THE EXISTENCE AND EXPONENTIAL BEHAVIOR OF SOLUTIONS TO TIME FRACTIONAL STOCHASTIC DELAY EVOLUTION INCLUSIONS WITH NONLINEAR MULTIPLICATIVE NOISE AND FRACTIONAL NOISE

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Abstract. This article is devoted to study time fractional stochastic evolution inclusions with infinite delays driven by a nonlinear multiplicative noise and a fractional Brownian motion with Hurst parameter $H \in \left( \frac{1}{2}, 1 \right)$. First of all, we investigate the local and global existence of mild solutions to such evolution inclusions by using the fractional resolvent operator theory and some new results on the measure of noncompactness for the stochastic integral term. Further, we prove that every mild solution decays exponentially to zero in the sense of mean-square topology.

1. Introduction. The study connected with stochastic partial differential inclusions with or without delays arises from the theory of stochastic controlled dynamical systems. In recent years, stochastic differential equations or inclusions have been extensively studied; see for example, [6, 7, 13, 21, 23, 34, 36]. More general inclusions, also with multi-valued diffusion terms, were considered in [1]. The connections between stochastic differential inclusions and set-valued stochastic differential equations have been investigated in [28].

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Fractional differential equations arise naturally in a wide variety of applications such as physics, fluid mechanics, viscoelasticity, heat conduction in material with memory, chemistry and engineering. It is worth mentioning that fractional differential equations with the tempered fractional derivative also appear in the transport equation of describing the time evolution of the PDF of a Lévy walk, which is a CTRW model with the spatiotemporal coupled PDFs of waiting time and jump length [15, 38]. Fractional partial differential equations were investigated in recent decades, see, e.g., [2, 3, 9, 10, 12, 20, 33, 40] and the references therein. The Cauchy problem associated with integro-differential inclusions of diffusion-wave type involving infinite delays has been studied in [31]. Fractional integro-differential inclusions with state-dependent delay have been considered in [5] and [41].

In this paper, we consider the following fractional stochastic evolution inclusions with infinite delay

\[
\begin{cases}
\frac{d}{dt}\mathcal{D}_t^\alpha y(t) \in A y(t) + F(t, y_t) + G(t, y_t) \frac{d\mathbb{B}_Q(t)}{dt} + h(t) \frac{d\mathbb{B}_Q^H(t)}{dt}, & t \geq 0, \quad \frac{1}{2} < \alpha < 1, \\
y(t) = \phi(t), & t \in (-\infty, 0],
\end{cases}
\]

(1.1)

where \(\mathcal{D}_t^\alpha\) is the left Caputo fractional derivative, and

\[
\begin{cases}
\frac{d}{dt}\mathcal{D}_t^{\alpha,\nu} y(t) \in A y(t) + F(t, y_t) + G(t, y_t) \frac{d\mathbb{B}_Q(t)}{dt} + h(t) \frac{d\mathbb{B}_Q^H(t)}{dt}, & t \geq 0, \quad \frac{1}{2} < \alpha < 1, \\
y(t) = \varphi(t), & t \in (-\infty, 0],
\end{cases}
\]

(1.2)

where \(\mathcal{D}_t^{\alpha,\nu}\) is the left Caputo tempered fractional derivative with \(\nu > 0\), \(\phi(\cdot)\) takes the values in the separable real Hilbert space \(\mathcal{H}\), \(y_t\) stands for the history of the state function up to time \(t\), i.e. \(y_t(s) = y(t+s)\) for \(s \in (-\infty, 0]\), \(A : D(A) \subset \mathcal{H} \to \mathcal{H}\) is a closed linear operator and \(A\) generates a uniformly bounded \(C_0\)-semigroup \(T(t)\) \((t \geq 0)\) in \(\mathcal{H}\). This means that there exists \(M \geq 1\) such that

\[M := \sup_{t \in [0, \infty)} \|T(t)\|_{\mathcal{L}(\mathcal{H})} < \infty,\]

where \(\|\cdot\|_{\mathcal{L}(\mathcal{H})}\) stands for the operator norm of bounded linear operators on \(\mathcal{H}\). Let \(\mathcal{W}\) be another separable real Hilbert space, and let \(\{\mathbb{B}_Q(t) : t \geq 0\}\) and \(\{\mathbb{B}_Q^H(t) : t \geq 0\}\), respectively, be cylindrical \(\mathcal{W}\)-valued Brownian motion and fractional Brownian motion with a finite trace nuclear covariance operator \(Q \geq 0\) defined on a filtered complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).

Stochastic functional differential inclusions or equations driven by a fractional Brownian motion have been investigated by [6] and [7]. Fractional partial differential equations driven by a fractional Brownian motion or space-time white noise have been studied in [22] and [32] (see [30] for asymptotic behavior of linear fractional stochastic differential equations as well). Approximate controllability of fractional stochastic inclusions with or without delay has been established in [4, 16, 17, 35]. Existence of mild solutions for fractional partial neutral stochastic integro-differential inclusions has been proved in [11] (see [3] for the state-dependent delay case as well). The existence and asymptotic stability of mild solutions for fractional stochastic differential inclusions were presented in [39].

The purpose of this paper is to investigate the local and global existence of mild solutions to the time fractional stochastic delay evolution inclusion (1.1) and to study the exponential asymptotic behavior of mild solutions to the time tempered fractional stochastic delay evolution inclusion (1.2) in the sense of mean-square...
topology. The novelty and the difficulties of this work are in three aspects: (i) The multi-valued nonlinear multiplicative noise term. In order to obtain the existence of local mild solutions, we establish new results on the measure of noncompactness for the stochastic integral term. It is worth to mention that the idea here is also applicable to stochastic partial differential inclusions with classical derivative. In addition, some new ideas for checking the upper semicontinuity of the multi-valued operator $F$ (see details in Section 3.1) are developed here to circumvent the difficulty caused by the multi-valued nonlinear multiplicative noise term. (ii) The time fractional derivative. The local existence result of deterministic functional partial differential inclusions with classical derivative given in [19] is developed by using the fractional resolvent operator theory [10, 12, 37, 40]. We also extend the global existence results for deterministic single-valued fractional partial differential equations given in [9] by using the general theory of multi-valued mappings and stochastic partial differential equations. (iii) The exponential asymptotic behavior of mild solutions to problem (1.2). Here we consider the time tempered fractional stochastic delay evolution inclusions.

The contents of this paper are as follows. In Section 2, we recall some basic definitions and preliminary results that are useful throughout the paper. In Section 3, we prove the local and global existence of mild solutions for problem (1.1) by using the fractional resolvent operator theory and some new results on the measure of noncompactness for the stochastic integral term. The exponential asymptotic behavior of mild solutions for problem (1.2) in the mean-square sense is investigated in Section 4.

2. Preliminaries.

2.1. Brownian motion and fractional Brownian motion. In this subsection, we introduce the fractional Brownian motion as well as the Wiener integral with respect to it; for more details, we refer to [7, 27, 29]. We denote by $\mathcal{H}$ a separable real Hilbert space with the inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. Let $\mathcal{U}$ be another separable real Hilbert space and $\mathcal{L}(\mathcal{U}, \mathcal{H})$ be the space of bounded linear operators from $\mathcal{U}$ into $\mathcal{H}$. Particularly, we denote by $\mathcal{L}(\mathcal{U}) = \mathcal{L}(\mathcal{U}, \mathcal{U})$. For convenience, we will use the same notation $\| \cdot \|$ to denote the norms in $\mathcal{U}$ and $\mathcal{L}(\mathcal{U}, \mathcal{H})$, and use $(\cdot, \cdot)$ to denote the inner product of $\mathcal{U}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$.

Definition 1. The two-sided one-dimensional fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a Gaussian process $\beta^H = \{\beta^H(t), t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, having the properties

(i) $\beta^H_0 = 0$,
(ii) $\mathbb{E}\beta^H_0 = 0$, $t \in \mathbb{R}$,
(iii) $\mathbb{E}\beta^H_t \beta^H_s = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, $t, s \in \mathbb{R}$.

Remark 2. For $H = \frac{1}{2}$, we set $\beta^H(t) = \beta(t)$, where $\beta$ is a standard Brownian motion, in this case the increments of the process are independent. On the contrary, for $H \neq \frac{1}{2}$ the increments are not independent.

We assume that there exists a complete orthonormal basis $\{e_k\}_{k \geq 1}$ in $\mathcal{U}$, and that $\mathbb{B}_Q = \{\mathbb{B}_Q(t)\}_{t \geq 0}$ and $\mathbb{B}^H_Q = \{\mathbb{B}^H_Q(t)\}_{t \geq 0}$, respectively, are cylindrical $\mathcal{U}$-valued Brownian motion and fractional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. 
fractional Brownian motions mutually independent on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) such that

\[
\mathbb{B}_Q(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \geq 0,
\]

and

\[
\mathbb{B}^H_2(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta^H_k(t) e_k, \quad t \geq 0.
\]

For \(\varphi, \psi \in \mathcal{L}(\mathcal{W}, \mathcal{H})\), we define \((\varphi, \psi)_Q = Tr(\varphi Q \psi^*)\), where \(\psi^*\) is the adjoint of the operator \(\psi\). Then we see that for any bounded operator \(\psi \in \mathcal{L}(\mathcal{W}, \mathcal{H})\),

\[
||\psi||_Q^2 = Tr(\psi Q \psi^*) = \sum_{k=1}^{\infty} ||\lambda_k e_k||_2^2.
\]

If \(||\psi||_Q^2 < \infty\), then \(\psi\) is called a \(Q\)-Hilbert-Schmidt operator. Denote by \(\mathcal{L}_Q^0(\mathcal{W}, \mathcal{H})\) the space of all \(\psi \in \mathcal{L}(\mathcal{W}, \mathcal{H})\) such that \(\psi\) is a \(Q\)-Hilbert-Schmidt operator equipped with the norm \(||\cdot||_Q\). Then by Lemma 2 in [7] and Proposition 2.8 in [13], we have the following properties.

**Lemma 3.** If \(f \in L^2(\Omega \times [0, T]; \mathcal{L}_Q^0(\mathcal{W}, \mathcal{H}))\) and \(\psi : [0, T] \rightarrow \mathcal{L}_Q^0(\mathcal{W}, \mathcal{H})\) satisfies

\[
\int_0^T ||\psi(s)||_Q^2 ds < \infty,
\]

then we have

\[
\mathbb{E} \left\| \int_0^t f(s) d\mathbb{B}_Q(s) \right\|^2 \leq \int_0^t \mathbb{E} \|f(s)||_Q^2 ds,
\]

and

\[
\mathbb{E} \left\| \int_0^t \psi(s) d\mathbb{B}^H_2(s) \right\|^2 \leq 2H^2 t^{2H-1} \int_0^t \|\psi(s)||_Q^2 ds.
\]

### 2.2. Fractional setting

The purpose of this subsection is to introduce some definitions and preliminary facts from the fractional calculus.

Let \(X\) be a Banach space with the norm \(\|\cdot\|\). Now, assume that \(T > 0\). We denote by \(C([0, T]; X)\) the Banach space of all continuous \(X\)-valued functions on \([0, T]\) equipped with its usual norm. For \(1 \leq p < \infty\), \(L^p([0, T]; X)\) denotes the Banach space of \(L^p\) integrable function \(u : [0, T] \rightarrow X\). \(W^{1,p}([0, T]; X)\) is the subspace of \(L^p([0, T]; X)\) consisting of functions such that the weak derivative \(u_t\) belongs to \(L^p([0, T]; X)\). Both spaces \(L^p([0, T]; X)\) and \(W^{1,p}([0, T]; X)\) are endowed with their standard norm. In the sequel, \(C\) denotes an arbitrary positive constant, which may be different from line to line and even in the same line.

For \(\alpha > 0\), define the function \(g_\alpha : \mathbb{R} \rightarrow \mathbb{R}\) by

\[
g_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}
\]

where \(\Gamma(\alpha)\) is the Euler Gamma function. For a function \(u \in L^1([0, T]; X)\), the left Riemann-Liouville fractional integral of order \(\alpha\) of \(u\) is given by

\[
o^\mu_0 J^\mu u(t) := g_\alpha \ast u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [0, T].
\]

Observe that the previous identification associates the properties of convolution with the definition of fractional integral operator.
Thus, based on the definition of left Riemann-Liouville fractional integral operator, we present the left Caputo fractional differential operator; for more details, we refer to the books [20, 33].

**Definition 4.** Let \( \alpha \in (0,1) \) and \( T > 0 \). Consider \( u \in C([0,T];X) \) such that the convolution \( g_{{\alpha}} u := u \ast \mathcal{J}_t^{{\alpha}}u \) is in \( W^{1,1}([0,T];X) \). The expression

\[
\mathcal{D}_t^{{\alpha}}u(t) := \frac{d}{dt} \left\{ \mathcal{J}_t^{{1-\alpha}}[u(t) - u(0)] \right\} = \frac{d}{dt} \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}[u(s) - u(0)]ds \right\}
\]

is called the left Caputo fractional derivative of order \( \alpha \) of the function \( u \).

If \( u \) is an abstract function with values in \( X \), then the integral which appears in Definition 4 is taken in Bochner’s sense. A measurable function \( u : [0, \infty) \to X \) is Bochner integrable if \( \|u\| \) is Lebesgue integrable.

In what follows, let us state some properties of the special function \( M_\alpha \) also called Mainardi function [9, 10, 26, 40]. This function is a particular case of the Wright type function introduced by Mainardi in [25] in order to characterize the fundamental solutions for some standard boundary value problems in physics. More precisely, for \( \alpha \in (0,1) \), the entire function \( M_\alpha : \mathbb{C} \to \mathbb{C} \) is given by

\[
M_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{n\Gamma(1-\alpha(1+n))}.
\]

**Proposition 5.** For \( \alpha \in (0,1) \) and \(-1 < r < \infty\), when we restrict \( M_\alpha \) to the positive real line, it holds that

\[
M_\alpha(t) \geq 0 \text{ for all } t \geq 0, \quad \text{and} \quad \int_0^\infty t^r M_\alpha(t)dt = \frac{\Gamma(r+1)}{\Gamma(\alpha r + 1)}.
\]

In the sequel, we introduce the Mittag-Leffler operators. To this end, suppose that \( A : D(A) \subset X \to X \) is a closed linear operator and \( A \) generates a uniformly bounded \( C_0 \)-semigroup \( \{T(t) : t \geq 0\} \) in \( X \). This means that there exists \( M \geq 1 \) such that \( M := \sup_{t \in [0, \infty)} \|T(t)\|_{\mathcal{L}(X)} < \infty \), where \( \mathcal{L}(X) \) stands for the Banach space of all linear and bounded operators on \( X \). Then, for each \( \alpha \in (0,1) \), we define the Mittag-Leffler families \( \{E_\alpha(t^\alpha A) : t \geq 0\} \) and \( \{E_{\alpha,\alpha}(t^\alpha A) : t \geq 0\} \) by

\[
E_\alpha(t^\alpha A) = \int_0^\infty M_\alpha(s)T(st^\alpha)ds,
\]

and

\[
E_{\alpha,\alpha}(t^\alpha A) = \int_0^\infty s \alpha M_\alpha(s)T(st^\alpha)ds.
\]

It is interesting to notice that the Mainardi functions act as a bridge between the fractional and the classical abstract theories; for more details see [10, 12, 37, 40]. The following lemma will be used throughout this paper.

**Lemma 6.** The operators \( E_\alpha(t^\alpha A) \) and \( E_{\alpha,\alpha}(t^\alpha A) \) are well defined from \( X \) to \( X \). Moreover, it holds

(i) For every \( x \in X \), \( E_\alpha(t^\alpha A)x |_{t=0} = x \) and \( E_{\alpha,\alpha}(t^\alpha A)x |_{t=0} = x \).

(ii) For every \( x \in X \), the functions \( t \to E_\alpha(t^\alpha A)x \) and \( t \to E_{\alpha,\alpha}(t^\alpha A)x \) are continuous from \([0, \infty)\) to \( X \).

(iii) For any fixed \( t \geq 0 \), \( E_\alpha(t^\alpha A) \) and \( E_{\alpha,\alpha}(t^\alpha A) \) are linear and bounded operators, i.e., for any \( x \in X \),

\[
\|E_\alpha(t^\alpha A)x\| \leq M\|x\|, \quad \|E_{\alpha,\alpha}(t^\alpha A)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\| = \frac{M}{\Gamma(\alpha)}\|x\|,
\]
where $M$ is a positive constant independent of $t \in [0, \infty)$.

(iv) The operators $E_\alpha(t^nA)(t \geq 0)$ and $E_{\alpha, \alpha}(t^n\alpha A)(t \geq 0)$ are strongly continuous, which means that for any $x \in X$ and $0 \leq t_1 < t_2 \leq T$, we have

$$|E_\alpha(t^n_2A)x - E_\alpha(t^n_1A)x| \to 0 \quad \text{and} \quad \|E_{\alpha, \alpha}(t^n_2\alpha A)x - E_{\alpha, \alpha}(t^n_1\alpha A)x\| \to 0 \quad \text{as} \quad t_2 - t_1 \to 0.$$

(v) If the semigroup $T(t)$ is compact, then for every $t > 0$, $E_\alpha(t^nA)$ and $E_{\alpha, \alpha}(t^n\alpha A)$ are also compact operators.

(vi) If the semigroup $T(t)$ is continuous by operator norm for every $t > 0$, then $E_\alpha(t^nA)$ and $E_{\alpha, \alpha}(t^n\alpha A)$ are continuous in $(0, \infty)$ by the operator norm.

(vii) For each fixed $x \in X$, if the semigroup $T(t)$ is analytic, then the functions $t \to E_\alpha(t^nA)x$ and $t \to E_{\alpha, \alpha}(t^n\alpha A)x$ are analytic from $[0, \infty)$ to $X$, and satisfies

$$\int_0^t E_\alpha(t^nA)x = AE_\alpha(t^nA)x, \quad t > 0.$$

2.3. Hausdorff measure of noncompactness and multi-valued mappings.

Now let us recapitulate the standard definition of the Hausdorff measure of noncompactness and its basic properties; see [18, 19, 24] for more details.

Let $X$ be a Banach space and $B(X)$ be the collection of all nonempty and bounded subsets of $X$.

**Definition 7.** A function $\chi : B(X) \to \mathbb{R}^+$ is called the Hausdorff measure of noncompactness (MNC) on $X$ if for every $\Xi \in B(X)$,

$$\chi(\Xi) = \inf \{\varepsilon : \Xi \text{ has a finite } \varepsilon \text{-net}\}.$$

**Lemma 8.** Let $X$ be a Banach space and $\chi$ be the Hausdorff MNC on $X$. Then

(i) if for each $\Xi_0, \Xi_1 \in B(X)$ such that $\Xi_0 \subset \Xi_1$, then $\chi(\Xi_0) \leq \chi(\Xi_1)$;

(ii) $\chi(\{a\} \cup \Xi) = \chi(\Xi)$ for any $a \in X, \Xi \in B(X)$;

(iii) $\chi(K \cup \Xi) = \chi(\Xi)$ for every relatively compact set $K \subset X, \Xi \in B(X)$;

(iv) $\chi(\Xi_0 + \Xi_1) \leq \chi(\Xi_0) + \chi(\Xi_1)$ for any $\Xi_0, \Xi_1 \in B(X)$;

(v) $\chi(\Xi) = 0$ is equivalent to the relative compactness of $\Xi$;

(vi) $\chi(\Xi) = \chi(\Xi)$, where $\Xi$ is the closure of $\Xi$;

(vii) $\chi(\operatorname{conv}\Xi) = \chi(\Xi)$, where $\operatorname{conv}\Xi$ is the closed convex hull of $\Xi$;

(viii) if $\Xi_0$ is a family of nonempty, closed and bounded sets defined for $t > r$ that satisfy $\Xi_t \subset \Xi_s$ whenever $s \leq t$ and $\chi(\Xi_t) \to 0$ as $t \to \infty$, then $\cap_{t > r} \Xi_t$ is a nonempty, compact set in $X$.

**Proposition 9.** Let $\chi$ be the Hausdorff MNC on $X$ and $\Xi \subset X$ be a bounded set. Then for every $\varepsilon > 0$, there exists a sequence $\{x_n\} \subset \Xi$ such that $\chi(\Xi) \leq 2\chi(\{x_n\}) + \varepsilon$.

**Proposition 10.** (see [19, Proposition 2.5].) Let $\Xi \subset L^1([0, T]; X)$ be such that

1. $|\xi(t)| \leq \vartheta(t)$ for all $\xi \in \Xi$ and for a.e. $t \in [0, T]$,
2. $\chi(\Xi(t)) \leq q(t)$ for a.e. $t \in [0, T]$,

where $\vartheta, q \in L^1([0, T])$. Then for all $t \in [0, T]$,

$$\chi\left(\int_0^t \Xi(s)ds\right) \leq 4 \int_0^t q(s)ds,$$

where $\int_0^t \Xi(s)ds = \{\int_0^t \xi(s)ds : \xi \in \Xi\}$.

We now recall some notions of set-valued analysis and a fixed point theorem for multi-valued mappings.
Let $X$, $Y$ be two metric spaces. In the whole paper we denote by $F : Y \to X$ a multi-valued mapping from $Y$ to $X$ with domain $Y$ and values being nonempty subsets of $X$.

**Definition 11.** A multi-valued mapping $F : Y \to X$ is said to be

(i) upper semicontinuous (u.s.c.) if $F^{-1}(V) := \{ y \in Y : F(y) \cap V \neq \emptyset \}$ is a closed subset of $Y$ for every closed set $V \subseteq X$;

(ii) closed if its graph $\Gamma_F := \{(y, z) : z \in F(y)\}$ is a closed subset of $Y \times X$;

(iii) compact if its range $F(Y)$ is relatively compact in $X$;

(iv) locally compact if every point $y \in Y$ has a neighborhood $V(y)$ such that the restriction of $F$ to $V(y)$ is compact.

**Definition 12.** A sequence $\{f_n\} \subset L^1([0, T]; X)$ is said to be semicompact if it is integrably bounded and $\{f_n(t)\} \subset \mathcal{K}(t)$ for a.e. $t \in [0, T]$, where $\mathcal{K}(t) \subseteq X$, $t \in [0, T]$, is a family of compact sets.

The following results will be used.

**Theorem 13.** (see [1, Theorem 1.1.5].) Let $F : Y \to X$ be a closed locally compact multi-valued mapping. Then $F$ is u.s.c.

**Proposition 14.** (see [14, Proposition 1.1].) Let $X$ be a Banach space and $\Xi$ be a nonempty subset of another Banach space. Assume that $F : \Xi \to X$ is a multi-valued mapping with compact values. Then $F$ is u.s.c. if and only if $\{x_n\}_{n=1}^{\infty} \subset \Xi$ with $x_n \to x_0 \in \Xi$ and $y_n \in F(x_n)$ implies $y_n \to y_0 \in F(x_0)$ up to a subsequence.

**Theorem 15.** (see [19, Lemma 3.2].) Let $X$ be a Banach space and $\Xi \subseteq X$ be a nonempty compact convex set. If the multi-valued operator $\mathcal{F} : \Xi \to \Xi$ is u.s.c. with closed convex values, then $\mathcal{F}$ has a fixed point.

2.4. Phase spaces and the Gronwall-Bellman type inequalities. Let $X$ be a Banach space. The collection of all strongly-measurable, square-integrable $X$-valued random variables, denoted $L^2(\Omega; X)$, is a Banach space equipped with the norm $(\mathbb{E}\|y\|^2)^{\frac{1}{2}}$, where the expectation $\mathbb{E}$ is defined by $\mathbb{E}y = \int_{\Omega} y(\omega)d\mathbb{P}$. We denote by $C([a, b]; L^2(\Omega; X))$ the Banach space of all continuous functions from $[a, b]$ into $L^2(\Omega; X)$ with the norm $(\sup_{t\in[a,b]} \mathbb{E}\|y(t)\|^2)^{\frac{1}{2}}$. In the following, we will use the spaces $L^2(\Omega; \mathcal{H})$, $L^2(\Omega; L^0_c(\mathcal{U}, \mathcal{H}))$, $C([a, b]; L^2(\Omega; \mathcal{H}))$ and $C([a, b]; L^2(\Omega; L^0_c(\mathcal{U}, \mathcal{H})))$ in our analysis, where $\mathcal{H}$ and $L^0_c(\mathcal{U}, \mathcal{H})$ are given in subsection 2.1. We define the abstract phase space $\mathcal{C}^\gamma$ by

$$
\mathcal{C}^\gamma = \left\{ \psi \in C((-\infty, 0]; L^2(\Omega; \mathcal{H})) : \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\psi(\theta)\|^2 < \infty \right\},
$$

where the parameter $\gamma > 0$. If $\mathcal{C}^\gamma$ is endowed with the norm

$$
\|\psi\|_{\mathcal{C}^\gamma} = \left( \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} \mathbb{E}\|\psi(\theta)\|^2 \right)^{\frac{1}{2}}, \quad \psi \in \mathcal{C}^\gamma,
$$

then $(\mathcal{C}^\gamma, \| \cdot \|_{\mathcal{C}^\gamma})$ is a Banach space.

We are now in a position to recall the Gronwall-Bellman type inequalities.

**Lemma 16.** Let $u(t)$ and $n(t)$ be real valued continuous functions for $t \geq 0$, and let $n(t)$ be nonnegative for $t \geq 0$. If $m \geq 0$ is a constant, $u(t)$ is nonnegative and
satisfies the integral inequality
\[ u(t) \leq m + \int_0^t n(s)u(s)ds, \quad t \geq 0, \]
then
\[ u(t) \leq m \exp\left( \int_0^t n(s)ds \right) \quad \text{for} \quad t \geq 0. \]

If the negative part of the real valued function \( m(t) \) is integrable on every closed and bounded subinterval of \([0, \infty)\) and \( u(t) \) satisfies the integral inequality
\[ u(t) \leq m(t) + \int_0^t n(s)u(s)ds, \quad t \geq 0, \]
then
\[ u(t) \leq m(t) + \int_0^t m(s)n(s)\exp\left( \int_s^t n(\tau)d\tau \right)ds \quad \text{for} \quad t \geq 0. \]

If, in addition, the function \( m(t) \) is nondecreasing, then
\[ u(t) \leq m(t)\exp\left( \int_0^t n(s)ds \right) \quad \text{for} \quad t \geq 0. \]

3. Existence of mild solutions. For \( \phi \in \mathcal{C}_\gamma \), we define the space
\[ \mathcal{C}_\phi = \{ v \in C([0, T]; \mathcal{L}^2(\Omega; \mathcal{H})) : v(0) = \phi(0) \}. \]
Then for any \( v \in \mathcal{C}_\phi \), we denote by \( v[\phi] \) the mapping from \(( -\infty, T] \) to \( \mathcal{L}^2(\Omega; \mathcal{H}) \)
defined by
\[ v[\phi](t) = \begin{cases} \phi(t), & -\infty < t \leq 0, \\ v(t), & t \in [0, T]. \end{cases} \]
Thus,
\[ v[\phi](\theta) = \begin{cases} \phi(t + \theta), & -\infty < \theta \leq -t, \\ v(t + \theta), & \theta \in [-t, 0]. \end{cases} \]

For \( v \in \mathcal{C}_\phi \), let us define \( \mathcal{P}_F(v) \) and \( \mathcal{P}_G(v) \) by setting
\[ \mathcal{P}_F(v) = \{ f \in L^2([0, T]; \mathcal{L}^2(\Omega; \mathcal{H})) : f(t) \in F(t, v[\phi], t) \text{ for a.e. } t \in [0, T] \}, \]
and
\[ \mathcal{P}_G(v) = \{ g \in L^2([0, T]; \mathcal{L}^2(\Omega; \mathcal{L}^0_0(\mathcal{H}, \mathcal{H}))) : g(t) \in G(t, v[\phi], t) \text{ for a.e. } t \in [0, T] \}. \]

By the arguments in [10, 12, 40] and references therein, the notion of mild solutions to problem (1.1) is given by a fractional variation of constants formula which involves the Mittag-Leffler families.

**Definition 17.** A measurable and \( \mathcal{F}_t \)-adapted stochastic process \( y : (-\infty, T] \rightarrow \mathcal{H} \)
is called a mild solution of (1.1), if \( y \in C(( -\infty, T]; \mathcal{L}^2(\Omega; \mathcal{H})) \), \( y(t) = \phi(t) \) for \( t \in ( -\infty, 0] \) with \( \phi \in \mathcal{C}_\gamma \), and, for \( t \in [0, T] \), there exist \( f \in \mathcal{P}_F(y) \) and \( g \in \mathcal{P}_G(y) \) such that the following integral equation holds
\[
y(t) = E_t(t^\alpha A)\phi(0) + \int_0^t (t-s)^{\alpha-1}E_{t-s}(t^\alpha A)f(s)ds
\]
\[
+ \int_0^t (t-s)^{\alpha-1}E_{t-s}(t^\alpha A)g(s)d\mathbb{B}_Q(s)
\]
\[
+ \int_0^t (t-s)^{\alpha-1}E_{t-s}(t^\alpha A)h(s)d\mathbb{B}_Q^H(s) \quad \text{P-a.s.} \tag{3.1}
\]
Now we establish some important results which will be used in the proof of our main theorems. First, we need the following useful result.

**Theorem 18.** (see [18, Theorem 4.2.1].) Let $X$ be a Banach space, and let the sequence of functions $\{f_n\} \subset L^1([0, T]; X)$ be integrably bounded:

$$|f_n(t)| \leq \vartheta(t), \quad \text{for all } n = 1, 2, \ldots \text{ and a.e. } t \in [0, T],$$

where $\vartheta \in L^1([0, T])$. Assume that

$$\chi(\{f_n(t)\}) \leq \nu(t)$$

for a.e. $t \in [0, T]$ where $\nu \in L^1([0, T])$. Then for every $\delta > 0$ there exist a compact set $K_\delta \subset X$, a set $m_\delta \subset [0, T]$, $\text{meas}(m_\delta) < \delta$ and a set of functions $G_\delta \subset L^1([0, T]; X)$ with values in $K_\delta$ such that for every $n \geq 1$ there exists $x_n \in G_\delta$ for which

$$\|f_n(t) - x_n(t)\| \leq 2\nu(t) + \delta, \quad t \in [0, T]\setminus m_\delta.$$

**Proposition 19.** Let $\mathcal{H}$ be a separable real Hilbert space, $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ be a closed linear operator, and $A$ generates a uniformly bounded $C_0$-semigroup $T(t)$ which is continuous by operator norm for every $t > 0$. If $\mathcal{X} \subset L^2([0, T]; L^2(\Omega; \mathcal{H}))$ is such that $E|x(t)|^2 \leq q(t)$, for all $x \in \mathcal{X}$ and a.e. $t \in [0, T]$, where $q \in L^p([0, T])$ with $p \in (\frac{2}{2n-1}, \infty)$, and $\Xi \subset L^2([0, T]; L^2(\Omega; L^0_Q(\mathcal{H}, \mathcal{H})))$ is such that $E|\xi(t)|^2 \leq \vartheta(t)$, for all $\xi \in \Xi$ and a.e. $t \in [0, T]$, where $\vartheta \in L^p([0, T])$ with $p \in (\frac{2}{2n-1}, \infty)$, then

(i) the operator $W$ defined by

$$W(f)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds$$

maps $\mathcal{X} \subset L^2([0, T]; L^2(\Omega; \mathcal{H}))$ into a equicontinuous set in $C([0, T]; L^2(\Omega; \mathcal{H}))$, and the operator $V$ defined by

$$V(g)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathbb{B}_Q(s)$$

maps $\Xi \subset L^2([0, T]; L^2(\Omega; L^0_Q(\mathcal{H}, \mathcal{H})))$ into a equicontinuous set in $C([0, T]; L^2(\Omega; \mathcal{H}))$;

(ii) moreover if $\{f_n\} \subset \mathcal{X}$ and $\{g_n\} \subset \Xi$ are semicompact sequences, then $\{W(f_n)\}$ and $\{V(g_n)\}$ are relatively compact in $C([0, T]; L^2(\Omega; \mathcal{H}))$.

**Proof.** Since the proof for the operator $W$ is similar, here we only need to verify the properties of $V$.

(i) Note that $E|x(t)|^2 \leq \vartheta(t)$ for all $x \in \Xi$ and a.e. $t \in [0, T]$. Then for any $x(\cdot) \in \Xi$ and $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, it follows from (2.1) and Lemma 6 (iii) that

$$E|V(x)(t_2) - V(x)(t_1)|^2$$

$$= E\left[ \int_0^{t_2} (t_2-s)^{\alpha-1} E_{\alpha,\alpha}((t_2-s)^\alpha A)x(s)d\mathbb{B}_Q(s) \right.$$}

$$- \left. \int_0^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}((t_1-s)^\alpha A)x(s)d\mathbb{B}_Q(s) \right]^2$$

$$\leq 3E\left[ \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) E_{\alpha,\alpha}((t_2-s)^\alpha A)x(s)d\mathbb{B}_Q(s) \right]^2$$

$$+ 3E\left[ \int_0^{t_1} (t_1-s)^{\alpha-1} (E_{\alpha,\alpha}((t_2-s)^\alpha A)x(s) - E_{\alpha,\alpha}((t_1-s)^\alpha A)x(s))d\mathbb{B}_Q(s) \right]^2$$
\[ + 3E \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} E_{\alpha,\alpha}((t_2 - s)^\alpha A)x(s)dE_Q(s) \right\|_Q^2 \]
\[ \leq 3 \int_0^{t_1} ((t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1})^2 E \left\| E_{\alpha,\alpha}((t_2 - s)^\alpha A)x(s) \right\|_Q^2 ds \]
\[ + 3 \int_0^{t_1} (t_1 - s)^{2\alpha - 2} E \left\| E_{\alpha,\alpha}((t_2 - s)^\alpha A)x(s) - E_{\alpha,\alpha}((t_1 - s)^\alpha A)x(s) \right\|_Q^2 ds \]
\[ + 3 \int_{t_1}^{t_2} (t_2 - s)^{2\alpha - 2} E \left\| E_{\alpha,\alpha}((t_2 - s)^\alpha A)x(s) \right\|_Q^2 ds \]
\[ \leq \frac{3M^2}{(\Gamma(\alpha))^2} \int_0^{t_1} ((t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1})^2 \vartheta(s) ds \]
\[ + 3 \int_0^{t_1} (t_1 - s)^{2\alpha - 2} \left\| E_{\alpha,\alpha}((t_2 - s)^\alpha A)x(s) - E_{\alpha,\alpha}((t_1 - s)^\alpha A) \right\|^2 \vartheta(s) ds \]
\[ + \frac{3M^2}{(\Gamma(\alpha))^2} \int_{t_1}^{t_2} (t_2 - s)^{2\alpha - 2} \vartheta(s) ds. \tag{3.4} \]

For \( t_1 = 0 \), by Hölder’s inequality, we find that
\[ E \left\| \mathcal{V}(x)(t_2) - \mathcal{V}(x)(t_1) \right\|^2 \leq \frac{3M^2}{(\Gamma(\alpha))^2} \int_0^{t_2} (t_2 - s)^{2\alpha - 2} \vartheta(s) ds \]
\[ \leq \frac{3M^2}{(\Gamma(\alpha))^2} \left( \int_0^{t_2} (t_2 - s)^{\frac{2\alpha - 2)p}{p-1}} ds \right)^{\frac{p-1}{p}} \left( \int_0^{t_2} \vartheta^p(s) ds \right)^{\frac{1}{p}} \]
\[ \leq \frac{3M^2}{(\Gamma(\alpha))^2} \left( \frac{p - 1}{2\alpha p - p - 1} \right)^{\frac{p-1}{p}} \left\| \vartheta \right\|_{L^p}^{2\alpha - 1 - \frac{1}{p}}, \tag{3.5} \]

where \( \left\| \vartheta \right\|_p = \left( \int_0^{t_1} (\vartheta(s))^p ds \right)^{\frac{1}{p}} \).

For \( 0 < t_1 < T \), in view of \( \vartheta \in L^p([0, T]) \), then for any \( \varepsilon > 0 \) there exists
\[ 0 < \delta < \left( \frac{2\alpha p - p - 1}{p - 1} \right)^{\frac{p-1}{p}} \min \left\{ \left( \frac{(\Gamma(\alpha))^2 \sqrt{\varepsilon}}{32M^2} \right)^{\frac{p}{3(p-1)}} \left( \frac{(\Gamma(\alpha))^2 \varepsilon}{24M^2 \left\| \vartheta \right\|_p} \right)^{\frac{p}{3(p-1)}} \right\} \]
such that for any set \( c \subset [0, T] \) with \( \text{meas}(c) < \delta \), we have
\[ \int_c \vartheta^p(s) ds < \varepsilon^\frac{p}{2}. \]

Choosing \( \zeta \) such that \( 0 < \zeta < \min\{\delta, t_1\} \) and using the continuity of \( E_{\alpha,\alpha}(t^\alpha A) \)
in the sense of the operator norm for \( t > 0 \), we deduce that there exists \( \delta_0 \) with \( 0 < \delta_0 < \delta \) such that
\[ \sup_{s \in [\zeta, t_1]} \left\| E_{\alpha,\alpha}((t + s)^\alpha A) - E_{\alpha,\alpha}(s^\alpha A) \right\|^2 < \left( \frac{p-1}{2\alpha p - p - 1} \right)^{\frac{p-1}{p}} \varepsilon, \quad \text{for all} \ 0 \leq \tau \leq \delta_0. \]

Thus for \( 0 < t_2 - t_1 < \delta_0 \), we obtain from (3.4) and Hölder’s inequality that
\[ E \left\| \mathcal{V}(x)(t_2) - \mathcal{V}(x)(t_1) \right\|^2 \]
\[ \leq \frac{3M^2}{(\Gamma(\alpha))^2} \int_0^{t_1} ((t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1})^2 \vartheta(s) ds \]
\[ + 3 \int_0^{t_1 - \zeta} (t_1 - s)^{2\alpha - 2} \left\| E_{\alpha,\alpha}((t_2 - s)^\alpha A) - E_{\alpha,\alpha}((t_1 - s)^\alpha A) \right\|^2 \vartheta(s) ds \]
\[ + 3 \int_{t_1 - \zeta}^{t_1} (t_1 - s)^{2p-2} \|E_{\alpha,\alpha}(t_2 - s)^{\alpha}A - E_{\alpha,\alpha}(t_1 - s)^{\alpha}A\|^2 \, ds \\
+ \frac{3M^2}{(\Gamma(\alpha))^2} \int_{t_1 - \zeta}^{t_2} (t_2 - s)^{2p-2} \vartheta(s) \, ds \\
\leq \frac{3M^2}{(\Gamma(\alpha))^2} \left( \int_0^{t_1} \left( (t_1 - s)^{\alpha - 1} - (t_2 - s)^{\alpha - 1} \right)^{\frac{2p}{p-1}} \, ds \right)^{\frac{p}{p-1}} |\vartheta||_p \\
+ 3 \sup_{s \in [0, t_1 - \zeta]} \|E_{\alpha,\alpha}(t_2 - s)^{\alpha}A - E_{\alpha,\alpha}(t_1 - s)^{\alpha}A\|^2 \int_{t_1 - \zeta}^{t_1 - \zeta} (t_1 - s)^{2p-2} \vartheta(s) \, ds \\
+ \frac{12M^2}{(\Gamma(\alpha))^2} \int_{t_1 - \zeta}^{t_1} (t_1 - s)^{2p-2} \vartheta(s) \, ds \\
+ \frac{3M^2}{(\Gamma(\alpha))^2} \int_{t_1 - \zeta}^{t_1} (t_1 - s)^{2p-2} \vartheta(s) \, ds \\
\leq \frac{3M^2}{(\Gamma(\alpha))^2} |\vartheta||_p \left( \int_0^{t_1} \left( (t_1 - s)^{\alpha - \frac{1}{2p}} - (t_2 - s)^{\alpha - \frac{1}{2p}} \right)^{\frac{p}{p-1}} \, ds \right)^{\frac{p}{p-1}} \\
+ \left( \frac{p-1}{2p^2 - 2p - 1} \right)^{\frac{p}{p-1}} \varepsilon \left( \int_0^{t_1} (t_1 - s)^{\frac{2p^2 - 2p - 1}{2p - 1}} \vartheta(s) \, ds \right)^{\frac{p}{p-1}} |\vartheta||_p \\
+ \frac{12M^2}{(\Gamma(\alpha))^2} \left( \frac{p-1}{2p^2 - 2p - 1} \right)^{\frac{p}{p-1}} \varepsilon \left( \int_0^{t_1} \vartheta(s) \, ds \right)^{\frac{1}{p}} \\
+ \frac{3M^2}{(\Gamma(\alpha))^2} \left( \frac{p-1}{2p^2 - 2p - 1} \right)^{\frac{p}{p-1}} \varepsilon \left( \int_0^{t_1} \vartheta(s) \, ds \right)^{\frac{1}{p}} \\
\leq \frac{3M^2}{(\Gamma(\alpha))^2} |\vartheta||_p \left( \frac{p-1}{2p^2 - 2p - 1} \right)^{\frac{p}{p-1}} \varepsilon \left( \frac{t_2 - t_1}{2p^2 - 2p - 1} \right)^{\frac{p}{p-1}} + \varepsilon + \frac{3\varepsilon}{4} < \varepsilon, \quad (3.6) \]

thanks to \( a^\theta - b^\theta \leq (a - b)^\theta \) if \( \theta \in (0, 1) \), and \( a^\theta - b^\theta \geq (a - b)^\theta \) if \( \theta \geq 1 \) for \( a > b > 0 \).

Hence, (3.5) and (3.6) imply that \( E|\mathcal{V}(x)(t_2) - \mathcal{V}(x)(t_1)|^2 \) tends to zero independently of \( x(\cdot) \in \Xi \) as \( t_2 - t_1 \to 0 \).

(ii) We first prove that \( \{\mathcal{V}(g_n(t))\} \) is relatively compact in \( \mathcal{L}^2(\Omega; \mathcal{H}) \) for each fixed \( t \in [0, T] \). Noticing that

\[ E\|g_n(t)\|_Q^2 < \vartheta(t), \quad \text{for all } n = 1, 2, \ldots \text{ and a.e. } t \in [0, T], \]

and by \( \vartheta \in \mathcal{L}^p([0, T]) \), we conclude that for any \( \eta > 0 \), there exists \( \delta_1 \in (0, \frac{\sqrt{2\gamma - T(\alpha)}}{2M^2 - \frac{1}{2}} \sqrt{\eta}) \) such that for every set \( m \subset [0, T] \) with \( \text{meas}(m) < \delta_1 \),

\[ \int_m \vartheta^p(s) \, ds < \frac{(\Gamma(\alpha))^{2p}(\frac{2p^2 - 2p - 1}{2p^2 - 2p - 1})^{p-1}}{8pM^2T^{2p^2 - 2p - 1}} (\eta)^p. \]

Invoking Theorem 18, in view of (3.7) and the semicompactness of \( \{g_n\} \), hence there exist a set \( m_{\delta_1} \subset [0, T] \) with \( \text{meas}(m_{\delta_1}) < \delta_1 \), a compact set \( K_{\delta_1} \subset \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{H}, \mathcal{H})) \)

and a set of functions \( G_{\delta_1} \subset \Xi \) with values in \( K_{\delta_1} \) such that for every \( n \geq 1 \) there exists \( y_n \in G_{\delta_1} \) for which

\[ (E\|g_n(t) - y_n(t)\|_Q^2)^{\frac{1}{2}} \leq \delta_1, \quad t \in [0, T]\backslash m_{\delta_1}. \]

By (2.1), (3.8)-(3.9), Lemma 6 (iii) and Hölder’s inequality, we obtain

\[ E\|\mathcal{V}(g_n)(t) - \mathcal{V}(y_n)(t)\|^2 \]
\[\begin{align*}
&= \mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)(g_n(s) - y_n(s))d\mathbb{B}_Q(s) \right\|^2 \\
&\leq \int_0^t (t-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A)\|^2 \mathbb{E}\|g_n(s) - y_n(s)\|_Q^2 ds \\
&= \int_{[0,t]} (t-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A)\|^2 \mathbb{E}\|g_n(s) - y_n(s)\|_Q^2 ds \\
&\quad + \int_{m_{\delta_1}} (t-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A)\|^2 \mathbb{E}\|g_n(s) - y_n(s)\|_Q^2 ds \\
&\leq \frac{M^2\delta_1^2}{(1-\gamma/2)^2} \int_0^t (t-s)^{2\alpha-2} ds + \frac{4M^2}{(1-\gamma/2)^2} \int_{m_{\delta_1}} (t-s)^{2\alpha-2} \vartheta(s) ds \\
&\leq \frac{M^2T^{2\alpha-1}}{(2\alpha-1)(1-\gamma/2)^2} + \frac{4M^2}{(1-\gamma/2)^2} \left( \int_{m_{\delta_1}} (t-s)^{2\alpha-2} ds \right)^{\frac{n}{p}} \left( \int_{m_{\delta_1}} \vartheta(s) ds \right)^{\frac{n}{p}} \\
&\leq \eta.
\end{align*}\]

Therefore, \(\{V(g_n(t))\}\) belongs to a \(\sqrt{\eta}\)-net of the set \(V(K_{\delta_1})(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)K_{\delta_1}d\mathbb{B}_Q(s)\). In order to prove the relative compactness of \(\{V(g_n(t))\}\) in \(L^2(\Omega; \mathcal{H})\), let us consider the relative compactness of \(V(K_{\delta_1})(t)\) in \(L^2(\Omega; \mathcal{H})\). Let \(w_n \in V(K_{\delta_1})(t)\) be given arbitrarily. Then there exists \(z_n \in K_{\delta_1}\) such that

\[w_n = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)z_n d\mathbb{B}_Q(s)\]

Since \(K_{\delta_1}\) is compact in \(L^2(\Omega; \mathcal{L}_Q^0(\mathcal{Y}, \mathcal{H}))\), there exist a subsequence \(\{z_{n_k}\}\) and \(z \in K_{\delta_1}\) which is the limit of \(z_{n_k}\). Applying Lemma 6 (iii), we deduce that

\[\mathbb{E}\|V(z_{n_k})(t) - V(z)(t)\|^2 = \mathbb{E}\left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)(z_{n_k} - z) d\mathbb{B}_Q(s) \right\|^2 \leq \frac{M^2}{(1-\gamma/2)^2} \int_0^t (t-s)^{2\alpha-2} \mathbb{E}\|z_{n_k} - z\|_Q^2 ds \leq \frac{M^2T^{2\alpha-1}}{(2\alpha-1)(1-\gamma/2)^2} \mathbb{E}\|z_{n_k} - z\|_Q^2 \to 0\]

as \(k \to \infty\). Therefore, \(V(K_{\delta_1})(t)\) is relatively compact in \(L^2(\Omega; \mathcal{H})\). This implies that \(\{V(g_n(t))\}\) is relatively compact in \(L^2(\Omega; \mathcal{H})\).

By the argument of (i), we get the equicontinuity of \(\{V(g_n(\cdot))\}\). In accordance with the Arzelà-Ascoli theorem it follows that the sequence of functions \(\{V(g_n)\}\) is relatively compact in \(C([0,T]; L^2(\Omega; \mathcal{H}))\).\]

**Proposition 20.** Let \(\mathcal{H}\) be a separable real Hilbert space, \(A : D(A) \subset \mathcal{H} \to \mathcal{H}\) be a closed linear operator and \(A\) generates a uniformly bounded \(C_0\)-semigroup \(T(t)\), and let \(\Xi \subset L^2([0,T]; L^2(\Omega; \mathcal{L}_Q^0(\mathcal{Y}, \mathcal{H})))\) be such that

\[
\begin{align*}
(1) & : \mathbb{E}\|\xi(t)\|_Q^2 \leq \vartheta(t), \text{ for all } \xi \in \Xi \text{ and a.e. } t \in [0,T], \\
(2) & : \chi_Q(\Xi(t)) \leq \nu(t), \text{ for a.e. } t \in [0,T],
\end{align*}
\]

where \(\vartheta, \nu \in L^p([0,T])\) with \(p > \frac{1}{2\alpha - 1}\), and \(\chi_Q\) is the Hausdorff MNC on \(L^2(\Omega; \mathcal{L}_Q^0(\mathcal{Y}, \mathcal{H}))\). Then for all \(t \in [0,T]\),

\[
\chi(\mathcal{V}(\Xi)(t)) \leq 4\sqrt{2} \left( \int_0^t (t-s)^{2\alpha-2} \|E_{\alpha,\alpha}((t-s)^\alpha A)\|^2 \nu(s) ds \right)^{\frac{1}{2}},
\]
here \( \chi \) is the Hausdorff MNC on \( L^2(\Omega; \mathcal{H}) \), and \( V(\Xi)(t) = \{ \int_0^t (t-s)^{a-1} E_{\alpha,a}((t-s)^aA)\xi(s)dB_Q(s) : \xi \in \Xi \} \).

**Proof.** For any \( \eta > 0 \), there exists a sequence \( \xi_n \in \Xi \) such that
\[
\chi(V(\Xi)(t)) \leq 2\chi(\{V(\xi_n)(t)\}) + \eta,
\]
thanks to Proposition 9 and the boundedness of \( V(\Xi)(t) \) in \( L^2(\Omega; \mathcal{H}) \). Since \( \xi_n \in \Xi \), we have
\[
E\|\xi_n(t)\|^2_Q \leq \vartheta(t), \quad \text{for all } n = 1, 2, \ldots \quad \text{and \ a.e. } t \in [0,T].
\]
By \( \vartheta \in L^p([0,T]) \), then there exists \( \delta_1 \in \left(0, \frac{\sqrt{2\alpha-1}\Gamma(\alpha)}{2MT^{\alpha-\frac{1}{2}}}\sqrt{\eta}\right) \) such that for every set \( m \subset [0,T] \) with \( \text{meas}(m) < \delta_1 \), we obtain
\[
\int_m (\vartheta(s))^p ds < \frac{(\Gamma(\alpha))^{2p}(2\alpha p - p - 1)^{p-1}}{8^p M^{2p} T^{2\alpha p - p - 1}} \eta^p.
\]
Invoking Theorem 18, in view of (3.13) and the assumption (2), we deduce that there exist a set \( m_{\delta_1} \subset [0,T] \) with \( \text{meas}(m_{\delta_1}) < \delta_1 \), a compact set \( K_{\delta_1} \subset L^2(\Omega; L^Q(\mathcal{H}, \mathcal{H})) \) and a set of functions \( G_{\delta_1} \subset \Xi \) with values in \( K_{\delta_1} \) such that for every \( n \geq 1 \) there exists \( x_n \in G_{\delta_1} \) for which
\[
(E\|\xi_n(t) - x_n(t)\|^2_Q)^{\frac{1}{2}} \leq 2\nu(t) + \delta_1, \quad t \in [0,T] \setminus m_{\delta_1}.
\]
Arguing as in the proof of (3.10), we obtain that
\[
E\|V(\xi_n)(t) - V(x_n)(t)\|^2_Q
\leq \int_{[0,t]} (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 E\|\xi_n(s) - x_n(s)\|^2_Q ds
\]
\[
+ \int_{m_{\delta_1}} (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 E\|\xi_n(s) - x_n(s)\|^2_Q ds
\]
\[
\leq 8 \int_0^t (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 (\nu(s))^2 ds
\]
\[
+ \frac{2M^2 \delta_1^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2} ds + \frac{4M^2}{(\Gamma(\alpha))^2} \int_{m_{\delta_1}} (t-s)^{2\alpha-2} \vartheta(s) ds
\]
\[
\leq 8 \int_0^t (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 (\nu(s))^2 ds + \frac{2M^2 T^{2\alpha-1}}{(2\alpha - 1)(\Gamma(\alpha))^2} \delta_1^2
\]
\[
+ \frac{4M^2}{(\Gamma(\alpha))^2} \left( \int_0^t (t-s)^{2\alpha-2} ds \right)^{\frac{p}{2}} \times \left( \int_{m_{\delta_1}} (\vartheta(s))^p ds \right)^{\frac{1}{p}}
\]
\[
\leq 8 \int_0^t (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 (\nu(s))^2 ds + \eta.
\]
On the other hand, similar to (3.11), we see that \( V(K_{\delta_1})(t) \) is relatively compact in \( L^2(\Omega; \mathcal{H}) \). Therefore, \( \{V(\xi_n)(t)\} \) belongs to a \( \sqrt{8 \int_0^t (t-s)^{2\alpha-2} \|E_{\alpha,a}((t-s)^aA)\|^2 ds} + \eta \)-net of the relatively compact set \( V(K_{\delta_1})(t) \). Since \( \eta \) is arbitrary, the conclusion follows immediately from (3.12). \]
3.1. **Existence of mild solutions.** In this subsection, we give the local and global existence of mild solutions to \((1.1)\). First we list the following conditions:

\((Y_0)\) The operator \(A : D(A) \subset \mathcal{H} \to \mathcal{H}\) is a closed linear operator, and \(A\) generates a uniformly bounded \(C_0\)-semigroup \(T(t)\) which is continuous by operator norm for every \(t > 0\).

\((Y_1)\) The multi-valued functions \(F : [0, \infty) \times \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{H})\) and \(G : [0, \infty) \times \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H}))\) have compact convex values; for each \(t\), the multi-valued functions \(F(t, \cdot) : \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{H})\) and \(G(t, \cdot) : \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H}))\) are u.s.c., and for each \(\psi \in \mathcal{E}^\gamma\), the multi-valued functions \(F(\cdot, \psi)\) and \(G(\cdot, \psi)\) are measurable; for each fixed \(T > 0\) and \(\psi \in \mathcal{E}^\gamma\), the sets

\[
\mathcal{P}_{F, \psi} = \{ f \in L^2([0, T]; \mathcal{L}^2(\Omega; \mathcal{H})) : f(t) \in F(t, \psi) \quad \text{for a.e. } t \in [0, T]\}
\]

and

\[
\mathcal{P}_{G, \psi} = \{ g \in L^2([0, T]; \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H}))) : g(t) \in G(t, \psi) \quad \text{for a.e. } t \in [0, T]\}
\]

are nonempty.

\((Y_2)\) There exist nonnegative functions \(\ell_0 \in L^p(\mathbb{R}^+)\) with \(p \in \left(\frac{1}{2\alpha - 1}, \infty\right)\) and \(\ell_1 \in L^\infty(\mathbb{R}^+)\) such that the multi-valued mapping \(F : [0, \infty) \times \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{H})\) satisfies

\[
\mathbf{E}|F(t, y)|^2 := \sup \{ \mathbf{E}|\xi|^2 : \xi \in F(t, y) \} \leq \ell_1(t)\|y\|^2_{\mathcal{E}^\gamma} + \ell_0(t), \quad \text{for all } t \geq 0, \; y \in \mathcal{E}^\gamma,
\]

and there exist nonnegative functions \(m_0 \in L^p(\mathbb{R}^+)\) with \(p \in \left(\frac{1}{2\alpha - 1}, \infty\right)\) and \(m_1 \in L^\infty(\mathbb{R}^+)\) such that the multi-valued mapping \(G : [0, \infty) \times \mathcal{E}^\gamma \to \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H}))\) satisfies

\[
\mathbf{E}|G(t, y)|^2 := \sup \{ \mathbf{E}|\zeta|^2 : \zeta \in G(t, y) \} \leq m_1(t)\|y\|^2_{\mathcal{E}^\gamma} + m_0(t), \quad \text{for all } t \geq 0, \; y \in \mathcal{E}^\gamma.
\]

\((Y_3)\) There exists a constant \(p \in \left(\frac{1}{2\alpha - 1}, \infty\right)\) such that the function \(\chi : [0, \infty) \to \mathcal{L}_{Q}^{0}(\mathcal{U}, \mathcal{H})\) satisfies

\[
\int_{0}^{\infty} |\chi(s)|^{2p}_{Q} \, ds = \Lambda < \infty.
\]

\((Y_4)\) There exist nonnegative functions \(\rho, \varrho \in L^\infty(\mathbb{R}^+)\) such that for all \(D \subset \mathcal{E}^\gamma\) and a.e. \(t \in \mathbb{R}^+\),

\[
\chi(F(t, D)) \leq \rho(t) \sup_{s \leq 0} \chi(D(s)),
\]

and

\[
\chi_{Q}(G(t, D)) \leq \varrho(t) \sup_{s \leq 0} \chi(D(s)),
\]

where \(\chi\) and \(\chi_{Q}\) are Hausdorff measures of noncompactness on \(\mathcal{L}^2(\Omega; \mathcal{H})\) and \(\mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{U}, \mathcal{H}))\), respectively.

We now define the multi-valued operator \(\mathcal{F} : \mathcal{E}_{\phi} \to \mathcal{E}_{\phi}\) as follows

\[
\mathcal{F}(v)(t) = \left\{ E_{\alpha}(t^{\alpha}A)\phi(0) + W(f)(t) + V(g)(t) + \int_{0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^{\alpha}A)h(s)d\mathbb{E}_{Q}(s) : f \in \mathcal{P}_{F}(v), g \in \mathcal{P}_{G}(v) \right\},
\]

\[(3.17)\]
where
\[
\mathcal{W}(f)(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds,
\]
(3.18)
\[
\mathcal{V}(g)(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathbb{B}_Q(s).
\]
(3.19)

It is easy to see that, \( v \in \mathcal{C}_\phi \) is a fixed point of \( \mathcal{F} \) iff \( v[\phi] \) is an integral solution of (1.1). Thanks to the formulation of the operators \( \mathcal{W} \) and \( \mathcal{V} \), \( \mathcal{F} \) can be rewritten as
\[
\mathcal{F}(v)(t) = E_{\alpha}(t^\alpha A)\phi(0) + \mathcal{W} \circ \mathcal{P}_F(v)(t) + \mathcal{V} \circ \mathcal{P}_G(v)(t)
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{H}_Q^H(s).
\]

**Theorem 21.** Assume that (T_0)-(T_4) hold. Then for every \( \phi \in \mathcal{C} \), (1.1) has at least one local mild solution \( y(t) \).

**Proof.** We divide the proof into three steps.

**Step 1.** There exists a closed convex set \( \Xi_0 \subset \mathcal{C}_\phi \) satisfying that \( \mathcal{F}(\Xi_0) \subset \Xi_0 \).

Let \( z \in \mathcal{F}(y) \). Then by (T_2)-(T_3), Lemmas 3 and 6, and Hölder’s inequality, we obtain from (3.17) that
\[
E[z(t)]^2 = E\left[ E_{\alpha}(t^\alpha A)\phi(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds \right]
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathbb{B}_Q(s)
+ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{H}_Q^H(s)\right]^2
\]
\[
\leq 4E[\mathcal{W}(\phi(0))]^2 + 4E\left[ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds \right]^2
+ 4E\left[ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathbb{B}_Q(s) \right]^2
+ 4E\left[ \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{H}_Q^H(s) \right]^2
\]
\[
\leq 4M^2E[\phi(0)]^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}E[f(s)]^2ds
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}E[g(s)]_Q^2ds + \frac{8HM^2t^{2H-1}}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}h(s)_Q^2ds
\]
\[
\leq 4M^2E[\phi(0)]^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}f_0(s)ds
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}f_1(s)ds + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}f_2(s)ds
+ \frac{8HM^2t^{2H-1}}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}m_0(s)ds
+ \frac{8HM^2t^{2H-1}}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}m_1(s)ds + \frac{8HM^2t^{2H-1}}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}m_2(s)ds
+ \frac{8HM^2t^{2H-1}}{(\Gamma(\alpha))^2} \left( \int_0^t (t-s)^{\frac{2\alpha-2-n}{p-1}}ds \right)^{\frac{p-1}{p}} \Lambda^\frac{1}{p}
where we have used the notations

\[ s \leq E |||M ||| y \leq E (\phi ||| 4 z) = 4 M \]

\[ s \leq E (t-s)^{2n-2} \|y\|_{\psi} ds. \quad (3.20) \]

Noticing that for all \( s \in [0, t] \),

\[
\|y\|_{\psi} = \sup_{\theta \in (-\infty, 0)} e^{\gamma \theta} E |y| \|\phi(s + \theta)\|^2
\]

\[
\leq e^{-\gamma s} \sup_{\theta \in (-\infty, -s]} e^{\gamma(s+\theta)} E |\phi(s + \theta)|^2 + \sup_{\theta \in [-s, 0]} e^{\gamma \theta} E |y(s + \theta)|^2
\]

\[
= e^{-\gamma s} |\phi|_{\psi} + \sup_{\theta \in [-s, 0]} e^{\gamma \theta} E |y(s + \theta)|^2 \leq e^{-\gamma s} |\phi|_{\psi}^2 + \sup_{\tau \in [0, s]} \|E\| \|y\|_{\tau}^2. \quad (3.21)
\]

Hence,

\[
E \|z(t)\|^2 \leq 4 M^2 E |\phi(0)|^2 + \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_0|_p + |m_0|_p + 2 H t^{2H-1} \Lambda^{\frac{1}{2}}))} \frac{t^{2n-1 - \frac{1}{p}}}{(2a - 1)^{p-1}}
\]

\[
+ \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_1|_\infty + |m_1|_\infty)} \int_0^t (t-s)^{2n-2} (e^{-\gamma s} |\phi|_{\psi}^2 + \sup_{\tau \in [0, s]} \|E\| \|y\|_{\tau}^2)ds
\]

\[
\leq 4 M^2 E |\phi(0)|^2 + \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_0|_p + |m_0|_p + 2 H t^{2H-1} \Lambda^{\frac{1}{2}}))} \frac{t^{2n-1 - \frac{1}{p}}}{(2a - 1)^{p-1}}
\]

\[
+ \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_1|_\infty + |m_1|_\infty)} |\phi|_{\psi}^2 \frac{t^{2n-1 - \frac{1}{p}}}{2a - 1} + \int_0^t (t-s)^{2n-2} \sup_{\tau \in [0, s]} \|E\| \|y\|_{\tau}^2 ds
\]

\[
\leq M^* + M_1 \int_0^t (t-s)^{2n-2} \sup_{\tau \in [0, s]} \|E\| \|y\|_{\tau}^2 ds, \quad (3.22)
\]

where we have used the notations

\[
M^* := 4 M^2 E \|\phi(0)\|^2 + \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_0|_p + |m_0|_p + 2 H t^{2H-1} \Lambda^{\frac{1}{2}}))} \frac{t^{2n-1 - \frac{1}{p}}}{(2a - 1)^{p-1}}
\]

\[
+ \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_1|_\infty + |m_1|_\infty)} |\phi|_{\psi}^2 \frac{t^{2n-1 - \frac{1}{p}}}{2a - 1},
\]

and

\[
M_1 := \frac{4 M^2}{(\Gamma(\alpha))^2 (|f_1|_\infty + |m_1|_\infty)}.
\]

Applying Hölder’s inequality to the last term of (3.22) yields

\[
E \|z(t)\|^2 \leq M^* + M_1 \left( \int_0^t (t-s)^{\frac{(2n-2)p}{p-1}} ds \right)^\frac{p}{p-1} \left( \int_0^t \left( \sup_{\tau \in [0, s]} \|E\| \|y\|_{\tau}^2 \right)^p ds \right)^\frac{1}{p}.
\]
\[ \leq M^* + M_1 \left( \frac{t^{2\alpha-1-\frac{1}{p}}}{(2\alpha-1)p-1} \right)^{\frac{p}{2}} \left( \int_0^t \left( \sup_{\tau \in [0,s]} E[|y(\tau)|^2] \right)^p ds \right)^{\frac{1}{p}}, \]

and thus

\[ \left( \sup_{\tau \in [0,t]} E[|z(\tau)|^2] \right)^p \]

\[ \leq 2^{p-1}(M^*)^p + 2^{p-1}(M_1)^p \left( \frac{t^{2\alpha p-1}}{(2\alpha-1)p-1} \right)^{p-1} \left( \sup_{\tau \in [0,s]} E[|y(\tau)|^2] \right)^p \int_0^t ds, \]

\[ \leq 2^{p-1}(M^*)^p + M_1^* \int_0^t \left( \sup_{\tau \in [0,s]} E[|y(\tau)|^2] \right)^p ds, \]

where we have used the notation

\[ M_1^* := 2^{p-1}(M_1)^p \left( \frac{t^{2\alpha p-1}}{(2\alpha-1)p-1} \right)^{p-1}. \]

Denote

\[ \Xi_0 = \left\{ y \in C_{\phi} : \left( \sup_{s \in [0,t]} E[|y(s)|^2] \right)^p \leq \varphi(t), t \in [0,T] \right\}, \]

where \( \varphi \) is the unique solution of the integral equation

\[ \varphi(t) = 2^{p-1}(M^*)^p + M_1^* \int_0^t \varphi(s) ds. \]

It is clear that \( \Xi_0 \) is a closed convex subset of \( C_{\phi} \) and (3.23) ensures that \( \mathcal{F}(\Xi_0) \subset \Xi_0 \).

**Step 2.** \( \Xi \) is a compact convex set.

Set

\[ \Xi_{k+1} = \text{conv} \mathcal{F}(\Xi_k), \quad k = 0, 1, 2, \ldots, \]

where \( \text{conv} \) stands for the closed convex hull of a subset in \( C_{\phi} \). We note that \( \Xi_k \) is closed, convex and \( \Xi_{k+1} \subset \Xi_k \) for all \( k \in \mathbb{N} \).

Let \( \Xi = \bigcap_{k=0}^{\infty} \Xi_k. \) Then \( \Xi \) is a closed convex subset of \( C_{\phi} \) and \( \mathcal{F}(\Xi) \subset \Xi \). We will show that \( \Xi \) is compact by using the Arzelà-Ascoli theorem. Indeed, for each \( k \geq 0, \mathcal{P}_F(\Xi_k) \) and \( \mathcal{P}_G(\Xi_k) \) are integrally bounded thanks to \((Y_2)\). Then Lemma 3, Proposition 19, \((Y_0)\) and \((Y_3)\) ensure that \( \mathcal{F}(\Xi_k) \) is equicontinuous. It follows that \( \Xi_{k+1} \) is equicontinuous for all \( k \geq 0. \) Therefore \( \Xi \) is equicontinuous as well.

In order to apply the Arzelà-Ascoli theorem, we have to prove that \( \Xi(t) \) is compact for each \( t \in [0,T] \). This will be done if we show that \( \chi(\Xi_k(t)) \to 0 \) as \( k \to \infty, \) where \( \chi \) is the Hausdorff MNC on \( L^2(\Omega; C). \)

By \((Y_4)\), Lemma 8, Propositions 10 and 20, we deduce that

\[ \chi(\Xi_{k+1}(t)) \]

\[ \leq \chi \left( E_\alpha(t^\alpha A)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{H}^H(s) \right) + \chi \left( \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}((t-s)^\alpha A)\mathcal{P}_F(\Xi_k)(s)ds \right) \]
\[
\begin{align*}
&+ \chi \left( \int_0^t (t-s) \alpha E_{\alpha,\alpha} ((t-s)^\alpha A) \mathcal{P}_G(\Xi_k)(s) d\mathcal{E}_G(s) \right) \\
&\leq 4 \int_0^t (t-s) \alpha E_{\alpha,\alpha} ((t-s)^\alpha A) \|\rho(s)\|_{\sup_{\tau \leq 0} \chi (\Xi_k(\phi)(s+\tau))} ds \\
&+ 4\sqrt{2} \left( \int_0^t (t-s) \alpha - 2 \theta(s)^2 \left( \sup_{\tau \leq 0} \chi (\Xi_k(\phi)(s+\tau)) \right)^2 ds \right)^{\frac{1}{2}}.
\end{align*}
\]

Let \( S \subset \mathbb{R} \). Then \( L^\infty (S; \mathbb{R}) \) denotes the Banach space of the essential bounded functions with the norm \( \| \cdot \|_\infty \). It follows from Lemma 6 and Hölder’s inequality that
\[
\chi (\Xi_{k+1}(t)) \leq \frac{4M}{\Gamma (\alpha)} \int_0^t (t-s) \alpha - 1 \rho(s) \chi \left( \sup_{\tau \leq 0} \chi (\Xi_k(\phi)(s+\tau)) \right) ds \\
+ \frac{4\sqrt{2}M}{\Gamma (\alpha)} \left( \int_0^t (t-s) \alpha - 2 \theta(s)^2 \left( \sup_{\tau \leq 0} \chi (\Xi_k(\phi)(s+\tau)) \right)^2 ds \right)^{\frac{1}{2}},
\]

\[
\leq 4M \left( \| \rho \|_\infty \sqrt{T} + \sqrt{2} |\phi|_\infty \right) \left( \int_0^t (t-s) \alpha - 2 \left( \sup_{\tau \leq 0} \chi (\Xi_k(\phi)(s+\tau)) \right)^2 ds \right)^{\frac{1}{2}}.
\]

\[
\leq \frac{4M}{\Gamma (\alpha)} \left( \| \rho \|_\infty \sqrt{T} + \sqrt{2} |\phi|_\infty \right) \left( \int_0^t (t-s)^{\alpha - 2} ds \right)^{\frac{\alpha - 1}{2}} \\
\times \left( \int_0^t \left( \sup_{\tau \in [0, s]} \chi (\Xi_k(\tau)) \right)^{2p} ds \right)^{\frac{1}{2p}},
\]

thanks to the fact that \( \Xi_k(\phi)(s+\tau) = \{ \phi(s+\tau) \} \) for \( \tau < -s \), which is a singleton. Therefore,
\[
\left( \sup_{\tau \in [0, t]} \chi (\Xi_{k+1}(\tau)) \right)^{2p} \leq \frac{16^pM^2p T^{2\alpha - p - 1}}{\Gamma (\alpha)^2p} \left( \| \rho \|_\infty \sqrt{T} + \sqrt{2} |\phi|_\infty \right)^{2p} \int_0^t \left( \sup_{\tau \in [0, s]} \chi (\Xi_k(\tau)) \right)^{2p} ds.
\]

Defining \( \sigma_k(t) = \left( \sup_{\tau \in [0, t]} \chi (\Xi_k(\tau)) \right)^{2p} \), then we have
\[
\sigma_{k+1}(t) \leq \frac{16^pM^2p T^{2\alpha - p - 1}}{\Gamma (\alpha)^2p} \left( \| \rho \|_\infty \sqrt{T} + \sqrt{2} |\phi|_\infty \right)^{2p} \int_0^t \sigma_k(s) ds.
\]

Let \( \sigma_\infty(t) = \lim_{k \to \infty} \sigma_k(t) \). Then by the Lebesgue majorant theorem, we get
\[
\sigma_\infty(t) \leq \frac{16^pM^2p T^{2\alpha - p - 1}}{\Gamma (\alpha)^2p} \left( \| \rho \|_\infty \sqrt{T} + \sqrt{2} |\phi|_\infty \right)^{2p} \int_0^t \sigma_\infty(s) ds.
\]
Hence Lemma 16 ensures that $\sigma_\infty(t) = 0$ for all $t \in [0, T]$. Since $0 \leq (\chi(\Xi_k(t)))^{2p} \leq \sigma_k(t) \to 0$ as $k \to \infty$, we have $\chi(\Xi_k(t)) \to 0$ as $k \to \infty$ as desired. Thanks to Lemma 8 (viii), it is clear that $\Xi(t)$ is compact for each $t \in [0, T]$.

**Step 3.** $\mathcal{F} : \Xi \to \Xi$ is u.s.c. with closed convex values.

In order to apply the fixed point principle given by Theorem 15, it remains to show that $\mathcal{F}$ is u.s.c. with convex closed values. Note that $\mathcal{P}_F$ and $\mathcal{P}_G$ have convex values, so $\mathcal{F}$ does.

We next show that $\mathcal{F}$ is u.s.c. with closed values. Thanks to Proposition 14, now it suffices to show that $\{y_n\} \subset \Xi$ with $y_n \to y^* \in \Xi$ and $v_n \in \mathcal{F}(y_n)$ implies $v_n \to v^* \in \mathcal{F}(y^*)$ up to a subsequence. Suppose not. Then there exists a neighborhood $O$ of $\mathcal{F}(y^*)$ such that

$$v_n \notin O, \quad \forall \ n \in \mathbb{N}. \quad (3.24)$$

By the definition of the multi-valued mapping $\mathcal{F}$, we obtain that for any $t \in [0, T]$,

$$v_n(t) \in E_\alpha(t^\alpha A)\phi(0) + W \circ \mathcal{P}_F(y_n)(t) + V \circ \mathcal{P}_G(y_n)(t) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{B}_Q^H(s).$$

Let $f_n \in \mathcal{P}_F(y_n)$ and $g_n \in \mathcal{P}_G(y_n)$. Then it follows that

$$v_n(t) = E_\alpha(t^\alpha A)\phi(0) + W(f_n(t)) + V(g_n(t)) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathbb{B}_Q^H(s). \quad (3.25)$$

By the assumptions $(\Upsilon_1)$-$(\Upsilon_2)$ and $y_n \to y^* \in \Xi$, we deduce from Propositions 14 and 19 that $\{W(f_n)\}$ and $\{V(g_n)\}$ are relatively compact in $C([0,T] ; \mathcal{L}^2(\Omega; \mathcal{H}))$. Without generality, we assume that there exist $\xi_1, \xi_2 \in C([0,T] ; \mathcal{L}^2(\Omega; \mathcal{H}))$ such that

$$\sup_{t \in [0,T]} E\|W(f_n)(t) - \xi_1(t)\|^2 \to 0 \quad \text{and} \quad \sup_{t \in [0,T]} E\|V(g_n)(t) - \xi_2(t)\|^2 \to 0 \quad (3.26)$$

as $n \to \infty$. Noticing that $(\Upsilon_2)$ and $y_n \to y^* \in \Xi$ ensure that $\{g_n\}$ is integrally bounded in $L^2([0,T] ; \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{W}) \mathcal{H}))$, hence $\{g_n\}$ is weakly compact in $L^2([0,T] ; \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{W}) \mathcal{H}))$. Then by Mazur’s lemma, there exists a sequence $\tilde{g}_n \in \text{Co}\{g_m : m \geq n\}$ such that $\tilde{g}_n \to g^*$ in $L^2([0,T] ; \mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{W}) \mathcal{H}))$ and consequently,

$$\tilde{g}_n(t) \to g^*(t), \quad \text{for a.e. } t \in [0,T]. \quad (3.27)$$

By the upper semicontinuity of $G(t, \cdot)$, we conclude from $y_n \to y^* \in \Xi$ that for $\varepsilon > 0$, $G(t, y_n[\phi]_t) \subset G(t, y^*[\phi]_t) + B_\varepsilon$ for all large $n$, here $B_\varepsilon$ denotes the closed ball in $\mathcal{L}^2(\Omega; \mathcal{L}_Q^0(\mathcal{W}) \mathcal{H}))$ at origin with radius $\varepsilon$. Therefore, $g_n(t) \in G(t, y^*[\phi]_t) + B_\varepsilon$ for a.e. $t \in [0,T]$. We observe that $G(t, y^*[\phi]_t) + B_\varepsilon$ is convex, so $\tilde{g}_n(t) \in G(t, y^*[\phi]_t) + B_\varepsilon$ for a.e. $t \in [0,T]$. This implies that $g^*(t) \in G(t, y^*[\phi]_t) + B_\varepsilon$ for a.e. $t \in [0,T]$. Since $\varepsilon$ is arbitrary, we have

$$g^*(t) \in G(t, y^*[\phi]_t), \quad \text{for a.e. } t \in [0,T]. \quad (3.28)$$

By (2.1), (3.27) and Lemma 6 (iii), we get that for each $t \in [0,T]$,

$$E\|V(\tilde{g}_n)(t) - V(g^*)(t)\|^2$$

$$= E\left\| \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}((t-s)^\alpha A)(\tilde{g}_n(s) - g^*(s))d\mathbb{B}_Q^H(s) \right\|^2$$
Theorem 22. Suppose that
\begin{equation}
\frac{M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2} E_0 y(s) - g^*(s) \|g\|Q ds \to 0 \tag{3.29}
\end{equation}
when \( n \to \infty \) thanks to the Lebesgue majorant theorem. Noticing that \( V \) is a linear operator, hence it follows from (3.26) and (3.28)-(3.29) that \( \xi_2 = V(g^*) \in V \circ \mathcal{P}_C(g^*) \).

In the similar way, we can deduce that there exists \( f^* \in L^2([0,T]; \mathcal{L}^2(\Omega; \mathcal{H})) \) such that \( \xi_1 = \mathcal{V}(f^*) \in \mathcal{V} \circ \mathcal{P}_F(g^*) \). Recall that \( v_n \in \mathcal{F}(y_n) \subset \Xi \), so there exist \( v^* \in \Xi \) and a subsequence which we relabel as \( \{v_n\} \) such that
\begin{equation}
\sup_{t \in [0,T]} E_0 \|v_n(t) - v^*(t)\|^2 \to 0 \tag{3.30}
\end{equation}
as \( n \to \infty \). By (3.25), we find that
\begin{equation*}
v^*(t) = E_0(tA)\phi(0) + \mathcal{V}(f^*)(t) + \mathcal{V}(g^*)(t) + \int_0^t (t-s)^{-1} E_0 A h(s) d\mathbb{E}_Q(s) (3.31)
\end{equation*}
and \( v^* \in \mathcal{F}(y^*) \). This contradicts (3.24).

Therefore, Theorem 15 gives the existence of a fixed point of \( \mathcal{F} \), which is a local mild solution of (1.1).

Now we establish the continuation of local mild solutions of (1.1).

**Theorem 22.** Suppose that \( (\Upsilon_0)-(\Upsilon_4) \) hold. If \( y(t) \) is a local solution to (1.1) in \([0,T]\), then there exists a continuation \( y^* \) of \( y \) in some interval \([0,T+\tau]\) with \( \tau > 0 \).

**Proof.** Let \( y : [0,T] \to \mathcal{H} \) be the local mild solution to (1.1) in \([0,T]\). Then there exist \( f \in \mathcal{P}_F(y) \) and \( g \in \mathcal{P}_C(y) \) such that
\begin{equation*}
y(t) = E_0(tA)\phi(0) + \int_0^t (t-s)^{-1} E_0 A f(s) ds + \int_0^t (t-s)^{-1} E_0 A g(s) d\mathbb{E}_Q(s) + \int_0^t (t-s)^{-1} E_0 A h(s) d\mathbb{E}_Q(s). \tag{3.31}
\end{equation*}
Fix \( R > 0 \) and consider
\begin{equation*}
\Xi_0 = \left\{ w \in C([0,T+\tau]; \mathcal{L}^2(\Omega; \mathcal{H})) : w(t) = y(t) \text{ for all } t \in [0,T] \right\}
\end{equation*}
and \( \left( \sup_{t \in [T,T+\tau]} E_0 \|w(t) - y(T)\|^2 \right)^{\frac{1}{2}} \leq R \), \( \tag{3.32} \)
where \( \tau > 0 \) will be chosen later. Then we define the multi-valued operator \( \mathcal{F} : \Xi_0 \to C([0,T+\tau]; \mathcal{L}^2(\Omega; \mathcal{H})) \) as follows
\begin{equation*}
\mathcal{F}(w)(t) = \left\{ E_0(tA)\phi(0) + \int_0^t (t-s)^{-1} E_0 A f(s) ds + \int_0^t (t-s)^{-1} E_0 A g(s) d\mathbb{E}_Q(s) + \int_0^t (t-s)^{-1} E_0 A h(s) d\mathbb{E}_Q(s) : f \in \mathcal{P}_F(w), \right. \tag{3.33}
\end{equation*}
and
\begin{equation*}
\bar{g} \in \mathcal{P}_C(w), \bar{f} = f(s) \text{ and } \bar{g} = g(s) \text{ for all } s \in [0,T] \}
\end{equation*}
We check that \( \mathcal{F}(\Xi^*_0) \subset \Xi^*_0 \).

(a) If \( w \in \Xi^*_0 \), then \( w(t) = y(t) \) for all \( t \in [0, T] \) with \( y \) the local mild solution to (1.1) in \([0, T]\). So, if \( t \in [0, T] \),

\[
\begin{align*}
\mathcal{F}(w)(t) &= E_\alpha(t^\alpha A)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) \tilde{f}(s) \, ds \\
& \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) \tilde{g}(s) \, dB_Q(s) \\
& \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) h(s) \, dB_Q^H(s) \\
& = E_\alpha(t^\alpha A)\phi(0) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) f(s) \, ds \\
& \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) g(s) \, dB_Q(s) \\
& \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A) h(s) \, dB_Q^H(s) = y(t).
\end{align*}
\]

(b) If \( t \in [T, T + \tau] \), then for any \( z \in \mathcal{F}(w) \) with \( w \in \Xi^*_0 \), we have

\[
E\|z(t) - y(T)\|^2 \leq CE\|E_\alpha(t^\alpha A)\phi(0) - E_\alpha(T^\alpha A)\phi(0)\|^2
\]

\[
+ CE \left\| \int_T^t (t-s)^{\alpha-1} E_{\alpha,A}((t-s)^\alpha A)( \tilde{f}(s) ds + \tilde{g}(s) dB_Q(s) + h(s) dB_Q^H(s) ) \right\|^2
\]

\[
+ CE \left\| \int_0^T (T-s)^{\alpha-1} (E_\alpha((t-s)^\alpha A) - E_\alpha((t-s)^\alpha A)) \\
\times (f(s) ds + g(s) dB_Q(s) + h(s) dB_Q^H(s)) \right\|^2
\]

\[
:= I_1 + I_2 + I_3 + I_4. \tag{3.34}
\]

For \( I_1 \), using Lemma 6, we can choose \( \tau > 0 \) sufficiently small such that for all \( t \in [T, T + \tau] \),

\[
I_1 = CE\|E_\alpha(t^\alpha A)\phi(0) - E_\alpha(T^\alpha A)\phi(0)\|^2 \leq \frac{1}{4} R^\frac{1}{2}. \tag{3.35}
\]

For \( I_2 \), arguing as in (3.20), then it follows from (\( \mathcal{Y}_2 \)-(\( \mathcal{Y}_3 \)), Lemmas 3 and 6, (3.31)-
(3.32) and H"older's inequality that when \( \tau > 0 \) is sufficiently small, we have for all \( t \in [T, T + \tau] \),

\[
I_2 \leq \frac{CM^2}{(\Gamma(\alpha))^2} \int_T^t (t-s)^{2\alpha-2} E\|\tilde{f}(s)\|^2 ds \\
+ \frac{CM^2}{(\Gamma(\alpha))^2} \int_T^t (t-s)^{2\alpha-2} E\|\tilde{g}(s)\|^2 dB_Q ds
\]

\[
\leq C \int_T^t (t-s)^{2\alpha-2} \ell_0(s) ds + C \int_T^t (t-s)^{2\alpha-2} \ell_1(s) ||w(\phi)||_{\mathcal{E}^-}^2 ds
\]

\[
+ C \int_T^t (t-s)^{2\alpha-2} m_0(s) ds + C \int_T^t (t-s)^{2\alpha-2} m_1(s) ||w(\phi)||_{\mathcal{E}^-}^2 ds
\]
\[
+ C(t - T)^{2H - 1} \left( \int_T^t (t - s)^{\alpha - 1} \frac{2(\alpha - 1)}{p - 1} ds \right)^{\frac{p - 1}{p}} \\
\leq C(t - T)^{2\alpha - 1} \left( \frac{1}{p - \frac{1}{2}} \right) + C(t - T)^{2H - 1}(t - T)^{2\alpha - 1 - \frac{1}{2}} + (C + CR^\frac{1}{2})(t - T)^{2\alpha - 1} \\
\leq \frac{1}{4} R^\frac{1}{2}. \quad (3.36)
\]

For \( I_3 \), similar to (3.36), by Lebesgue’s dominated convergence theorem, we can choose \( \tau > 0 \) sufficiently small such that for all \( t \in [T, T + \tau] \),

\[
I_3 \leq \frac{CM^2}{(\Gamma(\alpha))^2} \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 E|f(s)|^2 ds \\
+ \frac{CM^2}{(\Gamma(\alpha))^2} \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 E|g(s)|^2_Q ds \\
+ \frac{CHM^2T^{2H - 1}}{(\Gamma(\alpha))^2} \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 |h(s)|^2_Q ds \\
\leq C \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 \ell_0(s) ds \\
+ C \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 \ell_1(s) ||y[\phi]_s||^2_{L^2} ds \\
+ C \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 m_0(s) ds \\
+ C \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 m_1(s) ||y[\phi]_s||^2_{L^2} ds \\
+ CT^{2H - 1} \left( \frac{1}{2} \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^{2p} ds \right)^{\frac{p - 1}{p}} \\
\leq C \left( \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^{\frac{2p}{p - 1}} ds \right)^{\frac{p - 1}{p}} \\
+ C \int_0^T ((t - s)^{\alpha - 1} - (T - s)^{\alpha - 1})^2 ds \leq \frac{1}{4} R^\frac{1}{2}. \quad (3.37)
\]

Let \( 0 < \zeta < T \) be given arbitrarily. For \( I_4 \), by the similar way as above, in view of Lemma 6 and the arbitrariness of \( \zeta \), we can choose \( \tau > 0 \) sufficiently small such that for all \( t \in [T, T + \tau] \),

\[
I_4 \leq C \int_0^T (T - s)^{2\alpha - 2} |E_{a,a}((t - s)\alpha A) - E_{a,a}((T - s)^{\alpha} A)|^2 E|f(s)|^2 ds \\
+ C \int_0^T (T - s)^{2\alpha - 2} |E_{a,a}((t - s)\alpha A) - E_{a,a}((T - s)^{\alpha} A)|^2 E|g(s)|^2_Q ds \\
+ C HT^{2H - 1} \int_0^T (T - s)^{2\alpha - 2} |E_{a,a}((t - s)\alpha A) - E_{a,a}((T - s)^{\alpha} A)|^2 |h(s)|^2_Q ds \\
\leq C \int_0^T (T - s)^{2\alpha - 2} |E_{a,a}((t - s)\alpha A) - E_{a,a}((T - s)^{\alpha} A)|^2 \ell_0(s) ds \\
+ C \int_0^T (T - s)^{2\alpha - 2} |E_{a,a}((t - s)\alpha A) - E_{a,a}((T - s)^{\alpha} A)|^2 \ell_1(s) ||y[\phi]_s||^2_{L^2} ds
\]
\[ + C \int_0^T (T-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A) - E_{\alpha,\alpha}((T-s)^\alpha A)\|^2 m_0(s) ds \\
+ C \int_0^T (T-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A) - E_{\alpha,\alpha}((T-s)^\alpha A)\|^2 m_1(s)\|y(\cdot)|^2 \phi, ds \\
+ C T^{2H-1} \int_0^T (T-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A) - E_{\alpha,\alpha}((T-s)^\alpha A)\|^2 \|h(s)\|^2 ds \\
\leq C \left( \int_0^{T-\zeta} (T-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A) - E_{\alpha,\alpha}((T-s)^\alpha A)\|^2 \right)^{p-1} ds \\
+ C \left( \int_{T-\zeta}^T (T-s)^{2\alpha-2\alpha} ds \right)^{p-1} + C \int_0^T (T-s)^{2\alpha-2} ds \\
+ C \int_0^{T-\zeta} (T-s)^{2\alpha-2}\|E_{\alpha,\alpha}((t-s)^\alpha A) - E_{\alpha,\alpha}((T-s)^\alpha A)\|^2 ds \leq \frac{1}{4} R^2. \]

(3.38)

Hence, it follows from (3.34)-(3.38) that for any \( z \in \hat{\mathcal{F}}(w) \) with \( w \in \Xi_0^* \),

\[ \left( \sup_{t \in [T,T+\tau]} \mathbb{E}[|y(t)-y(t)|^2] \right)^p \leq R, \]

and consequently, \( \hat{\mathcal{F}}(\Xi_0^*) \subset \Xi_0^* \).

Noticing that \( \Xi_0^* \) is a closed convex subset of \( C([0,T+\tau]; L^2(\Omega; \mathcal{H})) \). Set

\[ \Xi_{k+1}^* = \text{conv} \hat{\mathcal{F}}(\Xi_k^*), k = 0, 1, 2, \ldots, \]

here the \( \text{conv} \) stands for the closed convex hull of a subset in \( C([0,T+\tau]; L^2(\Omega; \mathcal{H})) \). Let \( \Xi^* = \cap_{k=0}^\infty \Xi_k^* \). Arguing as in the proof of Theorem 21, we obtain that \( \Xi^* \) is a compact convex subset of \( C([0,T+\tau]; L^2(\Omega; \mathcal{H})) \) and \( \hat{\mathcal{F}}: \Xi^* \to \Xi^* \) is upper semi-continuous with closed convex values. Therefore, Theorem 15 ensures that there exists a fixed point \( y^* \) of \( \hat{\mathcal{F}} \). Since \( y^*(t) = y(t) \) for all \( t \in [0,T] \), we see that \( y^* \) is the continuation of \( y \) in \([0,T+\tau]\).

We finish Section 3 with a result on global existence.

**Theorem 23.** Suppose that \( (Y_0)-(Y_4) \) hold. Then for every initial data \( \phi \in \mathcal{C}^\gamma \), (1.1) has at least one global mild solution \( y(t) \) almost surely.

**Proof.** For any initial data \( \phi \in \mathcal{C}^\gamma \), Theorem 21 ensures that (1.1) has at least one local mild solution \( y \) in \([0,T]\). Consider

\[ \mathbb{H} := \{ \tau \in [0,\infty) : y \text{ has a continuation in } [0,T+\tau] \text{ with } \tau > 0 \}. \]

Let \( \sup_{\mathbb{H}} \mathbb{T} = T_{\text{max}} \). To show that \( y(\cdot) \) is a global mild solution, we need to prove that \( T_{\text{max}} = \infty \) almost surely.

For sufficiently large \( k \), let us define the stopping time

\[ t_k = \inf \{ t \in [0,T_{\text{max}}) : \|y(t)| > k \} \]

with the usual convention \( \inf \emptyset = \infty \), where \( \emptyset \) denotes the empty set. Clearly, \( t_k \) is a nondecreasing sequence and \( t_k \to t_\infty \leq T_{\text{max}} \) almost surely as \( k \to \infty \). If we can show that \( t_\infty = \infty \) a.s., then \( T_{\text{max}} = \infty \) a.s., which implies that \( y(t) \) is globally defined. Since the sequence \( t_k \) is increasing, \( t_\infty = \infty \) a.s. is equivalent to proving that for any \( T > 0 \), \( \mathbb{P}(t_k \leq T) \to 0 \) as \( k \to \infty \).
For any \( t \in [0, T] \), Lemmas 3 and 6 imply that
\[
E\|y(t \wedge t_k)\|^2 \leq 4E\|E_\alpha((t \wedge t_k)A)\phi(0)\|^2 \\
+ 4E\left\| \int_0^{t \wedge t_k} (t \wedge t_k - s)^\alpha E_{\alpha,\alpha}((t \wedge t_k - s)A)f(s)ds \right\|^2 \\
+ 4E\left\| \int_0^{t \wedge t_k} (t \wedge t_k - s)^\alpha E_{\alpha,\alpha}((t \wedge t_k - s)A)g(s)d\mathbb{E}_Q(s) \right\|^2 \\
+ 4E\left\| \int_0^{t \wedge t_k} (t \wedge t_k - s)^\alpha E_{\alpha,\alpha}((t \wedge t_k - s)A)h(s)d\mathbb{E}_G(s) \right\|^2 \\
\leq 4M^2E\|\phi(0)\|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}E\|f(s)\|^2ds \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}E\|g(s)\|^2_Qds \\
+ \frac{8M^2H}{(\Gamma(\alpha))^2} (t \wedge t_k)^{2H-1} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}\|h(s)\|^2_Qds,
\]
for \( f \in \mathcal{P}_T(y) \), \( g \in \mathcal{P}_Q(y) \).

First, we calculate the last term on the right-hand side of the above inequality, by (Y_3) and Hölder’s inequality we obtain
\[
\frac{8M^2H}{(\Gamma(\alpha))^2} (t \wedge t_k)^{2H-1} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}\|h(s)\|^2_Qds \\
\leq \frac{8M^2H}{(\Gamma(\alpha))^2} (t \wedge t_k)^{2H-1} \left( \int_0^{t \wedge t_k} \|h(s)\|^2_Qds \right)^{\frac{1}{p}} \cdot \left( \int_0^{t \wedge t_k} (t \wedge t_k - s)^{\frac{(2\alpha-2)p}{p-1}}ds \right)^{\frac{p-1}{p}} \\
\leq \frac{8M^2H}{(\Gamma(\alpha))^2} \Lambda^\frac{1}{p} \left( \frac{p-1}{2\alpha p - p - 1} \right)^{\frac{p-1}{p}} T^{2H+2\alpha-2-\frac{1}{p}} \equiv \Pi_{IT}.
\]

Then,
\[
E\|y(t \wedge t_k)\|^2 \leq \Pi_{IT} + 4M^2E\|\phi(0)\|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}E\|f(s)\|^2ds \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}E\|g(s)\|^2_Qds.
\]

Applying Hölder’s inequality to (3.40), in view of (Y_2), we get that
\[
E\|y(t \wedge t_k)\|^2 \leq \Pi_{IT} + 4M^2E\|\phi(0)\|^2 \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}(\ell_0(s) + \ell_1(s)|y|_p^2)ds \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}(\ell_0(s) + \ell_1(s)|y|_p^2)ds \\
\leq \Pi_{IT} + 4M^2E\|\phi(0)\|^2 \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} (\ell_0 + \ell_1)_{\tilde{P}} \left( \int_0^{t \wedge t_k} (t \wedge t_k - s)^{\frac{(2\alpha-2)p}{p-1}}ds \right)^{\frac{p-1}{p}} \\
+ \frac{4M^2}{(\Gamma(\alpha))^2} (\ell_1 + \ell_1)_{\tilde{P}} \int_0^{t \wedge t_k} (t \wedge t_k - s)^{2\alpha-2}y|\phi|_p^2ds,
By (3.21) and Hölder’s inequality, it follows from (3.41) that

\[
\begin{align*}
&\leq \Pi_{1T} + 4M^2E\|\phi(0)\|^2 + \frac{4M^2}{(\Gamma(\alpha))^2}(\|\ell_0\|_p + \|m_0\|_p)\left(\frac{p-1}{2\alpha p - p - 1}\right)^{\frac{p-1}{p}} T^{2\alpha p - p - 1} \\
&+ \frac{4M^2}{(\Gamma(\alpha))^2}(\|\ell_1\|_\infty + \|m_1\|_\infty) \int_0^{t \land t_k} (t \land t_k - s)^{2\alpha - 2} \|\gamma[\phi]_s\|_{\mathcal{F}}^2 ds \\
&:= \Pi_{2T} + 4M^2E\|\phi(0)\|^2 + \Pi_3 \int_0^{t \land t_k} (t \land t_k - s)^{2\alpha - 2} \|\gamma[\phi]_s\|_{\mathcal{F}}^2 ds,
\end{align*}
\]

where we have used the notations

\[
\Pi_{2T} := \Pi_{1T} + \frac{4M^2}{(\Gamma(\alpha))^2}(\|\ell_0\|_p + \|m_0\|_p)\left(\frac{p-1}{2\alpha p - p - 1}\right)^{\frac{p-1}{p}} T^{2\alpha p - p - 1},
\]

\[
\Pi_3 := \frac{4M^2}{(\Gamma(\alpha))^2}(\|\ell_1\|_\infty + \|m_1\|_\infty).
\]

By (3.21) and Hölder’s inequality, it follows from (3.41) that

\[
\begin{align*}
&\leq \Pi_{2T} + 4M^2E\|\phi(0)\|^2 + \Pi_3 \int_0^{t \land t_k} (t \land t_k - s)^{2\alpha - 2} \|\gamma[\phi]_s\|^2 ds \\
&+ \frac{\Pi_3}{2\alpha - 1} (t \land t_k)^{2\alpha - 1} \|\phi\|^2_{\mathcal{F}}, \\
&\leq \Pi_{2T} + 4M^2E\|\phi(0)\|^2 + \frac{\Pi_3}{2\alpha - 1} (t \land t_k)^{2\alpha - 1} \|\phi\|^2_{\mathcal{F}} \\
&+ \Pi_3 \left(\int_0^{t \land t_k} (t \land t_k - s)^{\frac{2\alpha - 2}{p - 1}} ds\right)^{\frac{p-1}{p}} \times \left(\int_0^{t \land t_k} \left(\sup_{\tau \in [0,s]} E\|y(\tau)\|^2\right)^p ds\right)^{\frac{1}{p}} \\
&\leq \Pi_{2T} + \left(4M^2 + \frac{\Pi_3}{2\alpha - 1} T^{2\alpha - 1}\right) \|\phi\|^2_{\mathcal{F}} \\
&+ \Pi_3 \left(\frac{p-1}{2\alpha p - p - 1}\right)^{\frac{p-1}{p}} T^{2\alpha p - p - 1} \left(\int_0^t \left(\sup_{\tau \in [0,s]} E\|y(\tau \land t_k)\|^2\right)^p ds\right)^{\frac{1}{p}}.
\end{align*}
\]

Thus,

\[
\left(\sup_{\tau \in [0,t]} E\|y(\tau \land t_k)\|^2\right)^p \leq 3^{p-1}(\Pi_{2T})^p + 3^{p-1}\left(4M^2 + \frac{\Pi_3}{2\alpha - 1} T^{2\alpha - 1}\right)^p \|\phi\|_{\mathcal{F}}^{2p} \\
+ 3^{p-1}(\Pi_3)^p \left(\frac{p-1}{2\alpha p - p - 1}\right)^{p-1} T^{2\alpha p - p - 1} \int_0^t \left(\sup_{\tau \in [0,s]} E\|y(\tau \land t_k)\|^2\right)^p ds
\]

where we have used the notations \(\Pi_{4T} := 3^{p-1}(\Pi_{2T})^p + 3^{p-1}(4M^2 + \frac{\Pi_3}{2\alpha - 1} T^{2\alpha - 1})^p \|\phi\|_{\mathcal{F}}^{2p}\) and \(\Pi_{5T} := 3^{p-1}(\Pi_3)^p \left(\frac{p-1}{2\alpha p - p - 1}\right)^{p-1} T^{2\alpha p - p - 1}\). From Lemma 16, we have for all \(t \in [0,T]\),

\[
\left(\sup_{\tau \in [0,t]} E\|y(\tau \land t_k)\|^2\right)^p \leq \Pi_{4T} e^{\Pi_{5T} t},
\]
and consequently,
\[
\left( \sup_{\tau \in [0,T]} E|y(\tau \wedge t_k)|^2 \right)^p \leq \Pi_{4T} e^{\Pi_{5T} T}.
\]
According to the definition of \(t_k\), \(|y(t_k)| = k\). This implies
\[
k^{2p} (\mathbb{P}(t_k \leq T))^p = (E(\|y(t_k)\|^2 1_{\{t_k \leq T\}}))^p \leq (E(\|y(T \wedge t_k)\|^2 1_{\{t_k \leq T\}}))^p \leq \Pi_{4T} e^{\Pi_{5T} T}.
\]
Note that \(\Pi_{4T}\) and \(\Pi_{5T}\) are independent of \(k\). Letting \(k \to \infty\), then we have
\[
\lim_{k \to \infty} \mathbb{P}(t_k \leq T) = 0,
\]
which implies that (1.1) has a global solution \(y(t)\) on \([0, \infty)\) almost surely.

4. Asymptotic behavior of mild solutions. In this section, we investigate the asymptotic behavior of mild solutions to the tempered fractional stochastic evolution inclusion with infinite delay (1.2).

First, we give some concepts of tempered fractional calculus; for more details, we refer to [8]. Assume that \(X\) is a Banach space and \(T > 0\).

**Definition 24.** For \(\alpha > 0, \nu > 0\) and \(u \in L^1([0,T]; X)\), the left tempered fractional integral of order \(\alpha\) of \(u\) is defined by
\[
0 \eta_{t}^{\alpha,\nu} [ \cdot ] : = e^{-\nu t} \int_{0}^{t} (t-s)^{-\alpha - 1} e^{-\nu (t-s)} u(s) ds, \quad t \in [0,T].
\]
Let \(0 < \alpha < 1\) and \(\nu > 0\). Consider \(u \in C([0,T]; X)\) such that \(0 \eta_{t}^{1-\alpha,\nu} [ e^{\nu t} u(t) ] \in W^{1,1}([0,T]; X)\). The expression
\[
0 \eta_{t}^{\alpha,\nu} u(t) := e^{-\nu t} \int_{0}^{t} (t-s)^{-\alpha - 1} e^{-\nu (t-s)} u(s) ds, \quad t \in [0,T].
\]
is called the left Caputo tempered fractional derivative of order \(\alpha\) of the function \(u\), where \(\Gamma(\cdot)\) is the Euler Gamma function, the left Riemann-Liouville fractional integral operator \(0 \eta_{t}^{1-\alpha,\nu}\) and the left Caputo fractional derivative operator \(0 \eta_{t}^{\alpha,\nu}\) are given in Section 2.

Let \(u(t) = e^{\nu t} y(t)\), in view of Definition 24, we rewrite Eq. (1.2) as
\[
\begin{cases}
0 \eta_{t}^{\alpha,\nu} u(t) \in Au(t) + e^{\nu t} F(t,y_t) + e^{\nu t} G(t,y_t) \dfrac{dB_{Q}(t)}{dt} + e^{\nu t} h(t) \dfrac{dB_{Q}(t)}{dt}, & t \geq 0, \quad \frac{1}{2} < \alpha < 1, \\
u t \in (-\infty,0].
\end{cases}
\]  
(4.1)

Thanks to Definition 17, we are now ready to provide the notion of mild solutions to problem (1.2).

**Definition 25.** A measurable and \(\mathcal{F}_{t}\)-adapted stochastic process \(y : (-\infty,T] \to \mathcal{H}\) is called a mild solution of (1.2), if \(y \in C((-\infty,T]; L^2(\Omega; \mathcal{H}))\), \(y(t) = \varphi(t)\) for
Then every mild solution $y$ of (1.2) with the initial condition $\varphi \in \mathcal{C}$, satisfies

$$
y(t) = e^{-\nu t}E_{\alpha}(t^\alpha A)\varphi(0) + \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathcal{B}_Q(s)$$

$$+ \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathcal{B}_Q^H(s) \quad \mathbb{P}\text{-a.s.}$$

Similar to the arguments of Theorems 21-23, we have the following result.

**Corollary 26.** Suppose that $(\Upsilon_0)-(\Upsilon_4)$ hold. Then for every $\varphi \in \mathcal{C}$, (1.2) has at least one global mild solution $y(t)$ almost surely.

We now show that the solutions are ultimately bounded in the mean-square sense.

**Lemma 27.** In addition to the assumptions $(\Upsilon_0)-(\Upsilon_2)$ and $(\Upsilon_4)$, suppose that $(\Upsilon_3)$ there exists a constant $p \in \left(\frac{1}{2\alpha-1}, \infty\right)$ such that the function $h : [0, \infty) \rightarrow L_0^p(\mathcal{M},\mathcal{H})$ satisfies

$$\int_0^\infty e^{2\nu t}\|h(s)\|^p_Qds = \Lambda_\nu < \infty,$$

and

$$\frac{\widetilde{P}_4}{p} < \nu < \frac{\gamma}{2}.$$  

Then every mild solution $y$ of (1.2) with the initial condition $\varphi \in \mathcal{C}$ verifies

$$\|y_t\|^2_{\mathcal{C}} \leq 4^{p-1}(\widetilde{P}_2)^p \left(1 + \frac{\widetilde{P}_4}{\nu p - \widetilde{P}_4}\right) + (4^{p-1} + 4^p M^2p)\|\varphi\|^2_{\mathcal{C}} \left(e^{-\nu p t} + e^{(\widetilde{P}_4 - \nu p)t}\right),$$

where the constants $\widetilde{P}_2$ and $\widetilde{P}_4$ will be given later.

**Proof.** Let $t > 0$. We consider the solution $y$ given by

$$y(t) = e^{-\nu t}E_{\alpha}(t^\alpha A)\varphi(0) + \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds$$

$$+ \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathcal{B}_Q(s)$$

$$+ \int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathcal{B}_Q^H(s),$$

where $f(t) \in F(t,y|\varphi)$ and $g(t) \in G(t,y|\varphi)$ for a.e. $t \in [0, \infty)$. By Lemmas 3 and 6, we have

$$\mathbb{E}[y(t)]^2 \leq 4e^{-2\nu t}\mathbb{E}[E_{\alpha}(t^\alpha A)\varphi(0)]^2$$

$$+ 4\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)f(s)ds\right\|^2$$

$$+ 4\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)g(s)d\mathcal{B}_Q(s)\right\|^2$$

$$+ 4\mathbb{E}\left\|\int_0^t (t-s)^{\alpha-1}e^{-\nu(t-s)}E_{\alpha,\alpha}((t-s)^\alpha A)h(s)d\mathcal{B}_Q^H(s)\right\|^2.$$
We first estimate the last term on the right-hand side of the above inequality. Using Hölder’s inequality, in view of (4.4), we find that there exists a positive constant \( \Pi_1 \) such that

\[
8M^2 H \left( \frac{2H-1}{2} \right) t^{2H-1} \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} |h(s)|^2 ds \\
\leq 8M^2 H \left( \frac{2H-1}{2} \right) t^{2H-1} e^{-2\nu t} \left( \int_0^t e^{2\nu(s-t)} h(s)^2 ds \right)^{\frac{1}{2}} \times \left( \int_0^t e^{2\nu(s-t)} (s^{\frac{2\alpha-2}{p-1}} ds \right)^{\frac{p-1}{p}} \\
\leq 8M^2 H \left( \frac{2H-1}{2} \right) t^{2H-1} e^{-2\nu t} \Pi_1 \left( \int_0^t e^{2\nu(s-t)} (s^{\frac{2\alpha-2}{p-1}} ds \right)^{\frac{p-1}{p}} \\
\leq \Pi_1, \quad \text{for all } t \geq 0.
\]

Hence,

\[
E\|y(t)\|^2 \leq \Pi_1 + 4M^2 e^{-2\nu t} E\|\varphi(0)\|^2 + 4M^2 \left( \frac{2H-1}{2} \right) t^{2H-1} \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} E\|f(s)\|^2 ds \\
+ 4M^2 \left( \frac{2H-1}{2} \right) t^{2H-1} \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} E\|g(s)\|^2 ds.
\]

(4.6)

Applying Hölder’s inequality, in view of \((\Upsilon_2)\) and (4.3), we obtain

\[
E\|y(t)\|^2 \leq \Pi_1 + 4M^2 e^{-2\nu t} E\|\varphi(0)\|^2 \\
+ 4M^2 \left( \frac{2H-1}{2} \right) t^{2H-1} \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} (f_0(s) + f_1(s))|y_s|_{E^\gamma}^2 ds \\
+ 4M^2 \left( \frac{2H-1}{2} \right) t^{2H-1} \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} (m_0(s) + m_1(s))|y_s|_{E^\gamma}^2 ds \\
\leq \Pi_1 + 4M^2 e^{-2\nu t} E\|\varphi(0)\|^2 \\
+ 4M^2 \left( \frac{2H-1}{2} \right) \left( \|f_0\|_p + \|m_0\|_p \right) \left( \int_0^t (t-s)^{\frac{2\alpha-2}{p-1}} e^{-2\nu(t-s)} ds \right)^{\frac{p-1}{p}} \\
+ 4M^2 \left( \frac{2H-1}{2} \right) \left( \|f_1\|_p + \|m_1\|_p \right) \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} |y_s|_{E^\gamma}^2 ds \\
\leq \Pi_1 + 4M^2 e^{-2\nu t} E\|\varphi(0)\|^2 \\
+ 4M^2 \left( \frac{2H-1}{2} \right) \left( \|f_0\|_p + \|m_0\|_p \right) \left( \frac{2\nu p}{p-1} \right)^{\frac{p-1}{p}} \left( \Gamma \left( \frac{2\alpha p - p - 1}{p - 1} \right) \right)^{\frac{p-1}{p}} \\
+ 4M^2 \left( \frac{2H-1}{2} \right) \left( \|f_1\|_p + \|m_1\|_p \right) \int_0^t (t-s)^{2\alpha-2} e^{-2\nu(t-s)} |y_s|_{E^\gamma}^2 ds
\]

(4.7)
By the assumption (4.3), we have

\[ \Phi_{\Pi_2} := \nu \int_0^t (t-s)^{2(\alpha - 2\nu - 4)(t+\theta - s)} ds, \]  

where we have used the notations

\[ \Pi_2 := \frac{4M^2}{\Gamma(\alpha)^2} L(\|\ell_0\|_p + \|\ell_1\|_\infty) \left( \left( \frac{2p\alpha - p - 1}{p - 1} \right) \right)^{\frac{p - 1}{p}}, \]

\[ \Pi_3 := \frac{4M^2}{\Gamma(\alpha)^2} \left( \|\ell_1\|_\infty + \|\ell_2\|_\infty \right). \]

By the assumption (4.3), we have \( \gamma > 2\nu \), and so \( e^{(\gamma - 2\nu)\theta} \leq 1 \) for \( \theta \leq 0 \). Multiplying (4.7) by \( e^{\gamma t} e^{-\gamma t} \) and replacing \( t \) by \( t + \theta \), it follows from Hölder’s inequality that

\[
\sup_{\theta \in [-\nu, t]} e^{\gamma t} E|y(t + \theta)|^2 \leq \Pi_2 + 4M^2 e^{-2\nu t} E|\varphi(0)|^2 \\
+ \Pi_3 \sup_{\theta \in [-\nu, t]} e^{\gamma t} \int_0^{t+\theta} (t + \theta - s)^{2(\alpha - 2\nu - 4)(t+\theta - s)} |y_s|^2 ds \\
\leq \Pi_2 + 4M^2 e^{-2\nu t} E|\varphi(0)|^2 \\
+ \Pi_3 \sup_{\theta \in [-\nu, t]} e^{\gamma t} \left( \int_0^{t+\theta} (t + \theta - s)^{2(\alpha - 2\nu - 4)(t+\theta - s)} ds \right)^{\frac{p - 1}{p}} \\
\times \left( \int_0^t e^{-\nu p(t-s)} |y_s|^2 ds \right)^{\frac{1}{2}} \\
\leq \Pi_2 + 4M^2 e^{-2\nu t} E|\varphi(0)|^2 + \Pi_3 \left( \frac{\nu p}{p - 1} \right) \left( \left( \frac{2p\alpha - p - 1}{p - 1} \right) \right)^{\frac{p - 1}{p}} \\
\times \left( \int_0^t e^{-\nu p(t-s)} |y_s|^2 ds \right)^{\frac{1}{2}}. \quad (4.8)
\]

Note that \( \gamma > 2\nu \), hence for all \( \theta \in (-\infty, -\nu] \),

\[ e^{\gamma t} E|y(t + \theta)|^2 \leq e^{-\gamma t} e^{(\gamma - \nu)\theta} E|\varphi(0)|^2 \leq e^{-\nu t} E|\varphi|_\gamma^2. \quad (4.9)\]

(4.8) and (4.9) imply that

\[ |y(t)|^2 \leq e^{-2\nu t} |\varphi|_\gamma^2 + \Pi_2 + 4M^2 e^{-2\nu t} E|\varphi(0)|^2 \]

\[ + \Pi_3 \left( \frac{\nu p}{p - 1} \right) \left( \left( \frac{2p\alpha - p - 1}{p - 1} \right) \right)^{\frac{p - 1}{p}} \left( \int_0^t e^{-\nu p(t-s)} |y_s|^2 ds \right)^{\frac{1}{2}}, \]

and consequently,

\[ e^{\nu p t} |y(t)|^2 \leq 4^{p-1} e^{-\nu p t} |\varphi|_\gamma^{2p} + 4^{p-1} (\Pi_2)^p e^{\nu p t} + 4M^2 e^{-\nu p t} E|\varphi(0)|^2 \\
+ \Pi_4 \int_0^t e^{\nu p s} |y_s|^2 ds \\
\leq 4^{p-1} (\Pi_2)^p e^{\nu p t} + (4^{p-1} + 4M^2) |\varphi|_\gamma^{2p} + \Pi_4 \int_0^t e^{\nu p s} |y_s|^2 ds, \quad (4.10)\]

where \( \Pi_4 := 4^{p-1} (\Pi_3)^p \left( \frac{\nu p}{p - 1} \right) \left( \left( \frac{2p\alpha - p - 1}{p - 1} \right) \right)^{p - 1} \).
Applying Lemma 16 to (4.10), in view of (4.3), we obtain that for all $t \geq 0$,
\[
|y_t|^{2p} \leq 4^{p-1}(\Pi_2)^p \left(1 + \frac{\Pi_1}{\nu p - \Pi_4}\right) + (4^{p-1} + 4^p M^{2p})\|\varphi\|_{L^\infty,G}^{2p} \left(e^{-\nu pt} + e^{(\Pi_4 - \nu)pt}\right).
\]
(4.11)
This completes the proof of this lemma. \qed

The following theorem shows a priori estimates which means the exponential decay to zero in the mean-square sense.

**Theorem 28.** In addition to the assumptions $(\Upsilon)_0$-$(\Upsilon)_2$ and $(\Upsilon)_4$, suppose that $(\Upsilon_5)$ there exists a constant $p \in (\frac{1}{2\alpha - 1}, \infty)$ such that the functions $h, \ell_0$ and $m_0$ satisfy
\[
\int_0^\infty e^{2\nu ps}||h(s)||^2_{Q^2} ds < \infty, \quad \int_0^\infty e^{2\nu ps}(\ell_0(s))^p ds < \infty, \quad \int_0^\infty e^{2\nu ps}(m_0(s))^p ds < \infty,
\]
and
\[
\frac{\Pi_2^*}{p} < \nu < \frac{\gamma}{2},
\]
(4.12)
Then every mild solution $y$ of (1.2) with the initial condition $\varphi \in L^\gamma$ verifies
\[
||y(t)||_{L^\gamma} \leq \left(4^{p-1}(1 + 4^p M^{2p})\|\varphi\|^2_{L^\gamma} + 4^{p-1}(\Pi_2^*)^p\right) e^{(\Pi_4^* - \nu)p},
\]
where the positive constants $\Pi_2^*$ and $\Pi_4^*$ will be given later.

**Proof.** Thanks to Lemmas 3 and 6, it follows from (4.4) that
\[
E\|y(t)\|^2 \leq 4M^2e^{-2\nu t} E|\varphi(0)|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} E\|f(s)\|^2_{Q^2} ds + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} E\|g(s)\|^2_{Q^2} ds + \frac{8M^2 H}{(\Gamma(\alpha))^2} t^{2H-1} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} \|h(s)\|^2_{Q^2} ds,
\]
(4.13)
where $f(t) \in F(t, y[\varphi], t)$ and $g(t) \in G(t, y[\varphi], t)$ for a.e. $t \in [0, \infty)$. By $(\Upsilon)_2$, $(\Upsilon)_5$ and Hölder’s inequality, in view of $p \in (\frac{1}{2\alpha - 1}, \infty)$ and $H \in (\frac{1}{2}, 1)$, we get
\[
E\|y(t)\|^2 \leq 4M^2e^{-2\nu t} E|\varphi(0)|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} (\ell_0(s) + \ell_1(s)) |\varphi_{\alpha-1}(s)|_{\gamma}^2 ds + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} (m_0(s) + m_1(s)) |\varphi_{\alpha-1}(s)|_{\gamma}^2 ds + \frac{8M^2 H}{(\Gamma(\alpha))^2} t^{2H-1} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} \|h(s)\|^2_{Q^2} ds
\]
\[
\leq 4M^2e^{-2\nu t} E|\varphi(0)|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{2\alpha-2}e^{-2\nu(t-s)} (\ell_0(s) + \ell_1(s)) |\varphi_{\alpha-1}(s)|_{\gamma}^2 ds + \frac{4M^2}{(\Gamma(\alpha))^2} \int_0^t (t-s)^{(2\alpha-2)p} ds \left(\int_0^t e^{2\nu ps}(\ell_0(s))^p ds\right)^\frac{1}{p}
\]
\[
+ \frac{4M^2}{(\Gamma(\alpha))^2} e^{-2\nu t} \left(\int_0^t (t-s)^{(2\alpha-2)p} ds\right) \left(\int_0^t e^{2\nu ps}(m_0(s))^p ds\right)^\frac{1}{p}
\]
\[ + \frac{8M^2 H}{(\Gamma(\alpha))^2} t^{2H-1} e^{-2\nu t} \left( \int_0^t (t-s)^{\frac{(2\alpha-2p)}{p}} ds \right)^{\frac{p-1}{p}} \left( \int_0^t e^{2\nu ps} |h(s)|^{2p} ds \right)^{\frac{1}{p}} \]

\[ + \frac{4M^2}{(\Gamma(\alpha))^2} \left( \|\ell_1\|_\infty + \|m_1\|_\infty \right) \left( \int_0^t (t-s)^{\frac{(2\alpha-2p)}{p}} e^{-\nu p(t-s)} ds \right)^{\frac{p-1}{p}} \times \left( \int_0^t e^{-\nu p(t-s)} |y_s|^{2p} ds \right)^{\frac{1}{p}} \]

\[ \leq 4M^2 e^{-2\nu t} E|\varphi(0)|^2 + \frac{4M^2}{(\Gamma(\alpha))^2} \left( \frac{p-1}{2p\alpha - p - 1} \right)^{\frac{p-1}{p}} e^{2\alpha - \frac{1}{p}e^{-2\nu t}} \times \left[ \left( \int_0^t e^{2\nu ps} |\ell_0(s)|^p ds \right)^{\frac{1}{p}} + \left( \int_0^t e^{2\nu ps} |m_0(s)|^p ds \right)^{\frac{1}{p}} \right] \]

\[ + 2H t^{2H-1} \left( \int_0^t e^{2\nu ps} |h(s)|^{2p} ds \right)^{\frac{1}{p}} \]

\[ \leq 4M^2 e^{-2\nu t} E|\varphi(0)|^2 + \Pi_1^* + \frac{4M^2}{(\Gamma(\alpha))^2} \left( \frac{\nu p}{p-1} \right)^{1-2\alpha + \frac{1}{p}} \left( \Gamma \left( \frac{2p\alpha - p - 1}{p-1} \right) \right)^{\frac{p-1}{p}} \times \left( \|\ell_1\|_\infty + \|m_1\|_\infty \right) \left( \int_0^t e^{-\nu p(t-s)} |y_s|^{2p} ds \right)^{\frac{1}{p}}, \quad (4.14) \]

where \( \Pi_1^* \) is a positive constant such that

\[ \frac{4M^2}{(\Gamma(\alpha))^2} \left( \frac{p-1}{2p\alpha - p - 1} \right)^{\frac{p-1}{p}} e^{2\alpha - \frac{1}{p}e^{-2\nu t}} \times \left[ \left( \int_0^t e^{2\nu ps} |\ell_0(s)|^p ds \right)^{\frac{1}{p}} + \left( \int_0^t e^{2\nu ps} |m_0(s)|^p ds \right)^{\frac{1}{p}} \right] \]

\[ \leq \Pi_1^*, \quad \text{for all } t \geq 0. \]

By the assumption (4.12), we have \( \gamma > 2\nu \), and so \( e^{(\gamma - 2\nu)t} \leq 1 \) for \( \theta \leq 0 \). Multiplying (4.14) by \( e^{\gamma \theta - \gamma \theta} \) and replacing \( t \) by \( t + \theta \), we obtain from H"older's inequality that

\[ \sup_{\theta \in [-t, 0]} e^{\gamma \theta} E|g(t + \theta)|^2 \leq 4M^2 e^{-2\nu t} E|\varphi(0)|^2 + \Pi_1^* + \frac{4M^2}{(\Gamma(\alpha))^2} \left( \frac{\nu p}{p-1} \right)^{1-2\alpha + \frac{1}{p}} \times \left( \Gamma \left( \frac{2p\alpha - p - 1}{p-1} \right) \right)^{\frac{p-1}{p}} \left( \|\ell_1\|_\infty + \|m_1\|_\infty \right) \left( \int_0^t e^{-\nu p(t-s)} |y_s|^{2p} ds \right)^{\frac{1}{p}}. \]

Combining this with (4.9), therefore

\[ \|y_t\|_{\mathcal{E}^p} \leq 4^{p-1} (1 + 4pM^2e^{-2\nu pt}) \|\varphi\|_{\mathcal{E}^p}^{2p} + 4^{p-1}(\Pi_1^*)^p + \frac{4^{2p-1}M^2}{(\Gamma(\alpha))^{2p}} \left( \frac{\nu p}{p-1} \right)^{p-2p\alpha + 1} \times \left( \Gamma \left( \frac{2p\alpha - p - 1}{p-1} \right) \right)^{p-1} \left( \|\ell_1\|_\infty + \|m_1\|_\infty \right)^p \int_0^t e^{-\nu p(t-s)} |y_s|^{2p} ds. \]
This implies that
\[ e^{\nu pt} \| y_t \|^{2p}_{\ell^\gamma} \leq 4^{p-1} (1 + 4^p M^{2p}) \| \varphi \|^{2p}_{\ell^\gamma} + 4^{p-1} (\Pi_1^*)^p + \Pi_2^* \int_0^t e^{\nu ps} \| y_s \|^{2p}_{\ell^\gamma} \, ds , \quad (4.16) \]
where
\[ \Pi_2^* := \frac{4^{p-1} M^{2p}}{(\Gamma(\alpha))^{2p}} \left( \frac{\nu p}{p - 1} \right)^{p-2p\alpha+1} \left( \frac{2p\alpha - p - 1}{p - 1} \right)^{p-1} (\| \ell_1 \|_{\infty} + \| m_1 \|_{\infty})^p . \]

Applying now Lemma 16 to (4.16), in view of (4.12), we deduce that for all \( t \geq 0 \),
\[ \| y_t \|^{2p}_{\ell^\gamma} \leq \left( 4^{p-1} (1 + 4^p M^{2p}) \| \varphi \|^{2p}_{\ell^\gamma} + 4^{p-1} (\Pi_1^*)^p \right) e^{(\Pi_2^* - \nu p)t} . \]
The proof is therefore complete. \( \square \)

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