FAITH’S PROBLEM ON \( R \)-PROJECTIVITY IS UNDECIDABLE

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Abstract. In [7], Faith asked for what rings \( R \) does the Dual Baer Criterion hold in \( \text{Mod–} R \), that is, when does \( R \)-projectivity imply projectivity for all right \( R \)-modules? Such rings \( R \) were called right testing. Sandomierski proved that if \( R \) is right perfect, then \( R \) is right testing. Puninski et al. [1] have recently shown for a number of non-right perfect rings that they are not right testing, and noticed that [17] proved consistency with ZFC of the statement ‘each right testing ring is right perfect’ (the proof used Shelah’s uniformization).

Here, we prove the complementing consistency result: the existence of a right testing, but not right perfect ring is also consistent with ZFC (our proof uses Jensen-functions). Thus the answer to the Faith’s question above is undecidable in ZFC. We also provide examples of non-right perfect rings such that the Dual Baer Criterion holds for ‘small’ modules (where ‘small’ means countably generated, or \( \leq 2^{\aleph_0} \)-presented of projective dimension \( \leq 1 \)).

1. Introduction

The classic Baer Criterion for Injectivity [3] says that a (right \( R \)-) module \( M \) is injective, if and only if it is \( R \)-injective, that is, each homomorphism from any right ideal \( I \) of \( R \) into \( M \) extends to \( R \). This criterion is the key tool for classification of injective modules over particular rings.

A module \( M \) is called \( R \)-projective provided that each homomorphism from \( M \) into \( R/I \) where \( I \) is any right ideal, factors through the canonical projection \( \pi : R \to R/I \) [2, p.184]. One can formulate the Dual Baer Criterion as follows: a module \( M \) is projective, if and only if it is \( R \)-projective. The rings \( R \) such that this criterion holds true are called right testing, [1, Definition 2.2].

Dualizations are often possible over perfect rings. Indeed, Sandomierski proved that each right perfect ring is right testing [15]. The question of existence of non-right perfect right testing rings is much harder. Faith [7, p.175] says that “the characterization of all such rings is still an open problem” – we call it the Faith’s problem here.

Note that if \( R \) is not right perfect, then it is consistent with ZFC + GCH that \( R \) is not right testing. Indeed, as observed in [1, 17 Lemma 2.4] (or [16]) implies that there is a \( \kappa^+ \)-presented module \( N \) of projective dimension 1 such that \( \text{Ext}^1_R(N, I) = 0 \) for each right ideal \( I \) of \( R \) (and hence \( N \) is \( R \)-projective, but not projective) in the extension of ZFC satisfying GCH and Shelah’s Uniformization Principle UP\( \kappa \) for an uncountable cardinal \( \kappa \) such that \( \text{card}(R) < \kappa \) and \( \text{cf}(\kappa) = \aleph_0 \).

In particular, attempts [4] to prove the existence of non-right perfect testing rings in ZFC could not be successful.

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Moreover, in the extension of ZFC + GCH satisfying UP$_\kappa$ for all uncountable cardinals $\kappa$ such that $\text{cf}(\kappa) = \aleph_0$ [6], all right testing rings are right perfect. So it is consistent with ZFC + GCH that all right testing rings are right perfect.

For many non-right perfect rings $R$, one can actually prove that $R$ is not right testing in ZFC: this is the case for all commutative noetherian rings [10, Theorem 1], all semilocal right noetherian rings [1, Proposition 2.11], and all commutative domains (see Lemma 2.1 below).

It is easy to see that all finitely generated $R$-projective modules are projective, that is, the Dual Baer Criterion holds for all finitely generated modules over any ring. So in order to find examples of $R$-projective modules which are not projective, one has to deal with infinitely generated modules. The task is quite complex in general: in Section 2, we will show that there exist non-right perfect rings such that the Dual Baer Criterion holds for all countably generated modules, or for all $\leq 2^{\aleph_0}$-presented modules of projective dimension $\leq 1$.

Some questions related to the vanishing of Ext, such as the Whitehead problem, are known to be undecidable in ZFC, cf. [5]. In Section 3.3, we will prove that this is also true of the existence of non-right perfect right testing rings. To this purpose, we will employ Gödel’s Axiom of Constructibility $V = L$, or rather its combinatorial consequence, the existence of Jensen-functions (see [5, §VI.1] and [8, §18.2]). Our main result, Theorem 3.3 below, says that the existence of Jensen-functions implies that a particular subring of $K^\omega$ (where $K$ is a field of cardinality $\leq 2^{\aleph_0}$) is testing, but not perfect.

For unexplained terminology, we refer the reader to [2], [5], [8] and [9].

2. $R$-PROJECTIVITY VERSUS PROJECTIVITY

It is easy to see that for each $R$-projective module $M$, each submodule $I \subseteq R^n$ and each $f \in \text{Hom}_R(M, R^n/N)$, there exists $g \in \text{Hom}_R(M, R^n)$ such that $f = \pi_N g$ where $\pi_N : R^n \to R^n/N$ is the projection (see e.g. [2, Proposition 16.12(2)]). In particular, all finitely generated $R$-projective modules are projective.

This not true of countably generated $R$-projective modules in general - for example, by the following lemma, the abelian group $\mathbb{Q}$ is $R$-projective, but not projective:

**Lemma 2.1.** Let $R$ be a commutative domain. Then each divisible module is $R$-projective. So $R$ is testing, iff $R$ is a field.

**Proof.** Assume $R$ is testing and possesses a non-trivial ideal $I$. Let $M$ be any divisible module. If $0 \neq \text{Hom}_R(M, R/I)$, then $R/I$ contains a non-zero divisible submodule of the form $J/I$ for an ideal $I \subseteq J \subseteq R$. Let $0 \neq r \in I$. The $r$-divisibility of $J/I$ yields $Jr + I = J$, but $Jr \subseteq I$, a contradiction. So $\text{Hom}_R(M, R/I) = 0$, and $M$ is projective. In particular, each injective module is projective, so $R$ is a commutative QF-domain, hence a field.

However, there do exist rings such that all countably generated $R$-projective modules are projective. We will now examine one such class of rings that will be relevant for proving the independence result in Section 3.

**Definition 2.2.** Let $K$ be a field, and $R$ the unital $K$-subalgebra of $K^\omega$ generated by $K^{(\omega)}$. In other words, $R$ is the subalgebra of $K^\omega$ consisting of all eventually constant sequences in $K^\omega$.

For each $i < \omega$, we let $e_i$ be the idempotent in $K^\omega$ whose $i$th component is 1 and all the other components are 0. Notice that $\{e_i \mid i < \omega\}$ is a set of pairwise orthogonal idempotents in $R$, so $R$ is not perfect.

First, we note basic ring and module theoretic properties of this particular setting:
Lemma 2.3. Let $R$ be as in Definition 2.2.

(1) $R$ is a commutative von Neumann regular semiartinian ring of Loewy length 2, with $\text{Soc}(R) = \sum_{i < \omega} e_i R = K(\omega)$ and $R/\text{Soc}(R) \cong K$.

(2) If $I$ is an ideal of $R$, then either $I = I_A = \sum_{i \in A} e_i R$ for a subset $A \subseteq \omega$ and $I$ is semisimple and projective, or else $I = f R$ for an idempotent $f \in R$ such that $f$ is eventually 1. In particular, $R$ is hereditary.

(3) $\{e_i R \mid i < \omega\} \cup \{S\}$ is a representative set of all simple modules, where $S = R/\text{Soc}(R)$.

(4) Let $M \in \text{Mod-}R$. Then there are unique cardinals $\kappa_i$ ($i < \omega$) and $\lambda$ such that $M \cong S^{(\kappa)} \oplus N$, $\text{Soc}(N) \cong \bigoplus_{i < \omega}(e_i R)^{\kappa_i}$, and $N/\text{Soc}(N) \cong S^{(\lambda)}$.

Proof. (1) Clearly, $R$ is commutative, and if $r \in R$, then all non-zero components of $r$ are invertible in $K$, so there exists $s \in R$ with $rsr = r$, i.e., $R$ is von Neumann regular.

For each $i < \omega$, $e_i R = e_i K$ is a simple projective module, whence $J = \sum_{i < \omega} e_i R \subseteq \text{Soc}(R)$. Moreover, $R/J \cong K$ is a simple non-projective module. So $R$ is semisimple of Loewy length 2, and $J = \text{Soc}(R)$ is a maximal ideal of $R$.

(2) If $I \subseteq \text{Soc}(R)$, then $I$ is a direct summand in the semisimple projective module $\text{Soc}(R)$. Since the simple projective modules $\{e_i R \mid i < \omega\}$ are pairwise non-isomorphic, $I \cong I_A = \sum_{i \in A} e_i R$, and hence $I = I_A$, for a subset $A \subseteq \omega$.

If $I \not\subseteq \text{Soc}(R)$, then there is an idempotent $e \in I \setminus \text{Soc}(R)$ and $e R + \text{Soc}(R) = R$. Note that $e$ is eventually 1, so in particular, $e R \supseteq \sum_{i \in B} e_i R$ where $B \subseteq \omega$ is the (cofinite) set of all indices $i$ such that the $i$th component of $e$ is 1. Then $I = e R \oplus (\sum_{i \in B} e_i R \cap I)$. The latter direct summand equals $I_A$ for a (finite) subset $A \subseteq \omega \setminus B$, and $I = f R$ for the idempotent $f = e + \sum_{i \in A} e_i$.

In either case, $I$ is projective, hence $R$ is hereditary.

(3) By part (2), the maximal spectrum $m\text{Spec}(R) = \{I_\omega\} \cup \{(1 - e_i) R \mid i < \omega\}$. The $\sum$-injectivity of all simple modules follows from part (1) and [9] Proposition 6.18. The simple module $S$ is not projective because $I_\omega$ is not finitely generated.

(4) These (unique) cardinals are determined as follows: $\kappa$ is the dimension of the $S$-homogenous component of $M$, and $\kappa_i$ the dimension of its $e_i R$-homogenous component ($i < \omega$). The semisimple module $M = M/\text{Soc}(M) \cong N/\text{Soc}(N)$ is isomorphic to a direct sum of copies of the unique non-projective simple module $S$; $\lambda$ is the ($S$-) dimension of $M$.

The final claim follows from the fact that $P = (\text{Soc}(R^{(\mu)}) + I)/I$ is a direct sum of projective simple modules, while $R^{(\mu)}/(\text{Soc}(R^{(\mu)}) + I)$ a direct sum of copies of $S$, so $\{0, P, N\}$ is the socle sequence of $N$.

Next we turn to $R$-projectivity:

Lemma 2.4. Let $R$ be as in Definition 2.2.

(1) A module $M$ is $R$-projective, if it is projective w.r.t. the projection $\pi : R \to R/\text{Soc}(R)$.

(2) The class of all $R$-projective modules is closed under submodules. If $M \in \text{Mod-}R$ is $R$-projective, then all countably generated submodules of $M$ are projective. In particular, the Dual Baer Criterion holds for all countably generated modules.
Proof. (1) First, note that by part (2) of Lemma 2.3, the only ideals $I$ such that $R/I$ is not projective, are of the form $I = IA$ where $A$ is an infinite subset of $\omega$ (and hence $I \subseteq \text{Soc}(R) = L_\omega$). So it suffices to prove that if $M$ is projective w.r.t. the projection $\pi : R \rightarrow R/\text{Soc}(R)$, then it is projective w.r.t. all the projections $\pi_{IA} : R \rightarrow R/IA$ such that $A \subseteq \omega$ is infinite.

Let $f \in \text{Hom}_R(M, R/IA)$. If $\text{Im}(f) \subseteq \text{Soc}(R)/IA$, then there exists a homomorphism $h \in \text{Hom}_R(\text{Soc}(R)/IA, \text{Soc}(R))$ such that $\pi_{IA}h = \text{id}$, whence $g = hf$ yields a factorization of $f$ through $\pi_{IA}$. Otherwise, let $\rho : R/IA \rightarrow R/\text{Soc}(R)$ be the projection. By assumption, there is $g \in \text{Hom}_R(M, R)$ such that $\rho f = \pi g$. So $\rho(f - \pi_{IA} g) = 0$, and $\text{Im}(f - \pi_{IA} g) \subseteq \text{Soc}(R)/IA$. Then $f - \pi_{IA} g$ factorizes through $\pi_{IA}$ by the above, and so does $f$.

(2) The closure of the class of all $R$-projective modules under submodules follows from part (1) and from the injectivity of $S = R/\text{Soc}(R)$ (see part (3) of Lemma 2.3). So it only remains to prove that each countably generated $R$-projective module is projective. However, as remarked above, for any ring $R$, each finitely generated $R$-projective module is projective. Since $R$ is hereditary and von Neumann regular, [17, Lemma 3.4] applies and gives that also all countably generated $R$-projective modules are projective.

We finish this section by presenting two more classes of non-right perfect rings over which small modules satisfy the Dual Baer Criterion.

In both cases, the rings will be von Neumann regular and right self-injective. Apart from classic facts about these rings from [9, §10], we will also need the following easy observation (valid for any right self-injective ring $R$, see [1, Proposition 2.6]): a module $M$ is $R$-projective, iff $\text{Ext}^1_R(M, I) = 0$ for each right ideal $I$ of $R$.

Example 2.5. Let $R$ be a right self-injective von Neumann regular ring such that $R$ has primitive factors artinian, but $R$ is not artinian (e.g., let $R$ be an infinite direct product of skew-fields). Then all $R$-projective modules are non-singular, and the Dual Baer Criterion holds for all countably generated modules.

For the first claim, let $M$ be $R$-projective and assume there is an essential right ideal $I \subsetneq R$ such that $R/I$ embeds into $M$. Let $J$ be a maximal right ideal containing $I$. By [9, Proposition 6.18], the simple module $R/J$ is injective, so the projection $\rho : R/I \rightarrow R/J$ extends to some $f \in \text{Hom}_R(M, R/J)$. The $R$-projectivity of $M$ yields $g \in \text{Hom}_R(M, R)$ such that $f = \pi g$ where $\pi : R \rightarrow R/J$ is the projection. Then $g$ restricts to a non-zero homomorphism from $R/I$ into the non-singular module $R$, a contradiction. Thus, $M$ is non-singular.

For the second claim, we recall from [11, Example 6.8], that for von Neumann regular right self-injective rings, non-singular modules coincide with the (flat) Mittag-Leffler ones. However, each countably generated flat Mittag-Leffler module (over any ring) is projective, see e.g. [8, Corollary 3.19]. Thus each countably generated $R$-projective module is projective.

Example 2.6. Let $R$ be a von Neumann regular right self-injective ring which is purely infinite in the sense of [9, Definition on p.116]. That is, there exists no central idempotent $0 \neq e \in R$ such that the ring $eRe$ is directly finite (where a ring $R$ is directly finite in case $xy = 1$ implies $yx = 1$ for all $x, y \in R$.)

For example, the endomorphism ring of any infinite dimensional right vector space over a skew-field has this property, see [9, p. 116].

We claim that the Dual Baer Criterion holds for all $\leq 2^{\aleph_0}$-presented modules $M$ of projective dimension $\leq 1$. Indeed, assume that such module $M$ is $R$-projective. By [9, Theorem 10.19], $R$ contains a right ideal $J$ which is a free module of rank $2^{\aleph_0}$. If the projective dimension of $M$ equals 1, then there is a non-split presentation
0 → K → L → M → 0 where K and L are free of rank ≤ 2^\aleph_0. Thus Ext_B^1(M, J) ≠ 0, in contradiction with the R-projectivity of M. This shows that M is projective.

In particular, if the global dimension of R is 2, and all right ideals of R are ≤ 2^\aleph_0-presented (which is the case when R is the endomorphism ring of a vector space of dimension \aleph_0 over a field of cardinality ≤ 2^\aleph_0 under CH - see [13]), then the Dual Baer Criterion holds for all ideals of R.

**Remark 2.7.** As mentioned in the Introduction, for any non-right perfect ring R, Shelah’s Uniformization Principle UP (κ) (for an uncountable cardinal κ such that card(R) < κ and cl(κ) = ℵ_0) and GCH imply the existence of a κ⁺-presented R-projective module N of projective dimension equal to 1.

If we choose R to be the endomorphism ring of a vector space of dimension < ℵω over a field of cardinality < ℵω, then we can take the minimal choice, κ = ℵω, so the module N above can be chosen ℵ^ω⁺-presented. Example 2.6 gives a lower bound for the possible size of N: it has to be > 2^\aleph_0-presented.

3. The consistency of existence of non-perfect testing rings

In this section, we return to the setting of Definition 2.2 so K will denote a field, and R the subalgebra of K^ω consisting of all eventually constant sequences in K^ω. In order to prove that it is consistent with ZFC that R is testing, we will employ the notion of Jensen-functions, cf. [12] and [8, §18.2].

**Definition 3.1.** Let κ be a regular uncountable cardinal.

1. A subset C ⊆ κ is called a club provided that C is closed in κ (i.e., sup(D) ∈ C for each subset D ⊆ C such that sup(D) < κ) and C is unbounded (i.e., sup(C) = κ). Equivalently, there exists a strictly increasing continuous function f : κ → κ whose image is C.

2. A subset E ⊆ κ is stationary provided that E ∩ C ≠ ∅ for each club C ⊆ κ.

3. Let A be a set of cardinality ≤ κ. An increasing continuous chain, \{A_\alpha | \alpha < \kappa\}, consisting of subsets of A of cardinality < κ such that A_0 = 0 and A = \bigcup_{\alpha < \kappa} A_\alpha, is called a κ-filtra tion of the set A.

4. Let E be a stationary subset of κ. Let A and B be sets of cardinality ≤ κ. Let \{A_\alpha | \alpha < \kappa\} and \{B_\alpha | \alpha < \kappa\} be κ-filtrations of A and B, respectively. For each \alpha < \kappa, let c_\alpha : A_\alpha → B_\alpha be a map. Then \{c_\alpha | \alpha < \kappa\} are called Jensen-functions provided that for each map c : A → B, the set E(c) = \{\alpha ∈ E | c ↾ A_\alpha = c_\alpha\} is stationary in κ.

Jensen [12] proved the following (cf. [8, Theorem 18.9]):

**Theorem 3.2.** Assume Gödel’s Axiom of Constructibility (V = L). Let κ be a regular infinite cardinal, E ⊆ κ a stationary subset of κ, and A and B sets of cardinality ≤ κ. Let \{A_\alpha | \alpha < \kappa\} and \{B_\alpha | \alpha < \kappa\} be κ-filtrations of A and B, respectively. Then there exist Jensen-functions \{c_\alpha | \alpha < \kappa\}.

Now, we can prove our main result:

**Theorem 3.3.** Assume V = L. Let K be a field of cardinality ≤ 2^\omega. Then all R-projective modules are projective.

**Proof.** Let M be an R-projective module. By induction on the minimal number of generators, κ, of M, we will prove that M is projective. For κ ≤ ℵ_0, we appeal to part (2) of Lemma 2.4 and for κ a singular cardinal, we apply [17, Corollary 3.11].

Assume κ is a regular uncountable cardinal. Let G = \{m_\alpha | \alpha < \kappa\} be a minimal set of R-generators of M. For each \alpha < \kappa, let G_\alpha = \{m_\beta | \beta < \alpha\}. Let M_\alpha be the submodule of M generated by G_\alpha. Then M = (M_\alpha | \alpha < \kappa) is a κ-filtra tion of the module M. Possibly skipping some terms of M, we can w.l.o.g. assume that M
has the following property for each $\alpha < \kappa$: if $M_\beta/M_\alpha$ is not $R$-projective for some $\alpha < \beta < \kappa$, then also $M_{\alpha+1}/M_\alpha$ is not $R$-projective. Let $E$ be the set of all $\alpha < \kappa$ such that $M_{\alpha+1}/M_\alpha$ is not $R$-projective.

We claim that $E$ is not stationary in $\kappa$. If our claim is true, then there is a club $C$ in $\kappa$ such that $C \cap E = \emptyset$. Let $f : \kappa \to \kappa$ be a strictly increasing continuous function whose image is $C$. Then $M_{f(\alpha+1)}/M_{f(\alpha)}$ is $R$-projective for each $\alpha < \kappa$. By the inductive premise, $M_{f(\alpha+1)}/M_f(\alpha)$ is projective for all $\alpha < \kappa$, whence $M$ is projective, too.

Assume our claim is not true. We will make use of Theorem 52 in the following setting. We let $A = G$ and $B = R$. The relevant $\kappa$-filtration of $A$ will be $(G_\alpha : \alpha < \kappa)$. For $B$, we consider any $\kappa$-filtration $(R_\alpha : \alpha < \kappa)$ of the additive group $(R, +)$ consisting of subgroups of $(R, +)$ (which exists since card($K$) $\leq \aleph_1$ implies card($R$) $\leq \aleph_1 < \kappa$; if card($K$) is countable, the filtration can even be taken constant $= R$). By Theorem 3.2 there exist Jensen-functions $c_\alpha : G_\alpha \to R_\alpha (\alpha < \kappa)$ such that for each function $c : G \to R$, the set $E(c) = \{ \alpha \in E \mid c_\alpha = c \upharpoonright G_\alpha \}$ is stationary in $\kappa$.

By induction on $\alpha < \kappa$, we will define a sequence $(g_\alpha : \alpha < \kappa)$ such that $g_\alpha \in \text{Hom}_R(M_\alpha, S)$ as follows: $g_0 = 0$; if $\alpha < \kappa$ and $g_\alpha$ is defined, we distinguish two cases:

1. $\alpha \in E$, and there exist $h_{\alpha+1} \in \text{Hom}_R(M_{\alpha+1}, S)$ and $y_{\alpha+1} \in \text{Hom}_R(M_{\alpha+1}, R)$, such that $h_{\alpha+1} \upharpoonright M_\alpha = g_\alpha$, $h_{\alpha+1} = \pi y_{\alpha+1}$ and $y_{\alpha+1} \upharpoonright G_\alpha = c_\alpha$. In this case we define $g_{\alpha+1} = h_{\alpha+1} + f_{\alpha+1} \rho_{\alpha+1}$, where $f_{\alpha+1} : M_{\alpha+1} \to M_{\alpha+1}/M_\alpha$ is the projection and $f_{\alpha+1} \in \text{Hom}_R(M_{\alpha+1}/M_\alpha, S)$ is chosen so that it does not factorize through $\pi$ (such $f_{\alpha+1}$ exists because $\alpha \in E$ by part (1) of Lemma 2.4). Note that $g_{\alpha+1} \upharpoonright M_\alpha = h_{\alpha+1} \upharpoonright M_\alpha = g_\alpha$.

2. If $\alpha$ is a limit ordinal, we let $g_\alpha = \bigcup_{\beta < \alpha} g_\beta$. Finally, we define $g = \bigcup_{\alpha < \kappa} g_\alpha$. We will prove that $g$ does not factorize through $\pi$. This will contradict the $R$-projectivity of $M$, and prove our claim.

Assume there is $x \in \text{Hom}_R(M, R)$ such that $g = \pi x$. Then the set of all $\alpha < \kappa$ such that $x \upharpoonright G_\alpha$ maps into $R_\alpha$ is closed and unbounded in $\kappa$, so it contains some element $\alpha \in E(x \upharpoonright G)$. For such $\alpha$, we have $g_{\alpha+1} = \pi x \upharpoonright M_{\alpha+1}$ and $x \upharpoonright G_\alpha = c_\alpha$, so $\alpha$ is in case (1) (this is witnessed by taking $h_{\alpha+1} = g_{\alpha+1}$ and $y_{\alpha+1} = x \upharpoonright M_{\alpha+1}$).

Let $z_{\alpha+1} = x \upharpoonright M_{\alpha+1} - y_{\alpha+1}$. Then $z_{\alpha+1} \upharpoonright G_\alpha = x \upharpoonright G_\alpha - y_{\alpha+1} \upharpoonright G_\alpha = c_\alpha - c_\alpha = 0$. So there exists $u_{\alpha+1} \in \text{Hom}_R(M_{\alpha+1}/M_\alpha, R)$ such that $z_{\alpha+1} = u_{\alpha+1} \rho_{\alpha+1}$. Moreover,

$$\pi u_{\alpha+1} \rho_{\alpha+1} + \pi z_{\alpha+1} = \pi u_{\alpha+1} = \pi x \upharpoonright M_{\alpha+1} - \pi y_{\alpha+1} = g_{\alpha+1} - h_{\alpha+1} = f_{\alpha+1} \rho_{\alpha+1}.$$

Since $\rho_{\alpha+1}$ is surjective, we conclude that $\pi u_{\alpha+1} = f_{\alpha+1}$, in contradiction with our choice of the homomorphism $f_{\alpha+1}$. \qed

**Corollary 3.4.** Let $K$ be a field of cardinality $\leq 2^\omega$. Then the statement ‘$R$ is a testing ring’ is independent of ZFC + GCH. Hence Faith’s problem is undecidable in ZFC + GCH.

**Proof.** Assume UP$_\kappa$ for some $\kappa$ such that card($R$) $< \kappa$ and cf($\kappa$) = $\aleph_0$. Then $R$ is not testing by [17] Lemma 2.4 (see also [1] Theorem 2.7]).

Assume $V = L$. Then $R$ is testing by Theorems 5.2 and 5.3. \qed

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