Determinantal and Pfaffian formulas of $K$-theoretic Schubert calculus

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Abstract

In this paper, we prove determinantal and Pfaffian formulas that describe the $K$-theoretic degeneracy loci classes for the Grassmann and the symplectic Grassmann bundles respectively. The former generalizes Kempf-Laksov’s determinantal formula and the latter generalizes Kazarian’s Pfaffian formula for the Chow rings. Our proof is based on the technique of Kazarian’s pushforward formula generalized to the $K$-theory of algebraic vector bundles. An application, we introduce the factorial $G\Theta$-functions representing the equivariant $K$-theoretic Schubert classes of the symplectic Grassmannians, generalizing the (double) theta polynomials of Buch-Kresch-Tamvakis and Wilson.

1 Introduction

The determinantal formula of Schubert calculus goes back to Giambelli [5]. It expresses an arbitrary Schubert class in the Chow ring of the Grassmannian variety as a determinant that is similar to the Jacobi-Trudi formula for a Schur function. In 1974, Kempf and Laksov [14] proved an extension of Giambelli’s formula to the degeneracy loci in the Grassmann bundle associated to a vector bundle. Their formula expresses the class of a degeneracy locus as a determinant whose entries are the Chern classes of the vector bundle and the tautological vector bundles. Analogous formulas when a vector bundle has a non-degenerate quadratic form have been also extensively studied ([13], [15], [19], [20], [23]). Most notable here is Kazarian’s work [13]. He proved a Pfaffian formula for the degeneracy loci of the both Lagrangian and maximal orthogonal subbundles.

The aim of this paper is to prove generalizations of these results from Chow ring to $K$-theory. The first main result is the $K$-theoretic Kempf-Laksov formula (Theorem 4.1) which describes the classes of the structure sheaves of the degeneracy loci in the classical Grassmann bundles. The second is the $K$-theoretic degeneracy loci formula for the symplectic Grassmann bundles, i.e. the Lagrangian Grassmann bundles and, more generally, the Grassmann bundles of submaximal isotropic subbundles of a vector bundle with a symplectic structure. The Lagrangian case of our formula is a direct generalization of Kazarian’s formula. The formula
for the submaximal symplectic case is given as a sum of Pfaffians (Theorem [6.5]), which extends a previous result [8] on the torus equivariant cohomology of the symplectic Grassmannian.

The main idea for the proof of the determinantal and Pfaffian (sum) formulas is to use the $K$-theoretic generalization of Kazarian’s pushforward formula. We desingularize the degeneracy locus through a tower of projective bundles and then compute the desired class by an iterated use of the pushforward formula. In order to obtain the determinantal or Pfaffian form ultimately, we develop a calculus of formal power series employed by Kazarian in the case of Chow ring.

As an application, we study the torus equivariant $K$-theory of the symplectic Grassmannian $SG^k(n)$ of $(n-k)$-dimensional isotropic subspaces in a symplectic vector space of dimension $2n$. We introduce a family of special functions called factorial $G\Theta$-functions, using the Pfaffian sum formula. These functions generalize the factorial $GQ$-functions defined in [11], which are relevant for the Lagrangian Grassmannian $LG(n) := SG^0(n)$. The factorial $G\Theta$-functions are canonically identified with the Schubert classes in the torus equivariant $K$-theory of the symplectic Grassmannian $SG^k(n)$.

In this paper, all varieties are assumed to be quasi-projective over the complex number field $\mathbb{C}$. For a pair of integers $a, b$ such that $a \leq b$, we denote by $[a, b]$ the set of integers $i$ such that $a \leq i \leq b$. Below we describe our main results in more detail.

### 1.1 Kempf-Laksov formula in $K$-theory

Our first result is an extension of the Kempf-Laksov formula to $K$-theory. Let $E$ be a vector bundle of rank $n$ over a smooth variety $X$. Let $\xi : Gr_d(E) \rightarrow X$ denote the Grassmann bundle parametrizing subbundles of rank $d$ in $E$.

Let $S$ be the tautological subbundle of $\xi^* E$ over $Gr_d(E)$. Fix a flag of subbundles of $E$

$$0 = F^0 \subset \cdots \subset F^2 \subset F^1 \subset E$$

such that the rank of $F^i$ is $n - i$. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be a non-increasing sequence of nonnegative integers (often called a partition in combinatorics). We assume that $\lambda_1 \leq n - d$. Consider the subvariety of $Gr_d(E)$ defined by the so-called Schubert condition:

$$\Omega_\lambda = \{ (x, S_x) \in Gr_d(E) \mid \dim(F^{\lambda_i-i+d}_x \cap S_x) \geq i, i = 1, \ldots, d \}.$$

We call $\Omega_\lambda$ the degeneracy locus associated to $\lambda$. We are interested in the structure sheaf $\mathcal{O}_{\Omega_\lambda}$ of $\Omega_\lambda$ in the Grothendieck ring $K(Gr_d(E))$ of algebraic vector bundles on $Gr_d(E)$.

Since we work with the connective $K$-theory throughout the paper, we briefly explain it now as a digression. It is an example of oriented cohomology theory in algebraic geometry, equipped with a natural notion of Chern classes (see [2]). The connective $K$-theory assigns to a smooth variety $Y$ a commutative graded algebra $CK^*(Y) = \bigoplus_{m \in \mathbb{Z}} CK^m(Y)$ over $\mathbb{Z}[\beta]$ where $\beta$ is a formal parameter of degree $-1$. If we specialize $\beta$ to $-1$, then $CK^*(Y)$ becomes the Grothendieck ring $K(Y)$. On the other hand, if $\beta = 0$, then $CK^*(Y)$ becomes the Chow ring $CH^*(Y)$. Note that we adopt the opposite sign convention for $\beta$ from the references [21] [2] (see [2] for details). One of the advantages of using $CK^*$ is that we can make the theory $\mathbb{Z}$-graded, whereas $K(Y)$ is naturally only a filtered algebra. The reader who is not familiar with the connective $K$-theory can just think of the case of $K$-theory assuming $\beta$ to $-1$. The
fundamental class $[W]_{CK^*}$ for any subvariety $W$ of $Y$ is also available in $CK^*(Y)$. In fact, we have $[W]_{CK^*} = i_*(1)$ where $i : W \to Y$ is the inclusion. This element corresponds to the class of the structure sheaf $\mathcal{O}_W$ in $K(Y)$ when we specialize $\beta$ to $-1$, while it gives the class of the algebraic cycle determined by $W$ in the Chow ring $CH^*(Y)$ when we specialize $\beta$ to $0$. In the sequel of this paper, we will simply denote $[W]_{CK^*}$ by $[W]$.

Our formula for $[\Omega_\lambda]$ is given in terms of certain characteristic classes, which we call $K$-theoretic Segre classes following Buch [1]. Let $m \in \mathbb{Z}$. According to a fundamental result due to Buch op.cit., the $m$-th Segre class $\mathcal{S}_m(E)$ is equal to the pushforward of a power of the 1st Chern class of the tautological quotient line bundle over the dual projective bundle $\mathbb{P}^*(E)$ of $E$ (§3.1). More generally, we define the $m$-th Segre class $\mathcal{S}_m(E - F)$ for any difference of vector bundles $E - F$ considered as an element in $K(X)$ and prove the analogous pushforward formula ($§3.2$). We denote by $V^\vee$ the dual of a vector bundle $V$. For a nonnegative integer $k$, let $\binom{m}{k} = m(m - 1) \cdots (m - k + 1)/k!$ be the generalized binomial coefficient.

**Theorem A** (Theorem 4.1). The class of $\Omega_\lambda$ in $CK^*(Gr_d(E))$ is given by the formula

$$[\Omega_\lambda] = \det \left( \sum_{k=0}^{\infty} \binom{i-j}{k} \beta^k \mathcal{S}_k(\lambda_i, j - i + k)(S^k - (E/F)^{\lambda_i - i + k}) \right)_{1 \leq i, j \leq d}. \tag{1.1}$$

Although the entries in the determinant appear to be infinite series in $\beta$, the formula is well-defined and it gives an element in $CK^*(Gr_d(E))$. Note that if we specialize $\beta$ to zero, then this formula agrees with Kempf-Laksov’s determinantal formula in the case of Chow ring.

It should be noticed that the class $[\Omega_\lambda]$ can be identified with the double Grothendieck polynomial of Grassmannian type, also known as the factorial Grothendieck polynomials. Therefore our formula should give a Jacobi-Trudi type expression for the factorial Grothendieck polynomials indexed by $\lambda$.

### 1.2 Symplectic degeneracy loci formula in $K$-theory

Let us now turn to the symplectic degeneracy loci. Let $E$ be a vector bundle of rank $2n$ over a smooth variety $X$. Suppose that we are given a symplectic structure on $E$, i.e., a nowhere degenerating section of $\Lambda^2 E^\vee$. For any subbundle $V$ of $E$, we denote by $V^\perp$ the orthogonal complement of $V$ with respect to the symplectic form. A subbundle $V$ of $E$ is isotropic if the symplectic form vanishes identically on $V$, that is to say $V_x \subset V_x^\perp$ for all $x \in X$. An isotropic subbundle of rank $n$ is called a Lagrangian subbundle. Fix a nonnegative integer $k < n$. Let $\xi : SG^k(E) \to X$ be the symplectic Grassmann bundle parametrizing $(n - k)$-dimensional isotropic subbundles over $X$. We denote the Lagrangian Grassmann bundle $SG^0(E)$ by $LG(E)$.

We introduce some notation to describe the symplectic degeneracy loci. A **partition** $\lambda$ is a sequence $(\lambda_1, \lambda_2, \ldots)$ of nonnegative integers such that $\lambda_i = 0$ for all sufficiently large $i$. We call each $\lambda_i$ a **part** of $\lambda$. The **length** of $\lambda$ is defined to be the number of the non-zero parts in $\lambda$. A partition $\lambda$ is **$k$-strict** if $\lambda_i > k$ implies $\lambda_i > \lambda_{i+1}$. Let $SP^k(n)$ be the set of all $k$-strict partitions $\lambda$ of length at most $n - k$ such that $\lambda_1 \leq n + k$. To each $k$-strict partition $\lambda$ in $SP^k(n)$, we associate a sequence of integers $\chi = (\chi_1, \ldots, \chi_{n-k}) \in \mathbb{Z}^{n-k}$ which we call the **characteristic index** of $\lambda$. See [7.4.2] or [8]. For $i \in [1, n-k]$, let

$$\gamma(i) = \# \{ s \in [1, i - 1] \mid \lambda_s + \lambda_i > 2k + i - s \}. \tag{1.2}$$
Define the characteristic index of \( \lambda \) by

\[
\chi_i = \lambda_i - i - k + \gamma(i) \quad \text{for} \quad i \in [1, n-k].
\]

In the Lagrangian case \( (k = 0) \), the partition \( \lambda \) is an honest strict partition and we have \( \chi_i = \lambda_i - 1 \).

Fix a flag of subbundles of \( E \)

\[
0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-n} = E,
\]

such that \( \text{rk} F^i = n - i \) for \( i \in [-n, n] \), and

\[
(F^i)^\perp = F^{-i} \quad \text{for} \quad i \in [0, n].
\]

This means that the bundle \( F^i \) is isotropic for \( i \geq 0 \), and in particular \( F^0 \) is Lagrangian. Let \( U \) be the tautological subbundle of \( \xi^* E \) over \( SG^k(E) \). Define the symplectic degeneracy locus \( \Omega_\lambda \subset SG^k(E) \) by

\[
\Omega_\lambda = \{(x, U_x) \in SG^k(E) \mid \dim(F^i_x \cap U_x) \geq i \quad \text{for} \quad i = 1, \ldots, n-k\}.
\]

Define \( D(\lambda) := \{(i, j) \mid 1 \leq i < j \leq n-k, \chi_i + \chi_j < 0\} \). There is an explicitly defined element \( \Lambda^\lambda_{i,j} \) in \( \text{ CK}^*(SG^k(E)) \) for \( 1 \leq i < j \leq 2n \) and for each subset \( I \subset D(\lambda) \), written in terms of the Segre classes of \( U, F^i, E \). See [6.2] for the precise definition.

Now we can state the second main result.

**Theorem B** (Theorem 6.3). The class of the symplectic degeneracy locus \( \Omega_\lambda \) in \( \text{ CK}^*(SG^k(E)) \) is given by

\[
[\Omega_\lambda] = \sum_{I \subset D(\lambda)} \text{PF} \left( \Lambda^\lambda_{i,j} \right)_{1 \leq i < j \leq 2m},
\]

where \( 2m \) is the smallest even number greater than or equal to the length of \( \lambda \).

Let us focus on the Lagrangian case \( k = 0 \). In this case \( D(\lambda) \) is empty and hence the sum reduces to a single Pfaffian. Define

\[
\mathcal{E}^{(\ell)}_s := \mathcal{S}_s(L^\ell - (E/F^\ell)^\vee) \in \text{ CK}^*(LG(E)) \quad \text{for} \quad \ell \in [-n, n] \quad \text{and} \quad s \in \mathbb{Z}.
\]

Here we denote the tautological Lagrangian subbundle by \( L \) instead of \( U \). The entry \( \Lambda^\lambda_{i,j} \) of the Pfaffian is given by

\[
\Lambda^\lambda_{i,j} = \phi \left( \tau^\lambda_{i,j} \left( 1 + \beta \tau_i \right)^{2m-i} \left( 1 + \beta \tau_j \right)^{2m-j} \frac{\bar{\tau}_j - \bar{\tau}_i}{\bar{\tau}_j + \bar{\tau}_i + \beta \tau_j \tau_i} \right),
\]

where \( \bar{\tau}_i := -\tau_i/(1 + \beta \tau_i) \) and \( \phi \) is a map from a certain ring of formal Laurent series in variables \( \tau_1, \ldots, \tau_n \) to \( \text{ CK}^*(LG(E)) \) sending \( \tau_1^{s_1} \cdots \tau_n^{s_n} \) to \( \mathcal{E}^{(\lambda^{s_1}-1)}_1 \cdots \mathcal{E}^{(\lambda^{s_n}-1)}_n \) (see [6.4]). If we let \( \beta = 0 \), the function in the parenthesis of (1.3) specializes to

\[
\tau^\lambda_{i,j} + 2 \sum_{s=1}^{\lambda_j} (-1)^s \tau^{\lambda_i+s}_i \tau^{\lambda_j-s}_j.
\]

This quadratic expression is exactly the one that describes the entries of Kazarian’s Pfaffian formula for the Lagrangian degeneracy loci.
1.3 Method of proofs

In order to prove the formulas (1.1) and (1.2), we use a resolution of singularities of the variety $\Omega_{\lambda}$ constructed through a tower of projective bundles. We can compute the class $[\Omega_{\lambda}]$ by pushing forward the class of the resolution along the projective bundles, and hence by the iterative use of a generalization (Proposition 3.6) of Buch’s push-forward formula. In particular, this generalization is exactly an extension of the fundamental push-forward formula due to Kazarian [13] to $K$-theory, with a help of the Segre classes. We also develop a calculus of formal power series employed by Kazarian and apply it to obtain a determinantal formula for the class of the degeneracy locus in $K$-theory (Theorem 4.1), and also the Pfaffian sum formulas for the symplectic degeneracy loci (Theorem 5.2 and Theorem 6.5).

1.4 From equivariant point of view

Here we add some remarks on the symplectic degeneracy loci from point of views of torus equivariant theory, and explain further results on the special functions representing the Schubert classes in the equivariant $K$-theory of the symplectic Grassmannian.

Kazarian’s Pfaffian formulas were reproved later but independently in the context of torus equivariant cohomology in [7] and [10] by using more combinatorial approaches. In fact, the equivariant Schubert classes are the counterparts of the degeneracy loci and they are identified with the *factorial* $Q$- and $P$-functions. These functions correspond to the Lagrangian and maximal orthogonal cases respectively. See [9, §8] for the precise comparison. It should be also noted that these results are equivariant extensions of Pragacz’s result [22] in the non-equivariant case.

In [11], the second and fourth authors introduced the *K-theoretic factorial* $Q$- and $P$-functions, which we denote by $GQ$ and $GP$. These functions are relevant in the torus equivariant $K$-theory of the maximal isotropic Grassmannians, and are the canonical representatives for the structure sheaves of the Schubert varieties for the Lagrangian and maximal orthogonal Grassmannians respectively. Therefore our result for the Lagrangian degeneracy loci (Theorem 5.2) gives the formula for the $GQ$-functions, as Kazarian’s result is related to the factorial $Q$-function (cf. [2]). We deal with this issue in a more general context of the symplectic Grassmannians below. As a consequence, we can prove that the $GQ$-functions are given in a Pfaffian formula. The details will be discussed elsewhere. A natural open question is about the orthogonal Grassmannians.

The $K$-theoretic Pfaffian sum formula (Theorem 6.5) suggests that we can introduce the special functions generalizing the factorial $GQ$-function to the submaximal symplectic case. In fact, we define the (factorial) $G\Theta$-functions in [7]. These functions are natural extensions of the *double theta polynomials* introduced by Wilson [26]. We prove that the factorial $G\Theta$-functions are the canonical representatives of the equivariant Schubert classes in torus equivariant connective $K$-theory (Theorem 7.13). The non-equivariant version of the $G\Theta$-functions are also new and give the canonical representatives for the Schubert classes in the $K$-theory of the submaximal symplectic Grassmannian.
2 Chern classes in connective $K$-theory

Connective $K$-theory denoted by $CK^*$ is an example of oriented cohomology theory introduced by Levine and Morel. It comes equipped with a contravariant functor, together with pushforwards for projective morphisms, satisfying some axioms. We refer to [2, 6], and [21] for the detailed construction. In this section, we recall some preliminary facts on $CK^*$, especially on the Chern classes.

Let $X$ be a smooth quasiprojective variety over $\mathbb{C}$. The connective $K$-theory of $X$ interpolates the Grothendieck ring $K(X)$ of the algebraic vector bundles on $X$ and the Chow ring $CH^*(X)$ of $X$. For any closed equidimensional subvariety $Y$ of $X$, there is the associated fundamental class $[Y]_{CK^*}$ in $CK^*(X)$. The connective $K$-theory assigns to $X$ a commutative graded algebra $CK^*(X)$ over the coefficient ring $\mathbb{Z}[\beta]$ where $\beta$ is a formal variable of degree $-1$. The $\mathbb{Z}[\beta]$-algebra $CK^*(X)$ specializes to the Chow ring $CH^*(X)$ and the Grothendieck ring $K(X)$ by setting $\beta$ equal to $0$ and $-1$ respectively. In particular, $[Y]_{CK^*}$ is specialized to the class $[Y]$ in $CH^*(X)$ and also to the class of the structure sheaf $\mathcal{O}_Y$ of $Y$ in $K(X)$. In the rest of the paper, we denote the fundamental class of $Y$ in $CK^*(X)$ by $[Y]$ instead of $[Y]_{CK^*}$.

As a feature of any complex oriented cohomology theory, the connective $K$-theory admits the theory of Chern classes. For line bundles $L_1$ and $L_2$ over $X$, we have their 1st Chern classes $c_1(L_i) \in CK^1(X)$ such that

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) + \beta c_1(L_1)c_1(L_2).$$

This fundamental law characterizes the theory. Note that the operation

$$(u, v) \mapsto u \oplus u := u + v + \beta uv$$

is an example of commutative one-dimensional formal group law, which is the key ingredient of oriented cohomology theories. We should stress here that the sign convention of $\beta$ is opposite from one in the literatures [21, 2, 6].

It is convenient to introduce the notation for the formal inverse:

$$u \ominus v = \frac{u - v}{1 + \beta v} \quad \text{and} \quad \bar{u} := \frac{-u}{1 + \beta u}.$$

It is easy to check that $u \ominus \bar{v} = u \ominus v$. Furthermore, if $u = c_1(L)$, then $\bar{u} = c_1(L^\vee)$.

In general, for a vector bundle $E \to X$ of rank $e$, the Chern classes $c_i(E)$ can be defined similarly to Grothendieck’s argument used for the Chow ring. Consider the dual projective bundle $\mathbb{P}^*(E) \to X$. We have the exact sequence of vector bundles on $\mathbb{P}^*(E)$

$$0 \to \mathcal{H} \to \pi^*E \to Q \to 0$$

where $\mathcal{H}$ is the rank $(e - 1)$ subbundle of $\pi^*E$ whose fiber at $x \in \mathbb{P}^*(E)$ is precisely the hyperplane represented by $x$ itself. We call $Q$ the universal quotient line bundle. The ring $CK^*(\mathbb{P}^*(E))$ is generated by $\tau := s^*s_s(1)$ as a $CK^*(X)$-algebra, where $s$ is the zero section of $Q$. One of the axioms of the oriented cohomology theory states that there is a relation

$$\sum_{i=0}^{e} (-1)^i c_i(E)\tau^{e-i} = 0 \quad (2.1)$$
for some \( c_i(E) \in \text{CK}^i(X) \), \( 0 \leq i \leq e \), and indeed this relation uniquely determines \( c_i(E) \)'s. Thus we can define these elements to be the Chern classes of \( E \). By definition the canonical generator \( \tau \) is naturally identified with \( c_1(\mathcal{Q}) \).

For computations it is convenient to combine the classes we just defined into a Chern polynomial

\[
c(E; u) := \sum_{i=0}^{e} c_i(E) u^i.
\]

A formal difference \( E - F \) of vector bundles \( E \) and \( F \) over \( X \) defines a virtual vector bundle. Two such virtual vector bundles \( E - F \) and \( E' - F' \) are considered to be identical if they are equal in the Grothendieck ring \( K(X) \) of vector bundles over \( X \). Let us set \( c(E - F; u) := c(E; u)/c(F; u) \). This is well-defined because of the Whitney formula: given a short exact sequence of bundles \( 0 \to F \to E \to W \to 0 \), one has \( c(E; u) = c(F; u)c(W; u) \).

**Remark 2.1.** Let \( x_1, \ldots, x_a \) and \( y_1, \ldots, y_b \) be the Chern roots of \( E \) and \( F \) respectively. The explicit expression of \( c_p(E - F) \) is given by

\[
c(E - F; u) = \frac{\prod_{i=1}^{a}(1 + x_i u)}{\prod_{j=1}^{b}(1 + y_j u)} = \sum_{p=0}^{\infty} \sum_{j=0}^{p} (-1)^j c_{p-j}((x_1, \ldots, x_a)h_j(y_1, \ldots, y_b)) \cdot u^p,
\]

where \( h_j \) is the \( j \)-th complete symmetric function.

The classical fact that the top Chern class is equal to the Euler class also holds in \( \text{CK}^e \).

We will use the next lemma which follows from [3, Example 14.1.1] and [21, Lemma 6.6.7] (See also [3, Example 3.2.16]).

**Lemma 2.2.** Let \( E \) be a vector bundle of rank \( e \) over \( X \) and \( s \) a section of \( E \). Let \( Z \) be the zero scheme of \( s \). If \( X \) is Cohen-Macaulay and the codimension of \( Z \) is \( e \), then \( s \) is regular and

\[
c_e(E) = [Z] \in \text{CK}^e(X).
\]

We conclude this section with the following formula.

**Lemma 2.3.** Let \( E \) be a vector bundle of rank \( e \) and \( L \) a line bundle. We have

\[
c_e(L \otimes E) = \sum_{p=0}^{e} c_p(E) \sum_{q=0}^{p} \binom{p}{q} \beta^q c_1(L)^{e-p+q}
\]

**Proof.** Let \( x_1, \ldots, x_e \) be Chern roots of \( E \). Let \( c_1 := c_1(L) \). Then

\[
c_e(L \otimes E) = (x_1 \oplus c_1) \cdots (x_e \oplus c_1) = (x_1 + c_1 + \beta x_1 c_1) \cdots (x_e + c_1 + \beta x_e c_1)
\]

\[
= (x_1(1 + \beta c_1) + c_1) \cdots (x_e(1 + \beta c_1) + c_1) = \sum_{p=0}^{e} e_p(x) c_1^{e-p}(1 + \beta c_1)^p
\]

\[
= \sum_{p=0}^{e} e_p(x) \sum_{q=0}^{p} \binom{p}{q} \beta^q c_1^{e-p+q}.
\]

\[\square\]
3 K-theoretic Segre classes

In this section, we define the K-theoretic Segre classes for a vector bundle, and more generally for a virtual vector bundle. The goal is to show that they are obtained as the pushforward of certain characteristic classes along a projective bundle.

3.1 Buch’s pushforward formula

Let $E$ be a vector bundle over $X$ of rank $e$. Let $\pi : \mathbb{P}^*(E) \to X$ be the dual projective bundle of $E$ and $Q$ its universal quotient line bundle of $\pi^*E$. Let $\tau = c_1(Q)$. For each integer $m \geq -e + 1$, define

$$J_m(E) := \pi_*(\tau^{m+e-1}).$$

Lemma 3.1. We have $\pi_*(\tau^{m+e-1}) = (-\beta)^{-m}$ for $-e + 1 \leq m \leq 0$.

Proof. Let $z_1, \ldots, z_e$ be the Chern roots of $E$. From Vishik’s formula [25, Proposition 5.29, p.548], we have

$$\pi_*(\tau^{-m+e-1}) = \sum_{i=1}^e \frac{z_i^{-m+e-1}}{\prod_{j \neq i}(z_i \otimes z_j)}.$$ 

The right hand side is a polynomial in $\beta$ of degree at most $e - 1$. Let us denote it by $F(\beta)$. It is easy to see that $F(-1/z_i)$ is equal to $z_i^{-m}$ for $1 \leq i \leq e$. This property uniquely determines $F(\beta)$ by degree reason. Obviously $(-\beta)^m$ also satisfies the same conditions, so we have $F(\beta) = (-\beta)^m$. \hfill \Box

For each $m < -e + 1$, define $J_m(E) := (-\beta)^{-m}$ formally (See Remark 3.4). Define

$$J(E; u) := \sum_{m \in \mathbb{Z}} J_m(E)u^m.$$ 

Then we can show the following theorem.

Theorem 3.2. We have

$$J(E; u) = \frac{1}{1 + \beta u^{-1}} \frac{c(E; \beta)}{c(E; -u)},$$

where $\frac{1}{1 + \beta u^{-1}}$ is expanded in the form $\sum_{i=0}^{\infty} (-\beta)^i u^{-i}$.

Proof. We multiply (3.1) by $\tau^{m-1}$ for $m \geq 1$, and pushforward it to obtain

$$\sum_{i=0}^{e} (-1)^i c_i(E) \pi_*(\tau^{m-i+e-1}) = 0.$$ 

One observes that by using Lemma 3.1 together with equations (3.2), we can uniquely determine $\pi_*(\tau^k)$ ($k \geq e$) successively. This recursion relation is equivalent to require $c(E; -u)J(E; u)$ has only non-positive powers in $u$. Thus the series $c(E; -u)J(E; u)$ equals to

$$\sum_{m \leq 0} \left( \sum_{i=0}^{e} (-1)^i c_i(E) (-\beta)^{i-m} \right) u^m = \left( \sum_{m \leq 0} (-\beta)^{-m} u^m \right) \left( \sum_{i=0}^{e} c_i(E) \beta^i \right) = (1 + \beta u^{-1})^{-1} c(E; \beta).$$ 

Hence we have $J(E; u) = (1 + \beta u^{-1})^{-1} c(E; \beta)/c(E; -u)$. This proves the theorem. \hfill \Box
Remark 3.3. Buch [1, Lemma 7.1] proved that $S_m(E) = G_m(z_1, \ldots, z_e)$ where $G_m(z_1, \ldots, z_e)$ stands for the Grothendieck polynomial and we regard $z_1, \ldots, z_e$ as Chern roots of $E$. Lemma 6.6 [1] defines the generating function of $G_m$ in the form of (3.1), and thus our proof of Theorem 3.2 gives a different proof of Buch’s result. See §8.

Remark 3.4. We can show that $S_m(E) = S_m(E \oplus O_X)$ for all $m \geq -e + 1$. With this fact, it is possible to define $S_m(E)$ := $S_m(E \oplus O_X \oplus nX)$ for all $m < -e + 1$ by taking $n$ sufficiently large.

Remark 3.5. It follows from a simple identity $\frac{1 + \beta x}{1 - ux} = \frac{1}{1 + (u + \beta)x}$ that

$$\frac{c(E; \beta)}{c(E; -u)} = \frac{1}{c(E^\vee; u + \beta)}.$$  

3.2 Extension to virtual bundles

It is natural to extend the definition of the $K$-theoretic Segre classes to the virtual vector bundles by the following generating function:

$$\mathcal{S}(E - F; u) := \sum_{m \in \mathbb{Z}} \mathcal{S}_m(E - F) u^m = \frac{1}{1 + \beta u^{-1} c_{-u}(E - F)}.$$  

(3.3)

Since the right hand side is written in Chern classes of virtual vector bundles, it depends only on the virtual vector bundle $E - F$, so the notion is well-defined. Since

$$\mathcal{S}(E - F; u) = \mathcal{S}(E; u)c(F^\vee; u + \beta),$$

we have

$$\mathcal{S}_m(E - F) = \sum_{p=0}^{\text{rk}(F)} c_p(F^\vee) \sum_{q=0}^{p} \binom{p}{q} \beta^q \mathcal{S}_{m-p+q}(E).$$  

(3.4)

Finally we prove that we can obtain $\mathcal{S}_m(E - F)$ also as the pushforward of certain Chern classes.

Proposition 3.6. Recall $\pi : \mathbb{P}^*(E) \to X$, $Q$, and $\tau = c_1(Q)$ from §3.1 Let $F$ be a vector bundle over $X$ of rank $f$. Denote its pullback to $\mathbb{P}^*(E)$ also by $F$. We have

$$\pi_* \left( \tau^* c_f(Q \otimes F^\vee) \right) = \mathcal{S}_{s+f-e+1}(E - F).$$

Proof. By using Lemma 2.3 and Theorem 3.2, we can compute

$$\pi_* \left( \tau^* c_f(Q \otimes F^\vee) \right) = \sum_{p=0}^{\text{rk}(F)} c_p(F^\vee) \sum_{q=0}^{p} \binom{p}{q} \beta^q \mathcal{S}_{s+f-e+1-p+q}(E).$$

Thus (3.4) implies the claim.

4 $K$-theoretic Kempf-Laksov formula

In this section, we prove the $K$-theoretic Kempf-Laksov formula, using a resolution of singularities of the degeneracy loci constructed by a tower of projective bundles.
4.1 Degeneracy loci of Grassmann bundle

Let $E$ be a vector bundle of rank $n$ over a smooth quasi-projective variety $X$. Let $\xi : Gr_d(E) \to X$ be the Grassmannian bundle of rank $d$ subbundles over $X$, i.e.

$$Gr_d(E) := \{(x, S_x) \mid x \in X, S_x \text{ is a } d\text{-dimensional subspace of } E_x\}.$$ 

Let $S$ be the tautological subbundle of $\xi^*E$ over $Gr_d(E)$. Fix a complete flag $0 = F^n \subset \cdots \subset F^1 \subset E$ where the superscript indicates the corank, i.e. $rk F^k = n - k$. We denote also by $E$ and $F^i$ the pullback of $E$ and $F^i$ along $\xi$ respectively.

Let $P_d$ be the set of all partitions $(\lambda_1, \ldots, \lambda_d)$ with at most $d$ parts. Let $P_d(n)$ be the set of all partitions such that $\lambda_i \leq n - d$ for all $i = 1, \ldots, d$. Consider the partial flag of $E$

$$F^*_\lambda : F_{\lambda_1-1+d} \subset F_{\lambda_2-2+d} \subset \cdots \subset F_{\lambda_i-i+d} \subset \cdots \subset F_{\lambda_d} \subset E.$$ 

Define the type $A$ degeneracy locus $\Omega_\lambda$ in $Gr_d(E)$ by

$$\Omega_\lambda := \{(x, S_x) \in Gr_d(E) \mid \dim(F^*_x - i+d \cap S_x) \geq i, i = 1, \ldots, d\}.$$ 

If $\lambda_i = 0$, then the condition $\dim(F^*_x - i+d \cap S_x) \geq i$ is vacuous.

Let $\binom{n}{k}$ be the generalized binomial coefficient, i.e., $(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$.

**Theorem 4.1** (K-theoretic Kempf-Laksov formula). Let $\lambda \in P_d(n)$ and $\Omega_\lambda$ its associated degeneracy locus in $Gr_d(E)$. Let $\mathcal{S}_m^{(f)}$ be the Segre class $\mathcal{S}_m(S^r - (E/F^f))$. The class associated to $\Omega_\lambda$ in $\mathcal{C}K^*(Gr_d(E))$ is given by

$$[\Omega_\lambda] = \det \left( \sum_{s=0}^{\infty} \binom{i-j}{s} \beta^s \mathcal{S}_s^{(1)} \mathcal{S}_s^{(\lambda_i-i+d)} \right)_{1 \leq i, j \leq d}.$$ 

We prove this theorem at the end of this section.

4.2 Resolution of singularities

Let $r$ be the length of $\lambda$, i.e., $\lambda_r \neq 0$ and $\lambda_{r+1} = 0$. Associated to the partial flag $F^*_{\lambda}$ is a generalized flag bundle

$$\varpi : Fl_{\lambda}(E) \to Gr_d(E),$$ 

where the fiber at $p \in Gr_d(E)$ consists of flags of subspaces $(D_1)_p \subset \cdots \subset (D_r)_p$ of $E_p$ with $\dim(D_i)_p = i$ and $(D_i)_p \subset F^i_p - i+d$. Let $D_1 \subset \cdots \subset D_r$ be the corresponding flag of tautological subbundles over $Fl_{\lambda}(E)$.

We construct a sequence of varieties $Y_r \subset \cdots \subset Y_1 \subset Y_0 := Fl_{\lambda}(E)$ inductively as follows. First let $Y_1 \subset Fl_{\lambda}(E)$ be the locus where the bundle map $D_1 \to E/S$ has rank 0, i.e. the corresponding section of $D_1^\vee \otimes E/S$ vanishes. Next let $i \geq 2$. Over $Y_{i-1}$, we have the bundle map $(D_i/D_{i-1})|_{Y_{i-1}} \to (E/S)|_{Y_{i-1}}$. Define $Y_i$ to be the locus where this bundle map has rank 0, i.e. the corresponding section of $(D_i/D_{i-1})^\vee \otimes E/S$ vanishes. It is easy to see that $Y_i$ is smooth of codimension $n - d$ in $Y_{i-1}$ and that $Y_r$ is birational to $\Omega_\lambda$ along $\varpi$ (cf. Lemma 6.6 [6.7 and Remark 6.8],

**Lemma 4.2.** The class of $\Omega_\lambda$ in $\mathcal{C}K^*(Gr_d(E))$ is given by

$$[\Omega_\lambda] = \varpi_*([Y_1]).$$
Proof. The degeneracy locus \( \Omega_{\lambda} \) has rational singularities \( [18] \) p.274, 8.2.2.Theorem (c)] (see also [4 Proof of Theorem 3]). Thus \( \varpi|_{Y_i} : Y_i \to \Omega_{\lambda} \) is a rational resolution, i.e. the higher direct images of the structure sheaf of \( Y_i \) vanish. Therefore it follows that in the connective K-theory, the pushforward of \([Y_i]\) is \([\Omega_{\lambda}]\) (see [2 or [6, Lemma 2.2]). Hence [4.2] implies the formula. \( \square \)

4.3 Tower of projective bundles — pushforward formula

Let \( \iota_i : Y_i \to Y_{i-1} \) be the inclusion for all \( i = 1, \ldots, r \). It follows from Lemma 2.2 that the fundamental class of \( Y_i \) in \( CK^*(Y_{i-1}) \) is given by

\[
\iota_{i*}(1) = (\iota_1 \cdots \iota_{i-1})^* c_{n-d}(D_1/D_{i-1})\bigvee E/S
\]

where we set \( D_0 = 0 \). The consecutive application of (4.1) and the projection formula implies that the class of \( Y_r \) in \( CK^*(Fl_\lambda(E)) \) is given by

\[
[Y_r] = (\iota_1 \cdots \iota_r)_*(1) = \prod_{i=1}^r c_{n-d}(D_i/D_{i-1})\bigvee E/S.
\]

For example, if \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \), then

\[
[Y_3] = (\iota_1 \iota_2 \iota_3)_*(1) = (\iota_1 \iota_2)_*(\iota_1^* \iota_2^* c_{n-d}(D_3/D_2)\bigvee E/S) = \iota_{1*}(\iota_1^* c_{n-d}(D_3/D_2)\bigvee E/S)\iota_1^* c_{n-d}(D_2/D_1)\bigvee E/S) = c_{n-d}(D_3/D_2)\bigvee E/S)c_{n-d}(D_2/D_1)\bigvee E/S)c_{n-d}(D_1)\bigvee E/S).
\]

One can obtain the flag bundle \( Fl_\lambda(E) \) as a tower of projective bundles

\[
\varpi : Fl_\lambda(E) = \mathbb{P}(F^{\lambda_0 - r + d}/D_{r-1}) \xrightarrow{\varpi_1} \mathbb{P}(F^{\lambda_1 - (r-1)+ d}/D_{r-2}) \xrightarrow{\varpi_2} \cdots \xrightarrow{\varpi_{r-1}} \mathbb{P}(F^{\lambda_{r-2} - 2d}/D_1) \xrightarrow{\varpi_r} \mathbb{P}(F^{\lambda_{r-1} + d}) \xrightarrow{\varpi_{r+1}} Gr_d(E).
\]

We regard \( D_i/D_{i-1} \) as the tautological line bundle of \( \mathbb{P}(F^{\lambda_i + d}/D_{i-1}) \).

Lemma 4.3 (The \( i \)-th stage pushforward formula). Let \( \tau_i := c_1((D_i/D_{i-1})\bigvee) \) and \( \alpha_i := c_{n-d}(D_i/D_{i-1})\bigvee E/S) \in CK^*(\mathbb{P}(F^{\lambda_i + d}/D_{i-1})\bigvee).

We have

\[
\varpi_{i*}(\tau_i^* \alpha_i) = \sum_{p=0}^{i-1} c_p(D_{i-1}) \sum_{q=0}^p \left( \frac{p}{q} \right) \beta^q \mathcal{S}_{\lambda_i + s - p + q}.
\]

Proof. By Proposition 3.6, we have

\[
\varpi_{i*}(\tau_i^* \cdot c_{n-d}(D_i/D_{i-1})\bigvee E/S)) = \mathcal{S}_{\lambda_i + s}((F^{\lambda_i + d}/D_{i-1} - E/S)\bigvee) = \mathcal{S}_{\lambda_i + s}((F^{\lambda_i + d} - E + S)\bigvee - D_{i-1}) = \sum_{p=0}^{i-1} c_p(D_{i-1}) \sum_{q=0}^p \left( \frac{p}{q} \right) \beta^q \mathcal{S}_{\lambda_i + s + p - q} (S\bigvee - (E/F^{\lambda_i + d})\bigvee)
\]

where we have used [3.4] for the last equality. \( \square \)
Example 4.4. We have
\[
\varpi_1(\tau_1 \Delta_1) = \partial \omega_{\lambda_1+1}^{[1]}, \quad \varpi_2(\tau_2^2 \Delta_2) = \partial \omega_{\lambda_2+2}^{[2]} + \tau_1(\partial \omega_{\lambda_2+3}^{[2]} + \beta \partial \omega_{\lambda_2+3}^{[2]}), \\
\varpi_3(\tau_3^3 \Delta_3) = \partial \omega_{\lambda_3+3}^{[3]} + \tau_1 + \tau_2(\partial \omega_{\lambda_3+4}^{[3]} + \beta \partial \omega_{\lambda_3+4}^{[3]} + \tau_1 \tau_2(\partial \omega_{\lambda_3+5}^{[3]} + 2 \beta \partial \omega_{\lambda_3+5}^{[3]} + \beta^2 \partial \omega_{\lambda_3+5}^{[3]})),
\]

Once we obtain the pushforward formula, we can in principle calculate the class \([\Omega_\lambda]\) in \(C^k\gamma(Gr_d(E))\).

Example 4.5. Let us consider the case \(r = 2\). The class \([\Omega_{\lambda_1,\lambda_2}]\) is given by
\[
\varpi_s([Y_2]) = \varpi_s(\alpha_1 \cdot \varpi_2(\alpha_2)) = \varpi_s(\alpha_1 \partial \omega_{\lambda_2}^{[2]} + \tau_1 \alpha_1(\partial \omega_{\lambda_2+1}^{[2]} + \beta \partial \omega_{\lambda_2+1}^{[2]})).
\]
Using expansion \(\bar{\tau}_s = -\tau_1 + \beta \tau_2^2 - \tau_1^3 + \cdots\), we have
\[
\partial \omega_{\lambda_1}^{[1]} \partial \omega_{\lambda_2}^{[2]} - \left( \partial \omega_{\lambda_1+1}^{[1]} + \beta \partial \omega_{\lambda_2}^{[2]} \right) \left( \partial \omega_{\lambda_2+1}^{[2]} + \beta \partial \omega_{\lambda_2+1}^{[2]} + \cdots \right) (\partial \omega_{\lambda_2+2}^{[2]} + \beta \partial \omega_{\lambda_2+2}^{[2]}).
\]
One notices that the result can be calculated by the following formal Laurent series:
\[
f(\tau_1, \tau_2) = \frac{\tau_1^{\lambda_1}}{\tau_2^{\lambda_2}} \left( \frac{\tau_2^{\lambda_2}}{\tau_1^{\lambda_2}} + \tau_1^{\lambda_1} \tau_2^{\lambda_2} \right).
\]
We can write it more compactly as
\[
f(\tau_1, \tau_2) = \tau_1^{\lambda_1} \tau_2^{\lambda_2} (1 - \tau_1^{\lambda_1} \tau_2^{\lambda_2}) = \frac{\tau_1^{\lambda_1} \tau_2^{\lambda_2}}{\tau_2^{\lambda_2} - \tau_1^{\lambda_1}}.
\]
The answer is given by replacing \(\tau_i^m\) in \(f(\tau_1, \tau_2)\) by \(\partial \omega_{\lambda_1}^{[i]}\).

Thus, in principle, we can calculate \([\Omega_\lambda] = \varpi_s([Y_r])\) from the consecutive applications of the pushforward formula in Lemma 4.5 to the product formula 4.12 for \([Y_r]\) (cf. 13 Appendix C). In order to obtain the final determinantal form, we use a manipulation using formal Laurent series in next section, a kind of umbral calculus.

4.4 Algebraic set up — Umbral calculus

This section is devoted to the formal algebraic set up needed for the calculation of iterated pushforward in the previous section. The same framework will be used also in the symplectic case later.

Let \(R = \bigoplus_{m \in \mathbb{Z}} R_m\) be a graded algebra over \(\mathbb{Z}[\beta]\). Let \(\tau_1, \ldots, \tau_r\) be indeterminates of degree 1. We use the multi-index notation \(\tau^s := \tau_1^{s_1} \cdots \tau_r^{s_r}\) for \(s = (s_1, \ldots, s_r) \in \mathbb{Z}^r\). A formal Laurent series
\[
f(\tau_1, \ldots, \tau_r) = \sum_{s \in \mathbb{Z}^r} a_s \tau^s
\]
is homogeneous of degree \(m \in \mathbb{Z}\) if \(a_s\) is zero unless \(a_s \in R_{m-|s|}\) with \(|s| = \sum_{i=1}^r s_i\). A series \(f(\tau_1, \ldots, \tau_r)\) is power series if \(a_s = 0\) if \(s_i < 0\) for some \(1 \leq i \leq r\). For \(1 \leq i \leq r\), Let \(R[[\tau_i, \ldots, \tau_i]]_m\) denote the set of all power series of degree \(m \in \mathbb{Z}\). We define
\[
R[[\tau_1, \ldots, \tau_i]]_m := \bigoplus_{m \in \mathbb{Z}} R[[\tau_1, \ldots, \tau_i]]_m.
\]
Then \(R[[\tau_1, \ldots, \tau_i]]_m\) is a graded \(\mathbb{Z}[\beta]\)-algebra called the ring of graded formal power series.

Fix a partition \(\lambda\) of length at most \(r\) throughout the section.
**Definition 4.6.** For each $m \in \mathbb{Z}$ and $1 \leq i \leq r$, fix a homogeneous element $\mathcal{A}_{m}^{[i]} \in R_{m}$. We assume that $\mathcal{A}_{m}^{[i]}$ is zero for all sufficiently large $m$. Define the following sequence of maps

$$R[[\tau_{1}, \ldots, \tau_{r}]]_{\text{gr}} \xrightarrow{p_{r}} R[[\tau_{1}, \ldots, \tau_{r-1}]]_{\text{gr}} \xrightarrow{p_{r-1}} \ldots \xrightarrow{p_{2}} R[[\tau_{1}]]_{\text{gr}} \xrightarrow{p_{1}} R,$$

by the formula (cf. [4.39])

$$p_{i}(\tau_{i}^{s}) = \sum_{p=0}^{i-1} e_{p}(\bar{\tau}_{1}, \ldots, \bar{\tau}_{r-1}) \sum_{q=0}^{p} \left( \begin{array}{c} p \\ q \end{array} \right) \beta^{q} \mathcal{A}_{m}^{[i]} \chi_{i+s-p+q} \quad \text{for } s \geq 0,$$

(4.4)

and by assuming $p_{i}$ to be an $R[[\tau_{1}, \ldots, \tau_{r-1}]]_{\text{gr}}$-module homomorphism. Here $e_{p}$ denotes the $p$-th elementary symmetric function.

**Remark 4.7.** The above definition is relevant for the Grassmannian bundle $\text{Gr}_{d}(E)$. We will use the same formulation for the symplectic Grassmannian later but only replacing $e_{p}$ by other special functions.

**Example 4.8.** Below we apply the above definitions to the case when $R = \text{CK}^{*}(\text{Gr}_{d}(X))$. We choose $\mathcal{A}_{m}^{[i]}$ to be

$$\mathcal{A}_{m}^{[i]} := \mathcal{L}_{m}(S^{\vee}(E/F)/(1+i+d)^{\vee}).$$

It is known that $\text{CK}^{*}(\text{Gr}_{d}(E))$ is bounded above, i.e., $\text{CK}^{m}(\text{Gr}_{d}(E)) = 0$ for all $m > \dim \text{Gr}_{d}(E)$. Therefore $\mathcal{A}_{m}^{[i]}$ is zero for all sufficiently large $m$. Note also that $\mathcal{A}_{m}^{[i]} = (-\beta)^{-m}$ for $m \leq 0$. It is worth stressing that we have

$$\mathcal{A}_{s}(X_{r}) = p_{1} \cdots p_{r}(1),$$

(5.5)

where we regard $\tau_{i}$ as $c_{1}(D_{i}/D_{i-1})^{\vee}$. In fact, one sees that $p_{i}$ in Definition 4.6 is the algebraic counterpart of $\mathcal{A}_{k_{s}}$.

In order to compute the element of the form [4.5], we use the following algebraic manipulation. We look at the intermediate step $p_{1} \cdots p_{r}(1)$. It is written in terms of $\mathcal{A}_{m}^{[s]}$ with $s \geq i$ and $\tau_{1}^{m_{1}}, \ldots, \tau_{r-1}^{m_{r-1}}$ with nonnegative powers. Here we can regard $\mathcal{A}_{m}^{[s]}$ as the elements that are “already integrated”. The idea is to regard $\mathcal{A}_{m}^{[s]}$ in $p_{1} \cdots p_{r}(1)$ as the image of $\tau_{s}^{m}$ under certain evaluation maps from formal Laurent series. In this way, we are “lifting” the element $p_{1} \cdots p_{r}(1)$ to an element in a ring of certain formal Laurent series. Then we can make a use of multiplications in a larger ring of formal Laurent series to obtain the desired formula.

**Definition 4.9.** Let $L_{R}$ denote the set of formal Laurent series $f(\tau_{1}, \ldots, \tau_{r}) = \sum_{s \in \mathbb{Z}^{r}} a_{s} \tau^{s}$ ($a_{s} \in R$) satisfying the following conditions:

1. $f(\tau_{1}, \ldots, \tau_{r})$ is a finite sum of homogeneous elements;

2. there are only finitely many non zero $a_{s}$’s such that $s$ does not satisfy the conditions

$$s_{1} \geq 0, \ s_{1} + s_{2} \geq 0, \ \cdots, \ s_{1} + \cdots + s_{r} \geq 0.$$

For each $m \in \mathbb{Z}$, define $L_{m}^{R}$ to be the space of all formal Laurent series of homogeneous degree $m$. Then $L_{R} := \bigoplus_{m \in \mathbb{Z}} L_{m}^{R}$ is a graded ring over $R$ with the obvious product.
**Definition 4.10.** For each $i = 1, \ldots, r$, let $L^{R,i}$ be the $R$-subring of $L^R$, consisting of series that do not contain the negative powers of $\tau_1, \ldots, \tau_{i-1}$. In particular, $L^{R,1} = L^R$. Define a graded $R[[\tau_1, \ldots, \tau_{i-1}]]_{gr}$-linear map

$$
\phi_i : L^{R,i} \to R[[\tau_1, \ldots, \tau_{i-1}]]_{gr}
$$

by $\phi_i(\tau_1^{s_1} \cdots \tau_r^{s_r}) = \tau_1^{s_1} \cdots \tau_{i-1}^{s_{i-1}} \beta_{s_i}^{[i]} \cdots \beta_{s_r}^{[r]}$. We denote $\phi_1$ also by $\phi$.

We have the commutative diagram

$$
\begin{array}{cccc}
L^{R,r} & \to & L^{R,r-1} & \to & \cdots & \to & L^{R,2} & \to & L^R, \\
\phi_r & \downarrow & \phi_{r-1} & \downarrow & \phi_3 & \downarrow & \phi_2 & \downarrow & \phi_1 & \\
R[[\tau_1, \ldots, \tau_{r-1}]]_{gr} & \to & R[[\tau_1, \ldots, \tau_{r-2}]]_{gr} & \to & \cdots & \to & R[[\tau_1, \tau_2]]_{gr} & \to & R[[\tau_1]]_{gr} & \to & R \\
\end{array}
$$

where the arrows in the first row are the obvious inclusion maps.

**Lemma 4.11.** For $s \geq 0$ we have

$$
p_i(\tau_i^s) = \phi_i \left( \tau_i^{\lambda_i+s} \prod_{j=1}^{i-1} \left(1 - \tau_j/\tau_i \right) \right).
$$

**Proof.** We compute

$$
p_i(\tau_i^s) = \sum_{p=0}^{i-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{i-1}) \sum_{q=0}^p \left( \binom{p}{q} \beta^q \beta_{\lambda_i+s-p+q}^{[i]} \right)
$$

$$
= \phi_i \left( \sum_{p=0}^{i-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{i-1}) \sum_{q=0}^p \left( \binom{p}{q} \beta^q \beta_{\lambda_i+s-p+q}^{[i]} \right) \right)
$$

$$
= \phi_i \left( \tau_i^{\lambda_i+s} \sum_{p=0}^{i-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{i-1}) \tau_i^{-p} \sum_{q=0}^p \left( \binom{p}{q} \beta^q \beta_{\lambda_i+s-p+q}^{[i]} \right) \right)
$$

$$
= \phi_i \left( \tau_i^{\lambda_i+s} \sum_{p=0}^{i-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{i-1}) \tau_i^{-p} \left(1 + \beta \tau_i\right)^p \right)
$$

$$
= \phi_i \left( \tau_i^{\lambda_i+s} \prod_{j=1}^{i-1} \left(1 - \tau_j/\tau_i \right) \right).
$$

**Proposition 4.12.** We have

$$
p_1 \cdots p_r(1) = \phi_1 \left( \tau_1^{\lambda_1} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} \left(1 - \tau_j/\tau_i \right) \right).
$$
Proof. By the commutativity of the diagram (4.6) and Lemma 4.11, we can proceed as follows:

\[ p_1 \cdots p_r(1) = p_1 \cdots p_{r-1} \phi_r \left( \tau_r^{r-1} \prod_{s=1}^{r-1} (1 - \bar{\tau}_s/\bar{\tau}_r) \right) \]

\[ = p_1 \cdots p_{r-2} \phi_{r-1} \left( \tau_r^{r-2} \prod_{s=1}^{r-2} (1 - \bar{\tau}_s/\bar{\tau}_{r-1}) \cdot \tau_r^{r-1} \prod_{s=1}^{r-1} (1 - \bar{\tau}_s/\bar{\tau}_r) \right) \]

\[ = \cdots = \phi_1 \left( \tau_1^{\lambda_1} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - \bar{\tau}_i/\bar{\tau}_j) \right). \]

\[
\text{Lemma 4.13. We have}
\[
\tau_1^{\lambda_1} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - \bar{\tau}_j/\bar{\tau}_i) = \det \left( \tau_1^{\lambda_i - j-i} \right)_{1 \leq i, j \leq r}.
\]

Proof. It follows from the Vandermonde determinant formula. \[\Box\]

\[
\text{Lemma 4.14. We have}
\]

\[
\phi_1(\tau_1^{\lambda_i - j-i}) = (-1)^{j-i} \sum_{k=0}^{\infty} \binom{i-j}{k} \omega_{j+i+k} \beta_k.
\]

(4.7)

Proof. We can compute the series in \( \tau_i \)'s:

\[
\tau_1^{\lambda_1 - j-i} = (-1)^{j-i} \tau_1^{\lambda_i + j-i} (1 + \beta_1)^{i-j} = (-1)^{j-i} \sum_{k=0}^{\infty} \binom{i-j}{k} \beta_k \tau_1^{\lambda_i + j-i+1}.
\]

\[\Box\]

The next lemma shows that the element \( p_1 \cdots p_r(1) \) only depends on the partition \( \lambda \), as long as the length of \( \lambda \) is at most \( r \).

\[
\text{Lemma 4.15. Assume that } \mathcal{D}_m^{[i]} = (-\beta)^{-m} \text{ for each } m \in \mathbb{Z}_{\leq 0} \text{ for all } i = 1, \ldots, r. \text{ If } \lambda_{s+1} = \cdots = \lambda_r = 0 \text{ for some } s < r, \text{ then we have } p_1 \cdots p_r(1) = p_1 \cdots p_s(1).
\]

Proof. It suffices to show that, if \( \lambda_r = 0 \), then \( p_r(1) = 1 \).

\[
p_r(1) = \sum_{p=0}^{r-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{r-1}) \sum_{q=0}^{\min(p, r)} \binom{p}{q} \beta^q \mathcal{D}_m^{[r]}_{-p+q}
\]

\[= \sum_{p=0}^{r-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{r-1}) \sum_{q=0}^{p} \binom{p}{q} \beta^q (-\beta)^{-q+p}
\]

\[= \sum_{p=0}^{r-1} e_p(\bar{\tau}_1, \ldots, \bar{\tau}_{r-1}) (-\beta)^p \sum_{q=0}^{p} \binom{p}{q} (-1)^q
\]

\[= e_0(\bar{\tau}_1, \ldots, \bar{\tau}_{r-1}) = 1
\]

where we used \( \sum_{q=0}^{p} \binom{p}{q} (-1)^q = 0 \) for all \( p > 0 \). \[\Box\]
4.5 Proof of Theorem 4.1

We use the notation from Example 4.8, i.e. \( R = CK^*(Gr_d(E)) \) and \( R_m^[[i]] = \mathcal{A}_m^[[i]] \).

By Lemma 4.2, 4.13 and Proposition 4.12 we have

\[
\Omega_\lambda = \varpi_*([Y_r]) = p_1 \cdots p_r(1) = \phi_1 \left( \tau_1^{\lambda_1} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} (1 - \bar{\tau}_i/\bar{\tau}_j) \right).
\] (4.8)

By Lemma 4.13 and 4.14 the right hand side equals to

\[
\det \left( \sum_{k=0}^{\infty} \binom{i-j}{k} \mathcal{A}_{\lambda_1+j-i+k} \right)_{1 \leq i, j \leq r}.
\]

Note that the sign \((-1)^{j-i}\) from Lemma 4.14 can be removed from the formula by some elementary operations on the determinant. Finally by Lemma 4.15 we have the determinant of size \( d \).

5 Pfaffian formula in Lagrangian case

The goal of this section is to prove the Pfaffian formula for the Lagrangian degeneracy loci classes (Theorem 5.2 below). Although all the contents of this section follow from §6 as a special case, we first treat this Lagrangian case separately in order to ease the burden of notation. The whole proof is analogous to the one for Theorem 4.1.

5.1 Lagrangian degeneracy loci

Let \( E \) be a symplectic vector bundle over a smooth quasi-projective variety \( X \) with rank \( 2n \), i.e., we are given a nowhere degenerating section of \( \wedge^2 E \). For a subbundle \( F \) of \( E \), we denote by \( F^\perp \) the orthogonal complement of \( F \) with respect to the symplectic form. Fix a complete flag \( F^\bullet \) of subbundles of \( E \):

\[
0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-n} = E,
\]

such that \( \text{rk} F^i = n - i \) and \( (F^i)^\perp = F^{-i} \) for all \( i \). The condition implies that \( F^i \) is isotropic for each \( i \geq 0 \). In particular, \( F^0 \) is Lagrangian. Let \( LG(E) \to X \) be the Lagrangian Grassmannian bundle over \( X \), i.e., the fiber at \( x \in X \) is the Grassmannian \( LG(E_x) \) of the Lagrangian subspaces of \( E_x \). We suppress the notation for the pullback of vector bundles as before. Let \( L \) be the tautological vector bundle over \( LG(E) \).

Let \( \mathcal{S} \) be the set of all strict partitions, i.e. \( \lambda \in \mathcal{S} \) is an infinite sequence \( (\lambda_1, \lambda_2, \cdots) \) of nonnegative integers such that all but finitely many \( \lambda_i \)'s are zero, and such that \( \lambda_i > 0 \) implies \( \lambda_i > \lambda_{i+1} \). The length of \( \lambda \) is the number of nonzero parts. The subset \( \mathcal{S}_n \subset \mathcal{S} \) consists of all strict partitions with at most \( r \) parts. We often write an element \( \lambda \) of \( \mathcal{S}_n \) by \( (\lambda_1, \ldots, \lambda_n) \). We also consider the subset \( \mathcal{S}_n(n) \subset \mathcal{S}_n \) consisting of all strict partitions such that \( \lambda_1 \leq n \).

Let \( \lambda \in \mathcal{S}_n(n) \). Suppose that the length of \( \lambda \) is \( r \). Consider the partial flag of \( F^\bullet \)

\[
F^\lambda_\bullet: F^{\lambda_1-1} \subset \cdots \subset F^{\lambda_r-1} \subset F^0.
\]
Definition 5.1. Define the Lagrangian degeneracy loci $\Omega_\lambda \subset LG(E)$ by

$$\Omega_\lambda = \{(L_x,x) \in LG(E) \mid \dim(F_x^\lambda - L_x^\lambda) \geq \} = \infty, \ldots, \nabla)\}.$$ 

To describe the formula for the class $[\Omega_\lambda]$, we first recall the definition of $\mathcal{L}^R$ and $\phi$ from Definition 4.1 Let $\mathcal{L}^R := \mathcal{L}_m(L^\lambda - (E/F)^\lambda) \subset CK^*(LG(E))$. Let $R := CK^*(LG(E))$ and define $\mathcal{L}_m^R$ to be $\mathcal{L}_m^{\lambda_i-1}$ for each $i = 1,2,\ldots,n$. Namely we have the $CK^*(LG(E))$-module homomorphism

$$\phi : \mathcal{L}^{CK^*(LG(E))} \rightarrow CK^*(LG(E))$$

sending $\tau_1^m \cdots \tau_n^m$ to $\mathcal{L}_m^{\lambda_1-1} \cdots \mathcal{L}_m^{\lambda_n-1}$. Now we can state the main result of this section.

Theorem 5.2. Let $\lambda \in SP(n)$ and $\Omega_\lambda$ its associated degeneracy locus in $LG(E)$. The class associated to $\Omega_\lambda$ in $CK^*(LG(E))$ is given by

$$[\Omega_\lambda] = \phi \left( \tau_1^{\lambda_1} \cdots \tau_n^{\lambda_n} \prod_{1 \leq i < j \leq n} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right).$$

Furthermore, for any $m$ such that $\lambda \in SP_{2m}$, we have

$$[\Omega_\lambda] = \text{Pf} \left( \phi \left( \tau_1^{\lambda_1} \cdots \tau_2^{\lambda_2} \left(1 + \beta \tau_2\right)^{2m} \left(1 + \beta \tau_j\right)^{2m} \frac{\tau_j - \tau_i}{\tau_j + \tau_i} \right) \right)_{1 \leq i < j \leq 2m}.$$

5.2 Tower of projective bundles

As in [10] we regard $E$ as a bundle over $LG(E)$ by pulling back. Associated to the partial flag $F_\lambda^*$ of $E$ is a generalized flag bundle $\varpi : Fl_\lambda(E) \rightarrow LG(E)$. It is worth stressing that we do not use the isotropic condition to define this generalized flag bundle $Fl_\lambda(E)$.

We construct a sequence of subvarieties $Y_r \subset \cdots \subset Y_1 \subset Y_0 := Fl_\lambda(E)$ inductively as follows. First let $Y_1 \subset Fl_\lambda(E)$ be the locus where the bundle map $D_1 \rightarrow E/L$ has rank 0, i.e. the corresponding section of $D_1^\vee \otimes E/L$ vanishes. Next let $i \geq 2$. By the natural inclusions $F_\lambda^{\lambda_i-1} \subset (F_\lambda^{\lambda_i-1})^\perp$ and $D_i \subset D_{i-1}^\perp$, we have the bundle map $(D_i/D_{i-1})|_{Y_{i-1}} \rightarrow (D_{i-1}^\perp/L)|_{Y_{i-1}}$ over $Y_{i-1}$. Let $Y_i$ be the locus where this bundle map has rank 0, i.e. the corresponding section of $(D_i/D_{i-1})^\vee \otimes D_{i-1}^\perp/L$ vanishes. It is easy to show that $Y_i$ is smooth of codimension $n-i+1$ and that $Y_r$ is birational to $\Omega_\lambda$ along $\varpi$ (cf. Remark 6.8).

Lemma 5.3. The class of $\Omega_\lambda$ in $CK^*(LG(E))$ is given by

$$[\Omega_\lambda] = \varpi_*([Y_i]).$$

Proof. The proof is exactly the same as Lemma 4.2. \hfill \Box

By Lemma 2.2 we find that $[Y_i] = c_{n-i+1}((D_i/D_{i-1})^\vee \otimes D_{i-1}^\perp/L) \in CK^*(Y_{i-1})$ for all $i = 1,\ldots,r$ where $D_0 = 0$. As in 4.3 the class of $Y_r$ in $CK^*(Fl_\lambda(E))$ is given by

$$[Y_r] = \prod_{i=1}^r c_{n-i+1}((D_i/D_{i-1})^\vee \otimes D_{i-1}^\perp/L) \in CK^*(Fl_\lambda(E)).$$
As in \[\text{Lemma 5.1}\] the flag bundle \( Fl_\lambda(E) \) can be obtained as a tower of projective bundles

\[
\varpi: Fl_\lambda(E) = \mathbb{P}(F^{\lambda_r-1}/D_{r-1}) \xrightarrow{\varpi_r} \mathbb{P}(F^{\lambda_{r-1}-1}/D_{r-2}) \xrightarrow{\varpi_{r-1}} \cdots \xrightarrow{\varpi_2} \mathbb{P}(F^{\lambda_2-1}/D_1) \xrightarrow{\varpi_1} \mathbb{P}(F^{\lambda_1-1}) \xrightarrow{\varpi_0} LG(E).
\]

**Lemma 5.4.** Let \( \tau_i := c_1((D_i/D_{i-1})^\vee) \). We have

\[
\varpi_{i*} \left( \tau_i^s c_{n-i+1}((D_i/D_{i-1})^\vee \otimes D_{i-1}^+/L) \right) = \sum_{p=0}^\infty c_p(D_{i-1} - D_{i-1}^+) \sum_{q=0}^{p} \binom{p}{q} \beta q \mathcal{A}_{\lambda_i+s-p+q}(L^\vee - (E/F^{\lambda_i-1})^\vee),
\]

where we have used the identity \( D_{i-1}^+ = E - D_{i-1}^\vee \) in the third equality.

**Proof.** We apply Lemma 3.6

\[
\varpi_{i*} \left( \tau_i^s c_{n-i+1}((D_i/D_{i-1})^\vee \otimes D_{i-1}^+/L) \right) = \mathcal{A}_{\lambda_i+s}((F^{\lambda_i-1}/D_{i-1})^\vee - (D_{i-1}^+/L)^\vee) = \mathcal{A}_{\lambda_i+s}((L - E/F^{\lambda_i-1})^\vee - (D_{i-1} - D_{i-1}^\vee)^\vee) = \sum_{p=0}^\infty c_p(D_{i-1} - D_{i-1}^+) \sum_{q=0}^{p} \binom{p}{q} \beta q \mathcal{A}_{\lambda_i+s-p+q}(L^\vee - (E/F^{\lambda_i-1})^\vee),
\]

5.3 **Algebraic pushforward in the Lagrangian case**

In this section, we generalize Kazarian’s algebraic formulation in the Lagrangian case to the context of connective K-theory. We use the same notation from Definition 4.10. We follow \[\text{Lemma 4.11}\] starting with replacing \[\text{Lemma 4.14}\] by \[\text{Lemma 5.1}\]. Then we will develop the results analogous to \[\text{Lemma 4.11}\], Proposition 4.12 and Lemma 4.15. Assume that \( R \) is a graded ring over \( \mathbb{Z}[\beta] \) as above. For each \( m \in \mathbb{Z} \) and \( 1 \leq i \leq r \), fix a homogeneous element \( \mathcal{A}_m^{[i]} \in R_m \). We assume that \( \mathcal{A}_m^{[i]} \) is zero for all sufficiently large \( m \). In this section, we regard \( \tau_i \)'s as indeterminates.

**Definition 5.5** (Algebraic pushforward in the Lagrangian case). For each \( \lambda \in SP_r \) and \( i = 1, \ldots, r \), we define a graded \( R[[\tau_1, \ldots, \tau_{i-1}]]_{gr} \)-module homomorphism

\[
p_i : R[[\tau_1, \ldots, \tau_i]]_{gr} \to R[[\tau_1, \ldots, \tau_{i-1}]]_{gr}
\]

by sending \( \tau_i^s \) to

\[
\sum_{p=0}^\infty \left( \sum_{s=0}^p (-1)^p e_{p-s}(\bar{\tau}_1, \ldots, \bar{\tau}_{i-1}) h_p(\tau_1, \ldots, \tau_{i-1}) \right) \sum_{q=0}^{p} \binom{p}{q} \beta q \mathcal{A}_i^{[i]} \lambda_i+s-p+q
\]

for all \( s \geq 0 \).

This definition of \( p_i \) should be compared with the pushforward formula in Lemma 5.2. See also Remark 2.1 and note that we regard \( \bar{\tau}_1, \ldots, \bar{\tau}_{i-1} \) and \( \tau_1, \ldots, \tau_{i-1} \) as Chern roots of \( D_{i-1} \) and \( D_{i-1}^\vee \) respectively.

**Lemma 5.6.** We have

\[
p_i (\tau_i^s) = \phi_i \left( \tau_i^{\lambda_i+s} \prod_{s=1}^{i-1} \frac{1 - \tau_s/\bar{\tau}_i}{1 - \tau_s/\bar{\tau}_i} \right)
\]

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Proof. It is similar to the proof of Lemma 4.11.

**Proposition 5.7.** We have  
\[ p_1 \cdots p_r(1) = \phi_1 \left( \tau_1^{\lambda_1} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right), \]

Proof. Similar to the proof of Proposition 4.12. The consecutive application of Lemma 5.6 implies the claim.

**Lemma 5.8.** Assume that \( D^i_m = (-\beta)^{-m} \) for each \( m \in \mathbb{Z}_{\geq 0} \) for all \( i = 1, \ldots, r \). If \( \lambda_{s+1} = \cdots = \lambda_r = 0 \) for some \( s < r \), then we have \( p_1 \cdots p_r(1) = p_1 \cdots p_s(1) \).

Proof. Similar to the proof of Lemma 4.15.

By Lemma 5.8 the element \( C_\lambda := p_1 \cdots p_r(1) \) does not depend on \( r \). In other words, we can define the element \( C_\lambda \) canonically for each \( \lambda \in S_P \). Finally we write \( C_\lambda \) as a Pfaffian. The next proposition follows from the lemma below.

**Proposition 5.9.** Let \( \lambda \in S_P \). For any \( m \) such that \( \lambda \in S_{P2m} \), we have  
\[ C_\lambda = \text{Pf} \left( \phi_1 \left( \tau_1^{\lambda_1} \tau_2^{\lambda_2} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right) \right)_{1 \leq i < j \leq 2m}. \]

**Lemma 5.10.** Let \( m \geq 0 \) and \( \lambda = (\lambda_1, \ldots, \lambda_{2m}) \) any sequence of integers.

\[ \prod_{1 \leq i < j \leq 2m} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} = \text{Pf} \left( \phi_1 \left( \tau_1^{\lambda_1} \tau_2^{\lambda_2} \cdots \tau_r^{\lambda_r} \prod_{1 \leq i < j \leq r} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right) \right)_{1 \leq i < j \leq 2m}. \]

Proof. Recall from [12] Ikeda-Naruse, Lemma 2.4 that  
\[ \text{Pf} \left( \frac{x_j - x_i}{x_j + x_i} \right)_{1 \leq i < j \leq 2m} = \prod_{1 \leq i < j \leq 2m} \frac{x_j - x_i}{x_j + x_i}. \]

Note that we multiplied \(-1\) to each factor in the original formula, but the equality still holds. We can check the following by a direct computation:

\[ (1 + \beta x)(1 + \beta \bar{x}) = 1: \quad \frac{1 - \beta x/y}{1 - x/y} = (1 + \beta x) \frac{y - \bar{x}}{y + \bar{x}}. \]

Now compute:

\[ \prod_{1 \leq i < j \leq 2m} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} = \prod_{1 \leq i < j \leq 2m} \frac{\bar{\tau}_j - \bar{\tau}_i}{\bar{\tau}_j + \bar{\tau}_i}, \]

\[ = \prod_{1 \leq i < j \leq 2m} \frac{(1 + \beta \bar{\tau}_i) \bar{\tau}_j - \bar{\tau}_i}{(1 + \beta \bar{\tau}_i) \bar{\tau}_j + \bar{\tau}_i}, \]

\[ = \prod_{1 \leq i < j \leq 2m} \frac{(1 + \beta \bar{\tau}_i) \bar{\tau}_j - \bar{\tau}_i}{(1 + \beta \bar{\tau}_i) \bar{\tau}_j + \bar{\tau}_i}, \]

\[ = \text{Pf} \left( \phi_1 \left( \frac{(1 + \beta \bar{\tau}_i) \bar{\tau}_j - \bar{\tau}_i}{(1 + \beta \bar{\tau}_i) \bar{\tau}_j + \bar{\tau}_i} \right) \right)_{1 \leq i < j \leq 2m}. \]
5.4 Proof of Theorem 5.2

From Lemma 5.3 and 5.4, combined with the results in §5.3 we obtain Theorem 5.2.

6 Pfaffian sum formula for the submaximal symplectic case

In this section, we fix a nonnegative integer $k \geq 0$. The case when $k = 0$ is exactly the Lagrangian case in the previous section.

6.1 $k$-strict partitions and its characteristic index

Let $SP^k$ be the set of all $k$-strict partitions, i.e. $\lambda \in SP^k$ is an infinite sequence $(\lambda_1, \lambda_2, \ldots)$ of non-increasing nonnegative integers such that all but finitely many $\lambda_i$’s are zero, and such that $\lambda_i > k$ implies $\lambda_i > \lambda_{i+1}$. Let $SP^k_r$ be the subset of $SP^k$ consisting of all $k$-strict partitions with length at most $r$. If $\lambda \in SP^k_r$, then we write $\lambda = (\lambda_1, \ldots, \lambda_r)$. We also consider the subset $SP^k_r(n)$ of $SP^k_{n-k}$, which consists of all $k$-strict partitions such that $\lambda_1 \leq n + k$.

For $\lambda \in SP^k$, define
\[
C(\lambda) = \{(i, j) \mid 0 < i < j, \lambda_i + \lambda_j > 2k + j - i\}
\]

For each $j > 0$, let
\[
\gamma(j) := \sharp\{i \mid 1 \leq i < j, \lambda_i + \lambda_j > 2k + j - i\}.
\]

The characteristic index $\chi = (\chi_1, \chi_2, \ldots)$ ([8, §1.4 and Equation (3.1)]) associated to $\lambda$ is defined by
\[
\chi_j := \lambda_j - j + \gamma(j) - k. \tag{6.1}
\]

If $\lambda = (\lambda_1, \ldots, \lambda_r) \in SP^k_r$, then the associated characteristic index is denoted by $\chi = (\chi_1, \ldots, \chi_r)$. Note that, for $i < j$, we have $\chi_i + \chi_j \geq 0$ if and only if $\lambda_i + \lambda_j > 2k + j - i$. See [8, Lemma 3.3].

Example 6.1. For example, consider $\lambda = (4, 3, 2)$ in $SP^2(5)$. Its characteristic index is $\chi = (1, 0, -3)$.

Remark 6.2. If $k = 0$, then $\chi_i = \lambda_i - 1$ for all $i = 1, \ldots, r$.

6.2 Submaximal symplectic degeneracy loci

Let $E$ be a symplectic vector bundle over a smooth quasi-projective variety $X$ with rank $2n$. Fix a complete flag $F^\bullet$ of subbundles of $E$
\[
0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-n} = E,
\]
such that $\text{rk} F^i = n - i$ and $(F^i)^\perp = F^{-i}$ for all $i$. Let $SG^k(E) \to X$ be the symplectic Grassmannian bundle over $X$ such that the fiber at $x \in X$ is the Grassmannian $SG^k(E_x)$ of $(n - k)$-dimensional isotropic subspaces of $E_x$. We suppress the notation for the pullback of vector bundles as before. Let $U$ be the tautological vector bundle over $SG^k(E)$.

Let $\lambda \in SP^k_r(n)$ and $\chi$ its characteristic index. Let $r$ be the length of $\lambda$. Consider the partial flag of $F^\bullet$
\[
F^\lambda_\chi : F^{\chi_1} \subset \cdots \subset F^{\chi_r} \subset E.
\]
Definition 6.3. Define the symplectic degeneracy loci $\Omega_\lambda \subset SG^k(E)$ by
\[
\Omega_\lambda = \{(x, U_x) \in SG^k(E) \mid \dim(F_x^{i_i} \cap U_x) \geq i, \ i = 1, \ldots, r\}. \tag{6.2}
\]

Let
\[
\mathcal{C}_m^\ell := \mathcal{C}_m(U^\vee - (E/F^\ell)^\vee) \in CK^*(SG^k(E)). \tag{6.3}
\]

To describe the formula for the class $[\Omega_\lambda]$, we recall the definition of $L^R$ and $\phi$ from Definition 4.10. Let $R := CK^*(SG^k(E))$ and define $\mathcal{R}_m^\ell$ to be $\mathcal{C}_m^\ell$ for each $i = 1, 2, \ldots, n - k$. Namely, we have the $CK^*(SG^k(E))$-module homomorphism
\[
\phi : L^{CK^*(SG^k(E))} \to CK^*(SG^k(E))
\]
sending $\tau_1^m \cdots \tau_{n-k}^m$ to $\mathcal{C}_m^{(\lambda_1)} \cdots \mathcal{C}_m^{(\lambda_{n-k})}$.

Remark 6.4. By (3.3) and Remark 3.5, these classes $\mathcal{C}_m^\ell$ are combined also into the following generating function:
\[
\sum_{m \in \mathbb{Z}} \mathcal{C}_m^\ell u^m = \frac{1}{1 + \beta u - 1} e(E - U - F^\ell; u + \beta). \tag{6.4}
\]

Now we can state the main result of this section.

Theorem 6.5. Let $\lambda \in SP^k(n)$, $\chi$ its characteristic index, and $\Omega_\lambda$ the associated degeneracy locus in $SG^k(E)$. The class associated to $\Omega_\lambda$ in $CK^*(SG^k(E))$ is given by
\[
[\Omega_\lambda] = \sum_{I \subset D(\lambda)} \phi \left( \tau_1^\lambda \cdots \tau_{n-k}^\lambda \left( \prod_{(i,j) \in I} (-\tau_i/\bar{\tau}_j) \right) \left( \prod_{(i,j) \in \Delta} \frac{1 - \tau_i/\bar{\tau}_j}{1 - \beta \tau_i/\bar{\tau}_j} \right) \right), \tag{6.5}
\]
where $\Delta = \{(i, j) \mid 1 \leq i < j \leq n - k\}$ and $D(\lambda) := \{(i, j) \in \Delta \mid \chi_i + \chi_j < 0\}$. Furthermore, for any $m$ such that $\lambda \in SP^k_2$, we have
\[
[\Omega_\lambda] = \sum_{I \subset D(\lambda)} \text{Pf} \left( \Lambda_{ij}^{\lambda, I} \right)_{1 \leq i < j \leq 2m}, \tag{6.6}
\]
where we write $\lambda = (\lambda_1, \ldots, \lambda_{2m})$ and
\[
\Lambda_{ij}^{\lambda, I}(i, j) = \phi \left( \tau_i^{\lambda_i + a_i} (-\bar{\tau}_i)^{-c_i} \tau_j^{\lambda_j + a_j} (-\bar{\tau}_j)^{-c_j} \right) \frac{1 - \lambda_i}{1 + \beta \tau_i/\bar{\tau}_j} \frac{1 - \lambda_j}{1 + \beta \tau_j/\bar{\tau}_i} \tag{6.7}
\]

6.3 Tower of projective bundles

As in [4.11] we regard $E$ as a bundle over $SG^k(E)$ by pullback and, associated to the partial flag $F_{\lambda}^\ast$ of $E$, there is a generalized flag bundle $\varpi : F_{\lambda}(E) \to SG^k(E)$. Here we do not use the isotropic condition to define this generalized flag bundle $F_{\lambda}(E)$. As in [4.11] the flag bundle $F_{\lambda}(E)$ can be obtained as a tower of projective bundles
\[
\varpi : F_{\lambda}(E) = \mathbb{P}(F^x/D_{r-1}) \xrightarrow{\varpi_1} \mathbb{P}(F^x/D_{r-2}) \xrightarrow{\varpi_2} \cdots \xrightarrow{\varpi_k} \mathbb{P}(F^x/D_1) \xrightarrow{\varpi_k} \mathbb{P}(F^x) \xrightarrow{\varpi_k} SG^k(E).
\]
We construct a sequence of subvarieties \( Y_r \subset \cdots \subset Y_1 \subset Y_0 := Fl_\lambda(E) \) inductively as follows. First define \( Y_1 \subset Fl_\lambda(E) \) to be the locus where the bundle map \( D_1 \to E/U \) has rank 0, i.e. the corresponding section of \( D_1^\perp \otimes (E/U) \) vanishes. Next let \( i \geq 2 \). If \( \chi_j + \chi_i \geq 0 \) with \( j < i \), then \( F^{\chi_i} \subset (F^{\chi_j})^\perp \), and hence we have \( D_i \subset D_j^\perp \). Furthermore, if \( \chi_j + \chi_i < 0 \), then \( D_i \) is not a subbundle of \( D_j^\perp \). Recall that
\[
\gamma(i) = \sharp\{j \mid 1 \leq j < i, \chi_j + \chi_i \geq 0\}.
\]

We have the bundle map \( (D_i/D_{i-1})|_{Y_{i-1}} \to (D_j^\perp/\gamma(i))/U_{i-1} \) over \( Y_{i-1} \). Let \( Y_i \) be the locus where this bundle map has rank 0, i.e. \( Y_i \) is the zero scheme of the corresponding section of \( (D_i/D_{i-1})^\gamma \otimes D_j^\perp/\gamma(i)/U \).

**Lemma 6.6.** For each \( i = 1, \ldots, r \), \( Y_i \) is irreducible and has at worst rational singularity. In particular, \( Y_r \) is birational to \( \Omega_\lambda \) through the projection \( \varpi \).

**Proof.** Consider the following \( r \)-step isotropic partial flag bundle \( Fl(C_{1,2,\ldots,r})(E) \) over \( X \): the fiber at \( x \in X \) consists of flags \( (C_\bullet)_x \) of isotropic subspaces \( (C_1)_x \subset \cdots \subset (C_r)_x \subset E_x \) such that \( \dim(C_i)_x = i \). Let \( Z \) be the following \( SG^k(C_2^{(n-r)}) \)-bundle over \( Fl(C_{1,2,\ldots,r})(E) \)
\[
Z = \{(G_\bullet)_x, X_x) \mid ((C_\bullet)_x, x) \in Fl(C_{1,2,\ldots,r})(E), X_x \in SG^k((U_r^+/U_r)_x)\}.
\]

Let \( W_{\lambda,i} \) be the degeneracy locus in \( Fl(C_{1,2,\ldots,r})(E) \) defined by
\[
W_{\lambda,i} := \{(G_\bullet)_x, x) \in Fl(C_{1,2,\ldots,r})(E) \mid \dim(F_x^s \cap (C_s)_x) \geq s, \quad s = 1, \ldots, i\}.
\]

Then we can show that the variety \( Y_i \) is exactly the total space \( Z\mid_{W_{\lambda,i}} \) of the restriction of the bundle \( Z \) to \( W_{\lambda,i} \). Indeed, we have
\[
Y_i = \{(D_\bullet)_x, x, U_x) \in Fl(E) \mid (D_\bullet)_x \subset U_x \cap F_x^s, s = 1, \ldots, i\},
\]
and the bijection is given by
\[
Z\mid_{W_{\lambda,i}} \to Y_i; \quad ((C_\bullet)_x, x, V_x) \mapsto ((C_\bullet)_x, x, \overline{V}_x),
\]
where the \((n-k)\)-dimensional isotropic subspace \( \overline{V}_x \subset E_x \) is defined as the preimage of \( V_x \) under the quotient map \( (U_r^+) \to (U_r^+/U_r)_x \). It follows from a well-known fact about the Schubert varieties that the variety \( W_{\lambda,i} \) is irreducible and has at worst rational singularity (cf. \cite[p.274, 8.2.2. Theorem (c)]{IS}). Therefore \( Y_r \) is irreducible and has at worst rational singularity as well.

The restriction of \( \varpi \) to \( Y_r \) maps to \( \Omega_\lambda \). Consider the the big cell
\[
\Omega_\lambda^r := \{(x, U_x) \in SG^k(E) \mid \dim(F_x^s \cap U_x) = i, \dim(F_x^{s+1} \cap U_x) = i - 1 \ (i = 1, \ldots, r)\}.
\]

Let \( Y_r^\varpi \) be the preimage of \( \Omega_\lambda^r \) by \( \varpi|_{Y_r} : Y_r \to \Omega_\lambda \). Then it is easy to see that \( \varpi|_{Y_r} : Y_r^\varpi \to \Omega_\lambda^r \) is bijective. Thus the irreducibility of \( Y_r \) and \( \Omega_\lambda \) implies that \( \varpi|_{Y_r} \) is birational.

**Lemma 6.7.** The codimension of \( Y_i \) in \( Y_{i-1} \) is \( n + k - \gamma(i) \).
Proof. It follows from Lemma 6.6 that the image $Y_i'$ of $Y_i$ under $\varpi_{i+1} \varpi_{i+2} \cdots \varpi_r$ is birational to $\Omega_{(\lambda_1, \ldots, \lambda_i)}$. Since the dimension of the fiber of $\varpi_i$ is $n - \chi_i - s$, we can compute that the codimension of $Y_i$ in $Fl_\lambda(E)$ is
\[
\sum_{s=1}^{i} (\lambda_s + n - \chi_s - s)
\]
Thus the codimension of $Y_i$ in $Y_{i-1}$ is $\lambda_i + n - \chi_i - i$ which equals to $n + k - \gamma(i)$ by the definition of $\chi$.

Remark 6.8. Recall that, if $k = 0$, then $\chi_i = \lambda_i - 1$ for all $i = 1, \ldots, r$. In this case, $F^{\lambda_i-1}$ is isotropic for all $i$. Therefore the projection from $W_{\lambda,i}$ to the generalized flag bundle over $X$ associated to the partial flag $F^{\lambda_i-1} \subset \cdots \subset F^{\lambda_1-1}$ is surjective and it defines a bundle with fibers isomorphic to $Fl_{(1,2,\ldots,r-i)}(\mathbb{C}^{2(n-i)})$. Thus $W_{\lambda,i}$ is smooth. This implies that $Z|_{W_{\lambda,i}}$ is smooth and so is $Y_i$.

Lemma 6.9. The class of $\Omega_\lambda$ in $CK^*(SG^k(E))$ is given by
\[
[\Omega_\lambda] = \varpi_*(\langle Y_r \rangle).
\]
Proof. By Hironaka’s theorem (see [21, Appendix]), there is a projective birational map $Z \to Y_r$ such that $Z$ is smooth. Since $\Omega_\lambda$ has at worst rational singularities ([13, Section 8.2.2, Theorem (c), p.274]) and $Y_r$ as well, we can conclude that $\varpi_*(\langle Y_r \rangle) = [\Omega_\lambda]$ (see [6, Lemma 2.2]).

By Lemma 2.2, we find that $[Y_i] = c_{n+k-\gamma(i)}((D_i/D_{i-1})^\vee \otimes D_{\gamma(i)}^\perp/U) \in CK^*(Y_{i-1})$ for all $i$. Similarly to (6.1), the class of $Y_r$ in $CK^*(Fl_\lambda(E))$ is given by
\[
[Y_r] = \prod_{i=1}^{r} c_{n+k-\gamma(i)}((D_i/D_{i-1})^\vee \otimes D_{\gamma(i)}^\perp/U).
\]

Lemma 6.10. Let $\tau_i := c_1((D_i/D_{i-1})^\vee)$. We have
\[
\varpi_*(\tau_i^{\vee} c_{n+k-\gamma(i)}((D_i/D_{i-1})^\vee \otimes D_{\gamma(i)}^\perp/U))
= \sum_{p=0}^{\infty} c_p(D_{i-1} - D_{\gamma(i)}^\perp) \sum_{q=0}^{p} \left(\frac{p}{q}\right) \beta_q \mathcal{S}_{\lambda,s-p+q}(U^\perp - (E/F^x)^\perp).
\]

Proof. By using the definition of $\chi$ in (6.1), the identity $D_{\gamma(i)}^\perp = E - D_{\gamma(i)}^\perp$, and Lemma 3.4, we can compute
\[
\varpi_*(\tau_i^{\vee} c_{n+k-\gamma(i)}((D_i/D_{i-1})^\vee \otimes D_{\gamma(i)}^\perp/U))
= \mathcal{S}_{\lambda,s}(F^x_i/D_{i-1})^\perp - (D_{\gamma(i)}^\perp/U)^\perp
= \mathcal{S}_{\lambda,s}(U - E/F^x)^\vee - (D_{i-1} - D_{\gamma(i)}^\perp)^\vee
= \sum_{p=0}^{\infty} c_p(D_{i-1} - D_{\gamma(i)}^\perp) \sum_{q=0}^{p} \left(\frac{p}{q}\right) \beta_q \mathcal{S}_{\lambda,s-p+q}(U^\perp - (E/F^x)^\perp).
\]

\]
6.4 Algebraic pushforward in the submaximal symplectic case

We generalize the results in [6.3] to the submaximal isotropic Grassmannian. We use the same notation from Definition 4.10 and follow [4.4]. Assume that \( R \) is a graded ring over \( \mathbb{Z}[\beta] \) as above. For each \( m \in \mathbb{Z} \) and \( 1 \leq i \leq r \), fix a homogeneous element \( R^{[i]}_m \in R_m \). We assume that \( R^{[i]}_m \) is zero for all sufficiently large \( m \). In this section, we regard \( \tau_i \)'s as indeterminates.

**Definition 6.11** (Algebraic pushforward for the submaximal symplectic case). For \( \lambda \in SP^k \), we define a graded \( R[[\tau_1, \ldots, \tau_{i-1}]]_{\text{gr}} \)-module homomorphism

\[
p_i : R[[\tau_1, \ldots, \tau_i]]_{\text{gr}} \to R[[\tau_1, \ldots, \tau_{i-1}]]_{\text{gr}}
\]

by sending \( \tau_i^s \) to

\[
\sum_{p=0}^{\infty} \left( \sum_{s=0}^{p} (-1)^p c_{p-s}(\tau_1, \ldots, \tau_{i-1}) h_p(\tau_1, \ldots, \tau_{\gamma(i)}) \right) \sum_{q=0}^{p} \left( \sum_{p-q} \right) \beta q \| \xi_{s-r-p} \| \quad (6.8)
\]

for all \( s \geq 0 \).

The definition of \( p_i \) should be compared with the pushforward formula in Lemma 6.10. See also Remark 2.1 and note that we regard \( \bar{\tau}_1, \ldots, \bar{\tau}_{i-1} \) and \( \tau_1, \ldots, \tau_{\gamma(i)} \) as Chern roots of \( D_{i-1} \) and \( D'_{i-1} \) respectively.

**Lemma 6.12.** We have

\[
p_i(\tau_i^s) = \phi_i \left( \tau_i^{\lambda_i+s} \prod_{s=1}^{i-1} \frac{(1 - \bar{\tau}_s/\bar{\tau}_1)}{1 - \tau_s/\tau_i} \right).
\]

**Proof.** Similar to the proof of Lemma 4.11. \( \square \)

**Proposition 6.13.** We have

\[
p_1 \cdots p_r(1) = \phi_1 \left( \tau_1^{\lambda_1} \cdot \tau_r^{\lambda_r} \prod_{(i,j) \in \Delta} \frac{(1 - \bar{\tau}_i/\bar{\tau}_j)}{1 - \tau_i/\tau_j} \right),
\]

where \( \Delta = \{(i, j) \mid 1 \leq i < j \leq r\} \) and \( C(\lambda) = \{(i, j) \in \Delta \mid \chi_i + \chi_j \geq 0\} \)

**Proof.** It is similar to the proof of Proposition 4.12. Indeed, by applying Lemma 6.12 consecutively, we have

\[
p_1 \cdots p_r(1) = p_1 \cdots p_{r-1} \phi_r \left( \tau_r^{\lambda_r} \frac{\prod_{s=1}^{r-1} (1 - \bar{\tau}_s/\bar{\tau}_r)}{\prod_{s=1}^{r} (1 - \tau_s/\tau_r)} \right)
\]

\[
= p_1 \cdots p_{r-2} \phi_{r-1} \left( \tau_{r-1}^{\lambda_{r-1}} \tau_r^{\lambda_r} \frac{\prod_{s=1}^{r-2} (1 - \bar{\tau}_s/\bar{\tau}_{r-1}) \prod_{s=1}^{r-1} (1 - \bar{\tau}_s/\bar{\tau}_r)}{\prod_{s=1}^{r-1} (1 - \tau_s/\tau_{r-1}) \prod_{s=1}^{r} (1 - \tau_s/\tau_r)} \right)
\]

\[
= \cdots = \phi_1 \left( \tau_1^{\lambda_1} \cdot \tau_r^{\lambda_r} \prod_{(i,j) \in \Delta} \frac{(1 - \bar{\tau}_i/\bar{\tau}_j)}{1 - \tau_i/\tau_j} \right).
\]

\( \square \)

**Lemma 6.14.** Assume that \( R^{[i]}_m = (-\beta)^{-m} \) for each \( m \in \mathbb{Z}_{\geq 0} \) for all \( i = 1, \ldots, r \). If \( \lambda_{s+1} = \cdots = \lambda_r = 0 \) for some \( s < r \), then we have \( p_1 \cdots p_r(1) = p_1 \cdots p_s(1) \).
Proof. Similar to the proof of Lemma 4.15.

By this lemma, the element $C_{\lambda} := p_1 \cdots p_r(1)$ does not depend on $r$. In other words, we can define the element $C_{\lambda}$ canonically for each $\lambda \in SP$.

The formula in Proposition 6.13 can also be written as

$$C_{\lambda} = \sum_{I \subseteq D(\lambda)} \phi_I \left( \prod_{(i,j) \in I} (-\tau_i/\tau_j) \right) \left( \prod_{(i,j) \in \Delta} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right)$$

where $D(\lambda) = \Delta \setminus C(\lambda)$. The following lemma generalizes Lemma 5.10 and allows us to write the formula in Proposition 6.13 as a sum of Pfaffians.

Lemma 6.15. Let $m \geq 0$ and $\lambda = (\lambda_1, \ldots, \lambda_{2m})$ any sequence of integers. Let $I$ be a subset of $\Delta_{2m} = \{(i,j) \mid 1 \leq i < j \leq 2m\}$. Let $a^I_{\lambda} := 2\{j \mid (i,j) \in I\}$ and $c^I_{\lambda} := 2\{i \mid (i,j) \in I\}$.

$$\tau_1^{\lambda_1} \cdots \tau_r^{\lambda_{2m}} \left( \prod_{(i,j) \in I} (-\tau_i/\tau_j) \right) \left( \prod_{(i,j) \in \Delta} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right) = \text{Pf} \left( \frac{-1}{1 + \beta \tau_i} \right)^{2m-i-1} \left( \frac{-1}{1 + \beta \tau_j} \right)^{2m-j} \left( \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right)^{1 \leq i < j \leq 2m}.$$

Proof. First note that $\prod_{(i,j) \in I} \tau_i(-\tau_j)^{-1} = \prod_{i=1}^{2m} \tau_i^{a^I_{\lambda} - c^I_{\lambda}}$.

$$\tau_1^{\lambda_1} \cdots \tau_r^{\lambda_{2m}} \left( \prod_{(i,j) \in I} (-\tau_i/\tau_j) \right) \left( \prod_{(i,j) \in \Delta} \frac{1 - \tau_i/\tau_j}{1 - \tau_i/\tau_j} \right) = \prod_{i=1}^{2m} \left( \frac{-1}{1 + \beta \tau_i} \right)^{2m-i} \prod_{1 \leq i < j \leq 2m} \frac{\tau_i - \tau_j}{\tau_i + \tau_j}.$$

6.5 Proof of Theorem 6.5

Theorem 6.5 follows from Lemma 6.9, Lemma 6.10 and the results in 6.4.
7 \textit{GΘ}-functions and the equivariant Schubert classes of $SG^k$

In this section, we introduce the \textit{factorial GΘ-functions} using the Pfaffian sum formula in terms of the \textit{equivariantly shifted one row functions} $\mathcal{G}^{(i)}_m$. We will show that they represent the Schubert classes in the torus equivariant connective $K$-theory of the symplectic Grassmannians. \textit{Throughout this section, we work over $\mathbb{Q}$ coefficients unless otherwise specified. In particular, $CK^*$ is always tensored with $\mathbb{Q}$.}

7.1 Torus equivariant $K$-theory of the symplectic Grassmannian

Let us consider $2n$-dimensional vector space $E = \mathbb{C}^{2n}$ with basis $\{e_i, e_i^* \mid 1 \leq i \leq n\}$ and the symplectic structure $\sum_{i=1}^n e_i^* \wedge e_i^*$, where $e_i^*$ and $e_i^*$ form the dual basis. Let $SG^k(n)$ be the symplectic Grassmannian of $n - k$ dimensional isotropic subspaces in $\mathbb{C}^{2n}$. One can regard $SG^k(n)$ as $SG^k(E)$ by setting $X$ to be a point and $E = \mathbb{C}^{2n}$. We can also realize $SG^k(n)$ as a homogeneous space of the symplectic group $Sp_{2n}(\mathbb{C})$. Let $T_n$ be a maximal torus of $Sp_{2n}(\mathbb{C})$.

We study the $T_n$-equivariant connective $K$-theory $CK^*_T(n) = (SG^k(n))$, following Krishna \cite{Krishna}. It is a graded algebra over the $T_n$-equivariant connective $K$-theory $CK^*_T(pt)$ of a point. First of all, we fix the identification of $CK^*_T(pt)$ with a ring of all graded formal power series as follows. Let $L_i := \mathbb{C} e_i$. Let $e_1, \ldots, e_n$ be the standard basis of the character group of $T_n$ such that $L_i$ is the $T_n$-equivariant line bundle over a point associated to $e_i$.

We have

$$CK^*_T(pt) \cong Q[[\beta]][[b_1, \ldots, b_n]]_\text{gr}; \quad c^*_T(L_i) \mapsto b_i$$

(7.1)

as algebras over $Q[[\beta]]_\text{gr}$ (Krishna \cite{Krishna} §2.6) where $c^T$ denote the $T_n$-equivariant Chern class.

\textbf{Remark 7.1.} After specializing at $\beta = 0$, $Q[[\beta]][b_1, \ldots, b_n]_\text{gr}$ becomes $Q[b_1, \ldots, b_n]$. If we specialize at $\beta = -1$, $Q[[\beta]][b_1, \ldots, b_n]_\text{gr}$ becomes $Q[[b_1, \ldots, b_n]]$ which can be identified with the completion of the representation ring $R(T_n) = K(BT_n)$ (cf. Krishna \cite{Krishna} Theorem 7.3).

Let $F^i$ be $\text{Span}_\mathbb{C}\{e_n, \ldots, e_{i+1}\}$ and $F^{i-1} = F^i_1$ for each $i = 0, \ldots, n$. Then $F^*$ is a flag of $E$ satisfying the condition in (5.1). We have $b_i = c^*_T(F^{i-1}/F^i) \in CK^*_T(SG^k(n))$ for all $i = 1, \ldots, n$.

For each $\lambda \in SP^k(n)$, the corresponding Schubert variety $\Omega_\lambda$ is defined by (6.2) and it is $T_n$-stable. Let $[\Omega_\lambda]_{T_n}$ denote the $T_n$-equivariant fundamental class of $\Omega_\lambda$. As $CK^*_T(pt)$-module, $CK^*_T(SG^k(n))$ is freely generated by $[\Omega_\lambda]_{T_n}, \lambda \in SP^k(n)$.

Let $U$ be the tautological isotropic bundle over $SG^k(n)$. For each $\ell \in \mathbb{Z}$, define the classes $c_m^{(l)} \in CK^*_T(SG^k(n))$ for all $m \in \mathbb{Z}$ by

$$\sum_{m \in \mathbb{Z}} c^{(l)}_m m = \frac{1}{1 + \beta u - 1} c^T(E - U - F^\ell, u + \beta).$$

(7.2)

We can regard $c_m^{(l)}$ as the equivariant Segre classes by Remark 6.4.

\textbf{Proposition 7.2.} $[\Omega_\lambda]_{T_n}$ is given by the formula (6.3) with $\mathcal{A}^{[i]} = c_m^{(\chi_i)}$ defined by (6.4).

\textbf{Proof.} Let $BT_n$ be the classifying space of $T_n$ and $ET_n \to BT_n$ the universal bundle. Let $\mathcal{E} := ET_n \times_{T_n} \mathbb{C}^{2n}$. Let $SG^k(\mathcal{E}) := ET_n \times_{T_n} SG^k(n)$. We can apply Theorem 6.5 to every finite approximation of $ET_n \to BT_n$, i.e. we let $X$ and $E$ be approximations of $BT_n$ and $\mathcal{E}$.

By the functoriality of Chern classes, we obtain the claim. \qed
7.2 Factorial $G\Theta$-functions

In this section, we first introduce a graded ring $G\Gamma_+$ defined by a $K$-theoretic cancelation property. It is a completion of the ring of $GQ$-functions defined in $[3]$. In an extension of this fundamental ring, we define rings $G\Gamma_+^{(k)}$ and $G\mathcal{R}_k^{(k)}$ which dominate the connective $K$-theory of the symplectic Grassmannian. Then we define the factorial $G\Theta$-functions. They are canonical elements of $G\mathcal{R}_k^{(k)}$ representing the equivariant $K$-theoretic Schubert classes of the symplectic Grassmannian.

7.2.1 The rings $G\Gamma_+, G\Gamma_+^{(k)}$, and $G\mathcal{R}_k^{(k)}$

Definition 7.3 ($G\Gamma_+$). The subalgebra $G\Gamma_+ \subset \mathbb{Q}[\beta][[x_1, x_2, \ldots]]_{gr}$ is defined by the following conditions: $f(x) \in G\Gamma_+$ if and only if

1. $f$ is symmetric in $x$
2. $t \oplus t$ divides $f(t, x_2, x_3, \ldots) - f(0, x_2, x_3, \ldots)$.
3. $f(t, t, x_3, x_4, \ldots) = f(0, 0, x_3, x_4, \ldots)$

Definition 7.4 ($G\Gamma_+^{(k)}$ and $G\mathcal{R}_k^{(k)}$). Define the operators $s_i^{(a)}$, $i = 1, 2, \ldots, k - 1$, acting $G\Gamma_+^{(k)}$ as follows: $s_i^{(a)}$, $i \geq 1$ only switches $a_i$ and $a_{i+1}$ and

$$s_i^{(a)} (f(x_1, x_2, \ldots, a_1, a_2, \ldots, a_k)) = f(a_1, x_1, x_2, \ldots, a_i, a_{i+1}, \ldots, a_k).$$

Define the graded subring $G\Gamma_+^{(k)} \subset G\Gamma_+[[a_1, \ldots, a_k]]_{gr}$ by the following condition: $f(x; a) \in G\Gamma_+^{(k)}$ if and only if $s_i^{(a)} f = f$ for all $i = 1, 2, \ldots, k - 1$. Let

$$G\mathcal{R}_k^{(k)} := G\Gamma_+^{(k)}[[b_1, b_2, \ldots]]_{gr}.$$

Remark 7.5. $G\mathcal{R}_k^{(k)}$ is the $K$-theoretic version of $G\mathcal{R}_k^{(k)}$ introduced in $[8]$. $G\mathcal{R}_k^{(k)}$ is the invariant subring of the ring $G\mathcal{R}_k^{(k)}$ of the double Schubert polynomials of type $C_k$ with respect to the right action of $W_k$ in $[9]$. It is natural to consider $G\mathcal{R}_k^{(k)} := G\Gamma_+^{(k)}[[a_1, a_2, \ldots; b_1, b_2, \ldots]]_{gr}$ in order to study torus equivariant connective $K$-theory of the flag variety of type $C$. We address this issue elsewhere.

7.2.2 Definition of $G\Theta_\lambda$

Let $x = (x_1, x_2, \ldots)$, $a = (a_1, a_2, \ldots, a_k)$, and $b = (b_1, b_2, \ldots)$ be sequences of indeterminates.

Definition 7.6 (Equivariantly shifted $G\Theta$-functions of one row shape). Fix $k \geq 0$ and $\ell \geq 0$. Define $kG\Theta_\ell^{(k)}(x, a|b)$ and $kG\Theta_{\ell-m}^{(k)}(x, a|b)$ by

$$
\sum_{m \in \mathbb{Z}} kG\Theta_\ell^{(k)}(x, a|b)u^m = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^{\infty} \frac{1 + (u + \beta)x_i}{1 + (u + \beta)\bar{x}_i} \prod_{i=1}^{k} (1 + (u + \beta)a_i) \prod_{i=1}^{\ell} (1 + (u + \beta)b_i);
$$

$$
\sum_{m \in \mathbb{Z}} kG\Theta_{\ell-m}^{(k)}(x, a|b)u^m = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^{\infty} \frac{1 + (u + \beta)x_i}{1 + (u + \beta)\bar{x}_i} \prod_{i=1}^{k} (1 + (u + \beta)a_i) \prod_{i=1}^{\ell} \frac{1}{1 + (u + \beta)b_i}.
$$

Remark 7.7. A direct computation shows that $kG\Theta_m(x, a|b) = (-\beta)^{-m}$ for each $m \leq 0$. 

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The following lemma is obvious from the definition.

**Lemma 7.8.** For \( \ell \in \mathbb{Z} \), we have \( kG\Theta_m^{(\ell)}(x, a|b) \in G\mathcal{R}_k^{(k)} \). In particular, we have \( kG\Theta_m(x, a) := kG\Theta_m^{(0)}(x, a|0) \in G\mathcal{R}_k^{(k)} \).

**Definition 7.9** (\( G\Theta_\lambda \)-functions). Let \( \lambda \in S\mathcal{P}_r^k \) and \( \chi \) its characteristic index. Choose an \( r \) such that \( \lambda \in S\mathcal{P}_r^k \). Consider the \( G\mathcal{R}_k^{(k)} \)-homomorphism

\[
\phi : L^{G\mathcal{R}_k^{(k)}} \to G\mathcal{R}_k^{(k)}
\]

defined by

\[
\sum_{\tau^s \in Z} a_{\tau^s} \tau^s \mapsto \sum_{\tau^s \in Z} a_{\tau^s} G\Theta_{s_1}^{(\lambda_1)} \cdots G\Theta_{s_r}^{(\lambda_r)}.
\]

This map is well-defined essentially because of the fact that \( \mathbb{Z}[\beta] \) is graded non-positively. Indeed, the coefficient of a given monomial \( x_1^{r_1} \cdots x_d^{r_d}b_1^{p_1} \cdots b_L^{p_L} \) is contributed from only finitely many \( s \) in the summation. Define

\[
G\Theta_\lambda(x, a|b) := \phi \left( \tau_1^{r_1} \cdots \tau_r^{r_r} \prod_{(i,j) \in \Delta} (1 - \bar{\tau}_i/\bar{\tau}_j) \right).
\]

By Lemma 6.14 and Remark 7.7, \( G\Theta_\lambda(x, a|b) \) does not depend on the choice of \( r \).

The next lemma is clear from the construction.

**Lemma 7.10.** \( G\Theta_\lambda \in G\mathcal{R}_k^{(k)} \).

**Remark 7.11.** It is worth stressing that \( G\Theta_\lambda \) depends on \( k \), as \( \lambda \) is considered as an element of \( S\mathcal{P}_r^k \).

**Remark 7.12.** By (6.6) and (6.7) (cf. Lemma 5.10), we can write \( G\Theta_\lambda(x, a|b) \) as a sum of Pfaffian, which is an analogue of the well-known identity due to Schur [24].

### 7.3 Statements of the results and the main idea of constructions

The following theorem is the main result of [7] and it shows that \( G\mathcal{R}_k^{(k)} \) is a universal ring with a distinguished \( \mathbb{Q}[\beta][[b]]_{\text{gr}} \)-basis consisting of \( G\Theta \)-functions that describe \( CK_{T_n}^*(SG^k(n)) \) for all \( n \) and their Schubert basis.

**Theorem 7.13.** There is a surjective homomorphism of \( \mathbb{Q}[\beta][[b]]_{\text{gr}} \)-algebras \( \pi_n : G\mathcal{R}_k^{(k)} \to CK_{T_n}^*(SG^k(n)) \) such that

\[
\pi_n(G\Theta_\lambda(x, a|b)) = [\Omega_\lambda]_{T_n} \quad \text{for all } \lambda \in S\mathcal{P}_r^k(n).
\]

We briefly explain how to construct the map \( \pi_n \). The details are in the rest of the section.

**Notation 7.14.** We denote \( CK_{T_n}^* := \mathbb{Q}[\beta][[b]]_{\text{gr}} \) and \( CK_{T_n}^* := CK_{T_n}^*(pt) \cong \mathbb{Q}[\beta][[b_1, \ldots, b_n]]_{\text{gr}} \) for simplicity.

Let \( SG^k(n)^{T_n} \) be the set of \( T_n \)-fixed points in \( SG^k(n) \). Let \( e_\lambda \) denote the \( T_n \) fixed point in \( SG^k(n) \) corresponding to \( \lambda \in S\mathcal{P}_r^k(n) \). Consider the inclusion \( \iota_n : SG^k(n)^{T_n} \hookrightarrow SG^k(n) \) and its pull-back homomorphism

\[
\iota_n^* : CK_{T_n}^*(SG^k(n)) \to CK_{T_n}^*(SG^k(n)^{T_n}).
\]
We often call this map the localization map. It is known that $\iota_n^*$ is injective and that the image is described by the so-called GKM condition (Goresky-Kottwitz-MacPherson). We identify $CK^*_{T_n}(SG^k(n))$ with $\text{Fun}(SP^k(n), CK^*_{T_n})$ through a natural bijection between $SG^k(n)$ and $SP^k(n)$. We endow the set $\text{Fun}(SP^k(n), CK^*_{T_n})$ of functions on $SP^k(n)$ valued in $CK^*_{T_n}$ with the $CK^*_{T_n}$-algebra structure by pointwise multiplication.

Consider the projection map

$$
\overline{\pi}_n : \text{Fun}(SP^k, CK^*_{T_n}) \to \text{Fun}(SP^k(n), CK^*_{T_n})
$$

defined by the restriction of the domain $SP^k$ to $SP^k(n)$ and the projection of the range $CK^*_{T_n}$ to $CK^*_{T_n}$. In Definition [7.17] we define a $CK^*_{T_n}$-algebra homomorphism

$$
\Phi^{(k)}_{\infty} : GR^{(k)}_{\infty} \to \text{Fun}(SP^k, CK^*_{T_n}),
$$

so that the following diagram commutes:

$$
\begin{array}{ccc}
GR^{(k)}_{\infty} & \xrightarrow{\Phi^{(k)}_{\infty}} & \text{Fun}(SP^k, CK^*_{T_n}) \\
\pi_n \downarrow & & \downarrow \overline{\pi}_n \\
CK^*_{T_n}(SG^k(n)) & \cong & \text{Fun}(SP^k(n), CK^*_{T_n})
\end{array}
$$

Here we denote the image of $\Phi^{(k)}_{\infty}$ and $\iota_n^*$ by $R^{(k)}_{\infty}$ and $R^{(k)}_n$ respectively.

### 7.4 GKM theory

In this section, we first recall the Weyl group of type $C_\infty$ and its relation to the $k$-strict partitions. Then we introduce the algebraic localization map $\Phi^{(k)}_{\infty}$ and study its images (the GKM ring $R^{(k)}_{\infty}$).

#### 7.4.1 Weyl Group of type $C_\infty$

Let $W_\infty$ be the infinite hyperoctahedral group which is defined by the generators $s_i, i = 0, 1, \ldots$, and the relations

$$
\begin{align*}
s_i^2 &= e \quad (i \geq 0), \\
s_is_j &= s_js_i \quad (|i-j| \geq 2), \\
s_0s_1s_0s_1 &= s_1s_0s_1s_0, \\
s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \quad (i \geq 1).
\end{align*}
$$

We identify $W_\infty$ with the group of all permutations $w$ of $\mathbb{Z}\setminus\{0\}$ such that $w(i) \neq i$ for only finitely many $i \in \mathbb{Z}\setminus\{0\}$, and $\overline{w(i)} = w(i)$ for all $i$ where $\overline{i} := -i$. The generators, often referred to as the simple reflections, are identified with the transpositions $s_0 = (1, \overline{1})$ and $s_i = (i+1, \overline{i})(\overline{i}, \overline{i+1})$ for $i \geq 1$. Let $W_n$ be the subgroup of $W_\infty$ generated by $s_0, s_1, \ldots, s_{n-1}$. Or equivalently, it consists of elements $w \in W_\infty$ such that $w(i) = i$ for all $i > n$. The one-line notation of an element $w \in W_\infty$ is the sequence $w = (w(1)w(2)w(3) \cdots)$. We often write the one-line notation of $w \in W_n$ by the finite sequence $(w(1)w(2) \cdots w(n))$. The length of $w \in W_\infty$ is denoted by $\ell(w)$.
7.4.2 The \( k \)-strict partitions as the coset representatives

Fix \( k \geq 0 \). Let \( W_{(k)} \) be the subgroup of \( W_\infty \) generated by all \( s_i, i \neq k \). Let \( W^{(k)}_\infty \) be the set of minimum length coset representatives for \( W_\infty / W_{(k)} \), and it is given by

\[
W^{(k)}_\infty = \{ w \in W_\infty \mid \ell(ws_i) > \ell(w) \text{ for all } i \neq k \}.
\]

We denote \( W^{(k)}_\infty \cap W_n \) by \( W^{(k)}_n \). An element of \( W^{(k)}_\infty \) is called \( k \)-Grassmannian and it is given by the following one-line notation:

\[
w = (v_1 \cdots v_k | \xi_1 \cdots \xi_s u_1 u_2 \cdots);
\]

\[
0 < v_1 < \cdots < v_k, \xi_1 < \cdots < \xi_s < 0 < u_1 < u_2 < \cdots.
\]

We insert a vertical line after \( v_k \) to indicate that \( w \) is regarded as a \( k \)-Grassmannian element. For example, \((134256 \cdots)\) is a \( 2 \)-Grassmannian element in \( W_\infty \). We list two useful facts.

**Proposition 7.15.** There is a bijection \( W^{(k)}_\infty \cong S^P_k \).

**Proposition 7.16.** If \( \chi \) is the characteristic index associated to a \( k \)-strict partition \( \lambda \), then it is given by

\[
\chi = (\chi_1, \chi_2, \ldots) = (\xi_1 - 1, \xi_2 - 1, \ldots, \xi_s - 1, -u_1, -u_2, \ldots).
\]

The map from \( W^{(k)}_\infty \) to \( S^P_k \) is given as follows. Let \( w \in W^{(k)}_\infty \) with the one-line notation \( w = (v_1 \cdots v_k | \xi_1 \cdots \xi_s u_1 u_2 \cdots) \). Define a partition \( \nu = (v_1, v_2, \ldots) \) by \( v_i = \# \{ p \mid v_p > u_i \} \). Then a \( k \)-strict partition \( \lambda \) defined by setting \( \lambda_i = \xi_i + k \) if \( 1 \leq i \leq s \) and \( \lambda_i = v_{i-s} \) if \( s + 1 \leq i \). See \[8\] for more details.

7.4.3 GKM ring \( \Phi^{(k)}_\infty \)

We define the algebraic localization map \( \Phi^{(k)}_\infty \). Recall that \( CK^*_{T_\infty} = Q[\beta][[b]]_{gr} \) (Notation \[7\]).

**Definition 7.17** (Algebraic localization map \( \Phi^{(k)}_\infty \)). Let \( \lambda \in S^P_k \) and

\[
w = (u_1, \ldots, u_k | \xi_1, \ldots, \xi_s, v_1, v_2, \ldots) \in W^{(k)}_\infty
\]

the corresponding \( k \)-Grassmannian element. First define the \( CK^*_{T_\infty} \)-algebra homomorphism \( \Phi^*_\lambda : GR^{(k)}_\infty \rightarrow CK^*_T_{\infty} \) by the substitution

\[
f(x_1, x_2, \ldots; a_1, a_2, \ldots) \mapsto f(\bar{b}_{\xi_1}, \ldots, \bar{b}_{\xi_s}, 0, 0, \ldots; \bar{b}_{u_1}, \ldots, \bar{b}_{u_k}).
\]

Define the homomorphism of \( CK^*_{T_\infty} \)-algebras

\[
\Phi^{(k)}_\infty : GR^{(k)}_\infty \rightarrow \text{Fun}(S^P_k, CK^*_{T_\infty}); \quad f \mapsto (\lambda \mapsto \Phi^*_\lambda(f)).
\]

**Remark 7.18.** We can show that \( \Phi^{(k)}_\infty \) is injective, cf. \[9\] Lemma 6.5. We do not need the fact here and so the details will be discussed elsewhere.

7.4.4 Description of the image of \( \Phi^{(k)}_\infty \)

Let \( \Delta^+ \) be the set of positive roots in \( L := \bigoplus_{i=1}^\infty \mathbb{Z}t_i \) defined by

\[
\Delta^+ := \{2t_i, 1 \leq i \} \cup \{t_j \pm t_i \mid 1 \leq i < j \}.
\]

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We define the map \( e : L \to CK_{T_n}^* \) satisfying
\[
e(t_i) = b_i, \quad e(-t_i) = b_i, \quad e(\alpha + \gamma) = e(\alpha) \oplus e(\gamma) \quad \text{and} \quad e(\alpha - \gamma) = e(\alpha) \ominus e(\gamma).
\]

For \( \alpha \in \Delta^+ \), let \( s_\alpha \in W_\infty \) be the simple reflection associated with the positive root \( \alpha \in \Delta^+ \). Note that \( W_\infty \) acts naturally on the set \( SP^k \) via the bijection \( SP^k \cong W_\infty / W(k) \).

**Definition 7.19.** Let \( R(k) \) be the subring of \( Fun(SP^k, CK_{T_n}^*) \) consisting of functions \( \psi \) such that
\[
\psi(s_\alpha \mu) - \psi(\mu) \in e(\alpha) \cdot CK_{T_n}^*, \quad \text{for all } \mu \in SP^k \quad \text{and } \alpha \in \Delta^+.
\]

**Proposition 7.20.** The image of \( \Phi_{\infty}^{(k)} \) lies in \( R(k) \).

**Proof.** The proof is similar to the one for Lemma 7.1 [11]. Let \( t_{ij} := s_{t_j - t_i}, \ s_{ij} := s_{t_i + t_j} \) for \( j > i \geq 1 \) and \( s_{ii} := s_{t_i} = s_{2t_i} \). Let \( f := f(x; a) \in GT^*_n \), then for each \( \mu \in SP^k \), we have
\[
\Phi_{t_{ij} \mu}(f) - \Phi_{ij \mu}(f) \in (b_j \oplus b_i) \cdot CK_{T_n}^* \quad \text{for } j > i \geq 1,
\]
\[
\Phi_{s_{ij} \mu}(f) - \Phi_{ij \mu}(f) \in (b_j \oplus b_i) \cdot CK_{T_n}^* \quad \text{for } j > i \geq 1,
\]
\[
\Phi_{s_{i} \mu}(f) - \Phi_{i \mu}(f) \in (b_i \oplus b_i) \cdot CK_{T_n}^* \quad \text{for } j > i \geq 1.
\]

The details are left to the readers. \( \square \)

**Remark 7.21.** We can show that the image coincides with \( R_{\infty}^{(k)} \). Since we do not need this fact here, we discuss it elsewhere.

### 7.5 Proof of Theorem 7.13 — Localization of the special classes

Let \( \Delta^+(n) := \Delta^+ \cap \text{Span}_\mathbb{Z}\{t_1, \ldots, t_n\} \) be the finite set of positive roots. We regard \( CK_{T_n}^* (SG^n(n)) \) as a \( CK_{T_n}^* \)-algebra via the homomorphism \( CK_{T_n}^* \to CK_{T_n}^* \) defined by \( b_i = 0 \) for all \( i > n \).

**Definition 7.22.** Let \( R_n^{(k)} \) be the \( CK_{T_n}^* \)-subalgebra of \( Fun(SP^k(n), CK_{T_n}^*) \) defined as follows: a map \( \psi : SP^k(n) \to CK_{T_n}^* \) is in \( R_n^{(k)} \) if and only if
\[
\psi(s_\alpha \mu) - \psi(\mu) \in e(\alpha) \cdot CK_{T_n}^* \quad \text{for all } \mu \in SP^k(n) \quad \text{and } \alpha \in \Delta(n)^+.
\]

The next theorem holds by Theorem 7.8 [10] and the standard fact about the flag varieties.

**Theorem 7.23.** The image of \( t_n^* \) is equal to \( R_n^{(k)} \).

Proposition 7.20 and Theorem 7.23 define the homomorphism of \( CK_{T_n}^* \)-algebras
\[
\pi_n : GR_{\infty}^{(k)} \xrightarrow{\Phi_{\infty}^{(k)}} R_{\infty}^{(k)} \xrightarrow{\pi_n} R_n^{(k)} \xrightarrow{(t_n^*)^{-1}} CK_{T_n}^* (SG^n(n)).
\]

The following is the key fact to prove Theorem 7.13.

**Proposition 7.24.** For all \(-n \leq \ell \leq n\), we have
\[
\pi_n (G\Theta_m^{(\ell)}(x, a|b)) = \mathcal{C}_m^{(\ell)}.
\]
Proof. Let $\lambda \in SP^k(n)$ and $w = (u_1, \ldots, u_k, u_{k+1}, \ldots, v_1, \ldots, v_{n-k-s}) \in W_n$ its corresponding permutation. Let $\iota^*_\lambda: CK^*_{T^n}(SG^k(n)) \to CK^*_{T^n}$ be the pullback by the inclusion map $\iota_\lambda: SG^k(n) \to \{e_\lambda\}$. The claim follows from the comparison of the localizations at $e_\lambda$, i.e. it suffices to show

$$\Phi_\lambda \left( \sum_{m \in \mathbb{Z}} G\Theta^{(\ell)}_m u^m \right) = \iota^*_\lambda \left( \sum_{m \in \mathbb{Z}} \mathcal{G}^{(\ell)}_m u^m \right)$$

for $\ell \in \mathbb{Z}$. One can check this identity as follows. We have

$$\Phi_\lambda \left( \sum_{m \in \mathbb{Z}} G\Theta^{(\ell)}_m u^m \right) = \left\{ \begin{array}{ll}
\frac{1}{1 + \beta u^{-1}} \prod_{i=1}^n \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=1}^k \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=0}^0 (1 + (u + \beta)b_i) & \text{if } \ell \geq 0, \\
\frac{1}{1 + \beta u^{-1}} \prod_{i=1}^n \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=1}^k \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=0}^\ell 1 + (u + \beta)b_i & \text{if } \ell \leq 0.
\end{array} \right.$$ 

On the other hand, if $z_{k+1}, \ldots, z_n$ are the Chern roots of $U$, we have

$$\sum_{m \in \mathbb{Z}} \mathcal{G}^{(\ell)}_m u^m = \left\{ \begin{array}{ll}
\frac{1}{1 + \beta u^{-1}} \prod_{i=1}^n \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=0}^0 (1 + (u + \beta)b_i) & \text{if } \ell \geq 0, \\
\frac{1}{1 + \beta u^{-1}} \prod_{i=1}^n \frac{1}{1 + (u + \beta)b_{\lambda_i}} \prod_{i=0}^\ell 1 + (u + \beta)b_i & \text{if } \ell \leq 0.
\end{array} \right.$$ 

Since $\iota^*_\lambda(z_i) = b_{w(i)}$ for $i \geq 1$, we have \eqref{eq:identity}. \hfill \Box

Proof Theorem \ref{thm:appendix}. Since $[\Omega_\lambda]$ and $G\Theta_\lambda$ are given by the same formula except that $\phi$ replaces $\tau^{\lambda_i}_m$ by $\mathcal{G}^{(\lambda_i)}_{m_i}$ in the case of $[\Omega_\lambda]$ and by $G\Theta^{(\lambda_i)}_{m_i}$ in the case of $G\Theta_\lambda$. Therefore Proposition \ref{prop:compare} shows \eqref{eq:identity}. The surjectivity follows from the fact that $\Omega^*_{T^n}, \lambda \in SP^k(n)$ form a basis of $CK^*_{T^n}(SG^k(n))$ as a module over $\mathbb{Q}[\beta][[b_1, \ldots, b_n]]_{\text{gr}}$. \hfill \Box

8 Appendix: Grothendieck polynomials

The main purpose of this section is to explain the relation between the $K$-theoretic Segre classes and the Grothendieck polynomials. The result stated in Theorem \ref{thm:appendix} is due to Buch. We include two proofs of the theorem. We also show a formula expressing a special Grothendieck polynomial in terms of the dual variables.

8.1 Generating functions of Grothendieck polynomials

Fix a positive integer $d$. Let $z = (z_1, \ldots, z_d)$ be a sequence of $d$ variables, and $b = (b_1, b_2, \ldots)$ an infinite sequence of variables. Let $P_d$ denote the set of all partitions with at most $d$ parts, i.e. all non-increasing sequences of $d$ nonnegative integers. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ be such a partition. The Grothendieck polynomials associated with $\lambda$ is defined by

$$G_\lambda(z) = \frac{\det \left( z_i^{\lambda_j} + d-j \right) 1 \leq i, j \leq d}{\prod_{1 \leq i < j \leq d} (z_i - z_j)}.$$ 

One can show that $G_\lambda(z)$ is a polynomial in $\mathbb{Z}[\beta][z_1, \ldots, z_d]$, symmetric in $z_1, \ldots, z_d$. 

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Remark 8.1. In the original paper by Lascoux-Schützenberger, $G_\lambda(z)$ is defined in terms of isobaric divided difference operators. Buch proved a combinatorial expression in terms of set-valued semistandard tableaux. A proof of the coincidence of the bi-alternant formula above and tableaux formula is given in [22].

For $m \geq 1$, we denote $G_{(m,0,\ldots,0)}(z)$ by $G_m(z)$. Set $G_{-m}(z) := (-\beta)^m$ for each $m \geq 0$. Buch showed the following formula for $G_m(z)$.

Lemma 8.2 (Lemma 6.6 [1]). For each $m \in \mathbb{Z}$,

$$G_m(z) = \sum_{i=0}^{d} \beta^i e_i(z) \sum_{j=0}^{\infty} (-\beta)^j h_{m+j}(z).$$  

(8.2)

The expression on the right hand side is defined in the ring of graded formal power series $\mathbb{Z}[[\beta, z]]_{gr} := \mathbb{Z}[[\beta, z_1, \ldots, z_d]]_{gr}$, although the left hand side is actually a polynomial in $\beta, z_1, \ldots, z_d$ as mentioned above. Furthermore, we can combine $G_m(z)$ for all $m \in \mathbb{Z}$ in a generating function and write

$$\sum_{m \in \mathbb{Z}} G_m(z) u^m = \frac{1}{1 + \beta z} \prod_{i=1}^{d} \frac{1 + \beta z_i}{1 - u z_i}$$  

(8.3)

where $\frac{1}{1 + \beta u^{-1}}$ is expanded in the form $\sum_{i=0}^{\infty} (-\beta)^i u^{-i}$. Indeed, the product on the right hand side is well-defined in the ring $\mathbb{Z}[u, u^{-1}][[\beta, z]]_{gr}$ of the graded formal power series with coefficients in $\mathbb{Z}[u, u^{-1}]$ where $\deg u = -1$. Now comparing it with Theorem 3.2, we derive the following result.

Theorem 8.3 ([1] Lemma 7.1). Let $G_m(z_1, \ldots, z_e)$ where we regard $z_1, \ldots, z_e$ as Chern roots of $E$. Then $\mathcal{S}_m(E) = G_m(E)$.

8.2 Second proof of Theorem 8.3

Let $z_1, \ldots, z_e$ be Chern roots of $E$. From Vishik’s formula [25, Proposition 5.29, p.548], we have

$$\pi_*(r^{m+e-1}) = \sum_{i=1}^{e} \frac{z_i^{m+e-1}}{\prod_{j \neq i}(z_i \otimes z_j)} = \sum_{i=1}^{e} \frac{z_i^{m+e-1} \prod_{j \neq i}(1 + \beta z_j)}{\prod_{j \neq i}(z_i - z_j)}$$

We can also write

$$\prod_{1 \leq i < j \leq e} (z_i - z_j) \sum_{i=1}^{e} (-1)^{i-1} z_i^{m+e-1} \prod_{j \neq i}(1 + \beta z_j) \prod_{1 \leq k < l \leq e} (z_k - z_l)$$

$$= \frac{1}{\prod_{1 \leq i < j \leq e} (z_i - z_j)} \det \begin{pmatrix}
    z_1^{m+e-1} & z_1^{e-2}(1 + \beta z_1) & \cdots & z_1(1 + \beta z_1)^{e-2} & (1 + \beta z_1)^{e-1}
    
    z_2^{m+e-1} & z_2^{e-2}(1 + \beta z_2) & \cdots & z_2(1 + \beta z_2)^{e-2} & (1 + \beta z_2)^{e-1}
    
    \vdots & \vdots & \ddots & \vdots & \vdots
    
    \vdots & \vdots & \ddots & \vdots & \vdots
    
    z_e^{m+e-1} & z_e^{e-2}(1 + \beta z_e) & \cdots & z_e(1 + \beta z_e)^{e-2} & (1 + \beta z_e)^{e-1}
\end{pmatrix}$$

This last expression is equal to $G_m(z)$ by [8.11]. On the other hand $\pi_*(r^{m+e-1})$ is by definition the $K$-theoretic Segre class $\mathcal{S}_m(E)$. \[\square\]
8.3 Grothendieck polynomials in terms of the dual variables

As an application of the generating function (8.3), we can show the following formula as well. We believe the result is new.

**Proposition 8.4.** For \( m > 0 \), we have

\[
G_m(z_1, \ldots, z_d) = \sum_{i \geq 0} \left( \binom{i + m + p - 1}{m + p - 1} \beta^i h_{m+i}(\bar{z}_1, \ldots, \bar{z}_d) \right).
\]

**Proof.** Since

\[
\sum_{m \in \mathbb{Z}} G_m(z)^m = \frac{1}{1 + \beta u^{-1}} \prod_{i=1}^{d} \frac{1}{1 + (u + \beta) \bar{z}_i},
\]

it follows from Lemma 8.5 below, which holds more generally.

**Lemma 8.5.** Let \( A \) be a ring over \( \mathbb{Z}[\beta] \). Consider \( F(u) = \sum_{i=0}^{\infty} F_i u^i \in \mathbb{Z}[u] \). For \( m \in \mathbb{Z} \), consider \( H_m \in A \) defined by \( \sum_{m \in \mathbb{Z}} H_m u^m = \frac{1}{1 + \beta u^{-1}} F(u + \beta) \). If \( m > 0 \) we have

\[
H_m = \sum_{i \geq 0} \left( \binom{i + m + p - 1}{m + p - 1} \beta^i F_{i+m} \right).
\]

If \( m \leq 0 \), \( H_m = (-\beta)^{-m} F_0 \).

**Proof.** We have

\[
\frac{1}{1 + \beta u^{-1}} F(u + \beta) = \frac{1}{1 + \beta u^{-1}} F_0 + u \sum_{i=1}^{\infty} (u + \beta)^{-1} F_i
\]

\[
= \sum_{m \leq 0} (-\beta)^{-m} F_0 u^m + \sum_{m > 0} \left( \sum_{e \geq 0} \binom{e + m - 1}{m - 1} \beta^e F_{m+e} \right) u^m,
\]

where in the first equality we have used

\[
\frac{1}{1 + \beta u^{-1}} (u + \beta) = \left( \sum_{i=0}^{\infty} (-1)^i \beta^i u^{-i} \right) (u + \beta) = u.
\]

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