explore this law for a closed universe filled with interacting $n$-components of fluid (in thermal equilibrium case) and with non-interacting dark matter and dark energy components (in thermal non-equilibrium case) on the apparent and event horizons and find conditions for its validity.

Keywords: Generalized second law of thermodynamics; Phantom energy.

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1 Introduction

Recent developments\(^1\) provide indication that our universe is expanding and has entered in the accelerating phase. In this respect, the unknown force called dark energy (DE) plays a crucial role. The value of the equation of state (EoS) parameter differentiates three different phases of DE according to the expansion rate of the universe such as quintessence \((-1 < \omega < -\frac{1}{3}\)), vacuum \((\omega = -1)\) and phantom \((\omega < -1)\). Nowadays, phantom cosmology has been taken into account with great deal of interest. According to it, the energy density of the phantom energy will turn out to be infinite in a finite time. This scenario corresponds to a type of future singularity named as big rip.

It is predicted that phantom energy with large negative pressure would unstable and disperse all the gravitationally bounded objects near big rip. This is also confirmed through numerical analysis on the milky way galaxy and solar system\(^5\). Babichev et al.\(^6\) modeled the scenario of accretion of phantom energy onto the Schwarzschild black hole (BH) and observed that BH mass would gradually decrease and disappear at the end near the big rip. Yurov et al.\(^7\) found a big hole phenomenon (where BH grows up rapidly and gains infinitely large size before big rip) for a specific choice of scale factor. Later, some people\(^5,9\) confirmed the results of Babichev et al.\(^6\) about the diminishing of mass of BH due to phantom energy accretion for different BHs, e.g., Reissner-Nördstrom\(^10\), Kerr-Newmann\(^9,11\), Schwarzschild-de Sitter\(^9\) etc. Jamil\(^12\) studied the accretion of phantom-like chaplygin gas models such as modified variable chaplygin gas and viscous generalized chaplygin gas onto the Schwarzschild BH. He showed that accretion process decreases the mass of BH in the violation of null energy condition. Bhadra and Debnath\(^13\) have extended this work for the Schwarzschild and Kerr-Newmann BHs by using new variable modified chaplygin gas and generalized cosmic chaplygin gas.

In addition, the loss of BH masses due to accreting phantom fluid causes decrease in the entropies since the area of BH is directly proportional to its entropy. In this scenario, the status of thermodynamical quantities like entropy as well as temperature and correspondingly the validity of the generalized second law of thermodynamics (GSLT) has become an interesting subject. Izquierdo and Pavón have discussed the GSLT for phantom dominated universe in the absence\(^14\) and presence of the Schwarzschild BH\(^15\). Further, Sadjadi found different conditions for the validity of GSLT for the phantom
dominated universe enclosed by the future event horizon in the absence\textsuperscript{16} and presence of the Schwarzschild BH\textsuperscript{17}. In the meantime, Pacheco and Horvath\textsuperscript{18} investigated GSLT of a system containing the phantom fluid and the Schwarzschild BH by using different technique.

On the other hand, GSLT of a system containing interacting DE with dark matter (DM) enclosed by either apparent or event horizons has been discussed widely in view of usual entropy of horizon. Karami et al.\textsuperscript{19} have investigated the validity of GSLT on the apparent and event horizons in non-flat FRW universe. Also, it was found\textsuperscript{20} that the GSLT always holds for a system having interaction of DE with DM and radiation in non-flat universe enclosed by apparent horizon. In the discussion of GSLT, it is realized that the logarithmic and power law corrections to the usual entropy-area relation of the expanding universe may provide better results. In this context, some authors\textsuperscript{21–23} have explored the validity of GSLT in closed modified Friedmann equations by using apparent horizon. A detailed review of DE phenomenon and thermodynamics in modified gravity from various cosmological aspects has been provided in the recent work\textsuperscript{24}.

Recently, Jamil et al.\textsuperscript{25} have studied the GSLT of phantom fluid accreting onto the Schwarzschild BH on the apparent and event horizons by using logarithmic and power law corrected entropies in flat FRW universe. Also, some people\textsuperscript{26,27} have explored GSLT in flat universe containing $n$-components of fluid comprising interactions with corrected entropies of horizon of the universe by using apparent and event horizons. Here, we extend the work\textsuperscript{25–27} in the closed FRW universe and modified generalized Chaplygin gas (MGCG).

The paper is organized as follows: In section 2, we discuss the GSLT for a system containing the Schwarzschild BH, phantom DE and also the corrected entropy on the apparent and event horizons. Section 3 is devoted to find GSLT constraints of phantom-like MGCG accretion onto the Schwarzschild BH with logarithmic and power law corrected entropies of horizon. In section 4, we investigate the GSLT for a system interacting $n$-components of fluid with logarithmic and power law corrected entropies on the apparent and event horizons. Section 5 explores the validity of GSLT for thermal non-equilibrium case. In the last section, we summarize our results.
2 Schwarzschild BH in Phantom Cosmology and GSLT

In this section, we discuss the GSLT of the system enclosed by apparent as well as event horizons in the closed universe. The non-flat FRW universe is given by

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]. \] (1)

Here \( a(t) \) is the cosmic scale factor which measures expansion of the universe and \( k = -1, 0, 1 \) represents the spatial curvature indicating the open, flat and closed universes, respectively, but we only assume the closed case. The corresponding equations of motion are

\[ H^2 + \frac{k}{a^2} = \frac{8\pi}{3} \rho, \] (2)

\[ \dot{H} + H^2 = -\frac{8\pi}{6} (\rho + 3p). \] (3)

We assume that the phantom DE accretes onto the Schwarzschild BH. This phantom DE is described by the perfect fluid whose equation of continuity is written as

\[ \dot{\rho}_{ph} + 3H(\rho_{ph} + p_{ph}) = 0. \] (4)

Here \( \rho_{ph} \) and \( p_{ph} \) indicate the energy density and pressure of the phantom energy that do not obey the null energy condition, i.e., \( \rho_{ph} + p_{ph} < 0 \). Babichev et al.\(^6\)) found that the phantom energy causes the reduction of the Schwarzschild BH mass. They calculated the following rate of change in the BH mass

\[ \dot{M} = \pi Z r_{bh}^2(\rho_{ph} + p_{ph}), \] (5)

where \( Z \) is a positive dimensionless constant and \( r_{bh} = 2M \) represents radius of the Schwarzschild BH.

Now we would like to check the validity of the GSLT of the system which contains BH and phantom energy enclosed by apparent and future event horizons. The GSLT is given in the form

\[ \dot{S}_{\text{tot}} \equiv \dot{S}_{\text{BH}} + \dot{S}_{\text{ph}} + \dot{S}_{h} \geq 0, \] (6)
where $S_{BH}$, $S_{ph}$ and $S_h$ show entropies due to BH horizon, phantom fluid and cosmological horizon, respectively. In order to discuss the GSLT, we need Gibb’s relation

$$TdS = d(pV) + pdV, \quad (7)$$

where $T$, $S$ denote temperature and entropy of the phantom fluid respectively, and $V = \frac{4}{3}\pi R_h^3$ ($R_h$ as the radius of the cosmological horizon) is the volume of the spherical system. Differentiating Eq.(7) with respect to time and using expression of volume, we get

$$\dot{S}_{ph} = 2\pi H^2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) R_h^3 (HR_h - \dot{R}_h), \quad (8)$$

where we have used temperature of the fluid as $T = \frac{1}{2\pi R_h}$ and $\Omega_k = \frac{k}{a^2 H^2}$.

There exist two types of horizon in the accelerated expansion of the universe for which we can associate entropy for collecting information behind them. The apparent horizon is the most natural choice given by

$$R_a = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \quad (9)$$

The event horizon is another choice of cosmological horizon defined as

$$R_e = a(t) \int_a^{\infty} \frac{da}{H a^2} = a(t) \int_t^{\infty} \frac{dt}{a(t)}. \quad (10)$$

It has more conceptual resemblance with the BH horizon. This horizon has attained much attention after description of the holographic DE with this horizon. For the sake of generality, we use these two horizons in our study. Differentiation of the above equation leads to

$$\dot{R}_e = H R_e - 1. \quad (11)$$

### 2.1 Quantum Corrections to Entropy

The entropy-area relation proposed by Bekenstein is

$$S_{bh} = \frac{k_B A}{4l_p^2}, \quad (12)$$
where $k_B$, $A$ and $l_{pl} = \sqrt{G \hbar c^3}$ represent Boltzmann constant, horizon area of BH and Planck’s length respectively. Here we take $c = G = \hbar = l_{pl} = k_B = 1$. We use $A = 4\pi r_{bh}^2$ as the area of BH horizon and $A = 4\pi R_h^2$ (where $R_h$ is either apparent horizon or event horizon) as the area of cosmological horizon (outer most boundary of cosmological system) and $R_h \gg r_{bh}$.

The logarithmic correction to Bekenstein’s relation is also obtained in the scenario of loop quantum gravity. This correction is found through infinite series expansion of entropy-area relation as

$$S_h = \frac{A}{4\hbar} + \tilde{\beta} \ln \left( \frac{A}{4\hbar} \right) - \tilde{\beta}_1 \left( \frac{4\hbar}{A} \right) - \tilde{\beta}_2 \left( \frac{16\hbar^2}{A^2} \right) - ...$$

where $\tilde{\beta}_l$ appear as finite constants, $S_0$ represents the usual entropy of BH, while the remaining terms involve as a quantum corrections. For cosmological analysis, the first order correction is worthy and its higher order corrections can be ignored due to smallness of $\hbar$. Thus, we assume first order logarithmic corrected entropy\(^{31-33}\)

$$S_h = \frac{A}{4} + \mu \ln \left( \frac{A}{4} \right) + \nu, \quad (13)$$

where $\mu$ and $\nu$ are constants. Cai et al.\(^{34}\) found that the logarithmic term involves in a model of entropic cosmology compensates the framework of the early inflation as well as present acceleration.

In the scenario of entanglement of entropy in the quantum fields, it is required to modify the entropy-area relation\(^{35}\). The entanglement entropy is stated as a measure of the information loss due to the spatial separation between the degrees of freedom inside and outside the horizon. It is observed that there exist several modes of gravitational fluctuations in the surroundings of BH which behaves like scalar fields and its entanglement entropy can be found by tracing degrees of freedom inside the horizon. Also, the authors\(^{35}\) showed that the Bekenstein’s relation of entropy works only in the ground state. In case of superposition of ground and excited states of a field, a correction in term of power law form is included as\(^{35}\)

$$S_h = \frac{A}{4} \left[ 1 - B\lambda A^{1-\frac{2}{3}} \right], \quad (14)$$
where \( B_\lambda = \frac{\lambda(4\pi)^{\frac{1}{2} - 1}}{(4 - \lambda)r_c^{2 - \lambda}} \), \( \lambda \) is a constant and \( r_c \) is the crossover scale. Firstly, we check the validity of GSLT for these two corrections on apparent and future event horizons.

**Logarithmic Entropy Correction**

The entropy of BH in logarithmic form becomes

\[
S_{BH} = 4\pi M^2 + \mu \ln(4\pi M^2) + \nu, \tag{15}
\]

here we use \( r_h = 2M \) in the horizon area of black hole. The rate of change of BH entropy with logarithmic correction is obtained through Eqs. (2), (4), (5) and (15)

\[
\dot{S}_{BH} = -8ZM H^2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ 4\pi M^2 + \mu \right], \tag{16}
\]

while for the horizon entropy of the universe on the apparent horizon, it follows that

\[
\dot{S}_A = \frac{2\pi}{H(1 + \Omega_k)^2} \left( -\frac{\dot{H}}{H^2} + \Omega_k \right) \left[ 1 + \frac{\mu H^2(1 + \Omega_k)}{\pi} \right]. \tag{17}
\]

It is noted that there are some possibilities for which \( \dot{S}_A \leq 0 \), i.e., \( \mu \leq 0 \) with \( \mu \leq -\frac{\pi}{H^2(1 + \Omega_k)} \) in quintessence case (\( \dot{H} < 0 \)). However, two sets of conditions exist for phantom case (\( \dot{H} > 0 \)) which are \( \Omega_k \geq \frac{\mu H^2}{\pi}, \mu < 0, \mu \geq -\frac{\pi}{H^2(1 + \Omega_k)} \) and \( \Omega_k \leq \frac{\mu H^2}{\pi}, \mu \geq 0 \). Using Eqs. (8), (9), (16) and (17), we have the total change in entropy on the apparent horizon as

\[
\dot{S}_{tot} = \frac{2\pi}{H(1 + \Omega_k)^3} \left( -\frac{\dot{H}}{H^2} + \Omega_k \right) \left[ \frac{\mu H^2(1 + \Omega_k)^2}{\pi} + \frac{4ZM H^3}{\pi} \right]
\times (1 + \Omega_k)^3(4\pi M^2 + \mu) + \left( -\frac{\dot{H}}{H^2} + \Omega_k \right). \tag{18}
\]

It has been found by Sadjadi\(^{16} \) that the derivative of the Hubble parameter is positive for phantom dominated universe and negative for quintessence
dominated universe. Since we have considered phantom dominated universe, so $\dot{H} > 0$. Thus the GSLT is respected for $\Omega_k > \frac{\dot{H}}{H^2}$ with $\mu \geq 0$ or $\mu \leq 0$ and

$$\frac{\dot{H}}{H^2} \leq \frac{\mu H^2 (1 + \Omega_k)^2}{\pi} + \frac{4 Z M_c H^3}{\pi} (1 + \Omega_k)^3 (4 \pi M_c^2 + \mu) + \Omega_k,$$

where $M_c$ is the critical mass of BH. Also, the GSLT follows for $\Omega_k < \frac{\dot{H}}{H^2}$ with $\mu \leq 0$, $M_c < \sqrt{-\frac{\mu}{4 \pi}}$ or $\mu \geq 0$ with

$$\frac{\dot{H}}{H^2} \geq \frac{\mu H^2 (1 + \Omega_k)^2}{\pi} + \frac{4 Z M_c H^3}{\pi} (1 + \Omega_k)^3 (4 \pi M_c^2 + \mu) + \Omega_k.$$

The rate of change of entropy of the phantom fluid and horizon entropy of the universe on the event horizon are

$$\dot{S}_{\text{ph}} = 2 \pi H^2 R_e^3 \left( \frac{\dot{H}}{H^2} - \Omega_k \right),$$

$$\dot{S}_e = 2 \pi (H R_e - 1) \left( R_e + \frac{\mu}{\pi R_e} \right).$$

It is observed that $\dot{S}_e \leq 0$ for phantom dominated case with $\mu \geq 0$ while for quintessence case with $\mu \leq -\pi R_e^2$.

Combining Eqs. (16), (19) and (20), the total entropy turns out to be

$$\dot{S}_{\text{tot}} = 2 \pi (H R_e - 1) \left( R_e + \frac{\mu}{\pi R_e} \right) - 2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) [4 Z M H^2 (4 \pi M^2 + \mu) - \pi H^2 R_e^3].$$

In the quintessence region, $\dot{H} < 0$ and $\dot{R}_e > 0$ ($H R_e > 1$), but this case is skipped because we consider phantom fluid as mentioned earlier. In the phantom region, $\dot{H} > 0$ and $\dot{R}_e < 0$, i.e., $H R_e < 1$, it is identified that the GSLT is satisfied under two possible choices for a closed universe as

1. $\mu \geq 0$ with

$$\dot{R}_e \geq \frac{\left( \frac{\dot{H}}{H^2} - \Omega_k \right) R_e}{\pi R_e^2 + \mu} [2 Z M_c H^2 (4 \pi M_c^2 + \mu) - \pi H^2 R_e^3].$$

2. $-\pi R_e^2 \leq \mu \leq 0$ with either $M_c < \sqrt{-\frac{\mu}{4 \pi}}$, $\Omega_k < \frac{\dot{H}}{H^2}$ or $\Omega_k > \frac{\dot{H}}{H^2}$, $\mu \geq \frac{\pi R_e^2}{2 Z M_c} - 4 \pi M_c^2$. 

8
Power Law Entropy Correction

In the similar way, we can evaluate the rates of change of BH entropy and horizon entropy of the universe (on the apparent horizon) with power law correction as

\[
\dot{S}_{BH} = -32\pi Z M^3 H^2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ 1 - \frac{\lambda}{2} \left( \frac{2M}{r_c} \right)^{2-\lambda} \right], \quad (22)
\]

\[
\dot{S}_A = \frac{2\pi}{H(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ 1 - \frac{\lambda}{2} \left( r_c H \sqrt{1 + \Omega_k} \right)^{\lambda-2} \right]. \quad (23)
\]

This rate of change of entropy turns out to be negative for \( \lambda \leq 0 \) in quintessence case (\( \dot{H} < 0 \)). However, two sets of condition exist for phantom case (\( \dot{H} > 0 \), i.e., \( \Omega_k \geq \frac{\dot{H}}{H^2} \), \( \lambda \leq 0 \) and \( \Omega_k \leq \frac{\dot{H}}{H^2} \), \( \mu \geq 0 \), \( \lambda \geq \frac{r_c H \sqrt{1 + \Omega_k}}{2} \). In view of Eqs. (8), (9), (22) and (23), the rate of change of total entropy on the apparent horizon leads to

\[
\dot{S}_{tot} = \frac{\pi}{H(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \lambda(r_c H \sqrt{1 + \Omega_k})^{\lambda-2} - 8M^3 Z H^3 \right]
\]

\[
\times \left( 1 - \frac{\lambda}{2} \left( \frac{2M}{r_c} \right)^{2-\lambda} \right) (1 + \Omega_k)^2 + \frac{2}{1 + \Omega_k} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \right]. \quad (24)
\]

This expression remains positive, i.e., \( \dot{S}_{tot} \geq 0 \) in the phantom regime as \( \Omega_k > \frac{\dot{H}}{H^2} \) with either \( \lambda \leq 0 \) or \( \lambda \geq 0 \) and

\[
\frac{\dot{H}}{H^2} < -\frac{\lambda}{2} \left( r_c H \right)^{\lambda-2} (1 + \Omega_k)^{\lambda} + 4M^3 Z H^3 (1 + \Omega_k) \left( 1 - \frac{\lambda}{2} \left( \frac{2M_c}{r_c} \right)^{2-\lambda} \right) + \Omega_k.
\]

Also, for \( \Omega_k < \frac{\dot{H}}{H^2} \) with either \( \lambda \geq 0 \), \( M_c \geq \frac{2}{\lambda} \left( \frac{4}{\lambda} \right)^{\frac{1}{\lambda-2}} \) or \( \lambda \leq 0 \) and

\[
\frac{\dot{H}}{H^2} > -\frac{\lambda}{2} \left( r_c H \right)^{\lambda-2} (1 + \Omega_k)^{\lambda} + 4M^3 Z H^3 (1 + \Omega_k) \left( 1 - \frac{\lambda}{2} \left( \frac{2M_c}{r_c} \right)^{2-\lambda} \right) + \Omega_k.
\]

For the event horizon, the rate of change of horizon entropy of the universe becomes

\[
\dot{S}_e = 2\pi R_e \dot{R}_e \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2-\lambda} \right]. \quad (25)
\]
This rate of change of horizon entropy would be negative for phantom case with $\lambda \leq 0$. With the help of Eqs. (19), (22) and (25), we obtain

$$
\dot{S}_{\text{tot}} = 2\pi R_e(H R_e - 1) \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2-\lambda} \right] - 2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \times \left[ 4\pi Z M^3 H^2 \left( 1 - \frac{\lambda}{2} \left( \frac{2M}{r_c} \right)^{2-\lambda} \right) - \pi R_e^3 H^2 \right].
$$

It turns out that the GSLT is respected in the phantom region when

1. $\lambda \geq 0$ with either $\Omega_k > \frac{\dot{H}}{H^2}$, $r_c \geq (\frac{2}{\lambda})^{-2+\lambda} R_e$, $r_c \geq (\frac{2}{\lambda})^{-2+\lambda} M_c$ or $\Omega_k < \frac{\dot{H}}{H^2}$, $2(\frac{R_c}{r_c})^{2-\lambda} \leq \lambda \leq 2(\frac{2M_c}{r_c})^{2-\lambda}$.

2. $\lambda \leq 0$ with

$$
\dot{R}_e \geq 2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ 4\pi Z M^3 H^2 \left( 1 - \frac{\lambda}{2} \left( \frac{2M}{r_c} \right)^{2-\lambda} \right) - \pi R_e^3 H^2 \right] \frac{1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2-\lambda}}{2\pi R_e \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2-\lambda} \right]}. \quad (26)
$$

3 **Accretion of Phantom-like Modified Generalized Chaplygin Gas and GSLT**

In this section, we extend the work of 25) with the most general EoS of MGCG\textsuperscript{36,37}) which is stated as

$$
p = b_1 \rho - \frac{b_2}{\rho^m}, \quad (27)
$$

where $b_1$, $b_2$ and $m$ are constant parameters. For $b_1 = 0$, it reduces to the generalized chaplygin gas, while we can recover the usual chaplygin gas by setting $b_1 = 0$ and $m = 1$. We assume that a phantom fluid is described by MGCG and a BH containing in a comoving volume $V \propto a^3$. The MGCG violates the null energy condition for $\rho^{1+m} < \frac{b_2}{1+b_1}$. However, the presence of BH creates distortions in the spacetime inside the cavity, which can be neglected up to first approximation. Here, we take the total entropy of the system in the form of logarithmic and power law corrections to BH entropy with the entropy of the phantom fluid.
Logarithmic Entropy Correction

Using the EoS of MGCG in continuity equation (4), we get new form of entropy as a sum of BH entropy and entropy of phantom energy

\[ S_n = [4\pi M^2 + \mu \log(4\pi M^2) + \nu] + \left[ \frac{1}{u} \left( \rho^{1+m} - \frac{b_2}{1+b_1} \right)^{(1+b_1)(1+m)} V \right] \], \quad (28)

where \( u \) is a constant of integration. Moreover, in a small interval of time \( \Delta t \), the BH mass changes by an amount \( \Delta M \) due to accretion process as well as change in energy density of the phantom fluid takes place. These variations lead to

\[ \Delta S_n = 8\pi M \left( 1 + \frac{\mu}{4\pi M^2} \right) \Delta M + \frac{\rho^m}{u(1+b_1)} \left[ -\frac{b_2}{u(1+b_1)} \right] \]

\[ + \ u^{-1}\rho^{1+m} \frac{1-(1+b_1)(1+m)}{1+b_1(1+m)} V \Delta \rho. \] \quad (29)

Notice that the homogeneous and time dependent scalar field with Lagrangian containing negative kinetic energy term can also interpret phantom energy. In terms of this scalar field model, the energy conservation is\(^{18}\)

\[ \Delta M = -\frac{1}{2} \Delta \dot{\phi}^2 V = -\frac{1}{2\rho^{1+m}} \left( (1+b_1)\rho^{1+m} + (1+m)b_2 \right) V \Delta \rho. \] \quad (30)

Inserting \( V \Delta \rho \) from this relation in Eq.(29), we get

\[ \Delta S_n = \left[ 8\pi M \left( 1 + \frac{\mu}{4\pi M^2} \right) - \frac{2\rho^{2m+1}}{u(1+b_1)((1+b_1)\rho^{1+m} + (1+m)b_2)} \right] \]

\[ \times \ \left( u^{-1}\rho^{1+m} - \frac{b_2}{u(1+b_1)} \right)^{(1+b_1)(1+m)} \Delta M. \] \quad (31)

On account of accretion of phantom energy, the mass of BH decreases as \( \Delta M < 0 \). For the validity of the GSLT, \( \Delta S_n > 0 \), we put

\[ 4\pi M_c \left( 1 + \frac{\mu}{4\pi M_c^2} \right) - \delta(\rho) \leq 0, \]

where

\[ \delta(\rho) = \frac{2u^{-1}(1+b_1)^{-1}\rho^{2m+1}}{((1+b_1)\rho^{1+m} + (1+m)b_2)} \left( u^{-1}\rho^{1+m} - \frac{b_2}{u(1+b_1)} \right)^{(1+b_1)(1+m)} \].
This leads to the following range of the critical mass of BH

\[ M_- \leq M_c \leq M_+ , \]

where

\[ M_\pm = \frac{1}{8\pi} \left[ \delta(\rho) \pm \sqrt{\delta^2(\rho) - 16\pi\mu} \right] , \]

indicating that the critical mass is bounded above and below by \( M_+ \) and \( M_- \) respectively. This also gives information about accretion process onto a BH that it cannot accrete above and below the critical values. It is observed that the range of the critical mass is physically acceptable for \( \mu \leq 0 \) or \( \mu \geq 0 \) with \( \delta^2(\rho) > 16\pi\mu \). If \( \mu \ll \frac{1}{16\pi}\delta^2(\rho) \), then the critical mass condition turns out to be \( M_c \leq \frac{\delta(\rho)}{4\pi} \).

**Power Law Entropy Correction**

In this case, the variation in the total entropy becomes

\[ \Delta S_n = \left[ 4\pi M \left( 1 - \frac{\lambda(2M)^{2-\lambda}}{2r_c^{2-\lambda}} \right) - \delta(\rho) \right] \Delta M . \tag{32} \]

The validity of GSLT leads to the constraint

\[ M_c \left( 1 - \frac{\mu(2M)^{2-\lambda}}{2r_c^{2-\lambda}} \right) - \frac{\delta(\rho)}{4\pi} < 0 . \tag{33} \]

It is a complicated equation in \( M_c \), so we handle it iteratively for different values of \( \lambda = 1, 2, 3, 4, 5 \).

- By inserting \( \lambda = 1 \) in Eq. (33), we obtain

\[ M_c^2 - r_c M_c + \frac{r_c \delta(\rho)}{4\pi} \geq 0 . \]

It gives the range of the critical mass of BH, i.e., \( M_- \leq M_c \leq M_+ \), where

\[ M_\pm = \frac{r_c}{2} \left[ 1 \pm \sqrt{1 - \frac{r_c \delta(\rho)}{\pi}} \right] . \]

Here, the critical values are physically acceptable if \( r_c < \frac{\pi}{\delta(\rho)} \).
• For $\lambda = 2$, we have $\frac{\delta(\rho)}{4\pi} \geq 0$, which is ruled out as we cannot find any critical value of the BH mass.

• For $\lambda = 3$, we obtain upper bound of the critical mass of BH

$$M_c < \frac{3r_c}{4} + \frac{\delta(\rho)}{4\pi}.$$  

• For $\lambda = 4$, it follows that

$$M_c^2 - \frac{\delta(\rho)}{4\pi}M_c - \frac{r_c^2}{2} < 0.$$  

It gives the following range of the critical mass of BH

$$M_- \leq M_c \leq M_+$$

where $M_\pm = \frac{1}{8\pi} \left[ \delta(\rho) \pm \sqrt{\delta^2(\rho) - 32\pi^2 r_c^2} \right]$. The roots are real if $r_c^2 < \frac{\delta(\rho)}{32\pi^2}$.

• For $\lambda = 5$, Eq.(33) turns out to be $M_c^3 - pM_c^2 - q < 0$ with

$$p = \frac{\delta(\rho)}{4\pi}, \quad q = \frac{5r_c^3}{16}.$$  

To solve this cubic equation, we introduce the variable $x = M_c - \frac{p}{3}$, which leads to

$$x^3 + lx + n < 0, \quad l = -\frac{p^2}{3}, \quad n = -(q + \frac{2}{27}p^3).$$  

This cubic equation has the following three distinct roots because $l < 0$

$$x_1 = 2\sqrt{-\frac{l}{3}} \cos(\varphi), \quad x_2 = 2\sqrt{-\frac{l}{3}} \cos\left(\frac{2\pi}{3} - \frac{\varphi}{3}\right),$$  

$$x_3 = 2\sqrt{-\frac{l}{3}} \cos\left(\frac{2\pi}{3} + \frac{\varphi}{3}\right),$$

where

$$\varphi = \frac{1}{3} \arccos\left(\frac{27q + 2p^3}{2p^3}\right).$$
Hence there exist two possibilities for the range of critical mass

\[ M_c \leq M_1, \quad M_2 < M_c < M_3, \]

where

\[ M_1 = \frac{\delta(\rho)}{6\pi} \left( \frac{1}{2} + \cos(\varphi) \right), \quad M_2 = \frac{\delta(\rho)}{6\pi} \left( \frac{1}{2} + \cos\left(\frac{2\pi}{3} + \frac{\varphi}{3}\right) \right), \]
\[ M_3 = \frac{\delta(\rho)}{6\pi} \left( \frac{1}{2} + \cos\left(\frac{2\pi}{3} - \frac{\varphi}{3}\right) \right). \]

4 Interacting Scenario and GSLT with Thermal Equilibrium

Here we consider perfect fluid of \( n \)-components such as DE, DM, radiation and so on that exist in the closed universe. Also, \( \rho \) and \( p \) represent the total energy density and pressure of the combined fluid, i.e., \( \rho_t = \sum_{i=1}^{n} \rho_i \) and \( p_t = \sum_{i=1}^{n} p_i \). We can rewrite Eq.(2) in the form of fractional energy densities as

\[ \Omega_t = 1 + \Omega_k, \]

where \( \Omega_t = \frac{\rho_t}{3H^2} \) and the equation of continuity in this scenario becomes

\[ \dot{\rho}_t + 3H(\rho_t + p_t) = 0. \tag{34} \]

We assume the interacting scenario of all fluid components\(^{38}\) but we are not introducing the specific form of interaction due to unknown nature of DE and DM. The interaction between the fluid components changes the nature of the corresponding equations of state. Moreover, the interaction makes possible transition from a quintessence state to phantom era and also helps in explaining the cosmic coincidence problem\(^{39-41}\). It has been obtained\(^ {42,43}\) that the interacting DE models favor 95% confidence limits with the data of the observational constraints. Thus Eq.(4) takes the following form for interacting fluid components

\[ \dot{\rho}_i + 3H(\rho_i + p_i) = \Upsilon_i, \quad i = 1, 2, 3, ..., n, \tag{35} \]

where \( \Upsilon_i \) indicate the interacting parameters which may be taken in terms of Hubble parameter or energy density of DM or DE or in terms of both\(^ {44,45}\).
This term plays the role of energy exchange between the components of perfect fluid and also it obeys \( \sum_{i=1}^{n} \gamma_i = 0 \).

In this scenario, Gibb’s relation takes the form

\[
T_i dS_i = d(p_i V) + p_i dV,
\]

for each component of fluid. Also, \( T_i \) and \( S_i \) denote temperature and entropy of the \( i \)th component of fluid respectively. Differentiating the above equation with respect to time and using the expression of volume, we obtain

\[
\dot{S}_i = \frac{4}{3} \pi R_h^3 \frac{\gamma_i}{T_i} + 4\pi R_h^2 (\dot{R}_h - H R_h) \frac{\rho_i + p_i}{T_i}.
\]

Further, the overall variation of entropy inside the horizon takes the form

\[
\dot{S}_I = \sum_{i=1}^{n} \dot{S}_i = \frac{4}{3} \pi R_h^3 \sum_{i=1}^{n} \frac{\gamma_i}{T_i} + 4\pi R_h^2 (\dot{R}_h - H R_h) \sum_{i=1}^{n} \frac{\rho_i + p_i}{T_i}.
\]

Here, we assume that the temperature of \( n \)-components of fluid is the same as the temperature of the horizon, i.e., \( \forall i, T_i = T \) (it is called thermal equilibrium condition). In general, it is not true because, in the present scenario of the universe, radiation temperature is larger than the non-relativistic cold DM.

### 4.1 Logarithmic Correction to Entropy

Here we discuss GSLT of the system by taking contribution of logarithmic correction to entropy into account on the above mentioned two cosmological horizons.

**On the Apparent Horizon**

In view of thermal equilibrium, \( \sum_{i=1}^{n} \gamma_i = 0 \) and Hawking temperature, one can get the change in entropy due to overall fluid inside the apparent horizon as

\[
\dot{S}_I = \frac{2\pi}{H(1 + \Omega_k)^3} \left( \frac{\dot{H}}{H^2} - \Omega_k \right)^2 + \frac{2\pi}{H(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right).
\]

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Consequently, by using Eqs. (17) and (39), the rate of change of the total entropy becomes

\[ \dot{S}_{\text{tot}} = \frac{2\pi}{H(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ -\frac{\mu H^2 (1 + \Omega_k)}{\pi} + \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \right]. \] (40)

It is observed that the GSLT always holds for \( \mu = 0 \). We also discuss two interesting possibilities \( \dot{H} < 0 \) (in quintessence region) and \( \dot{H} > 0 \) (in phantom region) for the validity of the GSLT. The condition \( \dot{H} < 0 \) implies that \( \frac{\dot{H}}{H^2} < \Omega_k \) and in this case, the GSLT favors for either \( \mu \geq 0 \) or \( \mu \leq 0 \) with \( \dot{R}_e > 0 \). The second condition provides two possible choices, i.e., \( \frac{\dot{H}}{H^2} < \Omega_k \) and \( \frac{\dot{H}}{H^2} > \Omega_k \). If \( \frac{\dot{H}}{H^2} < \Omega_k \), then the GSLT follows for either \( \mu \geq 0 \) or \( \mu \leq 0 \) with \( \dot{R}_e > 0 \).

On the Event Horizon

The entropy of the fluid enclosed by the event horizon is

\[ \dot{S}_I = 2\pi H^2 R_e^3 \left( \frac{\dot{H}}{H^2} - \Omega_k \right). \] (41)

The total rate of change of entropy can be obtained by combining Eqs. (20) and (41)

\[ \dot{S}_{\text{tot}} = 2\pi (HR_e - 1) \left( R_e + \frac{\mu}{\pi R_e} \right) + 2\pi H^2 R_e^3 \left( \frac{\dot{H}}{H^2} - \Omega_k \right). \] (42)

It is observed that the GSLT for logarithmic correction to entropy with future event horizon in quintessence regime (i.e., \( \dot{H} < 0 \) provided \( HR_e > 1 \)) is respected for \( \mu \geq 0 \) with \( \dot{R}_e \geq -\frac{H^2 R_e^3 (\frac{\mu}{\pi R_e} - \Omega_k)}{(R_e + \frac{\mu}{\pi R_e})} \) and for \( \mu \leq 0 \) with \( \pi R_e^2 \geq \dot{R}_e \geq -\frac{H^2 R_e^3 (\frac{\mu}{\pi R_e} - \Omega_k)}{(R_e + \frac{\mu}{\pi R_e})} \). In phantom regime (i.e., \( \dot{H} > 0 \) provided \( HR_e < 1 \)), it yields the following conditions

1. For \( \mu \geq 0 \) and \( \frac{\dot{H}}{H^2} \geq \Omega_k \) with \( \dot{R}_e \geq -\frac{H^2 R_e^3 (\frac{\mu}{\pi R_e} - \Omega_k)}{(R_e + \frac{\mu}{\pi R_e})} \), GSLT is satisfied and violated for \( \mu > 0 \) with \( \frac{\dot{H}}{H^2} \leq \Omega_k \). 


2. For $\mu \leq 0$ with $\pi R_e^2 \geq -\mu$, $\frac{\dot{H}}{H^2} \geq \Omega_k$, it holds.

Moreover, in order to get clear picture of result (42), we consider a model which corresponds to a super accelerated universe with big rip at a finite time. At this stage, $H$ gets very large value and this regime is named as phantom dominated universe. This model is also named as pole-like type scale factor which is given by

$$a(t) = a_0(t_s - t)^{-r}, \quad a_0 > 0, \quad r > 0, \quad t_s > t.$$  \hspace{1cm} (43)

By inserting this scale factor in relation (42), we have

$$\dot{S}_{tot} = \frac{2\pi (t_s - t)}{(r + 1)^2} \left[ -1 - \frac{\mu (r + 1)^2}{\pi (t_s - t)^2} + \frac{r}{r + 1} (1 - r \Omega_k) \right].$$ \hspace{1cm} (44)

The present time $t = t_0$ with $\Omega_{k_0} = 0.01$ provides the following condition for the validity of GSLT

$$t_s - t_0 \geq \sqrt{\frac{\pi}{\mu (r + 1)^2} \left[ 1 - \frac{r}{r + 1} \left( 1 - \frac{r}{100} \right) \right]}.$$ \hspace{1cm} (45)

The above result holds either for $\mu > 0$ or $\mu < 0$ with $r \geq 100$.

### 4.2 Power Law Correction to Entropy

By applying the previous procedure, we discuss the GSLT by using the horizon entropy with power law correction on the apparent and event horizons.

**On the Apparent Horizon**

The total rate of change of entropy on the apparent horizon becomes

$$\dot{S}_{tot} = \frac{\pi}{H(1 + \Omega_k)^2} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \lambda (r_c H \sqrt{1 + \Omega_k})^{\lambda - 2} + \frac{2}{1 + \Omega_k} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \right].$$ \hspace{1cm} (46)

Notice that the GSLT always holds for $\lambda = 0$. This shows that if $\dot{H} < 0$, then $\frac{\dot{H}}{H^2} < \Omega_k$ and the GSLT holds for either $\lambda \leq 0$ or $\lambda \geq 0$ with

$$\frac{\dot{H}}{H^2} < -\frac{\lambda}{2} (r_c H)^{\lambda - 2} (1 + \Omega_k)^{\lambda} + \Omega_k.$$
If $\dot{H} > 0$, then there are two possible constraints either $\frac{\dot{H}}{H^2} < \Omega_k$ or $\frac{\dot{H}}{H^2} > \Omega_k$. Thus the GSLT follows for $\frac{\dot{H}}{H^2} < \Omega_k$ either $\lambda \leq 0$ or $\lambda \geq 0$ with

$$\frac{\dot{H}}{H^2} < -\frac{\lambda}{2}(Hr_c)^{\lambda-2}(1 + \Omega_k)^{\lambda} + \Omega_k.$$  

For the second choice, we have constraints either $\lambda \geq 0$ or $\lambda \leq 0$ with $\frac{\dot{H}}{H^2} > -\frac{\lambda}{2}(Hr_c)^{\lambda-2}(1 + \Omega_k) + \Omega_k$. Otherwise, it is violated.

**On the Event Horizon**

The overall change of entropy takes the form

$$\dot{S}_\text{tot} = 2\pi R_e(HR_e - 1) \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2 - \lambda} \right] + 2\pi H^2 R_e^3 \left( \frac{\dot{H}}{H^2} - \Omega_k \right). \quad (47)$$

Here, $\dot{S}_\text{tot} \geq 0$ in the quintessence region as either $\lambda \geq 0$ with $r_c \geq (\frac{2}{\lambda})^{-2+\lambda} R_e$ and

$$\dot{R}_e \geq -\frac{2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \pi R_e^3 H^2}{2\pi R_e \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2 - \lambda} \right]}$$

or $\lambda \leq 0$ with the above condition. However, in the phantom region, it has the following validity conditions

1. $\lambda \geq 0$ with either $\Omega_k > \frac{\dot{H}}{H^2}$, $\lambda \geq 2 \left( \frac{R_e}{r_c} \right)^{2 - \lambda}$ or $\Omega_k < \frac{\dot{H}}{H^2}$, $2 \left( \frac{R_e}{r_c} \right)^{2 - \lambda} \leq \lambda \leq 2 \left( \frac{2M_c}{r_c} \right)^{2 - \lambda}$.

2. $\lambda \leq 0$ with $\dot{R}_e \geq -\frac{2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \pi R_e^3 H^2}{2\pi R_e \left[ 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2 - \lambda} \right]}$.

Substituting the scale factor (43), we obtain

$$\dot{S}_\text{tot} = \frac{2\pi(t_s - t)}{(r + 1)^2} \left[ \frac{\lambda(t_s - t)^{2 - \lambda}}{2(r_c(r + 1))^{2 - \lambda}} - \frac{1}{r + 1} - \frac{1}{1 + r^2 \Omega_k} \right]. \quad (48)$$

The GSLT is valid in the present time as

$$t_s - t_0 \geq r_c(r + 1) \left[ \frac{2}{\lambda(r + 1)} \left( 1 + \frac{r^2}{100} \right) \right]^{\frac{1}{\lambda - 1}}, \quad (49)$$

which holds for $\lambda > 0$. 

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5 GSLT with Thermal Non-Equilibrium

In order to explore the GSLT with non-equilibrium condition, we restrict the system into two non-interacting components such as DM and DE for getting more insights. In this scenario, two non-conserving equations turn out to be

\[ \dot{\rho}_{DM} + 3H(1 + \omega_{DM})\rho_{DM} = 0, \quad \dot{\rho}_{DE} + 3H(1 + \omega_{DE})\rho_{DE} = 0. \quad (50) \]

Here \( \rho_{DM}, \rho_{DE} \) are energy densities and \( \omega_{DM}, \omega_{DE} \) are EoS parameters of DM and DE respectively. Using Eqs.(1) and (50), we get the entropy of fluid in this scenario as

\[
\dot{S}_I = R_h^2(R_h - HR_h) \left[ 4\pi(1 + \omega_{DM})\rho_0 a^{-3(1+\omega_{DM})} \left( \frac{1}{T_{DM}} - 2\pi R_h \right) \right.
\]
\[ - 2\pi H^2 R_h \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left] \right. \] \[ , \quad (51) \]

where \( T_{DM} \) represents the temperature of DM while we use temperature \( T_{DE} = 2\pi R_h \) for DE component of fluid and \( \rho_0 \) is positive integration constant.

5.1 Logarithmic Correction to Entropy

Here we check the validity of GSLT in case of thermal non-equilibrium on the apparent and event horizons. The rate of change of total entropy on the apparent horizon can be obtained by using Eqs.(17) and (51) as

\[
\dot{S}_{tot} = -\frac{4\pi(1 + \omega_{DM})\rho_0}{H^2(1 + \Omega_k)^{\frac{3}{2}}} \left( 1 + \frac{\dot{H}}{H^2} - \Omega_k \right) \left( \frac{1}{T_{DM}} - \frac{2\pi}{H\sqrt{1 + \Omega_k}} \right) a^{-3(1+\omega_{DM})} \]
\[ + \frac{2\pi}{H(1 + \Omega_k)^{3}} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \sqrt{1 + \Omega_k} - \frac{\mu}{\pi^2} (1 + \Omega_k)^2 \right]. \quad (52) \]

This shows that GSLT is valid only if \( T_{DM} \geq \frac{H\sqrt{1 + \Omega_k}}{2\pi} \) along with \( \frac{H}{\pi^2} \geq \Omega_k \) and \( \mu \leq 0 \). The total rate of change of entropy on the event horizon turns
out to be
\[
\dot{S}_{\text{tot}} = -4\pi R_e^2 \left( \frac{1}{T_{DM}} - 2\pi R_e \right) (1 + \omega_{DM}) \rho_0 a^{-(1+\omega_{DM})} \\
+ 2\pi R_e \left[ H^2 R_e^2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}_e \left( 1 + \frac{\mu}{\pi R_e} \right) \right]. 
\]
(53)

Here, GSLT is respected for \( T_{DM} \geq \frac{1}{2\pi R_e} \) with \( \frac{\dot{H}}{H} \geq \Omega_k \), \( \dot{R}_e \geq 0 \) and \( \mu \geq 0 \).

### 5.2 Power Law Correction to Entropy

Now we find the validity conditions for power law correction on the apparent and event horizons. Equations (23) and (51) yield the following expression on the apparent horizon
\[
\dot{S}_{\text{tot}} = -4\pi(1 + \omega_{DM})\rho_0 \left( 1 + \frac{\dot{H}}{H^2} - \Omega_k \right) \left( \frac{1}{T_{DM}} - \frac{2\pi}{H \sqrt{1 + \Omega_k}} \right) \\
\times a^{-3(1+\omega_{DM})} + \frac{2\pi}{H(1 + \Omega_k)^3} \left( \frac{\dot{H}}{H^2} - \Omega_k \right) \left[ \frac{\lambda}{2} \left( r_c H \sqrt{1 + \Omega_k} \right)^{\lambda-2} \right] \\
\times (1 + \Omega_k) + \left( \frac{\dot{H}}{H^2} - \Omega_k \right). 
\]
(54)

The GSLT is respected for the same conditions as mentioned after Eq.(52) except \( \lambda > 0 \). In this correction, the total rate of change of entropy takes the following form
\[
\dot{S}_{\text{tot}} = -4\pi R_e^2 \left( \frac{1}{T_m} - 2\pi R_e \right) (1 + \omega_{DM}) \rho_0 a^{-(1+\omega_{DM})} \\
+ 2\pi R_e \left[ H^2 R_e^2 \left( \frac{\dot{H}}{H^2} - \Omega_k \right) + \dot{R}_e \left( 1 - \frac{\lambda}{2} \left( \frac{R_e}{r_c} \right)^{2-\lambda} \right) \right]. 
\]
(55)

which is positive and increasing for the same conditions as given after Eq.(53) except \( \lambda < 0 \).
6 Concluding Remarks

The evidence of spatial curvature in the universe comes from different observational data that predict the contribution of spatial curvature to the total energy density of the universe\(^ {46-52}\). Also, the early inflation era follows the flat universe but it is only possible for large number of e-folding\(^ {53}\). The evidences about the non-flat universe was also occurred through the data of the first year WMAP analysis\(^ {54}\). Moreover, the parameterizations of DE models allow the spatial curvature in the universe by using observations of type Ia supernovae (SNe Ia), baryon acoustic oscillation (BAO) and CMBR\(^ {55}\). By using WMAP five year data, it is found that \(\Omega_k\) remains in the range \((-0.2851, 0.0099)\) at 95% confidence level\(^ {56}\). This range is improved to \((-0.0181, 0.0071)\) by incorporating the data of BAO and SNe Ia. Recently, a little improvement appears in the range of \(\Omega_k\) by using latest WMAP 7 and 9-years data\(^ {57,58}\).

The GSLT is a basic ingredient of physics and its validity needs to be checked for thermal equilibrium and non-equilibrium frameworks in different cosmological systems (which includes either BH or n-components of fluid). The inclusion of BH entropy in the cosmological system is an interesting development. The earlier works\(^ {25-27}\) on this subject have been done for improving the shortcomings in the usual horizon entropy so that the GSLT is checked in general relativity through entropy corrections. These have been investigated with the help of phantom and quintessence regions through \(\dot{H}\) as suggested by Sadjadi\(^ {16}\) and set some physical conditions for the validity of GSLT. For the sake of consistency of the results, they have used apparent and event horizons. However, all the works have been done in flat FRW universe in this area.

The above discussion motivates us to study the GSLT in the closed FRW universe and we have discussed GSLT in three folds by assuming quantum corrected entropies to the horizon of the universe (logarithmic and power law versions). These corrections are specified by constant parameters \(\mu\) and \(\nu\) in logarithmic case and \(\lambda\) in power law case that play crucial role in the debate of results. We have investigated GSLT on apparent and event horizons for making our study more general.

- Firstly, we have checked the validity of the GSLT for a system containing the accretion of phantom fluid onto the Schwarzschild BH on the apparent and event horizons in closed universe. We have found
different possibilities for the validity of the GSLT for the logarithmic and power law forms of entropy in this scenario (see after Eqs. (18), (21) and Eqs. (24), (26) respectively). Since the entropy-area relation is same for any causal horizon, i.e., either it is BH or cosmological, hence for the sake of consistency, we have chosen quantum corrected entropies of these horizons.

- Secondly, we have developed the GSLT for a system containing only BH and phantom fluid described by MGCG (the most general EoS). We have found the critical values of the BH mass above which the phantom fluid cannot accrete onto a BH because of the violation of the GSLT. This is an interesting phenomenon which relates the entropy of fluid and mass of the BH. In accretion process, the entropy of the BH decreases but in order to keep the positivity of the total entropy, the entropy of fluid must increase enough to maintain it.

- Finally, we have investigated the GSLT for a system of $n$-components of fluid and quantum corrected entropies in the closed universe on the apparent and event horizons. We have made our analysis in thermal equilibrium and non-equilibrium (for non-interacting DM and DE components of fluid) and found the conditions under which the GSLT is valid. We have discussed the results by focusing on $\dot{H}$ (with $\lesssim 0$) and also pole-like scale factor in case of event horizon in equilibrium case.

The important description of this work is the inclusion of fractional curvature density which has led our results to more general as compared to past works. This density along with entropy corrections plays an important role in securing the validity of GSLT. It is also mentioned here that we have developed a thermal system by introducing MGCG which provides more insights about GSLT as compared to simple EoS. In conclusion, we have put some reliable conditions on different cosmological parameters in quintessence and phantom phases of the universe (for two systems including combination of BH and phantom fluid and $n$-components of fluid in thermal and non-thermal equilibrium) for the validity of GSLT. For a system having only BH and MGCG, we have mentioned lower and upper bounds of BH mass for which GSLT is respected.

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1) S. Perlmutter, G. Aldering, G. Goldhaber, R. A. Knop, P. Nugent, P. G. Castro, S. deustua, S. Fabbro, A. Goobar, D. E. Groom, I. M. Hook, A. G. Kim, M. Y. Kim, J. C. Lee, N. J. Nunes, R. Pain, C. R. Pennypacker and R. Quimby: Astrophys. J. **517** (1999) 565.
2) R. R. Caldwell and M. Doran: Phys. Rev. D **69** (2004) 103517.
3) T. Koivisto and D. F. Mota: Phys. Rev. D **73** (2006) 083502.
4) S. F. Daniel: Phys. Rev. D **77** (2008) 103513.
5) S. Nesseris and L. Perivolaropoulos: Phys. Rev. D **70** (2004) 123529.
6) E. Babichev, V. Dokuchaev and Y. Eroshenko: Phys. Rev. Lett. **93** (2004) 021102.
7) A. V. Yurov, P. M. Moruno and P. F. G. Diaz: Nuc. Phys. B **759** (2006) 320.
8) P. Martin-Moruno: Phys. Lett. B **659** (2008) 40.
9) P. Martin-Moruno, A. L. Marrakchi, S. Robles-Perez and P. F. G. Diaz: Gen. Relativ. Gravit. **41** (2009) 2797.
10) M. Jamil, M. A. Rashid and A. Qadir: Eur. Phys. J. C **58** (2008) 325.
11) E. Babichev, S. Chernov, V. Dokuchaev and Y. Eroshenko: Phys. Rev. D **78** (2008) 104027.
12) M. Jamil: Eur. Phys. J. C **62** (2009) 325.
13) J. Bhadra and U. Debnath: Eur. Phys. J. C **72** (2012) 1912.
14) G. Izquierdo and D. Pavón: Phys. Lett. B **633** (2006) 420.
15) G. Izquierdo and D. Pavón: Phys. Lett. B **639** (2006) 1.
16) H. M. Sadjadi: Phys. Rev. D **73** (2006) 063525.
17) H. M. Sadjadi: Phys. Lett. B **645** (2007) 108.
18) D. F. J. A. Pacheco and J. E. Horvath: Class. Quantum Grav. **24** (2007) 5427.
19) K. Karami, S. Ghaffari and M. M. Soltanzadeh: Class. Quantum Grav. **27** (2010) 205021.
20) M. Jamil, E. N. Saridakis and M. R. Setare: Phys. Rev. D **81** (2010) 023007.
21) N. Radicella and D. Pavón: Phys. Lett. B **704** (2011) 260.
22) K. Karami, A. Abdolmaleki, N. Sahraei and S. Ghaffari: JHEP **150** (2011) 1108.
23) K. Karami, A. Sheykhi, N. Sahraei and S. Ghaffari: Eur. Phys. Lett.
93 (2011) 29002.
24) K. Bamba, S. Capozziello, S. Nojiri, S. D. Odintsov: Astrophys. Space Sci. 337 (2012) 789.
25) M. Jamil, D. Momeni, K. Bamba and R. Myrzakulov: Int. J. Mod. Phys. D 21 (2012) 1250065.
26) H. M. Sadjadi and M. Jamil: Eur. Phys. Lett. 92 (2010) 69001.
27) U. Debnath, S. Chattopadhyay, I. Hussain, M. Jamil and R. Myrzakulov: Eur. Phys. J. C 72 (2012) 1875.
28) D. Bak and S. J. Rey: Class. Quantum Grav. 17 (2000) L83.
29) M. Li: Phys. Lett. B 603 (2004) 1.
30) J. D. Bekenstein: Phys. Rev. D 9 (1974) 3292.
31) R. Banerjee and S. K. Modak: JHEP 073 (2009) 0911.
32) H. Wei: Commun. Theor. Phys. 52 (2009) 743.
33) S. Banerjee, R. K. Gupta and A. Sen: JHEP 147 (2011) 1103.
34) Y. F. Cai, J. Liu and H. Li: Phys. Lett. B 690 (2010) 213.
35) S. Das, S. Shankaranarayanan and S. Sur: Phys. Rev. D 77 (2008) 064013.
36) H. B. Benoum: arXiv: 0205140.
37) Debnath, U., A. Banerjee and S. Chakraborty: Class. Quantum Grav. 21 (2004) 5609.
38) W. Zimdahl: Int. J. Mod. Phys. D 14 (2005) 2319.
39) M. R. Setare: Eur. Phys. J. C 50 (2007) 991.
40) K. Karami, S. Ghaffari and J. Fehri: Eur. Phys. J. C 64 (2009) 85.
41) H. M. Sadjadi and M. Jamil: Gen. Relativ. Gravit. 43 (2011) 1759.
42) J. Valiviita, R. Maartens and E. Majerotto: Mon. Not. R. Astron. Soc. 402 (2010) 2355.
43) H. Wei: Phys. Lett. B 691 (2010) 173.
44) H. M. Sadjadi and M. Honardoost: Phys. Lett. B 647 (2007) 231.
45) H. M. Sadjadi and M. Alimohammadi: Phys. Rev. D 74 (2006) 103007.
46) J. L. Sievers, J. R. Bond, J. K. Cartwright, C. R. Contaldi, B. S. Mason, S. T. Myers, S. Padin, T. J. Pearson, U.L. Pen, D. Pogosyan, S. Prunet, A. C. S. Readhead, M. C. Shepherd, P. S. Udomprasert, L. Bronfinman, W. L. Holzapfel, and J. May: Astrophys. J. 591 (2003) 599.
47) G. Efstathiou: Mon. Not. Roy. Astron. Soc. 343 (2003) L95.
48) J. P. Lumine: Nature 425 (2003) 593.
49) M. Tegmark, M. A. Strauss, M. R. Blanton, K. Abazajian, S. Dodelson, H. Sandvik, X. Wang, D. H. Weinberg, I. Zehavi, N. A. Bahcall, F. Hoyle, D. Schlegel, R. Scoccimarro, M. S. Vogeley, A. Berlind, T. Budavari, A. Con-
nolly, D. J. Eisenstein, D. Finkbeiner3, J. A. Frieman, J. E. Gunn, L. Hui, B. Jain, D. Johnston, S. Kent, H. Lin, R. Nakajima, R. C. Nicho, J. P. Ostriker, A. Pope, R. Scranton, U. Seljak, R. K. Sheth, A. Stebbins, A. S. Szalay, I. Szapudi, Y. Xu, J. Annis, J. Brinkmann, S. Burles, F. J. Castander, I. Csabai, J. Loveday, M. Doi, M. Fukugita, B. Gillespie, G. Hennessy, D. W. Hogg, Z. Ivezic, G. R. Knapp, D. Q. Lamb, B. C. Lee, R. H. Lupton, T. A. McKay, P. Kunszt, J. A. Munn, L. OConnell, J. Peoples, J. R. Pier, M. Richmond, C. Rockosi, D. P. Schneider, C. Stoughton, D. L. Tucker, D. E. Vanden Berk, B. Yanny, D. G. York: Phys. Rev. D 69 (2004) 103501.
50) Y. Gong, B. Wang and Zhang, Y.Z.: Phys. Rev. D 72 (2005) 043510.
51) U. Seljak, A. Slosar and P. McDonald: JCAP 10 (2006) 014.
52) D. N. Spergel, R. Bean, O. Dore, M. R. Nolta, C. L. Bennett, J. Dunkley, G. Hinshaw, N. Jarosik, E. Komatsu, L. Page, H. V. Peiris, L. Verde, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, E. L. Wright: Astrophys. J. Suppl. 170 (2007) 377.
53) Q. G. Huang and M. Li: JCAP 08 (2004) 013.
54) C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, D. N. Spergel, G. S. Tucker, E. Wollack, E. L. Wright, C. Barnes, M. R. Greason, R. S. Hill, E. Komatsu, M. R. Nolta, N. Odegard, H. V. Peiris, L. Verde, J. L. Weiland: Astrophys. J. Suppl. 148 (2003) 1.
55) K. Ichikawa, M. Kawasaki, T. Sekiguchi and T. Takahashi: JCAP 06 (2006) 005.
56) E. Komatsu, J. Dunkley, M. R. Nolta, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. Limon, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, S. S. Meyer, G. S. Tucker, J. L. Weiland, E. Wollack and E. L. Wright: Astrophys. J. Suppl. 180 (2009) 330.
57) E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright: Astrophys. J. Suppl. 192 (2011) 18.
58) G. Hinshaw, D. Larson, E. Komatsu, D. N. Spergel, C. L. Bennett, J. Dunkley, M. R. Nolta, M. Halpern, R. S. Hill, N. Odegard, L. Page, K. M. Smith, J. L. Weiland, B. Gold, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, G. S. Tucker, E. Wollack, E. L. Wright: Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results, arXiv: 1212.5226.

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