Impact of unphysical meson decays on the parameter fixing
in the three-flavour Nambu-Jona-Lasinio model
for different regularisation methods

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We investigate the influence of the allowed unphysical meson decay into a quark-antiquark pair of
the three flavour Nambu-Jona-Lasinio (NJL) model on its parameter fixing. This decay manifests
in a non-vanishing imaginary part of the corresponding meson propagator and its polarisation loop,
respectively. In order to handle the emerging divergent integrals, we focus on the Pauli-Villars,
three- and four momentum cutoff regularisation method. Here, we fix the parameters to certain
observables, which are effected by the non-vanishing imaginary part. The resulting parameter-sets
for each regularisation scheme are compared to each other with and without taking the imaginary
part of the polarisation loop into account.

I. INTRODUCTION

The NJL model is a well know and often used model
to investigate the properties of strongly interacting
matter in vacuum as well as under extreme conditions,
i.e. in hot and dense matter. Originally the model was
designed by Y. Nambu and G. Jona-Lasinio in 1961
to explain the high nucleon mass in accordance with
the partially conserved axial current in the pre-QCD
era [1, 2]. Later, it was reinterpreted for QCD calculations
in the low-energy regime, where gluon degrees of
freedom are supposed to be frozen-out. This leads to
a reduction of the complex structure of strong interactions
in the QCD Lagrangian to a point-like local interaction.
In particular, the NJL model shares the same global
symmetries with QCD and in this way also can describe
the chiral symmetry and its spontaneous breaking in the
vacuum as well. Moreover, the NJL model is often used
to study the thermodynamic properties of QCD like the
restoration of the chiral symmetry for high temperatures
and densities in the QCD phase diagram [3, 6], and has
even been used to investigate baryons, cf. [7, 8].

Due to the non-renormalizable character of the
model, an additional cutoff parameter Λ appears, which
has to be included in order to handle the upcoming
divergent integrals. Hereby, different methods can be
used like: (1) Introducing a sharp three-momentum
or four-momentum cutoff parameter. (2) Introducing
additional (counter-)terms to control the divergences
of the integrals for high momenta. After choosing an
appropriate regularisation method the other parameters
of the model can be fixed to physical observables. In our
case we fix to the masses of pion, kaon, η- and η’-meson
as well as to the pion decay constant [3, 6, 9].

Besides the non-renormalizable character of the NJL
model, another drawback is the lack of confinement
due to the missing gluonic interactions. While some
thermodynamic properties of QCD in the medium can
be simulated by including Polyakov loops [10, 11], the
meson decay into a quark-antiquark-pair is still allowed
and possible. By Choosing heavy enough up- and down-
quark masses, these decays can be circumvented for
pions and kaons. However, for reasonable quark masses
the decay is still possible for heavier mesons like the
η’-meson. This results in non-vanishing imaginary parts
of polarisation loops and, therefore, in complex-valued
meson propagators.

In this paper we create and compare parameter sets
for the three-flavour NJL model using the common
regularisation methods mentioned earlier. Hereby, we
focus on the treatment of the emerging imaginary part
of the meson propagator, which has not been studied in
detail so far. In order to understand the contribution
of the unphysical property of the model to allow decays
into quark-antiquark pairs, we take a look at parameter
sets with and without taking the imaginary part into
account. However, other authors tend to use parameter
sets from older publications, like [12, 13], or create
their own ones with limited information of the used fit
parameters or handle of imaginary parts.

This paper is organised as follows: At first we want
to introduce the NJL Lagrangian and shortly review

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the used methods in order to investigate the quark and mesonic properties of the model. Then, the method for the parameter creation for the different regularisation schemes is introduced. Hereby, we discuss the effect of the decay into a quark-antiquark pair as well as other emerging critical phenomena for the model. Of course, the parameter sets are presented and compared to each other.

We want to mention at this point that the NJL model in general have often been discussed in the literature and therefore we are not going into much detail. For further information we refer as an example to [3, 6, 14–17] and the literature mentioned in these works as well.

II. THE THREE-FLAVOUR NJL MODEL

In this paper we consider the following NJL Lagrangian:

\[ \mathcal{L}_\text{NJL} = \bar{\psi} \left( i \slashed{D} - m_0 \right) \psi + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} \]  

(1)

A NJL-type Lagrangian can be in general separated into two parts: The first one is the interaction-free part leading to the Dirac equation, which describes the kinematic of the system, while the second one represents the point-like interaction within the model.\[1\]

In our case with \( N_f = 3 \) flavours, the four-point interaction term is given by

\[ \mathcal{L}^{(4)} = G \sum_{a=0}^{8} \left[ (\bar{\psi} \tau_a \psi)^2 + (\bar{\psi} i \gamma_5 \tau_a \psi)^2 \right] , \]

(2)

with \( \tau_a \) denoting the eight Gell-Mann-matrices of the SU(3) plus \( \tau_0 = \sqrt{2/N_f} \).\[2\]

The latter term in equation [1] represents an instanton induced six-point interaction Lagrangian (‘t Hooft term) in order to break the otherwise remaining \( U_A(1) \) symmetry in the three flavour case explicitly. It is given by a determinate over flavour space \[18\]

\[ \mathcal{L}^{(6)} = K \left[ \det_f \left( \bar{\psi}(1 + \gamma_5)\psi \right) + \det_f \left( \bar{\psi}(1 - \gamma_5)\psi \right) \right] . \]

(3)

As one can see directly, the six-point interaction Lagrangian leads to a mixing of different quark flavours. We want to emphasise at this point that the upper given term is not unique and other interaction terms will lead also to an explicit breaking of the \( U_A(1) \) symmetry. However, equation [3] is the commonly used term.

In the upper equations, \( \psi \) denotes a Dirac spinor, representing a quark field. Additional indices for the colour and flavour space are for now omitted to increase readability. The mass \( m_0 \) denotes a diagonal matrix of dimension \( N_f \) containing the different current (or bare) masses of the quarks.

\[ A. \text{ Quark properties} \]

It is now convenient to introduce the quark condensate \( \langle \bar{\psi}_f \psi_f \rangle \) for a certain flavour \( f \) which is given by

\[ \langle \bar{\psi}_f \psi_f \rangle = - \int \frac{d^4k}{(2\pi)^4} \text{Tr} S_f(k) \]

\[ = -4N_c m_f I_1(m_f) , \]

(4)

where \( S_f \) denotes the quark propagator of flavour \( f \) and basically describes a closed quark-loop. A detailed discussion of the emerging integral \( I_1 \) can be found in appendix [A].

Note that in our approach non-scalar condensates and flavour-mixing ones vanish due to the trace in all three spaces. At first, the quark properties of the model is investigated in the mean-field approach by applying

\[ \langle \bar{\psi}_f \Gamma_f \psi \rangle \approx \langle \bar{\psi}_f \Gamma_f \psi \rangle + \delta \Gamma_f \]

(5)

to the interaction Lagrangian’s [2] and [3] and neglecting terms of order two in \( \delta \Gamma_f \). This leads to a linearisation of both Lagrangian’s such that the coupled gap-equations for a certain flavour can be read off directly

\[ m_i = m_{0,i} - 4G \langle \bar{\psi}_u \psi_i \rangle + 2K \langle \bar{\psi}_u \psi_j \rangle \langle \bar{\psi}_k \psi_k \rangle \]

(6)

with \( i \neq j \neq k \neq i \in \{ u, d, s \} \) and \( m_f \) denoting the so-called constituent quark mass.\[2\]

In case of the isospin limit (\( m_u = m_d \)), which will always be applied in this work, the upper equation further simplifies to

\[ m_u = m_{0,u} - 4G \langle \bar{\psi}_u \psi_u \rangle + 2K \langle \bar{\psi}_u \psi_u \rangle \langle \bar{\psi}_s \psi_s \rangle \]

\[ m_s = m_{0,s} - 4G \langle \bar{\psi}_s \psi_s \rangle + 2K \langle \bar{\psi}_u \psi_u \rangle \langle \bar{\psi}_s \psi_s \rangle \]

(7)

III. MESONIC SPECTRUM

A. Bethe-Salpeter equation and mesonic propagators

![](image)

FIG. 1. Self-consistent expression for the Bethe-Salpeter equation in RPA.

To obtain an expression for the meson propagator, and thus for the properties of the mesons, the self-consistent
Bethe-Salpeter equation (BSE) in random-phase approximation (RPA),
\[ i\mathcal{T}(p) = \mathcal{K} + i\mathcal{K}(-i\Pi(p))i\mathcal{T}(p), \]
diagrammatically represented in FIG. 1 has to be solved. Here, \( \mathcal{K} \) and \( \Pi \) denote the scattering kernel and the polarisation loop, respectively. \( \mathcal{T} \) denotes the so-called scattering matrix (T-matrix).

In random-phase approximation, where only quark-antiquark polarisation loops are included, the right-hand side of the BSE is the sum over all these loops. The leading order term therefore is the scattering kernel itself.

\[ \Pi_{\text{eff}}^{(4)} = \sum_{a=0}^{8} \left( G_{a}^{(-)}(\bar{\psi}\tau_{a}\psi)^{2} + G_{a}^{(+)}(\bar{\psi}i\gamma_{5}\tau_{a}\psi)^{2} \right) + \left[ G_{80}^{(-)}(\bar{\psi}\tau_{8}\psi)(\bar{\psi}\tau_{0}\psi) + G_{80}^{(+)}(\bar{\psi}i\gamma_{5}\tau_{8}\psi)(\bar{\psi}i\gamma_{5}\tau_{0}\psi) \right] + \left[ G_{08}^{(-)}(\bar{\psi}\tau_{0}\psi)(\bar{\psi}\tau_{8}\psi) + G_{08}^{(+)}(\bar{\psi}i\gamma_{5}\tau_{0}\psi)(\bar{\psi}i\gamma_{5}\tau_{8}\psi) \right], \]  

where the (redefined) flavour dependent coupling constant \( G_{a}^{\pm} \) can be expressed in terms of the quark condensate
\[ G_{0}^{(\pm)} = G \pm \frac{1}{3}K(\bar{\psi}_{s}\psi_{s}) + 2(\bar{\psi}_{u}\psi_{u}) \] \[ G_{1}^{(\pm)} = G_{2}^{(\pm)} = G_{3}^{(\pm)} = G \mp \frac{1}{2}K(\bar{\psi}_{s}\psi_{s}) \] \[ G_{4}^{(\pm)} = G_{5}^{(\pm)} = G_{6}^{(\pm)} = G_{7}^{(\pm)} = G \mp \frac{1}{2}K(\bar{\psi}_{u}\psi_{u}) \] \[ G_{8}^{(\pm)} = G \mp \frac{1}{6}K(\bar{\psi}_{s}\psi_{s}) - 4(\bar{\psi}_{u}\psi_{u}). \]

Since there is no flavour mixing contribution in the original four-point interaction Lagrangian, the terms related to this mixing come with the effective coupling strength
\[ G_{80,08}^{(\pm)} = \pm \frac{1}{6}K\sqrt{2}(\bar{\psi}_{s}\psi_{s} - \bar{\psi}_{u}\psi_{u}), \]  

which only depends on the coupling \( K \) of the t’Hooft determinant. Based on our assumption of degenerate up- and down-quark masses (isospin limit), we have no term in the Lagrangian that includes a mixing of \( \tau_{3} - \tau_{8} \) or \( \tau_{0} \), respectively. Hence, a mixing of the \( \pi \) and \( \eta \) modes does not appear. Moreover, this assumption leads to a degeneration of each of the three pions and four kaons as well.

Coming back to the BSE, the scattering kernel for the pseudo-scalar mesons, summarized in TABLE I are given by the effective interaction Lagrangian (9). The polarisation loop for a certain meson is given by
\[ -i\Pi_{ab}(p) \equiv - \int \frac{\text{d}^{4}k}{(2\pi)^{4}} \text{Tr} \left[ \Gamma^{N}\tau_{a}S(p + k)\Gamma^{N-1}\tau_{b}S(k) \right]. \]  

Here, \( \Gamma^{N} \) denotes the structure of the corresponding vertex in colour and Dirac-space and is given by
\[ \Gamma^{\text{PS}} = i\gamma_{5} \otimes 1_{c}, \]  

where PS denotes the pseudo-scalar channel. Explicit expressions for the relevant polarisation loops can be found in appendix [3]. Other combinations lead to a vanishing polarisation loop and therefore no bound-states can be obtained.

However, the right-hand side of the BSE gives rise to a self-consistent scalar function which can be interpreted as a meson propagator of a certain mode. However, in the vicinity of the pole, this function is expected to behave like a free scalar-particle propagator of mass \( m_{M} \). Hence, the left-hand side of (3) can be written as
\[ i\mathcal{T}_{ab}(p) = \left(-ig_{M}\bar{\psi}_{\psi}\right) \frac{i}{p^{2} - m_{M}^{2}} \left(-ig_{M}\bar{\psi}_{\psi}\right) \delta_{ab}, \]  

where \( g_{M}\bar{\psi}_{\psi} \) denotes the so-called meson-antiquark-quark coupling. In the context of the BSE as a description of a quark-antiquark scattering process, we can com-
table 1. Scattering kernel and corresponding Gell-Mann matrices for pseudo-scalar mesons.

| meson | $iK_{ab}^{PS}$ | $\tau_a$ | $\tau_b$ |
|-------|---------------|----------|----------|
| $\pi^0$ | $2G_3^{(+)}$ | $\frac{1}{\sqrt{2}}(\tau_1 + i\tau_2)$ | $\frac{1}{\sqrt{2}}(\tau_1 - i\tau_2)$ |
| $\pi^+$ | $2G_4^{(+)}$ | $\frac{1}{\sqrt{2}}(\tau_6 + i\tau_7)$ | $\frac{1}{\sqrt{2}}(\tau_6 - i\tau_7)$ |
| $K_0$, $\bar{K}_0$ | $2G_4^{(+)}$ | $\frac{1}{\sqrt{2}}(\tau_4 + i\tau_5)$ | $\frac{1}{\sqrt{2}}(\tau_4 - i\tau_5)$ |
| $\eta_8$ | $2G_0^{(+)}$ | $\tau_0$ | $\tau_0$ |
| $\eta_8'$ | $2G_8^{(+)}$ | $\tau_0$ | $\tau_0$ |

Based on the equations (8) and (14), the meson pole masses can be obtained by solving the eigenvalue problem

$$\det \left( 1 - K M(p^2) \right) = 0$$

for a certain meson $M$. We want to emphasise at this point, that the meson-antiquark-quark coupling in this work will only be evaluated for the pion because it arises in the description of the later introduced pion-decay constant, cf. [III C]. In the $\eta$-$\eta'$-channel equation (14) is more complicated due to the off-diagonal structure.

Before we want to start introducing the different regularisation methods, we will review the properties of the Pauli-Villars regularisation scheme with three regulators. However, the discussion in this subsection is not restricted to this method only but can be applied to all others as well. We have already seen in the previous sections that the $\eta$-propagator will be of particular interest since it gives rise to the $\eta$ and $\eta'$ mass as well.

Hence, we will choose the $\eta$-propagator as an example in order to show the general analytic properties of the propagators emerging in this work. After evaluating the determinant of the non-diagonal block-matrix in equation (16) which is related to $\eta_0$ and $\eta_8$ we find that the inverse $\eta$-propagator is given by

$$D_\eta^{-1}(p^2) \equiv (1 - K_{00} M(p^2))^{-1} = (1 - K_{00} M(p^2))^{-1} - K_{00} = (1 - K_{00} M(p^2))^{-1} - K_{00},$$

where we have hidden the dependency of $p$ on the right-hand side to increase overview and ignored its matrix structure. In FIG. 2 the real and imaginary part of the inverse propagator is shown as a function of the meson momentum ($p_0, \vec{p} = 0$). Note that the imaginary part of the inverse propagator only comes from the related polarisation loops, cf. appendix A. In fact, while the roots of the real part are in general related to bound states of the collective excitation, the imaginary part gives rise to decay processes: Since the NfL model does not contain confinement, mesons are allowed to decay into their constituent quarks. Hence, within the propagator a non-vanishing imaginary part shows up at $p_0 = 2m_u$ and leads to a kink in the real part. In the case of the pa-
The real and imaginary part of the inverse $\eta$-propagator as function of $p_0$ (parameter set R3B', cf. TABLE XVI). The vertical lines mark the masses of the $\eta$- and $\eta'$-mass, respectively.

The previously described interpretation of the real and imaginary part holds for all meson propagators within all regularisation methods. In particular, for the $\eta$ propagator one have to include the imaginary part of the polarisation loops in order to obtain the correct behaviour of the inverse propagator, cf. section VI.

We want to mention at this point, that without the t’Hooft interaction term no $\eta-\eta'$-splitting could be found since the $U_A(1)$-symmetry is not broken. This can be directly seen under virtue of the upper discussion: For $K = 0$ the flavour mixing coupling, i.e. equation (11), vanishes, while the other couplings reduces to the one in the original four-point interaction Lagrangian. Therefore, the off-diagonal elements of the interaction kernel in (19) also become zero. Note that the off-diagonal elements of the polarisation loop part do not vanish. Taking a closer look into equation (16), the matrix part related to $\eta_0$ and $\eta_8$ is still not fully diagonal. After some algebra the inverse propagator of $\eta$ reduces for $K = 0$ to

$$D^{-1}_{\eta} = (1 - 2G\Pi_\pi)(1 - 2G\Pi'_{\pi'}) ,$$  \hspace{1cm} (21)

where we recognize the polarisation loop of the pion $\Pi_\pi$. The other polarisation loop $\Pi'_\pi$ is equivalent to the pion one, except that it depends on the constituent mass of the strange quark $m_s$ rather than $m_u$. Compared to the inverse pion propagator (17), it follows from the upper equation, that one root is equal to the pion mass. Therefore, in the case of a not broken $U_A(1)$-symmetry the pion and $\eta$ state are degenerate as expected. The second root can be related to a pion-like state of two strange quarks and has no further meaning.

C. Pion decay constant

The observed weak decay of the charged pions into a muon and a muon-neutrino can be described as the tran-
sition probability of a pion decaying into the hadronic vacuum [19]. Although the electromagnetic decay of \( \pi^0 \) into two photons is favoured, the weak decay constant will be calculated with the vertex structure of the uncharged pion for simplicity and the fact that we make use of the isospin-limit. Due to parity conservation, the pion is only allowed to decay through the axial current. It follows that the decay constant \( f_\pi \), determining the strength of the chiral symmetry breaking [20], can be defined through

\[
\langle 0 | A_\mu^a(x) | \pi_\nu(p) \rangle = i f_\pi p^\mu \delta_{ab} \epsilon^{ipx},
\]

with the corresponding Feynman diagram shown in FIG. 4. After a carefull evaluation the pion decay constant reads

\[
f_\pi = -4 N_c g_\pi \bar{\psi} \psi m_u I_2(p^2, m_u) \bigg|_{p^2 = m_\pi^2},
\]

where integral \( I_2 \) defined in appendix A.

IV. REGULARISATION METHODS

A. Sharp momentum cutoff

1. Four momentum cutoff

In the four-momentum cutoff method the euclidean four-momentum integration is restricted to a certain cutoff \( \Lambda \). Therefore one has to perform a Wick-rotation in order to rewrite the integrand into euclidean space and transform it to spherical coordinates for a four-dimensional sphere. The result can then be treated with standard techniques, cf. appendix A.

2. Three momentum cutoff

Compared to a four-momentum cutoff, the a three-momentum cutoff does not restrict the integration domain of the four-momentum space completely rather than restrict the spatial component to a cutoff \( \Lambda \). Hence, the time component can be evaluated with the residue theorem directly. The remaining three dimensional integral then can be transformed into spherical coordinates without any issues. Here, the radial integration is restricted to radii \(|k| < \Lambda\), cf. appendix A. This scheme obviously breaks Lorentz invariance.

B. Pauli-Villars cutoff

Now, we want to introduce a method to handle divergent integrals described by Pauli and Villars in 1949 and give an overview of its application. Since we just wanted to give short introduction, a more detailed discussion can be found in references [3, 6, 21].

While the previous introduced methods use a sharp cutoff to handle the divergent integrals, PV regularisation is related to the proper time regularisation and adds additional terms that behave like the integrand itself for large momenta, but only have a small contribution for small momenta [22]. Therefore, the number of additional terms depends on the degree of divergence of the considered integral. Compared to the earlier introduced three-momentum cutoff, this method preserves the Lorentz invariance. Another advantage of PV regularisation compared to hard-cutoff methods is the possibility to make any substitutions within the integrand without taking care of the domain.

Overall, one replaces the original integrand by a weighted sum over new masses, i.e.

\[
\int \frac{d^4k}{(2\pi)^4} f(M_k) \longrightarrow \int \frac{d^4k}{(2\pi)^4} \sum_{j=0}^N c_j f(M_j, k),
\]

where \( N \) determines the highest order of divergence. In general one chooses the coefficient \( c_0 = 1 \) and \( M_0 = M \) in order to get the original version of the integral. The modified masses \( M_j(M, \Lambda) \) are given by

\[
M_j^2 = M^2 + j \cdot \Lambda^2.
\]

This choice is not unique, and one can of course make a different ansatz. Here, the (soft) cutoff \( \Lambda \) enters the model as a new free parameter.

To ensure that all divergences are treated, the coefficients \( c_j \) for the additional terms in equation (24) have to fulfil a set of equations given by [23]

\[
\sum_{j=0}^N c_j \left( M_j^2 \right)^n = 0 \text{ for } n = 0,1,\ldots,N-1.
\]

The set (26) determines the coefficients completely. In appendix A an example for a Pauli-Villars regularized integral is displayed.

In our case, where the highest order of divergence is quadratic, we need two additional regulators, i.e. \( N = 2 \), within the upper condition. This leads to two conditional equations for the coefficients of the PV method. However, in thermodynamic discussions of QCD within the NJL model framework three counter terms, i.e. \( N = 4 \), are needed typically to control the divergences of the thermodynamic potential. Then, an additional equation for \( n = 2 \) enters the set of equations for the \( c_j \). The resulting coefficients are shown in TABLE IV. Note that we will treat the three regulator case in the following discussions analogously to the two regulator one.

V. PARAMETER SETS FOR DIFFERENT REGULARISATION METHODS

In the three flavour NJL framework we have to fix the parameters of the model to certain physical observables.
In general, the parameters are the coupling $G$ and $K$ of the four-point and six-point interaction, respectively, the cutoff $\Lambda$ and the three current masses of the up-, down- and strange-quark as well. In our approach where up- and down-quarks are degenerate, we only have five free parameters.

The parameters are fitted to pion mass, kaon mass, $\eta$ and $\eta'$ masses

\[
\text{Re} \frac{1}{D^*_{\pi}(m_{\pi}^2)} = \text{Re} \frac{1}{D^*_{K}(m_{K}^2)} = 0 \quad (27)
\]

\[
\text{Re} \frac{1}{D^*_{\eta}(m_{\eta}^2)} = \text{Re} \frac{1}{D^*_{\eta'}(m_{\eta'}^2)} = 0 \quad (28)
\]

and pion decay constant $\langle \text{f} \rangle$. If not stated otherwise, the full complex character of the emerging integrals and therefore the inverse propagator will be used. The values of these physical observables are shown in TABLE III.

Note that within the related equations the constituent quark mass needs to be applied. Therefore, the two self-consistent gap-equations, cf. equation (7), have to be solved simultaneously as well. Of course, one can imagine other approaches where different observables can be fixed to create a set of parameters.

As a matter of fact, it is possible to find two such “perfect” parameter sets for each regularisation method, cf. TABLE IV. The corresponding values of $m_u$ are comparatively small. One parameter-set of each scheme is actually related to an unstable $\eta$-meson since $2m_u < m_\eta$. In the case of Pauli-Villars regularisation with three regulators the four-point couplings are even negative. Hence, these special parameter-sets will not always be appropriate which is why we decided to create various ones by varying one parameter. In our case we use the constituent mass of the up-quark $m_u$ as a free parameter between 275 MeV and 375 MeV.

First, we decided to fit the remaining four parameters to pion mass, kaon mass, $\eta$ mass and pion decay constant. The calculated $\eta'$ masses, besides the “perfect” solutions around $m_u = m_\eta/2$, lie well above the experimental value. Using only the real part of integrals leads to even higher masses, cf. section VI. Since the corresponding roots of the inverse $\eta$-propagator lie in every case in the unstable region, cf. section III B, we decided not to trust the calculated $\eta'$ masses. It seems to be more convenient for all regularisation methods to use it as a fixed point for the parameters. Then, the calculated mass of the $\eta$-meson always lies within the region of stability. For the different regularisation methods in case of fixed $\eta'$ masses, the corresponding mass of the $\eta$-meson is given.

For up-quark constituent masses between 275 MeV and 375 MeV the masses of the $\eta$-meson is relatively close to its experimental value.

Our results for different regularisation methods are presented in section VA and VB. All of them have been calculated numerically using Mathematica 12.0. The determination of parameter-sets has been done in two steps: First, we specified the value of $m_u$ and fixed the remaining parameters, i.e. $m_s$, $\Lambda$, $G$, and $K$, to $m_u$, $m_K$, $f_\pi$ and either $m_\eta$ or $m_{\eta'}$. Second, with the parameters at hand, we computed $m_{\eta',0}$, $m_{s,p}$ and either $m_{\eta'}$ or $m_{\eta}$. The couplings $G$ and $K$ are multiplied by appropriate powers of $\Lambda$ to obtain dimensionless numbers.

### A. Sharp momentum cutoff

#### 1. Four momentum cutoff

Using the four-momentum cutoff scheme, we find for given $m_u$ one solution when fixing the $\eta$ and two solutions when fixing the $\eta'$ mass. The corresponding parameter sets are displayed in TABLE VI and in TABLE VII and VIII, respectively. We decided not to display the calculated values for $m_{\eta'}$ for fixed $\eta$ mass because it lies in the unstable region of the propagator, cf. section III B.

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**TABLE II.** Coefficients of the PV regularisation method for two and three regulators.

| # regulator terms | $c_0$ | $c_1$ | $c_2$ | $c_3$ |
|-------------------|-------|-------|-------|-------|
| 2                 | 1     | -2    | 1     |       |
| 3                 | 1     | -3    | 3     | -1    |

**TABLE III.** Values for the physical observables on which the parameters are fixed.

| $m_\pi$ [MeV] | $m_K$ [MeV] | $m_\eta$ [MeV] | $m_{\eta'}$ [MeV] | $f_\pi$ [MeV] |
|---------------|-------------|-----------------|-------------------|---------------|
| 135           | 498         | 548             | 958               | 92.2          |

**TABLE IV.** Parameter sets for all regularisation schemes fixed to all physical observables. They are named after the regularisation schemes: C4 - four momentum cutoff, C3 - three momentum cutoff, R2 - PV with two regulators, R3 - PV with three regulators.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $G$ $^2$ | $K\Lambda$ $^2$ | $m_{\eta',u}$ [MeV] | $m_{\eta',s}$ [MeV] |
|-------------|-------------|-----------------|---------|-----------------|---------------------|---------------------|
| 262.694     | 397.595     | 770.037         | 0.706   | 124.5           | 6.004               | 164.824             |
| 261.941     | 368.908     | 736.846         | 0.622   | 139.422         | 6.418               | 172.497             |
| 277.028     | 384.93      | 701.415         | 0.634   | 175.442         | 7.72                | 207.3               |
The dependence of the couplings $G$ and $K$ on the constituent mass of the up-quark is shown in FIG. 5 and 6 in the upper left panels, respectively. Intersections of solutions correspond to “perfect” solutions for all physical observables, cf. parameter sets [C4i] and [C4ii] in TABLE IV.

Having a closer look at the solution for fixed $\eta$ mass, we find that the couplings show only marginal dependence on $m_u$ except for the domain around $m_u = m_\eta/2 = 274\,\text{MeV}$. For values $m_u < 274\,\text{MeV}$ the $\eta$-meson can decay into a quark-antiquark pair resulting in a kink in the real part of the meson propagator. This kink leads to the interesting behaviour of the couplings where, of course, it also emerges: While the six-point coupling just grows to large values in this domain the four-point coupling of the $\eta$-meson can decay into a quark-antiquark pair resulting in a kink in the real part of the meson propagator. This kink leads to the interesting behaviour of the couplings where, of course, it also emerges: While the six-point coupling just grows to large values in this domain the four-point coupling of the $\eta$-meson constantly scales. Since this observation holds for all used regularisation methods it already suggests that the $\eta'$-mesons should be used in order to fit parameters due to a broader range of its mass.

The authors grade the standard parameter set fixed to the $\eta$ mass as the most promising one. All couplings stay in reasonable intervals for a wide range of the constituent mass of the up-quark and show no kinks or discontinuities. Nevertheless, the calculated $\eta$ masses are roughly 100 MeV below the experimental value.

2. Three momentum cutoff

Using the three-momentum cutoff scheme, we find results similar to the four-momentum cutoff ones, i.e. one solution when fixing the $\eta$, cf. TABLE IX and two solutions when fixing the $\eta'$ mass, cf. TABLE X and XI. The dependence of the couplings $G$ and $K$ on the constituent mass of the up-quark is shown in FIG. 5 and 6.
Having a closer look at the solution for fixed $\eta$ mass, we observe only marginal dependence on $m_u$ except for the domain around $m_u = m_{\eta}/2$. Here, we observe an unique problem: For values close to $m_u = 274\text{ MeV}$, we were not able to find any solution when fixing the $\eta$ mass. The authors believe that this is just a numerical problem and expect a similar kink as in the four-momentum cutoff case.

In the case of fixed $\eta'$ mass the situation is different again: One solution shows only marginal dependence on $m_u$ over a wide range while the other one shows a strong dependence. In the following the second solution is referred to as alternative solution. For $m_u > 321\text{ MeV}$ the four-point coupling of the alternative solution becomes negative and tends fast towards highly negative values with increasing constituent mass of the up-quark. Of course, in order to compensate that the six-point coupling have to increase fast. This behaviour is similar to the four-momentum cutoff case and implies the same conclusions.

Here, the most promising parameter set seems to be the standard one fixed to the $\eta'$ mass again. Once more, the calculated $\eta$ masses are roughly 80 MeV below the experimental value.

Comparing the four and three-momentum cutoff schemes, we find that the four-point coupling $G$ is roughly twice as big in the four-momentum case over the whole range of the up-quark constituent mass. For the alternative solutions this statement holds only for small $m_u$. Making the same statement for the six-point coupling $K$ is more complicated, the factor is of order 10 but varies with $m_u$. Besides the discontinuity in case of fixed $\eta$ mass both schemes lead to similar parameter sets, which can also be seen in FIG. 7. The four-momentum cutoff scheme provides somewhat larger current quark masses.

### B. Pauli-Villars cutoff

#### 1. Two regulators

Analogously to the previous discussed sharp cutoff methods we have calculated parameter sets by fixing the $\eta$ and $\eta'$ mass, respectively with the Pauli-Villars regularisation method with two regulators. These parameter sets are shown in TABLE XIV again, by varying $m_u$.

Taking a look at FIG. 5 FIG. 6 and FIG. 7 (left lower panels) where the dependence of the coupling constants $G, K$ and $m_{\eta}$ (respectively $m_{\eta'}$) to the mass of the up-quark are shown, one notes that the behaviour is qualitatively nearly identical to the one for the four-momentum cutoff one. The only difference lies in the fact that the coupling of the six-point term, i.e. $K$, for the Pauli-Villars methods is roughly half as high as the ones we find for the four-momentum cutoff scheme. The four-point couplings are nearly the same for both methods. Therefore we want to refer to section VA.

Intersections correspond to “perfect” solutions again, cf. parameter sets [R2i] and [R2ii] in TABLE IV. However, it seems that the Pauli-Villars regularisation needs a stronger coupling in the six-point term, in order to ensure the $\eta$-$\eta'$-splitting within the given restrictions of the up-quark mass. This may be related to the fact that in the Pauli-Villars regularisation higher momenta of the quark-loops are allowed and not ignored completely because of a sharp cutoff. Besides the different ansatz of both methods, the four-momentum cutoff and the Pauli-Villars scheme even lead to quantitative similar parameter sets, cf. TABLE VII to XI and TABLE XII to XIV. Here, the $\eta$ mass is around 100 MeV lower than experimentally expected as discussed earlier.

Again, the alternative solution represents a set of parameters where the $\eta$ mass is in good agreement with the experimental value but corresponds to negative values of the four-point coupling and unreasonable high values for $K$ at some certain value for $m_u$.

#### 2. Three regulators

Beside the Pauli-Villars regularisation scheme with two additional regulator terms, we have also investigated this
method with three regulators. The latter one is often used in the NJL model for in-medium discussions and hence should be taken into account as well. The parameters sets, which have been calculated analogously to the previous ones, are shown in TABLE XIV to XV. Again, we fixed the parameters to the mass of the \( \eta \)-meson and \( \eta' \)-meson, respectively. In order to compare the results to the other methods the lower right panel of FIG. 5 and 8 shows the behaviour of the four-point and six-point couplings in dependence of the up-quark mass. Intersections correspond to “perfect” solutions, cf. parameter sets [R3i] and [R3ii] in TABLE IV.

One immediately notice the additional drop at around 350 MeV in the case where we fitted to the \( \eta \) mass. It turns out that we do not find reasonable parameter sets for \( m_u > 355 \text{ MeV} \). Therefore, in FIG. 7 no \( \eta' \)-mass can be calculated. Nevertheless, the fact that equal values of the couplings \( G \) and \( K \) do not have a critical impact on the related \( \eta' \)-mass before this point is strongly noticeable. It seems that compared to the other regularisation methods, the Pauli-Villars scheme with three regulators has the strongest dependence on the quark masses in terms of the \( \eta \)-meson. We assume this behaviour is related to the third PV regulator but further studies are required to confirm this conjecture.

The bare values we find for \( K \) are in this case way bigger compared to all other regularisation schemes.

| TABLE V. Comparing calculated results for \( m_{\eta'} \) using full and only real part of integrals (Pauli-Villars scheme with two regulators fit to \( \eta \) mass). |
|-----------------|--------|--------|--------|--------|--------|
| \( m_u \) [MeV] | 275.   | 300.   | 325.   | 350.   | 375.   |
| full integral   |        |        |        |        |        |
| \( m_{\eta'} \) [MeV] | 945.78 | 1047.06 | 1088.62 | 1127.75 | 1168.78 |
| only real part  |        |        |        |        |        |
| \( m_{\eta'} \) [MeV] | 1265.73 | 1292.40 | 1293.84 | 1299.50 | 1311.21 |

and neglected the imaginary part or just fitted the \( \eta' \) mass to high enough values of \( m_u \). While in the two flavour-case this shouldn’t be a problem because the emerging mesons, i.e. pions, only have small masses, this will not hold for the three flavour case anymore. In section III B we have seen that the \( \eta \) propagator of course is sensitive to a non-vanishing imaginary part of the integrals.

In TABLE V the parameter sets for various fixed \( m_u \) and \( m_{\eta'} \) are shown. Choosing the Pauli-Villars scheme with two regulators to be representative for all methods, cf. FIG. 8 it can be directly seen that the corresponding mass for the \( \eta' \)-meson is way to high if we only take the real part for the calculations. Therefore, no “perfect” solution can be found when using only the real part. This comes from the fact that the inverse \( \eta \) propagator contains multiple products of complex numbers, i.e. kinds of polarisation loops (cf. equation (20)). Hence, the imaginary part of a certain polarisation loop indeed contributes to the real part of the hole propagator. Since the root of the inverse \( \eta \)-propagator related to \( \eta' \) lies in the vicinity of the strange-anti-strange decay window, we have recalculated the parameter sets for the Pauli-Villars method with two regulators for fixed values of \( m_u \).

In order to increase the understanding TABLE XVIII in appendix D shows different parameter-sets for the two regulator Pauli-Villars regularisation method, where \( m_s \) has been fixed instead of \( m_u \). Due to the allowed decay into a strange-anti-strange quark pair for values of \( m_s < m_{\eta'}/2 \), the parameter sets do depend whether the full or only the real part of the integral has been taken into account. However, the decay into two up-quarks is in all of these sets allowed and therefore does not change the parameter sets at all. Despite the simplicity of this discussion, one has to keep the upper observation in mind when creating or using parameter sets for the three flavour NJL model.

VI. TREATMENT OF IMAGINARY PARTS

As already mentioned earlier, the imaginary part of the polarisation loops plays an important role for the inverse propagators of the mesons. In the literature we found that some authors, cf. reference [9, 12, 13] used the real part of the integrals.
VII. CONCLUSIONS AND OUTLOOK

All in all, we found for all regularisation methods three different sets of parameters where the mass of the up-quark has been treated as a free parameter. The different parameters sets for various regularisation methods, are displayed in appendix C. Here, one parameter set is fitted to the $\eta$ mass and two to the $\eta'$ mass. The ones fitted to $\eta$ show some interesting behaviour in terms of the couplings: While the fits to the $\eta'$ mass stay smooth, the ones fitted to $\eta$ show a dip around $m_u = m_{\eta}/2 = 274$ MeV. This is related to the allowed decay into a quark-antiquark pair which leads to a non-vanishing imaginary part of the meson propagator. Hence, the analytical behaviour of the propagator is different before and after this region.

It seems that the parameter fit to the $\eta'$-meson is more reasonable although its mass always lies outside the region of stability. Therefore, the related meson propagator always has a non-vanishing imaginary part. Nevertheless, under virtue of FIG. 5 and 6 the parameter sets coming from this ansatz are steady in terms of their charges over a wide range of given $m_u$. In addition, no negative values for $G$ have been found within the considered range of $m_u$. However, the parameter-sets of the so-called alternative solutions, which have been also fitted to $\eta'$, lead to negative values of the four-point coupling $G$ at a certain point.

Besides, we found that the Pauli-Villars regularisation method with three regulators has to be treated with some care when fitted to the $\eta$ mass. For $m_u > 355$ MeV it was not possible to find reasonable parameter. Here, further studies are required.

We have already mentioned the in-medium discussions of the NJL model and the corresponding thermodynamic potential, cf. section VI B 2. In fact, this ansatz can also be used to calculate parameter sets for the NJL model: Hereby one calculates the thermodynamic potential of the model via its partition function. The (physical) ground state of the model is then given by the global minimum of the thermodynamic potential. Of course one can do this in the limit of vanishing temperature and chemical potential, i.e. in the vacuum as well. Then the results can be compared with the ones of this work directly.

As an example, a short look at this method, already showed that the corresponding global minimum of the thermodynamic potential for the alternative solution is not as deep as the one for the other parameter sets. This holds for all regularisation methods. However, we want to emphasise at this point that this may not have any importance for a certain study within the NJL model framework, but can be seen as an indicator that the alternative solution have to be treated with some care.

The “perfect” parameter sets however, which reproduces the $\eta$ and $\eta'$ mass correctly, correspond to intersections of the fits to each mass separately, cf. TABLE XVIII and FIG. 5 and 6. Here, we have found that only intersections of the alternative solution and the fit to $\eta$ lead to results. Hence, two possible sets can be identified: Intersections before the dip in the alternative solutions corresponds to a set with a stable $\eta$-meson. The other intersections meanwhile corresponds to a set with an unstable $\eta$-meson.

Within the upper discussion we have investigated the impact of neglecting the imaginary part of the integrals and discussed it exemplary on the Pauli-Villars method with two regulators, cf. section VI When fitting to the $\eta'$ mass, for $m_u \geq m_{\eta'}/2$ the results do not depend whether one uses the full integral or only the real part. However, smaller $m_u$ will lead to different results, cf. TABLE XVIII In terms of $m_u$ this means that only for $m_u \geq 259$ MeV the obtained parameter sets do not depend on the treatment of the possible imaginary part of the integrals.

When fitting to the $\eta$ mass, neglecting the contribution of the imaginary part in the polarisation loop yields unreasonable high masses for the $\eta'$ meson, cf. TABLE XVIII In order to choose a wider range of constituent masses and to be sure to have consistent results the fully complex character of the polarisation loops should be taken into account.

Since the NJL model is often used as an effective model of QCD for in-medium studies, a comprehensive discussion of the different regularisation methods could be of particular interest. Especially a comparison of the obtained phase structure for each method and, moreover, with other approaches like FRG or lattice calculations, could give more insights in where the model boundaries lie. In addition other fit parameters like the mixing angle of the $\eta$ and $\eta'$-meson or the kaon decay constant can be used to fit the free parameters of the model. Alternatively one can also used the parameter sets of this work in order to calculate certain physical observables to verify their validity in the context of the NJL model as an effective model of QCD.

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Appendix A: Integrals

Evaluating quark condensate (4) and polarisation loops (12) will lead to the following three integrals

\[ I_1(M) \equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - M^2 + i\epsilon}, \quad \text{(A1)} \]

\[ I_2(p^2, M) \equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p+k)^2 - M^2 + i\epsilon} \left[ k^2 - M^2 + i\epsilon \right], \quad \text{(A2)} \]

\[ I_3(p^2, M_1, M_2) \equiv i \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p+k)^2 - M_1^2 + i\epsilon} \left[ k^2 - M_2^2 + i\epsilon \right]. \quad \text{(A3)} \]

Of course, these integrals are divergent and have to be regularized. E.g. for integral \( I_1 \), using the four-momentum cutoff scheme and Wick-rotating the integral, we can evaluate it directly and find

\[ I_1^{\text{C4}}(M, \Lambda) = \frac{1}{16\pi^2} \left( \Lambda^2 + M^2 \ln \left( \frac{M^2}{M^2 + \Lambda^2} \right) \right). \quad \text{(A4)} \]

For the other regularisation methods we perform the \( k_0 \) integration using Cauchy’s residue theorem. The remaining three-dimensional spherical integral can be computed directly. In the three-momentum cutoff case it reads

\[ I_1^{\text{C3}}(M, \Lambda) = \frac{1}{8\pi^2} \left( \Lambda \sqrt{M^2 + \Lambda}^2 + M^2 \ln \left( \frac{M}{\sqrt{M^2 + \Lambda^2} + \Lambda} \right) \right) \quad \text{(A5)} \]

and for Pauli-Villars regularisation scheme

\[ I_1^{PV}(M, \Lambda) = \frac{1}{16\pi^2} \sum_{j=0}^{N} c_j M_j^2 \ln(M_j^2), \quad \text{(A6)} \]

where \( N \in \{2, 3\} \) denotes the number of regulators, \( c_j \) and \( M_j(M, \Lambda) \) are defined in section IV B.

Integral (A3) remains real for \( p^2 > (M_1 + M_2)^2 \) and gets imaginary contributions for \( p^2 < (M_1 + M_2)^2 \).

Since we have to evaluate at e.g. \( p^2 = m_\eta^2 \) when fixing the \( \eta \) mass, the correspondence between upcoming imaginary part and allowed decay into quark-antiquark pair is obvious.
Appendix B: Polarisation loops

From equation (12) and TABLE I we can directly read of the definition of the pion polarisation loop

\[-i\Pi_\pi(p^2) \equiv -\int \frac{d^4k}{(2\pi)^4} \text{Tr} [i\gamma_5\tau_3 S(p + k)i\gamma_5\tau_3 S(k)] .\]  

Due to the isospin limit the polarisation loops for all three pions are the same. This justifies the above definition. A straightforward calculation leads to

\[\Pi_\pi(p^2) = 8N_c I_1(m_u) - 4N_c p^2 I_2(p^2, m_u) , \quad (B2)\]

where we recognize integral \(I_1\) and \(I_2\) (cf. appendix A). The kaon polarisation loop is defined as

\[-i\Pi_K(p^2) \equiv -\int \frac{d^4k}{(2\pi)^4} \text{Tr} [i\gamma_5\tau_1^+ S(p + k)i\gamma_5\tau_1^- S(k)] , \quad (B3)\]

where \(\tau_1^\pm \equiv \frac{1}{\sqrt{2}} (\tau_1 \pm i\tau_2)\). As in the pion case, all four kaon polarisation loops are degenerated and we find

\[\Pi_K(p^2) = 4N_c (I_1(m_u) + I_1(m_s)) + 4N_c \left((m_u - m_s)^2 - p^2\right) I_3(p^2, m_u, m_s) , \quad (B4)\]

where the more general integral \(I_3\) appears instead of \(I_2\). As discussed in section III A, the \(\eta\) and \(\eta'\) are composite particles. Therefore, also off-diagonal polarisation loops are relevant in this channel

\[-i\Pi_{00}(p^2) = -\int \frac{d^4k}{(2\pi)^4} \text{Tr} [i\gamma_5\tau_0 S(p + k)i\gamma_5\tau_0 S(k)] , \quad (B5)\]

\[-i\Pi_{08}(p^2) = -\int \frac{d^4k}{(2\pi)^4} \text{Tr} [i\gamma_5\tau_8 S(p + k)i\gamma_5\tau_8 S(k)] , \quad (B6)\]

\[-i\Pi_{08}(p^2) = -\int \frac{d^4k}{(2\pi)^4} \text{Tr} [i\gamma_5\tau_8 S(p + k)i\gamma_5\tau_8 S(k)] , \quad (B7)\]

where we dropped the superscript PS for simplicity. The third polarisation loop is symmetric, i.e. \(\Pi_{08} = \Pi_{00}\). An explicit calculation leads to

\[\Pi_{00}(p^2) = \frac{8}{3} N_c \left(2 I_1(m_u) + I_1(m_s)\right) - \frac{4}{3} N_c p^2 \left(2 I_2(p^2, m_u) + I_2(p^2, m_s)\right) , \quad (B8)\]

\[\Pi_{08}(p^2) = \frac{8}{3} N_c \left(I_1(m_u) + 2 I_1(m_s)\right) - \frac{4}{3} N_c p^2 \left(I_2(p^2, m_u) + 2 I_2(p^2, m_s)\right) , \quad (B9)\]

\[\Pi_{08}(p^2) = \frac{8}{3} \sqrt{2} N_c \left(I_1(m_u) - I_1(m_s)\right) - \frac{4}{3} \sqrt{2} N_c p^2 \left(I_2(p^2, m_u) - I_2(p^2, m_s)\right) . \quad (B10)\]

All other combinations \((a, b)\) in (12) lead to vanishing polarisation loops.
### TABLE VI. Parameter sets with four-momentum cutoff scheme, fit to $\eta$ mass.

| [C4A] | [C4B] | [C4C] | [C4D] | [C4E] |
|-------|-------|-------|-------|-------|
| $m_u$ [MeV] | 275. | 300. | 325. | 350. | 375. |
| $m_s$ [MeV] | 407.059 | 472.862 | 508.499 | 539.649 | 568.744 |
| $\Lambda$ [MeV] | 864.712 | 813.879 | 780.107 | 757.288 | 741.848 |
| $GA^2$ | 0.434 | 1.742 | 1.986 | 2.141 | 2.274 |
| $KA^5$ | 341.7 | 234.533 | 234.381 | 245.172 | 259.929 |
| $m_{0,u}$ [MeV] | 6.529 | 7.19 | 7.682 | 8.038 | 8.286 |
| $m_{0,s}$ [MeV] | 175.06 | 187.221 | 195.661 | 201.353 | 204.973 |
| $m_{\eta'}$ [MeV] | 948.594 | 1047.95 | 1090.13 | 1130.31 | 1172.48 |

### TABLE VII. Parameter sets with four-momentum cutoff scheme, fit to $\eta'$ mass.

| [C4A'] | [C4B'] | [C4C'] | [C4D'] | [C4E'] |
|-------|-------|-------|-------|-------|
| $m_u$ [MeV] | 275. | 300. | 325. | 350. | 375. |
| $m_s$ [MeV] | 496.977 | 520.78 | 543.293 | 565.059 | 586.331 |
| $\Lambda$ [MeV] | 864.712 | 813.879 | 780.107 | 757.288 | 741.848 |
| $GA^2$ | 3.487 | 3.711 | 3.946 | 4.195 | 4.458 |
| $KA^5$ | 63.349 | 66.819 | 71.1 | 53.75 | 79.777 |
| $m_{0,u}$ [MeV] | 6.529 | 7.19 | 7.682 | 8.038 | 8.286 |
| $m_{0,s}$ [MeV] | 175.765 | 187.475 | 195.645 | 201.178 | 204.741 |
| $m_{\eta}$ [MeV] | 454.361 | 455.479 | 457.737 | 460.34 | 462.866 |

### TABLE VIII. Parameter sets with three-momentum cutoff scheme, fit to $\eta$ mass - alternative solution.

| [C3A] | [C3B] | [C3C] | [C3D] | [C3E] |
|-------|-------|-------|-------|-------|
| $m_u$ [MeV] | 275. | 300. | 325. | 350. | 375. |
| $m_s$ [MeV] | 411.648 | 427.756 | 433.564 | 432.229 | 425.302 |
| $\Lambda$ [MeV] | 864.712 | 813.879 | 780.107 | 757.288 | 741.848 |
| $GA^2$ | 0.642 | -0.646 | -3.661 | -10.55 | -29.71 |
| $KA^5$ | 321.704 | 447.566 | 723.781 | 1350.18 | 2982.23 |
| $m_{0,u}$ [MeV] | 6.529 | 7.19 | 7.682 | 8.038 | 8.286 |
| $m_{0,s}$ [MeV] | 175.083 | 186.9 | 195.353 | 201.325 | 205.467 |
| $m_{\eta}$ [MeV] | 547.326 | 562.131 | 569.366 | 573.151 | 574.733 |

### TABLE IX. Parameter sets with three-momentum cutoff scheme, fit to $\eta'$ mass.

| [C3A'] | [C3B'] | [C3C'] | [C3D'] | [C3E'] |
|-------|-------|-------|-------|-------|
| $m_u$ [MeV] | 275. | 300. | 325. | 350. | 375. |
| $m_s$ [MeV] | 479.414 | 501.923 | 522.659 | 542.986 | 563.309 |
| $\Lambda$ [MeV] | 864.712 | 813.879 | 780.107 | 757.288 | 741.848 |
| $GA^2$ | 1.672 | 1.747 | 1.817 | 1.889 | 1.964 |
| $KA^5$ | 8.793 | 8.809 | 9.109 | 9.439 | 9.714 |
| $m_{0,u}$ [MeV] | 4.569 | 4.971 | 5.251 | 5.434 | 5.542 |
| $m_{0,s}$ [MeV] | 126.189 | 133.105 | 137.451 | 139.91 | 140.996 |
| $m_{\eta}$ [MeV] | 472.606 | 476.756 | 483.348 | 489.757 | 495.223 |

### TABLE X. Parameter sets with three-momentum cutoff scheme, fit to $\eta'$ mass - alternative solution.

| [C3a'] | [C3b'] | [C3c'] | [C3d'] | [C3e'] |
|-------|-------|-------|-------|-------|
| $m_u$ [MeV] | 275. | 300. | 325. | 350. | 375. |
| $m_s$ [MeV] | 382.951 | 403.442 | 415.309 | 419.04 | 425.302 |
| $\Lambda$ [MeV] | 864.712 | 813.879 | 780.107 | 757.288 | 741.848 |
| $GA^2$ | 0.591 | 0.389 | -0.108 | -1.298 | -4.878 |
| $KA^5$ | 45.679 | 51.758 | 67.236 | 103.062 | 207.964 |
| $m_{0,u}$ [MeV] | 4.569 | 4.971 | 5.251 | 5.434 | 5.542 |
| $m_{0,s}$ [MeV] | 130.042 | 137.256 | 142.28 | 145.783 | 148.23 |
| $m_{\eta}$ [MeV] | 546.266 | 558.499 | 564.17 | 567.338 | 568.317 |
TABLE XII. Parameter sets with Pauli-Villars scheme and two regulators, fit to $\eta$ mass.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 275. 300. | 325. 350. 375. | 633.813 | 1.583 | 1.685 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta'}$ [MeV] |
|-----------------|-----------------|-----------------|
| 6.392 | 7.039 | 6.071 |

TABLE XIII. Parameter sets with Pauli-Villars scheme and two regulators, fit to $\eta'$ mass.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 495.89 | 519.597 | 647.354 | 1.389 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta}$ [MeV] |
|-----------------|-----------------|-----------------|
| 6.392 | 7.039 | 6.071 |

TABLE XIV. Parameter sets with Pauli-Villars scheme and two regulators, fit to $\eta'$ mass - alternative solution.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 410.266 | 426.897 | 647.354 | 1.389 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta}$ [MeV] |
|-----------------|-----------------|-----------------|
| 6.392 | 7.039 | 6.071 |

TABLE XV. Parameter sets with Pauli-Villars scheme and three regulators, fit to $\eta$ mass.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 275. 300. | 325. 350. 375. | 633.813 | 1.583 | 1.685 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta'}$ [MeV] |
|-----------------|-----------------|-----------------|
| 8.002 | 8.896 | 9.596 |

TABLE XVI. Parameter sets with Pauli-Villars scheme and three regulators, fit to $\eta'$ mass.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 509.785 | 535.055 | 581.39 | 5.332 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta}$ [MeV] |
|-----------------|-----------------|-----------------|
| 8.002 | 8.896 | 9.596 |

TABLE XVII. Parameter sets with Pauli-Villars scheme and three regulators, fit to $\eta'$ mass - alternative solution.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 427.127 | 438.532 | 437.314 | 5.332 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta}$ [MeV] |
|-----------------|-----------------|-----------------|
| 8.002 | 8.896 | 9.596 |

TABLE XVIII. Parameter sets with Pauli-Villars scheme and three regulators, fit to $\eta'$ mass - alternative solution.

| $m_u$ [MeV] | $m_s$ [MeV] | $\Lambda$ [MeV] | $GA^2$ | $KA^3$ |
|------------|-------------|-----------------|--------|--------|
| 427.127 | 438.532 | 437.314 | 5.332 |

| $m_{0,u}$ [MeV] | $m_{0,s}$ [MeV] | $m_{\eta}$ [MeV] |
|-----------------|-----------------|-----------------|
| 8.002 | 8.896 | 9.596 |

TABLE XIX. Parameter sets with Pauli-Villars scheme and three regulators, fit to $\eta'$ mass - alternative solution.
Appendix D: Comparison between real part and full integrals

| $m_s$ [MeV] | 430. | 460. | 470. | 476. | 477. | 478. | 479. | 480. | 490. | 500. | 530. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| full integral | | | | | | | | | | | |
| $m_s$ [MeV] | 176.219 | 210.395 | 230.203 | 246.241 | 249.428 | 253.003 | 259.029 | 259.704 | 269.117 | 279.196 | 311.465 |
| $\Lambda$ [MeV] | 1496.79 | 1021.18 | 891.821 | 821.866 | 810.5 | 798.57 | 780.238 | 778.294 | 753.755 | 731.677 | 681.641 |
| $G\Lambda^2$ | 2.369 | 2.467 | 2.493 | 2.472 | 2.462 | 2.446 | 2.404 | 2.412 | 2.473 | 2.535 | 2.74 |
| $K\Lambda^5$ | 7.186 | 8.589 | 13.047 | 19.36 | 20.996 | 23.047 | 27.413 | 27.196 | 27.237 | 27.657 | 29.717 |
| $m_{0,u}$ [MeV] | 1.987 | 3.786 | 4.733 | 5.406 | 5.529 | 5.663 | 5.879 | 5.902 | 6.213 | 7.279 | |
| $m_{0,s}$ [MeV] | 70.834 | 118.212 | 139.549 | 153.507 | 155.961 | 158.598 | 162.769 | 163.227 | 169.135 | 174.688 | |
| $m_q$ [MeV] | 330.935 | 334.019 | 375.166 | 428.076 | 438.446 | 457.239 | 456.316 | 456.342 | 456.641 | 457.861 | |

| only real part | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_s$ [MeV] | 213.294 | 240.232 | 249.958 | 255.969 | 256.984 | 258.002 | 259.029 | 259.704 | 269.117 | 279.196 | 311.465 |
| $\Lambda$ [MeV] | 998.192 | 845.377 | 808.68 | 789.275 | 786.215 | 783.203 | 780.238 | 778.294 | 753.755 | 731.677 | 681.641 |
| $G\Lambda^2$ | 2.227 | 2.322 | 2.363 | 2.39 | 2.394 | 2.399 | 2.404 | 2.412 | 2.473 | 2.535 | 2.74 |
| $K\Lambda^5$ | 24.532 | 25.888 | 26.616 | 27.132 | 27.224 | 27.318 | 27.413 | 27.196 | 27.237 | 27.657 | 29.717 |
| $m_{0,u}$ [MeV] | 3.932 | 5.164 | 5.549 | 5.771 | 5.807 | 5.843 | 5.879 | 5.902 | 6.213 | 7.279 | |
| $m_{0,s}$ [MeV] | 121.503 | 148.491 | 156.303 | 160.674 | 161.379 | 162.078 | 162.769 | 163.227 | 169.135 | 174.688 | |
| $m_q$ [MeV] | 436.972 | 451.18 | 454.521 | 456.342 | 456.641 | 456.94 | 457.239 | 456.316 | 455.34 | 455.454 | 457.861 |

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