First passage time distribution of stationary Markovian processes

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received 2 October 2010; accepted in final form 22 November 2010
published online 4 January 2011

PACS 02.50.Ey – Stochastic processes
PACS 05.10.Gg – Stochastic analysis methods (Fokker-Planck, Langevin, etc.)
PACS 02.50.Ga – Markov processes

Abstract – We investigate how the correlation properties of a stationary Markovian stochastic process affect its First Passage Time Distribution (FPTD). With explicit examples, in this paper we show that the tail of the first passage time distribution crucially depends on the correlation properties of the process and it is independent of its stationary distribution. When the process includes an infinite set of time-scales bounded from above, the FPTD shows tails modulated by some exponential decay. In the case when the process is power-law correlated the FPTD shows power-law tails $1/t^{\nu}$ and therefore the moments $\langle t^n \rangle$ of the FPTD are finite only when $n < \nu - 1$. The existence of an infinite and unbounded set of time-scales is a necessary but not sufficient condition in order to observe power-law tails in the FPTD. Finally, we give a general result connecting the FPTD of an additive stochastic processes $x(t)$ to the FPTD of a generic process $y(t)$ related by a coordinate transformation $y = f(x)$ to the first one.

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Introduction. – Stochastic processes are used to model a great variety of systems in disciplines as disparate as physics [1–8], genomics [9,10], and finance [11,12]. Here we will devote our attention to the special class of stationary Markovian stochastic processes driven by Gaussian noise and that can be described by a (non-fractional) Fokker-Planck (FP) equation [3–5].

The simplest such stationary Markovian process is the Ornstein-Uhlenbeck (OU) one that is characterized by an exponential autocorrelation function $e^{-t/T}$ where $T$ is referred to as the time-scale of the process. For a general stationary Markovian additive process the methodology of eigenfunction expansion [4,5] allows us to write the autocovariance function as the infinite weighted sum of exponential functions $e^{-\lambda t}$, where $\lambda$ are eigenvalues of the FP operator. By analogy with respect to the OU process, for an additive stationary Markovian process the time-scales are thus defined as the inverse of the eigenvalues of the FP operator. There is therefore a strict link between the typology of time-scales included in a stochastic process and its autocorrelation properties. A possible generalization to multiplicative stationary Markovian processes has been considered in refs. [13,14].

The aim of this paper is to investigate how the inclusion of more and more time-scales into stationary Markovian stochastic processes affects the First Passage Time Distribution (FPTD). First passage time issues are a classical topic in stochastic processes research [15,16]. They also have relevant applications in other fields such as finance. For example, the assessment of the default risk for firms’ assets is essentially based on the evaluation of the FPT under the assumption that the asset value $V$ follows a geometric Brownian motion [17,18]. The pricing of some interest rate derivatives is based on the OU model [19]. Therefore, much effort has been devoted to obtaining the FPTD of the OU process [20] thus neglecting the implications of the correlation properties of the asset value on the FPTD and consequently on the risk assessment. Here we will start considering the effects of the correlation properties of a stationary Markovian stochastic process on the FPTD. Deliberately, our investigations will be at a theoretical level, with the aim of understanding what are the relevant aspects of the problem. The search for a more realistic model in the context of default risk or for other financial variables [21] will be left for the future.

We will show that the large time behaviour of the FPTD depends on the correlation properties of the process and it can be independent of its stationary distribution. Moreover, in the case when the process is power-law correlated the FPTD shows power-law tails $1/t^\nu$, thus implying that the moments $\langle t^n \rangle$ of the FPTD are finite only when $n < \nu - 1$. As we will see $\nu$ can be written in terms of the exponent of the power-law autocorrelation function. Finally, we will give a general result connecting the FPTD of an additive stochastic processes $x(t)$ to the
FPTD of a generic process $y(t)$ related by a coordinate transformation $y = f(x)$ to the first one.

The paper is organized as follows. We will first give explicit examples of stationary Markovian processes including multiple time-scales. We will then consider the mean FPT for such processes, showing how the simple knowledge of the mean FPT may be not enough to let us discern between processes with different correlation properties. We will then derive a general expression for the FPTD. By using explicit examples we will show how the correlation properties of the considered process affect the tails of the FPTD. Finally, we will draw our conclusions.

Multi-scale stationary Markovian processes. –

Let us consider stationary Markovian processes described by a FP equation [4]:

$$
\frac{\partial}{\partial t} W(x,t) = - \frac{\partial}{\partial x} \left( h(x) W(x,t) \right) + \frac{\partial^2}{\partial x^2} \left( g(x)^2 W(x,t) \right),
$$

(1)

where $h(x)$ and $g(x)$ are the drift and diffusion coefficient appearing in the associated Langevin equation: $\dot{x}(t) = h(x(t)) + \Gamma(t)$ with $\Gamma(t)$ a $\delta$-correlated Gaussian noise term. For a stochastic process with a constant diffusion coefficient, the eigenvalue spectrum of the FP equation describing a stationary process consists of a discrete part $\lambda_0 = 0, \lambda_1, \ldots, \lambda_p$ and a continuous part $|\lambda| > \infty$ (where $\lambda > \lambda_0$) associated with eigenfunctions $\varphi_\lambda$. The FP equation with constant diffusion coefficient can be transformed into a Schrödinger equation [4] with a quantum potential $V_\lambda(x) = h(x)^2/4 + \partial_\lambda h(x)/2$. The eigenvalue spectrum of the Schrödinger equation is equal to that of the FP equation.

For a stationary process such that $V_\lambda(x)$ only admits a null eigenvalue and a continuum part of the spectrum, one can show that

$$
\langle x(t + \tau) x(t) \rangle - \langle x(t) \rangle \langle x(t + \tau) \rangle = \int_{-\infty}^{\infty} c_\lambda^2 e^{-\lambda \tau} d\lambda,
$$

(2)

where $c_\lambda \equiv \int_{-\infty}^{\infty} dx x \varphi_\lambda(x)$. This shows that the autocovariance function is the infinite weighted sum of exponential functions $e^{-\lambda \tau}$ each characterized by a time-scale $\lambda^{-1}$, where the $\lambda$ values are the eigenvalues of the FP operator. A possible generalization to multiplicative stationary Markovian processes has been considered in refs. [13,14].

Below we will briefly illustrate the stochastic processes to be considered in this paper, see also refs. [13,14]. These processes are devised i) to be analytically tractable and ii) to allow for a comparative investigation of the role of both the pdf tails and the correlation properties of a stochastic process in determining its FPTD.

Process A. Let us consider the stochastic process described by the following Langevin equation [4]:

$$
\dot{x}(t) = -h(x(t)) + D \Gamma(t),
$$

$$
h(x) = \begin{cases} +k, & \text{if } x < 0, \\ -k, & \text{if } x > 0, \end{cases}
$$

(3)

where $k$ is a real constant. The diffusion coefficient $D$ will be set to unity hereafter. The stationary distribution of the process is $W_\lambda(x) = \frac{1}{K} \exp(-\frac{1}{K} |x|)$ [14]. By using eq. (2) it is possible to prove that for large time lags the autocovariance function $R_\lambda(t) = \langle x(t) x(0) \rangle$ of the above process behaves like: $R_\lambda(t) \approx \exp(-\frac{k^2}{4t}) e^{-\frac{t}{4\tau}}$ as $t \to \infty$ [14]. As a result, this is a short-range–correlated process characterized by an infinite set of time-scales bounded from above, i.e. $\lambda^{-1} < 4/k^2$.

Process B. Let us consider the stochastic process described by the following Langevin equation [13,14,22,23]:

$$
\dot{x} = h(x(t)) + \Gamma(t),
$$

$$
h(x) = \begin{cases} -2\sqrt{V_0} \tan(\sqrt{V_0} x), & \text{if } |x| \leq L, \\ -\alpha/x, & \text{if } |x| > L, \end{cases}
$$

(4)

$$
V_1 = L \sqrt{V_0} \tan(\sqrt{V_0} L) \left( 1 + L \sqrt{V_0} \tan(\sqrt{V_0} L) \right),
$$

$$
\alpha = \sqrt{1 + 4 V_1} - 1 = 2L \sqrt{V_0} \tan(\sqrt{V_0} L),
$$

where $L$ and $V_0$ are real arbitrary constants. For large $|x|$ values the stationary distribution of this process is

$$
W_\lambda(x) \approx N_L x^{\alpha}, \quad \text{if } |x| > L,
$$

(5)

where $N_L$ is a real constants that can be obtained by imposing that $W_\lambda(x)$ is continuous and normalized to unity, when $\alpha > 1$. In the present study we consider stochastic processes with finite variance which implies $\alpha > 3$. The quantum potential associated with this process is given by

$$
V_\lambda(x) = \int_{-L}^{L} V_0 - V_1 x^{\alpha}, \quad \text{if } |x| \leq L, \quad V_1 x^{\alpha}, \quad \text{if } |x| > L.
$$

(6)

Such potential admits a ground state with null eigenvalue and a continuum set of eigenvalues $\lambda > 0$ attached to the null eigenvalue. By using eq. (3) it is possible to prove that $\langle x(t + \tau) x(t) \rangle \propto \tau^{-\beta}$, where $\beta = (\alpha - 3)/2$ thus showing that we are dealing with a power-law–correlated stochastic process. In the range $3 < \alpha < 5$ the process is long-range correlated. As a result, this is a power-law–correlated process with power-law decaying pdf admitting an infinite and unbounded set of time-scales.

Process C. Let us now consider the process of eq. (3) and the coordinate transformation [14]:

$$
x \to y = f_s(x) = \sqrt{2s \tau} \exp[-(x^2/2s)],
$$

(7)

using the Itô lemma, in the coordinate space $y = f_s(x)$ one gets a multiplicative stochastic process whose stationary pdf is given by $W_\lambda(y) = \frac{1}{\sqrt{2\pi \tau}} \exp(-y^2/2\tau^2)$ with $s$ an additional arbitrary parameter. The autocovariance function of the process defined by eq. (3) with the coordinate transformation of eq. (7) is given by

$$
R_\lambda(\tau) = \langle y(t) y(t + \tau) \rangle = \int_{-\infty}^{\infty} d\lambda C_\lambda^2 e^{-\lambda \tau},
$$

(8)

$$
\lambda_c = k^2/4, \quad C_\lambda = \int_{-\infty}^{\infty} dx f_s(x) \psi_\lambda(x) \phi_\lambda(x),
$$

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where $\psi_0(x)$ and $\psi_\Lambda(x)$ are the eigenfunctions of the Schrödinger equation with potential $V_S(z) = k^2/4 - k \delta(x)$ associated to the stochastic process of eq. (3). As explained in ref. [14] the process considered here admits an infinite set of time-scales bounded from above and its autocorrelation function decays like $R_s(t) \approx \exp(-k^2/4t)t^{-3/2}$ for large time lags. As a result, this is a Gaussian-distributed, short-range–correlated process, characterized by an infinite set of time-scales bounded from above, i.e. $\lambda^{-1} < 4/k^2$.

**Process D.** Let us now consider again the process described by eq. (3). By performing the following coordinate transformation:

$$
x \mapsto f_p(x) = \begin{cases} (kL^\alpha)^{1/(\alpha-1)} e^{kx/(\alpha-1)}, & |x| > L/D, \\
Dx, & |x| \leq L/D,
\end{cases}$$  \hspace{1cm} (9)

with $L$ and $\alpha$ positive real constants and by using the Ito lemma, in the coordinate space $y = f_p(x)$ one gets a multiplicative stochastic process whose stationary pdf is given by:

$$W_p(y) = \begin{cases} \frac{kL^\alpha}{2D} e^{k\lambda L/D} \frac{1}{y^\alpha}, & |y| > L, \\
\frac{k}{2D} e^{-k\lambda y/D}, & |y| \leq L,
\end{cases}$$  \hspace{1cm} (10)

thus showing that the process admits a power-law decaying pdf. The drift and diffusion coefficients are given by:

$$H(y) = \begin{cases} -(\alpha - 2)/L^2 D^2 y, & |y| > L, \\
+ (\alpha - 1)/L D^2, & |y| \leq L,
\end{cases}$$  \hspace{1cm} (11)

$$G(y) = \begin{cases} D/L|y|, & |y| > L, \\
D, & |y| \geq L,
\end{cases}$$

and the requirement that the diffusion coefficient is continuous in $y = \pm L$ implies $k = (\alpha - 1)/L D/L$. The autocorrelation function of the process defined by eq. (3) with the coordinate transformation of eq. (9) is given by eq. (8) with $f_p(x)$ replaced by $f_p(x)$. The relevant integrations can be performed analytically. One can therefore show that such process admits an infinite set of time-scales bounded from above and its autocorrelation function decays like $R_s(t) \approx \exp(-k^2/4t)t^{-3/2}$ for large time lags. As a result, this is a short-range–correlated process with a power-law decaying stationary pdf, characterized by an infinite set of time-scales bounded from above, i.e. $\lambda^{-1} < 4/k^2$.

**Process E.** Let us now consider the process of section and the coordinate transformation [14]:

$$x \mapsto f_i(x) = \begin{cases} f_{\text{i1}} = \sqrt{2} \tanh^{-1}[1 - r(x)], & |x| > L, \\
0, & |x| \leq L,
\end{cases}$$

$$r(x) = \frac{2 N_f x^{\alpha - 1}}{\alpha - 1},$$  \hspace{1cm} (12)

 ainsi que $\Lambda$ et que la covariance du processus défini par eq. (3) avec la transformation coordonnée $x \mapsto f_i(x)$ est donnée par eq. (8) avec $f_p(x)$ remplacé par $f_i(x)$. Les intégrations pertinentes peuvent être effectuées analytiquement. On peut alors démontrer que de telles processus admettent un ensemble infini de temps-scales borné de l’extérieur et de la covariance fonction décroît comme $R_s(t) \approx \exp(-k^2/4t)t^{-3/2}$ pour de grandes écarts de temps. En conséquence, ce serait un processus de courte-valeurs – correlé avec une fonction pdf stationnaire de décroissance exponentielle, caractérisée par un ensemble infini de temps-scales borné de l’extérieur, i.e. $\lambda^{-1} < 4/k^2$.

**Process E.** Les processus considérés ici admettent un ensemble infini de temps-scales borné de l’extérieur et de la covariance fonction décroît comme $R_s(t) \approx \exp(-k^2/4t)t^{-3/2}$ pour de grandes écarts de temps. En conséquence, il s’agit d’un processus de courte-valeurs – correlé avec une fonction pdf stationnaire de décroissance exponentielle, caractérisée par un ensemble infini de temps-scales borné de l’extérieur, i.e. $\lambda^{-1} < 4/k^2$.

The mean first passage time. – In order to understand how the correlation properties of a stochastic process affect its FPTD we start investigating the mean FPT, which is the first moment of the FPTD. We will illustrate how processes with different correlation properties might show a similar behaviour in the mean FPT. Let us consider the mean time $T_\Lambda(\Lambda)$ is needed to reach for the first time position $x = \pm \Lambda$ starting from position $x$. This is obtained by solving the equation [5]:

$$h(x) \frac{\partial T_\Lambda(\Lambda)}{\partial x} + g(x)^2 \frac{\partial^2 T_\Lambda(\Lambda)}{\partial x^2} = -1$$

(13)

with boundary conditions $T_\Lambda(\Lambda) = 0$ when $x = \pm \Lambda$.

**Processes with a Gaussian pdf.** In this section we will compare the mean FPT of the two stochastic processes with Gaussian pdf considered in the previous section. Indeed, the results shown here have already been considered in ref. [14]. We recall here the main aspects.

For Process C the mean FPT can be analytically computed by using the drift and diffusion coefficients of the process into eq. (12). For instance, in the case $x = 0$ one can show that for large values of $\Lambda$ the mean FPT is given by

$$T_0(\Lambda) \approx \Lambda e^{z^2}, \quad z = \Lambda \sqrt{2 \alpha},$$

(14)

For the Process E the mean FPT cannot be computed by analytically solving eq. (12). We showed in ref. [14] that in the region $\Lambda > f_i(L)$ and for large values of $\Lambda$ one gets

$$T_0(\Lambda) \approx \Lambda e^{z^2 \frac{\alpha + 1}{\alpha}}, \quad z = \Lambda \sqrt{2 \alpha}.$$  \hspace{1cm} (15)

Equation (14) and eq. (15) clearly show that for large values of $\Lambda$ the power-law–correlated process has a mean FPT larger than the process including time-scales.
bounded from above. In the two examples considered here, the way the correlation properties of the process affect the mean FPT is clear. In the case of eq. (14) we have a linear growth in terms of the \( z e^z \)-function, while in the case of eq. (15) we have a power-law growth with exponent larger than unity.

**Processes with a power-law decaying pdf.** Let us now consider the mean FPT of the two stochastic processes with power-law decaying pdf considered in the previous section.

For Process D the mean FPT can be analytically computed by using the drift coefficient and the diffusion coefficient of eq. (11) into eq. (13). It is possible to obtain an analytical expression showing that for large values of \( \Lambda \) one gets

\[
T_0(\Lambda) \propto \Lambda^{\alpha-1},
\]

where \( \alpha \) is the exponent of the power-law decaying pdf tail. For small values of \( \Lambda \) one gets \( T_0(\Lambda) \approx \Lambda^2 \). For Process B the mean FPT can be analytically computed by using the drift and diffusion coefficients of eq. (4) into eq. (13). It is possible to obtain an analytical expression showing that for large values of \( \Lambda \) one gets

\[
T_0(\Lambda) \propto \Lambda^{\alpha+1}.
\]

For small values of \( \Lambda \) one gets \( T_0(\Lambda) \approx \Lambda^2 / 2 \).

Either in eq. (16) and in eq. (17) the parameter \( \alpha \) is the exponent of the pdf tail. Therefore, as for the processes with Gaussian pdf, the mean FPT of the process that includes more time-scales is larger than the other. However, in the two cases considered here a power-law dependence on \( \Lambda \) is found. The qualitative behaviour of the mean FPT in both processes is therefore very similar, although the correlation properties of the two processes are very different. In fact, we will show that the different correlation properties of the stochastic processes display their effect at the level of the higher order FPTD moments.

**The first passage time distribution.** Let us suppose to be interested in the distribution \( D_{a,b}(x_0,t) \) of the first times at which a process reaches the absorbing barriers \( x = a \) or \( x = b \) starting from a generic position \( x_0 \) with \( a < x_0 < b \). According to ref. [5] one gets

\[
D_{a,b}(x_0,t) = \partial_t G_{a,b}(x_0,t),
\]

where \( P(x,t|x_0,0) \) is the conditional probability of finding the process in position \( x \) at time \( t \), knowing that it was in position \( x_0 \) at time \( t = 0 \).

**A theoretical result.** According to the methodology of eigenfunction expansion [4,5], for an additive stationary Markovian process described by a FPE admitting only a continuum set of eigenvalues, the conditional probability is given by

\[
P(x,t|x_0,0) = \frac{\psi_0(x)}{\psi_0(x_0)} \int_{-\infty}^{\infty} d\lambda e^{-\lambda t} \psi_\lambda(x) \psi_\lambda(x_0),
\]

where the \( \lambda \)-integration is extended over the continuum part of the spectrum and \( \psi_\lambda(x) \) are the eigenfunctions of the quantum potential \( V_S \) associated to the stochastic process considered. By using the above expression for the conditional probability one gets

\[
G_{a,b}(x_0,t) = \int_{-\infty}^{\infty} d\lambda e^{-\lambda t} W_\lambda(x_0; a, b),
\]

where \( W_\lambda(x_0; a, b) \) can be considered as the weight with which each time-scale enters the FPTD. Let us now consider a stochastic process \( y(t) \) obtained by performing a coordinate transformation \( y = f(x) \) starting from a stochastic additive process \( x(t) \). The conditional probability is given by [14]:

\[
P(y, t|y_0, 0) = W(y) + \frac{1}{\partial f(x)/\partial x} \frac{\psi_0(x)}{\psi_0(x_0)} \int_{-\infty}^{\infty} d\lambda \psi_\lambda(x) \psi_\lambda(x_0) e^{-\lambda t},
\]

where \( W(y) \) is the stationary distribution of the process \( y(t) \) and \( \psi_0(x) \), \( \psi_\lambda(x) \) are the eigenfunctions of the Schrödinger equation with the potential \( V_S(x) \) associated to the additive process \( x(t) \). As a result, the FPTD \( D'_{a,b}(y_0,t) \) of the transformed process \( y(t) = f(x(t)) \) is given by

\[
D'_{a,b}(y_0,t) = \partial_t G'_{a,b}(y_0,t), \quad x_0 = f^{-1}(y_0),
\]

\[
G'_{a,b}(y_0,t) = \int_{-\infty}^{\infty} d\lambda e^{-\lambda t} W'_\lambda(y_0; a, b),
\]

A direct comparison of eq. (23) and eq. (20) shows that

\[
D'_{a,b}(y_0,t) = D_{f^{-1}(a),f^{-1}(b)}(f^{-1}(y_0),t). \quad (24)
\]

For a general stochastic process \( y(t) \), the problem of finding the FPTD can be therefore re-conducted to the problem of finding the FPTD of an appropriate additive process. When the additive process admits a quantum potential \( V_S(x) \) that is exactly solvable then eq. (20) can be used. If \( V_S(x) \) is not exactly solvable, in order to estimate the FPTD one might either take advantage of the approximations schemes widely used in quantum mechanics to evaluate the eigenfunctions or revert to numerical simulations.
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Fig. 1: The figure shows the function $G_{-\Lambda,\Lambda}(0,t)$ for Process B. The curves are obtained by using eq. (21). We have considered the following three set of parameters: i) $L = 1$ and $V_0 = 0.985$ (solid line) corresponding to $\alpha = 3.05$; ii) $L = 1$ and $V_0 = 1.16$ (dashed line) corresponding to $\alpha = 4.00$; iii) $L = 1$ and $V_0 = 1.42272$ (dash-dotted line) corresponding to $\alpha = 6.00$. For all the three sets we considered $\Lambda = 2L$. In the bottom part of the legend we report the result of a nonlinear fit performed in the range $t \in [20,100]$. These results are clearly in agreement with those of eq. (26).

Explicit examples. Let us now consider Process A. Moreover, let us again fix $x_0 = 0$ and $b = \Lambda$, $a = -\Lambda$. The integration over $x$ required in eq. (20) can be performed analytically. It gives that $W_\Lambda(0; -\Lambda,\Lambda) \approx \sqrt{\Lambda - k^2} t/4$ when $\lambda \to k^2/4$. Therefore the large time behaviour of the FPTD is given by [24]:

$$D_{-\Lambda,\Lambda}(0,t) \sim e^{-t^2/2t^{3/2}}, \quad t \to \infty. \quad (25)$$

According to the result of eq. (24), the same asymptotic behaviour is shared by the FPTD of Process C and Process D, although they have different pdfs.

Let us now consider Process B and $x_0 = 0$ and $b = \Lambda$, $a = -\Lambda$. The integration over $x$ required in eq. (20) can be performed analytically. One therefore obtains that $W_\Lambda(0; -\Lambda,\Lambda) \approx \lambda^{(\alpha-3)/2}$. Therefore, the asymptotic behaviour of the FPTD is given by

$$D_{-\Lambda,\Lambda}(0,t) \sim 1/t^2(t^{\alpha+1/2}), \quad t \to \infty. \quad (26)$$

According to the result of eq. (24), the same asymptotic behaviour is shared by the FPTD of Process E although it has a different pdf. In fig. 1 we show the function $G_{-\Lambda,\Lambda}(0,t)$ for Process B. The curves are obtained by using eq. (20). We have considered the following three set of parameters: i) $L = 1$ and $V_0 = 0.985$ (solid line) corresponding to $\alpha = 3.05$; ii) $L = 1$ and $V_0 = 1.16$ (dashed line) corresponding to $\alpha = 4.00$; iii) $L = 1$ and $V_0 = 1.42272$ (dash-dotted line) corresponding to $\alpha = 6.00$. For all the three sets we considered $\Lambda = 2L$. In the bottom part of the legend we report the result of a nonlinear fit performed in the range $t \in [20,100]$. These results are clearly in agreement with those of eq. (26).

The above results explicitly show that the tail of the FPTD is independent of the specific stationary distribution of the stochastic process considered. Rather, it is strictly related to the correlation properties of the process. In fact Process B and Process E have the same FPTD tail although their pdf is different. Similarly for Process A, Process C and Process D. Let us consider Process C and Process B. Both of them have a power-law decaying pdf $1/x^\alpha$. The mean FPT of Process C, see eq. (14), is larger than the mean FPT of Process B, see eq. (17). However, the probability of having large first passage times is larger for Process B, see eq. (26), than for Process C, see eq. (25). This gives an example of the non-trivial role played by correlations.

The existence of power-law tails in the FPTD of power-law–correlated stationary Markovian processes implies that the moments of the distribution $(t^n)$ are defined only for $n < (\alpha - 1)/2$. In the case of Process A, Process C and Process D the exponential cut-off of eq. (25) ensures that the moments of the FPTD are always finite. That marks a relevant difference between a process admitting a bounded set of time-scales and a process in which all time-scales are included. Such difference is not caught by looking at the mean FPT only. In fact both Process B and Process D, both characterized by a power-law decaying pdf, have a power-law growing mean FPT. However, in the case of Process D all the higher moments of the FPTD exist, while in the case of Process B only the moments $(t^n)$ such that $n < (\alpha - 1)/2$ are defined.

Finally, the result of eq. (26) shows that the tail of the FPTD for power-law–correlated stationary Markovian processes is different from what is expected for other non-Markovian processes with anomalous diffusion. For instance, in ref. [25] it is shown that for a fractional Brownian motion the tails of the FPTD decay like $t^{1-2/\gamma}$, where $\gamma \in [0,2]$ is defined as $\langle X(t)^2 \rangle \sim t^{\gamma}$ and $\langle X(t)^2 \rangle$ is the mean square displacement of the process. For Process B one gets $\gamma = (7-\alpha)/2$ and therefore eq. (26) can be rewritten as $D_{-\Lambda,\Lambda}(0,t) \sim t^{\gamma-4}$.

Another process with an infinite and unbounded set of time-scales. The fact that the process of section includes an infinite and unbounded set of time-scales is a necessary condition for the existence of power-law tails in its FPTD. This is ultimately due to the fact that the eigenvalues spectrum of the Schrödinger potential of eq. (6) is given by a continuum set of eigenvalues $\lambda > 0$ attached to the null eigenvalue associated to the ground state and therefore it includes an infinite and unbounded set of time-scales. We show below that such condition is necessary but not sufficient. In general, any Schrödinger potential that asymptotically decays like $1/x^\mu$ would give a stochastic process with an infinite and unbounded set of time-scales. In the case $\mu = 1$, we will show here that the corresponding FPTD tail does not display a power-law decay. Let us now consider the additive stochastic process associated to the Schrödinger potential [14]:

$$V_S(x) = \begin{cases} -V_0, & \text{if } |x| \leq L, \\ V_1/|x|, & \text{if } |x| > L, \end{cases} \quad (27)$$

where $L$, $V_0$ and $V_1$ are real positive constants. This quantum potential is exactly solvable, as shown in ref. [14].
The large time behaviour of $D_{-\Lambda,\lambda}(0, t)$ is determined by the small energy behaviour of $W_\lambda(0; -\Lambda, \Lambda)$ [24]. Following ref. [14], and by making use of eq. (13.6.8) and eq. (14.1.7) of ref. [26] one can explicitly show that in the limit when $\Lambda$ is small then

$$W_\lambda(0; -\Lambda, \Lambda) = A_0 \psi_\lambda(0) \frac{\lambda^3}{3} \int_L^\Lambda dy \sqrt{y} K_1(2V_1y) F_0 \left( \frac{V_1}{2\sqrt{\lambda}}, \sqrt{\lambda y} \right).$$

(28)

where $F_0$ is the regular Coulomb function with null index. By making use of eq. (14.1.1) of ref. [26] one gets

$$W_\lambda(0; -\Lambda, \Lambda) = A_0 V_1^{-1} A_\Lambda \psi_\lambda(0) \int_L^\Lambda dy \sqrt{y} K_1(2\sqrt{V_1y}) \times \sum_{k=1}^{\infty} b_k (2\sqrt{V_1y})^{k/2} I_k(2\sqrt{V_1y}),$$

where $I_k(\cdot)$ is the Bessel functions of order $k$ and the coefficients $b_k$ are defined in eq. (14.4.3) of ref. [26]. We have thus factorized the $\lambda$-dependences from the $y$-dependences. As a result, the large time behaviour of the FPTD is given by the small energy behaviour of $A_\Lambda \psi_\lambda(0) \approx e^{-\frac{\sqrt{t}}{2}}$. Using the results of ref. [24] one finally gets

$$D_{-\Lambda,\lambda}(0, t) \approx \frac{e^{-k t^{1/3}}}{t^{5/6}}, \quad \kappa = 3 \left( \frac{3\pi}{4} \right)^{2/3} V_1^{2/3}.$$

(29)

The above result shows that the mere existence of an infinite and unbounded set of time-scales is not enough to ensure an asymptotic power-law behaviour in $D_{-\Lambda,\lambda}(0, t)$. In passing, by using the same arguments it is also possible to prove that the stochastic process associated to the quantum potential of eq. (27) is not power-law correlated, even though it incorporates an infinite and unbounded set of time-scales.

**Conclusions.** We have shown that the tail of the FPTD crucially depends on the correlation properties of the process and it is rather independent of its stationary distribution. When the process includes an infinite set of time-scales bounded from above the FPTD shows tails modulated by some exponential decay. In the case when the process is power-law correlated then the FPTD shows power-law tails $1/t^\nu$. In this last case, the moments ($t^\nu$) of the FPTD are finite only when $n < \nu - 1$. We have also shown that such power-law behaviour is not merely due to the fact that the process includes an infinite and unbounded set of time-scales. Rather, it is also required that each time-scale enters the FPTD with weights $W_\lambda(x_0; a, b)$ that must be distributed according to a power-law for small values of $\lambda$, i.e. for large time-scales.

Moreover, we have given a general result showing that given i) an additive stochastic process $x(t)$ and ii) another process $y(t)$ related by a coordinate transformation $y = f(x)$ to the first one, the FPTD of $y(t)$ can be obtained from the one of the $x(t)$ process by using eq. (24). Therefore, for a general stochastic process $y(t)$, the problem of finding the FPTD can be re-conducted to the problem of finding the FPTD of an appropriate additive process.

Finally, the result of eq. (26) shows that the tail of the FPTD for power-law–correlated stationary Markovian processes is different from what is expected for other non-Markovian processes with anomalous diffusion, such as the fractional Brownian motion. This result might be used to test the Markovian property of a generic stochastic process.

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