AN ELEMENTARY PROOF OF THE POSITIVITY OF THE INTERTWINING OPERATOR IN ONE–DIMENSIONAL TRIGONOMETRIC DUNKL THEORY

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ABSTRACT. This note is devoted to the intertwining operator in the one–dimensional trigonometric Dunkl setting. We obtain a simple integral expression of this operator and deduce its positivity.

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1. Introduction

We use the lecture notes [6] as a general reference about trigonometric Dunkl theory. In dimension 1, this special function theory is a deformation of Fourier analysis on \( \mathbb{R} \), depending on two complex parameters \( k_1 \) and \( k_2 \), where the classical derivative is replaced by the Cherednik operator

\[
Df(x) = \left( \frac{d}{dx} \right) f(x) + \left\{ \frac{k_1}{1 - e^x} + \frac{2k_2}{1 - e^{-x}} \right\} \{ f(x) - f(-x) \} - \left( \frac{k_1}{2} + k_2 \right) f(x)
\]

and the Lebesgue measure by \( A(x) \, dx \), where

\[
A(x) = |2 \sinh \frac{x}{2}|^{2k_1} |2 \sinh x|^{2k_2},
\]

and the exponential function \( e^{i\lambda x} \) by the Opdam hypergeometric function

\[
G_{i\lambda}(x) = \frac{\varphi_{2\lambda}^{k_1+k_2-\frac{1}{2},k_2-\frac{1}{2}}(\frac{x}{2})}{\varphi_{2\lambda}^{k_1+k_2+\frac{1}{2},k_2+\frac{1}{2}}(\frac{x}{2})} \cdot \frac{\varphi_{2\lambda}^{k_1+k_2+1+i\lambda,k_2+1-i\lambda,k_1+k_2+\frac{1}{2},-\sinh^2\frac{x}{2}}}{\varphi_{2\lambda}^{k_1+k_2+1+i\lambda,k_2+1-i\lambda,k_1+k_2+\frac{1}{2},-\sinh^2\frac{x}{2}}}
\]

Here \( \varphi_{\lambda}^{\alpha,\beta}(x) \) denotes the Jacobi function and \( {}_2F\!_1(a,b;c;Z) \) the classical hypergeometric function.

In a series of papers (2, 5, 7, 9, 10, 11, ...), Trimeche and his collaborators have studied an intertwining operator \( \mathcal{V}: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}) \), which is characterized by

\[
\mathcal{V} \circ \left( \frac{d}{dx} \right) = D \circ \mathcal{V} \quad \text{and} \quad \delta_0 \circ \mathcal{V} = \delta_0,
\]

and the dual operator \( \mathcal{V}^* : C_c^\infty(\mathbb{R}) \rightarrow C_c^\infty(\mathbb{R}) \), which satisfies

\[
\int_{-\infty}^{+\infty} \mathcal{V}f(x) \, g(x) \, A(x) \, dx = \int_{-\infty}^{+\infty} f(y) \, \mathcal{V}^* g(y) \, dy.
\]

Let us mention in particular the following facts.

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• Eigenfunctions. For every $\lambda \in \mathbb{C}$,
$$V(x \mapsto e^{i\lambda x}) = G_{i\lambda}.$$

• Explicit expression. An integral representation of $V$ was computed in [2] (and independently in [1]), under the assumption that $k_1 \geq 0$, $k_2 \geq 0$ with $k_1 + k_2 > 0$.

• Analytic continuation. It was shown in [3] that the intertwining operator $V$ extends meromorphically with respect to $k \in \mathbb{C}^2$, with singularities in $\{k \in \mathbb{C}^2 | k_1 + k_2 + \frac{1}{2} \in -\mathbb{N}\}$.

• Positivity. On the one hand, the positivity of $V$ was disproved in [2], by using the above-mentioned expression of $V$ in the case $k_1 > 0$, $k_2 > 0$. On the other hand, the positivity of $V$ was investigated in [3], [9], [10], [11] by using the positivity of a heat type kernel in the case $k_1 \geq 0$, $k_2 \geq 0$.

In Section 2, we obtain an integral representation of $V$ and $V^t$ when $\text{Re} k_1 > 0$ and $\text{Re} k_2 > 0$. The expression is simpler and the proof is quicker than the previous ones in [2] or [1]. In Section 3, we deduce the positivity of $V$ and $V^t$ when $k_1 > 0$, $k_2 > 0$, and comment on the positivity issue.

2. INTEGRAL REPRESENTATION OF THE INTERTWining OPERATOR

In this section, we resume the computations in [2] Section 2] and prove the following result.

Theorem 2.1. Let $k = (k_1, k_2) \in \mathbb{C}^2$ with $\text{Re} k_1 > 0$ and $\text{Re} k_2 > 0$. Then
$$Vf(x) = \int_{|y| < |x|} K(x, y) f(y) \, dy \quad \forall \, x \in \mathbb{R}^n$$
and
$$V^t g(y) = \int_{|x| > |y|} K(x, y) g(x) A(x) \, dx,$$
where
$$K(x, y) = \frac{c}{4} A(x)^{-1} \int_{|y|}^{t|x|} \sigma(x, y, z) \left( \cosh \frac{z}{2} - \cosh \frac{y}{2} \right)^{k_1-1} \left( \cosh x - \cosh z \right)^{k_2-1} \left( \sinh \frac{z}{2} \right) \, dz,$$
(1)
with
$$c = 2^{3k_1+3k_2} \frac{\Gamma(k_1+k_2+1)}{\sqrt{\pi} \Gamma(k_1) \Gamma(k_2)}$$
(2)
and
$$\sigma(x, y, z) = (\text{sign} x) \left\{ e^{\frac{z}{2}} (2 \cosh \frac{y}{2}) - e^{-\frac{z}{2}} (2 \cosh \frac{y}{2}) \right\}.$$  (3)

Proof. As observed in [2] and [3],
$$Vf(x) = \int_{-|x|}^{+|x|} K(x, y) f(y) \, dy$$
is an integral operator, whose kernel
$$K(x, y) = \frac{1}{4} K(\frac{x}{2}, \frac{y}{2}) + (\text{sign} x) \left( \frac{k_1}{4} + \frac{k_2}{4} \right) A(x)^{-1} \tilde{K}(\frac{x}{2}, \frac{y}{2}) - (\text{sign} x) \frac{1}{4} A(x)^{-1} \frac{\cosh \frac{z}{2}}{\sinh \frac{z}{2}} \tilde{K}(\frac{x}{2}, \frac{y}{2})$$
(4)
can be expressed in terms of the kernel
$$K(x, y) = 2 c A(2x)^{-1} \left| \sinh 2x \right| \left( \cosh z - \cosh y \right)^{k_1-1} \left( \cosh 2x - \cosh 2z \right)^{k_2-1} \left( \sinh z \right) \, dz,$$
(5)
of the intertwining operator in the Jacobi setting (see [4] Subsection 5.3) and of its integral
\[
\hat{K}(x, y) = \int_{|y|}^{x} K(w, y) A(2w) \, dw 
\]
\[
= c \frac{k_2}{k_1} \int_{|y|}^{x} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2-1} (\sinh z) \, dz .
\] (6)

Let us integrate by parts (5) and differentiate the resulting expression with respect to \(y\). This way, we obtain
\[
\hat{K}(x, y) = \frac{4c}{k_1} \int_{|y|}^{x} (\cosh z - \cosh y)^{k_1} (\cosh 2x - \cosh 2z)^{k_2-1} (\sinh z) \, dz ,
\] (7)
and
\[
\hat{K}(x, y) = -4c (\sinh y) \int_{|y|}^{x} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2-1} \\
\times (\cosh z) (\sinh z) \, dz .
\] (8)

We conclude by substituting (5), (6), (7), (8) in (4) and more precisely (6), respectively (7) in
\[
(\text{sign } x) \frac{k_2}{2} A(x)^{-1} \hat{K}(\frac{x}{2}, \frac{y}{2}) , \quad \text{respectively } (\text{sign } x) \frac{k_1}{4} A(x)^{-1} \hat{K}(\frac{x}{2}, \frac{y}{2}) .
\]

**Remark 2.2.** Let \(x, y \in \mathbb{R}\) such that \(|x| > |y|\). The expression (11) extends meromorphically with respect to \(k \in \mathbb{C}^2\), with singularities in \(k \in \mathbb{C}^2 \mid k_1 + k_2 + \frac{1}{2} \in \mathbb{N}\), produced by the factor \(\Gamma(k_1 + k_2 + \frac{1}{2})\) in (2). In the limit cases where either \(k_1\) or \(k_2\) vanishes, (11) reduces to the following expressions, already obtained in (2) and (1):

- **Assume that** \(k_1 = 0\) **and** \(\text{Re } k_2 > 0\). **Then**
  \[
  K(x, y) = 2^{k_2-1} \frac{\Gamma(k_2 + \frac{1}{2})}{\sqrt{\pi} \Gamma(k_2)} |\sinh x|^{-2k_2} (\cosh x - \cosh y)^{k_2-1} (\text{sign } x) (e^x - e^{-y}) .
  \] (9)

- **Assume that** \(k_2 = 0\) **and** \(\text{Re } k_1 > 0\). **Then**
  \[
  K(x, y) = 2^{k_1-2} \frac{\Gamma(k_1 + \frac{1}{2})}{\sqrt{\pi} \Gamma(k_1)} |\sinh \frac{x}{2}|^{-2k_1} (\cosh \frac{x}{2} - \cosh \frac{y}{2})^{k_1-1} (\text{sign } x) (e^{\frac{x}{2}} - e^{-\frac{y}{2}}) .
  \] (10)

### 3. Positivity of the Intertwining Operator

**Corollary 3.1.** Assume that \(k_1 > 0\) and \(k_2 > 0\). Then the kernel (11) is strictly positive, for every \(x, y \in \mathbb{R}\) such that \(|x| > |y|\). Hence the intertwining operator \(\mathcal{V}\) and its dual \(\mathcal{V}'\) are positive.

**Proof.** Let us check the positivity of (3) when \(x, y, z \in \mathbb{R}\) satisfy \(|x| > z > |y|\). On the one hand, if \(x > 0\), then
\[
\sigma(x, y, z) = e^{\frac{z}{2}} (2 \cosh \frac{x}{2}) - e^{-\frac{z}{2}} (2 \cosh \frac{y}{2}) \\
> (e^{\frac{z}{2}} - e^{-\frac{z}{2}}) (2 \cosh \frac{y}{2}) > 0 .
\]

On the other hand, if \(x < 0\), then
\[
\sigma(x, y, z) = e^{-\frac{z}{2}} (2 \cosh \frac{x}{2}) - e^{\frac{z}{2}} (2 \cosh \frac{y}{2}) \\
> e^{-\frac{z}{2}} (2 \cosh \frac{y}{2}) - e^{\frac{z}{2}} (2 \cosh \frac{x}{2}) = e^{-y} - e^x > 0 .
\]
Remark 3.2. As already observed in [2], the positivity of (9), respectively (10), is immediate in the limit case where \( k_1 = 0 \) and \( k_2 > 0 \), respectively \( k_2 = 0 \) and \( k_1 > 0 \).

Remark 3.3. The positivity of \( V \) was mistakenly disproved in [2, Theorem 2.11] when \( k_1 > 0 \) and \( k_2 > 0 \). More precisely, by using a more complicated formula than (11), the density \( K_{p, x, y} \) was shown to be negative, when \( x > 0 \) and \( y \leq -x \). The error in the proof lies in the expression \( A_1 \), which is equal to \( \frac{k}{k^2} \frac{\sinh(2x)-\sinh(2|y|)}{E} \) and which tends to \( +2 \frac{k}{k^2} \frac{\cosh(2x)}{\sinh(2x)} = 0 \).

Remark 3.4. A different approach, based on the positivity of a heat type kernel, was used in [8], [9], [10] and [11] in order to tackle the positivity of \( V \). While [8] may be right, the same flaw occurs in [9], [10], [11], namely the cut-off \( 1_{\gamma} \) breaks down the differential–difference equations, which are not local.

In conclusion, the present note settles in a simple way the positivity issue in dimension 1 and hence in the product case. Otherwise, the positivity of the intertwining operator \( V \) and its dual \( V^t \), when the multiplicity function \( k \) is \( \geq 0 \), remains an open problem in higher dimensions.

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