QUERY COMPLEXITY AND THE POLYNOMIAL FREIMAN–RUZSA CONJECTURE

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ABSTRACT. We prove a query complexity variant of the weak polynomial Freiman–Ruzsa conjecture in the following form. For any $\epsilon > 0$, a set $A \subset \mathbb{Z}^d$ with doubling $K$ has a subset of size at least $K - \frac{4}{\epsilon} |A|$ with coordinate query complexity at most $\epsilon \log_2 |A|$.

We apply this structural result to give a simple proof of the “few products, many sums” phenomenon for integer sets. The resulting bounds are explicit and improve on the seminal result of Bourgain and Chang.

NOTATION AND PRELIMINARIES

The following notation is used throughout the paper. The expressions $X \gg Y$, $Y \ll X$, $Y = O(X)$, $X = \Omega(Y)$ all have the same meaning that there is an absolute constant $c$ such that $Y \leq cX$. Further, $X \gg Y$ means that there is a function $c(\cdot)$ such that $Y \leq c(\epsilon)X$, and the same convention applies for the $\gg, O, \Omega$-notation.

$X \geq Y^{c-o(1)}$ means that $X \gg Y^{c-\epsilon}$ for any $\epsilon > 0$.

If $X$ is a set then $|X|$ denotes its cardinality.

Let $G$ be an additive torsion-free group and $A, B \subset G$. For concreteness, we will assume henceforth that $G = \mathbb{Z}^d$ for some unspecified dimension $d$. We will also assume that $G$ is embedded into an ambient vector space $\mathbb{Q}^d$, making no distinction between $G$ and the embedding. In particular, we fix a standard basis $\{\vec{e}_1, \ldots, \vec{e}_d\}$ and define coordinate projections $\pi_i : G \to \mathbb{Z}$ by

$$\pi_i(n_1\vec{e}_1 + \cdots + n_d\vec{e}_d) = n_i.$$ 

The sumset $A + B$ is defined as the set of all pairwise sums

$$A + B := \{a + b : a \in A, b \in B\}.$$ 

The set of pairwise products (or the product set) $AB$ is defined mutatis mutandis with.

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The $\lambda_k$ constant of an integer set $A$ is defined as

$$\lambda_k(A) := \max \left\{ \left\| \sum_{n \in A} c_n e^{2\pi i n x} \right\|_{L_{2k}([0,1])}^2 \right\},$$

where max is taken over positive weights $\{c_n\}_{n \in A}$ with $\sum_n c_n^2 = 1$. It is related to a more commonly used notion of the additive energy or order $k$, defined as

$$E_k(A) := \left\| \sum_{n \in A} e^{2\pi i n x} \right\|_{L_{2k}([0,1])}^{2k}.$$ 

Note, the the sum above simply counts the number of $2k$-tuples $(a_1, \ldots, a_{2k}) \in A^{2k}$ such that $a_1 + \ldots + a_k = a_{k+1} + \ldots + a_{2k}$. By the Cauchy-Schwartz inequality, one immediately arrives at

$$\left| A + \ldots + A \right|_{k \text{ times}} \geq |A|^{2k} \frac{E_k(A)}{E_k(A)}.$$ 

More information on additive energies can be found in the book [13].

Taking $c_n = 1/\sqrt{|A|}$ in the definition of $\lambda_k(A)$, one arrives at the estimate

$$E_k^{1/k}(A) \leq |A|\lambda_k(A),$$

and indeed one can bound the additive energy of any $A' \subset A$ in a similar way.

It will be convenient to use the standard shortcut $e(x) := e^{2\pi i x}$.

1. Introduction

1.1. Weak polynomial Freiman–Ruzsa conjecture. One of the main research avenues of additive combinatorics is to extract structural information about sets with small doubling $K$ defined as

$$K := \frac{|A + A|}{|A|}.$$ 

A fundamental result, known as Freiman’s lemma [13], asserts that $A$ is always contained in an affine subspace of dimension at most $O(K)$.

A very rigid structure can be deduced when $K = o(\log |A|)$ using the much harder quantitative Freiman Theorem, see [12] for the state-of-the-art bounds and background.

However, very little is known in the regime $\log K \gg \log |A|$ and one of the central problems in the area is to close this gap (i.e. to obtain polynomial bounds). The Polynomial Freiman–Ruzsa conjecture predicts, informally, that there is a subset $A' \subset A$ of size at least $K^{-O(1)}|A|$, such that $A'$ (after a suitable transformation) is contained in a convex body of dimension $O(\log K)$ and volume $K^{O(1)}|A|$.

It turns out that for many applications (see [4]) it would suffice that the weaker form below holds true.
**Conjecture 1** (Weak Polynomial Freiman–Ruzsa Conjecture). *For any set $A$ with doubling $K$ there is a subset $A'$ of size $K^{-O(1)}|A|$ contained in an affine subspace of dimension $O(\log K)$.*

We make a step towards Conjecture 1 by replacing the rank condition with a weaker property of logarithmic coordinate query complexity. The definition we use is different from the one commonly used in computer science as we assume that a single query outputs an integer number rather than a $\{0,1\}$ bit. It is defined as follows.

Assume Alice and Bob agree on some large set $X \subset \mathbb{Z}^d$. Next, Alice chooses an element $x \in X$ and keeps it in secret. Bob tries to guess $x$ by probing the value of $\pi_i(x)$ for some $i$, one coordinate at a time. The coordinate query complexity of $X$ is then the maximal number of coordinate queries Bob should perform in order to recover $x$ in the worst case.

**Theorem 1.1** (Query-complexity PFR). *For any $\epsilon > 0$ the following holds. For any set $A \subset \mathbb{Z}^d$ with $|A + A| \leq K|A|$ there is a subset of size at least $K^{-2\epsilon}|A|$ with coordinate query complexity at most $\epsilon \log_2 |A|$.*

Note that Conjecture 1 would immediately imply Theorem 1.1. Indeed, assume $A' \subset A$ is contained in an affine subspace $V$ of dimension $s = O(\log K)$. Then it follows from basic linear algebra that there are $s$ coordinates $i_1, \ldots, i_s$ such that the map $v \mapsto (\pi_{i_1}(v), \ldots, \pi_{i_s}(v))$ is injective on $V$. Thus, Bob can recover any $a \in A'$ by probing at most $s$ coordinates.

At the same time Theorem 1.1 is, to our knowledge, the first result sensitive enough to detect a large structured piece inside a set $A$ with $\log K \gg \log |A|$.

### 1.2. Few products, many sums.

We apply Theorem 1.1 in the second part of the paper in order to give improved bounds for the “few products, many sums” phenomenon for integer sets, sometimes called the weak Erdős–Szemerédi conjecture. The state-of-the art bounds, due to Bourgain and Chang [1], were obtained using a *tour de force* induction on scales argument and are rather inefficient.

The sum-product problem is concerned with showing that either the set of sums or the set of products is always large. It was conjectured by Erdős and Szemerédi [6] that, for all $\epsilon > 0$ and any finite $A \subset \mathbb{Z}$,

$$\max\{|A + A|, |AA|\} \geq c(\epsilon)|A|^{2-\epsilon}$$  \hspace{1cm} (2)

where $c(\epsilon) > 0$ is an absolute constant. The same conjecture can also be made over the reals, and indeed other fields. The Erdős–Szemerédi conjecture remains open, and it appears to be a deep problem. Konyagin and Shkredov [8] proved that (2) holds with $\epsilon < 2/3$, and the current best bound, due to Rudnev and Stevens [11], has $\epsilon \leq 2/3 - 2/1167 + o(1)$. These bounds hold over real numbers, and their proofs are geometric in nature.
It turns out that geometric arguments are only efficient when \(|A + A|\) is small. Elekes and Ruzsa [5] proved that for any set \(A\) of real numbers
\[ |AA||A + A|^4 \gg |A|^{6-o(1)}, \]
thus confirming the Erdős–Szemerédi conjecture in the regime \(|A + A| \ll |A|^{1+o(1)}\). This particular case is known as the “few sums, many products” phenomenon.

Surprisingly enough, the dual “few products, many sums” case of the Erdős–Szemerédi conjecture remains open for sets of real numbers and is sometimes dubbed as the weak Erdős–Szemerédi conjecture. The best bound for real numbers is due to Murphy et al. [10], who proved that if \(|AA| \ll |A|^{1+o(1)}\) then
\[ |A + A| \gg |A|^{8/5-o(1)}. \]

The weak Erdős–Szemerédi conjecture has been resolved by Bourgain and Chang [1] for integer sets. They proved that, for any \(\epsilon > 0\) there is \(C(\epsilon)\) such that for any integer set \(A\)
\[ |A + A| \gg K^{C(\epsilon)}|A|^{2-\epsilon}, \]
with \(K = |AA|/|A|\). In other words, writing \(K = |A|^\delta\),
\[ |A + A| \gg |A|^{2-\epsilon(\delta)} \]
with \(\epsilon(\delta) \to 0\) as \(\delta \to 0\).

It was also proved in [1] that
\[ \max\{|\underbrace{A + \ldots + A}_{k \text{ times}}|, |\underbrace{A \ldots A}_{k \text{ times}}|\} \geq |A|^{b(k)} \]
with \(b(k) \gg \log^{1/4} k\).

The dependence \(C(\epsilon)\) in (3) is rather poor since the argument in [1] relies on an intricate induction on scales device. Theorem 1.1 applied to the prime valuation image of \(A\) allows one to bypass such complications since it is agnostic with regards to the dimension of the ambient space.

**Theorem 1.2** (Few products, many sums). For any \(1 > \epsilon > 0\) the following holds. Let \(A \subset \mathbb{Z}\) and
\[ K_* := \frac{|AA|}{|A|}. \]
Then
\[ |kA| \gg_k |A|^{k-2\epsilon k \log_2 k} K_*^{2k/\epsilon}. \]

**Remark 1.1.** Theorem 1.2 can be extended to sets \(A\) of algebraic numbers of degree \(O(\log |A|)\), applying almost verbatim the ideas of [2]. However, in order to resolve the weak Erdős–Szemerédi conjecture for sets of real (or complex) numbers one has to rule out the case when \(A\) consists of units in a number field of very large degree (cf. Proposition 10 of [2]). For example, a resolution of Conjecture 2.9 (“Log-span conjecture”) of [9] would provide such a tool.
To formulate Theorem 1.2 in its most general form we need to introduce
the notion of additive (or multiplicative) tripling introduced in [9].
Let \((G, +)\) be an abelian group. For a set \(U \subset G\) define
\[
\beta(U) := \inf_{A_1, A_2} \frac{|A_1 + A_2 + U|}{|A_1|^{1/2}|A_2|^{1/2}}.
\]

The following Lemma, proved in Statement 3.3 of [9], connects \(\beta\) with the
usual notion of additive doubling.

**Lemma 1.1** (\(\beta\) bounds additive doubling, [9]). Let \(A\) be an additive set and
write \(K_+ := |A + A|/|A|\). Then for any \(U \subset A\) holds
\[
\beta(U) \leq K_+^2. \ 
\tag{5}
\]

We invite the reader to consult [9] for a detailed treatment of \(\beta\) and related
quantities under the umbrella term “induced doubling”.

In what follows we are going to use \(\beta\) mostly with respect to multiplication,
and so to avoid confusion will write \(\beta^*\) in such cases.

**Theorem 1.3** (Few products, many sums for \(\beta^*\) and \(\lambda_k\)). For any \(1 > \epsilon > 0\)
the following holds. Let \(A \subset \mathbb{Z}\). Then
\[
\lambda_k(A) \leq 10 \beta^* |A|^{2 \epsilon \log_2 k},
\]
where
\[
\beta^*(A) := \inf_{B, C \subset \mathbb{Z}} \frac{|ABC|}{|B|^{1/2}|C|^{1/2}}. \ 
\tag{6}
\]

A corollary of Theorem 1.3 is the following \(k\)-fold sum-product estimate,
which improves on the state of the art bound in [1].

**Theorem 1.4** (Iterated sum-product). For all \(k \in \mathbb{N}, k > 2\) and \(A \subset \mathbb{Z}\) the
following holds. Let
\[
\delta := \log \beta^*(A) \log |A| \log \log |A|/\log k
\]
with \(\beta^*\) defined by (6). Then assuming \(|A|\) is large enough
\[
|kA| = |A + \ldots + A| \geq |A|^{k-10k\sqrt{\delta \log_2 k}}
\]
and
\[
|A^{(k)}| = |A \ldots A| \geq |A|^\delta \log_2 k.
\]

In particular, there is an absolute \(c > 0\) (\(c = 10^{-4}\) would do), such that
either
\[
|A^{(k)}| \geq |A|^b(k)
\]
or
\[
|kA| \geq |A|^b(k),
\]
with \(b(k) = \frac{\epsilon \log_2 k}{\log_2 \log_2 k}^\epsilon\).
This improves on [1] where a similar bound with $b(k)$ of order $\log^{1/4} k$ was obtained.

The growth of $b(k)$ is essentially the best one can hope for, as shown by the following example. The authors are indebted to an anonymous referee who brought this example to our attention.

**Proposition 1.5.** There is an absolute constant $C$ with the following property. Let $k \in \mathbb{N}$. Then there is a set $A \subset \mathbb{Z}$ such that $|kA| + |A^{(k)}| \leq |A|^C \log k / \log \log k$.

**Proof.** It is essentially due to Erdős and Szemerédi [6].

In what follows, $c_1, c_2, \ldots$ are absolute constants which we do not bother to specify explicitly. Assume $k$ is large and set

$$A := \{ \prod_{i} p_{i}^{e_{i}} : e_{i} \leq (\log k)^{1/2} \},$$

where the product is over the primes $p_{i}$ of size at most $(\log k)^{1/2}$. We have

$$e^{c_{1}(\log k)^{1/2}} \leq |A| \leq (\log k)^{\frac{1}{2} \pi((\log k)^{1/2})}.$$ 

Also,

$$\max A \leq (\prod_{i} p_{i})^{(\log k)^{1/2}} \leq e^{2(\log^{1/2} k \pi((\log k)^{1/2}) \log \log k} \leq k^{c_{2}},$$

and therefore

$$|kA| \leq k^{c_{2}+1} \leq |A|^{c_{3}(\log k)^{1/2}}.$$ 

On the other hand

$$A^{(k)} := \{ \prod_{i} p_{i}^{e_{i}} : e_{i} \leq k(\log k)^{1/2} \},$$

and so

$$|A^{(k)}| \leq (k(\log k))^{\frac{1}{2} \pi((\log k)^{1/2})} \leq e^{c_{4}(\log k)^{3/2} \log \log k} \leq |A|^{c_{5}(\log k)^{1/2} \log \log k}.$$ 

\[ \square \]

2. Proof of Theorem 1.4

Before moving forward, let’s deduce Theorem 1.4 assuming Theorem 1.3 holds true.

Write $\beta_{*}(A) = |A|^{\beta_{*}}$. First, let’s record the following one liner:

**Claim 2.1 (Iterated $\beta_{*}$).** For any integer $t > 1$

$$|A^{(2t-1)}| \geq \beta_{*}^{t}.$$ 

**Proof.** For $t = 2$ the claim follows from the definition of $\beta_{*}$. For $t > 2$ one has by induction

$$|A^{(2t-1)}| = |A^{2(t-1)-1}A^{2(t-1)-1}A| \geq \beta_{*}|A^{2(t-1)-1}| \geq \beta_{*}^{t}.$$ 

\[ \square \]
At the expense of decreasing the constants in $b(k)$, from now on let’s assume that $k = 2^t$ for some integer $t$. Henceforth $c > 0$ is assumed to be a small fixed constant to be defined in due course.

Then if $\delta \geq \frac{c}{\log 2^t}$, by Claim 2.1,

$$|A^{(k)}| = |A|^{2^t} \geq \beta_* |A|^t \geq |A|^c \log 2^t \geq |A|^{b(k)}.$$ 

Otherwise, from the definition of $\lambda_l$ with all the weights equal to $|A| - 1/2$ and the Cauchy-Schwartz inequality,

$$|lA| \geq |A|^l \lambda_l^l(A).$$

Applying Theorem 1.3 with $\beta_* (A) = |A|^\delta$ one has for any $\epsilon > 0$ and $l > 0$

$$|lA| \geq 10^{-l} |A|^l |A|^{-\delta l / \epsilon - 2c \log l}.$$ 

Taking $\epsilon = (\delta \log 2 l)^{-1/2}$ and assuming $|A|$ is large enough, we get

$$|lA| \geq |A|^{l - 10l \sqrt{\delta \log 2 l}} \geq |A|^{l - 10l \sqrt{\delta \log 2 l / \log 2 l}}.$$ 

Recalling that $k = 2^t$ we further estimate very crudely with $l = t$

$$|lA| \geq |tA| \geq |A|^{l - 10t c^{1/2}} \geq |A|^{t/2} \geq |A|^{b(k)}.$$ 

3. Background on $\beta$ and quasi-cubes

A quasi-cube is a generalization of the binary cube $\{0, 1\}^d$ and is defined recursively as follows.

**Definition 3.1 (Quasicubes).** We say that a set $H \subset \mathbb{Z}^d$ is a quasi-cube if there is a coordinate projection $\pi_i$ such that $|\pi_i(H)| = 2$ and either

1. $\pi_i$ is injective
2. $\pi_i(H) = \{x, y\}$ and both $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are quasi-cubes.

The following theorem was proved in [9], and a short self-contained proof can be found in [7].

**Theorem 3.2 (Subsets of quasi-cubes have large $\beta$, [9]).** Let $H$ be an arbitrary quasi-cube. Then for any $U \subset H$ holds

$$\beta(U) = |U|.$$ 

The power of Theorem 3.2 is that the estimate (7) depends neither on the dimension of the ambient space nor on the density of $U$ in $H$.

4. Branching depth and binary subtrees

Let $T$ be a rooted tree. We will write $L(T)$ for the set of leaves. We further define the following quantities.
Definition 4.1 (Branching depth). Let \( r \) be the root of \( T \) and write \( P(l \rightarrow r) \) for the set of vertices on the (unique) path from \( l \) to \( r \). Let \( d_l \) be the number of vertices in \( P(l \rightarrow r) \) with at least two children. Then the branch-depth of \( T \) is defined as

\[
d(T) := \max_{l \in L(T)} d_l.
\]

Definition 4.2 (Largest binary subtree). Let us call a rooted tree binary if each node has at most two children. Further, for a tree \( T \) write

\[
b(T) = \max_{\text{binary} \ T' \subset T} |L(T')|.
\]

The strategy for the rest of the argument is to prove that for any tree \( T \) either there is a large subtree \( T' \) with \( d(T') = \omega(\log |L(T)|) \) or \( \log b(T') \gg \log |L(T)| \).

Let \( T \) be a tree with \( N := |L(T)| \). Fix \( \epsilon > 0 \).

Definition 4.3 (Largest \( \epsilon \)-low subtree). Let us call a rooted tree \( T' \) \( \epsilon \)-low if \( d(T') \leq \epsilon \log_2 N \). Further, for a tree \( T \) write

\[
D_\epsilon(T) = \max_{\epsilon\text{-low} \ T' \subset T} |L(T')|.
\]

Lemma 4.1 (Low vs binary subtree alternative). For any tree \( T \) and \( 1 \geq \epsilon > 0 \)

\[
D_\epsilon(T)b^{1/\epsilon}(T) \geq |L(T)|.
\]

Proof. The proof is by induction on the height of \( T \). Write \( N := |L(T)| \). For a single root or a root with a single child the inequality is trivial. For a tree of height 1 with at least two children we have \( b(T) = 2 \). If \( \epsilon \log_2 N \geq 1 \), then the whole tree is \( \epsilon \)-low and we are done. Otherwise,

\[
b^{1/\epsilon}(T) \geq N.
\]

Now assume the height is larger. Without loss of generality we may assume that the root has at least two children. Let \( T_i \) be the subtrees rooted at the children, write \( D_i := D_\epsilon(T_i), \ N_i := |L(T_i)|, \ b_i := b(T_i) \) and \( b := b(T) \).

Call \( T_i \) small if \( N_i \leq 2^{-1/\epsilon} N \), otherwise call it big. Denote the families of these subtrees by \( S \) and \( B \), respectively.

Claim. If there are no big subtrees, i.e., \( \sum_{T_i \in B} N_i = 0 \), we are done. Indeed, in this case the branching depth of the tree constructed by attaching the maximal \( \epsilon \)-low trees in \( T_i \) to the root of \( T \) is at most

\[
\epsilon \max_i \log_2 N_i + 1 \leq \epsilon \log_2 N,
\]

so

\[
D_\epsilon(T) \geq \sum_i D_\epsilon(T_i) \geq \frac{\sum_i N_i}{\max_i b_i^{1/\epsilon}} = \frac{N}{\max_i b_i^{1/\epsilon}}.
\]

But clearly \( b^{1/\epsilon}(T) \geq \max_i b_i^{1/\epsilon} \) and the induction is closed.
Claim. If there are at least two big subtrees, we are also done. Indeed, let $T_i, T_j \in B$. Without loss of generality, $b_j \geq b_i$, so

$$b(T) \geq b_i + b_j \geq 2b_i.$$ 

Clearly, $D_\epsilon(T) \geq D_\epsilon(T_i)$ and thus

$$D_\epsilon(T)b^{1/\epsilon}(T) \geq D_\epsilon(T_i)2^{1/\epsilon}b_i^{1/\epsilon} \geq 2^{1/\epsilon}N_i \geq N.$$

So the remaining case is when $T_1$ is a single big subtree, so $N_1 > 2^{-1/\epsilon}N$. Write

$$b_M = \max \{b_i : T_i \text{ is small} \}.$$ 

Let $c > 0$ be such that $b_M = cb_1$. We have $b(T) \geq (1 + c)b_1 = (1 + \frac{1}{c})b_M$. Since $D_\epsilon(T) \geq D_1$, we are done unless

$$((1 + c)b_1)^{1/\epsilon}D_1 \leq N.$$ 

By induction, the left hand side is at least $(1 + c)^{1/\epsilon}N_1$, so it must be

$$(1 + c)^{1/\epsilon} \leq \frac{N}{N_1} < 2^{1/\epsilon}. \tag{9}$$

In particular, $c < 1$ and $b_M < b_1$.

Since attaching to the root of $T$ increases the branch-depth by at most one, it follows similarly to (8) that

$$D_\epsilon(T) \geq \sum_{T_i \in S} D_\epsilon(T_i) \geq \sum_{T_i \in S} \frac{N_i}{b_M^{1/\epsilon}}.$$ 

Thus,

$$D_\epsilon(T)b^{1/\epsilon}(T) \geq \frac{\sum_{T_i \in S} N_i}{b_M^{1/\epsilon}} b^{1/\epsilon}(T) \geq (1 + 1/c)^{1/\epsilon}(N - N_1).$$ 

We are done if the right hand side is at least $N$, that is if

$$(1 + 1/c)^{1/\epsilon}(N - N_1) \geq N.$$ 

Dividing by $N_1$, this is equivalent to

$$(1 + 1/c)^{1/\epsilon}(N/N_1 - 1) \geq \frac{N}{N_1}.$$ 

Rearranging for $N/N_1$ gives that we need

$$\frac{N}{N_1} \geq \frac{(1 + 1/c)^{1/\epsilon}}{(1 + 1/c)^{1/\epsilon} - 1}.$$ 

Using the left inequality from (9), it is sufficient to prove

$$(1 + c)^{1/\epsilon} \geq \frac{(1 + 1/c)^{1/\epsilon}}{(1 + 1/c)^{1/\epsilon} - 1} = 1 + \frac{1}{(1 + 1/c)^{1/\epsilon} - 1}.$$ 

The left hand side is decreasing in $\epsilon$ and the right hand side is increasing in $\epsilon$. Since they are equal for $\epsilon = 1$, the inequality holds for all $\epsilon \leq 1$. \qed
5. Proof of Theorem 1.1

The initial step to prove Theorem 1.1 is to transform the set $A$ in question into a rooted tree $T(A)$. We build $T(A)$ recursively.

Let $1 \leq i \leq d$ be some coordinate index and $\pi_i$ be the corresponding coordinate projection. Any set $X$ fibers with respect to $\pi_i$ in the sense that

$$X = \bigcup_{y \in \pi_i(X)} \pi_i^{-1}(y).$$

We call the disjoint sets $X_y := \pi_i^{-1}(y)$ fibers of $X$ above $y$.

Now let’s get back to the construction of $T(A)$. It has a root $v_0$, and it is the only node of $T(A)$ is $A$ is a singleton. If not, let $j$ be the minimal coordinate index such that $|\pi_j(A)| > 1$. We recursively attach to $v_0$ the trees $T(A_x)$ for each fiber $A_x$, $x \in \pi_j(A)$ induced the projection $\pi_j$. The root of $T(A_x)$ is labelled with the pair $(j, \pi_j(A_x))$. The process will terminate since the coordinate index always increases.

Since the process terminates when the fiber becomes a singleton set, the elements of $A$ are in one-to-one correspondence with the leafs of $T(A)$ endowed with the labels.

Now everything is set up for the proof of Theorem 1.1.

Proof. Let $\epsilon > 0$ be fixed and $A \subset G$ be a set with $K := |A + A|/|A|$.

Let $T_A$ be the rooted tree corresponding to $A$ and $T_B$ be a (one of possibly many) largest binary subtree of $T_A$.

Claim. We claim that the set $B \subset A$ which corresponds to the leaves $L(T_B)$ as described above is contained in a quasicube. The claim follows from a simple induction on the height of the tree $T_B$. Indeed, a single root or a binary tree of height one is clearly a quasicube subset. Otherwise, the root has either one or two children, and in both cases the claim follows from the definition of a quasicube.

By the hypothesis of the theorem and (5),

$$K \geq \beta^{1/2}(B) = |B|^{1/2}.$$  

It therefore follows that

$$b(T_A) = |L(T_B)| = |B| \leq K^2.$$  

We immediately conclude by Lemma 4.1 that

$$D_\epsilon(T_A) \geq b^{-1/\epsilon}(T_A)|L(T_A)| \geq K^{-\frac{2}{\epsilon}}|A|. \quad (10)$$

Thus, by definition, there is a subtree $T' \subset T_A$ with branching depth at most $\epsilon \log |A|$ and size at least $K^{-\frac{2}{\epsilon}}|A|$.

Let $A' \subset A$ be the subset corresponding to the leaves $L(T')$. In order to conclude the proof it remains to note the coordinate query complexity of $A'$ is at most the depth of $T'$. Let $x \in A'$. For any $j$ the $j$-coordinate query returns the value $\pi_j(x)$, which uniquely identifies the $\pi_j$-fiber of $x$. Thus, we can traverse $T(A')$ from the root to the unique leaf corresponding to $x$ each
time taking the branch corresponding to the coordinate query. The number of queries is going to be at most the depth of $T(A')$, and we are done by (10).

6. Proof of Theorem 1.2 and Theorem 1.3

6.1. Prime valuation mapping. Let $A$ be a set of integers, the goal is essentially to prove that either $\beta_+(A)$ (that is, $\beta(A)$ with respect to multiplication) is large or $E_+(A)$ is small.

The first step is to transform $A$ into a multidimensional set using the prime valuation map which is as follows. Let \{\(p_1, \ldots, p_D\)\} be the set of prime divisors of the elements in $A$. We consider the valuation map $\Pi : \mathbb{Z} \to \mathbb{Z}^D$:

$$\Pi(a) = (v_{p_1}(a), \ldots, v_{p_D}(a))$$

where $v_{p_i}(a)$ is the maximal power $\alpha$ such that $p_i^\alpha$ divides $a$.

Clearly for integer sets $\Pi(X) + \Pi(Y) = \Pi(XY)$ so $\beta_+(\Pi(A)) = \beta_+(\Pi(A)) = \beta_+(\Pi(A))$. 

Since $\Pi$ is one-to-one from now on we identify any $A$ with $A := \Pi(A) \subset \mathbb{Z}^D$. The convention is that calligraphic letters live in $\mathbb{Z}^D$ and capital italic live in $\mathbb{Z}$. We also follow that convention that $\pi_i$ is the one-dimensional projection in $\mathbb{Z}^D$ to the coordinate corresponding to the prime $p_i$.

6.2. Chang’s argument. Let’s recall Proposition 6 of [3].

**Proposition 6.1.** Let $p$ be a fixed prime, and let $F_j(x) \in \left\{ e^{2\pi ip^j nx} | (n, p) = 1 \right\}^+$. Then

$$\left( \| \sum_j F_j \|_{2k} \right)^2 \leq \left( \frac{2k}{2} \right) \sum_j \| F_j(x) \|_{2k}^2.$$ 

**Proof.** Expanding the brackets, one can write

$$\| \sum_j F_j \|_{2k}^2 = \int_0^1 | \sum_j F_j(x) |^{2k} dx$$

as a sum of terms of the form

$$\int_0^1 \prod_{j=1}^{2k} F_{j_1}(x) \ldots F_{j_k}(x) \overline{F_{j_k+1}(x)} \ldots \overline{F_{2k}(x)} dx.$$ 

(11)

We claim that if $\{j_i\}_{i=1}^{2k}$ are all distinct, the integral above is zero. Indeed, if this is the case, the integral above can be further broken down into a sum of integrals

$$C_{j_1, \ldots, j_{2k}} \int_0^1 e(p^{j_1} n_{j_1} + \ldots + p^{j_k} n_{j_k} - p^{j_{k+1}} n_{j_{k+1}} - \ldots - p^{j_{2k}} n_{j_{2k}}) dx.$$
Since \((n, p) = 1\), the expression in the brackets is non-zero, as the negative and the positive part are divisible by unequal powers of \(p\). Thus, the integral evaluates to zero.

It follows that only the terms with at least two equal indices survive. Let \(j = j_{i_1} = j_{i_2}\). There are three cases: \(i_1, i_2 \leq k, i_1 \leq k < i_2\) and \(k < i_1, i_2\).

For the terms of the first type, one can apply the Hölder inequality and estimate

\[
\left( \frac{k}{2} \right) \sum_j \int_0^1 F_j^2 \sum_{j_2, \ldots, j_{2k}} F_{j_2} F_{j_3} \ldots F_{j_k} \overline{f_{j_{k+1}}} \ldots \overline{f_{j_{2k}}} \, dx \leq \\
\left( \frac{k}{2} \right) \sum_j \int_0^1 \left| F_j \right|^2 \left| \sum_i F_i \right|^{2(k-1)} \, dx \leq \\
\left( \frac{k}{2} \right) \sum_j \left\| F_j \right\|_{2k}^2 \left( \sum_i F_i \right)^{2k-2}
\]

The remaining two cases can be treated similarly, arriving at the estimate

\[
\left\| \sum_j F_j \right\|_{2k}^2 \leq \left( k^2 + 2 \left( \frac{k}{2} \right) \right) \sum_j \left\| F_j \right\|_{2k}^2 \left( \sum_i F_i \right)^{2k-2} = \left( \frac{2k}{2} \right) \sum_j \left\| F_j \right\|_{2k}^2 \left( \sum_i F_i \right)^{2k-2}.
\]

The claim follows. \(\square\)

**Lemma 6.1.** Let \(A \subset \mathbb{Z}\). Then

\[
\lambda_k(A) \leq \left( \frac{2k}{2} \right)^{q(A)},
\]

where \(q(A)\) is the coordinate query complexity of \(A := \Pi(A)\).

**Proof.** Let \(\{w_a\}_{a \in A}\) be arbitrary positive weights with \(\sum_{a \in A} w_a^2 = 1\). Our goal is to show that

\[
\left\| \sum_{a \in A} w_a e(ax) \right\|_{2k}^2 \leq \left( \frac{2k}{2} \right)^{q(A)}.
\]

The strategy is to iteratively apply Proposition 6.1 with a suitable choice of the prime factor \(p\) until it is fully reduced to a sum of trivial terms \(C_a \|w_a e(a)\|_{2k}^2\) with some multiplicative factors \(C_a\). If then

\[
C_a \leq \left( \frac{2k}{2} \right)^{q(A)}
\]
for all $a$, we are done. Indeed, then one can simply write

$$\left\| \sum_{a \in A} w_a e(ax) \right\|_{2k}^2 \leq \binom{2k}{2}^q(A) \sum_a \left\| w_a e(ax) \right\|_{2k}^2 = \binom{2k}{2}^q(A).$$

It remains to show how to perform the reduction in such a way. Let $T(A)$ be the tree defined in Section 5 so that its depth is bounded by $q(A)$. Let $p_0$ be the prime assigned to the root of the tree. By the one-to-one correspondence between the branches of $T(A)$ and the subsets of $A$ we can decompose $A$ into a disjoint union

$$A = \bigcup_j p_0^j A_j$$

so that the elements of $A_j$ are comprime with $p_0$. Then one can apply Proposition 6.1 with $p = p_0$ and

$$F_j(x) := \sum_{a \in A_j} w_{p_0^j a} e(p_0^j ax).$$

It follows that

$$\left\| \sum_{a \in A} w_a e(ax) \right\|_{2k}^2 \leq \binom{2k}{2} \sum_j \left\| F_j \right\|_{2k}^2.$$

Now we apply the same reduction for each term $\| F_j \|_{2k}^2$ and the subtree of $T(A_j)$. However, by construction of $T(A)$, the depth of each subtree $T(A_j)$ is now at most $q(A) - 1$. Thus, by induction, it follows that the exponential sum will be fully reduced down to monomials with the multiplicative factors at most $\binom{2k}{2}^q(A)$.

\[\square\]

6.3. Concluding the proof. We will need the following standard estimate.

Lemma 6.2. Assume

$$A = \bigcup_i A_i.$$ 

Then

$$\lambda_k(A) \leq \sum_i \lambda_k(A_i).$$

Proof. Without loss of generality we assume that $A_i$ are disjoint. Let $w_a, a \in A$ be positive weights with $\sum_a w_a^2 = 1$. By the triangle inequality

$$\left\| \sum_{a \in A} w_a e(ax) \right\|_{2k} \leq \sum_i \left\| \sum_{a \in A_i} w_a e(ax) \right\|_{2k} \leq \sum_i \lambda^{1/2}_k(A_i) \left( \sum_{a \in A_i} w_a^2 \right)^{1/2}.$$

The claim follows by applying Cauchy-Schwarz to the last inequality and squaring both sides. \[\square\]
Now everything is set up for the proof of Theorem 1.3.
Let \( b := \beta_s(A) \) and \( \epsilon > 0 \) be fixed for the rest of the proof.

**Claim 6.2.** Assume \( \beta_s(A) \leq b \). Then there is a subset \( A' \subset A \) such that \( |A'| \geq |A|/2 \) and

\[
\lambda_k(A') \leq 2b^{1/\epsilon} \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A|}.
\]

**Proof.** Let \( T := T(A) \) be the tree defined in Section 5. By Theorem 3.2 the tree is free of binary subtrees of size \( b \). If \( |A|/2 \leq b^{1/\epsilon} \) then claim is trivially true for any \( A' \subset A \) of size \( |A|/2 \), so assume the opposite.

By Lemma 4.1, there’s a substree \( T' \subset T \) of size at least \( b^{-1/\epsilon} |A| \) and depth at most \( \epsilon \log_2 |A| \).

Let \( A'_0 \) be the subset of \( A \) corresponding to \( T' \). We have \( q(A') \leq \epsilon \log_2 |A| \) and so, by Lemma 6.1,

\[
\lambda_k(A'_0) \leq \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A|}.
\]

Let \( A_0 := A \setminus A'_0 \). If \( |A_0| \leq |A|/2 \), stop. If not, repeat the step above and find a set \( A'_1 \subset A_0 \) of size at least \( b^{-1/\epsilon} |A|/2 \) and the tree depth at most \( \epsilon \log_2 |A| \). We have

\[
\lambda_k(A'_1) \leq \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A|}.
\]

Reiterating, define \( i \) to be the first index such that \( |A_i| \leq |A|/2 \). Clearly \( i \leq b^{1/\epsilon} \) since \( |A'_j| \geq b^{-1/\epsilon} |A|/2 \) for \( 0 \leq j < i \). Put

\[
A' := \bigcup_{j=0}^{i} A'_j,
\]

and estimate

\[
\lambda_k(A') \leq \sum_j \lambda_k(A'_j) \leq (b^{1/\epsilon} + 1) \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A|} \leq 2b^{1/\epsilon} \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A|}.
\]

As \( |A'| > |A|/2 \) by the choice of \( i \), the claim follows. \( \square \)

Now everything is set up for the proof of Theorem 1.3.

**Proof.** Let \( A'_0 \) be the output of Claim 6.2. Put \( A_0 := A \setminus A'_0 \) and apply the claim again to \( A_0 \). Note that the hypothesis of Claim 6.2 still holds true as \( \beta(A_0) \leq \beta_s(A) = b \). We obtain \( A'_1 \) such that \( |A'_1| \geq |A'_0|/2 \) and

\[
\lambda_k(A'_1) \leq 2b^{1/\epsilon} \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A'_0|} \leq 2b^{1/\epsilon} \left( \frac{2k}{2} \right)^{\epsilon \log_2 |A| - 1}.
\]
Then reiterate with $A_1 := A_0 \setminus A'_1$ to obtain a finite sequence of sets $A'_i$ and $A_i := A_{i-1} \setminus A'_i$ with

$$
\lambda_k(A'_i) \leq 4b^{1/\epsilon} \left( \frac{2k}{2} \right) \epsilon \log_2 |A|^{-i}.
$$

Applying Lemma 6.2 to $\bigcup_i A'_i$ we can crudely estimate

$$
\lambda_k(A) \leq 2b^{1/\epsilon} \left( \frac{2k}{2} \right) \epsilon \log_2 |A| \sum_{i=0}^{\infty} 2^{-i} \leq 4b^{1/\epsilon} \left( \frac{2k}{2} \right) \epsilon \log_2 |A| \leq 10b^{2} |A|^{2 \epsilon \log_2 k}.
$$

\[ \Box \]

6.4. Proof of Theorem 1.2. Theorem 1.2 is a straightforward corollary of Theorem 1.3. By Lemma 1.1, $\beta_* \leq K_*^2$ and so

$$
|kA| \geq \frac{|A|^k}{\lambda_k(A)} \gg_k |A|^{k-2k \log_2 k K_*^{-2k/\epsilon}}.
$$

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