GLOBAL REGULARITY FOR THE 3D COMPRESSIBLE MAGNETOHYDRODYNAMICS WITH GENERAL PRESSURE

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ABSTRACT. We address the compressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^3$ and establish a blow-up criterion for the local strong solutions in terms of the density only. Namely, if the density is away from vacuum ($\rho = 0$) and the concentration of mass ($\rho = \infty$), then a local strong solution can be continued globally in time. The results generalise and strengthen the previous ones in the sense that there is no magnetic field present in the criterion and the assumption on the pressure is significantly relaxed. The proof is based on some new a priori estimates for three-dimensional compressible MHD equations.

1. Introduction

In this paper, we are concerned with the compressible magnetohydrodynamics (MHD) in three space dimensions. The fluid motion is described in the following system of partial differential equations (see Cabannes [Cab70] for a more comprehensive discussion on the system):

\begin{align*}
(1.1) & \quad \rho_t + \text{div}(\rho u) = 0, \\
(1.2) & \quad (\rho u^j)_t + \text{div}(\rho u^j u) + P(\rho)_{x_j} + \left(\frac{1}{2}|B|^2\right)_{x_j} - \text{div}(B^j B) = \mu \Delta u^j + \lambda \text{div} u_{x_j}, \\
(1.3) & \quad B^j_t + \text{div}(B^j u - u^j B) = \nu \Delta B^j, \\
(1.4) & \quad \text{div} B = 0.
\end{align*}

Here $\rho$, $u = (u^1, u^2, u^3)$ and $B = (B^1, B^2, B^3)$ are functions of $x \in \mathbb{R}^3$ and $t \geq 0$ representing density, velocity and magnetic field; $P = P(\rho)$ is the pressure; $\mu$, $\lambda$, $\nu$ are viscous constants. The system (1.1)-(1.4) is solved subjected to some given initial data:

\begin{equation}
(\rho, u, B)(x, 0) = (\rho_0, u_0, B_0)(x).
\end{equation}

The well-posedness of the MHD system (1.1)-(1.4) have been studied by many mathematicians in decades (see for example [CT10, HW10, HW08, Kaw84, LSW11, LSX16, LY11, Sar09, SH12, Sue20b] and the references therein), and we now give a brief review on the related results. When the initial data is taken to be close to a constant state in $H^3$, Kawashima [Kaw84] proved the existence of global-in-time $H^3$ solutions to the MHD system. Later, Hoff and Suen [SH12] generalised Kawashima’s results to obtain global smooth solutions when the initial data is taken to be $H^3$ but only close to a constant state in $L^2$. The existence of global weak solutions with large initial data was proved by Hu and Wang [HW10, HW08] and Sart [Sar09], which are extensions of Lions-type weak solutions [Lio98] for
the Navier-Stokes system. With initial $L^2$ data close to a constant state, Hoff and Suen \cite{SH12, Sue12, Sue20b} generalised Hoff-type intermediate weak solutions \cite{Hof95, Hof05, Hof06, LS16, Sue13b, Sue14, CS16, Sue20a} to obtain global solutions to the MHD system.

On the other hand, the global existence of smooth solution to \eqref{1.1} - \eqref{1.4} with arbitrary smooth data is still unknown, hence it is reasonable to consider the possibilities of blowing up of smooth solution. For the corresponding Navier-Stokes system, Xin \cite{Xin98} proved that smooth solution will blow up in finite time in the whole space when the initial density has compact support, while Rozanova \cite{Roz08} showed similar results for rapidly decreasing initial density. Fan-Jiang-Ou \cite{FJO10}, Sun-Wang-Zhang \cite{SWZ11} and Suen \cite{Sue20c} established some blow-up criteria for the classical solutions to 3D compressible flows, which were further extended by Lu-Du-Yao \cite{LDY12} for MHD system. For the isothermal case when $P(\rho) = K\rho$ for some $K > 0$, it was proved in \cite{Sue13a, Sue15} that without vacuum in the initial density, when the density and magnetic field are essential bounded, the smooth solutions to \eqref{1.1} - \eqref{1.4} can be extended globally in time.

The main goal of the present paper is to generalise and strengthen the corresponding results of \cite{Sue13a, Sue15}. The main novelties of this current work can be summarised as follows:

- We introduce some new type of estimates on functionals which are used for decoupling the velocity and magnetic field;
- We obtain a blow-up criterion for \eqref{1.1} - \eqref{1.4} for general pressure term $P(\rho)$ which is not restricted to the isothermal case;
- We assure that the blow-up criterion depends only on density, which is an improvement for the results obtained from \cite{Sue13a, Sue15} in which the magnetic field was present in the criterion;
- We do not impose any extra compatibility condition on the initial data, which is required in the work \cite{SWZ11, HLX11, Sue20c}.

We give a brief description on the analysis applied in this work. To extract the “hidden regularity” from the velocity $u$ and magnetic field $B$, we introduce an important canonical variable associated with the system \eqref{1.1} - \eqref{1.4}, which is known as the effective viscous flux. To see how it works, by the Helmholtz decomposition of the mechanical forces, we can rewrite the momentum equation \eqref{1.2} as follows (summation over $k$ is understood):

\begin{equation}
\rho \dot{u}^j + \left(\frac{1}{2} |B|^2 \right)_{x_j} - \text{div}(B^j B) = \text{F}_{x_j} + \mu \omega^{j,k}_{x_k},
\end{equation}

where $\dot{u}^j = u^j_t + u \cdot \omega^j$ is the material derivative on $u^j$ and the effective viscous flux $F$ is defined by

\begin{equation}
F = (\mu + \lambda) \text{div}(u) - P(\rho) + \rho \dot{\rho}.
\end{equation}

Differentiating \eqref{1.6}, we obtain the following Poisson equation

\begin{equation}
\Delta F = \text{div}(g),
\end{equation}

where $g^j = \rho \dot{u}^j + \left(\frac{1}{2} |B|^2 \right)_{x_j} - \text{div}(B^j B)$. The Poisson equation \eqref{1.8} can be viewed as the analog for compressible MHD of the well-known elliptic equation for pressure in incompressible flow. By exploring the Rankine-Hugoniot condition (see \cite{SH12}), one can deduce that the effective viscous flux $F$ is relatively more regular than $\text{div}(u)$ or $P(\rho)$, and it turns out to be crucial for the overall analysis in the following ways:
The equation (1.6) allows us to decompose the acceleration density \( \dot{\rho}u \) as the sum of the gradient of the scalar \( F \) and the divergence-free vector field \( \omega_{x^k} \) (we ignore those lower-order terms involving \( B \)). The skew-symmetry of \( \omega \) insures that these two vector fields are orthogonal in \( L^2(\mathbb{R}^3) \), so that \( L^2 \)-bounds for the terms on the left side of (1.6) immediately give \( L^2 \) bounds for the gradients of both \( F \) and \( \omega \). These in turn will be used for controlling \( \nabla u \) in \( L^4 \) when \( u(\cdot, t) \notin H^2 \), which are crucial for estimating different functionals in \( u \) and \( B \); see section 3 and the proof of Theorem 3.2.

(ii) One of the key step in obtaining higher order estimates on the solutions is to bound the time integral of \( \|\nabla u(\cdot, t)\|_{L^\infty} \). The key observation is to decompose \( u \) as \( u = u_F + u_P \), where \( u_F, u_P \) satisfy

\[
\begin{align*}
(\mu + \lambda)\Delta u_F^j &= F_{x^j} + (\mu + \lambda)\omega_{x^j}^k \quad \text{(i)} \\
(\mu + \lambda)\Delta u_P^j &= (P - P(\tilde{\rho}))_{x^j} \quad \text{(ii)}
\end{align*}
\]

Using the \textit{a priori} bounds on the effective viscous flux \( F \), the time integral of \( \|\nabla u_F(\cdot, t)\|_{L^\infty} \) can be estimated in terms of \( F \), which can be further estimated in terms of some \textit{a priori} bounds on \( \tilde{u} \) and \( \tilde{B} \). On the other hand, to control \( \int_0^T \|\nabla u_P(\cdot, s)\|_{L^\infty} ds \), by applying the analysis on Newtonian potentials given in [BC94], one can show that if \( \Gamma \) is the fundamental solution for the Laplace operator on \( \mathbb{R}^3 \), then \( u_F^j(\cdot, t) = (\mu + \lambda)^{-1} \Gamma(\rho)_{x^j} \ast (P(\rho(\cdot, t)) - \tilde{P}) \) is \textit{log-Lipschitz} provided that \( P(\rho(\cdot, t)) \in L^\infty \) holds. This is sufficient to guarantee that the integral curve \( x(\cdot, t) \) of \( u = u_F + u_P \) is H"older continuous under the assumption that \( u_F \) has enough regularity as claimed. If we assume that the initial density is H"older continuous, then using the mass equation (1.4), it implies that the density is also H"older continuous for positive time. Hence with such improved regularity on the density, it allows us to obtain the desired bound on \( u_F \); see section 3 and the proof of Theorem 3.1. Such method was also exploited in [Sue20b] for proving global-in-time existence and uniqueness of weak solutions to (1.1)-(1.4).

We now give a precise formulation of our results. First concerning the assumptions on the parameters, we have:

\[
P(\cdot) \text{ is a } C^2 \text{-function in } \rho \text{ with } P'(\rho) > 0 \text{ for all } \rho > 0;
\]

\[
(1.9) \quad \mu, \lambda, \nu > 0 \text{ with } \mu > 4\lambda.
\]

Given \( \tilde{\rho} > 0 \), for the initial data, we assume that

\[
(1.10) \quad \rho_0 - \tilde{\rho}, u_0, B_0 \in H^3(\mathbb{R}^3), \text{ with } \inf(\rho_0) > 0 \text{ and } \operatorname{div}(B_0) = 0.
\]

The following is the main result of this paper:

**Theorem 1.1.** Assume that the system parameters satisfy (1.9)-(1.10). Given \( \tilde{\rho} > 0 \), suppose \( (\rho_0 - \tilde{\rho}, u_0, B_0) \) satisfies (1.11). Assume that \( (\rho - \tilde{\rho}, u, B) \) is the smooth solution local-in-time solution to (1.1)-(1.4) as defined on \( \mathbb{R}^3 \times [0, T] \), and let \( T^* \geq T \) be the maximal existence time of the solution. If \( T^* < \infty \), then we have

\[
\lim_{t \to T^*} (\|\rho\|_{L^\infty((0,t) \times \mathbb{R}^3)} + \|\rho^{-1}\|_{L^\infty((0,t) \times \mathbb{R}^3)}) = +\infty.
\]

The rest of the paper is organised as follows. In section 2, we recall some known facts and inequalities which will be useful for later analysis. In section 3, we begin the proof of Theorem 3.1 with a number of \textit{a priori} bounds for smooth solutions, which are summarised in Theorem 3.2. Finally in section 4, we complete the proof.
of Theorem \([1.1]\) via a contradiction argument by deriving higher order \(H^3\)-bounds for smooth solutions.

2. Preliminaries

In this section, we recall some known facts and useful inequalities for our analysis. We first state a local existence theorem for \([1.1]-(1.4)\) proved by Kawashima \cite{Kaw84} pg. 34–35 and pg. 52–53:

**Theorem 2.1.** Assume that the system parameters satisfy \([1.9]-[1.10]\). Then given \(\hat{\rho} > 0\) and \(\hat{C} > 0\), there is a positive time \(T\) depending on \(\hat{\rho}, \hat{C}\) and the parameters \(\varepsilon, \lambda, \nu, P\) such that if the initial data \((\rho_0, u_0, B_0)\) is given satisfying \((1.11)\) and

\[\rho_0 < \hat{C},\]

then there is a solution \((\rho - \hat{\rho}, u, B)\) to \((1.1)\)-\((1.4)\) defined on \(\mathbb{R}^3 \times [0, T]\) satisfying

\[
\rho - \hat{\rho} \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3))
\]

and

\[
u, B \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^1(\mathbb{R}^3)) \cap L^2([0, T]; H^4(\mathbb{R}^3)).
\]

We make use of the following standard facts (see Ziemer \cite[Theorem 2.1.4, Remark 2.4.3, and Theorem 2.4.4]{Zie89} for example). First, given \(r \in [2, 6]\) there is a constant \(C(r)\) such that for \(w \in H^1(\mathbb{R}^3)\),

\[
\|w\|_{L^r(\mathbb{R}^3)} \leq C(r) \left( \|w\|_{L^2(\mathbb{R}^3)} \|\nabla w\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \right).
\]

For any \(r \in (1, \infty)\) there is a constant \(C(r)\) such that for \(w \in W^{1,r}(\mathbb{R}^3)\),

\[
\|w\|_{L^\infty(\mathbb{R}^3)} \leq C(r) \|w\|_{W^{1,r}(\mathbb{R}^3)}.
\]

3. A priori estimates

In this section we derive a priori estimates for the local solution \((\rho - \hat{\rho}, u, B)\) on \([0, t]\) with \(t \in [0, T^*]\) as described by Theorem \([1.1]\). Here \(T^*\) is the maximal time of existence which is defined in the following sense:

**Definition 3.1.** We call \(T^* \in (0, \infty)\) to be the maximal time of existence of a smooth solution \((\rho - \hat{\rho}, u, B)\) to \((1.1)-(1.4)\) if for any \(t \in [0, T^*]\), \((\rho - \hat{\rho}, u, B)\) solves \((1.1)-(1.4)\) in \([0, t] \times \mathbb{R}^3\) and satisfies \((2.1)-(2.2)\); moreover, the conditions \((2.1)-(2.2)\) fail to hold when \(t = T^*\).

We will prove Theorem \([1.1]\) using a contradiction argument. Therefore, for the sake of contradiction, we assume that

\[
\|\rho\|_{L^\infty((0, T^*) \times \mathbb{R}^3)} + \|\rho^{-1}\|_{L^\infty((0, T^*) \times \mathbb{R}^3)} \leq M_0.
\]

To facilitate our exposition, we first define some auxiliary functionals for \(t \in [0, T^*)\):

\[
\Phi_1(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|u|^2 + |B|^2) + \int_0^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |B_t|^2),
\]

\[
\Phi_2(t) = \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |B_t|^2) + \int_0^t \int_{\mathbb{R}^3} (|\nabla \dot{u}|^2 + |\nabla B_t|^2),
\]

where \(u, B\) satisfy \((1.1)-(1.4)\) with \(\rho = \rho_0 + \hat{\rho}\) and \(\rho_0 \in C\).
\[
\Phi_2(t) = \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^4 + |\nabla B|^4), \quad \Phi_3(t) = \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^3 + |\nabla B|^3).
\]

\[
\Phi_4(t) = \sup_{0 \leq s \leq t} \left( \int_{\mathbb{R}^3} (|\nabla B|^2|B|^2 + |\nabla u|^2|B|^2 + |\nabla B|^2|u|^2) \right).
\]

The following is the main theorem of this section:

**Theorem 3.2.** Assume that the hypotheses and notations in Theorem [1.1] are in force. Given \( M_0 > 0 \) and \( \tilde{\rho} > 0 \), assume further that \((\rho - \tilde{\rho}, u, B)\) satisfies [3.1]. Then for each \( t \in [0, T^*]\), there exists a positive constant \( C \) which depends on \( M_0, t, \tilde{\rho} \) and the system parameters \( P, \mu, \lambda, \nu \) such that

\[
(3.2) \quad \Phi_1(t) + \Phi_2(t) \leq C.
\]

We prove Theorem 3.2 in a sequence of lemmas. Throughout this section, for \( t \in [0, T^*] \), \( C \) always denotes a generic constant which depends only on \( \mu, \lambda, a, \nu, \tilde{\rho}, M_0, t \) and the initial data. For simplicity, we drop the symbols \( dx, ds \) or \( dx ds \) from the integrals.

We first give the following estimates on the effective vicious flux \( F \) which is defined in (1.7).

**Lemma 3.3.** Assume that \( \rho \) satisfies [3.1]. For each \( p > 1 \), there is a constant \( C > 0 \) such that for all \( t > 0 \), we have

\[
(3.3) \quad \|F(\cdot, t)\|_{L^p} \leq \left[ \|\nabla u(\cdot, t)\|_{L^p} + \|\rho - \tilde{\rho}(\cdot, t)\|_{L^p} \right],
\]

and

\[
(3.4) \quad \|\nabla F(\cdot, t)\|_{L^p} \leq \left[ \|\nabla u(\cdot, t)\|_{L^p} + \|B\nabla B(\cdot, t)\|_{L^p} \right].
\]

**Proof.** The assertion (3.3) follows immediately from the definition of \( F \), and the proof of (3.4) relies on the Poisson equation (1.8) and the Marcinkiewicz multiplier theorem (refer to Stein [Ste70], pg. 96). \( \square \)

Using the estimates (3.3)-(3.4) on \( F \), we have the following estimates on \( \nabla u \) and \( \nabla \omega \):

**Lemma 3.4.** Assume that \( \rho \) satisfies [3.1]. For each \( p > 1 \), there is a constant \( C > 0 \) depends on \( p \) such that for all \( t > 0 \), we have

\[
(3.5) \quad \|\nabla u(\cdot, t)\|_{L^p} \leq C \left[ \|F(\cdot, t)\|_{L^p} + \|\omega(\cdot, t)\|_{L^p} + \|(P - \tilde{P})(\cdot, t)\|_{L^p} \right],
\]

\[
(3.6) \quad \|\nabla \omega(\cdot, t)\|_{L^p} \leq C \left[ \|\nabla u(\cdot, t)\|_{L^p} + \|B\nabla B(\cdot, t)\|_{L^p} \right].
\]

**Proof.** By the definition (1.7) of \( F \),

\[
(\mu + \lambda)\Delta u^j = F_{x_j} + (\mu + \lambda)\omega_{x_j}^j + (P - \tilde{P})_{x_j}.
\]

Hence by differentiating and taking the Fourier transform on the above equation, we can apply Marcinkiewicz multiplier theorem and (3.3) follows.

For the case of \( \nabla \omega \), by direct computation, we have

\[
\mu \Delta \omega = (\rho \dot{u}^j)_{x_k} - (\rho \dot{u}^k)_{x_j} - (\nabla B^j \cdot B)_{x_k} + (\nabla B^k \cdot B)_{x_j},
\]

and using the same argument as for \( \nabla u \), (3.6) follows immediately. \( \square \)
Lemma 3.5. Assume that $\rho$ satisfies (3.11). For $t \in [0, T^*)$, we have
\[
(3.7) \quad \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \left( |\rho - \bar{\rho}|^2 + \rho |u|^2 + |B|^2 \right) + \int_0^t \int_{\mathbb{R}^3} \left( |\nabla u|^2 + |\nabla B|^2 \right) \leq C.
\]
\[
(3.8) \quad \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} \left( |u|^6 + |B|^6 \right) \leq C.
\]
Proof. The bound (3.7) follows from the standard energy balance equation and we refer to [Hof95] or [Sue20b] for related discussions. □

Next we derive the following $L^6$ bounds for $u$ and $B$. Such bounds are crucial for obtaining higher order estimates on $u$ and $B$.

Lemma 3.6. Assume that the hypotheses and notations of Theorem 3.2 are in force. Then for any $0 \leq t < T^*$,
\[
(3.9) \quad \Phi_1(t) + \Phi_4(t) \leq C.
\]
Proof. We follow the computations given in [SH12] and obtain, for $t \in [0, T^*)$,
\[
\int_{\mathbb{R}^3} \left( |u|^6 + |B|^6 \right)_{|s=0}^t + \int_0^t \int_{\mathbb{R}^3} \left( \mu |u|^2 |\nabla u|^2 + \nu |B|^2 |\nabla B|^2 \right)
+ (\rho - \bar{\rho}) \int_0^t \int_{\mathbb{R}^3} |u|^2 |\nabla(|u|^2)|^2 + 6 \int_0^t \int_{\mathbb{R}^3} |B|^2 |\nabla B|^2 |B|^2
\leq C \left[ \int_0^t \int_{\mathbb{R}^3} \rho - \bar{\rho} |\nabla(u)| u + \int_0^t \int_{\mathbb{R}^3} |u|^4 |u| |\nabla |B|^2| \right]
+ C \left[ \int_0^t \int_{\mathbb{R}^3} |u|^2 |u| |\nabla |B|^2| + \int_0^t \int_{\mathbb{R}^3} |B|^4 |B| \cdot |\nabla (Bu^T - uB^T)| \right].
\]
By the assumption (1.10), the term involving $(-24 \lambda + 6 \mu)$ is positive. The rest of the analysis follows by a Grönwall-type argument and we omit the details here. □

With the help of (3.7) and (3.8), we can start estimating the functionals $\Phi_1$ and $\Phi_2$ appeared in (3.2). We first consider $\Phi_1$ and $\Phi_4$:

Lemma 3.7. Assume that the hypotheses and notations of Theorem 3.2 are in force. Then for any $0 \leq t < T^*$,
\[
(3.9) \quad \Phi_1(t) + \Phi_4(t) \leq C.
\]
Proof. We multiply (1.2) by $\hat{u}_j$, sum over $j$ and integrate to get
\[
\int_{\mathbb{R}^3} |\nabla u|^2 + \int_0^t \int_{\mathbb{R}^3} \rho |\hat{u}|^2 \leq C + \int_0^t \int_{\mathbb{R}^3} \left| \hat{u} \cdot \nabla \left( \frac{1}{2} |B|^2 \right) - \hat{u} \cdot \nabla (BB^T) \right| + \int_0^t \int_{\mathbb{R}^3} |\nabla u|^3.
\]
Next we multiply (1.3) by $B_t$ and integrate,
\[
\int_{\mathbb{R}^3} |\nabla B|^2 + \int_0^t \int_{\mathbb{R}^3} |B_t|^2 \leq C + \int_0^t \int_{\mathbb{R}^3} B_t \cdot \nabla (uB^T - uB^T) \right|.
\]
Adding (3.10) and (3.11), we obtain
\[
\int_0^t \left( |\nabla u|^2 + |\nabla B|^2 \right) + \int_0^t \int_{\mathbb{R}^3} (\rho |\dot{u}|^2 + |B_t|^2)
\leq C + C\Phi_4 + C \int_0^t \int_{\mathbb{R}^3} (|\nabla B|^2 |B|^2 + |\nabla u|^2 |B|^2 + |\nabla B|^2 |u|^2).
\]

(3.12)

To bound the last inequality on the right side of (3.12), we bound the term \( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \) as an example. Using (2.23) and the bound (3.8), we arrive at
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \leq \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^3 \right)^{\frac{2}{3}} \left( \int_0^t \int_{\mathbb{R}^3} |B|^6 \right)^{\frac{1}{3}}
\leq C \left( \int_0^t \left( \int_{\mathbb{R}^3} |\Delta B|^2 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}} \right)
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{2}{7}} \left( \int_0^t \int_{\mathbb{R}^3} |B_t|^2 \right)^{\frac{1}{7}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
+ C \left( \int_0^t \left( \int_{\mathbb{R}^3} |\nabla B|^3 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |u|^6 \right)^{\frac{1}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}} \right)
+ C \left( \int_0^t \left( \int_{\mathbb{R}^3} |\nabla u|^3 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |B|^6 \right)^{\frac{1}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}} \right)
\leq C\Phi_1^{\frac{2}{7}} + C\Phi_1^{\frac{1}{7}}\Phi_4^{\frac{1}{7}}.
\]

The estimates on \( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 |B|^2 \) and \( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 \) are just similar, and we deduce that
\[
(3.13) \quad \Phi_1 \leq C\Phi_1^{\frac{2}{7}} + C\Phi_1^{\frac{1}{7}}\Phi_4^{\frac{1}{7}}.
\]

It remains to estimate the functional \( \Phi_4 \). Using (2.23), we can estimate the integral of \( |\nabla B|^3 \) as follows.
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^3 \leq C \int_0^t \left( \int_{\mathbb{R}^3} |\Delta B|^2 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
\leq C \int_0^t \left( \int_{\mathbb{R}^3} |B_t|^2 + |\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{2}{7}} \left( \int_0^t \int_{\mathbb{R}^3} |B_t|^2 \right)^{\frac{1}{7}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
+ C \int_0^t \left( \int_{\mathbb{R}^3} |\nabla B|^3 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |u|^6 \right)^{\frac{1}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
+ C \int_0^t \left( \int_{\mathbb{R}^3} |\nabla u|^3 \right)^{\frac{2}{7}} \left( \int_{\mathbb{R}^3} |B|^6 \right)^{\frac{1}{7}} \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{5}{7}}
\leq C\Phi_1^{\frac{2}{7}} + C\Phi_1^{\frac{1}{7}}\Phi_4^{\frac{1}{7}}.
\]
On the other hand, using (2.3), (3.5), (3.6) and (3.7), the integral of $|\nabla u|^3$ can be estimated as follows.

$$
\int_0^t \int_{\mathbb{R}^3} |\nabla u|^3 \leq C \int_0^t \int_{\mathbb{R}^3} (|F|^3 + |\omega|^3 + |P - \bar{P}|^3)
$$

$$
\leq C \int_0^t \left( \int_{\mathbb{R}^3} |F|^2 \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla F|^2 \right)^{\frac{3}{4}} + C \int_0^t \left( \int_{\mathbb{R}^3} |\omega|^2 \right)^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} |\nabla \omega|^2 \right)^{\frac{3}{4}} + C
$$

$$
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} + C \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B|^3 \right)^{\frac{1}{2}} + C,
$$

$$
\leq C \Phi_1^2 + C \Phi_2^2 + C \Phi_3^2 + C \Phi_4^2 + C.
$$

Therefore, we obtain the following bound on $\Phi_4$ in terms of $\Phi_1$:

$$
\Phi_4 \leq C \Phi_1^2 + C \Phi_3^2 + C \Phi_4^2 + C.
$$

Combining (3.13) with (3.14), the result (3.9) follows. \qed

Next, we derive the following estimates for $\Phi_2$ in terms of $\Phi_3$ and $\Phi_5$:

**Lemma 3.8.** Assume that the hypotheses and notations of Theorem 3.2 are in force. Then for any $0 \leq t < T^*$,

$$
\Phi_2(t) + \Phi_3(t) + \Phi_5(t) \leq C.
$$

**Proof.** Taking the convective derivative in the momentum equation (1.2), multiplying it by $\dot{u}^3$, summing over $j$ and integrating, we obtain

$$
\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\dot{u}|^2 + \int_0^t \int_{\mathbb{R}^3} |\nabla \dot{u}|^2 \leq C + C \Phi_3 + C \int_0^t \int_{\mathbb{R}^3} |B|^2(|B|^2 + |u|^2|\nabla B|^2).
$$

Next we differentiate the magnetic field equation (1.3) with respect to $t$, multiply by $B_t$ and integrate,

$$
\frac{1}{2} \int_{\mathbb{R}^3} |B_t|^2 \bigg|_0^t + \nu \int_0^t \int_{\mathbb{R}^3} |\nabla B_t|^2
$$

$$
= - \int_0^t \int_{\mathbb{R}^3} B_t \cdot [\text{div}(Bu^T - uB^T)]_t.
$$

Adding the above to (3.16) and absorbing terms,

$$
\Phi_2 \leq C \left[ \Phi_3 + \int_0^t \int_{\mathbb{R}^3} |B|^2 |u|^2 (|\nabla u|^2 + |\nabla B|^2) + 1 \right]
$$

$$
+ C \int_0^t \int_{\mathbb{R}^3} (|B|^2 |B_t|^2 + |B|^2 |\dot{u}|^2 + |B_t|^2 |u|^2).
$$

(3.17)
We first consider the last integral on the right side of (3.17). To bound the term \( \int_0^t \int_{\mathbb{R}^3} |B|^2 |B_t|^2 \), using the bound (3.9), we have

\[
\int_0^t \int_{\mathbb{R}^3} |B|^2 |B_t|^2 \\
\leq C \left( \left( \int_0^t \int_{\mathbb{R}^3} |B|^6 \right)^{\frac{1}{3}} \left( \int_0^t \int_{\mathbb{R}^3} |B_t|^3 \right)^{\frac{2}{3}} \right)^{\frac{3}{2}} \\
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |B_t|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |B|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} |\nabla B_t|^2 \right)^{\frac{1}{2}} \\
\leq C \Phi_1 \Phi_2^{\frac{3}{2}} \leq C \Phi_2^{\frac{5}{6}},
\]

and the terms \( \int_0^t \int_{\mathbb{R}^3} |u|^2 |B_t|^2 \) and \( \int_0^t \int_{\mathbb{R}^3} |B|^2 |\dot{u}|^2 \) can be treated in a similar way.

To bound the third integral on the right side of (3.17), we have

\[
\int_0^t \int_{\mathbb{R}^3} |B|^2 |u|^2 (|\nabla u|^2 + |\nabla B|^2) \\
\leq \int_0^t \int_{\mathbb{R}^3} (|\nabla u|^4 + |\nabla B|^4) + \int_0^t \int_{\mathbb{R}^3} (|B|^8 + |u|^8) \\
\leq \Phi_3 + \int_0^t \int_{\mathbb{R}^3} (|B|^8 + |u|^8).
\]

Using the bounds (2.3) and (2.4), the term \( \int_0^t \int_{\mathbb{R}^3} |u|^8 \) can be estimated as follows.

\[
\int_0^t \int_{\mathbb{R}^3} |u|^8 \\
\leq (\sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u|^4) \left( \int_0^t \|u\|_{L^\infty} \right) \\
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u|^2 \right)^{\frac{1}{2}} \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |u|^6 \right)^{\frac{1}{2}} \left( \int_0^t \int_{\mathbb{R}^3} (|u|^4 + |\nabla u|^4) \right) \\
\leq C \left[ \int_0^t \left( \int_{\mathbb{R}^3} |u|^2 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^{\frac{1}{2}} + \Phi_3 \right] \\
\leq C (\Phi_3 + 1).
\]

The term \( \int_0^t \int_{\mathbb{R}^3} |B|^8 \) can be estimated in a similar way to get

\[
\int_0^t \int_{\mathbb{R}^3} |B|^2 |u|^2 (|\nabla u|^2 + |\nabla B|^2) \leq C (\Phi_3 + 1),
\]

and we obtain from (3.17) that

\[
(3.18) \quad \Phi_2 \leq C (\Phi_3 + 1) + C \Phi_2^{\frac{5}{6}}.
\]

It remains to estimate the functionals \( \Phi_3 \) and \( \Phi_5 \). Using (3.5) and (3.7), we have

\[
(3.19) \quad \int_0^t \int_{\mathbb{R}^3} |\nabla u|^4 \leq C \int_0^t \int_{\mathbb{R}^3} (|F|^4 + |\omega|^4) + C.
\]
Using (3.9), the integral on $|F|^4$ can be bounded by
\[
\int_0^t \int_{\mathbb{R}^3} |F|^4 \leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |F|^2 \right)^2 \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla F|^2 \right)^2 \left( \int_0^t \int_{\mathbb{R}^3} |\nabla F|^2 \right)^2 \\
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla B|^2 |B|^2)^{1/2} \left( \int_0^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla B|^2 |B|^2) \right)^{1/2} \\
\leq C (\Phi_2 + \Phi_5)^{1/2},
\]
and the estimates on $\omega$ is just similar. For $\int_0^t \int_{\mathbb{R}^3} |\nabla B|^4$, we estimate it as follows.
\[
\int_0^t \int_{\mathbb{R}^3} |\nabla B|^4 \leq C \int_0^t \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^2 \left( \int_{\mathbb{R}^3} |\Delta B|^2 \right)^2 \\
\leq C \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\Delta B|^2 \right)^2 \left( \sup_{0 \leq s \leq t} \int_{\mathbb{R}^3} |\nabla B|^2 \right)^2 \left( \int_0^t \int_{\mathbb{R}^3} |\Delta B|^2 \right)^2 \\
\leq C (\Phi_2 + \Phi_5)^{1/2}.
\]
Hence we have
\[
(3.20) \quad \Phi_3 \leq C (\Phi_2 + \Phi_5)^{1/2} + C.
\]
To bound $\Phi_5$, for $r \in (3, 6)$, we apply (2.4) to obtain
\[
\int_{\mathbb{R}^3} |\nabla u|^2 |B|^2 \\
\leq C \|B(:, t)\|_{L^r}^2 \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right) \\
\leq C \left( \int_{\mathbb{R}^3} |B|^r + \int_{\mathbb{R}^3} |\nabla B|^r \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right).
\]
The term $\|B\|_{L^r}$ can be bounded by the $L^2 - L^6$ interpolation on $B$. Hence using (2.3) and the bound (3.9), we obtain
\[
\int_{\mathbb{R}^3} |\nabla u|^2 |B|^2 \leq C + C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right) \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{\frac{3r-6}{2r}} \left( \int_{\mathbb{R}^3} |\Delta B|^2 \right)^{\frac{3r-6}{2r}} \\
\leq C + C \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right) \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right)^{1-s'} \left( \int_{\mathbb{R}^3} |\Delta B|^2 \right)^{s'} \\
\leq C + C \left( \int_{\mathbb{R}^3} |B|^2 + \int_{\mathbb{R}^3} (|\nabla B|^2 |u|^2 + |\nabla u|^2 |B|^2) \right)^{s'} \\
\leq C + C \left( \Phi_2 + \Phi_5 \right)^{s'},
\]
where $s' := \frac{3r-6}{2r}$ and $s' \in (0, 1)$. By similar method, we obtain
\[
\int_{\mathbb{R}^3} |\nabla B|^2 |B|^2 \leq \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right) \|B(:, t)\|_{L^r}^2 \\
\leq C + C \left( \Phi_2 + \Phi_5 \right)^{s'}.
\]
For the term $\int_{\mathbb{R}^3} |\nabla B|^2 |u|^2$, we have
\[
\int_{\mathbb{R}^3} |\nabla B|^2 |u|^2 \leq \left( \int_{\mathbb{R}^3} |\nabla B|^2 \right) \left( \int_{\mathbb{R}^3} |u|^r + \int_{\mathbb{R}^3} |\nabla u|^r \right) \frac{2}{r}.
\]

Similar to the case for $B$, the term $\|u\|_{L^r}$ can be bounded by the $L^2-L^6$ interpolation on $u$. For the term $\|\nabla u\|_{L^r}$, we can apply (3.5) with the bounds (3.3)-(3.4) to get
\[
\left( \int_{\mathbb{R}^3} |\nabla u|^r \right) \frac{2}{s'} \leq C \left( \int_{\mathbb{R}^3} (|F|^r + |\omega|^r + |\rho - \tilde{\rho}|^r) \right) \frac{s'}{s'}
\]
\[
+ C \left( \int_{\mathbb{R}^3} |\omega|^2 \right) \frac{1-s'}{s'} \left( \int_{\mathbb{R}^3} |\nabla \omega|^2 \right) \frac{s'}{s'} + C \left( \int_{\mathbb{R}^3} |\rho - \tilde{\rho}|^r \right) \frac{2}{r'}.
\]

Therefore, we obtain the following bound on $\Phi_5$:
\[
\Phi_5 \leq C \left( \Phi_2 + \Phi_5 \right) \frac{s'}{s'}.
\]

We combine (3.18), (3.20) and (3.21) to conclude that
\[
\Phi_2 \leq C,
\]
which can be further applied on (3.20) and (3.21) to give
\[
\Phi_3 + \Phi_5 \leq C.
\]

Hence the result (3.15) follows.

**Proof of Theorem 3.2.** Using the results obtained from Lemma 3.7 and Lemma 3.8, the bound (3.2) follows immediately from the estimates (3.9) and (3.15). □

4. Higher Order Estimates and proof of Theorem 1.1

In this section we continue to obtain higher order estimates on the smooth local solution $(\rho - \tilde{\rho}, u, B)$ as described in section 3. Together with Theorem 3.2, we show that, under the assumption (3.1), the smooth local solution to (1.1)-(1.4) can be extended beyond the maximal time of existence $T^*$ as defined in the previous section, thereby contradicting the maximality of $T^*$. The following is the main theorem of this section:

**Theorem 4.1.** Assume that the hypotheses and notations in Theorem 3.2 are in force. Given $M_0 > 0$ and $\tilde{\rho} > 0$, assume further that $(\rho - \tilde{\rho}, u, B)$ satisfies (3.1). If $C > 0$ is the constant as obtained in Theorem 3.2, then for each $t \in [0, T^*)$, there exists a positive number $M$ which depends on $C$, $M_0$, $t$, $\tilde{\rho}$ and the system parameters $P$, $\mu$, $\lambda$, $\nu$ such that

\[
\sup_{0 \leq s \leq t} \|(\rho - \tilde{\rho}, u, B)\|_{H^3(\mathbb{R}^3)} + \int_0^t \|(u, B)(\cdot, s)\|_{H^4(\mathbb{R}^3)}^2 \leq M.
\]

(4.1)
We give the proof of Theorem 4.1 in a sequence of steps. Most of the details are reminiscent of [SH12] and [Sue20a], hence we omit some of those which are identical to arguments given in [SH12] or [Sue20a]. Throughout this section, $M$ denotes a generic constant which depends on $M_0$, $t$, $	ilde{\rho}$, $P$, $\mu$, $\lambda$, $\nu$, and it may be changed from line to line.

We first begin with the following estimates on the time integral of the velocity gradient $\nabla u$:

**Step 1:** The velocity gradient satisfies the following bound

$$\int_0^t ||\nabla u(\cdot, s)||_{L^\infty} \leq M. \tag{4.2}$$

**Proof of Step 1.** The proof is similar to the one given in [Sue20a], and we only sketch here. The key is to decompose $u$ as $u = u_F + u_P$, where $u_F, u_P$ satisfy

$$\begin{cases} 
(\mu + \lambda)\Delta(u_F)^j = F_{x_j} + (\mu + \lambda)(\omega)^{j,k}_x \\
(\mu + \lambda)\Delta(u_P)^j = (P - P(\tilde{\rho}))_{x_j}.
\end{cases} \tag{4.3}$$

In view of the decomposition (4.3), it suffices to bound the time integral of $||\nabla u_F||_{L^\infty}$ and $||\nabla u_P||_{L^\infty}$. Using (2.4), for $r > 3$, we have

$$\int_0^t ||\nabla u_F(\cdot, s)||_{L^r} \leq C(r) \int_0^t [||\nabla u_F(\cdot, s)||_{L^r} + ||D^2_s u_F(\cdot, s)||_{L^r}] .$$

and with the help of the bound (3.2), the right side of the above can be bounded by $M$. On the other hand, to bound the time integral of $\nabla u_P$, by the pointwise bound (4.1) on $\rho$, one can show that $u_P(\cdot, t)$ is, in fact, log-Lipschitz with bounded log-Lipschitz seminorm. This is crucial for proving that, the integral curve $x(y, t)$ as defined by

$$\begin{cases} 
\dot{x}(t) = u(x(t), t) \\
x(0) = y,
\end{cases}$$

is Hölder-continuous in $y$. Upon integrating the mass equation along integral curves $x(t, y)$ and $x(t, z)$, subtracting and recalling the definition (1.7) of $F$, we obtain that

$$\begin{align*}
|\log \rho(x(t, y), t) - \log \rho(x(t, z), t)|
\leq |\log \rho_0(y) - \log \rho_0(z)| + \int_0^t |P(\rho_0(x(s, y), s)) - P(\rho(x(s, z), s))| \\
+ \int_0^t |F(x(s, y), s) - F(x(s, z), s)|.
\end{align*} \tag{4.4}$$

Since $P$ is increasing, the second term of the above is dissipative and can be dropped out. Moreover, with the help of the estimate (3.4) on $F$ and the Hölder-continuity of $x(y, t)$, the third term can be bounded by $M$. Hence we can conclude from (4.4) that $\rho(\cdot, t)$ is $C^{\beta(t)}$ for some $\beta(t) > 0$ with bounded modulus. Finally, with the improved regularity on $\rho(\cdot, t)$, we can now make use of (4.3) again and apply properties of Newtonian potentials to conclude that the $C^{1+\beta(t)}(\mathbb{R}^3)$ norm of $u_P$ remains finite in finite time, thereby giving the required bound on $\int_0^t ||\nabla u_P||_{L^\infty}$. \hfill $\square$

**Step 2:** We further obtain

$$||D^2_s u(\cdot, t)||_{L^2} \leq M ||\rho u(\cdot, t)||_{L^2} + ||\nabla B \cdot B(\cdot, t)||_{L^2} + ||\nabla P(\cdot, t)||_{L^2}, \tag{4.5}$$
\[ \|D_x^2 u(\cdot, t)\|_{L^2} \leq M \left[ \|\nabla \rho \cdot \dot{u}(\cdot, t)\|_{L^2} + \|\rho \nabla \dot{u}(\cdot, t)\|_{L^2} + \|B \cdot D_x^2 B(\cdot, t)\|_{L^2} \right] \\
\quad + M \left[ \|\nabla B^2(\cdot, t)\|_{L^2} + \|D_x^2 P(\cdot, t)\|_{L^2} \right]. \tag{4.6} \]

**Proof of Step 2.** These follow immediately from the momentum equation (1.2) and the ellipticity of the Lamé operator \(\varepsilon \Delta + (\varepsilon + \lambda)\nabla \text{div} \); see [SWZ11] for related discussion. \(\square\)

**Step 3:** The following \(H^2\) bound for density holds
\[ \sup_{0 \leq s \leq t} \|\rho(\cdot, s)\|_{H^2} \leq M. \tag{4.7} \]

**Proof of Step 3.** We take the spatial gradient of the mass equation (1.1), multiply by \(\nabla \rho\) and integrate by parts to obtain
\[ \frac{\partial}{\partial t} \int_{\mathbb{R}^3} |\nabla \rho|^2 \leq M \left[ \int_{\mathbb{R}^3} |\nabla \rho|^2 + \int_{\mathbb{R}^3} |D_x^2 u|^2 \right]. \tag{4.8} \]

Thanks to (4.6), we have
\[ \int_0^t \int_{\mathbb{R}^3} |D_x^2 u|^2 \leq M \int_0^t \int_{\mathbb{R}^3} (|\dot{u}|^2 + |\nabla B \cdot B|^2 + |\nabla \rho|^2) \]
\[ \leq M + \int_0^t \int_{\mathbb{R}^3} |\nabla \rho|^2, \]

hence by applying the above to (4.8) and using the bound (4.2) on the time integral of \(\|\nabla u\|_{L^\infty}\), we conclude that
\[ \sup_{0 \leq s \leq t} \|\nabla \rho(\cdot, s)\|_{L^2} \leq M. \]

By repeating the argument, one can prove that \(\sup_{0 \leq s \leq t} \|D_x^2 \rho(\cdot, s)\|_{L^2} \leq M\) and (4.7) follows. \(\square\)

**Step 4:** The velocity and magnetic field satisfy
\[ \sup_{0 \leq s \leq t} \left( \|u(\cdot, s)\|_{H^2} + \|B(\cdot, s)\|_{H^3} \right) \leq M. \tag{4.9} \]

**Proof of Step 4.** Define the forward difference of quotient \(D_h^t\) by
\[ D_h^t(f)(t) = (f(t + h) - f(t))h^{-1} \]
and let \(E^j = D_h^t(u^j) + u \cdot \nabla u^j\). By applying \(E^j\) on the momentum equation (1.2) and differentiating, it gives
\[ \int_{\mathbb{R}^3} \rho |E^j|^2 + \int_0^t \int_{\mathbb{R}^3} (|\nabla E^j|^2 + |D_h^t(\text{div}(u^j)) + u \cdot \nabla (\text{div}(u^j))|^2) \]
\[ \leq M + \int_0^t \int_{\mathbb{R}^3} |\nabla E|^2 + O(h), \]

where \(O(h) \to 0\) as \(h \to 0\). Therefore by choosing \(h \to 0\), we conclude
\[ \sup_{0 \leq s \leq t} \|\nabla \dot{u}(\cdot, s)\|_{L^2} + \int_0^t \int_{\mathbb{R}^3} |D_h^t \dot{u}|^2 \leq M, \]

and the bound for \(\nabla B_t\) can be derived in a similar way. \(\square\)
Step 5: Finally we have the following bound

\begin{equation}
\int_0^t \int_{\mathbb{R}^3} (|D_x^4 u|^2 + |D_x^4 B|^2) \leq M \left[ 1 + \int_0^t \int_{\mathbb{R}^3} |D_x^2 \rho|^2 \right],
\end{equation}

(4.10)

\begin{equation}
\sup_{0 \leq s \leq t} \left( |D_x^3 \rho(\cdot, s)||_{L^2} + |D_x^3 B(\cdot, s)||_{L^2} \right) + \int_0^t \int_{\mathbb{R}^3} |D_x^4 u|^2 \leq M.
\end{equation}

(4.11)

Proof of Step 5. To prove (4.10), we differentiate (1.2) and (1.3) twice with respect to space, express the fourth derivatives of \( u \) and \( B \) in the terms second derivatives of \( \dot{u}, B_t, \nabla \rho \) and lower order terms, and apply the bounds in (3.1) and (4.9).

On the other hand, to prove (4.11), we apply two space derivatives and one spatial difference operator \( D^h_x \) defined by

\[ D^h_x(f)(t) = (f(x + he_j) - f(x))h^{-1} \]

such that

\[ \int_{\mathbb{R}^3} |D^h_x D_x D_x \rho|^2 \leq M + \int_0^t \int_{\mathbb{R}^3} (|D_x^4 u|^2 + |D^h_x D_x D_x \rho|^2) \]

\[ \leq M + \int_0^t \int_{\mathbb{R}^3} |D_x^2 \rho|^2. \]

Taking \( h \to 0 \) and applying Gronwall’s inequality, we obtain the required bound for the term \( ||D_x^3 \rho(\cdot, s)||_{L^2} \). \( \square \)

Proof of Theorem 1.1 Using Theorem 4.1, we can apply an open-closed argument on the time interval which is identical to the one given in Hoff and Suen [S H12] pp. 31 to extend the local solution \( (\rho - \bar{\rho}, u, B) \) beyond \( T^* \), which contradicts the maximality of \( T^* \). Therefore the assumption (5.1) does not hold and this completes the proof of Theorem 1.1. \( \square \)

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