Conformal transformations are obtained by demanding that the form of the metric change by a "conformal" factor. Nevertheless, in the literature, this transformation of the metric is not taken into account when a variation of the action is performed. As a consequence, it is obtained that massive particles are not invariant under the conformal transformations, and that the scale dimension \( d \) of the fields coincides with the natural dimension of the fields, in the sense of dimensional analysis. The basic purpose of this paper is to take the transformation of the metric into the variation of the action. When this is done, we obtain now that even massive particles are invariant under the conformal transformations. Also, the scale dimension \( d \) of the fields does not coincide with the natural dimension of the fields, but seems to be related to the tensorial character of the fields.

I. INTRODUCTION

Besides invariant under the Poincaré transformations (translations and Lorentz transformations), Maxwell’s electromagnetic action possesses a wider invariance, namely, invariance under the conformal transformations. They were first obtained by Cunningham and Bateman [1]. (For a historical review see [2].) Since then, the conformal transformations have been widely studied. (See [3, 4, 5, 6, 7, 8, 9] and references there in.) In all these works it is asserted that the masses of particles breaks dilatation, and consequently also conformal, invariance. It is a well known result too, that quantically, dilatation can not be a symmetry of nature, because it would imply that all masses vanishes, or that the mass spectrum is continuous [10].

Nevertheless, although the conformal transformations are just obtained imposing that the form of the metric change by a "conformal" factor, it seems strange that it is usually assumed that the metric does not change. In fact, in [3, 10] they give and argument to suppose that. In this paper we will not use this argument because it seems to violate the isometries of the metric. As is well known, an isometry is a coordinate transformation which leaves the form of the metric invariant [11]. For the case of the Minkowski metric, only the Poincaré transformations are the isometries of the metric (Killing vectors), while the conformal transformations are not.

Consequently, in this paper we will take the change of the metric seriously, and re-examine the question of the conformal transformations. For example, in the variation of the Lagrangian, besides the usual variation of the fields, we have to take also the variation of the metric. When the transformation is an isometry of the metric, this variation vanishes. But, for transformations that are not isometries, the variation of the metric introduces one extra term, which is proportional to the symmetric energy-momentum tensor. This extra term implies that Noether theorem no longer gives a conserved current even when the action is invariant under this transformation. Only when the trace of the symmetric energy-momentum tensor vanishes we obtain a conserved current. Also, massive particles turn out now to be invariant under the conformal transformations. This is because for massive particles the trace of the symmetric energy-momentum tensor has a mass term which cancels the usual mass term that would appear. The invariance condition of the action under the conformal transformations, implies that the scale dimension \( d \) is related to the tensorial character of the fields.

We organize the paper as follows: in Sec. II we give just a brief review of the conformal transformations. Then, in Sec. III we introduce the variation of the metric and re-obtain Noether theorem for this case. Based on this, we reconsider the conformal transformations in Sec. IV. We apply these new considerations to the massive scalar and spinor field, and to the electromagnetic field in Sec. V. Finally, we draw the main conclusions of the paper in Sec. VI.

II. CONFORMAL TRANSFORMATIONS

We will use the Greek alphabet \((\mu, \nu, \rho, \ldots = 0, \ldots, 3)\) to denote spacetime indices. In the following, we are going to work in the usual four-dimensional Minkowski spacetime, with the Minkowski metric \( \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1) \).

The conformal transformations are obtained by demanding that the metric changes its form by

\[
g_{\mu \nu}(x) = \lambda(x) \eta_{\mu \nu}. \tag{1}
\]

We easily see that this transformation of the metric leave the light-cone \( ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu = 0 \) invariant. Infinites-
In general, let $\lambda(x) = 1 + \Omega(x)$, the change in the form of the metric is

$$\delta \eta_{\mu\nu} = g_{\mu\nu}(x) - \eta_{\mu\nu} = \Omega(x) \eta_{\mu\nu}. \quad (2)$$

Now, under the infinitesimal coordinate transformation

$$x'_{\mu} = x_{\mu} + \delta x_{\mu}(x), \quad (3)$$

Minkowski metric $\eta_{\mu\nu}$ changes its form according to

$$\tilde{\delta} \eta_{\mu\nu} = g_{\mu\nu}(x) - \eta_{\mu\nu} = -\eta_{\rho\mu}\partial_{\rho}\delta x^{\rho} - \eta_{\rho\nu}\partial_{\rho}\delta x^{\rho}. \quad (4)$$

We should remark that although Minkowski metric $\eta_{\mu\nu}$ is constant, the coordinate transformed metric $g_{\mu\nu}$ is not necessarily equal to $\eta_{\mu\nu}$, nor even constant. Under an arbitrary coordinate transformation, the metric $g_{\mu\nu}$ will in general depend on the spacetime coordinates. Therefore, when we talk about the variation of the Minkowski metric, we mean according to (4), the difference at the same point between the coordinate transformed metric $g_{\mu\nu}$, and Minkowski metric $\eta_{\mu\nu}$. Then, substituting (2) in (4), we arrive at the following equation,

$$\eta_{\mu\nu}\partial_{\rho}\delta x^{\rho} + \eta_{\rho\nu}\partial_{\rho}\delta x^{\rho} = \frac{1}{2} \eta_{\mu\nu}\partial_{\rho}\delta x^{\rho}, \quad (5)$$

where $\Omega(x) = -(\frac{1}{2})\partial_{\rho}\delta x^{\rho}$. This is called the conformal Killing equation. The Poincaré transformations are a particular solution to this equation with $\Omega(x) = 0$, that is, they are solution of the Killing equation $\delta \eta_{\mu\nu} = 0$. The solutions of the conformal Killing equation (5) are the conformal transformations

$$\delta_{D} x^{\mu} = ax^{\mu}, \quad (6)$$

which are called the dilatations, and

$$\delta_{S} x^{\mu} = 2x^{\mu\nu}c_{\nu}x^{\nu} - c^{\mu}x_{\nu}x^{\nu}, \quad (7)$$

which are called the special conformal transformations. (For a good review on the conformal transformations see [10].) Therefore, the conformal transformations are coordinate transformations, as can be seen from the l.h.s. of (6), which changes the form of the metric according to (2), the r.h.s. of (6).

III. THE VARIATION OF THE METRIC AND NOETHER THEOREM

Now, comes two important and crucial questions: Is Minkowski metric $\eta_{\mu\nu}$ invariant under the conformal transformations? Secondly, if Minkowski metric $\eta_{\mu\nu}$ is not invariant under the conformal transformations, Should we take the variation of the metric when we vary the action?

Usually in the literature [3, 4, 7, 8, 9, 10], it is assumed that Minkowski metric $\eta_{\mu\nu}$ does not changes under the conformal transformations. In fact, in [3, 10] they use the Weyl rescaling together with the conformal transformations to impose that $\delta \eta_{\mu\nu} = 0$. As remarked in [3], without the Weyl rescaling we can not obtain that $\delta \eta_{\mu\nu} = 0$ under the conformal transformations. (Note that as we are considering coordinate transformations, by $\delta \eta_{\mu\nu}$ we mean the difference between $g_{\mu\nu}$ and $\eta_{\mu\nu}$, as explained below (4), and this difference will in general not vanishes.) Besides that, it seems strange that the Weyl rescaling is used just to the metric, and not to the other fields too. In this work we will not use the Weyl rescaling argument, and therefore Minkowski metric $\eta_{\mu\nu}$ is not invariant under the conformal transformations. This seems to be the natural answer, because the conformal transformations (6) and (7) satisfy the conformal Killing equation (5), and not a killing equation. Only translations and Lorentz transformations are the Killing vectors of $\eta_{\mu\nu}$ [11]. For example, substituting (6) in (4) we obtain that

$$\tilde{\delta}_{D} \eta_{\mu\nu} = -2a\eta_{\mu\nu}, \quad (8)$$

which is of the form (4) with $\Omega(x) = -2a$. Now, substituting (7) in (4) we obtain that

$$\tilde{\delta}_{S} \eta_{\mu\nu} = -4c_{\mu}x^{\rho}\eta_{\mu\nu}, \quad (9)$$

which is of the form (4) with $\Omega(x) = -4c_{\mu}x^{\rho}$. Although Minkowski’s metric is not a dynamical field, it changes under the conformal transformations, and we should take this change of the metric when we make a variation in the action.

Let us begin, then, giving the action of a general field $\Phi$,

$$A = \int d^4x \ L(\Phi, \partial_{\mu}\Phi), \quad (10)$$

where $L(\Phi, \partial_{\mu}\Phi)$ is the Lagrangian of the field $\Phi$, and it depends only on the field and its first derivative. Under the infinitesimal coordinate transformations (3), we have a transformation in the field $\Phi$ given by [11, 12]

$$\delta \Phi(x) = \Phi'(x') - \Phi(x) = \delta \Phi(x) + \delta x^{\rho}\partial_{\rho}\Phi(x), \quad (11)$$

where $\delta \Phi(x) = \Phi'(x) - \Phi(x)$ is a variation just in the form of the field. The transformations (3) and (11) induces the following transformation on the Lagrangian,

$$\delta L = \tilde{\delta} L + \delta x^{\mu}\partial_{\mu}L, \quad (12)$$

where

$$\tilde{\delta} L = \frac{\partial L}{\partial \Phi}\delta \Phi + \frac{\partial L}{\partial (\partial_{\mu}\Phi)}\delta (\partial_{\mu}\Phi) + \frac{\partial L}{\partial \eta_{\mu\nu}}\delta \eta_{\mu\nu}, \quad (13)$$

is the variation in the form of the Lagrangian. Note that we have also taken the variation in the form of the metric. This term just vanishes for isometries transformations. As is well known, the variation of the Lagrangian with respect to the metric is

$$\frac{\partial L}{\partial \eta_{\mu\nu}} = -\frac{1}{2}\sqrt{-\eta} \bar{T}^{\mu\nu}, \quad (14)$$

where
where $\eta = \text{det}(\eta_{\mu\nu})$, and $T^{\mu\nu}$ is the Symmetric Energy-Momentum Tensor (SEMT), that is, the Canonical Energy-Momentum Tensor (CEMT) symmetrized through the Belinfante procedure \[13\],

$$T^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial x^\rho} \Phi - \delta^{\mu\nu} L,$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \frac{\partial}{\partial x^\rho} \Phi \delta^{\mu\nu} - \delta^{\mu\nu} \partial^\rho \Phi,$$

is the definition of the CEMT,

$$\varphi^{\mu\nu} = -\varphi^{\mu\nu} = S^{\mu\nu} + S^{\mu\rho} - S^{\nu\rho},$$

with

$$S^{\rho\mu} = \frac{1}{2} \frac{\partial L}{\partial (\partial_\rho \Phi)} S_{\mu\nu},$$

the definition of the Spin Tensor, and $S_{\mu\nu}$ the spin generator in an appropriate representation to the field $\Phi$.\[10\]

As $\delta d^4 x = \partial_\rho \delta x^\rho dx^\lambda$, and using \[12\], the invariance of the action under the transformations \[13\] and \[11\] implies that \[10\]

$$\delta L = L \partial_\rho \delta x^\rho + \delta L + \delta x^\rho \partial_\rho L = 0.$$ \hspace{1cm} (19)

This is a condition involving the Lagrangian which must be satisfied if we want $\delta A = 0$ for some symmetry transformation. As $\delta (\partial_\rho \Phi) = \partial_\rho (\delta \Phi)$, then, doing a partial integration in \[13\] and substituting in \[14\], we obtain a similar form of the usual Noether theorem,

$$\frac{\delta L}{\delta \Phi} - \partial_\mu J^\mu = - \frac{\partial L}{\partial \eta_{\mu\nu}} \delta \eta_{\mu\nu},$$ \hspace{1cm} (20)

where

$$\frac{\delta L}{\delta \phi} = \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \Phi)} \right),$$ \hspace{1cm} (21)

is the Euler-Lagrange functional variation, and $J^\mu$ is the definition of the current,

$$J^\mu = - \frac{\partial L}{\partial (\partial_\mu \Phi)} \delta \Phi - \delta x^\mu.$$ \hspace{1cm} (22)

When the transformation is an isometry of the metric, we easily see that the r.h.s. in \[20\] vanishes. Then, provided that we are in the field equations, $\delta L/\delta \Phi = 0$, the current $J^\mu$ is conserved, $\partial_\mu J^\mu = 0$. Therefore, invariance of the action under some isometry transformation implies a conserved current \[14\]. For example, invariance of the action under translations implies that the conserved current is the Total Angular Momentum Tensor, \[10\], while invariance of the action under Lorentz transformations implies that the conserved current is the Canonical Energy-Momentum Tensor. But, for transformations that are not isometries of the metric, there is the extra term in the r.h.s. of \[20\]. Therefore, in this case, we can see from \[20\] that even if the transformation is a symmetry of the action, $\delta A = 0$, and when we are in the field equations, $\delta L/\delta \Phi = 0$, this does not necessarily means that we have a conserved current. As the conformal transformations are of this kind, let us see how the variation of the metric modifies the earlier considerations on conformal transformations.

IV. CONFORMAL TRANSFORMATIONS AND NOETHER THEOREM

A. Dilatations

Under the dilatations transformations \[4\], the general transformation law of the field $\Phi$ is \[8\], \[5\], \[6\], \[7\], \[11\], \[10\], \[12\], \[13\], \[11\], \[10\]

$$\tilde{\delta}_D \Phi = -a(x^\mu \partial_\mu \Phi + d\Phi),$$ \hspace{1cm} (24)

where $d$ is the scale dimension of the field $\Phi$. We can write \[8\] in the form of \[24\] provided that $d(\eta_{\mu\nu}) = 2$. Note that as remarked in \[8\] and \[5\], we should not confuse the scale dimension of the field with its natural dimension, in the sense of dimensional analysis. Therefore, the fact that we are choosing $d(\eta_{\mu\nu}) = 2$, does not mean that we are assigning a natural dimension to the metric. Then, substituting \[24\] in \[15\] and using \[11\] and \[8\], we obtain that

$$\tilde{\delta}_D L = -a x^\mu \partial_\mu L - ad \frac{\partial L}{\partial \Phi} \Phi - a(d+1) \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial_\mu \Phi + aT^{\mu \mu},$$ \hspace{1cm} (25)

where $T^{\mu \mu}$ is just the trace of the SEMT, and where we used that the action is invariant under translations \[8\], \[5\], \[11\], \[10\],

$$\partial_\mu L = \frac{\partial L}{\partial \Phi} \partial_\mu \Phi + \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial_\mu \Phi.$$ \hspace{1cm} (26)

Then, substituting \[25\] in \[14\] and using \[8\], the variation of the action is

$$\Delta_D L = 4aL - ad \frac{\partial L}{\partial \Phi} \Phi - a(d+1) \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial_\mu \Phi + aT^{\mu \mu}.$$ \hspace{1cm} (27)

Invariance of the action under dilatations, $\Delta_D L = 0$, implies that

$$4L = d \frac{\partial L}{\partial \Phi} \Phi + (d+1) \frac{\partial L}{\partial (\partial_\mu \Phi)} \partial_\mu \Phi - T^{\mu \mu}.$$ \hspace{1cm} (28)

Without the last term, this equation is usually interpreted as saying that only massless fields are invariant under dilatations \[8\], \[5\], \[6\]. But, as we will see below, the last term just cancels the mass terms, and we can say that
dilatation invariance tell us how we can “decompose” the Lagrangian. Now, substituting (28) in (25), we obtain
\[ \delta_D \mathcal{L} = -a x^\mu \partial_\mu \mathcal{L} - 4a \mathcal{L}. \]  
(29)
We see that if the action is invariant under dilatations, the variation of the Lagrangian can be written as in (21) with \( d(\mathcal{L}) = 4 \). Now, comes an important point. The number 4 appearing in the above variation came from the term \( \mathcal{L} \partial_\mu \delta x^\mu = L \partial_\mu (a x^\mu) = 4a \mathcal{L} \). Consequently, the number 4 that appears in the variation of the Lagrangian does not seem to be related with the natural dimension of the Lagrangian. The number 4 came from the dimension of the spacetime, which in this case is the four-dimensional Minkowski spacetime. Therefore, as we will see below, the parameter \( d \) has nothing to do with the natural dimension of the fields when we consider the variation of the metric.

Now, substituting (16), (24), (24) and (14) in (20), and assuming that we are in the field equations, we obtain that
\[ \partial_\mu D^\mu = T^\mu_\mu, \]  
(30)
where
\[ D^\mu = x^\nu \epsilon^\mu_\nu + d \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \Phi, \]  
(31)
is the dilatation current. Therefore, the dilatation current is not conserved unless the trace of the SEMT vanishes. As the trace of the SEMT is proportional to the mass of the field, we see, in this way, why massive fields do not have a conserved dilatation current.

B. Special conformal transformations

The general transformation law of the field \( \Phi \) under the special conformal transformations (17) is (4, 5)\note{4, 5, 8, 9, 10} to (17),
\[ \delta_S \Phi = -c^\phi (2 x^\mu \partial_\nu \Phi + 2 d x^\mu \partial_\nu \Phi - x^\mu \partial_\nu x^\rho \partial_\rho \Phi - 2ix^\nu S_{\mu \nu} \Phi). \]  
(32)
The transformation of the metric (17) can be written in this form, provided that we use the appropriate representation of the spin generator for second rank tensors (12). Substituting (32) and (17) in (19), we obtain
\[ \Delta_S \mathcal{L} = 2 \epsilon^\phi (2 x^\mu \partial_\nu \Phi + 2 d x^\mu \partial_\nu \Phi - x^\mu \partial_\nu x^\rho \partial_\rho \Phi - 2ix^\nu S_{\mu \nu} \Phi) \]  
(33)
where we used (16), (14) and that the Lagrangian is invariant under translations (16) and Lorentz transformations (5, 8, 10),
\[ \frac{\partial \mathcal{L}}{\partial \Phi} S_{\mu \nu} \Phi + i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\rho (S_{\nu \rho} \Phi) \]  
\[ - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} (\eta_{\rho \nu} \partial_\rho \Phi - \eta_{\nu \rho} \partial_\rho \Phi) = 0. \]  
(34)
Therefore, in order to have invariance under the special conformal transformations, \( \Delta_S \mathcal{L} = 0 \), we must have, first of all, invariance under dilatations transformations (28). Then, the remaining condition is just
\[ S^\mu_\nu - d \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} \eta_{\mu \nu} \Phi = 0, \]  
(35)
where we used the definition of the Spin tensor (18). Consequently, if we can write the “trace” of the Spin tensor in the above manner, then, the action will be invariant under the special conformal transformations.

Now, substituting (7) and (32) in (20), and using (9) and (14), we obtain that
\[ \partial_\mu K^\mu_\nu = 2 x_\nu T^\mu_\mu, \]  
(36)
is the special conformal current. Consequently, the special conformal current is not conserved unless the trace of the SEMT vanishes.

In (6), they imposed (35) to be a divergence, and constructed a new improved energy-momentum tensor. However, if we impose (35) to vanish, the improved energy-momentum tensor becomes, essentially, the SEMT (19). This is consistent with (30) and (36), where instead of the improved energy-momentum tensor, there appears the SEMT. In fact, if we substitute (17) in (16) and (31), and use (35), after a partial integration we obtain that
\[ D^\mu = x^\nu T^\mu_\nu + \partial_\rho \left( \frac{1}{2} x^\nu \phi^{\rho \nu} \right), \]  
(38)
and
\[ K^\mu_\nu = (2x_\nu x^\rho - x_\rho x^\lambda) \delta^\rho_\nu \]  
\[ + \partial_\sigma \left[ \left( x_\nu x^\rho - \frac{1}{2} x_\lambda x^\lambda \delta^\rho_\nu \right) \phi^{\sigma \mu} \right]. \]  
(39)
As the last terms in (38) and (39) are just the divergence of an anti-symmetric tensor, we can discard them because they do not affect the “conservation” of the currents. It is worth to note that we could also put the conformal currents in a simple manner, similar to that in (6), without the necessity of constructing an improved energy-momentum tensor. Consequently, it seems that it is the SEMT that plays a fundamental role in the conformal transformations.

V. CONFORMAL TRANSFORMATIONS OF THE FIELDS

Now, we will apply the above considerations to the massive scalar and spinor field, and to the electromagnetic field. Let us begin, then, with the massive scalar
field. The Lagrangian of the scalar field is

\[ \mathcal{L} = \frac{1}{2} \eta^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} m^2 \phi^2. \]  

(40)

As for the scalar field \( \mathcal{T}_{\mu}^\nu = -2\mathcal{L} + m^2 \phi^2 \), then, substituting (40) in (27) we obtain

\[ \Delta_D \mathcal{L} = -2a d \mathcal{L}. \]  

(41)

We see that the mass term does not appear in the variation of the action. This is because the usual mass term that would appear from the variation of the field \( \phi \) is canceled by the mass term that comes from the \textit{SEMT}, which is the variation with respect to the metric. Invariance of the scalar field action under dilatations, \( \Delta_D \mathcal{L} = 0 \), implies that

\[ d(\phi) = 0. \]  

(42)

Note that as the scale dimension is not related with the natural dimension, the fact that the scale dimension of the scalar field \( \phi \) vanishes does not mean that its natural dimension also vanishes. From (24) we see that the change of the scalar field under dilatations becomes

\[ \delta_D \phi = -a x^\mu \partial_\mu \phi = -\delta_D x^\mu \partial_\mu \phi, \]  

(43)

where we used (39). The last term is the usual transformation law of the scalar field under general coordinate transformations. To prove invariance under the special conformal transformations, we just need to show that (35) is valid. As \( S_{\mu\nu} \phi = 0 \), then, the \textit{Spin} tensor vanishes. But, as \( d(\phi) = 0 \), the second term in (35) vanishes too. Therefore, the massive scalar field is also invariant under the special conformal transformations. (This can be seen, too, by explicitly substituting (43) in (35).) The transformation of the scalar field under special conformal transformations is, then,

\[ \delta_S \phi = -c^\nu (2x_\nu x^\mu \partial_\mu \phi - x_\nu x^\mu \partial_\nu \phi) = -\delta_S x^\mu \partial_\mu \phi, \]  

(44)

where we used (4). Therefore, the transformation law of the scalar field under the conformal transformations can be obtained from the transformation law of the scalar field under general coordinate transformations.

The spinor field Lagrangian is

\[ \mathcal{L} = \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi - m \bar{\psi} \psi. \]  

(45)

As for the spinor field \( \mathcal{T}_{\mu}^\nu = -3\mathcal{L} + m \bar{\psi} \psi \), then, substituting (45) in (27) we obtain

\[ \Delta_D \mathcal{L} = -2a d \mathcal{L}. \]  

(46)

We also see that the mass term of the \textit{SEMT} canceled the usual mass term. Invariance under dilatations, \( \Delta_D \mathcal{L} = 0 \), requires that

\[ d(\psi) = 0. \]  

(47)

Again, while the scale dimension of the spinor field \( \psi \) vanishes, its natural dimension does not. Similar to the scalar field, the transformation of spinor field under dilatations is

\[ \delta_D \psi = -a x^\mu \partial_\mu \psi = -\delta_D x^\mu \partial_\mu \psi, \]  

(48)

since the spinor field transform as a scalar under general coordinate transformations too. To see the invariance under the special conformal transformations, we just need to show that (35) is satisfied. As the \textit{Spin} tensor of the spinor field is totally anti-symmetric in its three indices, its trace vanishes. But, as \( d(\psi) = 0 \), both terms in (35) vanishes. Therefore, the spinor field is also invariant under the special conformal transformations.

From (31) and (30), we can see that for \( d = 0 \),

\[ \partial_\mu D^\mu = t^\mu = \mathcal{T}_{\mu}^\nu. \]  

(49)

This is clearly true for the scalar field, because for this field the \textit{CEMT} coincides with the \textit{SEMT}. For the spinor field, the \textit{SEMT} is just the symmetrized version of the \textit{CEMT}, unless of a 1/2 factor, so that (39) is also true.

For the electromagnetic field \( A_\mu \), all the usual considerations are still valid, because for this field \( \mathcal{T}_{\mu}^\nu = 0 \). It is not difficult to show from (27) that the condition for invariance under dilatations requires that \( d(A_\mu) = 1 \). The scale dimension and the natural dimension of the electromagnetic field \( A_\mu \) just coincide. We can show, too, that the trace of the \textit{Spin} tensor of the electromagnetic field can be written as in (35), so that, the electromagnetic field is invariant under the special conformal transformations. As with the scalar field, we can obtain the transformation law of the electromagnetic field \( A_\mu \) under dilatations (24) and special conformal transformations (52) from the transformation law of vectors fields under general coordinate transformation,

\[ \delta A_\mu = -d x^\nu \partial_\nu A_\mu - A_\nu \partial_\nu d x^\mu, \]  

(50)

just substituting (6) and (7), respectively. It is interesting to note that as the metric is not invariant under dilatations, for example, we have that

\[ \delta_D A_\mu = \bar{D} (\eta^{\mu\nu} A_\nu) = -a x^\nu \partial_\nu A_\mu + a A_\mu, \]  

(51)

where we used that \( d(\eta^{\mu\nu}) = -2 \). This can be put in the form of (24) if \( d(A_\mu) = -1 \).

**VI. FINAL REMARKS**

According to the conformal Killing equation (35), we are looking for coordinate transformations of the metric, the \textit{l.h.s.} of (35), that changes the form of the metric according to (2), the \textit{r.h.s.} of (35). Therefore, the conformal transformations are not isometries of the metric, because they do not obey a Killing equation. Hence, not only any general field \( \Phi \), but also spacetime changes under the conformal transformations. In this sense, there
seems to be no reason to consider also a Weyl rescaling of the metric to impose that the metric does not varies, as is usually done [8, 10]. Not only because the metric changes indeed, but also because it is considered only a Weyl rescaling of the metric, and not of the other fields.

When we take the variation of the metric in the variation of the action, we acquire an extra term which is, basically, the SEMT. Then, through a similar procedure to the usual Noether theorem, we see that we no longer obtain a conserved current, as can be seen from [20]. When the symmetry transformation is an isometry of the metric, we do obtain conserved currents. For the case of the conformal transformations, only fields for which the trace of the SEMT vanishes have conserved currents.

Similar results on the “conservation” of conformal currents were obtained in [14] through the definition of a new improved energy-momentum tensor. Nevertheless, provided that (25) vanishes, instead of being a divergence, this improved energy-momentum tensor becomes essentially the SEMT. Then, there would be no need to modify General Relativity.

The variation of the metric introduces a mass term, through the definition of a new improved energy-momentum tensor. As is usually done [6, 10], not only because the metric changes indeed, but also because it is considered only a Weyl rescaling of the metric, and not of the other fields. The invariance condition under dilatations [27] requires that \( d = 0 \) for fields that transforms as scalars under general coordinate transformations, and \( d = 1 \) for fields that transforms as vectors under general coordinate transformations. As \( d(\eta_{\mu
u}) = 2 \), and \( \eta_{\mu\nu} \) is a second rank tensor, it seems that the scale dimension \( d \) is related to the tensorial character of the fields. We should remark that, as pointed in [6, 8], the scale dimension \( d \) is not related to the natural dimensions of the fields. They just used to coincide. Now, that we are taking the variation of the metric, we see that they do not coincide anymore. We should note that the Lagrangian \( \mathcal{L} \) is, in fact, a scalar density, \( \mathcal{L} = \sqrt{-\eta}L \). Then, from [8] we can show that \( \delta_{D} \sqrt{-\eta} = -4a \sqrt{-\eta} \), so that from [20] we see that

\[
\delta_{D} L = -ax^{\mu} \partial_{\mu} L = -\delta_{D} x^{\mu} \partial_{\mu} L. 
\] (53)

So, \( L \) transforms as a scalar.

Finally, we have seen that the transformation law of the fields under the conformal transformations can be obtained from the transformation law of the fields under general coordinate transformations. This is as it should be, because the conformal transformations are, in fact, coordinate transformations. The fact that the action of the fields turns out now to be invariant under these transformations should also be expected, because the action should be invariant under coordinate transformations.

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