The Generalised Colouring Numbers on Classes of Bounded Expansion*

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Abstract

The generalised colouring numbers $\text{adm}_r(G)$, $\text{col}_r(G)$, and $\text{wcol}_r(G)$ were introduced by Kierstead and Yang as generalisations of the usual colouring number, also known as the degeneracy of a graph, and have since then found important applications in the theory of bounded expansion and nowhere dense classes of graphs, introduced by Nešetřil and Ossona de Mendez. In this paper, we study the relation of the colouring numbers with two other measures that characterise nowhere dense classes of graphs, namely with uniform quasi-wideness, studied first by Dawar et al. in the context of preservation theorems for first-order logic, and with the splitter game, introduced by Grohe et al. We show that every graph excluding a fixed topological minor admits a universal order, that is, one order witnessing that the colouring numbers are small for every value of $r$. Finally, we use our construction of such orders to give a new proof of a result of Eickmeyer and Kawarabayashi, showing that the model-checking problem for successor-invariant first-order formulas is fixed-parameter tractable on classes of graphs with excluded topological minors.

1 Introduction

The colouring number $\text{col}(G)$ of a graph $G$ is the minimum $k$ for which there is a linear order $<_L$ on the vertices of $G$ such that each vertex $v$ has back-degree at most $k - 1$, that is, $v$ has at most $k - 1$ neighbours $u$ with $u <_L v$. The colouring number is a measure for uniform sparseness in graphs: we have $\text{col}(G) = k$ if and only if every subgraph $H$ of $G$ has a vertex of degree at most $k - 1$. Hence, provided $\text{col}(G) = k$, not only $G$ is sparse, but also every subgraph of $G$ is sparse. The colouring number minus one is also known as the degeneracy.

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Recently, Nešetřil and Ossona de Mendez introduced the notions of bounded expansion [12] and nowhere density [14] as very general formalisations of uniform sparseness in graphs. Since then, several independent and seemingly unrelated characterisations of these notions have been found, showing that these concepts behave robustly. For example, nowhere dense classes of graphs can be defined in terms of excluded shallow minors [14], in terms of uniform quasi-wideness [2], a notion studied in model theory, or in terms of a game [8] with direct algorithmic applications. The generalised colouring numbers $\text{adm}_r$, $\text{col}_r$, and $\text{wcol}_r$ were introduced by Kierstead and Yang [11] in the context of colouring and marking games on graphs. As proved by Zhu [17], they can be used to characterise both bounded expansion and nowhere dense classes of graphs.

The invariants $\text{adm}_r$, $\text{col}_r$, and $\text{wcol}_r$ are defined similarly to the classic colouring number: for example, the weak $r$-colouring number $\text{wcol}_r(G)$ of a graph $G$ is the minimum integer $k$ for which there is a linear order of the vertices such that each vertex $v$ can reach at most $k - 1$ vertices $w$ by a path of length at most $r$ in which $w$ is the smallest vertex on the path.

The generalised colouring numbers found important applications in the context of algorithmic theory of sparse graphs. For example, they play a key role in Dvořák's approximation algorithm for minimum dominating sets [4], or in the construction of sparse neighbourhood covers on nowhere dense classes, a fundamental step in the almost linear time model-checking algorithm for first-order formulas of Grohe et al. [8].

In this paper we study the relation between the colouring numbers and the above mentioned characterisations of nowhere dense classes of graphs, namely with uniform quasi-wideness and the splitter game. We use the generalised colouring numbers to give a new proof that every bounded expansion class is uniformly quasi-wide. This was first proved by Nešetřil and Ossona de Mendez in [13]; however, the constants appearing in the proof of [13] are huge. We present a very simple proof which also improves the appearing constants. Furthermore, for the splitter game introduced in [8], we show that splitter has a very simple strategy to win on any class of bounded expansion, which leads to victory much faster than in general nowhere dense classes of graphs.

Every graph $G$ from a fixed class $\mathcal{C}$ of bounded expansion satisfies $\text{wcol}_r(G) \leq f(r)$ for some function $f$ and all positive integers $r$. However, the order that witnesses this inequality for $G$ may depend on the value $r$. We say that a class $\mathcal{C}$ admits uniform orders if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for each $G \in \mathcal{C}$ there is one linear order that witnesses $\text{wcol}_r(G) \leq f(r)$ for every value of $r$. We show that every class that excludes a fixed topological minor admits uniform orders that can be computed efficiently.

Finally, based on our construction of uniform orders for graphs that exclude a fixed topological minor, we provide an alternative proof of a very recent result of Eickmeyer and Kawarabayashi [7], that the model-checking problem for successor-invariant first-order (FO) formulas is fixed-parameter tractable on such classes (we obtained this result independently of but later than [7]). Successor-invariant logics have been studied in database theory and finite model theory, and successor-invariant FO is known to be more expressive than plain FO [15]. The model-checking problem for successor-invariant FO is known to be fixed-parameter tractable parameterized by the size of the formula on any graph class that excludes a fixed minor [6]. Very recently, this result
was lifted to classes that exclude a fixed topological minor by Eickmeyer and Kawarabayashi [7]. The key point of their proof is to use the decomposition theorem for graphs excluding a fixed topological minor, due to Grohe and Marx [9]. Our approach is similar to that of [7]. However, we employ new constructions based on the generalised colouring numbers and use the decomposition theorem of [9] only implicitly. In particular, we do not construct a graph decomposition in order to solve the model-checking problem. Therefore, we believe that our approach may be easier to extend further to classes of bounded expansion, or even to nowhere dense classes of graphs.

2 Preliminaries

Notation. We use standard graph-theoretical notation; see e.g. [3] for reference. All graphs considered in this paper are finite, simple, and undirected. For a graph $G$, by $V(G)$ and $E(G)$ we denote the vertex and edge sets of $G$, respectively. A graph $H$ is a subgraph of $G$, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any $M \subseteq V(G)$, by $G[M]$ we denote the subgraph induced by $M$. We write $G - M$ for the graph $G[V(G) \setminus M]$ and if $M = \{v\}$, we write $G - v$ for $G - M$. For a non-negative integer $\ell$, a path of length $\ell$ in $G$ is a sequence $P = (v_1, \ldots, v_{\ell+1})$ of pairwise different vertices such that $v_iv_{i+1} \in E(G)$ for all $1 \leq i \leq \ell$. We write $V(P)$ for the vertex set $\{v_1, \ldots, v_{\ell+1}\}$ of $P$ and $E(P)$ for the edge set $\{v_iv_{i+1} : 1 \leq i \leq \ell\}$ of $P$ and identify $P$ with the subgraph of $G$ with vertex set $V(P)$ and edge set $E(P)$. We say that the path $P$ connects its endpoints $v_1, v_{\ell+1}$, whereas $v_2, \ldots, v_\ell$ are the internal vertices of $P$. The length of a path is the number of its edges. Two vertices $u, v \in V(G)$ are connected if there is a path in $G$ with endpoints $u, v$. The distance $\text{dist}(u, v)$ between two connected vertices $u, v$ is the minimum length of a path connecting $u$ and $v$; if $u, v$ are not connected, we put $\text{dist}(u, v) = \infty$. The radius of $G$ is $\min_{u \in V(G)} \max_{v \in V(G)} \text{dist}(u, v)$. The set of all neighbours of a vertex $v$ in $G$ is denoted by $N^G(v)$, and the set of all vertices at distance at most $r$ from $v$ is denoted by $N_r^G(v)$. A graph $G$ is $c$-degenerate if every subgraph $H \subseteq G$ has a vertex of degree at most $c$. A $c$-degenerate graph of order $n$ contains an independent set of order at least $n/(c + 1)$.

A graph $H$ with $V(H) = \{v_1, \ldots, v_n\}$ is a minor of $G$, written $H \prec G$, if there are pairwise disjoint connected subgraphs $H_1, \ldots, H_n$ of $G$, called branch sets, such that whenever $v_iv_j \in E(H)$, then there are $u_i \in H_i$ and $u_j \in H_j$ with $u_iu_j \in E(G)$. We call $(H_1, \ldots, H_n)$ a minor model of $H$ in $G$. The graph $H$ is a topological minor of $G$, written $H \preceq^t G$, if there are pairwise different vertices $u_1, \ldots, u_n \in V(G)$ and a family of paths $\{P_{ij} : v_iv_j \in E(H)\}$, such that each $P_{ij}$ connects $u_i$ and $u_j$, and paths $P_{ij}$ are pairwise internally vertex-disjoint.

Generalised colouring numbers. Let us fix a graph $G$. By $\Pi(G)$ we denote the set of all linear orders of $V(G)$. For $L \in \Pi(G)$, we write $u <_L v$ if $u$ is smaller than $v$ in $L$, and $u \leq_L v$ if $u <_L v$ or $u = v$. Let $u, v \in V(G)$. For a non-negative integer $r$, we say that $u$ is weakly $r$-reachable from $v$ with respect to $L$, if there is a path $P$ of length $\ell$, $0 \leq \ell \leq r$, connecting $u$ and $v$ such that $u$ is minimum among the vertices of $P$ (with respect to $L$). By $\text{WReach}_L[G, L, v]$ we denote the set of vertices that are weakly $r$-reachable from $v$ w.r.t. $L$.

Vertex $u$ is strongly $r$-reachable from $v$ with respect to $L$, if there is a path $P$ of length $\ell$, $0 \leq \ell \leq r$, connecting $u$ and $v$ such that $u \leq_L v$ and such that all internal vertices $w$ of $P$ satisfy $v <_L w$. 
Let \( \text{SReach}_r[G, L, v] \) be the set of vertices that are strongly \( r \)-reachable from \( v \) w.r.t. \( L \). Note that we have \( v \in \text{SReach}_r[G, L, v] \subseteq \text{WReach}_r[G, L, v] \).

For a non-negative integer \( r \), we define the \emph{weak \( r \)-colouring number} \( \text{wcol}_r(G) \) of \( G \) and the \emph{\( r \)-colouring number} \( \text{col}_r(G) \) of \( G \) respectively as follows:

\[
\text{wcol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{WReach}_r[G, L, v]|,
\]
\[
\text{col}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{SReach}_r[G, L, v]|.
\]

For a non-negative integer \( r \), the \emph{\( r \)-admissibility} \( \text{adm}_r[G, L, v] \) of \( v \) w.r.t. \( L \) is the maximum size \( k \) of a family \( \{P_1, \ldots, P_k\} \) of paths of length at most \( r \) that start in \( v \), end at a vertex \( w \) with \( w \leq_L v \), and satisfy \( V(P_i) \cap V(P_j) = \{v\} \) for all \( 1 \leq i < j \leq k \). As for \( r > 0 \) we can always let the paths end in the first vertex smaller than \( v \), we can assume that the internal vertices of the paths are larger than \( v \). Note that \( \text{adm}_r[G, L, v] \) is an integer, whereas \( \text{WReach}_r[G, L, v] \) and \( \text{SReach}_r[G, L, v] \) are vertex sets. The \emph{\( r \)-admissibility} \( \text{adm}_r(G) \) of \( G \) is

\[
\text{adm}_r(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} \text{adm}_r[G, L, v].
\]

The generalised colouring numbers were introduced by Kierstead and Yang [11] in the context of colouring and marking games on graphs. The authors also proved that the generalised colouring numbers are related by the following inequalities:

\[
\text{adm}_r(G) \leq \text{col}_r(G) \leq \text{wcol}_r(G) \leq (\text{adm}_r(G))^r. \tag{1}
\]

**Shallow minors, bounded expansion, and nowhere denseness.** A graph \( H \) with \( V(H) = \{v_1, \ldots, v_n\} \) is a \emph{depth-\( r \) minor} of \( G \), denoted \( H \preccurlyeq_r G \), if there is a minor model \( (H_1, \ldots, H_n) \) of \( H \) in \( G \) such that each \( H_i \) has radius at most \( r \). We write \( d(H) \) for the \emph{average degree} of \( H \), that is, for the number \( 2|E(H)|/|V(H)| \). A class \( \mathcal{C} \) of graphs has \emph{bounded expansion} if there is a function \( f : \mathbb{N} \to \mathbb{N} \) such that for all non-negative integers \( r \) we have \( d(H) \leq f(r) \) for every \( H \preccurlyeq_r G \) with \( G \in \mathcal{C} \). A class \( \mathcal{C} \) of graphs is \emph{nowhere dense} if for every real \( \epsilon > 0 \) and every non-negative integer \( r \), there is an integer \( n_0 \) such that if \( H \) is an \( n \)-vertex graph with \( n \geq n_0 \) and \( H \preccurlyeq_r G \) for some \( G \in \mathcal{C} \), then \( d(H) \leq n^{\epsilon} \).

Bounded expansion and nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez as models for uniform sparseness of graphs [12, 14]. As proved by Zhu [17], the generalised colouring numbers are tightly related to densities of low-depth minors, and hence they can be used to characterise bounded expansion and nowhere dense classes.

**Theorem 1 (Zhu [17]).** A class \( \mathcal{C} \) of graphs has bounded expansion if and only if there is a function \( f : \mathbb{N} \to \mathbb{N} \) such that \( \text{wcol}_r(G) \leq f(r) \) for all \( r \in \mathbb{N} \) and all \( G \in \mathcal{C} \).

Due to Inequality (1), we may equivalently demand that there is a function \( f : \mathbb{N} \to \mathbb{N} \) such that \( \text{adm}_r(G) \leq f(r) \) or \( \text{col}_r(G) \leq f(r) \) for all non-negative integers \( r \) and all \( G \in \mathcal{C} \).
Similarly, from Zhu’s result one can derive a characterisation of nowhere dense classes of graphs, as presented in [14]. A class \( \mathcal{C} \) of graphs is called hereditary if it is closed under induced subgraphs, that is, if \( H \) is an induced subgraph of \( G \in \mathcal{C} \), then \( H \in \mathcal{C} \).

**Theorem 2 (Nešetřil and Ossona de Mendez [14]).** A hereditary class \( \mathcal{C} \) of graphs is nowhere dense if and only if for every real \( \epsilon > 0 \) and every non-negative integer \( r \), there is a positive integer \( n_0 \) such that if \( G \in \mathcal{C} \) is an \( n \)-vertex graph with \( n \geq n_0 \), then \( \text{wcol}_r(G) \leq n^\epsilon \).

As shown in [4], for every non-negative integer \( r \), computing \( \text{adm}_r(G) \) is fixed-parameter tractable on any class of bounded expansion (parameterized by \( \text{adm}_r(G) \)). For \( \text{col}_r(G) \) and \( \text{wcol}_r(G) \) this is not known; however, by (1) we can use admissibility to obtain approximations of these numbers. On nowhere dense classes of graphs, for every \( \epsilon > 0 \) and every non-negative integer \( r \), we can compute an order that witnesses \( \text{wcol}_r(G) \leq n^\epsilon \) in time \( O(n^{1+\epsilon}) \) if \( G \) is sufficiently large [8], based on Nešetřil and Ossona de Mendez’s augmentation technique [12].

### 3 Uniform quasi-wideness and the splitter game

In this section we discuss the relation between weak \( r \)-colouring numbers and two notions that characterise nowhere dense classes: uniform quasi-wideness and the splitter game.

For a graph \( G \), a vertex subset \( A \subseteq V(G) \) is called \( r \)-independent in \( G \), if \( \text{dist}_G(a,b) > r \) for all different \( a,b \in V(G) \). A vertex subset is called \( r \)-scattered, if it is \( 2r \)-independent, that is, if the \( r \)-neighbourhoods of different elements of \( A \) do not intersect.

Informally, uniform quasi-wideness means the following: in any large enough subset of vertices of a graph from \( \mathcal{C} \), one can find a large subset that is \( r \)-scattered in \( G \), possibly after removing from \( G \) a small number of vertices. Formally, a class \( \mathcal{C} \) of graphs is uniformly quasi-wide if there are functions \( N : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) and \( s : \mathbb{N} \to \mathbb{N} \) such that for all \( m,r \in \mathbb{N} \), if \( W \subseteq V(G) \) for a graph \( G \in \mathcal{C} \) with \( |W| > N(m,r) \), then there is a set \( S \subseteq V(G) \) of size at most \( s(r) \) such that \( W \) contains a subset of size at least \( m \) that is \( r \)-scattered in \( G - S \).

The notion of quasi-wideness was introduced by Dawar [2] in the context of homomorphism preservation theorems. It was shown in [13] that classes of bounded expansion are uniformly quasi-wide and that uniform quasi-wideness characterises nowhere dense classes of graphs.

**Theorem 3 (Nešetřil and Ossona de Mendez [13]).** A hereditary class \( \mathcal{C} \) of graphs is nowhere dense if and only if it is uniformly quasi-wide.

It was shown by Atserias et al. in [1] that classes that exclude \( K_k \) as a minor are uniformly quasi-wide. In fact, in this case we can choose \( s(r) = k - 1 \), independent of \( r \) (if such a constant function for a class \( \mathcal{C} \) exists, the class is called uniformly almost wide). However, the function \( N(m,r) \) that was used in the proof is huge: it comes from an iterated Ramsey argument. The same approach was used in [13] to show that every nowhere dense class, and in particular, every class of bounded expansion, is uniformly quasi-wide. We present a new proof that every bounded
We distinguish two cases. First, suppose \( \text{Reach}_r(G, L, v) \) which is much simpler than the previously known proof.

**Theorem 4.** Let \( G \) be a graph and let \( r, m \in \mathbb{N} \). Let \( c \in \mathbb{N} \) be such that \( \text{wcol}_r(G) \leq c \) and let \( A \subseteq V(G) \) be a set of size at least \((c+1) \cdot 2^m\). Then there exists a set \( S \) of size at most \( c(c-1) \) and a set \( B \subseteq A \) of size at least \( m \) which is \( r \)-independent in \( G - S \).

**Proof.** Let \( L \in \Pi(G) \) be such that \( |\text{Reach}_r(G, L, v)| \leq c \) for every \( v \in V(G) \). Let \( H \) be the graph with vertex set \( V(G) \), where we put an edge \( uv \in E(H) \) if and only if \( u \in \text{Reach}_r(G, L, v) \) or \( v \in \text{Reach}_r(G, L, u) \). Then \( L \) certifies that \( H \) is \( c \)-degenerate, and hence we can greedily find an independent set \( I \subseteq A \) of size \( 2^m \) in \( H \). By the definition of the graph \( H \), we have that \( \text{Reach}_r(G, L, v) \cap I = \{v\} \) for each \( v \in I \).

**Claim.** Let \( v \in I \). Then deleting \( \text{Reach}_r(G, L, v) \setminus \{v\} \) from \( G \) leaves \( v \) at a distance greater than \( r \) (in \( G - (\text{Reach}_r(G, L, v) \setminus \{v\}) \)) from all the other vertices of \( I \).

**Proof.** Let \( u \in I \) and let \( P \) be a path in \( G \) that has length at most \( r \) and connects \( u \) and \( v \). Let \( z \in V(P) \) be minimal with respect to \( L \). Then \( z <_L v \) or \( z = v \). If \( z <_L v \), then \( z \in \text{Reach}_r(G, L, v) \) and hence the path \( P \) no longer exists after the deletion of \( \text{Reach}_r(G, L, v) \setminus \{v\} \) from \( G \). On the other hand, if \( z = v \), then \( v \in \text{Reach}_r(G, L, u) \), which contradicts the fact that both \( u, v \in I \).

We iteratively find sets \( B_0 \subseteq \ldots \subseteq B_m \subseteq I \), sets \( I_0 \supseteq \ldots \supseteq I_m \), and sets \( S_0 \subseteq \ldots \subseteq S_m \) such that \( B \) is \( r \)-independent in \( G - S \), where \( B := B_m \) and \( S := S_m \). We maintain the invariant that sets \( B_i, I_i, \text{ and } S_i \) are pairwise disjoint for each \( i \). Let \( I_0 = I, B_0 = \emptyset \) and \( S_0 = \emptyset \). In one step \( i = 1, 2, \ldots, m \), we delete some vertices from \( I_i \) (thus obtaining \( I_{i+1} \)), shift one vertex from \( I_i \) to \( B_i \) (obtaining \( B_{i+1} \)) and, possibly, add some vertices from \( V(G) \setminus I_i \) to \( S_i \) (obtaining \( S_{i+1} \)). More precisely, let \( v \) be the vertex of \( I_i \) that is the largest in the order \( L \). We set \( B_{i+1} = B_i \cup \{v\} \), and now we discuss how \( I_{i+1} \) and \( S_{i+1} \) are constructed.

We distinguish two cases. First, suppose \( v \) is connected by a path of length at most \( r \) in \( G - S_i \) to at most half of the vertices of \( I_i \) (including \( v \)). Then we remove these reachable vertices from \( I_i \), and set \( I_{i+1} \) to be the result. We also set \( S_{i+1} = S_i \). Note that \( |I_{i+1}| \geq |I_i|/2 \).

Second, suppose \( v \) is connected by a path of length at most \( r \) in \( G - S_i \) to more than half of the vertices of \( I_i \) (including \( v \)). We proceed in two steps. First, we add the at most \( c - 1 \) vertices of \( \text{Reach}_r(G, L, v) \setminus \{v\} \) to \( I_{i+1} \), that is, we let \( I_{i+1} = S_i \cup (\text{Reach}_r(G, L, v) \setminus \{v\}) \). (Recall here that \( \text{Reach}_r(G, L, v) \cap I = \{v\} \).) By Claim 1, this leaves \( v \) at a distance greater than \( r \) from every other vertex of \( I_i \) in \( G - S_{i+1} \). Second, we construct \( I_{i+1} \) from \( I_i \) by removing the vertex \( v \) and all the vertices of \( I_i \) that are not connected to \( v \) by a path of length at most \( r \) in \( G - S_i \), hence we have \( |I_{i+1}| \geq \lfloor |I_i|/2 \rfloor \).

Observe the construction above can be carried out for \( m \) steps, because in each step, we remove at most half of the vertices of \( I_i \) (rounded up) when constructing \( I_{i+1} \). As \( |I_0| = |I| = 2^m \), it is easy to see that the set \( I_i \) cannot become empty within \( m \) iterations. Moreover, it is clear from the construction that we end up with a set \( B = B_m \) that has size \( m \) and is \( r \)-scattered in \( G - S \),
where $S = S_m$. It remains to argue that $|S_m| \leq c(c-1)$. For this, it suffices to show that the second case cannot apply more than $c$ times in total.

Suppose the second case was applied in the $i$th iteration, when considering a vertex $v$. Every vertex $u \in I_i$ with $u <_L v$ that was connected to $v$ by a path of length at most $r$ in $G - S_i$ satisfies $\text{WReach}_r[G, L, v] \cap \text{WReach}_r[G, L, u] \neq \emptyset$. Thus, every remaining vertex $u \in I_{i+1}$ has at least one of its weakly $r$-reachable vertices deleted (that is, included in $S_{i+1}$). As the number of such vertices is at most $c - 1$ at the beginning, and it can only decrease during the construction, this implies that the second case can occur at most $c$ times.

As shown in [16], if $K_k \not\cong G$, then $\text{wcol}_k(G) \in \mathcal{O}(r^{k-1})$. Hence, for such graphs we have to delete only a polynomial (in $r$) number of vertices in order to find an $r$-independent set of size $m$ in a set of vertices of size single exponential in $m$.

We now implement the same idea to find a very simple strategy for splitter in the splitter game, introduced by Grohe et al. [8] to characterise nowhere dense classes of graphs. Let $\ell, r \in \mathbb{N}$. The simple $\ell$-round radius-$r$ splitter game on $G$ is played by two players, connector and splitter, as follows. We let $G_0 := G$. In round $i + 1$ of the game, connector chooses a vertex $v_{i+1} \in V(G_i)$. Then splitter picks a vertex $w_{i+1} \in N_r^{G_i}(v_{i+1})$. We let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. Splitter wins if $G_{i+1} = \emptyset$. Otherwise the game continues at $G_{i+1}$. If splitter has not won after $\ell$ rounds, then connector wins.

A strategy for splitter is a function $\sigma$ that maps every partial play $(v_1, w_1, \ldots, v_s, w_s)$, with associated sequence $G_0, \ldots, G_s$ of graphs, and the next move $v_{s+1} \in V(G_s)$ of connector, to a vertex $w_{s+1} \in N_r^{G_s}(v_{s+1})$ that is the next move of splitter. A strategy $\sigma$ is a winning strategy for splitter if splitter wins every play in which she follows the strategy $\sigma$. We say that splitter wins the simple $\ell$-round radius-$r$ splitter game on $G$ if she has a winning strategy.

**Theorem 5 (Grohe et al. [8]).** A class $C$ of graphs is nowhere dense if and only if there is a function $\ell : \mathbb{N} \to \mathbb{N}$ such that splitter wins the simple $\ell(r)$-round radius-$r$ splitter game on every graph $G \in C$.

More precisely, it was shown in [8] that $\ell(r)$ can be chosen as $N(2s(r), r)$, where $N$ and $s$ are the functions that characterise $C$ as a uniformly quasi-wide class of graphs. We present a proof that on bounded expansion classes, splitter can win much faster.

**Theorem 6.** Let $G$ be a graph, let $r \in \mathbb{N}$ and let $\ell = \text{wcol}_2r(G)$. Then splitter wins the $\ell$-round radius-$r$ splitter game.

**PROOF.** Let $L$ be a linear order that witnesses $\text{wcol}_2r(G) = \ell$. Suppose in round $i + 1 \leq \ell$, connector chooses a vertex $v_{i+1} \in V(G_i)$. Let $w_{i+1}$ (splitter’s choice) be the minimum vertex of $N_r^{G_i}(v_{i+1})$ with respect to $L$. Then for each $u \in N_r^{G_i}(v_{i+1})$ there is a path between $u$ and $w_{i+1}$ of length at most $2r$ that uses only vertices of $N_r^{G_i}(v_{i+1})$. As $w_i$ is minimum in $N_r^{G_i}(v_{i+1})$, $w_{i+1}$ is weakly $2r$-reachable from each $u \in N_r^{G_i}(v_{i+1})$. Now let $G_{i+1} := G_i[N_r^{G_i}(v_{i+1}) \setminus \{w_{i+1}\}]$. As $w_{i+1}$ is not part of $G_{i+1}$, in the next round splitter will choose another vertex which is weakly $2r$-reachable from every vertex of the remaining $r$-neighbourhood. As $\text{wcol}_2r(G) = \ell$, the game must stop after at most $\ell$ rounds. □
4 Uniform orders for graphs excluding a topological minor

If $\mathcal{C}$ is a class of bounded expansion such that $\text{wcol}_r(G) \leq f(r)$ for all $G \in \mathcal{C}$ and all $r \in \mathbb{N}$, the order $L$ that witnesses this inequality for $G$ may depend on the value $r$. We say that a class $\mathcal{C}$ admits uniform orders if there is a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for each $G \in \mathcal{C}$, there is a linear order $L \in \Pi(G)$ such that $|\text{WReach}_r(G, L, v)| \leq f(r)$ for all $v \in V(G)$ and all $r \in \mathbb{N}$. In other words, there is one order that simultaneously certifies the inequality $\text{wcol}_r(G) \leq f(r)$ for all $r$.

It is implicit in [16] that every class that excludes a fixed minor admits uniform orders, which can be efficiently computed. We are going to show that the same holds for classes that exclude a fixed topological minor. Our construction is similar to the construction of [16], in particular, our orders can be computed quickly in a greedy fashion. The proof that we find an order of high quality is based on the decomposition theorem for graphs with excluded topological minors, due to Grohe and Marx [9]. Note however, that for the construction of the order we do not have to construct a tree decomposition according to Grohe and Marx [9].

**Construction.** Let $G$ be a graph. We present a construction of an order of $V(G)$ of high quality. We iteratively construct a sequence $H_1, \ldots, H_\ell$ of pairwise disjoint and connected subgraphs of $G$ such that $\bigcup_{1 \leq i \leq \ell} V(H_i) = V(G)$. For $0 \leq i < \ell$, let $G_i := G - \bigcup_{1 \leq j \leq i} V(H_j)$. We say that a component $C$ of $G_i$ is connected to a subgraph $H_j$, $j \leq i$, if there is a vertex $u \in V(H_j)$ and a vertex $v \in V(C)$ such that $uv \in E(G)$. For all $i$, $1 \leq i < \ell$, we will maintain the following invariant. If $C$ is a component of $G_i$, then the subgraphs $H_{i_1}, \ldots, H_{i_s} \in \{H_1, \ldots, H_i\}$ that are connected to $C$ form a minor model of the complete graph $K_s$, where $s$ is their number.

To start, we choose an arbitrary vertex $v \in V(G)$ and let $H_1$ be the connected subgraph $G \{v\}$.

Clearly, $H_1$ satisfies the above invariant. Now assume that for some $i$, $1 \leq i < \ell$, the sequence $H_1, \ldots, H_i$ has already been constructed. Fix some component $C$ of $G_i$ and, by the invariant, assume that the subgraphs $H_{i_1}, \ldots, H_{i_s} \in \{H_1, \ldots, H_i\}$ with $1 \leq i_1 < \ldots < i_s \leq i$ that have a connection to $C$ form a minor model of $K_s$. For a vertex $v \in V(C)$, let $m(v)$ be the maximum cardinality of a family $\mathcal{P}$ of paths with the following properties: each path of $\mathcal{P}$ connects $v$ with a different subgraph $H_{i_j}$, the internal vertices of each path from $\mathcal{P}$ belong to $G_i$, and the paths of $\mathcal{P}$ are pairwise disjoint apart from sharing $v$. Note that $m(v)$ can be computed in polynomial time using any maximum flow algorithm. Pick $v$ to be a vertex of $C$ with maximum $m(v)$. Let $T$ be the tree of the breadth-first search in $G[C]$ that starts in $v$; thus, $T$ is rooted at $v$. We choose $H_{i+1}$ to be a minimal connected subtree of $T$ that contains $v$ and, for each $j$ with $1 \leq j \leq s$, at least one neighbour of $H_{i_j}$ in $C$.

From the construction it is easy to see that for every component $C'$ of $G_{i+1}$, the subgraphs $H'_{i_1}, \ldots, H'_{i_s} \in \{H_1, \ldots, H_{i+1}\}$ that are connected to $C'$ form the minor model of a complete graph, hence the invariant is again established. Having chosen $H_{i+1}$, we proceed to the next iteration. The construction stops when all vertices are part of some $H_i$, $1 \leq i \leq \ell$.

We construct an order $L$ of $V(G)$ as follows. Let $v \prec_L u$ if $v \in V(H_i)$ and $u \in V(H_j)$ for some $i < j$. Furthermore, we order the vertices within each $H_i$ arbitrarily. Obviously, the construction does not depend on $r$, hence the produced order is uniform for $G$.  

## Footnotes

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Analysis. From now on we assume that $G$ excludes $K_k$ as a topological minor, for some constant $k$. Furthermore, assume that the graphs $H_1, \ldots, H_l$ and a corresponding order $L$ have been constructed, as described above. We now show that the constructed order has good qualities. Our proof is based on the following two key lemmas. The first lemma states that for every component $C$ of $G_i$ arising after the construction of $H_1, \ldots, H_i$, every vertex $v$ of $C$ can reach only a bounded number of subgraphs among $H_1, \ldots, H_i$ by disjoint paths.

Lemma 7. There is a constant $\alpha$ (depending only on $k$) such that for all integers $i$, $1 \leq i < \ell$, if $C$ is a component of $G_i$, then for every vertex $v \in V(C)$, we have $m(v) \leq \alpha$, where $m(v)$ is defined as in the construction.

The second lemma states that from a vertex of $H_{i+1}$, we can reach only a bounded number of vertices of each $H_j$, $1 \leq j \leq i + 1$, by short disjoint paths in $G_i$.

Lemma 8. There is a constant $\beta$ (depending only on $k$) such that for all integers $i, j$, where $1 \leq j \leq i \leq \ell$, and all positive integers $r$, the following holds. Suppose $v \in V(H_i)$, and let $\mathcal{P}$ be any family of paths of length at most $r$ with the following properties: each path from $\mathcal{P}$ connects $v$ with a different vertex of $H_j$, the internal vertices of $\mathcal{P}$ belong to $G_j$, and paths from $\mathcal{P}$ are internally vertex disjoint. Then $\mathcal{P}$ has size not larger than $\beta \cdot r$.

It is easy to show that the above two lemmas guarantee that $L$ has the required properties.

Corollary 9. If $K_k \not\cong G$, then there exists a constant $\gamma$ (depending only on $k$) and a uniform order $L$ that witnesses $\text{adm}_r(G) \leq \gamma \cdot r$ for all non-negative integers $r$.

Proof. The $r$-admissibility of a vertex $v$ is determined in the construction when $v$ first appears in some $H_i$, $1 \leq i < \ell$. More precisely, $\text{adm}_r(G)$ is upper bounded by the maximum possible number of disjoint paths in $G_i$ of length at most $r$ from $v$ to vertices of $H_1 \cup \ldots \cup H_i$. By Lemma 7, there is a constant $\alpha$ such that $v$ can reach at most $\alpha$ distinct $H_j$, $1 \leq j < i$, via internally vertex disjoint paths in $G_i$. Additionally, $v$ can reach other vertices of $H_i$. By Lemma 8, there is a constant $\beta$ such that $v$ reach at most $\beta \cdot r$ vertices of each $H_j$ with $j \leq i$ by internally vertex disjoint paths of length at most $r$. Let $\gamma := (\alpha + 1) \cdot \beta$, then $v$ can reach at most $\gamma \cdot r$ smaller vertices by internally vertex disjoint paths of length at most $r$ whose internal vertices are larger than $v$ w.r.t. $L$.

The proof of Lemma 7 is based on the decomposition theorem for graphs with excluded topological minors of Grohe and Marx [9]. Recall that a tree decomposition of a graph $G$ is a pair $(T, \beta)$, where $T$ is a tree and $\beta : V(T) \to 2^V(G)$, such that for every vertex $v \in V(G)$ the set $\beta^{-1}(v) = \{ t \in V(T) : v \in \beta(t) \}$ is non-empty and connected in $T$, and for every edge $e \in E(G)$ there is a node $t \in V(T)$ such that $e \subseteq \beta(t)$. The width of $(T, \beta)$ is max{ $|\beta(t)| - 1 : t \in V(T)$ } and the adhesion of $(T, \beta)$ is max{ $|\beta(s) \cap \beta(t)| : st \in E(T)$ }.

For a node $t \in T$, we call $\beta(t)$ the bag at $t$. If $T' \subseteq T$, we write $\beta(T')$ for $\bigcup_{t' \in V(T')} \beta(t')$ and if $M \subseteq V(G)$, we write $\beta^{-1}(M)$ for $\bigcup_{v \in M} \beta^{-1}(v)$. Denote by $K[X]$ the complete graph on a vertex set $X$. The torso at $t$ is the graph $\tau(t) := G[\beta(t)] \cup \bigcup_{st \in E(T)} K[\beta(s) \cap \beta(t)]$. 

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Theorem 10 ([9]). For every \( k \in \mathbb{N} \), there exist constants \( a(k), c(k), d(k) \) and \( e(k) \) such that the following holds. Let \( H \) be a graph on \( k \) vertices. Then for every graph \( G \) with \( H \not\approx G \) there is a tree decomposition \((T,\beta)\) of adhesion at most \( a(k) \) such that for all \( t \in V(T) \) one of the following two alternatives hold.

1. The torso \( \tau(t) \) has at most \( c(k) \) vertices of degree larger than \( d(k) \), which we call the apex vertices of \( \tau(t) \). Such a node \( t \) will be called a bounded degree node.

2. The torso \( \tau(t) \) excludes the complete graph \( K_{e(k)} \) as a minor. Such a node \( t \) will be called an excluded minor node.

We will need the following well-known properties of trees and tree decompositions.

Lemma 11 (Helly-property for trees). Let \( T \) be a tree and let \((T_i)_{i \in I}\) be a family of subtrees of \( T \). If \( V(T_i) \cap V(T_j) \neq \emptyset \), for all \( i,j \in I \), then \( \bigcap_{i \in I} V(T_i) \neq \emptyset \).

Lemma 12. Let \((T,\beta)\) be a tree decomposition of a graph \( G \). Let \( e = st \) be an edge of \( T \) and let \( T_1, T_2 \) be the components of \( T - e \). Then \( \beta(s) \cap \beta(t) \) separates \( \beta(T_1) \) from \( \beta(T_2) \), that is, every path from a vertex of \( \beta(T_1) \) to a vertex of \( \beta(T_2) \) traverses a vertex of \( \beta(s) \cap \beta(t) \).

Lemma 13. If \( H \subseteq G \) is a connected subgraph of \( G \), then \( \beta^{-1}(V(H)) \) is connected in \( T \).

For the proof of Lemma 7, assume that \( G \) is decomposed as described by Theorem 10. Assume that \( H_1, \ldots, H_s \) have been constructed and let \( C \) be a component of \( G \) that has a connection to the subgraphs \( H_{i_1}, \ldots, H_{i_s} \). Recall that throughout the construction we guarantee that the subgraphs \( H_{i_1}, \ldots, H_{i_s} \) form the minor model of a complete graph \( K_s \). We first identify one bag of the decomposition as a bag which intersects many distinct branch sets of this minor model. The following lemma follows easily from the separator properties of tree decompositions, in particular Lemma 12.

Lemma 14. There can be at most one node \( t \) such that \( \beta(t) \) intersects strictly more than \( a(k) \) of the branch sets \( H_{i_j} \), for \( 1 \leq j \leq s \).

**Proof.** Assume there are two distinct nodes \( t_1 \) and \( t_2 \) with this property, and suppose \( \beta(t_1) \) intersects \( R^1_{q_1(k)+1}, \ldots, R^1_{q_2(k)+1} \in \{H_{i_1}, \ldots, H_{i_s}\} \), whereas \( \beta(t_2) \) intersects \( R^2_{q_1(k)+1}, \ldots, R^2_{q_2(k)+1} \in \{H_{i_1}, \ldots, H_{i_s}\} \). Note that some branch sets \( R^1_p \) and \( R^2_q \) may coincide, but by reorder if necessary, we may assume w.l.o.g. that \( R^1_p = R^2_q \) can happen only if \( p = q \). Observe that since \( H_{i_1}, \ldots, H_{i_s} \) form a minor model of a complete graph, the union of the vertex sets of \( R^1_p \) and \( R^2_p \) induces a connected subgraph of \( G \), for each \( p \) with \( 1 \leq p \leq a(k) + 1 \); note that this is true also if \( R^1_p = R^2_p \). Consequently, if \( e = s_1s_2 \) is any edge of \( T \) whose removal disconnects \( t_1 \) from \( t_2 \), then by Lemma 12 we have that \( \beta(s_1) \cap \beta(s_2) \) must contain at least one vertex from \( V(R^1_p) \cup V(R^2_p) \) for each \( p \) with \( 1 \leq p \leq a(k) + 1 \). Sets \( V(R^1_p) \cup V(R^2_p) \) are disjoint for distinct \( p \), hence we conclude that \( |\beta(s_1) \cap \beta(s_2)| \geq a(k) + 1 \). This contradicts the fact that \((T,\beta)\) has adhesion at most \( a(k) \). \( \square \)

We now show that there is a bag that intersects every branch set. The proof is a simple application of the Helly property of trees (Lemma 11) and Lemma 13.
**Lemma 15.** There is a node $t$ such that $\beta(t)$ intersects each $H_{ij}$, for $1 \leq j \leq s$.

**Proof.** As each $H_{ij}$ is connected, by Lemma 13 we have that $\beta^{-1}(V(H_{ij}))$ induces a subtree $T_{ij}$ in $T$. As the $\{H_{i1}, \ldots, H_{is}\}$ form a model of a complete graph, they are pairwise connected and hence, by the definition of tree decompositions, any two such subtrees intersect. By Lemma 11, they all intersect in one node $t$, which is as desired. □

Hence, provided $s > a(k)$, there is a node $t$ with $\beta(t)$ intersecting at least $a(k) + 1$ branch sets $H_{ij}$. By Lemma 14, this node is unique. We call it the core node of the minor model. Next we show that if the model is large, then its core node must be a bounded degree node. Shortly speaking, this is because the model $H_{i1}, \ldots, H_{is}$ trimmed to the torso of the core node is already a minor model of $K_s$ in this torso.

**Lemma 16.** If $s > \max\{a(k), e(k)\}$, then the core node of the minor model is a bounded degree node.

**Proof.** As $s > a(k)$, by Lemma 14 we can identify the unique core node $t$ whose bag intersects all the branch sets $H_{ij}$. Recall that $\tau(t)$ is the graph induced by the bag $\beta(t)$ in which all adjacent separators are turned into cliques. It is easy to see that the subgraphs $H'_{ij} := \tau(t)[V(H_{ij}) \cap \beta(t)]$ are connected in $\tau(t)$ and form a minor model of $K_s$. As $s > e(k)$, we infer that $t$ cannot be an excluded minor node, and hence it is a bounded degree node. □

For vertices outside the bag of the core node, the bound promised in Lemma 7 can be proved similarly as Lemma 14.

**Lemma 17.** Let $C$ be a component of $G_i$ that has a connection to the subgraphs $H_{i1}, \ldots, H_{is}$. If $s > a(k)$, then for every vertex $v \in V(C) \setminus \beta(t)$, where $t$ is the core node of the model, we have that $m(v) \leq a(k)$.

**Proof.** By the properties of a tree decomposition, there is an edge $e = tt'$ of $T$ such that $\beta^{-1}(v)$ is contained in the subtree of $T - e$ that contains $t'$. Suppose $\mathcal{P}$ is a family of paths that connect $v$ with distinct branch sets $H_{ij}$ and are pairwise disjoint apart from $v$. Recall that $\beta(t)$ intersects every branch set $H_{ij}$. Therefore, by extending each path of $\mathcal{P}$ within the branch set it leads to, we can assume w.l.o.g. that each path of $\mathcal{P}$ connects $v$ with a vertex of $\beta(t)$. By Lemma 12, this implies that each path of $\mathcal{P}$ intersects $\beta(t) \cap \beta(t')$. Paths of $\mathcal{P}$ share only $v$, which is not contained in $\beta(t) \cap \beta(t')$, and hence we conclude that $|\mathcal{P}| \leq |\beta(t) \cap \beta(t')|$. As $\mathcal{P}$ was chosen arbitrarily, we obtain that $m(v) \leq |\beta(t) \cap \beta(t')| \leq a(k)$. □

We now complete the proof of Lemma 7 by looking at the vertices inside the core bag.

**Proof.** (Proof of Lemma 7) We set $\alpha := a(k) + c(k) + d(k) + e(k)$. Assume towards a contradiction that for some $i$, $1 \leq i < \ell$, we have that some component $C$ of $G_i$ contains a vertex $v_1$ with $m(v_1) > \alpha$. Denote the branch sets that have a connection to $C$ by $H_{i1}, \ldots, H_{is}$, where $i_1 < i_2 < \ldots < i_s$. Let $\mathcal{P}$ be a maximum-size family of paths that pairwise share only $v_1$ and connect $v_1$ with different branch sets $H_{ij}$. As $m(v_1) > \alpha$, we have that $|\mathcal{P}| > \alpha$, and in particular $s > \alpha$. As $\alpha > a(k)$, by Lemma 14 and Lemma 15 we can identify the unique core node $t$ of the
minor model. As \( s > \max\{a(k), e(k)\} \), by Lemma 16 the core node is a bounded degree node. As \( m(v_1) > a(k) \), by Lemma 17 we have \( v_1 \in \beta(t) \). As \( \mathcal{P} \) contains more than \( d(k) \) disjoint paths from \( v \) to distinct branch sets, the degree of \( v_1 \) in \( G \) must be greater than \( d(k) \), hence \( v_1 \) is an apex vertex of \( \tau(t) \).

Since \( i_1 < i_2 < \ldots < i_s \), we have that the component \( C \) was created when \( H_{i_s} \) was removed from \( G_{i_s-1} \). Let \( C' \) be the component of \( G_{i_s-1} \) that contains \( C \) and \( H_{i_s} \) (and thus \( v_1 \)). Observe that \( C' \) is still connected to \( H_1, \ldots, H_{i_s-1} \), and possibly to some other branch sets. Recall that \( H_{i_s} \) was constructed as a subtree of the breadth-first search tree in \( G_{i_s} \) that started in a vertex \( v_2 \in V(C') \) which, at this point of the construction, had maximum \( m(v_2) \) among vertices in \( C' \). However, at this point vertex \( v_1 \) was also present in \( C' \), and \( \mathcal{P} \) certifies that it could send at least \( \alpha - 1 \) disjoint paths to different branch sets among \( H_1, \ldots, H_{i_s-1} \) (in \( \mathcal{P} \), at most one path leads to \( H_{i_s} \), and all the other paths are also present in \( C' \)). We infer that it held that \( m(v_2) \geq \alpha - 1 \) at the moment \( v_2 \) was taken. Since \( \alpha > a(k) + c(k) + d(k) + e(k) \geq a(k) + d(k) + e(k) + 1 \), the same reasoning as above shows that \( t \) is also the core vertex of the minor model formed by branch sets connected to \( C' \). Thus, by exactly the same reasoning we obtain that \( v_2 \) is also an apex vertex of \( \tau(t) \).

Since \( \alpha > a(k) + c(k) + d(k) + e(k) \), we can repeat this reasoning \( c(k) + 1 \) times, obtaining vertices \( v_1, \ldots, v_{C(k)+1} \), which are all apex vertices of \( \tau(t) \). This contradicts the fact that \( \tau(t) \) contains at most \( c(k) \) apex vertices. \( \square \)

**Proof. (Proof of Lemma 8)** We set \( \beta \) so that \( \beta \cdot r \geq (2r + 1) \cdot \alpha \), where \( \alpha \) is the constant given by Lemma 7. For the sake of contradiction, suppose there is a family of paths \( \mathcal{P} \) as in the statement, whose size is larger than \( (2r + 1) \cdot \alpha \).

Recall that \( H_j \) was chosen as a subtree of a breadth-first search tree in \( G_{j-1} \); throughout the proof, we treat \( H_j \) as a rooted tree. As \( H_j \) is a subtree of a BFS tree, every path from a vertex \( w \) of the tree to the root \( v' \) of the tree is an isometric path in \( G_{j-1} \), that is, a shortest path between \( w \) and \( v' \) in the graph \( G_{j-1} \). If \( P \) is an isometric path in a graph \( H \), then \( |N^H(v) \cap V(P)| \leq 2r + 1 \) for all \( v \in V(H) \) and all \( r \in \mathbb{N} \). As the paths from \( \mathcal{P} \) are all contained in \( G_{j-1} \), and they have lengths at most \( r \), this implies that the path family \( \mathcal{P} \) cannot connect \( v \) with more than \( 2r + 1 \) vertices of \( H_j \) which lie on the same root-to-leaf path in \( H_j \). Since \( |\mathcal{P}| > (2r + 1) \cdot \alpha \), we can find a set \( X \subseteq V(H_j) \) such that \( |X| > \alpha \), each vertex of \( X \) is connected to \( v \) by some path from \( \mathcal{P} \), and no two vertices of \( X \) lie on the same root-to-leaf path in \( H_j \). Recall that, by the construction, each leaf of \( H_j \) is connected to a different branch set \( H_{j'} \) for some \( j' < j \). Consequently, we can take the paths of \( \mathcal{P} \) leading to \( X \) and extend them within \( H_j \) to obtain a family of more than \( \alpha \) disjoint paths in \( G_{j-1} \) that connect \( v \) with different branch sets \( H_{j'} \) for \( j' < j \). This contradicts Lemma 7. \( \square \)

Observe that the order can be computed in time \( O(n^5) \): for each vertex, we compute by a standard flow algorithm in time \( O(n^3) \) whether it should be chosen as the next tree root to form a subgraph \( H_{ij} \). This choice has to be made at most \( n \) times.

Finally, we state one property of the construction that follows immediately from Lemma 7.

**Lemma 18.** Each constructed subgraph \( H_i \) has maximum degree at most \( \alpha + 1 \), where \( \alpha \) is the constant given by Lemma 7.

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5 Model-checking for successor-invariant first-order formulas

A finite and purely relational signature $\tau$ is a finite set $\{R_1, \ldots, R_k\}$ of relation symbols, where each relation symbol $R_i$ has an associated arity $a_i$. A finite $\tau$-structure $\mathcal{A}$ consists of a finite set $A$, the universe of $\mathcal{A}$, and a relation $R_i(A) \subseteq A^{a_i}$ for each relation symbol $R_i \in \tau$. If $\mathcal{A}$ is a finite $\tau$-structure, then the Gaifman graph of $\mathcal{A}$, denoted $G(\mathcal{A})$, is the graph with $V(G(\mathcal{A})) = A$ and there is an edge $uv \in E(G(\mathcal{A}))$ if and only if $u \neq v$ and $u$ and $v$ appear together in some relation $R_i(A)$ of $\mathcal{A}$. We say that a class $C$ of finite $\tau$-structures has bounded expansion if the graph class $G(C) := \{G(A) : A \in C\}$ has bounded expansion. Similarly, for $r \in \mathbb{N}$, we write $\text{adm}_r(\mathcal{A})$ for $\text{adm}_r(G(\mathcal{A}))$ etc.

Let $V$ be a set. A successor relation on $V$ is a binary relation $S \subseteq V \times V$ such that $(V, S)$ is a directed path of length $|V| - 1$. Let $\tau$ be a finite relational signature. A formula $\varphi \in \text{FO}[\sigma \cup \{S\}]$ is successor-invariant if for all $\tau$-structures $\mathcal{A}$ and for all successor relations $S_1, S_2$ on $V(\mathcal{A})$ it holds that $(\mathcal{A}, S_1) \models \varphi \iff (\mathcal{A}, S_2) \models \varphi$.

Successor-invariant logics have been studied in database theory and finite model theory in the past. It was shown by Rossman [15] that successor-invariant FO is more expressive than FO without access to a successor relation. It is known that successor-invariant FO (in fact even order-invariant FO) can express only local queries [10], however, the proof does not translate formulas into local FO-formulas which could be evaluated algorithmically. It was shown in [6] that the model-checking problem for successor-invariant first-order formulas is fixed-parameter tractable on any proper minor closed class of graphs. Very recently, the same result was shown for classes with excluded topological minors [7]. We give a new proof of the model-checking result of [7] which is based on the nice properties of the order we have constructed for graphs that exclude a topological minor.

Eickmeyer et al. [6] showed that on well-behaved classes of graphs one can apply the following reduction from the model-checking problem for successor-invariant formulas to the model-checking problem for plain first-order formulas.

Lemma 19 (Eickmeyer et al. [6]). Let $C$ be a class of $\tau$-structures such that for each $\mathcal{A} \in C$ one can compute in polynomial time a graph $H(\mathcal{A})$ such that

1. $V(H(\mathcal{A})) = V(G(\mathcal{A}))$ and $E(H(\mathcal{A})) \supseteq E(G(\mathcal{A}))$.
2. $H$ contains a spanning tree $T$ which can be computed in polynomial time and which is of maximum degree $d$ for some fixed integer $d$ depending on $C$ only.
3. The model-checking problem for first-order formulas on the graph class $\{H(\mathcal{A}) : \mathcal{A} \in C\}$ is fixed-parameter tractable.

Then the model-checking problem for successor-invariant first-order formulas is fixed-parameter tractable on $C$.

We remark that the original lemma from [6] refers to $k$-walks in $H$, which are easily seen to be
We now construct \( H \). We perform the construction of the subgraphs where \( \alpha \) is the constant given by Lemma 7.

**Lemma 20.** Let \( k \in \mathbb{N} \). There is a constant \( \delta \), depending only on \( k \), and a function \( f: \mathbb{N} \to \mathbb{N} \) such that the following holds. For every graph \( G \) with \( K_k \not\subseteq G \) we can compute in polynomial time a supergraph \( H \) with \( V(H) = V(G) \) and \( E(H) \supseteq E(G) \) such that \( \text{adm}_r(H) \leq f(r) \) for all \( r \in \mathbb{N} \) and such that \( H \) contains a spanning tree \( T \) with maximum degree at most \( \delta \); furthermore, such a spanning tree \( T \) can be also computed in polynomial time.

**Proof.** Without loss of generality, we assume that \( G \) is connected. Otherwise, we may apply the construction in each connected component separately, and then connect the components arbitrarily using single edges (added to \( H \)) in a path-like manner. It is easy to see that including the additional edges to the spanning tree increases its maximum degree by at most 2, while the admissibility of the graph also increases by at most 2.

We perform the construction of the subgraphs \( H_1, \ldots, H_\ell \) almost exactly as in Lemma 4. However, when constructing the \( H_i \)'s and the order \( L \), we put some additional restrictions that do not change the quality of \( L \). First, recall that when we defined \( H_{i+1} \), for some \( 0 \leq i < \ell \), we considered a tree of breadth-first search starting at \( v_{i+1} \) in a connected component \( C \) of \( G_i \).

Suppose that the subgraphs that are going to be included in some edges of \( H \) are ordered by \( L \). For each \( i \) that is going to be included in \( H \), we could order the vertices of \( H_i \) arbitrarily. Hence, we fix an order of \( H_i \) as a spanning edge, and add edges \( \{v_1, v_2, \ldots, v_h\} \) to \( H \), marking them as spanning edges as well.

**Claim.** The spanning edges form a spanning tree of \( H \) of maximum degree at most \( \alpha + 4 \), where \( \alpha \) is the constant given by Lemma 7.
Proof. Because the branch sets partition the graph, the spanning edges form a spanning subgraph of $H$. Because we connect the branch set $H_{i+1}$ only to the largest reachable branch set $H_i$, (and this set is never again the largest reachable branch set for $H_j$, $j > i$), the spanning subgraph is acyclic. It is easy to see that the spanning subgraph is also connected. By Lemma 18, we have that each $H_i$ has maximum degree at most $\alpha + 1$. Also, for every vertex $w \in V(G)$, at most 3 additional edges incident to $w$ in $H$ are marked as spanning (two edges are contributed by the path from $u_1$ to $u_h$ (only $u_1$ charges to a different vertex and has degree 1 on the path) and one edge may be added if a vertex is charged to it). In total, this means that $H$ has maximum degree bounded by $\alpha + 4$.

It remains to argue that $H$ has small admissibility. For this, it suffices to prove the following claim. The proof uses the additional restrictions we introduced in the construction.

**Claim.** Let $r$ be a positive integer. If the order $L$ satisfies $\max_{v \in V(G)} |SReach_{2r}[G, L, v]| \leq m$, that is, the order certifies $\text{col}_{2r}(G) \leq m$, then $\text{adm}_r(H) \leq m + 2$.

**Proof.** We verify that for each $r$, the order $L$ certifies that $\text{adm}_r(H) \leq m + 2$. For this, take any vertex $v \in V(H) = V(G)$, and let $\mathcal{P}$ be any family of paths of length at most $r$ in $H$ that start in $v$, end in distinct vertices smaller than $v$ in $L$, and are pairwise internally disjoint. We can further assume that all the internal vertices of all the paths from $\mathcal{P}$ are larger than $v$ in $L$. Let $i, 0 \leq i < \ell$, be such that $v \in V(H_{i+1})$. We distinguish two cases: either $v = w_{i+1}$ or $v \neq w_{i+1}$.

We first consider the case $v \neq w_{i+1}$; the second one will be very similar. By the construction of the order $L$, it follows that $w_{i+1} <_L v$. Consider any path $P \in \mathcal{P}$. Then $P$ is a path in $H$; we shall modify it to a walk $P'$ in $G$ as follows. Suppose $P$ uses some edge $e$ that is not present in $H$. By the construction of $H$, it follows that $e = u_1u_2$ is an edge connecting two vertices that are charged to the same vertex $w$; suppose w.l.o.g. $P$ traverses $e$ from $u_1$ to $u_2$. Define $P'$ by replacing the traversal of $e$ on $P$ by a path of length two consisting of $u_1w$ and $wu_2$, and making the same replacement for all other edges on $P$ that do not belong to $G$.

We claim that all the internal vertices of $P'$ are not smaller, in $L$, than $v$. For this, it suffices to show that whenever some edge $u_1u_2$ is replaced by a path $(u_1w, wu_2)$ as above, then we have that $v \leq_L w$. Aiming towards a contradiction, suppose that $u_1u_2$ is the first edge on $P$ for which we have $w <_L v$. By the construction, it must be that $(u_1, u_2) = (w_{j_1}, w_{j_2})$ for some $j_1, j_2 > i + 1$, and $w = w'_{j_1} = w'_{j_2}$. Let $j$ be such that $w \in H_j$. When constructing $H_{j+1}$, we chose $w = w'_{j_1}$ as the largest, w.r.t. $L$, vertex of $H_j$ which was adjacent to the connected component $C'$ of $G_{j-1}$ that contains $H_{j_1}$. Observe that the prefix of $P'$ up to $w_{j_1}$ is a path in $G$ that, by the choice of $u_1u_2$, contains only vertices not smaller in $L$ than $v$. This prefix has to access the connected component $C'$ from some vertex $q$, for which we of course have $v \leq_L q$. If $q \notin V(H_j)$ then, as $H_j$ is the last among subgraphs connected to $C'$, we have that $q \in V(H_{j'})$ for some $j' < j$ and, consequently, $v \leq_L q <_L w$. Otherwise, if $q \in V(H_j)$, then by the choice of $w = w'_{j_1}$ as the last, in $L$, vertex of the neighbourhood of $C'$ within $H_j$, we also have $v \leq_L q \leq_L w$. In both cases we conclude that $v \leq_L w$, a contradiction.

Hence, if we apply the above procedure to all the paths from $\mathcal{P}$, we obtain a family $\mathcal{P}'$ of walks in $G$ with the following properties: each walk of $\mathcal{P}'$ has length at most $2r$, it connects $v$ with a
different vertex that is smaller in $L$ than $v$, and all its internal vertices are not smaller than $v$. Note here that walks from $P'$ are not necessarily disjoint, but still their number must be bounded by $\max_{v \in V(G)} |S\text{Reach}_{2r}[G, L, v]| \leq m$. It follows that $|P| \leq m$.

Finally, we consider the second case $v = w_{i+1}$. Observe that in the construction of $H$, we added at most 2 additional edges incident to $v$ that connect $v$ with other vertices charged to the same vertex. At most two paths from $P$ can use these edges, and for the other paths we may apply exactly the same reasoning as in the first case. It follows that $|P| \leq m + 2$ in this case; this concludes the proof. \hfill ⊣

The statement of the lemma now directly follows from Claim 2 and Claim 3. \hfill \Box

Given a graph $G$ that excludes $K_k$ as a topological minor, let us write $H(G)$ for a graph constructed according to Lemma 20.

**Corollary 21.** The class $\{H(G) : K_k \not\sim t G\}$ has bounded expansion.

We can now use Theorem 19 to combine the following result of Dvořák et al. [5] with Lemma 20, to prove fixed-parameter tractability of successor-invariant FO on classes that exclude a fixed topological minor.

**Lemma 22 (Dvořák et al. [5]).** The model-checking problem for first-order formulas is fixed-parameter tractable on any class of bounded expansion.

**Corollary 23.** The model-checking problem for successor-invariant first-order formulas is fixed parameter tractable on any class of graphs that excludes a fixed topological minor.

6 Conclusions

In this work we gave several new applications of the generalised colouring numbers on classes of bounded expansion. In particular, we have shown that whenever a graph class $\mathcal{C}$ excludes some fixed topological minor, then any graph from $\mathcal{C}$ admits one ordering of vertices that certifies the boundedness of the generalised colouring numbers for all radii $r$ at once. It is tempting to conjecture that such an ordering exists for any graph class of bounded expansion.

Our construction of the uniform ordering proved to be useful in showing that model-checking successor-invariant FO is FPT on any graph class that excludes a fixed topological minor. We believe that our construction may be helpful in extending this result to any graph class of bounded expansion, since both the construction of the order, and the reasoning of Section 5, are oblivious to the fact that the graph class excludes some topological minor. The only place where we used this assumption is the analysis of the constructed order.
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