SOLUTIONS OF INITIAL VALUE PROBLEMS FOR A PAIR OF LINEAR FIRST ORDER ORDINARY DIFFERENTIAL SYSTEMS WITH INTERFACE-Spatial CONDITIONS

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(Received April, 1995; Revised August, 1995)

ABSTRACT

Solutions of initial value problems associated with a pair of ordinary differential systems \((L_1, L_2)\) defined on two adjacent intervals \(I_1\) and \(I_2\) and satisfying certain interface-spatial conditions at the common end (interface) point are studied.

Key words: Interface-Spatially Mixed Conditions, Ordinary Differential Systems, Equations, Initial Value Problems, Linearly Independent Solutions, Fundamental Systems.

AMS (MOS) subject classifications: 34AXX, 34A10, 34A15.

1. Introduction

In the studies of acoustic waveguides in ocean [1], optical fiber transmission [4], soliton theory [3], etc., we encounter a new class of problems of the type

\[ L_1 f_1 = \sum_{k=0}^{n} P_k \frac{d f_1^k}{dt^k} = \theta f_1 \text{ defined on an interval } I_1 \]

and

\[ L_2 f_2 = \sum_{k=0}^{m} Q_k \frac{d f_2^k}{dt^k} = \theta f_2 \text{ defined on an adjacent interval } I_2, \]

where \(\theta_1, \theta_2\) are constants, intervals \(I_1\) and \(I_2\) have common end (interface) point \(t = c\), and the functions \(f_1, f_2\) are required to satisfy certain interface conditions at \(t = c\). In most of the cases, the complete set of physical conditions on the system gives rise to self-adjoint eigenvalue problems associated with the pair \((L_1, L_2)\). In some cases, however, the physical conditions at the interface may be inadequate to describe the problem in a mathematically sound manner. In such a situation, when the problem is formulated mathematically, it becomes ill-posed, and therefore cannot be solved effectively (uniquely) using existing methods. With the introduction of interface-spatial conditions (entirely a new concept), we shall be able to convert these ill-posed problems into well-posed problems and this justifies their mathematical study.

In a series of papers, we wish to develop a unified approach to these interface-spatial problems for both the regular and the singular cases. In the present paper, for the first time, we
shall study the initial value problems (IVPs) for a pair of linear first order ordinary differential systems satisfying certain interface-spatial conditions.

Before proving the main theorems, we introduce a few notations and make some assumptions. For any compact interval $J$ of $\mathbb{R}$ and for any non-negative integer $k$, let $C^k(J)$ denote the space of $k$-times continuously differentiable complex-valued functions defined on $J$. If $I$ is a non-compact interval of $\mathbb{R}$, $C^k(I)$ denotes the collection of all complex-valued functions $f$ defined on $I$ whose restriction $f|_J$ to any compact subinterval $J$ of $I$ belongs to $C^k(J)$. Let $AC^k(I)$ denote the space of all complex-valued functions $f$ which have $(k-1)$ derivatives on $I$, and the $(k-t)$th derivative is absolutely continuous over each compact subinterval of $I$. Let $I_1 = (a,c], I_2 = [c,b)$, $-\infty < a < c < b < +\infty$, and let $f^{(j)}$ denote the $j$th derivative of $f$. For a matrix $A$, let $R(A)$ and $\rho(A)$ denote the range and rank of $A$. Let $\mathbb{C}^n$ denote the complex $n$-dimensional space.

Let $A_1(t)$ ($A_2(t)$) be matrix valued functions of order $n \times n$ ($m \times m$), whose entries belong to $C^0(I_1)$ ($C^0(I_2)$). Let $b_1(t)$ ($b_2(t)$) be a vector-valued function of order $n \times 1$ ($m \times 1$), whose entries are integrable over every compact subinterval of $I_1$ ($I_2$).

Let the functions $P_k \in C^k(I_1)$ ($k = 0, 1, \ldots, n$) $Q_k \in C^k(I_2)$ ($k = 0, 1, \ldots, m$) $P_n(t) \neq \emptyset$ on $I_1$ and $Q_m(t) \neq \emptyset$ on $I_2$. Let $g_1(g_2)$ be a measurable complex-valued function defined on $I_1$ ($I_2$) which is integrable over every compact subinterval of $I_1$ ($I_2$).

Without loss of generality, we assume $n \geq m$. Let $A$ and $B$ be $m \times n$ and $m \times m$ matrices with complex entries respectively, and $R(A) = R(B)$. Consequently, $\rho(A) = \rho(B) = d(\leq m)$. Let $N$ be a subspace of $R(A)$, and the dimension of $N$ equals $d'$. Let $t_i \in I_i$ ($i = 1, 2$), $C = \text{column } (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{C}^n$, and $D = \text{column } (d_0, d_1, \ldots, d_{m-1}) \in \mathbb{C}^m$. Let $Y_1 = \text{column } (y_{11}, y_{12}, \ldots, y_{1n})$ and $Y_2 = \text{column } (y_{21}, y_{22}, \ldots, y_{2m})$.

Consider the following interface-spatially mixed pair of linear first order ordinary differential systems:

\begin{align*}
Y_1' &= A_1(t)Y_1 + b_1(t), \quad t \in I_1, \\
Y_2' &= A_2(t)Y_2 + b_2(t), \quad t \in I_2, \\
AY_1(c) - BY_2(c) &\in N.
\end{align*}

Also, consider the initial conditions

\begin{align*}
Y_1(c) &= C, \\
Y_2(c) &= D.
\end{align*}

We call problems (1)-(3) and (4)((5)) the interface-spatially mixed initial value problems (IFIVP) (I) (II).

Consider the following interface-spatially mixed pair of linear ordinary differential equations (of orders $n$ and $m$):

\begin{align*}
L_1f_1 &= \sum_{k=0}^{n} P_k \frac{d^2 f_1}{dt^2} = g_1, \quad t \in I_1, \\
L_2f_2 &= \sum_{k=0}^{m} Q_k \frac{d^2 f_2}{dt^2} = g_2, \quad t \in I_2, \\
A\tilde{f}_1(c) - B\tilde{f}_2(c) &\in N,
\end{align*}

where

\[ \tilde{f}_1 = \text{column } (f_1, f_1^{(1)}, \ldots, f_1^{(n-1)}), \]

\[ \tilde{f}_2 = \text{column } (f_2, f_2^{(1)}, \ldots, f_2^{(m-1)}). \]
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and

\[ \tilde{f}_2 = \text{column } (f_2, f_2^{(1)}, \ldots, f_2^{(m-1)}). \]

Also consider the initial conditions

\[ f_1^{(j)}(t_1) = c_j \quad (j = 0, 1, \ldots, n-1), \quad (9) \]
\[ f_2^{(j)}(t_2) = d_j \quad (j = 0, 1, \ldots, m-1). \quad (10) \]

We call problems (6)-(8) and (9) (10) the interface-spatially mixed initial value problems (IFSIVP) (I) ((II')).

**Definition 1:** We call a pair of vector-valued functions \((Y_1, Y_2)\), defined on \(I_1 \times I_2\), an interface-spatially mixed (IFS) solution of (1)-(2) if

(i) \( Y_{1j} \in AC^1(I_1) \) \((j = 1, \ldots, n)\),

(ii) \( Y_1 \) satisfies equation (1) for almost all \( t \in I_1\),

(iii) \( Y_{2j} \in AC^1(I_2) \) \((j = 1, \ldots, m)\),

(iv) \( Y_2 \) satisfies equation (2) for almost all \( t \in I_2\),

and

(v) the pair \((Y_1, Y_2)\) satisfies relation (3).

**Definition 2:** We call a pair of complex-valued functions \((f_1, f_2)\), defined on \(I_1 \times I_2\), an interface-spatially mixed (IFS) solution of (6)-(7) if

(i) \( f_1 \in AC^m(I_1) \) and satisfies equation (6) for almost all \( t \in I_1\),

(ii) \( f_2 \in AC^m(I_2) \) and satisfies equation (7) for almost all \( t \in I_2\),

and

(iii) the pair \((f_1, f_2)\) satisfies relation (8).

**Definition 3:** We call a pair of vector-valued functions \((Y_1, Y_2)\), defined on \(I_1 \times I_2\), an interface-spatially mixed solution of IFSIVP(I) ((II)) if

(i) \( Y_{1}(Y_2) \) is an IFS solution of (1)-(2)

and

(ii) \( Y_1(Y_2) \) satisfies condition (4) (5).

**Definition 4:** We call a pair of complex-valued functions \((f_1, f_2)\), defined on \(I_1 \times I_2\), an interface-spatially mixed solution of IFSIVP(I') ((II')) if

(i) \( (f_1, f_2) \) is a IFS solution of (6)-(7)

and

(ii) \( (f_1, f_2) \) satisfies condition (9) (10).

**Definition 5:** We say that a collection of non-trivial pairs \((Y_{11}, Y_{12}), \ldots, (Y_{p1}, Y_{p2})\) are linearly independent if for any set of scalars \(\alpha_1, \ldots, \alpha_p\),

\[ \sum_{i=0}^{p} \alpha_i(Y_{1i}, Y_{2i}) = (0, 0) \]

implies that \(\alpha_1 = \alpha_2 = \ldots = \alpha_p = 0\).

Similarly, we define the linear independency of a collection of pairs \((f_{11}, f_{12}), \ldots, (f_{p1}, f_{p2})\).

**Definition 6:** By an IFS fundamental system for the IFSIVP(I) ((II)), we mean a set of linearly independent IFS solutions of IFSIVP(I) ((II)) which span the IFS solution space of IFSIVP(I) ((II)).

Similarly, we define a fundamental system for the IFSIVP(I') ((II')).
2. Main Theorems

Theorem 1: (a) If either \(b_1(t) \neq 0\), \(b_2(t) \neq 0\), or \(C\) is a nonzero vector, then the IFS IVP(\(I\)) has an IFS fundamental system consisting of \(m - d + d' + 1\) linearly independent IFS solutions of IFS IVP(\(I\)). If \(b_1(t) \equiv 0\), \(b_2(t) \equiv 0\), and \(C\) is a zero vector, then the IFS IVP(\(I\)) has an IFS fundamental system consisting of \(m - d + d'\) linearly independent IFS solutions of IFS IVP(\(I\)).

(b) If either \(b_1(t) \neq 0\), \(b_2(t) \neq 0\), or \(D\) is a nonzero vector, then the IFS IVP(\(II\)) has a fundamental system consisting of \(n - d + d' + 1\) linearly independent IFS solutions of IFS IVP(\(II\)). If \(b_1(t) \equiv 0\), \(b_2(t) = 0\), and \(D\) is a zero vector, then the IFS IVP(\(II\)) has an IFS fundamental system consisting of \(n - d + d'\) linearly independent IFS solutions of IFS IVP(\(II\)).

Proof: Since the components of \(b_1(t)\) are measurable complex-valued functions integrable on \(I_1\) by Theorem 2.1 [2], there exists a unique vector-valued function \(\phi(t) = \text{column}(\phi_1(t), \phi_2(t), \ldots, \phi_n(t))\) defined on \(I_1\) with \(\phi_j \in AC^1(I_1)\) such that

\[
\phi'(t) = A_1(t)\phi(t) + b_1(t), \quad t \in I_1,
\]

\[
\phi(t_1) = C.
\]

Let \(\phi(c) = \eta\). Since \(R(A) = R(B)\), there exists a vector \(\beta^0 \in C^n\) such that \(A\eta = B\beta^0\). If \(A\eta \neq 0\), \(\beta^0\) is a nonzero vector, and if \(A\eta = 0\), then we take \(\beta^0\) to be zero vector. Since \(\rho(B) = d\), there exist \((m - d)\) linearly independent vectors \(\beta^1, \beta^2, \ldots, \beta^{m - d} \in C^n\) which are solutions of \(B\beta = 0\). Clearly, \(\beta^0, \beta^0 + \beta^1, \ldots, \beta^0 + \beta^{m - d}\) are \((m - d + 1)\) or \((m - d)\) linearly independent solutions of \(A\eta = B\beta\), affected by \(A\eta \neq 0\) or \(A\eta = 0\).

Also, since the components of \(b_2(t)\) are measurable complex-valued functions integrable on \(I_2\), there exists a unique vector-valued function \(\psi_0(t) = \text{column}(\psi_{01}(t), \ldots, \psi_{0n}(t))\) defined on \(I_2\) with \(\psi_{0j} \in AC^1(I_2)\) such that

\[
\psi'(t) = A_2(t)\psi(t) + b_2(t), \quad t \in I_2,
\]

\[
\psi_0(c) = \beta^0.
\]

Let \(\psi_1(t) = \text{column}(\psi_{i1}(t), \ldots, \psi_{in}(t))\), defined on \(I_2\) with \(\psi_{ij} \in AC^1(I_2)\), be the unique vector-valued function such that

\[
\psi_1(t) = A_2(t)\psi_1(t), \quad t \in I_2,
\]

\[
\psi_1(c) = \beta^i, \quad i = 1, \ldots, m - d.
\]

Clearly, \(\psi_1, \ldots, \psi_{m - d}\) are linearly independent and if \(\beta^0 \neq 0\), then \(\psi_0, \ldots, \psi_{m - d}\) are also linearly independent.

Choose a basis \(\alpha^1, \ldots, \alpha^{d'}\) for \(N\), and let \(\beta = \beta^{m - d + i}\) be a solution of

\[-B\beta^{m - d + i} = \alpha^i \quad (i = 1, \ldots, d').\]

Since \(\alpha^i\) are linearly independent \(\beta^{m - d + d'}\)s are also linearly independent. In fact, \(\beta^1, \ldots, \beta^{m - d + d'}\) are linearly independent.

Again, let \(\psi_1(t)\), defined on \(I_2\), be a unique vector-valued function such that

\[
\psi_1(t) = A_2(t)\psi_1(t), \quad t \in I_2,
\]

\[
\psi_1(c) = \beta^i \quad (i = m - d + 1, \ldots, m - d + d').
\]

Clearly, \(\psi_1, \ldots, \psi_{m - d + d'}\) are linearly independent.

Now, define

\[
(Y_{01}, Y_{02}) = (\phi, \psi_0),
\]
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Clearly, each pair \((Y_{i1}, Y_{i2})\) \((i = 0, 1, \ldots, m-d+d')\) is an IFS solution of (1)-(2). Moreover, if \(b_1(t) \neq 0, b_2(t) \neq 0, or \exists C \neq 0,\) then the pair \((\phi, \psi_0)\) is nontrivial.

Claim: For \(b_1(t) \neq 0, b_2 \neq 0 or C \neq 0,\) \((Y_{i1}, Y_{i2}), i = 0, \ldots, m-d+d'\) is an IFS fundamental system for the IFSIVP(I).

Let \(\sum_{i=0}^{m-d+d'} a_i(Y_{i1}, Y_{i2}) = (0,0),\) where \(a_i\) are scalars. Then
\[
\sum_{i=0}^{m-d+d'} a_i Y_{i1} = 0 \quad \text{and} \quad \sum_{i=0}^{m-d+d'} a_i Y_{i2} = 0.
\]

Consequently, we get
\[
\sum_{i=1}^{m-d+d'} a_i [A\phi(c) - B(\psi_0(c) + \psi_1(c))] + a_0 (A\phi(c) - B\psi_0(c)) = 0,
\]
\[
\sum_{i=m-d+i}^{m-d+d'} a_i (-B\psi_1(c)) = 0,
\]
i.e.,
\[
\sum_{i=m-d+1}^{m-d+d'} a_i \alpha^i = 0, \quad \text{which implies that} \quad a_i = 0 \quad (i = m-d+1, \ldots, m-d+d').
\]

Hence, relation (11) becomes
\[
\sum_{i=0}^{m-d} a_i \phi = 0 \quad \text{and} \quad \sum_{i=1}^{m-d} a_i (\psi_0 + \psi_1) + A_0 \psi_0 = 0. \tag{12}
\]

Again, from relation (12), we get
\[
(\sum_{i=0}^{m-d} a_i) \psi_0(c) + \sum_{i=1}^{m-d} a_i \psi_1(c) = 0,
\]
i.e.,
\[
(\sum_{i=0}^{m-d} a_i) \beta^0 + \sum_{i=1}^{m-d} a_i \beta^i = 0. \tag{13}
\]

If \(\beta^0 \neq 0,\) then \(\beta^0, \beta^1, \ldots, \beta^{m-d},\) are linearly independent and hence \(a_i = 0 \quad (i = 0, 1, \ldots, m-d).\) If \(\beta^0 = 0,\) then relation (13) gives \(a_i = 0 \quad (i = 1, \ldots, m-d)\) and from relation (12) we get \(a_0(\phi, \psi_0) = (0,0),\) which implies that \(a_0 = 0.\) Thus, \((Y_{i1}, Y_{i2})\) \((i = 0, 1, \ldots, m-d+d')\) are linearly independent.

Now, let \((Y_1, Y_2)\) be any solution of the IFSIVP(I). We note that \(Y_1 = \phi.\)

Case (i): Suppose that \(AY_1(c) - BY_2(c) = 0.\) Furthermore, since \(A\phi(c) - B\psi_0(c) = 0,\) we get \(B(Y_2(c) - \psi_0(c)) = 0,\) which implies that \(Y_2(c) - \psi_0(c)\) belongs to the null space of \(B.\) Therefore, there exist constants \(a_i \quad (i = 1, \ldots, m-d)\) such that
\[
Y_2(c) = \sum_{i=1}^{m-d} a_i \beta^i,
\]
i.e.,
\[
Y_2(c) = \beta^0 + \sum_{i=1}^{m-d} a_i \beta^i = (1 - \sum_{i=1}^{m-d} a_i) \beta^0 + \sum_{i=1}^{m-d} a_i (\beta^0 + \beta^i)
\]
\[
= (1 - \sum_{i=1}^{m-d} a_i) \psi_0(c) + \sum_{i=1}^{m-d} a_i (\psi_0(c) + \psi_1(c))
\]
\[
= (1 - \sum_{i=1}^{m-d} a_i) Y_{o2}(c) + \sum_{i=1}^{m-d} a_i Y_{i2}(c).
\]
Thus, by the uniqueness of the solution of IVPs for a system of ordinary differential equations, we have

\[
(Y_1, Y_2) = (1 - \sum_{i=1}^{m-d} a_i)(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d} a_i(Y_{i1}, Y_{i2}).
\]

**Case (ii):** Suppose that \( AY_1(c) - BY_2(c) = \xi = \sum_{i=1}^{d'} a_{i+m-d} \alpha_i \), where \( a_i \)'s are scalars.

Define a pair \((K_1, K_2)\) by

\[
(K_1, K_2) = (1 - \sum_{i=1}^{m-d+d'} a_i)(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d+d'} a_i(Y_{i1}, Y_{i2}).
\]

Then \((K_1, K_2)\) is an IFS solution of IFSIVP(I). Consequently, we get

\[
B(Y_2(c) - K_2(c)) = 0.
\]

Therefore, there exist scalars \( a_i \) (\( i = 1, \ldots, m-d \)) such that

\[
Y_2(c) - K_2(c) = \sum_{i=1}^{m-d} a_i \beta^i,
\]

i.e.,

\[
Y_2(c) = K_2(c) + \sum_{i=1}^{m-d} a_i \beta^i
\]

\[
= K_2(c) - \left( \sum_{i=1}^{m-d} a_i \beta^0 + \sum_{i=1}^{m-d} a_i(\beta^0 + \beta^i) \right)
\]

\[
= K_2(c) - \left( \sum_{i=1}^{m-d} a_i \psi_0(c) + \sum_{i=1}^{m-d} a_i(\psi_0(c) + \psi_i(c)) \right)
\]

\[
(15)
\]

Thus,

\[
(Y_1, Y_2) = (K_1, K_2) - \sum_{i=1}^{m-d} a_i(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d} a_i(Y_{i1}, Y_{i2})
\]

\[
= (1 - \sum_{i=1}^{m-d+d'} a_i)(Y_{01}, Y_{02}) + \sum_{i=1}^{m-d+d'} a_i(Y_{i1}, Y_{i2}).
\]

Hence, the claim is proved. If \( b_1(t) \equiv 0, b_2(t) \equiv 0, \) and \( C = 0, \) then \((\phi, \psi_0)\) is a trivial pair and the pairs \((Y_{i1}, Y_{i2})\) (\( i = 1, \ldots, m-d+d' \)) form an IFS fundamental system for the IFSIVP(I).

This completes the proof of part (a). Part (b) can proved similarly.

**Theorem 2:** There exist exactly \( n + m - d + d' \) linearly independent (IFS) solutions of

\[
Y_1' = A_1(t)Y_1, \quad t \in I_1,
\]

\[
Y_2' = A_2(t)Y_2, \quad t \in I_2,
\]

satisfying the interface-spatial conditions

\[
AY_1(c) - BY_2(c) \in N.
\]

**Proof:** Since \( \rho(A) = \rho(B) = :d, \) there exists a basis \( \{\eta^1, \ldots, \eta^n\} \) for \( C^n \) such that \( \{\eta^1, \ldots, \eta^{n-d}\} \) forms a basis for the null-space of \( A, \) and a basis \( \{\beta^1, \ldots, \beta^m\} \) for \( C^m \) such that \( \{\beta^{d+1}, \ldots, \beta^m\} \) forms a basis for the null space of \( B. \)

Let \( \bar{Y}_{i1} \) (whose components belong to \( AC^1(I_1) \)) be the unique solution of
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\[ Y'_1 = A_1(t)Y_1, \quad t \in I_1, \]
\[ Y_1(c) = \eta^i \quad (i = 1, \ldots, n). \]

Since \( R(A) = R(B) \), for each \( i = n - d + 1, \ldots, n \), there exist scalars \( \theta_j^i \) \( (j = 1, \ldots, d) \) such that
\[ A\eta^i = \sum_{j=1}^{d} \theta_j^i B \beta_j^i. \]

Let \( \hat{Y}_{i2} \) (with components belonging to \( AC(I_2) \)) be the unique solution of
\[ Y'_2 = A_2(t)Y_2, \quad t \in I_2, \]
\[ Y_2(c) = B\beta_i - n + d \quad (i = n + 1, \ldots, n + m - d). \]

Let \( (\alpha^1, \ldots, \alpha^d) \) be a basis for \( N \) and choose \( \hat{\beta}^i \in \mathbb{C}^m \) such that
\[ -B\hat{\beta}^i = \alpha^i \quad (i = 1, \ldots, d'). \]

Let \( \hat{Y}_{i2} \) (with components belonging to \( AC^1(I_2) \)) be the unique solution of
\[ Y'_2 = A_2(t)Y_2, \quad t \in I_2, \]
\[ Y_2(c) = \hat{\beta}^i - n - m + d \quad (i = n + m - d + 1, \ldots, n + m - d + d'). \]

Define the pairs
\[ (Y_{i1}, Y_{i2}) = \begin{cases} (\hat{Y}_{i1}, 0) & (i = 1, \ldots, n - d), \\ (\hat{Y}_{i1}, \hat{Y}_{i2}) & (i = n - d + 1, \ldots, n), \\ (0, \hat{Y}_{i2}) & (i = n + 1, \ldots, n + m - d + d'). \end{cases} \]

Clearly each pair \( (Y_{i1}, Y_{i2}) \) is a nontrivial IFS solution of (16)-(18).

**Claim:** \( (Y_{i1}, Y_{i2}) \) \((i = 1, \ldots, n + m - d + d')\) form an IFS fundamental system for the IFS solutions of (16)-(18).

First, we shall show that the pairs \( (Y_{i1}, Y_{i2}) \) are linearly independent. To this end, let
\[ \sum_{i=1}^{n+m-d+d'} a_i (Y_{i1}, Y_{i2}) = (0, 0), \quad \text{where } a_i \text{ are scalars.} \]

Then,
\[ \sum_{i=1}^{n} a_i \hat{Y}_{i1} = 0 \quad \text{and} \quad \sum_{i=n-d+1}^{n+m-d+d'} a_i \hat{Y}_{i2} = 0. \quad (19) \]

Since \( \hat{Y}_{i1}(c) \) \((i = 1, \ldots, n + m - d + d')\) are linearly independent, from the first equation of relation (19) we get \( a_i = 0 \) \((i = 1, \ldots, n)\). Consequently, the second equation reduces to
\[ \sum_{i=n+1}^{n+m-d+d'} a_i Y_{i2} = 0. \quad (20) \]

Evaluating the above expression at \( t = c \) and then applying the matrix \( B \) to the resulting expression, we get
\[ \sum_{i=n+m-d+1}^{n+m-d+d'} a_i \alpha^i - n - m + d = 0, \]
which implies that \( a_i = 0 \), for \( i = n + m - d + 1, \ldots, n + m - d + d' \). Thus, relation (20) reduces to
\[ \sum_{i=n+1}^{n+m-d} a_i \hat{Y}_{i2} = 0, \]
and since \( \hat{Y}_{i2}(c) (i = n+1, \ldots, n+m-d) \) are linearly independent (this fact can be easily verified), it follows that
\[ a_i = 0 \ (i = n+1, \ldots, n+m-d). \]

This proves the linear independency of \( (Y_{i1}, Y_{i2}) \)s.

Next, let \( (Y_1, Y_2) \) be any IFS solution of (16)-(18). Choose scalars \( a_i \ (i = 1, \ldots, n) \) such that
\[ Y_1(c) = \sum_{i=1}^{n} a_i y_i^i. \]

**Case (1):** Suppose that \( AY_1(c) - BY_2(c) = 0. \)

Define the pair \( (K_1, K_2) = \sum_{i=1}^{n} a_i (Y_{i1}, Y_{i2}). \)

Then \( K_1(c) = \sum_{i=1}^{n} a_i Y_{i1}(c) = Y_1(c) \). Hence, \( Y_1 = K_1 \) and \( B(Y_2(c) - K_2(c)) = 0 \), which implies that
\[ Y_2(c) = K_2(c) + \sum_{i=n+1}^{n+m-d} a_i \beta^i - n + d \text{ for some scalars } a_i \text{s,} \]
i.e.,
\[ Y_2(c) = K_2(c) + \sum_{i=n+1}^{n+m-d} a_i Y_{i2}(c). \]

Thus,
\[ (Y_1, Y_2) = (K_1, K_2) + \sum_{i=n+1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \]
\[ = \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}). \]

**Case (2):** Suppose that \( A(Y_1(c) - K_1(c)) - B(Y_2(c) - K_2(c)) = 0 \).

Define \( (H_1, H_2) = \sum_{i=n+m-d+1}^{n+m-d} a_i (Y_{i1}, Y_{i2}). \)

Then \( A(H_1(c) - Y_1(c)) - B(H_2(c) - Y_2(c)) = 0 \), and therefore, by case (1),
\[ (Y_1 - H_1, Y_2 - H_2) = \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \text{ for some scalars } a_i \text{s.} \]

Thus,
\[ (Y_1, Y_2) = (H_1, H_2) + \sum_{i=1}^{n+m-d} a_i (Y_{i1}, Y_{i2}) \]
\[ = \sum_{i=1}^{n+m-d+d'} a_i (Y_{i1}, Y_{i2}). \]

This completes the proof.

**Remark 1:** The assumption \( d'' = d \) yields that there are no explicit boundary conditions at the interface point.

If \( d' = 0 \), then the interface-spatial condition becomes
\[ AY_1(c) - BY_2(c) = 0, \]
which is generally called the *interface condition*.

Since higher order ordinary differential equations can be converted into a system of first order
Theorem 3: (a) If either \( g_1 \neq 0, g_2 \neq 0, \) or \( c_{0}, c_{1}, \ldots, c_{n-1} \) are not all zeros, then the IFS \( (I') \) has a fundamental system consisting of \( "m-d+d'+1" \) linearly independent IFS solutions of IFS \( (I') \). If \( g_1 \equiv 0, g_2 \equiv 0, \) and \( c_{0}, c_{1}, \ldots, c_{n-1} \) are all zeros, then the IFS \( (I') \) has a IFS fundamental system consisting of \( "m-d+d'" \) linearly independent solutions of IFS \( (I') \).

(b) If either \( g_1 \neq 0, g_2 \neq 0, \) or \( d_{0}, d_{1}, \ldots, d_{n-1} \) are not all zeros, then the IFS \( (II') \) has a IFS fundamental system consisting of \( "n-d+d'+1" \) linearly independent IFS solutions of IFS \( (II') \). If \( g_1 \equiv 0, g_2 \equiv 0, \) and \( d_{0}, d_{1}, \ldots, d_{n-1} \) are all zeros, then the IFS \( (II') \) has an IFS fundamental system consisting of \( "n-d+d" \) linearly independent IFS solutions of IFS \( (II') \).

Theorem 4: There exist exactly \( "n+m-d+d'" \) linearly independent (IFS) solutions of
\[
L_1 f_1 = 0, \quad t \in I_1, \\
L_2 f_2 = 0, \quad t \in I_2, 
\]
satisfying the interface-spatial conditions
\[
A f_1(c) - B f_2(c) \in N. 
\]

Remark 4: For \( d' = d \), Theorems 3 and 4 reduce to Theorems 1 and 4 of [6].

For \( d' = 0 \), Theorems 3 and 4 reduce to Theorems 3 and 6 of [6].

For \( d' = 0 \) as well as for the \((m \times n)\) matrix \( A \) given by
\[
A = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0
\end{pmatrix}
\]
and \( B \) equal to the \((m \times m)\) identity matrix, Theorems 3 and 4 reduce to Theorems 2 and 5 of [6].

3. Physical Examples

Example 1 - Acoustic waveguides in ocean [1]: The following problem is encountered in the study of acoustic waves in the ocean consisting of two layers: an outer layer of finite depth and an inner layer of infinite depth:

\[
L_1 f_1 = \frac{d^2 f_1}{dt^2} + k_1^2 f_1 = \lambda f_1, \quad 0 \leq t \leq d_1, 
\]
\[
L_2 f_2 = \frac{d^2 f_2}{dt^2} + k_2^2 f_2 = \lambda f_2, \quad d_1 \leq t \leq +\infty, 
\]
together with the end point conditions given by
\[
f_1(0) = 0, \quad \lim_{t \to +\infty} f_1^{(1)}(t) = 0, 
\]
and the interface conditions given by
\[ f_1(d_1) = f_2(d_1), \quad 1/\rho_1 f_1^{(1)}(d_1) = 1/\rho_2 f_2^{(1)}(d_1). \]  

(25)

Here \( \rho_1, \rho_2 \) are constant densities of the two layers, \( k_1, k_2 \) are the constants which depend upon the frequency constant and the constant sound velocities \( c_1, c_2 \) of the two layers, respectively, \( \lambda \) is an unknown constant, \( d_1 \) denotes the depth of the outer layer, and \( f_1, f_2 \) stand for the depth eigenfunctions.

In this example, the interface conditions at \( t = d_1 \) of the two layers can be written in the matrix form

\[
\begin{pmatrix}
1 & 0 \\
0 & 1/\rho_1
\end{pmatrix}
\begin{pmatrix}
f_1(d_1) \\
f_1^{(1)}(d_1)
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 \\
0 & 1/\rho_1
\end{pmatrix}
\begin{pmatrix}
f_2(d_1) \\
f_2^{(1)}(d_1)
\end{pmatrix}.
\]

Here \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho_2 \end{pmatrix} \), rank \( A = \text{rank} B = 2 \), \( m = n = d = 2 \) and \( d' = 0 \).

Hence, by Theorem 3 and Remark 2, there exists a unique IFS solution for any IFSIVP associated with (22)-(23) and (25). Also, by Theorem 4 and Remark 2, there exist exactly two linearly independent IFS solutions of problems (22)-(23) and (25).

**Example 2 - Optical fiber transmission [4]:** In the study of wave optics of step index fiber, we encounter the following problem

\[
L_1 f_1 = \frac{d^2 f_1}{dt^2} + 1/t \frac{df_1}{dt} + (\eta_1 k_0^2 - \nu^2/t^2)f_1 = \beta^2 f_1, \quad 0 < t \leq a,
\]

(26)

\[
L_2 f_2 = \frac{d^2 f_2}{dt^2} + 1/t \frac{df_2}{dt} + (\eta_2 k_0^2 - \nu^2/t^2)f_2 = \beta^2 f_2, \quad a < t < +\infty,
\]

(27)

together with the interface conditions at \( t = a \), given by

\[
\lim_{t \to 0} |f_1(t)| < +\infty, \quad \lim_{t \to -\infty} |f_2(t)| = 0.
\]

(29)

Here \( \eta_1 \) and \( \eta_2 \) are the refractive indices of the core and cladding, respectively, \( \beta \) is the wave propagation constant, \( \nu \) is an integer \( k_0 = w/c \), \( c \) is the propagation velocity and \( w \) is the wave frequency and \( f_1 \) and \( f_2 \) are the field (electromagnetic) distributions of core and cladding, respectively.

In this example, relation (28) gives continuity conditions at \( t = a \). Here \( A \) and \( B \) are the \( 2 \times 2 \) identity matrices, \( n = m = d = 2 \) and \( d' = 0 \). Hence, by Theorem 3 and Remark 2, there exists a unique IFS solution for IFSIVP associated with (26)-(28). Also, by Theorem 4 and Remark 2, there exist exactly two linearly independent IFS (continuous) solutions of (26)-(28).

**Example 3 - One-dimensional scattering in quantum theorem [3]:** In quantum theory, the one-dimensional time-independent scattering problem with the delta function scattering potential is represented by the problem

\[
L_1 f_1 = \frac{d^2 f_1}{dt^2} + k^2 f_1 = 0, \quad -\infty < t \leq 0,
\]

(30)

\[
L_2 f_2 = \frac{d^2 f_2}{dt^2} + (k^2 - \nu_0)f_2 = 0, \quad 0 \leq t < +\infty,
\]

(31)

together with the interface conditions given by

\[
f_1(0) - f_2(0) = 0,
\]

(32)
where \( k^2 = 2mE/h^2 \), \( \nu_0 \) is a constant, and the functions \( f_1 \) and \( f_2 \) are associated with the flux density of the particle of the two regions, respectively. Here, \( m \) denotes the mass of the particle, \( E \) denotes its total energy, and \( h \) denotes the Planck constant divided by \( 2\pi \). In this example, the interface conditions at \( t = 0 \) of the two regions can be written in the matrix form

\[
\begin{pmatrix}
1 & 0 \\

\nu_0 & 1
\end{pmatrix}
\begin{pmatrix}
f_1(0) \\
f_2(0)
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
f_1^{(1)}(0) \\
f_2^{(1)}(0)
\end{pmatrix}.
\]

Here

\[
A = \begin{pmatrix}
1 & 0 \\
\nu_0 & 1
\end{pmatrix},
B = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

rank \( A = \text{rank} \, B = 2 \), \( m = n = d = 2 \), and \( d' = 0 \).

Hence, by Theorem 3 and Remark 2, there exists a unique IFS solution of any IFSIVP associated with (30)-(33). Also, by Theorem 4 and Remark 2, there exist exactly two linearly independent IFS solutions of (30)-(33).

**Example 4:** In this illustrative example, consider the following problem:

\[
\begin{align*}
L_1 f_1 &= \frac{d^2 f_1}{dt^2} - k_1^2 f_1 = 0, \quad a < t < c, \\
L_2 f_2 &= \frac{d^2 f_2}{dt^2} + k_2^2 f_2 = 0, \quad c < t < b,
\end{align*}
\]  

(34)

(35)

together with interface condition

\[ f_1(c) = f_2(c) \]  

(36)

and the end point conditions

\[ f_1(a) = 0 = f_2(b), \]  

(37)

where \( k_1 \) and \( k_2 \) are constants. Problems (34)-(37) can be thought of as the transverse vibrations of a string stretched between \( a \) and \( b \), fixed at \( a \) and \( b \), with different uniform linear densities (in the portion) between \( a \) and \( c \) and between \( c \) and \( b \), and plucked at the point \( t = c \).

In this example, there is only one condition at the interface (i.e., the continuity condition), and no definite relation between the derivatives is available. Therefore, we may take

\[ f_1^{(1)}(c) - f_2^{(1)}(c) = \alpha, \quad \alpha \in \mathbb{R}. \]  

(38)

We note that relation (38) is not at all a restriction on derivatives. Consequently, relation (36) and (38) can be written as

\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
f_1(c) \\
f_1^{(1)}(c)
\end{pmatrix} -
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
f_2(c) \\
f_2^{(1)}(c)
\end{pmatrix} \in N,
\]  

(39)

where \( N = \text{the linear span of } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Here, \( A = B = \text{the } (2 \times 2) \text{ identity matrix} \), \( n = m = d = 2 \), and \( d' = 1 \). Therefore, by Theorem 3, there exist one or two linearly independent IFS solutions of the IFSIVP associated with problems (34)-(36) depending on whether the initial data is zero or nonzero. Also, by Theorem 4, there exist three linearly independent IFS solutions of problems (34)-(36).
Remark 3: The results of this paper are used in studying the deficiency indices and self-adjoint boundary value problems associated with \((L_1, L_2)\) satisfying interface-spatial conditions which we shall establish elsewhere.

Acknowledgement

The authors dedicate the work to the chancellor of the Institute Bhagawan Sri Sathya Sai Baba.

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