A SPECTRAL SEQUENCE FOR SYMPLECTIC HOMOLOGY

MARK MCLEAN

Abstract. We construct a spectral sequence converging to symplectic homology of a Lefschetz fibration whose $E^1$ page is related to Floer homology of the monodromy symplectomorphism and its iterates. We use this to show the existence of fixed points of certain symplectomorphisms.

Contents

1. Introduction 1
2. Definitions of all our Floer homology groups 5
   2.1. Lefschetz fibrations with positive and negative ends 5
   2.2. Definitions of Floer cohomology for symplectomorphisms 8
   2.3. Defining another Floer homology group 10
   2.4. Definition of symplectic homology 17
3. Relationship between each of the Floer homology groups 18
   3.1. Construction of the main spectral sequence 18
   3.2. Group actions 22
   3.3. A long exact sequence between Floer homology groups 23
4. Applications of our spectral sequence 28
5. Appendix A: Stabilization 30
   5.1. Attaching a Weinstein $n$-handle 31
   5.2. Showing handle cancellation 36
6. Appendix B: A minimum principle 38
References 40

1. INTRODUCTION

Symplectic homology is a useful tool in symplectic geometry. It has been used in many areas. We are interested in symplectic homology of Liouville domains. A Liouville domain is a compact manifold $F$ with boundary and a 1-form $\lambda$ satisfying:

(1) $d\lambda$ is a symplectic form.
(2) $\alpha := \lambda|_{\partial F}$ is a contact form on the boundary of $F$.
(3) The boundary is positively oriented. This means that if we have an outward pointing vector field $V$ defined near $\partial F$, then the volume
form:
\[ i(V)(d\lambda)^n|_{\partial F} \]
has the same orientation as \( \alpha \wedge (d\alpha)^n \).

There exists a collar neighborhood \((1-\epsilon, 1] \times \partial F\) of \( \partial F \) such that \( \lambda = r\alpha \) where \( r \) parameterizes \((1-\epsilon, 1]\). The reason why this exists is because we have a natural vector field \( K \) transverse to the boundary which is the \( d\lambda \)-dual of \( \lambda \), and the collar neighborhood is constructed by flowing the boundary \( \partial F \) backwards along \( K \). We can form the completion of a Liouville domain by extending \((1-\epsilon, 1] \times \partial F\) by attaching a cylindrical end \([1, \infty) \times \partial F\) and extending \( \lambda \) by \( r\alpha \). We write \( \widehat{F} \) for the completion of a Liouville domain \( F \).

Symplectic homology was a tool defined in [Vit99]. Viterbo used it to give obstructions to various Lagrangian embeddings in cotangent bundles, and also used it to prove the Weinstein conjecture for subcritical Stein manifolds. It also has many other applications. Symplectic homology involves taking some time dependent Hamiltonian \( H : S^1 \times \widehat{F} \rightarrow \mathbb{R} \) such that \( H = r^2 \) near infinity, and creating a chain complex involving fixed points of the time 1 Hamiltonian symplectomorphism \( \phi_H^1 \). We write \( SH_*(\widehat{F}) \) for this symplectic homology group. See section 2.4 for a precise definition.

Let \( \phi : F \rightarrow F \) be a symplectomorphism which is the identity on the boundary \( \partial F \). We can assign a group \( HF^*(\phi) \) to it called Floer cohomology. This is basically a cohomology group whose chain complex is generated by fixed points of \( \phi \) away from the boundary. In particular if all the fixed points of \( \phi \) are non-degenerate, then we get that the number of fixed points of \( \phi \) is bounded below by the rank of \( HF^*(\phi) \). The aim of this paper is to relate this group with symplectic homology. We have to do this via another Floer homology group \( HF_*(\phi,k) \). The chain complex is generated by two copies of fixed points of \( \phi^k \) modulo an equivalence relation given by identifying a fixed point \( x \) with \( \phi^l(x) \) for some \( l \in \mathbb{Z} \).

Suppose that \( \pi : E \rightarrow C \) is a Lefschetz fibration with one positive end whose smooth fibers are symplectomorphic to \( \widehat{F} \). This is defined in section 2.1. This has a compactly supported monodromy map \( \phi' : \widehat{F} \rightarrow \widehat{F} \). Suppose that \( \phi'|_F = \phi \) and \( \phi' \) is the identity outside \( F \subset \widehat{F} \). This Lefschetz fibration is symplectomorphic to \( \widehat{M} \) for some Liouville domain \( M \).

The first main theorem says:

**Theorem 1.1.** If the dimension \( 2n \) of \( E \) is greater than 2 then there is a spectral sequence converging to \( SH_*(\widehat{M}) \) with \( E^1 \) page satisfying:
\[ E^1_{0,q} = H^{n-q}(M) \]
and for \( p > 1 \),
\[ E^1_{p,q} = HF_{q-p}(\phi,p). \]
For \( p < 0 \),
\[ E^1_{p,q} = 0. \]
The differential \( d^r \) on the \( r \)th page sends \( E^r_{p,q} \) to \( E^r_{p-r,q+r-1} \).
This is basically a slightly more refined statement of [McL09, Theorem 2.24]. Here is an example: Suppose we have a trivial Lefschetz fibration
\[ \pi : \hat{F} \times \mathbb{C} \to \mathbb{C} \]
where \( F \) is a Stein domain. We have that the monodromy map is the identity map and so
\[ HF_*(\phi, p) = H^{n-1-*}(\hat{F}) \oplus H^{n-*}(\hat{F}) \]
which gives us the following \( E_1 \) page:

| \( p = 0 \) | \( p = 1 \) | \( p = 2 \) |
|---|---|---|
| \( q = 3 \) | \( H^{n-3}(\hat{F}) \) | \( H^{n-3}(\hat{F}) \oplus H^{n-2}(\hat{F}) \) | \( H^{n-2}(\hat{F}) \oplus H^{n-1}(\hat{F}) \) | \( \cdots \) |
| \( q = 2 \) | \( H^{n-2}(\hat{F}) \) | \( H^{n-2}(\hat{F}) \oplus H^{n-1}(\hat{F}) \) | \( H^{n-1}(\hat{F}) \oplus H^n(\hat{F}) \) |
| \( q = 1 \) | \( H^{n-1}(\hat{F}) \) | \( H^{n-1}(\hat{F}) \oplus H^n(\hat{F}) \) | \( H^n(\hat{F}) \) |
| \( q = 0 \) | \( H^n(\hat{F}) \) | \( H^n(\hat{F}) \) | \( 0 \) |
| \( q = -1 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( q = -2 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( \cdots \) |

The differential \( d_{1,q}^1 \) has degree \((-1, 0)\). It is a natural projection map composed with a natural inclusion map. For instance it projects
\[ E_{2,3}^1 = H^{n-2}(\hat{F}) \oplus H^{n-1}(\hat{F}) \]
to \( H^{n-2}(\hat{F}) \) and then includes this into
\[ E_{1,3}^1 = H^{n-3}(\hat{F}) \oplus H^{n-2}(\hat{F}) \]
via the natural inclusion. It turns out that the spectral sequence in this case degenerates on the first page giving us \( \text{SH}_*(\hat{F} \times \mathbb{C}) = 0 \) basically by [Oan06, Theorem C].

We also have a relationship between \( HF_*(\phi, k) \) and \( HF_*(\phi^k, 1) \):

**Theorem 1.2.** Suppose that the coefficient field \( \mathbb{K} \) has characteristic 0 or characteristic \( p \) where \( p \) does not divide \( k \), then there exists a \( \mathbb{Z}/k\mathbb{Z} \) action \( \Gamma \) on \( HF_*(\phi^k, 1) \) such that \( HF_*(\phi, k) \cong HF_*(\phi^k, 1)^\Gamma \).

This is proven in section 3.2. Finally we can relate \( HF_*(\phi, 1) \) with \( HF^*(\phi) \):

**Theorem 1.3.** For any symplectomorphism \( \phi \), we have a long exact sequence
\[ \to HF_i^*(\phi) \to HF_i(\phi, 1) \to HF_{i-1}(\phi) \to . \]

This is proven is section 3.3. It is very similar to the Gysin exact sequence from [BO09]. These theorems might give us information about symplectic homology if we know a lot about Floer homology of symplectomorphisms. For instance the work from [CC09] enables us to calculate these groups if \( F \).
is a surface. In this paper, we go the other way around. We use symplectic homology \( SH_*(E) \) to give us information about \( \phi \).

We have the following application of the previous three theorems: For any symplectomorphism \( \phi \), we can define its positive stabilization. This involves adding a Weinstein \( n \)-handle to \( F \) to create \( F' \) and changing \( \phi \) to \( \tau_S \circ \phi \) where \( \tau_S \) is the Dehn twist about an exact Lagrangian sphere which intersects the cocore of the handle in exactly one point. This construction is described in more detail at the start of section 5.2. Also see [Sei03, Section 1] for a detailed discussion of Dehn twists and their relation with Lefschetz fibrations and [Cie02, Section 2.2] or [Wei91] for a description of Weinstein handle attaching.

**Corollary 1.4.** Suppose that \( \phi : F \to F \) is a symplectomorphism such that it is obtained by one or more stabilizations to the identity map \( id : F' \to F' \). Then if the Euler characteristic is odd, then \( HF^*(\phi^k, \mathbb{Q}) \neq 0 \) for infinitely many \( k \).

This implies that for infinitely many \( k \), any symplectomorphism \( \psi : F \to F \) which is the identity at the boundary and Hamiltonian isotopic through such symplectomorphisms to \( \phi^k \) has at least one fixed point away from the boundary. This corollary is proven in section 4. The key idea here is that \( \phi \) is the monodromy map of some Lefschetz fibration symplectomorphic to \( \mathbb{C} \times \hat{F} \). This has symplectic homology zero. We also use the fact that the \( E^\infty \) page of a spectral sequence is non-trivial if the total dimension of the \( E^1 \) pages is odd.

We also have the following corollary of Theorems 1.1, 1.2 and 1.3:

**Corollary 1.5.** Suppose \( E \) is the total space of a Lefschetz fibration with monodromy \( \phi \) such that \( SH_*(E, \mathbb{Q}) \) has infinite rank, then \( HF^*(\phi^k, \mathbb{Q}) \) is nonzero for infinitely many \( k \).

This is basically because \( HF^*(\phi^k) \) is finite dimensional for all \( k \), but symplectic homology is infinite dimensional. Hence \( HF^*(\phi^k) \) must be non-trivial for infinitely many \( k \) because we need \( \bigoplus_k HF^*(\phi^k) \) to be infinite dimensional. There are many examples of Liouville domains with infinite dimensional symplectic homology such as cotangent bundles of manifolds and also there are examples from [AM09]. Many of these examples admit Lefschetz fibrations. In fact all of these have boundaries admitting open book decompositions (see [Gir02] and [Akb10]), and we can generalize some of the above theorems in the following way:

Suppose \( p : \partial M \setminus B \to S^1 \) is an open book on the contact boundary of a Liouville domain \( M \). An open book is described as follows: The map \( p \) is a fibration whose fibers are symplectomorphic to the interiors of Liouville domains (in fact these are a special class of Liouville domains called Stein domains). The symplectic form on these fibers is the one induced from \( d\alpha_M \) where \( \alpha_M \) is the contact form on \( \partial M \). The closure of each fiber is a Liouville domain with Liouville form \( \alpha_M|_{\text{fiber}} \) and the boundary is \( B \). This has a
A SPECTRAL SEQUENCE FOR SYMPLECTIC HOMOLOGY 5

natural connection and a monodromy map coming from parallel transport around the base $S^1$. This monodromy map is some symplectomorphism $\phi : F \to F$ fixing the boundary. If we look at the statement of Theorems 1.1 and 1.5 then if we change $\phi$ so that it is now the monodromy map of our open book, then they are still true as long as we have the additional condition that the natural map $H^1(M) \to H^1(\partial M)$ is surjective. This is because every open book decomposition of $\partial M$ gives us a map $\pi : \tilde{M} \setminus K \to \mathbb{C}$ where $K$ is a compact subset of $\tilde{M}$ and such that $\pi$ looks like a Lefschetz fibration away from $K$. This map $\pi$ has a monodromy map equal to the monodromy map $\phi$ of the open book decomposition. The point is that when we prove Theorem 1.1 we do not use the interior of the Lefschetz fibration, we only use the boundary. Some of these issues will be dealt with in a future paper [McL].

Acknowledgments: I would like to thank Ivan Smith, Peter Albers and Paul Seidel for useful comments. The author was partially supported by NSF grant DMS-1005365.

2. Definitions of all our Floer homology groups

2.1. Lefschetz fibrations with positive and negative ends. Before we define other Floer homology groups we will define Lefschetz fibrations with positive and negative ends. These Lefschetz fibrations will also be used in the definition of the Floer homology groups $HF_\ast(\phi, k)$. Also we use these fibrations to construct our spectral sequence.

Let $\phi : \tilde{F} \to \tilde{F}$ be a compactly supported symplectomorphism. Any symplectomorphism $\phi$ is exact if $\phi^* (\theta_F) = \theta_F + df$ where $\theta_F$ is the Liouville form and $f : \tilde{F} \to \mathbb{R}$ is a function. We have that $\phi$ is isotopic to an exact symplectomorphism (see the proof of [BEE, Lemma 1.1]). So from now on by this deformation argument we can assume that $\phi$ is exact. Before we define Lefschetz fibrations, we will construct for each exact symplectomorphism $\phi$ a mapping torus $M_\phi$. The mapping torus is a fibration $\pi_\phi : M_\phi \to S^1$ with the following properties:

1. There exists a contact form $\alpha_\phi$ on $M_\phi$.
2. $d\alpha_\phi$ restricted to each fiber is a symplectic form. This means we have a connection on this fibration coming from the line field that is $d\alpha_\phi$ orthogonal to the fibers.
3. The monodromy map going positively around $S^1$ is Hamiltonian isotopic to $\phi$. This means that it is equal to $\phi$ composed with a compactly supported Hamiltonian symplectomorphism.
4. Near infinity, the fibration is equal to the product fibration $[R, \infty) \times \partial F \times S^1 \to S^1$ where $\alpha = d\theta + \theta_F$. Here $\theta$ is the angle coordinate.
These properties determine $M_\phi$ up to isotopy of contact structures (in fact up to contact isomorphism by Gray’s stability theorem even though the contact manifold is non-compact).

Here is an explicit construction of $M_\phi$: As a manifold, $M_\phi$ is equal to $[0,1] \times \hat{\mathcal{F}} / \sim$ where the equivalence relation $\sim$ identifies $(0,x)$ with $(1,\phi(x))$. We have a contact form $dt + \theta_F$ on the product $[0,2\pi] \times \hat{\mathcal{F}}$ where $t$ parameterizes $[0,2\pi]$. The problem is that this does not fit well with our equivalence relation so we cannot pass directly to the quotient. The symplectomorphism $\phi$ is exact, which means that $\phi^*\theta_F = \theta_F + df$. In our case we can assume that $f = 0$ outside a compact set. Choose a function $G : [0,2\pi] \times \hat{\mathcal{F}} \to \mathbb{R}$ such that $G(t,x) = 0$ near $t = 0$ and $G(t,x) = f(x)$ near $t = 2\pi$. We also require that $G = 0$ outside a large compact set. For a constant $C$ large enough, $Cdt + \theta_F + dG$ is a contact form. This also passes to the quotient and hence defines a contact form $\alpha_\phi'$. This satisfies all the properties stated above except possibly the last one. The contact form $\alpha_\phi'$ can be rescaled by a constant so that it satisfies property (4) for the following reason: Near infinity, we have that $\alpha_\phi'$ is equal to $Cdt + \theta_F$, where $\theta_F = r_F \alpha_F$ where $\alpha_F$ is the contact form on $\partial\mathcal{F}$. If we multiply $\alpha_\phi'$ by $\frac{1}{C}$, then it must look like $dt + \frac{1}{C}r_F \alpha_F$ near infinity. We can construct a diffeomorphism $\Phi : M_\phi \to M_\phi$ such that outside the region

$$\{r_F \geq 1\} = [1,\infty) \times \partial F \times S^1,$$

we have that $\Phi = id$ and inside this region $\Phi(x,y,z) = (h(x),y,z)$ for some $h : [1,\infty) \to [1,\infty)$. We define $h$ so that $h(r_F) = id$ near $r_F = 1$ and $h_F(r_F) = Cr_F$ for $r_F$ large. Then $\alpha_\phi := \Phi^*(\alpha_\phi')$ is equal to $dt + \theta_F$ near infinity. This satisfies all the required properties.

Let $S$ be an oriented surface with $s_+ + s_-$ punctures where we label $s_+$ of these punctures with a $+$ sign and the others with a $-$ sign. The $+$ punctures are called positive punctures and the $-$ punctures are called negative punctures. Around each positive puncture, we choose an orientation-preserving diffeomorphism from $[1,\infty) \times S^1$ to a neighborhood of this puncture. Also around each negative puncture we choose an orientation-preserving diffeomorphism from $(0,1] \times S^1$ to a neighborhood of this puncture. These diffeomorphisms are called framings around each puncture. On $S$ we choose a 1-form $\theta_S$ such that $d\theta_S$ is a symplectic form and such that $\theta_S = rd\theta$ around each puncture where $r$ and $\theta$ parameterizes $[1,\infty)$ (or $(0,1]$) and $S^1$ respectively. It turns out that for any oriented punctured surface with at least one positive labeled puncture, we can find such a structure and this structure is unique up to isotopy.

Let $\pi : E \to S$ be a smooth map with finitely many singularities such that the derivative of $\pi$ is surjective away from these singularities. We let $E$ have a symplectic form $\omega_E$ making all the smooth fibers into symplectic manifolds and that $\omega_E = d\theta_E$ for some 1-form $\theta_E$. We assume that $\pi$ has the following properties:
(1) The smooth fibers of $E$ are symplectomorphic to completion $\hat{F}$ of a
Liouville domain $(F, \theta_F)$.
(2) There exists a subset $E_h \subset E$ such that $(E \setminus E_h) \cap \pi^{-1}(p)$ is relatively
compact for all $p \in S$ and such that $\pi|_{E_h}$ is the trivial fibration
$$S \times \partial(F) \times [1, \infty) \to S.$$ Also on $E_h$, $\theta_E = \theta_S + \theta_F$.
(3) Around each positive puncture, $\pi$ is equal to the product
$$(\text{id}, \pi_\phi) : [1, \infty) \times M_\phi \to [1, \infty) \times S^1$$
where $\pi_\phi : M_\phi \to S^1$ is the mapping torus of some symplectomorphism $\phi$. We assume that $\theta_E$ is equal to $r \wedge \pi_\phi^* \vartheta + \alpha$ where $\alpha$ is a
contact form on $M_\phi$ and $\vartheta$ is the angle coordinate on $S^1$. This is
called a positive cylindrical end.
(4) We have a similar model $(0, 1] \times M_\phi$ around each negative puncture.
This is called a negative cylindrical end.
(5) Around each singularity $p$, there exists a complex structure $J_p$
compatible with the symplectic form, and another complex structure $j_p$
around $\pi(p)$ making $p$ holomorphic and equal to
$$(z_1, \cdots, z_n) \to \sum z_i^2$$
for some chosen holomorphic charts around $p$ and $\pi(p)$.

We say that an almost complex structures $J$ is compatible with a symplectic
form $\omega$ if:
(1) $\omega(\cdot, J(\cdot))$ is symmetric and positive definite.
(2) $\omega(J(\cdot), J(\cdot)) = \omega(\cdot, \cdot)$.

**Definition 2.1.** The above map $\pi : E \to S$ is called a Lefschetz fibration
with $s_-$ negative ends and $s_+$ positive ends. An isotopy of Lefschetz fibrations
is a smooth family of Lefschetz fibrations parameterized by $[0, 1]$. This
means that all the data such as the 1-form $\theta_E$, almost complex structure, the
choice of trivialization of $\pi|_{E_h}$ and the cylindrical ends varies smoothly.

We are interested in the Lefschetz fibrations up to isotopy. If $p$ is a regular
point of $\pi$, then the tangent space at $p$ splits up as $TE^v \oplus TE^h$ where $TE^v$
is the set of vectors tangent to the fiber at $p$ and $TE^h$ is the set of vectors
orthogonal to $TE^h$. Each Lefschetz fibration has a connection (well
defined away from the singularities) given by the plane distribution $TE^h$.
The parallel transport maps are exact symplectomorphisms. If we are given
a small circle around some puncture $p$ which winds in the direction where $\theta$
is increasing, then this induces a compactly supported symplectomorphism
$\phi_p : F \to F$ called the monodromy around $p$. This is unique up to isotopy.
2.2. Definitions of Floer cohomology for symplectomorphisms. We will use coefficients in a field \( \mathbb{K} \). In general our homology theory works if we have coefficients in a principal ideal domain. Here we define the Floer cohomology groups \( \widehat{H}F_\ast (\phi) \), where \( \phi : \widehat{F} \to \widehat{F} \) is a compactly supported symplectomorphism and where \( \widehat{F} \) is the completion of a Liouville domain \( (F, \theta_F) \). We write \( \omega_F := d\theta_F \). The boundary \( \partial F \) has a natural contact form \( \alpha_F := \theta_F |_{\partial F} \) and \( \theta_F = r_F \alpha_F \) on the cylindrical end \([1, \infty) \times \partial F\) where \( r_F \) parameterizes \([1, \infty)\). For simplicity we assume that the first Chern class of the symplectic manifold \( F \) is trivial. We will assume that \( \phi \) is an exact symplectomorphism. Any compactly supported symplectomorphism is isotopic through compactly supported symplectomorphisms to an exact symplectomorphism anyway so this does not really put any constraint on \( \phi \) (see the proof of [BEE, Lemma 1.1]). Let \( H : \widehat{F} \to \mathbb{R} \) be a Hamiltonian such that \( H|_F = 0 \) and \( H = h(r_F) \) on the cylindrical end of \( \widehat{F} \). On the manifold \( F_R := \{r_F = R\} \), the Hamiltonian flow of \( \partial F \cong \{r_F = R\} \) where \( X \) is the Reeb flow of \( \partial F \) is \( -R h'(r) X \) where \( X \) is the projection \([1, \infty) \times \partial F \to \partial F\). We assume that \( H = 0 \) on the region where \( \phi \) is non-trivial and that \( h'(r) \) is so small that \( H \) has no 1-periodic orbits in the region where \( h'(r) \neq 0 \). We also require that for \( r \gg 0 \), \( h'(r) > 0 \). Let \( \phi^1_H \) be the time 1 Hamiltonian flow of this Hamiltonian. This means that the symplectomorphism \( \phi \circ \phi^1_H \) has all its fixed points lying inside a compact subset of \( \widehat{F} \). We say that a point \( x \) is a non-degenerate fixed point if \( \phi(x) = x \) and \( D_x\phi : T_x M \to T_x M \) has no eigenvalues equal to 1. By using work from [DS94], we can perturb \( \phi \circ \phi^1_H \) by a generic compactly supported Hamiltonian symplectomorphism \( \phi^1_H \), so that all the periodic orbits of \( \phi' := \phi \circ \phi^1_H \circ \phi^1_H \) are non-degenerate. In particular this means that there are only finitely many 1-periodic orbits.

**Definition 2.2.** Any symplectomorphism \( \phi' \) constructed as above from \( \phi \) is called a standard perturbation of \( \phi \).

Let \( C_\ast (\phi') \) be the free \( \mathbb{K} \) vector space generated by the fixed points of \( \phi' \). This is a graded vector space, and its grading is the Conley-Zehnder index taken with negative sign. Choose an almost complex structure \( J \) on \( T\widehat{F} \). We have a complex line bundle \( \Lambda^n(T\widehat{F}, J) \) given by the highest exterior power of the complex vector bundle \( (T\widehat{F}, J) \).

In order to define the Conley-Zehnder index for fixed points of a symplectomorphism, we need to choose a smooth family of bundle trivializations as follows: Let \( J_t \) be a family of almost complex structures compatible with the symplectic form \( \omega_M \) on \( M \) parameterized by \( t \in [0, 1] \). We also assume that \( J_t \circ D\phi' = D\phi' \circ J_0 \) (i.e. \( \phi' \) is \((J_0, J_t))-holomorphic). These almost complex structures \( J_t \) must by cylindrical at infinity which means that outside a large compact set, \( dr_F \circ J_t = -\theta_F \). We also assume that \( J_t \) is independent of \( t \) near infinity. Our series of bundle trivializations are

\[
\text{Tr}(t) : \widehat{F} \times \mathbb{C} \to \Lambda^n(T\widehat{F}, J_t)
\]
parameterized by $t \in [0, 1]$. Because $\phi'$ is $(J_0, J_1)$-holomorphic, we have that $D\phi'$ induces a map $\Lambda^n D\phi'$ from $\Lambda^n(T\hat{F}, J_0)$ to $\Lambda^n(T\hat{F}, J_1)$ given by 

$$\Lambda^n D\phi'(v_1 \land v_2 \cdots \land v_n) := D\phi'(v_1) \land \cdots \land D\phi'(v_n).$$

We assume that the sequence of trivializations above fits into the following commutative diagram:

\[
\begin{array}{ccc}
\hat{F} \times \mathbb{C} & \xrightarrow{\text{Tr}(0)} & \Lambda^n(T\hat{F}, J_0) \\
\phi' \times \text{id} \downarrow & & \downarrow \Lambda^n D\phi' \\
\hat{F} \times \mathbb{C} & \xrightarrow{\text{Tr}(1)} & \Lambda^n(T\hat{F}, J_1)
\end{array}
\]

Another way of thinking about this family of trivializations is as follows: The family of almost complex structure $J_t$ in fact induces an almost complex structure on the contact distribution of the mapping torus $M_{\phi'}$. The above set of trivializations is equivalent to a trivialization of the highest complex exterior power of this contact distribution.

Let $x$ be a fixed point of $\phi'$. We wish to assign an index to this point. The trivializations $\text{Tr}(t)$ give us a sequence of maps 

$$\text{Tr}(t)_x : \mathbb{C} \to \Lambda^n(T_{x\hat{F}}, J_t).$$

We can find smooth family of complex linear maps 

$$a(t) : \mathbb{C}^n \to T_x(\hat{F}, J_t)$$

which are also symplectomorphisms where $\mathbb{C}^n$ has the standard symplectic structure such that 

1. the induced map 

$$\Lambda^n a(t) : \mathbb{C} \to \Lambda^n T_x(\hat{F}, J_t)$$

is equal to $\text{Tr}(t)_x$ 

2. $a(1) \circ a(0)^{-1} = D\phi'$.

Such a choice is unique up to homotopy relative to the endpoints. The maps $a(0)^{-1} \circ a(t)$ give us a path of symplectic matrices and then we can use the work of [RS93] to compute an index. For a fixed point $x$, we write $\text{ind}(x)$ for its index.

The index makes $C_\ast(\phi')$ into a graded vector space. The non-degeneracy assumption combined with the fact that all the fixed points are inside some compact set ensures that this is a finite dimensional vector space. We need to define a differential on this vector space. For a fixed point $x$ of index $i$ we define $\partial x$ as follows:

$$\partial x = \sum_{\text{ind}(y) = i-1} \#(M(x, y)/\mathbb{R}) y.$$

We will now define the manifold $M(x, y)$ together with its free $\mathbb{R}$ action.
The set $M(x, y)$ consists of maps $u : \mathbb{R} \times [0, 1] \to M$ which are $J_t$ holomorphic such that $u(s, t)$ tends to $x$ as $s$ tends to $-\infty$ and $y$ as $s$ tends to $+\infty$. The map $u$ is $J_t$ holomorphic if it satisfies the following equation:

$$u_s + J_t u_t = 0$$

where $(s, t)$ parameterizes $\mathbb{R} \times [0, 1]$. We also require that $u(1, t) = \phi(u(0, t))$.

The set $M(x, y)$ has an $\mathbb{R}$ action given by translation in the $s$ variable. We have that this set $M(x, y)/\mathbb{R}$ is a finite set for generic $J_t$. This space $M(x, y)/\mathbb{R}$ is a 0 dimensional compact oriented manifold. Each point comes with a positive or negative orientation. We define $#M(x, y)/\mathbb{R}$ to be equal to the number of positively oriented points minus the number of negatively oriented points.

**Theorem 2.3.** [DS94] We have $\partial^2 = 0$.

This group is independent of all choices made (except the choice of trivializations in order to defined the index, but we will suppress this choice here). This gives us our Floer cohomology group $HF^*(\phi)$ for a symplectomorphism $\phi$. In the introduction, our symplectomorphism was not a compactly supported symplectomorphism in $\hat{F}$, instead it was a symplectomorphism from $F$ to $F$ fixing its boundary. Because $\phi$ fixes the boundary, we have that $D\phi : TF|_{\partial F} \to TF|_{\partial F}$ is the identity. This is because there is a collar neighborhood $(1 - \epsilon, 1] \times \partial F$ of $\partial F$ such that $\lambda = r\alpha$ where $r$ parameterizes $(1 - \epsilon, 1]$, and $D\phi|_{\partial F}$ preserves the Reeb flow of $\alpha$. Hence it must also preserve $\frac{\partial}{\partial r}$ which is transverse to the boundary. This implies that we can extend $\phi$ to a $C^1$ symplectomorphism $\hat{\phi} : \hat{F} \to \hat{F}$ which is compactly supported. Hence we can define Floer homology for $\phi$ (maybe after perturbing it slightly so that it becomes $C^\infty$ without adding extra orbits). Hence we define $HF^*(\phi) := HF^*(\hat{\phi})$.

### 2.3. Defining another Floer homology group.

We will first describe a family of Hamiltonians which behave well with respect to the Lefschetz fibration $E$. Let $M_\phi$ be the mapping torus of $\phi$. The Reeb orbits of the contact form $\alpha_\phi$ are in one to one correspondence with fixed points of $\phi^k$ for each $k \in \mathbb{N}_{>0}$. Let $R > 0$ be a constant such that in the region $\{r_F \geq R\}$, we have that $\phi$ is the constant symplectomorphism. We denote $R_{\alpha_\phi}$ to be the set of Reeb orbits of $\alpha_\phi$ corresponding to fixed points of $\phi$ outside the region $\{r_F \geq R\}$. By using a similar argument to | can perturb the contact form $\alpha_\phi$ generically inside the region $\{r_F < R\}$ so that all the Reeb orbits inside this region are non-degenerate (the point here is that we consider the moduli space of contact forms $\alpha$ such that $\alpha = \alpha_\phi$ in the region $\{r_F \geq R\}$). The period spectrum is a subset of $\mathbb{R}$ corresponding to the set of lengths of Reeb orbits.

Let $E$ be a Lefschetz fibration as described in section 2.1. Each cylindrical end is either of the form $[1, \infty) \times M_\phi$ or $(0, 1] \times M_\phi$. 

For each function $g : (0, \infty) \rightarrow \mathbb{R}$ such that $g(r) = 0$ for $r$ near 1, we will construct a Hamiltonian $H_g : E \rightarrow \mathbb{R}$ as follows: Let $S$ be the base of the Hamiltonian. The region $E_h$ as described in section 2.1 is equal to $S \times [1, \infty) \times \partial F$. We choose a Hamiltonian $H : E_h \rightarrow \mathbb{R}$ such that $H = h(r_F)$ where $r_F$ parameterizes the interval $[1, \infty)$. We choose $h$ so that $h(r_F) = 0$ near $r_F = 1$, and $h'(r_F) > 0$ near infinity. We also ensure that $h'(r_F)$ is small so that in the region where $h'(r_F) \neq 0$, $H$ has no periodic orbits. We also require $h < 1$. The Hamiltonian $H$ can be extended to a Hamiltonian $\tilde{H} : E \rightarrow \mathbb{R}$ which is equal to 0 outside $E_h$. This is because $H = 0$ for near $r_F = 1$.

Let $I$ be the interval $[1, \infty)$ or $(0, 1]$. Let $C$ be a cylindrical end of $E$. This cylindrical end is of the form $I \times M_\phi$. We define $H_g^C$ as $g(s) + \tilde{H}$ on this cylindrical end for any $\tilde{H}$ described as above. We define $H_g$ to be equal to $\tilde{H}$ outside all the cylindrical ends, and for each cylindrical end $C$, we define $H_g$ to be equal to $H_g^C$. This function is smooth because $g(r) = 0$ for $r$ close to 1.

From now on we assume that $g'(r)$ is constant and not a multiple of $2\pi$ when $r \gg 0$. On any cylindrical end, the Liouville form is equal to $rd\theta + \alpha$ hence the Hamiltonian vector field of $g(r)$ is equal to the horizontal lift of the Hamiltonian vector field of $g(r)$ on $[1, \infty) \times S^1$ with symplectic form $dr \wedge d\alpha$. Hence $g(r)$ has no 1-periodic orbits for $r \gg 0$ because $g'(r)$ is not a multiple of $2\pi$. Also, we assume that $g'(r) > 0$ and very small (less than 2$\pi$ will do) for $r$ very small and also that $g'$ is constant on some open subset of this region. This ensures that $H_g$ has no 1-periodic orbits on the negative cylindrical ends when $r$ is small.

**Definition 2.4.** A Hamiltonian $H : S^1 \times E \rightarrow \mathbb{R}$ is Lefschetz admissible of slope $\lambda$ if there exists a $g : (0, \infty) \rightarrow \mathbb{R}$ as described above such that $H = H_g$ outside some compact set $K$.

A Hamiltonian $H$ is said to be non-degenerate if all the fixed points of the symplectomorphism $\phi^1_H$ are non-degenerate. The set of non-degenerate Lefschetz admissible Hamiltonians is generic among Lefschetz admissible Hamiltonians.

The map $\phi^1_{H_g}$ is a symplectomorphism so we can define $HF_s(\phi^1_{H_g})$ as above. The only difference is that the symplectomorphism $\phi^1_{H_g}$ is not compactly supported, but this does not affect the definition. For a Hamiltonian symplectomorphism, we can define $HF_s(\phi^1_{H_g})$ in a slightly different way. This definition will be useful for our purposes even though the definition is almost identical. There is a 1-1 correspondence between fixed points $x$ of $\phi^1_{H_g}$ and 1 periodic orbits $\bar{x} : S^1 \rightarrow E$ of the Hamiltonian flow $X_H$ (i.e. the vector field $X_H$ satisfies $\omega(X_H, \cdot) = dH$). Hence the chain complex $C_*(\phi^1_H)$ is generated by 1-periodic orbits of $X_H$. If we choose a trivialization of the line bundle $\Lambda^{n+1}TE$ (where the dimension of $E$ is $2n + 2$), then we have a well defined grading. We choose the trivialization as follows: Let $j$ be
the natural complex structure on \( \mathbb{R} \times S^1 \) where \( j(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t} \). Let \( J \) be an almost complex structure compatible with the symplectic form \( \omega_E \) making \( \pi_E \) holomorphic. The vertical tangent spaces \( TE^v \) are complex vector bundles because \( \pi \) is \((J, j)\)-holomorphic. The tangent bundle of the base \( \mathbb{R} \times S^1 \) of the Lefschetz fibration also has a natural trivialization, and hence the horizontal plane bundle \( TE^h \) has a natural trivialization. Combining these two trivializations gives us a trivialization of \( \Lambda^n TE^v \oplus TE^h \) and hence we get a trivialization of \( \Lambda^{n+1}TE \). This trivialization enables us to define for each 1-periodic orbit \( \bar{x} \) an index which we denote by \( \text{ind}(\bar{x}) \).

We now need to define a differential. In order to do this, we first need to describe the set of Lefschetz admissible complex structures \( J_t(\pi_E) \). Let \( J \) be an almost complex structure making \( \pi_E \) holomorphic. On the region \( E_h = S \times [1, \infty) \times \partial F \), we let \( J = j \oplus J_F \) where \( J_F \) is some almost complex structure on \( \tilde{F} \) that is cylindrical at infinity. We also assume that on each cylindrical end \( I \times M_\phi \), that \( J \) is invariant under translations in the \( r \) direction where \( r \) parameterizes \( I \).

**Definition 2.5.** An \( S^1 \) family of almost complex structures \( J_t \) compatible with \( \omega_E \) is Lefschetz admissible if outside some compact set \( K \), \( J_t = J \) for some \( J \) as described above.

The differential is a count of cylinders satisfying the perturbed Cauchy Riemann equations. We define the differential \( \partial \) as follows:

\[
\partial \bar{x} = \sum_{\text{ind}(\bar{y}) = \text{ind}(\bar{x})-1} (#M(\bar{x}, \bar{y})/\mathbb{R}) \bar{y}.
\]

Here \( M(\bar{x}, \bar{y}) \) is the set of maps \( u : \mathbb{R} \times S^1 \rightarrow E \) satisfying

\[
\partial_s u + J_t(\partial_t u - X_H) = 0
\]

where \( X_H \) is the Hamiltonian vector field of \( H \). We also require that \( u(s, t) \) tends to \( \bar{x}(t) \) as \( s \) tends to \(-\infty\) and to \( \bar{y}(t) \) as \( s \) tends to \(+\infty\). The \( \mathbb{R} \) action again is translations in the \( s \) coordinate. In order for \( \partial \) to be well defined and a differential, we need to ensure that these moduli spaces are compact. If all these solutions stay inside a compact subset of \( E \), then the results from \( \text{[BEH+03]} \) ensure that \( #M(\bar{x}, \bar{y})/\mathbb{R} \) is compact. The problem is that \( E \) is non-compact and these solutions \( u \) could escape to infinity. The maximum principle \( \text{[AS07, Lemma 7.2]} \) combined with \( \text{[McL09, Lemma 5.2]} \) and Lemma \( 6.1 \) ensures that this does not happen.

For generic \((H_t, J_t)\), the map \( \partial \) is well defined and satisfies \( \partial^2 = 0 \). If this is true then we say that the pair \((H_t, J_t)\) is regular (actually we say that this pair is regular if a particular linear map which won’t be described here is surjective). The homology of this chain complex is denoted by \( HF_*(\phi^1_{H_t}) \). For each element \( \alpha \in H_1(E) \), we have a subgroup \( HF_*(\phi^1_{H_t}) \) generated by 1-periodic orbits in the homology class \( \alpha \).
If we have two regular pairs \((H_1^t, J_1^t), (H_2^t, J_2^t)\) such that \(H_1^t \leq H_2^t\), then there is a natural map from \(HF_*(\phi_{H_1^1})\) to \(HF_*(\phi_{H_2^1})\). Again this also induces a map from \(HF_*^\alpha(\phi_{H_1^1})\) to \(HF_*^\alpha(\phi_{H_2^1})\). This is called the continuation map. The idea is that we choose an increasing homotopy \(K_s^t\) of Lefschetz admissible Hamiltonians such that \(K_s^t = H_1^t\) for \(s \ll 0\) and \(K_s^t = H_2^t\) for \(s \gg 0\). We also choose a smooth family of Lefschetz admissible almost complex structures \(J_s^t\) such that \(J_s^t = J_1^t\) for \(t \ll 0\) and \(J_s^t = J_2^t\) for \(t \gg 0\).

The continuation map sends an orbit \(\bar{x}\) of \(H_1^t\) to:

\[
\sum_{\text{ind}(\bar{y}) = \text{ind}(\bar{x})} \#M(\bar{x}, \bar{y}, J_s^t)\bar{y}.
\]

The space \(M(\bar{x}, \bar{y}, J_s^t)\) is the set of maps \(u : S^1 \times \mathbb{R} \to E\) satisfying

\[
\partial_s u + J_s^t(\partial_t u - X_{K_s^t}) = 0
\]

joining \(\bar{x}\) and \(\bar{y}\). This map is well defined only if we have a non-decreasing homotopy. The maximum principles [AS07] combined with [McL09, Lemma 5.2] and Lemma 6.1. If we have a different family of Hamiltonians \(K_s^t\) and almost complex structures \(J_s^t\) then we get a chain homotopic map. This means that we get the same map on homology. If we compose two continuation maps, then we get a continuation map.

For each orbit, we have its action which is defined as

\[
A(x) := \int_x \theta_E - \int_x H dt.
\]

The differential \(\partial\) sends an orbit of action \(A\) to a linear combination of orbits of action \(\leq A\). Hence we can define a subcomplex \(C_*^{(-\infty,a)}(H)\) generated by orbits of action strictly less than \(a\). We also have a quotient complex:

\[
C_*^{[b,a)}(H) := C_*^{(-\infty,b)}(H) / C_*^{(-\infty,a)}(H).
\]

The homology of this complex is denoted by \(HF_*^{[b,a)}(H)\). The continuation maps also decrease action, and hence they induce chain maps:

\[
C_*^{[b,a)}(H_1) \to C_*^{[b,a)}(H_2).
\]

For a Lefschetz fibration \(E\) and a subset \(K \subset E\), we have that the set of regular Lefschetz admissible pairs \((H, J)\) form a directed system with respect to the ordering \(\leq\) where \((H_1, J_1) \leq (H_2, J_2)\) if and only if \(H_1 \leq H_2\). This means that the groups \(HF_*^{[b,a)}(H, J)\) also form a directed system where the morphisms are continuation maps. We define

\[
SH_*^+(\pi_E, K) := \lim_{\to} HF_*^{[0,\infty)}(H, J).
\]

The direct limit is taken over Lefschetz admissible pairs \((H, J)\). We also define

\[
SH_* (\pi_E) := \lim_{\to} HF_*(H, J).
\]
Really these direct limits should be done on the chain level, but this produces the same result. If $\alpha \subset H_1(E)$, we can also define $\mathit{SH}_\alpha^*(E,K)$ by only considering orbits in the $H_1$ class is contained in $\alpha$.

If $K_1 \subset K_2$, we have a natural morphism $\mathit{SH}_\alpha(\pi_E,K_2) \to \mathit{SH}_\alpha(\pi_E,K_1)$. This is called a transfer map. This is because the directed system of pairs $(H,J)$ defining $\mathit{SH}_\alpha(\pi_E,K_1)$ is a subdirected system of the one defining $\mathit{SH}_\alpha(\pi_E,K_1)$.

Let $\phi$ be a symplectomorphism. Let $W_\phi$ be the Lefschetz fibration $(0,\infty) \times M_\phi$ with 1-form $\theta_\phi := sdp_\phi^* \vartheta + \alpha_\phi$ where $\alpha_\phi$ is the natural contact form on $M_\phi$ and $s$ parameterizes $(0,\infty)$. Also $\vartheta$ is the angle coordinate on $S^1$ and $p : M_\phi \to S^1$ is the natural mapping torus fibration map. The Lefschetz fibration map

$$\pi_\phi : W_\phi = \mathbb{R} \times M_\phi \to \mathbb{R} \times S^1$$

is given by $\pi_\phi(s,x) = (s,p(x))$. For each $k \in \mathbb{N}$ we define $\beta_k \subset H_1(W_\phi,\mathbb{Z})$ to be the subset $\phi_\phi^{-1}(\{1\})$ where $l \in H_1(\mathbb{R} \times S^1)$ is represented by the loop winding around the $S^1$ factor negatively $k$ times around (i.e. so that integrating $d\vartheta$ over the loop is negative).

**Definition 2.6.**

$$HF_\alpha^*(\phi,k) := \mathit{SH}_\alpha^*(\pi_\phi).$$

Here is an invariance result.

**Lemma 2.7.** If $\phi'$ is another exact symplectomorphism isotopic to $\phi$ through compactly supported exact symplectomorphisms, then $HF_\alpha^*(\phi,k)$ is isomorphic to $HF_\alpha^*(\phi',k)$.

We need some preliminary Lemmas first:

**Lemma 2.8.** We have that $(M_\phi,\alpha_\phi)$ is contactomorphic to $(M_{\phi'},\alpha_{\phi'})$. In fact there exists a function $f : M_\phi \to (0,\infty)$ such that $f = 1$ outside a large compact set and a diffeomorphism $P : M_{\phi} \to M_{\phi'}$ such that $P^* \alpha_{\phi'} = f \alpha_\phi$.

**Proof.** of Lemma 2.8. Let $\phi_t : F \to F$ be a path parameterized by $t \in [0,1]$ of compactly supported exact symplectomorphisms joining $\phi$ and $\phi'$. We have that $M_\phi$ is diffeomorphic to $M_{\phi'}$. So from now on we regard the contact form $\alpha_{\phi'}$ to be a contact form on $M_{\phi}$ making $(M_{\phi},\alpha_{\phi'})$ contactomorphic to $M_{\phi'}$. We can choose a series of contact forms $\alpha_{\phi_t}$ on $M_{\phi}$ such that $\alpha_{\phi_0} = \alpha_\phi$ and $\alpha_{\phi_1} = \alpha_{\phi'}$. We can also ensure that there exists a constant $R > 0$ such that the region $\{r_F \geq R\}$ with contact form $\alpha_{\phi_t}$ is contactomorphic to $[R,\infty) \times \partial F \times S^1$ with contact form $d\vartheta + r_F \vartheta$. Hence all the contact forms are equal near infinity, so we can use Grays stability theorem to show that they are in fact contactomorphic.

**Lemma 2.9.** Let $H : S^1 \times W_\phi \to \mathbb{R}$ be a Lefschetz admissible Hamiltonian of slope greater than $2\pi k$ then

$$HF_\alpha^*(\phi,k) := HF_\alpha^{3k}(H,J).$$
Because $H$ has a fixed slope, we have that $HF_*(\phi, k)$ is a finite dimensional vector space.

**Proof.** First of all note that if we have two Hamiltonians $H_1$ and $H_2$ of the same slope then $HF_*^{\beta_k}(H_1, J) \cong HF_*^{\beta_k}(H_2, J)$. This is because there exists a constant $C > 0$ such that $H_1 - C < H_2 < H_1 + C < H_2 + 2C$ and hence we have natural continuation maps:

$$HF_*^{\beta_k}(H_1 - C, J) \rightarrow HF_*^{\beta_k}(H_2, J) \rightarrow$$

$$HF_*^{\beta_k}(H_1 + C, J) \rightarrow HF_*^{\beta_k}(H_2 + 2C, J).$$

Composing any two of these continuation maps gives us an isomorphism because adding a constant to a Hamiltonian does not change the Floer equations or the orbits. This means we get isomorphisms

$$HF_*^{\beta_k}(H_2, J) \cong HF_*^{\beta_k}(H_1 + C, J) \cong HF_*^{\beta_k}(H_1, J).$$

Choose $K > 2k\pi$ which is not a multiple of $2\pi$. Let $h(r_F)$ be the function as defined at the start of this section. We assume that the derivatives of $h$ are so small that the only 1-periodic orbits are the constant orbits when $h = 0$.

Let $g_s : (0, \infty) \rightarrow \mathbb{R}$ be a family of functions parameterized by $s \in [0, \infty)$ with the following properties:

1. $g'_s(r) < \pi$ for $r \leq 1$.
2. $g'_s > 0, g''_s(s) \geq 0$.
3. $g'_s(r) = K$ for $1 \leq r \leq 2$.
4. $g'_s(r) = K + s$ for $s \geq 3$.
5. $g_s(r)$ tends to infinity pointwise, and $g'_s$ tends to 0 in the region $r \leq 1$ as $s$ tends to infinity.
6. $g_s(r) = g_0(r)$ for $r \leq 2$.

Let $J$ be an almost complex structure which is Lefschetz admissible and makes $\pi_{\phi}(J, j)$ holomorphic where $j$ is the complex structure on $(0, \infty) \times S^1$ given by quotienting the upper half plane in $\mathbb{C}$ by $\mathbb{Z}$. By definition, we get that

$$HF_*(\phi, k) = \lim_{s \rightarrow} HF_*^{\beta_k}(g_s(r) + h(r_F), J).$$

Let $\overline{\partial}$ be the horizontal lift of the vector field $\frac{\partial}{\partial \vartheta}$ where $\vartheta$ is the angle coordinate for $S^1$. The Hamiltonian $g_s(r) + h(r_F)$ has no 1-periodic orbits in the region $\{r_F \geq 1\}$ also all the orbits that wrap around the base $k$ times are situated in the region where $g'_s(r) = 2k\pi$. This is a subset of the region $r \leq 2$. Hence all the generators of $HF_*^{\beta_k}(g_s(r) + h(r_F))$ are contained in this region. The maximum principle \cite[Lemma 5.2]{McLO09} ensures that any Floer trajectory or continuation map trajectory connecting orbits of $g_{s_1} + h(r_F)$ and $g_{s_2} + h(r_F)$ for $s_1 \leq s_2$ is contained in the region $r \leq 2$. We have
that $g_s = g_0$ in this region and hence we get that $HF^\beta_\phi(g_s + h(r_F), J) \cong HF^\beta_\phi(g_0 + h(r_F), J)$ for all $J$ and all the continuation maps

$$HF^\beta_\phi(g_{s_1} + h(r_F), J) \rightarrow HF^\beta_\phi(g_{s_2} + h(r_F), J)$$

commute with this isomorphism. Hence

$$HF^\beta_\phi(g_0 + h(r_F), J) = \lim_{s \rightarrow \infty} HF^\beta_\phi(g_s(r) + h(r_F), J)$$

and so

$$HF^\beta_\phi(g_0 + h(r_F), J) \cong HF_\phi(k, k).$$

This Hamiltonian is of slope $K$ and hence we have proven the Lemma. □

**Proof.** of Lemma 2.7 Because $M_\phi$ is contactomorphic to $M_{\phi'}$, there is a fibrewise diffeomorphism $\Phi$ from $W_\phi$ to $W_{\phi'}$ such that $\Phi^*(r d\theta + \alpha_\phi) = rd\theta' + f\alpha_{\phi'}$ where $f : M_\phi \rightarrow (0, \infty)$. So from now on we will assume that $\pi_\phi$ and $\pi_{\phi'}$ are identical smooth fibrations and $\theta_{\phi'} = rd\theta + f\alpha_{\phi'}$. Let $\frac{\partial}{\partial r}$ be the horizontal lift of $\frac{\partial}{\partial \theta}$ to $M_\phi$. The function $\alpha_\phi(\frac{\partial}{\partial \theta})$ is positive and bounded so has image inside $[0, Q']$ for some $Q' > 0$. Note for any function $F : W_\phi \rightarrow (0, \infty)$ such that $F = 1$ outside some compact set and such that $\frac{\partial F}{\partial r} > -\frac{1}{Q'}$ we have that $\theta_F := rd\theta + F\alpha_\phi$ is a 1-form where

$$d\theta_F = dr \wedge \left( d\theta + \frac{\partial F}{\partial r} \alpha_\phi \right) \wedge \left( d(F|_{M_{\phi}}) \alpha_\phi \right)$$

is a symplectic form. Let $\frac{\partial}{\partial \theta_F}$ be the horizontal lift of the vector field $\frac{\partial}{\partial \theta}$ with respect to the symplectic form $d\theta_F$. If we have a Hamiltonian of the form $h(r)$ then its Hamiltonian vector field $X_{h(r)}$ (with respect to the symplectic form $d\theta_F$) is

$$-\frac{h'(r)}{1 + \frac{\partial F}{\partial r} \alpha_\phi(\frac{\partial}{\partial \theta_F})} \frac{\partial}{\partial \theta_F}.$$

From now on we assume that $h(r)$ is Lefschetz admissible of slope $K$.

By Lemma 2.9 there exists a $K \gg 0$ (not multiple of $2\pi$) such that for any Lefschetz admissible Hamiltonian $H$ of slope $K$, $HF^\beta_\phi(H, \theta_\phi) = HF_\phi(\phi, k)$ and $HF^\beta_\phi(H, \theta_{\phi'}) = HF_\phi(\phi', k)$. The function $\alpha_\phi(\frac{\partial}{\partial \theta_F})$ is positive and bounded so there is a constant $Q > 0$ such that the image of the function is contained in $[0, Q]$. Let $\epsilon > 0$ be smaller than the smallest distance between $K$ and any multiple of $2\pi$. If $|\frac{\partial F}{\partial r}| < \delta := \frac{\epsilon}{Q(K-\epsilon)}$ then the function

$$\frac{K}{1 + \frac{\partial F}{\partial r} \alpha_\phi(\frac{\partial}{\partial \theta_F})}$$

is never a multiple of $2\pi$. In particular the Hamiltonian $Kr$ has no 1-periodic orbits with respect to the symplectic form $d\theta_F$.

Let $G_s : W_\phi \rightarrow (0, \infty)$ be a smooth family of functions parameterized by $s \in [0, 1]$ with the following properties:
1. $\frac{\partial G}{\partial r} < \frac{\delta}{2}$.
2. In the region $r \leq 1$ or $r \gg 0$, $G_s = 1$.
3. $G_s = 1$ outside a compact set.
4. There is a constant $\Delta > 1$ such that $G_1 = f$ in the region $[\Delta, \Delta + 1]$.
5. $G_0 = 1$.

Let $h : (0, \infty) \to \mathbb{R}$ be a function such that

1. $h'(r) > 0, h''(r) \geq 0$.
2. $h'(r) < \pi$ in the region $\{r \leq \Delta + \frac{1}{3}\}$ and $h'$ is constant near $\{r = \Delta\}$.
3. $h'(r) = K$ for $r \geq \Delta + \frac{2}{3}$.

Let $J^s$ be a family of almost complex structures on $W_\phi$ parameterized by $(s, t) \in [0, 1] \times S^1$ such that $J^s$ makes $\pi_\phi$ a $(J, j)$ holomorphic map where $j$ is a standard complex structure on the base $(0, \infty) \times S^1 = \{\text{im}(z) > 0\}/\mathbb{Z}$. We also assume that $J^s$ is Lefschetz admissible with respect to the symplectic form $d\theta_G$, and the fibration $\pi_\phi$. All the 1-periodic orbits of the Hamiltonian $h(r)$ with respect to the symplectic form $d\theta_G$ are contained in $\{\Delta < r < \Delta + 1\}$. The maximum principle [McL09, Lemma 5.2] and Lemma 6.1 ensures that any Floer trajectory connecting these orbits is contained in this region as well. Also the orbits and Floer trajectories of $h(r)$ are contained in $\{\Delta < r < \Delta + 1\}$ if we use the Liouville form $\theta_f := r d\theta + f \alpha_\phi$. This implies that $HF^\beta \otimes (h(r), \theta_G)$ is isomorphic to $HF^\beta \otimes (h(r), \theta_f)$ which in turn is equal to $HF_*(\phi', k)$. Also because the family of 1-forms $\theta_G$ are all equal to each other outside a compact set, we get by a Moser theorem a compactly supported exact symplectomorphism $\Phi$ from $(W_\phi, \theta_\phi)$ to $(W_\phi, \theta_{G_1})$ (which is smoothly isotopic to the identity map). Hence a continuation argument tells us that

$$HF_*(\phi, k) = HF^\beta \otimes (h(r), \theta_\phi) \cong HF^\beta \otimes (\Phi^* h(r), \Phi^* \theta_{G_1})$$

$$\cong HF^\beta \otimes (h(r), \theta_{G_1}) = HF_*(\phi', k).$$

This proves the Lemma. \qed

2.4. **Definition of symplectic homology.** Let $M$ be a Liouville domain with Liouville form $\theta_M$, and let $\hat{M}$ be the completion of $M$. This has a cylindrical end diffeomorphic to $[1, \infty) \times \partial M$. We write $r_M$ for the coordinate parameterizing $[1, \infty)$. We write $\{r_F \leq R\}$ for the set:

$$M \cup \{(1, R) \times \partial M\} \subset \hat{M}.$$ 

The manifold $\partial M$ is a contact manifold with contact form $\alpha_M := \theta_M|_{\partial M}$. We can perturb $\theta_M$ slightly so that the period spectrum of the contact form $\alpha_M$ is discrete and injective.

**Definition 2.10.** We say that a Hamiltonian $H : \hat{M} \to \mathbb{R}$ is admissible of slope $\lambda$ if $H = \lambda r_F$ near infinity.

**Definition 2.11.** An $S^1$ family almost complex structures $J_t$ compatible with $d\theta_M$ is said to be admissible, if $dr_F \circ J_t = -\theta_M$. 


All admissible pairs \((H, J)\) form a directed system where \((H_1, J_1) \leq (H_2, J_2)\) if and only if \(H_1 \leq H_2\). We define symplectic homology as:

\[
SH_\ast(\hat{M}) := \lim_{\longleftarrow}^{H} HF_\ast(H, J).
\]

This is a symplectic invariant of \(\hat{M}\). It is invariant under exact symplectomorphisms by \([\text{Sei08}, \text{section 7b}]\). This implies that it is in fact invariant under general symplectomorphisms as every symplectomorphism is isotopic to an exact one by \([\text{BEE}, \text{Lemma 1}]\).

Let \(E\) be a Lefschetz fibration with one positive cylindrical end and no negative cylindrical ends. This is naturally symplectomorphic to the completion of a Liouville domain as explained in \([\text{McL09}, \text{Section 1}]\). Let \(C = [1, \infty) \times M_\phi \subset E\) be this positive cylindrical end. The following theorem relates symplectic homology to Lefschetz fibrations: Suppose that \(\hat{M}\) is symplectomorphic to \(E\).

**Theorem 2.12.** There is an isomorphism of groups:

\[
SH_\ast(\hat{M}) = SH_\ast(\pi).
\]

This theorem is proven in \([\text{McL09}, \text{Theorem 2.24}]\), where the right hand group is called Lefschetz symplectic homology.

3. **Relationship between each of the Floer homology groups**

3.1. **Construction of the main spectral sequence.** In this section we prove Theorem \([\text{L1}]\). Here is a statement of the main theorem of this section: Let \(E\) be a Lefschetz fibration with one positive cylindrical end and no negative cylindrical ends. This is symplectomorphic to the completion \(\hat{M}\) of a Liouville domain \(M\). Let \(M_\phi\) be the mapping torus associated to the cylindrical end. There is a spectral sequence converging to \(SH_\ast(\hat{M})\) with \(E^1\) page given by:

\[
E_{0,q} = H^{n-q}(E)
\]

And for \(p > 1\),

\[
E_{p,q} = HF_{q-p}(\phi, q).
\]

For \(p < 0\), these pages are 0.

Here is the idea of the proof of theorem \([\text{L1}]\). We carefully construct a cofinal family of Lefschetz admissible Hamiltonians for \(SH_\ast(E, E \setminus C)\) where \(C\) is the cylindrical end for \(E\). We ensure that these Hamiltonians \(H_i\) fix some smooth fiber \(\pi^{-1}(q)\). The base of the Lefschetz fibration is the plane \(C\). If we project these orbits to the base \(C\) of \(E\), then they either project to \(q\), or they wrap around \(q\) a non-negative number of times (i.e. any disk whose boundary is the projected orbit must intersect \(q\) a non-negative number of times). We have a natural filtration \(F_i\) of the chain complex for \(HF_{[0, \infty]}(H_i)\) where the subspace \(F_i\) consists of the subspace of orbits projecting to \(q\) or wrapping around \(q\) at most \(i\) times. We then show that
$H_*(F_i/F_{i-1}) \cong HF_*(\phi,i)$. This is because the cylindrical end of $E$ is the same as the cylindrical end for the fibration $(0,\infty) \times M_\phi$ used to define $HF_*(\phi,i)$ and $F_i/F_{i-1}$ is generated by the orbits which wrap around the base $(0,\infty) \times S^1$ exactly $i$ times around.

Before we prove Theorem 1.1, we need a preliminary Lemma: We will first deform our Lefschetz fibration $E$ slightly. This does not change $SH_*(\hat{M})$ as the deformation is compactly supported hence we can use a Moser theorem.

Lemma 3.1. Let $q \in \mathbb{C}$ be a regular value of the Lefschetz fibration map $\pi : E \to \mathbb{C}$. We can deform the Lefschetz fibration $E$ through a family of Lefschetz fibrations $E_t$, so that there exists a small neighborhood $U$ around $q$ where the fibration $\pi^{-1}(U)$ is equal to the product fibration

$\hat{F} \times U \to U$.

On this region $\theta_F$ splits as a product $\theta_F + \theta_U$ where $d\theta_U$ is some volume form on $U$. This deformation has compact support (i.e. the family of 1-forms $\theta_E$ inducing this deformation all agree outside some compact set).

Proof. Because $q$ is a regular value and $\pi$ has only finitely many singularities, there exists a small neighborhood $V$ where $\pi$ is regular on $\pi^{-1}(V)$. By using parallel transport techniques from [McL09, Lemma 8.6, Step 1], we can choose a smooth trivialization $\hat{F} \times V \to V$ of $\pi$ with $\theta_E = \theta_F + \theta_V + b \pm dR$ where $b$ is some 1-form which vanishes when restricted to each fiber $\hat{F} \times \{x\}$.

Let $\rho_t : \mathbb{C} \to \mathbb{R}$ be a sequence of functions parameterized by $t \in [0,1]$ such that $\rho_1 = 1$ near the boundary of $V$ and outside $V$, $\rho_0 = 1$ everywhere and $\rho_1$ is equal to 0 on a smaller neighborhood $U$ of $q$. We then deform $\theta_E$ through the family

$\theta^t_E = \theta_F + \theta_V + \rho_t b + d(\rho_t R)$

of one forms. These formulas makes sense because $\rho_t = 1$ outside $V$, so we just define $\theta^t_E$ to be equal to $\theta^t_E$ in this region.

This deformation has compact support because $b$ and $dR$ are equal to 0 for $r_F$ large. Also $\theta^1_E = \theta_F + \theta_V$ which is the product form we want. This completes our Lemma. $\square$

Proof. of Theorem 1.1 Let $E,q,U$ be as in Lemma 3.1. The Lefschetz fibration has one positive end $C = [1,\infty) \times M_\phi$. Let $r$ parameterize $[1,\infty)$ and let $\alpha$ be the contact form on $M_\phi$ such that $\theta_E = r d\vartheta + \alpha$ on this cylindrical end. Here $\vartheta$ is the pullback via the fibration map $p_\phi : M_\phi \to S^1$ of the angle coordinate of $S^1$. First of all, we can assume that $U$ is a small disk centered at $q$ (as we can shrink it). Let $s$ be the radial coordinate for this disk. We can also assume that $q$ is disjoint from the cylindrical end $C$ (either by shifting the cylindrical end or by moving $q$). We can also shrink $U$ so that it is also disjoint from $C$. 
Let $H_i$ be a family of Lefschetz Hamiltonians indexed by $i \in \mathbb{N}$ with the following properties:

1. If $(s, t)$ are radial coordinates for the disk $U$, $H_i = b_i(s) + H_F$ inside $U$ where $H_F : \hat{F} \to \mathbb{R}$ is some admissible Hamiltonian on the fiber. The 1-periodic orbits of $H_F$ are non-degenerate.
2. $b'_i(s) \geq 0$ and is 0 if and only if $s = 0$.
3. $b'_i(s)$ tends to 0 as $i$ tends to infinity.
4. All the 1-periodic orbits of $H_i$ are non-degenerate. For this to be true, $H_i$ really should be an $S^1$ family of Hamiltonians.
5. $H_i$ is cofinal, which means that $H_i$ tends to infinity pointwise in the region $\{ r > 0 \}$ and it tends to 0 pointwise everywhere else.
6. On the cylindrical end $C$, we have that in the region $\{ 1 < r < 1 + 1/i \}$, $H_i = h_i(r)$ where $h'_i$ is small so there are no periodic orbits in this region. We also assume that $h'_i = 1$ in the region $[1/i + 2, 1/i]$.

We also choose a sequence $J_i$ of Lefschetz admissible almost complex structures such that:

1. The pair $(H_i, J_i)$ is regular (i.e. so that $HF_*(H_i, J_i)$ is well defined). For this to be true, we really need an $S^1$ family of almost complex structures.
2. Let $j$ be the standard complex structure with respect to the radial coordinates $(s, t)$ in $U$. This means that $J_i(\partial/\partial s) = \partial/\partial t$. We assume that in $\pi^{-1}(U)$, $J_i = j + J_F$ where $J_F$ is an admissible almost complex structure for $\hat{F}$.
3. We require $\pi$ to be $(J, j)$ holomorphic in the region $1/2i < r < 1/i$. The reason why we need this condition is to ensure that we have control over cylinders satisfying the perturbed Cauchy-Riemann equations passing through this region.

To define symplectic homology, we don’t just need a cofinal family of Hamiltonians, we also need for two such Hamiltonians $H_j \leq H_i$, a smooth family of Hamiltonians joining them. We ensure that each element of this smooth family of Hamiltonians $H_{i,j}^s$ has exactly the same properties listed above as $H_i$ and $H_j$. The only extra condition is that in $\{ 1 < r < 1/i \}$ we require $H_{i,j}^s = h_i^s(r)$ where $\frac{\partial^2}{\partial s \partial r} h_i^s(r) > 0$. We also require similar condition $b_i^s$ inside $U$ where $b_i^s$ is the function from property (2). Let $CF_*(H_i, J_i)$ be the chain complex for $HF_*(H_i, J_i)$. Symplectic homology is the homology of the following complex:

$$\lim_{i} CF_*(H_i, J_i)$$

where the maps of this directed system $(H_i, J_i)$ come from continuation maps $H_{i,j}^s$ described above.

We can put a filtration on this chain complex as follows: We first start with filtrations $F_k^i$ on the chain complex $CF_*(H_i, J_i)$. The orbits $x$ of $H_i$
which project to \( q \) or whose projection to the base winds 0 times around \( q \) generate a subvector space \( F_0^i \subset CF_*(H_i, J_i) \).

All other projected orbits wrap positively around \( q \), so we define \( F_k^i \subset CF_*(H_i, J_i) \) to be the subspace generated by orbits whose projection winds around \( q \) at most \( k \) times. This means that \( \bigcup_k F_k^i = CF_*(H_i, J_i) \) and \( F_0^i \subset F_1^i \subset F_2^i \cdots \). We wish to show that if \( x \) is an orbit in \( F_k^i \), then \( \partial x \in F_k^i \) as well. This is true if we can show that each solution \( u : \mathbb{R} \times S^1 \to E \) satisfying the perturbed Cauchy Riemann equations intersects the fiber \( \pi^{-1}(q) \) positively. This is true because the Hamiltonian vector field \( X_{H_i} \) is tangent to \( \pi^{-1}(q) \) and because \( \pi^{-1}(q) \) is a holomorphic submanifold. This means that if we construct the mapping torus \( M_{\phi_{H_i}} \) of the symplectomorphism induced by the \( S^1 \) family of Hamiltonians \( H_i \), then solutions of the perturbed Cauchy Riemann equations correspond to holomorphic sections of the holomorphic fibration

\[
\mathbb{R} \times M_{\phi_{H_i}} \to \mathbb{R} \times S^1.
\]

The subset \( \pi^{-1}(q) \) becomes a holomorphic submanifold of \( \mathbb{R} \times M_{\phi_{H_i}} \) (i.e. the mapping torus \( \mathbb{R} \times M_{\phi_{H_i}}|_{\pi^{-1}(q)} \)) of complex codimension 1. Any holomorphic curve must intersect this positively. In particular all holomorphic sections must intersect this positively, and holomorphic sections correspond to Floer trajectories. Finally, if a holomorphic section intersects it positively, then the corresponding solution \( u \) of the perturbed Cauchy-Riemann equations must intersect \( \pi^{-1}(q) \) positively. This implies that \( F_0^i \subset F_1^i \subset \cdots \) is a filtration.

Solutions of the continuation map equations also must intersect the fiber \( \pi^{-1}(q) \) positively. Hence the natural continuation maps \( CF_*(H_i, J_i) \to CF_*(H_j, J_j) \) respect the filtration structure. This means that it sends \( F_k^i \) to \( F_k^j \). Hence on the chain complex \( \lim_{i \to -\infty} CF_*(H_i, J_i) \) has an induced filtration \( F_0^i \subset F_1^i \subset \cdots \) where \( F_k^i \) is the direct limit \( \varinjlim F_k^i \).

The spectral sequence we want is the one induced by this filtration. Hence in order to prove our result, we need to show that the homology of the chain complex \( F_k/F_{k-1} \) is equal to \( HF_{*+2k-1}(\phi, k) \). The chain complex \( F_k/F_{k-1} \) is the same as the chain complex generated by orbits wrapping \( k \) times around \( \pi^{-1}(q) \) and where the differential counts Floer trajectories that do not intersect this fiber. We have the Lefschetz fibration \( W_\phi = (0, \infty) \times M_\phi \). The region \([1, \infty) \times M_\phi \subset W_\phi \) is exactly the same as the cylindrical end \( C \) of \( E \). So we create another cofinal family of Lefschetz admissible Hamiltonians \( H'_i \) for \( HF_*(\phi, k) \) where \( H'_i = H_i \) in the region \( r > 1 + \frac{1}{2r} \), and \( H'_i = g(r) \) for \( r \leq 1 + \frac{1}{2r} \) where \( g'(r) \) has a very small positive derivative, so that \( H'_i \) has no periodic orbits in this region. Hence there is a \( 1-1 \) correspondence between orbits of \( H'_i \) that wrap around the base \( k \) times and orbits of \( H_i \) that wrap around \( \pi^{-1}(q) \) \( k \) times when projected to the base \( \mathbb{C} \). We also choose a Lefschetz admissible almost complex structure \( J'_i \) such that \( J'_i = J_i \) in the
region \( r > 1 + \frac{1}{2} \). The chain complex of the quotient \( F_k/F_{k-1} \) is generated by these orbits that wrap around \( k \) times. The differential counts cylinders satisfying the perturbed Cauchy-Riemann equations joining these orbits. All these orbits are in the region \( r \geq 1 + 1/i \), and also Lemma 6.1 ensures that any cylinder satisfying the perturbed Cauchy-Riemann equations stays in this region as well. The reason why Lemma 6.1 works here is because these cylinders do not intersect the fiber \( \pi^{-1}(q) \), hence these are subsets of the fibration

\[
E \setminus \pi^{-1}(q) \to \mathbb{C} \setminus q = C_+.
\]

The same argument also ensures that all the orbits and cylinders satisfying the perturbed Cauchy-Riemann equations in \( W_\phi \) stay inside the region \( r \geq 1/i \) as well. We then have \( F^i_k/F^i_{k-1} \) is chain isomorphic to \( CF^\alpha_{s+2k-1}(H'_i, J'_i) \), hence \( H_*(F^i_k/F^i_{k-1}) = HF^\alpha_{s+2k-1}(H'_i, J'_i) \). The shift in grading comes from the fact that we have two different trivializations of the tangent bundle of the base \( \mathbb{C}_+ \). The first trivialization comes from embedding \( \mathbb{C}_+ \) in \( \mathbb{C} \) and choosing the standard trivialization of \( TC \). The other trivialization comes from identifying \( \mathbb{C}_+ \) with \( \mathbb{C}/\mathbb{Z} \) where the \( \mathbb{Z} \) action is generated by the map \((x + iy) \to x + i(y + 1)\) and then trivializing \( TC \) in the standard way. Any such trivialization of \( TC_+ \) combined with an appropriate sequence of trivializations of the canonical bundle of the fiber \( \tilde{F} \) induces a trivialization of the canonical bundle of our respective Lefschetz fibrations. This is done by using the splitting of the tangent bundle into horizontal and vertical subspaces (See 2.3 earlier).

The continuation maps between these Floer homology groups are the same as well using a similar arguments, hence

\[
H_*(F^i_k/F^i_{k-1}) = \lim_{i} H_*(F^i_k/F^i_{k-1}) = \\
\lim_{i} HF^\alpha_{s+2k-1}(H'_i, J'_i) = HF_1(H_1, J_1) = HF_1(\phi, k).
\]

This gives us our spectral sequence. \( \square \)

3.2. Group actions. In this section we will prove Lemma 1.2 which says: Suppose that the coefficient field \( \mathbb{K} \) has characteristic 0 or characteristic \( p \) where \( p \) does not divide \( k \), then there exists a \( \mathbb{Z}/k\mathbb{Z} \) action \( \Gamma \) on \( HF_*(\phi^k, 1) \) such that \( HF_*(\phi, k) \cong HF_*(\phi^k, 1)^\Gamma \).

Proof. Let \( \pi_\phi : W_\phi \to \mathbb{R} \times S^1 \) be equal to the Lefschetz fibration \((0, \infty) \times M_\phi \). Then there is a natural \( k \) fold covering map: \( p_k : W_{\phi^k} \to W_\phi \). Here \( W_{\phi^k} \) is obtained as the pullback bundle of \( W_\phi \) via the map \( c : (0, \infty) \times S^1 \to (0, \infty) \times S^1 \) where \( c(s, t) = (s, kt) \) where we view \( S^1 \) as \( \mathbb{R}/\mathbb{Z} \).

Let \( \beta_k \subset H_1(W_\phi) \) be the set of \( H_1 \) classes represented by loops which project down to loops in \((0, \infty) \times S^1 \) which wrap around \( S^1 \) \( k \) times. Then \((p_k)_*^{-1}(\beta_k) \) is the generated by loops in \( W_{\phi^k} \) which wrap around the base
once. If we have a Lefschetz admissible pair \((H_i, J_i)\) on \(W_\phi\), then the preimage \((\tilde{H}_i, \tilde{J}_i)\) of \((H_i, J_i)\) under the covering map \(p_k\) is also Lefschetz admissible. Also if \((H_i, J_i)\) is regular for orbits wrapping \(k\) times around, then so is \((\tilde{H}_i, \tilde{J}_i)\). Here is a very brief sketch of why this statement is true: Non-degenerate orbits of \(H_i\) wrapping \(k\) times around the base lift to non-degenerate orbits wrapping once around the base (there is a choice of \(k\) lifts). Also if \(x, y\) are such orbits of \(H_i\) and \(\tilde{x}, \tilde{y}\) are choices of lifts of \(x\) and \(y\), then the moduli space of cylinders joining \(\tilde{x}\) and \(\tilde{y}\) is a clopen component of the moduli space of cylinders joining \(x\) and \(y\). Hence if \(J_i\) is regular for this moduli space then \(\tilde{J}\) must also be regular for this lifted moduli space because the linearized \(\partial_\phi\) operator is exactly the same.

The deck transformations of the covering map \(p_k\) preserve \((\tilde{H}_i, \tilde{J}_i)\). These transformations induce a \(\mathbb{Z}/k\mathbb{Z}\) action on \(CF^*_{\phi}([\tilde{H}_i, \tilde{J}_i])\). Let \(\Gamma : \mathbb{Z}/k\mathbb{Z} \to \text{Hom}(CF^*_{\phi}([\tilde{H}_i, \tilde{J}_i]))\) be the induced action on the chain complex. The quotient complex: \(CF^*_{\phi}[\tilde{H}_i, \tilde{J}_i]/(\mathbb{Z}/k\mathbb{Z})\) is canonically isomorphic to the chain complex \(CF^*_\phi(H_i, J_i)\). The \(\mathbb{Z}/k\mathbb{Z}\) action on the set of orbits of \(\tilde{H}_i\) (which generate the vector space \(CF^*_{\phi}[\tilde{H}_i, \tilde{J}_i]\)) is free, hence we can use work from [Hat02, Proposition 3G.1] to construct a transfer map from \(HF^*_{\phi}(\tilde{H}_i, \tilde{J}_i)\) to \(HF^*_\phi(H_i, J_i)\) (the subgroup consisting of elements invariant under the action of \(\Gamma\)). If the coefficient field \(\mathbb{K}\) has characteristic 0 or characteristic \(p\) not dividing \(k\), then this map is an isomorphism. The continuation maps are compatible with the group actions as well, hence taking direct limits gives us our result. This proves our lemma.

\[3.3.\text{A long exact sequence between Floer homology groups.}\] We will prove theorem \[1.3\]. Here the statement of this theorem: For any symplectomorphism \(\phi\), we have a long exact sequence 

\[\rightarrow HF^i(\phi) \to HF^i(\phi, 1) \to HF^{i-1}(\phi) \to.\]

Before we prove this Theorem, we need a preliminary Lemma. This is a correspondence between maps into a particular Lefschetz fibration and maps into its fiber. Suppose we have a non-degenerate symplectomorphism \(\phi : F \to F\), and let \(\tilde{W}_\phi\) be the mapping cylinder constructed as follows: Let \(\tilde{S} = (0, \infty) \times [0, 1]\) with the symplectic form \(ds \wedge dt\) where \(s\) parameterizes the first interval and \(t\) parameterizes the second one. We take \(S := \tilde{S}/\sim\) where \(\sim\) identifies \((t, 0)\) with \((s, 1)\). The symplectic form \(ds \wedge dt\) descends to \(S\). We define \(\tilde{W} := \tilde{S} \times F\) with the natural product symplectic structure and we define \(W := \tilde{W}/\sim\) where \(\sim\) identifies \((t, 0, f)\) with \((t, 1, \phi(f))\) where the first two coordinates are the natural coordinates parameterizing \(\tilde{S}\) and the third one is a point \(f \in F\). Again the symplectic form descends. Let \(p_1, p_2\) be the projection maps from \(\tilde{W}\) to \(\tilde{S}\) and \(F\) respectively. The map \(p_1\) descends to a map \(\pi_W\) from \(W\) to \(S\).
We have a natural connection on $\pi_W$ given by the $\omega$-orthogonal plane distribution to the fibers of $\pi_W$. If $H_l$ is any Hamiltonian on $S$ then the Hamiltonian vector field of $\pi_W^*H_l$ at a point $x$ is the unique horizontal lift at $x$ of $X_{H_l}$ in $TS_{\pi_W(x)}$. Let $H_l$ be an $S^1$ family of Hamiltonians on $S$ such that it has a non-degenerate orbit $l : S^1 \to S$ where $l$ is the natural injection $S^1 = \{1\} \times S^1 \hookrightarrow S$. We will also suppose that $\phi$ is a non-degenerate symplectomorphism. Let $J_T$, $T \in [0,1]$ be a family of almost complex structures parameterized by $[0,1]$ on $F$ such that $\phi_*J_0 = J_1$. Let $j_S$ be a complex structure on $S$. From these we can construct a new almost complex structure on $T\tilde{W} = T\tilde{S} \oplus TF$ given by $j_S \oplus J_l$ where $t$ is the coordinate parameterizing $[0,1]$ in $\tilde{S} = (0,\infty) \times [0,1]$. Conversely suppose we have some almost complex structure $J$ on $\tilde{W}$ which is compatible with the symplectic form and makes $\pi_W^*(J,j_S)$ holomorphic then we can construct a family of almost complex structures $J_t$ on $F$ such that $\phi_*J_0 = J_1$. The point is that $J$ must split up as $J_S \oplus J_l$ after we pull it back to $\tilde{W} = T\tilde{S} \oplus TF$ because the projection map to $\tilde{S}$ is $(J,j_S)$ holomorphic and compatible with the symplectic form.

Lemma 3.2. There is a natural 1-1 correspondence $\Psi_p$ between 1-periodic orbits of $\pi_W^*H_l$ which project to $l$ and fixed points of $\phi$. Let $D$ be a small disk around $l(0)$. If we have two such 1-periodic orbits $x,y$ of $\pi_W^*H_l$ then there is a natural 1-1 correspondence between smooth maps $u : \mathbb{R} \times S^1 \to \tilde{W}$ connecting $x,y$ such that

1. $\pi_W(u(0,0)) \in D$.
2. The loop $\pi_W(u(0,t))$ is homotopic to $l$

and pairs of smooth maps $u_1 : \mathbb{R} \times [0,1] \to F$, $u_2 : \mathbb{R} \times S^1 \to S$ such that

1. $u_1(s,1) = \phi(u_1(s,0))$.
2. $u_1(s,t)$ converges to $\Psi_p(x)$ as $s$ goes to $-\infty$. and converges to $\Psi_p(y)$ as $s$ goes to $+\infty$.
3. $u_2(s,t)$ converges to $l(t)$ as $s$ goes to $\pm\infty$.
4. $u_2(0,0) \in D$.
5. The loop $u_2(0,t)$ is homotopic to $l$.

Also the Cauchy-Riemann operator $\partial_{J,H}(u) := (Du + \pi_W^*X_{H_l} \otimes ds)^{(0,1)}$ maps naturally to the sum $\partial_1(u_1) + \partial_{J,H}(u_2)$.

Proof. of Lemma 3.2 Let $q : S^1 \to W$ be a 1-periodic orbit of $H_l$ which maps to $l$. Let $Q_W : \tilde{W} \to W$, $Q_S : \tilde{S} \to S$ be the natural quotient maps. We can lift $q$ to a map $\tilde{q} : [0,1] \to \tilde{W}$ which is the Hamiltonian flow of $Q_W^*\pi_W^*H_l = (p_1)^*Q_S^*H_l$. We define $\Psi_p(q)$ to be the point $p_2(\tilde{q}(0)) \in F$. We have a natural inverse as follows: If $x$ is a fixed point then we take the line given by the inclusion

$$v : [0,1] = \{1\} \times [0,1] \times \{x\} \hookrightarrow \tilde{W} = (0,\infty) \times [0,1] \times F.$$ 

This is a flow line for $(p_1)^*Q_S^*H_l$ and it satisfies $\phi(v(0)) = v(1)$ hence it descends to a 1-periodic orbit of $H_l$. This is exactly the inverse of $\Psi_p$. 
We now wish to find a natural 1-1 correspondence between maps $u$ and pairs $(u_1, u_2)$. We define $u_2$ as $\pi_W \circ u$ this satisfies $u_2(0, 0) \in \mathbb{D}$. Let $\tilde{W}'$ be the universal cover of $W$. This is symplectomorphic to the product $(0, \infty) \times \mathbb{R} \times F$ with the product symplectic form $ds \wedge dt + \omega_F$. The quotient map $\pi_W' : \tilde{W}' \to W$ is induced by the quotient of the $\mathbb{Z}$ group action where $1 \in \mathbb{Z}$ is the map: $(s, t, f) \to (s, t + 1, \phi(f))$. Choose some lift $\tilde{D}$ of $\mathbb{D}$ in $\tilde{W}'$. We now replace $u$ with $\tilde{u} : \mathbb{R} \times [0, 1] \to W$ which is the composition $u \circ p_{S'}$ where $p_{S'}$ is the natural projection map $\mathbb{R} \times [0, 1]$ to $\mathbb{R} \times S^1$ identifying $(t, 0)$ with $(t, 1)$. Because the domain of $\tilde{u}$ is contractible, we have a unique lift of $\tilde{u}$ to $\tilde{u}' : \mathbb{R} \times [0, 1] \to \tilde{W}'$ such that $\tilde{u}'(0, 0) \in \tilde{\mathbb{D}}$. This is because the disk $\tilde{\mathbb{D}}$ is small enough so that the $\mathbb{Z}$ action on $\tilde{W}'$ sends $\tilde{\mathbb{D}}$ to other sets disjoint from $\tilde{\mathbb{D}}$. Note that the projection maps $p_1$ an $p_2$ extend to $\tilde{W}'$ if we view $\tilde{W}$ as some subset of $\tilde{W}'$. We define $u_1$ to be $p_2 \circ \tilde{u}'$. Because the loop $\pi_W(u(s, t))$ is homotopic to $t$ for each $s$, we get that $u_1(s, 0) = \phi(u_1(s, 1))$.

Suppose we are now given $u_1$ and $u_2$, we wish to reconstruct $u$ from these maps. Let $\tilde{u}_2$ be the composition $u_2 \circ p_{S'}$. Choose a unique lift $\tilde{u}_2'$ of $\tilde{u}_2$ into the universal cover $\mathbb{R} \times \mathbb{R}$ such that $\tilde{u}_2'(0, 0) \in p_1(\tilde{\mathbb{D}})$. We have a natural map from $\mathbb{R} \times [0, 1]$ into $\tilde{W}'$ given by $(u_1, \tilde{u}_2')$. This projects to the map $u$.

We have $\partial_{J, H}(u) = (Du + X_{\pi_W^* H_t} \otimes ds)^{(0,1)}$, $\partial_{J}(u_1) = (Du_1)^{(0,1)}$ and $\partial_{J, H}(u_2) = (Du_2 + X_{H_t} \otimes ds)^{(0,1)}$. Also for any vector $X$, $(\pi_W)_*(X)$ is equal to $(\pi_W)_*(X^h)$ where $X^h$ is the horizontal component of $X$. We have

$$(\pi_W)_*(Du + X_{\pi_W^* H_t} \otimes ds)^{(0,1)}(Y) =$$

$$(\pi_W)_* \left( \left( Du(Y)^h + X_{\pi_W^* H_t}^h(ds(Y)) \right) + \left( J \circ Du(jY) \right)^h + J X_{\pi_W^* H_t}^h(ds(jY)) \right).$$

We have $X_{\pi_W^* H_t} = \tilde{X}_{H_t}$, where $\tilde{X}_{H_t}$ at $x$ is the unique horizontal lift of $X_{H_t}$ at $\pi_W(x)$. Also $\pi_W$ is $(J, J_{S'})$-holomorphic. Hence we have

$$\pi_W Du(Y)^h = Du_2(Y), \pi_W(\pi_{\pi_W^* H_t}^h X_{\pi_W^* H_t}) = X_{\pi_W^* H_t},$$

$$\left( J \circ Du(jY) \right)^h = \pi_{J} \circ Du(jY).$$

This implies $(\pi_W)_* \partial_{J, H}(u) = \partial_{J, H} u_2$.

The Cauchy-Riemann operator for the map $\tilde{u}$ is exactly the same as the one for $u$ (you just pull back everything via the map $p_{S'}$). Also if we consider the lift $\tilde{u}'$ of $\tilde{u}$ as described earlier, we still have exactly the same operator. This is because $\tilde{W}'$ is a cover of $W$ such that its projection map is holomorphic. The almost complex structure on $W$ is $(J_S, J_{\tilde{H}})$ on the tangent space $\mathbb{R} \times \mathbb{R} \times TF$ of $W'$ at the point $(s, t, f)$. We defined $T^v \tilde{W}'$ as the tangent spaces $TF$ in $\mathbb{R} \times \mathbb{R} \times TF$. These are the vertical tangent spaces of the fibration $p_1$. We have that:

$$(p_2)_*(Du' + X_{p_1^* H_t} \otimes ds)^{(0,1)}(Y) = (p_2)_*(Du' + X_{p_1^* H_t} \otimes ds)^{(0,1)}(Y)^v.$$
Because $X_{p_t^*H_t}$ vanishes on $T^\omega \tilde W'$, we get that this is equal to:

$$(p_2)_*(((D\tilde w(Y))^u)^{(0,1)} = ((D\tilde w'(Y))^u + J_t \circ (D\tilde w(jY))^u).$$

Because $(p_2)_*D_t^u(Y) = D(u_1)(Y)$, we get:

$$(p_2)_*\partial J_{p_t^*H_t}(\tilde u')(Y) = (D(u_1)(Y)) + J_t \circ D(u_1)(jY) = (\partial J(u_1))(Y).$$

Hence our correspondence between $u$ and $(u_1, u_2)$ also gives us a natural map between Cauchy-Riemann operators on $u$ and $(u_1, u_2)$ respectively.

Because we have a nice correspondence between between maps $u$ and $(u_1, u_2)$ and their respective Cauchy-Riemann operators, we have that the moduli space of maps $u$ satisfying the perturbed Cauchy-Riemann equations is the same as the moduli space of $J_t$ holomorphic maps $u_1$ and maps $u_2$ satisfying the perturbed Cauchy-Riemann equations. If we have regularity for one moduli space then we have regularity for the other and we can ensure that their orientations coincide. Because the maps $u_2$ satisfying the perturbed Cauchy-Riemann equations joining $l$ with $l$ have energy zero, they must map to the constant loop $l$. Hence we actually have a natural correspondence between maps $u$ satisfying the perturbed Cauchy-Riemann equations and $J_t$-holomorphic maps $u_1$.

Let $J_T, T \in [0, 1]$ be a smooth family of almost complex structures compatible with the symplectic form on $F$ such that $\phi_*^1J_0 = J_1$.

**Proof.** of Theorem 1.3. Let $\phi'$ be a standard perturbation for $\phi$ as in definition 2.2. Let $r_F$ be the cylindrical coordinate for $\tilde F$. Using this symplectomorphism $\phi'$ we will first carefully construct a Lefschetz fibration with one positive and one negative end whose monodromy is almost equal to $\phi'$.

The reason why it cannot be exactly equal to $\phi'$ is that the parallel transport maps are equal to the identity map for $r_F$ large, but this isn’t true for $\phi'$. Instead we produce a new symplectomorphism $\phi''$ as follows: There exists a small constant $\epsilon > 0$ and a large constant $R > 0$ such that in the region $r_F \geq R$, $\phi'$ is equal to the symplectomorphism $\phi^{1}_{r_F}$. Choose a smooth function $l : [R, \infty) \rightarrow \mathbb{R}$ with the following properties:

1. $l(r_F) = \epsilon r_F$ for $r_F$ near $R$
2. $0 < l'(r_F) \leq \epsilon$.
3. $l'(r_F) = 0$ for $r_F \geq R + 1$

Let $\phi''$ be the new symplectomorphism equal to $\phi'$ in the region $r_F < R$ and equal to $\phi^1_{l(r_F)}$ in the region $r_F \geq R$. In section 2.1 we constructed a mapping torus $M_\phi$ with an explicit contact form equal to $Cdt + \theta_F + dG$.

The good thing about this contact form is that the monodromy map around the base $S^1$ is exactly the same as $\phi$. Using this construction, let $M_{\phi''}$ be the mapping torus of $\phi''$ with contact form $\alpha_{\phi''}$ whose monodromy map is exactly equal to $\phi''$.

Using this contact form $\alpha_{\phi''}$ we can construct a Lefschetz fibration $W_{\phi''} = (0, \infty) \times M_{\phi''}$ with associated one form $\theta_{\phi''} = r d\theta + \alpha_{\phi''}$.
We will now construct a Lefschetz admissible Hamiltonian $H$ of slope $\beta$ as follows: Let $L : W_{\phi'} \to \mathbb{R}$ be equal to 0 in the region $\{r_F \leq R\}$ and equal to $\epsilon r_F - l(r_F)$ in the region $\{r_F \geq R\}$. Let $\kappa : (0, \infty) \to \mathbb{R}$ satisfy:

1. $\kappa'' \geq 0$ and $\kappa' > 0$.
2. $\kappa'(r) = \beta$ for $r$ large.
3. $\kappa'(r)$ is small for $r$ small.
4. $\kappa'(1) = 2\pi, \kappa''(1) > 0$.

Let $K : (0, \infty) \times S^1 \to \mathbb{R}$ be equal to $\kappa(r)$. Let $H = \pi'_{\phi'}K + L$. Any 1-periodic orbit of $H$ with $H_1$ class in $\beta_1$ projects isomorphically to the circle $\{r = 1\}$. Because $H$ is not time dependent, we have $S^1$ families of orbits. Choose a Morse function $f$ on $S^1 \cong \{r = 1\}$ with one maximum and one minimum. We can use work from [FH94], combined with the Morse function on each orbit to perturb $H$ to a non-degenerate Hamiltonian. In fact, we can perturb $K$ to a time dependent function $K_t$ so that $H_t := \pi'_{\phi'}K_t + L$ has non-degenerate orbits because the symplectomorphism $\phi'$ is non-degenerate.

The Hamiltonian $H_t$ has the following property: the 1-periodic orbits of $H_t$ are exactly 1-periodic orbits of $H$ that start and end at the critical points of the function $\pi'_{\phi'}f$. Let $J$ be an almost complex structure which is Lefschetz admissible and invariant under translations in the $r$ direction. We also assume that $\pi_{\phi'}$ is $(J, j)$ holomorphic.

We view the base $(0, \infty) \times S^1$ as a symplectic manifold with symplectic form $dr \wedge dt$. The Hamiltonian $K_t$ has two orbits that wrap around the $S^1$ factor of $\mathbb{R} \times S^1$ once. One orbit starts and ends at the maximum of the Morse function $f$ and the other starts and ends at the minimum. Let $o_m, o_M : S^1 \to (0, \infty) \times S^1$ be these two orbits corresponding to the minimum and maximum of the Morse function $f$ respectively. Let $p_m, p_M$ be the starting points of the orbits $o_m$ and $o_M$ respectively (these are fixed points of the time 1 Hamiltonian symplectomorphism $\phi_{H_t}^1$). Every fixed point of $\phi_{H_t}^1$, whose associated orbit wraps around $(0, \infty) \times S^1$ once maps to either $p_m$ or $p_M$ via $\pi_{\phi'}$. Define $F_m := \pi_{\phi'}^{-1}(p_m)$ and $F_M := \pi_{\phi'}^{-1}(p_M) \cong \hat{F}$. The time 1 flow of $H_t$ sends $F_m$ to $F_m$ and $F_M$ to $F_M$ and both these maps are exactly equal to $\phi'$ after identifying $F_m$ and $F_M$ with $\hat{F}$. This means that there is a bijection between fixed points of $\phi'$ and fixed points of $\phi_{H_t}^1$, that project to $p_m$. Similarly there is a bijection between fixed points of $\phi'$ and fixed points of $\phi_{H_t}^1$, that project to $p_M$. If $x$ is a fixed point of $\phi'$, we write $x_m, x_M$ for the corresponding fixed points of $\phi_{H_t}^1$ that project to $p_m$ and $p_M$ respectively. We also write $x_m, x_M$ for the corresponding orbits.

Let $x^1, x^2$ be two fixed points of $\phi$ such that they project down to the fixed point $p_m$. Using the orbit $o_m : S^1 \to (0, \infty) \times S^1$ we can construct a family of almost complex structures $J_t$ on $\hat{F} \cong F$ from $J$ as in the statement before Lemma 3.2.

Let $M$ be the moduli space of $J_t$ holomorphic maps $u : \mathbb{R} \times [0, 1] \to \hat{F}$ where $\phi'(u(1, t)) = u(0, t)$ joining $x^1$ and $x^2$. Let $M_m$ be the moduli space
of maps $u : \mathbb{R} \times S^1 \to \mathbb{R} \times M_\phi$ satisfying the perturbed Cauchy Riemann equations for $H$ joining $x^1_m$ and $x^2_m$. We have a natural bijections

$$\mathcal{M} \to \mathcal{M}_m$$

from 3.2. Also we can ensure that both are regular and that the orientation of these moduli spaces are the same. Similar reasoning ensures that we also have a bijection between $\mathcal{M}_M$ and $\mathcal{M}_m$.

Let $\partial$ be the differential for $HF^*(\phi', 1)$. Let $x, y$ be fixed points of $\phi'$. Regularity of $(K, j)$ ensures that there is no cylinder satisfying Floer’s equations for $H$ starting at $x_M$ and ending at $y_m$. This is because any cylinder $u$ satisfying such equations projects to another cylinder $u'$ satisfying Floer’s equations for $K$ on $(0, \infty) \times S^1$. But the cylinder $u'$ starts at an orbit of index 0 (the orbit $o_M$) and ends at an orbit of index 1 (the orbit $o_m$). This is impossible by regularity. Hence we have an increasing filtration $F_{M} \subset F_{M_m} \oplus F_{m} = C^*_*(H)$ where $F_{M}$ consists of fixed points of the form $x_M$ and $F_{m}$ consists of fixed points of the form $x_m$. This means that the differential is of the form:

$$\partial = \begin{pmatrix} \partial_{F_m} & 0 \\ \partial_{m,M} & \partial_{F_M} \end{pmatrix}.$$ 

Let Fix($\phi'$) be the set of fixed points of $\phi'$. There is a bijection between Fix($\phi'$) and orbits of the form $x_m$ given by the map $x \to x_m$. Similarly there is a bijection between Fix($\phi'$) and orbits of the form $x_M$. Using these bijections and the fact that we proved $M = M_m = M_M$ for all orbits $x^1, x^2$, we have that the chain complexes $(F_M, \partial_{F_M})$ and $(F_m, \partial_{F_m})$ are chain isomorphic to the chain complex for $HF_*(\phi) = HF_*(\phi')$. Using this fact and this filtration we get our long exact sequence. □

4. Applications of our spectral sequence

We will first prove Corollary 1.4. Here is a statement of this corollary: Suppose that $\phi : F \to F$ is a symplectomorphism such that it is obtained by one or more stabilizations to the identity map $id : F' \to F'$. Then if the Euler characteristic is odd, then $HF^*(\phi^k, Q) \neq 0$ for infinitely many $k$. From now on (from the comment at the end of Section 2.2) we will assume that $\phi$ is a compactly supported symplectomorphism $\phi : \hat{F} \to \hat{F}$. We first some preliminary Lemmas.

Lemma 4.1. Suppose that $HF_*(\phi, k, Q) \neq 0$, then $HF^*(\phi^k, Q)$ is non-trivial.

Proof. of Lemma 4.1 Theorem 1.2 tells us that there is a $\mathbb{Z}/k\mathbb{Z}$ action on $HF_*(\phi^k, 1, Q)$ whose fixed points give us $HF_*(\phi, k, Q)$. This means that if $HF_*(\phi, k, Q)$ is non-zero, then so is $HF_*(\phi^k, 1, Q)$. Then the long exact sequence from Theorem 1.3 tells us that $HF^*(\phi^k, Q)$ is non-zero. □
Lemma 4.2. \textit{Let }$E_{r,s}^\ast$\textit{ be a spectral sequence (with coefficients in some field $\mathbb{K}$). Suppose for some }$r \geq 0$, \textit{the total rank of }$E_{r,s}^\ast$\textit{ is odd (resp. even), then the total rank of }$E_{s,s}^\infty$\textit{ is odd (resp. even).}

\textit{Proof.} of Lemma 4.2. This is done by induction. Suppose for some $R \geq r$, $E_{R,s}^\ast$ has odd rank. Then there is a differential $\partial$ on $E_{R,s}^\ast$ such that $E_{R+1,s}^\ast = H_\ast(E_{R,s}^\ast, \partial)$. This must have odd rank, because the homology of an odd dimensional chain complex is odd dimensional. The reason why this is true is because: if the rank of the image of $\partial$ is odd, then the rank of the kernel of $\partial$ is even by the first isomorphism theorem. This means that the rank of $H_\ast(E_{R,s}^\ast, \partial)$ is an odd number minus an even number which is odd. Similar reasoning ensures that if the rank of the image of $\partial$ is even, then the homology has odd rank.

Exactly the same reasoning ensures that if $E_{r,s}^\ast$ has even rank, then so does $E_{R,s}^\ast$. This proves the Lemma. \hfill $\square$

Lemma 4.3. \textit{The rank of }$HF_\ast(\phi, k, \mathbb{Q})$\textit{ is even.}

\textit{Proof.} of Lemma 4.3. The idea here is to construct an explicit chain complex of even rank for $HF_\ast(\phi, k, \mathbb{Q})$. Then the same reasoning as in the previous Lemma 4.2 gives us our result. Let $W_\phi := \mathbb{R} \times M_\phi$ be the Lefschetz fibration associated to $\phi$. Let $H : W_\phi \to \mathbb{R}$ be a Lefschetz admissible Hamiltonian which is time independent. Using similar methods to [HS95, Theorem 3.1], a generic such $H$ has non-degenerate orbits. This means that for all fixed points $x$ of $\phi_1^1$, the linearized return map

$$D_x\phi_1^1 : T_xW_\phi \to T_xW_\phi$$

has at most one eigenvalue equal to 1. Let $\beta_k \subset H_1(W_\phi)$ be equal to $\pi_{\phi_1^{-1}}(l_k)$ where $l_k$ is the homology class represented by a loop wrapping $k$ times positively around the $S^1$ factor of the base $\mathbb{R} \times S^1$. All of the 1-periodic orbits whose $H_1$ class is in $\beta_k$ are non-trivial. Choose a Morse function $f$ on each 1-periodic orbit $o$ of $\phi_1^1$ whose $H_1$ class is in $\beta_k$ with one maximum and one minimum. We can do this because each of these orbits are non-trivial. We can perturb $H$ to a time dependent Hamiltonian

$$H' : S^1 \times W_\phi \to \mathbb{R}$$

using these Morse functions so that for each 1-periodic orbit $x$ of $H$ in the class $\beta_k$, we get two non-degenerate periodic orbits $x_m, x_M$ of $H'$ in the class $\beta_k$ corresponding to the minimum $m$ and maximum $M$ of the Morse function $f$ (see [CFHW96]). This implies that the chain complex $CF_{s}^{\beta_k}(H', J)$ is even dimensional, which means that $HF_\ast^{\beta_k}(H', J)$ is even dimensional. We can construct a cofinal family $H_1'$ of such Hamiltonians so that $HF_\ast(\phi, k, \mathbb{K}) = \lim_i HF_\ast^{\beta_k}(H_1', J)$. By Lemma 2.9, there exists an $i$ such that $HF_\ast(\phi, k, \mathbb{K}) = HF_\ast^{\beta_k}(H_1', J, \mathbb{K})$. This implies that the homology group $HF_\ast(\phi, k, \mathbb{K})$ has even rank. \hfill $\square$
Proof. of Corollary 1.4. In view of Lemma 4.1 we have to prove that \( HF_\ast(\phi, k, Q) \) is non-trivial for infinitely many \( k \). We have a Lefschetz fibration:
\[
\pi_{\text{prod}} : \hat{F}^i \times \mathbb{C} \rightarrow \mathbb{C}
\]
which is the natural projection whose monodromy map is the identity \( \text{id} : \hat{F}^i \rightarrow \hat{F}^i \). By [Oan06, Proposition 2], we get that \( SH_\ast(\hat{F}^i) = 0 \). Positive stabilization is some operation on a Lefschetz fibration which does not change the symplectomorphism type of the total space (see Theorem 5.5 in the appendix). In particular, any sequence of positive stabilizations of \( \pi_{\text{prod}} \) gives us a new Lefschetz fibration:
\[
\pi_E : E \rightarrow \mathbb{C}
\]
such that \( E \) is symplectomorphic to \( \hat{F}^i \times \mathbb{C} \). Hence \( SH_\ast(E) = 0 \). We will assume that the monodromy map of \( E \) is equal to \( \phi \).

Suppose for a contradiction that there exists a \( K > 0 \) such that that \( HF_\ast(\phi, k, Q) = 0 \) for all \( k \geq K \). Then by Theorem 1.1 there exists a spectral sequence converging to \( SH_\ast(E) = 0 \) with \( E^1 \) page:
\[
E^1_{0,q} = H^{n-*}(E)
\]
And for \( p > 1 \),
\[
E^1_{p,q} = HF_{q-p+1}(\phi, q).
\]
For \( p < 0 \),
\[
E^1_{p,q} = 0.
\]
The total rank \( \bigoplus_{p,q} E^1_{p,q} \) is finite and odd. This is because the rank of \( E^1_{0,q} = H^{n-*}(E, Q) \) is odd and the rank of \( E^1_{p,q} \) is even for \( p \neq 0 \) and \( E^1_{p,q} = 0 \) for \( |p|, |q| \gg 0 \). The reason why the rank of \( H^{n-*}(E, Q) \) is odd is because the Euler characteristic of \( E \) is odd. By Lemma 4.2 this implies that that the rank of \( \bigoplus_{p,q} E_{p,q}^\infty \approx SH_\ast(E) \) has odd rank. But \( SH_\ast(E) \) has even rank (equal to 0).

The above proof actually tells us the following fact: For any Lefschetz fibration \( E \), if the rank of \( SH_\ast(E) \) mod 2 is different from the rank of \( H^\ast(E) \) mod 2, then \( HF^\ast(\phi^k) \) does not vanish for infinitely many \( k \) where \( \phi \) is the monodromy map. Similar methods also show that if the rank of \( SH_\ast(E) \) is infinite, then \( HF^\ast(\phi^k) \) does not vanish for infinitely many \( k \). For instance \( SH_\ast(T^*M) \) has infinite rank if \( M \) is simply connected, hence the monodromy map of any Lefschetz fibration symplectomorphic to \( T^*M \) has the above property.

5. Appendix A: Stabilization

Let \( M \) be an exact symplectic manifold with boundary. Let \( d\lambda \) be the associated symplectic form on \( M \). A Weinstein function \( f : M \rightarrow \mathbb{R} \) is a function such that \( -i(X_f)\lambda > 0 \) away from the critical points of \( f \). A Weinstein cobordism is an exact symplectic manifold \( M \) as above with a Weinstein function \( f : M \rightarrow [c, d] \) where \( \partial M = \partial_- M \sqcup \partial_+ M \) with \( \partial_- = \partial_+ = \partial_0 \).
$f^{-1}(c)$ and $\partial_+ = f^{-1}(d)$ and such that $c$ and $d$ are regular values of $f$. We will assume that $M$ is compact.

5.1. **Attaching a Weinstein $n$-handle.** We will first describe how adding Dehn twists corresponds to attaching $n$-handles. We will do this in two parts. We will first describe the $n$-handle carefully and attach it to the Lefschetz fibration creating a Liouville domain. We will then deform this Liouville domain so that the Lefschetz fibration structure extends over the handle in such a way that we have a new critical point of our Lefschetz fibration whose vanishing cycle corresponds to the attaching Legendrian sphere.

**Definition 5.1.** Let $\pi : E \to S$ be a Lefschetz fibration as in Definition 2.1. As in the definition, there is a subset $E_h \subset E$ which is symplectomorphic to a product $S \times \partial F \times [1, \infty)$. Let $E^0_h \subset E^h$ be the interior of this set (i.e. the subset $E^h \setminus (S \times \partial F \times \{1\})$). Let $A$ be the union of all the positive and negative ends of $E$. Let $A^0$ be equal to the interior of $A$. Let $\bar{E} := E \setminus (E^0_h \cap A^0)$. Let $\bar{S}$ be a compact oriented surface which is equal to $\pi(\bar{E})$. This has a positive (resp. negative) boundary component for each positive (resp. negative) puncture of $S$. The map $\pi : \bar{E} \to \bar{S}$ is called a compact convex Lefschetz fibration associated to $E$.

This is also defined in [McL09, Definition 2.14]. We have that $\bar{E}$ is a manifold with corners. By [McL09, Theorem 2.15], we can assume (maybe after deforming the Lefschetz fibration without changing the monodromy maps) that $\bar{E}$ is a Liouville domain after smoothing the corners. From now on by abuse of notation, we will write $\bar{E}$ for the Liouville domain obtained by smoothing the corners of $\bar{E}$. Also if we have a smooth family of compact convex Lefschetz fibrations $E_t$, then the associated Lefschetz fibrations $E_t$ are all symplectomorphic. Let $\pi : E \to \mathbb{C}$ be a Lefschetz fibration and let $\pi : \bar{E} \to \mathbb{D}$ be the associated compact convex Lefschetz fibration whose base is the unit disk $\mathbb{D} \subset \mathbb{C}$. Let $F = \pi^{-1}(1)$. Let $L$ be an exact Lagrangian sphere in $F$. This means that $\theta_F|_L = d\kappa$ form some function $\kappa : L \to \mathbb{R}$ ($\theta_F$ is the Liouville form).

We can choose our Liouville form on $\bar{E}$ so that it becomes a Legendrian sphere in $\partial \bar{E}$. The reason is as follows: Because $L$ is an exact Lagrangian in $F$, we have a small neighborhood of $L$ diffeomorphic to a small neighborhood of the zero section of $T^*L$ such that $\theta_F$ is equal to $d\kappa' + \sum_j p_j dq_j$ where $p_j$ are momentum coordinates and $q_j$ are position coordinates. We can then modify $\theta_F$ so that $\kappa' = 0$ near $L$, hence $TL$ is in the kernel of $\theta_F$. The subset $\pi^{-1}(\partial \mathbb{D}) \subset \partial \bar{E}$ is a mapping torus of some symplectomorphism $\phi : F \to F$, and we can ensure that the contact form is equal to $C dt + \theta_F + dG$ as described in section 2.1. We can also ensure that $G$ is 0 near $F$. This then ensures that $L$ is a Legendrian with respect to this contact form.

**Theorem 5.2.** We can attach a Weinstein $n$ handle $H$ to $\bar{E}$ along $L$ creating a new Lefschetz fibration

\[ \pi' : \bar{E} \cup H \to \mathbb{D} \]
where:

1. If $\Delta$ is an arbitrarily small neighborhood of $1 \in \mathbb{D} \subset \mathbb{C}$, then we can ensure that $\pi'$ coincides with $\pi$ outside $\pi^{-1}(\Delta)$.
2. $\pi'$ has one extra singularity inside $H$ and its vanishing cycle is Hamiltonian isotopic to $L$ inside $F$.

Weinstein handles are described in [Wei91] and [Cie02]. We will start by describing the Weinstein $n$-handle $H$: Let $(p_1, p_2, \ldots, p_n, q_1, \ldots, q_n)$ be standard coordinates for $\mathbb{C}^n$ where the symplectic form $\omega_{\text{std}} = \sum_j dp_j \wedge dq_j$. We also assume that $z_j = p_j + iq_j$ are the standard complex coordinates for $\mathbb{C}^n$. Let $p : \mathbb{C}^n \to \mathbb{C}$ be defined by $p(z) = \sum_j z_j^2$. Let $p_R$ be the real part of $p$, and $p_i$ the complex part of $p$. We write $z = (z_1, \ldots, z_n)$. We have $p_R(z) = \sum_j (p_i^2 - q_i^2)$ and $p_C(z) = 2\sum_j (p_i q_i)$. Let $x = \sum_j p_j^2$, $y := \sum_j q_j^2$. We can view $p_R$ as a function of $x$ and $y$. This means that $p_R = x - y$. From now on let $0 < \delta \ll \epsilon \ll 1$. We will now construct a function $\psi_{\delta, \epsilon} : \mathbb{R}^2 \to \mathbb{R}$ which we will also view as a function of $x$ and $y$. This is constructed as follows: Let $V_\epsilon$ be a vector field which is 0 outside a ball of radius $\epsilon$. We also assume that $V_\epsilon$ is of the form $a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}$ where $a \leq 0$ and $b \geq 0$ and on the ball of radius $2\delta$ centered at 0, we set $a = -1, b = 1$. Let $\phi_t^\delta : \mathbb{R}^2 \to \mathbb{R}^2$ be the time $t$ flow of the vector field $V_\epsilon$. We define $\psi_{\delta, \epsilon} := p_R \circ \phi_t^\delta$. Here is a picture of the level curves of $\psi$ (solid lines) and $p_R$ (dotted lines):

![Level curves](image)

We let

$$H := B_{2\epsilon} \cap \{\psi_{\delta, \epsilon} \leq -\delta/2\} \cap \{p_R \geq -\delta/2\}$$
where $B_{2\epsilon}$ is a ball of radius $2\epsilon$. This set is our $n$-handle $H$. This has a Liouville vector field
\[
\frac{1}{2} \sum_j \left( p_j \frac{\partial}{\partial p_j} - q_j \frac{\partial}{\partial q_j} \right).
\]
We have that $\psi_{\delta,\epsilon}$ is a Weinstein function for this handle for $0 < \delta \ll \epsilon \ll 1$. We also define a function $p' : \mathbb{C}^n \to \mathbb{C}$ by $p' = \psi_{\delta,\epsilon} + ip$. This has a Lefschetz singular point at 0, and the map is smooth away from 0. Also if $\delta$ is small enough then the symplectic form $\omega_{\text{std}}$ is non-degenerate on all the vertical tangent spaces of $p'$ away from 0. The attaching region $H_-$ is the subset $\{p_\mathbb{R} = -\delta/2\} \cap H \subset H$ and $H_+$ is the subset $\{\psi_{\delta,\epsilon} = -\delta/2\} \cap H \subset H$. Let $S \subset p^{-1}(-\delta/2)$ be the vanishing cycle of the singular point 0 of $p$ along the path joining 0 and $-\delta/2$ along the real axis. We have that a small neighborhood of $H_-$ in $\{p_\mathbb{R} = -\delta/2\}$ is a codimension 0 submanifold of the product $T^*S^{n-1} \times \mathbb{R} = \{p_\mathbb{R} = -\delta/2\}$ containing the Legendrian sphere $S \times \{0\}$. Here $S$ is the zero section of $T^*S^{n-1}$. This is true because parallel transport maps for $p$ are well defined, so we can trivialize the fibration $p_\mathbb{R} |_{\{p_\mathbb{R} = -\delta/2\}}$. The Liouville vector field
\[
\frac{1}{2} \sum_j \left( p_j \frac{\partial}{\partial p_j} - q_j \frac{\partial}{\partial q_j} \right)
\]
is transverse to $\{p_\mathbb{R} = -\delta/2\}$, so it is a contact manifold. The map $p$ restricted to this contact manifold is smooth and has fibers that are symplectomorphic to $T^*S^{n-1}$.

We now need a Lemma describing how we can glue our handle while preserving our fibration structure. Let $U$ be a contact manifold and $q : U \to (-\epsilon', \epsilon') \subset \mathbb{R}$ be a fibration whose fibers are symplectic submanifolds with symplectic form $d\lambda$ (where $\lambda$ is the contact form). Similarly let $q' : U' \to (-\epsilon', \epsilon')$ be a map with the same properties as $q$. We also assume that $q^{-1}(0)$ is exact symplectomorphic to $q'^{-1}(0)$. Because the fibers are symplectic submanifolds, there is a natural connection on $q$ and $q'$. For $q$ this is the line field given by the kernel of $d\lambda$ and for $q'$ it is the kernel of $d\lambda'$. This means that we can do parallel transport. The problem is that it is possible for a point $p$ to be transported off the edge of $U$. We will therefore assume that this cannot happen for $q$ and $q'$. Let $t$ parameterize $(-\epsilon, \epsilon)$. We assume that $\lambda(\frac{\partial}{\partial t}) > 0$ and $\lambda'(\frac{\partial}{\partial t}) > 0$ where $\frac{\partial}{\partial t}$ is the horizontal lift of $\frac{\partial}{\partial t}$ for $q$ or $q'$.

**Lemma 5.3.** There is a diffeomorphism $C : U \to U'$ such that $q' \circ C = q$ and such that $C^*\lambda' = \lambda + dR$ for some function $R$. Also, there is a smooth family of contact forms $\lambda^s$ such that $d\lambda^s = d\lambda$ for all $s$, $\lambda^0 = \lambda$ and $\lambda^1 = C^*\lambda'$.

**Proof.** By using parallel transport maps, we have that $U$ is diffeomorphic to $q^{-1}(0) \times (-\epsilon', \epsilon')$ with $\lambda = \lambda|_{q^{-1}(0)} + dR$ where $R$ is a function with $\frac{\partial R}{\partial t} > 0$. Similarly we have $\lambda' = \lambda'|_{q'^{-1}(0)} + dR'$. We have a diffeomorphism
\(\Phi: q^{-1}(0) \rightarrow q^{-1}(0)\) such that \(\Phi^*\lambda = \lambda + dR_2\) for some function \(R_2\). We define \(C: U \rightarrow U'\) as \((\Phi, \text{id})\), then \(C^*\lambda = \lambda - dR_1 + d(R_1' \circ \Phi) + dR_2\). This proves the first part of the Lemma with \(R = -R_1 + R_1' \circ \Phi + R_2\). We have that \(\frac{\partial R_1}{\partial t} > 0\) and \(\frac{\partial(R_1' \circ \Phi + R_2)}{\partial t} > 0\) because \(\partial_t R_2 = 0\) and \(\partial_t R_1 \circ \Phi > 0\). So we can join these functions via a family of functions \(R^s\) with \(\frac{\partial R^s}{\partial t} > 0\). We define \(\lambda^s := \lambda|_{q^{-1}(0)} + dR^s\). These are contact forms because \(\lambda|_{q^{-1}(0)}\) is a symplectic form and \(\frac{\partial R^s}{\partial t} > 0\), and \(\lambda^0 = \lambda\) and \(\lambda^1 = \lambda + dR = C^*\lambda\). Also \(d\lambda^s = d\lambda\) for all \(s\). This proves the Lemma.

Let \(f_1: A_1 \rightarrow \mathbb{D}, f_2: A_2 \rightarrow \mathbb{D}\) be two smooth fibrations such that \(A_1\) and \(A_2\) have symplectic forms \(\omega_1\) and \(\omega_2\) respectively such that the fibers of \(f_1\) and \(f_2\) are symplectic submanifolds. These fibrations have a connection given by the plane field which is \(\omega_i\) orthogonal to the fibers. We also assume that this connection gives us well defined parallel transport maps (i.e. no points get transported off the edge of the manifold \(A_i\)).

**Lemma 5.4.** Suppose that there exists a line \(l \subset \partial \mathbb{D}\) and a fibrewise diffeomorphism \(\Phi\) from \(A_1\) to \(A_2\) such that it is a symplectomorphism from \(A_1|_l\) to \(A_2|_l\). We suppose that \(\mathbb{D}\) smoothly deformation retracts onto \(l\). Then we can deform \(\omega_2\) through symplectic forms \(\omega_2^t\) such that \(\omega_2^t\) restricted to each fiber is the same as \(\omega_2\) and such that there is a symplectic embedding of \(A_1\) into some arbitrarily small neighborhood of \(\pi_2^{-1}(l)\) with symplectic form \(\omega_2^t\). This symplectic embedding is equal to \(\Phi\) in the region \(\pi_1^{-1}(l)\) and is a fibrewise diffeomorphism. This deformation also fixes \(\omega_2^t\) in the region \(\pi_2^{-1}(l)\).

**Proof.** of Lemma 5.4. Fix a small neighborhood \((-1, 0] \times l \subset \mathbb{D}\) of \(l\). The line \(l\) is a line with boundary, so we should lengthen \(l\) slightly so that we get a neighborhood. We identify \(l\) with \(\{0\} \times l\) here. Let \(l(0)\) be a point in \(l\). Let \(F_i\) be the fiber \(\pi_i^{-1}(0)\) for \(i = 1, 2\). Let \(\omega_{F_i}\) \((i = 1, 2)\) be the corresponding symplectic forms on these fibers. For a path \(\mathcal{P}: [0, 1] \rightarrow \mathbb{D}\), let \(\Phi_{\mathcal{P}}: \pi_1^{-1}(\mathcal{P}(0)) \rightarrow \pi_1^{-1}(\mathcal{P}(1))\) be the corresponding parallel transport map for \(\pi_1\). For each point \(x = (x_0, x_1) \in (-1, 0] \times l\) we have a path \(\gamma_{x_0}\) joining \(l(0)\) with \(x\). This path first travels along \(l\) from \(l(0)\) to \((0, x_1) \in (-1, 0] \times l\) and then it joins \((0, x_1)\) to \((x_0, x_1)\) along the path \((-1, 0] \times \{x_1\}\). We have the following trivialization:

\[T_1: (-1, 0] \times l \times F_1 \rightarrow \pi_1^{-1}((-1, 0] \times l)\]

given by: \(T_1(x_0, x_1, c) = \Phi_{\gamma_{x_0}(x_1)}(c)\). The symplectic form on \((-1, 0] \times l \times F_1\) is the product \(\omega_{\mathbb{D}} \times \omega_{F_1}\) where \(\omega_{\mathbb{D}}\) is a symplectic form on \((-1, 0] \times l\). We have a similar trivialization \(T_2\). Because \(\omega_1\) and \(\omega_2\) agree on the region \(\pi_1^{-1}(l) = \pi_2^{-1}(l)\), we have that the symplectic fibrations are exactly the same via the symplectomorphism \(T_2 \circ T_1^{-1}\) (as long as we choose appropriate neighborhoods of \(l\) for \(T_1\) and \(T_2\) respectively). Hence we only need to embed the fibration \(\pi_1\) into the fibration given by the image of \(T_1\).

Let \(r_1: \mathbb{D} \rightarrow \mathbb{D}\) be the deformation retraction onto \(l\). Here we assume that \(r_1\) is a smooth embedding for \(t < 1\). We have that \(r_{1-\epsilon}\) maps \(\mathbb{D}\) to a small
neighborhood of \( l \) (so that it fits inside the neighborhood described above. We have a smooth fibrewise embedding \( \Psi \) from \( A_1 \) into \( \pi_1^{-1}((-1,0] \times l) \) given by \( \Phi_{r_t(\pi_1(x))}(x) \). Here \( r_t(\pi_1(x)) \) is the path

\[
a : [0, 1 - \epsilon] \to \mathbb{D} \\
a(t) = r_t(\pi_1(x)).
\]

This fibrewise embedding sends fibers to fibers symplectically. The problem is that it isn’t a symplectomorphism. If we look at \( \Psi^*(\omega_1) - \omega_1 \), it is a closed 2-form which vanishes on the fibers. Because \( \mathbb{D} \) is contractible, this means that \( \omega_1 - \Psi^*(\omega_1) = d\theta \). Choose a 1-form \( \theta' \) such that \( \Psi^*\theta' = \theta \) (we can choose our symplectic embedding \( \Psi \) appropriately so that this works). Let \( h_t : \mathbb{D} \to \mathbb{R} \) be a family of functions such that \( h_t = t \) near \( l \) and is 0 outside a small neighborhood of \( l \). In order for \( \Psi \) to be a symplectic embedding, we first consider the family of closed 2-forms \( \omega_1 + d(h_t \circ \pi_1)\theta' \). All these symplectic forms agree with \( \omega_1 \) when we restrict to each fiber, but they are not necessarily symplectic forms on the total space \( A_1 \). In order to make them into symplectic forms, we need them to be non-degenerate. This is done by pulling back a sufficiently large multiple of a time dependent volume form \( v_t \) for \( \mathbb{D} \). This volume form can be chosen so that it is equal to 0 for \( t = 0 \) and outside a small neighborhood of \( l \). We can also assume that for \( t = 1 \) it is equal to 0 on a small neighborhood of \( l \) as well. We have that \( \Psi \) is a symplectic embedding if we consider the symplectic form \( \omega_1 + d(h_t \circ \pi_1)\theta' \).

This proves the lemma.

\[\square\]

**Proof.** of theorem 5.2 We choose a small neighborhood \( NL \subset F \) of \( L \) which is exact symplectomorphic to the interior of the unit disk bundle \( D^*L \) associated to some metric \( g \) on \( L \). We also have a similar neighborhood \( NS \) of \( S \subset p^{-1}(-\delta/2) \) where \( p : \mathbb{C}^n \to \mathbb{C} \) is the map described earlier. Choose \( NL \) and \( NS \) so that they are exact symplectomorphic. The exponential map gives us a smooth embedding \( e : (-\frac{1}{2}, \frac{1}{2}) \to \partial \mathbb{D} \). Let \( P_t \) be the parallel transport map for \( \pi \) starting at \( e(0) \) and traveling along the path \( e(t) \). We define \( U := \bigcup_{|t| < \frac{1}{2}} P_t(NL) \). We let \( e' : (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{C} \) be the path defined by \( e'(t) = -\delta/2 + it \). Let \( P'_t \) be the parallel transport map for \( p \) starting at \( e'(0) \) and traveling along the path \( e' \) to \( e'(t) \). We define \( V \) to be \( \bigcup_{|t| < \frac{1}{2}} P'_t(NS) \subset \mathbb{C}^n \). By using Lemma 5.3 we have a diffeomorphism \( C : U \to V \) such that \( C^*p|_V = \pi|_U \) and such that \( C^*d\theta_H|_V = d\theta_E|_U \) where \( \theta_H \) is the Liouville form on the Handle and \( \theta_E \) is the Liouville form on the Lefschetz fibration \( E \). Also by the second part of this Lemma, we can deform the contact form on the boundary of \( E \) through contact forms so that \( C \) becomes a contactomorphism. We can ensure that this deformation of contact forms extends to a Liouville deformation of \( E \). The handle \( H \) depends on parameters \( \delta \) and \( \epsilon \), so for \( 0 < \delta \ll \epsilon \ll 1 \), we have that \( H_- \) is a subset of \( V \). Hence we can use \( C^{-1} \) to glue \( H_- \) to \( U \). We will write \( \tilde{E} \cup H \) for the Lefschetz fibration with the handle glued. We now wish to
extend the Lefschetz fibration structure over the handle $H$. The handle $H$
has two maps $p$ and $p'$ from $H$ to $C$. We will now enlarge $H$ to $\tilde{H}$ inside $C^n$
as follows: We use parallel transport to construct a subset $[-\delta, -\delta/2] \times H_-$
where the map $p_\mathbb{R}$ is the projection to $[-\delta, -\delta/2]$ and the map $p_\mathbb{C}$ is the
composition of the projection to $H_-$ with $p_\mathbb{C}|_{H_-}$.

We can use the same parallel transport trick to construct a neighborhood of $U$ in $E$
diffeomorphic to $[-\delta, -\delta/2] \times U$ as follows: We have a small neighborhood of
the image of $e$ inside the disk $\mathbb{D}$ is biholomorphic to the product $[-\delta, -\delta/2] \times (-\frac{1}{2}, \frac{1}{2})$. We can ensure that it is disjoint from the
singular values of $\pi$. Then we use parallel transport as before to construct a subset of $E$
diffeomorphic to $[-\delta, -\delta/2] \times U$ which projects via $\pi$ to $[-\delta, -\delta/2] \times (-\frac{1}{2}, \frac{1}{2})$. We have a smooth fibrewise embedding of
$[-\delta, -\delta/2] \times H_-$ into $[-\delta, -\delta/2] \times U$. Each inclusion $\{x\} \times H_- \hookrightarrow \{x\} \times U$ is a symplectic embedding. By Lemma 5.3, we can Liouville deform $\tilde{E}$ through
compact convex Lefschetz fibrations so that it becomes a symplectic embedding. The map $\pi$
coincides with $p$ in this region. Let $N \subset H_- \subset H$. Let $H^U$ be the region

$$(\{[-\delta, -\delta + \eta]\} \times H_-) \cup \{[-\delta, -\delta/2] \times N\} \subset H.$$ 

For $\eta$ small enough, the map $p'$ coincides with $p$ in the region $F$. This
means we can change $\pi$ to a new map $\pi'$ in the following way: We define $\pi'$
to be equal to $\pi$ outside the region $[-\delta, -\delta/2] \times H_-$ and set it equal to $p'$
inside this region. This map is smooth because $p'$ coincides with $\pi$ on the
region $F$. The map $\pi'$ also has one extra singularity whose vanishing cycle
is Lagrangian isotopic to the original Lagrangian sphere $L$. We also have
that $\tilde{E} \cup H$ is equal to $\tilde{E}$ with an $n$-handle attached. 

5.2. Showing handle cancellation. Let $F$ be a Liouville domain of
dimension $2n - 2$. Let $\phi : \tilde{F} \to \tilde{F}$ be a compactly supported symplectomor-
phism where $\tilde{F}$ is the completion of the Liouville domain $F$. We have that
$\{r_F \leq K\}$ is also a Liouville domain for any $K$ whose completion is also
symplectomorphic to $\tilde{F}$. The support of $\phi$ is contained in this set for large
enough $K$. Because of this we may as well redefine $\tilde{F}$ so that the support
of $\phi$ is contained inside $F$. Let $F'$ be obtained from $F$ by attaching an
$n - 1$-handle. The $n - 1$-handle $H$ has a natural Weinstein function given
by $\psi$ described in the previous section. This has exactly one critical point.
This has an unstable manifold. Let $L$ be a Lagrangian in $F'$ which intersects
the unstable manifold in exactly one point. Let $\tau_L$ be a symplectomorphism
which is a Dehn twist around $L$. Because $\phi$ is the identity near the boundary
of $F$, we can extend $\phi$ to a symplectomorphism $\phi'$ from $F'$ to $F'$ by making
$\phi$ equal to the identity on the handle. A Stabilization of $\phi$ is defined to be
$\tau_L \circ \phi$. We can extend this map to the completion $\tilde{F'}$ of $F'$ by making it
equal to the identity map outside $F'$. By abuse of notation, we will use the
same names $\tau_L$ and $\phi$ for the corresponding maps defined on the completion.
Theorem 5.5. Let \( \pi : \tilde{E} \rightarrow \mathbb{C} \) be a Lefschetz fibration. Let \( \phi : \tilde{F} \rightarrow \tilde{F} \) be the monodromy map. Then there exists another Lefschetz fibration \( \pi'' : \tilde{E''} \rightarrow \mathbb{C} \) whose monodromy map is \( \tau_{\lambda} \circ \phi : \tilde{F'} \rightarrow \tilde{F'} \) such that \( \tilde{E''} \) is symplectomorphic to \( \tilde{E} \).

Proof. of Theorem 5.5 The Lefschetz fibration \( \tilde{E} \) is the completion of some compact convex Lefschetz fibration \( E \) see [McL09, Definition 2.16]). The fiber of this Lefschetz fibration is some Weinstein domain \( F \). A small neighborhood of \( \partial F \) is symplectomorphic to \( (1 - \eta, 1] \times \partial F \) with Liouville form \( r_F \alpha_F \) where \( \alpha_F \) is the contact form on the boundary of \( F \) and \( r_F \) is the coordinate parameterizing \( (1 - \eta, 1] \). Because \( E \) is a Liouville domain (with corners), we have a Liouville vector field \( X \) on \( E \) which is transverse to \( \partial F \). Its boundary is a union of two manifolds \( E^h \) and \( E^u \) meeting in one corner \( \partial \) \( E \cap \partial \). The manifold \( E^x \) is the vertical boundary equal to \( \pi^{-1} (\partial \mathbb{D}) \) and \( E^h \) is the horizontal boundary. A small neighborhood of \( E^h \) is symplectomorphic to \( \mathbb{D} \times ((1 - \eta, 1] \times \partial F) \). We can ensure that the Liouville vector field \( X \) is equal to \( r \frac{\partial}{\partial r} + r_F \alpha \) in this region where \( r \) is the radial coordinate for \( \mathbb{D} \).

A small neighborhood of \( \partial E \) is of the form \( \{ r_F \geq 1 - \eta \} \cup \{ r \geq 1 - \eta \} \). Let \( g : (1 - \eta, 1] \rightarrow \mathbb{R} \) be a function which is equal to 0 on a small neighborhood of \( 1 - \eta \) and such that \( g(x) = x \) for \( x \) near 1. We can define the function \( g(r_F) + r^2 : E \rightarrow \mathbb{R} \). Because \( g(x) \) is equal to zero near \( 1 - \eta \), this function is well defined on all of \( E \) as we extend it by \( r^2 \) when \( r_F \) is ill defined. Here we write \( r \) instead of \( \pi^* r \) by abuse of notation. This is also a Weinstein function with respect to \( X \) on a neighborhood of \( \partial E \) (but not on all of \( E \)).

We then attach the \( n - 1 \) handle to \( F \) as above to create a new Liouville domain \( F' \). Let \( U \) be a small neighborhood of \( F' \setminus F \). We have a Weinstein function \( w : U \rightarrow \mathbb{R} \) with exactly one singular point of index \( n - 1 \) and such that \( w = r_F \) near \( \partial F \subset F' \). Let \( H \) be this handle. This is equal to the closure of \( F' \setminus F \) inside \( F' \). Let \( Y \) be the Liouville vector field described above for this handle. This is has one critical point of index \( n - 1 \) and \( w \) is a Weinstein function for this handle.

Let \( E' \) be a new compact convex Lefschetz fibration obtained by gluing \( \mathbb{D} \times H \) to the region \( v_E = \mathbb{D} \times \partial F \) where we identify the \( \mathbb{D} \) factor with itself and we attach \( H \) to \( \partial F \). The map \( \pi \) extends to a map \( \pi' : E' \rightarrow \mathbb{D} \) by setting \( \pi' \) equal to the projection map

\[
\mathbb{D} \times H \rightarrow \mathbb{D}
\]

outside \( E \). We can extend the function \( g(r_F) + r^2 \) to \( E' \) over the region \( \mathbb{D} \times H \) by the function \( r^2 + w \). Let \( w_E' : E' \rightarrow \mathbb{R} \) be this new function. Also, the Liouville vector field \( X \) extends to a Liouville vector field \( X' \) defined on \( E' \) by setting \( X' \) to be equal to

\[
r \frac{\partial}{\partial r} + Y
\]
on the region $\mathbb{D} \times \bar{H}$ and $X' = X$ elsewhere. The vector field $X'$ has exactly one critical point of index $n - 1$ in the region $E' \setminus E$. The unstable manifold of this critical point is of the form $\mathbb{D} \times U_Y$ inside $E' \setminus E$ where $U_Y$ is the unstable manifold of $Y$.

We have a Lagrangian $L \subset F'$. Without loss of generality, we can assume that 0 is a regular value of $\pi'$. Identify $F'$ with $\pi'^{-1}(0)$. We now attach our $n$ handle as in Theorem 5.2 along $L$. This means we need to modify the Liouville domain $E'$ on $\pi'^{-1}(U)$ where $U$ is an arbitrarily small neighborhood of 1. We can assume that $U$ is disjoint from all the singular values and also 0. Let $\pi'' : E'' := E' \cup H \to \mathbb{D}$ be this new Liouville domain, and let $X''$ be the new Liouville vector field on $E''$. We can extend the Weinstein function $w_{E''}$ over this handle so it becomes a Weinstein function $w_{E''}$ for $X''$. The unstable manifold $\mathbb{D} \times U_Y$ described above is also an unstable manifold for the same critical point of $X''$ away from $\pi'^{-1}(U)$. Let $A \subset E''$ be the unstable manifold of $X''$ at this critical point. The Liouville vector field $X''$ has one extra critical point in the handle of index $n$. The map $\pi''$ has one extra critical value $x \in \mathbb{D}$ near 1. Let $l$ be a path joining $x$ with 0 avoiding all the critical values of $\pi''$. Let $V \subset E''$ be the thimble associated to this path. I.e. it is the set of points in $E''$ that parallel transport along this path into the singularity of $\pi''$ inside the handle. This is a Lagrangian submanifold diffeomorphic to a ball (see [Sei03, Section 1]). Let $V_1 \subset E''$ be the stable manifold of the singularity of $X''$ in the handle $H$. Near the singularity $x$ we have that $V_1$ is the same as $V$. This is because we have that $\pi''$ is the same (after a translation) as $p : \mathbb{C}^n \to \mathbb{C}$. By symmetry (the antiholomorphic involution fixing $\mathbb{R}^n \subset \mathbb{C}^n$) we have that $V$ and $V_1$ are open subsets of $\mathbb{R}^n \subset \mathbb{C}^n$. We also have for some $\epsilon > 0$ small enough, that $\tau := w_{E''}^{-1}(w_{E''}(x) - \epsilon)$ is a contact submanifold which is isotopic through contact submanifolds to some smoothing of the boundary of $E' \subset E''$. Also because the Lagrangian sphere $(\pi'')^{-1}(l(0)) \cap V$ is Hamiltonian isotopic inside $(\pi'')^{-1}(l(0))$ to a Lagrangian sphere intersecting $A$ once, we have that the corresponding Legendrian sphere $\partial E' \cap V$ is Legendrian isotopic to a Legendrian sphere intersecting $A$ once. Hence $\tau$ is Legendrian isotopic to a Legendrian sphere intersecting $A$ once.

By work of Eliashberg [Eli97, Lemma 3.6 b] we can replace the Weinstein function $w_{E''}$ and the Liouville form with another Weinstein function $w'_{E''}$ such that $w'_{E''}|E = w_{E''}|E$ and such that $w'_{E''}$ has no critical points outside $E$. This ensures that the completion $\tilde{E}$ is symplectomorphic to $\tilde{E''}$.

6. Appendix B: A minimum principle

We have a Lefschetz fibration $\pi : E \to \mathbb{C}$. Let $[1, \infty) \times M_\phi$ be its cylindrical end where $r_\phi$ parameterizes the interval. The 1-form $\theta_E$ is equal to $r_\phi d\theta + \alpha_\phi$ in this region where $\alpha_\phi$ is the contact form on $M_\phi$ and $\theta$ is the pullback of the angle coordinate on $S^1$ to $M_\phi$. Let $H_{s,t}$ be a family of Hamiltonians parameterized by $\mathbb{R} \times S^1$ where $H_{s,t} = \kappa r_S + g(r_F)$ where $\kappa$ is a constant
near the level set \( r_S = \Delta \) and \( r_F \) is the fiber cylindrical coordinate. Let \( J_t \) be an \( S^1 \) family of almost complex structures where \( \pi \) is \( (J_t,j) \) holomorphic near \( r_S = \Delta \). Here \( j \) is the standard complex structure on the cylinder \([1,\infty) \times S^1\).

**Lemma 6.1.** Any Floer trajectory which does not intersect a regular fiber \( \pi^{-1}(q) \) connecting orbits in the region \( r_S > \Delta \) must be contained in the region \( r_S > \Delta \).

**Proof.** of Lemma 6.1. The proof of this is very similar to the proof of [AS07, Lemma 7.2]. A Floer trajectory connecting orbits is a cylinder:

\[
u : \mathbb{R} \times S^1 \to \mathbb{R}, \ u_s + J_t u_t = J_t X_{H_{s,t}}.
\]

After perturbing \( \Delta \) slightly we have that \( u \) is transverse to \( \{r = \Delta\} \) so \( u^{-1}(\{r_S \leq \Delta\}) \) is a compact codimension 0 submanifold \( \mathcal{S} \) of the cylinder \( \mathbb{R} \times S^1 \). There is a 1 form \( \beta \) on \( E \setminus \pi^{-1}(q) \) such that \( \beta|_{M_{\phi}} = d\vartheta \) where \( \vartheta \) is the angle coordinate for \( S^1 \). The Hamiltonian vector field associated to \( H_{s,t} \) near \( r = \Delta \) is equal to \(-\kappa \tilde{\partial} + X\) where \( \tilde{\partial} \) is the horizontal lift of \( \frac{\partial}{\partial \vartheta} \) and \( X \) is the Hamiltonian flow of \( g(r_F) \). Here \( X \) is tangent to the fibers of \( \pi \). We have:

\[
\int_{\partial \mathcal{S}} u^* \beta = \int_{\partial \mathcal{S}} u^* d\beta = 0.
\]

Hence

\[
0 = \int_{\partial \mathcal{S}} \kappa dt = \int_{\partial \mathcal{S}} u^* d\vartheta - \int_{\partial \mathcal{S}} \left(-\kappa \frac{\partial}{\partial \vartheta} + X\right) dt = \int_{\partial \mathcal{S}} u^* d\vartheta - \int_{\partial \mathcal{S}} d\vartheta X_{H_{s,t}} dt
\]

\[
= \int_{\partial \mathcal{S}} d\vartheta (u_s) ds + \int_{\partial \mathcal{S}} d\vartheta (u_t - X_{H_{s,t}}) dt = \int_{\partial \mathcal{S}} d\vartheta (J_t(X_{H_{s,t}}) - J_t u_t) ds + \int_{\partial \mathcal{S}} d\vartheta (J_t u_t) dt
\]

Now \( d\vartheta \circ J_t = dr_S \) and \( dr_S(X_{H_{s,t}}) = 0 \) along \( \{r_S = \Delta\} \) hence our integral becomes:

\[
\int_{\partial \mathcal{S}} dr_S(u_s) dt + dr_S(X_{H_{s,t}} - u_t) ds = \int_{\partial \mathcal{S}} dr_S(u_s) dt - dr_S(u_t) ds
\]

\[
= \int_{\partial \mathcal{S}} dr_S \circ du \circ j.
\]

Let \( \xi \) be a vector on \( \partial \mathcal{S} \) which is positively oriented then \( j(\xi) \) points inwards along \( \partial \mathcal{S} \) and because \( u \) is transverse to \( \{r = \Delta\} \) we get that \( dr_S \circ du \circ j(\xi) < 0 \) which implies that our integral satisfies:

\[
\int_{\partial \mathcal{S}} dr_S \circ du \circ j < 0
\]

which is a contradiction. \( \square \)
References

[Akb10] Akbulut, S. Lefschetz Fibrations on Compact Stein Manifolds. pages 1–19, 2010, arXiv:1003.2200.

[AM09] P. Albers and M. McLean. Non-displaceable contact embeddings and infinitely many leaf-wise intersections. pages 1–11, 2009, arXiv:0904.3564.

[AS07] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. pages 1–74, 2007, arXiv:0712.3177.

[BEE] F. Bourgeois, T. Ekholm, and Y. Eliashberg. Symplectic homology as Hochschild homology. arXiv:SG/0609037.

[BEH+03] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in symplectic field theory. Geom. Topol., 7:799–888, 2003, arXiv:SG/0308183.

[BO09] F. Bourgeois and A. Oancea. An exact sequence for contact- and symplectic homology. Invent. Math., 175(3):611–680, 2009.

[CC09] Andrew Cotton-Clay. Symplectic Floer homology of area-preserving surface diffeomorphisms. Geom. Topol., 13(5):2619–2674, 2009.

[CFHW96] K. Cieliebak, A. Floer, H. Hofer, and K. Wysocki. Applications of symplectic homology I: Stability of the action spectrum. Math. Z, 223:27–45, 1996.

[Cie02] K. Cieliebak. Handle attaching in symplectic homology and the chord conjecture. J. Eur. Math. Soc. (JEMS), 4:115–142, 2002.

[DS94] S. Dostoglou and D. Salamon. Self-dual instantons and holomorphic curves. Ann. of Math. (2), 139:581–640, 1994.

[Eli97] Y. Eliashberg. Symplectic geometry of plurisubharmonic functions, notes by M. Abreu, in: Gauge theory and symplectic geometry (Montreal 1995). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 488:49–67, 1997.

[FH94] A. Floer and H. Hofer. Symplectic homology I. Math. Z, 215:37–88, 1994.

[Gir02] Emmanuel Giroux. Géométrie de contact: de la dimension trois vers les dimensions supérieures. In Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), pages 405–414, Beijing, 2002. Higher Ed. Press.

[Hat02] A. Hatcher. Algebraic topology. Cambridge University Press, Cambridge, 2002.

[HS95] H. Hofer and D. A. Salamon. Floer homology and Novikov rings. In The Floer memorial volume, volume 133 of Progr. Math., pages 483–524. Birkhäuser, Basel, 1995.

[McL] M. McLean. Computability and the growth rate of symplectic homology. In preparation.

[McL09] M. McLean. Lefschetz fibrations and symplectic homology. Geom. Topol., 13(4):1877–1944, 2009.

[Oan06] A. Oancea. The Kühneth formula in Floer homology for manifolds with restricted contact type boundary. Math. Ann., 334:65–89, 2006, arXiv:SG/0403376.

[RS93] J. Robbin and D. Salamon. The Maslov index for paths. Topology, 32:827–844, 1993.

[Sei03] P. Seidel. A long exact sequence for symplectic Floer cohomology. Topology, 42:1003–1063, 2003, arXiv:SG/0105186.

[Sei08] P. Seidel. A biased view of symplectic cohomology. Current Developments in Mathematics, 2006:211–253, 2008.

[Vit99] C. Viterbo. Functors and computations in Floer homology with applications, part I. Geom. Funct. Anal., 9:985–1033, 1999.

[Wei91] A. Weinstein. Contact surgery and symplectic handlebodies. Hokkaido Math. J., 20(2):241–251, 1991.