Asymptotics of Solutions of a Perfect Fluid Coupled with a Cosmological Constant in Four-Dimensional Spacetime with Toroidal Symmetry

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Abstract

Asymptotics of solutions of a perfect fluid when coupled with a cosmological constant in four-dimensional spacetime with toroidal symmetry are studied. In particular, it is found that the problem of self-similar solutions of the first kind for a fluid with the equation of state, \( p = k \rho \), can be reduced to solving a master equation of the form,

\[
2F(q, k) \frac{q''(\xi)}{q'(\xi)} - G(q, k)q'(\xi) = \frac{4}{\xi},
\]

For \( k = 0 \) and \( k = -1/3 \) the general solutions are obtained and their main local and global properties are studied in detail.

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I. INTRODUCTION

Recently, self-similar solutions of the Einstein field equations have attracted much attention, not only because they can be studied analytically through simplification of the problem, but also because of their relevance in astrophysics [1] and critical phenomena in gravitational collapse [2–4]. In the latter case, as in statistical mechanics [5,6], the self-similar solutions represent the asymptotics of critical collapse in the intermediate regions.

Recently, self-similar solutions of a scalar field were studied and all such solutions were found analytically [7]. It was shown that some of the solutions can be interpreted as representing gravitational collapse of the scalar field. During the collapse, trapped surfaces never develop, and as a result, no black hole is formed. This is consistent with the general theorem presented in [8]. Although the collapse always ends in spacetime singularities, it was found that these singularities are spacelike and not naked [9].

In this paper, we shall study self-similar solutions of a perfect fluid with toroidal symmetry and the equation of state, $p = k \rho$, where $k$ is a constant, and $p$ and $\rho$ denote, respectively, the pressure and energy density of the fluid. In particular, in Sec. II we write the corresponding Einstein field equations and define apparent horizons and black holes in such spacetimes, while in Sec. III we present the dimensional analysis of the problem. In Sec. IV, following Barenblatt [5], we study the self-similar solutions of the first kind, and find general solutions for $k = 0$ and $k = -1/3$. For other cases the problem reduces to solving a master equation of the form,

$$2F(q,k)\frac{q''}{q'} - G(q,k)q' = \frac{4}{\xi}. \quad (1.1)$$

In Sec. V, we study the local and global properties of the solutions found in Sec. IV, while in Sec. VI, we present our main conclusions.
II. THE EINSTEIN-FLUID EQUATIONS

The general metric for a four-dimensional spacetime with toroidal symmetry can be cast in the form,

\[ ds^2 = g_{ab}(x^0, x^1) \, dx^a dx^b - L_0^2 e^{-\mu(x^0, x^1)} \left( d\theta^2 + d\varphi^2 \right), \quad (a, b = 0, 1), \]  

where \( x^a \in (-\infty, \infty) \) and all \( x^a \) have the dimension of length, \( L \). The coordinates \( x^A = \{ \theta, \varphi \} \), \( (A = 2, 3) \) are dimensionless with the surfaces \( x^A = 0 \) and \( x^A = 2\pi \) being identified. The constant \( L_0 \) has the dimension of length. The spacetimes usually have three Killing vectors, given, respectively, by

\[ \xi^{(1)} = \theta \frac{\partial}{\partial \varphi} - \varphi \frac{\partial}{\partial \theta}, \quad \xi^{(2)} = \frac{\partial}{\partial \theta}, \quad \xi^{(3)} = \frac{\partial}{\partial \varphi}. \]  

Clearly, the metric is invariant under the coordinate transformation,

\[ x^0 = x^0(x^0, x^1), \quad x^1 = x^1(x^0, x^1). \]  

The energy-momentum tensor (EMT) for a perfect fluid, , takes the form

\[ T_{\mu\nu} = (\rho + p) \, u_{\mu} u_{\nu} - pg_{\mu\nu}. \]  

where \( u_{\mu} \), \( \rho \) and \( p \) denote, respectively, the four-velocity, energy density and pressure of the fluid. Using gauge freedom (2.3) we choose the coordinates such that

\[ g_{01}(x^0, x^1) = 0, \quad u_{\mu} = (g_{00})^{1/2} \delta^0_{\mu}. \]  

Denoting \( x^0 \) and \( x^1 \) by \( t \) and \( z \), respectively, we find that the metric can be written as

\[ ds^2 = e^{\lambda(t, z)} \, dt^2 - e^{\nu(t, z)} \, dz^2 - L_0^2 e^{-\mu(t, z)} \left( d\theta^2 + d\varphi^2 \right). \]  

In this note we consider asymptotics of solutions of a perfect fluid. Using the remaining (trivial) coordinate transformation \( t' = t + t_0 \), where \( t_0 \) is a constant, we assume that a spacetime singularity always starts to form at \( t = 0 \) (if there is any). Then, it can be shown that the Einstein-fluid equations,
\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \Lambda g_{\mu \nu} = (\rho + p) u_{\mu} u_{\nu} - p g_{\mu \nu}, \quad (2.7) \]

can be cast in the form,
\[ 2 \mu_{,tz} - \mu_{,t} (\mu_{,z} + \lambda_{,z}) - \mu_{,z} \nu_{,t} = 0, \quad (2.8) \]
\[ e^{\lambda - \nu} \left\{ 2 (\lambda_{,zz} - \mu_{,zz}) + \mu_{,z} (\mu_{,z} + \lambda_{,z} + \nu_{,z}) + \lambda_{,z} (\lambda_{,z} - \nu_{,z}) \right\} \]
\[ -2 (\nu_{,tt} + \mu_{,tt}) - \mu_{,t} (2 \mu_{,t} + \nu_{,t} + \lambda_{,t}) + \nu_{,t} (\nu_{,t} - \lambda_{,t}) = 0, \quad (2.9) \]
\[ \rho = \frac{1}{4} \left\{ e^{\nu} [4 \mu_{,zz} - \mu_{,z} (3 \mu_{,z} + 2 \nu_{,z})] + e^{-\lambda} \mu_{,t} (\mu_{,t} - 2 \nu_{,t}) \right\} + \Lambda, \quad (2.10) \]
\[ p = \frac{1}{4} \left\{ e^{-\lambda} [4 \mu_{,tt} - \mu_{,t} (3 \mu_{,t} + 2 \lambda_{,t})] + e^{-\nu} \mu_{,z} (\mu_{,z} - 2 \lambda_{,z}) \right\} - \Lambda, \quad (2.11) \]

where \( \Lambda \) denotes the cosmological constant, which can be parameterized as
\[ \Lambda = \pm \frac{1}{l^2}. \quad (2.12) \]

To study the formation of apparent horizons and black holes in the spacetimes described by the metric (2.6), we first give the definition of such notions. Following [8,10] let us first introduce two null coordinates \( u \) and \( v \) via the relations
\[ du = J(t, z) \left( e^{\lambda/2} dt - e^{\nu/2} dz \right), \quad dv = K(t, z) \left( e^{\lambda/2} dt + e^{\nu/2} dz \right), \quad (2.13) \]

where \( J(t, z) \) and \( K(t, z) \) satisfy the integrability conditions,
\[ \frac{\partial^2 u}{\partial t \partial z} = \frac{\partial^2 u}{\partial z \partial t}, \quad \frac{\partial^2 v}{\partial t \partial z} = \frac{\partial^2 v}{\partial z \partial t}. \quad (2.14) \]

Without loss of generality, we assume that they are all strictly positive,
\[ J(t, z) > 0, \quad K(t, z) > 0. \quad (2.15) \]

Then, it is easy to show that, in terms of \( u \) and \( v \), the metric (2.6) takes the form
\[ ds^2 = 2 e^{2\sigma(u,v)} du dv - R^2(u,v) \left( d\theta^2 + d\varphi^2 \right), \quad (2.16) \]

where
\[ \sigma(u,v) = -\frac{1}{2} \ln (2JK), \quad R(u,v) = L_0 e^{-\mu/2}. \quad (2.17) \]
It should be noted that the metric (2.16) is invariant under the transformations

\[ u = u(\bar{u}), \quad v = v(\bar{v}). \] (2.18)

Using this gauge freedom, we assume that the metric has no coordinate singularities in the coordinates, \( u, v \) and \( x^A \). This, in particular, implies that \( \sigma \) is finite except at some points or on some surfaces where the spacetime is singular.

Introducing two null vectors, \( l_\mu \) and \( n_\mu \), by

\[ l_\lambda = \frac{\partial u}{\partial x^\lambda} = \delta^u_\lambda, \quad n_\lambda = \frac{\partial v}{\partial x^\lambda} = \delta^v_\lambda, \] (2.19)

which are orthogonal to the two-surfaces \( S = \{ x^\lambda : t, z = \text{Constant} \} \), we find that the expansions of the null rays \( u = \text{Constant} \) and \( v = \text{Constant} \) are given, respectively, by

\[ \theta_l \equiv l_\mu g^{\mu
u} = 2e^{-2\sigma} \frac{R_v}{R}, \]
\[ \theta_n \equiv n_\mu g^{\mu
u} = 2e^{-2\sigma} \frac{R_u}{R}. \] (2.20)

Definitions. A two-surface, \( S \), of constant \( u \) and \( v \) (or constant \( t \) and \( z \)) is trapped, marginally trapped, or untrapped, according to whether \( \theta_l \theta_n > 0 \), \( \theta_l \theta_n = 0 \), or \( \theta_l \theta_n < 0 \). Assuming that on the marginally trapped surfaces we have \( \theta_l|_S = 0 \), an apparent horizon is defined as a two-surface \( H \) foliated by marginally trapped surfaces, on which \( \theta_n|_H \neq 0 \).

The apparent horizon is outer, degenerate, or inner, according to whether \( \mathcal{L}_u \theta_l|_H < 0 \), \( \mathcal{L}_u \theta_l|_H = 0 \), or \( \mathcal{L}_u \theta_l|_H > 0 \), future if \( \theta_n|_H < 0 \), and past if \( \theta_n|_H > 0 \), where \( \mathcal{L}_v \) (\( \mathcal{L}_u \)) denotes the Lie derivative along the vector \( l^\mu \) (\( n^\mu \)). We define a black hole by the existence of a future outer or degenerate apparent horizon [8,11].

It is interesting to note that the above definitions can also be given in terms of the vector \( R_{,\lambda} \),

\[ R_{,\lambda} R^{,\lambda} = 2e^{-2\sigma} R_v R_u = \frac{1}{2} R^2 e^{2\sigma} \theta_l \theta_n. \] (2.21)

Thus, a two-surface, \( S \), of constant \( t \) and \( z \) is trapped, marginally trapped, or untrapped, according to whether \( R_{,\lambda} \) is timelike, null, or spacelike.
From Eq.(2.13) we find that
\[
\frac{\partial t}{\partial u} = \frac{1}{2J} e^{-\lambda/2}, \quad \frac{\partial t}{\partial v} = \frac{1}{2K} e^{-\nu/2}, \\
\frac{\partial z}{\partial u} = -\frac{1}{2J} e^{-\nu/2}, \quad \frac{\partial z}{\partial v} = \frac{1}{2K} e^{-\nu/2},
\]
(2.22)
from which we have
\[
\theta_t = 2 e^{-2\sigma} \frac{\mathcal{R}_v}{\mathcal{R}} = \frac{2K}{\mathcal{R}} \left( e^{-\lambda/2} \mathcal{R}_{,t} + e^{-\nu/2} \mathcal{R}_{,z} \right), \\
\theta_n = 2 e^{-2\sigma} \frac{\mathcal{R}_u}{\mathcal{R}} = \frac{2J}{\mathcal{R}} \left( e^{-\lambda/2} \mathcal{R}_{,t} - e^{-\nu/2} \mathcal{R}_{,z} \right).
\]
(2.23)

III. DIMENSIONAL ANALYSIS

Since \( t \) and \( z \) all have dimensions of length, and \( \lambda, \nu \) and \( \mu \) are dimensionless, we have
\[
[t] = [z] = L, \quad [\lambda] = [\nu] = [\mu] = 1.
\]
(3.1)
On the other hand, because the Ricci tensor has the dimension of \( L^{-2} \), then from Eq.(2.7) we find that
\[
[p] = [p] = L^{-2}, \quad [l] = L.
\]
(3.2)
Thus, from \( t, z \) and \( l \) we construct two dimensionless quantities,
\[
\xi \equiv -\frac{z}{t}, \quad \eta \equiv -\frac{t}{l}.
\]
(3.3)
Since the metric coefficients are functions of \( t, z \) and \( l \), and \( \lambda, \nu \) and \( \mu \) are dimensionless, we must have
\[
\lambda (t, z, l) = \lambda (\xi, \eta), \quad \nu (t, z, l) = \nu (\xi, \eta), \quad \mu (t, z, l) = \mu (\xi, \eta), \\
\rho (t, z, l) = \frac{1}{4l^2} \Phi (\xi, \eta), \quad p (t, z, l) = \frac{1}{4l^2} \Pi (\xi, \eta).
\]
(3.4)
For any given function \( f(\xi, \eta) \), we have
\[ f_{,t} = -\frac{1}{t} (\xi f_{,\xi} - \eta f_{,\eta}), \]
\[ f_{,z} = -\frac{1}{t} f_{,\xi}, \]
\[ f_{,tt} = \frac{1}{t^2} \left( \xi^2 f_{,\xi\xi} - 2\eta \xi f_{,\eta\xi} + \eta^2 f_{,\eta\eta} + 2\xi f_{,\xi} \right), \]
\[ f_{,tz} = \frac{1}{t^2} \left( \xi f_{,\xi\xi} - \eta f_{,\eta\xi} + f_{,\xi} \right), \]
\[ f_{,zz} = \frac{1}{t^2} f_{,\xi\xi}. \]  
(3.5)

Inserting these expressions into Eqs.(2.8)-(2.11), we find that

\[ 2 \left( \xi \mu_{,\xi} - \eta \mu_{,\eta} + \mu_{,\xi} \right) - \left( \mu_{,\xi} + \lambda_{,\xi} \right) \left( \xi \mu_{,\xi} - \eta \mu_{,\eta} \right) \]
\[ - \mu_{,\xi} \left( \xi \nu_{,\xi} - \eta \nu_{,\eta} \right) = 0, \]  
(3.6)

\[ 2 \left[ \xi^2 \left( \mu_{,\xi} + \nu_{,\xi} \right) - 2\eta \xi \left( \mu_{,\eta} + \nu_{,\eta} \right) + \eta^2 \left( \mu_{,\eta\eta} + \nu_{,\eta\eta} \right) + 2\xi \left( \mu_{,\xi} + \nu_{,\xi} \right) \right] \]
\[ - 2 \left( \xi \mu_{,\xi} - \eta \mu_{,\eta} \right) \left[ \xi \left( \mu_{,\xi} + \lambda_{,\xi} \right) - \eta \left( \mu_{,\eta} + \lambda_{,\eta} \right) \right] \]
\[ + \left[ \xi \left( \nu_{,\xi} - \lambda_{,\xi} \right) - \eta \left( \nu_{,\eta} - \lambda_{,\eta} \right) \right] \left[ \xi \left( \nu_{,\xi} - \mu_{,\xi} \right) - \eta \left( \nu_{,\eta} - \mu_{,\eta} \right) \right] \]
\[ + e^{\lambda - \nu} \left[ 2 \left( \mu_{,\xi\xi} - \lambda_{,\xi\xi} \right) - \mu_{,\xi} \left( \lambda_{,\xi} + \nu_{,\xi} \right) - \lambda_{,\xi} \left( \lambda_{,\xi} - \nu_{,\xi} \right) \right] = 0, \]  
(3.7)

\[ \Phi = e^{-\nu} \left[ 4 \mu_{,\xi\xi} - \mu_{,\xi} \left( 3\mu_{,\xi} + 2\nu_{,\xi} \right) \right] \]
\[ + e^{-\lambda} \left( \xi \mu_{,\xi} - \eta \mu_{,\eta} \right) \left[ \xi \left( \mu_{,\xi} - \nu_{,\xi} \right) - \eta \left( \mu_{,\eta} - 2\nu_{,\eta} \right) \right] \pm 4\eta^2, \]  
(3.8)

\[ \Pi = e^{-\lambda} \left\{ 4 \left( \xi^2 \mu_{,\xi\xi} - 2\eta \xi \mu_{,\eta\xi} + \eta^2 \mu_{,\eta\eta} + 2\xi \mu_{,\xi} \right) \right. \]
\[ - \left( \xi \mu_{,\xi} - \eta \mu_{,\eta} \right) \left[ \xi \left( 3\mu_{,\xi} + 2\lambda_{,\xi} \right) - \eta \left( 3\mu_{,\eta} + 2\lambda_{,\eta} \right) \right] \}
\[ + e^{-\nu} \mu_{,\xi} \left( \mu_{,\xi} - 2\lambda_{,\xi} \right) \pm 4\eta^2, \]  
(3.9)

where the + (-) sign corresponds to \( \Lambda > 0 \) (\( \Lambda < 0 \)).

Note that in the present case we have four equations, Eqs.(3.6) - (3.9), and five unknowns: \( \lambda, \nu, \mu, \Phi \) and \( \Pi \). Thus, to determine these five functions uniquely, we need to have one more equation, which is usually provided by the equation of state of the perfect fluid. In this paper, we consider the case

\[ p = k\rho, \]  
(3.10)

where \( k \) is a constant.
IV. SELF-SIMILAR SOLUTIONS OF THE FIRST KIND

In this section, we consider the asymptotic behavior of the previous equations as \( \eta \to 0 \).

According to Barenblatt [5] (See also [6]), self-similar solutions of the first kind are defined as the existence of the limits,

\[
\begin{align*}
\lambda_0(\xi) &= \lim_{\eta \to 0} \lambda(\xi, \eta), & \nu_0(\xi) &= \lim_{\eta \to 0} \nu(\xi, \eta), \\
\mu_0(\xi) &= \lim_{\eta \to 0} \mu(\xi, \eta), & \Phi_0(\xi) &= \lim_{\eta \to 0} \Phi(\xi, \eta, p), \\
\Pi_0(\xi) &= \lim_{\eta \to 0} \Pi(\xi, \eta).
\end{align*}
\]  

(4.1)

Substituting these expressions into Eqs. (3.6)-(3.9) and Eq. (3.10), we find that

\[
\begin{align*}
2 \left( \xi \mu' \right)' - \xi \mu' \left( \lambda' + \nu' + \mu' \right) &= 0, \\
2 \xi^2 \left( \mu'' + \nu'' \right) + \xi \left( \mu' + \nu' \right) \left[ \xi \left( \nu' - 2 \mu' - \lambda' \right) + 4 \right] \\
&+ e^{\lambda - \nu} \left[ 2 \left( \mu' - \lambda'' \right) - \nu' \left( \mu' - \lambda' \right) - \lambda' \left( \mu' + \lambda' \right) \right] = 0, \\
4 \left( \xi^2 \mu' \right)' - \xi^2 \mu' \left[ (3 + k) \mu' + 2 \left( \lambda' - k \nu' \right) \right] \\
&- e^{\lambda - \nu} \left\{ 4 k \mu'' - \mu' \left[ (1 + 3 k) \mu' - 2 \left( \lambda' - k \nu' \right) \right] \right\} = 0, \\
\Phi &= e^{-\nu} \left[ 4 \mu'' - \mu' \left( 3 \mu' + 2 \nu' \right) \right] + e^{-\lambda} e^{2 \xi^2 \mu'} \left( \mu' - 2 \nu' \right), \\
\Pi &= e^{-\lambda} \left[ 4 \left( \xi^2 \mu' \right)' - \xi^2 \mu' \left( 3 \mu' + 2 \lambda' \right) \right] + e^{-\nu} \mu' \left( \mu' - 2 \lambda' \right),
\end{align*}
\]  

(4.2)-(4.6)

where a prime denotes ordinary differentiation with respect to \( \xi \). In writing Eqs. (4.2)-(4.6), we have omitted all zero subscripts for simplicity, however, the functions \( \mu, \nu, \) and \( \lambda \) appearing in Eqs. (4.2)-(4.6) are functions of \( \xi \) only.

Integrating Eq. (4.2), we obtain

\[
\mu' = \frac{c_0}{\xi} e^{(\mu + \nu + \lambda)/2},
\]  

(4.7)

where \( c_0 \) is a non-zero and otherwise arbitrary constant. Inserting it into Eq. (4.4), we find that

\[
e^{\mu/2} = \frac{2 e^{-(\nu + \lambda)/2}}{c_0 (1 + k) (\xi^2 e^{2 \nu} - e^{\lambda})}
\times \left\{ \xi^2 e^{\nu} \left[ 2 + (1 + k) \xi \nu' \right] + e^{\lambda} \left[ 2 k - (1 + k) \xi \lambda' \right] \right\},
\]  

(4.8)
for \( k \neq -1 \). When \( k = -1 \), substituting Eq.(4.7) into Eq.(4.4) and considering Eq.(4.7) yields

\[
\lambda = -\frac{\mu}{2} + \ln \left( \frac{\xi^2 \mu'}{c_0} \right),
\]
\[
\nu = -\frac{\mu}{2} + \ln \left( \frac{\mu'}{c_0} \right), \quad (k = -1),
\]
where \( \mu \) is an arbitrary function of \( \xi \) only. It can be shown that for such solutions Eq.(4.3) is satisfied automatically, while Eqs.(4.5) and (4.6) yield

\[
\Phi = 0 = \Pi, \quad (k = -1),
\]
that is, the corresponding spacetime is vacuum. In fact, it is not only vacuum but also flat, as the corresponding Riemann tensor vanishes identically. Thus, no self-similar toroidal solutions of the first kind exist with the equation of state \( p = -\rho \). In the following we consider only the case where \( k \neq -1 \).

The combination of Eqs.(4.2), (4.3), (4.7) and (4.8) yields,

\[
\xi \left[ \xi^2 \left( 2e^\lambda - \xi^2 e^\nu \right) e^\nu - e^{2\lambda} \right] \lambda'
\]
\[ -k \xi^4 e^{2\nu} \left[ 3(1 + k)\xi \nu' + \xi \lambda' + 4 \right]
\]
\[ +ke^{2\lambda} \left[ (1 + k)\xi \nu' - (5 + 4k)\xi \lambda' + 4(2k + 1) \right]
\]
\[ +2k \xi^2 e^{\nu+\lambda} \left[ (1 + k)\xi \nu' + (2k + 3)\xi \lambda' - 4k \right] = 0,
\]
\[
\xi^4 e^{2\nu} \left\{ (1 + k)^2 \xi^2 \left( 2\nu'' - 3\nu'^2 - \nu' \lambda' \right) + (1 + k)\xi \left[ (5k - 8)\nu' - \lambda' \right] + 4(4k - 2) \right\}
\]
\[ -\xi^2 e^{\nu+\lambda} \left\{ (1 + k)^2 \xi^2 \left( 2\nu'' + 2\lambda'' + \nu'^2 + \lambda'^2 - 10\nu' \lambda' \right)
\]
\[ +2(1 + k)\xi \left[ (11k + 4)\nu' - 7\lambda' \right] + 8(2k + 1) \}
\]
\[ + e^{2\lambda} \left\{ (1 + k)^2 \xi^2 \left( 2\lambda'' - 3\lambda'^2 - \nu' \lambda' \right) + (1 + k)\xi \left[ k\nu' + (16k + 3) \lambda' \right]
\]
\[ -4k(4k + 1) \right\} = 0, \quad (k \neq -1).
\]

From the above, we see that the problem of solving the Einstein field equations now reduces to solving Eqs.(4.11) and (4.12) for the functions of \( \nu \) and \( \lambda \). Once they are found, Eqs.(4.7),
(4.5) and (4.6) will give the functions $\mu$, $\Phi$ and $\Pi$. Eqs.(4.11) and (4.12) are non-linear, and finding the solutions of such equations is, in general, very complicated. To start with, let us consider the case $k = 0$.

**A. $k = 0$**

When $k = 0$, Eq.(4.11) reduces to

$$\lambda' \left( \xi^2 e^\nu - e^\lambda \right) = 0,$$

which has two solutions,

(i) $\lambda = \lambda_0$,  
(ii) $\lambda = \nu + 2 \ln(\xi)$,

where $\lambda_0$ is a constant. However, it can be shown that in the latter case the corresponding spacetime is vacuum, $\Phi = 0 = \Pi$. Thus, in the following, we consider only the first case, for which we can set $\lambda_0 = 0$ without loss of generality. Then, Eq.(4.12) reduces to

$$2\xi^2 \left( \xi^2 e^\nu - 1 \right) \nu'' - \xi^2 \left( 3\xi^2 e^\nu + 1 \right) \nu^2 - 8 \left( \xi^2 e^\nu + 1 \right) (\xi \nu' + 1) = 0.$$

To solve this equation, we introduce the function $y(\xi)$ via the relation

$$y \equiv \xi^2 e^\nu - 1,$$

for which Eq.(4.15) takes the form,

$$\frac{y''}{y'} - \frac{5y + 4}{2y(1 + y)} y' = -\frac{2}{\xi}.$$

It can be shown that the general solution is

$$\frac{2Y}{Y^2 - 1} + \ln \left| \frac{Y - 1}{Y + 1} \right| = \frac{a}{\xi} + b,$$

where $a$ and $b$ are two integration constants, and

$$Y \equiv \xi e^{\nu/2}.$$
In terms of $Y$, Eqs.(4.5) and (4.8) yield
\[
\Phi = 2a \frac{(Y^2 - 1)^2}{\xi Y},
\]
\[
e^\mu = \left( \frac{a}{2c_0} \right)^2 (Y^2 - 1)^2, \quad (k = 0).
\]

(4.20)

**B. $k \neq 0, -1$**

When $\lambda = \nu + 2 \ln(\xi)$, spacetime is vacuum for any value of $k$. Then, for $k \neq 0, -1$ we consider the following ansatz,
\[
\lambda(\xi) = \nu(\xi) + 2 \ln(\xi) + q(\xi),
\]

(4.21)

where $q(\xi)$ is an arbitrary function of $\xi$. Substituting into Eq.(4.11) we obtain
\[
(1 + 3k) \nu' + \left( 1 - \frac{4k}{1 - e^q} e^q \right) q' + \frac{2(1 + 3k)}{(1 + k) \xi} = 0.
\]

(4.22)

To solve this equation, let us first consider the case $k = -1/3$.

1. $k = -1/3$

In this case Eq.(4.22) has the solution,
\[
q(\xi) = q_0,
\]

(4.23)

where $q_0$ is a constant. Inserting Eqs.(4.21) and (4.23) into Eq.(4.12), we find
\[
\xi^2 \nu'' - 2\xi^2 \nu'^2 - 9\xi \nu' - 12 = 0.
\]

(4.24)

There are two particular solutions,
\[
(1) \ \nu_s^{(1)} = -2 \ln(\xi), \quad (2) \ \nu_s^{(2)} = -3 \ln(\xi).
\]

(4.25)

To find the general solution of Eq.(4.24) we set $\nu = A(\xi) + \nu_s^{(1)}$, and then Eq.(4.24) reduces to
\[ A'' - 2A'^2 - \frac{1}{\xi} A' = 0, \quad (4.26) \]

which has the general solution,

\[ A(\xi) = -\frac{1}{2} \ln |A_1 - A_0 \xi^2|, \quad (4.27) \]

where \( A_0 \) and \( A_1 \) are two integration constants. Thus, for \( k = -1/3 \) we have the following general solutions,

\[
\begin{align*}
\lambda(\xi) &= q_0 - \frac{1}{2} \ln |A_1 - A_0 \xi^2|, \\
\nu(\xi) &= -\frac{1}{2} \ln \left| \xi^4 \left( A_1 - A_0 \xi^2 \right) \right|, \\
e^{\mu/2} &= \frac{2A_1 e^{-q_0/2\xi}}{c_0 (A_1 - A_0 \xi^2)^{1/2}}, \\
\Phi &= -3\Pi = \frac{12A_1 (1 - e^{q_0})}{e^{q_0} (A_1 - A_0 \xi^2)^{1/2}}. \quad (4.28)
\end{align*}
\]

2. \( k \neq -1/3 \)

In this case, Eq.(4.22) has the solution,

\[ \nu(\xi) = -\frac{2}{1 + k} \ln(\xi) - \frac{1}{1 + 3k} \left( q + 4k \ln(1 - e^q) \right) + \nu_0, \quad (4.29) \]

where \( \nu_0 \) is a constant. Inserting the above expression into Eq.(4.12), we find

\[ 2f(q)q'' - g(q)q'^2 - \frac{4}{\xi} h(q)q' = 0, \quad (4.30) \]

where

\[
\begin{align*}
f(q) &\equiv \left( 3k^4 + 10k^3 + 12k^2 + 6k + 1 \right) \left( 1 - e^{3q} \right) \\
&\quad + \left( 6k^5 + 23k^4 + 34k^3 + 24k^2 + 8k + 1 \right) e^{2q} \\
&\quad - \left( 3k^5 + 16k^4 + 32k^3 + 30k^2 + 13k + 2 \right) e^q, \\
g(q) &\equiv k \left( 7k^4 + 22k^3 + 24k^2 + 10k + 1 \right) e^{3q} \\
&\quad - \left( 10k^5 + 37k^4 + 50k^3 + 28k^2 + 4k - 1 \right) e^{2q}
\end{align*}
\]
\[ + \left(3k^5 + 12k^4 + 20k^3 + 18k^2 + 9k + 2\right) e^q \]

\[-3(1+k)^2 \left(1 - k^2\right), \]

\[h(q) \equiv k \left(3k^3 + 7k^2 + 5k + 1\right) e^{2q} \]

\[-\left(6k^4 + 17k^3 + 17k^2 + 7k + 1\right) e^{2q} \]

\[+ \left(3k^4 + 13k^3 + 19k^2 + 11k + 2\right) e^q \]

\[-\left(3k^3 + 7k^2 + 5k + 1\right). \quad (4.31)\]

It can be shown that when \(q' (\xi) = 0\) the corresponding solutions represent a vacuum space. When \(q' \neq 0\), Eq.(4.30) can be written as

\[2F(q) \left[\ln (q')\right]' - G(q)q' = \frac{4}{\xi}, \quad (4.32)\]

where

\[F(q) \equiv \frac{f(q)}{h(q)}, \quad G(q) \equiv \frac{g(q)}{h(q)}. \quad (4.33)\]

Solving Eq.(4.32) is not a trivial exercise. One may first try to find a particular solution of it, say, \(q_s(\xi)\). Once such a solution is known, setting \(q = q_s(\xi) + q_0(\xi)\), we find that Eq.(4.32) reduces to the following form for \(q_0(\xi)\),

\[\left[\ln (q_0')\right]' - H(q_0, q_s)q_0' = 0. \quad (4.34)\]

Unfortunately, we have not yet been able to find such a particular solution.

\textbf{V. PHYSICAL AND GEOMETRICAL INTERPRETATIONS OF THE SELF-SIMILAR SOLUTIONS}

In this section, we study the local, as well as the global, properties of the self-similar solutions for \(k = 0\) and \(-1/3\), obtained in the last section. Note that, although these solutions were found by taking the limit \(\eta \to 0\), in this section we extend them to any \(t \in (-\infty, 0)\).
A. Self-Similar Solutions With \( k = 0 \)

These are the solutions given by Eqs. (4.14)-(4.20). Rescaling the coordinates \( t \) and \( z \) and using the conformal transformation \( g'_{\mu\nu} = B^2 g_{\mu\nu} \), where \( B \) is a constant, without loss of generality, we set

\[
c_0 = \frac{1}{2} a, \quad A_1 = 1, \quad (5.1)
\]

for which the metric reads

\[
ds^2 = dt^2 - \frac{Y^2}{\xi^2} dz^2 - \frac{L_0^2}{(Y^2 - 1)^2} \left( d\theta^2 + d\varphi^2 \right). \quad (5.2)
\]

Then, the corresponding energy density is given by

\[
\rho = \frac{1}{4t^2} \Phi = \frac{Y^2 - 1}{2t^2 Y} I(Y), \quad (5.3)
\]

where

\[
I(Y) \equiv -b(Y^2 - 1) + 2Y + (Y^2 - 1) \ln \left| \frac{Y - 1}{Y + 1} \right|, \quad (5.4)
\]

\[
\xi(Y) = \frac{a(Y^2 - 1)}{I(Y)}. \quad (5.5)
\]

From the above expressions one can see that the spacetime is singular at \( t = 0 \) and \( \xi = -a/b \), where

\[
\xi(Y) = -\frac{a}{b}, \quad \text{when} \ Y = 0, \pm \infty. \quad (5.6)
\]

The singularity at \( t = 0 \) is always spacelike, while the nature of the singularity at \( \xi = -a/b \) depends on the values of \( Y \). The normal vector to the surface \( \xi = -a/b \) is given by

\[
n_\lambda \equiv \frac{\partial}{\partial x^\lambda} \left( z - \frac{a}{b} t \right) = -\frac{a}{b} \delta_\lambda^z + \delta_\lambda^t. \quad (5.7)
\]

Then,

\[
n_\lambda n^\lambda = \frac{a^2}{b^2} - \frac{\xi^2}{Y^2} = \begin{cases} \ a^2 > 0, & Y = \pm \infty, \\ \ -\infty < 0, & Y = 0. \end{cases} \quad (5.8)
\]
Therefore, when \( Y = 0 \) the corresponding spacetime singularity at \( x = -a/b \) is timelike, and when \( Y = \pm \infty \), it is spacelike. From Eqs.(2.17) and (5.2) we obtain

\[
\begin{align*}
\cal R &= L_0 e^{-\mu/2} = \pm \frac{L_0}{Y^2 - 1}, \\
\cal R_{,\lambda} &= \pm \frac{2L_0 Y Y_{,\xi}}{t (Y^2 - 1)^2} \left( \xi \delta_{\lambda}^t + \delta_{\lambda}^\xi \right). 
\end{align*}
\] (5.9)

Thus, we have

\[
\cal R_{,\lambda} R^{,\lambda} = \frac{L_0^2 I^2(Y)}{4t^2 (Y^2 - 1)}. 
\] (5.10)

From Eq.(5.5) we also find that

\[
\frac{d\xi(Y)}{dY} = \frac{4a}{I^2(Y)}, 
\] (5.11)

which shows that \( \xi(Y) \) is a monotonically increasing \((a > 0)\) or decreasing \((a < 0)\) function of \( Y \), depending on the sign of the constant \( a \). To have the energy density \( \rho \) non-negative, the solutions are restricted to the following regions for different \( a \),

\[
Y \xi = \begin{cases} 
\geq 0, & a > 0, \\
\leq 0, & a < 0,
\end{cases}
\] (5.12)

as one can see clearly from Eq.(4.20). To further study the properties of the solutions, it is convenient to consider the following four cases separately: (a) \( a > 0, b > 0 \); (b) \( a > 0, b < 0 \); (c) \( a < 0, b > 0 \); and (d) \( a < 0, b < 0 \).

**Case (a) \( a > 0, b > 0 \):** In this case we find that

\[
I(Y) = \begin{cases} 
-\infty, & Y = \pm \infty, \\
\pm b, & Y = 0, \\
\pm 2, & Y = \pm 1,
\end{cases}
\] (5.13)

and \( I(Y) = 0 \) has two real roots, \( Y_{\pm} \), with the properties

\[
Y_+ > 1, \quad -1 < Y_- < 0,
\] (5.14)

as can be seen from Fig. 1.
From Eqs. (5.4) and (5.5) we find that

\[
\xi(Y) = \begin{cases} 
-a/b, & Y = 0, \pm \infty, \\
0, & Y = \pm 1, \\
\infty, & Y = Y_{\pm}.
\end{cases}
\]  

(5.15)

Then, one can see that \(\xi(Y)\) must behave as that given by Fig. 2, from which, together with Eq. (5.12), we can see that the energy density is non-negative only in the following three regions,

(i) \(Y \in [-\infty, -1]\), or \(\xi \in [-a/b, 0]\),

(ii) \(Y \in (Y_-, 0]\), or \(\xi \in (-\infty, 0]\),

(iii) \(Y \in [1, Y_+]\), or \(\xi \in [0, \infty)\).

(5.16)
FIG. 2. The function $\xi(Y)$ versus $Y$ for $a > 0$ and $b > 0$.

In the region $Y \in [-\infty, -1]$ or $\xi \in [-a/b, 0]$, the spacetime is singular at $\xi = -a/b$ or $Y = -\infty$. From Eq.(5.8) we can see that this singularity is spacelike. In the $(t, z)$-plane, this region is between the two lines $\xi = -a/b$ and $\xi = 0(z = 0)$, as shown in Fig. 3, in which it is referred to as Region $I$. The metric is singular at $\xi = 0$ or $Y = -1$. However, this singularity is a coordinate one, as one can see from the expression of $\rho$, which is finite there. The nature of this surface is null. In fact, introducing the normal vector $N_\lambda$ to this surface by

$$N_\lambda = \frac{\partial z}{\partial x^\lambda} = \delta_\lambda^z,$$  \hspace{1cm} (5.17)

we find that

$$N_\lambda N^{\lambda} = \frac{\xi^2}{Y^2} \to 0,$$  \hspace{1cm} (5.18)

as $\xi \to 0$ and $Y \to -1$. On the other hand, from Eq.(5.10) we find that in this region $R_{\lambda}$ is always timelike, that is, the whole region is trapped.

Region $II$, where $Y \in (Y_-, 0]$ or $\xi \in (-\infty, 0]$, is the region in between the two spacetime singularities $t = 0$ and $\xi = -a/b$. In this region, $R_{\lambda}$ is always spacelike, that is, this region is untrapped. However, because of the two spacetime singularities, it is difficult to interpret this region physically.

In Region $III$, where $Y \in [1, Y_+)$, or $\xi \in [0, \infty)$, the vector $R_{\lambda}$ is always timelike, that is, all of this region is trapped. The spacetime is singular at $t = 0$. 

\[\begin{array}{c}
\text{Singularities} \\
(\theta, \theta_n < 0) \quad II \quad (\xi = -a/b) \\
(\theta, \theta_n > 0) \\
(\theta, \theta_n > 0) \\
0 \\
(0, 0) \quad I \\
(0, 0) \quad III
\end{array}\]
FIG. 3. The \((t, z)\)-plane for \(a > 0\) and \(b > 0\). The spacetime is singular on the lines \(\xi = -a/b\) and \(t = 0\).

**Case (b) \(a > 0, b < 0\):** In this case we find

\[
I(Y) = \begin{cases}
\infty, & Y = \pm\infty, \\
-|b|, & Y = 0, \\
\pm 2, & Y = \pm 1,
\end{cases}
\]

(5.19)

for which \(I(Y) = 0\) has also two real roots, \(Y_\pm\), but now with

\[
0 < Y_+ < 1, \quad Y_- < -1,
\]

(5.20)

as shown in Fig. 4.

![Graph of I(Y) versus Y](image)

FIG. 4. The function \(I(Y)\) versus \(Y\) for \(a > 0\) and \(b < 0\).

Then, it can be shown that

\[
\xi(Y) = \begin{cases}
a/|b|, & Y = 0, \pm\infty, \\
0, & Y = \pm 1, \\
\infty, & Y = Y_\pm,
\end{cases}
\]

(5.21)

and the curve of \(\xi(Y)\) versus \(Y\) is given by Fig. 5, from which we can see that the energy density is non-negative only in the regions,

\[
(i) \ Y \in \langle Y_-, -1\rangle, \quad \text{or} \quad \xi \in (-\infty, 0], \\
(ii) \ Y \in [1, \infty], \quad \text{or} \quad \xi \in [0, a/|b|].
\]

(5.22)
In the region $Y \in (Y_-, -1]$ or $\xi \in (-\infty, 0]$, the spacetime is singular at $t = 0$ or $Y = Y_-$. In the $(t, z)$-plane, this region is the one where $t \leq 0$ and $z \leq 0$, marked as Region $I$ in Fig. 6. The metric is singular at $\xi = 0$ or $z = 0$. As in the previous case, this singularity is a coordinate one, and the surface is null, as shown by Eq.(5.17). Moreover, in this region $\mathcal{R}_\lambda$ is always timelike, except for the line $t = 0$ or $Y = Y_-$. Thus, in the present case the whole region is trapped.

In Region $II$, where $Y \in [1, \infty]$ or $\xi \in [0, a/|b|]$, the spacetime is singular on the line $\xi = a/|b|$ or $z = -at/|b|$, and the nature of the singularity is spacelike, as can be seen from Eq.(5.8), considering the fact that now $\xi = a/|b|$ corresponds to $Y = \infty$. In addition, $\mathcal{R}_\lambda$ is also timelike, that is, Region $II$ in the present case is also trapped.

FIG. 5. The function $\xi(Y)$ versus $Y$ for $a > 0$ and $b < 0$.

In Region $I$, where $Y \in [1, \infty]$ or $\xi \in [0, a/|b|]$, the spacetime is singular on the line $\xi = a/|b|$ or $z = -at/|b|$, and the nature of the singularity is spacelike, as can be seen from Eq.(5.8), considering the fact that now $\xi = a/|b|$ corresponds to $Y = \infty$. In addition, $\mathcal{R}_\lambda$ is also timelike, that is, Region $II$ in the present case is also trapped.

FIG. 6. The $(t, z)$-plane for $a > 0$ and $b < 0$. The spacetime is singular on the lines $\xi = a/|b|$ and $t = 0$. 

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Case (c) $a < 0$, $b > 0$: In this case $I(Y)$ is given by Fig. 1, and $\xi(Y)$ behaves as that given by Fig. 7, from which we can see that the energy density is non-negative only in the regions,

(i) $Y \in [-\infty, -1]$, or $\xi \in [|a|/b, 0]$,

(ii) $Y \in (Y_-, 0]$, or $\xi \in [|a|/b, \infty)$,

(iii) $Y \in [1, Y_+]$, or $\xi \in (-\infty, 0]$.

\[(5.23)\]

![FIG. 7. The function $\xi(Y)$ versus $Y$ for $a < 0$ and $b > 0$.](image)

The corresponding three regions of Eq.(5.23) in the $(t, z)$-plane are shown in Fig. 8. In particular, the spacetime is singular on the lines $t = 0$ and $\xi = |a|/b$, and Regions I and III are trapped, while Region II is not. The metric is singular on the line $z = 0$ or $Y = \pm 1$, which is a null surface.

![FIG. 8. The $(t, z)$-plane for $a < 0$ and $b > 0$. The spacetime is singular on the lines $\xi = |a|/b$ and $t = 0$.](image)
Case (d) $a < 0$, $b < 0$: In this case $I(Y)$ is given by Fig. 4, and $\xi(Y)$ behaves as that given in Fig. 9, from which we can see that the energy density is non-negative only in the regions,

1. $Y \in (Y_-, -1]$, or $\xi \in [0, \infty)$,
2. $Y \in [0, Y_+)$, or $\xi \in (-\infty, -|a/b|]$,
3. $Y \in [1, \infty]$, or $\xi \in [-|a/b|, 0]$.

(5.24)

FIG. 9. The function $\xi(Y)$ versus $Y$ for $a < 0$ and $b < 0$.

The corresponding three regions of Eq.(5.24) in the $(t, z)$-plane are shown in Fig. 10, where the spacetime is also singular on the lines $t = 0$ and $\xi = -|a/b|$, and Regions I and III are trapped, while Region II is not. The metric is singular on the line $z = 0$ or $Y = \pm 1$, and this line is null.

FIG. 10. The $(t, z)$-plane for $a < 0$ and $b < 0$. The spacetime is singular on the lines $\xi = |a/b|$ and $t = 0$. 

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B. Self-Similar Solutions With \( k = -1/3 \)

These are the solutions given by Eq.(4.28). Without loss of generality, we set

\[
A_1 = 1, \quad c_0 = 2e^{-q_0/2}.
\] (5.25)

Then, the solutions are given by

\[
\begin{align*}
\lambda &= q_0 - \frac{1}{2} \ln \left(1 - \alpha \xi^2\right), \\
\nu &= -\frac{1}{2} \ln \left[\xi^4 \left(1 - \alpha \xi^2\right)\right], \\
\mu &= \ln \left(\frac{\xi^2}{1 - \alpha \xi^2}\right),
\end{align*}
\] (5.26)

for which we have

\[
\rho = -3p = \frac{\rho_0}{t^2 (1 - \alpha \xi^2)^{1/2}},
\] (5.27)

where \( \rho_0 \equiv (1 - e^{q_0})/(4e^{q_0}) \) and \( \alpha \equiv A_0 \). Thus, when \( \alpha > 0 \), the solutions are valid only in the region where \( |\xi| \leq \xi_0 \) as shown in Fig. 11, and the spacetime is singular at \( t = 0 \) and at \( \xi = \pm \xi_0 \equiv \pm \alpha^{-1/2} \). These singularities are all spacelike. This can be seen easily when \( t = 0 \). For the cases \( \xi = \pm \xi_0 \), we first introduce the normal vectors \( n_{\mu}^{\pm} \) by

\[
n_{\mu}^{\pm} \equiv \partial(z \pm \xi_0 t)/\partial x^\mu = \delta_{\mu}^{\pm} \pm \xi_0 \delta_{\mu}^{t}.
\] (5.28)

Then, we find that

\[
n_{\mu}^{\pm} n_{\nu}^{\pm} g^{\mu \nu} = e^{-(\nu + q_0)} \left(\frac{\xi_0^2}{\xi^2} - e^{q_0}\right) \sim e^{-(\nu(\xi_0)+q_0)} (1 - e^{q_0}) > 0.
\] (5.29)

Thus, the singularities on \( \xi = \pm \xi_0 \) are indeed spacelike. On the other hand, it can be shown that now we have

\[
\mathcal{R} = L_0 \frac{(1 - \alpha \xi^2)^{1/2}}{|\xi|},
\]

\[
\mathcal{R}_a \mathcal{R}^{a} = \frac{L_0^2 \xi_0 e^{-q_0}}{t^2 \xi^2 (\xi_0^2/\xi^2 - 1)^{1/2}} (1 - e^{q_0}) > 0.
\] (5.30)

Thus, the whole region \(-x_0 < \xi < x_0\) is trapped [cf. Fig. 11]. It should be noted that the metric is singular on \( z = 0 \) or \( \xi = 0 \), but it can be shown that it is a coordinate one.
FIG. 11. The spacetime in the \((t, z)\)-plane for \(\alpha > 0\), for which the solutions are valid only in the region \(|\xi| \leq \xi_0\). The spacetime is singular on the lines \(\xi = \pm \xi_0\), which are always spacelike. For \(\alpha \leq 0\), the solutions are valid in the whole half plane \(t \leq 0\), and the spacetime is singular only on the line \(t = 0\).

When \(\alpha \leq 0\), the solutions are valid in the whole half plane \(t \leq 0\), and the spacetime is singular only on \(t = 0\). It can also be shown that the whole region \(t \leq 0\) is now trapped, as one can show that \(\theta_1 \theta_n > 0\) for all time \(t < 0\).

VI. CONCLUSIONS

In this paper, we have studied the asymptotics of solutions of a perfect fluid when coupled with a cosmological constant in four-dimensional spacetimes with toroidal symmetry. We found that the problem for self-similar solutions of the first kind with the equation of state, \(p = k \rho\), can be reduced to solving a master equation of the form,

\[
2F(q, k) \frac{q''(\xi)}{q'(\xi)} - G(q, k)q'(\xi) = \frac{4}{\xi}.
\]

Although we were not able to solve this equation for all \(k\), we did obtain the general solutions for \(k = 0\) and \(k = -1/3\). The local and global properties of these solutions were studied in detail, and it was found that no apparent horizons develop during the evolution of the fluid, although trapped regions indeed exist. This is consistent with the general theorem obtained previously [8].
Finally we note that spacetimes with toroidal symmetry are *locally* indistinguishable from those with plane symmetry. In fact, by first unwrapping the angular coordinates $\theta$ and $\varphi$ and then extending the ranges to $x^A \in (-\infty, \infty)$, we obtain spacetimes with plane symmetry. With a different identification of coordinates, one may obtain topologies other than toroidal and plane geometries.
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