First preliminary tests of the general relativistic gravitomagnetic field of the Sun and new constraints on a Yukawa-like fifth force from planetary data

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Abstract

The general relativistic Lense-Thirring precessions of the perihelia of the inner planets of the Solar System are $\lesssim 10^{-3}$ arcseconds per century. Recent improvements in planetary orbit determination may yield the first observational evidence of such a tiny effect. Indeed, corrections to the known perihelion rates of $-0.0036 \pm 0.0050$, $-0.0002 \pm 0.0004$ and $0.0001 \pm 0.0005$ arcseconds per century were recently estimated by E.V. Pitjeva for Mercury, the Earth and Mars, respectively, on the basis of the EPM2004 ephemerides and a set of more than 317,000 observations of various kinds. The predicted relativistic Lense-Thirring precessions for these planets are $-0.0020$, $-0.0001$ and $-3 \times 10^{-5}$ arcseconds per century, respectively and are compatible with the determined perihelion corrections. The relativistic predictions fit better than the zero-effect hypothesis, especially if a suitable linear combination of the perihelia of Mercury and the Earth, which a priori cancels out any possible bias due to the solar quadrupole mass moment, is considered. However, the experimental errors are still large. Also the latest data for Mercury processed independently by Fienga et al. with the INPOP ephemerides yield preliminary insights about the existence of the solar Lense-Thirring effect. The data from the forthcoming planetary mission BepiColombo will improve our knowledge of the orbital motion of this planet and, consequently, the precision of the measurement of the Lense-Thirring effect. As a by-product of the present analysis, it is also possible to constrain the strength of a Yukawa-like fifth force to a $10^{-12} - 10^{-13}$ level at scales of about one Astronomical Unit ($10^{11}$ m).

Keywords: experimental tests of gravity, Lense-Thirring effect, orbital motions, planets, Solar System
1 Introduction

A satisfactorily empirical corroboration of a fundamental theory requires that as many independent experiments as possible are conducted by different scientists in different laboratories. Now, the general relativistic gravitomagnetic Lense-Thirring (LT) effect is difficult to test, also in the weak-field and slow-motion approximation, valid in our Solar System, both because such a relativistic effect is very small and the competing classical signals are often quite larger. Until now, a 6% LT test has recently been conducted in the gravitational field of Mars by using the data of the Mars Global Surveyor (MGS) spacecraft; other tests accurate to about $\frac{1}{10}$% have been performed by a different team in the gravitational field of the Earth by analyzing the data of the LAGEOS and LAGEOS II artificial satellites (see Section 1.2). We think it is worthwhile to further extend these efforts trying to use different laboratories, i.e. other gravitational fields, even if the outcomes of such tests should be less accurate than those conducted so far. To this aim, in this paper we investigate the LT effect induced by the Sun on the orbital motion of the inner planets of the Solar System in the context of the latest results in the planetary ephemerides field.

1.1 The Lense-Thirring effect

According to Einstein, the action of the gravitational potential $U$ of a given distribution of mass-energy is described by the coefficients $g_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, of the space-time metric tensor. They are determined, in principle, by solving the fully non-linear field equations of the Einsteinian General Theory of Relativity (GTR) for the considered mass-energy content. These equations can be linearized in the weak-field ($U/c^2 << 1$, where $c$ is the speed of light in vacuum) and slow-motion ($v/c << 1$) approximation (Mashhoon 2001; Ruggiero and Tartaglia 2002), valid throughout the Solar System, and look like the equations of the linear Maxwellian electromagnetism. Among other things, a noncentral, Lorentz-like force

$$F_{LT} = -2m \left(\frac{v}{c}\right) \times B_g$$  \hspace{1cm} (1)

\footnote{Other estimates point towards a 15 – 20% error.}
acts on a moving test particle of mass $m$. It is induced by the post-
Newtonian component $B_g$ of the gravitational field in which the particle
moves with velocity $v$. $B_g$ is related to the mass currents of the mass-
energy distribution of the source and comes from the off-diagonal compo-
nents $g_{0i}, i = 1, 2, 3$ of the metric tensor. Thanks to such an analogy, the
ensemble of the gravitational effects induced by mass displacements is also
named gravitomagnetism. For a central rotating body of mass $M$ and proper
angular momentum $L$ the gravitomagnetic field is

$$B_g = \frac{G[3r(r \cdot L) - r^2L]}{ct^5}. \tag{2}$$

One of the consequences of eq. (1) and eq. (2) is a gravitational spin–
orbit coupling. Indeed, if we consider the orbital motion of a particle in the
gravitational field of a central spinning mass, it turns out that the longitude
of the ascending node $\Omega$ and the argument of pericentre $\omega$ of the orbit
of the test particle are affected by tiny secular advances $\dot{\Omega}_{LT}, \dot{\omega}_{LT}$ (Lense
and Thirring 1918, Barker and O’Connell 1974, Cugusi and Proverbio 1978,
Soffel 1989, Ashby and Allison 1993, Iorio 2001)

$$\dot{\Omega}_{LT} = \frac{2GL}{c^2a^3(1-e^2)^{\frac{3}{2}}}, \quad \dot{\omega}_{LT} = -\frac{6GL \cos i}{c^2a^3(1-e^2)^{\frac{3}{2}}}, \tag{3}$$

where $a$, $e$ and $i$ are the semimajor axis, the eccentricity and the inclina-
tion, respectively, of the orbit and $G$ is the Newtonian gravitational con-
stant. Note that in their original paper Lense and Thirring (1918) used the
longitude of pericentre $\varpi \equiv \Omega + \omega$.

The gravitomagnetic force may have strong consequences in many astro-
physical and astronomical scenarios involving, e.g., accreting disks around
black holes (Thorne et al. 1986; Stella et al. 2003), gravitational lensing and
time delay (Sereno 2003; 2005a; 2005b). Unfortunately, in these contexts
the knowledge of the various competing effects is rather poor and makes
very difficult to reliably extract the genuine gravitomagnetic signal from the
noisy background. E.g., attempts to measure the LT effect around black
holes are often confounded by the complexities of the dynamics of the hot
gas in their accretion disks. On the contrary, in the solar and terrestrial
space environments the LT effect is weaker but the various sources of sys-
tematic errors are relatively well known and we have the possibility of using
various artificial and natural orbiters both to improve our knowledge of such
biases and to design suitable observables circumventing these problems, at
least to a certain extent.
1.2 The performed and ongoing tests

Up to now, all the performed and ongoing tests of gravitomagnetism were implemented in the weak-field and slow-motion scenarios of the Earth and Mars gravitational fields.

As far as the Earth is concerned, in April 2004 the GP-B spacecraft (Everitt 1974; Fitch et al. 1995; Everitt et al. 2001) was launched. Its aim is the measurement of another gravitomagnetic effect, i.e. the precession of the spins (Pugh 1959; Schiff 1960) of four superconducting gyroscopes carried onboard. The level of accuracy obtained so far is about $256 - 128\%$ (Muhlfelder et al. 2007), with the hope of reaching $2^{2} \approx 13\%$ in December 2007.

In regard to the LT effect on the orbit of a test particle, the idea of using the LAGEOS satellite and, more generally, the Satellite Laser Ranging (SLR) technique to measure it in the terrestrial gravitational field with the existing artificial satellites was put forth for the first time by Cugusi and Proverbio (1978). Attempts to practically implement such a strategy began in 1996 with the LAGEOS and LAGEOS II satellites (Ciufolini et al. 1996). The latest test was performed by Ciufolini and Pavlis (2004). They analyzed the data of LAGEOS and LAGEOS II by using an observable independently proposed by Pavlis (2002), Ries et al. (2003a, 2003b) and Iorio and Morea (2004). The error claimed by Ciufolini and Pavlis (2004) is 5-10% at 1-3 sigma, respectively. The assessment of the total accuracy of such a test raised a debate (Iorio 2005a; 2005b; 2006a; 2007; Ciufolini and Pavlis 2005; Lucchesi 2005).

Recently, a 6% LT test on the orbit of the Mars Global Surveyor (MGS) spacecraft in the gravitational field of Mars has been reported (Iorio 2006b); indeed, the predictions of general relativity are able to accommodate, on average, about 94% of the measured residuals in the out-of-plane part of the MGS orbit over 5 years.

Finally, it must be noted that, according to Nordtvedt (2003), the multi-decade analysis of the Moon’s orbit by means of the Lunar Laser Ranging (LLR) technique yields a comprehensive test of the various parts of order $O(c^{-2})$ of the post-Newtonian equation of motion. The existence of the gravitomagnetic interaction as predicted by GTR would, then, be inferred from the high accuracy of the lunar orbital reconstruction. A 0.1% test was recently reported (Murphy et al. 2007): a critical discussion of the real sensitivity of LLR to gravitomagnetism can be found in (Kopeikin 2007).

\(^{2}\text{See on the WEB StanfordNews 4/14/07 downloadable at http://einstein.stanford.edu/}\)
Table 1: Gravitomagnetic secular precessions of the longitudes of perihelion \( \varpi \) of Mercury, Venus, Earth and Mars in \( '' \text{cy}^{-1} \). The value \( (190.0 \pm 1.5) \times 10^{39} \text{kg m}^2 \text{s}^{-1} \) (Pijpers 1998; 2003) has been adopted for the solar proper angular momentum \( L_\odot \).

|       | Mercury | Venus | Earth | Mars      |
|-------|---------|-------|-------|-----------|
| Precession | -0.0020 | -0.0003 | -0.0001 | \(-3 \times 10^{-5}\) |

Also the radial motion of the LAGEOS satellite would yield another indirect confirmation of the existence of the gravitomagnetic interaction (Nordtvedt 1988).

2 The solar gravitomagnetic field

The action of the solar gravitomagnetic field on the Mercury’s longitude of perihelion was calculated for the first time by de Sitter (1916) who, by assuming an homogenous and uniformly rotating Sun, found a secular advance of \(-0.01 \text{arcseconds per century} \) (\( '' \text{cy}^{-1} \) in the following). This value is also quoted by Soffel (1989). Cugusi and Proverbio (1978) yield \(-0.02 '' \text{cy}^{-1} \) for the argument of perihelion of Mercury. Instead, recent determinations of the Sun’s proper angular momentum \( L_\odot = (190.0 \pm 1.5) \times 10^{39} \text{kg m}^2 \text{s}^{-1} \) from helioseismology (Pijpers 1998; 2003), accurate to 0.8%, yield a precessional effect one order of magnitude smaller for Mercury (Ciufolini and Wheeler 1995; Iorio 2005c). See Table 1 for the gravitomagnetic precessions of the four inner planets. As can be seen, they are of the order of \( 10^{-3} - 10^{-5} '' \text{cy}^{-1} \).

So far, the LT effect on the orbits of the Sun’s planets was believed to be too small to be detected (Soffel 1989). Iorio (2005c) preliminarily investigated the possibility of measuring such tiny effects in view of recent important developments in the planetary ephemerides generation. It is remarkable to note that the currently available estimate of \( L_\odot \) is accurate enough to allow, in principle, a genuine test of GTR. Moreover, it was determined in a relativity-free fashion from astrophysical techniques which do not rely on the dynamics of planets in the gravitational field of the Sun. Thus, there is no any a priori ‘memory’ effect of GTR itself in the adopted value of \( L_\odot \).
3 Compatibility of the determined extra-precessions of planetary perihelia with the LT effect

3.1 The Keplerian orbital elements

The Keplerian orbital elements like $\varpi$ are not directly observable quantities like right ascensions, declinations, ranges and range-rates which can be measured from optical observations, radiometric measurements, meridian transits, etc. They can only be computed from a state vector in rectangular Cartesian coordinates which also allows to compute predicted values of the observations. In this sense, speaking of an “observed” time series of a certain Keplerian element would mean that it has been computed from the machinery of the data reduction of the real observations. Keeping this in mind, it would be possible, in principle, to extract the LT signal from the planetary motions by taking the difference between two suitably computed time-series of the Keplerian elements in such a way that it fully accounts for the gravitomagnetic signature. Such ephemerides, which should share the same initial conditions, would differ in the fact that one would be based on the processing of the real data, which are presumed to fully contain also the LT signal, and the other one would, instead, be the result of a purely numerical propagation. The dynamical force models with which the data are to be processed and the numerical ephemeris propagated do not contain the gravitomagnetic force itself: only the general relativistic gravitoelectric terms must be present. Moreover, the astronomical parameters entering the perturbations which can mimic the LT signature should not be fitted in the data reduction process: they should be kept fixed to some reference values, preferably obtained in a relativity-independent way so to avoid ‘imprinting’ effects. Thus, in the resulting “residual” time series $\Delta \varpi_{\text{obs}}(t)$, the LT signature should be entirely present.

3.2 The EPM2004 ephemerides

A somewhat analogous procedure was recently implemented with the Ephemerides of Planets and the Moon EPM2004 (Pitjeva 2005a; 2005b) produced by the Institute of Applied Astronomy (IAA) of the Russian Academy of Sciences (RAS). They are based on a data set of more than 317,000 observations (1913-2003) including radiometric measurements of planets and space-

\footnote{In practice, this procedure cannot be always performed at all times because the observations collected at a given time not always allow for the computation of the entire state vector in rectangular Cartesian coordinates.}
craft, astrometric CCD observations of the outer planets and their satellites, and meridian and photographic observations. Such ephemerides were constructed by the simultaneous numerical integration of the equations of motion for all planets, the Sun, the Moon, 301 largest asteroids, rotations of the Earth and the Moon, including the perturbations from the solar quadrupolar mass moment $J_2^\odot$ and asteroid ring that lies in the ecliptic plane and consists of the remaining smaller asteroids. In regard to the post-Newtonian dynamics, only the gravitoelectric terms, in the harmonic gauge, were included (Newhall et al. 1983).

3.3 The measured extra-precessions of the planetary perihelia and the Lense-Thirring effect

As a preliminary outlook on the measurability of the Lense-Thirring perihelion precessions, let us make the following considerations. The magnitude of the gravitomagnetic shift of the Mercury’s perihelion over a 90-years time span like that covered by the EPM2004 data amounts to $0.0018''$. The accuracy in determining the secular motion of Mercury’s perihelion can be inferred from the results for the components of the eccentricity vector $k = e \cos \varpi$ and $h = e \sin \varpi$ reported in Table 4 by Pitjeva (2005b). Indeed, the formal standard deviations of $k$ and $h$ are 0.123 and 0.099 milliarcseconds, respectively. Thus, the formal error in measuring $\varpi$ is about $0.0007''$. An analogous calculation for the Earth yields an error in $\varpi$ of $8 \times 10^{-5}''$.

The EPM2004 ephemerides were used to determine corrections $\Delta \dot{\varpi}_{\text{obs}}$ to the secular precessions of the longitudes of perihelia of the inner planets as fitted parameters of a particular solution. In Table 3 by Pitjeva (2005a), part of which is reproduced in Table 2, it is possible to find their values obtained by comparing the model observations computed using the constructed ephemerides with actual observations. Note that in determining such extra-precessions the PPN parameters (Will 1993) $\gamma$ and $\beta$ and the solar even zonal harmonic coefficient $J_2^\odot$ were not fitted; they were held fixed to their GTR values, i.e. $\gamma = \beta = 1$, and to $J_2^\odot = 2 \times 10^{-7}$. Note also that the unit values of $\beta$ and $\gamma$ were measured in a variety of approaches which are independent of the gravitomagnetic force itself. Although the original purpose of the determination of such corrections was not the measurement

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4The goal by Pitjeva (2005a) was to make a test of the quality of the previously obtained general solution in which certain values of $\beta, \gamma$ and $J_2$, were obtained. If the construction of the ephemerides was satisfactory, very small “residual” effects due to such parameters should have been found. She writes: “At present, as a test, we can determine [...] the
Table 2: Determined extra-precessions $\Delta \tilde{\omega}_{\text{obs}}$ of the longitudes of perihelia of the inner planets, in $''$ cy$^{-1}$, by using EPM2004 with $\beta = \gamma = 1$, $J_2^\odot = 2 \times 10^{-7}$. The gravitomagnetic force was not included in the adopted dynamical force models. Data taken from Table 3 of (Pitjeva 2005a). It is important to note that the quoted uncertainties are not the mere formal, statistical errors but are realistic in the sense that they were obtained from comparison of many different solutions with different sets of parameters and observations (Pitjeva, private communication 2005a). The correlations among such determined planetary perihelia rates are very low with a maximum of about 20% between Mercury and the Earth (Pitjeva, private communication 2005b).

| Planet   | $\text{det. correction}$ |
|----------|---------------------------|
| Mercury  | $-0.0036 \pm 0.0050$      |
| Venus    | $0.53 \pm 0.30$          |
| Earth    | $-0.0002 \pm 0.0004$      |
| Mars     | $0.0001 \pm 0.0005$       |

of the LT effect, the results of Table 3 by Pitjeva (2005a) can be used to take first steps towards an observational corroboration of the existence of the solar gravitomagnetic force. Indeed, the uncertainties in the predicted values of the LT precessions induced by the error in $L_\odot$ (Pijpers 1998; 2003) amount to $1 \times 10^{-5}''$ cy$^{-1}$ for Mercury, $7 \times 10^{-7}''$ cy$^{-1}$ for the Earth and $2 \times 10^{-7}''$ cy$^{-1}$ for Mars: they are far smaller than the errors in Table 2, so that a genuine comparison with the measured precessions make sense.

By comparing Table 1 and Table 2 of this paper it turns out that the predictions of GTR for the LT effect are compatible with the small determined corrections to the secular motions of the planetary perihelia for Mercury ($-0.0086''$ cy$^{-1} < -0.0020''$ cy$^{-1} < 0.0014''$ cy$^{-1}$), the Earth ($-0.0006''$ cy$^{-1} < -0.0001''$ cy$^{-1} < 0.0002''$ cy$^{-1}$) and Mars ($-0.0004''$ cy$^{-1} < -3 \times 10^{-5}$ cy$^{-1} < 0.0006''$ cy$^{-1}$). In normalized units $\mu$ ($\mu_{\text{GTR}} = 1$) we have $\mu_{\text{Mercury}} = 1.8 \pm 2.5$, $\mu_{\text{Earth}} = 2 \pm 4$ and $\mu_{\text{Mars}} = -3.3 \pm 16$. Figure summarizes the obtained results. The discrepancies between the predicted and the determined values are reported in Table 3 of this paper. They are smaller than the measurement uncertainties, so that $\chi^2 = \sum \left( \frac{P - D}{\delta P} \right)^2 = 0.2$ can be corrections to the motions of the planetary perihelia, which allows us to judge whether the values of $\beta$, $\gamma$, and $J_2$ used to construct the ephemerides are valid.

5 In the case of Venus the discrepancy between the predicted and the measured values is slightly larger than the measurement error. For such a planet the perihelion is not a good observable because of the small eccentricity of its orbit ($e_{\text{Venus}} = 0.0066$).
Figure 1: The horizontal dash-dotted lines represent the predicted values of the LT secular precessions of the perihelia of Mercury, Venus, the Earth and Mars according to GTR. The vertical solid lines represent the values of the additional secular precessions of Mercury, Venus, the Earth and Mars determined by Pitjeva (2005a) along with their error bars. The predictions of the LT effect by GTR (1 in normalized units) are compatible with them for Mercury (1.8 ± 2.5), the Earth (2 ± 4) and Mars (−3.3 ± 16.6).
Table 3: Comparison between the predicted values (P) of the LT precessions of the perihelia of Mercury, the Earth and Mars (Table 1) and the determined values (D) of the extra-precessions of their perihelia (Table 2). Their differences are smaller than the errors (E). From them a $\chi^2 = 0.2$ can be obtained. Note that the $\chi^2$ for zero is 0.8.

| Planet | P-D ("cy$^{-1}$) | E ("cy$^{-1}$) |
|--------|------------------|----------------|
| Mercury| 0.0016           | 0.0050         |
| Earth  | 0.0001           | 0.0004         |
| Mars   | -0.0001          | 0.0005         |

obtained. It must be noted that the determined extra-precessions of Table 2 are also compatible with zero, but at a worse level. Indeed, $\chi^2 = 0.8$ in this case.

A way to improve the robustness and reliability of such a test would be to vary the adopted values for the solar oblateness within the currently accepted ranges and investigate the changes in the fitted values of the extra-precessions. Moreover, it would also be important to produce an analogous set of solutions with $\beta$, $\gamma$ and $J^2_{\odot}$ fixed in which the extra-precessions of the nodes are determined; in this way it would be possible to use only Mercury.

### 3.4 Some possible systematic errors due to other competing effects

In order to check our conclusion that the LT effect is the main responsible for the observed secular corrections to the planetary perihelia $\Delta \varpi_{\text{obs}}$ let us focus on Mercury and on the known perturbations which could induce a secular extra-perihelion advance due to their mismodelling.

The major sources of secular advances of the perihelia are the Schwarzschild gravitoelectric part of the solar gravitational field and the quadrupolar mass moment $J^2_{\odot}$ of the Sun. Their nominal effects on the longitudes of perihelion of the inner planets are quoted in Table 4 and Table 5 of this paper; the analytical expressions are

$$\dot{\varpi}_{\text{GE}} = \frac{3nGM}{c^2a(1-e^2)}, \quad (4)$$

$$\dot{\varpi}_{j_2} = \frac{3}{2} \frac{nJ_2}{(1-e^2)^2} \left( \frac{R}{a} \right)^2 \left( 1 - \frac{3}{2} \sin^2 i \right) \equiv \dot{\varpi}_{2}J_2, \quad (5)$$

where $n = \sqrt{GM/a^3}$ is the Keplerian mean motion and $R$ is the mean
Table 4: Nominal values of the secular post-Newtonian gravitoelectric precessions of the longitudes of perihelion $\varpi$ of Mercury, Venus, Earth and Mars in $"$ $cy^{-1}$. Their mismodelled amplitudes are fixed by the uncertainties in $\gamma$ and $\beta$ which are of the order of 0.01% (Pitjeva 2005a). However, deviations from their relativistic values are expected only at a $10^{-6} - 10^{-7}$ level (Damour and Nordtvedt 1993).

|       | Mercury | Venus | Earth | Mars |
|-------|---------|-------|-------|------|
| Value | 42.9812 | 8.6247 | 3.8387 | 1.3509 |

Table 5: Nominal values of the classical secular precessions of the longitudes of perihelion $\varpi$ of Mercury, Venus, Earth and Mars, in $"$ $cy^{-1}$, induced by the solar quadrupolar mass moment $J_2^\odot$. The value $J_2^\odot = 2 \times 10^{-7}$ used in (Pitjeva 2005a) has been adopted. Their mismodelled amplitudes are fixed by the uncertainty in $J_2^\odot$ which is of the order of $\sim 10\%$.

|       | Mercury | Venus | Earth | Mars |
|-------|---------|-------|-------|------|
| Value | 0.0254  | 0.0026 | 0.0008 | 0.0002 |

equatorial radius of the central body. In view of their large size with respect to the LT effect, one could legitimately ask if the determined extra-precessions are due to the systematic errors in such competing secular rates. An a-priori analytical analysis shows that it should not be the case.

3.4.1 The impact of the solar oblateness

In regard to $J_2^\odot$, which is still rather poorly known, only values measured in such a way that no a priori ‘imprinting’ effects occurred should be considered for our purposes.

E.g., the most recent determinations of the solar oblateness based on astrophysical techniques yield values close to $2.2 \times 10^{-7}$ (Paternò et al. 1996; Pijpers 1998; Mecheri et al. 2004) with discrepancies between the various best estimates of the same order of magnitude of their errors, i.e. $\sim 10^{-9}$. Let us see if such determinations are compatible with the determined extra-advances of perihelia. By assuming a correction of $\sim 10\%$ of the adopted reference value by Pitjeva, i.e. $\delta J_2^\odot = 2 \times 10^{-8}$, the resulting residual precession due to the solar oblateness would amount to $+0.0025"$ $cy^{-1}$ for Mercury. It falls outside the measured range.

A way to a priori cancel out any possible impact of the uncertainty in the solar oblateness consists in suitably combining the perihelia advances
of two planets so to de-correlate by construction the LT effect and the pre-
cessions due to $J_2$. This approach allows to extract the gravitomagnetic
signal independently of the solar quadrupolar mass moment. On the other
hand, it is also possible to measure a correction $\delta J_2^\odot$ to it independently
of the LT effect. In turn, such value for $\delta J_2^\odot$ can be used to check if it can
accommodate the determined extra-precessions of Table 2. Let us write

\[
\begin{aligned}
\Delta \dot{\omega}_{\text{obs}}^\text{Mercury} &= \Delta \dot{\omega}_2^\text{Mercury} \delta J_2^\odot + \dot{\omega}_\text{LT}^\text{Mercury} \mu, \\
\Delta \dot{\omega}_{\text{obs}}^\text{Earth} &= \Delta \dot{\omega}_2^\text{Earth} \delta J_2^\odot + \dot{\omega}_\text{LT}^\text{Earth} \mu.
\end{aligned}
\]  

(6)

By solving eq. (6) with respect to $\mu$ it is possible to obtain

\[
\mu = \frac{\Delta \dot{\omega}_{\text{obs}}^\text{Mercury}}{\dot{\omega}_\text{LT}^\text{Mercury}} + \frac{c_1 \Delta \dot{\omega}_{\text{obs}}^\text{Earth}}{\dot{\omega}_\text{LT}^\text{Earth}},
\]  

(7)

with

\[
c_1 = \frac{\Delta \dot{\omega}_2^\text{Mercury}}{\dot{\omega}_2^\text{Earth}} \sim - \left( \frac{a_{\text{Earth}}}{a_{\text{Mercury}}} \right)^{\frac{7}{2}} \left( \frac{1 - e_2^2_{\text{Earth}}}{1 - e_2^2_{\text{Mercury}}} \right)^2 = -30.1930,
\]  

(8)

and

\[
\dot{\omega}_\text{LT}^\text{Mercury} + c_1 \dot{\omega}_\text{LT}^\text{Earth} = 0.0013 ''cy^{-1}.
\]  

(9)

The combination of eq. (7) is not affected by the solar oblateness whatever
its real value is: indeed,

\[
\dot{\omega}_2^\text{Mercury} + c_1 \dot{\omega}_2^\text{Earth} = 0, \ \forall J_2^\odot.
\]  

(10)

By inserting the value of eq. (8), the figures of Table 1 and the results of
Table 2 in eq. (7) one obtains for such a combination $\mu_{\text{obs}} = 1.8 \pm 10$. As
can be noted, the best estimate for $\mu$ does not change with respect to the
case of Mercury’s perihelion only, as if departures of the solar oblateness
from the adopted reference value were of little importance. Indeed, if we
solve eq. (6) with respect to the correction to the Sun’s quadrupolar mass
moment the equation

\[
\delta J_2^\odot = \frac{\Delta \dot{\omega}_{\text{obs}}^\text{Mercury}}{\dot{\omega}_2^\text{Mercury}} + d_1 \frac{\Delta \dot{\omega}_{\text{obs}}^\text{Earth}}{\dot{\omega}_2^\text{Earth}},
\]  

(11)

\footnote{The error has been evaluated by summing in quadrature the errors of Table 2 according to eq. (7).}
with

\[ d_1 = \frac{\omega_{\text{Mercury}}}{\omega_{\text{Earth}}} \sim - \left( \frac{a_{\text{Earth}}}{a_{\text{Mercury}}} \right)^3 \left( \frac{1 - e_{\text{Earth}}^2}{1 - e_{\text{Mercury}}^2} \right)^{3/2} = -18.3864, \quad (12) \]

is obtained. Eq. (11) allows to measure the correction to the adopted value of \( J_2^\odot \), by construction, independently of the LT effect in the sense that

\[ \omega_{\text{Mercury}}^{\odot} + d_1 \omega_{\text{Earth}}^{\odot} = 0, \forall L_\odot. \quad (13) \]

The result is

\[ \delta J_2^\odot = ( +0.01 \pm 0.47 ) \times 10^{-7}. \quad (14) \]

Such value can be considered as a dynamical measurement of the solar oblateness independent of the general relativistic gravitomagnetic features of motion. It induces a “residual” precession of +0.0001 " cy\(^{-1}\) on Mercury’s perihelion, which is smaller than its observed extra-advance and the related error. For the Earth the “residual” effect of eq. (14) would amount to +4 \times 10^{-6} " cy\(^{-1}\).

It is important to note that the combination of eq. (7) yields \( \chi^2 = 0.007 \), while, by assuming zero extra-precessions, one has \( \chi^2 = 0.03 \): also in this case, the relativistic prediction of LT is in better agreement with data than the zero-effect hypothesis.

### 3.4.2 The post-Newtonian gravitoelectric precessions

Although the large nominal values of their precessions, the post-Newtonian gravitoelectric terms do not represent a problem. Indeed, they are fully included in the dynamical force models of EPM2004 in terms of the PPN parameters \( \beta \) and \( \gamma \) which are presently known at a \( 10^{-4} - 10^{-5} \) level (Pitjeva 2005a; Bertotti et al. 2003). Moreover, theoretical deviations from the GTR values are expected at a \( 10^{-6} - 10^{-7} \) level (Damour and Nordtvedt 1993).

### 3.4.3 The impact of the asteroids

As already noted, the dynamical force models adopted in EPM2004 also include the action of the major asteroids and of the ecliptic ring which accounts for the other minor bodies. Indeed, it has recently pointed out

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\[ \text{Note that Pitjeva (2005a) derived a negative correction } \delta J_2^\odot \text{ of order } 10^{-8} \text{ from the determined extra-advance of Mercury’s perihelion only, without taking into account the biasing impact of the LT effect which amounts to } \sim 8\% \text{ for Mercury, as can be inferred from Table 11 and Table 5.} \]
that their impact limits the accuracy of the inner planets’ ephemerides over time-scales of a few decades (Standish and Fienga 2002) in view of the relatively high uncertainty in their masses (Krasinsky et al. 2002; Pitjeva 2005b). Recently, Fienga and Simon (2005) have shown that also Mercury’s orbit is affected to a detectable level by secular perturbations due to the most important asteroids.

May it happen that the mismodeled part of such secular precessions could explain the observed $\Delta \dot{\omega}_{\text{Mercury}}^{\text{obs}}$?

From Table 3 by Fienga and Simon (2005) the nominal amplitude of the secular perturbations on $\omega_{\text{Mercury}}$ due to 295 major asteroids can be calculated. It turns out to be 0.0004 $"\text{cy}^{-1}$; even assuming a conservative $\sim 10\%$ uncertainty (Pitjeva 2005b), it is clear that the asteroids are not the cause of the determined extra-shift of Mercury’s perihelion.

### 3.4.4 The impact of non-Einsteinian effects

In regard to other possible sources of extra-secular precessions of the planetary perihelia outside the scheme of the Newton-Einstein gravity, recently it has been shown by Lue and Starkman (2003) that the multidimensional braneworld gravity model by Dvali, Gabadadze and Porrati (2000) predicts also a secular perihelion shift in addition to certain cosmological features. By postulating that the current cosmic acceleration is entirely caused by the late-time self-acceleration, constraints from Type 1A Supernovæ data yield a value of $\sim 0.0005"\text{cy}^{-1}$ for the Lue-Starkman planetary precessions. Also this effect is too small to accommodate the determined additional perihelion advance of Mercury.

### 3.5 Analysis of other independent data

The shift of the perihelion yields a variation of the planet’s range that can be expressed as (Nordtvedt 2000) $\Delta r = ea \Delta \omega$. In the case of Mercury the centennial variation due to the Lense-Thirring precession amounts to -115 m.

Fienga et al. (2005) used their numerical ephemerides, $^8$INPOP, recently produced at the Institute of mécanique céleste et de calcul des éphémérides (IMCCE), to fit different kinds of observations including also the radar-ranging data to the inner planets. They found for the Mercury’s range

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$^8$They include a complete suit of improved dynamical models, apart from just the post-Newtonian gravitomagnetic forces whose effects are, thus, fully present in the determined residuals.
residual the value $\delta r_{\text{meas}} = -95.6 \pm 784$ m. It is worth noting that in obtaining these results also the impact of the asteroids on Mercury’s orbital motion was accounted for. As it can be noted, also in this case the errors are large, but the general relativistic prediction for the Lense-Thirring effect are in better agreement with the data than the hypothesis of null effect ($([P - D]/E]^2 = 6 \times 10^{-4}$ and $([P - D]/E]^2 = 1 \times 10^{-2}$, respectively).

4 Constraints on a Yukawa-like fifth force

The differences between the determined extra-precessions and the predicted LT rates of Table 3 of this paper can also be used to strongly constrain, at planetary length-scales $10^{10} - 10^{11}$ m, departures from the inverse-square-law phenomenologically parameterized in terms of the magnitude $|\alpha|$ of the strength of a Yukawa-like fifth force (Fischbach et al. 1986; Adelberger et al. 2003). Indeed, a potential

$$U_{\text{Yukawa}} = -\frac{GM}{r} \left[1 + \alpha \exp \left(-\frac{r}{\lambda}\right)\right],$$

(15)

where $\lambda$ is the range of such a hypothesized force, can produce a secular perihelion advance over scales $\lambda$ comparable to $a$ (Lucchesi 2003)

$$\dot{\varpi}_{\text{Yukawa}} \propto \frac{\alpha n}{e}.$$  

(16)

By using the figures of Table 3 it is possible to constrain $\alpha$ to $\approx 10^{-12} - 10^{-13}$ level at $r = \lambda \approx 1$ A.U. The most recently published constraints in the planetary range are at $10^{-9} - 10^{-10}$ level (Bertolami and Paramos 2005; Reynaud and Jaekel 2005).

5 Discussion and conclusions

In this paper we discussed the possibility of performing new tests of post-Newtonian gravity in the Solar System. To this aim, we analyzed the estimated corrections to the secular rates of the perihelia of the inner planets of the Solar System recently determined by E.V. Pitjeva (Institute of Applied Astronomy, Russian Academy of Sciences). She used the EPM2004 ephemerides with a wide range of observational data spanning almost one century; in a particular solution, she solved also for the secular motions of the perihelia by keeping fixed the PPN parameters $\beta$ and $\gamma$ and the solar quadrupole mass moment $J^2_2$, and neglecting the gravitomagnetic force in the dynamical force models.
It turns out that the post-Newtonian LT secular precessions predicted by GTR are compatible with the determined extra-precessions for Mercury, the Earth and, to a lesser extent, Mars: in normalized units ($\mu = 1$ in GTR) we have $\mu_{\text{obs}} = 1.8 \pm 2.5$ for Mercury, $\mu_{\text{obs}} = 2 \pm 4$ for the Earth and $\mu_{\text{obs}} = -3.3 \pm 16.6$ for Mars. A suitable combination of the perihelia of Mercury and the Earth, which cancels out any possible bias by $J_2^\odot$, yields $\mu_{\text{obs}} = 1.8 \pm 10$. It must be noted that the errors are still large and the data are compatible also with the hypothesis of zero extra-precessions, but at a worse level with respect to the relativistic LT prediction. If confirmed by further, more extensive and robust data analysis by determining, e.g., the extra-precessions of the nodes as well, it would be the first observational evidence of the solar gravitomagnetic field. The processing of further amounts of data, along with those expected in future from the forthcoming planetary mission BepiColombo and, perhaps, Messenger and Venus Express as well, although to a lesser extent, will further improve the accuracy in determining the orbital motion of these planets and, consequently, the precision of the LT tests.

A by-product of the present analysis is represented by new, strong constraints ($10^{-12} - 10^{-13}$) on the strength of a Yukawa-like fifth force at scales of about one Astronomical Unit.

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