Hub Location with Protection under Link Failures

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Abstract. This paper introduces the Hub Location Problem under Link Failures, a hub location problem in which activated inter-hub links may fail with a given probability. Two different optimization models are studied, which construct hub backbone networks protected under hub links disruptions by imposing that for each commodity an additional routing path exists besides its original routing path. Both models consider the minimization of the setup costs of the activated hubs and inter-hub links plus the expected value of the routing costs of the original and alternative paths. The first model builds explicitly the alternative routing paths, while the second model guarantees that for each commodity at least one alternative path exists using a large set of connectivity constraints, although the alternative paths are not built explicitly. The results of extensive computational testing allow to analyze the performance of the two proposed models and to evaluate the extra cost required to design a robust backbone network under hub links failures. The obtained results support the validity of the proposal.

1. Introduction

Hub location lies in the intersection of Location Analysis and Network Design, and produces challenging optimization problems with multiple applications, mostly in the fields of distribution/logistics (parcel delivery, air transportation, etc.) and telecommunications (Farahani et al. 2013). The increasing attention they have received in the last decades is thus not surprising (see e.g. Campbell and O’Kelly 2012, Contreras and O’Kelly 2019). One of the current trends in hub location is the search of models suitable for dealing with different sources of uncertainty (Alumur et al. 2012). While some models in the literature consider uncertainty in demand (Contreras et al. 2011, Zetina et al. 2017), other models are concerned with the robustness of solution hub networks, by associating uncertainty with the possibility (probability) of disruption of the involved elements of the solution networks and looking for solutions that are robust under disruptions.

Some works have studied models in which it is assumed that activated hub nodes may (totally or partially) fail with a certain probability. An et al. (2015) propose a model in which two backup hub nodes are determined for each commodity. Rostami et al. (2019) assume that a finite set of hub breakdown scenarios is known and provide a two-stage formulation for the single allocation hub location problem with possible hub breakdown. Cui et al. (2010) provide a stochastic model to determine a subset of the activated hub nodes of given size through which each commodity can be routed, in such a way that if the cheapest route
fails, the commodity can be routed through the second cheapest, and so on, or through an emergency facility. The authors provide a Mixed Integer Linear Programming (MILP) formulation for the problem as well as an approximate Lagrangean relaxation scheme for its resolution based on the ideas in (Snyder and Daskin 2005) for the p-median problem. The planar version of this model is also analyzed there.

Kim and O’Kelly (2009) propose the reliable p-hub location problem (PHMR) and the p-hub mandatory dispersion (PHMD). In the PHMR the goal is to determine the location of p nodes based on the level of reliability to maximize the completed flows among the set of nodes. The PHMD imposes a certain minimum separation between the p selected hub nodes, and maximizes the reliability of the network. MILP formulations and heuristic approaches are provided for both problems both in the single and the multiple-allocation framework. Reliability of hub backbone networks has been also studied in (Zeng et al. 2010, Korani and Eydi 2021, Li et al. 2022).

Parvaresh et al. (2013) consider the multiple allocation p-hub median problem under intentional disruptions in which the goal is to identify the optimal strategy for the location of p hub nodes by minimizing the expected transportation cost in case the worst-case disruptions is minimized. A bilevel mixed integer formulation is provided as well as a simulated annealing heuristic for its resolution.

The problem of designing robust networks under edges/arcs failures has also been studied in the literature under different settings. Aneja et al. (2001) study the single-commodity maximum flow problem for the case when edge failures may occur by means of the maximal residual flow problem, whose goal is to determine the maximal flow in which the largest arc flow is as small as possible. This problem is closely related to the network interdiction problem that consists of determining a certain number of arcs whose removal from the network minimizes the maximum amount of flow that one can send through the network (see e.g. Altner et al. 2010, Cormican et al. 1998, Royset and Wood 2007, Wood 1993).

Ma et al. (2016) propose the Conditional Value-at-Risk Constrained Minimum Spanning k-Core Problem where the possibility of edge disruptions is prevented when trying to construct minimum cost subgraphs of a network with a minimum number of incident edges at each node. Andreas and Smith (2008) study shortest path problems under link failures by imposing that the probability that all arcs that successfully operate in at least one path greater than certain threshold value.

However, existing works dealing with potential failure of inter-hub links is very scarce (Mohammadi et al. 2019). This is precisely the focus of this work, where we introduce the Hub Location Problem under Link Failures (HLPLF), a hub location problem in which activated inter-hub links may fail with a given probability. This can be very useful in typical hub location applications in which the total failure of a hub is highly unlikely, whereas partial failures occur only affecting some of the links incident with the hubs (certain air connections, train lines, etc.) We point out that by protecting inter-hub links under failure we also partially protect hub nodes under failures.

For dealing with the HLPLF we propose two alternative models, which guarantee that solution networks are protected under disruption of inter-hub links, in the sense that for each commodity at least one alternative (backup) routing
path exists. The main difference between the two models is how backup paths are enforced.

In both cases we consider set-up costs for both activated hubs and activated inter-hub edges. Thus, we do not fix the number of hubs to activate, allowing for incomplete backbone networks. As usual, the routing costs apply a discount factor $\alpha$ to inter-hub arcs. We assume multiple allocation of nodes to activated hubs, although the allocation may be different in the original and backup paths. We further impose that the original routing path of each commodity $r = (o_r, d_r)$ contains exactly one inter-hub arc, which can be a loop. That is, the original routing path is of the form $(o_r, k, m, d_r)$, where $k$ and $m$ are activated hubs, and $(k, m)$ is an inter-hub arc, which reduces to a loop when $k = m$. We are also given the failure probabilities for each potential inter-hub edge. Then, the objective is to minimize the sum of the set-up costs of the activated hubs and inter-hub edges, plus the expected routing costs.

Our models can be seen as two-stage stochastic programming models in which the a priori solution is determined by the strategic decisions associated with the selection of activated hub nodes and inter-hub edges together with an original plan given by a set of feasible routing paths, one for each commodity, whereas the recourse action determines a backup plan, given by a set of alternative routing paths for the commodities, that can be used in case the inter-hub edge of the original plan fails. As already mentioned, the models that we propose differ on the way backup paths are constructed. The first model imposes that the alternative routing path of each commodity contains exactly one inter-hub arc (as in the original plan), which can be a loop, and builds it explicitly. The second model is more flexible, in the sense that it allows for arbitrarily large sequences of inter-hub arcs to be used in the alternative routing paths, although such paths are not built explicitly. This is achieved with a set of exponentially many (on the number of nodes of the network) constraints, by imposing that the cut-set of the backbone network contains at least $\lambda$ edges, for a given integer value of $\lambda \geq 2$. We study some properties of both models and propose a MILP in each case. For the second model, since it has exponentially many constraints, we also propose a branch-and-cut solution algorithm.

Extensive computational experiments have been carried out on a large set of benchmark instances based on the well-known CAB (O’Kelly 1987), AP (Ernst and Krishnamoorthy 1996), and TR (Tan and Kara 2007) datasets, for varying settings of the failure probabilities and other cost parameters. The obtained results are summarized and analyzed, comparing the computational performance of each of the models and the effect of the different parameters. Managerial insights are derived from the analysis of the characteristics of the solutions produced by each of the models and their comparison. In particular, we analyze the distribution of the costs among the different elements considered (hubs and links set-up costs and routing costs), the number of activated hub and links, and the density of the obtained backbone networks. Finally, an empirical analysis of the two proposed models has been carried out. We compare solutions obtained with the different models in terms of efficiency and robustness. For this analysis, multiple failure scenarios have been generated from optimal solutions to the underlying deterministic hub location model and their a posteriori capability.
to re-route the commodities, tested against that of the proposed models. The obtained results assess the validity of the proposal.

The remainder of this paper is structured as follows. In Section 2, we introduce the notation that we will use and formally define the HLPLF from a general perspective. Section 3 is devoted to the first HLPLF model that we study in which it is assumed that the alternative paths for the commodities contain exactly one inter-hub arc. We study some of its properties and propose a MILP formulation for it. The model in which we impose that the backbone network is $\lambda$-connected is studied in Section 4 where we also present a MILP formulation for it. Section 5 describes the computational experiments we have carried out and summarizes the obtained results. Some managerial insights from the analysis of the structure of the solution networks produced by each of the models are also derived in this section. Finally, Section 6 describes the empirical analysis that has been carried out in which multiple failure scenarios have been generated from optimal solutions to the underlying deterministic hub location model and their a posteriori capability of re-routing the commodities tested against that obtained with the proposed models. The paper closes in Section 7 with some conclusions.

2. Notation and definition of the problem

Consider a graph $N = (V, E)$, where the node set $V = \{1, 2, \ldots, n\}$ represents a given set of users and the edge set $E$ the existing connections between pairs of users. We assume that $N$ is a complete graph and that $E$ contains loops, i.e. for all $i \in V$, edge $\{i, i\} \in E$. We further assume that potential locations for hubs are placed at nodes of the graph and the set of potential locations coincides with $V$. For each potential location $k \in V$, we denote by $f_k$ the set-up cost for activating a hub at node $k$. Any pair of hub nodes can be connected by means of an inter-hub edge, provided that both endnodes $k$ and $l$ are activated as hub nodes as well.

The set $E$ will be referred to as the set of potential inter-hub edges or just as set of potential hub edges. Activated hub edges incur set-up costs as well; let $h_{kl} \geq 0$ be the set-up cost for activating hub edge $\{k, l\} \in E$. A set of activated hubs will be denoted by $H \subseteq V$, and a set of activated hub edges for $H$ by $E_H \subseteq E[H]$, where $E[H]$ is the set of edges with both endnodes in $H$. Note that the assumption that $N$ is a complete graph implies no loss of generality, since (i) arbitrarily large set-up costs can be associated with nodes that are not potential hubs; and, (ii) arbitrarily large activation costs can be associated with non-existing hub edges. Activating a hub edge allows to send flows through it in either direction. Let $A = \{(i, j) \cup (j, i) : \{i, j\} \in E, i, j \in V\}$ be the arc set. Arcs in the form $(i, i)$ will also be called loops and we will use $A_H \subseteq A$ to denote the set of inter-hub arcs induced by $E_H$.

Service demand is given by a set of commodities defined over pairs of users, indexed in a set $R$. Let $D = \{(o_r, d_r, w_r) : r \in R\}$ denote the set of commodities, where the triplet $(o_r, d_r, w_r)$ indicates that an amount of flow $w_r \geq 0$ must be routed from origin $o_r \in V$ to destination $d_r \in V$. The origin/destination pair associated with a given commodity will also be referred to as its OD pair. Commodities must be routed via paths of the form $\pi = (o_r, k_1, \ldots, k_s, d_r)$ with $k_i \in H$, $1 \leq i \leq s$. Similarly to most Hub Location Problems (HLPs), a routing path $\pi = (o_r, k_1, \ldots, k_s, d_r)$ is feasible if (1) it includes at least one hub node,
i.e. $s \geq 1$, and (2) the underlying edges of all traversed arcs other than the access and delivery arcs, $(o_r, k_1)$ and $(k_s, d_r)$, respectively, are activated inter-hub edges, i.e., $\{k_i, k_{i+1}\} \in E_H$, $1 \leq i \leq s - 1$. Note that this implies that in any feasible path all intermediate nodes are activated as hubs as well, i.e. $k_i \in H$, $1 \leq i \leq s$. In the following the set of feasible paths for a given commodity $r \in R$ will be denoted by $\Pi_{(H,E_H)}(r)$.

Most HLPs studied in the literature do not consider loops explicitly. Then, only proper arcs $(k,l)$ with $k \neq l$ can be considered as hub arcs. In this work we follow a slightly more general setting in which loops of the form $(k,k)$ can also be considered as hub arcs. Then, if a loop $(k,k)$ is used in a routing path, it is required not only that $k$ is activated as a hub node, but also that the loop $\{k,k\}$ is activated as a hub edge as well.

Throughout we assume multiple allocation of commodities to open hubs. That is, it is possible that two commodities with the same origin are routed using a different access hub. Routing flows through the arcs of a hub-and-spoke network incurs different types of costs. These costs, which may depend on the type of arc, account for transportation costs as well as for some additional collection/handling/distribution costs at the endnodes of the arcs. As usual in the literature, we assume that transportation costs of flows routed through inter-hub arcs are subjected to a discount factor $0 \leq \alpha \leq 1$. In this work we will denote by $c_{ij} \geq 0$ the unit routing cost through inter-hub arc $(i,j) \in A_H$, which includes discounted transportation costs and handling costs and we will denote by $\bar{c}_{ij} \geq 0$ the unit routing cost for an access or a delivery arc $(i,j) \in A \setminus A_H$, which could also incorporate different discounted access or delivery costs.

Thus, with the above notation, the routing cost of commodity $r \in R$ through a feasible path $\pi = (o_r, k_1, \ldots, k_s, d_r) \in \Pi_{(H,E_H)}(r)$ is:

$$C^r_\pi = w_r \left( c_{o_r,k_1} + \sum_{i=1}^{s-1} c_{k_i,k_{i+1}} + c_{k_s,d_r} \right),$$

where the first and last addends correspond to the access and delivery arcs, respectively, and the intermediate ones are the service costs through the backbone network $(H, E_H)$.

Broadly speaking, under the above assumptions, the goal of a HLP is to decide the location of the hub nodes $H$ and to select a suitable subset of hub edges $E_H$, to optimally route the commodities through the backbone network $(H, E_H)$ induced by the activated hub nodes and hub edges, so as to minimize the sum of the overall set-up costs for activating hub nodes and hub links, plus the commodities routing costs. With the above notation, this problem can be stated as:

$$\min_{H \subseteq V, E_H \subseteq E[H]} \sum_{k \in H} f_k + \sum_{e \in E_H} h_e + \sum_{r \in R} \min_{\pi \in \Pi_{(H,E_H)}(r)} C^r_\pi. \quad \text{(HLP)}$$

Most HLPs studied in the literature (that do not consider loops explicitly) restrict the set of potential paths for routing the commodities to those using at most one hub arc. When loops are also considered as potential hub arcs as we do, the analogous set of potential paths for routing the commodities is restricted to those containing exactly one hub arc. Thus, for $H$ and $E_H$ given, the set of potential paths for routing commodity $r \in R$ is given by
\( \Pi (H, E_H) (r) = \{ \pi = (o_r, k, l, d_r) : k, l \in H, \{ k, l \} \in E_H \} \). In such a case the routing cost of commodity \( r \) through path \( \pi = (o_r, k, l, d_r) \) reduces to

\[
C^r_{\pi} := C^r = w_r(c_{o_r k} + c_{kl} + c_{ld_r}),
\]

and the HLP simplifies to:

\[
\min_{H \subseteq V, E_H \subseteq E[H]} \sum_{k \in H} f_k + \sum_{e \in E_H} h_e + \sum_{r \in R, \{k, l\} \in A_H} \min_{\pi \in R} C^r_{\pi, k, l, d}. \tag{HLP^1}
\]

Most hub networks are sensitive to failures in their links, being the impact in some of them particularly harmful for the users. Examples of potential applications of the models that we study include the management of airlines and airport industries (Campbell et al. 2005), in which breakdowns in certain flight connections may occur, and passengers are directly affected. Also, in rapid delivery packing systems (Cetiner et al. 2010), where the users pay for fast services and failures in the hub network cause large delays.

In the remainder of this work we consider HLPs in which the (original) routing paths consist of exactly one hub arc (possibly a loop), and we assume that activated hub edges in \( E_H \) may fail. In case hub edge \( \{ k, l \} \in E_H \) fails, then the inter-hub arcs \( \{ k, l \}, (l, k) \in A_H \) can no longer be used for routing the commodities. In order to protect solution networks from failure we follow a policy that does not alter the strategic decisions on the activated hubs and inter-hub edges and focuses solely on the operational decisions concerning the re-routing of affected commodities. Accordingly, we impose that, for each commodity, the backbone network \( (H, E_H) \) contains, in addition to the original routing path, some substitute path connecting \( o_r \) and \( d_r \). Such a substitute path will be referred to as backup or alternative path.

For each edge \( e = \{ k, l \} \in E \), let \( X_{kl} \) denote the random variable modeling whether \( e \) fails. We assume that the random variable \( X_{kl} \) follows a Bernoulli distribution with probability \( p_{kl} \), for each \( \{ k, l \} \in E \). When \( k \neq l \) failure of edge \( e \) arises not only when the link \( \{ k, l \} \) can no longer be used, but also when, for any reason, the collection and redistribution services at any endnode of edge \( \{ k, l \} \) cannot be carried out. Thus \( p_{kl} \) represents the probability that any of these events happen. In case edge \( e \) is a loop, i.e., \( e = \{ k, k \} \), with \( k \in H \), then \( p_{kk} \) represents the probability that the handling process carried out when \( k \) is used as the unique intermediate hub fails.

Observe that when edges may fail with a given probability, the costs of feasible routing paths are also random variables, which will be denoted by \( C_r, r \in R \). Furthermore, the probability distribution of \( C_r, r \in R \), is dictated by the failure probability distribution of the involved inter-hub edges. In particular, when \( \pi_0 = (o_r, k, l, d_r) \) is the original routing path of a given commodity \( r \in R \), the expected routing cost of commodity \( r \) can be calculated as:

\[
E[C_r | \pi_0] = (1 - p_{kl}) C^r_{\pi} + p_{kl} C^r_{\pi_0},
\]

where \( \pi_0 \in \Pi ((H, E_H) \backslash \{ \pi_0 \}) \) is the backup path in case \( \{ k, l \} \) fails.

In the following we deal with the problem of finding hub networks protected against inter-hub edge failures under the above assumptions. For this, in the objective function, instead of considering the costs of the routing paths, we will
Hub Location with Protection under Link Failures

consider their expected routing cost. That is, the HLPLF can be stated as:

\[
\min_{H \subseteq V, E_H \subseteq E[H]} \sum_{k \in H} f_k + \sum_{e \in E_H} h_e + \min_{r \in R} E[C_r | \pi_0]. \quad \text{(HLPLF)}
\]

Indeed, multiple alternatives fall within the above generic framework which differ from each other in how the alternative routing paths are obtained. In the following sections we propose two alternative models for determining such backup paths, based on different assumptions, and provide mathematical programming formulations for each of them.

3. HLPLF with single inter-hub arc backup paths

The [HLPLF] that we address in this section enforces that the alternative paths for routing the commodities have the same structure as the original ones. That is, we assume that backup paths contain at least one hub and have exactly one inter-hub arc (possibly a loop). This avoids having to use many transshipment points in case of failure. This model will be referred to as (HLPLF-1BP).

Given a commodity \( r \in R \), the backbone network \((H, E_H)\) and the original routing path \( \pi_0 = (o_r, k, l, d_r) \), we assume that the backup path is in the form \( \pi_0 = (o_r, \bar{k}, \bar{l}, d_r) \), with \( \{k, l\} \neq \{\bar{k}, \bar{l}\} \).

Thus, the expected routing cost of commodity \( r \) is:

\[
E[C_r | \pi_0] = (1 - p_{kl}) C_{kl}^r + p_{kl} C_{\bar{k}\bar{l}}^r.
\]

Figure 1 illustrates the different situations that may arise in case a hub link fails. Figure 1a shows a backbone network with four hub nodes \((k, l, m, q)\) and five hub links, two of them corresponding to loops, \( \{m, m\} \) and \( \{l, l\} \), and rest ones corresponding to edges \( \{k, q\} \), \( \{k, l\} \) and \( \{q, m\} \). The figure also depicts the origin \((o_r)\) and destination \((d_r)\) of a given commodity \( r \in R \) and a possible path for this commodity through hub arc \((k, l)\). We assume that it is the original path for the commodity \( r \in R \). Access/distribution arcs are depicted as dashed lines and hub edges as solid lines.

Figures 1b and 1c show different single inter-hub arc backup paths for commodity \( r \in R \) in case the original one fails. In Figure 1b the backup path uses hub-arc \((q, m)\) to re-route the commodity, while in Figure 1c the backup path uses loop arc \((l, l)\).

Figure 1. A network with four nodes and one commodity \( r = (o_r, d_r, w_r)\): Original path, \( \pi_0 = (o_r, k, l, d_r) \), and different backup paths in case the original hub arc \((k, l)\) fails.

Next we develop a mathematical programming formulation for the above problem, first introducing the decision variables.
We use the following variables associated with the design decisions on the elements of the network that are activated, hubs and edges:

\[ z_k = \begin{cases} 1 & \text{if a hub is opened at the potential hub node } k, \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k \in V. \]

\[ y_{kl} = \begin{cases} 1 & \text{if hub edge } \{k, l\} \text{ is activated,} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \{k, l\} \in E. \]

The formulation uses two additional sets of variables, which respectively represent the original and alternative routing path for each commodity. In particular, for \( r \in R \) and \( (k, l) \in A \):

\[ x_{kl}^r = \begin{cases} 1 & \text{if the original routing path for commodity } r \text{ is } (o_r, k, l, d_r) \\ 0 & \text{otherwise,} \end{cases} \]

\[ \bar{x}_{kl}^r = \begin{cases} 1 & \text{if the alternative path for commodity } r \in R \text{ is } (o_r, k, l, d_r), \\ 0 & \text{otherwise.} \end{cases} \]

With these sets of decision variables the expected routing cost of commodity \( r \in R \) can be expressed as:

\[
\sum_{(k, l) \in A} x_{kl}^r \left[ C_{kl}^r (1 - p_{kl}) + p_{kl} \sum_{(k', l') \in A \setminus \{(k, l) \cup (l, k)\}} C_{k'l'}^r \bar{x}_{k'l'}^r \right]
\]

where the two addends in each term of the above expression correspond to the expected routing cost of the original and backup plan of commodity \( r \), respectively, both of which only apply if, in the original plan, the commodity is routed through the inter-hub edge corresponding to the term. In particular, the first addend gives the overall routing cost for commodity \( r \) in case the arc of the backbone network used for routing \( r \) in the original plan does not fail (multiplied by the probability of not failing). The second term computes the cost of the alternative routing path, multiplied by the probability of failure of the inter-hub edge of the original plan. Observe that in case \((k, l)\) is the arc used initially by commodity \( r \in R \) and \((k', l')\) is the backup arc for \( r \), one obtains the cost \((1 - p_{kl})C_{kl}^r + p_{kl}C_{k'l'}^r\).

Rearranging terms in the above expression one can rewrite the overall routing cost function for commodity \( r \) as:

\[
\sum_{(k, l) \in A} C_{kl}^r \left[ (1 - p_{kl})x_{kl}^r + p_{kl} \sum_{(k', l') \in A \setminus \{(k, l) \cup (l, k)\}} p_{k'l'} \bar{x}_{k'l'}^r \right],
\]

where it can be observed that the impact of a given arc \((k, l)\) in \( A \) in the routing cost of commodity \( r \) is either 0 (if it is not used neither in the original nor the alternative path); \((1 - p_{kl})C_{kl}^r\) if it is used in the original path; or \(C_{kl}^r p_{k'l'}\) in case arc \((k, l)\) is used in the alternative path and arc \((k', l')\) in the original one.

The above decision variables together with this routing cost function lead to the following Integer Nonlinear Programming formulation for (HLPLF-IBP):

\[
\min \sum_{k \in V} f_k z_k + \sum_{(k, l) \in E} h_{kl} y_{kl} + \sum_{(k, l) \in A} C_{kl}^r \left[ (1 - p_{kl})x_{kl}^r + p_{kl} \sum_{(k', l') \in A \setminus \{(k, l) \cup (l, k)\}} p_{k'l'} \bar{x}_{k'l'}^r \right]
\]
Hub Location with Protection under Link Failures

s.t. \[ \sum_{(k,l) \in A} x_{kl}^r = 1 \quad \forall r \in R \] \hfill (1.1)
\[ \sum_{(k,l) \in A} \bar{x}_{kl}^r = 1 \quad \forall r \in R \] \hfill (1.2)
\[ x_{ik}^r + x_{ik}^r + \bar{x}_{kl}^r + \bar{x}_{lk}^r \leq y_{kl} \quad \forall r \in R, \{k,l\} \in E, k \neq l \] \hfill (1.3)
\[ x_{kk}^r + \bar{x}_{kk}^r \leq y_{kk} \quad \forall r \in R, \{k,k\} \in E \] \hfill (1.4)
\[ y_{kl} \leq z_k \quad \forall \{k,l\} \in E \] \hfill (1.5)
\[ y_{kl} \leq z_l \quad \forall \{k,l\} \in E, k \neq l \] \hfill (1.6)
\[ x_{kk}^r + \sum_{l \in V \setminus k \neq k} (x_{kl}^r + \bar{x}_{lk}^r) \leq z_k \quad \forall r, \forall k \in V \] \hfill (1.7)
\[ \bar{x}_{kk}^r + \sum_{l \in V \setminus k \neq k} (\bar{x}_{kl}^r + \bar{x}_{lk}^r) \leq z_k \quad \forall r, \forall k \in V \] \hfill (1.8)
\[ x_{kl}^r, \bar{x}_{kl}^r \in \{0,1\} \quad \forall r \in R, (k,l) \in A \] \hfill (1.9)
\[ z_k \in \{0,1\} \quad \forall k \in V \] \hfill (1.10)
\[ y_{kl} \in \{0,1\} \quad \forall \{k,l\} \in E. \] \hfill (1.11)

where constraints (1.1) and (1.2) enforce that each commodity uses exactly one inter-hub arc both in the original and the backup path. Constraints (1.3) and (1.4) impose that the original and the backup path do not coincide. These constraints also guarantee that any used inter-hub edge is activated. Constraints (1.5) and (1.6) ensure that any endnode of an activated inter-hub edge must be activated as a hub. Constraints (1.7) and (1.8) are valid inequalities, already proposed in Marín et al. (2006), which reinforce the relationship between the routing variables and the hub activation variables. Finally, (1.9)-(1.11) are the domains of the decision variables.

3.1. Linearization of the objective function. The reader may have observed the non-linearity of the objective function term corresponding to the expected routing cost. As we explain below this term can be suitably linearized by introducing a new auxiliary variable \( P_{kl}^r \in \mathbb{R}_+ \), associated with each commodity \( r \in R \) and each arc \((k,l) \in A\).

Let \( P_{kl}^r = \bar{x}_{kl}^r \sum_{(k',l') \in A \setminus (k,l) \cup (l,k)} p_{kl} x_{k'\ell'}^r \) denote the probability of using inter-hub arc \((k,l)\) in the alternative path of commodity \( r \). Observe that because of the minimization criterion, the nonnegativity of the routing costs, and constraints (1.1)-(1.4), the value of \( P_{kl}^r \) can be determined by the following set of constraints:

\[ P_{kl}^r \geq \sum_{(k',\ell') \in A} p_{k'\ell'} x_{k'\ell'}^r + (\bar{x}_{kl}^r - 1), \quad \forall r \in R, \forall (k,l) \in A, \] \hfill (1.12)
\[ P_{kl}^r \geq 0, \quad \forall r \in R, \forall (k,l) \in A, \] \hfill (1.13)

and the objective function can be rewritten as:

\[ \sum_{k \in V} f_k z_k + \sum_{(k,l) \in E} h_{kl} y_{kl} + \sum_{r \in R} \sum_{(k,l) \in A} C_{kl}^r (1 - p_{kl}) \bar{x}_{kl}^r + P_{kl}^r. \]
We can also incorporate the following valid inequalities to reinforce our formulation:

\[ \sum_{(k,l) \in A} P_{kl}^r \leq \max_{(k,l) \in E} p_{kl}, \quad \forall r \in R. \]  

(1.14)

Therefore, we have the following MILP formulation for the problem:

\[
\begin{align*}
\min & \sum_{k \in V} f_k z_k + \sum_{(k,l) \in E} h_{kl} y_{kl} + \sum_{r \in R} \sum_{(k,l) \in A} C_{kl}^r \left( (1 - p_{kl}) x_{kl}^r + P_{kl}^r \right) \\
\text{s.t.} & \quad (1.1) - (1.14). 
\end{align*}
\]

(1.15)

Below we state some simple optimality conditions that can be used to reduce the set of decision variables.

**Proposition 3.1.** There is an optimal solution to (HLPLF-1BP) such that

\[ x_{lk}^r = \bar{x}_{lk}^r = P_{lk}^r = 0 \]

for all \( r \in R, \ (l,k) \in A \), with \( C_{kl}^r \leq C_{lk}^r \).

**Proof.** The proof is straightforward. Indeed, the value of any solution with \( x_{lk}^r = 1 \) where \( C_{kl}^r < C_{lk}^r \) will improve by changing the direction in which edge \( \{k,l\} \) is traversed, i.e., by doing \( x_{kl}^r = 0, \ x_{lk}^r = 1 \). When \( C_{kl}^r = C_{lk}^r \) the value of the new solution will not change.

The same argument can be applied for setting \( \bar{x}_{lk}^r = 0 \) and thus, \( P_{lk}^r = 0 \). \( \square \)

Note that the above result allows one to reduce by one half the number of decision variables.

In practice, it is likely that there are few possible values for failure probabilities, and the edges of the network are clustered in groups such that, within each group, all edges have the same failure probability. We next analyze such situation.

**Remark 3.1** (Clustered sets of edges). Let us assume that the edges in \( E \) are clustered in \( K \) groups \( E_1, \ldots, E_K \) such that all edges in \( E_s \) have the same failure probability \( \rho_s \in [0,1] \), for \( s = 1, \ldots, K \). Then, in the term of the objective function of (HLPLF-1BP) corresponding to the expected cost of the commodities, variables \( P_{kl}^r \) can be substituted by a new set of variables as follows. For \( r \in R, \ (k,l) \in A_s \), being \( A_s \) the arc set induced by \( E_s, \ s = 1, \ldots, K \), let \( \xi_{kl}^r \) be a binary variable that takes value one if and only if the original route of commodity \( r \) uses some hub arc in the \( s \)-th cluster (with failure probability \( \rho_s \)) and in the backup route it uses hub arc \( (k,l) \).

Then, the expected routing cost of commodity \( r \in R \) can be rewritten as:

\[ \sum_{s=1}^K \left[ (1 - \rho_s) \sum_{(k,l) \in A_s} C_{kl}^r x_{kl}^r + \rho_s \sum_{(k,l) \in A} C_{kl}^r \xi_{kl}^r \right]. \]

Using similar arguments as for the linearization of variables \( P_{kl}^r \), the values of the \( \xi_{kl}^r \) variables can be determined by the following sets of constraints:

\[ \xi_{kl}^r \geq \sum_{(k',l') \in A_s} x_{k'l'}^r + (\bar{x}_{kl}^r - 1) \quad \forall r \in R, (k,l) \in A_s, s = 1, \ldots, K \]

(2)

\[ \sum_{s=1}^K \xi_{kl}^r = \bar{x}_{kl}^r \quad \forall r \in R, (k,l) \in A \]

(3)
Hub Location with Protection under Link Failures

\[ \xi_{kl} \geq 0 \quad \forall r \in R, (k, l) \in A, s = 1, \ldots, K. \] (4)

The particular case of one single cluster \((K = 1)\) where all edges have the same failure probability, i.e., \(p_{kl} = \rho\) for all \(\{k, l\} \in E\), allows to further simplify the above formulation. Now the index \(s\) can be dropped from variables \(\xi\) and Constraints (2) - (4) are no longer needed, as Constraints (3) reduce to \(\xi_{kl} = \bar{x}_{kl}\), \((k, l) \in A\). Then, the expected routing cost of commodity \(r \in R\) simplifies to:

\[ \sum_{(k,l) \in A} C_r^{(1-\rho)x_{kl}^r + \rho\bar{x}_{kl}^r}. \]

4. HLPLF with \(\lambda\)-connected backbone networks

In this section we introduce a different model for the HLPLF, that will be referred to as \(\lambda\)-connected HLPLF (HLPLF-\(\lambda\)). Again we make the assumption that the original routing paths contain at least one hub node and exactly one inter-hub arc, although we follow a different modeling approach as for how to protect the backbone network \((H, E_H)\) against potential failures. On the one hand, we extend the set of alternative paths that can be used when hub edges in original paths fail, and allow for any arbitrarily long chain of arcs connecting the OD pair of each commodity, provided that all its intermediate arcs are activated inter-hub arcs. On the other hand, we no longer make explicit the alternative routing paths for the commodities. Instead, we impose that the backbone network is \(\lambda\)-connected, in the sense that it must contain at least \(\lambda\) routing paths connecting any pair of activated hubs \(k, l \in H\) with \(k \neq l\), where \(\lambda \geq 2\) is a given integer parameter. This implies that if some hub arc of the original path fails, then the backbone network contains at least \(\lambda - 1\) alternative paths connecting the activated hubs. Note that, this forces the backbone network to have at least \(\lambda\) activated hub nodes. The particular case of HLPLF-\(\lambda\) with \(\lambda = 2\), extends the HLPLF-1BP studied in the previous section, as it enforces at least one backup path in the backbone network in addition to the original one, which can be arbitrarily long.

We recall that for any non-empty subset of nodes \(S \subset V\), the cutset associated with \(S\) is precisely the set of edges connecting \(S\) and \(V \setminus S\) namely:

\[ \delta(S) = \{\{k, l\} \in E | k \in S, l \in V \setminus S\}. \]

Observe that the backbone network \((H, E_H)\) depicted in Figure 1a is \(2\)-connected since any cutset has at least two edges (possibly one of them being a loop).

Let us introduce the following additional notation. For a given indicator vector \(\bar{y} \in \{0, 1\}^{|E|}\):

\[ \bar{y}(\delta(S)) = \sum_{\{k,l\} \in \delta(S)} \bar{y}_{kl}. \]

That is \(\bar{y}(\delta(S))\) gives the number of edges in the cutset \(\delta(S)\), that are activated relative to vector \(\bar{y}\). When \(\bar{y} = y\), with \(y\) being the vector of hub edge decision variables as defined in HLPLF-1BP, then \(y(\delta(S))\) gives precisely the number of inter-hub edges in the cutset \(\delta(S)\).

Figure 2 shows different choices for backup paths in case the hub arc \((k, l)\), using in the original path for commodity \(r = (o_r, d_r, w_r)\), fails. Note that while the backup paths drawn in Figures 2a and 2c are also valid for HLPLF-1BP, the backup path shown in Figure 2b uses two inter-hub arcs, thus not being
valid for HLPLF-1BP. Similarly to HLPLF-1BP, loops are also counted for the \( \lambda \)-connectivity, as they can be used both in original and backup paths. Note also that in case the original path for commodity \( r = (o_r, d_r, w_r) \) uses the loop \((m, m)\), the backup paths in Figure 2 are also feasible.

![Figure 2](image-url)

**Figure 2.** Alternative possibilities for backup paths in the \( \lambda \)-connected model.

With the above notation, and taking into account that, by definition, \( \delta(S) \) contains no loop hub edges, the \( \lambda \)-connectivity of the backbone network can be stated by means of the following constraints, associated with each subset \( S \subset V \), and each pair of potential hubs \( k, l \in V \) with \( k \in S, l \notin S \):

\[
y(\delta(S)) + y_{kk} \geq \lambda(z_k + z_l - 1).
\]

The right hand side of the above constraint can take a strictly positive value only when \( k, l \) are activated hub nodes, that is, when the cutset \( \delta(S) \) is a cutset of the backbone network. In this case, the inequality imposes that \( \delta(S) \) contains at least \( \lambda - y_{kk} \) activated hub edges. As indicated, the loop \( \{k, k\} \) has also been taken into account as a potential hub edge since it can be used in routing paths. Hence, if the loop \( \{k, k\} \) is activated as hub edge, it should be discounted from the number of hub edges in \( \delta(S) \) that must be activated. Summarizing, the above constraint imposes that, if nodes \( k, l \), \( k \in S, l \notin S \), are activated hubs, then the number of hub edges in the cutset \( \delta(S) \) must be at least \( \lambda - 1 \) if the loop \( \{k, k\} \) is activated as a hub edge or \( \lambda \) otherwise. The \( \lambda \)-connectivity of singletons can be imposed by means of constraints \( y(\delta(k)) + y_{kk} \geq \lambda z_k, k \in V \), which have an analogous interpretation.

Below we develop a MILP formulation for the HLPLF-\( \lambda \), which incorporates \( \lambda \)-connectivity by means of the family of constraints \([5]\) introduced above. The formulation uses the same \( z, y, \) and \( x \) variables as before. Still, since backup paths are no longer made explicit, variables \( \bar{x} \) used in formulation HLPLF-1BP of the previous section are no longer needed. As explained, the \( y \) variables will be used to impose the \( \lambda \)-connectivity condition, which will be stated by means of an exponential set of constraints.

Given that backup routes are no longer made explicit, we no longer have closed expressions for their expected routing costs and we must estimate their values. Let us denote by \( \bar{C}_{kr}^r \) an estimation of the backup routing cost of commodity \( r \) when the hub arc \((k, l)\) of its original routing path fails. The resulting formulation for the HLPLF-\( \lambda \) is:
Hub Location with Protection under Link Failures

\[
\begin{aligned}
\min & \sum_{k \in V} f_k z_k + \sum_{k,l \in E} h_{kl} y_{kl} + \sum_{r \in R} \sum_{(k,l) \in A} [(1 - p_{kl})C_{kl}^p + p_{kl} C_{kl}^q] x_{kl}^r \\
\text{s.t.} & \quad \sum_{(k,l) \in A} x_{kl}^r = 1 & \quad \forall r \in R \\
& \quad x_{kl}^r + x_{lk}^r \leq y_{kl} & \quad \forall r \in R, \{k,l\} \in E, k \neq l \\
& \quad x_{kk}^r \leq y_{kk} & \quad \forall r \in R, k \in V \\
& \quad y_{kl} \leq z_k & \quad \forall \{k,l\} \in E \\
& \quad y_{kl} \leq z_l & \quad \forall \{k,l\} \in E, k \neq l \\
& \quad y(\delta(S)) + y_{kk} \geq \lambda(z_k + z_l - 1) & \quad \forall S \subset V, |S| \geq 2, k \in S, l \notin S \\
& \quad y(\delta(k)) + y_{kk} \geq \lambda z_k & \quad \forall k \in V \\
& \quad x_{kl} \in \{0,1\} & \quad \forall r \in R, (k,l) \in A \\
& \quad z_k \in \{0,1\} & \quad \forall k \in V \\
& \quad y_{kl} \in \{0,1\} & \quad \forall \{k,l\} \in E.
\end{aligned}
\]

where constraints (2.1)-(2.5) are similar to (1.1)-(1.6) but referring to the original path only, and (2.6) and (2.7) are the \(\lambda\)-connectivity constraints described above.

Note that once the hub backbone network \((H,E_H)\) is obtained by solving the above problem, one can explicitly compute a backup path for a commodity \(r \in R\), whose original path is \((o_r,k_r,l_r,d_r)\), by solving (in polynomial time) a shortest path problem from source \(o_r\) to destination \(d_r\) on the graph \(G_r = (V_r,E_r)\) with nodes \(V_r = \{o_r,d_r\} \cup H\) and edges \(E_r = \{(o_r,h) \in H \cup E_h \cup \{(h,d_r) : h \in H\} \setminus \{(k_r,l_r)\}\}

**Proposition 4.1.** Let \(S \subset V\) be a nonempty subset of nodes with \(|S| \leq \lambda - 1\). Then, the set of constraints (2.6) is dominated by the set of constraints, which are also valid for (HLPLF-\(\lambda\)):

\[
y(\delta(S)) + y_{kk} \geq \lambda z_k & \quad \forall k \in S.
\]

**Proof.** Indeed, (2.6') dominate (2.6), since \(z_l \leq 1\) implies that \(\lambda z_k \geq \lambda(z_k + z_l - 1)\).

We now see that (2.6) are valid for (HLPLF-\(\lambda\)). Taking into account that \(\lambda\)-connectivity with \(\lambda \geq 2\) implies that any feasible solution has at least \(\lambda\) open hubs and that \(|S| \leq \lambda - 1\), when \(k \in S\) is activated as a hub node (i.e. \(z_k = 1\)), there will be at least one more open hub \(l \notin S\) (i.e. \(z_l = 1\)). That is, when \(z_k = 1\) there will be at least one active constraint in the set (2.6) with right-hand-side value \(\lambda(z_k + z_l - 1) = \lambda z_k\). When \(k\) is not activated as a hub node (i.e. \(z_k = 0\)) none of the constraints (2.6), nor of the constraints (2.6') will be active. Therefore, the result follows.

Note that when \(S\) is a singleton, i.e., \(S = \{k\}\), the set of constraints (2.6') reduces precisely to (2.7). Thus, in what follows, we replace in (HLPLF-\(\lambda\)) both constraints (2.6) and (2.7) by (2.6').

**4.1. Incorporation of \(\lambda\)-cutset constraints: a branch-and-cut approach.**

As already mentioned, in (HLPLF-\(\lambda\)), the size of the family of constraints (2.6') is exponential in the number of potential hub nodes, \(n\). It is thus not possible
to solve the formulation directly with some off-the-shelf solver, even for medium size instances. In this section we present an exact branch-and-cut algorithm for this formulation in which, as usual, the family of constraints of exponential size (2.6') is initially relaxed. The strategy that we describe below is embedded within an enumeration tree and it is applied not only at the root node but also at all explored nodes. Our separation procedure is an adaptation of the separation procedure for classical connectivity constraints (Padberg and Grötschel 1985), and follows the same vein of those applied to more general connectivity inequalities in node and arc routing problems (see, e.g., (Belenguer and Benavent 1998, Aráoz et al. 2009, Rodríguez-Pereira et al. 2019) for further details).

The initial formulation includes all constraints (2.1)-(2.5), and the singleton version of (2.6'). Furthermore, all integrality conditions are relaxed.

Let \((x, y, z)\) be the solution to the current LP and let \(G(y) = (V(y), E(y))\) denote its associated support graph where \(E(y)\) consists of all the edges of \(E\) such that \(y_{kl} > 0\) and \(V(y)\) the set of endnodes of the edges of \(E(y)\). Each edge \((k, l) \in E(y)\) is associated with a capacity \(\bar{y}_{kl}\). The separation for inequalities (2.6') is to find \(S \subset V\), and \(k \in S\), with \(y(\delta(S)) < \lambda z_k - \bar{y}_{kk}\) or to prove that no such inequality exists. Note that, when they exist, violated \(\lambda\)-connectivity constraints (2.6') can be identified from a tree of min-cuts associated with \(G(y)\) relative to the capacities vector \(y\), \(T(y)\).

Therefore, to solve the above separation problem we proceed as follows. For each min-cut \(\delta(S)\) of \(T(y)\) of value \(y(\delta(S))\), we identify \(k \in \arg \max \{\lambda z_k - \bar{y}_{kk} : k \in S\}\). Then, if \(y(\delta(S)) < \lambda z_k - \bar{y}_{kk}\), the inequality (2.6') associated with \(S\) and \(k\) is violated by \(y\).

We use the procedure proposed by Gusfield (1993) to identify \(T(y)\). Such an algorithm computes \(V(y)\) max-flows in \(G(y)\), so its overall complexity is \(O(|V(y)| \times |V(y)|^3)\).

5. Computational Experience

In this section we report the results of an extensive battery of computational tests, which have been carried out to analyze the performance of the two modeling approaches for obtaining robust hub networks protected under inter-hub failures, discussed in the previous sections. For the experiments, we have used a large set of benchmark instances based on the well-known CAB (O’Kelly (1987)), AP (Ernst and Krishnamoorthy (1996)) and on the TR (Tan and Kara (2007)) datasets (taken from the phub datasets in ORLIB http://people.brunel.ac.uk/~mastjjb/jeb/orlib/ and https://ie.bilkent.edu.tr/~bkara/dataset.php), for varying settings of the failure probabilities and other parameters as described below. All instances were solved with the Gurobi 9.1.1 optimizer, under a Windows 10 environment on an Intel(R) Core(TM) i7-6700K CPU @ 4.00 GHz 4.01 GHz processor and 32 GB of RAM. Default values were used for all parameters of Gurobi solver and a computing time limit of 7200 seconds was set.

5.1. Instances generation. We have generated several instances based on the entire CAB, AP and TR datasets with a number of nodes \((n)\) initially ranging in \(\{10, 15, 20, 25\}\) for the instances based on the CAB and TR datasets and in \(\{10, 20, 25\}\) for the instances based on the AP dataset. Let \(c'_{kl}\) be the standard
unit transportation costs provided in ORLIB for CAB and AP instances or the travel distances provided for the TR instances.

The unit routing costs for access/distribution arcs \( (c_{ij}) \) and the inter-hub routing costs \( (c_{kl}) \) have been obtained as follows. We take the original costs as the unit routing cost through the access and delivery arcs, i.e., \( c_{ij} = c'_{ij} \). For the routing costs through the inter-hubs arcs, we assume that these costs include not only transportation costs but also some additional handling costs at the endnodes of the traversed arcs, associated with the collection (at the entering node) and redistribution (at the leaving node) of the routed commodity. Then, we define the unit routing costs through arc \((k,l) \in A \) as:

\[
c_{kl} = \alpha(a_k + c'_kl + d_l),
\]

where:

- \( \alpha \in [0,1] \) is the usual discount factor applied to routing costs through inter-hub arcs due to economies of scale. Three values for the discount factor \( \alpha \in \{0.2, 0.5, 0.8\} \) have been considered in our study.
- \( a_k \geq 0 \) and \( d_k \geq 0 \) are the unit collection and redistribution costs at node \( k \), respectively. Note that applying the discount factor \( \alpha \) to these terms implies no loss of generality. Note also that with this choice of costs, in case \( k = l \), the unit routing (service) cost through the loop \((k,k) \) reduces to \( c_{kk} = \alpha(a_k + d_k) \). In our computational study we define \( a_k = d_k = \min\{\min_{j \neq k} c'_{kj}, \min_{j \neq k} c'_{jk}\} \).

As usual in the literature (O’Kelly 1992), we have considered the same set-up costs for all potential hubs \( k \in V \), \( f_k = 100 \) for the CAB dataset, two types of set-up costs \( (T \) and \( L \) ) for the hub nodes provided with the AP dataset, and the set-up costs provided in the TR dataset. Service demand, \( w_r \), \( r \in R \), was also taken from the provided datasets.

As considered in the literature (see e.g., Alumur et al. (2009), Calık et al. (2009)), the set-up costs for activating hub edges for the CAB and the AP datasets were set:

\[
h_{kl} = \begin{cases} 
100 \frac{c_{kl}/w_{kl}}{\text{MAXW}} & \text{if } k \neq l, \\
100 \frac{c_{kl}/\bar{w}}{\text{MAXW}} & \text{if } k = l.
\end{cases}
\]

where \( w \) is the normalized vector of flows, \( \bar{w} \) is the mean of \( w \) and \( \text{MAXW} = \max\{\frac{c_{ij}}{w_{ij}} : i,j \in V, w_{ij} > 0\} \) and for TR those provided in the original dataset.

In formulation (HLPLF-\( \lambda \)), we have estimated the costs of backup paths as \( C'_{kl} = (1 + \beta)C_{kl} \) for two different values of \( \beta \in \{0.5, 1\} \). Observe that in this case the expected routing cost simplifies to:

\[
\sum_{r \in R} \sum_{(k,l) \in A} (1 + \beta p_{kl})C'_{kl}x_{kl}.\]

As for the failure probabilities \( p_{kl} \), \( \{k,l\} \in E \), we have considered three different scenarios:

**RP:** Random probabilities. The failure probability of each edge is randomly generated from a uniform distribution, i.e. \( p_{kl} \equiv U[0,1] \) for all \( \{k,l\} \in E \).
Instances | $n$ | $\alpha$ | $c_{kl}$ | $h_{kl}$ | $\lambda$
--- | --- | --- | --- | --- | ---
CAB | {10, 15, 20, 25} | {0.2, 0.5, 0.8} | $a_k + c_{kl} + d_l$ | $100\frac{c_{kl}/w_{kl}}{\max w_{kl}}$ if $k \neq l$, $100\frac{c_{kl}/\bar{w}_{kl}}{\max w_{kl}}$ if $k = l$. | {2, 3, 4} 
AP | {10, 20, 25, 40, 50} | Data file | Data file | Data file | Data file 
TR | {10, 15, 20, 25, 40, 50} | Data file | Data file | Data file | Data file

Failure probabilities

| Random probabilities (RP) | $p_{kl} \sim U[0, \rho]$, $\rho \in \{0.1, 0.3\}$ |
| Clustered probabilities (CP) | $p_{kl} \in \{0.1, 0.2, 0.3\}$ |
| Same probability (SP) | $p_{kl} = \rho$, $\rho \in \{0.1, 0.3\}$ |

Table 1. Summary of instances and parameters.

**CP**: Clustered probabilities. Edges are clustered into three groups, each of them with a different failure probability. For this, each edge $\{k, l\} \in E$ is randomly assigned a failure probability in $p_{kl} \in \{0.1, 0.2, 0.3\}$.

**SP**: Same probability. All edges have the same failure probability, i.e. $p_{kl} = \rho$, for all $\{k, l\} \in E$.

The values of the parameter $\rho$ we have used in RP and SP scenarios are $\rho \in \{0.1, 0.3\}$. The files of the randomly generated probabilities are available in the Github repository [https://github.com/vblancoDR/HLPLF](https://github.com/vblancoDR/HLPLF).

For each combination of parameters $n$, $\alpha$, $\rho$, and each dataset (CAB, AP with type T and L fixed set-up hub nodes costs, and TR) five different instances have been generated for scenario of failure probabilities RP and one instance has been considered for scenario SP. Five instances have been also generated for scenario CP and each combination of $n$, $\alpha$, and each dataset. Thus, (HLPLF-1BP), hereafter called M1, has been solved on a total of 714 instances.

Concerning formulation (HLPLF-$\lambda$), we considered three different values for the parameter $\lambda$, namely $\lambda = 2$, $\lambda = 3$ and $\lambda = 4$ (we call the corresponding models M2.2, M2.3 and M2.4, respectively). Thus, (HLPLF-$\lambda$) has been solved on a total of 4284 instances. Additionally, for comparative purposes, we have solved 42 instances of the Uncapacitated Hub Location Problem, in which no protection under failures is considered. This model will be referred to as M0.

Finally, to test the scalability of our formulations, a second experiment was carried out on a set of larger instances ($n \in \{40, 50\}$) based on the AP and TR datasets considering only (HLPLF-$\lambda$), which, as we will see, is the most promising formulation, for $\lambda \in \{2, 4\}$ and $\beta = 1$. We have solved a total of 612 instances in this second study. Overall, 5652 instances have been solved.

Table 1 summarizes the main characteristics of the testing instances and the selected parameters.

5.2. **Numerical results with (HLPLF-1BP) and (HLPLF-$\lambda$)**. The results obtained in our first computational study are summarized in Tables 2 and 3 for
(HLPLF-1BP) and (HLPLF-λ), respectively. In both tables, “RP”, “CP” and “SP” stand for the scenarios with random failure probabilities (with $\rho = 0.1$ and $\rho = 0.3$), clustered failure probabilities and same failure probability (with $\rho = 0.1$ and $\rho = 0.3$), respectively, as described in Section 5.1. The values of $n$, $\alpha$ and “Data” in both tables indicate the number of nodes in the network, the value for the discount factor applied to the routing cost through inter-hubs arcs, and the dataset that has been used to obtain the costs and the flows, respectively. “AP$^T$” and “AP$^L$” refer to AP dataset using type T and type L set-up costs for the hub nodes, respectively. In Table 2, for scenarios RP and CP, the information contained in each row refers to average values over the five instances with the corresponding combination of parameters, whereas for scenario SP the values of the entries correspond to the unique instance with this combination of parameters. In Table 2, we summarize the numerical results of (HLPLF-1BP) in three blocks of columns. Block “CPUTime” gives the computing times, in seconds, required to solve the instances, block “MIPGap” the percentage MIP gaps returned by Gurobi at termination, and block “%Solved” the percentage

| $\alpha$ | Data | $\alpha$ | Data | $\alpha$ | Data |
|---------|------|---------|------|---------|------|
| 0.2     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 4    | 7      | 15   | 1      | 1    |
| AP$^L$  | 7    | 93     | 13   | 2      | 3    |
| TR      | 143  | 5198   | TL   | 1      | 1    |
| 0.5     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 10   | 112    | 11   | 2      | 3    |
| AP$^L$  | 25   | 727    | TL   | 1      | 1    |
| TR      | 5    | 19     | 53   | 1      | 1    |
| 0.8     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 9    | 99     | 12   | 1      | 3    |
| AP$^L$  | 25   | 727    | TL   | 1      | 1    |
| TR      | 5    | 19     | 53   | 1      | 1    |
| CAB     | TL   | TL     | TL   | 2      | 7    |
| TR      | 51   | 71     | 287  | 4      | 6    |
| 15      | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 5    | 138    | 412  | 3      | 4    |
| AP$^L$  | 5     | 138    | 412  | 3      | 4    |
| TR      | 52   | 89     | 619  | 3      | 5    |
| CAB     | TL   | TL     | TL   | 4      | 6    |
| TR      | 51   | 138    | 412  | 3      | 4    |
| 20      | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 1520 | 1097   | 43   | 3      | 4    |
| AP$^L$  | TL   | TL     | TL   | 2      | 7    |
| TR      | 363  | 1850   | 5945 | 21     | 36   |
| CAB     | TL   | TL     | TL   | 4308   | 1257 |
| TR      | 363  | 1850   | 5497 | 21     | 36   |
| 25      | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 3609 | 7313   | 5780 | 40     | 33   |
| AP$^L$  | TL   | TL     | TL   | 2343   | 1809 |
| TR      | 422  | 1864   | 5625 | 19     | 33   |
| CAB     | TL   | TL     | TL   | 12     | 27   |
| TR      | 422  | 1864   | 5625 | 19     | 33   |
| 0.2     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 2774 | 6825   | 4713 | 128    | 129  |
| AP$^L$  | TL   | TL     | TL   | 2543   | 1809 |
| TR      | 232  | 119    | 169  | 1      |
| CAB     | TL   | TL     | TL   | 119    | 169  |
| TR      | 232  | 119    | 169  | 1      |
| 0.5     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 6389 | 4763   | 129  | 1      |
| AP$^L$  | TL   | TL     | TL   | 1      |
| TR      | 232  | 119    | 169  | 1      |
| CAB     | TL   | TL     | TL   | 1      |
| TR      | 232  | 119    | 169  | 1      |
| 0.8     | RP   | 0.4     | CP   | 0.4     | SP   |
| AP$^T$  | 6389 | 4763   | 129  | 1      |
| AP$^L$  | TL   | TL     | TL   | 1      |
| TR      | 232  | 119    | 169  | 1      |

Table 2. Average Results for (HLPLF-1BP).
instances solved to proven optimality within the time limit. An entry “TL” in the CPU Time block means that the time limit of 7200 seconds was reached in all five instances of the group. The “OoM” entry indicates that the flag “Out of memory” was the output of the solver in at least one of the instances in the row, and then, the remaining information of the row refers to average values over the solved instances only (even if none of these instances could be solved to proven optimality).

Table 3 is organized in three blocks, $\lambda = 2$, $\lambda = 3$ and $\lambda = 4$, for each of the three considered values of $\lambda$ in (HLPLF-$\lambda$). We have observed that the value of the parameter $\beta$ does not affect the results and thus, in this table the information contained in each row refers to average values of 10 instances ($5 \times 2$ different values of $\beta$) for RP and CP scenarios and refers to the average values of the two (different values of $\beta$) instances for SP scenario. Using formulation (HLPLF-$\lambda$), all instances have been solved to proven optimality for all three considered values of $\lambda$. For this reason, blocks “MIPGap” and “%Solved” have been omitted in Table 3 since MIPGap is 0.00 for all the instances and the percentage of solved instances is always 100%.

In Table 2 we observe a different performance of (HLPLF-1BP) among the instances corresponding to the different scenarios. Based on the computing times, MIPGAPs, and percentage of solved instances, scenario SP produces the easiest instances, for all configurations of parameters, as expected. One can observe that, for instances with the same probability, all instances generated from the CAB, AP$_T$, and TR datasets with up to $n = 25$, as well as the instances generated from AP$_L$ with up to $n = 20$ have been optimally solved within the time limit.

On the other hand, note that instances based on CAB dataset are more difficult to solve than instances based on TR and AP datasets. TR based instances are the easiest to solve: all instances with up to $n = 15$, 95% for $n = 20$ and 35% for $n = 25$ have been optimally solved within the time limit. Regarding AP based instances, all instances with $n = 10$ have been optimally solved within the time limit, although AP$_T$ instances consumed, in general, less computing time. The difference between AP$_T$ and AP$_L$ instances becomes more evident for $n > 10$, since approximately 50% of the AP$_T$ instances were optimally solved whereas none of the AP$_L$ instances with random and clustered probabilities (scenarios RP and CP) was solved to proven optimality within the time limit. As mentioned before, CAB instances are the most difficult ones. For $n = 10$, 30% of these instances could not be optimally solved within the time limit. This percentage increases up to 90% for $n = 20$ and up to 100% for $n = 25$. Additionally, for $n = 25$, the execution was stopped due to an Out of Memory flag with 30% of the CAB instances under the random probabilities (RP) scenario.

Comparing Table 2 with Table 3 we observe that (HLPLF-$\lambda$) is notably easier to solve than (HLPLF-1BP), which can be explained by its smaller number of decision variables. The difficulty of (HLPLF-$\lambda$) increases with the value of $\lambda$, as reflected by a decrease in its performance for higher values of this parameter. This could be expected, as (HLPLF-$\lambda$) becomes more restrictive as the value of $\lambda$ increases. When $\lambda = 2$, the average computing time over all the instances is approximately 64 seconds, being two seconds for the CAB instances, 70 seconds for the AP$_T$ instances, 89 seconds for the AP$_L$ instances, and 102 for the TR
Hub Location with Protection under Link Failures

| λ  | Data | λ = 2 | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 |
|----|------|-------|-----|-----|----|-----|-----|----|-----|-----|----|-----|-----|
| 0.2 | RP | 0.1 | 3 | 2 | 1 | 1 | 5 | 3 | 3 | 2 | 2 | 3 | 3 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 6 | 1 | 7 | 1 | 4 | 5 | 6 |
| 10.0 | AP | 2 | 3 | 4 | 2 | 4 | 3 | 4 | 2 | 7 | 4 | 2 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 2 | 3 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 3 | 2 | 3 | 2 | 2 | 3 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 3 | 2 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 3 | 2 | 3 | 2 | 2 | 3 |

| λ  | Data | λ = 3 | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 |
|----|------|-------|-----|-----|----|-----|-----|----|-----|-----|----|-----|-----|
| 0.2 | RP | 0.1 | 3 | 2 | 1 | 1 | 5 | 3 | 3 | 2 | 2 | 3 | 3 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 6 | 1 | 7 | 1 | 4 | 5 | 6 |
| 10.0 | AP | 2 | 3 | 4 | 2 | 4 | 3 | 4 | 2 | 7 | 4 | 2 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 2 | 3 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 3 | 2 | 3 | 2 | 2 | 3 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 3 | 2 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 3 | 2 | 3 | 2 | 2 | 3 |

| λ  | Data | λ = 4 | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 | CP | 0.1 | 0.3 |
|----|------|-------|-----|-----|----|-----|-----|----|-----|-----|----|-----|-----|
| 0.2 | RP | 0.1 | 3 | 2 | 1 | 1 | 5 | 3 | 3 | 2 | 2 | 3 | 3 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 6 | 1 | 7 | 1 | 4 | 5 | 6 |
| 10.0 | AP | 2 | 3 | 4 | 2 | 4 | 3 | 4 | 2 | 7 | 4 | 2 |
| 0.5 | CAB | 0.0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 2 | 3 |
| 1.0 | TR | 0.1 | 1 | 1 | 0 | 1 | 3 | 2 | 3 | 2 | 2 | 3 |

Table 3. Average CPU times for (HLPLF-λ).

Table 3. Average CPU times for (HLPLF-λ).

instances. Note that, unlike (HLPLF-1BP), CAB instances are less computationally demanding than AP and TR instances. This behavior was also observed for λ = 3 and λ = 4. For λ = 4 the average computing time over all the instances is approximately 218 seconds, being 30 seconds for the CAB instances, 168 seconds for the AP instances, 225 seconds for the AP instances, and 438 for the TR instances. Observe that the value of α also affects the performance of (HLPLF-λ), instances being more difficult for smaller α values, specially for the AP instances. We also note that, unlike (HLPLF-1BP), with (HLPLF-λ) there seem to be no noticeably differences among scenarios.

Finally, Table 4 summarizes the results of our second set of computational experiments, which was carried out considering only (HLPLF-λ) for λ ∈ {2, 4} and β = 1 on larger instances (n ∈ {40, 50}) based on the AP and TR datasets. In this second set of experiments we did not consider (HLPLF-1BP) as most instances could not be optimally solved within the time limit already for n = 25.

We can observe that, for λ = 2, all the TR instances, 99% of the AP instances, and 80% of the AP instances have been solved to proven optimality within the time limit, whereas for λ = 4 the percentage of solved instances was
70% of the AP₃ instances, 40% of the AP₁ instances, and 6% of the TR instances. This shows that (HLPLF-λ) is able to solve larger instances with up to $n = 50$, even if instances become more challenging as the value of the parameter $λ$ increases.

5.3. Managerial Insight . In this section we derive some managerial insight from the results obtained in our first set of experiments, i.e., $n \leq 25$ when the instances were solved with both formulations, as well from the solutions of these instances for M0 (the uncapacitated HLP with no protection under failures). Figure 3 shows the percentage contribution to the objective function value of the different types of costs: routing costs, set-up costs for activating hubs (Hubs_Costs) and set-up costs for activating inter-hub edges (Links_Costs).

We have observed that results are similar for AP₃ and AP₄ data sets and thus, for each formulation, we differentiate between datasets CAB, AP and TR, as well as among the three values of the $α$ parameter.

We can observe that the percent contribution of the set-up costs for activating inter-hubs edges varies from 0.1% for CAB instances to 13% for TR instances in M0. The percent contribution of hub set-up costs depends, as expected, on the value of the parameter $α$ but mainly on the dataset and on the model. For the instances based on the CAB dataset, the percent contribution of hub set-up costs varies from 20% with M0, M1, and M₂₂ for $α = 0.8$, to 40% with M₂₄ for
For the instances based on the AP dataset, the percent contribution of hub set-up costs varies from 45% with M0 and $\alpha \in \{0.5, 0.8\}$ to 75% with M2.4. Regarding the instances based on the TR dataset, the percent contribution of hub set-up costs varies from 21% with M0 and $\alpha \in \{0.5, 0.8\}$ to 69% with M2.4 for $\alpha = 0.2$. The percent contribution of routing costs also depends on the value of the parameter $\alpha$, on the dataset, and on the model. For the instances based on the CAB dataset, this percentage varies from 55% to 80%, with the highest values for M0, M1, and M2.2 and $\alpha = 0.8$. For the instances based on the AP dataset, the percent contribution of routing costs varies from 15% to 55%, corresponding the highest values to M0 with $\alpha = 0.8$. Finally, for the instances based on the TR dataset, the percent contribution of routing costs varies from 22% to 68%, corresponding again the highest values to M0 with $\alpha = 0.8$.

Information about the structure of the optimal backbone network can be found in Table 5 and Figures 4, 5, and 6. In Table 5, “#H”, “#Lk” and “#Lp” stand for the average number of hubs activated in the optimal backbone network, the average number of activated inter-hub edges (including loops), and the average number of activated loops, respectively. Two density indices have been studied. While the first index ($I_1$) indicates the density of the backbone network including loops, the second one ($I_2$) indicates the density of the backbone network when loops are not considered. That is, $I_1$ is the ratio between the number of activated inter-hub edges and the total number of links in a complete graph with $#H$ nodes if loops were included:

$$I_1 = \frac{2#Lk}{#H(#H+1)}.$$
Figure 4. Backbone network density for CAB-based instances

Figure 5. Backbone network density for AP-based instances

Figure 6. Backbone network density for TR-based instances
and $I_2$ the ratio between the number of non-loop inter-hub edges activated and
the number of links in a complete graph with $\#H$ nodes if loops were excluded:

$$I_2 = \frac{2(\#Lk - \#Lp)}{\#H(\#H - 1)}.$$ 

Note that values of the two indices range in $[0, 1]$, i.e., $0 \leq I_1, I_2 \leq 1$. Figures [4] and [5] show, for each formulation and value of the parameter $\alpha$, the percentage of instances that reach the corresponding index value, for the CAB, the AP and the TR instances, respectively.

We can observe in Table 5 that the number of open hubs depends mainly on the model and also on the dataset. For all AP instances the number of opens hubs is always one for M0, two for M1 and M2, three for M2, three and four for M2, four for M2, and five for M2, four for M2, and five for M2. For TR instances, the number of opens hubs is always one for M0, two for M1 and M2, and four for M2, four for M2, and five for M2. On the other hand, we can observe that the number of activated inter-hubs edges is smaller for M0 than for M1 and that, for M1, this number is similar to that for M2 but smaller than that for M2, $\lambda > 2$, since, as expected, this number increases with the value of $\lambda$. Additionally, we can observe that with M0 most of the activated inter-hubs edges are loops, mainly with the AP and the TR instances. This fact can be also observed in Figure [6] for the AP instances where for M0 and M1 the density index $I_1$ is close to 1, whereas the density index $I_2$ is close to 0. This indicates

| $\alpha$ | Data | $\# H$ | $\# Lk$ | $\# Lp$ | $\# H$ | $\# Lk$ | $\# Lp$ | $\# H$ | $\# Lk$ | $\# Lp$ | $\# H$ | $\# Lk$ | $\# Lp$ |
|---------|------|--------|--------|-------|--------|--------|--------|--------|--------|-------|--------|--------|-------|
| 0.2     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.06   | 0.06   | 4.00   | 6.00   | 0.00   |
| 0.5     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 0.00   | 4.00   | 6.00   | 0.00   |
| 0.8     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 0.00   | 4.00   | 6.00   | 0.00   |
| 1.0     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 0.00   | 4.00   | 6.00   | 0.00   |
| 1.5     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 0.00   | 4.00   | 6.00   | 0.00   |
| 2.0     | AP   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 3.00   | 4.00   | 10.00  | 4.00   |
|         | TR   | 1.00   | 1.00   | 1.00   | 2.00   | 2.00   | 2.00   | 3.00   | 3.00   | 0.00   | 4.00   | 6.00   | 0.00   |

Table 5. Average number of open hubs, links and loops.
Table 6. Average percent deviation of activated hubs and inter-hub set-up costs relative to those of M0.

| n   | α   | M1        | M2        | M2,3       | M2,4       |
|-----|-----|-----------|-----------|------------|------------|
| 10  | 0.2 | 68.57%    | 136.86%   | 235.11%    | 361.15%    |
|     | 0.5 | 68.57%    | 134.70%   | 235.11%    | 360.23%    |
|     | 0.8 | 68.57%    | 148.34%   | 240.17%    | 358.39%    |
| 15  | 0.2 | 68.57%    | 147.33%   | 233.01%    | 358.39%    |
|     | 0.5 | 68.57%    | 143.45%   | 233.01%    | 358.39%    |
|     | 0.8 | 68.57%    | 143.17%   | 234.23%    | 358.39%    |
| 20  | 0.2 | 61.52%    | 98.44%    | 191.38%    | 302.97%    |
|     | 0.5 | 61.93%    | 113.60%   | 184.87%    | 279.68%    |
|     | 0.8 | 61.52%    | 126.21%   | 184.87%    | 280.18%    |

6. The price of robustness

For assessing the robustness and reliability of the hub network models proposed in this paper, we evaluate the so-called price of robustness (see Bertsimas and Sim (2004)), defined as the extra cost incurred to design a robust network. In our case robustness translates into protecting the backbone network under inter-hub edge failures, which, essentially, is attained by incorporating additional inter-hub edges to the backbone network. We thus start out analysis by comparing the overall set-up cost of the activated inter-hub edges for each of the proposed models M1, M2, M2, M2, and M2, with that of the unprotected network obtained with M0. This information is summarized in Table 6 which, for each of the models M1, M2, M2, and M2, gives the average percent deviation of the inter-hub set-up costs with respect those of M0. For the sake of simplicity, in this table we only show the results for the TR dataset, although the behavior of the CAB and AP datasets is similar.

As expected, constructing networks that are robust under inter-hub edge failures has a significative impact in the design cost of the network. For the λ-connected models, the reported percent deviations increase with the value of λ, which can be easily explained as larger backbone networks are required as λ increases. Nevertheless, as shown by the results of the experiment that we report next, in case of failure, this increase in the design costs strengthens the possibility of being able of re-routing all the commodities of a posteriori solutions (after the occurrence of a failure in the inter-hub edges).

For this experiment, we have used all the n = 20 instances of the TR dataset, whose optimality is guaranteed for all the models. For each of the 255 instances generated for the TR dataset with n = 20, we have simulated the following
scenarios for potential failures of the inter-hub edges of the backbone networks produced by the different models:

- **Failure scenario 1 (FS1):** Only activated inter-hub edges may fail. Each activated hub edge \( \{k, l\} \) in a backbone network fails (and removed from the backbone network) according to a Bernoulli distribution with probability \( p_{kl} \).

- **Failure scenario 2 (FS2):** In this scenario failures are associated with hub nodes. Failure of a hub node implies the failure of all the inter-hub edges incident to the hub. Thus, each activated hub fails with probability \( p_{kk} \), and then all its incident inter-hub edges are removed from the backbone network.

- **Failure scenario 3 (FS3):** Failures are simulated for inter-hub edges of the backbone network similarly to FS1. In case an inter-hub edge fails, the failure probability of the loops at the extreme nodes of the edge is increased in 50%. Then, failures in the loop edges are simulated.

- **Failure scenario 4 (FS4):** First, failures are simulated for inter-hub edges of the backbone network similarly to FS1. The difference is that we now assume that the failure of a considerable number of inter-hub edges incident with any activated hub node, will provoke the failure of the hub node as well, and thus the failure of all its incident inter-hub edges. That is, for each activated hub node, we assume that if at least a given percentage \( \gamma \% \) of its incident inter-hub edges have failed, then the whole hub node fails, provoking that its remaining incident inter-hub edges also fail, which are also removed from the network. In our study we fix the value of the parameter \( \gamma \) to 75%, i.e., if 75% or more of the inter-hub edges incident to a hub node fails, then, the hub (and the remaining incident inter-hub edges) cannot be used any longer for routing the commodities.
Figure 7 illustrates with simple backbone networks \((H,E_H)\) the four failure scenarios that we consider. The backbone network has four hub nodes \(H = \{k,l,m,q\}\) and six inter-hub edges, of which two are loops \(E_H = \{\{k,l\}, \{k,q\}, \{l,m\}, \{q,m\}, \{l,l\}, \{m,m\}\}\). In FS1, hub edges \(\{k,l\}\) and \(\{m,m\}\) are chosen to fail, both depicted with dashed lines in the left picture. The network resulting after removing these inter-hub edges is shown in the right picture. In FS2, hub node \(m\) (in a gray circle) is chosen to fail, and then, all the hub edges incident with it (depicted with dashed lines), namely \(\{q,m\}, \{m,m\}\) and \(\{l,m\}\), removed from the network. In FS3 the interhub edge \(\{q,m\}\) is chosen to fail, and increases in 50% the failure probability of the loop \(\{m,m\}\), which is then randomly chosen to fail. Thus, we chose \(\{q,m\}\) and \(\{m,m\}\) (depicted with dashed lines), which are then removed from the network. Finally, in FS4, hub edges \(\{k,l\}, \{l,m\}\) and \(\{m,m\}\) are chosen to fail. Then, since the percentage of inter-hub edges incident to \(m\) that fail exceeds \(\gamma = 75\%\), all edges incident to \(m\) are removed. For the remaining hub nodes, such a percentage is not exceeded so no further inter-hub edges are removed.

We have carried out simulations for each of the above failure scenarios, all of which follow the same general structure, for a given backbone network. (i) We randomly generate the links that fail according to the corresponding failure scenario and obtain the after-failure network by removing from the backbone network the edges that fail. (ii) We try to re-route all the commodities through...
Hub Location with Protection under Link Failures

the after-failure network. (iii) Since it may happen that it is no longer possible to route some of the commodities in the after-failure network, we will analyze this circumstance in our study. For each failure scenario, each simulation is repeated 10000 times over each instance. The average results obtained for all the instances are reported in Figure 8. There we draw light blue bars to represent the results for model $M_0$, dark blue for $M_1$, orange for $M_{2,2}$, gray for $M_{2,3}$, and yellow for $M_{2,4}$.

As one can observe, the networks obtained with the proposed models ($M_1$ and $M_{2,\lambda}$) are clearly more robust under inter-hub failures than $M_0$. Specifically, in average, our models allow re-routing all the involved commodities in more than 90% of the failure occurrences while $M_0$ was only able to re-route 78% of them. The robustness of model $M_{2,4}$ is even more impressive, being the percentage of simulations in which re-routing is possible 99.5%.

On the other hand, analyzing the results of the failure scenario FS2, one can observe that models $M_{2,\lambda}$ are not only robust under inter-hub edges failures, but also under failures of the hub nodes. However, $M_0$ and $M_1$ have a similar behaviour under these scenarios with close to 20% of simulations, in average, in which the commodities could not be routed. This highlights the performance of $M_{2,2}$, $M_{2,3}$, and $M_{2,4}$, for which the percentage of simulations where some commodity could not be rerouted decreases to 12%, 5%, and 2%, respectively.

At each of the simulations, when all commodities can be routed in the after-failure network, we compute the overall a posteriori routing cost $R^F$. We denote by $\tau_F \in [0,1]$ the proportion of simulations for which this cost can be computed. In case a commodity $r \in R$ is not able to be routed through the after-failure backbone network, we assume that its routing cost is proportional to the cost of the direct connection $\bar{c}(r)$, i.e. the overall routing cost is $(1+q)\sum_{r \in R} w_r c_o(r)d(r)$.

The parameter $q > 0$ represents the extra percent cost (over the cost of the direct connection) when re-routing a commodity in case the backbone network cannot be used any longer to route it. Such a cost may represent the outsourcing cost of a direct delivery between the origin and destination of the commodity or the lost of opportunity cost of an unsatisfied user for which the service could not be provided. With this information, we compute the average set-up and routing cost for the network as:

$$\Phi(q) = \sum_{h \in H} f_h + \sum_{e \in E_H} h_e + \tau_F R^F + (1 - \tau_F)(1 + q)\sum_{r \in R} w_r c_o(r)d(r).$$

We summarize in Figure 9 the average behavior of this cost for all the simulations and all the failure scenarios. Each line represents the above cost, as a function of the parameter $q$, for each of the after-failure networks produced by the simulations constructed with the five different models ($M_0$, $M_1$, $M_{2,2}$, $M_{2,3}$, and $M_{2,4}$). One can observe that for small values of $q$ (the re-routing costs are a small factor of the direct costs from origin to destination) $M_0$ is more convenient. This is clear, since in case the re-routing costs are not very high, one may undertake these costs, even when these failures occur very often. As $q$ increases, the most convenient models are $M_1$, $M_{2,2}$, and $M_{2,3}$ (in this order). Model $M_{2,4}$ is clearly the most robust one, since the parameter $q$ almost
does not affect the cost (in this case the percentage of simulations for which the commodities cannot be routed is tiny), but their set-up costs are very high.

In Figure 10 we show the disaggregated results by failure scenario (FS1, FS2, FS3, and FS4). There, we observe that the behavior of $\Phi(q)$ is different for the failure scenario FS2, where we simulate failures in hub nodes. In FS2, models $M_0$, $M_2.3$, and $M_2.4$ outperform $M_1$ and $M_2.2$, on average, while in the remaining
failure scenarios models M0, M1 and M2.2 are more convenient for reasonable values of $q$.

We conclude this section by highlighting that the study that we have carried out allows the decision maker to determine the best model to construct the backbone network based on the expected extra cost that should be paid for not providing the service to commodities due to failures in the network.

7. Conclusions

In this paper we propose different models to construct robust hub networks under inter-hub links failures. The models that we develop ensure that an additional routing path exists besides its original routing path for all the commodities. In the first model, an explicit backup path using at most an inter-hub edge is constructed to be used in case of failure of the original path from which each of the commodities is routed. The second model assures the existence of backup paths (using an arbitrary number of inter-hub links) in case of failure of the original inter-hub edges by means of imposing $\lambda$-connectivity of the backbone network for a given value of $\lambda \geq 2$. The two models present advantages from the point of view of the robustness of the hub network. One the one hand, the first model guarantees that backup paths for the commodities are of the same nature than the original non-failing network, although the computational difficulty to obtain solutions is high. On the other hand, the second model, although ensuring also the construction of backup paths, exhibits a lower computational load than the first model.

Both models have been computationally tested on an extensive battery of experiments with three hub location benchmarks, namely AP, CAB and TR. Some conclusions are derived from this study. Furthermore, we have analyzed the robustness of the model by simulating different types of failures on the TR network, concluding the applicability of our models.

Future research on the topic includes the study of valid inequalities for both models in order to alleviate the computational complexity of the exact resolution of the model. For larger instances, it would be helpful to design heuristic approaches that assure good quality solution in smaller computing times.

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References

Altner, D. S., Ergun, Ö., and Uhan, N. A. (2010). The maximum flow network interdiction problem: valid inequalities, integrality gaps, and approximability. Operations Research Letters, 38(1):33–38.

Alumur, S., Nickel, S., and da Gama, F. S. (2012). Hub location under uncertainty. Transportation Research Part B: Methodological, 46(4):529–543.
Alumur, S. A., Kara, B. Y., and Karasan, O. E. (2009). The design of single allocation incomplete hub networks. *Transportation Research Part B: Methodological*, 43(10):936–951.

An, Y., Zhang, Y., and Zeng, B. (2015). The reliable hub-and-spoke design problem: Models and algorithms. *Transportation Research Part B: Methodological*, 77:103–122.

Andreas, A. K. and Smith, J. C. (2008). Mathematical programming algorithms for two-path routing problems with reliability considerations. *INFORMS Journal on Computing*, 20(4):553–564.

Aneja, Y. P., Chandrasekaran, R., and Nair, K. (2001). Maximizing residual flow under an arc destruction. *Networks: An International Journal*, 38(4):194–198.

Aráoz, J., Fernández, E., and Franquesa, C. (2009). The clustered prize-collecting arc routing problem. *Transportation Science*, 43:287–300.

Belenguer, J. M. and Benavent, E. (1998). The capacitated arc routing problem: Valid inequalities and facets. *Computational Optimization and Applications*, 10(2):165–187.

Bertsimas, D. and Sim, M. (2004). The price of robustness. *Operations Research*, 52(1):35–53.

Calık, H., Alumur, S. A., Kara, B. Y., and Karasan, O. E. (2009). A tabu-search based heuristic for the hub covering problem over incomplete hub networks. *Computers & Operations Research*, 36(12):3088–3096.

Campbell, J. and O’Kelly, M. (2012). Twenty-five years of hub location research. *Transportation Science*, 46(2):153–169.

Campbell, J. F., Ernst, A. T., and Krishnamoorthy, M. (2005). Hub arc location problems: part i—introduction and results. *Management Science*, 51(10):1540–1555.

Çetiner, S., Sepil, C., and Süral, H. (2010). Hubbing and routing in postal delivery systems. *Annals of Operations Research*, 181(1):109–124.

Contreras, I., Cordeau, J.-F., and Laporte, G. (2011). Stochastic uncapacitated hub location. *European Journal of Operational Research*, 212(3):518–528.

Contreras, I. and O’Kelly, M. (2019). Hub location problems. In Laporte, G., Nickel, S., and Saldanha da Gama, F., editors, *Location Science*, chapter 8, pages 177–205. Springer, 2 edition.

Cormican, K. J., Morton, D. P., and Wood, R. K. (1998). Stochastic network interdiction. *Operations Research*, 46(2):184–197.

Cui, T., Ouyang, Y., and Shen, Z.-J. M. (2010). Reliable facility location design under the risk of disruptions. *Operations Research*, 58(4):998–1011.

Ernst, A. T. and Krishnamoorthy, M. (1996). Efficient algorithms for the uncapacitated single allocation p-hub median problem. *Location Science*, 4(3):139 – 154.

Farahani, R. Z., Hekmatfar, M., Arabani, A. B., and Nikbakhsh, E. (2013). Hub location problems: A review of models, classification, solution techniques, and applications. *Computers & Industrial Engineering*, 64(4):1096–1109.

Gusfield, D. (1993). Very simple methods for all pairs network flow analysis. *SIAM Journal on Applied Mathematics*, 19(1):143–555.

Kim, H. and O’Kelly, M. E. (2009). Reliable p-hub location problems in telecommunication networks. *Geographical Analysis*, 41(3):283–306.

Korani, E. and Eydi, A. (2021). Bi-level programming model and KKT penalty function solution approach for reliable hub location problem. *Expert Systems with Applications*, 184:115505.
Hub Location with Protection under Link Failures

Li, Y., Li, X., Shu, J., Song, M., and Zhang, K. (2022). A general model and efficient algorithms for reliable facility location problem under uncertain disruptions. *INFORMS Journal on Computing*, 34(1):407–426.

Ma, J., Pajouh, F. M., Balasundaram, B., and Boginski, V. (2016). The minimum spanning k-core problem with bounded cvar under probabilistic edge failures. *INFORMS Journal on Computing*, 28(2):295–307.

Marín, A., Cánovas, L., and Landete, M. (2006). New formulations for the uncapacitated multiple allocation hub location problem. *European Journal of Operational Research*, 172(1):274–292.

Mohammadi, M., Jula, P., and Tavakkoli-Moghaddam, R. (2019). Reliable single-allocation hub location problem with disruptions. *Transportation Research Part E: Logistics and Transportation Review*, 123:90–120.

O’Kelly, M. E. (1987). A quadratic integer program for the location of interacting hub facilities. *European Journal of Operational Research*, 32(3):393–404.

O’Kelly, M. E. (1992). Hub facility location with fixed costs. *Papers in Regional Science*, 71(3):293–306.

Padberg, M. and Grötschel, M. (1985). Polyhedral computations. In *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, pages 307–360.

Parvaresh, F., Hashemi Golpayegany, S. A., Moattar Husseini, S. M., and Karimi, B. (2013). Solving the p-hub median problem under intentional disruptions using simulated annealing. *Networks and Spatial Economics*, 13(4):445–470.

Rodríguez-Pereira, J., Fernández, E., Laporte, G., Benavent, E., and Martínez-Sykora, A. (2019). The Steiner traveling salesman problem and its extensions. *European Journal of Operational Research*, 278:615–628.

Rostami, B., Kämmerling, N., Buchheim, C., and Clausen, U. (2019). Reliable single allocation hub location problem under hub breakdowns. *Computers & Operations Research*, 96:15–29.

Royset, J. O. and Wood, R. K. (2007). Solving the bi-objective maximum-flow network-interdiction problem. *INFORMS Journal on Computing*, 19(2):175–184.

Snyder, L. V. and Daskin, M. S. (2005). Reliability models for facility location: the expected failure cost case. *Transportation Science*, 39(3):400–416.

Tan, P. Z. and Kara, B. Y. (2007). A hub covering model for cargo delivery systems. *Networks: An International Journal*, 49(1):28–39.

Wood, R. K. (1993). Deterministic network interdiction. *Mathematical and Computer Modelling*, 17(2):1–18.

Zeng, B., An, Y., Zhang, Y., and Kim, H. (2010). A reliable hub-spoke model in transportation systems. In *Proceedings of the 4th international symposium on transportation network reliability, Minneapolis, Minnesota, USA*, pages 22–23.

Zetina, C., Contreras, I., Cordeau, J.-F., and Nikbaksh, E. (2017). Robust uncapacitated hub location. *Transportation Research Part B: Methodological*, 106:393–410.

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