THE CATEGORICAL WEIL REPRESENTATION

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Abstract. In a previous work the authors gave a conceptual explanation for the linearity of the Weil representation over a finite field \( k \) of odd characteristic: There exists a canonical system of intertwining operators between the Lagrangian models of the Heisenberg representation. This defines a canonical vector space \( \mathcal{H}(V) \) associated with a symplectic vector space \( V \) over \( k \). In this paper we prove a general theorem about idempotents in categories, and we use it to solve the sign problem, formulated by Bernstein and Deligne, on the compatibility between the associativity constraint and the convolution structure of the \( \ell \)-adic sheaf of canonical intertwining kernels. This sheaf governs—via the sheaf-to-function correspondence—the function theoretic system of intertwiners. As an application we define a canonical category \( \mathcal{C}(V) \) associated with the symplectic vector space variety \( V \), and we obtain the canonical model of the categorical Weil representation.

0. Introduction

0.1. The Weil representation. The Weil representation \([16]\) over a finite field \( k \) is the algebra object that governs the symmetries of the standard Hilbert space \( \mathcal{H}_L = L^2(L, \mathbb{C}) \) of complex valued functions on a finite-dimensional vector space \( L \) over \( k \). In this paper we will be interested only with the case where \( k \) is of odd characteristic. We have the (split) symplectic vector space \( V = L \times L^* \) with its standard symplectic form. We denote by \( Sp(V) \) the corresponding symplectic group. A nontrivial argument, due to Schur, implies that there exists a linear representation

\[ \rho_L : Sp(V) \to GL(\mathcal{H}_L), \]

obtaining the Weil representation. This representation depends also on a choice of additive character \( \psi : k \to \mathbb{C}^* \).

The Weil representation is a central object of modern harmonic analysis and the theory of the discrete Fourier transform. It has many applications in automorphic forms, number theory, the theory of theta functions, mathematical physics, coding theory, signal processing, and other domains of knowledge.

Probably, the most interesting fact, for researchers who are not familiar with the Weil representation, is that it can be thought of as a group of operators which includes the discrete Fourier transform. Indeed, for any nondegenerate symmetric bilinear form \( B \) on \( L \) we have \([3]\)

\[ \left[ \rho_L \left( \begin{pmatrix} 0 & -B^{-1} \\ B & 0 \end{pmatrix} \right) \right] f(x) = \frac{1}{G(B, \psi)} \sum_{y \in L} \psi(B(x, y)) f(y), \]

where \( G(B, \psi) \) is an appropriate Gauss sum normalization.

0.2. Canonical vector space. In \([10, 11]\) a conceptual explanation for the existence of the linear Weil representation was proposed. Specifically, it was shown that there exists an explicit quantization functor

\[ \mathcal{H} : \text{Symp} \to \text{Vect}, \]

(0.1)

where \( \text{Symp} \) denotes the (groupoid) category whose objects are finite dimensional symplectic vector spaces over the finite field \( k \), and morphisms are linear isomorphisms of symplectic vector spaces and \( \text{Vect} \) denotes the category of finite dimensional complex vector spaces.

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As a consequence, for a fixed symplectic vector space \( V \in \text{Symp} \), we obtain, by functoriality, a homomorphism
\[
\rho_V : \text{Sp}(V) \to GL(\mathcal{H}(V)),
\]
which is isomorphic to the Weil representation. We refer to the vector space \( \mathcal{H}(V) \) as the *canonical vector space* associated to \( V \), and to the representation (0.2) as the *canonical model* of the Weil representation of \( \text{Sp}(V) \).

0.3. Main results.

0.3.1. **Canonical category.** In this paper, we obtain the categorical analog of the functor (0.1), i.e., we define an explicit quantization lax 2-functor [9, 15]
\[
V \mapsto \mathcal{C}(V),
\]
associating certain category of \( \ell \)-adic sheaves \( \mathcal{C}(V) \) to any object \( V \) in the (groupoid) category \( \text{Symp} \) whose objects are finite dimensional symplectic vector spaces in the category of algebraic varieties over \( k \), and morphisms are linear isomorphisms of symplectic vector spaces. We will refer to the category \( \mathcal{C}(V) \) as the *canonical category* associated to \( V \).

0.3.2. **The categorical Weil representation.** In particular, for a fixed object \( V \in \text{Symp} \) we obtain, by functoriality, an action of the algebraic group \( \text{Sp} = \text{Sp}(V) \) on the category \( \mathcal{C}(V) \), forming the categorical analog of (0.2). For the sake of the introduction, it is enough to say that this action, which we will call the *categorical Weil representation*, is an explicit family of auto-equivalence functors
\[
\rho_V(g) : \mathcal{C}(V) \to \mathcal{C}(V), \quad g \in \text{Sp},
\]
which are induced from the action morphism \( \text{Sp} \times V \to V \).

0.3.3. **The idempotent theorem.** The main technical result of this paper is the proof of the idempotent theorem which is needed in order to define the category \( \mathcal{C}(V) \). This theorem proposes a solution to a problem, formulated by Bernstein [1] and Deligne [4], that we will call the *sign problem*. We devote the rest of the introduction to an intuitive description of the sign problem, its solution, and its implication to the definition of \( \mathcal{C}(V) \).

0.4. **Canonical system of intertwining operators.** The existence of the canonical vector space \( \mathcal{H}(V) \) is a manifestation of the existence of a canonical system of intertwining operators, between the models of the Heisenberg representation, associated with oriented Lagrangian subspaces in \( V \).

We denote by \( O\text{Lag} = O\text{Lag}(V) \) the set of oriented Lagrangian subspaces in \( V \), i.e., the set of pairs \( L^\circ = (L, \alpha_L) \) where \( L \subset V \) is a Lagrangian subspace and \( \alpha_L \) is a nonzero vector in the top wedge product \( \bigwedge^{\text{top}} L \). In addition, we denote by \( H = H(V) \) the Heisenberg group associated with \( V \), and by \( Z = Z(H) \cong k \) its center. Finally, let us choose a nontrivial additive character \( \psi : Z \to \mathbb{C}^\ast \).

The Stone–von Neumann theorem asserts that there exist a unique (up to isomorphism) irreducible representation \( \pi : H \to GL(\mathcal{H}) \) with \( \pi(z) = \psi(z) \cdot \text{Id}_{\mathcal{H}} \) for every \( z \in Z \). We will call the representation \( \pi, H, \mathcal{H} \) the *Heisenberg representation*. An important family of models of the Heisenberg representation is associated with oriented Lagrangian subspaces in \( V \). To every \( L^\circ \in O\text{Lag} \) we have the model \( (\pi_{L^\circ}, H, \mathcal{H}_{L^\circ}) \), where the vector space \( \mathcal{H}_{L^\circ} \) is the space \( \mathbb{C}(L \setminus H, \psi) \) of functions \( f : H \to \mathbb{C} \) such that \( f(z \cdot l \cdot b) = \psi(z) f(b) \), for every \( z \in Z, \ l \in L \), and the action \( \pi_{L^\circ} \) is given by right translation. The collection of models \( \{\mathcal{H}_{L^\circ}\} \) can be thought of as a vector bundle \( \mathcal{H} \) on the set \( O\text{Lag} \), with fibers \( \mathcal{H}_{L^\circ} \). The main technical result proved in [10, 11] is the strong Stone–von Neumann theorem, i.e., the vector bundle \( \mathcal{H} \) admits a canonical trivialization. Concretely, this means that there exists a canonical system of intertwining operators \( T_{M^\circ,L^\circ} \in \text{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}_{M^\circ}) \), for every \( M^\circ, L^\circ \in O\text{Lag} \), satisfying the following multiplicativity property:
\[
T_{N^\circ,M^\circ} \circ T_{M^\circ,L^\circ} = T_{N^\circ,L^\circ},
\]
for every \( N^\circ, M^\circ, L^\circ \in O\text{Lag} \).

**Remark 0.4.1.** Similar considerations with \( k = \mathbb{R} \) yield [13] a canonical system of intertwiners, which are multiplicative up to a sign \( \pm 1 \), referred to as the metaplectic sign.
The canonical vector space $\mathcal{H}(V)$ is the space of ”horizontal sections” of $\mathcal{H}$
\[ \mathcal{H}(V) = \Gamma_{hor}(\text{OLag},\mathcal{H}), \]
i.e., a vector in $\mathcal{H}(V)$ is a compatible system $(f_L \in \mathcal{H}_L; L^2 \in \text{OLag})$ such that $T_{M^2, L^2}(f_L) = f_{M^2}$, for every $M^2, L^2 \in \text{OLag}$. The symplectic group $Sp(V)$ acts on the vector space $\mathcal{H}(V)$ in an obvious manner and we obtain the model (0.2).

In order to formulate the sign problem [10], we will need the description of the system $\{T_{M^2, L^2}\}$ by kernels and their geometrization.

0.5. Canonical system of intertwining kernels. Every intertwining operator $T_{M^2, L^2}$ can be uniquely presented by a kernel function $K_{M^2, L^2} \in \mathbb{C}(M \setminus H/L, \psi)$. The collection of kernel functions $\{K_{M^2, L^2}\}$ can be thought of as a single function
\[ K \in \mathbb{C}(\text{OLag}^2 \times H), \quad (0.4) \]
given by $K \big( (M^2, L^2, -) \big) = K_{M^2, L^2}(\cdot, \cdot)$, for every $(M^2, L^2) \in \text{OLag}^2$. The multiplicativity property (0.3) transformed into the following convolution property with respect to the convolution $*$ of functions on the Heisenberg group:
\[ p_{3i}^*K * p_{2j}^*K = p_{31}^*K, \quad (0.5) \]
where $p_{ij} : \text{OLag}^3 \times H \to \text{OLag}^2 \times H$ are the projections given by $p_{ij}(L^3_1, L^2_2, L^1_3, h) = (L^3_i, L^i_1, h)$, for $1 \leq i < j \leq 3$.

We proceeds to the geometrization of the kernels.

0.6. The sheaf of canonical geometric intertwining kernels.

0.6.1. Geometrization. A general ideology due to Grothendieck is that any meaningful set-theoretic object is governed by a more fundamental algebra-geometric one. The procedure by which one translate from the set theoretic setting to algebraic geometry is called geometrization, which is a formal procedure by which sets are replaced by algebraic varieties and functions are replaced by certain sheaf-theoretic objects.

The precise setting consists of:
- A set $X = X(k)$ of rational points of an algebraic variety $X$ defined over $k$.
- A complex valued function $f \in \mathbb{C}(X)$ governed by an $\ell$-adic Weil sheaf $\mathcal{F}$ on $X$.

The variety $X$ is a space equipped with an endomorphism $F \in X \rightarrow X$, called Frobenius, such that the set $X$ is naturally identified with the set of fixed points $X = X^{F^r}$.

The sheaf $\mathcal{F}$ can be thought of as a vector bundle on the variety $X$, equipped with an endomorphism $\vartheta : \mathcal{F} \rightarrow \mathcal{F}$ which lifts $F$.

The relation between the function $f$ and the sheaf $\mathcal{F}$ is called Grothendieck’s sheaf-to-function correspondence: Given a point $x \in X$, the endomorphism $\vartheta$ restricts to an endomorphism $\vartheta_x : \mathcal{F}_{|x} \rightarrow \mathcal{F}_{|x}$ of the fiber $\mathcal{F}_{|x}$. The value of $f$ on the point $x$ is given by
\[ f(x) = f^\mathcal{F}(x) = Tr(\vartheta_x : \mathcal{F}_{|x} \rightarrow \mathcal{F}_{|x}). \]

0.6.2. The sheaf of canonical geometric intertwining kernels. The function $K$ (0.4) fits nicely to the geometrization procedure. Denote by $\text{OLag}^2 \times H$ the algebraic variety with $\text{OLag}^2 \times H = \text{OLag}^2 \times H(k)$. In [10] the authors defined a geometrically irreducible, (shifted) perverse, $\ell$-adic Weil sheaf $\mathcal{K}$ on $\text{OLag}^2 \times H$, that satisfies the following two properties:

1. Convolution. There exists a canonical isomorphism $\theta : \mathcal{K} \ast \mathcal{K} \rightarrow \mathcal{K}$.
2. Function. Applying sheaf-to-function procedure we recover $f^K = K$.

Here, the notation $\theta : \mathcal{K} \ast \mathcal{K} \rightarrow \mathcal{K}$ stands to simplify the more precise notation which is the geometric analogue of (0.5). We will call $\mathcal{K}$ the sheaf of canonical geometric intertwining kernels.

We are ready now to formulate the sign problem.
0.7. The sign problem. In an attempt to understand the nature of the ("disappearance" of the) metaplectic sign over finite fields of odd characteristic, we formulate \[1, 4\] the following sign problem. Let $K$ be the sheaf of canonical geometric intertwining kernels. Consider the commutative diagram

$$
\begin{array}{ccc}
(K \star K) \star K & \xrightarrow{\alpha} & K \star (K \star K) \\
\downarrow \theta \star \text{id} & & \downarrow \text{id} \star \theta \\
K \star K & \xrightarrow{C} & K
\end{array}
$$

where $\alpha$ is the associativity constraint for convolution $\star$ of sheaves on the Heisenberg group $H$, the morphism $\theta$ is the isomorphism appearing in the convolution property of the sheaf $K$, and $C$ is by definition the isomorphism that makes the diagram commutative.

The sheaf $K$ is geometrically irreducible, hence, $C = c \cdot \text{id}$ is a scalar morphism.

**Problem 0.7.1 (The sign problem).** Compute the scalar $c$.

The idea is \[1, 4\] that the value of $c$ might suggest a new understanding about the nature of the metaplectic sign over finite field, i.e., that it "moves" one hierarchy higher, becoming a property of the sheaf $K$ that cannot be observed on the level of the function $K$.

0.8. A solution to the sign problem. The main technical result of this paper is the proof of the following theorem:

**Theorem 0.8.1 (The idempotent theorem—particular case).** We have $c = 1$.

In fact, we will show that the idempotent theorem holds in a very general situation. Let $C$ be any category with an "operation", i.e., a functor $\otimes : C \times C \to C$ equipped with associativity constraint $\alpha$ \[6\], and suppose $P \in C$ is an "idempotent", i.e., equipped with an isomorphism $\theta : P \otimes P \to P$. Then we can form an analogue diagram to (0.6) and obtain an isomorphism $C : P \to P$. We will prove that under natural conditions $C = \text{id}$.

Let us describe shortly the application of Theorem 0.8.1 to the definition of the canonical category.

0.9. The canonical category. The idempotent theorem suggests the definition of a canonical category $C(V)$ associated with the symplectic vector space variety $V$ with $V = V(k)$. This category is the categorical analogue of the vector space $H(V)$, and it is the basic object behind the existence of the categorical Weil representation. The category $C(V)$ consists of $\ell$-adic sheaves on $\text{OLag} \times H$ satisfying geometric conditions that are analog of these satisfied by vectors in $H(V)$. The most important condition is that each $F \in C(V)$ is equipped with an isomorphism

$$
\eta : K \star p_1^* F \to p_2^* F,
$$

which is compatible with $\alpha$ and $\theta$. Here, $p_i : \text{OLag}^2 \times H \to \text{OLag} \times H$ are given by $p_i(L_i^2, L_i^0, h) = (L_i^0, h)$, $i = 1, 2$, and compatibility means the commutativity

$$
\begin{array}{ccc}
(K \star K) \star F & \xrightarrow{\alpha} & K \star (K \star F) \\
\downarrow \theta \star \text{id} & & \downarrow \text{id} \star \eta \\
K \star F & \xrightarrow{\eta} & K \star F \\
\downarrow \eta & & \downarrow \eta \\
F & \xrightarrow{id} & F
\end{array}
$$

This definition make sense only if $C = \text{id}$ in (0.6).
0.10. Structure of the paper. In Section 1 we recall the construction of the canonical vector space, and the Weil representation, using the canonical intertwining operators which we describe in the language of kernels. In Section 2 we recall the sheaf theoretic counterpart of the intertwining kernels. In Section 3 we define the notion of idempotent in a category, and we state the idempotent theorem (Theorem 3.1.2) on the compatibility between an idempotent structure and an associativity constraint for operations in categories. We describe also a generalization of the idempotent theorem which suggests a solution to the sign problem. In Section 4 we describe the following applications: We formulate and solve the sign problem; we define the canonical category which is a categorification of the canonical vector space; we obtain the canonical model of the categorical Weil representation; and we suggest a solution to the sign problem in the context of the geometric Weil representation sheaf. Finally, in Appendix A we suggest a proof for the idempotent theorem.

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1. The canonical vector space

We would like to recall the construction of the canonical vector space established in [10, 11], to describe the family of canonical intertwining operators and their explicit presentation as kernels.

1.1. The Heisenberg representation. Let \((V, \omega)\) be a \(2n\)-dimensional symplectic vector space over the finite field \(k = \mathbb{F}_q\), where \(q\) is odd. The Heisenberg group associated with \(V\) can be presented as the set \(H = V \times k\) with the multiplication given by

\[
(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v')).
\]

The center of \(H\) is \(Z = Z(H) = \{(0, z) : z \in k\}\).

Let \(\psi : Z \to \mathbb{C}^*\) be a nontrivial character of the center.

Theorem 1.1.1 (Stone–von Neuman). There exists a unique (up to isomorphism) irreducible unitary representation \((\pi, H, \mathcal{H})\) with the center acting by \(\psi\), i.e., \(\pi|_Z = \psi \cdot \text{Id}_H\).

The representation \(\pi\) which appears in the above theorem will be called the Heisenberg representation.

1.2. Oriented Lagrangian models. We are interested in a particular family of models of the Heisenberg representation which are associated with oriented Lagrangian subspaces in \(V\). Let \(\text{Lag}\) denote the set of Lagrangian subspaces in \(V\).

Definition 1.2.1. An oriented Lagrangian subspace in \(V\) is a pair \(L^\circ = (L, o_L)\), where \(L \in \text{Lag}\) and \(o_L \in \bigwedge^{top} L\) is a nonzero vector.

Let us denote by \(OL\text{ag}\) the set of oriented Lagrangian subspaces in \(V\). We associate with each oriented Lagrangian \(L^\circ \in O\text{Lag}\) a model \((\pi_{L^\circ}, H, \mathcal{H}_{L^\circ})\) of the Heisenberg representation as follows: The vector space \(\mathcal{H}_{L^\circ}\) is the space \(\mathbb{C}(L \setminus H, \psi)\) of functions \(f : H \to \mathbb{C}\) satisfying \(f(z \cdot l \cdot h) = \psi(z)f(h)\) for every \(z \in Z, l \in L, \) and the Heisenberg action is given by right translation \(\pi_{L^\circ}(h)[f](h') = f(h' \cdot h)\) for every \(f \in \mathcal{H}_{L^\circ}\).

1.3. System of canonical intertwining operators. The collection of models \(\{\mathcal{H}_{L^\circ}\}\) forms a vector bundle \(\mathcal{E} \to O\text{Lag}\) with fibers \(\mathcal{H}_{L^\circ} = \mathcal{H}_{L^\circ}\) acted upon by \(H\) via \(\pi_{L^\circ}\).

Definition 1.3.1. Let \(\mathcal{E} \to O\text{Lag}\) be an \(H\)-vector bundle. A trivialization of \(\mathcal{E}\) is a system of intertwining operators (isomorphisms) \(\{E_{M^\circ, L^\circ} \in \text{Hom}_H(\mathcal{E}_{L^\circ}, \mathcal{E}_{M^\circ}) : (M^\circ, L^\circ) \in O\text{Lag}^2\}\) satisfying the following multiplicativity condition

\[
E_{N^\circ, M^\circ} \circ E_{M^\circ, L^\circ} = E_{N^\circ, L^\circ},
\]

for every \(N^\circ, M^\circ, L^\circ \in O\text{Lag}\).
The main result of [10, 11] is the following:

**Theorem 1.3.2 (The strong S-vN property).** The $H$-vector bundle $\mathfrak{H}$ admits a natural trivialization $\{T_{M^o, L^o}\}$.

The intertwining operators $\{T_{M^o, L^o}\}$ in the above theorem will be referred to as the system of canonical intertwining operators.

1.4. **Explicit formulas for the canonical intertwining operators.** Let us denote by $U_2 \subset OLag(V)^2$ the subset consisting of pairs of oriented Lagrangians $(M^o, L^o) \in OLag(V)^2$ which are in general position, i.e., $L + M = V$. For every $(M^o, L^o) \in U_2$, we have [10]

$$T_{M^o, L^o} = A_{M^o, L^o} \cdot F_{M^o, L^o},$$

where $F_{M^o, L^o} : \mathcal{H}_{L^o} \rightarrow \mathcal{H}_{M^o}$ is the averaging morphism given by

$$F_{M^o, L^o} [f] (h) = \sum_{m \in M} f (m \cdot h),$$

for every $f \in \mathcal{H}_{L^o}$ and $A_{M^o, L^o}$ is a normalization constant given by

$$A_{M^o, L^o} = (G(\psi)/q)^n \sigma((-1)^{\frac{n}{2}}) \omega \Lambda (o_L, o_M), \quad (1.1)$$

where

- $\sigma$ is the unique quadratic character (also called the Legendre character) of the multiplicative group $G_m = k^*$.
- $G(\psi)$ is the one-dimensional Gauss sum

$$G(\psi) = \sum_{z \in k} \psi(\frac{1}{2}z^2).$$

- $\omega : \wedge^{top} L \times \wedge^{top} M \rightarrow k$ is the pairing induced by the symplectic form.

1.5. **The canonical vector space and the Weil representation.** Using Theorem 1.3.2, we can associate, in a functorial manner, a vector space $\mathcal{H}(V)$ to the symplectic vector space $V$ as follows: Define $\mathcal{H}(V)$ to be the space of “horizontal sections” of the trivialized vector bundle $\mathfrak{H} H = \Gamma_{hor} (OLag, \mathfrak{H})$,

$$\mathcal{H}(V) = \Gamma_{hor} (OLag, \mathfrak{H}),$$

where $\Gamma_{hor} (OLag, \mathfrak{H}) \subset \Gamma (OLag, \mathfrak{H})$ is the subspace consisting of sections $(f_{L^o} \in \mathcal{H}_{L^o} : L^o \in OLag)$ satisfying $T_{M^o, L^o} (f_{L^o}) = f_{M^o}$ for every $(M^o, L^o) \in OLag^2$. The vector space $\mathcal{H}(V)$ will be referred to as the canonical vector space associated with $V$.

**Remark 1.5.1.** The definition of the vector space $\mathcal{H}(V)$ depends on a choice of a central character $\psi$.

The group $Sp(V)$ acts on the set $OLag \times H$ and this induces an action on the canonical vector space. The representation obtained in this way

$$\rho_V : Sp(V) \rightarrow GL(\mathcal{H}(V)),$$

will be called the canonical model of the Weil representation.

1.6. **Construction using kernels of intertwiners.**
1.6.1. **Kernel presentation of an intertwining operator.** Fix a pair \((M^\circ, L^\circ) \in O\text{Lag}^2\) of oriented Lagrangians and let \(\mathbb{C}(M/H/L, \psi) \subset \mathbb{C}(H, \psi)\) be the subspace of functions \(K \in \mathbb{C}(H, \psi)\), satisfying the equivariance property \(K(m \cdot h \cdot l) = K(h)\), for every \(m \in M\) and \(l \in L\). We have an isomorphism of vector spaces

\[
I : \mathbb{C}(M/H/L, \psi) \to \text{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}_{M^\circ}),
\]

associating to a function \(K \in \mathbb{C}(M/H/L, \psi)\) the intertwining operator \(I[K] \in \text{Hom}_H(\mathcal{H}_{L^\circ}, \mathcal{H}_{M^\circ})\) defined by

\[
I[K](f) = K \ast f = m_1(K \boxtimes_{Z \cdot L} f),
\]

for every \(f \in \mathcal{H}_{L^\circ}\). Here, \(K \boxtimes_{Z \cdot L} f\) denotes the descent of the function \(K \boxtimes f \in \mathbb{C}(H \times H)\) to \(H \times_{Z \cdot L} H\)—the quotient of \(H \times H\) by the action \(x \cdot (h_1, h_2) = (h_1x, x^{-1}h_2)\) for \(x \in Z \cdot L\)—and \(m_1\) denotes the operation of summation along the fibers of the multiplication mapping \(m : H \times H \to H\). We call the function \(K\) an **intertwining kernel**.

It is easy to verify that for a triple \((N^\circ, M^\circ, L^\circ) \in O\text{Lag}^3\) and kernels \(K_2 \in \mathbb{C}(N/H/M, \psi)\) and \(K_1 \in \mathbb{C}(M/H/L, \psi)\), their convolution \(K_2 \ast K_1 = m_1(K_2 \boxtimes_{Z \cdot M} K_1)\) lies in \(\mathbb{C}(N/H/L, \psi)\). Moreover, the transform \(I\) sends convolution of kernels to composition of operators

\[
I[K_1 \ast K_2] = I[K_1] \circ I[K_2].
\]

1.6.2. **System of canonical intertwining kernels.** For every \((M^\circ, L^\circ) \in O\text{Lag}^2\), there exists a unique kernel \(K_{M^\circ, L^\circ} \in \mathbb{C}(M/H/L, \psi)\) such that \(T_{M^\circ, L^\circ} = I[K_{M^\circ, L^\circ}]\). We will refer to \(\{K_{M^\circ, L^\circ}\}\) as the system of **canonical intertwining kernels**. A reformulation of Theorem 1.3.2 is that the system \(\{K_{M^\circ, L^\circ}\}\) is multiplicative, in the sense that \(K_{N^\circ, M^\circ} \ast K_{M^\circ, L^\circ} = K_{N^\circ, L^\circ}\) for every triple \((N^\circ, M^\circ, L^\circ) \in O\text{Lag}^3\).

The system of kernels \(\{K_{M^\circ, L^\circ}\}\) can be equivalently thought of as a single function

\[
K \in \mathbb{C}(O\text{Lag}^2 \times H),
\]

defined by \(K(M^\circ, L^\circ, -) = K_{M^\circ, L^\circ}(-)\) satisfying the following multiplicativity relation on \(O\text{Lag}^3 \times H\)

\[
p_{32}^*K \ast p_{21}^*K = p_{31}^*K,
\]

where \(p_{ji}(L^\circ_3, L^\circ_2, L^\circ_1, h) = (L^\circ_j, L^\circ_i, h)\) are the projections on the \(j, i\) copies of \(O\text{Lag}\) and the left-hand side of (1.3) means fiberwise convolution

\[
p_{32}^*K \ast p_{21}^*K(L^\circ_3, L^\circ_2, L^\circ_1, -) = K(L^\circ_2, L^\circ_1, -) \ast K(L^\circ_3, L^\circ_2, -).
\]

1.6.3. **Formula.** In case \(M^\circ, L^\circ\) are in generic position, i.e., \((M^\circ, L^\circ) \in U_2\), the kernel \(K_{M^\circ, L^\circ}\) is given by the following explicit formula

\[
K_{M^\circ, L^\circ} = A_{M^\circ, L^\circ} \cdot \tilde{K}_{M^\circ, L^\circ},
\]

with \(\tilde{K}_{M^\circ, L^\circ} = \tau^*\psi\), where \(\tau = \tau_{M^\circ, L^\circ}\) is the inverse of the isomorphism given by the composition \(Z \hookrightarrow H \rightarrow M/H/L\), and \(A_{M^\circ, L^\circ}\) as in (1.1).

1.6.4. **Definition of the canonical vector space using kernels.** The canonical vector space \(\mathcal{H}(V)\) can be defined using the kernel function \(K \in \mathbb{C}(O\text{Lag} \times H)\) such that

- \(a_{Z}^*f = \psi \cdot f\), where \(a_{Z} : Z \times O\text{Lag} \times H \to O\text{Lag} \times H\) is the action map induced from the action of \(Z\) on \(H\).
- \(a_{S}^*f = p^*f\), where \(S \to O\text{Lag}\) is the tautological vector bundle with fiber \(S_{L^\circ} = L\), and \(p : a_{S} : S \times H \to O\text{Lag} \times H\) are the projection and the action map \(a_{S}(l, L^\circ, h) = (L^\circ, l \cdot h)\), respectively.
- We have

\[
K \ast p_i^*f = p_i^*f,
\]

where \(p_i : O\text{Lag}^2 \times H \to O\text{Lag} \times H\) are the projections \(p_i(L^\circ_2, L^\circ_1, h) = (L^\circ_i, h)\).
1.6.5. The Weil representation using the language of kernels. We consider the set $X = O\text{Lag} \times H$, the action map $a_{Sp} : Sp \times X \to X$, and the pullback operator
\[ a_{Sp}^* : C(X) \to C(Sp \times X). \] (1.4)

For what follows, it will be convenient for us to denote the canonical vector space $\mathcal{H}(V)$ by $\mathcal{H}(X)$. We have the vector spaces $\mathcal{H}(Sp \times X)$ and $\mathcal{H}(Sp \times Sp \times X)$ which are defined exactly in the same way (see Subsection [6.4] as $\mathcal{H}(X)$, i.e., by considering the conditions with respect to the $X$-coordinate. The function kernel $K$ satisfies $a_{Sp}^* K = p^* K$ where $a_{Sp}, p : Sp \times O\text{Lag}^\times \times H \to O\text{Lag}^\times \times H$ are the natural action map and projection map, respectively. Hence, (1.4) induces an operator

\[ \rho_{Sp} : \mathcal{H}(X) \to \mathcal{H}(Sp \times X). \]

Note that the homomorphism condition is now manifested via the equality

\[ (m \times id)^* \circ \rho_{Sp} = (id \times a_{Sp})^* \circ \rho_{Sp}, \]

between the two compositions of the following diagram:

\[
\begin{array}{ccc}
\mathcal{H}(X) & \xrightarrow{\rho_{Sp}} & \mathcal{H}(Sp \times X) \\
\rho_{Sp} \downarrow & & \downarrow (id \times a_{Sp})^* \\
\mathcal{H}(Sp \times X) & \xrightarrow{(m \times id)^*} & \mathcal{H}(Sp \times Sp \times X)
\end{array}
\]

where $m : Sp \times Sp \to Sp$ denotes the multiplication map.

The triple $(\rho_{Sp}, Sp, \mathcal{H}(X))$ is trivially identified with the canonical model of the Weil representation.

2. Geometric canonical intertwining kernels

In this section, we recall the definition [10] of the geometric counterpart to the set-theoretic system of canonical intertwining kernels.

2.1. Preliminaries from algebraic geometry. First, we need to use some space to recall notions and notations from algebraic geometry and the theory of $\ell$-adic sheaves.

2.1.1. Varieties. In the sequel, we are going to translate back and forth between algebraic varieties defined over the finite field $k$ and their corresponding sets of rational points. In order to prevent confusion between the two, we use bold-face letters to denote a variety $X$ and normal letters $x$ to denote its corresponding set of rational points $X = X(k)$. For us, a variety $X$ over the finite field is a quasi-projective algebraic variety, such that the defining equations are given by homogeneous polynomials with coefficients in the finite field $k$. In this situation, there exists a (geometric) Frobenius endomorphism $Fr : X \to X$, which is a morphism of algebraic varieties. We denote by $X$ the set of points fixed by $Fr$, i.e.,

\[ X = X(k) = X^{Fr} = \{ x \in X : Fr(x) = x \}. \]

The category of algebraic varieties over $k$ will be denoted by $\text{Var}_k$.

2.1.2. Sheaves. Let $D^b(X)$ denotes the bounded derived category of constructible $\ell$-adic sheaves on $X$ [2]. We denote by $\text{Perv}(X)$ the Abelian category of perverse sheaves on the variety $X$, that is the heart with respect to the autodual perverse t-structure in $D^b(X)$. An object $\mathcal{F} \in D^b(X)$ is called $N$-perverse if $\mathcal{F}[N] \in \text{Perv}(X)$. Finally, we recall the notion of a Weil structure (Frobenius structure) [5]. A Weil structure associated to an object $\mathcal{F} \in D^b(X)$ is an isomorphism

\[ \vartheta : Fr^* \mathcal{F} \overset{\sim}{\longrightarrow} \mathcal{F}. \]

A pair $(\mathcal{F}, \vartheta)$ is called a Weil object. By an abuse of notation we often denote $\vartheta$ also by $Fr$. We choose once an identification $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, hence all sheaves are considered over the complex numbers.

Remark 2.1.1. All the results in this section make perfect sense over the field $\overline{\mathbb{Q}}_\ell$, in this respect the identification of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ is redundant. The reason it is specified is in order to relate our results with the standard constructions.
Given a Weil object \((\mathcal{F}, Fr^*\mathcal{F} \simeq \mathcal{F})\) one can associate to it a function \(f^\mathcal{F} : X \rightarrow \mathbb{C}\) to \(\mathcal{F}\) as follows

\[
   f^\mathcal{F}(x) = \sum_i (-1)^i \text{Tr}(Fr|_{H^i(\mathcal{F})}).
\]

(2.1)

This procedure is called \textit{Grothendieck’s sheaf-to-function correspondence} \cite{8}.

2.2. Geometrization. We shall now start the geometrization procedure.

2.2.1. Replacing sets by varieties. The first step we take is to replace all sets involved by their geometric counterparts, i.e., algebraic varieties. We denote by \(k\) an algebraic closure of the field \(k\). The symplectic space \((V, \omega)\) is naturally identified as the set \(V = \mathbb{V}(k)\), where \(\mathbb{V}\) is a \(2n\)-dimensional symplectic vector space in \(\text{Var}_k\); the Heisenberg group \(H\) is naturally identified as the set \(H = \mathbb{H}(k)\), where \(\mathbb{H} = \mathbb{V} \times \mathbb{G}_a\) is the corresponding group variety; finally, \(\text{OLag} = \text{OLag}(k)\), where \(\text{OLag}\) is the variety of oriented Lagrangians in \(V\).

2.2.2. Replacing functions by sheaves. The second step is to replace functions by their sheaf-theoretic counterparts \cite{2}. The additive character \(\psi : k \rightarrow \mathbb{C}^*\) is associated via the sheaf-to-function correspondence to the Artin-Schreier sheaf \(\mathcal{L}_\psi\) on the variety \(\mathbb{G}_a\), i.e., we have \(f^\mathcal{L}_\psi = \psi\). The Legendre character \(\sigma\) on \(k^* \simeq \mathbb{G}_m(k)\) is associated to the Kummer sheaf \(\mathcal{L}_\sigma\) on the variety \(\mathbb{G}_m\). The one-dimensional Gauss sum \(G(\psi)\) is associated with the Weil object

\[
   G = \int_{\mathbb{G}_a} \mathcal{L}_\psi(z^2) \in D^b(\text{pt}),
\]

where, for the rest of this paper, \(\int = \int_!\) denotes integration with compact support \cite{2}. Grothendieck’s Lefschetz trace formula \cite{8} implies that, indeed, \(f^G = G(\psi)\). In fact, there exists a quasi-isomorphism \(G \cong_{\mathcal{D}} H^1(\mathcal{G})[-1]\) and \(\dim H^1(\mathcal{G}) = 1\), hence, \(\mathcal{G}\) can be thought of as a one-dimensional vector space, equipped with a Frobenius operator, sitting at cohomological degree 1.

2.2.3. Replacing the canonical intertwining kernels by a sheaf. We recall the geometrization of the function \(K\) \cite{12} obtained in \cite{10}.

Let \(U_2 \subset \text{OLag}^2\) be the open subvariety consisting of pairs \((M^o, L^o) \in \text{OLag}^2\) which are in general position. We define a sheaf "of kernels" \(\mathcal{K}\) on the variety \(U_2 \times H\) as follows:

\[
   \mathcal{K}_{U_2} = \mathcal{A} \otimes \widetilde{\mathcal{K}}_{U_2},
\]

where

- \(\widetilde{\mathcal{K}}_{U_2}\) is the sheaf of non-normalized kernels given by

\[
   \widetilde{\mathcal{K}}_{U_2}(M^o, L^o) = \tau^* \mathcal{L}_\psi,
\]

where \(\tau = \tau_{M^o, L^o}\) is the inverse of the isomorphism given by the composition

\[
   Z \hookrightarrow H \rightarrow M \setminus H/L.
\]

- \(\mathcal{A}\) is the "Normalization coefficient" sheaf given by

\[
   \mathcal{A}(M^o, L^o) = \mathcal{G}^\otimes n \otimes \mathcal{L}_\sigma \left( (-1)^{\binom{n}{2}} \omega \lambda (o_L, o_M) \right) [2n] (n).
\]

Let \(n_k = \dim(\text{OLag}^k) + n + 1\) for \(k \in \mathbb{N}\). Consider the projection morphisms \(p_{ji} : \text{OLag}^3 \times H \rightarrow \text{OLag}^2 \times H\) on the \(j, i\) coordinates of \(\text{OLag}^3\). The main geometric statement of \cite{10} is the following:

\textbf{Theorem 2.2.1} (Geometric canonical intertwining kernels). There exist a geometrically irreducible \([n_2]\)-perverse Weil sheaf \(K\) on \(\text{OLag}^2 \times H\) of pure weight \(w(K) = 0\) that satisfies the following three properties:

1. **Convolution.** There exists a canonical isomorphism \(\theta : p_{ii}^*K * p_{jj}^*K \simeq p_{ii}^*K\).
2. **Function.** Applying sheaf-to-function procedure we recover \(f^K = K\).
3. **Formula.** Restricting the sheaf \(K\) to the open subvariety \(U_2 \times H\) we have \(K|_{U_2 \times H} = K_{U_2}\).
3. The idempotent theorem

In order to formulate precisely the sign problem and to suggest an answer, we choose to work in the more general setting of idempotents in categories with operation $\otimes$.

3.1. Categorical statement. Let $C$ be a category and $\otimes : C \times C \to C$, $(A, B) \mapsto A \otimes B$, a functor. In this paper we will refer to $\otimes$ as an operation in $C$.

We have two induced functors $\otimes_l, \otimes_r : C \times C \times C \to C$, given by $\otimes_l(A, B, C) = (A \otimes B) \otimes C$ and $\otimes_r(A, B, C) = A \otimes (B \otimes C)$.

An associativity constraint for $\otimes$ is a natural isomorphism $\alpha : \otimes_l \sim \otimes_r$, such that, for every $A, B, C, D \in C$ the diagram

\[
\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) \\
\alpha \otimes \text{id} & & \alpha \\
(A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha} & A \otimes ((B \otimes C) \otimes D)
\end{array}
\]

is commutative. This is usually called the pentagon coherence for $\alpha$.

We would like to study a specific class of objects in $C$. An idempotent in $C$ is a pair $(P, \theta)$ where $P$ is an object of $C$ and $\theta$ is an isomorphism $\theta : P \otimes P \sim P$.

We will also refer to $\theta$ as an idempotent structure on $P$.

We would like now to study the relation between the idempotent structure and the associativity constraint.

We have the commutativity

\[
\begin{array}{ccc}
(P \otimes P) \otimes P & \xrightarrow{\alpha} & P \otimes (P \otimes P) \\
\theta \otimes \text{id} & & \text{id} \otimes \theta \\
P \otimes P & \xrightarrow{\theta} & P \otimes P
\end{array}
\]

(3.1)

where $C : P \to P$ is by definition the isomorphism which makes the diagram commutative.

**Definition 3.1.1.** We say that the idempotent structure $\theta$ on $P$ is compatible with the associativity constraint $\alpha$, if in (3.1) we have $C = \text{id}$.

The main technical result of the paper is the following:

**Theorem 3.1.2** (The idempotent theorem). Let $C$ be a category with an operation $\otimes$ equipped with an associativity constraint $\alpha$. Let $(P, \theta)$ be an idempotent in $C$. Let $C$ be the automorphism of $P$ defined in (3.1). If $\theta$ satisfies the commutativity of the following diagrams

\[
\begin{array}{ccc}
P \otimes P & \xrightarrow{\theta} & P \\
C \otimes \text{id} & & \text{id} \otimes C
\end{array}
\]

then $\theta$ is compatible with $\alpha$.  

\[
\begin{array}{ccc}
P & \xrightarrow{C} & P \\
\otimes \theta & & \otimes \theta
\end{array}
\]

(3.2)
For a proof of Theorem 3.1.2 see Appendix A.

For our purposes we will need a straightforward generalization of Theorem 3.1.2. We consider the sets $2 = \{1, 2\}$, $3 = \{1, 2, 3\}$, $4 = \{1, 2, 3, 4\}$, $5 = \{1, 2, 3, 4, 5\}$, and the following datum:

- For every $m = 2, \ldots, 5$ a category $C_m$ with an operation $\otimes_m$ equipped with an associativity constraint $\alpha_m$.
- For every order preserving embedding $\lambda : 1 \to m$, $1 \subsetneq m = 3, 4, 5$ a faithful functor $F_\lambda : C_1 \to C_m$.

We assume that the above functors respect the operations $\otimes_m$ and the associativity constraints $\alpha_m$, and are compatible in the strong sense that the diagram

$$
\begin{array}{ccc}
C_3 & \xrightarrow{F_2} & C_4 \\
\downarrow F_\lambda & & \downarrow F_2 \\
C_2 & \xrightarrow{F_\lambda} & C_5
\end{array}
$$

is commutative for $\mu = \sigma \circ \lambda$ and $v = \tau \circ \mu$.

For simplicity, let us denote the above datum by $(C_\bullet, \otimes_\bullet, \alpha_\bullet, F_\bullet)$. Then we can define the notion of an $F_\bullet$-idempotent. This is an object $P \in C_2$ equipped with an isomorphism in $C_3$

$$
\theta : F_{32}(P) \otimes_3 F_{21}(P) \cong F_{31}(P),
$$

where by $ji$ we mean the function from 2 to 3 given by $ji(1) = i$ and $ji(2) = j$, $1 \leq i < j \leq 3$. For such idempotent we have the commutative diagram in $C_4$

$$
\begin{array}{c}
(F_{43}(P) \otimes F_{32}(P)) \otimes F_{21}(P) \\
\downarrow \alpha \\
F_{42}(P) \otimes F_{21}(P)
\end{array}
\begin{array}{c}
F_{43}(P) \otimes F_{31}(P) \\
\downarrow \theta
\end{array}
\begin{array}{c}
F_{41}(P)
\end{array}
$$

(3.4)

where $ji$ means the function from 2 to 4 given by $ji(1) = i$ and $ji(2) = j$, $1 \leq i < j \leq 4$, $\otimes = \otimes_4$, $\alpha = \alpha_4$, and $\theta$ is an abbreviation for the various isomorphisms $F_\sigma(\theta)$ for the appropriate $\sigma$'s.

Exactly the same argument as in the proof of Theorem 3.1.2 yields the following theorem:

**Theorem 3.1.3** (The idempotent theorem—generalization). Let $(C_\bullet, \otimes_\bullet, \alpha_\bullet, F_\bullet)$ be a datum as defined above. Let $(P, \theta)$ be an $F_\bullet$-idempotent. If $\theta$ satisfies the analogue of (3.4), then $\theta$ is compatible with $\alpha_\bullet$, i.e., in (3.4) we have $C = \text{id}$.

4. Applications

We are ready to formulate and solve the sign problem, and to suggest several additional applications. The main application is the definition of the canonical category, which yields the canonical model of the categorical Weil representation.

4.1. **Application I: Solution to the sign problem.** We follow the setup of Sections 2 and 3.

Consider the following datum:

- For every $m = 2, \ldots, 5$ the category $C_m = D^b(\text{OLag}^m \times H)$ with the operation $\otimes_m$ which is induced by the convolution $*$ on $H$, and with the standard associativity constraint $\alpha_m$.
- For every order preserving embedding $\lambda : 1 \to m$, $1 \subsetneq m = 3, 4, 5$, a faithful functor $F_\lambda : C_1 \to C_m$ given by the pullback functor $p_\lambda^\ast$, where $p_\lambda : \text{OLag}^m \times H \to \text{OLag}^j \times H$ is the projection morphism induced by $\lambda$. 


\[\]
The datum \((C, \otimes, a, F)\) satisfies the compatibility of Diagram (3.3). In these terms the pair \((\mathcal{K}, \theta)\) where \(\mathcal{K}\) is the sheaf of canonical geometric intertwining kernels (see Theorem 2.2.1) is an \(F\)-idempotent. Recall that \(\mathcal{K}\) is geometrically irreducible. The precise formulation of the sign problem \([\text{II} 2]\) is the following:

**Problem 4.1.1** (The sign problem). Compute the value of the scalar \(c\) in the morphism \(C = c \cdot \text{id}\) that appears in diagram (3.4) with \(P\) replaced by the sheaf \(\mathcal{K}\).

Theorem 3.1.3 suggests the answer.

**Theorem 4.1.2** (The idempotent theorem—particular case). We have \(c = 1\).

4.2. **Application II: The canonical category.** Theorem 4.1.2 suggests the definition of a canonical category \(\mathcal{C}(V)\) associated with a symplectic vector space variety \((V, \omega)\) defined over \(k\). This category, and its definition, are the geometric analogue of the canonical vector space \(\mathcal{H}(V)\), and its definition, via the kernel function \(K\), as described in Subsection [1.6.4).

The canonical category \(\mathcal{C}(V)\) consists of sheaves \(\mathcal{F} \in D^b(\text{OLag} \times H)\) equipped with the following structure:

- An isomorphism \(\nu_Z : a_Z^* \mathcal{F} \to \mathcal{L}_\omega \otimes \mathcal{F}\), where \(a_Z : Z \times \text{OLag} \times H \to \text{OLag} \times H\) is the action morphism induced from the action of \(Z\) on \(H\).
- An isomorphism \(\nu_S^* : a_S^* \mathcal{F} \to p^* \mathcal{F}\), where \(S \to \text{OLag}\) is the tautological vector bundle with fiber \(S_{\omega} = L\), and \(p, a_S : S \times H \to \text{OLag} \times H\) are the projection morphism and the action morphism \(a_S(l, L^i, h) = (L^i, l \cdot h)\), respectively.
- An isomorphism \(\eta : \mathcal{K} \times p_1^* \mathcal{F} \to p_2^* \mathcal{F}\), where \(p_i : \text{OLag}^i \times H \to \text{OLag} \times H\), \(i = 1, 2\), are the projection morphisms \(p_i(L^i_2, L^i_1, h) = (L^i_1, h)\), and \(\mathcal{K}\) is the sheaf of canonical geometric intertwining kernels (Theorem 2.2.1).

The isomorphism \(\eta\) is required to be compatible with the associativity constraint \(\alpha\), induced from the convolution \(*\) on \(H\), and the idempotent structure \(\theta\) of \(\mathcal{K}\), i.e., the following diagram

\[
\begin{array}{ccc}
(p_{32}^* \mathcal{K} \ast p_{21}^* \mathcal{K}) \ast p_1^* \mathcal{F} & \xrightarrow{\alpha} & (p_{32}^* \mathcal{K} \ast p_1^* \mathcal{F}) \\
\downarrow \phi \ast \text{id} & & \downarrow \text{id} \ast \eta \\
p_{31}^* \mathcal{K} \ast p_1^* \mathcal{F} & & p_{32}^* \mathcal{K} \ast p_2^* \mathcal{F} \\
\downarrow \eta & & \downarrow \eta \\
p_3^* \mathcal{F} & \xrightarrow{\text{id}} & p_3^* \mathcal{F}
\end{array}
\]

is required to be commutative. Here, \(p_{ji} : \text{OLag}^j \times H \to \text{OLag}^i \times H\), \(1 \leq i < j \leq 3\), and \(p_i : \text{OLag}^3 \times H \to \text{OLag} \times H\), \(1 \leq i \leq 3\), denote the projection morphisms on the \(ji\) and \(i\) coordinates, respectively.

**Remark 4.2.1** (Important remark). The category \(\mathcal{C}(V)\) makes sense, i.e., it is nontrivial, only if \(c = 1\) in Theorem 4.1.2. The verification of this assertion, follows, more or less the same argument as in the proof of the idempotent theorem, and hence will be left for the reader.

**Remark 4.2.2.** Let us denote by \(\text{Symp}\) the (groupoid) category whose objects are finite dimensional symplectic vector spaces in the category of algebraic varieties over \(k\), and morphisms are linear isomorphisms of symplectic vector spaces. It is not hard to see that for each morphism \(f : U \to V\) in \(\text{Symp}\), we have an induced pullback functor \(f^* : \mathcal{C}(V) \to \mathcal{C}(U)\). Moreover, for any pair of morphisms \(U \to V \xrightarrow{\phi} W\) we have, in this case, the equality \((g \circ f)^* = f^* \circ g^*\). This means that we obtain the categorical analog of the functor \([L.17]\), i.e., we constructed an explicit quantization lax 2-functor \([9, 15]\)

\[V \mapsto \mathcal{C}(V)\],

associating the category of \(\ell\)-adic sheaves \(\mathcal{C}(V)\) to the object \(V\).
4.2.1. The subcategories of perverse and Weil objects. Let us consider the subcategories $\text{Perv}(\text{OLag} \times H)$ and $\mathcal{D}_W^b(\text{OLag} \times H)$, of $\mathcal{D}^b(\text{OLag} \times H)$, consisting of perverse sheaves, and Weil sheaves, respectively (see Section 2). These categories are preserved by the action of the kernel $K$. For the case of $\text{Perv}(\text{OLag} \times H)$, since $K$ is essentially a Fourier transform kernel, this follows from the Katz–Laumon theorem [14]. Hence, it makes sense to define in $\mathcal{C}(V)$ the subcategories $P(V)$, $\mathcal{C}_w(V)$, and $P_w(V) = P(V) \cap \mathcal{C}_w(V)$, of perverse sheaves, Weil sheaves, and perverse Weil sheaves, respectively, which satisfies the conditions of Subsection 4.2 above. In particular, using the sheaf-to-function procedure (2.1) we can recover the vector space $\mathcal{H}(V)$ from the categories $\mathcal{C}_w(V)$, or $P_w(V)$. This means that we obtained the geometrizations of the vector space $\mathcal{H}(V)$ by these categories.

4.3. Application III: The categorical Weil representation. The canonical vector space $\mathcal{H}(V)$ yields a natural model for the Weil representation of $Sp$. In the same spirit, the canonical category $\mathcal{C}(V)$ yields a model of, what we will call, the categorical Weil representation of the algebraic group $Sp = Sp(V, \omega)$.

We consider the variety $X = \text{OLag} \times H$, the action morphism $a_{Sp} : Sp \times X \to X$, and the pullback functor

$$a^*_{Sp} : \mathcal{D}^b(X) \to \mathcal{D}^b(Sp \times X).$$  \hspace{1cm} (4.1)

For what follows it will be convenient for us to denote the category $\mathcal{C}(V)$ by $\mathcal{C}(X)$. We have the categories $\mathcal{C}(Sp \times X)$ and $\mathcal{C}(Sp \times Sp \times X)$ which are defined exactly in the same way (see Subsection 4.2) as $\mathcal{C}(X)$, i.e., by considering the conditions with respect to the $X$-coordinate. The sheaf $K$ of geometric intertwining kernels (Theorem 2.2.1) is equipped with a natural isomorphism $a_{Sp}^* : a_{Sp}^* \mathcal{K} \to \rho^p \mathcal{K}$ where $a_{Sp} : Sp \times \text{OLag}^g \times H \to \text{OLag}^g \times H$ are the natural action morphism and projection morphism, respectively. Hence, the functor (4.1) induces a functor

$$\rho_{Sp} : \mathcal{C}(X) \to \mathcal{C}(Sp \times X).$$ \hspace{1cm} (4.2)

Note that, in our specific situation, we have an equality

$$(m \times id)^* \circ \rho_{Sp} = (id \times a_{Sp})^* \circ \rho_{Sp},$$

between the two compositions of the following diagram:

$$\begin{array}{ccc}
\mathcal{C}(X) & \xrightarrow{\rho_{Sp}} & \mathcal{C}(Sp \times X) \\
\rho_{Sp} \downarrow & & \downarrow (id \times a_{Sp})^* \\
\mathcal{C}(Sp \times X) & \xrightarrow{(m \times id)^*} & \mathcal{C}(Sp \times Sp \times X)
\end{array}$$

where $m : Sp \times Sp \to Sp$ denotes the multiplication morphism.

We will call the triple $(\rho_{Sp}, Sp, \mathcal{C}(X))$ the canonical model of the categorical Weil representation.

4.3.1. Action on subcategories. The categorical Weil representation $(\rho_{Sp}, Sp, \mathcal{C}(X))$ induces the categorical representations $(\rho_{Sp}, Sp, P(X))$, $(\rho_{Sp}, Sp, \mathcal{C}_w(X))$, and $(\rho_{Sp}, Sp, P_w(X))$ by restricting the functor $\rho_{Sp}$ (4.2) to the various subcategories introduced in Subsection 4.2.1. In particular, using the sheaf-to-function procedure (2.1) we can recover the canonical model $(\rho_{Sp}, Sp, \mathcal{H}(X))$ of the Weil representation (see Subsection 1.6.5) from the categorical representations $(\rho_{Sp}, Sp, \mathcal{C}_w(X))$ and $(\rho_{Sp}, Sp, P_w(X))$. This means that we obtained the geometrizations of the representation $(\rho_{Sp}, Sp, \mathcal{H}(X))$ by these categorical representations.

4.4. Application IV: The geometric Weil representation. In [12] the authors elaborated on [3], and developed the theory of invariant presentation of the Weil representation and its geometric analogue.

Consider the Heisenberg representation $(\pi, H, \mathcal{H})$ (Subsection 1.1), and the associated Weil representation $(\rho, Sp, \mathcal{H})$. We define the kernel of the Weil representation as the function $K(g, h) =$
The function $K$ determines the Weil representation $\rho$ completely, since we have $\pi(K(g, \bullet)) := \sum_{v \in V} K(g, v)$. $\pi(v) = \rho(g)$, and it satisfies the convolution property

$$p_1^K \ast p_2^K = (m \times \text{id})^* K,$$

where $m : Sp \times Sp \to Sp$ denotes the multiplication map, and $p_1, p_2 : Sp \times Sp \times V \to Sp \times V$ denote the projections on $Sp \times V$ via the first and second $Sp$ coordinates, respectively, and the $\ast$ denotes the operation on the function space $C(H, \psi^{-1}) = C(V)$ induced from the convolution operation on functions on the Heisenberg group $H$.

The two main contributions of [12] are:

- An explicit formula for the function $K$ as follows:
  $$K(g, v) = \frac{1}{\dim H} \cdot \text{Tr}(\rho(g) \circ \pi(h^{-1})),$$
  for every $g \in Sp$ such that $g - I$ is invertible, where $n = \dim(V)$, and $\sigma$ denotes the unique nontrivial quadratic character (Legendre character) of the multiplicative group $k^*$, and $\kappa(g) = \frac{\sigma(g)}{\sigma(1)}$ is the Cayley transform.

- The construction of the sheaf theoretic object that geometrizes the function $K$. This is the content of the following theorem (see Section [2] for notations and definitions):

**Theorem 4.4.1** (Geometric Weil representation). There exists a geometrically irreducible $[\dim Sp]$-perverse Weil sheaf $K$ of pure weight zero on $Sp \times V$ satisfying the following properties:

1. **Convolution.** There exists a canonical isomorphism $\theta : p_1^K \ast p_2^K \overset{\sim}{\to} (m \times \text{id})^* K$.
2. **Function.** We have $f^K = K$.
3. **Formula.** For every $g \in Sp$ with $\det(g - I) \neq 0$ we have
   $$K(g, v) = L_\sigma((-1)^n \cdot \det(g - I)) \otimes L_\psi \left(\frac{1}{\sqrt{\text{id}}} \cdot \kappa(g) v, v\right) [2n](n).$$

Here $m : Sp \times Sp \to Sp$ denotes the multiplication morphism, and $p_1, p_2 : Sp \times Sp \times V \to Sp \times V$ denote the projections on $Sp \times V$ via the first and second $Sp$ coordinates, respectively.

The following is similar to the sign problem. Consider the commutative diagram of isomorphisms in $D^b(Sp^3 \times V)$

$$
\begin{array}{ccc}
(p_1^K \ast p_2^K) \ast p_3^K & \xrightarrow{\alpha} & p_1^K \ast (p_2^K \ast p_3^K) \\
\downarrow_{\theta \ast \text{id}} & & \downarrow_{\text{id} \ast \theta} \\
(m_{12} \times \text{id})^* K \ast p_3^K & & p_1^K \ast (m_{23} \times \text{id})^* K \\
\downarrow_{\phi} & & \downarrow_{\theta} \\
m_{123}^* K & \xrightarrow{C} & m_{123}^* K
\end{array}
$$

where $\alpha$ is the associativity constraint [3] for convolution $\ast$ of sheaves on the Heisenberg group $H$, $\phi$ is the isomorphism appearing in the convolution property of the sheaf $K$, and $C$ is by definition the isomorphism that makes the diagram commutative. In addition $m_{ij} : Sp \to Sp$ are the morphisms defined by multiplication of the $i, j$ coordinates, $1 \leq i < j \leq 3$, and all the three coordinates, respectively, and $p_1, p_2 : Sp \times Sp \times V \to Sp \times V$ denote the projections on $Sp \times V$ via the first and second $Sp$ coordinates.

The sheaf $K$ is geometrically irreducible, hence, $C = c \cdot \text{id}$ is a scalar morphism.

**Problem 4.4.2** (The sign problem). Compute the scalar $c$.

A variant of Theorem [4.1.8] with the same proof, can be applied also in this case, yielding:
**Theorem 4.4.3** (The idempotent theorem—particular variant). We have $c = 1$.

**Appendix A. Proof of Theorem 4.1.2**

We have the commutativity

$$\alpha \circ \id \longrightarrow (P \otimes P) \otimes P \overset{\alpha}{\longrightarrow} (P \otimes P) \otimes (P \otimes P) \overset{\alpha}{\longrightarrow} P \otimes (P \otimes (P \otimes P))$$

$$(P \otimes (P \otimes P)) \otimes P \overset{\alpha}{\longrightarrow} P \otimes ((P \otimes P) \otimes P)$$

By successive application of $\theta$, each term in (A.1) is identified with $P$, and using (3.1), (3.2), and the naturality of $\alpha$, the arrows become

$$P \overset{C}{\longrightarrow} P \overset{C}{\longrightarrow} P \overset{C}{\longrightarrow} P$$

which implies that $C^3 = C^2$, i.e., $C = id$ as claimed.

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