Electromagnetic Radiation and Motion of Really Shaped Particle

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Abstract. Relativistically covariant form of equation of motion for real particle (neutral in charge) under the action of electromagnetic radiation is derived. Various formulations of the equation of motion in the proper frame of reference of the particle are used.

Main attention is devoted to the reformulation of the equation of motion in the general frame of reference, e.g., in the frame of reference of the source of electromagnetic radiation. This is the crucial form of equation of motion in applying it to motion of particles (cosmic dust, asteroids, ...) in the Universe if electromagnetic radiation acts on the particles.

General relativistic equation of motion is presented.

Key words: relativity theory, cosmic dust, asteroids

1. Introduction

Covariant form of equation of motion for a particle is required if one wants to be sure that the motion of the particle is correctly described. This is the crucial fact of relativity theory, firstly understood by Minkowski. Thus, if we want to derive equation of motion of a particle under the action of electromagnetic radiation, we have to bear in mind this crucial requirement of relativity theory. Bearing this requirement in mind, Klačka (2000a) has derived covariant form of equation of motion for really shaped dust particle under the action of electromagnetic radiation. However, even one year ago the general opinion...
was that its significance is negligible. Two conferences (US-European Celestial Mechanics Workshop, Poznan, Poland, July 2000; Light Scattering by Nonpherical Particles: halifax Contributions, Halifax, Canada, August/September 2000) have changed the opinion of several astronomers. Although the original papers Klačka (2000a, 2000b, 2000c) has not succeeded in journal publication, one consequence of Klačka (2000a) – derivation to the first order in $v/c$ (higher orders are neglected; $v$ is velocity of the particle, $c$ is the speed of light) – with application to real system in the Solar System is published (Klačka and Kocifaj 2001); accuracy to the first order in $v/c$ may be sufficient in many applications in practice.

Several authors have tried to derive equation of motion for really shaped dust particle under the action of electromagnetic radiation during the last several months. However, they have not succeeded, and, moreover, the paper Klačka (2000a) has not help them in correct understanding of the physics of the phenomenon. (According to their opinion publication in international journal is required, not only web-form.) This motivates to derive covariant form of equation of motion for particle under the action of electromagnetic radiation in the way which will stress the crucial steps. Thus, we will take equation of motion in the proper frame of reference as the initial point of our paper. We will rewrite it to covariant form, in this paper. We will use the results of Klačka (2000a, 2001). However, derivations will be more simple and a little generalized results will be obtained. Moreover, general relativistic equation of motion will be presented.

2. Generalized special Lorentz transformation

By the term “stationary frame of reference” we shall mean a frame of reference in which particle moves with a velocity vector $\mathbf{v} = v(t)$. The physical quantities measured in the stationary frame of reference will be denoted by unprimed symbols.

The term “stationary particle” will denote particle which does not move in a given inertial frame of reference. Primed quantities will denote quantities measured in the proper reference frame of the particle.

Our situation corresponds to the fact that we know equation of motion of the particle in its proper frame of reference. We want to derive equation of motion for the particle in the stationary frame of reference.

We have to use generalized special Lorentz transformation for the purpose of making transformation from proper frame of reference to stationary frame of reference.
If we have a four-vector $A^\mu = (A^0, A)$, where $A^0$ is its time component and $A$ is its spatial component, generalized special Lorentz transformation yields

$$A^0' = \gamma (A^0 - v \cdot A/c),$$

$$A' = A + [(\gamma - 1) v \cdot A/v^2 - \gamma A^0/c] v.$$  \hspace{1cm} (1)

The inverse generalized special Lorentz transformation is

$$A^0 = \gamma (A^0' + v \cdot A'/c),$$

$$A = A' + [(\gamma - 1) v \cdot A'/v^2 + \gamma A^0'/c] v.$$  \hspace{1cm} (2)

The $\gamma$ factor is given by the well-known relation

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$  \hspace{1cm} (3)

As for four-vectors we can immediately introduce four-momentum:

$$p^\mu \equiv (p^0, p) = (E/c, p).$$  \hspace{1cm} (4)

3. First derivation

3.1. Proper reference frame of the particle – stationary particle

As it was already stated, primed quantities will denote quantities measured in the proper reference frame of the particle.

Equation of motion of the particle in its proper frame of reference is taken in the form

$$\frac{d E'}{d \tau} = 0,$$

$$\frac{d p'}{d \tau} = \frac{1}{c} S' A' \{Q'_R S'_i + Q'_1 e'_1 + Q'_2 e'_2\},$$  \hspace{1cm} (5)

where $E'$ is particle’s energy, $p'$ its momentum, $\tau$ is proper time, $c$ is the speed of light, $S'$ is the flux density of radiation energy (energy flow through unit area perpendicular to the ray per unit time), $A'$ is geometrical cross-section of a sphere of volume equal to the volume of the particle, $Q'_R$, $Q'_1$ and $Q'_2$ represent effective factors of radiation pressure, unit vector $S'_i$ is directed along the path of the incident radiation (it is supposed that beam of photons propagate in parallel lines) and its orientation corresponds to the orientation of light propagation, the unit vectors $S'_i$, $e'_1$ and $e'_2$ used on the RHS of Eqs. (5) form an orthogonal basis.

Eqs. (5) describe equation of motion of the particle in the proper frame of reference due to its interaction with electromagnetic radiation. (It is supposed that the energy $E'$ of the particle is unchanged: the energy of the incoming radiation equals to the energy of the outgoing radiation, per unit time.)
3.2. Transformation of the required quantities

Our aim is to derive equation of motion for the particle in the stationary frame of reference. We will use the fact that we know this equation in the proper frame of reference – see Eqs. (5). We have to use generalized special Lorentz transformation for the purpose of making transformation from proper frame of reference to stationary frame of reference.

We have to rewrite Eqs. (5) into covariant form. We have to transform the corresponding primed quantities according to the generalized special Lorentz transformation, as it was described in the section 2.

As for the transformation of the LHS of Eqs. (5), there is no problem: Eqs. (2) and Eq. (4) yield

\[ \frac{d E}{d \tau} = \gamma v \frac{d \mathbf{p}'}{d \tau}, \]
\[ \frac{d \mathbf{p}}{d \tau} = \frac{d \mathbf{p}'}{d \tau} + (\gamma - 1) \left( v \cdot \frac{d \mathbf{p}'}{d \tau} \right) \frac{v}{v^2}. \]  

(6)

Using the fact that \( p^\mu = (h \nu, h \nu \mathbf{S}_i) \) for the incident photon, Lorentz transformation yields

\[ \nu' = \nu \, w, \]
\[ \mathbf{S}'_i = \frac{1}{w} \left\{ \mathbf{S}_i + \left[ (\gamma - 1) \, v \cdot \mathbf{S}_i/v^2 - \gamma/c \right] \, v \right\}, \]  

(7)

where abbreviation

\[ w \equiv \gamma (1 - v \cdot \mathbf{S}_i/c) \]  

(8)

is used.

As for the transformation of the flux density of radiation energy, we use derivation presented in Klačka (1992). We can write for monochromatic radiation

\[ S' = n' \, h \, \nu' \, c; \quad S = n \, h \, \nu \, c, \]  

(9)

where \( n \) and \( n' \) are concentrations of photons in corresponding reference frames. The continuity equation

\[ \partial_\mu \, j^{\mu} = 0, \quad j^{\mu} = (c \, n, c \, n \, \mathbf{S}_i), \]  

(10)

holds for the four-vector of the current density \( j^\mu \). Using Eqs. (1) one can easily obtain

\[ n' = w \, n. \]  

(11)

On the basis of Eqs. (7), (9) and (11) we finally obtain
\[ S' = w^2 S \].

How to find transformations for unit vectors \( e'_1 \) and \( e'_2 \)? The crucial point is what physics do these unit vectors describe. They (also, together with unit vector \( S'_i \)) describe the directions of propagation of the light after interaction with the particle. Thus, the abberation of light also exists for each of these unit vectors. Considerations analogous to those for vector \( S'_i \) immediately yield:

\[
e'_j = \frac{1}{w_j} \left\{ e_j + \left[ (\gamma - 1) v \cdot e_j / v^2 - \gamma / c \right] v \right\},
\]

\[
w_j = \gamma \left( 1 - v \cdot e_j / c \right), \quad j = 1, 2.
\]

On the basis of Kláčka (1992) we know that the quantity \( w^2 S A' \) is scalar.

3.3. Covariant form of equation of motion

Putting RHS of Eqs. (5) into Eqs. (6), and, using Eqs. (7), (12) and (13), one can obtain

\[
\frac{d E}{d \tau} = w^2 S A' \left\{ Q_R \left( c \frac{1}{w} - \gamma c \right) + \sum_{j=1}^{2} Q'_j \left( c \frac{1}{w_j} - \gamma c \right) \right\},
\]

\[
\frac{d p}{d \tau} = w^2 S A' \left\{ Q_R \left( c \frac{S_i}{w} - \gamma v \right) + \sum_{j=1}^{2} Q'_j \left( c \frac{e_j}{w_j} - \gamma v \right) \right\}.
\]

Comparisons of Eqs. (2), (7) and (13) yield that we have these three four-vectors:

\[
b^i_j = (1/w, S_i/w),
\]

\[
b^j_j = (1/w_j, e_j/w_j), \quad j = 1, 2.
\]

Considering the fact that \( p^\mu = (E / c, p) \) is four-momentum of the particle of mass \( m \)

\[
p^\mu = m w^\mu,
\]

four-vector of the world-velocity of the particle is

\[
u^\mu = (\gamma c, \gamma v)
\]

and that other four-vectors are defined by Eqs. (15), one easily obtains that Eqs. (14) may be rewritten in terms of four-vectors:

\[
\frac{d p^\mu}{d \tau} = w^2 S A' \left\{ Q_R \left( c b^\mu_i - u^\mu \right) + \sum_{j=1}^{2} Q'_j \left( c b^\mu_j - u^\mu \right) \right\}.
\]
Eq. (18) is covariant form of equation of motion for the particle moving in the field of electromagnetic radiation. Since we have not transformed the quantities $Q'_R$, $Q'_1$ and $Q'_2$, these quantities are scalars, invariants of the generalized special Lorentz transformation (mass $m$ of the particle is also scalar, invariant).

It can be easily verified that Eq. (18) yields $d m/d \tau = 0$.

4. Second derivation

4.1. Proper reference frame of the particle – stationary particle

Again, primed quantities will denote quantities measured in the proper reference frame of the particle.

Let the equation of motion of a particle in the electromagnetic radiation field is expressed in the form

$$\frac{dp'}{d\tau} = \frac{1}{c} S' \left( C' S'_i \right)$$
$$\frac{dE'}{d\tau} = 0 ;$$

(19)

$S'$ is the flux density of the radiation energy, $C'$ is pressure cross section $3 \times 3$ matrix, $S'_i$ is unit vector of the incident radiation, $\tau$ is proper time, $c$ is the speed of light.

We are interested in deriving equation of motion of the particle in the rest frame of the source: the particle moves with instantaneous velocity $v$ with respect to the source, the unit vector of the incident radiation is $S_i$ and other physical quantities measured in the rest frame of the source are also unprimed.

4.2. Reformulation of the initial equation of motion

We will be inspired by ideas presented in Klačka (2001) – the ideas will be a little generalized.

Let the components of the pressure cross section $C'$ $3 \times 3$ matrix are given in a basis of orthonormal vectors $e'_{b1}$, $e'_{b2}$, $e'_{b3}$. We may write, then

$$S'_i = \sum_{k=1}^{3} s'_k e'_{bk} .$$

(20)

Using Eq. (20) we have

$$C' S'_i = \sum_{k=1}^{3} \left( \sum_{l=1}^{3} C'_{kl} s'_l \right) e'_{bk} .$$

(21)

On the basis of Eq. (21) we can rewrite the first of Eqs. (19) to the form

$$\frac{dp'}{d\tau} = \frac{1}{c} S' \sum_{k=1}^{3} \left( e'^T_{bk} C' S'_i \right) e'_{bk} .$$

(22)
4.3. Covariant form of equation of motion

Comparison of Eq. (22) with Eq. (5) enables immediately write covariant form of equation of motion, on the basis of Eq. (18):

\[
\frac{dp^\mu}{d\tau} = \frac{w^2}{c^2} \sum_{j=1}^{3} \left( e'_{bj}^T C' S'_i \right) \left( c b^\mu_{bj} - u^\mu \right),
\]  

(23)

where \( p^\mu = m u^\mu, u^\mu = (\gamma c, \gamma v), w = \gamma (1 - v \cdot S_i / c), b^\mu_{bj} = (1 / w_{bj}, e_{bj} / w_{bj}), w_{bj} = \gamma (1 - v \cdot e_{bj} / c) \) (relations analogous to those given by Eqs. (13) hold for \( e_{bj} \) and \( e'_{bj}, j = 1 \) to 3).

4.4. Is everything correct?

Eq. (23) reduces to Eq. (7) in Klàcka (2001) for the case \( e'_{b1} = e'_1, e'_{b2} = e'_2, e'_{b3} = S'_i \).

These substitutions correspond to Eq. (18), also.

It is important to show that the final equation of motion does not depend on the choice of the set of orthonormal basis vectors. If this would not be true, then the results represented by Eqs. (5) and (18), or, by Eqs. (22) and (23), are physically incorrect.

Thus, let us consider two sets of orthonormal basis vectors: \( e'_{b1}, e'_{b2}, e'_{b3} \) and, \( e'_1, e'_2, e'_3 \) (we may take \( e'_3 = S'_i \), as an example).

First of all, we can immediately write

\[
C' S'_i = \sum_{k=1}^{3} \left( e'_{bk}^T C' S'_i \right) e'_{bk} = \sum_{k=1}^{3} \left( e'_{bk}^T C' S'_i \right) e'_k,
\]  

(24)

since the decomposition of a vector into a basis of orthonormal vectors is unique. This means that equation of motion in the proper frame of the particle does not depend on the basis vectors – this corresponds to correct physics.

Now, let us show that equation of motion given by Eq. (23) is also independent on the basis of orthonormal vectors. On the basis of definition of four-vectors

\[
b^\mu_{bj} = (1 / w_{bj}, e_{bj} / w_{bj}), w_{bj} = \gamma (1 - v \cdot e_{bj} / c),
\]

and generalized special Lorentz transformation represented by Eq. (2), we can write

\[
b^\mu_{bj} = 1 / w_{bj} = \gamma (1 + v \cdot e'_{bj} / c),
\]

\[
b_{bj} = e_{bj} / w_{bj} = e'_{bj} + \left[ (\gamma - 1) \frac{v \cdot e'_{bj}}{v^2} + \frac{1}{c} \right] v, \quad j = 1, 2, 3.
\]  

(25)

\[
\sum_{k=1}^{3} \left( e'_{bk}^T C' S'_i \right) e'_{bk} = \sum_{k=1}^{3} e'_k = C' S'_i
\]  

(24)
Let us calculate the important part of the RHS of Eq. (23), \( c b^\mu_{bj} - u^\mu \) – Eqs. (26) are used:

\[
\begin{align*}
    c b^0_{bj} - u^0 &\equiv c b^0_{bj} - \gamma c = \gamma (v \cdot e^\prime_{bj}), \\
    c b_{bj} - u &\equiv c b_{bj} - \gamma v = c e^\prime_{bj} + (\gamma - 1) \frac{c v}{v^2} (v \cdot e^\prime_{bj}), \quad j = 1, 2, 3. 
\end{align*}
\]  

(27)

Putting Eqs. (27) into Eqs. (23), we obtain

\[
\begin{align*}
    \frac{dE}{d\tau} &= \frac{w^2}{c} S \gamma v \cdot X, \\
    \frac{dp^\mu}{d\tau} &= \frac{w^2 S}{c^2} \left\{ c X + (\gamma - 1) \frac{c v}{v^2} (v \cdot X) \right\}, \\
    X &\equiv 3 \sum_{j=1}^3 \left( e^\prime_{bj} C' S'_i \right) e^\prime_{bj}.
\end{align*}
\]  

(28)

On the basis of Eq. (24) and Eqs. (28) we can conclude that equation of motion (Eq. (23)) does not depend on the chosen orthonormal basis of vectors. This is required by physics.

5. General relativity

The generally covariant equation of motion can be immediately written on the basis of Eq. (23):

\[
\frac{Dp^\mu}{d\tau} = \frac{w^2 S}{c^2} \sum_{j=1}^3 \left( e^\prime_{bj} C' S'_i \right) \left( c b^\mu_{bj} - u^\mu \right),
\]  

(29)

where the operator \( D / d\tau \) is the "total" covariant derivative in the general relativistic sense and includes gravitational effects.

6. Conclusion

We have derived equation of motion for real, arbitrarily shaped particle under the action of electromagnetic radiation. It is supposed that equation of motion is represented by Eqs. (5) in the proper frame of reference of the particle, or, that interaction between the particle and electromagnetic radiation is described by radiation pressure cross section \( C' \) \( 3 \times 3 \) matrix. The final covariant form is represented by Eq. (18), or, by Eq. (23), or, by Eq. (29).

Within the accuracy to the first order in \( v/c \), Eq. (18) yields

\[
\frac{d v}{d t} = \frac{S A'}{m c} \left\{ Q'_R \left[ (1 - v \cdot S_i/c) S_i - v/c \right] + \sum_{j=1}^2 Q'_j \left[ (1 - 2 v \cdot \hat{S}_i/c + v \cdot \hat{e}_j/c) \hat{e}_j - v/c \right] \right\}.
\]  

(30)
As for practical applications, the terms $v/c$ standing at $Q'$ are negligible for majority of real particles. (We want to stress that values of $Q'$—coefficients depend on particle’s orientation with respect to the incident radiation – their values are time dependent.)

Application to larger bodies, e. g., asteroids, may be found in Klačka (2000c).

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