Fractal Images as Number Sequences I

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Abstract

In general, a fractal is represented by its geometrical image. There are several mathematical ways to generate such figures but all systems generally transform an unordered set of fractals into another, unordered set of mathematical objects. All these methods require some mathematical knowledge for the generation of fractals. We describe a system that maps fractal images in arbitrary dimensions onto normalized, signed, integer sequences such that the correspondence is one-to-one. Such a set of sequences can easily be ordered; thus, the fractals can be easily catalogued and the sequence can be conveniently reverted to the corresponding figure. Using signed, integer sequences, we describe the isometries of the fractals using signed permutations that considerably simplify the substitutions. Such a correspondence may introduce a multitude of new sequences in the Online Encyclopedia of Integer Sequences and convert existing series to new fractals.

1 Introduction

We provide a short overview of the systems used for describing fractals that are most similar to the way we are proposing. In the 1960s, Lindenmayer [15] designed a system to formally describe the development of simple organisms such as bacteria, later known as L-systems. Mathematicians have used L-systems to describe all kinds of fractals as sequences of tokens, mainly letters and mathematical signs. L-systems became prominent, as the sequences could be fed into computers to generate fractal images.

In the 1980s, Dekking [6] led a thorough mathematical treatment of the subject, for which he used sequences of letters, eventually with indices. Although Dekking handled a variety of fractals, and endomorphisms played a significant role in his theory, the frequent use of indexed letters complicated matters more than necessary or desirable. For example, in (4.9) he used two indices, each to indicate a special homomorphism. Not only was that hard to follow, it also led to a mistake in his start sequence.
Arndt [2] further extended Dekking’s results from a later study (see [7]). He used L-systems to investigate all fractal-generating sequences that obeyed specific criteria. He briefly oversaw different notations such as numbers for directions or turns.

Ventrella [20], who was not a mathematician but an artist drawn to the beauty of fractals, developed a method to describe what he refers in [21], as “the taxonomy of fractals.” He showed the substitution graphically and then transformed it into a matrix of +1 and −1. He also investigated fractals on the Gaussian and Eisenstein integers and discovered various new ones.

It is noteworthy that these three authors restricted their approaches almost to two dimensions.

We now compare the descriptions mentioned above for the famous fractal, i.e., the Hilbert curve, as depicted in the original drawing in Figure 1, published in 1891 [10].

Figure 1: Hilbert’s original drawings, the first three approximants of his curve.

- Dekking writes in [6, p. 91]

“Let $S = \{a, b, c, d\}$, let $\sigma$ be the automorphism of $S^*$ defined by $\sigma(a) = b, \sigma(b) = c, \sigma(c) = d, \sigma(d) = a$, and let $\tau$ be the reversal map of $S^*$ defined by $\tau(s) = s, \tau(VW) = \tau(W)\tau(V)$ for $V, W \subset S^*$. Let the endomorphism $\Theta$ of $S^*$ be defined by $a \rightarrow baad$ and $\sigma \tau \Theta = \Theta \sigma \tau$. “ Then, he defines $f : S^* \rightarrow \mathbb{R}^2$ by $f(a) := (1, 0) := -f(c), f(b) := (0, 1) := -f(d)$. It can be observed that he makes extensive use of homomorphisms. This leads to $abbcbaadbaadddd$ for the second approximant and for the third to $baadabbcabcbddcccb abbcbaadbaadddd abbcbaadbaadddd dcbceddadcdbabaaad$

(we inserted spaces between each of its quadrants).

- Arndt writes in [2, p. 9]
“for the Hilbert curve, a possible L-system with axiom $L$ and (non-constant) maps $L \mapsto +Rt -LtL -tR +, R \mapsto -Lt +RtR +tL -$ can be used (only $t$ corresponds to an edge).” Therefore, this leads to

$$+(-Lt+RtR +tL -)t - (+Rt -LtL -tR +)t(+Rt -LtL -tR +)-t(-Lt+RtR +tL -)+$$

and a much longer expression (211 tokens) for the third Hilbert approximant.

• Ventrella [21, chap. 5] constructed the Hilbert curve practically the same way as Dekking, thereby conjecturing that “every edge-replacement curve that permits monohedral tiling (i.e., all norms are identical) has an associated self-avoiding node-replacement curve, and the nodes correspond to the centers of the curves’ tiles.”

Therefore, this leaves us with two constructions, one with an L-system and another with Dekking’s system. Notice that both the sequences, as in the original Hilbert drawings, are such that the first approximant is not the beginning of the second, and the second is not the beginning of the third. However, in all three cases, the odd and even approximants start with the former one, the third with the first, and the fourth with the second.

We describe our construction of the Hilbert curve extensively in Section 3.3 and its higher-dimensional analogs in Section 3.7. We translate Dekking’s parameters $a, b, c, d$ to 1, 2, −1, −2, indicating the directions $(1, 0), (0, 1), (-1, 0), (0, -1)$. Therefore, Dekking’s automorphism $\sigma$ becomes our rotation $\mu$ over $\pi/4$, denoted by signed permutation (Section 2.2) $\mu = [2, -1]$, with $\mu^3(1) = \mu^2(2) = \mu(-1) = -2$. Furthermore, we define $\iota = \sigma^0$ to be the identity. The signed permutation $\tau = [2, 1]$ is the reflection across the line $y = x$, as we do not use Dekking’s reverse mapping $\tau$ in this example. Our construction implies that the substitution $T : H_k \mapsto H_{k+1}$ between two succeeding approximants is given by $H_{k+1} = T(H_k) = \left(\tau(H_k), 1, H_k, 2, H_k, -1, -\tau(H_k)\right)$ with $T\tau = \tau T$, and for $k \in \{1, -1, 2, -2\}$, we get $T(k) = k$, following Hilbert’s original drawing. We write $T(\iota) = (\tau, 1, \iota, 2, \iota, -1, -\tau)$. The accompanying sequences for the odd and even approximants are $\langle 2, 1, -2, 1, 2, -1, 2, 1, 2, -1, -2, -1, 2, -1, 2, \ldots \rangle$ and $\langle 1, 2, -1, 2, 2, 1, -2, 1, 2, 1, -2, -2, -1, 2, 1, \ldots \rangle$, respectively. These sequences are not in the Online Encyclopedia of Integer Sequences (OEIS), which however need not be stated further. Only those fractal sequences that occur in the OEIS will be stated. Thus, we specify the main objective of this study: to describe a system in which fractal curves in all dimensions are represented as signed number sequences.

Remark 1. In this article, we use one numbering for figures, definitions, theorems, remarks, lemma’s, observations, and examples, and another one for (sub)sections. But there is no rule without exception: once in a while, we will use a framed box of text that we think is important enough to call this block an Intermezzo.
2 Axiomatic description

2.1 Digiset, sequences, substitution, and normalized

This section describes our application of the sequence theory. Instead of the usual alphabetical sequences, we use *signed integer sequences*. See [1] for a thorough treatment of this subject.

**Definition 2.** A digiset\(^1\) of size \(n\), for some \(n \in \mathbb{N}\), \(n > 0\), is \(\Delta_n = \{k | k \in \mathbb{Z}; 0 < |k| \leq n\}\), a subset of the integers, abbreviated to \(\Delta_n = \{±1, ±2, \ldots, ±n\}\). An infinite digiset has \(\Delta_\infty = \mathbb{Z} \setminus \{0\}\). For completeness, if we restrict ourselves to positive integers, the digiset is denoted by \(\oplus\Delta_n\).

**Definition 3.** A signed integer sequence, also known as a *sequence*, is a countable, ordered multiset with elements taken from a digiset \(\Delta_n\). Let \(\Delta_n^*\) constitute the set of all finite sequences, where each sequence is denoted by \(S = \langle s_1, s_2, \ldots \rangle\) with \(s_k \in \Delta_n\), i.e., with commas and angle brackets. In this notation, we represent the empty sequence by \(\langle \rangle = \epsilon\). We denote the length of a sequence by \(\|S\|\), indicating its number of elements. For \(k \geq 0\), let \(\Delta_n^k = \{S \in \Delta_n^*; \|S\| = k\}\) be the set of sequences with length \(k\), then \(\Delta_n^* = \bigcup_{k \geq 0} \Delta_n^k\).

For clarity, we use commas to separate items within a sequence because the integers may contain a minus sign and more than one digit.

**Observation 4.** The set \(\Delta_n^*\) is a *monoid* with concatenation (of sequences) as multiplication, denoted by a comma, and \(\langle \rangle = \epsilon\) as the identity element.

**Definition 5.** A mapping \(\phi: \Delta_n^* \rightarrow \Delta_n^*\) such that \(\phi(S, T) = (\phi(S), \phi(T))\), where \(S, T \in \Delta_n^*\) is called a *homomorphism*, or *morphism* for short.

**Definition 6.** The *reverse*, denoted by \(\mathcal{R}\), is a peculiar mapping \(\mathcal{R}: \Delta_n^* \rightarrow \Delta_n^*\) because this mapping is an *anti-homomorphism*, as we define \(\mathcal{R}(S, T) = (\mathcal{R}(T), \mathcal{R}(S))\) for \(S, T \in \Delta_n^*\), and \(\mathcal{R}(x) = \langle x \rangle\) for \(x \in \Delta_n\).

There is a natural embedding of \(\Delta_n\) into \(\Delta_n^*\) by the injection \(x \mapsto \langle x \rangle\); therefore, we identify \(\Delta_n\) with \(\Delta_n^1\).

If \(\alpha: \Delta_n \rightarrow \Delta_n^*\) is a mapping, then there is a natural extension to \(\alpha^*: \Delta_n^* \rightarrow \Delta_n^*\) by \(\alpha^*\langle s_1, s_2, \ldots, s_k \rangle = \langle \alpha(s_1), \alpha(s_2), \ldots, \alpha(s_k) \rangle\). We will, however, use \(\alpha\) instead of \(\alpha^*\). A bijection on \(\Delta_n\) extends to a bijection on \(\Delta_n^*\).

For a mapping \(\alpha: \Delta_n \rightarrow \Delta_n\), its extension \(\alpha: \Delta_n^* \rightarrow \Delta_n^*\) is *length-preserving*, i.e., \(\|\alpha(S)\| = \|S\|\).

The identity on \(\Delta_n\) we denote by \(\iota\). The *negation*, a rather important mapping, is indicated by \(-\iota\) or in expressions only by its sign \(\ominus\).

We use the *inverse* for concatenation to transform \(\Delta_n^*\) from a monoid to a *free group*.

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\(^1\) In sequence theory, a digiset is called an *alphabet*, and the sequences are called *words*.

\(^2\) Contrary to \(|S|\), which denotes the *absolute sequence* \(|S| = \langle |s_1|, |s_2|, \ldots \rangle\).
Definition 7. The inverse on $\Delta_n^*$ for concatenation, denoted by $S^{-1}$ for $S \in \Delta_n^*$, is defined as $(S, T)^{-1} = (T^{-1}, S^{-1})$ for $S, T \in \Delta_n^*$, and $(x)^{-1} = (-x)$ for $x \in \Delta_n$.

Observation 8. We observe $S^{-1} = -R(S)$ for $S \in \Delta_n^*$, where $R$ is the reverse from Definition 6. Hence, we use $-R$ as its inverse. The inverse is an anti-morphism because it is a combination of two mappings of which the reverse is an anti-homomorphism, and of which the negation is not.

Definition 9. A substitution is a morphism $T : \Delta_n^* \rightarrow \Delta_n^*$ that is expansive, i.e., for all $x \in \Delta_n$, we have $\|T(x)\| \geq 2$. Also, for every length-preserving morphism $\sigma$ and substitution $T$, we have $T\sigma = \sigma T$.

We call a sequence normalized if it shows a positive number from the digiset before its negation, and the positive numbers in the sequence occur in ascending order. More precisely:

Definition 10. For a sequence $S = \langle s_1, s_2, \ldots, \rangle$ and for $0 < k \in \Delta_n$, let $n_k$ be the lowest index $i$ such that $s_i = k$, and $n_k = \infty$ if $k \notin S$. This sequence is normalized if $|s_j| < k$ for all $1 \leq j < n_k$ and for all $0 < k$. Therefore, for a normalized sequence, we have $n_1 = 1$, and $n_{k-1} < n_k$ for all $k > 1, k \in \Delta_n$.

A finite sequence can be normalized in two ways because its reverse can also be normalized, for instance, $(1, 2, 1, 1)$. In this case, we prefer the smaller of the two. Therefore, $(1, 1, 2, 1)$ will be the minimal normalized version. See Definition 33 on page 45 for how we order sequences.

2.2 Signed permutations

Definition 11. A signed permutation is a bijection $\sigma$ on a digiset $\Delta_n$ with the property that $\sigma(-k) = -\sigma(k)$ for $k \in \Delta_n$.

Following Knuth in [12], we use perm to denote a signed permutation. Note that a signed permutation is a length-preserving morphism on $\Delta_n^*$ and commutes with a substitution.

In Cauchy’s two-line notation\(^3\), a permutation looks like

\[
\begin{array}{cccc}
x & y & z & \cdots \\
a & b & c & \cdots 
\end{array}
\]

where the first row contains elements from the domain, and the second row contains their respective images. As a signed permutation satisfies the property $\sigma(-x) = -\sigma(x)$, we use the one-line notation $[\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(n)]$, by which $\sigma$ is completely determined.\(^4\)

Examples are the identity $\iota = [1, 2, 3, \ldots, n]$ and its negation $-\iota = [-1, -2, \ldots, -n]$. In mathematics, a function operates from right to left. For $[-2, 4, -1, 3]\langle -3 \rangle$, the one-edge sequence $\langle -3 \rangle$ is transformed into $\langle 1 \rangle$, the negative of the third value within the signed

\(^3\) [23, p. 94], “Cauchy used his permutation notation – in which the arrangements are written one below the other and both are enclosed in parentheses – for the first time in 1815.”

\(^4\) Björner et al. [3, p. 246] called this a window notation. Section (8.1) of that work is devoted to the properties of signed permutations and their group.
the sign of a sequence. Therefore, if we must determine the product of two signed permutations, say, \( \sigma = [\sigma(1), \sigma(2), \ldots] \) and \( \tau = [\tau(1), \tau(2), \ldots] \), then
\[
\sigma \tau = \left[ \text{sign}(\tau(1)) \ast \sigma(\tau(1)), \text{sign}(\tau(2)) \ast \sigma(\tau(2)), \ldots \right].
\]

For instance, \([-2, 4, -1, 3][3, -1, 4, -2] = [-1, -(2), 3, (-4)] = [-1, 2, 3, -4]\).

We refer to the signed permutation \( \mu = [2, 3, 4, \ldots, n, -1] \) as the minimal rotation. If the digset only has positive integers, the minimal rotation is \( \mu = [2, 3, 4, \ldots, n, 1] \). Generally, a signed permutation is defined as a signed binary matrix, i.e., with elements 0, 1, and \(-1\). A signed permutation is defined as a signed binary matrix, i.e., with elements 0, 1, and \(-1\). As \( \sigma \) is a signed permutation, the corresponding matrix, i.e.,
\[
\sigma = \begin{pmatrix}
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

Therefore, in general, for the signed permutation \( \sigma = [\sigma(1), \sigma(2), \ldots] \), its corresponding matrix \( \omega(i, j) \) has \( \omega(\sigma(k), k) = \text{sign}(\sigma(k)) \) and \( \omega(m, k) = 0 \) for \( m \neq \sigma(k) \).

From the matrix representation of signed permutations, we know the determinant equals \( \pm 1 \). The permutations with positive determinants are rotations; those with negative determinants are rotation-reflections, i.e., the combination of rotation about an axis and reflection in a plane perpendicular to that axis ([19, p.84]). We now investigate whether a signed permutation contains a reflection by observing its one-line notation.

**Definition 12.** The parity of a signed permutation is equal to the determinant of the corresponding matrix, i.e., \(-1\) if the mapping is a rotation-reflection and \(+1\) if the mapping is a rotation.

Let \( \sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)] \) be a signed permutation. Then, we define \( \text{neg}(\sigma) \) and \( \text{inv}(\sigma) \) by \( \text{neg}(\sigma) = \left| \{1 \leq i \leq n : \sigma(i) < 0\} \right| \) and \( \text{inv}(\sigma) = \left| \{(i, j) : 1 \leq i < j \leq n, |\sigma(i)| > |\sigma(j)|\} \right| \).

**Theorem 13.** \( \text{parity}(\sigma) = -1^{\text{neg}(\sigma) + \text{inv}(\sigma)}. \)

**Proof.** We know that the number of inversions, \( \text{inv}(\sigma) \), is equal to modulo 2, the number of transpositions of two values in the one-line notation. Such a transposition, say \( \sigma(i) \) and \( \sigma(j) \), corresponds to swapping the two columns \( i \) and \( j \) in the corresponding matrix, which leads to an additional factor of \(-1\) in the determinant. Multiplying a value in the one-line notation by \(-1\) equals multiplying the corresponding column in the matrix with \(-1\). Therefore, if we add the inversions and the minus signs in the one-line notation, we get the exponent of \(-1\) in the determinant.

A sequence is not generally normalized (Definition 10), but we can easily construct a signed permutation to transform such an array into an isomorphic, normalized sequence. For this, we need the first occurrence \( |\Delta| = k \) in the series for each \( 0 < k \in \Delta_n \), along with the sign \( |\Delta| \) of that first occurrence. These first occurrences for all \( 0 < k \in \Delta_n \), in that order, constitute the inverse of the characteristic permutation of the original sequence.
Definition 14. Given a sequence $S = (s_1, s_2, \ldots)$ over $\Delta_n$, define $i_k$ for $1 \leq k \leq n$ such that $i_1 = 1$ and for all $k$ one has $\{ |s_j|; 1 \leq j \leq i_k \} = \{ |s_{i_1}|, |s_{i_2}|, \ldots, |s_{i_k}| \}$. Then, the characteristic permutation $\sigma$ is determined by $\sigma = \begin{bmatrix} s_{i_1} & s_{i_2} & \cdots & s_{i_n} \end{bmatrix}$, or $\sigma^{-1} = [s_{i_1}, s_{i_2}, \ldots, s_n]$.  

Definition 15. A constant substitution $\gamma^* : \Delta_n^* \to \Delta_n^*$ is a mapping such that for all $S \in \Delta_n^*$, we have $\gamma(S) = \langle c \rangle$. A constant morphism $\gamma : \Delta_n \to \Delta_n$ is such that for all $x \in \Delta_n$, we have $\gamma(x) = c$.

Henceforth, we use $\Delta$ instead of $\Delta_n$, if $n$ is evident from the context.

2.3 Fractals and approximants

Fractals are difficult to define. Mandelbrot, who coined the term in [16], described them as “(...) a rough or fragmented geometric shape that can be split into parts, each of which is - at least approximately - a reduced-size copy of the whole.” Falconer [8] stated “My personal feeling is that the definition of a ‘fractal’ should be regarded in the same way as a biologist regards the definition of ‘life’.” He refers to a fractal as an object with five different, explicit properties, not all of which he describes with sufficient precision. Finally, Kimberling [11] stated “A search of [17] for ‘fractal sequence’ reveals that in recent years, different kinds of sequences have been called ‘fractal’ and what many of them have in common is that they are SCSs.” (= Self-Contained Sequences).

We define a fractal as the limit of finite sequences of increasing length, analogous to an infinite sequence.

Definition 16. A fractal is the limit of an ordered, infinite set of sequences, each of finite length larger than the length of the previous sequence, which are called approximants. Each one is a concatenation of the images of the former approximants, usually only the previous one. Therefore, we have a countable set of finite sequences $\{ S_1, S_2, \ldots \}$ and a set of morphisms $\{ \alpha_{f(k,i)} : \Delta^* \to \Delta^*; i = 1, 2, \ldots, r; f(k,i) \in \mathbb{N} \}$, such that $S_{k+1} = T(S_k) = (\alpha_{f(k,1)}(S_{g(k,1)}), \alpha_{f(k,2)}(S_{g(k,2)}), \ldots, \alpha_{f(k,r)}(S_{g(k,r)}))$, with $g(k,j) \in \{ 1, 2, \ldots, k \}$; $j = 1, 2, \ldots, r$, then the fractal is $S = \lim_{k \to \infty} S_k$.

Notice that $\Delta^\mathbb{N}$ is the set of (right-)infinite sequences. We generally identify a fractal using one of its approximants.

A fractal is called self-similar if $f(k,j) = j$ and $g(k,j) = k$ for all $k > 0$ and all $j = 1, 2, \ldots, r$. Henceforth, we assume all our curves to be self-similar. Furthermore, without loss of generality, we assume the identity $\alpha_1 = \iota$ such that $S = (S_k, \ldots)$ for $k > 0$.

A substitution $T : \Delta^* \to \Delta^*$ implies a dual substitution $^mT : \Omega \to \Omega$, by $^mT(\sigma) = T \sigma$ for $\sigma \in \Omega$, which is the group of morphisms on $\Delta^*$. For a self-similar fractal, we write $T(S_k) = (S_k, \alpha_2(S_k), \ldots, \alpha_r(S_k))$ and denote this by $^mT = [\iota, \alpha_2, \ldots, \alpha_r]$. 

7
2.4 Grid, direction, and isometry

Definition 17. Let \( \{ u(1), u(2), \ldots, u(n) \} \subset \mathbb{R}^d \) be a set of vectors\(^5\), where \( d \leq n \) is a dimension such that the vectors span \( \mathbb{R}^d \). Furthermore, we have \( u(j) \neq \alpha \ast u(i) \) for all \( \alpha \in \mathbb{R} \) and all \( 1 \leq i \neq j \leq n \), i.e., every pair of vectors is independent. The set \( \Gamma_n = \left\{ \sum_{i=1}^{n} k_i \ast u(i) \right\} \) with \( k_i \in \mathbb{R} \) and \( |\{ k_i \notin \mathbb{Z} \}| \leq 1 \), is called a grid. A grid has \( 2n \) directions, i.e., its generators and their negations \( \{ \pm u(k) \} \). Each direction with its opposite forms a dimension.

The generators of a grid \( \Gamma_n \) relate to a digiset \( \Delta_n \) by the mapping \( u(k) \mapsto \langle k \rangle \).

Using the relation between the generators of the grid and the digiset, we have an association between number sequences and fractal images, i.e., subsets of the grid, as the title of this study suggests.

Example 18. The most crucial grid we encounter is the cubic grid \( \mathbb{Z}^d \) for \( 1 \leq d \), with the \( 2d \) directions \( \langle \pm 1 \rangle, \langle \pm 2 \rangle, \ldots, \langle \pm d \rangle \).

\[ \langle 1 \rangle \quad \langle 2 \rangle \quad \langle 3 \rangle \quad \langle 4 \rangle \]

Figure 2: Triangular and square-diagonal grids, with directions indicated by integers.

The triangular grid shown in Figure 2 is also essential. Notice that a grid can have more generators than its dimension, as observed in the triangular and square-diagonal grids. We determine a grid using a matrix in which the columns represent the generators in the appropriate order. The two grids in Figure 2 are given by \( \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \).

Contrary to a point lattice, which consists of only vertices, a grid is a graph with vertices and edges. To the best of our knowledge, in higher dimensions, only the cubic grid based on the cubic lattice is significant.

Example 19. The following grids are associated with the triangular and square-diagonal grids: the tri-hexagonal grid, hexagonal (honeycomb) grid, and truncated square grid, see Figure 3.

\(^5\) We identify points, vertices, and vectors in \( \mathbb{R}^d \).
Figure 3: Tri-hexagonal, Honeycomb, and Truncated square grids.

In these grids, we use directions from the triangular and square-diagonal grids, but with restrictions, as displayed in Figure 4.

Figure 4: Successor directions in the tri-hexagonal, honeycomb, and truncated square grids. The vertices indicate the incoming directions.

In Figure 4, the edges that can follow a specific direction in the central polygon are shown. In the tri-hexagonal grid, three directions correspond to one direction. By contrast, there are two possible directions at a vertex of the honeycomb or the truncated square grid.

Generally, fractal is represented as a geometrical figure, where the approximants undergo shrinking, such as \( F_{n+1} = \varphi \ast T(F_n) \) with \( 0 < \varphi < 1 \) and an (expanding) substitution \( T \). Therefore, the fractal is defined as \( F = \lim_{n \to \infty} F_n \). Our infinite sequence, with the integers interpreted as directions in a grid, becomes a geometrical object of infinite size, where the size or length of each approximant is less than that of the next one.

In our approach, the approximants of a fractal are paths, i.e., directed graphs with all vertices of degree two, except for the first and the last one, which have degree one, respectively.

Therefore, an entry (point) is the first vertex on the path, denoted by \( \circ \); an exit (point) is the last vertex on the path, denoted by \( \bullet \).
A fractal, being an infinite limit of finite paths, has a single entry. Therefore, we refer to a fractal as \textit{curve}, and replace the term “approximant” with \textit{k-curve} or \textit{curve of level \( k \)}.

We define a vertex as the 0-curve, or a curve without edges \([1]\), denoted by \( \epsilon = \langle \rangle \). A majority of the curves are generated from a substitution that uses the 1-curve, i.e., the first approximant, as the start.

**Definition 20.** The \textbf{orientation} of a \( k \)-curve is the vector directly from entry to exit.

**Remark 21.** A sequence represents subsequent edges, indicating graph-wise a difference between a single vertex and two successive edges in opposite directions. Hence, we do not use annihilation, i.e., \textit{not} identify \( \langle \rangle \) with \( (S, -R(S)) \) or \( \langle a, -a \rangle \).

**Definition 22.** An \textbf{isometry} is a distance-preserving transformation on the grid. If necessary, we denote two isometric sets by \( A \cong B \).

Notice that an isometry also \textit{preserves the angles} between vectors.

Whether a signed permutation is an isometry depends on the difference in the length of the generators, for instance, the square-diagonal grid in Figure 2.

The inverse \( -R \) of a (finite) path is a mapping from the path to itself, with entry and exit swapped, the ordered multiset of edges reversed, and the direction of each edge reversed. The reverse \( R \) only reverses the multiset of edges; hence, the entry and exit remain fixed. Therefore, if \( S = \langle e_1, e_2, \ldots, e_n \rangle \) are the edges of a path, then \( -R(S) = \langle -e_n, \ldots, -e_2, -e_1 \rangle \) and \( R(S) = \langle e_n, \ldots, e_2, e_1 \rangle \). See Intermezzo 1 on page 13.

### 2.5 Representing fractals uniquely

One of our objectives is to set up an encyclopedia of normalized fractals as an independent set. However, such an encyclopedia could partially be considered as a subset of OEIS. We order this list of normalized sequences according to Definition 33 on page 45.

We give the sequence, its index from OEIS, and describe the digiset, start sequence, substitution, and grid with its generators, and finally the figure of the geometric fractal, similar to the example below of the first sequence (Appendix B.3).

**B.3 Dekking’s Flowsnake**

**sequence:** \( \langle 1, 1, 2, -1, 2, 1, 2, -1, -1, 2, 1, 1, 1, 2, 1, -2, -2, -1, -2, -2, 1, 2, 1, -2, -2, 1, \ldots \rangle \)

**in OEIS:** Not present (18-01-2022)

**digiset:** \( \Delta = \{1, 2\} \)

**start sequence:** \( \langle 1 \rangle \)

**substitution:** \( T(\tau) = (t, t, \mu \tau, -\tau, \mu, t, \mu \tau, -\tau, -t, \mu \tau, t, t, \tau, \mu, \tau, -\mu, -\mu, -\tau, -\mu, -\mu \tau, \tau, \mu, t, -\mu \tau, -\mu \tau) \), where \( \tau = [1, -2] \) and, as usual, \( \mu = [2, -1] \)

**grid:** The square, plane grid.
generators: (1, 0); (1, 0)

**figure:** The 1-curve (first row left), two 2-curves (anti diagonal) and the 3-curve (last row right).

![Figure 5: First row is drawings of the 1st and 2nd approximants. Second row is the 2nd and 3rd approximants, separating the space into two parts, black and white, both with tree-like structures.](image)

3 Examples

The following examples highlight different aspects of representing fractals using signed sequences. We also investigate the relationship between sequences and their geometric pictures.
3.1 Ventrella’s Box 4

Suppose we have a fractal whose 1-curve is identical to the first image in Figure 6, under “identity.” If this is the image under the substitution of a horizontal unit line segment, then we investigate whether the images of the other line segments are horizontal or vertical. We can choose the transformations of the first curve, with similar entry and exit, as in the rest of Figure 6.

| \( \iota = [1, 2] \) | \( \tau_y = [1, -2] \) | \( \mathcal{R} \) | \( \mathcal{R}\tau_y = \mathcal{R}[1, -2] \) |
|----------------|----------------|---------|----------------|
| \( \langle 1, 2, 1, -2 \rangle \) | \( \langle 1, -2, 1, 2 \rangle \) | \( \langle -2, 1, 2, 1 \rangle \) | \( \langle 2, 1, -2, 1 \rangle \) |

Figure 6: Different directions of a 1-curve.

Ventrella suggested a flag-like arrow to indicate different oriented edges, which led to various images of those edges. We placed the flag at the center, and created the drawings in Figure 7, which are consistent with those in Figure 6.

Figure 7: Ventrella’s flags shifted to the center and its isometries.

In Figure 8, we observe the same figures as in Figure 6 and Figure 7, with their corresponding transformations, but we swapped the directions, just like the entry and exit. This completes the list of all the isometries of the original figure, except the (infinite number of) rotations.
\[ R \] inverse  
\[ R_{\tau x} = R[-1, 2] \] rotation over \( \pi \)  
\[ -\iota = [-1, -2] \] horizontal reflect.

| \(-R\) | \(R_{\tau x} = R[-1, 2]\) | \(-\iota = [-1, -2]\) | \(\tau_x = [-1, 2]\) |
|------|------------------|------------------|------------------|
| \(\langle 2, -1, -2, -1\rangle\) | \(\langle -2, -1, 2, -1\rangle\) | \(\langle -1, -2, -1, 2\rangle\) | \(\langle -1, 2, -1, -2\rangle\) |

Figure 8: Similar drawings as Figure 6 and Figure 7, other isometries swapping entry and exit.

Intermezzo 1
There is an issue in terms of the difference between reverse, negation, and rotation over \( \pi \), since we can swap the order of the edges, swap the direction of each edge, in which case the entry and exit are swapped as well, or both. In figure Figure 9, we illustrate different ways to revert a directed graph with entry and exit.

Both \( R =\) reverse and \( -\iota =\) negate produce a rotation over \( \pi \). The former preserves, whereas the latter swaps directions, including those of entry and exit.

\(-R =\) inverse, which annihilates the original by swapping everything, the order of edges, the directions of edges, and the entry and exit. When a rotated \( k \)-curve is used in the build-up of the next version, the order of the edges can be reversed without swapping the entry and exit, but one can also switch the edges themselves.

We refrain using the phrase “rotate over \( \pi \),” and use one of the different formulas \(-\iota\) or \( R \) for the sake of clarity.

We can decorate the original 1-curve by choosing for each edge, one of the flags from Figure 7 or Figure 8, to determine in which image of the 1-curve this edge can be substituted. The (open) question remains: how many different, i.e., non-isometric curves can be constructed using only normalized curves?
Ventrella [21] studied a fractal called “Box 4,” clearly indicated by his flags, conform the center drawing of the first row of Figure 10, where its 2-curve is at the right-hand side. We added the first two columns, “integers” and “transformations,” using his notation, and then a column using our notation, with \( \mathcal{R}, \tau_y, \) and \( \mu \), as in Figure 6, Figure 7, and Figure 8. In the second row, using our more informative flags, we transformed his curves by normalizing and extending, such that \( T(S_k) = S_{k+1} = (S_k, S'_k) \) for some sequence \( S'_k \).

We can perform this transformation in various ways. First, we can take Ventrella’s substitution \( V(\iota) = (\tau_d \mathcal{R}, \iota, -\mu \mathcal{R}, \tau_y) \)\(^6\), and apply the transformations \( \mathcal{R} \) and \( \tau_{-d} \), which produces our substitution \( T(S_k) = (S_k, \tau_{-d} \mathcal{R}(S_k), \tau_y(S_k), \mu \mathcal{R}(S_k)) \), abbreviated to \( T(\iota) = (\iota, \tau_{-d} \mathcal{R}, \tau_y, \mu \mathcal{R}) \).

Second, we can use the 2-curve (right-hand side of the first row in Figure 10) to determine the four transformations of the 1-curve using which we construct the 2-curve. First, we normalize Ventrella’s 2-curve by applying \( \mathcal{R} \), and setting \( \langle 1, 2, 1, -2 \rangle \) for the 1-curve and \( \langle 1, 2, 1, -2, 1, -2, 1, 2, -1, 2 \rangle \) for the 2-curve (right-hand side of the second row in Figure 10). We partition the 2-curve into discrete curves of length four and determine their isometric images of the 1-curve, the first one being \( \iota \). The other images are \( \langle 1, -2, 1, -2 \rangle = -\mu \langle 2, 1, -2, 1 \rangle = -\mu \tau_y \langle -2, 1, 2, 1 \rangle = \tau_{-d} \mathcal{R} \langle 1, 2, 1, -2 \rangle \), then \( \langle 1, -2, 1, -2 \rangle = \tau_y \langle 1, 2, 1, -2 \rangle \), and finally \( \langle 1, 2, -1, 2 \rangle = \mu \mathcal{R} \langle 1, 2, 1, -2 \rangle \), which brings the

\(^6\) In Appendix A, we described the group of isometries of the square grid with the signed permutations like \( \tau_d = [2, 1] \), the diagonal reflection, and \( \tau_{-d} = [-2, -1] \), the anti-diagonal reflection.
substitution of the isometries into $T(\iota) = (\iota, \tau_d \mathcal{R}, \tau_y, \mu \mathcal{R})$, which is the same as what we obtained before.

Finally, we construct higher-level approximants for Box 4 of Ventrella to find an easy substitution. This construction is done using our normalized version, starting with $\langle 1, 2, 1, -2 \rangle$; then, the final sequence becomes

$\langle 1, 2, 1, -2, 1, -2, -1, -2, 1, -2, 1, 2, -1, 2, 1, 2, -1, 2, -1, -2, -1, 2, -1, 2 \ldots \rangle$.

If we group this sequence into disjoint groups of four elements and match them with the corresponding first items of the sequence, we get the following substitution

$$T' = \left\{ \begin{array}{l}
1 \rightarrow \langle 1, 2, 1', -2' \rangle \\
1' \rightarrow \langle 1', -2', 1, 2 \rangle \\
2 \rightarrow \langle 1', -2', -1, -2 \rangle \\
2' \rightarrow \langle -1, -2, 1', -2' \rangle
\end{array} \right.$$

For this, we need $\{\pm 1, \pm 2\}$, and $\{\pm 1', \pm 2'\}$, where both $\langle x \rangle$ and $\langle x' \rangle$ indicate the same direction. As usual, $T'(-x) = -T'(x)$ for $x \in \{\pm 1, \pm 2, \pm 1', \pm 2'\}$. To make this substitution work, we need a perm $\sigma$ such that if $\sigma(1) = 2$, we have $\sigma(1, 2, 1', -2') = (1', -2', -1, -2)$. Therefore, we define $\tau_{-d} = \begin{bmatrix} 1 & 2 & 1' & 2' \\
-2 & -1 & -2' & -1 \end{bmatrix}$, then $\tau_{-d} \mathcal{R} \langle 1, 2, 1', -2' \rangle = \tau_{-d} \langle -2', 1', 2, 1 \rangle = \langle 1', -2', -1, -2 \rangle$. We further define the vertical reflection $\tau_y' = \begin{bmatrix} 1 & 2 & 1' & 2' \\
1' & -2' & 1 & -2 \end{bmatrix}$, the minimal rotation $\mu' = \begin{bmatrix} 1 & 2 & 1' & 2' \\
2' & -1' & 2 & -1 \end{bmatrix}$, and achieve $T' = (\iota, \tau_{-d} \mathcal{R}, \tau_y', \mu' \mathcal{R})$ or

$$T'(1) = \langle 1, 2, 1', -2' \rangle = (\iota(1), \tau_{-d} \mathcal{R}(1), \tau_y'(1), \mu' \mathcal{R}(1)).$$

Finally, we introduce an obstacle that did not occur in Ventrella’s approach. We not only want our sequences to be normalized but also to be “extending” (p. 14). If we apply our substitution $T(\iota) = (\iota, \tau_{-d} \mathcal{R}, \tau_y, \mu \mathcal{R})$ to the 0-curve, which is edge $\langle 1 \rangle$, we obtain for a 1-curve $\langle 1, -2, 1, 2 \rangle$, which is not the start of the 2-curve. Fortunately, we can resolve this by applying an additional $\tau_y$ after the substitution to get a $k$-curve, where $k$ is odd. Therefore, we get $S_k = \tau_y^k T(S_{k-1})$ because $\tau_y^2 = \iota$. Figure 11 shows a few approximations of the normalized and extending Box4-fractal.
Figure 11: \(k\)-curves for the Box $4$-fractal with $k = 3, 4, 5$, all with rounded corners.

Remark 23. The advantage of a transformation substitution over a number substitution is that, we can observe the different transformations involved from one approximant to the next.

A second observation is that the group of signed permutations in \(n\) dimensions is the hyper-octahedral group of order \((2n)!\)!! = \(2^n n!\). In two dimensions, this group is generated, among others, using minimal rotation \(\mu = [2, -1]\) and vertical reflection \(\tau_y = [1, -2]\), which we use for the square grid throughout this study. See Appendix A for the dihedral group D4 of transformations of the square grid.

### 3.2 Ventrella’s V1 Dragon

This example uses the square-diagonal grid, which is peculiar because not all directions have the same lengths; refer to the grid on the left-hand side in Figure 12. On comparing the two grids, the \(8^{th}\) roots of unity span the right one, where all directions have equal lengths.

![Square-diagonal and 8th-roots grids](image)

Figure 12: Square-diagonal and \(8^{th}\)-roots grids.

The minimal rotation in both grids over \(\pi/4\) is given by \(\mu = [2, 3, 4, -1]\), whereas the
vertical reflection is \( \tau_y = [1, -4, -3, -2] \). As the directions are mutually dependent, the two signed permutations do not generate the entire hyper-octahedral group of four dimensions but generate the symmetry group of the octagon, i.e., the dihedral group \( D_8 \).

We consider another example of Ventrella [21], called the “V1 Dragon.” We slightly altered his sample to make it normalized and extending. In Figure 13, we observe its 1- en 2-curves.

![Figure 13: 1- and 2-curves of Ventrella’s V1 Dragon.](image)

Similar to the previous Section 3.1, we can “read” the transformations involved from the first picture by using only reverse \( R \) and rotation \( \mu \) (and the identity \( \iota \)). Therefore, \( T(\iota) = (\iota, R\mu^2, \sqrt{2} \ast \mu^3) \). Here, we notice an important difference with previous fractals, which only had edges of length one. Apart from the transformations \( \mu \) and \( R \), we multiplied the length with \( \sqrt{2} \).

![Figure 14: Same 1- and 2-curves on the 8th-root grid.](image)

As the length of a fractal sequence cannot be determined from the directions in its representation, we introduce a separate sequence of lengths using its length substitution, which, completely independent from the directional substitution, determines the geometrical fractal from both series.
If we draw the same sequence in the 8th-roots grid, we have almost the same fractal, but with edges of length one, as shown in Figure 14. As this grid is dense in $\mathbb{R}^2$, the geometric picture is less pleasing than the one in the square-diagonal grid.

In Figure 15, we see the 4-curves of Ventrella’s V1 dragon, with the upper one on the 8th-roots grid and the lower one on the square-diagonal grid. Evidently, the lower one is larger because some of the edges have grown in length, and the vertices of this curve are on the lattice $\mathbb{Z}^2$. By contrast, the upper curve has edges that partially overlap which, to the best of our knowledge, is unseen in geometrical fractals. A fractal curve only shares vertices with itself, or edges, or neither of the two.

As we are more interested in fractals as number sequences than geometrical figures, we have the same sequence for directions in both grids, and a length sequence in the case of the square-diagonal grid. Therefore, we first split the substitution $T$ into two: $\tilde{d}T$ for the
directions and \( tT \) for the lengths, which lead to the sequences \( dS \) and \( tS \), respectively. Thus, we get
\[
T(t) = (\mu, R\mu^2, \sqrt{2} \ast \mu^3) = \begin{cases}
    dT(t) = (\mu, R\mu^2, \mu^3) \\
    iT(t) = (\mu, R, \sqrt{2})
\end{cases}
\]
Starting with \( S_0 = \langle 1 \rangle \) for both \( x = d, l \), this leads to the sequences
\[
dS = \langle 1, 3, 4, -2, -1, 3, 4, -2, -3, 1, -4, -2, -1, -3, -4, -2, -1, 3, 4, -2, -3, 1, -4, -2, \ldots \rangle
\]
and
\[
tS = \langle 1, 1, \sqrt{2}, \sqrt{2}, 1, 1, \sqrt{2}, \sqrt{2}, 2, 2, \sqrt{2}, \sqrt{2}, 1, 1, \sqrt{2}, \sqrt{2}, 2, 2, \sqrt{2}, \sqrt{2}, 2, 2, \ldots \rangle.
\]
The latter can be simplified by taking the \( \sqrt{2} \) logarithm, which gives
\[
\log_{\sqrt{2}}(tS) = \langle 0, 0, 1, 1, 0, 1, 1, 2, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 2, 1, 1, 2, 2, 3, 3, 2, 2, 1, 1, 2, 2, \ldots \rangle,
\]
which is the double (i.e., \( \langle x, x \rangle \) versus \( \langle x \rangle \)) of A062756 in [17].

\( dS \) is not normalized because the numbers 3 and 4 are used before 2, as observed from the first directions with different axes in our definition of V1 dragon, which are 1, 3, 4, -2. Refer to Figure 13 or 14. For this case, we have the characteristic permutation (c.f. Definition 14), which happens to be \([1, -4, 2, 3]\) and brings the sequence back to:
\[
dS' = \langle 1, 2, 3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -1, -2, -3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -2, 1, -4, 3 \ldots \rangle.
\]
Suppose we want this normalized sequence to represent the isometric image of V1 dragon. In this case, we must adjust the numbering of the directions as indicated by the characteristic permutation, which is the analog of a base transformation in a vector space.

Figure 16: Two grids are numbered differently than in Figure 12.

Suppose we change the numbering of the directions, such that 1, 3, 4, -2 becomes 1, 2, 3, 4, as in Figure 16, i.e., \([1, 3, 4, -2]^{-1} = [1, -4, 2, 3] = \sigma\). The transformations, derived from Figure 13, are the same; however, the minimal rotation is now represented by \( \mu' = [-4, 3, -1, -2] \), which is equal to \( \mu' = \sigma \mu \sigma^{-1} \), where \( \mu = [2, 3, 4, -1] \) is the minimal rotation in the original directions, and \( \sigma \) is the “base transformation.” Therefore, the new substitutions become \( dT'(t) = (\mu, R\mu'^2, \mu'^3) \) and \( iT' = iT \), and we get the same geometric pictures as Figure 15.
3.3 Hilbert’s original curve

In this section, we study one of the oldest and most famous fractals, the curve Hilbert presented in his two-page paper [10], with the drawing he depicted (Figure 17) as the primary explanation.

Figure 17: Hilbert’s original drawings, and his first, second, and third approximants.

We propose a new and thorough way of generating the fractal sequence that we discussed in the introduction. The Hilbert approximants lie on the square grid, as shown in Figure 18.

From Hilbert’s drawings, we observe that isometric images of the \(k\)-curves are present in each of the four corners of the next \(k+1\)-curve, and the only edges that connect these images are the edges of the 1-curve.

On further inspection, we observe the contrary. An isometric image of the 1-curve replaces each vertex of a \(k\)-curve, and the edges of the \(k\)-curve properly connect these images.

Therefore, we have the formula for the first approach and for some isometries \(\sigma_k\); \(k = 1, 2, 3, 4\):

\[
H(k + 1) = (\sigma_1(H(k)), 2, \sigma_2(H(k)), 1, \sigma_3(H(k)), -2, \sigma_4(H(k))).
\]

For the second approach, with \(H(k) = (s_1, s_2, \ldots, s_{4^{k-1}})\) as the edge-representation, \(H(k + 1) = (\sigma_1(H(1)), s_1, \sigma_2(H(1)), s_2, \ldots, \sigma_{4^{k-1}}(H(k)), s_{4^{k-1}}, \sigma_{4^k}(H(k))).\)

In the introduction, we described the first approach using \(\tau_d = [2, 1]\), the reflection in the line \(y = x\). We also observed that the substitution \(T : H(k) \mapsto H(k + 1)\) between two succeeding approximants is given by \(T(H(k)) = (\tau_d(H(k)), 1, H(k), 2, H(k), -1, -\tau_d(H(k)))\).
which corresponds to Hilbert’s original drawing. We write

\[ T(\iota) = (\tau_d, 1, \iota, 2, \iota, -1, -\tau_d) \]  

(1)

, where \( \iota \) is the identity, as usual.

As we prefer normalized, extending sequences, we start by redrawing the first approximant, such that \( H(k) \) is the first part of \( H(k+1) \) for all \( k = 1, 2, \ldots \), cf. Figure 19.

Figure 19: Hilbert’s 1-, 2, and 3-curves, normalized and extending, with directions for the isometries to be derived clearly.

Now, the substitution becomes

\[ T(H(k)) = \left(H(k), \tau_d^k(1), \tau_d(H(k)), \tau_d^k(2), \tau_d(H(k)), \tau_d^k(-1), -H(k)\right). \]

However, this formulation is a type of hybrid. We have isometries operating on \( k \)-curves; by contrast, they work on edges of the \( k \)-curve. When the edges are replaced with \( H(0) = \langle 1 \rangle \) or its images, the substitution looks unnatural. A better solution is to replace vertex replacement with edge replacement.

We show this implicitly in Figure 20, where we display the two versions of the first three approximants. The first approximant, \( H_1 \), is extended with one (dashed) edge before the entry, and one after the exit, such that both are isometric. However, these incoming and outgoing edges are redundant because we need only one edge to connect the \( k \)-curves to build a \( k+1 \)-curve; therefore, we omit the (dashed) entry-edge.

The two enlarged curves \( H_1 \) and \( \varphi(H_1) \) differ because the (extra) exit-edge in one case has the same direction as the orientation (see Definition 20) of the 1-curve. In the other case, the exit-edge is orthogonal to the orientation of the 1-curve. The other approximants are extended as well and denoted by \( H_k \), with the entry-edge in the same direction as the orientation, and by \( \varphi(H_k) \), with the entry-edge orthogonal to the orientation. As the

\[ \varphi(H_k), \text{ defined by } \varphi\langle s(1), s(2), \ldots, s(2^n - 1), s(2^n)\rangle = \langle s(1), s(2), \ldots, s(2^n - 1), \tau_d(s(2^n))\rangle. \]
approximants $H_k$ and $\varphi(H_k)$ are equal, except the last (exit) edge, the Hilbert $k$-curve is equal to $H_k$ or $\varphi(H_k)$, excluding that last edge.

Figure 20: Hilbert’s normalized, extending 1-curves $H_1$ and $\varphi(H_1)$ in the left, where the dashed extension is excluded. On the right, the layout of the 2-curves $H_2$ and $\varphi(H_2)$.

The substitution becomes relatively straightforward; however, we need two substitutions simultaneously. Note that with $\tau_d^2 = \iota = \varphi^2$, $T(A_k) = (H_k, \tau_d(H_k), \tau_d\varphi(H_k), -\varphi(A_k))$ with $A_k \in \{H_k, \varphi(H_k)\}$

The normalized Hilbert curve is extending. If we calculate the sequence with $H_0 = \langle 1 \rangle$ and $\varphi(H_0) = \langle 2 \rangle$, we get

$\langle 1, 2, -1, 2, 1, -2, 1, 2, 1, -2, -1, -2, 1, 1, 2, 1, -2, 1, 2, -1, -2, -1, -2, -1 \ldots \rangle$.

This sequence also has a simple substitution, where both $x$ and $x'$ have the same direction:

$$T = \begin{cases} 
1 & \to \langle 1, 2, -1', 2' \rangle \\
1' & \to \langle -2, -1, 2', 2 \rangle \\
2 & \to \langle 2, 1, -2', 1' \rangle \\
2' & \to \langle -1, -2, 1', 1 \rangle 
\end{cases}$$

An advantage of these enlarged Hilbert curves is that they can be generalized to higher dimensions to produce high dimensional (Hilbert) curves with special properties ([5]).

### 3.4 The $\beta, \Omega$ curves

In this section, we construct a pair of intertwining curves, which can be considered as the nephews of the Hilbert curve. Refer to the simple Equation (1) (page 21); except a few constants, we only use one isometry, i.e., the diagonal reflection $\tau_d = [2, 1]$, along with the identity $\iota$. Therefore, it is evident that there is scope for variations. We introduce the $\beta, \Omega$-curves, as referred by their inventor [22], see Figure 21.
Figure 21: On the left is the inventor’s drawing to justify the naming, in the middle is the corresponding artist’s impression, and on the right are the normalized $\beta$ and $\Omega$ 2-curves.  

As the first approximant is equal to that of the Hilbert curve (see the top row of Figure 22), the first substitutions, similar to Equation (1) (page 21), become $T_{\beta}(\iota) = (\iota, -1, -\tau_d, 2, \tau_d, 1, \tau_d)$ and $T_{\Omega}(\iota) = (\iota, -1, -\tau_d, 2, \tau_d, 1, \iota)$, where $\pm 1, \pm 2$ are the connecting edges. Notice the two substitutions only differ in terms of the last item.

Figure 22: The first row shows the two general forms of the $k$-curves of type $\beta$ and the general form of the $k$-curves of type $\Omega$. We display the modifications for the next generation underneath. See Appendix A for the dihedral group $D_4$ of transformations of the square grid.

The first approximants of the Hilbert curve have entry and exit on the vertices of the surrounding square, whereas the $k$-curves for $k \geq 2$ of the $\beta$- and $\Omega$-curve have entry and exit on (approximately) one-third of different edges of that square. Therefore, if we draw a

\footnote{Note that there can be another non-isomorphic normalization of the $\beta$-curve if you reverse the $\beta$-curve first.}
general picture of a $k$ curve for the $\beta$ and $\Omega$-curves, we should obtain something similar to the upper row of Figure 22.

**Intermezzo 2**

The extra curve $\beta'$ is required because of the asymmetry of the $\beta$-curve. $\beta'$ is approximately the inverse of the $\beta$-curve, as one can observe from the lower part of Figure 22: strip the last edge of $\beta'$, take the inverse of the rest, and glue the last edge diagonally reflected to the end of the part that was inversed. Or, if $\beta = \langle s(1), s(2), \ldots, s(n - 1), s(n) \rangle$, then $\beta' = \langle -s(n - 1), \ldots, -s(2), -s(1), -\tau_d(s(n)) \rangle$. Likewise, we could redefine the $\Omega$-curve: if $\beta = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ consists of four sub-curves of equal sizes, then $\Omega_k = (\alpha_1, \alpha_2, \alpha_3, (-\iota)^{k+1}\tau_d(\alpha_4))$. We neglect these more complicated isometries and prefer the alternate curves $\beta'$ and $\Omega$.

Therefore, we start with the 1-curves $\beta_1 = (1, 2, -1, -1); \beta'_1 = (1, -2, -1, -2)$ and $\Omega_1 = (1, 2, -1, 2)$, conform to the first row of Figure 22.\(^{10}\)

As shown in the bottom row of Figure 22, the next generation of the $\beta, \Omega$ curves is not normalized if the former one is. Thus, we apply the horizontal reflection $\tau_x$ alternately. To normalize the $k$-curves, we combine the construction in the lower part of Figure 22, and obtain the following substitutions for $k \geq 1$:

$$T_\beta(k + 1) = \tau_x^k(\beta_{k+1}); \quad \beta_{k+1} = (\tau_x(\beta_k), -\mu(\beta_k), \tau_d(\beta'_k), \mu(\Omega_k))$$
$$T_\beta'(k + 1) = \tau_x^k(\beta'_k+1); \quad \beta'_{k+1} = (\tau_d(\Omega_k), \tau_d(\beta_k), -\mu(\beta'_k), \tau_x(\beta'_k))$$
$$T_\Omega(k + 1) = \tau_x^k(\Omega_{k+1}); \quad \Omega_{k+1} = (\tau_x(\beta_k), -\mu(\beta_k), \tau_d(\beta'_k), -\iota(\beta'_k))$$

Refer to Appendix A for the dihedral group D4 of transformations of the square grid. For $k = 2$, we get the 2-curves, as in the right-hand side part of Figure 21:

$$\beta_2 = (1, 2, -1, -1, -2, -1, 2, 2, 1, -2, 1, 2, 1, -2, 1)$$
$$\beta'_2 = (2, -1, -2, -1, -1, -2, -2, -2, 1, 2, -1, 2, -1, -2)$$
$$\Omega_2 = (1, 2, -1, -1, -2, -1, 2, 2, 1, -2, 1, 2, 1, -1, 2)$$

The $\beta$ and $\Omega$ curves are also mutually dependent, similar to the Hilbert curve with its extended approximants. Their two approximants only differ in their last constituent.

---

\(^{10}\) In Figure 22, a non-normalized version of $\beta'$ is given and used further.
As $\lim_{k \to \infty} \beta_k = \lim_{k \to \infty} \Omega_k$, the corresponding sequence for $\beta$ suffices and equals
$$(1, 2, -1, -1, -2, -1, 2, 2, 1, -2, 1, 2, 1, -2, -2, -1, -2, 1, 1, 2, -1, 2, 1, \ldots).$$
There is a number-substitution possible, albeit not very simple. The one we found counted three variables in each direction, numbered $x_1, x_2, x_3$ for direction $x \in \{\pm 1, \pm 2\}$. Instead of choosing $x_1, x_2, x_3$, we could have chosen $x_1, x_2, x_3$, or even $x, x', x''$. Notice $T(-x) = -T(x)$.

$$T = \begin{cases} 
11 & \rightarrow (11, 23, -13, -12) \\
12 & \rightarrow (-23, -11, 22, -13) \\
13 & \rightarrow (-21, -11, 22, -13) \\
21 & \rightarrow (11, 23, -13, 22) \\
22 & \rightarrow (12, 23, -13, 22) \\
23 & \rightarrow (-23, -11, 22, 21) 
\end{cases}$$

### 3.5 Arndt’s Peano curve

Arndt [2] investigated all the fractal space-filling curves that can be constructed using a Lindenmayer system with only one variable, which corresponds to the minimal rotation of the grid, denoted by $\mu$. In this section, we discuss his case R9–1, the Peano curve on the square grid ([18]). This curve can also be drawn on the truncated square grid.
We observe that the substitution is rather simple because it only uses the minimal rotation \( \mu = [2, -1] \) over \( \pi/2 \), i.e., \( T(\iota) = (\iota, \mu, \iota, -\mu, -\iota, -\mu, \iota, \mu, \iota) \). Note that the last four transformations are equal to the first four in reverse order, which displays the folding characteristic of the curve. The normalized sequence is 

\[ \langle 1, 2, 1, -2, -1, -2, 1, 2, -1, 2, -1, 2, -2, 1, -2, 1, -2, 1, -2, \ldots \rangle \]

which has as simple substitution \( T \) with \( T(-x) = -T(x) \)

\[
T = \begin{cases} 
1 \rightarrow \langle 1, 2, 1, -2, -1, -2, 1, 2 \rangle \\
2 \rightarrow \langle 2, -1, 2, 1, -2, 1, 2 \rangle 
\end{cases}
\]

Figure 24: First two approximants of Arndt’s R9–1, the Peano curve, on the square grid.

This path is peculiar because the curve exhibits a higher degree of space-filling. Usually, similar to the Hilbert curve, a space-filler visits each vertex of the grid the curve lives on only once, and some edges of that grid are never visited; these are called space-filling curves and are in fact vertex-covering curves. But this Peano curve visits each edge of the square grid once and consequently each vertex exactly twice, and these curves are called edge-covering curves.

The author used the R9–1 square grid curve, Figure 24, to extend it to the truncated square grid, as depicted on the right-hand side of Figure 3 (page 9). This extension is relatively simple by considering the right-hand side drawing of Figure 4 (page 9). Here, we can observe that a direction in the square grid generates a pair of subsequent directions in the truncated square grid, such as \( \langle 1 \rangle \rightarrow \langle 1, 2 \rangle \), if the first edge was part of a pair \( \langle 1, 2 \rangle \), and \( \langle 1 \rangle \rightarrow \langle 1, -4 \rangle \), if the first edge was part of a pair \( \langle 1, -2 \rangle \).

Therefore, to get the fractal on the truncated square grid, we split the sequence for the square grid in overlapping pairs, such as \( \langle 1, 2 \rangle, \langle 2, 1 \rangle, \langle 1, -2 \rangle, \ldots \), and then apply the following substitution

\[
T' = \begin{cases} 
\langle 1, 2 \rangle \rightarrow \langle 1 \rangle \\
\langle 1, -2 \rangle \rightarrow \langle 1, 4 \rangle \\
\langle 2, 1 \rangle \rightarrow \langle 3, 2 \rangle \\
\langle 2, -1 \rangle \rightarrow \langle 3, -4 \rangle 
\end{cases}
\]

together with \( T'(-x, -y) = -T'(x, y) \)

26
and you get the sequence for the truncated square grid
\[(1, 2, 3, 2, 1, 4, -3, -2, -1, -2, -3, 4, 1, 2, 3, 2, 1, 2, 3, -4, -1, -4, 3, 2, 1, 4, -3, 4, 1, 2, 3, \ldots)\].

This sequence has the drawings in Figure 25 as the first two approximants, which are similar to those in [2, sec. 4.1.2].

![Figure 25: First two approximants of R9-1 on the truncated square grid.](image)

We could not obtain a simple substitution for this sequence because there are numerous variants for each direction.

### 3.6 Gray curve

The following example is distinctive: a fractal from which the \(k\)-curve exists in \(k\)-dimensional space and not in fewer dimensions. We call this resulting curve the Gray curve because the coordinates of the vertices of the curve form the binary reflected Gray code.

In Figure 26, we construct the binary reflected Gray code in dimension \(d\), an ordered set of the binary vertices of the unit cube.

First, we take the vertices in dimension \(d-1\) and suffix each vertex with a 0 coordinate. Second, we reflect the same \(d-1\)-dimensional vertices, i.e., place them in reverse order, and suffix them with a coordinate 1. Finally, we join these two sets of vertices with the last coordinate of 0 and 1 in that order, respectively.

Clearly, the Gray curves, as a set of vertices, are equal to the unit \(d\)-cube. Henceforth, we replace the term “binary reflected” with “Gray,” which translates to “binary reflected Gray.”

![Figure 26: bin. refl. Gray code](image)
Define the *Gray function* as \( g_d : \{1, 2, \ldots, 2^d - 1\} \to \{\pm1, \pm2, \ldots, \pm d\} \) such that if \( v(n - 1) \) and \( v(n) \) differ 1 in coordinate \( k \), then \( g_d(n) = k \), and if they differ \(-1\), then \( g_d(n) = -k \).\(^{11}\) This gives the sequence \( \langle g_d(1), g_d(2), \ldots, g_d(2^d - 1) \rangle = \langle 1, 2, -1, 3, \ldots, -1 \rangle \) and the next definition as a consequence.

**Definition 24.** The (binary reflected) *Gray sequence* \( G \) is an infinite-dimensional sequence in \( \Delta^N \), where \( \Delta = \mathbb{Z} \setminus \{0\} \). Its approximants \( G(d) \) are defined as \( G(0) = \langle \rangle \) and for \( d > 0 \),

\[
G(d) = \langle g_d(1), g_d(2), \ldots, g_d(2^d - 1) \rangle = \langle G(d - 1), d, -\mathcal{R}(G(d - 1)) \rangle,
\]

where \( -\mathcal{R} \) is the inverse.

Note that

\[
-\mathcal{R}(G(d)) = -\mathcal{R}\left\langle G(d - 1), d, -\mathcal{R}(G(d - 1)) \right\rangle = \langle G(d - 1), -d, -\mathcal{R}(G(d - 1)) \rangle;
\]

therefore, \( -\mathcal{R}(G(d)) = [1, 2, \ldots, d - 1, -d] G(d) \) as shown in Figure 27.

This Gray sequence appears under A164677 in [17]. It is normalized and starts with \( \langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, 5, 1, 2, -1, 3, 1, -2, -1, -4, 1, 2, -1, -3, 1 \ldots \rangle \).

Sloane A164677 observed that the Gray sequence is the paper-folding sequence \( \text{Fold}(1, 2, 3, 4, \ldots) \), mentioned in Exercise 15 in [1, p. 203]. This folding map is defined iteratively by \( \text{Fold}(x_1, \ldots, x_n) = \langle \text{Fold}(x_1, \ldots, x_n), x_{n+1}, -\mathcal{R}(\text{Fold}(x_1, \ldots, x_n)) \rangle \) and \( \text{Fold}(x_1) = \langle x_1 \rangle \), similar to our definition of the Gray sequence.

We notice that the absolute value of the Gray sequence is the *ruler function* in A001511 in [17].

There exist two substitutions that generate the Gray sequence: \( T_1 \) is *uniform* (of length

\[^{11}g_d \) is called a “delta” function by Knuth in [14, p. 293].
2), c.f. [1], that is, \( \|T_1(x)\| = \|T_1(y)\| \) for all \( x, y \in \Delta \), and \( T_2 \) is non-uniform.

\[
\begin{aligned}
T_1(x) &= \langle 1, x + \text{sign}(x) \rangle \quad \text{for } |x| = 1, \\
T_1(x) &= \langle -1, x + \text{sign}(x) \rangle \quad \text{for } |x| \neq 1
\end{aligned}
\]

\[
\begin{aligned}
T_2(1) &= \langle 1, 2, -1 \rangle; \quad T_2(-1) = -R(T_2(1)), \\
T_2(x) &= \langle x + \text{sign}(x) \rangle \quad \text{for } |x| \neq 1
\end{aligned}
\]

**Definition 25.** The (binary reflected) **Gray curve** \( G \) is the curve on \( \mathbb{Z}^N \), which has the Gray sequence as description and the Gray code (with subsequent vertices connected) as a graph; \( G(d) \), the \( d \)-th approximant, lives on \( \mathbb{Z}^d \).

Notice that \( G(d) \) is a Hamiltonian path on the unit cube \( C_d \), with the origin as entry and the last vertex of the Gray code, \((0,0,\ldots,0,1)\), as the exit. Therefore, adding orientation to \( G(d) \) transforms the Hamiltonian path to a Hamiltonian cycle.

Figure 28 shows the first few approximants, where the association with “paper folding” is evident. The 3-curve resembles a paperclip.

![Figure 28: First three approximants of the Gray curve.](image)

**Observation 26.** For \( k = 1, 2, \ldots \), any set of \( 2^k \) subsequent edges in a Gray curve spans a \((k + 1)\)-dimensional unit cube \( C_{k+1} \).

**Proof.** For no dimension \( d \), there is a vertex in the Gray code outside the unit cube \( C_d \) because all the vertices are in \( L_\infty \)-distance 1 from the origin and have non-negative coordinates. The number of vertices in the unit cube \( C_d \) is \( 2^d \), all traced by the Gray curve \( G(d) \). Hence, the number of edges in that path that trace each of these vertices only once is \( 2^d - 1 \). As we observe from Figure 27, there are different sub-curves \( H(j) \subset G(d) \) for \( 0 \leq j < d \) that are isometric with \( G(j) \) because each gray block is a \( G(j) \), and a black block is a \( -R(G(j)) \).

Let \( A = \langle a_1, a_2, \ldots, a_{2^k} \rangle \), of length \( ||A|| = 2^k \), be a (consecutive) sub-sequence \( A \subset G(d) \), with \( a \in A \) such that \( |a| = \max\{|a_i|; 1 = 1, 2, \ldots, 2^k\} \). Because \( ||A|| = 2^k > 2^k - 1 \), it follows that \( A \not\subset H(k) \), where \( H(k) \cong G(k) \) (isometric); thus, \( |a| > k \).

\[\text{This is the first of a series of uniform substitutions defined for } n > 1 \text{ by } T_n(x) = \langle G(n), x + n \ast \text{sign}(x) \rangle \text{ for } |x| = 1 \text{ and } T_n(x) = \langle R(G(n)), x + n \ast \text{sign}(x) \rangle \text{ for } |x| \neq 1\]
Therefore, we have \( A = \langle a_1, a_2, \ldots, a_m, a = a_{m+1}, a_{m+2}, \ldots, a_{m+n+1} = a_{2^k} \rangle \) with \( 0 \leq m < 2^k \) and \( m + n + 1 = 2^k \); thus, \( m \neq n \). Thus, either \( m < n \), in which case \( 2^k = m + n + 1 < 2n + 1 \) and \( 2^{k-1} \leq n \), or \( n < m \) and \( 2^{k-1} \leq m \). In the first case, \( (H(k-1), \langle k \rangle) \subseteq \langle a_{m+2}, \ldots, a_{m+n+1} \rangle \), and in the second case \( (\langle k \rangle, (H(k-1)) \subseteq \langle a_1, a_2, \ldots, a_m \rangle \). In both cases, \( A \setminus \{a\} \) counts \( k \) directions, and hence the number of directions in \( A \) equals \( k + 1 \).

If we consider \( k = 1 \), then every two subsequent edges in a Gray curve are mutually orthogonal.

**Definition 27.** A curve is \( n \)-hyper-orthogonal if for \( k = 1, 2, \ldots, n \) any set of \( 2^k \) subsequent edges in the curve span a \((k+1)\)-dimensional unit cube \( C_{k+1} \).

Notice that if a curve is \( n \)-hyper-orthogonal, so are its isometric images.

We say that a curve in \( \mathbb{R}^d \) is hyper-orthogonal if the curve is \((d-2)\)-hyper-orthogonal. In three dimensions, this implies that a curve is hyper-orthogonal if and only if all subsequent edges are orthogonal to each other.

Clearly, the Gray curve is \( k \)-hyper-orthogonal for all \( k > 0 \), and its approximant \( G(d) \) is \( k \)-hyper-orthogonal for all \( k \leq d - 1 \).

We can extend \( G(d) \) with additional edges preceding the entry and succeeding the exit without losing the \((d-1)\)-hyper-orthogonality by adding an edge \( \langle d \rangle \) before and after the curve. We can build a chain of \( G(d) \)s, mutually connected by the edge \( \langle d \rangle \), which is still \((d-1)\)-hyper-orthogonal.

We [5] used hyper-orthogonality to construct Hilbert curves with excellent properties, as discussed in the following section.

### 3.7 High-dimensional Hilbert curves

#### 3.7.1 The origin as an entry in 3D

In the introduction, we discussed the Hilbert curves confined only to a plane. Here, we extend them to higher dimensions; however, one could disagree whether these are “Hilbert” curves. We consider a curve in the higher dimensions to be a Hilbert curve if the same construction applied in two dimensions results in the original curve.

In Section 3.3, we proposed a construction in 2 dimensions as \( H(k+1) = T(H(k)) = (\tau(H(k)), 1, H(k), 2, H(k), -1, -\tau(H(k))) \), where \( \tau = [2, 1] \) is the reflection in the line \( y = x \) and \( T\tau = \tau T \). This reduces to \( T(\iota) = (\tau, 1, \iota, 2, \iota, -1, -\tau) \). The intermediate edges are the edges of the Hilbert 1-curve \( \langle 1, 2, -1 \rangle \).

From our observations, this Hilbert 1-curve is \( G(2) \), i.e., the 2-dimensional Gray curve.

**Definition 28.** In dimension \( d \), we call a curve a **Hilbert curve** if its 1-curve is equal to the Gray curve \( G(d) \).

From our considerations of hyper-orthogonality (at the end of Section 3.6), we discovered that the only curve in \( d \) dimensions that is \((d-1)\)-hyper-orthogonal is a chain of Gray
$d$-curves $G(d)$ connected by edges $\langle d \rangle$. This validates our definition of \textit{hyper-orthogonal as $(d - 2)$-hyper-orthogonal.}

**Observation 29.** The extension $G'(d)$ of $G(d)$, given by the concatenation $(\langle d \rangle, G(d), \langle -(d - 1) \rangle) = G'(d)$ is hyper-orthogonal.

\textit{Proof.} From our previous observations, the extra edge $\langle d \rangle$ does not perturb $(d - 1)$-hyper-orthogonality. Also, the $(d - 2)$-hyper-orthogonality holds: if the set of $2^{d-2}$ subsequent edges contains the first edge $\langle d \rangle$, then the set also contains $G(d - 2) \subset G(d)$, hence, there are $d - 1$ dimensions. If the last extra edge $\langle -(d - 1) \rangle$ is a part of the set of $2^{d-2}$ subsequent edges, then this set also contains $-R(G(d - 2)) \subset G(d)$, hence containing $d - 1$ dimensions in total. \hfill $\blacksquare$

**Observation 30.** The second hyper-orthogonal extended Gray curve is $(\langle d - 1 \rangle, G(d), \langle d \rangle) = G''(d)$, and both curves are isometric.

\textit{Proof.} Let $\omega = [1, 2, \ldots, (d - 1), -d]$ be the signed permutation that sends $d \mapsto -d$ and leaves all other directions unchanged. We observe that $-R(G(d)) = \omega(G(d))$. Therefore, $\omega\left( -R(\langle d - 1 \rangle, G(d), \langle d \rangle) \right) = \omega\left( \langle -d \rangle, \omega(G(d)), \langle -(d - 1) \rangle \right) = (\langle d \rangle, G(d), \langle -(d - 1) \rangle) = G'(d)$. \hfill $\blacksquare$

**Definition 31.** The \textbf{type of extension} of a Hilbert approximant is \textbf{one} or \textbf{two} if either its \textit{entry-edge} or \textit{exit-edge} has the same direction as its orientation.

Note that the type of $G'(d)$ is one, and that of $G''(d)$ is two. As entry- and exit-edge of $G(d)$ are always mutually orthogonal and the type gives the edge in the same direction as the orientation, the other one is orthogonal to the orientation. Therefore, we have two isometric extended Gray curves, which we can use as building blocks for Hilbert curves. The first one has its entry-edge in the direction of its orientation, while the second one does not (but it has its exit-edge in the direction of its orientation). For further details, we refer [5]; here, it is sufficient to use only the results.

We use isometries of both the extended Gray curves $G'(d)$ and $G''(d)$, but \textbf{without their entry-edge}, similar to our construction of the Hilbert curve in a plane, at the end of \textbf{Section 3.3}. If $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(d)]$ is a perm, then the orientation of $\sigma(G(d))$ is always $\langle \sigma_d \rangle$. Thus, the entry- and exit-edge of $\sigma(G'(d))$ equal $\langle \sigma(d) \rangle$ and $\langle -\sigma(d - 1) \rangle$, respectively, and those of $\sigma(G''(d))$ equal $\langle \sigma(d - 1) \rangle$ and $\langle \sigma(d) \rangle$, respectively.

\textsuperscript{13} See the note directly above Figure 27.
Figure 29: Type 1 and type 2 images of a Hilbert approximant with orientation between two given edges \(a\) and \(b\).

If we have two connected, orthogonal edges \(\langle a \rangle\) and \(\langle b \rangle\), cf. Figure 29, and we want an isometric image of \(G(d)\) in between, such that the entry and exit-edge are \(\langle a \rangle\) and \(\langle b \rangle\), then either \(\sigma(d) = \langle a \rangle\), where \(\sigma(G(d))\) is of type one, or \(\sigma(d) = \langle b \rangle\), where \(\sigma(G(d))\) is of type two.

In higher dimensions, we cannot visually illustrate the isometries we could use. As we have already discussed in [5], we only have to supply the details. For higher dimensions, say five or above, the number of isometries to be provided increases exponentially with the dimension. In this example, we restrict ourselves to lower dimensions.

Let us start with \(d = 3\) and construct a normalized, hyper-orthogonal Hilbert curve in three dimensions, which starts at the origin and ultimately fills \(\mathbb{Z}_3^{\geq 0}\). Recall that in 3D, hyper-orthogonal means mutually orthogonal subsequent edges.

Our construction, which is valid in other dimensions than three, is performed as demonstrated in [5]. For the next \((k + 1)\)-curve, given the \(k\)-curve, we inflate each vertex until it attains the shape of a unit cube, leaving all other edges of the grid \(\mathbb{Z}^d\) constant, as demonstrated in Figure 30.
First, we define the connecting curve, as in the first drawing. As shown in center illustration, this connecting curve becomes the curve from which we inflate the vertices. Then, we fill the cubes that replace the vertices with an isometric transformation of the 1-curve (= Gray-curve) so that their entry and exit glue properly\(^\text{15}\) to the edges of the connecting curve, as shown in the third drawing. We use the new curve as the next connecting curve.

Remark 32. However, we make a few remarks. First, we observe that the 2-curves in Figure 31 are normalized but not extending (i.e., start with the Hilbert 1-curve, which is \(G(3)\)).

\(^{14}\) The following description is derived from [5].

\(^{15}\) Properly, that is, while maintaining (hyper)orthogonality, and such that the exit of one inflated 1-curve plus the next connecting edge is equal to the entry of the next inflated 1-curve. Note that the entry plus the orientation of the 1-curve equals the exit.
Second, given a connecting curve with entry- and exit-edge in three dimensions, there exists only one possible way to fill all the cubes. This reasoning is rather simple. We observe from the inflating description (Figure 30) that a connecting edge \(\langle k \rangle; k > 0\) runs from the hyper-plane \(\{(x_1, \ldots, x_d) : x_k = 1 \mod 4\}\) to the plane \(\{(x_1, \ldots, x_d) : x_k = 2 \mod 4\}\), and the edge \(\langle -k \rangle\) connects the two hyper-planes in the reverse direction. We recall that the entry of a Hilbert curve plus its orientation equals its exit, and exit plus exit-edge equals the entry of the next Hilbert curve. From this, we deduce that if the exit-edge is \(\langle k \rangle; k > 0\) and the entry has \(x_k = 0 \mod 4\), then the orientation \(\sigma(d)\) of the 1-curve \(\sigma(G(d))\) has to be \(\langle k \rangle\) as well, and the curve is of type 2, similar to \(k < 0\), where the entry has \(x_k = 3 \mod 4\). In all other cases, \(\sigma(d - 1) = -k\), and the curve is of type 1. As \(\{|\sigma(d - 1)|, |\sigma(d)|\} = \{|entry-edge|, |exit-edge|\}\), in 3D, we have only \(\sigma(1)\) left, and this is equal to the axis that is orthogonal to the entry and exit-edge.

We [5] proved that in dimensions three and four, we have only one choice for the isometry of an inflated cube.

Before proceeding further, we first discuss the normalization of the Hilbert curve because it is evident that our construction does not automatically produce a normalized 2-curve from a normalized 1-curve. We begin with curves that start at the origin and are self-similar (c.f. Definition 16), extending, and normalized.

We exclude the entry-edge from the Hilbert curve and use it only in the construction. A \(k + 1\)-curve is normalized if and only if its first constituting \(k\)-curve is normalized, i.e., if the first Hilbert 1-curve is normalized, then that curve is the normalized Gray curve [1, 2, 3].

Owing to its hyper-orthogonality, the exit edge of the first 1-curve has to be \(\langle d \rangle\) or \(\langle -(d - 1) \rangle\), which is \(\langle 3 \rangle\) or \(\langle -2 \rangle\) in the 3D case. This is then the first edge of the connecting curve of the 2-curve, which is also a 1-curve, and so, it is also a Gray curve itself. Therefore, its last edge is equal to its first, both being either \(\langle 3 \rangle\) or \(\langle -2 \rangle\).

As this connecting curve cannot start with a negative edge because its entry is also the origin, it is one of two possible Gray curves given by the permutations \([3, 2, 1]\) and \([3, 1, 2]\) (shown in Figure 32), the first with exit edge \(\langle 1 \rangle\) (\(\langle -2 \rangle\) does not count), and the second with exit edge \(\langle 2 \rangle\) (as \(\langle -1 \rangle\) does not count).

![Figure 32: Graph with each of the six positive permutations connected to those whose first direction equals the perm’s last.](image)
Now, let us introduce a concise way of representing a 2-curve. This is done by representing a 1-curve by its isometry and adding its exit-edge and type, as in the next matrices Table 33. The first 1-curves in Figure 31 are denoted by \([2, 3, 1]; (1); 2\) and \([3, 2, 1]; (1); 2\). Therefore, the two 2-curves are described by the following matrices, in which the two signed permutations \(\sigma_1 = [3, -2, -1]\) and \(\sigma_2 = [-1, -3, 2]\), generate the group of 24 isometries.

| \(\sigma\) | perm | exit-edge | type |
|---|---|---|---|
| \(\sigma_6 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\) | [2, 3, 1] | (1) | 2 |
| \(\sigma_3 = \sigma_1^{-1}\sigma_2^{-1}\) | [3, 1, 2] | (2) | 2 |
| \(\sigma_3\) | [3, 1, 2] | (−1) | 1 |
| \(\sigma_4 = \sigma_2\sigma_3\sigma_2\) | [−2, −1, 3] | (3) | 2 |
| \(\sigma_4\) | [−2, −1, 3] | (1) | 1 |
| \(\sigma_5 = \sigma_1\sigma_2\) | [−3, 1, −2] | (−2) | 2 |
| \(\sigma_5\) | [−3, 1, −2] | (−1) | 1 |
| \(\sigma_7 = \sigma_1^{-1}\sigma_2^2\) | [−3, 2, −1] | (−2) | 1 |
| \(\sigma_8 = \sigma_2\sigma_1\sigma_2^2\) | [2, −3, −1] | (3) | 1 |

Table 33: Perms generating extending Hilbert 2-curves in 3D with the origin as entry.

Note that the central six permutations per 2-curve are equal for both curves, being \(\sigma_3, \sigma_4, \sigma_5\), and the first and last permutations are all different. Finally, the respective 1-curves are of types 2, 2, 1, 2, 1, 2, 1, 1 in both cases. The respective exit-edges constitute the connecting 1-curve.

It is relatively straightforward that we can continue with the construction depicted in Figure 30 by inflating the vertices of the newly obtained 2-curve, filling the cubes with the proper isometric images of the Hilbert 1-curve, and repeating this process. But it is easier, knowing that the above matrices represent the extending 2-curves \(H'(2)\) and \(H''(2)\) of type 1 and 2, respectively, to apply the following substitution with the perms from the matrices in Table 33.

\[
H'(k + 1) = \left( \sigma_6(H''(k)); \sigma_3(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \sigma_4(H'(k)); \sigma_5(H''(k)); \sigma_5(H'(k)); \sigma_7(H'(k)) \right)
\]

\[
H''(k + 1) = \left( \tau(H''(k)); \sigma_3(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \sigma_4(H'(k)); \sigma_5(H''(k)); \sigma_5(H'(k)); \sigma_8(H'(k)) \right)
\]

(2)

We now obtain the normalized, hyper-orthogonal Hilbert curves by first depicting the two 3-curves.
Figure 34: Two 3-dimensional, extending, and hyper-orthogonal Hilbert 3-curves of type 1 and 2, respectively. As in Figure 31, these two are also equal, except for their first and last octant. Here, they are observed from the left-hand side. Notice that only the curve on the right is normalized.

By applying the substitutions from Equation (2) on $H'(2)$ and $H''(2)$ from Figure 31, we generate these 3-curves. We observe that the first 1-curve in the $k$-curves of type 2 toggles between $\iota = [1, 2, 3]$, when $k$ is odd, and $\tau = [3, 2, 1]$, when $k$ is even, implied by the first term in the second line of Equation (2). The $k$-curves of type 1 are never normalized because their first 1-curve toggles between $\sigma_6 = [2, 3, 1]$ and $\sigma_6\tau = [1, 3, 2]$, respectively.

Therefore, a normalized approximant is produced by

$$H(k) = T(k) = [3, 2, 1]^{(k+1)} H''(k)$$

for $k > 1$

which generates the following normalized sequence

No simple substitution is found, as the simplest has eight substitutions per direction.

### 3.7.2 Non-origin entry in 3D

As we proved in [5], there are only two entries for hyper-orthogonal Hilbert curves in all dimensions $\geq 3$, which are $(x, x, \ldots, x, 0)$ and $(x, x, \ldots, x, 0, x)$ for type 1 and 2, respectively, with $x \in \{\frac{1}{3}, \frac{2}{3}, 0\}$ in the limit Hilbert curve, where the unit cube is the surrounding cube.\(^{17}\) This limit Hilbert curve is obtained by shrinking each $k$-curve with a factor of $2^k$. As the binary representation of $\frac{1}{3} = (0.01010101 \cdots)$, the respective entries of the $k$-curves are $(e_k, e_k, \ldots, e_k, 0)$ and $(e_k, e_k, \ldots, e_k, 0, e_k)$, where $e_k = 0, 1, 2, 5, 10, 21, \ldots$ (A000975) for $k = 1, 2, \ldots$. Using $\frac{2}{3} = (0.10101010 \cdots)$, the respective entries of the $k$-curves are $(e_k, e_k, \ldots, e_k, 0)$ and $(e_k, e_k, \ldots, e_k, 0, e_k)$, where $e_k = 1, 2, 5, 10, 21, \ldots$ for $k = 1, 2, \ldots$

\(^{16}\) We could have used a substitution with $H'(k+1)$ instead of $H''(k+1)$, but this is simpler.

\(^{17}\) Here, the ones with $x = \frac{1}{3}$ are isometric with the ones with $x = \frac{2}{3}$; hence, there are only two entries for different Hilbert curves.
Figure 35: Two 3-dimensional, extending, and hyper-orthogonal Hilbert 2-curves \( H'(2) \) and \( H''(2) \) of type 1 and 2, respectively, with the entry \((1,1,0)\) and \((1,0,1)\), plus the entry- and exit-edge.

These are the perms with their exit edges that appear from these constructions.

| \( \sigma \) | \( \text{perm} \) | \( \text{exit-edge} \) | \( \text{type} \) |
|----------|----------------|----------------|--------|
| \( \sigma_1 \) | \([-2, -1, 3]\) | \(1\) | \(1\) |
| \( \sigma_2 \) | \([-3, -2, 1]\) | \(2\) | \(1\) |
| \( \sigma_3 \) | \([-3, 2, -1]\) | \(-1\) | \(2\) |
| \( \sigma_4 \) | \([2, -3, -1]\) | \(3\) | \(1\) |
| \( \sigma_5 \) | \([2, 3, 1]\) | \(1\) | \(2\) |
| \( \sigma_6 \) | \([3, 2, 1]\) | \(-2\) | \(1\) |
| \( \sigma_7 \) | \([3, -2, -1]\) | \(-1\) | \(2\) |
| \( \sigma_8 \) | \([3, -1, -2]\) | \(-2\) | \(2\) |

| \( \sigma \) | \( \text{perm} \) | \( \text{exit-edge} \) | \( \text{type} \) |
|----------|----------------|----------------|--------|
| \( \sigma_9 \) | \([-3, -1, 2]\) | \(1\) | \(1\) |
| \( \sigma_2 \) | \([-3, -2, 1]\) | \(2\) | \(1\) |
| \( \sigma_3 \) | \([-3, 2, -1]\) | \(-1\) | \(2\) |
| \( \sigma_4 \) | \([2, -3, -1]\) | \(3\) | \(1\) |
| \( \sigma_5 \) | \([2, 3, 1]\) | \(1\) | \(2\) |
| \( \sigma_6 \) | \([3, 2, 1]\) | \(-2\) | \(1\) |
| \( \sigma_7 \) | \([3, -2, -1]\) | \(-1\) | \(2\) |
| \( \sigma_1 \) | \([-2, -1, 3]\) | \(3\) | \(2\) |

Table 36: Perms generating extending Hilbert 2-curves in 3D, not with the origin as entry.

Note that in the matrices in Table 36 the columns of types are \(1, 1, 2, 1, 2, 1, 2, 2\) on both sides with this entry. We [5] established the relation between the entry and the type sequence.

The normalized curve is then derived as

\[
H(k) = [-2, -1, 3]^{(k+1)} H'(k) \quad \text{for } k > 1
\]
based on

\[
H'(k + 1) = \left( \sigma_1(H''(k)); \sigma_2(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \\
\sigma_5(H'(k)); \sigma_6(H''(k)); \sigma_7(H'(k)); \sigma_8(H'(k)) \right)
\]

\[
H''(k + 1) = \left( \sigma_9(H''(k)); \sigma_2(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \\
\sigma_5(H'(k)); \sigma_6(H''(k)); \sigma_7(H'(k)); \sigma_1(H'(k)) \right)
\]

which gives rise to the following normalized sequence in three dimensions, which is different from the former, and without a simple substitution.

\[(1, 2, -1, 3, 1, -2, -1, -2, -3, 1, 3, -2, -3, -1, 3, -1, -3, -1, 3, 2, -3, 1, 3, 2, -1, -3, 1 \ldots)\]

### 3.7.3 The origin as the entry in 4D

In four dimensions, the two extending Hilbert 2-curves with the origin as entry are uniquely determined (c.f. [5]), which are obtained similarly as in three dimensions.

| perm       | exit-edge | type | perm       | exit-edge | type |
|------------|-----------|------|------------|-----------|------|
| [3, 2, 4, 1] | (1)       | 2    | [4, 2, 3, 1] | (1)       | 2    |
| [3, 4, 1, 2] | (2)       | 2    | [4, 3, 1, 2] | (2)       | 2    |
| [4, 3, 1, 2] | ⟨−1⟩     | 1    | [4, −2, −1, 3] | (3)       | 2    |
| [4, −2, −1, 3] | (3)       | 2    | [4, −2, −1, 3] | (1)       | 1    |
| [4, −2, −1, 3] | (1)       | 1    | [4, −2, −1, 3] | (1)       | 1    |
| [4, −3, 1, −2] | ⟨−2⟩     | 2    | [4, −3, 1, −2] | ⟨−2⟩     | 2    |
| [−3, 4, 1, −2] | ⟨−1⟩     | 1    | [−3, 4, 1, −2] | ⟨−1⟩     | 1    |
| [−3, 2, −1, 4] | ⟨4⟩       | 2    | [−3, 2, −1, 4] | ⟨4⟩       | 2    |
| [−3, 2, −1, 4] | ⟨1⟩       | 1    | [−3, 2, −1, 4] | ⟨1⟩       | 1    |
| [−3, −4, 1, 2] | ⟨2⟩       | 2    | [−3, −4, 1, 2] | ⟨2⟩       | 2    |
| [−4, −3, 1, 2] | ⟨−1⟩     | 1    | [−4, −3, 1, 2] | ⟨−1⟩     | 1    |
| [−4, −2, −1, −3] | ⟨−3⟩     | 2    | [−4, −2, −1, −3] | ⟨−3⟩     | 2    |
| [−4, −2, −1, −3] | ⟨1⟩       | 1    | [−4, −2, −1, −3] | ⟨1⟩       | 1    |
| [−4, 3, 1, −2] | ⟨−2⟩     | 2    | [−4, 3, 1, −2] | ⟨−2⟩     | 2    |
| [−4, 3, 1, −2] | ⟨−1⟩     | 1    | [3, −4, 1, −2] | ⟨−1⟩     | 1    |
| [−4, 2, 3, −1] | ⟨−3⟩     | 1    | [3, −2, −4, −1] | ⟨4⟩       | 1    |

Table 37: Perms generating extending Hilbert 2-curves in 4D with the origin as entry.

The 3\textsuperscript{rd} and 4\textsuperscript{th} perms \([4, 3, 1, 2]\) and \([4, −2, −1, 3]\) generate the group of 192 four-dimensional Hilbert isometries. In both cases, the types of the constituting 1-curves are as follows: the first two have type 2, the last two have type 1, and the 12 types in between are alternating, starting with 1.
We have a generalized version of the substitution for extending Hilbert curves with the entry as origin, where, except the first and last \(2^{d-3}\) perms, the perms \(\sigma'_k\) and \(\sigma''_k\) are equal.

\[
H'(k+1) = \left( \sigma'_1(H''(k)), \sigma'_2(H''(k)), \sigma'_3(H'(k)), \ldots, \sigma'_{2^d-2}(H''(k)), \sigma'_{2^d-1}(H'(k)), \sigma'_1(H'(k)) \right)
\]

\[
H''(k+1) = \left( \sigma''_1(H''(k)), \sigma''_2(H''(k)), \sigma''_3(H'(k)), \ldots, \sigma''_{2^d-2}(H''(k)), \sigma''_{2^d-1}(H'(k)), \sigma''_1(H'(k)) \right).
\]

The normalized curve in 4D is then easily derived

\[
H(k + 1) = [4, 2, 3, 1]^{(k \mod 2)} H''(k + 1)
\]

which generates the next normalized sequence in 4 dimensions.

\(\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, 4, 1, 3, -1, 4, 1, -3, -1, 2, 1, 3, -1, -4, 1, \ldots \rangle\)

### 3.7.4 Non-origin entry in 4D

The normalized sequence for the Hilbert curve with the entry \(\left( \frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3} \right)\) equals

\(\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, -3, 1, -4, -1, -2, 1, 4, -1, -3, 1, -4, -1, \ldots \rangle\)

which is derived from the normalized approximants

\[
H(k + 1) = [-3, -2, -1, 4]^{(k \mod 2)} H'(k + 1)
\]

In dimensions > 4, more than two hyper-orthogonal Hilbert curves exist per entry point.

### 3.8 Dekking’s Gosper-type curve

Dekking [6] gave an extensive treatment for the method we have applied here, with a more versatile notation. We apply the following translation table for the items he used in his example (4.9).

\[
\begin{align*}
s_{00} & \mapsto \langle 1 \rangle & s_{10} & \mapsto \langle 1' \rangle \\
s_{01} & \mapsto \langle 2 \rangle & s_{11} & \mapsto \langle 2' \rangle \\
s_{02} & \mapsto \langle -1 \rangle & s_{12} & \mapsto \langle -1' \rangle \\
s_{03} & \mapsto \langle -2 \rangle & s_{13} & \mapsto \langle -2' \rangle
\end{align*}
\]

Here \(\langle 1 \rangle \mapsto (1, 0) \leftrightarrow \langle 1' \rangle\), whereas \(\langle 2 \rangle \mapsto (0, 1) \leftrightarrow \langle 2' \rangle\). Hence, the used isometries are

\[
\rho = \begin{bmatrix} 1 & 2 & 1' & 2' \\ 1' & 2' & 1 & 2 \end{bmatrix}, \text{ which swaps the columns in the matrix above and geometrically is the identity, and } \\
\sigma = \begin{bmatrix} 1 & 2 & 1' & 2' \\ 2 & -1 & 2' & -1' \end{bmatrix}, \text{ which rotates each column one step downward (in}
\]

39
the matrix above) geometrically, and is the rotation over $\pi/2$. Finally, Dekking defined the substitution in [6] with a process that is greatly simplified using our proposed toolkit.

Figure 38: Construction of sequence and substitution of the 1-curve of Dekking’s flowsnake. See Figure 41 (p. 47) in Appendix B.3 for the next approximants

In Figure 38, we show the derivation of the sequence and the substitution for the 1-curve of Dekking’s flowsnake. In the drawing on the left, we observe that the primary edge, $\langle 1 \rangle$, covers the $5 \times 5$ square on its left side. Therefore, we determine the mapping between squares and edges in the center image by starting with the (13) dark gray squares with a unique edge. Each square contains the direction of its unique edge, without an accent if the square is at the left of the directed edge, or with an accent otherwise. Subsequently, the (4) edges in a unique light gray square are mapped to the corresponding squares. Finally, we similarly treat the (8) remaining unattached edges and white squares, as shown in the drawing rightward to Figure 38. This leads to the substitution $S\langle 1 \rangle = \{1, 1, 2', -1', 2, 1, 2', -1', -1, 2', 1, 1, 1', 2, 2', -2, -1', -2, -1', 2, 1, 2', -2'\}$.

If we replace Dekking’s $\rho$, the vertical flip, by $\tau_y = [1, -2]$, and his $\sigma$, the rotation over $\pi/2$ by $\mu = [2, -1]$, we see that $\langle 1' \rangle = \tau \langle 1 \rangle$ and $\langle 2 \rangle = \mu \langle 1 \rangle$. As for the substitution $S$, we have $S(\varphi) = \varphi S(\iota)$; for all isometries $\varphi$, we get $S(\iota) = (\iota, \iota, \mu \tau \iota, -\tau, \mu, \mu \tau \iota, -\tau, -\iota, \mu \tau \iota, \iota)$.

As Dekking’s flowsnake is normalized from the beginning, we have

$\langle 1, 1, 2, -1, 1, 2, 1, -1, -2, 1, 1, 1, 2, 1, -2, -1, -2, -1, 2, 1, 2, 1, -2, -1, 1, 1, 2, -1, 2, \ldots \rangle$.

There is a general way to apply this process to squares of different sizes, even if they are slightly rotated; see Figure 39. The simplest is $a = 2, b = 1$, as Mandelbrot described in [16]. Although he named these fractals “quintet” and “teragon,” we prefer to call them Mandelbrot flowsnakes. Dekking also described it in [6, Example 4.2].
In the 2-curve of Figure 39, we use \( a = 4, b = 3 \) and have \( 5^2 = 25 \) edges, which are denoted by their isometry, using \( \mu = [2, -1] \) and \( \tau_y = [1, -2] \). Hence, we get \( S(\iota) = (\iota, \mu \tau_y, \iota, \mu \tau_y, \iota, -\mu, \tau, -\mu, \tau, -\mu, -\mu \tau_y, -\iota, \mu, \mu \tau_y, -\tau, -\mu, -\tau, -\mu \tau_y, \iota, -\mu \tau_y, \iota, -\mu \tau_y) \), that gives rise to the normalized sequence 
\( \langle 1, 2, 1, 2, 1, 2, 1, -2, 1, -2, 1, -2, 1, 2, 2, -1, -2, 1, -2, 1, -2, 1, 2, 1, 2, 1, 2, 1, 2, 1 \ldots \rangle \).

As this fractal is a flowsnake, its boundary is also similar to the boundary of Gosper’s flowsnake. For this, we read the following substitution from the drawing at the right-hand side of Figure 39 \( S(\iota) = (\iota, -\mu, \iota, -\mu, \iota, -\mu, \iota) \), which requires the 1-curve \( \langle 1, 2, -1, -2 \rangle \) as a starting point. The resulting border of the Mandelbrot island is 
\( \langle 1, 2, 1, 2, 1, 2, 1, -2, 1, -2, 1, -2, 1, 2, 2, -1, -2, 1, -2, 1, -2, 1, 2, 1, 2, 1, 2, 1 \ldots \rangle \).

4 Considerations and conclusions

4.1 Conclusions

In this study, we have a digiset that we use to describe the fractal sequence. This digiset is our translation of the grid on which the fractal image exists. We explicitly map the set of geometrical line-fractals to the set of signed, integer sequences. We provide a substitution, and a starter sequence from which the fractal grows, via its approximants, which are finite approximations of the limit fractal.

The advantage of our approach is threefold. First, we can order the set of normalized, signed, integer sequences, which implies an ordering on the set of fractal images. Second, we can use the machinery of signed permutations as isometries of signed, integer sequences. Notably, the “reverse” appears to be an essential anti-morphism. Finally, third, the “coding” of a fractal image to an signed, integer sequence makes it sufficiently simple to obtain the image from that sequence.
We describe our finding using ten examples with fifteen sequences to illustrate the different peculiarities encountered when representing a fractal sequence as a signed, integer sequence.

Finally, we set up an inventory of the fifteen fractal integer sequences, most of which are unknown, in the Online Encyclopedia of Integers [17].

4.2 Considerations

It is evident that the fractal sequences we discussed here to illustrate the method of mapping a fractal to a signed, integer sequence are superficial. Therefore, the primary task should be to scan all publications describing fractals, convert them into integer sequences, and add them to the list provided in our study, which can build an ordered catalog of fractal sequences. The fractals in the publications of Mandelbrot, Dekking, Arndt, and Ventrella should be described first.

Subsequent research could use the method described here to generate new fractal sequences, comparable with what Arndt did with the Lindenmayer systems.

In further studies, emphasis should be given to fractals in higher dimensions, i.e., above two, as the only fractals known to us are on the cubic grid, like the Gray curves and their offspring, i.e., the Hilbert curves.

Finally, the fractals occurring in this study are line-fractals. We are anxious to know how fractals can be represented with higher dimensional structures, like planes or volumes.

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References

[1] Jean-Paul Allouche and Jeffrey Shallit, *Automatic Sequences*, Cambridge Univ. Press, 2003. 4, 10, 28, 29

[2] J. Arndt, Plane-filling curves on all uniform grids, http://arxiv.org/pdf/1607.02433, (2018). 2, 25, 27

[3] Anders Björner and Francesco Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics 231, Springer New York, 2005. 5

[4] A. Bos, Fractal Images and Number Sequences I,
[5] A. Bos and H. Haverkort, Hyper-orthogonal well-folded Hilbert curves, *J. Comput. Geom.*, **7** (2016), 145–190, http://jocg.org/index.php/jocg/article/view/269.

[6] F. M. Dekking, Recurrent sets, *Adv. Math.*, **44** (1982), 78–104.

[7] F. M. Dekking, Paper-folding morphisms, plane-filling curves, and fractal tiles, *Theoret. Comput. Sci.*, **414** (2012), 20–37.

[8] Kenneth Falconer, *Fractal Geometry*, John Wiley and Sons Ltd., ed. 3, 2014.

[9] Herman Haverkort, Harmonious Hilbert curves and other extra-dimensional space-filling curves, https://arxiv.org/abs/1211.0175v1, (2012).

[10] D. Hilbert, ”Über die stetige Abbildung einer Linie auf ein Flachenstück, *Math. Ann.*, **38**, (1891), 459–460.

[11] Clark Kimberling, Self-Containing Sequences, Fractal Sequences, Selection Functions, and Para-sequences, *Journal of Integer Sequences*, **25**, (2022), Article 22.2.1.

[12] D. E. Knuth, Efficient representation of perm groups, *Combinatorica*, **11**, (1991), 33–43.

[13] D. E. Knuth, Two notes on notation, *The American Mathematical Monthly*, **99**, (1992), 403–422, http://arxiv.org/abs/math/9205211.

[14] D. E. Knuth, *The Art of Computer Programming, Vol. 4A, Part 1*, Addison-Wesley, 2011.

[15] A. Lindenmayer, Mathematical models for cellular interaction in development, *J. Theoret. Biol.*, **18**, (1968), 280–315.

[16] B. Mandelbrot, *The Fractal Geometry of Nature*, W. H. Freeman and Company, 1982.

[17] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org.

[18] Hans Sagan, *Space-Filling Curves*, Springer-Verlag, 1994.

[19] David Salomon, *Computer Graphics and Geometric Modeling*, Springer, 1999.

[20] Jeffrey Ventrella, https://en.wikipedia.org/wiki/Jeffrey_Ventrella.

[21] Jeffrey Ventrella, *The Family Tree of Fractal Curves*, Eyebrain Books, 2019, http://fractalcurves.com/all_curves/4G_family.html.
Appendices

A Dihedral group D4

Figure 40: Cayley table of some transformations of the square grid with their pictorial description below and its results on flags. It indicates the reflections, such as \( \tau_x \) because the \( x \)-coordinate changes, \( \tau_d \) because the reflection is diagonal.
B Normalized Fractal Encyclopedia

B.1 Description

An (integer) sequence essentially determines its fractal, and because we cannot characterize an infinite sequence, we use a picture and need the determining digiset and its substitution. To depict the fractal, we need the generators of a grid and their relation with the digiset. What follows is a template of how we store a fractal.

**sequence:** We will provide approximately the first 20 numbers of the sequence.

**in OEIS:** its occurrence (signed) in the OEIS, plus the inspection date.

**digiset:** The digiset of the fractal.

**substitution:** The substitution of the fractal.

**grid:** The grid on which the fractal is depicted.

**generators:** The generators of the grid in the order of the positive elements of the digiset.

In most cases, we provide the graphs of some initial \(k\)-curves. In all cases, we use the following nomenclature

- The *identity isometry* \([1, 2, \ldots, d−1, d]\) is denoted by \(ι\).
- With \(μ\) the minimal rotation \([2, 3, 4, \ldots, d−1, d, −1]\) is represented.
- The notation \(−ι\) is shorthand for the isometry \([-1, -2, \ldots, -d]\).
- The *reverse* \(R\), only defined on a finite sequence, by \(R\langle s(1), s(2), \ldots, s(n−1), s(n)\rangle = \langle s(n), s(n−1), \ldots, s(2), s(1)\rangle\), is the anti-morphism, such as an approximant of a fractal.

**Definition 33.** The *ordering* of the sequences in the database is such that for the sequences \(S = \langle s(1), s(2), \ldots, s(k), s(k+1), \ldots \rangle\) and \(S' = \langle s(1), s(2), \ldots, s(k), s'(k+1), \ldots \rangle\) with \(s(k+1) \neq s'(k+1)\), we have

\[
S < S' \iff \frac{1}{s(k+1)} > \frac{1}{s'(k+1)}
\]

Notice that positive numbers come before negative numbers, but within these categories, the numbers are ordered naturally, i.e., smaller numbers first, like \(1, 2, 3, 4, \ldots, -4, -3, -2, -1\).

The *bold* number in a sequence is the first number that makes this sequence differ from the previous one.
B.2 Index of sequences

\begin{align*}
\langle 1, 1, 2, -1, 2, 1, 2, -1, -1, 2, 1, 1, 1, 2, 1, -2, -2, -1, -2, -2, \ldots \rangle, & \text{ c.f. B.3, Dekking's Flowsnake} \\
\langle 1, 2, 1, 2, 1, 2, 1, -2, 1, -2, 1, -2, -1, 2, -2, 1, -2, 1, \ldots \rangle, & \text{ c.f. B.4, Mandelbrot's Flowsnake} \\
\langle 1, 2, 1, 2, 1, 2, 1, -2, 1, -2, 1, -2, 1, 2, 1, 2, 1, 1, \ldots \rangle & \text{ c.f. B.5, Mandelbrot's Flowsnake Island} \\
\langle 1, 2, 1, -2, 1, -2, 1, -2, 1, -2, 1, -2, 1, 2, -2, 1, 2, \ldots \rangle, & \text{ c.f. B.6, Ventrella's Box4} \\
\langle 1, 2, 1, -2, 1, -2, 1, 2, 1, 2, 1, -2, 1, -2, 1, 2, 1, 2, \ldots \rangle, & \text{ c.f. B.7, Arndt's Peano} \\
\langle 1, 2, 3, 2, 1, 4, -3, -2, -1, -2, -3, 4, 1, 2, 3, 2, 1, 2, 3, -4, \ldots \rangle, & \text{ c.f. B.8, Arndt's Peano on truncated square grid} \\
\langle 1, 2, 3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -1, -2, -3, 4, -1, 2, 3, 4, \ldots \rangle, & \text{ c.f. B.9 and B.10, Ventrella's V1 Dragon} \\
\langle 1, 2, -1, 2, 1, -2, 1, 2, -2, 1, -2, 1, 2, 1, -2, 1, \ldots \rangle, & \text{ c.f. B.11, Hilbert's Original} \\
\langle 1, 2, -1, 3, 1, -2, -1, 3, 1, 3, -1, 2, 1, -3, 1, 2, 1, 3, -1, 2, \ldots \rangle, & \text{ c.f. B.12, First 3D Hilbert hyper-orthogonal curve} \\
\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -3, 1, -2, -1, 4, 1, 3, -1, 4, \ldots \rangle, & \text{ c.f. B.13, First 4D Hilbert hyper-orthogonal curve} \\
\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, 1, -3, 1, 2, -1, -3, 1, -4, -1, 2, \ldots \rangle, & \text{ c.f. B.14 Gray curve} \\
\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, -3, 1, -4, -1, 2, \ldots \rangle, & \text{ c.f. B.15 Second 4D Hilbert hyper-orthogonal curve} \\
\langle 1, 2, -1, 3, 1, -2, 1, 2, -1, -3, 1, 2, -1, -3, 1, -2, -1, -3, \ldots \rangle, & \text{ c.f. B.16, Second 3D Hilbert hyper-orthogonal curve.} \\
\langle 1, 2, -1, -1, -2, 1, -2, 2, 1, -2, 1, 2, 1, -2, 1, 2, 1, -2, -2, \ldots \rangle, & \text{ c.f. B.17, } \beta-\Omega \text{ curve.}
\end{align*}

B.3 Dekking’s Flowsnake

sequence: \(\langle 1, 1, 2, -1, 2, 1, 2, -1, -1, 2, 1, 1, 1, 2, 1, -2, -2, -1, -2, -2, 1, 2, 1, -2, -2, 1, \ldots \rangle\)

in OEIS: Not (18-01-2022)

digiset: \(\Delta = \{1, 2\}\)

start sequence: \(\langle 1 \rangle\)

substitution:

\[
T(t) = (t, t, \mu \tau_y, -\tau_y, \mu, t, \mu \tau_y, -\tau_y, -t, \mu t \tau_y, t, t, \tau_y, \mu, \\
\tau_y, -\mu, -\mu, -\tau_y, -\mu, -\mu \tau_y, \tau_y, \mu, t, -\mu t \tau_y, -\mu \tau_y),
\]

where \(\tau_y = -\tau_x = [1, -2]\) and, as usual, \(\mu = [2, -1]\)

grid: The square, plane grid.

generators: \((1, 0); (1, 0)\)
Figure 41: Drawings of the 1st and 2nd approximants in the first row, below the third and fourth approximants, separating two parts of space in black and white, both with tree-like structures.

B.4 Mandelbrot’s $4 \times 3$ Flowsnake

sequence: $\langle 1, 2, 1, 2, 1, 1, -2, 1, -2, 1, -2, 1, -2, -1, -2, -1, 2, 2, -1, -2, -1, 2, 1, -2, 1, -2, 1, \ldots \rangle$

in OEIS: Not (18-01-2022)
digiset: $\Delta = \{1, 2\}$

start sequence: $\langle 1 \rangle$

substitution:

$$T(i) = \left( t, \mu \tau_y, t, \mu \tau_y, t, \mu \tau_y, t, -\mu, \tau_y, -\mu, \tau_y, -\mu, -t, -\mu \tau_y, -t, \mu \tau_y, t, -\mu \tau_y, t, -\mu \tau_y, t, -\mu \tau_y \right)$$
with $\tau_y = [1, -2]$ and $\mu = [2, -1]$

**grid:** The square, plane grid.

**generators:** $(1, 0); (1, 0)$

---

**B.5 Mandelbrot’s $4 \times 3$ Flowsnake Island**

**sequence:** $\langle 1, 2, 1, 2, 1, 2, -1, 2, -1, 2, -1, 2, 1, 2, 1, 2, 1, 2, 1, 2, -1, 2, -1, 2, -1, 2, 1, \ldots \rangle$

**in OEIS:** Not (18-01-2022)

**digiset:** $\Delta = \{1, 2\}$

**start sequence:** $\langle 1, 2, -1, -2 \rangle$

**substitution:** $T(\iota) = (\iota, \mu, \iota, \mu, \iota, \mu, \iota)$, where $\mu = [2, -1]$

**grid and generators:** The square, plane grid and $(1, 0); (1, 0)$
B.6 Ventrella’s Box4

sequence: ⟨1, 2, 1, −2, 1, −2, −1, −2, 1, 2, 1, 2, −1, 2, 1, 2, −1, 2, −1, −2, −1, −2, . . .⟩
in OEIS: Not (18-01-2022)
digiset: Δ = {1, 2}
start sequence: VB₀ = ⟨1⟩
substitution: T(ι) = (ι, (−ι)ᵏ μτ_y R, τ_y, (−ι)ᵏ⁺¹ μR), where τ_y = [1, −2] and μ = [2, −1]
grid: The square grid.
generators: (1, 0); (1, 0)

Figure 43: The 1st approximant and the tiling of the islands.
Figure 44: The $k$-curves for $k = 1, 2, 3, 5$, with corners rounded for the last two.

**B.7 Arndt’s Peano curve**

sequence: $\langle 1, 2, 1, -2, -1, -2, 1, 2, 1, 2, -1, 2, 1, -2, 1, 2, -1, 2, 1, 2, 1, -2, -1, -2, 1, 2, 1, \ldots \rangle$

in OEIS: Not (18-01-22)

digiset: $\Delta = \{1, 2\}$

start sequence: $\langle 1, 2, 1, -2, -1, -2, 1, 2, 1 \rangle$

substitution: $T(\iota) = (\iota, \mu, \iota, \mu^{-1}, -, \mu^{-1}, \mu, \iota)$, where $\mu = [2, -1]$

grid: The square grid.

generators: $(1, 0); (1, 0)$

Figure 45: The first two approximants of Arndt’s R9-1, the Peano curve, on the square grid.

**B.8 Arndt’s Peano curve on the truncated square grid**

sequence: $\langle 1, 2, 3, 2, 1, 4, -3, -2, -1, -2, -3, 4, 1, 2, 3, 2, 1, 2, 3, -4, -1, -4, 3, 2, 1, 4, -3, \ldots \rangle$

in OEIS: Not (18-01-2022)

digiset: $\Delta = \{1, 2, 3, 4\}$

start sequence: subsequent pairs of B.7, Arndt’s Peano curve
substitution:

\[ T' = \begin{cases} 
(1, 2) & \rightarrow (1, 2) \\
(1, -2) & \rightarrow (1, 4) \\
(2, 1) & \rightarrow (3, 2) \\
(2, -1) & \rightarrow (3, -4) 
\end{cases} \]

together with \( T'(x, y) = -T'(-x, -y) \)

grid: The truncated square grid spanned by the 8th-roots of unity (c.f. Figure 12).
generators: \((1, 0); (\frac{1}{2} \sqrt{2}, \frac{1}{2} \sqrt{2}); (1, 0); (-\frac{1}{2} \sqrt{2}, \frac{1}{2} \sqrt{2})\)

Figure 46: The first two approximants of Arndt's Peano on the truncated square grid.

B.9 Ventrella's V1 Dragon on 8th-root plane grid

sequence: \((1, 2, 3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -1, -2, -3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -2, 1, \ldots)\)
in OEIS: Not (18-01-2022)
digiset: \(\Delta = \{1, 2, 3, 4\}\)
start sequence: \(V1_0 = (1)\)
substitution: \(T(\iota) = (\iota, R\mu^2, \mu^3)\), where \(\mu = [-4, 3, -1, -2]\)
grid: The 8th-root plane grid.
generators: \((1, 0); (1, 0); (-\frac{1}{2} \sqrt{2}, \frac{1}{2} \sqrt{2}); (-\frac{1}{2} \sqrt{2}, -\frac{1}{2} \sqrt{2})\)
Figure 47: The 1- and 2-curves on the 8<sup>th</sup>-root grid.

**B.10 Ventrella’s V1 Dragon on square-diagonal grid**

Same fractal but on the square-diagonal grid, together with its *length sequence*.  

sequence: \(\langle 1, 2, 3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -1, -2, -3, 4, -1, 2, 3, 4, -2, 1, -3, 4, -2, 1, \ldots \rangle\)  

in OEIS: Not (18-01-2022)  

digiset: \(\Delta = \{1, \sqrt{2}\}\)  

start sequence: \(V_{10} = \langle 1 \rangle\)  

substitution: \(dT(\iota) = (\iota, R\mu^2, \mu^3)\), where \(\mu = [-4, 3, -1, -2]\)  

grid: The square-diagonal grid.  

generators: \((1, 0); (1, 0); (-1, 1); (-1, -1)\)  

length sequence: \(\langle 1, 1, \sqrt{2}, \sqrt{2}, 1, 1, \sqrt{2}, 2, 2, \sqrt{2}, 1, 1, \sqrt{2}, \sqrt{2}, 1, 1, \sqrt{2}, 1, \sqrt{2}, \sqrt{2}, 2, 2, \ldots \rangle\)  

length substitution: \(lT(\iota) = (\iota, R, \iota \ast \sqrt{2})\)  

in OEIS: \(\log_{\sqrt{2}}(lS) = \langle 0, 0, 1, 1, 0, 0, 1, 1, 2, 2, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 2, 2, 1, 1, 2, 2, 3, 3, 2, \ldots \rangle\) that is the double (i.e., \langle x, x \rangle versus \langle x \rangle) of \textbf{A062756}  

Figure 48: The 1- and 2-curves of Ventrella’s V1 Dragon on the square-diagonal grid.
B.11 Hilbert’s original

sequence: \( \langle 1, 2, -1, 2, 2, 1, -2, 1, 2, 1, -2, -1, -2, 1, 1, 2, 1, -2, 1, 1, 2, -1, 2, 1, -1, \ldots \rangle \)

in OEIS: Not (18-01-2022)

digiset: \( \Delta = \{1, 2\} \)

start sequence: \( H_0 = \langle 1 \rangle \)

substitution: \( A_{k+1} = T(A_k) = (H_k, \tau_d(H_k), \tau_d(\varphi(H_k)), -\varphi(A_k)) \) with \( A_k \in \{H_k, \varphi(H_k)\} \)

and \( \varphi(s(1), s(2), \ldots, s(2^n)) = \langle s(1), s(2), \ldots, \tau_d(s(2^n)) \rangle \) with \( \tau_d = [2, 1] \).

grid: The square grid.

generators: \((1, 0); (1, 0)\)

\[ \text{Figure 49: The 1-, 2- and 3-curves of the original Hilbert curve.} \]

B.12 3D normalized, hyper-orthogonal Hilbert curve with origin as entry

sequence: \( \langle 1, 2, -1, 3, 1, -2, -1, 3, 1, 3, -1, 2, 1, -3, -1, 2, 1, -3, -1, -3, -2, \ldots \rangle \)

in OEIS: Not (17-1-2022)

digiset: \( \Delta = \{1, 2, 3\} \)

start sequence: \( H'_1 = \langle 3, 1, 2, -1, 3, 1, -2, -1, -2 \rangle = G'(3) \) (cf. Obs. 29, pg. 31);

\( H''_1 = \langle 2, 1, 2, -1, 3, 1, -2, -1, 3 \rangle = G''(3) \) (cf. Obs. 30)
substitution: with signed permutations as below:

\[
H(k) = T(k) = [3, 2, 1]^{(k+1)} H''(k) \text{ for the normalized version, with }
H'(k+1) = \left( \sigma_6(H''(k)); \sigma_3(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \sigma_4(H'(k)); \sigma_5(H''(k)); \sigma_5(H'(k)); \sigma_7(H'(k)) \right)
\]

\[
H''(k+1) = \left( \tau(H''(k)); \sigma_3(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k)); \sigma_4(H'(k)); \sigma_5(H''(k)); \sigma_5(H'(k)); \sigma_8(H'(k)) \right)
\]

| \(\sigma\) | \(perm\) | \(exit-edge\) | \(type\) |
|---------|--------|--------------|--------|
| \(\sigma_6 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\) | [2, 3, 1] | (1) | 2 |
| \(\sigma_3 = \sigma_1^{-1}\sigma_2^{-1}\) | [3, 1, 2] | \(-1\) | 1 |
| \(\sigma_4 = \sigma_2\sigma_1\sigma_2\) | [2, -1, 3] | (3) | 2 |
| \(\sigma_4\) | [2, -1, 3] | (1) | 1 |
| \(\sigma_5 = \sigma_1\sigma_2\) | [3, 1, 2] | (2) | 2 |
| \(\sigma_5\) | [3, 1, 2] | (1) | 1 |
| \(\sigma_7 = \sigma_1^{-1}\sigma_2^2\) | [3, 2, -1] | (2) | 2 |
| \(\sigma_8 = \sigma_2\sigma_1\sigma_2^2\) | [2, -3, -1] | (3) | 1 |

Table 50: Perms generating Hilbert curves in 3D with the origin as entry.

grid: The cubic grid.

generators: \((1, 0, 0); (0, 1, 0); (0, 0, 1)\)
Figure 51: 3-dimensional, extending, hyper-orthogonal, and normalized Hilbert 2-curve $H(2)$ of type 2, with the entry at the origin.

B.13 4D normalized, hyper-orthogonal Hilbert curve with origin entry

sequence: $\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -4, -1, 2, 1, 4, -1, -3, \ldots \rangle$

in OEIS: Not (17-1-2022)

digiset: $\Delta = \{1, 2, 3, 4\}$

start sequence:

- $H_1' = \langle 4, 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, -3 \rangle = G'(4)$ (cf. Obs. 29);
- $H_1'' = \langle 3, 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, 4 \rangle = G''(4)$ (cf. Obs. 30)

substitution: The particular perms to be used are
B.14 Gray curve

sequence: 〈1, 2, −1, 3, 1, −2, −1, 4, 1, 2, −1, −3, 1, −2, −1, −5, 1, 2, −1, 3, 1, −2, −1, −4, 1, . . .〉

in OEIS: A164677 (02-11-2021)

digiset: Δ = {1, 2, 3, . . .}
start sequence: \langle 1 \rangle

substitution:
\[
\begin{aligned}
T(x) &= \langle 1, x + \text{sign}(x) \rangle \text{ for } |x| = 1, \\
T(x) &= \langle -1, x + \text{sign}(x) \rangle \text{ for } |x| \neq 1
\end{aligned}
\]

grid: The \(d\)-dimensional cubic grid \(\mathbb{Z}_d \geq 0\)

generators: \(\langle 1, 2, 3, \ldots, d \rangle\), where \(\langle k \rangle = (I(j = k); j = 1, 2, \ldots, d) = (\delta_{jk}; j = 1, 2, \ldots, d)\).

Figure 53: The first three approximants of the Gray curve.

B.15 4D normalized, hyper-orthogonal Hilbert curve with non-origin entry

sequence: \(\langle 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, -3, 1, -4, -1, 2, 1, 4, -1, -3, \ldots \rangle\),
in OEIS: Not (17-1-2022)
digiset: \(\Delta = \{1, 2, 3, 4\}\)

start sequence:
\[
\begin{aligned}
H'_1 &= \langle 4, 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, -3 \rangle = G'(4) \text{ (cf. Obs. 29)}; \\
H''_1 &= \langle 3, 1, 2, -1, 3, 1, -2, -1, 4, 1, 2, -1, -3, 1, -2, -1, 4 \rangle = G''(4) \text{ (cf. Obs. 30)}
\end{aligned}
\]

substitution: with signed permutations as below:
\[ H(k + 1) = [-3, -2, -1, 4]^k H'(k + 1) \] for the normalized version with
\[ H'(k + 1) = \left( \sigma_1(H''(k)), \sigma_2(H''(k)), \sigma_3(H''(k)), \ldots , \sigma_{14}(H''(k)), \sigma_{15}'(H'(k)), \sigma_{16}'(H'(k)) \right) \]
\[ H''(k + 1) = \left( \sigma_1''(H''(k)), \sigma_2''(H''(k)), \sigma_3(H'(k)), \ldots , \sigma_{14}(H''(k)), \sigma_{15}''(H'(k)), \sigma_1(H'(k)) \right) \]

grid: The 4D cubic grid.
generators: \((1, 0, 0, 0); (0, 1, 0, 0); (0, 0, 1, 0); (0, 0, 0, 1)\)

**B.16 3D normalized, hyper-orthogonal Hilbert curve with non-origin entry**

sequence: \((1, 2, -1, 3, 1, -2, -1, 3, 1, 3, -1, 2, 1, -3, -1, 2, 1, -3, -1, -3, -2, \ldots)\)
in **OEIS:** Not (17-1-2022)
digiset: \(\Delta = \{1, 2, 3\}\)
start sequence: $H_1' = (3, 1, 2, -1, 3, 1, -2, -1, -2) = G'(3)$ (cf. Obs. 29, pg. 31); $H_1'' = (-2, 1, 2, -1, 3, 1, -2, -1, 3) = G''(3)$ (cf. Obs. 30)

substitution:

$$H(k) = T(k) = [-2, -1, 3]^{(k+1)}H'(k) \text{ for } k > 1 \text{ based on}$$

$$H'(k + 1) = \left(\sigma_1(H''(k)); \sigma_2(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k))\right)$$

$$H''(k + 1) = \left(\sigma_9(H''(k)); \sigma_2(H''(k)); \sigma_3(H'(k)); \sigma_4(H''(k))\right)$$

with signed permutations as below

| $\sigma$ | $perm$ | $exit$-edge | $\sigma$ | $perm$ | $exit$-edge |
|----------|--------|-------------|----------|--------|-------------|
| $\sigma_1$ | $[-2, -1, 3]$ | $\langle 1 \rangle$ | $\sigma_9$ | $[-3, -1, 2]$ | $\langle 1 \rangle$ |
| $\sigma_2$ | $[-3, -2, 1]$ | $\langle 2 \rangle$ | $\sigma_2$ | $[-3, -2, 1]$ | $\langle 2 \rangle$ |
| $\sigma_3$ | $[-3, 2, -1]$ | $\langle -1 \rangle$ | $\sigma_3$ | $[-3, 2, -1]$ | $\langle -1 \rangle$ |
| $\sigma_4$ | $[2, -3, -1]$ | $\langle 3 \rangle$ | $\sigma_4$ | $[2, -3, -1]$ | $\langle 3 \rangle$ |
| $\sigma_5$ | $[2, 3, 1]$ | $\langle 1 \rangle$ | $\sigma_5$ | $[2, 3, 1]$ | $\langle 1 \rangle$ |
| $\sigma_6$ | $[3, 2, 1]$ | $\langle -2 \rangle$ | $\sigma_6$ | $[3, 2, 1]$ | $\langle -2 \rangle$ |
| $\sigma_7$ | $[3, -2, -1]$ | $\langle -1 \rangle$ | $\sigma_7$ | $[3, -2, -1]$ | $\langle -1 \rangle$ |
| $\sigma_8$ | $[3, -1, -2]$ | $\langle -2 \rangle$ | $\sigma_1$ | $[-2, -1, 3]$ | $\langle 3 \rangle$ |

Table 55: Perms generating Hilbert curves in 3D, not with the origin as entry.

grid: The cubic grid.

generators: $(1, 0, 0); (0, 1, 0); (0, 0, 1)$
Figure 56: 3-dimensional, extending, and hyper-orthogonal Hilbert 2-curve $H(2)$ of type 1, with the entry at $(2, 2, 0)$.

B.17 The $\beta, \Omega$ curve

sequence: $(1, 2, -1, -1, -2, -1, 2, 2, 1, -2, 1, 2, 1, -2, 1, 2, 1, -2, -1, -2, 1, 1, 2, \ldots)$
in OEIS: Not (18-01-2022)
digiset: $\Delta = \{1, 2\}$
start sequences: $\beta_1 = (1, 2, -1, -1); \beta'_1 = (1, -2, -1, -2); \Omega_1 = (1, 2, -1, 2)$
substitution:

$$\beta_{k+1} = \left( \tau_x(\beta_k), -\mu(\beta_k), \tau_x\mu(\beta'_k), \mu(\Omega_k) \right)$$
$$\beta'_{k+1} = \left( \tau_x\mu(\Omega_k), \tau_x\mu(\beta_k), -\mu(\beta'_k), \tau_x(\beta'_k) \right)$$
$$\Omega_{k+1} = \left( \tau_x(\beta_k), -\mu(\beta_k), \tau_x(\beta'_k), -\iota(\beta'_k) \right)$$

for $k \geq 1$, where $\mu = [2, -1]$ is the minimal grid rotation and $\tau_x = [-1, 2]$ the horizontal reflection. Note that $\lim_{k \to \infty} \beta_k = \lim_{k \to \infty} \Omega_k$. The normalized curve is obtained by $T_{\beta}(k + 1) = \tau_x^k(\beta_{k+1})$.

grid: The square grid.
generators: $(1, 0); (1, 0)$
simple substitution: With $T(-x) = -T(x)$ and $\Delta = \{1, 1', 1'', 2, 2', 2''\}$, three tokens for
each of the two generators, we get

\[
T = \begin{cases}
1 & \rightarrow (1, 2'', -1'', -1') \\
1' & \rightarrow (-2'', -1, 2', -1'') \\
1'' & \rightarrow (-2, -1, 2', -1'') \\
2 & \rightarrow (1, 2'', -1'', 2') \\
2' & \rightarrow (1', 2'', -1'', 2') \\
2'' & \rightarrow (-2'', -1, 2', 2)
\end{cases}
\]

Figure 57: The 2, 3- and 4-curves of the \( \beta \) fractal, respectively, with entries (◦) and exits (●)
Figure 58: Drawings of the 3rd approximant of Mandelbrot’s flowsnake in black and white.

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