CYCLE LENGTHS IN SPARSE RANDOM GRAPHS

YAHAV ALON, MICHAEL KRIVELEVICH, AND EYAL LUBETZKY

Abstract. We study the set $\mathcal{L}(G)$ of lengths of all cycles that appear in a random $d$-regular $G$ on $n$ vertices for a fixed $d \geq 3$, as well as in Erdős–Rényi random graphs on $n$ vertices with a fixed average degree $c > 1$. Fundamental results on the distribution of cycle counts in these models were established in the 1980’s and early 1990’s, with a focus on the extreme lengths: cycles of fixed length, and cycles of length linear in $n$. Here we derive, for a random $d$-regular graph, the limiting probability that $\mathcal{L}(G)$ simultaneously contains the entire range $\{\ell, \ldots, n\}$ for $\ell \geq 3$, as an explicit expression $\theta_\ell = \theta_\ell(d) \in (0,1)$ which goes to 1 as $\ell \to \infty$.

1. Introduction

We study the set $\mathcal{L}(G)$ of cycle lengths appearing in a random graph $G$ on $n$ vertices with constant average degree under the classical random graph distributions: the random regular graph $\mathcal{G}(n,d)$ (the uniform distribution over $d$-regular simple graphs on $n$ vertices) and the (Erdős–Rényi) binomial random graph $\mathcal{G}(n,p)$ (each undirected edge $ij$ for $1 \leq i < j \leq n$ appears with probability $p$, independently of the other edges).

Much is known about the distribution of cycles in these random graph models (see §1.1 for a brief account), including (a) the convergence of the joint law of the variables $\{Z_k\}_{k \geq 3}$, counting the number of $k$-cycles in $G$, to the joint law of independent Poisson random variables with explicit means; and (b) typical existence of cycles of linear length in $G \sim \mathcal{G}(n,p)$ when $p = c/n$ for any $c > 1$, as well as Hamilton cycles in $G \sim \mathcal{G}(n,d)$.

Our results here demonstrate that the existence of cycle whose lengths are at the extreme ends of this spectrum dominate the behavior of the set $\mathcal{L}(G)$ of all cycle lengths appearing in these random graphs: We find the probability that $\mathcal{L}(G)$ contains the entire range from a given fixed $\ell$ all the way to $n$ in $\mathcal{G}(n,d)$, or to $(1 - \varepsilon)L$, where $L$ is the length of a longest cycle in $\mathcal{G}(n,p)$, converges to a limit $0 < \theta(c,\ell) < 1$ as $n \to \infty$.

Define the quantity $0 < \theta(c,\ell) < 1$ to be

$$\theta(c,\ell) := \prod_{k=\ell}^{\infty} (1 - e^{-c^{k}/(2k)}) \quad \text{for } c > 1 \text{ and } \ell \geq 3.$$

(We use $[a,b]$ to denote $\{k \in \mathbb{Z} : a \leq k \leq b\}$; an event $E_n$ holds with high probability (w.h.p.) if $\mathbb{P}(E_n) \to 1$.)

Theorem 1. For every fixed $d \geq 3$, the random regular graph $G \sim \mathcal{G}(n,d)$ satisfies that for every fixed $\ell \geq 3$,

$$\lim_{n \to \infty} \mathbb{P}(\{\ell, \ldots, n\} \subset \mathcal{L}(G)) = \theta(d - 1, \ell).$$

In particular, $G$ contains all cycle lengths between $\ell$ and $n$ with probability at least $1 - 2e^{-(d-1)^{\ell}/(2\ell)}$.

Taking $\ell \to \infty$ in the above theorem shows $[\omega_n, n] \subset \mathcal{L}(G)$ w.h.p. for every $\omega_n$ with $\lim_{n \to \infty} \omega_n = \infty$.

Theorem 2. There exists some $C_0 > 0$ so that, if $G \sim \mathcal{G}(n,p)$ where $p = \frac{c}{n}$ for almost every $c > C_0$ fixed, then for every fixed $\ell \geq 3$ and any fixed $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(\{\ell, (1 - \varepsilon)L_{\max}(G)\} \subset \mathcal{L}(G)) = \theta(c,\ell).$$

In addition, if $G \sim \mathcal{D}(n,p)$ where $p = \frac{c}{n}$ with $c > C_0$ fixed, then the analog of (1.3) holds true with respect to the modified quantity $\theta'(c,\ell) := \prod_{k=\ell}^{\infty} (1 - \exp(-c^{k}/k)).$

Taking $\ell \to \infty$ here yields $[\omega_n, (1 - \varepsilon)L_{\max}(G)] \subset \mathcal{L}(G)$ w.h.p. for every $\omega_n$ with $\lim_{n \to \infty} \omega_n = \infty$. 


Remark 1.1. Our results on $G(n,p)$, $D(n,p)$, though presented in Theorem 2 for $p = \frac{\lambda}{n}$ with a.e. $c > C_0$, address the entire supercritical regime $c > 1$. In lieu of the range $[\ell, (1 - o(1))L_{\max}(G)]$ in that theorem, one puts $[\ell, (1 - o(1))L_{\max}(G')]$ for an analogous random graph $G'$ with edge probability $p' = (1 - o(1))p$ (see Corollary 3.4). In particular, whenever $L_{\max}(G)/n$ converges in probability to a left continuous limit $f(c)$ (known to hold for a.e. $c > C_0$, see §1.1), one can further replace $L_{\max}(G')$ by $(1 - o(1))L_{\max}(G)$, as above.

Remark 1.2. Considering $G(n,p)$, $D(n,p)$ for $p = c/n$ with $c = 1 + \varepsilon$ for a sufficiently small $\varepsilon > 0$, and with the previous remark in mind, we can deduce that $G \sim G(n,p)$ has $P(\{f, (\frac{4}{3} - o(1))\varepsilon^2 n \in \mathcal{L}(G)\}) \to \theta(c, \ell)$ for every fixed $\ell \geq 3$, and the same holds for $G \sim D(n,p)$ with the limiting constant $\theta'(c, \ell)$. Furthermore, one can replace $\frac{4}{3}$ by any constant $\gamma_0$ that is known to lower bound $L_{\max}(G)/(\varepsilon^2 n)$ in probability; see Theorem 3.7.

1.1. Related work. Following is an account, by no means exhaustive, of related results on the distribution of cycles in random graphs. For more information, the reader is referred to [5, 15] and the references therein.

The random variables counting the number short cycles in $G(n,d)$ and $G(n,p)$ are well-known to be asymptotically independent Poisson. This was first established in $G \sim G(n,d)$ for $d \geq 3$ fixed by Bollobás [3] and by Wormald [26], where it was shown that for every integer $K$, the joint law of the random variables counting the number of $k$-cycles in a random graph $G$ converges weakly to the law of independent Poisson random variables $(Z_k)_{k=3}^{K}$ with respective means $\lambda_k = (d-1)^k/(2k)$. The analogous statement for $G(n,p)$ when $p = c/n$ for fixed $c > 0$ was shown by Bollobás in 1981 (see [5, §4.1]) and independently by Karoński and Ruciński [16] with respect to (w.r.t.) $\lambda_k = c^k/(2k)$. One can immediately recognize the limiting probabilities $\theta(d-1, \ell)$ in (1.2) and $\theta(c, \ell)$ in (1.3) as the probability that $\bigcap_{k=3}^{K} Z_k \neq 0$ in the respective limit as $K \to \infty$.

At the other extreme, longest cycles in $G(n,d)$ and $G(n,p)$ were the subject of intensive study. In $G(n,p)$ when $p = c/n$ for fixed $c > 1$, works by Bollobás [4] and by Bollobás, Fenner and Frieze [6] culminated in the result of Frieze [13] that w.h.p. there exists a cycle in $G$ going through all but $(1 + \varepsilon_c)\varepsilon n$ vertices, where $\varepsilon_c \to 0$ as $c \to \infty$ (note that w.h.p. $G$ has $(1 - o(1))\varepsilon c$ vertices of degree 1, thus this is sharp up to $\varepsilon n$). A directed analog of this result in $D(n,p)$ was derived by the last two authors and Sudakov in [18].

For $G(n,d)$, the longstanding conjecture that the graph is Hamiltonian w.h.p. for every fixed $d \geq 3$ was finally settled in the seminal works of Robinson and Wormald [23, 25], which introduced the small subgraph conditioning (ssc) method. These were followed by the paper of Janson [14], demonstrating how the ssc method allows one both to recover the distribution of the number of Hamilton cycles, and, remarkably, to show contiguity of models of random regular graphs. Our analysis of $G(n,d)$ will rely on these results.

Letting $L_{\max}(G)$ denote the length of a longest cycle in $G$ (its circumference), $L_{\max}(G)/n$ is expected to converge in probability when $p = c/n$ for every fixed $c > 1$, yet till recently this was not known for any $c > 1$. Anastos and Frieze [1] then proved that this holds when $c > C_0$ for some absolute constant $C_0$, and further identified the limit $f(c)$. The analogous result for $D(n,p)$ was thereafter obtained by the same authors in [2]. For $c = 1 + o(1)$ outside the critical window, $L_{\max}(G)$ is known up to constant factors [20]; see Remark 3.6.

For $G \sim G(n,d)$ in the denser regime, Cooper and Frieze [9] proved that if $np - \log n - \log \log n \to \infty$ then w.h.p. $\mathcal{L}(G) = [3,n]$, a property referred to as pancyclicity. Łuczak [21] obtained that if $np \to \infty$, then for every fixed $\varepsilon > 0$, the graph $G$ contains all cycle lengths up to $n - (1 + \varepsilon)\min(N_1, n)$ w.h.p., where $N_1$ is the number of vertices of degree 1 in $G$. Cooper [7, 8] later proved that if $np - \log n - \log \log n \to \infty$ then w.h.p. $G \sim G(n,p)$ contains a Hamilton cycle $H$ such that, for every $3 \leq \ell \leq n - 1$, one can construct a cycle of length $\ell$ in $G$ that contains only edges of $H$ and at most one additional edge.

Recently, Friedman and Krivelevich [12] studied $\mathcal{L}(G)$ for certain classes of expander graphs $G$ on $n$ vertices, showing that $\mathcal{L}(G)$ then contains an interval of $\bar{n}$ cycle lengths, for a constant $\delta > 0$ that depends on the expansion parameters. Combined with well-known results on expansion in random graphs, this implies that for every $\delta$ there exists $c_0$ such that for $c > c_0$, $G \sim G(n,c/n)$, $G \sim G(n,d)$ w.h.p. have that the set $\mathcal{L}(G)$ of cycle lengths contains an interval of length $(1 - \delta)n$ (see [12] for further details).

1.2. Proof techniques. For Theorem 1 when $n$ is even, the aforementioned contiguity results reduce the model to the union of a Hamilton cycle and an independent uniform perfect matching. Theorem 2.1 shows that such a random graph contains a cycle of length $O(n)$ with probability $1 - O(\exp(-c\min(\ell, n - \ell)))$, for an absolute constant $c > 0$. (A similar result holds for a union of two independent uniform Hamilton cycles, pertinent to the case of odd $n$). The proof relies on a switching argument akin to the approach of [21]; here, the matching edges play two roles: (I) a fraction of them, together with the Hamilton cycle, creates a large set of $(\ell - 1)$-paths; (II) another fraction of those is then used to close an $\ell$-cycle. Theorem 2 is proved by an analogous analysis for a union of a long cycle and $G(n, \alpha/n)$ (Theorem 3.1) or $D(n, \alpha/n)$ (Theorem 3.2).
2. RANDOM REGULAR GRAPHS

Our proof will be derived from the following ingredients via contiguity properties of random regular graphs. The first (and main) ingredient will treat cycles in \( \mathcal{G}(n,3) \) whose lengths are in the range \([\omega_n, n-\omega_n]\) for any \( \omega_n \gg 1 \), by means of studying the contiguous model \( \mathcal{H}(n) + \mathcal{G}(n,1) \), the 3-regular multigraph on \( n \) vertices obtained from the union of a Hamilton cycle and an independently and uniformly chosen perfect matching.

**Theorem 2.1.** Let \( G \sim \mathcal{H}(n) + \mathcal{G}(n,1) \) be the random cubic \( n \)-vertex multigraph (\( n \) even) which is the union of a Hamilton cycle and an independently chosen uniform perfect matching. There are absolute constants \( C, c > 0 \) so that, for any \( 4 \leq \ell \leq n/2 \), we have \( [\ell, n-\ell+4] \subset \mathcal{L}(G) \) with probability at least \( 1 - C \exp(-c \ell) \). The same holds for \( n \) odd when \( G \sim \mathcal{H}(n) + \mathcal{H}(n) \), a union of two independent uniform Hamilton cycle.

While the above theorem shows that \([4,n]\) is distributed as \( \mathcal{L}(G) \) with a probability that is uniformly bounded away from 0, its estimate on this probability is not sharp. To obtain the correct limiting probability for this event (and more generally, for the event \([\ell,n]\) for any fixed \( \ell \)), we must treat large cycles more carefully. Namely, the range \([n-\omega_n,n]\) is treated by the next theorem, proved via a reduction to a result of Robinson and Wormald [24] on Hamilton cycles avoiding a set of random edges while including another such set.

**Theorem 2.2.** Let \( G \sim \mathcal{G}(n,d) \) for \( d \geq 3 \) fixed. There exists some sequence \( \omega_n \) going to infinity with \( n \) (sufficiently slowly) such that \([n-\omega_n,n]\) is distributed as \( \mathcal{L}(G) \) w.h.p.

The above two theorems will be proved in Sections 2.1 and 2.2, resp.

**Proof of Theorem 1.** Let \( \tilde{\mathcal{G}}(n,d) \) denote the distribution over \( d \)-regular multigraphs obtained via the configuration model. It is well-known (see for instance [15, Thm. 9.30]) that \( \mathcal{H}(n) + \mathcal{G}(n,1) \) is contiguous to \( \tilde{\mathcal{G}}(n,3) \), the conditional distribution of \( \tilde{\mathcal{G}}(n,3) \) given there are no loops. With probability bounded away from 0, a graph distributed as \( \tilde{\mathcal{G}}(n,3) \) has no multiple edges (namely, with probability \( 1/e + o(1) \); see, e.g., [15, Thm. 9.5]), and on that event it is distributed as \( \tilde{\mathcal{G}}(n,3) \). Similarly, when \( n \) is odd, it is known that \( \mathcal{H}(n) + \mathcal{H}(n) \) is contiguous to \( \tilde{\mathcal{G}}(n,4) \) (see, e.g., [15, Thm. 9.41]), which is distributed as \( \mathcal{G}(n,4) \) conditional on having no multiple edges, an event whose probability is bounded away from 0 (namely, it is \( e^{-9/4} + o(1) \)).

Therefore, applying Theorem 2.1 for \( \ell \) going to infinity arbitrarily slowly, and using the monotonicity of \( \mathcal{G}(n,d) \) in \( d \) w.r.t. increasing properties that hold with probability \( 1 - o(1) \) (another consequence of contiguity of random regular graphs; see [15, Thm. 9.36(ii)]) we arrive at the conclusion that \( G \sim \mathcal{G}(n,d) \) has

\[
\Pr([\omega_n, n-\omega_n] \subset \mathcal{L}(G)) \to 1 \quad \text{for every sequence } \omega_n \text{ such that } \lim_{n \to \infty} \omega_n = \infty. \tag{2.1}
\]

The treatment of the regime of \([\ell,n]\) will follow immediately from the convergence of the short cycle distribution of \( \mathcal{G}(n,d) \) to asymptotically independent Poisson random variables. If \( Z_{n,k} \) counts the number of \( k \)-cycles in \( G \sim \mathcal{G}(n,d) \), it is well known that for every fixed \( k \geq 3 \), one has \( Z_{n,k} \xrightarrow{d} \text{Po}(\lambda_k) \) as \( n \to \infty \), where \( \lambda_k = (d-1)^k/(2k) \), and moreover, the joint law \( \{Z_{n,k}\}_{k \geq 3} \) weakly converges as \( n \to \infty \) to the joint law of independent Poisson random variables \( \{Z_{\infty,k}\}_{k \geq 3} \) where \( EZ_{\infty,k} = \lambda_k \) (see, for instance, [15, Cor. 9.6]).

With this in mind, fix \( \varepsilon > 0 \) and let \( \omega_n' \) be the maximal integer \( K \geq \ell \) satisfying

\[
|\Pr(Z_{N,\ell} > 0, \ldots, Z_{N,K} > 0) - \Pr(Z_{\infty,\ell} > 0, \ldots, Z_{\infty,K} > 0)| < \varepsilon \quad \text{for all } N \geq n. \tag{2.2}
\]

By the preceding discussion, \( \omega_n' \to \infty \) with \( n \), and so

\[
\lim_{n \to \infty} \Pr\left(\bigcap_{k=\ell}^{\omega_n'} \{Z_{\infty,k} > 0\}\right) = \prod_{k=\ell}^{\infty} (1 - e^{-\lambda_k}) = \theta(d-1, \ell).
\]

It therefore follows that

\[
\theta(d-1, \ell) - \varepsilon \leq \liminf_{n \to \infty} \Pr([\ell, \omega_n'] \subset \mathcal{L}(G)) \leq \limsup_{n \to \infty} \Pr([\ell, \omega_n'] \subset \mathcal{L}(G)) \leq \theta(d-1, \ell) + \varepsilon.
\]

Finally, let \( \omega_n'' \) be the sequence with \([n - \omega_n'', n] \subset \mathcal{L}(G) \) w.h.p., specified in the conclusion of Theorem 2.2. As \( \omega_n := \omega_n' \wedge \omega_n'' \to \infty \) with \( n \), we have \([\omega_n', n - \omega_n'']\) w.h.p. by (2.1); letting \( \varepsilon \downarrow 0 \) completes the proof. ■
2.1. Proof of Theorem 2.1. Let \( V = V(G) = \{v_0, v_1, \ldots, v_{n-1}\} \) be such that the Hamilton cycle specified in the definition of \( H(n) + G(n, 1) \) is the cycle \( C_n = (v_0, v_1, \ldots, v_{n-1}, v_0) \), and let \( M \sim G(n, 1) \) be the uniform independent perfect matching added to it to form the multigraph \( G \). We will show that
\[
\mathbb{P}\{\{\ell, n - \ell + 4\} \not\in \mathcal{L}(G)\} \leq C e^{-\ell} \quad \text{for every } 4 \leq \ell \leq n/2 + 2,
\]
with \( C, c > 0 \) being absolute constants, from which the main result easily follows by a union bound over \( \ell \).

Throughout this proof, identify an undirected edge \( e = (v_i, v_j) \) with the ordered pair \( (v_i, v_j) \), where the ordering is such that \( j - i \mod n \leq n/2 \). Define
\[
E_{\ell} := \left\{ e = (v_i, v_j) \in \binom{V}{2} : j - i \mod n \geq \ell/2 \right\},
\]
(2.4)
(since \( |E_{\ell}| = \binom{n}{2} \left( \frac{\ell}{2} - \left\lfloor \frac{\ell}{2} \right\rfloor \right) \)) and for every \( e = (v_i, v_j) \in E_{\ell} \), further define the \((\ell/2 - 1)\)-element subset
\[
F_{e,\ell} := \left\{ (v_{i+k}, \mod\, n), (v_{j+\ell-k-2}, \mod\, n) \in \binom{V}{2} : 1 \leq k \leq \ell/2 - 1 \right\}.
\]
(2.5)
(Note that \( F_{e,\ell} \) is not assumed to be a subset of \( E_{\ell} \), and indeed, \( F_{e,\ell} \not\subset E_{\ell} \) e.g. for \( \ell \sim n/2 \) and \( i = 1, j \sim \ell \).) These definitions are motivated by the next fact, owing to the classical switching principle (see Fig. 1).

Observation 2.3 (switching). If \( e \in E_{\ell} \) and \( f \in F_{e,\ell} \) then \( C_n \cup \{e, f\} \) has cycles of length \( \ell \) and \( n - \ell + 4 \).

Proof. Let \( e \in E_{\ell} \), assume w.l.o.g. that \( e = (v_0, v_j) \) for \( \ell/2 \leq j \leq n/2 \) and let \( f = (v_k, v_{j+\ell-k-2}) \) for some \( 1 \leq k \leq \ell/2 - 1 \). (Notice \( j + \ell - k - 2 \leq n - k \leq n \) by our assumption on \( k \).) The paths \( P_1 = (v_0, v_1, \ldots, v_k) \) and \( P_2 = (v_{j+\ell-k-2}, \ldots, v_{j+1}, v_j) \) are disjoint since \( j \geq \ell/2 \geq k \), whence the sequence \( P_1, f, P_2 \) is an \( \ell \)-cycle, while \( P_3, e, P_4, f \) forms an \((n - \ell + 4)\)-cycle for \( P_3 = (v_k, v_{k+1}, \ldots, v_{n-1}, v_0) \) and \( P_4 = (v_1, \ldots, v_{k+1}, v_k) \).

Next, for a matching \( M' \subset M \), we define the auxiliary graph \( X_{M'} \) on our original vertex set \( V \) via
\[
E(X_{M'}) = \bigcup\{F_{e,\ell} : e \in M' \cap E_{\ell}\}.
\]

Observation 2.4. For any matching \( M' \), the maximum degree of \( X_{M'} \) is at most \( \ell - 3 \).

Proof. Consider \( v_i \in V \) and assume w.l.o.g. that \( i = 0 \). Recall its definition in (2.5) that, for any \( e \in E_{\ell} \), the set \( F_{e,\ell} \) cannot contribute a neighbor to \( v_0 \) unless \( e \) is incident to some vertex among \( U = \{v_{n-\ell+3}, \ldots, v_{n-1}\} \) (in order to get \( i + k = n \) we must have that \( i \in [n - \ell/2 + 1, \ldots, n - 1]\) whereas to get \( j + \ell - k - 2 = n \) we must have that \( j \in [n - \ell + 3, n - \ell/2 + 1]\)). Since \( M' \) is a matching, it contains at most \( \ell - 3 \) edges \( e \) incident to \( U \), and each such \( F_{e,\ell} \) contributes at most one neighbor to \( v_0 \) (being itself a matching).}

We will expose the matching \( M \) in stages:

(I) Expose a matching \( M_1 \) containing \( t_1 := \lfloor n/16 \rfloor \) edges chosen uniformly at random out of \( M \).

(II) Expose a matching \( M_2 \) containing \( t_2 := \lfloor n/500 \rfloor \) additional edges by repeatedly revealing the (random) match of a vertex with maximum degree in the subgraph induced by \( X_{M_1} \) on the yet unmatched vertices.

(III) Reveal all other remaining edges of the perfect matching \( M \) (these will not be used by our argument).

We will prove the following bounds for the auxiliary graph w.r.t. the matching \( M_1 \) at the end of Stage (I).
Lemma 2.5. There exists $c > 0$ such that, for every sufficiently large $n$, with probability at least $1 - e^{-cn}$, the auxiliary graph $X_M$ has at least $nl/128$ edges, and its induced subgraph $Y_0$ on the set $V \setminus V(M)$ of unmatched vertices has at least $nl/200$ edges.

Modulo the above lemma, one can easily show that $M_1 \cup M_2 \cup C_n$ already contains cycles of lengths $\ell$ and $n - \ell + 4$ with probability $1 - O(e^{-c})$, implying the sought inequality (2.3). To see this, condition on $M_1$ and suppose that $|E(Y_0)| \geq nl/200$ as per the conclusion of the lemma. Denote by $f_1, \ldots, f_k$ the edges exposed in Stage (II), and let $Y_t$ ($t = 0, \ldots, t_2 - 1$) denote the induced subgraph of $X_M$ on the vertices yet unmatched after revealing $f_1, \ldots, f_t$. Recall that $f_{t+1}$ will match some vertex $u$ with a maximum degree in $Y_t$. By Observation 2.4, deleting $k$ (matched) vertices from $Y_0$ results in the removal of at most $2k\ell$ edges; thus, for large enough $n$,

$$|E(Y_t)| \geq |E(Y_0)| - 2t\ell \geq \frac{nl}{200} - 2t_2\ell \geq \frac{nl}{1000},$$

and so $\deg(u) \geq \ell/500$ in $Y_t$. As the match of $u$ is uniformly distributed over the other vertices of $Y_t$,

$$\mathbb{P}\left(\bigcap_{t=1}^{t_2} \{\exists f \in E(X_M)\} \right) \leq \mathbb{P}\left(\bigcap_{t=0}^{t_2-1} \{\exists f \in E(Y_t)\} \right) \leq \left(1 - \frac{\ell}{500n}\right)^{t_2} \leq e^{-c\ell}$$

for an absolute constant $c > 0$. The event $f_t \in E(X_M)$ implies that $f_t \in E(f, \ell)$ for some $e \in M_1 \cap E_{\ell}$, in which case Observation 2.3 yields the sought cycles. It thus remains the prove the above lemma.

Proof of Lemma 2.5. Consider time $t = 0, \ldots, t_1 - 1$, and let $M_1^{(t)} = \{e_1, \ldots, e_{t_1}\}$ denote the first $t$ edges exposed in $M_1$, and let $F_t = \sigma(M_1^{(t)})$ be the corresponding filtration. Further let

$$S_t = \left\{ e \in E_{\ell} \setminus M_1^{(t)} : F(e, \ell) \cap \left( \bigcup_{j=t}^{t_1} F(e_j, \ell) \right) = \emptyset \right\},$$

noting that whenever $e_{t+1} \in S_t$, this edge will contribute the entire edge set $F(e_{t+1}, \ell)$ as new edges to $X_M$.

Let $e = (v_i, v_j) \in E_{\ell}$; the edges in $F_{e,\ell}$ are a matching of consecutive pairs from $L_e = (v_{i+k} \mod n)_{k=1}^K$ and $R_e = (v_{j+k-2} \mod n)_{k=1}^K$, with $K = \lfloor \ell/2 \rfloor - 1$. So, if $f = (v_{i'}, v_{j'})$ is such that $F_{e,\ell}$ and $F_{f,\ell}$ intersect, it must be that for some $d \in \lfloor -K + 1, K - 1 \rfloor$, either $i' \equiv i + d$ or $j' \equiv j + K - 3 \equiv i + d$ (with $\equiv$ denoting equivalence modulo $n$). For any common edge obtained as the $k$-th matched pair in $F_{e,\ell}$ and the $k$-th pair in $F_{f,\ell}$, in the former case we would have $i' + k' \equiv i + k$ and $j' + \ell - k' - 2 \equiv j + \ell - k - 2$, so $i' \equiv j - d$. In the latter case we have $j' + \ell - k' - 2 \equiv i + k$ and $i' + k' \equiv j + \ell - k - 2$, and therefore $i' + j' \equiv i + j$. Altogether, in each case the $2(K - 1)$ choices for $d$ determine the edge $f$, and hence

$$\# \left\{ e \in E_{\ell} \setminus \{e_0\} : |F_{e,\ell} \cap F_{e_0,\ell}| \neq \emptyset \right\} \leq 4\left(\frac{\ell}{2} - 2\right) \leq 2\ell - 8 \quad \text{for every } e_0 \in E_{\ell}. \quad (2.6)$$

From this bound, we immediately deduce that for every $t = 0, \ldots, t_1 - 1$,

$$|S_t| \geq |E_{\ell} \setminus M_1^{(t)}| - (2\ell - 8)(t - 1) \geq |E_{\ell}| - 2t_2\ell \geq n(n - \ell)/2 - 2t_1 \geq (\frac{4}{29} - o(1))n^2 > n^2/6$$

for large enough $n$; thus, $\mathbb{P}(e_t \in S_{t-1} \setminus F_{t-1}) \geq |S_{t-1}|/(n^2/2) > \frac{1}{9}$ for all $1 \leq t \leq t_1$, and so the variable

$$N_t := \#\{1 \leq t \leq t_1 : e_t \in S_{t-1}\}$$

stochastically dominates a $\text{Bin}(t_1, \frac{1}{9})$ random variable. Therefore, for some absolute constant $c > 0$,

$$\mathbb{P}(N_t \leq \frac{4}{19} t_1) \leq \exp(-c t_1).$$

As every $e_{t+1} \in S_t$ adds all of its $\lfloor \ell/2 - 1 \rfloor$ edges to $X_M$, on the event $\{N_t \geq \frac{3}{10} t_1\}$ we have

$$|E(X_M)| \geq (\frac{3}{20} - o(1))t_1 > \frac{1}{128} nl,$$

as claimed. To bound $|E(Y_0)|$, define

$$D = \sum_{t=1}^{t_1} D_t \quad \text{where} \quad D_t = \# \left\{ f \in E(X_{M_1^{(t-1)}}) : e_t \text{ and } f \text{ are incident} \right\},$$
i.e., $D_t$ bounds the number of edges deleted from $Y_0$ when moving from $M_1^{(t-1)}$ to $M_1^{(t)}$ due to the edge $e_t$. Each edge in $M_1^{(t-1)} \cap E_2$ adds at most $\ell/2$ edges to $X_{M_1}^{(t)}$, so $|E(X_{M_1}^{(t-1)})| \leq (t-1)\ell/2$ holds deterministically. The probability that a fixed edge $f \in E(X_{M_1}^{(t-1)})$ is incident to $e_t$ is at most $2/(n - 2t)$, so

$$\mathbb{E}[D_t | F_{t-1}] \leq \frac{\ell(t-1)}{n - 2t}$$

for every $1 \leq t \leq t_1$.

and in particular, $Z_t = \sum_{k=1}^{t} (D_k - \frac{k - 1}{n - 2k})$ is a supermartingale with

$$D - Z_{t+1} \leq \ell \sum_{t=1}^{t_1} \frac{t-1}{n - 2t} \leq \frac{\ell^2}{2} \frac{\ell}{n - 2t_1} = \frac{1 + o(1)}{448} n\ell.$$

Recalling that $0 \leq D_t \leq 2\ell$ for every $t$ by Observation 2.4 about the maximum degree of $X_{M_i}^{(t)}$, whereas $-\ell/\sqrt{n-2t} \in [-\ell, 0]$, we have that $|Z_t - Z_{t-1}| \leq 2\ell$, hence Hoeffding’s inequality implies that, for $\delta = 10^{-4}$,

$$\mathbb{P}(D > n\ell/400) \leq \mathbb{P}(Z_t \geq \delta n\ell) \leq \exp \left( - \frac{(\delta n\ell)^2}{2(2\ell)^2 t_1} \right) = \exp(-2\delta^2 n)$$

Overall we obtained that, for some absolute constant $c > 0$, with probability $1 - O(e^{-cn})$ we have (by a union bound) both $|E(X_{M_i}^{(t)})| > n\ell/128$ and $D < n\ell/400$, implying that $|E(Y_0)| > n\ell/200$, as required. ■

This completes the proof of Theorem 2.1 for the cubic case of $\mathcal{H}(n) + \mathcal{G}(n, 1)$. When $n$ is odd, we appeal to the same argument by treating the second independent and uniform Hamilton cycle as a matching (ordering this cycle, whenever the argument asks for the random match of a vertex $u$ we reveal its successor $v$ on the cycle, then discard $u, v$ from the pool of unmatched vertices). Note that the proof did not need the matching to be perfect, and only utilized $[n/16]$ of its edges in Step (I) and $[n/500]$ of its edges in Step (II). ■

2.2. Proof of Theorem 2.2. The case of $\mathcal{G}(n, d)$ for $d \geq 3$ reduces to the case of $\mathcal{G}(n, 3)$ by the monotonicity of $\mathcal{G}(n, d)$ in $d$ w.r.t. increasing properties that hold asymptotically almost surely. Moreover, since we aim to prove that $[n - \omega_n, n]$ for a sequence $\omega_n$ tending to $\infty$ however slowly with $n$, it suffices to show that

$$\mathbb{P}(n - k \in \mathcal{L}(G)) \to 1$$

for $G \sim \mathcal{G}(n, 3)$ and every fixed $k \geq 1$ (2.7)

(where the case $k = 0$—Hamiltonicity—owes of course to the famous result by Robinson and Wormald [23]). The following theorem immediately implies (2.7), and will consequently establish Theorem 2.2.

Theorem 2.6. Let $k = k(n)$ be such that $1 \leq k = o(\sqrt{n})$. Then $G \sim \mathcal{G}(n, 3)$ has $n - k \in \mathcal{L}(G)$ w.h.p.

Proof. First consider the case where $k = 2\ell \leq 1$. Let $S_G$ be a uniformly chosen set of ordered edges in $G$, chosen in the following manner. If $V(G) = \{v_1, \ldots, v_n\}$ and the 3 half-edges of $v_i$ are denoted $(e_{i,j})_{j=1}^3$, we let $S$ be a uniform $\ell$-subset of all $e_{i,j}$’s. Clearly, these half-edges and their matches are together associated with $2\ell$ distinct vertices except with probability $1 - O(k^2/n) = 1 - o(1)$ by our assumption on $k$. Denote the vertices corresponding to the $i$-th pair of half-edges by $(u_i, u_i')$. Further let $x_{i,j}, y_{i,j}$ denote the other two half-edges matched in $G$ to $u_i$ and $u_i'$, respectively, once again pointing out that these half-edges are w.h.p. not part of $\bigcup_{i=1}^{\ell} \{u_i, u_i'\}$ by our assumption on $k$.

Next, define $H$ to be the graph obtained from $G$ by deleting $u_1, u_1', \ldots, u_\ell, u_\ell'$ and thereafter connecting the half-edges $x_{i,j}y_i$ and $x_{i,j}'y_i'$ for each $i = 1, \ldots, \ell$ (we add every such edge whenever the corresponding two half edges were not deleted as part of some $u_i$ or $u_i'$), denoting these newly added edges by $S_H$ (see Fig. 2).

Observe that, on the event $E_1$ that the edges matched to $S_G$ in $G$ form a matching (occurring w.h.p.), the distribution of $G$ condition on these edges is uniform over perfect matchings of the remaining $3n - 2\ell$ half-edges. Thereafter, on the event $E_2$ that the half-edges $\{x_i, y_i, x_i', y_i'\}_{i=1}^{\ell}$ do not belong to any of the vertices $\{u_i, u_i'\}_{i=1}^{\ell}$ (occurring w.h.p.), these $4\ell$ half-edges are uniformly distributed over all $3(n - 2\ell)$ half-edges. In conclusion, on the event $E_1 \cap E_2$, which occurs w.h.p., we have that $H \sim \mathcal{G}(n - 2\ell, 3)$, and furthermore the set $S_H$ is a uniform set of $2\ell$ ordered edges in $H$ (analogous to the above set $S_G$ in $G$). We now appeal to a result of Robinson and Wormald [24, Thm. 3(i)], stating that w.h.p. there exists a Hamilton cycle in $H$ which avoids a set of $2\ell = o(\sqrt{n})$ randomly chosen edges $S_H$. The same cycle belongs to $G$, hence $n - 2\ell \in \mathcal{L}(G)$.

For the case $k = 2\ell - 1$, we apply the same coupling and appeal to the same theorem of [24], showing that w.h.p. there exists a Hamilton cycle in $H$ that includes the edge $x_1y_1$, and yet avoids the edges $S_H \setminus \{x_1y_1\}$. This corresponds to a path on $n - 2\ell$ vertices in $G$, beginning in the vertex associated to $x_1$, ending in the
vertex associated with $y_1$, and avoiding $u_1$. Adding to this path the edges in $E(G) \setminus E(H)$ from $u_1$ to $x_1$ and from $u_1$ to $y_1$ closes it into a cycle of length $n - 2\ell + 1 = n - k$; hence, again $n - k \in \mathcal{L}(G)$, as required.

3. Binomial random graphs

In this section we derive Theorem 2, as well as a result addressing the regime $p = \frac{1+\varepsilon}{n}$ for small $\varepsilon > 0$ (Theorem 3.7), as immediate consequences of results addressing $\mathcal{L}(G)$ for $G$ the random graph/digraph obtained as a union of a binomial random graph and a Hamilton cycle.

Denote by $\mathcal{H}(n) \oplus \mathcal{G}(n,p)$ the random simple graph on the vertices $\{v_0, \ldots, v_{n-1}\}$, whose edges are the union of the cycle $C_n = (v_0, v_1, \ldots, v_{n-1}, v_0)$ and a random subset of all other undirected edges, each one present independently with probability $p$. Its directed analog, denoted $\mathcal{H} \oplus \mathcal{D}(n,p)$, has the same vertices, and its edges are the union of $\overline{C}_n$—the directed cycle whose edges are $\{(v_i, v_{i+1} \text{ (mod } n)) : i \in [0, n-1]\}$—and a random subset of the other (directed) edges, each one present according to an independent Bernoulli$(p)$ random variable. Following are the analogs of Theorem 2.1 for $\mathcal{H}(n) \oplus \mathcal{G}(n,p)$ and $\mathcal{H}(n) \oplus \mathcal{D}(n,p)$.

**Theorem 3.1.** Fix $\delta > 0$, and let $G \sim \mathcal{H}(n) \oplus \mathcal{G}(n,p)$ for $p = \delta/n$. There exist absolute constants $C, c > 0$ such that, for any $4 \leq \ell \leq n/2$, we have $[\ell, n - \ell + 4] \subset \mathcal{L}(G)$ with probability at least $1 - C \exp(-c(\delta^2 \wedge 1)/\ell)$.

**Theorem 3.2.** Fix $\delta > 0$, and let $G \sim \mathcal{H}(n) \oplus \mathcal{D}(n,p)$ for $p = \delta/n$. There exist absolute constants $C, c > 0$ such that, for any $4 \leq \ell \leq n/2$, we have $[\ell, n - \ell] \subset \mathcal{L}(G)$ with probability at least $1 - C \exp(-c(\delta^2 \wedge 1)/\ell)$.

3.1. **Proof of Theorem 3.1.** Assume w.l.o.g. that $0 < \delta < \frac{1}{7}$. Using the same approach as in Section 2.1 (cf. Eq. (2.3)), we will establish the theorem by showing that, for every sufficiently large $n$,

$$\Pr\{[\ell, n - \ell + 4] \not\subset \mathcal{L}(G)\} \leq 3e^{-\delta(8/\ell)^3\ell} \quad \text{for every } 4 \leq \ell \leq n/2 + 2,$$

implying the statement of the lemma via a union bound. To this end, define $E_\ell$ and $F_{e,\ell}$ for all $e \in E_\ell$ as in (2.4) and (2.5), recalling from Observation 2.3 that should $G$ contain a pair of edges $e, f$ such that $e \in E_\ell$ and $f \in F_{e,\ell}$, then together with $C_n$, these would give rise to cycles of lengths $\ell$ and $n - \ell + 4$, as desired.

Expose the $\mathcal{G}(n,p)$ part of $G$ in two stages, as $G' \cup G''$ for independent random graphs $G' \sim \mathcal{G}(n, p')$ and $G'' \sim \mathcal{G}(n, p'')$ with $p' = p/2$ and $p'' = p/(2-p) \geq p/2$. Letting

$$S' = \bigcup\{F_{e,\ell} : e \in E(G') \cap E_\ell\},$$

we will show that for every sufficiently large $n$,

$$\Pr(|S'| < \frac{1}{30}dn(\ell - 3)) \leq 2exp(-\delta(8/\ell)^3\ell),$$

which will establish (3.1) and complete the proof, since on the event $|S'| \geq \frac{1}{30}dn(\ell - 3)$ we will encounter a pair of edges $e, f$ with $e \in E_\ell$ and $f \in F_{e,\ell}$ via some $e \in E(G') \cap E_\ell$ and $f \in E(G'')$ except with probability

$$\Pr\{E(G'') \cap S' = \emptyset \mid G', |S'| \geq \frac{1}{30}dn(\ell - 3)\} = (1 - p'')^{\frac{|S'|}{\ell + 3}} \leq e^{-\delta(8/\ell)^3\ell + \delta^2 \leq 2e^{-\delta(8/\ell)^3\ell}}.$$

To prove (3.2), we reveal the indicators in $G'$ of potential edges from $E_\ell$ sequentially, in $t := \lceil \delta n/15 \rceil$ steps. Step $t = 1, \ldots, t$ will involve revealing a sequence of indicators, until finding the first one that appears in $G'$:

1. Let $A$ (resp. $R$) be the set of all edges of $G'$ found (resp. pairs in $E_\ell$ examined) in previous steps.
2. Let $B = \{f \in E_\ell \setminus A : F_{f,\ell} \cap F_{e,\ell} \neq \emptyset \text{ for some } e \in A\}$.
3. Order $E = E_\ell \setminus (B \cup R)$ in an arbitrary way, and reveal its indicators one by one:
   a) if a pair $f \in E$ corresponds to an edge of $G'$, let $A \mapsto A \cup \{f\}$ and $R \mapsto R \cup \{f\}$, then end step $t$. 

**Figure 2.** Coupling $G \sim \mathcal{G}(n, 3)$ (on left) to $H \sim \mathcal{G}(n - 2l, 3)$ (on right).
(b) if a pair $f \in E$ does not belong to $E(G')$, let $R \mapsto R \cup \{f\}$. If this results in $|R| > m := \lfloor n^2/5 \rfloor$, abort the entire process, marking it a failure. Otherwise, move on to examine the next edge in $E$.

Recalling (2.6), in each step $t$ we have

$$|B| \leq (2\ell - 8)|A| \leq (2\ell - 8)(t - 1) \leq \delta n^2/15 < n^2/45$$

(assuming that $\delta < \frac{1}{5}$). By construction, $|R_{t-1}| \leq m = \lfloor n^2/5 \rfloor$, whereas $|E_\ell| = \frac{1}{2}n(n-\ell) \geq n^2/4 - n$, and so

$$|E| \geq |E_\ell| - \frac{n^2}{15} - \frac{n^2}{5} \geq \frac{1 - o(1)}{36}n^2,$$

and in particular $E \neq \emptyset$ for large enough $n$. So, the only way the process could fail is if we had $|R| > m$. The latter event, in turn, occurs if and only if fewer than $t$ edges were found in the first $m$ exposed pairs. Thus,

$$\mathbb{P}(|R| > m) \leq \mathbb{P}(\text{Bin}(m, p) < t) \leq \exp(-\frac{1}{150}\delta n) \leq \exp(-\delta^2/8n),$$

using $\mathbb{P}(X - \mu < -a) \leq \exp(-\frac{a^2}{2\mu})$ for a binomial random variable with mean $\mu$ (e.g., [15, §2, Eq. (2.6)]).

Hence, with probability at least $1 - \exp(-\delta^2/8n)$, we arrive at a set $A \subset E_\ell \cap E(G')$ where the corresponding sets $\{F_{\epsilon, t}\}_{\epsilon \in A}$ are pairwise disjoint by construction, thus $|S'| \geq (\lfloor \ell/2 \rfloor - 1)t \geq \frac{1}{100}\delta n(\ell - 3)$, yielding (3.2). 

### 3.2. Proof of Theorem 3.2.
Assume w.l.o.g. that $0 < \delta < \frac{1}{5}$. For $4 \leq \ell \leq n - 4$, define:

$$E_\ell := \{\epsilon = (v_i, v_j) \in V \times V : j - i \pmod{n} \in [2, n - \ell]\}$$

(3.3)

and for every $\epsilon' = (v_i, v_j) \in E_\ell$ let

$$F_{\epsilon, t} := \{(v_{j+t-k-2} \pmod{n}, v_{i-k} \pmod{n}) \in E_\ell : k \in [0, \ell - 2]\}.$$  

(3.4)

(To see that $F_{\epsilon, t} \subset E_\ell$, w.l.o.g. let $\epsilon' = (v_0, v_j) \in E_\ell$ for $j \in [2, n - \ell]$, whereby every $f' = (v_{i'}, v_{j'}) \in F_{\epsilon, t}$ has $j' - i' = n - \ell + 2 - j \in [2, n - \ell]$.) In lieu of Observation 2.3, we use the following simple fact (see Fig. 3).

### Observation 3.3.
If $\epsilon' \in E_\ell$ and $f' \in F_{\epsilon, t}$ then $C_n \cup \{\epsilon', f'\}$ has a directed cycle of length $\ell$.

**Proof.** Let $\epsilon' \in E_\ell$, assuming w.l.o.g. that $\epsilon' = (v_0, v_j)$ for $j \in [2, n - \ell]$, and let $f' \in F_{\epsilon, t}$, denoting by $k$ the index corresponding to $f'$ in this edge set as per Eq. (3.4). Since $\epsilon' \in E_\ell$, the (possibly trivial) paths $P_1 = (v_j, v_{j+1}, \ldots, v_{j+t-k-2})$, $P_2 = (v_{n-k}, \ldots, v_{n-1}, v_0)$ are disjoint, so $(\epsilon', P_1, f', P_2)$ is an $\ell$-cycle. 

The bound on pairwise intersections of the sets $F_{\epsilon, t}$ needed for the proofs, mirroring (2.6), becomes

$$\# \{\epsilon' \in E_\ell \setminus \{\epsilon_0\} : F_{\epsilon, t} \cap F_{\epsilon_0, \ell} \neq \emptyset\} \leq 2\ell - 6$$

for every $\epsilon_0 \in E_\ell$.

(3.5)

which follows immediately from the fact that if $\epsilon_1 = (v_i, v_j) \in E_\ell$ and $\epsilon_2 = (v_i', v_j') \in E_\ell$ are distinct edges satisfying $F_{\epsilon_1, t} \cap F_{\epsilon_2, \ell} \neq \emptyset$, then it must be the case that $i - i' \equiv d$ and $j - j' \equiv d$ for some $d \in [-\ell + 3, \ell - 3] \setminus \{0\}$.

From this point, we proceed with the procedure described in the proof of Theorem 3.1, with parameters $t$ (number of steps) and $m$ (limit on the number of edges that may be exposed) given by

$$t = \left\lceil \frac{3}{4}\delta(n - \ell - 1) \right\rceil, \quad m = \left\lfloor \frac{3}{4}n(n - \ell - 1) \right\rfloor.$$

In every step, the set $B$ of edges we wish to avoid satisfies

$$|B| \leq (2\ell - 6)|A| \leq (2\ell - 6)(t - 1) \leq \delta(n - \ell - 1)\ell/2.$$
By definition $|R_{t-1}| \leq m$ and $|E_t| = n(n - \ell - 1)$, so, recalling that $\delta < \frac{1}{4}$, plugging in $\ell \leq n$ yields

$|E| \geq |E_t| - |B| - m \geq n(n - \ell - 1)(1 - \frac{\delta}{4} - \frac{1}{4}) \geq \frac{1}{16} n(n - \ell - 1) > 0$.

Now we have $\frac{4}{3}\delta(n - \ell - 2) \leq mp' \leq \frac{4}{3}\delta(n - \ell - 1)$ and $mp' - t \geq \frac{1}{4}\delta(n - \ell - 1) - 2$, so (again using $\delta < \frac{1}{4}$)

$\mathbb{P}(|R| > m) \leq \mathbb{P}(\text{Bin}(m, p') < t) \leq 2 \exp\left(-\frac{1}{4}\delta(n - \ell - 1)\right) \leq 2 \exp\left(-\frac{1}{16}\delta^2(n - \ell - 1)\right)$.

Since each edge $\vec{e} \in A$ contributes $|F_{\vec{e}, t}| = \ell - 1$ unique edges to $S' = \bigcup\{F_{\vec{e}}, \vec{e} \in A\}$, we have that $|S'| \geq \frac{1}{4}\delta(n - \ell - 1)$ on the event that the above procedure was successful, whence

$\mathbb{P}(E(G') \cap S' = \emptyset \mid G', |A| \geq t) \leq (1 - p'')|S'| \leq \exp\left[-\frac{1}{4}\delta^2(n - \ell - 1)/n\right]$.

For $\ell \leq n/2$ this is at most $\exp\left[-\frac{1}{16}\delta^2(n - 1)\right]$, whereas for $\ell > n/2$ this is at most $\exp\left[-\frac{1}{16}\delta^2(n - 1)\right]$. Altogether, we conclude that

$\mathbb{P}(\ell \notin L(G)) \leq 3 \exp\left[-(\delta/4)^2((\ell - 1) \wedge (n - \ell - 1))\right]$ for every $4 \leq \ell \leq n - 4$,

completing the proof.

3.3. Consequences for $G(n, p)$ and $D(n, p)$. From Theorems 3.1 and 3.2 we can readily infer the following.

**Corollary 3.4.** Fix $c > c' > 1$ and $\gamma > 0$, and let $G \sim G(n, p = \frac{c}{n})$ and $G' \sim G(n, p' = \frac{c}{n})$. If $\omega_n$ and $L_n'$ are sequences such that $\omega_n \rightarrow \infty$ with $n$ and $L_{\max}(G') \geq L_n' \geq \gamma n$ w.h.p., then $[\omega_n, L_n' - \omega_n] \subset L(G)$ w.h.p. The same conclusion holds when $G \sim D(n, p = \frac{c}{n})$ and $G' \sim D(n, p' = \frac{c}{n})$ for $c, c', \omega_n, L_n'$ as above.

**Proof.** Via standard sprinkling, draw $G \sim G(n, p)$ by exposing $G' \sim G(n, p')$, and then for each of the missing edges, independently adding it with probability $p'' := \frac{p' - p}{1 - p}$. Reveal $G'$, and suppose that it contains a cycle $C'_{\ell_n}$ of length $\ell_n' \geq L_n' \geq \gamma n$ (an event that occurs w.h.p. by our hypothesis). The induced subgraph of $G$ on $C'_{\ell_n}$ dominates a copy of $H(\ell_n') \oplus G'(\ell_n', p'')$, and $p'' \geq \frac{\gamma(c - c')}{\ell_n} \geq \frac{\gamma(c - c')}{\ell_n}$, so Theorem 3.1 (with $\delta = \gamma(c - c') > 0$) implies $[\omega_n, L_n' - \omega_n] \subset L(G)$ (hence also $[\omega_n, L_n' - \omega_n] \subset L(G)$) w.h.p. For $G \sim D(n, p)$, we use the same coupling of $G$ to $G' \cup G''$ for $G' \sim D(n, p')$ and $G'' \sim D(L_n', p'')$, whence Theorem 3.2 completes the proof.

**Proof of Theorem 2.** Beginning with $G \sim G(n, p)$, we appeal to the recent result of Anastos and Frieze [1] (see Theorem 1.3(a) in that work) that, for some absolute $C_0 > 0$, if $G \sim G(n, p)$ with $p = c/n$ for fixed $c > C_0$ then $L_{\max}(G)/n \rightarrow f(c)$ in probability for some function $f(c)$. As $\{L_{\max}(G) \geq k\}$ is a monotone increasing property, the limit $f(c)$ is necessarily monotone non-decreasing in $c$, and as such has countably many discontinuity points. Restricting our attention to every continuity point $c$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that at $p'' = (1 - \delta)p$ we have $L_{\max}(G') \geq (1 - \varepsilon)L_{\max}(G)$, whence Corollary 3.4 implies that

$\mathbb{P}(\omega_n, (1 - \varepsilon)L_{\max}(G)) \rightarrow 1$ for every sequence $\omega_n$ such that $\lim_{n \rightarrow \infty} \omega_n = \infty$. (3.6)

Let $Z_{n, k}$ ($k \geq 3$) be the number of $k$-cycles in $G \sim G(n, p = \frac{c}{n})$. It is well-known (see [5, Cor. 4.9]) that, for every fixed integer $K$, the joint law of the variables $\{Z_{n, k}\}_{k=3}^{K}$ converges to that of independent Poisson random variables $\{Z_{\infty, k}\}_{k=3}^{K}$, where $E(\lambda k) = \lambda_k = e^k/(2k)$. Analogously to (2.2), fix $\varepsilon' > 0$ and let $\omega'_n$ be the maximal $K$ such that

$|\mathbb{P}(\bigcup_{k=\ell}^{K} \{Z_{n, k} > 0\}) - \mathbb{P}(\bigcup_{k=\ell}^{K} \{Z_{\infty, k} > 0\})| < \varepsilon'$ for all $N \geq n$.

The aforementioned convergence result implies that $\omega'_n \rightarrow \infty$ with $n$, so

$\mathbb{P}(\lim_{n \rightarrow \infty} \{Z_{\infty, k} > 0\}) = \prod_{k=\ell}^{\infty} (1 - e^{-\lambda_k}) = \theta(c, \ell)$.

Combining this with (2.1) shows that $\mathbb{P}(\ell, (1 - \varepsilon)L_{\max}(G) \subset L(G))$ is within $\varepsilon + o(1)$ of $\theta(c, \ell)$, as required.

The analogous statement for $G \sim D(n, p)$ follows from Corollary 3.4 in exactly the same manner as argued above, except that now, rather than relying on [1], we appeal to the sequel by the same authors [2] for the fact that there exists some absolute $C_0 > 0$ such that, if $G \sim D(n, p)$ with $p = c/n$ for fixed $c > C_0$ then $L_{\max}(G)/n \rightarrow f(c)$ in probability for some (non-decreasing) function $f(c)$. Finally, the joint law of short cycles is again that of asymptotically independent Poisson random variables (e.g., via the same method-of-moments argument referenced above, and stated for arbitrary strictly balanced graphs in [5, Thm. 4.8]), yet now the automorphism group of a $k$-cycle in the directed graph $G$ has order $k$ rather than $2k$. ■
Remark 3.5. The weaker statement where the absolute constant \( C_0 > 0 \) from Theorems 2 is replaced by \( C_\varepsilon \) may be derived from Corollary 3.4 using much earlier works. Namely, consider the statement that for every \( \varepsilon > 0 \) there exists some \( C_\varepsilon \) so that, if \( \omega_n \) is any sequence going to \( \infty \) with \( n \), then

\[
\mathbb{P}\left( [\omega_n, n - (1 + \varepsilon) e^{-\varepsilon} c \omega_n] \subset \mathcal{L}(G) \right) \to 1 \quad \text{if } G \sim \mathcal{G}(n, p) \text{ with } p = \frac{\varepsilon}{n} \text{ for } c \geq C_\varepsilon \text{ fixed.}
\]

(The number of degree 1 vertices in \( G \sim \mathcal{G}(n, p) \)—which are not part of any cycle—is typically \((e^{-\varepsilon} + o(1))n\). This follows from combining Corollary 3.4 with the result of Frieze [13] that \( L_{\max}(G) \geq (1 - (1 + \varepsilon) e^{-\varepsilon})n \) w.h.p. for some sequence \( \varepsilon_n \) going to 0 as \( c \to \infty \). Similarly, one obtains the analogous statement for \( \mathcal{D}(n, p) \) (where there are typically \((2e^{-\varepsilon} + o(1))n\) vertices of 0 out-degree or 0 in-degree), namely that

\[
\mathbb{P}\left( [\omega_n, n - (2 + \varepsilon) e^{-\varepsilon} c \omega_n] \subset \mathcal{L}(G) \right) \to 1 \quad \text{if } G \sim \mathcal{D}(n, p) \text{ with } p = \frac{\varepsilon}{n} \text{ for } c \geq C_\varepsilon \text{ fixed},
\]

via Corollary 3.4 and the \( \mathcal{D}(n, p) \) analog of said result of [13], due to the last two authors and Sudakov [18].

We next address the setting of \( \mathcal{G}(n, p) \) and \( \mathcal{D}(n, p) \) when \( p = \frac{1+\varepsilon}{n} \) for a small \( \varepsilon > 0 \). Luczak [20] established the existence of constants \( 0 < \gamma_0 < \gamma_1 \) and \( \varepsilon_0 > 0 \) such that

\[
\mathbb{P}(L_{\max}(G) \in [\gamma_0 e^2 n, \gamma_1 e^2 n]) \to 1 \quad \text{if } G \sim \mathcal{G}(n, p) \text{ with } p = \frac{1+\varepsilon}{n} \text{ for } 0 < \varepsilon < \varepsilon_0 \text{ fixed.} \tag{3.7}
\]

Remark 3.6. This statement was proved in [20] for the constants \( \gamma_0 = \frac{1}{4} \) and \( \gamma_1 = \frac{1}{4}(1 + \log \frac{3}{\varepsilon}) \) < 1.874 for the slightly supercritical case \( \varepsilon = o(1), \varepsilon^3 n \to \infty \). These constants are easily explained via the description of the giant component of \( G \) as having a kernel \( K \sim \mathcal{G}(N, 3) \) with \( N \sim \frac{1}{3} \varepsilon^3 n \) vertices (and \( \sim 2\varepsilon^3 n \) edges), inflated into a 2-core by replacing every edge by a path of length i.i.d. Geometric(\( 1/\varepsilon \)) (see [10] for a formal statement of this description). In this case, the kernel is Hamiltonian w.h.p. by [30], giving the constant \( \gamma_0 \). A cycle may visit at most two of the edges incident to any vertex in the kernel, thus taking the longest \( \frac{2}{3} \) of the paths replacing the edges of \( K \), combined with the classical representation of order statistics for i.i.d. exponential variables, yields the constant \( \gamma_1 \). For improved constants replacing \( \gamma_0 \) and \( \gamma_1 \), see, e.g., [17]. The analogous description of the strictly supercritical giant component [11] extends (3.7) to \( 0 < \varepsilon < \varepsilon_0 \) fixed.

The elegant coupling argument of McDiarmid [22] immediately extends (3.7) to \( G \sim \mathcal{D}(n, p) \).

Theorem 3.7. Suppose that there exist absolute constants \( \gamma_0, \varepsilon_0 > 0 \) such that

\[
\mathbb{P}(L_{\max}(G) \geq \gamma_0 e^2 n) \to 1 \quad \text{if } G \sim \mathcal{G}(n, p) \text{ for } p = \frac{1+\varepsilon}{n} \text{ fixed } 0 < \varepsilon < \varepsilon_0.
\]

Then for every \( 0 < \varepsilon < \varepsilon_0 \) and every fixed \( \ell \geq 3 \), the random graph \( G \sim \mathcal{G}(n, p) \) with \( p = (1+\varepsilon)/n \) has

\[
\mathbb{P}\left( [\ell, (1 - \delta) \gamma_0 e^2 n] \subset \mathcal{L}(G) \right) \to \theta(c, \ell) \quad \text{for every fixed } \delta > 0.
\]

The same statement holds for \( G \sim \mathcal{D}(n, p) \) with \( p = \frac{1+\varepsilon}{n} \) when replacing \( \theta(c, \ell) \) by \( \theta'(c, \ell) \) as in Theorem 2.

Proof. Let \( G \sim \mathcal{G}(n, p) \) (the same argument will cover \( G \sim \mathcal{D}(n, p) \), as Corollary 3.4 holds for both models). Fix \( 0 < \varepsilon < \varepsilon_0 \), let \( 0 < \delta < 1 \), and define \( c' = 1+\varepsilon\sqrt{1-\delta} \). Then w.h.p. \( G' \sim \mathcal{G}(n, p') \) with \( p = c'/n \) has \( L_{\max}(G') \geq (1 - \delta) \gamma_0 e^2 n \) by assumption, so \([\omega_n, (1 - \delta) \gamma_0 e^2 n - \omega_n] \subset \mathcal{L}(G) \) w.h.p. by Corollary 3.4. The range \([\ell, \omega_n]\) is covered exactly as in the proof of Theorem 2, and produces the limiting probability \( \theta(c, \ell) \) (and its modified version \( \theta'(c, \ell) \) in the directed case \( G \sim \mathcal{D}(n, p) \)).

Remark 3.8. The machinery developed in Theorem 3.1 can be applied also to models of random graphs set by adding random edges to base graphs with a given property. For example, it implies through the results of [19] that adding \( \delta n \) random edges to a tree \( T \) on \( n \) vertices with maximum degree bounded by \( \Delta \) produces typically a graph \( G \) with the set \( \mathcal{L}(G) \) containing all cycle lengths in \( [\omega_n, cn] \) for \( c = c(\Delta, \Delta) > 0 \). The proof proceeds using the first portion of random edges to create w.h.p. a linearly long cycle \( C \) (see [19, Thm. 6]), and then by applying Theorem 3.1 to \( C \) with the second portion of random edges.

Acknowledgment. M.K. was supported in part by USA-Israel BSF grant 2018267 and ISF grant 1261/17. E.L. was supported in part by NSF grant DMS-1812095.
References

[1] M. Anastos and A. Frieze. A scaling limit for the length of the longest cycle in a sparse random graph. 2019. Preprint, arXiv:1907.03657.
[2] M. Anastos and A. Frieze. A scaling limit for the length of the longest cycle in a sparse random digraph. 2020. Preprint, arXiv:2001.06481.
[3] B. Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. European J. Combin., 1(4):311–316, 1980.
[4] B. Bollobás. Long paths in sparse random graphs. Combinatorica, 2(3):223–228, 1982.
[5] B. Bollobás. Random graphs, volume 73 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 2001.
[6] B. Bollobás, T. I. Fenner, and A. M. Frieze. Long cycles in sparse random graphs. In Graph theory and combinatorics (Cambridge, 1983), pages 59–64. Academic Press, London, 1984.
[7] C. Cooper. Pancyclic Hamilton cycles in random graphs. Discrete Math., 91(2):141–148, 1991.
[8] C. Cooper. 1-pancyclic Hamilton cycles in random graphs. Random Structures Algorithms, 3(3):277–287, 1992.
[9] C. Cooper and A. M. Frieze. Pancyclic random graphs. In Random graphs ’87 (Poznań, 1987), pages 29–39. Wiley, Chichester, 1990.
[10] J. Ding, J. H. Kim, E. Lubetzky, and Y. Peres. Anatomy of a young giant component in the random graph. Random Structures Algorithms, 39(2):139–178, 2011.
[11] J. Ding, E. Lubetzky, and Y. Peres. Anatomy of the giant component: the strictly supercritical regime. European J. Combin., 35:155–168, 2014.
[12] L. Friedman and M. Krivelevich. Cycle lengths in expanding graphs. 2020. Preprint, arXiv:1912.11011.
[13] A. M. Frieze. On large matchings and cycles in sparse random graphs. Discrete Math., 59(3):243–256, 1986.
[14] S. Janson. Random regular graphs: asymptotic distributions and contiguity. Combin. Probab. Comput., 4(4):369–405, 1995.
[15] S. Janson, T. Łuczak, and A. Ruciński. Random graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[16] M. Karoński and A. Ruciński. On the number of strictly balanced subgraphs of a random graph. In Graph theory (Lągów, 1981), volume 1018 of Lecture Notes in Math., pages 79–83. Springer, Berlin, 1983.
[17] G. Kemkes and N. Wormald. An improved upper bound on the length of the longest cycle of a supercritical random graph. SIAM J. Discrete Math., 27(1):342–362, 2013.
[18] M. Krivelevich, E. Lubetzky, and B. Sudakov. Longest cycles in sparse random digraphs. Random Structures Algorithms, 43(1):1–15, 2013.
[19] M. Krivelevich, D. Reichman, and W. Samotij. Smoothed analysis on connected graphs. SIAM J. Discrete Math., 29(3):1654–1669, 2015.
[20] T. Łuczak. Cycles in a random graph near the critical point. Random Structures Algorithms, 2(4):421–439, 1991.
[21] T. Łuczak. Cycles in random graphs. Discrete Math., 98(3):231–236, 1991.
[22] C. McDiarmid. Clutter percolation and random graphs. Math. Programming Stud., (13):17–25, 1980.
[23] R. Robinson and N. Wormald. Almost all cubic graphs are Hamiltonian. Random Structures Algorithms, 3(2):117–125, 1992.
[24] R. Robinson and N. Wormald. Hamilton cycles containing randomly selected edges in random regular graphs. Random Structures Algorithms, 19(2):128–147, 2001.
[25] R. W. Robinson and N. C. Wormald. Almost all regular graphs are Hamiltonian. Random Structures Algorithms, 5(2):363–374, 1994.
[26] N. C. Wormald. The asymptotic distribution of short cycles in random regular graphs. J. Combin. Theory Ser. B, 31(2):168–182, 1981.

Y. Alon
School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 6997801, Israel.
E-mail address: yahavalo@tauex.tau.ac.il

M. Krivelevich
School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 6997801, Israel.
E-mail address: krivelev@tauex.tau.ac.il

E. Lubetzky
Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA.
E-mail address: eyal@courant.nyu.edu