Discrete Modified Projection Method for Urysohn Integral Equations with Smooth Kernels

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Abstract

Approximate solutions of linear and nonlinear integral equations using methods related to an interpolatory projection involve many integrals which need to be evaluated using a numerical quadrature formula. In this paper, we consider discrete versions of the modified projection method and of the iterated modified projection method for solution of a Urysohn integral equation with a smooth kernel. For $r \geq 1$, a space of piecewise polynomials of degree $\leq r - 1$ with respect to an uniform partition is chosen to be the approximating space and the projection is chosen to be the interpolatory projection at $r$ Gauss points. The orders of convergence which we obtain for these discrete versions indicate the choice of numerical quadrature which preserves the orders of convergence. Numerical results are given for a specific example.

Key Words : Urysohn integral operator, Interpolatory projection, Gauss points, Nyström Approximation

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1 Introduction

Let $X = L^\infty[a, b]$, and consider a Urysohn integral operator

$$\mathcal{K}(x)(s) = \int_a^b \kappa(s, t, x(t)) dt, \quad s \in [a, b], \ x \in X,$$

(1.1)

where $\kappa(s, t, u)$ is a continuous real valued function defined on

$$\Omega = [a, b] \times [a, b] \times \mathbb{R}.$$

Then $\mathcal{K}$ is a compact operator from $L^\infty[a, b]$ to $C[a, b]$. Assume that for $f \in C[a, b]$,

$$x - \mathcal{K}(x) = f$$

(1.2)

has a unique solution $\varphi$.

We are interested in approximate solutions of the above equation. We consider projection methods associated with a sequence of interpolatory projections converging to the Identity operator pointwise.

For $r \geq 1$, let $X_n$ denote the space of piecewise polynomials of degree $\leq r - 1$ with respect to a uniform partition of $[a, b]$ with $n$ subintervals. Let $h = \frac{b - a}{n}$ denote the length of each subinterval of the above partition. Let $Q_n : C[a, b] \to X_n$ denote the interpolation operator at $r$ Gauss points. Then in the collocation method, (1.2) is approximated by

$$\varphi_n^C - Q_n \mathcal{K}(\varphi_n^C) = Q_n f.$$  (1.3)

The iterated collocation solution is defined as

$$\varphi_n^S = \mathcal{K}(\varphi_n^C) + f.$$  (1.4)

In Grammont et al [11] the following modified projection method is investigated:

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f,$$

(1.5)

where

$$\mathcal{K}_n^M(x) = Q_n \mathcal{K}(x) + \mathcal{K}(Q_n x) - Q_n \mathcal{K}(Q_n x).$$  (1.6)
It is a generalization of the modified projection method in the linear case, which was proposed in Kulkarni [13]. The iterated modified projection solution is defined as
\[
\tilde{\varphi}_n^M = \mathcal{K}(\phi_n^M) + f.
\] (1.7)

If \( \frac{\partial \kappa}{\partial u} \in C^r(\Omega) \) and \( f \in C^r[a,b] \), then the following orders of convergence can be obtained from Atkinson-Potra [6]:
\[
\| \varphi - \varphi_n^C \|_\infty = O(h^r), \quad \| \varphi - \varphi_n^S \|_\infty = O(h^{2r}).
\] (1.8)

The following estimates are obtained in Grammont et al [11]: Let \( f \in C^{2r}[a,b] \).

If \( \frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega) \) and then
\[
\| \varphi - \varphi_n^M \|_\infty = O(h^{3r}),
\] (1.9)

whereas if \( \frac{\partial \kappa}{\partial u} \in C^{3r}(\Omega) \), then
\[
\| \varphi - \tilde{\varphi}_n^M \|_\infty = O(h^{4r}),
\] (1.10)

In practice, it is necessary to replace the integral in the definition of \( \mathcal{K} \) by a numerical quadrature formula, giving rise to a discrete version of the above methods. The discrete version of the iterated collocation method is considered in Atkinson-Flores [4]. Our aim is to investigate a choice of a numerical quadrature rule which preserves the above orders of convergence in the discrete versions of the modified projection and the iterated modified projection methods.

We choose a composite numerical quadrature formula with a degree of precision \( d \) and with respect to a uniform partition of \([a,b]\) with \( m \) subintervals. Let \( \bar{h} \) denote the length of each subinterval of this partition. The discrete modified projection solution and the iterated discrete modified projection solution are denoted respectively by \( z_n^M \) and \( \tilde{z}_n^M \). We prove the following orders of convergence. Let \( \kappa \in C^d(\Omega) \), \( f \in C^d[a,b] \).

If \( \frac{\partial^2 \kappa}{\partial u^2} \in C^{2r}(\Omega) \), then
\[
\| \varphi - z_n^M \|_\infty = O \left( \max \left\{ \bar{h}^d, h^{3r} \right\} \right),
\] (1.11)

whereas if \( \frac{\partial^2 \kappa}{\partial u^2} \in C^{3r}(\Omega) \), then
\[
\| \varphi - \tilde{z}_n^M \|_\infty = O \left( h^r \max \left\{ \bar{h}^d, h^{3r} \right\} \right).
\] (1.12)
Thus, the orders of convergence in (1.9) and (1.10) are preserved provided the numerical quadrature rule is so chosen that \( \tilde{h}^d = h^{3r} \).

In Chen et al [7] discrete versions of the modified projection and the iterated modified projection methods for solutions of linear integral equations are considered. They consider a slightly restrictive case where the same uniform partition of \([a, b]\) is considered to define a composite numerical quadrature formula and an interpolatory projection, that is \( \tilde{h} = h \). We allow these partitions to be different. More precisely, we choose \( m = np, \ p \in \mathbb{N} \), instead of \( m = n \).

The emphasis of this paper is on the solution of Urysohn integral equations and we include the results about the linear case for the sake of completeness and for their use to treat the nonlinear case.

The paper has been arranged in the following way. In Section 2.1, we describe the Nyström approximation of \( \mathcal{H} \) obtained by replacing the integral by a numerical quadrature. In Section 2.2, this approximation is used to define the discrete versions of projection methods (1.3)-(1.7). Section 3 is devoted to approximate solutions of linear integral equations using the discrete versions of the modified projection and the iterated modified projection methods. The order of convergence for the discrete modified projection solution of a Urysohn integral equation is obtained in Section 4, whereas the discrete iterated modified projection method is investigated in Section 5. Numerical results are given in Section 6.

2 Methods of Approximation

2.1 Nyström Approximation

Assume that the kernel \( \kappa \) of \( \mathcal{H} \) defined by (1.1) is such that

\[
\frac{\partial^2 \kappa}{\partial u^2} \in C(\Omega).
\]

It is also assumed that 1 is not an eigenvalue of the compact linear operator \( \mathcal{H}'(\varphi) \) so that

\[
(I - \mathcal{H}'(\varphi))^{-1} : C[a, b] \rightarrow C[a, b] \]

is a bounded linear operator.
Let \( m \in \mathbb{N}, \) and consider the following uniform partition of \([a, b]\):

\[
a = s_0 < s_1 < \cdots < s_m = b. \tag{2.1}
\]

Let

\[
\tilde{h} = s_k - s_{k-1} = \frac{b - a}{m}, \quad k = 1, \ldots, m.
\]

Consider a basic quadrature rule

\[
\int_0^1 f(t)dt \approx \sum_{i=1}^{\rho} w_i f(\mu_i) \tag{2.2}
\]

which has a degree of precision \(2r - 1\) or higher.

Let

\[
\zeta^j_i = s_{j-1} + \mu_i \tilde{h}, \quad i = 1, \ldots, \rho, \quad j = 1, \ldots, m. \tag{2.3}
\]

A composite integration rule with respect to the partition \((2.1)\) is then defined as

\[
\int_a^b f(t)dt = \sum_{j=1}^{m} \int_{s_{j-1}}^{s_j} f(t)dt \approx \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i f(\zeta^j_i). \tag{2.4}
\]

The error in the numerical quadrature is assumed to be of the following form:

There is an integer \( d \geq 2r \) such that if \( f \in C^d[a, b] \), then

\[
\left| \int_a^b f(t)dt - \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i f(\zeta^j_i) \right| \leq C_1 \| f^{(d)} \|_\infty \tilde{h}^d, \tag{2.5}
\]

where \( f^{(d)} \) denotes \( d \) th derivative of \( f \) and \( C_1 \) is a constant independent of \( \tilde{h} \).

If the basic quadrature rule in \((2.2)\) is chosen to be the Gaussian quadrature with \( \rho = r \) or the Newton-Cotes quadrature with \( \rho = 2r \), then \((2.5)\) is satisfied with \( d = 2r \).

We replace the integral in \((1.1)\) by the numerical quadrature formula \((2.4)\) and define the Nyström operator as

\[
\mathcal{K}_m(x) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \kappa \left( s_j, \zeta^j_i, x \left( \zeta^j_i \right) \right). \tag{2.6}
\]

Then \( \mathcal{K}_m \) has the following properties.

1. \( \mathcal{K}_m : C[a, b] \to C[a, b] \) are completely continuous operators.

2. \( \mathcal{K}_m \) is a collectively compact family: \( \mathcal{B} \subset C([a, b]) \) bounded implies that the set \( \{ \mathcal{K}_m(\mathcal{B}) : m \geq 1 \} \) is relatively compact in \( C[a, b] \).
3. At each \( x \in C[a, b] \), \( \mathcal{K}_m(x) \to \mathcal{K}(x) \) as \( m \to \infty \).

4. \( \{ \mathcal{K}_m \} \) is an equicontinuous family at each \( x \in C[a, b] \).

5. For \( m \geq 1 \), \( \mathcal{K}_m \) are twice Fréchet differentiable:

\[
\mathcal{K}_m'(x)(v)(s) = \hat{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial \kappa}{\partial u}(s, \zeta_i^j, x(\zeta_i^j))v(\zeta_i^j), \quad s \in [a, b].
\]  

(2.7)

and

\[
\mathcal{K}_m''(x)(v_1, v_2)(s) = \hat{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial^2 \kappa}{\partial u^2}(s, \zeta_i^j, x(\zeta_i^j))v_1(\zeta_i^j)v_2(\zeta_i^j), \quad s \in [a, b].
\]  

(2.8)

Let \( \kappa \in C^d(\Omega) \) and \( f \in C^d[a, b] \). Then \( \mathcal{K} \) is a compact operator from \( L^\infty[a, b] \) to \( C^d[a, b] \) and the exact solution \( \varphi \) of (1.2) belongs to \( C^d[a, b] \). From (2.5) we conclude that

\[
\|\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\|_\infty = O\left(\hat{h}^d\right).
\]  

(2.9)

In the Nyström method, (1.2) is approximated by

\[
x_m - \mathcal{K}_m(x_m) = f.
\]  

(2.10)

We quote the following result from Atkinson [1].

Fix \( \delta > 0 \) and define

\[
B(\varphi, \delta) = \{ x : \| x - \varphi \|_\infty \leq \delta \}.
\]  

(2.11)

There exists a positive integer \( m_0 \) such that for \( m \geq m_0 \), the equation (2.10) has a unique solution \( \varphi_m \) in \( B(\varphi, \delta) \) and

\[
\| \varphi - \varphi_m \|_\infty = O\left(\hat{h}^d\right).
\]  

(2.12)
2.2 Discrete Projection Methods

We first define the interpolatory projection at \( r \) Gauss points.

Let \( n \in \mathbb{N} \) and consider the following uniform partition of \([a, b]\):

\[
a = t_0 < t_1 < \cdots < t_n = b. \tag{2.13}
\]

Let

\[
h = t_k - t_{k-1} = \frac{b - a}{n}, \quad k = 1, \ldots, n.
\]

Let \( r \) be a positive integer and let

\[
\mathcal{X}_n = \{ g \in L^\infty[a, b] : g|_{[t_{k-1}, t_k]} \text{ is a polynomial of degree } \leq r - 1, \ k = 1, \ldots, n \}. \tag{2.14}
\]

Let

\[
0 < q_1 < \cdots < q_r < 1
\]

denote the Gauss-Legendre zeros of order \( r \) in \([0, 1]\). The collocation nodes are chosen as follows:

\[
\tau^k_i = t_{k-1} + q_i h, \quad i = 1, \ldots, r, \quad k = 1, \ldots, n. \tag{2.15}
\]

Define the interpolation operator \( Q_n : C[a, b] \to \mathcal{X}_n \) as

\[
(Q_n x)(\tau^k_i) = x(\tau^k_i), \quad i = 1, \ldots, r, \quad k = 1, \ldots, n. \tag{2.16}
\]

Using the Hahn-Banach extension theorem, as in Atkinson et al [5], \( Q_n \) can be extended to \( L^\infty[a, b] \). Note that for \( x \in C[a, b], \ Q_n x \to x \) as \( n \to \infty \). As a consequence,

\[
\sup_n \| Q_n|_{C[a,b]} \| \leq q. \tag{2.17}
\]

Let

\[
p \in \mathbb{N} \text{ and } m = pn.
\]

Replacing the operator \( \mathcal{K} \) by the Nyström operator \( \mathcal{K}_m \) from \( \mathbf{2.6} \) in (1.3)-(1.7) we obtain discrete versions of various projection methods as given below.

Discrete Collocation method:

\[
z^C_n - Q_n \mathcal{K}_m(z^C_n) = Q_n f. \tag{2.18}
\]
Discrete Iterated Collocation method:

\[ z^S_n = \mathcal{K}_m(z^C_n) + f. \]  

(2.19)

The discrete modified projection operator is defined as

\[ \tilde{\mathcal{K}}_n^M(x) = Q_n \mathcal{K}_m(x) + \mathcal{K}_m(Q_n x) - Q_n \mathcal{K}_m(Q_n x). \]  

(2.20)

Discrete Modified Projection Method: We later show that for \( n \) and \( m \) big enough,

\[ x_n - \tilde{\mathcal{K}}_n^M(x_n) = f \]  

(2.21)

has a unique solution \( z^M_n \) in a neighbourhood of the exact solution \( \varphi \).

Discrete Iterated Modified Projection Method:

\[ z^M_n = \mathcal{K}_m(z^M_n) + f. \]  

(2.22)

Our aim is to show that \( z^M_n \to \varphi, \tilde{z}^M_n \to \varphi \) and obtain their orders of convergence. We first consider the case of a linear integral operator. The results proved for the linear integral equation are needed in the case of the Urysohn integral equation.

3 Linear Integral Equations

Let \( \kappa(\cdot, \cdot) \in C([a,b] \times [a,b]) \). Then the integral operator

\[ (\mathcal{K}x)(s) = \int_a^b \kappa(s,t)x(t)dt, \quad s \in [a,b] \]

is a compact linear operator from \( L^\infty[a,b] \) to \( C[a,b] \). Assume that 1 is in the resolvent set of \( \mathcal{K} \). Then for \( f \in C[a,b] \), the following Fredholm integral equation

\[ x - \mathcal{K}x = f \]  

(3.1)

has a unique solution in \( C[a,b] \), say \( \varphi \).

In this special case, the Nyström operator defined in (2.6) is given by

\[ (\mathcal{K}_m x)(s) = h \sum_{j=1}^m \sum_{i=1}^\rho w_i \kappa(s, \zeta_i^j)x(\zeta_i^j), \quad s \in [a,b], \quad x \in C[a,b]. \]  

(3.2)
Note that for $x \in C[a, b]$, $\mathcal{K}_m x$ converges to $\mathcal{K} x$ and $\{\mathcal{K}_m\}$ is a collectively compact family. Hence for all $m$ big enough, 1 is in the resolvent set of $\mathcal{K}_m$ and the following Nyström approximation

$$x_m - \mathcal{K}_m x_m = f$$

has a unique solution, say $\varphi_m$. Also,

$$\|(I - \mathcal{K}_m)^{-1}\| \leq C_2.$$

If the kernel $\kappa \in C^d([a, b] \times [a, b])$ and if the right hand side $f \in C^d[a, b]$, then the exact solution $\varphi$ of (3.1) is in $C^d[a, b]$ and

$$\|\varphi - \varphi_m\|_\infty = O \left( \tilde{h}^d \right). \quad (3.3)$$

(See Atkinson [3]).

### 3.1 Discrete Modified Projection Methods

Note that since $\mathcal{K}_m$ is a linear operator, the discrete modified projection operator defined by (2.20) can be written as

$$\mathcal{K}_n^M = Q_n \mathcal{K}_m + \mathcal{K}_m Q_n - Q_n \mathcal{K}_m Q_n$$

and

$$\|\mathcal{K}_m - \mathcal{K}_n^M\| = \|(I - Q_n) \mathcal{K}_m (I - Q_n)\|.$$  

Since $Q_n$ converges to the Identity operator pointwise on $C[a, b]$ and $\{\mathcal{K}_m\}$ is a collectively compact family, it follows that

$$\|(I - Q_n) \mathcal{K}_m\| \to 0 \text{ as } n \to \infty.$$  

Hence using the estimate (2.17), we obtain

$$\|\mathcal{K}_m - \mathcal{K}_n^M\| \leq (1 + q)\|(I - Q_n) \mathcal{K}_m\| \to 0 \text{ as } n \to \infty.$$
Thus, for $n$ and $m$ big enough, 1 belongs to the resolvent set of $\mathcal{K}_n^M$ and $x_n - \mathcal{K}_n^M x_n = f$ has a unique solution, say $z_n^M$. Also, 

$$\| (I - \mathcal{K}_n^M)^{-1} \| \leq 2C_2. \quad (3.4)$$

The Discrete Iterated Modified Projection solution is defined as

$$\tilde{z}_n^M = \mathcal{K}_m z_n^M + f.$$ 

### 3.2 Orders of Convergence

Let $p \in \mathbb{N}$ and choose $m = np$. Then $p\tilde{h} = h$. Note that the $m\rho = npp$ nodes in the composite quadrature rule (2.4) can be divided into $n$ groups of $p\rho$ nodes and each of the $p\rho$ nodes in a group lie precisely in one of the subinterval of the partition (2.13). Based on this observation, we rewrite the expression (3.2) for the Nyström operator $\mathcal{K}_m$ as follows.

$$(\mathcal{K}_m x)(s) = \tilde{h} \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \kappa \left( s, \zeta_{i}^{(k-1)p+\nu} \right) x \left( \zeta_{i}^{(k-1)p+\nu} \right), \quad s \in [a, b]. \quad (3.5)$$

Let

$$\Psi(t) = (t - q_1) \cdots (t - q_r), \quad t \in [0, 1],$$

where $q_1, \ldots, q_r$ are Gauss-Legendre zeros in $[0, 1]$. Then

$$\int_0^1 t^j \Psi(t) dt = \int_0^1 t^j (t - q_1) \cdots (t - q_r) = 0 \quad \text{for} \quad 0 \leq j \leq r - 1.$$

We prove below a discrete analogue of the above result when the integral is replaced by a composite numerical quadrature.

**Lemma 3.1.** Let $\{w_i, i = 1, \ldots, \rho\}$ and $\{\mu_i, i = 1, \ldots, \rho\}$ be respectively the weights and the node points in the basic quadrature formula defined in (2.2). Then

$$\sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \left( \frac{\nu - 1 + \mu_i}{p} \right)^j \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) = 0, \quad 0 \leq j \leq r - 1. \quad (3.6)$$
Proof. Note that
\[
\int_0^1 f(t)dt = \frac{1}{p} \sum_{\nu=1}^{p} \int_{\nu/p}^{\nu/p + 1} f(t)dt \approx \frac{1}{p} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i f \left( \frac{\nu - 1 + \mu_i}{p} \right).
\]
The quadrature rule (2.2) is assumed to be exact for polynomials of degree \( \leq 2r - 1 \). Hence
\[
\int_0^1 t^j \Psi(t) dt = \frac{1}{p} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \left( \frac{\nu - 1 + \mu_i}{p} \right)^j \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right), \quad 0 \leq j \leq r - 1.
\]
Since for \( 0 \leq j \leq r - 1 \),
\[
\int_0^1 t^j \Psi(t) dt = 0,
\]
the desired result follows.

For future reference, we quote the following interpolation error estimates from Conte-de-Boor [8]. For \( t \in [t_{k-1}, t_k] \),
\[
x(t) - (Q_n x)(t) = x[\tau^k_1, \ldots, \tau^k_r, t] (t - \tau^k_1) \cdots (t - \tau^k_r),
\]
where \( x[\tau^k_1, \ldots, \tau^k_r, t] \) denotes the divided difference of \( x \) based on \( \{\tau^k_1, \ldots, \tau^k_r, t\} \).

Substituting for \( \tau^k_i \) from (2.15) we obtain,
\[
x(t) - (Q_n x)(t) = x[\tau^k_1, \ldots, \tau^k_r, t] \left[ (t - t_{k-1} - q_1 h) \cdots (t - t_{k-1} - q_r h) \left( t - \tau^k_1 h \right) \cdots \left( t - \tau^k_r h \right) \right] h^r.
\]
If \( x \in C^r[a, b] \), then
\[
\|x - Q_n x\|_\infty \leq \frac{\|x^{(r)}\|_\infty}{r!} \|\Psi\|_\infty h^r = C_3 \|x^{(r)}\|_\infty h^r,
\]
where
\[
C_3 = \frac{\|\Psi\|_\infty}{r!}.
\]

We use the following notation:
\[
\|\kappa\|_{2r, \infty} = \max\{\|D^{(i,j)}\kappa\|_\infty : 0 \leq i + j \leq 2r\} \quad \text{and} \quad \|x\|_{2r, \infty} = \max\{\|x^{(j)}\|_\infty : 0 \leq j \leq 2r\},
\]
where \( x^{(j)} \) denotes the \( j \)th derivative of \( x \) and

\[
D^{(i,j)} \kappa(s,t) = \frac{\partial^{i+j} \kappa}{\partial s^i \partial t^j}(s,t), \quad a \leq s, t \leq b.
\]

We now prove some preliminary results which are needed to obtain the orders of convergence in the Discrete Modified Projection Method and the Discrete Iterated Modified Projection Method. The following proposition is crucial in what follows.

**Proposition 3.2.** If \( \kappa \in C^r([a, b] \times [a, b]) \) and \( x \in C^{2r}[a, b] \), then

\[
\| K_m(I - Q_n)x \|_{\infty} \leq C_4 \| \kappa \|_{r, \infty} \| x \|_{2r, \infty} h^{2r},
\]

(3.9)

where

\[
C_4 = \frac{1}{r!} 2^r (b - a) \| \Psi \|_{\infty} \left( \sum_{i=1}^{\rho} |w_i| \right)
\]

is a constant independent of \( n \) and of \( m \).

**Proof.** For \( s \in [a, b] \),

\[
(K_m(I - Q_n)x)(s) = \tilde{h} \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \kappa \left( s, \zeta^{(k-1)p+\nu}_i \right) \times \left[ x \left( \zeta^{(k-1)p+\nu}_i \right) - Q_n x \left( \zeta^{(k-1)p+\nu}_i \right) \right].
\]

Substituting from the relation (3.7), we obtain

\[
(K_m(I - Q_n)x)(s) = \tilde{h} h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \kappa \left( s, \zeta^{(k-1)p+\nu}_i \right) \times \left[ x \left[ \zeta^{(k-1)p+\nu}_1, \ldots, \zeta^{(k-1)p+\nu}_r, \zeta^{(k-1)p+\nu}_i \right] \Psi \left( \frac{\zeta^{(k-1)p+\nu}_i - t_{k-1}}{h} \right) \right].
\]

Note that

\[
\frac{\zeta^{(k-1)p+\nu}_i - t_{k-1}}{h} = \frac{(\nu - 1 + \mu_i) \tilde{h}}{p h} = \frac{\nu - 1 + \mu_i}{p}.
\]

(3.10)

Thus,

\[
(K_m(I - Q_n)x)(s) = \tilde{h} h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \kappa \left( s, \zeta^{(k-1)p+\nu}_i \right) \times \left[ x \left[ \zeta^{(k-1)p+\nu}_1, \ldots, \zeta^{(k-1)p+\nu}_r, \zeta^{(k-1)p+\nu}_i \right] \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) \right].
\]

(3.11)
For a fixed \( s \in [a, b] \), define
\[
g^k_s(t) = \kappa(s, t) x[r^k_1, \ldots, r^k_r, t] \quad \text{for} \quad t_{k-1} \leq t \leq t_k.
\]
Then
\[
(\mathcal{X}_m(I - Q_n)x)(s) = \tilde{h} \ h^r \sum_{k=1}^{n} \sum_{j=0}^{r-1} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \ g^k_s \left( \zeta^{(k-1)p+\nu}_i - t_{k-1} \right)^j \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) \tag{3.12}
\]
Since \( k \in C^r([a, b] \times [a, b]) \) and \( x \in C^{2r}[a, b] \), it follows that \( g^k_s \in C^r[t_{k-1}, t_k] \) for \( k = 1, \ldots, n \).

For \( t \in [t_{k-1}, t_k] \), we expand \( g^k_s(t) \) about \( t_{k-1} \) and obtain
\[
g^k_s(t) = \sum_{j=0}^{r-1} \frac{1}{j!} (t - t_{k-1})^j (g^k_s)^{(j)}(t_{k-1}) + \frac{1}{r!} (t - t_{k-1})^r (g^k_s)^{(r)}(\xi^k_{t_{k-1}, t_k}), \quad \xi^k_{t_{k-1}, t_k} \in (t_{k-1}, t_k).
\]
Substitute the above expression for \( g^k_s(t) \) in (3.12) to obtain
\[
(\mathcal{X}_m(I - Q_n)x)(s) = \tilde{h} \ h^r \sum_{k=1}^{n} \sum_{j=0}^{r-1} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \left( \zeta^{(k-1)p+\nu}_i - t_{k-1} \right)^j \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) + \frac{1}{r!} \tilde{h} \ h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \left( \zeta^{(k-1)p+\nu}_i - t_{k-1} \right)^r (g^k_s)^{(r)}(\xi^k_{t_{k-1}, t_k}) \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right).
\]
Using (3.10) the above equation reduces to
\[
(\mathcal{X}_m(I - Q_n)x)(s) = \tilde{h} \ h^r \sum_{k=1}^{n} \sum_{j=0}^{r-1} \frac{1}{j!} (g^k_s)^{(j)}(t_{k-1}) \tilde{h}^j \times \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i (\nu - 1 + \mu_i)^j \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) + \frac{1}{r!} \tilde{h}^{r+1} \ h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i (\nu - 1 + \mu_i)^r (g^k_s)^{(r)}(\xi^k_{t_{k-1}, t_k}) \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right).
\]
From Lemma 3.1 we see that the first term in the above expression vanishes. Hence
\[
\|\mathcal{X}_m(I - Q_n)x\|_\infty \leq \frac{1}{r!} \tilde{h}^{r+1} \ h^r \ n \ \max_{s \in [a, b]} \ \max_{1 \leq k \leq n} \left( \max_{t \in [t_{k-1}, t_k]} \left| (g^k_s)^{(r)}(t) \right| \right) \times \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} \left\{ |w_i| \ |\nu - 1 + \mu_i|^r \ \left| \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right) \right| \right\}.
\]
It can be seen that for \( s \in [a, b] \),
\[
\max_{t \in [t_{k-1}, t_k]} \left| (g^k_s)^{(r)}(t) \right| \leq 2^r \|\kappa\|_{r, \infty} \|x\|_{2r, \infty}.
\]
Since \( \mu_i \in [0, 1] \), it follows that

\[
\nu - 1 + \mu_i \leq p \quad \text{for} \quad 1 \leq \nu \leq p
\]

and hence

\[
\sum_{\nu=1}^{p} \sum_{i=1}^{\rho} \left\{ |w_i| \left| \nu - 1 + \mu_i \right|^p \right\} \leq p^{r+1} \|\Psi\|_\infty \left( \sum_{i=1}^{\rho} |w_i| \right).
\]

Since \( \tilde{h}p = h \) and \( hn = b - a \), it follows that

\[
\| \mathcal{K}_m(I - Q_n)x\|_\infty \leq C_4\|\kappa\|_{r,\infty}\|x\|_{2r,\infty} h^{2r},
\]

which completes the proof.

The following two results are proved using Proposition 3.2.

**Proposition 3.3.** If \( \kappa \in C^{2r}([a, b] \times [a, b]) \), then

\[
\| \mathcal{K}_m(I - Q_n)\mathcal{K}_m \| \leq C_5\|\kappa\|_{r,\infty}\|\kappa\|_{2r,\infty} h^{2r},
\]

where

\[
C_5 = C_4(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) = \frac{1}{r!} 2^r (b - a)^2 \|\Psi\|_\infty \left( \sum_{i=1}^{\rho} |w_i| \right)^2
\]

is a constant independent of \( n \) and of \( m \).

**Proof.** From Proposition 3.2, we have

\[
\| \mathcal{K}_m(I - Q_n)\mathcal{K}_m x\|_\infty \leq C_4\|\kappa\|_{r,\infty}\|\kappa\|_{2r,\infty} h^{2r}.
\]

From (3.2)

\[
(\mathcal{K}_m x)(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \kappa \left( s, \zeta_i^j \right) x \left( \zeta_i^j \right).
\]

Hence for \( \beta = 1, \ldots, 2r \),

\[
(\mathcal{K}_m x)^{(\beta)}(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i D^{(\beta,0)} \kappa \left( s, \zeta_i^j \right) x \left( \zeta_i^j \right)
\]

and

\[
\| (\mathcal{K}_m x)^{(\beta)} \|_\infty \leq \tilde{h}m \left( \sum_{i=1}^{\rho} |w_i| \right) \| D^{(\beta,0)} \kappa \|_\infty \| x \|_\infty.
\]
As a consequence,
\[ \| \mathcal{K}_m x \|_{2r,\infty} \leq (b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) \| \kappa \|_{2r,\infty} \| x \|_{\infty} . \]

Thus,
\[ \| \mathcal{K}_m (I - Q_n) x \|_{\infty} \leq C_4 (b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) \| \kappa \|_{r,\infty} \| x \|_{2r,\infty} h^{2r}. \]

The desired estimate follows by taking the supremum over the set \( \{ x \in C[a, b] : \| x \|_{\infty} \leq 1 \} \).

**Proposition 3.4.** If \( \kappa \in C^{2r}([a, b] \times [a, b]) \) and \( x \in C^{2r}[a, b] \), then
\[ \| (I - Q_n) \mathcal{K}_m (I - Q_n) x \|_{\infty} \leq C_3 C_4 \| \kappa \|_{2r,\infty} \| x \|_{2r,\infty} h^{3r}. \] (3.14)

If \( \kappa \in C^{3r}([a, b] \times [a, b]) \) and \( x \in C^{2r}[a, b] \), then
\[ \| \mathcal{K}_m (I - Q_n) \mathcal{K}_m (I - Q_n) x \|_{\infty} \leq (C_4)^2 \| \kappa \|_{r,\infty} \| \kappa \|_{3r,\infty} \| x \|_{2r,\infty} h^{4r}. \] (3.15)

**Proof.** From the estimate (3.8), we obtain
\[ \| (I - Q_n) \mathcal{K}_m (I - Q_n) x \|_{\infty} \leq C_3 \| (\mathcal{K}_m (I - Q_n) x)^{(r)} \|_{\infty} h^r. \]

Differentiating (3.11) \( r \) times with respect to \( s \), we obtain
\[ (\mathcal{K}_m (I - Q_n) x)^{(r)}(s) = \bar{h} h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} \sum_{i=1}^{\rho} w_i \frac{\partial^r \kappa}{\partial s^r} \left( s, \zeta_{i}^{(k-1)p+\nu} \right) \times \]
\[ x[\tau_1^k, \ldots, \tau_r^k, \zeta_{i}^{(k-1)p+\nu}] \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right). \]

Let
\[ \ell(s, t) = \frac{\partial^r \kappa}{\partial s^r}(s, t), \quad a \leq s, t \leq b. \]

If \( \kappa \in C^{2r}([a, b] \times [a, b]) \), then \( \ell \in C^r([a, b] \times [a, b]) \). Hence proceeding as in the proof of Proposition 3.2 we obtain
\[ \| (\mathcal{K}_m (I - Q_n) x)^{(r)} \|_{\infty} \leq C_4 \| \ell \|_{r,\infty} \| x \|_{2r,\infty} h^{2r} \leq C_4 \| \kappa \|_{2r,\infty} \| x \|_{2r,\infty} h^{2r}. \]

As a consequence,
\[ \| (I - Q_n) \mathcal{K}_m (I - Q_n) x \|_{\infty} \leq C_3 C_4 \| \kappa \|_{2r,\infty} \| x \|_{2r,\infty} h^{3r}, \]
which completes the proof of (3.14).

Let \( \kappa \in C^d([a, b] \times [a, b]) \) and \( x \in C^{2r}[a, b] \). Then \( \mathcal{K}_m(I - Q_n)x \in C^d[a, b] \). Using (3.9) we obtain,

\[
\| \mathcal{K}_m(I - Q_n)\mathcal{K}_m(I - Q_n)x \|_\infty \leq C_4 \| \kappa \|_{r, \infty} \| \mathcal{K}_m(I - Q_n)x \|_{2r, \infty} h^{2r}.
\] (3.16)

From (3.11) for \( j = 0, 1, \ldots, 2r \),

\[
(\mathcal{K}_m(I - Q_n)x)^{(j)}(s) = \tilde{h} h^r \sum_{k=1}^{n} \sum_{\nu=1}^{p} w_i \frac{\partial^j \kappa}{\partial \nu} \left( s, \zeta_i^{(k-1)p+\nu} \right) \times x[\tau_1^k, \ldots, \tau_r^k, \zeta_i^{(k-1)p+\nu}] \Psi \left( \frac{\nu - 1 + \mu_i}{p} \right).
\]

Note that \( \frac{\partial^j \kappa}{\partial \nu} \in C^{3r-j}([a, b] \times [a, b]), \) for \( j = 0, 1, \ldots, 2r \). Proceeding as in the proof of Proposition 3.2 we obtain

\[
\| (\mathcal{K}_m(I - Q_n)x)^{(j)} \|_\infty \leq C_4 \| \kappa \|_{j+r, \infty} \| x \|_{2r, \infty} h^{2r}
\]

and hence

\[
\| \mathcal{K}_m(I - Q_n)x \|_{2r, \infty} = \max \left\{ \| (\mathcal{K}_m(I - Q_n)x)^{(j)} \|_\infty : 0 \leq j \leq 2r \right\}
\]

\[
\leq C_4 \| \kappa \|_{3r, \infty} \| x \|_{2r, \infty} h^{2r}.
\]

As a consequence,

\[
\| \mathcal{K}_m(I - Q_n)\mathcal{K}_m(I - Q_n)x \|_\infty \leq (C_4)^2 \| \kappa \|_{r, \infty} \| \kappa \|_{3r, \infty} \| x \|_{2r, \infty} h^{4r},
\]

which completes the proof of (3.15). \( \square \)

We now prove our main result for linear integral equations.

**Theorem 3.5.** If \( \kappa \in C^d([a, b] \times [a, b]) \) and \( f \in C^d[a, b] \), then

\[
\| \varphi - z_n \|^M = O \left( \max \left\{ \tilde{h}^d, h^{3r} \right\} \right).
\] (3.17)

If \( \kappa \in C^{\max(3r,d)}([a, b] \times [a, b]) \) and \( f \in C^d[a, b] \), then

\[
\| \varphi - z_n \|^M = O \left( \max \left\{ \tilde{h}^d, h^{4r} \right\} \right).
\] (3.18)
Proof. Since

\[ \varphi - z_n^M = (I - \mathcal{K})^{-1} f - (I - \mathcal{K}_n^M)^{-1} f = (I - \mathcal{K}_n^M)^{-1} (\mathcal{K} - \mathcal{K}_n^M) \varphi, \]

using (3.4) we obtain,

\[ \| \varphi - z_n^M \|_\infty \leq 2C_2 \left( \| \mathcal{K} \varphi - \mathcal{K}_m \varphi \|_\infty + \| (\mathcal{K}_m - \mathcal{K}_n^M) \varphi \|_\infty \right) \]

\[ \leq 2C_2 \left( \| \mathcal{K} \varphi - \mathcal{K}_m \varphi \|_\infty + \| (I - Q_n) \mathcal{K}_m (I - Q_n) \varphi \|_\infty \right). \]

Using the estimates (2.9) and (3.14) we obtain,

\[ \| \varphi - z_n^M \|_\infty = O \left( \max \left\{ \tilde{h}^d, h^{3r} \right\} \right), \]

which completes the proof of (3.17).

Note that

\[ \varphi_m - z_n^M = (I - \mathcal{K}_m)^{-1} (\mathcal{K}_m - \mathcal{K}_n^M) z_n^M \]

\[ = (I - \mathcal{K}_m)^{-1} (I - Q_n) \mathcal{K}_m (I - Q_n) z_n^M. \]

Hence

\[ \varphi_m - \tilde{z}_n^M = \mathcal{K}_m \varphi_m - \mathcal{K}_m z_n^M \]

\[ = (I - \mathcal{K}_m)^{-1} \mathcal{K}_m (I - Q_n) \mathcal{K}_m (I - Q_n) z_n^M \]

\[ = (I - \mathcal{K}_m)^{-1} \mathcal{K}_m (I - Q_n) \mathcal{K}_m (I - Q_n) (z_n^M - \varphi) \]

\[ + (I - \mathcal{K}_m)^{-1} \mathcal{K}_m (I - Q_n) \mathcal{K}_m (I - Q_n) \varphi. \]

Thus,

\[ \| \varphi_m - \tilde{z}_n^M \|_\infty \leq C_2 (1 + \| Q_n \| \| \mathcal{K}_m (I - Q_n) \mathcal{K}_m \| \| z_n^M - \varphi \|_\infty \]

\[ + C_2 \| \mathcal{K}_m (I - Q_n) \mathcal{K}_m (I - Q_n) \varphi \|_\infty. \]

Using (3.13), (3.15) and (3.17) and the fact that \( d \geq 2r \), it can be seen that

\[ \| \varphi_m - \tilde{z}_n^M \|_\infty = O(h^{4r}). \]

Hence using (3.3) we obtain,

\[ \| \varphi - \tilde{z}_n^M \|_\infty \leq \| \varphi - \varphi_m \|_\infty + \| \varphi_m - \tilde{z}_n^M \|_\infty = O \left( \max \left\{ \tilde{h}^d, h^{4r} \right\} \right), \]

which completes the proof. \( \square \)
4 Discrete Modified Projection Method for Urysohn Integral Equations

In this section we consider approximation of the Urysohn integral equation (1.1)-(1.2) by the discrete version of the modified projection method. For a fixed $\delta > 0$, a closed neighbourhood $B(\varphi, \delta)$ of the exact solution $\varphi$ is defined in (2.11). First we prove a result about the Nyström operator $K_m$ defined in (2.6). Let $\frac{\partial^2 \kappa}{\partial u^2} \in C(\Omega)$ and define

$$C_6 = \max_{s,t \in [a,b]} \left| \frac{\partial^2 \kappa}{\partial u^2}(s,t,u) \right| .$$

(4.1)

Proposition 4.1. Let $\frac{\partial^2 \kappa}{\partial u^2} \in C(\Omega)$. Then for $v_1, v_2 \in B(\varphi, \delta)$ and for $s \in [a,b]$,

$$K_m(v_2)(s) - K_m(v_1)(s) - K_m'(v_1)(v_2 - v_1)(s) = R(v_2 - v_1)(s),$$

where

$$\|R(v_2 - v_1)\|_\infty \leq C_6(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) \|v_2 - v_1\|_\infty^2.$$  

Proof. If $v_1, v_2 \in B(\varphi, \delta)$, then by the generalized Taylor’s theorem,

$$K_m(v_2)(s) - K_m(v_1)(s) - K_m'(v_1)(v_2 - v_1)(s) = R(v_2 - v_1)(s), \quad s \in [a,b],$$

where

$$R(v_2 - v_1)(s) = \int_0^1 (1 - \theta) \kappa''(v_1 + \theta(v_2 - v_1)) (v_2 - v_1)^2(s) d\theta.$$  

(4.2)

For $s \in [a,b]$ and $\theta \in [0,1]$, define

$$(S_\theta(v_2 - v_1))(s) = \kappa''(v_1 + \theta(v_2 - v_1)) (v_2 - v_1)^2(s)$$

$$= \hat{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial^2 \kappa}{\partial u^2} \left( s, \zeta_j, v_1(\zeta_j) + \theta(v_2 - v_1)(\zeta_j) + \theta(v_2 - v_1)(\zeta_j) \right) (v_2 - v_1)^2(\zeta_j).$$  

(4.3)

Then

$$\|S_\theta(v_2 - v_1)\|_\infty \leq C_6(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) \|v_2 - v_1\|_\infty^2.$$
Since

\[ R(v_2 - v_1)(s) = \int_0^1 (1 - \theta)S_\theta(v_2 - v_1)(s)d\theta, \]

it follows that

\[ \|R(v_2 - v_1)\|_\infty \leq \frac{C_6(b-a)}{2} \left( \sum_{i=1}^p |w_i| \right) \|v_2 - v_1\|_\infty^2. \]

This completes the proof. \[ \square \]

Remark 4.2. Note that

\[ \mathcal{K}'(\varphi)v(s) = \int_a^b \frac{\partial K}{\partial u}(s, t, \varphi(t))v(t) \, dt, \quad (4.4) \]

whereas

\[ \mathcal{K}'_m(\varphi)v(s) = \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho w_i \frac{\partial K}{\partial u}(s, \zeta_j^i, \varphi(\zeta_j^i))v(\zeta_j^i), \quad s \in [a, b]. \quad (4.5) \]

Thus, \( \mathcal{K}'_m(\varphi) \) is the Nyström approximation of the linear operator \( \mathcal{K}'(\varphi) \) associated with a convergent quadrature formula. Hence \( \mathcal{K}'_m(\varphi) \) converges to \( \mathcal{K}'(\varphi) \) pointwise and \( \mathcal{K}'_m(\varphi) \) is a collectively compact family.

As before, now onwards we assume that

\[ m = np \quad \text{for some} \quad p \in \mathbb{N}. \]

It follows that

\[ \|(I - Q_n)\mathcal{K}'_m(\varphi)\| \to 0 \text{ as } n \to \infty. \quad (4.6) \]

It is assumed that \( (I - \mathcal{K}'(\varphi))^{-1} : C[a, b] \to C[a, b] \) is a bounded linear operator. Hence there exists a positive integer \( m_1 \geq m_0 \) such that for \( m \geq m_1 \), \( (I - \mathcal{K}'_m(\varphi))^{-1} \) exists and

\[ \|(I - \mathcal{K}'_m(\varphi))^{-1}\| \leq C_7. \quad (4.7) \]

See Atkinson [3].

We prove some preliminary results which are needed to obtain the order of convergence of the discrete modified projection solution \( z^M_n \).
Proposition 4.3. Let $\frac{\partial^2 K}{\partial u^2} \in C(\Omega)$. Then $\mathcal{K}_m'$ is Lipschitz continuous in $B(\varphi, \delta)$:

$$\|\mathcal{K}'_m(x) - \mathcal{K}'_m(y)\| \leq \gamma \|x - y\|_{\infty}, \quad x, y \in B(\varphi, \delta), \quad (4.8)$$

where $\gamma$ is a constant independent of $m$.

Proof. For $x, y \in B(\varphi, \delta)$,

$$\|\mathcal{K}'_m(x) - \mathcal{K}'_m(y)\| = \sup_{\|v\| \leq 1} \sup_{s \in [a,b]} \left| (\mathcal{K}'_m(x) - \mathcal{K}'_m(y)) v(s) \right|$$

For $s \in [a,b]$, we have

$$(\mathcal{K}'_m(x) - \mathcal{K}'_m(y)) v(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \left( \frac{\partial K}{\partial u}(s, \zeta_i^j, x(\zeta_i^j)) - \frac{\partial K}{\partial u}(s, \zeta_i^j, y(\zeta_i^j)) \right) v(\zeta_i^j).$$

By the Mean Value Theorem,

$$(\mathcal{K}'_m(x) - \mathcal{K}'_m(y)) v(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial^2 K}{\partial u^2}(s, \zeta_i^j, \eta_i^j) \left( x(\zeta_i^j) - y(\zeta_i^j) \right) v(\zeta_i^j),$$

where

$$\eta_i^j = \theta_i^j x(\zeta_i^j) + (1 - \theta_i^j) y(\zeta_i^j) \quad \text{for some} \ \theta_i^j \in (0,1).$$

Since $x, y \in B(\varphi, \delta)$, it follows that

$$|\eta_i^j| \leq \|\varphi\|_{\infty} + \delta, \quad i = 1, \ldots, \rho, \ j = 1, \ldots, m.$$ 

Hence for $s \in [a,b]$,

$$\left| (\mathcal{K}'_m(x) - \mathcal{K}'_m(y)) v(s) \right| \leq C_6 \tilde{h} m \left( \sum_{i=1}^{\rho} |w_i| \right) \|x - y\|_{\infty} \|v\|_{\infty}$$

$$\leq \gamma \|x - y\|_{\infty} \|v\|_{\infty},$$

where

$$\gamma = C_6 (b - a) \left( \sum_{i=1}^{\rho} |w_i| \right),$$

$C_6$ being defined in (4.1). Hence

$$\| (\mathcal{K}'_m(x) - \mathcal{K}'_m(y)) v \|_{\infty} \leq \gamma \|x - y\|_{\infty} \|v\|_{\infty}.$$

Taking the supremum over the set $\{ v \in C[a,b] : \|v\|_{\infty} \leq 1 \}$, the required result follows. $\square$
Proposition 4.4. Let $\frac{\partial^2 \kappa}{\partial u^2} \in C(\Omega)$. There exists a positive integer $n_1$ such that for $n \geq n_1$ and for $m \geq m_1$, $I - \left(\mathcal{K}_n^M\right)'(\varphi)$ is invertible and

$$\left\| I - \left(\mathcal{K}_n^M\right)'(\varphi) \right\|^{-1} \leq 2C_7. \quad (4.9)$$

Proof. Fix $m = np \geq m_1$. Then by (4.7)

$$\left\| \left( I - \mathcal{K}_m'(\varphi) \right)^{-1} \right\| \leq C_7.$$

From the definition of $\mathcal{K}_n^M(x)$ in (2.20), it follows that

$$\left(\mathcal{K}_n^M\right)'(x) = Q_n\mathcal{K}_m'(x) + (I - Q_n)\mathcal{K}_m'(Q_nx)Q_n.$$

Hence

$$\mathcal{K}_m'(\varphi) - \left(\mathcal{K}_n^M\right)'(\varphi) = (I - Q_n)\mathcal{K}_m'(\varphi) - (I - Q_n)\mathcal{K}_m'(Q_n\varphi)Q_n$$

$$= (I - Q_n)\mathcal{K}_m'(\varphi)(I - Q_n) + (I - Q_n)(\mathcal{K}_m'(\varphi) - \mathcal{K}_m'(Q_n\varphi))Q_n.$$

Since $\varphi \in C[a, b]$, it follows that

$$\|Q_n\varphi - \varphi\|_\infty \to 0 \text{ as } n \to \infty.$$

Hence there exists a positive integer $n_0$ such that

$$n \geq n_0 \Rightarrow Q_n\varphi \in B(\varphi, \delta).$$

By Proposition 4.3

$$\|\mathcal{K}_m'(\varphi) - \mathcal{K}_m'(Q_n\varphi)\| \leq \gamma\|\varphi - Q_n\varphi\|_\infty \to 0$$

and by (4.6)

$$\|(I - Q_n)\mathcal{K}_m'(\varphi)\| \to 0 \text{ as } n \to \infty.$$

Hence for $n \geq n_0$,

$$\left\| \mathcal{K}_m'(\varphi) - \left(\mathcal{K}_n^M\right)'(\varphi) \right\| \leq (1 + \|Q_n\|)(I - Q_n)\mathcal{K}_m'(\varphi) \right\| + \|Q_n\|(1 + \|Q_n\|)\|\mathcal{K}_m'(\varphi) - \mathcal{K}_m'(Q_n\varphi)\|$$

$$\leq (1 + q)\|(I - Q_n)\mathcal{K}_m'(\varphi)\| + q(1 + q)\|\varphi - Q_n\varphi\|_\infty.$$
Thus,

\[ \| \mathcal{X}_m'(\varphi) - \left( \mathcal{X}_n^M \right)'(\varphi) \| \to 0 \text{ as } n \to \infty. \]

Choose \( n_1 \geq n_0 \) such that

\[ n \geq n_1 \Rightarrow \left\| \left( \mathcal{X}_n^M \right)'(\varphi) - \mathcal{X}_m'(\varphi) \right\| \leq \frac{1}{2} C_7. \]

Since

\[ I - \left( \mathcal{X}_n^M \right)'(\varphi) = \left[ I - \left\{ \left( \mathcal{X}_n^M \right)'(\varphi) - \mathcal{X}_m'(\varphi) \right\} (I - \mathcal{X}_m'(\varphi))^{-1} \right] (I - \mathcal{X}_m'(\varphi)), \]

it follows that for \( n \geq n_1 \),

\[ \left\| \left( I - \left( \mathcal{X}_n^M \right)'(\varphi) \right)^{-1} \right\| \leq 2 \left\| (I - \mathcal{X}_m'(\varphi))^{-1} \right\| \leq 2 C_7. \]

This completes the proof. \( \Box \)

Remark 4.5. For \( n \geq n_1 \) and \( m \geq m_1 \), define

\[ B_n(x) = x - \left[ I - \left( \mathcal{X}_n^M \right)'(\varphi) \right]^{-1} \left\{ x - \mathcal{X}_n^M(x) - f \right\} \]  \hspace{1cm} (4.10)

Then

\[ B_n(x) = x \text{ if and only if } x - \mathcal{X}_n^M(x) = f. \]  \hspace{1cm} (4.11)

As in Grammont [10], it can be shown that there is a \( \delta_0 > 0 \) such that \( B_n \) has a unique fixed point \( z_n^M \) in \( B(\varphi, \delta_0) \) and that

\[ \| z_n^M - \varphi \|_{\infty} \leq \frac{3}{2} \left\| I - \left( \mathcal{X}_n^M \right)'(\varphi) \right\|^{-1} \left\| \mathcal{X}(\varphi) - \mathcal{X}_n^M(\varphi) \right\|_{\infty}. \]

Hence

\[ \| z_n^M - \varphi \|_{\infty} \leq 3 C_7 \left( \| \mathcal{X}(\varphi) - \mathcal{X}_m(\varphi) \|_{\infty} + \| \mathcal{X}_m(\varphi) - \mathcal{X}_n^M(\varphi) \|_{\infty} \right). \]  \hspace{1cm} (4.12)

Without loss of generality, assume that

\[ n \geq n_1 \Rightarrow Q_n \varphi \in B(\varphi, \delta_0) \text{ and } n \geq n_1, m \geq m_1 = n_1 p \Rightarrow \varphi_m \in B(\varphi, \delta_0), Q_n \varphi_m \in B(\varphi, \delta_0). \]
By (2.9),

\[ \| \mathcal{K}(\varphi) - \mathcal{K}_m(\varphi) \|_{\infty} = O(h^4). \]

In order to obtain the order of convergence for the term \( \| \mathcal{K}_m(\varphi) - \mathcal{K}_n^M(\varphi) \|_{\infty} \) in the estimate (4.12), we prove the following result.

**Proposition 4.6.** Let \( r \geq 1 \). If \( \frac{\partial^2 \kappa}{\partial u^2} \in C^r(\Omega) \) and \( f \in C^r[a,b] \), then for \( n \geq n_1 \),

\[ \| (I - Q_n)(\mathcal{K}_m(Q_n \varphi) - \mathcal{K}_m(\varphi) - \mathcal{K}_m'(\varphi)(Q_n \varphi - \varphi))\|_{\infty} = O(h^{3r}). \]  

(4.13)

**Proof.** Let \( \delta = \delta_0, v_1 = \varphi \) and \( v_2 = Q_n \varphi \) in Proposition 4.1. Then

\[ \mathcal{K}_m(Q_n \varphi) - \mathcal{K}_m(\varphi) - \mathcal{K}_m'(\varphi)(Q_n \varphi - \varphi) = R(Q_n \varphi - \varphi). \]

By (3.8)

\[ \| (I - Q_n)R(Q_n \varphi - \varphi)\|_{\infty} \leq C_3 \| (R(Q_n \varphi - \varphi))^{(r)}\|_{\infty} h^r. \]

We have

\[ (R(Q_n \varphi - \varphi))^{(r)}(s) = \int_0^1 (1 - \theta)(S_\theta(Q_n \varphi - \varphi))^{(r)}(s) d\theta, \]

where from (4.3),

\[ (S_\theta(Q_n \varphi - \varphi))^{(r)}(s) = \tilde{h} \sum_{j=1}^m \rho \sum_{i=1}^{n} w_i \frac{\partial^{r+2} \kappa}{\partial s^r \partial u^2}(s, \zeta_i^j, \varphi(\zeta_i^j) + \theta(Q_n \varphi - \varphi)(\zeta_i^j)) (Q_n \varphi - \varphi)^2(\zeta_i^j) \]

Let

\[ C_8 = \max_{s,t \in [a,b], u \leq \| \varphi \|_{\infty} + \delta_0} \left| \frac{\partial^{r+2} \kappa}{\partial s^r \partial u^2}(s, t, u) \right|. \]

Then

\[ \| (S_\theta(Q_n \varphi - \varphi))^{(r)}\|_{\infty} \leq C_8(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right) \| Q_n \varphi - \varphi \|_{\infty}^2 \]

and

\[ \| (R(Q_n \varphi - \varphi))^{(r)}\|_{\infty} \leq C_8(b - a) \frac{1}{2} \left( \sum_{i=1}^{\rho} |w_i| \right) \| Q_n \varphi - \varphi \|_{\infty}^2. \]

Since \( \kappa \in C^r(\Omega) \) and \( f \in C^r[a,b] \), it follows that \( \varphi \in C^r[a,b] \). Hence by (3.8),

\[ \| Q_n \varphi - \varphi \|_{\infty} \leq C_3 \| \varphi^{(r)}\|_{\infty} h^r. \]
Thus,
\[ \|(I - Q_n)R(Q_n\varphi - \varphi)\|_\infty \leq \frac{(C_3)^3C_5(b - a)}{2} \left( \sum_{i=1}^{\rho} |w_i| \right) \|\varphi^{(r)}\|_\infty^{2}\tilde{h}^{3r}, \]
which completes the proof of (4.13). \qed

In the following theorem we obtain the order of convergence in the discrete modified
projection method.

**Theorem 4.7.** Let \( r \geq 1, \kappa \in C^d(\Omega), \frac{\partial\kappa}{\partial u} \in C^{2r}(\Omega) \) and \( f \in C^d[a,b] \). Let \( \varphi \) be the unique
solution of (2.1) and assume that \( 1 \) is not an eigenvalue of \( \mathcal{K}'(\varphi) \). Let \( n \geq n_1 \) and \( m \geq m_1 \).
Let \( \mathcal{X}_n \) be the space of piecewise polynomials of degree \( \leq r - 1 \) with respect to the partition
(2.13) and \( Q_n : L^\infty[0,1] \to \mathcal{X}_n \) be the interpolatory projection at \( r \) Gauss points defined by
(2.16). Let \( z_n^M \) be the unique solution of (2.21) in \( B(\varphi, \delta_0) \). Then
\[ \|z_n^M - \varphi\|_\infty = O\left(\max\{\tilde{h}^d, \tilde{h}^{3r}\}\right). \] (4.14)

**Proof.** From (4.12),
\[ \|z_n^M - \varphi\|_\infty \leq 3C_7 \left( \|\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\|_\infty + \|\mathcal{K}_m(\varphi) - \mathcal{K}_n^M(\varphi)\|_\infty \right). \] (4.15)

From (2.9),
\[ \|\mathcal{K}(\varphi) - \mathcal{K}_m(\varphi)\|_\infty = O\left(\tilde{h}^d\right). \] (4.16)

Note that
\[ \mathcal{K}_m(\varphi) - \mathcal{K}_n^M(\varphi) = (I - Q_n)(\mathcal{K}_m(\varphi) - \mathcal{K}_m(Q_n\varphi)) \]
\[ = -(I - Q_n)(\mathcal{K}_m(Q_n\varphi - \mathcal{K}_m(\varphi) - \mathcal{K}_m(\varphi)(Q_n\varphi - \varphi)) \]
\[ - (I - Q_n)\mathcal{K}_m'(\varphi)(Q_n\varphi - \varphi). \]

Hence
\[ \|\mathcal{K}_m(\varphi) - \mathcal{K}_n^M(\varphi)\|_\infty \leq \|(I - Q_n)(\mathcal{K}_m(Q_n\varphi - \mathcal{K}_m(\varphi) - \mathcal{K}_m(\varphi)(Q_n\varphi - \varphi))\|_\infty \]
\[ + \|(I - Q_n)\mathcal{K}_m'(\varphi)(Q_n\varphi - \varphi)\|_\infty. \] (4.17)
By (4.13) of Proposition 4.6,

\[ \| (I - Q_n)(\mathcal{K}_m(Q_n\varphi) - \mathcal{K}_m(\varphi) - \mathcal{K}'_m(\varphi)(Q_n\varphi - \varphi)) \|_\infty = O(h^{3r}). \]  

(4.18)

Define

\[ \ell(s, t) = \frac{\partial \kappa}{\partial u}(s, t, \varphi(t)). \]

Then from (4.5),

\[ \mathcal{K}'_m(\varphi)v(s) = \tilde{h} m \sum_{j=1}^{\rho} \sum_{i=1}^{w} \ell(s, \zeta_j^i)v(\zeta_j^i), \quad s \in [a, b]. \]

By assumption, \( \frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega) \). Since \( f \in C^{2r}[a, b] \), it follows that \( \varphi \in C^{2r}[a, b] \). Hence \( \ell \in C^{2r}([a, b] \times [a, b]) \). Thus, Proposition 3.4 is applicable and we obtain

\[ \| (I - Q_n)\mathcal{K}'_m(\varphi)(Q_n\varphi - \varphi) \|_\infty = O(h^{3r}). \]  

(4.19)

From (4.17)-(4.19), it follows that

\[ \| \mathcal{K}_m(\varphi) - \tilde{\mathcal{K}}_M n(\varphi) \|_\infty = O(h^{3r}). \]  

(4.20)

The required result follows from (4.15), (4.16) and (4.20). \( \square \)

5 Discrete Iterated Modified Projection method for Urysohn Integral Equations

Recall from Remark 4.5 that there exists \( \delta_0 > 0 \) and a positive integer \( n_1 \) such that for \( n \geq n_1 \) equation (2.21) has a unique solution \( z^M_n \) in \( B(\varphi, \delta_0) \).

The discrete iterated modified projection solution is defined as

\[ \tilde{z}_n^M = \mathcal{K}_m(z^M_n) + f. \]  

(5.1)

In this section we show that \( \tilde{z}_n^M \rightarrow \varphi \) as \( n \rightarrow \infty \) and obtain its order of convergence.

Proposition 5.1. Let \( r \geq 1, \kappa \in C^d(\Omega), \frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega) \) and \( f \in C^d[a, b] \). Let \( \varphi_m \) denote the Nyström approximation of the exact solution \( \varphi \) of the Urysohn integral equation (1.1)-(1.2). Then for \( n \geq n_1 \) and \( m = np \),

\[ \tilde{z}_n^M - \varphi_m = \mathcal{K}'_m(\varphi_m)(z^M_n - \varphi_m) + O(\max\{\tilde{h}^d, h^{3r}\}^2). \]  

(5.2)
Proof. Since
\[ \varphi_m - \mathcal{H}_m(\varphi_m) = f, \]
it follows that
\[ z^M_n - \varphi_m = \mathcal{H}_m(z^M_n) - \mathcal{H}_m(\varphi_m). \]
In Theorem 4.7 we proved that
\[ \|z^M_n - \varphi\|_\infty = O(\max\{\tilde{h}^d, h^3r\}). \]
By Proposition 4.1 with \( \delta = \delta_0 \), for \( s \in [a, b] \),
\[ \mathcal{H}_m(z^M_n)(s) - \mathcal{H}_m(\varphi_m)(s) = \mathcal{H}_m'(\varphi_m)(z^M_n - \varphi_m)(s) + R(z^M_n - \varphi_m)(s), \]
where
\[ \|R(z^M_n - \varphi_m)\|_\infty \leq C_6 \frac{(b-a)}{2} \left( \sum_{i=1}^{\rho} |w_i| \right) \|z^M_n - \varphi_m\|_\infty \]
\[ = O(\max\{\tilde{h}^d, h^3r\}). \]
The required result follows from (5.3)-(5.5).

In order to obtain an error estimate for the first term in (5.2) we need to define a new operator which has \( z^M_n \) as a fixed point. For this purpose, we show that for all \( m \) large enough, \( I - \mathcal{H}_m'(\varphi_m) \) are invertible and are uniformly bounded.

Proposition 5.2. Let \( \frac{\partial^2 K}{\partial u^2} \in C(\Omega) \). There exists a positive integer \( m_2 \geq m_1 \) such that for \( m \geq m_2 \)
\[ \|(I - \mathcal{H}_m'(\varphi_m))^{-1}\| \leq 2 C_7. \]

Proof. Recall from (4.7) that for \( m \geq m_1 \),
\[ \|(I - \mathcal{H}_m'(\varphi))^{-1}\| \leq C_7. \]

By Proposition 4.3 with \( \delta = \delta_0 \),
\[ \|\mathcal{H}_m'(\varphi) - \mathcal{H}_m'(\varphi_m)\| \leq \gamma \|\varphi - \varphi_m\|_\infty \to 0 \text{ as } m \to \infty. \]
Choose \( m_2 \geq m_1 \) such that
\[
m \geq m_2 \Rightarrow \| \varphi - \varphi_m \|_\infty \leq \frac{1}{2C_7}. \]

Since
\[
I - \mathcal{K}_m(\varphi) = (I - (\mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi))(I - \mathcal{K}_m(\varphi))^{-1})(I - \mathcal{K}_m(\varphi)),
\]
it follows that for \( m \geq m_2 \),
\[
(I - \mathcal{K}_m(\varphi_m))^{-1} \text{ exists}
\]
and that
\[
\| (I - \mathcal{K}_m(\varphi_m))^{-1} \| \leq 2C_7.
\]
This completes the proof.

\[\square\]

**Remark 5.3.** In (4.10) we defined an operator \( B_n \) which has a unique fixed point \( z_n^M \) in \( B(\varphi, \delta_0) \).
Now we define another operator \( \tilde{B}_n \) which also has \( z_n^M \) as a fixed point.

From (2.20) recall that
\[
\tilde{\mathcal{K}}_n^M(x) = Q_n\mathcal{K}_m(x) + \mathcal{K}_m(Q_nx) - Q_n\mathcal{K}_m(Q_nx).
\]
For \( m \geq m_2 \), define
\[
\tilde{B}_n(x) = \varphi_m - [I - \mathcal{K}_m(\varphi_m)]^{-1} \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m)\varphi_m - \tilde{\mathcal{K}}_n^M(x) + \mathcal{K}_m(\varphi_m)x \right\}. \quad (5.7)
\]
Then
\[
\tilde{B}_n(x) = x \\
\Leftrightarrow (I - \mathcal{K}_m(\varphi_m))\varphi_m - \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m)\varphi_m - \tilde{\mathcal{K}}_n^M(x) + \mathcal{K}_m(\varphi_m)x \right\} \\
= (I - \mathcal{K}_m(\varphi_m))x \\
\Leftrightarrow x - \tilde{\mathcal{K}}_n^M(x) = \varphi_m - \mathcal{K}_m(\varphi_m) = f.
\]
Thus, \( x \) is a fixed point of \( \tilde{B}_n \) if and only if it satisfies the equation (2.21).

Note that for \( n \geq n_1 \) and \( m \geq m_2 \),
\[
z_n^M - \varphi_m = \tilde{B}_n(z_n^M) - \varphi_m \\
= - [I - \mathcal{K}_m(\varphi_m)]^{-1} \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m(\varphi_m)\varphi_m - \tilde{\mathcal{K}}_n^M(z_n^M) + \mathcal{K}_m(\varphi_m)z_n^M \right\}.
\]
Since $\mathcal{K}'(\varphi_m)$ and $[I - \mathcal{K}'(\varphi_m)]^{-1}$ commute, the first term in (5.2) can be written as
\[
\mathcal{K}'(\varphi_m)(z_n^M - \varphi_m)
= - [I - \mathcal{K}'(\varphi_m)]^{-1} \mathcal{K}'(\varphi_m) \left\{ \mathcal{K}(\varphi_m) - \mathcal{K}'(\varphi_m) \varphi_m - \tilde{\mathcal{K}}_n^M(z_n^M) + \mathcal{K}'(\varphi_m)z_n^M \right\}.
\]
We write
\[
\mathcal{K}'(\varphi_m)(z_n^M - \varphi_m)
= - [I - \mathcal{K}'(\varphi_m)]^{-1} \mathcal{K}'(\varphi_m) \left\{ \mathcal{K}(\varphi_m) - \mathcal{K}'(\varphi_m) \varphi_m - \tilde{\mathcal{K}}_n^M(z_n^M) + \mathcal{K}'(\varphi_m)z_n^M \right\}
+ [I - \mathcal{K}'(\varphi_m)]^{-1} \mathcal{K}'(\varphi_m) \left\{ \tilde{\mathcal{K}}_n^M(z_n^M) - \tilde{\mathcal{K}}_n^M(\varphi_m) - \left( \tilde{\mathcal{K}}_n^M \right)'(\varphi_m)(z_n^M - \varphi_m) \right\}
+ [I - \mathcal{K}'(\varphi_m)]^{-1} \mathcal{K}'(\varphi_m) \left\{ \left( \tilde{\mathcal{K}}_n^M \right)'(\varphi_m) - \mathcal{K}'(\varphi_m) \right\}(z_n^M - \varphi_m) \right\}.
\]
(5.8)

We now obtain error estimates for the quantities appearing in the above expression.

**Proposition 5.4.** If $\frac{\partial \kappa}{\partial u} \in C^{3r}(\Omega)$ and $x \in C^{2r}[a, b]$, then for $n \geq n_1$ and $m \geq m_2$,
\[
\|\mathcal{K}'(\varphi_m)(I - Q_n) \mathcal{K}'(\varphi_m)(I - Q_n)x\|_\infty = O(h^{4r}).
\]
(5.9)

**Proof.** We have
\[
\mathcal{K}'(\varphi_m)v(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial \kappa}{\partial u}(s, \zeta_i, \varphi_m(\zeta_i))v(\zeta_i), \quad s \in [a, b].
\]
Let
\[
\ell_m(s, t) = \frac{\partial \kappa}{\partial u}(s, t, \varphi_m(t)), \quad s, t \in [a, b].
\]
Then $\ell_m \in C^{3r}([a, b] \times [a, b])$ and Proposition 3.2 is applicable. Hence
\[
\|\mathcal{K}'(\varphi_m)(I - Q_n)x\|_\infty \leq C_4 \|\ell_m\|_{r, \infty} \|x\|_{2r, \infty} h^{2r}.
\]
Let
\[
C_9 = \max_{0 \leq j \leq r} \max_{s, t \in [a, b]} \frac{\partial^{i+j+1} \kappa}{\partial s^i \partial t^j \partial u}(s, t, u).
\]
Since for $m \geq m_2$, $\varphi_m \in B(\varphi, \delta_0)$, it follows that
\[
\|\ell_m\|_{r, \infty} \leq C_9.
\]
It then follows that

$$\| \mathcal{K}'(\varphi_m)(I - Q_n)x\|_\infty \leq C_4 C_9 \|x\|_{2r,\infty} h^{2r}. \quad (5.10)$$

Using the above estimate, we obtain

$$\| \mathcal{K}'(\varphi_m)(I - Q_n)\mathcal{K}'(\varphi_m)(I - Q_n)x\|_\infty \leq C_4 C_9 \|\mathcal{K}'(\varphi_m)(I - Q_n)x\|_{2r,\infty} h^{2r}. \quad (5.11)$$

For $0 \leq \beta \leq 2r$,

$$(\mathcal{K}'(\varphi_m)v)^{(\beta)}(s) = \tilde{h} \sum_{j=1}^{m} \sum_{i=1}^{p} w_i \frac{\partial^{\beta+1} \kappa}{\partial s^\beta \partial u}(s, \zeta_i, \varphi_m(\zeta_i)) v(\zeta_i), \quad s \in [a, b].$$

Note that for $0 \leq \beta \leq 2r$,

$$\frac{\partial^{\beta+1} \kappa}{\partial s^\beta \partial u}(s, t, \varphi_m(t)) = D^{(\beta, 0)} \ell_m(s, t) \in C^{3r-\beta}([a, b] \times [a, b]).$$

Let

$$C_{10} = \max_{0 \leq i+j \leq 3r} \max_{s,t \in [a,b]} \left| \frac{\partial^{i+j+1} \kappa}{\partial s^i \partial t^j \partial u}(s, t, u) \right|.$$ 

Then for $0 \leq \beta \leq 2r$,

$$\| D^{(\beta, 0)} \ell_m \|_{r,\infty} \leq C_{10}.$$ 

By Proposition 3.2 for $0 \leq \beta \leq 2r$, we obtain

$$\| (\mathcal{K}'(\varphi_m)(I - Q_n)x)^{(\beta)} \|_\infty \leq C_4 \| D^{(\beta, 0)} \ell_m \|_{r,\infty} \|x\|_{2r,\infty} h^{2r} \leq C_4 C_{10} \|x\|_{2r,\infty} h^{2r}. $$

Hence

$$\| \mathcal{K}'(\varphi_m)(I - Q_n)x\|_{2r,\infty} = \max \{ \| (\mathcal{K}'(\varphi_m)(I - Q_n)x)^{(\beta)} \|_\infty : 0 \leq \beta \leq 2r \} \leq C_4 C_{10} \|x\|_{2r,\infty} h^{2r}. \quad (5.12)$$

Thus, from (5.11) and (5.12) we obtain

$$\| \mathcal{K}'(\varphi_m)(I - Q_n)\mathcal{K}'(\varphi_m)(I - Q_n)x\|_\infty \leq (C_4)^2 C_9 C_{10} \|x\|_{2r,\infty} h^{4r},$$

which completes the proof.
Proposition 5.5. If $\frac{\partial \kappa}{\partial u} \in C^{3r}(\Omega)$, then for $n \geq n_1$ and $m \geq m_2$,

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m'(\varphi_m)\| = O(h^{2r}).$$

(5.13)

Proof. From (5.10)

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m'(\varphi_m)x\|_{2r,\infty} \leq C_4 C_9 \|\mathcal{K}_m'(\varphi_m)x\|_{2r,\infty} h^{2r}.\tag{5.16}$$

For $0 \leq \beta \leq 2r$,

$$\big(\mathcal{K}_m'(\varphi_m)x\big)^{(\beta)}(s) = \frac{h}{s^{\beta+1}} \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \frac{\partial^{\beta+1} \kappa}{\partial s^{\beta} \partial u}(s, \zeta_j^s, \varphi_m(\zeta_j^s))x(\zeta_j^s), \quad s \in [a, b].$$

Then

$$\|\mathcal{K}_m'(\varphi_m)x\|_{2r,\infty} \leq (b - a) C_{10} \left( \sum_{i=1}^{\rho} |w_i| \right) \|x\|_{\infty}.$$

Thus,

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m'(\varphi_m)x\|_{\infty} \leq (b - a) C_4 C_9 C_{10} \left( \sum_{i=1}^{\rho} |w_i| \right) \|x\|_{\infty} h^{2r}.$$

Taking the supremum over the set $\{x \in C[a, b] : \|x\|_{\infty} \leq 1\}$, we obtain

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)\mathcal{K}_m'(\varphi_m)\| \leq (b - a) C_4 C_9 C_{10} \left( \sum_{i=1}^{\rho} |w_i| \right) h^{2r},$$

which completes the proof.

Proposition 5.6. Let $r \geq 1$, $\kappa \in C^d(\Omega)$ and $f \in C^d[a, b]$. If $\frac{\partial^2 \kappa}{\partial u^2} \in C^{2r}(\Omega)$, then for $n \geq n_1$ and $m \geq m_2$,

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n) [\mathcal{K}_m(Q_n \varphi_m) - \mathcal{K}_m(\varphi_m) - \mathcal{K}_m'(\varphi_m)(Q_n \varphi_m - \varphi_m)]\|_{\infty} = O(h^{4r}).$$

(5.14)

Proof. Let

$$y_n = \mathcal{K}_m(Q_n \varphi_m) - \mathcal{K}_m(\varphi_m) - \mathcal{K}_m'(\varphi_m)(Q_n \varphi_m - \varphi_m) = R(Q_n \varphi_m - \varphi_m).$$

(5.15)

By (5.10)

$$\|\mathcal{K}_m'(\varphi_m)(I - Q_n)y_n\|_{\infty} \leq C_4 C_9 \|y_n\|_{2r,\infty} h^{2r}.\tag{5.16}$$
Recall from (4.2) and (4.3) that
\[ R(v_2 - v_1)(s) = \int_0^1 (1 - \theta) S_\theta(v_2 - v_1)(s)\,d\theta, \]
with
\[ S_\theta(v_2 - v_1)(s) = \tilde{h} \sum_{j=1}^m \sum_{i=1}^\rho w_i \frac{\partial^2 \kappa}{\partial u^2} \left( s, \zeta_j^i, v_1(\zeta_j^i) + \theta(v_2 - v_1)(\zeta_j^i) \right) (v_2 - v_1)^2(\zeta_j^i). \]

Let
\[ C_{11} = \max_{0 \leq \beta \leq 2r} \max_{s,t \in [a,b]} \left| \frac{\partial^{\beta+2} \kappa}{\partial s^\beta \partial u^2}(s, t, u) \right|. \]

Then for \(0 \leq \beta \leq 2r\),
\[ \| (R(v_2 - v_1))^{(\beta)} \|_\infty \leq \frac{C_{11}(b-a)}{2} \left( \sum_{i=1}^\rho |w_i| \right) \| v_2 - v_1 \|_\infty^2 \]
and
\[ \| y_n \|_{2r, \infty} = \| R(Q_n \varphi_m - \varphi_m) \|_{2r, \infty} \leq \frac{C_{11}(b-a)}{2} \left( \sum_{i=1}^\rho |w_i| \right) \| Q_n \varphi_m - \varphi_m \|_\infty^2. \quad (5.17) \]

From (2.12), (2.17) and (3.8),
\[ \| (Q_n - I)\varphi_m \|_\infty \leq \| (Q_n - I)\varphi \|_\infty + \| (Q_n - I)(\varphi_m - \varphi) \|_\infty \]
\[ \leq C_3 \| \varphi^{(r)} \|_\infty \theta^r + O \left( \tilde{h}^d \right) \]
\[ = O(\theta^r). \quad (5.18) \]

The required result then follows from (5.16) - (5.18). \( \square \)

**Proposition 5.7.** Let \( \kappa \in C^d(\Omega) \), \( \frac{\partial \kappa}{\partial u} \in C^{2r}(\Omega) \) and \( f \in C^d[a, b] \). Then for \( n \geq n_1 \) and \( m \geq m_2 \),
\[ \left\| \mathcal{X}_n^M(\varphi_m) - \mathcal{X}_n^M(\varphi_m) - \left( \mathcal{X}_n^M \right)'(\varphi_m)(z_n^M - \varphi_m) \right\|_\infty = O \left( \max \left\{ \tilde{h}^d, \tilde{h}^{3r} \right\} \right)^2. \quad (5.19) \]

**Proof.** Note that \( \varphi_m, z_n^M \in B(\varphi_m, \delta_0) \). By the generalized Taylor’s theorem,
\[ \mathcal{X}_n^M(\varphi_m) - \mathcal{X}_n^M(\varphi_m) - \left( \mathcal{X}_n^M \right)'(\varphi_m)(z_n^M - \varphi_m) \]
\[ = \int_0^1 (1 - \theta) \left( \mathcal{X}_n^M \right)''(\varphi_m + \theta(z_n^M - \varphi_m))(z_n^M - \varphi_m)^2(s)\,d\theta. \]
Hence
\[
\left\| \tilde{\mathcal{K}}_n^M(z_M^n) - \tilde{\mathcal{K}}_n^M(\varphi_m) - \left( \tilde{\mathcal{K}}_n^M \right)'(\varphi_m)(z_M^n - \varphi_m) \right\|_{\infty} \\
\leq \frac{1}{2} \max_{0 \leq \theta \leq 1} \left\| \left( \tilde{\mathcal{K}}_n^M \right)''(\varphi_m + \theta(z_M^n - \varphi_m)) \right\| \parallel z_M^n - \varphi_m \parallel^2. \tag{5.20}
\]

Note that for \( x \in B(\varphi, \delta_0) \),
\[
\left( \tilde{\mathcal{K}}_n^M \right)''(x) = Q_n \mathcal{K}'(x) + (I - Q_n) \mathcal{K}'(Q_n x)(Q_n \otimes Q_n),
\]
where \( Q_n \otimes Q_n : X \times X \to X \times X \) is defined as
\[
(Q_n \otimes Q_n)(v, w) = (Q_n v, Q_n w).
\]

From (2.8) for \( s \in [a, b] \),
\[
\left( \mathcal{K}_m \right)''(\varphi_m + \theta(z_M^n - \varphi_m)) (v_1, v_2)(s) \\
= h_i \sum_{j=1}^{m} \sum_{i=1}^{\rho} w_i \partial^2 \kappa \partial u^2 \left( s, \zeta^j_i, (\varphi_m + \theta(z_M^n - \varphi_m)) (\zeta^j_i) \right) v_1(\zeta^j_i) v_2(\zeta^j_i).
\]

Then for \( 0 \leq \theta \leq 1 \),
\[
\left\| \left( \mathcal{K}_m \right)''(\varphi_m + \theta(z_M^n - \varphi_m)) \right\| \leq C_6(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right),
\]
where \( C_6 \) is defined in (4.11) with \( \delta = \delta_0 \).

In a similar manner, it can be shown that for \( 0 \leq \theta \leq 1 \),
\[
\left\| \left( \mathcal{K}_m \right)''(Q_n \varphi_m + \theta(Q_n z_M^n - Q_n \varphi_m)) \right\| \leq C_6(b - a) \left( \sum_{i=1}^{\rho} |w_i| \right).
\]

Hence
\[
\max_{0 \leq \theta \leq 1} \left\| \left( \tilde{\mathcal{K}}_n^M \right)''(\varphi_m + \theta(z_M^n - \varphi_m)) \right\| \leq C_6(b - a)q(1 + q + q^2) \left( \sum_{i=1}^{\rho} |w_i| \right). \tag{5.21}
\]

By Theorem 4.7 and by (2.12),
\[
\parallel z_M^n - \varphi_m \parallel_{\infty} \leq \parallel z_M^n - \varphi \parallel_{\infty} + \parallel \varphi - \varphi_m \parallel_{\infty} = O \left( \max \left\{ h^d, h^{3r} \right\} \right),
\]

The required result follows from (5.20), (5.21) and the above estimate.
Proposition 5.8. Let $\kappa \in C^d(\Omega)$, $\frac{\partial \kappa}{\partial u} \in C^{3r}(\Omega)$ and $f \in C^d[a, b]$. Then for $n \geq n_1$ and $m \geq m_2$,

$$
\left\| \mathcal{K}'_m(\varphi_m) \left( (\tilde{\mathcal{K}}_M)^t(\varphi_m) - \mathcal{K}'_m(\varphi_m) \right) (z^M_n - \varphi_m) \right\|_\infty = O \left( h^r \max \left\{ \tilde{h}^d, h^{3r} \right\} \right).
$$

(5.22)

Proof. Note that

$$(\tilde{\mathcal{K}}_M)^t(\varphi_m) = Q_n \mathcal{K}'_m(\varphi_m) + (I - Q_n) \mathcal{K}'_m(Q_n \varphi_m) Q_n.$$

Hence

$$
\mathcal{K}'_m(\varphi_m) \left( (\tilde{\mathcal{K}}_M)^t(\varphi_m) - \mathcal{K}'_m(\varphi_m) \right) = -\mathcal{K}'_m(\varphi_m)(I - Q_n) \mathcal{K}'_m(\varphi_m)
$$

$$
+ \mathcal{K}'_m(\varphi_m)(I - Q_n) \mathcal{K}'_m(Q_n \varphi_m) Q_n
$$

$$
= \mathcal{K}'_m(\varphi_m)(I - Q_n)(\mathcal{K}'_m(Q_n \varphi_m) - \mathcal{K}'_m(\varphi_m)) Q_n
$$

$$
- \mathcal{K}'_m(\varphi_m)(I - Q_n) \mathcal{K}'_m(\varphi_m)(I - Q_n).
$$

By Proposition 5.5,

$$
\parallel \mathcal{K}'_m(\varphi_m)(I - Q_n) \mathcal{K}'_m(\varphi_m) \parallel = O(h^{2r}).
$$

By (4.8) and (5.18),

$$
\parallel \mathcal{K}'_m(Q_n \varphi_m) - \mathcal{K}'_m(\varphi_m) \parallel \leq \gamma \parallel Q_n \varphi_m - \varphi_m \parallel_\infty = O(h^r).
$$

Let

$$
C_{12} = \max_{s, t \in [a, b], \parallel u \parallel \leq \parallel \varphi \parallel_\infty + \delta_0} \frac{|\partial \kappa}{\partial u}(s, t, u).
$$

Then

$$
\parallel \mathcal{K}'_m(\varphi_m)x \parallel_\infty \leq C_{12}(b - a) \left( \sum_{i=1}^\rho |w_i| \right) \parallel x \parallel_\infty.
$$

Hence

$$
\parallel \mathcal{K}'_m(\varphi_m) \parallel \leq C_{12}(b - a) \left( \sum_{i=1}^\rho |w_i| \right).
$$

Thus,

$$
\left\| \mathcal{K}'_m(\varphi_m) \left( (\tilde{\mathcal{K}}_M)^t(\varphi_m) - \mathcal{K}'_m(\varphi_m) \right) \right\| = O(h^r).
$$

Since

$$
\parallel z^M_n - \varphi \parallel_\infty = O \left( \max \left\{ \tilde{h}^d, h^{3r} \right\} \right),
$$

the required result follows.
In the following theorem we obtain the order of convergence in the discrete iterated modified projection method.

**Theorem 5.9.** Let \( r \geq 1, \kappa \in C^d(\Omega), \frac{\partial^2 \kappa}{\partial u^2} \in C^{3r}(\Omega) \) and \( f \in C^d[a,b] \). Let \( \varphi \) be the unique solution of (1.2) and assume that 1 is not an eigenvalue of \( \mathcal{K}'(\varphi) \). Let \( \mathcal{X}_n \) be the space of piecewise polynomials of degree \( \leq r-1 \) with respect to the partition (2.13) and \( Q_n : L^{\infty}[0,1] \rightarrow \mathcal{X}_n \) be the interpolatory projection at \( r \) Gauss points defined by (2.10). Let \( z^M_n \) be the unique solution of (2.21) in \( B(\varphi, \delta_0) \). Let \( \tilde{z}^M_n \) be the discrete iterated modified projection solution defined by (5.1). Then

\[
\| \tilde{z}^M_n - \varphi \|_\infty = O(h^r \max\{\bar{h}^d, h^{3r}\}).
\]

(5.23)

**Proof.** Recall from (5.2) that

\[
z^M_n - \varphi_m = \mathcal{K}_m'(\varphi_m)(z^M_n - \varphi_m) + O(\max\{\bar{h}^d, h^{3r}\})^2
\]

and from (5.8) that

\[
\mathcal{K}_m'(\varphi_m)(z^M_n - \varphi_m) = - \left[ I - \mathcal{K}_m'(\varphi_m) \right]^{-1} \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_m(\varphi_m) - \mathcal{K}_m^M(\varphi_m) \right\}
\]

\[
+ \left[ I - \mathcal{K}_m'(\varphi_m) \right]^{-1} \mathcal{K}_m'(\varphi_m) \left\{ \mathcal{K}_m^M(z^M_n) - \mathcal{K}_m^M(\varphi_m) - \left( \mathcal{K}_m^M \right)'(\varphi_m)(z^M_n - \varphi_m) \right\}
\]

\[
+ \left[ I - \mathcal{K}_m'(\varphi_m) \right]^{-1} \mathcal{K}_m'(\varphi_m) \left\{ \left( \mathcal{K}_m^M \right)'(\varphi_m) - \mathcal{K}_m'(\varphi_m) \right\} (z^M_n - \varphi_m).
\]

(5.25)

From (5.6) we have

\[
\left\| \left[ I - \mathcal{K}_m'(\varphi_m) \right]^{-1} \right\| \leq 2C_7.
\]

Note that

\[
\mathcal{K}_m(\varphi_m) - \mathcal{K}_m^M(\varphi_m) = (I - Q_n)(\mathcal{K}_m(\varphi_m) - \mathcal{K}_m(Q_n \varphi_m))
\]

\[
= -(I - Q_n)(\mathcal{K}_m(Q_n \varphi_m) - \mathcal{K}_m(\varphi_m) - \mathcal{K}'_m(\varphi_m)(Q_n \varphi_m - \varphi_m))
\]

\[
= -(I - Q_n)\mathcal{K}_m'(\varphi_m)(Q_n \varphi_m - \varphi_m).
\]

By Proposition 5.6

\[
\| \mathcal{K}_m'(\varphi_m)(I - Q_n)(\mathcal{K}_m(Q_n \varphi_m) - \mathcal{K}_m(\varphi_m) - \mathcal{K}'_m(\varphi_m)(Q_n \varphi_m - \varphi_m) \|_\infty = O(h^{4r}).
\]

(5.26)
Since
\[
\begin{align*}
&\left\| \mathcal{H}_m'(\varphi_m)(I - Q_n)\mathcal{H}_m'(\varphi_m)(I - Q_n)\varphi_m \right\| \infty \\
&\quad \leq \left\| \mathcal{H}_m'(\varphi_m)(I - Q_n)\mathcal{H}_m'(\varphi_m)(I - Q_n)\varphi \right\| \infty \\
&\quad + (1 + q) \left\| \mathcal{H}_m'(\varphi_m)(I - Q_n)\mathcal{H}_m'(\varphi_m) \right\| \left\| \varphi_m - \varphi \right\| \infty,
\end{align*}
\]
by Proposition 5.4 Proposition 5.5 and the estimate (2.12), we obtain
\[
\left\| \mathcal{H}_m'(\varphi_m)(I - Q_n)\mathcal{H}_m'(\varphi_m)(I - Q_n)\varphi_m \right\| \infty = O(h^{4r}).
\]
It then follows that
\[
\left\| (I - \mathcal{H}_m'(\varphi_m))^{-1} \mathcal{H}_m'(\varphi_m) \left( \mathcal{H}_m(\varphi_m) - \mathcal{H}_m^M(\varphi_m) \right) \right\| \infty = O(h^{4r}).
\]
By Proposition 5.7
\[
\left\| \mathcal{H}_m^M(\varphi_m) - \mathcal{H}_m^M(\varphi_m) \left( (\mathcal{H}_n^M)'(\varphi_m)(\varphi_m - \varphi_m) \right) \right\| \infty = O \left( \max \{ \tilde{h}^d, h^{3r} \} \right)^2.
\]
whereas by Proposition 5.8
\[
\left\| \mathcal{H}_m'(\varphi_m) \left\{ \left( (\mathcal{H}_n^M)'(\varphi_m) - \mathcal{H}_m'(\varphi_m) \right)(\varphi_m - \varphi_m) \right\} \right\| \infty = O \left( h^r \max \{ \tilde{h}^d, h^{3r} \} \right).
\]
The required result follows from (5.24)-(5.30).

6 Numerical Results

For the sake of illustration, we quote the following results from Grammont et al [11].

Consider
\[
\varphi(s) - \int_0^1 \frac{ds}{s + t + \varphi(t)} = f(s), \quad 0 \leq s \leq 1,
\]
where \( f \) is so chosen that
\[
\varphi(t) = \frac{1}{t + c}, \quad c > 0,
\]
is a solution of (6.1).
We consider $X_n$ to be either the space of piecewise constant functions or piecewise linear functions with respect to the following uniform partition of $[0,1]$:
\[
0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1.
\]
(6.2)
The projection $Q_n$ is chosen to be the interpolatory projection at $r$ Gauss points with $r = 1$ or $r = 2$. Hence by Theorem 4.7 and Theorem 5.9
\[
\|z_n^M - \varphi\|_\infty = O(\max\{\tilde{h}^d, h^{3r}\}), \quad \|\tilde{z}_n^M - \varphi\|_\infty = O(h^r \max\{\tilde{h}^d, h^{3r}\}).
\]
(6.3)
If $X_n$ is the space of piecewise constant functions with respect to the partition (5.2), then we choose the composite Simpson rule with respect to the partition (6.2) to evaluate the integrals numerically. Then $\tilde{h} = h$ and $d = 4$. Thus,
\[
\|z_n^M - \varphi\|_\infty = O(h^3), \quad \|\tilde{z}_n^M - \varphi\|_\infty = O(h^4).
\]
(6.4)
If $X_n$ is the space of piecewise linear functions with respect to the partition (5.2), and the interpolation points are Gauss 2 points, then we choose the composite Gauss 2 point rule with respect to an uniform partition with $m = n^2$ subintervals as the approximate quadrature rule. Then, $\tilde{h} = h^2$ and $d = 4$. Hence
\[
\|z_n^M - \varphi\|_\infty = O(h^6), \quad \|\tilde{z}_n^M - \varphi\|_\infty = O(h^8).
\]
(6.5)
In the following tables $\delta_m$ and $\delta_{IM}$ denote the computed orders of convergence in the discrete modified projection method and the discrete iterated modified projection method, respectively.

Table 6.1

| $n$ | $\|\varphi - z_n^M\|_\infty$ | $\delta_M$ | $\|\varphi - \tilde{z}_n^M\|_\infty$ | $\delta_{IM}$ | $\|\varphi - z_n^M\|_\infty$ | $\delta_M$ | $\|\varphi - \tilde{z}_n^M\|_\infty$ | $\delta_{IM}$ |
|-----|------------------------------|------------|------------------------------|------------|------------------------------|------------|------------------------------|------------|
| 2   | $8.46 \times 10^{-4}$       | 2.38       | $10^{-5}$                   | 5.06       | $6.47 \times 10^{-5}$       | 5.06       | $6.47 \times 10^{-5}$       | 5.06       |
| 4   | $1.03 \times 10^{-4}$       | 3.04       | $1.37 \times 10^{-6}$      | 1.07       | $2.09 \times 10^{-7}$       | 8.27       | $2.09 \times 10^{-7}$       | 8.27       |
| 8   | $1.24 \times 10^{-5}$       | 3.05       | $8.18 \times 10^{-8}$      | 4.07       | $8.45 \times 10^{-10}$      | 7.95       | $8.45 \times 10^{-10}$      | 7.95       |
| 16  | $1.45 \times 10^{-6}$       | 3.09       | $4.99 \times 10^{-9}$      | 4.04       | $3.35 \times 10^{-12}$      | 7.98       | $3.35 \times 10^{-12}$      | 7.98       |
| 32  | $1.59 \times 10^{-7}$       | 3.19       | $3.08 \times 10^{-10}$     | 4.02       | $1.34 \times 10^{-14}$      | 7.96       | $1.34 \times 10^{-14}$      | 7.96       |
Table 6.2: \( \varphi(t) = \frac{1}{t+0.1} \)

| \( n \) | \( \| \varphi - z_n^M \|_\infty \) | \( \delta_M \) | \( \| \varphi - \tilde{z}_n^M \|_\infty \) | \( \delta_{IM} \) | \( \| \varphi - \hat{z}_n^M \|_\infty \) | \( \delta_{IM} \) |
|---|---|---|---|---|---|---|
| 2  | \( 3.64 \times 10^{-4} \) | 7.80 \times 10^{-6} | \( 9.39 \times 10^{-5} \) | 1.14 \times 10^{-4} | 9.97 \times 10^{-5} | 1.14 \times 10^{-4} |
| 4  | \( 6.29 \times 10^{-5} \) | 2.53 | \( 4.20 \times 10^{-7} \) | 4.21 | \( 1.19 \times 10^{-4} \) | -0.35 | \( 2.84 \times 10^{-7} \) | 8.65 |
| 8  | \( 6.85 \times 10^{-6} \) | 2.83 | \( 2.42 \times 10^{-8} \) | 4.12 | \( 3.30 \times 10^{-6} \) | 5.18 | \( 1.10 \times 10^{-9} \) | 8.01 |
| 16 | \( 1.12 \times 10^{-6} \) | 2.99 | \( 1.45 \times 10^{-9} \) | 4.06 | \( 4.99 \times 10^{-8} \) | 6.05 | \( 4.35 \times 10^{-12} \) | 7.99 |
| 32 | \( 1.27 \times 10^{-7} \) | 3.14 | \( 8.89 \times 10^{-11} \) | 4.03 | \( 7.00 \times 10^{-10} \) | 6.16 | \( 1.78 \times 10^{-14} \) | 7.93 |

It is seen that the computed orders of convergence match well with the expected orders of convergence in (6.4) and (6.5).

7 Conclusion

In this paper we consider approximate solutions of Urysohn integral equations with smooth kernels using discrete versions of the Modified Projection Method and of the Iterated Modified Projection Method associated with an interpolatory projection at \( r \) Gauss points. The interval \([a, b]\) is divided into \( m \) subintervals each of length \( \tilde{h} = \frac{b-a}{m} \) and a composite numerical quadrature with a degree of precision \( d \) is chosen to replace all the integrals. The range of the interpolatory projection at \( r \) Gauss points is a space of piecewise polynomials of degree \( \leq r - 1 \) with respect to a uniform partition of \([a, b]\) with \( n \) subintervals each of length \( h = \frac{b-a}{n} \). The exact solution is denoted by \( \varphi \) and the approximate solutions obtained by using the Discrete Modified Projection Method and the Discrete Iterated Modified Projection Method are denoted respectively by \( z_n^M \) and \( \tilde{z}_n^M \). We choose \( m = p \) \( n \) where \( p \in \mathbb{N} \). The following orders of convergence are proved:

\[
\| z_n^M - \varphi \|_\infty = O(\max\{\tilde{h}^d, h^{3r}\}), \quad \| \tilde{z}_n^M - \varphi \|_\infty = O(h^r \max\{\tilde{h}^d, h^{3r}\}).
\]

Since the errors in the Modified Projection Method and the Iterated Modified Projection Method are respectively of the order of \( h^{3r} \) and \( h^d \), these orders of convergence are preserved if the numerical quadrature formula is so chosen that \( \tilde{h}^d = h^{3r} \). Note that we have at our disposal \( \tilde{h} \), that is \( m \), and \( d \) to achieve this equality.
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