SINGULAR VALUES FOR PRODUCTS OF COMPLEX GINIBRE MATRICES WITH A SOURCE: HARD EDGE LIMIT AND PHASE TRANSITION

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ABSTRACT. The singular values squared of the random matrix product $Y = G_r G_{r-1} \cdots G_1(G_0 + A)$, where each $G_j$ is a rectangular standard complex Gaussian matrix while $A$ is non-random, are shown to be a determinantal point process with correlation kernel given by a double contour integral. When all but finitely many eigenvalues of $A^*A$ are equal to $bN$, the corresponding correlation kernel is shown to admit a well-defined hard edge scaling, in which a critical value is established and a phase transition phenomenon is observed. More specifically, the limiting kernel in the subcritical regime of $0 < b < 1$ is independent of $b$, and is in fact the same as that known for the case $b = 0$ due to Kuijlaars and Zhang. The critical regime of $b = 1$ allows for a double scaling limit by choosing $b = (1 - \tau/\sqrt{N})^{-1}$, and for this the critical kernel and outlier phenomenon are established, while a distribution corresponding to a finite product is proven to be the scaling limit in the supercritical regime of $b > 1$ with two distinct scaling rates. In the simplest case $r = 0$, which is closely related to non-intersecting squared Bessel paths, the latter gives rise to the finite LUE distribution. Similar results also hold true for the random matrix product $T_r T_{r-1} \cdots T_1(G_0 + A)$, with each $T_j$ being a truncated unitary matrix.

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. The squared singular values of a matrix $X$ are equal to the eigenvalues of the positive semi-definite Hermitian matrix $X^* X$, where $X^*$ denotes the Hermitian conjugate of $X$. An ensemble of random matrices of the form $X^* X$ may then contain $x = 0$ as the left boundary of support of the eigenvalues. Since the eigenvalue density is strictly zero for $x < 0$, $x = 0$ is then called a hard edge (see e.g. [21, Ch. 7]). As an explicit example, consider the ensemble of $n \times N$ ($n \geq N$) rectangular standard complex Gaussian random matrices, namely the joint density of elements being proportional to $\exp\{-\text{tr}(X^* X)\}$, and let $X$ be a matrix from this ensemble. Let $\{\lambda_j\}$ denote the eigenvalues of the scaled positive semi-definite matrix $N^{-1} X^* X$. In the limit $N \to \infty$ with $n - N$ fixed, the density of $\{\lambda_j\}$ has support $[0, 4]$. That the support is a finite interval gives rise to this particular scaling being referred to as global scaling, and the corresponding density as the global density. The explicit functional form of the global density is given by the so-called Marchenko-Pastur law (see e.g. [13])

$$\rho_{\text{MP}}^{(1)}(\lambda) = \frac{1}{2\pi} \sqrt{\frac{4 - \lambda}{\lambda}}, \quad 0 < \lambda \leq 4. \quad (1.1)$$
Note in particular the reciprocal square root singularity as the hard edge $\lambda = 0$ is approached from above, in contrast to the square root singularity as $\lambda \to 4^-$. The point $\lambda = 4$ is an example of what is termed a soft edge, since for finite $N$ the eigenvalue density is not strictly zero for $\lambda > 4$.

Continuing with this example, for large $N$ the eigenvalues in the neighbourhood of the hard edge have spacing $O(1)$ upon the introduction of the scaled variables $X_j = 4N^2\lambda_j$ ($j = 1, \ldots, N$) (see e.g. [21 §7.2.1]). This will be referred to as hard edge scaling. Moreover, in the limit $N \to \infty$, and with $\nu_0 = n - N$, the limiting state — referred to as the hard edge state — is an example of a determinantal point process, meaning that the $k$-point correlation function can be written in the form

$$
\rho_{(k)}(X_1, \ldots, X_k) = \det[K^h(X_j, X_i)]_{j=1,\ldots,k}
$$

with correlation kernel (see e.g. [21 Exercises 7.2 q.1])

$$
K^h(x, y) = \frac{1}{4} \int_0^1 J_{\nu_0}(\sqrt{xt})J_{\nu_0}(\sqrt{yt})\,dt.
$$

Our interest in this paper is in the functional form and analytic properties of the correlation kernel for the hard edge scaling of the squared singular values of the matrix product

$$
Y = G_r G_{r-1} \cdots G_1 (G_0 + A),
$$

where each $G_j$ is an $(N + \nu_j) \times (N + \nu_{j-1})$ standard complex Gaussian matrix with $\nu_{-1} = 0$ and $\nu_0, \ldots, \nu_r \geq 0$, while $A$ is of size $(N + \nu_0) \times N$ and fixed. In the case that all entries of $A$ are zero, the limiting hard edge state and its properties have been the subject of a number of recent works [3, 4, 37, 38, 47]. These studies form part of a fast paced and very recent literature relating to the integrability and exactly solvable properties of random matrix products. Works relating to this theme which have appeared on the electronic preprint archive over the past few months include [40, 27, 32, 33, 31, 50, 26]; we refer the reader to [2] for a recent survey article.

That the hard edge state in the case $A = 0$ depends on $r$, and thus is no longer described by the correlation kernel (1.3), can be anticipated by an analysis of the global density of the squared singular values [5, 25, 42, 40]. The global density is found to exhibit the hard edge singularity [46, 25 eq. (2.16) with $r = 1$ and $p \mapsto r + 2$]

$$
\frac{1}{\pi} \sin \frac{\pi}{r+2} \lambda^{-1+\frac{1}{r+2}} \quad \text{as} \quad \lambda \to 0^+,
$$

which has an $r$-dependent exponent. In fact with the eigenvalues of $Y^*Y$ scaled according to $X_j = Nx_j$ ($j = 1, \ldots, N$), the hard edge state in the case $A = 0$ forms a determinantal point process with correlation kernel

$$
K^{h,r}(x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} du \int \frac{dt}{\Sigma} \prod_{j=-1}^{r} \frac{\Gamma(\nu_j + u + 1) \sin \pi u x^t y^{-u-1}}{\Gamma(\nu_j + t + 1) \sin \pi t} \left| u \right|

= \int_0^1 G_{0,r+2}^{1,0}(0, -\nu_0, -\nu_1, \ldots, -\nu_r, \nu_0, \nu_1, \ldots, \nu_r, 0 \big| uy) \, du,
$$

(1.6)
where $G_{m,n}^{p,q}(a_1, \ldots, a_p, b_1, \ldots, b_q|z)$ denotes the Meijer G-function defined by the contour integral

$$G_{m,n}^{p,q}(a_1, \ldots, a_p, b_1, \ldots, b_q|z) = \frac{1}{2\pi i} \int_{\gamma} \prod_{j=1}^{m} \Gamma(b_j + s) \prod_{j=1}^{n} \Gamma(1 - a_j - s) \prod_{j=m+1}^{p} \Gamma(1 - b_j - s) \prod_{j=n+1}^{q} \Gamma(a_j + s) z^{-s} ds;$$

(1.7)

see [41, Sect. 5.2] for the choice of the contour $\gamma$ and elementary properties of G-functions, or [8] for a gentle introduction. These kernels were described in [38] and are named after Meijer G-kernels in [37]. They also appear in the hard edge scaling for products with inverses of Ginibre matrices [23], products of truncated unitary matrices [37], Cauchy two matrix models [11, 12, 24], and Muttaib-Borodin biorthogonal ensembles [14, 43] (cf. [37] for the relationship between Borodin’s expression and Meijer G-kernels).

As noted in [38, Sect. 5.3], in the case $r = 0$ the facts that

$$G_{0,2}^{1,0}(0, -\nu|ux) = (ux)^{-\nu/2}J_{\nu}(2\sqrt{ux}), \quad G_{0,2}^{1,0}(\nu, 0|uy) = (uy)^{\nu/2}J_{\nu}(2\sqrt{uy}),$$

(1.8)

show

$$K^{h,0}(x, y) = 4(y/x)^{\nu/2}K^{h}(4x, 4y).$$

(1.9)

The factor of $(y/x)^{\nu/2}$ cancels out of the determinant (1.2), while the factors of 4 are accounted for by this same factor being present in the scaling leading to (1.3); recall the text leading to this equation.

Consider now (1.4) with

$$A = \sqrt{bNI_{(N+\nu_0)\times N}},$$

(1.10)

where $I_{(N+\nu_0)\times N}$ denotes the $(N + \nu_0) \times N$ matrix with 1’s in the diagonal, and 0’s elsewhere. It was shown recently in [25, Remark 3.4] that there is a critical value of $b = 1$ for which as $N \to \infty$ the left hand edge of the support of the global scaled squared singular values equals 0 for the last time as $b$ increases from 0. Moreover, it was shown that the singularity of the global density has the leading form

$$\frac{1}{\pi} \sin \frac{2\pi}{2r + 3} \lambda^{-1 + \frac{2}{2r + 3}} \quad \text{as} \quad \lambda \to 0^+, \quad (1.11)$$

which gives rise to a different family of exponents to those in (1.3). We remark that the fractional part of the exponents, $1/(r + 2)$ and $1/(r + 3/2)$ respectively in (1.5) and (1.11), are the reciprocals of positive integers and half-integers, which given knowledge of the correlation kernel (1.6) and its analogue in relation to (1.11) to be established herein (see eqn. (1.22) below), is coincident with them being the simplest in terms of tractability of the general rational fractional exponents accessible in the Raney family (see e.g. [25, eqn. (2.16)]), so named due to the sequence formed by the moments of the global density.

Let $A$ be again given by (1.10), and consider the case $r = 0$ in (1.4) so that

$$Y = G_0 + \sqrt{bNI_{(N+\nu_0)\times N}}.$$ 

It is well known that the squared singular eigenvalues allow for an interpretation as the positions of non-intersecting particles on the half line evolving according to the squared Bessel process with parameter $d = 2(\nu_0 + 1)$ (see e.g. [29, 30]). A functional form of the hard edge scaled kernel in the critical case $b = 1$, generalised to a double scaling by setting $b = (1-\tau/\sqrt{N})^{-1}$, has recently been obtained in [36]. In the present paper an alternative functional form to that
in [36] is derived; see eqn. (1.23) below. The kernel (1.23), further specialised to \( \nu_0 = -1/2 \) reads

\[
\frac{1}{2\pi i} \left( \frac{1}{\xi \eta} \right)^{1/4} \int_0^\infty du \int_{i\mathbb{R}} dv \frac{e^{-\tau u + \psi^2 + \tau v + \frac{1}{2}v^2}}{u-v} \frac{\cos(2\sqrt{u\xi})}{u^{1/2}} \frac{\cos(2\sqrt{v\eta})}{v^{1/2}}.
\]

(1.12)

With \( \xi \) and \( \eta \) replaced by squared variables, (1.12) is identified in [36] as the symmetric Pearcey kernel found in the study [15]. Moreover, our method of derivation of this new functional form in the case \( r = 0 \) works equally as well for the double scaling of the critical kernel in the general \( r \) case, which is our main theme. The resulting explicit double contour integral expression is given in Theorem 1.2 below.

1.2. Main results. In preparation for the statement of our first key result, let us introduce two auxiliary functions. The first is defined to be

\[
\Psi(u; x) = \frac{1}{(2\pi i)^r} \frac{1}{\Gamma(\nu_0 + 1)} \int_{\gamma_1} dw_1 \cdots \int_{\gamma_r} dw_r \prod_{l=1}^r w_l^{-\nu_l-1} e^{w_l} \\
\times e^{r/(u_1 \cdots u_r)} \Phi(\nu_0 + 1; -ux/(u_1 \cdots u_r)),
\]

(1.13)

where \( \gamma_1, \ldots, \gamma_r \) are paths starting and ending at negative infinity and encircling the origin once in the positive direction, while the other reads

\[
\Phi(v; y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \ y^{-s} \phi(v; s) \prod_{l=1}^r \Gamma(\nu_l + s),
\]

(1.14)

where \( c > -\min\{\nu_0, \nu_1, \ldots, \nu_r\} \), and

\[
\phi(v; s) = \frac{1}{\Gamma(\nu_0 + 1)} \int_0^\infty dt \ t^{\nu_0 + s - 1} e^{-t} \Phi(\nu_0 + 1; -vt)
\]

(1.15)

\[
= \frac{\Gamma(\nu_0 + s)}{\Gamma(\nu_0 + 1)} \Phi(\nu_0 + s; \nu_0 + 1; -v).
\]

(1.16)

In the case \( r = 0 \) (1.13) is to be interpreted as

\[
\Psi(u; x) = \frac{1}{(2\pi i)^r} \frac{1}{\Gamma(\nu_0 + 1)} e^{r/0} \Phi(\nu_0 + 1; -ux),
\]

(1.17)

and a calculation shows that (1.13) simplifies to read

\[
\Phi(v; y) = \frac{1}{\Gamma(\nu_0 + 1)} y^{\nu_0} e^{-y} \Phi(\nu_0 + 1; -vy).
\]

(1.18)

The two auxiliary functions appear in a double contour integral expression for the correlation kernel, which we present next. Its significance is that it provides the starting point for further asymptotic analysis. The special case \( r = 0 \) was previously obtained by Desrosiers and one of the present authors; see [20, Prop. 5].

**Proposition 1.1.** Let \( Y \) be defined by (1.4), and suppose that all eigenvalues \( a_1, \ldots, a_N \) of \( A^*A \) are positive. The joint density of eigenvalues for \( Y^*Y \) can be written in the form

\[
P_N(x_1, \ldots, x_N) = \frac{1}{N!} \det[K_N(x_i, x_j)]_{i,j=1}^N
\]

(1.19)

with correlation kernel

\[
K_N(x, y) = \frac{1}{2\pi i} \int_0^\infty du \int_{c} dv \ u^{\nu_0} e^{-u+v} \Psi(u; x) \Phi(v; y) \frac{1}{u-v} \prod_{l=1}^N \frac{u + a_l}{v + a_l},
\]

(1.20)
where $C$ is a counterclockwise contour encircling $-a_1, \ldots, -a_N$ but not $u$.

Remark 1.1. When some of the parameters $a_j$'s are null, the double integral representation (1.20) remains valid provided that $\int_0^\infty du$ is interpreted as $\lim_{x \to 0+} \int_x^\infty du$, or for given $u > 0$ $C$ is chosen such that $\Re\{v\} < u$ with any $v \in C$.

One of the main results in the present paper concerns a double scaling limit near the critical point, which permits a new family of limiting kernels.

**Theorem 1.2** (Critical kernel). For the kernel (1.20), for $\tau \in \mathbb{R}$ let

\[ a_1 = \cdots = a_N = N(1 - \tau/\sqrt{N})^{-1}. \]

Then we have

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} K_N \left( \frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}} \right) = \frac{1}{2\pi i} \int_0^\infty \int_{\mathbb{R}} \left( \frac{u}{v} \right)^{\nu_0/2} e^{-\tau u - \frac{1}{2} u^2 + \tau v + \frac{1}{2} v^2} \frac{du}{u-v} \times G_{0,r+2}^{1,0} \left( 0, -\nu_0, -\nu_1, \ldots, -\nu_r \right) \left| u\xi \right| G_{0,r+2}^{1,0} \left( \nu_0, \nu_1, \ldots, \nu_r, 0 \right) \left| v\eta \right|
\]

\[
= K_{h,0}^{1,0}(\xi, \eta; \tau),
\]

valid uniformly for $\xi, \eta$ in a compact set of $(0, \infty)$ and for $\tau$ in a compact set of $\mathbb{R}$.

In the special case $r = 0$, upon making use of (1.18) we see from (1.22) that

\[
\left( \frac{\xi}{\eta} \right)^{\nu_0/2} K_{h,0}^{1,0}(\xi, \eta; \tau) = \frac{1}{2\pi i} \int_0^\infty \int_{\mathbb{R}} \left( \frac{u}{v} \right)^{\nu_0/2} e^{-\tau u - \frac{1}{2} u^2 + \tau v + \frac{1}{2} v^2} \frac{du}{u-v} \times J_{\nu_0}(2\sqrt{u\xi})J_{\nu_0}(2\sqrt{v\eta}),
\]

where the integral form on the RHS of the above equation is similar to (1.20) with $r = 0$ (cf. [20, Prop. 5]). In the study [36, displayed equation below (1.34)], this kernel was conjectured to be an equivalent functional form to that derived therein in the case $r = 0$. Our work thus provides a direct way of deriving (1.23) for the $r = 0$ critical kernel. Recently, an understanding of the resulting functional identity has been given in the preprint [18, Remark 2.27].

Theorem 1.2 quantifies the limiting correlation kernel for the situation that $a_k = N(1 - \tau/\sqrt{N})^{-1}$ $(k = 1, \ldots, N)$, which is shown to depend on $\tau$, thus justifying the term critical kernel. A variation on this setting is to have at most finitely many source eigenvalues, say $a_1, \ldots, a_m$, go to infinity at a smaller but appropriate scale and others remain at the same critical value. This gives rise to a multi-parameter deformation of the critical kernel (1.22).

**Theorem 1.3** (Deformed critical kernel). With the kernel (1.20), for a fixed non-negative integer $m$, let

\[ a_j = \sqrt{N}\sigma_j, \quad j = 1, \ldots, m \quad \text{and} \quad a_k = N(1 - \tau/\sqrt{N})^{-1}, \quad k = m+1, \ldots, N, \]

where $\tau \in \mathbb{R}$ and $\sigma_1, \ldots, \sigma_m > 0$. Then we have

\[
\lim_{N \to \infty} \frac{1}{\sqrt{N}} K_N \left( \frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}} \right) = \frac{1}{2\pi i} \int_0^\infty \int_{\mathbb{R}} \left( \frac{u}{v} \right)^{\nu_0/2} e^{-\tau u - \frac{1}{2} u^2 + \tau v + \frac{1}{2} v^2} \frac{du}{u-v} \times \prod_{j=1}^m \left( \frac{u + \sigma_j v}{v + \sigma_j} \right)^{\nu_0/2} G_{0,r+2}^{1,0} \left( 0, -\nu_0, -\nu_1, \ldots, -\nu_r \right) \left| u\xi \right| G_{0,r+2}^{1,0} \left( \nu_0, \nu_1, \ldots, \nu_r, 0 \right) \left| v\eta \right|
\]

\[
= K_{h,m}^{1,0}(\xi, \eta; \tau, \sigma),
\]
where \( 0 < c < \min \{ \sigma_1, \ldots, \sigma_m \} \).

In the simplest case \( r = 0 \), upon making use of (1.8) we see from (1.25) that

\[
\frac{(\xi, \eta)}{v_0^2} e^{\tau u - \frac{1}{2} u^2} e^{\tau v + \frac{1}{2} v^2} \prod_{j=1}^{m} \frac{u + \sigma_j}{v + \sigma_j} J_{\nu_0}(2\sqrt{u\xi}) J_{\nu_0}(2\sqrt{v\eta}).
\]  

Even in this special case, the kernel (1.25) appears to be new.

We remark that the inter-relationship between the interpolating kernel (1.25) and critical kernel (1.22) is similar in form to that between the interpolating Airy kernel and Airy kernel (see e.g. [9, 1]). Furthermore, as the parameter \( b \) displayed in eqn. (1.10) increases from zero, we will establish a phase transition at the hard edge from the Meijer G-kernel (cf. Theorem 3.1) to the critical and deformed critical kernels (cf. Theorems 1.2 and 1.3), then to the finite product kernel (cf. Theorem 3.2); see Section 3 for more details. A similar phase transition occurs in another random matrix product \( T_r T_{r-1} \cdots T_1 (G_0 + A) \), with each \( T_j \) being a truncated unitary matrix; see Section 4.

The paper is organized as follows. Section 2 is devoted to the joint eigenvalue probability density function (PDF) and a double contour integral representation for the correlation kernel of the squared singular values of the product (1.4). The proof of Proposition 1.1 will be given, and the formulas for the average of the ratio of characteristic polynomials and a single (inverse) characteristic polynomial are also derived. In Section 3 the hard edge limits of the kernel in different regimes are evaluated, which include the proofs of Theorems 1.2 and 1.3. Our methods are used to similarly analyse the product of \( r \) truncated unitary matrices and one shifted mean Ginibre matrix in Section 4. In Section 5 further discussions on asymptotics for large variables, and some open problems, are presented.

2. Eigenvalue PDF and double integral for correlation kernel

2.1. Correlation kernels. Consider (1.3) in the case \( r = 0 \). Let \( x_1, \ldots, x_N \) and \( a_1, \ldots, a_N \) denote the eigenvalues of \( Y^*Y \) and \( A^*A \) respectively. It is well known (see e.g. [21] Prop. 5, [21] §11.6) that the eigenvalue PDF of the random matrix \( Y^*Y \) is an example of a biorthogonal ensemble [14]

\[
Q_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\eta_i(x_j)]_{i,j=1}^N \det[\xi_i(x_j)]_{i,j=1}^N,
\]  

where \( \eta_i(x) = x^{i-1} \), \( \xi_i(x) = x^{\nu_0} e^{-x} F_1(\nu_0 + 1; a_i x) \), and \( Z_N \) denotes the normalisation. Our first task is to specify a functional form for the joint eigenvalue PDF of \( Y^*Y \) in the case of general \( r \). For this purpose use will be made of a recent result due to Kuijlaars and Stivigny [37].

**Proposition 2.1** (Special case of [37] Thm. 2.1). *Let \( W \) be an \( n \times n \) random matrix, and suppose that the eigenvalue PDF of \( W^*W \) can be written in the form*

\[
\prod_{1 \leq j < k \leq n} (x_k - x_j) \det[f_{k-1}(x_j)]_{j,k=1}^n
\]  

*then the eigenvalue PDF of \( Y^*Y \) can be written as a double integral of the form*

\[
\frac{(\xi, \eta)}{v_0^2} e^{\tau u - \frac{1}{2} u^2} e^{\tau v + \frac{1}{2} v^2} \prod_{j=1}^{m} \frac{u + \sigma_j}{v + \sigma_j} J_{\nu_0}(2\sqrt{u\xi}) J_{\nu_0}(2\sqrt{v\eta}).
\]  

**Proof.**
for some \( \{f_{k-1}(x)\}_{k=1,\ldots,n} \). For \( \nu \geq 0 \), let \( G \) be an \( (n + \nu) \times n \) standard complex Gaussian matrix. The squared singular values of \( GW \), or equivalently the eigenvalues of \((GW)^*GW\), then have their PDF proportional to
\[
\prod_{1 \leq j < k \leq n} (y_k - y_j) \det[g_{k-1}(y_j)]_{j,k=1}^n, \tag{2.3}
\]
where
\[
g_k(y) = \int_0^\infty x^e e^{-x} f_{k-1}(\frac{y}{x}) \frac{dx}{x}, \quad k = 0, \ldots, n-1. \tag{2.4}
\]

Let \( Y \) be defined in \( \text{(1.4)} \) and let \( a_1, \ldots, a_N \) denote the eigenvalues of \( A^*A \). Starting with \( \text{(2.1)} \), application of Proposition \( \text{2.1} \) means that with \( L \) this purpose, we require the fact (see e.g. \[7, \text{eqns. (6.2.15), (6.2.33) and (4.5.2)} \]) that with
\[
\text{the same as defined in (1.14)}, \quad \text{for which application of the Mellin transform gives}
\]
\[
\Phi(v; y) = \frac{1}{\Gamma(\nu + 1)} \int_0^\infty dt_1 \cdots \int_0^\infty dt_r \prod_{i=1}^r t_i^{\nu-1} e^{-t_i} (\frac{y}{T})^{\nu_0} e^{-\frac{y}{T}} F_1(\nu_0 + 1; -v \frac{y}{T}), \tag{2.6}
\]
valid for \( r \geq 1 \) (for \( r = 0 \) \( \xi(x) \) is defined as below \( \text{(2.1)} \)). Here \( \Phi(v; y) \) is actually the same as defined in \( \text{(1.14)} \), for which application of the Mellin transform gives
\[
\int_0^\infty y^{s-1} \Phi(v; y) dy = \phi(v; s) \prod_{l=1}^r \Gamma(\nu + s), \tag{2.7}
\]
while use of the inverse Mellin transform gives the sought expression. We stress that when some of the \( a_j \)’s in \( \text{(2.3)} \) coincide L’Hospital’s rule provides the appropriate eigenvalue density.

The significance of the structure \( \text{(2.5)} \) is that it provides a systematic way to compute the corresponding correlation functions (see \( \text{(1.1)} \) for the definition).

**Proposition 2.2** \([\text{14}, \text{Prop. 2.2}] \). With \( g_{i,j} := \int_0^\infty \eta_i(x) \xi_j(x) \, dx \), let \( [g_{i,j}]_{i,j=1}^N \) be invertible for each \( n = 1, 2, \ldots. \) Defining \( c_{i,j} \) by
\[
\left(c_{i,j}\right)_{i,j=1}^N = \left(g_{i,j}\right)_{i,j=1}^N \tag{2.8}
\]
and setting
\[
K_N(x, y) = \sum_{i,j=1}^N c_{i,j} \eta_i(x) \xi_j(y), \tag{2.9}
\]
we have that the \( k \)-point correlation function is given by
\[
\rho_{(k)}(x_1, \ldots, x_k) = \det[K_N(x_j, x_l)]_{j,l=1}^k. \tag{2.10}
\]

We are now ready to complete the proof of Proposition \( \text{1.1} \).

**Proof of Proposition \( \text{1.1} \).** Our first task is to compute \( g_{i,j} := \int_0^\infty \eta_i(x) \xi_j(x) \, dx \). For this purpose, we require the fact (see e.g. \([\text{14}, \text{eqns. (6.2.15), (6.2.33) and (4.5.2)} \]) that with \( L_n^{\nu_0}(y) \) denoting the Laguerre polynomial of degree \( n \), one has the Hankel pair
\[
L_n^{\nu_0}(y) = \frac{e^y}{n! \Gamma(\nu_0 + 1)} \int_0^\infty t^{\nu_0+n} e^{-t} F_1(\nu_0 + 1; -yt) \, dt \tag{2.11}
\]
and
\[
t^{\alpha} = \frac{n!e^{t}}{\Gamma(n_0 + 1)} \int_{0}^{\infty} y^{n_0} L_{n_0}^{\nu_0}(y)e^{-y}F_1(\nu_0 + 1; -ty) \, dy.
\] (2.12)

Combination of (2.7), (1.15) and (2.11) shows that
\[
g_{i,j} = (i - 1)!e^{a_j}L_{i-1}^{\nu_0}(-a_j) \prod_{l=1}^{r} \Gamma(\nu_l + i).
\] (2.13)

According to Proposition 2.2 we must now invert the matrix (2.13). With \(G = [g_{i,j}]_{i,j=1}^{N}\), let \(C = (G^{-1})^T\), the entries \(c_{i,j}\) of \(C\) then satisfy
\[
e^{a_k} \sum_{i=1}^{N} (i - 1)! L_{i-1}^{\nu_0}(u) \prod_{l=1}^{r} \Gamma(\nu_l + i) c_{i,j} = e^{-a_j} \prod_{l=1, l \neq j}^{N} \frac{-u - a_l}{a_j - a_l},
\] (2.14)

Without loss of generality we assume that \(a_1, \ldots, a_N\) are pairwise distinct. In this case the above equations imply
\[
\sum_{i=1}^{N} (i - 1)! L_{i-1}^{\nu_0}(u) \prod_{l=1}^{r} \Gamma(\nu_l + i) c_{i,j} = e^{-a_j} \prod_{l=1, l \neq j}^{N} \frac{-u - a_l}{a_j - a_l},
\] (2.15)

as can be verified by noting that both sides are polynomials of degree \(N - 1\) in \(u\) which are equal at \(N\) different points since (2.14) is satisfied. Using this implicit formula for \(\{c_{i,j}\}\) we now want to show that (2.9) implies the double contour integral formula (1.20).

Using the integral representation of the reciprocal Gamma function
\[
\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma} w^{-z}e^{w} \, dw,
\] (2.16)

we have from (2.9) that
\[
K_N(x, y) = \frac{1}{(2\pi\sqrt{-1})^r} \sum_{i,j=1}^{N} \xi_j(y) \int_{\gamma_1} \cdots \int_{\gamma_r} dw_1 \cdots dw_r \prod_{l=1}^{r} w_l^{-\nu_l - 1} e^{w_l}
\times (x \prod_{l=1}^{r} \Gamma(\nu_l + i) c_{i,j})
\]
\[
= \frac{1}{(2\pi\sqrt{-1})^r} \sum_{j=1}^{N} \xi_j(y) \int_{\gamma_1} \cdots \int_{\gamma_r} dw_1 \cdots dw_r \prod_{l=1}^{r} w_l^{-\nu_l - 1} e^{w_l} \prod_{i=1}^{N} (i - 1)! \prod_{l=1}^{r} \Gamma(\nu_l + i) c_{i,j}
\times \frac{1}{\Gamma(\nu_0 + 1)} e^{-y} \int_{0}^{\infty} du \, u^{\nu_0} L_{i-1}^{\nu_0}(u) e^{-u}F_1(\nu_0 + 1; -\frac{ux}{w_1 \cdots w_r})
\]
\[
= \frac{1}{(2\pi\sqrt{-1})^r} \sum_{j=1}^{N} \xi_j(y) \int_{\gamma_1} \cdots \int_{\gamma_r} dw_1 \cdots dw_r \prod_{l=1}^{r} w_l^{-\nu_l - 1} e^{w_l} e^{x \sum_{l=1}^{r} w_l}
\times \frac{1}{\Gamma(\nu_0 + 1)} \int_{0}^{\infty} du \, u^{\nu_0} e^{-u}F_1(\nu_0 + 1; -\frac{ux}{w_1 \cdots w_r}) e^{-a_j} \prod_{l=1, l \neq j}^{N} \frac{-u - a_l}{a_j - a_l}.
\] (2.17)

Here the formulae (2.12) and (2.15) have been made use of respectively in the second and third equalities.
Finally, with (2.6) and (1.13) in mind, these facts substituted into (2.17) imply that

\[ K_N(x, y) = \int_0^\infty du \, u^{\nu_0}e^{-u}\Psi(u;x)\sum_{j=1}^{N} \Phi(-a_j; y) e^{-a_j} \prod_{l \neq j} \frac{-u - a_j}{a_j - a_l}. \]  

(2.18)

We recognise the sum over \( j \) as the sum of the residues at \( \{a_l\} \) of

\[ \Phi(-v; y) \frac{1}{-u - v} \prod_{l=1}^{N} \frac{-u - a_l}{v - a_l}, \]  

(2.19)

considered as a function of \( v \). Applying the residue theorem and changing \( v \) to \(-v\), we thus arrive at the desired result. \( \square \)

**Remark 2.1.** The case in which each \( a_l = 0 \) has been analysed previously \[3, 4, 38\], but using different working. Thus instead of computing the inverse matrix (2.8), functions

\[ P_{j-1}(x) \in \text{Span} \{\eta_1(x), \ldots, \eta_j(x)\}, \quad Q_{j-1}(x) \in \text{Span} \{\xi_1(x), \ldots, \xi_j(x)\}, \]

with the biorthogonality property \( \int_0^\infty P_k(x)Q_l(x)\,dx = \delta_{k,l} \) were constructed. In terms of these functions (2.9) simplifies from a double sum to the single sum

\[ K_N(x, y) = \sum_{j=1}^{N} P_{j-1}(x)Q_{j-1}(y). \]  

(2.20)

Instead of (1.20) with each \( a_l = 0 \) (which strictly speaking is ill-defined due to the restriction on the contour \( C \), but can be well understood in a limiting sense, cf. Remark 1.1) this leads to the double integral formula \[38, \text{Prop. 5.1}\]

\[ K_N(x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i^\infty}^{-1/2+i^\infty} du \int_{\Sigma} dt \prod_{j=-1}^{r} \frac{\Gamma(\nu_j + u + 1)}{\Gamma(\nu_j + t + 1)} \times \frac{\Gamma(t - N + 1)}{\Gamma(u - N + 1)} x^u y^{-(u+1)} \]  

(2.21)

where \( \Sigma \) is a simple closed contour encircling anti-clockwise \( t = 0, 1, \ldots, N - 1 \) but not \( u \).

Next, we further investigate Proposition 1.1 and establish a corollary under the assumption that all but a fixed number of source parameters are equal to \( a \). Precisely, for \( m \geq 0 \) let \( a_{m+1} = \cdots = a_N = a \). More definitions are also needed. For \( k = 1, 2, 3, \ldots \) and \( n = 0, 1, 2, \ldots \), set

\[ L_n^{(k)}(x; a, a_1, \ldots, a_{k-1}) = \int_0^\infty du \, u^{\nu_0}e^{-u}\Psi(u;x)(u + a)^n \prod_{l=1}^{k-1} (u + a_l), \]  

(2.22)

and

\[ \tilde{L}_n^{(k)}(x; a, a_1, \ldots, a_k) = \frac{1}{2\pi i} \int_C dv \, e^{v}\Phi(v;x)(v + a)^{-n} \prod_{l=1}^{k} \frac{1}{v + a_l}, \]  

(2.23)

where \( C \) is a counterclockwise contour encircling \(-a, -a_1, \ldots, -a_k\).
Corollary 2.3. Let $K_N$ be the kernel (1.20), and for $m \geq 0$ let
\[ a_m = \cdots = a_N = a. \] (2.24)
Then we have
\[ K_N(x, y) = \frac{1}{2\pi i} \int_0^\infty du \int_j \frac{dv}{v} e^{-u+v}\Psi(u; x)\Phi(v; y) \frac{1}{u-v} (\frac{u+a}{v+a})^{N-m} \]
\[ + \sum_{k=1}^m E_{N-m}(x; a, a_1, \ldots, a_{k-1}) \tilde{E}_{N-m}(x; a, a_1, \ldots, a_k), \] (2.25)
where $C$ is a counterclockwise contour encircling $-a$ but not $u$.

Proof. We will use the identity
\[ \frac{1}{u-v} \prod_{l=1}^m \frac{u+a_l}{v+a_l} = \frac{1}{u-v} + \sum_{k=1}^m \prod_{l=1}^{k-1} (u+a_l) \prod_{l=k}^m (v+a_l) \] (2.26)
which has been proved by induction in [19]; see the equation (5.12) therein. A direct proof can be given as follows. Rewriting $u-v = u+a_k - (v+a_k)$, we have
\[ (u-v) \sum_{k=1}^m \prod_{l=1}^{k-1} \frac{u+a_l}{v+a_l} = \sum_{k=1}^m \prod_{l=1}^k \frac{u+a_l}{v+a_l} - \sum_{k=1}^m \prod_{l=1}^{k-1} \frac{u+a_l}{v+a_l} = \prod_{l=1}^m \frac{u+a_l}{v+a_l} - 1, \] (2.27)
and (2.26) follows.

Recalling (2.24), substituting (2.26) in (1.20), comparing the sought equation with (2.22) and (2.23), this completes the proof. \hfill \Box

2.2. Average of characteristic polynomials. Recall that a biorthogonal ensemble [14] refers to the joint density function
\[ Q_N(x_1, \ldots, x_N) = \frac{1}{Z_N} \det[\eta_i(x_j)]_{i,j=1}^N \det[\xi_i(x_j)]_{i,j=1}^N. \] (2.28)
Here we assume that all variables $x_1, \ldots, x_N$ lie in the same interval $I \subseteq \mathbb{R}$ and also that the matrix $G$ with elements $g_{ij} = \int_I \eta_i(x)\xi_j(x) \, dx$ is not singular.

For the special case $\eta_i = x^{i-1}$, the average ratio of characteristic polynomials under the density (2.28) can be expressed in terms of the correlation kernel; thus as a minor variant of [20, Prop. 1] we have

Proposition 2.4. With the same assumption and notation as in Proposition 2.2, let $\eta_j(x) = x^{j-1}$ for $j = 1, 2, \ldots$. Then, for $z \in \mathbb{C}\backslash I$
\[ \mathbb{E}\left[ \prod_{i=1}^N \frac{x-x_i}{z-x_i} \right] = \int_I \frac{dx}{z-x} K_N(x, u). \] (2.29)
Equivalently, if for $x \in \mathbb{R}$ we define the residue
\[ \text{Res}_{x=x} f(z) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \text{Im} f(x-i\epsilon), \] (2.30)
then
\[ K_N(x, y) = \frac{1}{x-y} \text{Res}_{x=x} \mathbb{E}\left[ \prod_{i=1}^N \frac{x-x_i}{z-x_i} \right]. \] (2.31)

In the case of the average of a single characteristic polynomial or its reciprocal, alternative expressions are also available; cf. Proposition 2 of [20].
Proposition 2.5. With the same assumption and notation as in Proposition 2.3, let \( \eta_j(x) = x^{j-1} \) for \( j = 1, 2, \ldots \). Then,

\[
\mathbb{E} \left[ \frac{1}{\prod_{l=1}^{N} (x-x_l)} \right] = \frac{N!}{Z_N} \left| \begin{array}{cccc}
g_{1,1} & g_{1,2} & \cdots & g_{1,N} \\
\vdots & \vdots & \ddots & \vdots \\
g_{N-1,1} & g_{N-1,2} & \cdots & g_{N-1,N} \\
\int_I du \frac{\xi(u)}{z-u} & \int_I du \frac{\xi(u)}{z-u} & \cdots & \int_I du \frac{\xi(u)}{z-u} \\
\end{array} \right| 
\]

where \( Z_N = N! \det [g_{i,j}]_N \), while \( c_{N,j} \) and \( c_{i,N+1} \) are elements of the inverses of matrices \([g_{i,j}]_N\) and \([g_{i,j}]_{N+1}\) respectively.

Proof. The formulas (2.32) and (2.34) have been proved in Proposition 2 of [20]. After noting the facts \( [g_{i,j}]_N [c_{i,j}]_N = I_N \) and \( N! / Z_N = \det [c_{i,j}]_N \), building on (2.32), simple manipulation gives (2.33). On the other hand, by \( [c_{i,j}]_N [g_{i,j}]_{N+1} = I_{N+1} \) and

\[
\frac{N!}{Z_N} = \frac{Z_{N+1}}{(N+1)Z_N} \frac{(N+1)!}{Z_{N+1}} = \frac{Z_{N+1}}{(N+1)Z_N} \det [c_{i,j}]_N \]

we have from (2.33) that

\[
\mathbb{E} \left[ \prod_{l=1}^{N} (x-x_l) \right] = \frac{Z_{N+1}}{(N+1)Z_N} \sum_{j=1}^{N+1} c_{j,N+1} x^{j-1},
\]

which further implies the sought equation (2.35) since it is a monic polynomial. \( \square \)

Application of the previous two propositions gives us explicit evaluation of averages of characteristic polynomials for the product of random matrices (1.4).

Proposition 2.6. For the eigenvalue PDF (2.5), the following hold true.

(i) Let \( K_N \) be the kernel (1.20), then for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\mathbb{E} \left[ \prod_{l=1}^{N} \frac{x-x_l}{x-x_l} \right] = \int_0^\infty du \frac{x-u}{z-u} K_N(x, u).
\]

(ii) Let \( \Phi \) be given by (1.1), then for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\mathbb{E} \left[ \prod_{l=1}^{N} \frac{1}{x-x_l} \right] = \frac{1}{2\pi i} \prod_{l=1}^{N} (\nu + N) \int_0^\infty dt \int_C du e^{\nu \Phi(v; t)} \frac{1}{z-u} \prod_{l=1}^{N} \frac{1}{v + a_l}
\]

where \( C \) is a counterclockwise contour encircling \(-a_1, \ldots, -a_N\).
Substituting $\Phi(-\cdot)$

Combination of (2.43), (2.44) and (2.35) completes the proof of (iii).

Proof. It is immediate that Proposition 2.3 implies (i). For (ii), noting that the leading term of the Laguerre polynomial is

$$n! L_n^0(x) = (-x)^n + \cdots,$$

dividing by $(-u)^{N-1}$ and taking the limit $u \to \infty$ in (2.15) we see that

$$c_{N,j} = \prod_{l=1}^{r} \frac{1}{\Gamma(\nu_l + N)} e^{-a_j} \prod_{l=1, l \neq j}^{N} \frac{1}{a_{N+1} - a_l}.$$  

(2.42)

Substituting $c_{N,j}$ in (2.33) and noting $\eta_j(u) = \Phi(-a_j; u)$, we obtain (2.39).

For (iii), we first introduce an auxiliary variable $a_{N+1}$ and set $\eta_{N+1}(u) = \Phi(-a_{N+1}; u)$. The fact that $([g_{i,j}]_{N+1})^{l'}[c_{i,j}]_{N+1} = I_{N+1}$ implies

$$c_{N+1,j} = \frac{\det[g_{i,j}]}{\det[g_{i,j}]_{N+1}} = \prod_{l=1}^{r} \frac{1}{\Gamma(\nu_l + N)} e^{-a_j} \prod_{l=1, l \neq j}^{N} \frac{1}{a_{N+1} - a_l}.$$  

(2.43)

Changing $N$ to $N + 1$ and using (2.43), as derived in (2.17) we obtain

$$\sum_{j=1}^{N+1} c_{j, N+1} x^{j-1} = \int_{0}^{\infty} du \nu_0 e^{\nu_0} \psi(u; x) e^{-a_{N+1}} \prod_{l=1}^{N} \frac{-u - a_l}{a_{N+1} - a_l}.$$  

(2.44)

Combination of (2.43), (2.44) and (2.35) completes the proof of (iii). \hfill \square

Again, for the eigenvalue PDF (2.5), let

$$Q_{N-1}(x) = \text{Res}_{z=x} E \left[ \prod_{l=1}^{N} \frac{1}{z - x_l} \right], \quad x \in (0, \infty),$$  

(2.45)

then use of Proposition 2.4 (ii) shows

$$Q_{N-1}(x) = \frac{1}{2\pi i} \prod_{l=1}^{r} \Gamma(\nu_l + N) \int_C dv e^v \psi(v; x) \prod_{l=1}^{N} \frac{1}{v + a_l};$$  

(2.46)

when $a_1, \ldots, a_N$ are pairwise distinct it is a special case of Proposition 2.20. Also, let

$$P_N(x) = E \left[ \prod_{l=1}^{N} (x - x_l) \right],$$  

(2.47)

combining Corollary 2.3 where $m$ is taken to be zero and Proposition 2.6 the correlation kernel $K_N$ given by (1.24) can be expressed as the single sum (2.20) in terms of $P_j(x)$ and $Q_j(x)$. Here, without loss of generality, it is assumed that $P_j(x)$ corresponds to the multi-parameters $a_1, \ldots, a_j$ while $Q_j(x)$ corresponds to $a_1, \ldots, a_{j+1}$.

Remark 2.2. In the special case $r = 0$, use of (1.17) shows that (2.40) reduces to

$$P_N(x) = \frac{(-1)^N e^x}{\Gamma(\nu_0 + 1)} \int_{0}^{\infty} u^{\nu_0} e^{-u} F_1(\nu_0 + 1; -ux) \prod_{l=1}^{N} (u + a_l) du.$$
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This same expression has been derived using combinatorial means in [22], and as the solution of a partial differential equation in [13]. Furthermore, in this case \( Q_{N-1}(x) \) and \( P_N(x) \) are so-called multiple functions of type I and II respectively, and [22] reduces to Corollary 7 in [20]; see [20] or [34] for more details, especially when the parameters \( a_j \)'s coalesce into \( D \) different values. For the case of \( a_1 = \cdots = a_N = 0 \) and general \( r \), \( Q_{N-1}(x) \) and \( P_N(x) \) are also multiple functions of type I and II associated with \( r+1 \) weights; see [38]. However, in the general case it remains as a challenge to identify a multiple orthogonal functions structure.

3. Hard edge limits

In this section we choose the source \( A \) such that all but possibly a fixed number \( m \) of the eigenvalues of \( A^*A \) are equal to \( bN \). Three regimes are distinguished: subcritical regime \( 0 < b < 1 \), critical regime \( b = 1 \) and supercritical regime \( b > 1 \); as to the former two regimes, see [35] and [36] for a relevant discussion on non-interacting Bessel paths which corresponds to the case \( r = 0 \). In the present paper we focus on the scaled hard edge limits in the three regimes and leave the bulk and soft-edge limits to a future work; for the case \( a_1 = \cdots = a_N = 0 \) the latter two limits have been established in [40]. The critical kernel results from a double scaling limit, and its functional form is our main result as stated in Section 1. As \( b \) increases from zero, we will describe a phase transition from the Meijer kernel (1.6) to the critical kernel (cf. Theorem 1.2), then to the finite product kernel (1.20) (cf. Theorem 3.2).

3.1. Limiting kernels. We first suppose that \( 0 < b < 1 \). The hard edge scaling in this parameter range is in fact independent of \( b \), and the hard edge correlation kernel (1.6) already known for the case \( b = 0 \) is reclaimed.

**Theorem 3.1 (Subcritical regime).** With the kernel (1.20), let

\[
a_1 = \cdots = a_N = bN.
\]

Then for \( 0 < b < 1 \), we have

\[
\lim_{N \to \infty} \frac{1}{(1-b)N} K_N \left( \frac{\xi}{(1-b)N}, \frac{\eta}{(1-b)N} \right) = K^{b,r}(\xi, \eta),
\]

where \( K^{b,r} \) is given by (1.7), valid uniformly for \( \xi, \eta \) in a compact set of \((0, \infty)\).

**Proof.** Introducing rescaled variables \( x = \xi/((1-b)N), y = \eta/((1-b)N) \) in (1.20) and substituting \( u, v \) by \( uN, vN \) respectively, we obtain

\[
\frac{1}{(1-b)N} K_N \left( \frac{\xi}{(1-b)N}, \frac{\eta}{(1-b)N} \right) = \int_0^\infty du \int_C \frac{e^{-N(f(u)-f(v))}}{2(1-b)\pi i} \frac{1}{u-v} \left( Nu^{\alpha} \Psi(Nu; \frac{\xi}{(1-b)N}) \Phi(Nv; \frac{\eta}{(1-b)N}) \right),
\]

where

\[
f(z) = z - \log(b + z).
\]

Consider now the exponent on the RHS of (3.3). Since

\[
f'(z) = 1 - \frac{1}{b + z},
\]

there is a saddle point \( z_0 = 1 - b \). We hereby deform the contour \( C \) into the union of two closed contours \( C_1 \cup C_2^- \) such that \( C_1 = \{ z : |z + b| = 1 \} \) and \( C_2^- \) is a clockwise
contour encircling the segment $[0, 1 - b]$ but not $-b$. For instance, we can choose $C_2$ as the union of two segments from $-0.5b$ to $-b + e^{\pm i\epsilon}$ respectively and an arc \( \{z : z = -b + e^{i\theta}, -\epsilon \leq \theta \leq \epsilon \} \) for some small positive \( \epsilon \). With such a choice, we divide the integration over $C$ into two parts, and furthermore rewrite the double integral on the RHS of (3.3) as a sum of two integrals, that is,

\[
\frac{1}{(1 - b)N} K_N \left( \frac{\xi}{(1 - b)N}, \frac{\eta}{(1 - b)N} \right) = \text{p.v.} \int_0^\infty du \int_{C_1} dv \Phi(\cdot) + \text{p.v.} \int_0^\infty du \int_{C_2} dv \Phi(\cdot)
\]

\[
:= I_1 + I_2.
\]

It is worth stressing that, from (3.3), we can put some restrictions on the range of $u, v$ in the above integrals such that $u \neq 1 - b$ and $v \neq \pm i$. This is done for the convenience of subsequent asymptotic analysis only.

As $N \to \infty$, we claim that the leading contribution of the double integral on the RHS of (3.3) comes from the range of $u \in (0, 1 - b)$ and $v \in C_2$. Actually, for $I_2$, when $u > 1 - b$ the $v$-integral vanishes by Cauchy’s theorem since the integrand does not have any singularities inside $C_2$, while for $0 < u < 1$ application of the residue theorem gives

\[
I_2 = \frac{1}{1 - b} \int_0^{1 - b} du (Nu)^{\nu_0} \Phi(Nu; \frac{\xi}{(1 - b)N}) \Phi(Nu; \frac{\eta}{(1 - b)N}).
\]

Using the asymptotic expansion of the function \( _1F_1 \) for the large argument (cf. Theorem 4.2.2 and Corollary 4.2.3, [7]), for large $N$ we have

\[
_1F_1(\nu_0 + s; \nu_0 + 1; -Nu) = \frac{\Gamma(\nu_0 + 1)}{\Gamma(1 - s)} (Nu)^{-\nu_0 - s} \left( 1 + O\left( \frac{1}{N} \right) \right), \quad \text{Re} \, v > 0,
\]

and

\[
_1F_1(\nu_0 + s; \nu_0 + 1; -Nu) = \frac{\Gamma(\nu_0 + 1)}{\Gamma(\nu_0 + s)} (-Nu)^{s-1} e^{-Nu} \left( 1 + O\left( \frac{1}{N} \right) \right), \quad \text{Re} \, v < 0.
\]

Keeping in mind (1.13) and (1.16), by definition of the Meijer G-function (1.7) we have from (3.8) that

\[
(Nu)^{\nu_0} \Phi(Nu; \frac{\eta}{(1 - b)N}) \sim G^{r+1,0}_{0,r+2}(\nu_0, \ldots, \nu_r, 0 \mid \frac{\eta}{1 - b}).
\]

Here and below we use the notation \( f_N \sim g_N \) to mean that \( \lim_{N \to \infty} f_N/g_N = 1 \).

On the other hand, consideration of the definition (1.13) shows

\[
\Phi(Nu; \frac{\xi}{(1 - b)N}) \sim G^{1,0}_{0,r+2}(0, -\nu_0, \ldots, -\nu_r \mid \frac{u\xi}{1 - b}),
\]

where use has been made of the identity (cf. eqn (14), [11 Sect. 5.2])

\[
G^{1,0}_{0,r+2}(0, -\nu_0, \ldots, -\nu_r \mid z) = \prod_{i=0}^r \frac{1}{\Gamma(\nu_i + 1)} {}_0F_{r+1}(\nu_0 + 1, \ldots, \nu_r + 1; -z).
\]

Combining (3.7), (3.10) and (3.11), and changing variables we get

\[
I_2 \sim K^{1-r}(\xi, \eta).
\]

Next, we deal with the integral $I_1$ and show that it is negligible. In this case because of different asymptotic forms of $_1F_1$ given in (3.8) and (3.9), we divide $I_1$
into two parts as

$$I_1 = \text{p.v.} \int_0^\infty du \int_{C_{1,+}} dv \frac{e^{-N(\mathcal{f}(u) - \mathcal{f}(v))}}{u - v} \left(\frac{u}{v}\right)_{\nu_0}^\nu,$$

where $C_{1,+} = C_1 \cap \{v : \text{Re} \, v > 0\}$ and $C_{1,-} = C_1 \cap \{v : \text{Re} \, v < 0\}$. Notice that for $0 < b < 1$ one can easily check that $\text{Re}\{\mathcal{f}(u)\}$ attains its global minimum at $u = 1 - b$ over $(0, \infty)$, while $\text{Re}\{\mathcal{f}(v)\}$ attains its global maximum at $v = 1 - b$ over $C_1$. Therefore, for $I_{11}$ combining (1.14), (1.16), (3.8) and (3.11) we have

$$I_{11} \sim \text{p.v.} \int_0^\infty du \int_{C_{1,+}} dv \frac{e^{-N(\mathcal{f}(u) - \mathcal{f}(v))}}{u - v} \left(\frac{u}{v}\right)_{\nu_0}^\nu G_{0,r+2}^{1,0}(0, -\nu_0, \ldots, -\nu_r, 1, \ldots, 1 - b).$$

For this, the standard steepest descent argument shows that the main contribution comes from the neighbourhood of the saddle point $z_0 = 1 - b$, namely,

$$I_{11} = O(1/\sqrt{N}).$$

Similarly, for $I_{12}$ combination of (1.14), (1.16), (3.9) and (3.11) then gives us

$$I_{12} \sim \int_0^\infty du \int_{C_{1,-}} dv \frac{e^{-N(\mathcal{f}(u) - \mathcal{f}(1-b))}}{u - v} G_{0,r+2}^{1,0}(0, -\nu_0, \ldots, -\nu_r, 1, \ldots, 1 - b)$$

$$\times u^{\nu_0} e^{-N(\log(b+v)+1-b)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \left(\frac{\eta}{1-b}\right)^{-s} (-v)^{s-1} N^{\nu_0+2s-1} \prod_{i=1}^r \Gamma(\nu_i + s),$$

for which the integrals of $u$ and $v$ respectively afford us bounds $O(1/\sqrt{N})$ and $O(N^{\nu_0+2c-1} e^{-1(1-b)N})$. Together, we obtain the exponential decay estimation

$$I_{12} = O(N^{\nu_0+2c-3/2} e^{-(1-b)N}).$$

Combining (3.13), (3.16) and (3.18), we arrive at the equation (3.22). Furthermore, it is clear that the previously derived estimates are valid uniformly for $\xi, \eta$ in a given compact set of $(0, \infty)$.

**Remark 3.1.** When all the parameters $a_i$'s are null, if we understand the double integral representation (1.20) as described in Remark 1.1, then the same argument as in the proof of Theorem 3.1 is also applicable. This gives another derivation of (1.6) different from that in [38].

We turn to proofs of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** Rescaling variables in (1.20), we have

$$\frac{1}{N} K_N \left(\frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}}\right) =$$

$$\frac{1}{N} \int_0^\infty du \int_C \frac{e^{-N(\mathcal{f}(u) - \mathcal{f}(v))}}{u - v} (Nu)^{\nu_0} \mathcal{P}(Nu; \xi/\sqrt{N}) \Phi(Nv; \eta/\sqrt{N}).$$

where $f(z) = z - \log(1 + z/b)$ with $b = (1 - \tau/\sqrt{N})^{-1}$. If $b$ is equal to the critical value 1, then the saddle point of $f(z)$ is $z_0 = 0$. This time, for small $\delta > 0$ we choose the contour as

$$C = \{z = -1 + (1 + 2\delta) e^{i\theta} : \theta_0 \leq |\theta| \leq \pi\} \cup C_{\mathcal{A}_- \mathcal{O}_A},$$

(3.20)
where
\[ \theta_0 = \arccos \frac{1 + \delta}{1 + 2\delta}, \quad A_\pm = -1 + (1 + 2\delta)e^{\pm i\theta_0}, \]
(3.21)
and \( \mathcal{L}_{A-OA} \) denotes the union of two line segments from the point \( A_- \) to the origin to the point \( A_+ \). It is clear that \( A_\pm = (\delta, \pm \sqrt{(2 + 3\delta)\delta}) \), and the intersections of the \( y \)-axis and the contour \( \mathcal{C} \) are \((0, \pm 2\sqrt{(1 + \delta)\delta})\). Moreover, the four points come close to the origin as \( \delta \to 0 \), which permits us to use the Taylor series expansion of \( f(v) \) for any \( v \in \mathcal{C}_+ \) defined below (3.22).

First, we divide the integral on the RHS of (3.19) into two parts
\[ \frac{1}{\sqrt{N}} K_N \left( \frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}} \right) = \int_0^\infty du \int_{\mathcal{C}_-} dv (\cdot) + \int_0^\infty du \int_{\mathcal{C}_+} dv (\cdot) := I_- + I_+, \]
(3.22)
where \( \mathcal{C}_- = \mathcal{C} \cap \{v : \text{Re} v < 0\} \) and \( \mathcal{C}_+ = \mathcal{C} \cap \{v : \text{Re} v > 0\} \). We claim that the dominant contribution to (3.19) comes from the neighbourhoods of \( u_0 = 0 \) and \( v_0 = 0 \), so we need to expand the function \( f(z) \) at \( z_0 = 0 \). With the double scaling (1.21), we obtain the Taylor series
\[ f(z) = \frac{\tau z}{\sqrt{N}} + \frac{1}{2} \left( 1 - \frac{\tau}{\sqrt{N}} \right) z^2 - \frac{1}{3} \left( 1 - \frac{\tau}{\sqrt{N}} \right)^3 z^3 + \cdots. \]
(3.23)
Therefore for \( I_+ \), combining (1.13), (1.14), (1.16) and (3.8), together with the relation (3.12) and the definition of Meijer G-function (1.7), we see that
\[ I_+ \sim \frac{\sqrt{N}}{2\pi i} \int_0^\infty du \int_{\mathcal{C}_+} dv \frac{e^{-N(f(u) - f(v))}}{u - v} \left( \frac{u}{v} \right)^{\nu_0} G_{0, r+2}^{1, 0} \left( \begin{array}{c} 0, -\nu_0, \ldots, -\nu_r \\ \sqrt{N} u \xi \end{array} \right) G_{0, r+2}^{r+1, 0} \left( \begin{array}{c} \nu_0, \ldots, \nu_r \\ \sqrt{N} v \eta \end{array} \right). \]
(3.24)
Substituting (3.23) into (3.24) and rescaling \( u, v \) by \( u/\sqrt{N}, v/\sqrt{N} \), we conclude that \( I_+ \) converges to the kernel defined by (1.22), uniformly for \( \xi, \eta \) in a compact set of \((0, \infty)\) and for \( \tau \) in a compact set of \( \mathbb{R} \).

Secondly, for the integral \( I_- \), combination of (1.13), (1.14), (1.16) and (3.9) yields
\[ I_- \sim \frac{1}{2\pi i} \int_0^\infty du \int_{\mathcal{C}_-} dv \frac{e^{-Nf(u)-N\log(1+\nu/b)}}{u - v} u^{\nu_0} G_{0, r+2}^{1, 0} \left( \begin{array}{c} 0, -\nu_0, \ldots, -\nu_r \\ \sqrt{N} u \xi \end{array} \right) \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \eta^{-s} \frac{(-v)^{s-1} N^{s-1}}{\prod_{l=1}^r \Gamma(\nu_l + s)}. \]
(3.25)

Since for sufficiently large \( N \),
\[ \text{Re}\{\log(1 + \nu/b)\} = \frac{1}{2} \log \left( (1 + 2\delta)^2 + \mathcal{O} \left( \frac{\tau}{\sqrt{N}} \right) \right) > \log(1 + \delta) \]
(3.26)
holds true uniformly for \( \tau \) in a compact set of \( \mathbb{R} \) and for \( v \in \mathcal{C}_- \), use of the steepest descent argument leads to an exponential decay
\[ I_- = \mathcal{O} \left( N^{\nu_0-1+1.5c} e^{-N\log(1+\delta)} \right). \]
(3.27)

Lastly, by combining the foregoing results for \( I_- \) and \( I_+ \), we then complete the proof. \( \square \)
Proof of Theorem 3.3. Rescaling variables in \( (1.20) \), we have

\[
\frac{1}{\sqrt{N}} K_N \left( \frac{\xi}{\sqrt{N}}, \frac{\eta}{\sqrt{N}} \right) = \frac{1}{2\pi i} \int_0^\infty du \int_c dv \, N^{\nu_0+1/2} u^{\nu_0} e^{-N(f(u) - f(v))} \frac{u-v}{u-v} \\
\times \prod_{j=1}^m \frac{(v+b)(u+\sigma_j/\sqrt{N})}{(u+b)(v+\sigma_j/\sqrt{N})} \Psi(Nu; \frac{\xi}{\sqrt{N}}) \Phi(Nv; \frac{\eta}{\sqrt{N}}),
\]

(3.28)

where \( f(z) = z - \log(1+z/b) \) with \( b = (1 - \tau/\sqrt{N})^{-1} \).

Proceeding as in the proof of Theorem 1.2, Taylor expanding \( f(z) \) at \( z = 0 \), and rescaling \( u, v \) by \( u/\sqrt{N}, v/\sqrt{N} \), we can complete the proof.

We next consider the supercritical case, that is \( b > 1 \). For \( r = 0 \), the limiting eigenvalue density has support \([L_1, L_2]\) with \( L_1 > 0 \) (thus the left-most end changes from the hard to the soft edge as \( b \) increases beyond unity; see e.g. \( [35] \)). However, when \( r > 0 \) considerations from free probability theory suggest that the support will include the origin for general \( b \). Nonetheless, a particular tuning and scaling of the supercritical case can be given which effectively separates a bunch of eigenvalues near the origin from the rescaled left-end support.

**Theorem 3.2 (Supercritical regime).** With the kernel \( (1.20) \), for a fixed positive integer \( m \) let

\[
a_j = \sigma_j b/(b-1), \quad j = 1, \ldots, m \quad \text{and} \quad a_k = bN, \quad k = m+1, \ldots, N,
\]

(3.29)

where \( b > 1 \) and \( \sigma_1, \ldots, \sigma_m > 0 \). Then we have

\[
\lim_{N \to \infty} (1 - \frac{1}{b}) K_N \left( \frac{1}{b} \xi, \frac{1}{b} \eta \right) = \frac{1}{2\pi i} \int_0^\infty du \int \gamma \, \left( \frac{u}{\kappa} \right)^{\nu_0} e^{-\nu u + v} \Psi(\nu; \kappa \xi) \Phi(\nu; \kappa \eta) \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j} =: \tilde{K}^{h,r}_m(\xi, \eta; \kappa; \sigma),
\]

(3.30)

where \( \kappa = (b-1)/b \), \( \Psi, \Phi \) are given by \( (1.13), (1.14) \), and \( \gamma \) is a contour in the left-half plane going from \( e^{i(\pi + \theta)} \cdot \infty \) to \( e^{i(\pi - \theta)} \cdot \infty \) with \( \theta \in (0, \pi/2) \) such that \( -\sigma_1, \ldots, -\sigma_m \) lie on its left side.

**Proof.** We have from \( (1.20) \) that

\[
(1 - \frac{1}{b}) K_N \left( \frac{1}{b} \xi, \frac{1}{b} \eta \right) = \frac{1}{2\pi i} \int_0^\infty du \int \gamma \, \left( \frac{u}{\kappa} \right)^{\nu_0} e^{-\nu u + v} \Psi(\nu; \kappa \xi) \Phi(\nu; \kappa \eta) \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j} \frac{1 + u/(bN)}{1 + v/(bN)}^{N-m}.
\]

(3.31)

Then by taking the limit, the desired result immediately follows from change of variables.

In the special case \( r = 0 \), upon making use of \( (1.17) \) and \( (1.18) \), we see from \( (3.30) \) that

\[
e^{i(\eta - \xi)} \tilde{K}^{h,0}_m(\xi, \eta; \kappa; \sigma) = \frac{1}{2\pi i} \frac{\eta^{\nu_0}}{\Gamma(\nu_0 + 1)^2} \int_0^\infty du \int \gamma \, \left( \frac{u}{\kappa} \right)^{\nu_0} e^{-\nu u + v} \Psi(\nu; \kappa \xi) \Phi(\nu; \kappa \eta) \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j}.
\]

(3.32)
The RHS is independent of $\kappa$, and so is the corresponding correlation functions since the factor $e^{\kappa_{\eta} - \xi}$ cancels out of the scaled analogue of (2.10). Moreover, comparison with (1.20) in the case $r = 0$, $N = m$, $\{a_i\} = \{\sigma_i\}$, and after substituting (1.17) and (1.18) shows that the RHS of (3.32) is equal to $K_{m}(\xi, \eta)|_{\{a_i\} = \{\sigma_i\}}$. For general $r \geq 1$ we expect the correlations implied by (3.30) to be the same as those for $K_{m}(\xi, \eta)|_{\{a_i\} = \{\sigma_i\}}$. Upon comparing (1.20) with (3.30), this is immediate for $\kappa = 1$. However, the mechanism which makes the correlations implied by (3.30) in the cases $r \geq 1$ remains to be clarified.

Remark 3.2. If we strengthen the results in Theorems 1.2, 1.3, 3.1 and 3.2 from uniform convergence into the trace norm convergence of the integral operators with respect to the correlation kernels, then as a direct consequence we have the limiting gap probabilities after rescaling, especially including the smallest eigenvalue distribution; see [21, Chapters 8 & 9]. We postpone more detailed discussion in Section 3.3 for the supercritical kernel. Since the proof of trace norm convergence is only a technical elaboration that confirms a well-expected result, we do not give the details.

3.2. Characteristic polynomials. In this subsection we want to evaluate scaling limits for the ratio of characteristic polynomials according to three different regimes.

Theorem 3.3. With the eigenvalue PDF (2.5), fix $m \in \{0, 1, 2, \ldots\}$ and let

$$a_m = \cdots = a_N = Nb.$$ 

(i) Set $a_j = Nb_j$ with $b_j > 0$ for $j = 1, \ldots, m$, if $0 < b < 1$, then for $\zeta \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N \to \infty} \frac{1}{(1 - b)N} \mathbb{E} \left[ \prod_{l=1}^{N} \frac{x_l - \xi/((1 - b)N)}{x_l - \zeta/((1 - b)N)} \right] = \int_{0}^{\infty} du \frac{\xi - u}{\zeta - u} K_{m}^{b, r}(\xi, u). \quad (3.33)$$

(ii) Set $a_j = \sqrt{N}\sigma_j$ with $\sigma_j > 0$ for $j = 1, \ldots, m$, if $b = 1/(1 - \tau/\sqrt{N})$ with $\tau \in \mathbb{R}$, then for $\zeta \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N \to \infty} \frac{1}{\sqrt{N}} \mathbb{E} \left[ \prod_{l=1}^{N} \frac{x_l - \xi/\sqrt{N}}{x_l - \zeta/\sqrt{N}} \right] = \int_{0}^{\infty} du \frac{\xi - u}{\zeta - u} \tilde{K}_{m}^{b, r}(\xi, u; \tau, \sigma). \quad (3.34)$$

(iii) Set $a_j = \sigma_j b/(b - 1)$ with $\sigma_j > 0$ for $j = 1, \ldots, m$, if $b > 1$ and $m \geq 1$, then for $\zeta \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N \to \infty} (1 - \frac{1}{b}) \mathbb{E} \left[ \prod_{l=1}^{N} \frac{x_l - (1 - \frac{1}{b})\xi}{x_l - (1 - \frac{1}{b})\zeta} \right] = \int_{0}^{\infty} du \frac{\xi - u}{\zeta - u} \tilde{K}_{m}^{b, r}(\xi, u; 1 - \frac{1}{b}; \sigma). \quad (3.35)$$

Proof. By Proposition 2.6 Theorems 3.1, 3.2 and 3.4 imply the sought results although a minor modification in the proof of Theorem 3.4 is required in relation to (3.33) (in the same circumstance the limiting subcritical kernel still holds true).

Likewise, based on Proposition 2.6 we can prove the following theorem concerning the average of one single characteristic polynomial or its inverse. For this purpose we introduce four sets of generalised multiple functions (we say generalised since only for $r = 0$ do we know the multiple polynomial system; recall Remark 2.2) of types II and I with $m$ parameters $\sigma_1, \ldots, \sigma_m > 0$. For $k = 1, 2, \ldots, m$, we
define two sets of generalised multiple functions by
\[
\Gamma^{(k)}(x; \sigma_1, \ldots, \sigma_{k-1}) = \int_0^\infty du u^{\nu_0} e^{-u - \frac{1}{2} u^2} \times G_{0,r+2}^{1,0}(0, -\nu_0, -\nu_1, \ldots, -\nu_r | xu) \prod_{j=1}^{k-1} (u + \sigma_j), \tag{3.36}
\]
and
\[
\tilde{\Gamma}^{(k)}(x; \sigma_1, \ldots, \sigma_k) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dv v^{1/2} e^{v} \times G_{0,r+2}^{1,0}(\nu_0, \nu_1, \ldots, \nu_r, 0 | xv) \prod_{j=1}^k \frac{1}{v + \sigma_j}. \tag{3.37}
\]
while for \(0 < \kappa \leq 1\) two sets of Laguerre-like generalised multiple functions are defined by
\[
\mathcal{L}^{(k)}(x; \kappa; \sigma_1, \ldots, \sigma_{k-1}) = \int_0^\infty du \frac{(u/\kappa)^{\nu_0} e^{-u} \Phi(u/\kappa; \kappa x)^k}{\prod_{l=1}^{k-1} (u + \sigma_l)}, \tag{3.38}
\]
and
\[
\tilde{\mathcal{L}}^{(k)}(x; \kappa; \sigma_1, \ldots, \sigma_k) = \frac{1}{2\pi i} \int_{\gamma} dv v \Phi(v/\kappa; \kappa x) \prod_{l=1}^k \frac{1}{v + \sigma_l}. \tag{3.39}
\]
Here \(\gamma\) is a path starting and ending at \(-\infty\), and encircling \(-\sigma_1, \ldots, -\sigma_m\) once in the positive direction.

**Theorem 3.4.** With the eigenvalue PDF (2.7), fix \(m \in \{0, 1, 2, \ldots\}\) and let
\[
a_m = \cdots = a_N = Nb.
\]
(i) Set \(a_j = Nb_j\) with \(b_j > 0\) for \(j = 1, \ldots, m\), if \(0 < b < 1\), then
\[
\lim_{N \to \infty} \frac{-\sqrt{N}}{\mathcal{Y}_N^{(\text{sub})}} \mathbb{E}\left[ \prod_{l=1}^N \frac{1}{x_l - \xi/((1 - b)N)} \right] = \int_0^\infty du \frac{1}{\zeta - u} G_{0,r+2}^{1,0}(\nu_0, \nu_1, \ldots, \nu_r, 0 | u), \tag{3.40}
\]
\[
\lim_{N \to \infty} \frac{\sqrt{N}}{\mathcal{Y}_N^{(\text{sub})}} \mathbb{E}\left[ \prod_{l=1}^N \frac{1}{x_l - \xi/((1 - b)N)} \right] = G_{0,r+2}^{1,0}(0, -\nu_0, -\nu_1, \ldots, -\nu_r | \xi) \tag{3.41}
\]
where
\[
\mathcal{Y}_N^{(\text{sub})} = (-1)^N \sqrt{2\pi} N^{\nu_0 + N} e^{-(1 - b)N} \prod_{l=1}^r \Gamma(\nu_l + N) \prod_{j=1}^m (1 - b + b_j). \tag{3.42}
\]
(ii) Set \(a_j = \sqrt{N} \sigma_j\) with \(\sigma_j > 0\) for \(j = 1, \ldots, m\), if \(b = 1/(1 - \tau/\sqrt{N})\), then
\[
\lim_{N \to \infty} \frac{-\sqrt{N}}{\mathcal{Y}_N^{(\text{cri})}} \mathbb{E}\left[ \prod_{l=1}^N \frac{1}{x_l - \xi/\sqrt{N}} \right] = \int_0^\infty du \frac{1}{\zeta - u} \tilde{\Gamma}^{(m)}(u; \sigma_1, \ldots, \sigma_m), \tag{3.43}
\]
\[
\mathcal{Y}_N^{(\text{cri})} = \frac{(-1)^N \sqrt{2\pi} N^{\nu_0 + N} e^{-(1 - b)N}}{\sqrt{N}} \prod_{l=1}^r \Gamma(\nu_l + N) \prod_{j=1}^m (1 - b + \sigma_j). \tag{3.44}
\]
\[
\lim_{N \to \infty} \sqrt{N} \prod_{l=1}^{r} (\nu_l + N) \mathcal{Y}_N^{(\text{cri})} \mathbb{E} \left[ \prod_{l=1}^{N} (x_l - \xi/\sqrt{N}) \right] = \Gamma^{(m+1)}(\xi; \sigma_1, \ldots, \sigma_m) \tag{3.44}
\]

where
\[
\mathcal{Y}_N^{(\text{cri})} = (-1)^N N^{N+(m_0-m)/2} e^{\sqrt{N} \tau + \tau^2/2} \prod_{l=1}^{r} \Gamma(\nu_l + N). \tag{3.45}
\]

(iii) Set \( a_j = \sigma_j b/(b-1) \) with \( \sigma_j > 0 \) for \( j = 1, \ldots, m \), if \( b > 1 \), then
\[
\lim_{N \to \infty} \frac{-b}{b-1} \mathcal{Y}_N^{(\text{sup})} \mathbb{E} \left[ \prod_{l=1}^{N} (x_l - (1-1/\xi)\xi) \right] = \int_0^\infty du \frac{1}{\xi - u} \widetilde{\mathcal{L}}^{(m)}(u; 1-1/b; \sigma_1, \ldots, \sigma_m), \tag{3.46}
\]
\[
\lim_{N \to \infty} \frac{b}{b-1} \prod_{l=1}^{r} (\nu_l + N) \mathcal{Y}_N^{(\text{sup})} \mathbb{E} \left[ \prod_{l=1}^{N} (x_l - (1-1/b)\xi) \right] = \mathcal{L}^{(m)}(\xi; 1-1/b; \sigma_1, \ldots, \sigma_m) \tag{3.47}
\]

where
\[
\mathcal{Y}_N^{(\text{sup})} = (-1)^N (bN)^{N-m} (b/(b-1))^m. \tag{3.48}
\]

3.3. Inter-relationships between scaled limits. We have established scaled limits at the hard edge: the subcritical kernel \( \mathcal{K}_{m,r}^{(h)}(\xi, \eta; \tau) \) given by (1.6), critical kernel \( \mathcal{K}_{m,r}^{(c)}(\xi, \eta; \tau) \) by (1.22), deformed critical kernel \( \mathcal{K}_{m,r}^{(d)}(\xi, \eta; \tau, \sigma) \) by (1.25) and supercritical kernel \( \mathcal{K}_{m,r}^{(s)}(\xi, \eta; \tau, \sigma) \) by (3.30), and also the generalised multiple functions defined by (3.30)\textendash{(3.39)}. We now explore their inter-relations, cf. Corollary 2.3.

Proposition 3.5. We have
\[
\mathcal{K}_{m,r}^{(h)}(x, y; \tau, \sigma) = \mathcal{K}_{m,r}^{(c)}(x, y; \tau) + \sum_{k=1}^{m} \mathcal{L}^{(k)}(x; \sigma_1, \ldots, \sigma_{k-1}) \mathcal{L}^{(k)}(y; \sigma_1, \ldots, \sigma_k), \tag{3.49}
\]
and
\[
\mathcal{K}_{m,r}^{(s)}(x, y; \tau, \sigma) = \sum_{k=1}^{m} \mathcal{L}^{(k)}(x; \tau, \sigma_1, \ldots, \sigma_{k-1}) \mathcal{L}^{(k)}(y; \tau, \sigma_1, \ldots, \sigma_k). \tag{3.50}
\]

Proof. By use of the relation (2.20), noting the definition of involved functions (3.30)\textendash{(3.39)}, term-by-term integration immediately implies the above two formulas. Here use has been made of \( \mathcal{K}_{m,r}^{(d)}(x, y; \tau, \sigma) = 0 \) for the second formula. \( \Box \)

The limiting correlation kernel (3.30) in the supercritical regime is closely related to the product (1.4) with the density (2.5) (we suspect that they are the same thing after being multiplied by the factor \( g(\kappa; \eta)/g(\kappa; \xi) \) for some properly chosen function \( g \)). In particular, when \( r = 0 \) it is equivalent to the kernel for the \( m \times m \) Laguerre Unitary Ensemble (LUE for short) with a source; cf. (3.32). Furthermore, as \( \sigma_1, \ldots, \sigma_m \) go to zero from the above, we have the finite LUE distribution. We summarize some properties of LUE in the following proposition.

Proposition 3.6. The eigenvalue PDF of the \( m \times m \) LUE with a parameter \( \nu_0 \) has the form
\[
f_{\text{LUE}}(x_1, \ldots, x_m) = \frac{1}{m!} \det[K_m(x_i, x_j)]_{i,j=1}^m, \tag{3.51}
\]
where
\[
K_m(x, y) = \frac{m!}{\Gamma(m + \nu_0)} (xy)^{\nu_0} e^{-\frac{1}{2}(x+y)} \frac{L_{\nu_0}^m(x)L_{\nu_0}^m(y) - L_{\nu_0}^m(x)L_{\nu_0}^m(y)}{y-x}. \tag{3.52}
\]
The smallest eigenvalue denoted by \( \lambda_{\min}^{(m)} \) satisfies the relation
\[
P(\lambda_{\min}^{(m)} > s) = \int_{s}^{\infty} \cdots \int_{s}^{\infty} f_{\text{LUE}}(x_1, \ldots, x_m) dx_1 \cdots dx_m, \tag{3.53}
\]
and in particular \( \lambda_{\min}^{(1)} \) is the gamma distribution with density
\[
\frac{1}{\Gamma(\nu + 1)} s^{\nu_0} e^{-s}, \quad 0 < s < \infty. \tag{3.54}
\]

**Proof.** (3.54) is a standard result in random matrix theory, for instance, see Proposition 5.1.3 and the equations (5.46)–(5.48) [21]. The smallest eigenvalue distribution immediately follows from the definition. \( \square \)

### 4. Product with truncated unitary matrices

The derivation of the double contour integral expression (1.20) for the correlation kernel is expected to be applicable to a wider class of biorthogonal ensembles, specifically to those characterized by the form of (2.1) with \( h(a_i, x) \) for some appropriate function of two variables \( h \) and \( N \) generic parameters \( a_1, \ldots, a_N \). For general \( h \), the related results will be reported elsewhere. In this section we consider the specific case of the biorthogonal ensemble corresponding to the product of \( r \) truncated unitary matrices and one shifted mean Ginibre matrix and derive a double integral representation of the correlation kernel and analyze the scaled limits at the hard edge. Other types of products \( X_r \ldots X_1 Z \), where each \( X_j \) is a Ginibre or truncated unitary matrix while \( Z \) is a spiked Wishart matrix of the form \( G_0 \Sigma \) or a triangular random matrix (cf. [10, 23], are presently under consideration [39].

Explicitly, instead of (1.24), we now consider the matrix product
\[
Y = T_r \cdots T_1 (G_0 + A), \tag{4.1}
\]
where each \( T_j \) is an \((N + \nu_j) \times (N + \nu_{j-1})\) truncation of a Haar distributed unitary matrix of size \( M_j \times M_j \) and \( G_0 \) is an \((N + \nu_0) \times N\) standard complex Gaussian matrix while \( A \) is of size \((N + \nu_0) \times N\) and fixed. Here \( \nu_{-1} = 0, \nu_0, \ldots, \nu_r \) are the nonnegative integers and \( \mu_j := M_j - N > \nu_j \) (for the general \( \nu_j > -1 \) the analysis below is also applicable). In the case that the matrix \( (G_0 + A) \) is absent, this product has been studied in a recent paper [32].

An analogue of Proposition 1.1 for the correlation kernel can be given. As in Proposition 1.1, two auxiliary functions are required, and so as to stress the structural similarities, similar notation is used. Specifically, with \( r = 1, 2, \ldots, \) and \( 0 \leq q \leq r \), the first is defined to be
\[
\Psi_q(u; x) = \frac{1}{(2\pi i)^r} \int_{0, \infty}^{\Gamma} dt \int dt \prod_{l=1}^{q} t_l^{\mu_l} e^{-t_l} \times \prod_{l=1}^{r} \int w_l^{\nu_l-1} e^{w_l} \exp \left\{ x \frac{t_1 \cdots t_q}{w_1 \cdots w_r} \right\} \left. _0 F_1 \right( \nu_0 + 1; -x \frac{t_1 \cdots t_q}{w_1 \cdots w_r} \right), \tag{4.2}
\]
where \( \Gamma = \gamma_1 \times \cdots \times \gamma_r \), and \( \gamma_1, \ldots, \gamma_r \) are paths starting and ending at \( -\infty \) and encircling the origin anticlockwise, while the other reads
\[
\Phi_q(v; y) = \frac{1}{2\pi i} \int_{e^{-i\infty}}^{e^{i\infty}} ds y^{-s} \phi(v; s) \prod_{l=1}^{q} \Gamma(\nu_l + s), \tag{4.3}
\]
where \( \phi(v; s) \) is given in (1.15) and \( c > - \min\{\nu_0, \nu_1, \ldots, \nu_r\} \).

**Proposition 4.1.** Let \( Y \) be defined by (2.1), and suppose that all eigenvalues \( a_1, \ldots, a_N \) of \( A^*A \) are positive. The eigenvalue PDF of \( Y^*Y \) can be written as

\[
P_N(x_1, \ldots, x_N) = \frac{1}{N!} \det [K_N(x_i, x_j)]_{i,j=1}^N \tag{4.4}
\]

with correlation kernel

\[
K_N(x, y) = \frac{1}{2\pi i} \int_0^\infty du \int_C \frac{dv}{v} e^{-u+v} \Psi_r(u; x) \Phi_r(v; y) \frac{1}{u} \prod_{l=1}^N \frac{u + a_l}{v + a_l}, \tag{4.5}
\]

where \( C \) is a counterclockwise contour encircling \(-a_1, \ldots, -a_N\) but not \( u \).

**Proof.** Starting with the eigenvalue PDF (2.1) of \((G_0 + A)^*(G_0 + A)\), application of [22] Corollary 2.4 \( r \) times in succession shows that the eigenvalue PDF of \( Y^*Y \) is proportional to

\[
\det[\eta_i(x_i)]_{i,j=1}^N \det[\xi_i(x_i)]_{i,j=1}^N, \tag{4.6}
\]

where \( \eta_i(x) = x^{t-1} \) and with \( T = t_1 \cdots t_r \),

\[
\xi_i(x) = \frac{1}{\Gamma(\nu_0 + 1)} \int_{(0, 1)^r} dt_1 \cdots dt_r \prod_{l=1}^r t_l^{\nu_l-1}(1-t_l)^{\mu_l-\nu_l-1} \left( \frac{y}{T} \right)^{\nu_0} e^{-\frac{y}{T} a_0} F_1(\nu_0 + 1; a_1 \frac{x}{T}). \tag{4.7}
\]

Next, we proceed as in the proof of Proposition 1.1. Our first task is to compute \( g_{i,j} := \int_0^\infty \eta_i(x) \xi_j(x) dx \). For this purpose, we note that application of the Mellin transform gives

\[
\int_0^\infty y^{s-1} \xi_j(y) dy = \phi(-a_j; s) \prod_{l=1}^r B(\nu_l + s, \mu_l - \nu_l), \tag{4.8}
\]

where the notation \( B(a, b) \) refers to the gamma function evaluation of the beta integral and \( \phi(v; s) \) is given in (1.15), while use of the inverse Mellin transform gives

\[
\xi_j(y) = \Phi_r(-a_j; y) \prod_{l=1}^r \Gamma(\mu_l - \nu_l), \tag{4.9}
\]

where \( \Phi_r \) is defined in (1.13) with \( q = r \). Combining (4.8), (1.15) and (2.1), we obtain

\[
g_{i,j} = (i-1)! e^{a_j} L_{i-1}^{\nu_0}(a_j) \prod_{l=1}^r B(\nu_l + i, \mu_l - \nu_l). \tag{4.10}
\]

According to Proposition 2.22, with \( G = [g_{i,j}]_{i,j=1}^N \) and \( C = (G^{-1})^t \), the entries \( c_{i,j} \) of \( C \) then satisfy

\[
e^{a_k} \sum_{i=1}^N (i-1)! L_{i-1}^{\nu_0}(-a_k) \prod_{l=1}^r B(\nu_l + i, \mu_l - \nu_l) c_{i,j} = \delta_{j,k}. \tag{4.11}
\]

Without loss of generality we assume that \( a_1, \ldots, a_N \) are pairwise distinct. The above equations imply

\[
\sum_{i=1}^N (i-1)! L_{i-1}^{\nu_0}(u) \prod_{l=1}^r B(\nu_l + i, \mu_l - \nu_l) c_{i,j} = e^{-a_j} \prod_{l=1, l \neq j}^N \frac{-u + a_l}{a_j - a_l}. \tag{4.12}
\]
which can be verified by noting that both sides are polynomials of degree \( N - 1 \) in \( u \) which are equal at \( N \) different points. Using this implicit formula for \( \{c_{i,j}\} \) and the integral representations

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma w^{-z}e^wdw,
\]

we have from (2.9) that with \( T = t_1 \cdots t_r \) and \( W = w_1 \cdots w_r \)

\[
K_N(x, y) = \frac{1}{(2\pi \sqrt{-1})^r} \sum_{\substack{i,j=1 \leq \nu}}^N \xi_i(y) \prod_{l=1}^r \frac{1}{\Gamma(\mu_l - \nu_l)} \int_{(0,\infty)^r} dt \int_\Gamma dw (xT/W)^{i-1} \times \\
\prod_{l=1}^r \left( (t_i^{\mu_l} - e^{w_l-1}e^{w_l-1} \prod_{l=1}^r B(\nu_l + i, \mu_l - \nu_l) c_{i,j} \right)
\]

\[
= \frac{1}{(2\pi \sqrt{-1})^r} \sum_{j=1}^N \Phi_j(-a_j; y) \int_{(0,\infty)^r} dt \int_\Gamma dw \prod_{l=1}^r \left( x^{\mu_l} - e^{w_l-1}e^{w_l-1} \right) \frac{e^{xT/W}}{\Gamma(\nu_l + 1)} \times \\
\sum_{i=1}^N (i-1)! \prod_{l=1}^r B(\nu_l + i, \mu_l - \nu_l) c_{i,j} \int_0^\infty du \ u^{\nu_0} \frac{\Gamma(\nu_0 + 1)}{\Gamma(\nu_l + 1)} \times \\
\int_0^\infty du \ u^{\nu_0} e^{-u} F_1(\nu_0 + 1; -uxT/W) e^{-a_j} \prod_{l \neq j} \frac{-u - a_l}{a_j - a_l}. \tag{4.14}
\]

Here the formulae (2.12) and (4.12) have been used in the second and third equalities respectively.

Finally, recalling (4.2) we can rewrite (4.14) as

\[
K_N(x, y) = \int_0^\infty du \ u^{\nu_0} e^{-u} \Psi_r(u; x) \sum_{j=1}^N \Phi_j(-a_j; y) e^{-a_j} \prod_{l \neq j} \frac{-u - a_l}{a_j - a_l}. \tag{4.15}
\]

If we recognise the sum over \( j \) as the sum of the residues at \( \{a_l\} \) of the \( v \)-function

\[
\Phi_r(-v; y) \frac{1}{-u - v} \prod_{l=1}^N \frac{-u - a_l}{v - a_l}, \tag{4.16}
\]

by changing \( v \) to \( -v \) we then arrive at the desired result. \( \square \)

At this stage it would be possible to develop the theory of the corresponding averaged characteristic polynomials and their reciprocals, and then proceed to analyse their hard edge limit; recall Sections 2.2 and 3.2. However we pass on this, and instead analyse the hard edge phase transition analogous to the workings in Section 3.1. Specifically, taking \( N \to \infty \), we keep all \( \nu_j \) fixed and simultaneously let some of \( \mu_1, \ldots, \mu_r \) go to \( \infty \). Without loss of generality, we suppose that for some \( 0 \leq q \leq r \) all \( \nu_1, \ldots, \nu_r, \mu_1, \ldots, \mu_q \) are constants, and moreover

\[
\mu_{q+1}, \ldots, \mu_r \to \infty \quad \text{as} \quad N \to \infty. \tag{4.17}
\]
Theorem 4.2 (Subcritical kernel). For the kernel (1.5), for a fixed nonnegative integer \( m \) let

\[
a_j = N \sigma_j, \quad j = 1, \ldots, m \quad \text{and} \quad a_k = bN, \quad k = m + 1, \ldots, N,
\]

where \( 0 < b < 1 \) and \( \sigma_1, \ldots, \sigma_m > 0 \). Set \( c_N = (1 - b)N \mu_{q+1} \cdots \mu_r \). Under the assumption (4.17) we have

\[
\lim_{N \to \infty} \frac{1}{c_N} K_N \left( \frac{\xi}{c_N}, \frac{\eta}{c_N} \right) = \int_0^1 G_{q+2}^{1, q} \left( \begin{array}{c} -\mu_1, \ldots, -\mu_q \\ 0, -\nu_0, \ldots, -\nu_r \end{array} \right) \frac{u \xi}{G_{q+2}^{r+1, 0} \left( \begin{array}{c} \mu_1, \ldots, \mu_q \\ \nu_0, \ldots, \nu_r, 0 \end{array} \right)} \ du.
\]

Proof. Substituting \( u, v \) by \( uN, vN \) respectively in (4.18), it can be treated as a finite rank perturbation of (1.6) in some sense.

We can complete the proof in much the same way as in that of Theorem 3.1. But this time we have to rescale variables \( t_{q+1}, \ldots, t_r \), that is,

\[
\Psi_r(Nu; \frac{\xi}{c_N}) = \frac{1}{(2\pi i)^r} \int_{(0, \infty)^r} dt \int_{c-i\infty}^{c+i\infty} ds \left( \frac{\eta}{(1-b)N} \right)^{-s} \phi(Nv; s) \times \prod_{l=q+1}^r \Gamma(\mu_j + s) \prod_{l=1}^q \frac{1}{\Gamma(\mu_j + s)} \left( \frac{t_1 \cdots t_q}{1-b \cdot w_1 \cdots w_r} \right),
\]

and

\[
\Phi_r(Nv; \frac{\eta}{c_N}) = \frac{1}{(2\pi i)^r} \int_{c-i\infty}^{c+i\infty} ds \left( \frac{\eta}{(1-b)N} \right)^{-s} \phi(Nv; s) \times \prod_{l=q+1}^r \Gamma(\mu_j + s) \prod_{l=1}^q \frac{1}{\Gamma(\mu_j + s)} \left( \frac{t_1 \cdots t_q}{1-b \cdot w_1 \cdots w_r} \right),
\]

then apply the saddle point analysis (see e.g. [49]) to the integrals over \( t_{q+1}, \ldots, t_r \) in \( \Psi_r \) near the saddle point \( t_0 = 1 \), or expand the integrand in \( \Phi_r \) by the Stirling approximation formula as \( \mu_{q+1}, \ldots, \mu_r \to \infty \). We leave the details to the reader. \( \square \)

The limiting kernel on the RHS of (4.19), with the parameter \( \nu_0 \) absent and \( r + 1 \) replaced by \( r \) first appeared in [32, Theorem 2.8] as the hard edge correlation kernel for a product of truncated unitary matrices. Clearly, it reduces to the Meijer G-kernel (1.6) in case \( q = 0 \). Furthermore, as remarked in [32] (cf. eqns (2.37) and (2.38) therein), it can treated as a finite rank perturbation of (1.6) in some sense.

For the critical and supercritical regimes, proceeding as in the proofs of Theorems 1.3 and 2.2 as for the proof of Theorem 4.2, the required working to establish the following theorems can be given.
Theorem 4.3 (Deformed critical kernel). For the kernel \([1.3]\), for a fixed nonnegative integer \(m\) let

\[ a_j = \sqrt{N} \sigma_j, \quad j = 1, \ldots, m \quad \text{and} \quad a_k = N(1 - \tau / \sqrt{N})^{-1}, \quad k = m + 1, \ldots, N, \]  

where \(\tau \in \mathbb{R}\) and \(\sigma_1, \ldots, \sigma_m > 0\). Set \(c_N = \sqrt{N} \mu_{q+1} \cdots \mu_r\). Under the assumption \([4.17]\) we have

\[
\lim_{N \to \infty} \frac{1}{c_N} K_N \left( \frac{\xi}{c_N}, \frac{\eta}{c_N} \right) = \frac{1}{2\pi i} \int_0^\infty \int_{-c+i\infty}^{-c-i\infty} \frac{\left( \frac{u}{v} \right)^{\nu_0} e^{-\tau u - \frac{1}{2} u^2 + \nu_0 + \frac{1}{2} v^2}}{u - v} \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j} G_{q,r+2}^1 \left( -\mu_1, \ldots, -\mu_q, 0, -\nu_0, \ldots, -\nu_r \middle| u \xi \right) G_{q,r+2}^{r+1,0} \left( \mu_1, \ldots, \mu_q \middle| \nu_0, \ldots, \nu_r, 0 \middle| \nu \eta \right),
\]

where \(0 < c < \min\{\sigma_1, \ldots, \sigma_m\}\).

We remark that the kernels on the RHS of \([4.24]\) reduce to the deformed critical kernels \(K_{m,r}^h\) in \([1.25]\) in case \(q = 0\). These are the most general form of critical kernels that we have derived in the present paper. Moreover, they are new except for the simplest case \(q = r = m = 0\), which as previously remarked corresponds to non-intersecting squared Bessel paths and has been studied in [18, 20, 36].

Theorem 4.4 (Supercritical kernel). With the kernel \([1.5]\), for a fixed positive integer \(m\) let

\[ a_j = \sigma_j b / (b - 1), \quad j = 1, \ldots, m \quad \text{and} \quad a_k = bN, \quad k = m + 1, \ldots, N, \]  

where \(b > 1\) and \(\sigma_1, \ldots, \sigma_m > 0\). Set \(c_N = \mu_{q+1} \cdots \mu_b (b - 1)\). Under the assumption \([4.17]\) we have

\[
\lim_{N \to \infty} \frac{1}{c_N} K_N \left( \frac{\xi}{c_N}, \frac{\eta}{c_N} \right) = \frac{1}{2\pi i} \int_0^\infty \int_0^{\gamma} \frac{\left( \frac{u}{v} \right)^{\nu_0} e^{-u + v}}{u - v} \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j} 
\]

\[ \times \Psi_q(u/\kappa; \kappa \xi) \Phi_q(v/\kappa; \kappa \eta) \prod_{j=1}^m \frac{u + \sigma_j}{v + \sigma_j}, \]

where \(\kappa = (b - 1)/b\), \(\Psi_q, \Phi_q\) are given by \([1.2], [1.3]\), and \(\gamma\) is a contour in the left-half plane going from \(e^{i(\pi + \theta)} \cdot \infty\) to \(e^{i(\pi - \theta)} \cdot \infty\) with \(\theta \in (0, \pi/2)\) such that \(-\sigma_1, \ldots, -\sigma_m\) lie on its left side.

5. ASYMPTOTICS FOR LARGE PARAMETERS AND VARIABLES

5.1. Limits for large parameters. The behavior of the critical kernel \([1.22]\) for large values of the parameters will be discussed, one of which is the confluent relation between correlation kernels. The first to be considered is when some of \(\nu_1, \ldots, \nu_r\), say \(\nu_{m+1}, \ldots, \nu_r\), go to infinity, and we have

Proposition 5.1. Let \(K_{m,r}^h(\xi, \eta; \tau)\) be the critical kernel \([1.22]\). If \(0 \leq m < r\), then as \(\nu_{m+1}, \ldots, \nu_r \to \infty\) we have

\[
(\nu_{m+1} \cdots \nu_r)K_{m,r}^h((\nu_{m+1} \cdots \nu_r) x, (\nu_{m+1} \cdots \nu_r) y; \tau) \longrightarrow K_{m}^h(x, y; \tau).
\]

Proof. This immediately follows from the identity \([5.12]\) for \(G_{0,r+2}^{1,0}\) and the definition \([1.7]\) for \(G_{r+1,0}^{r+1,0}\).

For the large \(\nu_0\), the following result holds true.
Proposition 5.2. Let $K^{h,r}(\xi, \eta; \tau)$ be the critical kernel \[\text{(1.22)}\]. For $r \geq 1$, we have
\[
\lim_{\nu_0 \to \infty} \sqrt{\nu_0} K^{h,r}(\sqrt{\nu_0} x, \sqrt{\nu_0} y; \tau) = K^{h,r-1}(x, y) \bigg|_{\nu_0 = \nu_1, \ldots, \nu_r},
\] (5.2)
where $K^{h,r-1}$ is given by \[\text{(1.2)}\].

Proof. Substituting $u, v$ by $\sqrt{\nu_0} u$ and $\sqrt{\nu_0} v$ respectively in \[\text{(1.22)}\], we get
\[
\sqrt{\nu_0} K^{h,r}(\sqrt{\nu_0} x, \sqrt{\nu_0} y; \tau) = \frac{1}{2\pi i} \int e^{-\nu_0 f(u) - f(v)} e^{-\tau \sqrt{nu_0 + \tau \nu_0}} \nu_0 G^{1,0}_{0,r+2}(0, \nu_0) G^{r+1,0}_{0,r+2}(\nu_0, \nu_1, \ldots, \nu_r, 0 | \nu \nu \nu) \bigg|_{\nu = \nu_0},
\] (5.3)
where $f(z) = -\log z + z^2/2$.

Choose one saddle point $\zeta_0 = 1$ from $f'(z) = 0$ and deform $i\mathbb{R}$ as the union of one closed clockwise contour $C$ encircling the interval $[0, 1)$ and the vertical line $x = 1$. Note that as $\nu_0 \to \infty$
\[
\nu_0 G^{1,0}_{0,r+2}(0, -\nu_0, -\nu_1, \ldots, -\nu_r | \nu_0) G^{r+1,0}_{0,r+2}(\nu_0, \nu_1, \ldots, \nu_r, 0 | \nu_0) \sim G^{r+1,0}_{0,r+2}(0, -\nu_1, \ldots, -\nu_r | \nu) G^{r,0}_{0,r+1}(\nu_1, \ldots, \nu_r, 0 | \nu),
\] (5.4)
proceeding as in the proof of Theorem 3.1 we can show that the dominant contribution comes from the range of $u \in [0, 1)$ and $v \in \mathcal{C}$. Finally, application of residue theorem gives the proof. \(\square\)

Similarly, for the large negative $\tau$, we observe a transition from the critical kernel to the Meijer G-kernel.

Proposition 5.3. Let $K^{h,r}(\xi, \eta; \tau)$ be the critical kernel \[\text{(1.22)}\]. Then we have
\[
\lim_{\tau \to -\infty} (-1/\tau) K^{h,r}(-x/\tau, -y/\tau; \tau) = K^{h,r}(x, y).
\] (5.5)

Proof. Substituting $u, v$ by $-\tau u$ and $-\tau v$ respectively in \[\text{(1.22)}\], we get
\[
(-1/\tau) K^{h,r}(-x/\tau, -y/\tau; \tau) = \frac{1}{2\pi i} \int e^{-\tau f(u) - f(v)} e^{-\tau \sqrt{nu_0 + \tau \nu_0}} \nu_0 G^{1,0}_{0,r+2}(0, \nu_0) G^{r+1,0}_{0,r+2}(\nu_0, \nu_1, \ldots, \nu_r, 0 | \nu) \bigg|_{\tau = \nu_0},
\] (5.6)
where $f(z) = -\log z + z^2/2$. Proceeding as in the proof of Proposition 5.2 the sought result follows. \(\square\)

Lastly, as to the critical kernel on the RHS of \[\text{(1.24)}\] with $m = 0$, the functions $\tilde{\Gamma}^{(1)}(x)$ and $\tilde{\Gamma}^{(0)}(x)$ defined in \[\text{(3.36)}\] and \[\text{(3.37)}\], there exists similar asymptotic behavior for large parameters as in the above three propositions, but we refrain from writing them down.
5.2. Conjectures and open problems. In the concluding section of [23] a num-
ber of questions, mostly relating to asymptotics, were posed in relation to the
kernel (1.6). As we will indicate, these all carry over to the critical kernel (1.22).
It is also the case that the conjectured behaviours are all closely related to ana-
logous expected asymptotic properties of the finite N kernel (1.20). Two classes
of asymptotic problems stand out.

The first is to establish the global scaling limit of the critical one-point function.
For this we expect

$$\lim_{N \to \infty} K_N(Nx, Nx) \big|_{a_l=0} = \frac{1}{\pi} \Im G(x - i0),$$

(5.7)

where $$w(z) := zG(z)$$, satisfies the algebraic equation

$$w^{r+3/2} - zw^{1/2} + z = 0.$$  

(5.8)

The latter is known to specify the Raney distribution with parameters ($3 + 2r$, 2),
which according to free probability theory is the global density for the matrix
(1.14) in the critical case (see e.g. [25] Remark 3.4). In the case of the global limit
(5.7) with $$a_l = 0$$ ($l = 1, \ldots, N$), a recent achievement [40] has been the use of the double
contour integral formula (2.21) to deduce that (5.7) with $$w(z) := zG(z)$$ satisfies
the algebraic equation

$$w^{r+2} - zw + z = 0.$$  

(5.9)

The latter specifies the Raney distribution with parameters ($r + 2, 1$), also known
as the Fuss-Catalan distribution with parameter $$r + 1$$ [40].

To see the relevance of (5.7) to the asymptotics of the density in the critical hard
edge scaled state, $$K^{h,r}(x, x)$$, we recall (cf. [25] Cor. 2.5) that it can be deduced
from (5.9) that for small x the global density has its leading asymptotics given by
(1.14). In keeping with the discussion in the concluding section of [23], this should
be the leading large $$x$$ asymptotic form of $$K^{h,r}(x, x)$$. The second is to compute the leading asymptotic form of the off diagonal ana-
logue of the LHS of (5.7), namely $$K_N(Nx, Ny)$$ for $$x \neq y$$. To see the interest in
this quantity, note from (2.18) that the truncated (or connected) two-point cor-
relation $$\rho_T(x_1, x_2) := \rho_{(2)}(x_1, x_2) - \rho(1)(x_1)\rho(1)(x_2)$$ is given by $$\rho_T(x_1, x_2) = 
-K_N(x_1, x_2)K_N(x_2, x_1)$$, so knowledge of the asymtotic form of $$K_N(Nx, Ny)$$ tells us
the asymptotics of $$\rho_T(Nx, Ny)$$. With $$G = \sum_{j=1}^{N} g(x_j)$$ denoting a linear statistic
in the bulk scaled system, in view of the formula (see e.g. [21] eqn. (14.38))

$$\Var G = N^2 \int_0^\infty dx_1 \int_0^\infty dx_2 \ g(x_1)g(x_2)\rho_T(Nx_1, Nx_2) + N \int_0^\infty g(x)\rho(1)(Nx) \ dx$$  

(5.10)

one sees that the non-oscillatory portion of the leading asymptotic form of $$\rho_T(Nx, Ny)$$
(sometimes referred to as a wide correlator; see e.g. [28]) essentially determines the
large N form of this fluctuation, which is expected to be $$O(1)$$ (see e.g. [21] §14.3).

As a concrete example of this second type of asymptotics, consider the simplest
case of (1.4), namely $$r = 0$$ and $$A = 0$$. The squared singular values correspond
to the eigenvalues of $$G_0^*G_0$$, where $$G_0$$ is a $$(N + \nu_0) \times N$$ standard complex Gaussian
matrix. This class of random matrices is referred to as the complex Wishart
ensemble (see e.g. [21] §3.2). For this ensemble it is a known result that [10]

$$N^2 \rho_T(Nx, Ny) \sim -\frac{1}{2\pi^2} \frac{1}{(x - y)^2} \frac{(L/2)(x+y) - xy}{(x(L-x)y(L-y))^{1/2}}, \quad x \neq y,$$  

(5.11)
with $L = 4$, and where the dot above the asymptotic sign denotes a restriction to non-oscillatory terms.

Suppose now that in the asymptotic form of $K_N(Nx, Ny)$ we introduce a scale factor $L$ and compute instead the asymptotic form of $(1/L)K_N(Nx/L, Ny/L)$. For the complex Wishart ensemble the RHS of (5.11) with $L$ a variable results. For general $r$, if the original leading asymptotic form of $N^2\rho_T(Nx, Ny)$ was $R(x, y)$, this will now equal $(1/L^2)R(x/L, y/L)$. Following [10] we expect that

$$
\lim_{L \to \infty} \frac{1}{L^2} R\left(\frac{x}{L}, \frac{y}{L}\right) \to R^b(x, y),
$$

(5.12)

where $R^b(x, y)$ is the leading non-oscillatory large $x$, large $y$ asymptotic form of the hard edge scaling of $\rho_T(x, y)$. In the context of the present setting this corresponds to seeking the large $x$, large $y$ form of $K_{h,r}(x, y)$. In the case of the complex Wishart ensemble, (5.12) applied to (5.11) predicts that

$$
\rho_{T(2)}^{h,T}(x, y) \sim -\frac{1}{4\pi^2} \frac{(x/y)^{1/2} + (y/x)^{1/2}}{(x - y)^2},
$$

(5.13)

which is in fact a known exact result (see e.g. [21, eqn. (7.75)]). The analogue of (5.13) is known for the case $r = 1$, $A = 0$ of (1.4) [23, eqn. (5.28)], but the analogue of (5.11) is yet to be obtained. As discussed in [23], knowledge of an asymptotic form such as (5.13) is of interest for the computation of the variance of a scaled linear statistic at the hard edge, $G_\alpha = \sum_{j=1}^\infty g(x_j/\alpha)$ when $\alpha \to \infty$, which is given by

$$
\lim_{\alpha \to \infty} \text{Var} G_\alpha := \lim_{\alpha \to \infty} \left( \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \right.
$$

$$
\times g(\lambda_1/\alpha)g(\lambda_2/\alpha)\rho_{T(2)}^{h,r}(\lambda_1, \lambda_2) + \int_0^\infty d\lambda g(\lambda/\alpha)\rho_{T(1)}^{h,r}(\lambda)\right).
$$

(5.14)

A number of challenges for future research present themselves from the above discussion. We conclude this section with a list of a few more.

- Under the assumption of $a_1 = \cdots = a_N = bN$ with $b > 0$, verify the sine-kernel in the bulk and Airy-kernel at the soft edge for (1.20) and (4.5) (see recent monographs [6, 17, 21, 48] for the sine and Airy kernels and [40] for recent progress on the random matrix products).
- Under the assumption of $a_m+1 = \cdots = a_N = bN$ with $b > 0$, by tuning the parameters $a_1, \ldots, a_m$ verify the BBP transition for (1.20) and (4.5) (cf. [9, 45]).
- Verify the transitions from the critical kernels (1.22) and (4.24) to the sine-kernel and to the Airy-kernel (cf. [21 Exercise 7.2] and [23]).

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References
1. Adler M., Delépine, J. and van Moerbeke, P.: Dyson’s nonintersecting Brownian motions with a new outliers, Comm. Pure Appl. Math. 62 (2008), 334–395.
2. Akemann, G. and Ipsen, J.R.: Recent exact and asymptotic results for products of independent random matrices, [arXiv:1502.01667].
3. Akemann, G., Ipsen, J. and Kieburg, M.: Products of rectangular random matrices: singular values and progressive scattering, Phys. Rev. E 88 (2013), 052118 [13pp].
4. Akemann, G., Kieburg, M. and Wei, L.: Singular value correlation functions for products of Wishart matrices, J. Phys. A 46 (2013), 275205 [22pp].
5. Alexeev, N., Götze, F. and Tikhomirov, A.: On the asymptotic distribution of singular values of products of large rectangular random matrices, [arXiv:1012.2580v2].
6. Anderson, G.W., Guionnet, A. and Zeitouni, O.: An Introduction to Random Matrices, Cambridge University Press, Cambridge (2009).
7. Andrews, G. E., Askey, R. and Roy, R.: Special Functions, Cambridge University Press (2000).
8. Beals R. and Szmigielski J.: Cauchy-Laguerre two-matrix model and the Meijer-G random point field, Comm. Math. Phys. 326 (2014), 111–144.
9. Blaizot, J.-P., Nowak, M.A. and Warchoł, P.: Universal shocks in the Wishart random-matrix ensemble. II. Non-trivial initial conditions, Phys. Rev. E 89 (2014), 042130.
10. Borodin, A.: Biorthogonal ensembles, Nuclear Phys. B 536 (1999), 704–732.
11. Borodin, A. and Kuan, J.: Random surface growth with a wall and Plancherel measures for $O(\infty)$, Comm. Pure Appl. Math. 63 (2010), 831–894.
12. Cheliotis, D.: Triangular random matrices and biorthogonal ensembles, [arXiv:1404.4730].
13. Deift, P.: Orthogonal Polynomials and Random Matrices: a Riemann–Hilbert approach, Courant Lecture Notes in Mathematics Vol. 3, Amer. Math. Soc., Providence R.I., 1999.
14. Desrosiers, P. and Forrester, P.J.: Asymptotic correlations for Gaussian and Wishart matrices with external source, Int. Math. Res. Notices (2006), ID 273295, 1–43.
15. Forrester, P.J.: Log-gases and random matrices, Princeton University Press (2010).
16. Forrester, P.J.: The averaged characteristic polynomial for the Gaussian and chiral Gaussian ensemble with a source, J. Phys. A 46 (2013), 345204.
17. Forrester, P.J.: Eigenvalue statistics for product complex Wishart matrices, J. Phys. A 47 (2014), 345202.
18. Forrester, P.J. and Kieburg, M.: Relating the Bures measure to the Cauchy two-matrix model, [arXiv:1410.6883].
19. Forrester, P.J. and Liu, D.-Z.: Raney distributions and random matrix theory, J. Stat. Phys. 158 (2015), 1051–1082.
20. Forrester, P.J. and Wang, D.: Muttalib–Borodin ensembles in random matrix theory—realisations and correlation functions, [arXiv:1502.07147v2].
21. Götze, F., Naumov, A. and Tikhomirov, A.: Distribution of linear statistics of singular values of the product of random matrices, [arXiv:1412.3314].
22. Itoi, C.: Universal wide correlators in non-Gaussian orthogonal, unitary and symplectic random matrix ensembles, Nucl. Phys. B 493 (1997), 651–659.
23. König, W. and O’Connell, N.: Eigenvalues of the Laguerre process as non-colliding squared Bessel process. Elec. Commun. Probab. 6 (2001), 107–114.
24. Katori, M. and Tanemura, H.: Noncolliding squared Bessel process, J. Stat. Phys. 142 (2011), 592–615.
25. Kieburg, K.: Supersymmetry for products of random matrices, [arXiv:1502.06550].
26. Kieburg, K., Kuijlaars, A.B.J. and Stivigny, D.: Singular value statistics of matrix products with truncated unitary matrices, [arXiv:1501.03910].
27. Kuijlaars, A.B.J.: Transformations of polynomial ensembles, [arXiv:1501.05506].
34. Kuijlaars, A.B.J.: Multiple orthogonal polynomial ensembles, *Recent trends in orthogonal polynomials and approximation theory*, Contemp. Math. 507 (2010), 155–176.
35. Kuijlaars, A.B.J., Martnez-Finkelshtein, A. and Wielonsky, F.: Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weights, Comm. Math. Phys. 286 (2009), 217–275.
36. Kuijlaars, A.B.J., Martnez-Finkelshtein, A. and Wielonsky, F.: Non-intersecting squared Bessel paths: critical time and double scaling limit, Comm. Math. Phys. 308 (2011), 227–279.
37. Kuijlaars, A.B.J. and Stivigny, D.: Singular values of products of random matrices and polynomial ensembles, Random Matrices: Theory and Appl. Vol. 3, No. 3 (2014) 1450011 (22 pages).
38. Kuijlaars, A.B.J. and Zhang, L.: Singular values of products of Ginibre random matrices, multiple orthogonal polynomials and hard edge scaling limits, Comm. Math. Phys. 332 (2014), 759–781.
39. Liu, D.-Z., in preparation, 2015.
40. Liu, D.-Z., Wang, D. and Zhang, L.: Bulk and soft-edge universality for singular values of products of Ginibre random matrices, [arXiv:1412.6777](https://arxiv.org/abs/1412.6777).
41. Luke, Y.L.: *The Special Functions and their Approximations*, Vol. 1, Academic Press, New York, 1969.
42. Neuschel, T.: Plancherel-Rotach formulae for average characteristic polynomials of products of Ginibre random matrices and the Fuss-Catalan distribution, Random Matrices: Theory and Appl. 03, No. 1 (2014), 1450003, 18pp.
43. Muttaibb, K.A.: Random matrix models with additional interactions, J. Phys. A 28 (1995), L159–164.
44. Pastur, L. and Shcherbina, M.: *Eigenvalue distribution of large random matrices*, American Mathematical Society, Providence, RI, 2011.
45. Peché, S.: The largest eigenvalue of small rank perturbations of Hermitian random matrices, Probab. Theory and Related Fields 134 (2006), 127–173.
46. Penson, K.A., Życzkowski, K.: Product of Ginibre matrices: Fuss-Catalan and Raney distributions, Phys. Rev. E 83 (2011) 061118, 9 pp.
47. Strahov, E.: Differential equations for singular values of products of Ginibre random matrices, J. Phys. A: Math. Theor 47 (2014), 325203 (27pp).
48. Tao, T.: *Topics in Random Matrix Theory*, Graduate Studies in Mathematics 132, Amer. Math. Society, Providence RI, 2012.
49. Wong, R.: *Asymptotic approximations of integrals*, vol. 34, SIAM, 2001.
50. Zhang, L.: Local universality in biorthogonal Laguerre ensembles, arXiv: 1502.03160.

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