SYNTHETIC MTW CONDITIONS AND THEIR EQUIVALENCE UNDER MILD REGULARITY ASSUMPTION ON THE COST FUNCTION

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Abstract. Loeper’s condition in [8] and the quantitatively quasi-convex condition (QQ-conv) from [4] are synthetic expressions of the analytic MTW condition from [11] since they only require $C^2$ differentiability of the cost function $c$. When the cost function $c$ is $C^4$, it is known that the two synthetic MTW conditions are equivalent to the analytic MTW condition. However, when the cost function has regularity weaker than $C^4$, it is not known that if the two synthetic MTW conditions are equivalent. In this paper, we show the equivalence of the synthetic MTW conditions when the cost function has regularity slightly weaker than $C^3$.

1. Introduction

In this paper, we will consider MTW conditions which are necessary and sufficient conditions for the Hölder regularity of solutions to the Monge-Ampère equations which arise from the optimal transportation problems. The Monge-Ampère equation is a fully non-linear elliptic PDE of the form

\[ \det(D^2u - A(x, Du)) = \phi(x, Du), \]

where $A(x, p) = -D^2_{\xi\eta}c(x, exp_{\xi}(p))$ is a matrix valued function defined using the cost function from the optimal transportation problem (see Definition 2.1) and $\phi : X \times \mathbb{R}^n \to \mathbb{R}$ depends on the cost function and given data. The MTW condition is a condition on a $(2,2)$ tensor called MTW tensor which first appeared in [9]. The MTW tensor is defined using the cost function $c$ from the optimal transportation problem :

\[ MTW = D_{pp}^2A(x, p). \]

In [11], the authors had to assume the following condition to show the regularity of the solution to the optimal transportation problem : for any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$, we have

\[ MTW[\xi, \xi, \eta, \eta] = D_{pp}^2A(x, p)[\xi, \xi, \eta, \eta] \geq 0. \]

This condition is called the MTW condition or (A3w). When we have strict inequality in the inequality (2), the condition is called (A3s). (A3w) and (A3s) conditions
are used to show many regularity results such as Hölder regularity of solutions to (1) in [1], [8], [3] and Sobolev regularity in [1]. Moreover, It is proved in [8] that the MTW condition is a necessary and sufficient condition for the differential regularity of the solutions to (1) when the cost function is $C^4$.

In [8] and [4], the MTW condition is used to show the Hölder regularity results, but it is used in slightly different forms. In [8], the author showed that the MTW condition is equivalent to Loeper’s condition (See Definition 3.2). However, in [4], the MTW condition is used in the form of quantitatively quasi-convexity (QQconv) (See Definition 3.3).

In the papers [8] and [4], it is shown that the two conditions, Loeper’s condition and quantitatively quasi-convex, are shown to be equivalent to the MTW condition (2) under the assumption that the cost function is $C^4$. In addition, in the recent work of G. Loeper and N. Trudinger [7], it is shown that a condition called (A3v), which is slightly weaker than Loeper’s condition, is equivalent to the MTW condition when the cost function is $C^4$. However, the cost function does not have to be $C^4$ to satisfy Loeper’s condition and QQconv. Therefore, we can say that Loeper’s condition and QQconv are synthetic MTW conditions and (A3w) is an analytic MTW condition.

It is unknown that if synthetic MTW conditions are equivalent under milder regularity assumption on the cost function (when the cost function is not $C^4$). In this paper, we show that the synthetic MTW conditions are equivalent under some regularity assumption on the cost function $c$ weaker than $C^3$. The main theorem of this paper is the following:

**Theorem 1.1 (Main Theorem).** Let $X$ and $Y$ be compact subsets of $\mathbb{R}^n$, and let $c : X \times Y \to \mathbb{R}$ be a measurable function called the cost function. Assume that $c$ is twice differentiable in the sense that there exist the mixed hessian $D^2_{xy}c$ and $D^2_{yx}c$, and the mixed hessian are transpose of each other $D^2_{xy}c^T = D^2_{yx}c$. Suppose the conditions (Twisted), (Twisted*), (Non-degenerate), (Lip hessian), (cDomConv), and (cDomConv*) hold. Then if $c$ satisfies Loeper’s condition, then $c$ satisfies QQconv.

The conditions that we use in the main theorem will be explained in section 2. In section 3, we define the synthetic MTW type conditions : Loeper’s condition and QQconv, and prove some useful lemmas. In section 4, we show that if the cost function satisfies Loeper’s condition, then the cost function should satisfy QQconv away from the boundary $\partial Y$. In section 5, we show that the cost function satisfying Loeper’s condition should satisfy QQconv near the boundary $\partial Y$, and we combine the lemmas from sections 3, 4, and 5 to complete the proof of the main theorem.
We close this section by introducing some results about the MTW conditions. As mentioned, the MTW tensor was first introduced in [9]. The synthetic MTW conditions were appear in [8] and [4]. In [6], the authors showed that the MTW tensor is related to the sectional curvature of the manifold $X \times Y$ with a metric called Kim-McCann metric. Another result that is similar to [6] can be found in [5]. In [5], the authors show that the MTW tensor is related to the curvature of the manifold $TM$ equipped with some Kahler metric.

2. Assumptions on the cost function and domains

Let $X$ and $Y$ be compact subsets of $\mathbb{R}^n$. Let $c : X \times Y \to \mathbb{R}$ be a measurable function that is $C^2$ up to the boundary in the sense that there exist the mixed hessian $c_{xy}$ and $c_{yx}$ that satisfy $D^2_{x,y} c = D^2_{y,x} c$. We impose the following conditions on the cost function $c$ :

- **(Twisted)** $-D_x c(x, \cdot) : Y \to T^*_x X$ is injective for any $x \in X$.
- **(Twisted*)** $-D_y c(\cdot, y) : X \to T^*_y Y$ is injective for any $y \in Y$.
- **(Non-degenerate)** The mixed hessian $D^2_{x,y} c(x,y)$ is invertible for any $(x, y) \in X \times Y$.

These conditions are commonly used in many optimal transportation problems. In this paper, we impose one more condition :

- **(Lip hessian)** The mixed hessian $D^2_{x,y} c$ is Lipschitz.

(Lip hessian) condition is satisfied when the cost function is more regular, for example (Lip hessian) is satisfied when $c$ is $C^3$ up to the boundary.

The conditions (Twisted) and (Twisted*) imply the existence of inverse functions of $-D_x c(x, \cdot)$ and $-D_y c(\cdot, y)$. We name these inverse functions $c$-exponential map and $c^*$-exponential map respectively.

**Definition 2.1.** Let $x \in X$ and $y \in Y$. We denote the image of $Y_x$ under the map $-D_x c(x, \cdot)$ and $-D_y c(\cdot, y)$ by $Y^*_x$ and $X^*_y$ respectively :

$$ Y^*_x = -D_x c(x, Y), \quad X^*_y = -D_y c(X, y). $$

We define $c$-exponential map $exp^c_x : Y^*_x \to Y$ and $c^*$-exponential map $exp^c_\ast_y : X^*_y \to X$ by the inverse function of $-D_x c(x, \cdot)$ and $-D_y c(\cdot, y)$ respectively :

$$ -D_x c(x, exp^c_x(p)) = p, \quad -D_y c(y, exp^c_\ast_y(q)) = q. $$
Note that (Non-degenerate) condition implies that the functions $-D_x c(x, \cdot)$ and $-D_y c(\cdot, y)$ are bi-Lipschitz:

$$\frac{1}{\lambda}|y_1 - y_0| \leq | - D_x c(x, y_1) + D_x c(x, y_0)| \leq \lambda |y_1 - y_0|,$$

$$\frac{1}{\lambda}|x_1 - x_0| \leq | - D_y c(x_1, y) + D_y c(x_0, y)| \leq \lambda |x_1 - x_0|.$$ Moreover, (Non-degenerate) condition with compactness of domain $X \times Y$ implies existence of some positive constant $\alpha$ such that

$$\frac{1}{\alpha} \leq \|D^2_{xy} c\| \leq \alpha.$$ With this constant $\alpha$ and (Lip hessian) condition, we obtain a constant $\Lambda > 0$ such that

$$|D^2_{xy} c(x_1, y_1) - D^2_{xy} c(x_0, y_0)| \leq \Lambda |(x_1, y_1) - (x_0, y_0)|,$$

and

$$\left|\left[D^2_{xy} c(x_1, y_1)\right]^{-1} - \left[D^2_{xy} c(x_0, y_0)\right]^{-1}\right| \leq \Lambda |(x_1, y_1) - (x_0, y_0)|.$$ i.e. the mixed hessian and its inverse are Lipschitz.

In addition to the conditions on the cost function $c$, we need to impose some geometric conditions on the domain $X$ and $Y$:

(cDomConv) \( Y^*_x \) is convex for any $x \in X$.

(cDomConv*) \( X^*_y \) is convex for any $y \in Y$.

In the following sections, we always assume the conditions explained in this section hold.

### 3. Synthetic MTW type conditions

**Definition 3.1.** The MTW tensor of the cost function $c : X \times Y \to \mathbb{R}$ is a $(2,2)$ tensor defined by

$$MTW = D^2_{pp} A(x, p),$$

where $A(x, p) = -D^2_{xx} c(x, \exp^c_x (p))$. The cost function $c$ said to satisfy (A3w) condition if

$$(3) \quad MTW[\xi, \xi, \eta, \eta] \geq 0$$

holds for any $\xi \perp \eta$. If the cost function satisfy the inequality \(3\) with strict inequality, then the cost function is said to satisfy (A3s).

The following two conditions in next two definitions are equivalent to (A3w) when the cost function is $C^4$. See [8] and [4] for the proofs. However, we do not need $C^4$ regularity for the definitions.
Definition 3.2. Let $x_0 \in X$ and $v_1, v_0 \in Y^*_x$, and let $v_t = (1-t)v_0 + tv_1, \forall t \in [0, 1]$ and $f_t(x) = -c(x, \exp^c_{x_0}(v_t)) + c(x_0, \exp^c_{x_0}(v_t))$. The cost function $c$ is said to satisfy Loeper’s condition if it satisfies

$$(4) \quad f_t(x) \leq \max\{f_1(x), f_0(x)\}$$

for any $x, x_0 \in X$ and $v_1, v_0 \in Y^*_x$.

Definition 3.3. Let $f_t$ be as in Definition 3.2. The cost function is said to be quantitatively quasi-convex (QQconv) if there exists $M \geq 1$ such that

$$(5) \quad f_t(x) - f_0(x) \leq Mt(f_1(x) - f_0(x))_+$$

holds for any $x, x_0 \in X$ and $v_1, v_0 \in Y^*_x$.

Remark 3.4. It is easy to see that QQconv implies Loeper’s condition. In fact, if $f_0(x) \geq f_1(x)$ then

$$f_t(x) \leq Mt(f_1(x) - f_0(x))_+ + f_0(x) = f_0(x).$$

If $f_0(x) < f_1(x)$, then set $v'_0 = v_1$ and $v'_1 = v_0$ and use (5) like above.

Hereafter, we use notation $v_t = (1-t)v_0 + tv_1$ and

$$(6) \quad F(v) = -c(x_1, \exp^c_{x_0}(v)) + c(x_0, \exp^c_{x_0}(v))$$

Then (4) becomes

$$(7) \quad F(v_t) \leq \max\{F(v_1), F(v_0)\}$$

which is quasi-convexity of the function $F : Y^*_x \to \mathbb{R}$. (5) can be expressed using $F$ as well:

$$(8) \quad F(v_t) - F(v_0) \leq Mt(F(v_1) - F(v_0))_+.$$

Remark 3.5. Suppose the cost function satisfies Loeper’s condition. If $F(v_1) \leq F(v_0)$, then Loeper’s condition implies $F(v_t) \leq F(v_0)$ so that (8) holds. Therefore, when we show the implication from Loeper’s condition to QQconv, we only need to check the case when $F(v_1) > F(v_0)$.

Remark 3.6. Suppose the cost function satisfies Loeper’s condition. By Remark 3.5 consider $F(v_1) > F(v_0)$. If we have opposite inequality of (8) i.e.

$$F(v_t) - F(v_0) > Mt(F(v_1) - F(v_0)),$$

then we get $t < \frac{1}{M}$. In fact, Loeper’s condition implies

$$Mt(F(v_1) - F(v_0)) < F(v_t) - F(v_0) \leq F(v_1) - F(v_0).$$

This shows $t \leq \frac{1}{M}$. 
Lemma 3.7. Suppose the cost function $c$ satisfies Loeper’s condition. If there exists $r > 0$ such that \( [S] \) holds for any $x_0, x_1 \in X$ and $v_0, v_1 \in Y_{x_0}^*$ with $|v_1 - v_0| \leq r$, then the cost function is QQconv.

Proof. Suppose we have such $r$, and let $v_1 \in Y_{x_0}^*$ such that $|v_1 - v_0| > r$. Let $M' > \frac{1}{r} \text{diam}(Y) \geq \frac{1}{r} \text{diam}(Y_{x_0}^*)$. If $v_1$ satisfies \( [S] \) for any $t \in [0, 1]$ with the constant $M'$ instead of $M$, then we are done. Therefore we assume that

\[
F(v_1) - F(v_0) > M't(F(v_1) - F(v_0)) > 0.
\]

By Remark 3.6, we have $|v_1 - v_0| = t|v_1 - v_0| \leq t\text{diam}(Y_{x_0}^*) < r$.

Now, we choose $t' \in (t, \frac{1}{M'})$ such that

\[
\frac{1}{t'}(F(v_{t'}) - F(v_0)) = M'(F(v_1) - F(v_0)).
\]

Note that this is possible because $s \mapsto \frac{1}{s}(F(v_s) - F(v_0))$ is continuous on $[t, \frac{1}{M'}]$ and the value of this map is bigger than $M'(F(v_1) - F(v_0))$ at $s = t$ by \( [S] \). In addition, Loeper’s condition implies $F(v_{1/M'}) \leq F(v_1)$ so that when $s = \frac{1}{M'}$, we have $M'(F(v_{1/M'}) - F(v_0)) \leq M'(F(v_1) - F(v_0))$. Note that

\[
|v_0 - v_{t'}| = t'|v_1 - v_0| \leq \frac{1}{M'}|v_1 - v_0| \leq r.
\]

Combining the assumption and \( [10] \), we obtain.

\[
F(v_1) - F(v_0) \leq M \left( \frac{t}{t'} \right) (F(v_{t'}) - F(v_0)) = MM't(F(v_1) - F(v_0)).
\]

Note that the constant $MM'$ is uniform over $x_0, x_1$ and $v_0, v_1$. Therefore the cost function is QQconv with constant $MM'$.

For fixed $x_1, x_0 \in X$ and for $v_0 \in Y_{x_0}^*$, we use notation

\[
B^+_t(v_0) = \{ v \in B_t(v_0) | \langle v - v_0, \nabla F(v_0) \rangle \geq 0 \}.
\]

For the level sets and sublevel sets of $F$, we use notations

\[
L_{v_0} = \{ v | F(v) = F(v_0) \}, \quad SL_{v_0} = \{ v | F(v) \leq F(v_0) \}.
\]

Since $F$ is quasi-convex when Loeper’s condition holds, $SL_{v_0}$ is convex. Moreover, since $\nabla F(v_0)$ is the outer normal vector of $SL_{v_0}$, we have

\[
SL_{v_0} \subset \{ v | \langle v - v_0, \nabla F(v_0) \rangle \geq 0 \}.
\]

which implies that $B^+_t(v_0) \subset \mathbb{R}^n \setminus SL_{v_0}$. Therefore, for any $v \in B^+_t(v_0)$, we have $F(v) \geq F(v_0)$. 


Lemma 3.8. Suppose the cost function \( c \) satisfies Loeper’s condition. If there exists \( r > 0 \) such that (8) holds for any \( x_0, x_1 \in X, v_0 \in Y_{x_0}^* \) and \( v_1 \in B_{r}^c(v_0) \cap Y_{x_0}^* \), then the cost function is QQconv.

Proof. Suppose we have such \( r \). Suppose \( v_0 \in Y_{x_0}^* \), then by the proof of Lemma 3.7, for any \( v_1 \in Y_{x_0}^* \cap \{ v | \langle v - v_0, \nabla F(v_0) \rangle \geq 0 \} \), we obtain the inequality (8) for some \( M \geq 1 \). Hence we need to show the inequality (8) when \( v_1 \in Y_{x_0}^* \cap \{ v | \langle v - v_0, \nabla F(v_0) \rangle < 0 \} \).

Note that by (7), the sublevel set \( SL_{v_0} \) is convex. In addition, by the definition of \( F \) (6), \( F \) is a \( C^1 \) function and (twisted) and (non-degenerate) condition imply that \( \nabla F \neq 0 \) unless \( x_1 = x_0 \). Therefore the level set \( L_{v_0} \) is \( C^1 \). Hence, when \( v_1 \in Y_{x_0}^* \cap \{ v | \langle v - v_0, \nabla F(v_0) \rangle < 0 \} \), there exists \( t' \in (0, 1) \) such that \( F(v_{t'}) = F(v_0) \).

In fact, from \( \langle v_1 - v_0, \nabla F(v_0) \rangle > 0 \), we know that \( F(v_0) < F(v_0) \) for some \( \epsilon > 0 \) small. Therefore \( v_\epsilon \in \text{Int}(SL_{v_0}) \) and we get \( F(v_{t'}) = F(v_0) \) for some \( t' \in (\epsilon, 1) \). Moreover, we have \( \langle v_1 - v_{t'}, \nabla F(v_{t'}) \rangle > 0 \) since \( v_{t'} \) is in the intersection of \( L_{v_0} \), the boundary of the convex set \( SL_{v_0} \), and a ray that starts from the interior point \( v_\epsilon \).

Then by assumption we have

\[
F(v_0) - F(v_1) = F(v_0) - F(v_{t'}) \\
\leq MS - t' \left( F(v_1) - F(v_{t'}) \right) \\
\leq MS(F(v_1) - F(v_0))
\]

for \( s \in [t', 1] \). If \( s \in [0, t') \), then \( F(v_s) \leq F(v_0) \) and by Remark 3.5 we get the inequality (8). Therefore the inequality (8) holds for \( v_1 \in Y_{x_0}^* \cap \{ v | \langle v - v_0, \nabla F(v_0) \rangle < 0 \} \) too, and obtain that the cost function \( c \) is QQconv. \( \square \)

Lemma 3.9. for any \( x_1, x_0 \in X \) and \( v_1, v_0 \in Y_{x_0}^* \), we have

\[
(13) \quad |\nabla F(v_1) - \nabla F(v_0)| \leq C|x_1 - x_0||v_1 - v_0|.
\]

Proof. By the definition of \( F \) (6), we can compute

\[
\nabla F(v) = [-D_{y \epsilon}^2 c(x_0, y)]^{-1} \left( -D_y c(x_1, y) + D_y c(x_0, y) \right)
\]

where \( y = \text{exp}_{x_0}^c(v) \). Note that we used \( D_{x, y}^2 c = D_{y, x}^2 c \). Hence we have

\[
\nabla F(v_1) - \nabla F(v_0) = [-D_{y \epsilon}^2 c(x_0, y_1)]^{-1} \left( -D_y c(x_1, y_1) + D_y c(x_0, y_1) \right) - [-D_{y \epsilon}^2 c(x_0, y_0)]^{-1} \left( -D_y c(x_1, y_0) + D_y c(x_0, y_0) \right)
\]

where \( y_i = \text{exp}_{x_0}^c(v_i) \). Let L1 and L2 be the second and third line in (15) so that \( \nabla F(v_1) - \nabla F(v_0) = L1 + L2 \).

Let 

\[
L1' = L1 - [-D_{y \epsilon}^2 c(x_0, y_0)]^{-1} \left( -D_y c(x_1, y_1) + D_y c(x_0, y_1) \right) \\
L2' = L2 + [-D_{y \epsilon}^2 c(x_0, y_0)]^{-1} \left( -D_y c(x_1, y_1) + D_y c(x_0, y_1) \right),
\]

respectively.
then $\nabla F(v_1) - \nabla F(v_0) = L1' + L2'$. We use Lipschitzness of the mixed hessian $D^2_{xy}c$ on $L1'$ to obtain

$$\begin{align*}
|L1'| &= \left( [D^2_{yx}c(x_0, y_1)]^{-1} - [D^2_{yx}c(x_0, y_0)]^{-1} \right) (-D_xc(x_1, y_1) + D_yc(x_0, y_1)) \\
&\leq \Lambda |y_1 - y_0| \times \lambda |x_1 - x_0| \\
&\leq \lambda^2 \Lambda |x_0 - x_1||v_0 - v_1|.
\end{align*}$$

To get an estimate for $L2'$, we do

$$\begin{align*}
|L2'| &= \left( [D^2_{yx}c(x_0, y_0)]^{-1} \left( -D_xc(x_1, y_1) + D_yc(x_0, y_1) - D_yc(x_0, y_0) \right) \\
&= \left( [D^2_{yx}c(x_0, y_0)]^{-1} \times \int_0^1 [D^2_{yx}c(x_s, y_0)]^{-1} \left( [D^2_{yx}c(x_s, y_1)] - [D^2_{yx}c(x_0, y_0)] \right) ds (q_1 - q_0) \\
&= \left( [D^2_{yx}c(x_0, y_0)]^{-1} \times \int_0^1 [D^2_{yx}c(x_s, y_0)]^{-1} \left( [D^2_{yx}c(x_s, y_1)] - [D^2_{yx}c(x_0, y_0)] \right) ds (q_1 - q_0)
\end{align*}$$

where $q_i = -D_yc(x_i, y_0)$ and $x_s = \exp_{y_0}^s((1 - s)q_0 + sq_1)$. We use Lipschitzness of $D_yc$ and $D^2_{yx}c$ to obtain

$$|L2'| \leq \alpha^2 \lambda \Lambda |x_1 - x_0||v_1 - v_0|.$$

We combine (16) and (17) to obtain the desired result. \hfill \square

**Remark 3.10.** From the equation (14), we can get

$$|\nabla F(v)| \sim |x_1 - x_0|.$$

In particular, there exists a constant $C_1$ such that

$$|\nabla F(v)| \geq C_1 |x_1 - x_0|.$$

4. **Interior QQconv**

For $v_0 \in Y_{x_0}^*$, we denote the cone with vertex $v_1$ and axis $\nabla F(v_0)$ by

$$C_{k, v_0} = \left\{ v | \langle v - v_0, \nabla F(v_0) \rangle \geq \frac{1}{k} |v - v_0| |\nabla F(v_0)| \right\}.$$

**Lemma 4.1.** Let $c$ be the cost function that satisfies Loeper’s condition and let $k \geq 1$. Then there exists $r_k > 0$ such that if $v_0 \in Y_{x_0}^*$ and $v_1 \in C_{k, v_0} \cap B_{r_k} (v_0)$, then

$$F(v_1) - F(v_0) \leq 5t(F(v_1) - F(v_0)).$$
Proof. Let \( r_k = \frac{C_1}{2k} \) where \( C_1 \) is from Remark 3.10 and \( C \) is from Lemma 3.9. Then, for \( v \in B_{r_k} \), we have

\[
|\nabla F(v) - \nabla F(v_0)| \leq C|x_1 - x_0||v - v_0| \leq \frac{C_1}{2k} |x_1 - x_0| \leq \frac{1}{2k} |\nabla F(v_0)|. \tag{20}
\]

Let \( v_1 \in (C_{k,v_0} \cap B_{r_k}(v_0)) \setminus C_{\frac{1}{2},v_0} \) and let \( u = \frac{v_1 - v_0}{|v_1 - v_0|} \). Then

\[
\langle \nabla F(v), u \rangle = \langle \nabla F(v) - \nabla F(v_1), u \rangle + \langle \nabla F(v_1), u \rangle \geq -\frac{1}{2k} |\nabla F(v_1)| + \frac{1}{k} |\nabla F(v_0)| = \frac{1}{2k} |\nabla F(v_0)|.
\]

Therefore,

\[
F(v_1) - F(v_0) = \int_0^{|v_1 - v_0|} \langle \nabla F(v_0 + su), u \rangle ds \geq \frac{1}{2k} |v_1 - v_0||\nabla F(v_0)|. \tag{21}
\]

Moreover, for \( v_t = v_0 + t(v_1 - v_0) \), we have

\[
\langle \nabla F(v_s), u \rangle = \langle \nabla F(v_s) - \nabla F(v_0), u \rangle + \langle \nabla F(v_0), u \rangle \leq \frac{1}{2k} |\nabla F(v_0)| + \frac{2}{k} |\nabla F(v_0)| = \frac{5}{2k} |\nabla F(v_0)|
\]

where we have used that \( v_1 \notin C_{\frac{1}{2},v_0} \). Therefore,

\[
F(v_t) - F(v_0) = \int_0^{|v_1 - v_0|} \langle \nabla F(v_s), u \rangle ds \leq t |v_1 - v_0| \times \frac{5}{2k} |\nabla F(v_0)|. \tag{22}
\]

Combining (21) and (22), we get the desired inequality.

Now note that \( r_k \) increases as \( k \) decreases. Moreover,

\[
C_{k,v_0} \cap B_{r_k}(v_0) = \bigcup_{i=0}^{\infty} \left( \left( C_{k^i,v_0} \cap B_{r_k}(v_0) \right) \setminus C_{\frac{1}{2^{i+1}},v_0} \right).
\]

Therefore, for any \( v_1 \in C_{k,v_0} \cap B_{r_k}(v_0) \), we can do the same argument with \( k \) replaced with \( k/2^i \) to get the desired inequality. \( \Box \)

Remark 4.2. From the equation (20), we know that if \( v \in B_{r_k}(v_0) \), then

\[
\nabla F(v) \in B_{\rho_0}(\nabla F(v_0))
\]

where \( \rho_0 = |\nabla F(v_0)| \). Let \( \nabla F(v) = \nabla F(v_0) + u \) where \( |u| \leq \frac{\rho_0}{2k} \), and consider \( C_{k',v_0} \). For any \( \nu \) such that \( \nu + v_0 \in C_{k',v_0} \) with \( |\nu| = 1 \), we have

\[
\langle \nu, \nabla F(v) \rangle = \langle \nu, \nabla F(v_0) \rangle + \langle \nu, u \rangle \geq \frac{1}{k'} |\nabla F(v_0)| - \frac{1}{2k} |\nabla F(v_0)|.
\]

Therefore, once we fix \( k \) and \( k' < 2k \), we get \( \langle \nu, \nabla F(v) \rangle \sim \nabla F(v_0) \).
Suppose $B_{r_k}(v_0) \subset Y^*_{v_0}$. Since $F$ is quasi-convex, the sublevel set $SL_{v_0}$ is convex and $\nabla F(v_0)$ is the outer normal vector. Moreover, for any $v' \in L_{v_0} \cap B_{r_k}(v_0)$, we know \((23)\) holds. Then
\[
SL_{v_0} \cap B_{r_k}(v_0) = \left( \bigcap_{v'} \{ v | \langle v - v', \nabla F(v') \rangle \leq 0 \} \right) \cap B_{r_k}(v_0)
\]
\[
\supset \left( \bigcap_{v'} \{ v | \langle v - v_0, \nabla F(v') \rangle \leq 0 \} \right) \cap B_{r_k}(v_0)
\]
\[
\supset \{ v | \langle v - v_0, u \rangle \leq 0, \forall u \in B_{\rho_k}(\nabla F(v_0)) \} \cap B_{r_k}(v_0).
\]
where $\bigcap_{v'}$ means the intersection over $v' \in L_{v_0} \cap B_{r_k}(v_0)$. In particular, this shows that $v_0 - \frac{r_k}{2\rho_0} \nabla F(v_0) \in SL_{v_0} \cap B_{r_k}(v_0)$. Now consider the cone
\[
(25) \quad \overline{C}_{k',v_0}(v_1) = \left\{ v | \langle v - v_1, \nabla F(v_0) \rangle \leq -\frac{1}{k'} |v - v_1| |\nabla F(v_0)| \right\}.
\]
Let $k' \geq 4$ then for any $v_1 \in B_{r_k}^+(v_0)$,
\[
\langle v_0 - \frac{r_k}{2\rho_0} \nabla F(v_0) - v_1, \nabla F(v_0) \rangle = -\langle v_1 - v_0, \nabla F(v_0) \rangle - \frac{r_k}{2\rho_0} \langle \nabla F(v_0), \nabla F(v_0) \rangle
\]
\[
\leq -\frac{r_k}{2} |\nabla F(v_0)| = -\frac{2r_k}{4} |\nabla F(v_0)|
\]
\[
\leq -\frac{1}{k'} |v_0 - \frac{r_k}{2\rho_0} \nabla F(v_0) - v_1| |\nabla F(v_0)|.
\]
Therefore $\overline{C}_{k',v_0}(v_1)$ contains $v_0 - \frac{r_k}{2\rho_0} \nabla F(v_0)$ for any $v_1 \in B_{r_k}^+(v_0)$, which implies that $\overline{C}_{k',v_0}(v_1) \cap L_{v_0} \cap B_{r_k}(v_0) \neq \emptyset$.

**Lemma 4.4.** Suppose $B_{r_k}(v_0) \subset Y^*_{v_0}$. Then, for any $v_1 \in B_{r_k}^+(v_0)$, we have
\[
F(v_1) - F(v_0) \leq M_{k'} t (F(v_1) - F(v_0))
\]
for some $k' > 0$ and $M_{k'}$ a constant depending on $k'$.

**Proof.** Note that by Lemma 4.3, we have the result when $v_1 \in C_{k,v_0}$. Let $4 \leq k' < k$. Let $v_1 \in B_{r_k}^+(v_0) \setminus \overline{C}_{k,v_0}$ and consider the cone $\overline{C}_{k',v_0}(v_1)$ (recall (25)). By Remark 4.3, we have that $\overline{C}_{k',v_0}(v_1) \cap L_{v_0} \cap B_{r_k}(v_0) \neq \emptyset$. Let $u_1 \in \overline{C}_{k',v_0}(v_1) \cap L_{v_0} \cap B_{r_k}(v_0) \neq \emptyset$ and let $\nu = (v_1 - u_1)/\|v_1 - u_1\|$. We claim that $v_t - s_t \nu \in L_{v_0}$ for some $s_t$. In fact, taking $s = t |v_1 - u_1|$, we get
\[
v_t - s_t \nu = tu_1 + (1 - t)v_0 \in SL_{v_0}.
\]
Therefore, $v_t - s_t \nu \in L_{v_0}$ for some $s_t \in [0, t |v_1 - u_1|]$ from $F(v_t) \geq F(v_0)$. Now, up to an isometry, we can set $\nu = -e_n$, $v_0 = 0$, and $v_1 = ae_1 + be_n$ for some $a, b \in \mathbb{R}$, $a > 0$. Then, noting that $\nu$ is not parallel to $v_0 - v_1$ by our choice of $v_1$, we can view the set $\{ v_t - s_t \nu \}$ as a graph of a function $g$ on $[0, ae_1]$. Moreover, convexity
of $SL_{v_0}$ implies that the epigraph of $g$ is convex i.e. $g$ is a convex function. Note that $s_t = g(at) - bt$ so that $s_t$ is a convex function of $t$ on $[0, 1]$, which implies
\begin{equation}
|v_t - u_t| = s_t \leq ts_1 = t|v_1 - u_1|
\end{equation}

where $u_t = v_t - s_t \nu \in L_{v_0}$. Moreover, recalling Remark 4.2, we obtain the following
\begin{equation}
F(v_t) - F(v_0) = F(v_t) - F(u_t)
\end{equation}
\begin{equation}
= \int_0^{s_t} \langle \nabla F(u_t + s \nu), \nu \rangle ds \leq s_t \frac{2k + 1}{2k} |\nabla F(v_0)|,
\end{equation}
and
\begin{equation}
F(v_1) - F(v_0) = F(v_1) - F(u_1)
\end{equation}
\begin{equation}
= \int_0^{s_1} \langle \nabla F(u_1 + s \nu), \nu \rangle ds \geq s_1 \left( \frac{1}{k'} - \frac{1}{2k} \right) |\nabla F(v_0)|.
\end{equation}

We combine (26), (27), and (28), and we obtain
\begin{equation}
F(v_t) - F(v_0) \leq 2s_t |\nabla F(v_0)|
\end{equation}
\begin{equation}
\leq 2ts_1 |F(v_0)| \leq 2k(F(v_1) - F(v_0))
\end{equation}
which gives the desired inequality with $M_k = 2k'$. \hfill \Box

**Remark 4.5.** Lemma 4.4 and the proof of Lemma 3.7 shows that if we have a set $Y' \subset Y$ such that $\text{dist}(Y', \partial Y) = d$, then the cost function is $QQ\text{conv}$ on $Y'$ with the constant $M \sim \frac{1}{d}$. Therefore, Lemma 4.4 is not enough to get $QQ\text{conv}$ of the cost function $c$ on $Y$.

**5. Near the Boundary**

One problem of the argument that we used in the proof of Lemma 4.4 is the following: if we fix $r_k$, the radius of the ball centered at $v_0$, then as we pick $v_0$ close to the boundary $\partial Y^*_x$, the ball $B_{r_k}(v_0)$ may intersect with the boundary $\partial Y^*_x$. Then the cone $\mathcal{C}_{k', v_0}(v_1)$ may not intersect with the level set $L_{v_0}$ and we cannot get the function $s_t$ in the proof of Lemma 4.4. To deal with this case, we introduce another argument to get (8).

**Lemma 5.1.** Let $v_0 \in Y^*_x$ and let $v_1 \in B^+_{r_k}(v_0) \setminus \mathcal{C}_{k', v_0}$. Let $k' < k$ and fix $\nu$ such that $\nu + v_0 \in \mathcal{C}_{k', v_0}$ and $|\nu| = 1$. Suppose for any $t \in [0, 1]$, $\exists w_t \in L_{v_1} \cap B^+_{r_k}(v_0)$ such that $w_t = v_t + s_t \nu$. Then
\begin{equation}
F(v_t) - F(v_0) \leq M_{k, k'}(F(v_1) - F(v_0))
\end{equation}
for some constant $M_{k, k'}$.

**Proof.** Up to an isometry, we can set $\nu = -e_n$, $v_0 = 0$ and $v_1 = ae_1 + be_n$ for some $a, b \in \mathbb{R}$, $a > 0$. Then the set $\{w_t \mid t \in [0, 1]\}$ can be viewed as a graph of a convex
function $g$ on $[0, a_1c]$ and $s_t = bt - g(t)$. Therefore $s_t$ is a concave function of $t$ and we obtain

\[(29) \quad |w_t - v_t| = s_t \geq (1 - t)|s_0| = (1 - t)|w_0 - v_0|.
\]

Recall that by Remark 4.2, we have (24). Therefore,

\[
F(v_t) - F(v_0) = (F(v_1) - F(v_0)) - (F(v_1) - F(v_t)) \\
= (F(w_0) - F(v_0)) - (F(w_t) - F(v_t)) \\
= \int_0^1 \langle \nabla F(v_0 + (s_0\nu)s), s_0\nu \rangle ds - \int_0^1 \langle \nabla F(v_t + (s_t\nu)s), s_t\nu \rangle ds \\
= \int_0^1 \langle \nabla F(v_0 + (s_0\nu)s) - \nabla F(v_t + (s_t\nu)s), s_0\nu \rangle ds \\
+ \int_0^1 \langle \nabla F(v_t + (s_t\nu)s), s_t\nu \rangle (s_0 - s_t) ds \\
=: I_1 + I_2.
\]

We estimate $I_2$ first. By (29) and (23), we have

\[(30) \quad I_2 \leq \frac{2k + 1}{2k} |\nabla F(v_0)| \times t s_0.
\]

To estimate $I_1$, we use (23)

\[
I_1 \leq \int_0^1 C|x_1 - x_0||v_0 - v_t + (s_0 - s_t)s\nu|s_0 ds \\
\leq \int_0^1 C|x_1 - x_0|(|v_0 - v_t| + (s_0 - s_t)s) s_0 ds \\
\leq \int_0^1 C|x_1 - x_0|(t|v_1 - v_0| + ts_0 s) s_0 ds \\
\leq 3Ct|x_1 - x_0|r_k s_0 \leq 3t \times \frac{2k + 1}{2k} |\nabla F(v_0)| s_0.
\]

We combine (30), (31), and (24) to obtain

\[
F(v_t) - F(v_0) \leq I_1 + I_2 \leq 4t \times \frac{2k + 1}{2k} |\nabla F(v_0)| s_0 \\
\leq 4t \times \frac{2k + 1}{2k} \frac{2k k'}{2k - k'} \int_0^{s_0} \langle \nabla F(v_0 + sv), \nu \rangle ds \\
= \frac{4k'(2k + 1)}{2k - k'} t(F(w_0) - F(v_0)) \\
= M_{k,k'}(F(v_1) - F(v_0)),
\]

which is the desired inequality. \[
\]

Remark 5.2. The idea of Lemma 6.1 is to see an opposite direction of the direction that we have used in Lemma 4.3. Since we look at the opposite direction in Lemma
and Lemma 5.1, we will be able to use one of these lemmas even when one of
them fails to work. We make this idea precise in the rest of the paper.

The next proposition is the local Lipschitzness of convex functions. Since it
is well known in the literature, we omit the proof here. For the proof, see the
Appendix of [2] or Theorem 24.7 of [10].

Proposition 5.3. Let $g : B_l(0) \to \mathbb{R}$ be a bounded convex function. Then for
$x, y \in B_{l/2}(0)$, we have

$$|g(x) - g(y)| \leq \frac{4\|g\|_{L^\infty}}{l}|x - y|.$$  (32)

With help of above proposition, we can show the following lemma.

Lemma 5.4. There exists $\rho > 0$ and $0 < \sigma < 1$ that satisfy the following: For any
$x_0 \in X$ and $p \in \partial Y_{x_0}^*$, there exists a unit vector $u$ such that for any $v_0 \in B_{\rho}(p) \cap Y_{x_0}^*$,
we have

$$\{v|\langle v - v_0, u \rangle \geq \sigma|v - v_0|, v \in B_{\rho}(p)\} \subset Y_{x_0}^*. \tag{33}$$

Proof. Let $y \in \text{Int}(Y)$ and let $B_l(y) \subset Y$. Then from the bi-Lipschitzness of $D_x c$, we get

$$B_{l/\lambda}(p_{x_0}) \subset -D_x c(x_0, B_l(y)) \subset Y_{x_0}^*$$

where $p_{x_0} = -D_x c(x_0, y)$. Let $H_p = (p_{x_0} - p)^2 + p_{x_0}$, a hyperplane perpendicular to $p_{x_0} - p$ and containing $p_{x_0}$. Let $B_{l/\lambda}^{-1}(p_{x_0}) \subset H_p$. Define $\mathcal{L}_q^- = \{q + s(p_{x_0} - p)|s \leq 0\}$, and consider

$$\mathcal{D}_{p,l} = \bigcup_{q \in B_{l/\lambda}^{-1}(p_{x_0})} \mathcal{L}_q^- \quad \text{and} \quad \mathcal{Y}_{p,l} = \mathcal{D}_{p,l} \cap \partial Y_{x_0}^*.$$  (34)

Since $\mathcal{L}_q^-$ is a ray that starts from the interior point $q$ of the compact convex set
$Y_{x_0}^*$, $\mathcal{L}_q^- \cap \partial Y_{x_0}^*$ is a singleton. Therefore, by setting $p_{x_0} = 0$ and $(p_{x_0} - p)/e_n$ using
an isometry, $\mathcal{Y}_{p,l}$ can be viewed as a graph of a function $g$ on $B_{l/2}^{-1}(0) \subset \mathbb{R}^{n-1}$. Moreover, convexity of $Y_{x_0}^*$ implies convexity of the epigraph of $g$. Hence, $g$ is a
convex function. In particular, it is locally Lipschitz by (32) with Lipschitz constant $L$ in $B_{l/(2\lambda)}(0)$ given by

$$\frac{2\|g\|_{L^\infty}}{l/2} \leq \frac{4\text{diam}(Y_{x_0}^*)}{l} \leq \frac{4\lambda\text{diam}(Y)}{l} = L.$$  (35)

This shows that for any $v_0 \in \mathcal{D}_{p,l/2} \cap Y_{x_0}^*$, the upper Lipschitz cone $\{v|\langle v - v_0, e_n \rangle L/\sqrt{L^2 + 1}\}$ does not intersect with $\mathcal{Y}_{p,l/2}$, the graph of $g$ on $B_{l/2}$ :

$$\{v|\langle v - v_0, e_n \rangle > L/\sqrt{L^2 + 1}\} \cap \mathcal{Y}_{p,l/2} = \emptyset.$$  (36)

Taking intersection with $B_{l/2}(p)$, we get the desired result with $\rho = l/2$, $\sigma = L/\sqrt{L^2 + 1}$, and $u = e_n$. □
In the next lemma we make precise argument that we discussed in Remark 5.2. Lemma 5.4 will help us to find a direction in which we can use one of the arguments in Lemma 4.4 and Lemma 5.1.

**Lemma 5.5.** Let \( v_0 \in Y^*_x \) and take \( k \gg 1 \) so that \( 2r_k \leq \rho \) where \( \rho \) is from Lemma 5.4. Moreover, we choose \( k' \) big enough so that we can find \( k' < k \) such that \( \frac{1}{k'^2} \leq 1 - \sigma^2 \) where \( \sigma \) is from Lemma 5.4. Suppose \( B_{r_k/4}(v_0) \cap \partial Y^*_x \neq \emptyset \). If \( v_1 \in B_{r_k/4}(v_0) \setminus C_{k,v_0} \), then

\[
F(v_t) - F(v_0) \leq M_{k,k'} t(F(v_1) - F(v_0))
\]

For some constant \( M_{k,k'} \).

**Proof.** Let \( K(u) = \{ v | \langle v, u \rangle \geq \sigma |v| \} \) and \( K(w) = \{ v | \langle v, w \rangle \geq \frac{1}{k'} |v| \} \) for some unit vectors \( u \) and \( w \) satisfying \( \langle u, w \rangle \geq 0 \). We claim that \( K(u) \cap K(w) \neq \emptyset \).

WLOG, we assume \( u = e_n \) and \( w = ae_1 + be_n \) for some \( a \geq 0 \). Then \( \langle u, w \rangle = b \geq 0 \).

If \( b \geq \frac{1}{k'} \), then \( \langle e_n, w \rangle \geq \frac{1}{k'} \) so that \( e_n \in K(u) \cap K(w) \neq \emptyset \). Otherwise, let \( w^\bot = -be_1 + ae_n \). Then we have \( \frac{1}{k'} w + \left( 1 - \frac{1}{k'^2} \right) w^\bot \in K(w) \). Moreover,

\[
\left( \frac{1}{k'} w + \left( 1 - \frac{1}{k'^2} \right) w^\bot, e_n \right) = \frac{1}{k'} b + \left( 1 - \frac{1}{k'^2} \right)^{1/2} a
\]

This is a concave function of \( b \) on \( [0, \frac{1}{k'}] \), therefore it has minimum value at the boundary \( b = 0 \) or \( b = \frac{1}{k'} \). A simple computation shows that the minimum value is \( \left( 1 - \frac{1}{k'^2} \right)^{1/2} > \sigma \). Therefore \( \frac{1}{k'} w + \left( 1 - \frac{1}{k'^2} \right)^{1/2} w^\bot \in K(u) \cap K(w) \neq \emptyset \), and the claim is proved.

Now, suppose \( B_{r_k/4}(v_0) \cap \partial Y^*_x \neq \emptyset \) and \( v_1 \in B_{r_k/4}(v_0) \setminus C_{k,v_0} \). Let \( p \in B_{r_k/4}(v_0) \cap \partial Y^*_x \). Then there is a unit vector \( u \) which satisfies (33). Let \( K(u') = \{ v | \langle v - u', u \rangle > \sigma |v| \} \). We divide the proof into two cases.

Case 1) Suppose \( \langle \nabla F(v_0), u \rangle \geq 0 \). Then by the claim, \( C_{k',v_0} \cap K(v_0) \neq \emptyset \). Let \( v \) be a unit vector such that \( v_0 + v \in C_{k',v_0} \cap K(v_0) \). Then by Lemma 5.4, \( v_t + \frac{1}{2} r_k \nu \in Y^*_x \cap B_{r_k(p)}, \forall t \in [0, 1] \). Moreover, noting that \( |v_t + \frac{1}{2} r_k \nu - v_0| \leq r_k \), we can use (23) to obtain

\[
F(v_t + \frac{1}{2} r_k \nu) - F(v_t) = \int_0^{r_k} \langle \nabla F(v_t + s \nu), \nu \rangle ds + \int_0^{r_k} \langle \nabla F(v_t + s \nu), \nabla F(v_0) \rangle ds
\]

(34)

\[
\geq \left( \frac{1}{2k'} - \frac{1}{4k} \right) |\nabla F(v_0)| r_k.
\]
Also, noting that \( v_1 \notin C_{k,v_0} \), we have

\[
F(v_t) - F(v_0) = \int_0^1 \langle \nabla F(v_t + s(v_t - v_0)), v_t - v_0 \rangle ds
\]

\[
= \int_0^1 \langle \nabla F(v_0), v_1 - v_0 \rangle ds
\]

\[
+ \int_0^1 \langle F(v_t + s(v_1 - v_0)) - F(v_0), v_1 - v_0 \rangle ds
\]

(35)

\[
\leq \left( \frac{1}{k} + \frac{1}{2k} \right) |\nabla F(v_0)||v_1 - v_t|
\]

\[
\leq \frac{3}{8k} |\nabla F(v_0)| r_k
\]

Since \( k' < k \), we obtain \( F(v_t + 1/2r_k \nu) > F(v_1) \) from (34) and (35). This implies that \( \exists w_t \in L_{v_t} \) such that \( w_t = v_t + s_t \nu \) for some \( s_t \). Therefore, we can apply Lemma 5.2.1 to obtain the desired inequality.

Case 2) Suppose \( \langle \nabla F(v_0), \nu \rangle \leq 0 \). Then by the claim, \( \mathcal{C}_{k',v_0}(v_1) \cap K(v_1) = \emptyset \). Let \( \nu \) be a unit vector such that \( v_1 - \nu \in \mathcal{C}_{k',v_0}(v_1) \cap K(v_1) \). Then by Lemma 5.4, \( v_t - \frac{1}{2}r_k \nu \in Y^*_0 \cap B_\rho(p), \forall t \in [0,1] \). Therefore, we use (23) to obtain

\[
F(v_t) - F(v_1 - 1/2r_k \nu) = \int_0^{1/2r_k} \langle \nabla F(v_t - s \nu), \nu \rangle ds
\]

(36)

\[
= \int_0^{1/2r_k} \langle \nabla F(v_0), \nu \rangle ds + \int_0^{1/2r_k} \langle \nabla F(v_t - s \nu) - \nabla F(v_0), \nu \rangle ds
\]

\[
\geq \left( \frac{1}{2k'} - \frac{1}{4k} \right) |\nabla F(v_0)| r_k.
\]

Also, from the choice of \( v_1 \), we know that \( v_0 \notin \mathcal{C}_{k,v_0}(v_1) \) so that \( \langle v_0 - v_1, \nabla F(v_0) \rangle < \frac{1}{k} \). Therefore,

\[
F(v_t) - F(v_0) = \int_0^1 \langle \nabla F(v_0 + s(v_t - v_0)), v_t - v_0 \rangle
\]

\[
= \int_0^1 \langle \nabla F(v_0), v_t - v_0 \rangle ds
\]

(37)

\[
+ \int_0^1 \langle \nabla F(v_0 + s(v_1 - v_0)) - \nabla F(v_0), v_t - v_0 \rangle ds
\]

\[
\leq \left( \frac{1}{k} + \frac{1}{2k} \right) |\nabla F(v_0)||v_t - v_0|
\]

\[
\leq \frac{3}{8k} |\nabla F(v_0)| r_k.
\]

Since \( k' < k \), we obtain \( F(v_t - 1/2r_k \nu) < F(v_0) \) from (36) and (37). This implies that \( w_t \in L_{v_t} \) such that \( w_t = v_t - s_t \nu \) for some \( s_t \). Therefore, we can apply the proof of Lemma 4.4 to obtain the desired inequality. \( \square \)
In section 4, we showed that Loeper’s condition implies QQconv for $v_0$ in the interior of $Y_{x_0}$ with the constant $M$ in (8) comparable to $\text{dist}(\partial Y_{x_0}, v_0)$. In this section, we have shown that the inequality (8) holds with some constant $M$ when $v_0$ is close to $\partial Y_{x_0}$ by Lemma 5.5. Therefore, by combining the lemmas from sections 3, 4, 5, we can prove the main theorem.

**Proof of the Main Theorem.** Choose $k$ and $k'$ so that the conditions in Lemma 5.5 are satisfied. Then for any $x_1, x_0 \in X$ and $v_0 \in Y_{x_0}$ such that $B_{4r_{4k}}(v_0) \subset Y_{x_0}$, we can apply Lemma 4.4 and obtain the following: For any $v_1 \in B_{4r_{4k}}(v_0)$ we have the inequality (8) with some constant $M = M_1$.

Now, note that $r_{4k} = \frac{1}{4}r_k$. Therefore, if $B_{rk}(v_0) \not\subset Y_{x_0}$, then $B_{rk/4}(v_0) \cap \partial Y_{x_0} \neq \emptyset$, and we can apply Lemma 5.5 to obtain the following: For any $v_1 \in B_{rk/4}(v_0) \cap C_{k,v_0}$, we have the inequality (8) with some constant $M = M_2$. Moreover, by Lemma 4.1, the inequality (8) with $M = 5$ for any $v_1 \in B_{rk/4}(v_0) \cap C_{k,v_0}$. As a result, we obtain the inequality (8) for any $v_0 \in Y_{x_0}$ and $v_1 \in B_{rk/4}(v_0) \cap Y_{x_0}$ with $M = \max\{M_1, M_2, 5\}$. Noting that $M$ is uniform over $x_0, x_1 \in X$ and $v_0, v_1 \in Y_{x_0}$, Lemma 3.8 implies that $c$ satisfies QQconv. □

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