Density and Graviton Perturbations in the Cosmic Microwave Background

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Abstract

We evaluate and compare the gravitational wave and density perturbation contributions to the cosmic microwave background radiation, on the basis of the same power law inflationary model. The inflation to radiation transition is treated in this paper as instantaneous, but a model is constructed to allow for a smooth transition from the radiation to the matter dominated eras. The equations are numerically investigated and integrated, without any basic approximations being made. Use is made of the synchronous gauge, with appropriate gauge invariant variables, thus eliminating any confusion arising from unphysical gauge modes. We find a non-negligible gravitational wave contribution, which becomes dominant for a power law expansion with exponent $q < 13$. We also explore the dependence of our results with the main characteristic of the transition region, its length.

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I. INTRODUCTION

In the inflationary scenario graviton (metric tensor) and density (metric scalar) perturbations arise from quantum fluctuations early in the inflationary era. In many models the subsequent development of these separate perturbations is governed by the same parameters of the given model. Thus the absolute magnitude of the perturbations presently observed in the CMBR and through other phenomena is predicted in terms of these parameters so that, among other things, a comparison of the metric tensor and metric scalar contributions can be made. Models with power law inflation are of this type. It is the purpose of this paper to evaluate and compare the two contributions for such models. In doing so we shall make the instantaneous approximation for the transition from the inflationary era to the radiation era, since it holds with good accuracy over the range of metric perturbation wavelengths of most physical interest; we do not make that approximation for the transition from the radiation era to the matter era.

We take power law inflation to be driven by a single canonically normalized scalar field with Einstein gravity. The assumption of a single scalar field is more widely encompassing than it seems; almost any current extended gravity theory, such as a higher order or scalar-tensor theory can be rewritten as Einstein gravity using a conformal transformation. Then, moreover, slow roll inflation solutions (giving the standard calculations of density perturbations) can be expanded about power law inflation solutions.

We use a synchronous gauge for the formalism of the computing programs, while extracting gauge invariant quantities for making physical comparisons and for delicate procedures of conveying information from one cosmic era to another. Thus we eliminate any confusion from unphysical synchronous gauge modes from our results.

II. THE INFLATIONARY ERA

We calculate with perturbations in a synchronous gauge so that the metric is

$$ds^2 = -a^2d\tau^2 + a^2(\delta_{ij} + h_{ij})dx^idx^j$$  \hspace{1cm} (1)

In the inflationary era, with scalar field $\phi$, the energy-momentum tensor is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi)]$$  \hspace{1cm} (2)

A. Metric Scalar Perturbations

In this section we shall first treat the basic formalism and then specialise to the case of power-law inflation.

We adopt the notations of Grishchuk so that for density perturbations with wave number $k$

$$h_{ij} = h(\tau)Q\delta_{ij} + h_i(\tau)k^{-2}Q_{,ij}$$  \hspace{1cm} (3)
where \( Q = \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x}) \).

If \( \rho_1 \) and \( p_1 \) are the perturbations to the energy and pressure densities then the Einstein equations yield, where \( \kappa = 8\pi G \),

\[
a^2 \kappa \rho_1 = 3\alpha h' + k^2 h - \alpha h'_t, \tag{4}
\]

\[
a^2 \kappa p_1 = -h'' - 2\alpha h', \tag{5}
\]

and also with the hypothesis of non-diagonal space-space components in the energy-momentum tensor, as with a scalar field or with a perfect fluid

\[
0 = h'' + 2\alpha h'_t - k^2 h \tag{6}
\]

where \( \alpha \equiv a'/a \).

These equations being valid with an energy-momentum tensor of a scalar field or a perfect fluid (or a mixture) are applicable through all the model eras we shall consider from inflation to the present.

We now consider the inflationary era case with the energy-momentum tensor given by Eq.(2). Then if the unperturbed value of \( \phi \) is \( \phi_0 \) and the perturbation to it is \( \phi_1 \) the Einstein equations yield

\[
\kappa \phi_1 = h'/\phi'_0 \tag{7}
\]

and also [6]

\[
\mu'' + \mu \left[ k^2 - \left( a\sqrt{\gamma} \right)'/(a\sqrt{\gamma}) \right] = 0 \tag{8}
\]

\[
\mu \equiv \frac{a}{\alpha \sqrt{\gamma}} (h' + \alpha \gamma h), \tag{9}
\]

\[
\gamma \equiv 1 - \frac{\alpha'}{\alpha^2}. \tag{10}
\]

For the particular case of power law inflation the scale factor

\[
a(\tau) \propto (\tau_i - \tau)^p \tag{11}
\]

where \( p(< -1), \tau_i \) are constants so that \( \gamma = (1 + p)/p = \text{constant} \). Our model is one of distinctly power-law, rather than exponential, inflation; that is we do not approach the de Sitter limit, \( p \to -1 \) and \( \gamma \to 0 \). If \( t \) is the appropriate cosmic time for the inflation era then \( a(t) \propto t^q \) where \( q = 1/\gamma \). An adequate amount of inflation requires \( q \geq 10 \); for larger \( q \) we limit ourselves to values where expressions such as Eq.(23) below are extremely good approximations and this is certainly true up to \( q = 100 \).

The solution for \( h \) is given in terms of that for \( \mu \) by Eq.(9) as

\[
h = \frac{\alpha}{a} \left\{ \sqrt{\gamma} \int_{\tau'}^{\tau} \mu(k, \tau')d\tau' + C_i \right\}, \tag{12}
\]
where $C_i$ is an integration constant subsuming, when $\gamma$ is constant, the lower limit of the integration. $h = \frac{\alpha}{a}C_i$ is a solution of $\mu = 0$ corresponding to an unphysical mode of the synchronous gauge [3] and we drop this term since, giving zero contribution to the gauge invariant variables, it does not carry information through into the radiation era, as related in the next section.

All the above development was non-quantum mechanical. We denote the corresponding quantum field theory quantities by a tilde:

\[
\tilde{h} = \frac{\alpha}{a}\sqrt{\gamma}\int^\tau \tilde{\mu}(k, \tau') d\tau'
\]

(13)

\[
\tilde{\mu} = N \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{2k}\left[c_k\mu_1(k, \tau) \exp(i k . x) + h.c.\right]
\]

(14)

where $c_k$ is a quantum annihilation operator, $[c_k, c_k^\dagger] = \delta^d(k - k')$, and $\mu_1(y)$, $y = |k(\tau_i - \tau)|$, is that solution of the Bessel equation (8) (with $a$, $\gamma$ specified by power-law inflation as above) such that

\[
\mu_1(y) \rightarrow e^{-iy}, \text{ } y \rightarrow \infty
\]

(15)

thus corresponding for large $k$ to the usual mode function of quantum mechanical plane waves.

$N$ is a normalization factor whose determination gives the absolute magnitude of the observed density perturbations (in terms of the model parameters) given the assumption that these come from a primordial vacuum with zero quantum occupation number. The determination of $N$ is as follows.

Using the methods of ref. [3] and Eqs.(7, 36) the gauge invariant scalar field, $\phi_{gi}^q \equiv \phi_1 - a^{-1}\phi_0^q k^{-2}h_i$ is given by the gauge invariant version of Eq.(7) as

\[
\kappa \phi_{gi}^q = \frac{\alpha}{a}\sqrt{\gamma}\left[-\mu + \alpha \gamma k^{-2}\{-\mu' + \mu(\alpha + \gamma/2\gamma)\}\right]/\phi_0'
\]

(16)

where, for power law inflation, $\sqrt{\kappa}\phi_0' = -\alpha\sqrt{2\gamma}$. So in the limit $k \rightarrow \infty$ it follows that $\sqrt{2\kappa}\phi_1^q \rightarrow \tilde{\mu}/a = N\tilde{\mu}/a$. This yields the result

\[
N = \sqrt{2\kappa} = \sqrt{16\pi G}
\]

(17)

since then

\[
\phi_1^q \rightarrow a^{-1}\int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{2k}\left[c_k\mu_1(k, \tau) \exp(i k . x) + h.c.\right]
\]

(18)

which is the appropriate quantization for a scalar field with the normalization (2). This is the same normalizing factor as that found by Grishchuk [3] using a similar argument. Also it is the same normalizing factor as that appropriate for the metric tensor case [4,5] given below.

We can now proceed with the evaluation of the metric scalar components at the end of the power-law inflation. The exact expression for the solution $\mu_1$ is, with $n = \frac{1}{2} - p$,
\[\mu_1(y) = \sqrt{\frac{\pi y}{2}} \left[J_n - iY_n\right] \exp \left[-i \left(\frac{1}{2}n\pi + \frac{1}{4}\pi\right)\right]\]  

(19)

As previously stated we shall assume a sudden transition at the end of inflation and the beginning of the radiation era; variables at that interface we denote by \(\tau = \tau_2, a = a_2, H = H_2 = -p((\tau_1 - \tau_2)a_2)^{-1} = (\tau_2a_2)^{-1}\). The values of \(k\) of interest are of the order of magnitude (the relevant meaning here is being within a few factors of ten) of

\[k_1 \equiv \frac{a_1'}{a_1} = a_1H_1\]  

(20)

where \(a_1, H_1\) denote the values of the scale factor and Hubble at the time, \(\tau_1\), when the matter era begins. Thus for such values of \(k\) the values of \(y = |k(\tau_1 - \tau_2)|\) at the end of the inflationary era, beginning of the radiation era, are of order \(10^{-n}, n > 10\). So then we can expand \(\mu\) in a power series with leading terms

\[\mu_1(y) = M(p)\left[y^p - \frac{y^{p+2}}{2(2p+1)}\right].....\]  

(21)

\[M(p) \equiv -2^{-p}\Gamma\left(\frac{1}{2} - p\right) \exp(-i(1 - p)/2)/\sqrt{\pi}\]  

(22)

The corresponding expansion of \(h\) from Eq.(12), with \(C_i = 0\), is

\[h = (\sqrt{\gamma}a)^{-1}M(p)\left[y^p - \frac{(p + 1)y^{p+2}}{2(2p+1)(p+3)}\right].....\]  

(23)

We shall need these expressions at the interface of the inflationary and radiation eras, \(\tau = \tau_2\).

**B. Metric Tensor Perturbations**

The mode function \(\mu/a\) for gravitons is given by

\[\mu'' + \mu(k^2 - a''/a) = 0.\]  

(24)

This holds for any cosmic era, whatever may be the dynamics responsible for the particular form of \(a(\tau)\). The quantum field theory expression for tensor perturbations is [4,5]

\[\tilde{h}_{ij} = \sqrt{16\pi G} \sum_{\lambda=1}^2 \int \frac{d^3k}{(2\pi)^3} \frac{2}{\sqrt{2k}\alpha(\tau)} \left[a_{\lambda k}\epsilon^\lambda_{ij}(k)\mu(k, \tau) \exp(i\mathbf{k}.\mathbf{x}) + h.c.\right],\]  

(25)

where \(a_{\lambda k}\) is the annihilation operator for the graviton with polarization \(\lambda\) and wave number \(k\),

\[\mu = \mu_1\]  

(26)

and the polarization tensor satisfies \(\sum_{ij}\epsilon^\lambda_{ij}(k)\epsilon^\lambda_{ij}(k) = 2\delta_{\lambda\lambda'}\).
III. INTO AND THROUGH THE RADIATION ERA

We take the instantaneous approximation to the transition from inflation to radiation. This is well justified because for the values of $k$ of interest (discussed above in II A) and with a time of transition of order $\tau_1 - \tau_2 = p/a_2 H_2$, then the parameter measuring the suddenness of the transition is, from Eq.(20), of order $k_1(\tau_1 - \tau_2) = p a_1 H_1/a_2 H_2 \approx a_2/a_1$, and is thus exceedingly small. We use the method of Deruelle and Mukhanov \[4\] to match the end of inflation to the beginning of radiation.

Our convention is that the conformal time $\tau$ is continuous from inflation through the matter era, and that in the pure radiation era the scale factor

$$a \propto \tau$$

(27)

It follows that in the radiation era, $\tau = (aH)^{-1}$. In the inflation era the scale factor is proportional to $(\tau_i - \tau)^p$ (Eq.\[1\]) and in the pure matter era to $(\tau - \tau_m)^2$ where $\tau_i, \tau_m$ and the constants of proportionality are determined from the continuities of $a, a'$, with the convention $a$(present) = 1.

A. Metric Scalar Perturbations

In the radiation era, treated as a relativistic perfect fluid phase

$$\nu'' + \frac{1}{3} k^2 \nu = 0, \nu \equiv \frac{a}{\alpha}(h' + \alpha \gamma h),$$

(28)

where

$$h = \frac{a}{\alpha} \int \nu d\tau, \gamma \equiv 1 - \alpha'/\alpha^2 = 2.$$  

(29)

The mode function $\mu$ of the inflationary era is replaced by $\nu$. We write the solution of Eq.(28) as

$$\nu = B_+ \cos(k(\tau - \tau_2)/\sqrt{3}) - B_- \sin(k(\tau - \tau_2)/\sqrt{3}).$$

(30)

The gauge invariant function $\Phi$ is given by

$$\Phi = h - (\alpha/k^2)h',$$

(31)

and using the Einstein equations, Eqs.(\[4\]), with $p_1 = \rho_1/3$, and Eq.(28)

$$\Phi = -(\nu'/\alpha - \nu)/ae^2, \epsilon = k/(a\sqrt{3}).$$

(32)

The matching conditions found in ref. \[2\], by deduction from the Lichnerowicz conditions \[4\], imply that $\Phi$ and the expression

$$\Gamma \equiv (\Phi'/\alpha + \Phi + \epsilon^2\Phi)/\gamma,$$

(33)
should both be continuous on the surface $\tau = \tau_2$. On the radiation side of the surface, $\tau = \tau_{2+}$:

$$\Phi = (\epsilon_2 B_- + B_+)/(a_2 \epsilon_2^2),$$  \hspace{1cm} (34)$$

$$\Gamma = -[B_+(1 - \epsilon_2^2) + \epsilon_2 B_- (1 - \frac{1}{2} \epsilon_2^2)]/(a_2 \epsilon_2^2).$$  \hspace{1cm} (35)$$

We now need the values of $\Phi$ and $\Gamma$ on the inflation side of the surface, $\tau = \tau_{2-}$. In the inflationary era the Einstein equations yield

$$h''_i = [h'' + h'((\alpha - 2\alpha'/\alpha - \gamma'/\gamma) + k^2 h)/\alpha$$  \hspace{1cm} (36)$$

Using Eq.(23) we find that to leading order at $\tau = \tau_{2-}$

$$\Phi = \frac{p + 1}{2p + 1} \Sigma, \Gamma = \frac{p}{2p + 1} \Sigma,$$  \hspace{1cm} (37)$$

$$\Sigma \equiv M(p) y_2^p \sqrt{\frac{p}{p + 1}/a},$$  \hspace{1cm} (38)$$

where the expansion parameter $y_2$ is given by

$$y_2 = k(\tau_i - \tau_2) = -pk/\alpha_2 = -pk\tau_2$$  \hspace{1cm} (39)$$

noting that the continuity of $a$ and $\alpha \equiv a'/a$ is part of the matching conditions [3].

Solving the simultaneous equations got by equating the different expressions for the $\Phi, \Gamma$ pair at $\tau_{2-}$ and $\tau_{2+}$ gives

$$B_+ = -\epsilon_2 B_- = 2a_2 \Sigma,$$  \hspace{1cm} (40)$$

$$B_- = 2p \sqrt{\frac{3p}{1 + p} M(p)y_2^{p-1}},$$  \hspace{1cm} (41)$$

and the extreme smallness of $\epsilon_2$ means that we can put $B_+ = 0$ and thus neglect the cosine term in Eq.(30) for $\nu$. Thus Eqs.(30,41) determine $\nu$ and in consequence the gauge invariant amplitude $\Phi$, Eq.(32), throughout the radiation era:

$$\Phi = B_- [\epsilon \cos(k(\tau - \tau_2)/\sqrt{3}) - \sin(k(\tau - \tau_2)/\sqrt{3})]/(ae^2).$$  \hspace{1cm} (42)$$
B. Metric Tensor Perturbations

For gravitons the matching is simpler \[^{[10,12]}\]. All that is required is that \(h_{ij}\) and its first time derivative be continuous.

We have, in both the inflation and radiation eras, Eq. (25) for \(\tilde{h}_{ij}\) with \(\mu = \mu_1\) for inflation, but in the radiation era given by \(\mu'' + \mu (k^2 - a''/a) = \mu'' + k^2 \mu = 0\) so that

\[
\mu = G_+ \cos(k(\tau - \tau_2)) - G_- \sin(k(\tau - \tau_2)).
\]  

(43)

Matching \((\mu/a)\) and \((\mu/a)'\) we find, to leading order in \(y_2\),

\[
G_+ = 0, G_- = pM(p)y_2^{p-1}.
\]  

(44)

For power law inflation \(p = -(1 + \delta)\) where \(0 < \delta \leq 1/10\). We can make a direct comparison of Eq. (44) with Eq. (41), which would suggest that the influence of density perturbations on the CMB will be greater than that of gravity waves. However there are more stages to go through.

IV. THE RADIATION TO MATTER TRANSITION

We shall assume that for \(\tau > \tau_1\) the universe can be characterized by matter with zero pressure for some value \(\tau_1\) of the conformal time. The transition time, \(\tau_{tr}\), from the radiation era, characterized by \(p = \frac{1}{3}\rho\), can be as large as \(\sim (a_1 H_1)^{-1}\). Thus for waves influential in the CMBR, section (II A), \(k\tau_{tr}\) is not necessarily small and then the instantaneous transition approximation is not reliable.

For the development of the matter components there have been many detailed studies \(^1\) using the Boltzmann equations for various components of the matter which might be present. These details are important for the smaller scale structure of the CMBR, but not for the larger scale structure. We wish to concentrate on this larger scale in our comparison of the gravity wave and density perturbation contributions, and thus to avoid questions of the components of matter. Consequently we shall approximate the transition from radiation, relativistic matter, to the non-relativistic matter domination era by a smooth change in an overall equation of state. The most significant parameter of such an approximation is the length of time that the transition takes, and while we can indeed make a reasonable estimate of this time, we shall also consider results as a function of the transition time.

A. Metric Scalar Perturbations

We postulate a smooth transition from the radiation era to the matter era in which we first parametrize the density as a function of the scale factor, \(a\), by

\[
\rho = \rho_1 e^{-sr},
\]  

(45)

\(^1\)See White et al. \[^{[11]}\], Ma and Bertschinger \[^{[12]}\] and references therein.
where $s = s(r), r = \ln(a/a_1), a_1 = a(\tau_1), \varrho_1 = constant$. The energy conservation equation gives the pressure and thus an implicit equation of state by

$$\frac{d}{da}(\rho a^3) = -3pa^2, \quad (46)$$

$$p/\rho = \frac{1}{3}(r \frac{ds}{dr} + s) - 1. \quad (47)$$

$s = 4$ corresponds to the radiation era, $s = 3$ to the matter dominated era; we can postulate an explicit form for $s$ as a sufficiently smooth and smoothly joining function of $r$ between those two constants. The one we adopt is given in the appendix. Given such a function we can find $\alpha \equiv r' = a'/a$ and the Hubble parameter $H = r'/a$ as functions of $a$, and also, consequently, $\tau = \tau(a)$ or equivalently $a = a(\tau)$, when given also the values of $H = H_1, a = a_1$ at the beginning of the pure matter era. Thus from the above equations:

$$\varrho_1 = 3(H_1a_1)^2/\kappa, \quad (48)$$

$$\alpha = r' = H_1a_1 \exp[(2 - s)r/2]. \quad (49)$$

Adopting the notation that the transition begins at $a = a_e, r = r_e, \tau = \tau_e$, we specify the length of the transition by

$$r_{trans} \equiv \frac{1}{2}(r_1 - r_e) = \frac{1}{2} \ln(a_1/a_e). \quad (50)$$

We now consider the density perturbations; these are given by the development of $\nu$, defined in Eq.(28), or equivalently of $u$ \cite{6} where

$$u = \frac{\alpha}{a} \nu = \alpha(\frac{dh}{dr} + \gamma h). \quad (51)$$

In the radiation era $\nu$ took a simple sinusoidal form but now evolves according to

$$\frac{d^2u}{dr^2} + \frac{du}{dr}[3 + C + \frac{dB}{dr}] + u[(kc_s/\alpha)^2 + (1 + \frac{dB}{dr})(C + 1) + 2\frac{d^2B}{dr^2} + 2(\frac{dB}{dr})^2] = 0. \quad (52)$$

where $c_s^2 = p_0/\rho_0, B = (2 - s)r/2, C = 3c_s^2 - \frac{d^2 z}{dr^2}/c_s^2$, with $p_0$ and $\rho_0$, the unperturbed pressure and density, given by Eqs.(15,17). The initial conditions for solution are that $u, \frac{du}{dr}$ are continuous with the corresponding radiation era quantities; the equation has to be solved by numerical methods and the result is that the physical information on the development of the perturbations is passed continuously from the radiation era to the pure matter era. From the evolution of $u$ by Eq.(52), the evolution throughout the transition of the gauge invariant metric scalar perturbation, $\Phi$, can be computed using Eqs.(31,51,6).

In the matter era $a \propto \eta^2$ where
\[ \eta = \tau - \tau_m, \tau_m = \text{constant}, \]  \hfill (53) 

with the pressure being equal to zero. Consequently Eqs. (5, 6) yield

\[ h = C_1 + C_m \eta^{-3}, \]  \hfill (54) 

\[ h' = C_1 k^2 \eta/5 + C_2 \eta^3/\eta^4 + C_m k^2 \eta^{-2}/2, \]  \hfill (55) 

where \( C_1, C_2, C_m \) are constants; the synchronous gauge mode, proportional to \( C_m \), being non-physical it is just \( C_1, C_2 \) that we require. The gauge invariant perturbation, \( \Phi \), is given by

\[ \Phi = h - \frac{\alpha}{k^2} h' = \frac{3}{5} C_1 - \frac{2\eta_1^3}{k^2 \eta^5} C_2, \]  \hfill (56) 

and we note that \( C_m \) does not appear. At the beginning of the pure matter era (\( \tau = \tau_1, \eta = \eta_1, \alpha = \alpha_1 \))

\[ \Phi(\tau_1) = \frac{3}{5} C_1 - \frac{1}{2} (\alpha_1/k)^2 C_2, \]  \hfill (57) 

\[ \Phi'(\tau_1) = \frac{5}{4} \alpha_1 (\alpha_1/k)^2 C_2 \]  \hfill (58) 

and the continuity of \( \Phi, \Phi' \), calculated through the radiation and transition eras, gives \( C_1, C_2 \) in terms of (i) the inflationary power \( p \) and \( \tau_2 \) (or equivalently \( a_2 \)) by Eqs. (41, 42) and of (ii) the parameters of the radiation to matter transition.

We can also investigate what the sudden transition approximation gives. Analogously to section III A we then have to enforce the continuity of \( \Phi, \Gamma \) at \( \tau = \tau_1; \Gamma \) is given by Eq. (33) where now \( \epsilon \equiv k/\alpha \sqrt{3} \) is evaluated at \( \tau = \tau_1, \epsilon = \epsilon_1 \). From this we find that if we denote \( \Gamma_1-, \Phi_1- \) to be the values at the radiation side of the interface then

\[ C_1 = \Phi_1- (1 - \frac{2}{3} \epsilon_1^2) + \Gamma_1- \]  \hfill (59) 

\[ C_2 = \frac{18}{5} \epsilon_1^2 \{ (1 - \frac{2}{3} \Phi_1- (1 + \epsilon_1^2) + \Gamma_1- \} \]  \hfill (60) 

Thus for the coefficient of the growing component we find

\[ C_1 = \{ B_+(2 \cos y - \epsilon_1 \sin y) - B_- (2 \sin y + \epsilon_1 \cos y) \}/6a \]  \hfill (61) 

where \( y \equiv k(\tau_1 - \tau_2)/\sqrt{3} \approx k\tau_1/\sqrt{3} = \epsilon_1 \). Here \( B_+ = 0 \) is, as we have found above in Eq. (10), a result good to many powers of 10 relative to \( B_- \). So we write

\[ C_1 = B_- (2 \sin y + \epsilon_1 \cos y)/6a \]  \hfill (62) 

and in the discussion section we shall compare this with the corresponding result of ref. [3].

An essential difference between this radiation era to matter era transition and the inflation to radiation era transition is that now we cannot necessarily expect, for values of \( k \) relevant to the CMBR fluctuations, that the sudden approximation is nearly exact. We investigate this quantitatively below, taking account of both the growing, \( C_1 \), and the decaying, \( C_2 \), mode.
B. Metric Tensor Perturbations

Throughout all cosmic eras in the FRW universe, gravitons of primordial origin are given by Eqs. (24, 25). Thus their development depends only on the evolution of the scale factor of the universe through the function $a''/a$. From the beginning of radiation to the present time the gravitational mode function $\mu$, or equivalently the Bogoliubov coefficients \[8,13\], evolve smoothly by Eq.(24) given any twice differentiable scale parameter $a(\tau)$. In the radiation era we have Eq.(43) for $\mu$ while in the matter era where $a''/a = 2\eta - 2\eta^2$ we express the solution in terms of spherical Bessel functions:

$$\mu = \sqrt{\frac{\pi z}{2}} [G_1 J_{\frac{3}{2}}(z) + G_2 J_{-\frac{3}{2}}(z)], \quad (63)$$

where $z = k\eta$ and $G_1, G_2$ are constants. In the radiation to matter transition $\mu$ is found by computation using the postulated smooth $a(\tau)$ (see Appendix) and thus $G_1, G_2$ are found in terms of the $G_-$ of Eq.(44) through the continuity of $\mu, \mu'$.

V. THE SACHS-WOLFE EFFECT

Both the scalar and tensor perturbations give rise to perturbations in the wavelength of the photons of the CMBR. The observation of apparent temperature fluctuations, to which the perturbations would give rise, is well established and continuing. Scalar perturbations are usually considered to be dominant and we shall first deal with these.

To be appropriate for observations nearly in the rest frame of the earth the calculation should be done in a comoving frame implying, \[9\], $T_0^i = -a^2 h^i Q_4 = 0$, and from Eq.(54) this requires $C_m = 0$ in the matter era. A synchronous coordinate system which is not comoving can be changed to comoving while remaining synchronous. So we thus complete the definition of our coordinate system, which is continuous from the beginning of the radiation era, as can be seen from the discussion at the end of IV A. In the matter era $h$ and $h_l$ are now given by Eqs.(54,55) with $C_m = 0$.

We consider reception of the CMBR at the present time, $\eta = \eta_0$, and emission at $\eta = \eta_E$ where $\eta_E \geq \eta_1$ is a time within the matter era. (For the matter era we use the more convenient conformal time of Eq.(53)). Then in the synchronous comoving coordinate system the CMBR fractional temperature variation as a function of the direction of observation specified by the unit vector $e^i$ is \[14\] in quantum mechanical form

$$\frac{\delta T}{T}(e) = \frac{1}{2} \int_0^{\omega_E} \frac{dh_l}{d\eta} e^i e^j d\omega = -\sqrt{\frac{G}{2\pi^2}} \int_0^{\omega_E} \left[ \int \frac{d^3k}{2k^2} \left\{ c_k \frac{dh_l}{d\eta} k e^{ikx} \exp(i k x) + c.c. \right\} \right] d\omega. \quad (64)$$

In the above equation, $\omega = \eta_0 - \eta$, the integration is along the light path $x^i = e^i \omega$ and the quantum mechanical annihilation and creation operators, $c_k$ and $c^\dagger_k$, are the same as those in Eq.(14). Only $h_l$ appears in the mode function in Eq.(14) because $dh/d\eta = 0$. 
The angular correlation function for different directions $\mathbf{e}_1$ and $\mathbf{e}_2$ is

$$K = \langle 0 \big| \frac{\delta T}{T}(\mathbf{e}_1) \frac{\delta T}{T}(\mathbf{e}_2) \big| 0 \rangle,$$  \hspace{1cm} (65)

where $|0\rangle$ is the vacuum state, appropriate to the operators $c_k$, of the universe at the beginning of inflation. We are working in the Heisenberg picture where the cosmic state vector is constant and the operators vary; in Eq.(64) the variation of the quantum operators is subsumed in the variation of the mode functions.

Grishchuk [1] has shown that, with $\cos \delta = \mathbf{e}_1 \cdot \mathbf{e}_2$,

$$K = K(\cos \delta) = \sum_{l=0}^{\infty} K_l P_l(\cos \delta),$$  \hspace{1cm} (66)

$$K_l = (2l+1)l!^2 \int_0^{\infty} \frac{dk}{2k} \int_0^{k \omega_E} \frac{dh}{dn} (\eta_R - \frac{x}{k}) L_l \bigg| ^2,$$  \hspace{1cm} (67)

$$L_l = \{(\frac{l(l-1)}{x^2} - 1)x^{-\frac{3}{2}} J_{l+1/2}(x) + 2x^{-\frac{3}{2}} J_{l+3/2}(x)\}.$$  \hspace{1cm} (68)

We can express this with more information given as follows: We have shown that the parameters $C_1, C_2$ of $h_l$ are linear functions of $B_+, B_-$, say

$$C_1 = C_{1-}B_- + C_{1+}B_+; \quad C_2 = C_{-2}B_- + C_{2+}B_+.$$  \hspace{1cm} (69)

That is, for example, $C_{1-}$ is the value of $C_1$ when calculated for $B_- = 1, B_+ = 0$. We know from Eqs.(60),(61) the value of $B_-, B_+$ being nearly zero, so that

$$K_l = (2l+1)l!^2 \int_0^{\infty} \frac{dk}{2k} \int_0^{k \omega_E} \frac{dh}{dn} (\eta_R - \frac{x}{k}) L_l \bigg| ^2 \times \int_0^{x_E} dx \left[ C_{1-}z^2 + C_{2-}(\frac{z}{z})^3 \right] \left[ 1 - L_l(x) \right]^2,$$  \hspace{1cm} (70)

where $z = k\eta, x = k\eta_0 - k\eta = k\omega$.

A similar procedure can be carried through for the gravitons, having the mode function of Eq.(63) in the matter era with $G_1 = D_{1-G_-} + D_{1+}G_+; G_2 = D_{2-G_-} + D_{2+}G_+$. Knowing the value of $G_-$, and that $G_+$ is approximately zero, from Eq.(64) we find [8]

$$K_l = (2l+1)l(l+1)l!^2 [l(l+1) - 2](\frac{l!}{\tau_2})^2 (\frac{\tau_2}{\tau_1})^{2p} M^2(p) \int_0^{\infty} \frac{dk}{2k} (-k\tau_1)^{2p} \times \int_0^{x_E} dx [D_{1-J_{5/2}(z)} - D_{2-J_{5/2}(z)}] \left[ \sqrt{\frac{\pi z}{2 a(\eta)}} x^{-\frac{3}{2}} J_{l+1/2}(x) \right]^2.$$  \hspace{1cm} (71)
VI. RESULTS

To produce numerical results we have to specify parameters giving (i) the length of the transition (ii) the end of the transition when the matter era begins (iii) the background radiation emission time for the Sachs-Wolfe calculation. We can conveniently specify using the scale factor, \( a \), or the red shift, \( z \), where \( z + 1 = a_0 / a \); our convention is that \( a_0 \equiv a(\text{present}) = 1 \).

For (iii) it is rather natural to take the time of last scattering to be at the beginning of the matter era and this we shall do. We can make a remark from the literature supporting a not too great sensitivity to this assumption. If the emission time were taken to be within the radiation to matter transition then the correction from including the integrated Sachs-Wolfe effect is expected to be about 4\% in the quadrupole moment from scalar perturbations and less for other moments [16].

For the beginning of the matter era (zero pressure) we consider two values, mainly \( z_1 = 10^3 \) but also with some illustrations from \( z_1 = 10^{3.3} \) to assess the sensitivity of the results to the choice of \( z_1 \). As stated above \( z_1 \) is also taken to be the emission red-shift in the Sachs-Wolfe calculation:

\[
    z_E = z_1
\]

The parameter for the length of the transition, from \( a = a_e \) to \( a_1 \), is given by \( r_{\text{trans}} \) of Eq.(50) so that \( r_{\text{trans}} \equiv \frac{1}{2} \ln(a_1 / a_e) = \frac{1}{2} \ln(z_e / z_1) \). We shall present some results for a range of values of \( r_{\text{trans}} \) including \( r_{\text{trans}} = 0 \), that being the sudden transition approximation which has often been used. We consider that a likely physical value is round about 5 e-foldings:

\[
    r_{\text{trans}} = 2.5
\]

This can be very roughly estimated from a likely range for \( z_{EQ} \), which can then be used to make an estimate from the known [3] analytic two fluid model for the transition.

Our results of course also depend on \( p \), the power of inflation, Eq.(11). We present results for a range of values of \( p \), and firstly for the ratio of the gravitational to the density perturbation contributions to the multipoles. We emphasize that the ratio is calculated with the same physical model for both contributions and with no significant approximations. For the ratio no other parameters enter other than the ones just stated. But in the absolute magnitude of each contribution, Eqs.(70,71), there is a common factor

\[
    F_2 = \left( \frac{l_p !}{\tau_2} \right)^2 \left( \frac{\tau_2}{\eta_1} \right)^{2p}
\]

containing the parameter \( \tau_2 \), the time when the radiation era begins. The choice of \( \tau_2 \) finally specifies the absolute magnitude of the multipoles. Within the model \( p, \tau_2 \) are the important and significant unknown parameters; \( r_{\text{trans}} \) and \( z_1 \) also have importance, as will be shown, but more is known about the possible values of these.

As outlined in [11] our calculation of the radiation to matter transition is good for larger angular scale. There is an approximate relation between the correlation angle \( \theta \) and the order of the multipole, \( l \), which would give the main contribution to such a correlation [13]:

\[ 13 \]
\[
\frac{\theta}{1^\circ} \approx \frac{60}{l}.
\]  

(75)

We give results for multipoles with \( l \leq 8 \) corresponding by Eq.(75) to angular scales \( \theta \geq 8^\circ \). First we present the ratios, \((T/S)_l\), of the gravitational wave to the density wave Legendre series coefficients, \( K_l \), of Eqs.(70,71); these ratios are independent of \( \tau_2 \). In Table 1 we show these for 4 different values of \( p \) with \( r_{\text{trans}} = 2.5 \) and \( z_E = z_1 = 10^3 \).

From Table 2 we see, that for any particular multipole, the variation of \( T/S \) with \( p \) is approximately \((p+1)/p\). This is not quite unexpected as the ratio of gravitational to density wave amplitudes in the radiation era is given by Eqs.(41,44) as \( \sqrt{(p+1)/12p} \). However Table 2 shows that the ratio in observable multipoles, \((T/S)_l\), is bigger than \((p+1)/12p\) by a factor of order 100.

Production of an adequate amount of inflation requires the cosmic time power \( q \geq 10 \). From Table 1 (and taking account of the proportionality to \((p+1)/p = 1/q\)) we see that for \( q < 100 \) gravitational waves make a non-negligible contribution and that for \( q \) of order 10 the contribution is similar to that of the density perturbations. Gravitational waves are proportionally most important in the quadrupole and octopole moments and Table 3 shows that these are the largest multipoles.

Table 3 exhibits the absolute magnitude of the density component of the multipoles, from which the graviton induced component can be found using Table 1. Since \( \tau_1 \) is approximately known the most important adjustable parameter of the model here is \( \tau_2 \), the conformal time at the beginning of the radiation era \( \approx \) end of inflation. In the limit as \( p \to -1 \), \( F_2 \to \frac{L\tau_1}{\tau_2^2} \). We can find an order of magnitude of \( \tau_2 \), or equivalently the more convention-free redshift at the beginning of the radiation era, \( z_2 + 1 = a_0/a_2 \), by taking the estimate from observation of the quadrupole moment [11,17]. If we insert \( Q_{\text{rms}} \approx 18 \mu K \) then with \( p = -1.112, z_1 = 10^3 \) we find, where \( h \) is the Hubble parameter \( z_2 \approx 2.2 \times 10^{27} h^{-47} \),

(76)

and inserting \( 6 \mu K \) gives 1.3 instead of 2.2 in the above.

Table 4 shows the absolute values in units of \( 10^4 F_2 \) of the scalar and tensor quadrupoles for a range of values of the transition length \( r_{\text{trans}} \) and \( p = -1.112, q = 10 \). The values for \( r_{\text{trans}} = 0 \) were obtained using the same sudden transition methods as those for inflation to radiation in [1]. For \( r_{\text{trans}} > 0.4 \) we use the smooth model transition of the Appendix. (For \( 0 < r_{\text{trans}} < 0.4 \) the results are model dependent.) We note the approximate constancy of these results when the number of e-folds (= 2\( r_{\text{trans}} \)) is between 1 and 6 but that the sudden transition results are somewhat different, \( T/S \) then being about 3/2 larger. If we were to let the range of values of \( k/\alpha \) to be even smaller than those relevant for the CMBR observations we would expect these numbers to approach nearer to equality. Using the comoving gauge Lyth [18] showed rather generally that, for small enough values of \( k/\alpha \), the development of the density perturbation from one well-defined cosmic era to another is independent of the type of transition provided it be reasonably well behaved.

\( \text{2} \)The amplitude \( Q \) is related to the correlation multipole by \( Q = T_0 \sqrt{K_2} \) [1].

14
All the results discussed above were for values of the relevant red-shifts given by $z_E = z_1 = 10^3$. If we increase this value to $10^{3.3}$, the scalar and tensor contributions to the multipole moments increase by a factor of around 4; an increase is expected as the Sachs-Wolfe calculation is applied to a longer photon path. The ratio $(T/S)_2$, with $p = -1.122$, changes from 1.3 to 1.2 with similar changes for the other multipoles.

VII. DISCUSSION

In this work we have used the same power law inflationary model to evaluate and compare the density perturbation and gravitational wave contributions to the CMBR fluctuations; the transition from radiation domination to matter domination has been appropriately treated as a gradual rather than a sudden transition, but avoiding particle-spectrum dependent details. This latter limits the precise validity of our calculation to multipoles influencing correlation angles greater than about 5 degrees. For power law inflation, $t^q$, the larger part of the quadrupole moment comes from gravitational waves if $q < 13.5$; the corresponding equality value for other multipoles is $\approx 10$. At $q = 20$ gravitational waves contribute about 1/3 to correlation Legendre coefficients of $l \leq 8$ and 10% at $q \approx 100$.

We followed the paper of Grishchuk [6] by evolving using the metric perturbations, and also by making computations in the synchronous gauge, but our usage of this gauge, and our tracking of the perturbations from one cosmic era to the next, quite differ from his. In particular we made no significant use of non-physical synchronous gauge modes; we used gauge invariant variables in critical calculations of transitions and also, where a transition was treated as sudden, we used the Lichnerowicz matching conditions [9] as interpreted by Deruelle and Mukhanov [2].

The message delivered by our work on the relative importance of gravitational waves appears to differ from that of Grishchuk [6]. Since the question can be raised as to where this difference comes from it may be of interest to identify the detailed reasons: (i) Firstly in ref. [6] a special form for a continuous transition from inflation to radiation is used. In principle, with an appropriate choice of form, there should be nothing wrong with such a treatment; there is no interface and all quantities are continuous, this being indeed the principle we have used in the radiation era to matter era transition. Trouble seems to arise in ref. [6] when the transition is taken to the zero transition time limit. This is particularly evident in the final treatment of $\gamma \equiv 1 - \alpha'/\alpha^2$ which increases from the small value $(p+1)/p$ (in our notation) to 2 during the transition. The nature of the final expressions obtained in ref. [6] for the equivalent of $B_+$ and $B_-$ and other quantities make it necessary, for a reasonable result, to rather arbitrarily put $\gamma = 2$ in these expressions [7]. The result we find is, on evaluation of Grishchuk’s expression for the dominating constant which we call $B_-$, that it is smaller by a factor $\sqrt{(p+1)/p}$ than our expression given by Eqs.(38),(40), (41). This makes a factor of $(p+1)/p$ smaller for the density perturbation contribution to the final result. (ii) Secondly there is the question of the evaluation of the constants $C_1$ and $C_2$ of

---

3This non-rigorous treatment of $\gamma$ has also been criticised by Deruelle and Mukhanov [2].
the matter era, which involves the transition from the radiation era. In expression eq.(82) of ref. [6] for $C_1$, this expression has been treated as sudden (analytic expressions such as those of ref. [6] cannot otherwise be obtained); there are also other approximations equivalent to taking only the first term in eq.(62). Eq.(82) [6] is greater by a factor $3/2$ than our first term in addition to having the smaller value of $B_-$ just related above. In our opinion this factor $3/2$ arises from the non-trivial usage of the radiation gauge mode which is determined in ref. [6] by an extra continuity condition not arising from Lichnerowicz matching. Thus we believe there are two parts of the density perturbation calculation of ref. [6] which are incorrect.

A number of papers [19] by various authors found results on the relative importance of density and gravity wave perturbations from inflation in the CMBR fluctuations. These used particular properties of the various models in fairly complicated careful deductive processes. (For a recent account of a large range of density perturbation calculations, also referring to gravity wave contributions to the CMBR, and with complete references, see ref. [20].) By contrast we have used a direct calculation following in a unitary way both density and gravitational wave contributions from birth to observation, in a certain type of inflationary model. And our conclusions are not in great disagreement with the general concensus, for power law inflation, of those previous papers.

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APPENDIX:

We adopt the following form for the function $s(r)$, $r = \ln(a/a_1)$, of Section IV A where

$$s(r_e) = 4, s(r_1) = 3,$$

and $s(r)$ is specified between $r = r_e$ where the radiation era ends and $r = r_1$ where the matter era begins. In the radiation and matter eras $\rho_0 \propto \exp(-4r)$ and $\exp(-3r)$ respectively so the join with the transition era can be made arbitrarily smooth by specifying that

$$\frac{d^m s}{d r^m} = 0; r = r_e, r_1,$$

(A2)

up to the necessary $m$. We take $m = 3$ and implement $s(r)$ by the polynomial form

$$s(r) = s_0 + c(r - r_0)[1 - x^2 + \frac{3}{5}x^4 - \frac{1}{7}x^6],$$

$$r_0 = (r_e + r_1)/2, s_0 = (s_e + s_1)/2 = 7/2,$$

$$c = 35(s_1 - s_e)/16(r_1 - r_e),$$

$$x = (r - r_0)/(r_1 - r_0).$$

(A3)

The degree of smoothness specified by A2 with $m = 3$, ensures the smooth joining at $r = r_e$ and $r_1$ of the coefficients.
REFERENCES

[1] L. Parker, Phys. Rev. 183, 1057 (1969); Phys. Rev. D3, 346 (1971); Ya. B. Zeldovitch and A. A. Starobinski, Sov. Phys. JETP 34, 1159 (1972); L. P. Grishchuk, Lett. Nuovo Cimento 12, 60 (1975), Zh. Eksp. Teor. Fiz. 67, 825 (1974) [Sov. Phys. JETP 40, 409 (1975)].

[2] N. Deruelle and V. F. Mukhanov, Phys. Rev. D52, 5549 (1995).

[3] R. R. Caldwell, Class. Quant. Grav. 13, 2437 (1995).

[4] I. J. Grivell and A. R. Liddle, "Accurate determination of inflationary perturbations" Preprint No. SUSSEX-AST 96/7-4, astro-ph/9607096 (1996), (to be published in Phys.Rev.D). D. H. Lyth and E. D. Stewart, Phys. Lett. B274, 168 (1992); ibid. B302, 171 (1993).

[5] V. F. Mukhanov, H. A. Feldman and R. H. Brandenburger, Phys. Rep. 215, 203 (1992).

[6] L. P. Grishchuk, Phys. Rev. D50, 7154 (1994).

[7] M. White, Phys. Rev. D46, 4198 (1992).

[8] L. P. Grishchuk, Phys. Rev. D48, 3513 (1993).

[9] A. Lichnerowicz, "Theories relativistes de la gravitation et de l’electromagnetism" Masson (Paris) 1955.

[10] L. F. Abbott and D. Harari, Nucl. Phys. B264, 487 (1986).

[11] M. White, D. Scott and J. Silk, Ann. Rev. Astron. Astrophys., 32, 319 (1994).

[12] C-P. Ma and E. Bertschinger, "Cosmological perturbation theory in the synchronous and conformal Newtonian gauges" astro-ph/9506072 (1995).

[13] R. G. Moorhouse, A. B. Henriques and L. E. Mendes, Phys. Rev. D50, 2600 (1994); L. E. Mendes, A. B. Henriques and R. G. Moorhouse, Phys. Rev. D52, 2083 (1995).

[14] R. Sachs and A. Wolfe, Astrophys. J. 147, 73 (1967).

[15] A. R. Liddle and D. H. Lyth, Phys. Rep. 231, 1, (1993).

[16] M. S. Turner and M. White, Phys. Rev. D53, 6822 (1996).

[17] A. J. Banday, K. M. Gorski, C. L. Bennett, A. Kogut, C. Lineweaver, G. F. Smoot and L. Tenorio, “RMS anisotropy in the COBE-DMR four-year sky maps”, astro-ph/9601063.

[18] D. H. Lyth, Phys. Rev. D31, 1792 (1983).

[19] R. L. Davis, H. M. Hodges, G. F. Smoot, P. J. Steinhardt and M. S. Turner, Phys. Rev. Lett. 69, 1856 (1992); D. S. Salopek, Phys. Rev. Lett. 69, 3602 (1992); A. R. Liddle and D. H. Lyth, Phys. Lett. B291, 391 (1992); J. E. Lidsey and P. Coles, Mon. Not. R. Astron. Soc. 258, 57p (1992); F. Lucchin, S. Matterrese and S. Mollerach, Astrophys. J. Lett. 401, 49 (1992); T. Souradeep and V. Sahni, Mod. Phys. Lett. A7, 3541 (1992).

[20] D. H. Lyth, "Models of inflation and the spectral index of the density perturbation" hep-ph/9609431.
TABLES

TABLE I. The ratio \((T/S)_l\) of tensor to scalar multipoles of the correlation function, \(K\), for inflation \(\propto t^q\), where \(t\) is cosmic time; \(q = p/(p + 1)\).

| \(p + 1\) | -0.05 | -0.08 | -0.112 | -0.20 |
|-----------|-------|-------|--------|-------|
| \(q\)     | 21.0  | 13.5  | 9.9    | 6.0   |
| \((T/S)_2\) | 0.64  | 0.99  | 1.33   | 2.14  |
| \((T/S)_3\) | 0.52  | 0.81  | 1.09   | 1.77  |
| \((T/S)_4\) | 0.41  | 0.64  | 0.88   | 1.45  |
| \((T/S)_5\) | 0.46  | 0.71  | 0.97   | 1.59  |
| \((T/S)_6\) | 0.44  | 0.69  | 0.95   | 1.58  |
| \((T/S)_7\) | 0.39  | 0.61  | 0.84   | 1.41  |
| \((T/S)_8\) | 0.43  | 0.68  | 0.93   | 1.57  |

TABLE II. \(\alpha_l \equiv (T/S)_l \frac{p}{p+1}\)

| \(p + 1\) | -0.05 | -0.08 | -0.112 | -0.20 |
|-----------|-------|-------|--------|-------|
| \(q\)     | 21.0  | 13.5  | 9.9    | 6.0   |
| \(\alpha_2\) | 13.4  | 13.4  | 13.2   | 12.8  |
| \(\alpha_3\) | 10.9  | 10.9  | 10.8   | 10.6  |
| \(\alpha_4\) | 8.6   | 8.6   | 8.7    | 8.7   |
| \(\alpha_5\) | 9.7   | 9.6   | 9.6    | 9.5   |
| \(\alpha_6\) | 9.2   | 9.3   | 9.4    | 9.5   |
| \(\alpha_7\) | 8.2   | 8.2   | 8.3    | 8.5   |
| \(\alpha_8\) | 9.0   | 9.2   | 9.2    | 9.4   |

TABLE III. Multipoles \(S_l = K_l\) of density perturbations in units \(10^4 F_2\)

| \(p + 1\) | -0.05 | -0.08 | -0.112 | -0.20 |
|-----------|-------|-------|--------|-------|
| \(q\)     | 21.0  | 13.5  | 9.9    | 6.0   |
| \(S_2\)   | 3.41  | 2.48  | 2.08   | 1.82  |
| \(S_3\)   | 2.29  | 1.62  | 1.32   | 1.07  |
| \(S_4\)   | 2.01  | 1.40  | 1.12   | 0.86  |
| \(S_5\)   | 1.43  | 0.98  | 0.78   | 0.58  |
| \(S_6\)   | 1.24  | 0.84  | 0.65   | 0.47  |
| \(S_7\)   | 1.20  | 0.80  | 0.62   | 0.43  |
| \(S_8\)   | 0.95  | 0.63  | 0.48   | 0.33  |
TABLE IV. Scalar and tensor quadrupoles (units $10^4 F_2$), of the correlation function $K$, and their ratio as functions of the radiation to matter transition length

| $r_{\text{trans}}$ | $S_2$ | $T_2$ | $(T/S)_2$ |
|---------------------|-------|-------|-----------|
| 0                   | 1.72  | 3.44  | 2.00      |
| 0.5                 | 2.08  | 2.78  | 1.34      |
| 2.0                 | 2.12  | 2.81  | 1.33      |
| 2.5                 | 2.08  | 2.77  | 1.33      |
| 3.0                 | 2.09  | 2.73  | 1.31      |