ON THE LOGICAL STRENGTH OF THE AUTOMORPHISM GROUPS OF FREE NILPOTENT GROUPS

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Abstract. Considering a particular case of a problem posed by S. Shelah, we prove that the automorphism group of an infinitely generated free nilpotent group of cardinality \( \lambda \) first-order interprets the full second-order theory of the set \( \lambda \) in empty language.

1. Introduction

In his paper [8] of 1976, S. Shelah suggested a general program of the study of the logical strength of first-order theories of the automorphism groups of free algebras. Recently he has again attracted attention to that program in his survey [9]. Namely, the problem 3.14 from [9] asks for which varieties \( V \) of algebras, letting \( F_\lambda \) for a free algebra in \( V \) with \( \lambda \geq \aleph_0 \) free generators, can we syntactically interpret in the first-order theory of \( \text{Aut}(F_\lambda) \) the full second-order theory of the set \( \lambda \) in empty language (possibly for sufficiently large cardinals \( \lambda \)). Recall that a theory \( T_0 \) in a logic \( L_0 \) is said to be syntactically interpretable in a theory \( T_1 \) in a logic \( L_1 \) if there is a mapping \( \chi \to \chi^* \) from the set of all \( L_0 \)-sentences to the set of all \( L_1 \)-sentences such that

\[ \chi \in T_0 \iff \chi^* \in T_1. \]

It should be pointed out that one the main results of [8] states that for any variety of algebras \( V \) the first-order theory of the endomorphism semi-group \( \text{End}(F_\lambda) \) syntactically interprets \( \text{Th}_2(\lambda) \) provided that a cardinal \( \lambda \) is greater than or equal to the power of the language of \( V \). The situation with the automorphism groups seems to be more difficult and the reason is obvious: despite being complicated in many cases, the endomorphism semi-groups of free algebras, as Shelah’s analysis in [8] demonstrates, can be viewed as combinatorial objects.

There exist only a few examples of varieties for which the Shelah’s problem is completely investigated: some varieties have the desired property (for instance, the variety of all vector spaces over an arbitrary division ring and the variety of all groups [10, 12]), some do not (the variety of all sets with no structure [6, 7]), the automorphism groups of free algebras are here symmetric.

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groups). To the best of the author’s knowledge, there are no general results on the subject (however, Shelah introduces in [9, § 3] a wide class of so-called Aut-decomposable varieties, which are in many ways analogous to the variety of all sets).

The purpose of the present paper is to prove that the automorphism groups \( \text{Aut}(F_\lambda) \) of free groups \( F_\lambda \) in all varieties of nilpotent groups \( \mathfrak{N}_s \) with \( s \geq 2 \) are logically strong enough to interpret by means of first-order logic the full second-order theory of \( \lambda \) for all infinite \( \lambda \). We also consider a number of related questions; it is proved, in particular, that the first-order theory of the automorphism group of a finitely generated free nilpotent group of class \( \geq 2 \) is unstable and undecidable. The author is grateful to Oleg Belegradek, Edward Formanek and Alexandre Iwanow for helpful discussions.

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2. Reducing nilpotency class

Suppose that \( N \) is a free nilpotent group of class \( s \geq 2 \) and let \( K_m(N) \), where \( m \) is a natural number, denote the kernel of the homomorphism from the group \( \text{Aut}(N) \) to the group \( \text{Aut}(N/N_{m+1}) \) induced by the natural homomorphism \( N \rightarrow N/N_{m+1} \), from \( N \) to the free nilpotent group \( N/N_{m+1} \) of nilpotency class \( m \). In particular, \( K_1(N) = \text{IA}(N) \), to the subgroup of so-called IA-automorphisms of \( N \), and \( K_s(N) = \{ \text{id} \} \).

Lemma 2.1. Suppose that \( \gamma \) is an IA-automorphism. Then \( \gamma \) commutes with every element of the subgroup \( K_m(N) \) modulo the subgroup \( K_{m+1}(N) \).

Proof. According to [1], the groups \( K_m(N) \) form the lower central series of the group \( K_1(N) = \text{IA}(N) \); every element of an arbitrary group \( G \) commutes with the elements of the \( k \)th term of the lower central series of \( G \) modulo the \((k + 1)\)th term [4, Section 5.3]. \( \square \)

Like in our previous papers [11, 13], any automorphism \( \theta \) of \( N \), which inverts all elements of some basis of \( N \) will be called a symmetry.

Lemma 2.2. Let \( \theta \) be a symmetry.

(a) Suppose that \( c \) is an element of \( N_m \). Then \( \theta \) either fixes \( c \) modulo \( N_{m+1} \) (when \( m \) is even), or inverts \( c \) modulo \( N_{m+1} \) (when \( m \) is odd);

(b) Suppose that \( \gamma \) is an element of \( K_m(N) \). Then the conjugate of \( \gamma \) by \( \theta \) either equals to \( \gamma \) modulo \( K_{m+1}(N) \) (when \( m \) is even) or to the inverse of \( \gamma \) modulo \( K_{m+1}(N) \) (when \( m \) is odd).

Proof. (a) Assume that \( B \) is a basis of \( N \) such that \( \theta \) sends each element of \( B \) to its inverse. Since the group \( N_m/N_{m+1} \) is abelian it suffices to prove that \( \theta \) acts in a prescribed way on generators \( \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\}N_{m+1} \), where
\[x_{i_1}, \ldots, x_{i_m}\] are elements of \(B\) [4 Section 5.3]. We have
\[
\theta[x_{i_1}, [x_{i_2}, \ldots, x_{i_m}]] \equiv [x_{i_1}^{-1}, [x_{i_2}, \ldots, x_{i_m}]^{(-1)^{m-1}}] \\
\equiv [x_{i_1}, [x_{i_2}, \ldots, x_{i_m}]^{(-1)^{m}}] \pmod{N_{m+1}}.
\]

(b) By (a).

Suppose that \(\varphi\) is an involution from \(\text{Aut}(N)\) and \(\varphi_1, \varphi_2, \ldots, \varphi_m, \ldots\) are arbitrary conjugates of \(\varphi\). For every \(\sigma\) in \(\text{Aut}(N)\) let us construct the sequence
\[
\{\sigma_m(\varphi_1, \varphi_2, \ldots, \varphi_m) : m \in \mathbb{N}\}
\]
of automorphisms of \(N\) as follows:
\[
\sigma_0 = \sigma, \\
\sigma_1 = \varphi_1\sigma_0\varphi_1\sigma_0^{-1}, \\
\sigma_2 = \varphi_2\sigma_1\varphi_2\sigma_1, \\
\sigma_3 = \varphi_3\sigma_2\varphi_3\sigma_2^{-1}, \\
\vdots
\]

More formally, for every \(m \geq 0\)
\[
(2.1) \\
\sigma_{m+1} = \begin{cases} \\
\varphi_{m+1}\sigma_m\varphi_{m+1}\sigma_m^{-1}, & \text{if } m \text{ is even,} \\
\varphi_{m+1}\sigma_m\varphi_{m+1}\sigma_m, & \text{if } m \text{ is odd.}
\end{cases}
\]

The following result generalizes the corresponding fact from [13] proved there for free nilpotent groups of nilpotency class 2.

**Proposition 2.3.** Let \(N\) be a free nilpotent group of nilpotency class \(s\). Then an involution \(\theta \in \text{Aut}(N)\) is a symmetry modulo \(\text{IA}(N)\) (that is, coincides with some symmetry modulo the group \(\text{IA}(N)\)) if and only if for every \(\sigma\) from \(\text{Aut}(N)\) and every tuple \(\theta_1, \theta_2, \ldots, \theta_s\) of conjugates of \(\theta\) the automorphism \(\sigma_s(\theta_1, \theta_2, \ldots, \theta_s)\) of \(N\) is trivial.

**Proof.** Suppose that \(\theta = \theta^*\gamma\), where \(\theta^*\) is a symmetry and \(\gamma\) is an IA-automorphism. Since \(\theta\) is an involution, then \(\theta^*\gamma = \gamma^{-1}\theta^*\). Any member \(\theta_k\) of the tuple \(\theta_1, \theta_2, \ldots, \theta_s\), a symmetry modulo \(\text{IA}(N)\), also has the form \(\theta^*\gamma_k\) for a suitable IA-automorphism \(\gamma_k\).

Let us prove by induction on \(m\) that the automorphism \(\sigma_m = \sigma_m(\theta_1, \ldots, \theta_m)\) is an element of \(K_m(N)\). This will follow the necessity part of the Proposition.

Indeed, if \(m = 1\), then \(\sigma_m\) is an IA-automorphism, that is a member of \(K_1(N)\). Assume that \(\sigma_m \in K_m(N)\) and let \(m\) be, for instance, even. We have by Lemma 2.2(b) and Lemma 2.1
\[
\sigma_{m+1} = \theta_{m+1}\sigma_m\theta_{m+1}\sigma_m^{-1} = \gamma_{m+1}^{-1}\theta^*\sigma_m\theta^*\gamma_{m+1}\sigma_m^{-1} \\
\equiv \gamma_{m+1}^{-1}\sigma_m\gamma_{m+1}\sigma_m^{-1} \equiv \text{id} \pmod{K_{m+1}(N)}.
\]
Let us prove the converse. It is well-known that every automorphism of the abelianization $N$ of $N$, the free abelian group $N/[N, N]$, can be lifted up to an automorphism of $N$ (see, for instance, [5, § 4] or [3, Section 3.1, Section 4.2]). Then it suffices to prove that for every involution of $\text{Aut}(N)$, which is not $-\text{id}$, there exist an infinite sequence of the form (2.1), constructed inside $\text{Aut}(N)$, which contains no trivial members.

It can be seen quite easily that every involution $f \in \text{Aut}(N)$, which is not $-\text{id}$, has two $f$-invariant direct summands $B, C$ of $N$ with $N = B \oplus C$ and rank $B = 2$; moreover, the action of $f$ on $B$ can chosen so that $f|_B$ is neither $\text{id}_B$, nor $-\text{id}_B$ ([13, Theorem 1.4], [2, Lemma 1]). This reduces the problem to the automorphism groups of two-generator free abelian groups; for the sake of simplicity we shall work with the group $\text{GL}(2, \mathbb{Z})$.

According to the just mentioned result from [2], every involution in $\text{GL}(2, \mathbb{Z})$ is conjugate either to the involution
\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\]
or to the involution
\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix};
\]

hence the group $\text{GL}(2, \mathbb{Z})$ has exactly two conjugacy classes of non-central involutions. One readily checks that for every integer $m$
\[
(2.2) \quad \begin{pmatrix}
1 & 0 \\
2m & -1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
1 & 0 \\
2m - 1 & -1
\end{pmatrix} \sim \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

where $\sim$ denotes the conjugacy relation.

Let $S$ be a non-central matrix from $\text{GL}(2, \mathbb{Z})$ and $m$ an integer number. Suppose that
\[
X(m) = \begin{pmatrix}
1 & 0 \\
2m & -1
\end{pmatrix} S \begin{pmatrix}
1 & 0 \\
2m & -1
\end{pmatrix},
\]
\[
Y(m) = \begin{pmatrix}
1 & 0 \\
2m & -1
\end{pmatrix} S \begin{pmatrix}
1 & 0 \\
2m - 1 & -1
\end{pmatrix} S^{-1}.
\]

There are no difficulties in the verification of the following fact: some element of the ‘general’ matrix $X$ (and $Y$) depends linearly on $m$. It follows that for a suitable integer $m$ the matrix $X(m)$ ($Y(m)$) is again non-central. This means that, starting with a non-central matrix, we can construct an infinite sequence of the form (2.1) having no central matrices; in particular, there will be no trivial matrices in this sequence. Exactly the same argument, using matrices of the form
\[
\begin{pmatrix}
1 & 0 \\
2m - 1 & -1
\end{pmatrix},
\]

proves the similar result for the second conjugacy class of non-central involutions in $\text{GL}(2, \mathbb{Z})$. □
Corollary 2.4. Symmetries modulo $\text{IA}(N)$ form a definable family in the group $\text{Aut}(N)$.

Let $T^-(N)$ denote the set of all automorphisms $\{\sigma\}$ of $N$ such that for every $\theta$, which is a symmetry modulo $\text{IA}(N)$, the conjugate of $\sigma$ by $\theta$ is equal to $\sigma^{-1}$. Similarly $T^+(N)$ denotes the set of all automorphisms of $N$, which commute with every symmetry modulo $\text{IA}(N)$.

Proposition 2.5. Let $N$ be a free nilpotent group of nilpotency class $s$. Then $K_{s-1}(N) = T^+(N) \cup T^-(N)$. Therefore $K_{s-1}(N)$, the kernel of a surjective homomorphism from $\text{Aut}(N)$ to the automorphism group of a free nilpotent group of nilpotency class $s-1$ and of the same rank as one of $N$, is a definable subgroup of $\text{Aut}(N)$.

Proof. An arbitrary element $\sigma$ from $T^+(N) \cup T^-(N)$ must commute with any product of two symmetries: if, for instance, $\sigma \in T^-(N)$, $\theta_1$ and $\theta_2$ are two symmetries then

$$\theta_1 \theta_2 \sigma (\theta_1 \theta_2)^{-1} = \theta_1 \theta_2 \sigma \theta_2 \theta_1 = \theta_1 \sigma^{-1} \theta_1 = \sigma.$$  

On the other hand, one finds among the automorphisms of $N$, which can be expressed as a product of two symmetries, conjugations (inner automorphisms of $N$) by primitive elements (that is, members of bases of $N$). This implies that $\sigma$ commutes with every element of $\text{Inn}(N)$. Hence $\sigma$ preserves each element of $N$ modulo the center of $N$. The center of $N$ is equal to the subgroup $N_s$ [3, Section 3.1], and therefore $\sigma \in K_{s-1}(N)$.

Let $\tau$ be a conjugation by a primitive element $x$ of $N$. We are going to represent $\tau$ as a product of two symmetries. The element $x$ is a member of some basis $B$ of $N$. Suppose $\theta_1$ is a symmetry, which inverts each element of $B$. Then if a symmetry $\theta_2$ is defined as follows

$$\theta_2 x = x^{-1},$$

$$\theta_2 y = x^{-1} y^{-1} x, \ \forall y \in B \setminus \{x\},$$

the product of $\theta_1 \theta_2$ of $\theta_1$ and $\theta_2$ is equal to $\tau$.

Conversely, according to Lemma 2.2(b) every element of $K_{s-1}(N)$ either lies in $T^+(N)$, or in $T^-(N)$.

3. Interpretations

Theorem 3.1. Let $N$ be a free nilpotent group of class $\geq 2$. Then the automorphism group of $N$ first-order interprets the automorphism group of a free nilpotent group of class 2 and of rank which is the same as one of $N$ (uniformly in $N$).

Proof. By Proposition 2.5. 

Until otherwise stated, we shall assume that $N$ is a free nilpotent group of class 2 and that $A$ denotes the abelianization $\overline{N}$ of $N$.

It can be shown that $\text{Inn}(N)$, the subgroup of all conjugations, is a $\emptyset$-definable subgroup of $\text{Aut}(N)$ [13, Corollary 3.2]. The group $\text{Inn}(N)$ is
isomorphic to the free abelian group \( A \). Thus, we can interpret in \( \text{Aut}(N) \) the free abelian group \( A \) and the automorphism group of \( A \) with the action on the elements of \( A \).

We can also interpret in \( \text{Aut}(N) \) the family \( \mathcal{D} \) of all direct summands of \( A \) with inclusion relation and a binary relation, say \( R \) such that

\[
R(B, C) \iff A = B \oplus C.
\]

One can prove that an involution \( f \) from \( \text{Aut}(A) \) is diagonalizable in some basis of \( A \) if only if there are no elements of order three in the set \( K(f)K(f) \), where \( K(f) \) denotes the conjugacy class of \( f \) (see proof of Proposition 2.4 in [13]). Hence the fixed-point subgroups of diagonalizable involutions can be used to interpret the direct summands. Having the group \( A \) interpreted in \( \text{Aut}(N) \), we can easily interpret the inclusion relation and the relation \( R \) on the family \( \mathcal{D} \).

Summing up, we see that the group \( \text{Aut}(N) \) first-order interprets the multi-sorted structure \( M \) with the following description:

- the sorts of \( M \) are the free abelian group \( A \), its automorphism group \( \text{Aut}(A) \) and the family \( \mathcal{D} \) of all direct summands of \( A \);
- all sorts carry their natural relations; the relations of \( \mathcal{D} \) are the inclusion relation and the relation \( R \);
- \( M \) has as one of the basic relations the membership relation on \( A \cup \mathcal{D} \);
- there are relations defining the action of \( \text{Aut}(N) \) on other sorts.

**Lemma 3.2.** Let \( A \) be of infinite rank. Then the first-order theory of the structure \( M \) syntactically interprets the full second-order theory of the set \(|A|\) (in empty language), uniformly in \( A \).

**Proof.** It follows from the results in Section 4 of [8], that the first-order theory of the endomorphism semi-group \( \text{End}(A) \) of \( A \) syntactically interprets \( \text{Th}_2(|A|) \) (for the sake of convenience the reader may refer to [10], where the very similar case of varieties of vector spaces is considered in some details in the proof of Proposition 10.1; an analysis of the proof shows that it works also for free \( \mathbb{Z} \)-modules, or, in other words, for free abelian groups).

To complete the proof, we could therefore interpret in \( M \) the endomorphism semi-group of \( A \). There is a (folklore) trick by which the endomorphisms can be interpreted in structures similar to \( M \) constructed over modules. This trick can be briefly characterized as follows: three submodules, such that any two of them are direct complements of each other, are used to interpret the endomorphism semi-group of one of them. A detailed description of the trick for infinite-dimensional vector spaces can be found in [10] (see the proof of Proposition 9.3); the reader is again referred to [10] to see that everything works for free \( \mathbb{Z} \)-modules as well. \( \square \)

The following result solves the problem posed by S. Shelah (see the Introduction) for all varieties of nilpotent groups \( \mathfrak{N}_s \), where \( s \geq 2 \).
Theorem 3.3. The first-order theory of the automorphism group of any infinitely generated free nilpotent group $N$ of class $\geq 2$ syntactically interprets the full second-order theory of the set $|N|$. The first-order theory of $\operatorname{Aut}(N)$ is therefore unstable and undecidable.

Proof. By Theorem 3.1 and Lemma 3.2.

We have also solved the problem of classification of elementary types of the automorphism groups of infinitely generated free groups from varieties $\mathfrak{N}_s$:

Theorem 3.4. Let $N_1$ and $N_2$ be infinitely generated free nilpotent groups of the same class $\geq 2$. Then the automorphism groups $\operatorname{Aut}(N_1)$ and $\operatorname{Aut}(N_2)$ are elementarily equivalent if and only if the sets $|N_1|$ and $|N_2|$ (with no structure) are equivalent in the full second-order logic.

Proof. By Theorem 3.1 and Lemma 3.2.

Let $N$ again denote a free nilpotent group of class 2 (recall that $A$ stands for the abelianization of $N$ and $\mathcal{M}$ is the multi-sorted structure constructed over $A$).

We are going to estimate the logical strength/complexity of the first-order theory of $\operatorname{Aut}(N)$ in the case, when $N$ is finitely generated.

Lemma 3.5. Let $A$ be of rank at least 2. Then the structure $\mathcal{M}$ first-order interprets (with parameters) the ring of integers $\mathbb{Z}$.

Proof. Let us consider two direct summands $B, C$ of $A$ such that

$$A = B \oplus C \quad \text{and} \quad \operatorname{rank} B = 2.$$

Write $G$ for the group of all automorphisms of $A$ which preserve $B$ and point-wise fix $C$. Clearly, the structure $\langle G, B \rangle$ with natural relations (that is, with all relations on sorts along with the action of $G$ on $B$) is isomorphic to the two-sorted structure $\langle \operatorname{GL}(2, \mathbb{Z}), \mathbb{Z}^2 \rangle$ (taken in the same language as one of $\langle G, B \rangle$). It is a well-known and simple result that the latter two-sorted structure first-order interprets the ring of integers $\mathbb{Z}$.

As an immediate corollary we have the following fact.

Theorem 3.6. Suppose that $N$ is a finitely generated free nilpotent group of class $\geq 2$. Then the first-order theory of the group $\operatorname{Aut}(N)$ is unstable and undecidable.

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