Initial-boundary value problems for linear equations of electrodynamics with nonlinear boundary conditions

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Abstract. A new class of electromagnetic (EM) problems is introduced, initial-boundary value problem for linear wave equation with nonlinear boundary conditions, and methods for their solution are elaborated. Theory of nonlinear 1D maps and difference-differential equations have been applied to solve the problem. It has been shown that spatiotemporal dynamics of EM field in a resonator with nonlinear reflecting surfaces may be either regular or chaotic in spite of the major part of the system is formed by linear media. The approach may be used for analysis of chaotic processes in many electromagnetic and electronic devices.

1. Introduction
In many physical systems and electronic devices the volume of the active nonlinear elements is essentially smaller than that occupied by electromagnetic (EM) field (see in [1]) and therefore interaction between the field and such a nonlinear medium may be treated as a multiple scattering process. In general case the analysis of such systems requires formulation of initial-boundary value problems for 3D wave equation with nonlinear boundary conditions for the EM fields which is really hard task even for computer integration. That’s why we restrict ourselves with considerations of simplest problem statements, enabling description of initial-boundary value problem for 1D wave equation. This will allow us to understand physical properties of the investigated processes, to choose analysis technique and to consider ways of solving more complicated problems of that class.

In the paper we have considered the problem of multiple scattering of nonstationary plane EM waves by a nonlinear reflecting surface, such as a thin layer containing currents of free and/or bound charges with their nonlinear dependences on the electric field. Appropriate boundary conditions are formulated and the expression for energy conservation law is derived. The simplest case of the multiple scattering of plane linear polarized wave at the normal reflection from an infinite plane nonlinear reflecting structure is considered as an example. It is shown that at rather strong nonlinearity oscillations with complicated dynamics and wideband frequency spectrum could be generated in the systems under consideration. Such oscillations can be interpreted as a turbulent state of EM field. Actually, we have formulated a new class of EM problems: initial-boundary value problem for linear wave equation with nonlinear boundary conditions and elaborated methods for their solving [2].
2. Boundary conditions and energy conservation law for electromagnetic fields in 1D resonator with nonlinear reflecting surface

Consider multiple scattering of a plane EM wave, propagating along OX axis, by nonlinearly reflecting structure, located at the plane \( x = D \). Figure 1 shows geometry of the problem.

Assume that a possible smallest space scale \( \lambda_{\text{min}} \) of the scattered field is much greater than the width \( d \) of the nonlinear layer: \( \lambda_{\text{min}} \gg d \). In this case interaction between the field and the layer can be described locally along the coordinate \( x \) and the scattering surface may be treated as a boundary with nonlinear reflection. For derivation of the above boundary condition for the EM field at this surface we use a well known approach. Magnetic field \( \vec{H} \) and electric field \( \vec{E} \) inside the layer obey the Maxwell equation

\[
\begin{align*}
\text{rot } \vec{E} &= -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}; \\
\text{rot } \vec{H} &= \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \left\{ \frac{\partial \vec{P}}{\partial t} + \vec{j} \right\}
\end{align*}
\]  

(1)

where \( \vec{j} \) is the current density of free charges; \( \vec{P} \) is the polarization vector of bound charges; \( c \) is the light velocity in vacuum.

In order to derive boundary condition for EM fields at the nonlinear reflecting surface we have to integrate Eq. (1) over the area \( \gamma \) that contains all three media (Figure 1) and after that turn the height \( h_0 \) of the area \( \gamma \) to zero. Applying standard reasoning and taking into account condition \( \lambda_{\text{min}} \gg d \) and perfect conductivity of the metal, we may simplify both sides of Eq.(1) to obtain

\[
[\vec{n}, \vec{H}]|_{x \in [D,D-d]} \approx \frac{4\pi}{c} \left\{ \frac{\partial \vec{P}^{\text{surf}}(E,y,z,t)}{\partial t} + \vec{j}^{\text{surf}}(E,z,y,t) \right\}
\]  

(2)

where volume current density and the polarization vector in Eq. (1) have been replaced with corresponding surface values in Eq. (2).

Boundary condition (2) is valid for an arbitrary EM field. It determines nonlinear relation between fields \( \vec{E} \) and \( \vec{H} \) at the considered boundary \( (x \approx D) \) that contains the layer with nonlinear currents.

Taking scalar product of Eq. (2) with vector \( c\vec{E}/4\pi \), making cyclic permutation of the vectors in the left hand side and integrating over a surface \( S_0 \), we obtain the following expression:

\[
S_{x}^{NL} \equiv \frac{c}{4\pi} \int_{S_0} [\vec{n}, \vec{H}] n dS = -\int_{S_0} \vec{E} \frac{\partial \vec{P}^{\text{surf}}}{\partial t} dS - \int_{S_0} \vec{j}^{\text{surf}} \vec{E} dS.
\]  

(3)

Figure 1. Geometry of 1D resonator with nonlinear reflecting surface
Eq. (3) may be treated as an energy conservation law for the system under consideration. Left hand term in Eq. (3) is the Pointing vector of EM wave describing the EM energy flow through the surface $S_0$ perpendicular to it, and the right one is the work of the electric field over the surface currents of bounded and free charges of the layer within the area $S_0$. The work of the electric field $E$ over the currents is positive if the layer absorbs its energy. If the layer has an active media, then its energy can be transferred to the EM field as it happens in self-oscillatory systems. Following Eq. (3), in the first case the energy flow is directed against the normal vector $\vec{n}$ to the surface, and we have an EM energy sink, while in the second case the EM energy flows along the opposite direction, and we have an EM energy source. Power reflection coefficient is determined by the ratio of $X$-components of Poynting vectors for the incident and reflected waves:

$$\Gamma = 1 + \frac{S_{NL}}{S_L},$$

where $S_{NL}$ is determined by Eq. (3); $S_L$ is $X$-component of the Poynting vector for the incident wave. It is clear from Eq. (3) that the reflection coefficient can exceed unity (active layer), be less than unity (partial absorbance) and can be equal to 0 (absolute absorbance) depending on the sign and the value of the $E$-field work over the layer current.

3. Multiple scattering of nonstationary waves

Consider linear polarized plane wave with components $H_z$ and $E_y$ at normal incidence angel on the nonlinear reflecting structure. Assume that there is no dependence of the current density and polarization vector in the layer on the coordinates $z$ and $y$. Then the scattered field will not depend on these coordinates either. If we’ll put at the plane $x = 0$ a perfect conductive surface, where general boundary condition is valid

$$E_y |_{x=0} = 0 ,$$

then the excited wave will scatter repeatedly on the nonlinearily reflecting structure. In this way a 1D resonator with nonlinear reflection of EM waves has been formed. Spatiotemporal dynamics of nonstationary plane waves in this resonator is described by equations, following from homogeneous Maxwell equations

$$\frac{\partial E_y}{\partial x} = \frac{1}{c} \frac{\partial H_z}{\partial t},$$
$$\frac{\partial E_z}{\partial x} = \frac{1}{c} \frac{\partial E_y}{\partial t}$$

being supplied with the proper initial conditions for electric and magnetic fields and also boundary condition Eq. (2) and Eq. (4)

Linear equations of that type Eq. (5) with nonlinear boundary conditions have been considered, for example, in [3], [4], [5] for analysis of elastic oscillations in a bar with nonlinear elastic fastening of its ends; waves in periodical structures with local insertions of nonlinear elements, and nonlinear oscillations in transmission lines, respectively. However those problems have been solved approximately using asymptotic methods.

Introducing Riemann invariants $u_{\pm} = H_z \pm E_y$ we may transform Eq. (5) to the following form

$$\frac{\partial u_{\pm}}{\partial \tau} \pm \frac{\partial u_{\pm}}{\partial \xi} = 0,$$

where $\xi = x/D$, $\tau = ct/D$. 

3
Initial-boundary value problem for the functions $u_{\pm}(x, t)$ can be formulated as follows: it’s necessary to find functions $u_{\pm}(x, t)$ obeying equations (6) and satisfying both the initial conditions

$$u_{\pm}(\xi, 0) = U_{\pm} \equiv H_0(\xi) \pm E_0(\xi)$$

and boundary conditions (2) and (4) rewritten for the functions $u_{\pm}(x, t)$:

$$\left.\left(\frac{u_+(1, \tau) + u_-(1, \tau)}{2}\right)\right|_{\xi=1} = \frac{4\pi}{c} \left\{ \frac{\partial P_{\text{surf}}}{\partial t}[(u_+ - u_-)/2] + j_y \text{surf}[(u_+ - u_-)/2] \right\}$$

$$u_+(0, \tau) = u_-(0, \tau). \quad (7)$$

For simplicity we neglect the dipole current of bound charges $\frac{\partial P_{\text{surf}}}{\partial t} = 0$ in the second Eq. (7). Then initial boundary problem (6) – (7) is reduced to initial problem for nonlinear difference equation with continuous time using the method of characteristics (see for instance [5]).

Second condition (7) may be rewrite as follows

$$u_-(1, \tau) = -\nu(1, \tau) - \frac{4\pi}{c} j_y \text{surf}[\nu(1, \tau)] \equiv \Psi(\nu) \quad (8)$$

where $\nu(1, \tau) \equiv \frac{1}{2}[u_+(1, \tau) - u_-(1, \tau)]$.

It follows from Eq. (8)

$$u_+(1, \tau) = u_-(1, \tau) + 2\varphi[u_-(1, \tau)] \equiv \varphi[u_-(1, \tau)] \quad (9)$$

where $\varphi(u_-)$ is the inverse function of $\Psi(\bullet)$.

Solving equation (9) with respect to $u_-(1, \tau)$, we obtain

$$u_-(1, \tau) = g[u_+(1, \tau)], \quad (10)$$

where $g(u_+)$ is the inverse function of $\varphi(u_-)$.

Solutions for Eq. (6) are constant at their characteristics $\frac{d\xi}{d\tau} = \pm 1$. Using this property of the solutions to Eq. (6) and also boundary conditions (7) and (10) we derive the following nonlinear difference equation with the continuous argument for the function $u_+(1, \tau)$:

$$u_+(1, \tau + 2) = g[u_+(1, \tau)]. \quad (11)$$

This equation relates two values of the function $u_+(\xi, \tau)$ at the nonlinear structure ($\xi = 1$) for two different instants of time separated by the delay interval $T = 2D/c$, which is equal to round-trip propagation of the wave between the resonator planes. Solution $f(\tau)$ to the difference equation (11) with initial conditions

$$f(\tau)|_{\tau\in[-1,1]} = U(\tau) \equiv \begin{cases} U_+ : \tau \in [-1, 0) \\ U_- : \tau \in [0, 1) \end{cases} \quad (12)$$

gives solutions for the initial-boundary value problem (6) – (7) in the following form

$$u_+(\xi, \tau) = f(\tau \mp \xi).$$

Electric and magnetic fields can be calculated from the formulae for Reimann invariants introduced before Eq. (6):

$$E_y(x; t) = \frac{1}{2} \left[ f(\omega t - hx) - f(\omega t + hx) \right]$$
Having solutions for difference equation (11) one can calculate exactly the system behavior of the EM fields in real systems with a known law of nonlinearity on the electric field of the wave. This allows to make qualitative conclusions about the behavior ν the topology of the map Ψ(ν) is homeomorphic to the Cantor’s set.

Thus, spatiotemporal dynamics of EM plane wave in 1D resonator with nonlinear reflecting boundary is fully determined by the solutions of difference equation (11). The solutions to Eq. (11) are determined by trajectories of 1D map g(u). In the theory of dynamical systems defined by 1D maps, fundamental results have been obtained which allowed to define the character of the system motion and predicting its behavior with varying of map parameter. This also enabled to specify criteria for identification of system motion modes: regular, pre-turbulent and turbulent, etc. One of the main results of the theory is the famous Sharkovsky’s theorem [6], [5]. According to this theorem any continuous map of an interval g(u) : J → J along with a cycle of period m includes also cycles of all previous periods m’ in the following order (Sharkovsky’s order):

\[ 1 \prec 2 \prec 2^2 \prec 2^3 \prec ... \prec 5 \cdot 2^2 \prec 3 \cdot 2^2 \prec ... 5 \cdot 2 \prec 3 \cdot 2 \prec ... 7 \times 5 \prec 3 \]

(14)

In particular, it means that if for certain value of the parameter, the map g(u) has a stable cycle of period 3, then it also has cycles with all periods defined in Sharkovsky’s order (14), which are unstable ones. So we obtain the mechanism for transition (with map parameter change) to chaotic solution of deterministic system via consequence of period-doubling bifurcations of cyclic motions that was obtained independently and properly studied much later by Fejhenbaum (1980).

Earlier considered problems of this type are usually reduced (approximately in most cases) to 1D maps with discrete time. Unlike these approaches the initial-boundary value problem under consideration has been reduced (mathematically rigorous) to the equivalent 1D map with continuous time, In particular this means that:

(i) Phase space of the system under consideration is infinite;
(ii) For an analysis of the system behavior one has to know the structure of the trajectories set for the map g(u) and its intersection with “initial set”, initial conditions (12);
(iii) Having solutions for difference equation (11) one can calculate exactly the system behavior at any instant of time at any point of the interval (0, D).

Numerical analysis of Eq. (11) for the map function g(u) = A u (1 − u^2) has been done in [8] when investigating generators with delayed feedback. Dependence of Eq. (11) solutions and their asymptotic behavior (at t → ∞) on a set of map g(u) trajectories and its intersection with “initial set” are specified in [5].

According to these papers the behavior of solution u(t) depends on what kinds of the attractive subintervals J_i ⊂ J of the set of map attractive cycles g(u) intersect the initial set. These intervals are specified by the combining of finite or countable number of opened nonintersecting intervals J\W, where W is the set of points dividing the interval J into the subintervals. This set has a property of repulsion trajectories. If map g(u) has cycles only with period 1 or 2, then W is either finite or countable. If g(u) has cycles, period of which differs from 1, 2 and T_n, then the set W is uncountable. In the last case W is a mixing repeller, which is homeomorphic to the Cantor’s set.

From the derivation of the map g(u) (see Eq. (8) – Eq. (11)) it is obvious that it preserves the topology of the map Ψ(ν), that is defined by nonlinear dependence of surface current density on the electric field of the wave. This allows to make qualitative conclusions about the behavior of the EM fields in real systems with a known law of nonlinearity j(E). In particular, a cubic nonlinearity of the current dependence j(E) is common for dielectrics with center of inversion,
Figure 2. Solutions for Eq. (11) with the map function (15):
a – regular periodic solution for $A = 2.15$; b – “dry turbulence” solution for $A = 2.46$

electron beams synchronous with a wave, semiconductor devices, plasmas, etc. So it will be of high interest to consider the following map

$$g[u(t)] = A u(t)[1 - u^2(t)],$$

(15)

where $u(t) \equiv u_+(1, t)$.

If parameter $A$ lays within the interval $0 < A < 2.6$ then Eq. (15) defines the map of the interval $[0,1]$ into itself. When $2 < A < 2.286$ the map Eq. (15) has a cycle of period 2 [8]. With further increase in $A$, bifurcations of period doubling and further bifurcations, defined by Sharkovskiy’s order take place. For $A = 2.46$ this mapping has the unique attracting cycle with the period 3 ($u_1 = 0.2439..., u_2 = 0.5643..., u_3 = 0.9461...$) and mixing repeller according to [5]. In this case exponential increasing of oscillation frequencies characterizes the solutions for this equation, and so called “dry turbulence” takes place as has been shown in [5].

Figure 2 shows examples of two qualitatively different solutions of Eq. (11) with the map (15) obtained for two different values of parameter $A$. Periodic solution was obtained for $A = 2.15$ and is shown in the Figure 2a. For $A = 2.58$ the solution has chaotic structure and depicted in Figure 2b.

It follows from Eq. (13) that behavior of the electric and magnetic fields of the scattered waves might be similar. Indeed using the solutions for Eq. (11) we may readily construct spatiotemporal dependences for electric and magnetic fields in the resonator applying Eq. (13). Figures 3 and 4 shows initial stage of spatiotemporal evolution of electric field, $E$, in the resonator.
under consideration with nonlinear reflecting wall is placed at \( x = 1 \) in the case of regular period 2 solution for Eq. (11) with cubic parabola map (15): \( A = 2.15 \). It is clearly seen formation of squared surfaces in the areas where the E and H-fields have maximum values and that those areas are in opposite walls of the resonator as it should be according to boundary conditions (7).

Figure 5 and 6 shows initial and developed stages, respectively, of spatiotemporal dynamics of magnetic field, for the case of “dry turbulent” solution of Eq. (11) with map (15), \( A = 2.46 \). It is seen as smooth spatiotemporal pattern of the resonator magnetic field is transforming into rather complicated irregular structure, which may be called as turbulent state of the electromagnetic field in the resonator with nonlinear reflections.

So, multiple/repeated scattering of plane EM waves by nonlinear reflecting structure has a complicated spatiotemporal dynamics and is accompanied with an exponential growth of the frequency spectrum.

4. Finite conductivity of metal and reflection inertia effects

The above results have been obtained for an idealized situation. At first, perfect conductivity of metal surfaces was assumed, that is equivalent to neglecting of EM energy dissipation and inertia of EM field reflections taking place in the system. EM energy losses and reflections inertia in the systems under consideration emerge because of finite conductivity of the metal surfaces, energy dissipation in nonlinear layer, and also as a result of wave radiation out of the resonator through holes in metal surfaces which also may depart from plane geometry. Due to Ohmic losses inside metal surfaces, the boundary condition (4) is to be changed to the impedance one. It has been shown in [9] that for nonstationary fields with relatively narrow Fourier spectrum
Figure 4. Spatiotemporal evolution of magnetic field, $H$, in 1D resonator with nonlinear reflecting wall ($x = 1$) for the case of regular periodic solution for Eq. (11) with cubic parabola map (15); cycle of period 2; $A = 2.15$; time interval $\{75 - 100\}$

Figure 5. Initial stage of spatiotemporal evolution of magnetic field, $H$, in 1D resonator with nonlinear reflecting wall ($x = 1$) for the case of “dry turbulence” solution for Eq. (11) with cubic parabola map (15); cycle of period 3; $A = 2.46$; time interval $\{0 - 16\}$
Figure 6. Spatiotemporal evolution of magnetic field, $H$, in 1D resonator with nonlinear reflecting wall ($x = 1$) at the stage of developed “dry turbulence” solution for Eq. (11) with cubic parabola map (15); cycle of period 3; $A = 2.46$; time interval $\{98 – 100\}$

bandwidth the following impedance boundary condition can be used:

$$E(0, y, z, t) = R H(0, y, z, t) + L \frac{dH(0, y, z, t)}{dt}$$

(16)

where $Z = R + i\omega L$ is a complex surface impedance; $R$ is surface resistance and $L$ is surface inductance of the reflecting surface. Let’s assume that Ohmic losses are sufficient only in the plane $x = 0$. Then, applying the method of characteristics, the problem of repeated wave scattering by nonlinear reflecting surface with a finite metal conductivity is reduced to the following differential-difference equation

$$\chi \left\{ \frac{du(\tau + 2)}{d\tau} + \frac{d\varphi}{du} \frac{du(\tau)}{d\tau} \right\} + \frac{1 - R}{2(1 + R)} u(\tau + 2) = \varphi[u(\tau)]$$

(17)

where $\chi = \frac{LD}{2\omega(1+R)}$.

Eq. (17) is a difference equation of neutral type. However, for sufficiently smooth maps the following condition is valid

$$\left| \frac{d\varphi}{du} \right| \ll |\varphi[u(\tau)]|,$$

and we may reduce Eq. (17) to the differential-difference equation with delay argument

$$\chi \frac{du(\tau + 2)}{d\tau} + R_s u(\tau + 2) = \varphi[u(\tau)]$$

(18)

where $R_s = \frac{1 - R}{2(1 + R)}$.

Eq. (18) is suitable for description and analysis of the spatiotemporal dynamics of EM fields in 1D resonators formed by planes with inertial nonlinear reflections and energy dissipation within the frame of the theory of ordinary time-delay difference equations elaborated in [5]). In particular, they have shown that for $\chi \ll 1$ and $\varphi = Au(1 - u)$ the existence of chaotic
solutions for this equation has been shown. Important to underline that general properties of
the solutions of Eq. (18) are defined by the solutions of the related difference equation (when
$\chi = 0$). Due to topological equivalence of two maps $\varphi = Au(1 - u)$ and $\tilde{\varphi} = Au(1 - u^2)$ they
have similar solutions, but for different values of parameter $A$. For instance, turbulent solutions
exist at $A = 3.83$ for function $\varphi$ while for the function $\tilde{\varphi}$ it appears at $A = 2.46$.

Boundary condition (16) can be also applied in case of EM losses in the system are due to
the field leakage through little holes coupling the resonator with outer space, etc. Indeed in
this case linear relation between time-harmonic components of the fields $E$ and $H$ on the slot
boundary is valid and may be expressed as follows:

$$E(\omega) = Z(\omega)H(\omega),$$ (19)

where $Z(\omega)$ is “equivalent impedance” which may be found from solution of the related
diffraction problem. For narrow band processes we may assume $\omega = \omega_0 + \delta\omega$ ($\delta\omega \ll \omega_0$) and applying inverse Fourier transform over variable $\delta\omega$ to the Eq. (19) we obtain the relation

$$E(t) \approx Z(\omega_0)H(t) + \frac{dZ(\omega_0)}{d\omega_0} \frac{dH(t)}{dt}$$ (20)

where $H(t) = \tilde{H}(t)\exp(-i\omega_0t)$ and $E(t) = \tilde{E}(t)\exp(-i\omega_0t)$, $\tilde{H}(t)$ and $\tilde{E}(t)$ are slowly varying amplitudes of the magnetic and electric fields, respectively. Boundary condition (20) is similar
to the impedance boundary condition (16) and may be applied accordingly.

5. Conclusions
Multiple scattering of the electromagnetic waves by the surface with nonlinear reflection is
rather common phenomenon in various physical experiments and electronic devices. We have
formulated a new class of electromagnetic problems: initial-boundary value problems for linear
wave equation with nonlinear boundary conditions. This problem has been reduced to the initial
problem for nonlinear difference equation or differential-difference equation with continuous
argument, which is much simpler of the primary problem, but still gives a complete description
of spatiotemporal dynamics of the EM field in such nonlinear resonator. In spite of its simplicity
solutions to this type of problems allow to reveal the main physical phenomenon, which are
applicable to realistic EM systems and electronic devices. It has been shown that in such
systems spatiotemporal behavior of EM field may be periodic, regular or deterministic, and also
chaotic one in spite of major part of the system is formed by linear media. The results obtained,
have shown applicability of this approach to analysis of nonstationary process in Diffraction
Radiation Generator (DRG) and other open resonator based devices with active elements, such
as microwave diodes, transistors, etc. Besides, this approach has been applied for description of
complex dynamics of currents formed by electrons and holes in semiconductor PNIPN- reversed
biased structure with nonlinear transformation of these currents one each other at the PN-
junctions due to avalanche multiplication of the carriers [10] and [11].

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