Analysis of optimal lockdown in integral economic–epidemic model

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Abstract
We analyze the optimal lockdown in an economic–epidemic model with realistic infectiveness distribution. The model is described by Volterra integral equations and accurately depicts the COVID-19 infectivity pattern from clinical data. A maximum principle is derived, and a qualitative dynamic analysis of the optimal lockdown problem is provided over finite and infinite horizons. We analytically prove and economically justify the possibility of an endemic scenario when the infection rate begins to climb after the lockdown ends.

Keywords Optimal lockdown · Epidemic control · Cost minimization · Volterra integral equations

JEL Classification C61 · H51 · I18

1 Introduction
The COVID-19 pandemic has initiated extensive research on the modeling and optimal control of COVID-19 spread worldwide and in specific countries, see, e.g., excellent literature reviews in Amir and Boucekkine (2022) and Boucekkine et al. (2021). Various directions to study the COVID-19 epidemic and combat its negative effects have been suggested. Well-developed economic frameworks have been combined with established SIR and SEIR epidemic models to analyze the impact of government actions on the epidemic spread and suggest efficient strategies to control it. Government and...
public health interventions range from travel restrictions, masks, sanitation, social distance to advanced medical treatment, vaccination, and related social processes. The most common restrictions are lockdowns that control the fraction of active workers in an economy. When a lockdown starts, the employment drops down, which hurts the economy but reduces social interactions and the rate of new infections. A proper assessment of these measures is critical. Mathematically, most of them affect the reproduction rate of COVID-19 virus in chosen epidemiologic models.

A prospective central planner framework pioneered by Acemoglu et al. (2020) and (Alvarez et al. 2020) focuses on controlled lockdown in a SIR model and minimizes the infinite-horizon total costs that include production losses, loss of life, cost of treatment, and related damages. Optimization modeling of such coupled economic–epidemic processes on real data is the scientific topic of enormous complexity. Correspondingly, different papers emphasize some relevant features of such processes, simplifying or omitting others. Thus, there are two extremes in related optimization research, from using general functional spaces with measurable functions to describe lockdown controls, e.g., (Fabbri et al. 2021), to assuming a constant in time lockdown policy (Bosi et al. 2021; Loertscher and Muir 2021; Federico and Ferrari 2021). Despite a large number of related economic-epidemiological studies, only a fraction of them simultaneously optimizes the timing, length, and intensity of lockdowns. Those models still widely differ in their assumptions and the level of covered details. We briefly review some of them below.

Alvarez et al. (2020) formulates a central planner problem to simultaneously minimize the discounted value of COVID-related deaths and the economic costs of the lockdown. The authors provide numerical simulations using available data on COVID-19. In all their scenarios, the optimal policy starts with a severe lockdown (of at least 60% of labor) shortly after outbreak and gradually diminishes to zero. The disease also disappears in the long run in all considered scenarios. Acemoglu et al. (2020) extend the central planner framework to a SIR model with three age groups (young, middle aged and old individuals) that suffer differently from the COVID-19 pandemic.

Gonzalez-Eiras and Niepelt (2021) investigate the optimal lockdown intensity and duration using a simplified SIR model with one state variable. Their infinite-horizon objective optimizes the difference between a stylized economic production function and a “health burden” function. The single control function is a “measure of economic activity” inversely related to lockdown and social distancing. Per provided analysis, the optimal lockdown starts immediately, monotonically decreases, and never ends.

Aspri et al (2021) investigate optimal containment policies balancing the effect of overall death and production losses. The pandemic is described by a SLIAR model (Arino and Portet 2020), which is more realistic as compared to SIR. The social planner minimizes a decreasing convex functional of production costs plus a linear function of the number of COVID-related deaths over a finite horizon until the (supposedly predictable) end date of the epidemic. One of novel outcomes is the appearance of multiple optimal controls at critical values of some parameters (notably, the social cost of COVID-19 death). As the authors note, the objective functional is not convex, and there are no reasons to expect uniqueness of the optimal control.

Caulkins et al (2021) combine the standard SIR model with very detailed economic modeling of dynamic lockdown controls and related economic costs. The model
allows for multiple lockowns with different intensities and lengths. The finite-horizon objective function minimizes total health losses from COVID-related deaths, economic losses caused by lockdown, and quadratic costs of adjustment to changing the employment intensity. The model produces a complex qualitative picture of optimal controls. The optimal strategy may include several sequential lockdowns. Also, different lockdown strategies can be optimal with the same choice of model parameters.

Some empiric papers also support the existence of multiple optimal strategies. Gollier (2020) used empiric data to analyze the impact of the age-specific confinement and testing policies on income and mortality in COVID pandemic. He numerically demonstrates the existence of two potentially optimal solutions: a strict four-month lockdown of 90% labor with fully eradicated pandemic, and a much milder (30%) five-month lockdown with endemic ending.

All mentioned above papers are based on SIR-family of epidemic models, whose main assumption is that recovered individuals develop a permanent immunity against the virus. Recently, growing evidence of waning COVID-19 immunity demonstrates that people recovered from COVID-19 can routinely get infected again. This new fact has initiated a departure in modeling literature from SIR toward the SIS (susceptible–infected–susceptible) models.

Bosi et al (2021) combine a SIS model with dynamic general equilibrium models of market economy. Their altruistic-motivated welfare is an increasing concave function of individual consumption and the share of non-infected individuals in population. In the efforts to obtain an exact analytic solution, the authors assume the lockdown intensity be constant in time, which is a big simplification even in the SIS model. Namely, the control variable is a constant lockdown rate that maximizes an infinite-horizon overall welfare. Goenka and Liu (2020) employs the SIS model with no immunity in a dynamic general equilibrium model, where households choose how much to invest in human and physical capital and in controlling the risk of infection. La Torre et al. (2021) formulate and analyze a finite-horizon optimization problem of balancing of the health benefits and quadratic economic costs using a single dynamic lockdown control in the SIS model. They find that the optimal strategy can involve an endemic state of the epidemic.

There are some important data-driven research findings that potentially affect the economic modeling of COVID pandemic. New evidence challenges the common assumption that governments can directly control socio-economic activities and social distancing. Thus, Goolsbee and Syverson (2021) estimated the causal effect of government lockdowns on the US economy and consumer behavior during COVID-19 pandemic using vast cell phone data on customer visits to individual businesses. They found that, while the overall consumer foot traffic fell by 60%, legal shutdown orders account for only a modest share (7%) of this decrease. It appeared that this decrease in customer visits was strongly correlated with the number of local COVID deaths. So, individual choices were far more impacted by fears of infection (inflated by the media) than by legal restrictions.

Makris (2021) used a SIR model to analyze interactions between private decisions on social distancing and government restrictions on socio-economic activity that slow the virus transmission during an epidemic. The governments imposed various restrictions on socio-economic activities, limit operations in some industries, impose a curfew
or restrict large gatherings. The author argued that, when facing stricter government policies on social distancing, most individuals seem to take more risks and distance themselves less. Other related behavioral changes are outlined in Borissov and Lambrecht (2009), Matthies and Toxvaerd (2022) and Saak and Hennessy (2018).

Bandyopadhyay et al (2021) complement the findings of Goolsbee and Syverson (2021) claiming that a lockdown creates useful habit formation (such as social distancing, hygienic habits) which lasts after the lockdown is lifted. On the other side, delaying a lockdown allows to develop a better data-driven lockdown strategy and sequentially decrease the lockdown cost. The authors suggest a three-period discrete model that describes the optimal lockdown timing, duration, intensity based on the interplay of these two independent aspects. However, their model misses some other important epidemiologic and economic processes.

The above literature review demonstrates that combining traditional economic frameworks with epidemiologic models leads to interesting theoretic insights potentially beneficial for COVID-related policy analysis. However, the weakness of the above papers is that the epidemiologic SIR and SEIR models do not adequately describe transmission patterns of real diseases, including COVID-19. This deficiency is well known in modern epidemiology (Martcheva 2015) and lead to development of various advanced models to address it (Lloyd 2001; Champredon et al. 2018; Arino and Portet 2020).

The goal of this paper is to apply the economic framework with controlled dynamic (time-dependent) lockdown (Acemoglu et al. 2020; Alvarez et al. 2020, and others) to an epidemiologic model with a realistic COVID-19 description and analyze how it will affect the qualitative picture of optimal lockdown. We consider a cost-minimizing optimization problem in the model (Hritonenko et al. 2022) with delays and specific COVID-19 infectiveness pattern. The model is described by the Volterra integral equations and contains SIR and some other popular epidemic models as special cases. Indeed, integral equations are more general and natural in epidemic modeling as compared to the ODEs (Breda et al 2012; Brauer et al 2019). Equations with delays are also relevant for various branches of economic theory (Aoyagi 1998; Benhabib and Radner 1992; Boucekkine et al 2018; Gale 1995). The employed model uses the COVID-19 infectivity distribution obtained from clinical data. It has shown a good resemblance to real data about several waves of the COVID-19 epidemic in USA (Hritonenko et al. 2022).

The novel contribution of the paper is in developing analytic optimization techniques for an integral epidemic-economic model of COVID pandemic and examining economic implications of obtained optimal lockdown scenarios. Our analysis focuses on locking down and reopening the economy as common government actions in mitigating COVID-19 spread (Eichenbaum et al. 2020; Fernández-Villaverde and Jones 2020; Garriga et al. 2020; Gori et al. 2022, and others). We follow the economic framework of (Alvarez et al. 2020) with two essential costs: the economic impact of lockdown and the number of deaths caused by epidemic. Our goal is to investigate the optimal timing, length and intensity of the lockdown. We use realistic shapes of the lockdown and opening dynamic controls (rather than simple instantaneous jumps). To abstract from well-known theoretic difficulties in the optimal control theory, we justify the use of simple classes of contender solutions, which is in line with (Caulkins
et al 2021) and (Goenka et al. 2021). Next, we derive the first-order condition for an extremum and find an approximate analytic solution of the dual system of Volterra integral equations. Using this solution, we prove our major results about the structure of optimal lockdown control (Propositions 1 and 2).

Finally, we compare our findings to similar economic studies of lockdown (Alvarez et al. 2020; Caulkins et al. 2021; Aspri et al. 2021) and discuss new features and relations between epidemic and economic variables. In particular, we analytically prove the existence of the endemic lockdown ending when the infection rate begins to climb after an optimal lockdown ends. Such optimal scenario first appeared in numeric simulation (Acemoglu et al. 2020) and was later reported by Caulkins et al. (2021), Bosi et al. (2021), and La Torre et al. (2021). We offer our own qualitative justification and interpretation of the endemic lockdown scenario.

The paper is as follows. Section 2 describes the economic-epidemiological model, formulates and justifies the optimization problem, and derives extremum conditions. A theoretic analysis of the structure and timing of optimal lockdown is provided in Sect. 3. Section 4 discusses optimal lockdown scenarios and highlights economic relevance and novelty of the obtained results. Section 5 concludes.

2 Statement of optimization problem and preliminaries.

This section describes how economic controls are incorporated in the integral epidemiologic model, formulates the optimization problem, and provides some preliminary results.

2.1 Integral epidemic model for COVID-19 control

As in the classic SIR model (Hethcote 2000), all individuals in an isolated population are split into three stages: $S$ (Susceptible), $I$ (Infectious), and $R$ (Recovered or removed). The dynamic interaction of these stages can be described by the following integral model (Hritonenko et al. 2022):

\begin{align*}
I(t) &= \int_{t-d}^{t} \beta(u)I(u)S(u)\tilde{\beta}(t - u)f(t - u)du, \\
S(t) &= 1 - \int_{d}^{t} \beta(u)I(u)S(u)\tilde{\beta}(t - u)du, \\
R(t) &= 1 - I(t) - S(t), \quad 0 \leq t < \infty,
\end{align*}

where $d > 0$ is a duration of the infectiousness period. By (3), the total population size $S(t) + I(t) + R(t)$ is normalized to one, and the unknown variables $S(t)$, $I(t)$, and $R(t)$ are the fractions of $S$, $I$, and $R$ individuals in the population. A certain part $0 < \delta(t) < 1$ of Recovered or removed dies:

\begin{equation}
D(t) = \delta(t)R(t), \quad 0 \leq t < \infty
\end{equation}

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The functions $S(t)$ and $I(t)$ are known over the prehistory interval $[-d, 0]$:

$$I(t) = \tilde{I}_0(t), \quad S(t) = \tilde{S}_0(t), \quad t \in [-d, 0]$$  \hspace{1cm} (5)

The key model functions are the transmission rate $\beta(u)\tilde{\beta}(t-u)$, $t-d < u < t$, at which one individual infected at time $u$ contaminates others at time $t$, and the fraction of individuals $f(t-u)$, $t<u$, infected at time $u$ and still infectious at time $t$. The function $\tilde{\beta}(t-u)$ determines the infectiousness distribution period, while the function $\beta(u)$ reflects the economic control of the epidemic (as described below in Sect. 2.2). The function $f$ decreases from $f(0) = 1$ to $f(d) = 0$.

The integral model (1)–(3) can describe any pattern $\beta(u)\tilde{\beta}(t-u)$ of infectivity and latency, which is not possible in ODE-based epidemic models. As shown in (Hritonenko et al. 2022), the model (1)–(3) includes other famous epidemic models as special cases. Specifically, at the assumptions $d = \infty$, $\beta(u) = \text{const}$, $f(s) = e^{-\gamma s}$, the model (1)–(3) is equivalent to the SIR model at $\tilde{\beta}(s) = \text{const}$, to the SEIR model (Hethcote 2000) at

$$\tilde{\beta}(s) = \beta \frac{\gamma \sigma}{\gamma - \sigma} \left(1 - e^{(\gamma - \sigma)s}\right),$$

and to the multi-compartment Erlang SIR model (Lloyd 2001; Champredon et al. 2018) at

$$\tilde{\beta}(s) = \frac{\beta \gamma}{n} \sum_{i=1}^{n} \frac{(\gamma s)^{i-1}}{(i-1)!}.$$  \hspace{1cm} (7)

The distribution $\tilde{\beta}(s)$ of infection intensity over the infectiousness period $[0, d]$ is critical for correct description of epidemics. Most contemporary epidemic models have been designed to portray more accurate infectiousness period distributions of real epidemics. As found in clinical research, e.g., (van Kampen et al. 2021), the infectiousness period distribution $\tilde{\beta}(s)$ for COVID-19 is described by the solid black curve in Fig. 1. This shape of $\tilde{\beta}(s)$ is common for influenza-like diseases, see Fig. 13.1 in (Martcheva, 2015). For comparison, our Fig. 1 also shows the theoretic infectiousness distributions for the SIR, SEIR, and the multi-compartment Erlang SIR model at $n = 2$ and 5. As seen in Fig. 1, the time-since-infection $\tilde{\beta}(s)$ is very small when $s > 14$ days. So, we choose the COVID-19 infectiousness interval of $d = 14$ days that includes both latent (four days) and infectious periods of the COVID-19 infected individuals.

### 2.2 Epidemic control and costs in the epidemic model

Several recent epidemic-economic models (Alvarez et al. 2020; Acemoglu et al. 2020; Fernández-Villaverde and Jones 2020; Caccavo 2020; Gori et al. 2022) use the controlled time-varying transmission rate $\beta(t)$ in the ODE-based SIR and SEIR models to capture and optimize government actions to mitigate epidemic, such as lockdown and related behavioral changes. Specifically, when a government introduces the shelter-in-place order to force a fraction of susceptible and infected individuals stay at home,
the related decrease of the transmission rate $\beta_0$ is

$$\beta(t) = \beta_0 (1 - \theta L(t))^2,$$

where the function $0 \leq L(t) < 1$ is the lockdown control, and a fixed parameter $0 < \theta < 1$ is the measure of lockdown effectiveness (Alvarez et al. 2020; Acemoglu et al. 2020; Stock 2020). The economic loss due to the lockdown $L(t)$ over the period $[0, T]$ is

$$C_L(t) = \int_0^T e^{-rt} AL(t)(S(t) + I(t))dt,$$

where $A > 0$ is the individual productivity.

We shall mention that the transmission rate $\beta(t)$ varies even without human interventions. More or less contagious or lethal virus variants have appeared in both Spanish flu and CODID-19 pandemics.

The control goal is to mitigate the epidemic spread. For analytic clarity, we consider only two major factors (losses) from the epidemic: the economic loss (9) and the number of epidemic related deaths. Then, the total cost of the epidemic is

$$\Lambda(t) = \int_0^T e^{-rt} (AL(t)(S(t) + I(t)) + \chi \delta(t) R(t))dt.$$

where $R(t) = 1 - I(t) - S(t)$ and $\chi > 0$ is the cost of death. Alvarez et al. (2020) and Acemoglu et al. (2020) estimate the parameter $\chi$ relating it to the well-known statistical value of a life.
2.3 Optimization problem

We consider the following nonlinear optimal control problem:

Find the unknown functions \( L(t), I(t), S(t), t \in [0, T], T \leq \infty \), that minimize

\[
\min_L \Lambda(t) = \min_L \int_0^T e^{-rt} (AL(t)(S(t) + I(t)) + \chi \delta(t)(1 - S(t) - I(t))) dt, \quad (11)
\]

subject to the constraints-equalities (1)–(2), initial conditions (5), and the constraint-inequality

\[
0 \leq L(t) \leq 1 \quad \text{at} \quad t \in [0, T). \quad (12)
\]

The unknown function \( L \) is taken as the independent control. Then, \( I(t) \) and \( S(t) \) are state variables. The given functions \( \tilde{S}_0(t), \tilde{I}_0(t), t \in (-T,0], \tilde{\beta}(v), \tilde{f}(v), v \in [0,T), \delta(t), t \in [0,\infty) \), are positive and continuous.

Many recent papers, notably, (Goenka et al 2021; Federico and Ferrari 2021; Aspri et al 2021; Caulkins et al 2021; La Torre et al 2021) use convex utility functions and/or quadratic costs of economic and health losses in their objective functions. Such functions add complexity but are difficult to assess and justify. Some nonlinearities in objective functions are not shown to essentially affect qualitative picture, at least, in considered central planner problems. In the rest of this paper, we stay with the discounted linear objective functional (11) à la (Alvarez 2020) to examine the clear tradeoff between economic lockdown costs and COVID-related deaths in the integral model (1)–(5). The resulting optimization problem is nonconvex and challenging. We consider both finite and infinite horizons. Qualitative analysis of finite-horizon optimization is usually more complex than of the infinite-horizon case because of anticipation of a future policy change at the end of planning horizon. At the same time, the finite horizon makes a more practical sense because of the urgency to suppress the epidemic. The infinite horizon \( T = \infty \) has a more theoretic value but helps in investigating the finite horizon case.

In the finite-horizon optimal control of such integral models (Corduneanu 1991; Hritonenko and Yatsenko 2013), a standard assumption about the control function is \( L \in L^\infty[0,T] \). In the infinite-horizon case, \( L \in L^\infty_{\text{loc}}[0,\infty) \), where \( L^\infty_{\text{loc}}[0,\infty) \) is the space of measurable on \([0,\infty)\) functions bounded almost everywhere (a.e.) on any finite subinterval of \([0,\infty)\) (Boucekkine et al 2014; Hritonenko et al. 2017).

To avoid technical challenges of the optimal control theory and focus on qualitative dynamics, we make a simper assumption in this paper. Namely, let the lockdown control \( L(t) \) be piecewise-continuous. In doing so, we are in line with Caulkins et al (2021) and Aspri et al (2021) in their handling of similar SIR-based problems of COVID optimal control. Particularly, Caulkins et al (2021) note that while “an exact optimal control problem would require a function with a discontinuous derivative, it is possible to find a continuously differentiable function which very closely approximates it”. They refer to such numerically generated solutions as ‘optimal’, but do not provides formal proof of this. Aspri et al (2021) implement a similar “direct approach” showing the existence of lockdown/opening controls that are optimal within the preassigned
class of realistic controls available to the social planner, that is, constrained piecewise linear controls that are constant over reasonably long time periods.

**Lemma 1:** (solvability of state equations). Let \( \tilde{S}_0 > 0, \tilde{I}_0 > 0, \tilde{S}_0 + \tilde{I}_0 < 1, t \in [-d, 0] \). Then, the nonlinear integral Eqs. (1)–(3) with initial conditions (5) have a unique continuous solution \( \{ I, S, R \} \), such that \( 0 < I(t), R(t), S(t) < 1 \) at \( t \in [0, T] \), \( T \leq \infty \), for any piecewise-continuous control \( L \) from (12).

The proof follows standard techniques for Volterra integral equations (Corduneanu 1991; Hritonenko and Yatsenko 2013). The existence and uniqueness of the solution is proven using the contraction mapping principle (Kirk and Kramsi 2001; Hritonenko et al. 2017). The range \((0,1)\) of the unknown functions \( I \) and \( S \) follow from epidemic context of the model (1)–(3). Lemma is proven.

By Lemma 1, the integral in (11) always converges in the case \( T = \infty \).

### 2.4 Reproduction number

The most fundamental parameter in epidemiological literature is the reproduction number that determines whether and how fast epidemics spread out. Following (Iannelli and Milner 2017; Hritonenko and Yatsenko 2005), the **reproduction number** in the integral model (1)–(3)

\[
R_t = \int_{t-d}^{t} S(u) \beta(u) \tilde{\beta}(t - u) f(t - u) du
\]

(13)

describes the expected number of new infections generated by one infected individual. The percentage \( I(t) \) of infected increases at \( R_t > 1 \) and decreases at \( R_t < 1 \) (Hritonenko et al. 2022). The **basic reproduction number**

\[
R_0 = \int_{t-d}^{t} \beta(u) \tilde{\beta}(t - u) f(t - u) du
\]

(14)

is the expected number of new infections caused by one infected at the beginning of epidemic, when \( S(t) \approx 1 \) and \( I(t) \ll 1 \).

Another important property of epidemic models is the **herd immunity**. It means that an uncontrolled epidemic at \( R_0 > 1 \) initially spreads but eventually extinguishes by itself when more people get infected and recover (or die) and the percentage \( S(t) \) of susceptible decreases to a certain value \( 0 < S_H < 1 \). Particularly, at \( \beta(t)\tilde{\beta}(s) = \text{const} \) and \( R_0 > 1 \), the solution \( I(t) \to 0 \), while \( S(t) \) tends to the **herd immunity level**:

\[
S(t) \to S_H = \frac{1}{R_0} \quad \text{at } t \to \infty.
\]

(15)

The reproduction number and herd immunity property are well known for the standard SIR model and its numerous ODE-based modifications, see the excellent survey of Hethcote (2000).

The basic qualitative properties of the integral model (1)–(3) at \( \beta = \text{const} \) are similar to the standard SIR model (Bohner et al 2019). More realistic description of
infectiveness period in the integral model becomes important when the transmission rate \( \beta(t) \) sharply changes. Hritonenko et al. (2022) analyze the model (1)–(3) with a jump in \( \beta(t) \) and find essential differences in both transition and asymptotic dynamics of the model (1)–(3) and the SIR model after lockdown. Specifically, using calibration parameters from (Alvarez et al. 2020; Acemoglu et al. 2020), the percentage of infected \( I(t) \) in model (1)–(3) decreases three times faster after a lockdown and is 10–15 times smaller after 60 days as compared to the SIR model. It is of obvious interest whether the optimal dynamic lockdown in the model (1)–(3) would retain dynamic features discovered by Alvarez et al. (2020) and Acemoglu et al. (2020). The optimization analysis of the model (1)–(3) is the subject of the present paper.

3 Analysis and results

The first step of our analysis is to derive the first-order extremum condition (a maximum principle) for the optimization epidemic-economic problem under study, which includes a dual system of linear integral equations. To perform a qualitative dynamic analysis of the optimization problem, we find an approximate analytic solution of the dual system. Using this solution, we prove our major theoretic results (Propositions 1 and 2) about timing and intensity of the optimal lockdown.

3.1 Extremum conditions

In this section we establish optimality conditions that will allow a dynamic analysis of the optimization problem (1)–(5), (11), (12). Here and thereafter, the kernel of the integral Eq. (1) is denoted as

\[
K(s) = \tilde{\beta}(s) f(s).
\]

Following standard optimization techniques (Boucekkine et al. 2014; Corduneanu 1991; Hritonenko and Yatsenko 2013), we derive the first-order necessary condition for an extremum:

**Lemma 2:** (maximum principle). Let \( L^* \) be a solution to the problem (1)–(5), (11), (12) then the following inequalities hold at \( 0 \leq t < T, T \leq \infty \):

\[
\Lambda'(t) \geq 0 \text{ at } L(t) = 0, \quad \Lambda'(t) = 0 \text{ at } 0 < L * (t) < 1, \quad \Lambda'(t) \leq 0 \text{ at } L(t) = 1, \quad (17)
\]

where

\[
\Lambda'(t) = e^{-rt} \left( A(S(t) + I(t)) - 2\beta_0 \theta (1 - \theta L(t)) S(t) I(t) \right.
\]

\[
\left. \min(d, T - t) \int_0^{\min(d, T - t)} e^{-rv}(\tilde{\beta}(v) \mu(t + v) - K(v) \lambda(t + v)) dv \right), \quad (18)
\]

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and the dual variables $\lambda(t)$ and $\mu(t)$ satisfy the dual equations

$$
\begin{align*}
\lambda(t) &= \chi \delta(t) - AL(t) \\
&\quad - \beta_0 (1 - \theta L(t))^2 S(t) \min(d, T-t) \int_0^\infty e^{-rv} (\tilde{\beta}(v) \mu(t + v) - K(v) \lambda(t + v)) dv,
\end{align*}
$$

(19)

$$
\begin{align*}
\mu(t) &= \chi \delta(t) - AL(t) \\
&\quad - \beta_0 (1 - \theta L(t))^2 I(t) \min(d, T-t) \int_0^\infty e^{-rv} (\tilde{\beta}(v) \mu(t + v) - K(v) \lambda(t + v)) dv.
\end{align*}
$$

(20)

The proof is in Appendix.

In Lemma 2, the function $\Lambda'(t)$ is the Freschet derivative (or gradient) of the functional (10) with respect to the control $L$. The dual (co-state) variables functions $\lambda$ and $\eta$ are associated with the state constraints-equalities (1) and (2) and represent the marginal cost of violating those constraints.

While the state Eqs. (1)–(3) are subject to the initial condition (5), the dual Eqs. (19)–(20) are solved backwards in time starting with the transversality conditions at the end of the planning horizon. At $T = \infty$, the tranversality conditions are

$$
\lim_{t \to \infty} e^{-rt} l(t) = \lim_{t \to \infty} e^{-rt} m(t) = 0
$$

(21)

At a finite $T < \infty$, the tranversality conditions directly follow from (19)–(20) as

$$
\lambda(T) = \mu(T) = \chi \delta(T) - AL(T).
$$

(22)

Combining (19) and (20), we obtain two useful formulas:

$$
\begin{align*}
\mu(t) - \lambda(t) &= \beta_0 (1 - \theta L(t))^2 (S(t) - I(t)) \min(d, T-t) \int_0^\infty e^{-rv} (\tilde{\beta}(v) \mu(t + v) - K(v) \lambda(t + v)) dv, \\
\mu(t)S(t) - \lambda(t)I(t) &= (\chi \delta(t) - AL(t))(S(t) - I(t)).
\end{align*}
$$

(23)

(24)

We base our qualitative analysis on the first-order extremum conditions of Lemma 1. Because of the interplay between epidemic and controlled economic dynamics, optimization problems in similar SIR models are non-convex and do not allow for sufficient conditions (Boucekkine et al. 2021). The challenges of formally establishing optimality in non-convex dynamic optimization problems in SIR epidemic models are now an active area of research (Aspri et al. 2021; Bosi et al. 2021; Caulkins et al. 2021; Goenka et al. 2021).
Lemma 3: (solvability of the dual system). At known functions $I(t)>0$, $S(t)>0$, $\delta(t)>0$, and $0 \leq L(t) < 1$, the system of dual integral Eqs. (19)–(20) has a unique piecewise-continuous solution $\lambda, \mu, t \in [0, T)$, $T \leq \infty$, such that:

(a) $\mu(t) \approx \chi \delta(t) - AL(t)$ at $I(t) \ll S(t)$;
(b) $\lambda(t) < \mu(t)$ at $I(t) < S(t)$;
(c) $\lambda(t) \to \mu(t)$ at $I(t) \to S(t)$.

Proof By (24), we have

$$\mu(t) = \lambda(t)I(t)/S(t) + (\chi \delta(t) - AL(t))(S(t) - I(t))/S(t). \quad (25)$$

Substituting (25) into the Eq. (20), we obtain one equation in $\lambda$:

$$\lambda(t) = \beta_0(1 - \theta L(t))^2 S(t) \int_0^{\min(d, T-t)} e^{-rv} \lambda(t + v) K(v) - \tilde{\beta}(v) I(t + v) / S(t + v) \right) dv
$$
$$+ \chi \delta(t) - AL(t) - \beta_0(1 - \theta L(t))^2 S(t)
$$
$$\int_0^{\min(d, T-t)} e^{-rv} \tilde{\beta}(v)(\chi \delta(t + v) - AL(t + v))(1 - I(t + v)/S(t + v)) dv. \quad (26)$$

Using the new integration variable $s = t + v$, the linear Volterra Eq. (26) can be rewritten as

$$\lambda(t) = \beta_0(1 - \theta L(t))^2 S(t) \int_t^{\min(t + d, T)} e^{-r(s-t)} \lambda(s) K(s - t) - \tilde{\beta}(s - t) I(s)/S(s) ds
$$
$$+ \chi \delta(t) - AL(t) - \beta_0(1 - \theta L(t))^2 S(t)
$$
$$\int_t^{\min(t + d, T)} e^{-r(s-t)} \tilde{\beta}(s - t)(\chi \delta(s) - AL(s))(1 - I(s)/S(s)) ds. \quad (27)$$

Case $T < \infty$. Then, the linear Volterra Eq. (27) has a unique solution $\lambda(t), t \in [0, T)$, that is determined from the point $t = T$ to the left, see, e.g., (Corduneanu 1991).

Case $T = \infty$. Using the contraction mapping principle, we can prove that the linear Volterra Eq. (27) has a unique solution $\lambda(t), t \in [0, \infty)$, that satisfies the transversality conditions (22).

The smoothness and properties (a)–(c) of the solution follow from (23) and (24). The Lemma is proven.
3.2 Structure of optimal lockdown

Here and thereafter, we assume that the basic reproduction number

\[ R_0 = \beta_0 \int_0^d K(v)dv > 1, \]  

(28)

which means that the epidemic is severe and spreads in the absence of any control, that is, \( I(t) \) increases at \( L \equiv 0 \). We also assume that \( I(t) < S(t) \) to restrict our analysis to practical situations.

In line with (Alvarez et al. 2020; Acemoglu et al. 2020; Aspri et al. 2021; Bosi et al. 2021; Eichenbaum et al. 2020; Fabbri et al. 2021; Federico and Ferrari 2021; Garriga et al. 2020; Gori et al. 2022; La Torre et al 2021; Loertscher and Muir 2021), we analyze the infinite-horizon optimization problem (1)–(5), (11), (12) at \( T = \infty \).

First, let us demonstrate that a positive lockdown is optimal, provided that the cost of death \( \chi \) in (11) is large enough. Then, the “no lockdown” control \( L(t) = 0, t \in [0, \infty) \), is not optimal. Indeed, using Lemma 2, the gradient (18) at \( L \equiv 0 \) leads to

\[ \Lambda'(t) \approx e^{-rt} \left( A(S(t) + I(t)) - 2\beta_0 \vartheta S(t) I(t) \int_0^d e^{-rv} \tilde{\beta}(v) \lambda(t + v) - K(v) \lambda(t + v) dv \right). \]  

(29)

Since \( \mu(t) \approx \chi \delta(t) \) and \( \lambda(t) \leq \chi \delta(t) \) by Lemma 3, (29) gives

\[ \Lambda'(t) \leq e^{-rt} \left( A(S(t) + I(t)) - 2S(t) I(t) \chi \delta(t) \beta_0 \int_0^d e^{-rv} \tilde{\beta}(v) (1 - f(v)) dv \right). \]  

(30)

The gradient (29) depends on the state variables \( I \) and \( S \). Assuming that \( \chi \delta(t) \gg A \), the RHS of (30) becomes negative when \( I(t) \) is not small compared to \( S(t) \). Then, the optimal \( L^*(t) > 0 \). We note that this case cannot occur at the beginning of epidemic when \( I(t) < 1 \).

To determine an accurate qualitative picture of the optimal lockdown control, we shall investigate the dual system (19)–(20) that does not have an analytic solution. To find an approximate analytic solution to Eqs. (19)–(20), we employ the idea of multiple scale analysis (Bohner et al. 2019). Indeed, \( d < < T \) in practical problems of epidemic control. Specifically, \( d = 8 - 14 \) days in the COVID case, while the planning horizon \( T \) is at least one year (Acemoglu et al. 2020). Below, we approximate the integral in...
the dual Eqs. (19)–(20) in a way that allows for their analytic solution. Specifically, applying the Mean Value Theorem to the integrals in (19)–(20), we obtain

\[
\int_0^d e^{-rv} (\tilde{\beta}(v)\mu(t+v) - K(v)\lambda(t+v))dv = \mu(t + \varsigma) \int_0^d e^{-rv} \tilde{\beta}(v)dv
\]

\[
- \lambda(t + \xi) \int_0^d e^{-rv} K(v)dv,
\]

where \(0 < \varsigma, \xi < d\). Let us assume that the dual variables \(\lambda(t)\) and \(\mu(t)\) are slowly changing (we will justify that fact below in Remark 2). Then, \(\mu(t + \varsigma) \approx \mu(t)\) and \(\lambda(t + \xi) \approx \lambda(t)\), so, the Eqs. (19)–(20) can be approximated by the following linear equations:

\[
\lambda(t) = \chi \delta(t) - AL(t) - \beta_0(1 - \theta L(t))^2 S(t) \left( \mu(t) \int_0^d e^{-rv} \tilde{\beta}(v)dv - \lambda(t) \int_0^d e^{-rv} K(v)dv \right), \tag{31}
\]

\[
\mu(t) = \chi \delta(t) - AL(t) - \beta_0(1 - \theta L(t))^2 I(t) \left( \mu(t) \int_0^d e^{-rv} \tilde{\beta}(v)dv - \lambda(t) \int_0^d e^{-rv} K(v)dv \right). \tag{32}
\]

The solution to the system (31)–(32) of two linear equations in \(\lambda(t)\) and \(\mu(t)\) is

\[
\hat{\lambda}(t) = \frac{1 - \beta_0(1 - \theta L(t))^2 (S(t) - I(t)) \int_0^d e^{-rv} \tilde{\beta}(v)dv}{1 - \beta_0(1 - \theta L(t))^2 \left( S(t) \int_0^d e^{-rv} K(v)dv - I(t) \int_0^d e^{-rv} \tilde{\beta}(v)dv \right)}, \tag{33}
\]

\[
\hat{\mu}(t) = \frac{1 - \beta_0(1 - \theta L(t))^2 (S(t) - I(t)) \int_0^d e^{-rv} K(v)dv}{1 - \beta_0(1 - \theta L(t))^2 \left( S(t) \int_0^d e^{-rv} K(v)dv - I(t) \int_0^d e^{-rv} \tilde{\beta}(v)dv \right)}. \tag{34}
\]
Remark 1 If the functions $\delta(t)$, $I(t)$, $S(t)$, and $L(t)$ are constant, then $\hat{\lambda}(t)$ and $\hat{\mu}(t)$ are also constant by (33)–(34) and coincide with the exact stationary solution of the dual system (19)–(20). We conjecture that then $\hat{\lambda}(t)$ and $\hat{\mu}(t)$ are close (in some formal sense) to exact solutions $\lambda$ and $\mu$ of the dual system (19)–(20) if the functions $\delta(t)$, $I(t)$, $S(t)$, and $L(t)$ are slowly changing. A posteriori evidence is given by below Propositions 1 and 2, which demonstrate that the optimal control $L(t)$ decreases to zero, the percentage $S(t)$ of susceptible decreases but remains above the level $S_H > 0$, and the percentage $I(t)$ of infected eventually decreases to zero. Therefore, the functions $L, S, \text{and} I$ are slowly changing on the most of interval $[0, \infty)$, and so, are the dual variables $\lambda$ and $\mu$ by (31)–(32). Then, the approximate solution (33)–(34) of the dual system (19)–(20) allows us to analyze qualitative properties of the optimization problem.

Using formulas (33)–(34), we can prove the following statement about the structure and timing of the optimal lockdown.

Proposition 1 (optimal lockdown control). Let

$$\beta_0 \int_0^d e^{-rv} K(v) dv > 1 \quad \text{and} \quad (1 - \theta)^2 \beta_0 \int_0^d e^{-rv} K(v) dv < 1. \quad (35)$$

Then, the optimization problem (1)–(5), (11), (12) has a unique solution $L^*(t)$, $0 \leq L^*(t) < L_{max}$,

$$L_{max} = \min(1, \chi \delta(t)/A) \quad (36)$$

The solution $L^*(t)$ is positive when the corresponding state variables $I(t)$ and $S(t)$ satisfy the following conditions:

$$\beta_0 \left( S(t) \int_0^d e^{-rv} K(v) dv - I(t) \int_0^d e^{-rv} \tilde{\beta}(v) dv \right) < 1, \quad (37)$$

$$1 - \beta_0 \left( S(t) \int_0^d e^{-rv} K(v) dv - I(t) \int_0^d e^{-rv} \tilde{\beta}(v) dv \right)$$

$$< \beta_0 \chi \delta(t) \frac{2S(t)I(t)}{S(t) + I(t)} \int_0^d e^{-rv}(\tilde{\beta}(v) - K(v)) dv. \quad (38)$$

Proof By (18) and (23),

$$\Lambda'(t) = e^{-rI}(A(S(t) + I(t)) - 2\theta S(t)I(t)(\mu(t) - \lambda(t))/(((S(t) - I(t))(1 - \theta L(t)))). \quad (39)$$

1 Similar steady-state solutions have been successfully used in integral models of economic and population dynamics (Boucekkine et al. 2014, Hritonenko et al. 2017).
Replacing $\lambda(t)$ and $\mu(t)$ in (39) with their approximations (33) and (34), we obtain

\[
\Lambda'(t) \approx e^{-rt} \left( A(S(t) + I(t)) - \frac{2\beta_0 S(t) I(t) (\chi - AL(t)) (1 - \theta_L(t))}{1 - \beta_0 (1 - \theta_L(t))^2} \int_0^d e^{-rv} (\ddot{\beta}(v) - K(v)) dv \right).
\]

(40)

If (37) fails, then $\Lambda'(t) > 0$ at $0 < L(t) < \chi \delta(t)/A$ by (40) and, therefore, the optimal $L^*(t) = 0$. Now, let $I(t)$ and $S(t)$ be such that both inequalities (37) and (38) hold true. Then, we obtain from (40) that $\Lambda'(t) < 0$ at $L(t) = 0$ at the condition (38) and, therefore, $L^*(t) > 0$.

At conditions (37) and (38), the optimal interior $0 < L^*(t) < \chi \delta(t)/A$ is determined from the equation $\Lambda'(t) = 0$, $0 < t < T$, which can be rewritten as

\[
f_1(L(t), t) = f_2(L(t), t),
\]

(41)

where

\[
f_1(L, t) \overset{df}{=} 1 - \beta_0 (1 - \theta_L) \int S(t) \int_0^d e^{-rv} K(v) dv - I(t) \int_0^d e^{-rv} \ddot{\beta}(v) dv,
\]

(42)

\[
f_2(L, t) \overset{df}{=} 2\beta_0 \theta \frac{S(t) I(t)}{S(t) + I(t)} \left( \frac{\chi \delta(t)}{A} - L \right) (1 - \theta_L) \int_0^d e^{-rv} (\ddot{\beta}(v) - K(v)) dv.
\]

(43)

Note that $f_1(L)$ and $f_2(L)$ depend on the state variables $S$ and $I$ and, therefore, on $t$. The shapes of quadratic polynomial functions $f_1(L)$ and $f_2(L)$ are shown in Fig. 2.

The function $f_1(L(t), t) > 0$ monotonically increases in $L$ at $0 < L < 1$. By (43), $f_2(L) > 0$ at $L \in [0, L_{\max}]$ and $f_2(L) < 0$ at $L \in (L_{\max}, 1]$, where $L_{\max}$ is given by (36). So, the function $f_2(L(t), t)$ is positive and monotonically decreases in $L$ for $0 < L < L_{\max}$.

By Eq. (41), at a fixed $t$, the curves $f_1$ and $f_2$ intersect, $f_1(L, t) = f_2(L, t)$, at a single point $0 < L^*(t) < L_{\max}$ under conditions (37) and (38). These conditions guarantee that $f_1(0) < f_2(0)$. The quadratic Eq. (41) in $L$ has a single positive solution on $(0, L_{\max})$.

The proposition is proven.

Remark 2 A simple sufficient condition for (37) is $S(t) - I(t) < 1/ R_0$. Indeed, (37) holds if

\[
\beta_0 \left( S(t) \int_0^d e^{-rv} K(v) dv - I(t) \int_0^d e^{-rv} \ddot{\beta}(v) dv \right) < \beta_0 (S(t) - I(t)) \int_0^d e^{-rv} K(v) dv < (S(t) - I(t)) R_0 < 1.
\]

\[\text{Springer}\]
Analysis of optimal lockdown in integral economic–epidemic model

The larger $R_0$ is, the smaller is the threshold difference $S - I$. Thus, $I(t) > S(t) - 0.8$ at $R_0 = 1.25$ and $I(t) > S(t) - 0.5$ at $R_0 = 2$.

**Proposition 2** (timing of lockdown). Let the conditions of Proposition 1 hold, $\theta \chi \delta(t) > 1$ at $0 \leq t < \infty$, and $S(0) \approx 1$. Then,

$$L^*(t) = 0 \text{ at } 0 < t \leq t_L, \quad L^*(t) > 0 \text{ at } 0 < t_L \leq t < t_E, \quad \text{and } L^*(t) = 0 \text{ at } t_E \leq t < \infty. \quad (44)$$

that is, the optimal lockdown starts at $t_L > 0$ and ends at a finite instant $t_E$. The optimal $L^*(t) > 0$ decreases when $I(t)$ decreases. If $r > 0$, then $I(t)$ remains small positive and increases, $I(t) > 0, I(t) < < 1$, and $S(t) > S_H$ at $t_E \leq t < \infty$.

**Proof** At small $I(t)$ and $S(t) \approx 1$, (37) does not hold by the first condition (35). Then, $L^*(t) = 0$. by Proposition 1, It corresponds to the case $f_1(0, t) > f_2(0, t)$ in Fig. 2, where.

$$f_1(0, t) = 1 - \beta_0 \left( S(t) \int_0^d e^{-rv} K(v)dv - I(t) \int_0^d e^{-rv} \tilde{\beta}(v)dv \right) \quad (45)$$

and

$$f_2(0, t) = \frac{2S(t)I(t)}{S(t) + I(t)} \frac{\beta_0 \theta \chi \delta(t)}{A} \int_0^d e^{-rv}(\tilde{\beta}(v) - K(v))dv.$$
As the above dynamic scenario of the optimal lockdown demonstrates, Remark 3

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SEIR models. We prove that a similar behavior occurs in the integral model (1)–(3) with
knowledge, the only papers that demonstrated such dynamics are based on SIR and
our optimization problem can increase, decrease, and fluctuate. At the best of our
The dynamic trajectory of the optimal lockdown and related epidemic variables in
discussion of results

finding S(tE) from the last formula, we obtain S(tE) = ˆS = 1/β0 d 0 e−rvK(v)dv.
Since ˆS > S ≤ 1, therefore, L(t) > 1 at t ≥ te, so, I(tE) > 0 and a positive I(t)
increases at t > tL.
The proposition is proven.

Remark 3 As the above dynamic scenario of the optimal lockdown demonstrates,
the optimal state trajectory S(t) monotonically decreases but remains above a certain
positive value, therefore, |S′(t)| decreases. Eventually, the derivative I′(t) becomes
negative and |I′(t)| decreases. Therefore, following Remark 1, ˆλ(t) and ˆμ(t) better
approximate the exact solution λ(t) and μ(t) of the dual system (19)–(20). Corre-
respondingly, it provides an a posteriori justification that our proofs are correctly reflect
the time-dependent picture of the optimal trajectories.

4 Discussion of results

The dynamic trajectory of the optimal lockdown and related epidemic variables in
our optimization problem can increase, decrease, and fluctuate. At the best of our
knowledge, the only papers that demonstrated such dynamics are based on SIR and
SEIR models. We prove that a similar behavior occurs in the integral model (1)–(3) with
more realistic COVID infectiveness distribution, which causes analytic difficulties
compared to the SIR case.

Let us start with epidemiological interpretation of the conditions of Propositions
1 and 2. The first condition (35) reveals that the virus spreads up and the epidemic
grows if it is not mitigated. Indeed, S(t) ≈ 1 at the beginning of epidemic, so, by (13),
the reproduction number Rt ≈ R0 = β0 d d 0 K(v)dv > β0 d d 0 e−rvK(v)dv > 1, therefore,
I(t) increases.

The second condition (35) implies that the maximal lockdown L = 1 produces the
reproduction number R* < 1. So, the lockdown can eventually mitigate the epidemic
and cause the number of infected to decrease.
4.1 Optimal timing, length and intensity of the lockdown

Propositions 1 and 2 validate the following picture of the lockdown timing, duration, and intensity. The lockdown includes several stages:

1. Initial stage of epidemic (no lockdown). The conditions (37) and (38) can hold only when an essential part $I$ of population is infected. At the beginning of epidemic, $I(t)$ is small, and $S(t) \approx 1$. Then, the optimal lockdown is nil: $L^*(t) = 0$, because (37) does not hold by the first condition (35).

2. Start of lockdown. In an uncontrolled epidemic at $R_0 > 1$, the percentage $I(t)$ of infected increases and the percentage $S(t)$ of susceptible decreases (see Sect. 2.3). If the epidemic is severe, then conditions (37) and (38) of Proposition 1 will be eventually satisfied and the lockdown will start. When the lockdown starts, the transmission rate $\beta(t)$ in the Eq. (1) jumps down at a certain instant $t_L > 0$ in accordance with (8). At $t > t_L$, $S(t)$ decreases and $R_t$ will eventually become less than 1 and lead to decreasing $I(t)$. However, it is not guaranteed that $R^*_t < 1$ and $I(t)$ will decrease immediately after the lockdown at the optimal $L^*(t) < 1$.

3. Late stages and the end of lockdown. As the percentage of susceptible $S(t)$ decreases, the epidemic approaches the herd immunity, and the lockdown is not needed anymore. Proposition 2 formally determines the finite lockdown end time $t_E$ when $L^*(t)$ becomes 0.

Two dynamic scenarios are possible during the late stage of lockdown:

(a) Monotonically decreasing lockdown intensity. It occurs when the after-lockdown reproduction number $\hat{R}_t = \beta_0(1 - \theta L^*(t_L))^2 \int_0^d K(v)dv < 1$ at $t \geq t_L$ close to $t_L$. Then, $I(t)$ starts decreasing immediately at $t > t_L$. By Proposition 2, the optimal lockdown control $L^*(t)$ decreases and becomes zero at a certain instant $t_E$ before $S(t)$ reaches the level of $\hat{S} > S_H$. Then $\hat{R}_t > 1$ at $t = t_E$, so, a small positive $I(t)$ starts to increase. So, the epidemic becomes endemic after the optimal lockdown ends. The lockdown ends before $S(t)$ reaches $S_H$ and, therefore, a resurgence of virus is likely.

(b) Inverted U-shaped lockdown intensity. Let the reproduction number be $\hat{R}_t > 1$ at $t = t_L$. Then, the percentage $I(t)$ of infected slows down as compared to the uncontrolled case, but still increases at $t \geq t_L$ close to $t_L$. It will start decreasing eventually when $S(t)$ decreases and becomes closer to the herd immunity level $S_H$. Then, the optimal control $L^*(t)$ decreases and further dynamics follows Scenario A.

As opposed to ODE-based epidemic models, the integral Eqs. (1)–(3) with finite delay can possess smooth or frequently fluctuating trajectories $I(t)$ and $S(t)$ depending on possible fluctuations of the given initial functions $\hat{I}_0(t)$ and $\hat{S}_0(t)$ on the prehistory interval $[-d, 0]$, see Hritonenko et al. 2022). Correspondingly, the optimal lockdown control $L^*(t)$ may possess such fluctuations as it depends on $I(t)$ and $S(t)$ by (40).
4.2 Endemic lockdown

By Proposition 2, the optimal lockdown ends prematurely when the reproduction number is still greater than 1 and the number of infected \( I(t) \) may be small but remains positive. So, the infection rate begins to climb after the lockdown ends.

The possibility of an endemic scenario has been already mentioned by other researchers. Acemoglu et al (2020) simulated and discussed an optimal lockdown policy when the herd immunity has not achieved, and the infection rate begins to climb again (see their Fig. 9); they characterize their endemic scenario as a “waiting for vaccine strategy”.

Caulkins et al (2021) numerically demonstrated that premature lockdown ending may lead to successive renewals of lockdowns, however they did not prove this effect analytically. Sometimes, their optimal solution involves locking down, ending the lockdown, and reinstating it, with the second lockdown being more severe. Moreover, even when a single strategy is optimal, a very different strategy can quickly become optimal when a certain parameter changes. The most sensitive parameter to trigger such abrupt changes is the cost of premature death.

Bosi et al (2021) and La Torre et al (2021) reveal the possibility of endemic optimal lockdown strategies in their SIS-based optimization models. Specifically, the optimal (constant) lockdown strategy in (Bosi et al 2021) is a positive or zero (nil) lockdown under calibrated values, and correspondingly the economy converges to the disease-free or to the endemic steady state (depending on the “degree of altruism”). Bosi et al (2021) assert that the reproduction number is a critical parameter that drives the convergence to endemic state.

We suggest a different reason of endemic lockdown ending. Our upshot is that the lockdown is followed by an endemic state because of the discount rate. It perfectly matches the common economic interpretation of the discount as “time preference” that refers to the desire of individuals to enjoy benefits now while deferring negative effects of doing so.

As demonstrated in the Proposition 2 proof, the economic-mathematical justification of endemic lockdown is caused by the presence of discounting in the objective functional (11). Namely, the discount rate \( r > 0 \) in (11) decreases the present value of the cost of COVID-related deaths in a distant future. It causes a premature lockdown end with a reproduction number \( R_t > 1 \), which leads to sequential increase of the infected percentage \( I(t) \).

The additional argument in favor of discounting as the key factor of endemic lockdown is as follows. If we formally consider the discount rate \( r = 0 \) (and disregard for a moment the divergence of the functional (1)), then we immediately obtain \( R_t = 1 \), \( I(t_E) = 0 \), \( S(t_E) = S = S_H \), and \( R_t < 1 \) at \( t > t_E \). Therefore, the lockdown stage is followed by a disease-free state at \( t > t_E \). A mathematically correct but more complicated way to prove that fact is possible if we take and analyze limits of all involved formulas at \( r \to 0 \). Therefore, the magnitude of the endemic lockdown ending declines (in an informal sense) as the discounting rate decreases. 2

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2 We are grateful to the Associate Editor for noticing this phenomenon.
In conclusion, we conjecture that similar endemic lockdown ending caused by discounting is possible in SIR-based models of controlled lockdown.

### 4.3 Dependence of optimal lockdown on model parameters

In all epidemic-economic models with controlled dynamic lockdown, the optimal containment strategies highly depend on model parameters because of the nonlinear relations between epidemic and controlled economic dynamics.

Optimization models of Caulkins et al (2021) and Aspri et al (2021) produce complex scenarios of optimal dynamic lockdown strategies, with several sequential lockdowns and multiple optimal controls occurring at critical values of some parameters. The most sensitive parameter to trigger sharp changes and multiple solutions in these models is the social cost of premature death. This parameter determines the optimal balance between overall death and loss of production.

The product of the death cost $\chi$ and the death rate $\delta(t)$ essentially affects the optimal lockdown in our optimization problem (1)–(5), (11), (12) By Proposition 1, the optimal lockdown is smaller than the value $\frac{\chi \delta(t)}{A}$.

As Caulkins et al (2021) mention, an extensive literature discusses the appropriate value for the cost of death $\chi$. There is a consensus about the order of magnitude, but considerable debates continue as to the particular value of $\chi$. Early estimates of the death rate $\delta(t)$ and the death parameter $\chi$ by Alvarez et al. (2020) and Acemoglu et al. (2020) are $\chi \approx 20A$ and $\delta = 0.01–0.04$. Then, $\frac{\chi \delta(t)}{A} < 1$ and the optimal lockdown in our model is lower than 100%. The lockdown eventually decreases following Proposition 2 and the empiric evidence about a decreasing death rate $\delta(t)$ (because of new achievements in medical treatment and vaccination). Different estimates of the cost of death $\chi$ are ranging between 10 and 150 $A$ in various sources.

New findings of this paper (Sect. 4.2) highlight the importance and impact of the discount rate on the endemic lockdown.

### 5 Conclusion

Given the severity of COVID pandemic, it is important to describe its dynamics as accurately as possible in order to evaluate effective government actions to combat the pandemic. The majority of research to integrate epidemic dynamics in macroeconomic models has been based on SIR or SEIR models that do not adequately describe the transmission of COVID-19 virus (Martcheva 2015; Lloyd 2001; Champredon et al. 2018; Arino and Portet 2020). In this paper, we propose and analyze an economic-epidemiological model in which the COVID-19 dynamics is described by Volterra integral equations based on an accurate infectivity pattern from clinical data.

After addressing certain analytic challenges, we demonstrate that the optimal lockdown in our model can considerably slow down the virus spread and decrease the numbers of infected and deaths. The model possesses novel features, such as the endemic ending of lockdown, which are interpretable in terms of policy context. The lockdown can mitigate the epidemic until better vaccines and other ways of disease mitigation become available.
treatment are discovered, which will decrease the infectivity and death rate and eventually end the epidemic. Naturally, such events will cause an essential alteration of the control problem under study.

Ongoing changes in the COVID virus behavior and social-economic reality make the economic modeling of COVID epidemics continuously evolving. Some actions, such as vaccines, appear to be much less relevant now than it was widely expected two years ago. Current stage of COVID-19 pandemic is characterized by the appearance of new virus variants with higher infectivity but lower mortality. The decreased mortality can be in part explained by improved ability of health care system to care for people with COVID.

Next, as shown by Goolsbee and Syverson (2021) on vast amounts of cellphone data, the individual choices based on fear of infection appear to be far more important than government legal orders. A new social phenomenon is the lockdown fatigue that pushed re-opening of the economies after lockdown (Caulkins 2021). These and other new findings argue in favor of more detailed models of government lockdowns than the direct control of the variable transmission rate as in (Alvarez et al. 2020; Atkeson 2020) and our model.

More evidence appears that there is no permanent immunity, which is the cornerstone of most economic-environmental models of controlled COVID. Correspondingly, some related processes require more modeling attention than in the past while others (notably, permanent immunity) become doubtful. A progress in research is always built on previous achievements. A potential extension of this paper is to modify the permanent immunity assumption in existing models and see how it will affect the modeling outcomes, such as endemic lockdown.

Finally, travel restrictions, population density, medical infrastructure, vaccine availability, and other social and environmental factors also play a significant role. So, economic optimization with epidemic models may be useful for theoretic insights but it should be supported by data-driven practical evidence in order to provide reliable numbers for government decisions.

6 Appendix

6.1 Proof of Lemma 1:

To derive a necessary extremum condition for the problem (1)–(5), (11), (12) we use the method of Lagrange multipliers adjusted to the integral dynamic models with finite memory in (Boucekkine et al. 2014; Hritonenko and Yatsenko 2013; Hritonenko et al. 2017). After excluding the variable $R(t)$ from the state Eqs. (1)–(3), the Lagrange functional for the problem (1)–(5), (11), (12) can be chosen as

$$\Lambda = \int_0^T e^{-rt} (AL(t)(S(t) + I(t)) + \chi \delta(t) (1 - S(t) - I(t))) dt$$

$$+ \int_0^T e^{-r(t)} \lambda(t) \left( I(t) - \int_{t-d}^t \beta_0(1 - \theta L(u))^2 I(u) S(u) K(t-u) du \right) dt$$
\begin{align}
  &+ \int_0^T e^{-rt} \mu(t) \left( S(t) - 1 + \int_{t-d}^t \beta_0(1 - \theta L(u))^2 I(u) S(u) \tilde{\beta}(t - u) du \right) dt,
  
  \tag{46}
\end{align}

where \( \lambda(t) \) and \( \mu(t) \) are the unknown Lagrange multipliers for equalities (6)–(7). Giving small variations \( \delta L(t), \delta I(t), \delta S(t), \delta \lambda(t), \) and \( \delta \mu(t) \) to the unknown variables and interchanging limits of integration in the double integrals, we obtain that the corresponding increment \( \delta L \) of functional \( L \) is

\begin{align}
  \Lambda &= \int_0^T e^{-rt} (A \delta L(t)(S(t) + I(t)) + (AL(t) - \chi \delta(t) (\delta S(t) + \delta I(t))) dt \\
  &+ \int_0^T e^{-rt} (\lambda(t) \delta I(t) + \mu(t) \delta S(t)) dt \\
  &+ \beta_0 \int_0^T \left( (1 - \theta L(u))^2 (I(u) \delta S(u) + S(u) \delta I(u)) - 2\theta (1 - \theta L(u)) I(u) S(u) \delta L(u) \right) \\
  &\times \int_{\min(u+d,T)}^{\min(u+T,T)} e^{-r(t)} \left( \tilde{\beta}(t - u) \mu(t) - K(t - u) \lambda(t) \right) dt du \\
  &+ o(\|\delta L\|, \|\delta S\|, \|\delta I\|, \|\delta \lambda\|, \|\delta \mu\|) \tag{47}
\end{align}

Combining terms with \( \delta L(.) \) in (47) gives the expression (18) for the gradient. Setting the sum of terms with \( \delta I(.) \) equal to zero gives the Eq. (19) for the dual variable \( \lambda \). The Eq. (20) for \( \mu \) is obtained similarly. Next, standard optimization reasoning leads to the presented form of maximum principle. The lemma is proven.

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