PRYM VARIETIES AND FOURFOLD COVERS

SEVÍN RECILLAS AND RUBÍ E. RODRÍGUEZ

Contents

1. Introduction 2
2. Prym varieties for covers of curves 3
3. Galois covers 8
4. Degree two covers 10
5. Covers of degree three 13
6. Covers of degree four 15
6.1. The cyclic case 15
6.2. The Klein case 17
7. The dihedral case 22
7.1. The bigonal construction 34
8. The alternating case 37
8.1. The trigonal construction for the case $A_4$ 43
9. The symmetric case 44
9.1. The classical case of the trigonal construction 55
9.2. Completion of the principally polarized $P(S/R)$ case 57
9.3. The general case 58
10. Other applications 63
10.1. Examples 63
10.2. Jacobians of curves isogenous to a product of Jacobians 64
10.3. Rigid Jacobians with $S_4$ actions 66
10.4. One dimensional families of curves with $S_4$ action 67
10.5. Prym varieties of genus seven double covers of genus three curves, isogenous 70

Appendix 70

References 77

This research was partially supported by CONACYT–CONICYT Exchange Program, by grants CONACYT 27962–E, Fondecyt 1000623 and Presidential Science Chair on Geometry.
1. Introduction

The problem of finding an isogenous decomposition of the Jacobian $J\tilde{C}$ of a smooth connected complete curve $\tilde{C}$ has been studied by many people, as well as its generalization for any abelian variety.

The first example of finding subtori $X$ and $Y$ of $J\tilde{C}$ such that $X \times Y$ is isogenous to $J\tilde{C}$ is due to Wirtinger [W]; the situation is a double cover of curves $f : \tilde{C} \to C$, where $J\tilde{C}$ is found to be isogenous to the product of the abelian subvariety $f^*(JC)$ and its natural complement $P(\tilde{C}/C)$, later called the Prym variety of the cover $f$, by Mumford in [M2].

In this last work one finds a careful discussion of the kernels of the isogeny and of $f^*$, which is then used to describe the type of polarization that $P(\tilde{C}/C)$ inherits from $J\tilde{C}$.

These ideas were generalized by Ries in [R] where he studied the Prym variety associated to a cyclic unramified cover of a hyperelliptic Riemann surface, and in [G] for the case of a curve with an automorphism of prime order.

In a more general context, their results may be viewed as particular cases of the isotypical decomposition of an abelian variety with a finite group acting on it. Such decompositions have been studied by several people in different settings (Donagi, Kanev, Mérindol, Lange and Recillas, among others), with applications to the theory of integrable systems and the moduli spaces of principal bundles of curves.

In another related direction, there are two constructions that have proved very useful in the study of relations between Jacobians and Prym varieties. The Recillas trigonal construction (c.f. [Rec1]) shows that the Jacobian of a tetragonal curve is isomorphic to the Prym variety of a double cover of a trigonal curve, and Pantazis showed in [P] that the Prym varieties associated to the bigonal construction of Donagi (c.f. [D1]) are dual to each other.

The aim of this work is the study of the structure of the Jacobian of the Galois extension $W \to T$ of a fourfold cover of smooth connected complete curves $X \to T$. We give explicit descriptions of the rational simple components of this Jacobian as generalized Pryms of intermediate covers of $W \to T$ and analyze the polarization types and kernels of the isogenies involved.

Even though the isotypical decomposition of $JW$ may be obtained faster from general methods (see, for instance, [D2], [Ka] and [M]), our calculations for the kernels of the natural isogenies that appear allow us to obtain new proofs of the bigonal and trigonal constructions in
purely group-theoretical terms, by considering $T = \mathbb{P}^1$ and particular
values of the ramification of $X \to T$ (see Remark 9.13 and Section 7.1
Remark 9.13 and Theorem 9.15).

Furthermore, by applying appropriate restrictions for the ramification
data of $X \to T$, we also obtain families of Jacobian varieties isogenous
to products of Jacobians and families of Prym varieties isogenous
to the product of elliptic curves.

When working with the Jacobian of a curve, we use additive or tensor
notation depending on whether the points of the Jacobian are con-
sidered as points of an abelian variety or as line bundles on the curve.

If $Y$ is a subvariety of an abelian variety, we denote by $Y^0$ its con-
ected component of the origin, and by $Y[d]$ the subgroup of $Y$ con-
sisting of its points of order $d$.

If $f : \tilde{C} \to C$ is an unramified cyclic cover of curves of degree $d$,
we will denote by $\eta_f$ a point in $JC[d]$ which determines the cover $f$;
equivalently, a generator for the kernel of $f^*$.

If $A$ is a polarized abelian variety with polarization $\lambda : A \to \hat{A}$,
where $\hat{A}$ is the dual abelian variety, then $K(\lambda)$ will denote the kernel
of $\lambda$. If $\lambda$ is of type $(d_1, \ldots, d_g)$ with $d_i$ dividing $d_{i+1}$ for $1 \leq i \leq g-1$,
we will denote by $\lambda_{\hat{A}} : \hat{A} \to \hat{A} = A$ the polarization on $\hat{A}$ such that
$\lambda_{\hat{A}} \circ \lambda$ is multiplication by $d_g$ on $A$.

If $f : A \to B$ is a morphism between abelian varieties, the dual map
will be denoted by $\hat{f} : \hat{B} \to \hat{A}$; if $B$ has a polarization $\lambda_B$, then $A$
ierits the induced polarization $\lambda_A$ defined by $\lambda_A = \hat{f} \circ \lambda_B \circ f$ (the
pullback of $\lambda_B$ via $f$).

2. Prym varieties for covers of curves

From now on, $\tilde{C}$ and $C$ are smooth connected complete curves over
the complex numbers $\mathbb{C}$, of respective genera $\tilde{g}$ and $g$, and $f : \tilde{C} \to C$
is a cover of degree $d$.

Let $(J, \lambda)$ and $(\tilde{J}, \tilde{\lambda})$ denote the Jacobians of $C$ and $\tilde{C}$, respectively.
The cover $f$ induces two homomorphisms between Jacobians, denoted
by $f^* : J \to \tilde{J}$ and $\text{Nm} f : \tilde{J} \to J$ respectively. Just as in [M2] for
the case $d = 2$, they are related as follows.

\begin{equation}
\text{Nm} f \circ f^* = d \cdot \text{Id} \mid_J \quad \text{and} \quad \tilde{\text{Nm}} f = \tilde{\lambda} \circ f^* \circ \lambda^{-1}
\end{equation}

Also note that $(\ker \text{Nm} f)^0 = (\ker (f^* \circ \text{Nm} f))^0$.

We will use the following characterization given in [L-B, p. 337].
Proposition 2.1. Given a cover of curves $f : \tilde{C} \to C$, the induced homomorphism $f^* : JC \to \tilde{JC}$ is not injective if and only if $f$ factors via a cyclic étale covering $f'$ of degree $\geq 2$ as is shown in the following diagram.

\[
\begin{array}{ccc}
\tilde{C} & \xrightarrow{f} & C \\
\downarrow & & \downarrow \\
\tilde{C}' & \xrightarrow{f''} & C'
\end{array}
\]

Remark 2.2. Note that by the proof of the above proposition, $\ker f'^* \text{ is cyclic of order equal to the degree of } f'$; hence if $f''^*$ is injective, then $\ker f^*$ is cyclic of order equal to the degree of $f'$.

The following result is well known.

**Lemma 2.3.** Let $f : \tilde{C} \to C$ be a cover of curves of degree $d$.

Then the number of connected components of $\ker \text{Nm}\, f$ is the cardinality of $\ker f^*$.

In particular, if $d$ is prime and $f$ is ramified or if $d$ is prime and $f$ is not a cyclic cover then $\ker \text{Nm}\, f$ is connected.

We now recall the definition of the Prym variety of a subtorus of a principally polarized abelian variety (p.p.a.v.) and some of its properties (c.f. [R] and [M2]).

Let $(A, \lambda)$ be a p.p.a.v., where $\lambda : A \to \tilde{A}$ is the principal polarization and let $X \hookrightarrow A$ be a subtorus. Then the Prym variety $P = P(A, \lambda, X)$ is defined by $P = \ker(\hat{i} \circ \lambda) = \lambda^{-1}(\ker \hat{i})$. Observe that $P$ is connected (since $\ker \hat{i}$ is injective) and its dimension is the codimension of $X$ in $A$.

If we denote by $j : P \to A$ the inclusion and by $\lambda_X$ and $\lambda_P$ the induced polarizations on $X$ and $P$, respectively, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & A & \xrightarrow{j} & P \\
\downarrow_{\lambda_X} & & \downarrow_{\lambda} & & \downarrow_{\lambda_P} \\
\hat{X} & \xrightarrow{i} & \hat{A} & \xrightarrow{j} & \hat{P}
\end{array}
\]

The following results appear in [R].

**Proposition 2.4.** Let $(A, \lambda)$ be a p.p.a.v., let $X$ be a subtorus of $A$ and let $P = P(A, \lambda, X)$ be its Prym variety. Then
i) The sum homomorphism
\[ \sigma : X \times P \longrightarrow A \]
\[ (x, y) \longmapsto x + y \]
is an isogeny with kernel given by
\[ \ker \sigma = \{ (x, -x) : x \in X \cap P \}. \]

ii) If we denote by \( \lambda_{X \times P} \) the pullback of the polarization \( \lambda \) to \( X \times P \) via \( \sigma \), then
\[ \lambda_{X \times P} = \hat{\sigma} \circ \lambda \circ \sigma = \begin{pmatrix} \lambda_X & 0 \\ 0 & \lambda_P \end{pmatrix} \]

iii) The kernels of the induced polarizations are related as follows.
\[ K(\lambda_{X \times P}) = K(\lambda_X) \times K(\lambda_P) \]
Furthermore, \( \ker \sigma \) is a maximal isotropic subgroup.

iv) \( K(\lambda_X) = X \cap P = K(\lambda_P) \).

We now apply these results to our cover of curves \( f : \tilde{C} \to C \). In this case the Prym Variety of \( f \) is defined by
\[ P = P(f) = P(\tilde{C}/C) = (\ker(\tilde{f}^* \tilde{\lambda}))^0 \]
and we can prove the following results.

**Theorem 2.5.** Let \( f : \tilde{C} \to C \) be a cover of curves of degree \( d \). Denote by \( \tilde{\lambda}_{f^*(J)} \) and by \( \tilde{\lambda}_P \) the polarizations induced by \( \tilde{\lambda} \) on \( f^*(J) \) and on \( P \), respectively. Then
\[ i) \ P = (\ker \mathrm{Nm} f)^0; \]
\[ ii) \ P = P(\tilde{J}, \tilde{\lambda}, f^*(J)); \ i.e., \ P \ is \ the \ Prym \ variety \ of \ the \ subtorus \ f^*(J) \ of \ the \ p.p.a.v. \ (\tilde{J}, \tilde{\lambda}); \]
\[ iii) \ the \ polarization \ on \ J \ given \ by \ the \ pullback \ of \ \tilde{\lambda}_{f^*(J)} \ via \ f^* \ is \ \mathrm{d}\lambda \ \text{and} \ H_0 := \ker f^* \subset J[d] \ \text{is isotropic with respect to the Weil form associated to \( \text{d}\lambda \);} \]
\[ iv) \ f^* \ induces \ an \ isomorphism \]
\[ H_0^* / H_0 \longrightarrow \ker(\lambda_{f^*J}) = \ker(\lambda_P) = f^*J \cap P \subseteq P[d] \]
where orthogonality is with respect to the skew-symmetric multiplicative form \( \epsilon_{\text{d}\lambda} \) on \( J[d] \), and
\[ \left| f^*J \cap P \right| = \frac{|J[d]|}{|\ker f^*|^2}. \]
Consider the isogeny
\[ \phi : J \times P \to \tilde{J}, \quad \phi(c, \tilde{c}) = f^*(c) + \tilde{c} \]
and the projection onto the first factor \( \pi_1 : J \times P \to J. \)
Set \( H_1 = \pi_1(\ker \phi). \) Then
\[ \ker \phi = \{(c, -f^*(c)) : c \in J[d] \text{ and } f^*(c) \in P\} \to H_1 \]
\[ (c, -f^*(c)) \mapsto c \]
is an isomorphism.
Furthermore, \( H_1 = H_0^\perp \) and
\[ |\ker \phi| = |H_1| = |H_0^\perp| = \frac{|J[d]|}{|\ker f^*|}. \]

Proof. i) follows from \( \lambda^{-1} \circ \tilde{f}^* \circ \tilde{\lambda} = \Nm f. \)
ii) holds since \( f^* \) has finite kernel.
iii) is just a computation involving the relations given in (2.1):
\[ \tilde{f}^* \circ \tilde{\lambda} \circ f^* = \lambda \circ \Nm f \circ \tilde{\lambda}^{-1} \circ \tilde{\lambda} \circ f^* = \lambda \circ \Nm f \circ f^* = d \lambda \]
As for iv), it follows from descent theory that \( f^* : J \to f^* J \) induces an isomorphism from \( H_0^\perp / H_0 \) to \( \ker(\lambda f^* J) \); therefore,
\[ H_0^\perp = (f^*)^{-1}(\ker \lambda f^* J). \]
But we already know from Proposition 2.1 that
\[ \ker(\lambda f^* J) = \ker(\lambda P) = f^* J \cap P. \]
Observing that \( y \in f^* J \cap P \) if and only if \( y \) is in \( P \) and \( y = f^*(x) \)
for some \( x \in J \), we obtain \( 0 = \Nm f(y) = \Nm f(f^*(x)) = dx, \) and therefore \( dy = 0. \)
The equality
\[ (2.3) \quad |H_0| = \frac{|J[d]|}{|H_0^\perp|} \]
will complete the proof of iv).

To prove it, consider \( J = V/L \) and \( f^* J = V/M, \) with \( L \subseteq M \) lattices in \( V, \) and recall that \( H_0 = M/L, H_0^\perp = M^\perp / L, J[d] = K(d \lambda) = L^\perp / L, \)
and that the index of \( L \) in \( M \) equals the index of \( M^\perp \) in \( L^\perp. \)
Then from \( L \subseteq M \subseteq M^\perp \subseteq L^\perp \) it follows that \( [L^\perp : L] = [L^\perp : M^\perp] \cdot [M^\perp : L]; \) since \( [L^\perp : M^\perp] = [M : L], \) we obtain (2.3) in the form
\[ [L^\perp : L] = [M : L] \cdot [M^\perp : L]. \]
As for v), it is clear that \( \ker \phi \subseteq J[d] \times P[d] \) and therefore \( \phi \) is an isogeny. It is also clear that \( s : H_1 \to \ker \phi \) defined by \( c \mapsto (c, -f^*(c)) \)
is a section of \( \pi_1 : \ker \phi \to H_1, \) so it follows that \( \pi_1 \) is an isomorphism.
But also $H_1 = \{ c \in J : f^*(c) \in P \} = (f^*)^{-1}(f^*J \cap P) = H_0^\perp$. Therefore $|\ker \phi| = |H_0^\perp|$, which together with equality (2.3) complete the proof of v) and of the Theorem.

Our next result gives the relation between the Prym variety of the composition of two covers of curves and the Prym varieties of the intermediate covers.

**Proposition 2.6.** Let $f : X \to Y$ and $g : Y \to Z$ be two covers of curves. Set $h = g \circ f : X \to Z$.

Then there are two isogenies given as follows.

$ps : P(Y/Z) \times P(X/Y) \to P(X/Z), \quad ps(y, x) = x + f^*y$

and

$\psi : JZ \times P(Y/Z) \times P(X/Y) \to JX, \quad \psi(z, y, x) = x + f^*y + h^*z$.

Furthermore,

i) The kernel of $ps$ is contained in $P(Y/Z)[\deg f] \times P(X/Y)[\deg f]$ and its cardinality is equal to

$$|\ker ps| = |P(Y/Z)[\deg f]| \frac{|g^*(JZ) \cap \ker f^*|}{|\ker f^*|}.$$

ii) The kernel of $\psi$ has cardinality equal to

$$|\ker \psi| = \frac{|JY[\deg f]| |JZ[\deg g]|}{|\ker f^*| |\ker g^*|} = \frac{|JZ[\deg h]| |P(Y/Z)[\deg f]|}{|\ker f^*| |\ker g^*|}.$$

**Proof.** By Theorem 2.5 we have isogenies

$\alpha : JY \times P(X/Y) \to JX, \quad (y, x) \to f^*y + x,$

$\beta : JZ \times P(Y/Z) \to JY, \quad (z, y) \to g^*z + y$,

and

$\gamma : JZ \times P(X/Z) \to JX, \quad (z, x) \to h^*z + x$

whose kernels have sizes $|\ker \alpha| = \frac{|JY[\deg f]|}{|\ker f^*|}, \quad |\ker \beta| = \frac{|JZ[\deg g]|}{|\ker g^*|}$

and $|\ker \gamma| = \frac{|JZ[\deg h]|}{|\ker h^*|}$ respectively.

A short computation and connectedness show that the image of $ps$ is contained in $P(X/Z)$.

Furthermore if $(y, x) \in \ker ps$, we have that $0 = f^*y + x$. But then $0 = Nm f(f^*y) + Nm f(x) = (\deg f) y$; hence $y \in P(Y/Z)[\deg f]$ and $x = -f^*y$; therefore $\ker ps \subseteq \{(y, -f^*y) : y \in P(Y/Z)[\deg f]\}$, which proves the first part of i).
From the commutative diagram

\[
\begin{array}{ccc}
JZ \times P(Y/Z) \times P(X/Y) & \xrightarrow{\{\beta, \text{id}_{P(X/Y)}\}} & JY \times P(X/Y) \\
(id_{JZ}, \psi) & \downarrow & \downarrow \\
JZ \times P(X/Z) & \xrightarrow{\gamma} & JX
\end{array}
\]

we obtain that \( \psi \) is an isogeny and also that

\[
|\ker \psi| = |\ker \text{id}_{P(X/Y)}| = |\ker \beta|.
\]

Therefore \( |\ker \psi| = |\ker \text{id}_{P(X/Y)}| \cdot |\ker \alpha| \cdot |\ker \beta| \).

and in order to finish the proof we only need to compute \( |\ker \text{id}_{P(X/Y)}| \).

But it is clear that

\[
|\ker \text{id}_{P(X/Y)}| = |\ker \beta|.
\]

and that

\[
\ker h^* = g^* -1(\ker \beta) \cap \ker \alpha.
\]

The proof is now complete. \(\square\)

3. Galois covers

Assume that the cover \( \tilde{\alpha} : \tilde{C} \to C \) is a Galois cover; i.e., that there exists a subgroup \( G \) of the automorphism group of \( \tilde{C} \) such that \( C = \tilde{C}/G \) and \( f \) is the canonical quotient map. For any \( g \in G \) we denote by the same symbol \( g \) the automorphism induced by \( g \) on \( \tilde{J} \), by \( \langle g \rangle \) the subgroup of \( G \) generated by \( g \), and by \( \tilde{J}^H \) the set of fixed points of \( H \) in \( \tilde{J} \), for any subgroup \( H \) of \( G \). The following results are immediate.

**Proposition 3.1.** Let \( \tilde{\alpha} : \tilde{C} \to C = \tilde{C}/G \) be a Galois cover of degree \( d \) and let \( P = P(f) \) denote the corresponding Prym variety.

Then

i) \( f^*(J) = (\tilde{J}^G)^0 \) and \( \tilde{J}^G = f^*(J) + P_0 \), where \( P_0 = P \cap \tilde{J}^G \subseteq P[d] \).
ii) If we define $\text{Nm}_G : \tilde{J} \to \tilde{J}$ by $\text{Nm}_G = \sum_{g \in G} g$, then $\text{Nm}_G = f^* \circ \text{Nm}_{f^*}$.

iii) $P = (\ker \text{Nm}_G)^0$.

**Corollary 3.2.** If $f : \tilde{C} \to C$ denotes a double cover given by the involution $\sigma : \tilde{C} \to \tilde{C}$ and $P$ denotes the corresponding Prym variety, then

\[
\tilde{J}(\sigma) = f^* J + P[2]
\]

**Proof.** From Proposition 3.1 i) we know that $\tilde{J}(\sigma) = f^* J + P_0$, where $P_0 = \tilde{J}(\sigma) \cap P \subseteq P[2]$.

Since $\sigma$ is an involution, it is clear that $P[2] \subseteq \tilde{J}(\sigma)$ and the result follows. \(\square\)

**Proposition 3.3.** Let $f : \tilde{C} \to C = \tilde{C}/\langle \alpha \rangle$ be a cyclic unramified cover.

Then $\tilde{J}(\alpha)$ is connected; i.e., $\tilde{J}(\alpha) = f^* J$.

**Proof.** Assume $f$ is of degree $d$; then $\tilde{J}(\alpha) = \ker(1 - \alpha)$.

If $\mathcal{L}$ is in $\ker(1 - \alpha)$, then there is an isomorphism $\phi : \mathcal{L} \to \alpha(\mathcal{L})$. But then $\alpha^{d-1}(\phi) \circ \ldots \alpha(\phi) \circ \phi$ is an isomorphism from $\mathcal{L}$ to itself, and therefore equal to a nonzero complex constant $c$; adjusting the constant, we obtain an isomorphism $\phi_1 = \frac{1}{\sqrt[d]{c}} \phi$ of $\mathcal{L}$ of order $d$.

Therefore $G$ acts on $\mathcal{L}$; i.e., $G$ is linearizable and it follows from [M1] that then $\mathcal{L}$ is in $f^* J$. \(\square\)

The next result appears in [Rec–Ro]; its corollary will be very useful later on.

**Lemma 3.4.** Let $f : \tilde{C} \to C = \tilde{C}/G$ be a Galois cover of degree $d$, let $H$ be a subgroup of $G$ and denote by $h : \tilde{C} \to \tilde{C}/H$ and $\varphi : \tilde{C}/H \to C$ the corresponding covers, as in the following diagram.

\[
\begin{array}{c}
\tilde{C} \\
\downarrow f \\
\tilde{C}/H \\
\downarrow h \\
C = \tilde{C}/G
\end{array}
\]

Then $f^*(\text{Nm} \varphi(z)) = g_1 h^* z + g_2 h^* z + \cdots + g_r h^* z$ for all $z \in J(\tilde{C}/H)$, where $\{g_1, g_2, \ldots, g_r\}$ is a complete set of representatives for $G/H$. 

Corollary 3.5. Under the hypothesis of Lemma 3.4,

\[ h^*(P(\varphi)) = \{ x \in \tilde{J}^H : \sum_{i=1}^{r} g_i(x) = 0 \}^\circ. \]

Proof. Let

\[ A = \{ x \in \tilde{J}^H : \sum_{i=1}^{r} g_i(x) = 0 \}^\circ. \]

If \( z \in P(\varphi) \) then \( \text{Nm}_f(h^*z) = |H| \text{Nm}_f(z) = 0; \) hence \( h^*(P(\varphi)) \subseteq (\ker \text{Nm}_f)^\circ = (\ker \text{Nm}_G)^\circ \) and \( \sum_{g \in G} g(h^*z) = 0. \)

But clearly \( h^*(P(\varphi)) \subseteq \tilde{J}^H, \) so the last equation can be written as

\[ |H| \sum_{i=1}^{r} g_i(h^*z) = 0, \] and we have proven that \( h^*(P(\varphi)) \subseteq A. \)

Conversely, let \( x \in A; \) then

\[ \text{Nm}_G(x) = \sum_{i} g_i \left( \sum_{k \in H} k \right)(x) = |H| \sum_{i} g_i(x) = 0 \]

and we obtain that \( A \subseteq (\ker \text{Nm}_G)^\circ = (\ker \text{Nm}_f)^\circ. \)

Therefore \( \text{Nm}_h(A) \subseteq (\ker \text{Nm}_f)^\circ = P(\varphi); \) then

\[ |H|A = \sum_{k \in H} k(A) = h^*(\text{Nm}_h(A)) \subseteq h^*(P(\varphi)) \]

and the result follows. \( \square \)

4. Degree two covers

Consider a degree two morphism of curves \( f : \tilde{C} \rightarrow C \) with total ramification degree \( \omega. \)

Set \( g = g_{\tilde{C}} \) and \( P = P(\tilde{C}/C). \) Then \( g_{\tilde{C}} = 2g - 1 + \frac{\omega}{2} \), \( \dim P = g - 1 + \frac{\omega}{2}, \) and \( |P[2]| = 2^{2g-2+\omega}. \)

In this section we describe \( P[2] \) explicitly in terms of the ramification of \( f. \)

By Theorem 2.5 \( |(\ker f^*)^\perp| = \frac{|J[2]|}{|\ker f^*|} \) and \( f^* \) induces an isomorphism

\[ (\ker f^*)^\perp / \ker f^* \cong f^*J \cap P \subseteq P[2]. \]

If \( \omega = 0, \) then \( |\ker f^*| = 2; \) hence \( |(\ker f^*)^\perp / \ker f^*| = 2^{2g-2} \) and \( f^*J \cap P = P[2]. \)

If \( \omega \neq 0, \) then \( |\ker f^*| = 0; \) hence \( f^* \) induces an isomorphism from \( J[2] \) to \( f^*(J[2]) \cap P \subseteq P[2]. \) Therefore \( f^*(J[2]) \subseteq P[2] \) and \( |P[2]/f^*(J[2])| = 2^{\omega-2}. \)
In particular if \( \omega = 2 \), then we have that \( J[2] \) is isomorphic to \( f^*(J[2]) = P[2] \).

Also note that it follows from Lemma 2.3 and from Proposition 3.1 that if \( \omega \neq 0 \), then \( \ker(\text{Nm}_f) = \ker(1 + \sigma) \) is connected, where \( \sigma \) is the involution of \( \tilde{C} \) associated to the cover \( f \).

The cases \( \omega = 0 \) and \( \omega = 2 \) of above are the “Classical Pryms” described in Mumford’s beautiful paper [M2].

We now consider \( \omega > 2 \). Let \( P_1, \ldots, P_\omega \in \tilde{C} \) be the ramification points of \( f \), and for \( i \in \{1, \ldots, \omega \} \), let \( Q_i = f(P_i) \in C \).

For \( j \in \{2, \ldots, \omega \} \), choose \( m_j \in \text{Pic}^0(C) \) such that \( m_j \otimes 2 = \mathcal{O}_C(Q_1 - Q_j) \) and consider

\[
\mathcal{F}_j = \mathcal{O}_{\tilde{C}}(P_j - P_1) \otimes f^*(m_j).
\]

Then each \( \mathcal{F}_j \) is in \( P[2] \) and we have the following result.

**Theorem 4.1.** Let \( f : \tilde{C} \to C \) be a degree two cover with total ramification degree \( \omega > 2 \).

Then for any \( L \in P[2] \) there exist \( m \in J[2] \) and unique \( \nu_j \in \{0, 1\} \) such that

\[
L = \mathcal{F}_2^{\nu_2} \otimes \ldots \otimes \mathcal{F}_{\omega-1}^{\nu_{\omega-1}} \otimes f^*(m).
\]

In other words,

\[
P[2] = f^*(J[2]) \oplus_{j=2}^{\omega-1} \mathcal{F}_j \mathbb{Z}/2\mathbb{Z}.
\]

**Proof.** Let \( D \in \text{Pic}^{\tilde{C}}(C) \) be the divisor class defining the cover \( f : \tilde{C} \to C \); that is,

\[
D^{\otimes 2} = \mathcal{O}_C(Q_1 + \ldots + Q_\omega) \quad \text{and} \quad f^*(D) = \mathcal{O}_{\tilde{C}}(P_1 + \ldots + P_\omega).
\]

Consider the Abel-Jacobi map

\[
\tilde{C}(\mathbb{Z}) \xrightarrow{J} \mathcal{J}
\]

\[
D \mapsto \mathcal{O}_{\tilde{C}}(D - \tilde{g}P_1).
\]

Then, given \( L \in P[2] \), \( \sigma \) acts on \( f^{-1}(L) = |D| \); hence there exists \( D \in |D| \) such that \( \sigma D = D \), and therefore \( D \) must be of the form \( D = f^*(E) + \sum_{i=1}^{r} P_i \), for some effective divisor \( E \) on \( C \).

It follows that \( L = \mathcal{O}_{\tilde{C}}(\sum_{i=1}^{r}(P_i - P_1)) \otimes f^*(n) \), for some \( n \) in \( \text{Pic}^0(C) \). But \( L^{\otimes 2} = \mathcal{O}_{\tilde{C}} \) and, since \( f^* \) is injective, it follows that \( n^{\otimes 2} = \mathcal{O}_C(\sum_{i=1}^{r}(Q_1 - Q_i)) \).

Therefore

\[
L = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_r \otimes f^*(m),
\]

where \( m = n \otimes m_1^{-1} \otimes \ldots \otimes m_r^{-1} \in J[2] \).

The following result completes the proof. \( \square \)
Lemma 4.2. Under the conditions of Theorem 4.1, we have
\[ F_{i_1} \otimes \ldots \otimes F_{i_k} \in f^*(J[2]), \text{ with } 2 \leq i_1 < \ldots < i_k \leq \omega \]
if and only if \( \{i_1, \ldots, i_k\} = \{2, \ldots, \omega\} \).

Proof. Let \( D \in \text{Pic}^2(C) \) be as in the proof of Theorem 4.1 and assume \( \{i_1, \ldots, i_k\} = \{2, \ldots, \omega\} \).

Then
\[ F_2 \otimes \ldots \otimes F_\omega = O_C(P_2 + \ldots + P_\omega - (\omega - 1)P_1) \otimes f^*(m_2 \otimes \ldots \otimes m_\omega) \]
\[ = f^*(D(-\frac{\omega}{2}Q_1) \otimes m_2 \otimes \ldots \otimes m_\omega). \]

But
\[ (D(-\frac{\omega}{2}Q_1) \otimes m_2 \otimes \ldots \otimes m_\omega)^{\otimes 2} = O_C \]
i.e.,
\[ F_2 \otimes \ldots \otimes F_\omega \in f^*(J[2]). \]

To show the other implication we may assume, without loss of generality, that \( \{i_1, \ldots, i_k\} = \{2, 3, \ldots, r\} \). That is, it suffices to show that if \( F_2 \otimes \ldots \otimes F_r = f^*(n) \), with \( n \in J[2] \), then \( r = \omega \).

Indeed, \( F_2 \otimes \ldots \otimes F_r = f^*(n) \) with \( n \in J[2] \), is equivalent to
\[ O_C(\sum_{i=2}^{r} P_i - (r - 1)P_1) = f^*(n \otimes m_2^{-1} \otimes \ldots \otimes m_r^{-1}). \]

Now consider the case \( r \) even and define
\[ F = n \otimes m_2^{-1} \otimes \ldots \otimes m_r^{-1} \otimes O_C(\frac{r}{2}Q_1). \]

Then \( F^{\otimes 2} = O_C(\sum_{i=2}^{r} Q_i) \) and \( f^*(F) = O_C(\sum_{i=2}^{r} P_i) \). But this is possible by the definition of \( D \) only if \( r = \omega \).

If \( r \) were odd, then we would define
\[ F = n \otimes m_2^{-1} \otimes \ldots \otimes m_r^{-1} \otimes O_C(\frac{r - 1}{2}Q_1) \]
and have that \( F^{\otimes 2} = O_C(\sum_{i=2}^{r} Q_i) \) and that \( f^*(F) = O_C(\sum_{i=2}^{r} P_i) \), which is impossible since \( r < \omega \).

We now give a different set of generators for \( P[2]/f^*(J[2]) \), which will be useful later on.

Corollary 4.3. Under the conditions of Theorem 4.1, choose \( n_i \) and \( s_j \) in \( \text{Pic}^0(C) \) such that \( n_i^{\otimes 2} = O_C(Q_{2i-1} - Q_{2i}) \) and that \( s_j^{\otimes 2} = O_C(Q_{2j} - Q_{2j+1}) \). Also let
\[ L_i = O_C(P_{2i} - P_{2i-1}) \otimes f^*(n_i), \text{ for } 1 \leq i \leq \omega/2 \]
and let
\[ G_j = \mathcal{O}_C(P_{2j+1} - P_{2j}) \otimes f^*(s_j), \quad \text{for} \quad 1 \leq j \leq \omega/2 - 1 \]
respectively.

Then
\[ P[2] = f^*(J[2]) \oplus \bigoplus_{i=1}^{\omega/2-1} L_i \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{i=1}^{\omega/2-1} G_i \mathbb{Z}/2\mathbb{Z}. \]

Proof. Modulo \( f^*(J[2]) \) we may write
\[ F_{2i} = \bigoplus_{i} L_i \otimes G_i \quad \text{and} \quad F_{2i+1} = (\bigoplus_{i} L_i \otimes G_i)^{\tau}, \quad \text{for} \quad 1 \leq i \leq \omega/2 - 1. \]

Remark 4.4. Note that
\[ L_1 \otimes L_2 \otimes \ldots \otimes L_{\omega/2} = F_2 \otimes F_3 \otimes \ldots \otimes F_{\omega} \in f^*(J[2]). \]

5. Covers of degree three

For degree three covers, the following results appear in [Rec–Ro].

**Theorem 5.1.** Let \( u : Z \rightarrow X \) be a non-Galois cover of degree three.

Denote by \( g \) the genus of \( X \), by \( \alpha \) the number of total ramification points and by \( \beta \) the number of simple ramification points.

Then there is a curve \( W \) admitting the symmetric group on three letters \( S_3 = \langle \tau, \sigma : \tau^3 = \sigma^2 = (\tau \sigma)^2 = 1 \rangle \) as group of automorphisms and a commutative diagram of curves and covers as follows

\[\begin{array}{ccc}
W & \xrightarrow{\psi} & Y \\
\downarrow{\ell} & \quad & \downarrow{v} \\
Z & \xrightarrow{\gamma} & Y \\
\uparrow{u} & \quad & \uparrow{v} \\
X & \end{array}\]

where \( Y = W/\langle \tau \rangle \), \( Z = W/\langle \sigma \rangle \) and \( X = W/S_3 \).

Furthermore, there is an \( S_3 \)-equivariant isogeny
\[ R : JX \times P(Y/X) \times P(Z/X) \times P(Z/X) \rightarrow JW \]
given by \( R(x, y, z_1, z_2) = \gamma^* x + \psi^* y + \ell^* z_1 + \tau \ell^* z_2 \) where the action of \( S_3 \) is trivial on the first factor of the left hand side, the alternating action on the second factor, and the unique irreducible complex representation of degree two of \( S_3 \) on the product of the last two factors.

The cardinality of its kernel is given by
\[ |\ker R| = \begin{cases} 2^{2g-1} \cdot 3^{6g-3+\alpha} & \text{if } \beta = 0 \\ 2^{2g} \cdot 3^{6g-3+\alpha+\beta} & \text{if } \beta \neq 0 \end{cases}. \]

Remark 5.2. Observe that there are some natural restrictions on the data. Since \( \beta = |B(Y \rightarrow X)| \), we must have \( \beta \equiv 0 \pmod{2} \). Also, if \( g_X = 0 \), then \( Y \) and \( Z \) connected imply \( \beta \geq 2 \) and \( 2\alpha + \beta \geq 4 \).
The calculation of $|\ker R|$ mentioned above depends upon the following lemma (c.f. [Rec–Ro, p. 136]), similar in spirit to the description of the points of order two in a Prym variety for a degree two cover given in Section 4.1; we include it here for the sake of completeness.

**Lemma 5.3.** Let $W$ denote a curve with an $S_3$ action and associated Diagram (5.4).

Let

$$L = \{ z \in P(Z/X)[3] : \ell^* z = \tau \ell^* z \}$$

for $\tau$ an element of order three in $S_3$, and let $\{P_1, \ldots, P_\alpha \} \subseteq Z$ denote the set of total ramification points of $u : Z \to X$.

Also, if $\alpha$ is greater than or equal to two, for each $j$ in $\{2, \ldots, \alpha\}$ choose $m_j \in \Pic(X)$ such that $m_j^{\otimes 3} = \mathcal{O}_X(u(P_1) - u(P_j))$ and consider the element of $L$ given by

$$\mathfrak{f}_j = \mathcal{O}_Z(P_j - P_1) \otimes u^*(m_j).$$

Then $u^* : JX \to JZ$ is injective and

i) $u^*(JX[3]) \subseteq L$;

ii) for $\alpha$ equal to zero or one, $u^*(JX[3]) = L$. Thus in this case $L$ is isomorphic to $JX[3]$;

iii) for $\alpha$ greater than one, $L = u^*(JX[3]) \bigoplus_{j=2}^\alpha \mathfrak{f}_j Z/3Z$.

**Remark 5.4.** One can find in [R] the particular instance of Theorem 3.1 corresponding to the case $v : Y \to X = \mathbb{P}^1$ is a hyperelliptic cover and $\psi : W \to Y$ is a cyclic unramified cover of odd prime degree (in our notation, this is equivalent to $\alpha = 0$ and $X = \mathbb{P}^1$).

There he obtains an $S_3$–equivariant isomorphism

$$JZ \times JZ \to P(W/Y)$$

From our point of view, this situation is part of the more general $S_3$–equivariant isogeny $P(Z/X) \times P(Z/X) \to P(W/Y)$ described in [Rec–Ro]; together with the result that the group $L$ is isomorphic to the kernel of the isogeny

$$r : P(Z/X) \times P(Z/X) \to P(W/Y)$$

given by

$$r(z_1, z_2) = \ell^* z_1 + \tau \ell^* z_2$$

where $\tau$ is any element of order three in $S_3$, also in [Rec–Ro], and under the above assumptions, we also obtain that the isogeny $r$ is an isomorphism in this case.

In [Rec2] one can find the case when $u : Z \to X = \mathbb{P}^1$ is a simple trigonal cover (in our notation, this is equivalent to $\beta = 0$ and $X = \mathbb{P}^1$).
There, an $S_3$-equivariant isogeny is obtained

$$JY \times JZ \times JZ \to JW,$$

and statement ii) of the lemma is shown for the case $\alpha = 0$ (and $X = \mathbb{P}^1$).

6. Covers of degree four

In this section we study covers $f : X \to T$ of degree four.

There are five possibilities for $f$: if it is a Galois cover, then it may be either cyclic or given by the action of the Klein group on $X$; if $f$ is non Galois, then its corresponding Galois cover may be given either by the action of the dihedral group of order eight or by the alternating group on four letters or by the symmetric group on four letters.

We will see that the first three cases give the bigonal construction as a particular instance, whereas the last two imply the trigonal construction.

For each possibility of the Galois cover $W \to T$ with Galois group $G$, associated to $f : X \to T$, we will give a geometric decomposition of $JW$. Such decomposition is a $G$-equivariant isogeny between $JW$ and a product of appropriate Jacobians and Pryms of intermediate covers.

Moreover, each of the complex irreducible representations of $G$ corresponds to the action of $G$ on precisely one of the factors in the product.

We will also compute the kernel of each isogeny.

6.1. The cyclic case. If $X$ is a curve such that

$$\mathbb{Z}/4\mathbb{Z} = \langle g : g^4 = 1 \rangle \subseteq \text{Aut}(X)$$

then $T$ will denote the quotient $X/\langle g \rangle$ and $F$ will denote the quotient $X/\langle g^2 \rangle$; the corresponding maps will be denoted by $a : X \to F$ and $b : F \to T$.

We then have the following diagram of curves and covers.

(6.6)

\[
\begin{array}{c}
X \\
\downarrow a \\
F \\
\downarrow b \\
T
\end{array}
\]

Theorem 6.1. Let $X$ be a curve such that $\mathbb{Z}/4\mathbb{Z} = \langle g : g^4 = 1 \rangle \subseteq \text{Aut}(X)$ with associated Diagram (6.6).
Denote by $\delta$ the number of fixed points of $g$ in $X$ (i.e., the number of total ramification points of the cover $X \to T$), and by $\gamma$ the number of fixed points of $g^2$ in $X$ not fixed by $g$.

Then:

i) $g_F = 2g_T - 1 + \frac{\delta}{2}$ and $g_X = 2g_F - 1 + \frac{\gamma + \delta}{2} = 4g_T - 3 + \frac{\gamma + 3\delta}{2}$.

Furthermore, the cardinality of the ramification locus is: $\delta + \gamma$, for $a : X \to F$, and $\delta$, for $b : F \to T$.

In particular, the signature type of $T$ is $(g_T; 2, \ldots, 2, 4, \ldots, 4)$.

ii) There is a $\mathbb{Z}/4\mathbb{Z}$-equivariant isogeny

$\phi_{\mathbb{Z}/4\mathbb{Z}} : JT \times P(F/T) \times P(X/F) \to JW$

defined by

$\phi_{\mathbb{Z}/4\mathbb{Z}}(t, f, x) = a^*(b^*(t)) + a^*(f) + x$

where the action of $\mathbb{Z}/4\mathbb{Z}$ is: the trivial action on $JT$, the action given by the representation $g \to -1$ on $P(F/T)$, and, on $P(X/F)$, the sum of the other two irreducible representations of $\mathbb{Z}/4\mathbb{Z}$ given by $g \to I = \sqrt{-1}$ and $g \to -I$, respectively.

iii) The cardinality of the kernel of $\phi_{\mathbb{Z}/4\mathbb{Z}}$ is given by

$|\ker \phi_{\mathbb{Z}/4\mathbb{Z}}| = \begin{cases} 
2^{6g_T - 2 + \delta}, & \text{if } \delta > 0; \\
2^{6g_T - 3}, & \text{if } \delta = 0 \text{ and } \gamma > 0; \\
2^{6g_T - 4}, & \text{if } \delta = \gamma = 0.
\end{cases}$

Before proving the theorem, we state an elementary remark which will be useful in the sequel.

**Remark 6.2.** Let us observe that i) of the theorem imposes the conditions

$\gamma \equiv \delta \equiv 0 \pmod{2}$.

Moreover, if $g_T = 0$ we must have $\delta \geq 2$ in order that the corresponding covers be connected.

**Proof.** For the first statement, the cardinality of the ramification loci is clear, thus the formulae for the genera follow from the Riemann-Hurwitz formula.

For the second and third statements, note that by Proposition 2.6 and Remark 2.2 we have that if $\delta > 0$, all induced homomorphisms between Jacobians are injective, that if $\delta = 0$ and $\gamma > 0$, then $a^*$ is injective and $|\ker(b \circ a)^*| = |\ker b^*| = 2$, finally that if $\delta = \gamma = 0$, then $|\ker a^*| = |\ker b^*| = 2$, and then apply Proposition 2.6. □
6.2. The Klein case. Let $X$ be a curve such that the Klein group

$$\mathcal{K} = \langle \sigma, \tau : \sigma^2 = 1, \tau^2 = 1, (\sigma\tau)^2 = 1 \rangle$$

is contained in $\text{Aut}(X)$.

We let $T$ denote the quotient $X/\mathcal{K}$ and, for $k \in \mathcal{K}$, $k \neq 1$, we will let $X_k$ denote the quotient $X/\langle k \rangle$; the corresponding quotient maps will be denoted by $a_k : X \to X_k$, $b_k : X_k \to T$, and $f : X \to T$.

\begin{equation}
X \xleftarrow{a_\sigma} X_\sigma \xrightarrow{a_\tau} X_{\tau} \xrightarrow{a_{\sigma\tau}} X_{\sigma\tau} \xrightarrow{b_{\sigma\tau}} T \xleftarrow{b_\sigma} X_\sigma \xrightarrow{b_\tau} X_{\tau} \xrightarrow{b_{\sigma\tau}} X_{\sigma\tau}
\end{equation}

**Theorem 6.3.** Let $X$ be a curve such that $\mathcal{K} \subseteq \text{Aut}(X)$ with associated Diagram (6.7).

We also let $2r$ (resp. $2s$, $2t$) denote the number of fixed points in $X$ of $\sigma\tau$ (resp. $\sigma$, $\tau$) (or equivalently, the cardinality of the ramification locus of $a_{\sigma\tau}, a_\sigma, a_\tau$, respectively). Then:

i) If $g$ denotes the genus of $T$, the genera of the intermediate covers and the cardinality of the corresponding ramification loci are given in the following table; in particular, the signature type of $T$ is $(g; 2, \ldots, 2, 2, \ldots, 2, 2, \ldots, 2)$. 
| genus | order of ramification $|B|$ |
|-------|------------------|
| $g_X = 4g - 3 + s + t + r$ | $|B(X \to X_\sigma)| = 2s$ |
| $g_{X_\sigma} = 2g - 1 + \frac{r + t}{2}$ | $|B(X \to X_\sigma)| = 2t$ |
| $g_{X_\tau} = 2g - 1 + \frac{s + r}{2}$ | $|B(X \to X_\tau)| = 2t$ |
| $g_{X_{\sigma\tau}} = 2g - 1 + \frac{s + t}{2}$ | $|B(X \to X_{\sigma\tau})| = 2r$ |

For $j, k, l \in \{\sigma, \tau, \sigma\tau\}$ all different, there are respective isogenies

$\phi_j : P(X_k/T) \times P(X_l/T) \to P(X/X_j)$

defined by

$\phi_j(x_1, x_2) = a_k^*(x_1) + a_l^*(x_2).$

Furthermore, $\ker \phi_j \subseteq P(X_k/T)[2] \times P(X_l/T)[2]$ and its cardinality is given as follows, where $B_j = |B(X \to X_j)|$.

$$|\ker \phi_j| = \begin{cases} 2^{2g_{\sigma\tau} - 2}, & \text{if } B_j = B_k = B_l = 0; \\ 2^{2g_{\sigma\tau} - 1}, & \text{if } B_j = 0 \text{ and exactly one of } B_k, B_l \text{ is zero}; \\ 2^{2g_{\sigma\tau}}, & \text{if } B_j = 0 \text{ and } B_kB_l > 0; \\ 2^{2g_{\sigma\tau} - 1 + B_j/2}, & \text{if } B_j > 0. \end{cases}$$

iii) There is an isogeny

$\varphi : P(X_{\sigma\tau}/T) \times P(X_\sigma/T) \times P(X_\tau/T) \to P(X/T)$

defined by

$\varphi(x_1, x_2, x_3) = a_{\sigma\tau}^*(x_1) + a_\sigma^*(x_2) + a_\tau^*(x_3).$
Moreover, \( \ker \varphi \subseteq P(X_{\sigma \tau}/T)[2] \times P(X_{\sigma}/T)[2] \times P(X_{\tau}/T)[2] \) and its cardinality is given by

\[
| \ker \varphi | = \begin{cases} 
2^{4g_T-4}, & \text{if all covers are unramified;} \\
2^{4g_T-3+r+s+t}, & \text{otherwise.}
\end{cases}
\]

iv) There is a \( K \)-equivariant isogeny

\[
\phi_K : JT \times P(X_{\sigma \tau}/T) \times P(X_{\sigma}/T) \times P(X_{\tau}/T) \to JX
\]

defined by

\[
\phi_K(t, x_1, x_2, x_3) = f^*(t) + a^*_\sigma(t_1) + a^*_\sigma(t_2) + a^*_\sigma(t_3)
\]

where the action of \( K \) is: the trivial one on \( JT \), the action of the irreducible representation of \( K \) given by \( \sigma \to -1 \) and \( \tau \to -1 \) on \( P(X_{\sigma \tau}/T) \), the action of the irreducible representation of \( G \) given by \( \sigma \to 1 \) and \( \tau \to -1 \) on \( P(X_{\sigma}/T) \), and, on \( P(X_{\tau}/T) \), the action of the irreducible representation of \( K \) given by \( \sigma \to -1 \) and \( \tau \to 1 \).

The cardinality of the kernel of \( \phi_K \) is given by

\[
| \ker \phi_K | = \begin{cases} 
2^{8g_T-6}, & \text{if all covers are unramified;} \\
2^{8g_T-4+r+s+t}, & \text{if exactly two of} \ r, s, t \ \text{are equal to zero;} \\
2^{8g_T-3+r+s+t}, & \text{if exactly one or if none of} \ r, s, t \ \text{is equal to zero.}
\end{cases}
\]

Remark 6.4. The restrictions on the data in this case are

\[
r \equiv s \equiv t \equiv 0 \quad (2)
\]

and if \( g_T = 0 \) then \( r + s, r + t \) and \( s + t \geq 2 \).

Proof. For any involution \( k \in K \), denote \( P_k = P(X_k/T) \).

Note that \( f^*(t) \in JX^K \), for any \( t \in JT \), and that if \( k \) is any involution in \( K \), then \( a^*_k(x) \in JX^{(k)} \), for any \( x \in JX_k \).

Moreover if \( h \) is any of the other two involutions in \( K \), then \( h \) induces the involution \( \tilde{h} \) on \( X_k \) which gives the cover \( b_k : X_k \to T \). By Corollary 3.5 we have that \( P_k \) is the connected component of the identity of \( \ker(1 + \tilde{h}) \). Therefore \( h(a^*_k(x)) = a^*_k(\tilde{h}(x)) = -a^*_k(x) \), for every \( x \in P_k \).

This remark proves that the homomorphism \( \phi_K \) is \( K \)-equivariant.

Consider the following homomorphisms

\[
\text{ps} = \text{ps}_{\sigma \tau} : P_{\sigma \tau} \times P(X/X_{\sigma \tau}) \to P(X/T), \ \text{ps}(y, x) = a^*_\sigma(y) + x
\]

and

\[
\phi : JT \times P(X/T) \to JX, \ \phi(t, x) = f^*(t) + x.
\]
By Propositions 2.6 and 2.5 we have that \( ps \) and \( \phi \) are isogenies with kernels of cardinalities
\[
|\ker ps| = |P_{\sigma\tau}[2]| \frac{|b^*_{\sigma\tau}JT \cap \ker a^*_{\sigma\tau}|}{|\ker a^*_{\sigma\tau}|}
\]
and
\[
|\ker \phi| = \frac{|JT[4]|}{|\ker f^*|} = \frac{4^{2g_T}}{|\ker f^*|}
\]
respectively.

By Remark 2.2 we have
\[
|\ker f^*| = \begin{cases} 
4, & \text{if all covers are unramified;} \\
2, & \text{if exactly two of } r, s, t \text{ are equal to zero;} \\
1, & \text{if exactly one or if none of } r, s, t \text{ is equal to zero.}
\end{cases}
\]

Now the homomorphism \( \varphi \) may be factored as \( \varphi = ps \circ (id_{P_{\sigma\tau}}, \phi_{\sigma\tau}) \), and the homomorphism \( \phi_K \) as \( \phi_K = \phi \circ (id_{JT}, \varphi) \).

Hence if we show that \( \phi_{\sigma\tau} \) is an isogeny, it will follow that \( \varphi \) and \( \phi_K \) are isogenies.

Assuming this, it also follows that
\[
|\ker \varphi| = |\ker ps| \cdot |\ker \phi_{\sigma\tau}| = |\ker \phi_{\sigma\tau}|P_{\sigma\tau}[2]|b^*_{\sigma\tau}JT \cap \ker a^*_{\sigma\tau}| \frac{|P_{\sigma\tau}[2]|}{|\ker a^*_{\sigma\tau}|}
\]
and that
\[
|\ker \phi_K| = |\ker \phi| \cdot |\ker \varphi| = \frac{4^{2g_T}}{|\ker f^*|} |\ker \phi_{\sigma\tau}|P_{\sigma\tau}[2]|b^*_{\sigma\tau}JT \cap \ker a^*_{\sigma\tau}| \frac{|P_{\sigma\tau}[2]|}{|\ker a^*_{\sigma\tau}|}.
\]

To complete the proof we therefore have to show that the homomorphism \( \phi_{\sigma\tau} : P(X_{\sigma}/T) \times P(X_{\tau}/T) \to JX \) is an isogeny onto \( P(X/X_{\sigma\tau}) \) and compute \( |\ker \phi_{\sigma\tau}|, |P_{\sigma\tau}[2]| \) and \( \frac{|b^*_{\sigma\tau}JT \cap \ker a^*_{\sigma\tau}|}{|\ker a^*_{\sigma\tau}|} \).

To show that \( \phi_{\sigma\tau}(P(X_{\sigma}/T) \times P(X_{\tau}/T)) \) is contained in \( P(X/X_{\sigma\tau}) \) it suffices to prove that \( (1 + a_{\sigma\tau})(\phi_{\sigma\tau}(x_1, x_2)) = 0 \) for all \( (x_1, x_2) \) in \( P(X_{\sigma}/T) \times P(X_{\tau}/T) \), since \( P(X_{\sigma}/T) \times P(X_{\tau}/T) \) is connected and since \( P(X/X_{\sigma\tau}) = (\ker(1 + a_{\sigma\tau}))^\circ \) by Corollary 3.5.

But it follows from the remark at the beginning of the proof of this theorem that
\[
(1 + a_{\sigma\tau})(\phi_{\sigma\tau}(x_1, x_2)) = (1 + a_{\sigma\tau})(a^\sigma_\tau(x_1) + a^\tau_\sigma(x_2)) = 0.
\]

Hence we have proven that \( \phi_{\sigma\tau} \) is a homomorphism from \( P(X_{\sigma}/T) \times P(X_{\tau}/T) \) to \( P(X/X_{\sigma\tau}) \). But these two varieties have the same dimension by i); therefore to show that \( \phi_{\sigma\tau} \) is an isogeny, it suffices to show that its kernel is finite, which is what we prove next.
The fact that if \((y, x) \in \ker ps\), then \(y \in P_{\sigma\tau}[2]\) follows from the proof of Proposition 2.6.

It follows that if \((x_1, x_2, x_3) \in \ker \varphi\), then we have \(x_1 \in P_{\sigma\tau}[2]\) by the factorization for \(\varphi\) given above. By symmetry we obtain the inclusion

\[\ker \varphi \subseteq P_{\sigma\tau}[2] \times P_{\sigma}[2] \times P_{\tau}[2].\]

Hence we may write

\[
\ker(\phi_{\sigma\tau}) = \{(x_1, x_2) \in P_{\sigma}[2] \times P_{\tau}[2] : a_{\sigma}^*(x_1) = a_{\tau}^*(x_2)\}
\]

\[= (a_{\sigma}^*, a_{\tau}^*)^{-1}\{(x, x) : x \in a_{\sigma}^*(P_{\sigma}[2]) \cap a_{\tau}^*(P_{\tau}[2])\}\]

which says it is finite, since

\[|\ker(\phi_{\sigma\tau})| = \deg a_{\sigma}^*|_{P_{\sigma}} \cdot \deg a_{\tau}^*|_{P_{\tau}} \cdot |a_{\sigma}^*(P_{\sigma}[2]) \cap a_{\tau}^*(P_{\tau}[2])|.
\]

The numbers appearing on the right side of the last equality and also the number \(\ker ps_{\sigma\tau} = |P_{\sigma\tau}[2]| \frac{|b_{\sigma\tau}^*JT \cap \ker a_{\sigma\tau}^*|}{|\ker a_{\sigma\tau}^*|}\) are computed in the Appendix, thus the proof is now complete.

\[\square\]

Remark 6.5. Given a double cover of a double cover \(X \to Y \to T\) which corresponds to a Galois four-fold cover, we have seen that in both possible cases - a cyclic cover or a Klein group action cover - there is another such object \(X' \to Y' \to T\) (a double cover of a double cover of \(T\)) which is naturally associated to the original one: the same one for the cyclic case and any of the other two in Diagram (6.7) for the Klein case.

In other words, the cyclic and the Klein constructions together give the bigonal construction (see [D1]) for the orientable cover case.

In another section (see 7.1) we will complete the bigonal construction to include the non-orientable case.
7. The dihedral case

Let $W$ be a curve such that the dihedral group of order eight $D_4 = \langle r, s : r^4 = 1, s^2 = 1, (rs)^2 = 1 \rangle$ is contained in $\text{Aut}(W)$.

We let $T$ denote the quotient $W/D_4$, and for $d \in D_4$, $d \neq 1$, we will let $W_d$ denote the quotient $W/\langle d \rangle$.

Let $K_s$ (resp. $K_{rs}$) denote the Klein subgroup of $D_4$ generated by $r^2$ and $s$ (resp. $r^2$ and $rs$), and let $W_{K_s}$ (resp. $W_{K_{rs}}$) denote the quotient $W/K_s$ (resp. $W/K_{rs}$).

The corresponding quotient maps will be denoted by: $\gamma : W \to T$; for $n \in \{s, r^2s, rs, rs^3\}$, $a_n : W \to W_n$; for $n \in \{s, r^2s, r^2\}$, $b_n : W_n \to W_{K_s}$; for $n \in \{rs, r^3s, r^2\}$, $c_n : W_n \to W_{K_{rs}}$; for $n \in \{K_s, K_{rs}, r^2\}$, $d_n : W_n \to T$; and $e : W_{r^2} \to W_r$.

Then we have the following diagram of curves and covers.

\[(7.9)\]

\[
\begin{array}{ccccccc}
  & & W & & W & & W & & W \\
  & a_{r^2} & & a_{r^2} & & a_{r^2} & & a_{r^2} \\
W_{r^2s} & & W_s & & W_{r^2} & & W_{rs} & & W_{r^2s} \\
    & b_s & & e & & c_{rs} & & d_{K_s} \\
    & W_{K_s} & & W_r & & W_{K_{rs}} & & T \\
\end{array}
\]

**Theorem 7.1.** Let $W$ be a curve such that $D_4 \subseteq \text{Aut}(W)$ with associated Diagram (7.9).

We let $2\delta$, $2\alpha$, $2\gamma_1$, $4\gamma_2$ denote the number of fixed points in $W$ of $r$, $s$ (or $r^2s$), $rs$ (or $r^3s$), $r^2$ not fixed by $r$, respectively.

Then:

i) If $g$ denotes the genus of $T$, the genera of the intermediate covers and the cardinality of the corresponding ramification loci are given in the following table.

In particular, the signature type of $T$ is

\[ (g; 4, \ldots, 4, 2, \ldots, 2, 2, \ldots, 2, 2, \ldots, 2) \].
PRYM VARIETIES AND FOURFOLD COVERS

\begin{equation*}
\begin{array}{|c|c|}
\hline
\text{genus} & \text{order of ramification } |B| \\
\hline
\text{g}_{W} = 8g - 7 + 2\alpha + 2\gamma_{1} + 2\gamma_{2} + 3\delta & |B(W \to W_{s})| = 2\alpha = \\
\hline
\text{g}_{W_{r,2s}} = 4g - 3 + \frac{\alpha + 3\delta}{2} + \gamma_{1} + \gamma_{2} = & |B(W \to W_{r^{2s}})| \\
\hline
\text{g}_{W_{r,s}} = 4g - 3 + \frac{\gamma_{1} + 3\delta}{2} + \alpha + \gamma_{2} = & |B(W \to W_{rs})| = 2\gamma_{1} = \\
\hline
\text{g}_{W_{r,2}} = 4g - 3 + \alpha + \gamma_{1} + \delta = & |B(W \to W_{r^{2}})| = 4\gamma_{2} + 2\delta \\
\hline
\text{g}_{W_{K,s}} = 2g - 1 + \frac{\gamma_{1} + \delta}{2} = & |B(W_{s} \to W_{Ks})| = \alpha + 2\gamma_{2} + \delta \\
\text{} & |B(W_{r,2} \to W_{Ks})| = 2\alpha \\
\hline
\text{g}_{W_{K,rs}} = 2g - 1 + \frac{\alpha + \delta}{2} = & |B(W_{rs} \to W_{K,rs})| = \gamma_{1} + 2\gamma_{2} + \delta \\
\text{} & |B(W_{r,2} \to W_{K,rs})| = 2\gamma_{1} \\
\hline
\text{g}_{W_{r}} = 2g - 1 + \frac{\alpha + \gamma_{1}}{2} = & |B(W_{r,2} \to W_{r})| = 2\delta \\
\hline
\text{} & |B(W_{K,s} \to T)| = \gamma_{1} + \delta \\
\text{} & |B(W_{r} \to T)| = \alpha + \gamma_{1} \\
\text{} & |B(W_{K,rs} \to T)| = \alpha + \delta \\
\hline
\end{array}
\end{equation*}

ii) There is a \(D_{4}\)-equivariant isogeny

\(\phi_{D_{4}}: JT \times P(W_{r}/T) \times (P(W_{K,s}/T) \times P(W_{K,rs}/T) \times 2P(W_{s}/W_{Ks}) \to JW\)

given by

\(\phi_{D_{4}}(t, w_{1}, w_{2}, w_{3}, y_{1}, y_{2})) = \gamma^{*}(t) + (e \circ a_{r^{2}})^{*}(w_{1}) + (b_{s} \circ a_{s})^{*}(w_{2}) + (c_{rs} \circ a_{rs})^{*}(w_{3}) + a_{s}^{*}(y_{1}) + ra_{s}^{*}(y_{2})\)
where the action of $D_4$ is: the trivial one on $JT$, the action of the irreducible representation of $D_4$ given by $r \to 1$ and $s \to -1$ on $P(W_r/T)$, the action of the irreducible representation of $D_4$ given by $r \to -1$ and $s \to 1$ on $P(W_{Ks}/T)$, the action of the irreducible representation of $D_4$ given by $r \to -1$ and $s \to -1$ on $P(W_{Krs}/T)$, and, on $2P(W_s/W_{Ks})$, the action of the unique irreducible representation of degree two of $D_4$.

The kernel of $\phi_{D_4}$ has cardinality as follows.

| $|\ker \phi_{D_4}|$ | Case |
|------------------|------|
| $2^{20g-17}$     | if $\alpha = \gamma_1 = \gamma_2 = \delta = 0$ |
| $2^{20g-15+4\gamma_1}$ | if $\alpha = \gamma_2 = \delta = 0$ and $\gamma_1 > 0$ |
| $2^{20g-15}$     | if $\alpha = \gamma_1 = \delta = 0$ and $\gamma_2 > 0$ |
| $2^{20g-14+3\alpha}$ | if $\gamma_1 = \gamma_2 = \delta = 0$ and $\alpha > 0$ |
| $2^{20g-13+4\delta}$ | if $\alpha = \gamma_1 = 0$ and $\delta > 0$, for any $\gamma_2$ |
| $2^{20g-13+4\gamma_1}$ | if $\alpha = \delta = 0$ and $\gamma_1 \cdot \gamma_2 > 0$ |
| $2^{20g-12+4\delta+4\gamma_1}$ | if $\alpha = 0$ and $\delta \cdot \gamma_1 > 0$, for any $\gamma_2$ |
| $2^{20g-13+3\alpha+4\gamma_1+2\gamma_2}$ | if $\delta = 0$, $\alpha > 0$, and $\gamma_1 = 0$, $\gamma_2 > 0$ or $\gamma_2 = 0$, $\gamma_1 > 0$ |
| $2^{20g-12+3\alpha+4\gamma_1+2\gamma_2+5\delta}$ | if $\alpha \cdot \gamma_1 \cdot \gamma_2 > 0$, for any $\delta$ |

iii) There is a natural isogeny (the bigonal construction)

$$\text{Nm}_{a_{rs}} \circ a_{s}^{*}|_{P(W_{rs}/W_{Ks})} : P(W_{s}/W_{Ks}) \to P(W_{rs}/W_{Krs}).$$
If we denote by $G$ the Klein subgroup of $JW_{K,s}[2]$ giving the covers $c_n : W_n \rightarrow W_{K,s}$ for $n \in \{rs, r^2s, r^2\}$ for the case $\delta = \gamma_1 = \gamma_2 = 0$, then the kernel of the isogeny has cardinality given as follows.

$$\ker Nm a_{rs} \circ a_s^*|_{P(W_s/W_{K,s})} = \begin{cases} 2^{2g_T - 5 + 2\delta}, & \text{if } \delta > 0 \text{ and } \alpha = \gamma_1 = \gamma_2 = 0; \\ 2^{2g_T - 4 + 2\delta + \gamma_1 + \gamma_2}, & \text{if } \delta > 0 \text{ and} \\
& \text{either } (\gamma_1 > 0 \text{ and } \alpha = \gamma_2 = 0) \\
& \text{or } (\alpha > 0 \text{ and } \gamma_1 = \gamma_2 = 0) \\
& \text{or } (\gamma_2 > 0, \gamma_1 = 0 \text{ and any } \alpha); \\ 2^{2g_T - 3 + 2\delta + \gamma_1 + \gamma_2}, & \text{if } \gamma_1 > 0 \text{ and} \\
& \text{either } (\delta = \alpha = 0 \text{ and any } \gamma_2) \\
& \text{or } (\delta > 0 \text{ and } \gamma_2 = 0); \\ 2^{2g_T - 2 + \delta + \gamma_1 + \gamma_2}, & \text{if } \delta = 0 \text{ and} \\
& \text{either } (\gamma_1 \alpha > 0 \text{ and any } \gamma_2) \\
& \text{or } (\gamma_1 = \alpha = \gamma_2 = 0 \text{ and } G \text{ isotropic}) \\
& \text{or } (\gamma_1 = \alpha = 0 \text{ and } \gamma_2 > 0); \\ 2^{2g_T - 1 + \gamma_2}, & \text{if } \delta = \gamma_1 = 0 \text{ and} \\
& \text{either } (\gamma_2 \alpha > 0) \\
& \text{or } (\gamma_2 = \alpha = 0 \text{ and } G \text{ non isotropic}) \\
& \text{or } (\gamma_2 = 0, \alpha > 0 \text{ and } G \text{ isotropic}); \\ 2^{2g_T}, & \text{if } \delta = \gamma_1 = \gamma_2 = 0, \\
& \alpha > 0 \text{ and } G \text{ non isotropic.} \end{cases}$$

Remark 7.2. The conditions on the data in this case are the following.

$$\gamma_1 + \delta, \alpha + \delta, \alpha + \gamma_1 \equiv 0 \quad (2).$$

Moreover, if $g = g_T = 0$ then $\gamma_1 + \delta, \alpha + \delta$ and $\alpha + \gamma_1 \geq 2$.

Proof. Consider the following two subdiagrams of (7.9):

\begin{equation}
(7.10)
\end{equation}
They correspond to actions of the Klein group, thus we can apply Theorem 6.3 to obtain isogenies

$$\phi_K : JT \times P(W_r/T) \times P(W_{K_s}/T) \times P(W_{K_{s'}}/T) \to JW_{r^2}$$

and

$$\phi_{r^2} : P(W_s/W_{K_s}) \times P(W_{r^2s}/W_{K_s}) \to P(W/W_{r^2})$$

defined by

$$\phi_K(t, w_1, w_2, w_3) = (d_r \circ e)^*(t) + e^*(w_1) + b_{r^2}^*(w_2) + c_{r^2}^*(w_3)$$

and

$$\phi_{r^2}(z_1, z_2) = a_s^*(z_1) + a_{r^2s}^*(z_2)$$

respectively.

Furthermore, we have that the cardinality of their kernels is given by

(7.11)

$$|\ker \phi_K| = \begin{cases} 2^{8gT-6}, & \text{if } \alpha = \delta = \gamma_1 = 0; \\ 2^{8gT-4+\alpha+\delta+\gamma_1}, & \text{if exactly two of } \alpha, \delta, \gamma_1 \text{ are zero;} \\ 2^{8gT-3+\alpha+\delta+\gamma_1}, & \text{if exactly one or none of } \alpha, \delta, \gamma_1 \text{ is zero;} \end{cases}$$

and

$$|\ker \phi_{r^2}| = \begin{cases} 2^{16gT-14+4\gamma_1}, & \text{if } \alpha = \delta = \gamma_2 = 0; \\ 2^{16gT-12+4\gamma_1+2\gamma_2+5\delta}, & \text{if } \alpha = 0 \text{ and } \gamma_2 + \delta > 0 \\ 2^{16gT-11+2\alpha+4\gamma_1+2\gamma_2+5\delta}, & \text{if } \alpha > 0. \end{cases}$$

Now since $$r (r^2s) = s r$$, the automorphism $$r : JW \to JW$$ (induced by $$r : W \to W$$) induces an isomorphism $$\tilde{r} : JW_s \to JW_{r^2s}$$, therefore

$$r|_{a_s^*(P(W_s/W_{K_s}))} : a_s^*(P(W_s/W_{K_s})) \to a_{r^2s}^*(P(W_{r^2s}/W_{K_s}))$$

is an isomorphism.

In particular, it follows that $$a_{r^2s}^*(\tilde{r}(y)) = ra_s^*(y)$$, for any $$y \in P(W_s/W_{K_s})$$. By composing with $$\phi_{r^2}$$ we obtain an isogeny

$$g : 2P(W_s/W_{K_s}) \to P(W/W_{r^2})$$

defined by

$$g(y_1, y_2) = a_s^*(y_1) + ra_s^*(y_2)$$

whose kernel has cardinality given by

(7.12) $$|\ker g| = \begin{cases} 2^{16gT-14+4\gamma_1}, & \text{if } \alpha = \delta = \gamma_2 = 0; \\ 2^{16gT-12+4\gamma_1+2\gamma_2+5\delta}, & \text{if } \alpha = 0 \text{ and } \gamma_2 + \delta > 0 \\ 2^{16gT-11+2\alpha+4\gamma_1+2\gamma_2+5\delta}, & \text{if } \alpha > 0. \end{cases}$$
By Proposition 2.5 we also have an isogeny
\[ \phi : JW_r^2 \times P(W_r^2/T) \to JW \]
given by
\[ \phi(y,x) = a_r^*(y) + x \]
whose kernel has cardinality
\[ |\ker \phi| = \frac{|JW_r^2[2]|}{|\ker a_r^*|} = \frac{2^{8gT-6+\delta+\gamma_1}}{|\ker a_r^*|}. \]

By Remark 2.2 we know that \(|\ker a_r^*| = 2\), if \(\delta = \gamma_2 = 0\), and that \(|\ker a_r^*| = 1\), otherwise.

Therefore
\[ (7.13) \quad |\ker \phi| = \begin{cases} 2^{8gT-7+\alpha+2\gamma_1}, & \text{if } \delta = \gamma_2 = 0; \\ 2^{8gT-6+\delta+\alpha+2\gamma_1}, & \text{if } \delta + \gamma_2 > 0. \end{cases} \]

Now note that
\[ \phi_{D_4} = \phi \circ (\phi_K, g) \]
and it follows that \(\phi_{D_4}\) is an isogeny whose kernel has cardinality
\[ |\ker \phi_{D_4}| = |\ker \phi| \cdot |\ker \phi_K| \cdot |\ker g|. \]

Combining this equality with (7.13), (7.11), and (7.12) we obtain (7.12 ii).

Concerning the equivariance, note that the action of the irreducible representation of order two of \(D_4\) is given by \(r(x,y) = (-y,x)\) and \(s(x,y) = (x,-y)\).

For statement iii), we first prove that \(\Nm a_{rs}(a_r^*(P(W_s/W_{K_s})))\) is contained in \(P(W_{rs}/W_{K_{rs}})\); then we show that \(\Nm a_{rs} \circ a_r^*\) restricted to \(P(W_s/W_{K_s})\) is an isogeny and finally we compute the cardinality of its kernel.

Let us denote by
\[ h = (\Nm a_{rs} \circ a_r^*)|_{P(W_s/W_{K_s})} \]
the restriction of \(\Nm a_{rs} \circ a_r^*\) to \(P(W_s/W_{K_s})\).

Given \(x \in P(W_s/W_{K_s})\) we know from the Klein case that \(a_r^*(x)\) is in \(P(W/W_{rs})\) and hence \(\Nm a_{rs}(a_r^*(x)) = 0\).

Therefore
\[ \Nm c_{rs}(h(x)) = (\Nm c_{rs} \circ \Nm a_{rs})(a_r^*(x)) = (\Nm c_{r,t} \circ \Nm a_{r,t})(a_r^*(x)) = 0 \]
and it follows that \(h(P(W_s/W_{K_s})) \subseteq P(W_{rs}/W_{K_{rs}}).\)
To show that $h$ is an isogeny, denote by 

$$A = a^*_s(P(W_s/W_{K_s}))$$

and by 

$$B = a^*_{rs}(P(W_{rs}/W_{K_{rs}}))$$

Then, by Corollary 3.5 iii), we know that 

$$A = \{ z \in JW^{(s)} : z + r^2sz = 0 \}^\circ$$

and that 

$$B = \{ w \in JW^{(rs)} : w + r^2w = 0 \}^\circ.$$ 

Moreover, the endomorphisms of $JW$ given by $1 + rs$ and $1 + s$ induce morphisms which fit in the following commutative diagram.

\[ \begin{array}{c}
A \\
\downarrow h \\
P(W_s/W_{K_s}) \\
\downarrow a^*_{rs} \\
P(W_{rs}/W_{K_{rs}}) \\
\downarrow \text{Nm}_{a_{rs}} \\
A
\end{array} \quad \begin{array}{c}
1+rs \\
\downarrow a^*_{rs} \\
P(W_s/W_{K_s}) \\
\downarrow \text{Nm}_{a_{rs}}a^*_{rs} \\
P(W_{rs}/W_{K_{rs}}) \\
\downarrow \text{Nm}_{a_{rs}} \\
P(W_s/W_{K_s}) \\
\downarrow a^*_{s} \\
A
\end{array} \]

and a short computation shows that $(1 + s) \circ (1 + rs) = 2_A$ and $(1 + rs) \circ (1 + s) = 2_B$, respectively. Therefore $h$ is an isogeny.

We now compute the cardinality of the kernel $K$ of $h$ through the next steps.

1) Since 

$$\left| \ker ((1 + rs) \circ a^*_s)_{P(W_s/W_{K_s})} \right| = \left| \ker (a^*_{rs} \circ h) \right|,$$

we have that 

$$|K| = \frac{| \ker ((1 + rs) \circ a^*_s)_{P(W_s/W_{K_s})} |}{| \ker (a^*_{rs})_{P(W_{rs}/W_{K_{rs}})} |}$$

Now from the rightmost diagram in (7.10) we compute 

(7.16) 

$$|\ker (a^*_{rs})_{P(W_{rs}/W_{K_{rs}})}| = \left\{ \begin{array}{ll} 
1, & \text{if } \gamma_1 > 0; \\
2, & \text{if } \gamma_1 = 0 \text{ and } \gamma_2 + \delta > 0; \\
1, & \text{if } \gamma_1 = \gamma_2 = \delta = 0 \text{ and } G \text{ is non-isotropic}; \\
2, & \text{if } \gamma_1 = \gamma_2 = \delta = 0 \text{ and } G \text{ is isotropic}. 
\end{array} \right.$$ 

if $G$ denotes the Klein subgroup of $JW_{K_{rs}}[2]$ giving the covers in the diagram for the case when they are all unramified.
2) Since $a_s^*$ and $a_{rs}^*$ are isogenies and since $(1 + s) \circ (1 + rs) = 2$, it follows that $\text{Nm} a_s \circ a_{rs}^* \circ h = 2P(W_s/W_{K_s})$, so $\ker h \subseteq P((W_s/W_{K_s})[2]$. The same holds for

$$\Gamma = \ker((1 + r) \circ a_{s}^*|_{P(W_s/W_{K_s})} \subseteq P(W_s/W_{K_s})[2]$$

In fact, we see immediately that

$$\Gamma = \{x \in P(W_s/W_{K_s})[2] : a_{s}^*(x) \text{ is } D_4 \text{ -- invariant}\}$$

3) We compute the cardinality of $\Gamma$ by decomposing it into two parts: one coming from $JW_{W_{K_s}}$ and one coming from the ramification of $b_s : W_s \to W_{K_s}$. Therefore we define

$$\Gamma_1 = b_s^*(JW_{W_{K_s}}^{(\sigma)}) \cap P(W_s/W_{K_s})[2]$$

where $\sigma : W_{K_s} \to W_{K_s}$ denotes the involution induced by $rs$ and we will show that

$$\Gamma = \Gamma_1 \oplus \text{part coming from the ramification of } b_s^*$$

In particular, if $b_s$ is unramified then it is clear that $\Gamma = \Gamma_1$.

4) To compute $\Gamma_1$ we first apply Corollary 3.2 to the cover $d_{K_s} : W_{K_s} \to T$ to obtain

$$JW_{W_{K_s}}^{(\sigma)} = d_{K_s}^*(JT) + P(W_{K_s}/T)[2]$$

and therefore

$$JW_{W_{K_s}}^{(\sigma)}[2] = (d_{K_s}^*(JT))[2] + P(W_{K_s}/T)[2]$$

But we also know (see [M2] and Section 4) that

$$P(W_{K_s}/T)[2] \subseteq (d_{K_s}^*(JT))[2] \text{ if } d_{K_s} \text{ is unramified}$$

and

$$(d_{K_s}^*(JT))[2] = d_{K_s}^*(JT[2]) \subseteq P(W_{K_s}/T)[2] \text{ if } d_{K_s} \text{ is ramified}$$
Combining this information with (7.18) we obtain

\[
\begin{array}{|c|c|c|}
\hline
 JW_{K_s}^{(\sigma)}[2] & | JW_{K_s}^{(\sigma)}[2] | & \text{Case} \\
\hline
 (d_{K_s}^*(JT))[2] & 2^{2g_T} & \delta = \gamma_1 = 0 \\
\hline
 P(W_{K_s}/T)[2] & 2^{2g_T-2+\delta+\gamma_1} & \delta + \gamma_1 > 0 \\
\hline
\end{array}
\]

On the other hand we can now prove the following.

**Claim 7.3.** The following is always true.

\[
JW_{K_s}^{(\sigma)} \cap \ker b_s^* = \{0\}
\]

In particular,

\[
P(W_{K_s}/T)[2] \cap \ker b_s^* = \{0\}.
\]

Proof of the claim: if \(b_s : W_s \to W_{K_s}\) is ramified, then \(b_s^* : JW_{K_s} \to JW_s\) is injective, by Proposition 2.1 and the claim follows.

If \(b_s : W_s \to W_{K_s}\) is unramified then all maps on the rightmost diagram of (7.10) are unramified.

Let \(\ker b_s^* = \{0, \eta_{b_s}\} \subseteq JW_{K_s}[2]\) the element defining the cover \(b_s\). Also, let \(\sigma : W_{K_s} \to W_{K_s}\) denote the involution induced by \(rs : W \to W\).

¿From the following commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{r_s} & W \\
\downarrow{a_z} & & \downarrow{a_s} \\
W_{rz} & & W_s \\
\downarrow{b_z} & & \downarrow{b_s} \\
W_{K_s} & \xrightarrow{\sigma} & W_{K_s}
\end{array}
\]

we see that \(\sigma^*(\eta_{b_s})\) defines the cover \(b_{rz} : W_{rz} \to W_{K_s}\) and, in particular, \(\sigma^*(\eta_{b_s}) \not= \eta_{b_s}\); i.e., \(\eta_{b_s} \not\in JW_{K_s}^{(\sigma)}\), and the first part of the claim follows in this case.

The last part follows from (7.18), since it shows that \(P(W_{K_s}/T)[2] \subseteq JW_{K_s}^{(\sigma)}\).
We now continue the proof of the Theorem.
The claim just proved shows that \( b^* (JW_{K_s}^{(\sigma)}) \) is isomorphic to \( JW_{K_s}^{(\sigma)} \).

5) Computation of \( \Gamma_1 \), for the case \( b_s \) ramified: i.e., \( \alpha + 2\gamma_2 + \delta > 0 \).

In this case it follows from [M2] and Section 4 that \( b^*_s \) injects \( JW_{K_s}[2] \) into \( P(W_s/W_{K_s})[2] \) and from Claim 7.3 that

\[
\begin{align*}
    b^*_s (JW_{K_s}^{(\sigma)})[2] &= b^*_s (JW_{K_s}^{(\sigma)}[2] ) \subseteq P(W_s/W_{K_s})[2] \\

\end{align*}
\]

Therefore

\[
\Gamma_1 = b^*_s (JW_{K_s}^{(\sigma)}[2] ) \cong JW_{K_s}^{(\sigma)}[2]
\]

and from (7.19) we obtain the following.

| \( \Gamma_1 \) | \( |\Gamma_1| \) | Case |
|------------------|----------------|------|
| \( b^*_s ((dK_s^*(JT))[2]) \) | 2^\( 2gT \) | \( \delta = \gamma_1 = 0 \) and \( \alpha + 2\gamma_2 > 0 \) |
| \( b^*_s (P(W_{K_s}/T)[2]) \) | 2^\( 2gT-2+\delta+\gamma_1 \) | \( \delta + \gamma_1 > 0 \) and \( \alpha + 2\gamma_2 + \delta > 0 \) |

(7.20)

6) Computation of \( \Gamma_1 \), for the case \( b_s \) unramified: i.e., \( \alpha = \gamma_2 = \delta = 0 \).

Note that in this case

\[
\Gamma = \Gamma_1
\]

It follows from [M2] and Section 4 that if \( \ker b^*_s = \{ 0, \eta_{b_s} \} \subseteq JW_{K_s}[2] \) then

\[
\{ 0, \eta_{b_s} \}^\perp / \{ 0, \eta_{b_s} \} \cong b^*_s (\{ 0, \eta_{b_s} \}^\perp ) = P(W_s/W_{K_s})[2]
\]

Let us consider the map

\[
ps : P(W_{K_s}/T) \times P(W_s/W_{K_s}) \to P(W_s/T) \\
(x, y) \to b^*_s (x) + y
\]

whose kernel has order \( \frac{1}{2} |P(W_{K_s}/T)[2]| \), from Proposition 2.6.
It is clear that this kernel is isomorphic to 

\[ F = \{ x \in P(W_K/T)[2] : b_s^*(x) \in P(W_s/W_K)[2] \} \]

via the map \( F \rightarrow \ker ps \) given by \( x \rightarrow (x, -b_s^*(x)) \).

We now subdivide into two cases, according to Table 7.19.

6.1) If \( d_{K_s} \) is ramified, we know that \( JW^{(\sigma)}_{K_s}[2] = P(W_K/T)[2] \).

Then (from Claim 7.3) we have

\[ F \cong b_s^*(F) = b_s^*(JW^{(\sigma)}_{K_s}[2]) \cap P(W_s/W_K)[2] = \Gamma_1 \]

Therefore we have proved that

\[
(7.21) \quad |\Gamma_1| = 2^{2\dim P(W_s/T) - 1} = 2^{2g - 3 + \gamma_1},
\]

if \( \alpha = \gamma_2 = \delta = 0 \) and \( \gamma_1 > 0 \)

6.2) If \( d_{K_s} \) is unramified, then \( JW^{(\sigma)}_{K_s}[2] = (d_{K_s}^*JT)[2] \).

Since \( b_s^*(\{0, \eta_{bs}\}^\perp) = P(W_s/W_K)[2] \) we can write

\[ F = \{0, \eta_{bs}\}^\perp \cap P(W_K/T)[2] \text{ of index two in } P(W_K/T)[2] \, . \]

Therefore \( P(W_K/T)[2] \) is not contained in \( \{0, \eta_{bs}\}^\perp \); since \( P(W_K/T)[2] \subseteq (d_{K_s}^*JT)[2] \) we obtain that

\[ (d_{K_s}^*JT)[2] \nsubseteq \{0, \eta_{bs}\}^\perp . \]

Moreover, \( \{0, \eta_{bs}\}^\perp \) is of index two in \( JW_K[2] \), and hence

\[ (d_{K_s}^*JT)[2] \cap \{0, \eta_{bs}\}^\perp \text{ is of index two in } (d_{K_s}^*JT)[2] \, . \]

But

\[ b_s^*((d_{K_s}^*JT)[2] \cap \{0, \eta_{bs}\}^\perp) = b_s^*((d_{K_s}^*JT)[2]) \cap b_s^*(\{0, \eta_{bs}\}^\perp) = \Gamma_1 \]

where the first equality holds since \( z = b_s^*(x) = b_s^*(y) \) with \( x \in (d_{K_s}^*(JT))[2] \) and \( y \in (\ker b_s^*)^\perp \) implies that \( x = y + w \) with \( w \in \ker b_s^* \), which shows that \( x \in (\ker b_s^*)^\perp \) and therefore \( z \in b_s^*((d_{K_s}^*JT)[2] \cap \{0, \eta_{bs}\}^\perp) \).

It follows that \( |\Gamma_1| = \frac{1}{2}|(d_{K_s}^*JT)[2]| = \frac{1}{2}|JT[2]| \).

Therefore we have proved the following.

\[
(7.22) \quad |\Gamma_1| = 2^{2g - 1}, \text{ if } \alpha = \gamma_2 = \delta = \gamma_1 = 0.
\]
Putting together (7.21) and (7.22) we obtain

\[(7.23)\]

| \( \Gamma = \Gamma_1 \) | \( |\Gamma| \) | Case |
|----------------------|-------------|------|
| \( b_s^* (P(W_{K_s} / T)[2]) \cap P(W_s/W_{K_s}) [2] \) | \( 2^{2g_T - 3 + \gamma_1} \) | \( \alpha = \gamma_2 = \delta = 0 \) and \( \gamma_1 > 0 \) |
| \( b_s^* ((d_{K_s}^* JT)[2] \cap \{0, \eta b_s \}^\perp) \) | \( 2^{2g_T - 1} \) | \( \alpha = \gamma_2 = \delta = \gamma_1 = 0 \) |

7) We now complete the description of \( \Gamma \) (in the case \( b_s \) ramified: \( \alpha + 2\gamma_2 + \delta > 0 \)) by looking for those elements coming from the ramification of \( b_s \).

Denote by \( \{Q_1, \ldots, Q_{2\gamma_2}, \ldots, Q_{2\gamma_2 + \delta + \alpha}\} \) the ramification points of \( b_s \) ordered such that the corresponding branch points in \( T \)

\[ \{d_{W_{K_s}} (b_s(Q_i)) = d_{W_{K_s}} (b_s(Q_{i+1}))\}_{i \in \{1,3,\ldots,2\gamma_2-1\}} \text{ are of type } \gamma_2; \]

those of type \( \delta \) are

\[ \{d_{W_{K_s}} (b_s(Q_i))\}_{2\gamma_2 + 1 \leq i \leq 2\gamma_2 + \delta}; \]

and finally, those of type \( \alpha \) are

\[ \{d_{W_{K_s}} (b_s(Q_i))\}_{2\gamma_2 + \delta + 1 \leq i \leq 2\gamma_2 + \delta + \alpha}. \]

Now choose \( m_i \in \text{Pic}^0 W_{K_s} \) such that

\[ m_i^{\otimes 2} = \mathcal{O}_{W_{K_s}} (b_s(Q_{i+1}) - b_s(Q_i)) \]

and set

\[ \mathcal{F}_i = \mathcal{O}_{W_s} (Q_i - Q_{i+1}) \otimes b_s^*(m_i), \]

for \( 1 \leq i \leq 2\gamma_2 + \delta + \alpha - 1 \).

Again, by Proposition 4.1 we obtain the description
7.1. The bigonal construction. Giving a curve \( W \) with a \( \mathcal{D}_4 \) action and associated Diagram (7.9) is equivalent to giving a non-Galois

| \( P(W_s/W_{K_s})[2] \) | Case |
|-----------------|------|
| \( b_*(JW_{K_s}[2]) \oplus \bigoplus_{i=1}^{2\gamma_2-1} F_i\mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{i=1}^{\delta+\alpha-1} F_{2\gamma_2+i}\mathbb{Z}/2\mathbb{Z} \) | \( \gamma_2 \cdot (\delta + \alpha) > 0 \) |
| \( b_*(JW_{K_s}[2]) \oplus \bigoplus_{i=1}^{\gamma_2-2} F_i\mathbb{Z}/2\mathbb{Z} \) | \( \delta = \alpha = 0 \) and \( \gamma_2 > 0 \) |
| \( b_*(JW_{K_s}[2]) \oplus \bigoplus_{i=1}^{\delta+\alpha-2} F_i\mathbb{Z}/2\mathbb{Z} \) | \( (\delta + \alpha) > 0 \) and \( \gamma_2 = 0 \) |

Noting that the \( a_i^*(F_i) \) are \( \mathcal{D}_4 \)-invariant precisely for \( i \in \{1, 3, \ldots, 2\gamma_2 - 1\} \) and for \( 2\gamma_2 + 1 \leq i \leq 2\gamma_2 + \delta - 1 \), we obtain that

\[
\Gamma = \begin{cases} 
\Gamma_1 \oplus \bigoplus_{i=1}^{\gamma_2} F_{2i-1}\mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{i=1}^{\delta-1} F_{2\gamma_2+i}\mathbb{Z}/2\mathbb{Z} & \text{if } \gamma_2 > 0 \text{ and } \delta > 0; \\
\Gamma_1 \oplus \bigoplus_{i=1}^{\gamma_2} F_{2i-1}\mathbb{Z}/2\mathbb{Z} & \text{if } \gamma_2 > 0, \alpha > 0 \text{ and } \delta = 0; \\
\Gamma_1 \oplus \bigoplus_{i=1}^{\gamma_2-1} F_{2i-1}\mathbb{Z}/2\mathbb{Z} & \text{if } \gamma_2 > 0 \text{ and } \alpha = \delta = 0; \\
\Gamma_1 \oplus \bigoplus_{i=1}^{\gamma_2} F_{2\gamma_2+i}\mathbb{Z}/2\mathbb{Z} & \text{if } \gamma_2 = 0 \text{ and } \alpha \cdot \delta > 0; \\
\Gamma_1 \oplus \bigoplus_{i=1}^{\gamma_2-1} F_{2\gamma_2+i}\mathbb{Z}/2\mathbb{Z} & \text{if } \gamma_2 = 0 \text{ and } \alpha = 0 \text{ and } \delta > 0; \\
\Gamma_1 & \text{if } \gamma_2 = \delta = 0 \text{ and } \alpha > 0. 
\end{cases}
\]

Combining this expression with Table 7.20 we obtain

\[
(7.24) \quad |\Gamma| = \begin{cases} 
2^{2\gamma_2-4+2\delta+\gamma_1} & \text{if } \alpha = \gamma_2 = 0 \text{ and } \delta > 0; \\
2^{2\gamma_2-3+2\delta+\gamma_1+\gamma_2} & \text{if } (\delta > 0 \text{ and } \alpha + \gamma_2 > 0) \\
or (\alpha = \delta = 0 \text{ and } \gamma_1\gamma_2 > 0); \\
2^{2\gamma_2-2+\gamma_1+\gamma_2} & \text{if } \delta = 0 \text{ and } \alpha \gamma_1 > 0; \\
2^{2\gamma_2-1+\gamma_2} & \text{if } \delta = \alpha = \gamma_1 = 0 \text{ and } \gamma_2 > 0; \\
2^{2\gamma_2+\gamma_2} & \text{if } \delta = \gamma_1 = 0 \text{ and } \alpha > 0. 
\end{cases}
\]

8) The proof of Theorem 7.1 iii) is completed by using (7.11) and putting together (7.10), (7.23) and (7.24).

\[\square\]
degree four cover $f : X \to T$ which factorizes through two covers of degree two $X \xrightarrow{\pi} Y \xrightarrow{g} T$.

In the Diagram, $X$ corresponds to any one of the intermediate “first level” quotients of $W$ except for $W_{r^2}$. Without loss of generality, let $X$ correspond to $W_s$; then $Y$ corresponds to $W_{Ks}$.

In other words, we have associated to any nonorientable double cover of a double cover, say given by

$W_s \xrightarrow{\pi} W_{Ks} \xrightarrow{g} T$

another such double cover of a double cover, given by

$X' = W_{rs} \to Y' = W_{Krs} \to T$.

That is, we have recovered the bigonal construction (see [D1]) for the nonorientable case, which together with Remark 6.5 gives the bigonal construction in general.

Furthermore, we have shown that $P(X/Y)$ is always isogenous to $P(X'/Y')$ and we have described explicitly the kernel of the isogeny in each case.

The case studied in [P] is the bigonal construction applied to a ramified double cover $K \to K_0$ of a hyperelliptic curve $K_0$ to obtain the related cover $C \to C_0 \to \mathbb{P}^1$; the main result there is that the two Prym $P(K/K_0)$ and $P(C/C_0)$ in this special case are dual abelian varieties.

We will now obtain this result from our previous work on the action of $D_4$.

Let $K_0$ be a hyperelliptic curve of genus $g$ and let $f : K_0 \to \mathbb{P}^1$ be the morphism given by the $g_1^1$ with branch points $\{a_1, \ldots, a_{2g+2}\} \subseteq \mathbb{P}^1$.

Consider a ramified cover of degree two $\pi : K \to K_0$, and assume the ramification points $\{p_1, \ldots, p_{2h+2}\}$ satisfy the following condition

$(7.25) \quad \pi(p_i) + \pi(p_j) \notin g_1^1$ for all $i, j \in \{1, \ldots, 2h+2\}$.

Observe that $K$ has genus $2g + h$.

We claim that under the conditions just given, the Galois cover associated to the fourfold cover $F = f \circ \pi : K \to K_0 \to \mathbb{P}^1$ has Galois group $D_4$, with $\delta = \gamma_2 = 0$ and $\alpha \cdot \gamma_1 > 0$ where $\alpha, \delta, \gamma_1$ and $\gamma_2$ refer to the numbers in Theorem 7.1.

Proof of the claim: Let $b_i = F(p_i)$ for $1 \leq i \leq 2h+2$ and denote by

$\Sigma = \mathbb{P}^1 \setminus \{a_1, \ldots, a_{2g+2}, b_1, \ldots, b_{2h+2}\}$ and by $\sigma_1, \ldots, \sigma_{2g+2}, \tau_1, \ldots, \tau_{2h+2}$ the corresponding generators of $\Pi_1(\Sigma)$.

Then the cover $F : K \to \mathbb{P}^1$ corresponds to a representation

$F_1 : \Pi_1(\Sigma) \to S_4$

Let $G = F_1(\Pi_1(\Sigma))$. We know that $G$ is a transitive subgroup of $S_4$ and condition (7.25) implies that it is generated by transpositions.
$F_1(\tau_j)$ and products of two disjoint transpositions $F_1(\sigma_k)$ for $1 \leq j \leq 2h + 2$ and $1 \leq k \leq 2g + 2$.

But furthermore, the associated diagram of cover maps

$$\tilde{K} \xrightarrow{\pi} \tilde{K}_0 \xrightarrow{f} \Sigma$$

where

$$\tilde{K} = K - \{F^{-1}(a_i), F^{-1}(b_j)\} \text{ and } \tilde{K}_0 = K_0 - \{f^{-1}(a_i), f^{-1}(b_j)\}$$

corresponds to a chain of subgroups of index two

$$S_3 \cap G \subseteq H \subseteq G$$

where $S_3$ is the subgroup of permutations of $S_4$ fixing the fourth symbol say, and the correspondence is given by

$$\Pi_1(\tilde{K}) = F_1^{-1}(S_3 \cap G) \subseteq \Pi_1(\tilde{K}_0) = F_1^{-1}(H) \subseteq \Pi_1(\Sigma) = F_1^{-1}(G).$$

The existence of this chain of subgroups and the fact that $G$ contains transpositions and products of two disjoint transpositions shows that $G$ must be isomorphic to $D_4$ and the type for its generators proves that the numbers $\alpha$, $\delta$, $\gamma_1$ and $\gamma_2$ are as given.

In fact, recalling the notation of Theorem 7.1, without loss of generality we may assume that $K = W_s$ and $K_0 = W_{K_s}$; it follows that then $C = W_{rs}$ and $C_0 = W_{K_{rs}}$; furthermore, $\alpha = 2h + 2$ and $\gamma_1 = 2g + 2$ are positive and $\delta = \gamma_2 = 0$.

Under these conditions, observe that $a^*_s : JK \to JW$, $a^*_rs : JC \to JW$ and $b^*_s : JK_0 \to JK$ are injective.

Furthermore, part iii) of the Theorem shows that the isogeny

$$P_s = P(W_s/W_{K_s}) = P(K/K_0) \to P_{rs} = P(W_{rs}/W_{K_{rs}}) = P(C/C_0)$$

given by $H = \text{Nm} a_{rs} \circ a^*_s|_{P_s}$ has as kernel $b^*_s(P(W_{K_s}/T)[2]) = \pi^*(JK_0[2])$, of cardinality $2^{\alpha-2} = 2^{2h}$.

It also follows from Diagram (7.14) that the map $\tilde{H} \circ H : P_s \xrightarrow{\tilde{H}} P_{rs} \xrightarrow{H} P_s$ is multiplication by 2 in $P_s$, where $P_{rs} \xrightarrow{\tilde{H}} P_s$ is given by $\text{Nm} a_{s} \circ a^*_s|_{P_{rs}}$ and therefore $\ker(\tilde{H} \circ H) = P_s[2]$.

But the polarization $\lambda_{P_s} : P_s \to \hat{P}_s$ induced on $P_s$ by the principal polarization of $JK$ has as kernel $\ker \lambda_{P_s} = \pi^*(JK_0) \cap P_s = \pi^*(JK_0[2])$ by Proposition 2.3 and therefore $\ker \lambda_{P_s} = \ker \tilde{H}$.

Similarly, $\ker \lambda_{P_{rs}} = \ker \tilde{H} = b^*_rs(P(W_{K_{rs}}/T)[2]) = b^*_rs(JC_0[2])$, of cardinality $2^{\alpha-2} = 2^{2h}$.
Therefore, there are respective isomorphisms $\Gamma: \hat{P}_s \rightarrow P_{rs}$ and $\Delta: \hat{P}_{rs} \rightarrow P_s$ making the following diagram commutative.

If we denote by $\lambda$ the polarization on $\hat{P}_s$ induced by $\lambda_{P_{rs}}$ via $\Gamma$, we may complete the above to the following commutative diagram.

But then it follows that

$$\ker(\lambda \circ \lambda_{P_s}) = \lambda_{P_s}^{-1}(\ker \lambda) = H^{-1}(\ker \lambda_{P_{rs}})$$

$$= H^{-1}(\ker \tilde{H}) = \ker(\tilde{H} \circ H) = P_s[2]$$

and therefore

$$\lambda = \lambda_{\hat{P}_s}$$

showing that $(P_s, \lambda_{P_s})$ and $(P_{rs}, \lambda_{P_{rs}})$ are dual abelian varieties.

8. The alternating case

Let $W$ be a curve such that the alternating group on four letters $A_4$ is contained in $\text{Aut}(X)$.

Let $\Delta = W/A_4$ and $C = W/\langle \sigma \rangle$ for $\sigma \in A_4$ an element of order two.

Let $U = W/K$ where $K$ denotes the Klein subgroup of $A_4$, and let $Y = W/\langle \tau \rangle$ where $\tau \in A_4$ is an element of order three.

The corresponding cover maps will be denoted by $\gamma: W \rightarrow \Delta$, $\varepsilon: W \rightarrow U$, $\nu: W \rightarrow C$, $c: C \rightarrow U$, $\varphi: U \rightarrow \Delta$, $\psi: W \rightarrow Y$ and $h: Y \rightarrow \Delta$. 
Then we have the following diagram of curves and covers.

(8.26)

\[ \begin{array}{ccc}
W & \xrightarrow{\psi} & Y \\
\downarrow & & \downarrow h \\
C & \xrightarrow{\gamma} & \Delta \\
\downarrow & & \downarrow \epsilon \\
U & \xrightarrow{\varphi} & \Delta \\
\end{array} \]

**Theorem 8.1.** If \( W \) is a curve such that \( A_4 \subseteq \text{Aut}(X) \) with associated Diagram (8.26), let \( \beta \) denote the number of fixed points of \( \tau \), and let \( 2\gamma_1 \) denote the number of fixed points of \( \sigma \).

Then:

i) If \( g \) denotes the genus of \( \Delta \), the genera of the intermediate covers and the cardinality of the corresponding ramification loci are given in the following table. In particular, the signature type of \( \Delta \) is \( (g; 3, \ldots, 3, 2, \ldots, 2) \).

| genus | order of ramification \( |B| \) |
|-------|------------------|
| \( g_W = 12g - 11 + 4\beta + 3\gamma_1 \) | |
| \( g_C = 6g - 5 + 2\beta + \gamma_1 \) | \( |B(W \to C)| = 2\gamma_1 \) |
| \( g_Y = 4g - 3 + \beta + \gamma_1 \) | \( |B(W \to Y)| = 2\beta \) |
| \( g_U = 3g - 2 + \beta \) | \( |B(C \to U)| = 2\gamma_1 \) |
| | \( |B(U \to \Delta)| = 2\beta \) |
| | \( |B(Y \to \Delta)| = 2\beta + 2\gamma_1 \) |
ii) There is an $\mathcal{A}_4$-equivariant isogeny  
\[ \phi_{\mathcal{A}_4} : J\Delta \times P(U/\Delta) \times 3P(C/U) \to JW \]
defined by
\[ \phi_{\mathcal{A}_4}(d, u, c_1, c_2, c_3) = \gamma^*(d) + \varepsilon^*(u) + \nu^*(c_1) + \tau\nu^*(c_2) + \tau^2\nu^*(c_3) \]
where the action of $\mathcal{A}_4$ is: the trivial one on $J\Delta$, the action of the sum of the other two irreducible representations of degree one of $\mathcal{A}_4$ on $P(U/\Delta)$, and, on $3P(C/U)$, the action of the irreducible representation of degree three of $\mathcal{A}_4$.

The cardinality of the kernel of $\phi_{\mathcal{A}_4}$ is given by
\[ |\ker \phi_{\mathcal{A}_4}| = \begin{cases} 2^{24g-22}3^{2g-1}, & \text{if all covers are unramified;} \\ 2^{24g-22+8\beta}3^{2g}, & \text{if } \gamma_1 = 0 \text{ and } \beta \text{ is positive;} \\ 2^{24g-19+3\gamma_1}3^{2g-1}, & \text{if } \beta = 0 \text{ and } \gamma_1 \text{ is positive;} \\ 2^{24g-19+8\beta+3\gamma_1}3^{2g}, & \text{if } \beta \cdot \gamma_1 \text{ is positive.} \end{cases} \]

iii) There is a natural isogeny
\[ \text{Nm}\psi \circ \nu^*|_{P(C/U)} : P(C/U) \to P(Y/\Delta) \]
whose kernel is contained in $P(C/U)[2]$ and has cardinality given as follows:
\[ |\ker(\text{Nm}\psi \circ \nu^*|_{P(C/U)})| = \begin{cases} 2^{4g-6+2\beta}, & \text{if } \gamma_1 = 0 \text{ and } P(U/\Delta)[2] \notin (\ker c^*)^\perp; \\ 2^{4g-5+2\beta+\gamma_1}, & \text{otherwise}. \end{cases} \]

Remark 8.2. The conditions on the data in this case are the following: if $g_\Delta = 0$ then $\beta \geq 2$ and $\beta + \gamma_1 \geq 3$ since $U$ and $Y$ must be connected covers.

Proof. Statement i) is immediate.
For statement ii), consider the commutative diagram
\[ (8.27) \]

where $C_j = W/\langle (1 j)(k l) \rangle$ with $\{j, k, l\} = \{2, 3, 4\}$.
Since it corresponds to an action of the Klein group, we may apply Theorem 6.3 to obtain an isogeny
\[ \phi_K : JU \times P(C_2/U) \times P(C_3/U) \times P(C_4/U) \to JW \]
given by
\[ \phi_K(u_1, c_2, c_3, c_4) = \varepsilon^*(u_1) + \nu_2^*(c_2) + \nu_3^*(c_3) + \nu_4^*(c_4). \]
Now we assume that \( C = C_2 \) and that \( \tau \) is such that
\[ \tau(12)(34) = (13)(24)\tau. \]
Then, if we let \( \tilde{\tau} \) denote the induced isomorphism from \( JC \) to \( JC_3 \), we obtain an isomorphism
\[ n : 3P(C/U) \to P(C_2/U) \times P(C_3/U) \times P(C_4/U) \]
defined by
\[ n(c_1, c_2, c_3) = (c_1, \tilde{\tau}(c_2), \tilde{\tau}^2(c_3)). \]
We also have the natural isogeny
\[ \phi_U : J\Delta \times P(U/\Delta) \to JU \]
\[ (d, u) \to \gamma^*(d) + u. \]
Hence we can write
\[ \phi_{A_4}(d, w, c_1, c_2, c_3) = \phi_K(\phi_U(d, w), n(c_1, c_2, c_3)), \]
and therefore \( \phi_{A_4} \) is an isogeny.
It also follows that its kernel has cardinality given by
\[ |\ker \phi_{A_4}| = |\ker \phi_K| \cdot |\ker \phi_U|. \]
But
\[ |\ker \phi_K| = \begin{cases} 2^{2g-22+8\beta}, & \text{if } \gamma_1 = 0; \\ 2^{2g-19+8\beta+3\gamma_1}, & \text{if } \gamma_1 > 0, \end{cases} \]
by Theorem 6.3 and
\[ |\ker \phi_U| = \begin{cases} 3^{2g-1}, & \text{if } \beta = 0; \\ 3^{2g}, & \text{if } \beta > 0. \end{cases} \]
by Remark 2.2.
Concerning iii), we first prove that \( \text{Nm } \psi(\nu^*(P(C/U))) \) is contained in \( P(Y/\Delta) \); then we show that \( \text{Nm } \psi \circ \nu^* \) restricted to \( P(C/U) \) is an isogeny and finally we compute the cardinality of its kernel.
Denote by \( H \) the restriction of \( \text{Nm } \psi \circ \nu^* \) to \( P(C/U) \).
Given \( x \) in \( P(C/U) \) we obtain
\[ \text{Nm } h(H(x)) = \text{Nm } \varphi(\text{Nm } c(\text{Nm } \nu(\nu^*(x)))) \]
\[ = 2 \text{Nm } \varphi(\text{Nm } c(x)) = 2 \text{Nm } \varphi(0) = 0. \]
and it follows that $H(P(C/U)) \subseteq P(Y/\Delta)$.

We now show that $H$ is an isogeny. Let $A = \nu^*(P(C/U))$ and $B = \psi^*(P(Y/\Delta))$.

By Corollary 3.5 iii) we have that

$$A = \{ z \in JW : z = \sigma(z), z + \sigma'(z) = 0 \}$$

where $\sigma'$ is any element of order two in $A_4$ different from $\sigma$ and that

$$B = \{ w \in JW : w = \tau(w), \sum_{k \in K} k(w) = 0 \}.$$  

It is then clear that the endomorphisms of $JW$ given by $1 + \tau + \tau^2$ and $1 + \sigma$ induce respective isogenies $1 + \tau + \tau^2 : A \to B$ and $1 + \sigma : B \to A$ such that $(1 + \sigma) \circ (1 + \tau + \tau^2) = 2_A$.

Now observe that the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1+\tau+\tau^2} & B \\
\downarrow{\nu^*} & & \downarrow{\psi^*} \\
P(C/U) & \xrightarrow{H} & P(Y/\Delta) & \xrightarrow{\nu^*} & P(C/U) \\
\end{array}
\]

shows that $(\text{Nm} \nu \circ \psi^*) \circ H = 2_{P(C/U)}$, since $\nu^*$ is an isogeny.

In particular, $\ker H \subseteq P(C/U)[2]$.

Since it follows from $(1 + \sigma) \circ (1 + \tau + \tau^2) = 2_A$ that $\ker(1 + \tau + \tau^2)|_A \subseteq A[2]$ and since $\deg \psi = 3$ it follows that any $z \in P(Y/\Delta) \cap \ker \psi^*$ satisfies $2z = 0$ and $3z = 0$, hence $z = 0$.

Hence $\ker \psi^*|_{P(Y/\Delta)} = \{0\}$ and therefore $\ker(1 + \tau + \tau^2)|_A = \ker(\text{Nm} \psi)|_A$, so the description of the kernel of the isogeny in iii) is as follows.

$$\ker H = \{ z \in P(C/U)[2] : (1 + \tau + \tau^2)(\nu^*(z)) = 0 \}.$$  

We now compute the cardinality of this kernel.

Observe that all the cover maps in Diagram 8.27 have ramification indices $2 \gamma_1$, hence our calculation will be divided into two cases: $\gamma_1 = 0$ and $\gamma_1 > 0$.

Since $K$ is a normal subgroup of $A_4$, the action of $\tau$ on $W$ descends to an action on $U$, also denoted by $\tau$; recall that $P(U/\Delta)$ is the connected component of the identity of $\ker(1 + \tau + \tau^2)$.

**Claim 8.3.** If $\varepsilon : W \to U$ denotes the cover map $c_2 \circ \nu_2$, then $\varepsilon \tau = \tau \varepsilon$.

We also have:

i) $P(U/\Delta)[2] = (\ker(1 + \tau + \tau^2))[2]$,

ii) $\ker \varepsilon^* \subseteq (\ker(1 + \tau + \tau^2))[2]$.
Proof of the claim: The commutativity of $\varepsilon$ and $\tau$ is clear.
As for i), it follows from [R, p. 61] that $\ker(1 + \tau + \tau^2) = P(U/\Delta) + \varphi^*J\Delta[3]$. Therefore, if $x \in (\ker(1 + \tau + \tau^2))[2]$ then $x = u + z$, with $u \in P(U/\Delta)$ and $z \in \varphi^*J\Delta[3]$. But then $0 = 2x = 2u + 2z = 2u - z$, so $z \in P(U/\Delta)$ and hence $x \in P(U/\Delta)$. This proves i).
To prove ii) observe that
\[
\ker \varepsilon^* = \begin{cases} 
0 & \text{if } \gamma_1 \neq 0 \\
\{0, \eta_{c_2}, \eta_{c_3}, \eta_{c_4} = \eta_{c_2} + \eta_{c_3}\} & \text{if } \gamma_1 = 0
\end{cases}
\]
and that, in the second case, $\tau(\eta_{c_2}) = \eta_{c_3}$ and $\tau(\eta_{c_3}) = \eta_{c_4}$. \qed

We continue the calculation of $\ker H$.

Case I: $\gamma_1 = 0$. In this case all the cover maps in Diagram 8.27 are étale, hence $\ker \varepsilon^* = \{0, \eta_c, \eta_{c_3}, \eta_{c_4}\} \subseteq P(U/\Delta)[2]$.
We also have $c^*\{0, \eta_c\}\perp = P(C/U)[2]$.
We further subdivide Case I into two subcases: Case I.a: $g(= g_\Delta) = 0$ and Case I.b: $g > 0$.

Case I.a: $g = 0$. In this case $P(U/\Delta)[2] = JU/2$ and therefore $(1 + \tau + \tau^2)m = 0$ for all $m \in JU/2$ (by Claim 8.3); in particular, for all $m \in \{0, \eta_c\}\perp$.
But this implies that any $z \in P(C/U)[2]$ satisfies $(1 + \tau + \tau^2)(\nu^*(z)) = 0$, since $P(C/U)[2] = c^*\{0, \eta_c\}\perp$ and since $\varepsilon^*\tau = \tau\varepsilon^*$.
We have thus proven that in this case $\ker H = P(C/U)[2]$ and therefore
\[
|\ker H| = 2^{2\beta-6} \text{ if } \gamma_1 = g = 0.
\]

Case I.b: $g > 0$. Now let $z \in \ker H$; that is, let $z$ be in $P(C/U)[2]$ such that $(1 + \tau + \tau^2)(\nu^*(z)) = 0$.
Choose any $m \in \{0, \eta_c\}\perp$ such that $c^*(m) = z$. Then $m + \tau m + \tau^2 m$ belongs to $\ker \varepsilon^* \subseteq P(U/\Delta)[2]$.
But $m + \tau m + \tau^2 m$ also belongs to $JU^{(\tau)} \cap P(U/\Delta) \subseteq P(U/\Delta)[3]$, by Proposition 3.1 i), and therefore $m + \tau m + \tau^2 m = 0$; that is, $m$ belongs to $P(U/\Delta)[2] \cap \{0, \eta_c\}\perp$.
But, since $\{0, \eta_c\}\perp$ is a subgroup of index two of $JU/2$, we have
\[
|P(U/\Delta)[2] \cap \{0, \eta_c\}\perp| = \begin{cases} 
2^{2g\nu - 2g\Delta}, & \text{if } P(U/\Delta)[2] \subseteq \{0, \eta_c\}\perp; \\
2^{2g\nu - 2g\Delta - 1}, & \text{otherwise};
\end{cases}
\]
and therefore we obtain
\[
|\ker H| = \begin{cases} 
2^{4g + 2\beta - 5}, & \text{if } g > 0, \ \gamma_1 = 0 \ \text{and } P(U/\Delta)[2] \subseteq \{0, \eta_c\}\perp; \\
2^{4g + 2\beta - 6}, & \text{if } g > 0, \ \gamma_1 = 0 \ \text{and } P(U/\Delta)[2] \not\subseteq \{0, \eta_c\}\perp.
\end{cases}
\]
Case II: $\gamma_1 > 0$. In this case all of the maps $\nu_j^*$ and $c_j^*$ are injective.

To compute the cardinality of the kernel we first describe $P(C/U)[2]$. We let $\{Q_1, \ldots, Q_{\gamma_1}, \sigma Q_1, \ldots, \sigma Q_{\gamma_1}\}$ in $W$ denote the ramification points of $\nu_3 : W \rightarrow C_3$, where $\sigma$ is the involution of $W$ giving the cover $\nu = \nu_2 : W \rightarrow C = C_2$.

Then the ramification points of $c : C \rightarrow U$ are $\{P_1 = \nu(Q_1), P_2 = \nu(\tau(Q_1)), \ldots, P_{2\gamma_1-1} = \nu(Q_{\gamma_1}), P_{2\gamma_1} = \nu(\tau(Q_{\gamma_1}))\}$.

We now apply Corollary 4.3 to the cover $c : C \rightarrow U$ observing that $O_U(\varepsilon Q_i - \varepsilon \tau Q_i)$ belongs to $\ker(1 + \tau + \tau^2) \subseteq JU$, hence for $i \in \{1, \ldots, r\}$ we can choose $n_i \in \ker(1 + \tau + \tau^2)$ such that $n_i^{\otimes 2} = O_U(\varepsilon Q_i - \varepsilon \tau Q_i)$.

Now, setting $L_i = O_C(P_{2i-1} - P_{2i-2}) \otimes c^*(n_i) \in P(C/U)[2]$, we have that each $L_i$ clearly satisfies $(1 + \tau + \tau^2)\nu^*L_i = 0$, and since $L_1 \otimes L_2 \otimes \cdots \otimes L_{\gamma_1} \in c^*JU[2] \cap \ker(1 + \tau + \tau^2)$, we obtain $L_1 \otimes L_2 \otimes \cdots \otimes L_{\gamma_1} \in c^*P(U/\Delta)[2]$.

Letting $G_1, \ldots, G_{\gamma_1-1}$ be as in Corollary 4.3, we obtain

$$P(C/U)[2] = c^*JU[2] \oplus_{i=1}^{\gamma_1-1} L_i/\mathbb{Z}/2\mathbb{Z} \oplus_{i=1}^{\gamma_1-1} G_i/\mathbb{Z}/2\mathbb{Z}.$$  

To complete our computation, observe that $c^*JU[2] \cap \ker H = \{z \in c^*JU[2] : (1 + \tau + \tau^2)(\nu^*z) = 0\}$ is isomorphic (via $c^*$) to $JU[2] \cap \ker(1 + \tau + \tau^2) = P(U/\Delta)[2]$ and that no nontrivial combination of the $G_i$ is in $\ker H$.

Therefore we have obtained the following result: if $\gamma_1 > 0$, then $\ker H = c^*P(U/\Delta)[2] \oplus_{i=1}^{\gamma_1-1} L_i/\mathbb{Z}/2\mathbb{Z}$, and we have completed the proof.  

8.1. The trigonal construction for the case $A_4$. As a corollary of Theorem 8.1, we obtain a particular case of the trigonal construction as follows.

Let $h : Y \rightarrow \mathbb{P}^1$ be a tetragonal curve such that all its branch points come from triple ramification points, with the possible exception of one which is then of type $(2, 2)$.

If we denote by $\gamma : W \rightarrow \mathbb{P}^1$ the corresponding Galois extension, it has group $A_4$ and we are in the situation of Diagram 8.26 with $\Delta = \mathbb{P}^1$ and $\gamma_1 = 0$ or 1, respectively.

Moreover, $P(C/U)$ is a classical Prym.

We can now prove the following.

**Corollary 8.4.** Let $Y$ be a tetragonal curve such that all its branch points come from triple ramification points, with the possible exception of one which is then of type $(2, 2)$.

Then the isogeny $H : P(C/U) \rightarrow JY$ from Theorem 8.1 iii) induces an isomorphism between the principally polarized abelian varieties $(JY, \lambda_{JY})$ and $(\overline{P(C/U)}, \lambda_{P(C/U)})$. 

Proof. Let $P = P(C/U)$.

Under our hypothesis the isogeny given in Theorem 8.1 iii) by $H = \text{Nm} \psi \circ \nu^* : P \to JY$ has as kernel $P[2]$, which coincides with the kernel of $\lambda_P = 2\lambda$ on $P$.

Therefore $H$ factorizes as follows, with $F$ an automorphism.

We will now show that the isomorphism of complex tori $F : \hat{P} \to JY$ is also an isomorphism of p.p.a.v.’s.

If we denote by $\lambda_1$ the polarization on $\hat{P}$ induced via $F$ by $\lambda_{JY}$, we may complete the above diagram to the following one.

It now follows from the commutativity of the above diagram that $\lambda_1$ is principal and that $\ker(\lambda_1 \circ \lambda_P) = P[2]$; therefore, $\lambda_1 = \lambda_{\hat{P}}$, as claimed. \qed

9. The symmetric case

Let $W$ be a curve such that the symmetric group on four letters $\mathcal{S}_4$ is contained in $\text{Aut}(W)$.

Let $T = W/\mathcal{S}_4$ and $\Delta = W/A_4$.

For $\{j, k, l\} = \{2, 3, 4\}$ and $\sigma_j = (1 j)(k l) \in \mathcal{S}_4$, let $C_j = W/\langle \sigma_j \rangle$ and $U = W/\mathcal{K}$ where $\mathcal{K} = \{1, \sigma_2, \sigma_3, \sigma_4\}$ is the normal Klein subgroup of $\mathcal{S}_4$.

Also, let $Z_{kl} = W/\langle(k l)\rangle$, $S_j = W/\langle\sigma_j, (k l)\rangle$ the quotient by a non-normal Klein subgroup, $V_j = W/\langle(1 k j l)\rangle$ the quotient by a cyclic group of order four and $R_j = W/\langle(1 k j l), (k l)\rangle$ the quotient by $\mathcal{D}_4$.

For $n \in \{1, 2, 3, 4\}$ and $\{j, k, l, n\} = \{1, 2, 3, 4\}$, let $Y_n = W/\langle(j k l)\rangle$ be the quotient by a cyclic group of order three, and let $X_n = W/\langle(j k l), (j k)\rangle$ denote the quotient of $W$ by the corresponding $\mathcal{S}_3$. 

Theorem 9.1. Let $W$ be a curve such that $S_4 \subseteq \text{Aut}(W)$ with associated Diagram \((9.29)\).

We let $\gamma : W \to T = W/S_4$ denote the quotient map, and let $\tau$ denote any element of order three in $S_4$.

Let $2\alpha, 2\beta, 2\delta, 4\gamma$ denote the number of fixed points in $W$ of, respectively, any transposition, any element of order three, any element of order four, the square of any element of order four not fixed by the element of order four.

Then

i) If $g$ denotes the genus of $T$, the genera of the intermediate covers and the cardinality of the corresponding ramification loci are given in the following table. In particular, the signature type of $T$ is $(g; 2, \ldots, 2, 3, \ldots, 3, 2, \ldots, 2, 4, \ldots, 4)$. 

Then we have a diagram of curves and covers as follows; the sub-indices will be used as needed. 

\begin{equation}
(9.29)
\end{equation}
| genus                | order of ramification $|B|$ |
|----------------------|----------------------|
| $g_W = 24g - 23 + 6\alpha + 8\beta + 6\gamma + 9\delta$ |                       |
| $g_C = 12g - 11 + 3\alpha + 4\beta + 2\gamma + 4\delta$ | $|B(W \to C)| = 4\gamma + 2\delta$ |
| $g_Z = 12g - 11 + \frac{5\alpha + 9\delta}{2} + 4\beta + 3\gamma$ | $|B(W \to Z)| = 2\alpha$ |
| $g_Y = 8g - 7 + 2\alpha + 2\beta + 2\gamma + 3\delta$ | $|B(W \to Y)| = 4\beta$ |
| $g_U = 6g - 5 + 3\frac{\alpha + \delta}{2} + 2\beta$ | $|B(C \to U)| = 4\gamma + 2\delta$ |
| $g_S = 6g - 5 + \alpha + 2\beta + \gamma + 2\delta$ | $|B(Z \to S)| = \alpha + 2\gamma + \delta$ $|B(C \to S)| = 2\alpha$ |
| $g_X = 4g - 3 + \frac{\alpha + 3\delta}{2} + \beta + \gamma$ | $|B(Y \to X)| = 2\alpha$ $|B(Z \to X)| = 2\alpha + 2\beta$ |
| $g_V = 6g - 5 + 3\frac{\alpha + \delta}{2} + 2\beta + \gamma$ | $|B(C \to V)| = 2\delta$ |
| $g_R = 3g - 2 + \frac{\alpha + \delta}{2} + \beta$ | $|B(U \to R)| = \alpha + \delta$ $|B(S \to R)| = 2\gamma + 2\delta$ $|B(V \to R)| = \alpha + 2\gamma + \delta$ |
| $g_\Delta = 2g - 1 + \frac{\alpha + \delta}{2}$ | $|B(Y \to \Delta)| = 4\beta + 4\gamma + 2\delta$ $|B(U \to \Delta)| = 4\beta$ |
|                       | $|B(R \to T)| = \alpha + 2\beta + \delta$ $|B(X \to T)| = \alpha + 2\beta + 2\gamma + 3\delta$ $|B(\Delta \to T)| = \alpha + \delta$ |
ii) There is an $S_4$-equivariant isogeny

$$\phi_{S_4} : JT \times P(\Delta/T) \times 2P(R/T) \times 3P(S/R) \times 3P(V/R) \to JW$$

given by

$$\phi_{S_4}(t, d, r_1, r_2, s_1, s_2, s_3, v_1, v_2, v_3) = \gamma^*(t)$$

$$+ (h \circ \psi)^*(d) + (p \circ \pi \circ \ell)^*(r_1) + \tau(p \circ \pi \circ \ell)^*(r_2)$$

$$+ (a \circ \nu)^*(s_1) + \tau(a \circ \nu)^*(s_2) + \tau^2(a \circ \nu)^*(s_3)$$

$$+ (b \circ \nu)^*(v_1) + \tau(b \circ \nu)^*(v_2) + \tau^2(b \circ \nu)^*(v_3)$$

where the action of $S_4$ is: the trivial one on $JT$, the alternating action on $P(\Delta/T)$, the action of the unique irreducible representation of degree two of $S_4$ on $2P(R/T)$, the standard action (of degree three) of $S_4$ on $3P(S/R)$, and, on $3P(V/R)$, the other irreducible action of degree three of $S_4$.

The cardinality of the kernel of $\phi_{S_4}$ is given by

$$| \ker \phi_{S_4} | = \begin{cases} 
2^{6g-65+22}\alpha \cdot 3^{6g-3+\beta} & \text{if } \alpha = \gamma = \delta = 0; \\
2^{6g-61+15}\alpha + 22\beta \cdot 3^{6g-3+\alpha + \beta} & \text{if } \gamma = \delta = 0 \text{ and } \alpha > 0; \\
2^{6g-59+22}\alpha + 12\gamma \cdot 3^{6g-3+\beta} & \text{if } \alpha = \delta = 0 \text{ and } \gamma > 0; \\
2^{6g-58+15}\alpha + 22\beta + 12\gamma + 21\delta \cdot 3^{6g-3+\alpha + \beta + \delta} & \text{otherwise.}
\end{cases}$$

iii) For any $C$ and $Y$ there is a natural isogeny

$$\Nm \psi \circ \nu^*_{|P(C/U)} : P(C/U) \to P(Y/\Delta).$$

Its kernel is contained in $P(C/U)[2]$ and has cardinality given as follows:

$$| \ker (\Nm \psi \circ \nu^*_{|P(C/U)}) | = \begin{cases} 
2^{8g-10+4\beta+2\alpha} & \text{if } \gamma = \delta = 0 \text{ and } P(U/\Delta)[2] \not\subseteq (\ker c^*)^\perp; \\
2^{8g-9+4\beta+2\alpha+2\gamma+3\delta} & \text{otherwise.}
\end{cases}$$

iv) For any $Z$ and $C$ not covering the same $S$ there is a natural isogeny

$$\Nm \nu \circ \ell^*_{|P(Z/S)} : P(Z/S) \to P(C/U).$$

Moreover, if for the case $\gamma = \delta = 0$ we denote by $G$ the Klein subgroup of $JU$ giving the covers $c_j : C_j \to U$ for $j \in \{2, 3, 4\}$, then the kernel of the isogeny has cardinality given as follows:
Remark 9.2. Equivalently, we could start with a degree four cover $f : X \to T$ whose Galois group is $S_4$.

In this case, $\alpha$ is the number of simple ramification points of $f$, $\beta$ is the number of ramification points of order three, $\delta$ is the number of
total ramification points, and \(2\gamma\) is the number of the remaining type of ramification points.

**Remark 9.3.** The conditions on the data are

\[\alpha + \delta \equiv 0 \quad (2)\]

and if \(g = 0\) then \(\alpha + \delta \geq 2\), \(\beta \geq 1\) and \(\gamma + \delta \geq 1\).

**Proof.** Using the Riemann-Hurwitz formula, statement i) is immediate.

For the proof of ii), we follow the idea behind the method of “little groups” and apply it to the decomposition of \(S_4\) given by the semi-direct product of the abelian normal subgroup \(K\) and a subgroup \(S_3\).

Parts iii) and iv) are immediate consequences of Theorem 8.1 iii) and Theorem 7.1 iii), respectively.

For iii), just note that \(\gamma_1\) for the case \(A_4\) corresponds to \(2\gamma + \delta\) for the case \(S_4\) and \(\beta\) for the case \(A_4\) corresponds to \(2\beta\) for the case \(S_4\).

As for case iv) note that \(\alpha\) for the case \(D_4\) coincides with \(\alpha\) for the case \(S_4\), as well as \(\delta\), that \(\gamma_2\) for the case \(D_4\) coincides with \(\gamma\) for the case \(S_4\) and that \(\gamma_1\) for the case \(D_4\) corresponds to \(2\gamma + \delta\) for the case \(S_4\).

The isogenies in statements v) through vii) are suggested by comparing statement ii) with the geometric decompositions of \(JW\) obtained by respectively applying Theorems 6.3, 7.1, 8.1 and the results of [Rec–Ro] to the actions of appropriate subgroups of \(S_4\) on \(W\).

Concerning ii), by Proposition 2.5 we know that there is an isogeny

\[\phi_U : JU \times P(W/U) \to JW, \quad \phi_U(u, w) = \left(\nu \circ c\right)^*(u) + w\]

whose kernel has cardinality given by

\[|\ker \phi_U| = \begin{cases} 2^{2g - 22 + 6a + 8\beta}, & \text{if } \gamma = \delta = 0; \\ 2^{2g - 20 + 6a + 8\beta + 6\delta}, & \text{otherwise}. \end{cases}\]

Note that this isogeny is \(S_4\)-equivariant with the corresponding natural actions of \(S_4\) on each factor on the left side.

We now decompose each such factor.

Since the action of \(K \subseteq S_4\) is trivial on \(JU = J(W/K)\), there is a natural \(S_4/K = S_3\) action on \(JU\); under this condition and from [Rec–Ro], it follows that there is a natural \(S_3\)-equivariant isogeny

\[\phi_{S_3} : JT \times P(\Delta/T) \times 2P(R/T) \to JU\]

defined by

\[\phi_{S_3}(t, d, r_1, r_2) = (r \circ g)^*(t) + \psi^*(d) + r^*(r_1) + \tilde{\tau}r^*(r_2)\]

where \(\tilde{\tau}\) is the isomorphism induced on \(JU\) by \(\tau\) and the action of \(S_3\) on the domain is given by: the trivial representation of \(S_3\) on \(JT\),
the nontrivial representation of degree one on \( P(\Delta/T) \), and the unique irreducible complex representation of degree two on \( 2P(R/T) \).

Note that each of these actions induces the corresponding irreducible actions of \( S_4 \) on each factor.

Also by [Rec–Ro], the kernel of \( \phi_{S_3} \) has cardinality given by

\[
| \ker \phi_{S_3} | = \begin{cases} 
2^{2g-1} \cdot 3^{6g-3+\beta}, & \text{if } \alpha = \delta = 0; \\
2^{2g} \cdot 3^{6g-3+\alpha+\beta+\delta}, & \text{otherwise.}
\end{cases}
\] (9.30)

Concerning the factor \( P(W/U) \), there is a Klein-equivariant isogeny

\[
\nu^* + \tau \nu^* + \tau^2 \nu^* : 3P(C/U) \to P(W/U)
\]

whose kernel has cardinality given by

\[
| \ker (\nu^* + \tau \nu^* + \tau^2 \nu^*) | = \begin{cases} 
2^{24g-24+6\alpha+8\beta}, & \text{if } \gamma = \delta = 0, \\
2^{24g-23+6\alpha+8\beta+6\gamma+9\delta}, & \text{otherwise,}
\end{cases}
\] (9.31)

as in the proof of Theorem 8.1 and by Theorem 6.3.

Remark that this isogeny is, in fact, \( S_4 \)-equivariant.

To obtain the decomposition into irreducible representations, we further decompose \( P(C/U) \) as follows: Note that the following piece of Diagram (9.29) corresponds to an action of a Klein group on \( C \)

\[
\begin{array}{ccc}
C & \Downarrow & C \\
\Downarrow & & \Downarrow \\
S & \xrightarrow{a} & V \\
\Downarrow & & \Downarrow \\
R & \xrightarrow{b} & U
\end{array}
\]

By Theorem 6.3 ii) we have a natural Klein-equivariant isogeny defined by

\[
a^* + b^* : P(S/R) \times P(V/R) \to P(C/U).
\]

Furthermore, the cardinality of its kernel is given by

\[
| \ker (a^* + b^*) | = \begin{cases} 
2^{6g-6+2\beta}, & \text{if } \alpha = \gamma = \delta = 0; \\
2^{6g-5+\alpha+2\beta+2\gamma+2\delta}, & \text{otherwise.}
\end{cases}
\] (9.32)

Note that, then, the isogeny

\[
3P(S/R) \times 3P(V/R) \to P(W/U)
\]

given by the composition of the last two isogenies is \( S_4 \)-equivariant, where the action on \( 3P(S/R) \) is the standard one and the action on \( 3P(V/R) \) is the other irreducible action of degree three.
Combining the given isogenies we obtain part ii).

Concerning v), let $Z$ be the common cover of $S$ and $X$ via $\pi : Z \to S$ and $u : Z \to X$ respectively.

We first prove that $\Nm u(\pi^* (P(S/R)))$ is contained in $P(X/T)$; then we show that $\Nm u \circ \pi^*$ restricted to $P(S/R)$ is an isogeny and finally we compute the cardinality of its kernel.

Let $x \in P(S/R)$ and denote by $H$ the restriction of $\Nm u \circ \pi^*$ to $P(S/R)$; we need to show that $H(x) \in P(X/T)$. Since $P(S/R)$ is connected, it is enough to show that $\Nm f(H(x)) = 0$. But

$$
\Nm f(H(x)) = \Nm g(\Nm p(\Nm(\pi^*(x))))
= \Nm g(\Nm p(2x))
= 2 \Nm g(0)
= 0
$$

We now show that $H$ is an isogeny.

We may assume that $S = S_j = W/(1, j, (k l))$ and that $X = W/(\tau, (1, j))$ where $\tau$ denotes an element of order three of $S_4$. Then $Z = Z_{1_j}$ is a common cover of $S$ and $X$, via $\pi : Z \to S$ and $u : Z \to X$.

Let $A = \ell^* (\pi^*(P(S/R)))$ and let $B = \ell^*(u^*(P(X/T)))$. Also, let $\sigma = (1 k)(j l)$ denote an element of $K \subseteq S_4$ which induces the involution $S \to S$ giving the cover $S \to R$.

By Corollary 3.5 iii) we have that

$$
A = \{ z \in JW^{(1, j, (k l))} : z + \sigma z = 0 \}^\circ
$$

and that

$$
B = \{ w \in JW^{(\tau, (1, j))} : \sum_{k \in K} k(w) = 0 \}^\circ.
$$

It is then clear that the endomorphisms of $JW$ given by $1 + \tau + \tau^2$ and $1 + (k l)$ induce respective isogenies $1 + \tau + \tau^2 : A \to B$ and $1 + (k l) : B \to A$ such that $(1 + (k l)) \circ (1 + \tau + \tau^2) = 2_A$.

We also have the following commutative diagram

(9.33)

which shows that $H$ is an isogeny.
We are interested in computing the kernel of this isogeny.

For this, first note that it follows from Claim 7.3 in the proof of Theorem 7.1 iii) that $\pi^*$ restricted to $P(S/R)$ is injective; therefore, the two external vertical arrows of Diagram 9.33 are isomorphisms. That the middle vertical line is also an isomorphism follows from Lemma 5.3 (see also [Rec–Ro, p. 136]).

We can now prove the following result, which will be fundamental to complete the proof of the Theorem.

**Claim 9.4.** The composition

$$P(S/R) \xrightarrow{H} P(X/T) \xrightarrow{\text{Nm} \pi u^*} P(S/R)$$

is multiplication by 2.

Proof of the claim: Since the topmost line of Diagram (9.33) is multiplication by 2 on $A$ and since $\ell^*$ is an isogeny, it follows that the middle line is multiplication by 2 on $\pi^*(P(S/R))$. But then the claim follows since we already know that the vertical arrows are isomorphisms. $\square$

A similar proof shows that we also have the following result.

**Lemma 9.5.** The composition

$$u^*(P(X/T)) \xrightarrow{\pi^* \text{Nm} \pi} \pi^*(P(S/R)) \xrightarrow{u^* \text{Nm} u} u^*(P(X/T))$$

is multiplication by 2.

An immediate consequence of Claim 9.4 is that $\ker H \subseteq P(S/R)[2]$. Now it follows from Diagram (9.33) that $\ker H$ is contained in the following set.

$$F = \{ x \in P(S/R)[2] : (1 + \tau + \tau^2)\ell^* \pi^*(x) = 0 \}$$

In fact, we can be more precise.

**Remark 9.6.** Recall that if $\alpha = 0$, it follows from the description given in the table in part i) of Theorem 9.1 that the covers $\ell : W \to Z$, $v : Y \to X$ and $a : C \to S$ are unramified of degree two, and therefore the kernels of the respective $\ast$-induced maps on Jacobians are non trivial subgroups of order two of $JZ[2]$, $JX[2]$ and $JS[2]$ respectively.

If moreover $\delta = 0$ then we must have $T \neq \mathbb{P}^1$ for $\Delta \to T$ to be connected.

**Claim 9.7.** If $\alpha = 0$ then there exists $s \in F$ such that $u^*(H(s)) = \eta_\ell$. In particular, $\eta_\ell \in u^*(P(X/T))$. 
Proof. Note that $u^*$ is always injective and that $u^*(\eta_v) = \eta_v$; hence it is enough to show that there is $s \in P(S/R)[2]$ such that $H(s) = \eta_v$ because if this holds then $0 = \ell^*(u^*(\eta_v)) = \ell^*(u^*(H(s))) = (1 + \tau + \tau^2)(\ell^*(\pi^*(s)))$ and therefore $s \in F$.

Furthermore, note that

$$\Nm u(\eta_v) = \Nm u(u^*(\eta_v)) = 3\eta_v = \eta_v$$

and therefore

$$u^*(\Nm u(\eta_v)) = \eta_v$$

We also observe that since $f^*: JT \to JX$ is always injective (because it does not factor), we always have $f^*(JT[4]) \subseteq P(X/T)[4]$.

We consider two cases.

i) Case 1: $\delta > 0$: In this case it follows from Case III in the Appendix that $\eta_a \in P(S/R)[2]$. Furthermore, it is clear that $\pi^*(\eta_a) = \eta_v$, by commutativity of the following subdiagram and by the injectivity of $\pi^*$.

ii) Case 2: $\delta = 0$: In this case $\eta_{\delta}$ is a point of order two in $JT$ and therefore $f^*(\eta_{\delta}) = \eta_v \in P(X/T)$.

But we also have

$$\ell^*(\pi^* \circ \Nm \pi(\eta_v)) = (1 + (k l))(\ell^*(\eta_v)) = 0$$

where the first equality follows from Diagram 9.33 and hence $\pi^* \circ \Nm \pi(\eta_v) \in \ker \ell^* = \{0, \eta_v\}$.

We will now show that $\pi^* \circ \Nm \pi(\eta_v) = 0$: if not, then

$$(\pi^* \circ \Nm \pi)(\eta_v) = \eta_v$$

and it follows that

$$(u^* \circ \Nm u) \circ (\pi^* \circ \Nm \pi)(\eta_v) = (u^* \circ \Nm u)(\eta_v) = \eta_v.$$

But we also have that $(u^* \circ \Nm u) \circ (\pi^* \circ \Nm \pi)$ is multiplication by 2 on $u^*(P(X/T))$, from Lemma 9.5, which is a contradiction.
Proof. Assume there exists \( \ker \) from Claim 9.7.

Proposition 9.8. \( \ker H = F \) if and only if \( \alpha \neq 0 \).

Moreover, if \( \alpha = 0 \) then \( [F : \ker H] = 2 \).

Proof. Assume there exists \( s \in F \) with \( s \notin \ker H \). Then \( u^*(H(s)) \) is the non-zero element of \( \ker \ell^* \) in \( u^*(P(X/T)) \), because \( u^* : JX \rightarrow JZ \) is injective as we know from Lemma 5.3 and by the commutativity of Diagram 9.33. Hence \( [F : \ker H] = 2 \) in this case. The result now follows from Claim 9.7.

To compute the cardinality of \( \ker H \) it now suffices to compute the cardinality of \( F \).

Towards this goal we now prove the following result.

Claim 9.9. With the notation of Diagram (9.29) we have the following description.

\[
r^*-1(P(U/\Delta)[2]) = P(R/T)[2] = \{x \in JR[2] : (1+\tau+\tau^2)\ell^*\pi^*p^*x = 0 \}.
\]

Proof of the claim: We recall from the theory of \( S_3 \)-actions (see \cite{Rec–Ro}) that \( r^* : P(R/T) \rightarrow P(U/\Delta) \) is injective and also that \( \ker \{P(R/T) \times P(R/T) \xrightarrow{r^*+\tau r^*} P(U/\Delta)\} = \{(x,y) : r^*x = \tau r^*y, x \in P(R/T)[3]\} \).

Hence \( r^*P(R/T)[2] \cap \tau r^*P(R/T)[2] = \{0\} \) and therefore

\[
r^*P(R/T)[2] + \tau r^*P(R/T)[2] = P(U/\Delta)[2]
\]

(by counting cardinalities).

The first equality in the claim follows now from the injectivity of \( r^* \) on \( P(R/T) \). Noting that \( (1+\tau+\tau^2)\ell^*\pi^*p^* = (1+\tau+\tau^2)\varepsilon^*r^* = \varepsilon^*(1+\tau+\tau^2)r^* \) and using the first equality, we obtain that \( P(R/T)[2] \subseteq \{x \in JR[2] : (1+\tau+\tau^2)\ell^*\pi^*p^*x = 0 \} \).

Conversely, given \( x \in \{x \in JR[2] : (1+\tau+\tau^2)\ell^*\pi^*p^*x = 0 \} \), then \( (1+\tau+\tau^2)r^*(x) \in \ker \varepsilon^* \).
Since $\ker \varepsilon^* \subseteq \ker(1 + \tau + \tau^2)$ by Claim 8.3 in the proof of Theorem 8.1, it follows that $(1 + \tau + \tau^2)(1 + \tau + \tau^2)r^*(x) = 0$.

But $(1 + \tau + \tau^2)(1 + \tau + \tau^2) = 3(1 + \tau + \tau^2)r^*(x) = (1 + \tau + \tau^2)r^*(3x) = (1 + \tau + \tau^2)r^*(x)$ since $3x = x$. Using the first equality again, the claim is proved.

An immediate corollary of this claim is the following result.

**Claim 9.10.** If $p : S \to R$ is unramified, then $\eta_p$ is in $P(R/T)[2]$.

**Proof of the claim:** Since $p$ unramified is equivalent to $2\gamma + 2\delta = 0$, in this case we have that all maps $c_i : C_i \to U$ are unramified.

Furthermore, it is clear that $r^*\eta_p = \eta_{c_i}$ for the unique $i$ such that $C_i$ covers $S$. But then $r^*\eta_p$ belongs to $\ker \varepsilon^* \subseteq P(U/\Delta)[2]$ and this claim follows from Claim 9.9.

**Remark 9.11.** Recall that we are interested in describing the set $F$ defined in (9.34).

Note that $p^*(P(R/T)[2]) \cap P(S/R)[2]$ is contained in $F$, by Claim 9.9.

The general philosophy to complete the description is based on proving that the complementary part of $F$ arises from some specific elements of $P(S/R)[2]$ which come from the ramification of $p : S \to R$.

In particular, if $p$ is unramified we should already have a description of $F$.

Our next result shows that this is the case, even if $p$ has two ramification points.

**Proposition 9.12.** If $2\gamma + 2\delta = 0$ or 2, then

$$F = p^*(P(R/T)[2]) \cap P(S/R)[2]$$

**Proof.** If $x$ is any element of $F$, then $x \in P(S/R)[2]$ and $(1 + \tau + \tau^2)\ell^*\pi^*(x) = 0$.

If $2\gamma + 2\delta = 0$ we know that $P(S/R)[2] = p^*(\{0, \eta_p\})$ and if $2\gamma + 2\delta = 2$ then $P(S/R)[2] = p^*(JR[2])$.

In both cases there exists $y \in JR[2]$ such that $p^*(y) = x$; it now follows from Claim 9.9 that $y \in P(R/T)[2]$ and the result is proved.

We will analyze a further special case in the next section.

9.1. The classical case of the trigonal construction. Since the trigonal construction has been very useful in the theory of Prym varieties of unramified double covers, we devote this paragraph to it.

We say that we are in the classical case if the curve $W$ with $S_4$-action as in Diagram (9.29) is such that
i) $W/S_4 = T = \mathbb{P}^1$, and such that

ii) the canonical polarization on $JS$ induces $\lambda_P = 2\lambda$ twice a principal polarization on $P(S/R)$.

In this case $P(X/T) = JX$ is also a principally polarized abelian variety.

Note that condition ii) occurs precisely when $p : S \to R$ is unramified or when it has two ramification points; equivalently, when $2\gamma + 2\delta = 0$ or 2.

Condition i) forces the double cover $d : \Delta \to \mathbb{P}^1$ to have at least two points of ramification, since $\Delta$ must be connected; equivalently, $\alpha + \delta$ must be even and greater or equal to two.

Observe that both conditions together exclude the possibility $\alpha = 0$ for the classical case, and also that they imply that the triple cover $R \to T = \mathbb{P}^1$ has at least one simple ramification point.

**Remark 9.13.** We could also say that the classical case corresponds to starting with a tetragonal curve $X$ with at least one simple ramification point and either no total ramification points nor points of type $(2,2)$, or no total ramification point and one ramification point of type $(2,2)$, or with one total ramification point and no ramification point of type $(2,2)$.

Or, equivalently, to a double cover, either unramified or with two ramification points, of a trigonal curve with at least one simple ramification point: $S \to R \to \mathbb{P}^1$.

The trigonal construction (see [Rec2]) shows that these two situations are equivalent and, furthermore, that then $P(S/R)$ and $JX$ are isomorphic as principally polarized abelian varieties.

In both cases the corresponding Galois cover is given by the group $S_4$ and we are in the situation of Theorem 9.7 with $T = \mathbb{P}^1$, $\alpha > 0$ and $2\gamma + 2\delta = 0$ or 2; i.e., in the classical case.

We will now prove that we can also obtain from our methods that the two principally polarized abelian varieties are isomorphic.

We will first compute $\ker H$ for the classical case.

**Proposition 9.14.** If $T = \mathbb{P}^1$ and either $2\gamma + 2\delta = 0$ or 2, then the kernel of the morphism $H = \text{Nm} u \circ \pi^* : P(S/R) \to JX$ is $P(S/R)[2]$.

**Proof.** Note first that if $T = \mathbb{P}^1$, it follows from Claim 9.9 that $JR[2] = \{x \in JR[2] : (1 + \tau + \tau^2)\pi^*p^*x = 0\}$. Therefore $p^*(JR[2]) \subseteq F$; applying Proposition 9.12 we obtain $F = P(S/R)[2]$. It follows from $\alpha \neq 0$ and Proposition 9.8 that

$$\ker H = P(S/R)[2]$$
Now we can prove the following result.

**Theorem 9.15.** If \( T = \mathbb{P}^1 \) and either \( 2\gamma + 2\delta = 0 \) or \( 2 \), then the morphism \( H = Nm \circ \pi^* |_{P(S/R)} : P(S/R) \rightarrow JX \) induces an isomorphism between the principally polarized abelian varieties \((JX, \lambda_{JX})\) and \((\hat{P}(S/R), \lambda_{\hat{P}(S/R)})\).

**Proof.** If we denote by \( \lambda_P \) the polarization on \( P = P(S/R) \) induced by the natural principal polarization on \( JS \), it follows from our hypothesis that there exists a principal polarization \( \lambda \) on \( P \) such that \( \lambda_P = 2\lambda \).

Then note that, since \( \ker \lambda_P = P[2] = \ker \left( P \xymatrix{ Nm \circ \pi^* \ar[r] & JX } \right) \), there exists an isomorphism \( F : \hat{P} \rightarrow JX \) such that the following diagram commutes.

\[
\begin{array}{ccc}
P & \xymatrix{ \rightarrow \ar[rr]^{Nm \circ \pi^*} & & JX } \\
\lambda_P = 2\lambda & \approx \ & F \\
\hat{P} & \downarrow \ar[rr]^{F} & & JX \\
\end{array}
\]

We now show that the isomorphism \( F \) of tori is also an isomorphism of principally polarized abelian varieties. If we denote by \( \lambda_1 \) the polarization on \( \hat{P} \) induced via \( F \) by \( \lambda_{JX} \), we may complete the above diagram to the following one.

\[
\begin{array}{ccc}
P & \xymatrix{ \rightarrow \ar[rr]^{Nm \circ \pi^*} & & JX } \\
\lambda_P & \approx \ & F \\
\hat{P} \uparrow \ar[rr]^{\lambda_1} & & \hat{JX} \\
P & \downarrow \ar[rr]^{\hat{F}} & & JX \\
\end{array}
\]

It now follows from the commutativity of the above diagram that \( \lambda_1 \) is principal and that \( \ker(\lambda_1 \circ \lambda_P) = P[2] \); therefore, \( \lambda_1 = \lambda_{\hat{P}} \), as claimed. \( \square \)

9.2. **Completion of the principally polarized \( P(S/R) \) case.** In this section we compute \( \ker H \) for the remaining principally polarized cases: \( T \neq \mathbb{P}^1 \) and either \( 2\gamma + 2\delta = 0 \) or \( 2 \).
Proof. Recall that \( F = p^*(P(R/T)[2]) \cap P(S/R)[2] \) holds under our hypothesis, from Proposition 9.12. If \( 2 \gamma + 2 \delta = 2 \), we know that \( p^* \) is injective and \( P(S/R)[2] = p^*(JR[2]) \); it follows that \( F = p^*(P(R/T)[2]) \), with \(|F| = 2^{4g-4+\alpha+2\beta+\delta} \). We may now apply Proposition 9.16 to conclude that if \( \alpha \) is positive then \( \ker H = F \) and if \( \alpha \) is zero then \( [F : \ker H] = 2 \). If \( 2 \gamma + 2 \delta = 0 \) we know that \( \eta_p \in P(R/T) \), from Claim 9.10 and also that \( P(S/R)[2] = p^*(\{0, \eta_p\}^\perp) \) and is isomorphic to \( \{0, \eta_p\}^\perp / \{0, \eta_p\} \).

Therefore we have to analyze two separate cases:

1. \( P(R/T)[2] \subseteq \{0, \eta_p\}^\perp \), or
2. \( P(R/T)[2] \not\subseteq \{0, \eta_p\}^\perp \), which means \( P(R/T)[2] \cap \{0, \eta_p\}^\perp \) is of \( \mathbb{Z}/2\mathbb{Z} \)-codimension 1 in \( P(R/T)[2] \).

In Case 1) we obtain \( F = p^*(P(R/T)[2]) \), which is isomorphic to \( P(R/T)[2] / \{0, \eta_p\} \), and therefore \(|F| = 2^{4g-5+\alpha+2\beta+\delta} \). In Case 2) \( F = p^*(P(R/T)[2] \cap \{0, \eta_p\}^\perp) \) is isomorphic to \( P(R/T)[2] \cap \{0, \eta_p\}^\perp / \{0, \eta_p\} \), and therefore \(|F| = 2^{4g-6+\alpha+2\beta+\delta} \). Now each of the two cases splits into two more, depending on whether \( \alpha > 0 \) (with \(|\ker H| = |F| \)) or \( \alpha = 0 \), with \(|\ker H| = |F|/2 \).

\[ \square \]

9.3. The general case. We may now assume that \( 2 \gamma + 2 \delta > 2 \); we will continue the description of \( F \) by constructing the elements of \( P(S/R)[2] \) coming from the ramification. Then we will decide which of those lie in \( F \).

First note that there are four types of points in \( T \) over which \( f: X \to T \) ramifies: the \( \alpha, \beta, \delta \) and \( \gamma \) points, corresponding to the images of simple, triple, total or \((2,2)\) type of ramification points, respectively. Their preimages via \( g: R \to T \) are the places over which \( p: S \to R \) may ramify. Careful consideration of the group actions involved shows
that the $\alpha$ and the $\beta$ points do not contribute and that the $\delta$ and $\gamma$ points do contribute, in the following way.

The $\gamma$ points: If $\gamma_t \in T$ is a $\gamma$ point we will denote

$$g^*(\gamma_t) = \gamma_1' + \gamma_2' + \gamma_3' , \quad \gamma_i' \in R$$

where

$$p^*(\gamma_1') = 2\gamma_1, p^*(\gamma_2') = 2\gamma_2$$

and $p^*(\gamma_3') = \gamma_5 + \gamma_6 , \quad \gamma_j \in S$.

With respect to $r : U \to R$, we choose $n \in JR$ such that

$$n \otimes^2 = \mathcal{O}_R(\gamma_1' - \gamma_2')$$

and $r^*(n) \in \ker(1 + \tau + \tau^2)$

Then

$$\mathcal{G} = \mathcal{O}_S(\gamma_2 - \gamma_1) \otimes p^*(n)$$

is in $P(S/R)[2]$ and we also have

$$(1 + \tau + \tau^2)\ell^*\pi^*(\mathcal{G}) = 0;$$

that is, we have constructed an element of $F$.

Therefore, if we enumerate the $\gamma$ points of $T$ as $\gamma_1', \ldots, \gamma_\gamma'$, we have that the corresponding points $\gamma_1^1, \gamma_2^1, \ldots, \gamma_1^\gamma, \gamma_2^\gamma$ in $S$ are ramification points of $p : S \to R$, and as above we construct, for each $i$ in $\{1, \ldots, \gamma\}$, elements in $F$ given as follows

$$\mathcal{G}_{2i-1} = \mathcal{O}_S(\gamma_2^i - \gamma_1^i) \otimes p^*(n_i)$$

with $n_i \in JR$, $n_i \otimes^2 = \mathcal{O}_R(\gamma_1^i' - \gamma_2^i')$ and $(1 + \tau + \tau^2)r^*(n_i) = 0$.

Similarly, we construct elements of $P(S/R)[2]$ as follows.

$$\mathcal{G}_{2i} = \mathcal{O}_S(\gamma_2^{i+1} - \gamma_2^i) \otimes p^*(m_i)$$

with $m_i \in JR$, $m_i \otimes^2 = \mathcal{O}_R(\gamma_2^i' - \gamma_1^i)$.

The $\delta$ points:

If $\delta_t \in T$ is a $\delta$ point we will denote

$$g^*(\delta_t) = \delta_1' + 2\delta_2' , \quad \delta_i' \in R$$

and

$$p^*(\delta_1') = 2\delta_1, p^*(\delta_2') = 2\delta_2 , \quad \delta_j \in S.$$  

Next choose $n \in JR$ such that

$$n \otimes^2 = \mathcal{O}_R(\delta_2' - \delta_1')$$

with $r^*(n) \in \ker(1 + \tau + \tau^2)$.

If we define $\mathcal{D}$ as follows

$$\mathcal{D} = \mathcal{O}_S(\delta_1 - \delta_2) \otimes p^*(n)$$

then it is in $P(S/R)[2]$ and we also have

$$(1 + \tau + \tau^2)\ell^*\pi^*(\mathcal{D}) = 0;$$
that is, we have constructed an element of \( F \).

Therefore, if we enumerate the \( \delta \) points of \( T \) as \( \delta_1, \ldots, \delta_\ell \), we have that the corresponding points \( \delta_1, \delta_{23}, \ldots, \delta_1, \delta_{23} \) in \( S \) are ramification points of \( p : S \to R \), and as above we construct, for each \( i \) in \( \{1, \ldots, \delta\} \), elements in \( F \) given as follows

\[
D_{2i-1} = \mathcal{O}_S(\delta_1^i - \delta_{23}^i) \otimes p^*(n_i)
\]

with \( n_i \in JR, n_i^{\otimes 2} = \mathcal{O}_R(\delta_1^i - \delta_{23}^i) \) and \((1 + \tau + \tau^2)r^*(n_i) = 0\).

Similarly, we construct elements of \( P(S/R)[2] \) as follows.

\[
D_{2i} = \mathcal{O}_S(\delta_{23}^{i+1} - \delta_1^i) \otimes p^*(m_i)
\]

with \( m_i \in JR, m_i^{\otimes 2} = \mathcal{O}_R(\delta_{23}^i - \delta_{23}^i) \).

Finally, if \( \gamma \delta > 0 \) we consider one more sheaf which links both cases:

\[
\mathcal{L} = \mathcal{O}_S(\delta_1^1 - \delta_2^\delta) \otimes p^*(m)
\]

with \( m \in JR \) and

\[
m^{\otimes 2} = \mathcal{O}_R(\delta_2^{\gamma} - \delta_1^{\gamma'}). \]

Now we can apply Corollary \( 4.3 \) to give a description of \( P(S/R)[2] \) in this case: \( 2\gamma + 2\delta > 2 \).

We are now ready to describe \( F \) for the case \( 2\gamma + 2\delta > 2 \).
Proposition 9.17. With the above notation, \( F \) is described as follows.

\[
F = \begin{cases}
p^*(P(R/T)[2]) \oplus \bigoplus_{j=1}^{\gamma} G_{2j-1}Z/2Z \oplus \bigoplus_{j=1}^{\delta-1} D_{2j-1}Z/2Z & \text{\( \delta \gamma > 0 \) and \( 2\gamma + 2\delta > 2 \)} \\
p^*(P(R/T)[2]) \oplus \bigoplus_{j=1}^{\gamma-1} G_{2j-1}Z/2Z & \delta = 0 \text{ and } \gamma \geq 2 \\
p^*(P(R/T)[2]) \oplus \bigoplus_{j=1}^{\delta-1} D_{2j-1}Z/2Z & \delta \geq 2 \text{ and } \gamma = 0
\end{cases}
\]

Proof. Since we are assuming \( 2\gamma + 2\delta > 2 \), \( p^*: JR \to JS \) is injective; hence \( p^*(P(R/T)[2]) \subseteq P(S/R)[2] \) and therefore the factor of \( F \) not coming from the ramification is \( p^*(P(R/T)[2]) \).

For \( \delta \gamma > 0 \), let \( F = \{G_{2j-1}, D_{2i-1}\} \) with \( 1 \leq j \leq \gamma \) and \( 1 \leq i \leq \delta \); if \( \gamma \geq 2 \) and \( \delta = 0 \), let \( F = \{G_{2j-1}\} \) with \( 1 \leq j \leq \gamma \); and if \( \delta \geq 2 \) and \( \gamma = 0 \), let \( F = \{D_{2i-1}\} \) with \( 1 \leq i \leq \delta \).

Note that, in each case, the elements of the collection \( F \) span the sheaves \( S \) which come from the ramification and such that \( (1 + \tau + \tau^2)\ell^*\pi^*(S) = 0 \); i.e., those elements of \( F \) coming from the ramification.

Also, there is exactly one relation among them (c.f. Remark 4.4). \( \Box \)

In this way we have obtained that \( |F| = 2^{2(g_R-g_T)+\gamma+\delta-1} = 2^{4g-5+\alpha+2\beta+\gamma+2\delta} \) whenever \( 2\gamma + 2\delta > 2 \).

Now we can compute \( |\ker H| \) for this case.

Proposition 9.18. If \( 2\gamma + 2\delta > 2 \), then the cardinality of \( \ker H \) is given as follows.

\[
|\ker H| = \begin{cases}
2^{4g-5+\alpha+2\beta+\gamma+2\delta}, & \text{if } 2\gamma + 2\delta > 2 \text{ and } \alpha > 0; \\
2^{4g-6+\alpha+2\beta+\gamma+2\delta}, & \text{if } 2\gamma + 2\delta > 2 \text{ and } \alpha = 0.
\end{cases}
\]

Proof. We know from Proposition 9.8 that if \( \alpha \) is positive, then \( |\ker H| = |F| \) and that if \( \alpha = 0 \), then \( |\ker H| = |F|/2 \). \( \Box \)

We have thus completed the proof of \( v) \) in Theorem 9.1.

As for \( vi) \), let us fix the notation: \( V = V_3 \), \( R = R_3 \), \( Y = Y_4 \) and \( X = X_4 \) and let

\[ T_0 JW = U \oplus U' \oplus V_2 \oplus V_3 \oplus V_3' \]
denote the isotypical decomposition of the tangent space to $JW$ at the origin, where $U = U^{n_0}$, $U' = U^{n_1}$, $V_2 = V_2^{n_2}$, $V_3 = V_3^{n_3}$, $V_3' = V_3^{n_4}$ with $U, U', V_2, V_3, V_3' = V_3 \otimes U'$ the complex irreducible representations of $S_4$ of respective degrees 1, 1, 2, 3 and 3.

Following [SA], we compare actions to obtain the isogeny, as follows. A short computation shows that

\[
(d(b \circ \nu)^*)_0(T_0P(V/R)) = V_3'^{(1,3)(2,4)},
\]
\[
(d(\psi \circ h)^*)_0(T_0P(\Delta/T)) = U'
\]
\[
(d(\psi^*)_0(T_0P(Y/X)) = U' \oplus V_3'^{(1,2,3)}).
\]

Therefore it follows from the second and third equalities that

\[
T_0P(Y/X) = (dh^*)_0(T_0P(\Delta/T)) \oplus (d\psi^*_0)^{-1}V_3'^{(1,2,3)}
\]

But we can also prove that

\[
(1 + \tau + \tau^2)V_3'^{(1,3)(2,4)} = V_3'^{(1,2,3)},
\]

hence

\[
(1 + \tau + \tau^2)(d(b \circ \nu)^*)_0(T_0P(V/R)) = (1 + \tau + \tau^2)V_3'^{(1,3)(2,4)} = V_3'^{(1,2,3)}
\]

If we now observe that on $JW$ we have $1 + \tau + \tau^2 = \psi^* \circ Nm \psi$, we obtain

\[
T_0P(Y/X) = (dh^*)_0(T_0P(\Delta/T)) \oplus d(Nm \circ (b \circ \nu)^*)_0(T_0P(V/R))
\]

proving vi).

As for vii), let us fix $Z = Z_{13}$, $S = S_3$, $R = R_3$ and $X = X_4$. Since $|P(Z/S) \cap \pi^*(JS)| < \infty$ then

\[
|P(Z/S) \cap \pi^* \circ p^*(P(R/T))| < \infty.
\]

On the other hand, some computations show that

\[
T_0(\ell^* \circ \pi^* \circ p^*P(R/T)) = V_2^{(1,3)}
\]
\[
T_0(\ell^* \circ u^*JX) = V_3^{(1,2),(1,3)} \oplus U
\]

and

\[
T_0\ell^*JZ = V_3^{(1,3)} \cap T_0(\ell^*P(Z/X)) \oplus V_3^{(1,3)}
\]
\[
\oplus V_2^{(1,3)} \oplus V_3^{(1,2),(1,3)} \oplus U.
\]

These equalities imply that $\pi^* \circ p^*P(R/T) \subseteq P(Z/X)$. But

\[
\dim P(R/T) + \dim P(Z/S) = \dim P(Z/X)
\]

which together with (9.37) complete the proof of vii) and of Theorem 9.1. \hfill \Box
10. Other applications

10.1. Examples. Throughout the paper we did put some obvious restrictions to the ramification data in some formulae.

Here we will actually construct curves with given ramification data and given $G$–action, where $G$ is one of the groups associated to non-Galois fourfold covers, as considered in this paper: $S_4$, $A_4$ or $D_4$.

For this we recall the general construction: consider an $n$–fold cover between complex curves

$$f : X \to T.$$ 

If we denote by $B(f) = \{P_1, \ldots, P_\omega\} \subseteq T$ the branch locus of $f$, we have an induced homomorphism

$$f^\# : \Pi_1(T - B(f)) \to S_n$$

By the Monodromy theorem and the Riemann extension theorem, we know that the covers $f$ (up to isomorphism) are classified by the homomorphisms $f^\#$ with transitive image (up to inner automorphisms).

Moreover, one knows that $f^\#(S_{n-1}) \approx \Pi_1(X - f^{-1}(B(f)))$ and that $\ker f^\# \approx \Pi_1(W - \gamma^{-1}(B(f)))$, where $\gamma : W \to T$ is the corresponding Galois extension of $f$, with group $G = \text{Im} f^\# \subseteq S_n$.

Also recall that

$$\Pi_1 = \Pi_1(T - B(f)) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \sigma_1, \ldots, \sigma_\omega :$$

$$\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} = \sigma_1 \ldots \sigma_\omega \rangle$$

where $g =$ genus of $T$, $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are canonical generators for $\Pi_1(T)$ and each $\sigma_j$ is represented by a trajectory going from the base point to near $P_j$, around it once in the appropriate direction, and back.

So, to construct $W$ with $G = S_4$ action and given values of $\alpha$, $\beta$, $\gamma$ and $\delta$, where $\omega = \alpha + \beta + \gamma + \delta$, is equivalent to the construction of a four-fold cover of curves $f : X \to T$ with those ramification values, and therefore also equivalent to the construction of a surjective homomorphism

$$f^\# : \Pi_1 \to S_4$$

such that

\begin{align*}
\begin{cases}
 f^\#(\sigma_1), \ldots, f^\#(\sigma_\alpha) & \text{are transpositions;} \\
 f^\#(\sigma_{\alpha+1}), \ldots, f^\#(\sigma_{\alpha+\beta}) & \text{are three-cycles;} \\
 f^\#(\sigma_{\alpha+\beta+1}), \ldots, f^\#(\sigma_{\alpha+\beta+\gamma}) & \text{are products of two disjoint transpositions;} \\
 f^\#(\sigma_{\alpha+\beta+\gamma+1}), \ldots, f^\#(\sigma_{\alpha+\beta+\gamma+\delta}) & \text{are four-cycles;}
\end{cases}
\end{align*}
For the other cases $G = A_4$ or $G = D_4$ the condition that $f^#$ be surjective changes to $\text{Im } f^# = G$ is a transitive group of $S_4$, whereas \((10.38)\) stays the same.

10.2. Jacobians of curves isogenous to a product of Jacobians. 

The equivariant isogeny $\phi_{S_4}$ of Theorem 9.1 induces an isogeny

$$\phi : JU \times 3P(S/R) \times 3P(V/R) \rightarrow JW$$

Hence we obtain curves $W$ with $S_4$ action whose Jacobian $JW$ is isogenous to a product of Jacobians if $g_R = 0$.

Since $g_R = 3g_T - 2 + \frac{\alpha + \delta}{2} + \beta$, this is equivalent to $g_T = 0$ and $\alpha + \delta + 2\beta = 4$.

We now describe the cases for which such covers actually exist and the dimension of the corresponding moduli.

Recall from the previous section that we are looking for surjective homomorphisms

\[(10.39) \quad f^# : \Pi_1 = \langle \sigma_1, \ldots, \sigma_{\omega} : \sigma_1 \ldots \sigma_{\omega} = 1 \rangle \rightarrow S_4 \]

that satisfy \((10.38)\) with $\alpha + \delta + 2\beta = 4$ and such that

\[(10.40) \quad f^#(\sigma_1) \cdots f^#(\sigma_{\alpha+\beta+\gamma+\delta}) = 1\]

Many a priori possibilities for $\alpha$, $\beta$, $\gamma$ and $\delta$ are excluded by our conditions; for instance we already know that $g_T = 0$ implies $\alpha + \delta > 0$ and even.

A final observation before we actually give all possible cases is that under our conditions we obtain $g_U = g_\Delta$, hence in fact we are considering the isogeny of Theorem 9.1 ii) in the special cases

$$\phi_{S_4} : J\Delta \times 3JS \times 3JV \rightarrow JW$$

Let us observe that when $T = P^1$ then $P(S/R)$ is isogenous to $JX$, so apparently we only need to impose the condition that $P(V/R)$ be isogenous to a Jacobian. The only way we know to do this at the moment is that $\dim P(V/R) = 1$, but this is equivalent to

$$\alpha + \beta + \gamma + \delta = 4$$

so all possible cases are included in our next result.

**Theorem 10.1.** The Jacobian of a curve $W$ with $S_4$ action is isogenous to a product of Jacobians, via the isogeny $\phi_{S_4}$ of Theorem 9.1 ii) in the form

$$J\Delta \times 3JS \times 3JV \rightarrow JW$$

if the ramification data satisfies the following.
Furthermore, in the following table we list the genera of the corresponding curves, the degree of the isogeny $\phi_{S_4}$ and the number of respective moduli.

| Case | $g_{\Delta}$ | $g_\Delta$ | $g_\mathcal{V}$ | $g_W$ | $\deg \phi$ | moduli |
|------|--------------|-------------|---------------|-------|-------------|--------|
| $I$  | 1            | $\gamma - 1$ | $\gamma + 1$ | $6\gamma + 1$ | $2^{12\gamma + 23}$ | $\gamma + 1$ |
| $II$ | 0            | $\gamma - 1$ | $\gamma$     | $6\gamma - 3$ | $2^{12\gamma - 6}$ | $\gamma$ |
| $III$| 1            | $\gamma + 1$ | $\gamma + 1$ | $6\gamma + 7$ | $2^{12\gamma + 143}$ | $\gamma + 1$ |
| $IV$ | 1            | $\gamma + 2$ | $\gamma + 1$ | $6\gamma + 10$ | $2^{12\gamma + 203}$ | $\gamma + 1$ |
| $V$  | 0            | $\gamma$     | $\gamma$     | $6\gamma$     | $2^{12\gamma}$     | $\gamma$ |
| $VI$ | 1            | $\gamma + 3$ | $\gamma + 1$ | $6\gamma + 13$ | $2^{12\gamma + 263}$ | $\gamma + 1$ |
| $VII$| 0            | $\gamma + 1$ | $\gamma$     | $6\gamma + 3$ | $2^{12\gamma + 6}$ | $\gamma$ |
| $VIII$| 1            | $\gamma$     | $\gamma + 1$ | $6\gamma + 4$ | $2^{12\gamma + 83}$ | $\gamma + 1$ |

**Proof.** The proof of this result is a direct application of Theorem 0.1 but we must show that the given covers exist and why other cases are excluded.

In particular, since $\alpha + \delta$ must be positive and even, it is either 4 or 2, in which cases $\beta$ must be 0 or 1 respectively. From here we obtain eight cases for the values of $\alpha$, $\beta$, and $\delta$ as in the first table; the next step is to show that in the first two cases the value $\gamma = 0$ is excluded and that all the other possibilities actually exist.

This is done by either constructing surjective homomorphisms

$$f^\#: \Pi_1 = \langle \sigma_1, \ldots, \sigma_\omega : \sigma_1 \ldots \sigma_\omega = 1 \rangle \to S_4$$
as in (10.39) with \( \omega = \alpha + \beta + \gamma + \delta \), where \( \alpha + 2\beta + \delta = 4 \), and satisfying (10.38) and (10.40), or showing that they cannot exist, depending on the values of the first table.

First of all it is clear that \( S_4 \) cannot be generated by either 4 transpositions with trivial product nor by two transpositions with product a 3-cycle; therefore, if \( \alpha = 4 \) and \( \beta = \delta = 0 \), or if \( \alpha = 2, \beta = 1 \) and \( \delta = 0 \), then \( \gamma > 0 \).

The proof is completed by writing down specific homomorphisms for the other cases.

We will illustrate with a couple of examples, as follows.

Case I: \( \alpha = 4, \beta = \delta = 0 \) and \( \gamma \geq 1 \). Then \( f^\# \) is given by

\[
\begin{align*}
\sigma_1 &\rightarrow (1 \ 2) \\
\sigma_2 &\rightarrow (2 \ 3) \\
\sigma_3 &\rightarrow (3 \ 4) \\
\sigma_4 &\rightarrow (1 \ 3)
\end{align*}
\]

and, for \( \gamma \equiv 1 \ (2) \)

\[
\begin{align*}
\sigma_5 &\rightarrow (1 \ 4)(2 \ 3) \\
\sigma_6, \ldots, \sigma_{4+\gamma} &\rightarrow (1 \ 2)(3 \ 4), \text{ only if } \gamma > 1
\end{align*}
\]

and, for \( \gamma \equiv 0 \ (2) \)

\[
\begin{align*}
\sigma_5 &\rightarrow (1 \ 2)(3 \ 4) \\
\sigma_6 &\rightarrow (1 \ 3)(2 \ 4) \\
\sigma_7, \ldots, \sigma_{4+\gamma} &\rightarrow (1 \ 2)(3 \ 4), \text{ only if } \gamma > 2
\end{align*}
\]

\( \square \)

10.3. Rigid Jacobians with \( S_4 \) actions. This time we have to look for surjective homomorphisms as follows.

\[
(10.41) \quad f^\# : \Pi_1(\mathbb{P}^1 - \{P_1, P_2, P_3\}) \rightarrow S_4
\]

that satisfy (10.38) with \( \alpha + \beta + \gamma + \delta = 3 \) and such that

\[
(10.42) \quad f^\#(\sigma_1) \cdots f^\#(\sigma_{\alpha+\beta+\gamma+\delta}) = 1
\]

Again \( g = g_T = 0 \) implies \( \alpha + \delta \) even and positive, and hence \( \alpha + \delta = 2 \); therefore \( \beta + \gamma = 1 \).

One verifies then that the only cases that do appear are the following two special cases from the previous section, the other possibilities not being realizable.

Case V with \( \gamma = 0 \): i.e., \( \alpha = \beta = \delta = 1 \).
In this case $g_W = 0$; that is we obtain a rational function

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1$$

of degree 24, which corresponds to the quotient map by the action of $S_4$ (see [K]).

Case VII with $\gamma = 0$: i.e., $\alpha = 0, \beta = 1, \delta = 2$.

In this case $g_W = 3$ and $S = E$ is an elliptic curve with an isogeny

$$E \times E \times E \to JW$$

of degree $2^6$.

10.4. One dimensional families of curves with $S_4$ action. Here we have to look for surjective homomorphisms

$$f^\# : \Pi_1 \to S_4$$

where either

(I) $\Pi_1 = \Pi_1(\mathbb{P}^1 - \{P_1, P_2, P_3, P_4\})$

respectively

(II) $\Pi_1 = \Pi_1(E - \{P\})$

where $E$ is an elliptic curve, and such that $f^\#$ satisfies (10.38) with

(I) $\alpha + \beta + \gamma + \delta = 4$ with $f^\#(\sigma_1) \cdots f^\#(\sigma_4) = 1$

respectively

(II) $\alpha + \beta + \gamma + \delta = 1$ with $f^\#(\alpha_1)f^\#(\beta_1)f^\#(\alpha_1^{-1})f^\#(\beta_1^{-1}) = f^\#(\sigma_1)$

The next result gives all possible cases.
Theorem 10.2. The one–parameter families of curves $W$ with $S_4$ action correspond exactly to those in the following table, where $g_T = 0$ except in the last case, where $g_T = 1$.

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
Case & $\alpha$ & $\beta$ & $\gamma$ & $\delta$ & $g_R$ & $g_S$ & $g_V$ & $g_X$ & $g_W$ \\
\hline
I & 3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 4 \\
II & 2 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 5 \\
III & 2 & 0 & 0 & 2 & 0 & 1 & 1 & 1 & 7 \\
IV & 2 & 1 & 1 & 0 & 0 & 1 & 0 & 3 & \\
V & 1 & 0 & 0 & 3 & 0 & 2 & 1 & 2 & 10 \\
VI & 1 & 2 & 0 & 1 & 2 & 2 & 1 & 8 & \\
VII & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 6 & \\
VIII & 0 & 2 & 0 & 2 & 1 & 3 & 2 & 2 & 11 \\
IX & 0 & 1 & 1 & 2 & 0 & 2 & 1 & 2 & 9 \\
X & 0 & 0 & 0 & 4 & 0 & 2 & 1 & 3 & 13 \\
XI & 0 & 1 & 0 & 0 & 2 & 3 & 3 & 2 & 9 \\
\hline
\end{tabular}
\end{center}

Furthermore, in each case we have respective isogenies to $JW$ as follows.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
Case & Isogeny & Degree of Isogeny \\
\hline
I & $3E_0 \times E_1$ & $2^3$ \\
II & $3E_0 \times 2E_4$ & $2^{16}$ \\
III & $3E_0 \times 3E_0 \times E_1$ & $2^{14}$ \\
IV & $3E_0$ & $2^6$ \\
V & $3JS \times 3E_0 \times E_1$ & $2^{22}$ \\
VI & $3E_0 \times 3E_3 \times 2E_4$ & $2^{12}$ \\
VII & $3E_0 \times 3E_0$ & \\
VIII & $3JS \times 3E_0 \times 2E_4$ & \\
IX & $3JS \times 3E_0$ & \\
X & $E \times 2E_0 \times 3E_5 \times 3E_3$ & $2^{25} \cdot 3^4$ \\
XI & with $E_0 = V$, $E_1 = \Delta$, $E_2 = S$, $E_3 = P(V/R)$, $E_4 = R$, $E_5 = P(S/R)$, $E_6 = P(R/T = E)$ & \\
\hline
\end{tabular}
\end{center}

10.5. Prym varieties of genus seven double covers of genus three curves, isogenous to a product of elliptic curves. In [B-C-V] it is shown that the Prym varieties of genus 7 double covers of genus three curves, branched at 4 points, are dense in the the moduli space of abelian fourfolds of type (1, 2, 2, 2).

Here we describe a one parameter family of such Prym varieties which are, moreover, isogenous to the product of elliptic curves.

Our example corresponds to Case III of the previous Section, which can also be described as follows.

Let $E$ be an elliptic curve and consider $E : \dashrightarrow \mathbb{P}^3$ its projective normal embedding as a degree four space curve. Take two hyper-osculating points
$P_1$ and $P_2$ of $E$; i.e., there exist planes $H_1$ and $H_2$ in $\mathbb{P}^3$ such that $H_i \cdot E = 4P_i$ for $i = 1, 2$.

Now consider the degree four meromorphic function $f : E \to \mathbb{P}^1$ obtained by projection from the line $L = H_1 \cap H_2$; since the construction depends upon one parameter, we have that the ramification data for $f$ is given by $\alpha = \delta = 2$ and $\beta = \gamma = 0$ (the construction actually depends only on the corresponding cyclic subgroup of order 4 of $E[4]$).

If we denote by $V = E_0$, $\Delta = E_1$, $S = E_2$ and $U = E_7$ the corresponding elliptic curves in this case and with a superscript the genus of the other curve, we obtain the following version of Diagram (9.29).

By the general trigonal construction we know that

$$\ker(Nm \circ \pi^* : E_2 \to E) = E_2[2]$$

hence $E_2$ and $E$ are isomorphic.

From the theory of the Klein group action we know that the isogeny

$$a^* + b^* : E_2 \times E_0 \to P(C/E_7)$$

is of degree $2^2$ (Case IV in the Appendix).

With the notation of Theorem 8.1 and again from a Klein action we have that the isogeny

$$\nu_2^* + \nu_3^* : P(C_2/U) \times P(C_3/U) \to P(W/C_4)$$

is of degree $2^4$.

Therefore we have an isogeny

$$E_2 \times E_0 \times E_2 \times E_0 \to P(W/C)$$

of degree $2^6$. 
In this section we complete the proof of Theorem 6.3 by computing the following quantities for \( \{j, k, l\} = \{\sigma, \tau, \tau\} \) and \( P_k = P(X_k/T) \).

\[
|\ker(\phi_j)| = \deg a^*_k|_{P_k} \cdot \deg a^*_l|_{P_l} |a^*_k(P_k[2]) \cap a^*_l(P_l[2])|
\]

and

\[
|\ker(ps_j)| = |P_j[2]| \frac{|b^*_j JT \cap \ker a^*_j|}{|\ker a^*_j|}
\]

where

\[
\phi_j : P_k \times P_l \to P(X/X_j) , \quad \phi_j(x_1, x_2) = a^*_k(x_1) + a^*_l(x_2)
\]

and

\[
ps_j : P_j \times P(X/X_j) \to P(X/T) , \quad ps_j(y, x) = a^*_j(y) + x.
\]

We will compute by analyzing the different possible cases for the ramifications of the covers appearing in the following Diagram.

(10.45)

The possibilities are as follows. **Case I**: All covers in Diagram (10.45) are unramified: \( r = s = t = 0 \). **Case II**: Exactly two of the top covers in Diagram (10.45) are unramified. **Case III**: Exactly two of the top covers in Diagram (10.45) are ramified. **Case IV**: All covers in Diagram (10.45) are ramified: \( rst \neq 0 \).

**Case I :** \( r = s = t = 0 \). In this case, all induced morphisms between corresponding Jacobians are non-injective.

Let \( H_j = \ker b^*_j = \{0, \eta_{b_j}\} \subseteq JT[2] \). Then \( H_l = \ker b^*_l = \{0, \eta_{b_l} + \eta_{b_k}\} \) and

\[
\ker \gamma^* = H_j + H_k = \{0, \eta_{b_\sigma}, \eta_{b_\tau}, \eta_{b_\sigma} + \eta_{b_\tau}\}.
\]

By [M2] we have, for each \( j \), induced isomorphisms \( H_j^+ / H_j \to b^*_j(H_j^+) = P_j[2] \).

Therefore

\[
|P_\sigma[2]| = |P_\tau[2]| = |P_{\sigma \tau}[2]| = 2^{2q_r - 2} \quad \text{if } r = s = t = 0.
\]

We now distinguish two subcases, according to the value of the Weil pairing \( (\eta_{b_\sigma}, \eta_{b_\tau}) \) of \( \eta_{b_\sigma} \) and \( \eta_{b_\tau} \).
Case a): Assume \((\eta_{b_r}, \eta_{b_r}) = 0\); or, equivalently, \(\ker \gamma^* \subseteq H_j^+ \cap H_k^+\) for some pair (equivalently, each pair) \(j, k\).

Then \(\ker a^*_l = b^*_l (\ker \gamma^*) \subseteq P_l[2]\) for each \(l\) and it follows that for each \(l \in \{\sigma \tau, \sigma, \tau\}\) we have
\[
\deg a^*_l |_{P_l} = 2 , \quad \text{if } r = s = t = 0 \text{ and } (\eta_{b_r}, \eta_{b_r}) = 0 .
\]

On the other hand,
\[
|a^*_j(P_j[2]) \cap a^*_k(P_k[2])| = \left| \frac{H_j^+}{H_j + H_k} \cap \frac{H_k^+}{H_j + H_k} \right|
\]
but our assumption for this case (\(\ker \gamma^* = H_j + H_k \subseteq H_j^+ \cap H_k^+\)) implies that
\[
\frac{H_j^+}{H_j + H_k} \cap \frac{H_k^+}{H_j + H_k} = \frac{(H_j + H_k)^\perp}{H_j + H_k}
\]
from where it follows that for each pair \(j, k\) we have
\[
|a^*_jP_j[2] \cap a^*_kP_k[2]| = |(H_j + H_k)^\perp / (H_j + H_k)| = 2^{2g_r - 4} .
\]

Hence we have obtained that
\[
|\ker(\phi_j)| = \deg a^*_l |_{P_k} \cdot \deg a^*_l |_{P_l} \cdot |a^*_j(P_k[2]) \cap a^*_k(P_l[2])| = 2^{2g_r - 2}
\]
in this case.

We also have \(\ker a^*_j = b^*_j (\ker \gamma^*) \subseteq b^*_jJT\) and therefore
\[
\ker a^*_j \cap b^*_jJT = \ker a^*_j \text{ for each } j .
\]

It follows that
\[
|\ker(\psi_j)| = |P_j[2]| \frac{|b^*_jJT \cap \ker a^*_j|}{|\ker a^*_j|} = |P_l[2]| = 2^{2g_r - 2} .
\]

Case b): Assume \((\eta_{b_r}, \eta_{b_r}) \neq 0\); or, equivalently, \((H_j + H_k) \cap H_j^+ = H_j\) for some pair (equivalently, any pair) \(j, k\).

Then it follows that \(a^*_l |_{P_l}\) is injective for every \(l\).

Now \(H_j^+\) is of index two in \(JT[2]\) and hence
\[
\gamma^*(H_j^+) = \gamma^*JT[2] ;
\]
therefore,
\[
|a^*_jP_l[2] \cap a^*_kP_k[2]| = |\gamma^*(H_j^+) \cap \gamma^*(H_k^+)| = |\gamma^*JT[2]| = 2^{2g_r - 2} .
\]

So again we obtain
\[
|\ker(\phi_j)| = 2^{2g_r - 2} \text{ if } r = s = t = 0
\]
and
\[
|\ker(\psi_j)| = 2^{2g_r - 2} \text{ if } r = s = t = 0
\]
which complete the proof for the unramified case.
Case II: Without loss of generality, in this paragraph we assume \( r = t = 0 \) and \( s > 0 \), which imply \( s \) is even.

In this case \( a^*_\sigma, b^*_\tau \) and \( b^*_\sigma\tau \) are the only injective induced homomorphisms between Jacobians.

Let \( \ker \gamma^* = \ker b^*_\sigma \) = \( \{ 0, \eta_0 \} \subseteq JT[2] \). Then

\[
\ker a^*_j = \{ 0, b^*_j(\eta_0) \} \subseteq b^*_j(JT[2]) \subseteq P_j[2] \quad \text{for} \quad j \neq \sigma
\]

where the last inclusion follows from \( M2 \) since \( b^*_j \) is injective.

It follows that \( \deg a^*_j|_{P_j} = 2 \) for \( j \neq \sigma \).

It also follows that \( |\ker ps_j| = |P_j[2]| \) for \( j \neq \sigma \), whereas \( |\ker ps_\sigma| = |P_\sigma[2]| \) follows from the injectivity of \( a^*_\sigma \). Therefore in this case we have

\[
|\ker ps_k| = \begin{cases} 
2^{2gT-2+s} & \text{if } k \neq \sigma \text{ and } r = t = 0, s > 0; \\
2^{2gT-2} & \text{if } k = \sigma \text{ and } r = t = 0, s > 0 
\end{cases}
\]

Now from \( P_\sigma[2] = b^*_\sigma(\{ 0, \eta_0 \}^\perp) \) we obtain \( a^*_\sigma(P_\sigma[2]) = \gamma^*(\{ 0, \eta_0 \}^\perp) \subseteq \gamma^*(JT[2]) \).

But \( b^*_j(JT[2]) \subseteq P_j[2] \) for \( j \neq \sigma \), and therefore \( \gamma^*(JT[2]) \subseteq a^*_j(P_j[2]) \), from where \( a^*_\sigma(P_\sigma[2]) \cap a^*_j(P_j[2]) = a^*_\sigma(P_\sigma[2]) \) and \( |a^*_\sigma(P_\sigma[2]) \cap a^*_j(P_j[2])| = 2^{2gT-2} \).

In order to compute \( |a^*_\sigma(P_\sigma[2]) \cap a^*_j(P_j[2])| \) we use Proposition 4.1 as follows: let \( S_1, \ldots, S_s, \tau S_1, \ldots, \tau S_s \) denote the ramification points of \( a_\sigma \) in \( X \).

Then \( a_\tau(S_i) = a_\tau(\tau(S_i)) \) for \( i \in \{ 1, \ldots, s \} \) are the ramification points of \( b_\tau \) in \( X_\tau \) and \( a_{\sigma\tau}(S_i) = a_{\sigma\tau}(\sigma\tau(S_i)) \) for \( i \in \{ 1, \ldots, s \} \) are the ramification points of \( b_{\sigma\tau} \) in \( X_{\sigma\tau} \).

For \( i \in \{ 2, \ldots, s \} \) we let \( u_i \) in \( \text{Pic}^0(T) \) be such that

\[
u_i \otimes 2 = O_T(\gamma(S_1) - \gamma(S_i))
\]

and we define

\[
S_i^j = O_X_j(a_j(S_i) - a_j(S_1)) \otimes b^*_j(u_i) \quad \text{for} \quad j \in \{ \tau, \sigma\tau \} \quad \text{and} \quad 2 \leq i \leq s;
\]

then Proposition 4.1 implies that

\[
P_j[2] = b^*_j(JT[2]) \oplus_{i=2}^{s-1} S_i^j \otimes \mathbb{Z}/2\mathbb{Z}.
\]

But \( \ker a^*_j \subseteq b^*_j(JT[2]) \) and hence

\[
a^*_j(P_j[2]) = \gamma^*(JT[2]) \oplus_{i=2}^{s-1} a^*_j S_i^j \otimes \mathbb{Z}/2\mathbb{Z}.
\]

Now, since \( \sigma\tau(S_k) = \tau(S_k) \) for all \( k \), we obtain

\[
a^*_\tau(S_i^\tau) = O_X(S_i + \tau(S_i) - S_1 - \tau(S_1)) \otimes a^*_\tau b^*_\tau(u_i) \\
= O_X(S_i + \sigma\tau(S_i) - S_1 - \sigma\tau(S_1)) \otimes a^*_\sigma b^*_\sigma(u_i) \\
= a^*_\sigma(S_i^\sigma) \\
\]

and therefore

\[
|a^*_\tau P_\tau[2] \cap a^*_\sigma P_\sigma[2]| = |\gamma^*(JT[2]) \oplus_{i=2}^{s-1} a^*_\sigma S_i^\sigma \otimes \mathbb{Z}/2\mathbb{Z}| = 2^{2gT+s-3}.
\]
In this way we have obtained that
\[|\ker \phi_k| = \begin{cases} 
2^{2g_T-1} & \text{if } k \neq \sigma \text{ and } r = t = 0, s > 0; \\
2^{2g_T-1+s} & \text{if } k = \sigma \text{ and } r = t = 0, s > 0
\end{cases}\]
which concludes case II.

Case III: Without loss of generality, in this paragraph we assume \(s = 0\) and \(rt > 0\), which imply \(r\) and \(t\) are even.

In this case \(a_\sigma^*\) is the only non-injective induced homomorphism between Jacobians.

In particular \(\eta_\alpha \notin b_\sigma^*(JT)\). This proves that \(\ker ps_\sigma\) is a subgroup of index two of \(P_\sigma[2]\), and hence
\[|\ker ps_\sigma| = \begin{cases} 
2^{2g_T-3+r+t} & \text{if } k = \sigma \text{ and } rt > 0, s = 0; \\
2^{2g_T-2+r} & \text{if } k = \tau \text{ and } rt > 0, s = 0; \\
2^{2g_T-2+t} & \text{if } k = \sigma \tau \text{ and } rt > 0, s = 0
\end{cases}\]

In order to compute \(\deg a_\sigma^*|_{P_\sigma}\), we now show that \(\eta_\alpha \in P_\sigma[2]\): if \(\tilde{\tau} : X_\sigma \to X_\sigma\) denotes the involution induced by \(\tau : X \to X\), then \(a_\sigma^*(\tilde{\tau}(\eta_\alpha)) = \tau(a_\sigma^*(\eta_\alpha)) = 0\); hence \(\gamma^*(\Nm b_\sigma(\eta_\alpha)) = a_\sigma^*(\eta_\alpha + \tilde{\tau}\eta_\alpha) = 0\).

But \(\gamma^*\) is injective and therefore \(\Nm b_\sigma(\eta_\alpha) = 0\). Since \(\Nm b_\sigma\) and \(b_\sigma^*\) are dual morphisms (see \(\text{[2]}\) and since \(\ker b_\sigma^* = \{0\}\) has only one connected component, \(\ker \Nm b_\sigma\) must have only one connected component; i.e., \(\ker \Nm b_\sigma = P_\sigma\).

The claim follows, and we obtain \(\deg a_\sigma^*|_{P_\sigma} = 2\).

In order to compute \(|a_\sigma^*(P_\sigma[2]) \cap a_\sigma^*(P_\sigma[2])|\) we use Corollary \(\text{[3]}\) as follows:

Let \(\{a_\sigma(\sigma_1), \ldots, a_\sigma(\sigma_t)\}\), \(\{\sigma_1, \ldots, \sigma_t, \sigma_1, \ldots, \sigma_t\}\)

Then \(\{a_\sigma(\sigma_1), \ldots, a_\sigma(\sigma_t)\}\) and \(\{a_\sigma(\sigma_1), \ldots, a_\sigma(\sigma_t)\}\) denote the ramification points of \(a_\sigma\) and \(a_\sigma\) in \(X_\sigma\) respectively.

Then \(\{a_\sigma(\sigma_1), \ldots, a_\sigma(\sigma_t)\}\) are the ramification points of \(\sigma_1, \ldots, \sigma_t\) in \(X_\sigma\), \(\{a_\sigma(\sigma_1), \ldots, a_\sigma(\sigma_t)\}\) are the ramification points of \(b_\tau\) in \(X_\tau\)

For \(i \in \{2, \ldots, r + t\}\) we let \(\lambda_i \in \Pic^0(T)\) be such that
\[
(1) \quad \lambda_i^{\otimes 2} = O_T(\gamma(R_i) - \gamma(R_i+i)), \quad 1 \leq i \leq r + t - 1;
(2) \quad \lambda_i^{\otimes 2} = O_T(\gamma(R_i) - \gamma(T_i)), \quad i = r;
(3) \quad \lambda_i^{\otimes 2} = O_T(\gamma(T_i) - \gamma(T_i+i)), \quad 1 \leq i \leq t - 1.
\]

and we define
\[
(1) \quad F_i^r = O_{X_\sigma}(a_\sigma(R_{i+1}) - a_\sigma(R_i)) \otimes b_\sigma^*(\lambda_i) \in P_\sigma[2] \quad \text{and}
F_i^r = O_{X_\tau}(a_\sigma(R_{i+1}) - a_\sigma(R_i)) \otimes b_\sigma^*(\lambda_i) \in P_\sigma[2], \quad \text{for } 1 \leq i \leq r + t - 1;
(2) \quad F_i^r = O_{X_\sigma}(a_\sigma(T_i) - a_\sigma(R_i)) \otimes b_\sigma^*(\lambda_i) \in P_\sigma[2], \quad \text{for } i = r;
(3) \quad F_i^r = O_{X_\sigma}(a_\sigma(T_{i+1}) - a_\sigma(T_i)) \otimes b_\sigma^*(\lambda_{r+i}) \in P_\sigma[2] \quad \text{and}
F_i^r = O_{X_\tau}(a_\sigma(T_{i+1}) - a_\sigma(T_i)) \otimes b_\sigma^*(\lambda_{r+i}) \in P_\sigma[2], \quad \text{for } 1 \leq i \leq t - 1.
\]

Then Corollary \(\text{[4,3]}\) implies that
\[
P_\sigma[2] = b_\sigma^*(JT[2]) \oplus_{i=1}^{r+t-2} F_i^r \Z/2\Z,
\]
and we know that
\[ \eta \]

we obtain
\[ (10.48) \]

and
\[ (10.47) \]

Observe that \( a_1^* \) and \( a_2^* \) are injective, applying them to \( (10.47) \) and \( (10.48) \) respectively we obtain
\[ (10.49) \]

Since \( \ker a_\sigma^* = \{0, \eta_\alpha, \} \subseteq P_\sigma[2] \), in order to describe \( a_\sigma^*(P_\sigma[2]) \) we first have to express \( \eta_\alpha \) in terms of the given generators for \( P_\sigma[2] \) by Lemma 4.2 we know that
\[ \mathcal{F}_1^\sigma \otimes \ldots \otimes \mathcal{F}_{r-1}^\sigma = b_\sigma^* (m) \text{ for some } m \in JT[2] \]

and that
\[ \mathcal{F}_2^\sigma \otimes \ldots \otimes \mathcal{F}_{r-1}^\sigma \in P_\sigma[2] \text{ is not in } b_\sigma^* JT[2]. \]

This implies that
\[ \mathcal{F}_1^\sigma \otimes \ldots \otimes \mathcal{F}_{r-1}^\sigma \otimes b_\sigma^*(m^{-1}) \in P_\sigma[2] - \{0\} \]

but
\[
a_\sigma^* (\mathcal{F}_1^\sigma \otimes \ldots \otimes \mathcal{F}_{r-1}^\sigma \otimes b_\sigma^*(m^{-1})) = \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_{r-1} \otimes \gamma^*(m^{-1}) = a_\tau^* (\mathcal{F}_1^\tau \otimes \ldots \otimes \mathcal{F}_{r-1}^\tau \otimes b_\tau^*(m^{-1})) = a_\tau^* (0) = 0 \]

and therefore
\[ \eta_\alpha = \mathcal{F}_1^\sigma \otimes \ldots \otimes \mathcal{F}_{r-1}^\sigma \otimes b_\sigma^*(m^{-1}). \]

We may now apply \( a_\sigma^* \) to \( (10.46) \) to obtain
\[ (10.50) \]

Equality \( (10.51) \) shows that there is no dependence relation between \( \mathcal{F}_1, \ldots, \mathcal{F}_{r-2} \) and \( \mathcal{F}_{r+1}, \ldots, \mathcal{F}_{r+t-2} \) and comparing we obtain
\[ (10.51) \]
\[ a_\tau^* (P_\tau[2]) \cap \sigma_\tau^*(P_\sigma[2]) = \gamma^*(JT[2]) \]
\[ a_\tau^* (P_\tau[2]) \cap \sigma_\alpha^* (P_\alpha[2]) = \gamma^*(JT[2]) \oplus \mathcal{F}_i Z/2Z \]

Equality \( (10.51) \) shows that there is no dependence relation between \( \mathcal{F}_1, \ldots, \mathcal{F}_{r-2} \) and \( \mathcal{F}_{r+1}, \ldots, \mathcal{F}_{r+t-2} \) and comparing we obtain
\[ (10.51) \]
\[ a_\tau^* (P_\tau[2]) \cap \sigma_\tau^*(P_\sigma[2]) = \gamma^*(JT[2]) \]
\[ a_\tau^* (P_\tau[2]) \cap \sigma_\alpha^* (P_\alpha[2]) = \gamma^*(JT[2]) \oplus \mathcal{F}_i Z/2Z \]
and

\[ a_\sigma^*(P_\sigma[2]) \cap a_{\sigma \tau}^*(P_{\sigma \tau}[2]) = \gamma^*(JT[2]) \oplus \bigoplus_{i=1}^{t-2} F_{r+i} \mathbb{Z}/2\mathbb{Z}. \]

Thus we have proven

\[
| \ker \phi_k | = \begin{cases} 
2^{2g_t-1+t} & \text{if } k = \tau \text{ and } rt > 0, s = 0; \\
2^{2g_t-1+r} & \text{if } k = \sigma \tau \text{ and } rt > 0, s = 0; \\
2^{2g_t} & \text{if } k = \sigma \text{ and } rt > 0, s = 0
\end{cases}
\]

which concludes case III.

**Case IV:** All covers in Diagram 10.45 are ramified: \( rst \neq 0 \).

In this case all induced homomorphisms between Jacobians are injective and therefore

\[
| \ker ps_k | = |P_k[2]| = \begin{cases} 
2^{2g_t-2+s+r} & \text{if } k = \tau \text{ and } rst > 0; \\
2^{2g_t-2+s+t} & \text{if } k = \sigma \tau \text{ and } rst > 0; \\
2^{2g_t-2+t+r} & \text{if } k = \sigma \text{ and } rst > 0
\end{cases}
\]

We also have \( \deg a_j^*|_{P_j} = 1 \) for every \( j \); hence all that is left is to compute

\[
| \ker \phi_j | = |a_k^*(P_k[2]) \cap a_l^*(P_l[2])|.
\]

By the symmetries involved, it will be enough to compute one of them: we will compute

\[
| \ker \phi_{\sigma \tau} | = |a_\sigma^*(P_\sigma[2]) \cap a_{\tau}^*(P_{\tau}[2])|.
\]

The computation for this case is subdivided into two cases, according to whether \( r + t = 2 \) or \( r + t > 2 \).

**Case a):** \( r = t = 1 \). In this case \( P_\sigma[2] = b_\sigma^*(JT[2]) \) and therefore

\[ \gamma^*(JT[2]) = a_\sigma^*(P_\sigma[2]). \]

On the other hand, \( b_\tau^*(JT[2]) \subseteq P_\tau[2] \) and therefore \( a_\sigma^*(P_\sigma[2]) \subseteq a_\tau^*(P_{\tau}[2]) \). Thus

\[ a_\sigma^*(P_\sigma[2]) \cap a_\tau^*(P_{\tau}[2]) = \gamma^*(JT[2]) \]

and in this case we obtain

\[
| \ker \phi_{\sigma \tau} | = 2^{2g_t} \quad \text{if } r = t = 1, s > 0.
\]

**Case b):** \( r + t > 2 \). Without loss of generality, we will assume \( r > 1 \).

Let \( \{S_1, \ldots, S_s, \tau(S_1), \ldots, \tau(S_s)\}, \{T_1, \ldots, T_t, \sigma(T_1), \ldots, \sigma(T_t)\} \) and \( \{R_1, \ldots, R_r, \sigma(R_1), \ldots, \sigma(R_r)\} \) denote the ramification points of \( a_\sigma, a_\tau \) and \( a_{\sigma \tau} \) respectively.

Then the ramification points for \( b_\sigma, b_\tau \) and \( b_{\sigma \tau} \) are as follows.
If we choose $\mathcal{F}_i^\sigma \in P_\sigma[2]$ for $i \in \{1, \ldots, r + t - 1\}$ as in the proof of Case III, then (10.46) holds.

Furthermore, we choose $\mathcal{F}_j^\tau \in P_\tau[2]$ for $1 \leq j \leq r - 1$ and $\mathcal{F}_{r+j}^\sigma \in P_\sigma[2]$ for $1 \leq j \leq t - 1$ as in the proof of Case III.

Also, for $i \in \{1, \ldots, s - 1\},$ choose $m_i$ in $\text{Pic}^0(T)$ such that

$$m_i^{\sigma_2} = O_T(\gamma(S_i) - \gamma(S_{i+1}))$$

and define

$$G_i^\sigma = O_{X_\sigma}(a_{\sigma}(S_{i+1}) - a_\sigma(S_i)) \otimes b_{\sigma}(m_{r+i}) \in P_\sigma[2]$$

and

$$G_{r+i} = O_{X_\tau}(a_\tau(S_{i+1}) - a_\tau(S_i)) \otimes b_\tau(m_i) \in P_\tau[2].$$

With these generators and Corollary 4.3 we obtain the following descriptions.

\begin{align*}
(10.52) & \quad P_\tau[2] = b_\tau^*(JT[2]) \oplus_{i=1}^{r-1} \mathcal{F}_i^\tau \mathbb{Z}/2\mathbb{Z} \oplus_{i=1}^{s-1} G_i^\tau \mathbb{Z}/2\mathbb{Z}, \\
(10.53) & \quad P_\sigma[2] = b_{\sigma_2}^*(JT[2]) \oplus_{i=1}^{s-1} G_i^\sigma \mathbb{Z}/2\mathbb{Z} \oplus_{i=1}^{t-1} \mathcal{F}_{r+i}^\sigma \mathbb{Z}/2\mathbb{Z}.
\end{align*}

Note that $a_{\sigma}^*(G_i^\sigma) = a_\tau^*(G_{r+i})$ for $i \in \{1, \ldots, s - 1\};$ the common value will be denoted by $G_i.$ Applying $a_{\sigma}^*$, $a_\tau^*$ and $a_{\sigma_2}^*$ to (10.46), (10.52) and (10.53) respectively, we obtain the following.

\begin{align*}
(10.54) & \quad a_{\sigma}^*(P_\sigma[2]) = \gamma^*(JT[2]) \oplus_{i=1}^{r+t-1} \mathcal{F}_i \mathbb{Z}/2\mathbb{Z}, \\
(10.55) & \quad a_\tau^*(P_\tau[2]) = \gamma^*(JT[2]) \oplus_{i=1}^{r-1} \mathcal{F}_i \mathbb{Z}/2\mathbb{Z} \oplus_{i=1}^{s-1} G_i \mathbb{Z}/2\mathbb{Z}, \\
(10.56) & \quad a_{\sigma_2}^*(P_\sigma[2]) = \gamma^*(JT[2]) \oplus_{i=1}^{s-1} G_i \mathbb{Z}/2\mathbb{Z} \oplus_{i=1}^{t-1} \mathcal{F}_{r+i} \mathbb{Z}/2\mathbb{Z}.
\end{align*}

It follows from equalities (10.51), (10.55) and (10.56) that

\begin{align*}
a_{\sigma}^*P_\sigma[2] & \cap a_{\sigma}^*P_\tau[2] = \gamma^*JT[2] \oplus_{i=1}^{r-1} \mathcal{F}_i \mathbb{Z}/2\mathbb{Z}, \\
a_\tau^*P_\tau[2] & \cap a_{\sigma}^*P_\sigma[2] = \gamma^*JT[2] \oplus_{i=1}^{s-1} G_i \mathbb{Z}/2\mathbb{Z}, \\
a_{\sigma}^*P_\sigma[2] & \cap a_{\sigma_2}^*P_\sigma[2] = \gamma^*JT[2] \oplus_{i=1}^{t-1} \mathcal{F}_{r+i} \mathbb{Z}/2\mathbb{Z}.
\end{align*}
Therefore, we have proven that in this case we have
\[
\ker \phi_k = \begin{cases} 
2^{2g_T-1+t} & \text{if } k = \tau \text{ and } r > 1, st > 0; \\
2^{2g_T-1+r} & \text{if } k = \sigma \tau \text{ and } r > 1, st > 0; \\
2^{2g_T-1+s} & \text{if } k = \sigma \text{ and } r > 1, st > 0;
\end{cases}
\]
Thus the proof of Theorem 6.3 is now complete.

References

[B-C-V] F. Bardelli, C. Ciliberto and A. Verra, Curves of minimal genus on a general abelian variety Compositio Math. 96 (1995), no. 2, 115–147.

[D1] R. Donagi, The Fibers of the Prym Map, Curves, Jacobians, and abelian varieties (Amherst, MA, 1990), pp. 55–125. Contemp. Math., 136, Amer. Math. Soc., Providence, RI, 1992.

[D2] R. Donagi, Decomposition Of Spectral Covers, Journées de Géométrie Algébrique d’Orsay (Orsay, 1992). Astérisque 218 (1993), 145–175.

[G] E. Gómez-González, Prym varieties of curves with an automorphism of prime order, Aportaciones Mat. 13 (1998), 103–116.

[Ka] Spectral Curves, simple Lie Algebras, and Prym-Tjurin varieties, Theta functions – Bowdoin 1987, Part 1,627–645, Proc. Sympos. Pure Math. 49 Part 1, Amer. Math. Soc., 1989.

[K] F. Klein, Lectures on the Icosahedron. Dover Publications, New York, 1913.

[L-B] H. Lange and C. Birkenhake, Complex Abelian Varieties, Grundlehren der Mathematischen Wissenschaften 302. Springer-Verlag, Berlin, 1992.

[M] J-Y. Mérindol, Variétés de Prym d’un revêtement galoisien, J. Reine Angew. Math. 461 (1995), 49–61.

[M1] D. Mumford, Abelian Varieties, Tata Institute of Fundamental Research, Bombay: Oxford University Press, 1970.

[M2] D. Mumford, Prym Varieties I, Contributions to analysis (a collection of papers dedicated to Lipman Bers), pp. 325–350. Academic Press, New York, 1974.

[P] S. Pantazis, Prym Varieties and the Geodesic Flow on SO(n), Mathematische Annalen 273 (1986), 297–315.

[Rec1] S. Recillas, Jacobians of curves with $g_3^1$’s are the Prym’s of trigonal curves, Bol. Soc. Mat. Mexicana (2) 19 (1974), no. 1, 9–13. 19 (1974), 9–13.

[Rec2] S. Recillas, La Jacobiana de la extensión de Galois de una curva trigonal, Aportaciones Matemáticas de la Soc Mat. Mexicana 14 (1994), 159–167.

[Rec–Ro] S. Recillas and R. E. Rodríguez, Jacobians and Representations of $S_3$, Aportaciones Mat. Inv. 13 (1998), 117–140.

[R] J. Ries, The Prym variety for a cyclic unramified cover of a hyperelliptic Riemann surface, J. Reine Angew. Math. 340 (1983), 59–69.

[SA] A. Sánchez-Argáez, Acciones del grupo $A_5$ en variedades jacobianas, Aportaciones Mat. Com. 25 (1999), 99–108.

[W] W. Wirtinger, Untersuchungen über Theta Funktionen. Teubner, Berlin (1895).
Instituto de Matemáticas, UNAM Campus Morelia, Morelia, Mich. 58190, México
E-mail address: sevin@unam.mx

CIMAT, Callejón Jalisco s/n, Valenciana, Guanajuato, México
E-mail address: sevin@fractal.cimat.mx

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile
E-mail address: rubi@mat.puc.cl