Structural hysteresis model of transmitting mechanical systems

M Ruderman and T Bertram
Institute of Control Theory and Systems Engineering, TU-Dortmund, Dortmund, Germany
E-mail: mykhaylo.ruderman@tu-dortmund.de

Abstract. We present a structural hysteresis model which describes the dynamic behavior of transmitting mechanical systems with a hysteretic spring and damped bedstop element, both connected in series. From the application point of view this approach can be used for predicting the transmitted mechanical force based only on the known kinematic excitation. Using the case study of an elastic gear transmission we show and identify a hysteresis response which multivariate behavior depends on an internal state of the bedstop motion.

1. Introduction
The mechanical transmission elements can exhibit strongly pronounced nonlinear characteristics, in particular in relation to the lightweight structures, complex mechanical assemblies, and frictional interfaces. A phenomenological and system-oriented modeling of such mechanical structures frequently leads back to the use of elementary and compound nonlinear operators like saturation, dead-zone, Prandtl-Ishlinskii play and stop type, Preisach, and others. Several of them are of the hysteresis type and exhibit a memory effect. For more details to the hysteresis type operators we refer to [1],[2],[3],[4]. Besides the mentioned hysteresis operators the differential models of hysteresis have gained a wide acceptance, in particular in the filed of structural mechanics. One of them, which is particularly suitable to describe the hysteresis restoring forces, is the model originally proposed by Bouc [5] and elaborated by Wen [6]. A survey on the past and recent developments and implementations of the Bouc-Wen hysteresis model is provided in [7]. For a compact analytical representation of the frictional interfaces in mechanical systems a number of physics- and empirically-motivated dynamic friction models has been proposed over the last two decades. For an overview of the development of friction models and characterization of friction force dynamics we refer to [8].

Our recent work is concerned with developing and evaluating a structural hysteresis model of transmitting mechanical systems. The model we aim to introduce is case-specific and is targeted towards the class of mechanical systems which include the damped bedstop motion and nonlinear (hysteresis) spring(s) connected in series. The proposed modeling approach aims to predict the transmitted mechanical force based only on the kinematic excitation. Often, the latter constitutes the single known value measured on the interface to the system environment.
2. Force transmitting mechanical structure with hysteresis

2.1. General consideration

A transmitting mechanical system with the damped bedstop element and nonlinear hysteresis spring is shown in Figure 1. Both mechanical transmission elements are connected in series through an internal rigid interface which degree of freedom is denoted by $z$. We note that a particular affiliation of the mechanical transmission elements and the total number of degrees of freedom in the system depend on the application case. However, the principal structure of a system which includes bedstop and hysteresis nonlinearities can be assumed like this. Further, we will provide a case study which matches the model structure presented in Figure 1.

![Figure 1. Mechanical structure of the force transmitting system with a bedstop motion and hysteresis spring. The kinematic excitation $x$ and transmitted force $F$ are the single input and output values measured in the system](image)

The overall transmitting system undergoes a kinematic excitation in terms of a time-variant relative displacement $x(t)$ which acts on the left-hand side interface from the environment. The right-hand side interface is connected to the ground of the basic system which takes on, or more specifically consumes, the transmitted force $F(t)$. Both physical quantities constitute the system input and output and are indicated in Figure 1 as measurable. Such mechanical structures (substructures) can be found in numerous applications like vehicle and bogie suspension systems, wheel-tire systems, active dampers [9], gear mechanisms [10], and others. The principle difference to the common Maxwell model, which is represented by a purely viscous damper and elastic spring connected in series, are the nonlinear characteristics of both mechanical transmission elements. On the one hand, the damped relative motion is subject to the bedstop constraints. That is the kinematic excitation of the bedstop element transforms into the force excitation of the hysteresis spring up to the system attains a dead stop state. We note that the internal state $z(t)$ provides a transition between the kinematic and force excitation. On the one hand, the nonlinear hysteresis spring exhibits multiple equilibrium states and can dissipate energy by itself without additional dampers.

Examining in details the mechanical structure shown in Figure 1, we can infer that the composite system provides both the forwards and feedback couplings. An interplay between the dynamics of both nonlinearities occurs through the friction on a sliding interface on the one hand, and through the dead stop state of a rigid contact on the other hand.

2.2. Elastic gear transmission

As a case study we consider an elastic gear transmission which serves to boost the actuator drive torque and to provide a reduced angular motion to the output load. Such systems can be found predominantly in robotics [11] but also in machinery, aeronautic, and medical engineering. Since evermore lightweight and compliant structures are deployed in technology the torsion related issues have to be regarded when analyzing the dynamic system behavior.

The experimental setup of an elastic gear transmission is shown in Figure 2. Approaching the general mechanical structure introduced previously in Figure 1 the output shaft behind the gear transmission is locked. The free-motion torque sensor applied between the gear unit and locked-load mechanism allows to measure immediately the transmitted torque. The kinematic excitation at the free (left-hand side) end of the drive train is provided by a BLDC (brushless
direct current) motor through the controlled angular displacement of the motor shaft. We note that for the constrained motion experiments with the locked output shaft no measurements using the output encoder are required. The input angular displacement $x$ and the transmitted output torque (further as force) $F$ are the single system observations we are interested in. In between these values the complex mechanical assembly, including the couplings and gear mechanisms, constitutes a grey-box system for which the structure depicted in Figure 1 is assumed.

In Figure 3, we report the experimental results of the force response obtained at the periodic kinematic excitation of different frequencies. The measured transmitted force is shown over the angular displacement. For the sake of clarity, the angular displacement is denoted in the coordinates of the gear output, i.e. with a nominal gear reduction of 160:1. Three sinusoidal sequences of 0.05 Hz, 0.5 Hz, and 5 Hz have been applied consecutively, with a short break between each experiment. The measured hysteresis curves are symmetric to the point of origin and feature an apparent major loop obtained at 0.5 Hz excitation. The hysteresis response at 5 Hz is shown for two different initial states I and II. The transmitted force $F$ is measured between the gear output and locked rotary shaft.
relaxation range of the mechanical structure or to a presence of secondary backlash effects which are not captured by the assumed bedstop motion.

3. Modeling

3.1. Bouc-Wen like hysteresis model

To represent a nonlinear mechanical spring with hysteresis the overall restoring force \( H \) has to be computed in an analytical way as a function of the relative displacement \( z \). One of the most widely accepted models for mechanical structures is a differential model proposed by Bouc [5] and generalized by Wen [6]. This “semi-physical” model connects the relative displacement (originally deformation) and restoring force through a first-order nonlinear differential equation implying hysteresis. By choosing the appropriate values of unspecified parameters it is possible to vary substantially the shape of the hysteresis loops. The application specific tuning of the hysteresis shape parameters and numerous extensions of the Bouc-Wen hysteresis model are reported in the literature. For more details and a survey we refer to [12, 7].

In this work, the inelastic restoring force is described using a Bouc-Wen like hysteresis model

\[
H(z, t) = w S(z) z(t) + (1 - w) S(z) h(t) .
\] (1)

The equation (1) constitutes a weighted superposition of an elastic (\( z \)-dependent) and a hysteretic (\( h \)-dependent) contribution. The weighting coefficient \( 0 < w < 1 \) provides the relation between a purely elastic \( w = 1 \) and purely hysteretic \( w = 0 \) force response. The hysteresis state \( h \) is given in its differential form by

\[
\dot{h}(t) = \alpha \dot{x}(t) - \beta |\dot{x}(t)|^n h(t) - \gamma x(t) |h(t)|^n .
\] (2)

The parameters \( \alpha, \beta, \) and \( \gamma \) constitute the basic hysteresis shape control. Note that once the control parameters \( \beta \) and \( \gamma \) are zero the state equation (2) becomes the simple elastic case which represents a linear spring. The exponential parameter \( n \geq 1 \) assigns the smoothness of transition from the elastic to the post-elastic branch. For the small values (\( 1 < n < 2 \)) the transition is smooth while for large \( n \) values the transition is abrupt. Over the years, the original Bouc-Wen model has been extended several times. Many new parameters have been added to fit a variety of hysteretic shapes [12]. However, it has been found that several parameters of Bouc-Wen like hysteresis models are functionally redundant. Even for the earliest version of the Bouc-Wen model, which is given by equation (2), a redundancy can be removed by setting \( \alpha = 1 \).

Another modeling issue than the modification of the Bouc-Wen hysteresis equation (2) in concerned with mapping of the stiffness characteristic curve. The original form as well as several later formulations of the Bouc-Wen like hysteresis models (see [7] for survey) imply the single stiffness coefficient instead of \( S(\cdot) \) in equation (1). However, the underlying elastic stiffness characteristics can be nonlinear, for which the most common hardening effect is one of the typical phenomena. Therefore, we stress that a nonlinear stiffness map \( S(z) \) can be directly integrated into a Bouc-Wen like hysteresis model as provided in equation (1). In this case the displacement-dependent stiffness map is captured by a piecewise linear function with multiple characteristic sampling points \( \{ z_i \} \) over the total domain \( \{ z_i \} \in \mathbb{Z} \subset \mathbb{R} \). An alternative way to reshape the overall inelastic restoring force is to use an appropriate polynomial function \( S(z) : z \mapsto \mathbb{R}_+ \) like in [13].

3.2. Modified Maxwell-Slip representation of pre-sliding friction

On the sliding interface of the bedstop element the friction is considered mainly as a function of the relative displacement. Assuming a limited motion range of the bedstop slider no velocity- and acceleration-dependent effects of the transient friction are taken into consideration. At these simplifying conditions the mechanisms of the pre-sliding friction [14] are predominant that
includes the Coulomb friction and the pre-sliding hysteresis behavior. The latter means that the friction-displacement hysteresis curves arise at the motion onset and motion reversals. The hysteresis branches saturate at a constant level of the friction force which, in the simplest case, is represented by a Coulomb friction. Note that considering a time-invariant normal load and adhesion coefficient a straightforward assumption of the constant Coulomb friction force is made.

One of the ways to describe the pre-sliding friction behavior bases on the Prandtl-Ishlinskii stop-type (PIS) operator [2, 4] which is widely used for modeling systems with hysteresis. Given the Prandtl-Ishlinskii play-type (PIP) operator [2]

\[ y(t) = p[x](t) , \]

also known as mechanical backlash, the PIS operator is obtained by

\[ y(t) = s[x](t) = x(t) - p[x](t) . \]

The transfer characteristics of the PIS operator are shown in Figure 4. The single threshold parameter of the PIP operator characterizes the saturation level of the PIS one. Note that using additional scaling factors for \( x \) and \( y \) the branches of the transfer characteristic can be reshaped.

In context of modeling the pre-sliding friction the PIS operator describes a simplified elastoplastic characteristic of contacting surfaces, given in force-displacement coordinates. That is the surface asperities in contact undergo first a linear elastic deformation before breaking loose. Afterwards, the sliding surfaces behave like a plastic-type element until the next motion reversal. We note that the PIS transfer characteristics are also referred to as Maxwell-slip element when modeling the dynamic friction [8]. The Maxwell-slip (elastoplastic) element is represented by a massless block placed on a rubbing surface and connected to the input through a linear spring. The block is either sticking or slipping depending on the relative displacement \( x \) applied to the opposite end of connecting spring. The stiffness coefficient of the spring and maximum (break-away) force characterize the single Maxwell-slip element and determine the horizontal lines and slopes of the transfer characteristics shown in Figure 4. Connecting in parallel multiple Maxwell-slip elements with individual stiffness and break-away properties one obtains a hysteresis map with nonlocal memory characteristics. Note that a larger number of elastoplastic elements is assumed higher accuracy of mapping the pre-sliding hysteresis behavior can be achieved. However, this can claim a higher effort for identifying parameters of the distributed model. The Maxwell-slip structure also denoted as Maxwell-slip model has been successfully applied in several dynamic friction models like Leuven and Generalized Maxwell-slip (GMS) one [8].

For the sake of convenience and in terms of keeping the number of free parameters as low as possible we apply the so called Modified Maxwell-slip (MMS) model [15] of pre-sliding friction. The MMS model implies the single elastoplastic element connected to the input through a spring, in a similar way to the elementary Maxwell-slip element. However, the connecting spring is no longer linear and exhibits a saturating elastoplastic deflection until the next motion reversal. The displacement-dependent stiffness is assumed to decrease exponentially thus providing a smooth hysteresis transition to the plastic sliding. The transfer characteristics of the MMS element are shown in Figure 5. We note that each motion reversal gives rise to a branching transfer. The MMS model in its differential form is given by

\[ \dot{y}(t) = |\Omega| \dot{x}(t) K \exp(-K|x_{r}(t)|) \]

with

\[ x_{r}(t) = \begin{cases} 0, & \text{when motion reverses at } t_{r} \\ t \int_{t_{r}}^{t} \dot{x}(t) \, dt, & \text{else} \end{cases} \]
Figure 4. Prandtl-Ishlinskii stop-type (PIS) operator (in context of friction modeling also known as Maxwell-slip element). The slope transitions are reversible.

Figure 5. Modified Maxwell-slip (MMS) element for representation of pre-sliding friction. Each input reversion induces a novel (exponentially) converging curve.

The variable stiffness capacity

$$\Omega(t_r) = \text{sgn}(\dot{x}(t)) F_c - y(t_r)$$

(7)

memorizes the last reversal state each time the relative motion changes the direction. The equation (7) insures that the hysteresis branch converges at the constant Coulomb friction $F_c$ independent of the reversal state $y(t_r)$. The parameterizable initial stiffness $K$ determines the exponential rate with which the friction force proceeds towards the plastic sliding. We note that the MMS model has a local and neither non-local memory characteristic owing to the $\Omega$ property. It means that only the last motion reversal is stored into the memory. The knowledge about previous friction states is erased once the motion direction changes. Nevertheless, the applied MMS model allows to map the smooth hysteresis trajectories in the force-displacement coordinates and to describe the pre-sliding friction using only two parameters to be identified.

3.3. Force transmission

The bedstop motion introduced in Section 2 constitutes a combination of the kinematic backlash and damped relative motion due to the friction. Since the bedstop element is connected in series with the hysteresis spring, according to Figure 1, the question is which amount of the force is transmitted to the output of overall mechanical structure. We note that due to nonlinear and constrained behavior of the bedstop element and hysteresis spring neither of classical approaches for modeling the viscoelasticity, like Maxwell model, Kelvin-Voigt model, or combination of them, can be efficiently used. Instead, the Bouc-Wen like hysteresis model and MMS model of pre-sliding friction defined so far are arranged in the manner that accomplishes the structure provided in Figure 1.

Let us consider two different modes of the executed bedstop motion as schematically shown in Figures 6 and 7. While the dead stop is not reached and the plunger moves at the applied kinematic excitation $x$ the friction force $F_f$ arises on the sliding interface. The same amount of the reactive force is transmitted to the output of the bedstop element and furthermore to the spring, so that $F_{\text{out}} = -F_f$ can be assumed. At the same time, the reactive force $F_{\text{out}}$ acts upon the hysteresis spring so that the relative position $z(t)$ changes. Once the dead stop is reached (see Figure 7), i.e. $z(t) = x(t)$, the kinematic excitation acts immediately upon the hysteresis spring so that $F_{\text{out}} = H(z)$. When the direction of the kinematic excitation changes the bedstop element remains first in the dead stop until the hysteresis spring becomes fully relaxed. Afterwards, the bedstop element sets in motion and the friction force occurs in the
opposite direction, similar as depicted in Figure 6. An interplay of the bedstop motion and hysteresis spring dynamics provides the overall transfer characteristics of the system.

![Figure 6](image1.png)  ![Figure 7](image2.png)

**Figure 6.** Schematic representation of the force transmission at bedstop motion. **Figure 7.** Schematic representation of the force transmission at dead stop.

We note that the backlash size of the system is unknown and has to be identified together with other model parameters. The backlash serves as a saturation operator of the kinematic excitation when computing the friction force on the basis of relative velocity

\[ F_f(t) = \text{MMS} \left( \dot{x}(t) - \dot{z}(t) \right). \]  

At the same time, the backlash appears as a dead-zone (DZ) operator [4] when considering the kinematic excitation of the hysteresis spring so that

\[ F_{out}(t) = H \left( \text{DZ}[x](t), t \right). \]  

The dead-zone size is the same as that of the backlash. However, the left and right dead-zone limits are time-variant and depend on the instantaneous \( z(t) \) value.

4. Identification of model parameters

The derived model of transmitting mechanical system provides several degrees of freedom concerning the bedstop motion dynamics and shape of the hysteresis spring. A full decomposition of the overall system response in the associated terms appears as not directly feasible, due to an internal feedback and the lack of internal state measurements. Therefore, an appropriate system excitation aims to reduce as possible the impact of one of the transfer elements while inducing a characteristic behavior of the other. The system response obtained in this way can be considered as an approximated decomposition of both nonlinearities.

The identification of free model parameters is accomplished in two stages. As a first step, the parameters of the Bouc-Wen like hysteresis model are identified using the experimental data of the main hysteresis loop measured at 0.5 Hz excitation (see Figure 3).

The experimental data of an increasing hysteresis branch shown in Figure 8 is used to determine the stiffness characteristic curve \( S(\cdot) \) described by a piecewise linear function. A compact set of overall twelve sampling points \( \{x_i, F_i\} \), depicted in Figure 8 over the measurement, is selected with respect to the curvature of the hysteresis branch. Following, the stiffness characteristic curve is determined in due consideration of the force bias \( F_1 \) related to the hysteresis loop so that

\[ \left\{ |z_i| = x_i, F_i - F_1 \right\} \in S(z). \]  

Note that a symmetric hysteresis spring is assumed so that an absolute \( z \) value is included in equation (10). The \( S(z) \) values between the sampling points are obtained using the linear
interpolation. We note that the major hysteresis loop measured at a relatively slow smooth excitation does not include the internal branching attributed to the bedstop motion. Under these conditions the hysteresis spring is assumed to be solely contributing to the output force.

In order to determine the residual parameters of the Bouc-Wen like hysteresis model the experimental time series of multiple hysteresis cycles obtained at the same 0.5 Hz excitation are used. Applying the standard Levenberg-Marquardt nonlinear fit algorithm the parameters are identified in a least-squares sense. In Figure 9, the overall response of the modeled hysteresis spring is shown over the measurement. It is worth noting that since the hysteresis produced by the Bouc-Wen like model is rate-independent the single frequency excitation is sufficient to determine the full set of the model parameters.

The parameters of the modeled bedstop motion, among of the MMS friction model and the backlash size, are determined from the experiment with a variable excitation frequency. For this purpose, a linear down-chirp excitation is applied which covers the frequency range between 5 Hz and 0.1 Hz. We note that since neither frictional conditions nor backlash size of the system are known in advance a continuous variation of the excitation frequency aims to disclose the interaction between the bedstop element and hysteresis spring. The measured force response is shown in Figure 10 as a time series together with the response of the overall identified model. The residual model parameters are identified from the data in a least-squares sense.
Figure 10. Identified and measured torque response with hysteresis spring and bedstop motion. The linear down-chirp excitation with a bandwidth 5–0.1 Hz is applied.

It can be seen that the impact of the pre-sliding friction is predominant at times below 10 seconds. At this stage the hysteresis spring is marginally stressed so that no significant restoring force is contributing to the overall system output. In particular, we can infer that the hysteresis spring response becomes perceptible once the applied frequency is low enough to drive the system predominantly in a dead stop state (see Figure 7). Further, it is worth noting that no knowledge of the initial hysteresis state, which is $h(0)$ in terms of the equations (1) and (2), contributes to the model uncertainties which are visible in Figure 10 at times between 10 and 15 seconds.

5. Conclusions
We have proposed and evaluated in experiments a novel structural hysteresis model of transmitting mechanical systems. The problem of predicting the mechanical force transmitted through a structure with the damped bedstop motion and nonlinear hysteresis spring has been addressed. We have shown that the computation of the output force based only on the known kinematic excitation can be realized in an analytical form. It has been shown that both system nonlinearities are interacting with each other in a forward and feedback manner. It has also been emphasized that an internal not-measurable state gives rise to a multivariate hysteresis behavior observable at the system output. We suggest that the identification of the model parameters has to be carried out using such system excitations which decompose as possible the impact of both nonlinearities. The results of this paper can be of use for an accurate modeling and simulation of transmitting mechanical structures in various physics and engineering fields.

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