Quasi Exactly Solvable $2 \times 2$ Matrix Equations

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Abstract

We investigate the conditions under which systems of two differential eigenvalue equations are quasi exactly solvable. These systems reveal a rich set of algebraic structures. Some of them are explicitly described. An example of quasi exactly system is studied which provides a direct counterpart of the Lamé equation.
1. **Introduction**

Quasi exactly solvable (QES) equations are differential, spectral equations (typically Schrödinger eigenvalue equations) for which a finite number of eigenvalues and eigenvectors can be found by algebraic methods. In this respect, they contrast with the so called completely solvable equations (like the quantum harmonic oscillator or the Coulomb problem) where the whole discrete spectrum can be computed algebraically.

After their introduction, a few years ago, by A. V. Turbiner [1] and A. G. Ushveridze [2], the quasi exactly solvable equations received many new developments (see Refs. [3,4] for reviews).

Scalar QES equations in one variable are now well understood [1-5]. The basic ingredient is that, in a good variable, the linear differential operator defining them can be expressed in terms of a projectivized representation of the generators of the group SL(2, IR). Scalar equations in two variables, one real and one Grassmann, were investigated in Refs. [3,4,6]. All results obtained in this context can be reformulated in terms of systems of two equations in one real variable. The relevant algebraic structure behind this type of equations is the graded algebra osp(2,2).

In this paper, we want to revisit the systems of two coupled QES equations from a more general point of view. The understanding of these equations requires the knowledge of all $2 \times 2$ matrix differential operators that leave invariant the doublets of polynomials of given degrees, say $m$ and $n$. Such an investigation is reported in Sect. 2, where we put the main emphasis on the algebraic aspects of these sets of differential operators.

In Sect.3 we describe the possible forms of systems of two quasi exactly and completely solvable equations. We compare the number of free parameters they can depend on. Finally, in Sect.4 we present an explicit example of such system. It is related to the stability of sphaleron solutions in the two dimensional Abelian-Higgs model.

Before to start with the systems, we briefly recall the basic features of the scalar QES equations in one real variable. They are strongly related to the linear, differential operators that preserve $\mathcal{P}_n$: the space of polynomials of degree $n$ in a variable, say $x$. Among these operators, the three given by

$$
J_n^+ = x^2 \frac{d}{dx} - nx, \quad J_n^0 = x \frac{d}{dx} - \frac{n}{2}, \quad J_n^- = \frac{d}{dx}
$$

play a considerable role. They obey the commutation relations of the generators of the group SL(2, IR):

$$
[J_n^+, J_n^-] = -2J_n^0, \quad [J_n^+ J_n^0] = \mp J_n^\pm
$$

(2)
and therefore constitute an irreducible representation of dimension \( n + 1 \) of the algebra of this group. Using the irreductibility of the representation it is easy to show that the linear differential operators preserving the space \( \mathcal{P}_n \) are the elements of the envelopping algebra generated by the three operators (1). Accordingly, if \( P \) denotes a polynomial in three variables, then the eigenvalue equation

\[
P(J^+_n, J^0_n, J^-_n) \cdot F(x) = \lambda F(x)
\]

is equivalent to a system of \( n + 1 \) algebraic equations. If \( P \) is quadratic, it is possible, formally, to rephrase Eq. (3) as a Schrödinger equation by mean of a change of variable and of a change of function.

2. Algebraic Structure of QES Systems

The purpose of this section is to describe the algebraic features of quasi exactly solvable systems of two equations. We denote by \( \mathcal{P}_{m,n} \) the space of doublets of polynomials in \( x \), the first (resp. second) component being of degree \( m \) (resp.\( n \)). Given the integers \( m, n \), we will consider the set of operators defined by

\[
T^+ = \begin{pmatrix} J^+_m & 0 \\ 0 & J^+_n \end{pmatrix}, \quad T^0 = \begin{pmatrix} J^0_m & 0 \\ 0 & J^0_n \end{pmatrix}, \quad T^- = \begin{pmatrix} J^-_m & 0 \\ 0 & J^-_n \end{pmatrix},
\]

\[
J = \frac{1}{2} \begin{pmatrix} n + \Delta & 0 \\ 0 & n \end{pmatrix}, \quad \Delta \equiv n - m
\]

\[
\bar{Q}_\alpha = \left( \prod_{j=0}^{\Delta - \alpha} \left( x \frac{d}{dx} - (n + 1 - \Delta) - j \right) \right) \left( \frac{d}{dx} \right)^{\alpha - 1} \sigma^+, \quad \alpha = 1, 2, \cdots \Delta + 1
\]

\[
Q_\alpha = x^{\alpha - 1} \sigma^-, \quad \alpha = 1, 2, \cdots \Delta + 1
\]

It is easy to show that these operators are linearly independent and that each of them preserves the space \( \mathcal{P}_{m,n} \). Moreover, reasoning component by component, one can convince that any linear operator preserving \( \mathcal{P}_{m,n} \) can be constructed polynomially from the \( 6+2\Delta \) generators (4). So, the operators (4) play for systems of two equations the same role as the operators (1) do for scalar equations.

We want to demonstrate that the operators (4) close under (anti-) commutation, independently of \( m \) and \( n \). The subalgebra sustained by \( T^0, T^\pm \) is isomorphic to the algebra of \( \text{SL}(2,\mathbb{R}) \). The \( \text{SU}(2) \) generators, say \( T_k \) can be recovered by the combinations

\[
T_1 = \frac{1}{2} (T^- - T^+) , \quad T_2 = \frac{i}{2} (T^- + T^+) , \quad T_3 = T^0
\]
The commutation relations and the Casimir operator associated to this (reducible) representation are respectively

\[ [T_j, T_k] = i\epsilon_{jkl}T_l \]  
\[ C \equiv T_1^2 + T_2^2 + T_3^2 = \frac{1}{4} \begin{pmatrix} m(m+2) & 0 \\ 0 & n(n+2) \end{pmatrix} \]  

The operators (4) behave differently under the commutation with the matrix \( J \) (whose form in (4) is defined for later convenience):

\[ [J, T_k] = 0 \], \[ [J, Q_\alpha] = -\frac{\Delta}{2} Q_\alpha \], \[ [J, \bar{Q}_\alpha] = \frac{\Delta}{2} \bar{Q}_\alpha \]  

From now on we will refer to \( T_\alpha, J \) as to bosonic operators and to \( Q \) (resp. \( \bar{Q} \)) as to fermionic (resp. anti-fermionic) operators. The \( J \)-weight of bosonic operators is zero while that of fermionic (resp. antifermionic) ones is \(-\Delta/2\) (resp.\(+\Delta/2\)).

For the commutation rules between bosonic and fermionic operators we obtain

\[ [Q_\alpha, T^+] = (1 - \alpha + \Delta)Q_{\alpha+1} \]  
\[ [Q_\alpha, T^0] = (1 - \alpha + \frac{\Delta}{2})Q_\alpha \]  
\[ [Q_\alpha, T^-] = (1 - \alpha)Q_{\alpha-1} \]  
\[ [\bar{Q}_\alpha, T^+] = -(1 - \alpha)\bar{Q}_{\alpha-1} \]  
\[ [\bar{Q}_\alpha, T^0] = -(1 - \alpha + \frac{\Delta}{2})\bar{Q}_\alpha \]  
\[ [\bar{Q}_\alpha, T^-] = -(1 - \alpha + \Delta)\bar{Q}_{\alpha+1} \]  

demonstrating that the sets of \( Q_\alpha \)'s and \( \bar{Q}_\alpha \)'s transform according to the \( s = \frac{\Delta}{2} \) representation of the \( \text{SL}(2,\mathbb{R}) \) subalgebra. This statement becomes more transparent if we redefine these operators according to

\[ \hat{Q}_\rho = \left( \frac{2s}{s + \rho} \right)^{1/2} Q_{s+1+\rho} \; , \; -s \leq \rho \leq s \; , \; s \equiv \frac{\Delta}{2} \]  

The computation of the anti-commutation relations between fermionic and anti-fermionic operators is more involved. Formally, we obtain

\[ \{\bar{Q}_\alpha, Q_\beta\} = \mathcal{M}_{\alpha\beta}(T^-)_{\alpha-\beta} \; , \; \text{if} \; \alpha \geq \beta \]  
\[ = (T^+)_{\beta-\alpha} \mathcal{M}_{\alpha\beta} \; , \; \text{if} \; \alpha \leq \beta \]
with
\[ M_{\alpha\beta} = \prod_{j=0}^{\Delta-\alpha} (T^0 + J_c - j) \prod_{j=b_1+1}^c (T^0 - X - j) \prod_{j=0}^{\beta-2} (T^0 + J - k) \prod_{k=0}^{\Delta + 1 - \alpha} P_1 \] (12)
and
\[ J_c = (\Delta - 1) \prod_{j=0}^\beta, \quad P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \] (13)
In the definitions (12) it is understood that the first (resp. second) product is \( \mathbb{I} \) if \( \alpha \geq \Delta \) (resp. if \( \beta \leq 2 \)). The form (11) is not satisfactory because the projectors \( P_1, P_2 \) entering in \( M_{\alpha\beta} \) are not expressible in terms of \( J \) and of \( C \) (the Casimir) in an \( n \)-independent way. These matrices however obey the identity
\[ (J + (1 - \frac{\Delta}{2})) P_2 = -\frac{1}{2\Delta} (J^2 + (1 - 2\Delta)J + \Delta(\Delta - 1) - C) \] (14)
In order to demonstrate that \( M_{\alpha\beta} \) is independent on \( n, P_1, P_2 \) we write formally
\[ M_{\alpha\beta} = A_{\alpha\beta} P_2 + B_{\alpha\beta} \] (15)
Using the fact that \( P_1, P_2 \) are projectors, a direct evaluation of the matrix \( A_{\alpha\beta} \) yields
\[ A_{\alpha\beta} = \prod_{j=a}^{b_1} (T^0 + X - j) \prod_{j=b_1+1}^c (T^0 - X - j) \prod_{j=0}^{b_2} (T^0 + X - j) \prod_{j=b_2+1}^c (T^0 - X - j) \] (16)
with
\[ X \equiv J + \left(1 - \frac{\Delta}{2}\right) \] (17)
and
\[ a \equiv \frac{1 - \Delta}{2}, \quad c = \beta - \alpha + \frac{\Delta - 1}{2}, \quad b_1 = -\alpha + \frac{\Delta + 1}{2}, \quad b_2 = \beta - \frac{\Delta + 3}{2} \] (18)
From Eq. (16), it is clear that the matrix \( A \) contains \( X \) as a factor. Then the use of (16) together with the identity (14) demonstrates the statement that \( M \) are independent on \( n \) and on the \( P \)'s. For completeness we also mention the relations
\[ \{Q_\alpha, Q_\beta\} = 0, \quad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0 \] (19)
So, Eqs.(9)-(19) finally show that the \( 6+2\Delta \) generators (4) close under (anti)-commutation to a family of operator algebras indexed by the integer \( \Delta \). In order to find the abstract algebras for which the above operator algebras provide the representations
we have to check whether all Jacobi identities are fulfilled on abstract level and, if not, to modify some of the commutation rules.

It appears that only the Jacobi identities for \( Q, \bar{Q} \) and \( \bar{Q} \bar{Q} \) or for \( \bar{Q} \bar{Q} \) and \( Q \) impose further constraints on generators. This suggests that just the anticommutation rules (19) should be changed. The anticommutator \( \{ Q_\alpha, Q_\beta \} \) has weight 2 with respect to \( J \). Therefore its value should also be quadratic in \( Q \)'s (we exclude the more complicated objects like \( QQ\bar{Q}\bar{Q} \) etc.). By analysing the Jacobi identity for \( Q_\alpha, Q_\beta, \bar{Q}_\gamma \), we arrive at the conclusion that the coefficients of the quadratic form on the right hand side cannot depend on \( T_i \)'s and on \( J \); i.e. they have to be numerical. Therefore the only possibility is that we decompose \( \{ Q_\alpha, Q_\beta \} \) into SU(2) irreducible components and keep only some of them on the right hand side. We solved this problem for \( \Delta = 2 \) and 3. It appeared that in order to satisfy the Jacobi identity it is sufficient to keep only the lowest non vanishing representation. The resulting relation can be viewed as a quadratic constraint on the \( Q \)'s. After this constraint is imposed, no further constraint is implied by the Jacobi identities.

We now collect the cases where an abstract algebra was found.

Case \( \Delta = 0 \)

In the case \( \Delta = 0 \), we have an ambiguity in defining the algebra. We can treat it according to the general scheme above (in this case \( J \propto I \)). Alternatively, we can also put \( J = \sigma_3 \) and impose the commutation rule on \( Q, \bar{Q} \). In this case one obtains the the Lie algebra of \( \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R}) \)

Case \( \Delta = 1 \)

This case was considered at length in Refs. [3,4,6]. The generators (4) constitute a representation of the graded algebra \( \text{osp}(2,2) \). The anticommutators of \( Q \) and \( \bar{Q} \) close into linear combinations of \( T^a \) and \( J \) (see Eq.(9)). So, the relevant algebra is still a Lie (super) algebra; this is not the case any longer when \( \Delta > 1 \)

Case \( \Delta = 2 \)

The algebra associated with the case \( \Delta = 2 \) is better handled when we express the operators \( Q_\alpha \) in the vector basis, i.e.
\[ \Theta_1 = \frac{Q_3 - Q_1}{2}, \quad \Theta_2 = \frac{Q_3 + Q_1}{2i}, \quad \Theta_3 = -Q_2 \]
\[ \bar{\Theta}_1 = \frac{Q_3 - Q_1}{2}, \quad \bar{\Theta}_2 = -\frac{Q_3 + Q_1}{2i}, \quad \bar{\Theta}_3 = \bar{Q}_2 \]  

(20)

In this basis, the relations (8),(9),(10) become respectively

\[ [J, \Theta_j] = -\Theta_j, \quad [J, \bar{\Theta}_j] = \bar{\Theta}_j \]  
\[ [T_i, \Theta_j] = i\epsilon_{ijk} \Theta_k, \quad [T_i, \bar{\Theta}_j] = i\epsilon_{ijk} \bar{\Theta}_k \]  
\[ \{\Theta_i, \Theta_j\} = 2\delta_{jk}(\Theta^2_1 + \Theta^2_2 + \Theta^2_3) \]  
\[ \{\Theta_i, \bar{\Theta}_j\} = 2\delta_{jk}(\bar{\Theta}^2_1 + \bar{\Theta}^2_2 + \bar{\Theta}^2_3) \]  

(21) (22) (23) (24)

In the last relation, the right hand side is decomposed according to irreducible representations of SU(2). Using such a decomposition for the anticommutators (19) and keeping only the scalar part on the right hand side, we obtain instead

\[ \{\Theta_j, \Theta_k\} = \frac{2}{3}\delta_{jk}(\Theta^2_1 + \Theta^2_2 + \Theta^2_3) \]  

(25)

and a similar relation for \( \bar{\Theta} \). We checked that all the Jacobi identities are fulfilled with the choice (24).

Case \( \Delta = 3 \)

In this case, we found it convenient to express the operators \( Q_\alpha, \bar{Q}_\alpha \) (\( \alpha = 1, 2, 3, 4 \)) in a spin-tensor basis. It consists in linear combinations of the \( Q \), say \( Q_{ia} \), where the index \( i = 1, 2, 3 \) (resp. \( a = 1, 2 \)) transform according to the \( s = 1 \) (resp. \( s = 1/2 \)) representation of SU(2). The components of the spin-tensor \( Q_{ia} \) read

\[ Q_{3\frac{1}{2}} = \sqrt{2}Q_3, \quad \bar{Q}_{3\frac{1}{2}} = \sqrt{2}\bar{Q}_3 \]
\[ Q_{3\frac{1}{2}} = \sqrt{2}Q_2, \quad Q_{3\frac{-1}{2}} = -\sqrt{2}\bar{Q}_2 \]
\[ Q_{1\frac{1}{2}} = \frac{1}{\sqrt{2}}(Q_2 - Q_1), \quad \bar{Q}_{1\frac{-1}{2}} = \frac{-1}{\sqrt{2}}(\bar{Q}_2 - \bar{Q}_1) \]
\[ Q_{1\frac{-1}{2}} = \frac{1}{\sqrt{2}}(Q_1 - Q_3), \quad Q_{1\frac{-1}{2}} = \frac{1}{\sqrt{2}}(\bar{Q}_1 - \bar{Q}_3) \]
\[ Q_{2\frac{1}{2}} = \frac{i}{\sqrt{2}}(Q_2 + Q_4), \quad Q_{2\frac{1}{2}} = \frac{i}{\sqrt{2}}(\bar{Q}_2 - \bar{Q}_4) \]
\[ Q_{2\frac{-1}{2}} = \frac{i}{\sqrt{2}}(Q_1 + Q_3), \quad Q_{2\frac{-1}{2}} = \frac{-i}{\sqrt{2}}(\bar{Q}_1 - \bar{Q}_3) \]  

(25)
and obey the identities

\[ Q_{ia}(\sigma_i)_{ab} = 0 \quad , \quad (\sigma_i)_{ab}\bar{Q}_{ib} = 0 \]  

which guarantee that only the components corresponding to the representation \( s = 3/2 \) are selected.

The advantage of the spin tensor basis is that the commutators (9) write in a manifestly covariant way:

\[
[T_i, Q_{ja}] = i\epsilon_{ijk}Q_{ka} + \frac{1}{2}Q_{jb}(\sigma_i)_{ba}
\]

\[
[T_i, \bar{Q}_{ja}] = i\epsilon_{ijk}\bar{Q}_{ka} - \frac{1}{2}(\sigma_i)_{ab}\bar{Q}_{jb}
\]  

as well as the anticommutators (11):

\[
\{Q_{ia}, Q_{jb}\} = 2T_i(T \cdot \sigma)_{ab}T_j - \frac{2}{3}C\delta_{ij}(T \cdot \sigma)_{ab}
\]

\[
+ \frac{i}{3}\epsilon_{ijk}T_kC\delta_{ab} - \frac{1}{3}C(T_i\sigma_j + T_j\sigma_i)_{ab}
\]

\[
+ 2(J - 1)(T_iT_j - \frac{2}{3}\delta_{ij}C)\delta_{ab}
\]

\[
+ i(J - \frac{3}{2})\epsilon_{ijk}\{T_k, T_\ell\}(\sigma_\ell)_{ab}
\]

\[
+ \frac{i}{2}((T_j, T_k)\epsilon_{ik\ell} - (T_i, T_k)\epsilon_{jik\ell})(\sigma_\ell)_{ab}
\]

\[
- \frac{i}{3}(J - 1)\epsilon_{ijk}C(\sigma_k)_{ab}
\]

\[
+ \frac{i}{3}(J - 2)(5J - 3)\epsilon_{ijk}T_k\delta_{ab}
\]

\[
+ \frac{1}{3}J(J - 2)(T_i(\sigma_j)_{ab} + T_j(\sigma_i)_{ab} - 4\delta_{ij}(T \cdot \sigma)_{ab})
\]

\[
+ \frac{1}{3}J(J - 1)(J - 2)(i\epsilon_{ijk}(\sigma_k)_{ab} - 2\delta_{ij}\delta_{ab})
\]

Keeping again the lowest dimensional representation appearing in the decomposition of anticommutator \( \{Q_{ia}, Q_{jb}\} \) (in this case it is the vector representation because the scalar piece identically vanishes), i.e.

\[
\{Q_{ia}, Q_{jb}\} = \frac{1}{2}(\sigma_2)_{cd}\{Q_{id}, Q_{jc}\}(\sigma_2)_{ab} + \frac{2}{5}\delta_{ij}\{Q_{ka}, Q_{kb}\}
\]

\[
- \frac{i}{20}\left((\sigma_2\sigma_i)_{ab}\epsilon_{jkl} + (\sigma_2\sigma_j)_{ab}\epsilon_{ikl}\right)(\sigma_2)_{cd}\{Q_{kd}, Q_{kc}\}
\]
we have checked that all Jacobi identities are fullfilled.

3. **Systems of QES equations**

The $2 \times 2$ differential matrix operators which are quasi exactly solvable can be expressed in terms of elements of the envelopping algebras constructed from the generators (4). Using the same notations as in Ref.[6], these operators have the following form

$$ T_k(x) = \sum_{i=0}^{k} a_{k,i}(x) \frac{d^i}{dx^i} $$

(30)

where the coefficient $a_{k,i}(x)$ are $2 \times 2$ matrices with polynomial entries. Their degrees are indicated here in the square brackets

$$ a_{k,i}(x) = \begin{pmatrix} A_{k,i}^{[k+i]} & B_{k,i}^{[k+i-\Delta]} \\ C_{k,i}^{[k+i+\Delta]} & D_{k,i}^{[k+i]} \end{pmatrix} $$

(31)

Off course not all the polynomials $A_{k,i}, B_{k,i} \cdots$ are arbitrary: for $k \geq \Delta$ and for generic $n$, the most general operator $T_k(x)$ depends on

$$ 4(k+1)^2 , \quad ((k+1)^2) $$

(32)

arbitrary parameters, independly on $\Delta$ (the corresponding number for scalar equations is given in parenthesis). For $k < \Delta$ the situation is more complicated because some constraints may become dependant.

For the operator $T_k(x)$ to be exactly solvable, the degree of the polynomials in $a_{k,i}$ need to be as follows

$$ a_{k,i}(x) = \begin{pmatrix} A_{k,i}^{[i]} & B_{k,i}^{[i-\Delta]} \\ C_{k,i}^{[i+\Delta]} & D_{k,i}^{[i]} \end{pmatrix} $$

(31')

they are not constrained, so that these equations depend on

$$ 2(k+1)(k+2) + \frac{\Delta(\Delta - 1)}{2} \quad \text{for} \quad k \geq \Delta $$

$$ \frac{3}{2} (k+1)(k+2) + (k+1)\Delta \quad \text{for} \quad k < \Delta $$

(32')

free parameters (this number $(k+1)(k+2)/2$ for scalar equations).
4. Example

In this section, we discuss a system of two coupled equations which admits algebraic solutions. This example arises in the study of the stability about sphalerons [7] (i.e. unstable classical solutions) in the Abelian gauge-Higgs model in 1+1 dimension [8]. The relevant Schrödinger equation reads

\[
\left( \frac{d^2}{dz^2} + \lambda - \theta^2 k^2 \text{sn}^2 - 2 \theta k \text{cn} \cdot \text{dn} - 2 \theta k \text{cn} \cdot \text{dn} \right) \left( \frac{d^2}{dz^2} + \lambda + 1 + k^2 - (\theta^2 + 2) k^2 \text{sn}^2 \right) \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} = 0 \tag{33}
\]

and is considered on the Hilbert space of periodic functions over \([0, 4K(k)]\) (\(K(k)\) is the complete elliptic integral of the second type). The three elliptic functions \(\text{sn}(z, k), \text{cn}(z, k), \text{dn}(z, k)\) are periodic with periods \(4K(k), 4K(k), 2K(k)\) respectively. The spectral parameter \(\lambda\) represents the mode eigenvalue: negative \(\lambda\)'s correspond to the unstable modes of the sphalerons.

The parameter \(\theta\) depends on the coupling constants of the model; it represents the mass ratio \(2M_H/M_W\) where \(M_W\) (resp. \(M_H\)) is the mass of the gauge (resp. Higgs) boson. The system above admits algebraic solutions if these masses are such that

\[\theta^2 = N(N + 1) \quad \text{or} \quad M_H^2 = \frac{N(N + 1)}{4} M_W^2 \quad \text{with} \quad N \text{ integer} \quad \tag{34}\]

Following a similar scenario as for the Lamé equation [10]

\[
\left( \frac{d^2}{dz^2} + \lambda - N(N + 1)k^2 \text{sn}^2 \right) f(z) = 0 \tag{35}
\]

the algebraic solutions of Eq.(33) occurs in four sectors of the Hilbert space [11]. In order to construct them, we need to set the system in the forms discussed previously. For this purpose, we perform the change of variable

\[x = \text{sn}(z, k)^2 \tag{36}\]

accompanied by a change to new functions \(P(x), Q(x)\) defined through

\[f(z) = F(z)P(x), \quad g(z) = G(z)Q(x) \tag{37}\]

where \(F(z), G(z)\) are some products of \(\text{sn}, \text{cn}, \text{dn}\). Then we have to determine \(F(z)\) and \(G(z)\) so that the new equations for \(P(x), Q(x)\) admit polynomial solutions in the variable \(x\). The factors \(F(z), G(z)\) allowing for an algebratisation are different according
to the parity of $N$; the eight different possibilities are reported in the first two columns of Table 1.

Even if the changes of variable (36) and of function (37) are performed, the equation for $P, Q$ is still not in a form that preserves $P_{m,n}$: an additional change of basis of the form

$$
\left( \begin{array}{c} \tilde{P}(x) \\ \tilde{Q}(x) \end{array} \right) = V^{-1} \left( \begin{array}{c} P(x) \\ Q(x) \end{array} \right)
$$

(38)

need to be done. In the eight cases, the change of basis $V$ can be constructed from the following matrices:

$$
\alpha = \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{\theta}{N} \end{array} \right), \quad \beta(k) = \left( \begin{array}{cc} 1 & 0 \\ 0 & k \end{array} \right), \quad \gamma(x) = \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right), \quad \delta = \left( \begin{array}{cc} 1 & 1 + N \\ -1 & N \end{array} \right)
$$

(39)

They are presented in the third column of Table 1 and the dimensions $m, n$ of the invariant space $P_{m,n}$ are reported in the last two columns. These results demonstrate that Eq.(33) admits four types of algebraisation, yielding a total of $4N + 2$ algebraic solutions, if $N$ is an integer (this number is $2N + 1$ for the Lamé equation). The first (resp. last) four lines in the table correspond to the solutions available for $N$ odd (resp. even).

The analogy between Eqs. (33) and (35) is also present when doubly periodic solutions [10] are considered, i.e. solutions of period $8K(k)$. In order to discuss this issue we consider the following changes of functions:

$$
\begin{align*}
    f(z) &= \sqrt{dn(z) \pm cn(z)}(\pm Y(x)cn(z) + Z(x)dn(z)) \\
    g(z) &= \sqrt{dn(z) \pm cn(z)}(\pm V(x)cn(z) + W(x)dn(z))
\end{align*}
$$

(40)

and

$$
\begin{align*}
    f(z) &= \sqrt{dn(z) \pm cn(z)}(\pm Y(x)cn(z)dn(z) + Z(x)) \\
    g(z) &= \sqrt{dn(z) \pm cn(z)}(\pm V(x)cn(z)dn(z) + W(x))
\end{align*}
$$

(41)

Due to the square root factor, these functions have $8K(k)$ as period, the functions with + and – correspond to each other by a translation $z \to z + 2K(k)$.

Inserting the ansatzes (40) into Eq.(33) and identifying to zero the coefficients of the factors $cn$, $dn$, yields a system of four equations in $Y, Z, V, W$. To put this system into a canonical form; we must perform an additional change of basis. In this case, it reads

$$
\left( \begin{array}{c} \tilde{Y} \\ \tilde{W} \\ \tilde{Z} \\ \tilde{V} \end{array} \right) = \left( \begin{array}{cccc} 1 & 1 & 0 & 0 \\ \frac{k\theta}{N+1} & -\frac{k\theta}{N} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{k\theta}{N+1} & -\frac{k\theta}{N} \end{array} \right) \left( \begin{array}{c} Y \\ W \\ Z \\ V \end{array} \right);
$$

(42)
then the differential operator associated with the ansatz (40) preserves the space (using obvious notations)

\[ P_{\tilde{N}+1,\tilde{N}-1,\tilde{N}} \quad \text{if} \quad N = \frac{1}{2} + 2\tilde{N}, \quad \tilde{N} \text{ integer} \quad (43) \]

Similarly, after the change of basis defined by

\[
\begin{pmatrix}
\tilde{Y} \\
\tilde{W} \\
\tilde{Z} \\
\tilde{V}
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{k\theta}{N} x & 1 & 0 & 0 \\
0 & 0 & 1 & \frac{k\theta}{N+1} x \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
Y \\
W \\
Z \\
V
\end{pmatrix} \quad (44)
\]

we obtain the operator associated with the ansatz (41) in a form that preserves the space

\[ P_{\tilde{N},\tilde{N},\tilde{N},\tilde{N}} \quad \text{if} \quad N = \frac{3}{2} + 2\tilde{N}, \quad \tilde{N} \text{ integer} \quad (45) \]

This demonstrates that \( 4N + 2 \) algebraic solutions are available also when \( N \) is a semi-integer once the degeneracy \( \pm \) in Eqs.(40,41) is taken into account (both signs yield solutions of equal eigenvalues).

5. Conclusions

Quasi-exactly-solvable equations constitute an attractive bridge between group theory and spectral equations. While ordinary QES equations are related to usual Lie algebra, systems of coupled QES equations are related to more sophisticated structures. For systems of two coupled equations, the relevant operators are those that preserve the spaces \( P_{n,m} \). These operators can be perceived as projectivised representations of some graded algebra. For our equations, the relevant graded algebra have finite numbers of generators and our results demonstrate that they admit representations of arbitrary finite dimension. The anti-commutators between the fermionic generators close into \( n - m \)-powers of the bosonic generators.

Applications of quasi exactly solvable systems can be found in the framework of quantum mechanics, namely in the study of coupled channels. Here, we illustrate the relevance of such systems within another domain, namely in the study of unstable modes about sphalerons in a simple field theory. This example suggests that the stability analysis about other classical solutions (solitons, kinks....) might also be related to QES systems (see, however, Ref.[12]).
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| $F(z)$     | $G(z)$   | $V$             | $m$     | $n$     |
|-----------|---------|-----------------|---------|---------|
| 1         | cn dn   | $\alpha \beta(1/k) \gamma$ | $\frac{N-1}{2}$ | $\frac{N-1}{2}$ |
| cn dn     | 1       | $\alpha \beta(k) \gamma^t$  | $\frac{N-1}{2}$ | $\frac{N-1}{2}$ |
| sn cn     | sn dn   | $\alpha \beta(1/k) \delta$ | $\frac{N-3}{2}$ | $\frac{N-1}{2}$ |
| sn dn     | cn cn   | $\alpha \beta(k) \delta$   | $\frac{N-3}{2}$ | $\frac{N-1}{2}$ |
| sn        | sn cn dn| $\alpha \beta(1/k) \gamma$ | $\frac{N-2}{2}$ | $\frac{N-2}{2}$ |
| sn cn dn  | sn      | $\alpha \beta(k) \gamma^t$ | $\frac{N-2}{2}$ | $\frac{N-2}{2}$ |
| cn        | dn      | $\alpha \beta(1/k) \delta$ | $\frac{N-2}{2}$ | $\frac{N}{2}$   |
| dn        | cn      | $\alpha \beta(k) \delta$   | $\frac{N-2}{2}$ | $\frac{N}{2}$   |

The factors $F(z), G(z)$ allowing for an algebraisation of the system (33) are given in the first two columns. The third column contains the change of basis (38) and the last one the degree of the invariant polynomial space $P_{m,n}$.