Sources of Radiation in the Early Universe: The Equation of Radiative Transfer and Optical Distances

D. I. Nagirner, S. L. Kirusheva

Sobolev Astronomical Institute, St. Petersburg State University, Bibliotechnaya pl. 2, Petrodvorets, 198904 Russia

Abstract

We have derived the radiative-transfer equation for a point source with a specified intensity and spectrum, originating in the early Universe between the epochs of annihilation and recombination, at redshifts $z_s = 10^8 \div 10^4$. The direct radiation of the source is separated from the diffuse radiation it produces. Optical distances from the source for Thomson scattering and bremsstrahlung absorption at the maximum of the thermal background radiation are calculated as a function of the redshift $z$. The distances grow sharply with decreasing $z$, approaching asymptotic values, the absorption distance increasing more slowly and reaching their limiting values at lower $z$. For the adopted $z$ values, the optical parameters of the Universe can be described in a flat model with dusty material and radiation, and radiative transfer can be treated in a grey approximation.

1 Formulation of the problem

Let the early Universe be described by the standard model, i.e. by the Friedmann equations (see, for example, [1]). Let us suppose that a source of radiation is switched on at some time between the epochs of annihilation and recombination, that this source has a specified geometry, spectrum, and total luminosity, and that it radiates during some time interval. The intensity and spectrum of the source may be time-dependent. Such sources could occur in a number of processes, for example, during the amplification or damping of fluctuations in the matter distribution, in the formation of primary black holes, etc.

The action of the source causes the thermodynamical equilibrium in its vicinity to be violated, since the source radiation is either nonequilibrium (for example, it has a power-law spectrum) or corresponds to a higher temperature. The radiation can experience bremsstrahlung absorption and Thomson scattering. In [2], it was suggested that such sources could affect the background radiation and hence be observed via perturbations of this background, contrary to the prevalent opinion that the energy of such sources is totally dissipated. Both possibilities require careful verification using the methods of radiative-transfer theory.

We assume that the source is pointlike. The
point source problem is also of independent interest: it is fundamental, since all other types of sources can be reduced to a set of point sources.

If the source radiation is intense, it affects the surrounding matter in several ways. First, the radiation creates an additional pressure, which affects the local expansion of space. Second, this same pressure gives rise to inhomogeneity of the expansion. Third, a shock and, in the case of a periodic source, matter oscillations may be formed. Fourth, the source can distort the metric of the surrounding space. In all these cases, the geometry of the source is the key factor.

Here, we will assume that the source radiation is moderately intense and does not affect the expansion of the surrounding space. To start with, we will present the model for the Universe at the considered epoch.

2 Two-component model of the Universe: dustlike matter and radiation

2.1 Basic Relations

We consider a model with a homogeneous and isotropic expanding Universe that contains equilibrium plasma and Planck radiation in the stage when the plasma is radiation-dominated; more exactly, between the epochs of annihilation ($z < 10^8$) and recombination ($z > 10^3$). In this stage of the evolution of the Universe, the cosmological term does not play a marked role.

At the temperatures corresponding to these redshifts, the matter can be assumed to be nonrelativistic, and the conditions of thermodynamical balance with a common temperature for the matter and radiation are fulfilled [1]. The pressure of the matter may be neglected, i.e., the matter can be considered to be dustlike, in spite of its nearly total ionization.

Under these conditions, in the usual notation, the evolution of the expansion is described by the equation

$$\dot{R}^2 = \frac{8\pi G}{3} \rho_u R^2 - kc^2, \quad (1)$$

where a dot denotes a derivative with respect to the time $t$, the total mass density of the matter is equal to the sum of the matter and radiation densities, $\rho_u = \rho_d + \rho_r$, and the parameter $k$ assumes the values

$$k = \begin{cases} 
1, & \text{for a closed model,} \\
0, & \text{for a flat model,} \\
-1, & \text{for an open model.}
\end{cases}$$

The matter density decreases in inverse proportion to the volume, while the expression for the radiation density contains another power of the radius of curvature in the denominator, resulting from the cosmological redshift:

$$\rho_d = \rho_0 d a^3, \quad \rho_r = \rho_0 r R_0^4 a^4.$$

Here and below, zero superscripts will denote cosmological values at the current epoch.

We will use the dimensionless time coordinate $\eta$ and dimensionless radius of curvature (scale factor) $a(\eta)$:

$$R(t)d\eta = c dt, \quad a(\eta) = \frac{R(t)}{R_0}.$$ \quad (2)

With these variables, Eq. (1) can be transformed into

$$\left( \frac{da}{d\eta} \right)^2 = \frac{8\pi G}{3c^2} (\rho_0^d a + \rho_0^r) R_0^2 - ka^2. \quad (3)$$

This model was first considered by Chernin [3].

Let us also introduce the Hubble function and the critical density expressed in terms of this function:

$$H = \frac{\dot{R}}{R} = \frac{\dot{a}}{a} = \frac{c}{a R} \frac{da}{d\eta} = \frac{c}{R_0} a'(\eta),$$

$$\rho_c = \frac{3H^2}{8\pi G}.$$ 

Along with the density itself, we will use the critical parameters

$$\Omega_d = \frac{\rho_d}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}, \quad \Omega_u = \Omega_d + \Omega_r, \quad (4)$$

which can be used to rewrite (1) in another form:

$$R^2 H^2 (1 - \Omega_u) = -kc^2. \quad (5)$$
2.2 Auxiliary Functions

Let us define several mutually related functions, in terms of which the solutions and relations of the model will be written. The two main functions are \( \text{sn}_k(\eta) \) and \( \text{sc}_k(\eta) \), which are defined by the equalities

\[
\text{sn}_k(\eta) = \begin{cases} 
\sin \eta & \text{when } k = 1, \\
\eta & \text{when } k = 0, \\
\sinh \eta & \text{when } k = -1,
\end{cases}
\]

\[
\text{sc}_k(\eta) = \begin{cases} 
1 - \cos \eta & \text{when } k = 1, \\
\eta^2 / 2 & \text{when } k = 0, \\
\cosh \eta - 1 & \text{when } k = -1,
\end{cases}
\]

Two functions are related to function (6). One of these,

\[
\text{cs}_k(\eta) = \text{sn}'_k(\eta) = \begin{cases} 
\cos \eta & \text{when } k = 1, \\
1 & \text{when } k = 0, \\
\cosh \eta & \text{when } k = -1,
\end{cases}
\]

is its derivative, while the other,

\[
\text{ar}_k(y) = \begin{cases} 
\arcsin y, & k = 1, \\
y, & k = 0, \\
\operatorname{arsh} y = \ln(y + \sqrt{1 + y^2}), & k = -1,
\end{cases}
\]

\[
\frac{d\text{ar}_k(y)}{dy} = \frac{1}{\sqrt{1 - ky^2}}
\]

is the inverse of \( \text{sn}_k(\chi) \), i.e., \( \text{ar}_k(\text{sn}_k(\eta)) = \eta \). \( \text{sn}_k(\text{ar}_k(y))) = y \).

Another function,

\[
\text{cn}_k(\eta) = \begin{cases} 
\eta - \sin \eta & \text{when } k = 1, \\
\eta^3 / 6 & \text{when } k = 0, \\
\sinh \eta - \eta & \text{when } k = -1,
\end{cases}
\]

is the integral of \( \text{sc}_k(\eta) \), i.e., \( \text{cn}'_k(\eta) = \text{sc}_k(\eta) \).

It can easily be verified that the following relations between the introduced functions are satisfied:

\[
\begin{align*}
\text{cs}_k^2 + k \text{sn}_k^2 &= 1, \\
\text{sn}_k^2 + k \text{sc}_k^2 &= 2 \text{sc}_k, \\
\text{cs}_k + k \text{sc}_k &= 1.
\end{align*}
\]

Let us consider the ratio

\[
y(\eta) = \frac{\text{sn}_k(\eta)}{\text{sc}_k(\eta)} = \begin{cases} 
\cotg \frac{\eta}{2} & \text{when } k = 1, \\
\frac{\eta}{2} & \text{when } k = 0, \\
\coth \frac{\eta}{2} & \text{when } k = -1.
\end{cases}
\]

Its derivative with respect to \( \eta \) assumes a simple form if we use the relation between the derivatives and the functions and equalities (8):

\[
y'(\eta) = \frac{\text{sc}_k - k \text{sc}_k^2 - 2 \text{sc}_k + k \text{sc}_k^2}{\text{sc}_k^2} = -\frac{1}{\text{sc}_k}. \tag{10}
\]

Using the second of the equalities, this ratio can, in turn, be expressed in terms of \( y \):

\[
\frac{dy}{d\eta} = -\frac{y^2 + k}{2}. \tag{11}
\]

To obtain the correct solution, we must apply the appropriate initial condition: \( y = \infty \) when \( \eta = 0 \). The arguments of the functions in relations (8), (10), and (11) have been omitted for brevity.

2.3 Solution of the Equation of Motion

It was shown in [3] that the solution of (3) in implicit form is

\[
a(\eta) = a_r \text{sn}_k(\eta) + a_d \text{sc}_k(\eta),
\]

\[
t = \frac{R_0}{c} [a_r \text{sc}_k(\eta) + a_d \text{cn}_k(\eta)]. \tag{12}
\]

Indeed, we can easily verify using relations (8) that the function \( a(\eta) \) satisfies (3) if we assume

\[
a_r = \sqrt{\frac{8\pi G}{3} \rho_0}, \quad a_d = \frac{4\pi G}{3c^2} \rho_0. \tag{13}
\]

The obtained coefficients are expressed in terms of the critical parameters. Let us first derive an expression for the current radius of curvature [1]. To this end, we apply formula (5) to the current epoch:

\[
H_0^2 R_0^2 = -\frac{k e^2}{1 - \Omega_0}, \quad \frac{H_0 R_0}{c} = \frac{1}{\sqrt{|1 - \Omega_0|}}.
\]
We now have, in accordance with the definitions (4),
\[ a_d = \frac{4\pi G}{3c^2} \rho_c^0 \Omega_d^0 R_0^2 = \frac{4\pi G}{3c^2} \frac{3H_0}{8\pi G} \frac{\Omega_d^0 R_0^2}{2[1 - \Omega_d^0]} , \]
\[ a_k = \frac{8\pi G}{3c^2} \rho_c^0 \Omega_k^0 R_0^2 = \frac{8\pi G}{3c^2} \frac{3H_0}{8\pi G} \frac{\Omega_k^0 R_0^2}{2[1 - \Omega_u^0]} , \]
The inverse formulas have the form:
\[ \Omega_d^0 = \frac{2a_d}{2a_d + a_k^2 - k} , \quad \Omega_k^0 = \frac{a_k^2}{2a_d + a_k^2 - k} , \]
\[ \Omega_u^0 = \frac{2a_d + a_k^2}{2a_d + a_k^2 - k} . \]
In the following subsections (to Subsection 9), we will assume that \( k \neq 0 \), although some functions are calculated using formulas that include the special case of a flat space.

### 2.4 Relation to the Redshift

Let us derive expressions for the quantities associated with the solution (12) in terms of the redshift \( z \). By definition \( a(\eta) = \frac{R(\eta)}{R_0} = \frac{1}{1 + \eta} \). Successively substituting in (12) the expression for \( s_nk \) or \( sc_k \) in terms of the other, and also using the last relation in (8), we obtain
\[ s_nk(\eta) = \frac{Q - B(1 - kA)}{1 + kB^2} , \]
\[ sc_k(\eta) = \frac{B^2 + A - BQ}{1 + kB^2} , \]
\[ cs_k(\eta) = \frac{1 - kA + kBQ}{1 + kB^2} . \]
Here, we have introduced the notation
\[ A = \frac{1}{a_d} \frac{1}{1 + z} = \frac{2[1 - \Omega_u^0]}{\Omega_d^0} \frac{1}{1 + z} , \]
\[ B = \frac{a_k}{a_d} = \frac{2\sqrt{\Omega_u^0[1 - \Omega_u^0]}}{\Omega_d^0} , \]
\[ Q = \sqrt{B^2 + 2A - kA^2} . \]

The derivative \( a'(\eta) \) and the Hubble function can be expressed in terms of the redshift in this same way:
\[ a'(\eta) = a_r cs_k(\eta) + a_d sc_k(\eta) , \]
\[ H = \frac{c}{H_0} a'(\eta) = \frac{a'(\eta)}{a^2(\eta)} \sqrt{1 - \Omega_u^0} . \]
The ratio (9) and the product of this ratio with \( B \) can also be expressed in terms of \( z \):
\[ y(\eta) = \frac{kAB - B + Q}{B^2 + A - BQ} , \]
\[ X = By = \frac{kAB - B + Q}{B + A/B - Q} . \]
The critical parameters at the current epoch (\( B \) is constant) can likewise be expressed in terms of the quantities in (14):
\[ \Omega_d^0 = \frac{2A_0}{Q_0^2} \Omega_r^0 = \frac{B^2}{Q_0^2} ; \]
\[ \Omega_u^0 = \frac{2A_0 + B^2}{Q_0^2} , \]
\[ 1 - \Omega_u^0 = -kA_0^2 . \]
It is easy to verify that the identities \( a(\eta_0) = 1 \) and \( a'(\eta_0) = H_0 R_0 / c = 1 / \sqrt{1 - \Omega_u^0} \) are satisfied.

### 2.5 Specific Models

According to the above model, the curvature of space can be arbitrary (constant). However, recent observations lead to the conclusion that space was flat over the entire history of the Universe. In this context, we will consider two specific models. One is open, close to flat, and does not take into account the vacuum component (we will call this the nonvacuum model), while the other has three components and is strictly flat, but with the vacuum neglected at \( z > 10^3 \).

We will adopt for the Hubble constant \( H_0 = 65 \text{ km s}^{-1} \text{Mpc}^{-1} \), the corresponding critical density \( \rho_c^0 = 7.940 \times 10^{-30} \text{ g/cm}^3 \), and the critical parameters for the baryon component \( \Omega_b^0 = 0.025 \) and dust-like component \( \Omega_d^0 = 0.25 \). The baryon density is then \( \rho_b^0 = 4.684 \times 10^{-31} \text{ g/cm}^3 \), the density of dust-like matter is \( \rho_d^0 = 4.684 \times 10^{-30} \text{ g/cm}^3 \), the mass
density of the radiation corresponding to the temperature $T_0 = 2.7277$ K, is $\rho^0_\gamma = 4.63 \times 10^{-34}$ g/cm$^3$, and $\Omega_\gamma^0 = 5.83 \times 10^{-5}$.

2.5.1 Nonvacuum model.

For the adopted model parameters, $1 - \Omega^0_u = 1 - \Omega^0_d - \Omega^0 = 0.7499$; i.e., this model is open ($k = -1$), which is natural, since there is no vacuum component. The other parameters are $a_r = 8.869 \times 10^{-3}$ and $a_d = 0.1667$.

2.5.2 Strictly flat model.

For $k = 0$, we also have $1 - \Omega_u = 0$, so that many of the introduced values become equal to zero or infinity. In this case, the limit transition is complicated, and it is easier to consider this case individually.

We will assume that the most plausible model of the Universe for redshifts less than $10$–$100$, in particular for the current epoch, is a flat model with three noninteracting components: dustlike matter, radiation, and the vacuum. Based on this model, we will also construct a two-component flat model for the considered epoch of radiation-dominated plasma in which the two components have the same densities, but there is no vacuum component. All values corresponding to this model will be denoted by a tilda.

The contribution of the vacuum can be neglected for the redshifts considered ($10^3 \leq z \leq 10^9$), so that the Hubble functions in the two-component and three-component flat models differ only slightly:

$$H = \sqrt{\frac{8\pi G}{3} \rho^0_d(1+z)^3 + \rho^0_\gamma(1+z)^4 + \rho^0},$$

$$\dot{H} = \sqrt{\frac{8\pi G}{3} \rho^0_d(1+z)^3 + \rho^0_\gamma(1+z)^4}.$$  

We will use the same critical parameters $\Omega^0_d$ and $\Omega^0$ as above; the critical parameter for the vacuum is then $\Omega^0_\Lambda = 1 - \Omega^0_d - \Omega^0$. In the adopted model, these values are:

$$\dot{H}_0 = H_0 \sqrt{\frac{\rho^0_d + \rho^0_\gamma}{\rho^0_d + \rho^0_\gamma + \rho^0_\Lambda}} = H_0 \sqrt{1 - \Omega^0_\Lambda}, \quad (14)$$

$$\dot{\rho}^0_c = \rho^0_c(1 - \Omega^0_\Lambda), \quad (15)$$

$$\Omega^0_d = \frac{\Omega^0_\gamma}{1 - \Omega^0_\Lambda}, \quad \Omega^0 = \frac{\Omega^0_\gamma}{1 - \Omega^0_\Lambda}. \quad (16)$$

For brevity, let us set $\tilde{\Omega}_d = \Omega$, in which case $\tilde{\Omega}_d = 1 - \Omega$, so that $\tilde{\Omega}_d^0 = \Omega_0$ and $\tilde{\Omega}_d^0 = 1 - \Omega_0$.

The radius of curvature $R$ and its current value $R_0$ are meaningless in a flat Universe (they are infinite, and can be selected arbitrarily in the formulas), and should not arise in expressions for physical values. Indeed, the solution (12) can be written in the form

$$a(\eta) = a_0 + a_d \eta^2 = \sqrt{1 - \Omega_0 \frac{\dot{H}_0 R_0}{c} \eta^2} + \Omega_0 \frac{\dot{H}_0^2 R_0^2}{4c^2} \eta^2 = 2 \sqrt{1 - \Omega_0 \zeta + \Omega_0 \zeta^2},$$

where we have introduced a new time variable that is linearly related to the previous time variable and is expressed in terms of $a(\eta)$ and the redshift:

$$\zeta = \frac{\dot{H}_0 R_0}{2c} \eta = \frac{a}{\sqrt{1 - \Omega_0 (1 - a) + \sqrt{1 - \Omega_0}}}$$

$$= \sqrt{1 + \frac{z}{1 + (1 - \Omega_0)/\Omega_0}} \left(1 + \Omega_0/\Omega_0 \right)$$

3 Radiative-transfer equation in the early Universe

3.1 General Form of the Radiative-Transfer Equation for a Point Source

Let us assume the source to be pointlike and isotropic. In this case, the problem is spherically symmetrical, so that, generally speaking, the approach of the Tolman-Bondi model should be used. However, to first approximation, we can assume that the source radiation does not affect the metric of space and its expansion, and the evolution of the source radiation can be considered against the background of the standard cosmological two-component model described in Section 2.

We will describe the radiation with the mean occupation number of photon states $n$, rather then the intensity. The advantage of this quantity is
that it is dimensionless and relativistically invariant (scalar). Due to the symmetry of our problem, the mean occupation number of photon states will depend on the time $t$ or time variable $\eta$, the distance of the point considered from the coordinate origin (described by the parameter $\chi$), the angle $\theta$ between the radial direction and the ray along which the radiation propagates, and the dimensionless frequency $x$: $n = n(\eta, \chi, \theta, x)$. The radiative-transfer equation has the form

$$\frac{dn}{dt} = \frac{\partial n}{\partial \eta} + \frac{\partial n}{\partial x} \frac{dx}{dt} + \frac{\partial n}{\partial \chi} \frac{d\chi}{dt} + \frac{\partial n}{\partial \theta} \frac{d\theta}{dt} = I_c,$$

where $I_c$ is the collision integral, i.e., the difference between the numbers of photons with specified parameters entering and leaving the state.

We will use the parameter $\eta$ instead of the time; the derivatives will be calculated with respect to $\eta$ using relation (2), so that

$$\frac{\partial}{\partial t} = c \frac{\partial}{\partial \eta}.$$

The derivative of the frequency with respect to the time is the easiest to find. Due to the redshift,

$$\nu = \nu_0 \frac{R_0}{R}, \quad x = \nu = \nu_0 \frac{R_0}{\nu_*} \frac{x_0}{R}, \quad \frac{dx}{dt} = -x_0 \frac{R_0}{R} \frac{\dot{R}}{R^2} = -xH,$$

where $H = \frac{\dot{R}}{R}$ is the Hubble function and $\nu_*$ is some specified frequency.

To determine the other derivatives, we must analyze the variations of the corresponding values along the ray.

### 3.1.1 Left-hand Side of the Transfer Equation

Let us consider the ray along which the radiation propagates, having, in general, already been scattered. An arbitrary ray can be described using several parameters, one of which can be chosen to be the coordinates of the point on the ray closest to a point source at the coordinate origin. We will call this the initial point. Due to the symmetry of the problem, it is sufficient to specify this distance using the coordinate $\chi_o$. Let a photon pass through the initial point at time $t_o = t(\eta_o)$. Its position on the ray will be described by the parameter $\chi_*$ associated with this time and measured from time $t_o$. Let us determine the derivatives with respect to $\chi_*$ of the variables describing the position of the photon on the ray.

![Fig. 1: The curvilinear triangle.](image)

To this end, let us consider the curvilinear triangle presented in Fig. 1, with its sides extending from the source $S$ to the initial point, from the source to the photon at time $\eta$, and from the initial point to the photon. Since all the lengths are proportional to the radius of curvature, we will consider dimensionless distances described by the coordinates $\chi_0$, $\chi$, and $\chi_*$, respectively; the corresponding dimensionless lengths are $a_o = sn_k(\chi_o)$, $sn_k(\chi)$, and $sn_k(\chi_*)$. In the same order, the angles opposing these sides are those between the ray and radial direction, $\theta$, between the radial direction toward the initial point and the ray (a right angle), and between the radial direction toward the initial point and the direction toward the photon’s position, $\theta_*$.

We can write the geometrical integrals of the transfer equation for the constructed right-angle curvilinear triangle:

$$a_o = sn_k(\chi_o) = sn_k(\chi) \sin \theta,$$

$$\frac{sn_k(\chi_o)}{cs_k(\chi_o)} = sn_k(\chi_*) \tan \theta,$$

$$r_o = cs_k(\chi_o) = \frac{sn_k(\chi)}{sn_k(\chi_*)} \cos \theta = \frac{\cos \theta}{\sin \theta_*}.$$
CSOURCE OF RADIATION IN THE EARLY UNIVERSE

\[ \text{cs}_k(\chi) = r_0 \text{cs}_k(\chi_0), \]
\[ \text{sn}_k(\chi) \cos \theta = \frac{\text{sn}_k(\chi_0)}{\text{cs}_k(\chi_0)}, \]
\[ \cos \theta_0 = \text{cs}_k(\chi_0) \sin \theta. \]

The functions \( \text{sn}_k \) and \( \text{cs}_k \) are specified by equalities (6) and (7). Relation (20) follows from the two preceding expressions (18) and (19). These relations are trivial in the case of a flat space. For a space with positive curvature, the triangle can be placed on a unit sphere in three-dimensional space, with the subsequent use of spherical geometry. To make the translation to the case of negative curvature, the trigonometric functions expressing the sides of the triangle must be changed to hyperbolic functions.

The following derivatives can easily be obtained from these relations:

\[ \frac{\mathrm{d}\chi}{d\chi_0} = \cos \theta, \tag{21} \]
\[ \frac{\mathrm{d}\theta}{d\chi_0} = -\sin \theta \frac{\text{cs}_k(\chi)}{\text{sn}_k(\chi)}, \tag{22} \]
\[ \frac{\mathrm{d}\theta_0}{d\chi_0} = \frac{\sin \theta}{\text{sn}_k(\chi)}. \tag{23} \]

The equation of motion of a photon along a ray is \( \chi = \eta - \eta_0 \), so that the derivatives (21), (22), and (23) are simultaneously the derivatives with respect to the time coordinate \( \eta \).

Thus, the transfer equation (17) takes the form

\[ \frac{c}{R(\eta)} \left[ \frac{\partial n}{\partial \eta} + \cos \theta \frac{\partial n}{\partial \chi} - \sin \theta \frac{\text{cs}_k(\chi)}{\text{sn}_k(\chi)} \frac{\partial n}{\partial \theta} \right] \]
\[ - xH \frac{\partial n}{\partial x} = c \mathcal{I}_c. \tag{24} \]

For a flat space, when \( k = 0 \), \( \text{sn}_0(\chi) = \chi \), and \( \text{cs}_0(\chi) = 1 \), the equation takes the usual form.

### 3.2 Right-hand Side of the Transfer Equation

In the problem we are considering, bremsstrahlung absorption and emission, as well as Thomson scattering, are important processes in the interaction between the matter and point-source radiation.

Bremsstrahlung absorption by thermal nonrelativistic electrons with a Maxwellian velocity distribution and the temperature \( T \) is described by the formula [4]

\[ \alpha_{cc}^* = n_e n_i^+ k_{cc}(\nu, T), \]
\[ \alpha_{cc} = (1 - e^{-h\nu/k_B T}) \alpha_{cc}^*, \]

where the cross section is

\[ k_{cc}(\nu, T) = \frac{k^0_{cc}}{T^{1/2} \nu^3}, \]

\[ k^0_{cc} = 8\pi \frac{e^6}{c^3 h} \frac{k_B}{(6\pi m k_B)^{3/2}} = 3.69 \times 10^8 \text{cm}^5 \text{K}^{1/2} \text{s}^{-3}. \]

The absorption coefficient has been corrected for stimulated emission. The Gaunt factor is taken to be unity. The bremsstrahlung coefficient in the occupation numbers is related to the absorption coefficient by the Kirchhoff-Planck relation:

\[ \epsilon_{cc} = \alpha_{cc} e^{-h\nu/k_B T}. \tag{25} \]

The absorption coefficient associated with Thomson scattering is \( \alpha_T = n_e \sigma_T \), where the cross section is

\[ \sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 = 6.66 \times 10^{-25} \text{ cm}^2. \]

The indicative for Thomson scattering is the Rayleigh index. Since the radiation field does not depend on the azimuth, the equation will contain the azimuthaveraged indicatrix

\[ p(\theta, \theta') = 1 + \frac{1}{2} P_2(\cos \theta) P_2(\cos \theta'), \]

where \( P_2(\mu) \) is the second Legendre polynomial.

Thus, the right-hand side of the transfer equation has the form

\[ \mathcal{I}_c = -[\alpha_{cc}(\nu, T) + \alpha_T n(\eta, \chi, \theta, x)] \]
\[ + \epsilon_{cc}(\nu, T) + \alpha_T \mathcal{T} + \epsilon_*(\nu, \chi, \theta, x), \tag{26} \]

\[ \mathcal{T} = \frac{1}{2} \int_0^\pi \sin \theta' p(\theta, \theta') d\theta' n(\eta, \chi, \theta', x) \]

where \( \epsilon_* \) describes the primary radiation from the point source.
The integral in (26) can be divided into two terms containing two angular moments \( n(\eta, \chi, \theta, x) \):

\[
n_0(\eta, \chi, x) = \frac{1}{2} \int_0^\pi \sin \theta' d\theta' n(\eta, \chi, \theta', x),
\]

\[
n_2(\eta, \chi, x) = \frac{5}{2} \int_0^\pi \sin \theta' P_2(\cos \theta') d\theta' n(\eta, \chi, \theta', x).
\]

Let us now specify the source function \( \epsilon_* \) and separate the direct from the diffuse radiation.

### 3.3 Direct Radiation

The entire space is filled by homogeneous and isotropic equilibrium cosmological radiation with the mean occupation number

\[
n_\epsilon(\eta, x) = \frac{1}{e^{h\nu_\epsilon x/(k_B T)} - 1} = \frac{1}{e^{h\nu_\epsilon x_0/(k_B T_0)} - 1}.
\]  

(27)

It is straightforward to verify that this function satisfies (24) with the right-hand side (26) without the last term. In fact, both the left-hand and right-hand sides vanish, the left-hand side because the mean occupation number for the background radiation does not depend on anything, and the right-hand side due to the Kirchhoff-Planck relation (25). This is the equilibrium background radiation. In spite of the fact that it interacts with matter, overall, the matter and radiation do not affect one another during this interaction; only a common temperature is established. After the epoch of recombination, the electron density becomes extremely small (it can be assumed to be zero), and the right-hand side of the transfer equation no longer contains terms describing the interaction between radiation and matter (the radiation is separated from the matter), while the left-hand side corresponds to the free propagation of the radiation. The quantity (27) remains constant and satisfies the transfer equation.

Let us now consider the direct radiation of the source. We will first find an expression for \( \epsilon_* \). The relation between the total energy \( L \) emitted by the source (for all time, in all space, in all directions and frequencies) and \( \epsilon_* \) is

\[
L = \frac{2h\nu^4}{c^2} \int_0^\infty dt \int d^3 r \int_0^\infty x^3 dx \epsilon_*(\eta, \chi, \theta, x).
\]

The factor before the integrals is introduced to translate the occupation number into intensity. Expressing time in terms of \( \eta \) and writing the integral over space in terms of integrals over the three variables, taking into account the volume element \( d^3 r = R^3(\eta) \sin \theta d\theta d\varphi \), we obtain

\[
L = 2\pi \frac{2h\nu^4}{c^3} \int_0^\infty R^4(\eta)d\eta \int_0^\infty \sin^2(\chi)d\chi \times \int_0^\pi \sin \theta d\theta \int_0^\infty x^3 dx \epsilon_*(\eta, \chi, \theta, x).
\]

(28)

It follows from the expression for the right-hand side of the transfer equation (26) that the units of \( \epsilon_* \) are inverse length.

If the power of the source radiation at the time \( t_s \) with the time coordinate \( \eta_s \) at frequency \( x_s \) is \( L\delta(t - t_s) \), we can write

\[
\epsilon_*(\eta, \chi, \theta, x) = L \frac{c^3}{2h\nu^4} \frac{\delta(\eta - \eta_s)}{R^4(\eta)} \frac{\delta(\chi)}{\sin^2(\chi)} \times \frac{\delta(\cos \theta - 1)}{\delta(x - x_s)} \frac{\delta(x - x_s)}{2\pi x_s^3},
\]

and relation (28) is satisfied.

Further, let us determine the mean occupation number for the direct radiation from the source. It is specified by the transfer equation (24) with the right-hand side (26) without any terms describing scattering:

\[
\frac{\partial}{\partial \eta} + \frac{\cos \theta}{\partial \chi} - \frac{\sin \theta}{\sin \chi} \frac{c_{sl}(\chi)}{n_l(\chi)} \frac{\partial}{\partial \theta} - \frac{a(\eta)}{a(\eta)} \frac{\partial}{\partial x} \left[ n_*(\eta, \chi, \theta, x) \right.
\]

\[
- \left[ \frac{1}{n^+} k_{cc}(\nu, T) + \sigma_T \right] n_*(\eta, \chi, \theta, x) + \epsilon_*(\eta, \chi, \theta, x) R(\eta).
\]
Direct substitution verifies that the solution of this equation is given by the function
\[
\begin{align*}
n_s(\eta, \chi, \theta, x) &= L \frac{c^3}{2\hbar \nu_s^2} \frac{\delta(\chi - \eta + \eta_0)}{R^3(\eta_0)} \Theta(\chi) \\
&\quad \times \frac{\delta(xa(\eta)/a(\eta_0) - x_s)}{x_s^3} \frac{\delta(\cos \theta - 1)}{2\pi} e^{-\tau},
\end{align*}
\]
where the optical distance from the source to the point is (see below)
\[
\tau = \tau(\eta, \eta - \eta_0, x, a(\eta)/a(\eta_0)) = \tau_T(\eta) - \tau_T(\eta_0) + \left(1 - e^{-\hbar \nu_s x/a(\eta)}\right) R_0 K_0^0 [E(\eta) - E(\eta_0)] \nu_s^2 x_s^3 T_s^{1/2} A_s^{2/2}(\eta_s).
\]

### 3.4 Discriminating between Direct and Diffuse Radiation

Let us represent the total occupation number for the photon states created by the source as the sum of two terms:
\[
n(\eta, \chi, \theta, x) = n_s(\eta, \chi, \theta, x) + n_d(\eta, \chi, \theta, x),
\]
where \(n_d\) describes the diffuse part of the radiation. This function satisfies the same transfer equation (24) as the initial function, but with the "source" term \(\epsilon_s\) replaced by \(\epsilon_s(\eta, \chi, \theta, x)\) — the part of the emission coefficient due to scattering of the direct radiation from the point source. This quantity is specified by the formula
\[
\epsilon_s(\eta, \chi, \theta, x) = \alpha_T \left[1 + \frac{1}{2} P_2(\cos \theta)\right] n_s^0(\eta, \chi, x),
\]
where the zeroth moment of the direct intensity is
\[
n_s^0(\eta, \chi, x) = L \frac{c^3}{2\hbar \nu_s^2} \frac{\delta(\chi - \eta + \eta_0)}{R^3(\eta_0)} \Theta(\chi) \\
&\quad \times \frac{\delta(xa(\eta)/a(\eta_0) - x_s)}{x_s^3} \frac{\delta(\cos \theta - 1)}{2\pi} e^{-\tau}.
\]
The second moment of the direct radiation exceeds the zeroth moment by a factor of five, since \(P_2(1) = 1\).

For a noninstantaneous and nonmonochromatic source, the emitted energy may depend on \(\eta_s\) and \(x_s\), and all the equations and their solutions must be integrated over these parameters. This does not present a problem, due to the presence of \(\delta\) functions in these expressions.

### 3.5 A Planar Source

For comparison, a layered planar source can be considered. This is only possible if the geometry of the space itself is flat, \(k = 0\). As always in a planar geometry, the equation of radiative transfer does not contain an angular derivative, since the angle \(\theta\) between the direction of propagation of the radiation and the normal to the source plane along the ray remains constant. Thus, the transfer equations for a planar geometry can be obtained from the corresponding equations for spherical geometry, simply by omitting the term with the derivative with respect to the angle \(\theta\). In the expression for the source term and the intensity of the direct radiation, the combination \(\delta(\chi)/\text{sn}^2(\eta)\) must be replaced by \(\delta(\chi - \chi_s)\).

Thus, the transfer equation acquires the form
\[
\frac{c}{R(\eta)} \left[ \frac{\partial n}{\partial \eta} + \cos \theta \frac{\partial n}{\partial \chi} \right] - xH \frac{\partial n}{\partial x} = cI_c,
\]
where the collision integral \(I_c\) is specified by the same formula (26). In this case, the source term is
\[
\epsilon_s(\eta, \chi, \mu, x) = L \frac{c^3}{2\hbar \nu_s^2} \frac{\delta(\chi - \eta + \eta_0)}{R^3(\eta_0)} \frac{\delta(\chi - \chi_s)}{2\pi} \frac{\delta(\cos \theta - \cos \theta_s)}{x_s^3} d(x - x_s).
\]
The source is assumed to be located at the level with \(\chi_s\). Here, we calculate its luminosity per area of the planar boundary; for this reason, the denominator contains a second power of the radius of curvature. We assume that the source radiation does not depend on azimuth and propagates at an angle \(\theta_s\) relative to a normal to the layers.

### 4 Optical Distances

#### 4.1 Optical Distance from the Source

Under the adopted conditions, the densities of electrons \(n_e\) and protons \(n^+\) depend only on time (and, of course, the cosmological model). In the case of total ionization, \(n^+ = n_e = n_e^0/\alpha(\eta)\); in the case of partial ionization, the ionization equation must be
solved. We will assume that the matter consists of completely ionized hydrogen.

Expressions for the coefficient of bremsstrahlung absorption by thermal nonrelativistic electrons and the absorption coefficient for Thomson scattering were given in Section 2.

Since the scattering coefficient does not depend on frequency, the corresponding optical distance can be determined very easily. If a photon was emitted from the source at time $\eta_s$, its equation of motion is $\chi = \eta - \eta_s$. In the course of its motion, the optical distance between the photon and the source will increase according to the relation

$$\tau_T(\eta, \eta_s) = \int_0^\chi \alpha_T(\eta_s + \chi') R(\eta_s + \chi') d\chi'$$

$$= \int_{\eta_s}^\eta \alpha_T(\eta') R(\eta') d\eta' = \tau_T(\eta) - \tau_T(\eta_s),$$

where the universal Thomson opacity is

$$\tau_T(\eta) = \int_{\eta_s}^\eta \alpha_T(\eta) R(\eta) d\eta$$

$$= \sigma_T \int_{\eta_s}^\eta n_e(\eta) R(\eta) d\eta. \quad (29)$$

Since only differences in the optical distances are of interest, the lower integration limit in this formula can be chosen arbitrarily.

The situation with the bremsstrahlung mechanism is more complicated, since in this case, the absorption cross section depends on both the frequency and temperature. The time dependence of the frequency is described as follows. Let the frequency of a photon emitted from the source at time $t_s = t(\eta_s)$ be $\nu_s$. Then, its frequency at time $t = t(\eta)$, will be $\nu = \nu_s a(\eta_s)/a(\eta)$. The temperature also depends on the time, according to the usual formula, $T = T_0/a(\eta) = T_s a(\eta_s)/a(\eta)$, where $T_s = T_0/a(\eta_s)$ is the temperature when the photon is emitted. The ratio $\nu/T = \nu_s/T_s$ is time independent.

Since the particle densities, frequency, and temperature depend on time in a known way, the absorption coefficient is a known function of $\eta$ or $z$:

$$\tau_{cc}(\eta, \eta_s, x) = \left(1 - e^{-\hbar \nu_s/a(\eta_s) T_s}\right) \int_0^\chi n_e(\eta_s + \chi')$$

$$\times n^+(\eta_s + \chi') \frac{k_{cc}}{\nu^2_a \chi_a \alpha^{7/2}(\eta_s) T_s^{1/2}} (\eta_s + \chi')$$

$$\times R(\eta_s + \chi') d\chi' = \left(1 - e^{-\hbar \nu_s/a(\eta_s) T_s}\right) R_0$$

$$\times k_{cc}(\nu_s a(\eta_s), T_s a(\eta_s)) [E(\eta) - E(\eta_s)],$$

where

$$E(\eta) = \int_{\eta_s}^\eta n_e(\eta)n^+(\eta) a^{9/2}(\eta) d\eta \quad (30)$$

is a quantity similar to the emission measure. The total optical distance from the source is

$$\tau(\eta, \eta_s, x) = \tau_T(\eta, \eta_s) + \tau_{cc}(\eta, \eta_s, x).$$

### 4.2 Calculation of the Scattering Optical Distance

Let us choose for the initial times in (29) and (30) the times $t_i = t(\eta_i)$ for which the ratio $y = s_{\eta_i}(\eta)/s_{\eta_i}(\eta)$

assumes the values from the table. The table also presents the products $X = \frac{a_T}{a_d} y = By$ for the same times. Here, we have chosen simple values for this ratio; in fact, the values of $\eta_i$ are maximum for $k = 0$

and $k = -1$, $\eta_i < \eta_i$, so that the optical distance (29) will be negative. This does not cause any problems, since only the differences of these quantities, which are positive, will be needed. When $k = 1$, the value of $\eta_i$ is finite, since the range of variations of this variable is finite in a closed model.

We will rewrite the integral (29), substituting the time dependence of the electron density, $n_e = n_e^0/a^3(\eta) = (\rho^0_a/m_H)/a^3(\eta)$, where $\rho^0_a$ is the current baryon density:

$$\tau_T(\eta) = \sigma_T \int_{\eta_i}^\eta n_e R(\eta) d\eta = \sigma_T \rho^0_a/m_H B_0 \int_{\eta_i}^\eta \frac{d\eta}{a^2(\eta)}.$$

Introducing the new variable of integration $X = \frac{a_T}{a_d} y = By$, where $y$ is specified by (9), and using
the characteristics of this ratio enables us to reduce the integral to the form

\[
\int_{\eta}^{\ast} \frac{dy}{a^2(\eta)} = \frac{1}{2} \int_{y_0}^{y_\ast} \frac{y^2 + 2}{(a_2 y + a_1)^2} dy
\]

\[
= -\frac{1}{2} \frac{a_4}{a_1^2} f_T(X, B, k),
\]

\[
f_T(X, B, k) = \frac{X^2 + k B^2}{(X + 1)^2} dX_1.
\]

Calculating the integral within the limits indicated in the table yields for \(k = 0\) and 1:

\[
f_T(X, B, k) = X \frac{X^2 + k B^2}{X + 1} - 2 \ln(X + 1).
\]

When \(k = -1\), the constant \(B(B - 2) + 2 \ln(1 + B)\) is added, so that, in an open model, this function can also be written in a form that explicitly vanishes for the limiting value of the argument, \(X = B\):

\[
f_T(X, B, -1) = (X - B) \frac{X - B + 2}{X + 1} - 2 \ln \frac{X + 1}{B + 1}.
\]

For arguments close to the limiting value, \(X \to 0\), the function (31) is equivalent to \(f_T(X, B, k) \sim k B^2 X(1 - X) + (1/3 + k B^2) X^3\), while for an open model, we obtain as \(X \to B\) then \(f_T(X, B, -1) \sim (X - B)^2 \frac{B + 1/3 - B}{B + 1} (X - B)\).

Thus, the optical distance based on Thomson scattering is given by the formula

\[
\tau_T(\eta, \eta_0) = \frac{\sigma_T}{4} \frac{\rho_0^0}{m_H} \left( \frac{8 \pi G}{3 c^2 \rho_0^0} \right)^{1/2} \times [f_T(X_0, B, k) - f_T(X, B, k)],
\]

where \(B = \frac{a_r}{a_d}, X = B y = B \frac{\sin_k(\eta)}{\sin_k(\eta)},\) and \(X < X_s\) for \(\eta > \eta_s\). The factor before the bracket can also be written as \(\frac{\sigma_T \rho_0^0}{4} \frac{\Omega_0^0}{m_H} \frac{c}{H_0}\).

### 4.3 Calculation of the Absorption Optical Distance

Let us calculate the function (30). After making the same substitutions as for the Thomson-scattering distance calculation, we obtain

\[
E(\eta) = \left( \frac{\rho_0^0}{m_H} \right)^2 \int_{\eta}^{\ast} \frac{dy}{a^{3/2}(\eta)}
\]

\[
= - \left( \frac{\rho_0^0}{m_H} \right)^2 \frac{1}{\sqrt{2}} \int_{y_0}^{y_\ast} \frac{\sqrt{y^2 + k}}{(a_1 y + a_3)^{3/2}} dy
\]

\[
= - \left( \frac{\rho_0^0}{m_H} \right)^2 \frac{1}{\sqrt{2}} \frac{a_1^{1/2}}{a_1^2} f_c(X, B, k),
\]

where

\[
f_c(X, B, k) = \int_{X_0}^{X} \frac{\sqrt{X^2 + k B^2}}{(X + 1)^{3/2}} dX_1.
\]

Only for \(B = 0\) (i.e. for \(k = 0\)) can the integral be expressed in terms of elementary functions:

\[
f_c(X) = f_c(X, 0, 0) = 2 \frac{\sqrt{X + 1}}{\sqrt{X + 1}} + \frac{2}{\sqrt{X + 1}} - 4 = 2 \frac{\sqrt{X + 1} - 1}{\sqrt{X + 1}}^2,
\]

\[
= 2 \frac{X^2}{\sqrt{X + 1}} (\sqrt{X + 1} + 1)^2.
\]

If \(B \neq 0\), the integral can be expressed in terms of elliptical integrals, but it is easier to calculate it numerically.

### 4.4 Calculations of Optical Distances for the Nonvacuum Model

For the values in (32), we will obtain \(B = 5.321 \times 10^{-2}\).
Let us consider optical distances in the selected model. We will first calculate the dimensionless functions $f_T$ and $f_c$ for redshifts $z$ from $10^8$ to $10^2$. Assuming that the photon was emitted from the source at time $t_0$, the Thomson-scattering distance at time $\eta$ is specified by (32). The factor before the bracket in this formula is equal to 156.2.

The optical distance based on absorption can be determined in a similar way. After substituting expressions for its terms, the product $R_0 E(\eta)$ is shown to be

$$R_0 E(\eta) = -\frac{1}{2} \left( \frac{\rho_0}{m_H} \right)^2 \frac{\sqrt{\rho_0^2}}{\rho_0^2} \left( \frac{8\pi G}{3c^2} \right)^{1/2}$$

$$\times f_c(X, B, -1) = -\frac{1}{2} \left( \frac{\rho_0}{m_H} \right)^2 \frac{\sqrt{\Omega_0^2}}{\Omega_0^2} \frac{c}{H_0}$$

$$\times f_c(X, B, -1).$$

Let us estimate the optical distance for the frequency at which the Planck function reaches its maximum at a certain cosmological time, $\nu_m = c_W k_B T/h$, $c_W = 2.821438$. For this distance,

$$\tau_{cc} = (1 - e^{-c_W}) \left( \frac{\rho_0}{m_H} \right)^2 \frac{k_{cc}^2}{T_0^{7/2}} \left( \frac{h}{c_W k_B} \right)^3 \frac{\sqrt{\Omega_0^2}}{\Omega_0^2}$$

$$\times \frac{c}{2H_0} [f_c(X_s, B, -1) - f_c(X, B, -1)]. \quad (34)$$

### 4.5 Calculation of Optical Distance for a Flat Model

The optical distances needed for our model have already been determined. Let us rewrite them using new notation. The scattering optical distance is

$$\tau_T = \frac{\sigma_T \rho_0}{m_H} R_0 \frac{2c}{H_0 R_0} \frac{\Omega_0}{8(1 - \Omega_0)^{3/2}}$$

$$\times \left[ f_T \left( \frac{\sqrt{1 - \Omega_0}}{\Omega_0} + \frac{2}{\zeta_s} \right) - f_T \left( \frac{\sqrt{1 - \Omega_0}}{\Omega_0} \right) \right]$$

$$= \frac{\sigma_T \rho_0^2}{4 m_H H_0} \frac{c}{(1 - \Omega_0)^{3/2}}$$

$$\times [f_T(X_s) - f_T(X)]. \quad (35)$$

We have obtained the same formula, (32); the function $f_T$ coincides with that determined from (31), but, since $B = 0$ and $k = 0$, it does not contain any parameters:

$$f_T(X) = X \frac{X + 2}{X + 1} - 2 \ln(X + 1).$$

Taking into account the relations between the Hubble constant and critical parameters (14), (15), and (16), we will see that the coefficients before the brackets in (35) and (32) also exactly coincide.

The absorption distance is calculated in exactly the same way. This product is equal to

$$R_0 E(\eta) = -\frac{1}{2} \left( \frac{\rho_0}{m_H} \right)^2 \frac{\sqrt{\Omega_0^2}}{1 - \Omega_0} \frac{c}{H_0} f_c(X),$$

where the function $f_c(X)$ is specified by the previous formula (33). The resulting formula for the absorption distance is

$$\tau_{cc} = (1 - e^{-c_W}) \left( \frac{\rho_0}{m_H} \right)^2 \frac{k_{cc}^2}{T_0^{7/2}} \left( \frac{h}{c_W k_B} \right)^3$$

$$\times \frac{\sqrt{\Omega_0^2}}{1 - \Omega_0} \frac{c}{2H_0} [f_c(X_s) - f_c(X)]. \quad (36)$$

Here, the coefficients in (34) and (36) also coincide.

### 4.6 Comparing the Two Models

The optical distances in the two models considered differ only in the functions $f$ in the flat model, $B = 0$, while this parameter is $B \approx 0.05$ in the open model, so that the difference is insignificant. Indeed, this difference

$$f_T(X) - f_T(X, B, -1) = B^2 \frac{X}{X + 1}$$

$$- B(B - 2) - 2 \ln(B + 1)$$

is small for both small and large $X$. The term that is independent of $X$ disappears when the difference of the functions is taken. The difference between the functions $f_c$ is more difficult to estimate, since these functions have different domains:

$$f_c(X) - f_c(X, B, -1) = \int_0^B \frac{X_1}{(X_1 + 1)^{3/2}} dX_1$$
\[+ \int_{B}^{X} \frac{X_1 - \sqrt{X_1^2 - B^2}}{(X_1 + 1)^{3/2}} \, dX_1 = \frac{2 B^2}{\sqrt{1 + B} (1 + \sqrt{1 + B})^2} + B^2 \int_{B}^{X} \frac{1}{(X_1 + 1)^{3/2} X_1 + \sqrt{X_1^2 - B^2}} \, dX_1.\]

The order of the difference is \(B^2\). For small \(X \sim B\), this can be seen directly, while, in the case of larger \(X\), the contribution of large values of \(X_1\) to the integral is small.

We can describe the behavior of our functions for small and large \(X\). When \(X \rightarrow 0\),

\[f_T(X) = X^3 \sum_{n=0}^{\infty} (-1)^n \frac{n + 1}{n + 3} X^n,\]

\[f_c(X) = X^2 \sum_{n=0}^{\infty} (-1)^n \frac{(2n + 1)!}{2^n n!(n + 2)}.\]

For large \(X\),

\[f_T(X) = X - 2 \ln X + 1 - \frac{1}{X} \sum_{n=0}^{\infty} (-1)^n \frac{n + 3}{n + 1} X^n,\]

\[f_c(X) = 2X^{1/2} \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2n-3)! (2n+1)}{(2n)!} X^n - 4.\]

Here, as usual, we assume that \((-3)! = -1, (-1)! = 1.\)

For the large redshifts of interest for us or, more exactly, for \((1 - \Omega_0) z \gg 1\), the parameters \(\zeta, y,\) and \(X\) behave as follows:

\[\zeta \sim \frac{1}{2\sqrt{1 - \Omega_0} z},\]

\[y = \frac{1}{\zeta} \sim 2\sqrt{1 - \Omega_0} z,\]

\[X = 2\sqrt{1 - \Omega_0} \frac{1 - \Omega_0}{\Omega_0} y \sim 4 \frac{1 - \Omega_0}{\Omega_0} z.\]

Accordingly, the functions \(f\) are easy to determine for such \(z\) values. The scattering optical distances calculated for the open and flat models are only different for the largest \(z\), and even then only slightly: for \(z = 10^8\), in the fourth significant figure. The absorption distances display larger but also insignificant differences, in the third significant figure for all \(z\).

Figure 2 presents the results of these calculations. The starting value for the curves is \(z = z_0\); \(\lg z_0 = n_s = 4(1)8.\) To present the curves on comparable scales, the \(\tau_T\) values on the curve with \(\lg z_0 = n_s\) have been multiplied by \(10^{1-n_s}\), while the \(\tau_{cc}\) curves have been multiplied by \(10^{(17-n_s)/2}\). The difference between these factors reflects the fact that the absorption coefficient is proportional to \(T^{-1/2}\).

\[\text{Fig. 2: Scattering and absorption optical distances as a function of the photon redshift in the course of the photon’s motion away from the source. The } \tau_T \text{ values are multiplied by } 10^{z_0}, \text{ and the } \tau_{cc} \text{ values by } 10^{-8.5z_0^{-1/2}}.\]

We can see from the figures that the optical distances initially increase rapidly with decreasing redshift; this growth is slower for the absorption distances at the frequency of the maximum of the Planck function. If the source’s redshift is \(z = z_s\), then, for \(z_s/3 \leq z \leq z_s\), these distances are

\[\tau_T \sim \sigma_T \frac{\rho_0}{m_H} \frac{c}{H_0} \sqrt{T - 1},\]

\[\tau_{cc} \sim (1 - e^{-cW}) \left( \frac{\rho_0}{m_H} \right)^2 \frac{k_0^3}{T_0^{7/2}} \times \left( \frac{h}{cWk_B} \right)^3 \frac{c}{H_0} \sqrt{T - 1} \left( 1 - \frac{1}{z_s} \right) \frac{1 - z}{z_s} \].

For the adopted model parameters, \(\tau_T \approx 0.13(z_s - z)\), and \(\tau_{cc} \approx 1.8 \times 10^{-9} z_s^{-1/2}(z_s - z)\). The optical dis-
tances increase more slowly with decreasing $z$, gradually reaching their asymptotic values $\tau_T \sim 0.14 z_s$ and $\tau_{cc} \sim 2.5 \times 10^{-9} z_s^{1/2}$. The distances $\tau_{cc}$ reach their limiting values at smaller $z$, and these values are substantiilly smaller (by a factor of $5.6 \times 10^6 z_s^{1/2}$) than those for the scattering distances. Since the coefficient $k_{cc}$ is inversely proportional to the third power of the frequency, this difference will be smaller at lower frequencies.

4.7 The Grey Approximation

Since the scattering coefficient substantially exceeds the absorption coefficient, the latter can be neglected relative to the source radiation. Equation (24) with the right-hand side (26) can then be integrated over frequency, as is done in calculations of model atmospheres [4]. Let us denote the frequency moments for the mean occupation number ($s \geq 0$ is not necessarily integer, so that this is essentially the Mellin transform):

$$n^{(s)}(\eta, \chi, \mu) = \int_0^\infty x^s n(\eta, \chi, \mu, x) dx.$$ Integration over the frequency with the weights $x^s$ yields

$$\frac{\partial n^{(s)}}{\partial \eta} + \mu \frac{\partial n^{(s)}}{\partial \chi} + (1 - \mu^2) c_{sk}(\chi) \frac{\partial n^{(s)}}{\partial \mu} + (s + 1) \frac{a'(\eta)}{a(\eta)} n^{(s)} = R(\eta) \mathcal{I}^{(s)}_c,$$

where $\mathcal{I}^{(s)}_c$ is the result of integrating the right-hand side. The integral of the term with the frequency derivative was calculated via integration by parts. The frequency moments of the collision integral are

$$\mathcal{I}^{(s)}_c = -\alpha_T \left[ n^{(s)}(\eta, \chi, \mu) - n_0^{(s)}(\eta, \chi) \right] - \frac{1}{10} P_2(\mu) n_2^{(s)}(\eta, \chi) + \epsilon^{(s)}(\eta, \chi, \mu),$$

where the moments of the source power are

$$\epsilon^{(s)}(\eta, \chi, \mu) = \int_0^\infty \epsilon_*(\eta, \chi, \mu, x) x^s dx.$$

After separating out the direct radiation, the function $\epsilon_*$ should be substituted by $\epsilon_s$.

4.8 CONCLUSION

We have derived an equation describing the evolution of the radiation of a source whose intensity exceeds that of the surrounding equilibrium background radiation at the epoch of radiation-dominated plasma, between the epochs of annihilation and recombination. A cosmological model for this period was adopted. We have separated the diffuse and direct radiation, and calculated the opacities of the matter due to Thomson scattering and bremsstrahlung absorption. The latter mechanism can be neglected relative to the source radiation, and this fact was used to obtain an equation for the frequency moments.

This is the first paper in the series of studies that will be concerned with elucidating how sources of radiation affect the evolution of the Universe and how their presence is reflected in the thermal background radiation, taking into account the fact that the accuracy of observations is growing steadily. We hope to solve this problem in our forthcoming studies.

References

[1] Ya. B. Zel’dovich and I. D. Novikov, Structure and Evolution of the Universe(Nauka, Moscow, 1975; University of Chicago Press, 1983).
[2] V. K. Dubrovich, Pis’ma Astron. Zh. 29, 9(2003)[Astron. Lett.29, 6(2003)].
[3] A. D. Chernin, Astron.Zh. 42, 1124 (1965)[Sov. Astron. 9, 871 (1965)].
[4] V. V. Sobolev, A Course in Theoretical Astrophysics(Nauka, Moscow, 1985) [in Russian].