THE h-POLYNOMIAL AND THE ROOK POLYNOMIAL OF SOME POLYOMINOES

MANOJ KUMMINI AND DHARM VEER

ABSTRACT. Let $X$ be a convex polyomino such that its vertex set is a sublattice of $\mathbb{N}^2$. Let $k[X]$ be the toric ring (over a field $k$) associated to $X$ in the sense of Qureshi, J. Algebra, 2012. Write the Hilbert series of $k[X]$ as $(1 + h_1 t + h_2 t^2 + \cdots)/(1 - t)^{\dim(k[X])}$. For $k \in \mathbb{N}$, let $r_k$ be the number of configurations in $X$ with $k$ pairwise non-attacking rooks. We show that $h_2 < r_2$ if $X$ is not a thin polyomino. This partially confirms a conjectured characterization of thin polyominoes by Rinaldo and Romeo, J. Algebraic Combin., 2021.

1. Introduction

A polyomino is a finite union of unit squares with vertices at lattice points in the plane that is connected and has not finite cut-set [Sta97, 4.7.18]. (Definitions are in Section 2.) A. A. Qureshi [Qur12] associated a finitely generated graded algebra $k[X]$ (over a field $k$) to polyomino $X$. For $k \in \mathbb{N}$, a $k$-rook configuration in $X$ is an arrangement of $k$ rooks in pairwise non-attacking positions. The rook polynomial $r(t)$ of $X$ is $\sum_{k \in \mathbb{N}} r_k t^k$ where $r_k$ is the number of $k$-rook configurations in $X$. The $h$-polynomial of $k[X]$ is the (unique) polynomial $h(t) \in \mathbb{Z}[t]$ such that the Hilbert series of $k[X]$ is $h(t)/(1 - t)^d$ where $d = \dim k[X]$. A polyomino is thin if it does contain a $2 \times 2$ square of four unit squares (such as the one shown in Figure 2).

G. Rinaldo and F. Romeo [RR21, Theorem 1.1] showed that if $X$ is a simple thin polyomino, then $h(t) = r(t)$ and conjectured [RR21, Conjecture 4.5] that this property characterises thin polyominoes. In this paper, we prove this conjecture in the following case:

**Theorem 1.1.** Let $X$ be a convex polyomino such that its vertex set $V(X)$ is a sublattice of $\mathbb{N}^2$. Let $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ be the $h$-polynomial of $k[X]$ and $r(t) = 1 + r_1 t + r_2 t^2 + \cdots$ be the rook polynomial of $X$. If $X$ is not thin, then $h_2 < r_2$. In particular $h(t) \neq r(t)$.

Its proof proceeds as follows: we first observe that $k[X]$ is the Hibi ring of the distributive lattice $V(X)$ and that the Hilbert series of the Hibi ring of a distributive lattice and of the Stanley-Reisner ring of its order complex are the same. We then use the results of [BGS82] to relate the $h$-polynomial to descents in maximal chains of $V(X)$, and find an injective map from the set of maximal chains of $V(X)$ to the rook configurations in $X$, to conclude that $h_k \leq r_k$ in general. We then show that if $X$ is not thin, this map is not surjective to show that $h_2 < r_2$. In Corollary 3.4 we extend our result to $L$-convex polyominoes.

Section 2 contains the definitions and preliminaries. Proof of the theorem is given in Section 3.

ACKNOWLEDGEMENTS

The computer algebra systems Macaulay2 [M2] and SageMath [Sage] provided valuable assistance in studying examples.

2. Preliminaries

**Definition 2.1.** A cell in $\mathbb{R}^2$ is a set of the form $\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq a + 1, b \leq y \leq b + 1\}$ where $(a, b) \in \mathbb{Z}^2$. We identify the cells of $X$ by their top-right corners: For $\nu \in \mathbb{Z}^2$, $C(\nu)$ is the cell whose top-right corner is $\nu$. A polyomino $X$ is a finite union of cells that is connected and has no finite cut-set (i.e., removing finite sets from $X$ leaves $X$ connected) [Sta97, 4.7.18]. We say that a polyomino $X$ is

MK was partly supported by the grant CRG/2018/001592 from Science and Engineering Research Board, India and by an Infosys Foundation fellowship. DV was partly supported by an Infosys Foundation fellowship.
horizontally convex if for every line segment $t$ parallel to the $x$-axis with end-points in $X$, $t \subseteq X$. Similarly we define vertically convex polyominoes. We say that a polyomino $X$ is convex if it is horizontally convex and vertically convex. The set of cells of $X$ is denoted by $C(X)$. The vertex set $V(X)$ of $X$ is $X \cap \mathbb{Z}^2$. By the left-boundary vertices of $X$, we mean the elements of $\mathbb{Z}^2 \cap \partial X$ that are top-left vertices of the cells of $X$; the bottom-boundary vertices of $X$ are the elements of $\mathbb{Z}^2 \cap \partial X$ that are bottom-right vertices of the cells of $X$.

Qureshi [Qur12] associated a toric ring to a polyomino.

**Definition 2.2.** Let $X$ be a convex polyomino. Let $R = \mathbb{k}\{x_a \mid a \in V(X)\}$ be a polynomial ring. An interval in $X$ is a subset of $X$ of the form $[a, b] := \{c \in V(X) \mid a \leq c \leq b\}$ where $a \leq b \in V(X)$ and $c \leq b$ is the partial order on $\mathbb{R}^2$ given by componentwise comparison: $a = (a_1, a_2) \leq b = (b_1, b_2)$ if $a_1 \leq b_1$ and $a_2 \leq b_2$. Let $I_X$ be the $R$-ideal generated by the binomials of the form $x_a x_b - x_c x_d$ where $a \leq b \in V(X)$ and $c, d \in V(X)$ are the other two corners of the interval $[a, b]$. Let $\mathbb{k}[X] = R/I_X$.

**Setup 2.3.** Let $X$ be a convex polyomino such that $V(X)$ is a sublattice of $\mathbb{H}^2$. Let $J(X)$ be the poset of join-irreducible elements of $V(X)$. After a suitable translation, if necessary, we assume that $(0, 0)$ and $(m, n)$ are the elements $\hat{0}$ and $\hat{1}$ of $V(X)$. Hence $|J(X)| = m + n$.

**Definition 2.4.** Let $L$ be a finite distributive lattice. Let $R = \mathbb{k}\{x_a \mid a \in L\}$. The Hibi ideal [Hib87] $I_L$ of $L$ is the $R$-ideal generated by the binomials of the form $x_a x_b - x_c x_d$ where $a, b \in L$ and $c$ and $d$ are the join and the meet of $a$ and $b$. The Hibi ring of $L$ is $\mathbb{k}[L] := R/I_L$.

**Definition 2.5.** Let $R$ be a standard graded $\mathbb{k}$-algebra. The $h$-polynomial of $R$ is the polynomial $h(t)$ such that the Hilbert series of $R$ is $h(t)/(1 - t)^d$ where $d = \text{dim} R$.

**Remark 2.6.** When $X$ is as in Setup 2.3, the polyomino ring $\mathbb{k}[X]$ is the Hibi ring $\mathbb{k}[V(X)]$. Hence we are interested in the $h$-polynomial of the Hibi ring of a distributive lattice. Let $L$ be a distributive lattice. The order complex $\Delta(L)$ is the simplicial complex whose faces are the chains of $L$. The Stanley-Reisner ring $\mathbb{k}[\Delta(L)]$ of $\Delta(L)$ is the quotient of $\mathbb{k}\{x_a \mid a \in L\}$ by the ideal generated by $\{x_a x_b \mid a, b \text{ incomparable}\}$. There is a flat deformation from $\mathbb{k}[L]$ to $\mathbb{k}[\Delta(L)]$; see, e.g., [BH93, Section 7.1], after noting that Hibi rings are ASLs. Hence the $h$-polynomials of $\mathbb{k}[X]$ and of $\mathbb{k}[\Delta(V(X))]$ are the same. We use the results of [BG82] to relate the $h$-polynomial of $\Delta(L)$ to the descents in the maximal chains of $L$.

**Discussion 2.7.** We follow the discussion of [BG82, Section 1]. Let $\omega : J(I) \rightarrow \{1, \ldots, m + n\}$ be a (fixed) order-preserving map. Let $M(X)$ be the set of maximal chains of $V(X)$. Let $\mu \in M(X)$. We first write $\mu$ as a chain of order ideals of $J(X)$: $\hat{0} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m+n} = \hat{1}$. Then $|I_i \setminus I_{i-1}| = \{p_i\}$ for some $p_i \in J(I)$. Define $\omega(\mu) = (\omega(p_1), \ldots, \omega(p_{m+n}))$. For $1 \leq i \leq m + n - 1$, we say that $i$ is a descent of $\mu$ if $\omega(p_i) > \omega(p_{i+1})$. The descent set $\text{Des}(\mu)$ of $\mu$ is $\{i \mid 1 \leq i \leq m + n - 1, \ i \text{ is a descent of } \mu\}$. For $k \in \mathbb{N}$, define $M_k(X) = \{\mu \in M(X) : |\text{Des}(\mu)| = k\}$.

We now think of $\mu$ as a lattice path from $(0, 0)$ to $(m, n)$ consisting of horizontal and vertical edges. Label the vertices of $\mu$ as $(0, 0) = \mu_0, \mu_1, \ldots, \mu_{m+n} = (m, n)$, with $\mu_i - \mu_{i-1}$ a unit vector (when we think of these as elements of $\mathbb{R}^2$) pointing to the right or upwards. Then, if $i \in \text{Des}(\mu)$, then the direction of $\mu$ changes at $\mu_i$, i.e., the vectors $\mu_i - \mu_{i-1}$ and $\mu_{i+1} - \mu_i$ are perpendicular to each other. Hence $\mu_{i-1}$ and $\mu_{i+1}$ are the bottom-left and top-right vertices of a cell (the cell $C(\mu_{i+1})$ in our notation) of $X$. Thus we get a function

$$\psi : M(X) \rightarrow \text{Pow}(C(X)), \quad \mu \mapsto \{C(\mu_{i+1}) \in C(X) \mid i \in \text{Des}(\mu)\}.$$ \hfill \Box

**Proposition 2.9.** When $X$ is as in Setup 2.3. Write $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ for the $h$-polynomial of $\mathbb{k}[X]$. Then $h_1 = |M_1(X)|$.

**Proof.** Use [BG82, Theorems 4.1 and 1.1] with standard grading (i.e. setting $t_i = t$ for all $i$) to see that the $h$-polynomial of the Stanley Reisner ring of $\Delta(V(X))$ is

$$\sum_{t \in \mathbb{N}} |M_t(X)| t^i.$$ 

The proposition now follows from Remark 2.6. \hfill \Box
Discussion 2.10. Let $X$ be as in Setup 2.3. Left-boundary vertices and bottom-boundary vertices are join-irreducible. Let $p \in V(X)$; assume that $p$ is not a left-boundary vertex or a bottom-boundary vertex. If $p \notin \partial X$ then it is the top-right vertex of a cell in $X$, and hence is not join-irreducible. If $p \in \partial X$ then $p$ is the bottom-left vertex of the unique cell containing it (i.e., the bottom element $\hat{0}$ of $V(X)$) or the top-right vertex of the unique cell containing it (i.e., the top element $\hat{1}$ of $V(X)$); hence $p \notin \mathcal{J}(X)$. Thus we have established that $\mathcal{J}(X)$ is the union of the set of the left-boundary vertices and of the set of the bottom-boundary vertices. The sets of the left-boundary vertices and of the bottom-boundary vertices are totally ordered in $V(X)$. Therefore if $(p, p')$ is a pair of incomparable elements of $\mathcal{J}(X)$, then one of them is a left-boundary vertex and the other is a bottom-boundary vertex. \hfill \qed

3. Proof of the theorem

Proposition 3.1. Let $\mu \in \mathcal{M}(X)$ and $i \in \text{Des}(\mu)$. Write $\mu$ as a chain of order ideals $\hat{\mu} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m+n} = \hat{1}$ and $[I_i \setminus I_{i-1}] = \{p_i\}$ with $p_i \in \mathcal{J}(X)$. Then

(a) $p_i$ and $p_{i+1}$ are incomparable;
(b) $i + 1 \notin \text{Des}(\mu)$.

Proof. (a): Assume, by way of contradiction, that they are comparable. Then $p_i < p_{i+1}$. Hence $\omega(p_i) < \omega(p_{i+1})$, contradicting the hypothesis that $i \in \text{Des}(\mu)$.

(b): By way of contradiction, assume that $i + 1 \in \text{Des}(\mu)$. Then, by (a), $p_{i+1}$ and $p_{i+2}$ are incomparable.

We see from Discussion 2.10 and the definition of the $p_i$ that $p_i < p_{i+1}$. Therefore $\omega(p_i) < \omega(p_{i+2})$ contradicting the hypothesis that $\omega(p_i) > \omega(p_{i+1}) > \omega(p_{i+2})$. \hfill \qed

Proposition 3.2. The function $\psi$ of (2.8) is injective.

Proof. Let $\mu, \nu \in \mathcal{M}(X)$ be such that $\psi(\mu) = \psi(\nu)$. As earlier, write $\mu$ and $\nu$ as chains of order ideals $\mathcal{J}(X)$:

$$
\begin{align*}
\mu & : \hat{\mu} = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m+n} = \hat{1}; \\
\nu & : \hat{\nu} = I'_0 \subseteq I'_1 \subseteq \cdots \subseteq I'_{m+n} = \hat{1}.
\end{align*}
$$

For $1 \leq i \leq m+n$, write $I_i \setminus I_{i-1} = \{p_i\}$ and $I'_i \setminus I'_{i-1} = \{p'_i\}$ with $p_i, p'_i \in \mathcal{J}(X)$. We will prove by induction on $i$ that $I_i = I'_i$ for all $0 \leq i \leq m + n$. Since $I_0 = I'_0$, we may assume that $i > 0$ and that $I_j = I'_j$ for all $j < i$.

Assume, by way of contradiction, that $I_i \neq I'_i$. Then $I_{i-1}$ (which equals $I'_{i-1}$) is the bottom-left vertex of a cell $C$. Without loss of generality, we may assume that $I_i$ is the top-left vertex of $C$ and that $I'_i$ is the bottom-right vertex of $C$. (In other words, $\mu$ goes up and $\nu$ goes to the right from $I_{i-1}$, or equivalently, $p_i$ is a left-boundary vertex and $p'_i$ is a bottom-boundary vertex.)

Let

$$
\begin{align*}
i_1 & = \min\{j > i : p'_i \in I_j\} - 1; \\
i_2 & = \min\{j > i : p_i \in I'_j\} - 1.
\end{align*}
$$

Then the edge $(I_{i-1}, I_i)$ is vertical while $(I'_i, I_{i+1})$ is horizontal; this is the first time $\mu$ turns horizontal after $I_{i-1}$. Let $C_1$ be the cell with $I_{i-1}, I_i$ and $I_{i+1}$ as the bottom-left, the top-left and the top-right vertices respectively. Similarly the edge $(I'_{i-1}, I'_i)$ is vertical while $(I'_{i}, I'_{i+1})$ is horizontal; this is the first time $\nu$ turns vertical after $I'_{i-1}$. Let $C_2$ be the cell with $I'_{i-1}, I'_i$ and $I'_{i+1}$ as the bottom-left, the bottom-right and the top-right vertices respectively. (The possibility that $C_1 = C$ or $C_2 = C$ has not been ruled out.) See Figure 1 for a schematic showing the cells $C, C_1$ and $C_2$ and the chains $\mu$ and $\nu$.

We now prove a sequence of statements from which the proposition follows.

(a) If $C_1 \notin \psi(\mu)$, then $C_2 \in \psi(\nu)$. Proof: Note that $p_{i+1} = p'_i$ and $p'_{i+1} = p_i$. Since $C_1 \notin \psi(\mu)$, we see that

$$
\omega(p'_i) = \omega(p_{i+1}) > \omega(p_i) = \omega(p_i)
$$

where the last inequality follows from noting that \( p_i < \cdots < p_i \) since they are left-boundary vertices. Therefore, in the chain \( \nu \), we have

\[
\omega(p_i') \geq \omega(p_i') > \omega(p_i) = \omega(p_{i+1}'),
\]
i.e., \( i_2 \in \text{Des}(\nu) \). Hence \( C_2 \in \psi(\nu) \).

(b) If \( C_2 \notin \psi(\nu) \), then \( C_1 \in \psi(\mu) \). Immediate from (a).

(c) If \( C_1 \notin \psi(\mu) \) and \( C_2 \notin \psi(\nu) \). Proof: Note that \( \mu \) does not pass through the top-right vertex of \( C \) and that \( \nu \) does not pass through the bottom-left vertex of \( C_1 \).

(d) If \( C_2 \notin \psi(\nu) \) and \( C_2 \notin \psi(\mu) \). Proof: Note that \( \nu \) does not pass through the top-right vertex of \( C \) and that \( \mu \) does not pass through the bottom-left vertex of \( C_1 \).

(e) If \( C_1 \notin \psi(\mu) \), then \( \psi(\mu) \not\subseteq \psi(\nu) \). Proof: If \( C_1 \in \psi(\mu) \), use (c) to see that

\[
C_1 \in \psi(\mu) \setminus \psi(\nu).
\]

Now assume that \( C_1 \notin \psi(\mu) \). Then \( C_2 \in \psi(\nu) \) by (a). If \( C_2 = C \), then \( C_2 \notin \psi(\mu) \) by (c); otherwise, \( C_2 \notin \psi(\mu) \) by (d).

(f) If \( C_2 \notin \psi(\nu) \), then \( \psi(\mu) \not\subseteq \psi(\nu) \). Proof: If \( C_2 \in \psi(\nu) \), use (d) to see that

\[
C_2 \in \psi(\nu) \setminus \psi(\mu).
\]

Now assume that \( C_2 \notin \psi(\nu) \). Then \( C_1 \in \psi(\mu) \) by (b). If \( C_1 = C \), then \( C_1 \notin \psi(\nu) \) by (d); otherwise, \( C_1 \notin \psi(\nu) \) by (c).

(g) \( C \) belongs to at most one of \( \psi(\mu) \) and \( \psi(\nu) \). Proof: Suppose \( C \in \psi(\mu) \). Then \( i = i + 1, p_i = p_i' \) and \( \omega(p_i) \geq \omega(p_i') \). For \( C \) to belong to \( \psi(\nu) \), we need that \( I_{i+1} \) \( I_{i+1} \) (i.e., \( \mu \) and \( \nu \) are the same up to \( i + 1 \), except at \( i \)); for this to hold, it is necessary that \( p_i' = p_i \), but then \( i \notin \text{Des}(\nu) \). The other case is proved similarly.

(h) If \( C_1 = C_2 = C \) then \( \psi(\mu) \not\subseteq \psi(\nu) \). Proof: By (g), it suffices to show that \( C \in \psi(\mu) \) or \( C \in \psi(\nu) \). This follows from (a) and (b).

The proposition is proved by (e), (f), and (h). \( \Box \)

**Proposition 3.3.** Let \( k \in \mathbb{N} \) and \( \mu \in \mathcal{M}_k(X) \). Then \( \psi(\mu) \) is a k-rook configuration in \( X \).

**Proof.** Since \( |\psi(\mu)| = k \), it suffices to note that the cells of \( \psi(\mu) \) are in distinct rows and columns. This follows from Proposition 3.1(b). \( \Box \)

**Proof of Theorem 1.1.** For each \( i \in \mathbb{N} \), \( h_i = |\mathcal{M}_i(X)| \) by Proposition 2.9. By Propositions 3.2 and 3.3 we see that \( h_i \leq r_i \) for all \( i \). Since \( X \) is not thin, \( X \) contains a 2-rook configuration as in Figure 2. Such a rook configuration cannot be in the image of \( \psi \). Hence \( h_2 < r_2 \). \( \Box \)
Using results of [EHQR21], we can extend our result to $L$-convex polyominoes as follows. Let $X$ be an $L$-convex polyomino. Then there exists a polyomino $X^*$ (the Ferrer diagram projected by $X$, in the sense of [EHQR21]) such that

(a) $X^*$ is a convex polyomino such that $V(X^*)$ is a sublattice of $\mathbb{N}^2$ (since $X^*$ is a Ferrer diagram); 
(b) If $X$ is not thin, then $X^*$ is not thin; 
(c) $X$ and $X^*$ have the same rook polynomial [EHQR21, Lemma 2.4]; 
(d) $\mathbb{k}[X]$ and $\mathbb{k}[X^*]$ are isomorphic to each other [EHQR21, Theorem 3.1], so they have the same $h$-polynomial.

Thus we get:

**Corollary 3.4.** Let $X$ be an $L$-convex polyomino that is not thin. Let $h(t) = 1 + h_1 t + h_2 t^2 + \cdots$ be the $h$-polynomial of $\mathbb{k}[X]$ and $r(t) = 1 + r_1 t + r_2 t^2 + \cdots$ be the rook polynomial of $X$. Then $h_2 < r_2$.

**References**

[BGS82] A. Björner, A. M. Garsia, and R. P. Stanley. An introduction to Cohen–Macaulay partially ordered sets. In *Ordered sets (Banff, Alta., 1981)*, volume 83 of NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., pages 583–615. Reidel, Dordrecht-Boston, Mass., 1982.

[BEH93] W. Bruns and J. Herzog. *Cohen-Macaulay rings*, volume 39 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.

[EHQR21] V. Ene, J. Herzog, A. A. Qureshi, and F. Romeo. Regularity and the Gorenstein property of $L$-convex polyominoes. *Electron. J. Combin.*, 28(1):Paper No. 1.50, 23, 2021.

[Hib87] T. Hibi. Distributive lattices, affine semigroup rings and algebras with straightening laws. In *Commutative algebra and combinatorics (Kyoto, 1985)*, volume 11 of Adv. Stud. Pure Math., pages 93–109. North-Holland, Amsterdam, 1987.

[M2] D. R. Grayson and M. E. Stillman. Macaulay 2, a software system for research in algebraic geometry, 2006. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[Qur21] A. A. Qureshi. Ideals generated by 2-minors, collections of cells and stack polyominoes. *J. Algebra*, 357:279–303, 2012.

[RR21] G. Rinaldo and F. Romeo. Hilbert series of simple thin polyominoes. *J. Algebraic Combin.*, 54:607 – 624, 2021.

[Sage] The Sage Developers. SageMath, the Sage Mathematics Software System (Version 8.6), 2019. [https://www.sagemath.org](https://www.sagemath.org).

[Sta97] R. P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.