Decomposition of variance in terms of conditional means

Alessandro Figà Talamanca ∗ Angelo Guerriero †
Alberto Leone ‡ Gian Piero Mignoli § Enrico Rogora ¶

Abstract

We test against two different sets of data an apparently new approach to the analysis of the variance of a numerical variable which depends on qualitative characters. We suggest that this approach be used to complement other existing techniques to study the interdependence of the variables involved. According to our method the variance is expressed as a sum of orthogonal components, obtained as differences of conditional means, with respect to the qualitative characters. The resulting expression for the variance depends on the ordering in which the characters are considered. We suggest an algorithm which leads to an ordering which is deemed natural. The first set of data concerns the score achieved by a population of students, on an entrance examination, based on a multiple choice test with 30 questions. In this case the qualitative characters are dyadic and correspond to correct or

*Dept. of Mathematics, University of Rome "La Sapienza", sandroft@mat.uniroma1.it
†Alma Laurea, guerrier@stat.unibo.it
‡Alma Laurea, alberto.leone@almalaurea.it
§Alma Laurea, gianpiero.mignoli@almalaurea.it
¶Dept. of Mathematics, University of Rome "La Sapienza", rogora@mat.uniroma1.it
incorrect answer to each question. The second set of data concerns the delay in obtaining the degree for a population of graduates of Italian universities. The variance in this case is analyzed with respect to a set of seven specific qualitative characters of the population studied (gender, previous education, working condition, parent’s educational level, field of study, etc.)

1 Introduction and methodology

Let $X = (x_1, \ldots, x_N)$ be a numerical variable defined on a population $P$ of $N$ individuals. We may think of $X$ as an element of a real vector space $\mathbf{L}$ of dimension $N$. We equip $\mathbf{L}$ with a real, normalized scalar product: for $X, Y \in \mathbf{L}$, and $Y = (y_1, \ldots, y_N)$, we define:

$$< X, Y > = \frac{1}{N} \sum_{i=1}^{N} x_i y_i.$$

The length or norm of a vector is defined in terms of the scalar product:

$$\|X\|^2 = < X, X >.$$

The mean value of a vector $X$ is of course the scalar

$$\overline{X} = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

We may also think of the mean value as a vector $E_0(X)$ of $\mathbf{L}$ having all its components equal to the scalar $\overline{X}$. In this context $E_0$ may be thought of as a linear operator defined on $\mathbf{L}$ and mapping $\mathbf{L}$ into the subspace of constant vectors. The variance of $X$ can be written then as:

$$V(X) = \|X - E_0(X)\|^2 = < X - E_0(X), X - E_0(X) >.$$
We now suppose that the indices $i = 1, \ldots, N$, correspond to individuals of a population $P$, and that $X$ is a numerical variable defined on the population $P$. We further suppose that $\pi$ is a partition of the population $P$ into $q$ disjoint classes $P_1, P_2, \ldots, P_q$. Denote by $|P_j|$ the number of elements of $P_j$, so that $N = |P_1| + \cdots + |P_q|$. We can then define a vector $E_\pi(X)$ with components:

$$E_\pi(X)_i = \frac{1}{|P_k|} \sum_{j \in P_k} x_j, \quad (i \in P_k). \quad (1)$$

Observe that two components of this vector are identical if their indices belong to the same class $P_k$ of the partition $\pi$. The trivial identity:

$$X - E_0(X) = E_\pi(X) - E_0(X) + X - E_\pi(X),$$

implies

$$V(X) = \|X - E_0(X)\|^2 = \|E_\pi(X) - E_0(X)\|^2 + \|X - E_\pi(X)\|^2,$$

because, as it is easily seen, $E_\pi(X) - E_0(X)$ and $X - E_\pi(X)$ are orthogonal vectors.

Suppose now that $\pi_1, \pi_2, \ldots, \pi_n$ is a finite sequence of partitions of the population $P$, into respectively $q_1, q_2, \ldots, q_n$, classes. Suppose further that each partition $\pi_j$ is a refinement of the partition $\pi_{j-1}$. (This means that each class of the partition $\pi_j$ is contained in a class of the partition $\pi_{j-1}$). Define for completeness the trivial partition $\pi_0$ consisting of the full population $P$. Let $P^j_k$, for $k = 1, \ldots, q_j$ be the disjoint classes of the population $P$ relative to the partition $\pi_j$. With reference to the partition $\pi_j$ define the operator

$$E_j(X) = E_{\pi_j}(X).$$

In this fashion (1) reads:

$$E_{\pi_j}(X)_i = \frac{1}{|P^j_k|} \sum_{h \in P^j_k} x_h, \quad (i \in P^j_k).$$
Observe that this definition makes sense also in the case \( j = 0 \). The trivial identity
\[
X - E_0(X) = \sum_{j=1}^{n} [E_j(X) - E_{j-1}(X)] + X - E_n(X),
\] (2)
implies, because of the orthogonality of the terms on the right hand side of (2),
\[
V(X) = \sum_{j=1}^{n} \|E_j(X) - E_{j-1}(X)\|^2 + \|X - E_n(X)\|^2.
\] (3)
We are interested in the case in which the sequence of partitions \( \pi_j \) is defined by a sequence of qualitative characters \( C_1, C_2, \ldots, C_n \) of the population \( P \). We can define the partition \( \pi_j \) by considering the classes of the population formed by individuals with identical values of the first \( j \) characters. In this case the first \( n \) summands on the right hand side of (3) represent the contributions to the variance of the \( n \) qualitative characters \( C_1, \ldots, C_n \) within the population considered.

Observe however that, while the sum of the first \( n \) terms of the right hand side of (3) is independent of the order in which the characters \( C_1, \ldots, C_n \) are considered, the operators \( E_j \), for \( 0 < j < n \) are defined with respect to partitions which strongly depend on the order in which the characters are taken. As an obvious consequence, the value of each term \( \|E_j(X) - E_{j-1}(X)\|^2 \) also depends on the order of the characters. In a different order the characters would define a different set of partitions; only \( \pi_0 \) and \( \pi_n \), and consequently \( E_0 \) and \( E_n \) are independent of the chosen order.

We are led therefore to look for a natural order of the qualitative characters considered. We propose an ordering based on systematic, step by step, comparisons of the conditional means with respect to the variables considered. This ordering, which we call Stepwise Optimal Ordering (SOO) is defined as follows:
We choose the character $C_1$ and the corresponding partition $\pi_1$ which maximizes $\|E_1(X) - E_0(X)\|^2$. If $C_1, \ldots, C_k$ are chosen, the character $C_{k+1}$ is chosen so that it refines the partition $\pi_k$ into the partition $\pi_{k+1}$ in such a way that the value $\|E_{k+1}(X) - E_k(X)\|^2$ is largest.

The order $C_1, \ldots, C_n$ determined in this fashion may be considered as a ranking of the variables. One should be aware, however, that this ranking cannot be interpreted in terms of relative importance in determining the phenomenon measured by the variable $X$. As will be seen in the applications below, the qualitative characters considered may be far from independent. This may imply that a character which is recognized as a primary cause of the intensity of the phenomenon measured by $X$, may be mediated by other characters to whom it is associated, and therefore appear in the last positions of the ranking.

We do not propose a clear cut interpretation of the significance of the ranking obtained by our method, nor of the relative size of the first $n$ addends which appear in (3), when the qualitative characters are ordered according to our prescription. On the contrary, rather than expecting straight answers, we expect that both the ranking and the relative size of the addends in the expression (3) would solicit questions concerning the dependence of the variable $X$ on the qualitative variables and the interdependence of the qualitative variables themselves (with all the cautions regarding the possibility to consider causal relations between the variables, [2, 6, 7, 8]).

Nevertheless, in the very special case considered in the simulated experiment of Section 4, our method yields a ranking that reflects the relative weight of the characters.

In the following two sections we apply our method and discuss the ”ranking” of the qualitative characters, thus obtained to the two sets of data mentioned.
in the abstract. The fourth section is dedicated to a simulated experiment. We should mention that the ideas contained in Chapter 8D of [5] were influential in the inception of this work, which started as an attempt to apply Diaconis’ ideas to the case of tree-structured data, under the action of the group of tree-automorphisms. Under this action the ranges of the operators $E_j - E_{j-1}$ turn out to be irreducible subspaces of $L$.

2 The score on an entrance examination

Entering students of the University of Rome "La Sapienza" in scientific and technical fields take a multiple choice test in mathematics, which consists of 30 questions. The test, in Italian, may be downloaded at [1]. At the moment the purpose of the test is to discourage students who do not have an adequate background, and to make students aware of their potential weaknesses. We consider a population of 2,451 students who took the test in 2005, and we let $X$ be the score achieved by each student, that is the number of correct answers. The variable $X$ depends on the 30 dyadic characters, corresponding to the correct or incorrect answer to each question. Of course, in this case, $E_{30}(X) = X$, and

$$V(X) = \|X - E_0(X)\|^2 = \sum_{j=1}^{30} \|E_j(X) - E_{j-1}(X)\|^2.$$ 

The variable $X$ takes values between 0 and 30. Its mean value is 12.9 and the variance is $V(X) = 29.8$. The histogram of $X$ is in Fig. 1.

An application of our method shows that just ten questions, chosen according to the ranking we propose, "explain" 88% of the variance. In other words, if
we write

\[ V(X) = \sum_{j=1}^{10} \|E_j(X) - E_{j-1}(X)\|^2 + \|X - E_{10}\|^2, \]

the remainder term \( \|X - E_{10}\|^2 = 3.58 \) amounts to just 12% of \( V(X) = 29.8 \).

We presently list the remainders \( \|X - E_k(X)\|^2 \), for \( k = 1, \ldots, 10 \), obtained by applying our method, as percentage of \( V(X) \). To wit the values \( c_k = \|X - E_k(X)\|^2/V(X) \),

\[
\begin{align*}
c_1 &= \frac{75}{100}, \quad c_2 = \frac{59}{100}, \quad c_3 = \frac{48}{100}, \quad c_4 = \frac{40}{100}, \quad c_5 = \frac{34}{100}, \\
c_6 &= \frac{29}{100}, \quad c_7 = \frac{25}{100}, \quad c_8 = \frac{20}{100}, \quad c_9 = \frac{16}{100}, \quad c_{10} = \frac{12}{100}.
\end{align*}
\]

We do not claim, of course, that our method necessarily chooses the 10 characters for which \( \|X - E_{10}(X)\|^2 \) is lowest. In general, with arbitrary data, this may not be the case.

However, in this particular case, our choice compares well with other possible choices, as shown by the experiment which we presently describe. We se-

Figure 1: The histogram of the score.
Figure 2: Histogram of the values of residual variance (4), as percentage of total variance for 300 randomly selected subsets of 10 questions.

Selected, at random, 300 subsets of ten elements of the original thirty questions and we computed the conditional mean $E_\pi(X)$ with respect to the partition $\pi$ obtained by grouping together the students with identical performance on each of the ten question chosen. We computed then

$$\|X - E_\pi(X)\|^2,$$

relative to each ten element choice. The results are summarized in Fig. 2.

Observe that the lowest value of quantity (4) achieved by one of the 300 subsets we selected, is higher than 0.14, while with our choice of a subset of characters, we achieved a value of 0.12.

The experiment shows that the algorithm we propose performs decidedly better than a random choice if we want to choose ten out of thirty questions,
in such a way that the total variance of the variable $X$ is best explained. In conclusion there is at least some experimental evidence that our method may be used to select a small number of characters which account for most of the variance.

It is interesting to compare our results with the results obtained with linear regression. We found that the agreement between the results is strong. Eight of the ten variables selected by SOO are among the ten most important variables in terms of linear regression. Furthermore, the order of the first five variables coincides with both methods. We also observed that the variables selected according to SOO have the properties of discriminating the students (the differences of percentages of correct and incorrect answers is small).

3 The variable ”delay in completing a degree”

The Italian system of higher education is characterized by the marked difference between the time employed by most students to complete a degree and the number of years formally required to graduate. The average delay in completing a degree is well above two years for most fields of study\footnote{The recent reform of the university system may hopefully change this in the near future.}. In this section we consider a population of Italian university graduates obtained using the data bank “AlmaLaurea” which collects data of university graduates from a set of Italian universities\footnote{AlmaLaurea Consortium is an association of 49 Italian universities which, since 1994 collects statistical data about the scholastic and employment records of university graduates\cite{alma1, alma2}. The data bank of AlmaLaurea is also made available, under certain conditions, to prospective employers.}. The population amounts to 58,091.
graduates of 27 universities in 2003. On this population the variable $X$ represents the delay in completing the degree, computed in years, starting from a conventional date (November 1st) in which according to formal regulations the degree should have been completed. We excluded delays above ten years, which should be better interpreted as leaving and resuming the studies after several years. We study the dependence of $X$ on seven possible characters, which are the following:

(UN) University where the degree was obtained

(PE) Parent’s level of education

(HS) Type of high school attended

(GD) Grade in the final year of high school

(MA) Degree major

(WO) Working or not working during the studies

(GN) Gender

Proceeding as outlined in the introduction, we obtain the following ranking of the seven variables:

GD, UN, MA, HS, PE, WO, GN.

Accordingly we consider the operators

$$E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7,$$

and write

$$V(X) = \sum_{j=1}^{7} \|E_j(X) - E_{j-1}(X)\|^2 + \|X - E_7(X)\|^2$$  \hspace{1cm} (5)
The variance of the variable $X$ is $V(X) = 4.61$, while the residual variance, not "explained" by the qualitative variables under consideration is $\|X - E_7(X)\|^2 = 1.94$. The decomposition of the variance is:

$$4.61 = (0.30 + 0.28 + 0.49 + 0.45 + 0.51 + 0.33 + 0.31) + 1.94 = 2.67 + 1.94$$

Thus 2.67 represents the portion of the variance which is "explained" by the characters considered. We may say, therefore, that these characters explain 62% of the variance.

In this case the ranking obtained by our method is relatively "robust". Indeed if we omit consideration of one of the characters, the relative ranking of the other characters remains unchanged. We do not claim of course that this type of "robustness" is inherent in our method. It may very well occur, with different data, that omitting one character would determine a change in the order of the remaining characters.

We compared our results with the results obtained by using the binomial logistic regression. The delay in obtaining the degree becomes dicotomic assigning value zero to the population of graduates with a delay less than one year (34.1%) and value one to the others (65.9%). The results of our computations are shown in Table 1.

We observe that also in this case the rank in terms of size of the variances coincides, except for one inversion, with the ranking obtained by SOO. It should be noted however that the application of binomial logistic regression implies an arbitrary dicotomization of the variable. Moreover, it is questionable in this case that the binomial logistic regression would add information of inferential value, because its application leads to many classes which are empty or with few individuals.
| Variable | Variance | var, GD=100 |
|----------|----------|-------------|
| GD       | 0.0119   | 100.0       |
| UN       | 0.0076   | 63.6        |
| MA       | 0.0097   | 81.7        |
| HS       | 0.0036   | 30.2        |
| PE       | 0.0012   | 10.3        |
| WO       | 0.0010   | 8.4         |
| GN       | 0.0001   | 0.5         |

Table 1: Binomial logistic regression of the seven variables. In the column “Variance” is computed the variance of probability variation.

Figure 3: Histogram of the delay.
4 A simulated experiment

In order to better understand the properties of our Stepwise Optimal Order, we performed a simulation, repeating 20 times the following experiment. First we constructed 10 vectors \( x_1, \ldots, x_{10} \) each of 100 components and each component extracted from a simulated Bernoulli variable. Then we considered the variable

\[
x = c_1 x_1 + c_2 x_2 + \cdots + c_{10} x_{10} + \epsilon
\]

with \( c_1 = 1, c_2 = 0.9, \ldots, c_{10} = 0.1 \) and \( \epsilon \) consisting of 100 independent realizations of a simulated Gaussian variable with mean 0 and standard deviation 0.03.

In 18 cases out of the 20 observed experiments, SOO was exactly 1, 2, 3, \ldots, 10, i.e. for the variable \( x \) this order reflected, most of the time, the size of the coefficients \( c_1, \ldots, c_{10} \) which enter formula (6). In the remaining two case the difference between SOO and the increasing order was just one inversion.

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