Asymptotic Behaviors of Non-variational Elliptic Systems

Szu-yu Sophie Chen *

Abstract

We use a method, inspired by Pohozeev’s work, to study asymptotic behaviors of non-variational elliptic systems in dimension \( n \geq 3 \). As an application, we prove removal of an apparent singularity in a ball and uniqueness of the entire solution. All results apply to changing sign solutions.

A classical work by Gidas and Spruck [6] asserts that any nonnegative solution to \( \Delta u + |u|^\alpha - 2 u = 0 \) in \( \mathbb{R}^n \) with \( 2 < \alpha < \frac{2n}{n-2} \) (subcritical case) is trivial. For \( \alpha = \frac{2n}{n-2} \), Caffarelli, Gidas and Spruck [4] proved that any nonnegative solution in \( \mathbb{R}^n \) is of the form \( u = (a + b|x|^2)^{-\frac{n-2}{2}} \), where \( a, b \) are constants. Such problem for elliptic systems are also studied, for example, in the studies of Lane-Emden type systems; see Zou [17] and Polacik, Quittner and Souplet [10] and the references therein.

On the other hand, the behaviors of changing sign solutions are more delicate. For example, there exists a sequence of changing sign solutions to \( \Delta u + |u|^\alpha - 2 u = 0 \) in \( \mathbb{R}^n \) with \( 2 < \alpha < \frac{2n}{n-2} \); see [7]. In this paper, we study under what circumstances a solution to an elliptic system in an exterior domain is asymptotic to \( |x|^{-(n-2)} \) at the infinity. Such decay is optimal in the sense that the infinity is a regular point in the inverted coordinates. It is known [7] that there exist solutions to \( \Delta u + u^{\alpha - 1} = 0 \) in \( \mathbb{R}^n \) which decay slower than \( |x|^{-(n-2)} \). Thus, a suitable integrability condition is necessary to exclude such case.

While the study of changing sign solutions to elliptic systems is interesting by itself, the problem is well-motivated by differential geometry. For example, the decay of curvature tensors was studied for Yang-Mills fields [16], Einstein metrics [1] and other generalizations [15], [5], just to name a few. A typical system for the problem is of the form

\[
\Delta(Rm)_{ijkl} = Q_{ijkl}(Rm, Rm),
\]

where \( Rm \) is the Riemannian curvature tensor and \( Q \) is a quadratic in \( Rm \). A natural geometric assumption is that \( |Rm| \) is in \( L^{\frac{n}{2}} \). Therefore, \( |Rm| \) vanishes at infinity and the problem is to find out the decay rate. The study of geometrical systems is more subtle as \( (Rm)_{ijkl} \) satisfies an extra relation, the Bianchi identity, and the underlying spaces are not Euclidean.

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The technique we use in this paper is based on the method developed in [5] on asymptotically flat manifolds, where a special geometric setting is considered. In this paper, we study general non-variational elliptic systems of the reaction-diffusion type. Our result applies to changing sign solutions and includes the supercritical case (i.e., $\Delta u + Cu^{\alpha-1} = 0$ with $\alpha > \frac{2n}{n-2}$, where $C$ is a constant).

Let $V = (V_1, \cdots, V_m)$ and $f^i : \mathbb{R}^m \to \mathbb{R}$. Consider the system of equations

$$
\sum_{j=1}^{m} A_{ij} \Delta V_j = f^i(V),
$$

where $A$ is a constant positive definite symmetric matrix and $i = 1, \cdots, m$. The system (1) describes the steady states of the reaction-diffusion systems. The matrix $A_{ij}$ represents the diffusion rate and $f^i(V)$ is the reaction term. Note that $f^i(V)$ or $V_i$ may have no sign.

We assume the following structure conditions

(A1) $|f^i(V)| \leq C|V|^q$,

(A2) $|\nabla f^i(V)| \leq C|V|^{q-1}$.

Let $K$ be a compact subset in $\mathbb{R}^n$.

**Theorem 1.** Let $q > \frac{n+2}{n}$ and $p = \frac{n}{q}(q-1)$. Suppose that $f^i$ satisfies (A1) and (A2). Let $V \in L^p(\mathbb{R}^n \setminus K)$ be a solution to (1) in $\mathbb{R}^n \setminus K$. Then $|V| = O(|x|^{-(n-2)})$ and $|\nabla V| = O(|x|^{-(n-1)})$ at the infinity.

An immediate consequence is a result on singularity removal for affine invariant equations. For scalar equations, the problem was studied in [6], [3], [4].

Let $B_1$ be the unit ball centered at the origin.

**Corollary 1.** Suppose $f^i$ are homogeneous functions of degree $\frac{n+2}{n}$. Let $V \in L^2_n(B_1)$ be a solution to (1) in $B_1 \setminus \{0\}$. Then $V$ can be extended to a smooth solution to (1) in $B_1$.

By performing a linear transformation $W_i = \sum_j A_{ij}V_j$, the system (1) can be reduced to an equation of the diagonal form $\Delta W = \hat{f}(W)$. The assumptions (A1)-(A2) and other conditions on $V$ or $f^i$ equivalently hold for $W$ and $\hat{f}$. Therefore, for Theorem 1 and Corollary 1 we may assume without loss of generality the equation is of the diagonal form.

We turn to study the uniqueness of entire solutions for variational systems. Let $P(V)$ be a homogeneous function of degree $q+1$. Suppose that $f^i = \frac{\partial P}{\partial V^i}$ in (1). Note that (1) can not be reduced to a variational system of the form $\Delta W = \hat{f}(W)$ now; it can still be reduced to one with diagonal matrix $A_{ij}$. For scalar equations, there is a large literature on the uniqueness problem; e.g. Gidas and Spruck [6], Bidaut-Veron [2] and Serrin and Zou [14]; see also [12] and the references therein. For systems, when $P(V) \leq 0$ and $q > \frac{n+2}{n-2}$ (supercritical case), the problem was studied by Pucci and Serrin [11] under some asymptotic assumption of $V$. Their result also holds for non-homogeneous function $P$ (and more general $P(x, V, \nabla V)$) satisfying some inequality.
**Theorem 2.** Let $q > \frac{n+2}{n}$, $q \neq \frac{n+2}{n-2}$ and $p = \frac{n}{2}(q-1)$. Suppose $P(V)$ is a homogeneous function of degree $q+1$. Let $V \in L^p(\mathbb{R}^n)$ be a solution to (1) in $\mathbb{R}^n$ with $f^i = \frac{\partial P}{\partial V^i}$. Then $V \equiv 0$.

We give the outline of proofs. To fix notations, we denote by $dx$ the volume element in $\mathbb{R}^n$ and by $dS$ the area element of a hypersurface in $\mathbb{R}^n$. Let $B_r(x)$ and $S_r(x)$ be the ball of radius $r$ and sphere of radius $r$ centered at $x$, respectively. When $x$ is at the origin, we simply denote by $B_r$ and $S_r$.

The idea of proof of Theorem 1 is to compare the size of $\int_{\mathbb{R}^n} |\nabla V|^2dx$ (as a function of $r$) to its derivative $-\int_{S_r} |\nabla V|^2dS$. Then by ordinary differential inequality lemma, we get the optimal decay of $|\nabla V|$ and as a consequence the decay of $|V|$. In order to relate above two integrands, we use some version of Pohozaev’s identity for non-variational systems. Pohozaev’s ingenious idea [9] is to use a conformal killing field to prove uniqueness in a star-shaped domain. This idea was generalized nicely by Pucci and Serrin [11] to general variational systems. Our use of the identity is different from the original one. We apply the identity to an unbounded domain (the complement of a large ball) and use only the size of $|f^i|$. Therefore, our method can be applied to non-variational systems.

The proof of Theorem 2 is a combination of Theorem 1 and Pohozaev’s original idea. Since the solution decays fast enough at infinity, no terms from infinity contribute to the main integrand. We use the identity differently such that we obtain the uniqueness also in the subcritical case, in contrast to the problem in star-shaped regions where one has to restrict to the supercritical case.

Finally, we show that the assumptions in above Theorems are sharp.

Example 1: Consider the equation $\Delta u + u^q = 0$ in $\mathbb{R}^n$. By [4], nonnegative solutions are of the form $u = (a + b|x|^2)^{-\frac{n-2}{2}}$. Therefore, $u$ decays as $|x|^{-(n-2)}$ at the infinity. This example shows that in Theorem 2 the assumption $q \neq \frac{n+2}{n-2}$ is necessary. Consider instead the equation in $B_1 \setminus \{0\}$. There exists a nonnegative radial singular solution with the blow up rate $|x|^{-\frac{n-2}{2}}$ near the origin. Therefore, in Corollary 1 the condition $V \in L^{p}^{\frac{2n}{n-2}}(B_1)$ is sharp.

Example 2: Consider $\Delta u + u^q = 0$ in $\mathbb{R}^n$. For $q > \frac{n+2}{n-2}$, there exists a solution asymptotic to $|x|^{-\frac{n-2}{q}}$ at the infinity; see [7]. Hence, in Theorem 1 the conditions $q = \frac{2p+n}{n}$ and $V \in L^p$ are sharp. Moreover, in Theorem 2 the condition $q = \frac{2p+n}{n}$ is also sharp.

## 1 Preliminaries

We collect some standard results in elliptic regularity theory and ordinary differential equations. Lemma 1-3 follow by an argument similar to [1], section 4.

Let $C_s$ be the Sobolev constant and $\gamma = \frac{n}{n-2}$. Suppose that the nonnegative function $u \in C^{0,1}$ satisfies $\Delta u + C_0u^q \geq 0$ weakly in the sense that

$$\int (-\langle \nabla u, \nabla \phi \rangle + C_0 u^q \phi) dx \geq 0$$
for all $0 \leq \phi \in C_0^\infty$. Let $\varphi \geq 0$ be a function with compact support and $s > 1$. Then by Cauchy inequality,

$$\int \varphi^2 u^{q+s-1} \, dx \geq C_0^{-1} \int \left( \frac{4(s-1)}{s^2} |\varphi \nabla u^\varphi|^2 + \frac{4 \varphi u^\varphi (\nabla \varphi, \nabla u^\varphi) \right) \, dx \geq C_0^{-1} \int \left( \frac{2}{s^2} (s-1) |\varphi \nabla u^\varphi|^2 - \frac{2}{(s-1)} |\nabla \varphi|^2 u^\varphi \right) \, dx.$$

By Sobolev inequality, we have

$$\left( \int (\varphi^2 u^\varphi)^\gamma \, dx \right)^{\frac{1}{\gamma}} \leq C \int \left( \frac{s^2 C_0}{2(s-1)} \varphi^2 u^{q+s-1} + (1 + \frac{s^2}{(s-1)^2}) |\nabla \varphi|^2 u^\varphi \right) \, dx,$$

where $C = C(n, C_s, C_0)$.

In Lemma 13 $u$ is a $C^{0,1}$ function.

**Lemma 1.** Let $p > 1$ and $q = \frac{2p+n}{n}$. Suppose that the nonnegative function $u \in L^p(B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in $B_r$. Then there exists $\epsilon > 0$ such that if $\int_{B_r} u^p dx < \epsilon$, then $\sup_{B_{3r}} u \leq C r^{-\frac{n}{2}} \|u\|_{L^p(B_r)}$, where $C = C(n, p, C_s, C_0)$.

**Proof.** Let $s = p$ in (2). Then

$$\left( \int (\varphi^2 u^p)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} \leq C \int \left( \int_{\text{supp } \varphi} u^p + |\nabla \varphi|^2 u^p \right) \, dx \leq C \int_{\text{supp } \varphi} \left( \int (\varphi^2 u^p)^{\gamma} \, dx \right)^{\frac{1}{\gamma}} + C \int |\nabla \varphi|^2 u^p \, dx.$$

$\varphi$ is chosen to be a cutoff function such that $\varphi = 1$ in $B_{r/2}$ and $\varphi = 0$ outside $B_r$ with $|\nabla \varphi| \leq C r^{-1}$. We get

$$\left( \int_{B_{r/2}} u^p \, dx \right)^{\frac{1}{p}} \leq \frac{C}{r^2} \int_{B_r} u^p \, dx.$$

Choose a sequence $r_k = (2^{-1} + 2^{-k})r$. Apply (and rescale) the above inequality for $B_{r_k}$ and $B_{r_{k+1}}$ with $p_k = p \gamma^{k-1}$. By Moser iteration, we have $\sup_{B_{3r}} u \leq C r^{-\frac{n}{2}} \|u\|_{L^p(B_r)}$. \hfill $\square$

**Lemma 2.** Let $p > \frac{n}{n-2}$ and $q = \frac{2p+n}{n}$. Suppose that the nonnegative function $u \in L^p(\mathbb{R}^n \setminus B_r)$ satisfies $\Delta u + C_0 u^q \geq 0$ weakly in $\mathbb{R}^n \setminus B_r$. Then there exists $\epsilon > 0$ such that if $\int_{\mathbb{R}^n \setminus B_r} u^p \, dx < \epsilon$, then $u = O(|x|^{-\lambda})$ for all $\lambda < n-2$ as $|x| \to \infty$.

**Proof.** By Lemma 1 $u = O(|x|^{-n/p})$. Let $s = p \frac{n-2}{n} > 1$ in (2). Then

$$\left( \int \varphi^2 u^p \, dx \right)^{\frac{1}{p}} \leq C \int_{\text{supp } \varphi} \left( \int (\varphi^2 u^{p - \frac{2}{n}})^{\gamma} \, dx \right)^{\frac{1}{\gamma}} + C \int |\nabla \varphi|^2 u^{p - \frac{2}{n}} \, dx.$$
ϕ is chosen to be a cutoff function such that ϕ = 1 in B_{r'} \setminus B_2 \text{ and } ϕ = 0 \text{ outside } B_2 \setminus B_r \text{ with } |∇ϕ| ≤ C(1/r + 1/r'). Let r' → ∞. Then

\( \left( \int ϕ^2 u^p dx \right)^{\frac{1}{p}} \leq C \left( \int |∇ϕ|^n dx \right)^{\frac{2}{n}} \left( \int \{\text{supp } ∇ϕ\} u^p dx \right)^{\frac{1}{p}}. \)

And thus,

\( \left( \int_{\mathbb{R}^n \setminus B_{2r}} u^p dx \right)^{\frac{1}{p}} \leq C \left( \int_{B_{2r} \setminus B_r} u^p dx \right)^{\frac{1}{p}}. \)

This gives \( \int_{\mathbb{R}^n \setminus B_r} u^p = O(r^{-\delta}) \) for some small \( \delta > 0 \). Therefore, by Lemma 1, \( u = O(|x|^{-\lambda}) \). Let \( λ_0 = \sup\{λ : u = O(|x|^{-λ})\} \). By iteration and a contradiction argument, we get that \( λ_0 = n - 2 \).

Suppose that \( h ≥ 0 \) is a \( C^0 \) function. The nonnegative function \( u ∈ C^0,1 \) satisfies \( ∆u + C_0hu ≥ 0 \) weakly if

\( \int (-⟨∇u, ∇φ⟩ + C_0huφ) dx ≥ 0 \)

for all \( 0 ≤ φ ∈ C^∞_c \).

**Lemma 3.** Let \( p > 1 \) and \( t > \frac{n}{2} \). Suppose that the nonnegative function \( h ∈ L^t(B_r) \) satisfies \( ∫_{B_r} h^t dx ≤ \frac{C}{r^{2t-n}} \). Suppose also that the nonnegative function \( u ∈ L^p(B_r) \) satisfies \( ∆u + C_0hu ≥ 0 \) weakly in \( B_r \). Then \( \sup_{B_{2r}} u ≤ Cr^{-\frac{n}{2}}∥u∥_{L^p(B_r)}, \) where \( C = C(n, p, C_s, C_0, C_1) \).

**Proof.** The proof is by standard Moser iteration. See Morrey [8]. □

The following is a basic result in ordinary differential equations; see [3].

**Lemma 4.** Suppose that \( f(r) ≥ 0 \) satisfies \( f(r) ≤ -\frac{r}{a}f'(r) + C_2r^{-b} \) for some \( a, b > 0 \).

(a) \( a ≠ b \). Then there exists a constant \( C_3 \) such that

\( f(r) ≤ C_3r^{-a} + \frac{aC_2}{a-b}r^{-b}. \)

Therefore, \( f(r) = O(r^{-\min(a,b)}) \) as \( r → ∞ \).

(b) \( a = b \). Then there exists a constant \( C_3 \) such that

\( f(r) ≤ C_3r^{-a} + aC_2r^{-a} \ln r. \)

Therefore, \( f(r) = O(r^{-a} \ln r) \) as \( r → ∞. \)
2 Proof of Theorem 1

Proof of Theorem 1. We first derive a version of Pohozaev’s identity for non-variational systems. Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( N \) be the unit outer normal on \( \partial \Omega \). We will perform integration by parts repeatedly.

\[
\int_{\Omega} \sum_{k,l} f^k(V) x_i D_i V_k dx = \int_{\Omega} \sum_{i,j,k,l} A_{kj} \Delta V_j x_i D_l V_k dx
\]

\[
= \int_{\Omega} - \sum_{i,j,k,l} A_{kj} D_i V_j D_i(x_l D_l V_k) dx + \int_{\partial \Omega} \sum_{i,j,k,l} A_{kj} D_i V_j x_i D_l V_k N_l dS
\]

\[
= \int_{\Omega} \left( - \sum_{i,j,k} A_{kj} D_i V_j D_i V_k - \sum_{i,j,k,l} D_l(A_{kj} D_i V_j D_l V_k) \frac{x_l}{2} \right) dx + \int_{\partial \Omega} \sum_{i,j,k,l} A_{kj} D_i V_j x_i D_l V_k N_l dS
\]

\[
= \left( \frac{n}{2} - 1 \right) \int_{\Omega} \sum_{i,j,k} A_{kj} D_i V_j D_i V_k dx - \int_{\partial \Omega} \frac{1}{2} \sum_{i,j,k,l} A_{kj} D_i V_j D_l x_i N_l dS + \int_{\partial \Omega} \sum_{i,j,k,l} A_{kj} D_i V_j x_i D_l V_k N_l dS.
\]

(3)

It is worth mentioning that \( x_i D_i \) is a conformal killing field in \( \mathbb{R}^n \).

As we explained in the introduction, without loss of generality we may assume the equation is of the diagonal form, i.e.,

\[
\Delta V_i = f^i(V).
\]

(4)

Note that \(|V|\) and \(|\nabla V|\) are \(C^{0,1}\) functions. By (1) and (A1)- (A2), we have

\[
\Delta |V| \geq -C |V|^q;
\]

\[
\Delta |\nabla V| \geq -C |V|^{q-1} |\nabla V|
\]

weakly. Since \( V \in L^p(\mathbb{R}^n \setminus K) \), there exists a large number \( R \) such that \( \int_{\mathbb{R}^n \setminus B_R} |V|^p dx < \epsilon \), where \( \epsilon \) is as in Lemma [1]. Applying Lemma 1 to \( B_r(x_0) \) where \( |x_0| \geq 2r \geq 2R \), we get \( |V| = O(|x|^{-\frac{n}{p}}) \).

Case 1. If \( \frac{n+2}{n} \leq q \leq \frac{n}{n-2} \) (or equivalently, \( 1 \leq p \leq \frac{n}{n-2} \)), then \( \frac{n}{p} \geq n-2 \). By Lemma [1] we have \( |V| = O(|x|^{-n/p}) \). Let \( \varphi \) be a cutoff function such that \( \varphi = 1 \) in \( B_r \) and \( \varphi = 0 \) outside \( B_{2r} \) with \( |\nabla \varphi| \leq Cr^{-1} \). Applying \( \varphi V_i \) to (4) and integrating gives

\[
\int_{B_r(x_0)} |\nabla V|^2 dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1} dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2 dx = O(r^{n-2-\frac{2n}{p}}) \leq O(r^{-n+2}),
\]

where \( |x_0| \geq 2r \gg 1 \). By Lemma [3] with \( h = |V|^{q-1} \), we obtain \( |\nabla V| = O(|x|^{-(n-1)}) \) and thus \( |V| \leq O(|x|^{-(n-2)}) \).

Case 2. If \( \frac{n}{n-2} < q \) (or equivalently \( p > \frac{n}{n-2} \)), by Lemma [2] \( |V| = O(|x|^{-\lambda}) \) for all \( \lambda < n-2 \). Therefore,

\[
\int_{B_r(x_0)} |\nabla V|^2 dx \leq C \int_{B_{2r}(x_0)} |V|^{q+1} dx + \frac{C}{r^2} \int_{B_{2r}(x_0)} |V|^2 dx = O(r^{n-2-2\lambda}),
\]
Note that where $\int \nabla G \cdot \nabla \phi = 0$. Hence, $\int |V|^q \leq C \frac{r}{q}$. Choose $q > \frac{2p}{p-1}$. We can then find $q' > \frac{n}{2}$ such that

$$\int_{B_r(x_0)} (|V|^{q})^{q'} \, dx \leq C \frac{r}{q}$$

where $|x_0| > 2r \gg 1$. This is possible because $\lambda$ is close to $n - 2$. By Lemma 3, we obtain

$$\sup_{B_r(x_0)} |\nabla V| \leq \frac{C}{r^2} \|\nabla V\|_{L^2(B_r(x_0))} = O(r^{-\lambda - 1}),$$

where $|x_0| > 2r \gg 1$.

Let $\Omega = B_R \setminus B_r$ in (3). We have

$$\int \sum_{k,l} f^k(x_1) \frac{D_1 V_k dx}{(x_1)_k} = \left(\frac{n}{2} - 1\right) \int \nabla V \cdot \nabla \phi - \int_{\partial \Omega} \frac{1}{2} \sum_{l} \nabla V \cdot \nabla N_l dS$$

$$+ \int \sum_{i,j,l} D_i V_j \frac{D_l V_k dx}{(x_1)_k}.$$  \hspace{1cm} (5)

Note that

$$\lim_{R \to \infty} \int_{S_R} R |\nabla V|^2 dS = \lim_{R \to \infty} O(R^{-2\lambda - 2 + n}) = 0.$$  \hspace{1cm} (6)

Let $R \to \infty$ in (6). Then there is no boundary term coming from the infinity. We can choose $\Omega = \mathbb{R}^n \setminus B_r$. The boundary terms only occur on $S_r$. On $\partial \Omega$, $N = -\frac{x_1}{r}$. Hence,

$$\int_{\mathbb{R}^n \setminus B_r} \sum_{k,l} f^k(x_1) \frac{D_1 V_k dx}{(x_1)_k} = \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n \setminus B_r} \nabla V \cdot \nabla \phi - \int_{S_r} \frac{1}{2} |\nabla V|^2 dS - r \int_{S_r} |\nabla N|^2 dS$$

$$\geq \left(\frac{n}{2} - 1\right) \int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx - \int_{S_r} \frac{1}{2} |\nabla V|^2 dS.$$  \hspace{1cm} (7)

Let $G(r) := \int_{\mathbb{R}^n \setminus B_r} |\nabla V|^2 dx$. Since $G'(r) = -\int_{S_r} |\nabla V|^2 dS$, the previous formula becomes

$$G(r) \leq -\frac{r}{n-2} G'(r) + \frac{2}{n-2} \int_{\mathbb{R}^n \setminus B_r} \sum_{k,l} f^k(x_1) \frac{D_1 V_k dx}{(x_1)_k}.$$  \hspace{1cm} (8)

The key idea is to compare the size of $G(r)$ to that of $G'(r)$. The coefficient in front of $G'(r)$ plays an important role. Here is the only place we use the condition of $|f^k|$. We have

$$\int_{\mathbb{R}^n \setminus B_r} \sum_{k,l} f^k(x_1) \frac{D_1 V_k dx}{(x_1)_k} \leq \int_{\mathbb{R}^n \setminus B_r} |V|^q |x| |\nabla V| dx = O(r^{-\lambda(q+1) + n}).$$  \hspace{1cm} (9)

Thus,

$$G(r) \leq -\frac{r}{n-2} G'(r) + C r^{-\lambda(q+1) + n}.$$  \hspace{1cm} (10)
Since \( q > \frac{n}{n-2} \) and \( \lambda \) is close to \( n-2 \), we have \( \lambda(q+1) - n > n-2 \). By Lemma \( \ref{lemma1} \) this implies \( G(r) = O(r^{-(n-2)}) \). By Sobolev inequality, we get 
\[
\int_{B_{2r} \setminus B_r} |V|^\frac{2n}{n-2} dx = O(r^{-n}).
\]
Finally, by Lemma \( \ref{lemma1} \) and \( \ref{lemma3} \) we obtain \( |V| = O(|x|^{-(n-2)}) \) and \( |\nabla V| = O(|x|^{-(n-1)}) \).

### 3 Proofs of Corollary \( \ref{corollary1} \) and Theorem \( \ref{theorem2} \)

**Proof of Corollary \( \ref{corollary1} \)** Since the equation is invariant under inversion, we transform the solution to \( \mathbb{R}^n \setminus B_1 \) and apply Theorem \( \ref{theorem1} \).

Let \( y = \frac{x}{|x|^2} \). Define \( U_i(y) = \frac{1}{|y|} V_i(\frac{x}{|x|^2}) \). This is called the Kelvin transform with the property that 
\[
\Delta_y U_i(y) = \frac{1}{|y|^{n+2}} \Delta_x V_i(x).
\]
This can also be viewed as the conformal change formula of the conformal Laplacian with zero scalar curvature. Therefore, \( U_i(y) \) satisfies 
\[
\sum_j A_{ij} \Delta_y U_i(y) = \frac{1}{|y|^{n+2}} f^j(|y|^{n-2} U(y)) = f^j(U(y))
\]
in \( \mathbb{R}^n \setminus B_1 \), where we use that \( f^j \) is homogeneous of degree \( \frac{n+2}{n-2} \). Moreover,
\[
\int_{\mathbb{R}^n \setminus B_1} |U|^{\frac{2n}{n-2}} dy = \int_{\mathbb{R}^n \setminus B_1} (|V||y|^{-n+2})^{\frac{2n}{n-2}} dy = \int_{B_1 \setminus \{0\}} (|V||x|^{-n+2})^{\frac{2n}{n-2}} |x|^{-2n} dx
\]
\[
= \int_{B_1 \setminus \{0\}} |V|^{\frac{2n}{n-2}} dx < +\infty.
\]
Now we apply Theorem \( \ref{theorem1} \) with \( p = \frac{2n}{n-2} \) and \( q = \frac{n+2}{n-2} \). We get \( |U| = O(|y|^{-(n-2)}) \) and \( |\nabla U| = O(|y|^{-(n-1)}) \). Hence, \( |V| = O(1) \) and \( |\nabla V| = O(|x|^{-1}) \). As a result, \( V \in L^\infty(B_1) \) and \( \nabla V \in L^p(B_1) \) for all \( p < n \).

We show that \( V \) is a weak solution to \( \ref{equation1} \) in \( B_1 \). Let \( \varphi \in H^1_0(B_1, \mathbb{R}^m) \). Let \( \eta_k(|x|) \) be a compactly supported function in \( B_1 \setminus \{0\} \) such that \( \eta_k \to 1 \) a.e. in \( B_1 \) and \( \|\eta_k\|_{L^n(B_1)} \to 0 \) as \( k \to \infty \). (Such functions were used by Serrin \( \ref{13} \).) Then 
\[
\int_{B_1} \eta_k \sum_{i,j,l} A_{ij} D_i \varphi_j D_l V dx = \int_{B_1} - \sum_i f^i(V) \varphi_i \eta_k dx - \int_{B_1} \sum_{i,j,l} D_i \eta_k A_{ij} \varphi_j D_l V dx.
\]
The last term can be estimated as follows.
\[
|\int_{B_1} \sum_{i,j,l} D_i \eta_k A_{ij} \varphi_j D_l V dx| \leq C \|\varphi\|_{L^{\frac{2n}{n-2}}(B_1)} \|\nabla V\|_{L^2(B_1)} \|\eta_k\|_{L^n(B_1)} \leq C \|\eta_k\|_{L^n(B_1)} \to 0
\]
as \( k \to \infty \). Hence, in the limit
\[
\int_{B_1} \sum_{i,j,l} A_{ij} D_i \varphi_j D_l V_l dx = \int_{B_1} - \sum_i f^i(V) \varphi_i dx.
\]
Thus, \( V \) is a weak solution in \( B_1 \). It follows by elliptic regularity that \( V \in C^\infty(B_1) \).

**Proof of Theorem 2.** Let \( \Omega = B_R \). Therefore, \( N = \frac{2}{R} \) in (3). We get
\[
\int_{B_R} \sum_{k,l} f^k(V) x_l D_l V_k dx = \left( \frac{n}{2} - 1 \right) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx - \int_{S_R} \frac{R}{2} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dS + R \int_{S_R} \sum_{j,k} A_{kj} D_N V_j D_N V_k dS.
\]

Since \( f^k = \frac{\partial p}{\partial V_k} \), we have
\[
\int_{B_R} -nP(V) dx = \left( \frac{n}{2} - 1 \right) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx - \int_{S_R} \frac{R}{2} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dS + R \int_{S_R} \sum_{j,k} A_{kj} D_N V_j D_N V_k dS - \int_{S_R} RP(V) dS. \tag{6}
\]

On the other hand, we also have
\[
\int_{B_R} (q + 1)P(V) dx = \int_{B_R} \sum_k \frac{\partial p}{\partial V_k} V_k dx
\]
\[
= - \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx + \int_{S_R} \sum_{j,k} A_{kj} D_N V_j V_k dS, \tag{7}
\]
where we use the Euler formula for homogeneous functions.

Case 1. \( n \geq 4 \).

By Theorem 1, when \( R \to \infty \), (6) becomes
\[
\int_{B_R} -nP(V) dx = \left( \frac{n}{2} - 1 \right) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx + O(R^{-(n-2)}) + O(R^{-(q+1)(n-2)+n})
\]
\[
= \left( \frac{n}{2} - 1 \right) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx + o(1),
\]
where we use conditions on \( p, q \) and \( n \geq 4 \) to get \( (q + 1)(n - 2) - n > 0 \). Similarly, (7) gives
\[
\int_{B_R} (q + 1)P(V) dx = - \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_l V_k dx + O(R^{-(n-2)}).
\]
Combining these two formulas and noting that \( q + 1 \neq \frac{2n}{n-2} \), we finally arrive at

\[
\int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_i V_k \, dx = o(1).
\]

Since \( A \) is positive definite, we have \( |\nabla V| \equiv 0 \) and hence \( V \equiv 0 \).

Case 2. \( n = 3 \).

Note that \( \sup |V| \leq \frac{C}{|x|^{n/p}} \|V\|_{L^p} \). Combining this fact with Theorem\( \square \) we have \( |V| = O(|x|^{-\lambda}) \), where \( \lambda = \max\{1, \frac{2}{p}\} \). Therefore,

\[
\lambda(q+1) - 3 \geq \max\{q - 2, \frac{3}{p}(q+1) - 3\} \geq \max\{-1 + \frac{2p}{3}, -1 + \frac{6}{p}\} > 0.
\]

Then (6) becomes

\[
\int_{B_R} -3P(V) \, dx = (\frac{3}{2} - 1) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_i V_k \, dx + O(R^{-1}) + O(R^{-\lambda(q+1) + 3})
\]

\[
= (\frac{3}{2} - 1) \int_{B_R} \sum_{i,j,k} A_{kj} D_i V_j D_i V_k \, dx + o(1)
\]

as in Case 1. The rest of proof is the same as in Case 1. \( \square \)

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Institute for Advanced Study, Princeton, NJ
Email address: sophie@math.ias.edu