Similarity dark energy models in Bianchi type - I space-time

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Abstract

We investigate some new similarity inhomogeneous solutions of anisotropic dark energy and perfect fluid in Bianchi type-I space-time. Three different equation of state parameters along the spatial directions are introduced to quantify the deviation of pressure from isotropy. We consider the case when the dark energy is minimally coupled to the perfect fluid as well as direct interaction with it. The Lie symmetry generators that leave the equation invariant are identified and we generate an optimal system of one-dimensional subalgebras. Each element of the optimal system is used to reduce the partial differential equation to an ordinary differential equation which is further analyzed. We solve the Einstein field equations, described by a system of non-linear partial differential equations (NLPDEs), by using the Lie point symmetry analysis method. The geometrical and kinematical features of the models and the behavior of the anisotropy of dark energy, are examined in detail.

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1 Introduction

The overall energy budget of the universe is dictated by dark matter and dark energy with a minor contamination from baryonic matter, where dark energy is supposed to dominate the cosmic landscape causing the acceleration of universe. Since 1998, we have witnessed astrophysical observations, still we are struggling to find the suitable candidates for dark energy from fundamental physics. Many cosmologists believe that the simplest candidate for the dark energy is the cosmological constant (Λ) since it fits well with observational data. During the cosmological evolution, the Λ-term has the density density and pressure

\[ p^{(de)} = -\rho^{(de)}. \]

However, one has the reason to dislike Λ-term because it always suffers from the “fine-tuning” problems and “cosmic coincidence” puzzles [1] on the theoretical ground.

In this regard we note that the universe is mostly observed to be flat and isotropic and supports the predictions of ΛCDM model. However, observations of high resolution CMB radiation data from Wilkinson Microwave Anisotropy Probe (WMAP) showing some large angle anomalies of the large scale structure of the universe with an asymmetric expansion [2, 3]. Plank data also show a slight redshift of the primordial power spectrum of curvature perturbation from exact scale invariant [4]. These observations obviously hint towards the presence of some anisotropy energy source in the universe with anisotropic pressures. The issue of global anisotropy can be settled if anisotropy can be incorporated to be FRW models as a sort of small perturbation. In order to address the issue of smallness in the angular power spectrum, some anisotropic models have been proposed in recent times [5, 6]. These models have a similarity to the Bianchi morphology [7]. Spatially homogeneous Bianchi I model is more general than the FRW model and have anisotropic spatial sections. The Bianchi type models provide an opportunity to consider asymmetric expansion along different spatial sections. Very recently a study have been carried out [8] in Bianchi I space time with dominance of dark energy. Some other recent works on anisotropic dark energy are also available in the literature [9–12].

Lie group of transformations has been extensively applied to linear and nonlinear differential equations in the area of theoretical physics such as: general relativity, particle physics and cosmology [13, 14]. The method of Lie symmetry group is one of the most useful tools for finding exact solutions for the Einstein field equations.
described by a system of NLPDEs. Recently we have developed a formalism to solve non linear Einstein’s field equations in general relativity. In our earlier work, we have proposed the invariant solution of dark energy (DE) model in cylindrically symmetric space-time while in this paper, we confine ourselves to investigate the similarity solution of anisotropic DE model in Bianchi I space-time which is entirely different from. In the Astrophysical community, the inhomogeneous cosmological models have gained interest due to exact perturbation of FRW model and are more often employed to study cosmological phenomenon. That is why, here, we consider inhomogeneous Bianchi I space-time. The paper is organized as follows: In section 2, the basic formalism for anisotropic DE has been discussed for an anisotropic and inhomogeneous Bianchi I space-time. Similar formalism has been already developed in our earlier work. In section 3, Lie group analysis method is developed for Bianchi I space-time. Section 4 and 5 deal with the optimal system and similarity solutions of the models. The Physical viability of the discussed dark energy model is tested with graphical analysis of cosmological parameters in section 6. Finally the conclusions are summarized in section 7.

2 The metric and field equations

The Bianchi type-I space-time is given by

\[ ds^2 = A^2 dx^2 + B^2 dy^2 + C^2 dz^2 - dt^2 , \]

where the metric potentials \( A, B \) and \( C \) are functions of \( x \) and \( t \). Einstien’s field equations in the case of a mixture of perfect fluid and anisotropic dark energy are given by

\[ G^i_j = R^i_j - \frac{1}{2} g^i_j = -T^{(pf)i}_j - T^{(de)i}_j , \]

with

\[ T^{(pf)i}_j = \text{diag}[-\rho^{(pf)}, p^{(pf)}, p^{(pf)}, p^{(pf)}] = \text{diag}[-1, \omega^{(pf)}, \omega^{(pf)}, \omega^{(pf)}] \rho^{(pf)} \]

and

\[ T^{(de)i}_j = \text{diag}[-\rho^{(de)}, p^{(de)}, p^{(de)}, p^{(de)}] = \text{diag}[-1, \omega^{(de)}, \omega^{(de)}, \omega^{(de)}] \rho^{(de)} \]

where \( g^i_j \) are the metric tensor with \( g_{ij} u^i u^j = -1 \); \( u^i \) is the flow vector; \( R^i_j \) is the Ricci tensor; \( R = R^i_i \) is the Ricci scalar; \( p^{(pf)} \), \( \rho^{(pf)} \) and \( \rho^{(de)} \) are, respectively the pressure and energy density of the perfect fluid and dark energy components; \( \omega^{(pf)} \) is the EoS parameter of perfect fluid with \( \omega^{(pf)} \geq 0 \); \( \omega^{(de)} \) and \( \omega^{(de)} \) are the deviation-free EoS parameters of dark energy, respectively, on the \( x, y \) and \( z \) axis.

In co-moving coordinate system, the field equation, for the inhomogeneous space-time, read as

\[ p^{(pf)} + \omega^{(de)} \rho^{(de)} = \frac{B'C'}{A^2BC} \frac{B}{B} \frac{C'}{C} - \frac{\hat{A}}{A} \frac{\hat{A}C}{AC} - \frac{\hat{C}}{C} , \]

\[ p^{(pf)} + \omega^{(de)} \rho^{(de)} = \frac{1}{A^2} \left[ \frac{C''}{C} - \frac{A'C'}{AC} \right] - \frac{\hat{A}}{A} \frac{\hat{A}C}{AC} - \frac{\hat{C}}{C} , \]

\[ p^{(pf)} + \omega^{(de)} \rho^{(de)} = \frac{1}{A^2} \left[ \frac{B''}{B} - \frac{A'B'}{AB} \right] - \frac{\hat{A}}{A} \frac{\hat{A}B}{AB} - \frac{\hat{B}}{B} , \]

\[ p^{(pf)} + \rho^{(de)} = \frac{1}{A^2} \left[ \frac{B''}{B} + \frac{B'C'}{BC} + \frac{C''}{C} - \frac{A'B'}{AB} - \frac{A'C'}{AC} \right] - \frac{\hat{A}B}{AB} - \frac{\hat{A}C}{AC} - \frac{\hat{B}C}{BC} , \]

\[ \frac{\hat{C'}}{C} + \frac{\hat{B}}{B} = \frac{\hat{A}}{A} \left[ \frac{C'}{C} + \frac{B'}{B} \right] , \]

where \( A' = \frac{dA}{dx} \), \( \hat{A} = \frac{dA}{dt} \) and so on.

The velocity field \( u^i \) is ir-rotational. The scalar expansion \( \Theta \), shear scalar \( \sigma^2 \) and volume scalar \( V \) are respectively have the form.
\[\Theta = u^i_i = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},\]  
\[\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{\Theta^2}{3} - \frac{\dot{A}B}{AB} - \frac{\dot{A}C}{AC} - \frac{\dot{B}C}{BC},\]  
\[V = \sqrt{-g} = AB C,\]

where \(g\) is the determinant of the metric \(g\). The shear tensor is

\[\sigma_{ij} = u_{(ij)} + \dot{u}_{i} u_{j} - \frac{1}{3} \Theta (g_{ij} + u_{i} u_{j}).\]

and the non-vanishing components of the \(\sigma_i^i\) are

\[
\begin{align*}
\sigma_1^1 &= \frac{1}{3} \left( 2 \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right), \\
\sigma_2^2 &= \frac{1}{3} \left( \frac{2\dot{B}}{B} - \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right), \\
\sigma_3^3 &= \frac{1}{3} \left( 2 \frac{\dot{C}}{C} - \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right), \\
\sigma_i^i &= 0.
\end{align*}
\]

The Einstein’s field equations (5)-(9) constitute a system of five highly NLPDEs with six unknowns variables, \(A, B, C, p^{(pf)}, \rho^{(pf)}\) and \(\rho^{(de)}\). Therefore, one physically reasonable conditions amongst these parameters are required to obtain explicit solutions of the field equations. Let us assume that the metric potential function \(A\) is a function of the time only, i.e., \(A(x,t) = A(t)\). If we substitute the metric function \(A(x,t) = A(t)\) in the Einstein field equations, the equations (13) - (14) transform to the NLPDEs of the coefficients \(B\) and \(C\) only, as the following new form:

\[
E_1 = \frac{\dot{A}B}{AB} - \frac{\dot{C}}{C} - \frac{B''}{A^2 B^2} + \left( \frac{\omega_y^{(de)} - \omega_y^{(de)}}{\omega_x^{(de)} - \omega_y^{(de)}} \right) \left[ \frac{\dot{A}}{A} - \frac{\dot{B}C}{BC} - \frac{\dot{B}C'}{A^2 BC} \right] + \left( \frac{\omega_x^{(de)} - \omega_y^{(de)}}{\omega_y^{(de)}} \right) \left[ \frac{\dot{B}}{B} - \frac{\dot{A}C}{AC} + \frac{C''}{A^2 C} \right] = 0,
\]

\[
E_2 = \frac{\dot{C}}{C} + \frac{\dot{B}'}{B} - \frac{\dot{A}}{A} \left( \frac{C''}{C} + \frac{B''}{B} \right) = 0,
\]

where

\[
p^{(pf)}(x,t) = \frac{\omega_x^{(de)}}{\omega_x^{(de)} - \omega_y^{(de)}} \left( \frac{C''}{A^2 C} - \frac{\dot{A}C}{AC} - \frac{\dot{A}}{A} \right) - \frac{\omega_y^{(de)}}{\omega_x^{(de)} - \omega_y^{(de)}} \left( \frac{B'C'}{BC} - \frac{\dot{B}C}{BC} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right),
\]

\[
\rho^{(pf)}(x,t) = \frac{1}{\omega_x^{(de)} - \omega_y^{(de)}} \left( \frac{\dot{B}C}{BC} + \frac{\dot{B}}{B} + \frac{C''}{AC} - \frac{\dot{A}}{A} - \frac{\dot{A}C}{AC} - \frac{B'C'}{A^2 BC} \right) + \frac{1}{A^2} \left( \frac{B'C'}{BC} + \frac{B''}{B} + \frac{C''}{AC} - \frac{\dot{A}B}{AB} - \frac{\dot{A}C}{AC} - \frac{B'C'}{A^2 BC} \right),
\]

\[
\rho^{(de)}(x,t) = \frac{1}{\omega_x^{(de)} - \omega_y^{(de)}} \left( \frac{B'C'}{A^2 BC} + \frac{\dot{A}}{A} + \frac{\dot{A}C}{AC} - \frac{\dot{B}C}{BC} - \frac{\dot{B}}{B} - \frac{C''}{A^2 C} \right).
\]
3 Lie point symmetry

We consider a one-parameter Lie group of transformations

\[
\begin{align*}
x^*_i &= x_i + \epsilon \xi_i(x_j, u_\beta) + o(\epsilon^2), \\
u^*_\alpha &= u_\alpha + \epsilon \eta_\alpha(x_j, u_\beta) + o(\epsilon^2),
\end{align*}
\]

with a small parameter \( \epsilon \ll 1 \), where \( x_1 = x, x_2 = t, u_1 = B \) and \( u_2 = C \). The coefficients \( \xi_1, \xi_2, \eta_1 \) and \( \eta_2 \) are the functions of \( x, t, B \) and \( C \). The system (15)-(16) is invariant under the transformations given in Eq. (20) and the corresponding infinitesimal generator of Lie groups (symmetries)

\[
X = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{2} \eta_\alpha \frac{\partial}{\partial u_\alpha},
\]

must satisfy the invariance conditions:

\[
\text{Pr}^{(2)} X \left( E_m \right) |_{E_m=0} = 0,
\]

where \( E_m = 0, m = 1, 2 \) are the system (15)-(16) under study and \( \text{Pr}^{(2)} \) is the second prolongation of the symmetries \( X \). The detail of Lie point symmetry has been already given in [22]. Finally the characteristic equations are given by

\[
\frac{dx}{a_1 x + a_2} = \frac{dt}{a_3 t + a_4} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.
\]

4 Optimal system of subalgebras

The general Lie point symmetries (21) becomes

\[
X = \sum_{i=1}^{6} a_i X_i,
\]

where, the non-linear Einstein field equations (15)-(16) admits the 6-dimensional Lie algebra spanned by the independent symmetries shown below:

\[
\begin{align*}
X_1 &= x \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= t \frac{\partial}{\partial t}, \\
X_4 &= \frac{\partial}{\partial t}, \\
X_5 &= B \frac{\partial}{\partial B}, \\
X_6 &= C \frac{\partial}{\partial C}.
\end{align*}
\]

The forms of the symmetries \( X_i, i = 1, \ldots, 6 \) suggest their significations: \( X_2, X_3 \) generate the symmetry of space translation, \( X_1, X_3, X_5, X_6 \) are associated with the scaling transformations. When the Lie algebra of these symmetries is computed, the only non-vanishing relations are:

\[
[X_1, X_2] = -X_2, \quad [X_3, X_4] = -X_4.
\]

Following Yadav and Ali [22], we acquire an optimal system of one-dimensional subalgebras to be those spanned by:

\[
\{ X^{(1)} = X_1 + a_3 X_3 + a_5 X_5 + a_6 X_6, \quad X^{(2)} = X_1 + a_4 X_4 + a_5 X_5 + a_6 X_6, \}
\]

\[
X^{(3)} = X_2 + a_3 X_3 + a_5 X_5 + a_6 X_6, \quad X^{(4)} = X_2 + a_4 X_4 + a_5 X_5 + a_6 X_6,
\]

\[
X^{(5)} = X_3 + a_5 X_5 + a_6 X_6, \quad X^{(6)} = X_4 + a_5 X_5 + a_6 X_6, \quad X^{(7)} = X_5 + a_6 X_6, \quad X^{(8)} = X_6 \}
\]

5 Similarity solutions

If we considered the symmetries \( X^{(5)} \) or \( X^{(6)} \) or \( X^{(7)} \) or \( X^{(8)} \), then \( a_1 = a_2 = 0 \), we shall analyze the similarity solutions associated with the optimal systems of symmetries \( X^{(1)}, X^{(2)}, X^{(3)} \) and \( X^{(4)} \) only as the following:

**Solution (I):** The symmetries \( X^{(1)} \) has the characteristic equations:

\[
\frac{dx}{x} = \frac{dt}{a_3 t} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.
\]
Then the similarity variable and the similarity transformations takes the form:

\[ \xi = \frac{x^a}{t}, \quad B(x, t) = x^b \Psi(\xi), \quad C(x, t) = x^c \Phi(\xi), \]  

(29)

where \( a = a_3, b = a_5 \) and \( c = a_6 \) are an arbitrary constants. In this case, we have

\[ A(t) = dt^{1 - \frac{a}{2}}, \quad \omega_y^{(de)}(t) = \omega_x^{(de)}(t) + q t^\frac{\alpha}{2} - 2, \quad \omega_z^{(de)}(t) = \omega_x^{(de)}(t) + r t^\frac{\beta}{2} - 2, \]

(30)

where \( d = a_7 a_3^\frac{1 - \alpha}{a}, q = a_8 a_3^\frac{1 - \beta}{a} \) and \( r = a_9 a_3^\frac{1 - \gamma}{a} \) are an arbitrary constants. Substituting the transformations [29] in the field Eqs. (15)–(16) lead to the following system of ordinary differential equations:

\[ (2a + b - 1) \frac{\Psi'}{\Psi} + (2a + c - 1) \frac{\Phi'}{\Phi} + a \xi \left[ \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} \right] = \frac{(a - 1)(b + c)}{a \xi}, \]

(31)

\[ a^3 \xi^{1 - \frac{a}{2}} \left[ \left( \alpha_1 + a (r - q) \xi \frac{\Phi'}{\Phi} \right) \frac{\Psi'}{\Psi} + a q \xi \frac{\Psi''}{\Psi} \right] - a d^2 \xi \left[ \left( \alpha_2 + a (r - q) \xi \frac{\Phi'}{\Phi} \right) \frac{\Psi'}{\Psi} + a r \xi \frac{\Phi''}{\Phi} \right] 
+ \frac{d^2}{(a - 1)(q - r) + a \xi} \left[ \alpha_3 \frac{\Phi'}{\Phi} + a q \xi \frac{\Phi''}{\Phi} \right] - a^2 \xi^{\frac{a}{2}} \left[ \alpha_4 + a \xi \left( \alpha_5 \frac{\Phi'}{\Phi} + a r \xi \frac{\Phi''}{\Phi} \right) \right] = 0, \]

(32)

where

\[
\begin{align*}
\alpha_1 &= c r + q \left( a + 2 b - c - 1 \right), \\
\alpha_2 &= q \left( 1 - a \right) + 2 a r, \\
\alpha_3 &= 2 a q + r \left( 1 - a \right), \\
\alpha_4 &= b q \left( c - b + 1 \right) + c r \left( c - b - 1 \right), \\
\alpha_5 &= b q + r \left( a - b + 2 c - 1 \right).
\end{align*}
\]

(33)

The equations (31) and (32) are non-linear ordinary differential equations. One cannot solve these equations in general. However, in a special cases, one can find a solution. Now, we propose the following conditions:

\[ \Psi(\xi) = \beta_1 \xi^{\beta_2}, \quad \Phi(\xi) = \beta_5 + \beta_3 \xi^{\beta_4}, \quad a = -\frac{2}{\beta_4}, \]

(34)

where \( \beta_1, \beta_2, \beta_3, \beta_4 \) and \( \beta_5 \) are an arbitrary constants. Substitute (34) in (31), we have the following equation:

\[
\beta_5 \left[ 2 \beta_2 \left( 2 \beta_2 + 2 + \beta_4 - b \beta_3 \right) - \beta_4 \left( 2 + \beta_4 \right) \left( b + c \right) \right] \
+ \beta_3 \left[ 4 \beta_2 \left( 1 + \beta_4 \right) - 2 \beta_4 \left( 6 + c - 2 + b \beta_2 - \beta_2 \right) - \beta_4^2 \left( b + 3 c - 6 \right) \right] \xi^{\beta_4} = 0.
\]

(35)

The coefficients of \( \xi^{\beta_4} \) and the absolute value must be equal zero. Solving the two resulting conditions with respect to \( b \) and \( c \), we have:

\[
b = \frac{2 \beta_2}{\beta_4} - \frac{(2 + \beta_4)(2 + 3 \beta_4)}{\beta_4 (2 + 2 \beta_2 + \beta_4)}, \quad c = 3 + \frac{2}{\beta_4},
\]

(36)

Therefore the (32) becomes

\[ \gamma_0 + \gamma_1 \xi^{\beta_4} + \gamma_2 \xi^{2 \beta_4} = 0, \]

(37)

where

\[
\begin{align*}
\gamma_0 &= d^2 \beta_4^2 \beta_5 \left( 2 \beta_2 - \beta_4 \right) \left( 2 + 2 \beta_2 + \beta_4 \right)^2 \left[ r \left( 2 + 2 \beta_2 + \beta_4 \right) - q \left( 2 + \beta_4 \right) \right], \\
\gamma_1 &= d^2 \beta_3 \beta_4^2 \left( 2 \beta_2 + \beta_4 \right) \left( 2 + 2 \beta_2 + \beta_4 \right)^2 \left[ r \left( 2 + 2 \beta_2 + \beta_4 \right) - q \left( 2 + 3 \beta_4 \right) \right]
+ 4 \beta_5 \left( 2 + 3 \beta_4 \right) \left( 2 + 2 \beta_2 + \beta_4 \right) \left[ 4 \beta_2 \left( 1 + \beta_4 \right) + (2 + \beta_4) \left( 4 + 5 \beta_4 \right) \right] \\
&- q \left( 2 + \beta_4 \right) \left[ (2 + \beta_4) (4 + 7 \beta_4) + 4 \beta_2 \left( 1 + 2 \beta_4 \right) \right], \\
\gamma_2 &= 4 \beta_3 \left( 2 + \beta_4 \right) \left( 2 + 2 \beta_2 + \beta_4 \right) \left[ 4 \beta_2 + (2 + \beta_4) \left( 4 + 3 \beta_4 \right) \right]
- q \left( 2 + 3 \beta_4 \right) \left[ 4 \beta_2 \left( 1 + \beta_4 \right) + (2 + \beta_4) \left( 4 + 5 \beta_4 \right) \right].
\end{align*}
\]

(38)
By using the solution (3) in (39), (36), (34), (30) and (29), we obtain the solution of the Einstein field equations as the following:

\[
\begin{align*}
(1) : & \quad r = \frac{q \left(16 + 48 \beta_4 + 42 \beta_4^2 + 9 \beta_4^3\right)}{2 \left(8 + 8 \beta_4 + 3 \beta_4^2\right)}, \quad \beta_2 = -\frac{\beta_4}{2}, \quad \beta_5 = 0, \\
(2) : & \quad r = -\frac{q \left(4 + 3 \beta_4\right)}{2 + 3 \beta_4}, \quad \beta_2 = -\frac{3 \left(1 + \beta_4\right) \left(2 + \beta_4\right)}{4 + 3 \beta_4}, \quad \beta_5 = \frac{d^2 \beta_3 \beta_1^2 (12 + 14 \beta_4 + 3 \beta_4^2)}{4 \left(4 + 3 \beta_4\right)^2}, \\
(3) : & \quad r = \frac{q \left(16 + 56 \beta_4 + 62 \beta_4^2 + 21 \beta_4^3\right)}{16 + 40 \beta_4 + 30 \beta_4^2 + 6 \beta_4^3}, \quad \beta_2 = \frac{\beta_4}{2}, \quad \beta_5 = \frac{4 d^2 \beta_3 \beta_1^2 (1 + \beta_4)^3}{64 + 272 \beta_4 + 408 \beta_4^2 + 256 \beta_4^3 + 57 \beta_4^4}.
\end{align*}
\]

Here, we consider only one of the above solutions. By using the solution (3) in (39), (36), (34), (30) and (29), we obtain the solution of the Einstein field equations as the following:

\[
\begin{align*}
A(t) & = dt^{1+\frac{\beta_4}{2}}, \quad B(x,t) = \beta_1 x^{-\frac{(2+\beta_4)(2+3 \beta_4)}{2 \beta_4 (1+\beta_4)}} t^{1+\frac{2}{\beta_4}}, \quad C(x,t) = \beta_3 x^{1+\frac{2}{\beta_4}} \left(t^{-\beta_4} - d_0^2 x^2\right), \\
\omega_y(t) & = \omega_x(t) + q t^{-2-\beta_4}, \quad \omega_z(t) = \omega_x(t) + r t^{-2-\beta_4}, \quad d^2 = \frac{d_0^2 \left(64 + 272 \beta_4 + 408 \beta_4^2 + 256 \beta_4^3 + 57 \beta_4^4\right)}{4 \beta_1^2 (1 + \beta_4)^3},
\end{align*}
\]

where \(d_0, q, \) and \(\beta_4\) are an arbitrary constants, while \(\omega_x\) is an arbitrary function of \(t\).

It is observed from equations (10), the line element (11) can be written in the following form:

\[
ds^2 = d_0^2 t^{2+\beta_4} \, dx^2 + \beta_1^2 x^{-\frac{(2+\beta_4)(2+3 \beta_4)}{2 \beta_4 (1+\beta_4)}} t^{2+\frac{2}{\beta_4}} \, dy^2 + \beta_3^2 x^{2+\frac{2}{\beta_4}} \left(t^{-\beta_4} - d_0^2 x^2\right)^2 \, dz^2 - dt^2.
\]

**Remark:** In the above solution, we can replace \(t \) by \(t + \zeta_1\) and \(x \) by \(x + \zeta_2\) without loss of generality, where \(\zeta_1\) and \(\zeta_2\) are some arbitrary constants.

**Solution (II):** The symmetries \(X^{(2)}\) has the characteristic equations:

\[
\frac{dx}{x} = \frac{dt}{a_4}, \quad \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.
\]

Then the similarity variable and the similarity transformations takes the form:

\[
\xi = x \exp \left[a t\right], \quad B(x,t) = \Psi(\xi) \exp \left[b t\right], \quad C(x,t) = \Phi(\xi) \exp \left[c t\right],
\]

where \(a = -\frac{1}{a_4}, b = a_5 \) and \(c = a_6\) are an arbitrary constants. In this case, we have

\[
A(t) = d \exp \left[a t\right], \quad \omega_y^{(de)}(t) = \omega_x^{(de)}(t) + q \exp \left[-2 a t\right], \quad \omega_z^{(de)}(t) = \omega_x^{(de)}(t) + r \exp \left[-2 a t\right],
\]

where \(d = a_7, q = a_8, \) and \(r = a_9\) are an arbitrary constants. Substituting the transformations (43) in the field Eqs. (13)-(14), we can get the following system of ordinary differential equations:

\[
\frac{b \Psi'}{\Psi} + \frac{c \Phi'}{\Phi} + a \xi \left(\frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi}\right) = 0,
\]

\[
\alpha_1 - \alpha_2 \xi \frac{\Phi'}{\Phi} + \frac{\Psi'}{\Psi} \left[\alpha_3 \xi - (q - r) \left(a^2 d^2 \xi^2 - 1\right) \frac{\Phi'}{\Phi} + \left(a^2 d^2 r \xi^2 - q\right) \frac{\Psi''}{\Psi} - \left(a^2 d^2 q \xi^2 - r\right) \frac{\Phi''}{\Phi}\right] = 0,
\]

where

\[
\begin{align*}
\alpha_1 & = d^2 \left(a + b + c\right) \left[(a - c) q + (b - a) r\right] n (1 + q) - 1, \\
\alpha_2 & = a d^2 \left[(a + b + c) q + (a - b) r\right], \\
\alpha_3 & = a d^2 \left[(a - c) q + (a + 2 b + c) r\right].
\end{align*}
\]
\textbf{Solution (III):} The symmetries $X^{(3)}$ has the characteristic equations:

$$\frac{dx}{\Gamma} = \frac{dt}{a_3 t} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.$$  

(48)

Then the similarity variable and the similarity transformations takes the form:

$$\xi = t \exp [a x], \quad B(x,t) = \Psi(\xi) \exp [b x], \quad C(x,t) = \Phi(\xi) \exp [c x],$$

(49)

where $a = -\frac{1}{a_3}, b = a_5$ and $c = a_6$ are an arbitrary constants. In this case, we have

$$A(t) = d t, \quad \omega^{(de)}_y(t) = \omega^{(de)}_x(t) + \frac{d}{\tau^2}, \quad \omega^{(de)}_z(t) = \omega^{(de)}_x(t) + \frac{r}{\tau^2},$$

(50)

where $d = a_7, q = a_8$ and $r = a_9$ are an arbitrary constants. Substituting the transformations \textbf{18} in the field Eqs. (15)-(16), we obtain the following system of ordinary differential equations:

$$\frac{b_1 \Psi'}{\Psi} + \frac{c_1 \Phi'}{\Phi} + a \xi \left( \frac{\Psi'' + \Phi''}{\Psi + \Phi} \right) = \frac{b_1 + c_1}{\xi},$$

(51)

$$\alpha_4 + \xi \left( \frac{\alpha_1 \Psi'}{\Psi} + \frac{\alpha_5 \Phi'}{\Phi} \right) + \xi^2 \left( \frac{\alpha_3 \Psi''}{\Psi} + \frac{\alpha_2 \Psi'\Phi' + \alpha_6 \Phi''}{\Phi} \right) = 0,$$

(52)

where

$$\begin{cases}
\alpha_1 = q \left[ d^2 - 2 a b - 1 \right] + a c (q - r), \\
\alpha_2 = (a - d) (a + d) (q - r), \\
\alpha_3 = r d^2 - a^2, \\
\alpha_4 = (c - b) \left[ b q + c r \right], \\
\alpha_5 = a b q - r d^2 + a r (a - b + c), \\
\alpha_6 = a^2 r - d^2 q.
\end{cases}$$

(53)

Equation (51) can be written in the following form:

$$\Psi'' = \Psi \left[ \frac{b_1 + c_1}{a \xi^2} - \frac{\Phi''}{\Phi} - \frac{1}{a \xi} \left( \frac{b_1 \Psi'}{\Psi} + \frac{c_1 \Phi'}{\Phi} \right) \right].$$

(54)

Substituting $\Psi''$ into (52), we obtain the following equation:

$$a \left( a^2 - d^2 \right) \xi \left[ \frac{(q + r) \Phi''}{\Phi} + \frac{(q - r) \Psi' \Phi'}{\Psi \Phi} \right] = \frac{\alpha_6}{\xi} + \frac{\alpha_7 \Psi'}{\Psi} + \frac{\alpha_8 \Phi'}{\Phi},$$

(55)

where

$$\begin{cases}
\alpha_6 = (b + c) \left( a^2 q - d^2 r \right) + a (b - c) \left( b q + c r \right), \\
\alpha_7 = a \left( a^2 - d^2 \right) + d^2 b r + a^2 \left( b q - c q + c r \right), \\
\alpha_8 = a r \left( d^2 - a^2 \right) + c r \left( d^2 - 2 a^2 \right) - a^2 \left( b q + c q - b r \right).
\end{cases}$$

(56)

It is easy to solve the above equation when $d = a$ or $\frac{(q + r) \Phi''}{\Phi} + \frac{(q - r) \Psi' \Phi'}{\Psi \Phi} = 0$. When $d = a$, the equation (55) takes the following form:

$$\frac{a (b + c) (q - r) + (b - c) (b q + c r)}{a \xi} = \left[ c (r - q) + b (r + q) \right] \frac{\Psi'}{\Psi} + \left[ b (q - r) + c (q + r) \right] \frac{\Phi'}{\Phi}.$$  

(57)

The solution of the above equation is

$$\Psi(\xi) = \beta_1 \xi \frac{a (b + c) (r - q) - (b - c) (b q + c r)}{a c (r - q) + b (r + q)} \frac{\Phi'(r - q) + c (q + r)}{\Phi(r - q) + c (r + q)} (\xi),$$

(58)

where $\beta_1$ is an arbitrary constant of integration.

Substituting the above equation into (51), we get the following equation

$$\gamma_1 + \frac{\alpha \gamma_2 \xi \Phi'}{\Phi} + 2 \alpha^2 \xi^2 \left[ \gamma_3 \Phi'^2 \Phi + \gamma_4 \Phi'' \Phi \right] = 0,$$

(59)
where
\[
\begin{align*}
\gamma_1 &= 2a^2 b q (b + c) (q - r) - r (b - c) (b^2 + c^2) (b q + c r) + a \left[ b^3 q (q - 3 r) + c^3 q (r - q) + b c^2 (q^2 - q r - 2 r^2) - b^2 c (q^2 + q r + 2 r^2), \\
\gamma_2 &= 2a (b + c) (r - q) [b (q - r) + c (q + r)] - (b^2 + c^2) [b (q - r)^2 - c (q^2 + 3 r^2)], \\
\gamma_3 &= b^2 r (r - q) + c^2 q (q + r) + b c (q^2 - 2 q r - r^2), \\
\gamma_4 &= c^2 r (r - q) + b^2 q (q + r) + b c (r^2 + 2 q r - q^2).
\end{align*}
\]  
(60)

The general solution of the equation (63) is read as
\[
\Phi(\xi) = \beta_2 \xi^{\gamma_5} \left( \beta_3 + \xi^{\gamma_6} \right)^{\gamma_7},
\]
(61)
where \( \beta_2 \) and \( \beta_3 \) are arbitrary constants of integration and
\[
\begin{align*}
\gamma_5 &= \frac{a q (q - r) - r (b q + c r)}{a (q^2 + r^2)}, \\
\gamma_6 &= \frac{2 a (b r + c q) + (b^2 + c^2) (q + r)}{2 a (b q + c r)}, \\
\gamma_7 &= \frac{(b q + c r) [c (r - q) + b (r + q)]}{(b^2 + c^2) (q^2 + r^2)}.
\end{align*}
\]  
(62)

Substituting (61) in (68), we have
\[
\Psi(\xi) = \beta_4 \xi^{\gamma_8} \left( \beta_3 + \xi^{\gamma_9} \right)^{\gamma_9},
\]
(63)
where \( \beta_4 = \beta_1 \beta_2^{\frac{b (q - r) + c (q + r)}{b (q - r) + c (q + r)}} \) is the new arbitrary constant and
\[
\begin{align*}
\gamma_8 &= \frac{a r (r - q) - q (b q + c r)}{a (q^2 + r^2)}, \\
\gamma_9 &= \frac{(b q + c r) [b (q - r) + c (q + r)]}{(b^2 + c^2) (q^2 + r^2)}.
\end{align*}
\]  
(64)

Without loss of generality, we can take \( r = 0 \). Now using the solution (61) and (63) in (39), (36), (34), (39) and (29), we can find the solution of the Einstein field equations as the following:
\[
\begin{align*}
A(t) &= a t, \quad \omega_y^{(de)}(t) = \omega_x^{(de)}(t) + \frac{q}{t^2}, \quad \omega_z^{(de)}(t) = \omega_z^{(de)}(t), \\
B(x, t) &= \beta_4 t^{-b/a} \left[ b (b c + c) \right]^{\frac{b (b c + c)}{2 b,a}} x^{\frac{b (b c + c) - 1}{2 b,a}}, \\
C(x, t) &= \beta_2 t e^{(a + c) x} \left[ b (b c + c) \right]^{\frac{b (b c + c) - 1}{2 b,a}} x^{\frac{b (b c + c) - 1}{2 b,a}},
\end{align*}
\]
(65)

where \( a, b, c, \beta_2, \beta_3, \beta_4 \) and \( q \) are an arbitrary constants, while \( \omega_x \) is an arbitrary function of \( t \). It is observed from equations (65), the line element (1) can be written in the following form:
\[
\begin{align*}
ds_3^2 &= a^2 t^2 d x^2 + \beta_4 t^{-b/a} \left[ b (b c + c) \right]^{\frac{b (b c + c) - 1}{2 b,a}} x^{\frac{b (b c + c) - 1}{2 b,a}} d y^2 \\
&+ \beta_2^2 t^2 e^{(a + c) x} \left[ b (b c + c) \right]^{\frac{b (b c + c) - 1}{2 b,a}} x^{\frac{b (b c + c) - 1}{2 b,a}} d z^2 - dt^2.
\end{align*}
\]  
(66)
Solution (IV): The symmetries \( X^{(4)} \) has the characteristic equations:

\[
\frac{dx}{1} = \frac{dt}{a_4} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}
\]

Then the similarity variable and the similarity transformations takes the form:

\[
\xi = t - a x, \quad B(x,t) = \Psi(\xi) \exp [b t], \quad C(x,t) = \Phi(\xi) \exp [c t],
\]

where \( a = a_4, b = \frac{a_5}{a_4} \) and \( c = \frac{a_6}{a_4} \) are an arbitrary constants. In this case, we have

\[
A(t) = d, \quad \omega_y^{(de)}(t) = \omega_x^{(de)}(t) + q, \quad \omega_z^{(de)}(t) = \omega_x^{(de)}(t) + r,
\]

where \( d = a_7, q = a_8 \) and \( r = a_9 \) are an arbitrary constants.

Substituting the transformations (43) in the field Eqs. (15)-(16), we can get the following system of ordinary differential equations:

\[
\frac{b \Psi'}{\Psi} + \frac{c \Phi'}{\Phi} + \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} = 0,
\]

\[
d^2 (b + c) [b r - c q] + d^2 [2 b r + c (r - q)] \frac{\Psi'}{\Psi} + d^2 [b (r - q) - 2 c r] \frac{\Phi'}{\Phi} + (a^2 - d^2) (q - r) \frac{\Psi''}{\Psi} + (d^2 r - a^2 q) \frac{\Phi''}{\Phi} = 0.
\]

### 6 Physical and geometrical properties of the models

**For the Model (41):**

The expressions of \( p^{(pf)} \), \( \rho^{(pf)} \) and \( \rho^{(de)} \) for the model (41), are given by:

\[
p^{(pf)}(x,t) = \frac{\beta_4}{4 f(x,t)} \left[ 8 q (1 + \beta_4)^2 (2 + \beta_4)^2 (2 + 3 \beta_4) t^{-2 - \beta_4} + q d_0^3 [320 + 1728 \beta_4 + 3600 \beta_4^2 + 3616 \beta_4^3 + 1750 \beta_4^4 + 327 \beta_4^5] x^2 t^{-2} - q d_0^4 (2 + \beta_4) (4 + 3 \beta_4) \left[ 16 + 56 \beta_4 + 60 \beta_4^2 + 19 \beta_4^3 \right] x^2 t^{-2 + \beta_4} + 4 (1 + \beta_4) (8 + 12 \beta_4 + 3 \beta_4^2) \left[ 2 (1 + \beta_4) (2 + \beta_4) + d_0^2 (4 + 10 \beta_4 + 5 \beta_4^2) x^2 t^{\beta_4} \right] \omega_x^{(pf)}(t) \right],
\]

\[
\rho^{(pf)}(x,t) = \frac{\beta_4}{4 f(x,t)} \left[ 4 (1 + \beta_4) (8 + 12 \beta_4 + 3 \beta_4^2) \left[ 2 (1 + \beta_4) (2 + \beta_4) + d_0^2 (4 + 10 \beta_4 + 5 \beta_4^2) x^2 t^{\beta_4} \right] - 4 q (1 + \beta_4) (2 + \beta_4) (8 + 24 \beta_4 + 26 \beta_4^2 + 9 \beta_4^3) t^{-2 - \beta_4} + q d_0^3 [320 + 1344 \beta_4 + 1968 \beta_4^2 + 1160 \beta_4^3 + 194 \beta_4^4 - 27 \beta_4^5] x^2 t^{-2} + q d_0^4 (2 + \beta_4) (4 + 3 \beta_4) \left[ 16 + 56 \beta_4 + 60 \beta_4^2 + 19 \beta_4^3 \right] x^2 t^{-2 + \beta_4} \right],
\]

\[
\rho^{(de)}(x,t) = \frac{\beta_4 (1 + \beta_4) (8 + 12 \beta_4^2 + 3 \beta_4^3)}{f(x,t)} \left[ 2 (1 + \beta_4) (2 + \beta_4) + d_0^2 (4 + 10 \beta_4 + 5 \beta_4^2) x^2 t^{\beta_4} \right],
\]

where

\[
f(x,t) = q d_0^3 (4 + 3 \beta_4) \left[ 16 + 56 \beta_4 + 60 \beta_4^2 + 19 \beta_4^3 \right] \left( d_0^2 x^2 t^{\beta_4} - x^2 \right).
\]

The volume element is

\[
V = \beta_1 \beta_3 x^{-\frac{2 + \beta_4}{2 + \beta_4}} t^{1 - \beta_4} \left( d_0^2 x^2 t^{\beta_4} - 1 \right).
\]
The expansion scalar is given by:
\[ \Theta = \frac{1}{t} \left( 1 + \frac{\beta_4}{d_0^2 x^2 t^{\beta_4} - 1} \right) \]  
(76)

The non-vanishing components of the shear tensor, \( \sigma^i_j \), are:
\[ \sigma^1_1 = \frac{d_0^2 (4 + 3 \beta_4) x^2 t^{\beta_4} - 4 - 5 \beta_4}{6 \left[ \beta_4 - 1 + d_0^2 x^2 t^{\beta_4} \right]} \]  
(77)

\[ \sigma^2_2 = \frac{2 + \beta_4 - d_0^2 (4 + 3 \beta_4) x^2 t^{\beta_4}}{6 \left[ \beta_4 - 1 + d_0^2 x^2 t^{\beta_4} \right]} \]  
(78)

\[ \sigma^3_3 = \frac{1 + 2 \beta_4 - d_0^2 x^2 t^{\beta_4} - 4 - 5 \beta_4}{3 \left[ \beta_4 - 1 + d_0^2 x^2 t^{\beta_4} \right]} \]  
(79)

The shear scalar is:
\[ \frac{\sigma^2}{\Theta^2} = \frac{4 + 10 \beta_4 + 7 \beta_4^2 - 2 d_0^2 (4 + 8 \beta_4 + 3 \beta_4^2) x^2 t^{\beta_4} + d_0^4 (4 + 6 \beta_4 + 3 \beta_4^2) x^4 t^{2 \beta_4}}{12 \left[ \beta_4 - 1 + d_0^2 x^2 t^{\beta_4} \right]^2} \]  
(80)

The deceleration parameter is given by [24, 25]
\[ q = \frac{\left( \beta_4 - 1 + d_0^2 x^2 t^{\beta_4} \right) \left( 1 - \beta_4 \right) \left( 2 + \beta_4 \right) - d_0^2 (4 - \beta_4 - 3 \beta_4^2) x^2 t^{\beta_4} + 2 d_0^4 x^4 t^{2 \beta_4}}{t^4 \left[ d_0^2 x^2 t^{\beta_4} - 1 \right]^4} \]  
(81)

Figure 1: Variation of energy density (upper left panel), dark energy density (upper right panel), volume (lower left panel) and DP (lower right panel) versus time for model (41)
For the Model (66):

The expressions of $p^{(pf)}$, $\rho^{(pf)}$ and $\rho^{(de)}$ for the model (66), are given by:

$$p^{(pf)}(x,t) = \frac{b c q \left[ b^2 + c^2 + 2 a (b + c) \right] \left( t e^{a x} \right)^{\frac{b^2 + 2 a c + c^2}{2 a b}} - \beta_3 \left[ b^2 + c^2 \right] \left[ b^2 q + (b^2 + 2 a c + c^2) \omega_t^{(de)}(t) \right]}{g(x,t)},$$

$$\rho^{(pf)}(x,t) = \frac{1}{g(x,t)} \left[ q \left( b^2 + b c + c^2 \right) \left[ b^2 + c^2 + 2 a (b + c) \right] \left( t e^{a x} \right)^{\frac{b^2 + 2 a c + c^2}{2 a b}} \right. + \beta_3 \left( b^2 + c^2 \right) \left[ q \left[ c^2 + 2 a (b + c) \right] - \left( b^2 + 2 a c + c^2 \right) t^2 \right] \right],$$

$$\rho^{(de)}(x,t) = \frac{\beta_3 \left( b^2 + c^2 \right) \left( b^2 + 2 a c + c^2 \right) t^2}{g(x,t)},$$

where $g(x,t) = a^2 q \left( b^2 + c^2 \right) t^2 \left[ \beta_3 + \left( t e^{a x} \right)^{\frac{b^2 + 2 a c + c^2}{2 a b}} \right]$. The volume element is

$$V = a \beta_2 \beta_4 t^{2-b/a} e^{(a+c)x} \left[ \beta_3 + \left( t e^{a x} \right)^{\frac{b^2 + 2 a c + c^2}{2 a b}} \right]^{\frac{2 b^2}{b^2 + c^2}}.$$ 

Figure 2: Variation of energy density (upper left panel), dark energy density (upper right panel), volume (lower left panel) and DP (lower right panel) versus time for model (66).
The expansion scalar is given by:

\[
\Theta = \frac{1}{(a^2 + c^2) t} \left( 2 (b^2 + b c + c^2) - \frac{b \beta_3 (b^2 + 2 a c + c^2)}{a \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}} \right),
\]

(86)

The non-vanishing components of the shear tensor, \(\sigma_1^a\), are:

\[
\sigma_1^a = \frac{\beta_3 (a + b) (b^2 + c^2) + a (b - c) \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}}{3 \beta_3 (2 a - b) (b^2 + c^2) + 6 a (b^2 + b c + c^2)^2 (t e^a)^{t^{2+2 a c+c^2}}},
\]

(87)

\[
\sigma_2^a = \frac{4 \beta_3 (a + b) (b^2 + c^2) + (b - c) \left[(2 a (2 b + c) + 3 (b^2 + c^2) \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}}{6 \beta_3 (2 a - b) (b^2 + c^2) - 12 a (b^2 + b c + c^2)^2 (t e^a)^{t^{2+2 a c+c^2}}},
\]

(88)

\[
\sigma_3^a = \frac{2 \beta_3 (a + b) (b^2 + c^2) + (b - c) \left[(2 a (b + 2 c) + 3 (b^2 + c^2) \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}}{6 \beta_3 (2 a - b) (b^2 + c^2) + 12 a (b^2 + b c + c^2)^2 (t e^a)^{t^{2+2 a c+c^2}}},
\]

(89)

The shear scalar is:

\[
\frac{\sigma^2}{\Theta^2} = \frac{1}{h(x, t)} \left(4 \beta_3^2 (a + b) (b^2 + c^2)^2 \right.
\]

\[
+ 2 \beta_3 (a + b) (b - c) (b^2 + c^2) \left[(2 a (2 b + c) + 3 (b^2 + c^2) \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}}
\]

\[
+ (b - c)^2 \left[6 a (b + c) (b^2 + c^2) + 3 (b^2 + c^2)^2 + 4 a (b^2 + b c + c^2) \right] \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}
\]

(90)

where

\[
h(x, t) = 12 \left[\beta_3 (2 a - 2 b) (b^2 + c^2) - 2 a (b^2 + b c + c^2)^2 (t e^a)^{t^{2+2 a c+c^2}}\right]^2.
\]

(91)

The deceleration parameter is given by:

\[
q = \frac{1}{k(x, t)} \left(2 \beta_3^2 (a + b) (2 a - b) (b^2 + c^2)^2 - \beta_3 (b^2 + c^2)^2 \right.
\]

\[
+ 4 a^2 (c^2 + b c - 2 b^2) - 2 a (b^2 - 2 b c) \left[(a \beta_3 + (t e^a)^{t^{2+2 a c+c^2}})ight]^{2+2 a c+c^2}
\]

\[
+ 4 a^2 (b - c)^2 (b^2 + b c + c^2) \left[(t e^a)^{t^{2+2 a c+c^2}}\right]
\]

\[
x \left(\beta_3 (b^2 + 2 b c + c^2) + 2 a (b^2 + b c + c^2)^2 (t e^a)^{t^{2+2 a c+c^2}}\right)^2
\]

(92)

where

\[
k(x, t) = 2 a^4 (b^2 + c^2)^4 t^4 \left[\beta_3 + (t e^a)^{t^{2+2 a c+c^2}}\right]^{4+2 a c+c^2}.
\]

(93)
7 Conclusion

In this paper, we have investigated some models of accelerating universe with minimal interaction between normal matter and anisotropic DE in anisotropic and inhomogeneous Bianchi I space-time. Generally, the models behave like an expanding, shearing and non-rotating universe in which flow vector is geodetic. On the basis of optimal systems of symmetries $X^{(1)}$ and $X^{(2)}$, we obtain two models (41) and (66) respectively. The main features of the work are as follows:

- The models are based on similarity solution of field equation and we have obtained the new class of exact solution.
- We have, in general discussed several physical features and geometrical properties of the models. However, as a special case, most notable aspect of the solutions have been studied that are non-singular in nature. All figures depicts interesting features of the present cosmological models in terms of DP and other physical parameters.
- In the derived models, the matter energy density and dark energy density remains positive. Therefore, the WEC and NEC are satisfied, which in turn imply that the derived models are physically realistic.
- As $t \rightarrow \infty$, $V \rightarrow \infty$ but $\rho^{(de)} \rightarrow 0$, hence the volume increases with passage of time while DE density decreases.
- The derived models seem to describes the dynamics of universe from big bang to present epoch and DE dominates the universe at present time which may be attributed to the current accelerated expansion of universe.
- Hypothetical DE is the most reasonable way of explaining why the universe is expanding at an ever increasing rate. DE plays a massive part in shaping our reality; however, it is to be noted that no body seems certain of what the dang stuff actually is. Future space mission hope to solve this mystery and shake up our current understanding of the universe. To our knowledge, this work is the first study of minimally interacting anisotropic DE with normal matter in Bianchi - I inhomogeneous space-time and its general form.

It is important to note that $x = \text{constant}$ removes the inhomogeneity from the derived models and they simply represent model of universe based on power law cosmology. Numerous cosmological models with power law expansion exist in the literature. Thus the solutions, presented in this paper generalize the solution obtained by numerous authors in spatially homogeneous and anisotropic Bianchi I space-time, in particular Yadav and Saha \cite{8}.

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