Quantum Entanglement and
Conditional Information Transmission

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Abstract

We propose a new measure of quantum entanglement. Our measure is defined in terms of conditional information transmission for a Quantum Bayesian Net. We show that our measure is identically equal to the Entanglement of Formation in the case of a bipartite (two listener) system occupying a pure state. In the case of mixed states, the relationship between these two measures is not known yet. We discuss some properties of our measure. Our measure can be easily and naturally generalized to handle $n$-partite ($n$-listener) systems. It is non-negative for any $n$. It vanishes for conditionally separable states with $n$ listeners. It is symmetric under permutations of the $n$ listeners. It decreases if listeners are merged, pruned or removed. Most promising of all, it is intimately connected with the Data Processing Inequalities. We also find a new upper bound for classical mutual information which is of interest in its own right.
1 Introduction

Quantum entanglement is at the very heart of Quantum Mechanics so there is a vast amount of literature on the subject. Of particular interest to workers in the field of Quantum Information Theory are the issues of quantification and manipulation of entanglement. An important step in that direction was taken in Refs. [1]-[3]. These references introduced measures of entanglement called entanglement of formation and of distillation. Since Refs. [1]-[3], the implications of these two measures have been explored and clarified considerably by many workers [4]. And yet, the quantification of entanglement for mixed states and for more than two listeners is still not well understood.

The goal of this paper is to shed some light on the quantification of entanglement by approaching it from a new perspective, that of Quantum Bayesian Nets and conditional information transmission. For a review of Quantum Information Theory from the point of view of quantum Bayesian nets, see Ref. [5]. Henceforth, we will assume that the reader is familiar with the notation of Ref. [5].

![Figure 1: CB net in which a and b are conditionally independent.](image)

For motivation, consider the CB net of Fig. 1. This net satisfies

$$P(a, b, \lambda) = P(a|\lambda)P(b|\lambda)P(\lambda).$$  \hspace{1cm} (1.1)

Summing the last equation over $\lambda$, one gets

$$P(a, b) = \sum_{\lambda} P(a|\lambda)P(b|\lambda)P(\lambda).$$  \hspace{1cm} (1.2)

One says that $a$ and $b$ are conditionally independent. Eq. (1.2) is often used as the starting point in the derivation of Bell Inequalities [6]. In that context, $\lambda$ represents the hidden variables. As shown in Ref. [5], Eq. (1.1) implies

$$H((a : b) | \lambda) = 0.$$  \hspace{1cm} (1.3)

As we shall see in what follows, $S_\rho((a : b) | \lambda)$, the quantum mechanical counterpart of $H((a : b) | \lambda)$, is NOT generally zero for a QB net with the graph of Fig. 1. Thus, $S_\rho((a : b) | \lambda)$ appears to be a good measure of quantum entanglement, which is a
phenomenon that does not occur classically. This paper is devoted to discussing $S_{\rho}(a:b|\Lambda)$ and its generalizations.

## 2 Entanglement of Formation

In this section, we will give a very brief review of the most basic aspects of the Entanglement of Formation.

Consider two Hilbert spaces $H_x$ and $H_y$ which need not have the same dimension. Without loss of generality, we will assume that the dimension $N_x$ of $H_x$ is less than or equal to the dimension $N_y$ of $H_y$.

The entanglement of formation $E_F$ for a bipartite pure state $|\psi\rangle \in H_x \otimes H_y$ is defined by

$$E_F(|\psi\rangle) = S[\text{tr}_y(|\psi\rangle\langle\psi|)] .$$  \hspace{1cm} (2.1)

Consider any density matrix $\rho$. If $E = \{(w_a, |\psi_a\rangle) | \forall a\}$ satisfies

$$\rho = \sum_a w_a |\psi_a\rangle \langle\psi_a| ,$$  \hspace{1cm} (2.2)

then we say $E$ is a $\rho$-ensemble. (This clearly defines an equivalence relationship). Ref.[7] characterizes all $E$ belonging to a given $\rho$. The entanglement of formation $E_F$ for a bipartite mixed state with density matrix $\rho$ acting on $H_x \otimes H_y$ is defined by

$$E_F(\rho) = \min_E \left\{ \sum_a w_a E_F(|\psi_a\rangle) \right\} ,$$  \hspace{1cm} (2.3)

where the minimum is taken over all ensembles $E = \{(w_a, |\psi_a\rangle) | \forall a\}$ which are $\rho$-ensembles.

First, let us consider $E_F$ for pure states. Let $\psi$ be the rectangular matrix with entries $\psi_{xy} = \langle x, y | \psi \rangle$. We will often denote $E_F(|\psi\rangle)$ by $E_F(\psi)$ or $E_F(\psi_{xy})$. Thus,

$$E_F(\psi_{xy}) = S(\psi\psi^\dagger) .$$  \hspace{1cm} (2.4)

There always exist unitary matrices $U$ and $V$ such the

$$U \psi V^\dagger = \tilde{\psi} ,$$  \hspace{1cm} (2.5)

where the rectangular matrix $\tilde{\psi}$ is “diagonal”, in the sense that $\tilde{\psi}_{xy} = 0$ if $x \neq y$. Eq.(2.5) is called the Singular Value Decomposition[8] of $\psi$. Define $p_i$ for $0 \leq i \leq N_x - 1$ by

$$U \psi \psi^\dagger U^\dagger = \tilde{\psi} \tilde{\psi}^\dagger = \text{diag}(p_0, p_1, \ldots, p_{N_x - 1}) .$$  \hspace{1cm} (2.6)

Since $\langle \alpha | \tilde{\psi} \tilde{\psi}^\dagger | \alpha \rangle \geq 0$ for any $|\alpha\rangle \in H_x$, the $p_i$'s are non-negative numbers. Furthermore, since $\text{tr}(U \psi \psi^\dagger U^\dagger) = \sum_{x,y} |\psi_{xy}|^2 = 1$, the $p_i$'s add up to one. Note that
\[ |\psi\rangle = \sum_{x,y} \psi_{xy} |x, y\rangle , \quad (2.7) \]

\[ |\tilde{\psi}\rangle = \sum_x \sqrt{p_x} |x = x, \bar{y}\rangle . \quad (2.8) \]

Eq. (2.8) is called the Schmidt Representation of \(|\psi\rangle\). It follows directly from the Singular Value Decomposition of \(\psi\). By Eq. (2.4) and (2.6),

\[ E_F(\psi_{xy}) = E_F(\tilde{\psi}_{x,y}) = -\sum_x p_x \log_2 p_x . \quad (2.9) \]

For the remainder of this section, we will restrict our attention to the special case where \(\underline{x}\) and \(\underline{y}\) have just two states, 0 and 1. In this case, \(E_F(\psi_{xy}) = h(p_0)\), where \(h\) is the binary entropy function, and where \(p_0\) and \(p_1 = 1 - p_0\) are the eigenvalues of \(\psi\psi^\dagger\). Define complex numbers \(K_0, K_1\) and \(K\) by

\[ \psi\psi^\dagger = \begin{bmatrix} K_0 & K \\ K^* & K_1 \end{bmatrix} . \quad (2.10) \]

Thus,

\[ K_0 = |\psi_{00}|^2 + |\psi_{01}|^2 , \quad (2.11a) \]

\[ K_1 = |\psi_{10}|^2 + |\psi_{11}|^2 , \quad (2.11b) \]

\[ K = \psi_{00}\psi_{10}^* + \psi_{01}\psi_{11}^* . \quad (2.11c) \]

The two eigenvalues of \(\psi\psi^\dagger\) are

\[ p_0 = \frac{1 + \sqrt{1 - t}}{2} , \quad p_1 = 1 - p_0 , \quad (2.12a) \]

where

\[ t = 4(K_0K_1 - |K|^2) = 4|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}|^2 . \quad (2.12b) \]

The Bell Basis is defined by

\[ |B_f\rangle = \frac{i^{f_0 + f_1}}{\sqrt{2}} (|0, f_0\rangle + (-1)^{f_1}|1, \bar{f}_0\rangle) , \quad (2.13) \]

for \(f = (f_0, f_1) \in BOOL^2\). (\(\bar{0} = 1\) and \(\bar{1} = 0\).) If \(x, y \in Bool\), then

\[ \langle x, y | B_f \rangle = \frac{i^{f_0 + f_1}}{\sqrt{2}} (\delta_{x, y \bar{0}, f_0} + (-1)^{f_1}\delta_{x, y \bar{1}, f_0}) . \quad (2.14) \]

Let \(\alpha_j\) for \(j \in Z_{0, 3}\) be the components of \(|\psi\rangle\) in the Bell Basis:
\[ |\psi\rangle = \alpha_0 |B_{00}\rangle + \alpha_1 |B_{01}\rangle + \alpha_2 |B_{10}\rangle + \alpha_3 |B_{11}\rangle. \]  

(2.15)

Then

\[ \psi_{00} = \frac{1}{\sqrt{2}} (\alpha_0 + i\alpha_1), \]  

(2.16)

\[ \psi_{01} = \frac{1}{\sqrt{2}} (i\alpha_2 + \alpha_3), \]  

(2.17)

\[ \psi_{10} = \frac{1}{\sqrt{2}} (i\alpha_2 - \alpha_3), \]  

(2.18)

\[ \psi_{11} = \frac{1}{\sqrt{2}} (\alpha_0 - i\alpha_1). \]  

(2.19)

Substituting these equations into the definition Eq. (2.12b) of \( t \) yields

\[ t = \left| \sum_{j=0}^{3} \alpha_j^2 \right|^2. \]  

(2.20)

Suppose that \( Q_j = |\alpha_j^2| \) and \( \theta_j = \text{phase}(\alpha_j^2) \) for \( j \in \mathbb{Z}_{0,3} \). Then \( \sum_{j=0}^{3} Q_j = 1 \) and \( t = |\sum_{j=0}^{3} Q_je^{i\theta_j}|^2 \). Thus \( 0 \leq t \leq 1 \) and \( t = 1 \) iff the \( \theta_j \)'s are all zero (i.e., the \( \alpha_j^2 \)'s are all real). \( t = 0 \) iff \( E_F(\psi_{xy}) = 0 \), and \( t = 1 \) iff \( E_F(\psi_{xy}) \) is maximum. This is why. From Fig. 2, it is clear that \( h(p_0(t)) = E_F(\psi_{xy}) \) is a monotonically increasing function of \( t \) which goes from 0 to 1 as \( t \) goes from 0 to 1.

![Plot of functions \( p_0(t) \) and \( h(p_0) \).](image)

Figure 2: Plot of functions \( p_0(t) \) and \( h(p_0) \).

So far we have discussed \( E_F \) for pure states. There are still many unsolved mysteries about \( E_F \) for mixed states. An example for which definition Eq. (2.3) has been evaluated is when \( \rho \) is diagonal in the Bell basis:

\[ \rho = \sum_a w_a |B_a\rangle\langle B_a|, \]  

(2.21)

where the \( w_a \)'s are non-negative numbers that add up to one. Ref. [3] shows that for this \( \rho \),
\[
E_F(\rho) = \begin{cases} 
0 & \text{if } W \leq \frac{1}{2} \\
\ln \left( \frac{1+\sqrt{1-4(W-p)^2}}{2} \right) & \text{otherwise} 
\end{cases},
\]
(2.22)
where
\[
W = \max_a(w_a).
\]

### 3 Some Definitions

In this section, we will define our measure of entanglement. Future sections will explore the properties of our measure, and how it compares with \( E_F \).

Consider either a QB or CB net with \( N \) nodes \((\mathcal{E}.)_{Z_1,N}\). Suppose that \( L_1, L_2, \ldots, L_n \) and \( E \) are non-empty disjoint node collections of the net. For a CB net, we define the \( H \)-tanglement \( HT \) for \( n \) listeners (or receivers) \( L_1, L_2, \ldots, L_n \) and a speaker (or sender) \( E \) by
\[
HT(L_1 : L_2 : \ldots : L_n | E) = \sum_{i=1}^{n} H(L_i | E) - H(L_1, L_2, \ldots, L_n | E).
\]
(3.1)
Analogously, for a QB net we define the \( S \)-tanglement \( ST \) by
\[
ST_\rho(L_1 : L_2 : \ldots : L_n | E) = \sum_{i=1}^{n} S_\rho(L_i | E) - S_\rho(L_1, L_2, \ldots, L_n | E).
\]
(3.2)
Here \( \rho \) is any density matrix obtained by reducing the meta density matrix of the net, but such that the nodes in \( L_1, L_2, \ldots, L_n \) and \( E \) haven't been reduced. We will also use the term \textit{max} \( S \)-tanglement to refer to \( ST \) maximized over all local unitary operations on the \( L_i \)'s. If \( ST_\rho \neq 0 \) for a QB net but \( HT = 0 \) for its parent CB net, we will describe this situation by saying that there is \textit{non-classical tanglement}. \( H(L_1 : L_2 : \ldots : L_n) \) (or \( S(L_1 : L_2 : \ldots : L_n) \)) will be called an \( H \) (or \( S \)) \textit{mutual information} for \( n \) parts. \( H(L_1 : L_2 : \ldots : L_n | E) \) (or \( S(L_1 : L_2 : \ldots : L_n | E) \)) will be called an \( H \) (or \( S \)) \textit{conditional mutual information} (c.m.i.) for \( n \) listeners. When there are two listeners, tanglement equals a c.m.i.. As we shall see later, this is no longer the case for more than two listeners.

Recall from Ref.[5] that a node collection with more than one node is said to be \textit{compound}. Likewise, a listener or speaker with more than one node will be said to be compound.

Suppose that \( \mathcal{X} \) and \( \mathcal{Y} \) are non-empty disjoint node collections of either a CB or a QB net. For a CB net, we will say that \( \mathcal{X} \) and \( \mathcal{Y} \) are (probabilistically) \textit{independent} (also called separable or uncorrelated) if
for all possible $X$ and $Y$. For a QB net, suppose $\rho_{X,Y}$ is a density matrix acting on $\mathcal{H}_{X,Y}$ and obtained by reducing the meta density matrix of the net. We will say that $\overline{X}$ and $\overline{Y}$ are independent (or separable) if

$$\rho_{X,Y} = \rho_{\overline{X}} \rho_{\overline{Y}}.$$  \hfill (3.4)

Suppose that $\overline{X}$, $\overline{Y}$ and $\overline{E}$ are non-empty disjoint node collections of either a CB or a QB net. For a CB net, we will say that $\overline{X}$ and $\overline{Y}$ are conditionally independent (or conditionally separable) if

$$P(X, Y) = \sum_{E} P(X|E)P(Y|E)P(E),$$ \hfill (3.5)

for all possible $X$ and $Y$. For a QB net, suppose $\rho_{X,Y,E}$ is a density matrix acting on $\mathcal{H}_{X,Y,E}$ and obtained by reducing the meta density matrix of the net. We will say that $\overline{X}$ and $\overline{Y}$ are conditionally independent (or conditionally separable) if

$$\rho_{X,Y,E} = \sum_{E} \rho_{\overline{X}}^{(E)} \rho_{\overline{Y}}^{(E)} |E\rangle \langle E|,$$ \hfill (3.6)

where $\{|E\rangle|\forall E\}$ is orthonormal basis corresponding to $\overline{E}$, $\rho_{\overline{X}}^{(E)} \geq 0$ for all $E$, $\sum_{E} w_{E} = 1$, $\rho_{\overline{X}}^{(E)}$ acts on $\mathcal{H}_{\overline{X}}$, and $\rho_{\overline{Y}}^{(E)}$ acts on $\mathcal{H}_{\overline{Y}}$.

4 \hspace{1em} \textit{ST for 2 Single-node Listeners and a Pure State}

In this section, we will discuss S-tanglement for 2 single-node listeners and a pure state. We will show that it equals $E_{F}$ if we maximize it over all local unitary transformations on the two listeners.

Consider the QB net of Fig. 3, where

![Figure 3: Net for 2 single-node listeners and a pure state.](image)
\[
\begin{array}{|c|c|c|c|}
\hline
\text{nodes} & \text{states} & \text{amplitudes} & \text{comments} \\
\hline
\epsilon & e = (e_1, e_2) & \psi(e) = \sum_{x', y'} U_{e_1, x'} \psi^0(x', y') V_{y_2}^\dagger & \sum_{x, y} |\psi^0(x, y)|^2 = 1, \\
\hline
x & x \in S_x & \delta(x, e_1) & \sum_a U_a^* U_{ax} = \delta_{x'}^x, \\
\hline
y & y \in S_y & \delta(y, e_2) & \sum_b V_{by}^* V_{by'} = \delta_{y'}^y. \\
\hline
\end{array}
\]

We will sometimes write \( \psi_{xy} \) instead of \( \psi(x, y) \). Without loss of generality, we will assume that \( N_x \) (the size of set \( S_x \)) is less than or equal to \( N_y \).

The meta density matrix \( \mu \) of this net is

\[
\mu = |\psi_{\text{meta}}\rangle\langle \psi_{\text{meta}} |, \tag{4.1}
\]

where

\[
|\psi_{\text{meta}}\rangle = \sum_{ri} \psi(x, y)|\epsilon = (x, y), x, y \rangle. \tag{4.2}
\]

Define \( \rho \) by

\[
\rho = \text{tr}_\epsilon (\mu) = \sum_{ri} \psi(x, y)\psi^*(x, y)|x, y\rangle \langle x, y |. \tag{4.3}
\]

One has that

\[
S_\mu(x : y|\epsilon) = S_\mu(x, \epsilon) + S_\mu(y, \epsilon) - S_\mu(x, y, \epsilon) - S_\mu(\epsilon). \tag{4.4}
\]

But \( \mu \) is a pure state acting on \( \mathcal{H}_{x,y} \), so

\[
S_\mu(x, \epsilon) = S_\mu(y), \tag{4.5a}
\]

\[
S_\mu(y, \epsilon) = S_\mu(x), \tag{4.5b}
\]

\[
S_\mu(x, y, \epsilon) = 0, \tag{4.5c}
\]

\[
S_\mu(\epsilon) = S_\mu(x, y). \tag{4.5d}
\]

Substituting Eqs.(4.5) into Eq.(4.4) yields

\[
S_\mu(x : y|\epsilon) = S_\mu(x : y) = S_\rho(x : y). \tag{4.6}
\]

Note that \( \rho \) is diagonal in the \( |x, y\rangle \) basis so Eq.(4.6) can be simplified further. Let

\[
P(x, y) = |\psi(x, y)|^2. \tag{4.7}
\]

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With this $P(x, y)$, one can calculate $H(x : y)$. Eq. (4.6) reduces to

$$S_\mu(x : y|e) = H(x : y).$$  \hspace{1cm} (4.8)

Henceforth, we will often abbreviate $P(x, y)$ by $P_{xy}$, $P(x) = \sum_y P(x, y)$ by $P_x$, and $P(y) = \sum_x P(x, y)$ by $P_y$.

When $S_\mu = S_\nu = \text{Bool}$, the unitary matrices $U$ and $V$ mentioned in the above table determine what spin direction is measured at the nodes $x$ and $y$. The above table and the following one

| nodes | states  | amplitudes | comments |
|-------|---------|------------|----------|
| $e$   | $e = (e_1, e_2)$ | $\psi^0(e)$ |          |
| $x$   | $x \in S_x$ | $U_{e_1,x}$ |          |
| $y$   | $y \in S_y$ | $V_{y,e_2}^\dagger$ |          |

do not yield the same $S_\mu(x : y|e)$. In the first table, node $e$ upon which we condition has knowledge of $U$ and $V$, whereas in the second it doesn’t. We will call the $U$ and $V$ in the first (ditto, second) table a priori (ditto, a posteriori) local unitary transformations on $x$ and $y$. In this section, we are interested in the case of the first table, where $U$ and $V$ refer to a priori transformations.

Suppose $\psi$ (ditto, $\psi^0$) is the rectangular matrix with entries $\psi_{xy}$ (ditto, $\psi^0_{xy}$). Then

$$\psi = U\psi^0V^\dagger.$$ \hspace{1cm} (4.9)

Let us consider the special case that $U$ and $V$ make $\psi$ diagonal. Such a $U$ and $V$ always exist by the Singular Value Decomposition Theorem. Suppose that

$$\psi\psi^\dagger = \text{diag}(p_0, p_1, \ldots, p_{N_x-1}).$$ \hspace{1cm} (4.10)

The $p_x$’s must be non-negative numbers that add up to one. Then

$$H(x : y) = \sum_{x,y} P_{xy} \log_2 \frac{P_{xy}}{P_x P_y} = \sum_x p_x \log_2 \frac{1}{p_x} = E_F(\psi^0_{xy}) = E_F(\psi_{xy}).$$ \hspace{1cm} (4.11)

Combining the last equation and Eq. (4.8) yields

$$S_\mu(x : y|e) = E_F(\psi_{xy}).$$ \hspace{1cm} (4.12)

for the special case that $U$ and $V$ make $\psi$ diagonal.

In Appendices A and B, we show the following inequalities:

$$E_F(|\psi_{xy}|) \leq E_F(\psi_{xy}),$$ \hspace{1cm} (4.13)
Combining these inequalities and Eq. (4.8) yields

\[ S_\mu(\vec{x} : \vec{y} | \xi) \leq E_F(\psi_{xy}) . \]  

(4.15)

From the argument leading up to Eq. (4.12), we see that there exists a pair of unitary matrices \( U \) and \( V \) so that the S-tanglement \( ST \) equals the corresponding entanglement of formation \( E_F \). From the argument leading up to Eq. (4.15), we see that for any \( U \) and \( V \), \( ST \) is less than or equal to the corresponding \( E_F \). Therefore, if \( ST \) is maximized over all a priori local unitary transformations \( U \) and \( V \) on its two listeners, then it equals \( E_F \).

5 **ST for 2 Single-Node Listeners and a Mixed State**

In this section, we will discuss S-tanglement for 2 single-node listeners and a mixed state. We will show that it vanishes for a conditionally separable state. We will also calculate \( ST \) for any \( \rho \) which is diagonal in the Bell basis.

Suppose \( q_1, q_2, e \) are nodes of a QB net. Suppose

\[ \rho = \sum_a w_a \rho_a^{(1)} \rho_a^{(2)} , \]

(5.1)

where \( w_a \geq 0 \) for all \( a \) and \( \sum_a w_a = 1 \), and where for all \( a \) and for \( \lambda = 1,2 \), \( \rho_a^{(\lambda)} \) is a density matrix acting on \( H_{q_\lambda} \). For such a \( \rho \), \( E_F(\rho) = 0 \) [3]. To calculate \( S_\rho(q_1 : q_2 | a) \), we need a \( \rho \) that acts on a space \( H_{q_1,q_2,a} \) or larger, so the \( \rho \) in Eq. (5.1) will not do. Suppose we consider instead the following \( \rho \):

\[ \rho = \sum_a w_a |a\rangle \langle a| \rho_a^{(1)} \rho_a^{(2)} , \]

(5.2)

where \( \{|a\rangle | \forall a \} \) is the orthonormal basis for node \( a \). For this \( \rho \), one has

\[ S_\rho(q_1 : q_2 | a) = S_\rho(q_1, a) + S_\rho(q_2, a) - S_\rho(q_1, q_2, a) - S_\rho(a) , \]

(5.3)

where

\[ S_\rho(q_\lambda, a) = H(\vec{\omega}) + \sum_a w_a S(\rho_a^{(\lambda)}) \quad \text{for } \lambda = 1,2 , \]

(5.4)

\[ S_\rho(q_1, q_2, a) = H(\vec{\omega}) + \sum_a w_a \{ S(\rho_a^{(1)}) + S(\rho_a^{(2)}) \} , \]

(5.5)

\[ S_\rho(a) = H(\vec{\omega}) , \]

(5.6)

so
Figure 4: Net that implements a 2 choice conditionally separable density matrix.

\[ S_\rho(q_1 : q_2 | a) = 0 . \]  

(5.7)

Note that the \( \rho \) defined by Eq.(5.2) can be implemented by the QB net of Fig.4, where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \( \tilde{j} \) | \( \tilde{j} = (\tilde{j}^1, \tilde{j}^2) \) | \( \sqrt{w_{\tilde{j}}} \delta(\tilde{j}^1, \tilde{j}^2) \) | \( \sum_{\tilde{j}} w_{\tilde{j}} = 1 \) |
| \( \tilde{a} \) | \( \tilde{a} \) | \( \delta(a, \tilde{j}^1) \) | |
| \( \tilde{r} \) | \( \tilde{r} \) | \( \delta(\tilde{r}, \tilde{j}^2) \) | |
| \( j_\lambda \) for \( \lambda \in Z_{1,2} \) | \( j_\lambda = (j_\lambda^1, j_\lambda^2) \) | \( \alpha_\lambda(j_\lambda | a) \) | \( \sum_{j_\lambda} |\alpha_\lambda(j_\lambda | a)|^2 = 1 \) |
| \( q_\lambda \) for \( \lambda \in Z_{1,2} \) | \( q_\lambda \) | \( \delta(q_\lambda, j_\lambda^1) \) | |
| \( r_\lambda \) for \( \lambda \in Z_{1,2} \) | \( r_\lambda \) | \( \delta(r_\lambda, j_\lambda^2) \) | |

The meta density matrix \( \mu \) of this net is

\[ \mu = |\psi_{\text{meta}}\rangle \langle \psi_{\text{meta}}| , \]  

(5.8)

where

\[ |\psi_{\text{meta}}\rangle = \sum_{r_\lambda} \sqrt{w_{\lambda}} \left[ \prod_{\lambda=1}^{2} \alpha_\lambda(q_\lambda, r_\lambda | a) \right] |\tilde{j}_\lambda = (q_\lambda, r_\lambda, q_\lambda, r_\lambda) \rangle |\tilde{j} = (a, a), a, \tilde{r} = a \rangle . \]  

(5.9)

Define \( \rho \) by
\[
\rho = \mathbb{E}_{\bar{Z} \bar{\mathbb{Z}} \bar{\mathbb{Z}}} \text{tr}_{\bar{Z} \bar{\mathbb{Z}} \bar{\mathbb{Z}}} (\mu) . \tag{5.10}
\]

Then
\[
\rho = \sum_a w_a |a\rangle \langle a| \rho_a^{(1)} \rho_a^{(2)} , \tag{5.11}
\]

where
\[
\rho_a^{(\lambda)} = \sum_{all/a,\lambda} \alpha_\lambda(q_\lambda, r_\lambda |a\rangle \alpha_\lambda^*(q'_\lambda, r_\lambda |a\rangle |q_\lambda\rangle \langle q'_\lambda| \tag{5.12}
\]

for all \(a\) and for \(\lambda = 1, 2\).

![Figure 5: Net for 2 single-node listeners and a mixed state.](image)

Next consider the QB net of Fig. 5, where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \(f\) | \(f\)   | \(\sqrt{w_f}\) | \(\sum_f w_f = 1\) |
| \(e\) | \(e = (e_1, e_2)\) | \(\langle e|\psi_f e\rangle = \psi_f (e)\) | \(\sum_e |\psi_f (e)|^2 = 1\) |
| \(x\) | \(x\)   | \(\delta(x, e_1)\) |                      |
| \(y\) | \(y\)   | \(\delta(y, e_2)\) |                      |

The meta density matrix \(\mu\) of this net is
\[
\mu = |\psi_{\text{meta}}\rangle \langle \psi_{\text{meta}}| , \tag{5.13}
\]

where
\[
|\psi_{\text{meta}}\rangle = \sum_{r_i} \sqrt{w_f} \psi_f (x, y) |f, e = (x, y), x, y\rangle . \tag{5.14}
\]

Define \(\sigma\) by
\[
\sigma = \text{tr}_f (\mu) = \sum_{r_i} w_f \psi_f(x,y) \psi_f^*(x',y') |e = (x,y), x, y \rangle \langle e = (x',y'), x', y'| .
\]

(5.15)

We wish to calculate \( S_\sigma (x : y | e) \). Let

\[
P(x, y) = \sum_f w_f |\psi_f(x,y)|^2 .
\]

(5.16)

We can define a density matrix \( \rho(y) \) for each \( y \in S_y \) by

\[
\rho(y) = \frac{\sum_f w_f \psi_f(x,y) \psi_f(x',y) |x\rangle \langle x'|}{P(y)} .
\]

(5.17)

In an analogous manner, we can define a density matrix \( \rho(x) \) for each \( x \in S_x \). It is also convenient to define \( \rho \) by

\[
\rho = E \sum_\mu \text{tr}_f (\mu) = \sum_f w_f |\psi_f\rangle \langle \psi_f| .
\]

(5.18)

One has that

\[
S_\sigma (x : y | e) = S_\sigma (x, e) + S_\sigma (y, e) - S_\sigma (x, y, e) - S_\sigma (e) .
\]

(5.19)

Using the observations of Appendix C, one gets

\[
S_\sigma (x, e) = S \left[ \sum_y P(y) |y\rangle \langle y| \rho(y) \right] = H(y) + \sum_y P(y) S[\rho(y)] .
\]

(5.20)

Likewise,

\[
S_\sigma (y, e) = H(x) + \sum_x P(x) S[\rho(x)] .
\]

(5.21)

Furthermore,

\[
S_\sigma (x, y, e) = S(\rho) ,
\]

(5.22)

and

\[
S_\sigma (e) = H(x, y) .
\]

(5.23)

Therefore,

\[
S_\sigma (x : y | e) = H(x : y) + \sum_x P(x) S[\rho(x)] + \sum_y P(y) S[\rho(y)] - S(\rho) .
\]

(5.24)

Note that if \( w_f = \delta(f,0) \), then \( \rho(x), \rho(y) \) and \( \rho \) are all pure states so the right-hand side of the last equation reduces to \( H(x : y) \). This is what the previous section on pure states would lead us to expect.
Now consider the case that \( S_x = S_y = \text{Bool} \). Let \( w_{x-} = \sum_{y=0}^{1} w_{xy} \), and \( w_{-y} = \sum_{x=0}^{1} w_{xy} \). If we specialize Eq.\((5.24)\) by assuming that the states \(|\psi_f\rangle\) are the Bell Basis states (defined by Eq.\((2.13)\) ), then we obtain

\[
S_\alpha(x : y | E) = h(w_{0-}) + 1 - H(\bar{w}).
\] (5.25)

The last equation gives \( ST \) for a Bell diagonal mixture. \( E_F(\rho) \) for this same state was given in Eq.\((2.22)\). I'm not sure yet how these two results are connected. Also, note that Eq.\((5.25)\) is not yet maximized over all a priori local unitary transformations, and one should perform this maximization before comparing it with \( E_F(\rho) \), if one is to follow the same rules that were used in the pure state case.

## 6 Properties of Tanglement and C.M.I.

In this section we will discuss various properties satisfied by tanglements and c.m.i.’s.

The following notation will be used henceforth.

Often, after stating something about the classical entropy \( H \) or the classical tanglement \( HT \), we will append to the end of the statement the symbol \([H \rightarrow S]\) to indicate that the statement is also valid if one replaces \( H \) by \( S \) everywhere. Likewise, the symbol \([S \rightarrow H]\) will indicate that the previous statement is also valid if we replace \( S \) by \( H \) everywhere.

For any set \( S \), its power set \( \text{Pow}(S) \) is the set of all subsets of \( S \), including the null set. For example, \( \text{Pow}\{1, 2\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \). If \( S \) has \(|S|\) elements, then \( \text{Pow}(S) \) has \( 2^{|S|} \) elements. For this reason \( \text{Pow}(S) \) is often denoted by \( 2^S \).

We will also use \( \text{Pow}(S)_j \) for any \( j \in \mathbb{Z}_0^{\langle S \rangle} \) to denote the set of all subsets of \( S \) which contain \( j \) elements. For example, \( \text{Pow}(Z_{1,3})_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \).

Clearly, \( \text{Pow}(S) = \bigcup_{j=0}^{|S|} \text{Pow}(S)_j \).

For any set \( S = \{a_1, a_2, \ldots, a_n\} \), let \((; a \in S a) = (_; j=1^n a_j) = a_1 : a_2 : \ldots : a_n\).

Suppose \( E, X_1, X_2, \ldots, X_n \) with \( n \geq 2 \) are non-empty disjoint node collections of a Bayesian net, and \( \Gamma_\alpha \) for \( \alpha \in Z_{1,n} \) are non-empty disjoint subsets of \( Z_{1,n} \). We will sometimes use the following \( \tau, \mu \) shorthand for tanglement and c.m.i.:

\[
\tau(\Gamma_1 : \Gamma_2 : \ldots : \Gamma_m) = HT[(X_1)_{r_1} : (X_2)_{r_2} : \ldots : (X_m)_{r_m} | E] \quad (H \rightarrow S)
\] \hspace{0.5cm} (6.1)

\[
\mu(\Gamma_1 : \Gamma_2 : \ldots : \Gamma_m) = H[(X_1)_{r_1} : (X_2)_{r_2} : \ldots : (X_m)_{r_m} | E] \quad (H \rightarrow S)
\] \hspace{0.5cm} (6.2)

For example,

\[
\tau(1 : 2 : (3, 4)) = HT(X_1 : X_2 : (X_3, X_4) | E) \quad (H \rightarrow S)
\] \hspace{0.5cm} (6.3)
\[ \mu(1 : 2 : (3, 4)) = H(X_1 : X_2 : (X_3, X_4)|E) . \] (6.4)  

Sometimes, we will put the argument of \( \tau \) or \( \mu \) in a subscript (e.g., \( \tau_{1:2} \)), while other times we will put it in parentheses (e.g., \( \tau(1 : 2) \)).  

In discussing the following properties, we will use \( E, X_1, X_2, \ldots, X_n \) with \( n \geq 2 \) to denote non-empty disjoint node collections of a Bayesian net.  

(1) **Symmetry**  

\( H \text{ tanglement and c.m.i. are symmetric under permutations of their listeners.} \)  

(2) **Sign of tanglement**  

One has that  

\[ H(X_1 : X_2|E) = H(X_1|E) + H(X_2|E) - H(X_1, X_2|E) = H(X_1|E) - H(X_1|E, X_2) \geq 0 . \] (6.5)  

where the inequality follows by strong subadditivity.  

Tanglement is non-negative for any number of listeners, not just two. Indeed, an \( n \)-listener tanglement can always be expressed as a sum of 2-listener tanglements. For example, for 4 listeners, one has  

\[ \tau(1 : 2 : 3 : 4) = \tau(1 : 2) + \tau((1, 2) : 3) + \tau((1, 2, 3) : 4) \geq 0 . \] (6.6)  

(3) **Decomposition of c.m.i.**  

In discussing tanglements, c.m.i.'s often arise. Next we will show how to express a c.m.i. as a sum of \( \pm \) non-mutual informations.  

For 2 listeners  

\[ H(X_1 : X_2|E) = H(X_1|E) + H(X_2|E) - H(X_1, X_2|E) . \] (6.7)  

\[ H(X_1 : X_2|E) = H(X_1, E) + H(X_2, E) - H(X_1, X_2, E) - H(E) . \] (6.8)  

For 3 listeners,
\[ H(X_1 : X_2 : X_3 | E) = \begin{cases} H(X_1 | E) + H(X_2 | E) + H(X_3 | E) \\
- H(X_1, X_2 | E) - H(X_1, X_3 | E) - H(X_2, X_3 | E) \\
+ H(X_1, X_2, X_3 | E) \end{cases} , \quad H \rightarrow S \] (6.9)

\[ H(X_1 : X_2 : X_3 | E) = \begin{cases} H(X_1, E) + H(X_2, E) + H(X_3, E) \\
- H(X_1, X_2, E) - H(X_1, X_3, E) - H(X_2, X_3, E) \\
+ H(X_1, X_2, X_3, E) \end{cases} . \quad H \rightarrow S \] (6.10)

For 4 listeners,

\[ H(X_1 : X_2 : X_3 : X_4 | E) = \begin{cases} \sum_{\alpha=1}^{4} H(X_\alpha | E) \\
- \sum_{1 \leq \alpha < \beta < \gamma < \delta} H(X_\alpha, X_\beta, X_\gamma, X_\delta | E) \\
+ \sum_{1 \leq \alpha < \beta < \gamma < \delta} H(X_\alpha, X_\beta, X_\gamma, X_\delta | E) \\
- H(X_1, X_2, X_3, X_4, X_\alpha, X_\beta, X_\gamma, X_\delta) \end{cases} , \quad H \rightarrow S \] (6.11)

\[ H(X_1 : X_2 : X_3 : X_4 | E) = \begin{cases} \sum_{\alpha=1}^{4} H(X_\alpha, E) \\
- \sum_{1 \leq \alpha < \beta < \gamma < \delta} H(X_\alpha, X_\beta, X_\gamma, E) \\
+ \sum_{1 \leq \alpha < \beta < \gamma < \delta} H(X_\alpha, X_\beta, X_\gamma, E) \\
- H(X_1, X_2, X_3, X_4, X_\alpha, X_\beta, X_\gamma, E) \end{cases} . \quad H \rightarrow S \] (6.12)

One can show by induction that for \( n \geq 2 \) listeners,

\[ H(\lambda_1 X_\lambda | E) = \sum_{\lambda=1}^{n} (-1)^{\lambda+1} \sum_{\Gamma \in \text{Pow}(Z_{1,n})_\lambda} H[(X_\lambda)_\Gamma | E] , \quad H \rightarrow S \] (6.13)

\[ H(\lambda_1 X_\lambda | E) = \sum_{\lambda=1}^{n} (-1)^{\lambda+1} \sum_{\Gamma \in \text{Pow}(Z_{1,n})_\lambda} H[(X_\lambda)_\Gamma | E] . \quad H \rightarrow S \] (6.14)

For the quantum case, a simple consequence of the above decomposition of c.m.i. is as follows. For 2 listeners,

\[ S_\rho(X_1, X_2, E) = 0 \quad \text{implies} \quad S_\rho(X_1 : X_2 | E) = S_\rho(X_1, X_2) . \] (6.15)

For 3 listeners,

\[ S_\rho(X_1, X_2, X_3, E) = 0 \quad \text{implies} \quad S_\rho(X_1 : X_2 : X_3 | E) = -S_\rho(X_1 : X_2 : X_3) . \] (6.16)
One can show that for \( n \geq 2 \) listeners,

\[
S_\rho(X_1, X_2, \ldots, X_n; E) = 0 \implies S_\rho(X_1 : X_2 : \ldots : X_n; E) = (-1)^n S_\rho(X_1 : X_2 : \ldots : X_n).
\]

(6.17)

(4) Sign of c.m.i.

The c.m.i. \( H(X_1 : X_2 : \ldots : X_n; E) \) is non-negative for \( n = 2 \), because in that case it equals the tanglement \( HT(X_1 : X_2; E) \). However, for more than 2 listeners, the c.m.i. may be positive or negative, as the following example shows.\[9\] A 3 listener c.m.i. will be positive if one of the 3 listeners drops out so that there are effectively 2 listeners. Let us construct an example of a 3 listener c.m.i. that is negative. Assume the listeners are independent of the speaker \( E \) so that we can omit the conditioning on \( E \). Eq. (6.9) can be rewritten as

\[
H(X_1 : X_2 : X_3) = Pos + Neg,
\]

where

\[
Pos = H(X_1) - H(X_1 | X_2) = H(X_1 : X_2),
\]

and

\[
Neg = -\{H(X_1 | X_3) - H(X_1 | X_2, X_3)\} = -H[(X_1 : X_2) | X_3].
\]

As their names suggest, \( Pos \) and \( Neg \) are positive and negative, respectively. The idea is to make \( X_1 \) and \( X_2 \) independent so that \( Pos \) vanishes. The following probability distribution fits that bill:

\[
P(X_1, X_2, X_3) = \frac{1}{4} [\delta^{X_1}_0 \delta^{X_2}_0 + \delta^{X_1}_1 \delta^{X_2}_1],
\]

where \( X_1, X_2, X_3 \in \text{Bool}, \bar{0} = 1 \) and \( \bar{1} = 0 \). This distribution gives \( Pos = 0 \) and \( Neg = -1 \).

(5) Duality between tanglement and c.m.i.

We wish to express tanglements in terms of c.m.i.’s and vice versa. For 2 listeners, one finds

\[
\tau_{1:2} = \mu_{1:2}, \quad [H \rightarrow S]
\]

(6.22)

For 3 listeners, one finds
\[ \tau_{1:2:3} = \mu_{1:2} + \mu_{1:3} + \mu_{2:3} - \mu_{1:2:3}, \quad H \rightarrow S \] (6.23)

\[ \mu_{1:2:3} = \tau_{1:2} + \tau_{1:3} + \tau_{2:3} - \tau_{1:2:3}. \quad H \rightarrow S \] (6.24)

For 4 listeners, one finds

\[ \tau(1:2:3:4) = \sum_{\Gamma \in \text{Pow}(Z_{1,4})_2} \mu(j \in \Gamma) - \sum_{\Gamma \in \text{Pow}(Z_{1,4})_3} \mu(j \in \Gamma) + \mu(1:2:3:4), \quad H \rightarrow S \] (6.25)

\[ \mu(1:2:3:4) = \sum_{\Gamma \in \text{Pow}(Z_{1,4})_2} \tau(j \in \Gamma) - \sum_{\Gamma \in \text{Pow}(Z_{1,4})_3} \tau(j \in \Gamma) + \tau(1:2:3:4). \quad H \rightarrow S \] (6.26)

One can show by induction that for \( n \geq 2 \) listeners

\[ \tau(1:2:\ldots:n) = \sum_{\lambda=2}^{n} (-1)^{\lambda} \sum_{\Gamma \in \text{Pow}(Z_{1,n})_\lambda} \mu(j \in \Gamma) \quad H \rightarrow S \] (6.27)

\[ \mu(1:2:\ldots:n) = \sum_{\lambda=2}^{n} (-1)^{\lambda} \sum_{\Gamma \in \text{Pow}(Z_{1,n})_\lambda} \tau(j \in \Gamma) \quad H \rightarrow S \] (6.28)

An interesting aspect of Eqs. (6.27) and Eqs. (6.28) is that they transform into each other when one exchanges the symbols \( \tau \) and \( \mu \). Therefore, we will call such equations *duality equations*, and say that they describe a *duality* between tanglement and c.m.i.

(6) Merging two listeners

It is easy to check that for \( n \geq 2 \),

\[ HT(X_1 : X_2 : \ldots : X_n : X_{n+1} | E) - HT(X_1 : X_2 : \ldots : X_{n-1} : (X_n, X_{n+1}) | E) = HT(X_n : X_{n+1} | E) \geq 0 \quad H \rightarrow S \] (6.29)

In \( \tau \) notation,

\[ \tau[1:2:\ldots:n-1 : (n,n+1)] \leq \tau[1:2:\ldots:n : n+1] \quad H \rightarrow S \] (6.30)

For example,

\[ \tau_{1:2:3} \leq \tau_{1:2:3} \quad H \rightarrow S \] (6.31)

Thus, “merging” two listeners decreases tanglement. Since tanglement is non-negative, if the right-hand side of this inequality is zero, so is the left-hand side.

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(7) Pruning or removing a listener

It is easy to check that for \( n \geq 2 \),

\[
HT(X_1 : X_2 : \ldots : X_{n-1} : (X_n, X_{n+1}) | E) - HT(X_1 : X_2 : \ldots : X_n | E) = HT[(X_1, X_2, \ldots, X_{n-1}) : X_{n+1} | X_n, E] \geq 0
\]

(6.32)

In \( \tau \) notation,

\[
\tau[1 : 2 : \ldots : n - 1 : n] \leq \tau[1 : 2 : \ldots : n - 1 : (n, n + 1)]
\]

(6.33)

For example,

\[
\tau_{1:2} \leq \tau_{1:(2,3)}
\]

(6.34)

Thus, “pruning” a listener (i.e., removing some but not all of its nodes) decreases tanglement. Since tanglement is non-negative, if the right-hand side of this inequality is zero, so is the left-hand side.

And what happens if we remove all the nodes of a listener? It is easy to check that for \( n \geq 2 \),

\[
HT(X_1 : X_2 : \ldots : X_n : X_{n+1} | E) - HT(X_1 : X_2 : \ldots : X_n | E) = HT[(X_1, X_2, \ldots, X_n) : X_{n+1} | E] \geq 0
\]

(6.35)

In the \( \tau \) notation,

\[
\tau(1 : 2 : \ldots : n) \leq \tau(1 : 2 : \ldots : n : n + 1)
\]

(6.36)

For example,

\[
\tau_{1:2} \leq \tau_{1:2:3}
\]

(6.37)

Thus, completely “removing” a listener also decreases tanglement. Since tanglement is non-negative, if the right-hand side of this inequality is zero, so is the left-hand side.

Note that if \( \tau_{1:2:\ldots:n} = 0 \) for some \( n \), then \( \mu_{1:2:\ldots:n} = 0 \). Indeed, by the duality equations, \( \mu_{1:2:\ldots:n} \) can be expressed as a sum of \( \pm \tau \)'s obtained from \( \tau_{1:2:\ldots:n} \) by removing some of its listeners. But all such \( \tau \) must be zero because \( \tau_{1:2:\ldots:n} = 0 \) and removing listeners decreases tanglement.

(8) Decomposing compound listeners of tanglement and c.m.i.
It is easy to check that
\[
\tau_{1:2:3:4:5:6} = \tau_{1:2:3:4} - \tau_{2:3:4:5:6} \quad (6.40)
\]

Note that compound listeners in the left-hand side are “split” in the right-hand side.

More generally, suppose that \( E, X_1, X_2, \ldots, X_n \) for some \( n \geq 2 \) are non-empty disjoint node collections of a Bayesian net, and \( \Gamma_\alpha \) for \( \alpha \in Z_1, m \) are non-empty disjoint subsets of \( Z_{1:n} \). Then

\[
HT_{\alpha=1} (X_\alpha | E) = HT_{j \in \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m} (X_j | E) - \sum_{\alpha=1}^{m} HT_{j \in \Gamma_\alpha} (X_j | E) \quad (6.41)
\]

where we define \( HT_{j \in \Gamma_\alpha} (X_j | E) = 0 \) if \( \Gamma_\alpha \) has only one element. In \( \tau \) notation,

\[
\tau(\Gamma_1 : \Gamma_2 : \cdots : \Gamma_m) = \tau(\Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m) - \sum_{\alpha=1}^{m} \tau(\Gamma_\alpha) \quad (6.42)
\]

where we define \( \tau(\Gamma_\alpha) = 0 \) if \( \Gamma_\alpha \) has only one element. Thus, any tanglement which has compound listeners can be expressed as a sum of \( \pm \) tanglements whose listeners are smaller(i.e., have fewer nodes).

Note that given a c.m.i. with compound listeners, one can: (1) use the duality equations to express the c.m.i. as a sum of \( \pm \) tanglements; (2) use the results of this section to express the tanglements obtained in step 1 as a sum of \( \pm \) tanglements which have smaller listeners; (3) use the duality equations to express the tanglements obtained in step 2 as sum of \( \pm \) c.m.i.’s. For example,

\[
\mu_{1:2:3} = \tau_{1:2:3} = \tau_{1:2:3} - \tau_{2:3:4} = \mu_{1:2} + \mu_{1:3} - \mu_{1:2:3} \quad (6.43)
\]

Thus, any c.m.i. which has compound listeners can expressed as a sum of \( \pm \) c.m.i.’s whose listeners are smaller.

Another way of decomposing the compound listeners of a c.m.i. is by using the following “chain rule”:

\[
H[X_1 : (X_2, X_3, \ldots, X_n) | E] = \sum_{\lambda=2}^{n} H[X_1 : X_{\lambda-1} | (X_{\lambda+1}, \ldots, X_n, E)] \quad (6.44)
\]

For example,
\[ H[X_1 : (X_2, X_3, X_4) | E] = +H[X_1 : X_2, X_3, X_4, E] + H[X_1 : X_4 | X_3, X_4, E] + H[X_1 : X_4 | E]. \]  

This rule is also valid for more than 2 listeners. For example, it can be used to decompose the listeners of \( \mu((1, 2) : (3, 4) : (5, 6, 7)) \).

(9) Conditionally separable states

Suppose

\[ P(X_1, X_2, \ldots, X_n, E) = P(X_1 | E) P(X_2 | E) \ldots P(X_n | E) P(E) \]  

for all values of \( X_1, X_2, \ldots, X_n, E \). Then \( HT(X_1 : X_2 : \ldots : X_n | E) = 0 \). If the speaker \( E \) is a single node \( e \), and for each \( \lambda \), the listener \( X_\lambda \) is a single node \( x_\lambda \), then Eq. (6.46) is satisfied by the CB net in Fig. 6.

Figure 6: Net with one speaker and \( n \) listener nodes.

So far we’ve only considered the classical case. The analogous result in the quantum case is as follows. Suppose that \( \rho \) is defined by

\[ \rho = \sum_E w_E |E\rangle \langle E| \rho_E^{(1)} \rho_E^{(2)} \ldots \rho_E^{(n)} , \]  

where the \( w_E \)'s are non-negative numbers that add up to one, where \( \{|E\rangle \forall E\} \) is an orthonormal basis for \( \mathcal{H}_E \), and where for all \( \lambda \in Z_{1,n} \) and for all \( E \), \( \rho_E^{(\lambda)} \) acts on \( \mathcal{H}_{X_\lambda} \). The Hilbert spaces \( \mathcal{H}_{X_\lambda} \) for all \( \lambda \) and \( \mathcal{H}_E \) are different spaces. Then \( ST_\rho(X_1 : X_2 : \ldots : X_n | E) = 0 \). If the speaker \( E \) is a single node \( a \), and for each \( \lambda \), the listener \( X_\lambda \) is a single node \( x_\lambda \), then the \( \rho \) of Eq. (6.47) can be implemented by a QB net with a graph like the one in Fig. 4, but such that \( a \) has \( n \) branches instead of just 2.

We showed previously that \( t_{1,2,\ldots,n} = 0 \) implies \( \mu_{1,2,\ldots,n} = 0 \). The converse statement is not true (for \( n \) larger than 2). Next we will give an example of a situation in which the c.m.i. is always zero but the tanglement may be non-zero.

Suppose \( n \geq 2 \) and \( \Gamma_1, \Gamma_2 \) are non-empty disjoint sets such that \( \Gamma_1 \cup \Gamma_2 = Z_{1,n} \). In the classical case, assume
for all values of $X_1, X_2, \ldots, X_n, E$. In the quantum case, assume
\[
\rho = \sum_E w_E |E\rangle \langle E| \rho_E^{(1)} \rho_E^{(2)},
\]
where the $w_E$'s are non-negative numbers that add up to one, and where for $\lambda \in \mathbb{Z}_{1,2}$ and for all $E$, $\rho_E^{(\lambda)}$ acts on $\mathcal{H}(X)_{\Gamma_{1,2}}$. Then $\mu_{1:2:3:n} = 0$. We won’t give a completely general proof of this theorem. We will only prove it for $n = 4$.

One of the duality equations is:
\[
\mu_{1:2:3:4} = A - B + C , \quad H \rightarrow S
\] (6.50)
where
\[
A = \tau_{1:2} + \tau_{1:3} + \tau_{1:4} + \tau_{2:3} + \tau_{2:4} + \tau_{3:4} , \quad H \rightarrow S
\] (6.51)
\[
B = \tau_{1:2:3} + \tau_{1:2:4} + \tau_{1:3:4} + \tau_{2:3:4} , \quad H \rightarrow S
\] (6.52)
\[
C = \tau_{1:2:3:4} , \quad H \rightarrow S
\] (6.53)
First suppose that $\Gamma_1 = \{1, 2\}$ and $\Gamma_2 = \{3, 4\}$. Then $\tau(\Gamma_1 : \Gamma_2) = 0$. If $\Gamma_1'$ (ditto, $\Gamma_2'$) is a non-empty subset of $\Gamma_1$ (ditto, $\Gamma_2$), then, because removing listeners decreases tanglement, $\tau(\Gamma_1' : \Gamma_2') = 0$. Using Eq.(6.42) to decompose the compound listeners of $\tau(\Gamma_1' : \Gamma_2')$, one gets
\[
\tau(\{j \in \Gamma_1' \cup \Gamma_2' \}) = \tau(\{j \in \Gamma_1' \}) + \tau(\{j \in \Gamma_2' \}) . \quad H \rightarrow S
\] (6.54)
Using Eq.(6.54), one gets
\[
A = \tau_{1:2} + \tau_{3:4} , \quad H \rightarrow S
\] (6.55)
\[
B = 2(\tau_{1:2} + \tau_{3:4}) , \quad H \rightarrow S
\] (6.56)
\[
C = \tau_{1:2:3:4} , \quad H \rightarrow S
\] (6.57)
so
\[
\mu_{1:2:3:4} = -\tau_{1:2} - \tau_{3:4} + \tau_{1:2:3:4} = 0 . \quad H \rightarrow S
\] (6.58)
Next suppose that $\Gamma_1 = \{1\}$ and $\Gamma_2 = \{2, 3, 4\}$. Using Eq.(6.54), one gets
\[
A = \tau_{2:3} + \tau_{2:4} + \tau_{3:4} , \quad H \rightarrow S
\] (6.59)
\[ B = \tau_{2:3} + \tau_{2:4} + \tau_{3:4} + \tau_{2:3:4}, \quad \text{H} \rightarrow \text{S} \]  
\[ C = \tau_{2:3:4}, \quad \text{H} \rightarrow \text{S} \]  
so
\[ \mu_{1:2:3:4} = 0, \quad \text{H} \rightarrow \text{S} \]  

(10) A posteriori local unitary transformations

In Section 4, we distinguished between a priori and a posteriori local unitary transformations, and we maximized \( ST \) over all a priori transformations. Next we will show that \( ST \) is in fact invariant under a posteriori local unitary transformation. For definiteness, we will calculate \( ST \) for a pure state and 2 single-node listeners, but analogous conclusions hold for a mixed state and \( n \geq 2 \) either single-node or compound listeners.

![Net with one speaker node and 2 branches, each branch with 2 nodes.](image)

Consider the QB net of Fig.7, where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \( e \) | \( e = (e_1, e_2) \) | \( \psi(e) \) | \( \sum_e |\psi(e)|^2 = 1 \) |
| \( x \) | \( x \) | \( \delta(x, e_1) \) | |
| \( y \) | \( y \) | \( \delta(y, e_2) \) | |
| \( a \) | \( a \) | \( U_{ax} \) | \( \sum_a U_{ax}^* U_{ax'} = \delta_{x'}^x \) |
| \( b \) | \( b \) | \( U_{by} \) | \( \sum_b U_{by}^* U_{by'} = \delta_{y'}^y \) |

Let \( \mathcal{N}^Q \) be the QB net which contains all the nodes shown in Fig.7. Let \( \mathcal{N}_0^Q \) be the sub-net which contains only nodes \( e, x \) and \( y \).

The meta density matrix \( \mu_0 \) of \( \mathcal{N}_0^Q \) is
\[ \mu_0 = |\psi_{meta}^0 \rangle \langle \psi_{meta}^0 | , \] (6.63)

where

\[ |\psi_{meta}^0 \rangle = \sum_{ri} \psi(x, y)|e = (x, y), x, y \rangle . \] (6.64)

The meta density matrix \( \mu \) of \( \mathcal{N}^Q \) is

\[ \mu = |\psi_{meta} \rangle \langle \psi_{meta} | , \] (6.65)

where

\[ |\psi_{meta} \rangle = \sum_{ri} U_{ax} V_{by} \psi(x, y)|e = (x, y), x, y, a, b \rangle . \] (6.66)

This last equation can be rewritten as

\[ |\psi_{meta} \rangle = \sum_{ri} \psi(x, y)|e = (x, y), x, y \rangle |\phi_a(x) \rangle |\phi_b(y) \rangle , \] (6.67)

where

\[ |\phi_a(x) \rangle = \sum_a U_{ax} |a \rangle , \] \[ |\phi_b(y) \rangle = \sum_b V_{by} |b \rangle . \] (6.68)

The \( |\phi_a(x) \rangle \)'s (ditto, \( |\phi_b(y) \rangle \)'s ) are an orthonormal basis in \( \mathcal{H}_a \) (ditto, \( \mathcal{H}_b \)) labelled by the indices \( x \) (ditto, \( y \)).

Define \( \rho \) by

\[ \rho = \Sigma_{x, y} (\mu) = \sum_{ri} \psi(x, y)\psi^*(x', y')|e = (x, y), \phi_a(x), \phi_b(y) \rangle \langle e = (x', y'), \phi_a(x'), \phi_b(y') | . \] (6.69)

The only difference between \( \rho \) and \( \mu_0 \) is that the \( \phi_a(x) \) and \( \phi_b(y) \) indices in \( \rho \) are replaced by \( x \) and \( y \) in \( \mu_0 \). Thus,

\[ S_{\rho}(a : b | e) = S_{\mu_0}(x : y | e) . \] (6.70)

In other words, \( ST \) for net \( \mathcal{N}^Q \), density matrix \( \rho \) and listeners \( a \) and \( b \) equals \( ST \) for sub-net \( \mathcal{N}_0^Q \), density matrix \( \mu_0 \) and listeners \( x \) and \( y \). Note that in the definition Eq.(6.69) of \( \rho \), we e-summed \( \mu \) over \( x \) and \( y \). Consider a density matrix \( \sigma \) defined by trace-ing instead of e-summing over \( x \) and \( y \):

\[ \sigma = \text{tr}_{x,y} (\mu) = \sum_{ri} \psi(x, y)\psi^*(x, y)|e = (x, y), \phi_a(x), \phi_b(y) \rangle \langle e = (x, y), \phi_a(x), \phi_b(y) | . \] (6.71)

It is easy to show that
Thus, e-summing over \( x \) and \( y \) (which corresponds to not measuring those nodes) gives the same \( ST \) as if the local transformations at nodes \( a, b \) had not occurred. On the other hand, trace-ing over \( x \) and \( y \) (which corresponds to measuring those nodes in a particular way) gives zero \( ST \), just as in the classical case.

(11) Conditional Data Processing Inequalities

An introduction to Data Processing (DP) Inequalities for CB and QB nets may be found in Ref.[5]. Here, we will prove a new version of these inequalities which we call Conditional DP Inequalities. The Conditional DP Inequalities are conditioned on a speaker. Thus, they are closely linked to the phenomenon of tanglement. Consider the net of Fig.7. What we will show is that

\[
H(a : b|e) \leq H(x : y|e) . \tag{6.73}
\]

In the quantum case, we’ve shown in the previous section entitled “A posteriori local unitary transformations” that if nodes \( a \) and \( b \) correspond to unitary transformations and nodes \( x \) and \( y \) to delta functions, then equality is attained in inequality Eq.(6.73). No such assumptions about the nature of the transition matrices of the nodes will be made in this section. Our assumptions are only that the QB net has a particular topology, that of Fig.7.

Clearly, the Conditional DP Inequalities of this section can be greatly generalized in the same way that Ref.[10] generalizes DP Inequalities from a simple Markov chain to arbitrary CB or QB nets. In this section, we will discuss only the simplest case of the Conditional DP Inequalities. More general cases will be discussed in a future paper dedicated exclusively to this subject.

Eq.(6.73) has a simple interpretation, as all DP inequalities do. It says that the conditional information transmission between \( x \) and \( y \) is larger than that between \( a \) and \( b \) because the first pair of nodes is “closer”. Alternatively, one can say that the probabilistic dependency of \( x \) on \( y \) is larger than that between \( a \) and \( b \) because the first pair of nodes is “closer”.

First note that the graph of Fig.7 satisfies

\[
H(a|e, y, b) = H(a|e, y) . \tag{6.74}
\]

In the classical case, this follows because \( P(a|e, y, b) = P(a|e, y) \). By virtue of Eq.(6.74) and strong subadditivity,

\[
H(a|e, y) = H(a|e, y, b) \leq H(a|e, b) . \tag{6.75}
\]

Subtracting \( H(a|e) \) from each term of the last equation and multiplying the resulting equation by \(-1\) gives
\[ H(a : y|e) \geq H(a : b|e) \tag{6.76} \]

Now note that the graph of Fig. 7 satisfies
\[ H(y|x, a) = H(y|x, e) \tag{6.77} \]
In the classical case, this follows because \( P(y|x, a) = P(y|x, e) \). By virtue of Eq. (6.77) and strong subadditivity,
\[ H(y|x, e) = H(y|x, a) \leq H(y|a) \tag{6.78} \]
Subtracting \( H(y|e) \) from each term of the last equation and multiplying the resulting equation by \(-1\) gives
\[ H(y : x|e) \geq H(y : a|e) \tag{6.79} \]
Combining Eqs. (6.76) and (6.79) gives
\[ H(a : b|e) \leq H(a : y|e) \leq H(x : y|e) \tag{6.80} \]
QED.

![Diagram](image)

Figure 8: Net with one speaker node and \( n \) branches, each branch with 2 nodes.

Eq. (6.73) can be easily generalized to \( n \geq 2 \) listeners. Consider the graph of Fig. 8. Next we will show that for this graph,
\[ HT(a_1 : a_2 : \ldots : a_n|e) \leq HT(x_1 : x_2 : \ldots : x_n|e) \tag{6.81} \]
The proof is by induction on \( n \geq 2 \). Eq. (6.81) has been proven for \( n = 2 \). If it is true for all \( n \in Z_{2,n_0} \), then is must be true for \( n = n_0 + 1 \). Here is why. By virtue of the induction hypothesis, the following two inequalities must be true:
\[ HT[(a_1, a_2, \ldots, a_{n_0}) : a_{n_0 + 1}|e] \leq HT[(x_1, x_2, \ldots, x_{n_0}) : x_{n_0 + 1}|e], \tag{6.82} \]
\[ HT(a_1 : a_2 : \cdots : a_{n+1} \mid \varepsilon) \leq HT(x_1 : x_2 : \cdots : x_{n+1} \mid \varepsilon) . \]  

The sum of the left-hand sides (ditto, right-hand sides) of these two inequalities equals \( HT(a_1 : a_2 : \cdots : a_{n+1} \mid \varepsilon) \) (ditto, \( HT(x_1 : x_2 : \cdots : x_{n+1} \mid \varepsilon) \)) \( \text{H} \rightarrow \text{S} \). QED

**A  Proof that** \( E_F(|\psi_{xy}|) \leq E_F(\psi_{xy}) \)

We will first prove this inequality for the case that \( S_x = S_y = \text{Bool} \). Define the function \( p_0(t) \) for \( t \in [0,1] \) by

\[ p_0(t) = 1 + \sqrt{1 - t^2} . \]  

(A.1)

From Eqs.(2.9) and (2.12),

\[ E_F(\psi_{xy}) = h(p_0(t)) , \]  

(A.2)

where

\[ t = 4|\psi_{00}\psi_{11} - \psi_{01}\psi_{10}|^2 . \]  

(A.3)

Let

\[ t' = 4(|\psi_{00}\psi_{11} - |\psi_{10}\psi_{10}||^2 . \]  

(A.4)

Note that

\[ E_F(|\psi_{xy}|) = h(p_0(t')) . \]  

(A.5)

By the triangle inequality,

\[ t' \leq t . \]  

(A.6)

From Fig.2, \( h(p_0(t)) \) is a monotonically increasing function of \( t \). Thus

\[ E_F(\psi_{xy}) = h(p_0(t')) \leq h(p_0(t)) = E_F(\psi_{xy}) . \]  

(A.7)

Now consider the case of arbitrary \( N_x, N_y \) such that \( N_x \leq N_y \). Recall

\[ E_F(\psi_{xy}) = S(\rho) , \]  

(A.8)

where

\[ \rho = \psi\psi^\dagger . \]  

(A.9)

For all \( x, y \), define \( \theta_{xy} \) to be the phase of \( \psi_{xy} \). Then

27
\[
\rho_{xx'} = \sum_y \psi_{x'y} \psi_{x'y}^* = \sum_y e^{i(\theta_{x'y} - \theta_{x'y})} |\psi_{x'y}\psi_{x'y}^*|.
\]  
(A.10)

Suppose we vary the angles \(\theta_{xy}\). Then

\[
\delta S(\rho) = -\delta \text{tr} \left[ \rho \ln \rho \right] = -\text{tr} \left[ \frac{\delta \rho}{\ln 2} \ln (\rho + 1) \right],
\]  
(A.11)

where

\[
\delta \rho_{xx'} = \sum_y i(\delta \theta_{xy} - \delta \theta_{x'y}) \psi_{xy} \psi_{x'y}^*.
\]  
(A.12)

When \(\theta_{xy} = 0\) for all \(x\) and \(y\), \(\delta \rho_{xx'}\) is antisymmetric and \(\rho_{xx'}\) is symmetric under the exchange of \(x\) and \(x'\). If \(A\) and \(S\) are, respectively, an antisymmetric and a symmetric \(N \times N\) matrix, then \(\text{tr}(A) = \text{tr}(AS) = 0\). Thus, \(\text{tr}(\delta \rho) = \text{tr}(\delta \rho \ln \rho) = 0\). Thus, \(\delta S(\rho) = 0\) when \(\theta_{xy} = 0\) for all \(x\) and \(y\). I don’t know how to show for general values of \(N_x\) and \(N_y\) that this extremum of \(S(\rho)\) is a global minimum.

**B  Proof that** \(H(x : y) \leq EF(\sqrt{P_{xy}})\)

In this appendix, we will prove an inequality which gives an upper bound for the classical mutual information \(H(x : y)\). From \(H(x : y) = H(x) - H(x|y)\) and \(H(x|y) \geq 0\), it follows that

\[
H(x : y) \leq \min \{H(x), H(y)\}.
\]  
(B.1)

What we seek here is a tighter upper bound for \(H(x : y)\).

Suppose \(x\) (ditto, \(y\)) is a random variable that can assume values in a set \(S_x\) (ditto, \(S_y\)) which contains \(N_x\) (ditto, \(N_y\)) elements. Let \(P_{xy}\) be the joint probability distribution of \(x\) and \(y\). Let \(P_{x-} = \sum_y P_{xy}\) and \(P_{-y} = \sum_x P_{xy}\). Without loss of generality, we will assume that \(N_x \leq N_y\).

Define \(\Psi\) to be the rectangular matrix with entries

\[
\Psi_{xy} = \sqrt{P_{xy}}.
\]  
(B.2)

Note that

\[
\text{tr}(\Psi \Psi^T) = \sum_{x,y} \Psi_{xy}^2 = \sum_{x,y} P_{xy} = 1.
\]  
(B.3)

Let

\[
\tilde{\Psi} = U \Psi V^T,
\]  
(B.4)

where \(U\) and \(V\) are (real) orthogonal matrices. Define
\[ \tilde{P}_{xy} = \tilde{\Psi}_{xy}^2. \] (B.5)

Then
\[ \sum_{x,y} \tilde{P}_{xy} = \text{tr}(\tilde{\Psi}\tilde{\Psi}^T) = \text{tr}(\Psi\Psi^T) = 1. \] (B.6)

Define \( \eta \) by
\[ \eta = \sum_{x,y} \tilde{P}_{xy} \ln \frac{\tilde{P}_{xy}}{\tilde{P}_x - \tilde{P}_y}. \] (B.7)

Note that
\[ H(x:y) = \frac{\eta}{\ln 2} \bigg|_{U=V=1}, \] (B.8)
where the right-hand side is evaluated at \( U = V = 1 \). Our goal is to show that: (1) \( \eta \) has a global maximum when it varies over the spaces of all orthogonal \( N_x \times N_x \) matrices \( U \) and all orthogonal \( N_y \times N_y \) matrices \( V \); (2) the maximum occurs when \( \tilde{U} \) and \( \tilde{V} \) make \( \tilde{\Psi} \) diagonal. (Such a \( \tilde{U} \) and \( \tilde{V} \) exist by the Singular Value Decomposition Theorem). When \( \tilde{\Psi} \) is diagonal,
\[ \frac{\eta}{\ln 2} = \sum_x \tilde{P}_{xx} \log \frac{1}{\tilde{P}_{xx}} = E_F(\tilde{\Psi}_{xy}) = E_F(\Psi_{xy}) = E_F(\sqrt{P_{xy}}). \] (B.9)

Therefore, if \( \eta \) has a global maximum when \( \tilde{\Psi} \) is diagonal, then
\[ H(x:y) \leq E_F(\sqrt{P_{xy}}). \] (B.10)

Suppose we vary each \( \tilde{P}_{xy} \) by \( \delta \tilde{P}_{xy} \) in such a way that
\[ \sum_{x,y} \delta \tilde{P}_{xy} = 0. \] (B.11)
(And therefore also \( \sum_x \delta \tilde{P}_{x-} = \sum_y \delta \tilde{P}_{-y} = 0. \)) Then
\[ \delta \eta = \sum_{x,y} (\delta \tilde{P}_{xy}) \ln \left( \frac{\tilde{P}_{xy}}{\tilde{P}_x - \tilde{P}_y} \right) + \text{nil}, \] (B.12)
where
\[ \text{nil} = \sum_{x,y} \left( \delta \tilde{P}_{xy} - \frac{\tilde{P}_{xy}}{\tilde{P}_x - \tilde{P}_y} \delta \tilde{P}_{x-} - \frac{\tilde{P}_{xy}}{\tilde{P}_y - \tilde{P}_-} \delta \tilde{P}_{-y} \right) \] (B.13)
Because of Eq. (B.11), \( \text{nil} = 0 \).

\( \tilde{U} \) and \( \tilde{V} \) are orthogonal and we will vary them so that \( \tilde{U} + \delta \tilde{U} \) and \( \tilde{V} + \delta \tilde{V} \) are also orthogonal. Thus, \( \sum_{xy} (\tilde{P}_{xy} + \delta \tilde{P}_{xy}) = 1 \). Thus, Eq. (B.11) is satisfied.
For $N_x = N_y = 2$, $U$ and $V$ can be parameterized by expressing them as

$$U = \begin{bmatrix} c_1 & s_1 \\ -s_1 & c_1 \end{bmatrix}, \quad V = \begin{bmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{bmatrix},$$

where $c_j = \cos \theta_j$, $s_j = \sin \theta_j$ for $j = 1, 2$. Then we can vary $U$ and $V$ by varying the angles $\theta_1, \theta_2$. For general $N_x$ and $N_y$, we can express $U$ and $V$ as $U = e^\alpha$ and $V = e^\beta$, where $\alpha$ and $\beta$ are antisymmetric matrices. Then we can vary $U$ and $V$ by varying the components of $\alpha$ and $\beta$ that lie above their main diagonal.

One gets

$$\delta \tilde{P}_{xy} = 2 \tilde{\Psi}_{xy} \delta \tilde{\Psi}_{xy},$$

and

$$\delta \tilde{\Psi} = (\delta U) \Psi V^T + U \Psi (\delta V^T) = A \tilde{\Psi} + \tilde{\Psi} B,$$

where

$$A = (\delta U) U^T, \quad B = V \delta V^T.$$

Because $U U^T = 1$, $(\delta U) U^T + U \delta U^T = 0$, which can be expressed in terms of $A$ as $A = -A^T$, Thus, $A$ must be antisymmetric. $B$ must be antisymmetric too.

Next we will show that if $U$ and $V$ are such that $\tilde{\Psi}$ is diagonal, then $\delta \tilde{P}_{xy} = 0$ for all $x$ and $y$, and therefore, by Eq. (B.12), $\delta \eta = 0$. Consider some $x, y$ such that $x \neq y$; for example, $x = 0, y = 1$. Since $\tilde{\Psi}_{01} = 0$, Eq. (B.15) implies $\delta \tilde{P}_{01} = 0$. Consider some $x, y$ such that $x = y$; for example, $x = y = 0$. $\sum_a A_{0a} \tilde{\Psi}_{a0} = 0$ because when $a = 0$, $A_{00} = 0$, and when $a \neq 0$, $\tilde{\Psi}_{a0} = 0$. Likewise, $\sum_b \tilde{\Psi}_{b0} B_{b0} = 0$. Thus, by Eq. (B.16), $\delta \tilde{\Psi}_{00} = 0$. Since $\delta \tilde{\Psi}_{00} = 0$, Eq. (B.15) implies $\delta \tilde{P}_{00} = 0$.

So far we have shown that $\delta \eta = 0$ when $\tilde{\Psi}$ is diagonal. It remains for us to show that this extremum is a global maximum. I don’t know how to show this. However, my Monte Carlo tests support this claim. Furthermore, the following argument shows that the extremum is at least a local maximum. One has

$$\delta^2 \eta = \sum_{x,y} (\delta^2 \tilde{P}_{xy}) \ln \left( \frac{\tilde{P}_{xy}}{P_{x\bar{y}} - P_{y\bar{x}}} \right) + \text{nil}' ,$$

where

$$\text{nil}' = \sum_{x,y} \frac{(\delta \tilde{P}_{xy})^2}{\tilde{P}_{xy}} - \sum_x \frac{(\delta \tilde{P}_{x\bar{y}})^2}{P_{x\bar{y}}} - \sum_y \frac{(\delta \tilde{P}_{y\bar{x}})^2}{P_{y\bar{x}}} .$$

If $\tilde{\Psi}$ is diagonal, then $\delta \tilde{P}_{xy} = 0$ for all $x$ and $y$ so $\text{nil}' = 0$. One has

$$\delta^2 \tilde{P}_{xy} = \delta [2 \tilde{\Psi}_{xy} \delta \tilde{\Psi}_{xy}] = 2 (\delta \tilde{\Psi}_{xy})^2 + 2 \tilde{\Psi}_{xy} \delta^2 \tilde{\Psi}_{xy} .$$
If $\tilde{\Psi}$ is diagonal, then $\delta^2 \tilde{P}_{xy} = 2(\delta \tilde{\Psi}_{xy})^2 \geq 0$ for any $x \neq y$. But $\tilde{P}_{xy} = 0$ for $x \neq y$ so $\delta^2 \eta \to -\infty$. Thus $\eta$ has a local maximum when $\Psi$ is diagonal. In fact, $\eta$ has a cusp there. The cusp is on the boundary of the region on which $\tilde{P}_{xy}$ is defined.

### C Entropy of Density Matrix with Repeated Index Pairs

Often in this paper we need to evaluate the entropy of a density matrix such as

$$R = \sum_{a,a'} R_{a,a'} |a = a, b = a\rangle \langle a = a', b = a'|,$$  \hfill (C.1)

where the nodes $a$ and $b$ have the same states ($S_a = S_b$). By an “index pair” of a matrix $M$ we mean the row and column indices of an entry of $M$. The index pair $(a, a')$ is repeated in $R$. Consider the smaller density matrix

$$\rho = \sum_{a,a'} R_{a,a'} |a\rangle \langle a'|.$$  \hfill (C.2)

Next we will show that $S(R) = S(\rho)$. Thus, for the purpose of evaluating its entropy, one can replace the density matrix $R$ by the smaller $\rho$. The proof consists of showing that $R$ and $\rho$ have the same non-zero eigenvalues. Indeed, suppose $|\phi\rangle \in \mathcal{H}_a$ is an eigenvector of $\rho$:

$$\rho |\phi\rangle = \lambda |\phi\rangle.$$  \hfill (C.3)

Then $|\Phi\rangle$ defined by

$$|\Phi\rangle = \sum_{a'} |a = a', b = a'\rangle \langle a = a'|\phi\rangle$$  \hfill (C.4)

is an eigenvector of $R$ with the same eigenvalue $\lambda$. Indeed,

$$R|\Phi\rangle = \sum_{a,a'} R_{a,a'} |a = a, b = a\rangle \langle a = a'|\phi\rangle = \lambda |\Phi\rangle.$$  \hfill (C.5)

Thus, the set of eigenvalues of $R$ contains the set of eigenvalues of $\rho$. From the matrix representation of $R$, it is clear that any eigenvalue of $R$ which is not an eigenvalue of $\rho$ must be zero.

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