UNRAMIFIED BRAUER GROUP OF THE MODULI SPACES OF PGL\(_r(\mathbb{C})\)-BUNDLES OVER CURVES

INDRANIL BISWAS, AMIT HOGADI, AND YOGISH I. HOLLA

Abstract. Let \(X\) be an irreducible smooth complex projective curve of genus \(g\), with \(g \geq 2\). Let \(N\) be a connected component of the moduli space of semistable principal PGL\(_r(\mathbb{C})\)-bundles over \(X\); it is a normal unirational complex projective variety. We prove that the Brauer group of a desingularization of \(N\) is trivial.

1. Introduction

Let \(X\) be an irreducible smooth complex projective curve, with genus \(\text{genus}(X) = g \geq 2\). For a fixed line bundle \(\mathcal{L}\) over \(X\), let \(M_X(r, \mathcal{L})\) be the coarse moduli space of semistable vector bundles over \(X\) of rank \(r\) and determinant \(\mathcal{L}\). It is a normal unirational complex projective variety, and if \(\text{degree}(\mathcal{L})\) is coprime to \(r\), then \(M_X(r, \mathcal{L})\) is known to be rational [Ne], [KS]. Apart from these coprime case, and the single case of \(g = r = \text{degree}(\mathcal{L}) = 2\) when \(M_X(r, \mathcal{L}) = \mathbb{P}^3\), the rationality of \(M_X(r, \mathcal{L})\) is an open question in every other case. See [Ho] for rationality of some other types of moduli spaces associated to \(X\).

We consider the coarse moduli space \(N_X(r, d)\) of semistable principal PGL\(_r(\mathbb{C})\)-bundles of topological type \(d\) over \(X\). Recall that a PGL\(_r(\mathbb{C})\) bundle \(P/X\) is said to be of topological type \(d\) if the associated \(\mathbb{P}^{r-1}\)-bundle is isomorphic to \(\text{Proj}(E)\) for some rank \(r\) vector bundle \(E\) whose degree is congruent to \(d\) modulo \(r\). This \(N_X(r, d)\) is an irreducible normal unirational complex projective variety. This paper is a sequel to [BHH], where we investigate the Brauer group of desingularization of moduli spaces attached to curves. This Brauer group is a birational invariant of the space and its vanishing is a necessary condition for the space involved to be rational.

In this paper we prove that the Brauer group of a desingularization of \(N_X(r, d)\) is zero (see Theorem 8).

When \(g = 2\), the moduli space \(N_X(2, 0)\) is a quotient of \(\mathbb{P}^3\) by a faithful action of the abelian group \((\mathbb{Z}/2\mathbb{Z})^4\). In this special case it follows the quotient is rational.

2. Preliminaries

We continue with the above set–up and notation. Let \(N_X(r, d)\) denote the coarse moduli space of \(S\)-equivalence classes of all semistable principal PGL\(_r(\mathbb{C})\)-bundles of topological type \(d\) over \(X\). For notational convenience, \(N_X(r, d)\) will also be denoted by \(N\).

Let \(M_X(r, \mathcal{L}_X)\) be the coarse moduli space of \(S\)-equivalence classes of semistable vector bundles over \(X\) of rank \(r\) and determinant \(\mathcal{L}_X\). Let \(\Gamma\) be the group of all isomorphism
classes of algebraic line bundles $\tau$ over $X$ such that $\tau^{\otimes r} = \mathcal{O}_X$. This group $\Gamma$ has the following natural action on $M_X(r, \mathcal{L}_X)$: the action of any $\tau \in \Gamma$ sends any $E \in M_X(r, \mathcal{L}_X)$ to $E \otimes \tau$. The moduli space $N$ is identified with the quotient variety $M_X(r, \mathcal{L}_X)/\Gamma$. Let

\[ f : M_X(r, \mathcal{L}_X) \longrightarrow M_X(r, \mathcal{L}_X)/\Gamma = N \]

be the quotient morphism.

For notational convenience, the moduli space $M_X(r, \mathcal{O}_X)$ will also be denoted by $M_{\mathcal{L}_X}$.

Let

\[ M^\text{st}_{\mathcal{L}_X} \subset M_{\mathcal{L}_X} \quad \text{and} \quad N^\text{st} \subset N \]

be the loci of stable bundles. The above action of $\Gamma$ on $M_O$ preserves $M^\text{st}_{\mathcal{O}_X}$, and

\[ f(M^\text{st}_{\mathcal{O}_X}) = N^\text{st}. \]

3. THE ACTION OF $\Gamma$

Consider the action of $\Gamma$ on $M_O$ defined in Section 2. For any primitive $\tau \in \Gamma$, i.e. an element of order $r$, let

\[ M^\tau_O = \{ E \in M_O \mid E \otimes \tau = E \} \subset M_O \]

be the fixed point locus.

Take any nontrivial line bundle $\tau \in \Gamma$ of order $r$. Let

\[ \phi : Y \longrightarrow X \]

be the étale cyclic covering of degree $r$ given by $\tau$. We recall the construction of $Y$ as the spectral cover associated to the equation $\tau^r \cong \mathcal{O}_X$.

Let

\[ \beta : Y \longrightarrow Y \]

be a nontrivial generator of the Galois group $\text{Gal}(\phi) = \mathbb{Z}/r\mathbb{Z}$ of the covering $\phi$. The homomorphism $\xi \longmapsto \beta^* \xi$ defines an action of $\text{Gal}(\phi)$ on $\text{Pic}^d(Y)$ for any $d$.

Let

\[ \phi^* : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(Y) \]

be the pullback homomorphism $L \longmapsto \phi^* L$. Let $K$ denote the kernel of $\phi^*$; it is a group of order $r$ generated by $\tau$. Let

\[ \text{Nm} : \text{Pic}^d(Y) \longrightarrow \text{Pic}^d(X) \]

and

\[ N : \text{Pic}^d(Y) \longrightarrow \text{Pic}^d(X) \]

be the norm homomorphism and the twisted norm morphism. We recall that Nm takes a line bundle $\xi$ to the descent of $\otimes(\beta^* \xi)$ and $N$ sends a line bundle $\xi$ to $\text{Nm}(\xi) \otimes \tau^{(r-1)/2}$.

The group $\Gamma$ has a natural action on $\text{Pic}^d(Y)$; any $\sigma \in \Gamma$ acts as the automorphism $\xi \longmapsto \xi \otimes \phi^* \sigma$. Therefore, $\phi^*$ in (3.3) is $\Gamma$–equivariant, and the kernel $K$ acts trivially on $\text{Pic}^d(Y)$. The morphism $N$ in (3.5) factors through the quotient morphism $\text{Pic}^d(Y) \longrightarrow \text{Pic}^d(Y)/\Gamma$. The action of $\Gamma$ on $\text{Pic}^d(Y)$ clearly commutes with the action of $\text{Gal}(\phi)$ defined earlier.
Let
\[(3.6) \quad U_{\mathcal{L}_X} := N^{-1}(\mathcal{L}_X) \setminus (N^{-1}(\mathcal{L}_X))^{\text{Gal}(\phi)} \subset Nm^{-1}(\mathcal{L}_X)\]
be the complement of the fixed point locus for the action of Gal(\phi). It is a \(\Gamma\)-invariant open subscheme.

Now we state a well-known result (cf. [NR2, Lemma 3.4]).

**Lemma 1.** Take any primitive line bundle \(\tau \in \Gamma\). The reduced closed subscheme
\[(M^s_{\mathcal{L}_X})^\tau := M^s_{\mathcal{L}_X} \cap M^s_{\mathcal{L}_X} \subset M^s_{\mathcal{L}_X}\]
(see (3.1) and (2.2)) is \(\Gamma\)-equivariantly isomorphic to the quotient scheme
\[U_{\mathcal{L}_X}/\text{Gal}(\phi)\].

**Lemma 2.** The norm map as defined in (3.5) is surjective, and there is a bijection of the set of connected components \(\pi_0(N^{-1}(\mathcal{L}_X))\) with the Cartier dual \(K^\vee := \text{Hom}(K, \mathbb{C}^*) = \mathbb{Z}/r\mathbb{Z}\), where \(K := \ker(\phi^*) = \mathbb{Z}/r\mathbb{Z}\).

Lemma 2 is proved in [NR2] (see [NR2, Proposition 3.5]).

Let \(V_0\) be the connected components of \(N^{-1}(\mathcal{O}_X)\), with \(\mathcal{O}_Y \in V_0\). Since \(Nm^{-1}(\mathcal{O}_X)\) is smooth, both \(V_0\) is irreducible.

**Lemma 3.** The action of Gal(\phi) on \(N^{-1}(\mathcal{O}_X)\) preserves the connected component \(V_0\). For the action of Gal(\phi) on \(N^{-1}(\mathcal{L}_X)\) the quotient \(N^{-1}(\mathcal{L}_X)/\text{Gal}(\phi)\) has exactly \((r, d)\) components which are smooth, here \((r, d)\) is the greatest common divisor of \(r\) and \(d\).

**Proof.** The point \(\mathcal{O}_Y \in \text{Pic}^0(Y)\) is fixed by Gal(\phi); hence \(V_0\) is fixed by Gal(\phi). Therefore, the other component, namely \(V_1\), is also fixed by Gal(\phi). See [NR2, Proposition 3.5] for the proof of the second statement.

**Lemma 4.** Let \(r = 2\). The set of all points in the complement \(M_{\mathcal{O}_X} \setminus M^s_{\mathcal{O}_X}\) (see (2.2)) fixed by \(\tau\) is finite.

**Proof.** Take any point \(x \in M_{\mathcal{O}_X} \setminus M^s_{\mathcal{O}_X}\). Let \(E = L \oplus L^*\), with \(L \in \text{Pic}^0(X)\), be the unique polystable vector bundle representing the point \(x \in M_{\mathcal{O}_X}\). The action of \(\tau\) takes the point \(x\) to the point represented by the polystable vector bundle \((L \otimes \tau) \oplus (L^* \otimes \tau)\).

Assume that \(\tau \cdot x = x\). Then the two vector bundles \(L \oplus L^*\) and \((L \otimes \tau) \oplus (L^* \otimes \tau)\) are isomorphic. This implies that
\[(3.7) \quad L \otimes \tau \cong L^*\]
(recall that \(\tau\) is nontrivial; so \(L \neq L \otimes \tau\)). From (3.7) it follows that \(L^{\otimes 2} = (L^{\otimes 2})^*\).

Consequently, isomorphism classes of all line bundles \(L \in \text{Pic}^0(X)\) satisfying (3.7), for a given \(\tau\), is a finite subset. Therefore, there are only finitely many points of \(M_{\mathcal{O}_X} \setminus M^s_{\mathcal{O}_X}\) that are fixed by \(\tau\).

**Remark 5.** When genus of \(X\) equals 2, \(\dim(\text{Pic}^0(Y)) = 3\). It follows from Lemma 1 that \((M^s_{\mathcal{O}_X})^\tau\) is one dimensional and hence Lemma 4 implies that \(M^s_{\mathcal{O}_X}\) is of codimension two in \(M_{\mathcal{O}_X} \cong \mathbb{P}^3\).
Let $\sigma \in \Gamma$ be another primitive element such that $\sigma$ and $\tau$ are $\mathbb{Z}/r\mathbb{Z}$ linearly independent. The subgroup of $\Gamma$ generated by $\sigma$ and $\tau$ will be denoted by $A$. So $A$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{\oplus 2}$.

We note that $\Gamma \subset \text{Pic}(X)$ is identified with $H^1(X, \mathbb{Z}/r\mathbb{Z}) \subset H^1(X, \mathbb{G}_m)$ under the natural inclusion. Let $\xi$ be the pairing given by the cup product $\cdot \cup \cdot$. It is known that this $\xi$ coincides with the Weil pairing (see [Mu1, p. 183]).

**Proposition 6.** Let $\sigma$ and $\tau$ be two primitive elements of $\Gamma$ such that they generate a subgroup $A = (\mathbb{Z}/r\mathbb{Z})^{\oplus 2}$. If the pairing $\xi(\sigma, \tau) = 0$ then there exists a nonempty closed irreducible $A$–invariant subset of $M^a_{\mathfrak{O}_X}$ which is fixed pointwise by $\tau$.

**Proof.** By [BP2], Proposition 4.5, it follows that under the condition $\xi(\sigma, \tau) = 0$ there is a stable bundle $E$ of rank $r$ and determinant $\mathcal{O}_X$ such that $E \otimes \sigma = E \otimes \tau = E$.

The condition $E \otimes \tau = E$ implies the existence of a line bundle $\xi \in N^{-1}(\mathcal{O}_X)$ such that $\phi_* (\xi) = E$. The condition $E \otimes \sigma = E$ implies that there is a $\beta \in \text{Gal}(\phi)$ such that $\xi \otimes \phi^* \sigma = \beta^* \xi$. Hence $\phi^* \sigma = (\beta^* \xi) \otimes \xi^{-1}$ lies in $V_0$, because $\beta^* \xi$ and $\xi^{-1}$ lie in the same component (see both parts of the Lemma 3).

One observes that $N^{-1}(\mathcal{L}_X)$ is $\Gamma$ equivariantly isomorphic to the translate $L \cdot N^{-1}(\mathcal{O}_X)$ by any line bundle $L$ such that $N(L) = \mathcal{L}_X$. This along with the above fact that $\phi^* \sigma \in V_0$ implies that the translation by $\phi^* \sigma$ preserves the connected components of $N^{-1}(\mathcal{L}_X)$.

Hence we conclude that any connected component of the quotient $\mathcal{U}_{\mathcal{L}_X}/\text{Gal}(\phi) \subset (M^a_{\mathfrak{O}_X})^r$ is a closed irreducible $A$–invariant subscheme of $M^a_{\mathfrak{O}_X}$ which is fixed pointwise by $\tau$. This completes the proof of the proposition. $\square$

## 4. Brauer group of a desingularization of $N$

In this section we identify the second cohomology $H^2(\Gamma, \mathbb{C}^*)$ with the space of alternating bi-multiplicative maps from $\Gamma$ to $\mathbb{C}^*$ (see [Ra, p. 215, Proposition 4.3]); the group $H^2(\Gamma, \mathbb{C}^*)$ is isomorphic to the dual of the second exterior power of $(\mathbb{Z}/2\mathbb{Z})^{2g}$.

Recall that under the identification of $\Gamma$ with $H^1(X, \mathbb{Z}/r\mathbb{Z})$, the Weil pairing coincides with the intersection pairing. Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be a symplectic basis for $H^1(X, \mathbb{Z}/r\mathbb{Z})$. In other words we have $e(a_i, a_j) = 0 = e(b_i, b_j)$ for all $i$ and $j$, and $e(a_i, b_j) = \delta_{i,j}$.

Let $G \subset H^2(\Gamma, \mathbb{C}^*)$ be defined by

$G := \{ b \in H^2(\Gamma, \mathbb{C}^*) \mid e(\sigma_1, \sigma_2) = 0 \Rightarrow b(\sigma_1, \sigma_2) = 0 \}$.

Let $H$ be the subgroup of $G$ of order two generated by the Weil pairing $e$.

**Lemma 7.** The group $G$ coincides with the subgroup $H$.

**Proof.** Fix an $i$ and $j$ such that $i \neq j$ then one checks that $e(a_i + b_j, a_j - b_i) = e(a_j, b_j) - e(a_i, b_i) = 0$ hence if $f \in G$ then $0 = f(a_i + b_j, a_j - b_i) = f(a_j, b_j) - f(a_i, b_i)$. This implies that $f$ is a multiple of $e$. $\square$
Our main theorem is the following.

**Theorem 8.** Let \( \tilde{N} \) be a desingularization of the moduli space \( N \). Then the Brauer group \( \text{Br}(\tilde{N}) = 0 \)

**Proof.** We first assume that either \( g \geq 3 \) or when \( g = 2 \) rank \( r > 2 \). The case of \( g = 2 \) and rank \( r = 2 \) will be treated separately.

It is enough to prove the theorem for some desingularization \( \tilde{N} \) of \( N \) because the Brauer group is a birational invariant for the smooth projective varieties. We choose a \( \Gamma \)–equivariant desingularization

\[
p : \tilde{M}_\mathcal{O} \longrightarrow M_\mathcal{O}
\]

which is an isomorphism over \( M^\text{rst} \); so \( \tilde{M}_\mathcal{L} \) is equipped with an action of \( \Gamma \) given by the action of \( \Gamma \) on \( M_\mathcal{L} \). Define

\[
\tilde{N} := \tilde{M}_\mathcal{L}/\Gamma.
\]

Let

\[
\tilde{N} \longrightarrow \tilde{N}
\]

be a desingularization of \( \tilde{N} \) which is an isomorphism over the smooth locus. So \( \tilde{N} \) is also a desingularization of \( N \).

A stable principal PGL\(_r\)(\( \mathbb{C} \))–bundle \( E \) on \( X \) is called *regularly stable* if

\[
\text{Aut}(E) = e
\]

(by \( \text{Aut}(E) \) we denote the automorphisms of the principal bundle \( E \) over the identity map of \( X \)). It is known that the locus of regularly stable bundles in \( N \), which we will denote by \( N^\text{rst} \), coincides with the smooth locus of \( N \) \([BH, \text{Corollary 3.4}]\). Define

\[
M^\text{rst} := f^{-1}(N^\text{rst}),
\]

where \( f \) is the morphism in (2.1). We note that the action of \( \Gamma \) on \( M_\mathcal{L} \) preserves \( M^\text{rst} \), because \( f \) is an invariant for the action of \( \Gamma \). The action of \( \Gamma \) on \( M^\text{rst} \) can be shown to be free. Indeed, if \( E = E \otimes \tau \), where \( \tau \) is nontrivial, any isomorphism of \( E \) with \( E \otimes \tau \) produces a nontrivial automorphism of \( \mathbb{P}(E) \), because \( \mathbb{P}(E \otimes \tau) = \mathbb{P}(E) \). Hence such a vector bundle \( E \) cannot lie in \( M^\text{rst} \).

Consequently, the projection \( f \) in (2.1) defines a principal \( \Gamma \)–bundle

\[
M^\text{rst} \xrightarrow{f} N^\text{rst}.
\]

Since \( N \) is normal, and \( N^\text{rst} \) is its smooth locus, it follows that the codimension of the complement \( N \setminus N^\text{rst} \) is at least two. Therefore, the codimension of the complement of \( M^\text{rst} \subset M_\mathcal{O} \) is at least two. Hence

\[
H^0(M^\text{rst}, \mathbb{G}_m) = \mathbb{C}^*.
\]

The Serre spectral sequence for the above principal \( \Gamma \)–bundle gives an exact sequence

\[
\text{Pic}(N^\text{rst}) \xrightarrow{\delta} \text{Pic}(M^\text{rst})^\Gamma \longrightarrow H^2(\Gamma, \mathbb{C}^*) .
\]

We have \( \text{Pic}(M^\text{rst})^\Gamma/\text{image}(\delta) = \mathbb{Z}/l\mathbb{Z} \) \([BH]\) (see (3.5) in \([BH]\) and lines following it) where \( l = (r, d) \). Hence we get an inclusion

\[
\mathbb{Z}/l\mathbb{Z} \hookrightarrow H^2(\Gamma, \mathbb{C}^*) ,
\]
where the generator of $\mathbb{Z}/r\mathbb{Z}$ maps to the Weil pairing $e$ (see the proof of Proposition 9.1 in [BLS, p. 203]).

For the chosen desingularization $\hat{N} \to N$, we have

\[(4.4) \quad \text{Br}(\hat{N}) \subset \text{Br}(N_{\text{rst}}) = \text{Br}(M_{\text{rst}}/\Gamma)\]

using the inclusion of $N_{\text{rst}}$ in $\hat{N}$. The Brauer group $\text{Br}(N_{\text{rst}})$ is computed in [BH].

The Serre spectral sequence for the principal $\Gamma$–bundle in (4.2) gives the following exact sequence:

\[(4.5) \quad H^2(\Gamma, \mathbb{C}^*) \overset{\rho}{\longrightarrow} H^2(M_{\text{rst}}/\Gamma, \mathbb{G}_m) \longrightarrow H^2(M_{\text{rst}}, \mathbb{G}_m).\]

Let $\mathcal{S}$ be the set of all bicyclic subgroups $A \subset \Gamma$ of the form $(\mathbb{Z}/r\mathbb{Z})^\oplus 2$ satisfying the condition that there is some closed irreducible subvariety $Z$ of $M_{O_X}$ preserved by the action of $A$ such that a primitive element of $A$ fixes $Z$.

Define the subgroup

$$G' := \bigcap_{A \in \mathcal{S}} \ker(H^2(\Gamma, \mathbb{C}^*) \to H^2(A, \mathbb{C}^*)) \subset H^2(\Gamma, \mathbb{C}^*)$$

Using a theorem of Bogomolov, [Bo, p. 288, Theorem 1.3], we have

\[(4.6) \quad \rho^{-1}(H^2(\hat{N}, \mathbb{G}_m)) \subset G'.\]

We will show that $G'$ is a subgroup of $G$ in $H^2(\Gamma, \mathbb{C}^*)$. This will prove that the image $\rho(\rho^{-1}(H^2(\hat{N}, \mathbb{G}_m))) = 0$

$b \in G'$. We need to check that $b(\sigma, \tau) = 0$ whenever $e(\sigma, \tau) = 0$ for a pair of primitive elements generating the subgroup $A = \mathbb{Z}/r\mathbb{Z}^\oplus 2$. Since $e(\sigma, \tau) = 0$, by Proposition 6, there is an irreducible closed subscheme $Z \subset M_{O_X}^{st}$ which is $A$–invariant and fixed pointwise by $\tau$.

Since the $\Gamma$–equivariant desingularization $p$ in (4.1) is an isomorphism over $M_{O_X}^{st}$, we conclude that the closure of $Z$ in $\tilde{M}_{O_X}$ is an $A$–invariant closed irreducible subscheme which is fixed pointwise by $\tau$. Hence the action of $A$ on this closure is cyclic. This implies that $A \in \mathcal{S}$, and hence $b(\sigma, \tau) = 0$. Therefore $G' \subset G$.

Consequently, we have shown that $H^2(\hat{N}, \mathbb{G}_m) \cap \text{image}(\rho) = 0$.

This proves that the composition

\[(4.7) \quad H^2(\hat{N}, \mathbb{G}_m) \to H^2(M_{\text{rst}}/\Gamma, \mathbb{G}_m) \to H^2(M_{\text{rst}}, \mathbb{G}_m)\]

is injective. We will prove that this composition is zero (these homomorphisms are induced by the inclusion $M_{\text{rst}}/\Gamma \hookrightarrow \hat{N}$ and the quotient map to $M_{\text{rst}}/\Gamma$).

Consider the diagram

$$\begin{array}{c}
\hat{M} \\
\downarrow \\
\hat{N} \\
\downarrow \\
\tilde{N}
\end{array}$$
where $\hat{M}$ is a $\Gamma$–equivariant desingularization of the closure of

$$M^{\text{rst}} = \hat{N} \times_{N^{\text{rst}}} M^{\text{rst}}$$

in the fiber product $\hat{N} \times_{\hat{N}} \hat{M}_X$. This gives an action of $\Gamma$ on the smooth projective variety $\hat{M}$ which has a $\Gamma$–invariant open subscheme $M^{\text{rst}}$ with the quotient $M^{\text{rst}}/\Gamma$ being the Zariski open subset $N^{\text{rst}}$ of $\hat{N}$. Using the commutativity of $\Gamma$–actions we obtain a commutative diagram of homomorphisms

$$\begin{array}{cc}
H^2(\hat{N}, \mathbb{G}_m) & \longrightarrow H^2(M^{\text{rst}}, \mathbb{G}_m) \\
\downarrow & \downarrow \\
H^2(\hat{M}, \mathbb{G}_m) & \longrightarrow H^2(M^{\text{rst}}, \mathbb{G}_m)
\end{array}$$

Since $\hat{M}$ is also a desingularization of $M$, we conclude by [Ni, p. 309, Theorem 1] (see also [BHH, Theorem 1]) that

$$H^2(\hat{M}, \mathbb{G}_m) = 0.$$ 

Hence from (4.8) it follows that the image of $H^2(\hat{N}, \mathbb{G}_m)$ in $H^2(M^{\text{rst}}, \mathbb{G}_m)$ by the composition in (4.7) is zero. This completes the proof when $g \geq 3$.

Now assume that $g = 2$ and rank $r = 2$. So $M_X$ is already smooth. We take $\hat{M}_X = M_X$. Let $M^{\text{free}} \subset M_X$ be the largest Zariski open subset where the action of $\Gamma$ is free. It follows from Remark 5 that the complement of $M^{\text{free}}$ is of codimension two. The entire argument above works in this case after replacing $M^{\text{rst}}$ by $M^{\text{free}}$ and $N^{\text{rst}}$ by $M^{\text{free}}/\Gamma$. □

References

[AM] M. Artin and D. Mumford: Some elementary examples of unirational varieties which are not rational, Proc. London Math. Soc. 25 (1972), 75–95.

[BLS] A. Beauville, Y. Laszlo and C. Sorger: The Picard group of the moduli of $G$-bundles on a curve, Compos. Math. 112 (1998), 183–216.

[BH] I. Biswas and N. Hoffmann: A Torelli theorem for moduli spaces of principal bundles over a curve, Ann. Inst. Fourier (to appear), http://arxiv.org/abs/1003.4061

[BH] I. Biswas and A. Hogadi: Brauer group of moduli spaces of $\text{PGL}(r)$-bundles over a curve, Adv. Math. 225 (2010), 2317–2331.

[BHH] I. Biswas, A. Hogadi and Y. I. Holla: The Brauer group of desingularization of moduli spaces of vector bundles over a curve, Cent. Euro. Jour. of Math. 10 (2012), 1300–1305.

[BP] I. Biswas and M. Poddar: The Chen–Ruan cohomology of some moduli spaces, Int. Math. Res. Not. IMRN (2008), article ID rnm104.

[BP2] I. Biswas and M. Poddar: Chen-Ruan cohomology of some moduli spaces, II. Internat. J. Math. 21 (2010), no. 4, 49752.

[Bo] F. A. Bogomolov: Brauer groups of fields of invariants of algebraic groups, Math. USSR Sbornik 66 (1990), 285–299.

[CG] C. H. Clemens and P. A. Griffiths: The intermediate Jacobian of the cubic threefold, Ann. of Math. 95 (1972), 281–356.

[Ho] N. Hoffmann: Rationality and Poincaré families for vector bundles with extra structure on a curve, Int. Math. Res. Not. (2007), no. 3, Art. ID rnm010.

[IM] V. A. Iskovskih and Y. I. Manin: Three-dimensional quartics and counterexamples to the Lüroth problem, Mat. Sb. (N.S.) 86 (1971), 140–166.

[KS] A. King and A. Schofield: Rationality of the moduli of vector bundles on curves, Indag. Math. 10 (1999), 519–535.
[Mu1] D. Mumford: *Abelian varieties*, Oxford University Press, London, 1970.

[Mu2] D. Mumford: Theta characteristics of an algebraic curve, *Ann. Sci. École Norm. Sup.* 4 (1971), 181–192.

[NR1] M. S. Narasimhan and S. Ramanan: Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.* 89 (1969), 14–51.

[NR2] M. S. Narasimhan and S. Ramanan: Generalised Prym varieties as fixed points, *Jour. Indian Math. Soc.* 39 (1975), 1–19.

[Ne] P. E. Newstead: Rationality of moduli spaces of stable bundles, *Math. Ann.* 215 (1975), 251–268.

[Ni] N. Nitsure: Cohomology of desingularization of moduli space of vector bundles, *Compos. Math.* 69 (1989), 309-339.

[Ra] M. S. Raghunathan: Universal central extensions (Appendix to “Symmetries and quantization”), *Rev. Math. Phy.* 6 (1994), 207–225.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: indranil@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: amit@math.tifr.res.in

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

E-mail address: yogi@math.tifr.res.in