Stable determination of an elastic medium scatterer by a single far-field measurement and beyond

Zhengjian Bai · Huaian Diao · Hongyu Liu · Qingle Meng

Received: 24 November 2021 / Accepted: 31 May 2022 / Published online: 24 June 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We are concerned with the time-harmonic elastic scattering due to an inhomogeneous elastic material inclusion located inside a uniformly homogeneous isotropic medium. We establish a sharp stability estimate of logarithmic type in determining the support of the elastic scatterer, independent of its material content, by a single far-field measurement when the support is a convex polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$. Our argument in establishing the stability result is localized around a corner of the medium scatterer. This enables us to further establish a byproduct result by proving that if a generic medium scatterer, not necessary to be a polyhedral shape, possesses a corner, then there exists a positive lower bound of the scattered far-field patterns. The latter result indicates that if an elastic material object possesses a corner on its support, then it scatters every incident wave stably and invisibility phenomenon does not occur.

Mathematics Subject Classification 35Q60 · 78A46 · 35P25

Communicated by Y. Giga.

Huaian Diao
diao@jlu.edu.cn
Zhengjian Bai
zjbai@xmu.edu.cn
Hongyu Liu
hongyu.liuip@gmail.com ; hongyuliu@hkbu.edu.hk
Qingle Meng
mengql2021@foxmail.com

1 School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
2 School of Mathematics, Jilin University, Changchun 130012, China
3 Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, China
1 Introduction

1.1 Mathematical setup

We are mainly concerned with the time-harmonic elastic wave scattering due to the impingement of an incident field on an inhomogeneous isotropic medium scatterer as well as the associated inverse problem of determining the scatterer from the corresponding far-field measurement. We first introduce the mathematical formulation of our study.

Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with a connected complement $\mathbb{R}^n \setminus \overline{\Omega}$, $n = 2, 3$. In the physical setup, $\Omega$ is the support of an inhomogeneous elastic scatterer embedded in a uniformly homogeneous background space. The elastic medium parameters are characterised by $\rho$ and $\lambda, \mu$, which are respectively referred to as the density and the bulk moduli. It is assumed that $\lambda$ and $\mu$ are real constants satisfying the strong convexity conditions:

$$
\mu > 0 \quad \text{and} \quad n \lambda + 2 \mu > 0.
$$

It is also assumed that $\rho \in L^\infty(\mathbb{R}^n)$ and $\Omega = \text{supp}(1 - \rho)$. That is, by normalisation, we assume that the density of the homogeneous background space is 1, whereas the scatterer is characterised by the inhomogeneous density $\rho$ in $\Omega$.

To introduce the Lamé system that describes the elastic scattering, we let $\omega \in \mathbb{R}^+$ signify the angular frequency of the time-harmonic elastic wave. Denote $\kappa_s = \omega \sqrt{1/\mu}$ and $\kappa_p = \omega \sqrt{1/(\lambda + 2\mu)}$ by the shear and compressional wave numbers, respectively. Let $u^i$ denote the time-harmonic plane incident wave of the following form:

$$
u^i = \alpha_1 d e^{i\kappa_p x \cdot d} + \alpha_2 d \perp e^{i\kappa_s x \cdot d}, \quad \alpha_1, \alpha_2 \in \mathbb{C}, \quad |\alpha_1| + |\alpha_2| \neq 0, \quad (1.1)
$$

where $d \in S^{n-1}$ is called the incident direction, and $d \perp$ is a unit vector orthogonal to $d$. Due to the interaction between the incident elastic wave $u^i$ and the elastic scatter $\Omega$, the scattered elastic wave $u^s \in \mathbb{C}^n$ is generated. This induces the total elastic wave $u \in \mathbb{C}^n$, which is the superposition of $u^i$ and $u^s$, and satisfies the Navier equation

$$
\Delta u + \rho \omega^2 u = 0 \quad \text{in} \quad \mathbb{R}^n, \quad \Delta^* u := \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u). \quad (1.2)
$$

Clearly, $u^i$ is an entire solution to (1.2) with $\rho = 1$. By the Helmholtz decomposition, any solution $u \in H^2_{loc}(\mathbb{R}^n)$ to (1.2) can be decomposed as follows:

$$
\nu = \nu_p + \nu_s, \quad (1.3)
$$

where $\nu_p$ and $\nu_s$ satisfy the equations:

$$
\begin{cases}
(\Delta + \kappa_p^2) \nu_p = 0, \quad \nabla \times \nu_p = 0, \\
(\Delta + \kappa_s^2) \nu_s = 0, \quad \nabla \cdot \nu_s = 0.
\end{cases} \quad (1.4)
$$

The scattered wave $u^s$ satisfies the Kurpradze radiation condition

$$
\begin{cases}
\lim_{|x| \to \infty} |x|^{(n-1)/2} \left( \frac{\partial u^s}{\partial |x|} - i \kappa_p u^p \right) = 0, \\
\lim_{|x| \to \infty} |x|^{(n-1)/2} \left( \frac{\partial u^s}{\partial |x|} - i \kappa_s u^p \right) = 0.
\end{cases} \quad (1.5)
$$
and admits the following asymptotic expansion (cf. [26]):

\[ u^i(x) = \frac{\exp(i\kappa_s|\mathbf{x}|)}{|\mathbf{x}|^{(n-1)/2}} U_p(\mathbf{x}) + \frac{\exp(i\kappa_1|\mathbf{x}|)}{|\mathbf{x}|^{(n-1)/2}} U_s(\mathbf{x}) + O(|\mathbf{x}|^{-n+1/2}) \quad \text{as} \quad |\mathbf{x}| \to \infty, \quad (1.6) \]

which holds uniformly in all directions, \( \hat{\mathbf{x}} \in S^{n-1} \), where \( i := \sqrt{-1} \) is the imaginary unit. The vector fields \( U_p \) and \( U_s \) are abstractly given as:

\[ \mathcal{F}(\Omega) = U(\hat{x}; u^i), \quad \hat{x} \in S^{n-1}. \quad (1.7) \]

It is emphasized that in our study of (1.7), the material content of \( \Omega \), namely \( (\lambda, \mu, \rho) \), is not required to be known in advance though it belongs to a certain general a-priori class as shall be introduced in what follows. Moreover, we shall consider the case that only a fixed \( u^i \) is used, namely \( u^i \) is given in (1.1) with fixed \( \eta_1, \eta_2, d \) and \( d_\perp \). In such a case, \( U(\hat{x}; u^i) \) is said to be a single far-field measurement. The inverse scattering problem with a single far-field measurement constitutes a longstanding problem in the literature. Finally, we remark that by direction verifications, the inverse problem (1.7) is nonlinear and it is formally-determined with a single far-field measurement.

The second focus of our study is under what conditions, \( U \equiv 0 \). This is another aspect of the inverse scattering problem (1.7) and in such a case, the underlying object \( (\Omega; \lambda, \mu, \rho) \) is invisible with respect to the far-field measurement.

1.2 Statement of the main stability result and its implication to invisibility

First, we introduce the a-priori class of elastic scatterers for our study.

**Definition 1.1** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) with a connected complement \( \mathbb{R}^n \setminus \Omega \), \( n = 2, 3 \). In addition, \( \Omega = \text{supp}(1 - \rho) \subset B_R \), where \( B_R \) signifies a central ball of radius \( R \). We say \( (\Omega; \rho) \in \mathcal{K} \) if the following conditions are satisfied:

(a) \( \Omega \subset \mathbb{R}^2 \) is a convex polygon and the opening angle at each vertex of \( \Omega \) is in \( (2\alpha_m, 2\alpha_M) \), \( \alpha_m > 0 \) and \( \alpha_M < \pi/2 \);

(b) \( \Omega \subset \mathbb{R}^3 \) is a convex polyhedron;

(c) The distances of any vertex of \( \Omega \) to its non-adjacent edges are at least \( l_0, 0 < l_0 \leq 1 \);

(d) \( \rho(x) \) is a uniformly \( \theta \)-Hölder continuous function in \( \overline{\Omega} \), \( 0 < \theta \leq 1 \). In addition, \( |\rho(x_0) - 1| \geq \epsilon_0 > 0 \) at any vertex \( x_0 \) of \( \Omega \).

**Definition 1.2** We say \( \rho \) is called an admissible density function if for a time-harmonic plane incident wave \( u^i \) of the form (1.1), the forward scattering system (1.1)–(1.5) admits a unique
solution $u \in H^2_{\text{loc}}(\mathbb{R}^n)$ such that the scattered wave $u^s = u - u^i$ is the radiating solution which fulfills $\|u\|_{H^2(B_{2R})^n} \leq \mathcal{N}$ and $|u(x)| \geq \zeta_0$ for $x \in S_{\Omega}$, where $\mathcal{N}$ and $\zeta_0$ are a-priori positive constants, and $S_{\Omega}$ denotes the set of vertices of $\Omega$.

**Remark 1.1** The admissible conditions on $\rho(x)$ and $u$ in Definitions 1.1 and 1.2 can be fulfilled in generic physical scenarios. For example, when $\rho(x)$ is given by two distinct constants in $\overline{\Omega}$ and $\mathbb{R}^n \setminus \overline{\Omega}$, say

$$\rho(x) = \begin{cases} \alpha \neq 1, & x \in \overline{\Omega}, \\ 1, & x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases}$$

one readily sees that $\rho(x)$ satisfies the assumption (d) in Definitions 1.1. Furthermore, by the well-posedness of the direct scattering problem (1.2), there exists a unique solution $u \in H^1_{\text{loc}}(\mathbb{R}^n)$ to (1.2) (cf. [39, Chapter 4]). By further noting that the jump singularity is only attached to the lower order term $\rho$ and the standard elliptic interior regularity estimate (cf. [39, Theorem 4.16]), one has $u \in H^2_{\text{loc}}(\mathbb{R}^n)$. Regarding the technical conditions $\|u^i\|_{H^2(B_{2R})^n} \leq \mathcal{N}$ and $|u(x)| \geq \zeta_0$ for $x \in S_{\Omega}$, we can consider a specific case $\omega \cdot \text{diam}(\Omega) \ll 1$, where $\omega$ is the angular frequency and $\text{diam}(\Omega)$ is the diameter of the scatterer $\Omega$. In the physical setup, this means the scatterer is much small in size compared to the operating wavelength. Under this situation, the incident wave $u^i$ given by (1.1) is dominant in the total wave field $u$ and the energy of the scattered wave field $u^s$ is small (cf. [13, 29]). Hence the assumption $\|u^s\|_{H^2(B_{2R})^n} \leq \mathcal{N}$ can be satisfied for an a-priori positive constant $\mathcal{N}$ and the total wave field $u(x)$ is $\geq \zeta_0$ for $x \in S_{\Omega}$, for a certain a-priori positive constant $\zeta_0$. In fact, $\|u^s\|_{H^2(B_{2R})^n} \leq \mathcal{N}$ is clearly a generic physical condition by the well-posedness of the forward scattering problem, whereas for the technical condition $|u(x)| \geq \zeta_0$ for $x \in S_{\Omega}$, we also believe it holds generically. However, this is beyond the scope of the current study and we shall explore this point in our future work.

In what follows, $(\Omega, \rho)$ is said to be an admissible polyhedral scatterer if it fulfills the conditions in Definitions 1.1 and 1.2. The parameters $\{R, \mathcal{N}, l_0, \epsilon_0, \zeta_0, \alpha_m, \alpha_M\}$ in the above admissibility definitions are referred to as the a-priori parameters. It is remarked that the admissibility requirements in Definitions 1.1 and 1.2 will be needed in our stability study and can be fulfilled in certain general scenarios, which shall become clearer in our subsequent discussion.

Next, we present the logarithmic stability estimate for the inverse problem (1.7) on the shape determination of the elastic medium scatterer by a single far-field pattern.

Henceforth, we define the Hausdorff distance of two medium scatterers $(\Omega; \rho)$ and $(\Omega'; \rho')$ as follows,

$$d_H(\Omega, \Omega') = \max \left\{ \sup_{x \in \Omega} \text{dist}(x, \Omega'), \sup_{x \in \Omega'} \text{dist}(x, \Omega) \right\}. \quad (1.8)$$

**Theorem 1.1** Let $(\Omega; \rho)$ and $(\Omega'; \rho')$ be two admissible polyhedral scatterers. Let $u^i$ be a common time-harmonic plane incident wave of the form (1.1). Assume that $U$ and $U'$ are the far-field patterns of the scattered waves $u^s$ and $u'^s$ by the medium scatterers $(\Omega; \rho)$ and $(\Omega'; \rho')$, respectively. For sufficiently small $\varepsilon \in \mathbb{R}_+$, if

$$\|U - U'\|_{L^2(S^{n-1}, \mathbb{C}_x \times \mathbb{C}^n)} \leq \varepsilon,$$

then

$$d_H(\Omega, \Omega') \leq C(\ln(\mathcal{N}/\varepsilon))^{-\gamma}, \quad (1.9)$$
where $N$ is given in Definition 1.2, and $C$ and $\gamma$ are positive constants, depending only on the a-priori parameters involved in Definitions 1.1 and 1.2 as well as the Lamé constants $\lambda, \mu$.

**Remark 1.2** In Theorem 1.1, the shape of the scattering inhomogeneity is defined by the discontinuity interface of the density parameter $\rho$, which appears in the lower order term in the Lamé system (1.2). As also discussed in Remark 1.1, this can guarantee that the solution $u^s \in H^2_{\text{loc}}(\mathbb{R}^n)$, which is a key technical ingredient in deriving the stability estimate (1.9). It would be interesting to consider the case that the material discontinuity, say e.g. jump singularity, appears in $\lambda$ and $\mu$. In such a case, since the singularity is attached to the coefficients in the leading-order term of the Lamé system, one generically does not have $H^2$-regularity, which shall pose significant challenges. We shall explore along that direction in our future work.

**Remark 1.3** In (1.9), the stability is of a double logarithmic type, which indicates the ill-posedness nature of the inverse scattering problem (1.7). In our proof of Theorem 1.1 in what follows, the logarithmic estimates arise in two steps: first, the propagation of the data from the far field to the near field yields one logarithmic-type ill-posedness, and then the further propagation of the data from the near field to the boundary of the scatterer gives rise to another logarithmic-type ill-posedness. It is emphasized that the two-phase propagation is necessary since the analytic continuation, namely the three-sphere inequalities in what follows, degenerates when approaching the boundary of the scatterer. Our argument on this issue follows the spirit of the one developed in [5], where a related inverse scattering problem associated with the Schrödinger operator $-\Delta + q$ was studied. In a different but related physical context where the elastic scatterer is an impenetrable obstacle, a different approach of continuing the data, especially from the near field to the boundary, was developed in [46]; see also [31, 32, 44] for related studies for the inverse acoustic and electromagnetic scattering problems. It seems plausible that one might be able to increase the stability estimate to a certain extent by adapting the data continuation strategy in [46] to the current setup. However, this will be much technically involved and we choose to explore along that direction in our future work as well.

The argument in proving Theorem 1.1 is localized around a corner of the underlying polyhedral scatterer. As an interesting byproduct, we can establish another stability result where the medium scatterer is not necessarily polyhedral as long as it possesses a corner. The full technical details of the result will be given in Theorems 5.1 and 5.2, and we only provide a rough summary in the following theorem.

**Theorem 1.2** Consider the scattering problem (1.1)–(1.5) associated with a general medium scatterer $(\Omega, \rho)$, which is not necessarily polyhedral. Suppose that $\partial \Omega$ possesses a corner as described in Theorem 5.2. Then it holds that

$$\|U\|_{L^2(\mathbb{S}^{n-1}, C^0 \times C^0)} \geq C,$$

(1.10)

where $C$ is a positive constant depending on a certain set of a-priori parameters.

Theorem 1.2 indicates that for a general medium scatterer, if it possesses a corner, then it generically scatters every incident field stably, i.e. invisibility phenomenon cannot occur.
1.3 Connection to existing studies and discussion

Determining the shape of an inhomogeneous object by minimal/optimal scattering measurements has been a longstanding problem in the literature with a long and colorful history; see [16, 30, 37] for reviews and surveys. Uniqueness for the shape determination of inhomogeneous elastic medium caused by the mass density based on many far-field measurements can be found in [25, 27]. The single-measurement uniqueness result in determining a polygonal elastic medium scatter with a generalized transmission boundary condition can be found in [24]. Recently, several qualitative uniqueness results in determining convex polyhedral medium scatterers were established in [34] for electrostatics, [6, 12, 22] for acoustic scattering, [8, 36] for electromagnetic scattering and [20, 21, 24] for elastic scattering. In [7, 35], the shape determination of medium scatterers whose boundaries possess high-curvature parts was also considered for electrostatics and acoustic scattering. In [5, 33], quantitative stability estimates of double-logarithmic type were established in determining the convex polyhedral shape of an acoustic medium scatterer. The related stable determination of obstacle scatterers by finite measurements were established in [1, 19, 31, 38, 40, 48] for acoustic scattering, [31, 32] for electromagnetic scattering, and [28, 41, 42, 46] for elastic scattering.

Another issue of significant physical interest is the occurrence of invisibility, namely the scattering pattern is identically zero. Generically, it is believed that geometric singularities on the support of a generic medium scatterer prevents the occurrence of the invisibility phenomenon. We refer to [2, 3, 5–9, 11, 22, 30, 36, 43, 47] for related studies on this intriguing topic in different physical contexts. In [7], a geometrically singular point (say, e.g. a corner point) on the boundary of a shape is treated as with infinite curvature, and it is further shown that a medium scatterer whose smooth shape possesses a sufficiently high curvature point can also prevent the occurrence of invisibility. All of the aforementioned results are qualitative and in [5] sharp estimates were established by showing that there exist positive lower bounds of the scattering patterns, which quantify the non-invisibility phenomena due to the presence of the shape singularities. It is noted that the lower scattering bounds have been derived only for the acoustic scattering.

Our study in this article extends the related studies in [5, 33, 34] for the electrostatics and acoustic scattering to the elastic scattering which possesses more complicated and technical nature, both in physics and mathematics. It is pointed out that as remarked in [5], the double-logarithmic stability estimates are generically optimal for inverse scattering problems. Finally, we would like to mention in passing a closely related topic on the geometric structures of transmission eigenfunctions which is beyond the scope of this article [4, 14, 15, 17, 18, 22–24, 33].

The rest of the paper is organized as follows. In Section 2, we derive a critical auxiliary result about the propagation of smallness from far field to the boundary of the scatterer. Section 3 is devoted to a micro-local analysis of the scattering solution around a corner. The full details of the proof of Theorem 1.1 is presented in Section 4. In Section 5, we present the full technical details of Theorem 1.2.

2 Propagation of smallness from far-field to boundary

The main goal of this section is to show how the smallness from the far-field pattern propagates to the boundary of the elastic medium scatterer, which is a key ingredient in the stability proof.
of Theorem 1.1. Throughout the rest of the paper, we let \( u \) and \( u' \), respectively, denote the total wave fields corresponding to the scatterers \( (\Omega; \rho) \) and \( (\Omega'; \rho') \) in Theorem 1.1.

2.1 Stability estimates: from far-field to near-field

In this subsection, our aim is to estimate the difference of \( u \) and \( u' \) in \( B_{2R} \setminus B_R \) close to the convex hull of \( \Omega \) and \( \Omega' \). To begin with, we need to generalise Theorem 4.1 in [45] and Proposition 5.2 in [5] to the elastic inhomogeneous medium scatterer case.

**Lemma 2.1** Fix \( t \in (1, 2) \) and \( T > 0 \). Let \( u \in H^2_{\text{loc}}(\mathbb{R}^n)^n \) be a solution to the Navier equation (1.2) in \( \mathbb{R}^n \setminus B_R \) with \( \rho = 1 \), which satisfies the Kupradze radiation condition; see Definition 1.2 and Remark 1.1. Let \( U \) be the far field pattern of \( u \) with the norm \( \varepsilon = ||U||_{L^2(S^n-1, C^n \times \mathbb{R}^n)} \). Assume that \( ||u||_{L^2(B_{2R}\setminus B_R)^n} \leq T, = p, s \), where \( u_p \) and \( u_s \) are the longitudinal and the transversal parts of \( u \), respectively. Then there exists a constant \( C = C(\kappa_p, \kappa_s, T, n, R, t) > 0 \) such that the following stability estimate holds for \( \varepsilon \leq \frac{T}{c} \):

\[
||u||_{L^2(B_{2R}\setminus B_R)^n} \leq C e^{-\frac{c}{\varepsilon} \sqrt{\ln \frac{T}{c}}},
\]

where \( \kappa = \min\{\kappa_p, \kappa_s\} \) and \( \hat{c} \leq \ln t \sqrt{\frac{ekR}{2}} \).

**Proof** Let \( u^j \) and \( u'^j \) denote the \( j \)-th components of \( u \) and \( u' \), respectively, where \( = p, s \). From (1.3) and (1.4), we obtain that \( u^j \) satisfies the Helmholtz equation

\[
(\Delta + \kappa^2)u^j = 0, \quad j = 1, 2, \ldots, n.
\]

Then by [5, Proposition 5.2], we know that if \( \varepsilon \leq \frac{T}{c} \), then

\[
||u^j||_{L^2(B_{2R}\setminus B_R)} \leq C T e^{-\frac{c}{\varepsilon} \sqrt{\ln \frac{T}{c}}},
\]

where \( C = C(\kappa, n, R, t) > 0 \), = p, s, \( \hat{c} \leq \ln t \sqrt{\frac{ekR}{2}} \) and \( \kappa = \min\{\kappa_p, \kappa_s\} \).

When \( \varepsilon \leq \frac{T}{\sqrt{2n(C_p^2 + C_s^2)}} \), we can derive

\[
||u||_{L^2(B_{2R}\setminus B_R)^n} = \sqrt{\sum_{j=1}^{n} ||u^j||_{L^2(B_{2R}\setminus B_R)}^2} \leq \sqrt{\sum_{j=1}^{n} \sum_{p,s} ||u^j||_{L^2(B_{2R}\setminus B_R)}^2} \leq \sqrt{2n(C_p^2 + C_s^2)} e^{-\frac{c}{\varepsilon} \sqrt{\ln \frac{T}{c}}}.\]

The proof is complete. \( \square \)

With the help of elliptic interior regularity, we can derive the following stability estimate of the near-fields in \( B_{2R} \setminus B_R \).

**Proposition 2.1** Fix \( t \in (1, 2) \) and \( T > 0 \). Assume that \( u \in H^2_{\text{loc}}(\mathbb{R}^n)^n \) satisfies the Navier equation \( (\Delta^* + \omega^2)u = 0 \) in \( \mathbb{R}^n \setminus B_R \) and the Kupradze radiation condition; see Definition 1.2 and Remark 1.1. Let \( U \) be the far field pattern of \( u \) with the norm \( \varepsilon = ||U||_{L^2(S^n-1, C^n \times \mathbb{R}^n)} \). Assume that \( ||u||_{L^2(B_{2R}\setminus B_R)^n} \leq T, = p, s \), where \( u_p \) and \( u_s \) are the longitudinal and the transversal parts of \( u \), respectively. Let \( K \subset B_{2R} \setminus B_{\frac{1}{2}R} \) be a domain. Then there exists a
positive constant $C = C(\kappa_p, \kappa_s, n, R, r, t)$ such that the following stability estimate holds for $\varepsilon \leq \frac{T}{r}$:

$$\|u\|_{H^r(K)^n} \leq C T e^{-\hat{c} \sqrt{\ln \frac{T}{r}}},$$  \hspace{1cm} (2.2)

where $r \in \mathbb{N}$ is an order of the Sobolev space, $\kappa = \min\{\kappa_p, \kappa_s\}$ and $\hat{c} \leq \ln t \sqrt{\frac{\varepsilon R}{t}}$.

**Proof** Note that $u$ is the superposition of $u_p$ and $u_s$, which are vector-valued weak solutions to the Helmholtz equation with wave numbers $\kappa_p$ and $\kappa_s$, respectively. We use $u^j$ to denote the $j$-th component of $u$, where $= p, s$. Given two domains $\bar{K}$ and $\bar{K}'$ with $\bar{K} \subset K' \subset B_{2R} \setminus \overline{B_{r}R}$ and $\text{dist}(\partial \bar{K}, \partial K') > 0$. For any $s \in \mathbb{R}$ and the smooth cutoff function $\tilde{\phi} \in C_0^\infty(K)$ with $\tilde{\phi} \equiv 1$ in $\bar{K}$, it is not difficult to prove the two subsequent properties:

$$\|u^j\|_{H^{r+2}(\mathbb{R}^n)} = \|(1 + \kappa^2)u^j - (\Delta + \kappa^2)u^j\|_{H^{r}(\mathbb{R}^n)}$$
and

$$\|(\Delta + \kappa^2)(\tilde{\phi} u^j)\|_{H^{r+2}(\mathbb{R}^n)} = \|2 \nabla \tilde{\phi} \cdot \nabla u^j + u^j \Delta \tilde{\phi}\|_{H^{r}(\mathbb{R}^n)} \leq C \|u^j\|_{H^{r+1}(\Omega)},$$

where $C$ depends on $\tilde{\phi}$. Thus we obtain

$$\|u\|^2_{H^{r+2}(\bar{K})^n} = \|u_p + u_s\|^2_{H^{r+2}(\bar{K})^n} \leq \sum_{p, s, j=1}^n \|u^j\|^2_{H^{r+2}(\bar{K})^n}$$

$$\leq 2 \sum_{p, s, j=1}^n \|\tilde{\phi} u^j\|^2_{H^{r+2}(\mathbb{R}^n)} \leq \sum_{p, s, j=1}^n C_{\kappa, \phi}^2 \|u^j\|^2_{H^{r}(K')}$$

$$\leq C_{\kappa_p, \kappa_s, n, \tilde{\phi}}^2 \|u\|^2_{H^{r+1}(K')^n},$$

which implies that

$$\|u\|_{H^{r+2}(\bar{K})^n} \leq C_{\kappa_p, \kappa_s, n, \tilde{\phi}} \|u\|_{H^{r+1}(K')^n}. \hspace{1cm} (2.3)$$

By fixing $r \in \mathbb{N}$, there exists a subdomain sequence $\{K_j, \phi_j\}_{j=0}^\infty$ such that

$$\begin{cases} 
K_j \subset K_{j-1}, \quad \text{dist}(\partial K_j, \partial K_{j-1}) > 0, \\
\phi_j \in C_0^\infty(K_{j-1}), \quad \phi_j \equiv 1 \quad \text{in} \; K_j, \\
K_0 = B_{2R} \setminus \overline{B_{r}R}, \quad K_r = K.
\end{cases}$$

By using (2.3) and Lemma 2.1 repeatedly, we have

$$\|u\|_{H^{r}(\Omega)^n} \leq C_{\kappa_p, \kappa_s, \phi_1} \|u\|_{H^{r-1}(\Omega)^n} \leq \cdots \leq C_{\kappa_p, \kappa_s, \phi_r, \ldots, \phi_1} \|u\|_{L^2(K)^n}$$

$$\leq C_{\kappa_p, \kappa_s, \phi_r, \ldots, \phi_1} T e^{-\hat{c} \sqrt{\ln(T/r)}}.$$

The proof is complete. \( \square \)

### 2.2 Stability estimates: from near-field to boundary

In this subsection, we establish the propagation of smallness from the near-field to the boundary of $Q$ which is the convex hull of $\Omega$ and $\Omega'$. We are mainly interested in estimating the sum of $|u - u'|$ and $|\nabla u - \nabla u'|$ on $\partial Q$. To begin with, we introduce the following *three-spheres inequalities* which may be found, for instance, in [10].
Lemma 2.2 [44] There exist positive constants \( \hat{R} \), \( C \) and \( c \), \( 0 < c < 1 \), depending on \( \kappa \) only. Let \( 0 < r_1 < r_2 < r_3 < \hat{R} \), and \( u \) be a solution to \( \Delta u + \kappa^2 u = 0 \) in \( B_{r_3} \). For any \( s \in (r_2, r_3) \), we have

\[
\|u\|_{L^\infty(B_{r_2})} \leq C \left( 1 - \frac{r_2}{s} \right)^{-3/2} \|u\|_{L^\infty(B_{r_3})}^{1/2} \|u\|_{L^\infty(B_{r_1})}^\beta,
\]

where \( \beta \) satisfies the following inequality,

\[
c \frac{\log \frac{r_2}{s}}{\log \frac{r_2}{r_1}} \leq \beta \leq 1 - c \frac{\log \frac{r_2}{s}}{\log \frac{r_2}{r_1}}.
\]

From now on, we fix \( r \in \mathbb{R}_+ \) and set \( r_1 = r \), \( r_2 = 2r \), \( r_3 = 4r \), \( s = 2\sqrt{2}r \).

Lemma 2.3 Let \( V \subset \mathbb{R}^n \) be a bounded connected domain and \( \Gamma \subset V \) be a non-self-intersected curve with endpoints \( \mathbf{x} \) and \( \mathbf{x}' \). Fix positive constants \( T \) and \( \hat{R} \) such that \( 4r < \hat{R} \) and \( T \geq 1 \). Let \( B(\Gamma, 4r) = \bigcup_{\mathbf{y} \in \Gamma} B_{4r}(\mathbf{y}) \subset V \). Moreover, we assume that \( u \in L^\infty(V)^n \) satisfies

\[
\\{(\Delta^* + \omega^2)u = 0, \quad \|u\|_{L^\infty(V)^n} \leq T, \quad u_{p}, u_{s}\}
\]

max \\{ \|u_{p}\|_{L^\infty(B_r(\mathbf{x}))^n}, \|u_{s}\|_{L^\infty(B_r(\mathbf{x}))^n} \} \leq 1,

where \( u_p \) and \( u_s \) represent the longitudinal and the transversal parts of \( u \), respectively. Then the following result holds

\[
\|u\|_{L^\infty(B_r(\mathbf{x}'))^n} \leq CT \|u\|_{L^\infty(B_r(\mathbf{y}))^n}^{d_{r}+1}, = p, s,
\]

where \( d_{r} \) is the distance measured along \( \Gamma \).

**Proof** Let \( N = \lceil d_{r}/r \rceil \). There exists a sequence of balls, each of radius \( r \) and centred respectively at \( \mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{N} = \mathbf{x}' \), such that \( \mathbf{x}_k \in \Gamma \) and \( |\mathbf{x}_{k+1} - \mathbf{x}_k| \leq d_{r}(\mathbf{x}_k, \mathbf{x}_{k+1}) \leq r \), where \( d_{r}(\mathbf{x}_k, \mathbf{x}_{k+1}) \) signifies the length along the curve \( \Gamma \) between the points \( \mathbf{x}_k \) and \( \mathbf{x}_{k+1} \). Clearly, \( B_r(\mathbf{x}_k) \subset B_{2r}(\mathbf{x}_{k-1}) \). Let \( u^j \) denote the \( j \)-th component of \( u \), \( = p, s \). Then we have

\[
\|u\|_{L^\infty(B_r(\mathbf{x}'))^n} \leq \sum_{j=1}^{n} \|u^j\|_{L^\infty(B_r(\mathbf{x}'))^n} \leq \sum_{j=1}^{n} \left\{ C_1 T^{1-\beta} \|u^j\|_{L^\infty(B_r(\mathbf{x}_n))}^{\beta} \right\} \\
\leq \cdots \\
\leq \sum_{j=1}^{n} \left\{ C_N T^{(1-\beta)(1+\beta+\cdots+\beta^{N-1})} \|u^j\|_{L^\infty(B_r(\mathbf{x}))}^{\beta^N} \right\}.
\]

Note that \( 1 + \beta + \cdots + \beta^N \leq 1/(1 - \beta) \) and \( N \leq d_{r}/r + 1 \), hence we have the following claim

\[
\|u\|_{L^\infty(B_r(\mathbf{x}'))^n} \leq CT \|u\|_{L^\infty(B_r(\mathbf{x}))^n}^{d_{r}+1} \leq CT \|u\|_{L^\infty(B_r(\mathbf{x}))^n}^{\beta^{d_{r}+1}}, = p, s.
\]

The proof is complete. \( \square \)

We now give the stability estimations of \( u_p \) and \( u_s \) near the scatterer \( (\Omega; \rho) \).
Proposition 2.2 Given positive constants $T$, $\hat{R}$ and $\theta$, $\beta \in (0, 1)$ such that $T \geq 1$. Let $Q \subset B_R$ be a convex polytope. Let $u \in H^2_{loc}(B_2 R)^n$ satisfy the Navier equation $\Delta^B u + \omega^2 u = 0$ in $B_2 R \setminus Q$. Moreover, we assume that

$$\|u\|_{C^0(B_{3/2} R)^n} \leq T, \quad \|u\|_{L^\infty(B_{7/4} R \setminus B_{5/4} R)^n} \leq \delta < 1, \quad \alpha = p, s,$$

where $u_p$ and $u_s$ are the longitudinal and the transversal parts of $u$, respectively. Let $r = \frac{9 R |\ln \beta|}{4 (1-\theta) \ln |\ln \delta|}$. If $\delta < \left( \exp \left\{ \frac{4 r \ln |\ln \beta|}{\min\{\hat{R}, R/2\}} \right\} \right)^{-1}$,

then

$$\|u\|_{L^\infty(\partial Q)^n} \leq M (\ln |\ln \delta|)^{-\theta} T, \quad = p, s. \quad (2.5)$$

Here $Q' = \{x \in B_{3/2} R | \text{dist}(x, \partial Q) \leq 4 r \}$ and $M = M(\omega, \theta, \hat{R}, \beta) > 1$.

Proof Let $Q_1 = \{x \in B_{2 R} | \text{dist}(x, \partial Q) < 4 r \}$. It is easy to reduce that $4 r < \hat{R}$ and $2 r < \frac{R}{2}$ from the upper bound on $\delta$ and the expression of $r$. For any point $x' \in B_{2 R} \setminus Q_1$, there always exists a ray from $x'$ into $B_{2 R} \setminus B_{5/4} R$.

We consider a special segment of the ray with endpoints $x'$ and $y$, where $y \in B_{7/4} R \setminus B_{5/4} R$ and $\text{dist}(y, \partial B_{5/4} R) = r$. One can see that $\|u\|_{L^\infty(B_r(y))^{n}} \leq \delta, = p, s$. The length of that segment is smaller than $\frac{5 R}{2} + r$. By Lemma 2.3, we have

$$\|u\|_{L^\infty(B_r(y))^{n}} \leq C T \|u\|_{L^\infty(B_r(y))^{n}}^{\frac{5 R}{2} + 2} \leq C T \delta^{\frac{5 R}{2} + 2}, \quad = p, s.$$

Then

$$\|u\|_{L^\infty(B_r(x'))^{n}} \leq C T \delta^{\frac{5 R}{2} + 2}, \quad = p, s.$$

For any point $x' \in Q_1$, there must be $y \in \partial Q$ such that $|x' - y| \leq 4 r$. By the convexity of $Q$, there exists $x \in \mathbb{R}^n \setminus Q$ such that $\text{dist}(x, Q) = |x - y| = 4 r$. The upper bound of $\delta$ implies $4 r \leq \frac{5 R}{2}$, and thus

$$|x| \leq |x - y| + |y| \leq 4 r + R \leq \frac{3 R}{2}.$$

Moreover, $|x' - x| \leq |x' - y| + |y - x| \leq 8 r$. From the $\theta$-Hölder continuity of $u$ and proposition 5.7 in [5], we can obtain that

$$\|u\|_{L^\infty(Q_1)^n} \leq C \sum_{j=1}^{n} \|u_j\|_{L^\infty(Q_1)^n} \leq \sum_{j=1}^{n} \left( \|u_j\|_{C^0(B_{3/2} R)} |x - x'|^\theta + \|u_j\|_{L^\infty(Q_1)^n} \right) \leq \sum_{j=1}^{n} C T (8 r)^\theta + \delta^{\frac{5 R}{2} + 2} \leq \tilde{C} T (8 r)^\theta + \delta^{\frac{5 R}{2} + 2}, \quad (2.6)$$

where $\tilde{C}$ is a positive constant depending on $\kappa$ and $n$.

Combining (2.4) with $|\ln \delta| > 1$, we have

$$r^\theta = \frac{(9 R |\ln \beta|)^\theta}{((4 - 4 \theta) \ln |\ln \delta|)^\theta}, \quad \frac{5 R}{2 r} = \frac{10 (1 - \theta) \ln |\ln \delta|}{9 |\ln \beta|}. \quad (2.7)$$
and
\[ \delta^\frac{5\epsilon}{2\beta + 2} = e^{-|\ln \delta| \beta^\frac{5\epsilon}{2\beta + 2}} = e^{-\beta^2 |\ln \delta|} \leq e^{-\beta^2 (|\ln \delta|)^{-\theta}}. \] (2.8)

Plugging (2.7) and (2.8) into (2.6), it yields that
\[ \| u \|_{L^\infty(Q')} \leq \| u \|_{L^\infty(Q)} \leq M (|\ln \delta|)^{-\theta} T, \quad = p, s, \]
where \( M = \widetilde{C}[2^\theta R^\theta |\ln \beta|^\theta (1 - \theta) + \beta^{-2}] \).

The proof is complete. \( \square \)

The next proposition states the propagation of smallness from the near-field to the boundary of \( Q \), which is the convex hull of \( \Omega \) and \( \Omega' \). Firstly, we introduce the following lemma to derive Proposition 2.3.

**Lemma 2.4** Let \( n = 2, 3 \) and \( \rho \in L^\infty(B_{2R}) \) be supported in the ball \( B_R \) with \( R > 0 \). Let \( \mathbf{w} \in H^2(B_{2R}) \) be the solution to (1.2) in \( B_{2R} \). Then, \( \mathbf{w} \) belongs to \( C^{1,\frac{1}{2}}(\overline{B}_{3R}/2) \) and has the following estimate
\[ \| \mathbf{w} \|_{C^{1,\frac{1}{2}}(\overline{B}_{3R}/2)} \leq C(1 + \| \rho \|_{L^\infty(R^n)})\| \mathbf{w} \|_{H^2(B_{2R})}, \] (2.9)
where \( C \) is a positive constant relying on \( R, n, \kappa_s, \kappa_p, \lambda, \mu \).

**Proof** By virtue of the Helmholtz decomposition (1.3), each component of \( \mathbf{w}_p \) and \( \mathbf{w}_s \) satisfies the corresponding scalar Helmholtz equation associated with the shear and compressional wave numbers respectively, where \( \mathbf{w}_s \) and \( \mathbf{w}_p \) are the shear and compressional parts of \( \mathbf{w} \) respectively. Therefore, using a similar argument to that in the proof of [5, Lemma 5.9], we can prove this lemma. \( \square \)

**Proposition 2.3** Fix \( n = 2, 3 \) and the a-prior parameters \( \mathcal{M}, \mathcal{P} \) and \( \mathcal{N} \) greater than 1. Let \( (\Omega; \rho) \) and \( (\Omega'; \rho') \) be two medium scatterers associated with two admissible density functions \( \rho \) and \( \rho' \), where \( \max\{\| \rho \|_{L^\infty(\mathbb{R}^n)}, \| \rho' \|_{L^\infty(\mathbb{R}^n)}\} \leq \mathcal{M} \). Let \( \mathbf{u}^i \) be an incident wave with the form (1.1) and \( \| \mathbf{u}^i \|_{H^2(B_{2R})} \leq \mathcal{P} \). Assume that \( \mathbf{U} \) and \( \mathbf{U}' \) are the far field patterns of the scattered waves \( \mathbf{u}^s \) and \( \mathbf{u}^{s'} \) by the medium scatterers \( (\Omega; \rho) \) and \( (\Omega'; \rho') \), respectively, where \( \mathbf{u}^s, \mathbf{u}^{s'} \) can be bounded by \( \mathcal{N} > 1 \) in \( H^2(B_{2R}) \). Let \( \epsilon \) be a sufficiently small constant. If
\[ \| \mathbf{U} - \mathbf{U}' \|_{L^2(\mathbb{R}^{n-1}, C^n \times C^n)} \leq \epsilon, \]
then \( \mathbf{u} - \mathbf{u}' \) and \( \nabla \mathbf{u} - \nabla \mathbf{u}' \) are continuous in \( B_R \) and moreover, we have
\[ \sup_{Q} (\| \mathbf{u} - \mathbf{u}' \| + \| \nabla \mathbf{u} - \nabla \mathbf{u}' \|) \leq C (\ln \ln (\mathcal{N}/\epsilon))^{-1/2}, \] (2.10)
where \( Q \) is the convex hull of \( \Omega \) and \( \Omega' \), and \( C = C(\kappa_p, \kappa_s, n, R, \mathcal{P}, \mathcal{M}, \mathcal{N}) > 0 \).

**Proof** Denote \( \epsilon_0 = \| \mathbf{U} - \mathbf{U}' \|_{L^2(\mathbb{R}^{n-1}, C^n \times C^n)} \) and \( \mathbf{v} = \mathbf{u} - \mathbf{u}' = \mathbf{u}^s - \mathbf{u}^{s'} \). One can easily see that \( \| \mathbf{v} \|_{H^2(B_{2R})} \leq 2\mathcal{P} \). Let \( \Omega = B_{7/4R} \setminus B_{5/4R} \). From Proposition 2.1 and the Sobolev embedding \( H^2(\Omega) \rightarrow L^\infty(\Omega) \), there exists a positive constant \( C_1 = C_1(\kappa_p, \kappa_s, n, R) \) such that the following estimates hold for sufficiently small \( \epsilon \),
\[ \| \mathbf{v} \|_{L^\infty(\Omega)} \leq \| \mathbf{v} \|_{H^2(\Omega)} \leq C_1\mathcal{N}e^{-c_0\sqrt{\ln \mathcal{N}/\epsilon_0}} \leq C_1\mathcal{N}e^{-c_0\sqrt{\ln \mathcal{N}/\epsilon}}, \]
\[ \| \nabla \mathbf{v} \|_{L^\infty(\Omega)} \leq \| \mathbf{v} \|_{H^2(\Omega)} \leq C_1\mathcal{N}e^{-c_0\sqrt{\ln \mathcal{N}/\epsilon_0}} \leq C_1\mathcal{N}e^{-c_0\sqrt{\ln \mathcal{N}/\epsilon}}, \]
where \( \kappa = \min\{\kappa_p, \kappa_s\} \) and \( c_0 = \ln(\frac{5}{4})\sqrt{\frac{|eR|}{2}} \). According to Lemma 2.4, \( \|\rho\|_{L^\infty}(R^n) \leq M, \|u^i\|_{H^2(B_{2R})^n} \leq P, \|u^\sigma\|_{H^2(B_{2R})^n} \leq N', \) and \( \|u^a\|_{H^2(B_{2R})^n} \leq N' \), we obtain

\[
\|u\|_{C^{1, \frac{1}{2}}(B_{3/2R})^n} \leq C(1 + M)(P + N), \quad = p, s,
\]

\[
\|u\|_{C^{1, \frac{1}{2}}(B_{3/2R})^n} \leq C(1 + M)(P + N), \quad = p, s,
\]

where \((u_s, u_p)\) and \((u'_s, u'_p)\) are transverse and longitudinal wave pairs of \( u \) and \( u' \), respectively. Thus \( v \) and \( \partial_j v \in C^{1, \frac{1}{2}}(B_{3/2R})^n, \) \( j = 1, 2, \ldots, n. \) This shows that \( u - u' \) and \( \nabla u - \nabla u' \) are continuous in \( B_R. \)

Fix \( \beta \in (0, 1) \) and denote \( Q' = \{ x \in B_{3/2R} \mid \text{dist}(x, \partial Q) \leq 4 \cdot \frac{9R|\ln \beta|}{2|\ln |\delta||} \} \). We choose \( \delta(\varepsilon) = C_2 N e^{-c_0 \sqrt{\ln(N/\varepsilon)}} \) satisfying (2.4). According to Proposition 2.2, we know that

\[
\|v\|_{L^\infty(Q')^n} \leq C(1 + M)(2P + N)(\ln |\ln \delta|)^{-\frac{1}{2}}, \quad = p, s.
\]

Hence, one can deduce that

\[
\|v\|_{L^\infty(Q')^n} \leq \max\{C_p, C_s\}(1 + M)(2P + N)(\ln |\ln \delta|)^{-\frac{1}{2}}
\]

\[
\leq \max\{C_p, C_s\}(1 + M)(2P + N)\left[ \ln \left( \frac{c_0}{2} \ln(N/\varepsilon) \right) \right]^{-\frac{1}{2}}
\]

\[
\leq \max\{C_p, C_s\}(1 + M)(2P + N)(\ln(N/\varepsilon)^{\frac{1}{2}}) \frac{1}{2}
\]

\[
\leq 2 \max\{C_p, C_s\}(1 + M)(2P + N)(\ln(N/\varepsilon)^{\frac{1}{2}})^{\frac{1}{2}}.
\]

(2.11)

Similar to the upper bound of \( \|v\|_{L^\infty(Q')^n} \) in (2.11), we can obtain the upper bound of \( \partial_j v \) by following a similar argument. Hence, we can prove (2.10).

The proof is complete. \( \square \)

3 Micro-local analysis of corner scattering

The present section is devoted to analyzing the quantitative behaviours of the scattered field locally around a corner. We first present two auxiliary lemmas.

Lemma 3.1 [5, Lemma 8.2] Fix \( n = 2. \) Let \( Q \) be the convex hull of \( \Omega \) and \( \Omega' \), where \( \Omega \) and \( \Omega' \) are two open bounded convex polygons. If \( x_0 \) is a vertex of \( \Omega \) such that \( \text{dist}(x_0, \Omega') = \text{dist}_H(\Omega, \Omega'), \) then \( x_0 \) is also a vertex of \( Q. \) Let \( \sigma \) denote the angle of \( K \) at \( x_0, \) then the angle \( \sigma' \) of \( Q \) at \( x_0 \) satisfies the inequality \( \sigma < \sigma' \leq (\sigma + \pi)/2 < \pi. \)

Lemma 3.2 [5, Lemma 8.3] Fix \( n = 3. \) Let \( Q \) be the convex hull of \( \Omega \) and \( \Omega' \), where \( \Omega \) and \( \Omega' \) are open convex polyhedral cones. If \( x_0 \) is a vertex of \( \Omega \) such that \( \text{dist}(x_0, \Omega') = \text{dist}_H(\Omega, \Omega'), \) then \( x_0 \) is also a vertex of \( Q. \) And \( Q \) can fit inside such an open spherical cone \( A \) that \( x_0 \) is the vertex of \( A \) whose open angle \( \sigma \) at the vertex \( x_0 \) is smaller than \( \pi, \) where \( \sigma \) does not depend on \( \Omega, \Omega' \) and their locations.

Next, we establish an integral identity which follows from the Betti’s second formula.
Proposition 3.1 Let \( \omega > 0 \) and \( \rho(x) \in C^0(S) \), where \( S \subset \mathbb{R}^n \) is a bounded Lipschitz domain and \( \theta \in (0, 1) \). If \( u, u' \) and \( u_0 \in H^2(S)^n \) satisfy the Navier equations

\[
\begin{align*}
\Delta^* u + \rho \omega^2 u &= 0, \\
\Delta^* u' + \omega^2 u' &= 0, \\
\Delta^* u_0 + \omega^2 u_0 &= 0
\end{align*}
\]

in \( S \), then

\[
\omega^2 \int_S (\rho - 1) u \cdot u_0 \, dx = \int_{\partial S} u_0 \cdot T_{\hat{\nu}}(u^j - u) - (u^j - u) \cdot T_{\hat{\nu}}(u_0) \, d\sigma. 
\]

(3.1)

Here, \( \hat{\nu} \) denotes the outward unit normal to the boundary of \( S \) and the the conormal derivative \( T_{\hat{\nu}}(u) \) with \( u^j \) being the \( j \)-th component of \( u \) is defined by

\[
T_{\hat{\nu}}(u) = \begin{cases} 
2\mu \partial_3 u + \lambda \hat{\nu} (\nabla \cdot u) + \mu (\partial_2 u^1 - \partial_1 u^2) \hat{\nu}^\perp, & n = 2, \\
2\mu \partial_3 u + \lambda \hat{\nu} (\nabla \cdot u) + \mu \hat{\nu} \times (\nabla \times u), & n = 3.
\end{cases}
\]

(3.2)

The following complex geometric optics (CGO) solution \( u_0 \) is introduced in [26]:

\[
u_0(x) = e^{\xi \cdot (x-x_0)} \eta,
\]

where

\[
\xi = \tau p + i \sqrt{\kappa_s^2 + \tau^2} \hat{p} \perp, \quad \eta = \hat{p} \perp - i \sqrt{1 + \kappa_s^2/\tau^2} p.
\]

(3.3)

\[
\tau > \kappa_s, \quad p \cdot \hat{p} \perp = 0, \quad \hat{p} \perp, \quad p \in S^{n-1}.
\]

(3.4)

It is directly verified that

\[
\Delta^* u_0 + \omega^2 u_0 = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

(3.5)

Note that

\[
|\xi| = \sqrt{2\tau^2 + \kappa_s^2} \quad \text{and} \quad |\eta| = \sqrt{2 + \kappa_s^2/\tau^2} \leq \sqrt{3}.
\]

(3.6)

In Proposition 3.2 we shall present asymptotic analysis of the volume integral of the CGO solution \( u_0 \) over a polyhedral cone with respect to \( \tau \) as \( \tau \to +\infty \). Before proving that, we now introduce a special cone pair \((A, C)\) as follows.

Definition 3.1 Let \( n \in \{2, 3\} \), \( 0 < \alpha_m < \alpha_M < \frac{\pi}{2} \), \( \omega > 0 \). Assume that \( A, C \subset \mathbb{R}^n \) are, respectively, an open polyhedral cone and an open spherical cone. We say \((A, C) \in \mathcal{F}(\alpha_m, \alpha_M, n)\) if the following conditions are fulfilled:

(a) \( A \subset \mathbb{R}^n \) is a convex polyhedral cone;
(b) \( A, C \) both take \( x_0 \in \mathbb{R}^n \) as the apex and \( A \subset C \);
(c) the opening angle of \( C \) at the apex \( x_0 \) is no more than \( \pi \);
(d) the opening angle of \( A \subset \mathbb{R}^2 \) at the apex \( x_0 \) is in \((2\alpha_m, 2\alpha_M)\).

Proposition 3.2 Fix \( n \in \{2, 3\} \), \( 0 < 2\alpha_m < 2\alpha_M < \pi, \omega > 0 \). Let \( u \) be the solution to (1.2) with \( \Omega = \text{supp}(\rho - 1) \). In addition, \( \Omega \) takes \( x_0 \) as its apex and \( u(x_0) \) is a nontrivial constant complex vector. Let \( A \) and \( C \) be an open polyhedral cone and an open spherical cone, respectively, they both take \( x_0 \) as their vertex. If \((A, C) \in \mathcal{F}(\alpha_m, \alpha_M, n)\), then there exists a vector \( \xi \) defined by (3.4) such that

\[
p \cdot (x-x_0) \leq -\delta_0 |x-x_0| \quad \text{for any} \quad x \in A.
\]

(3.7)
where $\delta_0$ is a positive constant only depending on $p$, $\Omega$ and $A$. Moreover, we have
\begin{equation}
| \int_A e^{i(x-x_0)} \, dx | \geq C \tau^{-n}, \tag{3.8}
\end{equation}
and
\begin{equation}
|u(x_0) \cdot \eta| \geq C_0 > 0, \tag{3.9}
\end{equation}
where $\tau = |\Re \xi|$, $\eta$ is defined in (3.4), $p^\perp \in S^{n-1}$ satisfies $p \cdot p^\perp = 0$, and $C_0$ is a positive constant only depending on $\xi_0$ and $A$.

**Proof**  The above conclusions can be obtained directly from [5, Lemma 6.3] and [8, Lemma 2.2] except that we need to prove the inequality (3.9). Note that

\begin{equation}
\Re u(x_0) \cdot \eta = i(\Im u(x_0) \cdot p^\perp - \sqrt{1 + \kappa^2/\tau^2} \Re u(x_0) \cdot p)
+ \Re u(x_0) \cdot p^\perp + \sqrt{1 + \kappa^2/\tau^2} \Im u(x_0) \cdot p, \tag{3.10}
\end{equation}

where $\Re u(x_0)$ and $\Im u(x_0)$ stand for the real and imaginary parts of $u(x_0)$, respectively. Since $u(x_0)$ is a nontrivial constant complex vector, without loss of generality, we assume that $\Im u(x_0) \neq 0$. For simplicity, we only consider three-dimensional scenarios and the two-dimensional conclusion can be similarly proved. In fact, once $p$ is fixed, $p^\perp$ belongs to $\Pi$, where $\Pi$ represents the plane defined by $\Pi = \{x \in \mathbb{R}^3 \mid x \cdot p = 0\}$. We can easily prove (3.9) by virtue of choosing a special unit vector $p^\perp$ as follows:

(I) Case 1: $\Im u(x_0) \cdot p < 0$. In order to prove (3.9), we consider the lower bound for the real part of $u(x_0) \cdot \eta$.

Let us distinguish two separate situations. If $\Re u(x_0) \in \Pi$ (including $\Re u(x_0) \parallel \Pi$), we can take $p^\perp \in \Pi$ such that $\angle(p^\perp, \Re u(x_0)) \in (\pi/2, \pi)$, where $\angle(p^\perp, \Re u(x_0))$ is the angle between $\Re u(x_0)$ and $p^\perp$. If $\Re u(x_0) \notin \Pi$, it is clear that there must exist a unit vector $p^\perp \in \Pi$ such that $\angle(p^\perp, \Re u(x_0)) \in (\pi/2, \pi)$. Hence, we always can choose the unit vector $p^\perp \in \Pi$ fulfilling the following inequality,

\begin{equation}
\Re u(x_0) \cdot p^\perp < 0.
\end{equation}

It can be directly seen that the signs of $\Re u(x_0) \cdot p^\perp$ and $\Im u(x_0) \cdot p$ are the same. Therefore, we have

\begin{align}
|u(x_0) \cdot \eta| & \geq |\Re u(x_0) \cdot p^\perp + \sqrt{1 + \kappa^2/\tau^2} \Im u(x_0) \cdot p| \\
& \geq |\Re u(x_0) \cdot p^\perp + \Im u(x_0) \cdot p| := C_0 > 0. \tag{3.11}
\end{align}

(II) Case 2: $\Im u(x_0) \cdot p = 0$. In order to prove (3.9), we consider the lower bound for the imaginary part of $u(x_0) \cdot \eta$.

Let us investigate the following two situations. If $\Re u(x_0) \in \Pi$ (including $\Re u(x_0) \parallel \Pi$), then $\Re u(x_0) \cdot p = 0$. Thus we can take $p^\perp \in \Pi$ such that $\angle(p^\perp, \Re u(x_0)) \neq \pi/2$. If $\Re u(x_0) \notin \Pi$, let us suppose that $\Re u(x_0) \cdot p > 0$, we can choose a unit vector $p^\perp$ belonging to $\Pi$ such that $\angle(p^\perp, \Re u(x_0)) \in (\pi/2, \pi)$, which implies $\text{sign}(\Im u(x_0) \cdot p^\perp) = -\text{sign}(\Re u(x_0) \cdot p)$. Here $\text{sign}(a)$ is the sign of a real number $a$. Therefore, we obtain

\begin{align}
|u(x_0) \cdot \eta| & \geq |\Im u(x_0) \cdot p^\perp - \sqrt{1 + \kappa^2/\tau^2} \Re u(x_0) \cdot p| \\
& \geq |\Im u(x_0) \cdot p^\perp - \Re u(x_0) \cdot p| := C_0 > 0.
\end{align}
Case 3: If \( \Re u(x_0) \cdot p > 0 \). Similar to Case 1, in order to prove (3.9), we can consider the lower bound for the real part of \( u(x_0) \cdot \eta \).

Similar to the above two cases, let us consider the following two situations. If \( \Re u(x_0) \in \Pi \) (including \( \Re u(x_0) // \Pi \)), we can take \( p^\perp \in \Pi \) such that \( \angle(p^\perp, \Re u(x_0)) \in [0, \pi/2) \). If \( \Re u(x_0) \notin \Pi \), there must exist a unit vector \( p^\perp \in \Pi \) such \( \angle(p^\perp, \Re u(x_0)) \in [0, \pi/2) \). Hence, for the above two situations, we can always choose a unit vector \( p^\perp \) fulfilling the following inequality as \( p^\perp \):

\[
\Re u(x_0) \cdot p^\perp \geq 0.
\]

By the same deduction for Case 1, we can also obtain the inequality (3.11).

The proof is complete. \( \Box \)

A critical asymptotic estimate with respect to \( \Re \xi \) for the volume integral associated with the CGO solution \( u_0 \) and the total wave field \( u \) near a vertex of \( \Omega \) is presented in the following proposition, which is the key ingredient to derive the stability result in Theorem 1.1.

**Proposition 3.3** Let \( u \) and \( u' \) be the solutions to (1.2) with \( \Omega, \Omega', \rho, \rho' \) satisfying the assumptions in Theorem 1.1. Let \( x_0 \) be a vertex of \( \Omega \) such that \( \text{dist}(x_0, \Omega') = \text{dist}(\nu, \Omega, \Omega') \). Fix \( \omega, \tau > 0 \) and \( h \in (0, 1) \) such that \( B_h(x_0) \cap \Omega' = \emptyset \). Denote \( S_h = B_h(x_0) \cap \Omega \) and \( D_h = B_h(x_0) \cap Q \), where \( Q \) is the convex hull of \( \Omega \) and \( \Omega' \). Let \( u(x_0) \) be a nontrivial constant complex vector. Assume that \( A \) and \( B \) represent the polyhedron cone generated by \( K \) and \( Q \) at the vertex \( x_0 \), respectively. Let \( \rho \in C^{\theta_1}(S_h) \) and \( u \in C^{\theta_2}(S_h)^n \), \( \theta_j \in (0, 1) \), \( j = 1, 2 \), which yield that

\[
\rho(x) = 1 + \rho(x_0) + \rho_1(x), \quad |\rho_1(x)| \leq M|x - x_0|^\theta_1, \quad \theta_1 \in (0, 1),
\]

\[
u(x) = u(x_0) + u_1(x), \quad |u_1(x)| \leq R|x - x_0|^\theta_2, \quad \theta_2 \in (0, 1),
\]

where \( M = \|\rho_1\|_{C^{\theta_1}(\Omega)} \) and \( R = \|u_1\|_{C^{\theta_2}(\Omega)} \).

Then we have the integral identity

\[
(\rho(x_0) - 1) \int_A e^{\xi \cdot (x - x_0)} \eta \cdot u(x_0) dx = \frac{1}{\omega^2} \int_{\partial D_h} u_i \cdot T_\nu (u^i - u) - (u^i - u) T_\nu (u_0) d\nu
\]

\[
+ (\rho(x_0) - 1) \int_{A \setminus S_h} e^{\xi \cdot (x - x_0)} \eta \cdot u(x_0) dx
\]

\[
- \int_{S_h} e^{\xi \cdot (x - x_0)} (\rho_1 + 1) \eta \cdot u(x_0) dx
\]

\[
- \int_{S_h} e^{\xi \cdot (x - x_0)} (\rho - 1) \eta \cdot u_1 dx,
\]

and the estimate

\[
C |(\rho(x_0) - 1) \int_A e^{\xi \cdot (x - x_0)} \eta \cdot u(x_0) dx| \leq |\Re \xi|^{-n} e^{-\delta_0 |\Re \xi|h/2} + |\Re \xi|^{-n-\min(\theta_1, \theta_2)}
\]

\[
+ h^{n-1} \sup_{|\partial S_h \cap B_h(x_0)|} \left\{ |\nabla u - \nabla u'| + |u - u'| \right\}
\]

\[
+ h^{n-1} e^{-\delta_0 |\Re \xi|h} \left\{ \|u_s\|_{H^2(B_{2R})^n} + \|u_p\|_{H^2(B_{2R})^n} + \|u_p\|_{H^2(B_{2R})^n} \right\}.
\]

where \( \delta_0 \) coincides with the one in (3.7) and \( C \) is a positive constant depending on the a-priori parameters.
Proof According to (3.3) and (3.5), the integral identity (3.14) follows directly from (3.1) by utilizing (3.12) and (3.13).

To estimate each term in the right-hand side of (3.14), let us recall the incomplete Gamma functions \( \gamma(\cdot, \cdot) \) and \( \Gamma(\cdot, \cdot) \) (cf. [5]) which are defined by

\[
\gamma(x, y) = \int_y^{\infty} e^{-t} t^{x-1} dt, \quad \Gamma(x, y) = \int_y^{\infty} t^{x-1} e^{-t} dt.
\]

From the fact \( e^{-t} \leq e^{-t/2} e^{-x/2} \) and a variable substitution \( t' = t/2 \), it is easy to get the estimation:

\[
\gamma(s, x) \leq \Gamma(s) \leq [s - 1], \quad \Gamma(s, x) \leq 2^s \Gamma(s) e^{-x/2},
\]

where \([s]\) the largest integer satisfying \([s] \leq s\), and \(\Gamma(s)\) is the complete Gamma function of \(s\) bounded by \([s]!\).

To estimate the first four terms in the right-hand side of (3.14), in the following we derive some important asymptotic inequalities with respect to the parameter \(|\Re \xi|\) in the CGO solution \(u_0\). By using the polar coordinate transformation, as well as (3.7), (3.16) and (3.17), we can obtain

\[
\left| \int_{A \setminus S_h} e^{\xi \cdot (x-x_0)} \right| \leq \sigma(S_{n-1}) \int_0^\infty e^{-h|\Re \xi|} r^{n-1} dr
\]

\[
\leq \sigma(S_{n-1}) \int_0^\infty e^{-h|\Re \xi|} r^{n-1} \left( \frac{e^{|\Re \xi|}}{|\Re \xi|} \right)^{n-1} dt
\]

\[
= \sigma(S_{n-1}) (e^{|\Re \xi|})^{-n} \Gamma(n, \delta_0 |\Re \xi|/h)
\]

\[
\leq C_{\delta_0, n} |\Re \xi|^{-n} e^{-\delta_0 |\Re \xi|/2}
\]

as \(|\Re \xi| \to +\infty\). Similarly, we have

\[
\left| \int_{S_h} e^{\xi \cdot (x-x_0)} \right| \leq \sigma(S_{n-1}) \int_0^h e^{-h|\Re \xi|} r^{n-1} dr
\]

\[
\leq \sigma(S_{n-1}) \int_0^h e^{-h|\Re \xi|} r^{n-1} \left( \frac{e^{|\Re \xi|}}{|\Re \xi|} \right)^{n-1} dt
\]

\[
= \sigma(S_{n-1}) (e^{|\Re \xi|})^{-n} \Gamma(n, \delta_0 |\Re \xi|/h)
\]

\[
\leq C'_{\delta_0, n} |\Re \xi|^{-n}
\]

as \(|\Re \xi| \to +\infty\), where \(\sigma(S_{n-1})\) is the measure of \(S_{n-1}\). For scalar functions \(f\) and \(g\), which admit \(|f| \leq A|x-x_0|^p\) with \(A\) and \(B\) being positive constants, and \(g \in L^q(S_h)\) with \(1/p + 1/q = 1\). Combining the Hölder inequality, (3.7), (3.16) with (3.17), we obtain

\[
\left| \int_{S_h} e^{\xi \cdot (x-x_0)} f g \right| \leq A \left( \int_{S_h} e^{\Re \xi \cdot (x-x_0) p} |x-x_0|^B p d x \right)^{1/p} \|g\|_{L^q(S_h)}
\]

\[
\leq A \sigma(S_{n-1}) \left( \frac{e^{|\Re \xi|}}{|\Re \xi|} \right)^{-B-n/p} \left( \gamma(Bp + n, \delta_0 |\Re \xi|/h) \right)^{1/p} \|g\|_{L^q(S_h)}
\]

\[
\leq C_{A, B, \delta_0, p} |\Re \xi|^{-B-n/p} \|g\|_{L^q(S_h)}
\]
as $|\Re \xi| \to +\infty$. By similar arguments, we can derive the following estimations for vector-type functions $\mathbf{f}$ and $\mathbf{g}$ satisfying $|\mathbf{f}| \leq A |\mathbf{x} - \mathbf{x}_0|^B$ with $A$ and $B$ being positive constants and $\mathbf{g} \in L^q(S_h)^n$ with $1/p + 1/q = 1$,

$$\left| \int_{S_h} e^{\xi \cdot (\mathbf{x} - \mathbf{x}_0)} \mathbf{f} \cdot \mathbf{g} \, d\mathbf{x} \right| \leq A \left( \int_{S_h} e^{N \xi \cdot (\mathbf{x} - \mathbf{x}_0)^p} |\mathbf{x} - \mathbf{x}_0|^B \, d\mathbf{x} \right)^{1/p} \|\mathbf{g}\|_{L^q(S_h)^n}$$

$$\leq A \sigma (\Re \xi)^{-B - n/p} \gamma(Bp + n, \delta_0 |\Re \xi| h)^{1/p} \|\mathbf{g}\|_{L^q(S_h)^n}$$

$$\leq C_{A,B,\delta_0,p} |\Re \xi|^{-B - n/p} \|\mathbf{g}\|_{L^q(S_h)^n}$$

(3.21)

as $|\Re \xi| \to +\infty$.

By virtue of (3.12) and (3.13), from (3.18)-(3.21), we can deduce that

$$R_1 := \left| \int_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \mathbf{u}_0 \cdot T_\nu (\mathbf{u}' - \mathbf{u}) - (\mathbf{u}' - \mathbf{u}) T_\nu (\mathbf{u}_0) \, d\mathbf{x} \right|$$

$$\leq \sqrt{\sigma} (\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)) \left( \int_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \left| \mathbf{u}_0 \cdot T_\nu (\mathbf{u}' - \mathbf{u}) - (\mathbf{u}' - \mathbf{u}) T_\nu (\mathbf{u}_0) \right|^2 \, d\mathbf{x} \right)^{1/2}$$

$$\leq \sqrt{\sigma} (\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)) \left( \int_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \left( |\mathbf{u}_0| + |T_\nu (\mathbf{u}_0)| \right)^2 \, d\mathbf{x} \right)^{1/2} T_1,$$

(3.23)

where $T_1 = \sup_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \{|T_\nu (\mathbf{u}' - \mathbf{u})|, |\mathbf{u} - \mathbf{u}'|\}$. By virtue of (3.2) and (3.3), one can directly derive that

$$T_\nu (\mathbf{u}_0) = \mu e^{\xi \cdot (\mathbf{x} - \mathbf{x}_0)} [\xi \cdot \nu + (\eta \cdot \nu) \xi] \quad \text{and} \quad |T_\nu (\mathbf{u} - \mathbf{u}')| \leq C_{\lambda,\mu} |\nabla (\mathbf{u} - \mathbf{u}')|,$$

where $\nu$ is the outward unit normal vector to $\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)$ and $C_{\lambda,\mu}$ is a positive constant depending on $\lambda$ and $\mu$. From (3.23), we obtain that

$$I_1 \leq \sqrt{\sigma} (\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)) \left[ \int_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \left( |e^{\xi \cdot (\mathbf{x} - \mathbf{x}_0)}| + |\mu e^{\xi \cdot (\mathbf{x} - \mathbf{x}_0)} (\xi \cdot \nu) \eta \right. \right.$$

$$\left. + (\eta \cdot \nu) \xi \right)^2 \, d\mathbf{x} \right]^{1/2} T_1$$

$$\leq \sqrt{\sigma} (\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)) \left[ \int_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \left( |\eta| + \mu |\eta (\xi \cdot \nu) + (\eta \cdot \nu)\xi| \right)^2 \, d\mathbf{x} \right]^{1/2} T_1$$

$$\leq C_{\lambda,\mu} \sigma (\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)) \sup_{\partial \mathcal{B} \cap \mathcal{B}_h(\mathbf{x}_0)} \{|\nabla (\mathbf{u} - \mathbf{u}')|, |\mathbf{u} - \mathbf{u}'|\},$$

(3.24)
where \( C'_{\lambda, \mu} \) is a positive constant depending on \( \lambda \) and \( \mu \). There exist finite half-spaces \( \{ H_j \}_{j=1}^N \) through \( x_0 \) with a codimension of 1 such that

\[
\partial B \cap B_h(x_0) \subset \bigcap_{j=1}^N \left( H_j \cap B_h(x_0) \right).
\]

Hence, we have

\[
\sigma(\partial B \cap B_h(x_0)) \leq C_{N,n} \sigma(\partial H_j \cap B_h(x_0)) \leq C_{N,n} h^{n-1}, \tag{3.25}
\]

where \( C_{N,n} \) is a positive constant depending only on \( n \) and \( N \). Substituting (3.25) into (3.24), one has

\[
I_1 \leq C h^{n-1} \sup_{\partial B \cap B_h(x_0)} \left\{ |\nabla(u - u')|, |u - u'| \right\}, \tag{3.26}
\]

where \( C \) is an a-priori positive constant depending on \( \lambda, \mu, N \) and \( n \).

For the boundary integral over \( S = \partial S_h(x_0) \setminus (\partial B \cap B_h(x_0)) \), by the Cauchy-Schwarz inequality and the fact that \( \sigma(\partial B \cap S_h(x_0)) \leq C h^{n-1} \), where \( C \) is a positive constant depending on \( n \) and \( \pi \), one has

\[
I_2 := \left| \int_S u_0 T_\hat{\nu}(u - u') - (u - u') T_\hat{\nu}(u_0) \, d\sigma \right|
\leq \sqrt{\sigma(S)} \left( \int_S |u_0 \cdot T_\hat{\nu}(u' - u) - (u' - u) T_\hat{\nu}(u_0)|^2 \, d\sigma \right)^{1/2} T_2
\leq \sqrt{\sigma(S)} \left( \int_S (|u_0|^2 + |T_\hat{\nu}(u_0)|^2) \, d\sigma \right)^{1/2} T_3
\leq C'_{\lambda, \mu, n} \sigma(S) e^{-\delta_0 |\mathbb{N}^\xi| h} T_3
\leq C'_{\lambda, \mu, n} h^{n-1} e^{-\delta_0 |\mathbb{N}^\xi| h} T_4, \tag{3.27}
\]

where

\[
T_2 = \sup_S |\nabla(u - u')| + \sup_S |u - u'|,
T_3 = \|u_s - u'_s\|_{C^1(\overline{B_h(x_0)})}^n + \|u_p - u'_p\|_{C^1(\overline{B_h(x_0)})}^n,
T_4 = \|u_s\|_{H^2(B_{2R})}^n + \|u_p\|_{H^2(B_{2R})}^n + \|u'_s\|_{H^2(B_{2R})}^n + \|u'_p\|_{H^2(B_{2R})}^n.
\]

Finally, it is easy to see that

\[
C \left| (\rho(x_0) - 1) \int_A e^{\xi \cdot (x - x_0)} \eta \cdot u(x_0) \, dx \right| \leq R_1 + R_2 + R_3 + I_1 + I_2, \tag{3.28}
\]

where \( C = C(\kappa_p, \kappa_s, \mathcal{M}, \mathcal{R}, \theta_1, \theta_2, n, \delta_0) > 0 \). Substituting (3.22a) to (3.22d), (3.26) and (3.27) into (3.28), we prove (3.15).

\[\square\]

4 Proof of theorem 1.1

In this section, we shall give the proof of Theorem 1.1.
Proof of Theorem 1.1 Let \( x_0 \) be a vertex of \( \Omega \) such that \( \text{dist}(x_0, \Omega') = \dist(\Gamma, \Omega') \) and \( Q \) be the convex hull of \( \Omega \) and \( \Omega' \). From Lemma 3.1 and Lemma 3.2, we know \( x_0 \) must be a vertex of \( Q \). Let \( B \) be the open polyhedral cone generated by \( Q \) at \( x_0 \). Then there must be an open spherical cone \( D \) whose opening angle at \( x_0 \) is in \( (2\alpha_M, \pi) \subset (2\alpha_M, \pi) \). Let \( A \) denote the open polyhedral cone generated by \( \Omega\) at \( x_0 \) and \( C \) be an open spherical cone such that \( (A, C) \in \mathcal{F}(\alpha_m, \alpha_M, n) \) (cf. Definition 3.1). Denote \( h = \text{dist}(x_0, \Omega') < 1 \) and choose \( h \leq \min\{l_0, h\} \) such that

\[
\begin{align*}
\Delta^* u' + \omega^2 u' &= 0 \quad \text{in} \quad S_h \subset D_h, \\
S_h \cap \Omega' &= D_h \cap \Omega' = \emptyset,
\end{align*}
\]

where \( S_h = \Omega \cap B_h(x_0) = A \cap B_h(x_0) \) and \( D_h = Q \cap B_h(x_0) = B \cap B_h(x_0) \).

From the proof of Proposition 2.3, we can derive that

\[
\|u\|_{H^2(B_{2R})'} \leq C_{k_p, k_s, R, S, \epsilon} = p, s.
\]

Thus

\[
u \in H^2(B_{2R})' \quad \text{and} \quad u' \in H^2(B_{2R})'.
\]

The a-priori admissibility assumptions of \( \rho \) and the local \( H^2 \)-regularity of \( u \) near \( S_h \) imply the following splittings

\[
\rho(x) = 1 + \rho_0(x_0) + \rho_1(x), \quad |\rho_1(x)| \leq \|\rho_1\|_{C^0_1(K \cap B_h(x_0))} |x - x_0|^{\theta_1}, \quad \theta_1 \in (0, 1),
\]

\[
u(x) = u(x) + u_1(x), \quad |u_1(x)| \leq \|u_1\|_{C^0_1(K \cap B_h(x_0))} |x - x_0|^{\theta_2}, \quad \theta_2 \in (0, 1),
\]

where we use the Sobolev embedding theorem from \( H^2 \) to \( C^{\theta_2} \). So the conditions of Proposition 3.3 are all satisfied. We absorb all the constants into the left-hand side, which depend only on the \( a-priori \) parameters, then (3.15) reduces to

\[
C \left| (\rho(x_0) - 1) \int_A e^{\xi \cdot (x-x_0)} \eta \cdot u(x_0) \, dx \right| \leq |\eta \xi|^{-\eta} e^{-\delta_0|\eta \xi| h/2} + \langle |\eta \xi|^{-\eta} \min(\theta_1, \theta_2) \\
+ h^{n-1} \sup_{\partial B_h(x_0)} \left( |u - u'| + |\nabla u - \nabla u'| \right)^{-1/2} \\
+ h^{n-1} e^{-\delta_0|\eta \xi| h}.
\]

From Proposition 2.3 and the fact that \( \partial Q \supset \partial \Omega \cap B_h(x_0) \), we have

\[
\sup_{\partial B_h(x_0)} \left( |u - u'| + |\nabla u - \nabla u'| \right) \leq C' \left( \ln \ln(\mathcal{N}/\epsilon) \right)^{-1/2}.
\]

Let \( \delta(\epsilon) = \left( \ln \ln(\mathcal{N}/\epsilon) \right)^{-1/2} \), where \( C' \) depends only on those \( a-priori \) parameters.

On the other hand, note that \( (A, C) \in \mathcal{F}(\alpha_m, \alpha_M, n) \). We can apply Proposition 3.2 to obtain an estimate of the term on the left-hand side of (3.14) as follows

\[
\left| \int_A e^{\xi \cdot (x-x_0)} \eta \cdot u(x_0) \, dx \right| \geq C'_{A} \tau^{-n}.
\]

where \( C'_{A} \) is a positive constant only depending on \( A \) and \( \zeta_0 \).

From Definition 1.1, we have \( |\rho(x_0) - 1| \neq 0 \). Combining (4.1) with (4.2), we have

\[
C_1 |(\rho(x_0) - 1)| \leq e^{-\delta_0|\eta \xi| h/2} + \tau^{\eta} \min(\theta_1, \theta_2) + h^{n-1} \tau^n \delta(\epsilon) + h^{n-1} \tau^n e^{-\delta_0|\eta \xi| h},
\]
where $C_1$ is a positive constant depending only on those a-priori parameters and $u(x_0)$. Using the fact that $e^{-x} \leq \frac{1}{x^2}$ and $e^{-x} \leq \frac{(n+4)!}{x^{n+4}}$ for all $x > 0$, we get

$$C_2 \left| (\rho(x_0) - 1) \right| \leq \tau^{-1} h^{-1} + \tau^{-\min(\theta_1, \theta_2)} + h^{n-1} \tau^{n+1} \delta(\varepsilon) + h^{-5} \tau^{-3} \leq C_0 (\tau^{-m} h^{-5} + \tau^{n+1} \delta(\varepsilon) h^{n-1}) \leq C_0 h^{n-1} [\tau^{-m} h^{n-4} + \tau^{n+1} \delta(\varepsilon)].$$

Here, $m = \min\{1, \theta_1, \theta_2\}$. Note that $\tau^4 > 1$, $h^{-n} > 1$ and $0 < h^{n-1} < 1$. Dividing the both sides of the above inequality by $h^{n-1}$, we obtain the following inequality:

$$C_3 \left| (\rho(x_0) - 1) \right| \leq \tau^{-m} h^{-5} + \tau^{n+5} \delta(\varepsilon). \quad (4.3)$$

To determinate a minimum modulo constants of the right-hand side of (4.3). Set

$$\tau = \tau_e = \left[ \frac{1}{h^{n+5} \delta(\varepsilon)} \right]^{1/\frac{m+n-1}{n+5}}.$$  \quad (4.4)

we have

$$C_4 \left| (\rho(x_0) - 1) \right| \leq 2 \delta(\varepsilon) \frac{m}{n+5} \frac{h^{\frac{(n+5)^2}{m+n+1}}}{h} \leq C_5 \delta(\varepsilon) h^{\frac{(n+5)^2}{m+n+1}}.$$

Note that $\tau_e \geq \tau_e h^{\frac{n+5}{m+n+1}} = \delta(\varepsilon) \frac{1}{h^{n+5}} \geq \tau_0$ and $h \leq h \leq l_0$ for sufficiently small $\varepsilon$. From (4.5), we have

$$h \leq C_5 \delta(\varepsilon) h^{\frac{m}{n+5}} \left| \rho(x_0) \right| \leq C_5 \delta(\varepsilon) \frac{m}{n+5} \left| \varepsilon_0 + 1 \right| \left[ \frac{m}{(n+5)^2} - \frac{1}{n+5} \right] \leq C_6 (\ln \ln(N/\varepsilon))^{-\frac{m}{2(n+5)^2}},$$

and the claim readily follows.

The proof is complete. \quad $\square$

5 Related results about invisibility

The present section is devoted to the technical details of Theorem 1.2.

**Theorem 5.1** Let $(\Omega; \rho)$ be an admissible polyhedral scatterer. The plane incident wave $u^i$ of the form (1.1) is a nontrivial entire solution to (1.2) with $\rho = 1$. Let $U$ be the far-field pattern of the scattered wave $u^s$ by medium scatterer $(\Omega; \rho)$. Then there exists a positive constant $C$ only depending on the a-priori parameters such that the following estimate holds:

$$\|U\|_{L^2(\mathbb{R}^n, C^\infty \times C^\infty)} \geq C > 0. \quad (5.1)$$

In fact, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, can be relaxed to a general bounded Lipschitz domain, which admits a convex polygon/polyhedral point on its boundary, namely, there exit $x_0 \in \partial \Omega$ such that $\Omega \cap B_l(x_0) \subset \mathbb{R}^n$ is a plane sector for $l_0 > l > 0$ and $\Omega \cap B_l(x_0) \subset \mathbb{R}^n$ is a polyhedral cone with a spherical bottom surface for $l_0 > l > 0$. The general result is stated as follows.

**Theorem 5.2** Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded Lipschitz domain, with a convex polygonal/polyhedral point $x_0 \in \partial \Omega$. Let $\rho(x)$ be a uniformly $\theta$-Hölder continuous function in $\Omega$, $0 < \theta \leq 1$. The plane incident wave $u^i$ of the form (1.1) is a nontrivial entire solution to (1.2) with $\rho = 1$. Let $U$ be the far-field pattern of the scattered wave $u^s$ by medium scatterer $(\Omega; \rho)$. Suppose that the following conditions are fulfilled:
(1) \( \Omega = \text{supp}(1 - \rho) \subset B_R \) for some \( R > l > 0 \);
(2) \( \| u^I \|_{H^2(B_R)'} \leq \mathcal{N} \) for \( \mathcal{N} > 0 \);
(3) \( |\rho(x_c) - 1| \geq \varepsilon_0 > 0 \) and \( \| \rho \|_{C^0(\Omega)} \leq \mathcal{M} \) for some \( \mathcal{M} > 0 \);
(4) For \( n = 2, \) \( 0 < 2\alpha_m \leq \alpha \leq 2\alpha_M \leq \pi \), where \( \alpha \) signifies the opening angle at \( x_c \).

Then the following estimate holds:

\[
\| U \|_{L^2(\mathbb{S}^{n-1}, \mathbb{C}^n \times \mathbb{C}^n)} \geq C > 0,
\]

where \( C \) is a positive constant, which depends on the a-priori parameters \( \theta, l, \mathcal{N}, \mathcal{M}, \varepsilon_0, \zeta_0 \) and \( R \).

**Proof** We follow similar arguments in the proof of Theorem 1.1 with some necessary modifications. We take \( \Omega' = \emptyset \) and \( \rho' \equiv 1 \) in \( \mathbb{R}^n \). Clearly, the incident wave \( u^I \) of the form (1.1) is an entire solution to \( \Delta^2 u^I + \omega^2 u^I = 0 \), which yields \( u^I \equiv 0 \) and \( U' \equiv 0 \). Denote \( \varepsilon = \| U \|_{L^2(\mathbb{S}^{n-1}, \mathbb{C}^n \times \mathbb{C}^n)} \). In fact, we only consider the effect caused by \( x_c \) in its micro-local area \( \Omega \cap B_R(x_c) \), and therefore we can reduce the corresponding reasoning in the proof Theorem 1.1. Here we take \( h = l \). After a series of derivations similar to Theorem 1.1, we can obtain the following inequality

\[
C_1 |(\rho(x_c) - 1)| \leq \tau^{-m} l^{-5} + \tau^{n+1} \delta(\varepsilon) l^{n-1},
\]

where \( \delta(\varepsilon) = (\ln(\mathcal{N}/\varepsilon))^{-1/2} \) and \( C_1 > 0 \) does not depend on \( l \).

Set

\[
\tau = \tau_{e} = \left[ \frac{1}{l^{n+5} \delta(\varepsilon)} \right]^{1/(m+n+1)},
\]

we have

\[
C_2 |(\rho(x_c) - 1)| \leq 2 \delta(\varepsilon) \frac{m}{m+n+5} l^{-\frac{(n+5)^2}{m+n+5}}.
\]

It is straightforward to verify that for sufficient small \( \varepsilon \), one has

\[
\tau_{e} \geq \tau_{e} l^{n+5} = \frac{C_3}{\delta(\varepsilon)^{m+n+5}} = C_3 (\ln(\mathcal{N}/\varepsilon))^{(n+5)^2/(2m+n+5)} \geq \tau_0,
\]

Using \( \tau = \tau_{e} \) in (5.4), \( |\rho(x_c) - 1| \geq \varepsilon_0 > 0 \) and \( \delta(\varepsilon) = (\ln(\mathcal{N}/\varepsilon))^{-1/2} \),

\[
\varepsilon \geq \mathcal{N} \left[ \exp \left\{ C_3 l^{-\frac{(n+5)^2}{m}} |(\rho(x_c) - 1) - \frac{m+n+5}{m} | \right\} \right]^{-1} \\
\geq \mathcal{N} \left[ \exp \left\{ C_3 l^{-\frac{(n+5)^2}{m}} \varepsilon_0 \frac{m+n+5}{m} \right\} \right]^{-1}.
\]

That is,

\[
\| U \|_{L^2(\mathbb{S}^{n-1}, \mathbb{C}^n \times \mathbb{C}^n)} \geq C > 0,
\]

where \( C = \mathcal{N} \left[ \exp \left\{ C_3 l^{-\frac{(n+5)^2}{m}} \varepsilon_0 \frac{m+n+5}{m} \right\} \right]^{-1} \).

The proof is complete. \( \square \)
Acknowledgements The authors would like to thank three anonymous referees for their constructive comments and suggestions, which have led to significant improvement on the earlier version of this paper. Particular, anonymous referees reminded us to pay attentions to the related stability study [46] on the rigid obstacle in linear elasticity. The research of Z Bai was partially supported by the National Natural Science Foundation of China under grant 11671337 and the Natural Science Foundation of Fujian Province of China under grant 2021J01033. The work of H Diao was supported in part by the NSFC/RGC Joint Research Fund (project 12161160314) and the startup fund from Jilin University. The work of H Liu was supported by the startup fund from City University of Hong Kong and the Hong Kong RGC General Research Fund (projects 12301420, 11300821 and 12301218), and the NSFC/RGC Joint Research Fund (project N_CityU101/21).

References

1. Alessandrini, G., Rondi, L.: Determining a sound-soft polyhedral scaterer by a single far-field measurement. Proc. Aner. Math. Soc. 35, 1685–1691 (2005)
2. Blåsten, E.: Nonradiating sources and transmission eigenfunctions vanish at corners and edges. SIAM J. Math. Anal. 50(6), 6255–6270 (2018)
3. Blåsten, E., Lin, Y.-H.: Radiating and non-radiating sources in elasticity. Inverse Prob. 35(1), 015005 (2019)
4. Blåsten, E., Liu, H.: On vanishing near corners of transmission eigenfunctions. J. Funct. Anal. 273, 3616–3632 (2017)
5. Blåsten, E., Liu, H.: On corners scattering stably and stable shape determination by a single far-field pattern. Indiana Univ. Math. J. 70(3), 907–947 (2021)
6. Blåsten, E., Liu, H.: Recovering piecewise-constant refractive indices by a single far-field pattern. Inverse Prob. 36, 085005 (2020)
7. Blåsten, E., Liu, H.: Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems. SIAM J. Math. Anal. 53(4), 3801–3837 (2021)
8. Blåsten, E., Liu, H., Xiao, J.: On an electromagnetic problem in a corner and its applications. Analysis & PDE 14(7), 2207–2224 (2021)
9. Blåsten, E., Päivärinta, L., Sylvester, J.: Corners always scatter. Comm. Math. Phys. 331(2), 725–753 (2014)
10. Brummelhuis, R.: Three-spheres theorem for second order elliptic equations. J. Anal. Math. 65, 179–206 (1995)
11. Cakoni, F., Vogelius, M.: Singularities almost always scatter: regularity results for non-scattering inhomogeneities, arXiv:2104.05058
12. Cao, X., Diao, H., Liu, H.: Determining a piecewise conductive medium body by a single far-field measurement. CSIAM Trans. Appl. Math. 1, 740–765 (2020)
13. Challa, D.P., Sini, M.: The Foldy-Lax approximation of the scattered waves by many small bodies for the Lamé system. Math. Nachr. 288(16), 1834–1872 (2015)
14. Chow, Y.T., Deng, Y., He, Y., Liu, H., Wang, X.: Surface-localized transmission eigenstates, super-resolution imaging and pseudo surface plasmon modes. SIAM J. Imaging Sci. 14(3), 946–975 (2021)
15. Chow, Y. T., Deng, Y., Liu, H., Sunkula, M.: Surface concentration of transmission eigenfunctions, arXiv:2109.14361
16. Colton, D., Kress, R.: Looking back on inverse scattering theory. SIAM Rev. 60(4), 779–807 (2018)
17. Deng, Y., Jiang, Y., Liu, H., Zhang, K.: On new surface-localized transmission eigenmodes. Inverse Problems and Imaging 16(3), 595–611 (2022). https://doi.org/10.3934/ipi.2021063
18. Deng, Y., Liu, H., Wang, X., Wu, W.: On geometrical properties of electromagnetic transmission eigenfunctions and artificial mirage. SIAM J. Appl. Math. 82(1), 1–24 (2021)
19. Di Cristo, M., Rondi, L.: Example of exponential instability for inverse inclusion and scattering problems. Inverse problems 19(3), 685–701 (2003)
20. Diao, H., Liu, H., Wang, L.: Further results on generalized Holmgren’s principle to the Lamé operator and applications. J. Differ. Equ. 309, 841–882 (2022)
21. Diao, H., Liu, H., Wang, L.: On generalized Holmgren’s principle to the Lamé operator with applications to inverse elastic problems. Calc. Var. Partial. Differ. Equ. 59, 50 (2020)
22. Diao, H., Cao, X., Liu, H.: On the geometric structures of transmission eigenfunctions with a conductive boundary condition and application. Comm. Partial Differential Equations 46(4), 630–679 (2021)
23. Diao, H., Liu, H., Wang, X., Yang, K.: On vanishing and localizing around corners of electromagnetic transmission resonance. Partial Differ. Equ. 2, 78 (2021)
24. Diao, H., Liu, H., Sun, B.: On a local geometric structure of generalized elastic transmission eigenfunctions and application. Inverse Prob. 37, 105015 (2021)
25. Hähner, P.: A uniqueness theorem in inverse scattering of elastic waves. IMA J. Appl. Math. 51, 201–215 (1993)
26. Hähner, P.: On acoustic, electromagnetic, and elastic scattering problems in inhomogeneous media, Universität Göttingen, Habilitation Thesis (1998)
27. Hähner, P.: On uniqueness for an inverse problem in inhomogeneous elasticity. IMA J. Appl. Math. 67, 127–143 (2002)
28. Higashimori, N.: A conditional stability estimate for indentifying a cavity by an elstatostatic measurement, Ph. D. Thesis, Graduate School of Informatics, Kyoto University, (2003)
29. Hu, G., Liu, H.: Nearly cloaking the elastic wave fields. J. Math. Pures Appl. 104(9)(6), 1045–1074 (2015)
30. Liu, H.: On local and global structures of transmission eigenfunctions and beyond. J. Inverse and Ill-posed Problems 30(2), 287–305 (2022). https://doi.org/10.1515/jiip-2020-0099
31. Liu, H., Petrini, M., Rondi, L., Xiao, J.: Stable determination of sound-hard polyhedral scattereres by a minimal number of scattering measurements. J. Differential Equations 262(3), 1631–1670 (2017)
32. Liu, H., Rondi, L., Xiao, J.: Mosco convergence for $H(curl)$ spaces, higher integrability for Maxwell’s equations, and stability indirect and inverse EM scattering problems, J. Eur. Math. Soc(JEMS), 21(10), 2945–2993 (2019)
33. Liu, H., Tsou, C.H.: Stable determination by a single measurement, scattering bound and regularity of transmission eigenfunction. Calc. Var. Partial. Differ. Equ, 61, 91 (2022)
34. Liu, H., Tsou, C.H.: Stable determination of polygonal inclusions in Calderón’s problem by a single partial boundary measurement. Inverse Prob. 36, 085010 (2020)
35. Liu, H., Tsou, C.H., Yang, W.: On Calderón’s inverse inclusion problem with smooth shapes by a single partial boundary measurement. Inverse Prob. 37, 055005 (2021)
36. Liu, H., Xiao, J.: On electromagnetic scattering from a penetrable corner. SIAM J. Math. Anal. 49(6), 5207–5241 (2017)
37. Liu, H., Zou, J.: On uniqueness in inverse acoustic and electromagnetic obstacle scattering problems. Journal of Physics: Conference Series 124(1), 012006 (2008)
38. Mandache, N.: Exponential instability in an inverse problem for the Schrö equation. Inverse Problems 17(5), 1435–1444 (2001)
39. Mclean, W.: Strongly Elliptic Systems and Boundary Integral Equation. Cambridge University Press, Cambridge (2000)
40. Menegatti, G., Rondi, L.: Stability for the acoustic scattering problem for sound-hard scatterers. Inverse Probl. Imaging 7(3), 1307–1329 (2013)
41. Morassi, A., Rosset, E.: Stable determination of cavities in elastic bodies. Inverse Problems 20(2), 453–480 (2004)
42. Morassi, A., Rosset, E.: Uniqueness and stability in determining a rigid inclusion in an elastic body. Mem. Amer. Math. Soc. 20(938), viii+5888 (2009)
43. Päiväranta, L., Salo, M., Vesalainen, E.V.: Strictly convex corners scatter. Revista Matematica Iberoamericana 33(4), 1369–1396 (2017)
44. Rondi, L.: Stable determination of sound-soft polyhedral scatterers by a single measurement. Indiana Univ. Math. J. 57(3), 1377–140 (2008)
45. Rondi, L., Sini, M.: Stable determination of a scattered wave from its far-field pattern: the high frequency asymptotics. Arch. Ration. Mech. Anal. 218(1), 1–54 (2015)
46. Rondi, L., Sincich, E., Sini, M.: Stable determination of a rigid scatterer in elastodynamics. SIAM J. Math. Anal. 53(2), 2660–2689 (2021)
47. Salo, M., Shahgholian, H.: Free boundary methods and non-scattering phenomena. Res. Math. Sci. 8, 58 (2021). https://doi.org/10.1007/s40687-021-00294-z
48. Sincich, E., Sini, M.: Local stability for soft obstacles by a single measurement. Inverse Probl. Imaging 2(2), 301–315 (2008)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.